ANTICONCENTRATION AND BERRY–ESSEEN BOUNDS FOR RANDOM TENSORS

PANDELIS DODOS AND KONSTANTINOS TYROS

ABSTRACT. We obtain estimates for the Kolmogorov distance to appropriately chosen gaussians, of linear functions
\[ \sum_{i \in [n]^d} \theta_iX_i \]
of random tensors \( X = \langle X_i : i \in [n]^d \rangle \) which are symmetric and exchangeable, and whose entries have bounded third moment and vanish on diagonal indices. These estimates are expressed in terms of intrinsic (and easily computable) parameters associated with the random tensor \( X \) and the given coefficients \( \langle \theta_i : i \in [n]^d \rangle \), and they are optimal in various regimes.

The key ingredient—which is of independent interest—is a combinatorial CLT for high-dimensional tensors which provides quantitative non-asymptotic normality under suitable conditions, of statistics of the form
\[ \sum_{(i_1, \ldots, i_d) \in [n]^d} \zeta(i_1, \ldots, i_d, \pi(i_1), \ldots, \pi(i_d)) \]
where \( \zeta : [n]^d \times [n]^d \to \mathbb{R} \) is a deterministic real tensor, and \( \pi \) is a random permutation uniformly distributed on the symmetric group \( S_n \). Our results extend, in any dimension \( d \), classical work of Bolthausen who covered the one-dimensional case, and more recent work of Barbour/Chen who treated the two-dimensional case.

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Part 1. Introduction

1. Normal approximation of linear functions of random tensors

1.1. Framework. We will be working with tensors, and as such it is useful to begin by introducing some notation. In what follows, let $n, d$ be positive integers with $n \geq 2d$.

We denote by $[n]^d$ the set of all sequences of length $d$ taking values in the discrete interval $[n] := \{1, \ldots, n\}$. We view, however, every element of $[n]^d$ also as a function from $[d]$ to $[n]$. Thus, for every $i \in [n]^d$ and every $r \in [d]$ by $i(r)$ we denote the $r$-th element of $i$; moreover, if $s$ is a positive integer with $s \leq d$ and $p \in [n]^s$, then we write $p \subseteq i$ to denote the fact that $i$ extends $p$, that is, if $p(r) = i(r)$ for every $r \in [s]$. We also denote by $[n]^d_{\text{Inj}}$ the set of all one-to-one functions in $[n]^d$ (equivalently, the set of all finite sequences in $[n]^d$ with distinct entries).

1.1.1. Our objects of study are random tensors $X = \langle X_i : i \in [n]^d \rangle$ whose entries satisfy

$$E[X_i] = 0, \quad E[X_i^2] \leq 1 \quad \text{and} \quad E[|X_i|^3] < \infty,$$

and which are

— symmetric, that is, $X_{i_1, \ldots, i_d} = X_{i_{\tau(1)}, \ldots, i_{\tau(d)}}$ for every $(i_1, \ldots, i_d) \in [n]^d$ and every permutation $\tau$ of $[d]$, and

— exchangeable, that is, for every permutation $\pi$ of $[n]$ the random tensors $X$ and $X_{\pi} := \langle X_{\pi i} : i \in [n]^d \rangle$ have the same distribution.

(Here and in the rest of this paper, we set $\pi \circ i := (\pi(i_1), \ldots, \pi(i_d)) \in [n]^d$ for every $i = (i_1, \ldots, i_d) \in [n]^d$ and every permutation $\pi$ of $[n]$.) This is, arguably, a large class of random tensors which encompasses the following examples.

• It includes random tensors of the form $X_i = h(\xi_{i(1)}, \ldots, \xi_{i(d)})$, where $(\xi_k)$ is a sequence of i.i.d. random variables which take values in a measurable space $\mathcal{E}$, and $h : \mathcal{E}^d \to \mathbb{R}$ is a measurable symmetric function; see [Ald83, Kal05]. These random tensors are, of course, ubiquitous in probability and statistics.

• It also includes random tensors of the form $X_i = \prod_{\ell=1}^d \xi_{i(\ell)} - E[\prod_{\ell=1}^d \xi_{i(\ell)}]$, where $(\xi_1, \ldots, \xi_n)$ is an exchangeable random vector which take values in $[0,1]^n$. An important
special case of this class of examples is obtained by considering boolean random vectors
\((ζ_1, \ldots, ζ_n)\) which are uniformly distributed on a “slice” \(\binom{n}{k} := \{A \subseteq [n] : |A| = k\}\); see,
\(e.g., [FKMW18]\).

Remark 1.1. Note that the class of symmetric and exchangeable random tensors is closed
under mixtures. This is a basic property which is significant both from a theoretical as
well as an applied point of view.

1.1.2. Now let \(θ = ⟨θ_i : i ∈ [n]^d⟩\) be a (not necessarily symmetric) deterministic real
tensor whose diagonal terms vanish—that is, \(θ_i = 0\) if \(i \notin [n]_{\text{Inj}}\)—and consider the random
variable
\[
⟨θ, X⟩ := \sum_{i∈[n]^d} θ_i X_i.
\]

Our goal is to estimate the quantity
\[
d_K(⟨θ, X⟩, N(0, σ^2))
\]
where \(σ^2\) denotes the variance of \(⟨θ, X⟩\) and \(N(0, σ^2)\) denotes the normal random variable
with zero mean and variance \(σ^2\); here, and in the rest of this paper, for a pair \(X, Y\) of
real-valued random variables, by \(d_K(X, Y)\) we denote their Kolmogorov distance
\[
d_K(X, Y) := \sup_{x ∈ \mathbb{R}} |P(X ≤ x) − P(Y ≤ x)|.
\]

Notice that, since \(X\) is symmetric, we may write
\[
⟨θ, X⟩ = ⟨a, Y⟩
\]
where \(a = ⟨a_i : i ∈ [n]^d⟩\) is a symmetric real tensor whose diagonal terms vanish, and
\(Y = ⟨Y_i : i ∈ [n]^d⟩\) is the random tensor defined by setting \(Y_i = X_i\) if \(i ∈ [n]_{\text{Inj}}\), and
\(Y_i = 0\) otherwise. (In particular, \(Y\) is also symmetric and exchangeable.) Thus, in what
follows, we make the following basic assumptions on \(X\) and \(θ\).

(A1) We have \(E[X_i] = 0, E[X_i^2] ≤ 1\) and \(E[|X_i|^3] < ∞\) for every \(i ∈ [n]^d\).

(A2) The random tensor \(X\) is symmetric, exchangeable and its diagonal terms vanish.

(A3) The real tensor \(θ\) is symmetric and its diagonal terms vanish.

1.2. Relevant parameters. We shall estimate the distance in (1.2) using some intrinsic
parameters which are associated with \(θ\) and \(X\) respectively.

1.2.1. First, for every \(s ∈ \{0, 1, \ldots, d\}\) we set
\[
|||θ|||_s := \left( \sum_{j∈[n]^s} \left( \sum_{j⊆i∈[n]^d} θ_i \right)^2 \right)^{1/2}
\]
with the convention that the first sum vanishes if \(s = 0\). Thus, \(|||·|||_s\) is a seminorm which
interpolates between the summing seminorm \((s = 0)\) and the euclidean\(^1\) norm \((s = d)\).

\(^1\)In this context, the euclidean norm is also referred to as the Hilbert–Schmidt norm.
Moreover, for every \( s \in [d] \) we set

\[
\delta_s = \delta_s(X) := \mathbb{E}[X_{(1,\ldots,d)}X_{(1,\ldots,d+1,\ldots,2d-s)}]
\]

and

\[
\delta_0 = \delta_0(X) := \mathbb{E}[X_{(1,\ldots,d)}X_{(d+1,\ldots,2d)}].
\]

Notice that, since \( X \) is exchangeable, the quantities \( \delta_0, \delta_1, \ldots, \delta_d \) completely determine the correlation matrix of \( X \).

In order to see the relevance of the parameters introduced so far, we shall use them to compute the variance of \( \langle \theta, X \rangle \); the proof is given in Subsection 8.2.

**Proposition 1.2.** Let \( X, \theta \) which satisfy (A1), (A2) and (A3). Then we have

\[
\operatorname{Var}(\langle \theta, X \rangle) = \sum_{s=0}^{d} \binom{d}{s}^2 s! \left( \sum_{t=0}^{s} \binom{s}{t} (-1)^{s-t} \delta_t \right) \|\theta\|^2_s.
\]

1.2.2. Next, for every \( s \in \{0, 1, \ldots, d\} \) set

\[
\Sigma_s = \Sigma_s(X) := \sum_{t=0}^{s} \binom{s}{t} (-1)^{s-t} \delta_t.
\]

By (1.8), it is clear that these quantities are related to the variance of \( \langle \theta, X \rangle \), thought their role is most transparently seen in the case of random tensors that admit a Hoeffding decomposition. That said, we have the following information for general exchangeable random tensors; see Lemma 10.5 for a more precise result.

**Fact 1.3.** Let \( X \) be a random tensor which satisfies (A1) and (A2). Then, we have

\[
\Sigma_s \geq -\frac{8d^2d}{n} \quad \text{for every} \ s \in \{0, \ldots, d\}.
\]

1.2.3. We will need one last parameter which is associated with the random tensor \( X \). Specifically, we define the oscillation of \( X \) by

\[
\operatorname{osc}(X) := \left\| \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{n^{d-1}} \sum_{i \in [n]^d \atop i(1) = j} X_i \right)^2 - \delta_1 \right\|_{L^1}.
\]

The oscillation of random vectors appeared, for instance, in recent work of Bobkov, Chistyakov and Götze [BCG18] albeit with different terminology. In higher dimensions, it can be thought of as a quantitative measure of dissociativity; of course, in order to estimate the distance in (1.2), some information of this form is necessary. The advantage of the oscillation is that it leads—more often than not—to optimal results and, more importantly, it can be fairly easily estimated for several classes of random tensors, thus making our task computationally feasible. We shall discuss these issues in detail in Section 3.
1.3. Main estimate. We are now ready to state the first main result of this paper.

**Theorem 1.4.** Let $X, \theta$ which satisfy $(A1), (A2)$ and $(A3)$, and such that $\|\theta\|_1 = 1$. Set $\kappa = \kappa(d) := 20d^3 18^d (2d)!$ and $B := \|\frac{1}{n^d} \sum_{i \in [n]^d} X_i \|_{L_2}$, and let $\alpha \in (0, 1)$ such that the following non-degenericity condition holds true

\[(1.11) \quad \delta_1 \geq \max \left\{ \text{osc}(X)^\alpha, B^\alpha, \left(\frac{\kappa}{n}\right)^\alpha \right\}.\]

Then, setting $\sigma^2 := \text{Var}(\langle \theta, X \rangle)$, we have

\[(1.12) \quad d_K(\langle \theta, X \rangle, N(0, \sigma^2)) \leq E_1 + E_2 + E_3\]

where

\[(1.13) \quad E_1 := 5\text{osc}(X)^{1-\alpha} + 5|\delta_0|^{1-\alpha} + \left| \frac{\delta_0}{d^2 \delta_1} \left( \|\theta\|_0^2 - 1 \right) \right| + \frac{6\kappa}{n^{1-\alpha}} + 4 \frac{\|\theta\|_0^2}{n}\]

\[(1.14) \quad E_2 := 2^{3n} \frac{\mathbb{E}[|X(1, \ldots, d)|^3]}{\delta_1^{3/2}} \left( \sum_{j=1}^{n} \left| \sum_{i \in [n]^d} \theta_i \right|^3 \right)\]

\[(1.15) \quad E_3 := 3\kappa \frac{1}{d\sqrt{\delta_1}} \sum_{s=2}^{d} \frac{d^s}{s} \sqrt{s!} \sum_{i(1) = j} \frac{16d^{2d^2}}{n} \|\theta\|_s.\]

**Remark 1.5.** The proof of Theorem 1.4 actually yields a better estimate by comparing the Kolmogorov distance of $\langle \theta, X \rangle$ with a gaussian whose variance is an appropriately selected approximation of $\text{Var}(\langle \theta, X \rangle)$; see Proposition 11.1 for details.

**Remark 1.6.** The assumption in Theorem 1.4 that $\|\theta\|_1 = 1$ is, of course, a normalization and it can always be achieved by rescaling $\theta$.

**Remark 1.7.** The dependence of the constant $\kappa$ on $d$ is, most likely, non-optimal. It is an interesting problem to determine its dependence on the dimension $d$.

**Remark 1.8.** We note that a non-degenericity condition like (1.11) is natural at this level of generality; indeed, Theorem 1.4 can be seen as a quantitative generalization of Hoeffding’s CLT for degenerate symmetric U-statistics [Ho48]. (See Paragraph 12.2.1 for more details.) On the other hand, classical results from the theory of U-statistics—see, e.g., [Se80, Section 5.5]—suggest that, in the degenerate case, the random variable $\langle \theta, X \rangle$ is typically far from normal. That said, we note that normal approximation for degenerate statistics is also heavily investigated; see, e.g., [NP12] and the references therein.

1.3.1. We close this section by briefly discussing the nature of the bound (1.12).

The first term of $E_1$ is, essentially, the oscillation of $X$. The second and third terms are quantitative measures of the correlation of the entries of $X$, and they can be absorbed in the rest of the error terms if $\delta_0 = O(1/n)$. The fourth term is related to the non-degeneracy assumption (1.11), and it can also be absorbed in the rest of the error terms.
if the parameter $\alpha$ is less than or equal to $1/2$. The last term in $E_1$ is more subtle, and it is related to the extendability\(^2\) of $X$.

In order to appreciate the error terms $E_2$ and $E_3$, consider the important special case of a random tensor $X$ whose entries are functions of i.i.d. random variables. In this case, applying Hoeffding’s decomposition, we may decompose the random variable $(\theta, X)$ as a linear term $L$ plus a remainder $R$; then the bound $E_2$ is proportional to the classical Berry–Esseen bound for the linear term $L$, while the bound $E_3$ is proportional to the ratio of the standard deviations of $R$ and $L$. For this particular case, this bound is known and it follows (with better constants) from the powerful nonlinear Berry–Esseen theorem of Chen/Shao [CS07]. Theorem 1.4 essentially asserts that one can reach similar conclusions by merely assuming exchangeability instead of independence (as long as one can control the oscillation) despite the fact that in this context there is no canonical decomposition like Hoeffding’s decomposition. Thus, we may loosely describe Theorem 1.4 as a “nonlinear Berry–Esseen theorem without independence”.

2. Main tool: combinatorial CLT for high-dimensional tensors

The starting observation of the proof of Theorem 1.4 is that the distribution of the random variable $(\theta, X)$ can be expressed as a mixture of certain tensor permutation statistics. (See Subsection 11.1 for a more detailed high-level overview of the argument.) Thus, Theorem 1.4 is effectively reduced to a quantitative combinatorial central limit theorem for high-dimensional tensors which we are about to describe.

2.1. Tensor permutation statistics. Let $n, d$ be positive integers with $n \geq d$, and let $\xi: [n]^d \times [n]^d \to \mathbb{R}$ be a (deterministic) real tensor. With $\xi$ we associate the statistic

$$Z = \sum_{i \in [n]^d} \xi(i, \pi \circ i)$$

where $\pi$ is a random permutation which is uniformly distributed on the symmetric group $S_n$. (Recall that for every $\pi \in S_n$ and every $i = (i_1, \ldots, i_d) \in [n]^d$ we set $\pi \circ i := (\pi(i_1), \ldots, \pi(i_d)) \in [n]^d$.) These statistics are classical—see, e.g., [Da44]—and appear in a variety of disciplines with pure as well as applied orientation.

One drawback of tensor permutation statistics is the computational difficulty of their basic parameters, like the variance. This defect can be fixed by restricting our attention to the following class of tensors.

**Definition 2.1** (Hoeffding tensor). Let $n, d$ be positive integers with $n \geq d$, and let $\xi: [n]^d \times [n]^d \to \mathbb{R}$. We say that $\xi$ is a Hoeffding tensor if for every $r \in [d]$, every $j_0, q_0 \in [n]^d \setminus \{r\}$ and every $i_0, p_0 \in [n]^d$ we have

$$\sum_{j_0 \leq i \leq p} \xi(i, p_0) = 0 \quad \text{and} \quad \sum_{q_0 \leq p \leq n} \xi(i_0, p) = 0$$

\(^2\)See Paragraph 12.2.1 for the definition of extendability.
where \([n]^{d \setminus \{r\}}\) denotes the set of all maps from \([d] \setminus \{r\}\) to \([n]\). (See also Section 4.)

In order to see the relevance of Hoeffding tensors in this context note that, by applying an appropriate decomposition which goes back to Hoeffding [Ho51], the statistic (2.1) can be written as

\[
W = \sum_{s=1}^{d} \sum_{i \in [n]^{s}_{\text{Inj}}} \xi_s(i, \pi \circ i)
\]

where \(\xi_s: [n]^s \times [n]^s \to \mathbb{R}\) is a Hoeffding tensor for every \(s \in [d]\). (For the class of tensors which are relevant to Theorem 1.4, we describe this transformation in Section 9.) Therefore, in what follows we shall focus on statistics of the form (2.3).

2.2. Permutation statistics of order one. Classical results for matrix permutation statistics were obtained by Wald/Wolfowitz [WW44] and Hoeffding [Ho51] who established asymptotic normality under general conditions.

The problem of establishing quantitative, non-asymptotic, normality of \(W\)-statistics of order one, was more delicate. The optimal result in this direction was eventually obtained by Bolthausen [Bo84] who showed that

\[
d_K\left(\frac{1}{n} \sum_{i=1}^{n} \xi(i, \pi(i)), \mathcal{N}(0,1)\right) \leq C_1 \sum_{i,j=1}^{n} |\xi(i,j)|^3
\]

for every Hoeffding tensor \(\xi: [n] \times [n] \to \mathbb{R}\) which satisfies \(\sum_{i,j=1}^{n} \xi(i,j)^2 = n - 1\); here, \(C_1 \geq 1\) is an absolute constant. (It was shown in [CF15] that we can take \(C_1 = 451\).) Bolthausen’s work was one of the earliest and most successful applications of Stein’s method of normal approximation [CGS11, St82, St86].

2.3. Permutation statistics of order two. \(W\)-statistics of order two are also studied systematically in the literature; see, e.g., [BE86, ZBCL97] and the references therein. The strongest quantitative normal approximation was obtained by Barbour and Chen [BC05] who showed that if \(\xi_1: [n] \times [n] \to \mathbb{R}\) and \(\xi_2: [n]^2 \times [n]^2 \to \mathbb{R}\) are Hoeffding tensors with \(\sum_{i,j=1}^{n} \xi_1(i,j)^2 = n - 1\), and \(W\) is the statistic associated with \(\xi_1, \xi_2\) via (2.3), then

\[
d_K(W, \mathcal{N}(0,1)) \leq \frac{aC_1}{n} \sum_{i,j=1}^{n} |\xi_1(i,j)|^3 + C_2 \left(\frac{1}{n^2} \sum_{i,p \in [n]^2} \xi_2(i,p)^2\right)
\]

where \(C_1 \geq 1\) is the same constant as in (2.4), and \(a, C_2 \geq 1\) are absolute (and effective) constants. The work of Barbour and Chen was based on the Stein/Chen method of normal approximation via concentration, and it also used Bolthausen’s estimate (2.4). Note that (2.5) should be interpreted as a perturbation result, and when viewed as such, it is essentially optimal; see [BC05, Section 1] for a detailed discussion.

---

3Note that \(\pi \circ i := (\pi(i_1), \ldots, \pi(i_s)) \in [n]^{s}_{\text{Inj}}\) for every \(\pi \in S_n\) and every \(i = (i_1, \ldots, i_s) \in [n]^{s}_{\text{Inj}}\).

4This result was not explicitly isolated in [BC05], but it follows fairly straightforwardly from the methods developed therein.
2.4. Permutation statistics of arbitrary order. High-dimensional statistics related to (2.1) and (2.3) have been studied, for instance, in \([BG93, BoG02, Lo96]\). That said, however, so far no high-dimensional analogue of (2.4) and (2.5) has been obtained. The following theorem—which is the second main result of this paper—fills in this gap, and extends these estimates to \(W\)-statistics of arbitrary order.

**Theorem 2.2.** Let \(n, d\) be positive integers such that \(n \geq 4d^2\). For every \(s \in [d]\) let \(\xi_s : [n]^s \times [n]^s \to \mathbb{R}\) be a Hoeffding tensor, and set

\[
(2.6) \quad \beta_s := \sum_{i,p \in [n]^s} \xi_s(i,p)^2.
\]

Assume that \(\beta_1 = n-1\), and let \(W\) be the statistic associated with \(\xi_1, \ldots, \xi_d\) via (2.3). Then we have

\[
(2.7) \quad d_K(W, \mathcal{N}(0,1)) \leq \frac{2^{18}C_1}{n} \sum_{i,j=1}^{n} |\xi_1(i,j)|^3 + C_d \sum_{s=2}^{d} \sqrt{\frac{\beta_s}{n^s}}
\]

where \(C_1 \geq 1\) is as in (2.4), and \(C_d\) is a positive constant that depends only on \(d\). In fact, we can take \(C_d = 5d^2e^{d(2d)!}\).

As expected, the proof of Theorem 2.2 relies on the work of Bolthausen, and Barbour/Chen. In higher dimensions, the additional difficulty is to obtain tight estimates for the \(L_2\) distance of certain exchangeable pairs of random variables, a task that becomes more and more combinatorially intricate as the dimension \(d\) increases. These estimates are, in fact, responsible for the scaling \(n^s\) that appears in (2.7); this scaling is, in turn, important for the proof of Theorem 1.4 and its applications.

3. Applications

3.1. One-dimensional examples: exchangeable random vectors. The simplest instance of Theorem 1.4 concerns random vectors. Specifically, let \(n \geq 2\) be an integer, let \(X = (X_1, \ldots, X_n)\) be an exchangeable random vector whose entries satisfy (A1), and let \(\theta = (\theta_1, \ldots, \theta_n)\) be a vector in \(\mathbb{R}^n\). (Assumption (A3) is superfluous if \(d = 1\).) It is straightforward to check that

(i) \(\|\theta\|_0 = |\theta_1 + \cdots + \theta_n|\),

(ii) \(\|\theta\|_1 = \|\theta\|_{\ell_2} = (\theta_1^2 + \cdots + \theta_n^2)^{1/2}\),

(iii) \(\delta_0 = \mathbb{E}[X_1X_2]\) and \(\delta_1 = \mathbb{E}[X_1^2]\).

Specializing the bound obtained by Theorem 1.4 to random vectors, we obtain the following corollary. (See Subsection 12.1 for the proof.)

**Corollary 3.1.** Let \(X\) be an exchangeable random vector in \(\mathbb{R}^n\) (\(n \geq 2\)) which satisfies

\[
(3.1) \quad \mathbb{E}[X_1] = 0, \quad \mathbb{E}[X_1^2] = 1 \quad \text{and} \quad \mathbb{E}[|X_1|^3] < \infty.
\]
Then for every unit vector \( \theta \in \mathbb{R}^n \), setting \( \sigma^2 := \text{Var}(\theta_1 X_1 + \cdots + \theta_n X_n) \), we have

\[
d_K\left( \sum_{i=1}^n \theta_i X_i, N(0, \sigma^2) \right) \leq 5\text{osc}(X) + 6|\delta_0| + \left( \frac{\kappa_1}{n} + \frac{1}{n} \right) \cdot \left( \sum_{i=1}^n \theta_i \right)^2 + \\
+ \kappa_1 \mathbb{E}[|X_1|^3] \sum_{i=1}^n |\theta_i|^3.
\]

Here, \( \kappa_1 \geq 1 \) is an absolute constant. In fact, we can take \( \kappa_1 = 4320 \).

It follows, in particular, that if \( X \) is as in Corollary 3.1 and it satisfies \( |\delta_0| \leq C/n \) for some positive constant \( C \), then for every positive integer \( k \leq n \) we have

\[
d_K\left( \frac{X_1 + \cdots + X_k}{\sqrt{k}} , N(0, 1) \right) \leq 5\text{osc}(X) + (8C + \kappa_1 + 1) \frac{k}{n} + \kappa_1 \mathbb{E}[|X_1|^3] \frac{1}{\sqrt{k}}.
\]

This bound is optimal up to universal constants; see Paragraph 12.1.1 for details.

