An Asymptotically Optimal Policy for Finite Support Models in the Multiarmed Bandit Problem

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Abstract

Multiarmed bandit problem is an example of a dilemma between exploration and exploitation in reinforcement learning. This problem is expressed as a model of a gambler playing a slot machine with multiple arms. A policy chooses an arm so as to minimize the number of times that arms with inferior expectations are pulled. We propose minimum empirical divergence (MED) policy and prove asymptotic optimality of the policy for the case of finite support models. In a setting similar to ours, Burnetas and Katehakis have already proposed an asymptotically optimal policy. However we do not assume knowledge of the specific support except for the upper and lower bounds of the support. Furthermore, the criterion for choosing an arm, minimum empirical divergence, can be computed easily by a convex optimization technique. We confirm by simulations that MED policy demonstrates good performance in finite time in comparison to other currently popular policies.

1 Introduction

The multiarmed bandit problem is a problem based on an analogy with playing a slot machine with more than one arm or lever. Each arm has a reward distribution and the objective of a gambler is to maximize the collected sum of rewards by choosing an arm to pull for each round. There is a dilemma between exploration and exploitation, namely the gambler can not tell whether an arm is optimal unless he pulls it many times, but it is also a loss to pull an inferior (i.e. non-optimal) arm many times.

We consider an infinite-horizon $K$-armed bandit problem. There are $K$ arms $\Pi_1, \ldots, \Pi_K$ and arms are pulled infinite number of times. $\Pi_j$ has a probability distribution $F_j$ with the expected value $\mu_j$ and the player receives a reward according to $F_j$ independently.
in each round. If the expected values are known, it is optimal to always pull the arm with the maximum expected value \( \mu^* = \max_j \mu_j \). A policy is an algorithm to choose the next arm to pull based on the results of past rounds.

This problem is first considered by Robbins [16]. Since then, many studies have been conducted for the problem [2, 8, 15, 18, 19, 21]. There are also many extensions for the problem. For example, Auer et al. [4] removed the assumption that rewards are stochastic, and for the stochastic setting, the case of non-stationary distributions [10, 11, 12], or the case of infinite (possibly uncountable) arms [11, 13] have been considered.

In our setting, Lai and Robbins [14] established a theoretical framework for determining optimal policies, and Burnetas and Katehakis [6] extended their result to multi-parameter or non-parametric models. Consider a model \( \mathcal{F} \), a generic family of distributions. The player knows \( \mathcal{F} \) and that \( F_j \) is an element of \( \mathcal{F} \). Let \( T_j(n) \) denote the number of times that \( \Pi_j \) has been pulled over the first \( n \) rounds. A policy is consistent on model \( \mathcal{F} \) if 

\[
E[T_i(n)] = o(n^a) \quad \text{for all inferior arms } \Pi_i \text{ and all } a > 0.
\]

Burnetas and Katehakis [6] proved the following lower bound for any inferior arm \( \Pi_i \) under consistent policy:

\[
T_i(n) \geq \left( \frac{1}{\inf_{G \in \mathcal{F} : E(G) > \mu^*} D(F_i \| G)} + o(1) \right) \log n \tag{1}
\]

with probability tending to one, where \( E(G) \) is the expected value of distribution \( G \) and \( D(\cdot \| \cdot) \) denotes the Kullback-Leibler divergence. Under mild regularity conditions on \( \mathcal{F} \),

\[
\inf_{G \in \mathcal{F} : E(G) > \mu^*} D(F_i \| G) = \min_{G \in \mathcal{F} : E(G) \geq \mu^*} D(F_i \| G)
\]

and we write

\[
D_{\min}(F, \mu) = \min_{G \in \mathcal{F} : E(G) \geq \mu^*} D(F \| G)
\]

in the following.

A policy is asymptotically optimal if the expected value of \( T_j(n) \) achieves the right-hand side of (1) as \( n \to \infty \). In [14] and [6], policies achieving the above bound are also proposed. These policies are based on the notion of upper confidence bound. It can be interpreted as the upper confidence limit for the expectation of each arm with the significance level \( 1/n \).

Although policies based on upper confidence bound are optimal, upper confidence bounds are often hard to compute in practice. Then, Auer et al. [3] proposed some policies called UCB. UCB policies estimate the expectation of each arm in a similar way to upper confidence bound. They are practical policies for their simple form and fine performance. Especially, “UCB-tuned” is widely used because of its excellent simulation results. However, UCB-tuned has not been analyzed theoretically and it is unknown whether the policy has consistency. Theoretical analyses of other UCB policies have been given, but their coefficients of the logarithmic term do not necessarily achieve the bound [11].
In this paper we propose minimum empirical divergence (MED) policy. We prove the asymptotic optimality of MED when the model \( \mathcal{F} \) is the family of distributions with a finite bounded support, denoted by \( \mathcal{A} \). This model consists of all distributions with finite supports over a given interval, e.g. \([-1, 0]\). It is larger than the model used in [6], which assumes a specific finite support. We also demonstrate simulation results of MED policy comparable to UCB policies.

Our MED policy is motivated by the observation of (1). When a policy achieving (1) is used, an inferior arm \( \Pi_i \) waits roughly \( \exp(n_i D_{\min}(F_i, \mu^*)) \) rounds to be pulled after the \( n_i \)-th play of \( \Pi_i \). Then, it can be expected that a policy pulling \( \Pi_i \) with probability \( \exp(-n_i D_{\min}(F_i, \mu^*)) \) will achieve (1). MED policy is obtained by plugging \( \hat{F}_i, \hat{\mu}^* \) into \( F_i, \mu^* \) in \( D_{\min} \), where \( \hat{F}_i \) is the empirical distribution of rewards from \( \Pi_i \) and \( \hat{\mu}^* \) is the current best sample mean.

MED policy requires a computation of \( D_{\min}(\hat{F}_i, \hat{\mu}^*) = \min_{G \in \mathcal{A} : E(G) \geq \hat{\mu}^*} D(\hat{F}_i || G) \) at each round whereas upper confidence bound requires the computation of

\[
\max_{G \in \mathcal{A} : D_{\min}(\hat{F}_i || G) \leq \frac{\log n}{n_i}} E(G).
\]

Equations (1) and (2) are quantity dual to each other but the former has two advantages in practical implementation. First, \( D_{\min}(\hat{F}_i, \hat{\mu}^*) \) is smooth in \( \hat{\mu}^* \) which converges to \( \mu^* \). Therefore the value in the previous round can be used as a good approximation of \( D_{\min} \) for the current round. On the other hand (2) continues to increase according to \( n \) and it has to be computed many times. Second, as shown in Theorem 5 below, \( D_{\min} \) can be expressed as a univariate convex optimization problem for our model \( \mathcal{A} \). Although (2) is also a convex optimization problem, the nonlinear constraint \( D(\hat{F}_i || G) \leq \frac{\log n}{n_i} \) is harder to handle.

MED policy is categorized as a probability matching method (see, e.g. [19] for classification of policies). In this method each arm is pulled according to the probability reflecting how likely the arm is to be optimal. For example, Wyatt [20] proposed probability matching policies for Boolean and Gaussian models by Bayesian approach with prior/posterior distributions. In our approach the probability assigned to each arm is determined by (normalized) maximum likelihood instead of posterior probability.

This paper is organized as follows. In Section 2 we give definitions used throughout this paper and show the asymptotic bound by [6], which is satisfied by any consistent policy. In Section 3 we propose MED policy and prove that it is asymptotically optimal for finite support models. We also discuss practical implementation issues of minimization problem involved in MED. In Section 4 some simulation results are shown. We conclude the paper with some remarks in Section 5.

## 2 Preliminaries

In this section we introduce notation of this paper and present the asymptotic bound for a generic model, which is established by [6].
Let $\mathcal{F}$ be a generic family of probability distributions on $\mathbb{R}$ and let $F_j \in \mathcal{F}$ be the distribution of $\Pi_j$, $j = 1, \ldots, K$. $P_F[\cdot]$ and $E_F[\cdot]$ denotes the probability and the expectation under $F \in \mathcal{F}$, respectively. When we write e.g. $P_F[X \in A]$ ($A \subset \mathbb{R}$) or $E_F[\theta(X)]$ ($\theta(\cdot)$ is a function $\mathbb{R} \rightarrow \mathbb{R}$), $X$ denotes a random variable with distribution $F$. We define $F(A) \equiv P_F[X \in A]$ and $E(F) \equiv E_F[X]$.

A set of probability distributions for $K$ arms is denoted by $\mathbf{F} \equiv (F_1, \ldots, F_K) \in \mathcal{F}^K \equiv \prod_{j=1}^K \mathcal{F}$. The joint probability and the expected value under $\mathbf{F}$ are denoted by $P_{\mathbf{F}}[\cdot]$, $E_{\mathbf{F}}[\cdot]$, respectively.

