Ellipse, Hyperbola and Their Conjunction

Arkadiusz Kobiera

Warsaw University of Technology

Abstract

The article presents simple analysis of cones which are used to generate a given conic curve by section by a plane. It was found that if the given curve is an ellipse, then the locus of vertexes of the cones is a hyperbola. The hyperbola has focuses which coincidence with the ellipse vertexes. Similarly, if the given curve is the hyperbola, the locus of vertex of the cones is the ellipse. In the second case, the focuses of the ellipse are located in the hyperbola’s vertexes. These two relationships create kind of conjunction between the ellipse and the hyperbola which originate from the cones used for generation of these curves. The presented conjunction of the ellipse and hyperbola is perfect example of mathematical beauty which may be shown by use of very simple geometry. As in the past the conic curves appear to be very interesting and fruitful mathematical beings.
Introduction

The conical curves are mathematical entities which have been known for thousands years since the first Menaechmus’ research around 250 B.C. [2]. Anybody who has attempted undergraduate course of geometry knows that ellipse, hyperbola and parabola are obtained by section of a cone by a plane. Every book dealing with this subject has a sketch where the cone is sectioned by planes at various angles what produces different kinds of conics. Usually authors start considerations from the cone to produce the conic curve by section. Then, they use it to prove some facts about the conics. Many books focus on the curves themselves and their features. Even books which describe the conics theory in quite comprehensive way [2, 4, 1] abandon the cone after the first couple paragraphs or go to quite complex analysis of quadratics. We may find hundreds theorems about the curves but the relation between the cone and the conics is left to exercise part at best [4] or Authors quickly go to more complex systems of conics in three-dimensional space [2]. Probably the cone seems to be too simple to spent time on this topic, however we will show that the cone (strictly speaking family of cones) may have interesting properties as well. Apart of pure geometry, celestial mechanics is the second field where conics are important – the orbits are conic curves. Unfortunately, the books about celestial mechanics say only a few words about the the cone if any at al. [3, 5]. In this short paper we would like to focus on the cone and its relation to conic curves which is surprisingly omitted in books but interesting.
Ellipse and the Cones

Let us consider following problem: Given is an ellipse $\mathcal{E}$ defined by two focus points $F_1$ and $F_2$ and vertex $A$. This ellipse is created by section of the cone $\mathbf{S}$ by plane $\rho$. It is shown in Figure 1.

Our task is to find the vertex $E$ of the cone $\mathbf{S}$. Apart of the focuses, the ellipse has also two characteristics points: the vertexes $A$ and $B$. The distances from the vertexes to one of the focus e.g. $F_1$ will be noted as $r_a = |F_1A|$, and $r_b = |F_1B|$. The semi-axes of this ellipse are $a = |AJ|$ and $b = |H_1J|$ where $J$ is the center of the ellipse. The distance between focuses
is \( c = |F_1F_2| \). The radii \( r_a \) and \( r_b \) define the eccentricity:

\[
e = \frac{r_a - r_b}{r_a + r_b} = \frac{c}{a}.
\]

(1)

Obviously, we may use any set of these parameters to define the ellipse \( E \), however we will prefer the radii and focus \( F_1 \).

The first question is about the cone: "Is the cone \( S \) unique?" The answer is in the following lemma:

**Lemma 1** If the ellipse \( E \) lays on plane \( \rho \) and it is defined by two vertexes \( A, B \), and focus \( F_1 \) (or focuses \( F_1, F_2 \) and vertex \( A \)) then it may be generated by infinite number of cones \( S \) sectioned by the plane \( \rho \).

**Proof:**

The proof will be explained rather in quite informal manner. To solve this exercise let’s reduce the three-dimensional problem to a two-dimensional problem by considering plane \( \tau \) which is defined by cone’s axis and focuses (or vertexes) of the ellipse. It is shown in Figure 2. We put line \( a \) on plane \( \tau \). The line coincidences with the ellipse vertexes \( A \) and \( B \) and the focuses \( F_1, F_2 \) as well. The line \( a \) is also an intersection of planes \( \tau \) and \( \rho \). Note that the focus points (e.g. \( F_1 \)) are points of tangency of sphere of center \( O \) with the plane \( \rho \). This sphere is called Dandelin’s sphere and it is simultaneously tangent to the cone. The tangency points of the sphere and the cone create circle which define plane \( \omega \). Intersection of the planes \( \omega \) and \( \rho \) is line \( f \). We crate also additional line \( b \) on plane \( \omega \) which is perpendicular to \( f \) and goes through the axis of the cone. The intersection of the Dandelin’s sphere by the plane \( \tau \) is the circle with center \( O \). The circle is tangent to lines \( t_1 \) and
$t_2$ which are two elements of the cone. These lines are obtained by cutting the cone by plane $\tau$. They meet line $a$ at points $A$ and $B$.