3.1.1. **Estimating the oscillation.** By Corollary 3.1, the problem of establishing normal approximation of the random variable \( \theta_1 X_1 + \cdots + \theta_n X_n \) reduces to that of estimating the oscillation \( \text{osc}(X) \) of the exchangeable random vector \( X \).

To this end, we first observe that if the entries of \( X \) have zero mean, unit variance and finite fourth moment, then we have

\[
\text{osc}(X) \leq \sqrt{\mathbb{E}[X_1^2 X_2^2] - 1} + \frac{\mathbb{E}[X_1^4]^{1/2}}{\sqrt{n}}.
\]

(See Fact 12.2.) The quantity \( |\mathbb{E}[X_1^2 X_2^2] - 1| \) that appears in (3.4) is very natural in this context: it controls the variance \( \text{Var}(\|X\|_2^2) \) of the square of the euclidean norm of \( X \); see, e.g., [BCG18]. On the other hand, as we shall see in Proposition 12.3, without further assumptions on the existence of moments, the basic hypothesis (3.1) yields that

\[
\text{osc}(X) \leq \sqrt{\mathbb{E}[X_1^2 X_2^2] - 1} + \frac{4\mathbb{E}[|X_1|^3]}{\sqrt{n}}.
\]

3.1.2. **Exchangeable and isotropic random vectors.** Combining (3.2) and (3.4), we see that if \( X \) is an exchangeable and isotropic\(^5 \) random vector in \( \mathbb{R}^n \) whose entries have finite fourth moment, then, setting \( \tau = \tau(X) := n|\mathbb{E}[X_1^2 X_2^2] - 1| \), for every unit vector \( \theta \in \mathbb{R}^n \) we have

\[
d_K\left( \sum_{i=1}^n \theta_i X_i, N(0, 1) \right) \leq (5\sqrt{\tau} + 5\mathbb{E}[X_1^4]^{1/2} + \kappa_1) \frac{1}{\sqrt{n}} + \frac{1}{n} \left( \sum_{i=1}^n \theta_i \right)^2 + \\
+ 2\kappa_1 \mathbb{E}[|X_1|^3] \sum_{i=1}^n |\theta_i|^3.
\]

---

\(^5\)Recall that a random vector \( X \) in \( \mathbb{R}^n \) is called isotropic if its entries are uncorrelated random variables with zero mean and unit variance.
The estimate (3.6) extends and improves a result of Bobkov [Bob04, Proposition 6.1]. It shows, among others, that under the thin-shell condition $|E[X_1^2 X_2^2] - 1| = O(1/n)$ we have the Berry–Esseen bound

$$d_K\left(\frac{X_1 + \cdots + X_k}{\sqrt{k}}, \mathcal{N}(0, 1)\right) = O\left(\frac{1}{\sqrt{k}}\right)$$

in the regime $k = O(n^{2/3})$. This range is also optimal; see Example 12.5.

3.2. High-dimensional examples. We proceed to discuss the general case of high-dimensional random tensors which satisfy (A1) and (A2). It is clear that the remaining task is to estimate their oscillation. Not surprisingly, to this end we will need an analogue of the quantity $E[X_1^2 X_2^2] - 1$ that appears in (3.4) and (3.5).

Specifically, let $n, d$ be positive integers with $n \geq 4d$, let $X = \langle X_i : i \in [n]^d \rangle$ be a random tensor, and define the parallelepipedal correlation of $X$ by

$$pc(X) := E[X_{(1),...,d} X_{(1,d+1,...,2d-1)} X_{(2d,...,3d-1)} X_{(2d,3d,...,4d-2)}] - \delta_1^2].$$

For example, if $d = 2$, then $pc(X) = E[X_{(1,2)} X_{(1,3)} X_{(4,5)} X_{(4,6)}] - \delta_1^2].$

![Figure 1. The parallelepipedal correlation.](image)

The parameter $pc(X)$ has already appeared—though, not explicitly isolated—in [EW78, Theorem 4]. Notice that if $X$ is a random vector, then $pc(X) = |E[X_1^2 X_2^2] - \delta_1^2]|$; in particular, the parallelepipedal correlation $pc(X)$ of random vectors which satisfy (3.1) coincides with the aforementioned quantity $|E[X_1^2 X_2^2] - 1]$.

3.2.1. Random tensors whose entries have bounded fourth moment. We first observe that if a random tensor $X$ satisfies (A1) and (A2) and its entries have bounded fourth moment, then we have the following analogue of (3.4),

$$osc(X) \leq \sqrt{pc(X)} + \frac{5d}{\sqrt{n}} \left(1 + E[X_{(1,...,d)}^4]\right)^{1/2}.$$

(See Fact 12.6 for a slightly more precise estimate.)
3.2.2. Dissociated random tensors and their mixtures. Another natural class of random
tensors whose oscillation can be effectively estimated is that of exchangeable and dis-
sociated random tensors. Recall that a random tensor \( X = (X_i : i \in [n]^d) \) is called
dissociated if for every pair \( K, L \) of disjoint subsets of \([n]\) with \(|K|, |L| \geq d\), the random
tensors \( X_K := (X_i : i \in K^d) \) and \( X_L := (X_i : i \in L^d) \) are independent. Dissociativ-
ity plays an important role in the general theory of exchangeable random tensors; see
\[Ald81, Hoo79, Kal05\]. To see its relevance in this context, notice that \( \text{pc}(X) = 0 \) for
every exchangeable and dissociated random tensor. That said, we have
\[
\text{osc}(X) \leq \frac{16d \left( E \left[ |X_{(1, \ldots, d)}|^3 \right] + 1 \right)}{\sqrt{n}}
\]
for every dissociated random tensor \( X \) which satisfies (A1) and (A2); more generally,
if \( X \) is a mixture of exchangeable, symmetric and dissociated random tensors and the
entries of \( X \) have finite third moment, then
\[
\text{osc}(X) \leq \sqrt{\text{pc}(X)} + \frac{16d \left( E \left[ |X_{(1, \ldots, d)}|^3 \right] + 1 \right)}{\sqrt{n}}
\]
(See Proposition 12.7). These estimates together with Theorem 1.4 cover a wide range
of exchangeable random tensors, including all infinitely extendible random tensors which
satisfy (A1) and (A2)—see Paragraph 12.2.1 for more details.

3.3. Anticoncentration of polynomials of boolean random variables. Recall that,
given a real-valued random variable \( X \), its anticoncentration
\[
\sup_{x \in \mathbb{R}} \mathbb{P}(X = x)
\]
is a quantitative measure of its discreteness. A closely related, and more informative,
quantity is the Lévy concentration function of \( X \),
\[
\mathcal{L}_X(\varepsilon) := \sup_{x \in \mathbb{R}} \mathbb{P}(x \leq X \leq x + \varepsilon)
\]
which bounds the probability that \( X \) lies in an interval of length \( \varepsilon > 0 \). The first results
on anticoncentration were discovered by Littlewood/Offord [LO43] and Erdős [Erd45]
who obtained optimal anticoncentration of linear functions of random vectors with i.i.d.
Rademacher entries.

Much more recently, Costello, Tao and Vu [CTV06] have put forth\(^6\) a higher-degree
version of the classical Littlewood–Offord theory whose main goal is to understand
the anticoncentration of random variables of the form \( f(\xi) \) where \( f : \mathbb{R}^n \to \mathbb{R} \) is a polynomial
with real coefficients, and \( \xi \) is a random vector in \( \mathbb{R}^n \) with a well-behaved distribution.
Examples of random vectors which have been studied in this context include:
- random vectors with i.i.d. gaussian/Rademacher/Bernoulli entries, and
- random vectors which are uniformly distributed on a slice;

\(^6\)We note that closely related questions have been studied earlier—see, e.g., [RS96].
see, e.g., [CTV06, CW01, FKMW18, FM19, MNV16, MOO10, KST19, LLTTY17, RV13].
The topic is quite diverse, and it has found a variety of applications in analysis, combinatorics, discrete probability and theoretical computer science; we refer the reader to [NV13, Vu14] for recent expositions.

3.3.1. The Lévy concentration function of polynomials of boolean random variables. Theorem 1.4 can be used to estimate the Lévy concentration function of homogeneous polynomials of an important class of random vectors with boolean but not independent entries.

Specifically, let $n \geq d$ be positive integers, and let $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x_1, \ldots, x_n) = \sum_{F \in \binom{[n]}{d}} a_F \prod_{i \in F} x_i$$

be a homogeneous multilinear polynomial of degree $d$, with no constant term and real coefficients $a = \langle a_F : F \in \binom{[n]}{d} \rangle$. For every $s \in \{0, 1, \ldots, d\}$ we set

$$\|a\|_s := \left( \sum_{G \subseteq F \in \binom{[n]}{d}} (\sum_{a_F})^2 \right)^{1/2}$$

where we use again the convention that the first sum vanishes if $s = 0$; these seminorms are, of course, the analogues of the seminorms introduced in (1.5).

We have the following theorem. (The proof is given in Subsection 12.3.)

**Theorem 3.2.** Let $n, d$ be positive integers with $n \geq (4k)^{2d}$ where $k = 20d^3 18^d (2d)!$ is as in Theorem 1.4, and let $f : \mathbb{R}^n \to \mathbb{R}$ be as in (3.13). Also let $k$ be a positive integer with

$$(2k)n^{-1/2} \leq k \leq n - (2k)n^{-1/2}$$

and let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ be a random vector in $\{0, 1\}^n$ which is uniformly distributed\(^7\) on $\binom{[n]}{k}$. Then, setting $p := k/n$ and $\sigma^2 := \text{Var}(f(\boldsymbol{\xi}))$, we have

$$\sigma^2 - \sum_{s=1}^d p^{2d-s}(1-p)^s \|a\|_s^2 \leq \frac{12d^2 2^d}{(1-p)^2} \cdot \frac{\|a\|_0^2}{n} + \frac{12d^2 2^d}{p(1-p)^2} \cdot \frac{\sigma^2}{n}$$

and, for every $\varepsilon > 0$\(^8\),

$$\mathcal{L}_{f(\boldsymbol{\xi})}(\varepsilon) \leq \frac{\varepsilon}{\sqrt{2\pi} \sigma} + \frac{16k}{\sqrt{p}} \cdot \frac{1}{\sqrt{n}} + \frac{12d^2}{(1-p)} \cdot \frac{\|a\|_1^2}{n} + \frac{2^{39} p^{1/2}}{p^{3d}(1-p)^{3/2} \|a\|_1^3} \cdot \sum_{j=1}^n \left| \sum_{f \in \binom{[n]}{d}} a_F \right|^3 +$$

$$+ \frac{16k p^{1/2}}{p^{3d}(1-p)^{1/2} \|a\|_1^2} \cdot \sum_{s=2}^d \sqrt{p^{2d-s}(1-p)^s \|a\|_s}.$$  

\(^7\)Here, we identify $\binom{[n]}{k}$ with the set of all $x \in \{0, 1\}^n$ which have exactly $k$ nonzero coordinates.

\(^8\)We follow the convention that $\frac{1}{0} = \infty$.  

Remark 3.3. Using Proposition 1.2, it is easy to see that the left-hand-side of (3.16) is equal to $\text{Var}(f(\xi)) - \text{Var}(f(b))$ where $\xi$ is as in Theorem 3.2 and $b$ is a random vector in $\{0,1\}^n$ with i.i.d. Bernoulli entries with expectation $p$.

In order to put Theorem 3.2 into context, fix $f$ as in (3.13), and recall that if $G$ is a random vector in $\mathbb{R}^n$ with independent standard normal entries such that $\text{Var}(f(G)) = 1$, then the Carbery–Wright inequality [CW01] yields that for every $\varepsilon > 0$,

\begin{equation}
L_f(G)(\varepsilon) = O(\varepsilon^1/d).
\end{equation}

Using their invariance principle, Mossel, O’Donnell and Oleszkiewicz [MOO10] extended this estimate to general random vectors with i.i.d. entries with an extra term on the right-hand-side of (3.18) depending on the influence of the coefficients of $f$; more recently, Filmus, Kindler, Mossel and Wimmer [FKMW18] also covered the case of random vectors $\xi$ as in Theorem 3.2 under an extra harmonicity assumption on the polynomial $f$. (See, also, [FM19] for related results).

On the other hand, if $\xi$ is as in Theorem 3.2 with, say, $\frac{n}{10} \leq k \leq \frac{9n}{10}$ and $\text{Var}(f(\xi)) = 1$, then, by (3.16) and (3.17), for every $0 < \varepsilon \leq 1/2$ we have

\begin{equation}
L_f(\xi)(\varepsilon) = O(\varepsilon)
\end{equation}

provided that $n$ is sufficiently large and

\[
\sum_{j=1}^{n} \left| \sum_{f \in \binom{[n]}{k}} a_f^3 \right| \frac{\|a\|_2^2}{n}, \|a\|_2, \ldots, \|a\|_d \leq c_d \varepsilon
\]

where $c_d$ is a positive constant that depends only on $d$. In other words, we have an improvement over (3.18) as long as the coefficients of $f$ are balanced and not too dense, and the contribution of the seminorms $\|a\|_2, \ldots, \|a\|_d$ in $\text{Var}(f(\xi))$ is relatively small.

4. Notation

In this section we collect all pieces of notation that are used throughout this paper. Working with tensors (and, in general, with high-dimensional objects) necessitates some notation which, unfortunately, is not completely standardized. Most of this notation will be used in Sections 5, 7 and 9. The reader should have in mind this remark, and consult this particular section for any possible clarification while reading the paper.

As we have mentioned, for every integer $n \geq 1$ we set $[n] := \{1, \ldots, n\}$; moreover, for every $k \in \{0, \ldots, n\}$ by $\binom{n}{k}$ we denote the collection of all subsets of $[n]$ of cardinality $k$.

4.1. For every positive integer $n$ and every (possibly empty) set $S$ by $[n]^S$ we denote the set of all functions from $S$ into $[n]$, and by $[n]^S_{\text{inj}}$ the set of all one-to-one functions from $S$ into $[n]$. In particular, if $S$ is empty, then $[n]^S$ contains only the empty function which shall be denoted by $\emptyset$; on the other hand, if $S = [d]$ for some positive integer $d$, then the corresponding sets of functions shall be denoted simply by $[n]^d$ and $[n]^d_{\text{inj}}$. Finally, if $S$ is
nonempty, then by PartInj(S,[n]) we denote the set of all nonempty, partial, one-to-one maps from S into [n]. (We emphasize, again, the nonemptiness.)

4.2. Given a function f, by dom(f) we shall denote its domain and by Im(f) its image. For every (possibly empty) subset F of dom(f), by f ↾ F we denote the restriction of f on F. If g is another function, then we write f ⊆ g provided that dom(f) ⊆ dom(g) and g↾dom(f) = f. Finally, if Im(f) ⊆ dom(g), then by g ⋙ f we denote the composition of g with f, that is, the map dom(f) ⊇ i ↦ g(f(i)) ∈ Im(g).

In particular, if n,d are positive integers, then for every i = (i₁,...,iₙ) ∈ [n]^d, every permutation π of [n] and every permutation τ of [d] we have
\[(4.1) \quad \pi ⋙ i := (\pi(i₁),...,\pi(iₙ)) ∈ [n]^d \quad \text{and} \quad i ⋙ τ := (i_{τ(1)},...,i_{τ(d)}) ∈ [n]^d;\]
moreover, i ↾ F ∈ [n]^{F} for every subset F of [d].

4.3. Let n,d be positive integers, and let i,j,p,q ∈ [n]^d. We say that the pairs (i,j) and (p,q) are equivalent if there exists a permutation π of [n] such that p = π ⋙ i and q = π ⋙ j. We shall write (i,j) ∼ (p,q) to denote the fact that (i,j) and (p,q) are equivalent.

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Part 2. Combinatorial CLT for high-dimensional tensors

5. A TOOL FOR COMPUTING THE VARIANCE

The following proposition is the first step of the proof of Theorem 2.2. It is the main tool that enables us to compute the variance of several random variables associated with W-statistics.

**Proposition 5.1.** Let n ≥ s be positive integers, and let ξ: [n]^s × [n]^s → ℜ be a Hoeffding tensor. Also let \( t \in \{0\} \cup \text{PartInj}([s],[n]) \) and \( r \in \{\text{Im}(t)|,\ldots,s\} \). Then,
\[(5.1) \quad \sum_{(i,j,p,q)\in\mathcal{M}_{t,r}} \xi(i,p)\xi(j,q) ≤ e^{2s}((2s)!)^2 n^{s-r} \sum_{i,p\in[n]^s}_{i\subseteq i} \xi(i,p)^2 \]
where \( \mathcal{M}_{t,r} = \{(i,j,p,q) \in ([n]^s)^4 : t \subseteq i, t \subseteq j, (i,j) \sim (p,q) \text{ and } |\text{Im}(i) \cap \text{Im}(j)| = r\} \).

(Recall that, for a quadruple (i,j,p,q) of elements of [n]^s, we write (i,j) ∼ (p,q) to denote the fact (i,j) and (p,q) are equivalent; see Subsection 4.3.) For the proof of Proposition 5.1 we need to do some preparatory work. In what follows, let n,s,ξ,t,r be as in Proposition 5.1.

---

9That is, t is either the empty map, or a nonempty, partial, one-to-one map from [s] into [n].
5.1. **Partitions.** Consider the set \( \{0,1\} \times [s] \) which we view as a partially ordered set equipped with the lexicographical order \(<_{\text{lex}}\) defined by setting \((\mu, a) <_{\text{lex}} (\nu, b)\) if either \(\mu < \nu\), or \(\mu = \nu\) and \(a < b\). We denote by \(\Pi\) the set of all partitions (with nonempty parts) of \(\{0,1\} \times [s]\); we shall use the letters \(P, Q, R, S\) to denote partitions. Moreover, with every \((i,j) \in [n]^* \times [n]^*\) we associate a partition \(P(i,j) \in \Pi\) defined by
\[
\{ (0,u) \in \{0\} \times [s] : i(u) = x \} \cup \{ (1,v) \in \{1\} \times [s] : j(v) = x \} : x \in \text{Im}(i) \cup \text{Im}(j) \}.
\]

We isolate, for future use, the following elementary fact.

**Observation 5.2.** If \(i,j,p,q \in [n]^*\), then \((i,j) \sim (p,q)\) if and only if \(P(i,j) = P(p,q)\).

Next, we set
\[
\mathcal{I} := \{ P(i,j) : (i,j) \in [n]^*_\text{Inj}_1 \times [n]^*_\text{Inj}_2, |\text{Im}(i) \cap \text{Im}(j)| = r, t \subseteq i \text{ and } t \subseteq j \};
\]
notice that for every \(P \in \mathcal{I}\), its cells have cardinality 1 or 2, and it has exactly \(r\) cells with cardinality 2. We also set
\[
\mathcal{F} := \{ P \in \Pi : \text{every cell of } P \text{ has cardinality at least } 2 \}.
\]
We view \(\mathcal{I}\) as the set of “initial” partitions, and \(\mathcal{F}\) as the set of “final” partitions.

5.2. **Ordering partitions.** We about to define a partial order \(\preceq\) on the set \(\Pi\). To this end, first, for every \(P \in \Pi \setminus \mathcal{F}\) let \(X(P) \in P\) denote the \(<_{\text{lex}}\)-minimal element of \(\{X \in P : |X| = 1\}\); namely, among all cells of \(P\) which are singletons, \(X(P)\) is the lexicographically least. Next, for every \(P \in \Pi \setminus \mathcal{F}\) and every \(Q \in \Pi\) we write \(Q \preceq_1 P\) if there exist \(X_1 \in Q\) and \(X_2 \in P\) such that

- \((P1)\) \(X_2 \neq X(P)\) and \(X_1 = X_2 \cup X(P)\), and
- \((P2)\) for every \(X \in Q\) with \(X \neq X_1\) we have \(X \in P\).

Notice, in particular, that if \(Q \preceq_1 P\), then \(|P| = |Q| + 1\).

Finally, we define \(\preceq\) to be the transitive closure of \(\preceq_1\); more precisely, given \(P, Q \in \Pi\), we write \(Q \preceq P\) if either \(P = Q\), or there exists a finite sequence \((P_0, P_1, \ldots, P_\ell)\) in \(\Pi\) such that \(Q = P_\ell \preceq_1 \cdots \preceq_1 P_0 = P\). We shall refer to such a sequence \((P_0, P_1, \ldots, P_\ell)\) as a path (from \(P\) to \(Q\)) and we call the positive integer \(\ell\) as the length of the path; moreover, by \(\alpha(P, Q)\) we shall denote the number of paths from \(P\) to \(Q\). We will also need the following observation. (Its proof is straightforward.)

**Observation 5.3.** If \(Q \preceq P\), then all paths from \(P\) to \(Q\) have the same length.

5.3. **Successors.** For every \(P \in \Pi\) we set
\[
\text{Succ}(P) := \{ Q \in \Pi : Q \preceq P \} \quad \text{and} \quad \text{SuccFin}(P) := \text{Succ}(P) \cap \mathcal{F}.
\]
Moreover, for every \(P \in \Pi \setminus \mathcal{F}\) set \(\text{Succ}_1(P) := \{ Q \in \Pi : Q \preceq_1 P \}\). In what follows, we will be mainly interested in the set
\[
\mathcal{A} := \bigcap_{P \in \mathcal{I}} \text{Succ}(P) \subseteq \Pi.
\]
5.4. Coding pairs. For every \((i, j) \in [n]^* \times [n]^*\) let \(P(i, j)\) be as in (5.2) and define 
\(f_{i,j} : P(i, j) \to [n]\) by

\[
(5.7) \quad f_{i,j}(X) := \begin{cases} 
   i(u) & \text{if } (0, u) \in X, \\
   j(v) & \text{if } (1, v) \in X.
\end{cases}
\]

To spell it out in more detail, for every \(u \in [s]\) we have that \(f_{i,j}(X) = i(u)\) where \(X\) is the unique cell of the partition \(P(i, j)\) that contains \((0, u)\); respectively, for every \(v \in [s]\) we have that \(f_{i,j}(X) = j(v)\) where \(X\) is the unique cell of \(P(i, j)\) that contains \((1, v)\). Observe that, by definition, we have that \(f_{i,j} \in [n]^{P(i, j)}\).

Our next goal is to show that the process of obtaining \(P(i, j)\) and \(f_{i,j}\) from the pair \((i, j)\), can be reversed. More precisely, for every \(P \in \Pi\) and every \(f \in [n]^{P(i, j)}\) we define \(i_f, j_f : [s] \to [n]\) as follows. For every \(u \in [s]\) we set \(i_f(u) = f(X)\) where \(X\) is the unique cell of \(P\) that contains \((0, u)\); respectively, for every \(v \in [s]\) we set \(j_f(v) = f(X)\) where \(X\) is the unique cell of \(P\) that contains \((1, v)\). It is clear that for every \(P \in \Pi\) and \(f \in [n]^{P(i, j)}\) we have \(P(i_f, j_f) = P\) and \(f_{i_f, j_f} = f\). Moreover, for every \((i, j) \in [n]^* \times [n]^*\) the map

\[
(5.8) \quad [(i, j)]_\sim \ni (p, q) \to f_{p,q} \in [n]^{P(i, j)}
\]

is a bijection, and its inverse is given by

\[
(5.9) \quad [n]^{P(i, j)} \ni f \to (i_f, j_f) \in [(i, j)]_\sim
\]

where \([(i, j)]_\sim = \{(p, q) \in [n]^* \times [n]^* : (p, q) \sim (i, j)\}\) denotes the equivalence class of \((i, j)\).

5.5. The restriction of \(t\). Recall that in the left-hand-side of (5.1) the sum is over all \((i, j)\) which both extend the given partial map \(t\). Having this constrain in mind, for every \(P \in \mathcal{A}\) we set

\[
(5.10) \quad \mathcal{M}(P) := \{ f \in [n]^{P(i, j)} : t \subseteq i_f \text{ and } t \subseteq j_f \}.
\]

The set \(\mathcal{M}(P)\) can be alternatively described as follows. Notice that the map which sends each \(k \in \text{dom}(t)\) to the unique cell \(X_k \in P\) that contains \((0, k)\), is one-to-one. Then \(\mathcal{M}(P)\) consists of all \(f \in [n]^{P(i, j)}\) such that \(f(X_k) = t(k)\) for every \(k \in \text{dom}(t)\).

