STABILIZER RIGIDITY IN IRREDUCIBLE GROUP ACTIONS

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Abstract. We consider irreducible actions of locally compact product groups, and of higher rank semi-simple Lie groups. Using the intermediate factor theorems of Bader-Shalom and Nevo-Zimmer, we show that the action stabilizers, and all irreducible invariant random subgroups, are co-amenable in some normal subgroup. As a consequence, we derive rigidity results on irreducible actions that generalize and strengthen the results of Bader-Shalom and Stuck-Zimmer.

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1. Introduction

Let $G$ be a locally compact second countable (lcsc) group. An invariant random subgroup (IRS) of $G$ is a random variable that takes values in $\text{Sub}_G$, the space of closed subgroups of $G$, and whose distribution is invariant to conjugation by any element of $G$ [1]. IRSs arise naturally...
as stabilizers of probability measure preserving (pmp) actions. Moreover, any IRS is the stabilizer of some pmp action (see [2, Theorem 2.4] and also [1, 8]).

A subgroup $H \leq G$ is said to be co-amenable in $G$ if there exists a $G$-invariant mean on $G/H$ [17]. A normal subgroup $N \triangleleft G$ is co-amenable in $G$ if and only if $G/N$ is amenable.

We say that an IRS $K$ is co-amenable in $G$ if it is almost surely co-amenable in $G$. Likewise, if $K$ almost surely has some property (e.g., trivial, normal, co-finite), we say succinctly that $K$ has this property.

Let $G = G_1 \times G_2$ be a product of two lcsc groups. A pmp action $G \curvearrowright (X, m)$ is irreducible (with respect to the decomposition $G = G_1 \times G_2$) if the actions of both $G_1$ and $G_2$ are ergodic. Likewise, a pmp action of a semi-simple Lie group is said to be irreducible if the action of each simple factor is ergodic. An IRS $K$ is irreducible if $G \curvearrowright (\text{Sub}_G, \lambda)$ is irreducible, where $\lambda$ is the distribution of $K$ and $G$ acts on $\text{Sub}_G$ by conjugation [2]. Note that if an action is ergodic or irreducible then so is the associated stabilizer IRS, while the opposite direction is not true in general.

Our main results concern irreducible IRSs of product groups and of semi-simple Lie groups. These are generalizations of the theorems of Bader-Shalom [3, Theorem 1.6] and of Stuck-Zimmer [20]; we do not require $G$ to have property (T). Nevertheless, in both cases, we rely on the corresponding Intermediate Factor Theorems: Bader-Shalom [3] for products and Nevo-Zimmer [18] for semi-simple Lie groups.

**Theorem 1.** Let $G = G_1 \times G_2$ be an lcsc group, and let $K$ be an irreducible IRS in $G$. Then there exist closed normal subgroups $N_1 \triangleleft G_1$ and $N_2 \triangleleft G_2$ such that $K$ is co-amenable in $N_1 \times N_2$.

**Theorem 2.** Let $G$ be a connected semi-simple Lie group with finite center, no compact factors and $\mathbb{R}$-rank $\geq 2$. Let $K$ be an irreducible IRS in $G$. Then $K$ is either equal to a closed normal subgroup, or else $K$ is co-amenable in $G$.

In particular, these theorems are rigidity results on the irreducible IRSs of groups such as $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, to which the theorems of Bader-Shalom and Stuck-Zimmer do not apply.

The fact that every pmp action gives rise to an IRS and vice versa allows us to derive a number of corollaries regarding irreducible actions. We assume throughout that $G$ acts measurably on standard measurable spaces.

Given a pmp action $G \curvearrowright (X, m)$ and a normal subgroup $N \triangleleft G$, denote by $N \setminus (X, m)$ the space of $N$-ergodic components of $(X, m)$. 
Corollary 1.1. Let $G = G_1 \times G_2$ be an lcsc group, and let $G \curvearrowright (X, m)$ be an irreducible pmp action.

Then there exists a closed normal subgroup $N = N_1 \times N_2$, with $N_1 \triangleleft G_1$ and $N_2 \triangleleft G_2$, such that $G/N$ acts essentially freely on $N \backslash (X, m)$, and such that there exists an $N$-invariant mean on $m$-almost every $N$-orbit $N x$.

If furthermore $N$ has property (T), then the $N$-action on each $N$-ergodic component is essentially transitive.

The conclusions of Theorem 1 and Corollary 1.1 can be strengthened when more constraints are imposed on $G$. In particular, we consider the following notions: An lcsc group is said to be just non-compact if every closed normal subgroup is co-compact. An lcsc group is said to be just non-amenable if every closed normal subgroup is co-amenable. Note that if a group is just non-compact or simple then it is also just non-amenable.

Corollary 1.2. Let $G = G_1 \times G_2$ be an lcsc group, let both $G_1$ and $G_2$ be just non-amenable. Then every irreducible IRS is either co-amenable in $G$ or equal to a normal subgroup.

In the following corollaries we show that when one of the factors has property (T), then all IRSs are either co-finite or equal to a normal subgroup. In fact, co-finite IRSs admit some more structure: any ergodic co-finite IRS is supported on a single orbit $\{H^g\}_{g \in G}$, for some co-finite $H \leq G$ (see Corollary 4.4).

