Fermi edge resonances in non-equilibrium states of Fermi gases

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Abstract
We formulate the problem of the Fermi edge singularity in non-equilibrium states of a Fermi gas as a matrix Riemann–Hilbert problem with an integrable kernel. This formulation is the most suitable for studying the singular behavior at each edge of non-equilibrium Fermi states by means of the method of steepest descent, and also reveals the integrable structure of the problem. We supplement this result by extending the familiar approach to the problem of the Fermi edge singularity via the bosonic representation of the electronic operators to non-equilibrium settings. It provides a compact way to extract the leading asymptotes.

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1. Introduction

The FES (Fermi edge singularity) [1–4] is observed in absorption of x-rays in metals as a power law peak at the Fermi edge of a degenerate Fermi gas. There, a sudden removal of a localized electron from a hard-core atomic shell creates a potential which disturbs the electronic gas, thus producing a power law spectrum of electronic soft modes. In recent years, the FES has also been demonstrated in tunneling experiments [5–8]. There, a single electron can change the capacity of a contact producing a similar disturbance to the electronic gas as a localized hole. As a result, one observes a power law in tunneling current versus the bias voltage: 

\[ I(V) \sim V^{-2\pi k a^2} \]  

where \( \delta = \pi a \) is the scattering phase of the ensuing potential and \( k \) is the number of scattering channels. In the case of an attractive potential \( (a > 0) \), the current peaks at the Fermi edge.

One of the reasons of interest in the FES, and our own motivation in studying it, is that the origin of the FES may be found to be ascribable solely to Fermi statistics. It has been studied over at least five decades, has been well understood, and is considered as one of the fundamental quantum phenomena in electronic physics. Early theoretical papers [1–3] on the FES and the related phenomenon of orthogonality catastrophe [10] have proven to be...
In a degenerate Fermi gas, anything but the leading power asymptotes is rarely of any interest. The reason for it is that everything else, except the leading power, depends on details of band structure, tunneling contacts, etc, and lacks universal character.

A different situation occurs in a non-equilibrium Fermi gas. There the energy scale of non-equilibrium features can be much smaller than the Fermi scale and can be seen in the spectrum of absorption or tunneling. In this case, additional features become universal.

As a prototype of a non-equilibrium state, we consider a state where the Fermi distribution consists of steps at $E_1 < E_2 < \cdots < E_{2n+1}$ such that no states are occupied in the interval between energies $E_{2i-1}, E_{2i}$, where $i = 1, \ldots, n$. We denote by $E_0$ the bottom of the filled Fermi sea, which is assumed to be far away (figure 1). Assuming that the scattering phase does not change within a wide range of the conducting band, say, between $E_0 = 0$ and $E_{2n} = \Lambda$, the spectrum will be a transcendental universal function of $\nu/E_{ij}$, where $E_{ij} = E_i - E_j$.

These structured non-equilibrium states described above inevitably appear in the evolution of an arbitrary semiclassical Fermi state [11]. They were also realized in some nanoscale devices (see e.g. [12]).

In a non-equilibrium state, the absorption spectrum (or tunneling current) is a transcendental function and elementary methods can determine its asymptotic expansion in various regimes. In this communication, we develop a framework aiming to characterize the universal part of the spectrum. We derive a $2 \times 2$ matrix Riemann–Hilbert (RH) problem derived from an inversion problem of an integrable kernel. Similar matrix RH problems appeared in studies of various fermionic correlation functions and correlation functions of eigenvalues of random matrices [13]. One of the proven advantages of this formulation is that the RH problem is the most suitable for the application of the steepest descent method. The latter prompts the leading asymptotes at Fermi edges. Besides the fact that the RH problem connects FES to a number of physically unrelated but mathematically equivalent (often well studied) problems, we think that the links and analytical apparatus they bring to physics of the FES are important for a deeper understanding of quantum non-equilibrium phenomenon. A somewhat alternative but essentially equivalent approach is to establish a set of nonlinear integrable equations [14]. We do not discuss this approach here.

