A Bernstein–von-Mises theorem for the Calderón problem with piecewise constant conductivities

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Abstract
This note considers a finite dimensional statistical model for the Calderón problem with piecewise constant conductivities. In this setting it is shown that injectivity of the forward map and its linearisation suffice to prove the invertibility of the information operator, resulting in a Bernstein–von-Mises theorem and optimality guarantees for estimation by Bayesian posterior means.

Keywords: Bayesian inverse problems, Calderón problem, severe ill-posedness

1. Introduction
This article is concerned with the statistical analysis of nonlinear inverse problems modelled on parameter spaces \( \Theta \subset \mathbb{R}^D (D \in \mathbb{N}) \). For mildly ill-posed problems, recent years have seen great progress in establishing theoretical guarantees for estimation and uncertainty quantification in a high-dimensional setting, where \( D \to \infty \), as the number of measurements increases; see e.g. the articles [13, 14, 16, 23, 24, 31, 32] and [6, 25, 30, 32], which analyse inverse parameter identification problems for several representative partial differential equations (PDE). In the framework of Bayesian inverse problems it has been of particular interest to gain an understanding of the average ‘posterior landscape’ and the latter group of references addresses this issue by approximating posteriors with simpler types of distributions. Key conditions here include invertibility results for the infinite-dimensional information operator to obtain Bernstein–von-Mises type approximations by Gaussians [24, 30], or the weaker requirement of polynomial-in-\( D \) eigenvalue lower bounds for the \( D \)-dimensional information operator to...
Figure 1. The unknown conductivity $\gamma$ is assumed to be constant on $\Omega_1$, $\Omega_2$, $\ldots$, and equal to 1 in a region $\Omega_0$ near the boundary. Measurements are taken at electrodes $J_1, J_2, \ldots$ on $\partial \Omega$.

obtain relevant log-concave approximations [6, 32]. These conditions can be viewed as part of a spectrum with progressively weaker invertibility requirements on the information operator.

For severely ill-posed problems, results comparable to the above do not exist in a high-dimensional setting ($D \to \infty$). This is exemplified by the notorious Calderón problem (see the surveys in [7, 36, 37]), which is well known to exhibit only logarithmic stability in an infinite-dimensional setting—as a consequence, non-parametric estimators converge no better than at logarithmic rates in the inverse noise level [1] and high-dimension posterior approximations as above cannot be expected.

This brief note is concerned with the Calderón problem for piecewise constant conductivities. This is a widely studied variant of the problem (see e.g. [2, 3, 11, 15, 17]) that yields a model with parameter space of fixed dimension $D$. Postponing precise definitions to section 1.2, our setting can be described as follows: we take the viewpoint of electrical impedance tomography (EIT), where one seeks to recover an unknown conductivity $\gamma$ inside a body $\Omega \subset \mathbb{R}^d$ from voltage-to-current measurements $\Lambda_\gamma$ at the boundary $\partial \Omega$. We make the assumption that $\gamma$ is piecewise constant with respect to a given partition $\Omega_1 \cup \cdots \cup \Omega_D \subset \Omega$ and obeys bounds $0 < \gamma_{\min} \leq \gamma \leq \gamma_{\max}$, such that it can be described by a parameter

$$\theta \in \Theta = [\gamma_{\min}, \gamma_{\max}]^D \subset \mathbb{R}^D. \quad (1.1)$$

In applications this assumption can be reasonable when part of the interior structure of $\Omega$ or the required resolution is known a priori. The operator $\Lambda_\gamma$ is then measured at a finite collection of electrodes $J_1, \ldots, J_M \subset \partial \Omega$, assumed to be sufficiently small and densely distributed (see also figure 1). This setting allows to consider the Calderón problem in a classical i.i.d. noise model.

The contribution of this note lies in exhibiting, in the just described setting, estimators for $\theta$ with optimal convergence guarantees, as well as a Bernstein–von-Mises theorem, that is, a Gaussian approximation result for Bayesian posterior distributions. These results constitute final and definitive guarantees for the Bayesian method for EIT, advocated for in [11], in the setting of piecewise constant conductivities. In particular, as a consequence of the Bernstein–von-Mises theorem, estimation of piecewise constant conductivities $\gamma$ by ‘posterior means’ is
optimal in an asymptotic minimax sense (see remark 1.3)—at the same time, it is unclear whether classical reconstruction procedures (e.g. based on Nachman’s method [17–19]) enjoy similar statistical optimality properties for piecewise constant conductivities.

Our results also highlight the fact that high-dimensionality introduces significant challenges to the statistical understanding of inverse problems which are independent of the type of ill-posedness—indeed, as our results illustrate, many difficulties arising in the context of the first paragraph disappear for fixed dimension $D$, even for the severely ill-posed Calderón problem. Further, for a given severely ill-posed problem, it is of significance to identify models in which the severe ill-posedness disappears and we do this here from a statistical point of view for the Calderón problem by exhibiting a natural piecewise constant model.

The statistical conclusions of this article heavily rely on an analytical observation that can be paraphrased as follows: if $\gamma$ is restricted to an appropriate finite-dimensional space of conductivities, then finitely many noiseless measurements of the operator $\Lambda_\gamma$ already suffice to stably determine $\gamma$. Statements of this form were proved e.g. in [2, 3, 15] and in fact hold true for a larger class of inverse problems, essentially as long as the forward map (here $\gamma \mapsto \Lambda_\gamma$, mapping between appropriate spaces) and its linearisation are known to be injective. One aim of this article is to bridge the gap between the just mentioned advances in deterministic inverse problems and the statistical literature; this comes with the following message: if there are good reasons to model an inverse problem on a parameter space of fixed finite dimension $D$, then under natural injectivity assumptions, the analysis of the noisy problem falls into a well-understood regime, where excellent statistical guarantees—including Le Cam’s version of the Bernstein–von-Mises theorem—become available. At the same time we want to emphasise that outside of the severely ill-posed setting, where relevant stability constants typically grow exponentially in the model dimension $D$ (see also remark 3.4), it is not necessary to fix the dimension $D$ to obtain relevant statistical guarantees—this is illustrated by the results in the first paragraph.

Outline. In section 1.1 we introduce the statistical framework of the article and state our main results with an emphasis on their prototypical character pertaining to a more general class of inverse problems. The Calderón problem is fully introduced in section 1.2, where we discuss in detail how it fits into the preceding framework, subject to suitable a priori conditions. In section 2 we collect several analytical results on the Calderón problem. In section 3 we derive the required a priori conditions from section 1.2. Finally, section 4 contains the proofs of the main theorems.

1.1. Main results

The statistical set-up of our main theorems fits into the general framework of [25, 29, 30]; there the inverse problem is encoded in a forward map

$$G : \Theta \to L_2^2(\mathcal{X}, V), \quad \theta \mapsto G_\theta$$

(1.2)

and one seeks to make inference on $\theta$, given noisy measurements of $G_\theta$. Here $L_2^2(\mathcal{X}, V)$ is the space of $L^2$-integrable functions on a probability space $(\mathcal{X}, \lambda)$, taking values in a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. In the context of Calderón’s problem, we will have

$$\Theta \equiv [\gamma_{\min}, \gamma_{\max}]^D \subset \mathbb{R}^D, \quad \mathcal{X} = \{1, \ldots, M\}, \quad \text{and} \quad V = \mathbb{R}^M$$

for fixed $D \in \mathbb{N}$, $0 < \gamma_{\min} < \gamma_{\max}$, and $M = M(D, \gamma_{\min}, \gamma_{\max}) \in \mathbb{N}$; the precise definitions will be laid out in section 1.2, in particular, see definition 1.6. Note that while the Calderón problem is usually posed in terms of an operator valued forward map $\gamma \mapsto \Lambda_\gamma$, the restriction to a $D$-dimensional parameter space allows for a vector valued formulation as in (1.2). In
the context of EIT, one can think of $\mathcal{X}$ as a index set for a collection of electrodes, and of $G_\theta(x)$ as list of current measurements, when a standard voltage is applied at the electrode with index $x \in \mathcal{X}$.