Corollary 1.3. Let $G = G_1 \times G_2$ be an lcsc group, and let $G_1$ be just non-amenable and have property (T). Let $G \curvearrowright (X, m)$ be a faithful irreducible pmp action.

Then the action $G \curvearrowright (X, m)$ is either essentially free or essentially transitive. It follows that the associated stabilizer IRS is either trivial or co-finite in $G$.

This constitutes a strengthening of the Essentially Free Actions Theorem of Bader-Shalom; they require that both $G_1$ and $G_2$ have property (T) and be just non-compact.

A corollary of Theorem 2 is the following strengthening of the Stuck-Zimmer theorem [20].

Corollary 1.4. Let $G$ be a connected semi-simple Lie group with finite center, no compact factors and $\mathbb{R}$-rank $\geq 2$. Assume that one of the simple factors of $G$ has property (T). Then any faithful irreducible pmp $G$-action is either essentially free or essentially transitive, and its associated stabilizer is either trivial or a lattice in $G$. 

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Stuck-Zimmer derive the same conclusions but require every simple factor to have property (T).

Recently, Creutz [7] proved this result independently, using a different approach. In the same paper he also generalizes Bader-Shalom’s theorem to the case that \( G_1 \) has property (T) and both \( G_1 \) and \( G_2 \) are simple.

Creutz and Peterson [8] prove similar rigidity results for irreducible lattices and commensurators of lattices in semi-simple Lie groups, and also for product groups with the Howe-Moore property and property (T).

In [2] it is shown that in the setting of Corollary 1.4 if \( G \) has property (T) then every irreducible IRS is either equal to a normal subgroup or is a lattice.

1.1. Intermediate factor theorems. Theorems 1 and 2 are consequences of Theorem 3 below, which is a result on any IRS that satisfies an intermediate factor theorem (IFT). The original proofs of Stuck-Zimmer and of Bader-Shalom are also each based on a corresponding IFT. In this work (in particular, in Theorem 3) we give a more general result of IRS rigidity, given an IFT. In particular, our proof does not require property (T). This gives a partial answer to a question asked in Stuck-Zimmer [20, page 731].

Let \( \Pi(G, \mu) \) be the Poisson boundary of a group \( G \) with a measure \( \mu \), and let \( G \acts (X, m) \) be a pmp action. A \( G \)-quasi-invariant probability space \( (Y, \eta) \) is a \((G, \mu)\)-intermediate factor over \((X, m)\) if there exist \( G \)-factors

\[
\Pi(G, \mu) \times (X, m) \overset{\longrightarrow}{\longrightarrow} (Y, \eta) \overset{\longrightarrow}{\longrightarrow} (X, m)
\]

such that the composition is the natural projection

\[
\Pi(G, \mu) \times (X, m) \longrightarrow (X, m).
\]

An example of an intermediate factor is \((Y, \eta) \cong (C, \xi) \times (X, m)\), where \((C, \xi)\) is some \((G, \mu)\)-boundary, or, equivalently, where \((C, \xi)\) is a \( G \)-factor of the Poisson boundary. In this case, the intermediate factor \((Y, \eta)\) is said to be standard [11].

Zimmer [23] proves an intermediate factor theorem, which was generalized (and had its proof corrected) by Nevo and Zimmer [18]: When \( G \) is a connected semi-simple Lie group with finite center, no compact factors and higher rank, then there exists an admissible \( \mu \) such that any \((G, \mu)\)-intermediate factor over an irreducible pmp space is standard.

Bader and Shalom [3] prove the same result for intermediate factors over irreducible actions of product groups.

The following definition is inspired by these results.
Definition. Let $G$ be an lcsc group. A pmp action $G \acts (X, m)$ is said to be an IFT action if there exists an admissible $\mu$ on $G$ such that any $(G, \mu)$-intermediate factor over $(X, m)$ is standard.

Our main Theorems 1 and 2 are consequences of the following theorem and the above mentioned intermediate factor theorems.

**Theorem 3 (Co-amenable IRSs).** Let $G$ be an lcsc group. Let $K \leq G$ be an IRS with distribution $\lambda$ such that $G \acts (\text{Sub}_G, \lambda)$ is an IFT action. Then there exists a closed normal subgroup $N \triangleleft G$ such that $K$ is co-amenable in $N$.

The remainder of the paper is organized as follows. In Section 2 we construct the normal closure of an IRS and the IRS coset space, and discuss random walks on groups and coset spaces, Poisson boundaries and Bowen spaces. In Section 3 we prove Theorem 3. Finally, in Section 4 we prove Theorems 1 and 2 and their corollaries.

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2. Preliminaries

2.1. **The normal closure of an IRS.** Let $G$ be an lcsc group. Equip the space of closed subgroups of $G$, $\text{Sub}_G$, with the Chabauty topology $[5]$. Then the $G$ action on $\text{Sub}_G$ by conjugation is continuous. Denote by IRS$(G)$ the set of conjugation invariant probability Borel measures on $\text{Sub}_G$. These are the distributions of the IRSs of $G$.

**Definition 2.1.** For $\lambda \in \text{IRS}(G)$, let the normal closure $\langle \lambda \rangle$ be the unique minimal subgroup that is the closure of a subgroup generated by a $\lambda$-full measure set of subgroups.

