Exact Quantization of Einstein-Rosen Waves Coupled to Massless Scalar Matter

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We show in this letter that gravity coupled to a massless scalar field with full cylindrical symmetry can be exactly quantized by an extension of the techniques used in the quantization of Einstein-Rosen waves. This system provides a useful testbed to discuss a number of issues in quantum general relativity such as the emergence of the classical metric, microcausality, and large quantum gravity effects. It may also provide an appropriate framework to study gravitational critical phenomena from a quantum point of view, issues related to black hole evaporation, and the consistent definition of test fields and particles in quantum gravity.

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Symmetry reductions of general relativity have been used as model systems to extract information about quantum gravity. They usually allow the discussion of specific problems without the difficulties present in the full theory. Some of the most popular choices in this regard (Bianchi models) have only a finite number of degrees of freedom and, hence, are not suitable to address some of the more nagging questions posed by the study of quantum gravity (diffeomorphism invariance or issues related to the presence of an infinite number of local degrees of freedom such as perturbative non-renormalizability). Fortunately there are other symmetry reductions that retain these features while still being exactly solvable both at the classical and quantum levels. Chief among them are the Einstein-Rosen waves [1,2], obtained by requiring that space-time metrics have two commuting, spacelike, and hypersurface orthogonal Killing vector fields (one translational and the other rotational). This model has been extensively studied in the past [3,4] and some intriguing results have been derived, in particular the appearance of unexpected large quantum gravity effects [5] and a detailed picture of the emergence of the causal structure of space-time in the classical limit [6,7]. An improvement that would increase the usefulness of this system as a toy model for quantum gravity would be the coupling of matter. The availability of a solvable model with matter would open up a host of interesting possibilities deserving a careful investigation. We show in this letter that such a model exists and discuss how it can be exactly quantized. Specifically we will consider here the quantization of Einstein-Rosen waves coupled to a cylindrically symmetric massless scalar field.

To our knowledge this system was first discussed from a classical point of view by Chandrasekhar [8] who showed that a full solution to the Einstein field equations can be found in this case. The specific form of this solution suggests that a Hamiltonian treatment of the system would lead to a description very similar to the one found by Ashtekar, Pierri, and Varadarajan [3,4] for pure gravity in the asymptotically flat case. This is a strong indication that the model is amenable to quantization by a suitable extension of known techniques. The main point of this letter is to show that this is indeed the case.

The possible applications of such a model are manifold and can be classified in several different categories. First of all there is the issue of extracting information about geometry in quantized gravity. We do not expect to find here the kind of precise geometric information offered by loop quantum gravity in the form of geometric observables such as areas or volumes. Nevertheless our approach gives some indications about the validity of a metric description in the realm of quantum gravity. This has already been considered for pure gravity by studying expectation values of metric components. Here we propose to follow a different philosophy; instead of obtaining some approximate semiclassical metric by taking expectation values of metric components. Here we propose to follow a different philosophy; instead of obtaining some approximate semiclassical metric by taking expectation values of metric components. Here we propose to follow a different philosophy; instead of obtaining some approximate semiclassical metric by taking expectation values of metric components. Here we propose to follow a different philosophy; instead of obtaining some approximate semiclassical metric by taking expectation values of metric components. Here we propose to follow a different philosophy; instead of obtaining some approximate semiclassical metric by taking expectation values of metric components.

A second set of questions that can possibly be addressed within this framework in the quantum regime are...
related to critical phenomena in gravitational collapse and problems in black hole physics\textsuperscript{1}. These issues have been recently considered by Wang in his study of critical collapse of a cylindrically symmetric scalar field in four dimensions.\textsuperscript{2} This system displays some rich and non-trivial behavior, in particular, the possibility of forming solutions with future, spacelike singularities by the collapse of massless scalar matter and the appearance of a critical metric separating solutions with different singular behavior. Notice that having an exact solution, backreaction effects are automatically taken into account without any approximation. Finally we want to point out other possible uses of this model such as the discussion of the validity of the usual perturbative schemes in quantum gravity, the development of new ones, the discussion of issues in QFT in curved spacetimes, and the application to other useful symmetry reductions of these type (Gowdy models) that are similar to Einstein-Rosen waves and, hence, can also be solved after coupling massless scalar fields.