5.6. Using the fact that \(\xi\) is a Hoeffding tensor. For every \(P, Q \in \Pi\) with \(Q \preceq P\) and every \(g \in [n]^{P(X(P))}\) we define \(T^{P \to Q}(g) \in [n]^{Q(X(Q))}\) as follows. Let \(X_1 \in Q\) and \(X_2 \in P\) be the unique cells which satisfy \((P1)\) and \((P2)\). Then we set

\[
(5.11) \quad T^{P \to Q}(g)(X_1) = g(X_2)
\]

and \(T^{P \to Q}(g)(X) = g(X)\) for every \(X \in Q \setminus \{X_1\}\).

Using the material introduced so far and invoking the fact that \(\xi\) is a Hoeffding tensor, we see that for every \(P \in \Pi \setminus \mathcal{F}\), every \(g \in [n]^{P(X(P))}\) and every \((p, q) \in [n]^* \times [n]^*\),

\[
(5.12) \quad \sum_{\substack{f \in [n]^{P(i, j)} \\ f \notin P \setminus \{X(P)\} = g}} \xi(i_f, p) \xi(j_f, q) = - \sum_{\substack{Q \in \text{Succ}_1(P) \\ Q \neq P \setminus \{X(P)\}}} \xi(i_{T^{P \to Q}(g)}, p) \xi(j_{T^{P \to Q}(g)}, q)
\]
\begin{align}
&\sum_{f \in [n]_{\text{fin}}^g} \xi(p, i_f) \xi(q, j_f) = - \sum_{Q \in \text{Succ}_1(P)} \xi(p, i_{TP \rightarrow Q(g)}) \xi(q, j_{TP \rightarrow Q(g)}). \\
\end{align}

Therefore, by (5.10) and (5.12), for every $P \in A \setminus F$ and every $(p, q) \in [n]^* \times [n]^*$ we have
\begin{align}
&\sum_{f \in M(P)} \xi(i_f, p) \xi(j_f, q) = - \sum_{Q \in \text{Succ}_1(P)} \sum_{g \in M(Q)} \xi(i_g, p) \xi(j_g, q); \\
\end{align}
respectively, by (5.10) and (5.13), for every $R \in \Pi \setminus F$ and every $(p, q) \in [n]^* \times [n]^*$,
\begin{align}
&\sum_{f \in [n]_{\text{inj}}^R} \xi(p, i_f) \xi(q, j_f) = - \sum_{S \in \text{Succ}_1(R)} \sum_{g \in [n]_{\text{inj}}^S} \xi(p, i_g) \xi(q, j_g).
\end{align}
We have the following claim. (Recall that for every $P, Q \in \Pi$ by $\alpha(P, Q)$ we denote the number of paths from $P$ to $Q$.)

**Claim 5.4.** For every $P \in A$ and every $(p, q) \in [n]^* \times [n]^*$ we have
\begin{align}
&\sum_{f \in M(P)} \xi(i_f, p) \xi(j_f, q) = \sum_{Q \in \text{Succ}_1(P)} (-1)^{|P| - |Q|} \alpha(P, Q) \sum_{g \in M(Q)} \xi(i_g, p) \xi(j_g, q); \\
\end{align}
respectively, for every $R \in \Pi$ and every $(p, q) \in [n]^* \times [n]^*$ we have
\begin{align}
&\sum_{f \in [n]_{\text{inj}}^R} \xi(p, i_f) \xi(q, j_f) = \sum_{S \in \text{Succ}_1(R)} (-1)^{|R| - |S|} \alpha(R, S) \sum_{g \in [n]_{\text{inj}}^S} \xi(p, i_g) \xi(q, j_g).
\end{align}

**Proof.** We will only give the proof of (5.16); equality (5.17) follows with identical arguments. Fix $P \in A$, and for every $Q \in \text{Succ}(P)$, we set $\text{dist}(P, Q) = |P| - |Q|$; by Observation 5.3, this quantity coincides with the length of an arbitrary path from $P$ to $Q$. Moreover, for every positive integer $\ell$ we set
\begin{align}
&\text{Succ}_\ell(P) := \{ Q \in \Pi : Q \preceq P \text{ and } \text{dist}(P, Q) = \ell \}.
\end{align}
Clearly, we have $Q \in \text{Succ}_1(P)$ if and only if $Q \preceq_1 P$. On the other hand, using Observation 5.3 once again, we see that for every positive integer $\ell$ and every $Q \in \text{Succ}_{\ell+1}(P)$,
\begin{align}
&\alpha(P, Q) = \sum_{Q \preceq_1 R \in \text{Succ}_\ell(P)} \alpha(P, R).
\end{align}
Finally, using (5.14), (5.19) and proceeding by induction on $\ell$, we obtain that
\begin{align}
&\sum_{f \in M(P)} \xi(i_f, p) \xi(j_f, q) = \sum_{Q \in \text{Succ}_1(P)} (-1)^{|P| - |Q|} \alpha(P, Q) \sum_{g \in M(Q)} \xi(i_g, p) \xi(j_g, q) + \\
&+ \sum_{Q \in \text{Succ}_1(P)} (-1)^{|P| - |Q|} \alpha(P, Q) \sum_{g \in M(Q)} \xi(i_g, p) \xi(j_g, q).
\end{align}
Equality (5.16) follows from (5.20) for $\ell = |P|$.
\[\square\]
5.7. Estimating the number of paths. Notice, first, that the number of partitions of \( \{0, 1\} \times [s] \)—known as the Bell number \( B_{2s} \)—is at most \( (2s)^{2s} \); also observe that every path has length at most \( 2s \). Thus, we immediately obtain that \( \alpha(P, Q) \leq (2s)^{4s^2} \) for every \( P, Q \in \Pi \) with \( Q \preceq P \). We can improve this estimate as follows.

**Fact 5.5.** For every \( P \in \Pi \) and every \( Q \in \text{SuccFin}(P) \) we have \( \alpha(P, Q) \leq e^s \).

**Proof.** The result is a consequence of some rigidity properties of the partial order \( \preceq \). Indeed, fix \( P \in \Pi \) and \( Q \in \text{SuccFin}(P) \), and notice that the cells of \( Q \) can be categorized according to whether they contain “large” cells of \( P \). More precisely, let \( Y \subseteq \Pi \) be arbitrary. First, assume that \( Y \) contains a cell \( X \in P \) with \( |X| \geq 2 \); then the definitions of \( \preceq_1 \) and \( \preceq_2 \) in Section 5.2 imply that, in any possible path from \( P \) to \( Q \), there exists a unique way to glue the cells \( \{X \in P : X \subseteq Y\} \) and arrive at the cell \( Y \). Next, suppose that every cell of \( P \) which is contained in \( Y \) is a singleton; in this case, observe that there are precisely \( |Y| - 1 \) ways to glue the cells \( \{X \in P : X \subseteq Y\} \). Taking these remarks into account, we see that the number of paths from \( P \) to \( Q \) is at most

\[
\alpha(P, Q) \leq e^s.
\]

A straightforward computation shows that this number is at most \( \exp\left(\frac{2s^2}{e}\right) \leq e^s \). \( \square \)

5.8. Proof of Proposition 5.1. By (5.16), (5.17), Fact 5.5 and the Cauchy–Schwarz inequality, for every \( P \in \Pi \) and every \( R \in \Pi \) we have

\[
(5.22) \quad \sum_{f, g \in [n]_{\text{inj}}} \xi(i_f, i_g) \xi(j_f, j_g) = \sum_{Q \in \text{SuccFin}(P)} \left( \sum_{S \in \text{SuccFin}(R)} (-1)^{|P|+|R|-|Q|-|S|} \alpha(P, Q) \alpha(R, S) \times \sum_{f', g' \in [n]_{\text{inj}}} \xi(i_{f'}, i_{g'}) \xi(j_{f'}, j_{g'}) \right)^{1/2} \sum_{f', g' \in [n]_{\text{inj}}} \xi(j_{f'}, j_{g'})^{1/2}.
\]

For every \( P \in \Pi \) set \( X_0(P) := \{X \in P : X \subseteq \{0\} \times [s]\} \); \( X_1(P) := \{X \in P : X \subseteq \{1\} \times [s]\} \); \( \mathcal{X}_0(0) := P \setminus (X_0(P) \cup \mathcal{X}_1(P)) \); and notice that if \( Q \in \text{SuccFin}(P) \), then

- \( (P3) \) \( |X_0(Q)| \leq |X_0(P)|/2 \) and \( |X_1(Q)| \leq |X_1(P)|/2 \).
- \( (P4) \) for every \( X \in \mathcal{X}_0(1) \) there exists \( Y \in \mathcal{X}_0(1) \) such that \( X \subseteq Y \).

Also observe that for every \( (i, j) \in [n]_{\text{inj}} \times [n]_{\text{inj}} \) such that \( |\text{Im}(i) \cap \text{Im}(j)| = r \) we have \( |X_0(P(i, j))| = |X_1(P(i, j))| = s - r \). Therefore, using the bijection described in (5.8)
and (5.9), we obtain that

\[
(5.23) \quad \sum_{(i,j,p,q)\in \mathcal{F}_{t,r}} \xi(i,p)\xi(j,q) = \sum_{P\in \mathcal{I}} \sum_{f\in M(P)} \sum_{g\in [n]|_{I}} \xi(i_f,i_g)\xi(j_f,j_g)
\]

\[
(5.22) \quad e^{2s} \sum_{P\in \mathcal{I}, Q\in \text{SuccFin}(P)} \left( \sum_{f'\in M(Q)} \xi(i_{f'},i_{g'})^2 \right)^{1/2} \left( \sum_{f'\in M(Q)} \xi(j_{f'},j_{g'})^2 \right)^{1/2} \leq e^{2s} \sum_{P\in \mathcal{I}, Q\in \text{SuccFin}(P)} \left( n|X_1(Q)|+|X_1(S)| \sum_{i,p\in [n]} \xi(i,p)^2 \right)^{1/2} \times
\]

\[
\times \left( n|X_0(Q)|+|X_0(S)| \sum_{j,q\in [n]} \xi(j,q)^2 \right)^{1/2}
\]

\[
\leq e^{2s} n^{s-r} \sum_{P\in \mathcal{I}, Q\in \text{SuccFin}(P)} \sum_{i,p\in [n]} \xi(i,p)^2 \leq e^{2s} \left( 2s! \right)^2 n^{s-r} \sum_{i,p\in [n]} \xi(i,p)^2
\]

where in the last inequality we have used the fact that

\[
(5.24) \quad |\mathcal{I}| \leq \left( \frac{s!}{(s-r)!} \right)^2 \quad \text{and} \quad |\text{SuccFin}(P)| \leq (2s-r-1)!
\]

for every \( P \in \mathcal{I} \). The proof of Proposition 5.1 is completed.

6. An Exchangeable Pair of Random Permutations

Let \( X, Y \) be random variables which are defined on a common probability space and take values in a common measurable space, and recall that the pair \((X, Y)\) is called exchangeable if \((X, Y)\) and \((Y, X)\) have the same distribution. This notion was introduced by Stein [St86], and it is a key concept in his method for normal approximation as well as in several related developments (see, e.g., [Ch14]).

An important ingredient of the proof of Theorem 2.2 is an exchangeable pair of random permutations which also appears in the works of Bolthausen [Bo84] and Barbour/Chen [BC05]. Specifically, given an integer \( n \geq 2 \), we fix a triple \( I_1, I_2, \pi_1 \) of independent random variables such that \( I_1, I_2 \) are uniformly distributed on \( S_n \), and we set

\[
(6.1) \quad J_1 := \pi_1(I_1), \quad J_2 := \pi_1(I_2) \quad \text{and} \quad \pi_2 := \pi_1 \circ t(I_1, I_2)
\]

where for every \( i_1, i_2 \in [n] \) by \( t(i_1, i_2) \in S_n \) we denote the transposition\(^{10}\) of \( i_1 \) and \( i_2 \).

We will need the following basic (and well-known) properties of this construction.

- **(E1)** \( \pi_1 \) and \( (I_1, I_2) \) are independent.
- **(E2)** \( \pi_2 \) and \( (I_1, I_2) \) are independent.

\(^{10}\)That is, \( t(i_1, i_2) \) is the unique permutation which maps \( i_1 \) to \( i_2 \) and \( i_2 \) to \( i_1 \), and it is the identity on \([n] \setminus \{i_1, i_2\}\).
(E3) \( \pi_1 \) and \( \pi_2 \) are uniformly distributed on \( S_n \).

(E4) \((\pi_1, \pi_2)\) is an exchangeable pair.

Property (E1) is an immediate consequence of the relevant definitions. Properties (E2) and (E3) follow from the fact that for every \( \ell_1, \ell_2 \in [n] \) and every \( \pi \in S_n \) we have that

\[
P(\pi_2 = \pi | (I_1, I_2) = (\ell_1, \ell_2)) = \frac{1}{n!};
\]

indeed, observe that

\[
(6.2) \quad P(\pi_2 = \pi | (I_1, I_2) = (\ell_1, \ell_2)) = P(\pi_1 \circ \ell(\ell_1, \ell_2) = \pi | (I_1, I_2) = (\ell_1, \ell_2))
\]

\[
= P(\pi_1 = \pi \circ \ell(\ell_1, \ell_2)^{-1} | (I_1, I_2) = (\ell_1, \ell_2)) = \frac{1}{n!}.
\]

Finally, for property (E4) notice that, by (E1), (E2) and (E3), the pairs \((\pi_1, (I_1, I_2))\) and \((\pi_2, (I_1, I_2))\) have the same distribution; consequently, the pairs \((\pi_1, \pi_1 \circ \ell(I_1, I_2))\) and \((\pi_2, \pi_2 \circ \ell(I_1, I_2))\) also have the same distribution. By (6.1) we have \(\pi_2 = \pi_1 \circ \ell(I_1, I_2)\) which implies, in particular, that \(\pi_2 \circ \ell(I_1, I_2) = \pi_1 \circ \ell(I_1, I_2) \circ \ell(I_1, I_2) = \pi_1\). Combining the previous observations, we conclude that \((\pi_1, \pi_2)\) is an exchangeable pair, as desired.

7. PROOF OF THEOREM 2.2

We have already pointed out that the one-dimensional case of Theorem 2.2 is due to Bolthausen, while the two-dimensional case is due to Barbour/Chen. That said, the proof for all high-dimensional cases is uniform, and so, in what follows we fix two positive integers \(n, d\) with \(d \geq 2\) and \(n \geq 4d^2\). Let \(\xi_1, \ldots, \xi_d\) be as in Theorem 2.2 and recall that \(\beta_1 = n - 1\), where \(\beta_s\) is as in (2.6) for every \(s \in [d]\); moreover, let \(\pi_1, \pi_2\) be the random permutations described in Section 6 and set

\[
(7.1) \quad \Xi_s = \sum_{i \in [n]} \xi_s(i, \pi_1 \circ i) \quad \text{and} \quad \Xi'_s = \sum_{i \in [n]} \xi_s(i, \pi_2 \circ i).
\]

Note that, by property (E4), the pair \((\Xi_s, \Xi'_s)\) is exchangeable for every \(s \in [d]\). We also set

\[
(7.2) \quad \Lambda := \sum_{i,j=1}^n |\xi_1(i, j)|^3.
\]

In the following lemma we collect some known properties of the exchangeable pair \((\Xi_1, \Xi'_1)\); for a proof, see [BC05, Bo84, ZBCL97].

**Lemma 7.1.** We have

\[
(7.3) \quad \mathbb{E}[\Xi_1^2] = 1
\]

\[
(7.4) \quad \mathbb{E}[\Xi_1 - \Xi'_1 | \pi_1] = \frac{2}{n} \Xi_1
\]

\[
(7.5) \quad \mathbb{E}[(\Xi_1 - \Xi'_1)^2] = \frac{4}{n}
\]

\[
(7.6) \quad \mathbb{E}[|\Xi_1 - \Xi'_1|^3] \leq 64 \frac{\Lambda}{n^2}.
\]
The next lemma complements Lemma 7.1 and provides analogous estimates for the rest of the exchangeable pairs \((\Xi_s, \Xi_s')\). We note that it is precisely in the proof of this lemma where Proposition 5.1 is applied.

**Lemma 7.2.** For every \(s \in \{2, \ldots, d\}\) we have

\[
E[\Xi_s^2] \leq 2e^{2s}((2s)!)^2 \frac{\beta_s}{n^s} \\
E[(\Xi_s - \Xi_s')^2] \leq 24s^4 e^{2s}((2s)!)^2 \frac{\beta_s}{n^{s+1}}.
\]

**Proof.** Fix \(s \in \{2, \ldots, d\}\). For every \(r \in \{0, \ldots, s\}\) set

- \(\mathcal{F}_r^1 := \{(i, j) \in [n]_{s1} \times [n]_{s1} : |\text{Im}(i) \cap \text{Im}(j)| = r\}\) and
- \(\mathcal{F}_r^2 := \{(i, j, p, q) \in ([n]_{s2})^4 : |\text{Im}(i) \cap \text{Im}(j)| = r\) and \((i, j) \sim (p, q)\}\).

Then observe that

\[
E[\Xi_s^2] = E\left[ \sum_{i,j \in [n]_{s1}} \xi_s(i, \pi_1 \circ i) \xi_s(j, \pi_1 \circ j) \right] = \sum_{r=0}^{s} \sum_{(i,j) \in \mathcal{F}_r^1} E[\xi_s(i, \pi_1 \circ i) \xi_s(j, \pi_1 \circ j)] = \sum_{r=0}^{s} \sum_{(i,j,p,q) \in \mathcal{F}_r^2} \frac{(n - 2s + r)!}{n!} \xi_s(i, p) \xi_s(j, q).
\]

Also notice that for every \(k \leq \sqrt{n}\) we have

\[
\frac{(n - k)!}{n!} \leq \frac{1}{\left(\frac{k}{n}\right)} \leq \frac{2}{n^k};
\]

since \(2s \leq 2d \leq \sqrt{n}\), by (7.9) we obtain that

\[
E[\Xi_s^2] \leq 2 \sum_{r=0}^{s} \frac{1}{n^{2s-r}} \left| \sum_{(i,j,p,q) \in \mathcal{F}_r^2} \xi_s(i, p) \xi_s(j, q) \right|.
\]

Finally, plugging in the right-hand-side of (7.11) the estimate in (5.1) applied for \(t = \emptyset\), we conclude that (7.7) is satisfied.

We proceed to the proof of inequality (7.8). For every \(\ell_1, \ell_2 \in [n]\) set

- \(\mathcal{F}_{\ell_1, \ell_2} := \{i \in [n]_{s1} : \text{Im}(i) \cap \{\ell_1, \ell_2\} \neq \emptyset\}\).

We start by observing that

\[
E[(\Xi_s - \Xi_s')^2] = E\left[ 1_{\{\ell_1 \neq \ell_2\}} \left( \sum_{i \in \mathcal{F}_{\ell_1, \ell_2}} \xi_s(i, \pi_1 \circ i) - \xi_s(i, \pi_2 \circ i) \right)^2 \right] \leq \leq 2E\left[ 1_{\{\ell_1 \neq \ell_2\}} \left( \sum_{i \in [n]_{s2}, \text{Im}(i) \cap \{\ell_1, \ell_2\} \neq \emptyset} \xi_s(i, \pi_1 \circ i) \right)^2 \right] \leq 2E\left[ 1_{\{\ell_1 \neq \ell_2\}} \left( \sum_{i \in [n]_{s2}} \xi_s(i, \pi_2 \circ i) \right)^2 \right].
\]
By (E1), (E2) and (E3), we see that the pairs \((\pi_1, (I_1, I_2))\) and \((\pi_2, (I_1, I_2))\) have the same distribution. Therefore, by (7.12), we have
\[
\mathbb{E}[(\Xi_s - \Xi_s')] \leq 4\mathbb{E}\left[\mathbf{1}_{\{I_1 \neq I_2\}} \left( \sum_{i \in [n]_{\text{in}}} \xi_s(i, \pi_1 \circ i) \right)^2 \right]
\]
which implies, by (E1) and the fact that \((I_1, I_2)\) is uniformly distributed on \([n]^2\), that
\[
\mathbb{E}[(\Xi_s - \Xi_s')] \leq 4 \sum_{\ell_1, \ell_2 \in [n]} \mathbb{E}\left[\left( \sum_{i \in [n]_{\text{in}}} \xi_s(i, \pi_1 \circ i) \right)^2 \right]
\]
\[
= \frac{4}{n^2} \sum_{\ell_1, \ell_2 \in [n]} \mathbb{E}\left[\left( \sum_{t \in \text{PartInj}([s], \{\ell_1, \ell_2\})} \left( \sum_{t \subseteq i \in [n]_{\text{in}}} \xi_s(i, \pi_1 \circ i) \right)^2 \right)\right]
\]
\[
= \frac{4}{n^2} \sum_{\ell_1, \ell_2 \in [n]} \mathbb{E}\left[\left( \sum_{t \in \text{PartInj}([s], \{\ell_1, \ell_2\})} (-1)^{|t|+1} \sum_{t \subseteq i \in [n]_{\text{in}}} \xi_s(i, \pi_1 \circ i) \right)^2 \right].
\]
Now observe that
\[
|\text{PartInj}([s], \{\ell_1, \ell_2\})| = 2s + 2 \left(\frac{s}{2}\right) = s(s + 1).
\]
Hence, by (7.14), (7.15) and the Cauchy–Schwarz inequality,
\[
\mathbb{E}[(\Xi_s - \Xi_s')] \leq \frac{4s(s + 1)}{n^2} \sum_{\ell_1, \ell_2 \in [n]} \mathbb{E}\left[\left( \sum_{t \subseteq i \in [n]_{\text{in}}} \xi_s(i, \pi_1 \circ i) \right)^2 \right]
\]
\[
= \frac{4s(s + 1)}{n^2} \sum_{\ell_1, \ell_2 \in [n]} \sum_{t \in \text{PartInj}([s], \{\ell_1, \ell_2\})} \sum_{t \subseteq i \in [n]_{\text{in}}} \mathbb{E}[\xi_s(i, \pi_1 \circ i) \xi_s(j, \pi_1 \circ j)]
\]
\[
= \frac{4s(s + 1)}{n^2} \sum_{\ell_1, \ell_2 \in [n]} \sum_{t \in \text{PartInj}([s], \{\ell_1, \ell_2\})} \sum_{r = |\text{Im}(t)|}^{s} \frac{(n - (2s - r))!}{n!} \sum_{i, j, p, q \in [n]_{\text{in}}} \xi_s(i, p) \xi_s(j, q)
\]
\[
\leq \frac{8s(s + 1)}{n^2} \sum_{\ell_1, \ell_2 \in [n]} \sum_{t \in \text{PartInj}([s], \{\ell_1, \ell_2\})} \sum_{r = |\text{Im}(t)|}^{s} \frac{1}{n^{2s-r}} \sum_{i, j, p, q \in [n]_{\text{in}}} \xi_s(i, p) \xi_s(j, q).
\]
Setting \(K_d := 8s(s + 1) e^{2s} ((2s)!)^2\) and applying (5.1) for every \(t \in \text{PartInj}([s], \{\ell_1, \ell_2\})\) and every \(r \in \{|\text{Im}(t)|, \ldots, s\}\), by (7.16) we have
\[
\mathbb{E}[(\Xi_s - \Xi_s')] \leq \frac{K_d}{n^{s+2}} \sum_{\ell_1, \ell_2 \in [n]} \sum_{t \in \text{PartInj}([s], \{\ell_1, \ell_2\})} \sum_{r = |\text{Im}(t)|}^{s} \sum_{t \subseteq i} \xi_s(i, p)^2
\]
By double-counting, we see that the right-hand-side of (7.17) is equal to
\[
\frac{K_d}{n^{s+2}} (2(n-1)s(s-1) \sum_{F \in \binom{[r]}{1}} \sum_{x \in [n]^r} \sum_{i \in F} \xi_s(i, p)^2 + 2(s-2) \sum_{H \in \binom{[r]}{2}} \sum_{y \in [n]^r} \sum_{i \in H} \sum_{i \in H} \xi_s(i, p)^2)
\]
which is at most, since \( s \leq d \leq n \),
\[
(7.18) \quad 2 K_d s(s-1) \frac{\beta_s}{n^{s+1}} + K_d s(s-1)(s-2) \frac{\beta_s}{n^{s+2}} \leq 3 K_d s(s-1) \frac{\beta_s}{n^{s+1}}.
\]
By (7.17), (7.18) and the choice of \( K_d \), we conclude that (7.8) is satisfied. \( \square \)

Now set
\[
(7.19) \quad \Theta := \sum_{s=2}^{d} \Xi_s \quad \text{and} \quad \Theta' := \sum_{s=2}^{d} \Xi_s'.
\]
The last ingredient of the proof of Theorem 2.2 is the following lemma which is taken from [BC05, Lemma 2.2].

