BIGRATED LIE ALGEBRAS RELATED TO MZVS

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ABSTRACT. We prove that the dihedral Lie coalgebra $D_{\bullet \bullet} := \oplus_{k \geq m \geq 1} D_{m,k}$ corresponding to $\widehat{\mathfrak{g}}_{\bullet \bullet}(G)$ of [Gon98] for $G = \{ e \}$ (also denoted by $D_{\bullet \bullet}(G)$ in [Gon01]) is the bigraded dual of the linearized double shuffle Lie algebra $\mathfrak{ls} := \oplus_{k \geq m \geq 1} \mathfrak{ls}_m^k \subset \mathbb{Q}(x,z)$ of [Bro13] whose Lie bracket is the Ihara bracket initially defined over $\mathbb{Q}(x,z)$. This by constructing an explicit isomorphism $D_{\bullet \bullet} \to \mathfrak{ls}^\vee$, where $\mathfrak{ls}^\vee$ is the Lie coalgebra dual (in the bigraded sense) to $\mathfrak{ls}$. The work leads to a new proof of the fact that $\mathfrak{ls}$ is preserved by the Ihara bracket. We also prove folklore results from [Bro13] and [IKZ06] (that apparently have no written proofs in the literature) stating that (for $k \geq m \geq 2$) the linear map $f_m : \mathbb{Q}(x,z)_m \to \mathbb{Q}[x_1, \ldots, x_m]$ ($\mathbb{Q}(x,z)_m$ is the space linearly generated by monomials of $\mathbb{Q}(x,z)$ of degree $m$ with respect to $z$) given by $x^{n_1}z \cdots z^{n_m} z \mapsto x_1^{n_1} \cdots x_m^{n_m}$ and $x^{n_1}z \cdots z^{n_m} z x^{n_m+1} \mapsto 0$ for $n_1, \ldots, n_{m+1} \in \mathbb{N}$, maps $\mathfrak{ls}_m^k$ isomorphically to the double shuffle space $\text{Dsh}_m(k-m) \subset \mathbb{Q}[x_1, \ldots, x_m]$ of [IKZ06] (stated in [Bro13]) and that $D_{m,k}$ is isomorphic to $\text{Dsh}_m(k-m)$ (stated in [IKZ06]). Here, we give an explicit isomorphism between $D_{m,k}$ and the dual space $\text{Dsh}_m(k-m)^\vee$ of $\text{Dsh}_m(k-m)$ and we show the statement concerning $f_m$. In fact, we have three explicit compatible isomorphisms $D_{\bullet \bullet} \to \mathfrak{ls}^\vee$, $D_{m,k} \to \text{Dsh}_m^\vee(k-m)$ and $\tilde{f}_m : \mathfrak{ls}_m^k \to \text{Dsh}_m^\vee(k-m)$, where $\tilde{f}_m$ is the restriction of $f_m$.

INTRODUCTION

Context and main results.

Multiple zeta values (MZVs) are real numbers generalising the values of the Riemann zeta function at positive integers. These numbers were first studied by Euler then reappeared recently in geometry, knot theory, quantum algebra and arithmetic geometry. The study of relations between MZVs over $\mathbb{Q}$ is a well known problem subject to many conjectures.

A MZV is a real number of the form:

$$\zeta(k) = \sum_{n_1 > \cdots > n_m \geq 1} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}},$$

where $\underline{k} = (k_1, \ldots, k_m) \in (\mathbb{N}^*)^m$ with $k_m \geq 2$ and $m \geq 1$. We say that $\zeta(\underline{k})$ is of depth $m$ and weight $k = k_1 + \cdots + k_m$. In the literature, we find various polynomial relations between MZVs over $\mathbb{Q}$. For instance double shuffle relations ([IKZ06]) are $\mathbb{Q}$-linear relations between MZVs. One can also obtain relations between MZVs by comparing different regularisations of divergent MZVs ($\zeta(k)$ with $k_m = 1$). These new relations with the double shuffle relations are called the regularised (or also known as extended) double shuffle relations ([IKZ06]). One of the main conjectures on MZVs is that the extended double shuffle relations suffice to describe the $\mathbb{Q}$-subalgebra $\mathcal{Z}$ of $\mathbb{R}$ generated by the MZVs ([IKZ06]). We describe briefly the conjecture. Denote by $\mathcal{Z}^f$ the $\mathbb{Q}$-algebra linearly spanned by $1$ and the symbols $\zeta^f$ (for $\zeta$ running over all the MZVs) equipped with a product mimicking the shuffle product ([IKZ06]).
of MZVs. Now consider the quotient $\mathcal{Z}^f$ of $\hat{\mathcal{Z}}^f$ by the formal analogues of the extended double shuffle relations. The main conjecture is equivalent to the following statement:

the mapping $\mathcal{Z}^f \to \mathcal{Z}$ given by $\zeta^f \mapsto \zeta$ is an isomorphism of algebras.

The algebra $\mathcal{Z}^f$ described here is the algebra $R_{EDS}$ of [IKZ06] and the above statement corresponds to conjecture 1 of the same paper.

Denote by $\mathcal{Z}_+$ the vector subspace of $\mathcal{Z}$ generated by the MZVs and denote by $\mathcal{Z}_k^{(m)}$ (for $k \geq m \geq 1$) the subspace of $\mathcal{Z}$ generated by MZV of weight $k$ and depth less than or equal to $m$ (we also set $\mathcal{Z}_k^{(0)} = 0$). The standard conjectures lead to the study of:

$$\mathcal{M} := \bigoplus_{k \geq m \geq 1} \mathcal{M}_{k,m}, \quad \mathcal{M}_{k,m} := \mathcal{Z}_k^{(m)}/(\mathcal{Z}_k^{(m-1)} + \mathcal{Z}_k^{(m)} \cap \mathcal{Z}_+^2).$$

Conjecturally, the space $\mathcal{M}$ should be isomorphic to the associated graded of $\mathcal{Z}_+/\mathcal{Z}_+^2$ with respect to the weight filtration and the dimension $C_{k,m}$ of $\mathcal{M}_{k,m}$ should correspond to the number of generators of $\mathcal{Z}$ of depth $m$ and weight $k$. A conjectural formula for $C_{k,m}$ is known (appendix of [IKZ06]).

For $G$ an abelian group, Goncharov introduces in [Gon98] a bigraded Lie coalgebra called the dihedral Lie coalgebra $D_{\bullet\bullet}(G)$ also denoted by $\hat{\mathcal{D}}_{\bullet\bullet}(G)$ in [Gon01]. In this paper, we mean by the dihedral Lie coalgebra the coalgebra $\hat{\mathcal{D}}_{\bullet\bullet}(G)$ for $G = \{e\}$ which we denote by $D_{\bullet\bullet}$. The bigraded vector space $\mathcal{M}$ is a quotient vector space of $D_{\bullet\bullet}$. Goncharov computes the dimension of the bidegree $(m,k)$ part $D_{m,k}$ (of weight $k$ and depth $m$) of $D_{\bullet\bullet}$, for $m = 1, 2, 3$. He also proves that $D_{m,k} = 0$ if $k + m$ is odd (parity result). From theses results he deduces that as conjectured $C_{k,m} = 0$ if $k + m$ is odd and gives upper bounds for $C_{k,1}, C_{k,2}, C_{k,3}$ corresponding to the conjectural values of these numbers. The cohomology of $D_{\bullet\bullet}$ plays an important role in the computations, this explains the importance of the Lie structure of $D_{\bullet\bullet}$.

In ([Rac02]) Racinet introduces a prounipotent group scheme $DMR_0$ whose Lie algebra is $\mathfrak{dmr}_0$ which Lie bracket is the Ihara bracket. The Lie algebra $\mathfrak{dmr}_0$ is a complete graded (for weight) Lie algebra equipped with a depth decreasing filtration compatible to the "grading". One can consider a Lie subalgebra $\mathfrak{s}_0$ of $\mathfrak{dmr}_0$ graded for weight and whose degree completion (for weight) is isomorphic to $\mathfrak{dmr}_0$. The coordinate ring $\mathcal{O}(DMR_0)$ of $DMR_0$ and $\bar{\mathcal{Z}}^f := \mathcal{Z}^f/\zeta(2)^f \mathcal{Z}^f$ are isomorphic algebras and therefore the graded dual $\mathcal{U}(\mathfrak{s}_0)^\lor$ (for weight) of the enveloping algebra $\mathcal{U}(\mathfrak{s}_0)$ of $\mathfrak{s}_0$ is isomorphic to the algebra $\bar{\mathcal{Z}}^f$ (EL16). This implies that the graded dual $\mathfrak{s}_0^\lor$ (for weight) of $\mathfrak{s}_0$ is isomorphic to $\bar{\mathcal{Z}}^f/\bar{\mathcal{Z}}^f_2$, where $\bar{\mathcal{Z}}^f_2 \subset \bar{\mathcal{Z}}^f$ is the ideal generated by the symbols $\zeta^f$.

A bigraded version $\mathcal{I}_s$ (linearized double shuffle space) of the vector space $\mathfrak{dmr}_0$ (or $\mathfrak{s}_0$) is defined by Brown ([Bro13]). The space $\mathcal{I}_s$ is a bigraded subspace of the free associative algebra $\mathbb{Q}\langle x, z \rangle$ on the indeterminants $x, z$. Here $\mathbb{Q}\langle x, z \rangle$ is bigraded with respect to the total degree (weight) and the partial degree with respect to $z$ (depth). One has a natural injective map from the associated graded of $\mathfrak{s}_0$ with respect to depth filtration to $\mathcal{I}_s$. This map should be an isomorphism except in $(1,1)$ ([Bro13]). In the same paper Brown states that one can show that $\mathcal{I}_s$ is preserved by the Ihara bracket (defined over $\mathbb{Q}\langle x, z \rangle$) by adapting the
work of [Rac02] for $\mathfrak{dm}r_0$. Schneps proves in [Sch15] (see theorem 3.4.3 and its proof) that $\mathfrak{ls}$ is preserved by the Ihara bracket. The proof uses the theory of (bi)moulds introduced by Écalle (see for example [Écalle04]) and an analogy between the theory of (bi)moulds and series in non commutative variables $x, z$ established by Racinet in [Rac00].

In this paper we show that:

(a) The Lie colagebra $D_{\bullet \bullet}$ is the bigraded dual of the bigraded Lie algebra $(\mathfrak{ls}, \{-, -\})$ (subsection 7.1). We construct an explicit isomorphism of bigraded Lie colagebra between $D_{\bullet \bullet}$ and the bigraded Lie coalgebra dual (in the bigraded sense) to $(\mathfrak{ls}, \{-, -\})$. The work leads to a new proof of:

(b) The bigraded space $\mathfrak{ls}$ is preserved by the Ihara bracket $\{-, -\}$ (subsection 7.1).

Our proof of (b) is different from the one suggested by Brown and is independent of the proof of Schneps. Here, the proof is based on the dihedral Lie algebra.

In [IKZ06], the authors construct, for $m \geq 2$, the double shuffle subspace:

$$D\text{sh}_m = \bigoplus_{k \geq m} D\text{sh}_m(k - m) \subset \mathbb{Q}[x_1, \ldots, x_m].$$

The space $\mathcal{M}_{k,m}$ is naturally a quotient of the dual of the double shuffle space $D\text{sh}_m(k - m)$. It is conjectured in [IKZ06] that $D\text{sh}_m(k - m)$ and $\mathcal{M}_{k,m}$ have the same dimensions for $k \geq m \geq 2$. The authors also compute the dimension of $D\text{sh}_m(d)$ for $m = 1, 2$ (give estimates for $m = 3$) and they obtain a parity result similar to the one obtained for $D_{\bullet \bullet}$ in [Gon01]. These results give the same upper bounds for $C_{k,m}$ (for $m = 1, 2$) obtained by Goncharov and also imply the parity result for $C_{k,m}$.

In [IKZ06], footnote page 335, it is stated (without a proof) that the spaces $D_{m,k}$ and $D\text{sh}_m(k - m)$ are isomorphic. Therefore $D_{m,k}$ and $\mathcal{M}_{k,m}$ are conjecturally isomorphic for $k \geq m \geq 2$. In subsection 7.2 we prove that:

(c) For $m \geq 2$, the depth $m$ part $D_{m,\bullet}$ of $D_{\bullet \bullet}$ is isomorphic as a graded space (for weight) to the graded dual $D\text{sh}_m^\vee$ of $D\text{sh}_m = \bigoplus_{k \geq m} D\text{sh}_m(k - m)$ (subsection 7.2). Or equivalently, for $k \geq m \geq 2$, the part $D_{m,k}$ of weight $k$ and depth $m$ of $D_{\bullet \bullet}$ is naturally isomorphic to the dual of the double shuffle space $D\text{sh}_m(k - m)$.

To do so we construct an explicit isomorphism of graded spaces $D_{m,\bullet} \rightarrow D\text{sh}_m^\vee$.

In [Bro13], Brown also states without giving a proof that the map $f_m : \mathbb{Q}\langle x, z \rangle_m \rightarrow \mathbb{Q}[x_1, \ldots, x_m]$ (defined in (d) below) restricts to an isomorphism of graded spaces between the depth $m$ part $\mathfrak{ls}_m$ of $\mathfrak{ls}$ and $D\text{sh}_m$ (for $m \geq 2$). In 7.2 we show that we have the following result:

(d) For $m \geq 2$, the linear map $f_m : \mathbb{Q}\langle x, z \rangle_m \rightarrow \mathbb{Q}[x_1, \ldots, x_m], x^{n_1}z \cdots x^{n_m}z \mapsto \delta_m x_1^{n_1} \cdots x_m^{n_m}$, where $\delta_m$ is the Kronecker symbol and $\mathbb{Q}\langle x, z \rangle_m$ is the depth $m$ part of $\mathbb{Q}\langle x, z \rangle$, restricts to an isomorphism $\mathfrak{ls}_m \rightarrow D\text{sh}_m$ of weight graded spaces compatible to the isomorphisms constructed to show (a) and (c) (subsection 7.2).

By combining the isomorphisms constructed to prove (a) and (c) we get an isomorphism of graded spaces between $\mathfrak{ls}_m$ and $D\text{sh}_m$. Result (d) allows to understand this isomorphism. Note that the duality between the underlying bigraded vector spaces of $D_{\bullet \bullet}$ and $\mathfrak{ls}$ (in depth $m \geq 2$) can be deduced from (c) and (d).
The following table gives a clearer view of the results in this paper and the contributions of other authors:

| Result | Stated in | Proved in |
|--------|-----------|-----------|
| (a)    | This paper | This paper |
| (b)    | [Bro13]   | [Sch15], this paper. |
| (c)    | [IKZ06]   | This paper |
| (d)    | [Bro13]   | This paper |

Outline of the paper.

Through the paper $k$ is a field of characteristic zero. The spaces $D_{\bullet\bullet}$, $\mathcal{I}S$ and $D_{\text{sh}}m$ are originally $\mathbb{Q}$-vector spaces and we should take $k = \mathbb{Q}$, but the definitions of these spaces and the results can be easily given for any field of characteristic zero.

The first section contains reminders on free associative algebras, pairings of bigraded spaces, adjoints and duality between bigraded Lie algebras and Lie coalgebras that will be used in the sequel of the paper.

In the second section we recall the definition of the linearized double shuffle Lie algebra ([Bro13]). In subsection 2.1 we introduce the bigraded space $\mathcal{I}S \subset k\langle x, z \rangle$. We then define in subsection 2.2 the Ihara Lie bracket $\{-, -\} : k\langle x, z \rangle \otimes^2 \to k\langle x, z \rangle$ that preserves $\mathcal{I}S$. The proof of Schneps discussed before is sketched in this subsection (see the proof of proposition 2.6). We then give in subsection 2.3 the definition of the double shuffle space $D_{\text{sh}}m$ (for $m \geq 2$) of [IKZ06].

Section 3 is devoted to the dihedral Lie coalgebra $D_{\bullet\bullet}$. We give alternative definitions of $D_{\bullet\bullet}$ more suitable for the paper. In subsection 3.1 we introduce a free bigraded vector space $V = W \oplus U$. We define using generating series $\{t_1 : \cdots : t_{m+1}\}$ and $\{t_1, \ldots, t_m\}$ (for $m \geq 1$) similar to those in [Gon01], different bigraded subspaces of $W$; their sum is denoted by $W_R$. The key points of the subsection appear in proposition 3.1; we construct a diagram of bigraded isomorphisms $D_{\bullet\bullet} \xrightarrow{\eta} W/W_R \xrightarrow{i} V/F$, where $F = W_R \oplus U$, $i$ is induced by the natural inclusion $W \subset V$ and $\eta$ is defined by formula (18) in the proof of the proposition. In subsection 3.2 we define a map $\bar{\delta} : W \to W \otimes W$ inducing a Lie cobracket $\delta$ over $W/W_R$ for which $\eta : D_{\bullet\bullet} \to W/W_R$ is an isomorphism of Lie coalgebras. The construction of $\delta$ is simply an adaptation of the construction of the Lie cobracket of $D_{\bullet\bullet}$ given by Goncharov.

In the fourth section, we introduce a family of series $\{Q_m\}_m$ in commuting variables with values in the free associative algebra $k\langle x, z \rangle$ on two indeterminates $x, z$. We show that the
shuffle product (a product defined over $\mathbb{k}\langle x, z \rangle$ denoted by $\shuffle$) of $Q_p$ and $Q_q$ is given by:

$$Q_p(t_1, \ldots, t_p) \shuffle Q_q(t_{p+1}, \ldots, t_{p+q}) = \sum_{\sigma \in S(p,q)} Q_{p+q}(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(p+q)}),$$

where $S(p,q)$ is the set of $(p,q)$-shuffles. This formula is a formal analogue of formula (15) of [Gon98] for generating series of iterated integrals in the case where the iterated integrals represent MZVs and it will be used in section 5.

In section 5 we prove that $F$ is the orthogonal of $\mathfrak{I}S \subset \mathbb{k}\langle x, z \rangle_+$ (the subscript "+" is for elements with no constant term) with respect to a perfect pairing $\langle \cdot, \cdot \rangle_\phi : \mathbb{V} \otimes \mathbb{k}\langle x, z \rangle_+ \rightarrow \mathbb{k}$ (theorem 5.1). The pairing is given by $\langle \cdot, \cdot \rangle_\phi = \langle \cdot, \cdot \rangle_+ \circ (\phi \otimes \text{id})$ where $\phi : \mathbb{V} \rightarrow \mathbb{k}\langle x, z \rangle_+$ is a bigraded isomorphism introduced in proposition 5.6 of subsection 5.2 and $\langle \cdot, \cdot \rangle_+$ is a perfect pairing over $\mathbb{k}\langle x, z \rangle_+$ for which the unitary monomials form an orthonormal basis. To prove the result we compute in the first subsection the orthogonal complement $\mathfrak{I}S^\perp_+$ of $\mathfrak{I}S$ with respect to $\langle \cdot, \cdot \rangle_+$. We then introduce in subsection 5.2 the morphism $\phi$ and show that the image of $F$ by $\phi$ corresponds to $\mathfrak{I}S^\perp_+$. The orthogonality between $F$ and $\mathfrak{I}S$ follows immediately from these results. For completeness, we write down the proof in subsection 5.3.

In section 6 we compare the co-Lie structure of $D_{\bullet \bullet}$ with the one obtained from the adjoint $\{\cdot, \cdot\}_{\delta}^*$ (that we call Ihara cobracket) of $\{-, -\}_{\delta}$ (the restriction of the Ihara bracket to $\mathbb{k}\langle x, z \rangle_+$) with respect to $\langle \cdot, \cdot \rangle_+$ and $\langle \cdot, \cdot \rangle_{\delta}^\otimes 2$ where the latter is a perfect bigraded pairing over $\mathbb{k}\langle x, z \rangle_{\delta}^\otimes 2$ naturally obtained out of $\langle \cdot, \cdot \rangle_+$. The aim of the section is to show theorem 6.4 stating that $F$ is a coideal for $\phi^*\{-, -\}_{\delta}^*$, the pullback of $\{-, -\}_{\delta}^*$ by $\phi$ and that we have a diagram of isomorphisms of bigraded Lie coalgebras:

$$D_{\bullet \bullet} \xrightarrow{\gamma} (W, \delta) \xrightarrow{i} (V/F, \delta_{V/F}),$$

where $\delta_{V/F}$ is induced by $\phi^*\{-, -\}_{\delta}^*$ and $\eta, i$ are the isomorphisms of bigraded vector spaces of section 3. We mainly decompose the Ihara bracket into different operators and study these operators. This allows us to show that $U$ is a coideal for $\phi^*\{-, -\}_{\delta}^*$ (proposition 6.8) and that (proposition 6.13) the image of $W$ under $\phi^*\{-, -\}_{\delta}^* - \delta$ lies in $F \otimes V + V \otimes F$, where $\delta : W \otimes W \otimes W$ is the map inducing $\delta$. We then derive theorem 6.4 from these results. More details on the content of the section, that we skip here to avoid technical details, are given in the introduction of the section.

In the last section we prove the main results announced in the previous subsection of the introduction. Since $F$ is the orthogonal of $\mathfrak{I}S$ with respect to $\langle \cdot, \cdot \rangle_\phi$ and $\phi^*\{-, -\}^*_\delta$ is the adjoint of $\{-, -\}^*_\phi$ with respect to $\langle \cdot, \cdot \rangle_\phi$ and $\langle \cdot, \cdot \rangle_{\phi}^\otimes 2$ (naturally obtained out of $\langle \cdot, \cdot \rangle_\phi$), we conclude in the first subsection using a general argument that (theorem 7.1) $\mathfrak{I}S$ is preserved by the Ihara bracket $\{-, -\}$ and we construct (theorems 7.2, 7.3) explicit isomorphisms of bigraded Lie coalgebras between the isomorphic Lie coalgebras $(V/F, \delta_{V/F}) \simeq D_{\bullet \bullet}$ and the bigraded Lie coalgebra dual to $(\mathfrak{I}S, \{-, -\})$. This corresponds to results (a) and (b) of the previous subsection of this introduction.