Our starting point is the four dimensional action for a massless scalar $\Phi_s$ coupled to gravity with cylindrical symmetry

$$4S = \frac{1}{16\pi G_N} \int_{M \times I} d^4x \sqrt{|g|} \left[ R - \frac{1}{2} g^{ab} \nabla_a \Phi_b \nabla_b \Phi_s \right] + \frac{1}{8\pi G_N} \int_{\partial(M \times I)} d^3x \left( \sqrt{|h|} K - \sqrt{|h^0|} K^0 \right).$$

Here we have included the surface terms necessary to have a well-defined variational principle, $I \equiv [z_1, z_2]$ is a closed interval in the direction of the translational Killing vector $\partial_z$, and $K, K^0$ are the extrinsic curvatures of the boundary defined by the dynamical metric $g_{ab}$ and a fiducial metric $g^0_{ab}$ that we choose as Minkowski in the following (we denote the induced metrics on the boundary as $^3h_{ab}$ and $^3h^0_{ab}$). The Geroch formalism (and a subsequent conformal transformation) allows us to reduce the previous action to the following three dimensional one by taking advantage of the translational symmetry

$$3S = \frac{1}{16\pi G_3} \int_M d^3x \sqrt{|g|} \left[ 3R - \frac{1}{2} g^{ab} \nabla_a \phi_g \nabla_b \phi_g \right] - \frac{1}{2} g^{ab} \nabla_a \phi_s \nabla_b \phi_s \right] + \frac{1}{8\pi G_3} \int_{\partial M} d^2x \left( \sqrt{|h|} K - \sqrt{|h^0|} K^0 \right).$$

Here $g_{ab}$ is a 3-dimensional metric and $3R$ the corresponding scalar curvature, $\phi_g$ is the scalar field that encodes the local gravitational degrees of freedom of the model $\phi_s$ is the massless matter scalar field in three dimensions, and $G_3$ is the gravitational constant per unit length along the symmetry axis (with dimensions of inverse energy; in the following we choose units such that $\hbar = c = 8G_3 = 1$). The integration is extended to a 3-manifold $\mathcal{M}$ with boundary $\partial \mathcal{M}$ with the appropriate topology. At this point it could be argued that the inclusion of the massless scalar is a rather trivial addition to the system because it just plays the role of an extra field of the same type of the gravitational scalar already present in the 2+1 dimensional description of Einstein-Rosen waves. However this is the most important and unexpected feature of this system because \textit{both} the gravitational and matter degrees of freedom are described by the same type of term in the three-dimensional Lagrangian in spite of their very different meaning in the original action, the Geroch reduction, and the conformal transformation used to arrive at this. It is also striking that they couple only through the metric and not directly (there are no cross terms).

As we are interested in the quantization of the system it is necessary to obtain the Hamiltonian corresponding to this. Although the final answer turns out to be quite simple it is not completely obvious, and it has some surprising features, so we provide some details on its derivation. To this end we choose a foliation of $\mathcal{M}$ with time-like unit normal $n^a$, a radial unit vector $\hat{r}^a$, and denote as $\sigma^\alpha$ the azimuthal, hypersurface orthogonal, Killing field (notice that this is not a unit vector). We further introduce two additional vector fields $t^a$ and $r^a$ defined as $t^a = Nn^a + N^r \hat{r}^a$ and $r^a = e^{\gamma/2} \hat{r}^a$, where $N$ is the lapse function, $N^r$ the radial shift, and $\gamma$ is an additional field. It is possible to find conditions that ensure that $t^a$, $r^a$, and $\sigma^\alpha$ are coordinate vectors. If we define $\partial_r \equiv \sigma^\alpha \nabla_a$, $\partial_t \equiv \sigma^\alpha \nabla_a$, and $\partial_\gamma \equiv t^a \nabla_a$ these are the following:

