The Complexity of Proportionality Degree in Committee Elections

Łukasz Janeczko  
AGH University  
ljaneczk@agh.edu.pl

Piotr Faliszewski  
AGH University  
faliszew@agh.edu.pl

July 11, 2022

Abstract

Over the last few years, researchers have put significant effort into understanding of the notion of proportional representation in committee election. In particular, recently they have proposed the notion of proportionality degree. We study the complexity of computing committees with a given proportionality degree and of testing if a given committee provides a particular one. This way, we complement recent studies that mostly focused on the notion of (extended) justified representation. We also study the problems of testing if a cohesive group of a given size exists and of counting such groups.

1 Introduction

If we consider a parliamentary election where about 45% voters support party $A$, 30% support party $B$, and the remaining 25% support party $C$, then there are well-understood ways of assigning seats to the parties in a proportional manner. However, if instead of naming the supported parties the voters can approve each candidate individually, e.g., depending on some nonpartisan agendas, then the situation becomes less clear. While such free-form elections are not popular in political settings, they do appear in the context of artificial intelligence. For example, they can be used to organize search results [Skowron et al., 2017], assure fairness in social media [Chakraborty et al., 2019], help design online Q&A systems [Israel and Brill, 2021], or suggest movies to watch [Gawron and Faliszewski, 2021]. As a consequence, seeking formal understanding of proportionality in multi-winner elections is among the most active branches of computational social choice [Lackner and Skowron, 2020a]. In this paper we extend this line of work by analyzing the computational complexity of one of the recent measures of proportionality, the proportionality degree, introduced by [Aziz et al., 2018].

We consider the model of elections where, given a set of candidates, each of the $n$ voters specifies which candidates he or she approves, and the goal is to choose a size-$k$ subset of the candidates,
called the winning committee. Since the committee is supposed to represent the voters proportionally, Aziz et al. [2017] proposed the following requirement: For each positive integer $\ell$ and each group of $\ell \cdot n/k$ voters who agree on at least $\ell$ common candidates, the committee should contain at least $\ell$ candidates approved by at least one member of the group. In other words, such a group, known as an $\ell$-cohesive group, deserves at least $\ell$ candidates, but it suffices that a single member of the group approves them. If this condition holds, then we say that the committee provides extended justified representation (provides EJR; if we restrict our attention to $\ell = 1$, then we speak of providing justified representation, JR). The key to the success of this notion is that committees providing EJR always exist and are selected by voting rules designed to provide proportional representation, such as the proportional approval voting rule (PAV) of Thiele [1895]. Many researchers followed Aziz et al. [2017] either in defining new variants of the justified representation axioms [Sánchez-Fernández et al., 2017, Peters et al., 2020] or in analyzing and designing rules that would provide committees satisfying these properties [Aziz et al., 2018, Sánchez-Fernández et al., 2021, Brill et al., 2017]. Others study these notions experimentally [Bredereck et al., 2019] or analyze the restrictions that they impose [Lackner and Skowron, 2020b].

Yet, JR and EJR are somewhat unsatisfying. After all, to provide them it suffices that a single member of each cohesive group approves enough committee members, irrespective of all the other voters. To address this issue, Aziz et al. [2018] introduced the notion of proportionality degree (PD). They said that the satisfaction of a voter is equal to the number of committee members that he or she approves and they considered average satisfactions of the voters in cohesive groups. More precisely, they said that a committee has PD $f$, where $f$ is a function from positive integers to nonnegative reals, if for each $\ell$-cohesive group, the average satisfaction of the voters in this group is at least $f(\ell)$. So, establishing the PD of a rule gives quantitative understanding of its proportionality, whereas JR and EJR only give qualitative information. That said, Aziz et al. [2018] did show that if a committee provides EJR then it has PD at least $f(\ell) = \ell - 1/2$, so the two notions are related. Further, they showed that PAV committees have PD $f(\ell) = \ell - 1$ and Skowron [2021] established good bounds on the PDs of numerous other proportionality-oriented rules. In particular, these results allow one to order the rules according to their theoretical guarantees, from those providing the strongest ones to those providing the weakest.

We extend this line of work by studying the complexity of problems pertaining to the proportionality degree:

1. We show that, in general, deciding if a committee with a given PD exists is both NP-hard and coNP-hard and we suspect it to be NP-complete (but we only show membership). Nonetheless, we find the problem to be NP-complete for (certain) constant PD functions. These results contrast those for JR and EJR, for which analogous problems are in P.

2. We show that verifying if a given committee provides a given PD is coNP complete, which is analogous to the case of EJR. We also provide ILP formulations that may allow one to compute PDs in practice (thus, one could use them to establish an empirical hierarchy of proportionality among multiwinner voting rules).

3. We show that many of our problems are polynomial-time solvable for the candidate interval and voter interval domains of preferences [Elkind and Lackner, 2015] and are fixed-
parameter tractable for the parameterizations by the number of candidates or the number of voters.

We also study the complexity of finding and counting cohesive groups.

2 Preliminaries

For an integer $t$, we write $[t]$ to denote the set $\{1, \ldots, t\}$. An election $E = (C, V)$ consists of a finite set $C$ of candidates and a finite collection $V$ of voters. Each voter $v \in V$ is endowed with a set $A(v) \subseteq C$ of candidates that he or she approves. Analogously, for each candidate $c$ we write $A(c)$ to denote the set of voters that approve $c$; value $|A(c)|$ is known as the approval score of $c$. The elections considered in the $A(\cdot)$ notation will always be clear from the context.

Multiwinner Voting Rules. A multiwinner voting rule is a function that given an election $E = (C, V)$ and a committee size $k \leq |C|$ outputs a family of winning committees, i.e., a family of size-$k$ subsets of $C$. (While in practice some form of tie-breaking is necessary, theoretical studies usually disregard this issue.) Generally, we do not focus on specific rules, but the following three provide appropriate context for our discussions (we assume that $E = (C, V)$ is some election and we seek a committee of size $k$):

1. Multiwinner Approval Voting (AV) selects size-$k$ committees whose members have highest total approval score. Intuitively, AV selects committees of individually excellent candidates.

2. The Approval-Based Chamberlin–Courant rule (CC) selects those size-$k$ committees that maximize the number of voters who approve at least one member of the committee. Originally, the CC rule was introduced by Chamberlin and Courant [1983] and its approval variant was discussed, e.g., by Procaccia et al. [2008] and Betzler et al. [2013]. CC selects committees of diverse candidates, that cover as many voters as possible.

3. Proportional Approval Voting (PAV) selects those size-$k$ committees $S$ that maximize the value $\sum_{v \in V} w(|S \cap A(v)|)$, where for each natural number $t$ we have $w(t) = \sum_{j=1}^{t} 1/j$. PAV selects committees that, in a certain sense, represent the voters proportionally; see, e.g., the works of Brill et al. [2018] and Lackner and Skowron [2021]. The rule is due to Thiele [1895].

AV, CC, and PAV are examples of the so-called Thiele rules [Thiele 1895, Lackner and Skowron 2021], but there are also many other rules, belonging to other families. For more details, we point the reader to the survey of Lackner and Skowron [2020a]. Classifying multiwinner rules as focused on individual excellence, diversity, or proportionality is due to Faliszewski et al. [2017].

(Extended) Justified Representation. Let $E$ be an election with $n$ voters and let $k$ be the committee size. For an integer $\ell \in [k]$, called the cohesiveness level, we say that a group of voters forms an $\ell$-cohesive group if (a) the group consists of at least $\ell \cdot n/k$ voters, and (b) there are at least $\ell$ voters in $V$ who approve at least $\ell$ members of the group.
candidates approved by each member of the group. Intuitively, $\ell$-cohesive groups are large enough to demand representation by at least $\ell$ candidates (as they include a large-enough proportion of the voters) and they can name these $\ell$ candidates (as there are at least $\ell$ common candidates that they approve). Thus many proportionality axioms focus on satisfying such demands. In particular, we are interested in the notions of (extended) justified representation, due to\cite{Aziz2017}.