**Lemma 7.3** (Barbour/Chen). For every \( z \in \mathbb{R} \) we have
\[
(7.20) \quad \max \left\{ \mathbb{P}(z - |\Theta| \leq \Xi_1 \leq z), \mathbb{P}(z \leq \Xi_1 \leq z + |\Theta|) \right\} \leq \frac{2^5 \Lambda}{n} + 217 \Lambda^2 n^2 + 5d^2 e^{d(2d)!} \sum_{s=2}^{d} \frac{\sqrt{\beta_s}}{n^s}
\]
where \( \Lambda \) is as in (7.2) and \( \beta_s \) is as in (2.6) for every \( s \in \{2, \ldots, d\} \).

Since the setting in [BC05] is not identical with ours, for completeness we give the proof of Lemma 7.3 in Appendix A.

We are now in a position to complete the proof of the theorem. Notice that (2.7) is automatically satisfied if \( \frac{\Lambda}{n} \geq 1 \). Thus, in what follows, we may assume that \( \frac{\Lambda}{n} < 1 \). By Bolthausen’s theorem [Bo84], we have
\[
(7.21) \quad \sup_{z \in \mathbb{R}} |\mathbb{P}(\Xi_1 \leq z) - \mathbb{P}(N(0, 1) \leq z)| \leq C_1 \frac{\Lambda}{n}
\]
where \( C_1 \geq 1 \) is as in (2.4). Since \( \Theta + |\Theta| \geq 0 \), for every \( z \in \mathbb{R} \) we have
\[
(7.22) \quad \mathbb{P}(W \leq z) = \mathbb{P}(\Xi_1 + \Theta + |\Theta| \leq z + |\Theta|) \leq \mathbb{P}(\Xi_1 \leq z + |\Theta|)
\]
and so,
\[
(7.23) \quad \mathbb{P}(W \leq z) = \mathbb{P}(\Xi_1 + \Theta - |\Theta| \leq z - |\Theta|) \geq \mathbb{P}(\Xi_1 \leq z - |\Theta|)
\]
\[
(7.24) \quad \mathbb{P}(W \leq z) - \mathbb{P}(\Xi_1 \leq z) \leq \mathbb{P}(z < \Xi_1 \leq z + |\Theta|)
\]
and
\[
(7.25) \quad \mathbb{P}(W \leq z) - \mathbb{P}(\Xi_1 \leq z) \geq -\mathbb{P}(z - |\Theta| < \Xi_1 \leq z).
\]
Hence, by Lemma 7.3 and the fact that \( \frac{\Lambda}{n} < 1 \), we obtain that

\[
(7.26) \sup_{z \in \mathbb{R}} |\mathbb{P}(W \leq z) - \mathbb{P}(\Xi_1 \leq z)|
\leq \sup_{z \in \mathbb{R}} \max \{\mathbb{P}(z < \Xi_1 \leq z + |\Theta|), \mathbb{P}(z - |\Theta| < \Xi_1 \leq z)\}
\leq (2^5 + 2^{17}) \frac{\Lambda}{n} + 5d^2 e^d (2d)! \sum_{s=2}^{d} \sqrt{\frac{\beta_s}{n^s}}.
\]

Inequality (2.7) follows from (7.21), (7.26), the triangle inequality and the choice of the constant \( C_d \) in Theorem 2.2. The proof is completed.

**Remark 7.4 (Towards a group-theoretic combinatorial CLT).** Let \( I \) be a nonempty finite set, and let \( G \) be a subgroup of \( S_I \) which acts transitively on \( I \). With every matrix \( \zeta : I \times I \to \mathbb{R} \) we associate the statistic

\[
(7.27) Z_G = \sum_{i \in I} \zeta(i, \pi(i))
\]

where \( \pi \) is a random permutation which is uniformly distributed on \( G \). The case \( G = S_I \) is, of course, the setting of the classical combinatorial CLT; on the other hand, it is easy to see that the statistics covered by the work of Barbour/Chen [BC05] and Theorem 2.2 are also included\(^{11}\) in this general framework. It is thus natural to ask\(^{12}\) for which finite transitive permutation groups \( G \) we have quantitative normal approximation of the statistics \( Z_G \) under suitable conditions on the matrix \( \zeta \). The case of alternating groups is already interesting.

**Remark 7.5 (Finite population statistics).** This remark has been kindly communicated to us by the referee. Let \( n \geq d \geq 2 \) be positive integers, and let \( t : [n]^d \to \mathbb{R} \) be a (deterministic) symmetric real tensor. With \( t \) we associate the statistic

\[
(7.28) T := t(\pi(1), \ldots, \pi(d))
\]

where \( \pi \) is a random permutation which is uniformly distributed on \( S_n \). These statistics have been studied by several authors; see, e.g., [BG93, BoG02, ZC90] and the references therein. It was shown by Bloznelis and Götze [BoG01] that \( T - \mathbb{E}[T] \) can be written as

\[
(7.29) T - \mathbb{E}[T] = \sum_{s=1}^{d} \sum_{1 \leq i_1 < \cdots < i_s \leq d} g_s(\pi(i_1), \ldots, \pi(i_s))
\]

where, for every \( s \in [d] \), \( g_s : [n]^s \to \mathbb{R} \) is a symmetric, Hoeffding\(^{13}\) tensor whose diagonal terms vanish. Using this representation, it is easy to see that the random variable \( T - \mathbb{E}[T] \)
can also be expressed as a \( W \)-statistic of order \( d \). More precisely, for every \( s \in \{2, \ldots, d\} \) define \( \xi_s : [n]^s \times [n]^s \to \mathbb{R} \) by setting \( \xi_s(i,j) = \alpha_s \mathbb{1}_{i=0} \cdot (\mathbb{1}_{i=0} - \frac{d}{s}) g_s(j) \) where \( \alpha_s := \frac{1}{s!} \) and \( \alpha_s^* \) is as in (2.4), and \( \xi_1 : [n] \times [n] \to \mathbb{R} \) by \( \xi_1(i,j) = (\mathbb{1}_{i=0} - \frac{d}{s}) g_s(j) \). Then observe that the tensors \( \xi_1, \ldots, \xi_d \) are Hoeffding in the sense of Definition 2.1, and the \( W \)-statistic associated with them via (2.3) coincides with \( T - \mathbb{E}[T] \). In particular, if \( n \geq 4d^2 \), then, by Theorem 2.2, we have

\[
(7.30) \quad d_K(T, N(\mu, \sigma^2)) \leq \frac{2^{19} C_1}{\sqrt{d n}} \left( \frac{\|g_1\|_{\ell_2}}{\|g_1\|_{\ell_2}} \right)^3 + 2C_d \sum_{s=2}^{d} \frac{1}{s!} \frac{d}{\sqrt{n}} \frac{s-1}{s} \|g_s\|_{\ell_2} \|g_1\|_{\ell_2}
\]

where \( \mu := \mathbb{E}[T] \) and \( \sigma^2 := \frac{d}{n-1} \|g_1\|_{\ell_2}^2 \). (Recall that \( C_1 \geq 1 \) is as in (2.4), and \( C_d = 5d^2 e^d(2d)! \).)

Part 3. Proofs of Theorem 1.4 and its applications

8. Computation of the variance: proof of Proposition 1.2

This section is devoted to the proof of Proposition 1.2. We start by introducing some auxiliary quantities which will also be used in Section 9.

8.1. The parameters \( \gamma_{s,r} \). For every \( s, r \in \{0, 1, \ldots\} \) with \( s \leq r \),

- let \( \mathcal{X}_{s,r} \) denote the set of all strictly increasing maps \( u : [\ell] \to \mathbb{Z} \) for some \( \ell \in \mathbb{N} \) which satisfy \( u(1) = s \) and \( u(\ell) = r \) (thus, \( \ell \leq r - s + 1 \)),

and set

\[
(8.1) \quad \gamma_{s,r} := \sum_{u \in \mathcal{X}_{s,r}} (-1)^{|\text{dom}(u)|+1} \prod_{i=2}^{|\text{dom}(u)|} \left( \frac{u(i)}{u(i-1)} - 1 \right) = \sum_{u \in \mathcal{X}_{s,r}} (-1)^{|\text{dom}(u)|+1} \left( s, u(2) - u(1), \ldots, u(|\text{dom}(u)|) - u(|\text{dom}(u)| - 1) \right)
\]

with the convention that the above product is equal to 1 if \( i > |\text{dom}(u)| \). Observe that

\[
(8.2) \quad \gamma_{s,s} = 1.
\]

Also notice that for every \( s, r \in \{0, 1, \ldots\} \) with \( s \leq r \) we have

\[
(8.3) \quad \mathcal{X}_{s-1,r} = \bigcup_{x=s}^r \{f_{s-1,u} : u \in \mathcal{X}_{x,r} \}
\]

where the union above is a disjoint union; here, \( f_{s-1,u} : [\text{dom}(u)|+1] \to \mathbb{Z} \) denotes the function defined by \( f_{s-1,u}(1) = s - 1 \) and \( f_{s-1,u}(i) = u(i-1) \) if \( i \in \{2, \ldots, |\text{dom}(u)| + 1\} \).

By (8.1), it follows in particular that for every \( s, r \in \{1, 2, \ldots\} \) with \( s \leq r \) we have

\[
(8.4) \quad \gamma_{s-1,r} = - \sum_{x=s}^r \left( x \right) \gamma_{x,r}.
\]

\[\text{Here, for every tensor } g : [n]^s \to \mathbb{R} \text{ and every } p > 1 \text{ we set } \|g\|_{\ell_p} := (\sum_{i \in [n]^s} |g(i)|^p)^{1/p}.\]
8.1.1. We proceed to compute these quantities. More precisely, we claim that for every \( s, r \in \{0, 1, \ldots \} \) with \( s \leq r \) we have

\[
\gamma_{s,r} = (-1)^{r-s} \binom{r}{s}.
\]

The case “\( s = r \)” follows, of course, from (8.2). So, let \( s, r \in \{0, 1, \ldots \} \) with \( s < r \). Notice that for every \( u \in \mathcal{X}_{s,r} \) we have \( 2 \leq \lvert \text{dom}(u) \rvert \leq r - s + 1 \). For every \( \ell \in [r - s] \) set

- \( \mathcal{X}_{s,r}^{\ell} := \{ u \in \mathcal{X}_{s,r} : \lvert \text{dom}(u) \rvert = \ell + 1 \} \) and \( \mathcal{Y}_{\ell} := \{ y \in [r-s]^\ell : \sum_{i=1}^{\ell} y(i) = r-s \} \).

Then we have

\[
\gamma_{s,r} = \binom{r}{s} \sum_{\ell=1}^{r-s} (-1)^{\ell+2} \sum_{u \in \mathcal{X}_{s,r}^{\ell}} (u(2) - u(1), \ldots, u(\ell + 1) - u(\ell))
\]

\[
= \binom{r}{s} \sum_{\ell=1}^{r-s} (-1)^{\ell} \sum_{y \in \mathcal{Y}_{\ell}} (y(1), \ldots, y(\ell))
\]

\[
= \binom{r}{s} \sum_{\ell=1}^{r-s} (-1)^{\ell} \binom{\ell}{r-s} \sum_{j=0}^{\ell-1} (-1)^{j} \binom{\ell}{j} (\ell - j)^{r-s} = \binom{r}{s} \sum_{0 \leq j \leq \ell \leq r-s} (-1)^{\ell-j} \binom{\ell}{j} (\ell - j)^{r-s}
\]

which yields, by setting “\( t = \ell - j \)”, that

\[
\gamma_{s,r} = \binom{r}{s} \sum_{t=0}^{r-s-t} (-1)^t t^{r-s} \sum_{j=0}^{t} \binom{t}{j} t + j.
\]

Noticing that \( \sum_{t=0}^{r-s-t} t^{r-s} = \binom{r-s+1}{r-s-t+1} \) for every positive integer \( t \), by (8.7) and an additional change of variable “\( q = t + 1 \)”, we obtain that

\[
\gamma_{s,r} = \binom{r}{s} \sum_{t=0}^{r-s-t} (-1)^t t^{r-s} \frac{r-s+1}{t+1} = - \binom{r}{s} \sum_{q=1}^{r-s+1} (-1)^q (q-1)^{r-s} \frac{r-s+1}{q}.
\]

By a classical formula which goes back to Euler—see, e.g., [GKP94, Eq. (5.42)]—and the fact that the degree of the polynomial \((x-1)^{r-s}\) is strictly smaller than \(r-s+1\), we also obtain that \( \sum_{q=0}^{r-s+1} (-1)^q (q-1)^{r-s} \frac{r-s+1}{q} = 0 \). Therefore, by (8.8), we conclude that (8.5) is satisfied.

8.2. Proof of Proposition 1.2. In what follows, let \( n, d \) be positive integers with \( n \geq 2d \), and fix \( X = (X_i : i \in [n]^d) \) and \( \theta = (\theta_i : i \in [n]^d) \) which satisfy (A1), (A2) and (A3). We define \( a = (a_F : F \in \binom{[n]}{d}) \) by setting \( a_F = \theta_{i(1), \ldots, i(d)} \) for every \( F = \{i_1 < \cdots < i_d\} \in \binom{[n]}{d} \).

By (A2), we see that \( a_F = \theta_i \) for every \( F \in \binom{[n]}{d} \) and every \( i \in [n]^d \) with \( \text{Im}(i) = F \). Moreover, by (A1), we have \( E[\langle \theta, X \rangle] = 0 \). Therefore,

\[
\text{Var}(\langle \theta, X \rangle) = \sum_{i,j \in [n]^d} \theta_i \theta_j E[X_i X_j] = \sum_{s=0}^{d} \delta_s \sum_{i,j \in [n]^d} \theta_i \theta_j = (d!)^2 \sum_{s=0}^{d} \delta_s \sum_{F,G \in \binom{[n]}{d} |F \cap G| = s} a_F a_G.
\]
For every $s \in \{0, 1, \ldots, d\}$ set

\begin{equation}
\kappa_s := \sum_{F,G \in \binom{[n]}{d-s}} a_F a_G \quad \text{and} \quad \lambda_s := \sum_{H \in \binom{[n]}{d-s}} \left( \sum_{F \in \binom{[n]}{d-s} \setminus H} a_{F \cup H} \right)^2.
\end{equation}

Notice that $\kappa_d = \lambda_d$; moreover, for every $s \in \{0, \ldots, d-1\}$ we have

\begin{equation}
\kappa_s = \sum_{H \in \binom{[n]}{s}} \sum_{K,L \in \binom{[n]}{d-s}} a_{H \cup K} a_{H \cup L} = \lambda_s - \sum_{t=1}^{d-s} \sum_{K,L \in \binom{[n]}{d-s}} a_{H \cup K} a_{H \cup L} = \lambda_s - \sum_{t=1}^{d-s} \binom{s+t}{s} \kappa_{s+t} = \lambda_s - \sum_{r=s+1}^{d} \binom{r}{s} \kappa_r.
\end{equation}

Proceeding by backwards induction and using the fact that $\kappa_d = \lambda_d$, (8.11), (8.2) and (8.4), we see that for every $s \in \{0, 1, \ldots, d\}$,

\begin{equation}
\kappa_s = \sum_{r=s}^{d} \gamma_{s,r} \lambda_r.
\end{equation}

By (8.10), the description of the variance of $\langle \theta, X \rangle$ in (8.9) and (8.12), we obtain that

\begin{equation}
\text{Var}(\langle \theta, X \rangle) = (dl)^2 \sum_{s=0}^{d} \delta_s \sum_{r=s}^{d} \gamma_{s,r} \lambda_r = (dl)^2 \sum_{r=0}^{d} \lambda_r \sum_{s=0}^{r} \gamma_{s,r} \delta_s
\end{equation}

which in turn implies, by (8.5), that

\begin{equation}
\text{Var}(\langle \theta, X \rangle) = (dl)^2 \sum_{r=0}^{d} \lambda_r \left( \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \delta_s \right).
\end{equation}

Equality (1.8) follows from (8.14) after observing that $\|\theta\|_s^2 = s! (d-s)!^2 \lambda_s$ for every $s \in \{0, \ldots, d\}$. The proof of Proposition 1.2 is completed.

9. From Z-statistics to W-statistics

We have pointed out in Section 2 that every statistic of the form (2.1) can be written in the form (2.3). In the one-dimensional case, this transformation is classical; see [Ho51]. The two-dimensional case is also well-known; see, e.g., [ZBCL97]. These arguments can be extended to the higher-dimensional case though, to the best of our knowledge, this has not appeared in the literature. Since this step is needed for the proof of Theorem 1.4 and, more important, it affects its quantitative aspects15, in this section we shall describe this transformation for those tensor permutation statistics which are relevant to Theorem 1.4. That said, we note that some of the combinatorial constructions that appear in this section are related to the classical work of de Jong [deJ90] as well as the more recent work of Döbler, Kasprzak and Peccati [DKP22].

15Specifically, it affects the constant $\kappa$. 
We also point out that a slightly different type of decomposition of permutation statistics—which is more akin to the standard Hoeffding decomposition [Ho48]—has been considered by various authors, including Zhao/Chen [ZC90] and Bloznelis/Götze [BoG01], and culminating in the work of Peccati [P04] (see, also, [PP11]).

9.1. Summing, averaging and Hoeffding operators. Let \( n \geq s \) be positive integers, and let \( a: [n]^s \to \mathbb{R} \) be a real tensor. For every (possibly empty) subset \( F \) of \( [s] \) we introduce the tensors \( S_F[a]: [n]^F \to \mathbb{R} \) and \( A_F[a]: [n]^F \to \mathbb{R} \) by setting for every \( i \in [n]^F \),

\[
S_F[a](i) := \sum_{i \subseteq j \in [n]^s} a(j) \quad \text{and} \quad A_F[a](i) := \frac{1}{n^{s-|F|}} \sum_{i \subseteq j \in [n]^s} a(j).
\]

Note that if “\( F = \emptyset \)”, then \([n]^0 = \{\emptyset\}\) and, consequently, \( S_\emptyset[a] \) and \( A_\emptyset[a] \) are tensors with a single entry which is the sum and the average of the entries of \( a \) respectively. Also observe that \( S_s[a] = A_s[a] = a \).

Moreover, we define the Hoeffding tensor \( \mathcal{H}[a]: [n]^s \to \mathbb{R} \) associated with \( a \) by setting for every \( i \in [n]^s \),

\[
\mathcal{H}[a](i) := \sum_{F \subseteq [s]} (-1)^{s-|F|} A_F[a](i \mid F).
\]

9.1.1. Doubly-indexed tensors. We will also work with averaging and Hoeffding operators associated with doubly-indexed tensors. More precisely, let \( n \geq s \) be positive integers, and let \( \zeta: [n]^s \times [n]^s \to \mathbb{R} \). For every pair \( F,G \) of (possibly empty) subsets of \([s]\) we define the tensor \( A_{F,G}[\zeta]: [n]^F \times [n]^G \to \mathbb{R} \) by setting for every \( i \in [n]^F \) and every \( p \in [n]^G \),

\[
A_{F,G}[\zeta](i,p) := \frac{1}{n^{2s-|F|-|G|}} \sum_{i \subseteq j \in [n]^s \atop p \subseteq q \in [n]^s} \zeta(j,q).
\]

Again, observe that \( A_{s,s}[\zeta] = \zeta \); moreover, the tensor \( A_{\emptyset,\emptyset}(\zeta) \) contains a single entry which is equal to the average of the entries of \( \zeta \).

Finally, we define the Hoeffding tensor \( \mathcal{H}[\zeta]: [n]^s \times [n]^s \to \mathbb{R} \) associated with \( \zeta \) by setting for every \( i,p \in [n]^s \),

\[
\mathcal{H}[\zeta](i,p) := \sum_{F,G \subseteq [s]} (-1)^{|F|+|G|} A_{F,G}[\zeta](i \mid F,p \mid G).
\]

We have the following elementary—though important—fact.

**Fact 9.1.** For every \( \zeta: [n]^s \times [n]^s \to \mathbb{R} \), the tensor \( \mathcal{H}[\zeta] \) associated with \( \zeta \) is Hoeffding in the sense of Definition 2.1.

**Remark 9.2.** It is clear that every tensor \( \zeta: [n]^s \times [n]^s \to \mathbb{R} \) can be naturally identified with a tensor \( a_\zeta: [n]^{2s} \to \mathbb{R} \). Then observe that the Hoeffding tensor \( \mathcal{H}[\zeta] \) associated with \( \zeta \) via (9.4) coincides with the Hoeffding tensor \( \mathcal{H}[a_\zeta] \) associated with \( a_\zeta \) via (9.2).
9.2. **Decomposition.** Let \( n, r \) be integers with \( n \geq r \geq 2 \), and let \( \zeta: [n]^r \times [n]^r \to \mathbb{R} \) be a real tensor. In what follows, we make the following assumptions on \( \zeta \).

(A5) **(Symmetry)** We have \( \zeta(i, j) = \zeta(i \circ \tau, j \circ \rho) \) for every \( i, j \in [n]^r \) and \( \tau, \rho \in S_r \).

(A6) **(Vanishing diagonal)** We have \( \zeta(i, j) = 0 \) if either \( i \notin [n]^r_{\text{Inj}} \) or \( j \notin [n]^r_{\text{Inj}} \).

As we shall shortly see, this class includes—among others—all those tensors which appear in the proof of Theorem 1.4.

Next, for every \( s \in \{0, 1, \ldots, r\} \) we define \( \zeta_s: [n]^s \times [n]^s \to \mathbb{R} \) by

\[
\zeta_s := \mathcal{A}_{[s], [s]}[\zeta]
\]

where \( \mathcal{A}_{[s], [s]} \) is the averaging operator introduced in (9.3); in particular, \( \zeta_0 = \mathcal{A}_{\emptyset, \emptyset}(\zeta) \) is the average of the entries of \( \zeta \). Moreover, for every \( s \in [r] \) set

\[
R_s := \sum_{i \in [n]_{\text{Inj}}} \mathcal{H}(\zeta_s)(i, \pi \circ i)
\]

where \( \pi \) is a random permutation uniformly distributed on \( S_n \); here \( \mathcal{H}(\zeta_s) \) denotes the Hoeffding tensor associated with \( \zeta_s \) via (9.4).

9.2.1. **The weights** \( w_{s,r} \). For every pair of integers \( h, s \) with \( 0 \leq h < s \leq r \) set

\[
\mu_{h,s} := \left( \frac{s}{h} \right)^{s-h} \sum_{f=0}^{s-h} 2^{s-h-f} (-1)^f \binom{s-h}{f} \binom{n-s+f}{n-s} \frac{(n-s+f)!}{n^f}.
\]

Moreover, for every \( s \in \{0, \ldots, r-1\} \),

- let \( \mathcal{X}_{s,r} \) be as in Subsection 8.1, namely, \( \mathcal{X}_{s,r} \) is the set of all strictly increasing maps \( u: [\ell] \to \mathbb{Z} \) for some \( \ell \in [n] \) such that \( u(1) = s \) and \( u(\ell) = r \).

Given \( u \in \mathcal{X}_{s,r} \) set

\[
\mu_u := \frac{1}{|\text{dom}(u)|-1} \prod_{i=1}^{\text{dom}(u)-1} \mu_{u(i), u(i+1)}
\]

and for every \( s \in \{0, \ldots, r-1\} \) define

\[
w_{s,r} := (-1)^{r-s+1} \sum_{u \in \mathcal{X}_{s,r}} (-1)^{|\text{dom}(u)|} \mu_u.
\]

Also set

\[
w_{r,r} := 1.
\]

We have the following estimate for these quantities.