**Claim 2.2.** $\langle \lambda \rangle$ is well defined and is equal to the closure of the subgroup generated by all the subgroups in $\text{supp} \lambda$.

Clearly, $\langle \lambda \rangle$ is a closed normal subgroup of $G$.

**Proof.** Let $T$ be the closure of the subgroup that is generated by $\text{supp} \lambda$. We prove the claim by showing that every closed subgroup $S$ that is the closure of a subgroup generated by a full measure set, contains $T$.

Let $S$ be the closure of the subgroup generated by a set $A$ of subgroups, with $\lambda(A) = 1$, and, without loss of generality, let $A \subseteq \text{supp} \lambda$. Then the closure of $A$ equals $\text{supp} \lambda$, and so for every $g \in T$ there
exists a sequence $g_n \to g$ with $g_n \in S$. Since $S$ is closed, it follows that $g \in S$. \qed

2.2. IRS coset spaces.

**Definition 2.3.** Given $\lambda \in \text{IRS}(G)$, let $G/\lambda$ be the locally compact space of closed subsets of $G$ given by

$$G/\lambda = \{ gH : H \in \text{supp} \lambda, g \in G \},$$

endowed with the Chabauty topology and equipped with the left and right $G$-actions on subsets of $G$.

Note that $\lambda \setminus G = \{ Hg : H \in \text{supp} \lambda, g \in G \}$ is equal to $G/\lambda$ as a $G$-space of subsets of $G$, since $gH = gHg^{-1}g = H^g g$ and $\text{supp} \lambda$ is conjugation invariant.

2.3. Random walks on groups and Bowen spaces. Let $G$ be a lcsc group, and let $\mu$ be an *admissible* measure on $G$. Namely, let some convolution power of $\mu$ be absolutely continuous with respect to the Haar measure, and let $\text{supp} \mu$ generate $G$ as a semi-group.

A $\mu$-random walk on a group is a measure $\mathbb{P}\mu$ on $G^\mathbb{N}$ given by the push-forward of $\mu^\mathbb{N}$ under the map $(g_1, g_2, g_3, \ldots) \mapsto (g_1, g_1g_2, g_1g_2g_3, \ldots)$. Equivalently, let $\{X_n\}_{n \in \mathbb{N}}$ be i.i.d. random variables with measure $\mu$, and let $Z_n = X_1 \cdots X_n$. Then a $\mu$-random walk is the distribution of $(Z_1, Z_2, \ldots)$. The (Furstenberg-)Poisson boundary $[9,10]$ of a $\mu$-random walk, denoted by $\Pi(G, \mu)$, is Mackey’s point realization of the shift-invariant sigma-algebra of $(G^\mathbb{N}, \mathbb{P}\mu)$ (see, e.g., [22]), also known as the space of shift-ergodic components.

Let $H \leq G$ be a closed subgroup. The map $(g_1, g_2, \ldots) \mapsto (Hg_1, Hg_2, \ldots)$ pushes forward $\mathbb{P}\mu$, the $\mu$-random walk on $G$, to $\mathbb{P}\mu^H$, the $\mu$-random walk on $H \setminus G$. We denote by $\Pi(G, \mu, H) = (B_H, \nu_H)$ the Poisson boundary of this random walk, which is here equal to Mackey’s point realization of the shift-invariant sigma-algebra of $((H \setminus G)^\mathbb{N}, \mathbb{P}\mu^H)$. Equivalently, $\Pi(G, \mu, H)$ is equal to $H \setminus \Pi(G, \mu)$, the space of $K$-ergodic components of $\Pi(G, \mu)$.

If $H$ is a normal subgroup, then the $H$-invariant sigma-algebra is $G$-invariant and hence $\Pi(G, \mu, H)$ is a $G$-space. In fact, it is a $(G, \mu)$-boundary, which is isomorphic to the Poisson boundary of the group $G/H$, equipped with the projection of $\mu$ (see [3] Lemma 2.15).

L. Bowen [4] introduces what we shall call *Bowen spaces* (see also earlier work by Kaimanovich [16]).
Definition 2.4. Let \( \lambda \in \text{IRS}(G) \). Consider the space

\[
\widetilde{\text{Sub}}_G = \{(H; Hg_1, Hg_2, \ldots) \mid H \in \text{Sub}_G, g_n \in G\}
\]

with the topology induced from \( (2^G)^\mathbb{N} \) and equipped with the measure

\[
d^\mathbb{P}_\mu^\lambda(H; Hg_1, Hg_2, \ldots) = d^\mathbb{P}_\mu(Hg_1, Hg_2, \ldots)d\lambda(H).
\]

\( G \) acts on \( \widetilde{\text{Sub}}_G \) by \( g(H; Hg_1, Hg_2, \ldots) = (H^g; H^g g_1, H^g g_2, \ldots) \), and the natural shift maps \( (H; Hg_1, Hg_2, \ldots) \) to \( (H; Hg_2, \ldots) \).

Denote by \( B_\mu(\lambda) \) Mackey’s point realization of the shift-invariant sigma algebra. We shall refer to \( B_\mu(\lambda) \) as a Bowen space.

Note that the \( G \) action on \( \widetilde{\text{Sub}}_G \) commutes with the shift, and so \( B_\mu(\lambda) \) is a \( G \)-space.