In the second subsection, we give some reminders before proving the results (c) (paragraphe 7.2.1) and (d) (paragraphe 7.2.2) announced before. In paragraphe 7.2.1 we introduce for $m \geq 2$ an isomorphism $h_m : W^m \rightarrow \mathbb{k}[x_1, \ldots, x_m]$ mapping $W_R^m$ (where $A^m$ designates
the depth $m$ part of the vector space $A$) to the orthogonal complement of $D_{sh,m}$ with respect to a perfect pairing $(-,-)_m$ (see proposition 7.6). This proves that the mapping $W^m \to k[x_1, \ldots, x_m]^\vee, w \mapsto (h_m(w), -)_m$ induces an isomorphism of graded vector spaces $D_{m,\bullet} \simeq W^m/W^m_R h_m \to D_{sh,m}^\vee$ (see corollaries 7.7, 7.8). We then prove (d) in paragraph 7.2.2: we first prove that the isomorphism $f_m: k\langle x, z \rangle_m \to k[x_1, \ldots, x_m]$ induces an isomorphism $\tilde{f}_m: ls_m \to D_{sh,m}$ (proposition 7.11), we then prove (proposition 7.13) that $\tilde{f}_m^\vee$ is compatible with the isomorphisms $D_{\bullet,\bullet} \to ls^\vee$ ($ls^\vee$ is the bigraded dual of $ls$) and $D_{m,\bullet} \to D_{sh,m}^\vee$ constructed previously.

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1. Algebra Reminders

The section contains some algebra reminders. In the first subsection we recall facts on free associative algebras. Most of the material in the first subsection can be found in [Reu93]. The second subsection contains basic reminders on bigraded pairings, adjoints, adjoints of Lie brackets and duality between Lie algebras and Lie coalgebras.
1.1. **Free associative algebras.** We fix a field \( \mathbb{k} \) of characteristic zero. Let \( A \) be a set and \( X \) be the collection of indeterminants \( \{x_a\}_{a \in A} \). We denote by \( \mathbb{k} \langle X \rangle \) the \( \mathbb{k} \)-free associative algebra over the indeterminants \( \{x_a\}_{a \in A} \).

1.1.1. **Shuffle product.** We recall that for \((p, q) \in \mathbb{N}^2\) a \((p, q)\)-shuffle is a permutation \( \sigma \) of the set \([1, p + q] \) such that the restrictions \( \sigma|_{[1, p]} \) and \( \sigma|_{[p+1, p+q]} \) are increasing functions. The set of all \((p, q)\)-shuffles will be denoted by \( S(p, q) \).

The shuffle product \( \shuffle \) is an associative commutative product on \( \mathbb{k} \langle X \rangle \) defined by:

\[
x_{a_1} \cdots x_{a_p} \shuffle x_{a_{p+1}} \cdots x_{a_{p+q}} = \sum_{\sigma \in S(p, q)} x_{a_{\sigma^{-1}(1)}} \cdots x_{a_{\sigma^{-1}(p+q)}},
\]

for \((a_1, \ldots, a_{p+q}) \in A^{p+q}\).

We can also define inductively the shuffle product as follows:

\[
1 \shuffle w = w \shuffle 1 = w,
\]

\[
w_1 x_{a_1} \shuffle w_2 x_{a_2} = (w_1 \shuffle w_2 x_{a_2}) x_{a_1} + (x_{a_1} w_1 \shuffle w_2)x_{a_2},
\]

for \(x_{a_1}, x_{a_2} \in X\) and \(w, w_1, w_2\) unitary monomials of \( \mathbb{k} \langle X \rangle \).

Finally, the involution \( I : \mathbb{k} \langle X \rangle \to \mathbb{k} \langle X \rangle \) given by:

\[
I(x_{a_1} \cdots x_{a_m}) = x_{a_m} \cdots x_{a_1},
\]

for \((a_1, \ldots, a_m) \in A^m\) is a morphism for the shuffle product. Hence:

\[
x_{p} \cdots x_1 \shuffle x_{p+q} \cdots x_{p+1} = \sum_{\sigma \in S(p, q)} x_{\sigma^{-1}(p+q)} \cdots x_{\sigma^{-1}(1)}.
\]

1.1.2. **Coproducts.** There exists a unique morphism of algebras \( \Delta_X : \mathbb{k} \langle X \rangle \to \mathbb{k} \langle X \rangle \otimes \mathbb{k} \langle X \rangle \) for the canonical product of \( \mathbb{k} \langle X \rangle \) satisfying:

\[
\Delta_X(x_a) = 1 \otimes x_a + x_a \otimes 1, \text{ for } a \in A.
\]

We call \( \Delta_X \) the shuffle coproduct of \( \mathbb{k} \langle X \rangle \). One can explicitly describe the coproduct of a monomial of \( \mathbb{k} \langle X \rangle \) using shuffle permutations:

\[
\Delta_X(x_{a_1} \cdots x_{a_m}) = \sum_{p+q=m} \sum_{\sigma \in S(p, q)} x_{a_{\sigma(1)}} \cdots x_{a_{\sigma(p)}} \otimes x_{a_{\sigma(p+1)}} \cdots x_{a_{\sigma(p+q)}},
\]

for \(m \geq 1\) and \((a_1, \ldots, a_m) \in A^m\).

We will give another formula for the coproduct using the shuffle product and a canonical perfect pairing on \( \mathbb{k} \langle X \rangle \):

The canonical canonical perfect pairing denoted by \( \langle -, - \rangle_X : \mathbb{k} \langle X \rangle \otimes \mathbb{k} \langle X \rangle \to \mathbb{k} \), is given by the datum:

\[
\langle w_1, w_2 \rangle_X = \begin{cases} 
1 & \text{if } w_1 = w_2 \\
0 & \text{otherwise}
\end{cases}
\]

\[\text{7}\]

\[\text{10}\]
for \( w_1, w_2 \) two unitary monomials of \( \mathbb{k}(X) \).

Now let \( M_X \) be the set of unitary monomials of \( \mathbb{k}(X) \). One can express the coproduct \( \Delta_X(w) \) of \( w \in \mathbb{k}(X) \) as follows:

\[
\Delta_X(w) = \sum_{w_1,w_2 \in M_X} \langle w, w_1 \shuffle w_2 \rangle_X w_1 \otimes w_2.
\]

(11)

The terms \( \langle w, w_1 \shuffle w_2 \rangle \) (in the above formula) are nonzero for only finite number of monomials \( w_1, w_2 \) and the formula make sense in \( \mathbb{k}(X) \).

One can also define the deconcatenation coproduct \( \Delta^X_{\shuffle} : \mathbb{k}(X) \rightarrow \mathbb{k}(X) \otimes \mathbb{k}(X) \) as the linear map given by:

\[
\Delta^X_{\shuffle}(w) = \sum_{w_1,w_2 = w} w_1 \otimes w_2,
\]

(12)

where \( w, w_1, w_2 \) are monomials. The coproduct \( \Delta^X_{\shuffle} \) is a morphism for the shuffle product.

**Proposition 1.1.** Denote by \( \mu_X : \mathbb{k}(X)^{\otimes 2} \rightarrow \mathbb{k}(X) \) the canonical product of \( \mathbb{k}(X) \). The deconcatenation coproduct \( \Delta^X_{\shuffle} \) of \( \mathbb{k}(X) \) is the adjoint of \( \mu_X \) with respect to the pairings \( \langle -, - \rangle_X \) and \( \langle -, - \rangle^{\otimes 2}_X : (\mathbb{k}(X) \otimes \mathbb{k}(X))^{\otimes 2} \rightarrow \mathbb{k} \), where \( \langle -, - \rangle^{\otimes 2}_X \) is given by:

\[
\langle x_1 \otimes x_1', x_2 \otimes x_2' \rangle^{\otimes 2}_X = \langle x_1, x_2 \rangle_X \langle x_1', x_2' \rangle_X,
\]

for \( x_1, x_2, x_1', x_2' \in \mathbb{k}(X) \).

1.1.3. **Derivations.** Let \( d \) be a derivation of the free associative algebra \( \mathbb{k}(X) \) (for the canonical product). Applying the Leibniz rule to the product \( 1^2 = 1 \cdot 1 \) we see that a derivation maps the unit to zero. Also, by applying the Leibniz rule we get:

\[
d(x_{a_1} \cdots x_{a_m}) = \sum_{i \in [1, m]} x_{a_1} \cdots x_{a_{i-1}} d(x_{a_i}) x_{a_{i+1}} \cdots x_{a_m},
\]

(13)

For \( m \in \mathbb{N}^* \) and \( (a_1, \ldots, a_m) \in A^m \). This shows that a derivation of \( \mathbb{k}(X) \) is entirely determined by its values on the indeterminants \( x_a \in X \). Conversely, given a family \( \{y_a\}_{a \in A} \subset \mathbb{k}(X) \), one can define using (13) a unique linear map \( d' \) of \( \mathbb{k}(X) \) sending 1 to zero and \( x_a \) to \( y_a \) for \( x_a \in X \). Since \( d' \) satisfies (13), we have for \( w_1 = x_{a_1} \cdots x_{a_m} \) and \( w_2 = x_{a_{m+1}} \cdots x_{a_{m+m'}} \):

\[
d(w_1 w_2) = \sum_{i \in [1, m+m']} x_{a_1} \cdots x_{a_{i-1}} d(x_{a_i}) x_{a_{i+1}} \cdots x_{a_{m+m'}}
\]

\[
\quad = d(w_1)w_2 + w_1 d(w_2),
\]

and \( d' \) is a derivation. This proves the following result:

**Proposition 1.2.** Let \( \{y_a\}_{a \in A} \) be a subset of \( \mathbb{k}(X) \). There exists a unique derivation of \( \mathbb{k}(X) \) sending the indeterminant \( x_a \) to \( y_a \) for \( a \in A \).

1.2. **Parings, adjoints, duality between Lie algebras and Lie coalgebras.** Throughout this subsection \( A \) and \( B \) are two biradegred vector spaces:

\[
A = \bigoplus_{m,n \geq 0} A_{m,n} \quad \text{and} \quad B = \bigoplus_{m,n \geq 0} B_{m,n}.
\]

The spaces \( A_{m,n} \) and \( B_{m,n} \) are called homogenous elements of bidegree \((m, n)\) of \( A \) and \( B \) respectively. A pairing \( \langle-, - \rangle_{A,B} : A \otimes B \rightarrow \mathbb{k} \) is a bigraded pairing if \( A_{m,n}, B_{k,l} = 0 \) for
\[(m, n) \neq (k, l). \text{ Given a bigraded paring } (-, -)_{A,B} : A \otimes B \to k \text{ one can construct a bigraded pairing } (-, -)_{A,B}^{\otimes 2} : (A^{\otimes 2}) \otimes (B^{\otimes 2}) \to k \text{ given by:} \]
\[\begin{align*}
(a \otimes a', b \otimes b')_{A,B}^{\otimes 2} &= (a, b)_{A,B}(a', b')_{A,B},
\end{align*}\]  
for \(a, a' \in A\) and \(b, b' \in B\).

We assume that \(A\) and \(B\) are equipped with bigraded pairings \((-,-)_A : A \otimes A \to k\) and \((-,-)_B : B \otimes B \to k\). For \(f : A \to B\) a linear map the adjoint \(f^*\) of \(f\) with respect to \((-,-)_A\) and \((-,-)_B\) is the linear map \(f^* : B \to A\) satisfying:
\[ (f(a), b)_B = (a, f^*(b))_B, \quad \text{for } a \in A \text{ and } b \in B. \]
The linear map \(f^*\) is unique. Furthermore, if \(f\) is a bigraded map then \(f^*\) is also a bigraded map.

A bigraded Lie algebra (resp. coalgebra) is Lie algebra (resp. coalgebra) whose Lie bracket (resp. cobracket) is a graded map.

**Proposition 1.3.** Let \(g : A \otimes A \to A\) be a bigraded Lie bracket.

1) We assume that \(A\) is equipped with a perfect bigraded pairing \((-,-)_A : A \otimes A \to k\). The adjoint \(g^* : A \to A \otimes A\) of \(g\) with respect to \((-,-)_A\) and \((-,-)_{A,B}^{\otimes 2}\) is a bigraded Lie cobracket.

2) We denote by \(A^\vee\) and \((A \otimes A)^\vee\) the bigraded duals of \(A\) and \(A \otimes A\) respectively and by \(\varphi_{A,2}\) the canonical isomorphism of bigraded spaces \((A \otimes A)^\vee \to A^\vee \otimes A^\vee\). The space \(A^\vee\) equipped with \(g^\vee = \varphi_{A,2} \circ g^\vee\) (\(g^\vee\) is the bigraded dual map of \(g\)) is a bigraded Lie coalgebra.

**Proof.** We prove (1). Denote by \(\text{Cyc}\) the isomorphism of \(A^{\otimes 3}\) mapping \(a \otimes a' \otimes a''\) to \(a \otimes a' \otimes a'' + a'' \otimes a \otimes a' + a' \otimes a'' \otimes a\) and by \((-,-)^{\otimes 3}_A : A^{\otimes 3} \otimes A^{\otimes 3} \to k\) the pairing given by:
\[ (a_1 \otimes a'_1 \otimes a''_1, a_2 \otimes a'_2 \otimes a''_2)^{\otimes 3}_A = (a_1, a_2)^{\otimes 3}_A(a'_1, a'_2)^{\otimes 3}_A(a''_1, a''_2), \]
for \(a_1, a'_1, a''_1, a_2, a'_2, a''_2\) in \(A\). One checks that the adjoints \((\text{id} \otimes g)^*\) and \(\text{Cyc}^*\) of \(\text{id} \otimes g\) (with respect to \((-,-)_A\) and \((-,-)^{\otimes 3}_A\)) and \(\text{Cyc}\) (with respect to \((-,-)^{\otimes 3}_A\)) respectively are:
\[ (\text{id} \otimes g)^* = \text{id} \otimes g^* \quad \text{and} \quad \text{Cyc}^* = \text{Cyc}. \]
Since \(g\) is a Lie bracket we have \(g \circ (\text{id} \otimes g) \circ \text{Cyc} = 0\). Therefore:
\[ \text{Cyc} \circ (\text{id} \otimes g^*) \circ g^* = 0. \]
This shows that \(g^*\) satisfies the cocycle relation. On the other hand \(g \circ (\text{id} + \tau) = 0\) where \(\tau\) is the isomorphism of \(A^{\otimes 2}\) sending \(a \otimes a'\) to \(a' \otimes a\). Since \(\tau^* = \tau\) (with respect to \((-,-)^{\otimes 2}_A\)) and therefore \(g^*\) is antisymmetric:
\[ (\text{id} + \tau) \circ g^* = 0. \]
We have show that \(g^*\) is antysymetric and satisfies the cocycle relation, therefore \(g^*\) is a Lie cobracket. Finally, \(g^*\) is bigraded because \(g\) is bigraded. This proves (1). The proof of (2) is quite similar to the proof of (1). \(\square\)
We say that a bigraded Lie coalgebra \((C, \gamma)\) is the bigraded dual of a bigraded Lie algebra \((A, g)\) if \((C, \gamma)\) is isomorphic (as bigraded Lie coalgebra) to \((A^\vee, g^\vee)\), where \(g^\vee = \varphi_{A,2} \circ g^\vee\) as in the previous proposition.

**Proposition 1.4.** Let \((C, \gamma)\) be a bigraded Lie coalgebra and \((A, g)\) be a bigraded Lie algebra both with finite dimensional homogenous elements. We assume that \((-,-)_{C,A} : C \otimes A \to \k\) is a prefect bigraded pairing and that \(\gamma\) is the adjoint of \(g\) with respect to \((-,-)_{C,A}\) and \((-, -)^{\otimes 2}_{C,A}\).

1. The linear map \(\varepsilon : C \to A^\vee\) given by \(c \mapsto \varepsilon(c) : a \mapsto (c,a)_{C,A}\) for \(c \in C\) and \(a \in A\) is an isomorphism of bigraded Lie coalgebras, where \(A^\vee\) is equipped with \(g^\vee\).
   In particular, the Lie coalgebra \((C, \gamma)\) is the bigraded dual of the bigraded Lie algebra \((A, g)\).

2. If \(C_0 \subset C\) is a bigraded coideal for \(\gamma\) (i.e. \(\gamma(C_0) \subset C_0 \otimes C + C \otimes C_0\)) then:
   a) The orthogonal \(A_0\) of \(C_0\) with respect to \((-,-)_{C,A}\) is a bigraded subspace of \(A\) preserved by \(g\).
   b) The map \(\varepsilon\) induces a bigraded isomorphism \(\overline{\varepsilon} : C/C_0 \to A_0^\vee\) defined for \(\overline{c} \in C/C_0\) and \(a_0 \in A_0\) by:
      \[\overline{\varepsilon}(\overline{c})(a_0) = (c,a)_{C,A},\]
      where \(c\) is any lift of \(\overline{c}\) to \(c\).
   c) The map \(\overline{\varepsilon}\) is an isomorphism of bigraded Lie coalgebras between \(C/C_0\) equipped with the Lie cobracket \(\overline{\gamma}\) induced by \(\gamma\) and \(A_0^\vee\) equipped with \(g^\vee_{A_0}\). In particular, the Lie coalgebra \((C/C_0, \overline{\gamma})\) is the bigraded dual of the Lie algebra \((A_0, g)\).

**Proof.** We first prove (1). The map \(\varepsilon\) is linear bigraded isomorphism because \((-,-)_{C,A}\) is a perfect bigraded pairing. Furthermore, one checks the following equations for \(c \in C\) and \(a,a' \in A\):

\[
\begin{align*}
(c, g(a \otimes a'))_{C,A} &= \varepsilon(c)(g(a \otimes a')) = g^\vee(\varepsilon(c))(a \otimes a') \\
(\gamma(c), a \otimes a')^{\otimes 2}_{C,A} &= (\varphi_{A,2}^{-1} \circ (\varepsilon \otimes \varepsilon) \circ \gamma)(c)(a \otimes a')
\end{align*}
\]

where \(\varphi_{A,2}\) is the canonical isomorphism \((A \otimes A)^\vee \to A^\vee \otimes A^\vee\). Since \(\gamma\) is the adjoint of \(g\), we get by combining the last two equations:

\[
(\varphi_{A,2}^{-1} \circ (\varepsilon \otimes \varepsilon) \circ \gamma)(c)(a \otimes a') = g^\vee(\varepsilon(c))(a \otimes a'),
\]

For all \(c \in C\) and \(a,a' \in A\). This proves that \((\varepsilon \otimes \varepsilon) \circ \gamma = \varphi_{A,2}^{-1} \circ g^\vee \circ \varepsilon = g^\vee \circ \varepsilon\) and therefore \(\varepsilon\) is an isomorphism of bigraded Lie coalgebras between \((C, \gamma)\) and \((A^\vee, g^\vee)\). We now prove (2). We start by (a). The map \(\gamma\) is the adjoint of \(g\) and therefore:

\[
(c_0, g(a_0 \otimes a_0'))_{C,A} = (\gamma(c_0), a_0 \otimes a_0')^{\otimes 2}_{C,A}.
\]

for all \(c_0 \in C_0\) and \(a_0, a_0' \in A_0\). Since \(C_0\) is a coideal and \(A_0\) is orthogonal to \(C_0\) the right hand side of the above equation is equal to 0 and therefore for \(a_0, a_0' \in A_0\) the element \(g(a_0 \otimes a_0')\) lies in the orthogonal to \(C_0\) (with respect \((-,-)_{C,A}\)) which is \(A_0\) by hypothesis. This proves that \(A_0\) is preserved by \(g\). The space \(A_0\) is a bigraded subspace of \(A\) because the pairing is bigraded and \(C_0\) is a bigraded subspace of \(C\). We have proved (a). To prove (b) and (c), one
first checks that $(-,-)_{C,A}$ induces a perfect bigraded pairing $(-,-)_0 : C/C_0 \otimes A_0 \rightarrow k$ and that $\gamma$ is the adjoint of $g_\lambda A_0 \otimes A_0$ with respect to $(-,-)_0$ and $(-,-)_0^{\otimes 2}$. One then applies (1) to $(C/C_0, \gamma), (A_0, g_\lambda A_0 \otimes A_0)$ and $(-,-)_0$ to conclude. We have proved the proposition. \smallqed

2. The linearized double shuffle Lie algebra $ls$ and the double shuffle space

Given a field $k$ of characteristic zero, we recall the definition of the linearized double shuffle Lie algebra $ls$ (in \cite{Bro13} $ls$ is defined over $\mathbb{Q}$) and the definition of the double shuffle subspace $Dsh_m = \bigoplus_{k,m \geq 2} Dsh_m(k - m)$ of the polynomial algebra $k[x_1, \ldots, x_m]$ (introduced in \cite{IKZ06} for $k = \mathbb{Q}$) for $m \geq 2$. The Lie algebra $ls$ is a bigraded Lie subalgebra of the free associative algebra $k\langle x, z \rangle$ on two indeterminates $x, z$ equipped with the Ihara Lie bracket and a depth-weight bigrading. We introduce the bigraded vector space $ls$ in the first subsection, we then recall the definition of the Ihara bracket \{-,-\} in the second subsection. The third subsection contains the definition of the double shuffle subspace $Dsh_m$.

2.1. The bigraded space $ls$. In order to define $ls$ we use two algebras: the free associative algebra $k\langle x, z \rangle$ on two indeterminates $x, z$ and the free associative algebra $k\langle Y \rangle$ on the set of indeterminates $Y = \{y_i | i \in \mathbb{N}^*\}$.

We define the weight of a monomial $w \in k\langle x, z \rangle$ as the total degree of $w$ and the depth of $w$ as the degree of $w$ with respect to the variable $z$. The algebra $k\langle x, z \rangle$ is a bigraded algebra with respect to depth and weight. Similarly, we define the depth of $w_Y := y_1 \cdots y_m \in k\langle Y \rangle$ as the total degree of $w_Y$ and the weight of $w_Y$ as the number $n_1 + \cdots + n_m$.

The shuffle coproducts of $k\langle x, z \rangle$ and $k\langle Y \rangle$ as defined in \cite{Bro13} section I will be denoted by $\Delta$ and $\Delta_Y$, respectively. The coproducts $\Delta$ and $\Delta_Y$ respect the bigradings defined above.