**Definition 1.** Let $E = (C, V)$ be an election, let $k$ be a committee size, and let $S$ be some committee:

1. We say that $S$ provides justified representation (JR) if each 1-cohesive group contains at least one voter who approves at least one member of $S$.

2. We say that $S$ provides extended justified representation (EJR) if for each $\ell \in [k]$, each $\ell$-cohesive group contains at least one voter that approves $\ell$ members of $S$.

Researchers also consider other proportionality axioms, such as the notion of proportional justified representation (PJR), due to\cite{Sanchez2017}, and the recently introduced axiom of fully justified representation (FJR), due to\cite{Peters2020}. JR is the weakest of these (in the sense that if a committee satisfies any of the other ones then it also provides JR), followed by PJR, EJR, and FJR. We focus on JR and EJR as they will suffice for our purposes. For every election and every committee size there always exists at least one committee providing EJR (thus, also JR). Indeed, all CC committees provide JR and all PAV committees also provide EJR \cite{Aziz2017}, but AV committees may fail to provide (E)JR.

**Proportionality Degree.** Our main focus is on the notion of a proportionality degree of a committee, introduced by\cite{Aziz2018}. Let us consider some voter $v$ and a committee $S$. We define $v$’s satisfaction with $S$ as $|A(v) \cap S|$, i.e., the number of committee members that $v$ approves.

**Definition 2.** Let $E$ be an election, let $S$ be a committee of size $k$, and let $f : [k] \to \mathbb{R}$ be a function. We say that $S$ has proportionality degree $f$ if for each $\ell$-cohesive group of voters $X$ (where $\ell \in [k]$) the average satisfaction of the voters in $X$ is at least $f(\ell)$.

In other words, if a committee has a certain proportionality degree $f$ for a given election, then members of the cohesive groups in this election are guaranteed at least a certain average level of satisfaction. We are interested in several special types of proportionality degree (PD) functions:

1. We say that $f$ is a nonzero PD if $f(\ell) > 0$ for all $\ell$.

2. We say that $f$ is a unit PD if $f(\ell) = 1$ for all $\ell$.

3. We say that $f$ is nearly perfect PD if $f(\ell) = \ell - 1$ for all $\ell$.

4. We say that $f$ is a perfect PD if $f(\ell) = \ell$ for all $\ell$. 
One can verify that every CC committee (or, in fact, every JR committee) has nonzero PD, and Aziz et al. [2018] have shown that every PAV committee has nearly perfect PD. It is also known that if a committee provides EJR then, at least, it has proportionality degree $f$ such that $f(\ell) = \ell^{-1/2}$ [Sánchez-Fernández et al., 2017]. Yet there exist elections for which no committee has unit PD or perfect PD [Aziz et al., 2018]. For a detailed analysis of proportionality degrees of various multiwinner rules, we point to the work of Skowron [2021]. Finally, we note that by saying that a certain rule has PD $f$ we only indicate a lower bound on its performance. Consequently, for many rules we can say that they provide several different proportionality degrees, as in the case of PAV, which provides both nearly perfect PD and some nonzero PD (nonzero PD offers a stronger guarantee than the nearly perfect one for $\ell = 1$).

Computational Complexity. We assume knowledge of classic and parameterized computational complexity theory, including classes P and NP, the notions of hardness and completeness for a given complexity class, and FPT algorithms. Occasionally, we also refer to the coNP class and to higher levels of the Polynomial Hierarchy. Given a problem $X$ from NP, where we ask if a certain mathematical object exists, we write $\#X$ to denote its counting variant, where we ask for the number of such objects. Counting problems belong to the class $\#P$ and it is commonly believed that if a counting problem is $\#P$-complete then it cannot be solved in polynomial time. We mention that $\#P$-completeness is defined via Turing reductions [Valiant, 1979] and not many-one reductions, as in the case of NP-completeness. A counting problem $\#X$ Turing-reduces to a counting problem $\#Y$, denoted $\#X \leq_{T} \#Y$, if there is an algorithm that solves $\#X$ in polynomial time, provided it has access to $\#Y$ as an oracle (i.e., provided that it has a subroutine for solving $\#Y$ in constant time).

Computational Aspects of JR, EJR, and PD. There are polynomial-time algorithms that given an election and a committee size compute committees which provide JR [Aziz et al., 2017] or EJR [Aziz et al., 2018]. There is also a polynomial-time algorithm that given a committee verifies if it provides JR. The same task for EJR is coNP-complete [Aziz et al., 2017]. In this paper we answer analogous questions for the case of the proportionality degree.

3 Finding and Counting Cohesive Groups

As cohesive groups lay at the heart of JR, EJR, and PD, we start our discussion by analyzing the hardness of finding them. More precisely, we consider the following problem.

Definition 3. An instance of the COHESIVE-GROUP problem consists of an election $E$, a committee size $k$, and a positive integer $\ell$. We ask if $E$ contains an $\ell$-cohesive group.

Somewhat disappointingly, this problem is NP-complete. This follows via a reduction inspired by that provided by Aziz et al. [2017] to show that testing if a given committee provides EJR is coNP-complete (we include the proof for the sake of completeness, as some of our further hardness proofs follow by reductions from COHESIVE-GROUP).
**Theorem 3.1. Cohesive-Group is NP-complete**

**Proof.** We observe that Cohesive-Group is in NP: Given an election $E$ with $n$ voters, committee size $k$, and cohesiveness level $\ell$, it suffices to nondeterministically guess a group of at least $\ell \cdot n/k$ voters and check that the intersection of their approval sets contains at least $\ell$ candidates.

To show NP-hardness, we give a reduction from the NP-complete Balanced-Biclique problem [Johnson 1987]. The input for Balanced-Biclique consists of a bipartite graph $G$ and a nonnegative integer $k$. The vertices of $G$ are partitioned into two sets, $L(G)$ and $R(G)$, and we write $E(G)$ to denote the set of $G$’s edges; each edge connects a vertex from $L(G)$ with a vertex from $R(G)$. We ask if there is a size-$k$ subset of $L(G)$ and a size-$k$ subset of $R(G)$ such that each vertex from the former is connected with each vertex from the latter. Such two sets are jointly referred to as a $k$-biclique of $G$.

Given an instance of Balanced-Biclique, we form an instance of Cohesive-Group as follows. We construct an election $E'$, where $R(G)$ is the set of candidates and $L(G)$ is a collection of voters. A voter $\ell_i \in L(G)$ approves a candidate $r_j \in R(G)$ if $\ell_i$ and $r_j$ are connected in $G$. We extend $E'$ by adding $\max(|L(G)| - |R(G)|, 0)$ candidates not approved by any voter, we set the committee size to be $k' = |L|$, and we let the desired cohesiveness level be $\ell' = k$. This completes the construction.

Note that each $\ell'$-cohesive group in our election consists of at least $\ell'|\frac{n|}{k'} = k|\frac{n|}{k'} = k$ voters who approve at least $k$ common candidates. Focusing on exactly $k$ voters and $k$ candidates, we see that such a group exists if and only if $G$ has a $k$-biclique. This completes the proof.

On the positive side, Aziz et al. [2017] gave a polynomial-time algorithm for deciding if an election contains a 1-cohesive group (we refer to this variant of the problem as One-Cohesive-Group): It suffices to check if there is a candidate $c$ for whom $|A(c)| \geq n/k$, where $n$ is the total number of voters and $k$ is the committee size. If such a candidate $c$ exists, then the voters from $A(c)$ form a 1-cohesive group; otherwise, there are no 1-cohesive groups.

**Corollary 3.2 (Aziz et al. [2017]).** One-Cohesive-Group is in P.

We complement the above results by considering the complexity of #Cohesive-Group, i.e., the problem of counting cohesive groups. If we had an efficient algorithm for this problem, then we could also derive an efficient procedure for sampling cohesive groups uniformly at random [Jerrum et al. 1986], which would be quite useful. Indeed, we could use it, e.g., to experimentally study the distribution of cohesive groups in elections. Naturally, #Cohesive-Group is intractable—namely, #P-complete—since even deciding if a single cohesive group exists is hard. More surprisingly, the same holds for 1-cohesive groups.