As far as the main asymptote is concerned, we will show that in the setting described above, the tunneling current at voltage close to a Fermi edge $\epsilon_i(eV - E_i) \ll |E_{i,i+1}|$ reads

\begin{equation}
I(V) \propto A_i |eV - E_i|^{(2n-1)a^2-2c_i}.
\end{equation}
Figure 2. A schematic plot of tunneling current for small \( a < 0 \) (left panel) and \( a > 0 \) (right panel), solid lines show computed power law asymptotes. Dashed lines interpolate between resonances.

\[
A_i = \text{const} \prod_{l<m}^{2n} E_{im}^{2\epsilon_l E_m a^2} \prod_{0 \leq j \neq i}^{2n} E_{ij}^{-2\epsilon_j a}, \tag{2}
\]

where \( \epsilon_i = \pm 1 \), if \( E_i \) is an lower/upper edge of an occupied band, respectively.

Some noticeable features of this result are as follows: (i) the exponent depends on the total number of bands, but stays the same for all upper (lower) edges of occupied intervals \( \frac{dI}{dV} = (2na^2 - (a - \epsilon_i)^2)|V - E_i/e|^{-1} \); (ii) if the potential is attractive \( a > 0 \) (which is the common case) the current features a peak with a power law decay at upper edges toward increasing energy; and in contrast the current is suppressed at lower edges; if the potential is repulsive, \( a < 0 \), and sufficiently small, a peak appears at the edge \( E_{2k} \) with a power law to the left of the edge (figure 2); (iii) the logarithm of the factor \( A_i \) can be treated as the electrostatic energy of particles with alternating charges \( \pm a \) positioned at the upper/lower edges with an insertion of a unit \( \epsilon_i \) charge at the edge \( E_i \).

Fermi edge singularities with a structured Fermi distribution have been studied before. In 1984, Combescot and Tanguy [15–17] considered a situation where the interval \((E_1, E_2)\) of the band \((0, \Lambda)\) is occupied, while the interval \((0, E_1)\) starting from the bottom of the band is unoccupied. Later in 2004, Abanin and Levitov [18] considered FES with a two-step Fermi distribution. A more general situation has been considered in the recent papers by Gutman et al [19], and by the authors of [11], where equation (1) has been obtained. The approach employed in these papers (with the exception of [11]) is based on the expression of the tunneling current as a determinant of a Fredholm operator. A basis of particle–hole excitations of the structured Fermi sea has been used to write the Fredholm operator in early papers [15–17]. This approach has been developed by Othaka and Tanebe [4] in 1984. In contrast, a one-particle basis in an empty vacuum has been used in later articles [18, 19]. Naturally the basis of particle–hole excitations of the Fermi sea, employed in earlier papers [4, 15–17], which already captures the many-body physics of the Fermi sea, is a step forward in obtaining the desired results. We employ this approach here. In addition, for illustrative purposes (appendix B), we also present a simple compact method to capture a leading singularity developed in [11]. That method is based on the bosonic representation of electronic operators in a Fermi state with multiple edges.

2. Tunneling current

The tunneling setting of FES is as follows: a Fermi gas is in contact with a localized resonant level (a quantum dot). It is initially uncharged and provides no scattering to electrons. When
an electron tunnels to the dot, it suddenly charges the dot, switching-on a small potential $H \to H' = H + U$ localized at the dot [9]. We assume no further interaction, no dissipation, and we ignore spin and channels.

The tunneling current is given by the golden rule [1, 9]. In units of the tunneling amplitude, $I(\omega) \big|_{\hbar \omega = eV}$ reads

$$I(\omega) \propto \text{Re} \int_0^\infty e^{i\omega(t_1 - t_2)} G(t_1, t_2) \, dt,$$

(3)

$$G(t_1, t_2) = \langle \Omega | e^{iHt_2} c e^{iH'(t_1 - t_2)} c^\dagger e^{-iHt_1} | \Omega \rangle.$$

(4)

Here, $c = \sum_\epsilon c_\epsilon$ is an electronic operator at the position of the dot, assumed to be the origin, while $c_\epsilon$ is an electronic mode with energy $\epsilon$ and $| \Omega \rangle$ is the structured Fermi state of interest. We also assume that the bias voltage is with respect to $E_1$.

