Independence, Conditionality and Structure of Dempster-Shafer Belief Functions

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Abstract

Several approaches of structuring (factorization, decomposition) of Dempster-Shafer joint belief functions from literature are reviewed with special emphasis on their capability to capture independence from the point of view of the claim that belief functions generalize Bayes notion of probability. It is demonstrated that Zhu and Lee’s [12] logical networks and Smets’ [11] directed acyclic graphs are unable to capture statistical dependence/independence of Bayesian networks [5]. On the other hand, though Shenoy and Shafer’s hypergraphs can explicitly represent Bayesian network factorization of Bayesian belief functions, they disclaim any need for representation of independence of variables in belief functions. Cano et al. [1] reject the hypergraph representation of Shenoy and Shafer just on grounds of missing representation of variable independence, but in their frameworks some belief functions factorizable in Shenoy/Shafer framework cannot be factored. The approach in [4] on the other hand combines the merits of both Cano et al. and of Shenoy/Shafer approach in that for
Shenoy/Shafer approach no simpler factorization than that in [4] approach exists and on the other hand all independences among variables captured in Cano et al. framework and many more are captured in [4] approach.

1 Introduction

The Dempster-Shafer Theory or the Mathematical Theory of Evidence (MTE) [6], [2] shows one of possible ways of application of mathematical probability for subjective evaluation and is intended to be a generalization of bayesian theory of subjective probability [8]. Belief functions are deemed to generalize (finite discrete) probability functions in that belief functions assign basic belief mass to (non-empty) subsets of set of elementary events, whereas probability functions assign basic belief mass only to elementary events. It is frequently claimed that, though they comprise something more than just probabilistic uncertainty, MTE belief function behavior reduces to behavior of probability if probabilities are available [11]. That is if a belief function assigns non-zero basic belief mass only to subsets of cardinality 1 of the set of elementary events, then it is called bayesian belief function and considered as equivalent to probability function.

A known method of representation of joint probability distribution in (many) discrete variables are so-called bayesian networks (as described e.g. in [5], [3]). The joint probability distribution \( Pr(x_1, ..., x_n) \) in variables \( X_1, X_2, ..., X_n \), where for a node \( X_i \) only variables with indices from the set \( \pi(i) \) directly influence the value of \( X_i \), is expressed as:

\[
Pr(x_1, ..., x_n) = \prod_{i=1}^{n} Pr(x_i|x_{\pi(i)})
\]

It is assumed that if we form a directed graph with nodes representing variables and directed edges are (all and only) of the form \( (X_k \rightarrow X_i) \) with \( k \in \pi(i) \) then this graph is acyclic. As a direct representation of a joint probability distribution in 15 discrete variables, with a domain with cardinality four each, would require more than 1 000 000 000 storage cells, bayesian network representation will immensely contribute to reduction of storage requirement if every variable is directly influenced by only a few other. Beside this, however, they are known to represent qualitatively many (conditional)
independences among variables as well as to capture (a part of) causal relations among variables.

Structuring is much more urgently needed for belief functions. If we have a MTE belief distribution in 3 discrete variables, each with a domain of cardinality 4, then the joint belief distribution will be non-zero for possibly $2^{4^3} - 1 > 1000000000000000000$ points! A much more elaborated handling of structure of a joint belief distribution may be needed, but let us restrict ourselves to the modest requirement to structure at least as good as bayesian networks do for probability.

Several concepts of structuring belief functions have been proposed. Shenoy and Shafer [9] have proposed factorizations of belief functions along hyper-graphs. Smets [11] and Cano at al. [1] proposed factorization of belief function along directed acyclic graphs (Both proposals are radically different, nonetheless). Zhu and Lee [12] proposed amendment of a logical network connectives with specialized belief functions. Still another type of belief network, based on directed acyclic graphs, has been proposed in [4].

Within this paper, let us look at these proposals from the point of view of the benchmark established by bayesian networks. Especially we will ask whether or not these proposals cover structuring of probability distributions into a bayesian network and whether they are capable of expressing dependence independence relations among variables, especially for special case of probability distribution.

Basic definitions from the MTE are given in Appendix.

2 Logical Networks of Zhu and Lee

Zhu and Lee [12] propose representation of a knowledge base (thus the joint belief) as a set of rules (and facts) amended by MTE-styled truth probability intervals. Though a name ”logical network” is never used in their paper, it is clearly an intention of the authors to have one as they consider forward and backward propagation for each type of basic logical connector (negation, and, or, material implication). For a rule $r : A \rightarrow B$ its probability interval would be $[r_L, r_U]$. This implies immediately $m(\overline{A} \lor B) = r_L$, $m(A \land \overline{B}) = 1 - r_U$, $m(X) = r_U - r_L$, where $X=\{\text{true, false}\}$ is the logical universe considered in the paper. The authors claim the necessity of permanent search for common frame of discernment for combined rules and facts to be major disadvantage
of general MTE framework. Therefore they seek a way around by restricting themselves to logical values of rules and facts. They derive formulas for forward and backward propagation of uncertainty as well as for combination of evidence.

On page 347 they propose e.g. the following ”Modus Ponens”:

\[
\begin{align*}
A \rightarrow B &:: [r_L, r_U] \\
A &:: [a_L, a_U] \\
B &:: [b_L, b_U]
\end{align*}
\]

with

\[
\begin{align*}
b_L &= \min\{1, \max\{0, (r_L + a_U - 1)/a_U\}\} \\
b_U &= \min\{1, \max\{0, [r_U + a_U - a_L(r_L + a_U - 1)/a_U - 1]/(a_U - a_L)\}\}
\end{align*}
\]

One could feel impressed by the simplicity of this and other formulas if not the way they are derived. On page 345 we find the table for joint belief distribution of A and B (Table II, proposed conjunction procedure for \(A \land B\)):

| \(A\) | \(B\) | \(B\) | \(B\) | \(B\) |
|-----|-----|-----|-----|-----|
| \{t\},a_L | \{t\},b_L | \{f\},1-b_U | \{t,f\},b_U-b_L |
| \{f\},1-a_U | \{f\},m_{21} | \{f\},m_{22} | \{f\},m_{23} |
| \{t,f\},a_U-a_L | \{t\},m_{31} | \{f\},m_{32} | \{t,f\},m_{33} |

Out of this table the basic belief assignment for implication is (presumably) derived as follows:

\[
m_{A \rightarrow B}(\{true\}) = r_L = m_{11} + m_{21} + m_{22} + m_{23} + m_{31}
\]

\[
m_{A \rightarrow B}(\{false\}) = 1 - r_U = m_{12} + m_{13} + m_{32}
\]

\[
m_{A \rightarrow B}(\{true, false\}) = r_U - r_L = m_{33}
\]

Up to this point one can little complain (beside e.g. typing error in \{f\} column header of B). But subsequently authors assume independence(!) of A and B, just: \(m_{11} = a_L b_L\), \(m_{12} = a_L(1-b_U)\), \(m_{13} = a_L(b_U - b_L)\), \(m_{21} = (1-a_U)b_L\), \(m_{22} = (1-a_U)(1-b_U)\), \(m_{23} = (1-a_U)(b_U - b_L)\), \(m_{31} = (a_U - a_L)b_L\), \(m_{32} = (a_U - a_L)(1-b_U)\), \(m_{33} = (a_U - a_L)(b_U - b_L)\).

