PICARD NUMBER OF THE GENERIC FIBER OF AN ABELIAN FIBERED HYPERKÄHLER MANIFOLD

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Abstract. We shall show that the Picard number of the generic fiber of an abelian fibered hyperkähler manifold over the projective space is always one. We then give a few applications for the Mordell-Weil group. In particular, by deforming O’Grady’s 10-dimensional manifold, we construct an abelian fibered hyperkähler manifold of Mordell-Weil rank 20, which is the maximum possible among all known ones.

1. Introduction

By a hyperkähler manifold (HK manifold, for short), we mean a simply connected compact Kähler manifold $X$ admitting an everywhere non-degenerate holomorphic 2-form $\sigma_X$ such that $H^0(X, \Omega_X^2) = \mathbb{C}\sigma_X$. Note that $X$ is even dimensional. Put $\dim X = 2n$. We call a surjective morphism $f : X \to B$ a fibered HK manifold if in addition $B$ is normal, projective and $f$ has connected fibers. A fundamental result of Matsushita (M1, M2) says that each fiber of $f$ is Lagrangian. Especially smooth fibers are complex tori of dimension $n$. They are also projective and hence abelian varieties (see e.g. [Og2]). We call $f$ an abelian fibered HK manifold if in addition $f$ admits a bimeromorphic section, i.e., a subvariety $O \subset X$ such that the induced morphism $f|O : O \to B$ is bimeromorphic. Then, $X$ is projective (see e.g. [Og2]) and we can work in the category of algebraic varieties over $\mathbb{C}$. In particular, we can speak of the generic fiber $X_\eta$ of $f$ in the scheme theoretic sense. This is an abelian variety with origin $O|X_\eta$, defined over the function field $\mathbb{C}(B)$. Concerning the base space of a fibered HK manifold, there is a conjecture that it is always the projective space $\mathbb{P}^n$. Hwang [Hw] has shown that this is true if $X$ is projective and $B$ is smooth.

In this note, we study an abelian fibered HK manifold over the projective space. Our main results are Theorems 1.1, 1.2, 1.3 and 1.4.

Theorem 1.1. Let $f : X \to \mathbb{P}^n$ be an abelian fibered HK manifold. Let $K = \mathbb{C}(\mathbb{P}^n)$ and let $A_K$ be the generic fiber of $f$. Then, $\rho(A_K) = 1$. Here $\rho(A_K)$ is the Picard number of $A_K$ over $K$.

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It can happen that $\rho(X_t) \geq 2$ for all smooth closed fiber of $f$ (see the first step of the proof of Theorem 1.4). The statement is of arithmetical nature. Geometrically, it means that two horizontal divisors on $X$ are proportional in $NS(X)$ up to vertical divisors. Key of our proof is to compare two different ways to discuss deformations of $f$; Matsushita’s way [M3] and Voisin’s way [Vo]. The argument is very short, but it is quite mysterious, at least for me, that such considerations yield a strong information about the Picard number of the generic fiber. It may be interesting, if possible, to give a more natural proof based on the monodromy representation around the discriminant locus $D \subset \mathbb{P}^n$, which is of pure codimension one ([HO], [Hw]).

The set $A_K(K)$ of $K$-rational points, or equivalently, the set of rational sections of $f$, forms an abelian group called the *Mordell-Weil group* of $f$. We denote it by $MW(f)$. When $MW(f)$ is finitely generated, we call $mw(f) := \text{rank } MW(f)$ the *Mordell-Weil rank*. Mordell-Weil group and Mordell-Weil rank are important invariants of an abelian fibered variety. Combining Theorem 1.1 with [HPW] (or [Kh], [Og2]), we obtain:

**Theorem 1.2.** Let $f : X \to \mathbb{P}^n$ be as in Theorem 1.1. Let $D = \bigcup_{i=1}^k D_i$ be the irreducible decomposition of the discriminant divisor $D \subset \mathbb{P}^n$ and let $m_i$ be the number of prime divisors in $X$ lying over $D_i$. Then the Mordell-Weil group $MW(f)$ is a finitely generated abelian group of rank

$$mw(f) = \rho(X) - 2 - \sum_{i=1}^k (m_i - 1).$$

In particular, $2 \leq \rho(X) \leq b_2(X) - 2$ and $mw(f) \leq \rho(X) - 2$.

