Quasiconformal mappings and curvatures on metric measure spaces

Jialong Deng

Abstract

In an attempt to develop higher-dimensional quasiconformal mappings on metric measure spaces with curvature conditions, i.e. from Ahlfors to Alexandrov, we show that a non-collapsed RCD$(0, n)$ space $(n \geq 2)$ with Euclidean volume growth is an $n$-Loewner space and satisfies the infinitesimal-to-global principle.

Originating in cartography that represents the regions of the surface of the earth on a Euclidean piece of paper and beginning in the works of Tissot [Tis80], Pap [Pap17], Pap [Pap20], Grötzsch [Grö28], AP [AP20b], Lavrentieff [Lav20], AP [AP20a] and others [Pap18], the study of quasiconformal mappings on the Euclidean spaces $\mathbb{E}^n$ has a rich history of over one hundred years, see the books [Ahl06], [GMP17] and the references therein.

Beginning with Alexandrov’s insight in the 1940s [Ale48], the geometry of metric (measure) spaces became an important part of modern geometry [BBI01], [Vil09]. Since the metric structure plays an essential role in the theory of higher-dimensional quasiconformal mappings, it is natural to see how this theory behaves on the metrics with curvature conditions. For example, according to Heinonen and Koskela, it is a fundamental fact that a quasiconformal homeomorphism of the Euclidean space $\mathbb{E}^n$ with $n \geq 2$ is quasisymmetric, if it maps bounded sets to bounded sets [HK98]. We argue that the Euclidean metrics on $\mathbb{R}^n$ with $n \geq 2$ satisfy the infinitesimal-to-global principle. What are the other metrics $\mathbb{R}^n$ satisfy the infinitesimal-to-global principle is a matter of interest.

We will show in the following that the metrics $d$ that make $(\mathbb{R}^n, d, \mathcal{H}^n)$ to be non-collapsed RCD$(0, n)$ spaces with Euclidean volume growth also satisfy the principle.

Theorem 1. A quasiconformal homeomorphism $f$ of a non-collapsed RCD$(0, n)$ space with Euclidean volume growth and $n \geq 2 (n \in \mathbb{N})$ is quasisymmetric, if it maps bounded sets to bounded sets.

The idea of the proof came from Heinonen-Koskela [HK98]. That is, we need the following result to prove the above theorem.

Theorem 2. Non-collapsed RCD$(0, n)$ spaces $(X, d, \mathcal{H}^n)$ with $n \geq 2 (n \in \mathbb{N})$ and Euclidean volume growth are $n$-Loewner spaces.

The definitions and details will be given later. As an application of the two aforementioned theorems, we can show the distortion volume inequality of quasisymmetric mappings of a non-collapsed RCD$(0, n)$ space $(X, d, \mathcal{H}^n)$ with Euclidean volume growth.
Remark 3. The note is a step to answering the question of what facts of the classical theory on $\mathbb{E}^n$ are applicable to quasiconformal/quasiregular mappings on metric measure spaces with curvature conditions. More results are expected to be found in this direction.

The paper is organized as follows. In Section 1 we introduce the notions of quasiconformality and RCD spaces. Section 2 is devoted to the main results. In Section 3 we give two applications of the theorems and two further questions.

Acknowledgment: The author appreciates the anonymous reviewer’s constructive feedback. This note is a part of my proposal that was submitted to MathJobs for a postdoctoral position in May 2021. The funding is derived from a postdoctoral fellowship of Yau Mathematical Sciences Center, Tsinghua University. This note was dedicated to the mathematicians who were working in Ukraine during 2022.

1 Preliminaries

In this section, we recall the definition of Alexandrov spaces, non-collapsed RCD($0, n$) spaces and quasiconformality for the reader’s convenience.

A metric space $(X, d)$ is said to be quasiconvex if there is a constant $C > 0$ so that every pair of points $x$ and $y$ in $X$ can be joined by a curve $\gamma$, whose length satisfies $l(\gamma) \leq Cd(x, y)$. A metric space $(X, d)$ is a length metric space if the distance between each pair of points equals the infimum of the lengths of curves joining the points. Thus, a locally compact and complete length space is quasiconvex.

Throughout this section, a metric space $(X, d)$ refers to a locally compact and complete length metric space. For $1 \leq n \in \mathbb{N}$, $H^n$ refers to the $n$-dimensional Hausdorff measure of $(X, d)$. A metric measure space $(X, d, H^n)$ is a locally compact and complete length space with the full support of $n$-dimensional Hausdorff measure $H^n$.

Recall that a model space in Riemannian geometry is a simply connected complete Riemannian surface with the constant sectional curvature $\kappa$.

A geodesic triangle $\triangle$ in $X$ with geodesic segments as its sides is said to satisfy the CAT($\kappa$)-inequality if it is slimmer than its comparison triangle in the model space. That is, if there is a comparison triangle $\triangle'$ in the model space with its sides of the same length as the sides of $\triangle$ such that the distance between the points on $\triangle$ is less than or equal to the distance between the corresponding points on $\triangle'$. A length metric $d$ on $X$ is said to be a locally CAT($\kappa$)-metric if every point in $X$ has a geodesically convex neighborhood, in which every geodesic triangle satisfies the CAT($\kappa$)-inequality.

Similarly, an Alexandrov space $(X, d)$ with nonnegative curvature means that every point in $X$ has a geodesically convex neighborhood, in which every geodesic triangle is fatter than the comparison triangle in the Euclidean plane. More details about comparison geometry can be found in the book [BBI01] and the references therein.

