ANOMALY AND CONDENSATE IN THE LIGHT-CONE
SCHWINGER MODEL

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Abstract

The axial anomaly and fermion condensate in the light cone Schwinger model are studied, following path integral methods. This formalism allows for a simple and direct calculation of these and other vacuum dependent phenomena.
1. Introduction

Light-cone quantization has been a very successful formalism for calculating a number of physical quantities, mainly within perturbation theory [1]. In fact, in bound state calculations it has clear advantages over the usual equal-time formalism. In light-cone quantization all constituents have positive longitudinal momentum and energy $p^\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_3)$, which means that the ground state of the system can have no constituents and it is then trivial. Thus a Fock-state expansion, in which one builds for example a constituent quark model in QCD which includes gluon and quark-antiquark emission, makes sense. Interactions which connect the perturbative vacuum which has $p^+ > 0$ to states with particles (each of these particles with positive $p^+$), are just not present. On the other hand, in equal-time quantization the vacuum is an infinite sea of constituents, and one cannot identify these excitations with a single quark-antiquark pair or a single triplet of quarks separate from the sea.

The quantization of gauge theories in the light cone has been extensively studied in the last years in connection with the possible understanding of non-perturbative phenomena [2], [3]. The question immediately arises, however, of how it is possible that a trivial vacuum can give rise to phenomena, such as chiral symmetry breaking, condensate formation, and others, that are generally associated to a non-trivial vacuum structure. This issue has been studied recently, and one usually finds that it is necessary to introduce an infrared regulator, but it is difficult to find a regulator that does not automatically remove the vacuum structure [4].

In this paper we propose a simple direct method for defining a quantum theory in the light-cone, through a light-cone functional integral [5]. We will show that this formalism allows for a direct calculation of several non-perturbative vacuum associated phenomena. We consider the massless Schwinger model which is a simple model where many non-perturbative properties, such as the axial anomaly and the appearance of a fermion condensate, can be studied [6].

The Schwinger model is described by the following action

$$S_{SM} = \int d^2x [\bar{\psi}i\gamma^\mu D_\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}], \quad (1)$$
with the covariant derivative $D_\mu$ defined as

$$D_\mu = \partial_\mu - ieA_\mu.$$  

(2)

The Dirac matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$  

(3)

with $\eta^{\mu\nu} = \text{diag}(1,-1)$ and the $\gamma$-matrices are represented by $\gamma^0 = \sigma_1, \gamma^1 = -i\sigma_2$ and $\gamma^5 = \gamma^0\gamma^1$.

In terms of light cone coordinates

$$x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^1),$$  

(4)

the $\gamma$-matrices and the covariant derivative become

$$\gamma^\pm = \frac{1}{\sqrt{2}} (\gamma^0 \pm \gamma^1), \quad D^\pm = \frac{1}{\sqrt{2}} (D_0 \pm D_1).$$  

(5)

Using this notation, the Lagrangian becomes

$$\mathcal{L} = \bar{\psi}(i\gamma_- D_+ + i\gamma_+ D_-)\psi - \frac{1}{2} F_{+-}^2,$$  

(6)

and the formal partition function for this system is

$$Z = \int \mathcal{D}A_+ \mathcal{D}A_- \mathcal{D}\bar{\psi}\mathcal{D}\psi \ e^{iS_M(\bar{\psi},\psi,A_-,A_+)}.$$  

(7)

Here we are using a Lagrangian quantization in order to right down the functional integral, which can be shown explicitly to be equivalent to a Hamiltonian functional integral quantization [6], [7].

