SURPRISES FOR QCD AT NONZERO CHEMICAL POTENTIAL

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In this lecture we compare different QCD-like partition functions with bosonic quarks and fermionic quarks at nonzero chemical potential. Although it is not a surprise that the ground state properties of a fermionic quantum system and a bosonic quantum system are completely different, the behavior of partition functions with bosonic quarks does not follow our naive expectation. Among other surprises, we find that the partition function with one bosonic quark only exists at nonzero chemical potential if a conjugate bosonic quark and a conjugate fermionic quark are added to the partition function.

Keywords: QCD at nonzero chemical potential, bosonic quarks

1. Introduction

The QCD phase diagram in the chemical potential temperature plane has far reaching phenomenological implications ranging from heavy ion collisions to the interior of neutron stars. Unfortunately, first principle lattice simulations are only possible at zero chemical potential, and our knowledge of the phase diagram mainly relies on model calculations (see for example 1, 2). Physically, we know that at zero temperature the baryon density is zero below a chemical potential equal to $m_N/3$. Therefore, in the thermodynamic limit, the QCD free energy and its derivatives, such as for example the chiral condensate, do not depend on the chemical potential for $\mu < m_N/3$. Since the Dirac operator depends on the chemical potential this requires miraculous cancellations in the microscopic theory a problem that was coined\footnote{3} as the The Silver Blaze Problem. This problem becomes particularly manifest in terms of the eigenvalues of the Dirac operator which are distributed homogeneously in a strip\footnote{4} with a width that increases a function
critical endpoint

Fig. 1. Possible phases of the QCD partition function at nonzero temperature and chemical potential.

of $\mu$. In this lecture we will mainly focus on the zero temperature axis of the phase diagram. To better understand the effect of the baryon chemical potential in QCD, we will consider four different partition functions listed in Table 1 which were discussed in 5 and 6. Although, the first partition function is physically the most relevant, the other partition functions have important applications. Because lattice QCD simulations of full QCD at nonzero chemical potential are not possible, one sometimes uses the phase

| Theory | Number of Charged Goldstone Modes for $\mu < \mu_c$ | Critical Chemical Potential |
|--------|---------------------------------------------------|----------------------------|
| $\langle \det(D + \mu \gamma_0 + m) \rangle$ | 0 | $\mu_c = \frac{1}{3} m_N$ |
| $\langle |\det(D + \mu \gamma_0 + m)|^2 \rangle$ | 2 | $\mu_c = \frac{1}{2} m_\pi$ |
| $\langle \frac{1}{\det(D + \mu \gamma_0 + m)} \rangle$ | 4 | $\mu_c = \frac{1}{2} m_\pi$ |
| $\langle \frac{1}{|\det(D + \mu \gamma_0 + m)|^2} \rangle$ | na | $\mu_c = 0$ |

Table 1. Summary of properties of low energy QCD at nonzero chemical potential and zero temperature. These partition functions will be denoted by $Z_{N_f=1}$, $Z_{n=1}$, $Z_{N_f=-1}$, $Z_{n=-1}$, in this order.
quenched approximation where
\[
\langle \det^2(D + \mu\gamma_0 + m) \rangle \rightarrow \langle |\det(D + \mu\gamma_0 + m)|^2 \rangle,
\]
which can be interpreted as a partition function of quarks and conjugate quarks. Then Goldstone bosons made out of quarks and conjugate antiquarks have nonzero baryon number resulting in a critical chemical potential of \( m_\pi/2 \) instead of \( m_N/3 \).

The bosonic partition function occurs in the formula for the quenched spectral density in the microscopic domain of QCD which is given by
\[
\rho_{\text{quen}}(z, \mu) = \left| z \right|^2 2 Z_{n=1}(z, \mu)Z_{n=-1}(z, \mu).
\]

In a future publication we will consider the expectation value of the phase of the fermion determinant given by
\[
\langle e^{2i\theta} \rangle = \left\langle \frac{\det(D + \mu\gamma_0 + m)}{\det(-D + \mu\gamma_0 + m^*)} \right\rangle.
\]

This partition function is not among the above list, but based on our insights from the bosonic partition function, we will be able to predict its phase diagram.

