A new rate of convergence estimate for homogeneous
discrete-time nonlinear Markov chains

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Abstract

In the paper, we study a new rate of convergence estimate for homogeneous discrete-
time nonlinear Markov chains based on the Markov-Dobrushin condition. This result
generalizes the convergence estimates for any positive number of transition steps. An ex-
ample of a class such a process provided indicates that such types of estimates considering
several transition steps may be applicable when one transition can not guarantee any con-
vergence. Moreover, a better estimate can be obtained for a higher number of transitions
steps. A law of large numbers is presented for a class of ergodic nonlinear Markov chains
with finite state space that may serve as a basis for nonparametric estimation and other
statistics.

Keywords: 60J05; Nonlinear Markov chains; ergodicity; rate of convergence; law of large
counts

1 Introduction

The article considers estimating the rate of convergence to an invariant measure for a
specific class of nonlinear Markov chains. The work develops the approach presented in
the article [4], which initially adopts the idea of [3] for two-step transition kernels. In
these works the Markov-Dobrushin type condition plays an essential role

\[
\sup_{\mu,\nu \in \mathcal{P}(E)} \|P_\mu(x, \cdot) - P_\nu(y, \cdot)\|_{TV} \leq 2(1 - \alpha), \quad 0 < \alpha < 1, \quad x, y \in E,
\]

as well as the following restrictive condition

\[
\|P_\mu(x, \cdot) - P_\nu(x, \cdot)\|_{TV} \leq \lambda \|\mu - \nu\|_{TV}, \quad \lambda \in [0, 1], \quad x \in E, \quad \mu, \nu \in \mathcal{P}(E).
\]

The current article further develops the approach from [4], checking the ergodicity
conditions for discrete-time homogeneous nonlinear Markov chains by taking k-step tran-
sition probabilities, where k is an arbitrary positive number. An example of such a process
is presented, illustrating that the estimate taken on multiple steps is applicable while the
one-step estimate is not. This example also provided the rate of convergence calculation
for k = 2 and k = 3, indicating that estimates for the higher number of steps can be
slightly better. Finally, the law of large numbers is deduced for the nonlinear Markov
chains for the case of finite state space.

The rest of the article is organized as follows. In Section 2, the results generalizing
estimates for k-step are provided. Section 3 provides an example of such a discrete-time
homogeneous nonlinear Markov chain with an explicit rate of convergence calculation. The final section establishes the law of large numbers for nonlinear Markov chains that satisfy the conditions for the existence and uniqueness of invariant measure.

2 Rate of convergence estimate for k-step transitions

Let the process \((X^n_n)_{n \in \mathbb{Z}^+}\) be a nonlinear Markov chain defined on the finite state space \((E, \mathcal{E})\), initial distribution \(\mu = \text{Law}(X^n_0)\), \(\mu \in \mathcal{P}(E)\) and transition probabilities \(P^n_{\mu_n}(x,B) = \mathbb{P}_{\mu_n}(X^n_{n+1} \in B | X^n_n = x)\), where \(x \in E, B \in \mathcal{E}, n \in \mathbb{Z}^+\) and \(\mu_n := \text{Law}(X^n_n)\).

Let \(\mu, \nu \in \mathcal{P}(E)\), then the total variation distance between two probability measures may be defined as follows:

\[
\|\mu - \nu\|_{TV} = 2 \sup_{A \in \mathcal{E}} |\mu(A) - \nu(A)| = \int_E |\mu(dx) - \nu(dx)|.
\]

According to the [3] results, the nonlinear Markov chain is a uniformly ergodic process, and the existence of a unique invariant measure \(\pi\) is guaranteed if the conditions (1) and (2) are satisfied.

