The cluster value problem in spaces of continuous functions.†

W. B. Johnson‡ and S. Ortega Castillo‡

June 27, 2013

Abstract

We study the cluster value problem for certain Banach algebras of holomorphic functions defined on the unit ball of a complex Banach space \( X \). The main results are for spaces of the form \( X = C(K) \).

1 Preliminaries.

A cluster value problem for a complex Banach space \( X \) is a weak version of the corona problem for the open unit ball \( B \) of \( X \), which is a long-standing open problem in complex analysis when \( X \) has dimension at least 2. Instead of studying when \( B \) is dense in the spectrum of a uniform algebra \( H \) of bounded analytic functions on \( B \) in the weak topology induced by \( H \) (corona problem), the cluster value problem investigates the following situation:

Let \( \bar{B}^* \) be the closed unit ball of the bidual \( X^{**} \), and let \( M_H \) be the spectrum (i.e. maximal ideal space) of a uniform algebra \( H \) of norm continuous functions on \( B \) with \( H \supset X^* \). Then \( \pi : M_H \to \bar{B}^* \), given by \( \pi(\tau) = \tau|_{X^*} \) for \( \tau \in M_H \), is surjective (as a consequence of the results in Chapter 2 and 5 of [10]). For each \( x^{**} \in \bar{B}^* \), \( M_{x^{**}}(B) = \pi^{-1}(x^{**}) \) is called the fiber of \( M_H \) over \( x^{**} \). Aron, Carando, Gamelin, Lasalle and Maestre observed in [4] that for every \( x^{**} \in \bar{B}^{**} \) we have the inclusion

\(^{†}\)Primary subject classification: Several complex variables and analytic spaces.

\(^{‡}\)Secondary subject classification: Functional analysis.

\(^{‡‡}\)Supported in part by NSF DMS 10-01321
\[ \text{Cl}_B(f, x^{**}) \subset \hat{f}(M_{x^{**}}(B)), \forall f \in H, \quad (1) \]

where \( \text{Cl}_B(f, x^{**}) \), the cluster set of \( f \) at \( x^{**} \), stands for the set of all limits of values of \( f \) along nets in \( B \) converging weak-star to \( x^{**} \), while \( \hat{f} \) represents the Gelfand transform of \( f \). There they formulated the cluster value problem for \( H \): for which Banach spaces \( X \) is there equality in (1) for all \( x^{**} \in \bar{B}^{**} \)? When there is equality in (1) for a certain \( x^{**} \in \bar{B}^{**} \), we say \( X \) satisfies the cluster value theorem for \( H \) at \( x^{**} \).

As was pointed out in [4], it is easy to check that the cluster value theorem for \( H \) at all points in \( \bar{B}^{**} \) is indeed weaker than the corona problem for \( B \) and \( H \): Given \( x^{**} \in \bar{B}^{**} \), if \( \tau \in M_{x^{**}}(B) \) were the weak-star limit of the net \( (x_\alpha) \subset B \), then \( \lim_{\alpha} f(x_\alpha) = \hat{f}(\tau) \) for all \( f \in H \), and in particular \( \lim_{\alpha} x^*(x_\alpha) = \hat{x}^*(\tau) = x^*(x^{**}) \) for all \( x^* \in X^* \), i.e. \( x^{**} \) would be the weak-star limit of \( (x_\alpha) \), and so \( \hat{f}(\tau) \in \text{Cl}_B(f, x^{**}) \).

In an effort to research the corona problem, we investigate conditions that guarantee the simpler cluster value theorem for a Banach algebra of analytic functions defined on the unit ball of a complex Banach space \( X \). In particular, we generalize some of the results in [4]. Our main results are for the spaces of the form \( X = C(K) \), including a translation result of a cluster value problem, from any point in \( B^{**}_{C(K)} \) to the origin.

We thank Richard Aron and Manuel Maestre for their communications.

2 Cluster value problem in finite-codimensional subspaces.

In [4], the authors obtain a cluster value theorem at the origin for Banach spaces with shrinking 1-unconditional bases for the algebra \( H = A_u(B) \) of bounded analytic functions on \( B \) that are also uniformly continuous. Slight modifications of their arguments in Section 3 of [4] yield the following:

**Proposition 1.** Let \( S \) be a finite rank operator on \( X \), so that \( P = I - S \) has norm one. If \( \phi \in M_0(B) \), then \( \hat{f}(\phi) = \hat{f} \circ \hat{P}(\phi) \), for all \( f \in A_u(B) \).
Proposition 2. Suppose that for each finite dimensional subspace $E$ of $X^*$ and $\epsilon > 0$ there exists a finite rank operator $S$ on $X$ so that $\| (I - S^*)|_E \| < \epsilon$ and $\| I - S \| = 1$. Then the cluster value theorem holds for $A_u(B)$ at 0.

