Fundamental Commutators in a Gravitational Field

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Abstract

We show how an induced invariance of the massless particle action can be used to construct an extension of the Heisenberg canonical commutation relations in a non-commutative space-time.

1 Introduction

The issue of non-commutativity of the space-time coordinates has become a central one in all discussions about the description of physics at the Planck scale. The large amounts of momentum and energy necessary to probe the physics at this length scale must necessarily modify the space-time geometry, and this modification affects the outcome of quantum measurement processes, introducing additional new uncertainties among the canonical variables. In particular, position measurements are expected to non-commute. A rigorous way to find and incorporate these additional new uncertainties in a definite physical theory is presently unknown. The issue of non-commutative space-time coordinates also makes its appearance in discussions concerning formulations of quantum gravity. Quantum gravity has an uncertainty principle \cite{1} which prevents one from measuring positions to better accuracies than the Planck length. This effect would then be modeled by a non-vanishing commutation relation between the space-time coordinates. In these two related \cite{2} situations space-time non-commutativity introduces non-locality, and it is far from clear that Poincaré invariance can survive in a non-local, non-commutative, space-time geometry.

The point of view we explore in this work is that non-locality brings with it deep conceptual issues which have not yet been well understood. It is therefore useful to try to understand these issues in the simplest examples first, before proceeding to a more realistic theory of quantum gravity. The simplest example of a generally covariant theory is relativistic particle theory. The relativistic particle action is therefore the simplest theoretical laboratory where to search.
for the dynamical origin of a non-commutative space-time geometry, which is what ultimately causes the non-locality.

The ADM construction [3] of general relativity made it clear that the mass of a particle comes from its interactions with fields other than gravitational. A massless particle couples only to the gravitational field. Because of the existence of the holographic principle, massless particles also play important roles in the description of the low and high energy physics of M-theory [4]. By considering the dynamics of massless particles, we can access at least some part of the full physics of M-theory. For these reasons, in this work we concentrate only in the massless sector of relativistic particle theory. Our contribution to the literature is to present a theoretical path to new canonical commutation relations which incorporate uncertainties arising from gravitational effects. We find here that we can use the special-relativistic orthogonality condition between the velocity and the acceleration to induce the appearance of a new type of local scale invariance in the massless particle action functional. In the transition to the Hamiltonian formalism this new scale invariance causes the appearance of new canonical commutator relations between the transformed canonical variables. These new commutators obey all Jacobi identities among the canonical variables and preserve the structure of the Poincaré algebra. The results contained in this work are therefore important for the following two reasons. First, they make it evident that the Heisenberg canonical commutator relations are not the adequate canonical commutator relations to be used in quantum mechanical calculations when strong gravitational fields are present. Second, they make it evident that the adequate canonical commutator relations preserve the structure of the Poincaré space-time algebra. These observations can perhaps, in the future, solve the problem of the apparent incompatibility between general relativity and quantum mechanics.

2 Relativistic Particles

A relativistic particle describes is space-time a one-parameter trajectory \( x^\mu(\tau) \). A possible form of the action is the one proportional to the arc length traveled by the particle and given by

\[
S = -m \int ds = -m \int d\tau \sqrt{-\dot{x}^2} \tag{2.1}
\]

In this work we choose \( \tau \) to be the particle’s proper time, \( m \) is the particle’s mass and \( ds^2 = -\delta_{\mu\nu} dx^\mu dx^\nu \). We work in an Euclidean space-time and so the index \( \mu \) takes the values 1,2,3,4, with \( x^4 = i\epsilon t \). A dot denotes derivatives with respect to \( \tau \) and we use units in which \( \hbar = c = 1 \).

Action (2.1) is invariant under the Poincaré transformation

\[
\delta x^\mu = a^\mu + \omega_{\mu\nu} x^\nu \tag{2.2}
\]

where \( a^\mu \) is a constant vector and \( \omega_{\mu\nu} = -\omega_{\nu\mu} \). As a consequence of the invariance of action (2.1) under transformation (2.2), the following vector field
can be defined in space-time

\[ V = i a^\mu p_\mu - \frac{i}{2} \omega^{\mu\nu} M_{\mu\nu} \]  

where \( p_\mu = -i \partial_\mu \) and \( M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \). Introducing the fundamental Heisenberg commutators

\[ [p_\mu, p_\nu] = 0 \]  
\[ [x_\mu, p_\nu] = i \delta^{\mu\nu} \]  
\[ [x_\mu, x_\nu] = 0 \]

we find that the generators of the vector field \( V \) obey the algebra

\[ [p_\mu, p_\nu] = 0 \]  
\[ [p_\mu, M_{\nu\lambda}] = i \delta^{\mu\nu} p_\lambda - i \delta_{\mu\lambda} p_\nu \]  
\[ [M_{\mu\nu}, M_{\rho\lambda}] = i \delta_{\nu\lambda} M_{\mu\rho} + i \delta_{\mu\rho} M_{\nu\lambda} - i \delta_{\nu\lambda} M_{\mu\rho} - i \delta_{\mu\rho} M_{\nu\lambda} \]  

