Inverse Function Theorems for Generalized Smooth Functions

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1 Introduction

Since its inception, category theory has underscored the importance of unrestricted composition of morphisms for many parts of mathematics. The closure of a given space of “arrows” with respect to composition proved to be a key foundational property. It is therefore clear that the lack of this feature for Schwartz distributions has considerable consequences in the study of differential equations [28, 14], in mathematical physics [4, 6, 8, 10, 18, 21, 25, 41, 42, 43, 46], and in the calculus of variations [30], to name but a few.

On the other hand, Schwartz distributions are so deeply rooted in the linear framework that one can even isomorphically approach them focusing only on this aspect, opting for a completely formal/syntactic viewpoint and without requiring any functional analysis, see [49]. So, Schwartz distributions do not have a notion of pointwise evaluation in general, and do not form a category, although it is well known that certain subclasses of distributions have meaningful notions of pointwise evaluation, see e.g. [35, 36, 47, 45, 16, 15, 51].

This is even more surprising if one takes into account the earlier historical genesis of generalized functions dating back to authors like Cauchy, Poisson, Kirchhoff, Helmholtz, Kelvin, Heaviside, and Dirac, see [29, 53, 34, 50]. For them, this “generalization” is simply accomplished by fixing an infinitesimal or infinite parameter in an ordinary smooth function, e.g. an infinitesimal and invertible standard deviation in a Gaussian probability density. Therefore, generalized functions are thought of as some kind of smooth set-theoretical functions defined and valued in a suitable

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non-Archimedean ring of scalars. From this intuitive point of view, they clearly have point values and form a category.

This aspect also bears upon the concept of (a generalized) solution of a differential equation. In fact, any theory of generalized functions must have a link with the classical notion of (smooth) solution. However, this classical notion is deeply grounded on the concept of composition of functions and, at the same time, it is often too narrow, as is amply demonstrated e.g. in the study of PDE in the presence of singularities. In our opinion, it is at least not surprising that also the notion of distributional solution did not lead to a satisfying theory of nonlinear PDE (not even of singular ODE). We have hence a wild garden of flourishing equation-dependent techniques and a zoo of counter-examples. The well-known detaching between these techniques and numerical solutions of PDE is another side of the same question.

One can say that this situation presents several analogies with the classical compass-and-straightedge solution of geometrical problems, or with the solution of polynomial equations by radicals. The distinction between algebraic and irrational numbers and the advent of Galois theory were essential steps for mathematics to start focusing on a different concept of solution, frequently nearer to applied problems. In the end, these classical problems stimulated more general notions of geometrical transformation and numerical solution, which nowadays have superseded their origins. The analogies are even greater when observing that first steps toward a Galois theory of nonlinear PDE are arising, see [5, 7, 38, 39].

Generalized smooth functions (GSF) are a possible formalization of the original historical approach of the aforementioned classical authors. We extend the field of real numbers into a natural non-Archimedean ring \( \tilde{\rho}R \) and we consider the simplest notion of smooth function on the extended ring of scalars \( \tilde{\rho}R \). To define a GSF \( f : X \longrightarrow Y, X \subseteq \tilde{\rho}R^n, Y \subseteq \tilde{\rho}R^d \), we simply require the minimal logical conditions so that a net of ordinary smooth functions \( f_\varepsilon \in C^\infty(\Omega_\varepsilon, \mathbb{R}^d), \Omega_\varepsilon \subseteq \mathbb{R}^n \), defines a set-theoretical map \( X \longrightarrow Y \) which is infinitely differentiable; see below for the details. This freedom in the choice of domains and codomains is a key property to prove that GSF are closed with respect to composition. As a result, GSF share so many properties with ordinary smooth functions that frequently we only have to formally generalize classical proofs to the new context. This allows an easier approach to this new theory of generalized functions.

It is important to note that the new framework is richer than the classical one because of the possibility to express non-Archimedean properties. So, e.g., two different infinitesimal standard deviations in a Gaussian result in infinitely close Dirac-delta-like functionals but, generally speaking, these two GSF could have different infinite values at infinitesimal points \( h \in \tilde{\rho}R \). For this reason, Schwartz distributions are embedded as GSF, but this embedding is not intrinsic and it has to be chosen depending on the physical problem or on the particular differential equation we aim to solve.

In the present work, we establish several inverse function theorems for GSF. We prove both the classical local and also some global versions of this theorem. It is remarkable to note that the local version is formally very similar to the classical one, but with the sharp topology instead of the standard Euclidean one. We also show
the relations between our results and the inverse function theorem for Colombeau
functions established by using the discontinuous calculus of [2, 3].

The paper is self-contained in the sense that it contains all the statements of
results required for the proofs of the new inverse function theorems. If proofs of
preliminaries are omitted, we give references to where they can be found.

\section{1.1 Basic notions}

The ring of generalized scalars

In this work, \(I\) denotes the interval \((0,1] \subseteq \mathbb{R}\) and we will always use the variable \(\varepsilon\)
for elements of \(I\); we also denote \(\varepsilon\)-dependent nets \(x_\varepsilon \in \mathbb{R}^I\) simply by \((x_\varepsilon)\). By \(\mathbb{N}\) we
denote the set of natural numbers, including zero.

We start by defining the non-Archimedean ring of scalars that extends the real
field \(\mathbb{R}\). For all the proofs of results in this section, see [19, 18].

**Definition 1.** Let \(\rho = (\rho_\varepsilon) \in \mathbb{R}^I\) be a net such that \(\lim_{\varepsilon \to 0^+} \rho_\varepsilon = 0^+\), then

(i) \(\mathcal{S}(\rho) := \{(\rho_\varepsilon^{-n})_{n=0} \mid a \in \mathbb{R}_{>0}\}\) is called the asymptotic gauge generated by \(\rho\).

(ii) If \(\mathcal{P}(\varepsilon)\) is a property of \(\varepsilon \in I\), we use the notation \(\forall^0 \varepsilon : \mathcal{P}(\varepsilon)\) to denote

\[\exists \varepsilon_0 \in I \forall \varepsilon \in (0, \varepsilon_0) : \mathcal{P}(\varepsilon)\]. We can read \(\forall^0 \varepsilon\) as for \(\varepsilon\) small.

(iii) We say that a net \((x_\varepsilon) \in \mathbb{R}^I\) is \(\rho\)-moderate, and we write \((x_\varepsilon) \in \mathbb{R}_\rho\) if \(\exists(J_\varepsilon) \in \mathcal{S}(\rho) : x_\varepsilon = O(J_\varepsilon)\) as \(\varepsilon \to 0^+\).

(iv) Let \((x_\varepsilon), (y_\varepsilon) \in \mathbb{R}^I\), then we say that \((x_\varepsilon) \sim_\rho (y_\varepsilon)\) if \(\forall(J_\varepsilon) \in \mathcal{S}(\rho) : x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1})\) as \(\varepsilon \to 0^+\). This is an equivalence relation on the ring \(\mathbb{R}_\rho\) of

moderate nets with respect to pointwise operations, and we can hence define

\[\rho\mathbb{R} := \mathbb{R}_\rho / \sim_\rho\]

which we call Robinson-Colombeau ring of generalized numbers. [43] [3]. We
denote the equivalence class \(x \in \rho\mathbb{R}\) simply by \(x := [x_\varepsilon] := [(x_\varepsilon)]_{\sim_\rho} \in \rho\mathbb{R}\).

In the following, \(\rho\) will always denote a net as in Def. [1] and we will use the simpler
notation \(\mathbb{R}\) for the case \(\rho_\varepsilon = \varepsilon\). The infinitesimal \(\rho\) can be chosen depending on the
class of differential equations we need to solve for the generalized functions we are
going to introduce, see [20]. For motivations concerning the naturality of \(\rho\mathbb{R}\), see [18]. We also use the notation \(d\rho := [d\rho_\varepsilon] \in \rho\mathbb{R}\) and \(d\varepsilon := [d\varepsilon] \in \rho\mathbb{R}\).

We can also define an order relation on \(\rho\mathbb{R}\) by saying \([x_\varepsilon] \leq [y_\varepsilon]\) if there exists \((z_\varepsilon) \in \mathbb{R}^I\) such that \((z_\varepsilon) \sim_\rho 0\) (we then say that \((z_\varepsilon)\) is \(\rho\)-negligible) and \(x_\varepsilon \leq y_\varepsilon + z_\varepsilon\)
for \(\varepsilon\) small. Equivalently, we have that \(x \leq y\) if and only if there exist representatives \((x_\varepsilon), (y_\varepsilon)\) of \(x, y\) such that \(x_\varepsilon \leq y_\varepsilon\) for all \(\varepsilon\). Clearly, \(\rho\mathbb{R}\) is a partially ordered ring.
The usual real numbers \(r \in \mathbb{R}\) are embedded in \(\rho\mathbb{R}\) considering constant nets \([r] \in \rho\mathbb{R}\).