2. The Anomaly

In order to compute the axial anomaly we will introduce into the partition function an auxiliary field $A_{\pm}^5$, defined through the replacement

$$D_\pm \to \mathcal{D}_\pm = D_\pm - iA_\pm^5 \gamma^5.$$  

(8)
Thus, the axial current can be calculated by deriving the fermionic partition function with respect to \( A^5_{\pm} \) and then setting this field to zero at the end of the calculation. However, before proceeding this way the fermionic partition must be properly normalized such that the zero modes are cancelled out. Therefore, instead of (7) our starting point will be the partition function

\[
Z = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ e^{iS(\bar{\psi},\psi,\gamma_5,\gamma_5)}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ e^{iS(\bar{\psi},\psi)}},
\]

which can be directly evaluated to give the ratio of two determinants. They are in turn expanded and the constant part left out. Therefore, after using the explicit form of \( D_{\pm} \) and \( D_{\pm} \), one finds

\[
Z = \text{Tr}\{(e\gamma_5 - A_+ + \gamma_5) \cdot S_F \}
- \frac{1}{2} \text{Tr}\{(e\gamma_5 A_+ + \gamma_5) \cdot S_F \}
\times (e\gamma_5 - A_+ + \gamma_5) \cdot S_F, \tag{10}
\]

where \( S_F^{-1} = i(\gamma_\partial_+ + \gamma_\partial_-) \) and the \( \text{Tr} \) symbol means an integration over space coordinates and a trace over gamma matrices. Higher order terms should not contribute to the anomaly according to the Adler - Bardeen conjecture [8].

The divergence of the axial current is given by

\[
< \partial \cdot j^5 > = \partial_+ \frac{\delta Z}{\delta A^5_+} \bigg|_{A^5_+ = 0} + \partial_- \frac{\delta Z}{\delta A^5_-} \bigg|_{A^5_- = 0}, \tag{11}
\]

where \( \partial \cdot j^5 = \partial_+ j^5_- + \partial_- j^5_+ \).

One readily observes that by translation invariance the first term in eq. (11) does not contribute to the anomaly. Therefore, by going to Fourier space one finds that

\[
< ik \cdot j^5 > = -e \int \frac{dp_+ dp_-}{(2\pi)^2} \text{Tr}\left[ \gamma^5_{+} \frac{1}{\gamma_{+} p_{-} + \gamma_{-} p_{+}[\gamma_{-} A_{+} + \gamma_{+} A_{-}] \frac{1}{\gamma_{+} (p_{-} - k_{-}) + \gamma_{-} (p_{+} - k_{+})} \times [\gamma_{-} k_{+} + \gamma_{+} k_{-}]} \right],
\]

\[
= e \int \frac{dp_+ dp_-}{(2\pi)^2} \left\{ \text{Tr}\left[ \gamma^5_{+} \frac{1}{\gamma_{+} p_{-} + \gamma_{-} p_{+} [\gamma_{-} A_{+} + \gamma_{+} A_{-}]} \right] - \text{Tr}\left[ \gamma^5_{+} \frac{1}{\gamma_{+} (p_{-} - k_{-}) + \gamma_{-} (p_{+} - k_{+}) [\gamma_{-} A_{+} + \gamma_{+} A_{-}]} \right] \right\}, \tag{12}
\]
where the identity

$$\gamma_- k_+ + \gamma_+ k_- = -\gamma_+ (p_- - k_-) - \gamma_- (p_+ - k_+) + \gamma_- p_+ + \gamma_+ p_-$$ (13)

has been used. The integrals that appears in (12) are linearly divergent and as is well known one cannot make a shift in the momenta. The integrals can be performed by expanding the second term around $k = 0$, i.e.

$$\frac{1}{\gamma_+ (p_- - k_-) + \gamma_- (p_+ - k_+)} = \frac{1}{\gamma_+ p_- + \gamma_- p_+} + \frac{1}{\gamma_+ p_- + \gamma_- p_+} \frac{1}{(\gamma_- k_+ + \gamma_+ k_-)} \frac{1}{\gamma_+ p_- + \gamma_- p_+},$$ (14)

where we have not written the higher order terms because they do not contribute to the integrals (12). In addition those integrals are infrared divergent and must be regularized by adding a mass term in the propagator.