Hence

\[
m_{A \rightarrow B}(\{true\}) = a_L b_L + (1-a_U)b_L + (1-a_U)(1-b_U) + (1-a_U)(b_U - b_L) + (a_U - a_L)b_L =
\]
\[ a_L b_L + a_L (b_U - b_L) + (1 - a_U) + (a_U - a_L) b_L = \]
\[ = a_L b_L + 1 - a_U + a_U b_L - a_L b_L = \]
\[ = a_L b_L + 1 - a_U + a_U b_L - a_L b_L = \]
\[ = 1 - a_U + a_U b_L = r_L \]

In a similar way we obtain an expression for

\[ m_{A \rightarrow B} (\{true, false\}) = (a_U - a_L) \cdot (b_U - b_L) = \]
\[ = a_U \cdot b_U - a_U \cdot b_L - a_L \cdot b_U + a_L \cdot b_L = \]
\[ = r_U - r_L \]

Hence

\[ r_U = a_U b_U - a_U b_L - a_L b_U + a_L b_L + 1 - a_U + a_U b_L = \]
\[ = a_U b_U - a_L b_U + a_L b_L + 1 - a_U \]

These are the formulas for \([r_L, r_U]\) interval of a rule \(R\) as presented on page 346. Equations for \(r_L, r_U\) are solved to obtain \(b_L\) and \(b_U\) as given at the beginning of this section. Let us assume that logical formulas \(A\) and \(B\) have the following joint distribution of probability of truth:

| A   | \{t\}, \(a_L = 0.4\) | \{f\}, \(1 - a_U = 0.6\) | \{t,f\}, \(a_U - a_L = 0\) |
|-----|---------------------|---------------------|---------------------|
| \{t\} | 0.1                 | 0.2                 | 0                   |
| \{f\} | 0.3                 | 0.4                 | 0                   |
| \{t,f\} | 0                   | 0                   | 0                   |

Hence \(a_L = a_U = 0.4\) and \(b_L = b_U = 0.3\). So \(r_L = 1 - 0.4 + 0.3 \cdot 0.4 = 0.48\) and \(r_U = 0.4 \cdot 0.3 - 0.4 \cdot 0.3 + 0.4 \cdot 0.3 + 1 - 0.4 = 0.48\).

Let us assume we know from somewhere that \(A\) is true nearly for sure that is that probability of truth of \(A\) is \(a_L = 0.999, a_U = 1\). Then from Zhu and Lee formulas follows:

\[ b_L = \min\{1, \max\{0, (0.48 + 1 - 1)/1\}\} = 0.48 \]
\[ b_U = \min\{1, \max\{0, [0.48 + 1 - 0.999(0.48 + 1 - 1)/1 - 1]/(1 - 0.999)\}\} = \]
\[ = \min\{1, \max\{0, [0.48 + 0.999 \cdot 0.48]/(1 - 0.999)\}\} = 0.48 \]
But if we look at the data then it is clear that the probability of truth of B given truth of A is 0.25 and given the narrow uncertainty bound on truth of A the conditional (obtained by Jeffrey’s rule) will not exceed 0.26.

This example demonstrates in a clear way that the interpretation of DST proposed in [12] in no way supports the generally expressed claim that DST can capture bayesian reasoning as a special case. Also the source of the bug is obvious. If one assumes a priori the independence of facts and hypotheses (both in bayesian and DST sense) then one shall not wonder that the rule of inference tells nothing meaningful about the relationship between variables considered.

3 Directed Acyclic Graphs of Smets

In his paper [11] Smets attempts to generalize the bayesian theorem (being foundation of probability propagation in bayesian networks, e.g. [5]) in such a way as to enable propagation of beliefs in a directed networks. For this purpose he introduces a special notion of conditional beliefs (page 6)

\[ \text{bel}(B; A) = \text{bel}(B \cup \overline{A}) - \text{bel}(\overline{A}) \quad \forall B \subseteq \Omega \]

\( \Omega \) - set of all elementary events.

On page 5 he states that \( \text{bel}(\emptyset) = 0 \). Let \( x \) be a subset of the set \( X \), \( \theta \) a subset of the set \( \Theta \). Then on page 8 he states that \( \text{pl}_X(x; \theta) = \text{pl}_{X \times \Theta}(\text{cyl}(x); \text{cyl}(\theta)) \). On page 9 he writes that \( \text{pl}(A) = \text{bel}(\Omega) - \text{bel}(\overline{A}) \).

Though not explicitly stated, we expect that also \( \text{pl}_X(A; \theta) = \text{bel}_X(X; \theta) - \text{bel}_X(x; \theta) \) should hold. So

\[ \text{bel}(B; A) = \text{bel}(B \cup \overline{A}) - \text{bel}(\overline{A}) \]

\[ \text{bel}(\Omega; A) = \text{bel}(\Omega \cup \overline{A}) - \text{bel}(\overline{A}) \]

Hence

\[ \text{bel}(\Omega; A) - \text{bel}(B; A) = \text{bel}(\Omega \cup \overline{A}) - \text{bel}(\overline{A}) - \text{bel}(B \cup \overline{A}) + \text{bel}(\overline{A}) = \]

\[ = \text{bel}(\Omega) - \text{bel}(B \cup \overline{A}) = \]
Hence
\[
pl_X(x; \theta) = pl_{X \times \Theta}(cyl(x); cyl(\theta)) = bel_{X \times \Theta}(cyl(X); cyl(\theta)) - bel_{X \times \Theta}(\overline{cyl(x)}; cyl(\theta))
\]

But then we easily derive that \( bel_X(x; \theta) = bel_{X \times \Theta}(cyl(x); cyl(\theta)) \).