Even if the order \(\leq\) is not total, we still have the possibility to define the infimum
\([x_\varepsilon] \wedge [y_\varepsilon] := \min(x_\varepsilon, y_\varepsilon)\), and analogously the supremum function \([x_\varepsilon] \vee [y_\varepsilon] := \max(x_\varepsilon, y_\varepsilon)\) and the absolute value \([|x_\varepsilon|] := [|x_\varepsilon|] \in \rho\mathbb{R}\). Our notations for intervals
are: \([a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}\), \([a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}\), and analogously for segments \([x, y] := \{x + r \cdot (y - x) | r \in [0, 1]\} \subseteq \mathbb{R}^n\) and \([x, y]_{\mathbb{R}^n} := [x, y] \cap \mathbb{R}^n\). Finally, we write \(x \approx y\) to denote that \(|x - y|\) is an infinitesimal number, i.e. \(|x - y| \leq r\) for all \(r \in \mathbb{R}_{>0}\).

This is equivalent to \(\lim_{\epsilon \to 0^+} |x_\epsilon - y_\epsilon| = 0\) for all representatives \((x_\epsilon), (y_\epsilon)\) of \(x, y\).

**Topologies on** \(\mathbb{R}^n\)

On the \(\mathbb{R}^n\)-module \(\mathbb{R}^n\), we can consider the natural extension of the Euclidean norm, i.e. \(|[x_\epsilon]| := |[x]| \in \mathbb{R}^n\), where \([x_\epsilon] \in \mathbb{R}^n\). Even if this generalized norm takes values in \(\mathbb{R}^n\), it shares several properties with usual norms, like the triangular inequality or the property \(|y \cdot x| = |y| \cdot |x|\). It is therefore natural to consider on \(\mathbb{R}^n\) topologies generated by balls defined by this generalized norm and suitable notions of being “strictly less than a given radius”:

**Definition 2.** Let \(c \in \mathbb{R}^n\) and \(x, y \in \mathbb{R}^n\), then:

(i) We write \(x < y\) if \(\exists r \in \mathbb{R}_{>0}: r \) is invertible, and \(r \leq y - x\)

(ii) We write \(x < y\) if \(\exists r \in \mathbb{R}_{>0}: r \leq y - x\).

(iii) \(B_y(c) := \{x \in \mathbb{R}^n | |x - c| < r\}\) for each \(r \in \mathbb{R}_{>0}\).

(iv) \(B_y^\epsilon(c) := \{x \in \mathbb{R}^n | |x - c| < r\}\) for each \(r \in \mathbb{R}_{>0}\).

(v) \(B_y^\epsilon(c) := \{x \in \mathbb{R}^n | |x - c| < r\}\), for each \(r \in \mathbb{R}_{>0}\), denotes an ordinary Euclidean ball in \(\mathbb{R}^n\).

The relations \(<, <_\epsilon\) have better topological properties as compared to the usual strict order relation \(a \leq b\) and \(a \neq b\) (that we will never use) because both the sets of balls \(\{B_y(c) | r \in \mathbb{R}_{>0}, c \in \mathbb{R}^n\}\) and \(\{B_y^\epsilon(c) | r \in \mathbb{R}_{>0}, c \in \mathbb{R}^n\}\) are bases for two topologies on \(\mathbb{R}^n\). The former is called sharp topology, whereas the latter is called Fermat topology. We will call sharply open set any open set in the sharp topology, and large open set any open set in the Fermat topology; clearly, the latter is coarser than the former. The existence of infinitesimal neighborhoods implies that the sharp topology induces the discrete topology on \(\mathbb{R}\). This is a necessary result when one has to deal with continuous generalized functions which have infinite derivatives. In fact, if \(f'(x_0)\) is infinite, only for \(x \approx x_0\) we can have \(f(x) \approx f(x_0)\).

The following result is useful to deal with positive and invertible generalized numbers (cf. [24][40]).

**Lemma 1.** Let \(x \in \mathbb{R}^n\). Then the following are equivalent:

(i) \(x\) is invertible and \(x \geq 0\), i.e. \(x > 0\).

(ii) For each representative \((x_\epsilon) \in \mathbb{R}_\rho\) of \(x\) we have \(\forall \epsilon > 0: x_\epsilon > 0\).

(iii) For each representative \((x_\epsilon) \in \mathbb{R}_\rho\) of \(x\) we have \(\exists m \in \mathbb{N} \forall \epsilon > 0: x_\epsilon > p_\epsilon^m\)
**Internal and strongly internal sets**

A natural way to obtain sharply open, closed and bounded sets in $\mathbb{R}^n$ is by using a net $(A_\varepsilon)$ of subsets $A_\varepsilon \subseteq \mathbb{R}^n$. We have two ways of extending the membership relation $x_\varepsilon \in A_\varepsilon$ to generalized points $x_\varepsilon \in \mathbb{R}^n$:

**Definition 3.** Let $(A_\varepsilon)$ be a net of subsets of $\mathbb{R}^n$, then

(i) $[A_\varepsilon] := \{ [x_\varepsilon] \in \mathbb{R}^n | \forall \varepsilon : x_\varepsilon \in A_\varepsilon \}$ is called the internal set generated by the net $(A_\varepsilon)$. See [44] for the introduction and an in-depth study of this notion.

(ii) Let $(x_\varepsilon)$ be a net of points of $\mathbb{R}^n$, then we say that $x_\varepsilon \in_A A_\varepsilon$, and we read it as $(x_\varepsilon)$ strongly belongs to $(A_\varepsilon)$, if $\forall \varepsilon : x_\varepsilon \in A_\varepsilon$ and if $(x_\varepsilon') \sim_\delta (x_\varepsilon)$, then also $x_\varepsilon' \in A_\varepsilon$ for $\varepsilon$ small. Moreover, we set $\langle A_\varepsilon \rangle := \{ [x_\varepsilon] \in \mathbb{R}^n | x_\varepsilon \in A_\varepsilon \}$, and we call it the strongly internal set generated by the net $(A_\varepsilon)$.

(iii) Finally, we say that the internal set $K = [A_\varepsilon]$ is sharply bounded if there exists $r \in \mathbb{R}_{>0}$ such that $K \subseteq B_r(0)$. Analogously, a net $(A_\varepsilon)$ is sharply bounded if there exists $r \in \mathbb{R}_{>0}$ such that $[A_\varepsilon] \subseteq B_r(0)$.

Therefore, $x \in [A_\varepsilon]$ if there exists a representative $(x_\varepsilon)$ of $x$ such that $x_\varepsilon \in A_\varepsilon$ for $\varepsilon$ small, whereas this membership is independent from the chosen representative in the case of strongly internal sets. Note explicitly that an internal set generated by a constant net $A_\varepsilon = A \subseteq \mathbb{R}^n$ is simply denoted by $[A]$.

The following theorem shows that internal and strongly internal sets have dual topological properties:

**Theorem 1.** For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then we have

(i) $[x_\varepsilon] \in [A_\varepsilon]$ if and only if $\forall q \in \mathbb{R}_{>0} \forall \varepsilon : d(x_\varepsilon, A_\varepsilon) \leq \rho^q_\varepsilon$. Therefore $[x_\varepsilon] \in [A_\varepsilon]$ if and only if $[d(x_\varepsilon, A_\varepsilon)] = 0 \in \mathbb{R}^n$.

(ii) $[x_\varepsilon] \in (A_\varepsilon)$ if and only if $\exists q \in \mathbb{R}_{>0} \forall \varepsilon : d(x_\varepsilon, A_\varepsilon^c) > \rho^q_\varepsilon$, where $A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon$. Therefore, if $(d(x_\varepsilon, A_\varepsilon^c)) \in \mathbb{R}_0$, then $[x_\varepsilon] \in (A_\varepsilon)$ if and only if $[d(x_\varepsilon, A_\varepsilon^c)] > 0$.

(iii) $[A_\varepsilon]$ is sharply closed and $(A_\varepsilon)$ is sharply open.

(iv) $[A_\varepsilon] = [\text{cl}(A_\varepsilon)]$, where $\text{cl}(S)$ is the closure of $S \subseteq \mathbb{R}^n$. On the other hand $\langle A_\varepsilon \rangle = \langle \text{int}(A_\varepsilon) \rangle$, where $\text{int}(S)$ is the interior of $S \subseteq \mathbb{R}^n$.

We will also use the following:

**Lemma 2.** Let $(\Omega_\varepsilon)$ be a net of subsets in $\mathbb{R}^n$ for all $\varepsilon$, and $(B_\varepsilon)$ a sharply bounded net such that $[B_\varepsilon] \subseteq (\Omega_\varepsilon)$, then

$\forall \varepsilon : B_\varepsilon \subseteq \Omega_\varepsilon$.