It is useful to realize the Bowen space \( B_\mu(\lambda) \) as the space of pairs

\[
B_\mu(\lambda) = \{(H, \xi) : H \in \text{supp}\lambda, \xi \in \Pi(G, \mu, H)\},
\]
equipped with the measure \( d\nu_\lambda(H, \xi) = d\lambda(H)d\nu_H(\xi) \). Therefore, \( B_\mu(\lambda) \) is the fiber bundle over \( \text{supp}\lambda \) in which the fiber over \( H \) is \( H\backslash\Pi(G, \mu) \). Hence the Bowen space \( B_\mu(\lambda) \) is an intermediate factor over \( \text{supp}(G, \mu) \):

\[
\Pi(G, \mu) \times (\text{Sub}_G, \lambda) \longrightarrow B_\mu(\lambda) \longrightarrow (\text{Sub}_G, \lambda).
\]

We encourage the reader to study the details in Bowen’s original paper [4]. For further discussion and another application of Bowen spaces see [13].

Consider the process on \( \lambda\backslash G \) that first chooses an element according to \( \lambda \), and then applies a \( \mu \)-right random walk. Formally, this is the Markov chain \( \{KZ_n\}_{n \in \mathbb{N}} \) where \( K \sim \lambda \) and \( Z_n \sim \mu^n \). It follows from the definitions that

Claim 2.5. The Bowen space \( B_\mu(\lambda) \) is the Poisson boundary of the Markov chain \( \{KZ_n\}_{n \in \mathbb{N}} \).

The Poisson boundary here is the space of shift-ergodic components of the Markov chain measure on \( (\lambda\backslash G)^\mathbb{N} \).

3. Proof of the Co-amenable IRSs Theorem

In this section we prove the following.

Proposition 3.1. Let \( K \) be an IRS in \( G \) with distribution \( \lambda \in \text{IRS}(G) \) such that \( G \backslash (\text{Sub}_G, \lambda) \) is an IFT action. Then \( K \) is co-amenable in \( \langle \lambda \rangle \).
Note that Theorem 3 is a direct consequence, since \( \langle \lambda \rangle \) is a closed normal subgroup of \( G \).

Let \( \mu \) be an admissible measure on \( G \) that is given by the IFT property, so that any \((G, \mu)\)-intermediate factor over \((\text{Sub}_G, \lambda)\) is standard. Fix \( \lambda \in \text{IRS}(G) \), and denote \( T = \langle \lambda \rangle \).

Recall that \( B_\mu(\lambda) \) is the fiber bundle over \( \text{supp} \lambda \) in which the fiber over \( H \) is \( H \setminus \Pi(\text{Sub}_G, \lambda) \). It follows that when the Bowen space \( B_\mu(\lambda) \) is standard - that is, when it is equal to \((C, \xi) \times (\text{Sub}_G, \lambda)\) for some boundary \((C, \xi)\) - then \( H \setminus \Pi(G, \mu) \cong (C, \xi) \), for \( \lambda \)-almost every \( H \). Note that while a priori \( H \setminus \Pi(\text{Sub}_G, \lambda) \) is not a \( G \)-space, we can now conclude that it is.

It follows that there exists a \( \lambda \)-full measure \( A \subset \text{Sub}_G \) such that \((C, \xi)\) is \( H \)-invariant for every \( H \in A \). Without loss of generality we may take a compact model for \( \Pi(\text{Sub}_G, \lambda) \) such that \( G \curvearrowright \Pi(\text{Sub}_G, \lambda) \) is continuous. It follows that \((C, \xi)\) is \( T \)-invariant. We have therefore proven the following claim.

**Claim 3.2.** Let \( B_\mu(\lambda) = (C, \xi) \times (\text{Sub}_G, \lambda) \) be standard. Then \( H \setminus \Pi(G, \mu) = T \setminus \Pi(G, \mu) \) for \( \lambda \)-almost every \( H \), and \((C, \xi)\) is \( T \)-invariant. Moreover, \((C, \xi)\) is the Poisson boundary of the quotient group \( G/T \), equipped with the projection of \( \mu \).

The second part of the claim follows from the fact that \( T \setminus \Pi(G, \mu) \) is the Poisson boundary of the projection of the \( \mu \)-random walk on \( G \) on the quotient group \( G/T \).

Given the fact that \( H \setminus \Pi(G, \mu) = T \setminus \Pi(G, \mu) \), it should perhaps be not very surprising that \( H \) is co-amenable in \( T \), as Proposition 3.1 claims. Indeed, the fact that \( G \curvearrowright \Pi(G, \mu) \) is an amenable action [22] implies the following result, to which Kaimanovich [15] gives a simple, elementary proof.

**Theorem 3.3.** Let \( N_1 \leq N_2 \) be two normal subgroups of \( G \) such that \( N_1 \setminus \Pi(G, \mu) = N_2 \setminus \Pi(G, \mu) \). Then \( N_1 \) is co-amenable in \( N_2 \).

In this sense, we show that invariant random subgroups behave like normal subgroups, as has been observed in other contexts [1].