Note that the two inequalities in Theorem 1.2 are valid even if the base space were singular [Og2]. It is natural to ask if these inequalities are optimal. This is also a problem of arithmetical nature, but again by geometry, we obtain the following:

**Theorem 1.3.** Let $f : X \to \mathbb{P}^n$ be as in Theorem 1.1. Assume that $X$ admits at least one (not only rational but also) holomorphic section, i.e., a subvariety $O \subset X$ such that $f|O : O \to \mathbb{P}^n$ is an isomorphism. Then, for each integer $\rho$ such that $2 \leq \rho \leq b_2(X) - 2$, there is an abelian fibered HK manifold $f_\rho : X_\rho \to \mathbb{P}^n$ which is deformation equivalent to $X$ and which satisfies $\rho(X_\rho) = \rho$ and $mw(f_\rho) = \rho - 2$. Moreover, for each $\rho$, such abelian fibered HK manifolds are dense around $X$ in the Kuranishi space with respect to Euclidean topology.

This is a generalization of our earlier result [Og1] about Mordell-Weil groups of Jacobian K3 surfaces (see also [Og2] for an intermediate result).

As an application, we obtain:
Theorem 1.4. (1) Let \( \rho \) be an integer such that \( 2 \leq \rho \leq 21 \) and let \( n \geq 2 \) be an integer. Then, for each such \( \rho \) and \( n \), there is an abelian fibered HK manifold \( f : X \to \mathbb{P}^n \) such that \( X \) is deformation equivalent to the Hilbert scheme \( S^{[n]} \) of \( n \)-points of a K3 surface \( S \), \( \rho(X) = \rho \) and \( mw(f) = \rho - 2 \).

(2) Let \( \rho \) be an integer such that \( 2 \leq \rho \leq 22 \). Then, for each such \( \rho \), there is an abelian fibered HK manifold \( f : X \to \mathbb{P}^5 \) such that \( X \) is deformation equivalent to O'Grady's 10-dimensional HK manifold \( \mathcal{M}_4 \) [OG], \( \rho(X) = \rho \) and \( mw(f) = \rho - 2 \). In particular, the case \( \rho = 22 \) gives the largest record of the Mordell-Weil rank 20, among all the currently known abelian fibered HK manifolds.

Note that \( S^{[n]} \) is a HK manifold of dim \( S^{[n]} = 2n \) and \( b_2(S^{[n]}) = 23 \) [Be], and that \( \mathcal{M}_4 \) is a HK manifold with largest known second Betti number, \( b_2(\mathcal{M}_4) = 24 \) [Ra2].

We show Theorems 1.1, 1.2 in Section 2 and Theorems 1.3, 1.4 in Section 3.

2. Proof of Theorems 1.1 and 1.2

Let \( \mathcal{P} = \{[\sigma] \in \mathbb{P}(H^2(X, \mathbb{C})) \mid (\sigma, \sigma) = 0, (\sigma, \bar{\sigma}) > 0 \} \) be the period domain of \( X \). By the local Torelli theorem for HK manifolds (see eg. [GHJ] Proposition 22.11), we can (and will) regard the Kuranishi space \( \mathcal{K} \) of \( X \) as a small neighbourhood of \( [\sigma_X] \in \mathcal{P} \).

Let \( F \) be a smooth closed fiber of \( f \) and let \( \iota : F \to X \) be the inclusion map. As \( f \) is fibered over \( \mathbb{P}^n \), the deformations of \( X \) that preserve the fibration are of codimension 1 in \( \mathcal{K} \) by Matsushita [M3] Corollary 1.7 (see also [Sa2]). More precisely, it is a hypersurface defined by \( (h, \sigma) = 0 \), where \( h = f^*H \) is the pullback of the hyperplane class \( H \) of \( \mathbb{P}^n \). Here the fact that the base space is \( \mathbb{P}^n \) is important. As fibers of a fibered HK manifold are Lagrangian [M2], this deformation is part of deformations of \( X \) that keep \( F \) Lagrangian. By the easier direction of Voisin’s result [Vo], the latter space is of codimension \( r \) in \( \mathcal{K} \), where \( r = \operatorname{rank} \Im(\iota^* : H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})) \). Thus we have \( r = 1 \).

