String cosmology with Hermitian structure

F. Naderi∗, A. Rezaei-Aghdam† and F. Darabi‡

Department of Physics, Azarbaijan Shahid Madani University
53714-161, Tabriz, Iran

April 30, 2015

Abstract

We consider a string cosmological model related to a σ-model whose antisymmetric tensor field is constructed as a geometrical structure from complex structure on the background manifold, specially on a manifold $R \times N$, $N$ being a complex manifold. As an example, we consider an homogeneous anisotropic (1+4) string cosmological model where space part of the background is a four dimensional complex manifold. Then, by solving the related one-loop β-functions we obtain a static solution and then by reduction of this solution to (1+3) dimensions we obtain a cosmological model where the matter part is effectively interpreted as a barotropic matter with two components: one as a homogeneous and isotropic component $\rho$ and the other as a inhomogeneous and anisotropic component $\rho_n(r)$. It is shown that the isotropic part corresponds to a cosmic string, and the inhomogeneous part violates the dominant energy condition. Finally, the T-dual backgrounds of the solution is investigated and it is shown that the duality transformation and reduction processes commute with each other.

1 Introduction

String theory is the most consistent theory for unifying the fundamental interactions, incorporating gravity. The cosmological implications of string theory have recently received considerable attention [1]. In this context, the string theory has been used in the study of the physical situation at very early universe [2].

The string theory in curved background, generally known as a conformal σ-model, is concerned with the massless modes of the string including metric, a rank two antisymmetric tensor field $B_{\mu\nu}$ and a dilaton field $\phi$. In the σ-model approach, the conformal invariance condition is the vanishing of the β-functions of the fields which may be interpreted as field equations of effective space-time action [3, 4, 5].

An interesting class of manifolds in the study of (super)string theory includes complex manifolds in which the set of solutions of string equations is based on complex non-Kähler manifolds [6, 7, 8, 9]. In Ref. [10], Witten has constructed a topological sigma model where the complex structure $J$ of target space appeared in the Lagrangian in an interaction term like Kalb-Ramond field, through a supersymmetry. From another but related point of view, the flux $H$ could be related to the almost complex structure by imposing supersymmetry as $dJ \sim *H$ [11, 12, 13].

In this paper, we are interested in the idea of our recent paper to establish a physical interpretation for complex structures [14]. We know that in general relativity the gravitational effects are characterized by a geometrical structure, so called metric, on a Riemannian manifold. But, on a complex manifold there is an additional geometrical structure, namely the complex structure $J$, which may be capable of carrying physical significance. In this regard, we have already presented a matter interpretation for the almost complex structure [14]. Here, we will investigate the solution of β-functions of a σ-model with a non-zero central charge deficit where the role of antisymmetric tensor field $B$ is played by the complex structure (or more clearly by the fundamental antisymmetric kähler form $\Omega$ which is related to the complex structure with a hermitian metric). In this way, similar to the metric, the antisymmetric tensor field in the context of σ-model will be a geometrical structure of the manifold. We will assume that the background space-time is a (1 + 4) dimensional manifold

*e-mail: f.naderi@azaruniv.edu
†Corresponding author. e-mail: rezaei-a@azaruniv.edu
‡e-mail: f.darabi@azaruniv.edu

Here * is Hodge star and is expressed as $*H_{ijk} = \frac{1}{2!} \varepsilon_{ijk} H^{mnp}$. 

1 Here * is Hodge star and is expressed as $*H_{ijk} = \frac{1}{2!} \varepsilon_{ijk} H^{mnp}$. 

1
such that the 4-dimensional space part is a complex manifold carrying a complex structure. Then, the $B$-field defined on the space-time is considered to be related to the complex structure of 4-dimensional part.

The organization of the paper is as follows. In section 2, we start by collecting some preliminaries about string cosmology and Hermitian structure. In section 3, we introduce the special backgrounds of $(1 + 4)$ dimensional $\sigma$-model consisting of a hermitian metric and a complex structure as $B$-field. Then, the solution of $\beta$-functions are investigated. In section 4, dimensionally reduction of the static $(1 + 4)$ dimensional solutions to $(1 + 3)$ dimensional space-time is performed. Then, we discuss the nature of derived cosmological solutions, with the particular anisotropic and inhomogeneous form of energy momentum tensor given by dilaton and $B$-field. In section 5, $T$-dual solutions are investigated. The paper ends with a conclusion.

## 2 Review on string cosmology and Hermitian structure

Consider the string in a curved background $M$, where its action is expressed by the two dimensional $\sigma$-model

$$I = \frac{1}{4 \pi \alpha'} \int_{\Sigma} \sqrt{h} d^2 z \left( h^{\alpha \beta} \ddot{g}_{\alpha \beta} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu + \epsilon^{\alpha \beta} \dot{B}_{\alpha \beta} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu + \alpha' R^{(2)}(\phi) \right),$$

such that $h$ denotes the metric of worldsheet $\Sigma$, $\ddot{g}_{\alpha \beta}$ is the metric of D-dimensional space-time $M$, $\phi$ and $\dot{B}_{\alpha \beta}$ are dilaton and antisymmetric tensor known as Kalb-Ramond field on $M$, respectively.

In the leading-order of string coupling $\alpha'$, the conditions of conformal (Weyl) invariance of $\sigma$-model [1], i.e. the one-loop $\beta$-functions, are given by [3] [5]

$$\beta_{\mu \nu} (\ddot{g}) = \ddot{R}_{\mu \nu} + \frac{1}{4} \dddot{H}_{\mu \nu} - \nabla_{\mu} \nabla_{\nu} \phi,$$

$$\beta_{\mu \nu} (\dot{B}) = \nabla^{\mu} (e^\phi \dddot{H}_{\mu \nu} \phi),$$

$$\beta (\ddot{\phi}) = -\dddot{R} - \frac{1}{12} \dddot{H}^2 + 2 \nabla_{\mu} \nabla^{\mu} \ddot{\phi} + (\partial_{\nu} \ddot{\phi})^2 + \Lambda.$$  