2. Gauge Invariance and the Phases of QCD at \( \mu \neq 0 \)

The principle that underlies the independence of the free energy on the chemical potential is gauge invariance. The Dirac operator can be written as
\[
D + \mu\gamma_0 + m = e^{-\mu\tau}(D + m)e^{\mu\tau},
\]
which implies that the \( \mu \)-dependence can be transformed into the boundary conditions. A \( \mu \)-independent free energy is possible in a phase that is not sensitive to the boundary conditions. This is the case for \( \mu < \mu_c \) when the quarks do not loop around the torus in the time direction.

Although the partition function \( Z_{n=-1} \) is naively gauge invariant it turns out that the regulator of the partition function breaks gauge invariance so that a \( \mu \)-independent phase cannot exist. The need for regularization is best seen by writing the partition function in terms of eigenvalues
\[
Z_{n=-1} = \int_{\mathbb{C}/C_m(\epsilon)} \prod_k d^2z_k \prod_k \frac{\rho(\{z_k\})}{(z_k^2 - m^2)(z_k^2 - m^2)},
\]
where \( \mathbb{C}/C_m(\epsilon) \) is the complex plane except two small spheres with radius \( \epsilon \) around \( \pm m \). Because of the complex conjugated pole the integral diverges.
as \( \log \epsilon \). Instead of this regularization we prefer to regularize the partition function as\(^1\) \( \epsilon \) (also known as hermitization\(^2\))

\[
Z_{n=-1} = \left\langle \det^{-1} \left( \begin{array}{cc} \epsilon & D + \mu \gamma_0 + m \\ -D + \mu \gamma_0 + m^* & \epsilon \end{array} \right) \right\rangle.
\]  

(6)

Since the matrix inside the determinant is Hermitian, this partition function can be written as a convergent bosonic integral. However, \( \epsilon \) breaks gauge invariance, and for \( \epsilon \neq 0 \) it is not possible to gauge away the chemical potential. We thus find that \( \mu_c = 0 \) in this case.

Let us now consider the partition function with one bosonic flavor. This partition function cannot be written as a convergent bosonic integral and therefore cannot be interpreted in terms of bosonic quarks only. The correct interpretation is to rewrite this partition function as\(^6\)

\[
Z_{N_f=-1} = \left\langle \frac{\det^*(D + \mu \gamma_0 + m)}{\det(D + \mu \gamma_0 + m)\det^*(D + \mu \gamma_0 + m)} \right\rangle.
\]  

(7)

and regulate the denominator as in (6). However, contrary to the case of a pair of conjugate bosonic quarks, this partition function does not diverge for \( \epsilon \to 0 \), and it is possible to gauge away the chemical potential. In this case the free energy will be \( \mu \)-independent in the thermodynamic limit below the lightest particle with nonzero baryon number which is a Goldstone boson made out of a bosonic quark and a conjugate bosonic anti-quark.

### 3. Low Energy Limit of QCD

The low-energy limit of the partition functions in Table 1 uniquely follows from chiral symmetry and gauge invariance. In Table 2 and Fig. 2 we compare the bosonic and fermionic (see 7, 10, 13, 14, 15) partition functions with a pair of conjugate flavors.

For \( N_f = -1 \) the partition function is given by the ratio in eq. (7). In this case the partition function is finite for vanishing regulator and the gauge symmetry (4) is not obstructed. Therefore, we have a \( \mu \)-independent phase for \( \mu < \mu_c \). In this phase the chiral condensate is given by

\[
\langle \bar{\psi} \psi \rangle_{N_f=-1} = \frac{1}{V} \left( \sum_k \frac{1}{z_k + m} + \sum_k \frac{1}{z_k^* + m} - \sum_k \frac{1}{z_k + m} \right) \quad \text{for} \quad \mu < \mu_c
\]

\[
= 2\Sigma - \Sigma = \Sigma.
\]  