### 3.0.1. Proof of Proposition 3.1

Let \( \mu \) be a measure on \( G \) such that \( G \curvearrowright (\text{Sub}_G, \lambda) \) is an IFT action, so that the Bowen space \( B_\mu(\lambda) \) is standard. Let \( K \) have distribution \( \lambda \).

To prove that \( K \) is co-amenable in \( T \), we first construct a sequence of asymptotically \( T \)-left invariant measures on \( \lambda \setminus G \), in Lemma 3.4. Then, we push this sequence to \( \lambda \setminus T \) in Claim 3.5. Finally, in Claim 3.6 we
use this sequence to show the existence of a $T$-left invariant mean on $T/H$, for $\lambda$-almost every $H$.

Denote by $\phi : \lambda \backslash G \to \text{Sub}_G$ the map $Hg = gH \theta^{-1} \mapsto H \theta^{-1}$. As a map $G/\lambda \to \text{Sub}_G$, (recall that $\lambda \backslash G = G/\lambda$), $\phi$ is the natural projection $gH \mapsto H$. Denote by $| \cdot |$ the total variation norm.

**Lemma 3.4.** There exists a sequence of measures $\theta_n$ on $\lambda \backslash G$ such that $|t\theta_n - \theta_n| \to 0$ for all $t \in T$, and $\phi_* g\theta_n = \lambda$ for all $n \in \mathbb{N}$ and $g \in G$.

**Proof.** Recall $\{KZ_n\}_{n \in \mathbb{N}}$, our Markov chain on $\lambda \backslash G$ given by $K \sim \lambda$ and $Z_n \sim \mu^n$. Denote by $\eta_n = \lambda \ast \mu^n$ the distribution of $KZ_n$. Let $\theta_n$ be given by

$$\theta_n = \frac{1}{n} \sum_{k=0}^{n-1} \eta_k.$$ 

Equip the space $\lambda \backslash G$ with a $\sigma$-finite measure $\alpha$ such that $\theta_n \prec \alpha$ for all $n$. Then $g\theta_n \prec \alpha$ for all $g \in G$ and $n$, since $\mu$ is generating.

By [14, Lemma 2.9], it follows that $|g\theta_n - \theta_n| \to |\nu_{g\lambda} - \nu_\lambda|$, where $\nu_{g\lambda}$ and $\nu_\lambda$ are the measures on the Poisson boundaries corresponding to the initial distributions $\lambda$ and $g\lambda$. Note that here, as $\lambda$ is a measure on $\lambda \backslash G$, the $G$-action is the left multiplication action rather than conjugation. Hence the initial distributions $g\lambda$ and $\lambda$ are in general different.

By Claim 2.5, $\nu_\lambda$ is the measure of the Bowen space $B_\mu(\lambda)$. By the definition of the $G$ action on Bowen spaces, $\nu_{g\lambda} = g\nu_\lambda$. Hence we get that

$$|g\theta_n - \theta_n| \to |g\nu_\lambda - \nu_\lambda|.$$ 

Finally, by Claim 3.2, $\nu_\lambda$ is $T$-invariant, and so $|t\theta_n - \theta_n| \to 0$ for all $t \in T$.

We show that $\phi_* g\theta_n = \lambda$ by showing that $\phi_* g\eta_n = \lambda$. Let $KZ_n \sim \eta_n$, where $K \sim \lambda$ and $Z_n \sim \mu^n$. But $gKZ_n = gZ_n KZ_n^{-1}$, and since $KZ_n^{-1} \sim \lambda$ by the conjugation invariance of $\lambda$, we get that $\phi_* g\eta_n = \lambda$. \hfill $\square$

Since $T$ includes every $H \in \text{supp} \lambda$, $\lambda$ can be considered to be an element in $\text{IRS}(T)$. In analogy to the definition of $\lambda \backslash G$, we can define $\lambda \backslash T$. We define $\phi : \lambda \backslash T \to \text{Sub}_G$ likewise.

Choose a $T$-equivariant measurable map $G \to T$ (recall that $T$ is normal in $G$). It induces a natural map $\lambda \backslash G \to \lambda \backslash T$. We can now push forward the measures $\theta_n$ from Lemma 3.4 above, to get the following.

**Claim 3.5.** There exists a sequence of measures $\theta_n$ on $\lambda \backslash T$ such that $|t\theta_n - \theta_n| \to 0$ for all $t \in T$, and $\phi_* s\theta_n = \lambda$ for all $n \in \mathbb{N}$ and $s \in T$.

We are now ready to take the last step in the proof of Proposition 3.1.
Claim 3.6. \( K \) is co-amenable in \( T \).

Proof. Let \( \theta_n \) be a sequence of probability measures on \( \lambda \setminus T \) given by Claim 3.5. Thinking of \( \theta_n \) as a measure on \( T/\lambda \), and thinking of \( \phi : T/\lambda \to \text{Sub}_G \) as the natural projection \( tH \mapsto H \), we disintegrate \( \theta_n \) with respect to \( \phi \):

\[
\theta_n = \int_{\text{Sub}_G} \theta_n^H d\lambda(H),
\]

so that \( \theta_n^H \) is a measure on \( T/H \). We likewise disintegrate \( t\theta_n \) into the fiber measures \( (t\theta_n)^H \). Note that \( \phi(tsH) = \phi(sH) \) for every \( t, s \in T \). Hence both \( \theta_n^H \) and \( t\theta_n^H \) are supported on the same fiber, and we get that \( (t\theta_n)^H = t\theta_n^H \).