**Theorem 3.3. #One-Cohesive-Group is #P-complete**

An intuition as to why finding a single 1-cohesive group is easy but counting them is hard is as follows. Using the argument from Corollary 3.2 for each candidate we can count (in polynomial time) the number of 1-cohesive groups whose members approve this candidate. Yet, if we simply

1Formally, an approximate counting algorithm would suffice to obtain a nearly uniform sampling procedure. Our results do not preclude existence of such an algorithm, but we leave studies in this direction for future work.
added these values, then some groups could be counted multiple times. If we used the inclusion-exclusion principle, then we would get the correct result, but doing so would take exponentially many arithmetic operations.

To give a formal proof of Theorem 3.3 we use the following intermediate problem, which captures counting cohesive groups that consist of a given number of voters.

**Definition 4.** In the $\#\text{FIXED-SIZE-COHESIVE-GROUP}$ problem (the $\#\text{FSCG}$ problem) we are given an election $E = (C, V)$, a committee size $k$, and two positive integers, $\ell$ and $x$. We ask how many $\ell$-cohesive groups that consist of exactly $x$ voters are there in $E$.

Given an instance $(E, k, \ell, x)$ of $\#\text{FSCG}$, by $\#\text{LCG}(E, k, \ell, x)$ we mean the number of $\ell$-cohesive groups of size $x$ from election $E$ for committee size $k$. If we omit parameter $x$, then we mean to total number of $\ell$-cohesive groups, irrespective how many voters they include. In the next proposition we show that $\#\text{FSCG}$ is computationally equivalent to $\#\text{COHESIVE-GROUP}$ (the proof is in Appendix A).

**Proposition 3.4.** $\#\text{COHESIVE-GROUP} \leq_{fp} \#\text{FSCG}$ and $\#\text{FSCG} \leq_{fp} \#\text{COHESIVE-GROUP}$.

We define problem $\#\text{ONE-FSCG}$ by fixing $\ell = 1$ in the definition of $\#\text{FSCG}$. Proposition 3.4 also holds for the case of $\#\text{ONE-FSCG}$ and $\#\text{ONE-COHESIVE-GROUP}$ (indeed, we never modify $\ell$ in the proposition’s proof). We use the computational equivalence of $\#\text{ONE-FSCG}$ and $\#\text{ONE-COHESIVE-GROUP}$ to prove Theorem 3.3.

**Proof of Theorem 3.3.** A problem belongs to $\#\text{P}$ if its value can be expressed as the number of accepting paths of a polynomial-time nondeterministic Turing machine. For $\#\text{ONE-COHESIVE-GROUP}$ it suffices that such a machine guesses a group of voters, verifies if they form an $\ell$-cohesive group (which can be done deterministically in polynomial time), and accepts if so. Hence, $\#\text{ONE-COHESIVE-GROUP}$ is in $\#\text{P}$. To show $\#\text{P}$-hardness of $\#\text{ONE-COHESIVE-GROUP}$, we give a reduction from $\#\text{SET-COVER}$ to $\#\text{ONE-FSCG}$. The latter problem is well-known to be $\#\text{P}$-complete, and the former is computationally equivalent to $\#\text{ONE-COHESIVE-GROUP}$.

Let $((U, S, k)$ be an instance of the $\#\text{SET-COVER}$ problem, where $U = \{u_1, \ldots, u_n\}$ is a universe, $S = \{S_1, \ldots, S_m\}$ is a family of subsets of $U$, and $k$ is a positive integer. The question is how many combinations of at most $k$ subsets from $S$ sum up to the universe $U$. We create an instance of $\#\text{ONE-FSCG}$ with an election $E’ = (C’, V’)$, such that the candidates correspond to the elements of $U$ and the voters correspond to the elements of $S$ (hence, we can speak both of a universe element $u_i$ and a candidate $u_i$, or of a set $S_j$ and voter $S_j$). For each candidate $u_i$ and voter $S_j$, $u_i$ is approved by voter $S_j$ if element $u_i$ does not belong to the set $S_j$. Further, we extend $E’$ by adding $\max(m - n, 0)$ new candidates not approved by anyone. Altogether, the number of candidates in $E’$ is at least $m$. We set the size of the final committee to be $k’ = m$. We also write $n’$ to denote the number of voters in $E’$; naturally, we have $n’ = m$. Due to the definition of a 1-cohesive group, its size must be at least $n’/k’ = m/m = 1$. Thus, every group of voters that approve at least one common candidate is a 1-cohesive group in our election.

Let us consider a 1-cohesive group of size $x’ \leq k$ and let us call it $T$. By definition, there is at least one candidate approved by all members of $T$. Let us call her $c’$. This means that $c’$ is not included in any set $S_j$ corresponding to the voters from $T$. Hence, the union of these sets is different
from $U$. On the other hand, if a group $R$ of $x' \leq k$ voters does not form an 1-cohesive group, then the sets corresponding to the voters from this group do sum up to the universe $U$. Indeed, for each candidate $c$ there is a voter in $R$ who does not approve $c$, which means that the corresponding set includes her. As a consequence, each member of $U$ belongs to at least one set corresponding to a voter from $R$.

Above observations mean that families of subsets from $S$ sum up to the universe $U$ if and only if the voter groups that correspond to these families are not 1-cohesive. Since for a positive integer $x$ there are $\binom{|S|}{x} = \binom{m}{x}$ size-$x$ families of sets from $S$, we conclude that the answer for our instance of \#SET-COVER is $\sum_{x=1}^{k} (\binom{|S|}{x} - \#LCG(E', 1, x))$. This completes the proof.

4 Computing a Committee with a Given PD

In this section we focus on the complexity of deciding if a committee with a given proportionality degree exists. At first, this problem may seem trivial as for each election there is a committee with nearly perfect PD [Aziz et al., 2018]. Yet, we find that the answer is quite nuanced. This stands in sharp contrast to analogous decision questions for JR and EJR, which are trivial (a committee with the desired property always exists so the algorithm always accepts). Formally, we consider the following problem.

**Definition 5.** In the PD-COMMITTEE problem we are given an election $E$, a committee size $k$, and a function $f : [k] \to \mathbb{Q}$, specified by listing its values. We ask if $E$ has a size-$k$ committee with proportionality degree at least $f$.

We find that PD-COMMITTEE is both \textsc{coNP}-hard and \textsc{NP}-hard. For the former result, we use the fact that for a given $\ell$, the $f(\ell)$ value of a PD function is binding only if the given election contains $\ell$-cohesive groups.

**Theorem 4.1.** PD-COMMITTEE is both \textsc{NP}-hard and \textsc{coNP}-hard

**Proof.** We will show \textsc{NP}-hardness in Theorem 4.4 and here we focus on \textsc{coNP}-hardness. To this end, we give a reduction from COHESIVE-GROUP to the complement of PD-COMMITTEE. Let $(E, k, \ell)$ be our input instance, where $E = (C, V)$ is an election, $k$ is the committee size, and $\ell$ is the cohesiveness level. The question is if there exists an $\ell$-cohesive group for election $E$ with committee size $k$. For convenience, we set $n = |V|$, and $m = |C|$.

We create an instance of the complement of PD-COMMITTEE as follows. Let $s$ be the smallest integer such that $s \cdot k > m$. We form an election $E'$ by first copying $E$ and then adding (a) $s \cdot k$ new candidates who are not approved by any voters and (b) $(s - 1) \cdot n$ new voters who do not approve any candidates. Altogether, in $E'$ we have $n' = s \cdot n$ voters, and $m' = m + s \cdot k$ candidates. Further, we set the committee size to be $k' = s \cdot k$ and we let the PD function $f$ be such that for $i < \ell$ we have $f(i) = 0$ and for $i \geq \ell$ we have $f(i) = k'$. This completes the construction.