Figure 2: Section of cone $S$ by plane $\tau$.

The problem was reduced to a problem of finding the point $E$ which is vertex of triangle $ABE$ circumscribed on circle $O$. While the ellipse $\mathcal{E}$ is given the three points $A$, $B$ and $F_1$ are fixed. Point $E$ is a point of intersection of lines $t_1$ and $t_2$. These lines are defined by points $A$ and $B$ and the circle $O$ which is tangent to the lines. If the radius $r$ is smaller than certain limit $r_{\text{max}}$ then the two lines meet at point $E$ (this fact seems to be quite obvious so we skip this part of the proof). The maximum radius $r_{\text{max}}$ is determined by the case when the lines $t_1$ and $t_2$ are parallel. In such case the lines $t_1$ and $t_2$ become element lines of cylinder as it is a limiting case of the cone.
when the point $E$ goes to infinity. In this case the ellipse $E$ is obtained as a section of the cylinder. One may show that the limiting radius is equal to minor semi-axes of the ellipse

$$r_{\text{max}} = a\sqrt{1 - e^2}. \tag{2}$$

If the radius $r$ can be of any length between 0 and $r_{\text{max}}$ then the location of point $E$ is not unique and its position depends on radius $r$. Hence, one can construct infinite number of cones which may be used to generate the ellipse $E$.

□

If the cone $S$ is not unique, the next question is: "What is the locus of the cone vertexes $E$?" First, we calculate the distance from the cone vertex $E$ to the ellipse vertex $B$

$$|EB| = |BD| + |ED| = r_b + |ED| \tag{3}$$

Second equality results from the fact that $BD$ and $BF_1$ are tangent to circle $O$ and they have common endpoint $B$. Obviously, the angles $\angle F_1BO$ and $\angle DBO$ are equal and right triangles $OF_1B$ and $OBD$ are congruent. Then segments $F_1B$ and $BD$ are of the same length $r_b$. One can write similar equations for segment $EA$

$$|EA| = |HA| + |EH| = r_a + |ED|. \tag{4}$$

Here we use the fact that triangles $HOE$ and $DOE$ are congruent and tri-
angles $HOA$ and $F_1OA$ are congruent as well. Comparison of the above equation lead to following proposition:

**Proposition 1** If the ellipse $\mathcal{E}$ defined by two vertexes $A$, $B$ and focus $F_1$ (or focuses $F_1, F_2$ and vertex $A$) is generated by section of the cones $\mathcal{S}$ by the plane $\rho$ then the locus of vertexes $E$ of the all possible cones $\mathcal{S}$ is a hyperbola $\mathcal{H}$. The focuses of the hyperbola $\mathcal{H}$ are points $A$ and $B$, vertexes are points $F_1$ and $F_2$ (ellipse focuses).

**Proof:**

Let us calculate the difference of length of two segments $EB$ and $EA$

$$|EB| - |EA| = r_b + |ED| - (r_a + |ED|) = r_b - r_a = \text{const}, \quad (5)$$

$$|EB| - |EA| = |F_1F_2|. \quad (6)$$

This difference is a constant number because $r_a$ and $r_b$ are constant as ellipse parameters, also the distance $|F_1F_2|$ is obviously constant. This directly agrees with definition of a hyperbola which focuses are located in points $A$ and $B$ (see Figure 3). It is also clear that vertexes of this hyperbola $\mathcal{H}$ are points $F_1$ and $F_2$.

$\square$

Indeed, the hyperbola $\mathcal{H}$ contains all the possible locations of vertexes $E$. Left branch contains vertexes where the Dandelin’s sphere is tangent to focus $F_1$. The upper part of the branch represents the case when the sphere is above the plane $\rho$, lower part is for opposite position of the sphere. The right
branch is for the case where the sphere is tangent at point $F_2$. If the radius $r$ of the sphere vanishes to 0, the point $E$ goes toward focuses $F_1$ or $F_2$. If the sphere’s radius $r$ goes to the maximum value $r_{max}$ the point $E$ goes to infinity on the hyperbola’s branches. Asymptotic lines $s_1$, $s_2$ are the axes of cylinders which are limiting cases of the cones with vertex in infinity.