Proof. Suppose that $0 \notin Cl_B(f,0)$. We must show that $0 \notin \hat{f}(M_0)$. Since Proposition 2 builds on Proposition 1, one naturally wonders if Proposition 2 can be extended to the larger algebra $H^\infty(B)$ of all bounded analytic functions on $B$. The answer is no in general, as shown by the following example of Aron.

Example 1. There exists a finite rank operator $S$ on $\ell_2$ so that $P = I - S$ has norm one, and there exist $\phi \in M_0(B_{\ell_2})$ as well as $f \in H^\infty(B_{\ell_2})$ so that $\hat{f} (\phi) \neq \hat{f} (P(\phi))$.

Proof. Let $S: \ell_2 \to \ell_2$ be given by $S(x) = (x_1, 0, 0, \cdots)$. Clearly $S$ is a finite rank operator and $P = I - S$ has norm one.

Let $(r_j)$ and $(s_j)$ be sequences of positive real numbers, such that $(r_j) \downarrow 0$ and $(s_j) \uparrow 1$ in such a way that each $r_j^2 + s_j^2 < 1$ and $r_j^2 + s_j^2 \to 1^-$. For each $j = 1, 2, 3, \cdots$, let $\delta_{r_1e_1+s_1e_j}$ be the usual point evaluation homomorphism from $H^\infty(B_{\ell_2}) \to \mathbb{C}$. Let $\phi : H^\infty(B_{\ell_2}) \to \mathbb{C}$ be an accumulation point of $\{ \delta_{r_1e_1+s_1e_j} \}$ in the spectrum of $H^\infty(B_{\ell_2})$. Let $f : B_{\ell_2} \to \mathbb{C}$ be the $H^\infty$ function given by

$$
\hat{f}(x) = \frac{x_1}{\sqrt{1 - \sum_{j=2}^{\infty} x_j^2}},
$$

where the square root is taken with respect to the usual logarithm branch. Then $\phi(f) = 1$. However $\phi(f \circ P) = 0$ since $f \circ P \equiv 0$. 

\[\blacksquare\]
When a Banach space has a shrinking reverse monotone finite dimensional decomposition (FDD), that is, a shrinking FDD so that the natural projections are at distance one from the identity operator, we have that the condition in Proposition 2 holds, and therefore we obtain a cluster value theorem:

**Corollary 1.** If $X$ is a Banach space with a shrinking reverse monotone FDD, then the cluster value theorem holds for $A_u(B)$ at 0.

The operators $P$ considered in Propositions 1 and 2 have finite-codimensional rank, which suggests that the cluster value problem at the origin of a Banach space can be studied by considering the same problem in its finite-codimensional subspaces. As we conjectured, this turns out to be the case:

**Proposition 3.** [Aron, Maestre] If $Y$ is a closed finite-codimensional subspace of $X$ and $f \in A_u(B)$, then $Cl_B(f, 0) = Cl_{B_Y}(f|_Y, 0)$, where $B_Y$ is the unit ball of $Y$.

**Proof.** $A_u(B)$ coincides with the uniform limits on $\bar{B}$ of continuous polynomials on $X$ (see Theorem 7.13 in [11] and p. 56 in [3]), where polynomials are finite linear combinations of symmetric $m$-linear mappings restricted to the diagonal. Thus, by passing to the uniform limit on $\bar{B}$, we may assume $f$ is an $m$-homogeneous polynomial, with associated symmetric $m$-linear functional $F$. Let $(x_\alpha)$ be a weakly null net in $B$ such that $F(x_\alpha, \cdots, x_\alpha) \to \lambda$.