We consider a random design regression model with Gaussian noise: for $\theta \in \Theta$ define $P_\theta$ as the law of $(Y, X) \in \mathbb{R}^M \times \mathcal{X}$, where $Y$ satisfies the regression equation

$$Y = G_\theta(X) + \epsilon,$$

with $X \sim \lambda$ (uniform measure) and $\epsilon \sim \mathcal{N}(0, I)$ drawn independently. For EIT, this describes a statistical experiment, where an electrode is chosen at random (draw of $X$) and currents are measured at all $M$ electrode locations (recorded in $Y$); again we refer to section 1.2 for details. Independent repetition of this procedure yields a sequence of statistical experiments

$$\mathcal{P}_N = \{P_\theta^N : \theta \in \Theta\}, \quad N = 1, 2, \ldots,$$

where $P_\theta^N = P_\theta \otimes \cdots \otimes P_\theta$ is the $N$-fold product measure on $(\mathbb{R}^M \times \mathcal{X})^N$. The corresponding expectation operator is denoted $\mathbb{E}_\theta^N$ and we use the notation $Z_N = ((Y_i, X_i) : i = 1, \ldots, N)$ for a random vector of law $P_\theta^N$ for some $\theta \in \Theta$. Finally, for $N = 1$ we write $\mathcal{P} = \mathcal{P}_1$ and $Z = (Y, X) = Z_1$.

The experiment $\mathcal{P}$ (arising from any sufficiently smooth $G$, in particular in the case of the Calderón problem) satisfies a natural regularity condition, called differentiability in quadratic mean, which allows the definition of score function and information operator, denoted by

$$s_\theta(Y, X) \in (\mathbb{R}^D)^* \quad \text{and} \quad \mathbb{N}_\theta \in \mathbb{R}^{D \times D}, \quad \theta \in \Theta \setminus \partial \Theta,$$

respectively. Here $\partial \Theta$ denotes the boundary of $\Theta$ in $\mathbb{R}^D$. For precise definitions in the present context we refer to definition 4.2 and for a general overview of the importance of $\mathbb{N}_\theta$ in estimation problems we refer to the monograph [38].

**Theorem 1.1.** Assume that $G$ (and hence $\mathcal{P}$) arise from the Calderón problem for piecewise constant conductivities as described in section 1.2. Then the information matrix $\mathbb{N}_\theta$ is invertible for all $\theta \in \Theta \setminus \partial \Theta$.

The proof of theorem 1.1 is a consequence of the general theory in [2, 3, 15] and can be adapted to parametric models of other inverse problems, as long as injectivity and compactness properties similar to (P1)–(P3) from section 1.2 are available.

Amongst the consequences of theorem 1.1 are optimal consistency and uncertainty quantification results in a Bayesian setting. The Bayesian approach to inverse problems, see [10, 11, 34] as well as the recent lecture notes [29], uses the data $Z_N$ to update a chosen prior measure $\Pi$ on $\Theta$ to a posterior measure $\Pi(\cdot|Z_N)$. The posterior can in principle be computed from evaluating the forward map $G$ alone and is then used to make inference on $\theta$ by deriving point estimators and credible regions.

As prior we may choose here any probability measure $\Pi$ on $\Theta$ (defined on the Borel $\sigma$-algebra of $\Theta$) with continuous and positive Lebesgue density $\pi$, that is,

$$d\Pi(\theta) = \pi(\theta)d\theta \quad \text{(1.3)}$$

for a continuous function $\pi : \Theta \to (0, \infty)$ with integral $\int_{\Theta} \pi \ d\theta = 1$. Given data $Z_N$ in $(\mathbb{R}^M \times \mathcal{X})^N$, define the log-likelihood function

$$\ell_N(\theta|Z_N) = -\frac{1}{2} \sum_{i=1}^N |G_\theta(X_i) - Y_i|^2_{\mathbb{R}^M}, \quad \theta \in \Theta,$$
and the resulting posterior measure by
\[ \Pi(A|Z_N) = \frac{\int_\Theta e^{N(\theta|Z_N)}d\Pi(\theta)}{\int_\Theta e^{N(\theta|Z_N)}d\Pi(\theta)} A \subset \Theta \text{ Borel measurable.} \]

Our next result is a Bernstein–von-Mises theorem for the Calderón problem:

**Theorem 1.2.** Assume that \( G \) (and hence \( P_N \)) arise from the Calderón problem for piecewise constant conductivities as described in section 1.2. Let \( \Pi \) be a prior as in (1.3) and let \( \theta_0 \in \Theta \setminus \partial \Theta \). Then, if \( \vartheta \sim \Pi(\cdot|Z_N) \) is a posterior draw, as \( N \to \infty \),
\[ \| L \left( \sqrt{N}(\vartheta - \Psi_{\theta_0,N})|Z_N \right) - N_D(0,N_{\theta_0}^{-1}) \|_{TV} \to 0 \quad \text{in } P_{\theta_0}^N. \]

Here \( L(\cdot) \) denotes the law of a random variable in \( \mathbb{R}^D \), \( \| \cdot \|_{TV} \) is the total variation norm of Borel measures on \( \mathbb{R}^D \), and the re-centring \( \Psi_{\theta_0,N} \) is given by
\[ \Psi_{\theta_0,N} = \theta_0 + \frac{1}{N} \sum_{i=1}^N N_{\theta_0}^{-1} \Delta \theta_i(Y_i, X_i). \quad (1.4) \]

**Remark 1.3. (Posterior means).** If the prior \( \Pi \) is assumed to have finite moments to all orders (e.g. if \( \Pi \) is a uniform prior), then—arguing as in step VII of the proof of theorem 8 in [28]—the theorem also holds true when the re-centring \( \Psi_{\theta_0,N} \) is replaced by the more tractable posterior mean
\[ \bar{\theta}_N \equiv E^{\Pi}[\vartheta|Z_N]. \]

By the same proof, for \( Z_N \sim P_{\theta_0}^N (\theta_0 \in \Theta \setminus \partial \Theta) \) and as \( N \to \infty \),
\[ \sqrt{N}(\bar{\theta}_N - \theta_0) \to N_D(0,N_{\theta_0}^{-1}) \quad \text{in distribution,} \quad (1.5) \]
see also theorem 10.8 in [38]. In particular, posterior means achieve the asymptotic minimax optimal variance for estimation of \( \theta \) under square loss. That is, the posterior means \( T_N = \theta_N \) obtain the lower bound in the asymptotic risk estimate
\[ \lim_{\delta \to 0} \liminf_{N \to \infty} \sup_{|\theta - \theta_0| < \delta} \text{Cov}_{\theta_0}^N \left[ \sqrt{N}(T_N - \theta) \right] \geq N_{\theta_0}^{-1}, \quad (1.6) \]
understood in the sense of the Loewner order\(^1\), which is valid for all estimator sequences \( (T_N : (\mathbb{R}^d \times X)^N \to \Theta : N = 1, 2, \ldots) \) (cf theorem 8.11 in [38]). This follows from a version of (1.5) that is locally uniform in \( \theta_0 \in \Theta \setminus \partial \Theta \)—we omit a detailed proof.