Let $\pi : k\langle x, z \rangle \rightarrow k\langle Y \rangle$ be the projection of bigraded spaces, given by:

$$k\langle x, z \rangle x \mapsto 0, \quad 1 \mapsto 1 \quad \text{and} \quad x^{n_1-1}z^{n_2-1}z \cdots x^{n_m-1}z \mapsto y_1y_2\cdots y_m, \quad (15)$$

for $m \geq 1$ and $n_1, \ldots, n_m \geq 1$.

We will introduce a bigraded space $ls'$ containing $ls$ before reminding the definition of $ls$.

**Definition 2.1.** The space $ls'$ is the vector space of elements $\psi \in k\langle x, z \rangle$ satisfying the equations:

$$\Delta(\psi) = 1 \otimes \psi + \psi \otimes 1 \quad \text{and} \quad \Delta_Y(\pi(\psi)) = 1 \otimes \pi(\psi) + \pi(\psi) \otimes 1.$$

**Proposition 2.2.** The space $ls'$ is bigraded for depth and weight and can be decomposed as follows:

$$ls' = k \cdot x \bigoplus \bigoplus_{k \geq m \geq 1} ls'_m,$$

where $ls'_m$ is the vector space of elements of depth $m$ and weight $k$ in $ls'$.

**Proof.** The space $ls'$ is bigraded since the maps $\Delta$, $\Delta_Y$ and $\pi$ are bigraded linear maps and we have the following decomposition:

$$ls' = k \cdot x \bigoplus \bigoplus_{k \geq m \geq 1} ls'_m.$$
Furthermore, an element of $k\langle x, z \rangle$ of depth zero is of the form $c + c'x^k$ with $c, c' \in k$ and
\[
\Delta(c + c'x^k) = c \otimes 1 + c'\Delta(x)^k = c \otimes 1 + c'(1 \otimes x + x \otimes 1)^k.
\]
Therefore, the elements $a \in k\langle x, z \rangle$ satisfying $\Delta(a) = 1 \otimes a + a \otimes 1$ are of the form $c'x$. From this, the fact that $\pi(x) = 0$ and the definition of $l_s'$ we deduce that the depth 0 part of $l_s'$ is equal to $k \cdot x$. We have proved the proposition. 

\[\square\]

**Definition 2.3** ([Bro13]). For $k \geq m \geq 1$ set:
\[
l^*_m = \begin{cases} 0 & \text{if } m = 1 \text{ and } k \text{ is even} \\ l^*_m & \text{otherwise} \end{cases}.
\]

The linearised double shuffle space is the bigraded vector space $l_s := \bigoplus_{k \geq m \geq 1} l^*_m \subset k\langle x, z \rangle$.

**Remark 2.4.** We can decompose $l_s'$ as follows:
\[
l_s' = l_s \oplus k \cdot x \oplus (l_s')_{\text{even}},
\]
where $(l_s')_{\text{even}} = \bigoplus_{n \geq 1} l_{2n}^*1$.

2.2. **Ihara’s bracket.** Let $[\cdot, \cdot]$ be the canonical Lie bracket of the free associative $k\langle x, z \rangle$, i.e:
\[
[w, w'] = ww' - w'w,
\]
for $w, w' \in k\langle x, z \rangle$. Given $w \in k\langle x, z \rangle$, we denote by $d_w$ the unique derivation of $k\langle x, z \rangle$ such that:
\[
d_w(x) = 0 \quad \text{and} \quad d_w(z) = [z, w].
\]
The existence and uniqueness of $d_w$ follows from Proposition 1.2 of section 1.

**Proposition 2.5.** The mapping (the Ihara bracket) $\{\cdot, \cdot\} : k\langle x, z \rangle^{\otimes 2} \rightarrow k\langle x, z \rangle$ given by:
\[
\{w, w'\} = d_w(w') - d_{w'}(w) + [w, w'].
\]
for $w, w' \in k\langle x, z \rangle$ defines a (bigraded) Lie bracket on $k\langle x, z \rangle$.

**Proof.** It is clear that the above map is antistymmetric. One can prove that $\{\cdot, \cdot\}$ satisfies the Jacobi identity by direct computations using the identity:
\[
d_w \circ d_{w'} - d_{w'} \circ d_w = d_{\{w, w'\}},
\]
for $w, w' \in k\langle x, z \rangle$. \[\square\]

Brown affirms in [Bro13] that $l_s$ is preserved by the Ihara bracket and that this can be proven by adapting the work of [Rac02] for $\mathfrak{d}m_{r_0}$. Schneps (Theorem 3.4.3 of [Sch15] and its proof) gives a proof using the theory of (bi)moulds introduced by Écalle (see for example [Écalle04]) and a correspondence between the work of Racinet and Écalle established by Racinet in [Rac00]. As mentioned in the introduction of the paper, we give a new proof using the dihedral Lie coalgebra of Goncharov (see Theorem 7.1, section 7).

**Proposition 2.6** ([Bro13], [Sch15]). The Linearized double shuffle space $l_s$ is preserved by the Ihara bracket.
2.3. The double shuffle space. For \( m \geq 2 \) and \( \sigma \in \mathfrak{S}_m \) we denote by \( S_m \) and \( T_\sigma \) the automorphisms of the polynomial ring \( \mathbb{k}[x_1, \ldots, x_m] \) given by:
\[
S_m(x_i) = x_i + \cdots + x_m, \quad T_\sigma(x_i) = x_{\sigma^{-1}(i)}, \quad \text{for } i \in [1, m]
\]
and we define for \( l \in [1, m-1] \) the linear endomorphisms \( T_{m,*}^{(l)} \) and \( T_{m,\,\hat{\imath}}^{(l)} \) of \( \mathbb{k}[x_1, \ldots, x_m] \) by the following:
\[
T_{m,*}^{(l)} = \sum_{\sigma \in S(l, m-l)} T_\sigma, \quad T_{m,\,\hat{\imath}}^{(l)} = T_{m,*}^{(l)} \circ S_m,
\]
where \( S(p, q) \) denotes the set of \((p, q)\) shuffles, as in the previous sections.

**Definition 2.7** ([IKZ06]).

1) The double shuffle subspace \( D_{sh,m} \) of \( \mathbb{k}[x_1, \ldots, x_m] \) (for \( m \geq 2 \)) is the intersection of the kernels of the endomorphisms \( T_{m,*}^{(l)} \) and \( T_{m,\,\hat{\imath}}^{(l)} \) for \( l \in [1, m-1] \).

2) For \( m \geq 2 \) and \( d \geq 1 \), the double shuffle space \( D_{sh,m}(d) \) is the vector subspace of polynomials of total degree \( d \) lying in \( D_{sh,m} \).

The maps \( T_{m,*}^{(l)} \) and \( T_{m,\,\hat{\imath}}^{(l)} \) respect the total degree grading of \( \mathbb{k}[x_1, \ldots, x_m] \). Hence, the total degree induces a grading of \( D_{sh,m} \). Here, we endow \( D_{sh,m} \) with a weight grading corresponding to a shift by \(+m\) of the total degree grading:

**Definition 2.8.** The weight grading of \( D_{sh,m} \) is given by:
\[
D_{sh,m} = \bigoplus_{k \geq m} D_{sh,m}^k,
\]
where the weight \( k \) part \( D_{sh,m}^k \) of \( D_{sh,m} \) is the double shuffle space \( D_{sh,m}(k-m) \).

3. The dihedral Lie coalgebra \( D_{\ast\ast} \)

Given a field \( \mathbb{k} \) of characteristic zero, we give an alternative definition of the dihedral Lie coalgebra \( D_{\ast\ast} \) (\( \mathfrak{D}_{\ast\ast}(\{e\}) \) of [Gon01] originally defined over \( \mathbb{Q} \)). In the first subsection, we introduce bigraded \( \mathbb{k} \)-vector spaces \( W_R \subset W \subset V \supset F \) such that the natural inclusion \( W \subset V \) induces an isomorphism of bigraded spaces \( W/W_R \to V/F \). The space \( W_R \) (and \( F \)) is defined using generating series \( \{t_1 : \cdots : t_{m+1}\} \) (similar to those in [Gon01]). We show (proposition 3.1) that the underlying bigraded vector space of the dihedral Lie coalgebra \( D_{\ast\ast} \) is isomorphic to the bigraded spaces \( W/W_R \) and \( V/F \) by constructing an isomorphism of bigraded spaces \( \eta : D_{\ast\ast} \to W/W_R \). In the second subsection, we define a map \( \bar{\delta} : W \to W \otimes W \) inducing a bigraded Lie cobracket \( \delta \) on \( W/W_R \) for which \( \eta : D_{\ast\ast} \to W/W_R \) is an isomorphism of bigraded Lie coalgebra. The construction of \( \delta \) is a transcription of the work of [Gon01].

3.1. The bigraded space \( D_{\ast\ast} \). Let \( V \) be the free \( \mathbb{k} \) vector space with basis the elements of the sets:
\[
I := \{I(n_1, \ldots, n_m) | m \geq 1, n_i \in \mathbb{N}^* \text{ for } i \in [1, m]\},
\]
and
\[
I' := \{I(n_0, n_1, \ldots, n_m) | m \geq 0, n_i \in \mathbb{N}^* \text{ for } i \in [0, m]\},
\]
where \([i, j]\) denotes the set of integers \(i, i+1, \ldots, j\) for \( i \leq j\).
For an element $a \in I \cup I'$ we define the depth $d(a)$ and the weight $\omega(a)$ of $a$ by the following:

$$d(I(n_1, \ldots, n_m)) = d(I'(n_0, \ldots n_m)) = m,$$

$$\omega(I(n_1, \ldots, n_m)) = n_1 + \cdots + n_m; \quad \omega(I'(n_0, \ldots n_m)) = n_0 + \cdots + n_m.$$

Depth and weight induce a natural bigrading of $A$ for a vector space $V$.

$$V = \bigoplus_{m,k \geq 0} V_{m,k},$$

where $V_{m,k}$ is the subspace of $V$ generated by the elements of $I \cup I'$ of depth $m$ and weight $k$.

For a vector space $A$ and $m \in \mathbb{N}$, we denote by $A[[t_1, \ldots, t_{m+1}]]$ the vector space of series in the commuting variables $t_1, \ldots, t_{m+1}$ with values in $A$.

For $m \geq 1$, we define the multivariable series with values in $V$:

$$\{t_1 : \cdots : t_m : t_{m+1}\} = \sum_{n_1, \ldots, n_m \geq 1} I(n_1, \ldots, n_m)(t_1 - t_{m+1})^{n_1-1}(t_m - t_{m+1})^{n_m-1}$$

and the series:

$$\{t_1, \ldots, t_m\} = \{t_1 : t_1 + t_2 : t_1 + t_2 + t_3 : \cdots : t_1 + t_2 + \cdots + t_m : 0\}.$$

These series are similar to those used by Goncharov (see (54) and (56) in [Gon01] for $G = \{e\}$).

Let $V_1$ be the smallest subspace of $V$ such that for all $(p, q) \in (\mathbb{N}^*)^2$:

$$\sum_{\sigma \in S(p,q)} \{t_{\sigma^{-1}(1)} : \cdots : t_{\sigma^{-1}(p+q)} : t_{m+1}\},$$

is mapped to 0 by the natural map $V[[t_1, \ldots, t_{m+1}]] \mapsto (V/V_1)[[t_1, \ldots, t_{m+1}]]$ ($S(p,q)$ is the set of $(p,q)$-shuffle). Similarly, we define $V_2$ as the smallest subspace of $V$ such that for all $(p, q) \in (\mathbb{N}^*)^2$, the image of the series:

$$\sum_{\sigma \in S(p,q)} \{t_{\sigma^{-1}(1)} : \cdots : t_{\sigma^{-1}(p+q)}\},$$

with respect to $V[[t_1, \ldots, t_{m+1}]] \mapsto (V/V_2)[[t_1, \ldots, t_{m+1}]]$ is 0.

We denote by $V'$ the subspace of $V$ generated by the elements of $I' \setminus \{I'(1)\}$ and by $V''$ the subspace of $V$ generated by the elements $I(2n)$ for $n \geq 1$.

The subspaces $V_1, V_2, V'$ and $V''$ are bigraded subspaces of $V$.

**Proposition 3.1.** Denote by $W$ the subspace of $V$ generated by the elements of the set $I$ and set $W_R := V_1 + V_2 + V''$, $F := W_R \oplus U$, with $U := \mathbb{k} \cdot I'(1) + V'$ ($U$ is the subspace of $V$ generated by the elements of $I'$) and $V_1, V_2, V''$ as in the previous paragraph.
1) The inclusion $W \subset V$ induces an isomorphism of bigraded vector spaces $\tilde{\iota} : W/W_R \rightarrow V/F$, where the bigradings are those induced by weight and depth.

2) We have a natural isomorphism of bigraded vector spaces $\eta : D_{\bullet \bullet} \rightarrow W/W_R$. The isomorphism $\eta$ is constructed in the proof below.

Proof. The fact that the inclusion $W \rightarrow V$ induces and isomorphism $W/W_R \simeq V/V_R$ is clear since $F = W_R \oplus U$ and $V = W \oplus U$. We show that $W/W_R$ is isomorphic $D_{\bullet \bullet}$. As mentioned before $D_{\bullet \bullet}$ denotes the space $\hat{D}_{\bullet \bullet}(\{e\})$ of [Gon01]. In [Gon01], Goncharov first introduces a space $\mathcal{D}_{\bullet \bullet}(G)$ (section 4) defined (over $\mathbb{Q}$, here we take a field $k$ of characteristic zero) by generators and relations:

- the generators are the symbols $I_{n_1,\ldots,n_m}(g_0, \ldots, g_m)$ for $m \geq 1$, $(g_0, \ldots, g_m) \in G^m$ and $(n_1, \ldots, n_m) \in (\mathbb{N}^*)^m$,
- the relations are: (i) homogeneity relations, (ii) double shuffle relations, (iii) distribution relations, and the relation (iv) $I_1(e : e) = 0$.

He then introduces the space $\hat{D}_{\bullet \bullet}(G)$ (see page 430) generated by the same generators of $\mathcal{D}_{\bullet \bullet}(G)$ but the relations are relations (i), (ii) and (iii), equivalently he does not impose the relation $I_1(e : e) = 0$ in $\hat{D}_{\bullet \bullet}(G)$ and $\hat{D}_{\bullet \bullet}(G) = \mathcal{D}_{\bullet \bullet}(G) \oplus k \cdot I_1(e, e)$. When $G = \{e\}$, the generators are the symbols $I_{n_1,\ldots,n_m}(e, \ldots, e)$ and the homogeneity relations are empty. We will show that there exists a linear isomorphism:

$$\eta : \hat{D}_{\bullet \bullet}(\{e\}) = D_{\bullet \bullet} \rightarrow W/W_R$$

$$I_{n_1,\ldots,n_m}(e, \ldots, e) \mapsto I(n_1, \ldots, n_m).$$

The correspondence $I_{n_1,\ldots,n_m}(e, \ldots, e) \mapsto I(n_1, \ldots, n_m)$ allows to identify the series defined by formulas (54) and (56) in section 4 of [Gon01] to the series $\{t_1 : \cdots : t_{m+1}\}$ and $\{t_1, \ldots, t_m\}$ defined in this section and therefore we obtain a correspondence between the double shuffle relations (relations (ii)) of Goncharov and the subspace $V_1 + V_2$. To prove the existence of $\eta$ we still have to identify the distribution relations to $V''$ modulo $V_1 + V_2$. For $G = \{e\}$, the distribution relations are given by the equations:

$$\{t_1 : \cdots : t_{m+1}\} = \{-t_1, \cdots, -t_{m+1}\}.$$  

for $m \geq 1$. These are the distribution relations for $l = -1$ ( $l$ is the notation used in [Gon01]).

As mentioned in the beginnig of subsection 4.2 of [Gon01] the distribution relations for $l = -1$ correspond to the inversion relations (see formula (66) of [Gon01]). The distribution relations are part of the dihedral symmetrie relations ([Gon01] section 4.2 formulas (64), (65) and (66) ). Goncharov shows (theorem 4.1 section 4.2 of [Gon01]) that the double shuffle relations imply the dihedral symmetry relations for $m \geq 2$. In particular, the double shuffle relations imply the distribution relations for $m \geq 2$. Since, the double shuffle relations correspond to $V_1 + V_2$, we only need to identify the subspace of $V$ corresponding to the distribution relations for $m = 1$. One can readily check that the distribution relations for $m = 1$ correspond to $V''$ via $I_{n_1,\ldots,n_m}(e, \ldots, e) \mapsto I(n_1, \ldots, n_m)$. We have therefore identified the set of relations defining $\hat{D}_{\bullet \bullet}(\{e\})$ to the subspace $W_R = V_1 + V_2 + V''$. This proves the existence of the isomorphism $\eta$ of (18). The weight and depth of $I(n_1, \ldots, n_m)$ defined here correspond to the weight and depth of $I_{n_1,\ldots,n_m}(e : \cdots : e)$ used by Goncharov.
Hence \( \eta \) is a bigraded isomorphism and point (2) of the proposition is proved. We have shown the proposition. \( \square \)

**Remark 3.3.** The series \( \{t_1 : \cdots : t_{m+1}\} \) and \( \{t_1, \cdots, t_n\} \) are with values in \( W \).

**Remark 3.3.** The cyclic symmetric relation:

\[
\{t_1 : \cdots : t_{m+1}\} = \{t_{m+1} : t_1 : \cdots : t_m\}, \quad \text{for } m \geq 1
\]

is satisfied in \( W/W_R \). Indeed, the cyclic symmetry relation correspond via \( \eta \) to formula (64) of \cite{Gon01} which is satisfied in \( \mathcal{D}_\bullet(\{e\}) \) (denoted by \( D_\bullet \) in this paper). The fact that (64) of \cite{Gon01} is satisfied in \( \mathcal{D}_\bullet(\{e\}) \) is a consequence of theorem 4.1 (\cite{Gon01}) and the distribution relations (satisfied by definition in \( \mathcal{D}_\bullet(\{e\}) \)) for \( m = 1 \) and \( l = -1 \) following the notations of \cite{Gon01}.

This cyclic symmetric relation can be interpreted in \( V \) (or \( W \)) as follows:

\[
\{t_1 : \cdots : t_{m+1}\} = \{t_{m+1} : t_1 : \cdots : t_m\} + c_m,
\]

in \( V \) (or \( W \)) with \( c_m \in W_R[[t_1, \cdots, t_{m+1}]] \) (\( W_R = V_1 + V_2 + V'' \subset F \)).

### 3.2. The Lie cobracket of \( D_\bullet \)

For \( A \) a vector space and \( f \) an element of \( A[[t_1, \cdots, t_{m+1}]] \) (\( m \in \mathbb{N}^* \)), we define \( \text{Cycle}_{m+1}(f) \in A[[t_1, \cdots, t_{m+1}]] \) by:

\[
\text{Cycle}_{m+1}(f(t_1, \cdots, t_{m+1})) = \sum_{j=0}^{m} f(t_{1+k}, \cdots, t_{m+1+k})
\]

where the indices of the variables \( t_{1+k}, \cdots, t_{m+1+k} \) are modulo \( m + 1 \).

Let \( \tilde{\delta} : W \to W \otimes W \) be the linear map given by:

\[
\tilde{\delta}(\{t_1 : \cdots : t_{m+1}\}) = - \sum_{k=2}^{m} \text{Cycle}_{m+1}(\{t_1 : \cdots : t_{k-1} : t_{m+1}\} \wedge \{t_k : \cdots : t_{m+1}\}),
\]

for \( m \geq 2 \) (where \( a \wedge b = a \otimes b - b \otimes a \)) and

\[
\tilde{\delta}(\{t_1 : t_2\}) = 0.
\]

The following theorem is a transcription of the work of \cite{Gon01} concerning the cobracket of \( D_\bullet \):

**Theorem 3.4.**

1) The map \( \tilde{\delta} \) induces a map \( \delta : W/W_R \to W/W_R \otimes W/W_R \), providing a bigraded Lie coalgebra structure on \( W/W_R \).

2) The isomorphism of bigraded spaces \( \eta : D_\bullet \to W/W_R \) (see proposition 3.1 and formula (18) of its proof) is an isomorphism of Lie coalgebra with \( W/W_R \) equipped with \( \delta \) and \( D_\bullet \) equipped with its Lie cobracket defined in \cite{Gon01}.

**Proof.** Note that \( \delta(\{t_1 : t_2\}) = 0 \) is a necessary condition so that \( \delta \) respects the bigrading since \( D_\bullet \) has no depth 0 part. As mentioned before \( D_\bullet \) denotes the Lie coalgebra \( \mathcal{D}_\bullet(\{e\}) \) of \cite{Gon01}. We have seen in the proof of proposition 3.1 that \( \mathcal{D}_\bullet(\{e\}) \) is generated by the symbols \( I_{n_1, \cdots, n_m}(e : \cdots : e) \) and that the isomorphism \( \eta : D_\bullet \to W/W_R \) is given by
\[ \eta(I(n_1, \ldots, n_m)) = I_{n_1, \ldots, n_m}(e : \cdots : e). \]

Using \( \eta \), the statements in the theorem follow readily from (b) of theorem 4.3 of [Gon01] for \( G = \{ e \} \). 

\[ \square \]

**Remark 3.5.** The space \( W_R \) is a coideal for \( \tilde{\delta} \) i.e. \( \tilde{\delta}(W_R) \subset W_R \otimes W + W \otimes W_R \).

4. Series with values in the free associative algebra on two indeterminants

We fix a field \( k \) of characteristic zero. We define in formula (21), for \( n \geq 1 \), a series \( Q_n \) in \( n \) commuting variables. The series \( Q_n \) is a generating series for the unitary monomials of \( k\langle x, z \rangle \) of depth \( n \). We then introduce (definition 4.1) for \( s \geq 1 \) an algebra \( k\langle x, z| T_s \rangle \) of series in the commuting variables \( t_1, \ldots, t_s \) which is linearly generated by the unit 1 and the series \( Q_n(t_{a_1}, \ldots, t_{a_n}) \) for \( n \geq 1 \) and \( \{a_1, \ldots, a_n\} \subset [1, s] \). This construction is used in order to give a formula describing the shuffle product of two series \( Q_n, Q_m \) (proposition 4.7) that will be used in section 5. The formula is a formal analogue of formula (15) of [Gon98] for generating series of iterated integrals in the case where the iterated integrals represent MZVs.