Next we assume that the potential is regular within a wide energy range which exceeds the energy range of features of the structured Fermi state and that the tunneling time is sufficiently small. Under this assumption, the energy dependence of the scattering phase $\delta$ caused by potential $U$ can be dropped. This amounts to a downward shift of the energy levels by a constant amount $a$ (in units of level spacing): $\epsilon \to \epsilon - a$.

In [3] it has been shown that the vertex operator $e^{a\phi}$ implements a shift of momenta: the perturbed Hamiltonian and perturbed states are seen as a similarity transformation of the unperturbed ones $H' = e^{-a\phi} H e^{a\phi}$ and $| \Omega' \rangle = e^{a\phi} | \Omega \rangle$. Here, an operator $\phi$ is a chiral canonical Bose field related to the chiral part of the electronic density

$$\varphi(t) = \hbar \sum_{\epsilon \neq 0} e^{i\hbar \epsilon t} \rho_{\epsilon}, \quad \rho_{\epsilon} = \sum_\epsilon c_\epsilon^\dagger c_{\epsilon+\epsilon}.$$

(5)

Then, Green’s function reads

$$G(t_1, t_2) = \langle \Omega | e^{i\phi(t_2)} e^{-i\phi(t_1)} c^\dagger(t_1) | \Omega \rangle,$$

(6)

where $c(t) = \sum_\epsilon e^{i\epsilon t} c_\epsilon$. This formula is standard.

3. Fredholm determinants

Following [1], Green’s function can be understood as consisting of three multiplicative factors $G(t_1, t_2) = |\langle \Omega' | \Omega \rangle|^2 e^C \cdot L$—an overall normalization:

$$|\langle \Omega' | \Omega \rangle|^2, \quad \text{where } | \Omega' \rangle = e^{a\phi(0)} | \Omega \rangle,$$

(7)

closed loops

$$e^C = \frac{\langle \Omega | e^{-\phi(t_2)} e^{\phi(t_1)} | \Omega \rangle}{|\langle \Omega' | \Omega \rangle|^2},$$

(8)

and open lines

$$L = \frac{\langle \Omega | c(t_2) e^{-\phi(t_2)} e^{\phi(t_1)} c^\dagger(t_1) | \Omega \rangle}{\langle \Omega | e^{-\phi(t_2)} e^{\phi(t_1)} | \Omega \rangle} \frac{|\langle \Omega' | \Omega \rangle|^2}{\langle \Omega | e^{-\phi(t_2)} e^{\phi(t_1)} | \Omega \rangle}.$$

(9)

The latter two objects can be cast in the form of the Fredholm determinant by means of the Wick theorem. We recall the major formulas.

Consider coherent states of $G\!l(\infty)$. These states are obtained by transforming the ground state of the Fermi gas $| A \rangle = g(A) | 0 \rangle$ by an exponent of a bilinear form of Fermi operators $g(A) = e^{\sum_\epsilon A_{\epsilon,\epsilon} c_{\epsilon}^\dagger c_{\epsilon}}$, where $A_{\epsilon,\epsilon}$ is an arbitrary $g\!l(\infty)$ matrix. Our structured Fermi state is a coherent state.
For arbitrary coherent states $A_1, A_2$ and arbitrary $g\ell(\infty)$ matrices $B, C$, the following holds:

$$\frac{\langle A_1 | g(B)g(C) | A_2 \rangle}{\langle A_1 | g(B) | A_2 \rangle \langle A_1 | g(C) | A_2 \rangle} = \det(I + K), \quad (10)$$

where

$$K_{\epsilon_1, \epsilon_2} = \sum_{\epsilon} M_{\epsilon_1, \epsilon}(A_1, B, A_3)M_{\epsilon, \epsilon_2}^\dagger(A_3, C, A_2) \quad (11)$$

and

$$M_{\epsilon, \eta}(A_1, B, A_3) = \langle A_1 | g(B) c_\epsilon^\dagger c_\eta | A_3 \rangle \quad (12)$$

are matrix elements of the operator $g(B)$ between the state $\langle \Omega(A) |$ and a state $c_\epsilon^\dagger c_\eta | A_3 \rangle$, where a particle–hole pair is added to an arbitrary chosen coherent state $| A_3 \rangle$. The result does not depend on the choice of $A_3$.