Note that the minimum size of an $\ell$-cohesive group in $E'$ is equal to the minimum size of an $\ell$-cohesive group in $E$, because $(n')/k' = (s \cdot n)/k = (n)/k$. Thus every $\ell$-cohesive group from $E$ is also an $\ell$-cohesive group for $E'$ and vice versa. Further, each size-$k'$ committee must contain at least one
new candidate, because \( k' = s \cdot k > m \). Yet, the new candidates are not approved by any voter and, so, if \( E' \) has some \( \ell \)-cohesive group, then its average satisfaction must be strictly below \( f(\ell) = k' \). This means that if \( E' \) has a committee with PD \( f \) then there are no \( \ell \)-cohesive groups in \( E' \) (and, thus, there are no cohesive groups in \( E \)). In other words, the answer for the \textsc{PD-Committee} instance is “yes” if and only if the answer for the \textsc{Cohesive-Group} instance is “no.” Since, by Theorem 3.1 the latter is \text{NP}-complete, the former is \text{coNP}-hard.

Since \textsc{PD-Committee} is both \text{NP}-hard and \text{coNP}-hard, it is unlikely that it is complete for either of these classes (we would have \( \text{NP} = \text{coNP} \) if it were). Indeed, we suspect that it is complete for \text{NP} and we show that it belongs to this class. An \text{NP}-hardness result remains elusive, unfortunately.

**Theorem 4.2.** \textsc{PD-Committee} is in \text{NP}.

**Proof.** Consider an instance \((E, k, f)\) of \textsc{PD-Committee}. It is a “yes”-instance exactly if there exists a size-\( k \) committee such that for every \( \ell \in [k] \), every \( \ell \)-cohesive group has average satisfaction at least \( f(\ell) \). We can verify that this holds by first nondeterministically guessing the committee and then asking the oracle if there is a cohesive group for which the constrained implied by the PD function is failed (since computing an average satisfaction of a given cohesive group can be done in polynomial time, this task belongs to \text{NP}). We accept if the oracle answers “yes” and we reject otherwise.

While \textsc{PD-Committee} seems very hard in general, for some classes of PD functions it is significantly easier. As an extreme example, for nearly perfect ones it is trivially in \text{P} because PAV winning committees always have nearly perfect PD. We consider the following restricted variants of \textsc{PD-Committee}: In \textsc{Constant-PD-Committee} we require the desired PD functions to be constant, in \textsc{Unit-PD-Committee} we require them to take value 1 for each argument, and in \textsc{Perfect-PD-Committee} we require them to be perfect. We find that both \textsc{Constant-PD-Committee} and \textsc{Unit-PD-Committee} are \text{NP}-complete and, thus, likely much easier than the general variant. To establish these results, it suffices to show membership in \text{NP} for the former and \text{NP}-hardness for the latter.

**Theorem 4.3.** \textsc{Constant-PD-Committee} is in \text{NP}.

**Proof.** Consider an instance \((E, k, f)\) of \textsc{Constant-PD-Committee}, where \( E = (C, V) \) is an election, \( k \) is the committee size, and \( f \) is a constant PD function. Since \( f \) is a constant function, there is a value \( x \) such that for each \( \ell \in [k] \) we have \( f(\ell) = x \). To show that \textsc{Constant-PD-Committee} is in \text{NP}, we give a polynomial-time algorithm that given such an instance and size-\( k \) committee \( W \) verifies if \( W \) has PD \( f \).

Let \( n = |V| \) be the number of voters. For each candidate \( c \in C \), we define \( \text{sat}(c) \) to be the average satisfaction of \( \left\lceil \frac{n}{k} \right\rceil \) members of \( A(c) \) that are least satisfied with \( W \); if \( A(c) \) contains fewer than \( \frac{n}{k} \) voters then we set \( \text{sat}(c) = +\infty \). We set \( y = \min_{c \in C} \text{sat}(c) \). If \( y = +\infty \) then election \( E \)
has no cohesive groups and \( W \) has PD \( f \) trivially. Otherwise, \( y \) is the smallest average satisfaction that a 1-cohesive group from \( E \) has for \( W \) (indeed, every 1-cohesive group must have at least \( \lceil \frac{n}{k} \rceil \) members and for each \( c \in C \), each 1-cohesive group whose members approve \( c \) has satisfaction at least \( \text{sat}(c) \)). For each \( \ell \in [k] \), each \( \ell \)-cohesive group also has satisfaction at least \( y \) (each such group also is a 1-cohesive group and, so, also has average satisfaction at least \( y \)). Thus, if \( y \geq x \) then we accept and otherwise we reject. This algorithm runs in polynomial time. \( \square \)

**Theorem 4.4.** \textsc{Unit-PD-Committee} is NP-hard

**Proof.** We give a reduction from a variant of the classic X3C problem, which we call RX3C and which is well-known to be NP-complete [Gonzalez 1985]: An instance of RX3C consists of a universe set \( U = \{u_1, u_2, ..., u_{3k}\} \) and a family \( S = \{S_1, S_2, ..., S_{3k}\} \) of size-3 subsets of \( U \), each element from \( U \) belongs to exactly three sets from \( S \), and we ask if there exist \( k \) subsets from \( S \) which sum up to the universe \( U \).

We form an instance of \textsc{Unit-PD-Committee} with an election \( E \), committee size \( k \), and unit PD function. We let the sets from \( S \) be the candidates in \( E \), and we let the universe elements be the voters. A voter \( u_i \) approves a candidate \( S_j \) if \( u_i \in S_j \). This completes the construction.

We note that all cohesive groups in \( E \) contain exactly three voters and have cohesiveness level one. This holds because each candidate is approved by exactly three voters and this is also the lower bound on the size of 1-cohesive groups in \( E \) (indeed, \( \frac{3k}{k} = 3 \)).

It is clear that if there exist \( k \) subsets from \( S \) which sum up to \( U \), then the corresponding candidates form a committee which has average satisfaction at least 1. Indeed, for each voter there is at least one candidate in the committee that he or she approves (in fact, exactly one). Otherwise the selected sets would not sum up to \( U \). As a consequence, the average satisfaction of each (1-)cohesive group with the committee is at least 1.

Next, let us show that if there exists a committee \( W \) of size \( k \) such that each cohesive group has average satisfaction at least 1, then there is a collection \( T \) of \( k \) sets from \( S \) that sum up to \( U \) (i.e., there is an exact cover of \( U \)). Let \( B \) be the sum of the total satisfactions of all the \( 3k \) 1-cohesive groups in \( E \). Since each 1-cohesive group has average satisfaction at least one, its total satisfaction is at least 3. There are \( 3k \) such groups, so we have that \( B \) is at least \( 9k \). Moreover, \( B \) is equal to \( 9k \) exactly if each 1-cohesive group has average satisfaction equal to 1. However, each committee member is approved by exactly three voters, and each of these voters belongs to exactly three 1-cohesive groups. Hence \( B = 9k \) and each 1-cohesive group has average satisfaction equal to 1.

Consider some set \( S_j = \{u_{j_1}, u_{j_2}, u_{j_3}\} \) such that candidate \( S_j \) is a member of committee \( W \). Naturally, \( \{u_{j_1}, u_{j_2}, u_{j_3}\} \) is a 1-cohesive group, all its member approve \( S_j \), and, so, its average satisfaction is at least 1. Indeed, by previous discussion we know that it is exactly 1. Hence, for each voter in \( \{u_{j_1}, u_{j_2}, u_{j_3}\} \), candidate \( S_j \) is the only member of \( W \) that he or she approves. If we repeat this reasoning for every member of \( W \), we find that each of them is approved by exactly three voters and no two of them are approved by the same voters. This means that \( W \) corresponds to an exact cover of \( U \). The proof is complete. \( \square \)

**Corollary 4.5.** Both \textsc{Constant-PD-Committee} and \textsc{Unit-PD-Committee} are NP-complete.
As all the cohesive groups in the election constructed in the proof of Theorem 4.4 have cohesiveness level 1, we have a stronger result: Given a PD function \( f \) such that \( f(1) = 1 \), it is NP-hard to decide if there is a committee with proportionality degree \( f \). In particular, we have the next corollary.