Each $x_\alpha$ can be written uniquely as $y_\alpha + u_\alpha$, where $y_\alpha \in Y$ and $u_\alpha$ is from a fixed finite dimensional complement of $Y$ in $X$. Then

$$f(x_\alpha) = F(x_\alpha, \cdots, x_\alpha) = f(y_\alpha) + m F(y_\alpha, \cdots, y_\alpha, u_\alpha) + [m(m-1)/2] F(x_\alpha, \cdots, x_\alpha, u_\alpha, u_\alpha) + \cdots + f(u_\alpha).$$

Now, since $(x_\alpha)$ is weakly null, the same holds for $(y_\alpha)$ and $(u_\alpha)$. However, since $(u_\alpha)$ belongs to a finite dimensional space, it follows that $||u_\alpha|| \to 0$. Thus $F(y_\alpha, \cdots, y_\alpha, u_\alpha), F(x_\alpha, \cdots, x_\alpha, u_\alpha, u_\alpha), \cdots, f(u_\alpha)$ all go to 0. Thus $f(y_\alpha) \to \lambda$. Finally, since $\lim \sup ||y_\alpha|| \leq 1$, we can take a sequence of scalars $(t_\alpha)$ such that $||t_\alpha y_\alpha|| < 1$ for all $\alpha$ and $t_\alpha \to 1$, and consequently, $\lim f(t_\alpha y_\alpha) = \lim t_\alpha^m f(y_\alpha) = \lambda$. Hence $\lambda \in Cl_{B_Y}(f|_Y, 0)$.

As a consequence we obtain that the cluster sets of an element $f$ of $A_u(B)$ at 0 can be described in terms of the Gelfand transforms of $f|_{B_Y}$ as $Y$ ranges over finite-codimensional subspaces of $X$:
Proposition 4. For every Banach space $X$,
\[
\text{Cl}_B(f, 0) = \bigcap_{Y \subset X, \dim(X/Y) < \infty} f|_{B_Y}(M_0(B_Y)), \forall f \in A_u(B).
\]

Proof. From Proposition 3 and the inclusion in (1), for every finite-codimensional subspace $Y$ of $X$,
\[
\text{Cl}_B(f, 0) = \text{Cl}_{B_Y}(f|_{B_Y}, 0) \subset \hat{f}|_{B_Y}(M_0(B_Y)).
\]

For the reverse inclusion, suppose $0 \notin \text{Cl}_B(f, 0)$. Then there are $\epsilon > 0$ and a weak neighborhood $U$ of 0 such that $|f| > \epsilon$ on $U \cap B$. $U$ contains a closed finite-codimensional subspace $Y_0$ of $X$, so $|f|_{B_{Y_0}} > \epsilon$. Hence $\hat{f}|_{B_{Y_0}}$ is invertible, which implies that $0 \notin \hat{f}|_{B_{Y_0}}(M_0(B_{Y_0}))$.

Going back to Proposition 2 we see that having the cluster value property at 0 only requires the existence of a certain type of finite rank operators at distance one from the identity operator. However simple this condition may seem, it is impossible in the case of the Banach space $c$ of continuous functions on $\omega$, also seen as the subspace of $l^\infty$ of convergent sequences:

Example 2. Let $L \in B_{c^*}$ be given by
\[
L((c_n)_n) = \lim_{n \to \infty} c_n.
\]

If $S : c \to c$ is a finite rank operator with $\|(S^*-I_c)L\| < \epsilon$, then $\|S-I_c\| \geq 2-\epsilon$.

Proof. For each $k \in \mathbb{N}$, consider $L_k \in B_{c^*}$ given by
\[
L_k((c_n)_n) = (\lim_{n \to \infty} c_n - c_k)/2.
\]

Let us show that $\|S^*(L_k)\| \to 0$ as $k \to \infty$. For every $x \in B_c$, $S^*(L_k)x = L_k(Sx) \to 0$ as $k \to \infty$. Moreover, since $S$ has finite rank, $\{Sx : x \in B_c\}$ is pre-compact. Thus $S^*L_k = L_k \circ S$ converges to zero uniformly on $B_c$, i.e. $\|S^*L_k\| \to 0$ as $k \to \infty$.

Now note that $\|L - 2L_k\| = 1$ for each $k$, so
\[
\|S^*-I_c\| \geq \|(S^*-I_c)(L-2L_k)\| \geq \|2L_k-2S^*(L_k)\| - \epsilon \geq 2-\epsilon - 2\|S^*(L_k)\|.
\]

Since $S^*(L_k) \to 0$, then $\|S - I_c\| = \|S^*-I_c\| \geq 2-\epsilon$.

\[\]
The reader may check that the condition is also impossible for $L_p$, $1 \leq p \neq 2 < \infty$.