In order to obtain this formula, one inserts a superposition of an arbitrary number of particle–hole excitations into a chosen coherent state $| A_3 \rangle | A_3 \rangle$, apply the Wick theorem to each term and sum them up.

The variation of (10) $\delta \log \det(I + K) = \text{tr}[(I + K)^{-1} \delta K]$ gives another known formula

$$\frac{\langle A_1 | c_\epsilon^\dagger g(B) g(C) c_\eta | A_2 \rangle}{\langle A_1 | g(B) g(C) | A_2 \rangle} = \text{tr}[(I + K)^{-1} P], \quad (13)$$

where

$$P_{\epsilon, \eta}(\epsilon_1, \epsilon_2) = \tilde{M}_{\epsilon_1, \eta}(A_1, B, A_2)\tilde{M}_{\eta, \epsilon_2}^\dagger(A_1, C, A_2), \quad (14)$$

and

$$\tilde{M}_{\eta}(A_1, B, A_2) = \langle A_1 | c_\eta g(B) c_\eta^\dagger | A_2 \rangle. \quad \text{Specification of these formulas } \langle A_1 \rangle = \langle A_2 \rangle = \langle \Omega \rangle, \quad g(B) = g(C) = e^{-a\phi(t)} \text{ prompts a determinantal representation of the current}$$

$$e^C = \det(I + K), \quad L = \text{tr}[(I + K)^{-1} P] \quad (15)$$

with a kernel

$$K(\epsilon_1, \epsilon_2) = \sum_{\eta} M_{\epsilon_1, \eta}(t_1)M_{\eta, \epsilon_2}^\dagger(t_2), \quad (16)$$

$$P_{\epsilon, \eta} = M_{\epsilon}(t_1)M_{\eta}(t_2), \quad (17)$$

where

$$M_{\epsilon, \eta}(t) = \frac{\langle \Omega | e^{-a\phi(t)} c_\epsilon^\dagger c_\eta | \Omega \rangle}{\langle \Omega | \Omega \rangle}, \quad (18)$$

$$M_{\epsilon}(t) = \frac{\langle \Omega | c(t) e^{-a\phi(t)} c_\eta^\dagger | \Omega \rangle}{\langle \Omega | \Omega \rangle} \quad (19)$$

are matrix elements of the vertex operator between states where a particle–hole pair, or just one particle, is added to the state $| \Omega \rangle$. These are general formulas valid for any coherent state $| \Omega \rangle$.

The formulas are further specified since the structured Fermi state we are considering is an eigenvalue of the Hamiltonian. In this case,

$$M_{\epsilon, \eta}(t) = e^{i(\eta - \epsilon)\int_{\Omega}^{\Omega} \langle \Omega | \Omega ; \epsilon, \eta \rangle}, \quad (20)$$

3 Formulas equivalent to (16) for the ground state can be found in [4].
\( M_{\epsilon\eta}(t) = e^{-i\epsilon t} \frac{\langle (\epsilon; \Omega') | \Omega; \epsilon \rangle}{\Omega'} \).  

(21)

\( M_{\epsilon\eta} \) is an overlap between the state \( |\Omega'\rangle \) which appears after the shake-up and a particle–hole excitation of the state \( |\Omega\rangle \) before shake-up with energy \( \epsilon \notin \Omega \) and \( \eta \in \Omega \), where we denote \( \Omega = \bigcup_{n=0}^{n-1} (E_2, E_{2+1}) \) as the set of occupied single particle states in \( |\Omega\rangle \). \( \Omega; \epsilon, \eta \) are states where an extra particle or a particle–hole pair is added into \( |\Omega\rangle \). Similarly \( M_{\epsilon} \) is the overlap of states \( |\Omega\rangle \) and the state \( |\Omega'\rangle \) with an added particle with energy \( \epsilon \).