Let us discuss some further consequence of the preceding theorem. First, for any \( \theta_0 \in \Theta \setminus \partial \Theta \) there exists a constant \( C > 0 \) such that, as \( N \to \infty \),
\[ \Pi \left( \theta : |\theta - \theta_0| \leq CN^{-1/2}|Z_N \right) \to 1 \quad \text{in } P_{\theta_0}^N. \]
This means that the posterior is consistent at \( \theta_0 \), in the sense that it is likely to concentrate its mass in small balls around the truth, at the optimal rate of \( N^{-1/2} \). Recall from [1] that in infinite

\(^1\)One says that two symmetric matrices \( A, B \) satisfy \( A \preceq B \) in the Loewner order, if their difference \( B - A \) is positive semi-definite.
dimensions (say, for $C^\beta$-regular conductivities for some $\beta > 0$), the posterior only concentrates about $\theta_0$ with rate $(\log N)^{-\delta}$ for some $\delta > 0$.

Moreover, using the strengthened version of the theorem from remark 1.3, one obtains an uncertainty quantification as follows: for a confidence level $0 < \alpha < 1$, define credible regions $C_N \subset \Theta$ and quantiles $R_N > 0$ by

$$C_N = \{ \theta \in \Theta : |\theta - \bar{\theta}_N| < R_N \}, \quad \Pi(C_N|Z_N) = 1 - \alpha.$$ 

Note that finding $C_N$ and $R_N$ does not require knowledge of the information matrix $N_{\theta_0}$, but can be computed approximately via Markov chain Monte Carlo methods. Arguing as in section 2.5 of [24] (see also section 4.1.3 in [29]), the Bernstein–von-Mises theorem implies that, as $N \to \infty$,

$$P_N^N (\theta_0 \in C_N) \to 1 - \alpha,$$

showing that $C_N$ has valid frequentist coverage of the true parameter $\theta_0$ at level $\alpha$.

### 1.2. A statistical Calderón problem

We now give a brief account of the Calderón problem and explain how it can be recast as an inverse problem with a forward map $G$ as in (1.2). Throughout, we employ the following notational conventions:

- If not stated otherwise, function spaces contain $\mathbb{R}$-valued functions and all Banach spaces are real;
- for two Banach spaces $E$ and $F$, the space of bounded linear operators is denoted $\mathcal{B}(E,F)$ and the operator norm is written $\| \cdot \|_{\mathcal{B}(E,F)}$ or $\| \cdot \|_{E \to F}$. If $E = F$, then we write $\mathcal{B}(E)$ instead of $\mathcal{B}(E,E)$;
- the indicator function of a set $S$ is denoted with $1_S$.

Let $\Omega \subset \mathbb{R}^d (d \geq 2)$ be a bounded domain with smooth boundary $\partial \Omega$. Let $L_+^\infty(\Omega)$ be the space of $\gamma \in L^\infty(\Omega)$ which satisfy a lower bound $\gamma \geq c$ for some $c \in (0, \infty)$, almost everywhere in $\Omega$. Given $\gamma \in L_+^\infty(\Omega)$, the boundary value problem

$$\text{div}(\gamma\nabla u) = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega$$

has a unique solution $u = u^g$ for any given boundary datum $g : \partial \Omega \to \mathbb{R}$ of sufficient regularity. The Dirichlet-to-Neumann operator of $\gamma$ is then formally defined by

$$\Lambda_\gamma g = \gamma \partial_\nu u^g |_{\partial \Omega},$$

where $\partial_\nu$ is the (outward pointing) normal derivative. Denoting with $H^s = H^s(\partial \Omega)$ the Sobolev space of order $s \in \mathbb{R}$, the operator $\Lambda_\gamma$ extends to a bounded linear map $H^{1/2} \to H^{-1/2}$. This gives rise to a nonlinear map

$$L_+^\infty(\Omega) \to \mathcal{B}(H^{1/2},H^{-1/2}), \quad \gamma \mapsto \Lambda_\gamma,$$

and Calderón’s problem asks to establish injectivity of this map on large classes of conductivities, or indeed to reconstruct $\gamma$ from $\Lambda_\gamma$. We refer to [5, 9, 20, 26, 27, 35] for some of the most influential articles on this topic.

In EIT, the function $\gamma$ models an unknown electrical conductivity inside $\Omega$ and $\Lambda_\gamma g$ is the electrical current caused by applying voltage $g$ on $\partial \Omega$; the conductivity $\gamma$ is then to be determined from voltage-to-current measurements. We want to pose a statistical inverse problem of
the following form: after fixing electrode positions \( J_1, \ldots, J_M \subset \partial \Omega \), one electrode \( J_k \) is chosen at random and a constant voltage \( g = 1_{J_k} \) is applied. The resulting current is then measured at all electrodes, corrupted by statistical noise, yielding a data vector \( Y \) in \( \mathbb{R}^M \). The random vector \((Y, X) \in \mathbb{R}^M \times \{1, \ldots, M\} \) can then be interpreted as data vector in a statistical regression model, as described in section 1.1.

To pass to a setting as in section 1.1, we make several \textit{a priori} assumptions, which will inform the precise definition of the forward map \( G \) in (1.2). In essence, we require that \( \gamma \) is piecewise constant and satisfies a bound \( \gamma_{\text{min}} \leq \gamma \leq \gamma_{\text{max}} \). In this case, the electrodes \( J_1, \ldots, J_M \) can be chosen such that \( \gamma \) is determined from finitely many \textit{noiseless} measurements at these positions (cf. remark 1.7); this is the setting in which our statistical analysis takes place (see also figure 1 above).

We now describe the assumptions on \( \gamma \) and the forward map \( G \) in detail. First, we assume that \( \gamma \) belongs to a known, finite dimensional class \( E_D \) of conductivities. Let \( \Omega_0, \ldots, \Omega_D \subset \Omega \) be a collection of pairwise disjoint domains, satisfying

\[
\Omega = \Omega_0 \cup \cdots \cup \Omega_D \quad \text{and} \quad \Omega_i \subset \Omega \quad \text{for} \quad i = 1, \ldots, D,
\]

and having piecewise smooth boundaries. Denoting the associated characteristic functions by \( 1_{\Omega_0}, 1_{\Omega_1}, \ldots \), we define

\[
E_D := \left\{ \gamma \in L^\infty(\Omega) : \gamma = \gamma_\theta \equiv 1_{\Omega_\theta} + \sum_{i=1}^D \theta_i 1_{\Omega_i} \text{ for } (\theta_1, \ldots, \theta_D) \in \mathbb{R}^D \right\}, \quad (1.7)
\]

with tangent space \( E_D^0 = \{ \kappa \in L^\infty(\Omega) : \kappa + 1_{\Omega_0} \in E_D \} \). The space \( E_D \) satisfies the following properties (see theorem 2.5):

(P1) \( E_D \cap L^\infty_1(\Omega) \ni \gamma \mapsto \Lambda_\gamma \) is injective;

(P2) For all \( \gamma \in E_D \cap L^\infty_1(\Omega) \), the differential \( E_D \ni \kappa \mapsto d\Lambda_\gamma(\kappa) \) is injective.