However, note that since $c_0$ is one-codimensional in $c$, Proposition 3 implies that for all $f \in A_u(B_c)$,

$$Cl_{B_c}(f, 0) = Cl_{B_{c_0}}(f|_{B_{c_0}}, 0).$$

Also, Propositions 1.59 and 2.8 of [8] imply that all functions in $A_u(B_{c_0})$ can be uniformly approximated on $B_{c_0}$ by polynomials in the functions in $X^*$, which in turn implies that each fiber at $x \in \bar{B}^{**}$ consists only of $x$, so the cluster value theorem for $A_u(B_{c_0})$ holds, and in particular

$$Cl_{B_{c_0}}(f|_{B_{c_0}}, 0) = \hat{f}(M_0(B_{c_0})), \quad \forall f \in A_u(B_c).$$

Hence we are left to compare $\hat{f}(M_0(B_{c_0}))$ with $\hat{f}(M_0(B_c))$ for $f \in A_u(B_c)$.

Note that an inclusion is evident:

**Proposition 5.** For a Banach space $X$ and $Y$ a subspace of $X$,

$$\hat{f}|_{B_Y}(M_0(B_Y)) \subset \hat{f}(M_0(B)), \quad \forall f \in A_u(B).$$

**Proof.** Let $f \in A_u(B)$ and $\tau \in M_0(B_Y)$. Since $\phi_1 : A_u(B) \rightarrow A_u(B_Y)$ given by $\phi(g) = g|_Y$ for all $g \in A_u(B)$ is a continuous homomorphism that maps $A(B)$ into $A(B_Y)$, the mapping $\tau : A_u(B) \rightarrow \mathbb{C}$ given by $\tau(g) = \tau(g|_Y)$ for all $g \in A_u(B)$ is in the fiber $M_0(B_Y)$. Moreover,

$$\hat{f}|_Y(\tau) = \hat{f}(\tau).$$

The reverse inclusion is unclear. However, the space $c$ also has the property of being isomorphic to $c_0$, which implies, as we will see, that $c$ has the cluster value property too.

Let $P(X)$ denote the continuous polynomials on $X$, $P_f(X)$ the polynomials in the functions of $X^*$ (known as finite type polynomials), and $A(B_X)$ the uniform algebra of uniform limits of elements in $P_f(X)$.

**Lemma 1.** Let $X$ be a Banach space so that $A_u(B_X) = A(B_X)$. If the Banach space $Y$ is isomorphic to $X$, then also $A_u(B_Y) = A(B_Y)$. 

6
Proof. Let $T : Y \to X$ be the Banach space isomorphism between $Y$ and $X$. Let $f \in A_u(B_Y)$. Then there exist a sequence of polynomials $P_n \in \mathcal{P}(Y)$ such that $||P_n - f||_{B_Y} \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $P_n \circ T^{-1} \in \mathcal{P}(X)$, so there exists a polynomial $Q_n \in \mathcal{P}_f(X)$ such that $||P_n \circ T^{-1} - Q_n||_{B_X} < \frac{1}{n||T||}$, and consequently $||P_n - Q_n \circ T||_{B_Y} < \frac{1}{n}$, where $Q_n \circ T \in \mathcal{P}_f(Y)$.

Consequently, the sequence of polynomials $Q_n \circ T \in \mathcal{P}_f(Y)$ converges to $f$ uniformly on $B_Y$, so $f \in A(B_Y)$.

\[ \square \]

Corollary 2. The Banach space $c$ satisfies the cluster value theorem for $A_u(B_c)$ at all points in $\overline{B_c}^{**}$.

3 Cluster value problem in $C(K) \not\cong c$.

Bessaga and Pełczyński proved in [6] that, when $\alpha \geq \omega$ is a countable ordinal, $C(\alpha)$ is not isomorphic to $c = C(\omega)$. Therefore we no longer can use Lemma 1 to obtain a cluster value theorem on such spaces of continuous functions.

Nevertheless, for $\alpha$ a countable ordinal, the intervals $[1, \alpha]$ are always compact, Hausdorff and dispersed (they contain no perfect non-void subset). The compact, Hausdorff and dispersed sets $K$ satisfy, from the Main Theorem in [12], that $X = C(K)$ contains no isomorphic copy of $l_1$. Moreover, from Theorem 5.4.5 in [11], $X = C(K)$ has the Dunford-Pettis property. Therefore, for dispersed $K$, the continuous polynomials on $X = C(K)$ are weakly (uniformly) continuous on bounded sets by Corollary 2.37 in [8].

Moreover, since $X^* = l_1(K)$ has the approximation property, Proposition 2.8 in [8] now yields that all continuous polynomials on $X$ can be uniformly approximated, on bounded sets, by polynomials of finite type. Thus the elements of $A_u(B)$ can be approximated, uniformly on $B$, by polynomials of finite type. Hence $A_u(B) = A(B)$, so each fiber at $x \in \hat{B}^{**}$ is the singleton $\{x\}$, and then $X$ satisfies the cluster value theorem for the algebra $A_u(B)$.