Then in case when \(\lambda < \alpha\) we have an exponential rate of convergence

\[
\|\mu_n - \pi\|_{TV} \leq 2(1 - (\alpha - \lambda)^n), \quad n \in \mathbb{Z}^+,\]

while in case \(\lambda = \alpha\) there is a linear convergence

\[
\|\mu_n - \pi\|_{TV} \leq \frac{2}{\lambda n}, \quad n \in \mathbb{Z}^+.
\]

Otherwise, when \(\lambda > \alpha\), there may be either an infinite number of invariant measures or no invariant measure at all.

The paper [4] proposes a generalization of this approach to a two-step transition probability kernel. The current article elaborates the idea of this paper and mainly follows along the same lines by using \(k\)-step transition probabilities, \(k > 0\).

**Theorem 1** (Existence and uniqueness of an invariant measure). Let the process \(X\) have a \(k\)-step transition probabilities \(Q_{\mu_n}(x,A) := P^n_{\mu_n}(X^n_{n+k} \in A | X^n_n = x)\) and satisfies the following conditions:

\[
\sup_{\mu, \nu \in \mathcal{P}(E)} \|Q_{\mu}(x, \cdot) - Q_{\nu}(y, \cdot)\|_{TV} \leq 2(1 - \alpha_k), \quad \text{(3)}
\]

where \(0 < \alpha_k < 1\), \(x,y \in E\),

\[
\|Q_{\mu}(x, \cdot) - Q_{\nu}(x, \cdot)\|_{TV} \leq \lambda_k \|\mu - \nu\|_{TV}, \quad \text{(4)}
\]

where \(\lambda_k \in [0, \alpha_k]\), \(x \in E\), \(\mu, \nu \in \mathcal{P}(E)\),

\[
\|P^n_{\mu}(x, \cdot) - P^n_{\nu}(x, \cdot)\|_{TV} \leq \lambda_1 \|\mu - \nu\|_{TV}, \quad \lambda_1 < \infty. \quad \text{(5)}
\]

Then the process \(X\) has a unique invariant measure \(\pi\) and for any probability measure \(\mu \in \mathcal{P}(E)\) the convergence is true:

\[
\|\mu_n - \pi\|_{TV} \leq \|\mu_0 - \pi\|_{TV}(1 - \alpha_k + \lambda_k)^{\lfloor n/k \rfloor}(1 + \lambda_1)^{\lfloor n \bmod k \rfloor}, \quad \text{(6)}
\]

while in case \(\lambda_k = \alpha_k\)

\[
\|\mu_n - \pi\|_{TV} \leq \frac{\|\mu_0 - \pi\|_{TV}}{2 + \lambda_k \nu}\|\mu_0 - \pi\|_{TV}(1 + \lambda_1)^{\lfloor n \bmod k \rfloor}. \quad \text{(7)}
\]
In order to prove this theorem, we need the following theorem on the convergence of any two initial probability measures for the process under consideration.

**Theorem 2.** Let the process \( X \) have a \( k \)-step transition probabilities \( Q_\mu(x, B) \) and satisfies conditions (3), (4) and (5) of Theorem 1. Then for any pair of probability measures \( \mu, \nu \in \mathcal{P}(E) \) the following convergence is true:

\[
\|\mu_n - \nu_n\|_{TV} \leq \|\mu_0 - \nu_0\|_{TV} (1 - \alpha_k + \lambda_k)^{(n/k)} (1 + \lambda_1)^{(n \mod k)}. \quad (8)
\]

while in case \( \lambda_k = \alpha_k \)

\[
\|\mu_n - \nu_n\|_{TV} \leq \frac{\|\mu_0 - \nu_0\|_{TV}}{2 + \lambda_k n} \|\mu_0 - \nu_0\|_{TV} (1 + \lambda_1)^{(n \mod k)}. \quad (9)
\]

Let us prove the theorem 2.

