BÖTTCHER COORDINATES AT SUPERATTRACTING FIXED POINTS OF HOLOMORPHIC SKEW PRODUCTS

KOHEI UENO

Abstract. Let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) be a germ of holomorphic skew product with a superattracting fixed point at the origin. If it has a suitable weight, then we can construct a Böttcher coordinate which conjugates \( f \) to the associated monomial map. This Böttcher coordinate is defined on an open set whose interior or boundary contains the origin.

1. Introduction

Let \( p : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) be a holomorphic germ with a superattracting fixed point at the origin. Taking an affine conjugate, we may write \( p(z) = z^\delta + O(z^\delta+1) \), where \( \delta \geq 2 \). Böttcher’s theorem [2] asserts that there is a conformal function \( \varphi_p \) defined near the origin, with \( \varphi_p \sim id \), that conjugates \( p \) to \( p_0 \). Here \( \varphi_p \sim id \) means that the ratio of \( \varphi_p \) and \( id \) tends to 1 as \( z \) tends to 0. This function is called the Böttcher coordinate for \( p \), and obtained as the limit of the compositions of \( p_0^{-n} \) and \( p^n \), where \( p^n \) denotes the \( n \)-th iterate of \( p \). The branch of \( p_0^{-n} \) is taken as \( p_0^{-n} \circ p_0^n = id \), where \( \circ \) denotes the composition of functions.

Böttcher’s theorem does not extend to higher dimensions entirely as stated in [5]. For example, let \( f(z, w) = (z^2, w^2 + z^4) \). Then it has a superattracting fixed point at the origin, but there is no neighborhood of the origin on which \( f \) is conjugate to \( f_0(z, w) = (z^2, w^2) \) because the critical orbits of \( f \) and \( f_0 \) behave differently. However, we can completely understand the dynamics of \( f \) because it is semiconjugate to \( g(z, w) = (z^2, w^2 + 1) \) by \( \pi(z, w) = (z, z^2w) : \pi \circ g = f \circ \pi \). In particular, from the one-dimensional Böttcher coordinate of \( w \to w^2 + 1 \) near infinity, one can construct a biholomorphic map defined on \( \{|z| < r \} \) for small \( r \) that conjugates \( f \) to \( f_0 \). This domain is not a neighborhood of the origin, but its boundary contains the origin. In this paper, we analyze such phenomena for holomorphic skew products with superattracting fixed points at the origin in \( \mathbb{C}^2 \). By assigning suitable weights, we obtain an analogue of the one-dimensional Böttcher coordinates; see Theorems 1 and 2 below. The idea of this study is the same.

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as that of our previous study [8], in which we obtained similar results on Böttcher coordinates for polynomial skew products near infinity. Moreover, our results are closely related to Theorems C and 5.1 in [4]. Favre and Jonsson [4] have established a systematic way to study the dynamics of all holomorphic germs with superattracting fixed points in dimension two; see also Section 8 in a survey article [6].

For other studies on Böttcher’s theorem in higher dimensions, we refer to [9], [7] and [3]: they dealt with holomorphic germs with superattracting fixed points at the origin in dimension two or more. Ushiki [9] and Ueda [7] gave different classes of germs that have the Böttcher coordinates on neighborhoods of the origin. Buff, Epstein and Koch [3] gave criteria, in terms of vector fields, for an certain class of germs to have the Böttcher coordinates on neighborhoods of the origin. We also refer to a survey article [1]. Besides theorems for the superattracting case, Abate [1] collected major theorems on local dynamics of holomorphic germs with fixed points of several types in one and higher dimensions.

Let $f : (C^2, 0) \to (C^2, 0)$ be a holomorphic germ of the form $f(z, w) = (p(z), q(z, w))$, which is called a holomorphic skew product in this paper. We assume that it has a superattracting fixed point at the origin; that is, $f(0) = 0$ and $Df(0)$ is the zero matrix. Then we may write $p(z) = z^\delta + O(z^{\delta+1})$, where $\delta \geq 2$. On the other hand, let

$$q(z, w) = bz^\gamma w^d + \sum b_j z^{n_j} w^{m_j},$$

where $b \neq 0$, $n_j \geq \gamma$, and $m_j > d$ if $n_j = \gamma$. Since the origin is superattracting, $\gamma + d \geq 2$ and $n_j + m_j \geq 2$. If $d \geq 2$, then we may assume that $b = 1$; this case is studied in Sections 3 and 4. We also consider the case $d = 1$ in Section 5. In this paper, we say that $f$ is trivial if $m_j \geq d$ for any $j$. For this case, we prove that the Böttcher coordinate for $f$ exists on a neighborhood of the origin, and the proof is rather easy. On the other hand, we say that $f$ is non-trivial if $m_j < d$ for some $j$. This case is the difficult part, in which we need the idea of assigning a suitable weight.