Sharply bounded internal sets (which are always sharply closed by Thm. 1(iii)) serve as compact sets for our generalized functions. For a deeper study of this type of sets in the case $\rho = (\varepsilon)$ see [44][17]; in the same particular setting, see [19] and references therein for (strongly) internal sets.
Generalized smooth functions

For the ideas presented in this section, see also e.g. [19] [18].

Using the ring $\mathbb{R}_p$, it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by $f(x, \sigma)$, and if we set $\sigma = [\sigma_e] \in \mathbb{R}_{>0}$, where $\sigma \approx 0$, we obtain the net of smooth functions $(f(-, \sigma_e))_{e \in E}$. This is the basic idea we develop in the following.

**Definition 4.** Let $X \subseteq \mathbb{R}_p^n$ and $Y \subseteq \mathbb{R}_p^d$ be arbitrary subsets of generalized points. Then we say that

$$f : X \longrightarrow Y$$ is a generalized smooth function

if there exists a net of functions $f_e \in \mathcal{C}^\infty(\Omega_e, \mathbb{R}_p^d)$ defining $f$ in the sense that $X \subseteq \{\Omega_e\}$, $f([x_e]) = [f_e(x_e)] \in Y$ and $(\partial^\alpha f_e(x_e)) \in \mathbb{R}_p^d$ for all $x = [x_e] \in X$ and all $\alpha \in \mathbb{N}^n$.

The space of GSF from $X$ to $Y$ is denoted by $\mathcal{P}(X, Y)$.

Let us note explicitly that this definition states minimal logical conditions to obtain a set-theoretical map from $X$ into $Y$ and defined by a net of smooth functions. In particular, the following Thm. 2 states that the equality $f([x_e]) = [f_e(x_e)]$ is meaningful, i.e. that we have independence from the representatives for all derivatives $[x_e] \in X \mapsto [\partial^\alpha f_e(x_e)] \in \mathbb{R}_p^d$, $\alpha \in \mathbb{N}^n$.

**Theorem 2.** Let $X \subseteq \mathbb{R}_p^n$ and $Y \subseteq \mathbb{R}_p^d$ be arbitrary subsets of generalized points. Let $f_e \in \mathcal{C}^\infty(\Omega_e, \mathbb{R}_p^d)$ be a net of smooth functions that defines a generalized smooth map of the type $X \longrightarrow Y$, then

(i) $\forall \alpha \in \mathbb{N}^n \forall (x_e), (x'_e) \in \mathbb{R}_p^n : [x_e] = [x'_e] \in X \Rightarrow (\partial^\alpha u_e(x_e)) \sim_p (\partial^\alpha u_e(x'_e))$.

(ii) $\forall [x_e] \in X \forall \alpha \in \mathbb{N}^n \exists q \in \mathbb{R}_{>0} \forall \epsilon \in \mathbb{R}_p : \sup_{y \in \mathbb{P}_{x_e}(x_e)} |\partial^\alpha u_e(y)| \leq e^{-q}$.

(iii) For all $\alpha \in \mathbb{N}^n$, the GSF $g : [x_e] \in X \mapsto [\partial^\alpha f_e(x_e)] \in \mathbb{R}_p^d$ is locally Lipschitz in the sharp topology, i.e. each $x \in X$ possesses a sharp neighborhood $U$ such that $|g(x) - g(y)| \leq L|x - y|$ for all $x, y \in U$ and some $L \in \mathbb{R}_p$.

(iv) Each $f \in \mathcal{P}(\mathcal{C}^\infty(X, Y))$ is continuous with respect to the sharp topologies induced on $X, Y$.

(v) Assume that the GSF $f$ is locally Lipschitz in the Fermat topology and that its Lipschitz constants are always finite: $L \in \mathbb{R}$. Then $f$ is continuous in the Fermat topology.

(vi) $f : X \longrightarrow Y$ is a GSF if and only if there exists a net $v_e \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ defining a generalized smooth map of type $X \longrightarrow Y$ such that $f = [v_e(-)]|x$.

(vii) Subsets $S \subseteq \mathbb{R}_p^n$ with the trace of the sharp topology, and generalized smooth maps as arrows form a subcategory of the category of topological spaces. We will call this category $\mathcal{P}(\mathcal{C}^\infty)$, the category of GSF.

The differential calculus for GSF can be introduced showing existence and uniqueness of another GSF serving as incremental ratio. For its statement, if $\mathcal{P}(h)$ is a property of $h \in \mathbb{R}_p$, then we write $\forall h : \mathcal{P}(h)$ to denote $\exists r \in \mathbb{R}_{>0} \forall h \in B_r(0) : \mathcal{P}(h)$ and $\forall h : \mathcal{P}(h)$ for $\exists r \in \mathbb{R}_{>0} \forall h \in B_r(e) : \mathcal{P}(h)$. 

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Theorem 3. Let \( U \subseteq \mathbb{R}^n \) be a sharply open set, let \( v = [v_x] \in \mathbb{R}^n \), and let \( f \in \mathcal{G}^I(U, \mathbb{R}) \) be a generalized smooth map generated by the net of smooth functions \( f_\epsilon \in C^I(\Omega_\epsilon, \mathbb{R}) \). Then

(i) There exists a sharp neighborhood \( T \) of \( U \times \{0\} \) and a generalized smooth map \( r \in \mathcal{G}^I(T, \mathbb{R}) \), called the generalized incremental ratio of \( f \) along \( v \), such that

\[
\forall x \in U \forall h : f(x + hv) = f(x) + h \cdot r(x, h).
\]

(ii) If \( \bar{r} \in \mathcal{G}^I(S, \mathbb{R}) \) is another generalized incremental ratio of \( f \) along \( v \) defined on a sharp neighborhood \( S \) of \( U \times \{0\} \), then

\[
\forall x \in U \forall h : \bar{r}(x, h) = f(x, h).
\]

(iii) We have \( r(x, 0) = \left[ \frac{\partial f}{\partial x} (x_\epsilon) \right] \) for every \( x \in U \) and we can thus define \( \frac{\partial f}{\partial x} : r(x, 0) \), so that \( \frac{\partial f}{\partial x} \in \mathcal{G}^I(U, \mathbb{R}) \).

If \( U \) is a large open set, then an analogous statement holds replacing \( \forall h \) by \( \forall \rho \) and sharp neighborhoods by large neighborhoods.

Note that this result permits to consider the partial derivative of \( f \) with respect to an arbitrary generalized vector \( v \in \mathbb{R}^n \) which can be, e.g., infinitesimal or infinite.

Using this result we obtain the usual rules of differential calculus, including the chain rule. Finally, we note that for each \( x \in U \), the map \( D f(x) : v := \frac{\partial f}{\partial x} (x_\epsilon) \in \mathbb{R}^d \) is \( \mathbb{R} \)-linear in \( v \in \mathbb{R}^n \). The set of all the \( \mathbb{R} \)-linear maps \( \mathbb{R}^n \rightarrow \mathbb{R}^d \) will be denoted by \( L(\mathbb{R}^n, \mathbb{R}^d) \). For \( A = [A_\epsilon(-)] \in L(\mathbb{R}^n, \mathbb{R}^d) \), we set \( |A| := |A_\epsilon| \), the generalized number defined by the operator norms of the matrices \( A_\epsilon \in L(\mathbb{R}^n, \mathbb{R}^d) \).

Embedding of Schwartz distributions and Colombeau functions

We finally recall two results that give a certain flexibility in constructing embeddings of Schwartz distributions. Note that both the infinitesimal \( \rho \) and the embedding of Schwartz distributions have to be chosen depending on the problem we aim to solve.

A trivial example in this direction is the ODE \( y' = y/d\epsilon \), which cannot be solved for \( \rho = (\epsilon) \), but it has a solution for \( \rho = (\epsilon^{-1}/\epsilon) \). As another simple example, if we need the property \( H(0) = 1/2 \), where \( H \) is the Heaviside function, then we have to choose the embedding of distributions accordingly. This corresponds to the philosophy followed in [26]. See also [20] for further details.

If \( \phi \in \mathcal{D}(\mathbb{R}^n) \), \( r \in \mathbb{R}_{>0} \) and \( x \in \mathbb{R}^n \), we use the notations \( r \circ \phi \) for the function \( x \in \mathbb{R}^n \mapsto \frac{1}{r} \cdot \phi \left( \frac{x}{r} \right) \in \mathbb{R}^n \) and \( x \oplus \phi \) for the function \( y \in \mathbb{R}^n \mapsto \phi(y - x) \in \mathbb{R}^n \). These notations permit to highlight that \( \circ \) is a free action of the multiplicative group \( (\mathbb{R}_{>0}, \cdot, 1) \) on \( \mathcal{D}(\mathbb{R}^n) \) and \( \oplus \) is a free action of the additive group \( (\mathbb{R}_{>0}, +, 0) \) on \( \mathcal{D}(\mathbb{R}^n) \). We also have the distributive property \( r \circ (x \oplus \phi) = rx \oplus r \circ \phi \).