Let us show that \( \rho(A_K) = 1 \). For this, it suffices to show that two elements \( L_1 \) and \( L_2 \) of \( NS(A_K) \) are proportional. As \( A_K \) is projective, we may (and will) assume that both are represented by very ample divisors on \( A_K \). We can then write \( L_i = [H_i]A_K \) (\( i = 1, 2 \)). Here \( H_i \) (\( i = 1, 2 \)) are effective Cartier divisors on \( X \) and \( [H_i] \in NS(X) \) are their classes.

Let \( U \subset \mathbb{P}^n \) be a dense Zariski open subset of \( \mathbb{P}^n \) such that \( f \) is smooth over \( U \), the rational section \( O \) is holomorphic over \( U \) and both \( H_i \) are flat over \( U \). Let \( X_U = f^{-1}(U) \). Then, one can speak of the relative Picard scheme \( \operatorname{Pic}(X_U/U) \) and its identity component \( \operatorname{Pic}^0(X_U/U) \). Note that both \( \operatorname{Pic}(X_U/U) \) and \( \operatorname{Pic}^0(X_U/U) \) satisfy the base change property. (For the Picard scheme, see [Gr] or [FK], Chapter 9.) For simplicity, we denote the line bundles corresponding to \( H_i|X_U \) (\( i = 1, 2 \)) by the same letters. Then
$H_i|X_U \in \text{Pic}(X_U/U)(U)$. For a pair of integers $(n, m)$, we set, under the additive notation,

$$H(n, m) = nH_1|X_U - mH_2|X_U \in \text{Pic}(X_U/U)(U).$$

By the base change property of $\text{Pic}(X_U/U)$, the element $H(n, m)$ naturally defines a holomorphic section, say $s(n, m)$, of $\text{Pic}(X_U/U)$ over $U$. Explicitly, $s(n, m)$ is defined by:

$$U \ni t \mapsto H(n, m)|F_t \in \text{Pic}(F_t).$$

Here $t$ is not necessarily a closed point.

Let $F_b$ be the fiber of $f$ over a closed point $b$ of $U$. Then, by $r = 1$, the two classes $[H_i|F_b]$ $(i = 1, 2)$ are linearly dependent in $\text{NS}(F_b)$. Here we used the fact that $[H_i]$ $(i = 1, 2)$ are elements of $\text{NS}(X)$ and $[H_i|F_b] = \iota^*([H_i])$ $(i = 1, 2)$, where $\iota$ is a natural inclusion map of $F_b$ into $X$. Thus, there is a pair of integers $(n_1, n_2) \neq (0, 0)$ such that $[H(n_1, n_2)|F_b] = 0$ in $\text{NS}(F_b)$. Hence $s(n_1, n_2)(b) = H(n_1, n_2)|F_b \in \text{Pic}^0(F_b)$.

As $\text{Pic}^0(X_U/U)$ is a connected component of $\text{Pic}(X_U/U)$ and $s(n_1, n_2)(b) \in \text{Pic}^0(F_b)$, it follows that $s(n_1, n_2)(t) \in \text{Pic}^0(F_t)$ for every point $t$ of $U$. Now, taking $t = \eta$, we have $H(n_1, n_2)|A_K = s(n_1, n_2)(\eta) \in \text{Pic}^0(A_K)$. Hence $n_1L_1 - n_2L_2 = 0$ in $\text{NS}(A_K)$. This completes the proof of Theorem 1.1.

Substituting $\rho(P^n) = \rho(A_K) = 1$ into the formula [Og2] Proposition 2.6 and Lemma 2.4, or [Kh]), we obtain Theorem 1.2.

### 3. Proof of Theorems 1.3 and 1.4

Let $f : X \to P^n$ and $O \simeq P^n$ be as in Theorem 1.3 and let $F$ be a general closed fiber of $f$. Then $O$ and $F$ are Lagrangian submanifolds of $X$. Consider the deformations $\mathcal{X} \to \mathcal{B} \subset \mathcal{K}$ of $f : X \to P^n$ which keep $F$ and $O$ Lagrangian. Let $U \subset \mathcal{K}$ be a small neighborhood of $[\sigma_X]$.