We define the rational number $\alpha$ associated with $f$ as

$$\min \left\{ a \geq 0 \left| \begin{array}{c}
ay + d \leq \delta \text{ and } ay + d \leq an_j + m_j \text{ for any integers } n_j \text{ and } m_j \text{ s.t. } z^{n_j} w^{m_j} \text{ is a term in } q \text{ with a nonzero coefficient}
\end{array} \right. \right\}$$

if $f$ is non-trivial, and as 0 if $f$ is trivial. Let $U_r = U_r^\alpha = \{|z| < r|w|^\alpha, |w| < r\}$. Although $\alpha$ may not be well-defined, the benefit of $\alpha$ is presented in the following lemma.

**Lemma 1.** Let $d \geq 2$. If $\alpha$ is well-defined, then $f(z, w) \sim (z^\delta, z^\gamma w^d)$ on $U_r$ as $r \to 0$, and $f(U_r) \subset U_r$ for small $r$. 
The notation \( f \sim f_0 \) means that both the first and second components of \( f \) and \( f_0 \) tend to 1 on \( U_r \) as \( r \) tends to 0. As the one-dimensional case, this lemma induces a Böttcher coordinate for \( f \).

**Theorem 1.** Let \( d \geq 2 \). If \( \alpha \) is well-defined, then there is a biholomorphic map \( \phi \) defined on \( U_r \), with \( \phi \sim \text{id} \) on \( U_r \) as \( r \to 0 \), that conjugates \( f \) to \((z, w) \to (z^\delta, z^{\gamma} w^d)\).

As the one-dimensional case, the Böttcher coordinate \( \phi \) is obtained as the limit of the compositions of \( f_0^{-n} \) and \( f^n \), where \( f_0(z, w) = (z^\delta, z^{\gamma} w^d) \).

Our idea is useful even for the case \( d = 1 \). With the additional condition \( \alpha < (\delta - 1)/\gamma \), we obtain the same results as above.

**Lemma 2.** Let \( d = 1 \). If \( \alpha \) is well-defined and \( \alpha < (\delta - 1)/\gamma \), then \( f(z, w) \sim (z^\delta, b z^{\gamma} w) \) on \( U_r \) as \( r \to 0 \), and \( f(U_r) \subset U_r \) for small \( r \).

**Theorem 2.** Let \( d = 1 \). If \( \alpha \) is well-defined and \( \alpha < (\delta - 1)/\gamma \), then there is a biholomorphic map \( \phi \) defined on \( U_r \), with \( \phi \sim \text{id} \) on \( U_r \) as \( r \to 0 \), that conjugates \( f \) to \((z, w) \to (z^\delta, b z^{\gamma} w)\).

Our results also hold for the nilpotent case. We say that the point \( x \) is nilpotent if the eigenvalues of \( Df(x) \) are both zero. If the origin is a nilpotent fixed point of \( f \), then it is superattracting for \( f^2 \). Hence Lemmas 1 and 2 hold for \( f^2 \); or these lemmas hold for \( f \) on \( U_r \cap \{|z| \leq r_1, |w| \leq r_2\} \), where \( r_1 \) is enough smaller than \( r_2 \). Consequently, Theorems 1 and 2 hold for \( f \) itself.

Moreover, we can perturb \( f \) slightly so that it is not skew product but our results hold. Let \( \tilde{p}(z, w) = z^\delta + \sum a_l z^{n_l} w^{m_l} \), where \( n_l \geq \delta \), and \( m_l \geq 1 \) if \( n_l = \delta \), and let \( q \) be the same as above. Then, for a holomorphic germ \( \tilde{f} = (\tilde{p}, q) \), we have the same lemma and theorem as the skew product case.

The organization of the paper is as follows. In Section 2 we introduce the interval \( \mathcal{I}_f \) in \( \mathbb{R} \) associated with \( f \), which is closely related to \( \alpha \), and provide a lemma that implies Lemma 1. The proof of this lemma exhibits where the conditions in the definition of \( \alpha \) come from. Assuming that \( d \geq 2 \), we prove that the composition \( f_0^{-n} \circ f^n \) is well-defined and converges uniformly to \( \phi \) on \( U_r \) in Section 3, and that \( \phi \) is biholomorphic in Section 4. The case \( d = 1 \) is studied in Section 5. Finally, we slightly generalize our results to holomorphic germs in Section 6.