Alexandrov geometry is a generalized Riemannian manifold with sectional curvature bounded below (or above). Similarly, Lott-Sturm-Villani theory on metric measure spaces is a synthetic generalized Ricci curvature bounded below, see [LV09], [Stu06] and [Vil09]. Later, the Riemannian curvature-dimensional condition was introduced as a refinement in order to single out “Riemannian structures” from the “possibly Finslerian CD structures” so that the study of RCD($K, N$) spaces became active, see Ambrosio’s survey [Amb18].
and the references therein. Recently, a generalized scalar curvature bounded below on metric measure spaces was studied by the author in [Den21a], [Den21b] and [Den21c].

The curvature-dimension condition $\text{CD}(K, N)$ of Lott–Sturm–Villani was defined through optimal transport theory and the convexity properties of $N$-dimensional entropy on the space of all Borel probability measures over a metric spaces at the beginning. Thanks to the works [EKS15], [AMS19], [CM21], we can choose an equivalently shorter version in the following way.

Let $(X, d, m)$ be a metric measure space with the full Borel measure $m$ and the parameters $K \in \mathbb{R}$ (lower bound on Ricci curvature) and $N \in (1, \infty)$ (upper bound on dimension) will be kept fixed. Define the Cheeger energy $\text{Ch} : L^2(X, m) \to [0, \infty]$ by

$$\text{Ch}(f) := \inf \left\{ \liminf_{i \to \infty} \int_X \mathcal{L}^2 f_i dm \mid f_i \in \text{Lip}_{b}(X, d) \cap L^2(X, m), \|f_i - f\|_{L^2} \to 0 \right\},$$

where $\text{Lip}_{b}(X, d)$ is the set of all bounded Lipschitz functions on $X$ and

$$\text{lip} f(x) := \lim_{r \to 0^+} \sup_{y \in B(x) \setminus \{x\}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

if $x$ is not isolated, $\text{lip} f(x) := 0$ otherwise. Given $f \in L^2(X, m)$, a function $g \in L^2(X, m)$ is called a relaxed gradient if there exists a sequence $\{f_n\} \subset \text{Lip}(X, d)$ and $\tilde{g} \in L^2(X, m)$ so that

1. $f_n \to f$ in $L^2(X, m)$ and $\text{lip} f_n$ converge weakly to $\tilde{g} \in L^2(X, m).
2. $g \geq \tilde{g}$ m-a.e..

A minimal relaxed gradient is a relaxed gradient that is minimal in $L^2$-norm in the family of relaxed gradients of $f$. If this family is non-empty, the minimal relaxed gradient which is denoted by $|\nabla f|$ exists and is unique m-a.e..

Then the Sobolev space $H^{1,2} = H^{1,2}(X, d, m)$ is defined as the finiteness domain of $\text{Ch}$ in $L^2(X, m)$ and it is a Banach space equipped with the norm $\|f\|_{H^{1,2}}^2 := \|f\|_{L^2}^2 + \text{Ch}(f)$. $(X, d, m)$ is said to be \textit{infinitesimally Hilbertian} if $H^{1,2}(X, d, m)$ is a Hilbert space. In particular, for all $f_i \in H^{1,2}$ ($i = 1, 2$),

$$\langle \nabla f_1, \nabla f_2 \rangle := \lim_{t \to 0} \frac{\|\nabla (f_1 + tf_2)\|^2 - \|\nabla f_1\|^2}{2t} \in L^2(X, m).$$

is well defined. Then, by using the infinitesimally Hilbertian condition, $f \in H^{1,2}$ is said to be in the domain of the Laplacian ($f \in D(\Delta)$) if there exists $\Delta f \in L^2(X, m)$ so that

$$\int_X g \Delta f dm + \int_X \langle \nabla g, \nabla f \rangle dm = 0$$

for any $g \in H^{1,2}$. The following definition of RCD($K, N$) spaces comes from [Hon20] Definition 2.1.

**Definition 4** (RCD($K, N$) spaces). Let $(X, d, m)$ be a metric measure space, let $K \in \mathbb{R}$ and let $N \in (1, \infty)$. We say $(X, d, m)$ is an RCD($K, N$) spaces if the following hold:

1. (Volume growth condition) There exist $x \in X$ and $C > 1$ such that $m(B_r(x)) \leq C e^{Cr^2}$ for all $r \in (0, \infty)$. 

(2). (Riemannian structure) The Sobolev space $H^{1,2} = H^{1,2}(X, d, m)$ is a Hilbert space.

(3). (Sobolev-to-Lipschitz property) Any function $f \in H^{1,2}$ satisfying $|\nabla f|(y) \leq 1$ for $m$-a.e. $y \in X$ has a 1-Lipschitz representative.

(4). (Bochner inequality) For all $f \in D(\Delta)$ with $\Delta f \in H^{1,2}$,

$$\frac{1}{2} \int_X |\nabla f|^2 \Delta \varphi \, dm \geq \int_X \varphi \left(\frac{(\Delta f)^2}{N}\right) + <\nabla f, \nabla \Delta f> + K|\nabla f|^2 \, dm$$

for all $\varphi \in D(\Delta) \cap L^\infty(X, m)$ with $\varphi \geq 0$ and $\Delta \varphi \in L^\infty(X, m)$.

In fact, a metric measure space $(X, d, m)$ is an RCD$(K, N)$ space iff $(X, d, m)$ satisfies the CD$(K, N)$ condition and is infinitesimally Hilbertian [Den20 Theorem 2.14]. Typical examples of RCD spaces are measured Gromov–Hausdorff limit spaces of Riemannian manifolds with Ricci bounds from below and dimension bounds from above, so-called Ricci limit spaces. RCD$(0, n)$ spaces come out naturally as the metric cone of RCD$(n-2, n-1)$ spaces.