By [Sa2] (see also [Og2] Proof of Proposition 3.3), the deformation $\mathcal{X} \to \mathcal{B}$ forms a projective family of abelian fibered HK manifolds $\tilde{f} : \mathcal{X} \to P^n \times \mathcal{B}$, over $\mathcal{B}$, with holomorphic section $\mathcal{O} \simeq P^n \times \mathcal{B}$. The base space $\mathcal{B}$ is a complex submanifold of codimension 2 in $\mathcal{K}$, of the form $(a, \sigma) = (s, \sigma) = 0$, where $a$ and $s$ are linearly independent elements in $H^2(X, \mathbb{Q})$. For this, we need the harder direction of [Vo] and the fact that $O \simeq P^n$.

Let $\tilde{f}_b : \mathcal{X}_b \to P^n$ $(b \in \mathcal{B})$ be the fibration induced by $\tilde{f}$. By the shape of equations defining $\mathcal{B}$ and by the Lefschetz (1, 1)-Theorem, we see that $\rho(\mathcal{X}_b) \geq 2$ for all $b \in \mathcal{B}$ and $\rho(\mathcal{X}_b) = 2$ for generic $b \in \mathcal{B}$. Here, the word “generic” means that the condition is satisfied outside a countable union of hypersurfaces. In fact, $\rho(\mathcal{X}_b) \geq \rho$ if and only if $\dim \{ \ell \in H^2(X, \mathbb{Q}) | (\ell, [\omega_{\mathcal{X}_b}]) = 0 \} \geq \rho$, i.e., there are at least $\rho$ linearly independent rational hyperplanes passing through the point $[\sigma_{\mathcal{X}_b}]$ corresponding to $\mathcal{X}_b$. Thus, we have in fact a much stronger statement; in any open neighborhood of $[\sigma_X] \in \mathcal{B}$, the subset
\{b \mid \rho(X_b) = \rho\} is also dense with respect to Euclidean topology, for each integer \(\rho\) such that \(2 \leq \rho \leq b_2(X) - 2\). (See \cite{OG1} Theorem 1.1 and its proof for more details).

Let \(D \subset \mathbb{P}^n \times B\) be the discriminant locus of \(\tilde{f}\). The discriminant locus of \(\tilde{f}_b (b \in B)\) is then \(D_b = D \cap (\mathbb{P}^n \times \{b\})\). This is of pure codimension 1 in \(\mathbb{P}^n\) \cite{HO} Proposition 3.1 (2), \cite{Hw} Proposition 4.1. Let \(k(b)\) be the number of the irreducible components of \(D_b\) in \(\mathbb{P}^n\). Then \(k(b)\) is upper-semicontinuous on \(B\) with respect to Zariski topology. Choose \(p \in B \cap U\) such that \(k(p)\) is minimal, say \(k\). There is then an open neighbourhood \(p \in V \subset B \cap U\) such that the number of irreducible components of \(D_b (b \in V)\) is \(k\), in particular, constant. For the argument from now, we may (and will) assume that the irreducible decomposition of \(D\) is \(D = \cup_{i=1}^k D_i\) and \(D_b = \cup_{i,b}^k D_{i,b}\) is the irreducible decomposition of \(D_b (b \in V)\).

Take one general reference point \(o \in V\) such that \(\rho(X_o) = 2\). Then, by Theorem \ref{thm1.2} \(\tilde{f}_o^{-1}(D_{i,o})\) are all irreducible. Moreover, they are reduced as \(\tilde{f}\) admits a holomorphic section \(O\). Take then a resolution of singularities \(\tilde{D}_i\) of \(\tilde{f}^{-1}(D_i)\) and denote by \(g : \tilde{D}_i \longrightarrow V\) the natural morphism. Recall that smoothness of fibers is a non-empty open condition with respect to Zariski topology for a proper morphism from a smooth variety (in characteristic 0). Then, by generality of \(o\) and by the fact that \(\tilde{f}_o^{-1}(D_{i,o})\) is irreducible and reduced, we see that \(g^{-1}(o)\) is also irreducible and smooth, and the same holds over some Zariski open neighbourhood \(W\) of \(o\). Thus \(\tilde{f}_b^{-1}(D_{i,b})\) are irreducible and reduced for all \(b \in W\). Thus, from Theorem \ref{thm1.2} we have \(mw(\tilde{f}_b) = \rho(X_b) - 2\) for all \(b \in W\). On the other hand, as we already remarked, the set \(W_{\rho} = \{b \in W \mid \rho(X_b) = \rho\}\) is dense in \(W\) (with respect to Euclidean topology) for each \(\rho\) with \(2 \leq \rho \leq b_2(X) - 2\). Hence, for each such \(\rho\), there is \(b \in W \subset U\) such that \(\rho(X_b) = \rho\) and \(mw(\tilde{f}_b) = \rho - 2\). This completes the proof of Theorem \ref{thm1.3}.