The expected value of $\Pi_j$ is denoted by $\mu_j \equiv E(F_j)$. We denote the optimal expected value by $\mu^* \equiv \max_j \mu_j$. Let $J_n$ be the arm chosen in the $n$-th round. Then

$$T_j(n) = \sum_{m=1}^n 1[J_m = j],$$

where $1[\cdot]$ denotes the indicator function. For notational convenience we write $T_j'(n) \equiv T_j(n - 1)$, which is the number of times the arm $\Pi_j$ has been pulled prior to the $n$-th round.

Let $\hat{F}_{j,t}$ and $\hat{\mu}_{j,t} \equiv E(\hat{F}_{j,t})$ be the empirical distribution and the mean of the first $t$ rewards from $\Pi_j$, respectively. Similarly, let $\hat{F}_j(n) \equiv \hat{F}_{j,T_j'(n)}$ and $\hat{\mu}_j(n) \equiv \hat{\mu}_{j,T_j'(n)}$ be the empirical distribution and mean of $\Pi_j$ after the first $n - 1$ rounds, respectively.

$\hat{\mu}^*(n) \equiv \max_j \hat{\mu}_j(n)$ denotes the highest empirical mean after $n - 1$ rounds. We call $\Pi_j$ a current best if $\hat{\mu}_j(n) = \hat{\mu}^*(n)$.

Let $\Omega$ denote the whole sample space. For an event $A \subset \Omega$, the complement of $A$ is denoted by $A^c$. The joint probability of two events $A$ and $B$ under $\mathbf{F}$ is written as $P_{\mathbf{F}}[A \cap B]$. For notational simplicity we often write, e.g., $P_{\mathbf{F}}[J_n = j \cap T'_j(n) = t]$ instead of the more precise $P_{\mathbf{F}}[\{J_n = j\} \cap \{T'_j(n) = t\}]$.

Finally we define an index for $F \in \mathcal{F}$ and $\mu \in \mathbb{R}$

$$D_{\inf}(F, \mu, \mathcal{F}) \equiv \inf_{G \in \mathcal{F}, E(G) > \mu} D(F || G)$$

where Kullback-Leibler divergence $D(F || G)$ is given by

$$D(F || G) \equiv \begin{cases} E_F[ \log \frac{dF}{dG} ] & \text{if } \frac{dF}{dG} \text{ exists}, \\ +\infty & \text{otherwise.} \end{cases}$$

$D_{\inf}$ represents how distinguishable $F$ is from distributions having expectations larger than $\mu$. If $\{ G \in \mathcal{F} : E(G) > \mu \}$ is empty, we define $D_{\inf}(F, \mu, \mathcal{F}) = +\infty$. We adopt Lévy distance $L(F, G)$ for distance between two distributions $F, G$. We use only the fact that the convergence of the Lévy distance $L(F, F_n) \rightarrow 0$ is equivalent to the weak convergence of $\{F_n\}$ to distribution $F$ and we write $F_n \rightarrow F$ in this sense.

Lai and Robbins [14] gave a lower bound for $E[T_i(n)]$ for any inferior $\Pi_i$ when a consistent policy is adopted. However their result was hard to apply for multiparameter models and more general non-parametric models. Later Burnetas and Katehakis [6] extended the bound to general non-parametric models. Their bound is given as follows.
Theorem 1. [6, Proposition 1] Fix a consistent policy and \( F \in \mathcal{F}^K \). If \( \mathbb{E}(F_i) < \mu^* \) and \( 0 < D_{\text{inf}}(F_i, \mu^*, \mathcal{F}) < \infty \), then for any \( \epsilon > 0 \)
\[
\lim_{N \to \infty} P_F \left[ T_i(N) \geq \frac{(1 - \epsilon) \log N}{D_{\text{inf}}(F_i, \mu^*, \mathcal{F})} \right] = 1.
\]

Consequently
\[
\liminf_{N \to \infty} \frac{\mathbb{E}_F[T_i(N)]}{\log N} \geq \frac{1}{D_{\text{inf}}(F_i, \mu^*, \mathcal{F})}. \tag{3}
\]

3 Asymptotically Optimal Policy for Finite Support Models

Let \( \mathcal{A} \equiv \{ F : |\text{supp}(F)| < \infty, \text{supp}(F) \subset [a, b] \} \) be the family of distributions with a finite bounded support, where supp\((F)\) is the support of distribution \( F \) and \( a, b \) are constants known to the player. We assume \( a = -1, b = 0 \) without loss of generality. We write \( \text{supp}'(F) \equiv \{0\} \cup \text{supp}(F) \) and \( \mathcal{A}_X \equiv \{ G \in \mathcal{A} : \text{supp}(G) \subset X \} \) where \( X \) is an arbitrary subset of \([-1, 0]\).

We consider \( \mathcal{A} \) as a model \( \mathcal{F} \) and propose a policy which we call the minimum empirical divergence (MED) policy in this section. We prove in Theorem 3 that the proposed policy achieves the bound given in the previous section. Then, we describe a univariate convex optimization technique to compute \( D_{\text{min}} \) used in the policy.

Note that the finiteness of the support can not be determined from finite samples and every policy for \( \mathcal{A} \) is applicable also for \( \{ F : \text{supp}(F) \subset [a, b] \} \). However our proof of the optimality in this paper is for the above \( \mathcal{A} \). The advantage of assuming the finiteness is that we can employ the method of types in the large deviation technique. This enables us to consider all empirical distributions obtained from each arm.

In this model it is convenient to use
\[
D_{\text{min}}(F, \mu, \mathcal{A}) \equiv \min_{G \in \mathcal{A} : \mathbb{E}(G) \geq \mu} D(F \| G)
\]
instead of \( D_{\text{inf}}(F, \mu, \mathcal{A}) \equiv \inf_{G \in \mathcal{A} : \mathbb{E}(G) > \mu} D(F \| G) \). Properties of the minimizer \( G^* \) of the right-hand side will be discussed in Section 3.2.

Lemma 2. \( D_{\text{min}}(F, \mu, \mathcal{A}) = D_{\text{inf}}(F, \mu, \mathcal{A}) \) holds for all \( F \in \mathcal{A} \) and \( \mu < 0 \).

Proof. We will prove in Lemma 3 that \( D_{\text{min}}(F, \mu, \mathcal{A}) \) is continuous in \( \mu < 0 \). \( D_{\text{min}}(F, \mu, \mathcal{A}) = D_{\text{inf}}(F, \mu, \mathcal{A}) \) follows easily from the continuity. \( \square \)

3.1 Optimality of the Minimum Empirical Divergence Policy

We now introduce our MED policy. In MED an arm is chosen randomly in the following way:
[Minimum Empirical Divergence Policy]

Initialization. Pull each arm once.

Loop. For the $n$-th round,

1. For each $j$ compute $\hat{D}_j(n) \equiv D_{\min}(\hat{F}_j(n), \bar{\mu}^*(n), \mathcal{A})$.
2. Choose arm $\Pi_j$ according to the probability

$$p_j(n) \equiv \frac{\exp(-T'_j(n)\hat{D}_j(n))}{\sum_{i=1}^K \exp(-T'_i(n)\hat{D}_i(n))}.$$  

Note that

$$\frac{1}{K} \leq p_j(n) \leq 1$$  

for any currently best $\Pi_j$ since $\hat{D}_j(n) = 0$. As a result, it holds for all $j$ that

$$\frac{1}{K} \exp(-T'_j(n)\hat{D}_j(n)) \leq p_j(n) \leq \exp(-T'_j(n)\hat{D}_j(n)).$$

Intuitively, $p_j(n)$ for a currently not best arm $\Pi_j$ corresponds to the maximum likelihood that $\Pi_j$ is actually the best arm. Therefore in MED an arm $\Pi_j$ is pulled with the probability proportional to this likelihood.

Now we present the main theorem of this paper.

**Theorem 3.** Fix $\mathbf{F} \in \mathcal{A}^K$ satisfying $\mu_j = \mu^*$ and $\mu_i < \mu^*$ for all $i \neq j$. Under MED policy, for any $i \neq j$ and $\epsilon > 0$ it holds that

$$\mathbb{E}_\mathbf{F}[T_i(N)] \leq \frac{1 + \epsilon}{D_{\min}(F_i, \mu^*, \mathcal{A})} \log N + O(1).$$

Note that we obtain

$$\limsup_{N \to \infty} \frac{\mathbb{E}_\mathbf{F}[T_i(N)]}{\log N} \leq \frac{1}{D_{\min}(F_i, \mu^*, \mathcal{A})},$$

by dividing both sides by $\log N$, letting $N \to \infty$ and finally letting $\epsilon \downarrow 0$. In view of (3) we see that MED policy is asymptotically optimal. We give a proof of Theorem 3 in Section 3.3.