**Lemma 9.3.** If \( n \geq 6r^2 \), then for every \( s \in \{0, 1, \ldots, r\} \),

\[
w_{s,r} - \binom{r}{s} \leq \frac{r^3 18^r r!}{n}.
\]

We postpone the proof of Lemma 9.3 to the end of this section.

\[\text{\footnote{Recall that for every } i = (i_1, \ldots, i_r) \in [n]^r \text{ and every } \tau \in S_r \text{ we set } i \circ \tau = (i_{\tau(1)}, \ldots, i_{\tau(r)}) \in [n]^r.}\]
9.2.2. Main result. We have the following proposition.

**Proposition 9.4.** Let \( \zeta : [n]^r \times [n]^r \to \mathbb{R} \) be a tensor which satisfies \((A5)\) and \((A6)\), and let \( Z \) be the statistic associated with \( \zeta \) via \((2.1)\). Then we have

\[
Z = \sum_{s=1}^{r} n^{r-s} w_{s,r} R_s + n^r w_{0,r} \zeta_0
\]

where \( R_s \) and \( w_{s,r} \) are as in \((9.6)\) and \((9.9)\) respectively, and \( \zeta_0 \) is the average of the entries of \( \zeta \).

**Proof.** For every \( s \in \{0, \ldots, r\} \) let \( Z_s \) denote the \( Z \)-statistic associated with \( \zeta_s \) via \((2.1)\). (In particular, \( Z_0 \) is constant, and takes the value \( \zeta_0 \).) Clearly, we have \( Z_r = Z \). Next, for every \( s, p \in \{0, \ldots, r\} \) with \( s < p \),

- let \( X_{s,p,r} := \{ u \in X_{s,r} : u(\text{dom}(u) - 1) \geq p \} \) where \( X_{s,r} \) is as in Subsection 8.1, and set

\[
D_{s,p} := \sum_{u \in X_{s,p,r}} (-1)^{|\text{dom}(u)|} \mu_u.
\]

Since \( X_{s,r} = X_{s,s+1,r} \) for every \( s \in \{0, \ldots, r-1\} \), by \((9.9)\) we have

\[
w_{s,r} = (-1)^{r-s+1} D_{s,s+1}.
\]

Moreover, using a decomposition as in \((8.3)\), we see that for every \( s, p \in \{0, \ldots, r-1\} \) with \( s < p \),

\[
D_{s,p} = D_{s,p+1} - D_{p,p+1} \mu_{s,p}.
\]

Next, we claim that for every \( s \in [r] \) we have

\[
Z_s = R_s + \sum_{h=0}^{s-1} \mu_{h,s} n^{s-h} (-1)^{s-h-1} Z_h.
\]

Indeed, fix \( s \in [r] \). Then, upon conditioning on the value of \( \pi \),

\[
Z_s(\pi) \overset{(2.1),(46)}{=} \sum_{i \in [n]} \zeta(i, \pi \circ i) \overset{(9.6)}{=} R_s(\pi) + \sum_{i \in [n]} (\zeta(i, \pi \circ i) - \mathcal{H}(\zeta_s)(i, \pi \circ i)) \overset{(9.4)}{=} R_s(\pi) - \sum_{F \subseteq [s]} (-1)^{|F|+|G|} \sum_{i \in [n]} A_{F,G}[\zeta_s](i \mid F, \pi \circ i \mid G) \overset{(9.17)}{=} R_s(\pi) - \sum_{F \subseteq [s]} (-1)^{|F|+|G|} \sum_{i \in [n]} A_{F,G}[\zeta_s](i \mid F, \pi \circ i \mid G) \]

\[= R_s(\pi) - \sum_{F \subseteq [s]} (-1)^{|F|+|G|} \binom{n-|F \cup G|}{n-s} \frac{(n-|F \cup G|)!}{(n-s)! n(|F \cup G|)!} \times \sum_{i \in [n]} A_{F,G}[\zeta_s](i \mid F, \pi \circ i \mid G) \]
where the complements above are taken with respect to \([s]\). Therefore, after observing that \(|(F \cup G)^2| + |F \setminus G| + |G \setminus F| = |(F \cap G)^2|\), we obtain that

\[
Z_s(\pi) = R_s(\pi) - \sum_{F,G \subseteq [s], |F|+|G| \leq 2s} (-1)^{|F|+|G|} n^{|(F \cup G)^2|} \frac{(n - |F \cup G|)!}{(n-s)! n^{|(F \cup G)^2|}} \times \sum_{i \in [n]} A_{F \cup G,F \cap G} [\zeta_s](i, \pi \circ i)
\]

\[
= R_s(\pi) - \sum_{H \subseteq [s]} n^{s-|H|} \sum_{i \in [n]} A_{H,H} [\zeta_s](i, \pi \circ i) \times \sum_{F,G \subseteq [s], F \cup G = H} (-1)^{|F \cup G|} \frac{(n - |F \cup G|)!}{(n-s)! n^{|(F \cup G)^2|}}.
\]

Fix a proper subset \(H\) of \([s]\) and set \(h = |H|\). Then we have

\[
Z_s(\pi) = R_s(\pi) - \sum_{h=0}^{s-1} n^{s-h} \sum_{H \subseteq [s]} \sum_{i \in [n]^{|H|}} A_{H,H} [\zeta_s](i, \pi \circ i) \times \sum_{f=0}^{s-h} \binom{s-h}{f} 2^{s-h-f} (-1)^{s-h-f} \frac{(n - s + f)!}{(n-s)! n^f}
\]

where the last equality follows from the change of variable \(f = s - h - k\). Thus, by (9.18) and (9.19), we conclude that

\[
Z_s(\pi) = R_s(\pi) - \sum_{h=0}^{s-1} n^{s-h} \sum_{H \subseteq [s]} \sum_{i \in [n]^{|H|}} A_{H,H} [\zeta_s](i, \pi \circ i) \times \sum_{f=0}^{s-h} \binom{s-h}{f} 2^{s-h-f} (-1)^{s-h-f} \frac{(n - s + f)!}{(n-s)! n^f}
\]

By the definition of \(\zeta_s\) in (9.5) and (A5), for every \(H \subseteq [s]\), setting \(h := |H|\), we have

\[
\sum_{i \in [n]^{|H|}} A_{H,H} [\zeta_s](i, \pi \circ i) = \sum_{i \in [n]^{|H|}} \zeta_h(i, \pi \circ i) = Z_h(\pi);
\]

hence, (9.20) yields that

\[
Z_s(\pi) = R_s(\pi) - \sum_{h=0}^{s-1} n^{s-h} \binom{s}{h} Z_h(\pi) \times \sum_{f=0}^{s-h} \binom{s-h}{f} 2^{s-h-f} (-1)^{s-h-f} \frac{(n - s + f)!}{(n-s)! n^f}.
\]

Thus, (9.16) follows from (9.21) and (9.7).
Next, proceeding by backwards induction, we will show that for every \( p \in [r] \) we have
\[
Z_r = R_r + \sum_{q=p}^{r-1} n^{r-q} w_{q,r} R_q + \sum_{q=0}^{p-1} (-1)^{r-q-1} n^{r-q} D_{q,p} Z_q
\]
with the convection that a sum of the form \( \sum_{i=a}^{b} \) with \( b < a \) is equal to zero. For the case “\( p = r \)”, notice that for every \( q \in \{0, \ldots, r-1\} \) the set \( X_{q,r,r} \) is a singleton which contains the map \( u: [2] \to \mathbb{Z} \) with \( u(1) = q \) and \( u(2) = r \); in particular, for every \( q \in \{0, \ldots, r-1\} \) we have \( D_{q,r} = \mu_{q,r} \), and so (9.22) for “\( p = r \)” follows from (9.16) for “\( s = r \)”. Next, let \( p \in [r-1] \) and assume that equality (9.22) holds true for \( p + 1 \), that is,
\[
Z_r = R_r + \sum_{q=p+1}^{r-1} n^{r-q} w_{q,r} R_q + \sum_{q=0}^{p} (-1)^{r-q} n^{r-q} D_{q,p+1} Z_q.
\]
Plugging in (9.23) equality (9.16) for “\( s = p \)”, we see that
\[
Z_r = R_r + \sum_{q=p+1}^{r-1} n^{r-q} w_{q,r} R_q + (-1)^{r-p-1} n^{r-p} D_{p,p+1} R_p + \sum_{q=0}^{p-1} (-1)^{r-q} n^{r-q} (D_{q,p+1} - D_{p,p+1} \mu_{q,p}) Z_q.
\]
Using (9.14) for “\( s = p \)”, (9.15) for “\( s = q \)” and (9.24), we conclude that (9.22) also holds true for \( p \). This completes the proof of (9.22).

Finally, after recalling that \( w_{r,r} = 1 \), the desired equality (9.12) follows from (9.22) applied for “\( p = 1 \)” and (9.14) applied for “\( s = 0 \)”. The proof of Proposition 9.4 is completed.

9.3. Proof of Lemma 9.3. Our first goal is to approximate the quantities \( \mu_{q,p} \). To this end, fix \( q, p \in \{0, 1, \ldots, r\} \) with \( q < p \). Then, by (9.7), we have
\[
\mu_{q,p} = \left( \frac{p}{q} \right) \sum_{f=0}^{p-q} 2^{p-q-f} (-1)^f \left( \frac{p-q}{f} \right) \prod_{u=1}^{f} \left( 1 - \frac{p-u}{n} \right);
\]
moreover,
\[
\sum_{f=0}^{p-q} 2^{p-q-f} (-1)^f \left( \frac{p-q}{f} \right) = 1.
\]
Also observe that for every \( f \in \{0, \ldots, p-q\} \) we have
\[
1 \geq \prod_{u=1}^{f} \left( 1 - \frac{p-u}{n} \right) \geq \prod_{v=1}^{r} \left( 1 - \frac{v}{n} \right) \geq 1 - \sum_{v=1}^{r} \frac{v}{n} \geq 1 - \frac{r^2}{n}.
\]
On the other hand, we have
\[
\sum_{f=0}^{p-q} 2^{p-q-f} (-1)^f \left( \frac{p-q}{f} \right) = \sum_{f=0}^{p-q} 2^{p-q-f} \left( \frac{p-q}{f} \right) = 2^{p-q} \leq 3^r
\]
By (9.25)–(9.28), we obtain that

\[(9.29) \quad |\mu_{q,p} - \left(\frac{p}{q}\right)| \leq \left(\frac{p}{q}\right) 3^2r^2 \frac{n}{2} \leq 6^r r^2 \frac{n}{n}.
\]

Next, we shall approximate the quantities \(\mu_u\). So, fix \(u \in \bigcup_{s=0}^{r-1} A_{s,r}\) and, for notational simplicity, set \(\ell := |\text{dom}(u)|\). Then,

\[(9.30) \quad |\mu_u - \prod_{i=2}^{\ell} \left( \frac{u(i)}{u(i-1)} \right) | = \left| \prod_{i=2}^{\ell} \mu_{u(i-1),u(i)} - \prod_{i=2}^{\ell} \left( \frac{u(i)}{u(i-1)} \right) \right| \leq \left(\frac{9r^3}{n}\right) \prod_{i=2}^{\ell-1} \mu_{u(i),u(i+1)} \cdot \frac{6^r r^2}{n} \cdot \prod_{i=j+1}^{\ell-1} \left( u(i+1) \right) u(i) \]

with the usual convention that a product of the form \(\prod_{i=a}^{b} x_i\) with \(b < a\) is equal to 1. By (9.29) and the fact that \(n \geq 6^r r^2\), we see that \(|\mu_{u(i),u(i+1)}| \leq \frac{3}{2}(u(i+1))\) for every \(i \in \{2, \ldots, \ell\}\). Hence, by (9.30),

\[(9.31) \quad |\mu_u - \prod_{i=2}^{\ell} \left( \frac{u(i)}{u(i-1)} \right) | \leq \frac{9r^3}{n} \prod_{i=2}^{\ell} \left( \frac{u(i)}{u(i-1)} \right) \leq \frac{12r^3 r^!}{n}.
\]

Finally, let \(s \in \{0, \ldots, r-1\}\). Since \(|A_{s,r}| = 2^{r-s-1} \leq 2^r\), by (9.9), (8.1) and (9.31),

\[(9.32) \quad |w_{s,r} - (-1)^{r-s} \gamma_{s,r}| \leq \sum_{u \in A_{s,r}} \left| \mu_u - \prod_{i=2}^{|\text{dom}(u)|} \left( \frac{u(i)}{u(i+1)} \right) \right| \leq \frac{18r^3 r!}{n}.
\]

By (8.5) and (9.32), we conclude that (9.11) is satisfied, as desired.

### 10. Preparatory lemmas

Our goal in this section is to present some preparatory lemmas which are needed for the proof of Theorem 1.4 but are not related to the main argument.

The first lemma is a simple comparison result of gaussians.

**Lemma 10.1.** Let \(\mu_1, \mu_2 \in \mathbb{R}\), and let \(\sigma_1, \sigma_2 \geq 0\) such that \(\max\{\sigma_1, \sigma_2\} > 0\). Then,

\[(10.1) \quad d_K(\mathcal{N}(\mu_1, \sigma_1^2), \mathcal{N}(\mu_2, \sigma_2^2)) \leq \frac{1}{\max\{\sigma_1, \sigma_2\}} \left( |\mu_1 - \mu_2| + |\sigma_1 - \sigma_2| \right).
\]

**Proof.** Clearly, we may assume that \(\sigma_1 \geq \sigma_2\). (In particular, we have \(\sigma_1 > 0\).) Set \(\sigma := \sigma_2/\sigma_1\) and notice that \(0 \leq \sigma \leq 1\). We first observe that

\[(10.2) \quad d_K(\mathcal{N}(\mu_1, \sigma_1^2), \mathcal{N}(\mu_2, \sigma_2^2)) = d_K(\mathcal{N}(0,1), \mathcal{N}\left(\frac{\mu_2 - \mu_1}{\sigma_1}, \sigma^2\right)) \leq d_K(\mathcal{N}(0,1), \mathcal{N}\left(\frac{\mu_2 - \mu_1}{\sigma_1}, 1\right)) + d_K(\mathcal{N}(0,1), \mathcal{N}(0, \sigma^2)).
\]
The first distance in the right-hand-side of (10.2) can be estimated by

\[
d_K\left(\mathcal{N}(0,1),\mathcal{N}\left(\frac{\mu_2 - \mu_1}{\sigma_1}, 1\right)\right) = \sup_{x \in \mathbb{R}}\left|\mathbb{P}(\mathcal{N}(0,1) \leq x) - \mathbb{P}\left(\mathcal{N}(0,1) \leq x - \frac{\mu_2 - \mu_1}{\sigma_1}\right)\right|
\leq \mathbb{P}\left(-\frac{|\mu_1 - \mu_2|}{2\sigma_1} \leq \mathcal{N}(0,1) \leq \frac{|\mu_1 - \mu_2|}{2\sigma_1}\right) \leq \frac{|\mu_1 - \mu_2|}{\sigma_1}.
\]

Next, notice that if \(\sigma \leq 1/2\), then (10.1) follows immediately by (10.2) and (10.3). Thus, in order to estimate the second distance in the right-hand-side of (10.2), we may assume that \(\sigma > 1/2\). Then, upon recalling that \(\sigma \leq 1\), we obtain that

\[
d_K(\mathcal{N}(0,1),\mathcal{N}(0,\sigma^2)) = \sup_{x \in \mathbb{R}}\left|\mathbb{P}(\mathcal{N}(0,1) \leq x) - \mathbb{P}(\mathcal{N}(0,\sigma^2) \leq x)\right|
= \sup_{x > 0} \left[\mathbb{P}(\mathcal{N}(0,1) \leq x \sigma) - \mathbb{P}(\mathcal{N}(0,1) \leq x)\right]
= \sup_{x > 0} \frac{1}{\sqrt{2\pi}} \int_{x}^{x + (\frac{1}{\sqrt{2}} - 1)x} e^{-\frac{t^2}{2}} dt \leq \sup_{x > 0} \frac{2|\sigma - 1|}{\sqrt{2\pi}} xe^{-\frac{x^2}{2}} \leq |\sigma - 1| = \frac{|\sigma_1 - \sigma_2|}{\sigma_1}.
\]

By (10.2)–(10.4), we conclude that (10.1) is satisfied. 

The second lemma is a variant of Lemma 10.1 and refers to mixtures of gaussians.

**Lemma 10.2.** Let \(\sigma > 0\), and let \(Y\) be a real-valued random variable defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with zero mean and finite second moment. Let \(Y_\omega\) denote a realization of \(Y\), and let \(M\) be the mixture with respect to \((\Omega, \mathcal{F}, \mathbb{P})\) of \(\mathcal{N}(Y_\omega, \sigma^2)\), that is, for every \(x \in \mathbb{R}\) we have

\[
\mathbb{P}(M \leq x) = \mathbb{E}[\mathbb{P}(\mathcal{N}(Y, \sigma^2) \leq x)]
\]

where \(\mathbb{E}\) denotes expectation with respect to \(\mathbb{P}\). Then we have

\[
d_K(M, \mathcal{N}(0, \sigma^2)) \leq \frac{\mathbb{E}[Y^2]}{2\sqrt{2\pi e \sigma^2}}.
\]

**Proof.** Let \(x \in \mathbb{R}\) be arbitrary. Then, setting \(P = [Y \geq 0]\) and \(N = [Y < 0]\), we have

\[
\mathbb{P}(M \leq x) - \mathbb{P}(\mathcal{N}(0,\sigma^2) \leq x) = |\mathbb{E}[\mathbb{P}(\mathcal{N}(Y, \sigma^2) \leq x)] - \mathbb{P}(\mathcal{N}(0,\sigma^2) \leq x)|
= |\mathbb{E}[\mathbb{P}(\mathcal{N}(0,\sigma^2) \leq x - Y)] - \mathbb{P}(\mathcal{N}(0,\sigma^2) \leq x)|
= |\mathbb{E}[1_P(x - Y < \mathcal{N}(0,\sigma^2) \leq x) - 1_N(x < \mathcal{N}(0,\sigma^2) \leq x - Y)]|.
\]

Define \(F_{\sigma,x}: \mathbb{R} \to \mathbb{R}\) by

\[
F_{\sigma,x}(s) := \frac{1}{\sqrt{2\pi} \sigma} \int_{x-s}^{x-s} e^{-\frac{t^2}{2\sigma^2}} dt
\]

with the usual convention that \(\int_{a}^{b} g(t) dt = -\int_{b}^{a} g(t) dt\) if \(a > b\). Using Taylor approximation, it is easy to see that for every \(s \in \mathbb{R}\) we have

\[
\left|F_{\sigma,x}(s) - \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{s^2}{2\sigma^2}} s\right| \leq \frac{s^2}{2\sqrt{2\pi e \sigma^2}}.
\]
Moreover, by (10.7), (10.8) and the fact that \( \mathbb{E}[Y] = 0 \),
\begin{align}
\left| \mathbb{P}(M \leq x) - \mathbb{P}(N(0, \sigma^2) \leq x) \right| &= \left| \mathbb{E}[1_P F_{\sigma,x}(Y) + 1_N F_{\sigma,x}(Y)] \right| \\
&= \left| \mathbb{E}[F_{\sigma,x}(Y)] \right| \overset{(10.9)}{\leq} \frac{\mathbb{E}[Y^2]}{2 \sqrt{2 \pi} \sigma^2}.
\end{align}

Remark 10.3. It is easy to see that if the random variable \( Y \) in Lemma 10.2 does not have zero mean, then we have \( d_K(M, N(0, \sigma^2)) \leq \frac{\mathbb{E}[Y^2]}{\sigma} + \frac{\text{Var}(Y)}{2 \sqrt{2 \pi} \sigma^2} \) where \( M \) denotes the mixture associated with \( Y \) via (10.6).

The third lemma is a coherence property of Hoeffding’s operators, and it concerns doubly-indexed tensors whose entries are products of entries of tensors.

Lemma 10.4. Let \( n \geq s \) be positive integers, let \( \mathbf{a}, \mathbf{b} : [n]^s \to \mathbb{R} \) be tensors, and define \( \zeta : [n]^s \times [n]^s \to \mathbb{R} \) by setting \( \zeta(i, p) := \mathbf{a}(i) \cdot \mathbf{b}(p) \). Then for every \( i, p \in [n]^s \),
\begin{equation}
\mathcal{H}[\zeta](i, p) = \mathcal{H}[\mathbf{a}](i) \cdot \mathcal{H}[\mathbf{b}](p)
\end{equation}
where \( \mathcal{H}[\zeta] \) is the Hoeffding tensor associated with \( \zeta \) via (9.4), and \( \mathcal{H}[\mathbf{a}] \) and \( \mathcal{H}[\mathbf{b}] \) are the Hoeffding tensors associated with \( \mathbf{a} \) and \( \mathbf{b} \), respectively, via (9.2).

Proof. By (9.1), (9.3) and the definition of \( \zeta \), for every \( F, G \subseteq [s] \) and \( i, p \in [n]^s \) we have
\begin{equation}
\mathcal{A}_{F, G}[\zeta](i \upharpoonright F, p \upharpoonright G) = \mathcal{A}_{F}[\mathbf{a}](i \upharpoonright F) \cdot \mathcal{A}_{G}[\mathbf{b}](p \upharpoonright G).
\end{equation}
Therefore,
\begin{align}
\mathcal{H}[\zeta](i, p) &= \sum_{F, G \subseteq [s]} (-1)^{|F| + |G|} \mathcal{A}_{F, G}[\zeta](i \upharpoonright F, p \upharpoonright G) \\
&= \sum_{F, G \subseteq [s]} (-1)^{s-|F|}(-1)^{s-|G|} \mathcal{A}_F[\mathbf{a}](i \upharpoonright F) \cdot \mathcal{A}_G[\mathbf{b}](p \upharpoonright G) \\
&= \left( \sum_{F \subseteq [s]} (-1)^{s-|F|} \mathcal{A}_F[\mathbf{a}](i \upharpoonright F) \right) \cdot \left( \sum_{G \subseteq [s]} (-1)^{s-|G|} \mathcal{A}_G[\mathbf{b}](p \upharpoonright G) \right) = \mathcal{H}[\mathbf{a}](i) \cdot \mathcal{H}[\mathbf{b}](p). \quad \Box
\end{align}

The fourth, and last, result in this section is a representation of the parameters \( \Sigma_s \) defined in (1.9). Specifically, let \( \mathbf{X} \) be a random tensor which satisfies (A1) and (A2), and let \((\Omega, \mathcal{F}, \mathbb{P})\) denote the underlying probability space. Every realization \( \mathbf{X}_\omega \) of \( \mathbf{X} \) is a (deterministic) real tensor \( \mathbf{X}_\omega : [n]^s \to \mathbb{R} \) and, consequently, we may associate with \( \mathbf{X}_\omega \) the average and Hoeffding tensors defined in Subsection 9.1. The following lemma relates their statistical behavior with the parameters \( \Sigma_s \).