These equations are identical with the equations obtained by variation of the following one-loop string effective action with respect to the fields $\ddot{g}_{\mu \nu}$, $\ddot{B}_{\mu \nu}$ and $\ddot{\phi}$, respectively [4] [5]

$$S(\ddot{g}_{\mu \nu}, \ddot{B}_{\mu \nu}, \ddot{\phi}) = \frac{1}{2} \int d^D x \sqrt{-g} e^{\phi} \left( \dddot{R} - \frac{1}{12} \dddot{H}_{\mu \nu \rho} \dddot{H}^{\mu \nu \rho} + \partial_{\nu} \ddot{\phi} \partial^{\nu} \ddot{\phi} - \Lambda \right).$$

Here, $\dddot{H}_{\mu \nu \rho} = \partial_{\nu} \partial_{\rho} \ddot{H} - \partial_{\mu} \ddot{H}$, $\dddot{H} = \partial_{\nu} \partial^{\nu} \ddot{H}$, where the field strength $\dddot{H}$ of $B$-field is defined as follows

$$\dddot{H}_{\mu \nu \rho} = \partial_{\nu} \ddot{B}_{\rho \mu} + \partial_{\rho} \ddot{B}_{\nu \mu} + \partial_{\mu} \ddot{B}_{\nu \rho}.$$  

The constant $\Lambda$ is related to the central charge deficit of the original theory [15] [16] and is negligible in the large curvature or large kinetic of $\ddot{B}$ or $\ddot{\phi}$. The $\Lambda$ plays the role of cosmological constant [17].

The above formalism is in the string frame. Another useful frame, in which the string effective action [5] appears as the $D$-dimensional Einstein-Hilbert action of general relativity, is the Einstein frame. The new metric is called Einstein metric and is obtained by a rescaling of the string metric as follows [15]

$$\tilde{g}_{\mu \nu} = e^{\frac{\phi}{2}} \ddot{g}_{\mu \nu}.$$  

So, the Einstein frame form of action will be

$$S = \frac{1}{2} \int d^D x \sqrt{-g} \left( \tilde{\dddot{R}} - \frac{1}{12} e^{4\phi} \tilde{H}_{\mu \nu \rho} \tilde{H}^{\mu \nu \rho} - \frac{1}{D - 2} \partial_{\mu} \ddot{\phi} \partial^{\mu} \ddot{\phi} - e^{\frac{\phi}{2}} \ddot{\phi}^2 \Lambda \right).$$

In this frame, the equations $\beta(\ddot{g}, \ddot{B}, \ddot{\phi}) = 0$ in (2) - (4) are transformed to the Einstein equations as follows

$$\tilde{R}_{\mu \nu} - \frac{1}{2} \tilde{\dddot{R}} \tilde{g}_{\mu \nu} = \kappa^2 (T^{(\tilde{\phi})}_{\mu \nu} + T^{(\dddot{\phi})}_{\mu \nu}),$$

where $T^{(\tilde{\phi})}_{\mu \nu}$ and $T^{(\dddot{\phi})}_{\mu \nu}$ are the energy-momentum and the stress-energy tensor of $\tilde{\phi}$, respectively.
where the energy-momentum tensors are given by

\[
\kappa^2 T_{\mu\nu}^{(\phi)} = \frac{2}{D-2} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} (\partial \phi)^2 g_{\mu\nu} - \Lambda e^{-\phi} g_{\mu\nu} \right),
\]

and

\[
\kappa^2 T_{\mu\nu}^{(H)} = \frac{1}{4} \left( \hat{H}_{\mu\nu\lambda} \hat{H}^{\lambda}_{\rho\sigma} - \frac{1}{6} \hat{H}^2 g_{\mu\nu} \right).
\]

### 2.1 String cosmology with Hermitian structure

We consider a σ-model on a manifold \(M\), at least one part of which is a complex manifold. The manifold carries another geometrical structure beside the metric, namely the complex structure \(J\). Following the idea of giving a physical role to the complex structure, it is intriguing to investigate coupling of the complex structure to the string, included in the \(B\)-field interaction in the second term of the σ-model [1].

Before attempting to investigate such a model, let us collect some preliminaries about the complex structure. Let \(M\) be an even dimensional manifold, an almost complex Hermitian structure \((g, J)\) on \(M\) with coordinates \(\{x^i\}\) consisting of a Riemannian metric \(g = g_{ij} dx^i \otimes dx^j\) and a structure \(J = J^i_j dx^i \otimes \partial_j\) satisfying [18]

\[
J^i_j J^j_k = -\delta^i_k,
\]

\[
g_{ij} J^i_m J^j_n = g_{mn},
\]

which are definitions of the almost complex structure and Hermitian metric, respectively. These two conditions introduce a Hermitian structure. Fundamental two form \(\Omega\) is defined by \(\Omega = g(\cdot, J \cdot) = \frac{1}{2} \Omega_{ij} dx^i \wedge dx^j\), whose components are given by \(\Omega_{ij} = g_{ik} J^k_j\). Then from [12] and [13], we have

\[
\Omega_{ij} = -\Omega_{ji}.
\]

Nijenhuis tensor of an almost Hermitian manifold is defined by [19]

\[
N^i_{jk} = -J^i_m \partial_j J^m_k + J^i_j \partial_k J^m_j - J^m_j \partial_m J^i_j + J^m_j \partial_j J^i_m.
\]

An almost complex structure \(J\) on a manifold \(M\) is integrable if and only if \(N^i_{jk} = 0\); in this case the almost complex structure is called complex structure [18].

As mentioned above, we want to describe a string cosmology model on a special manifold where the \(B\)-field is related to the complex structure in a special case. For this propose, an appropriate part of the \(D\)-dimensional manifold must be a complex manifold equipped with a complex structure \(J\). On the other hand, noting the fact that the components of \(J\) should be real themselves, through the equations [12] and [13], a diagonal metric with Minkowski signature is not capable of being a Hermitian metric associated with a complex structure. This persuases us to consider a model on a manifold of the form \(M = R \times N\), where \(N\) is an even dimensional manifold which is demanded to be a complex manifold. The manifold \(N\) will describes the spatial part and therefore the signature problem will be removed. Summarizing, if the manifold \(M\) is denoted by hatted indices and \(N\) by the \(i, j, \ldots\) indices, we are going to consider an antisymmetric \(B\)-field of the form

\[
\hat{B} = \frac{1}{2} \hat{B}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{\gamma}{2} \Omega_{ij} dx^i \wedge dx^j, \quad \hat{B}_{0i} = 0,
\]

where \(\gamma\) is a coupling constant and \(\Omega_{ij} = g_{jk} J^k_i\) is the Hermitian kähler two form associated to \(g_{ij}\).