Throughout this section, we denote by \( k\langle x, z\rangle[[t_1, \ldots, t_n]] \) (for \( n \geq 1 \)) the space of series in the commuting variables \( t_1, \ldots, t_n \) with values in \( k\langle x, z \rangle \). and we set:

\[
\frac{1}{1 - xt} := \sum_{n \geq 0} x^n t^n,
\]

for \( t \) a formal variable.

For \( n \geq 1 \), we define \( Q_n(t_1, \ldots, t_n) \in k\langle x, z\rangle[[t_1, \ldots, t_n]] \) by:

\[
Q_n(t_1, \ldots, t_n) := \frac{1}{1 - xu_n} z \frac{1}{1 - xu_{n-1}} z \cdots \frac{1}{1 - xu_1} z,
\]

where \( u_k = t_1 + \cdots + t_k \), for \( k \in [1, n] \).

**Definition 4.1.** For \( s \geq 1 \), we define the algebra \( k\langle x, z| T_s \rangle \) as the vector subspace of \( k\langle x, z\rangle[[t_1, \ldots, t_s]] \) linearly generated by the unit 1 and the series \( Q_n(t_{a_1}, \ldots, t_{a_n}) \) for \( n \geq 1 \) and \( \{a_1, \ldots, a_n\} \subset [1, s] \) equipped with the associative unital product \( \circ \) given by:

\[
Q_m(t_{a_{n+1}}, \ldots, t_{a_{n+m}}) \circ Q_n(t_{a_1}, \ldots, t_{a_n}) = Q_{n+m}(t_{a_1}, \ldots, t_{a_{n+m}}),
\]

for \( n, m \geq 1 \) and \( \{a_1, \ldots, a_{n+m}\} \subset [1, s] \).

**Remark 4.2.** The series \( Q(t_1), \ldots, Q(t_s) \) generate the algebra \( k\langle x, z| T_s \rangle \). Indeed,

\[
Q_m(t_{a_1}, \ldots, t_{a_m}) = Q_1(t_{a_m}) \circ Q_1(t_{a_{m-1}}) \circ \cdots \circ Q_1(t_{a_1}),
\]

for \( m \geq 1 \) and \( \{a_1, \ldots, a_{n+m}\} \subset [1, s] \).

We denote by \( \shuffle \) and \( \Delta_{\shuffle} \) the shuffle product and the deconcatenation coproduct of \( k\langle x, z \rangle \) respectively (see section 1 formulas (2) and (12)).

**Lemme 4.3.** Let \( t \) and \( t' \) be two commuting variables. One has the following formulas:

\[
\frac{1}{1 - xt} \shuffle \frac{1}{1 - xt} = \frac{1}{1 - x(t + t')} , \quad \Delta_{\shuffle}(\frac{1}{1 - xt}) = \frac{1}{1 - xt} \otimes \frac{1}{1 - xt},
\]

where we extend naturally \( \shuffle \) and \( \Delta_{\shuffle} \) to \( k\langle x, z\rangle[[t, t']] \).

**Proof.** The formulas can be easily obtained by direct computations. \[ \square \]
Proposition 4.5. For $a, b, c$ unitary monomials of $k<x, z>$ and $y \in \{x, z\}$. One has:

$$a \uplus byc = \sum_{(a_1, a_2) \in E} (a_1 \uplus b)y(a_2 \uplus c),$$

where $E$ is the set of pairs of unitary monomials $(a_1, a_2)$ such that $a_1a_2 = a$ (note that $\Delta_m(a) = \sum_{(a_1, a_2) \in E} a_1 \otimes a_2$).

Proof. One can prove the result using formula (2) in section I.

Proposition 4.5. For $a_0 \in [1, s]$ with $n \geq 1$, and $R \in k<x, z|T_s>$, we have:

$$R \circ Q_1(t_{a_0}) = \left( \frac{1}{1 - xt_{a_0}} \uplus R \right) z$$

Proof. To prove the proposition, we will prove inductively that:

$$Q_n(t_{a_1}, \ldots, t_{a_n}) \circ Q_1(t_{a_0}) = \left( \frac{1}{1 - xt_{a_0}} \uplus Q_n(t_{a_1}, \ldots, t_{a_n}) \right) z, \quad (H_n)$$

for $n \geq 0$ with the convention $Q_0(t_{a_1}, \ldots, t_{a_0}) := 1$. The equation $(H_0)$ clearly holds. Assume now that $(H_n)$ holds, let us prove $(H_{n+1})$. We have:

$$Q_{n+1}(t_{a_1}, \ldots, t_{a_{n+1}}) = \frac{1}{1 - xv_{n+1}} z Q_n(t_{a_1}, \ldots, t_{a_n}),$$

where $v_{n+1} = t_{a_1} + \cdots + t_{a_{n+1}}$. Hence,

$$\left( \frac{1}{1 - xt_{a_0}} \uplus Q_{n+1}(t_{a_1}, \ldots, t_{a_{n+1}}) \right) z = \left( \frac{1}{1 - xt_{a_0}} \uplus \left( \frac{1}{1 - xv_{n+1}} z Q_n(t_{a_1}, \ldots, t_{a_n}) \right) \right) z.$$

By applying to the right hand side of the equation lemma 4.4 for $a = \frac{1}{1 - xt_{a_0}}, b = \frac{1}{1 - xv_{n+1}}, y = z$ and $c = Q_n(t_{a_1}, \ldots, t_{a_n})$, then simplifying the expression using lemma 4.3 we get the following:

$$\left( \frac{1}{1 - xt_{a_0}} \uplus Q_{n+1}(t_{a_1}, \ldots, t_{a_{n+1}}) \right) z = \frac{1}{1 - x(t_{a_0} + v_{n+1})} z \left( \frac{1}{1 - xt_{a_0}} \uplus Q_n(t_{a_1}, \ldots, t_{a_n}) \right) z.$$

From this we derive using $(H_n)$ then the definition of the product $\circ$ the equation:

$$\left( \frac{1}{1 - xt_{a_0}} \uplus Q_{n+1}(t_{a_1}, \ldots, t_{a_{n+1}}) \right) z = \frac{1}{1 - x(t_{a_0} + v_{n+1})} z Q_{n+1}(t_{a_0}, \ldots, t_{a_n}).$$

This proves $(H_{n+1})$, since:

$$\frac{1}{1 - x(t_{a_0} + v_{n+1})} z Q_{n+1}(t_{a_0}, \ldots, t_{a_n}) = Q_{n+2}(t_{a_0}, \ldots, t_{a_{n+1}}) = Q_{n+1}(t_{a_1}, \ldots, t_{a_{n+1}}) \circ Q_1(t_{a_0}).$$

We have proved the proposition by induction. \hfill \Box

The formula of proposition 4.5 allows to extend the action of the vector space (as an abelian group) $\text{vect}(Q_1(t_1), \ldots, Q_1(t_s))$ on $k<x, z|T_s>$ given by

$$(R, \lambda Q_1(t_i)) \mapsto \lambda(R \circ Q_1(t_i)),$$

for $i \in [1, s]$ and $\lambda \in k$, into an action on $k<x, z|[t_1, \ldots, t_s]]$ given by:

$$(R', \lambda Q_1(t_i)) \mapsto R' \circ \lambda Q_1(t_i) := \lambda\left( \frac{1}{1 - xt_i} \uplus R' \right) z,$$

(27)
for \( R' \in \mathbb{k}\langle x, z \rangle[[t_1, \ldots, t_s]], i \in [1, \ldots, s] \) and \( \lambda \in \mathbb{k} \).

**Proposition 4.6.** Let \( K\langle x_1, \ldots, x_s \rangle \) be the free associative algebra on the indeterminates \( x_1, \ldots, x_s \) and \( f : \mathbb{k}\langle x_1, \ldots, x_s \rangle \to \mathbb{k}\langle x, z|T_s \rangle \) be the unique algebra morphism mapping \( x_i \) to \( Q_1(t_i) \) for \( i \in [1, s] \).

1) For \( w, w' \in \mathbb{k}\langle x_1, \ldots, x_s \rangle \) we have:

\[
f(w \shuffle w') = f(w) \shuffle f(w'),
\]

where the shuffle product \( f(w) \shuffle f(w') \) is the natural shuffle product in \( \mathbb{k}\langle x, z \rangle[[t_1, \ldots, t_s]] \).

2) The algebra \( \mathbb{k}\langle x, z|T_s \rangle \) is stable under the shuffle product of \( \mathbb{k}\langle x, z \rangle[[t_1, \ldots, t_s]] \) and \( f \) is a homomorphism of algebras with both \( \mathbb{k}\langle x_1, \ldots, x_s \rangle \) and \( \mathbb{k}\langle x, z|T_s \rangle \) equipped with shuffle products.

**Proof.** We first prove (1). The result is clear for \( w \) or \( w' \) equal to 1. Assume that \( w = a'x_i \) and \( w' = b'x_j \) are two monomials. Therefore \( f(w) = a \circ Q_1(t_i) \) and \( f(w') = b \circ Q_1(t_j) \) where \( a = f(a') \) and \( b = f(b') \). By proposition 4.5, we have:

\[
f(w) = a \circ Q_1(t_i) = (\frac{1}{1 - xt_i} \shuffle a)z \quad \text{and} \quad f(w') = b \circ Q_1(t_j) = (\frac{1}{1 - xt_j} \shuffle b)z.
\]

Hence using formula (5) of section 1 we find:

\[
(a \circ Q_1(t_i)) \shuffle (b \circ Q_1(t_j)) = ((a \circ Q_1(t_i)) \shuffle (\frac{1}{1 - xt_j} \shuffle b))z + ((\frac{1}{1 - xt_i} \shuffle a) \shuffle (b \circ Q_1(t_j)))z.
\]

Since the shuffle product is associative and commutative, we get by applying formula (27) to the right hand side:

\[
(a \circ Q_1(t_i)) \shuffle (b \circ Q_1(t_j)) = ((a \circ Q_1(t_i)) \shuffle b) \circ Q_1(t_j) + (a \shuffle (b \circ Q_1(t_j))) \circ Q_1(t_i).
\]

Thus we prove (1).

An therefore:

\[
f(a'x_i) \shuffle f(b'x_j) = (f(a'x_i) \shuffle f(b')) \circ f(x_j) + (f(a') \shuffle f(b'x_j)) \circ f(x_i).
\]

Using the last equation and formula (5) for the shuffle product (section 1), one can prove that \( f(w \shuffle w') = f(w) \shuffle f(w') \) for \( w, w' \) monomials by induction on the total degree of \( ww' \). We still have to prove (2).

We have seen (remark 4.2) that the series \( Q(t_1), \ldots, Q(t_s) \) generate the algebra \( \mathbb{k}\langle x, z|T_s \rangle \). Hence, the map \( f \) is a linear surjective map and we can deduce (2) from (1).

We now give a formula describing the shuffle product of two series \( Q_n, Q_m \). The formula is a formal analogue of formula (15) of Gon98 for generating series of iterated integrals in the case where the iterated integrals represent MZVs.

**Proposition 4.7.** For \( (p, q) \in (\mathbb{N}^*)^2 \), we have:

\[
Q_p(t_1, \ldots, t_p) \shuffle Q_q(t_{p+1}, \ldots, t_{p+q}) = \sum_{\sigma \in S(p,q)} Q_{p+q}(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(p+q)}).
\]
Proof. For $m \geq 1$, we have:

$$Q_m(t_{a_1}, \ldots, t_{a_m}) = Q_1(t_{a_m}) \circ Q_1(t_{a_{m-1}}) \circ \cdots \circ Q_1(t_{a_1}).$$

Therefore, by the definition of the morphism $f$ and (1) of proposition 5.6:

$$Q_p(t_1, \ldots, t_p) \sqcup Q_q(t_{p+1}, \ldots, t_{p+q}) = f(x_p \cdots x_1 \sqcup x_{p+q} \cdots x_{p+1}).$$

Applying to the right hand side of the above equation: formula (7) of section I the definition of $f$ then the first equation in this proof we find:

$$Q_p(t_1, \ldots, t_p) \sqcup Q_q(t_{p+1}, \ldots, t_{p+q}) = f(x_p \cdots x_1 \sqcup x_{p+q} \cdots x_{p+1}) = \sum_{\sigma \in S(p,q)} f(x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(p+q)}) = \sum_{\sigma \in S(p,q)} Q_1(t_{\sigma^{-1}(1)} \cdots t_{\sigma^{-1}(p+q)}).$$

We have proved the proposition. \qed

5. Orthogonality between $F$ and $ls$ with respect to $\langle -, - \rangle_{\phi}$

In this section we prove theorem 5.1 below stating that $F \subset V$ (of proposition 3.1) is the orthogonal of $ls$ with respect to a perfect pairing $\langle -, - \rangle_{\phi} : V \otimes k(x, z)_{+} \to k$, where $k(x, z)_{+}$ the subspace of elements of $k(x, z)$ with zero constant term. In subsection 5.1 we describe in proposition 5.5 the orthogonal complement $ls_{+}$ of $ls$ with respect to the restriction $\langle -, - \rangle_{+}$ of the canonical paring $\langle -, - \rangle$ of $k(x, z)$ to $k(x, z)_{+}$. In subsection 5.2 we introduce in proposition 5.6 a linear isomorphism of bigraded spaces $\phi : V \to k(x, z)_{+}$ where $V$ is the space defined in subsection 3.1. We then show (see corollary 5.9) that the image of $F$ by $\phi$ is $ls_{+}$. From these results we prove in subsection 5.3 the theorem:

**Theorem 5.1.** The space $F$ is the orthogonal of $ls$ with respect to the bigraded pairing $\langle -, - \rangle_{\phi} : V \otimes k(x, z)_{+} \to k$ given by:

$$\langle v, w \rangle_{\phi} = \langle \phi(v), w \rangle_{+}$$

for $v \in V$ and $w \in k(x, z)_{+}$.

Note that the theorem implies that $V/F$ and the bigraded dual $ls^\vee$ of $ls$ are isomorphic bigraded vector spaces. Since $D_{\bullet}$ and $V/F$ are isomorphic bigraded spaces (see proposition 3.1), we can deduce that $D_{\bullet}$ and $ls^\vee$ are isomorphic as bigraded vector spaces.

5.1. The orthogonal complement of $ls$. The linear map $i : k(Y) \to k(x, z)$, defined by:

$$i(1) = 1 \quad \text{and} \quad i(y_{n_1} \cdots y_{n_m}) = x^{n_1-1}z \cdots x^{n_m-1}z,$$

(28)

for $k \in k$, $m \geq 1$ and $(n_1, \ldots, n_m) \in (\mathbb{N})^m$, is a linear section to the projection $\pi$ defined by (15) subsection 2.1. We have:

$$\pi \circ i = \text{id}_{k(Y)} \quad \text{and} \quad i \circ \pi|_{k(x, z)_{-}} = i \circ \pi|_{k(x, z)_{+}} = \text{id}_{k(x, z)_{+}}.$$

(29)

The canonical perfect pairing $k(x, z)$ and $k(Y)$ as defined in (10) section II will be denoted by $\langle -, - \rangle$ and $\langle -, - \rangle_{Y}$, respectively. One checks that:

$$\langle w, w' \rangle_{Y} = \langle i(w), i(w') \rangle,$$

(30)
for \( w, w' \in \mathbb{k}\langle Y \rangle \).

We denote by \( \shuffle \) and \( \shuffle_Y \) are the shuffle products of \( \mathbb{k}\langle x, z \rangle \) and \( \mathbb{k}\langle Y \rangle \) respectively (see formula (2) and (5), section 1). Let \( \mathbb{k}\langle x, z \rangle_+ \) and \( \mathbb{k}\langle Y \rangle_+ \) be the vector spaces of elements with zero constant term of \( \mathbb{k}\langle x, z \rangle \) and \( \mathbb{k}\langle Y \rangle \), respectively:

**Proposition 5.2.** For \( w \) in \( \mathbb{k}\langle x, z \rangle \), we have the following equivalences:

1) \( \Delta(w) = w \otimes 1 + 1 \otimes w \) if and only if \( w \in (\mathbb{k} \oplus \mathbb{k}\langle x, z \rangle_+^{w2})^\perp \),

2) \( \Delta_Y(\pi(w)) = \pi(w) \otimes 1 + 1 \otimes \pi(w) \) if and only if \( w \in i(\mathbb{k} \oplus \mathbb{k}\langle Y \rangle_+^{w2})^\perp \),

where \( A^\perp \) is the orthogonal complement of \( A \) with respect to \( \langle -, - \rangle \) for \( A \subset \mathbb{k}\langle x, z \rangle \) and \( B^* = B \ast B \) for \( \ast \) a given product.

**Proof.** The proof follows from the formula (11) of section 1 and formula (30) of this section. \( \square \)

Both spaces \( \mathfrak{ls}' \) and \( \mathfrak{ls} \) are subspaces of \( \mathbb{k}\langle x, z \rangle_+ \). We denote by \( \langle -, - \rangle_+ \) the restriction of \( \langle-, - \rangle \) to \( \mathbb{k}\langle x, z \rangle_+ \). The map \( \langle -, - \rangle_+ \) is a perfect pairing.

**Corollary 5.3.** Let \( \mathfrak{ls}^l_+ \) be the orthogonal complement of \( \mathfrak{ls}' \subset \mathbb{k}\langle x, z \rangle_+ \) with respect to \( \langle -, - \rangle_+ \).

\[ \mathfrak{ls}^l_+ = \mathbb{k}\langle x, z \rangle_+^{w2} + i(\mathbb{k}\langle Y \rangle_+^{w2})^\perp, \]

where \( B^* = B \ast B \) for \( \ast \) a given product.

**Proof.** The corollary is a direct consequence of proposition 5.2 and the definition of \( \mathfrak{ls}' \) (definition 2.1). \( \square \)

**Lemme 5.4.** Denote by \( \mathbb{k}\langle x, z \rangle_1 \) the depth one part of \( \mathbb{k}\langle x, z \rangle \) and by \( A \) the depth one part of \( \mathbb{k}\langle x, z \rangle_+^{w2} \). We have:

\[ \mathbb{k}\langle x, z \rangle_1 = K \oplus A = \mathfrak{ls}'_1 \oplus A, \]

where \( K := \bigoplus_{n \geq 0} \mathbb{k} \cdot x^n z \).

**Proof.** We first prove that \( \mathbb{k}\langle x, z \rangle_1 = K + A \). To do so we show by induction on \( m \) the hypothesis:

\[ x^n z x^m \equiv_A \lambda_{m,n} x^{n+m} z, \quad (H_m), \]

for \( \lambda_{m,n} \in \mathbb{k} \) and where \( w \equiv_A w' \) means \( w - w' \in A \). The hypothesis \((H_0)\) is true. We assume that \((H_m)\) is true. Applying lemma 4.4 for \( a = x, b = x^n, c = x^m \) and \( y = z \) we find:

\[ x^n z x^m \shuffle x = (n+1)x^{n+1} z x^m + (m+1)x^n z x^{m+1}. \]

Therefore

\[ x^n z x^{m+1} \equiv_A -\frac{n+1}{m+1} x^{n+1} z x^m (H_m) \equiv_A -\frac{n+1}{m+1} \lambda_{m,n+1} x^{n+m+1} z. \]

This shows \((H_{m+1})\). We have proved that the hypothesis \((H_k)\) is true for \( k \in \mathbb{N} \) as a consequence we have \( \mathbb{k}\langle x, z \rangle_1 = K + A \). It follows from the decomposition \( \mathfrak{ls}^l_+ = \mathbb{k}\langle x, z \rangle_+^{w2} + i(\mathbb{k}\langle Y \rangle_+ \) (corollary 5.3) that the depth 1 part \( \mathfrak{ls}'_1 \) of \( \mathfrak{ls}' \) and \( A \) are in direct summand. Indeed, the depth 1 part of \( i(\mathbb{k}\langle Y \rangle_+ \) is trivial. From the decomposition \( \mathbb{k}\langle x, z \rangle_1 = K + A \) we deduce
that the weight $n$ part $Is_1^n$ of $Is_1'$ is of dimension at most 1 since the weight $n$ part of $K$ is of dimension 1. It is not hard to check that $\text{ad}(x)^n(y) \in Is_1^n$ and therefore the dimension of $Is_1^n$ is equal to 1. From this, the fact that $Is_1'$ and $A$ are in direct summand, the decomposition $k \langle x, z \rangle_1 = K + A$, and the fact that the dimension of the part of depth $n$ of $K$ is 1 we deduce the lemma.

□

Proposition 5.5. Set $K_{\text{even}} = \bigoplus_{n \geq 1} k \cdot x^{2n-1} z$ and denote by $Is_{\perp}$ the orthogonal complement of $Is \subset k \langle x, z \rangle_+$ with respect to $\langle -, - \rangle_+$.

$$Is_{\perp} = (k \cdot x + K_{\text{even}} + k \langle x, z \rangle_+^{\perp}) + i(k \langle Y \rangle_+^{\perp} + 2).$$

Proof. We have seen in remark 2.4 that:

$$Is' = Is \oplus k \cdot x \oplus (Is_1')_{\text{even}},$$

where $(Is_1')_{\text{even}} = \bigoplus Is_1^{2n}$. This summand is an orthogonal summand. Hence:

$$Is_{\perp} = Is_{\perp} \oplus k \cdot x \oplus (Is_1')_{\text{even}} \cap \text{corollary } 5.3 = k \cdot x \oplus (Is_1')_{\text{even}} \oplus (k \langle x, z \rangle_+^{\perp} + i(k \langle Y \rangle_+^{\perp} + 2). \quad (31)$$

On the other hand we deduce from lemma 5.4 that $K_{\text{even}} + k \langle x, z \rangle_+^{\perp} = (Is_1')_{\text{even}} + k \langle x, z \rangle_+^{\perp}$. We prove the proposition by replacing in (31) the summand $(Is_1')_{\text{even}} \oplus k \langle x, z \rangle_+^{\perp}$ by $K_{\text{even}} \oplus k \langle x, z \rangle_+^{\perp}$.