Lemma 10.5. Let \( \mathbf{X} \) which satisfies (A1) and (A2). Then for every \( s \in \{0, \ldots, d\} \),
\begin{equation}
\left| \sum_{p \in [n]^s} \mathbb{E} \left[ \mathcal{A}_{[s]}[\mathbf{X}_\omega](p)^2 \right] - n^s \Sigma_s \right| \leq n^s \frac{8d^2 2^d}{n}
\end{equation}
where \( \Sigma_s \) is as in (1.9). In particular, we have \( \Sigma_s \geq -\frac{8d^2 2^d}{n} \).
Proof. Assume that \( s \in [d] \), and notice that
\[
\begin{aligned}
\sum_{p \in [n]^*} E[H[A_s][X_\omega](p)^2] &\overset{(9.1),(9.2)}{=} E\left[ \sum_{p \in [n]^*} \sum_{F,G \subseteq [s]} (-1)^{|F|+|G|} A_F[X_\omega](p \mid F) A_G[X_\omega](p \mid G) \right] \\
&= \sum_{F,G \subseteq [s]} (-1)^{|F|+|G|} E\left[ n^s \sum_{p \in [n]^{|F\cap G|}} A_F[X_\omega](p \mid F) A_G[X_\omega](p \mid G) \right] \\
&= \sum_{F,G \subseteq [s]} (-1)^{|F|+|G|} E\left[ n^s \sum_{p \in [n]^{|F\cap G|}} A_{F\cap G}[X_\omega](p) A_{F\cap G}[X_\omega](p) \right] \\
&= n^s \sum_{F,G \subseteq [s]} (-1)^{|F|+|G|} \frac{1}{n^{|F\cap G|}} \sum_{p \in [n]^{|F\cap G|}} E[A_{F\cap G}[X_\omega](p)^2].
\end{aligned}
\]
Thus, after observing that for every \( H \subseteq [s] \) we have
\[
(-1)^{|s|-|H|} = \sum_{K,L \subseteq [s]\setminus H} (-1)^{|H|} (-1)^{|L|},
\]
by (10.15) and (42) we obtain that
\[
\sum_{p \in [n]^*} E[H[A_s][X_\omega](p)^2] = n^s(-1)^s \sum_{h=0}^s \binom{s}{h} \frac{(-1)^h}{n^h} \sum_{p \in [n]^h} E[A_{[h]}[X_\omega](p)^2]
\]
for every \( s \in [d] \). On the other hand, if \( "s = 0" \), then (10.17) is trivially valid. Hence, in what follows, we have at our disposal identity (10.17) for every \( s \in \{0, 1, \ldots, d\} \).

We proceed to estimate \( E[A_{[h]}[X_\omega](p)^2] \). By (42), this quantity is 0 if \( p \notin [n]^h \).

So, let \( p \in [n]^h \) for some \( h \in \{0, 1, \ldots, s\} \) and notice that
\[
E[A_{[h]}[X_\omega](p)^2] = \frac{1}{n^{2(d-h)}} \sum_{q_1,q_2 \in [n]_{[h]}} E[X_{q_1} X_{q_2}]
\]

\[
= \frac{1}{n^{2(d-h)}} \sum_{t_0}^{d-h} \frac{(n-h)!}{(n-2d+h+t)!} \cdot \frac{1}{t!} \cdot \left( \frac{(d-h)!}{(d-h-t)!} \right)^2 \delta_{h+t}
\]

where we have used (42) and the definition of \( \delta_0, \delta_1, \ldots, \delta_d \) in Paragraph 1.2.1. We will need the following elementary fact.

**Fact 10.6.** Let \( n, d \) be positive integers with \( n \geq 2d \). Then for every \( k, \ell \in \{0, \ldots, d\} \),
\[
\left| \frac{1}{n^\ell} \frac{(n-k)!}{(n-k-\ell)!} - 1 \right| \leq \frac{(k+\ell)^2}{n}. 
\]

By (41) and the Cauchy–Schwarz inequality, we see that \( \delta_t \leq 1 \) for every \( t \in \{0, \ldots, d\} \).

Using this observation, (10.18) and Fact 10.6, we obtain that
\[
\left| E[A_{[h]}[X_\omega](p)^2] - \delta_h \right| \leq \frac{4d^2}{n} + \sum_{t=1}^{d-h} \frac{1}{t!} \cdot \frac{(d-h)!}{(d-h-t)!}^2 \leq \frac{7d^2}{n}.
\]
This, in turn, implies that for every $h \in \{0, \ldots, d\}$ we have

$$
1 \leq \frac{1}{n^h} \sum_{p \in [n]^h} \mathbb{E} [\mathcal{A}_{[n]} [X_\omega](p)^2] - \delta_h \leq \frac{8d^2}{n}. 
$$

Plugging inequality (10.21) in the right-hand-side of (10.17), we conclude that

$$
\sum_{p \in [n]^s} \mathbb{E} \left[ \mathcal{H} \left( \mathcal{A}_{[n]} [X_\omega](p)^2 \right) - n^s \Sigma_s \right] \leq n^s \frac{8d^2 2^d}{n}.
$$

for every $s \in \{0, 1, \ldots, d\}$. The proof of Lemma 10.5 is thus completed.

11. Proof of Theorem 1.4

11.1. Overview of the argument. As we have noted in Section 2, Theorem 1.4 follows by expressing the distribution of $\langle \theta, X \rangle$ as a mixture of distributions of simpler random variables, and then integrating the Kolmogorov distance of each component of the mixture to an appropriately chosen gaussian. Of course, in order to implement this strategy, the selection of the mixture is quite important. Usually, in arguments of this sort, the components satisfy some independence-type property; however, these are not available in our context since we only have at our disposal the fact that the distribution of the random tensor $X$ is invariant under certain symmetries.

The key observation in the proof of Theorem 1.4 is that the distribution of $\langle \theta, X \rangle$ can be expressed as a mixture of $Z$-statistics which can then be handled effectively by Theorem 2.2. In retrospect, this maneuver is very natural, so much so that sometimes we tend to view Theorems 1.4 and 2.2 as a single result.

11.2. A preliminary estimate. Set $K := \mathbb{E} \left[ |X_{(1, \ldots, d)}|^3 \right]$ and recall that $\kappa = 20d^3 18^d (2d)!$ and $B = \|\frac{1}{n^d} \sum_{i \in [n]^d} X_i \|_{L_2^2}$; in what follows, we will use these constants without further notice. The following proposition is the main estimate of the proof of Theorem 1.4.

**Proposition 11.1.** Let the notation and assumptions be as in Theorem 1.4. Assume that $n \geq \kappa$ and $\|\theta\|_0 < \sqrt{n}$, and set

$$
\sigma_1^2 := d^2 \left( 1 - \frac{\|\theta\|_0^2}{n} \right) \delta_1 > 0
$$

Then

$$
d_K (\langle \theta, X \rangle, \mathcal{N}(0, \sigma_1^2)) \leq E'_1 + E'_2 + E'_3.
$$
where, setting \( \theta_+ := \sum_{i \in [n]^d} \theta_i \), we have

\[
E_1' := 5 \text{osc}(X)^{1-\alpha} + 5B^{1-\alpha} + 5\left(\frac{K}{n}\right)^{1-\alpha} + \frac{\|\theta\|_0^2 B}{\sqrt{2\pi e \sigma_1^2}}
\]

\[
E_2' := 2^{22}C_1d^3 \frac{E[|X(1, \ldots, d)|^3]}{\sigma_1^3} \left( \sum_{i=1}^n \left( \sum_{i \in [n]^d \atop i(1)=j} \theta_i \right)^3 \right)
\]

\[
E_3' := 2\sqrt{2}C_d \frac{d}{\sigma_1} \sum_{s=2}^d \binom{d}{s} \sqrt{\sum_{s=2}^d \binom{s}{d} \|H[S_s[\theta]]\|_s}.
\]

Here, \( H[S_s[\theta]] \) denotes the Hoeffding tensor associated with the tensor \( S_s[\theta] \) for every \( s \in \{2, \ldots, d\} \). (See Subsection 9.1.) Moreover, the constant \( C_1 \geq 1 \) is as in (2.4), and \( C_d = 5d^{2e^2(2d)!} \) is as in Theorem 2.2.

**Proof.** Let \( Z \) denote the random variable \( \sum_{i \in [n]^d} \theta(i) X(\pi \circ i) \) where \( \pi \) is a random permutation, independent of \( X \), which is uniformly distributed on \( S_n \). (Here, \( X(\pi \circ i) \) denotes the \((\pi \circ i)\)-entry of \( X \).) Since \( X \) is exchangeable, we see that \( \langle \theta, X \rangle \) and \( Z \) have the same distribution. Consequently, we have

\[
d_K \left( \langle \theta, X \rangle, N(0, \sigma_1^2) \right) = d_K \left( Z, N(0, \sigma_1^2) \right).
\]

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) denote the underlying probability space on which the random tensor \( X \) is defined; as we have already pointed out, every realization \( X_\omega \) of \( X \) is a deterministic real tensor. It follows from its definition, that \( Z \) is a mixture of the \( Z \)-statistics \( Z_\omega \) associated with the tensors \( \zeta_\omega : [n]^d \times [n]^d \rightarrow \mathbb{R} \) defined by

\[
\zeta_\omega(i, p) := \theta(i) \cdot X_\omega(p).
\]

Notice that \( \zeta_\omega \) satisfies (A5) and (A6). Moreover, for every \( s \in [d] \) and every \( i, p \in [n]^s \),

\[
A[s][\zeta_\omega](i, p) = A[s][\theta](i) \cdot A[s][X_\omega](p)
\]

and so, by Lemma 10.4,

\[
H[A[s][\zeta_\omega]](i, p) = H[A[s][\theta]](i) \cdot H[A[s][X_\omega]](p).
\]

Therefore, setting

\[
X_\omega := \frac{1}{n^d} \sum_{i \in [n]^d} X_\omega(i),
\]
by Proposition 9.4 and Lemma 9.3, we obtain that

\[ Z_\omega = \sum_{s=1}^{d} n^{-s} \left( \frac{d}{s} \right) \left( 1 + \frac{\lambda_s}{n} \right) \sum_{i \in [n]} \mathcal{H}[A_{[s]}(\Theta)](i) \cdot \mathcal{H}[A_{[s]}(X_\omega)](\pi \circ i) + \left( 1 + \frac{\lambda_0}{n} \right) \theta \cdot X_\omega \]

(11.11)

where \( |\lambda_s| \leq d^3 18^d d! \leq \kappa/20 \) for every \( s \in \{0, \ldots, d\} \); in particular, since \( n \geq \kappa \),

\[ \frac{19}{20} \leq 1 + \frac{\lambda_s}{n} \leq \frac{21}{20} \quad \text{for every } s \in \{0, 1, \ldots, d\}. \]

Set \( Y := (1 + \frac{\lambda_0}{n}) \theta \cdot X_\omega \), and let \( M \) denote the mixture in (10.5) associated with \( Y \) and \( \sigma = \sigma_Y \). By (11.6), Lemma 10.2 and the triangle inequality for the Kolmogorov distance, we have

\[ d_K((\Theta, X), \mathcal{N}(0, \sigma_Y^2)) \leq d_K(Z, M) + d_K(M, \mathcal{N}(0, \sigma_Y^2)) \leq \]

\[ \leq E[d_K(Z, \mathcal{N}(Y, \sigma_Y^2))] + \frac{1}{2\sqrt{2\pi e \sigma_Y^2}} E[Y^2] \leq \frac{11d}{20} E[d_K(Z, Y, \mathcal{N}(0, \sigma_Y^2))] + \frac{\|\Theta\|_1^2 B}{\sqrt{2\pi e \sigma_Y^2}}. \]

Let \( \sigma_Y^2 \) denote the variance of the one dimensional component,

\[ d \left( 1 + \frac{\lambda_1}{n} \right) \sum_{i \in [n]} \mathcal{H}[S_{[1]}(\Theta)](i) \cdot \mathcal{H}[A_{[1]}(X_\omega)](\pi \circ i), \]

of \( Z_\omega - Y_\omega \). Since \( \|\Theta\|_1 = 1 \), we have

\[ \sigma_\omega^2 = d^2 \left( 1 + \frac{\lambda_1}{n} \right)^2 \frac{1}{n-1} \left( \sum_{i \in [n]} \mathcal{H}[S_{[1]}(\Theta)](i)^2 \left( \sum_{p \in [n]} \mathcal{H}[A_{[1]}(X_\omega)](p)^2 \right)^2 \right) \]

\[ = d^2 \left( 1 + \frac{\lambda_1}{n} \right)^2 \frac{1}{n-1} \left( 1 - \frac{\Theta_Y^2}{n} \right) \left( \sum_{p \in [n]} \mathcal{H}[A_{[1]}(X_\omega)](p)^2 \right)^2 \]

\[ = d^2 \left( 1 + \frac{\lambda_1}{n} \right)^2 \frac{1}{n} \left( 1 - \frac{\Theta_Y^2}{n} \right) \left( \sum_{p \in [n]} \mathcal{H}[A_{[1]}(X_\omega)](p)^2 \right)^2 + d^2 \left( 1 + \frac{\lambda_1}{n} \right)^2 \frac{1}{n(n-1)} \left( 1 - \frac{\Theta_Y^2}{n} \right) \left( \sum_{p \in [n]} \mathcal{H}[A_{[1]}(X_\omega)](p)^2 \right)^2 \]

\[ = d^2 \left( 1 - \frac{\Theta_Y^2}{n} \right) \left( \frac{1}{n} \sum_{p \in [n]} A_{[1]}(X_\omega)(p)^2 \right)^2 + d^2 \left( 1 - \frac{\Theta_Y^2}{n} \right) \frac{1}{n} E_\omega \]

where

\[ E_\omega := \left( \frac{1}{n} \sum_{p \in [n]} \mathcal{H}[A_{[1]}(X_\omega)](p)^2 \right) \left[ \frac{n}{n-1} \left( 1 + \frac{\lambda_1}{n} \right)^2 + 2\lambda_1 + \frac{\lambda_1^2}{n} \right] - n X_\omega^2. \]
Claim 11.2. We have

\begin{equation}
\mathbb{E}[E_\omega] \leq nB + \kappa. \tag{11.16}
\end{equation}

Proof of Claim 11.2. By Lemma 10.5 and (1.9),

\begin{equation}
\frac{1}{n} \sum_{p \in [n]} \mathbb{E}[\mathcal{H}[\mathcal{A}_1[\mathcal{X}_\omega]](p)^2] \leq \delta_1 - \delta_0 + \frac{8d^2 2^d}{n} \tag{11.17} \leq 2 + \frac{8d^2 2^d}{n}.
\end{equation}

Since \( n \geq \kappa \) and \( |\lambda_1| \leq \kappa/20 \), (11.16) follows from (11.17), the definition of \( \mathcal{X}_\omega \) in (11.10) and the choice of the constant \( B \).

By Claim 11.2, (11.14), the definition of oscillation in (1.10) and the choice of \( \sigma_1^2 \) in (11.1), we obtain that

\begin{equation}
\mathbb{E}[|\sigma_\omega^2 - \sigma_1^2|] \leq d^2 \left( 1 - \frac{\|\theta\|_0^2}{n} \right) \left[ \text{osc}(\mathcal{X}) + B + \frac{\kappa}{n} \right]. \tag{11.18}
\end{equation}

Define the event

\begin{equation}
\Gamma := \left\{ |\sigma_\omega^2 - \sigma_1^2| \leq d^2 \left( 1 - \frac{\|\theta\|_0^2}{n} \right) \max \left\{ \text{osc}(\mathcal{X})^\alpha, B^\alpha, \left( \frac{\kappa}{n} \right)^\alpha \right\} \right\}. \tag{11.19}
\end{equation}

By Markov’s inequality and (11.18), we see that

\begin{equation}
P(\Gamma^c) \leq 4\text{osc}(\mathcal{X})^{1-\alpha} + 4B^{1-\alpha} + 4\left( \frac{\kappa}{n} \right)^{1-\alpha}. \tag{11.20}
\end{equation}

Moreover, by (1.11) and (11.1), for every \( \omega \in \Gamma \) we have

\begin{equation}
\sigma_\omega \geq \frac{\sigma_1}{2}. \tag{11.21}
\end{equation}

On the other hand, setting for every \( \omega \in \Gamma \) and every \( s \in \{2, \ldots, d\} \)

\begin{align}
\Lambda_\omega &:= d^3 \left( 1 + \frac{\lambda_1}{n} \right)^3 \left( \sum_{i \in [n]} |\mathcal{H}[\mathcal{S}_i[\theta]](i)|^3 \right) \left( \sum_{p \in [n]} |\mathcal{H}[\mathcal{A}_1[\mathcal{X}_\omega]](p)|^3 \right) \tag{11.22} \\
\beta_{s,\omega} &:= \left( \frac{d}{s} \right)^2 \left( 1 + \frac{\lambda_1}{n} \right)^2 \left( \sum_{i \in [n]^s} |\mathcal{H}[\mathcal{S}_s[\theta]](i)^2 \right) \left( \sum_{p \in [n]^s} |\mathcal{H}[\mathcal{A}_s[\mathcal{X}_\omega]](p)^2 \right), \tag{11.23}
\end{align}

by (11.11), the choice of \( Y \), the fact that \( n \geq \kappa \geq 4d^2 \), Theorem 2.2 and (11.21), we have

\begin{equation}
d_K(Z_\omega - Y_\omega, \mathcal{N}(0, \sigma_\omega^2)) \leq \frac{2^{21}C_1\Lambda_\omega}{\sigma_1^2 n} + \frac{2Cd}{\sigma_1} \sum_{s=2}^d \sqrt{\frac{\beta_{s,\omega}}{n^s}}, \tag{11.24}
\end{equation}

where \( C_1 \geq 1 \) is as in (2.4) and \( C_d = 5d^2 e^d(2d)! \).

Claim 11.3. We have

\begin{equation}
\mathbb{E}[1_{\Gamma \Lambda_\omega}] \leq 2d^3 K n \left( \sum_{i=1}^n \left( \sum_{i(1)=j} \theta_i \right) - \frac{\theta_+}{n} \right)^3. \tag{11.25}
\end{equation}
Proof of Claim 11.3. By the definition of $\Lambda_\omega$ in (11.22) and using (9.1), (9.2), (A1) and the triangle inequality, we obtain that

(11.26) \[ E[1_\Gamma \Lambda_\omega] \leq d^3 \left(1 + \frac{\lambda_1}{n}\right)^3 K n \left( \sum_{i=1}^n \left| \left( \sum_{i \in [n] \atop i(1)=j} \theta_i \right) - \frac{\theta_+}{n} \right|^3 \right). \]

The claim follows from this estimate and (11.12). \hfill \Box

Claim 11.4. For every $s \in \{2, \ldots, d\}$ we have

(11.27) \[ E[1_\Gamma \beta_{s,\omega}] \leq 2n^s \left( \frac{d}{s} \right)^2 \left( \sum_{i \in [n]^s} \|\mathcal{H} [S_{[s]} | \theta] \|^2 \right). \]

Proof of Claim 11.4. We first observe that, by (11.12) and Lemma 10.5, we have

(11.28) \[ \left( \frac{d}{s} \right)^2 \left( 1 + \frac{\lambda_n}{n} \right)^2 E \left[ \left( \sum_{i \in [n]^s} \mathcal{H} [A_{[s]} | X_\omega] \right) (p^2) \right] \leq 2n^s \left( \frac{d}{s} \right)^2 \left( \sum_{i \in [n]^s} \frac{8d^2}{n} \right). \]

Moreover, by the definition of the seminorm $\|\cdot\|_s$ in (1.5),

(11.29) \[ \sum_{i \in [n]^s} \mathcal{H} [S_{[s]} | \theta] (i)^2 = \|\mathcal{H} [S_{[s]} | \theta] \|^2. \]

Inequality (11.27) follows by combining (11.28) and (11.29). \hfill \Box

We are ready for the last step of the argument. By (11.13), we have

(11.30) \[ d_K ((\theta, X), \mathcal{N}(0, \sigma_\theta^2)) \leq E[1_\Gamma d_K (Z - Y, \mathcal{N}(0, \sigma_\theta^2))] + \frac{\|\theta\|^2 B}{\sqrt{2\pi e \sigma_\theta^2}}. \]

Moreover, by the triangle inequality for the Kolmogorov distance and Lemma 10.1,

(11.31) \[ E[1_\Gamma d_K (Z - Y, \mathcal{N}(0, \sigma_\theta^2))] \leq E[1_\Gamma d_K (Z - Y, \mathcal{N}(0, \sigma_\omega^2))] + E\left[1_\Gamma \left| \sigma_1 - \sigma_\omega \right| \max_{\sigma_1, \sigma_\omega} \right] \]

\[ \leq E[1_\Gamma d_K (Z - Y, \mathcal{N}(0, \sigma_\omega^2))] + \frac{1}{\sigma_1^2} E[\left| \sigma_1^2 - \sigma_\omega^2 \right|] \]

\[ \overset{(11.1), (11.18)}{\leq} E[1_\Gamma d_K (Z - Y, \mathcal{N}(0, \sigma_\omega^2))] + \frac{1}{\sigma_1} \left( \operatorname{osc}(X) + B + \frac{\kappa}{n} \right) \]

\[ \overset{(11.11)}{\leq} E[1_\Gamma d_K (Z - Y, \mathcal{N}(0, \sigma_\omega^2))] + \operatorname{osc}(X)^{1-\alpha} + B^{1-\alpha} + \left( \frac{\kappa}{n} \right)^{1-\alpha}. \]

By (11.24), Claims 11.2–11.4 and the Cauchy–Schwarz inequality,

(11.32) \[ E[1_\Gamma d_K (Z - Y, \mathcal{N}(0, \sigma_\omega^2))] \leq \frac{2^{22} C_1 d^3 K}{\sigma_1^3} \left( \sum_{i=1}^n \left| \left( \sum_{i \in [n]^s \atop i(1)=j} \theta_i \right) - \frac{\theta_+}{n} \right|^3 \right) + \]

\[ + \frac{2 \sqrt{2} C_2 d}{\sigma_1} \sum_{s=2}^d \left( \frac{d}{s} \right)^3 \sum_{i=1}^n \left| \left( \sum_{i \in [n]^s \atop i(1)=j} \theta_i \right) - \frac{\theta_+}{n} \right|^3 + \]

\[ + \frac{2 \sqrt{2} C_2 d}{\sigma_1} \sum_{s=2}^d \left( \frac{d}{s} \right)^3 \sum_{i=1}^n \left| \left( \sum_{i \in [n]^s \atop i(1)=j} \theta_i \right) - \frac{\theta_+}{n} \right|^3. \]

The desired estimate (11.2) follows by combining (11.30), (11.20), (11.31) and (11.32), and invoking the definition of $E_1, E_2, E_3$. The proof of Proposition 11.1 is completed. \hfill \Box
11.3. Completion of the proof. We are finally in a position to complete the proof of Theorem 1.4. To this end, we first observe that we may assume that \( n \geq \kappa \) and \( \|\theta\|_0 \leq \sqrt{n}/2 \). (Otherwise, (1.12) is straightforward.) By (11.1), the last assumption implies, in particular, that

\[
\sigma_1^2 \geq \frac{3d^2\delta_1}{4}.
\]

Next observe that, by the triangle inequality for the Kolmogorov distance, Lemma 10.1, (11.33) and Proposition 1.2, we have

\[
d_K\left(\mathcal{N}(0, \sigma_1^2), \mathcal{N}(0, \sigma^2)\right) \leq d_K\left(\mathcal{N}(0, d^2\delta_1), \mathcal{N}(0, \sigma^2)\right) + \frac{\|\theta\|^2_0}{n}
\]

\[
\leq \left|\frac{\delta_0}{d^2\delta_1} \left(\|\theta\|^2_0 - 1\right)\right| + \frac{1}{d^2\delta_1} \sum_{s=2}^{d} \binom{d}{s}^2 s! |\Sigma_s| \cdot \|\theta\|^2_s + \frac{\|\theta\|^2_0}{n}.
\]

(Recall that \( \sigma^2 \) denotes the variance of \( \langle \theta, X \rangle \).) On the other hand, by \((A1), (A2), (1.7),\) Fact 10.6 and the definition of \( B \), we have

\[
B = \frac{1}{n^{2d}} \sum_{i,j \in [n]^d} E[X_iX_j] \leq |\delta_0| + \frac{4d^2}{n}.
\]

Moreover, by Hölder’s inequality,

\[
\sum_{j=1}^{n} \left|\left(\sum_{i \in [n]^d} \theta_i \right) - \frac{\theta}{n}\right|^3 \leq 2^4 \sum_{j=1}^{n} \left|\sum_{i \in [n]^d} \theta_i \right|^3
\]

Finally, by the Cauchy–Schwarz inequality, for every \( s \in \{2, \ldots, d\} \) we have

\[
\|\mathcal{H}[S_s]_o[\theta]\|_s^2 = \sum_{i \in [n]^s} \mathcal{H}[S_s[i][\theta]](i)^2 = n^{2d-2s} \sum_{i \in [n]^s} \mathcal{H}[A_s[i][\theta]](i)^2
\]

\[
\overset{(9.2)}{=} n^{2d-2s} \sum_{i \in [n]^s} \left(\sum_{F \subseteq [s]} (-1)^{s-|F|} A_F[i][\theta](i \upharpoonright F)\right)^2
\]

\[
\leq n^{2d-2s} 2^s \sum_{F \subseteq [s]} \sum_{i \in [n]^s} A_F[i][\theta](i \upharpoonright F)^2
\]

\[
\leq n^{2d-2s} 2^s \sum_{F \subseteq [s]} \sum_{i \in [n]^s} \frac{1}{n^{s-|F|}} \sum_{i \in F \subseteq [s]} A_s[i][\theta](j)^2
\]

\[
= n^{2d-2s} 2^s \sum_{F \subseteq [s]} \sum_{j \in [n]^s} A_s[i][\theta](j)^2 \overset{(1.5)}{=} 2^s \times \|\theta\|^2_s \overset{(\text{Fact 1.3})}{\leq} 12s! \|\theta\|^2_s
\]

where in the last inequality we have used the fact that \( 2^s \leq 12s! \) for every integer \( s \geq 2 \).