### 3 A static solution of string cosmology on (1 + 4) dimension

This section is devoted to find a solution of the \(\beta\)-function equations [2] - [3] in a special case in which the \(B\)-field includes a complex structure. Before attempting to do this, let us remark some requirements. We are interested in an integrable complex structure, i.e. vanishing Nijenhuis tensor, with \(\nabla J \neq 0\).
In the particular dimension of \((1 + 4)\), we take a general metric ansatz (in string frame) of the form
\[
ds^2 = g_{\bar{\mu} \bar{\nu}} dx^\bar{\mu} dx^\bar{\nu} = -N(t) \, dt^2 + g_{ij} dx^i dx^j
\]
\[
= -N(t) \, dt^2 + \frac{\alpha^2(t)}{1 - kr^2} \, dr^2 + b^2(t) \, r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + c^2(t) \, dx^2,
\]
where
\[
\{\bar{\mu}\} = \{0, 1, 2, 3, 4\}, \quad \{i\} = \{1, 2, 3, 4\},
\]
and \(x^4 \equiv x\) being an extra space-like dimension.

The 4-dimensional space will be required to be a complex manifold carrying a complex structure \(J\), and in this context the \(g_{ij}\) is to be the associated Hermitian metric. So, the first step toward finding a solution for the \(\beta\)-function equations \((2)\) with the proposed \(B\)-field in \((16)\) is to determine the \(\Omega_{ij}\) (or equivalently the \(J_{i}^{\perp}\)). Thus, we must solve the equations of complex structure \((12)\) and Hermitian metric condition \((13)\) along with the integrability condition, \(N_{ij}^{k} = 0\), in \((15)\). Solving them with the above metric ansatz gives the following Hermitian structure and \(B\)-field
\[
J = J_{i}^{\perp} dx^i \otimes \partial_j
\]
\[
= \frac{c(t)\sqrt{(1 - kr^2)}}{a(t)} dx \otimes \partial_r + \frac{a(t)}{c(t)\sqrt{(1 - kr^2)}} dr \otimes \partial_x - \frac{1}{\sin \theta} d\theta \otimes d\varphi - \sin \theta d\varphi \otimes d\theta,
\]
\[
\dot{B} = \frac{\gamma}{2} (\sqrt{(1 - kr^2)} - 1) a(t) c(t) \, dr \wedge dx + b(t) r^2 \sin \theta d\theta \wedge d\varphi.
\]
The predicted time dependent functions in the above structure will be fixed by solving the \(\beta\)-function equations. According to \((19)\), the non-zero components of field strength tensor \(H_{\mu \nu \rho}\) can be found as follows
\[
\dot{H}_{014} = -\gamma \sqrt{(1 - kr^2)^{-1}} (\dot{a} t + a(t) \dot{b}(t)),
\]
\[
\dot{H}_{123} = -2 \gamma c^2(t) r \sin \theta,
\]
\[
\dot{H}_{023} = -2 \gamma c(t) \dot{c}(t) r^2 \sin \theta.
\]
The Weyl anomaly coefficients, the \(\beta\)-function equations of the above metric and \(B\)-field with a consistent dilaton field of type \(\phi(t, r)\) are given in the Appendix.

After solving the \(\beta\)-function equations, we get a solution of the type
\[
k = 0, \quad \gamma = \frac{\sqrt{2}}{2}, \quad \dot{\phi}(r, t) = \ln(r) + F(t), \quad N(t) = \frac{\dot{F}(t)^2}{\Lambda},
\]
where \(F(t)\) is an arbitrary function of time, and all the scale factor functions which are anticipated in space part of the metric \((17)\) are fixed as constants with the following relations
\[
a(t) = \frac{\sqrt{3}}{3q_1}, \quad b(t) = \sqrt{3} q_1, \quad c(t) = q_1,
\]
where \(q_1\) is an arbitrary constant.

Consequently, the explicit form of \((1 + 4)\) dimensional backgrounds achieved by considering a complex structure \(J\) in the space part of \(B\)-field, has the following form
\[
ds^2_{\text{string frame}} = -\frac{\dot{F}(t)^2}{\Lambda} \, dt^2 + q_1^2 (3dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) + \frac{1}{3q_1^2} \, dx^2,
\]
\[
\dot{B} = -\frac{\sqrt{2}}{4} (q_1 \sin \theta d\theta \wedge d\varphi + dr \wedge dx).
\]
4 Dimension reduction to (1 + 3) dimensions

In this section, we are going to dimensionally reduce the (1 + 4) theory (with coordinates \((t, r, \theta, \varphi, x)\)) to the physical (1+3) space-time (with coordinates \((t, r, \theta, \varphi)\)). By using the standard technique of Scherk and Schwarz [20] one can parameterize the 5-bein as follows [21]

\[
\hat{e}_\mu^a = \begin{bmatrix} e_\mu^a \\ qA_\mu \\ 0 \end{bmatrix}.
\] (27)