Lemma 3. Let \( b \in \mathbb{R}_\rho \) be a net such that \( \lim_{\epsilon \rightarrow 0^+} b_\epsilon = +\infty \). Let \( d \in (0, 1) \). There exists a net \( (\psi_\epsilon)_{\epsilon \in I} \) of \( \mathcal{D}(\mathbb{R}^n) \) with the properties:
∀ \psi \in \mathcal{B}_1(0) \text{ for all } \epsilon \in I.\\
(iii) \forall \alpha \in \mathbb{N}^n \exists p \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} |\partial^\alpha \psi(x)| = O(b^p_\epsilon) \text{ as } \epsilon \to 0^+.\\
(iv) \forall j \in \mathbb{N}^n \exists \epsilon : 1 \leq |\alpha| \leq j \Rightarrow \int x^\alpha \cdot \psi(x)dx = 0.\\
(v) \forall \eta \in \mathbb{R}_{>0} \exists \epsilon : \int |\psi_\epsilon| \leq 1 + \eta.\\
(vi) If n = 1, then the net \((\psi_\epsilon)_{\epsilon \in I}\) can be chosen so that \(\int_0^\infty \psi_\epsilon = d.\\

If \psi_\epsilon satisfies (i) - (vi) then in particular \(\psi_\epsilon^b := b^{-1}_\epsilon \circ \psi_\epsilon\) satisfies (ii) - (v).

Concerning embeddings of Schwartz distributions, we have the following result, where \(\Omega := \{x \in \Omega \mid \exists K \in \mathbb{N}^d \text{ s.t. } x \in K\}\) is called the set of compactly supported points in \(\Omega \subseteq \mathbb{R}^n.\\

Theorem 4. Under the assumptions of Lemma 3, let \(\Omega \subseteq \mathbb{R}^n\) be an open set and let \((\psi_\epsilon^b)\) be the net defined in Lemma 3. Then the mapping

\[
\iota_\Omega^b : T \in \mathcal{D}'(\Omega) \mapsto \left[(T \ast \psi_\epsilon^b)(-\epsilon)\right] \in \mathcal{D}'\mathcal{E}^{\infty}(\rho \tilde{\Omega}, \rho \mathbb{R})
\]

uniquely extends to a sheaf morphism of real vector spaces

\[
\iota^b : \mathcal{D}' \rightarrow \mathcal{D}'\mathcal{E}^{\infty}(\rho \tilde{\Omega}, \rho \mathbb{R}),
\]

and satisfies the following properties:

(i) If \(b \geq \rho^{-a}\) for some \(a \in \mathbb{R}_{>0}\), then \(\iota^b \mid _{\mathcal{D}'(\Omega)} : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'\mathcal{E}^{\infty}(\rho \tilde{\Omega}, \rho \mathbb{R})\) is a sheaf morphism of algebras.

(ii) If \(T \in \mathcal{D}'(\Omega)\) then supp\((T) = \text{ supp}(\iota^b_\Omega(T))\).

(iii) \(\lim_{\epsilon \to 0^+} \iota^b_\Omega(T) = \iota^b_\Omega(\phi)\) for all \(\phi \in \mathcal{D}(\Omega)\) and all \(T \in \mathcal{D}'(\Omega)\).

(iv) \(\iota^b\) commutes with partial derivatives, i.e. \(\partial^\alpha \left(\iota^b_\Omega(T)\right) = \iota^b_\Omega \left(\partial^\alpha T\right)\) for each \(T \in \mathcal{D}'(\Omega)\) and \(\alpha \in \mathbb{N}\).

Concerning the embedding of Colombeau generalized functions, we recall that the special Colombeau algebra on \(\Omega\) is defined as the quotient \(\mathcal{D}'(\Omega) : = \mathcal{E}_M(\Omega)/\mathcal{N}^\times(\Omega)\) of moderate nets over negligible nets, where the former is

\[
\mathcal{E}_M(\Omega) := \{(u_\epsilon) \in \mathcal{E}^{\infty}(\Omega) \mid \forall K \in \Omega \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-N})\}
\]

and the latter is

\[
\mathcal{N}^\times(\Omega) := \{(u_\epsilon) \in \mathcal{E}^{\infty}(\Omega) \mid \forall K \in \Omega \forall \alpha \in \mathbb{N}^n \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^m)\}.
\]

Using \(\rho = (\epsilon)\), we have the following compatibility result:

Theorem 5. A Colombeau generalized function \(u = (u_\epsilon) + \mathcal{N}^\times(\Omega)\) defines a generalized smooth map \(u : [x_\epsilon] \in \tilde{\Omega}_c \rightarrow [u_\epsilon(x_\epsilon)] \in \mathbb{R}^d\) which is locally Lipschitz on the same neighborhood of the Fermat topology for all derivatives. This
assignment provides a bijection of $\mathcal{G}^{s}(\Omega)^d$ onto $\mathcal{G}^{s}(\Omega)$, for every open set $\Omega \subseteq \mathbb{R}^n$.

For GSF, suitable generalizations of many classical theorems of differential and integral calculus hold: intermediate value theorem, mean value theorems, Taylor formulas in different forms, a sheaf property for the Fermat topology, and the extreme value theorem on internal sharply bounded sets (see [18]). The latter are called functionally compact subsets of $\rho \mathbb{R}^n$ and serve as compact sets for GSF. A theory of compactly supported GSF has been developed in [17], and it closely resembles the classical theory of LF-spaces of compactly supported smooth functions. It results that for suitable functionally compact subsets, the corresponding space of compactly supported GSF contains extensions of all Colombeau generalized functions, and hence also of all Schwartz distributions. Finally, in these spaces it is possible to prove the Banach fixed point theorem and a corresponding Picard-Lindelöf theorem, see [37].

2 Local inverse function theorems

As in the case of classical smooth functions, any infinitesimal criterion for the invertibility of generalized smooth functions will rely on the invertibility of the corresponding differential. We therefore note the following analogue of [24, Lemma 1.2.41] (whose proof transfers literally to the present situation):

**Lemma 4.** Let $A \in \rho \mathbb{R}^{n \times n}$ be a square matrix. The following are equivalent:

(i) $A$ is nondegenerate, i.e., $\xi \in \rho \mathbb{R}^n$, $\xi^t A \eta = 0 \forall \eta \in \rho \mathbb{R}^n$ implies $\xi = 0$.

(ii) $A : \rho \mathbb{R}^n \to \rho \mathbb{R}^n$ is injective.

(iii) $A : \rho \mathbb{R}^n \to \rho \mathbb{R}^n$ is surjective.

(iv) $\det(A)$ is invertible.

**Theorem 6.** Let $X \subseteq \rho \mathbb{R}^n$, let $f \in \mathcal{G}^{s}(\rho \mathbb{R}^n)$ and suppose that for some $x_0$ in the sharp interior of $X$, $Df(x_0)$ is invertible in $L(\rho \mathbb{R}^n, \rho \mathbb{R}^n)$. Then there exists a sharp neighborhood $U \subseteq X$ of $x_0$ and a sharp neighborhood $V$ of $f(x_0)$ such that $f : U \to V$ is invertible and $f^{-1} \in \mathcal{G}^{s}(\rho \mathbb{R}^n)$.

**Proof.** Thm. 2.(vi) entails that $f$ can be defined by a globally defined net $f_\varepsilon \in \mathcal{G}^{s}(\mathbb{R}^n, \mathbb{R}^n)$. Hadamard’s inequality (cf. [11] Prop. 3.43) implies $|\det(Df(x_0))^{-1}| \geq n!C^{-n} \det(Df(x_0))$, where $C \in \mathbb{R}_{>0}$ is a universal constant that only depends on the dimension $n$. Thus, by Lemma 4 and Lemma 1 $\det(Df(x_0))$ and consequently also $a := |Df(x_0)|$ is invertible. Next, pick positive invertible numbers $b, r \in \rho \mathbb{R}$ such that $ab < 1$, $B_{2r}(x_0) \subseteq X$ and

$$|Df(x_0) - Df(x)| < b$$
for all $x \in B_{2r}(x_0)$. Such a choice of $r$ is possible since every derivative of $f$ is continuous with respect to the sharp topology (see Thm. 2(iv) and Thm. 3(iii)). Pick representatives $(a_ε), (b_ε)$ and $(r_ε)$ of $a$, $b$ and $r$ such that for all $ε \in I$ we have $b_ε > 0$, $a_ε b_ε < 1$, and $r_ε > 0$. Let $(x_0ε)$ be a representative of $x_0$. Since $[B_{r_ε}(x_0ε)] \subseteq B_{2r}(x_0)$, by Lemma 2 we can also assume that $B_{r_ε}(x_0ε) \subseteq Ω_ε$, and $|D f_ε(x_0ε) - D f_ε(x)| < b_ε$ for all $x \in U_ε := B_{r_ε}(x_0ε)$. Now let $c_ε := [cε] > 0$ and by [1, Th. 6.4] we obtain for each $ε \in I$:

(a) For all $x \in U_ε := B_{r_ε}(x_0ε)$, $D f_ε(x)$ is invertible and $|D f_ε(x)^{-1}| \leq c_ε$.