He gives also the formula that given two belief distributions \( bel_1, bel_2 \) and \( bel_{12} = bel_1 \ominus bel_2 \) (\( \ominus \) is a version of \( \oplus \) which is not normalized) we have
\[
m_{12}(A) = \sum_{B \subseteq \Omega} m_1(A; B)m_2(B)
\]

On page 12 he defines that two variables \( X \) and \( Y \) are said to be independent iff
\[
bel_X(A; y) = bel_X(A; y'), \forall A \subseteq X, \forall y, y' \in Y, y \neq y'
\]
and
\[
bel_Y(B; x) = bel_Y(B; x'), \forall B \subseteq Y, \forall x, x' \in X, x \neq x'
\]
Furthermore, for the set (of contexts) \( \Theta = \{\theta_i, i = 1, \ldots, n\} \) he defines that when two observations are independent whatever the context \( \theta_i \), then they are called conditionally independent.

Then on page 16 he requires that there is a \( bel_{\Theta}(.; x; y) \) such that
\[
bel_{\Theta}(.; x; y) = bel_{\Theta}(.; x) \ominus bel_{\Theta}(.; y)
\]

Let us consider the consequences. Let \( X \), and \( Y \) be sets \( \{x_p, x_q\} \) and \( \{y_p, y_q\} \) resp. Let \( bel_{X \times Y} \) be a bayesian belief distribution, that is \( m_{X \times Y}(A) > 0 \) for some \( A \subseteq X \times Y \) with card(A)=1 (singletons) and elsewhere equal zero.

Let be given the following distribution - basic belief assignment

| Y   | {y_p} | {y_q} |
|-----|-------|-------|
| X   | {x_p} | \( m_{pp} \) \( m_{pq} \) |
|     | {x_q} | \( m_{qp} \) \( m_{qq} \) |
Under which conditions are \(X\) and \(Y\) "cognitively independent"? Smets requires that \(\text{bel}_X(x_p; y_p) = \text{bel}_X(x_p; y_q), \text{bel}_X(x_q; y_p) = \text{bel}_X(x_q; y_q), \text{bel}_Y(y_p; x_p) = \text{bel}_Y(y_p; x_q), \text{bel}_Y(y_q; x_p) = \text{bel}_Y(y_q; x_q)\).

Following Smets’ notation \(\text{cyl}(x)\) shall denote cylindric (vacuous) extension of set \(x\). Now

\[
\text{bel}_X(x_p; y_p) = \text{bel}_X(x_p; \text{cyl}(y_p)) = \\
= \text{bel}_X(\Theta; \text{cyl}(x_p) \cup \text{cyl}(y_p)) - \text{bel}_X(\Theta; \text{cyl}(y_p)) = \\
= (m_{pp} + m_{pq} + m_{qq}) - (m_{pq} + m_{qq}) = m_{pp}
\]

\(\text{bel}_X(x_p; y_q) = m_{pq}, \text{bel}_X(x_q; y_p) = m_{qp}, \text{bel}_X(x_q; y_q) = m_{qq}, \text{bel}_Y(y_p; x_p) = m_{pp}, \text{bel}_Y(y_p; x_q) = m_{qp}, \text{bel}_Y(y_q; x_p) = m_{pq}, \text{bel}_Y(y_q; x_q) = m_{qq}\). But \(\text{bel}_X(x_p; y_p) = \text{bel}_X(x_p; y_q)\) implies \(m_{pp} = m_{pq}\). \(\text{bel}_Y(y_p; x_p) = \text{bel}_Y(y_p; x_q)\) implies \(m_{qp} = m_{qq}\), \(\text{bel}_Y(y_q; x_p) = \text{bel}_Y(y_q; x_q)\) implies \(m_{pp} = m_{pq}\).

Hence both are independent only if

\[
\begin{array}{ccc|c|c|c}
X & \{y_p\} & \{y_q\} \\
\hline
\{x_p\} & m & m \\\n\{x_q\} & m & m \\
\end{array}
\]

\((m - \text{a constant equal } 1/4)\).

That is cognitive independence of Smets does not cover statistical independence for bayesian belief functions, but rather is a very special case of it (for uniform distributions).

Let us consider now three variables \(X, Y, \Theta\) for conditional independence of \(X, Y\) on \(\Theta\). Let \(X, Y\) have domains as above, let \(\Theta = \{\theta_p, \theta_q\}\). Let the joint belief distribution basic belief assignment be as follows:

\[
\begin{array}{ccc|c|c|c}
X & \{y_p\} & \{y_q\} \\
\hline
\{x_p\} & m_{pp} & m_{pq} \\\n\{x_q\} & m_{qp} & m_{qq} \\
\end{array}
\]

\[
\begin{array}{ccc|c|c|c}
X & \{y_p\} & \{y_q\} \\
\hline
\{x_p\} & m_{ppq} & m_{pqq} \\\n\{x_q\} & m_{qpq} & m_{qqq} \\
\end{array}
\]

When may \(X\) and \(Y\) be conditionally independent given \(\Theta\)? Smets requires that among others

\[
m_{\Theta}(\theta_p; x_p, y_p) = m_{\Theta}(\theta_p; x_p) \cdot m_{\Theta}(\theta_p; y_p)
\]
One can easily check that for bayesian belief functions \( bel(B \cup \overline{A}) = bel(B \cap A) + bel(\overline{A}) \). Then \( bel(B; A) = bel(B \cap A) \). But \( bel(B \cap A) \) is the sum of m-function values for all singleton subsets of \( B \cap A \).

This actually means that:

\[
m_\Theta(\theta_p; x_p, y_p) = m_\Theta(\theta_p; x_p) \cdot m_\Theta(\theta_p; y_p)
\]

\[
m_\Theta(\theta_p; x_q, y_p) = m_\Theta(\theta_p; x_q) \cdot m_\Theta(\theta_p; y_p)
\]

\[
m_\Theta(\theta_p; x_q, y_q) = m_\Theta(\theta_p; x_q) \cdot m_\Theta(\theta_p; y_q)
\]

\[
m_\Theta(\theta_q; x_p, y_p) = m_\Theta(\theta_q; x_p) \cdot m_\Theta(\theta_q; y_p)
\]

\[
m_\Theta(\theta_q; x_q, y_q) = m_\Theta(\theta_q; x_q) \cdot m_\Theta(\theta_q; y_q)
\]

\[
m_\Theta(\theta_q; x_p, y_q) = m_\Theta(\theta_q; x_p) \cdot m_\Theta(\theta_q; y_q)
\]

\[
m_\Theta(\theta_q; x_q, y_p) = m_\Theta(\theta_q; x_q) \cdot m_\Theta(\theta_q; y_p)
\]

\[
m_\Theta(\theta_q; x_q, y_q) = m_\Theta(\theta_q; x_q) \cdot m_\Theta(\theta_q; y_q)
\]

