Non-linear Realisation of the $\mathcal{N} = 2, \, D = 6$ Supergravity

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Abstract

We have applied the method of dualisation to construct the coset realisation of the bosonic sector of the $\mathcal{N} = 2, \, D = 6$ supergravity which is coupled to a tensor multiplet. The bosonic field equations are regained through the Cartan-Maurer equation which the Cartan form satisfies. The first-order formulation of the theory is also obtained as a twisted self-duality condition within the non-linear coset construction.

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1 Introduction

The coset construction of the bosonic sectors of the maximal supergravities obtained from the $D = 11$ supergravity [1] by dimensional reduction has been given in [2]. The non-linear coset construction is based on the doubled formalism of the theories in which a dual field is introduced for each bosonic field of the theory. The results of [2] have been extended to the matter coupled supergravities in [3, 4, 5]. This has been possible since in [6, 7] a general coset construction is performed for the symmetric space sigma models of generic scalar coset manifolds. Also in [8] a general construction is presented for a matter coupled scalar coset.

The coset constructions or the non-linear realisations of the bosonic sectors of the supergravity theories are important for the understanding of the global symmetries of these theories. The global symmetry of the bosonic sector; in particular the scalar sector can be extended over the other fields to be the rigid global symmetry of the entire theory [9, 10]. Besides a restriction of the global symmetry group $G$ of a supergravity theory to the integers is conjectured to be the U-duality symmetry of the relative string theory whose low energy effective theory becomes the supergravity theory at hand [11, 12].

In [13] the non-linear coset realisation of the pure $\mathcal{N} = 4$, $D = 5$ supergravity is given. In this work we construct the coset realisation of the $\mathcal{N} = 2$, $D = 6$ supergravity which is coupled to a tensor multiplet [14, 15]. The coupling of the tensor multiplet is necessary to be able to write a Lorentz covariant lagrangian since for the pure $\mathcal{N} = 2$, $D = 6$ graviton multiplet a canonical Lorentz covariant lagrangian can not be constructed owing to the existence of an anti-self-dual three-form field strength [14, 15]. When one introduces a tensor multiplet as it will be clear in section two one can lift the constraint of (anti) self-duality by combining the fields of the graviton and the tensor multiplets. In section two we will derive the field equations and then we will give the locally integrated first-order field equations. In section three following the introduction of the coset map we will derive the algebra
which parameterizes this map by using the dualisation method of [2]. We will denote that the first-order field equations can be obtained from the Cartan form which is induced by the coset map as a twisted self-duality condition satisfied by it [16].

2 The $\mathcal{N} = 2, D = 6$ Supergravity

The field content of the pure $\mathcal{N} = 2, D = 6$ supergravity multiplet [14, 15] can be given as

$$(e^m_\mu, \psi^i_\mu, A^{(-)}_{\mu\nu}),$$ (2.1)

where $e^m_\mu$ is the vielbein, $A^{(-)}_{\mu\nu}$ is an anti-self-dual two-form field and $\psi^i_\mu$ for $i = 1, 2$ are the gravitini. Due to the anti-self-duality constraint on the two-form gauge field $A$ there is no way of constructing a canonical Lorentz covariant and unconstrained lagrangian for the pure $\mathcal{N} = 2, D = 6$ supergravity [14, 15, 17, 18, 19]. The minimal coupling which enables the construction of a Lorentz covariant lagrangian is the coupling of a matter tensor multiplet whose field content is

$$(\lambda_i, A^{(+)}_{\mu\nu}, \phi),$$ (2.2)

where $A^{(+)}_{\mu\nu}$ is a self-dual two-form field, $\phi$ is a scalar and $\lambda_i$ for $i = 1, 2$ are symplectic Majorana-Weyl spinors. By field redefinitions one may combine the anti-self-dual two-form $A^{(-)}_{\mu\nu}$ of the supergravity multiplet and the self-dual two-form $A^{(+)}_{\mu\nu}$ of the tensor multiplet into a single two-form field $A_{\mu\nu}$ which is unconstrained and this enables the construction of the Lorentz covariant and unconstrained lagrangian for the $\mathcal{N} = 2, D = 6$ supergravity coupled to a tensor multiplet. Thus the field content of the theory can be given as

$$(e^m_\mu, \psi^i_\mu, A_{\mu\nu}, \lambda_i, \phi).$$ (2.3)
We assume the signature of the space-time metric as

\[ \eta_{AB} = \text{diag}(-, +, +, +, +, +). \]  

(2.4)

The bosonic lagrangian of the \( N = 2, D = 6 \) supergravity coupled to a tensor multiplet can be given as \([14, 15]\)

\[ \mathcal{L} = -\frac{1}{2} R * 1 - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} * F \wedge F, \]  