We now consider the cluster value problem on $X$ for the algebra of all bounded analytic functions $H^\infty(B)$. Following the line of proof of Theorem 5.1 in [11], we still get a cluster value theorem:
Theorem 1. If $X$ is the Banach space $C(K)$, for $K$ compact, Hausdorff and dispersed, then the cluster value theorem holds for $H^\infty(B)$ at every $x \in B^{**}$.

Proof. Fix $f \in H^\infty(B)$ and $w = (w_t)_{t \in K} \in B^{**}$ (where $C(K)^{**} = l_\infty(K)$). Suppose $0 \notin Cl_B(f, w)$. It suffices to show that $0 \notin \hat{f}(M_w)$.

Since 0 is not a cluster value of $f$ at $w$, there exists a weak-star neighborhood $U$ of $w$ such that $0 \notin f(U \cap B)$, where

$$U \cap B \supset \bigcap_{i=1}^{n} \{ z \in B : | (z - w), x_i^* | < \epsilon \},$$

for some $\epsilon > 0$ and $x_1^*, \ldots, x_n^* \in X^* = l_1(K)$.

We have that $x_i^* = (x_i^*(t))_{t \in K}$ has countably many nonzero coordinates for $i = 1, \ldots, n$. Thus,

$$U \cap B \supset \bigcap_{i=1}^{n} \{ z \in B : | \sum_{t \in K} (z_t - w_t) x_i^*(t) | < \epsilon \},$$

and there is a finite set $F \subset \bigcup_{i=1}^{n} F_i$ so that $\sum_{t \notin F} |x_i^*(t)| < \epsilon/4$, for $i = 1, \ldots, n$. Then,

$$U \cap B \supset \bigcap_{t \in F} \{ z \in B : |z_t - w_t| < \delta \},$$

where

$$\delta = \min_{1 \leq i \leq n, t \in F} \left( \frac{\epsilon}{2 |F| |x_i^*(t)|} \right).$$

In summary, there exist $c > 0$, $\delta > 0$ and a finite set $F \subset K$ such that if $z \in B$ satisfies $|z_t - w_t| < \delta$ for $t \in F$ then $|f(z)| \geq c$. Relabel the indices in $F$ as $t_1, \ldots, t_m$, where $m = |F|$. Then proceed as in the proof of Theorem 5.1 in [4].

For $0 \leq k \leq m - 1$, define $U_k = \{ z \in B : |z_j - w_j| < \delta, k + 1 \leq j \leq m \}$, and set $U_m = B$. Note that $1/f$ is bounded and analytic on $U_0$.

We claim that for each $k$, $1 \leq k \leq m$, there are functions $g_k$ and $h_{k,j}$, $1 \leq j \leq k$, in $H^\infty(U_k)$ that satisfy

$$f(z)g_k(z) = 1 + (z_{t_1} - w_{t_1})h_{k1}(z) + \cdots + (z_{t_k} - w_{t_k})h_{kk}(z), \quad z \in U_k.$$  \hfill (2)
Once this claim is established, the proof is easily completed as follows. The functions \( g_m \) and \( h_{mj} \) belong to \( H^\infty(B) \) and satisfy

\[
\hat{f}\hat{g}_m = \hat{f} + \sum_{j=1}^{m} (\hat{z}_{t_j} - \hat{w}_{t_j})\hat{h}_{mj}.
\]

Since each \( \hat{z}_{t_j} - \hat{w}_{t_j} \) vanishes on \( M_w \) (by the definition of \( M_w \)), we obtain \( \hat{f}\hat{g}_m = 1 \) on \( M_w \), and consequently \( \hat{f} \) does not vanish on \( M_w \), as required.