(b) $V_ε := f_ε(B_{r_ε}(x_0ε))$ is open in $\mathbb{R}^n$.

(c) $f_ε|U_ε : U_ε \rightarrow V_ε$ is a diffeomorphism, and

(d) setting $y_0ε := f_ε(x_0ε)$, we have $B_{c_ε}(y_0ε) \subseteq f_ε(B_{r_ε}(x_0ε))$.

The sets $U := (U_ε) = B_r(x_0) \subseteq X$ and $V := (V_ε)$ are sharp neighborhoods of $x_0$ and $f(x_0)$, respectively, by [1] and so it remains to prove that $[f_ε|U_ε|] \subseteq p^{G C}(V, U)$.

We first note that by (a) $|D f_ε(x)^{-1}| \leq c_ε$ for all $x \in B_{r_ε}(x_0ε)$, which by Hadamard’s inequality implies

$$|\det(D f_ε(x))| \geq \frac{1}{C \cdot c_ε^m} \quad (x \in B_{r_ε}(x_0ε)). \quad (1)$$

Now for $[y_ε] \in V$ and $1 \leq i, j \leq n$ we have (see e.g. [1] (3.15))

$$∂_j (f_ε)^{-1}(y_ε) = \frac{1}{|\det(D f_ε(y_ε))|} \cdot P_{ij}(\partial_s (f_ε)^{-1}(y_ε))_{r,s}, \quad (2)$$

where $P_{ij}$ is a polynomial in the entries of the matrix $D f_ε$. Since $[f_ε|U_ε|] \subseteq U \subseteq X$, it follows from (1) and the fact that $f|U \in p^{G C}(U, p^{\mathbb{R}^n})$ that

$$· (f_ε)^{-1}(y_ε)) \in \mathbb{R}_p^\mathbb{R}.$$  

Higher order derivatives can be treated analogously, thereby establishing that every derivative of $g_ε := f_ε|U_ε|$ is moderate. To prove the claim, it remains to show that $[g_ε(y_ε)] \in U = (U_ε)$ for all $[y_ε] \in V = (V_ε)$. Since $g_ε : V \rightarrow U_ε$, we only prove that if $(x_ε) \sim (y_ε)$, then also $x_ε \in U_ε$ for $ε$ small. We can set $y_ε' := f_ε(x_ε)$ because $f_ε$ is defined on the entire $\mathbb{R}^n$. By the mean value theorem applied to $f_ε$, we get

$$|y_ε' - y_ε| = |f_ε(x_ε) - f_ε(y_ε)| \leq p^{\mathbb{R}^n} |x_ε - y_ε| \cdot (x_ε - y_ε|).$$

Therefore $(y_ε') \sim (y_ε)$ and hence $y_ε' \in V_ε$ and $g_ε(y_ε') = x_ε \in U_ε$ for $ε$ small. \ □

From Thm. 2(iv) we know that any generalized smooth function is sharply continuous. Thus we obtain:

**Corollary 1.** Let $X \subseteq p^{\mathbb{R}^n}$ be a sharply open set, and let $f \in ^c G C(X, p^{\mathbb{R}^n})$ be such that $D f(x)$ is invertible for each $x \in X$. Then $f$ is a local homeomorphism with respect to the sharp topology. In particular, it is an open map.
Any such map \( f \) will therefore be called a local generalized diffeomorphism. If \( f \in \mathcal{G}^m(X, Y) \) possesses an inverse in \( \mathcal{G}^m(Y, X) \), then it is called a global generalized diffeomorphism.

Following the same idea we used in the proof of Thm. 5, we can prove a sufficient condition to have a local generalized diffeomorphism which is defined in a large neighborhood of \( x_0 \):

**Theorem 7.** Let \( X \subseteq \mathbb{R}^n \), let \( f \in \mathcal{G}^m(X, \mathbb{R}^n) \) and suppose that for some \( x_0 \) in the Fermat interior of \( X \), \( Df(x_0) \) is invertible in \( L(\mathbb{R}^n, \mathbb{R}^n) \). Assume that \( |Df(x_0)|^{-1} \) is finite, i.e. \( |Df(x_0)|^{-1} \leq k \) for some \( k \in \mathbb{R}_{>0} \), and \( Df \) is Fermat continuous. Then there exists a large neighborhood \( U \subseteq X \) of \( x_0 \) and a large neighborhood \( V \) of \( f(x_0) \) such that \( f : U \rightarrow V \) is invertible and \( f^{-1} \in \mathcal{G}^m(V, U) \).

**Proof.** We proceed as above, but now we have \( r = r \in \mathbb{R}_{>0} \), \( b = b \in \mathbb{R}_{>0} \) because of our assumptions. Setting \( c_0 := \frac{a_{b,c}}{1 - a_{b,c}} \), we have that \( c := [c_0] \in \mathbb{R}_{>0} \) is finite. Therefore, there exists \( s \in \mathbb{R}_{>0} \) such that \( s < \frac{r}{2} \). We can continue as above, noting that now \( B_r(x_0) \subseteq U = B_r(x_0) \subseteq X \) and \( B_s(y_0) \subseteq B_s(y_0) \subseteq V \) are large neighborhoods of \( x_0 \) and \( f(x_0) \) respectively.

**Example 1.**

(i) Thm. 4 for \( n = 1 \), shows that \( \delta(x) = [b_c \psi(x) (b_c x)] \) is, up to sheaf isomorphism, the Dirac delta. This also shows directly that \( \delta \in \mathcal{G}^m(\mathbb{R}, \mathbb{R}) \). We can take the net \( (\psi_c) \) so that \( \psi_c(0) = 1 \) for all \( c \). In this way, \( H'(0) = \delta(0) = b \) is an infinite number. We can thus apply the local inverse function theorem 6 to the Heaviside function \( H \) obtaining that \( H \) is a generalized diffeomorphism in an infinitesimal neighborhood of 0. This neighborhood cannot be finite because \( H'(r) = 0 \) for all \( r \in \mathbb{R}_{>0} \).

(ii) By the intermediate value theorem for GSF (see [18, Cor. 42]), in the interval \([0, 1/2]\) the Dirac delta takes any value in \([0, \delta(0)]\). So, let \( k \in [0, 1/2] \) such that \( \delta(k) = 1 \). Then by the mean value theorem for GSF (see [18, Thm. 43]) \( \delta(\delta(1)) - \delta(\delta(k)) = \delta(0) - \delta(1) = b - 0 = (\delta \circ \delta)'(c) \cdot (1 - k) \) for some \( c \in [k, 1] \). Therefore \( (\delta \circ \delta)'(c) = \frac{b - \delta(0)}{1 - k} \in \mathbb{R}_{>0} \), and around \( c \) the composition \( \delta \circ \delta \) is invertible. Note that \( (\delta \circ \delta)'(r) = b \) for all \( r \in \mathbb{R}_{>0} \), and \( (\delta \circ \delta)'(h) = 0 \) for all \( h \in \mathbb{R} \) such that \( \delta(h) \) is not infinitesimal.

Now, let \( r \in \mathbb{R}_{>0} \) be an infinitesimal generalized number, i.e. \( r \approx 0 \).

(iii) Let \( f(x) := r \cdot x \) for \( x \in \mathbb{R}_{>0} \). Then \( f'(x_0) = r \approx 0 \) and Thm. 6 yields \( f^{-1} : y \in B_r(x_0) \mapsto y/r \in \mathbb{R}_{>0} \) for some \( s \in \mathbb{R}_{>0} \). But \( y/r \) is finite only if \( y \) is infinitesimal, so that \( s \approx 0 \). This shows that the assumption in Thm. 7 on \( |Df(x_0)|^{-1} | \) being finite is necessary.

(iv) Let \( f(x) := \sin \frac{x}{2} \). We have \( f \in \mathcal{G}^m(\mathbb{R} \setminus \{2\pi k \} \cup \{2\pi k + \pi \} \mathbb{R} \) and \( f'(x) = \frac{1}{2} \cos \frac{x}{2} \), which is always an infinite number e.g. if \( \exists \lim_{x \rightarrow 0^+} x \neq r(2k + 1) \frac{\pi}{2} \approx 0, k \in \mathbb{Z} \). By Thm. 6 we know that \( f \) is invertible e.g. around \( x = 0 \). It is easy to recognize that \( f \) is injective in the infinitesimal interval \((-\frac{\pi}{2} r + \frac{\pi}{2} r) \). In [11, Exa. 3.9], it is proved that \( f \) is not injective in any large neighborhood of \( x = 0 \). Therefore,
\( f \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \) is a GSF that cannot be extended to a Colombeau generalized function.