To formulate a third property, recall that an operator \( T \) on \( \partial \Omega \) is called \textit{smoothing} if it extends to a bounded linear operator \( T : H^s \to H^t \) for all choices of \( s, t \in \mathbb{R} \). As functions in \( E_D \) are constant near \( \partial \Omega \), also the following holds true:

(P3) For all \( \gamma \in E_D \cap L^\infty_1(\Omega) \) and \( \kappa \in E_D^0 \), the operator \( d\Lambda_\gamma(\kappa) \) is smoothing and the map \( (\gamma, \kappa) \mapsto \|d\Lambda_\gamma(\kappa)\|_{H^s \to H^t} \) is locally bounded for any \( s, t \in \mathbb{R} \).

\textbf{Remark 1.4.} The preceding three properties also hold for other finite dimensional classes \( E_D \) (e.g. containing piecewise analytic functions [15]) and we could have chosen any such class; for the sake of concreteness we will only work with piecewise constant conductivities as in (1.7).

\textbf{Remark 1.5.} Property (P3) guarantees certain compactness properties that allow for the passage to "finite measurements" in the sense of [3]. One can dispense with (P3), if one passes to a different measurement model and uses e.g. the Neumann-to-Dirichlet operators—this route is chosen in the just cited article.

Now suppose that \( J = \{J_1, \ldots, J_M\} \) is a collection of pairwise disjoint, measurable subsets \( J_1, \ldots, J_M \subset \partial \Omega \) of positive measure. We define the quantity

\[
\Delta(J) := \left| \partial \Omega \setminus \bigcup_{k=1}^M J_k \right|^{1/2} + \sup_{k=1,\ldots,M} \text{diam}(J_k) > 0, \quad (1.8)
\]
where $|\cdot|$ denotes the surface measure on $\partial \Omega$ and $\text{diam} J = \sup_{x, y \in J} |x - y|$ is the diameter. Given a priori bounds $\gamma_{\text{min}} \leq \gamma \leq \gamma_{\text{max}}$, it will suffice to test $\Lambda_\gamma$ at the indicator functions $1_{J_1}, \ldots, 1_{J_M}$, provided $\Delta(J)$ is sufficiently small. Precisely, we make the assumption that

$$\Delta(J) \leq \delta = \delta(E_D, \gamma_{\text{min}}, \gamma_{\text{max}}),$$

(1.9)

where $\delta > 0$ is specified in theorem 3.5 below. For technical reasons we now pass to the normalised Dirichlet-to-Neumann operator

$$\tilde{\Lambda}_\gamma = \Lambda_\gamma - \Lambda_1,$$

(1.10)

which extends to an $L^2$-bounded (in fact, smoothing) operator and contains the same information as $\Lambda_\gamma$; this ensures that the $L^2$-pairing $\langle \tilde{\Lambda}_\gamma 1_{J_i}, 1_{J_j} \rangle$ is well defined.

**Definition 1.6.** Given a priori choices of $D \in \mathbb{N}$ and $0 < \gamma_{\text{min}} \leq \gamma_{\text{max}}$, as well as a choice of $J = \{J_1, \ldots, J_M\}$ satisfying (1.9), we make the following definitions:

$$\Theta := \{\theta \in \mathbb{R}^D : \gamma_{\text{min}} \leq \theta_i \leq \gamma_{\text{max}} \text{ for all } i = 1, \ldots, D\},$$

$$\Theta_j := \{\theta \in \Theta : \theta_j \neq 0\},$$

where $\gamma_\theta$ is the piecewise constant conductivity from (1.7). Finally, with $(\mathcal{X}, \lambda) = (\{1, \ldots, M\}, \text{uniform})$, a forward map $G$ as in section 1.1 is defined by

$$G : \Theta \to L^2_2(\mathcal{X}, \mathbb{R}^M), \quad G_\theta(x) = (G_\theta^{(j)} : j = 1, \ldots, M) \in \mathbb{R}^M \quad (x \in \mathcal{X}).$$

(1.12)

**Remark 1.7.** In order for the results of section 1.1 to apply, the electrodes have to be chosen in accordance with requirement (1.9), which is of course difficult to verify in practice. The threshold $\delta$ will in particular depend on the precise Lipschitz stability constants for the forward map and its linearisation on $E_D$, as well as a domain dependent constant—see also remark 4 and example 4 in [3].

### 2. Analytical aspects

In this section we collect some results, mostly well known, on the forward map $\gamma \mapsto \Lambda_\gamma$ and its linearisation $d \Lambda_\gamma$. In particular, we show that properties (P1)–(P3) from section 1.2 are satisfied on spaces of piecewise constant conductivities. Throughout, $\Omega$ is a bounded, smooth domain $\Omega \subset \mathbb{R}^d$. We will write $\int_\Omega$ and $\int_{\partial \Omega}$ for integration on $\Omega$ and $\partial \Omega$, understood with respect to the Lebesgue measures of dimension $d$ and $d - 1$, respectively; the corresponding $L^2$-pairing (as usual, extended as duality pairing between distributions) is denoted with $\langle \cdot, \cdot \rangle$ in both cases.

#### 2.1. Linearisation

The linearisation of $\gamma \mapsto \Lambda_\gamma$ was already computed in Calderón’s original article [9], we summarise his result (supplemented by a statement on Fréchet differentiability, which is easy to check) in the following proposition.

**Proposition 2.1.** Given $g \in H^{1/2}(\partial \Omega)$, the map $\gamma \mapsto \langle \Lambda_\gamma, g \rangle$ is real analytic on $L^\infty_\gamma(\Omega)$. Its differential at $\gamma \in L^\infty_\gamma(\Omega)$ in direction $\kappa \in \mathbb{R}^\infty(\Omega)$ is given by
\[ \langle d\Lambda_\gamma(\kappa)g, g \rangle := \frac{d}{dt} \bigg|_{t=0} \langle \Lambda_{\gamma + t\kappa}g, g \rangle = \int_\Omega |\nabla u|^2, \quad (2.1) \]

where \( u = u^\gamma \) is the unique solution in \( H^1(\Omega) \) to

\[ \text{div}(\gamma \nabla u) = 0 \quad \text{on} \quad \Omega \quad \text{and} \quad u = g \quad \text{on} \quad \partial \Omega. \]

Equation (2.1) defines a bounded linear operator \( d\Lambda_\gamma \) from \( L^\infty(\Omega) \) to \( \mathcal{B}(H^{1/2}, H^{-1/2}) \), depending continuously on \( \gamma \in L^\infty(\Omega) \). Moreover, the map \( \gamma \mapsto \Lambda_\gamma \) is Fréchet differentiable as map from \( L^\infty(\Omega) \) into \( \mathcal{B}(H^{1/2}, H^{-1/2}) \), with Fréchet derivative \( d\Lambda_\gamma \). \( \Box \)

The next proposition is an instance of a well-known phenomenon: if \( \gamma \) agrees with a smooth conductivity \( \gamma' \) in a neighbourhood of \( \partial \Omega \), then \( \Lambda_\gamma - \Lambda_{\gamma'} \) is a smoothing operator. For the linearisation \( d\Lambda_\gamma \) this has the following consequence:

**Proposition 2.2.** Let \( K \subset \Omega \) be compact and \( s, t \in \mathbb{R} \). Then for all \( \gamma \in L^\infty(\Omega) \) with \( \gamma = 1 \) on \( \Omega \setminus K \) and all \( \kappa \in L^\infty(\Omega) \) with \( \kappa = 0 \) on \( \Omega \setminus K \) the differential \( d\Lambda_\gamma(\kappa) \) extends to a bounded linear operator from \( H^s \) to \( H^t \), with operator norm

\[ \|d\Lambda_\gamma(\kappa)\|_{H^s \to H^t} \leq C \]

for a constant \( C \) depending only on \( s, t, K, \|\kappa\|_{L^\infty(\Omega)} \) and \( \inf \Omega \gamma \).