**Proof.** Let \( P : E \times E \to [0,1] \) be a transition kernel, \( \varphi : E \to \mathbb{R} \) be a measurable function and probability measure \( \mu \in \mathcal{P}(E) \); denote \( \mu P := \int_E P(x, dt)\mu(dx) \); in case when \( P \) depends on measure \( \mu \), we have: \( \mu_1(\mu) := \int_E P_\mu(x, dt)\mu(dx) \), then \( k \)-step transition probability kernel

\[
Q_\mu(x, dy) = \int P_\mu(x, dx_1)P_{\mu_1(\mu)}(x_1, dx_2) \ldots P_{\mu_{k-1}(\mu)}(x_{k-1}, dy).
\]

Consider the total variation distance between measures after applying the \( k \)-step transition kernel. For any probability measure \( \mu, \nu \in \mathcal{P}(E) \) denote

\[
d\eta = ((d\mu/d\nu) \land 1)d\nu
\]

and apply the triangle inequality to obtain

\[
\|\mu Q_\mu - \nu Q_\nu\|_{TV} = \|(\eta + (\mu - \eta))Q_\mu - (\eta - (\nu - \eta))Q_\nu\|_{TV} = \int_E |\eta Q_\mu(dx) + (\mu - \eta)Q_\mu(dx) - \eta Q_\nu(dx) - (\nu - \eta)Q_\nu(dx)| \leq \int_E |\eta Q_\mu(dx) - \eta Q_\nu(dx)| + \int_E |(\mu - \eta)Q_\mu(dx) - (\nu - \eta)Q_\nu(dx)| \leq \|\eta Q_\mu - \eta Q_\nu\|_{TV} + \|(\mu - \eta)Q_\mu - (\nu - \eta)Q_\nu\|_{TV}.
\]

Consider the first term, applying Jensen’s inequality and (4) to it, and using the following fact: \( \eta(E) = 1 - \|\mu - \nu\|_{TV}/2 \). We get

\[
\|\eta Q_\mu - \eta Q_\nu\|_{TV} = \int_E \left| \int Q_\mu(x, dy)\eta(dx) - \int Q_\nu(x, dy)\eta(dx) \right| \leq \int_E \int E |Q_\mu(x, dy) - Q_\nu(x, dy)|(\eta(dx) \leq \lambda_k \|\mu - \nu\|_{TV} \left( 1 - \frac{1}{2}\|\mu - \nu\|_{TV} \right).
\]

Then the second term

\[
\|(\mu - \eta)Q_\mu - (\nu - \eta)Q_\nu\|_{TV} = \int_E |(\mu - \eta)Q_\mu(dx) - (\nu - \eta)Q_\nu(dx)| = \int_E \left| \int Q_\mu(x, dy)(\mu - \eta)(dx) - \int Q_\nu(x', dy)(\nu - \eta)(dx') \right|.
\]
Lemma 1. Let \( a_0, a_1, \ldots \) be a sequence of positive numbers. Assume that \( 0 < a_0 \leq 1 \) and the following estimate is true

\[
a_{n+1} \leq a_n(1 - \psi(a_n)), \quad n \in \mathbb{Z}_+,
\]

where \( \psi : [0, \infty) \to [0, 1] \) is a continuous non-decreasing function with \( \psi(0) = 0 \) and \( \psi(x) > 0 \) as \( x > 0 \). Then

\[
a_n \leq g^{-1}(n)
\]

for all \( n \in \mathbb{Z}_+ \), where

\[
g(x) = \int_x^{a_0} \frac{dt}{t \psi(t)}, \quad 0 < x \leq 1.
\]

Remind that \( \mu_n = \text{Law}(X_n^p) \), \( \nu_n = \text{Law}(X_n^\nu) \), denote \( p_0 = \|\mu_0 - \nu_0\|_{TV}/2 \), assuming \( p_0 > 0 \) (if \( p_0 = 0 \), then \( p_k = 0 \) etc.).