The following corollary shows that the optimality of MED policy given in Theorem 3 is a generalization of the optimality in [6].
Corollary 1. Let $\mathcal{X} \subset [-1, 0]$ be an arbitrary subset of $[-1, 0]$ such that $0 \in \mathcal{X}$. Fix $F' \in \mathcal{A}_F^{\mathcal{X}}$ satisfying $\mu_j = \mu^*$ and $\mu_i < \mu^*$ for all $i \neq j$. Under MED policy, for any $i \neq j$ and $\epsilon > 0$ it holds that

$$E_F[T_i(N)] \leq \frac{1 + \epsilon}{D_{\min}(F_i, \mu, \mathcal{X})} \log N + O(1). \tag{6}$$

Proof. We prove in Lemma 4 that $D_{\min}(F, \mu, \mathcal{A}) = D_{\min}(F, \mu, \mathcal{A}_{\supp'(F)})$. On the other hand, $D_{\min}(F, \mu, \mathcal{A}_{\supp'(F)}) \geq D_{\min}(F, \mu, \mathcal{A}_F)$ holds from $\mathcal{A}_{\supp'(F)} \subset \mathcal{A}_F$. Then we obtain (6) from Theorem 3. \qed

Note that (6) is achieved also by the policy used in the [3] if $\mathcal{X}$ is fixed and assumed to be known. Our result establishes the same bound without this assumption.

3.2 Computation of $D_{\min}$ and Properties of the Minimizer

For implementing MED policy it is essential to efficiently compute the minimum empirical divergence $D_{\min}(\hat{F}_j(n), \mu^*(n), \mathcal{A})$ for each round. In this subsection, we clarify the nature of the convex optimization involved in $D_{\min}(\hat{F}_j(n), \mu^*(n), \mathcal{A})$ and show how the minimization can be computed efficiently. In addition, for proofs of Lemma 2 and Theorem 3 we need to clarify the behavior of $D_{\min}(F, \mu, \mathcal{A})$ as a function of $\mu$.

First we prove that it is sufficient to consider $\mathcal{A}_{\supp'(F)}$ for the computation of $D_{\min}(F, \mu, \mathcal{A})$:

Lemma 4. $D_{\min}(F, \mu, \mathcal{A}) = D_{\min}(F, \mu, \mathcal{A}_{\supp'(F)})$ holds for any $F \in \mathcal{A}$.

Proof. Take an arbitrary $G \in \mathcal{A} \setminus \mathcal{A}_{\supp'(F)}$ such that $E(G) \geq \mu$ and $G(\supp'(F)) = p < 1$. Define $G' \in \mathcal{A}_{\supp'(F)}$ as

$$G'(\{0\}) = \frac{G(\{0\}) + (1 - p)}{x = 0} \quad G'(\{x\}) = \frac{G(\{x\})}{x \neq 0, x \in \supp(F)} \quad 0$$

otherwise.

Since $D(F||G') \leq D(F||G)$ and $E(G') \geq E(G)$, we obtain

$$\min_{G \in \mathcal{A}: E(G) \geq \mu} D(F||G) \geq \min_{G' \in \mathcal{A}_{\supp'(F)}: E(G') \geq \mu} D(F||G').$$

The converse inequality is obvious from $\mathcal{A}_{\supp'(F)} \subset \mathcal{A}$. \qed

In view of this lemma, we simply write $D_{\min}(F, \mu)$ instead of $D_{\min}(F, \mu, \mathcal{A}) = D_{\min}(F, \mu, \mathcal{A}_{\supp'(F)})$ when the third argument is obvious from the context.

Let $M \equiv |\supp'(F)|$ and denote the finite symbols in $\supp'(F)$ by $x_1, \ldots, x_M$, i.e. $\{0\} \cup \supp(F) = \{x_1, \ldots, x_M\}$. We assume $x_1 = 0$ and $x_i < 0$ for $i > 1$ without loss of generality and write $f_i \equiv F(\{x_i\})$. 7
Now the computation of $D_{\min}(F, \mu)$ is formulated as the following convex optimization problem for $G = (g_1, \ldots, g_M)$ from Lemma [4]:

$$\text{minimize} \quad \sum_{i=1}^{M} f_i \log \frac{f_i}{g_i}$$

subject to

$$-g_i \leq 0, \quad \forall i, \quad \mu - \sum_{i=1}^{M} x_i g_i \leq 0, \quad \sum_{i=1}^{M} g_i = 1,$$

(7)

where we define $0 \log 0 \equiv 0$, $0 \log \frac{0}{0} \equiv 0$, and $\frac{1}{0} \equiv +\infty$.

It is obvious that $G = F$ is the optimal solution with the optimal value 0 when $0 \geq \mathbb{E}(F) \geq \mu$. Also $G = \delta_0$, the unit point mass at 0, is the unique feasible solution if $\mu = 0$. For $\mu > 0$ the problem is infeasible. Since these cases are trivial, we consider the case $\mathbb{E}(F) < \mu < 0$ in the following.

Define $h(\nu)$ and its first and second order derivatives as

$$h(\nu) \equiv \mathbb{E}_F[\log(1 - (X - \mu)\nu)] = \sum_{i=1}^{M} f_i \log(1 - (x_i - \mu)\nu),$$

(8)

$$h'(\nu) \equiv \frac{\partial}{\partial \nu} h(\nu) = -\sum_{i=1}^{M} \frac{f_i (x_i - \mu)}{1 - (x_i - \mu)\nu},$$

(9)

$$h''(\nu) \equiv \frac{\partial^2}{\partial \nu^2} h(\nu) = -\sum_{i=1}^{M} \frac{f_i (x_i - \mu)^2}{(1 - (x_i - \mu)\nu)^2}.$$  

(10)

Now we show in Theorem [5] that the computation of $D_{\min}$ is expressed as maximization of $h(\nu)$. Since $h(\nu)$ is concave, it is a univariate convex optimization problem. Therefore $D_{\min}$ can be computed easily by iterative methods such as Newton’s method (see, e.g., [5] for general methods of convex programming).

**Theorem 5.** Define $\mathbb{E}_F[\mu/X] = \infty$ for the case $F(\{0\}) = f_1 > 0$. Then following three properties hold for $\mathbb{E}(F) < \mu < 0$:

(i) $D_{\min}(F, \mu)$ is written as

$$D_{\min}(F, \mu) = \max_{0 \leq \nu \leq \frac{1}{\mu}} h(\nu)$$

(11)

and the optimal solution $\nu^* \equiv \arg\max_{0 \leq \nu \leq \frac{1}{\mu}} h(\nu)$ is unique.

In particular for the case $\mathbb{E}[\mu/X] \leq 1$, $\nu^* = -1/\mu$ and (11) is simply written as

$$D_{\min}(F, \mu) = h\left(\frac{1}{-\mu}\right) = \sum_{i=2}^{M} f_i \log(x_i/\mu).$$

(12)

On the other hand for the case $\mathbb{E}[\mu/X] \geq 1$, (11) is written as an unconstrained optimization problem

$$D_{\min}(F, \mu) = \max_{\nu} h(\nu).$$

(13)
(ii) \( \nu^* \) satisfies
\[
\nu^* \geq \frac{\mu - \mathbb{E}(F)}{-\mu(1 + \mu)}.
\]

(iii) \( D_{\text{min}}(F, \mu) \) is differentiable in \( \mu \in (\mathbb{E}(F), 0) \) and
\[
\frac{\partial}{\partial \mu} D_{\text{min}}(F, \mu) = \nu^*.
\]

We give a proof of Theorem 5 in Section 3.3.

### 3.3 Proofs of Theorem 3 and 5

In this section we give proofs of Theorem 3 and 5. Actually we prove Theorem 3 using Theorem 5 and prove Theorem 5 independently of Theorem 3.

We first show Lemmas 6 and 7 on properties of \( D_{\text{min}} \) to prove Theorem 3.

**Lemma 6.** \( D_{\text{min}}(F, \mu) \) is monotonically increasing in \( \mu \) and possesses following continuities: (1) lower semicontinuous in \( F \in \mathcal{A} \), that is, 
\[
\liminf_{F' \to F} D_{\text{min}}(F', \mu) \geq D_{\text{min}}(F, \mu).
\]

(2) continuous in \( \mu < 0 \).