**Corollary 4.6.** **PERFECT-PD-COMMITTEE** is NP-hard.

We can extend Theorem 4.4 to work for any positive integer constant \( x \) and functions \( f \) such that \( f(1) = x \). For example, for \( x = 2 \) it suffices to extend the constructed election with three voters that do not approve anyone and with a single candidate who is approved by all the other voters. It would also be interesting to consider functions \( f \) such that \( f(1) \) is a constant between 0 and 1, but we leave it for future work. The above results are nicely aligned with existing polynomial-time algorithms for computing committees with guarantees on their PD. For example, there are polynomial-time algorithms for computing EJR committees, and EJR committees are guaranteed to have PD \( f \) such that \( f(\ell) = \frac{\ell - 1}{2} \) [Sánchez-Fernández et al., 2017]. As we see, \( f(1) = 0 \) (though this could be improved very slightly\(^3\)). As we have shown, extending the algorithm to find committees with PD functions \( f \) such that \( f(1) = 1 \) (whenever such committees exist) would not be possible in polynomial time (assuming P \( \not= \) NP).

## 5 Computing the PD of a Given Committee

Sometimes, instead of computing a committee with a specified PD, we would like to establish the PD of an already existing one. For example, this would be the case if we wanted to experimentally compare how well the committees provided by various voting rules represent the voters.

One way to proceed would be as follows: For a given election \( E \) and committee \( W \), consider each cohesiveness level \( \ell \) and, using binary search, find value \( f(\ell) \), \( 0 \leq f(\ell) \leq |W| \), such that each \( \ell \)-cohesive group has average satisfaction at least \( f(\ell) \), but for every \( \varepsilon > 0 \) there exists an \( \ell \)-cohesive group with average satisfaction below \( f(\ell) + \varepsilon \) (or there are no \( \ell \)-cohesive groups in this election). Using binary search to compute this value is possible because in an election with \( n \) voters and committee size \( k \), there are at most \( O(kn^2) \) different average satisfaction values of cohesive groups (each cohesive group can have total satisfaction between 0 and \( nk \), and each cohesive group can have at most \( n \) voters). Running such binary search requires the ability to solve the following problem.

**Definition 6.** In the PD-\textsc{Failure} problem we are given an election \( E \), a committee \( W \), a cohesiveness level \( \ell \), and a nonnegative rational threshold \( y \leq k \). We ask if \( E \) contains an \( \ell \)-cohesive group whose average satisfaction for \( W \) is lower than \( y \).

As one may expect, this problem is NP-complete. Membership in NP follows by nondeterministically guessing an \( \ell \)-cohesive group and checking if its average satisfaction is below \( y \). To

---

\(^3\)Since the committee provides EJR, and thus JR, this zero could be replaced by \( \frac{1}{\frac{n}{k}} = \frac{k}{n} \), where \( n \) is the number of voters and \( k \) is the committee size. This follows from the fact that in each 1-cohesive group of size \( \frac{n}{k} \) there is at least one candidate who approves at least one voter.  


show NP-hardness, we note that setting \( y \) to an impossible-to-achieve value makes the problem equivalent to testing if an \( \ell \)-cohesive group exists.

**Theorem 5.1.** PD-FAILURE is NP-complete.

**Proof.** Membership in NP was already argued in the paragraph above the theorem statement. To prove NP-hardness, we show a reduction from COHESIVE-GROUP to PD-FAILURE. Let \( (E, k, \ell) \) be an instance of the COHESIVE-GROUP problem, where \( E \) is an election consisting of candidates \( C \) and voters \( V \), \( k \) is the size of the final committee, and \( \ell \) is the cohesiveness level. We ask if there exists an \( \ell \)-cohesive group for election \( E \) with committee size \( k \).

To create a PD-FAILURE instance, we use the same election \( E \), the same committee size \( k \), and the same cohesiveness level \( \ell \), but we add \( k \) fresh candidates who are not approved by any voter and select them to the committee \( W' \). Further, we set the PD threshold \( y = k \). From the above observation we conclude that if there exists any valid \( \ell \)-cohesive group, then its average satisfaction with \( W' \) is 0, which is strictly less than \( y \). Therefore if there exists an \( \ell \)-cohesive group with average satisfaction lower than \( y \), then this group is also an \( \ell \)-cohesive group for the COHESIVE-GROUP instance. Further, if there are no \( \ell \)-cohesive groups for the PD-FAILURE instance, then the COHESIVE-GROUP instance doesn’t have any \( \ell \)-cohesive groups either.

Since the COHESIVE-GROUP problem is NP-complete, the PD-FAILURE problem is in NP and we reduced the COHESIVE-GROUP problem to the PD-FAILURE problem, the PD-FAILURE problem is also NP-complete.

### 5.1 ILP Formulation

Fortunately, in practice we may be able to solve instances of our problem by expressing them as integer linear programs (ILPs) and solving them using off-the-shelf software. Specifically, let us consider an instance of PD-FAILURE with election \( E = (C, V) \), committee \( W \), cohesiveness level \( \ell \), and threshold \( y \). We set \( m = |C| \), \( n = |V| \), and \( k = |W| \). For convenience, let \( A \) be the binary matrix of approvals for \( E \), that is, we have \( a_{ij} = 1 \) if the \( i \)-th voter approves the \( j \)-th candidate, and we have \( a_{ij} = 0 \) otherwise. We note that if there is an \( \ell \)-cohesive group \( X \) whose satisfaction for \( W \) is below \( y \), then there is such a group of size exactly \( s = \lceil \ell \cdot n/k \rceil \) (e.g., consider \( X \) and remove sufficiently many voters who approve the most members of \( W \)).

To form our ILP instance, we first specify the variables:

1. For each \( i \in [n] \), we have a binary variable \( x_i \), with the intention that \( x_i = 1 \) if the \( i \)-th voter is included in the sought cohesive group, and \( x_i = 0 \) otherwise.

2. For each \( j \in [m] \), we have a binary variable \( y_j \), with the intention that \( y_j = 1 \) if all the voters in the group specified by variables \( x_1, \ldots, x_n \) approve the \( j \)-th candidate, and \( y_j = 0 \) otherwise.

We refer to the voters (to the candidates) whose \( x_i \) (\( y_j \)) variables are set to 1 as selected. Next, we specify the constraints. Foremost, we ensure that we select exactly \( s \) voters and at least \( \ell \) candidates:

\[
\sum_{i=1}^{n} x_i = s, \quad \text{and} \quad \sum_{j=1}^{m} y_j \geq \ell.
\]
Then, we ensure that each selected voter approves all the selected candidates. For each $j \in [m]$, we form constraint:

$$\sum_{i=1}^{n} a_{ij} \cdot x_i \geq s \cdot y_j.$$  

If the $j$-th candidate is not selected, then this inequality is satisfied trivially. However, if the $j$-th candidate is selected, then the sum on the left-hand side must be at least $s$, i.e., there must be at least $s$ selected voters who approve the $j$-th candidate. Since there are exactly $s$ selected voters, all of them must approve the $j$-th candidate.

Finally, we ensure that the average satisfaction of the selected voters is below $y$, by adding constraint $\frac{1}{s} \sum_{i=1}^{n} \sum_{j \in W} a_{ij} \cdot x_i < y$. If there is an assignment that satisfies these constraints, then the selected voters form an $\ell$-cohesive group with average satisfaction below $y$. Otherwise, no such group exists.

### 5.2 Verification

For a comparison with previous studies regarding JR and EJR, we also consider the following verification problem.

**Definition 7.** In the PD-VERIFICATION problem we are given an election $E$, a committee $W$, a PD function $f$, and we ask if $W$ has proportionality degree $f$.

As PD-VERIFICATION is very closely related to the complement of PD-FAILURE, we find that it is coNP-complete (we give the formal proof in Appendix B).

**Theorem 5.2.** PD-VERIFICATION is coNP-complete.

Testing if a committee provides EJR is coNP-complete as well [Aziz et al., 2017], so in this respect PD and EJR are analogous. There is also a polynomial-time algorithm for testing if a committee provides JR, and in the PD world this corresponds to a polynomial-time algorithm for checking if a committee admits a given constant PD function. Such an algorithm was included as part of the proof of Theorem 4.3.

