ON AUTOMORPHISMS GROUP OF SOME $K3$ SURFACES.

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Abstract. In this paper we study the automorphisms group of some $K3$ surfaces which are double covers of the projective plane ramified over a smooth sextic plane curve. More precisely, we study some particular case of a $K3$ surface of Picard rank two.

Introduction

$K3$ surfaces which are double covers of the plane ramified over a plane sextic are classical objects. In this paper we determine the automorphisms group of some of these surfaces. More precisely, we restrict to the case of Picard rank two. We study the case of a $K3$ surface with Picard lattice of rank two with quadratic form given by $Q_d := \begin{pmatrix} 2 & d \\ d & 2 \end{pmatrix}$ and we obtain that the automorphism group is infinite and isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$.

The automorphisms of a $K3$ surface are given by the Hodge isometries of the second cohomology group that preserve the Kähler cone (see [3, VIII.11]). Thus, the strategy to study the automorphism of a $K3$ surface $X$ is to determine its Kähler cone and the Hodge isometries of $H^2(X, \mathbb{Z})$ which preserve it. To do this, one can determine the isometries of the Néron-Severi lattice $\text{NS}(X)$ which preserve the Kähler cone and satisfy a "gluing condition" with those of the transcendental lattice $T(X)$.

In the preliminary Section 1 we introduce some basic material on lattices and $K3$ surfaces. To illustrate the method, in section 2 we analyze explicitly a geometric example. We determine the Kähler cone in Prop. 2.1 and the automorphisms group of the Néron-Severi lattice in Prop. 2.2 using some basic facts on generalized Pell equations. Using similar techniques we obtain the result about surfaces with Neron-Severi lattice of rank two with quadratic form $Q_d$ in section 3.

1. Preliminaries

1.1. Lattices. A lattice is a free $\mathbb{Z}$-module $L$ of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form $<,>$. A lattice is called even if the quadratic form associated to the bilinear form has only even values, odd otherwise. The discriminant $d(L)$ is the
determinant of the matrix of the bilinear form. A lattice is called non-degenerate if the discriminant is non-zero and unimodular if the discriminant is ±1. If the lattice \( L \) is non-degenerate, the pair \( (s_+, s_-) \), where \( s_{\pm} \) denotes the multiplicity of the eigenvalue ±1 for the quadratic form associated to \( L \otimes \mathbb{R} \), is called signature of \( L \). Finally, we call \( s_+ + s_- \) the rank of \( L \).

Given a lattice \( (L, <, >) \) we can construct the lattice \( (L(m), <, >) \), that is the \( \mathbb{Z} \)-module \( L \) with form \( <x, y> = m <x, y> \).

An isometry of lattices is an isomorphism preserving the bilinear form. Given a sublattice \( L \hookrightarrow L' \), the embedding is primitive if \( L \) is free. Two even lattices \( S, T \) are orthogonal if there exist an even unimodular lattice \( L \) and a primitive embedding \( S \hookrightarrow L \) for which \( (S) \cong T \). The discriminant group of a lattice \( L \) is the abelian group \( A_L = L^* \cong \{ x \in L \otimes \mathbb{Q} / <x, l> \in \mathbb{Z} \forall l \in L \} \).

1.2. \( K^3 \) surfaces. A \( K^3 \) surface is a compact Kähler surface with trivial canonical bundle and such that its first Betti number is equal to zero. Let \( U \) be the lattice of rank two with quadratic form given by the matrix \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] and let \( E_8 \) be the lattice of rank eight whose quadratic form is the Cartan matrix of the root system of \( E_8 \). It is an even, unimodular and positive definite lattice.