Note that the continuity in \( \mu < 0 \) is not trivial at \( \mu = \mathbb{E}(F) \) because the differentiability in Theorem 5 is valid only for the case \( \mathbb{E}(F) < \mu < 0 \) and \( D_{\text{min}}(F, \mu) \) may not be differentiable at \( \mu = \mathbb{E}(F) \).

**Proof.** The monotonicity is obvious from the definition of \( D_{\text{min}} \).

(1) Fix an arbitrary \( \epsilon > 0 \). From (11) and the continuity of \( h(\nu) \), there exists \( \nu_0 \in [0, -1/\mu) \) such that \( \mathbb{E}_F[\log(1 - (X - \mu)\nu)] \geq D_{\text{min}}(F, \mu) - \epsilon \). Then we obtain
\[
\liminf_{F' \to F} D_{\text{min}}(F', \mu) \geq \liminf_{F' \to F} \mathbb{E}_F[\log(1 - (X - \mu)\nu)] \\
= \mathbb{E}_F[\log(1 - (X - \mu)\nu)] \\
\geq D_{\text{min}}(F, \mu) - \epsilon. \tag{14}
\]

Note that \( \log(1 - (x - \mu)\nu) \) is continuous and bounded in \( x \in [-1, 0] \) and (14) follows from the definition of weak convergence. The lower semicontinuity holds since \( \epsilon \) is arbitrary.

(2) The continuity is obvious for \( \mu > \mathbb{E}(F) \) from the differentiability in Theorem 5. The case \( \mu < \mathbb{E}(F) \) is also obvious since \( D_{\text{min}}(F, \mu) = 0 \) holds for \( \mu \leq \mathbb{E}(F) \). Then it is sufficient to show
\[
\lim_{\mu \downarrow \mathbb{E}(F)} D_{\text{min}}(F, \mu) = D_{\text{min}}(F, \mathbb{E}(F)) = 0. \tag{15}
\]

From (11) and the concavity of \( h(\nu) \), it holds that
\[
h(0) \leq D_{\text{min}}(F, \mu) \leq h(0) + h'(0) \frac{1}{-\mu} \\
\iff 0 \leq D_{\text{min}}(F, \mu) \leq \frac{\mathbb{E}(F)}{\mu} - 1
\]
for \( \mu > \mathbb{E}(F) \). (15) is obtained by letting \( \mu \downarrow \mathbb{E}(F) \).
Lemma 7. Fix arbitrary $\mu, \mu' \in (-1, 0)$ satisfying $\mu' < \mu$. Then there exists $C(\mu, \mu') > 0$ such that

$$D_{\min}(F, \mu) - D_{\min}(F, \mu') \geq C(\mu, \mu').$$

for all $F \in A$ satisfying $E(F) < \mu'$.

Proof. Since $D_{\min}(F, \mu)$ is differentiable in $\mu > E(F)$ from Theorem 5, we have

$$D_{\min}(F, \mu) - D_{\min}(F, \mu') = \int_{\mu'}^{\mu} \frac{\partial}{\partial u} D_{\min}(F, u) du \geq \int_{\mu'}^{\mu} \frac{u - \mu'}{-u(1 + u)} du \geq \int_{\mu'}^{\mu} \frac{u - \mu'}{-\mu'(1 + \mu)} du = \frac{(\mu - \mu')^2}{-2\mu'(1 + \mu)} (=: C(\mu, \mu')).$$

Proof of Theorem 3. We define more notation used in the following proof. We fix $j = 1$ and let $L \equiv \{2, \ldots, K\}$. Then, $\mu^* = \mu_1$ and $\mu_k < \mu_1$ for $k \in L$. For notational convenience we denote $J_n(i) \equiv \{J_n = i\}$ which is the event that the arm $\Pi_i$ is pulled at the $n$-th round.

We simply write $E[\cdot], P[\cdot]$ as an expectation and a probability under $F$ and the randomization in the policy. Now we define events $A_n, B_n, C_n, D_n$ as follows:

$$A_n \equiv \left\{ \hat{D}_i(n) \geq \frac{D_{\min}(F_i, \mu^*)}{1 + \epsilon/2} \right\},$$

$$B_n \equiv \{\hat{\mu}_1(n) \geq \mu_1 - \delta\};$$

$$C_n \equiv \{\hat{\mu}_1(n) < \mu_1 - \delta \land \max_{k \in L} \hat{\mu}_k(n) < \mu_1 - \delta\};$$

$$D_n \equiv \{\hat{\mu}_1(n) < \mu_1 - \delta \land \max_{k \in L} \hat{\mu}_k(n) \geq \mu_1 - \delta\};$$

where $\delta > 0$ is a constant satisfying $\max_{k \in L} \mu_k < \mu_1 - \delta$ which is set sufficiently small in the evaluation on $B_n$. Note that $B_n \cup C_n \cup D_n = \Omega$ and each $\mathbb{I}[J_n(i)]$ in the sum $T_i(N) = \sum_{n=1}^{N} \mathbb{I}[J_n(i)]$ is bounded from above by

$$\mathbb{I}[J_n(i)] \leq \mathbb{I}[J_n(i) \cap A_n] + \mathbb{I}[J_n(i) \cap C_n] + \mathbb{I}[J_n(i) \cap A_n^c \cap B_n] + \mathbb{I}[J_n(i) \cap D_n]. \ (16)$$

In the following Lemmas 8-11 we bound the expected values of sums of the four terms on the right-hand side of (16) in this order and they are sufficient to prove Theorem 3.

Lemma 8. Fix an arbitrary $\epsilon > 0$. Then it holds that

$$E\left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n] \right] \leq \frac{1 + \epsilon}{D_{\min}(F_i, \mu^*)} \log N + o(1).$$
Lemma 9.

\[
E \left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap C_n] \right] = O(1).
\]

Lemma 10.

\[
E \left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n^C \cap B_n] \right] = O(1).
\]

Lemma 11.

\[
E \left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap D_n] \right] = O(1).
\]

Before proving these lemmas, we give intuitive interpretations for these terms.

\( A_n \) represents the event that the estimator \( \hat{\mu}_n = D_{\min}(F_i, \mu^*) \) is already close to \( D_{\min}(F_i, \mu^*) \) and \( \Pi_i \) is pulled with a small probability. After sufficiently many rounds \( A_n \) holds with probability close to 1 and the term \( \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n] \) is the main term of \( T_i(N) \).

Other terms of (16) represent events that \( \Pi_i \) is pulled when each estimator is not yet close to the true value. The term involving \( C_n \) is essential for the consistency of MED.

\( A_n^C \cap B_n \) represents the following event: \( \hat{\mu}_n \) has not converged because \( \hat{F}_n(i) \) is not close to \( F_i \) although \( \hat{\mu}_n(i) \) is already close to \( \mu_1 \). In this event \( \Pi_i \) is pulled and therefore \( \hat{F}_n(i) \) is updated more frequently. As a result, \( A_n^C \cap B_n \) happens only for a few \( n \).

Similarly, \( D_n \) represents the event that \( \hat{\mu}_n \) is large for some \( k \in L \). Also in this event \( \hat{F}_k(n) \) is updated more frequently and \( D_n \) happens only for a few \( n \).

On the other hand, \( C_n \) represents the event that \( \hat{\mu}_1 \) is not yet close to \( \mu_1 \). It requires many rounds for \( \Pi_1 \) to be pulled since \( \Pi_1 \) seems to be inferior in this event. Therefore \( C_n \) may happen for many \( n \).

**Proof of Lemma 9**. By partitioning \( \mathbb{I}[J_n(i) \cap A_n] \) according to the number of occurrences \( \sum_{m=1}^{n-1} \mathbb{I}[J_m(i) \cap A_m] \) of the event \( J_m(i) \cap A_m \) before the \( n \)-th round, we have

\[
\sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n] \\
\leq \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} + \sum_{n=1}^{N} \mathbb{I} \left[ J_n(i) \cap A_n \cap \left\{ \sum_{m=1}^{n-1} \mathbb{I}[J_m(i) \cap A_m] > \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} \right\} \right].
\]

Since \( \sum_{m=1}^{n-1} \mathbb{I}[J_m(i) \cap A_m] \leq \sum_{m=1}^{n-1} \mathbb{I}[J_m(i)] = T_i'(n) \), we obtain

\[
\sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n] \leq \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} + \sum_{n=1}^{N} \mathbb{I} \left[ J_n(i) \cap A_n \cap T_i'(n) > \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} \right].
\]
Taking the expected value we have

\[
\mathbb{E}\left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n] \right]
\]

\[
\leq \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} + \sum_{n=1}^{N} P \left[ J_n(i) \cap A_n \cap T'(n) > \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} \right].
\]

\[
\leq \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} + \sum_{n=1}^{N} P \left[ J_n(i) \cap A_n \cap T'_i(n) > \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} \right]
\]

\[
\leq \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} + N \exp \left( -\frac{(1 + \epsilon) \log N D_{\min}(F_i, \mu^*)}{1 + \epsilon/2} \right) \quad \text{(by (15))}
\]

\[
= \frac{(1 + \epsilon) \log N}{D_{\min}(F_i, \mu^*)} + N^{-\frac{1 + \epsilon}{1 + \epsilon/2} + 1}
\]

The lemma is proved since \( N^{-\frac{1 + \epsilon}{1 + \epsilon/2} + 1} = o(1) \).