Inequality (1.12) now follows from (11.2), (11.33)–(11.37), Fact 1.3 and the estimate \( C_1 \leq 451 \) obtained in [CF15]. The proof of Theorem 1.4 is completed.
12. Proofs of Applications

12.1. Exchangeable random vectors. We start by giving the proof of Corollary 3.1.

Proof of Corollary 3.1. First observe that we may assume that $\text{osc}(X) \leq 1$ and $n \geq \kappa_1$. (Otherwise, (3.2) is straightforward.) Therefore, by (3.1), the non-degeneracy condition (1.11) is satisfied for every $\alpha \in (0, 1)$. Taking into account this remark, the result follows by Theorem 1.4 albeit with a different absolute constant. The fact that in (3.2) we can select the constant $\kappa_1 = 4320$ follows after noticing that, for the one-dimensional case, in the proof of Proposition 11.1 it suffices to apply (2.4) with $C_1 = 451$. □

Remark 12.1. It is instructive to compare Corollary 3.1 with the classical work of Diaconis and Freedman [DF80] which estimates the total variation distance between the marginals of the law of an exchangeable random vector, and the corresponding marginals of a mixture of product measures. The results in [DF80] are very informative, but unfortunately they become quantitatively less useful for exchangeable random vectors whose entries are not finite-valued. On the other hand, by considering a statistically less rigid quantity, Corollary 3.1 provides estimates of equal quantitative strength for all permissible distributions.

We proceed to estimate the oscillation of exchangeable random vectors. We first treat the case of random vectors whose entries have finite fourth moment.

Fact 12.2. Let $X$ be an exchangeable random vector in $\mathbb{R}^n$ ($n \geq 2$) whose entries have zero mean, unit variance and finite fourth moment. Then we have

$$\text{osc}(X) \leq \sqrt{|E[X_1^2 X_2^2] - 1|} + \frac{E[X_1^4]^{1/2}}{\sqrt{n}}.$$ \hfill (12.1)

Proof. By the Cauchy–Schwarz inequality and our assumptions, we have

$$\text{osc}(X) \overset{(1.10)}{=} \left\| \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 1 \right\|_{L_1} \leq \left\| \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 1 \right\|_{L_2}$$

$$= \left( \frac{1}{n^2} \sum_{i,j=1}^{n} E[X_i^2 X_j^2] + 1 - \frac{2}{n} \sum_{i=1}^{n} E[X_i^4] \right)^{1/2}$$

$$= \left( \frac{E[X_1^4]}{n} + E[X_1^4 X_2^2] - 1 \right)^{1/2} \leq \sqrt{|E[X_1^2 X_2^2] - 1|} + \frac{E[X_1^4]^{1/2}}{\sqrt{n}}. \quad \Box$$

The following proposition supplements Fact 12.2, and it estimates the oscillation of exchangeable random vectors whose entries have finite third moment.

Proposition 12.3. Let $X$ be an exchangeable random vector in $\mathbb{R}^n$ ($n \geq 2$) whose entries have zero mean, unit variance and finite third moment. Then we have

$$\text{osc}(X) \leq \sqrt{|E[X_1^2 X_2^2] - 1|} + \frac{4E[X_1^3]}{\sqrt{n}}.$$ \hfill (12.3)
The necessity of the error term—the oscillation—is shown in the next example.

Let \( n \geq 2 \) be an arbitrary integer. Fix \( 0 < \varepsilon \leq 1/2 \), and let \( \mathbf{X} \) be a random vector in \( \mathbb{R}^n \) whose distribution satisfies

\[
\mathbb{P}(\mathbf{X} \in A) = \varepsilon \mathbb{P}(\mathbf{Z} \in A) + (1 - \varepsilon) \mathbb{P}\left(\frac{R}{\sqrt{1 - \varepsilon}} \in A\right)
\]
for every Borel subset $A$ of $\mathbb{R}^n$, where $Z$ is a random vector in $\mathbb{R}^n$ with constant zero entries and $R$ is a random vector in $\mathbb{R}^n$ with independent Rademacher entries. Notice that $X$ is exchangeable, isotropic and bounded; in particular, it satisfies (12.7). Moreover,

$$\text{osc}(X) = 2\varepsilon.$$ \hspace{1cm} (12.10)

Finally, for every positive integer $k \leq n$ we have

$$d_K\left(\frac{X_1 + \cdots + X_k}{\sqrt{k}}, \mathcal{N}(0, 1)\right) \geq \frac{1}{2} \mathbb{P}\left(\frac{X_1 + \cdots + X_k}{\sqrt{k}} = 0\right) \geq \frac{\varepsilon}{2} \geq \frac{\text{osc}(X)}{4}.$$ \hspace{1cm} (12.11)

The necessity of the error term $O(k/n)$ is the content of the following example.

**Example 12.5.** It is a slight modification of the results of Diaconis and Freedman in [DF80, Section 4]. (See also [DF87] for closely related results.) Specifically, let $n$ be a sufficiently large positive even integer, and let $\xi = (\xi_1, \ldots, \xi_n)$ be a random vector which is uniformly distributed on the set of all $x \in \{-1, 1\}^n$ whose coordinates sum-up to 0. Notice that $\xi$ is exchangeable. Using a non-asymptotic version of Stirling’s approximation—see [Ro55]—and elementary computations, it is easy to verify that for every integer $k \in \{16, \ldots, \frac{n}{4}\}$ and every integer $a \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ we have

$$\mathbb{P}(S_k = a) \geq \mathbb{P}(R_k = a) \cdot \left(1 + \frac{1}{32} \cdot \frac{k^2}{n}\right)$$ \hspace{1cm} (12.12)

where $S_k := \xi_1 + \cdots + \xi_k$ and $R_k := r_1 + \cdots + r_k$ with $r_1, \ldots, r_k$ independent Rademacher random variables. This estimate and the classical Berry–Esseen theorem in turn imply that for every\(^{17}\) $n \geq k \geq n^{2/3}$ we have

$$d_K\left(\frac{\xi_1 + \cdots + \xi_k}{\sqrt{k}}, \mathcal{N}(0, 1)\right) \geq \frac{k}{n};$$ \hspace{1cm} (12.13)

on the other hand, by anticoncentration considerations, if $k = O(n^{2/3})$, then we have

$$d_K\left(\frac{\xi_1 + \cdots + \xi_k}{\sqrt{k}}, \mathcal{N}(0, 1)\right) \geq \frac{1}{\sqrt{k}} \geq \frac{k}{n}.$$ \hspace{1cm} (12.14)

In other words, we have the desired lower bound in the regime $k = O(n)$. Notice, however, that the random vector $\xi$ is not isotropic, but this is not a serious problem and it can be easily fixed by appropriately perturbing $\xi$.

Specifically, let $A_0, A_1, \ldots, A_n$ be pairwise disjoint events which are independent of $\xi$ and satisfy $\mathbb{P}(A_0) = \frac{1}{n-1}$ and $\mathbb{P}(A_1) = \cdots = \mathbb{P}(A_n) = \frac{n-2}{n(n-1)}$. For every $i \in [n]$ set

$$E_i := 1_{A_0} - \frac{n-2}{n-1} 1_{A_i}, \quad X_i := \frac{1}{\sigma}(\xi_i + E_i)$$ \hspace{1cm} (12.15)

where $\sigma^2 := \mathbb{E}[\xi_1^2] = 1 + \frac{2}{n-2}$. Using the fact that $\mathbb{E}[^2] = -\frac{1}{n-1}$, it is straightforward to check that the random vector $X = (X_1, \ldots, X_n)$ is exchangeable, isotropic

\(^{17}\)Here, given two positive quantities $a$ and $b$, we write $a \geq b$ to denote the fact that $a \geq Cb$ for some positive universal constant $C$. 

and bounded—hence, it satisfies (12.7)—and, moreover,

\begin{equation}
osc(X) = O\left( \frac{1}{n} \right) \quad \text{and} \quad |E[X_1^2 X_2^2] - 1| = O\left( \frac{1}{n} \right).
\end{equation}

Next let \( k \leq n \) be a positive integer. Note that the random variable \( \frac{1}{\sqrt{k}}(E_1 + \cdots + E_k) \) is independent of \( \frac{1}{\sqrt{k}}(\xi_1 + \cdots + \xi_k) \) and it is highly concentrated around zero: it takes the value \( \sqrt{k} \) with probability \( \frac{1}{n-1} \), the value \( -\frac{1}{\sqrt{k}} \cdot \frac{n-2}{n-1} \) with probability \( \frac{k(n-2)}{n(n-1)} \) and the value 0 everywhere else. Taking into account these remarks and using (12.12) and the classical Berry–Esseen theorem, we obtain that for every \( n \gg k \gg \frac{n^2}{9} \),

\begin{equation}
d_K\left( \frac{X_1 + \cdots + X_k}{\sqrt{k}}, \mathcal{N}(0, 1) \right) \geq \frac{k}{n}.
\end{equation}

After observing that the analogue of (12.14) for the random vector \( X \) is also valid if \( k = O(n^{2/3}) \), we see that for every \( k = O(n) \),

\begin{equation}
d_K\left( \frac{X_1 + \cdots + X_k}{\sqrt{k}}, \mathcal{N}(0, 1) \right) \geq \frac{k}{n}.
\end{equation}

Finally, by replacing \( X \) with a subvector of appropriate length\(^{18}\) if necessary, we conclude that (12.18) holds true for every positive integer \( k \leq n \); in particular, by (12.8), (12.16) and (12.18), it follows that the Berry–Esseen bound

\begin{equation}
d_K\left( \frac{X_1 + \cdots + X_k}{\sqrt{k}}, \mathcal{N}(0, 1) \right) = O\left( \frac{1}{\sqrt{k}} \right)
\end{equation}

is satisfied only in the regime \( k = O(n^{2/3}) \).

12.2. High-dimensional examples. Our goal in this subsection is twofold: to estimate the oscillation of high-dimensional exchangeable random tensors, and to present related applications. We first deal with random tensors whose entries have finite fourth moment. Specifically, we have the following analogue of Fact 12.2. (The proof is left to the reader.)

**Fact 12.6.** Let \( X \) which satisfy (A1) and (A2). Assume that the entries of \( X \) have finite fourth moment. Then we have

\begin{equation}
\text{osc}(X) \leq \sqrt{p c(X)} + \frac{5d}{\sqrt{n}}\left( |\delta_1| + E[X_1^4 \cdots d] \right)^{1/2}.
\end{equation}

The following proposition estimates the oscillation of dissociated, exchangeable and symmetric random tensors, as well as the oscillation of their mixtures.

**Proposition 12.7.** Let \( n, d \) be positive integers with \( n \geq 4d \), and let \( X = \langle X_i : i \in [n]^d \rangle \) be an exchangeable and symmetric random tensor whose entries have finite third moment.

1. If \( X \) is dissociated, then

\begin{equation}
\text{osc}(X) \leq \frac{16d \left( E[|X_1 \cdots d^3]| + 1 \right)}{\sqrt{n}}.
\end{equation}

\(^{18}\)More precisely, of length \( cn \) where \( c > 0 \) is a sufficiently small constant.
(2) If $X$ is a mixture of dissociated, exchangeable and symmetric random tensors, then
\[
\text{osc}(X) \leq \sqrt{\text{pe}(X)} + \frac{16d}{\sqrt{n}} \left( \mathbb{E}\left[|X_{1,\ldots,d}|^3\right] + 1 \right).
\]

Proof. We start with the proof of part (1). Set $K := \mathbb{E}\left[|X_{1,\ldots,d}|^3\right]$. By Hölder’s inequality, we have $\mathbb{E}\left[|X_iX_p|\right] \leq K^{2/3}$ for every $i, p \in [n]^d$; in particular, $\delta_1 \leq K^{2/3}$.

As in the proof of Proposition 12.3, we will use a truncation argument. Let $\lambda > 0$ be a cut-off parameter which will be specified later. For every $i \in [n]^d$ set
\[
G_i := X_i \mathbf{1}_{|X_i| \leq \lambda} \quad \text{and} \quad B_i := X_i \mathbf{1}_{|X_i| > \lambda}
\]
and observe that
\[
\|B_i\|_{L_1} \leq \frac{K}{\lambda^2} \quad \text{and} \quad \|B_i\|_{L_2} \leq \sqrt{\frac{K}{\lambda}}.
\]
Moreover, for every $j \in [n]$ set
\[
Y_j := \frac{1}{n^{d-1}} \sum_{i \in [n]^d : i(1) = j} X_i \quad \text{and} \quad Z_j := \frac{1}{n^{d-1}} \sum_{i \in [n]^d : i(1) = j} G_i.
\]
By (12.24) and the Cauchy–Schwarz inequality, we have
\[
\mathbb{E}\left[|Y_j^2 - Z_j^2|\right] \leq \frac{1}{n^{2d-2}} \sum_{i,p \in [n]^d \atop i(1) = p(1) = j} \left( \mathbb{E}[|G_iB_p|] + \mathbb{E}[|B_iG_p|] + \mathbb{E}[|B_iB_p|] \right) \leq \frac{3K}{\lambda}
\]
which yields that
\[
\text{osc}(X) \stackrel{(1.10)}{=} \left\| \frac{1}{n} \sum_{j=1}^n Y_j^2 - \delta_1 \right\|_{L_1} \leq \left\| \frac{1}{n} \sum_{j=1}^n Z_j^2 - \delta_1 \right\|_{L_1} + \frac{3K}{\lambda} \leq \left\| \frac{1}{n} \sum_{j=1}^n Z_j^2 - \delta_1 \right\|_{L_2} + \frac{3K}{\lambda} = \left( \frac{1}{n^2} \sum_{j_1,j_2=1}^n Z_{j_1}^2 Z_{j_2}^2 + \delta_1^2 - 2\delta_1 \frac{1}{n} \sum_{j=1}^n Z_j^2 \right)^{1/2} + \frac{3K}{\lambda}.
\]
Using (12.24) and Hölder’s inequality, it is easy to see that for every $j \in [n]$, \begin{align}
\mathbb{E}[Z_j^2] - \delta_1 & \leq \frac{3K}{\lambda} + \frac{4d^2 K^{2/3}}{n} \quad \text{and} \quad \mathbb{E}[Z_j^4] \leq K \lambda.
\end{align}
Next, set $i_1 := (1, 2, \ldots, d), i_2 := (1, d+1, \ldots, 2d-1), i_3 := (2d, 2d+1, \ldots, 3d-1)$ and $i_4 := (2d, 3d, \ldots, 4d-2)$ if $d \geq 2$; otherwise, set $i_1 = i_2 = 1$ and $i_3 = i_4 = 2$. By the exchangeability of $X$ and Hölder’s inequality, for every $j_1, j_2 \in [n]$ with $j_1 \neq j_2$ we have
\[
\mathbb{E}[Z_{j_1}^2 Z_{j_2}^2] - \delta_1^2 \leq \mathbb{E}[G_{i_1} G_{i_2} G_{i_3} G_{i_4}] - \delta_1^2 + \frac{8d^2 K^{4/3}}{n} + K \lambda \frac{8d^2}{n};
\]
on the other hand, by the dissociativity and exchangeability of $X$, \begin{align}
\mathbb{E}[G_{i_1} G_{i_2} G_{i_3} G_{i_4}] = \mathbb{E}[G_{i_1} G_{i_2}] \cdot \mathbb{E}[G_{i_3} G_{i_4}] = \mathbb{E}[G_{i_1} G_{i_2}]^2.
\end{align}
Since $X_{i_1}X_{i_2} = G_{i_1}G_{i_2} + G_{i_1}B_{i_2} + B_{i_1}G_{i_2} + B_{i_1}B_{i_2}$ and $E[X_{i_1}X_{i_2}] = \delta_1$, by (12.24) and Hölder’s inequality, we obtain that

\begin{equation}
|E[G_{i_1}G_{i_2}G_{i_3}G_{i_4}] - \delta_1^2| \leq \frac{6K^{5/3}}{\lambda}.
\end{equation}

By (12.27)–(12.31) and setting $\lambda = \sqrt{n}$, we conclude that (12.21) is satisfied.

We proceed to the proof of part (2). Let $(\Omega, \mathcal{F}, P)$ be a probability space, and assume that $X$ is the mixture with respect to $(\Omega, \mathcal{F}, P)$ of the dissociated, exchangeable and symmetric random tensors $\langle X_\omega : \omega \in \Omega \rangle$, that is,

\begin{equation}
P(X \in A) = E[P(X_\omega \in A)]
\end{equation}

for every Borel subset $A$ of $\mathbb{R}^{[n]^d}$, where $E$ denotes expectation with respect to $P$. Since the entries of $X$ have finite third moment, the entries of $X_\omega$ also have finite third moment $P$-almost surely. For every $i \in [n]^d$ let $X_\omega(i)$ denote the $i$-entry of $X_\omega$. Set

\begin{equation}
Z_\omega := \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{n^{d-1}} \sum_{i \in [n]^d, i(j) = j} X_\omega(i) \right)^2
\end{equation}

and observe that

\begin{equation}
\text{osc}(X) = E[\|Z_\omega - \delta_1\|_{L_1}].
\end{equation}

If $\delta_{1,\omega} := \delta_1(X_\omega)$ is the parameter in (1.6) associated with the random tensor $X_\omega$ and $K_\omega$ denotes the absolute third moment of its entries, then, by (12.34), the triangle inequality for the $L_1$-norm and the Cauchy–Schwarz inequality,

\begin{equation}
\text{osc}(X) \leq E[\|Z_\omega - \delta_{1,\omega}\|_{L_1}] + E[|\delta_{1,\omega} - \delta_1|]
\end{equation}

\begin{align*}
&= E[\text{osc}(X_\omega)] + E[|\delta_{1,\omega} - \delta_1|] \overset{\text{(12.21)}}{\leq} E[|\delta_{1,\omega} - \delta_1|] + \frac{16d(E[K_\omega] + 1)}{\sqrt{n}}
\end{align*}

\begin{equation}
\leq \sqrt{E[\delta_{1,\omega}^2]} - \delta_1^2 + \frac{16d(E[|X_{(1,\ldots,d)}|^3] + 1)}{\sqrt{n}}.
\end{equation}

Finally, since the random tensors $\langle X_\omega : \omega \in \Omega \rangle$ are exchangeable and dissociated, we have

\begin{equation}
\text{pc}(X) = |E[\delta_{1,\omega}^2] - \delta_1^2|.
\end{equation}

Combining (12.35) and (12.36) the result follows. \hfill \Box

Remark 12.8. Note that the proof of Proposition 12.7 actually yields the validity of (12.21) under the slightly weaker hypothesis that the subtensor $X_{[4d-2]} = \langle X_i : i \in [4d-2]^d \rangle$ of $X$ is dissociated.

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19Here, we assume that this process is sufficiently measurable.
12.2.1. Infinitely extendible random tensors. Recall that an exchangeable random tensor $X = \{X_i : i \in [n]^d\}$ is called infinitely extendible if there exists an exchangeable (infinite) random tensor $Y = \{Y_i : i \in \mathbb{N}^d\}$ such that $Y_n := \{Y_i : i \in [n]^d\}$ and $X$ have the same distribution. It is a classical observation that exchangeable random tensors might fail to be infinitely extendible. Nevertheless, many exchangeable random tensors that appear in practice are infinitely extendible; e.g., all exchangeable random tensors whose entries are functions of i.i.d. random variables are infinitely extendible.

Arguably, infinitely extendible exchangeable random tensors are better-behaved; for instance, it is easy to see that if $X$ satisfies (A1) and (A2) and it is infinitely extendible, then the parameters

$$\delta_0(X), \ldots, \delta_d(X) \quad \text{and} \quad \Sigma_0(X), \ldots, \Sigma_d(X)$$

associated with $X$ via (1.6), (1.7) and (1.9), are all nonnegative. That said, we have the following version of Theorem 1.4 for this class of random tensors.

**Corollary 12.9.** Let $X, \theta$ which satisfy (A1), (A2), (A3) with $n \geq 4d$, and assume that $X$ is infinitely extendible. Assume, moreover, that $|||\theta|||_1 = 1$, and let $\alpha \in (0, 1)$ such that the following non-degenericity condition holds true

$$\delta_1 > \max\{pc(X)^{1/2}, \delta_0^\alpha\}$$

where $pc(X)$ is as in (3.7). Then, setting $\sigma^2 := \text{Var}(\langle \theta, X \rangle)$, we have

$$d_K(\langle \theta, X \rangle, \mathcal{N}(0, \sigma^2)) \leq E_1'' + E_2'' + E_3''$$

where

$$E_1'' := 5pc(X)^{1/2} + 5\delta_0^{1-\alpha} + \frac{\delta_0}{d^2\delta_1} \left| \|\theta\|_0^2 - 1 \right|$$

$$E_2'' := 2^{36} \frac{\mathbb{E}|||X(1, \ldots, d)|||^3}{\delta_1^{3/2}} \left( \sum_{j=1}^n \left( \sum_{i \in [n]^d \cap \{1\} = j} \theta_i \right)^3 \right)$$

$$E_3'' := 3\kappa \left( \frac{1}{d\sqrt{\delta_1}} \sum_{s=2}^d \binom{d}{s} \right) \sqrt{s!} \sqrt{\sum_{s} \|\theta\|_s}.$$  

Here, $\kappa = 20d^318^d(2d)!$ is as in Theorem 1.4.

The asymptotic behavior of the normalized averages $\frac{1}{n^{d-1/2}} \sum_{i \in [n]^d} X_i$ which correspond to a random tensor $X = \{X_i : i \in [n]^d\}$ as in Corollary 12.9, is well-studied in the literature; see, e.g., [BCRT58, DDG21, EW78, Si76]. For this particular case, Corollary 12.9 yields Berry–Esseen bounds under the conditions

$$\delta_0 = 0, \quad pc(X) = 0 \quad \text{and} \quad \delta_1 > 0.$$  

These conditions also cover the case of several popular statistics (e.g., non-degenerate, possibly weighted, U-statistics).
Proof of Corollary 12.9. Let $Y = \langle Y_i : i \in \mathbb{N}^d \rangle$ be an exchangeable (infinite) random tensor such that $X$ and $Y_n$ have the same distribution, where for every integer $N \geq d$ by $Y_N$ we denote the subtensor $\langle Y_i : i \in [N]^d \rangle$ of $Y$. By (42), we see that $Y$ is also symmetric and its diagonal terms vanish. Therefore, by the Aldous–Hoover theorem [Ald81, Hoo79], $Y$ is a mixture of exchangeable, dissociated and symmetric infinite random tensors whose diagonal terms vanish. This property is inherited, of course, to the subtensors of $Y$. In particular, for every integer $N \geq n$ the finite tensor $Y_N$ is a mixture of exchangeable, dissociated and symmetric random tensors and, consequently, part (2) of Proposition 12.7 can be applied to $Y_N$; also notice that

- the entries of $Y_N$ and $X$ have the same absolute third moment,
- $\delta_s(X) = \delta_s(Y_N)$ and $\Sigma_s(X) = \Sigma_s(Y_N)$ for every $s \in \{0, \ldots, d\}$, and
- $\text{pc}(X) = \text{pc}(Y_N)$.