It is well known that \( H^2(X, \mathbb{Z}) \) is an even lattice of rank 22 and signature \( (3, 19) \) isomorphic to the lattice \( \Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \), that we will call, from now on, the \( K^3 \) lattice. Denote with \( \Delta \) the set of the classes of the \( (−2) \)-curves in \( NS(X) \) and with \( C \subset NS(X) \otimes \mathbb{R} \) the connected component of the set of elements \( x \in NS(X) \otimes \mathbb{R} \) with \( x^2 > 0 \) which contains an ample divisor. The Kähler cone is the convex subcone of \( C \) defined as

\[
C^+ = \{ y \in C : (y, D) > 0 \text{ for all } D \in NS(X), D \text{ effective} \}.
\]

We will also use the following
Proposition 1.2. [3, VIII 3.8.] The Kähler cone is given by
\[ C^+ = \{ w \in C : wN > 0, \text{ for all } N \in \Delta \}. \]

1.3. Automorphisms. Let \( L \) be a lattice, an element \( \varphi \in O(L) \) gives naturally an automorphism \( \overline{\varphi} \) of the discriminant group. Let \( X \) be a K3 surface, let \( O_{C^+}(NS(X)) \) be the set of the isometries of the Neron-Severi lattice which preserve the Kähler cone and \( O_{\omega X}(T(X)) \) be the set of isometries of the transcendental lattice which preserve the period \( \omega_X \) of the K3 surface \( (H^{2,0}(X) = \langle \omega_X \rangle) \). From Nikulin [8] we have that
\[ \text{Aut}(X) \cong \{(\varphi, \psi) \in O_{C^+}(NS(X)) \times O_{\omega X}(T(X)) / \varphi = \psi \} \]
The fact \( \varphi = \psi \) is the so called "glueing condition".

In the remainder we consider the general case, so we can assume that the only Hodge isometries of the transcendental lattice are \( \pm Id \).

2. An easy geometric example.

It is well known that a surface which is a double cover of the projective plane ramified over a smooth sextic plane curve is a K3 surface. We restrict to the case when the Néron-Severi lattice has rank two and such that there is a rational curve of degree \( d \) which is tangent to the sextic. Let \( X_d \) be such a surface. We suppose that \( X_d \) is general. The Néron-Severi lattice of \( X_d \) has quadratic form given by
\[ Q_d := \begin{pmatrix} 2 & d \\ d & -2 \end{pmatrix}. \]
We denote this lattice \( L_d \). It is has segnature \((1, 1)\).

Our aim is to compute the automorphisms group of \( X_d \). We start with the case \( d = 3 \).

2.1. Case \( d = 3 \). We want to study now the automorphisms group of a K3 surface \( X_3 \) of rank two which is a double cover of the plane ramified over a smooth sextic which has a rational tritangent cubic. Such a surface has a Néron-Severi lattice \( L_3 \) given by the matrix
\[ Q_3 := \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}. \]

2.2. The Kähler cone. Denote with \( C^+_3 \) the Kähler cone of \( X_3 \). We have the following

Proposition 2.1. Let \( X_3 \) be a surface with Néron-Severi lattice isomorphic to \( L_3 \). Then there is an isomorphism of lattices \( NS(X_3) \otimes \mathbb{R} \cong L_3 \otimes \mathbb{R} \cong \mathbb{R}^2 \) such that:
\[ C^+_3 = \{(x, y) \in \mathbb{R}^2 : 3x - 2y > 0\} \cap \{(x, y) \in \mathbb{R}^2 : 3x + 11y > 0\}. \]
Proof. We have first to determine the classes of the \((-2)\)-curves on \(X_3\) that is the set \(\Delta \subset NS(X_3) : \)
\[
\Delta = \{ D \in NS(X_3) : D > 0, D^2 = -2, D \text{ irreducible} \}.
\]
The condition \(D^2 = -2\) means that we have to determine the integer solutions of the equation
\[
(1) \quad x^2 + 3xy - y^2 = -1.
\]
We write
\[
x^2 + 3xy - y^2 = (x - \alpha y)(x - \bar{\alpha} y),
\]
with
\[
\alpha = \frac{-3 + \sqrt{13}}{2}, \quad \text{and} \quad \bar{\alpha} = \frac{-3 - \sqrt{13}}{2}.
\]
Thus \(\Delta\) corresponds to the set
\[
\{ u \in \mathbb{Z}[\alpha] : u\bar{u} = -1 \}.
\]
It is known that the invertible elements in \(\mathbb{Z}[\alpha]\) are \(\mathbb{Z}[\alpha]^* = < \eta >\) where \(\eta = \frac{3 + \sqrt{13}}{2} = \alpha + 3 \) and \(\eta\bar{\eta} = -1\). Thus, solutions of \((1)\) are given by the odd powers of \(\eta\) and \(\bar{\eta}\).