4. Matrix elements

The following formula helps evaluating the matrix elements (20) and (21). If \( |\epsilon'\rangle \) is a single particle eigenstate of the perturbed Hamiltonian and \( |\epsilon\rangle \) is a one-particle eigenstate of an unperturbed state, then their overlap (in units of level spacing) is \( \langle \epsilon' | \epsilon \rangle = \frac{\sin(\pi a)}{\pi (\epsilon - \epsilon')} \). An extension of this formula to many-particle states \( \langle \epsilon | = \langle \epsilon_1, \ldots, \epsilon_N | \) and \( |\epsilon'\rangle = |\epsilon'_1, \ldots, \epsilon'_N\rangle \) gives a Cauchy determinant

\[
\left( \frac{\pi}{\sin(\pi a)} \right)^N \langle \epsilon' | \epsilon \rangle = \det \frac{1}{\epsilon_i - \epsilon_j} = \prod_{i>j} (\epsilon_i - \epsilon_j)(\epsilon'_i - \epsilon'_j) \prod_{i,j} (\epsilon'_i - \epsilon_j).
\]

(22)

With the help of this formula, the matrix elements in (16) can be computed in a manner similar to [4]. While computing, one must take into account that the set of occupied single particle levels in \( \Omega \) and \( \Omega' \) is shifted by \( a \) with respect to each other. Then, the problem is reduced to an electrostatic problem of placing a dipole or a charge into a Coulomb plasma confined in the intervals \( \bigcup_{n=0}^{n-1} (E_2, E_{2+1}) \). It gives the overlap between states \( \langle \Omega | \Omega' \rangle \) generalizing the orthogonality catastrophe formula [10]. Up to an \( E_i \)-independent constant factor, it reads

\[
\langle \Omega | \Omega' \rangle \sim \Delta^{(n+1)a^2} \prod_{i>j} (E_{ij})^a \times \prod_{i} (\epsilon_i - E_i)^{-a} \times \prod_{i} (\epsilon'_i - E_i)^a.
\]

(23)

The results for matrix elements are

\[
M_{\epsilon\eta} = e^{-i(\eta - \epsilon)t} r(\epsilon) s(\eta), \quad M_{\epsilon} = e^{-i\epsilon t} r(\epsilon),
\]

(24)

\[
r(\epsilon) = \prod_{i=1}^{2n} (\epsilon - E_i)^{a}, \quad \epsilon \notin \Omega; \]

\[
s(\eta) = \frac{\sin(\pi a)}{\pi} \prod_{i=1}^{2n} (E_i - \eta)^{-a}, \quad \eta \in \Omega
\]

(25)

where \( \epsilon \notin \Omega, \eta \in \Omega \) are energies of particles and holes. A short sketch of these calculations is found in appendix A.

Summing up, the kernel reads

\[
K(\epsilon_1, \epsilon_2) = e^{-i(\epsilon_2-\epsilon_1)t} r(\epsilon_1) r(\epsilon_2) \frac{Q(\epsilon_2) - Q(\epsilon_1)}{\epsilon_1 - \epsilon_2},
\]

(26)

\[
Q(\epsilon, \tau) = \int_{\eta \in \Omega} e^{i\eta \tau} s(\eta) \frac{d\eta}{\epsilon - \eta}, \quad \tau = t_1 - t_2.
\]

(27)
5. Integrable kernel

The next step is to invert the Fredholm kernel $K$. It can be done in a straightforward manner similar to [4, 15–17] employing the Wiener–Hopf method at every edge. However, calculations become more structured if we use the integrable property of the kernel. Integrability is the general property of free fermion correlators (see [13, 14, 20–22]).

A kernel is called integrable if it has the form

$$K(\epsilon_1, \epsilon_2) = \sum_{\alpha=1}^{l} \alpha f_{\alpha}(\epsilon_1) g_{\alpha}(\epsilon_2)\epsilon_1 - \epsilon_2,$$

$$\sum_{\alpha=1}^{l} f_{\alpha}(\epsilon) g_{\alpha}(\epsilon) = 0.$$  

In the case of a structured Fermi sea $l = 2$ and as follows from (16) and (24):

$$g_1(\epsilon) = Q(\epsilon, \tau)g_2(\epsilon), \quad f_2(\epsilon) = -Q(\epsilon, \tau)f_1(\epsilon),$$

$$g_2(\epsilon) = e^{i\bar{h} \epsilon \tau r}(\epsilon), \quad f_1(\epsilon) = e^{-i\bar{h} \epsilon \tau r}(\epsilon).$$  