An RCD$(K, N)$ space $(X, d, m)$ is noncollapsed if $n$ is a natural number and $m = H^n$. Noncollapsed RCD$(K, n)$ spaces give a natural intrinsic generalization of noncollapsing Ricci limit spaces. Furthermore, the class of RCD$(K, n)$ spaces strictly contains the noncollapsed Ricci limit spaces. A metric cone (resp. a spherical suspension) over $\mathbb{R}P^2$ is an example of a noncollapsed RCD$(0, 2)$ (resp. RCD$(1, 3)$) space. A convex body in $\mathbb{E}^n$ with boundary cannot arise as a noncollapsed Ricci limit of manifolds without boundary. However, this is a noncollapsed RCD$(0, n)$ space [KM21 Theorem 1.10].

Petrunin shows that an $n$-dimensional Alexandrov space with non-negative curvature and equipped with the induced Hausdorff measure satisfies CD$(0, n)$ condition, see [Pet11] and [Den21a Section 2]. A finite dimensional Alexandrov space with curvature bounded below is infinitesimally Hilbertian. Thus an $n$-dimensional Alexandrov space with non-negative curvature and equipped with the induced Hausdorff measure is a non-collapsed RCD$(0, n)$ space.

The reason why we focus on the subject of noncollapsed RCD$(0, n)$ spaces is because the measures are determined by the metrics. In the following, we will elaborate on the metric definition of quasiconformality, which is one of the three commonly adopted definitions in the existing literature.

**Definition 5** (Quasiconformal). A homeomorphism $f : X \to Y$ between the metric spaces $(X, d_X)$ and $(Y, d_Y)$ is said to be $K$-quasiconformal if there is a constant $0 < K < \infty$ so that

$$\limsup_{r \to 0} \frac{L_f(x, r)}{l_f(x, r)} \leq K$$

for all $x \in X$, where

$$L_f(x, r) := \sup_{d_X(x, y) \leq r} d_Y(f(x), f(y))$$

and

$$l_f(x, r) := \inf_{d_Y(x, y) \geq r} d_Y(f(x), f(y)).$$
Quasiconformal homeomorphisms arise not only in cartography and geometric functional theory, but also in some parts of dynamics system, topology, and Riemannian geometry, see the survey [Sul87] and the references therein.

The theory of quasiconformal mappings plays a role in applied mathematics. For example, combining quasiconformal Teichmüller theory [GL00] with scientific computing techniques, computational quasiconformal geometry has various applications in engineering and medical imaging, see [LWZ+12], [GZL+12] and the references therein.

**Definition 6 (Quasisymmetric).** A homeomorphism \( f : X \to Y \) between the metric spaces \((X, d_X)\) and \((Y, d_Y)\) is said to be quasisymmetric if there is a constant \( H_f < \infty \) so that

\[
H_f(x, r) := \frac{L_f(x, r)}{l_f(x, r)} \leq H_f
\]

for all \( x \in X \) and all \( r > 0 \).

Namely,

\[
d_X(x, a) \leq d_X(x, b) \quad \text{implies} \quad d_Y(f(x), f(a)) \leq H_f d_Y(f(x), f(b))
\]

for each triple \( x, a, b \) of points of \( X \).

A quasiconformal homeomorphism is a local and infinitesimal condition, whereas a quasisymmetric homeomorphism is a global condition that imposes a uniform requirement on the relative metric distortion of any triple of points.

**Definition 7 (\( \eta \)-quasisymmetric).** A homeomorphism \( f : X \to Y \) between the metric spaces \((X, d_X)\) and \((Y, d_Y)\) is said to be \( \eta \)-quasisymmetric if there is a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) so that

\[
d_X(x, a) \leq \tau d_X(x, b) \quad \text{implies} \quad d_Y(f(x), f(a)) \leq \eta(\tau) d_Y(f(x), f(b))
\]

for each \( \tau > 0 \) and for each triple \( x, a, b \) of points of \( X \).

Though \( \eta \)-quasisymmetry implies quasisymmetry, these two notions are not equivalent in general. However, if \( X \) and \( Y \) are path-connected doubling metric spaces, then these two notions are equivalent.

\( \eta \)-quasisymmetric mappings naturally arise in one variable complex dynamics, see [MSS83]. \( \eta \)-quasisymmetric mappings can be used in geometric group theory. For instance, quasi-isometric mappings of Gromov hyperbolic spaces are in one-to-one correspondence with \( \eta \)-quasisymmetric mappings on the Gromov boundaries, see [Pan89].

**Definition 8 (Ahlfors \( Q \)-regular).** A metric space \((X, d)\) is said to be Ahlfors \( Q \)-regular if there is a constant \( C \geq 1 \) so that

\[
C^{-1} R^Q \leq \mathcal{H}^Q(B(R)) \leq C R^Q
\]

for all \( R \)-balls \( B(R) \) in \( X \) of radius \( R \) less than the diameter of \( X \) (the diameter may be infinite).
Let \((M, g)\) be a complete open Riemannian \(n\)-manifold, then the asymptotic volume ratio of \((M, g)\) is defined as following:

\[
\text{AVR}_g := \lim_{r \to \infty} \frac{\text{Vol}_g(B_x(r))}{\omega_n r^n},
\]

where \(\omega_n\) is the volume of the Euclidean unit ball in \(\mathbb{R}^n\). The manifold \(M\) is said to have Euclidean volume growth when \(\text{AVR}_g > 0\). The constant \(\text{AVR}_g\) is a global geometric invariant of \(M\), i.e., it is independent of the base point. Also, when \(\text{AVR}_g > 0\), we have \(\text{Vol}_g(B_x(r)) \geq \text{AVR}_g \omega_n r^n\) for all \(x \in M\) and for all \(r > 0\).