$$\partial_r N = \partial_r N^r = \partial_r \gamma = 0; \quad [\sigma, \hat{r}]^2 = [\sigma, n]^2 = 0; \quad n^r \partial_r N + \hat{r}^a (\partial_r N^r - \partial_r e^{\gamma/2}) + N e^{\gamma/2} \hat{r}^a [\hat{r}, n]^a = 0.$$ 

Writing the metric as $\tilde{g}_{ab} = -n_an_b + \hat{r}_a \hat{r}_b + \frac{1}{\sqrt{|g|}} \sigma^a \sigma_b$ (with $R^2 = g_{ab} \sigma^a \sigma^b$) we obtain the line element in these coordinates

$$ds^2 = (N^r - N^2) dt^2 + 2e^{\gamma/2} N^r dt dr + e^{\gamma} dr^2 + R^2 d\sigma^2.$$ 

The action can be rewritten now as

$$3S = \int_{t_1}^{t_2} dt \int_{0}^{\hat{r}} dr \left\{ N e^{-\gamma/2} (\gamma^r R^r - 2 R^r)\right. \left. - \frac{1}{N} (e^{\gamma/2} \hat{r}^2 - 2 N^r) (\dot{R} - e^{-\gamma/2} N^r R^r) + \frac{R}{2N} \left( e^{\gamma/2} \dot{\phi}^2 - 2 N^r \dot{\phi} \phi_g + e^{-\gamma/2} (N^r - N^2) \phi^2_g \right) \right. \left. + \frac{R}{2N} \left( e^{\gamma/2} \dot{\phi}_s^2 - 2 N^r \dot{\phi}_s \phi_s + e^{-\gamma/2} (N^r - N^2) \phi^2_s \right) \right. \right. \right. \right.$$

$$+ 2 \int_{t_1}^{t_2} dt (Ne^{-\gamma/2} R^r - 1),$$

\textsuperscript{1} This would require dropping the radial asymptotic flatness condition. By doing this we can have the self-similar solutions needed to discuss critical collapse or escape the conclusions of the about the absence of compact trapped surfaces in the asymptotically flat case.
where we have denoted $\partial_t$ with a dot, $\partial_r$ with a prime. The Hamiltonian when we take $r \to \infty$ is

$$H = \int_0^\infty dr \left\{ N^* e^{-\gamma/2} \left[ p_R R' - p_\gamma' + p_\gamma \gamma' + \phi_\theta \phi_g + \phi_\gamma \phi_s \right] + N e^{-\gamma/2} \left[ 2R'' - \gamma' R' - p_R p_\gamma + \frac{1}{2} p_\gamma^2 + \frac{R}{2} \phi_g^2 + \frac{1}{2} R^2 \phi_s^2 + \frac{R}{2} \phi_s^2 \right] \right\} + 2(1 - e^{-\gamma/2}),$$

where $p_R, p_\gamma, p_g$, and $p_s$ are the momenta canonically conjugate to $R, \gamma, \phi_g$, and $\phi_s$ respectively, $\gamma_\infty \equiv \lim_{r \to \infty} \gamma(r)$, and the fall-off of the fields, that ensures asymptotic flatness in 2+1 dimensions and implies $N \to 1$ and $R' \to 1$, is the one used in $[3]$. All fields are chosen to be regular in the axis. From the previous expression the Hamiltonian of the system and the constraints can be immediately read. To proceed ahead we fix the gauge with the conditions $R(r) = r$ and $p_\gamma(r) = 0$ (the same as in the absence of matter). It is straightforward to show that they are admissible. After fixing the gauge and solving the constraints we get