Estimating the expression \( \|\mu_k - \nu_k\|_{TV} \) from above.

\[
\|\mu_k - \nu_k\|_{TV} \leq \lambda_k \|\mu_0 - \nu_0\|_{TV} \left(1 - \frac{1}{2}\|\mu_0 - \nu_0\|_{TV}\right)
\]

\[
+ p_0 \int \left| \int Q_\mu(x, dy) \frac{(\mu_0 - \eta_0)(dx)}{\lambda_k} - \int Q_\nu(x', dy) \frac{(\nu_0 - \eta_0)(dx')}{\lambda_k} \right|
\]

\[
= 2p_0 \lambda_k (1 - p_0) + p_0 \int \int \left| (Q_\mu(x, dy) - Q_\nu(x', dy)) \frac{(\mu_0 - \eta_0)(dx)}{\lambda_k} \right|
\]

\[
\leq 2p_0 \lambda_k (1 - p_0) + 2(1 - \alpha_k)p_0 \int \left| (\mu_0 - \eta_0)(dx) \right|
\]

Thus, iterating the estimate for

\[
\text{if } \lambda_k < \alpha_k, \text{ we obtain}
\]

\[
\|\mu_k - \nu_k\|_{TV} \leq \|\mu_0 - \nu_0\|_{TV}(1 - \alpha_k + \lambda_k),
\]

while in the case \( \lambda_k = \alpha_k \) we get

\[
\|\mu_k - \nu_k\|_{TV} \leq 2p_0 (1 - \lambda_k p_0),
\]

or

\[
p_k \leq p_0 (1 - \lambda_k p_0).
\]

In case \( kn + i, 0 < i < k \) we have:

\[
\|\mu_{kn+i} - \nu_{kn+i}\|_{TV} \leq \lambda_i \|\mu_{kn+i} - \nu_{kn+i}\|_{TV} \left(1 - \frac{1}{2}\|\mu_{kn+i} - \nu_{kn+i}\|_{TV}\right)
\]

\[
+ p_{kn} \int \left| \int P_{\mu}(x, dy) \frac{(\mu_{2n} - \eta_{2n})(dx)}{\lambda_{n+i}} - \int P_{\nu}(x', dy) \frac{(\nu_{2n} - \eta_{2n})(dx')}{\lambda_{n+i}} \right|
\]

\[
= 2p_{2n} \lambda_i (1 - p_{2n}) + p_{2n} \int \left| (P_{\mu}(x, dy) - P_{\nu}(x', dy)) \frac{(\mu_{2n} - \eta_{2n})(dx)}{\lambda_{n+i}} \right|
\]

\[
\leq 2p_{2n} \lambda_i (1 - p_{2n}) + p_{2n} \int \left| (P_{\mu}(x, dy) - P_{\nu}(x', dy)) \frac{(\mu_{2n} - \eta_{2n})(dx)}{\lambda_{n+i}} \right|
\]

\[
\leq 2p_{2n} (\lambda_i (1 - p_{2n}) + 1) = (1 + \lambda_i) \|\mu_{2n} - \nu_{2n}\|_{TV}.
\]

Hence,

\[
\|\mu_{2n+1} - \nu_{2n+1}\|_{TV} \leq (1 + \lambda_i) \|\mu_{2n} - \nu_{2n}\|_{TV}.
\]

Thus, iterating the estimate for \( \lambda_k < \alpha_k \), we obtain by induction

\[
\|\mu_n - \nu_n\|_{TV} \leq \|\mu_0 - \nu_0\|_{TV}(1 - \alpha_k + \lambda_k)^{[n/k]}(1 + \lambda_1)^{[n \mod k]}.
\]
This lemma and its proof are given in a slightly different version in [8 Lemma 1.4.2]. Since we use a slightly modified version with a different upper limit in the integral, the proof is presented for the reader’s convenience, even though it coincides with the source.

**Proof.** Notice that function $g^{-1}$ exists, as $g$ is unbounded, non-negative and strictly decreasing. Since $\psi(x) > 0$, then $a_{n+1} \leq a_n$ for any $n \in \mathbb{N}$. Therefore we can find $s \in [a_{n+1}, a_n]$ such that

$$g(a_{n+1}) - g(a_n) = g'(s)(a_{n+1} - a_n) = \frac{a_{n+1} - a_n}{s\psi(s)} \geq \frac{a_n\psi(a_n)}{s\psi(s)} \geq 1.$$  

Thus, $g(a_n) \geq n$ and $a_n \leq g^{-1}(n)$.