Let $\vec{F} = (1 + K)^{-1} \vec{f}$ (we denote $\vec{f} = (f_1, f_2)$) be a solution of the singular integral equation

$$\vec{F}(\epsilon_1) + \int_{\epsilon \in \Omega} K(\epsilon_1, \epsilon_2) \vec{F}(\epsilon_2) d\epsilon_2 = \vec{f}(\epsilon_1), \quad \epsilon_1 \notin \Omega.$$  

The time derivative of closed loops’ contribution and contribution of open lines (15) are expressed through the solutions $\vec{F} = (F_1, F_2)$:

$$\frac{dC}{d\tau} = \text{tr} \left[ (1 + K)^{-1} \frac{dK}{d\tau} \right] = \frac{i}{\hbar} \int_{\epsilon \in \Omega} (g_1 F_1 - g_2 F_2) d\epsilon$$

$$L = \int_{\epsilon \notin \Omega} g_2 F_1 d\epsilon.$$  

6. Matrix Riemann–Hilbert problem

The Fredholm equation (29) is sufficient to obtain the singular behavior at Fermi edges. However, it is instructive to cast the FES problem as a matrix RH problem along the lines described in [13]. In that form, the FES problem falls in the general scheme of integrable problems. In addition, the RH problem is the most suitable for analysis near edges [20].

The central object of the RH problem is a matrix-valued functions $m(\epsilon)$ analytic in a complex $\epsilon$-plane cut along the unoccupied intervals $\cup(E_{2i-1}, E_{2i})$ (figure 3) defined such that at infinity $m$ approaches the unit matrix, and that its boundary value on the cuts $m_{\pm} = m(\epsilon \pm i0)$ connects the vector $\vec{F}$ to the vector $\vec{f}$ as

$$\vec{F}(\epsilon) = m_+(\epsilon) \vec{f}(\epsilon), \quad \epsilon \in \Omega.$$  

In [13] it has been shown that the matrix is a solution of the RH problem:

$$m_+ v = m_-, \quad v_{\alpha\beta} = \delta_{\alpha\beta} - 2\pi i f_{\alpha} g_{\beta}.$$  

In the case of FES,

$$v(\epsilon) = 1 + 2\pi i e^{-i\bar{h} \epsilon \tau r}(\epsilon) \begin{pmatrix} Q & 1 \\ -Q^2 & -Q \end{pmatrix}.$$  

Equations (30)–(34) constitute the matrix RH problem for FES. As typical for other integrable RH problems, a similarity transformation can be found to reduce the jump matrix $v(\epsilon)$ to a constant matrix, such that the analytic behavior in the energy dependence of the kernel will be
Unoccupied electronic states are between edges $E_{2k-1}$ and $E_{2k}$, $k = 1, \ldots, n$. The matrix $m$ of the RH problem (33) jumps on segments of real axis corresponding to unoccupied states (solid line). The steepest descent contour goes down vertically (dashed line) in the lower half-plane.

Figure 3.

translated into the analytic nature of singularities of the solution at infinity. We do not do this here.

Being specified for a one-edge problem ($n = 0$), in units of upper and lower cutoffs,

$$r^{(0)}(\epsilon) = e^{i\epsilon a}, \quad \epsilon_k \epsilon > 0;$$

$$Q^{(0)}(\epsilon, \tau) = \left(\frac{\sin \pi a}{\pi}\right)^2 \int_{\bar{E}_{\eta} < 0} e^{i\bar{h} \eta (\epsilon - \eta) - 2\epsilon a \eta} d\eta,$$

where we count energy from the edge. In this case, the RH problem is solved by elementary means. In fact, technically it is easier to proceed directly through the integral equation following [4].

7. Method of steepest descent and the leading singularity

The asymptotic behavior at the edges can be found by the steepest descent method described in [20].