**Proof of Lemma 9.** First we have

\[
\sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap C_n] \leq \sum_{n=1}^{N} \mathbb{I}[J_n \in L \cap C_n]
\]

\[
\leq \sum_{n=1}^{N} \sum_{m=1}^{\infty} \mathbb{I}[J_n \in L \cap T'_1(n) = t \cap C_n], \quad (17)
\]

From the technique of type [7 Lemma 2.1.9], it holds for any type \( Q \in \mathcal{A} \) that

\[
P_{F_1}[\hat{F}_{1,t} = Q] \leq \exp(-tD(Q||F_1)) \leq \exp(-tD_{\min}(Q, \mu_1)). \quad (18)
\]

Let \( R = (R_1, \ldots, R_m) \) be the smallest \( m \) integers in \( \{n : T'_1(n) = t \cap C_n\} \). \( R \) is well defined on the event \( m \leq \sum_{n=1}^{\infty} \mathbb{I}[J_n \in L \cap T'_1(n) = t \cap C_n] \). Let \( r = (r_1, \ldots, r_m) \in \mathbb{N}^m \) be a realization of \( R \). Here recall that we write an event e.g. \( \cdots \cap R = r \cap \hat{F}_{1,t} = Q \) instead of \( \cdots \cap \{R = r\} \cap \{\hat{F}_{1,t} = Q\} \). Then we obtain for any \( r \) that

\[
P\left[ \left\{ \sum_{n=1}^{\infty} \mathbb{I}[J_n \in L \cap T'_1(n) = t \cap C_n] \geq m \right\} \cap R = r \cap \hat{F}_{1,t} = Q \right]
\]

\[
= P \left[ \bigcap_{l=1}^{m} \{J_{r_l} \in L\} \cap R = r \cap \hat{F}_{1,t} = Q \right]
\]

\[
= P_{F_1}[\hat{F}_{1,t} = Q] \prod_{l=1}^{m} \left( P \left[ R_l = r_l \ \bigg| \ \bigcap_{k=1}^{l-1} \{J_{r_k} \in L \cap R_k = r_k\} \cap \hat{F}_{1,t} = Q \right] \right)
\]

\[
\times P \left[ J_{r_l} \in L \ \bigg| \ R_l = r_l \ \bigcap_{k=1}^{l-1} \{J_{r_k} \in L \cap R_k = r_k\} \cap \hat{F}_{1,t} = Q \right]
\]

\[
\leq \prod_{l=1}^{m} \left( \exp(-tD(Q||F_1)) \right)
\]

\[
\leq \prod_{l=1}^{m} \left( \exp(-tD_{\min}(Q, \mu_1)) \right)
\]

\[
= \exp\left( -t \sum_{l=1}^{m} D(Q||F_l) \right)
\]

\[
= \exp\left( -t D(Q||F_1) \right)
\]

\[
= \exp\left( -t D_{\min}(Q, \mu_1) \right)
\]
\[ \leq P_{F_1}[\hat{F}_{1,t} = Q] \prod_{l=1}^{m} \left( P \left[ R_l = r_l \ \bigg| \ \bigcap_{k=1}^{l-1} \{ J_{r_k} \in L \cap R_k = r_k \} \cap \hat{F}_{1,t} = Q \right] \right. \\
\times \left( 1 - \frac{1}{K} \exp(-tD_{\min}(Q, \mu_1 - \delta)) \right) \) \quad \text{(by (15) and } \hat{\mu}^*(R_l) < \mu_1 - \delta) \\
= P_{F_1}[\hat{F}_{1,t} = Q] \left( 1 - \frac{1}{K} \exp(-tD_{\min}(Q, \mu_1 - \delta)) \right)^m \\
\times \prod_{l=1}^{m} P \left[ R_l = r_l \ \bigg| \ \bigcap_{k=1}^{l-1} \{ J_{r_k} \in L \cap R_k = r_k \} \cap \hat{F}_{1,t} = Q \right]. \]

By taking the disjoint union of \( R \), we have

\[ P \left[ \left\{ \sum_{n=1}^{\infty} \mathbb{I}[J_n \in L \cap T_1'(n) = t \cap C_n] \geq m \right\} \cap \hat{F}_{1,t} = Q \right] \]
\[ \leq P_{F_1}[\hat{F}_{1,t} = Q] \left( 1 - \frac{1}{K} \exp(-tD_{\min}(Q, \mu_1 - \delta)) \right)^m. \tag{19} \]

Then we have

\[ E \left[ \sum_{n=1}^{\infty} \mathbb{I}[J_n \in L \cap T_1'(n) = t \cap C_n] \right] \]
\[ = \sum_{Q:E(Q) < \mu_1 - \delta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P \left[ \left\{ \sum_{n=1}^{\infty} \mathbb{I}[J_n \in L \cap T_1'(n) = t \cap C_n] \geq m \right\} \cap \hat{F}_{1,t} = Q \right] \]
\[ \leq \sum_{Q:E(Q) < \mu_1 - \delta} \sum_{m=1}^{\infty} \exp(-tD_{\min}(Q, \mu_1)) \left( 1 - \frac{1}{K} \exp(-tD_{\min}(Q, \mu_1 - \delta)) \right)^m \]
\[ \leq K \sum_{Q:E(Q) < \mu_1 - \delta} \exp \left( -t \left( D_{\min}(Q, \mu_1) - D_{\min}(Q, \mu_1 - \delta) \right) \right) \quad \text{(by (18) and (19))} \]
\[ \leq K \sum_{Q:E(Q) < \mu_1 - \delta} \exp(-tC(\mu_1, \mu_1 - \delta)) \quad \text{(by Lemma 7)} \]
\[ \leq K(t + 1)^{|\text{supp}(F_1)|} \exp(-tC(\mu_1, \mu_1 - \delta)). \tag{20} \]

The last inequality holds since there are at most \((t + 1)^{|\text{supp}(F_1)|}\) combinations as a type of \( t \) samples from \( F_1 \).

Finally we obtain from (17), (20) and \( C(\mu_1, \mu_1 - \delta) > 0 \) that

\[ E \left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap C_n] \right] \leq \sum_{t=1}^{N} K(t + 1)^{|\text{supp}(F_1)|} \exp(-tC(\mu_1, \mu_1 - \delta)) = O(1) \]

and the proof is completed. \( \square \)
In the proofs of remaining two lemmas, we use [7, Theorem 6.2.10] on the empirical distribution:

**Theorem 12** (Sanov’s Theorem). For every closed set $\Gamma$ of probability distributions
\[
\limsup_{t \to \infty} \frac{1}{t} \log P_F[\hat{F}_t \in \Gamma] \leq -\inf_{G \in \Gamma} D(G||F).
\]
where $\hat{F}_t$ is the empirical distribution of $t$ samples from $F$.

**Proof of Lemma 10.** We apply Sanov’s Theorem with $F = F_i$ and
\[
\Gamma = \{G \in A : L(F_i, G) \geq \delta_1\}
\]
where $\delta_1 > 0$ is a constant. Since $\inf_{G \in \Gamma} D(G||F_i) > 0$, there exists a constant $C_1 > 0$ such that
\[
P_{F_i}[\hat{F}_{i,t} \in \Gamma] \leq \exp(-C_1 t) \tag{21}
\]
for sufficiently large $t$.