2. **Weights**

We introduce the interval \( \mathcal{I}_f \) associated with \( f \). If \( \mathcal{I}_f \) is non-empty, then \( \alpha \) is well-defined and coincides with \( \max(\inf \mathcal{I}_f, 0) \). Together with the proof of Lemma 3 below, the definition of \( \mathcal{I}_f \) illustrates where the conditions in the definition of \( \alpha \) come from. We also display charts of \( \mathcal{I}_f \) and \( \alpha \), which
differ whether \( \gamma \) is zero or not and depend on the magnitude relation of \( \delta \) and \( d \).

We define the interval \( I_f \) associated with \( f \) as

\[
I_f = \left\{ a \in \mathbb{R} \mid a(ay + d) \leq a\delta \text{ and } ay + d \leq an_j + m_j \right\}
\]

for any integers \( n_j \) and \( m_j \) s.t. \( z^{n_j\gamma/m_j} \) is a term in \( q \) with a nonzero coefficient.

Let \( U^a_r = \{ |z| < r_1|w|^a, |w| < r_2 \} \cap \{ |z| < r_2 \} \). We remark that, unlike the definition of \( U_r \) in the introduction, this set need to be intersected with \( \{ |z| < r_2 \} \) because \( a \) can be negative. Although we assume that \( d \geq 2 \) in the following lemma, the same claim holds for \( d = 1 \) with the condition \( a < (\delta - 1)/\gamma \) as roughly explained in Section 5.

**Lemma 3.** Let \( d \geq 2 \). For any number \( a \) in \( I_f \), it follows that \( q(z, w) \sim z^{\gamma w^d} \) on \( U^a_r \) as \( r_1 \) and \( r_2 \to 0 \), and \( f(U^a_r) \subset U^a_r \) for small \( r_1 \) and \( r_2 \).

**Proof.** Let \( |z| = |cw|^a \) for any \( a \) in \( I_f \). Then \( U^a_r \subset \{ \gamma < r_1, |w| < r_2 \} \). For a term \( bjz^{n_j\gamma/m_j} \) in \( q \) which is not \( z^{\gamma w^d} \),

\[
\left| \frac{z^{n_j\gamma/w^d}}{z^{\gamma w^d}} \right| = \left| \frac{(cw^a)^{n_j\gamma/m_j}}{(cw^a)^{\gamma w^d}} \right| = \left| \frac{c^{n_j\gamma/m_j} + m_j}{c^{\gamma w^d}} \right| = |c|^{n_j\gamma/m_j} |w|^{(an_j + m_j)/(ay + d)}.
\]

The condition \( an_j + m_j \geq ay + d \) ensures that this ratio tends to 0 on \( U^a_r \) as both \( r_1 \) and \( r_2 \) tend to 0. More precisely, at least one of the inequalities \( n_j > \gamma \) or \( an_j + m_j > ay + d \) holds since \( n_j \geq \gamma \), and \( m_j > d \) if \( n_j = \gamma \). Hence \( q(z, w) \sim z^{\gamma w^d} \) on \( U^a_r \).

For the invariance of \( U^a_r \), it is enough to show that \( |p(z)| < r_1|q(z, w)|^a \) for any \( (z, w) \) in \( U^a_r \). Because

\[
\frac{|p(z)|}{|q(z, w)|^a} \sim \frac{z^{\delta}}{(z^{\gamma w^d})^a} = \frac{(cw^a)^{\delta}}{(cw^a)^{\gamma w^d}} = |c|^{\delta - ay} |w|^{a\delta - a(ay + d)}
\]

on \( U^a_r \), we need the conditions \( \delta - ay \geq 0 \) and \( a\delta \geq a(ay + d) \). However, the condition \( \delta - ay \geq 0 \) follows from the condition \( a\delta \geq a(ay + d) \) because \( d \geq 2 \). In fact, it follows that \( \delta - ay \geq 2 \); if \( a \leq 0 \) then \( \delta - ay \geq \delta \geq 2 \), and if \( a > 0 \) then \( \delta - ay \geq d \). Hence \( |p(z)|/q(z, w)|^a \leq C \cdot |c|^{\delta - ay} \leq C \cdot |c|^{\delta} \leq |c| \leq r_1 \) for some constant \( C \) and sufficiently small \( r_1 \).