Similarly, the asymptotic volume ratio can be defined for a metric measure space \((X, d, \mathcal{H}^n)\). That is

\[
\text{AVR}_d := \lim_{r \to \infty} \frac{\mathcal{H}^n(B_x(r))}{\omega_n r^n}.
\]

### 2 Quasiconformality VS. Quasisymmetry on \(\mathbb{R}^n\)

In this section, we search metrics with curvature conditions on \(\mathbb{R}^n\) to satisfy the condition that quasiconformal mappings are quasisymmetric.

A complete open nonnegatively curved Riemannian manifold may not have Euclidean volume growth in general. For example, the product metric of a standard spherical metric and a Euclidean metric is a complete Riemannian metric with nonnegative sectional curvature and without Euclidean volume growth. However, a complete open non-negatively curved Riemannian manifold with Euclidean volume growth is diffeomorphic to \(\mathbb{R}^n\), since its asymptotic cone at infinity has its dimension strictly smaller than \(n\) if the soul is not a point, while manifolds with Euclidean volume growth have asymptotic cones of dimension \(n\). Non-negatively curved Alexandrov spaces with Euclidean volume growth do not have to be smooth or even topological manifolds. For example, a metric cone over any \((n-1)\)-dimensional Alexandrov space of curvature \(\geq 1\) is Alexandrov of curvature \(\geq 0\) and has Euclidean volume growth.

Assume \((M, g)\) is a complete open Riemannian \(n\)-manifold with non-negative Ricci curvature and \(\text{AVR}_g = 1\), then the Bishop-Gromov inequality implies that \(M\) is isometric to \(\mathbb{R}^n\). It is known that if \(n = 3\) and \(\text{AVR}_g > 0\), then \(M\) is contractible [Zhu93]; if \(n = 4\) and \(\text{AVR}_g > 0\), then the manifolds may have infinite topological types [Men00]. However, Perelman shows that if \(M\) has maximal Euclidean volume growth, i.e. there exists a small positive constant \(a(n)\) such that \(\text{AVR}_g \geq 1 - a(n) > 0\), then \(M\) is contractible and homeomorphic to \(\mathbb{R}^n\) [Per94, Theorem 2]. Here \(a(n)\) only depends on the dimension \(n \geq 2\) of the manifold. Furthermore, Cheeger and Colding show that \(M\) is indeed \(C^{1,\alpha}\)-diffeomorphic to \(\mathbb{R}^n\) on the same assumption of Perelman’s theorem in [CC97, Theorem A.1.11].

**Remark 9.** However, there exists Riemannian manifolds with non-negative Ricci curvature and linear volume growth, thus it cannot be \(Q\)-regular for \(Q > 1\). Furthermore, there exists Riemannian metrics \(g\) with non-negative Ricci curvature and arbitrarily small \(\text{AVR}_g\) on \(\mathbb{R}^n\). For example, let \(n \geq 3\) and \(f : [0, \infty) \to [0, 1]\) be a smooth nonincreasing function such that \(f(0) = 1\) and

\[
\lim_{s \to \infty} f(s) = a \in (0, 1],
\]
then the warped product metric $g := dr^2 + F(r)^2 d\theta^2$ is the rotationally invariant metric on $\mathbb{R}^n$. Here

$$F(r) := \int_0^r f(s) \, ds$$

and $d\theta^2$ is the standard metric on the sphere $S^{n-1}$. If $x = (x_1, \theta_1)$ and $\tilde{x} = (x_2, \theta_2)$ are in $\mathbb{R}^n$, then one has $d_g(x, \tilde{x}) \geq \|x_1 - x_2\|$ and it implies that the metric $g$ is complete. One can show that the sectional curvature of $g$ is nonnegative and $\text{AVR}_g = a^{n-1}$, see [BK22 Example 2.4].

A conformal deformation of the Euclidean metric on $\mathbb{R}^n$ with infinite volume and under $L^{n/2}$ scalar curvature bounds has Euclidean volume growth, see [Car20 Theorem 2.8].

Inspired by Perelman’s theorem, Kapovitch and Mondino show that there exists a small positive constant $\epsilon(n)$ such that if a metric measure space $(X, d, H^n)$ is a non-collapsed RCD$(0, n)$ space and $\text{AVR}_d \geq 1 - \epsilon(n) > 0$, then $X$ is homeomorphic to $\mathbb{R}^n$ in [KM21 Theorem 1.3], also see [HH22 Proposition 7.1]. Kapovitch-Mondino theorem does not hold for CD$(0, n)$ space in general. Notice that there exists CD$(0, n)$ spaces that are not non-collapsed RCD$(0, n)$ spaces. For instance, since $\mathbb{R}^n$ that is endowed with the Lebesgue measure and the distance coming from a norm is a CD$(0, n)$ space, one can equip $\mathbb{R}^n$ with the $L^\infty$-norm such that it is not a non-collapsed RCD$(0, n)$ space.