Applying the lemma [1] for $a_{n+k}$ and $\psi(t) = \lambda_k t$ we obtain

$$p_n \leq g^{-1}(n) = \frac{1}{\lambda_k n + \frac{1}{p_0}} = \frac{p_0}{1 + p_0 \lambda_k n}.$$  

Generalizing the result for arbitrary $n$ using (10) we get

$$\|\mu_n - \nu_n\|_{TV} \leq \frac{\|\mu_0 - \nu_0\|_{TV}}{2 + \lambda_k n \|\mu_0 - \nu_0\|_{TV}} (1 + \lambda_1)^{\lfloor n \mod k \rfloor}.$$  

Next, we proceed to the proof of Theorem [1].

**Proof.** Consider a sequence of probability measures $(\mu_n)_{n \in \mathbb{N}}$. According to the Theorem [2] by virtue of [8] and [9], for any $m, n \in \mathbb{N}$

$$\|\mu_n - \mu_{n+m}\|_{TV} \leq \frac{\|\mu_0 - \mu_m\|_{TV}}{2 + \lambda_k n \|\mu_0 - \mu_m\|_{TV}} (1 + \lambda_1)^{\lfloor n \mod k \rfloor}.$$  

Then $(\mu_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in complete metric space $(\mathcal{P}(E), \|\cdot\|_{TV})$ and we can find $\pi \in \mathcal{P}(E)$ such that $\lim_{n \to \infty} \|\mu_n - \pi\|_{TV} = 0$.

Let us show that the limiting measure $\pi$ is invariant. For this, we can use the triangle inequality and the condition (10) as $n \to \infty$:

$$\|\pi P_\pi - \mu_{n+1}\|_{TV} = \|\pi P_\pi - \mu_n P_{\mu_n}\|_{TV} \leq (1 + \lambda_1)\|\pi - \mu_n\|_{TV} \to 0,$$  

while $\mu_{n+1} \to \pi$, we have

$$\|\pi P_\pi - \pi\|_{TV} \leq \|\pi P_\pi - \mu_n\|_{TV} + \|\mu_{n+1} - \pi\|_{TV} \to 0.$$  

From where we get $\pi = \pi P_\pi$.

To prove the uniqueness of the invariant measure $\pi$, assume that $\nu \in \mathcal{P}(E)$ is such that $\nu \neq \pi$ and $\nu = \nu P_\nu$, then iteratively applying the results (8) and (9), for a sufficiently large $n$ we obtain a contradiction

$$\|\nu - \pi\|_{TV} = \|\nu Q_n \pi - \pi Q_n\|_{TV} = \|\nu Q_n^n - \pi Q_n^n\|_{TV} < \|\nu - \pi\|_{TV}.$$  

Thus, the process $(X_n^\mu)_{n \in \mathbb{Z}}$ has a unique invariant measure $\pi$.

In section [3] we show an example of a discrete-time nonlinear Markov chain satisfying such ergodic conditions.
3 An example of ergodic nonlinear Markov chain

Let us show that this result can be used in cases where the one-step estimate is inapplicable, and, at the same time, violation of [3] conditions does not prevent exponential convergence for some nonlinear Markov chains. Consider the following discrete nonlinear Markov chain $X^p_n$ with state space $(\mathcal{E}, \mathcal{F}) = ([1, 2, 3, 4], 2^{[1, 2, 3, 4]})$, the initial distribution $\mu_0$ and the transition probability matrix $P_{\mu_0}(i, j)$, defined as follows:

$$
\mu_0 = (\mu(\{1\}) \quad \mu(\{2\}) \quad \mu(\{3\}) \quad \mu(\{4\})) \quad \text{and}
$$

$$
P_{\mu_0}(i, j) = \begin{pmatrix}
0 & \gamma \mu(\{1\}) & 0.5 - \gamma \mu(\{1\}) & 0.5 \\
0.5 & 0 & 0 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.5 & 0 & 0.5
\end{pmatrix},
$$

where $0 \leq \gamma \leq 0.5$.