etc. Hence

\[
m_\Theta \times X \times Y (\theta_p; x_p, y_p) = m_\Theta \times X \times Y (\theta_p; x_p, y_p)^2 + m_\Theta \times X \times Y (\theta_p; x_p, y_q) \cdot m_\Theta \times X \times Y (\theta_p; x_p, y_p) +
\]

\[
+ m_\Theta \times X \times Y (\theta_p; x_p, y_p) \cdot m_\Theta \times X \times Y (\theta_p; x_q, y_p)
\]

etc. Hence

\[
m_\Theta \times X \times Y (\theta_p; x_p, y_p) + m_\Theta \times X \times Y (\theta_p; x_p, y_q) + m_\Theta \times X \times Y (\theta_p; x_q, y_p) + m_\Theta \times X \times Y (\theta_p; x_q, y_q) =
\]

\[
= m_\Theta \times X \times Y (\theta_p; x_p, y_p)^2 + m_\Theta \times X \times Y (\theta_p; x_p, y_p) \cdot m_\Theta \times X \times Y (\theta_p; x_p, y_q) +
\]

\[
+ m_\Theta \times X \times Y (\theta_p; x_p, y_q) + m_\Theta \times X \times Y (\theta_p; x_q, y_p) +
\]

\[
+ m_\Theta \times X \times Y (\theta_p; x_q, y_q)^2 + m_\Theta \times X \times Y (\theta_p; x_p, y_q) \cdot m_\Theta \times X \times Y (\theta_p; x_q, y_p) +
\]

\[
+ m_\Theta \times X \times Y (\theta_p; x_q, y_q) \cdot m_\Theta \times X \times Y (\theta_p; x_q, y_p) +
\]

\[
+ m_\Theta \times X \times Y (\theta_p; x_q, y_q) \cdot m_\Theta \times X \times Y (\theta_p; x_q, y_q)
\]
+m_{θ×X×Y}(θ_p, x_p, y_p) + m_{θ×X×Y}(θ_p, x_q, y_q) +
+m_{θ×X×Y}(θ_p, x_q, y_p)^2 + m_{θ×X×Y}(θ_p, x_q, y_p) \cdot m_{θ×X×Y}(θ_p, x_p, y_p) +
+m_{θ×X×Y}(θ_p, x_q, y_p) + m_{θ×X×Y}(θ_p, x_q, y_p) +
+m_{θ×X×Y}(θ_p, x_q, y_p) \cdot m_{θ×X×Y}(θ_p, x_p, y_p) +
+m_{θ×X×Y}(θ_p, x_q, y_q) + m_{θ×X×Y}(θ_p, x_p, y_q) +
+m_{θ×X×Y}(θ_p, x_q, y_q) \cdot m_{θ×X×Y}(θ_p, x_p, y_q) +
+m_{θ×X×Y}(θ_p, x_q, y_q) + m_{θ×X×Y}(θ_p, x_p, y_q)

Hence

m_{θ×X×Y}(θ_p, x_p, y_p) + m_{θ×X×Y}(θ_p, x_q, y_q) +
+m_{θ×X×Y}(θ_p, x_q, y_p) + m_{θ×X×Y}(θ_p, x_q, y_q) =
= (m_{θ×X×Y}(θ_p, x_p, y_p) + m_{θ×X×Y}(θ_p, x_p, y_q) +
+m_{θ×X×Y}(θ_p, x_q, y_p) + m_{θ×X×Y}(θ_p, x_q, y_q))^2

That is \( bel_{θ×X×Y}(θ_p) = bel_{θ×X×Y}(θ_p)^2 \). Shall we take seriously the assumption from page that \( \sum_{A⊆Ω} m(A) = 1 \), then either \( bel_{θ×X×Y}(θ_p) = 1 \) and \( bel_{θ×X×Y}(θ_q) = 0 \) or \( bel_{θ×X×Y}(θ_p) = 0 \) and \( bel_{θ×X×Y}(θ_q) = 1 \). But this actually means that variable \( Θ \) does not influence the joint distribution of \( X,Y \) at all. Just the notion of conditional independence for bayesian belief networks of Smets is devoid of any meaning.

4 Hypergraphs of Shenoy and Shafer

In [9], Shenoy and Shafer proposed a general framework for uncertainty propagation if uncertainty is structured along a hypergraph. Their framework covers both probability and belief functions.

Hypergraphs: A nonempty set \( H \) of nonempty subsets of a finite set \( S \) be called a hypergraph on \( S \). The elements of \( H \) be called hyperedges. Elements of \( S \) be called vertices. \( H \) and \( H' \) be both hypergraphs on \( S \), then we call a hypergraph \( H' \) a reduced hypergraph of the hypergraph \( H \), iff for every \( h' ∈ H' \) also \( h' ∈ H \) holds, and for every \( h ∈ H \) there exists such a \( h' ∈ H' \) that \( h ⊆ h' \). A hypergraph \( H \) covers a hypergraph \( H' \) iff for every \( h' ∈ H' \) there exists such a \( h ∈ H \) that \( h' ⊆ h \).

Hypertrees: \( t \) and \( b \) be distinct hyperedges in a hypergraph \( H \), \( t ∩ b ≠ ∅ \), and \( b \) contains every vertex of \( t \) that is contained in a hyperedge of \( H \) other than \( t \); if \( X ∈ t \) and \( X ∈ h \), where \( h ∈ H \) and \( h ≠ t \), then \( X ∈ b \). Then
we call $t$ a twig of $H$, and we call $b$ a branch for $t$. A twig may have more than one branch. We call a hypergraph a hypertree if there is an ordering of its hyperedges, say $h_1, h_2, ..., h_n$ such that $h_k$ is a twig in the hypergraph $\{h_1, h_2, ..., h_k\}$ whenever $2 \leq k \leq n$. We call any such ordering of hyperedges a hypertree construction sequence for the hypertree. The first hyperedge in the hypertree construction sequence be called the root of the hypertree construction sequence.

Please refer to the paper of Shenoy and Shafer [9] on notions of Markov trees, variables ($V$), valuations ($VV$), valuations on a set of variables $h$ ($VV_h$), and proper valuations, combination operator $\oplus : VV \times VV \rightarrow VV$, marginalization operator $\downarrow h : \bigcup\{VV_g|g \subseteq h\} \rightarrow VV_h$, the axiomatic framework and the local computation method of Shenoy and Shafer. We recall here only the definitions of:

**Factorization:** Suppose $A$ is a valuation on a finite set of variables $V$, and suppose $HV$ is a hypergraph on $V$. If $A$ is equal to the combination of valuations of all hyperedges $h$ of $HV$ then we say that $A$ factors on $HV$.