**Corollary 5.3.** PD-VERIFICATION for a constant PD functions (provided as input) is in P.

### 6 Dealing With Computational Hardness

In this section we consider circumventing the computational hardness of our problems by studying their parameterized complexity and by considering structured elections.

#### 6.1 Fixed-Parameter Tractability

Our two main problems, PD-COMMITTEE and PD-FAILURE, are fixed-parameter tractable with respect to the number of candidates and the number of voters.

For PD-FAILURE and the parameterization by the number of candidates, we proceed similarly as in the proof of Theorem 4.3. Namely, for each set $R$ of candidates we consider the set $V(R)$ of all
the voters that approve members of $R$, one-by-one remove from this set the voters with the highest satisfaction, and watch if at any point we obtain an $\ell$-cohesive group with average satisfaction below the required value. Using a similar approach, and trying every possible committee, we also obtain an algorithm for PD-COMMITTEE.

For the parameterization by the number of voters, we solve our problems by forming ILP instances and solving them using the classic algorithm of [Lenstra, Jr. 1983]. This is possible because with $n$ voters there are at most $2^n$ cohesive groups and each candidate has one of $2^n$ types (where the type of a candidate is the set of voters that approve him; candidates with the same type are interchangeable).

**Theorem 6.1.** There are FPT algorithms for PD-COMMITTEE and PD-FAILURE both for the parameterization by the number of candidates and for the parameterization by the number of voters.

**Proof.** Let us first consider the parameterization by the number of candidates and the PD-FAILURE problem. Our input consists of an election $E = (C, V)$, committee $W$ of size $k$, cohesiveness level $\ell$, and rational threshold $y$. Let $m$ be the number of candidates and let $n$ be the number of voters. For each subset of $\ell$ candidates, we find a group of $\ell \cdot \frac{n}{k}$ voters who are least satisfied with $W$. If the lowest satisfaction among such groups is below $y$ then we accept and otherwise we reject. The correctness and fixed-parameter tractability follow immediately.

For parameterization by the number of candidates and the PD-COMMITTEE problem, it suffices to try all committees and for each of them (and each cohesiveness level) use the algorithm for PD-FAILURE to check if it indeed achieves required PD.

Next let us move on to the parameterization by the number of voters and the PD-FAILURE problem. We use the same notation as in the argument above for parameterization by the number of candidates. It suffices to consider every subset of voters, check if it is an $\ell$-cohesive group, and verify if its average satisfaction is below $y$.

For the case of PD-COMMITTEE and parameterization by the number of voters, we employ integer linear programming. Let $E = (C, V)$ be the input election with $m$ candidates and $n$ voters. We seek a committee of size $k$, with PD $f$. There are $2^n$ subsets of the voters, and we order them in some way, so for each $i \in [2^n]$ we can speak of the $i$-th subset. For each such subset, we say that a candidate has type $i$ if she is approved exactly by the voters from the $i$-th subset (and only by them). For each $i \in [2^n]$ we let $c_i$ be the number of type-$i$ candidates and we form a variable $x_i$, with the intended meaning that $x_i$ is the number of type-$i$ candidates in the committee. We introduce the following constraints:

1. For each $i$, we require that $x_i \leq c_i$, i.e., that we do not select more type-$i$ candidates than available.
2. We require that $\sum_{i \in [2^n]} x_i = k$, i.e., we ensure that we select a committee of size exactly $k$.
3. For each $\ell \in [k]$ and each $\ell$-cohesive group $S$ of voters (due to our parameterization, we can enumerate them all), we form the following constraint:

$$\sum_{v \in S} \sum_{i \in [2^n]} x_i \cdot [v \text{ approves type}-i \text{ candidates}] \geq |S| \cdot f(\ell),$$

$$14$$
where we use the Iverson bracket notation (i.e., for a true/false statement $F$, by $[F]$ we mean 1 if $F$ is true and we mean 0 otherwise). This constraint ensures that each cohesive group has required level of average satisfaction.

We solve this ILP instance using the classic algorithm of Lenstra, Jr. [1983]. Since the number of variables is $2^n$ and $n$ is the parameter, doing so is possible in FPT time. This completes the proof.

Testing if a committee provides EJR is also fixed-parameter tractable for the parameterizations considered in Theorem 6.1. So, from this point of view, dealing with PD is not harder than dealing with EJR.

Finally, the problem of counting cohesive groups (and, thus, also the problem of deciding if groups with particular cohesiveness level exist) also is fixed-parameter tractable for our parameters. For parameterization by the number of candidates, we can use the inclusion-exclusion principle, and for the parameterization by the number of voters we can explicitly look at each subset of voters.

**Theorem 6.2.** There are FPT algorithms for $(\#)$COHESIVE-GROUP, for the parameterizations by the number of candidates and by the number of voters.

**Proof.** Let us first consider the parameterization by the number of voters and the #COHESIVE-GROUP problem. Our input consists of an election $E = (C, V)$, committee $W$ of size $k$, and cohesiveness level $\ell$. Let $m$ be the number of candidates and let $n$ be the number of voters. Initially, we have a counter set to zero. For each subset of at least $\ell \cdot \frac{n}{k}$ voters, we compute the set of candidates that are approved by all these voters. If this set has size at least $\ell$ then we increase the counter and otherwise we do not. At the end, the counter contains the desired answer. For the COHESIVE-GROUP problem it is enough to check if the counter is above 0.

For the parameterization by the number of candidates and the COHESIVE-GROUP problem, for each subset of $\ell$ candidates we calculate the number of voters that approve all these candidates and accept if it is at least $\ell \cdot \frac{n}{k}$, we reject if we do not accept for any subset. For the #COHESIVE-GROUP problem, we can use the inclusion-exclusion principle.

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### 6.2 Structured Preferences

Next we consider two domains of structured preferences, introduced by Elkind and Lackner [2015]. Such domains are interesting because, on the one hand, they capture some realistic scenarios, and, on the other hand, by assuming them it is often possible to provide polynomial time algorithms for problems that in general are intractable.

**Definition 8** (Elkind and Lackner [2015]). An election $E = (C, V)$ has candidate interval (CI) preferences (voter interval preferences, VI) if it is possible to order the candidates (the voters) so that for each voter $v$ (for each candidate $c$) the set $A(v)$ (the set $A(c)$) is an interval w.r.t. this order.
For an example of CI preferences, consider a political election where candidates are ordered according to the left-to-right spectrum of opinions and the voters approve ranges of candidates whose opinions are close enough to their own. \cite{ElkindLackner2015} gave algorithms for deciding if a given election has CI or VI preferences, and for computing appropriate orders of candidates or voters. Thus, for simplicity, we assume that these orders are provided together with our input elections. We mention that a number of other preference domains are considered in the literature—see, e.g., the works of \cite{Yang2019} and \cite{GodziszewskiEtAl2021}—but the CI and VI ones are by far the most popular. For a very detailed discussion of structured domains, albeit in the world of ordinal preferences, we point to the survey of \cite{ElkindEtAl2022}.

Unfortunately, even for CI and VI elections we do not know how to solve the PD-COMMITTEE problem in polynomial-time. Nonetheless, we do have polynomial-time algorithms for the PD-FAILURE problem.

**Theorem 6.3.** PD-FAILURE restricted to either CI or VI elections is in \( P \).

**Proof.** First, we give an algorithm for the CI case. Our input consists of an election \( E = (C, V) \), where \( C = \{ c_1, \ldots, c_m \} \) and \( V = (v_1, \ldots, v_n) \), a size-\( k \) committee \( W \), cohesiveness level \( \ell \), and threshold value \( y \). Without loss of generality, we assume that \( E \) is CI with respect to the order \( c_1 \preceq c_2 \preceq \cdots \preceq c_m \).

