On Frobenius functionals of the Lie algebra $M_3(\mathbb{R}) \oplus g_{3}(\mathbb{R})$

Henti 1), E Kurniadi 1,a), E Carnia 1

1Department of Mathematics, Universitas Padjadjaran, Bandung, Indonesia
a)Corresponding author: edi.kurniadi@unpad.ac.id

Abstract. In the present paper, we study the Lie algebra written in the semi-direct sum formula of the vector space $M_3(\mathbb{R})$ and the Lie algebra $g_3(\mathbb{R})$ whose both contain $3 \times 3$ real matrices. We denote it by $g_3 := M_3(\mathbb{R}) \oplus g_{3}(\mathbb{R})$. The aim of this research is to study the existence of a linear functional such that $g_3$ is the Frobenius Lie algebra of dimension 18. Such the linear functional is called the Frobenius functional. We applied the literature reviews to achieve this result, particularly we study the notion of Frobenius Lie algebra in Ooms and Rais results. The main result of our research is the proof that $g_3$ is Frobenius Lie algebra. For the future research, the existence of a Frobenius functional is still an open problem to study for higher dimensional Lie algebras.

1. Introduction

In this paper, all Lie algebras are determined over the set of real numbers $\mathbb{R}$. Roughly speaking, a real finite-dimensional Lie algebra whose index equals 0 is said to be Frobenius ([4], [9], [11]). Many studies of Frobenius Lie algebras have been done over the years. For instance, we found the construction and classifications of 4-dimensional Frobenius Lie algebras over a field with characteristics $\neq 2$ and 6-dimensional Frobenius Lie algebras over an algebraically closed field [3] and an abelian nilradical in a solvable Frobenius Lie algebra [2]. We noticed some facts about Frobenius Lie algebras. Firstly, we found that a Lie group of a Frobenius Lie algebra is always non-unimodular. Secondly, we also found that a Frobenius Lie algebra has even dimension. In the second fact, we can understand by definition of Frobenius Lie algebras that there exists a linear functional implying a stabilizer of a Frobenius Lie algebra is trivial. Namely, let $g$ be a Frobenius Lie algebra and let $g^*$ be its dual vector space of $g$. In other words, $g^*$ consists of real valued linear functionals on $g$. Let $\mu_0 \in g^*$ be a linear functional, then we define an alternating bilinear form

$$B_{\mu_0}: g \times g \to \mathbb{R}$$

given in the following equation

$$B_{\mu_0}(x, y) = \mu_0([x, y]) = \langle \mu_0, [x, y] \rangle.$$ \hspace{1cm} (2)

We observe that a symplectic form on the Frobenius Lie algebra $g$ in the equation (2) induce a left-symmetric algebra on $g$. To see this, let us define a bilinear product in the Lie algebra $g$ by $\alpha \beta := \alpha * \beta$ for all $\alpha, \beta \in g$. Then we define the induced left-symmetric algebra from the symplectic form in the equation (2) by $B_{\mu_0}(\alpha \beta, y) := -B_{\mu_0}(\beta, [\alpha, y]) = -\langle \mu_0, [\beta, [\alpha, y]] \rangle$ for all $\alpha, \beta, y \in g$ [5]. Indeed, the alternating bilinear form $B_{\mu_0}$ defined in the equation (2) is non-degenerate. Further observation, the
non-degenerate condition happened because of the existence of the linear functional \( \mu_0 \in \mathfrak{g}^* \). We shall called this \( \mu_0 \) as the Frobenius functional. Therefore, the dimension of a Frobenius Lie algebra is always even.

Our explanations above show the importance of a Frobenius functional corresponding to a Frobenius Lie algebra. In the other hand, we have the notation of semi-direct sum of the vector space \( \mathbb{M}_{n,p}(\mathbb{R}) \) consisting of \( n \times p \) real matrices and the Lie algebra \( \mathfrak{gl}_n(\mathbb{R}) \) of \( n \times n \) matrices which is denoted by \( \mathfrak{g}_{n,p} := \mathbb{M}_{n,p}(\mathbb{R}) \oplus \mathfrak{gl}_n(\mathbb{R}) \). The notion of the Lie algebra \( \mathfrak{g}_{n,p} \) was introduced by Rais [12]. In this paper, we offer a construction how to find a Frobenius functional that is a linear functional in \( \mathfrak{g}_{n,p}^* \) such that the bilinear form in the equation (2) is non-degenerate. In particular, we shall consider some Frobenius functionals for special case \( n = p = 3 \) of the Lie algebra \( \mathfrak{g}_{n,p} := \mathbb{M}_{n,p}(\mathbb{R}) \oplus \mathfrak{gl}_n(\mathbb{R}) \). Namely, we shall choose Frobenius functionals for the Lie algebra \( \mathfrak{g}_3 := \mathbb{M}_3(\mathbb{R}) \oplus \mathfrak{gl}_3(\mathbb{R}) \) in order \( \mathfrak{g}_3 \) is the Frobenius Lie algebra.