Here, it is assumed that the fields are independent of the \(x\) coordinate, i.e. there is a killing vector \(\hat{q}_\mu\) defined by [21]

\[
\hat{q}_\mu \partial_\mu = \partial_x,
\] (28)

where \(x\) is the flat version of the \(x\) coordinate and \(q = \sqrt{\hat{q}_\mu \hat{q}_\mu}\). If the \(\mu\) and \(\nu\) indices run over \(\{0, 1, 2, 3\}\) then 5-dimensional fields will be decomposed as following

\[
\hat{g}_{xx} = \eta_{xx} q^2, \quad \hat{B}_{x\mu} = B_{\mu},
\]

\[
\hat{g}_{x\mu} = \eta_{x\mu} q^2 A_{\mu}, \quad \hat{B}_{\mu\nu} = B_{\mu\nu} + A_{[\mu} B_{\nu]},
\]

\[
\hat{g}_{\mu\nu} = g_{\mu\nu} + \eta_{x\mu} q^2 A_{\mu} A_{\nu}, \quad \hat{\phi} = \phi + \frac{1}{2} \ln q,
\] (29)

where the \(\{g_{\mu\nu}, B_{\mu\nu}, \phi, A_{\mu}, B_{\mu} ,q\}\) are 4-dimensional fields. So, the expressions of them in terms of 5-dimensional fields are written as follows

\[
g_{\mu\nu} = \hat{g}_{\mu\nu} - \frac{\hat{g}_{x\mu} \hat{g}_{x\nu}}{\hat{g}_{xx}}, \quad B_{\mu} = \hat{B}_{x\mu},
\]

\[
B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{\hat{g}_{x[\mu} \hat{B}_{\nu]x}}{\hat{g}_{xx}}, \quad \phi = \hat{\phi} - \frac{1}{4} \ln |\hat{g}_{xx}|,
\]

\[
A_{\mu} = \frac{\hat{g}_{x\mu}}{\hat{g}_{xx}}, \quad q = |\hat{g}_{xx}|^{1/2}.
\] (30)

Applying these on the (1 + 4)-dimensional metric and \(B\)-field in [25] and [26], we get the backgrounds of the 4-dimensional space-time as

\[
\text{ds}_{\text{stringframe}}^2 = -\frac{\dot{F}^2}{\Lambda} \, dt^2 + q_t^2 (3dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\varphi^2),
\] (31)

\[
\phi(r, t) = \ln (r) + F(t) + \frac{1}{4} \ln (3q_t^2),
\] (32)

\[
B = -\frac{\sqrt{2}}{4} q_t^2 r^2 \sin(\theta) \, d\theta \wedge d\varphi,
\] (33)

\[
A_{\mu} = 0, \quad B_{\mu} = \frac{\sqrt{5}}{2},
\] (34)

where \(A_{\mu}\) and \(B_{\mu}\) fields have no kinetic terms. In turn, the dimensionally reduced action will contain the same terms as in [3] in which the (1 + 4)-dimensional fields are replaced by (1 + 4)-dimensional ones [31], [33] and (32).

The spatial part of the metric (31) seems not having the known spherical symmetry in flat spacetime due to the term \(3dr^2\). Comparing with \((1 - kr^2)^{-1}dr^2\) in the non-flat Robertson-Walker metric, we find that the metric (31) has spherical symmetry in a non-flat background. It is worth noting that at very small values of \(\theta\), \(\sin(\theta) \approx \theta\), we have

\[
\text{ds}_{\text{stringframe}}^2 = -\frac{\dot{F}^2}{\Lambda} \, dt^2 + q_t^2 (3dr^2 + r^2d\theta^2 + r^2 \theta^2 d\varphi^2),
\] (35)
whose Ricci scalar is 
\[ R = \frac{\dot{F}^2}{\Lambda} dt^2 + 3q_1^2 (dr^2 + r^2 d\theta^2 + r^2 \theta^2 d\varphi^2). \] (36)

Hence, for small values of \( \theta \) the metric has almost spherical symmetry in flat space-time. Moreover, considering just the radial degree of freedom, namely \( d\theta = d\varphi = 0 \) in the metric (31), we have
\[ ds^2_{\text{string frame}} = -\frac{\dot{F}^2}{\Lambda} dt^2 + q_1^2 (3dr^2), \] (37)
which by the following definitions
\[ \frac{\dot{F}^2}{\Lambda} (t) = 3q_1^2 \text{ or } F(t) = \sqrt{3|\Lambda|} q_1 t, \] (38)
becomes a metric, conformally related to a flat metric, with light cone structure
\[ ds^2_{\text{string frame}} = 3q_1^2 (-dt^2 + dr^2). \]

In order to study the above result in the Einstein frame we may use a conformal transformation to obtain the metric in Einstein frame. So, in 4 dimensions
\[ ds^2_{\text{Einstein frame}} = e^{\phi(t,r)} ds^2_{\text{string frame}}, \] (39)
or equivalently
\[ \tilde{g}_{\mu\nu} = e^{\phi(t,r)} g_{\mu\nu}, \] (40)
whose Ricci scalar is
\[ \tilde{R} = \frac{3^{3/4} \left( 9r^2 q_1^2 \Lambda - 1 \right)}{18r^3 (q_1)^{5/2} e^{F(t)}}. \] (41)

Obviously, in the limit \( r \to \infty \) we have \( \tilde{R} \to 0 \), which accounts for the asymptotically flatness. The corresponding Kretschmann scalar is obtained as
\[ \tilde{K} = \frac{e^{2F(t)} \sqrt{3} (36 \Lambda^2 q_1^4 r^4 - 21 r^2 q_1^2 \Lambda + 44)}{72 q_1^3 r^2}, \] (42)
which shows an essential singularity at \( r \to 0 \).

Now, we consider the Einstein equations (9) for this 4-dimensional space-time in order to construct an effective matter source. The 4-dimensional Einstein equations are as follows
\[ \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} = \kappa^2 (T^{(\phi)}_{\mu\nu} + T^{(H)}_{\mu\nu}), \] (43)
where
\[ \kappa^2 T^{(\phi)}_{\mu\nu} = \frac{1}{2} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} (\partial \phi)^2 \tilde{g}_{\mu\nu} - \Lambda e^{-\phi} \tilde{g}_{\mu\nu} \right), \] (44)
\[ \kappa^2 T^{(H)}_{\mu\nu} = \frac{1}{4} \left( H_{\mu\nu\lambda} H^{\lambda}_{\rho\lambda} - \frac{1}{6} H^2 \tilde{g}_{\mu\nu} \right). \] (45)

Thus, the corresponding energy-momentum tensor is given by
\[ T_{\mu\nu} = \kappa^2 (T^{(\phi)}_{\mu\nu} + T^{(H)}_{\mu\nu}) = \rho(r) e_0^\mu e_0^\nu + P_1(r) e^1_\mu e^1_\nu + P_2(r) e^2_\mu e^2_\nu + P_3(r) e^3_\mu e^3_\nu + \Sigma (r) (e_0^\mu e_1^\nu + e_1^\mu e_0^\nu). \] (46)