Just as in \[4\], the claim is established by induction on \( k \). The first step, the construction of \( g_1 \) and \( h_{11} \), is as follows. We regard \( 1/f((z_t)_{t \in K}) \) as a bounded analytic function of \( z_t \) for \( |z_t| < 1 \) and \( |z_t - w_t| < \delta \), with \( z_t \), \( t \in K - \{t_1\} \), as analytic parameters in the range \( |z_t| < 1 \) for \( t \in K - \{t_1\} \), and \( |z_t - w_t| < \delta \) for \( 2 \leq j \leq m \). According to lemma 5.3 in \[4\], we can express

\[
\frac{1}{f(z)} = g_1(z) + (z_t - w_t)h(z), \quad z \in U_0,
\]

where \( g_1 \in H^\infty(U_1) \) and \( h \in H^\infty(U_0) \). We set

\[
h_{11}(z) = [f(z)g_1(z) - 1]/(z_t - w_t), \quad z \in U_1,
\]

so that (2) is valid for \( k = 1 \). Note that \( h_{11} = -hf \) on \( U_0 \). Consequently \( h_{11} \) is bounded and analytic on \( U_0 \). The defining formula then shows that \( h_{11} \) is analytic on all of \( U_1 \), and since \( |z_t - w_t| \geq \delta \) on \( U_1 - U_0 \), \( h_{11} \) is bounded on \( U_1 \).

Now suppose that \( 2 \leq k \leq m \), and that there are functions \( g_k \) and \( h_{k-1,j} \) \( (1 \leq j \leq k - 1) \) that satisfy (2) and are appropriately analytic. We apply lemma 5.3 in \[4\] to these as functions of \( z_{t_k} \), with the other variables regarded as analytic parameters, to obtain decompositions

\[
g_{k-1}(z) = g_k(z) + (z_{t_k} - w_{t_k})G_k(z)
\]

and

\[
h_{k-1,j}(z) = h_{k,j}(z) + (z_{t_k} - w_{t_k})H_{k,j}(z), \quad 1 \leq j \leq m - 1,
\]

where \( g_k \) and the \( h_{k,j} \)'s are in \( H^\infty(U_k) \), and \( G_k \) and the \( H_{k,j} \)'s are in \( H^\infty(U_{k-1}) \).

From the identity (2), with \( k \) replaced with \( k - 1 \), we obtain

\[
f g_k = 1 + \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})h_{k,j} + (z_{t_k} - w_{t_k})\left[-fG_k + \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})H_{k,j}\right]
\]
on $U_{k-1}$. We define
\[ h_{kk} = (fg_k - 1 - \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})h_{kj})/(z_{t_k} - w_{t_k}), \quad z \in U_k. \]
Then (2) is valid. On $U_{k-1}$ we have
\[ h_{kk} = -fG_k + \sum_{j=1}^{k-1} (z_{t_j} - w_{t_j})H_{kj}, \]
so that $h_{kk}$ is bounded and analytic on $U_{k-1}$. Since $|z_{t_k} - w_{t_k}| \geq \delta$ on $U_k - U_{k-1}$, we see from the defining formula that $h_{kk} \in H^\infty(U_k)$. This establishes the induction step, and the proof is complete.

We do not know the answer to the cluster value problem for other spaces $C(K)$.

Consider the following cluster value problem: Given $f_0^{**} \in B^{**}$, the cluster value problem for $H^\infty(B)$ over $A_u(B)$ at $f_0^{**}$ asks whether for all $\psi \in H^\infty(B)$ and $\tau \in M_{H^\infty(B)}$ such that $\tau|_{A_u(B)} = f_0^{**}$ (that we denote by $\tau \in M_{f_0^{**}}(B)$), can we find a net $(f_\alpha) \subset B$ such that $\psi(f_\alpha) \to \tau(\psi)$ and $f_\alpha$ converges to $f_0^{**}$ in the polynomial-star topology, as defined in p. 200 in [10] (that we denote by $\tau(\psi) \in Cl_B(\psi, f_0^{**})$)?

The previous problem seems to be highly nontrivial. Since for every infinite compact Hausdorff space $K$, $C(K)$ contains a subspace $Y$ isometric to $c_0$ (Proposition 4.3.11 in [1]), the fiber $M_0(B_{C(K)})$ is huge (and from Lemma 6.6 in [7]), also each fiber $M_{f_0}(B_{C(K)})$ for $f_0 \in B_{C(K)}$. Indeed, according to Theorem 6.6 in [7], there is a family of distinct characters $\{\tau_\alpha\}_{\alpha \in B_{C(K)}}$, such that each $\tau_\alpha : H^\infty(B_Y) \to \mathbb{C}$ satisfies $\delta_0 = \tau_\alpha|_{A(B_Y)} = \tau_\alpha|_{A_u(B_Y)}$ (because $Y$ is isometric to $c_0$, so $A(B_Y) = A_u(B_Y)$). Hence $\{\tau_\alpha\}_{\alpha \in B_{C(K)}} \subset M_0(B_Y)$ and therefore $\{\tau_\alpha \circ R\}_{\alpha \in B_{C(K)}} \subset M_0(B_{C(K)})$, where $R$ is the restriction mapping $R : H^\infty(B_{C(K)}) \to H^\infty(B_Y)$, which is clearly a homomorphism. Note that the characters $\{\tau_\alpha \circ R\}_{\alpha \in B_{C(K)}}$ are all distinct due to Theorem 1.1 in [2], because $\ell_\infty$ is an isometrically injective space (Proposition 2.5.2 in [1]), so there exists a norm-one linear map $S : C(K) \to \ell_\infty$ such that $S|_{c_0} = I_{c_0}$.