For this, we require two auxiliary lemmas.

**Lemma 2.3.** For all \( s \in (-\infty, 1/2] \) and all \( \gamma \in L^\infty(\Omega) \) with \( \gamma = 1 \) on \( \Omega \setminus K \) there exists a continuous linear Poisson operator

\[ P_\gamma : H^s(\partial\Omega) \to H^{s+1/2}(\Omega) \cap H^1_{\text{loc}}(\Omega), \]

as follows: if \( g \in H^s(\partial\Omega) \), then \( u = P_\gamma g \) solves the boundary value problem

\[ \text{div}(\gamma \nabla u) = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega \quad (2.3) \]

in a sense made precise in the proof. Moreover, the difference operator \( P_\gamma - P_1 \) maps \( H^s(\partial\Omega) \) into \( H^1_{\text{loc}}(\Omega) \), with norm

\[ \|P_\gamma - P_1\|_{H^s(\partial\Omega) \to H^1_{\text{loc}}(\Omega)} \leq C(s, K, \inf \Omega \gamma). \]

This is essentially proved in [1, lemma 19]; for the convenience of the reader, we give a brief recap of the proof, referring there for further details.

**Proof of Lemma 2.3.** For \( \gamma = 1 \), the existence and mapping properties of \( P_1 \) follow from the standard solution theory of the Laplace equation, as summarised e.g. in remark 7.2 of [22]. For general \( \gamma \in L^\infty(\Omega) \) with \( \gamma = 1 \) in \( \Omega \setminus K \), we say that \( u \in H^{s+1/2}(\Omega) \cap H^1_{\text{loc}}(\Omega) \) is a solution of (2.3), if \( w = u - P_1 g \) satisfies

\[ w \in H^1_0(\Omega), \quad \langle \gamma \nabla w, \nabla \varphi \rangle = F(\varphi) \quad \text{for all} \quad \varphi \in H^1_0(\Omega), \]

where \( F(\varphi) = \langle -\gamma \nabla P_1 g, \nabla \varphi \rangle \) for \( \varphi \in C_0^\infty(\Omega) \). One shows, using the mapping properties of \( P_1 \) and that \( \text{supp}(1 - \gamma) \subset K \), that \( F \) extends to a continuous linear functional on \( H^1_0(\Omega) \) with norm \( \|F\|_{H^{-1}(\Omega)} \leq C\|g\|_{H^s(\partial\Omega)} \), where \( C = C(s, K, \inf \Omega \gamma) > 0 \). Existence and \textit{a priori} bounds on \( w \) then follow from Lax–Milgram theory and the lemma is proved upon setting \( P_\gamma g = w + P_1 g \). \( \Box \)
The next lemma is an application of elliptic regularity theory for the laplace equation, see remark 7.2 in [22] and lemma 22 in [1].

**Lemma 2.4.** For $U \subset \Omega$ open and $s \in \mathbb{R}$, consider the space of harmonic functions $H^s(\Omega) = \{u \in H^s(\Omega) \mid \Delta u = 0\}$, equipped with the $H^s(\Omega)$ norm. Then:

(a) for all $s \in \mathbb{R}$, the operator $P_1 : H^s(\partial \Omega) \rightarrow H^{s+1/2}(\Omega)$ is an isomorphism;
(b) for all $s, t \in \mathbb{R}$, and any open set $U \subset \Omega$, restriction $u \mapsto u|_U$ defines a bounded linear map $H^s(\Omega) \rightarrow H^s(U)$. □

**Proof of Proposition 2.2.** Let $g, h \in C^\infty(\partial \Omega)$ and suppose that $U$ is an open set with $K \subset U \subset \Omega$. Consider $u = P_1g$ and $v = P_1 h$, both elements in $H^1(\Omega)$. Then by the H"older inequality,

$$|\langle d\Lambda_*(\kappa)g, h \rangle| \leq \|\nabla u\|^2_{H^s(\Omega)} \leq \|\kappa\|_{L^\infty(\Omega)} \|u\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)},$$

because $\text{supp} \kappa \subset U$. Then, for any $s \in (\infty, 1/2]$ we have

$$\|u\|_{H^s(\Omega)} \leq \|(P_1 - P_1)g\|_{H^s(\Omega)} + \|P_1g\|_{H^s(\Omega)} \leq C \left\{ \|g\|_{H^s(\Omega)} + \|P_1g\|_{H^{s+1/2}(\Omega)} \right\} \leq C \|g\|_{H^s(\Omega)}$$

with constants $C, C' > 0$, depending at most on $s, K, U$, and $\inf_{\Omega} \gamma$. Here we used lemmas 2.3 and 2.4(b) for the second estimate and part (a) of that lemma for the third one. Bounding $v$ in the same way, we see that for all $s, r \in (-\infty, 1/2]$, $r \neq s$, $\|d\Lambda_*(\kappa)g, h \rangle| \leq C'' \|g\|_{H^s(\Omega)} \|h\|_{H^r(\Omega)}$, (2.4)

with $C''$ depending at most on $s, r, K, U$ and $\inf_{\Omega} \gamma$. This shows that $d\Lambda_*(\kappa)$ extends to a bounded operator from $H^s(\partial \Omega)$ to $H^r(\partial \Omega)$, with norm $\leq C''$. This encompasses the desired mapping property for all $s, t \in \mathbb{R}$. □

### 2.2. Piecewise constant conductivities

**Theorem 2.5.** The space $E_D$ of piecewise constant conductivities from (1.7) satisfies properties (P1)–(P3).

**Proof of (P1).** Injectivity results on piecewise constant conductivities go back at least to [20]; for a setting that encompasses piecewise smooth boundaries $\partial \Omega_0, \ldots, \partial \Omega_D$ we refer to [4, theorem 2.7]. (In $d = 2$, article [5] settles the injectivity question on all of $L^\infty(\Omega)$ and regularity assumptions on $\partial \Omega_0, \ldots, \partial \Omega_D$ become obsolete.)