As both \( t\theta_n \) and \( \theta_n \) are projected by \( \phi \) to \( \lambda \), we can disintegrate \( t\theta_n - \theta_n \) to get

\[
t\theta_n - \theta_n = \int_{\text{Sub}_G} ((t\theta_n)^H - \theta_n^H) d\lambda(H)
= \int_{\text{Sub}_G} (t\theta_n^H - \theta_n^H) d\lambda(H),
\]

and in particular,

\[
|t\theta_n - \theta_n| = \int_{\text{Sub}_G} |t\theta_n^H - \theta_n^H| d\lambda(H).
\]

But for any \( t \in T \) we have that \( |t\theta_n - \theta_n| \to 0 \), and therefore \( |t\theta_n^H - \theta_n^H| \to 0 \) for \( \lambda \)-almost every \( H \). Finally, the existence of asymptotically invariant measures on \( T/H \) implies the existence of a \( T \)-left invariant mean on \( T/H \). This, in turn, implies that \( H \) is co-amenable in \( T \) \[12\].

This completes the proof of Proposition 3.1.

4. Proofs of main theorems and corollaries

Proof of Theorem 4. Let \( \lambda \in \text{IRS}(G) \) be the distribution of \( K \), and denote \( T = \langle \lambda \rangle \). By the Bader-Shalom IFT, Proposition 3.1 implies that \( K \) is co-amenable in \( T \).

Let \( \mu = \mu_1 \times \mu_2 \), where \( \mu_i \) is an admissible measure on \( G_i \). Then

\[
\Pi(G, \mu) = \Pi(G_1, \mu_1) \times \Pi(G_2, \mu_2),
\]

and so, if we denote by \( N_i \triangleleft G_i \) the projection of \( T \) on \( G_i \) we get that \( T \setminus \Pi(G, \mu) = N_1 \setminus \Pi(G_1, \mu_1) \times N_2 \setminus \Pi(G_2, \mu_2) = (N_1 \setminus N_2) \setminus \Pi(G, \mu) \).

By Theorem 3.3 it follows that \( T \) is co-amenable in \( N_1 \times N_2 \). As \( K \) is co-amenable in \( T \), we conclude that \( K \) is co-amenable in \( N_1 \times N_2 \). \( \square \)
Proof of Theorem 2. By the Nevo-Zimmer IFT, Proposition 3.1 implies that $K$ is co-amenable in its normal closure $T$. If $T$ is central then $K$ is equal to a normal subgroup, and we are done.

Otherwise, we show that $T = G$: Let $S$ be such that $G = TS$ and $T$ and $S$ commute [19]. It follows that $S$ acts trivially on $K$, and so, by the irreducibility of $K$, $S$ is trivial and $T = G$. □

Before proving Corollary 1.1 we make note of a few facts and prove a lemma.

Let $G$ be an lcsc group, and let $H \leq G$. By definition, if $H$ is co-amenable in $G$ then there exists a $G$-invariant mean on the homogeneous space $G/H$. If $G$ additionally has property (T), the existence of this invariant mean implies that there exists a $G$-invariant probability measure on $G/H$. That is, if $G$ has property (T) then the co-amenable subgroups are co-finite.

We next show that it follows from Varadarajan’s ergodic decomposition theorem [21] that the existence of invariant measures supported on orbits can be used to show that an action is essentially transitive.

Definition 4.1. Let $G$ be an lcsc group acting on a standard measurable space $X$. Let $E(X)^G$ denote the space of $G$-invariant, ergodic probability measures on $X$. A decomposition map is a measurable map $\beta : X \to E(X)^G$ with the following properties.

1. $\beta$ is $G$-invariant. I.e., denoting $x \to \beta_x$, it holds that $\beta_{gx} = \beta_x$ for all $g \in G$ and $x \in X$.

2. For every $\eta \in E(X)^G$, it holds that $\eta(\beta^{-1}(\eta)) = 1$.

3. For every $G$-invariant measure $\theta$ it holds that

\begin{equation}
\theta = \int_X \beta_x d\theta(x).
\end{equation}

Theorem 4.2 (Varadarajan). For every action of an lcsc group $G$ on a standard measurable space $X$, there exists a decomposition map $\beta$. Furthermore, $\beta$ is essentially unique, in the sense that if $\beta$ and $\beta'$ are decomposition maps then $\theta(\{x \in X : \beta_x \neq \beta'_x\}) = 0$ for any $G$-invariant probability measure $\theta$.

Lemma 4.3. Let $G$ be an lcsc group, and let $G \actson (X, m)$ be a pmp action on a standard measurable space. Assume that there exists a $G$-invariant probability measure on $m$-almost every $G$-orbit. Then the action $G \actson (X, m)$ is essentially transitive.

Proof. Let $\beta : X \to E(X)^G$ be a decomposition map of $X$ with respect to the $G$-action. Let $x \in X$ be such that there exists a $G$-invariant and ergodic probability measure $\eta_x$ with $\eta_x(Gx) = 1$. 
By the second property of decomposition maps, there exists an element \( y \in Gx \) (in fact a \( \eta_x \)-full measure set of such elements) for which \( \beta_y = \eta_x \). Since \( \beta \) is \( G \)-invariant, we get that \( \beta_x = \eta_x \) and in particular \( \beta_x \) is supported on \( Gx \).