In addition, even though Rais proved that \( \mathfrak{g}_{n,p} \) is the Frobenius Lie algebra for \( p \) devides \( n \) [12], but in this work we give another alternative proof to show \( \mathfrak{g}_3 \) is the Frobenius Lie algebra. In other words, we give complete computations to find Frobenius functionals in \( \mathfrak{g}_3^* \) in order the Lie algebra \( \mathfrak{g}_3 \) is Frobenius. We believe for the future research that our result can be generalized for general case \( \mathfrak{g}_{n,p} \).

Moreover, we are going to prove our main proposition as follows:

**Proposition 1.** Let \( \mathfrak{g}_3 := \mathbb{M}_3(\mathbb{R}) \oplus \mathfrak{gl}_3(\mathbb{R}) \) be the vector space of dimension 18. Then there exists Lie brackets on \( \mathfrak{g}_3 \) defined as the matrix commutator such that \( \mathfrak{g}_3 \) is the Lie algebra. Furthermore, there exists a Frobenius functional \( \mu \in \mathfrak{g}_3^* \) such that \( \mathfrak{g}_3 \) is the Frobenius Lie algebra.

Let us review briefly some basic notions that shall be useful in our paper. We are going to introduce the notions of Lie algebras, Frobenius Lie algebras, and Frobenius functionals. For more details about these notions, readers can read more particularly in [7] and [8].

**Definition 1** [8]. Let \( \mathfrak{g} \) be a real vector space. A bilinear map \( [, , ] : \mathfrak{g} \times \mathfrak{g} \ni (a, b) \mapsto [a, b] \in \mathfrak{g} \) is called a Lie bracket if the following conditions satisfied:

1. \( [x, x] = 0 \); for \( x \in \mathfrak{g} \).
2. \( [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \); for \( x, y, z \in \mathfrak{g} \).

The latter equation is called the Jacobi identity. The real vector space \( \mathfrak{g} \) equipped by Lie brackets is called a Lie algebra.

It is well known that every associative algebra \( \mathfrak{A} \) whose Lie brackets are the matrix commutator is a Lie algebra. Namely, Lie brackets for an associate algebra \( \mathfrak{A} \) are given by

\[
[x, y] := xy - yx,
\]

(3)

For all \( x, y \in \mathfrak{A} \). Furthermore, it is well known that vector spaces of matrices \( \mathfrak{gl}(n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{R}), \mathfrak{o}(n, \mathbb{R}), \mathfrak{se}(n, \mathbb{R}) \) are Lie algebras with respect to the commutator matrix in the equation (3).

Let \( \mathfrak{g} \) be a Lie algebra and \( \mathfrak{g}^* \) its dual vector space of \( \mathfrak{g} \). Let \( f \) be a real valued linear functional on \( \mathfrak{g} \). A representation of a Lie algebra \( \mathfrak{g} \) in the space \( \mathfrak{g}^* \) is denoted by \( \text{ad}^* \) whose value on \( \mathfrak{g} \) is determined by following formula:

\[
\langle \text{ad}^*(x)f, y \rangle = \langle f, \text{ad}(-x)y \rangle = \langle f, [y, x] \rangle.
\]

(4)

for \( f \in \mathfrak{g}^*, x, y \in \mathfrak{g} \).
A stabilizer of \( g \) at the point \( \mu \in g^* \) is given in the following form:

\[
g^\mu := \{ x \in g \ ; \ \text{ad}^*(x)\mu = 0 \}.
\] (5)

The stabilizer \( g^\mu \) in the equation (5) is nothing but a kernel of alternating bilinear form defined in the equation (2).

**Definition 2** [3]. A real Lie algebra is said to be Frobenius if there exist a linear functional \( \mu \in g^* \) such that \( g^\mu = \{0\} \).