Let us show Theorem \ref{thm1.4} Let \(f : S \longrightarrow \mathbb{P}^1\) be an elliptic K3 surface with a section. Then \(f\) induces an abelian fibration \(\varphi : S^{[n]} \longrightarrow \mathbb{P}^n\) with holomorphic section. Thus, Theorem \ref{thm1.4} (1) follows from Theorem \ref{thm1.3} Note that \(\rho(F) \geq 2\) for each smooth close fiber \(F\) of \(\varphi\), as \(F\) is the product of elliptic curves (cf. Remark after Theorem \ref{thm1.1}).

For Theorem \ref{thm1.4} (2), it suffices to find one abelian HK manifold \(f : X \longrightarrow \mathbb{P}^5\) such that \(X\) is deformation equivalent to O’Grady’s 10-dimensional HK manifold \(M_4\) and \(f\) admits a holomorphic section.

Let \(S\) be a generic algebraic K3 surface of degree 2. Then, \(\text{Pic} S = \mathbb{Z} H\), \(H\) is ample, and \((H^2) = 2\). We denote by \(\pi : S \longrightarrow \mathbb{P}^2\) the finite double cover induced by \(|H|\). Let \(\overline{Y} := M((0, [2H], 2))\) be the moduli space of semi-stable sheaves with Mukai vector \((0, [2H], 2)\) on \(S\). The space \(\overline{Y}\) is singular, but it admits a symplectic resolution \(Y\) \cite{LS}. This \(Y\) is a HK manifold and is birational to \(M_4\) \cite{OG} Section 4. Hence \(Y\)
is deformation equivalent to $\mathcal{M}_4$ by a fundamental result of Huybrechts [Hu]. For each $I \in M((0,[2H],2))$, the fitting ideal of $I$ is the ideal sheaf of some $C \in |2H|$ and vice versa ([Ra] Remark 2.1.1. See also [Sa1]). Thus, we have a natural surjective morphism $\overline{g} : \overline{Y} \longrightarrow \mathbb{P}^5 \cong |2H|$. Let $U \subset |2H|$ be the open subset consisting of smooth members. If $[C] \in U$, then $C$ is a smooth curve of genus 5 and the fiber $\overline{g}^{-1}([C])$ is isomorphic $\operatorname{Pic}^0(C)$, which is an abelian variety of dimension 5. The same is true for the induced fibration $g : Y \longrightarrow \mathbb{P}^5 \cong |2H|$. As $C$ admits a unique $g_2^1(C)$ corresponding to the double cover $\pi|C$, the map defined by $C \mapsto 3g_2^1(C) \in \operatorname{Pic}^0(C)$ gives a section of $g$ over $U$.

However, it is unclear if this section will be extended to a holomorphic section of $g$. The following idea to replace $Y$ further is due to Yasunari Nagai.

Similarly, one has the moduli space $\overline{X} := M((0,[2H],-4))$ of semi-stable sheaves with Mukai vector $(0,[2H],-4)$ on $S$ and its symplectic resolution $X$ [LS]. For the same reason as before, we have a fibration $\overline{f} : \overline{X} \longrightarrow \mathbb{P}^5 \cong |2H|$, and the induced fibration $f : X \longrightarrow \mathbb{P}^5 \cong |2H|$. The fiber over $[C] \in U$ is $\operatorname{Pic}^0(C)$ for both $\overline{f}$ and $f$. As $g$ admits a section over $U$, we have $f^{-1}(U) \cong g^{-1}(U)$ over $U$. Thus, $X$ is birational to $Y$, whence, to $\mathcal{M}_4$. So, $X$ is also deformation equivalent to $\mathcal{M}_4$ again by [Hu].