**Proof of (P2).** We use the technique of [21], where linearised injectivity was proved in a slightly different context. Assume that $d\Lambda_*(\kappa) = 0$ for some $\gamma \in E_D \cap L^\infty(\partial \Omega)$ and some $\kappa \in E_D$ with $\kappa \neq 0$. Then by a simple geometric argument, and after replacing $\kappa$ with $-\kappa$ if necessary, there exists a subdomain $U \subset \Omega$ such that

$$0 \neq \kappa \geq 0 \quad \text{on} \quad U, \quad U \cap \partial \Omega \neq \emptyset \quad \text{and} \quad U \text{ is connected.} \quad (2.5)$$

Let $V_1 \subset U$ be a small ball on which $\kappa > 0$ and put $V_2 = \Omega \setminus U$. Then $V_1$ and $V_2$ satisfy the properties of [12, theorem 2.7] and as any $\gamma \in E_D$ has the unique continuation property, the just
cited theorem yields a sequence \((u_n) \subset H^1(\Omega)\) of solutions to \(\text{div}(\gamma \nabla u) = 0\) on \(\Omega\), with energies satisfying \(\int_{V_1} |\nabla u_n|^2 \to \infty\) and \(\int_{V_2} |\nabla u_n|^2 \to 0\) as \(n \to \infty\). This leads to a contradiction, as \(g_n = u_n|_{\partial \Omega} \in H^{1/2}(\partial \Omega)\) satisfies
\[
0 = \langle d\Lambda_\gamma(\kappa) g_n, g_n \rangle = \int_{\Omega} \kappa |\nabla u_n|^2 \geq \min_{V_1} \kappa \int_{V_1} |\nabla u_n|^2 - \|\kappa\|_\infty \int_{V_2} |\nabla u_n|^2 \to \infty.
\]
Here we used the representation of \(d\Lambda_\gamma(\kappa)\) from proposition 2.1.

**Proof of (P3).** This is a direct consequence proposition 2.2 above. \(\square\)

### 3. Finite number of electrodes

This section is based on the articles [2, 3], which develop a general theory of infinite-dimensional inverse problems with finite (noiseless) measurements. In our context, this will allow to formulate sufficient *a priori* conditions on the set of electrodes \(J\) in terms of the quantity \(\Delta(J)\).

We will start by introducing appropriate *projection operators* that map the operator \(\Lambda_\gamma\) to a finite dimensional matrix and that are adapted to the concrete measurement setup discussed in section 1.2. The articles cited in the preceding paragraph allow for more general projection operators (see e.g. the discussion in section 2.3 of [3]) and the results of this article could be generalised accordingly—the choice for the present type of projection operators was made for the sake of concreteness and because there is a natural noise model for the resulting measurements.

#### 3.1. Projection operators

Let \(J = \{J_1, \ldots, J_M\}\) be a collection of pairwise disjoint, measurable subsets of \(\partial \Omega\) of positive measure. Writing \(L^2\) for \(L^2(\partial \Omega)\), we define the following projection operators:
\[
P_J : L^2(\partial \Omega) \to L^2(\partial \Omega), \quad P_J g = \sum_{k=1}^{M} \frac{1_{J_k}}{|J_k|} \int_{J_k} g,
\]
\[
Q_J : B(L^2) \to B(L^2), \quad Q_J T = P_J T P_J
\]
We seek out norm bounds for \(Q_J\) and \(I - Q_J\) in terms of \(\Delta(J)\) from (1.8). To simplify the statements below, we assume that the norms \(\|\cdot\|_{H^s}\) on \(H^s(\partial \Omega)\) are chosen to be increasing in \(s\), and such that
\[
\|g\|_\infty + \sup_{x,y \in \partial \Omega, x \neq y} \frac{|g(x) - g(y)|}{|x - y|} \leq \|g\|_{H^s}, \quad \text{for } g \in H^s(\partial \Omega), s \geq d/2 + 1,
\]
which is possible by the Sobolev embedding theorem.

**Lemma 3.1.** For all \(T \in B(L^2)\) we have
\[
\|Q_J T\|_{L^2 \to L^2}^2 \leq \sum_{i,j=1}^{M} T^2_{ij}, \quad \text{where } T_{ij} = \frac{1}{(|J_i||J_j|)^{1/2}} \langle T 1_{J_i}, 1_{J_j} \rangle_{L^2}.
\]
Proof. Note that the functions $\psi_i = 1_{J_i}/|J_i|^{1/2}$ ($i = 1, \ldots, M$) are orthonormal and that $P_ig = \sum_{i=1}^M \psi_i(g, \psi_i)_L$ for all $g \in L^2$. Hence, by Cauchy–Schwarz,

$$\|Q_1Tg\|_{L^2}^2 = \sum_{i=1}^M \langle \psi_i, TP_ig \rangle_{L^2}^2 = \sum_{i=1}^M \left( \sum_{j=1}^M \langle \psi_i, T\psi_j \rangle_{L^2} \langle g, \psi_j \rangle_{L^2} \right)^2 \leq \sum_{i=1}^M \left( \sum_{j=1}^M T_{ij} \right) \left( \sum_{j=1}^M \|\psi_j\|^2_{L^2} \right) \leq \|g\|_{L^2}^2 \sum_{i,j=1}^M T_{ij}^2.$$  

□

Lemma 3.2. Let $s = d/2 + 1$.

(a) Let $s \geq s^*$ and suppose that $g \in H^s(\partial\Omega)$. Then

$$\|(I - P_1)g\|_{L^2(\partial\Omega)} \leq (1 + |\partial\Omega|^{1/4}) \|g\|_{H^s} \cdot \Delta(J).$$

(b) Suppose that $T$ is a smoothing operator on $\partial\Omega$, such that

$$\|T\| = (1 + |\partial\Omega|^{1/4}) \left(\|T\|_{L^2(H^s \rightarrow H^s)} + \|T\|_{H^{-s} \rightarrow L^2} \right) < \infty.$$

Then

$$\|(I - Q_1)T\|_{L^2(\partial\Omega)} \leq \Delta(J) \cdot \|T\|.$$  

Proof. Let $J_0 = \partial\Omega \setminus \bigcup_{k=1}^M J_k$ and choose points $x_k \in J_k$ such that $\int_{J_k} g = g(x_k)|J_k|$ for all $k = 1, \ldots, M$. The indicator functions of $J_0, \ldots, J_M$ are orthogonal and hence

$$\|(I - P_1)g\|_{L^2(\partial\Omega)}^2 = \|g\|_{L^2(J_0)}^2 + \sum_{k=1}^M \|g - g(x_k)\|_{L^2(J_k)}^2.$$

For $x \in J_k$ we have $|g(x) - g(x_k)| \leq \text{diam}(J_k) \cdot \|g\|_{H^s}$ by (3.3) and thus we can bound the previous display by

$$\leq |J_0| \cdot \|g\|_{H^s}^2 + \|g\|_{H^s}^2 \cdot \sup_{k=1,\ldots,M} \text{diam}(J_k)^2 \cdot \sum_{k=1}^M |J_k| \leq (1 + |\partial\Omega|) \cdot \Delta(J)^2 \cdot \|g\|_{H^s}^2,$$

which proves (a). Next, note that $\|P_1\|_{L^2 \rightarrow L^2} \leq 1$ and write $c_J = (1 + |\partial\Omega|)^{1/4} \Delta(J)$. Then for $g \in L^2(\partial\Omega)$ it holds that

$$\|(I - Q_1)Tg\|_{L^2} \leq \|(I - P_1)Tg\|_{L^2} + \|P_1T(I - P_1)g\|_{L^2} \leq c_J \|Tg\|_{H^s} + \|T\|_{H^{-s} \rightarrow L^2} \|(I - P_1)g\|_{H^{-s}}.$$  

Now $\|(I - P_1)g\|_{H^{-s}} = \sup \|(I - P_1)g, h\|_{L^2},$ where the supremum is taken over $h \in H^s(\partial\Omega)$ with $\|h\|_{H^s} \leq 1$. Using self-adjointness of $(I - P_1)$, the Cauchy–Schwarz inequality and part (a), this can be bounded by $c_J \|g\|_{L^2}$. Hence the previous display continues with

$$\leq c_J \left( \|(Tg\|_{H^s} + \|T\|_{H^{-s} \rightarrow L^2} \cdot \|g\|_{L^2} \right),$$

as desired. □
3.2. Stability estimates and finite measurements

In this section $E_D \subset L^\infty(\Omega)$ is an arbitrary $D$-dimensional linear subspace.