Let \( g \) be a Lie algebra of dimension 4 whose basis is \( S = \{e_1, e_2, e_3, e_4\} \). The non-zero brackets for \( g \) are given by

\[
[e_1, e_2] = \frac{1}{2} e_2 + e_3, \quad [e_1, e_3] = \frac{1}{2} e_3, \quad [e_1, e_4] = e_4, \quad [e_2, e_3] = e_4. \] (6)

We observe that at the point \( \mu = x_4^* \), the stabilizer \( g^{x_4^*} \) is trivial. Therefore, the Lie algebra \( g \) is Frobenius of dimension 4 [11]. In the other hand, the Heisenberg Lie algebra of dimension \((2n + 1)\) is not a Frobenius Lie algebra.

2. Methods

The method of this research was based on literature reviews, particularly we study the notion of Frobenius Lie algebra in Ooms and Rais work ([9], [12]). Let \( \{x_i\}_{i=1}^n \) be a basis for \( g \) and \( \mathcal{R} \) be an \( n \times n \) matrix whose \((i,j)\)th entry is determined by a bracket \([x_i, x_j]\). The determinant of \( \mathcal{R} \) is considered as an element of a symmetric algebra \( S(g) \) which is identified with the polynomial algebra \( \text{Pol}(g^*) \) on \( g^* \). Let \( \mathcal{R}(f) \) be a matrix defined by \( \langle \mu, [x_i, x_j] \rangle_{1 \leq i, j \leq n} \) where \( \mu \in g^* \). The determinant \( \mathcal{R}(f) \) is equal to the determinant of a representation matrix of the alternating bilinear form \( B_\mu \) in the equation (2). In the case that a Lie algebra \( g \) is Frobenius then \( \det \mathcal{R} \neq 0 \) if and only if \( \det \mathcal{R}(f) \neq 0 \). Indeed, the latter statement is equivalent to non-degeneracy of the alternating bilinear form \( B_\mu \). We applied this construction to prove that \( g_3 := M_3(\mathbb{R}) \oplus gl_3(\mathbb{R}) \) is a Frobenius Lie algebra.

3. Result and Discussion

In this session we discuss that the vector space \( g_3 \) as a semi-direct sum of the space \( M_3(\mathbb{R}) \) and the Lie algebra \( gl_3(\mathbb{R}) \) is the Lie algebra. Indeed, the Lie algebra \( gl_3(\mathbb{R}) \) acts on the space \( M_3(\mathbb{R}) \). Furthermore, we show that there exists a linear functional \( \mu_0 \) contained in \( g_3^* \), which is called the Frobenius functional, such that the Lie algebra \( g_3 \) is the Frobenius Lie algebra. We mention here that the existence of a Frobenius functional is not unique. In this result, we also give another choice of a Frobenius functional. Formally, our result is written in Proposition 1 above.

Now we are going to prove Proposition 1 as already mentioned in introduction as follows.

**Proof of Proposition 1.** Let \( g_3 := M_3(\mathbb{R}) \oplus gl_3(\mathbb{R}) \) be the vector space with basis \( S = \{x_i\}_{i=1}^{18} \). We realize the Lie algebra \( g_3 \) as a sub-Lie algebra of \( gl_6(\mathbb{R}) \). We can write the elements of \( g_3 := M_3(\mathbb{R}) \oplus gl_3(\mathbb{R}) \) in the following matrix form of dimension \( 6 \times 6 \).
Moreover, the equation

\[
\begin{pmatrix}
\alpha & \beta & \gamma \\
\delta & \epsilon & \zeta \\
\theta & \iota & \kappa
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{pmatrix}
= \begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{pmatrix}
\] \quad (7)

We equip Lie brackets in \( g_3 \) by matrix commutator in the equation (3). To see this commutator is Lie brackets, we just prove that it satisfies identity Jacobi. Let \( x, y, z \in g_3 \) and we observe that

\[ [x, yz] = xyz - yza = (xy - yx)z + y(xz - zx) = [x, y]z + y[x, z]. \] \quad (8)

Moreover, the equation (8) implies that

\[ [x, [y, z]] = [x, y]z + y[x, z] - [x, z]y - z[x, y] = [[x, y], z] + [y, [x, z]] \] \quad (9)

as desired. Therefore, \( g_3 \) is the Lie algebra. The complete computations of the Lie brackets for \( g_3 \) with respect to the basis \( S \) is the following form.