**Hyperbola and the Cones**

Now we can ask reversed question: *What is the locus of vertexes $G$ of cones $Z$ which generate the given hyperbole $\mathcal{H}$.* One can consider the hyperbola $\mathcal{H}$ which was found in the previous part. This will not reduce generality of
our reasoning. We will keep same plane $\tau$ where four points are defined $A$, $B$, $F_1$ and $F_2$. They also define the hyperbola $\mathcal{H}$ on the plane $\tau$. Figure 4 shows the situation where the hyperbola is created by sectioning the cone $Z$ by plane $\tau$. We state following lemma by analogy to the case of ellipse:

**Figure 4:** Hyperbola $\mathcal{H}$ created by section of the cone $Z$.

**Lemma 2** If the hyperbola $\mathcal{H}$ lays on plane $\tau$ and it is defined by two focuses $A$, $B$, and vertex $F_1$ (or two vertexes $F_1, F_2$ and focus $A$) then it may be generated by infinite number of cones $Z$ sectioned by plane $\tau$.

**Proof:**

The proof is analogous to proof of Lemma 1. First, we reduce the problem to planimetry by considering the plane $\rho$ (see Figure 5).
By lemma assumption we have points $A$, $B$ and $F_1$ given, then also the point $F_2$ is established because it is the vertex of the hyperbola. The vertex $G$ of the cone is defined by section of two lines $t_3$ and $t_4$ which are elements of cone $Z$. The lines lay on plane $\rho$ and go through points $F_1$ and $F_2$ and are tangent to two circles $O_3$ and $O_4$ respectively. The circles are sections of Dandelin’s spheres (Figure 4) which are tangent to the plane $\tau$. They are also tangent to line $a$ at points $A$ and $B$ which are focuses of the hyperbola $\mathcal{H}$. Let $r_1$ be the radius of the circle $O_3$. Then the point $G$ is not unique and its position depends on the radius $r_1$. The radius $r_1$ may vary from zero to infinity: $0 < r_1 < \infty$. Hence, there exists infinite number of cones $Z$ which generate the hyperbola if they are sectioned by the plane $\tau$.
The next step is finding the locus of vertexes $G$. By analogy to the Proposition 1 we write following proposition:

**Proposition 2** If the hyperbola $\mathcal{H}$ defined by two vertexes $F_1, F_2$ and focus $A$ (or focuses $A, B$ and vertex $F_1$) is generated by section of the cones $Z$ by plane $\tau$ then the locus of vertexes $G$ of the cones $Z$ is an ellipse $E$. The focuses of the ellipse $E$ are points $F_1$ and $F_2$, vertexes are points $A$ and $B$ (ellipse focuses).

**Proof:**

We will look for relationship between the distances from the vertex $G$ to the points $F_1$ and $F_2$. First, we will consider the right triangle $O_3PO_4$ (Figure 5). Point $P$ is normal projection of point $O_4$ onto segment $AO_3$. By using Pythagoras theorem we have

$$|O_3O_4|^2 = |AB|^2 + (r_1 - r_2)^2.$$  \hspace{1cm} (7)

where $r_2$ is radius of the circle $O_3$. The triangles $GKO_3$ and $GMO_4$ are also right triangles because the points $K$ and $N$ are points of tangency of the lines $t_3$ and $t_4$ to the circles $O_3$ and $O_4$. Hence, one can write

$$|O_3G|^2 = |GK|^2 + r_1^2,$$  \hspace{1cm} (8)

$$|O_4G|^2 = |GM|^2 + r_2^2.$$  \hspace{1cm} (9)
Recalling that

\[ |O_3O_4| = |O_3G| + |O_4G|. \]  

(10)

and substituting equations (8), (9) and (10) to equation (7) following equation is obtained

\[
(|KG| + |MG|)^2 - 2|KG||MG| + 2|O_3G||O_4G| = |AB|^2 - 2r_1r_2 =
\]

\[
= |AB|^2 - 2|O_4M||O_3K| \tag{11}
\]

When the sides of equation (11) are divided by \(|O_4M||O_3K|\) we get

\[
\frac{(|KG| + |MG|)^2}{2r_1r_2} - \left( \frac{|KG|}{|O_3K|} \frac{|GM|}{|O_4M|} - \frac{|O_3G|}{|O_3K|} \frac{|O_4G|}{|O_4M|} \right) = \frac{|AB|^2}{2r_1r_2} - 1. \tag{12}
\]