On $E_D$, the stability estimates required in the setting of [2, 3] are immediate corollaries of the corresponding injectivity results. This is a consequence of the compactness argument from [8] (extended to non-convex compacts in [2]); precisely:

**Proposition 3.3.** Suppose properties (P1) and (P2) are satisfied and $K \subset E_D \cap L^\infty(\Omega)$ is compact. Then for all $\gamma, \gamma' \in K$ and $\kappa \in E_D$, it holds that

$$
\|\gamma - \gamma'\|_\infty \leq S_1 \|\Lambda_\gamma - \Lambda_{\gamma'}\|_{H^{1/2}-H^{-1/2}}
$$

(3.4)

$$
\|\kappa\|_\infty \leq S_2 \|d\Lambda_\gamma(\kappa)\|_{H^{1/2}-H^{-1/2}}
$$

(3.5)

for constants $S_1, S_2 > 0$ only depending on $K$ and $E_D$.

**Proof.** The nonlinear stability estimate is provided by theorem 1 in [2], noting that $\gamma \mapsto \Lambda_\gamma$ is of class $C^1$ in view of proposition 2.1. The linear stability estimate follows from (P2), by taking an infimum over the compact set $K \times \{\kappa \in E_D : \|\kappa\| \leq 1\}$. \qed

**Remark 3.4.** Note that the stability constants $S_1$ and $S_2$ are obtained by compactness arguments and are thus not quantitative—however, they should be expected to grow exponentially in $D$ as $D \to \infty$, as verified in some special cases in [33].

The following contains the main theorem from [3], applied to our setting:

**Theorem 3.5.** In the setting of the previous proposition, assume in addition that $K$ is convex and that property (P3) is satisfied. Then:

(a) there exists constant $A = A(K) > 0$ such that $Q_\gamma$ from (3.2) satisfies

$$
\sup_{\gamma \in K} \| (1 - Q_\gamma) d\Lambda_\gamma \|_{E_D^* \otimes B(\Omega)} \leq A(K) \cdot \Delta(J);
$$

(b) with $S_1, S_2$ and $A$ as in proposition 3.3 and part (a), define

$$
\delta(E_D, K) := \frac{1}{2A \max(S_1, S_2)}
$$

and suppose that $\Delta(J) \leq \delta(E_D, K)$. Then for all $\theta, \theta' \in \Theta$ with $\gamma_{\theta}, \gamma_{\theta'} \in K$ (defined as in (1.7)) the matrices $G_{\theta}, G_{\theta'}$ from (1.11) satisfy

$$
\|\theta - \theta'\|_\infty \leq 2S_1 \|G_\theta - G_{\theta'}\|_{R^{M \times M}}.
$$

(3.6)

Moreover, if $\theta \in \Theta$ with $\gamma_{\theta} \in K$ and $h \in \mathbb{R}^D$, then

$$
\|h\|_\infty \leq 2S_2 \|dG_\theta(h)\|_{R^{M \times M}},
$$

(3.7)

where $dG_\theta(h) = Q_\gamma d\Lambda_{\gamma_{\theta}}(h_11_{\Omega_1} + \ldots + h_D1_{\Omega_D})$, viewed as element in range($Q_N$) $\equiv \mathbb{R}^{M \times M}$.

**Proof.** By lemma 3.2, for all $\gamma \in E_D$ and $\kappa \in E_D$, it holds that

$$
\|(I - Q_\gamma) d\Lambda_\gamma(\kappa)\|_{B(\Omega)} \leq \Delta(J) \cdot \|d\Lambda_\gamma(\kappa)\|.
$$

Set $A(K) = \sup \|d\Lambda_\gamma(\kappa)\|$, where the supremum is taken over all $(\gamma, \kappa)$ with $\gamma \in K$ and $\|\kappa\| \leq 1$; this is finite by property (P3) and (a) is proved.
Next, by theorem 2(ii) in [3], if \( \Delta(J) \leq 1/(2AS) \) and \( \gamma, \gamma' \in K \), then
\[
\| \gamma - \gamma' \|_{L^\infty} \leq 2S_1 \| Q_J \tilde{\Lambda} \gamma - Q_J \tilde{\Lambda} \gamma' \|_{L^2 \rightarrow L^2}
\]
If \( \gamma = \gamma_0 \) and \( \gamma' = \gamma_0 \), then the norm on the right-hand side equals \( \| G_\theta - G_\theta' \|_{R^M \rightarrow R^M} \) by lemma 3.1 and (3.6) follows. Let \( \kappa = h_1 1_{\Omega_1} + \cdots + h_D 1_{\Omega_D} \). Then by lemma 3.1 we have
\[
\| dG_\theta(h) \|_{R^{M \times M}} \geq \| Q_J d\Lambda_\gamma(\kappa) \|_{R^{L^2 \rightarrow L^2}}
\]
\[
\geq \| d\Lambda_\gamma(\kappa) \|_{R^{L^2 \rightarrow L^2}} - \| (I - Q_J) d\Lambda_\gamma(\kappa) \|_{R^{L^2 \rightarrow L^2}}.
\]
By proposition 3.3 and part (a), this can be bounded from below by
\[
\geq (S_2^{-1} - A \Delta(J)) \| h \|_{L^\infty}
\]
and if \( \Delta(J) \leq 1/(2AS_2) \), we get the desired estimate (3.7).
\[\square\]

**Remark 3.6.** In fact, the preceding proof requires a slight variant of [3, theorem 2], differing as follows: first, it is not necessary for the forward map to be globally Lipschitz if one is only interested in stability for \( \gamma, \gamma' \in K \)—in this case the additional approximation step in the proof becomes obsolete. Second, hypothesis 1 of that theorem is only needed to obtain part (a) of the theorem and we achieve the corresponding bound by means of lemma 3.2. Part (b) of the theorem, which is needed above, is proved independently and (in their notation) for fixed \( N \) such that \( s_N \leq 1/2C \). We apply the result for fixed \( J \) such that \( A \Delta(J) \leq 1/2S_1 \), which is nothing but a change in notation.

## 4. Proof of the main theorems

In this section, we connect to the statistical setting and prove the main theorems.

### 4.1. DQM property & information operator

Suppose a statistical experiment \( P = (P_\theta : \theta \in \Theta) \) with parameter space \( \Theta \subset \mathbb{R}^D \) describes a random design regression model with Gaussian noise, arising from a forward map
\[
G : \Theta \rightarrow L_2^2(\mathcal{X}, V), \quad \theta \mapsto G_\theta
\]
as in section 1.1—at this point there is no need to specify to the Calderón problem. Then differentiability of \( G \) can be leveraged to show that \( P \) is differentiable in quadratic mean (DQM) (that is, (4.4) below). For \( h \in \mathbb{R}^D \) we write \( \| h \| \) for any of the equivalent norms of \( \mathbb{R}^D \).