$$\gamma(R) = \frac{1}{2} \int_0^R dr \left[ \phi_g'^2 + \frac{p_g^2}{r^2} + \phi_s'^2 + \frac{p_s^2}{r^2} \right],$$

the three dimensional line element can be written as

$$ds^2 = e^\gamma [-e^{-\gamma} dt^2 + dR^2] + R^2 d\sigma^2$$ (2)

and the reduced Hamiltonian is

$$H = 2(1 - e^{-\gamma/2}).$$

This is a function of the sum of the Hamiltonians for two massless cylindrically symmetric fields evolving in a fictitious Minkowskian background. For every solution to the field equations $\gamma_\infty$ is a constant of motion. Taking advantage of this we can introduce an auxiliary, solution-dependent, time variable as in $[3]$ defined according to $T = e^{-\gamma/2}t$, that allows us to simplify the form of the field equations to get

$$\partial_t^2 \phi_g - \phi_g' - \frac{1}{R} \phi_g' = 0, \quad \partial_t^2 \phi_s - \phi_s' - \frac{1}{R} \phi_s' = 0.$$ (3)

Equations (3) describe two massless, cylindrically symmetric scalar fields in 2+1 dimensions. Classically this is a time redefinition that amounts to a change of the coordinate $t$; once we pick a certain solution to (3) we can choose to write (3) either in terms of $t$ or $T$. Quantum mechanically the situation is more complicated because the evolution of wave packets generically involves the superposition of Hilbert space vectors with energy dependent phases so a change in the functional form of the energy completely changes the evolution of the states. It is very important to notice that the form of the Hamiltonian means that the model is not free. The two fields that appear are coupled in a non trivial way.

In order to quantify the system we define field and momenta operators $\hat{\phi}_g, \hat{\phi}_s(R), \hat{p}_g, \hat{p}_s(R)$ satisfying the commutation relations $[\hat{\phi}_g, \hat{p}_g(R)] = i\delta(R, R')$ and introduce creation and annihilation operators as usual according to

$$\phi_g, \phi_s(R) = \frac{1}{\sqrt{2}} \int_0^\infty dk J_0(Rk) [a_g, a_g^\dagger(k)],$$

$$p_g, p_s(R) = \frac{iR}{\sqrt{2}} \int_0^\infty dk k J_0(Rk) [a_g^\dagger(k) - a_g(k)],$$

with non-zero commutators given by

$$[a_g(k), a_g^\dagger(q)] = \delta(k, q), \quad [a_g(k), a_s^\dagger(q)] = \delta(k, q).$$

These operators are defined in a Hilbert space built as a tensor product of two Fock spaces $H_g$ and $H_s$, $H = H_g \otimes H_s$ with a vacuum state $|\Omega\rangle = |0\rangle^g \otimes |0\rangle^s$ defined in terms of the vacua annihilated by $a_g, a_s$. States with a fixed number of quanta of “gravitational” or “scalar” type are obtained by repeated action of the corresponding creation operators $|k, g, s\rangle \equiv A_g^\dagger(k) A_s^\dagger(k) |\Omega\rangle$, where we have written $A_g^\dagger(k) \equiv a_g^\dagger(k) \otimes 1_s, A_s^\dagger(k) \equiv 1_g \otimes a_s^\dagger(k)$.

The quantum Hamiltonian in $H$ is

$$\hat{H} = 2 \left( 1 - \exp \left[ -\frac{1}{2} \int_0^\infty dk k [A_g^\dagger(k) A_g(k) + A_s^\dagger(k) A_s(k)] \right] \right).$$ (4)

We have normal ordered the exponent to remove the zero point energy of the vacuum. This Hamiltonian is a non-linear and bounded function of the sum of the Hamiltonians for two massless, cylindrically symmetric scalar fields in 2+1 dimensions, $H_g^0$ and $H_s^0$. It is an observable of the system (the energy) and the generator of time evolution in the time variable $t$ (from $t_1$ to $t_2$)

$$U(t_2 - t_1) = \exp \left[ -2i(t_2 - t_1) \left( 1 - e^{-\frac{1}{2}([H_g^0 + H_s^0])} \right) \right].$$ (4)