This is the Poincaré space-time algebra. Action (2.1) is also invariant under the reparametrization

\[ \tau \to \tau' = f(\tau) \]  

where \( f \) is an arbitrary continuous function of \( \tau \). As a consequence of its invariance under transformation (2.6), the particle action (2.1) defines the simplest possible generally covariant physical system.

Action (2.1) is obviously inadequate to study the massless limit of particle theory and so we must find an alternative action. Such an action can be easily computed by treating the relativistic particle as a constrained system. In the transition to the Hamiltonian formalism action (2.1) gives the canonical momentum

\[ p_\mu = \frac{m}{\sqrt{-\dot{x}^2}} \dot{x}_\mu \]  

and this momentum gives rise to the primary constraint

\[ \phi = \frac{1}{2} (p^2 + m^2) = 0 \]  

In this work we follow Dirac’s [5] convention that a constraint is set equal to zero only after all calculations have been performed. The canonical Hamiltonian corresponding to action (2.1), \( H = p \dot{x} - L \), identically vanishes. Dirac’s Hamiltonian for the relativistic particle is then

\[ H_D = H + \lambda \phi = \frac{1}{2} \lambda (p^2 + m^2) \]
where $\lambda(\tau)$ is a Lagrange multiplier, to be interpreted as an independent variable. The Lagrangian that corresponds to (2.9) is

$$L = p\dot{x} - \frac{1}{2} \lambda(p^2 + m^2)$$  \hspace{1cm} (2.10)$$

Solving the equation of motion for $p_\mu$ that follows from (2.10) and inserting the result back in it, we obtain the particle action

$$S = \int d\tau \left( \frac{1}{2} \lambda^{-1} \dot{x}^2 - \frac{1}{2} \lambda m^2 \right)$$  \hspace{1cm} (2.11)$$

In action (2.11), $\lambda(\tau)$ can be associated \cite{6} to a “world-line metric” $\gamma_{\tau\tau}$, $\lambda(\tau) = [-\gamma_{\tau\tau}(\tau)]^{\frac{1}{2}}$, such that $d\tau^2 = \gamma_{\tau\tau} d\tau d\tau$. In (2.11), $\lambda(\tau)$ is an “einbein” field. In more dimensions, the “vielbein” $e^a_\mu$ is an alternative description of the metric tensor \cite{4}. In this context, the particle mass $m$ plays the role of a (0+1)-dimensional “cosmological constant”. Action (2.11) is classically equivalent to action (2.1). This can be checked in the following way. If we solve the classical equation of motion for $\lambda(\tau)$ that follows from (2.11) we get the result $\lambda = \pm (\sqrt{-\dot{x}^2}/m)$. Inserting the solution with the positive sign in (2.11), it becomes identical to (2.1). The great advantage of action (2.11) is that it has a smooth transition to the $m = 0$ limit.

The general covariance of action (2.11) manifests itself through invariance under the transformation

$$\delta x^\mu = \epsilon \dot{x}^\mu$$  \hspace{1cm} (2.12a)$$

$$\delta \lambda = \frac{d}{d\tau} (\epsilon \lambda)$$  \hspace{1cm} (2.12b)$$

where $\epsilon(\tau)$ is an arbitrary infinitesimal parameter. Varying $x^\mu$ in (2.11) we obtain the classical equation for free motion

$$\frac{d}{d\tau} (\dot{x}_\mu \lambda) = \frac{dp_\mu}{d\tau} = 0$$  \hspace{1cm} (2.13)$$

Now we make a transition to the massless limit. This limit is described by the action

$$S = \frac{1}{2} \int d\tau \lambda^{-1} \dot{x}^2$$  \hspace{1cm} (2.14)$$

Action (2.14) is invariant under the Poincaré transformation (2.2) with $\delta \lambda = 0$, and under the infinitesimal reparametrization (2.12). The classical equation of motion for $x^\mu$ that follows from action (2.14) is identical to (2.13). The equation of motion for $\lambda$ gives the condition $\dot{x}^2 = 0$, which tells us that a massless particle moves at the speed of light. As a consequence of this, it becomes impossible to solve for $\lambda(\tau)$ from its equation of motion. In the massless theory the value of $\lambda(\tau)$ is completely arbitrary.