(8)

For \( \mu > \mu_c \) the bosonic contribution to the chiral condensate rotates into a pion condensate (see Fig. 3) so that for \( \mu \gg \mu_c \) only the fermionic contribution remains:

\[
\langle \bar{\psi} \psi \rangle = -\Sigma \quad \text{for} \quad \mu \gg \mu_c.
\]  

(9)
\[
\frac{1}{\det(D + \mu \gamma_0 + m)^2} \quad \frac{1}{\det(D + \mu \gamma_0 + m)^{-2}}
\]

Table 2. Comparison of the \(n = 1\) and the \(n = -1\) partition function.

4. Chiral Symmetry Breaking at \(\mu \neq 0\) and the Dirac Spectrum

In the domain where the kinetic term of the chiral Lagrangian factorizes from the partition function, i.e. for \(\mu \ll 1/L\) and \(|z| \ll 1/L^2\), the quenched spectral density satisfies the relation (2). This offers the possibility to test our results for \(Z_{n=1}\) by means of lattice QCD simulations. Calculations both with staggered fermions and overlap fermions show an impressive
agreement\textsuperscript{16} with (2).

In the thermodynamic limit the eigenvalues are distributed homogeneously inside the strip

\[ |\text{Re}(z)| < \frac{2\mu^2 F^2}{\Sigma}. \]

Therefore, inside this strip, the quenched chiral condensate goes to zero linearly. The behavior of the chiral condensate for full QCD is quite different. In that case the chiral condensate remains nonzero for \( m \to 0 \). On the other hand, the eigenvalues still spread out in the complex plane. To explain\textsuperscript{5} this so called “Silver Blaze Problem” we introduce the “spectral density”

\[ \rho^{\text{full}}(z, \mu) = \frac{\langle \det(D + \mu \gamma_0 + m) \sum_k \delta^2(z - z_k) \rangle}{\langle \det(D + \mu \gamma_0 + m) \rangle}. \]

Because of the phase of the fermion determinant \( \rho^{\text{full}}(z, \mu) \) is in general complex. Its microscopic limit is known analytically\textsuperscript{17} and can be decomposed as

\[ \rho^{\text{full}}(z, \mu) = \rho^{\text{quen}}(z, \mu) + \rho^{\text{osc}}(z, \mu), \]

where \( \rho^{\text{osc}} \) is complex with oscillations with a period of \( O(1/V) \) and an amplitude that diverges exponentially with the volume. For \( V \to \infty \) it vanishes outside a region with \( m < |\text{Re}(z)| < \frac{8}{3} \mu^2 F^2 / \Sigma - \frac{m}{3} \). The chiral condensate follows the same decomposition

\[ \Sigma^{\text{full}} = \Sigma^{\text{quen}} + \Sigma^{\text{osc}}. \]
Fig. 4. The chiral condensate in quenched QCD ($\Sigma_{\text{quen}}$) and in full QCD ($\Sigma_{\text{full}}$) as a function of the mass. The support of the Dirac spectrum is in between the vertical lines.

In Fig. 4 we show the behavior of the different contributions in the thermodynamic limit. This shows that a nonzero chiral condensate for $m \to 0$ is due to the oscillatory contribution to the spectral density\(^5\). Therefore these oscillations solve the Silver Blaze Problem.

5. Conclusions

The behavior of bosonic partition functions at nonzero chemical potential is quite different from what could be expected naively. This surprising behavior can be understood from gauging the chemical potential into the
boundary conditions. In particular, this shows that the partition function with a pair of conjugate bosonic quarks has no $\mu$-independent phase. The free energy of the theory with one bosonic quark, on the other hand is $\mu$-independent for $\mu < \frac{m_\pi}{2}$. However, this partition function only exists as a partition function of a pair of conjugate bosonic quarks and a fermionic quark with the same mass. Finally, an analysis along the lines of this paper\textsuperscript{8} shows that the expectation value of the phase of the fermion determinant, eq. (3), behaves as in Fig. 5.

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