\[
\begin{align*}
[x_1, x_2] &= x_2 \\
[x_1, x_3] &= x_3 \\
[x_1, x_4] &= -x_4 \\
[x_1, x_7] &= -x_7 \\
[x_1, x_{10}] &= x_{10} \\
[x_1, x_{11}] &= x_{11} \\
[x_1, x_{12}] &= x_{12} \\
[x_2, x_1] &= -x_2 \\
[x_2, x_4] &= x_1 - x_5 \\
[x_2, x_5] &= x_2 \\
[x_2, x_6] &= x_3 \\
[x_2, x_7] &= -x_8 \\
[x_2, x_{13}] &= x_{10} \\
[x_2, x_{14}] &= x_{11} \\
[x_2, x_{15}] &= x_{12} \\
[x_3, x_1] &= -x_3 \\
[x_3, x_4] &= -x_6 \\
[x_3, x_7] &= x_1 - x_9 \\
[x_3, x_8] &= x_2 \\
[x_3, x_9] &= x_3 \\
[x_3, x_{16}] &= x_{10} \\
[x_3, x_{17}] &= x_{11} \\
[x_3, x_{18}] &= x_{12} \\
[x_4, x_5] &= -x_4 \\
[x_4, x_8] &= -x_7 \\
[x_4, x_{10}] &= x_{13} \\
[x_4, x_{11}] &= x_{14} \\
[x_4, x_{12}] &= x_{15} \\
[x_5, x_2] &= -x_2 \\
[x_5, x_4] &= x_4 \\
[x_5, x_6] &= x_6 \\
[x_5, x_7] &= x_7 \\
\end{align*}
\]
Furthermore, we shall prove that the Lie algebra \( g_3 \) is Frobenius. Firstly, we consider the \( 18 \times 18 \) matrix \( \mathcal{R} \) whose \((i,j)\)th entry is determined by the Lie bracket \([x_i,x_j]\) where \(1 \leq i,j \leq 18\). With respect to Lie brackets in the equation (10), then the determinant of \( \mathcal{R} \) can be written as following form

\[
\det \mathcal{R} = (x_{10}x_{14}x_{18} - x_{10}x_{15}x_{17} - x_{11}x_{13}x_{18} + x_{11}x_{15}x_{16} + x_{12}x_{13}x_{17} - x_{12}x_{14}x_{16})^6. \tag{11}
\]

Since \( S \) is basis for \( g_3 \), then \(\det \mathcal{R} \neq 0\). Furthermore, we claim there exists a Frobenius functional \( \mu_0 \in g^*_3 \) corresponds to each terms in the equation (11). In such a case, we choose \( \mu_0 := (x_{10}^* + x_{14}^* + x_{18}^*) \in g^* \). The value of \( \mu_0 \) at a point \( x_i \) where \(1 \leq i \leq 18\) is defined by

\[
\mu_0(x_i) = \begin{cases} 1, & \text{if } i = 10,14,18, \\ 0, & \text{elsewhere}. \end{cases} \tag{12}
\]

Let \( \mathcal{R}(\mu_0) = (\langle \mu_0, [x_i, x_j] \rangle)_{1 \leq i,j \leq 18} \) be the matrix whose \((i,j)\)th entry of \( \mathcal{R}(\mu_0) \) is given by \(\langle \mu_0, [x_i, x_j] \rangle\) \(1 \leq i,j \leq 18\). Combining the equation (12), then we get the determinant of \( \mathcal{R}(\mu_0) \) is equal to 1. Thus, there exists a Frobenius functional \( \mu_0 := (x_{10}^* + x_{14}^* + x_{18}^*) \in g^* \) such that \( g_3 \) is a Frobenius Lie algebra.

The choice a Frobenius functional is not unique. In other words, there is another Frobenius functional implies that \( g_3 \) is Frobenius Lie algebra. In similar way, we can chose another a linear functional \( \psi_0 \). Let us claim that \( \psi_0 = (x_{10}^* + x_{15}^* + x_{17}^*) \in g^* \) is a Frobenius functional. Let us defined that the value of \( \psi_0 \) at a point \( x_i \) in basis \( S \) where \(1 \leq i \leq 18\) is given by

\[
\psi_0(x_i) = \begin{cases} 1, & \text{if } i = 10,15,17, \\ 0, & \text{elsewhere}. \end{cases} \tag{13}
\]

We compute that the determinant of the matrix \( \mathcal{R}(\psi_0) = (\langle \psi_0, [x_i, x_j] \rangle)_{1 \leq i,j \leq 18} \) is equal to 1. This means that there exists a Frobenius functional \( \psi_0 \) such that we have already shown again \( g_3 \) is a Frobenius Lie algebra.