In the transition to the Hamiltonian formalism action (2.14) gives the canonical momenta

$$p_\lambda = 0$$  \hspace{1cm} (2.15)$$
\[ p_\mu = \frac{\dot{x}_\mu}{\lambda} \]  
\hspace{1cm} (2.16)

and the canonical Hamiltonian

\[ H = \frac{1}{2} \lambda p^2 \]  
\hspace{1cm} (2.17)

Equation (2.15) is a primary constraint. Introducing the Lagrange multiplier \( \xi(\tau) \) for this constraint we can write the Dirac Hamiltonian

\[ H_D = \frac{1}{2} \lambda p^2 + \xi p_\lambda \]  
\hspace{1cm} (2.18)

Requiring the dynamical stability of constraint (2.15), \( \dot{p}_\lambda = \{p_\lambda, H_D\} = 0 \), we obtain the secondary constraint

\[ \phi = \frac{1}{2} p^2 = 0 \]  
\hspace{1cm} (2.19)

Now we may use the fact that we are dealing with a special-relativistic system. Special relativity has the dynamical feature [7] that the relativistic velocity is always orthogonal to the relativistic acceleration, \( \dot{x} \ddot{x} = 0 \). We can use this orthogonality to induce the invariance of the massless particle action (2.14) under the transformation

\[ x^\mu \rightarrow \tilde{x}^\mu = \exp\{\beta(\dot{x}^2)\} x^\mu \]  
\hspace{1cm} (2.20a)

\[ \lambda \rightarrow \exp\{2\beta(\dot{x}^2)\} \lambda \]  
\hspace{1cm} (2.20b)

where \( \beta \) is an arbitrary function of \( \dot{x}^2 \). Transformation (2.20) is a new type of local scale invariance of the action for a massless particle. Action (2.14) is invariant under (2.20) because the orthogonality condition forces \( \dot{x}^\mu \) to transform as \( \exp\{\beta(\dot{x}^2)\} \dot{x}^\mu \) when \( x^\mu \) transforms as in (2.20a). We emphasize that although the orthogonality condition must be used to get the invariance of action (2.14) under transformation (2.20), this condition is not an external ingredient in the theory. In fact, the orthogonality between the relativistic velocity and acceleration is an unavoidable condition here. It is an imposition of special relativity.

Consider now the commutator structure that transformation (2.20a) induces on the canonical operators. From the definition (2.16) of the canonical momentum we find that the momenta must transform as

\[ p_\mu \rightarrow \tilde{p}_\mu = \exp\{-\beta(\dot{x}^2)\} p_\mu \]  
\hspace{1cm} (2.21)

when \( x^\mu \) transforms as in (2.20a). It can be verified that transformations (2.20) and (2.21) together leave invariant the \( m = 0 \) limit of the first order Lagrangian (2.10), and the massless particle Hamiltonian (2.17). These observations confirm that transformation (2.20) is a true invariance of the massless particle action (2.14).
Taking $\beta(\dot{x}^2) = \beta(\lambda^2 p^2)$ in transformations (2.20a) and (2.21), and retaining only the linear terms in $\beta$ in the exponentials, we find that the new transformed canonical variables ($\tilde{x}_\mu, \tilde{p}_\mu$) obey the commutators

$$[\tilde{p}_\mu, \tilde{p}_\nu] = 0 \quad (2.22a)$$

$$[\tilde{x}_\mu, \tilde{p}_\nu] = (1 + \beta)\{i\delta_{\mu\nu}(1 - \beta) - [x_\mu, \beta]p_\nu\} \quad (2.22b)$$

$$[\tilde{x}_\mu, \tilde{x}_\nu] = (1 + \beta)\{x_\mu[\beta, x_\nu] - x_\nu[\beta, x_\mu]\} \quad (2.22c)$$

written in terms of the old canonical variables. These commutators obey the non trivial Jacobi identities $(\tilde{x}_\mu, \tilde{x}_\nu, \tilde{x}_\lambda) = 0$ and $(\tilde{x}_\mu, \tilde{p}_\nu, \tilde{p}_\lambda) = 0$. They also reduce to the usual Heisenberg commutators when constraint (2.19) is imposed. We see from commutator (2.22c) that transformation (2.20a) induces a transition to a non-commutative space-time geometry.