After this regularization one finds that

$$<ik \cdot j^5> = -e \int \frac{dp_+ dp_-}{(2\pi)^2} \frac{1}{\gamma_+ p_- + \gamma_- p_+} \left\{ \frac{1}{\gamma_+ p_- + \gamma_- p_+} \right\} \left\{ \gamma_+ A_+ + \gamma_- A_- \right\}$$

$$\times (\gamma_+ p_- + \gamma_- p_+) (\gamma_+ A_+ + \gamma_- A_-) \left\{ \gamma_+ k_- + \gamma_- k_+ \right\} + m^2 tr \left\{ \gamma_5 (\gamma_+ k_- + \gamma_- k_+) (\gamma_+ A_- + \gamma_- A_+) \right\},$$ (15)

and using standard identities for the $\gamma$-matrices (E5) reads

$$<ik \cdot j^5> = -e \int \frac{dp_+ dp_-}{(2\pi)^2} \frac{1}{(2p_+ p_- - m^2)^2} \left\{ 4(p_+^2 k_+ A_+ - p_-^2 k_- A_-) + 2m^2 (k_+ A_- - k_- A_+) \right\}.$$ (16)

The first term on the RHS vanishes by symmetry considerations. In fact, by writing

$$\int \frac{dp_+ dp_-}{(2\pi)^2} \frac{(p_+^2 k_+ A_+ - p_-^2 k_- A_-)}{(2p_+ p_- - m^2)^2} = (k_+ A_- - k_- A_+) I,$$ (17)

with
\[
I = \int \frac{dp_+ dp_-}{(2\pi)^2} \frac{p_+^2}{(2p_+ p_- - m^2)^2}, \tag{18}
\]

from Eqs. (17) and (18) one notes that by interchanging \((k_+, A_+) \leftrightarrow (k_-, A_-)\) one gets
\[
(k_+ A_+ - k_- A_-)I = -(k_+ A_+ - k_- A_-)I = 0. \tag{19}
\]

The integral on the second term on the RHS in (16) can be directly evaluated to give
\[
\int \frac{dp_+ dp_-}{(2\pi)^2} \frac{1}{(2p_+ p_- - m^2)^2} = \frac{i}{4\pi m^2}, \tag{20}
\]
and therefore the expectation value of the divergence of the axial current in momentum space is
\[
< k \cdot j^5 > = \frac{e}{\pi} (k_+ A_- - k_- A_+), \tag{21}
\]
or in coordinate space
\[
< \partial \cdot j^5 > = -\frac{e}{\pi} F_{+-}, \tag{22}
\]
which is the axial anomaly.

2. The Condensate and Mass Gap

The same arguments presented above remain valid when one computes the fermion condensate. In fact, the central point is to find an argument that permits us to compute \(< \bar{\psi} \psi >\) independently of \(x\), but as is well known, this quantity can be computed in quantum field theory by looking at an equivalence between \(< \bar{\psi} \psi >\) and \(G(x, 0)\) defined as
\[
G(x, 0) = < 0 | \bar{\psi}(x) \psi(x) \bar{\psi}(0) \psi(0) | 0 >, \tag{23}
\]
where \(|0 >\) is the perturbative vacuum. Then, following standard arguments, one evaluates \(G(x, 0)\) when \(x\) goes to infinity \(^1\) and finds that

\(^1\)Here one should note that such a limit is well defined in euclidean space. There \(x_+ \to z = ix_0 + x_1\)
and \(x_- \to \bar{z} = -ix_0 + x_1\).
\[
\lim_{x \to \infty} G(x, 0) = \frac{1}{2} |< \bar{\psi}\psi >|^2. \tag{24}
\]

This last formula is heavily dependent of a non-trivial vacuum and the problem is, how can one correlate it with the light cone vacuum. A possible answer to this question has been given in references [4,9]. Essentially these authors find that the above formulae remain valid in the light cone and here we suppose that this is true.

The path integral representation of \( G(x, 0) \) for the Schwinger model is

\[
G(x, 0) = \frac{\int D\bar{A}_\mu D\bar{\psi} D\psi \bar{\psi}(x) \psi(x) \bar{\psi}(0) \psi(0) e^{iS_{SM}}}{\int D\bar{A}_\mu D\bar{\psi} D\psi e^{iS_{SM}}}, \tag{25}
\]

where \( S_{SM} \) is defined in (1).