The \( e_a^\mu \) in non-coordinate basis are satisfying the relation
\[ g_{\mu\nu} e_a^\mu e_a^\nu = \delta_{ab}, \quad \eta_{ab} = \text{diag}(-1, 1, 1, 1). \]

In the metric (31) we have \( e_a^\mu \) as \( e_0^0 = \sqrt{\frac{\Lambda}{q_1}}, e_1^1 = \sqrt{\frac{\Lambda}{q_1}}, e_2^2 = \frac{1}{q_1}, e_3^3 = \frac{1}{q_1 r \sin(\theta)}. \)
where, $P_1(r)$, $P_2(r)$ are pressures and $\Sigma_r(r)$ is the Pointing vector in radial direction. In a comprehensive view, this energy-momentum tensor represents an anisotropic inhomogeneous non-barotropic matter with the following components with a positive $q_1$

$$\rho(r) = \frac{9 \Lambda q_1^6 r^2 + q_1^4 + 2}{12q_1^6 r^2}, \quad (47)$$

$$P_1(r) = \frac{-3 \Lambda q_1^6 r^2 + q_1^4 + 2}{12q_1^6 r^2}, \quad (48)$$

$$P_2(r) = \frac{-3 \Lambda q_1^6 r^2 - q_1^4 + 2}{12q_1^6 r^2}, \quad (49)$$

$$\Sigma_r(r) = \frac{\sqrt{3 \Lambda}}{6r q_1}. \quad (50)$$

In the $r \to \infty$ limit, all components are tending to a finite value, i.e. $\rho = -3P_1 = -3P_2 = \frac{3\Lambda}{4}$ and $\Sigma_r(r) = 0$. The pressures $P_1$ and $P_2$ are negative in the regions $r > \sqrt{\frac{3\Lambda(q_1^2 + 2)}{3q_1^2}}$ and $r > \sqrt{\frac{3\Lambda(-q_1^2 + 2)}{3q_1^2}}$, respectively. Thus, the pressures have sign change in some specific radiiuses. This argument may be justified if the above effective matter is considered as two component matter including a perfect fluid and an anisotropic and inhomogeneous matter source, as investigated in Ref [22]. Then, the decomposed energy-momentum tensor will be

$$T_{\mu\nu} = (\rho + \rho_{in}(r)) e^0_\mu e^0_\nu + (P + P_r(r)) e^1_\mu e^1_\nu + (P + P_l(r)) e^2_\mu e^2_\nu + (P + P_1(r)) e^3_\mu e^3_\nu + \Sigma_r(r) (e^0_\mu e^1_\nu + e^1_\mu e^0_\nu). \quad (51)$$

So, the space is filled with an isotropic and homogeneous fluid with density $\rho$ and pressure $P$, and an anisotropic inhomogeneous matter source with $\rho_{in}(r)$, $P_r(r)$, $P_l(r)$ and $\Sigma_r(r)$ denoting for energy density, radial and lateral pressures and Pointing vector, respectively. In this decomposition, the first component which originated from the dilaton contribution in (44) is defined by

$$\rho = -3P = \frac{3\Lambda}{4}, \quad (52)$$

which represents a perfect fluid with negative pressure and state parameter of $w = -1/3$ (i.e. $p = \rho w$). This special state parameter corresponds to the cosmic string network case [23]. The second component is defined by

$$\rho_{in}(r) = \frac{q_1^4 + 2}{12q_1^6 r^2}, \quad (53)$$

$$P_r(r) = \frac{q_1^4 + 2}{12q_1^6 r^2}, \quad (54)$$

$$P_l(r) = \frac{-q_1^4 + 2}{12q_1^6 r^2}, \quad (55)$$

$$\Sigma_r(r) = \frac{\sqrt{3 \Lambda}}{6r q_1}. \quad (56)$$

By imposing the condition $q_1 < \sqrt{2}$ on the arbitrary parameter $q_1$, the pressure $P_l(r)$ becomes positive everywhere. Note that there are divergences in the pressures and component of the Pointing vector at $r = 0$ which may be interpreted by the presence of an effective charge introduced by the $B$-field and dilaton at $r = 0$. In other words, in this example the geometry is so deformed that there is effectively a charge at $r = 0$.

The effective matter (44) is decomposed into two matter components consisting of an isotropic negative pressure matter and an anisotropic inhomogeneous positive pressure matter. Now, the second component obeys the barotropic equations of state as follows

$$P_r(r) = w_r \rho_{in}(r), \quad P_l(r) = w_l \rho_{in}(r), \quad (57)$$

where two positive constants are given by $w_r = 1$ and $w_l = \frac{q_1^4 + 2}{q_1^4 + 2}$.\footnote{In fact, this divergence is appeared in both $B$-field and dilaton energy momentum tensor. The divergence which caused by the dilaton is originally the result of the particular form of dilaton filed as a logarithmic function of $r$ (32) in this model. This type of dilaton field specially appears in black hole solution in string theory [28].}
5 Energy conditions

It is worth investigating the energy conditions for the above matters. The first fluid simply satisfies all energy conditions. The second fluid which has an off-diagonal energy-momentum tensor has the following properties.

1. The null energy condition (NEC) is \( T_{\alpha\beta}k^\alpha k^\beta \geq 0 \), with arbitrary null vector \( k^\alpha \). Based on (21), this implies

\[
\rho + P_r - 2\Sigma_r(r) \geq 0, \quad \rho + P_t \geq 0,
\]

which are always true for a positive \( \Lambda \).

2. The strong energy condition (SEC) requires \( (T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta})\nu^\alpha \nu^\beta \geq 0 \), where \( \nu^\alpha \) is any future-directed, normalized, time-like vector. This accounts for the following conditions

\[
\rho \geq 0, \quad \rho + P_r - 2\Sigma_r > 0, \quad \rho + P_t > 0,
\]

which are always true for a positive \( \Lambda \).