Let us describe \( I_f \) more practically. Let \( \alpha_0 = (\delta - d)/\gamma \), which is derived from the first condition. The second condition \( ay + d \leq an_j + m_j \) implies that

\[
a \geq \frac{d - m_j}{n_j - \gamma}
\]
if \( n_j > \gamma \). We define \( m_f \) as

\[
\sup \left\{ \frac{d - m_j}{n_j - \gamma} \left| z^{n_j} w^{m_j} \text{ is a term in } q \text{ with a non-zero coefficient such that } n_j > \gamma \right. \right\}
\]

if \( n_j > \gamma \) for some \( j \), and as \(-\infty\) if \( n_j = \gamma \) for any \( j \). Note that \( I_f \subset [m_f, \infty) \).

If \( f \) is trivial, then \( m_f \leq 0 \). If \( f \) is non-trivial, then \( m_f > 0 \) and, moreover, we can replace the supremum to the maximum in the definition of \( m_f \).

If \( f \) is trivial, then we can chart \( I_f \) as follows, where \( m_f \leq 0 \).

| \( f \) trivial | \( \gamma = 0 \) | \( \gamma \neq 0 \) |
|-----------------|-----------------|-----------------|
| \( \delta > d \) | \( [0, \infty) \) | \( [0, \alpha_0] \) |
| \( \delta = d \) | \( [m_f, \infty) \) | \( [0] \) |
| \( \delta < d \) | \( [m_f, 0] \) | \( \max\{m_f, \alpha_0\}, 0 \) |

In particular, \( I_f \) is always non-empty if \( f \) is trivial.

If \( f \) is non-trivial, then we can chart \( I_f \) as follows, where \( m_f > 0 \).

| \( f \) non-trivial | \( \gamma = 0 \) | \( \gamma \neq 0 \) |
|---------------------|-----------------|-----------------|
| \( \delta > d \)     | \( [m_f, \infty) \) | \( [m_f, \alpha_0] \) or \( \emptyset \) |
| \( \delta = d \)     | \( [m_f, \infty) \) | \( \emptyset \) |
| \( \delta < d \)     | \( \emptyset \)   | \( \emptyset \) |

Note that \( I_f \) can be empty if \( f \) is non-trivial. For the case \( \delta > d \) and \( \gamma \neq 0 \), the interval \( I_f \) is equal to \( [m_f, \alpha_0] \) if \( m_f \leq \alpha_0 \) and is empty if \( m_f > \alpha_0 \).

We may restrict our attention to non-negative weights for our theorems, although negative weights make sense as in Lemma 3. Then the assumption \( a \geq 0 \) reduces the condition \( a(ay + d) \leq a\delta \) to the condition \( ay + d \leq \delta \) unless \( a = 0 \), which induces the definition of \( \alpha \). The interval of non-negative numbers that satisfy the conditions in the definition of \( \alpha \), coincides with \( I_f \cap [0, \infty) \) if \( \delta \geq d \). For any case, it follows that \( \alpha \) is well-defined if and only if \( I_f \) is not empty, and that

\[
\alpha = \min I_f \cap [0, \infty) = \max\{\inf I_f, 0\}
\]

if it is well-defined. Let us denote \( \alpha \) more practically, which follows from the charts of \( I_f \). If \( f \) is trivial, then \( \alpha = 0 \). If \( f \) is non-trivial, then \( m_f > 0 \) and we can chart \( \alpha \) as follows.

| \( f \) non-trivial | \( \gamma = 0 \) | \( \gamma \neq 0 \) |
|---------------------|-----------------|-----------------|
| \( \delta > d \)     | \( m_f \) | \( m_f \) or \( \emptyset \) |
| \( \delta = d \)     | \( m_f \) | \( \emptyset \) |
| \( \delta < d \)     | \( \emptyset \) | \( \emptyset \) |
The notation $m_f$ in the chart means that $\alpha$ is well-defined and coincides with $m_f$. The notation $\not\exists$ means that $\alpha$ is not well-defined.

3. Existence of the Limit $\phi$ for the Case $d \geq 2$

In this section we show that $\phi_n$ is well-defined and converges uniformly to $\phi$ on $U_r$ for the case $d \geq 2$, where $\phi_n = f_0^{-n} \circ f^n$. Since the proof is the same as [8], we only give the outline of the proof; we omit some calculations. The biholomorphicity of $\phi$ will be proved in the next section, which completes the proof of Theorem I.