The study of Frobenius Lie algebras encourages us to study types of Lie algebras, for instance \( j \)-algebras and Lie algebra of affine Lie group. Corresponding to Lie group of Frobenius Lie algebras, in the case of dimension 4, we have that irreducible unitary representations of these Lie groups are square-integrable. We need further to investigate for Lie groups of Frobenius Lie algebras of dimension \( \geq 6 \). These will be interested problems for future research. Indeed, our result can be generalized for \( g_{n,p} := M_{n,p}(\mathbb{R}) \oplus g_{n}(\mathbb{R}) \) of dimension \( n(n + p) \). The reader can reformulate this result to show the existence of some Frobenius functionals to show that the Lie algebra \( g_{n,p} \) is Frobenius where \( p \) devides \( n \). Recent results of Frobenius Lie algebras can be seen, for instance in \([2],[4],[5],[6],[10]\). Eventhought Frobenius Lie algebras are not nilpotent but it can be constructed from a nilpotent Lie algebra. To do this we can find a 1-dimensional split torus for any given nilpotent Lie algebra \([1]\).
4. Conclusion
We conclude that the vector space $\mathfrak{g}_3 := M_3(\mathbb{R}) \oplus \mathfrak{gl}_3(\mathbb{R})$ with basis $S = \{x_i\}_{i=1}^{18}$ is the Lie algebra with concrete Lie brackets. Furthermore, we proved that there exist some linear functional $f$ on $\mathfrak{g}_3^*$ which is called a Frobenius functional such that $\mathfrak{g}_3$ is the Frobenius Lie algebra. Even though Rais proved that $\mathfrak{g}_3$ is Frobenius Lie algebra but our result is different from Rais’ work. We give detail computations of Frobenius functionals of $\mathfrak{g}_3^*$ is order $\mathfrak{g}_3$ is Frobenius. In our contructions, we proved that the determinant of the matrix $\mathcal{R}(\mu_0) = (\langle \mu_0, [x_i, x_j]\rangle)_{1 \leq i, j \leq 18}$ is equal to one and this implies that $\mathfrak{g}_3$ is Frobenius. For the future research, the existence of Frobenius functionals is still an open problem to study for higher dimensional Lie algebras.

Acknowledgments
The first author would like to take this opportunity sincerely thank my supervisor Edi Kurniadi, Ph.D and Dr. Ema carnia, M.Si for warm support and thoughtful guidance. We sincerely thank to reviewers for worth advice. We also thank Universitas Padjadjaran who has funded the work through Riset Percepatan Lektor Kepala (RPLK) year 2020 with contract number 1427/UN6.3.1/LT/2020.

References
[1] Ayala V, Kizil E and Tribuzy I D A 2012 On an algorithm for finding derivations of Lie algebras Proyecciones J Math. 31(1) 81–90
[2] Alvarez M A et al 2018 Contact and Frobenius solvable Lie algebras with abelian nilradical Comm Algebra 46 4344–4354
[3] Csikós B and Verhóczki L 2007 Classification of Frobenius Lie algebras of dimension ≤ 6 Publ Math 70(3–4) 427–451
[4] Diatta A and Manga B 2014 On properties of principal elements of Frobenius lie algebras J Lie Theory 24(3) 849–864
[5] Diatta A, Manga B and Mbaye A 2020 On systems of commuting matrices, Frobenius Lie algebras and Gerstenhaber’s theorem arXiv:2002.08737
[6] Gerstenhaber M and Giaquinto A 2009 The principal element of a frobenius Lie algebra Lett. Math Phys 88(1–3) 333–341
[7] Hilgert J and Neeb K -H 2012 Structure and Geometry of Lie Groups (New York: Springer Monographs in Mathematics Springer )
[8] Humphreys J 1972 Introduction to Lie ALgebra and its representation Third Print (New York Heidelberg Berlin: Springer-Verlag)
[9] Ooms A I 1980 On Frobenius Lie algebras Comm Algebra 8 13–52
[10] Ooms A I 2009 Computing invariants and semi-invariants by means of Frobenius Lie algebras J Algebra. 321 1293–1312
[11] Pham D N 2016 G-Quasi-Frobenius Lie Algebras Arch Math 52(4) 233–262
[12] Rais M 1978 La representation du groupe affine Ann Inst Fourier Grenoble 26 207–237