**Proposition 4.1.** Assume that \( U = \sup_{\theta \in \Theta} \| G_\theta \|_{L^\infty} < \infty \) and let \( \theta \) be an interior point of \( \Theta \). Suppose there exists a bounded linear operator \( I_\theta : \mathbb{R}^D \rightarrow L_2^2(\mathcal{X}, V) \) such that for all \( x \in \mathcal{X} \) we have
\[
| G_{\theta+h}(x) - G_\theta(x) - I_\theta[h]|_V = o(\| h \|) \quad \text{as} \quad \| h \| \to 0.
\]
Suppose further that there exists constants \( \epsilon, B > 0 \) such that
\[
\| G_{\theta+h} - G_\theta \|_{L^\infty} \leq B \| h \| \quad \text{for} \quad \| h \| < \epsilon.
\]
Then \( P \) satisfies the DQM property at \( \theta \) in the following sense: define
\[ A_\theta : \mathbb{R}^D \rightarrow L^2(V \times \mathcal{X}, P_\theta), \quad A_\theta[h](y, x) = \langle y - G_\theta(x), I_\theta[h](x) \rangle_V, \quad (4.3) \]

then
\[ \int_{V \times \mathcal{X}} \left[ dP_{\theta+h}^{1/2} - dP_\theta^{1/2} - \frac{1}{2} A_\theta[h]dP_\theta^{1/2} \right]^2 = o(\|h\|^2), \quad \text{as } \|h\| \rightarrow 0. \quad (4.4) \]

**Definition 4.2.** The map \( A_\theta : \mathbb{R}^D \rightarrow L^2(V \times \mathcal{X}, P_\theta) \) from (4.3) is called the score operator of \( \mathcal{P} \) at \( \theta \). Further, the information operator/matrix of \( \mathcal{P} \) at \( \theta \) is defined as
\[ \mathbb{N}_\theta = A_\theta^* A_\theta \in \mathcal{B}(\mathbb{R}^D) \equiv \mathbb{R}^{D \times D}, \]
where the adjoint of \( A_\theta \) is formed with respect to the natural inner products on \( \mathbb{R}^D \) and \( L^2(V \times \mathcal{X}, P_\theta) \).

**Remark 4.3.** For \( (Y, X) \sim P_\theta \) we may consider \( \hat{A}_\theta(Y, X) \) as random element in \((\mathbb{R}^D)^*\) (the dual space of \( \mathbb{R}^D \), say, viewed as ‘row vectors’). The transpose \( \hat{A}_\theta^T(Y, X) \in \mathbb{R}^D \) equals the score \( \hat{I}_\theta \equiv \hat{I}_\theta(Y, X) \) from [38]; another description of the information matrix is thus
\[ \mathbb{N}_\theta = E_\theta[\hat{I}_\theta(Y, X)\hat{I}_\theta^T(Y, X)], \]
which coincides with the definition on p 39 of [38].

**Remark 4.4.** Proposition 4.1 extends to the infinite-dimensional setting from [30] and yields a result similar to theorem 1 in that article—the difference is that here we consider total instead of partial derivatives. Moreover, proposition 1 of [30] also applies here, yielding yet another representation of the information matrix: with \( \mathbb{I}_\theta : \mathbb{R}^D \rightarrow L^2_{\text{loc}}(\mathcal{X}, V) \) as in (4.1), it holds that
\[ \mathbb{N}_\theta = \mathbb{I}_\theta^* \mathbb{I}_\theta, \]
again taking adjoints with respect to the natural inner products.

**Proof of Proposition 4.1.** Let
\[ f_\theta(y, x) := \log \left( \frac{dP_{\theta+h}^{1/2}}{dP_\theta^{1/2}}(y, x) \right) \]
\[ = \frac{1}{2} \langle y, G_{\theta+h}(x) - G_\theta(x) \rangle_V - \frac{1}{4} \left( |G_{\theta+h}(x)|_V^2 - |G_\theta(x)|_V^2 \right). \]

The proposition is proved, once we show that the integral
\[ \int_{V \times \mathcal{X}} \left[ e^{\delta_{\theta+h}(x)} - 1 + \frac{1}{2} A_\theta[h] \right]^2 dP_\theta \]
vanishes in the limit \( \|h\| \rightarrow 0 \). For fixed \((y, x) \in V \times \mathcal{X}\), the integrant vanishes in the limit, as \( h \mapsto \exp(f_\theta(y, x)) \) is Fréchet differentiable at 0 with derivative \( \frac{1}{2} A_\theta(h, y, x) \); this follows from (4.1) and the chain rule. Further, by the convexity of exp, the Cauchy–Schwarz inequality and property (4.2), it holds for all \( 0 < \|h\| < \epsilon \) that
\[ \frac{|e^{\delta_{\theta+h}(x)} - 1 - \exp(f_\theta(y, x))|}{\|h\|} \leq \exp \left( \frac{|f_\theta(y, x)|}{\|h\|} \right) + \exp \left( \frac{B}{2} |y|_V + \frac{1}{2} U^2 \right), \]
which is \( L^2 \)-integrable with respect to \( P_\theta \). Also \( A_\theta[h] \) is in \( L^2(P_\theta) \), with norm \( \lesssim \|h\| \). The result then follows by the dominated convergence theorem. \( \square \)
4.2. Proof of the main theorems

We make the identifications
\[ \text{range}(P_J) \equiv \mathbb{R}^M, \quad \text{range}(Q_J) \equiv \mathbb{R}^{M \times M} \equiv L^2_\lambda(\mathcal{X}, \mathbb{R}^M), \]
where the projections \( P_J \) and \( Q_J \) are defined as in section 3.1 and the \( L^2 \)-space is as in definition 1.6. With \( \gamma_\theta \) as in (1.7), the forward map \( G \) can then be written as
\[ G : \Theta \to L^2_\lambda(\mathcal{X}, \mathbb{R}^M), \quad G_\theta = Q_J \tilde{\Lambda} \gamma_\theta, \]
where \( \tilde{\Lambda} \) is the normalised Dirichlet-to-Neumann operator from (1.10). By proposition 2.1 it follows that \( \theta \mapsto G_\theta \) is differentiable in the sense of proposition 4.1, with
\[ I_\theta(h) \equiv dG_\theta(h) = Q_J d\tilde{\Lambda} \gamma_\theta(\kappa), \quad \kappa = h_1 \Omega_1 + \cdots + h_D \Omega_D. \]
In particular, the model \( \mathcal{P} \) defined in section 1.1 is DQM and score and information operators are well defined.

**Proof of Theorem 1.1.** By theorem 3.5(b), the derivative \( I_\theta : \mathbb{R}^D \to L^2_\lambda(\mathcal{X}, \mathbb{R}^M) \equiv \mathbb{R}^{M \times M} \) is injective and thus also \( I^*_\theta I_\theta \in \mathbb{R}^{D \times D} \) is injective. By remark 4.4 this equals \( N_\theta \) and as we are in finite dimension, the information operator must be invertible. □

**Proof of Theorem 1.2.** Under the identification from remark 4.3, the conclusion of the theorem equals that of theorem 10.1 in [38]; it thus remains to check the prerequisites of the latter theorem. The DQM property and invertibility of the information matrix follow from proposition 4.1 and theorem 1.1. The separation condition is satisfied in view of lemma 10.6 in [38] (see also the discussion preceding it), the compactness of \( \Theta \) and theorem 3.5(a), which implies identifiability of the model. □

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Data availability statement

No new data were created or analysed in this study.

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