Let us recall the outline of the proof, assuming that $d \geq 2$ and that $\alpha$ is well-defined. Let $p(z) = z^\delta(1 + \zeta(z))$ and $q(z, w) = z^\gamma w^\alpha(1 + \eta(z, w))$; Lemma I implies that $\zeta$ and $\eta$ converge to 0 on $U_r$ as $r$ tends to 0. Then $f^n$ is written as

$$f^n = \left(z^\delta \prod_{j=1}^n (1 + \zeta(p^{j-1}(z))), \ z^\gamma w^\alpha \prod_{j=1}^n (1 + \eta(f^{j-1}(z, w)))\right),$$

where $\gamma_n = \sum_{j=1}^n \delta^{n-j} d^{j-1} \gamma$. Formally, $f_0^n(z, w) = (z^{1/\delta^n}, z^{-\gamma_n/\delta^n} w^{1/d^n})$ and we can define $\phi_n$ as

$$\phi_n(z) = \left(z \prod_{j=1}^n \delta^j 1 + \zeta(p^{j-1}(z)), w \prod_{j=1}^n \delta^j 1 + \eta(f^{j-1}(z, w))\right).$$

It follows from Lemma I that $\phi_n$ is well-defined and so holomorphic on $U_r$.

In order to prove the uniform convergence of $\phi_n$, we lift $f$ and $f_0$ to $F$ and $F_0$ by the exponential product $\pi(z, w) = (e^z, e^w); \pi \circ F = f \circ \pi$ and $\pi \circ F_0 = f_0 \circ \pi$. More precisely, we define

$$F(Z, W) = (\delta Z + \log(1 + \zeta(e^Z)), \gamma Z + dW + \log(1 + \eta(e^Z, e^W)))$$

and $F_0(Z, W) = (\delta Z, \gamma Z + dW)$. By Lemma II we may assume that

$$|F - F_0| < \varepsilon$$

for any small $\varepsilon > 0$, taking $r$ small enough. Similarly, we can lift $\phi_n$ to $\Phi_n$ so that the equation $\Phi_n = F_0^{-n} \circ F^n$ holds; thus

$$\Phi_n(Z, W) = \left(\frac{1}{\delta^n} P_n(Z), \frac{1}{d^n} Q_n(Z, W) - \frac{\gamma_n}{\delta^n d^n} P_n(Z)\right),$$

where $(P_n(Z), Q_n(Z, W)) = F^n(Z, W)$. Let $\Phi_n = (\Phi_n^1, \Phi_n^2)$. Then

$$|\Phi_1^{n+1} - \Phi_1^n| = \left|\frac{P_{n+1}}{\delta^{n+1}} - \frac{P_n}{\delta^n}\right| = \left|\frac{P_{n+1} - \delta P_n}{\delta^{n+1}}\right| < \frac{1}{\delta^{n+1}} \varepsilon$$

and

$$|\Phi_2^{n+1} - \Phi_2^n| = \left|\frac{Q_{n+1}}{d^{n+1}} \frac{\gamma_n}{\delta^{n+1} d^{n+1}} P_{n+1} - 1\right| = \left|\frac{Q_{n+1}}{d^{n+1}} - \frac{\gamma_n P_n}{\delta^n d^n}\right|.$$
\[
\begin{align*}
\frac{Q_{n+1}}{d^{n+1}} - \frac{\gamma P_n}{d^{n+1}} - \frac{Q_n}{d^n} + \frac{\gamma_{n+1} P_{n+1}}{\delta^{n+1} d^{n+1}} - \frac{\gamma_n P_n}{\delta^n d^n} - \frac{\gamma P_n}{d^{n+1}} &= \bigg| \frac{Q_{n+1}}{d^{n+1}} - (\gamma P_n + dQ_n) \bigg| \\
&= \frac{\gamma_{n+1}|P_{n+1} - \delta P_n|}{\delta^{n+1} d^{n+1}} < \frac{1}{d^{n+1}} \varepsilon + \frac{\gamma_{n+1}}{\delta^{n+1} d^{n+1}} \varepsilon.
\end{align*}
\]

Hence \( \Phi_n \) converges uniformly to \( \Phi \). In particular, if \( \delta \neq d \), then
\[
|\Phi - id| < \max \left\{ \frac{1}{\delta - 1}, \frac{1}{d - 1} + \frac{\gamma}{\delta - d} \left( \frac{1}{d - 1} - \frac{1}{\delta - 1} \right) \right\} \varepsilon.
\]

If \( \delta = d \), then \( \gamma = 0 \) and so \( |\Phi - id| < \max\{(\delta - 1)^{-1}, (d - 1)^{-1}\} \varepsilon \). By the inequality \( |e^z - e^w| < |z - w|e^{\min\{z,w\}} \), the uniform convergence of \( \Phi_n \) translates into that of \( \phi_n \). Therefore, \( \phi \) is holomorphic on \( U_r - \{ z = 0 \} \), which extends to \( U_r \) by Riemann’s removable singularity theorem. In particular, if \( |\Phi - id| < \varepsilon \), then \( |\phi - id| < \varepsilon e^\varepsilon |id| \). Therefore, \( \phi \sim id \) on \( U_r \) as \( r \to 0 \).