**Definition 10** (Heinonen-Koskela). A metric measure space $(X, d, \mu)$ is a $p$-Loewner space if there exists a decreasing function $\phi : (0, \infty) \to (0, \infty)$ so that

$$\text{Mod}_p \Gamma(E, F) \geq \phi(\Delta(E, F))$$

for all disjoint compact connected subsets $E, F \subset X$. Here $\Gamma(E, F)$ denotes the collection of curves joining $E$ to $F$, the $p$-modulus ($p \geq 1$) of $\Gamma(E, F)$ is defined as

$$\text{Mod}_p \Gamma(E, F) := \inf \int_X \phi^p d\mu$$

where the infimum takes over all nonnegative Borel functions $\phi : X \to [0, \infty]$ satisfying

$$\int_\gamma \phi \, ds \geq 1$$

for all locally rectifiable curves $\gamma \in \Gamma(E, F)$. Note that by definition the modulus of all curves in $X$ that are not locally rectifiable is zero.

And here

$$\Delta(E, F) := \frac{d(E, F)}{\min\{\text{diam}(E), \text{diam}(F)\}}$$

denotes the relative distance between $E$ and $F$ and $\text{diam}(E)$ denotes the diameter of $E$.

We recap the two main theorems mentioned at the beginning of the paper and recall the definition of non-branching geodesic space for convenience. Let Geo$(X)$ be the space of constant speed geodesics in the geodesic space $(X, d)$, i.e.,

$$\text{Geo}(X) := \{ \gamma \in C([0, 1]; X) : d(\gamma(t), \gamma(s)) = |t - s|d(\gamma(0), \gamma(1)), \text{ for any } t, s \in [0, 1] \}.$$ 

The geodesic space $(X, d)$ is non-branching iff for any $\gamma_1, \gamma_2 \in \text{Geo}(X)$ we have: if there exists $t \in (0, 1)$ such that $\gamma_1(s) = \gamma_2(s)$ for all $s \in [0, t]$, then $\gamma_1 = \gamma_2$. 


Theorem 11. Non-collapsed RCD(0, n) spaces \((X, d, H^n)\) with \(n \geq 2\) \((n \in \mathbb{N})\) and Euclidean volume growth are \(n\)-Loewner spaces.

Proof. Since an RCD(0, n) space with \(n \geq 2\) is non-branching [Den20, Theorem 1.3], non-collapsed RCD(0, n) spaces with \(n \geq 2\) satisfy (1, 1)-Poincare inequality [Vil09, Theorem 30.26]. In addition, non-collapsed RCD(0, n) spaces with Euclidean volume growth are proper, doubling and quasiconvex, then Heinonen-Koskela theorem implies that these are \(n\)-Loewner spaces [HK98, Theorem 5.7].

Remark 12. The non-collapsed condition is not necessary for this theorem to hold. Having said that, we state the theorem with non-collapsed condition because our target is the metrics on \(\mathbb{R}^n\).

Theorem 13. A quasiconformal homeomorphism \(f\) of a non-collapsed RCD(0, n) space with Euclidean volume growth and \(n \geq 2\) \((n \in \mathbb{N})\) is quasisymmetric, if it maps bounded sets to bounded sets.

Proof. Since a non-collapsed RCD(0, n) space \((X, d, H^n)\) satisfies Bishop-Gromov inequality, \((X, d, H^n)\) with Euclidean volume growth is Ahlfors \(n\)-regular. On the other hand, Theorem [11] implies that it is an \(n\)-Loewner space. Then the quasiconformal homeomorphisms that maps bounded sets to bounded sets between \(n\)-regular Loewner spaces \((n \geq 2)\) are quasisymmetric [HK98, Corollary 4.8].

Thus, the homeomorphism \(f\) is also \(n\)-quasisymmetric. We do not need any of the priori regularity assumptions on the metrics or homeomorphisms to obtain global bounds. Since the classic Liouville theorem shows that the unit balls in \(\mathbb{R}^n\) with Euclidean metric \((n \geq 3)\) can be conformally equivalent to half-spaces, the condition that maps bounded sets to bounded sets in Theorem [13] is necessary.

Since non-collapsed RCD(K, n) spaces \((K > 0)\) are compact according to the generalized Bonnet-Meyer theorem, a quasiconformal homeomorphism of a non-collapsed RCD(K, n) space \((X, d, H^n)\) is quasisymmetric. The proof of Theorem [13] shows that a quasiconformal homeomorphism from non-collapsed RCD(0, n) spaces \((X, d, H^n)\) with maximal Euclidean volume growth and \(n \geq 2\) to \(\mathbb{E}^n\) is quasisymmetric.

Remark 14. A separable metric space is said to be purely \(n\)-dimensional if every non-empty open subset has the topological dimension \(n\). Then a purely \(n\)-dimensional, proper, geodesically complete CAT(0) space \((X, d)\) with \(\text{AVR}_d(X) < 3\omega_n/2\) is homeomorphic to \(\mathbb{R}^n\) [Nag22, Theorem 1.2]. It is not clear to the author whether quasiconformal homeomorphism of the space satisfies the infinitesimal-to-global principle.

Remark 15. The hyperbolic \(n\)-plane \(\mathbb{H}^n\) is an \(n\)-Loewner space. However, the volume of balls in \(\mathbb{H}^n\) increases exponentially with respect to the radius of the ball rather than polynomially as in the Euclidean space such that \(\mathbb{H}^n\) is not \(n\)-regular.

Remark 16. A complete open Riemannian \(n\)-manifold with uniformly positive scalar curvature is also not \(n\)-regular in general. Because \(\mathbb{R}^3\) can be given a complete Riemannian metric \(g_r\) with scalar curvature greater than 2, one can take the metric product of \(\mathbb{H}^n\) and \((\mathbb{R}^3, g_r)\). Then the scalar curvature of the product manifold is bounded below by 1 and the volume of the balls in the product manifold has exponential growth.
3 Final Remarks

Since the second-order differential calculus is developed on RCD spaces \cite{Gig18}, \cite{GP20}, one can also develop a distortion theory of quasiconformal mappings in a non-collapsed RCD(0, n) space \((X, d, \mathcal{H}^n)\) with Euclidean volume growth, which deals with the estimates for the modulus of continuity and change of distances under these mappings. We will give two generalized results in this section.