The triangles \(O_3KG\) and \(O_4MG\) are similar because they are right triangles and angles \(\angle KGO_3\) and \(\angle MGO_4\) are equal. The second statement is true because the triangles \(O_4MG\) and \(O_4NG\) are congruent and angles \(\angle KGO_3\) and \(\angle NGO_4\) are congruent as well (points \(G, O_3\) and \(O_4\) lays on the axis of the cone then the segments \(O_3G\) and \(O_4G\) are co-linear). Let the measure of angles \(\angle KO_3G\) and \(\angle MO_4G\) be \(\chi\). Simple trigonometrical relations based on Figure 5 yield:

\[
\frac{|KG|}{|O_3K|} = \tan \chi = \frac{|GM|}{|O_4M|}. \tag{13}
\]

\[
\frac{|O_3G|}{|O_3K|} = \frac{1}{\cos \chi} = \frac{|O_4G|}{|O_4M|}. \tag{14}
\]

The second term of left hand side of equation (12) can be simplified by use
of the two relationship stated above

\[ \left( \frac{|KG|}{|O_3K|} \frac{|GM|}{|O_4M|} - \frac{|O_3G|}{|O_3K|} \frac{|O_4G|}{|O_4M|} \right) = (\tan \chi)^2 - \frac{1}{(\cos \chi)^2} = -1. \quad (15) \]

Finally, we got simple equation:

\[ (|GK| + |GM|)^2 = |AB|^2 - 4r_1r_2. \quad (16) \]

Next step is finding the product \( r_1r_2 \). Let’s note that the angle \( \angle AF_1O_3 \) is equal to \( \angle AF_1O_3 \) and it is \( \pi/2 - \psi \). We may say the same about \( \angle KF_1O_3 \). This fact leads to conclusion that the angle \( \angle NF_1B \) is equal to

\[ \angle NF_1B = \pi - 2\angle AF_1O_3 = \pi - 2(\pi/2 - \psi) = 2\psi. \quad (17) \]

Obviously, the line \( F_1O_4 \) is the bisector of this angle. Hence, the angle \( \angle O_4F_1B \) is equal to \( \psi \). Triangles \( F_1AO_3 \) and \( F_1BO_4 \) are similar and we may write following proportion:

\[ \frac{|O_3A|}{|F_1A|} = \frac{|F_1B|}{|O_4B|}. \quad (18) \]

Length of segment \( O_3A \) is \( r_1 \), length of segment \( O_4B \) is \( r_2 \). We may rewrite above equation as

\[ \frac{r_1}{r_a} = \frac{r_b}{r_2}. \quad (19) \]

Hence, \( r_1r_2 \) is equal to

\[ r_1r_2 = r_ar_b. \quad (20) \]
We successfully arrived to conclusion that the sum of length of segments GK and GM is constant

\[(|GK| + |GM|)^2 = |AB|^2 - 4ra rb = const. \tag{21}\]

|AB| is equal to \(r_a + r_b\) then

\[(|GK| + |GM|)^2 = (r_a + r_b)^2 - 4ra rb = (r_b - r_a)^2. \tag{22}\]

The fact that \(r_a = |AF_1| = |F_1K| = |F_2M| = |BF_2|\) and the equation \[22\] allow calculating the sum of the distances between vertex \(G\) and the focuses \(F_1\) and \(F_2\)

\[|GF_1| + |GF_2| = |GK| + r_a + |GM| + r_a = (|GK| + |GM|) + 2r_a = r_b + r_a. \tag{23}\]

Hence, the sum of distances of the vertex \(G\) from focuses \(F_1\) and \(F_2\) is constant \(|AB| = r_a + r_b\)

\[|GF_1| + |GF_2| = |AB|. \tag{24}\]

This equation is the simplest form of definition of the ellipse and we proved the proposition.

\[\square\]
Conclusions

We just showed that we were able to discover very interesting relationship between ellipse and hyperbola by use of very simple geometry. It was shown that ellipse and hyperbola are conjugate. This conjunction is created by locus of vertexes of cones which generate the two conics. However, it seems to be very basic property of the conics, surprisingly it is not mentioned even in some books devoted to conics geometry only. On the other hand it is wonderful that such simple mathematics may lead to such interesting results and express the beauty of geometry what is imperfectly shown in Figure 6.

Figure 6: Conjugate ellipse $\mathcal{E}$ i hyperbola $\mathcal{H}$ as curves generated by section of cones which vertexes are located on these curves.
References

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