(v) Similarly, \( f(x) := r \sin x, x \in \mathbb{R}^n \), has an inverse function which cannot be extended outside the infinitesimal neighborhood \((-r, r)\).

(vi) Thm. [3] cannot be applied to \( f(x) := x^3 \) at \( x_0 = 0 \). However, if we restrict to \( x \in (-\infty,-r) \cup (r, +\infty) \), then the inverse function \( f^{-1}(y) = y^{1/3} \) is defined in \( y \in (-\infty,-r^3) \cup (r^3, +\infty) \) and has infinite derivative at each infinitesimal point in its domain.

In [2], Aragona, Fernandez and Juriaans introduced a differential calculus on spaces of Colombeau generalized points based on a specific form of convergence of difference quotients. Moreover, in [3], an inverse function theorem for Colombeau generalized functions in this calculus was established. In the one-dimensional case it was shown in [19] that any GSF is differentiable in the sense of [2, 3], with the same derivative. Below we will show that this compatibility is in fact true in arbitrary dimensions and that Theorem 6 implies the corresponding result from [3]. In the remaining part of the present section, we therefore restrict our attention to the case \( r = \varepsilon \), the gauge that is used in standard Colombeau theory (as well as in [2, 3]), and hence \( \mathbb{R}^n = \mathbb{R}^l \) and \( \mathbb{R} \mathcal{C}_\varepsilon = \mathbb{R} \).

First, we recall the definition from [2]:

**Definition 5.** A map \( f \) from some sharply open subset \( U \) of \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is called differentiable in \( x_0 \in U \) in the sense of [2] with derivative \( A \in L(\mathbb{R}^n, \mathbb{R}^m) \) if

\[
\lim_{x \to x_0} \frac{|f(x) - f(x_0) - A(x-x_0)|_e}{|x-x_0|_e} = 0,
\]

where

\[
v : (x_\varepsilon) \in \mathbb{R}^n_{(\varepsilon)} \mapsto \sup \{ b \in \mathbb{R} \mid |x| = O(\varepsilon^b) \} \in (-\infty, \infty]
\]

\[
| - |_e : x \in \mathbb{R}^n \mapsto \exp(-v(x)) \in [0, \infty).
\]

The following result shows compatibility of this notion with the derivative in the sense of GSF.

**Lemma 5.** Let \( U \) be sharply open in \( \mathbb{R}^n \), let \( x_0 \in U \) and suppose that \( f \in G^\omega(U, \mathbb{R}^m) \). Then \( f \) is differentiable in the sense of [2] in \( x_0 \) with derivative \( Df(x_0) \).

**Proof.** Without loss of generality we may suppose that \( m = 1 \). Let \( f \) be defined by the net \( f_\varepsilon \in G^\omega(\mathbb{R}^n, \mathbb{R}) \) for all \( \varepsilon \). Since \( (D^2 f_\varepsilon(x_\varepsilon)) \) is moderate, it follows from Thm. [2(ii)] that there exists some \( q > 0 \) such that \( \sup_{y \in \mathbb{B}^\varepsilon_{x_\varepsilon}(x_\varepsilon)} |D^2 f_\varepsilon(y)| \leq \varepsilon^{-q} \) for \( \varepsilon \) small. Then by Taylor’s theorem we have

\[
f_\varepsilon(x_\varepsilon) - f_\varepsilon(x_0\varepsilon) - Df_\varepsilon(x_0\varepsilon)(x_\varepsilon - x_0\varepsilon) =
\]

\[
= \sum_{|\alpha| \geq 2} \frac{|\alpha|!}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \partial^\alpha f_\varepsilon(x_0\varepsilon + t(x_\varepsilon - x_0\varepsilon)) \, dt \cdot (x_\varepsilon - x_0\varepsilon)^\alpha.
\]
For \( x_1 \in B_{d_{\text{def}}}(x_0) \) this implies that
\[
|f(x) - f(x_0) - Df(x_0)(x - x_0)|_e \leq e^9|x - x_0|_e^2,
\]
thereby establishing (3) with \( A = Df(x_0) \), as claimed.

\[ \square \]

It follows that any \( f \in \mathcal{G}^\infty(U, \mathbb{R}^m) \) is in fact even infinitely often differentiable in the sense of [2].

Based on these observations we may now give an alternative proof for [3, Th. 3]:

**Theorem 8.** Let \( \Omega \subseteq \mathbb{R}^n \) be open, \( f \in \mathcal{G}^\omega(\Omega)^n \), and \( x_0 \in \bar{\Omega}_c \) such that \( \det Df(x_0) \) is invertible in \( \mathbb{R} \). Then there are sharply open neighborhoods \( U \) of \( x_0 \) and \( V \) of \( f(x_0) \) such that \( f : U \to V \) is a diffeomorphism in the sense of [2].

**Proof.** By Thm. [5], \( f \) can be viewed as an element of \( \mathcal{G}^\omega(\bar{\Omega}_c, \mathbb{R}^n) \). Moreover, \( \bar{\Omega}_c \) is sharply open, which together with Lemma [4] shows that all the assumptions of Thm. [6] are satisfied.

We conclude that \( f \) possesses an inverse \( f^{-1} \) in \( \mathcal{G}^\omega(V, U) \) for a suitable sharp neighborhood \( V \) of \( f(x_0) \). Finally, by Lemma [5] both \( f \) and \( f^{-1} \) are infinitely differentiable in the sense of [2].

\[ \square \]

## 3 Global inverse function theorems

The aim of the present section is to obtain statements on the global invertibility of generalized smooth functions. For classical smooth functions, a number of criteria for global invertibility are known, and we refer to [2][51] for an overview.

The following auxiliary result will repeatedly be needed below:

**Lemma 6.** Let \( f \in \mathcal{G}^\omega(X, Y) \) be defined by \( (f_e) \), where \( X \subseteq \mathbb{R}^n \) and \( Y \subseteq \mathbb{R}^d \). Assume that \( \emptyset \neq [a_e] \subseteq X \). Let \( b : \mathbb{R}^d \to \mathbb{R} \) be a set-theoretical map such that \( \tilde{b} : [y_e] \in Y \mapsto [b(y_e)] \in \mathbb{R} \) is well-defined (e.g., \( b(x) = |x| \)). If \( f \) satisfies
\[
\forall x \in X : \tilde{b}[f(x)] > 0, \tag{4}
\]
then

(i) \( \exists q \in \mathbb{R}_{>0} \forall x \in A_e : b(f_e(x)) > \rho^q \).

(ii) For all \( K \subseteq \mathbb{R}^n \), if \( [K] \subseteq X \) then \( \forall \epsilon \exists x \in K : b(f_e(x)) > 0 \).

**Proof.** In fact, suppose to the contrary that there was a sequence \((\epsilon_k) \downarrow 0\) and a sequence \( x_k \in A_{\epsilon_k} \) such that \( b(f_{\epsilon_k}(x_k)) \leq \rho_{\epsilon_k}^q \). Let \( A_{\epsilon} \neq \emptyset \) for \( \epsilon \leq \epsilon_0 \), and pick \( a_\epsilon \in A_{\epsilon} \). Set
\[
\epsilon_k := \begin{cases} x_k & \text{for } \epsilon = \epsilon_k \\ a_\epsilon & \text{otherwise}. \end{cases} \tag{5}
\]

It follows that \( x := [x_\epsilon] \in [A_{\epsilon}] \subseteq X \), and hence \( \tilde{b}[f(x)] > 0 \) by (4). Therefore, \( b(f_{\epsilon_k}(x_k)) > \rho_{\epsilon_k}^q \) for some \( p \in \mathbb{R}_{>0} \) by Lemma [1] and this yields a contradiction. The second part follows by setting \( A_{\epsilon} = K \) in the first one and by noting that \( \rho_{\epsilon} > 0 \). \[ \square \]
After these preparations, we now turn to generalizing global inverse function theorems from the smooth setting to GSF. We start with the one-dimensional case. Here it is well-known that a smooth function \( f : \mathbb{R} \to \mathbb{R} \) is a diffeomorphism onto its image if and only if \( |f'(x)| > 0 \) for all \( x \in \mathbb{R} \). It is a diffeomorphism onto \( \mathbb{R} \) if in addition there exists some \( r > 0 \) with \( |f'(x)| > r \) for all \( x \in \mathbb{R} \). Despite the fact that \( \rho_c \) is non-Archimedean, there is a close counterpart of this result in GSF.