We prove in Corollary 3 that if the latter cluster value problem has an affirmative answer at some point of $B_{C(K)}$, then it has an affirmative answer at
all points of $B_{C(K)}$. For that let us first establish the following lemmas, the
first of which is a folklore result mentioned e.g. in [14] and [5], but inasmuch
there seems to be no proof in the literature we will sketch the proof.

**Lemma 2.** Let $f_0 \in B = B_{C(K)}$. $T : B \to B$ given by

$$T(f) = \frac{f - f_0}{1 - f_0 \cdot f} \quad \forall f \in B,$$

is biholomorphic.

**Proof.** Set $\delta_0 = ||f_0||$.

Let us start by showing that $T$ is well defined, i.e. $||Tf|| < 1$ when $||f|| < 1$.

Let $f \in B$. We can find $\delta \in (\delta_0, 1)$ such that $||f|| \leq \delta$.

For every $t_0 \in K$, let $z = f(t_0)$ and $c = f_0(t_0)$, so that $T(f)(t_0) = \frac{z - c}{1 - f_0}$.

Let $\Delta$ denote the open unit disk in the complex plane $C$.

Since $\sigma : (\delta \cdot \Delta) \times (\delta_0 \cdot \Delta) \to \Delta$ given by $\sigma(z, c) = \frac{z - c}{1 - f_0}$ is continuous, then $\sigma((\delta \cdot \Delta) \times (\delta_0 \cdot \Delta))$ is a compact subset of $\Delta$, so there exists $\delta_1 < 1$ so that $\sigma((\delta \cdot \Delta) \times (\delta_0 \cdot \Delta)) \subset \delta_1 \Delta$.

Thus $||Tf|| \leq \delta_1 < 1$.

Let us now show that $T$ is also holomorphic, or equivalently, $C$-differentiable.

For $f \in B$ fixed, the linear mapping $L : C(K) \to C(K)$ given by $L(h) = \frac{1 - |f_0|^2}{(1 - f_0 h)^2} h$ satisfies that, for $h \neq 0$ small enough,

$$\frac{T(f + h) - T(f) - L(h)}{||h||} = \frac{f + h - f_0}{1 - f_0 (f + h)} - \frac{f - f_0}{1 - f_0 f} - \frac{1 - |f_0|^2}{(1 - f_0 h)^2} h/||h||$$

$$= \frac{1 - |f_0|^2}{(1 - f_0 h)^2} \frac{h}{1 - f_0 (f + h)} - \frac{1 - |f_0|^2}{(1 - f_0 h)^2} h/||h||$$

$$= \frac{f_0 h}{(1 - f_0 h)^2 (1 - f_0 (f + h))} (1 - |f_0|^2) h/||h||,$$

which goes to zero as $h \to 0$. Thus $T$ is holomorphic.

Since $T$ clearly has a necessarily holomorphic inverse ($S(f) = \frac{f + f_0}{1 + f_0 f}$), we
have that $T$ is a biholomorphic function on $B$ that sends $f_0$ to the function identically zero.

$\blacksquare$
Lemma 3. The biholomorphic function $T$ from the previous lemma induces a mapping $\hat{T}$ on the spectrum $M_{H(B)}$, where $H$ denotes either the algebra $A_u$ or the algebra $H^\infty$, that maps $M_{f_0}(B)$ onto $M_0(B)$.

Proof. Note that $T$ is a Lipschitz function. Indeed, if $f, g \in B$,

$$\|T(f) - T(g)\| = \frac{\| (1 - |f_0|^2)(f - g) \|}{(1 - \|f_0\|)(1 - \|f_0\|)} \leq \frac{1}{(1 - \|f_0\|)^2} \|f - g\|.$$

Thus for every $\psi \in H(B)$, $\psi \circ T \in H(B)$. So $\hat{T} : M_{H(B)} \to M_{H(B)}$, given by

$$\hat{T}(\psi) = \psi(\circ T), \ \forall \psi \in H(B),$$

is well defined. Moreover, given $\tau \in M_{f_0}(B)$ and $\psi \in A_u(B)$,

$$\hat{T}(\psi) = \tau(\psi \circ T) = (\psi \circ T)(f_0) = \psi(0),$$

i.e. $\hat{T}(\tau) \in M_0(B)$, for every $\tau \in M_{f_0}(B)$.