The element \(\eta\) represents the solution \((0,1)\) of the equation \((1)\) and \(\bar{\eta} = 3 - \eta\) represents \((3, -1)\). Now, we can determine \(C^+\) that is, following Prop.\(1.2\) the set
\[
C^+ = \{ w \in \mathcal{C} : Q_3(w, D) > 0, \text{ for all } D \in \Delta \}.
\]
This means that we are looking for the elements \(w = (x, y)\) such that \(Q_3(w, \eta) > 0\) and \(Q_3(w, \bar{\eta}) > 0\), thus we obtain the statement. \(\square\)

2.3. The automorphisms group. Denote with \(T_3\) the transcendental lattice of \(X_3\). We start studying the isometries of \(L_3\), then we’ll identify the ones preserving the ample cone and finally we’ll analyze the glueing conditions on \(T_3\).

Proposition 2.2. The automorphisms group \(\text{Aut}(L_3)\) is isomorphic to the group \(\mathbb{Z}_2 \ast \mathbb{Z}_2\).

Proof. The group of isometries of \(L_3\) are given by
\[
O(L_3) = \{ M \in GL_2(\mathbb{Z}) : M^t Q_3 M = Q_3 \}
\]
By direct computations one obtains matrices of the following form
\[
P_{(a,b)}^\pm := \begin{pmatrix} 11b \mp 3a & -3b \pm a \\ -3b \pm a & 2 \end{pmatrix}, \quad Q_{(a,b)}^\pm := \begin{pmatrix} -b & -3b \mp a \\ -3b \pm a & 2 \end{pmatrix}
\]
where the \((a,b)\) are solutions of the generalized Pell equation
\[
a^2 - 13b^2 = -4
\]
A standard result on Pell equations and on fundamental units, see for example [11] and [4], says that the solutions are \((\pm a_n, \pm b_n)\) with 
\[
\frac{a_n + \sqrt{13}b_n}{2} = \left(\frac{a_0 + \sqrt{13}b_0}{2}\right)^{2n+1},
\]
\(n \in \mathbb{N}\) and \((a_0, b_0)\) is the pair of smallest positive integers satisfying the equation. In our case the pair of smallest positive integers that satisfy the Pell equation (2) is \((a_0, b_0) = (3, 1)\). Notice that 
\[
a_0 + \sqrt{13}b_0 = \eta
\]
and then we can obtain solutions \((a_n, b_n)\) by recurrence multiplying by \(\eta^2\). By direct computations:

\[
\begin{align*}
a_{n+1} &= \frac{11a_n + 39b_n}{2} \\
b_{n+1} &= \frac{3a_n + 11b_n}{2}.
\end{align*}
\]
Moreover, if \((a_n, b_n)\) gives rise to the matrices \(P_{(a_n, b_n)}^\pm, Q_{(a_n, b_n)}^\pm\), then the couples \((a_n, -b_n), (-a_n, b_n), (-a_n, -b_n)\) give rise to the matrices
\[
(-P_{(a_n, b_n)}^\pm, -Q_{(a_n, b_n)}^\pm), (P_{(a_n, b_n)}^\pm, -Q_{(a_n, b_n)}^\pm), (-P_{(a_n, b_n)}^\pm, -Q_{(a_n, b_n)}^\pm)
\]
respectively. We write \(P_n^\pm := P_{(a_n, b_n)}^\pm\) and \(Q_n^\pm := Q_{(a_n, b_n)}^\pm\). For \((3, 1)\) one obtains the matrices
\[
P_0^+ = I, \quad P_0^- = \begin{pmatrix} 10 & -3 \\ -3 & 1 \end{pmatrix}, \quad Q_0^+ = \begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix}, \quad Q_0^- = \begin{pmatrix} -1 & -3 \\ 0 & 1 \end{pmatrix}.
\]
The matrices \(Q_{n+1}^+, Q_0^+\) are non commuting involutions and \(P_0^{-} = Q_0^{-} Q_0^{+}\). The matrices \(P_n^+, Q_n^+\) are obtained by multiplication
\[
P_n^+ = (P_0^+)^n = (P_0^-)^{-n}, \quad P_n^- = (P_0^-)^{n+1}, \quad Q_n^+ = P_0^{-} Q_n^+, \quad Q_n^- = Q_n^{-} P_0^-\]
and \((P_0^-)^{-1} = P_1^+\). Set \(p\) and \(q\) for the automorphism of \(L_3\) corresponding to \(Q_0^+\) and \(Q_0^-\) respectively. Thus we have showed that the group \(O(L_3)\) can be described as \(\langle p \rangle * \langle q \rangle\).

**Theorem 2.3.** The automorphisms group of \(X_3\) is isomorphic to \(\mathbb{Z}_2\).

**Proof.** We are looking for Hodge isometries of \(H^2(X_3, \mathbb{Z})\) which preserve the ample cone. From the generality of \(X_3\) we may assume that the only Hodge isometries of \(T_3\) are \(\pm I\). Thus, we have first to identify the elements in \(Aut(L_3)\) which preserve the Kähler cone and then we impose a gluing condition on \(T_3\), since the isometries we are looking for have to induce \(\pm I\) on \(T_3\). Note first that \(-I \in Aut(L_3)\) can not preserve the ample cone. We have from Prop 2.1 that the Kähler cone is isomorphic to the chamber delimited by \(H_{D_q} \cong \mathbb{R}^+(v)\) and \(H_{D_q} \cong \mathbb{R}^+(w)\) where \(v = (2, 3)\) and \(w = (11, -3)\). An easy computation shows that the elements in \(Aut(L_3)\) having this property are the ones generated by \(-q\) which forms a \(\mathbb{Z}_2\). A direct computations gives that \(-q\) satisfy the gluing condition on \(T_3\). \(\square\)
2.4. Case $d$ odd. In this case we have a $K3$ surface with Néron-Severi lattice $L_d$ of rank two given by the matrix

$$Q_d := \begin{pmatrix} 2 & d \\ d & -2 \end{pmatrix}.$$ 

Set $X_d$ for the $K3$ surface having $NS(X_d) \cong L_d$. Such a $K3$ is a double cover of the plane ramified over a smooth sextic tangent to a rational curve of degree $d$. Following the same strategy adopted for the case $d = 3$, we obtain

**Theorem 2.4.** If $d$ is odd the automorphisms group of $X_d$ is isomorphic to $\mathbb{Z}_2$.

**Proof.** We compute

$$O(L_d) = \{ M \in \text{GL}_2(\mathbb{Z}) : \ tMQ_dM = Q_d \}$$

and we obtain matrices of the following form

$$R^\pm_{(a,b)} := \begin{pmatrix} (2 + d^2)b \mp da & -db \pm a \\ -db \pm a & 2 \end{pmatrix}, \quad S^\pm_{(a,b)} := \begin{pmatrix} -b & -da \pm a \\ -db \pm a & 2b \end{pmatrix}.$$ 

where the $(a,b)$ are solutions of the Pell equation

$$(3) \quad a^2 - (d^2 + 4)b^2 = -4.$$ 

When $d$ is odd, by theory on Pell equation the situation is analogous to the one of Prop.2.2 that is, all solutions can be generated from the minimal positive solution.