**Theorem 9.** Let \( f \in \rho^G\mathcal{C}^\infty(\rho_c, \rho_c) \) and suppose that there exists some \( r \in \mathbb{R}_{>0} \) such that \( |f'(x)| > r \) for all \( x \in \rho_c \). Then

(i) \( f \) has a defining net \( (\tilde{f}_\varepsilon) \) consisting of diffeomorphisms \( f_\varepsilon : \mathbb{R} \to \mathbb{R} \).

(ii) \( f \) is a global generalized diffeomorphism in \( \rho^G\mathcal{C}^\infty(\rho_c, \rho_c) \).

(iii) If \( r > 0 \), then \( f(\rho_c) = \rho_c \), so \( f \) is a global generalized diffeomorphism in \( \rho^G\mathcal{C}^\infty(\rho_c, \rho_c) \).

**Proof.** Let \( (f_\varepsilon) \) be a defining net for \( f \) such that \( f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \) for each \( \varepsilon \) (cf. Thm. 2(vi)). Since \( |f'(x)| > 0 \) for every \( x \in \rho_c \), Lemma 6 implies that for each \( n \in \mathbb{N} \) there exists some \( \varepsilon_n > 0 \) and some \( q_n > 0 \) such that for each \( \varepsilon \in (0, \varepsilon_n) \) and each \( x \in [-n, n] \) we have \( |f'_\varepsilon(x)| > \rho_\varepsilon^{q_n} \). Clearly we may suppose that \( \varepsilon_n \to 0 \), \( q_{n+1} > q_n \) for all \( n \) and that \( \rho_\varepsilon^{q_n} \to 1 \). Now for any \( n \in \mathbb{N}_{\geq 0} \) let \( \phi_n : \mathbb{R} \to [0, 1] \) be a smooth cut-off function with \( \phi_n \equiv 1 \) on \([-n-1, n-1]\) and \( \text{supp} \phi_n \subseteq [-n, n] \). Supposing that \( f'_\varepsilon(x) > 0 \) on \([-n, n]\) (the case \( f'_\varepsilon(x) < 0 \) on \([-n, n]\) can be handled analogously), we set

\[
v_{\varepsilon n}(x) := f'_\varepsilon(x) \phi_{\varepsilon n}(x) + 1 - \phi_{\varepsilon n}(x) \quad (x \in \mathbb{R})
\]

\[
\tilde{f}_\varepsilon(x) := f_\varepsilon(0) + \int_0^x v_{\varepsilon n}(t) \, dt \quad (x \in \mathbb{R}, \varepsilon_{n+1} < \varepsilon \leq \varepsilon_n),
\]

and \( \tilde{f}_\varepsilon := f_\varepsilon \) for \( \varepsilon \in (\varepsilon_0, 1) \). Then \( \tilde{f}_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \) for each \( \varepsilon \), and for each \( x \in \mathbb{R} \) and each \( \varepsilon \in (\varepsilon_{n+1}, \varepsilon_n) \), we have \( \tilde{f}'_\varepsilon(x) = f'_\varepsilon(x) \phi_{\varepsilon n}(x) + 1 - \phi_{\varepsilon n}(x) > \rho_\varepsilon^{q_n} \) if and only if \( f'(x)(1 - f'_\varepsilon(x)) < 1 - \rho_\varepsilon^{q_n} \). The latter inequality holds if \( x \notin [-n, n] \) or if \( f'_\varepsilon(x) \geq 1 \).

Otherwise, \( \phi_{\varepsilon n}(x) \leq 1 < \frac{1 - \rho_\varepsilon^{q_n}}{1 - f'_\varepsilon(x)} \) because \( 1 > f'_\varepsilon(x) > \rho_\varepsilon^{q_n} \). Any such \( \tilde{f}_\varepsilon \) therefore is a diffeomorphism from \( \mathbb{R} \) onto \( \mathbb{R} \). Also, \( \tilde{f}_\varepsilon(x) = f_\varepsilon(x) \) for all \( x \in [-n, n] \) as soon as \( \varepsilon \leq \varepsilon_{n+1} \). Hence also \( (\tilde{f}_\varepsilon) \) is a defining net for \( f \). This proves (i).

For each \( \varepsilon \leq \varepsilon_0 \), let \( g_{\varepsilon} \) be the global inverse of \( \tilde{f}_\varepsilon \). We claim that \( g := [g_\varepsilon] \) is a GSF from \( f(\rho_c) \) onto \( \rho_c \) that is inverse to \( f \). For this it suffices to show that whenever \( y = \{y_\varepsilon\} \in f(\rho_c) \), then for each \( k \in \mathbb{N} \), the net \( (g_{\varepsilon}^{(k)}(y_\varepsilon)) \) is \( \rho \)-moderate. To see this, it suffices to observe that for \( y = f(x) \), \( f \) satisfies the assumptions of the local inverse function theorem (Thm. 6) at \( x \), and so the proof of that result shows that \( g \) is a GSF when restricted to a suitable sharp neighborhood of \( y \). But this in particular entails the desired moderateness property at \( y \), establishing (ii).

Finally, assume that \( r > 0 \). The same reasoning as in the proof of (i) now produces a defining net \( (\tilde{f}_\varepsilon) \) with the property that \( |\tilde{f}'_\varepsilon(x)| > r \) for all \( \varepsilon \leq \varepsilon_0 \) and all \( x \in \mathbb{R} \). Again, each \( \tilde{f}_\varepsilon \) is a diffeomorphism from \( \mathbb{R} \) onto \( \mathbb{R} \), and we denote its inverse by \( g_\varepsilon : \mathbb{R} \to \mathbb{R} \). Due to (ii) it remains to show that \( f : \rho_c \to \rho_c \) is onto.
To this end, note first that \( |g'_\varepsilon(y)| < 1/r \) for all \( \varepsilon \leq \varepsilon_0 \) and all \( y \in \mathbb{R}^n \). Also, since \( f \in \Phi^\varepsilon G^m(\mathcal{R}^n_{\mathcal{R}^n_{\varepsilon}}, \mathcal{R}^n_{\varepsilon}) \), there exists some real number \( C > 0 \) such that \( |f_\varepsilon(0)| \leq C \) for \( \varepsilon \) small. For such \( \varepsilon \) and any \( y_\varepsilon \in \mathcal{R}^n_{\varepsilon} \) we obtain by the mean value theorem

\[
|g_\varepsilon(y_\varepsilon)| = |g_\varepsilon(y_\varepsilon) - g_\varepsilon(f_\varepsilon(0))| \leq \frac{1}{r}|y_\varepsilon - f_\varepsilon(0)| \leq \frac{1}{r}(|y_\varepsilon| + C),
\]

so that \( g_\varepsilon(y_\varepsilon) \) remains in a compact set for \( \varepsilon \) small. Based on this observation, the same argument as in (2) shows that, for any \( \varepsilon \) small. For such \( \varepsilon \) and any \( y_\varepsilon \in \mathcal{R}^n_{\varepsilon} \) we obtain by the mean value theorem

\[
|g_\varepsilon(y_\varepsilon)| = |g_\varepsilon(y_\varepsilon) - g_\varepsilon(f_\varepsilon(0))| \leq \frac{1}{r}|y_\varepsilon - f_\varepsilon(0)| \leq \frac{1}{r}(|y_\varepsilon| + C),
\]

so that \( g_\varepsilon(y_\varepsilon) \) remains in a compact set for \( \varepsilon \) small. Based on this observation, the same argument as in (2) shows that, for any \( \varepsilon \) small.

Turning now to the multi-dimensional case, we first consider Hadamard’s global inverse function theorem. For its formulation, recall that a map between topological spaces is called proper if the inverse image of any compact subset is again compact. As is easily verified, a continuous map \( \alpha : \mathbb{R}^n \to \mathbb{R}^m \) is proper if and only if

\[
|\alpha(x)| \to \infty \quad \text{as} \quad |x| \to \infty.
\]

**Theorem 10.** (Hadamard) A smooth map \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a global diffeomorphism if and only if it is proper and its Jacobian determinant never vanishes.

For a proof of this result we refer to [22].

The following theorem provides an extension of Thm. 10 to the setting of GSF.

**Theorem 11.** Suppose that \( f \in \Phi^\varepsilon G^m(\mathcal{R}^n_{\mathcal{R}^n_{\varepsilon}}, \mathcal{R}^n_{\varepsilon}) \) possesses a defining net \( f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \) such that:

1. \( \forall x \in \mathbb{R}^n \forall \varepsilon \in I : Df_\varepsilon(x) \) is invertible in \( L(\mathbb{R}^n, \mathbb{R}^n) \), and for each \( x \in \mathcal{R}^n_{\varepsilon} \), \( Df(x) \) is invertible in \( L(\mathbb{R}^n, \mathbb{R}^n) \).
2. There exists some \( \varepsilon' \in I \) such that \( \inf_{\varepsilon \in (0, \varepsilon')} |f_\varepsilon(x)| \to +\infty \) as \( |x| \to \infty \).