4. Biholomorphicity of \( \phi \) for the case \( d \geq 2 \)

We continue the proof of Theorem 1. In the previous section we showed that \( \phi \) is well-defined and so holomorphic on \( U_r \). However, unlike the one-dimensional case, the biholomorphicity of \( \phi \) does not follow immediately because the domain \( U_r \) may not be a neighborhood of the origin. In this section we prove that \( \phi \) is actually biholomorphic on \( U_r \), shrinking \( r \) from that of Lemma 1. More precisely, the property \( \phi \sim id \) suggests the biholomorphicity of \( \phi \), which is ensured by the following Rouché’s theorem.

**Theorem 3** (Rouché). Let \( g \) and \( h \) be holomorphic functions on a simply connected region \( D \). Let \( \Gamma \) be a smooth, simply closed curve in \( D \). If \( |g - h| < |h| \) on \( \Gamma \), then the numbers of zero points of \( g \) and \( h \) are the same in the region surrounded by \( \Gamma \).

Let \( \alpha > 0 \) and \( \phi = (\phi_1, \phi_2) \). We may assume that the function \( \phi_1 \) in \( z \) is conformal for simplicity, because it is well-defined at the origin. Let us fix small \( \varepsilon, r_1 \) and \( r_2 \) such that \( |\varepsilon| = |q/z^\gamma w^d - 1| < \varepsilon \) on \( U_r \) and \( f(U_r) \subset U_r \), where \( U_r = \{|z| < r_1|w|^\alpha, |w| < r_2\} \). Then \( |F - F_0| < \log(1 + \varepsilon) \) on \( \pi^{-1}(U_r) \), where \( F \) is the lift of \( f \) by \( \pi(Z, W) = (e^Z, e^W) \) and \( \pi^{-1}(U_r) = \{ \text{Re}(Z - \alpha W) < \log r_1, \text{Re}W < \log r_2 \} \). We know that the lift \( \Phi \) of \( \phi \) is well-defined and holomorphic on \( \pi^{-1}(U_r) \). Let \( \Phi(Z, W) = (\phi_1(Z), \phi_2(W)) \). The conformality of \( \phi_1 \) derives that of \( \Phi_1 \) because \( \Phi_1 \sim id \). We prove the conformality of \( \Phi_2 \) in Proposition 1 below; then the biholomorphicity of \( \Phi \) derives that of \( \phi \) because \( \Phi \sim id \). Recall that \( |\Phi_2 - id| < C\varepsilon \), where \( \varepsilon = \log(1 + \varepsilon) \) and
\[
C = \frac{1}{d-1} + \frac{\gamma}{\delta - d} \left( \frac{1}{d-1} - \frac{1}{\delta - 1} \right) \quad \text{or} \quad \tilde{C} = \frac{1}{d-1}
\]
if \( \delta \neq d \) or \( \delta = d \). Let \( V_Z = V \cup C_Z \) and \( V'_Z = V' \cup C_Z \), where

\[
\begin{align*}
V &= \pi^{-1}(U_r) = \left\{ \frac{\Re Z}{\alpha} - \frac{\log r_1}{\alpha} < \Re W < \log r_2 \right\} \quad \text{and} \\
V' &= \left\{ \frac{\Re Z}{\alpha} - \frac{\log r_1}{\alpha} + 2C\varepsilon < \Re W < \log r_2 - 2C\varepsilon \right\} \subset V.
\end{align*}
\]

**Proposition 1.** Let \( \alpha > 0 \). Then \( \Phi_Z \) is conformal on \( V'_Z \) for any fixed \( Z \).

*Proof.* Let \( W_1 \) and \( W_2 \) be two points in \( V'_Z \) such that \( \Phi_Z(W_1) = \Phi_Z(W_2) \), and show that \( W_1 = W_2 \). Define \( g(W) = \Phi_Z(W) - \Phi_Z(W_1) \) and \( h(W) = W - \Phi_Z(W_1) \). Then \( |g - h| = |\Phi_Z - id| < C\varepsilon \) on \( V_Z \). By the definition of \( V_Z \) and \( V'_Z \), there is a smooth, simply closed curve \( \Gamma \) in \( V_Z \) whose distances from \( W_1 \) and \( W_2 \) are greater than \( C\varepsilon \). Hence \( |h| \geq \text{dist}(\Phi_Z(\Gamma), \delta V_Z) \geq 2C\varepsilon - C\varepsilon = C\varepsilon \) on \( \Gamma \). Therefore, \( |g - h| < |h| \) on \( \Gamma \). Rouché’s theorem implies that the number of zero points of \( g \) is exactly one in the region surrounded by \( \Gamma \); thus \( W_1 = W_2 \). \( \square \)