The non-commutative space-time geometry is completely determined by the choice $\beta(\dot{x}^2) = \beta(\lambda^2 p^2) = \lambda^2 p^2$. This is because commutators satisfy the property $[A^n, B] = n A^{n-1}[A, B]$. If we consider, for instance, $\beta = (\lambda^2 p^2)^2$ and compute $[\beta, x^\mu]$ we will find $[\beta, x^\mu] = 2\lambda^2 p^2[\lambda^2 p^2, x^\mu]$, and this vanishes when constraint (2.19) is imposed. Similarly, all higher order terms will vanish when (2.19) is imposed, and the space-time geometry is completely determined by the case $\beta = \lambda^2 p^2$. Computing the commutators (2.22b) and (2.22c) for this form of $\beta$, and finally imposing constraint (2.19), we arrive at the canonical commutation relations

$$[\tilde{p}_\mu, \tilde{p}_\nu] = 0 \quad (2.23a)$$

$$[\tilde{x}_\mu, \tilde{p}_\nu] = i\delta_{\mu\nu} - i\lambda^2 p_\mu p_\nu \quad (2.23b)$$

$$[\tilde{x}_\mu, \tilde{x}_\nu] = -2i\lambda^2(x_\mu p_\nu - x_\nu p_\mu) \quad (2.23c)$$

while the transformation equations (2.20a) and (2.21) become

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu \quad (2.24a)$$

$$p_\mu \rightarrow \tilde{p}_\mu = p_\mu \quad (2.24b)$$

We can then write down the canonical commutators

$$[p_\mu, p_\nu] = 0 \quad (2.25a)$$

$$[x_\mu, p_\nu] = i\delta_{\mu\nu} - i\lambda^2 p_\mu p_\nu \quad (2.25b)$$

$$[x_\mu, x_\nu] = -2i\lambda^2(x_\mu p_\nu - x_\nu p_\mu) \quad (2.25c)$$
Notice that, after imposing constraint (2.19), we can not return to a commutative space-time by performing the inverse transformation because the transformations (2.20a) and (2.21) become the identity transformation, as given by (2.24). Now the only way to return to a commutative space-time is with the vanishing of the (0+1)-dimensional gravitational field described by $\lambda(\tau)$. This makes it evident that the Heisenberg canonical commutation relations (2.4) are adequate only in the presence of vanishing, or at best very weak, gravitational fields.

Commutators (2.25) obey all Jacobi identities among the canonical variables. It can be also verified that if we compute the algebra of the generators of the vector field (2.3) using the commutators (2.25), instead of the Heisenberg commutators (2.4), the same expressions (2.5) are reproduced. Commutators (2.25) preserve the structure of the Poincaré space-time algebra. Since in the massless theory the value of $\lambda(\tau)$ is arbitrary, as we saw above, we can use the reparametrization invariance (2.12) to choose a gauge in which $\lambda = 1$. We then end up with the convenient commutators

\[ [p_\mu, p_\nu] = 0 \quad (2.26a) \]

\[ [x_\mu, p_\nu] = i\delta_{\mu\nu} - ip_\mu p_\nu \quad (2.26b) \]

\[ [x_\mu, x_\nu] = -2i(x_\mu p_\nu - x_\nu p_\mu) \quad (2.26c) \]

Commutators (2.26) form a consistent set of quantum commutators on which an entirely new formulation of quantum mechanics in a gravitational field can be based. Perhaps this new formulation can have the power to bring the apparent incompatibility between general relativity and quantum mechanics to an end.

3 Conclusion

In this work we deduced a new set of canonical commutation relations for the massless relativistic particle. This new set generalizes the usual Heisenberg canonical commutators to the case when a gravitational field is present. The origin of these new commutators is a new type of local scale invariance of the massless particle action. This new invariance manifests itself when the special-relativistic orthogonality condition between the velocity and the acceleration is imposed. The new commutators obey all Jacobi identities among the canonical variables and, although they bring in a certain amount of non-locality, this non-locality does not conflict with the Poincaré invariance of the theory because the new commutators leave the Poincaré space-time algebra invariant. The new commutators therefore form a consistent set of quantum commutators which could solve the apparent incompatibility between general relativity and quantum mechanics.
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