In order to complete the proof, we need the following slightly technical lemma. Note that it is unclear if the structure sheaf of a curve is stable unless the curve is irreducible and reduced. This is pointed out by the referee.

**Lemma 3.1.** For each $C$ in $|2H|$, the structure sheaf $\mathcal{O}_C$ is stable with respect to the polarization $H$.

**Proof.** As $\operatorname{Pic} S = \mathbf{Z}H$, an element $C$ of $|2H|$ falls into one of the following types (i), (ii), (iii): (i) an irreducible and reduced curve; (ii) a reduced curve of the form $C_1 + C_2$, where $C_1, C_2 \in |H|$; (iii) a non-reduced curve of the form $2C_1$, where $C_1 \in |H|$. We put $C_2 = C_1$ for a curve of type (iii).

Let $F$ be a proper subsheaf of $\mathcal{O}_C$. It suffices to show that $p(F,m) < p(\mathcal{O}_C,m)$, where $p(\ast,m)$ is the reduced Hilbert polynomial of a coherent sheaf $\ast$ with respect to the polarization $H$ (see [HL], pages 10-11). In what follows, we denote $F \otimes \mathcal{O}_S(mH)$ by $F(m)$ and so on. When $C$ is of type (i), the result just follows from the exact sequence $0 \rightarrow F(m) \rightarrow \mathcal{O}_C(m) \rightarrow \mathcal{O}_C/F \rightarrow 0$, where $\mathcal{O}_C/F$ is a non-zero skyscraper sheaf.

Let us consider the case where $C$ is of type (ii) or (iii).

Let $I_{C_1}$ be the ideal sheaf of $C_1$ in $\mathcal{O}_S$. Then $I_{C_1}/I_{C_1}I_{C_2} \cong \mathcal{O}_{C_2}(-C_1)$ when $C$ is of type (ii) and $I_{C_1}/I_{C_1}^2 \cong \mathcal{O}_{C_1}(-C_1)$ when $C$ is of type (iii). We have then the following exact sequence called the decomposition sequence (see e.g. [BHPV] page 62):

$$0 \rightarrow \mathcal{O}_{C_2}(-C_1) \xrightarrow{\alpha} \mathcal{O}_C \xrightarrow{\beta} \mathcal{O}_{C_1} \rightarrow 0.$$
Recall that \((H^2) = 2\). Then, by the exact sequence
\[
0 \to \mathcal{O}_S(m - 2) \to \mathcal{O}_S(m - 1) \to \mathcal{O}_{C_2}(-C_1)(m) \to 0
\]
and by the Riemann-Roch theorem on \(S\), we have
\[
\chi(\mathcal{O}_{C_2}(-C_1)(m)) = \chi(\mathcal{O}_S(m - 1)) - \chi(\mathcal{O}_S(m - 2)) = (m - 1)^2 - (m - 2)^2 = 2m - 3.
\]
Similarly, \(\chi(\mathcal{O}_C(m)) = 4m - 4\) and \(\chi(\mathcal{O}_{C_1}(m)) = 2m - 1\), and hence \(p(\mathcal{O}_C, m) = m - 1\).

Put \(G = \text{Im}(\beta|F)\) and \(K = \text{Ker}(\beta|F)\).

First we consider the case where \(G \neq 0\) and \(K \neq 0\). In this case, both \(\mathcal{O}_{C_2}(-C_1)/K\) and \(\mathcal{O}_{C_1}/G\) are skyscraper sheaves. We have then
\[
\chi(K(m)) = 2m - 3 - \ell(\mathcal{O}_{C_2}(-C_1)/K), \quad \chi(G(m)) = 2m - 1 - \ell(\mathcal{O}_{C_1}/G),
\]
where \(\ell(\ast)\) is the length of a skyscraper sheaf \(\ast\). Thus
\[
\chi(F(m)) = \chi(K(m)) + \chi(G(m)) = 4m - 4 - (\ell(\mathcal{O}_{C_2}(-C_1)/K) + \ell(\mathcal{O}_{C_1}/G)),
\]
and hence
\[
p(F, m) = m - 1 - \frac{\ell(\mathcal{O}_{C_2}(-C_1)/K) + \ell(\mathcal{O}_{C_1}/G)}{4}.
\]
As, \(F \neq \mathcal{O}_C\), we have \(\ell(\mathcal{O}_{C_2}(-C_1)/K) + \ell(\mathcal{O}_{C_1}/G) > 0\). Therefore \(p(F, m) < p(\mathcal{O}_C, m)\).