We can notice that for a given process the conditions [1] and [2] do not guarantee convergence to an invariant measure, since $\lambda > \alpha$, since $\alpha = 0$ and $\lambda = \gamma$.

However, if we consider the corresponding three-step transition probabilities matrix $Q_{\mu_0}(i, j)$, we may obtain the following result. We have $\lambda_3 < \alpha_3$, as $\lambda_3 = \gamma/4$, while $\alpha_3$ reaches its minimum for a pair of states $\{3, 4\}$ with the value in range $[0.75; 0.75 + 0.125\gamma]$, thus $\alpha_3 = 0.75$. Thus, the proposed estimate can guarantee exponential convergence in some cases when the existing [3] result does not work. And we may also compare the rate of convergence on this example by taking estimates for two [4] and three steps. We may consider the part of estimate $(1 - \alpha_k + \lambda_k)^{[n/k]}(1 + \lambda_1)^{[n \mod k]}$ for the case $\alpha_k < \lambda_k$, since we have straightforward results in case $\alpha_k = \lambda_k$. In Table 1 we calculate the rate of convergence for $k = 2$ and $k = 3$ in limit values of the parameter $\gamma \in [0, 1/2]$. Here we omit the multiplier $(1 + \lambda_k)^{[n \mod k]}$ for the case $\gamma = 1/2$, since it does not impact much the convergence itself.

Table 1: Rate of convergence calculation, $(1 - \alpha_k + \lambda_k)^{[n/k]}$.

| $k$ | $\gamma = 0$ | $\gamma = 1/2$ |
|-----|---------------|----------------|
| 2   | $0.707107^n$  | $0.866025^n$   |
| 3   | $0.629961^n$  | $0.721125^n$   |

We can see that the result obtained using three steps tend to be better given that the remainders $n \mod k$ are the same.

4 Law of large numbers

This section presents a particular result of the law of large numbers for nonlinear Markov chains. In literature, there exist similar results, for example, [1], still, the model under consideration is quite different. However, such a result can be helpful in nonparametric estimation and serves as the basis for further statistics.
Theorem 3. Let \( X_k^\mu \) be a nonlinear Markov chain defined on a measurable finite state space \((E, \mathcal{E})\) that satisfies the conditions of Theorem 1 and \( f, g : E \to \mathbb{R} \) be a measurable bounded continuous functions. The function \( f \) is also non-negative and should have uniform modulus of continuity. Denote \( S_n = \sum_{k=0}^{n-1} g(X_k^\mu) \), then the sequence \( S_n \) satisfies the law of large numbers, such as, \( n \to \infty \)

\[
\left( \frac{S_n}{n} \right) \xrightarrow{p} f(E[g(X_0^\mu)]),
\]

where \( E[g(X_0^\mu)] = \int g(x) \pi(dx) \), \( X_k^\mu \) is a copy of \( X_k^\mu \) with initial distribution equal to the invariant measure \( \pi \).

Proof. First, we can notice that \( X_k^\mu \), which is, in fact, a homogeneous “linear” Markov chain that does not depend on the measure, and it is also \( \beta \)-mixing due to the (3). Denote \( \xi_k = g(X_k^\mu) - E[g(X_k^\mu)] \) and consider the sequence of sums \( \tilde{S}_n^\mu = \sum_{k=0}^{n-1} \xi_k \).