Now we show
\[
\{A^C_n \cap B_n\} \subset \{\hat{F}_i(n) \in \Gamma\} \tag{22}
\]
or equivalently $\{\hat{F}_i(n) \notin \Gamma \cap B_n\} \subset A_n$ for sufficiently small $\delta_1$. If $\hat{F}_i(n) \notin \Gamma_1$ and $B_n$, then
\[
D_{\min}(\hat{F}_i(n), \hat{\mu}^*(n)) \geq D_{\min}(\hat{F}_i(n), \mu^* - \delta)
\]
from $\hat{\mu}^*(n) \geq \hat{\mu}_1(n) \geq \mu_1 - \delta = \mu^* - \delta$ and the monotonicity of $D_{\min}$ in $\mu$. Since $D_{\min}(F_i, \mu^* - \delta) > 0$, for sufficiently small $\delta_1$ we obtain
\[
D_{\min}(\hat{F}_i(n), \mu^* - \delta) \geq \frac{D_{\min}(F_i, \mu^* - \delta)}{1 + \epsilon/3}
\]
from the lower semicontinuity in $F$ of $D_{\min}$ in Lemma [8]. Moreover, from the continuity of $D_{\min}$ in $\mu$, it holds for sufficiently small $\delta$ that
\[
\frac{D_{\min}(F_i, \mu^* - \delta)}{1 + \epsilon/3} \geq \frac{D_{\min}(F_i, \mu^*)}{1 + \epsilon/2}.
\]
Then $A_n$ holds and (22) is proved.
From (22) we obtain
\[
\left\{ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n^C \cap B_n] \geq m \right\} \\
\subset \left\{ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap \hat{F}_i(n) \in \Gamma] \geq m \right\} \\
= \left\{ \sum_{t=1}^{N} \mathbb{I} \left[ \bigcup_{n=1}^{N} \{ J_n(i) \cap T'_i(n) = t \cap \hat{F}_i \in \Gamma \} \right] \geq m \right\} \\
\subset \left\{ \sum_{t=1}^{N} \mathbb{I} \left[ \hat{F}_{i,t} \in \Gamma \right] \geq m \right\} \\
\subset \bigcup_{l=m}^{N} \{ \hat{F}_{i,t} \in \Gamma \}. \tag{23}
\]

(23) follows because there is at most one \(n\) such that \(J_n(i) \cap T_i(n) = t\).

Finally, from (21) and (24) we obtain
\[
E \left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n^C \cap B_n] \right] = \sum_{n=1}^{N} P \left[ \sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap A_n^C \cap B_n] \geq m \right] \\
\leq \sum_{m=1}^{N} \sum_{l=m}^{N} P_F[\hat{F}_{i,t} \in \Gamma] \\
= O(1). \tag{24}
\]

Proof of Lemma 11. First we simply bound \(\sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap D_n]\) by
\[
\sum_{n=1}^{N} \mathbb{I}[J_n(i) \cap D_n] \leq \sum_{n=1}^{\infty} \mathbb{I}[D_n].
\]

Since \(D_n \subset \bigcup_{k \in L} \{ \hat{\mu}_k(n) = \hat{\mu}^*(n) > \mu_1 - \delta \}\), it holds that
\[
\sum_{n=1}^{\infty} \mathbb{I}[D_n] \leq \sum_{k \in L} \sum_{n=1}^{\infty} \mathbb{I}[\hat{\mu}_k(n) = \hat{\mu}^*(n) > \mu_1 - \delta] \\
= \sum_{k \in L} \sum_{t=1}^{\infty} \sum_{n=1}^{\infty} \mathbb{I}[\hat{\mu}_{k,t} = \hat{\mu}^*(n) > \mu_1 - \delta \cap T'_k(n) = t]. \tag{25}
\]

Now we use a reasoning similar to (19). Let \(R = (R_1, \ldots, R_m)\) be the smallest \(m\) integers in \(\{ n : T'_k(n) = t \cap \hat{\mu}_{k,t} = \hat{\mu}^*(n) > \mu_1 - \delta \}\). \(R\) is well defined on the event
m \leq \sum_{n=1}^{\infty} \mathbb{I}[T_k'(n) = t \cap \hat{\mu}_{k,t} = \hat{\mu}^*(n) > \mu_1 - \delta]. \text{ Then we have}

\begin{align*}
P \left[ \sum_{n=1}^{\infty} \mathbb{I}[T_k'(n) = t \cap \hat{\mu}_{k,t} = \hat{\mu}^*(n) > \mu_1 - \delta] \geq m \right] \\
&= P_{F_k} [\hat{\mu}_{k,t} > \mu_1 - \delta] P \left[ \sum_{n=1}^{\infty} \mathbb{I}[T_k'(n) = t \cap \hat{\mu}_{k,t} = \hat{\mu}^*(n)] \geq m \mid \hat{\mu}_{k,t} > \mu_1 - \delta \right] \\
&\leq P_{F_k} [\hat{\mu}_{k,t} > \mu_1 - \delta] P \left[ \prod_{i=1}^{m-1} \{J_{R_i} \neq k\} \mid \hat{\mu}_{k,t} > \mu_1 - \delta \right] \\
&\leq P_{F_k} [\hat{\mu}_{k,t} > \mu_1 - \delta] \left( 1 - \frac{1}{K} \right)^{m-1}
\end{align*}

from \( \hat{\mu}_k(R_i) = \hat{\mu}^*(R_i) \) and (14). Therefore we obtain

\begin{align*}
E \left[ \sum_{n=1}^{\infty} \mathbb{I}[\hat{\mu}_{k,t} = \hat{\mu}^*(n) > \mu_1 - \delta \cap T_k'(n) = t] \right] \\
&= \sum_{m=1}^{\infty} P \left[ \sum_{n=1}^{\infty} \mathbb{I}[T_k'(n) = t \cap \hat{\mu}_{k,t} = \hat{\mu}^*(n) > \mu_1 - \delta] \geq m \right] \\
&\leq K P_{F_k} [\hat{\mu}_{k,t} > \mu_1 - \delta]. \quad (26)
\end{align*}

On the other hand, it holds from Sanov’s theorem that for a constant \( C_2 > 0 \)

\[ P_{F_k} [\hat{\mu}_{k,t} > \mu_1 - \delta] = O(\exp(-C_2t)) \quad (27) \]

by setting \( F = F_k \) and \( \Gamma = \{ G \in \mathcal{A} : E(G) \geq \mu_1 - \delta \} \). From (25), (26) and (27), we obtain

\[ E \left[ \sum_{n=1}^{N} \mathbb{I}[D_n] \right] \leq \sum_{k \in L} \sum_{t=1}^{\infty} K O(\exp(-C_2t)) = O(1) \]

\( \square \)

**Proof of Theorem 2** (i) \( h''(\nu) = 0 \) holds only for the degenerate case that \( f_i = 1 \) at \( x_i = \mu \) and this case does not satisfy the assumption \( E(F) < \mu \). Therefore \( h''(\nu) < 0 \) and \( h(\nu) \) is strictly concave. \( \nu^* \) is unique from the strict concavity.

Now we show (11), (12) and (13) by the technique of Lagrange multipliers. The Lagrangian function for (17) is written as

\[ \sum_{i=1}^{M} f_i \log \frac{f_i}{g_i} - \sum_{i=1}^{M} \lambda_i g_i + \nu \left( \mu - \sum_{i=1}^{M} x_i g_i \right) + \xi \sum_{i=1}^{M} g_i. \]
Then there exists a Kuhn-Tucker vector \((\lambda_1^*, \cdots, \lambda_M^*, \nu^*, \xi^*)\) for the problem (7) from [17, Theorem 28.2]. On the other hand it is obvious that the problem (7) has an optimal solution \(G^* = (g_1^*, \cdots, g_M^*)\). From [17, Theorem 28.3], \((g_1^*, \cdots, g_M^*)\) is an optimal value and \((\lambda_1^*, \cdots, \lambda_M^*, \nu^*, \xi^*)\) is a Kuhn-Tucker vector if and only if the following Kuhn-Tucker conditions are satisfied:

\[
-f_i \frac{g_i^*}{g_i} - \lambda_i^* - x_i \nu^* + \xi^* = 0, \quad \forall i
\]

\[
g_i^* \geq 0, \quad \lambda_i \geq 0, \quad g_i \lambda_i = 0, \quad \forall i,
\]

\[
\sum_{i=1}^{M} x_i g_i^* \geq \mu, \quad \nu^* \geq 0, \quad \nu^* \left( \mu - \sum_{i=1}^{M} x_i g_i^* \right) = 0,
\]

\[
\sum_{i=1}^{M} g_i^* = 1.
\]

First we consider the case \(E_F[\mu/X] \leq 1\). In this case, it is easily checked that

\[
g_i^* = \begin{cases} \frac{\mu}{f_i} \\ \frac{\mu}{1 - \sum_{i=2}^{M} \frac{f_i}{x_i}} \end{cases} \quad i \neq 1
\]

\[i = 1,\]

\[
\lambda_i^* = 0, \quad \nu^* = -\frac{1}{\mu} \quad \text{and} \quad \xi^* = 0
\]

satisfy Kuhn-Tucker conditions since \(f_1 = 0\) and \(f_i > 0\) for \(i \neq 1\). Therefore (12) is obtained. (11) follows from \(h'(-1/\mu) \geq 0\) and the concavity of \(h(\nu)\).