This means that $\mathbb{Z}[\sqrt{d^2 + 4}]^+$ is generated by $\eta = \frac{d}{2} + \frac{\sqrt{d^2 + 4}}{2}$ and that if $(a_n, b_n)$ is a solution of (3), then $(a_{n+1}, b_{n+1})$ is obtained by multiplying by $\eta^2$. By direct computations, the solutions are obtained by recurrence

$$\begin{cases} a_{n+1} = \frac{a_n d^2 + 2a_n b_n d^3 + 4 b_n d}{2} \\ b_{n+1} = \frac{a_n d + b_n d^2 + 2 b_n}{2} \end{cases}.$$ 

The pair of smallest positive integers that satisfy the Pell equation (2) is $(a_0, b_0) = (d, 1)$.

We write $R^\pm_n := R^\pm_{(a_n,b_n)}$ and $S^\pm_n := S^\pm_{(a_n,b_n)}$. For $(d, 1)$ one obtains the matrices

$$R_0^+ = I, \quad R_0^- = \begin{pmatrix} 1 + d & d \\ -d & 1 \end{pmatrix}, \quad S_0^+ = \begin{pmatrix} -1 & 0 \\ -d & 1 \end{pmatrix}, \quad S_0^- = \begin{pmatrix} -1 & -d \\ 0 & 1 \end{pmatrix}.$$ 

The matrices $S_0^+, S_0^-$ are non commuting involutions and $R_0^- = S_0^- S_0^+$. The matrices $R^\pm_{n+1}, S^\pm_{n+1}$ are obtained by multiplication

$$R^+_n = (R_1^+)^n = (R_0^-)^{n}, \quad S^+_{n+1} = R_0^- S^+_n, \quad S^-_{n+1} = S^-_n R_0^-.$$
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and \((R_0^-)^{-1} = R_1^+\).

Set \(r\) and \(s\) for the automorphism of \(L_3\) corresponding to \(S_0^+\) and \(S_0^-\) respectively. The group \(O(L_d)\) can be described as \((r) * (s)\). We obtain that the Kähler cone is isomorphic to

\[ C^+_d = \{(x, y) \in \mathbb{R}^2 : dx - 2y > 0\} \cap \{(x, y) \in \mathbb{R}^2 : dx + (d^2 + 2)y > 0\} \]

and, as before, the only automorphism of the Néron-Severi lattice which preserves the cone is \(-s\) and it satisfies the gluing condition on \(T_d\).

\[ \square \]

3. Automorphisms of a family of K3 surfaces of Picard rank two

We study the case of a K3 surface having Néron-Severi lattice of rank two with quadratic form given by

\[ Q'_d := \begin{pmatrix} 2 & d \\ d & 2 \end{pmatrix} \]

\((d > 0, \text{odd})\). We indicate this lattice with \(M_d\) and we denote by \(Y_d\) a K3 surface with Néron-Severi lattice \(NS(Y_d)\) isomorphic to \(M_d\).

Lemma 3.1. There exists a K3 surface with Néron-Severi lattice isomorphic to \(M_d\).

Proof. This follows from the fact that there is an embedding, unique up to isometry of \(M_d\) in the K3 lattice \(\Lambda_{K3} \cong U^3 \oplus E_8(-1)^{\oplus 2}\). In fact, every even lattice of signature \((1, \rho - 1)\) occurs as the Neron-Severi group of some algebraic K3 surface and the primitive embedding \(M_d \hookrightarrow \Lambda\) is unique (see Theorem [13]).

We first try to determine the classes of 0-curves and \((-2)\)-curves. The class of a 0 (or a \(-2\))-curve is represented by an integer solution of the equation \(x^2 + y^2 - dxy = 0\) (or \(x^2 + y^2 - dxy = -2\) respectively). This corresponds to find solutions for the Pell’s equations

\[ q^2 = d^2 - 4, \quad q^2 - (d^2 - 4)x^2 = -4. \]

In both cases one verifies that there are no solutions (see [14], [15]). This means that \(Aut(Y_d)\) is not finite. Indeed, we have from [9] (pag. 581) that the automorphism group of a K3 surface of Picard rank two is infinite if and only if there are no 0-curves nor \(-2\)-curves.