Assume that \(f\) is an \(\eta\)-quasisymmetric homeomorphism of a non-collapsed RCD(0, n) space \((X, d, \mathcal{H}^n)\) with Euclidean volume growth and \(n \geq 2\), then one can define the volume derivative of \(f\) at \(x \in X\) as

\[
\mu_f(x) := \lim_{r \to 0} \frac{\mathcal{H}^n(f(B_x(r)))}{\mathcal{H}^n(B_x(r))}.
\]

This limit exists according to the Lebesgue-Radon-Nikodym theorem and it is finite for almost every \(x \in X\). The function \(\mu_f\) is an \(\mathcal{H}^n\)-measurable function on \(X\) and is known as the generalized Jacobian of \(f\).

Recall that a doubling Borel measure \(\rho\) is \(A_\infty\)-related to the Hausdorff measure \(\mathcal{H}^n\) in \(X\) if for each \(\varepsilon > 0\) there is \(\delta > 0\) such that \(\mathcal{H}^n(E) < \delta\mathcal{H}^n(E)\) implies \(\rho(E) < \varepsilon\rho(E)\). The function \(\mu_f\) can be related to the pull-back measure \(f^*\mathcal{H}^n(E) := \mathcal{H}^n(f(E))\) in the following way.

**Theorem 17.** Let \(f\) be an \(\eta\)-quasisymmetric homeomorphism of a non-collapsed RCD(0, n) space \((X, d, \mathcal{H}^n)\) with Euclidean volume growth and \(n \geq 2\), then the pull-back measure \(f^*\mathcal{H}^n\) is \(A_\infty\)-related to \(\mathcal{H}^n\) in \(X\). Moreover, \(d f^*\mathcal{H}^n = \mu_f d \mathcal{H}^n\) with \(\mu_f > 0\) for \(\mathcal{H}^n\)-almost every \(x \in X\), and there is \(\varepsilon > 0\) such that

\[
(\int_B \mu_f^{1+\varepsilon} d \mathcal{H}^n(1)^{\frac{1}{1+\varepsilon}} \leq C \int_B \mu_f d \mathcal{H}^n
\]

for all balls \(B \subset X\), quantitatively.

**Proof.** Since non-collapsed RCD(0, n) spaces with \(n \geq 2\) satisfy (1, 1)-Poincare inequality, then the pull-back measure \(f^*\mathcal{H}^n(E)\) is \(A_\infty\)-related to the Hausdorff measure \(\mathcal{H}^n\) in \(X\) \cite[Theorem 7.11]{HK98}, \cite[Theorem 1.0.4]{KZ08}.

Therefore, Theorem 17 implies that \(\mathcal{H}^n(E) = 0\) if and only if \(\mathcal{H}^n(f(E)) = 0\) for an \(\mathcal{H}^n\)-measurable subset \(E \subset X\). That is, \(f\) and its inverse are absolutely continuous. Theorem 17 also implies that the \(\eta\)-quasisymmetric homeomorphism \(f\) preserves the dimensions of the sets of Hausdorff dimension \(n\). Furthermore, we can bound the measure of the image of a set by the measure of the set in the following way.

**Corollary 18.** Let \(f\) be an \(\eta\)-quasisymmetric homeomorphism of a non-collapsed RCD(0, n) space \((X, d, \mathcal{H}^n)\) with Euclidean volume growth and \(n \geq 2\), let \(\varepsilon\) be the constant in Theorem 17 and let \(F\) be a compact subset of \(X\), then for each \(a \in (0, \varepsilon/1 + \varepsilon)\) there exists a constant \(b\) such that

\[
\mathcal{H}^n(f(E)) \leq b \mathcal{H}^n(E)^{\varepsilon/a}
\]

for each \(\mathcal{H}^n\)-measurable subset \(E\) of \(F\).
Proof. Fix $a \in (0, \frac{1}{1+e})$ and let $q = \frac{1}{1-a} \in (1, 1+e)$. Since $\mu_f$ is locally $L^q$-integrable in $X$ by Theorem 17 then

$$b := (\int_E \mu_f^q dH^n)^{\frac{1}{q}} < \infty,$$

and for each measurable $E \subset F$, we have

$$H^n(f(E)) = \int_E \mu_f dH^n \leq (\int_F \mu_f^q dH^n)^{\frac{1}{q}} H^n(E)^a = b H^n(E)^a.$$

\qed

However, based on the existence of the Cantor sets on $\mathbb{E}^n$, Gehring and Väisälä [GV73] show that quasiconformal homeomorphisms of $\mathbb{E}^n$, $n \geq 2$, can distort the Hausdorff dimensions of the subsets, whose Hausdorff dimensions are not zero or $n$. Thus, the quasiconformal homeomorphisms of $\mathbb{E}^n$ can distort the perimeters of the subsets.

It is not clear to the author whether or not quasiconformal homeomorphisms of a non-collapsed RCD(0, $n$) space $(X, d, H^n)$ with Euclidean volume growth and $n \geq 2$ can distort the Hausdorff dimension of subsets. Recall that the perimeter of $E$ on the metric measure space $(X, d, H^n)$ can be defined as the relaxed Minkowski content of $E$ for a finite $H^n$-measurable set $E$ [ADMG17, Theorem 3.6].