Next we consider the case where \(G = 0\). In this case, \(F = K\) and \(\mathcal{O}_{C_2}(-C_1)/K\) is a skyscraper sheaf. Thus \(\chi(F(m)) = 2m - 3 - \ell(\mathcal{O}_{C_2}(-C_1)/K)\), and hence we have \(p(F, m) \leq m - 3/2\). Therefore \(p(F, m) < p(\mathcal{O}_C, m)\).

It remains to consider the case where \(K = 0\). When \(C\) is of type (ii), we can interchange the roles of \(C_1\) and \(C_2\) to reduce the problem to the case where \(K \neq 0\), and we are done. So, we may assume that \(C\) is of type (iii). As \(F \simeq G\), we have
\[
\chi(F(m)) = \chi(G(m)) = 2m - 1 - \ell(\mathcal{O}_{C_1}/G)
\]
and hence
\[
p(F, m) = m - \frac{1 + \ell(\mathcal{O}_{C_1}/G)}{2}.
\]
Thus \(p(F, m) < p(\mathcal{O}_C, m)\) unless \(\ell(\mathcal{O}_{C_1}/G) = 0\) or \(1\). Let us show \(\ell(\mathcal{O}_{C_1}/G) \neq 0, 1\) by argue by contradiction.

Assume that \(\ell(\mathcal{O}_{C_1}/G) = 0\). Then \(G = \mathcal{O}_{C_1}\) and the decomposition sequence splits as
\[
\mathcal{O}_C \simeq \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_1}(-C_1)
\]
as \(\mathcal{O}_C\)-modules. However, this is impossible, because the right hand side is annihilated by the subsheaf \(I_{C_1}/I_{C_1}^2\) of \(\mathcal{O}_C\), while the left hand side is not.

Assume that \(\ell(\mathcal{O}_{C_1}/G) = 1\). Then \(G\) is the maximal ideal sheaf \(\mathfrak{m}_{C_1, P}\) of some point \(P\) in \(C_1\). Notice the following natural exact sequence:
\[
0 \to (f)/(f^2) \to (x, y)/(f^2) \to (x, y)/(f) \to 0,
\]
where \((x, y)\) is the local coordinates of \(S\) at \(P\) and \(f\) is the defining equation of \(C_1\) at \(P\). Then, by restricting the decomposition sequence to the maximal ideal \(m_{C,P}\) of \(P\) in \(C\), we have the following exact sequence of \(O_C\)-modules:

\[
0 \to O_{C_1}(-C_1) \xrightarrow{\alpha} m_{C,P} \xrightarrow{\beta} m_{C_1,P} \to 0.
\]

As \(\beta|F : F \simeq G = m_{C_1,P}\), we have then

\[
m_{C,P} \simeq m_{C_1,P} \oplus O_{C_1}(-C_1)
\]

as \(O_C\)-modules. However, for the same reason as before, this is impossible.

This completes the proof of Lemma 3.1. □

Let us return back to the proof of Theorem 1.4 (2). Let \(B\) be the closed subset of \(\overline{X}\) that parametrizes the \(S\)-equivalence classes of semistable but unstable sheaves. Then, by Lemma 3.1, the map \(C \mapsto O_C\) gives a holomorphic section \(s : \mathbb{P}^5 \to P \subset \overline{X}\) of \(\overline{f}\) that satisfies \(s(\mathbb{P}^5) \cap B = \emptyset\). As \(X\) is just a blow-up of \(\overline{X}\) along \(B\) \([\text{LS}]\), it follows that \(s\) gives rise to a holomorphic section of \(f : X \to \mathbb{P}^5\). Now, Theorem 1.4 (2) follows from Theorem 1.3.

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