Let \( A \) be an \( m \)-full measure set of \( x \in X \) for which there exists a \( G \)-invariant, ergodic measure \( \eta_x \) on \( Gx \), and for which, by the above, \( \beta_x = \eta_x \) is supported on a \( G \)-orbit. Then

\[
m = \int_X \beta_x dm(x) = \int_A \beta_x dm(x),
\]
and it follows by the ergodicity of \( m \) that it is equal to some \( \beta_x \). Hence \( m \) is supported on a \( G \)-orbit, or, equivalently, the action is essentially transitive. \( \square \)

The following is a corollary of Lemma 4.3.

**Corollary 4.4.** Let \( G \) be an lcsc group, and let an ergodic IRS \( K \) in \( G \) be almost surely co-finite. Then its distribution is supported on a single orbit \( \{ H^g : g \in G \} \).

In other words, the action of an lcsc group on its co-finite subgroups is tame. In Stuck-Zimmer this is proven for the case of Lie groups \([20, \text{Corollary 3.2}]\).

We are now ready to prove Corollary 1.1.

**Proof of Corollary 1.1.** Let \( K \) be the stabilizer IRS associated with the action \( G \curvearrowright (X, m) \).

By Theorem 1 we have that there exists a normal subgroup \( N \triangleleft G \) such that \( K \) is co-amenable in \( N \). Since \( N \) includes the stabilizers of \( m \)-almost every \( x \in X \), it follows that \( G/N \) acts essentially freely on \( N \backslash (X, m) \).

Let \((X, m_0)\) be an \( N \)-ergodic component of \( m \). Since the stabilizer \( N_x \) of \( x \) is \( m_0 \)-almost surely co-amenable in \( N \), there exists an \( N \)-invariant mean on \( N/N_x \), for \( m_0 \)-almost every \( x \). Since the orbit \( Nx \) can be identified with \( N/N_x \), it follows that there exists an \( N \)-invariant mean on the orbit \( N \).

If in addition \( N \) has property (T) then, because \( K \) is co-amenable in \( N \), it is in fact co-finite in \( N \). Hence, by again identifying each orbit \( Nx \) with \( N/N_x \), it follows that there exists an \( N \)-invariant measure on the orbit \( N \) for \( m_0 \)-almost every \( x \). Therefore, by Lemma 4.3 \( N \) acts essentially transitively on \((X, m_0)\). \( \square \)

**Remark 4.5.** For discrete groups, it is straightforward to see that when an action’s stabilizers are co-amenable, then the action is weakly...
amenable (in the sense of Zimmer [22]), or, equivalently, that the induced orbital equivalence relation is amenable [6].

Proof of Corollary 1.2. By Theorem 1, there exist subgroups $N_1 \triangleleft G_1$ and $N_2 \triangleleft G_2$ such that $K$ is co-amenable in $N = N_1 \times N_2$. Since $K$ is irreducible, if either $N_1$ or $N_2$ is trivial then $K$ must equal $N_1 \times N_2$. Otherwise, because $G_1$ and $G_2$ are just non-amenable, $N_1$ is co-amenable in $G_1$ and $N_2$ is co-amenable in $G_2$. Hence $N$ is co-amenable in $G$, and so $K$ is co-amenable in $G$. □

Before proving Corollaries 1.3 and 1.4 we prove an analogue of Lemma 4.3 for almost direct products. $G$ is said to be an almost direct product of the subgroups $G_1$ and $G_2$ if the two groups commute and $G = G_1 G_2$.

Lemma 4.6. Let $G = G_1 G_2$ be an lcsc almost direct product, and let $G \acts (X, m)$ be a pmp action on a standard measurable space that is $G_1$-ergodic. Assume that there exists a $G_1$-invariant probability measure on $m$-almost every $G$-orbit. Then the action $G \acts (X, m)$ is essentially transitive.

Proof. Let $\beta^1 : X \to E(X)^{G_1}$ be a decomposition map of the $G_1$-action on $X$. Note that, since $G_1$ and $G_2$ commute, it holds for every $\eta \in E(X)^{G_1}$ and $g \in G$ that $g \eta \in E(X)^{G_1}$, and so $G$ acts on $E(X)^{G_1}$. Hence one can consider the question of whether $\beta^1$ commutes with $G$. In fact, in order to follow the same arguments of Lemma 4.3 we will require that $\beta^1$ be essentially $G$-equivariant. That is, that there exists an $A \subseteq X$ with $\theta(A) = 1$ for any $G_1$-invariant measure $\theta$, and such that $g \beta^1_x = \beta^1_{gx}$ for all $g \in G$ and $x \in A$. Assume first that $\beta^1$ satisfies this condition.

Since $m$ is $G_1$-invariant, $m(A) = 1$. Hence there exists an $m$-full measure set $A' \subseteq A$ such that there exists a $G_1$-invariant and ergodic measure $\eta_x$ on $G x$ for all $x \in A'$. Fix some $x \in A'$. Since $\eta_x(G x) = 1$, we can find some $y = gx \in G x$ such that $\beta^1_y = \eta_x$. Then

$$g \beta^1_x = \beta^1_{gx} = \beta^1_y = \eta_x$$

and so we conclude that $\beta^1_x$ is supported on $G x$ for every $x \in A'$.