3. The weak energy condition (WEC), \( T_{\alpha\beta}\nu^\alpha \nu^\beta \geq 0 \), leads to

\[
\rho - \Sigma_r \geq 0, \quad |\rho - \Sigma_r| \geq |P_r - \Sigma_r|, \quad \rho \geq \sqrt{P_t^2 + \Sigma_r^2}.
\]

The first inequality is violated for \( r > \frac{\sqrt{Q(g^2 + 2) \Lambda^2}}{\sqrt{N_{d_f}}} \) while the last one is violated for \( r > \frac{\sqrt{6}}{3\sqrt{N_{dp}}} \).

6 \( T \)-dual solutions

\( T \)-duality (target space duality) \( [29] \) can be generally used in the \( \sigma \)-model context to generate a new class of solutions and background of string theory. The presented example at the previous section is independent of the two \( x \) and \( \varphi \) coordinates, which will be regarded as isometry coordinates. In this section, we investigate the \( T \)-duality with respect to these coordinates.

In 5-dimensions, Buscher’s \( T \)-duality transformations \( [29] \) with respect to the isometry direction of \( x \) have the following form:

\[
\begin{align*}
\tilde{g}_{xx} &= \frac{1}{g_{xx}}, \quad \tilde{g}_{xi} = \frac{\hat{B}_{xi}}{g_{xx}}, \quad \tilde{B}_{xx} = \frac{\hat{g}_{xi}}{g_{xx}}, \\
\tilde{B}_{ij} &= \frac{\hat{B}_{ij}}{g_{xx}}, \\
\tilde{g}_{ij} &= \frac{\hat{g}_{ij}}{g_{xx}}, \\
\tilde{\phi} &= \phi - 1/2 \ln |\hat{g}_{xx}|.
\end{align*}
\]

where \( \tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu} \) and \( \tilde{\phi} \) are metric, antisymmetric tensor and dilaton fields of the dual model

\[
I = \frac{1}{4\pi\alpha'} \int_{\Sigma} \sqrt{h}d^2z(h^{\alpha\beta}\tilde{g}_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta}\tilde{B}_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \alpha'R^{(2)} \tilde{\phi}).
\]

For the model \( [25, 26] \) after using transformation (58) one can find the following dual metric

\[
d\tilde{s}^2_{\text{string frame}} = \frac{\tilde{g}_{\mu\nu}}{F(t)^2} dt^2 + \frac{q_1^2}{2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 + 3 dx^2 + \frac{3\sqrt{2}}{2} dr dx,
\]

where...
and dual antisymmetric and dilaton fields

\[ \tilde{B} = -\frac{\sqrt{2}}{4} q_1 r^2 \sin(\theta) d\theta \wedge d\varphi, \quad (61) \]

\[ \tilde{\phi} = \ln(r) + F(t) + 1/2 \ln(3 q_1^2). \quad (62) \]

The other isometry coordinate is the \( \varphi \) coordinate. Similar to the (68), the \( T \)-dual transformation with respect to \( \varphi \) in 5-dimensions gives the following \( T \)-dual metric, antisymmetric and dilaton fields respectively as follows

\[
d\tilde{s}_{\text{string frame}}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -\frac{\dot{F}(t)^2}{\Lambda} dt^2 + 3 q_1^2 dr^2 + \frac{3}{2} q_1^2 r^2 d\theta^2 + \frac{1}{q_1^2 r^2 \sin^2(\theta)} d\varphi^2 + \frac{\sqrt{2}}{2 \sin(\theta)} d\theta d\varphi + \frac{1}{3 q_1^2} dx^2, \quad (63) \]

\[ \tilde{\phi} = 3 \ln(r) + F(t) + 2 \ln(\sin \theta) + 1/4 \ln(3) + 5/2 \ln(q_1). \quad (65) \]

In 4-dimensions, the only isometry coordinate is \( \varphi \). We may apply two procedures here: we can find the \( T \)-dual backgrounds in 5-dimensions and then dimensionally reduce the \( x \) coordinate; or, we may reduce the \( x \) coordinate and then find the \( T \)-dual solution with respect to \( \varphi \). In the second procedure, after dimension reduction of \( x \) coordinate, Buscher’s \( T \)-duality transformation with respect to \( \varphi \) coordinate are as follows

\[ \tilde{g}_{\varphi\varphi} = \frac{1}{g_{\varphi\varphi}}, \quad \tilde{g}_{\varphi\mu} = \frac{B_{\varphi\mu}}{g_{\varphi\varphi}}, \quad \tilde{B}_{\varphi\mu} = \frac{g_{\varphi\mu}}{g_{\varphi\varphi}}, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{(g_{\varphi\mu} B_{\nu\varphi} - g_{\varphi\nu} B_{\mu\varphi})}{g_{\varphi\varphi}}, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{(g_{\varphi\mu} g_{\varphi\nu} - B_{\varphi\mu} B_{\varphi\nu})}{g_{\varphi\varphi}}, \quad \tilde{\phi} = \phi - 1/2 \ln |g_{\varphi\varphi}|. \quad (66) \]

Then using (61) - (62), the \( T \)-dual answers with respect to \( \varphi \) coordinate are given as follows

\[
d\tilde{s}_{\text{string frame}}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -\frac{\dot{F}(t)^2}{\Lambda} dt^2 + 3 q_1^2 dr^2 + \frac{3}{2} q_1^2 r^2 d\theta^2 + \frac{1}{q_1^2 r^2 \sin^2(\theta)} d\varphi^2 + \frac{\sqrt{2}}{2 \sin(\theta)} d\theta d\varphi, \quad (67) \]

\[ \tilde{\phi} = 3 \ln(r) + F(t) + 2 \ln(\sin \theta) + 1/4 \ln(3) + 5/2 \ln(q_1), \quad (68) \]

such that the \( T \)-dual \( B \)-Field is zero. On the other hand, if we apply the dimensionality reduction according to (30) on the \( T \)-dual solutions (63) - (64), we will obtain the same results as (67), (68) and similar to it, the reduced \( B \)-field would be zero, i.e. the dimension reduction procedure and \( T \)-duality transformation are consistent with each other.