Next fix a real tensor $\theta = \langle \theta_i : i \in [n]^d \rangle$ which satisfies (43) and with $\|\theta\|_1 = 1$. Let $N \geq n$ be an arbitrary integer. We extend $\theta$ to a real tensor $\theta_N = \langle \theta_i : i \in [N]^d \rangle$ by setting $\theta_i = 0$ if $i \notin [n]^d$; clearly we have $\|\theta\|_* = \|\theta_N\|_*$ for every $s \in \{0, \ldots, d\}$.

Moreover, the random variables $(\theta, X)$ and $(\theta_N, Y_N)$ have the same distribution. By the previous observations and (12.22), we have

$$\text{osc}(Y_N) \leq \sqrt{\text{pc}(X)} + \frac{16d (\mathbb{E}[\|X_{(1,\ldots,d)}\|_3^3] + 1)}{\sqrt{N}}$$

while, by (11.35),

$$B_N = \left\| \frac{1}{N^d} \sum_{i \in [N]^d} Y_i \right\|_{L_2}^2 \leq \delta_0 + \frac{4d^2}{N^2}.$$  

By (12.43), (12.44) and (12.37), we see that the non-degeneracy condition (1.11) for the random tensor $Y_N$ is satisfied for every sufficiently large positive integer $N$. The result follows by applying Theorem 1.4 to $Y_N, \theta_N$ and using (12.43), and then taking the limit as $N$ goes to infinity.  

\[\square\]

12.3. Anticoncentration of polynomials: proof of Theorem 3.2. Let $n, d, \kappa, k$ and $\xi = (\xi_1, \ldots, \xi_n)$ be as in the statement of the theorem, and fix a polynomial $f$ as in (3.13). We define a random tensor $X = (X_i : i \in [n]^d)$ and a real tensor $\theta = (\theta_i : i \in [n]^d)$ by setting $X_i = \theta_i = 0$ if $i \notin [n]_{\text{inj}}$, and

$$X_i := \prod_{r=1}^d \xi_{i_r} - \mathbb{E} \left[ \prod_{r=1}^d \xi_{i_r} \right] \quad \text{and} \quad \theta_i := \frac{a(i_1,\ldots,i_d)}{d!}$$

for every $i = (i_1,\ldots,i_d) \in [n]_{\text{inj}}^d$, where $\alpha = \langle a_F : F \in \binom{[n]}{d} \rangle$ are the coefficients of $f$. It is clear that with these choices we have $\langle \theta, X \rangle = f(\xi) - \mathbb{E}[f(\xi)]$ and, consequently,

- $\text{Var}(\langle \theta, X \rangle) = \text{Var}(f(\xi))$, and
- $\mathcal{L}_{\langle \theta, X \rangle}(\varepsilon) = \mathcal{L}_{f(\xi)}(\varepsilon)$ for every $\varepsilon > 0$.  

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Also notice that $X$ and $\theta$ satisfy the basic assumptions \((A1), (A2)\) and \((A3)\). Moreover, by a tedious but fairly straightforward computation, it is not hard to see that

\[(12.46) \quad |\delta_s - p^{2d-s}(1-p^s)| \leq p^{2d-s} \frac{12d^2}{k} \text{ for every } s \in \{0, \ldots, d\}\]

\[(12.47) \quad |\Sigma_s - p^{2d-s}(1+p)^s| \leq p^{2d-s}(1+p)^s \frac{12d^2}{k} \text{ for every } s \in \{1, \ldots, d\}\]

\[(12.48) \quad \text{osc}(X) \leq p^{2d-1} \frac{15d^2}{k}\]

\[(12.49) \quad B = \left\| \frac{1}{n} \sum_{i \in [n]} X_i \right\|^2_{L_2} = 0\]

where $p = k/n$. By \((3.15), (12.46), (12.48)\) and \((12.49)\), it follows that the non-degeneracy condition \((1.11)\) is satisfied with $\alpha = 1/2$. Therefore, after appropriately normalizing, by Theorem 1.4 we obtain that

\[(12.50) \quad d_K(\langle \theta, X \rangle, \mathcal{N}(0, \sigma^2)) \leq \frac{8k}{\sqrt{p}} \frac{1}{\sqrt{n}} + \frac{6}{(1-p)\|\theta\|_1^2} \frac{\|\theta\|_0^2}{n} + \frac{2^{38}p^{3/2}}{p^{3d} (1-p)^{3/2} \|\theta\|_1^3} \left( \sum_{j=1}^n \sum_{i \in [n]^{(1)} \setminus (1) = j} \theta_i \right)\]

\[+ \frac{8kp^{1/2}}{dp^d (1-p)^{1/2} \|\theta\|_1} \sum_{s=2}^d \left( \frac{d}{s} \right) \sqrt{s!} \sqrt{p^{2d-s}(1-p)^s} \|\theta\|_s\]

where we have used \((3.15)\) and \((12.46)\)–\((12.49)\). Next observe that for every $\varepsilon > 0$,

\[(12.51) \quad \mathcal{L}_{\langle \theta, X \rangle}(\varepsilon) \leq \mathcal{L}_{\mathcal{N}(0, \sigma^2)}(\varepsilon) + 2d_K(\langle \theta, X \rangle, \mathcal{N}(0, \sigma^2))\]

\[\leq \frac{\varepsilon}{\sqrt{2\pi} \sigma} + 2d_K(\langle \theta, X \rangle, \mathcal{N}(0, \sigma^2))\]

On the other hand, by the choice of the random tensor $\theta$ in \((12.45)\) and the definitions of the relevant seminorms in \((1.5)\) and \((3.14)\), for every $s \in \{0, \ldots, d\}$ we have

\[(12.52) \quad \|\theta\|_s = \frac{(d-s)!}{d!} \frac{1}{\sqrt{s!}} \|\alpha\|_s\]

Combining \((12.50), (12.51)\) and \((12.52)\), we see that \((3.17)\) is satisfied.

Finally, in order to verify the estimate for the variance $\sigma^2$ of $f(\xi)$ observe that

\[(12.53) \quad \sigma^2 = \text{Var}(\langle \theta, X \rangle) = \sum_{s=0}^d \left( \frac{d}{s} \right)^2 s! \Sigma_s \|\theta\|_s^2 = \sum_{s=0}^d \Sigma_s \|\alpha\|_s^2\]

Thus, \((3.16)\) follows from \((12.53)\) together with \((12.46), (12.47)\) and \((3.15)\).

**Appendix A. Proof of Lemma 7.3**

The argument is an interesting and nontrivial instance of the Stein/Chen method of normal approximation via concentration (see, e.g., [BC14, Section 6]).
A.1. Recall that \( n, d \) are positive integers with \( d \geq 2 \) and \( n \geq 4d^2 \). We start by setting
\[
\delta := \frac{16 \Lambda}{n} \quad \text{and} \quad \eta_\delta := \frac{n}{4} \mathbb{E}[[\Xi_1 - \Xi_1'] \cdot \min\{|\Xi_1 - \Xi_1'|, \delta\} | \pi_1].
\]
Notice that, by (7.6), we have
\[
\delta \geq \frac{n}{4} \mathbb{E}[|\Xi_1 - \Xi_1'|^3].
\]
Using (7.5), (7.6), (A.2) and the fact that \( \min(\alpha, \beta) \geq \alpha - \frac{\alpha^2}{4\beta} \) for every \( \alpha \geq 0 \) and \( \beta > 0 \), we obtain that
\[
\mathbb{E}[\eta_\delta] = \frac{n}{4} \mathbb{E}[|\Xi_1 - \Xi_1'| \cdot \min\{|\Xi_1 - \Xi_1'|, \delta\}] \\
\geq \frac{n}{4} \mathbb{E}[|\Xi_1 - \Xi_1'|^2] - \frac{n}{16\delta} \mathbb{E}[|\Xi_1 - \Xi_1'|^3] = 1 - \frac{n^2}{16^2\Lambda} \mathbb{E}[|\Xi_1 - \Xi_1'|^3] \geq \frac{3}{4}.
\]

**Sublemma A.1.** We have
\[
\text{Var}(\eta_\delta) \leq 2^9 \delta^2.
\]
We postpone the proof of Sublemma A.1 to the end of this appendix.

A.2. Now let \( z \in \mathbb{R} \) be arbitrary. We will show that the probabilities \( \mathbb{P}(z - |\Theta| \leq \Xi_1 \leq z) \) and \( \mathbb{P}(z \leq \Xi_1 \leq z + |\Theta|) \) are both upper-bounded by the quantity appearing in the right-hand-side of (7.20); clearly, this is enough to complete the proof. The argument is symmetric, and so we shall focus on bounding the probability \( \mathbb{P}(z \leq \Xi_1 \leq z + |\Theta|) \).

Let \( f_\Theta : \mathbb{R} \rightarrow \mathbb{R} \) be defined by
\[
f_\Theta(x) := \begin{cases} 
-\left(\frac{1}{2}|\Theta| + \delta\right) & \text{if } x \leq z - \delta, \\
-\frac{1}{2}|\Theta| + x - z & \text{if } z - \delta < x < z + |\Theta| + \delta, \\
\frac{1}{2}|\Theta| + \delta & \text{if } x \geq z + |\Theta| + \delta.
\end{cases}
\]
By (E4), the pair \( (\pi_1, \pi_2) \) is exchangeable. Hence, by (7.1) and (7.19), we have
\[
\mathbb{E}\left[(\Xi_1' - \Xi_1)(f_\Theta(\Xi_1') + f_\Theta(\Xi_1))\right] = 0
\]
which in turn implies, after adding \( 2 \mathbb{E}\left[(\Xi_1 - \Xi_1)f_\Theta(\Xi_1)\right] \) and multiplying by \( \frac{n}{4} \), that
\[
\frac{n}{2} \mathbb{E}[f_\Theta(\Xi_1)] = \frac{n}{4} \mathbb{E}[(\Xi_1' - \Xi_1)(f_\Theta(\Xi_1) - f_\Theta(\Xi_1))].
\]
Moreover, by (7.4), we have
\[
\frac{n}{2} \mathbb{E}[(\Xi_1' - \Xi_1)f_\Theta(\Xi_1)] = \frac{n}{2} \mathbb{E}[f_\Theta(\Xi_1) \mathbb{E}[\Xi_1' - \Xi_1 | \pi_1]] = \mathbb{E}[f_\Theta(\Xi_1) \Xi_1]
\]
and so, by (7.3), the Cauchy–Schwarz inequality and the fact that \( \|f_\Theta\|_{L^\infty} \leq \delta + \frac{1}{2}|\Theta| \),
\[
\frac{n}{2} \mathbb{E}[(\Xi_1' - \Xi_1)f_\Theta(\Xi_1)] \leq \delta \mathbb{E}[|\Xi_1|] + \frac{1}{2} \mathbb{E}[|\Xi_1\Theta|] \leq \delta + \frac{1}{2} \|\Theta\|_{L^2}.
\]
Moreover, for every

(A.10) \[ \left| \frac{n}{2} \mathbb{E}[(\Xi'_1 - \Xi_1)f_{\Theta}(\Xi_1)] \right| \leq \delta + e^d(2d)! \sum_{s=2}^{d} \sqrt{\beta_s n^s}. \]

Next observe that \( f_{\Theta} - f_{\Theta}' \|_{L_\infty} \leq \frac{1}{2} ||\Theta| - |\Theta'|| \leq \frac{1}{2} ||\Theta - \Theta'||. \) Hence,

(A.11) \[ \left| \frac{n}{4} \mathbb{E}[(\Xi'_1 - \Xi_1)(f_{\Theta}(\Xi_1) - f_{\Theta}(\Xi_1))] \right| \leq \frac{n}{8} \mathbb{E}[|\Xi'_1 - \Xi_1| \cdot |\Theta - \Theta'|] \leq \frac{\sqrt{n}}{4} \sum_{s=2}^{d} \|\Xi_2 - \Xi_1\|_{L_2} \leq 2d^2e^d(2d)! \sum_{s=2}^{d} \sqrt{\beta_s n^s} \]

where we have used the Cauchy–Schwarz inequality, the triangle inequality and (7.8).

From this point on, the proof proceeds exactly as in [BC05, Lemma 2.2]. More precisely, in order to estimate the second term of the right-hand-side of (A.7), by the fundamental theorem of calculus, we have

(A.12) \[ A := \frac{n}{4} \mathbb{E}[\Xi'_1 - \Xi_1)](f_{\Theta}(\Xi_1) - f_{\Theta}(\Xi_1)) = \frac{n}{4} \mathbb{E}[(\Xi'_1 - \Xi_1) \int_0^{\Xi'_1 - \Xi_1} f_{\Theta}(\Xi_1 + t) \, dt] \]

\[ = \frac{n}{4} \mathbb{E}[(\Xi'_1 - \Xi_1) \int_{-\infty}^{t_0} f_{\Theta}(\Xi_1 + t) \left(1_{[0,\Xi'_1 - \Xi_1]}(t) - 1_{[\Xi'_1 - \Xi_1,0]}(t)\right) \, dt] \]

with the convention that 1_{[0,\alpha]} is constantly 0 if \( \alpha < \beta \). Also observe that, by (A.5), we have \( f_{\Theta} = 1_{[z-\delta,z+\delta+|\Theta|]} \) which implies that for every \( t \in \mathbb{R}, \)

- \( (\Xi'_1 - \Xi_1) f_{\Theta}'(\Xi_1 + t) \left(1_{[0,\Xi'_1 - \Xi_1]}(t) - 1_{[\Xi'_1 - \Xi_1,0]}(t)\right) \geq 0. \)

Therefore,

(A.13) \[ A \geq \frac{n}{4} \mathbb{E} \left[ \int_{|t| \leq \delta} (\Xi'_1 - \Xi_1) \left(1_{[z-\delta,z+\delta+|\Theta|]}(\Xi_1 + t) \left(1_{[0,\Xi'_1 - \Xi_1]}(t) - 1_{[\Xi'_1 - \Xi_1,0]}(t)\right) \right) \, dt \right]. \]

Moreover, for every \( |t| \leq \delta \) we have

- \( (\Xi'_1 - \Xi_1) \left(1_{[0,\Xi'_1 - \Xi_1]}(t) - 1_{[\Xi'_1 - \Xi_1,0]}(t)\right) \geq 0 \) and
- \( 1_{[z-\delta,z+\delta+|\Theta|]}(\Xi_1 + t) \geq 1_{[z,\delta+|\Theta|]}(\Xi_1), \)

and so, by (A.13),

(A.14) \[ A \geq \frac{n}{4} \mathbb{E} \left[ \left(1_{[z,\delta+|\Theta|]}(\Xi_1) \right) (\Xi'_1 - \Xi_1) \int_{|t| \leq \delta} \left(1_{[0,\Xi'_1 - \Xi_1]}(t) - 1_{[\Xi'_1 - \Xi_1,0]}(t)\right) \, dt \right] \]

\[ = \frac{n}{4} \mathbb{E} \left[ \left(1_{[z,\delta+|\Theta|]}(\Xi_1) \right) |\Xi'_1 - \Xi_1| \cdot \min \left\{ \delta, |\Xi'_1 - \Xi_1| \right\} \right] \]

\[ \overset{(A.1)}{\geq} \mathbb{E} \left[ \left(1_{[z,\delta+|\Theta|]}(\Xi_1) \right) \eta_\delta \right] \geq \mathbb{E} \left[ \left(1_{[z,\delta+|\Theta|]}(\Xi_1) \right) \mathbb{E}[\eta_\delta] - \mathbb{E} \left[ \left. \left(1_{[z,\delta+|\Theta|]}(\Xi_1) \right) (\eta_\delta - \mathbb{E}[\eta_\delta]) \right| \Xi_1 \right] \right] \]

\[ \overset{(A.3)}{\geq} \frac{3}{4} \mathbb{P}(\zeta \leq \Xi_1 \leq \zeta + |\Theta|) - \mathbb{E} \left[ \left(1_{[z,\delta+|\Theta|]}(\Xi_1) \right) (\eta_\delta - \mathbb{E}[\eta_\delta]) \right]. \]
On the other hand, by Sublemma A.1 and inequality $\alpha \beta \leq \frac{1}{2}(\alpha^2 + \beta^2)$ applied for $\alpha = \frac{1}{\sqrt{2}} 1_{[z, z+|\Theta|]}(\Xi_1)$ and $\beta = \sqrt{2}(\eta_\delta - \mathbb{E}[\eta_\delta])$, we see that
\begin{equation}
\mathbb{E}[1_{[z, z+|\Theta|]}(\Xi_1)(\eta_\delta - \mathbb{E}[\eta_\delta])] \leq \frac{1}{2} \left( \frac{1}{2} \mathbb{P}(z \leq \Xi_1 \leq z + |\Theta|) + 2\text{Var}(\eta_\delta) \right)
\leq \frac{1}{4} \mathbb{P}(z \leq \Xi_1 \leq z + |\Theta|) + 2^9 \delta^2.
\end{equation}

Thus, by (A.14) and (A.15), we have
\begin{equation}
A \geq \frac{1}{2} \mathbb{P}(z \leq \Xi_1 \leq z + |\Theta|) - 2^9 \delta^2.
\end{equation}

Invoking the choice of $\delta$ in (A.1) and the choice of $A$ in (A.12) and combining (A.7), (A.10), (A.11) and (A.16), we conclude that
\begin{equation}
\mathbb{P}(z \leq \Xi_1 \leq z + |\Theta|) \leq 2^{5} \frac{A}{n} + 2^{17} \frac{A^2}{n^2} + 5d^2e^d(2d)! \sum_{s=2}^{d} \sqrt{\frac{\beta_s}{n^s}}.
\end{equation}

### A.3. Proof of Sublemma A.1.

Let $\zeta: [n]^2 \times [n]^2 \to \mathbb{R}$ be defined by setting
\begin{equation}
\zeta(i, j, p, q) := |\xi_1(i, p) + \xi_1(j, q) - \xi_1(i, q) - \xi_1(j, p)| \times \\
\min \{ |\xi_1(i, p) + \xi_1(j, q) - \xi_1(i, q) - \xi_1(j, p)|, \delta \}
\end{equation}

for every $i, j, p, q \in [n]$ with $i \neq j$ and $p \neq q$; otherwise, set $\zeta(i, j, p, q) = 0$. Then observe that, by the choice of $\eta_\delta$ in (A.1), we have
\begin{equation}
\eta_\delta = \frac{n}{4} \mathbb{E}[|\Xi_1 - \Xi'_1| \cdot \min \{ |\Xi_1 - \Xi'_1|, \delta \} | \pi_1] = \frac{1}{4n} \sum_{i,j \in [n], i \neq j} \zeta(i, j, \pi_1(i), \pi_1(j)).
\end{equation}

Also recall that, by (E3), $\pi_1$ is uniformly distributed on $S_n$. Thus, (A.19) asserts the random variable $\eta_\delta$ can be expressed as the $Z$-statistic of order two associated with the tensor $\zeta$ which has a rather special form.

The proof is based on a specific decomposition of this $Z$-statistic. This decomposition, also introduced by Barbour/Chen [BC05], is less symmetric than the decomposition described in Section 9 (yet it has enough symmetries in order to be computationally useful), but it has the advantage that its non-constant components are mean zero random variables. Of course, this property is very useful for computing the variance. For the convenience of the reader we shall briefly recall this decomposition; for more information we refer to [BC05].

Specifically, for every $i, j, p, q \in [n]$ set
\begin{align*}
\zeta(i, j, \cdot, q) &:= \frac{1}{n - 1} \sum_{r \in [n], r \neq q} \zeta(i, j, r, q), & \zeta(i, j, \cdot, p) &:= \frac{1}{n - 1} \sum_{r \in [n], r \neq p} \zeta(i, j, r, p), \\
\zeta(i, j, \cdot, \cdot) &:= \frac{1}{n(n - 1)} \sum_{r, s \in [n], r \neq s} \zeta(i, j, r, s), & \zeta(i, \cdot, p, \cdot) &:= \frac{1}{(n - 1)^2} \sum_{r, s \in [n], r \neq i, s \neq p} \zeta(i, r, p, s),
\end{align*}
\[ \zeta(i, \cdot, \cdot, \cdot) := \frac{1}{n(n-1)^2} \sum_{r, s \in [n], \ell \neq i, r \neq s} \zeta(i, \ell, r, s), \quad \zeta(\cdot, j, \cdot, \cdot) := \frac{1}{(n-1)^2} \sum_{r, s \in [n], r \neq j, s \neq q} \zeta(r, j, s, q), \]

\[ \zeta(\cdot, \cdot, j, \cdot) := \frac{1}{n(n-1)^2} \sum_{r, s \in [n], \ell \neq j, r \neq s} \zeta(\ell, j, r, s), \quad \zeta(\cdot, \cdot, \cdot, \cdot) := \frac{1}{n^2(n-1)^2} \sum_{r, s, k \in [n], r \neq j, s \neq q, k \neq \ell} \zeta(r, s, k, \ell). \]

Then, for every \( i, j, p, q \in [n] \) we define

\begin{align*}
\zeta_D(i, j, p, q) & := \zeta(i, j, p, q) - \zeta(i, j, \cdot, \cdot) - \zeta(i, \cdot, p, \cdot) + \zeta(i, \cdot, \cdot, \cdot) \\
\zeta_D(i, \cdot, p, \cdot) & := \zeta(i, \cdot, p, \cdot) - \zeta(i, \cdot, \cdot, \cdot) \\
\zeta_D(\cdot, j, \cdot, q) & := \zeta(\cdot, j, \cdot, q) - \zeta(\cdot, \cdot, j, \cdot).
\end{align*}

Notice that for every \( i, p, q \in [n] \) we have \( \zeta_D(i, i, p, q) = 0 \); moreover, for every \( i, j \in [n] \),

\[ \sum_{p, q \in [n]} \zeta_D(i, j, p, q) = 0, \quad \sum_{p=1}^{n} \zeta_D(i, \cdot, p, \cdot) = 0 \quad \text{and} \quad \sum_{p=1}^{n} \zeta_D(\cdot, i, \cdot, p) = 0. \]

By (A.19) and the previous definitions, we may decompose \( \eta \) as

\[ \eta = \frac{1}{4n} \sum_{i, j \in [n]} \zeta_D(i, j, \pi_1(i), \pi_1(j)) + \frac{n-1}{4n} \sum_{i=1}^{n} \zeta_D(\cdot, i, \cdot, \pi_1(i)) + \frac{n-1}{4n} \sum_{i=1}^{n} \zeta_D(i, \cdot, \cdot, \cdot) + \frac{n}{4n} \zeta_D(\cdot, \cdot, \cdot, \cdot). \]

Invoking (A.23) and the fact that \( \pi_1 \) is uniformly distributed on \( S_n \) we see, in particular, that \( \mathbb{E}[\eta] = \frac{n-1}{4} \zeta(\cdot, \cdot, \cdot, \cdot) \). Therefore,

\begin{align*}
\text{(A.25) } \quad \text{Var}(\eta) & \leq \frac{3}{2^4 n^2} \mathbb{E} \left[ \left( \sum_{i, j \in [n], \neq j} \zeta_D(i, j, \pi_1(i), \pi_1(j)) \right)^2 \right] + \\
& \quad + \frac{3}{2^4} \mathbb{E} \left[ \left( \sum_{i=1}^{n} \zeta_D(\cdot, i, \cdot, \pi_1(i)) \right)^2 \right] + \frac{3}{2^4} \mathbb{E} \left[ \left( \sum_{i=1}^{n} \zeta_D(i, \cdot, \cdot, \pi_1(i)) \right)^2 \right] \\
& \quad := E_1 + E_2 + E_3.
\end{align*}

Finally, using the Cauchy–Schwarz inequality, (A.23) and arguing\(^{20}\) as in Subsection 5.6, it is not hard to verify that

\[ E_1 \leq \left( \frac{20^3}{n} + 2^6 3 \right) \delta^2, \quad E_2 \leq 2^4 3 \delta^2 \quad \text{and} \quad E_3 \leq 2^3 3 \delta^2. \]

By (A.25), (A.26) and the fact that \( n \geq 3 \), we conclude that (A.4) is satisfied.

\(^{20}\)The argument in this context is actually simpler.
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Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece

Email address: pdodos@math.uoa.gr

Department of Mathematics, University of Athens, Panepistimiopolis 157 84, Athens, Greece

Email address: ktyros@math.uoa.gr