Then, according to the results of Ibragimov and Linnik [7, Theorem 18.5.3], we have that \( E[|\xi_k|^{2+\delta}] < \infty \) for some \( \delta > 0 \) and \( \sum_{n=1}^{\infty} \alpha(n) \delta/(2+\delta) < \infty \), where \( \alpha(n) \) is a mixing coefficient. The variance of \( \tilde{S}_n^\mu \) is finite, \( \sigma^2 = \text{Var}(\tilde{S}_n^\mu) = \sum_{j=-\infty}^{\infty} \text{cov}(\tilde{\xi}_j, \tilde{\xi}_j) < \infty \). Then, the sequence \( \tilde{S}_n^\mu \) satisfies the central limit theorem, \( \tilde{S}_n^\mu \sim \mathcal{N}(0, \sigma^2/n) \). The case \( \sigma^2 = 0 \) can be interpreted as a degenerate Gaussian distribution, or Dirac delta distribution concentrated at the point 0.

Therefore, the process \( \tilde{S}_n^\mu \) also satisfies the central limit theorem, implying the weak law of large numbers,

\[
\frac{S_n^\mu}{n} \xrightarrow{p} E[g(X_0^\mu)].
\]

Next, we need to show that \( S_n/n - S_n^\mu/n \) converges to 0. In order to prove this convergence, a coupling construction technique for nonlinear Markov chains similar to the method presented in [2, 5, 6] can be developed.

Alternatively, we can use the estimates for convergence from Theorem 1 to show that for \( n \to \infty \), \( \forall f \in \mathcal{C}_0 \), \( f \geq 0 \), \( E[f(S_n/n)] \to f(E[g(X_0^\mu)]) \).

Let

\[
E[f(S_n/n)] = \int \cdots \int f \left( \frac{x_0 + x_1 + \cdots + x_{n-1}}{n} \right) \mu_0(x_0)P_{x_0,x_1}P_{x_1,x_2} \cdots P_{x_{n-2},x_{n-1}}.
\]

From the Theorem 1 we get that \( \|\mu_n - \pi\|_{TV} \leq 2e^{-Cn} \), where \( C \) is a constant that does not depend on \( n \). Then assume that all elements of \( P_{x,y}^\mu \) are bounded away from zero uniformly with respect to the measure,

\[
\frac{\|\mu_n - \pi\|_{TV}}{2P_{x,y}^\mu} \leq \frac{P_{x,y}^\mu - P_{x,y}^\pi}{P_{x,y}^\pi} \leq \frac{\|\mu_n - \pi\|_{TV}}{2P_{x,y}^\pi},
\]

hence we deduce

\[
1 - e^{\ln K - Cn} \leq 1 - e^{-Cn} \leq \frac{P_{x,y}^\mu}{P_{x,y}^\pi} \leq 1 + e^{-Cn} \leq 1 + e^{\ln K - Cn}.
\]

Then, we can assume that starting from some moment \( n_0 + 1 \) such that \( n_0 \ll n \), the transition kernels \( P_{x,y}^\mu \) are close to the invariant transition kernel \( P_{x,y}^\pi \). We may define a transition kernel for \( n_0 \) steps, \( Q_{x_0, dx_{n_0}}^{\mu_0,n_0} \), such that

\[
\mu_n = \mu_0 Q_{x_0, dx_{n_0}}^{\mu_0,n_0}(dx_{n_0}) = \int \mu_0(dx_0)Q_{x_0, dx_{n_0}}^{\mu_0,n_0}(x_0, dx_{n_0}).
\]
Then we can denote $\rho_f$ the modulus of continuity of function $f$, $|\sum_{i=0}^{n} x_i/n| \leq \delta$ and rewrite

$$E[f(S_n/n)] = \int \cdots \int f \left( \frac{x_0 + \cdots + x_n}{n} + \frac{x_{n+1} + \cdots + x_{n-1}}{n} \right) \mu_0 Q_{n_0} (dx_{n_0}) \prod_{k=n_0}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k},$$

$$\leq \rho_f(\delta) + \int \cdots \int f \left( \frac{x_{n+1} + \cdots + x_{n-1}}{n} \right) \left( 1 + e^{\ln K - C(n_o - 1)} \right) \pi(dx_{n_0}) \prod_{k=n_0}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k},$$

$$\leq \rho_f(\delta) + \prod_{k=n_0-1}^{n-2} \left( 1 + e^{\ln K - Ck} \right) \prod_{k=n_0}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k} \cdot \left( \int \cdots \int f \left( \frac{\sum_{i=n_0+1}^{n-1} x_i}{n} \right) \pi(dx_{n_0}) \prod_{k=n_0+1}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k} \right).$$