Since $m$ is $G_1$-invariant we can write

$$m = \int_X \beta^1_x dm(x) = \int_{A'} \beta^1_x dm(x)$$

and, by the $G_1$-ergodicity of $m$, $m = \beta^1_x$ for some $x \in A'$ and in particular $m(G x) = 1$. Hence the action $G \acts (G, m)$ is essentially transitive.
Finally, we argue that $\beta^1$ is essentially $G$-equivariant. We first show the simple case that $G_2$ is countable, and then prove the more general case below. Let $\theta$ be a $G_1$-invariant measure, and let $g \in G$. Applying the decomposition (4.1) to $g\theta$ yields

$$\theta = \int_X g^{-1} \beta^1_{gy} d\theta(x).$$

which, by the essential uniqueness of $\beta^1$, implies that there exists an $A_g \subset X$ such that $g\beta^1_x = \beta^1_{gy}$ for every $x \in A_g$, and such that $A_g$ has full measure with respect to any $G_1$-invariant measure. Hence, if $G_2$ is countable, we get that $\beta^1$ is equivariant on $A = \bigcap_{g \in G_2} A_g$, where $A$ also has full measure with respect to any $G_1$-invariant measure.

We are left with the case that $G_2$, and so $G$, is not countable. Following Varadarajan, let $\mathcal{F}$ be the Banach space of all bounded measurable functions on $X$. Let $U^1 : \mathcal{F} \to \mathcal{F}$ be the operator defined by $U^1(f)(x) = \int_X f(y) d\beta^1_{gy}(y)$, for any $f \in \mathcal{F}$. Note that by definition, the equivariance of $\beta^1$ is equivalent to the equivariance of $U^1$.

Now, for any $G_1$-invariant measure $\theta$, $U^1$ is the conditional expectation defined by the factor $(X, \theta) \to G_1 \backslash (X, \theta)$:

$$U^1 : L^\infty(X, \theta) \to L^\infty(G_1 \backslash (X, \theta)).$$

Since the actions of $G_1$ and $G_2$ on $X$ commute, $U^1$, as the conditional expectation map, is $G_2$-equivariant. Hence $U^1 : \mathcal{F} \to \mathcal{F}$ is $G$-equivariant on a $\theta$-full measure set. Since this holds for any $G_1$-invariant measure $\theta$, we conclude that $U^1$, and so the associated decomposition map $\beta^1$, are essentially $G$-equivariant, and the proof is complete.

Proof of Corollary 1.3. Let $K$ be the associated stabilizer IRS. By Theorem 1 and Corollary 1.1, there exists a normal subgroup $N = N_1 \times N_2 \triangleleft G$ such that $K$ is co-amenable in $N$, $G/N$ acts essentially freely on $N \backslash (X, m)$, and $m$-almost every orbit $Nx$ admits an $N$-invariant mean.

Since $G_1$ is just non-amenable, either $N_1$ is co-amenable in $G_1$ or else $N_1$ is trivial. In the latter case we have that $K$ is contained (in fact, co-amenable) in $\{e\} \times N_2$, and since $K$ is irreducible, it must almost surely equal $\{e\} \times N'_2$, for some $N'_2 \triangleleft G_2$. Since the action $G \actson (X, m)$ is faithful, $N'_2 = \{e\}$ and the action is essentially free.

We are therefore left with the case that $N_1$ is co-amenable in $G_1$. Then since $K$ is co-amenable in $N_1 \times N_2$ we get that $K$ is co-amenable in $G_1 \times N_2$, and, identifying each $G_1 \times N_2$-orbit with $(G_1 \times N_2)/(G_1 \times N_2)_x$, we have that $m$-almost every $G_1 \times N_2$-orbit admits a $G_1 \times N_2$-invariant
mean. In particular, it admits a $G_1$-invariant mean. Since $G_1$ has property (T), $m$-almost every $G_1 \times N_2$-orbit admits a $G_1$-invariant probability measure, and so each $G$-orbit admits a $G_1$-invariant measure. It follow from Lemma 4.6 that $m$ is essentially transitive.

Proof of Corollary 1.4. Assume that $G_1$ is a simple factor of $G$ with property (T), and note that by Theorem 2 $K$ is either almost surely equal to a normal subgroup $N$, or $K$ is co-amenable in $G$.

In the former case, since the action is faithful, the only possibility is $N = \{e\}$, and then the action is essentially free.

In the latter case, as in the proof of Corollary 1.3, we can identify each orbit $Gx$ with $G/G_x$, and so, by the co-amenability of $G_x$, there exists a $G_1$-invariant mean on $m$-almost every orbit $Gx$. Since $G_1$ has property (T), $m$-almost every orbit $Gx$ admits a $G_1$-invariant measure. It then follows from Lemma 4.6 that $m$ is essentially transitive. The identification of $G_x$ with $G/G_x$ now implies that $K$ is almost surely co-finite in $G$. Finally, it follows from the Borel Density Theorem that $K$ is almost surely a lattice in $G$. 

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