In the light cone, the gauge field can be decomposed as

\[
A_\pm = \mp \partial_\pm S + \partial_\pm \varphi, \tag{26}
\]

where \( S \) and \( \varphi \) are pseudoscalar and scalar fields respectively. Then, if one makes a chiral transformation

\[
\psi = e^{-i\gamma_5 S} \psi', \quad \bar{\psi} = \bar{\psi'} e^{-i\gamma_5 S}, \tag{27}
\]

the functional measure is modified, \( i.e. \)

\[
D\bar{\psi} D\psi \rightarrow D\bar{\psi} D\psi \ e^{\pm \int dx_+ dx_- S} \partial_+ \partial_- S. \tag{28}
\]

If in addition one performs a gauge transformation

\[
\psi = e^{i\varphi} \psi', \quad \bar{\psi} = \bar{\psi'} e^{-i\varphi}, \tag{29}
\]

one easily sees that the fermionic fields decouple. Therefore, the effective action that appears from these operations is given by

\[
S_{eff} = \int dx_+ dx_- \left( \frac{2}{e^2} (\partial_+ \partial_- S)^2 + \frac{1}{\pi} S (\partial_+ \partial_- S) + \bar{\psi} i \gamma_\cdot \partial \psi \right). \tag{30}
\]

In addition, the product of the four fermions \( \bar{\psi}(x) \psi(x) \bar{\psi}(0) \psi(0) \) is changed by the chiral transformation to
\[ \bar{\psi}(x) e^{-2i\gamma^5 S(x)} \psi(x) \bar{\psi}(0) e^{-2i\gamma^5 S(0)} \psi(0), \] (31)

and as a consequence, \( G(x, 0) \) becomes

\[ G(x, 0) = \int D\bar{\psi} D\psi \bar{\psi}(x) e^{-2i\gamma^5 S(x)} \psi(x) \bar{\psi}(0) e^{-2i\gamma^5 S(0)} \psi(0) e^{iS_{\text{eff}}} \]

\[ \int D\bar{\psi} D\psi e^{iS_{\text{eff}}} , \] (32)

where a trace over the fermionic indices is implied. Here one should note that the longitudinal modes \( (\varphi) \) are exactly cancelled in the light cone without having used a gauge fixing condition as is mandatory in the standard approach.

The integration over the fermions can be straightforwardly done and we find that

\[ G(x, 0) = -\frac{1}{4\pi^2 (x_+ x_-)} \int D\bar{\psi} D\psi e^{iS_{\text{eff}}} \cos (2[S(x) - S(0)]) \]

\[ \int D\bar{\psi} D\psi e^{iS_{\text{eff}}} . \] (33)

These integrals are Gaussian and can be best evaluated in Euclidean Fourier space,

\[ G(|z|, 0) = \frac{1}{4\pi^2 |z|^2} e^{P(|z|)} , \] (34)

where

\[ P(|z|) = \int \frac{dp d\bar{p}}{(2\pi)^2 |p|^2 (2|p|^2 + m^2)} \frac{|e^{ip\cdot z} - 1|^2}{|p|^2} \]

\[ = 2\{\gamma_E + \frac{1}{2} \ln(m \sqrt{2|z|^2}) + K_0(m \sqrt{2|z|^2})\} \] (35)

with \( m^2 = e^2 / \pi \) the mass of the photon (mass gap). \( \gamma_E \) is the Euler constant and \( K_0 \) the associated Bessel function.

Therefore, in the limit \( |z| \to \infty \) the fermion condensate is

\[ |<\bar{\psi}\psi>|^2 = \frac{m^2}{4\pi^2} e^{2\gamma_E} . \] (36)

The same result can also be obtained by summing fermion-antifermion diagrams as is discussed in \[10\] in the standard approach.

In conclusion we have discussed a functional integral approach to light cone gauge theories. This approach has technical advantages with respect to the standard light cone approach because it allows a direct calculation of the anomaly and the fermion condensate, avoiding the technicalities present when one uses canonical light-cone quantization.
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