It is important to realize at this point that this is the physical evolution. The free Hamiltonians $H_g^0$ and $H_s^0$ are indeed observables but are not directly related to the time evolution of the system. It is necessary to take this fact into account in the search for semiclassical states because coherent states should display a classical behavior under the evolution given by (3) rather than under the one that would be defined by the “free” Hamiltonian $H_g^0 + H_s^0$. The expectation value of the field and momenta operators should evolve in $t$ according to the classical field equations in terms of $t$; notice that they are not (3).

The unitary evolution given by $U(t)$ defines the $S$-matrix of the system (the $S$-matrix in QFT is basically the evolution operator in the limit $t \to \infty$). Its matrix elements in $n$-particle states can be computed in a straightforward way because they are eigenstates of $H_g^0$ and $H_s^0$. The only non-zero matrix elements on states with a definite number of both quanta (i.e. gravitational
and matter) are those connecting state vectors with the same number of particles of each type; hence there is no conversion of quanta of one type into the other. It is necessary at this point to stress that the split of the Hilbert space as a tensor product of two Hilbert spaces should not immediately lead us to interpret one of them as “gravitational” and the other as “matter”; in fact the classical metric depends on both the gravitational scalar and the matter scalar and semiclassical approximations of it obtained by computing expectation values of a metric operator would also depend on both the “gravitational” and the “matter” part of the state. In this sense a vector such as $|0\rangle^g \otimes |\Phi\rangle^s$ should not be interpreted as an approximate description of a pure matter state $|\Psi\rangle^s$ on some quantum approximation of the Minkowski metric. In fact the quantum state that most closely resembles the approximate plane of the vacuum expectation of the metric scalar and the matter scalar. As in somewhere else [6, 7] the inclusion of scalar matter in the system cannot be obtained. Quite on the contrary some scalar field in a 2+1-dimensional Minkowskian background. This happens in the limit when spatial distances are less scalar field in a 2+1-dimensional Minkowskian back-ground. This happens in the limit when spatial distances and time intervals are large in comparison with the length scale $\hbar G_3$. Finally it is interesting to point out that even though the diagonal matrix elements of the commutator of $\hat{\phi}_g(R, t)$ and $\hat{\phi}_s(R, t)$ are zero, the non-diagonal ones are generically different from zero for $t_2 \neq t_1$. This is a clear indication that both fields interact in a non-trivial way. The availability of a matter field allows us to explore how the quantization of gravity reflects on the geometric properties of spacetime by using its particle-like excitations as quantum test particles. Though the details of this will appear elsewhere we want to explain here how it can be done. A possible approach to the problem is to interpret $\langle \Omega | \hat{\phi}_g(R_2, t_2) \hat{\phi}_s(R_1, t_1) | \Omega \rangle$ as the probability amplitude for a particle created at the radial distance $R_1$ in the instant $t_1$ to be found at $(R_2, t_2)$. As in ordinary QFT in Minkowski spacetime this interpretation is only approximate (i.e. valid only above a certain distance scale) because the $R$-dependent states $\hat{\phi}_s(R)|\Omega\rangle$ do not constitute an orthonormal basis. One can, however, introduce the analogous of the Newton-Wigner localized states for this system, define one-particle wave functions depending on the radial coordinate, and study their time evolution under the dynamics given by $\mathcal{H}$. If we choose appropriately peaked states it should be possible to study to what extent the evolution of wave packets follows the null geodesics of some cylindrical spacetime metric and the length scales in which a metric gives an accurate description of spacetime geometry.

Finally we want to point out that once the massless case is understood it could be possible to use it as a guide to find a consistent way to introduce other types of test fields (such as massive scalars or electromagnetic fields) that further improve our ability to explore quantum geometry and quantum gravity. This will be the focus of our attention in the near future.

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