□

5.2. A map $\phi$ sending $F$ to the orthogonal complement of $Is$. We recall that the vector space $V$ introduced in section 3 is freely generated by the elements of the sets:

$$I = \{I(n_1, \ldots, n_m) | m \geq 1, n_1, \ldots, n_m \geq 1\}$$

and

$$I' = \{I'(n_0, n_1, \ldots, n_m) | m \geq 0, n_0, \ldots, n_m \geq 1\}.$$

Proposition 5.6. The linear map $\phi : V \to k \langle x, z \rangle_+$ given by:

$$\phi(I(n_1, \ldots, n_m)) = x^{n_1-1}z \cdots x^{n_{m-1}z} \cdots x^{n_{m-1}z} x \cdots x^{n_0-1z} z, \quad \phi(I'(n_0, n_1, \ldots, n_m)) = x^{n_1-1z} \cdots x^{n_{m-1}z} x^{n_0-1z} z,$$

for $m \geq 1, (n_0, \ldots, n_m) \in (\mathbb{N}^*)^{m+1}$ and

$$\phi(I'(n_0)) = x^{n_0-1z} z,$$

for $n_0 \in \mathbb{N}^*$ is an isomorphism of bigraded vector spaces.

Proof. For $k \in \mathbb{N}$, set $W_k = (k \oplus k \langle x, z \rangle) x^k$ for $k \geq 1$ and $W_0 = k \langle x, z \rangle$. We have:

$$k \langle x, z \rangle_{> 0} = \bigoplus_{k \geq 1} W_k.$$

Let $L$ and $M$ be the endomorphisms of $k \langle x, z \rangle$ defined by:

$$L(w x^k) = k w x^k, \quad M(w x^k) = w x^{k-1} \bigcup x,$$

and $L(w) = M(w) = w$,

for $w \in k \oplus k \langle x, z \rangle z, k \in \mathbb{N}^*$. The map $L$ is an isomorphism stabilising $W_k$ (for $k \geq 0$) and we can verify the following:

$$(M - L)(W_0) = 0, \quad \text{and} \quad (M - L)(W_k) \subset W_{k-1}.$$
for $k \geq 1$. Therefore, $ML^{-1}$ is a unipotent endomorphism (with respect to the increasing flag obtained out of the spaces $W_k$) and both $ML^{-1}$ and $M$ are invertible. Notice that 

$$\phi = M \circ \phi',$$

where $\phi'$ is given by:

$$\phi'(I(n_1, \ldots, n_m)) = \phi(I(n_1, \ldots, n_m)), \quad \phi'(I'(n_0, n_1, \ldots, n_m)) = x^{n_1} \cdots x^{n_m} y^{n_0},$$

for $m \geq 1$ and $(n_0, \ldots, n_m) \in (\mathbb{N}^*)^{m+1}$ and

$$\phi'(I'(n_0)) = x^{n_0},$$

for $n_0 \in \mathbb{N}^*$. The family $I \cup I'$ is a basis of $V$ and one checks readily that the family $

\{ \phi'(v) \}_{v \in I \cup I'}$ is a basis of $\mathbb{k}(x, z)_+$. Therefore $\phi' : V \to \mathbb{k}(x, z)_+$ is an isomorphism. Since $M$ is also an isomorphism and $\phi = M \circ \phi'$, we conclude that $\phi$ is an isomorphism. This proves the proposition.

\[\square\]

**Corollary 5.7.** We can decompose an element $w \in \mathbb{k}(x, z)_+$ into the sum:

$$w = w_1 z + w_2 \mathbb{1} x,$$

with $w_1$ and $w_2$ in $\mathbb{k}(x, z)$. Moreover, the decomposition is unique.

**Proposition 5.8.** Let $\phi$ be the isomorphism of proposition 5.6.

1) The image of $V_1$ with respect to $\phi$ is $i(\mathbb{k}(Y)^{\mathfrak{m}^2})$.

2) The image of $V_2$ with respect to $\phi$ is $i(\mathbb{k}(x, z)_+^2)$.

3) The image of $V_2 + V'$ with respect to $\phi$ is $\mathbb{k}(x, z)_+^{\mathfrak{m}^2}$.

**Proof.** Let $T$ be a series with values in a vector space $A$ and $f : A \to B$ a morphism of vector spaces. We denote by $T^f$ the series with values in $B$ obtained by applying $f$ to the coefficients of $T$. We recall that $V_1$ is the smallest subspace of $V$ such that for all $(p, q) \in (\mathbb{N}^*)^2$, the image of series:

$$M_{p, q} := \sum_{\sigma \in S(p, q)} \{ t_{\sigma^{-1}(1)} : \cdots : t_{\sigma^{-1}(p+q)} : t_{p+q+1} \},$$

in $V/V_1[[t_1, \ldots, t_{p+q+1}]]$ is 0. Hence, we can characterize the space $\phi(V_1)$ as the smallest subspace $X_1$ of $\mathbb{k}(x, z)_+$ such that for all $(p, q) \in (\mathbb{N}^*)^2$ $M_{p, q}^\phi$ is sent to 0 in $\mathbb{k}(x, z)_+/X_1[[t_1, \ldots, t_{p+q+1}]]$.

Note that $M_{p, q}^\phi$ is with values in $\mathbb{k}(x, z)$. Hence by (23) $M_{p, q} = M_{p, q}^{\text{fin}}$. Let us first compute $M_{p, q}^{\pi}$. Set $m = p + q$. We have:

$$M_{p, q}^{\pi} = \sum_{\sigma \in S(p, q)} \sum_{n_1, \ldots, n_m \geq 1} y_{n_m} \cdots y_{n_1} (t_{\sigma^{-1}(1)} - t_{m+1})^{n_1} \cdots (t_{\sigma^{-1}(m)} - t_{m+1})^{n_m}

= \sum_{\sigma \in S(p, q)} \sum_{k_1, \ldots, k_m \geq 1} y_{k_{\sigma^{-1}(m)}} \cdots y_{k_{\sigma^{-1}(1)}} (t_1 - t_{m+1})^{k_1} \cdots (t_m - t_{m+1})^{k_m},$$

where the second equality is obtained by setting $k_i = n_{\sigma(i)}$. Since:

$$\sum_{\sigma \in S(p, q)} y_{k_{\sigma^{-1}(m)}} \cdots y_{k_{\sigma^{-1}(1)}} = y_{k_m} \cdots y_{k_{p+1}} \mathbb{1} y_{k_p} \cdots y_{k_1},$$
(see [7], section 11), we find:
\[ M_{p,q}^\pi = \sum_{k_1, \ldots, k_m \geq 0} (y_{k_1} \cdots y_{k_{p-1}} \uplus y_p \cdots y_{k_1})(t_1 - t_{m+1})^{k_1} \cdots (t_m - t_{m+1})^{k_m} \]
and since \( M_{p,q} = M_{p,q}^{\pi \circ \phi} \), we have:
\[ M_{p,q} = M_{p,q}^{\pi \circ \phi} = (M_{p,q}^\pi)^i = \sum_{k_1, \ldots, k_m \geq 0} i(y_{k_1} \cdots y_{k_{p-1}} \uplus y_p \cdots y_{k_1})(t_1 - t_{m+1})^{k_1} \cdots (t_m - t_{m+1})^{k_m}. \]

From this and the characterization of \( \phi(V_1) \) given before we deduce that \( \phi(V_1) = i(\kappa(Y)_{+}^{\pi \circ \phi}^2) \).
This proves (1).

We now show (2). The subspace \( V_2 \) was defined as the smallest subspace of \( V \) such that for all \( (p, q) \in (\mathbb{N}^*)^2 \), the image of the series:
\[ N_{p,q} := \sum_{\sigma \in S(p,q)} \{ t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(p+q)} \}, \]
is 0 in \( V/V_2[[t_1, \ldots, t_{p+q}]] \). One checks that:
\[ N_{p,q}^\phi = \sum_{\sigma \in S(p,q)} Q(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(p+q)}), \]
where \( Q \) is the series as in of formula (24) of section 4. Hence, by proposition 14.7
\[ N_{p,q}^\phi = Q(t_1, \ldots, t_p) \uplus Q(t_{p+1}, \ldots, t_{p+q}) \]
\[ = \sum_{n_1, \ldots, n_m \geq 0} (x^{n_p}z \cdots x^{n_1}z \uplus x^{n_p+1}z \cdots x^{n_p+q}z) R_p(n_1, \ldots, n_{p+q}), \]
where \( R_p(n_1, \ldots, n_{p+q}) = R_1(n_1, \ldots, n_p) R_{p+1}(n_{p+1}, \ldots, n_{p+q}) \) with
\[ R_1(n_1, \ldots, n_p) = t_1^{n_1}(t_1 + t_2)^{n_2} \cdots (t_1 + \cdots + t_p)^{n_p}, \]
and
\[ R_{p+1}(n_{p+1}, \ldots, n_{p+q}) = t_{p+1}^{n_{p+1}}(t_{p+1} + t_{p+2})^{n_{p+2}} \cdots (t_{p+1} + \cdots + t_{p+q})^{n_{p+q}}. \]
Since the monomials \( R_p(n_1, \ldots, p+q) \) are linearly independent, we deduce from (33) by reasoning as in the proof of (1), that \( \phi(V_2) = (\kappa(x, z)z)^{\pi \circ \phi^2} \). We have proved (2).

We now prove (3). We have shown that \( \phi(V_2) = (\kappa(x, z)z)^{\pi \circ \phi^2} \) and it is easy to check that \( \phi(V') = \kappa(x, z)_+ \uplus x \). Hence:
\[ \phi(V_2 + V') = (\kappa(x, z)z)^{\pi \circ \phi^2} + \kappa(x, z)_+ \uplus x. \]
This shows that \( \phi(V_2 + V') \) is a subset of \( \kappa(x, z)^{\pi \circ \phi^2} \). We show that the reverse inclusion holds.
Let us decompose two elements \( w, w' \in \kappa(x, z)_+ \) into sums as in corollary 5.7
\[ w = w_1z + w_2 \uplus x, \quad w' = w'_1z + w'_2 \uplus x, \]
with \( w_1, w_2, w'_1, w'_2 \in \kappa(x, z) \). Using these decompositions we get:
\[ w \uplus w' = w_1z \uplus w'_1z + b \uplus x, \]
where \( b = w_2 \uplus w'_2 \uplus x + w_2 \uplus w'_1z + w_1z \uplus w'_2 \). Since \( b \in \kappa(x, z)_+ \), we deduce that:
\[ w \uplus w' \in \kappa(x, z)^{\pi \circ \phi^2} + \kappa(x, z)z \uplus x = \phi(V_2 + V'), \]
and therefore \( \kappa(x, z)^{\pi \circ \phi^2} \subset \phi(V_2 + V') \). The point (3) is proved and so is the proposition. \( \square \)
Corollary 5.9. Let $F$ be the subspace of $V$ as in proposition 5.1. The image of $F$ with respect to $\phi$ is $I_{s}^{\perp +}$.

Proof. We recall that $F = V_{1} + V_{2} + V' + V'' + \mathbb{k} \cdot x$. We deduce from proposition 5.8 that $\phi(V_{1} + V_{2} + V') = \mathbb{k} \langle x, z \rangle_{+}^{\omega} + i(\mathbb{k} \langle Y \rangle_{+}^{\omega} \cdot 2)$. On the other hand, we have $\phi(V'') = K_{\text{even}} (K_{\text{even}}$ was introduced in proposition 5.5) and $\phi(\mathbb{k} \cdot I'(1)) = \mathbb{k} \cdot x$. Hence

$$\phi(F) = \mathbb{k} \langle x, z \rangle_{+}^{\omega} + i(\mathbb{k} \langle Y \rangle_{+}^{\omega} \cdot 2) + K_{\text{even}} + \mathbb{k} \cdot x.$$ 

The last summand is exactly $I_{s}^{\perp +}$ by proposition 5.5. This proves the corollary. 

5.3. Proof of the orthogonality between $F$ and $I_{s}$. We recall that the pairing $\langle - , - \rangle_{\phi} : V \otimes \mathbb{k} \langle x, z \rangle_{+} \rightarrow \mathbb{k}$ of theorem 5.1 is given by:

$$\langle v , w \rangle_{\phi} = \langle \phi(v) , w \rangle_{+}$$

for $v \in V$ and $w \in \mathbb{k} \langle x, z \rangle_{+}$. The orthogonal $I_{s}^{\perp \phi}$ of $I_{s}$ with respect to $\langle - , - \rangle_{\phi}$ is the preimage by $\phi$ of the orthogonal complement $I_{s}^{\perp +}$ of $I_{s}$ with respect to $\langle - , - \rangle_{+}$. Since $\phi : V \rightarrow \mathbb{k} \langle x, z \rangle_{+}$ is an isomorphism (proposition 5.6) and $\phi(F) = I_{s}^{\perp +}$ (corollary 5.9) we have $I_{s}^{\perp \phi} = F$. This proves theorem 5.1.

6. The dihedral Lie cobracket and the pullback by $\phi$ of the Ihara cobracket

Throughout this section, we denote by $\langle - , - \rangle_{+}$ and $\langle - , - \rangle_{+}$ the restrictions of the Ihara bracket $\langle - , - \rangle$ and the canonical pairing $\langle - , - \rangle$ to $\mathbb{k} \langle x, z \rangle_{+}$. We also denote by $\langle - , - \rangle^{*}_{+}$ (the Ihara cobracket) the adjoint of $\langle - , - \rangle_{+}$ with respect to $\langle - , - \rangle_{+}$ and $\langle - , - \rangle^{*}_{+}$ where the latter is the perfect bigraded pairing over $\mathbb{k} \langle x, z \rangle^{*}_{2}$ obtained out of $\langle - , - \rangle$ using formula (14) in section 11. In this section we prove the following theorem:

Theorem 6.1. 1) $F$ is a coideal for $\phi^{*} \langle - , - \rangle^{*}_{+}$, the pullback of $\langle - , - \rangle^{*}_{+}$ by $\phi$:

$$\phi^{*} \langle - , - \rangle^{*}_{+}(F) \subset F \otimes V + V \otimes F$$

and $\phi^{*} \langle - , - \rangle^{*}_{+}$ induces a Lie cobracket $\delta_{V/F} : V/F \rightarrow V/F \otimes V/F$.

2) We have the following isomorphisms of bigraded Lie coalgebras:

$$D_{\bullet \cdot} \eta_{Y} (W, \delta) \overset{\tilde{i}}{\rightarrow} (V/F, \delta_{V/F})$$

where $\eta, \tilde{i}$ are the isomorphisms of bigraded vector spaces of proposition 6.4.

The theorem will be used to show in the next section the main results (a) and (b) announced in the introduction. In subsection 6.1 we decompose the adjoint $\langle - , - \rangle^{*}$ of $\langle - , - \rangle$ with respect to $\langle - , - \rangle$ and $\langle - , - \rangle^{*}_{2}$ (the latter is naturally obtained out of $\langle - , - \rangle$) into a "sum" of linear maps and we describe these maps. We use this decomposition in subsection 6.2 to show that $U = \mathbb{k} \langle x, z \rangle_{2} \cdot \{ x \}$ is a coideal for $\phi^{*} \langle - , - \rangle^{*}_{+}$ (corollary 6.9). In subsection 6.3 we split $\phi$ into two automorphisms $M : \mathbb{k} \langle x, z \rangle \rightarrow \mathbb{k} \langle x, z \rangle$, $\phi_{0} : V \rightarrow \mathbb{k} \langle x, z \rangle_{+}$ and we introduce for $m \geq 1$ the series $(t_{1} : \cdots : t_{m+1})$ corresponding to the image of $(t_{1} : \cdots : t_{m+1})$ (used in the previous sections) by $\phi$. We also compute for technical reasons $M^{-1}((t_{1} : \cdots : t_{m+1}))$ up to elements in $\phi_{0}(U)$. In subsection 6.4 we compute the image of the series $(t_{1} : \cdots : t_{m+1})$ (up to elements in $H_{0} = \phi_{0}(U) \otimes \mathbb{k} \langle x, z \rangle + \mathbb{k} \langle x, z \rangle_{+} \otimes \phi_{0}(U)$) with respect to the pullbacks by $M$ of the maps in the decomposition of $\langle - , - \rangle^{*}$ given in the first subsection to deduce a
formula for $M^*\{−,−\}^*((t_1:⋅⋅⋅:t_{m+1}))$ up to elements in $H_0$. We use these computations and the fact that $U$ is a coideal for $φ^*\{−,−\}^+_+$ (from subsection 6.2) in subsection 6.5 in order to compare $φ^*\{−,−\}^+_+$ to $δ : W → W ⊗ W$ the map inducing the Lie cobracket $δ$ over $W/W_W → W/W_W ⊗ W/W_W$ and to prove theorem 6.1.

In this section, the adjoints will be taken with respect to the pairings $⟨−,−⟩$ and $⟨−,−⟩^{⊙2}$ ($⟨−,−⟩^{⊙2}$ is naturally obtained out of $⟨−,−⟩$ as in Formula (14), section 1 or their restrictions depending on the linear map we consider. The adjoint of a linear map $f$ is denoted by $f^*$.

6.1. Decomposition the Ihara cobracket.

**Definition 6.2.** 1) For $w = x^m z \cdots x^n z x^{n_0}$ and $1 ≤ i$, we define four monomials $w_i^R, w_i^L$ and $w_{i-1}^R, w_{i-1}^L$:

$$w_i^R = x^{m-i} z \cdots x^n z x^{n_0}, \quad w_i^L = x^m z \cdots x^{n-i+1} z,$$

$$w_{i-1}^R = z x^{n-i} z \cdots x^n z x^{n_0}, \quad w_{i-1}^L = x^m z \cdots x^{n+i} z x^{n_i}, \quad \text{if } m ≥ i,$$

and

$$w_i^R = w_{i-1}^L = w_{i+1}^R = w_{i-1}^L = 0, \quad \text{otherwise.}$$

2) For $i ≥ 1$, we define the linear maps $d_i^+, d_i^-$ from $k(x, z)^{⊙2}$ to $k(x, z)$ by the following:

$$d_i^+(w' ⊗ w) = w_i^L w' w_i^R, \quad d_i^-(w' ⊗ w) = w_i^L w' w_i^R$$

for $w, w'$ unitary monomials of $k(x, z)$.

**Remark 6.3.** Consider the linear map $d : k(x, z)^{⊙2} → k(x, z), w ⊗ w' → d_w(w')$ where $d_w$ is the derivation of $k(x, z)$ given by $d_w(x) = 0, d_w(z) = [z, w]$ (as in subsection 2.2). We can show using Formula (13) section 7 that:

$$d(w' ⊗ w) = \sum_{1 ≤ i ≤ m} (d_i^+ - d_i^-)(w' ⊗ w),$$

for $w ⊗ w'$ of depth $m$.

We denote by $τ$ the involution of $k(x, z) ⊗ k(x, z)$ sending $w ⊗ w'$ to $w' ⊗ w$, by $Δ_w$ the deconcatenation prooduct of $k(x, z)$ (see [12], section 11 for the definition of $Δ_w$) and by $[−,−] : k(x, z)^{⊙2} → k(x, z)$ the canonical Lie bracket on $k(x, z)$ ($[w, w'] = ww' - w'w$).

**Lemme 6.4.** Take $τ, Δ_w$ and $[−,−]$ as in the previous paragraph:

1) Let $[−,−]^*$ be the adjoint of $[−,−]$. We have:

$$[−,−]^* = (id - τ) ∘ Δ_w.$$

2) The adjoint of the Ihara bracket can be expressed as follows:

$$[−,−]^* = (id - τ) ∘ (Δ_w + d^*)$$

where $d : k(x, z)^{⊙2} → k(x, z)$ is as in the previous remark.
Proposition 6.5. For \( d \) where \( k \) the free associative algebra \( (2) \), this proves (1). We prove 1.1 (section 1), we have:
\[
\{−, −\} = [−, −] + d \circ (id − τ),
\]
where \( d \) is as in the proposition. Hence:
\[
\{−, −\}^* = [−, −]^* + (d \circ (id − τ))^* = [−, −]^* + (id − τ) \circ d^*.
\]
We derive readily (2) from the equation above using (1).

\[\square\]

Proposition 6.6. For \( w \in k\langle x, z \rangle \) of depth \( m \geq 1 \), we have:
\[
\{−, −\}^*(w) = (id − τ) \circ (\Delta_\omega + \sum_{1 \leq i \leq m} d^*_{i, +} − d^*_{i, −})(w),
\]
where \( τ \) and \( \Delta_\omega \) are as in the previous lemma and \( d_{i, +}, d_{i, −} \) are the maps of definition 6.2.

Proof. We have seen in remark 6.3 that:
\[
d(w' \otimes w) = \sum_{1 \leq i \leq m} (d_{i, +} − d_{i, −})(w' \otimes w),
\]
for \( w \otimes w' \) of depth \( m \). Since the linear maps \( d, d_{i, +}, d_{i, −} \) (for \( i \geq 1 \)) are depth graded maps, we have:
\[
d^*(w'') = \sum_{1 \leq i \leq m} (d^*_{i, +} − d^*_{i, −})(w''),
\]
for \( w'' \) of depth \( m \). We derive from this the equation of the proposition using (2) of lemma 6.4 We have proved the proposition.

Proposition 6.6. For \( i \geq 1 \) and \( w \) a monomial of \( k\langle x, z \rangle_+ \), we have:
\[
d^*_{i, +}(w) = (1 \otimes w^L_{i, +})\Delta_\omega(w^R_{i, +}), \tag{34}
\]
\[
d^*_{i, −}(w) = \Delta^\text{op}_\omega(w^L_{i, −})(1 \otimes w^R_{i, −}), \tag{35}
\]
where \( d_{i, +}, d_{i, −}, w^L_{i, +}, w^R_{i, +}, w^L_{i, −} \) and \( w^R_{i, −} \) are as in definition 6.2.