The valuation for MTE is simply the belief function. For MTE the fact that a belief function $Bel$ defined for the set of variables $V$ factors over a hypergraph $HV$ means that it may be represented as

$$Bel = \bigoplus_{h: h \in HV} Bel^h$$

where $Bel^h$ is a (different) belief function defined over the set of variables $h$.

**Conditioning:** Suppose $Bel$ is a belief distribution, and $Bel_E$ is an indicator potential capturing the evidence $E$ (that is $Bel_E$ is such a belief function that $M_E(E) = 1$ and for any $A$ different from $E$ $m(A) = 0$). Then conditional belief function conditioned on $E$, $Bel(.||E)$, is defined as $Bel(.||E) = Bel \oplus Bel_E$.

Actually the propagation of belief in the hypergraph via Shenoy/Shafer propagation of uncertainty means for a given hypothesis variable $V_i$ calculation of $Bel(.||E)^{VV_i} (\downarrow V_i$ means projection of belief function onto the subspace of the single variable $V_i$) for the given belief function $Bel$, the factorization of which along a hypergraph is known, and for the evidence $E$ in such a way as to avoid the calculation of the complete function $Bel$ and $Bel(.||E)$ as a in-between result - because both functions may consume too much memory. To manage it, the underlying hypergraph is first transformed to a hypertree.
and thereafter the computations are quick.

Shenoy and Shafer consider it unimportant whether or not the factors $Bel^h$ factorization of $Bel$ should refer to any notion of conditionality. In fact, one may easily construct a belief function factorization in which no proper subset of factors can tell anything about marginal distribution of any variable belonging to the factors of this subset.

The hypergraph makes the impression of apparent greater generality than bayesian networks of Pearl [5] because in case of bayesian belief functions simpler factorization is possible (less factors, fewer variables) than that along a bayesian network. However, in [4] it has been shown that this is only a superficial effect because the real propagation is run on hypertrees, and not in general type hypergraphs, and then the generality of Shenoy & Shafer factorization gives nothing beyond that of bayesian network factorization (see also below).

5 Cano’s et al. A Priori Conditionals in Directed Acyclic Graphs

Cano et al. in [1] proposed a generalization of Pearl’s bayesian networks to capture DS belief distributions instead of Shenoy/Shafer hypergraphs. They argue: ”graphical structures used to represent relationships among variables in our work are Pearl’s causal networks [5], not Shenoy/Shafer’s hypergraphs, because the former are more appropriate to represent independence relationships among variables in a direct way.” (p.257). On page 262 (Definition 2) they define a belief function $Bel$ (a priori) conditional belief function conditioned on variable set $h$ by requiring $Bel^h$ to be a vacuous belief function.

It is easily checked that this notion of conditional belief functions allows to represent statistically conditionally independent variables of a bayesian belief network as a priori conditionally independent variables in Cano’s et al. sense.

However, it cannot handle other belief functions which could be expressed in terms of a Dempster Rule of Combination.
As an example please verify, that the belief function \( Bel_{12} \)

\[
Bel_{12} = Bel_1 \oplus Bel_2
\]

with focal points for \( Bel_1, Bel_2 \) (\( Bel_1 \) defined for variables \( X,Y \), \( Bel_2 \) for variables \( X,Z \), domains of variables: \( X: \{x_1, x_2\}, Y: \{y_1, y_2\}, Z: \{z_1, z_2\} \))

| set                                      | \( m_1(set) \) | set                                      | \( m_2(set) \) |
|-----------------------------------------|----------------|-----------------------------------------|----------------|
| \{\( (x_1, y_1) \), \( (x_2, y_2) \)\} | 0.1            | \{\( (x_1, z_1) \), \( (x_1, z_2) \)\} | 0.2            |
| \{\( (x_2, y_1) \)\}                   | 0.2            | \{\( (x_1, z_1) \)\}                   | 0.2            |
| \{\( (x_2, y_2) \)\}                   | 0.15           | \{\( (x_1, z_2) \)\}                   | 0.3            |
| \{\( (x_1, y_1) \)\}                   | 0.3            | \{\( (x_2, z_1) \)\}                   | 0.25           |

cannot be represented in a structured manner as a product of a normal and conditional belief function in sense of Cano et al. In this sense it is immediately visible, that Shenoy-Shafer hypergraphs allow for more efficient structuring of belief functions than Cano’s et al. directed acyclic graph representation.

### 6 Generalized belief networks

The axiomatization system of Shenoy/Shafer refers to the notion of factorization along a hypergraph. However, the actual propagation algorithm operates on hypertrees. We investigate below implications of this disagreement.

**Definition 1** \([4]\) We define a mapping \( \ominus : VV \times VV \rightarrow VV \) called decombination such that: if \( Bel_{12} = Bel_1 \ominus Bel_2 \) then \( Bel_1 = Bel_2 \oplus Bel_{12} \).

In case of probabilities, decombination means memberwise division: \( Pr_{12}(A) = Pr_1(A)/Pr_2(A) \). In case of DS pseudo-belief functions it means the operator \( \ominus \) yielding a DS pseudo-belief function such that: whenever \( Bel_{12} = Bel_1 \ominus Bel_2 \) then \( Q_{12}(A) = c \cdot Q_1/Q_2 \). Both for probabilities and for DS belief functions decombination may be not uniquely determined. Moreover, for DS belief functions not always a decombined DS belief function will exist. Hence we extend the domain to DS pseudo-belief functions which is closed under this operator. We claim here without a proof (which is simple) that
DS pseudo-belief functions fit the axiomatic framework of Shenoy/Shafer. Moreover, we claim that if an (ordinary) DS belief function is represented by a factorization in DS pseudo-belief functions, then any propagation of uncertainty yields the very same results as when it would have been factored into ordinary DS belief functions.

**Definition 2** [4] By anti-conditioning of a belief function Bel on a set of variables $h$ we understand the transformation: $Bel^h = Bel \ominus Bel^{\uparrow h}$.