Since \( E \) is a CI election, we observe that if \( X \) is some cohesive group whose all members approve some two candidates \( c_i \) and \( c_j, i \leq j \), then all members of \( X \) also approve candidates \( c_{i+1}, \ldots, c_{j-1} \). For each \( i \leq m - \ell + 1 \), let \( X(i) \) be the set of all voters who approve each of the candidates \( c_i, \ldots, c_{i+\ell-1} \). By the preceding argument, we see that every \( \ell \)-cohesive group can be obtained by taking some set \( X(i) \) and (possibly) removing some of its members.

Our algorithm proceeds as follows. For each set \( X(i) \), we form a set \( Y(i) \) by taking \( X(i) \) and removing all but \( \lceil \ell \cdot \frac{n}{k} \rceil \) voters that are least satisfied with \( W \). (If a given \( X(i) \) contains fewer than \( \lceil \ell \cdot \frac{n}{k} \rceil \) voters then we set \( Y(i) = \emptyset \) and we assume that the average satisfaction of its voters is \(+\infty\).) If there is some \( i \) such that the average satisfaction of the voters in \( Y(i) \) is below \( y \), then we accept (indeed, we have just found an \( \ell \)-cohesive group with average satisfaction below \( y \)). If there is no such \( Y(i) \), then we reject (we do so because each nonempty \( Y(i) \) has the lowest average satisfaction among all the \( \ell \)-cohesive groups that can be obtained by removing voters from \( X(i) \)). Correctness and polynomial running time follow immediately.

Now let us consider the VI case. We use the same notation as before, except that we assume that \( E \) is VI with respect to the voter order \( v_1 \preceq v_2 \preceq \cdots \preceq v_n \). We use the same algorithm as in the CI case, but for the \( X(i) \) sets defined as follows (let \( s = \lceil \ell \cdot \frac{n}{k} \rceil \)): For each \( i \in [n - s + 1] \), we let \( X(i) = \{ v_i, v_{i+1}, \ldots, v_j \} \), where \( j \) is the largest value such that \( |A(v_i) \cap A(v_j)| \geq \ell \) (if \( v_i \) approves fewer than \( \ell \) candidates then \( X(i) \) is empty). The algorithm remains correct because, as in the CI case, every \( \ell \)-cohesive group is a subset of some \( X(i) \).

Similar reasoning and observations as in the above proof also give the algorithms for counting cohesive groups (and, thus, for deciding their existence).

\footnote{In particular, the approach of \cite{PetersLackner2020} based on solving totally unimodular ILP instances does not seem to work here.}
Theorem 6.4. \((\#)\text{COHESIVE-GROUP restricted to either CI or VI elections is in P.}\)

We prove Theorem 6.4 via the following two theorems (they suffice due to Proposition 3.4).

Theorem 6.5. There is a polynomial-time algorithm for the \#FSCG problem under the VI restriction.

Proof. Let \((E, k, \ell, x)\) be a \#FSCG VI instance, where \(E\) is an election with candidates \(C\) and voters \(V, k\) is the committee size, \(\ell\) is the cohesiveness level, and \(x\) is the size of cohesive groups. We ask how many \(\ell\)-cohesive groups of size \(x\) are there in election \(E\). We assume that \(V = (v_1, \ldots, v_n)\) and the election is VI for this order of the candidates.

We observe that if voters \(v_i\) and \(v_j\) approve candidate \(c_p\), then each voter \(v_k\) between \(v_i\) and \(v_j\) also approves \(c_p\), because under VI each candidate is approved by a consecutive segment of voters. As a result, if \(v_i\) and \(v_j\) have at least \(\ell\) common candidates, then each voter \(v_k\) between \(v_i\) and \(v_j\) also approves these candidates.

By \(\text{smallestCG}(v_i, \ell, x)\) we mean the number of \(\ell\)-cohesive groups of size \(x\) in which voter \(v_i\) has the smallest index. Then, the sum of the \(\text{smallestCG}\) values over all the voters is the final answer. Now let us show how to calculate \(\text{smallestCG}(v_i, \ell, x)\).

Given a voter \(v_i\), a group cohesiveness level \(\ell\), and an integer \(x\), we find the greatest index \(j\) such that voter \(v_i\) still has at least \(\ell\) common candidates with \(v_i\). If \(v_j\) does not exist or the number of voters in range \([v_i, v_j]\) is lower than \(x\), then return 0. Otherwise, we select the voter \(v_i\) and \(x - 1\) other voters from \([v_i + 1, v_j]\); we can do it in \(\binom{x-1}{x-1}\) ways and this is the value we output. This completes the proof.

Theorem 6.6. There is a polynomial-time algorithm for the \#FSCG problem under the CI restriction.

Proof. Let \((E, k, \ell, x)\) be a \#FSCG CI instance, where \(E\) is an election with candidates \(C\) and voters \(V, k\) is the committee size, \(\ell\) is the cohesiveness level, and \(x\) is the size of cohesive groups. We ask how many \(\ell\)-cohesive groups of size \(x\) are there in election \(E\). We assume that \(C = \{c_1, \ldots, c_m\}\) and the election is CI for candidate order \(c_1, c_2, \ldots, c_m\).

We observe that if candidates \(c_i\) and \(c_j\) are approved by voter \(v_p\), then each candidate \(c_k\) between \(c_i\) and \(c_j\) is also approved by \(v_p\), because under CI each voter approves a consecutive segment of candidates. As a result, for each \(\ell\)-cohesive group the candidates approved by all its members form a consecutive segment.

By \(\text{smallestCG}(c_j, \ell, x)\) we mean the number of \(\ell\)-cohesive groups of size \(x\) in which candidate \(c_j\) is the commonly approved candidate with the smallest index. Then, the sum of the \(\text{smallestCG}\) values through all the candidates is the final answer. Now let us show how to calculate the function \(\text{smallestCG}(c_j, \ell, x)\).

Assume we are given candidate \(c_j\), group cohesiveness level \(\ell\), and an integer \(x\). Let \(L_1\) be the set of voters that approve all the candidates from \(\{c_j, c_{j+1}, \ldots, c_{j+\ell-1}\}\) and at least one candidate which has index strictly smaller than \(j\). Similarly, let \(L_2\) be the set of voters that approve all the candidates from \(\{c_j, c_{j+1}, \ldots, c_{j+\ell-1}\}\) and do not approve any candidate whose index is strictly smaller than \(j\). Both values can be calculated in polynomial-time by a single iteration through election \(E\). Now let us point out that each \(\ell\)-cohesive group accounted for in \(\text{smallestCG}(c_j, \ell, x)\)
must consist of at least one voter included in $L_2$ and some voters included in $L_1$. As we do not know how many voters we should take from the first part, we iterate through all possible partition sizes. Thus, the number of $\ell$-cohesive groups of size $x$ whose members’ smallest index of a commonly approved candidate is $j$, is equal to:

$$\sum_{k=1}^{\min(|L_2|,x)} \binom{|L_2|}{k} \cdot \binom{|L_1|}{x-k}$$

This completes the proof.

Similar approach shows that testing if a committee provides EJR can be done in polynomial time for CI or VI elections (to the best of our knowledge, this is a folk result).

**Perfect PD in CI/VI Elections?** [Aziz et al. 2018] have shown that for each election and each committee size there is a committee with a nearly perfect PD, but there are scenarios where committees with perfect PDs do not exist. Unfortunately, this remains true even if the elections are CI and VI at the same time.