Some of the examples of the distortion theory of quasiconformal mappings in $\mathbb{E}^n$ are the quasiconformal counterparts of the Schwarz Lemma by Grötzsch [Grö32], the classical Schwarz-Pick-Ahlfors Lemma [Ahl38], [Oss99] and Mori-Fehlmann-Vuorinen theorem [Mor56], [FV88].

**Theorem 19** (Mori-Fehlmann-Vuorinen Theorem). Let $\mathbb{B}^n$ be the unit ball of the Euclidean space $(\mathbb{E}^n, d_{\mathbb{E}^n})$, $n \geq 2$, and $f$ be a $K$-quasiconformal mapping of $\mathbb{B}^n$ onto $\mathbb{B}^n$ with $f(0) = 0$. Then,

$$d_{\mathbb{E}^n}(f(x), f(y)) \leq M(n, K)d_{\mathbb{E}^n}(x, y)^{\frac{1}{1-n}}$$

for all $x, y \in \mathbb{B}^n$ and the constant $M(n, K)$ has the following three properties:

1. $M(n, K) \to 1$ as $K \to 1$, uniformly in $n$;
2. $M(n, K)$ remains bounded for fixed $K$ and varying $n$;
3. $M(n, K)$ remains bounded for fixed $n$ and varying $K$.

Motivated by the similarity of the inequalities in Corollary 18 and Mori-Fehlmann-Vuorinen Theorem, one could ask the following question:

**Question 20.** Do the classical Schwarz-Pick-Ahlfors Lemma and Mori-Fehlmann-Vuorinen Theorem hold for CAT(−1) spaces?

If we do not require the homeomorphisms mapping in Definition 5 (of quasiconformality), then we get the definition of quasiregular mappings on metric spaces. To answer Zorich’s question, Rickman shows that a nonconstant quasiregular mapping $f : \mathbb{E}^n \to \mathbb{E}^n$ can only omit finite values for $n \geq 3$ [Ric80]. Zorich [Zor67] shows that any locally injective quasiconformal mapping $f : \mathbb{E}^n \to \mathbb{E}^n$ for $n \geq 3$ is globally injective.

**Question 21.** Can those two classic theorems be extended to non-collapsed RCD(0, $n$) spaces $(X, d, H^n)$ with Euclidean volume growth and $n \geq 3$?
References

[ADMG17] Luigi Ambrosio, Simone Di Marino, and Nicola Gigli. Perimeter as relaxed Minkowski content in metric measure spaces. *Nonlinear Anal.*, 153:78–88, 2017.

[Ahl38] Lars V. Ahlfors. An extension of Schwarz’s lemma. *Trans. Amer. Math. Soc.*, 43(3):359–364, 1938.

[Ahl06] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard.

[Ale48] A. D. Aleksandrov. *Vnutrennyaya Geometriya Vypuklyh Poverhnostei*. OGIZ, Moscow-Leningrad, 1948.

[Amb18] Luigi Ambrosio. Calculus, heat flow and curvature-dimension bounds in metric measure spaces. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 301–340. World Sci. Publ., Hackensack, NJ, 2018.

[AMS19] Luigi Ambrosio, Andrea Mondino, and Giuseppe Savaré. Nonlinear diffusion equations and curvature conditions in metric measure spaces. *Mem. Amer. Math. Soc.*, 262(1270):v+121, 2019.

[AP20a] Vincent Alberge and Athanase Papadopoulos. A commentary on Lavrentieff’s paper “Sur une classe de représentations continues”. In *Handbook of Teichmüller theory. Vol. VII*, volume 30 of *IRMA Lect. Math. Theor. Phys.*, pages 441–451. Eur. Math. Soc., Zürich, [2020] ©2020.

[AP20b] Vincent Alberge and Athanase Papadopoulos. On five papers by Herbert Grötzsch. In *Handbook of Teichmüller theory. Vol. VII*, volume 30 of *IRMA Lect. Math. Theor. Phys.*, pages 393–415. Eur. Math. Soc., Zürich, [2020] ©2020.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[BK22] Zoltán M. Balogh and Alexandru Kristály. Sharp isoperimetric and sobolev inequalities in spaces with nonnegative ricci curvature. *Mathematische Annalen*, March 2022.

[Car20] Gilles Carron. Euclidean volume growth for complete Riemannian manifolds. *Milan J. Math.*, 88(2):455–478, 2020.

[CC97] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.
[CM21] Fabio Cavalletti and Emanuel Milman. The globalization theorem for the curvature-dimension condition. Invent. Math., 226(1):1–137, 2021.

[Den20] Qin Deng. Hölder continuity of tangent cones in RCD(K,N) spaces and applications to non-branching. arXiv e-prints, page arXiv:2009.07956, September 2020.

[Den21a] Jialong Deng. Curvature-dimension condition meets Gromov’s n-volumic scalar curvature. SIGMA Symmetry Integrability Geom. Methods Appl., 17:013, 20 pages, 2021.

[Den21b] Jialong Deng. Enlargeable length-structure and scalar curvatures. Ann. Global Anal. Geom., 60(2):217–230, 2021.

[Den21c] Jialong Deng. Foliated Positive Scalar Curvature. PhD thesis, University of Goettingen, 2021.

[EKS15] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm. On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces. Invent. Math., 201(3):993–1071, 2015.

[FV88] Richard Fehlmann and Matti Vuorinen. Mori’s theorem for n-dimensional quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A I Math., 13(1):111–124, 1988.

[Gig18] Nicola Gigli. Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. Mem. Amer. Math. Soc., 251(1196):v+161, 2018.

[GL00] Frederick P. Gardiner and Nikola Lakic. Quasiconformal Teichmüller theory, volume 76 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000.