Notably, anti-conditioning means in case of probability functions proper conditioning. Notice that due to the fact that $\ominus$ does not provide with a unique result, so also anti-conditioning may yield many pseudo-belief functions none of which is particularly distinguished. Let us define now the general notion of belief networks:

**Definition 3** [4] A belief network is a pair $(D, Bel)$ where $D$ is a DAG (directed acyclic graph) and $Bel$ is a belief distribution called the underlying distribution. Each node $i$ in $D$ corresponds to a variable $X_i$ in $Bel$, a set of nodes $I$ corresponds to a set of variables $X_I$ and $x_i, x_I$ denote values drawn from the domain of $X_i$ and from the (cross product) domain of $X_I$ respectively. Each node in the network is regarded as a storage cell for any distribution $Bel^{\downarrow \{X_i\} \cup X_{\pi(i)} \mid X_{\pi(i)}}$ where $X_{\pi(i)}$ is a set of nodes corresponding to the parent nodes $\pi(i)$ of $i$. The underlying distribution represented by a belief network is computed via:

$$Bel = \bigoplus_{i=1}^{n} Bel^{\downarrow \{X_i\} \cup X_{\pi(i)} \mid X_{\pi(i)}}$$

Please notice the local character of valuation of a node: to evaluate the node $i$ corresponding to variable $X_i$ only the marginal $Bel^{\{X_i\} \cup X_{\pi(i)} \mid X_{\pi(i)}}$ needs to be known (e.g. from data) and not the entire belief distribution.

There exists a straightforward transformation of a belief network structure into a hypergraph, and hence of a belief network into a hypergraph: for every node $i$ of the underlying DAG define a hyperedge as the set $\{X_i \} \cup X_{\pi(i)}$; then the valuation of this hyperedge define as $Bel^{\downarrow \{X_i\} \cup X_{\pi(i)} \mid X_{\pi(i)}}$. We say that the hypergraph obtained in this way is induced by the belief network.

Let us consider now the inverse operation: transformation of a valuated hypergraph into a belief network. As the first stage we consider structures of a hypergraph and of a belief network (the underlying DAG), we say that a belief
network is *compatible* with a hypergraph iff the reduced set of hyperedges induced by the belief network is identical with the reduced hypergraph.

**Example 1** Let us consider the following hypergraph (see Fig.1.a)): \{\{A,B,C\}, \{C,D\}, \{D,E\}, \{A, E\}\}. The following belief network structures are compatible with this hypergraph: \{A, C \rightarrow B, C \rightarrow D, D \rightarrow E, E \rightarrow A\} (see Fig.2.a)), \{A, C \rightarrow B, D \rightarrow C, D \rightarrow E, E \rightarrow A\} (see Fig.2.b)), \{A, C \rightarrow B, D \rightarrow C, E \rightarrow D, E \rightarrow A\} (see Fig.2.c)), \{A, C \rightarrow B, D \rightarrow C, E \rightarrow D, A \rightarrow E\} (see Fig.2.d)).

**Example 2** Let us consider the following hypergraph (see Fig.1.b)): \{\{A,B,C\}, \{C,D\}, \{D,E\}, \{A, E\}\}. No belief network structure is compatible with it.

The missing compatibility is connected with the fact that a hypergraph may represent a cyclic graph. Even if a compatible belief network has been found we may have troubles with valuations. In Example 1 an unfriendly valuation of hyperedge \{A,C,B\} may require an edge AC in a belief network representing the same distribution, but it will make the hypergraph incompatible (as e.g. hyperedge \{A,C,E\} would be induced). This may be demonstrated as follows:
Figure 2: An example of belief networks corresponding to a hypergraph from Fig. 1.a)
Definition 4 [4] If \( X_J, X_K, X_L \) are three disjoint sets of variables of a distribution \( \text{Bel} \), then \( X_J, X_K \) are said to be conditionally independent given \( X_L \) (denoted \( I(X_J, X_K|X_L)_{\text{Bel}} \)) iff

\[
\text{Bel}^i_{X_J \cup X_K \cup X_L|X_L} \oplus \text{Bel}^i_{X_J \cup X_L|X_L} \oplus \text{Bel}^i_{X_K \cup X_L|X_L} \oplus \text{Bel}^i_{X_L}
\]

\( I(X_J, X_K|X_L)_{\text{Bel}} \) is called a conditional independence statement

Let \( I(J, K|L)_D \) denote d-separation in a graph [3].

**Theorem 1** [4] Let \( \text{Bel}_D = \{ \text{Bel} | (D, \text{Bel}) \text{ is a belief network} \} \). Then: \( I(J, K|L)_D \) iff \( I(X_J, X_K|X_L)_{\text{Bel}} \) for all \( \text{Bel} \in \text{Bel}_D \).

Now we see in the above example that nodes D and E d-separate nodes A and C. Hence within any belief network based on one of the three DAGs mentioned A will be conditionally independent from C given D and E. But one can easily check that with general type of hypergraph valuation nodes A and C may be rendered dependent. The sad result is, that

**Theorem 2** [4] Hypergraphs considered by Shenoy/Shafer [9] may for a given joint belief distribution have simpler structure than (be properly covered by) the closest hypergraph induced by a belief network.

Notably, though the axiomatic system of Shenoy/Shafer refers to hypergraph factorization of a joint belief distribution, the actual propagation is run on a hypertree (or more precisely, on one construction sequence of a hypertree, that is on Markov tree) covering that hypergraph. Let us look closer at the outcome of the process of covering with a reduced hypertree factorization, or more precisely, at the relationship of a hypertree construction sequence and a belief network constructed out of it in the following way: If \( h_k \) is a twig in the sequence \( \{h_1, \ldots, h_k\} \) and \( h_{i_k} \) its branch with \( i_k < k \), then let us span the following directed edges in a belief network: First make a complete directed acyclic graph out of nodes \( h_k - h_{i_k} \). Then add edges \( Y_i \to X_j \) for every \( Y_i \in h_k \cap h_{i_k} \) and every \( X_j \in h_k - h_{i_k} \). (see Fig.3). Repeat this for every \( k = 2, \ldots, n \). For \( k = 1 \) proceed as if \( h_1 \) were a twig with an empty set as a branch for it.

It is easily checked that the hypergraph induced by a belief network structure obtained in this way is in fact a hypertree (if reduced, then exactly the original reduced hypertree). Let us turn now to valuations (Fig.4.a).
Let $Bel_i$ be the valuation originally attached to the hyperedge $h_i$. Then $Bel = Bel_1 \oplus \ldots \oplus Bel_n$. What conditional belief is to be attached to $h_n$? First marginalize: $Bel'_n = Bel_1^{\downarrow h_1 \cap h_n} \oplus \ldots \oplus Bel_{n-1}^{\downarrow h_{n-1} \cap h_n} \oplus Bel_n$. (Fig. 4.b,c) Now calculate: $Bel''_n = Bel'_n^{\downarrow h_n \cap h_{i_n}}$, and $Bel'''_n = Bel'_n^{\downarrow h_n \cap h_{i_n}}$. Let $Bel_{*k} = Bel_k \oplus Bel_k^{\downarrow h_n}$ for $k=1,\ldots,i_{n-1},i_n+1,\ldots,(n-1)$, (Fig. 4.d) and let $Bel_{*i_n} = (Bel_{i_n} \oplus Bel_{i_n}^{\downarrow h_n}) \oplus Bel''_n$. Obviously, $Bel = Bel_{*1} \oplus \ldots \oplus Bel_{s(n-1)} \oplus Bel''_n$ (Fig. 4.e). Now let us consider a new hypertree only with hyperedges $h_1,\ldots,h_{n-1}$, and with valuations equal to those marked with asterisk (*), and repeat the process until only one hyperedge is left, the new valuation of which is considered as $Bel''_1$. In the process, a new factorization is obtained: $Bel = Bel''_1 \oplus \ldots \oplus Bel''_n$.