**Example 1.** Consider an election $E = (C, V)$, where $C = \{c_1, \ldots, c_7\}$, and $V = (v_1, \ldots, v_{15})$. The committee size is $k = 5$ and the approval sets are as follows:

|     | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| $v_1$| 1     | -     | -     | -     | -     | -     | -     |
| $v_2$| 1     | -     | -     | -     | -     | -     | -     |
| $v_3$| 1     | 1     | -     | -     | -     | -     | -     |
| $v_4$| -     | 1     | -     | -     | -     | -     | -     |
| $v_5$| -     | 1     | 1     | -     | -     | -     | -     |
| $v_6$| -     | -     | 1     | -     | -     | -     | -     |
| $v_7$| -     | -     | 1     | 1     | -     | -     | -     |
| $v_8$| -     | -     | -     | 1     | -     | -     | -     |
| $v_9$| -     | -     | -     | 1     | 1     | -     | -     |
| $v_{10}$| -     | -     | -     | -     | 1     | -     | -     |
| $v_{11}$| -     | -     | -     | -     | 1     | 1     | -     |
| $v_{12}$| -     | -     | -     | -     | -     | 1     | -     |
| $v_{13}$| -     | -     | -     | -     | -     | -     | 1     |
| $v_{14}$| -     | -     | -     | -     | -     | -     | 1     |
| $v_{15}$| -     | -     | -     | -     | -     | -     | 1     |

Clearly, the election is both CI and VI. We see that $n/k = 3$ and, thus, for each $i \in [7]$, voters $v_{2i-1}, v_{2i}, v_{2i+1}$ form a cohesive group (for candidate $c_i$).

Now consider some size-$k$ committee. If it does not contain some candidate $c_i$, then the 1-cohesive group $\{v_{2i-1}, v_{2i}, v_{2i+1}\}$ must have average satisfaction below 1. Indeed, altogether members of this group give at most five approvals, of which three go to $c_i$. Thus, without $c_i$, the average satisfaction is at most $2/3 < 1$. However, since the committee size is five and there are seven candidates, for each committee there is some 1-cohesive group with satisfaction below 1. Thus there is no committee with a perfect PD for this election and committee size five.
7 Conclusions and Future Work

We have shown that computing committees with a given proportionality degree is, apparently, more difficult than computing EJR committees, but verification problems for these two notions have the same complexity. Two most natural directions of future work would be to establish the exact complexity of the PD-COMMITTEE problem and experimentally analyze PDs of committees provided by various voting rules.

Acknowledgments

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 101002854).

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## Appendix

### Proof of Proposition 3.4

We prove Proposition 3.4 via Lemmas A.1 and A.2 below.

**Lemma A.1.** \( \#\text{COHESIVE-GROUP} \leq^{fp}_{\mathbb{P}} \#\text{FSCG} \)

**Proof.** Let \((E, k, \ell)\) be an instance of \#COHESIVE-GROUP, where \(E\) is an election instance with \(m\) candidates and \(n\) voters, \(k\) is the committee size, and \(\ell\) is the cohesiveness level.

To count the number of \(\ell\)-cohesive groups, it suffices to sum up the number of \(\ell\)-cohesive groups of each possible size \(x\). From the definition of an \(\ell\)-cohesive group, we know that its size must be at least \(\ell \cdot \frac{n}{k}\). It is also clear that its size cannot exceed the number of voters. Thus, \(\#\text{LCG}(E, \ell, k) = \sum_{x=\lceil \ell \cdot \frac{n}{k} \rceil}^{n} \#\text{LCG}(E, \ell, k, x)\). This concludes the argument. \(\square\)

We also have a reduction in the reverse direction, thus obtaining computational equivalence between \#COHESIVE-GROUP and \#FSCG.
Lemma A.2. \#FSCG $\leq_{T^P} \text{\#COHESIVE-GROUP}$

Proof. Let $(E, k, \ell, x)$ be an instance of \#FSCG, where $E$ is an election with $m$ candidates and $n$ voters, $k$ is the size of the final committee, $\ell$ is the cohesiveness level, and $x$ is the size of the considered groups. We assume that $\ell \cdot \frac{n}{k} \leq x \leq n$ and $1 \leq \ell \leq m$ as otherwise we would immediately output zero as the answer.

We create an election $E'$ which, initially, is a copy of $E$. Next, we extend $E'$ by adding $n \cdot x \cdot (x+1) - n$ new voters that do not approve any candidates and $n \cdot x \cdot (x+1) \cdot m - m$ new candidates that are not approved by any voter. Thus, in $E'$ we have $n' = n \cdot x \cdot (x+1)$ voters and $m' = n \cdot x \cdot (x+1) \cdot m$ candidates. The aim of adding the new voters and candidates is to establish the lower bound on the size of the cohesive groups.

If we were able to count the number of $\ell$-cohesive groups with the size greater or equal to $x$ and those with the size strictly greater than $x$, then the difference between these two values would be the number of $\ell$-cohesive groups with size $x$.

To use the idea from the preceding paragraph, we define $k'_1 = \ell \cdot n \cdot x$ and $k'_2 = \ell \cdot n \cdot (x+1)$. It should be clear that $1 \leq k'_1 < k'_2 \leq m'$. Further, we note that $\ell \cdot \frac{n'}{k'_1} = \ell \cdot \frac{n \cdot x \cdot (x+1)}{\ell \cdot n \cdot x} = x + 1$ and, so, \#LCG($E', k'_1$) is equal to the number of $\ell$-cohesive groups in $E'$ with size at least $x + 1$. Similarly, as $\ell \cdot \frac{n'}{k'_2} = \ell \cdot \frac{n \cdot x \cdot (x+1)}{\ell \cdot n \cdot (x+1)} = x$, we have that \#LCG($E', k'_2$) is equal to the number of $\ell$-cohesive groups in $E'$ with size at least $x$. Furthermore, as newly created voters in $E'$ do not approve any candidates and newly created candidates in $E'$ are not approved by any voter, each $\ell$-cohesive group in $E'$ consists of the voters from $E$ approving only the candidates from $E$, so it is also a valid $\ell$-cohesive group in $E$. Thus, \#LCG($E, k, \ell$) = \#LCG($E', k'_1, \ell$) - \#LCG($E, k'_2, \ell$).

This completes the proof as we have shown a polynomial-time algorithm that solves \#FSCG using oracle access to \#COHESIVE-GROUP.

\[\square\]

B Proof of Theorem 5.2

Theorem 5.2. PD-VERIFICATION is coNP-complete.

Proof. Let us show that PD-VERIFICATION is in NP. Given a PD-VERIFICATION instance, we guess a group of voters and a cohesiveness level $\ell$. We can verify in polynomial time whether these voters form an $\ell$-cohesive group. Then we calculate the average satisfaction of the group and compare it with the given threshold. If the voters form an $\ell$-cohesive group and their average satisfaction is lower than the threshold, then the selected voters witness PD-VERIFICATION. Therefore PD-VERIFICATION is in NP and PD-VERIFICATION is in coNP.

We show a reduction from the PD-FAIL problem to the complement of the PD-VERIFICATION problem. Let $(E, W, k, \ell, y)$ be a PD-FAIL instance, where $E$ is an election, $W$ is a final committee of size $k$, $\ell$ is the cohesiveness level, and $y$ is a nonnegative real threshold $y \leq k$. We ask if there exists an $\ell$-cohesive group whose average satisfaction is lower than $y$.

We create a PD-VERIFICATION instance as follows. We have the same election $E$ and the same committee $W$. We set the PD function to be $f(\ell) = y$, and 0 otherwise.
Suppose that the answer for the PD-VERIFICATION instance is “no”. Then, for some $\ell'$ there exists an $\ell'$-cohesive group $S$ whose average satisfaction is lower than $f(\ell')$. It is quite clear that each $\ell'$-cohesive group has average satisfaction at least $0$, regardless of a selected committee. It means that $\ell'$ must be equal to $\ell$. Therefore $S$ is an $\ell$-cohesive group and has average satisfaction lower than $f(\ell') = f(\ell) = y$. Thus the answer for the PD-FAILURE instance is “yes”.

Suppose that the answer for the PD-VERIFICATION instance is “yes”. Then, for each $\ell'$ and each $\ell'$-cohesive group its average satisfaction is at least $f(\ell')$. In particular, it means that each $\ell$-cohesive group has average satisfaction at least $f(\ell) = y$. Therefore there does not exist an $\ell$-cohesive group that has average satisfaction lower than $y$. Thus the answer for the PD-FAILURE instance is “no”.

Since the PD-VERIFICATION problem is in coNP, the PD-FAILURE problem is NP-complete, and we reduced the PD-FAILURE problem to the complement of the PD-VERIFICATION problem, the PD-VERIFICATION problem is coNP-complete.