Proof. We first prove (1). Take two monomials \( a, b \) of \( k\langle x, z \rangle \) and \( w \) as in the proposition. We have \( \langle d_{i, +}(a \otimes b), w \rangle = \langle b^L_{i, +}ab^R_{i, +}, w \rangle \). Since \( b^L_{i, +} \) and \( w^L_{i, +} \) are both of depth \( i \) ending with \( z \) we have:
\[
\langle d_{i, +}(a \otimes b), w \rangle \otimes^2 = \langle b^L_{i, +}w^L_{i, +}, ab^R_{i, +}w^R_{i, +} \rangle.
\]
By proposition 6.1 (section 1), \( \mu^* = \Delta_\omega \) and therefore \( \langle ab^R_{i, +}, w^R_{i, +} \rangle = \langle a \otimes b^R_{i, +}, \Delta_\omega(w^R_{i, +}) \rangle \otimes^2 \). Using this expression of \( \langle ab^R_{i, +}, w^R_{i, +} \rangle \), we can derive:
\[
\langle d_{i, +}(a \otimes b), w \rangle \otimes^2 = \langle b^L_{i, +}w^L_{i, +}, (a \otimes b^R_{i, +}, \Delta_\omega(w^R_{i, +})) \rangle \otimes^2
\]
\[
= \langle a \otimes b^R_{i, +}b^R_{i, +}, (1 \otimes w^L_{i, +})\Delta_\omega(w^R_{i, +}) \rangle \otimes^2
\]
\[
= \langle a \otimes b, (1 \otimes w^L_{i, +})\Delta_\omega(w^R_{i, +}) \rangle \otimes^2
\]
\[
27
This proves (34). We now prove (35). Take two monomials $a, b$ and $w$ as in the proposition. We have:

$$\langle d_{i,-}(a \otimes b), w \rangle^{\otimes 2} = \langle b^{L}_{i,-}a b^{R}_{i,-}, w^{L}_{i,-}w^{R}_{i,-} \rangle^{\otimes 2}. $$

Note that $w^{R}_{i,-}$ and $b^{R}_{i,-}$ are both monomials of depth $i$ and they both start with $z$. We also have $\langle b^{L}_{i,-}a, w^{L}_{i,-} \rangle = (a \otimes b^{L}_{i,-}, \Delta^{op}_{\omega}(w^{L}_{i,-}))^{\otimes 2}$ (indeed $\mu^{*} = \Delta_{\omega}$). Therefore:

$$\langle d_{i,-}(a \otimes b), w \rangle^{\otimes 2} = \langle b^{L}_{i,-}a, w^{L}_{i,-} \rangle \langle b^{R}_{i,-}, w^{R}_{i,-} \rangle $$

$$= (a \otimes b^{L}_{i,-}, \Delta^{op}_{\omega}(w^{L}_{i,-}))^{\otimes 2} \langle b^{R}_{i,-}, w^{R}_{i,-} \rangle $$

$$= (a \otimes b, \Delta^{op}_{\omega}(w^{L}_{i,-})(1 \otimes w^{R}_{i,-}))^{\otimes 2}. $$

This proves (35). We have proved the proposition.  

6.2. A coideal for the Ihara cobracket.

**Proposition 6.7.** For $w \in \mathbb{k}\langle x, z \rangle$ and $i > 1$, we have:

$$\Delta_{\omega}(w \shuffle x) = \Delta_{\omega}(w) \shuffle (1 \otimes x + x \otimes 1).$$  \hspace{1cm} (36)

$$d^{*}_{i,+}(w \shuffle x) = d^{*}_{i,+}(w) \shuffle (1 \otimes x + x \otimes 1). $$  \hspace{1cm} (37)

$$d^{*}_{i,-}(w \shuffle x) = d^{*}_{i,-}(w) \shuffle (1 \otimes x + x \otimes 1). $$  \hspace{1cm} (38)

**Proof.** The equation (36) is due to the fact that $\Delta_{\omega}$ is morphism for the shuffle product and that $\Delta_{\omega}(x) = 1 \otimes x + x \otimes 1$. We now prove (37). For $a, b \in \mathbb{k}\langle x, z \rangle$, one can show that $ab \shuffle x = (a \shuffle x)b - axb + a(b \shuffle x)$. Hence, $w \shuffle x = (w^{L}_{i,+} \shuffle x - w^{L}_{i,+}x)w^{R}_{i,+} + w^{L}_{i,+}(w^{R}_{i,+} \shuffle x)$. Since $w^{L}_{i,+} \shuffle x - w^{L}_{i,+}x$ is a sum of monomials of depth $i$ ending with $z$ and the map $(a, b) \rightarrow (1 \otimes a)\Delta_{\omega}(b)$ is multilinear we deduce from (34):

$$d^{*}_{i,+}(w \shuffle x) = (1 \otimes w^{L}_{i,+} \shuffle x - w^{L}_{i,+}x)\Delta_{\omega}(w^{R}_{i,+}) + (1 \otimes w^{L}_{i,+})\Delta_{\omega}(w^{R}_{i,+} \shuffle x). $$

Using that $\Delta_{\omega}$ is a morphism for shuffle product and that $\Delta_{\omega}(x) = x \otimes 1 + 1 \otimes x$, we derive:

$$d^{*}_{i,+}(w \shuffle x) = (1 \otimes (w^{L}_{i,+} \shuffle x - w^{L}_{i,+}x))\Delta_{\omega}(w^{R}_{i,+}) + (1 \otimes w^{L}_{i,+})(\Delta_{\omega}(w^{R}_{i,+} \shuffle (x \otimes 1 + 1 \otimes x)))$$  \hspace{1cm} (39)

Now applying the formula $ab \shuffle x = (a \shuffle x)b - axb + a(b \shuffle x)$ to the sum:

$$C := (1 \otimes (w^{L}_{i,+} \shuffle x - w^{L}_{i,+}x))\Delta_{\omega}(w^{R}_{i,+}) + (1 \otimes w^{L}_{i,+})(\Delta_{\omega}(w^{R}_{i,+} \shuffle 1 \otimes x), $$

we conclude that $C = ((1 \otimes w^{L}_{i,+})\Delta_{\omega}(w^{R}_{i,+})) \shuffle 1 \otimes x$. Replacing $C$ by $((1 \otimes w^{L}_{i,+})\Delta_{\omega}(w^{R}_{i,+})) \shuffle 1 \otimes x$ in (39), we find:

$$d^{*}_{i,+}(w \shuffle x) = ((1 \otimes w^{L}_{i,+})\Delta_{\omega}(w^{R}_{i,+})) \shuffle (1 \otimes x + x \otimes 1). $$

This proves (37) since $d^{*}_{i,+}(w) = (1 \otimes w^{L}_{i,+})\Delta_{\omega}(w^{R}_{i,+})$ by (34) (proposition 6.6).

We still have to prove (38). For that we use the equation:

$$x \shuffle w = (x \shuffle w^{L}_{i,-})w^{R}_{i,-} + w^{L}_{i,-}(x \shuffle w^{R}_{i,-} - xw^{R}_{i,-}). $$  \hspace{1cm} (A)
Since $x \sqcup w^R_{i,-} - x w^R_{i,-}$ is a sum of monomials starting with $z$ and $(a, b) \mapsto \Delta^{op}(a)(1 \otimes b)$ is multilinear we can apply \[35\] to right hand side of the above equation to derive the following:

$$d_{i,-}^\ast(w \sqcup x) = \Delta^{op}_w(x \sqcup w^L_{i,-})(1 \otimes w^R_{i,-}) + \Delta^{op}_w(w^L_{i,-})(1 \otimes (x \sqcup w^R_{i,-} - x w^R_{i,-}))$$

$$= (1 \otimes x + x \otimes 1) \sqcup \Delta^{op}_w(w^L_{i,-}))(1 \otimes w^R_{i,-}) + \Delta^{op}_w(w^L_{i,-}))(1 \otimes (x \sqcup w^R_{i,-} - x w^R_{i,-})).$$

Where we use to prove (B) the fact that $\Delta^{op}_w$ is a morphism for shuffle product and $\Delta^{op}_w(x) = 1 \otimes x + x \otimes 1$. Using(A), we find that:

$$(1 \otimes x) \sqcup (\Delta^{op}_w(w^L_{i,-}))(1 \otimes w^R_{i,-}) + \Delta^{op}_w(w^L_{i,-}))(1 \otimes (x \sqcup w^R_{i,-} - x w^R_{i,-})) = (1 \otimes x \sqcup (\Delta^{op}_w(w^L_{i,-}))(1 \otimes w^R_{i,-})).$$

Furthermore, we have:

$$(x \otimes 1) \sqcup (\Delta^{op}_w(w^L_{i,-}))(1 \otimes w^R_{i,-}) = (x \otimes 1) \sqcup (\Delta^{op}_w(w^L_{i,-}))(1 \otimes w^R_{i,-}).$$

combining the last three equations, we get:

$$d_{i,-}^\ast(w \sqcup x) = (\Delta^{op}_w(w^L_{i,-}))(1 \otimes w^R_{i,-}) \sqcup (x \otimes 1 + 1 \otimes 1).$$

We derive \[38\] from this equation using \[35\] (proposition 6.6). The proposition is proved. \[\square\]

**Proposition 6.8.** The space $\phi(U) = k\langle x, z \rangle \sqcup x$ is a coideal for $\{-, -\}_+$, i.e:

$$\{-, -\}_+^\ast(\phi(U)) = \phi(U) \otimes k\langle x, z \rangle_+ + k\langle x, z \rangle_+ \otimes \phi(U).$$

**Proof.** To prove the proposition we show that:

$$\{-, -\}_+^\ast(w \sqcup x) \in H := \phi(U) \otimes k\langle x, z \rangle_+ + k\langle x, z \rangle_+ \otimes \phi(U), \quad (A)$$

for all monomials $w$ of $k\langle x, z \rangle$. If $w$ is of depth 0, $w \sqcup x$ is of depth 0. Since $\{-, -\}_+^\ast$ is a map of bigraded spaces with respect to weight and depth $\{-, -\}_+^\ast(w \sqcup x)$ lies in the depth 0 part $K' := k[x]_+ \otimes k[x]_+$ of $k\langle x, z \rangle \otimes^2$, where $k[x]_+$ is the subspace (of $k\langle x, z \rangle$) of polynomials in $x$ with no constant term. Since $K'$ is a subset of $H$, this shows (A) for $w$ of depth 0.

We now show (A) for $w$ of depth $m \geq 1$. Applying the decomposition of $\{-, -\}_+$ in proposition 6.5 we find:

$$\{-, -\}_+^\ast(w \sqcup x) = (id - \tau)(\Delta_w + \sum_{i=1}^m d_{i,+} - d_{i,-})(w \sqcup x).$$

Moreover, $L(w \sqcup x) = w \sqcup x$ for $L \in \{\Delta_w, d_{i,+}^\ast, d_{i,-}^\ast\}$ (see proposition 6.7). Therefore $\{-, -\}_+^\ast(w \sqcup x)$ lies in $H_0 = \phi(U) \otimes k\langle x, z \rangle + k\langle x, z \rangle \otimes \phi(U)$. One checks that $\{-, -\}_+^\ast(w \sqcup x)$ is the image of $\{-, -\}_+^\ast(w \sqcup x)$ by the orthogonal projection $\pi_0 : k\langle x, z \rangle \otimes^2 \rightarrow k\langle x, z \rangle_+ \otimes^2$ (with respect to $\langle, \rangle_2$) with kernel $k \otimes k\langle x, z \rangle + k\langle x, z \rangle \otimes k$. Since $\pi_0(H_0) = H$ and $\{-, -\}_+^\ast(w \sqcup x) \in H_0$, we deduce that $\{-, -\}_+^\ast(w \sqcup x) \in H$. This shows (A) for $w$ of depth $m \geq 1$.

We have shown (A) for $w$ of depth $m \geq 0$, proving the proposition. \[\square\]

**Corollary 6.9.** $U$ is a coideal for $\phi^\ast\{-, -\}_+^\ast$. \[29\]
6.3. Splitting of \( \phi \) and generating series. Define the linear isomorphism \( \phi_0 : V \to k\langle x, z \rangle \) by:
\[
\phi_0(I(n_1, \ldots, n_m)) = \phi(I(n_1, \ldots, n_m)) = x^{n_m}y \cdots x^{n_1}y x^{n_0},
\]
for \( m \geq 1 \) and \((n_0, \ldots, n_m) \in (N^*)^{m+1}\) and
\[
\phi_0(I'(n_0)) = x^{n_0},
\]
for \( n_0 \in N^* \). The isomorphism \( \phi \) of proposition \( \ref{6.6} \) splits into \( \phi = M \circ \phi_0 \), where \( M \) is the automorphism of \( k\langle x, z \rangle \) given by:
\[
M(w) = w \quad \text{and} \quad M(wx^k) = wx^{k-1} \| x, \quad (40)
\]
for \( w \in k \oplus k\langle x, z \rangle z \) and \( k \in N^* \). We note that this splitting was used in the proof of \( \ref{5.6} \).

For \( m \geq 1 \), set:
\[
(t_1 : \ldots : t_{m+1}) = \sum_{n_1, \ldots, n_m \geq 0} x^{n_m}y \cdots x^{n_1}y (t_1 - t_{m+1})^{n_1} \cdots (t_m - t_{m+1})^{n_m}. \quad (41)
\]
The series \((t_1 : \ldots : t_{m+1})\) is obtained by applying the isomorphism \( \phi \) (or \( \phi_0 \)) to the coefficients of the series \( \{t_1 : \cdots : t_{m+1}\} \).

**Notation 6.10.** Let \( T \) and \( T' \) be series with values in \( k\langle x, z \rangle \) (or \( k\langle x, z \rangle \oplus 2 \)). We use the notation \( T \equiv_x T' \) if \( T - T' \) is a series with values in \( k\langle x, z \rangle x \) (or \( k\langle x, z \rangle x \oplus k\langle x, z \rangle x \oplus k\langle x, z \rangle x \)).

**Proposition 6.11.** For \( m \geq 0 \) and \((l_1, \ldots, l_m) \in N^m \), set \( w(l_1, \ldots, l_m) := x^{l_m} \cdots x^{l_1}y \) if \( m \geq 1 \) and \( w(l_1, \ldots, l_m) = 1 \) otherwise.

1) For \( k \geq 0, m \geq 0 \) and \((n_1, \ldots, n_m) \in N^m \) we have:
\[
M^{-1}(w(n_1, \ldots, n_m)x^k) = w_x + (-1)^k \sum_{k_1 + \cdots + k_m = k} C_{k_1}^{n_1+k_1} \cdots C_{k_m}^{n_m+k_m} w(n_1 + k_1, \ldots, n_m + k_m),
\]
where \( w_x \) is an element of \( k\langle x, z \rangle x \).

2) For \( s \geq 1 \), set:
\[
U_s := \sum_{n_0, n_1, \ldots, n_s \geq 0} x^{n_s}y \cdots x^{n_1}y x^{n_0} u_{s+1}^{n_1} u_1^{n_1} \cdots u_s^{n_s},
\]
where the \( u_1, \ldots, u_{s+1} \) are commuting formal variables. We have:
\[
M^{-1}(U_s) \equiv_x (u_1 : \cdots : u_{s+1}),
\]
where \( \equiv_x \) is as in notation 6.10.

**Proof.** We first show (1) by induction on \( k \). The formula is true for \( k = 0 \). Assume the formula true for \( k = l \). One has:
\[
w x^{l+1} = (l + 1) w x^{l+1} + \sum_{1 \leq i \leq m} (n_i + 1) w_i^{l+1} x^l,
\]
where \( w = w(n_1, \ldots, n_m) \) and \( w_i^{l+1} = w(n_1, \ldots, n_i + 1, \ldots, n_m) \). Hence:
\[
M^{-1}(w x^{l+1}) = c_x - \frac{1}{l+1} \sum_{1 \leq i \leq m} (n_i + 1) M^{-1}(w_i^{l+1} x^l). \quad (A)
\]
Proposition 6.12. \(c_x = \frac{1}{l+1} M^{-1}(wx^l \square x) = \frac{1}{l+1} wx^l\) is an element of \(\mathbb{k}(x, z)x\). Applying the induction hypothesis to the term \(M^{-1}(w_i^{x+1}x^l)\) we get:

\[
M^{-1}(w_i^{x+1}x^l) = (-1)^l \sum_{l_1 + \cdots + l_m = l} C_i^i w(n_1 + l_1, \ldots, n_i + l_i + 1, \ldots, n_m + l_m)x^l,
\]

with \(C_i^i = C_{l_1}^{n_1 + l_1} \cdots C_{l_m}^{n_m + l_m} \cdots C_{l_n}^{m_m + l_m}\). Since \((n_i + 1)C_{l_i}^{n_i + l_i + 1} = (l_i + 1)C_{l_i + 1}^{n_i + l_i + 1}\), we get by setting \(a_i = l_i + 1\) and \(a_j = l_j\) for \(j \neq i\):

\[
(n_i + 1)M^{-1}(w_i^{x+1}x^l) = (-1)^l \sum_{a_1 + \cdots + a_m = l + 1} a_i C_{a_1}^{n_1 + a_1} \cdots C_{a_m}^{n_m + a_m} w(n_1 + a_1, \ldots, n_m + a_m)x^l.
\]

Therefore the equation (A) can be reduced to:

\[
M^{-1}(wx^{l+1}) = c_x + (-1)^{l+1} \sum_{a_1 + \cdots + a_m = l + 1} a_i C_{a_1}^{n_1 + a_1} \cdots C_{a_m}^{n_m + a_m} w(n_1 + a_1, \ldots, n_m + a_m)x^l
\]

with \(c_x \in \mathbb{k}(x, z)\). We have shown (1) by induction.

We now show (2). Using (1) we get:

\[
M^{-1}(U_s) \equiv_x U(x) + \sum_{n_0, n_1, \ldots, n_s \geq 0} (-1)^{n_0} \sum_{k_1 + \cdots + k_s = n_0} C_{k_1}^{n_1 + k_1} \cdots C_{k_s}^{n_s + k_s} w(n_1 + k_1, \ldots, n_s + k_s) u_1^{n_1} \cdots u_s^{n_s},
\]

where \(C_{k_s}^{n_s + k_s} = C_{k_1}^{n_1 + k_1} \cdots C_{k_s}^{n_s + k_s}\). Now setting \(l_i = n_i + k_i\) for \(1 \leq i \leq s\), we get:

\[
M^{-1}(U_s) \equiv_x \sum_{l_1, \ldots, l_s \geq 0} w(l_1, \ldots, l_s) \sum_{k_1 \leq l_1, \ldots, k_s \leq l_s} P_{k_1}(u_1, \ldots, u_{s+1}),
\]

where

\[
P_{k_1}(u_1, \ldots, u_{s+1}) = (-1)^{k_1 + \cdots + k_s} C_{k_1}^{l_1} \cdots C_{k_s}^{l_s} u_1^{k_1 - l_1} \cdots u_s^{k_s - l_s}.
\]

This proves (2). Indeed, we have:

\[
\sum_{k_1 \leq l_1, \ldots, k_s \leq l_s} P_{k_1}(u_1, \ldots, u_{s+1}) = (u_1 - u_{s+1})^{l_1} \cdots (u_s - u_{s+1})^{l_s}.
\]

We have shown the proposition. \(\square\)

6.4. **An intermediate pullback of the Ihara cobracket.** We recall that the pullback \(f^*Z : A \to A \otimes A\) of a linear map \(Z : B \to B \otimes B\) by an isomorphism of vector spaces \(f : A \to B\) is given by:

\[
f^*Z = (f^{-1} \otimes f^{-1}) \circ Z \circ f.
\]  

Proposition 6.12. Take \(m \geq 1\) and set \(T_m^1 = (t_1 : \cdots : t_{m+1}) \otimes 1\) and \(T_m^2 = 1 \otimes (t_1 : \cdots : t_{m+1})\), we have:

\[
M^* \Delta_m t_1 \cdots t_{m+1} (t_1 : \cdots : t_{m+1}) \equiv_x T_m^1 + T_m^2 + \sum_{1 \leq i < m} (t_{i+1} : \cdots : t_{m} : t_i) \otimes (t_{1} : \cdots : t_{i} : t_{m+1})
\]
We obtain the (2) of proposition 6.6 that:

\[
\sum_{a_i=0}^{m} (t_{i+1} : \cdots : m+1)x^{a_i}(t_i - t_{m+1})^{a_i} \equiv x + (t_{i+1}: \cdots : m : t_i).
\]

Since \( M(1) = 1 \) and \( M(x^s) = x^{s-1} \), \( x = s!x^s \) for \( s \geq 1 \), we have:

\[
M^{-1}(x^s) = \frac{x^s}{s!}
\]

We obtain the (2) by first applying \( M^{-1} \otimes M^{-1} \) to both sides of (44) and then simplifying the right hand side using (45) and (46).

**Proposition 6.13.** Take \( i \geq 1 \). For \( m \leq i \):

\[
M^*d_{i,+}^s((t_1 : \cdots : t_{m+1})) = \begin{cases} 
1 \otimes (t_1 : \cdots : t_{m+1}) & \text{if } m = i \\
0 & \text{otherwise},
\end{cases}
\]

and for \( m > i \):

\[
M^*d_{i,+}^s((t_1 : \cdots : t_{m+1})) \equiv x \sum_{1 \leq j < m-i} (t_{j+1} : \cdots : t_{m-i}: t_j) \otimes (t_1 : \cdots : t_j : t_{m-i+1} : \cdots : t_m) + 1 \otimes (t_1 : \cdots : t_{m+1}) + (t_1 : \cdots : t_{m-i} : t_{m+1}) \otimes (t_{m-i+1} : \cdots : t_{m+1}).
\]

**Proof.** For \( w \) a unitary monomial of \( \mathbb{k}(x, z) \) of weight \( m \) less than \( i \), we find by applying (34) of proposition 6.6 that:

\[
d_{i,+}^s(w) = \begin{cases} 
1 \otimes w & \text{if } m = i \\
0 & \text{otherwise},
\end{cases}
\]

We can readily derive (47) from this since \( M((t_1 : \cdots : t_{m+1})) = (t_1 : \cdots : t_{m+1}) \).