Now we consider the second case \(E_F[\mu/X] \geq 1\). Since \(h'(0) > 0\), \(h'(-1/\mu) \leq 0\) and \(h(\nu)\) is concave,

\[
\max_{0 \leq \nu \leq \frac{1}{\mu}} h(\nu) = \max_{\nu} h(\nu) \quad \text{(28)}
\]

holds and \(\nu^* = \arg \max_{0 \leq \nu \leq -1/\mu} h(\nu)\) satisfies

\[
-h'(\nu^*) = \sum_{i=1}^{M} f_i \frac{x_i - \mu}{1 - (x_i - \mu)\nu^*} = 0. \quad \text{(29)}
\]

From (29) we obtain

\[
\sum_{i=1}^{M} \frac{f_i}{1 - (x_i - \mu)\nu^*} = \sum_{i=1}^{M} f_i \frac{1 - (x_i - \mu)\nu^*}{1 - (x_i - \mu)\nu^*} + \nu^* \sum_{i=1}^{M} f_i \frac{x_i - \mu}{1 - (x_i - \mu)\nu^*} = 1 \quad \text{(30)}
\]

and

\[
\sum_{i=1}^{M} \frac{f_i x_i}{1 - (x_i - \mu)\nu^*} = \sum_{i=1}^{M} f_i \frac{x_i - \mu}{1 - (x_i - \mu)\nu^*} + \mu \sum_{i=1}^{M} f_i \frac{f_i}{1 - (x_i - \mu)\nu^*} = \mu. \quad \text{(31)}
\]
From (30) and (31), it is easily checked that
\[
\begin{align*}
g_i^* &= \begin{cases} 
\frac{f_i}{1-(x_i-\mu)\nu^*} & f_i > 0 \\
0 & f_i = 0,
\end{cases} \\
\lambda_i^* &= \begin{cases} 
0 & f_i > 0 \\
1-(x_i-\mu)\nu^* & f_i = 0,
\end{cases}
\end{align*}
\]
\[\xi^* = 1 + \mu \nu^* \] and \[\nu^*\] satisfy Kuhn-Tucker conditions and (11) is obtained. (13) follows immediately from (28).

(ii) The claim is obviously true for the case \[\mathbb{E}_F[\mu/X] \leq 1\] and we consider the case \[\mathbb{E}_F[\mu/X] \geq 1\].

Define
\[
w(x, \nu) \equiv \frac{x-\mu}{1-(x-\mu)\nu}.
\]
For any fixed \[\nu \in [0, -1/\mu]\], \(w(x, \nu)\) is convex in \(x \in [-1, 0]\). Therefore
\[
h'(\nu) = -\sum_{i=1}^{M} f_i w(x_i, \nu) \\
\geq -\sum_{i=1}^{M} f_i \left(-x_i w(-1, \nu) + (1+x_i)w(0, \nu)\right) \\
= \mathbb{E}(F)w(-1, \nu) - (1 + \mathbb{E}(F))w(0, \nu).
\]
The right-hand side of (32) is 0 for \(\nu = (\mu - \mathbb{E}(F))/(-\mu (1+\mu))\) and therefore
\[
h'\left(\frac{\mu - \mathbb{E}(F)}{-\mu (1+\mu)}\right) \geq 0.
\]
Since \(h'(\nu)\) is monotonically decreasing, \(\nu^* \geq (\mu - \mathbb{E}(F))/(-\mu (1+\mu))\) is proved.

(iii) It is obvious that \[\frac{\partial}{\partial \mu} D_{\min}(F, \nu) = \nu^* = -1/\mu\] for \(\mathbb{E}_F[\mu/X] < 1\) and
\[
\lim_{\epsilon \downarrow 0} \frac{D_{\min}(F, \mu + \epsilon) - D_{\min}(F, \mu)}{\epsilon} = \frac{1}{-\mu}
\]
for \(\mathbb{E}_F[\mu/X] = 1\).

Define \(D'_{\min}(F, \mu) \equiv \max_\nu h(\nu)\). Then \(D_{\min}(F, \mu) = D'_{\min}(F, \mu)\) for the case \(\mathbb{E}_F[\mu/X] \geq 1\). From [9, Corollary 3.4.3], \(D'_{\min}(F, \mu)\) is differentiable in \(\mu\) with
\[
\frac{\partial}{\partial \mu} D'_{\min}(F, \nu) = \frac{\partial}{\partial \mu} h(\nu) \bigg|_{\nu = \nu^*} = \nu^*.
\]
Therefore we obtain
\[
\frac{\partial}{\partial \mu} D_{\min}(F, \nu) = \frac{\partial}{\partial \mu} D'_{\min}(F, \nu) = \nu^*.
\]
for $E_F[\mu/X] > 1$ and
\[
\lim_{\epsilon \downarrow 0} \frac{D_{\min}(F, \mu - \epsilon) - D_{\min}(F, \mu)}{-\epsilon} = \lim_{\epsilon \downarrow 0} \frac{D'_{\min}(F, \mu - \epsilon) - D'_{\min}(F, \mu)}{-\epsilon} = \nu^* = \frac{1}{-\mu}
\]
for $E_F[\mu/X] = 1$.

4 Experiments

In this section, we present some simulation results on our MED and UCB policies in [3].

First we give an algorithm for computing $\nu^*$ and $D_{\min}(F, \mu)$ with parameters $r, \nu_0$, which we denote by $D_{\min}(F, \mu; r, \nu_0)$. Here $r$ is a repetition number and $\nu_0$ is an initial value of $\nu$ for the optimization in Theorem 5. Recall that $h, h', h''$ are defined in (8), (9) and (10).

**[Computation of $D_{\min}(F, \mu; r, \nu_0)$]**

Require: $r > 0$, $\nu_0 \geq 0$;
if $f_1 = 0$ and $\mu \sum_{i \neq 1} \frac{f_i}{x_i} \leq 1$ then
return $\left( h\left( \frac{1}{\mu'} \right), \frac{1}{-\mu} \right)$;
end if

$\underline{\nu}, \nu := \frac{\mu - E(F)}{-\mu(1+\mu)}$, $\overline{\nu} := \frac{1}{-\mu}$;
if $\nu_0 \in (\underline{\nu}, \overline{\nu})$ then
$\nu := \nu_0$;
end if

for $t := 1$ to $r$ do
if $h'(\nu) > 0$ then
$\underline{\nu} := \nu$;
else
$\overline{\nu} := \nu$;
end if
$\nu := \nu - h'(\nu)/h''(\nu)$;
if $\nu \notin (\underline{\nu}, \overline{\nu})$ then
$\nu := \frac{\underline{\nu} + \overline{\nu}}{2}$;
end if
end for
return $\left( \max_{\nu' \in (\underline{\nu}, \overline{\nu})} h(\nu'), \arg\max_{\nu' \in (\underline{\nu}, \overline{\nu})} h(\nu') \right)$;

In this algorithm, a lower and an upper bound of $\nu^*$ are given by $\underline{\nu}$ and $\overline{\nu}$, respectively. In each step, the next point is determined based on Newton’s method by $\nu := \nu - h'(\nu)/h''(\nu)$. When $\nu$ does not improve the bounds $\underline{\nu}$, $\overline{\nu}$, the next point is determined by bisection method, $\nu := (\underline{\nu} + \overline{\nu})/2$. The complexity of the algorithm is given by $O(r |\text{supp}(F)|)$.
The complexity \( O(r |\text{supp}(F)|) \) is not very small when \( |\text{supp}(F)| \) is large. Especially it requires \( O(r T_i(n)) (\approx O(r \log n)) \) computations when it is adopted for a continuous support model since \( |\text{supp}(\hat{F}_i,F)| \leq t \). On the other hand, \( D_{\text{min}}(F,\mu) \) is differentiable in \( \mu \) (with slope \( \nu^* \)) and the argument \( \mu \) converges to \( \mu^* \) after sufficiently many rounds. Therefore it is reasonable to approximate \( D_{\text{min}}(F,\mu) \) by past value of \( D_{\text{min}}(F,\mu;\nu_0,r) \) until the variation of \( \mu \) is small. In this point of view, we implemented our MED policy for our simulations in the following way:

[An implementation of MED policy]

Parameter: Integer \( r > 0 \) and real \( d > 0 \).

Initialization:

1. Pull each arm once.
2. Set \((\hat{D}_i, \nu_i) := D_{\text{min}}(\hat{F}_i, 1, \hat{\mu}^*(K + 1); 0, r) \) and \( m_i := \hat{\mu}^*(K + 1) \) for each \( i = 1, \cdots, K \).