7 Conclusion

We have developed a \( \sigma \)-model on a manifold \( R \times N \) whose space part \( N \) is a complex manifold. Specially, following the idea of giving a physical implication to the complex structure, we have considered the complex structure of the complex manifold as the antisymmetric tensor field of the \( \sigma \)-model in the form of \( \tilde{B} = \frac{1}{2} g_{k\ell} J_{k}^{i} dx^{i} \wedge dx^{j} \).
Therefor, since the metric field in $\sigma$-model is a geometrical structure, the role of antisymmetric filed is played by a geometrical structure on the manifold. A static solution is found for the $\beta$-functions in $(1+4)$ dimensions with a non-zero central charge deficit $\Lambda$ and then dimensionally reduction of it to $(1+3)$ dimensional physical space-time is performed. We have seen that the two component energy-momentum tensor generated in this solution corresponds to a cosmic string and an anisotropic inhomogeneous isotropic matter. The second matter satisfies the energy conditions except the dominant energy condition which is violated in some range of radius. There is a divergence in the energy density and pressure at $r = 0$ which is interpreted by the presence of an effective charge. Finally, $T$-dual solutions are constructed in both $(1+4)$ and physical $(1+3)$ dimensions. We show that in $(1+3)$ space-time the $T$-duality transformation and dimension reduction procedures are compatible.

Acknowledgment

This research has been supported by Azarbaijan Shahid Madani university by a research fund No. 401.231.

Appendix

Here we give the non zero components of the $\beta$-functions $\beta_{\mu \nu}$ related to the metric given by (17) and the B-field given by (19), as follows:\footnote{The overdot and prime stand for differentiation $\dot{\phi}$ and $\phi'$, respectively.}

$$\begin{align*}
\beta_{00}^\beta &= -\left\{ \dot{\phi}(t, r) + \ddot{\ln}a(t) + 2 \dddot{b}(t) + \dddot{\ln}c(t) \right\} \dddot{N}(t) - \gamma^2 N(t) \left\{ (\dddot{\ln}a(t) + \dddot{\ln}c(t))^2 + 4 \dddot{\ln}b(t)^2 \right\} a(t)^2 c(t)^2 b(t)^2 \\
&+ 2 c(t)^2 N(t) a(t) b(t) [\dot{\phi}(t, r) a(t) b(t) + 2 a(t) \dddot{b}(t) + b(t) \dddot{a}(t)] + 2 c(t) a(t)^2 \dddot{c}(t) b(t)^2 N(t), \\
\beta_{11}^\beta &= 2 c(t)^2 b(t) N(t) \left\{ r^2 [N(t) (kr^2 - 1) \phi''(t, r) + kr \phi'(t, r)] N(t) + 1/2 \dot{a}(t)^2 \gamma^2 + a(t) \dddot{a}(t) \phi(t, r) + a(t) \dddot{a}(t) \right\} \\
&+ 2 [a(t) r^2 \dddot{a}(t) \left( \frac{1}{2} (\gamma^2 + 1) \dddot{\ln}c(t) + \dddot{\ln}b(t) - \frac{1}{4} \ddot{\ln}N(t) \right) + \frac{1}{4} \ddot{\ln}(t)^2 r^2 \gamma^2 a(t)^2 + N(t) \left( k (\gamma^2 + 1) r^2 - \gamma^2 \right)] \right\},
\end{align*}$$

$$\begin{align*}
\beta_{10}^\rho &= r \phi''(t, r) b(t) a(t) \dddot{a}(t) + \dddot{a}(t) \phi'(t, r) b(t) + 2 \dddot{a}(t) b(t) + 2 \dddot{b}(t) a(t) \left( \gamma^2 + 1 \right), \\
\beta_{22}^\beta &= N(t) c(t) [b(t)^2 r \left\{ (kr^2 - 1) N(t) \phi'(t, r) + \dddot{\ln}b(t) \left( \frac{\dot{\phi}(t, r) + \dddot{\ln}b(t) (2 \gamma^2 + 1)}{ra(t)^2} \right) + a(t)^2 \dddot{b}(t) + \dddot{b}(t) r^2 \right\] \\
&+ a(t)^2 c(t) N(t) b(t) \dddot{b}(t) r^2 \left[ \frac{1}{2} \dddot{\ln}N(t) + \dddot{\ln}a(t) + \dddot{\ln}c(t) \right] + 2 \left\{ (k (\gamma^2 + 1) r^2 - \gamma^2 - \frac{1}{2}) b(t)^2 + \frac{1}{2} a(t)^2 r^2 N(t)^2 c(t), \\
\beta_{33}^\beta &= 1/2 \sin^2(\theta) \beta_{22}^\beta, \\
\beta_{44}^\beta &= a(t)^2 b(t) N(t) \left\{ \gamma^2 \dddot{c}(t)^2 + 2 c(t) \dddot{c}(t) + 2 c(t) \dddot{c}(t) [\dot{\phi}(t, r) + (\gamma^2 + 1) \dddot{\ln}a(t) + 2 \dddot{\ln}b(t) - 1/2 \dddot{\ln}N(t)] \right\} \\
&+ c(t)^2 b(t) N(t) \dot{a}(t)^2 \gamma^2, \\
\beta_{14}^B &= 2 N(t) a(t) b(t) \left\{ c(t)^2 a(t) \dddot{\ln}a(t)^2 + a(t) [c(t) \dddot{\ln}a(t) + \dddot{\ln}c(t)] \right\} \dddot{\phi}(t, r) - c(t)^2 \dddot{a}(t) - c(t) a(t) \dddot{c}(t) \\
&+ c(t) N(t) a(t) b(t) \left\{ a(t) \dddot{\ln}N(t) - 4 \dddot{\ln}b(t) + 2 \dddot{\ln}c(t) \dddot{c}(t) + c(t) \dddot{a}(t) \dddot{\ln}N(t) - 4 \dddot{\ln}b(t) \right\}, \\
\beta_{14}^B &= \left\{ (kr^3 - r) \phi'(t, r) - 4 kr^2 + 2 \right\} \left( \dot{a}(t) c(t) + a(t) \dddot{c}(t) \right), \\
\beta_{23}^B &= c(t) b(t)^2 N(t) \left\{ r^2 a(t)^2 \left[ \frac{1}{2} \dddot{\ln}N(t) + \dddot{\ln}a(t) + \dddot{\ln}c(t) \right] \dddot{b}(t) + r N(t) (kr^2 - 1) \phi'(t, r) \right\} \\
&- \dddot{\ln}b(t)^2 r^2 a(t)^2 + \dddot{\ln}b(t) r^2 a(t)^2 \dddot{\phi}(t, r) + N(t) \right\} + \dddot{b}(t) N(t) c(t) b(t) a(t)^2 r^2, \\
\end{align*}$$
\[
\beta^\phi = -4c(t)N(t)b(t)a(t)r^2[b(t)a(t)c(t) + c(t)\left(\ddot{a}(t)b(t) + 2a(t)\dot{b}(t)\right)] \\
-4[(kr^2 - 1)N(t)\phi''(t, r) + a(t)^2\phi'(t, r) + 1/2N(t)(kr^2 - 1)\phi'(t, r)^2 + 1/2\phi(t, r)^2 a(t)^2]r^2b(t)^2c(t)N(t) \\
-4N(t)c(t)\{rb(t)^2(3kr^2 - 2)\phi'(t, r) + [-1/2r^2 + k(a(t)^2 + k(\gamma^2 + 1/3)r^2 - \gamma^2 - 1)b(t)^2 + 1/3 a(t)^2]\} \\
-8b(t)c(t)^2a(t)^2N(tr^2\{1/4b(t)[-2\ln(a(t) + \ln N(t) - 4\ln b(t) - 2\ln c(t)]\phi(t, r) \\
+1/8b(t)\ln(a(t)^2\gamma + (1/4b(t)(\gamma^2 + 2)\ln(t) - 1/4b(t)\ln N(t) + \dot{b}(t))\ln(a(t) \\
-1/2b(t)\left[-1/4\ln c(t)^2\gamma^2 + [1/2\ln N(t) - 2\ln b(t)]\ln(c(t) + \ln(b(t))(-\gamma^2 - 1)\ln(b(t) + \ln N(t))\right].
\]