**Proposition 2.** Let \( \alpha > 0 \). Then \( \phi \) is biholomorphic on

\[
\left\{ \frac{|z|}{|w|^\alpha} < \frac{r_1}{(1 + \varepsilon)^{2\alpha C}}, \quad |w| < \frac{r_2}{(1 + \varepsilon)^{2C}} \right\}.
\]

*Proof.* By Proposition\[1\] \( \Phi \) is biholomorphic on \( V' \). Hence \( \phi \) is biholomorphic on \( \pi(V') \) because \( \Phi \sim id \), where \( \pi(V') = |z/w^\alpha| < r_1', |w| < r_2' \) for some constants \( r_1' \) and \( r_2' \). Indeed, \( r_1' = r_1/(1 + \varepsilon)^{2\alpha C} \) and \( r_2' = r_2/(1 + \varepsilon)^{2C} \) since \( (\log r_1')/\alpha = (\log r_1)/\alpha - 2C\varepsilon \) and \( \log r_2' = \log r_2 - 2C\varepsilon \). \( \square \)

**Remark 1.** By similar arguments, it follows that \( F \) is biholomorphic on

\[
\left\{ \frac{\Re Z}{\alpha} - \frac{\log r_1}{\alpha} + \frac{2\varepsilon}{d} < \Re W < \log r_2 - \frac{2\varepsilon}{d} \right\}.
\]

Hence \( F^n, F_{\lambda}^{-n} \circ F^n \) and \( \Phi \) are biholomorphic on the same region. This region is bigger than \( V' \) since \( C \geq 1/(d - 1) > 1/d \). Therefore, we have a bigger region that ensures the biholomorphicity of \( \phi \).

5. The case \( d = 1 \)

We extend our ideas and results for the case \( d \geq 2 \) to the case \( d = 1 \); we prove Lemma\[2\] and Theorem\[2\]. The proof of the uniform convergence of \( \phi_n \) is different from the previous case. At the end of this section we exhibit an example that does not satisfy the condition \( \alpha < (\delta - 1)/\gamma \).

Let us give an outline of the proof of Lemma\[2\]. As in Lemma\[1\] if \( \alpha \) is well-defined, then \( f(z, w) \sim (z^\delta, b^\gamma w) \) on \( U_r \) as \( r \to 0 \). On the other hand, the invariance of \( U_r \) does not follow from the same argument. Recall that
Proposition 3. \(|p/q^\alpha| \sim |c|^{\delta - \alpha \gamma}w^{\alpha - \alpha \gamma + d}\) on \(U_r\), where \(|c| = |z/w^\alpha|\), and that \(\delta - \alpha \gamma \geq \min\{\delta, d\} \geq 2\) if \(d \geq 2\). However, \(\delta - \alpha \gamma\) can be 1 if \(d = 1\). To obtain the invariance of \(U_r\), we add the condition \(\delta - \alpha \gamma > 1\), which is equivalent to \(\alpha < (\delta - 1)/\gamma\).

For the proof of the uniform convergence of \(\phi_n\), we cannot use the same argument as the case \(d \geq 2\); the sum of \(d^{-n}\) does not converge anymore. Since the investigation of the second components of maps is the essential part for proofs, we sometimes omit the expressions of the first components hereafter. Recall that \(|Q(Z) - Q_0(Z)| = |\log(1 + \xi^Z)|\). Since we may assume that \(|\eta| < 1\), it follows that

\[|Q - Q_0| \leq \log(1 + |\eta|) \leq |\eta|\]

and so \(|Q(F^n) - Q_0(F^n)| \leq |\eta(F^n)|\).

We prove the uniform convergence of \(\phi_n\) by estimating \(|\eta(F^n)|\), which is equal to \(|\eta(F^n)|\), appropriately. First, note that \(f^n\) contracts a small bidisk rapidly as follows.

**Lemma 4.** Let \(d = 1\). If \(\alpha\) is well-defined and \(\alpha < (\delta - 1)/\gamma\), then

\(f^n(\{|z| < r, |w| < r\}) \subset \{|z| < r/2^n, |w| < r/2^n\}\).