Now, given $\tau \in M_0(B)$ it is clear that $\hat{\tau} : H(B) \to C$ given by

$$\hat{\tau}(\psi) = \tau(\psi \circ T^{-1}), \ \forall \psi \in H(B),$$

is in $M_{H(B)}$, actually in $M_{f_0}(B)$, and $\forall \psi \in H(B)$,

$$\hat{\tau}(\psi) = \tau(\psi \circ T) = \tau(\psi),$$

i.e. $\hat{T}(\hat{\tau}) = \tau$. □

The reader can easily check that the previous mapping $\hat{T}$ is actually a homeomorphism.

Corollary 3. The cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at 0 is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at every $f_0 \in B$.

Proof. Let $f_0 \in B$ and set $T$ as in Lemma 2. Then, $\forall \psi \in H^\infty(B)$,

$$\hat{\psi}(M_0(B)) = \hat{\psi} \circ \hat{T}(M_{f_0}(B)) = \hat{\psi} \circ T(M_{f_0}(B)),$$

$$C_B(\psi, 0) = C_B(\psi \circ T, f_0),$$

because $\psi \circ T \in H^\infty(B)$ too, and $T^{-1}(f) = (f + f_0) \sum_{n=0}^{\infty} (-\overline{f_0}f)^n \ \forall f \in B_{C(K)}$ is polynomially-star continuous, because sums and norm limits of polynomially-star continuous maps are polynomially-star continuous, as well as multiplication by a fixed element of $C(K)$. □
References

[1] F. Albiac, N. J. Kalton, *Topics in Banach Space Theory*, Graduate Texts in Mathematics, 233. Springer, New York, 2006.

[2] R. M. Aron, P. D. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. math. France 106 (1978), pp. 3-24.

[3] R. M. Aron, B. J. Cole, T. W. Gamelin, *Spectra of algebras of analytic functions on a Banach space*, J. reine angew. Math 415 (1991), pp. 51-93.

[4] R. M. Aron, D. Carando, T. W. Gamelin, S. Lasalle, M. Maestre, *Cluster Values of Analytic Functions on a Banach space*, Math. Ann. 353 (2012), pp. 293-303.

[5] R. Braun, W. Kaup, H. Upmeier, *On the automorphisms of circular and Reinhardt domains in complex Banach spaces*, Manuscripta Math. 25 (1978), pp. 97-133.

[6] C. Bessaga, A. Pełczyński, *Spaces of continuous functions (IV) (On isomorphical classification of spaces of continuous functions)*, Studia Math. 19 (1960), pp. 53-62.

[7] , B. J. Cole, T. W. Gamelin, W. B. Johnson, *Analytic Disks in Fibers over the Unit Ball of a Banach Space*, Michigan Math. J. 39 (1992), pp. 551-569.

[8] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monographs in Mathematics, Springer Verlag, London, 1999.

[9] J. Duncan, S. A. R. Hosseiniun, *The second dual of a Banach algebra*, Proceedings of the Royal Society of Edingburg 84A (1979), pp. 309-325.

[10] T. Gamelin, *Analytic functions on Banach spaces*, in *Complex Function Theory*, edited by Gauthier and Sabidussi, Kluwer Academic Publisher, 1994, pp. 187-233.

[11] J. Mujica, *Complex Analysis in Banach Spaces*, North Holland Mathematics Studies, 120, Amsterdam, 1986.

[12] A. Pełczyński, Z. Semadeni, *Spaces of continuous functions (III)*, Studia Math. 18 (1959), pp. 211-222.
[13] S. Sakai, *C*-algebras and *W*-algebras, Springer Verlag, Berlin, 1971.

[14] J.-P. Vigué, *Le groupe des automorphismes analytiques d'un domaine borné d'un espace de Banach complexe. Application aux domaines bornés symétriques*, Ann. Scient. Éc. Norm. Sup. 4e série 9, 1976.

W. B. JOHNSON, DEPT. OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, E-MAIL: JOHNSON@MATH.TAMU.EDU

S. ORTEGA CASTILLO, DEPT. OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, E-MAIL: ORTEGA@MATH.TAMU.EDU