(2.5)

where \( F = dA \). In the next section we will workout the coset construction of the bosonic sector of the theory excluding the gravity sector; for this reason we are interested in only the equations of motion for the fields \( A \) and \( \phi \). If we vary the lagrangian with respect to the fields \( A \) and \( \phi \) we find the corresponding second-order field equations respectively as

\[ d(e^{2\phi} * F) = 0, \]  

\[ d(*d\phi) = -e^{2\phi} * F \wedge F. \]  

(2.6)

By using the fact that locally a closed differential form is an exact one we can locally integrate the second-order field equations to obtain the local first-order ones. Thus if we introduce the three-form \( \tilde{A} \) and the four-form \( \tilde{\phi} \), by eliminating an exterior derivative operator on both sides of the field equations in (2.6) we can write down the first-order equations as

\[ e^{2\phi} * F = d\tilde{A}, \]  

\[ *d\phi = d\tilde{\phi} + d\tilde{A} \wedge A. \]  

(2.7)

If one takes the exterior derivative of the equations in (2.7) one would obtain the second-order field equations (2.6) which are free of the lagrange multiplier.
fields $\tilde{A}$ and $\tilde{\phi}$.

3 The Doubled-formalism and the Coset Structure

In this section we will construct a coset structure for the bosonic sector of the $\mathcal{N} = 2$, $D = 6$ supergravity that is coupled to a tensor multiplet such that the second-order field equations of (2.6) can be realized by the Cartan form of the coset map in the Cartan-Maurer equation [2, 3, 4, 5, 8]. The coset map will be parameterized by a Lie superalgebra and our aim will be to derive the structure of this algebra. We first assign the generators

$$(Y, K),$$

(3.1)

to the original bosonic fields $A$ and $\phi$ respectively. We have already introduced the dual fields $\tilde{A}$ and $\tilde{\phi}$ in (2.7) therefore we also define the dual generators

$$(\tilde{Y}, \tilde{K}),$$

(3.2)

which will couple to the dual fields in the construction of the coset map. Since the Lie superalgebra of the generators of any coset construction has a $\mathbb{Z}_2$ grading the generators are chosen to be odd if the corresponding coupling field is an odd degree differential form and otherwise even [2]. For our case all the generators defined in (3.1) and (3.2) are even according to the above general scheme. In the construction of the coset map we will make use of the differential graded algebra structure of the differential forms and the field generators given in (3.1) and (3.2). The details of this algebra can be referred in [2, 6, 7, 8, 13]. Before giving the definition of the full coset map which includes the dual fields and the dual generators as well we define

$$\nu = e^{\phi K} \epsilon^{AY}.$$  

(3.3)
If one assumes a matrix representation for the algebra generated by the generators $K$ and $Y$, by using the matrix identities

$$de^X e^{-X} = dX + \frac{1}{2!}[X, dX] + \frac{1}{3!}[X, [X, dX]] + \cdots,$$  

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \cdots,$$  

one can calculate the Cartan form

$$\mathcal{G} = d\nu \nu^{-1},$$  

as

$$\mathcal{G} = d\phi K + e^\phi dAY,$$  

where the only non-vanishing commutator of the algebra of $K$ and $Y$ is defined as

$$[K, Y] = Y.$$  

When we insert the Cartan form (3.6) in the Cartan-Maurer equation

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0,$$  

whose validity originates from the definition of $\nu$ in (3.3) we see that we obtain the trivial Bianchi identities

$$d(e^\phi dA) = -e^\phi dA \wedge d\phi,$$

$$d(d\phi) = 0,$$  

for the field strengths $e^\phi dA$ and $d\phi$ of the potentials $A$ and $\phi$ respectively. Thus we observe that in order to obtain the realisation of the field equations (2.6) of the theory one has to construct the coset map by using the dual fields.
and the dual generators together with the original ones. Therefore next we propose the map
\[ \nu' = e^{\phi K} e^{\Delta Y} e^{\tilde{\phi} \tilde{K}}. \]  
(3.10)