Then, for the invariant distribution we have

$$E[f(S_n^*/n)] = \int \cdots \int f \left( \frac{x_0 + \cdots + x_n}{n} + \frac{x_{n+1} + \cdots + x_{n-1}}{n} \right) \pi(dx_{n_0}) \prod_{k=n_0}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k},$$

$$\geq - \rho(\delta) + \int \cdots \int f \left( \frac{\sum_{i=n_0+1}^{n-1} x_i}{n} \right) \pi(dx_{n_0}) \prod_{k=n_0+1}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k} \cdot \left( \int \cdots \int f \left( \frac{\sum_{i=n_0+1}^{n-1} x_i}{n} \right) \pi(dx_{n_0}) \prod_{k=n_0+1}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k} \right).$$

Hence, we may obtain

$$E[f(S_n/n)] \leq \rho_f(\delta) + (\rho_f(\delta) + E[f(S_n^*/n)]) \prod_{k=n_0-1}^{n-2} \left( 1 + e^{\ln K - Ck} \right).$$

Analogically we can derive the following result for the lower bound of $E[f(S_n/n)]$, that is

$$E[f(S_n/n)] \geq - \rho_f(\delta) + \int \cdots \int f \left( \frac{x_{n+1} + \cdots + x_{n-1}}{n} \right) \left( 1 - e^{\ln K - C(n_o - 1)} \right) \pi(dx_{n_0}) \prod_{k=n_0}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k},$$

$$\geq - \rho_f(\delta) + \prod_{k=n_0-1}^{n-2} \left( 1 - e^{\ln K - Ck} \right) \prod_{k=n_0+1}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k} \cdot \left( \int \cdots \int f \left( \frac{\sum_{i=n_0+1}^{n-1} x_i}{n} \right) \pi(dx_{n_0}) \prod_{k=n_0+1}^{n-2} P_{x_k, dx_{k+1}}^{\pi_k} \right).$$

Next, we can consider the bounds for $E[f(S_n/n)]$,

$$- \rho_f(\delta) + (E[f(S_n^*/n)] - \rho_f(\delta)) \prod_{k=n_0-1}^{n-2} \left( 1 - e^{\ln K - Ck} \right) \leq E[f(S_n/n)] \leq \rho_f(\delta) + (\rho_f(\delta) + E[f(S_n^*/n)]) \prod_{k=n_0-1}^{n-2} \left( 1 + e^{\ln K - Ck} \right).$$
Notice that $\rho_f(\delta)$ as $\delta \to 0$, and that both multipliers $\prod_{k=n_0-1}^{n-2} (1 - e^{K-C_k})$ and $\prod_{k=n_0-1}^{n-2} (1 + e^{K-C_k})$ approach 1 as $n_0 \to \infty$ uniformly over $n > n_0$, while $\mathbb{E}[f(S_n/n)]$ itself does not depend on $n_0$. Thus, due to (12) we obtain for any non-negative $f \in C_b$,

$$
\limsup_{n \to \infty} \mathbb{E}[f(S_n/n)] \leq f(\mathbb{E}[g(X_0^\pi)]) ,
$$

and

$$
\liminf_{n \to \infty} \mathbb{E}[f(S_n/n)] \geq f(\mathbb{E}[g(X_0^\pi)]) ,
$$

which implies

$$
\lim_{n \to \infty} \mathbb{E}[f(S_n/n)] = f(\mathbb{E}[g(X_0^\pi)]) ,
$$

which is equivalent to the convergence in probability (11) as required. Theorem 3 is proved.

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