We now prove (48). Take \( m > i \). We can check that:

\[
(t_1 : \cdots : t_{m+1}) = (t_{m-i+1} : \cdots : t_m : t_{m+1})(t_1 : \cdots : t_{m-i} : t_{m+1}).
\]
The coefficients of $(t_1 : \cdots : t_m : t_{m+1})$ are of depth $i$ and end with $z$ so we can apply formula (34) to obtain:

$$d^*_{i,+}(t_1 : \cdots : t_{m+1}) = (1 \otimes (t_{m-i+1} : \cdots : t_m : t_{m+1})) \Delta_w((t_1 : \cdots : t_{m-i} : t_{m+1})).$$

Since $(t_{a_1} : \cdots : t_{a_j})$ is with values in $k(x, z)z$, $M$ fixes the elements of $k(x, z)$ and $\Delta_w((t_{a_1} : \cdots : t_{a_j}))$ is with values in $k(x, z) \otimes (k \oplus k(x, z))$ we have:

$$M^*d^*_{i,+}(t_1 : \cdots : t_{m+1}) = (1 \otimes (t_{m-i+1} : \cdots : t_m : t_{m+1})) M^* \Delta_w((t_1 : \cdots : t_{m-i} : t_{m+1})). \quad (51)$$

Applying proposition 6.12, we get:

$$M^* \Delta_w((t_1 : \cdots : t_{m-i} : t_{m+1})) \equiv x (t_m-i) + \sum_{1 \leq j < m-i} (t_{j+1} : \cdots : t_{m-i} : t_j) \otimes (t_1 : \cdots : t_j : t_{m+1}).$$

where $T_{m-i} = (t_1 : \cdots : t_{m-i} : t_{m+1}) \otimes 1 + 1 \otimes (t_1 : \cdots : t_{m-i} : t_{m+1})$. We then derive (48) by first combining the last two equations and then by simplifying the result using () for adequate variables. We proved the proposition. \(\square\)

**Proposition 6.14.** Take $i \geq 1$, for $m < i$:

$$M^*d^*_{i,-}(t_1 : \cdots : t_{m+1}) = 0, \quad (52)$$

and for $m \geq i$:

$$M^*d^*_{i,-}(t_1 : \cdots : t_{m+1}) \equiv x \sum_{i+1 \leq j \leq m} (t_{i+1} : \cdots : t_i : t_i) \otimes (t_1 : \cdots : t_{i-1} : t_i : \cdots : t_{m+1})$$

$$+ 1 \otimes (t_1 : \cdots : t_{m+1}), \quad (53)$$

where we use exceptionly the convention

$$(t_1 : \cdots : t_{i-1} : t_i : \cdots : t_{m+1}) := (t_i : \cdots : t_{m+1}) \quad , \quad$$

for $i = 1$.

**Proof.** Equation (52) can be obtained using the definition of $d_{i,-}$. We show (54). We assume that $m \geq i$. We can check that:

$$(t_1 : \cdots : t_{m+1}) = \left( \sum_{n_i \geq 0} (t_{i+1} : \cdots : t_m : t_{m+1}) x^{n_i} (t_i - t_{m+1})^{n_i} \right) (z(t_1 : \cdots : t_{i-1} : t_{m+1})), \quad (54)$$

with the conventions $(t_1 : \cdots : t_{i-1} : \cdots : t_{m+1}) := 1$ for $i = 1$ and $(t_{i+1} : \cdots : t_m : t_{m+1}) = 1$ for $m = i$. The coefficients of $z(t_1 : \cdots : t_{i-1} : t_{m+1})$ start with $z$ and are of depth $i$ so we can apply formula (35) to obtain:

$$d^*_{i,-}(t_1 : \cdots : t_{m+1}) = \left( \sum_{n_i \geq 0} A^i_{n_i} (t_i - t_{m+1})^{n_i} \right) (1 \otimes z(t_1 : \cdots : t_{i-1} : t_{m+1})), \quad (55)$$

where

$$A^i_{n_i} = \begin{cases} 
\Delta_w^{op}(x^{n_i}) & \text{if } m = i, \\
\Delta_w^{op}(t_{i+1} : \cdots : t_m : t_{m+1}) x^{n_i} & \text{if } m > i.
\end{cases}$$

One the other hand we can verify that:

$$\Delta_w^{op}(x^{n_i}) = \sum_{a_m + b_n = n_m} x^b \otimes x^a.$$
and that:

\[ \Delta_{i}^{op}(t_{i+1} : \cdots : t_{m+1}) = + \sum_{a_{m} \geq 0} (t_{i+1} : \cdots : t_{m+1})x^{a_{m}} (t_{m} - t_{m+1})^{a_{m}} \]

\[ + \sum_{l=i+1}^{m-1} \sum_{a_{l} \geq 0} (t_{i+1} : \cdots : t_{l} : t_{m+1})x^{a_{l}} (t_{l} - t_{m+1})^{a_{l}} \]

\[ + \sum_{a_{i} + b_{i} = n_{i}} x^{b_{i}} \otimes (t_{i+1} : \cdots : t_{m+1})x^{a_{i}}, \]

for \( m > i \).

Now by replacing \( A_{n_{i}}^{i} \) in (55) by the right hand side of one of the last two equations (depending on if \( m = i \) or \( m > i \)) we find:

\[ d_{i}^{*}((t_{1} : \cdots : t_{m+1})) = \sum_{b_{i} \geq 0} x^{b_{i}} (t_{m} - t_{m+1})^{b_{i}} \otimes (t_{1} : \cdots : t_{m} : t_{m+1}), \] (56)

if \( m = i \) and

\[ d_{i}^{*}((t_{1} : \cdots : t_{m+1})) = \sum_{l=i+1}^{m} \sum_{a_{l} \geq 0} (t_{i+1} : \cdots : t_{l} : t_{m+1})x^{a_{l}} (t_{l} - t_{m+1})^{a_{l}} \otimes (t_{1} : \cdots : t_{l-1} : t_{l} : \cdots : t_{m+1}) \]

\[ + \sum_{c_{i} \geq 0} x^{c_{i}} \otimes (t_{1} : \cdots : t_{m+1}). \] (57)

if \( m > i \).

Since \( M(t_{1}, \cdots, t_{m+1}) = (t_{1}, \cdots, t_{m+1}) \) and:

\[ M^{-1}(x^{r}) = \frac{x^{r}}{r!}. \] (58)

(see equation (46)), we deduce from (56) that:

\[ M^{*}d_{i}^{*}((t_{1} : \cdots : t_{m+1})) \equiv x^{1} \otimes (t_{1} : \cdots : t_{m+1}), \]

if \( m = i \). This shows () for \( m = i \).

We still have to prove () for \( m > i \). Take \( m > i \). Setting \( u_{1} = t_{i+1} - t_{m+1}, \ldots, u_{s} = t_{l} - t_{m+1} \) and \( t_{s+1} = t_{i} - t_{m+1} \) in (2) of proposition 6.11 we obtain for \( l \in [1, m] \) the equation:

\[ M^{-1}(\sum_{n_{i} \geq 0} (t_{i+1} : \cdots : t_{l} : t_{m+1})x^{n_{i}} (t_{l} - t_{m+1})^{n_{i}}) \equiv x^{(t_{i} : \cdots : t_{l}).} \] (59)

Since \( M((t_{a_{1}}, \cdots, t_{a_{m+1}})) = (t_{a_{1}}, \cdots, t_{a_{m+1}}) \), we obtain () for \( m > i \) by combining (57), (58) and (59). The proposition is proved.

**Proposition 6.15.** For \( m > 1 \) we have:

\[ M^{*}\{-, -\}^{*}((t_{1} : \cdots : t_{m+1})) \equiv x^{m+1} \sum_{k=1}^{m+1} B_{k}, \]
where

\[
B_k = \sum_{j=1}^{k-1} (t_{j+1} : \cdots : t_k : t_j) \land (t_1 : \cdots : t_j : t_{k+1} : \cdots : t_{m+1})
+ \sum_{j=k+2}^{m} (t_1 : \cdots : t_k : t_j : \cdots : t_{m+1}) \land (t_{k+2} : \cdots : t_j : t_{k+1})
+ (t_1 : \cdots : t_k : t_{m+1}) \land (t_{k+1} : \cdots : t_{m+1}),
\]

for \( k \in [1, m-2] \),

\[
B_{m-1} = \sum_{j=1}^{m-2} (t_{j+1} : \cdots : t_{m-1} : t_j) \land (t_1 : \cdots : t_j : t_{m+1})
+ (t_1 : \cdots : t_{m-1} : t_{m+1}) \land (t_m : t_{m+1}),
\]

\[
B_m = \sum_{j=1}^{m-1} (t_{j+1} : \cdots : t_m : t_j) \land (t_1 : \cdots : t_j : t_{m+1}),
\]

and

\[
B_{m+1} = \sum_{j=2}^{m} (t_j : \cdots : t_{m+1}) \land (t_2 : \cdots : t_j : t_1)
\]

**Proof.** One can derive using propositions 6.13 and 6.14 the following equations:

\[
(id - \tau) \circ (M^* d_{m-k, +}^* - M^* d_{k+1, -}^*)( (t_1 : \cdots : t_{m+1}) ) \equiv_x B_k, \quad \text{for } k \in [1, m-2],
\]

\[
(id - \tau) \circ (M^* d_{1, +}^* - M^* d_{1, -}^*)( (t_1 : \cdots : t_{m+1}) ) \equiv_x B_{m+1} + B_{m-1},
\]

and

\[
(id - \tau) \circ (d_{m, +}^* - d_{m, -}^*)( (t_1 : \cdots : t_{m+1}) ) \equiv_x 0.
\]

We have also seen (proposition 6.12) that:

\[
M^* ((id - \tau) \circ \Delta_{w})( (t_1 : \cdots : t_{m+1}) ) \equiv_x B_m.
\]

Since \( \{-, -\}^*(w) = (id - \tau) \circ (\Delta_{w} + \sum_{i=1}^{m} d_{i, +}^* - d_{i, -}^*)(w) \) for \( w \in \mathbb{k}(x, z) \) of depth \( m \) (see proposition 6.5), we get by adding up the last four equations that:

\[
M^* \{ - , - \}^*( (t_1 : \cdots : t_{m+1}) ) \equiv_x \sum_{k=1}^{m+1} B_k.
\]

Indeed, \( M^{-1} \otimes M^{-1} \) commutes with \( (id - \tau) \) and \( M((t_1 : \cdots : t_{m+1})) = (t_1 : \cdots : t_{m+1}) \). We have proved the proposition. \( \square \)
6.5. Correspondance between the dihedral cobracket and the Ihara cobracket.

**Lemma 6.16.** Set
\[
A := \bigcup_{i=1}^{m-2} \{(i,j)| j \in [0,i-1]\} \cup \bigcup_{i=m+2}^{2m-1} \{(i,j)| j \in [i-m+1,m]\} \cup \bigcup_{i=m-1}^{m+1} \{(i,j)| j \in [i-m+1,i-1]\}.
\]
The map \( h : \mathbb{Z}^2 \to \mathbb{Z}^2 \) given by \((i,j) \to (k := i - j + 1, j)\) is a bijection sending \( A \) to \([2,m] \times [0,m]\).

**Proof.** We leave the proof to the reader. \( \square \)

Recall that we have introduced in subsection 3.2 a map \( \tilde{\delta} : W \to W \otimes W \) inducing a Lie cobracket \( \delta : W/W_R \to W/W_R \).

**Lemma 6.17.** For two series \( T, T' \) with values in \( V \otimes V \) we write \( T \equiv_F T' \) if \( T - T' \) takes values in \( F \otimes V + V \otimes F \). For \( m > 1 \), we have:
\[
\tilde{\delta}(\{t_1 : \cdots : t_{m+1}\}) \equiv_F \sum_{k=1}^{m+1} \phi^{-1}_0 \otimes \phi^{-1}_0(B_k)
\]
where \( B_1, \ldots, B_{m+1} \) are the series defined in proposition 6.15.

**Proof.** Take \( m > 1 \). By definition (formula (22) of subsection 3.2):
\[
\tilde{\delta}(\{t_1 : \cdots : t_{m+1}\}) = - \sum_{k=2}^{m} \text{Cycle}_{m+1}(\{t_1 : \cdots : t_{k-1} : t_{m+1}\} \land \{t_k : \cdots : t_{m+1}\}), \quad (65)
\]
for \( m \geq 2 \). Hence:
\[
\tilde{\delta}(\{t_1 : \cdots : t_{m+1}\}) = \sum_{j=0}^{m} \sum_{k=2}^{m} B'_{j,k},
\]
with \( B'_{j,k} = \{t_j : \cdots : t_{k+j-1}\} \land \{t_{k+j} : \cdots : t_{m+j+1}\} \) where the indices are modulo \( m + 1 \) and they are taken in \([1, m+1]\). Now setting \( i = k + j - 1 \) we can decompose \( \tilde{\delta}(\{t_1 : \cdots : t_{m+1}\}) \) using lemma 6.16 as follows:
\[
\tilde{\delta}(\{t_1 : \cdots : t_{m+1}\}) = \sum_{i=1}^{m+1} B'_i,
\]
where
\[
B'_i = \sum_{j=i-1}^{i-1} B'_{j,i-j+1} + \sum_{j=i+2}^{m} B'_{j,i-j+m+2} + B'_{0,i+1}, \quad \text{for } i \in [1, m-2],
\]
and
\[
B'_i = \sum_{j=i-m-1}^{i-1} B'_{j,i-j+1}, \quad \text{for } i \in [m-1, m+1].
\]
Note that \( \phi^{-1}_0((t_{a_1} : \cdots : t_{a_i})) = \{t_{a_1} : \cdots : t_{a_i}\} \). Using this, one checks that up to cyclic symmetrie (see (20) of remark 3.3 for the definition of the cyclic symmetrie), the following equation holds:
\[
B'_k = \phi^{-1}_0 \otimes \phi^{-1}_0(B_k),
\]
for \( k \in [1, m+1] \). Since \( W_R \subset F \), it follows from the interpretation of the cyclic symmetrie in \( V \) (see (21) of remark 3.3) that:
\[
B'_k \equiv_F \phi^{-1}_0 \otimes \phi^{-1}_0(B_k),
\]
36
for $k \in [1, m + 1]$. This proves the lemma, since $\tilde{\delta}(\{t_1 : \cdots : t_{m+1}\}) = \sum_{i=1}^{m+1} B'_i$. \hfill \square

**Proposition 6.18.** The image of $W$ by $(\phi^*\{-,-\}^*_+ - \tilde{\delta})$ lies in $F \otimes V + V \otimes F$.

**Proof.** Denote by $M_+$ the automorphism of $\mathbb{k}\langle x, z \rangle_+$ obtained by restricting $M$. One can deduce from proposition 6.15 that for $m > 1$:

$$M^*_+\{-,-\}^*_+((t_1 : \cdots : t_{m+1})) = R + \sum_{k=1}^{m+1} B_k,$$

where $R \in \phi_0(U) \otimes \mathbb{k}\langle x, z \rangle_+ + \mathbb{k}\langle x, z \rangle_+ \otimes \phi_0(U)$ ($\phi_0(U) = \mathbb{k}\langle x, z \rangle x$). Since $\phi_0(\{t_1 : \cdots : t_{m+1}\}) = (t_1 : \cdots : t_{m+1})$, $U \subset F$, $\phi = M \circ \phi_0 = M_+ \circ \phi_0$ and both $M_+$ and $\phi_0$ are isomorphisms we deduce from the above equation that:

$$\phi^*\{-,-\}^*_+((t_1 : \cdots : t_{m+1})) \equiv_F \sum_{k=1}^{m+1} \phi_0^{-1} \otimes \phi_0^{-1}(B_k),$$

for $m > 1$ and therefore by lemma 6.17:

$$(\phi^*\{-,-\}^*_+ - \tilde{\delta})(\{t_1 : \cdots : t_{m+1}\}) \equiv_F 0,$$

for $m > 1$. Since $\{t_1 : \cdots : t_{m+1}\}$ (for $m \geq 1$) is a generating series of the basis $\{I(n_1, \ldots, n_m)\}_{n_1, \ldots, n_m \geq 1}$ of the depth $m$ part of $W$, we deduce from the above equation that:

$$(\phi^*\{-,-\}^*_+ - \tilde{\delta})(W_{>1}) \equiv_F 0,$$

where $W_{>1}$ is the subspace of $W$ generated by elements of depth strictly greater than 1. We still have to show that $(\phi^*\{-,-\}^*_+ - \tilde{\delta})(w) \equiv_F 0$ for $w \in W$ of depth 1. But this is true since $V$ has no depth zero part and $(\phi^*\{-,-\}^*_+ - \tilde{\delta})$ is graded for the depth. We have proved the proposition. \hfill \square

**Proof of theorem 6.1**

We recall that $F = W_R \oplus U$. We first show (1). For that, we prove that $\phi^*\{-,-\}^*_+(C) \subset F \otimes V + V \otimes F$ for $C = U, F$. Since $U$ is a coideal for $\phi^*\{-,-\}^*_+$ (corollary 6.9), we have $\phi^*\{-,-\}^*_+(U) \subset F \otimes V + V \otimes F$. In section 5 we have seen (remark 3.5) that $W_R$ is a coideal for $\tilde{\delta} : W \rightarrow W \otimes W$. Combining this with proposition 6.18 we deduce that $\phi^*\{-,-\}^*_+(W_R) \subset F \otimes V + V \otimes F$. We have proved (1) of theorem 6.1. We show (2).

In section 3.1 we have seen that $D_+ : (W/W_R, \delta)$ is an isomorphism of Lie coalgebras (theorem 3.4). We have also seen that $\tilde{i} : W/W_R \rightarrow V/F$ induced by the inclusion $W \subset V$ is an isomorphism of bigraded spaces (proposition 3.1). From this and proposition 6.18 we deduce that $(W/W_R, \delta) \tilde{i} \rightarrow (V/F, \delta_{V/F})$ (with $\delta_{V/F}$ the Lie cobracket induced by $\phi^*\{-,-\}^*_+$) is an isomorphism of bigraded Lie coalgebras proving (2). We have proved the theorem.

**7. PROOF OF THE MAIN RESULTS**

In this section we prove the main results announced in the introduction of the paper. In the first subsection we prove (theorem 7.1) that $\mathfrak{ls}$ is preserved by the Ihara bracket $\{-,-\}$ and that (theorems 7.2, 7.3) the isomorphic Lie coalgebras $(V/F, \delta_{V/F})$ and $D_+$ are bigraded duals of the bigraded Lie algebra $\mathfrak{ls}$. We give explicit isomorphisms between the Lie coalgebras $V/F \simeq D_+$ and the bigraded Lie coalgebra dual to $(\mathfrak{ls}, \{-,-\})$. These results correspond to results (a) and (b) of the introduction and are easily obtained out if
the material of the previous sections. The other results announced in the introduction of the paper ((c) and (d)) are proved in the second subsection. We start the second subsection by reminders on algebra. In paragraph 7.2.1, we construct \( f \) or \( D_{sh} \) and show the result (c): the depth \( m \) of \( D_{sh} \) is defined for \( \overline{\cdot} \) as stated in Theorem 7.2. We have an isomorphism of bigraded Lie coalgebras. In particular, the dihedral Lie coalgebra \( D_{sh} \) is isomorphic to the graded dual \( D_{sh}^\vee \) of \( D_{sh} \). We then show (proposition 7.13) that the induced isomorphism \( ls \) of \( D_{sh} \) is compatible with the isomorphisms \( D_{sh} \rightarrow D_{sh}^\vee \) and \( D_{sh} \rightarrow D_{sh}^\vee \) mentioned previously (result (d)).

7.1. Proof of main results (a) and (b). We will prove the next three theorems:

**Theorem 7.1.** The space \( ls \) is preserved by the Ihara bracket.

Denote by \( \{-,-\}_ls \) the (bigraded) dual cobracket of the Ihara bracket restricted to \( ls \), the bigraded dual \( ls^\vee \) of \( ls \) equipped with \( \{-,-\}_ls \) is a bigraded Lie coalgebra (see (2) of proposition 1.3 section 1). We have shown that \( \phi^*\{-,-\}_+ \) induces a Lie cobracket \( \delta_{V/F} : V/F \rightarrow V/F \otimes V/F \).

**Theorem 7.2.** We have an isomorphism of bigraded Lie coalgebras \( (V/F, \delta_{V/F}) \rightarrow (ls^\vee, \{-,-\}_ls^\vee) \) defined for \( v \in V/F \) and \( w \in ls \) by:

\[
\theta(v)(w) = \langle v, w \rangle_{\phi},
\]

where \( v \) is any lift of \( \overline{v} \) to \( V \) and \( \langle -,-\rangle_{\phi} \) is the pairing of theorem 5.4.

**Theorem 7.3.** The map \( D_{sh} \rightarrow (ls^\vee, \{-,-\}_ls^\vee) \) is an isomorphism of bigraded Lie coalgebras. In particular, the dihedral Lie coalgebra \( D_{sh} \) is the bigraded dual of the bigraded Lie algebra \( ls \) (equipped with the Ihara bracket).

**Proof of the theorems:**

We denote by \( \langle -,-\rangle_{\phi} : V \otimes V \rightarrow k(x,z) \otimes k(x,z) \) the perfect bigraded pairing given by:

\[
\langle v \otimes v', w \otimes w' \rangle_{\phi} = \langle v, w \rangle_{\phi} \langle v', w' \rangle_{\phi},
\]

for \( v, v' \in V \) and \( w, w' \in k(x,z) \) with \( \langle -,-\rangle_{\phi} = \langle \phi(-),-\rangle_{\phi} \) is the pairing of theorem 5.1 of section 5. One can readily check that \( \phi^*\{-,-\}_+ \) is the adjoint of \( \{-,-\}_+ \) with respect to \( \langle -,-\rangle_{\phi} \) and \( \langle -,-\rangle_{\phi}^\otimes \). Moreover, we have seen (theorem 5.1) that the orthogonal of the bigraded subspace \( F \subset V \) with respect to \( \langle -,-\rangle_{\phi} \) is \( ls \), that \( F \) is a coideal for \( \phi^*\{-,-\}_+ \) (theorem 6.1) and we have denoted by \( \delta_{V/F}^\phi \) the cobracket induced by \( \phi^*\{-,-\}_+ \). These facts prove that the hypothesis of (2) of proposition 6.1 (section 1) are satisfied for \( (C,\gamma) = (V, \phi^*\{-,-\}_+), (A,g) = (k(x,z), \{-,-\}) \) and \( C_0 = F \) and we can apply the proposition to deduce theorems 7.4 and 7.2. Theorem 7.3 is an immediate consequence of (2) of theorem 6.1 and theorem 7.2. We have proved the three theorems of this subsection.
7.2. Proof of main results (c) and (d). Let \( A := \bigoplus_{k \geq k_0} A_k \) and \( B := \bigoplus_{k \geq k_0} B_k \) be graded \( \mathbb{k} \)-vector spaces with finite dimensional homogeneous elements. We denote by \( A \otimes B \) the completion of \( A \otimes B \) with respect to its grading and by \( f_A \otimes f_B \) the unique endomorphism of \( A \otimes B \) extending \( f_A \otimes f_B \) for \( f_A \) and \( f_B \) given endomorphisms of \( A \) and \( B \). If \( A \) and \( B \) are isomorphic graded space, we assign to a family of basis \( \{ f \} \) of \( A \) that is an immediate consequence of (1).