The Cartan form \( \mathcal{G}' = d\nu'/\nu'^{-1} \) induced by this coset map will also satisfy the Cartan-Maurer equation
\[ d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0, \]  
(3.11)
canonically. Following the general method of the coset construction [2, 3, 4, 5, 6, 7, 8, 13] we will demand that when we calculate the Cartan form \( \mathcal{G}' = d\nu'/\nu'^{-1} \) and insert it in the Cartan-Maurer equation (3.11) we should reach the second-order field equations (2.6) of the theory. One immediately observes that the calculation of the Cartan form \( \mathcal{G}' \) starting from the definition of the coset map (3.10) needs the specification of the algebra structure of the generators \( Y, K, \tilde{Y}, \tilde{K} \). As a matter of fact this is the mechanism we need to derive the algebra structure of the non-linear realisation. If one calculates the Cartan form \( \mathcal{G}' \) in terms of the unknown structure constants of the algebra of the generators by using the identities (3.4) and then inserts it in (3.11); by comparing the result with the second-order field equations (2.6) one can read the desired structure constants. We should remark that to be able to use the identities (3.4) we assume that we choose a matrix representation for the algebra generated by the generators of (3.1) and (3.2). Performing the above mentioned calculation we find that the only non-vanishing commutators of the algebra of the generators (3.1) and (3.2) are
\[ [K, Y] = Y, \quad [K, \tilde{Y}] = -\tilde{Y}, \]
\[ [Y, \tilde{Y}] = \tilde{K}. \]  
(3.12)

Now by using the matrix identities (3.4), also the commutators of (3.12) we can calculate the Cartan form \( \mathcal{G}' = d\nu'/\nu'^{-1} \) of the coset map (3.10) explicitly.
as
\[ G' = d\nu' \nu'^{-1} \]

\[ = d\phi K + e^\phi dY + e^{-\phi} d\tilde{A} \tilde{Y} \]

\[ + (d\tilde{\phi} + A \wedge d\tilde{A}) \tilde{K}. \]

(3.13)

The coset construction of the supergravities also produces the first-order formulation of these theories \([2, 16]\). The locally integrated first-order field equations are encoded in the doubled Cartan form \(G'\) as a twisted self-duality condition

\[ *G' = SG', \]

(3.14)

which it satisfies \([3, 4, 5, 13]\). In (3.14) \(S\) is a pseudo-involution of the algebra generated by the generators in (3.1) and (3.2). For our construction we define its action on the generators as

\[ SY = \tilde{Y}, \quad SK = \tilde{K}, \]

\[ S\tilde{Y} = Y, \quad S\tilde{K} = K. \]

(3.15)

The general construction of \(S\) for a generic coset formulation can be referred in \([2, 6, 7]\). Now, by using the definition of \(S\) given in (3.15); inserting (3.13) in (3.14) gives

\[ e^\phi * dA = e^{-\phi} d\tilde{A}, \]

\[ *d\phi = d\tilde{\phi} + A \wedge d\tilde{A}. \]

(3.16)

We see that these are the same equations with (2.7) of section two which have been obtained from the second-order field equations (2.6) by differential
algebraic integration. However, here they are obtained through the coset construction which includes the definition of the coset map (3.10), the algebra structure of (3.12) whose generators parameterize the coset map and the matrix representation chosen for this algebra.

As discussed in [14, 15] the $\mathcal{N} = 2$, $D = 6$ supergravity coupled to a tensor multiplet is the minimal extension of the $\mathcal{N} = 2$, $D = 6$ supergravity multiplet to write an invariant lagrangian. Accordingly we conclude that from the coset construction point of view the algebra derived in (3.12) is a minimal one thus it plays a special role in the covariant Lagrangian formulation of the theory.

4 Conclusion

We have obtained the coset formulation of the bosonic sector of the $\mathcal{N} = 2$, $D = 6$ supergravity which is coupled to a tensor multiplet [14, 15]. We have derived the algebra structure which is used to parameterize a coset map such that the induced Cartan form realizes the second-order field equations in the Cartan-Maurer equation. Thus the bosonic field equations of the $\mathcal{N} = 2$, $D = 6$ supergravity coupled to a tensor multiplet are obtained within the geometrical construction of the non-linear sigma model. The first-order formulation of the theory is also encoded in the coset construction. The locally integrated first-order field equations can be found through a twisted self-duality condition satisfied by the Cartan form [2, 16].

Our main objective in this work was to construct the algebra which parameterizes the coset map and which generates the field equations. The group theoretical structure of our coset realisation can further be studied separately. In section three we have stated that the algebra we have constructed can be considered as a minimal one. This fact may be linked to the minimality of the tensor multiplet coupling to write an invariant lagrangian. One may inspect the role of the generators of the two-form field and its dual
in the algebra constructed in section three. After studying the group theoretical construction of the coset one may question what it means algebraically and geometrically to introduce the other generators of the algebra to construct a model which will enable a Lorentz covariant and an unconstrained lagrangian.

The non-linear coset construction of this work can be extended to include the gravity and the fermionic sectors. The dualisation of the $\mathcal{N} = 2, D = 6$ supergravity which is coupled to other multiplets can be studied to orient the position of the $\mathcal{N} = 2, D = 6$ supergravity in the general dualisation scheme of the supergravity theories and to improve our knowledge of the global symmetries of these theories.

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