Then \( f \) is a global generalized diffeomorphism in \( \Phi^\varepsilon G^m(\mathcal{R}^n_{\mathcal{R}^n_{\varepsilon}}, \mathcal{R}^n_{\varepsilon}) \).

**Proof.** By Thm. 10 each \( f_\varepsilon \) is a global diffeomorphism \( \mathbb{R}^n \to \mathbb{R}^n \) for each \( \varepsilon \leq \varepsilon' \) and we denote by \( g_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \) the global inverse of \( f_\varepsilon \). In order to prove that the net \( (g_\varepsilon)_{\varepsilon \leq \varepsilon'} \) defines a GSF, we first note that, by (ii), the net \( (f_\varepsilon)_{\varepsilon \leq \varepsilon'} \) is ‘uniformly proper’ in the following sense: Given any \( M \in \mathbb{R}_{\geq 0} \) there exists some \( M' \in \mathbb{R}_{\geq 0} \) such that when \( |x| \geq M' \) then \( \forall \varepsilon \leq \varepsilon' : |f_\varepsilon(x)| \geq M \).

Hence, for any \( K \subseteq \mathbb{R}^n \), picking \( M > 0 \) with \( K \subseteq B_M(0) \) it follows that \( g_\varepsilon(K) \subseteq B_M(0) =: K' \subseteq \mathbb{R}^n \) for all \( \varepsilon \leq \varepsilon' \). Thereby, the net \( (g_\varepsilon)_{\varepsilon \leq \varepsilon'} \) maps \( \mathcal{R}^n_{\mathcal{R}^n_{\varepsilon}} \) into itself, i.e.

\[
\forall [y_\varepsilon] \in \mathcal{R}^n_{\mathcal{R}^n_{\varepsilon}} : [g_\varepsilon(y_\varepsilon)] \in \mathcal{R}^n_{\mathcal{R}^n_{\varepsilon}}.
\]

Moreover, for each \( K \subseteq \mathbb{R}^n \), assumption (i) and Lemma 1 yield

\[
\exists q \in \mathbb{R}_{>0} \forall \varepsilon \forall x \in K : |\det Df_\varepsilon(x)| > q^2.
\]
From (8) and (9) it follows as in (2) that, for any \( y = [y_\varepsilon] \in \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon \) and any \( |\beta| \geq 1 \), \( \langle \partial^\beta g_\varepsilon (y_\varepsilon) \rangle \) is moderate, so \( g := [y_\varepsilon] \mapsto [g_\varepsilon (y_\varepsilon)] \in \mathring{\rho}\mathcal{G}\mathcal{C}^\infty (\mathring{\rho}\mathring{\mathbb{R}}_\varepsilon, \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon) \). Finally, that \( g \) is the inverse of \( f \) on \( \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon \) follows as in Thm. [9].

The next classical inversion theorem we want to adapt to the setting of generalized smooth functions is the following one:

**Theorem 12.** *(Hadamard-Levy)* Let \( f : X \rightarrow Y \) be a local diffeomorphism between Banach spaces. Then \( f \) is a diffeomorphism if there exists a continuous non-decreasing function \( \beta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) such that

\[
\int_0^\infty \frac{1}{\beta(s)} \, ds = +\infty, \quad |Df(x)^{-1}| \leq \beta(|x|) \quad \forall x \in X.
\]

This holds, in particular, if there exist \( a, b \in \mathbb{R}_{\geq 0} \) with \( |Df(x)^{-1}| \leq a + b|x| \) for all \( x \in X \).

For a proof, see [9].

We can employ this result to establish the following global inverse function theorem for GSF.

**Theorem 13.** Suppose that \( f \in \mathring{\rho}\mathcal{G}\mathcal{C}^\infty (\mathring{\rho}\mathring{\mathbb{R}}_\varepsilon, \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon) \) satisfies:

(i) \( f \) possesses a defining net \( f_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( \forall x \in \mathbb{R}^n \forall \varepsilon \in I : Df_\varepsilon (x) \) is invertible in \( L(\mathbb{R}^n, \mathbb{R}^n) \), and for each \( x \in \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon, Df(x) \) is invertible in \( L(\mathring{\rho}\mathring{\mathbb{R}}_\varepsilon, \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon) \).

(ii) There exists a set of continuous non-decreasing functions \( \beta_\varepsilon : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) such that \( \forall \varepsilon \forall x \in \mathbb{R}^n : |Df_\varepsilon (x)^{-1}| \leq \beta_\varepsilon(|x|) \) and

\[
\int_0^\infty \frac{1}{\beta_\varepsilon(s)} \, ds = +\infty.
\]

Then \( f \) is a global generalized diffeomorphism in \( \mathring{\rho}\mathcal{G}\mathcal{C}^\infty (\mathring{\rho}\mathring{\mathbb{R}}_\varepsilon, f(\mathring{\rho}\mathring{\mathbb{R}}_\varepsilon)) \).

If instead of (iii) we make the stronger assumption

(iii) \( \exists C \in \mathbb{R}_{\geq 0} : \forall \varepsilon \forall x \in \mathbb{R}^n : |Df_\varepsilon (x)^{-1}| \leq C \),

then \( f \) is a global generalized diffeomorphism in \( \mathring{\rho}\mathcal{G}\mathcal{C}^\infty (\mathring{\rho}\mathring{\mathbb{R}}_\varepsilon, \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon) \).

In particular, (ii) applies if there exist \( a, b \in \mathbb{R}_{\geq 0} \) that are finite (i.e., \( a_\varepsilon, b_\varepsilon < R \) for some \( R \in \mathbb{R} \) and \( \varepsilon \) small) with \( |Df_\varepsilon (x)^{-1}| \leq a_\varepsilon + b_\varepsilon |x| \) for \( \varepsilon \) small and all \( x \in \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon \).

**Proof.** From (ii) it follows by an \( \varepsilon \)-wise application of Thm. [12] that there exists some \( \varepsilon_0 > 0 \) such that each \( f_\varepsilon \) with \( \varepsilon < \varepsilon_0 \) is a diffeomorphism: \( \mathbb{R}^n \rightarrow \mathbb{R}^n \). We denote by \( g_\varepsilon \) its inverse. Using assumption (iii) it follows exactly as in the proof of Thm. [9](ii) that \( g := [g_\varepsilon] \) is an element of \( \mathring{\rho}\mathcal{G}\mathcal{C}^\infty (f(\mathring{\rho}\mathring{\mathbb{R}}_\varepsilon), \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon) \) that is inverse to \( f \).

Assuming (iii) for any \( [y_\varepsilon] \in \mathring{\rho}\mathring{\mathbb{R}}_\varepsilon \) and \( \varepsilon \) small, we have

\[
|Dg_\varepsilon (y_\varepsilon)| = |(Df_\varepsilon (g_\varepsilon (y_\varepsilon)))^{-1}| \leq C,
\]

so the mean value theorem yields
which is uniformly bounded for $\varepsilon$ small since $f \in \rho \mathcal{G} \mathcal{C}_\infty(\tilde{\rho} R^n_\sigma, \tilde{\rho} R^n_\sigma)$. We conclude that $(g_\varepsilon)$ satisfies (8). From here, the proof can be concluded literally as in Thm. 11.

Remark 1. By Thm. 5, for $\rho = (\varepsilon)$, the space $\rho \mathcal{G} \mathcal{C}_\infty(\tilde{\rho} R^n_\sigma, \tilde{\rho} R^n_\sigma)$ can be identified with the special Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$. In this picture, $\rho \mathcal{G} \mathcal{C}_\infty(\tilde{\rho} R^n_\sigma, \tilde{\rho} R^n_\sigma)$ corresponds to the space of $c$-bounded Colombeau generalized functions on $\mathbb{R}^n$ (cf. 32, 24). Therefore, under the further assumption that $f(\rho \tilde{\rho} R^n_\sigma) = \rho \tilde{\rho} R^n_\sigma$, theorems 11 and 13 can alternatively be viewed as global inverse function theorems for $c$-bounded Colombeau generalized functions.

4 Conclusions

Once again, we want to underscore that the statement of the local inverse function theorem 6 is the natural generalization to GSF of the classical result. Its simplicity relies on the fact that the sharp topology is the natural one for GSF, as explained above. This natural setting permits to include examples in our theory that cannot be incorporated in an approach based purely on Colombeau generalized functions on classical domains (cf. Example 1 and 11).

Moreover, as Thm. 7 shows, the concept of Fermat topology leads, with comparable simplicity, to sufficient conditions that guarantee solutions defined on large (non-infinitesimal) neighborhoods.

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