[GMP17] Frederick W. Gehring, Gaven J. Martin, and Bruce P. Palka. An introduction to the theory of higher-dimensional quasiconformal mappings, volume 216 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017.

[GP20] Nicola Gigli and Enrico Pasqualetto. Lectures on nonsmooth differential geometry, volume 2 of SISSA Springer Series. Springer, Cham, [2020] ©2020.

[Grö28] H. Grötzsch. Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes. Berichte Leipzig 80, 503-507 (1928)., 1928.

[Grö32] H. Grötzsch. Über möglichst konforme Abbildungen von schlichten Bereichen. Berichte Leipzig 84, 114-120 (1932)., 1932.

[GV73] F. W. Gehring and J. Väisälä. Hausdorff dimension and quasiconformal mappings. J. London Math. Soc. (2), 6:504–512, 1973.
[GZL+12] David Xianfeng Gu, Wei Zeng, Lok Ming Lui, Feng Luo, and Shing-Tung Yau. Recent development of computational conformal geometry. In *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, volume 2 of *AMS/IP Stud. Adv. Math.*, 51, pt. 1, pages 515–560. Amer. Math. Soc., Providence, RI, 2012.

[HH22] Hongzhi Huang and Xian-Tao Huang. Almost splitting maps, transformation theorems and smooth fibration theorems. *arXiv e-prints*, page arXiv:2207.10029, July 2022.

[HK98] Juha Heinonen and Pekka Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998.

[Hon20] Shouhei Honda. New differential operator and noncollapsed RCD spaces. *Geom. Topol.*, 24(4):2127–2148, 2020.

[KM21] Vitali Kapovitch and Andrea Mondino. On the topology and the boundary of $N$-dimensional $RCD(K,N)$ spaces. *Geom. Topol.*, 25(1):445–495, 2021.

[KZ08] Stephen Keith and Xiao Zhong. The Poincaré inequality is an open ended condition. *Ann. of Math. (2)*, 167(2):575–599, 2008.

[Lav20] Mikhaïl Lavrentieff. On a class of continuous representations. In *Handbook of Teichmüller theory. Volume VII*, pages 417–439. Berlin: European Mathematical Society (EMS), 2020.

[LV09] John Lott and Cédric Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[LWZ+12] Lok Ming Lui, Tsz Wai Wong, Wei Zeng, Xiangfeng Gu, Paul M. Thompson, Tony F. Chan, and Shing-Tung Yau. A survey on recent development in computational quasi-conformal geometry and its applications. In *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, volume 2 of *AMS/IP Stud. Adv. Math.*, 51, pt. 1, pages 697–717. Amer. Math. Soc., Providence, RI, 2012.

[Men00] X. Menguy. Noncollapsing examples with positive Ricci curvature and infinite topological type. *Geom. Funct. Anal.*, 10(3):600–627, 2000.

[Mor56] Akira Mori. On an absolute constant in the theory of quasi-conformal mappings. *J. Math. Soc. Japan*, 8:156–166, 1956.

[MSS83] R. Mane, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Supér. (4)*, 16:193–217, 1983.

[Nag22] Koichi Nagano. Asymptotic topological regularity of CAT(0) spaces. *Ann. Global Anal. Geom.*, 61(2):427–457, 2022.

[Oss99] Robert Osserman. From Schwarz to Pick to Ahlfors and beyond. *Notices Amer. Math. Soc.*, 46(8):868–873, 1999.
Pierre Pansu. Dimension conforme et sphère à l’infini des variétés à courbure négative. (Conformal dimension and the ideal boundary of manifolds with negative curvature). *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, 14(2):177–212, 1989.

Athanase Papadopoulos. Nicolas-Auguste Tissot: a link between cartography and quasiconformal theory. *Arch. Hist. Exact Sci.*, 71(4):319–336, 2017.

Athanase Papadopoulos. Quasiconformal mappings, from Ptolemy’s geography to the work of Teichmüller. In *Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds & Picard-Fuchs equations*, volume 42 of *Adv. Lect. Math. (ALM)*, pages 237–314. Int. Press, Somerville, MA, 2018.

Athanase Papadopoulos. A note on Nicolas-Auguste Tissot: at the origin of quasiconformal mappings. In *Handbook of Teichmüller theory. Vol. VII*, volume 30 of *IRMA Lect. Math. Theor. Phys.*, pages 289–299. Eur. Math. Soc., Zürich, [2020] ©2020.

G. Perelman. Manifolds of positive Ricci curvature with almost maximal volume. *J. Amer. Math. Soc.*, 7(2):299–305, 1994.

Anton Petrunin. Alexandrov meets Lott-Villani-Sturm. *Münster J. Math.*, 4:53–64, 2011.

Seppo Rickman. On the number of omitted values of entire quasiregular mappings. *J. Analyse Math.*, 37:100–117, 1980.

Karl-Theodor Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.

Dennis Sullivan. Quasiconformal homeomorphisms in dynamics, topology, and geometry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 1216–1228. Amer. Math. Soc., Providence, RI, 1987.

A. Tissot. Mémoire sur la représentation des surfaces et les projections des cartes géographiques. Nouv. Ann. (2) XVIII. 337-356, 385-397, 532-548, 1879; Nouv. Ann. (2) XIX. Suppl. 1-40, 1880 (1880)., 1880.

Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

Shun-Hui Zhu. A finiteness theorem for Ricci curvature in dimension three. *J. Differential Geom.*, 37(3):711–727, 1993.

V. A. Zorich. Homeomorphism of quasiconformal space mappings. *Sov. Math., Dokl.*, 8:1039–1042, 1967.