If now for a hyperedge $h_k$ $\text{card}(h_k - h_{i_k}) = 1$, then we assign $Bel''_k$ to the node of the belief network corresponding to $h_k - h_{i_k}$. If for a hyperedge $h_k$ $\text{card}(h_k - h_{i_k}) > 1$, then we split $Bel''_k$ as follows: Let $h_k - h_{i_k} = \{X_{k1}, X_{k2}, \ldots, X_{km}\}$ and the indices shall correspond to the order in the belief network induced by the above construction procedure. Then

$$Bel''_k = Bel^{\downarrow h_k \cap h_{i_k}} = \bigoplus_{j=1}^{m} Bel^{\downarrow (h_k \cap h_{i_k}) \cup \{X_{k1}, \ldots, X_{kj}\}} \cup \{X_{k1}, \ldots, X_{kj}\} - \{X_{kj}\}$$

and we assign valuation $Bel^{\downarrow (h_k \cap h_{i_k}) \cup \{X_{k1}, \ldots, X_{kj}\}} \cup \{X_{k1}, \ldots, X_{kj}\} - \{X_{kj}\}$ to

Figure 3: An example of a) a twig in hypertree and b) its fragment of belief network
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a) twig: $\text{Bel}_t$

branch: $\text{Bel}_b$

other hyperedge: $\text{Bel}_o$

b) twig: $\text{Bel}_t$

branch: $(\text{Bel}_b \oplus \text{Bel}_b^{\cup \land \neg}) \oplus \text{Bel}_o^{\cup \land \neg}$

other hyperedge: $(\text{Bel}_o \oplus \text{Bel}_o^{\cup \land \neg}) \oplus \text{Bel}_o^{\cup \land \neg}$

c) twig: $\text{Bel}_t' = \text{Bel}_t \oplus \text{Bel}_b^{\cup \land \neg} \oplus \text{Bel}_o^{\cup \land \neg}$

branch: $\text{Bel}_b \oplus \text{Bel}_b^{\cup \land \neg}$

other hyperedge: $\text{Bel}_o \oplus \text{Bel}_o^{\cup \land \neg}$
Figure 4: An example of valuation transformation
the node corresponding to $X_{kj}$ in the network structure. It is easily checked that:

**THEOREM 3** [4] (i) The network obtained by the above construction of its structure and valuation from hypertree factorization is a belief network. 

(ii) This belief network represents exactly the joint belief distribution of the hypertree

(iii) This belief network induces exactly the original reduced hypertree structure

The above theorem implies that any hypergraph suitable for propagation must have a compatible belief network. Hence seeking for belief network decompositions of joint belief distributions is sufficient for finding any factorization suitable for Shenoy/Shafer propagation of uncertainty.

7 Discussion

We presented here selected concepts of structuring of DS belief functions with a special emphasis on their capability to capture independence from the point of view of the claim that belief functions generalize bayes notion of probability.

It is demonstrated that Zhu and Lee’s [12] logical networks and Smets’ [11] directed acyclic graphs are unable to capture statistical dependence/independence of bayesian networks [5]. Zhu and Lee’s [12] just assume conditional independence of premise and conclusion of an implication and hence assume that the dominant value of premise implies the dominant value of conclusion. This is wrong from the logical point of view because the implication may be inverse in the data, but the presence of cases for which the rule is not applicable will distort the conclusions of expert system based on rules like those proposed by Zhu and Lee’s [12]. Smets [11] goes to other extreme and assumes that all variables but in very restricted cases are dependent. His definition of cognitive independence applies to bayesian belief functions only in those rare cases of uniform joint probability distribution. Smets’ notion of conditional independence for bayesian belief functions means that the conditioning variable is degenerate from statistical point of view: it takes only one value and hence is usually ommitted from any statistical analysis. So conditional
independence again reduces to uniform distribution of conditioned variables.

On the other hand, though Shenoy and Shafer’s hypergraphs can explicitly represent bayesian network factorization of bayesian belief functions, they disclaim any need for representation of independence of variables in belief functions. Cano et al. [1] reject the hypergraph representation of Shenoy and Shafer just on grounds of missing representation of variable independence, but in their frameworks some belief functions factorizable in Shenoy/Shafer framework cannot be factored. The approach in [4] on the other hand combines the merits of both Cano et al. and of Shenoy/Shafer approach in that for Shenoy/Shafer approach no simpler factorization than that in [4] approach exists and on the other hand all independences among variables captured in Cano et al. framework and many more are captured in [4] approach.

Two new operators for MTE have been introduced in connection with the approach of [4]: decombination $\ominus$ and anti-conditioning $\mid$. We shall stress that operators in this sense have been also introduced previously by Shenoy in [10] in connection with his valuation networks. We shall emphasize however one important difference making our approach more general. Shenoy insists that belief functions suitable for anti-conditioning should have unique unity item which means that the belief function shall have a focal point equal to the universe. This excludes bayesian belief functions so that Shenoy’s framework could not represent many of independences explicited within our framework.