The steepest descent contour starts from Fermi edges and extends to $i\infty$ in the lower half-plane as in figure 3. Along this contour, the rapidly falling exponential factor $e^{-i\bar{h} \tau}$ in (34) suppresses the jump of the matrix $m$ except at small segments near the Fermi edges, where $r(\epsilon)$ and $Q(\epsilon)$ are singular. At energy close to an edge, say $E_i$, we estimate $r(E_i + \epsilon) \approx r^{(0)}(\epsilon) \prod_{j \neq i} (E_{ij})^{\epsilon_j a}$ and $Q(E_i + \epsilon, \tau) \approx e^{iE_i \tau} \prod_{j \neq i} (E_{ij})^{-2\epsilon_i a} Q^{(0)}(\epsilon)$, and the problem reduces by a similarity transformation to the one-edge problem

$$\tilde{f}(E_i + \epsilon) = Z_i \tilde{f}^{(0)}(\epsilon); \quad \tilde{g}(E_i + \epsilon) = Z_i^{-1} \tilde{g}^{(0)}(\epsilon);$$

$$m(E_i + \epsilon) = Z_i m^{(0)}(\epsilon) Z_i^{-1};$$

$$Z_i = e^{-\frac{1}{2} E_i (I - \sigma_3 \tau)} \prod_{j \neq i} (E_{ij})^{\epsilon_j a},$$

where script $0$ indicates entities of the one-edge problem, $t = t_1 + t_2$, and $I$ is the $2 \times 2$ unit matrix.

The reduction of the multi-edge problem to the one-edge problem using the RH steepest descent method is what allows us to solve the problem. Indeed, the contribution of each edge is well known. In units of spacing and up to a constant factor, they are

$$C^{(0)}_i \sim \tau^{-a^2}, \quad L^{(0)}_i \sim e^{i \epsilon \tau 2a^{-1}}.$$

We note that Ohtaka and Tanebe [4] showed how (38) (originally obtained in [1, 2] by different means) follows from the integral equation (29).

The leading asymptote for the structured Fermi sea can be obtained by a combination of the one-edge problem and a similarity transformation. The similarity transformation does not
affect the contribution of closed loops. Therefore, each edge contributes equally. Summing them up, we obtain \( e^C \approx \sum C_{ij} \).

The similarity transformation and (31) give the contribution of open lines as a sum of one-edge open lines each weighted by its own amplitude

\[
L = \sum_{i=1}^{2n+1} e^{-iE_i\tau} \prod_{j \neq i} E_{ij}^{-2\epsilon_j^a} E_i^{(0)}.
\]

Combining the normalization of the closed loop and the open line, we obtain

\[
G(\tau) \sim \tau^{-(2n+1)a^2} \left( \prod_{i<j} E_{ij}^{2\epsilon_i^j a^2} \right) e^{\sum_m \epsilon_m \tau 2^{a-1} e^{-iE_m\tau} \prod_{n \neq m} E_{mn}^{-2\epsilon_m a}},
\]

where a multiplicative constant depending only on cutoffs has been omitted. Fourier transforming, this formula prompts the main result (1).

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**Appendix A. Computation of (23) and (24)**

We sketch the computation of the matrix element (23). They are not much different from similar calculations for a single-edge problem [4] . We start from a general formula (22) describing an overlap of arbitrary electronic states consisting of a finite number \( N \) of occupied levels. It can be seen as the exponent of the electrostatic interaction energy of log-interacting particles with charges of size \(+a\) at \( \epsilon_i \in \Omega \)’s and \(-a\) at \( \epsilon_i' \in \Omega - a \). Thus, we have a regular alternating pattern of charges, the negative charges at an offset of \(-a\) with respect to the positive ones; the pattern starts at \( E_0 \) and persists up to \( E_{2n+1} \), with gaps at \([E_{2i-1}, E_{2i}]\), \( i = 1, \ldots, n \). The electrostatic energy is

\[
\sum_{i \neq j, \epsilon_i, \epsilon_j \in \Omega} \log \left( \frac{\epsilon_i - \epsilon_j}{\epsilon_i - \epsilon_j + a} \right) - N \log a.
\]

This sum can be evaluated in terms of Barnes functions, but we are interested only in the limit of large number of levels. There we may think in terms of a density of dipoles with a polarization vector \( a \). The energy of this system is equivalent to the energy of charges \( \epsilon_i a \) placed at edge \( E_i \). The latter (up to \( a\)-dependent constant) is \( \sum_{i<j} \epsilon_i \epsilon_j a^2 \log |E_i - E_j| \). The exponent of the energy yields (23).