Loop: For the \( n \)-th round,

1. Update variables for each \( i \):
   - If \( J_{n-1} \neq i \) and \(|\hat{\mu}^*(n) - m_i| < d\) then \( \hat{D}_i := \hat{D}_i + \nu_i(\hat{\mu}^*(n) - m_i) \).
   - Otherwise \((\hat{D}_i, \nu_i) := D_{\text{min}}(\hat{F}_i(n), \hat{\mu}^*(n); \nu_i, r) \) and \( m_i := \hat{\mu}_i(n) \).
2. Choose arm \( \Pi_j \) according to the probability \( p_j(n) \equiv \frac{\exp(-T'^j(n)\hat{D}_j)}{\sum_{i=1}^{K} \exp(-T'^i(n)\hat{D}_i)} \).

Now we describe the setting of our experiments. We used MED, UCB-tuned and UCB2. Each plot is an average over 1,000 different runs. The parameter \( \alpha \) for UCB2 is set to 0.001, the choice of which is not very important for the performance (see [3]). First we check the effect of the choice of the parameters \( r \) and \( d \). Then MED and UCB policies are compared.

In the following simulations, we use the model where the support is included in \([0,1]\). Note that in the computation of \( D_{\text{min}}(F,\mu;\nu, r) \) we assumed that the support is included in \([-1,0]\) for computational convenience. Then, all rewards are passed to computation after 1 is subtracted from them in MED.

Table 1 gives the list of distributions used in the experiments. They cover various situations on the computation of \( D_{\text{min}} \) and how distinguishable the optimal arm is. Distributions 1-4 are examples of 2-armed bandit problems. In Distribution 1, \( \nu^* \geq (\mu - E(F))/(-\mu(1+\mu)) \) in Theorem [1] always holds with equality since \( \text{supp}(F) \subseteq \{0,1\} \). Therefore the exact solution can be obtained by \( D_{\text{min}}(F,\mu;\nu, r) \) regardless of \( r \). Also in Distribution 2, \( D_{\text{min}}(F,\mu;\nu, r) \) does not require the repetition after sufficiently many rounds since \( E_{F}[\mu_1/X] < 1 \). On the other hand in Distribution 3, the maximization [13] is necessary in almost all rounds since \( E_{F}[\mu_1/X] > 1 \). Distribution 4 is an example of
Table 1: Distributions for experiments.

| Distribution | Notation | Value |
|--------------|----------|-------|
| Distribution 1: | $F_1(\{0\}) = 0.45, F_1(\{1\}) = 0.55$ | $E(F_1) = 0.55$ |
|               | $F_2(\{0\}) = 0.55, F_2(\{1\}) = 0.45$ | $E(F_2) = 0.45$ |
| Distribution 2: | $F_1(\{0.4\}) = 0.5, F_1(\{0.8\}) = 0.5$ | $E(F_1) = 0.6$ |
|               | $F_2(\{0.2\}) = 0.5, F_2(\{0.6\}) = 0.5$ | $E(F_2) = 0.4$ |
| Distribution 3: | $F_1(\{x\}) = 0.08$ for $x = 0, 0.1, \ldots, 0.9, F_1(\{1\}) = 0.2$ | $E(F_2) = 0.56$ |
|               | $F_2(\{x\}) = \frac{1}{11}$ for $x = 0, 0.1, \ldots, 0.9, 1$ | $E(F_2) = 0.5$ |
| Distribution 4: | $F_1(\{0\}) = 0.99, F_1(\{1\}) = 0.01$ | $E(F_1) = 0.01$ |
|               | $F_2(\{0.008\}) = 0.5, F_2(\{0.009\}) = 0.5$ | $E(F_2) = 0.0085$ |
| Distribution 5: | $F_1(\{x\}) = 0.08$ for $x = 0, 0.1, \ldots, 0.9, F_1(\{1\}) = 0.2$ | $E(F_1) = 0.56$ |
|               | $F_i(\{x\}) = \frac{1}{11}$ for $x = 0, 0.1, \ldots, 0.9, 1$ | $E(F_i) = 0.5$ for $i = 2, 3, 4, 5$ |
| Distribution 6: | $F_1 = \text{Be}(0.9, 0.1)$ | $E(F_1) = 0.9$ |
|               | $F_2 = \text{Be}(7, 3)$ | $E(F_2) = 0.7$ |
|               | $F_3 = \text{Be}(0.5, 0.5)$ | $E(F_3) = 0.5$ |
|               | $F_4 = \text{Be}(3, 7)$ | $E(F_4) = 0.3$ |
|               | $F_5 = \text{Be}(0.1, 0.9)$ | $E(F_5) = 0.1$ |

A difficult problem where the optimal arm is hard to distinguish since the inferior arm appears to be optimal at first with high probability. Distribution 5 and 6 are examples of more general problems where the numbers of arms $K$ and the support sizes are large. $\text{Be}(\alpha, \beta)$ ($\alpha, \beta > 0$) in Distribution 6 denotes beta distribution which has the density function

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} \quad \text{for } x \in [0, 1]$$

where $B(\alpha, \beta)$ is beta function. Note that beta distributions have continuous support and are not included in $\mathcal{A}$ and therefore the performance of MED is not assured theoretically. However, MED is still formally applicable since the supports are bounded.

The labels of each figure are as follows. "regret" denotes $\sum_{i: \mu_i < \mu^*} (\mu^* - \mu_i) T_i(n)$, which
is the loss due to choosing suboptimal arms. “% best arm played” is the percentage that the best arm is chosen, that is, $100 \times T_1(n)/n$ in these problems. “Dmin” stands for the asymptotic bound for a consistent policy, $\sum_{i: \hat{\mu}_i < \mu^*} (\mu^* - \mu_i) \log n / D_{\min}(F_i, \mu^*)$. The asymptotic slope of the regret (in the semi-logarithmic plot) of a consistent policy is more than or equal to that of “Dmin”.

Figure 1 shows an experiment on the choice of the parameters $r$ and $d$ of MED for Distribution 3. Our implementation of MED approaches the ideal MED as $d \to 0$ and $r \to \infty$. However, we see from the figure that the performance is not sensitive to the choice of $r, d$. This may be understood as follows: (1) the linear approximation for the case $|\hat{\mu}^*(n) - \mu_i| < d$ is accurate, (2) the initial value $\nu_i$ in $D_{\min}(\hat{F}_i(n), \hat{\mu}^*(n); \nu_i, r)$ seems to be a good approximation of $\nu^*$ and the repetition number does not have to be large. We use $r = 2$ and $d = 0.01$ in the remaining experiments based on this result.

Now we summarize the remaining experiments on the comparison of the policies (Figure 2–7).

- MED always seems to be achieving the asymptotic bound even for continuous support distributions, since the asymptotic slope of the regret is close to that of “Dmin”.

- MED performs best except for Distribution 1 where MED performs worst. However, the consistency of UCB-tuned is not proved unlike MED and UCB2. It appears that UCB-tuned might not be consistent, because the asymptotic slope of $T_2(n)$ seems to be smaller than that of “Dmin”. Note that the theoretical logarithmic term of the regret is very near between MED and UCB2 for Distribution 1 ($4.983 \log n$ and $5.025 \log n$, respectively). Therefore this result can be interpreted as follows: MED achieves the asymptotic bound but needs some improvement in the constant term of the regret compared to UCB2.
Figure 2: Simulation result for Distribution 1 (Bernoulli distributions).

Figure 3: Simulation result for Distribution 2 (uniform distributions with different supports).

Figure 4: Simulation result for Distribution 3 (distributions where $D_{\text{min}}$ is computed by repetitions).
Figure 5: Simulation result for Distribution 4 (very confusing distributions).

Figure 6: Simulation result for Distribution 5 (5 arms with a wide support).

Figure 7: Simulation result for Distribution 6 (beta distributions).
5 Concluding remarks

We proposed a policy, MED, and proved that our policy achieves the asymptotic bound for finite support models. We also showed that our policy can be implemented efficiently by a convex optimization technique.

In the theoretical analysis of this paper, we assumed the finiteness of the support although MED worked nicely also for distributions with continuous bounded support in the simulation. We conjecture that the optimality of MED holds also for the continuous bounded support model. In addition, there are many models that \( D_{\min} \) can be computed explicitly, such as normal distribution model with unknown mean and variance. We expect that our MED can be extended to these models. Furthermore, our MED is a randomized policy and the theoretical evaluation of the expectation includes randomization in the policy. We may be able to construct a deterministic version of MED.

In addition to the above theoretical analyses, it is also important to consider the finite horizon case. Then it is necessary to derive a finite-time bound of MED for this case. Especially, MED policy itself should be improved when the number of rounds is given in advance. In this setting, the value of “exploration” becomes smaller and a current best arm is to be pulled more often as the number of remaining rounds becomes smaller.

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