Inversion of Shafer’s conditioning $Bel(\mid\mid B)$ was also considered by Cano et al. [1], Smets [11] and Shafer [6]. The ’a priori conditioning’ of Cano et al. [1] is a special case of our anti-conditioning in that it requires marginalization on conditioning variables to yield a vacuous belief function. On the other hand deconditioning of Smets [11] and conditional embedding of Shafer [6] are entirely different in nature. Our approach starts with joint absolute belief distribution and tries to remove the impact of absolute distribution of anti-conditioning variables from the picture of relation between the anti-conditioning and the anti-conditioned variable. Deconditioning and conditional embedding start with selected conditional beliefs try to establish a joint absolute belief distribution of conditioned and conditioning variables. Hence the concept of anti-conditioning is orthogonal to that of conditional embedding and deconditioning.
8 Conclusions

For many approaches to structuring of belief functions the claim of generalization of bayesian probabilities by Dempster-Shafer belief functions is actually illusive. Especially

- Zhu and Lee’s [12] logical networks and Smets’ [11] directed acyclic graphs are unable to capture statistical dependence/independence of bayesian networks
- Though Shenoy and Shafer’s hypergraphs can explicitly represent bayesian network factorization of bayesian belief functions, they disclaim any need for representation of independence of variables in belief functions.
- In Cano et al. [1] frameworks some belief functions factorizable in Shenoy/Shafer framework cannot be factored.
- The approach in [4] on the other hand combines the merits of both Cano et al. (representation of independences) and of Shenoy/Shafer approach (Shenoy/Shafer approach yields no simpler factorization than that in [4] approach)

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Appendix: Basic Definitions of MTE

Definition 5 Let $\Xi$ be a finite set of elements called elementary events. Any subset of $\Xi$ be a composite event. $\Xi$ be called also the frame of discernment. A basic probability assignment function $m: 2^\Xi \rightarrow [0, 1]$ such that

$$\sum_{A \in 2^\Xi} |m(A)| = 1$$
$$m(\emptyset) = 0$$
$$\forall A \in 2^\Xi \quad 0 \leq \sum_{A \subseteq B} m(B)$$

$|.|$ - absolute value.

A belief function be defined as $\text{Bel}: 2^\Xi \rightarrow [0, 1]$ so that

$$\text{Bel}(A) = \sum_{B \subseteq A} m(B)$$

A plausibility function be $\text{Pl}: 2^\Xi \rightarrow [0, 1]$ with

$$\forall A \in 2^\Xi \quad \text{Pl}(A) = 1 - \text{Bel}(\Xi - A)$$

A commonality function be $\text{Q}: 2^\Xi \rightarrow [0, 1]$ with

$$\forall A \in 2^\Xi \quad \text{Q}(A) = \sum_{A \subseteq B} m(B)$$

For a belief function, any set $A$ such that $m(A)$ differs from zero, is called focal point. A belief function where every focal point is a set with cardinality 1 (singleton) is called bayesian belief function.

Furthermore, a Rule of Combination of two Independent Belief Functions $\text{Bel}_1, \text{Bel}_2$ Over the Same Frame of Discernment (the so-called Dempster-Rule), denoted

$$\text{Bel}_{E_1, E_2} = \text{Bel}_{E_1} \oplus \text{Bel}_{E_2}$$

is defined as follows: 

$$m_{E_1, E_2}(A) = c \cdot \sum_{B, C: A = B \cap C} m_{E_1}(B) \cdot m_{E_2}(C)$$
Furthermore, let the frame of discernment \( \Xi \) be structured in that it is identical to cross product of domains \( \Xi_1, \Xi_2, \ldots, \Xi_n \) of \( n \) discrete variables \( X_1, X_2, \ldots, X_n \), which span the space \( \Xi \). Let \((x_1, x_2, \ldots, x_n)\) be a vector in the space spanned by the variables \( X_1, X_2, \ldots, X_n \). Its projection onto the subspace spanned by variables \( X_{j_1}, X_{j_2}, \ldots, X_{j_k} \) (\( j_1, j_2, \ldots, j_k \) distinct indices from the set \( 1, 2, \ldots, n \)) is then the vector \((x_{j_1}, x_{j_2}, \ldots, x_{j_k})\). \((x_1, x_2, \ldots, x_n)\) is also called an extension of \((x_{j_1}, x_{j_2}, \ldots, x_{j_k})\). A projection of a set \( A \) of such vectors is the set \( A \downarrow_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}} \) of projections of all individual vectors from \( A \) onto \( X_{j_1}, X_{j_2}, \ldots, X_{j_k} \). \( A \) is also called an extension of \( A \downarrow_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}} \). \( A \) is called the vacuous extension of \( A \downarrow_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}} \) iff \( A \) contains all possible extensions of each individual vector in \( A \downarrow_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}} \). The fact, that \( A \) is a vacuous extension of \( B \) onto space \( X_1, X_2, \ldots, X_n \) is denoted by \( A = B \uparrow_{X_1, X_2, \ldots, X_n} \).

**Definition 6** Let \( m \) be a basic probability assignment function on the space of discernment spanned by variables \( X_1, X_2, \ldots, X_n \). \( m \downarrow_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}} \) is called the projection of \( m \) onto subspace spanned by \( X_{j_1}, X_{j_2}, \ldots, X_{j_k} \) iff

\[
m\downarrow_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}}(B) = c \cdot \sum_{A; B = A\downarrow_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}}} m(A)
\]

\((c - \text{normalizing factor})\)

**Definition 7** Let \( m \) be a basic probability assignment function on the space of discernment spanned by variables \( X_1, X_2, \ldots, X_n \). \( m \uparrow_{X_1, X_2, \ldots, X_n} \) is called the vacuous extension of \( m \) onto superspace spanned by \( X_1, X_2, \ldots, X_n \) iff

\[
m \uparrow_{X_1, X_2, \ldots, X_n}(B \uparrow_{X_1, X_2, \ldots, X_n}) = m(B)
\]

and \( m \uparrow_{X_1, X_2, \ldots, X_n}(A) = 0 \) for any other \( A \).

We say that a belief function is vacuous iff \( m(\Xi) = 1 \) and \( m(A) = 0 \) for any \( A \) different from \( \Xi \).

Projections and vacuous extensions of Bel, Pl and Q functions are defined with respect to operations on \( m \) function. Notice that by convention if we want to combine by Dempster rule two belief functions not sharing the frame of discernment, we look for the closest common vacuous extension of their frames of discernment without explicitly notifying it.
Definition 8 (See [7]) Let $B$ be a subset of $\Xi$, called evidence, $m_B$ be a basic probability assignment such that $m_B(B) = 1$ and $m_B(A) = 0$ for any $A$ different from $B$. Then the conditional belief function $Bel(.\mid B)$ representing the belief function $Bel$ conditioned on evidence $B$ is defined as: $Bel(.\mid B) = Bel \oplus Bel_B$.

Notice: Vacuous extension is also called cylindric extension. In Smets’ interpretation of Dempster-Shafer theory names bel, pl and q are used instead of Bel, Pl and Q. The justification is that Smets allows for $\emptyset$ to be a focal point (open world assumption).

In this paper, the notion of pseudo-belief functions is also used (last section before discussion). A pseudo-belief function differs from proper belief function in that $m$ is allowed to take also negative values, but only in such a way as to ensure that $Q$ remains non-negative. However, Bel and Pl may get negative for pseudo-belief functions.