3.1. The Kähler cone of \(Y_d\). We determine now the Kähler cone \(C^+ \subset NS(Y_d) \otimes \mathbb{R}\). By [3], Chapter VIII, Cor.3.8. follows that in this case the Kähler cone is spanned (over \(\mathbb{R}_{>0}\)) by the vectors \(u := \begin{pmatrix} 2 \\ -d + \sqrt{d^2 - 4} \end{pmatrix}\) and \(v := \begin{pmatrix} -2 \\ d + \sqrt{d^2 - 4} \end{pmatrix}\).
3.2. Automorphisms group. We use the presentation of \( \text{1.3} \) to find \( \text{Aut}(Y_d) \). We start computing the group \( O(\text{NS}(Y_d)) = O(M_d) \), where

\[
O(M_d) = \{ M \in GL_2(\mathbb{Z}) : \, ^tM Q'_d M = Q'_d \}.
\]

We obtain matrices of the following form

\[
A^\pm := \begin{pmatrix}
\frac{(2 - d^2)b \pm ad}{bd \mp a} & -bd \pm a \\
\frac{2}{bd \mp a} & b
\end{pmatrix}, 
B^\pm := \begin{pmatrix}
-b & -bd \pm a \\
\frac{2}{bd \mp a} & b
\end{pmatrix}
\]

\[
X = \begin{pmatrix}
d & 1 \\
-1 & 0
\end{pmatrix}, 
Y := \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

where \( (a, b) \) are solutions of the Pell’s equation

\[
a^2 - (d^2 - 4)b^2 = 4
\]

As before the solutions are \( (\pm a_n, \pm b_n) \) with

\[
a_n + b_n\sqrt{d^2 - 4} = \left( \frac{a_0 + b_0\sqrt{d^2 - 4}}{2} \right)^n,
\]

\( n \in \mathbb{N} \) and \( (a_0, b_0) \) is the pair of smallest positive integers satisfying the equation.

In our case the pair of smallest positive integers that satisfy the Pell equation is \( (a_0, b_0) = (d, 1) \). By direct computations, the solutions are obtained by recurrence

\[
\begin{cases}
a_{n+1} = a_n d + b_n(d^2 - 4) \\
b_{n+1} = \frac{a_n + b_n d^2}{2}
\end{cases}
\]

Using this recurrence, one can see that the group \( O(M_d) \) is generated by the matrices \( X, Y, -Id, P, Q \) where

\[
P := \begin{pmatrix}
-1 & 0 \\
d & 1
\end{pmatrix}, 
Q := \begin{pmatrix}
-1 & -d \\
0 & 1
\end{pmatrix}.
\]

We observe that \( P, Q, Y, -Id \) are involutions and the relations \( P \cdot Q = -X^2, \, Q \cdot Y = -Y \cdot P \) hold. We can prove then the following

**Theorem 3.2.** The automorphism group \( \text{Aut}(Y_d) \cong \mathbb{Z} \ast \mathbb{Z}_2 \)

**Proof.** It is easy to check that the automorphisms of the Picard lattice represented by the matrices \( P, -Q, X, Y \) preserve the Kähler cone. Since in our case we assumed that \( O(T_{Y_d}) = \pm Id \) we look for automorphisms \( \varphi \) such that \( \varphi = \pm \mathbb{T_d} \). We obtain that the automorphisms satisfying the gluing conditions are \( P, -Q \) and \( X^2 \) since

\[
\mathbb{T}_{P} = -Q = \mathbb{T}_d, \, X^2 = \mathbb{T}_d.
\]

\( P \) and \( X \) doesn’t commute and \( P \cdot Q = -X^2 \) so we have \( \text{Aut}(Y_d) = \langle (P, -Id), (X^2, Id) \rangle \cong \mathbb{Z} \ast \mathbb{Z}_2 \). \( \square \)
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