We leave the proof to the reader.

**Proposition 7.5.** Let \( A_T \) be the smallest subspace of \( A \) such that the image of \( (\text{id}_A \otimes T)(S_{A,B}) \) in \( (A/A_T) \otimes B \) is null.

1) The space \( A_T \) is a graded subspace of \( A \) corresponding to the image \( \text{Im}(T^*) \) of \( T^* \).

2) The image of \( A_T \) with respect to \( h_{A,B} \) is the orthogonal complement \( \ker(T)^\perp \) of the kernel \( \ker(T) \) of \( T \) with respect to the pairing \( (h_{A,B}^{-1}, -)_{A,B} \).

**Proof.** We prove (1). Using the previous lemma, we find that:

\[
(\text{id}_A \otimes T)(S_{A,B}) = \sum_{l \geq 1} \sum_{i_l \in I_l} T^*(a_{i_l}) \hat{\otimes} b_{i_l}
\]

The above series is null in \( (A/A_T) \otimes B \) if and only if \( T^*(a_{i_l}) \in A_T \) for all \( a_{i_l} \), with \( l \geq m \) and \( i_l \in I_l \). Since the elements \( a_{i_l} \) generate \( A \) and \( A_T \) is the smallest subspace possible, we deduce that \( A_T = \text{Im}(T^*) \). Since \( T \) is graded and the pairing is compatible to the gradings, \( T^* \) is also graded and as a consequence \( A_T = \text{Im}(T^*) \) is a graded subspace of \( A \). The assertion (2) is an immediate consequence of (1).

7.2.1. **Proof of (c):** We recall (see proposition 3.1 and the paragraph preceding it) that we have an isomorphism of bigraded spaces \( \eta : D_{\ast \ast} \rightarrow W/W_R \). The spaces \( W \) and \( W_R \) are bigraded (for weight and depth) subspaces of \( V \). We denote by \( W^m \) and \( W^m_R \) the depth \( m \) parts of \( W \) and \( W_R \) respectively. More precisely, \( W^m \) is the \( \mathbb{k} \)-vector space with basis \( \{ I(n_1, \ldots, n_m) | n_1, \ldots, n_m \geq 1 \} \).

In the sequel, the depth \( m \) part \( \mathbb{k}\langle x, z \rangle_m \) of \( \mathbb{k}\langle x, z \rangle \) is equipped with a perfect pairing \( \langle -, -, - \rangle_m \) obtained by restricting the canonical (perfect) pairing \( \langle -, -, - \rangle_m \) of \( \mathbb{k}\langle x, z \rangle \) used in the previous sections; the space \( \mathbb{k}[x_1, \ldots, x_m] \) is endowed with a weight grading (corresponding to a shift
by \(+m\) of the total degree) and a perfect (graded) paring \((-,-)_m\) for which the unitary monomials form an orthonormal basis.

**Proposition 7.6.** Take \(m \geq 2\). Let \(h_m : W^m \to \mathbb{k}[x_1, \cdots, x_m]\) be the weight graded isomorphism mapping \(I(n_1, \ldots, n_m)\) to \(x_1^{n_{m-1}}x_2^{n_{m-2}}\cdots x_m^{n_1}\). The image of \(W^m_R\) by \(h_m\) is the orthogonal complement \(\text{Dsh}_m^\perp\) of \(\text{Dsh}_m\) with respect to \((-,-)_m\).

**Proof.** We first recall that for \(I\),\(\text{morphism mapping}\) of the polynomial ring \(\mathbb{k}[x_1, \ldots, x_m]\) given by:

\[
S_m(x_i) = x_i + \cdots + x_m, \quad T_\sigma(x_i) = x_{\sigma^{-1}(i)}, \quad \text{for } i \in [1, m],
\]

and that:

\[
T_{m,*}^{(l)} = \sum_{\sigma \in S(l,m-l)} T_\sigma, \quad T_{m,\|l\|}^{(l)} = T_{m,*}^{(l)} \circ S_m,
\]

for \(l \in [1, m-1]\), where \(S(p,q)\) denotes the set of \((p,q)\)-shuffles, as in the previous sections. We will apply proposition 7.5. We take \(A = W^m\) and \(B = \mathbb{k}[x_1, \ldots, x_m]\) equipped with their weight grading. For \(k \geq k_0 = m\), set \(I_k = \{(n_1, \ldots, n_m) | n_1 + \cdots + n_m = k, n_i \geq 1\ \text{for } i \in [1, m]\}\) and take:

\[
A_k = \{I(n_1, \ldots, n_m) | (n_1, \ldots, n_m) \in I_k\} \quad \text{and} \quad B_k = \{x_1^{n_{m-1}}x_2^{n_{m-2}}\cdots x_m^{n_1} | (n_1, \ldots, n_m) \in I_k\},
\]

as basis for \(A_k\) and \(B_k\) (the weight \(k\) parts) respectively. The map, the paring and the \(\text{series assigned (as in the paragraph preceding the lemma 7.4)}\) to these basis are given by:

\[
S_{A,B} = \sum_{n_1, \ldots, n_m \geq 1} I(n_1, \ldots, n_m) \hat{\otimes} x_1^{n_{m-1}}\cdots x_m^{n_1},
\]

\[
h_{A,B} = h_m \quad \text{and} \quad (h_{A,B}^{-1} (-,-))_{A,B} = (-,-)_m.
\]

We recall that \(W_R = V_1 + V_2 + V''\) (see proposition 3.1 and the paragraph preceding it), where \(V_1, V_2\) and \(V''\) are bigraded subspaces of \(W\) and \(V''\) consist of elements of depth 1. Hence, for \(m \geq 2\):

\[
W^m_R = V_1^m + V_2^m
\]

where \(V_1^m\) and \(V_2^m\) are the depth \(m\) parts of \(V_1\) and \(V_2\) respectively. Let \(R_m\) be the algebra automorphism (involution) of \(\mathbb{k}[x_1, \ldots, x_m]\) mapping \(x_i\) to \(x_{m+1-i}\) for \(i \in [1, m]\). From the definition of \(V_1\) (see the paragraph of formula 10), one can readily deduce the following statement:

\[
V_1^m \text{ is the smallest subspace of } W^m \text{ such that the image of } (\text{id}_{W^m} \hat{\otimes} T_{m,*}^{(l)} R_m)(S_{A,B}),
\]

\[
\text{for } l \in [1, m-1], \text{ is 0 in } W^m/V_1^m \hat{\otimes} \mathbb{k}[x_1, \ldots, x_m],
\]

with \(S_{A,B}\) as in the first formula of this proof. We deduce from the above statement, by applying (2) of proposition 7.5 for \(T = T_{m,*}^{(l)} R_m \ (l \in [1, m-1])\) with the identifications of \(66\) that:

\[
\text{for } m \geq 2, \quad h_m(V_1^m) = (\cap_{l \in [1, m-1]} \text{Ker}(T_{m,*}^{(l)} R_m))^\perp,
\]

where the orthogonal complement in the right hand side of the equation is taken with respect to \((-,-)_m\).
Now, for \( l \in [1, m - 1] \), set \( \overline{T}^{(l)}_{m,\bar{\omega}} = T^{(l)}_{m,\ast} \circ S_{m}^{\ast} \), with \( S_{m}^{\ast} \) the automorphism of \( \mathbb{k}[x_{1}, \ldots, x_{m}] \) mapping \( x_{i} \) to \( x_{1} + \cdots + x_{i} \) (for \( i \in [1, m] \)). With a similar reasoning to the one applied to \( V^{m}_{1} \), one can show using the endomorphisms \( \overline{T}^{(l)}_{m,\bar{\omega}} R_{m} \) for \( l \in [1, m - 1] \) and the series \( S_{A,B} \) that:

\[
\text{for } m \geq 2, \quad h_{m}(V^{m}_{2}) = (\bigcap_{l \in [1, m-1]} \text{Ker}(\overline{T}^{(l)}_{m,\bar{\omega}} R_{m}))^{\perp}.
\]

combining the above equation with the one obtained just before for \( V^{m}_{1} \) and (67) we get:

\[
h_{m}(W^{m}_{R}) = (\bigcap_{l \in [1, m-1]} (\text{Ker}(T^{(l)}_{m,\ast} R_{m}) \cap \text{Ker}(\overline{T}^{(l)}_{m,\bar{\omega}} R_{m})))^{\perp} \quad \text{(68)}
\]

One can check that:

\[
R_{m} S_{m}^{\ast} R_{m} = S_{m}, \quad R_{m} T^{(l)}_{m,\ast} R_{m} = T^{(m-l)}_{m,\ast},
\]

for \( l \in [1, m - 1] \) and therefore \( (R_{m} \) is an involution):

\[
R_{m} \overline{T}^{(m-l)}_{m,\bar{\omega}} R_{m} = T^{(l)}_{m,\bar{\omega}}.
\]

Since \( R_{m} \) is an automorphism, these equations allow to reduce (68) to:

\[
h_{m}(W^{m}_{R}) = (\bigcap_{l \in [1, m-1]} (\text{Ker}(T^{(l)}_{m,\ast} R_{m}) \cap \text{Ker}(T^{(l)}_{m,\bar{\omega}} R_{m})))^{\perp} = \text{Dsh}_{m}^{\perp},
\]

where the second line of the equation holds by definition of \( \text{Dsh}_{m} \). \( \square \)

**Corollary 7.7.** We have an isomorphism \( \overline{h}_{\ast} \) of weight graded spaces between \( W^{m}/W^{m}_{R} \) and the graded dual \( \text{Dsh}_{m}^{\ast} \) of \( \text{Dsh}_{m} \), given by:

\[
\overline{h}_{\ast}(\overline{v})(P) = (h_{m}(v), P),
\]

for \( P \in \text{Dsh}_{m}, \overline{v} \in W^{m}/W^{m}_{R} \) and \( v \in W \) lifting \( \overline{v} \) to \( W \).

*Proof.* We know from the previous proposition that the kernel of the mapping \( h_{\ast} : W \to \text{Dsh}_{m}^{\ast}, w \mapsto (h_{m}(v), -)|_{\text{Dsh}_{m}} \) is \( W_{R} \). Hence, \( h_{\ast} \) factors through \( W^{m}/W^{m}_{R} \) giving the isomorphism \( \overline{h}_{\ast} \). \( \square \)

**Corollary 7.8.** Denote by \( D_{m,\bullet} \) the depth \( m \) part of \( D_{\bullet} \) and by \( \eta_{\ast} : D_{m,\bullet} \to W^{m}/W^{m}_{R} \) the restriction of the bigraded isomorphism \( \eta : D_{\bullet} \to W/W_{R} \) of proposition [5.7] to depth \( m \) parts. The map \( \eta_{\ast} \circ h_{\ast} : D_{m,\bullet} \to \text{Dsh}_{m}^{\ast} \) is an isomorphism of weight graded spaces.

*Proof.* This corollary is an immediate consequence of the previous one. \( \square \)

**Remark 7.9.** The above corollary with theorem 7.5 imply that \( \text{ts}_{m} \) and \( \text{Dsh}_{m}^{\ast} \) are isomorphic graded spaces.

7.2.2. *Proof of (d):* In this subsection, we equip \( \mathbb{k}(x, z)_{m} \) with the restriction \( \langle -,- \rangle_{m} \) of the canonical pairing over \( \mathbb{k}(x, z) \). For \( m \geq 2 \), we define the linear map \( f_{m} : \mathbb{k}(x, z)_{m} \to \mathbb{k}[x_{1}, \ldots, x_{m}], s_{m} : \mathbb{k}[x_{1}, \ldots, x_{m}] \to \mathbb{k}(x, z)_{m} \) and \( \pi_{m}^{z} : \mathbb{k}(x, z)_{m} \to \mathbb{k}(x, z)_{m-1}z \) by the following:

\[
f_{m}(x_{1}^{n_{1}} \cdots x_{m}^{n_{m}} z^{m_{1}+1}) = \begin{cases} x_{1}^{n_{1}} x_{2}^{n_{2}} \cdots x_{m}^{n_{m}} & \text{if } n_{m+1} = 0, \\ 0 & \text{otherwise} \end{cases}, \quad (69)
\]

\[
s_{m}(x_{1}^{n_{1}} x_{2}^{n_{2}} \cdots x_{m}^{n_{m}}) = x_{1}^{n_{1}} \cdots x_{m}^{n_{m}} z, \quad (70)
\]

\[
\pi_{m}^{z} = s_{m} \circ f \quad (71)
\]
for \( n_1, \ldots, n_m \geq 0 \). The map \( s_m \) is an isometry and \( \pi^z_m \) is the orthogonal projection (with respect to \( \langle -, - \rangle_m \)) on the subspace generated by monomials ending with \( z \). Furthermore:

\[
\kappa(x, z)_m = \text{Im}(\pi^z_m) \bigoplus \text{Ker}(\pi^z_m) = \text{Im}(s_m) \bigoplus \text{Ker}(\pi^z_m),
\]

where \( \text{Im}(\pi^z_m) \) and \( \text{Ker}(\pi^z_m) = \kappa(x, z)x \) are the image and the kernel of \( \pi^z_m \) respectively and \( \text{Im}(s_m) \) is the image of \( s_m \).

**Lemma 7.10.** Take \( m \geq 2 \).

1) The image of \( D_{sh}^\perp_m \) by \( s_m \) is a subspace of the orthogonal complement \( l_{s}^\perp_m \) of \( l_s_m \) in \( \kappa(x, z)_m \).

2) Denote by \( f_m^\vee \) the graded dual of the map \( f_m \). We have:

\[ f_m^\vee((P, -)_m) = \langle s_m(P), - \rangle_m, \]

for \( P \in \kappa[x_1, \ldots, x_m] \).

**Proof.** Take \( m \geq 2 \). We first prove (1). Let \( \phi|_W \) be the restriction of \( \phi \) of proposition 5.6 to \( W \). One can readily check that the following diagram commutes:

\[
\begin{align*}
W^m & \xrightarrow{\phi|_W} \kappa(x, z)_m, \\
& \downarrow h_m \quad \downarrow s_m \\
\kappa[x_1, \ldots, x_m] & \xrightarrow{s_m} \kappa(x, z)_m,
\end{align*}
\]

with \( h_m \) and \( s_m \) as defined in proposition 7.6 and formula (70). Since \( h_m(W^m_R) = D_{sh}^\perp_m \) (proposition 7.6), we deduce from the above diagram that:

\[
\text{Im}(s_m(D_{sh}^\perp_m)) = \phi|_W(W^m_R). \tag{72}
\]

On the other hand \( W_R \) is a subset of \( F \) (see proposition 3.1) and \( \phi(F) \) is the orthogonal complement of \( l_s \) in \( \kappa(x, z)_+ \) by corollary 5.9. Therefore \( \phi|_W(W^m_R) \subset l_{s}^\perp_m \subset \kappa(x, z)_m \). Combining this and equation (72) we obtain (1).

We now prove (2). We have seen in the paragraph preceding the lemma that \( (*) \) \( s_m \) is an isometry, that \( (**) \) \( s_m \circ f_m = \pi^z_m \) and that \( (***) \) \( \kappa(x, z)_m = \text{Im}(\pi^z_m) \bigoplus \text{Ker}(\pi^z_m) \) = \( \text{Im}(s_m) \bigoplus \text{Ker}(\pi^z_m) \). Using these facts we get the following, for \( P \in \kappa[x_1, \ldots, x_m] \):

\[
\begin{align*}
 f_m^\vee((P, -)_m) &= (P, f_m(-))_m \\
&= (s_m(P), (s_m \circ f_m)(-))_m \\
&= (s_m(P), \pi^z_m(-))_m \\
&= (s_m(P), -).
\end{align*}
\]

This proves (2). The lemma is proved. \( \square \)

**Proposition 7.11.** For \( m \geq 2 \), the map \( f_m \) defined by (69) induces (by restriction) an isomorphism of weight graded spaces between \( l_s_m \) and \( D_{sh}^\perp_m \).
Proof. The injective map $f_m$ is a graded map for weight. Hence, we only need to show that $f_m(\mathfrak l_m) = Dsh_m$. We have seen (remark 7.10) that $\mathfrak l_m^\vee$ and $Dsh_m^\vee$ are weight graded isomorphic spaces. Since the weight parts of both spaces are finite dimensional, the inclusion $f_m(\mathfrak l_m) \subset Dsh_m$ implies that $f_m(\mathfrak l_m) = Dsh_m$ and proves the proposition. We prove the inclusion $f(\mathfrak l_m) \subset Dsh_m$ (and therefore the proposition) by showing that:

$$f_m^\vee(Dsh_m') \subset \mathfrak l_m'$$

where $f_m^\vee$ is the graded dual of the map $f_m$, $Dsh_m'$ is the annihilator of $Dsh_m$ in $k[x_1, \ldots, x_m]^\vee$ the graded dual for weight of $k[x_1, \ldots, x_m]$ and $\mathfrak l_m'$ is the annihilator of $\mathfrak l_m$ in $k\langle x, z \rangle_m^\vee$ the graded dual for weight of $k\langle x, z \rangle_m$. Take $\varphi \in Dsh_m'$. Since the pairing $(-, -)_m$ is perfect there exists a $P_\varphi \in Dsh_m^\perp \subset k[x_1, \ldots, x_m]$ such that $\varphi = (P_\varphi, -)_m$. From this we deduce using (2) then (1) of lemma 7.10 that:

$$f_m^\vee(P_\varphi) = (s_m(P_\varphi), -)_m = \langle b, - \rangle_m, \quad \text{with } b \in \mathfrak l_m^\perp.$$

This shows that $f_m^\vee(Dsh_m') \subset \mathfrak l_m'$ and therefore proves the inclusion $f_m(\mathfrak l_m) \subset Dsh_m$. As mentioned at the beginning of the proof this inclusion implies the result of proposition. We have proved the proposition. \hfill $\square$

We have seen (proposition 3.1) that the inclusion $W \subset V$ induces an isomorphism of bigraded spaces $\tilde t : W/W_R \to V/F$ and we have constructed an isomorphism $\theta : V/F \to \mathfrak l_m^\vee$ (theorem 7.2). We denote by $\beta_m : W^m/W_R^m \to \mathfrak l_m^\vee$ the restriction of the map $\theta \circ \tilde t$ to depth $m$ parts. Denote by $\tilde I(n_1, \ldots, n_m)$ the image in $W/W_R$ of $I(n_1, \ldots, n_m) \in W$, the map $\beta_m$ is a weight graded isomorphism given by:

$$\beta_m(\tilde I(n_1, \ldots, n_m))(w) = \langle x_1^{n_{m-1}} \cdots x_1^{n_{1-1}}, w \rangle_m,$$

for $I(n_1, \ldots, n_m) \in W^m$ and $w \in \mathfrak l_m$.

**Lemme 7.12.** Take $m \geq 2$ and denote by $\bar f_m : \mathfrak l_m \to Dsh_m$ the isomorphism induced by $f_m$ (see the previous proposition). The following diagram of weight graded isomorphisms commutes:

$$\begin{array}{ccc}
W^m/W_R^m & \xrightarrow{\bar h_m} & Dsh_m^\vee \\
\downarrow{\beta_m} & & \downarrow{\bar f_m^\vee} \\
\mathfrak l_m^\vee & \xrightarrow{\bar f_m} & Dsh_m
\end{array}$$

where $\bar h_m$ is the map of proposition 7.6 and $\beta_m$ is defined in the previous paragraph.

**Proof.** Take $I(n_1, \ldots, n_m) \in W$ and denote by $\tilde I(n_1, \ldots, n_m)$ its image in $W^m/W_R^m$. By definition $\bar h_m(\tilde I(n_1, \ldots, n_m))(Q) = (x_1^{n_{m-1}} \cdots x_1^{n_{1-1}}, Q)_m$ for $Q \in Dsh_m$. Hence, by (2) of lemma 7.10

$$(\bar f_m^\vee \circ \bar h_m)(\tilde I(n_1, \ldots, n_m))(w) = (s_m(x_1^{n_{m-1}} \cdots x_1^{n_{1-1}}), w)_m = \langle x_1^{n_{m-1}} \cdots x_1^{n_{1-1}}, w \rangle_m,$$

for $w \in \mathfrak l_m$. By comparing the above formula to the one defining $\beta_m$ in the paragraph preceding the lemma, we deduce that $\beta_m = \bar f_m^\vee \circ \bar h_m$. This proves the lemma. \hfill $\square$
Proposition 7.13. For $m \geq 2$, the following diagram of weight graded isomorphisms commutes:

\[
\begin{array}{c}
\text{D}_{m} \xrightarrow{h_{m} \circ \eta_{m}} \text{DSh}_{m} \\
\downarrow \theta'_{m} \quad \downarrow f'_{m}
\end{array}
\]

where $\theta'_{m}$ is the restriction of the bigraded isomorphisms $\theta' : \text{D} \rightarrow \text{ls}^\vee$ of theorem 7.3 to depth $m$ parts.

**Proof.** This follows from the previous lemma, since by definition $\theta'_{m} = \beta_{m} \circ \eta_{m}$. □

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