(78)

References

[1] See the Home-Page, [http://www.to.infn.it/gasperin](http://www.to.infn.it/gasperin).
[2] G. Veneziano, Nucl. Phys. B (Proc. Suppl.) 55, 134 (1997); N. A. Batakis, Phys. Lett. B 353, 39 (1995); I. Florakisa, K. Koumasta, H. Partoucheb, N. Toumbasc, Nucl. Phys. B 844, 89 (2011).
[3] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B 158, 316 (1985).
[4] E. S. Fradkin and A. Tseytlin, Nucl. Phys. B 261, 1 (1985).
[5] C. G. Callan, E. J. Martinec, M. J. Perry, and D. Friedan, Nucl. Phys. B 262, 593 (1985).
[6] A. Strominger, Nucl. Phys. B 274, 253 (1986); K. Becker, M. Becker, K. Dasgupta, P. S. Green and E. Sharpe, Nucl. Phys. B 678, 19 (2004), [arXiv:hep-th/0310058](http://arxiv.org/abs/hep-th/0310058).
[7] A. R. Frey and M. Lippert, Phys. Rev. D 72, 126001 (2005), [hep-th/0507202](http://arxiv.org/abs/hep-th/0507202).
[8] G. L. Cardoso, G. Curio, G. DallAgata and D. Lust, JHEP 0310, 004 (2003), [arXiv:hep-th/0306088](http://arxiv.org/abs/hep-th/0306088).
[9] M. Klaput, A. Lukas, C. Matti and E. E. Svanes, JHEP 1301, 015 (2013), [arXiv:1210.5933](http://arxiv.org/abs/1210.5933).
[10] E. Witten, Commun. Math. Phys. 118, 411 (1988).
[11] A.R. Frey, M. Lippert, Phys. Rev. D 72, 126001 (2005).
[12] K. Becker, M. Becker, K. Dasgupta, and P. S. Green, JHEP 04 (2003) 007, [hep-th/0301161](http://arxiv.org/abs/hep-th/0301161).
[13] G. L. Cardoso, G. Curio, G. DallAgata, and D. Lust, JHEP 10 (2003) 004, [hep-th/0306088](http://arxiv.org/abs/hep-th/0306088).
[14] F. Naderi, A. Rezaei-Aghdam and F. Darabi, Int. J. Mod. Phys. A 30, 1550015 (2015), arXiv:hep-th/1403.3916.
[15] N. A. Batakis and A. A. Kehagias, Nucl. Phys. B 449, 248 (1995).
[16] E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. D 50, 4868 (1994).
[17] J. Maharana, H. Singh, Phys. Lett. B 368, 64 (1996).
[18] M. Nakahara, *Geometry, topology and physics*, CRC Press (2003).
[19] A. Frohlicher and A. Nijenhuis, Nederl. Acad. Wetensch Proc. Ser. A 59, 338 (1956); A. Nijenhuis, Nederl. Acad Wetensch Proc. Ser. A 58, 390 (1955).
[20] J. Scherk, J. H. Schwarz. Nucl. Phys. B 153, 61 (1979).
[21] E. Bergshoeff, R. Kallosh and T. Ortín, Phys. Rev. D 51, 3009 (1995).
[22] M. Cataldo, P. Meza, Phys. Rev. D 87, 064012 (2013); M. Cataldo, S. d. Campo, Phys. Rev. D 85, 104010 (2012).
[23] V. Sahni, Class. Quant. Grav. 19, 3435 (2002); astro-ph/0202076.
[24] A. Strominger, Nucl. Phys. B 274, 253 (1986).
[25] K. Becker, M. Becker, K. Dasgupta, P. S. Green, E. Sharpe, Nucl. Phys. B 678, 19 (2004); hep-th/0310058.
[26] K. Becker, M. Becker, K. Dasgupta, and P. S. Green, JHEP 0304, 007 (2003); hep-th/0301161.
[27] G. L. Cardoso, G. Curio, G. Dall’Agata, and D. Lust, JHEP 0310, 004 (2003); hep-th/0306088.
[28] D. Gershon, Phys. Rev. D 51, 4387 (1995); hep-th/9202005.
[29] T.H. Buscher, Phys. lett. B 201, 466 (1988).