The configuration of charges corresponding to \( M_{\epsilon,\eta} \) is a set of dipoles discussed above minus a charge \(+1\) at \( \epsilon \in \Omega \) plus a charge \(-1\) at \( \eta \notin \Omega \). The energy acquires the interaction energy between charges at \( \eta \) and \( \epsilon \) and charges \( \pm a \) sitting at edges \( E_0 \). This addition is \( \sum \epsilon_i a (\log |E_i| - \log |\eta - E_i|) \) plus an \( a\)-dependent constant. The origin of the constant is the energy of dipoles in the vicinity of the hole at \( \epsilon \). It is \( \sum_{\epsilon_i \in \Omega} \log |\epsilon_i - \eta - a| \). Assuming that \( \eta \) is far from edges, the contribution goes from levels close to \( \epsilon \). It gives log \( \left( \frac{\sin(\pi a)}{\pi} \right) \). All together it yields (24). Computation of \( M_{\epsilon} \) is similar. In that case, only interactions between the extra charge at \( \epsilon \notin \Omega \) and residual charges at edges contribute. It does not incur a constant factor.
Appendix B. Computation through the bosonic representation

The leading singularity can be understood using a bosonic formalism. First we separate fast oscillatory modes at each edge:

\[ c(t, x_0) = \sum_i e^{i \tilde{E}_i t} \psi_i(t). \]

Then, we represent slow modes through components of the Bose field \( \partial_x \psi_i \) as

\[ \psi_i \propto \left( \epsilon_i \prod_{j \neq i} E - \epsilon_i \epsilon_j \right)^{1/2} e^{-i \epsilon_i \phi_i}. \]  \hspace{1cm} (B.1)

The Bose field \( \phi \) is a sum of its components \( \phi = \sum \psi_i \). Components of the Bose field represent particle–hole excitations close at each edge. At \( \hbar / \tau \gg E_{ij} \) they can be treated as independent canonical Bose fields. Their variances \( D_i(t_1, t_2) = -\frac{1}{2} \left\langle \phi_i(t_1) - \phi_i(t_2) \right\rangle^2 \) are not difficult to compute. As follows from (B.1), \( D_{2k-1} \) are sums of \( \cos \epsilon \tau - 1 / \epsilon \) over all possible energy of a particle–hole excitation provided that a particle is placed in the ‘gap’ \( (E_{2k-1}, E_{2k}) \). Similarly \( D_{2k} \) is the sum over energies of a hole–particle excitation provided that a hole is placed to the band \( (E_{2k}, E_{2k+1}) \). Computing these integrals at \( \tau \gg \hbar / |E_{ij}| \) one obtains

\[ D_i(\tau) = -\log \tau + \epsilon_i \sum_{j \neq i} \epsilon_j \log |E_{ij}|. \]  \hspace{1cm} (B.2)

The time-independent term in (B.2) explains the prefactor in (B.1); the correlator \( \left\langle \psi_i^\dagger (t_2) \psi_i(t_2) \right\rangle \) then also has to be obtained for the Bose field. This yields \( \left( \prod_{j \neq i} E^{-\epsilon_i \epsilon_j} \right) e^{D_i} \).

In the Bose representation, Green’s function \( G(t_1, t_2) \) is a sum of edge components:

\[ G(t_1, t_2) = \sum_i \left( \epsilon_i \prod_{j \neq i} E^{-\epsilon_i \epsilon_j} \right) e^{i \tilde{E}_i t} G_i(t_1, t_2), \]

\[ G_i = \left\langle e^{\epsilon_i (a(\psi_i(t_2) - \psi_i(t_1)))} \prod_{j \neq i} \left\langle e^{-a(\psi_j(t_2) - \psi_j(t_1)))} \right\rangle. \]  \hspace{1cm} (B.3)

Computing this, we obtain Green’s function (40).

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