**Proof.** Since the origin is superattracting, we may assume that \(|p| < c_1 r^2\) and \(|q| < c_2 r^2\) on \(\{|z| < r, |w| < r\}\) for some constants \(c_1\) and \(c_2\), taking \(r\) small enough. Hence the inclusion relation holds if \(r < \min\{c_1^{-1}, c_2^{-1}\}/2\). \(\square\)

**Remark 2.** Restricting on \(U_r\), we can obtain the following sharper estimate:

\(f^n(U(r/B, r/B) \cap U_r) \subset U(r^{\delta_n}/B, r^{\gamma_n+1}/B) \cap U_r\),

where \(U(r_1, r_2) = \{|z| < r_1, |w| < r_2\}\) and \(B = 2 \max(|b|, 1)\). In particular,

\(f^n(U(r/B, r/B) \cap U_r) \subset U(r^{2^n}/B, r^{2^n}/B) \cap U_r\).

Lemma 4 derives the uniform estimate of \(|\eta(f^n)|\) on \(U_r\).

**Lemma 5.** Let \(d = 1\). If \(\alpha\) is well-defined and \(\alpha < (\delta - 1)/\gamma\), then

\(|\zeta(p^n)| \leq C_1 r/2^n\) and \(|\eta(f^n)| \leq C_2 r/2^n\) on \(U_r\) for some constants \(C_1\) and \(C_2\).

**Proof.** Since \(\eta(z, w) = c_1 z + c_2 w + \sum c_i z^{n_i} w^{m_i}\), where \(n_i + m_i \geq 2\), it follows from Lemma 4 that \(|\eta(f^n)| \leq |c_1| r/2^n + |c_2| r/2^n + |c_3| r/2^n = (|c_1| + |c_2| + |c_3|) r/2^n\) on \(U_r\). \(\square\)

Now we are ready to prove the uniform convergence of \(\phi_n\).

**Proposition 3.** Let \(d = 1\). If \(\alpha\) is well-defined and \(\alpha < (\delta - 1)/\gamma\), then \(\phi_n\) converges uniformly to \(\phi\) on \(U_r\), and \(\phi \sim \text{id} on U_r\) as \(r \to 0\).
Proof. It is enough to show the uniform convergence of \( \Phi_n^2 \). By Lemma 5,

\[
|\Phi_{n+1}^2 - \Phi_n^2| \leq \frac{|Q(F^n) - Q_0(F^n)|}{d^{n+1}} + \frac{\gamma_n+1|P(P^n) - P_0(P^n)|}{\delta^{n+1}d^{n+1}}.
\]

\[\leq |\eta(F^n)| + \frac{\gamma}{\delta - 1}|\xi(P^n)| < \left( C_2 + \frac{\gamma}{\delta - 1}C_1 \right) \frac{r}{2^n}.
\]

□

The proof of the biholomorphicity of \( \phi \) is the same as the case \( d \geq 2 \).

**Proposition 4.** Let \( d = 1 \). If \( \alpha \) is well-defined and \( \alpha < (\delta - 1)/\gamma \), then \( \phi \) is biholomorphic on \( U_r \).

If \( \alpha \) is well-defined, then \( \alpha \leq (\delta - 1)/\gamma \) by definition. Hence the case \( \alpha = (\delta - 1)/\gamma \) is left. We exhibit such an example, which suggests that the left case is not the same as the case \( \alpha < (\delta - 1)/\gamma \).

**Example 1.** Let \( f(z, w) = (z^2, \lambda zw + z^2) \). Then \( \alpha = (\delta - 1)/\gamma = 1 \).

(i) If \( \lambda = 1 \), then \( f \) is semiconjugate to \( g(z, w) = (z^2, w + 1) \) by
\[
\pi(z, w) = (z, zw) \colon \pi \circ g = f \circ \pi.
\]

(ii) If \( \lambda \neq 1 \), then \( f \) is semiconjugate to \( g(z, w) = (z^2, \lambda w + 1) \) by
\[
\pi(z, w) = (z, zw) \colon \pi \circ g = f \circ \pi. \quad \text{Moreover,} \quad f \text{ is conjugate to} \quad \tilde{f}(z, w) = (z^2, \lambda zw) \text{ by } h_f, \quad \text{and} \quad g \text{ is conjugate to} \quad \tilde{g}(z, w) = (z^2, \lambda w) \text{ by } g_h, \quad \text{where} \quad h_f(z, w) = (z, w + z/(1 - \lambda)) \quad \text{and} \quad h_g(z, w) = (z, w + 1/(1 - \lambda)).
\]

The case (i) shows the parabolic phenomena, and \( f \) does not seem to be conjugate to \( (z, w) \to (z^2, zw) \). Although \( f \) is conjugate to \( \tilde{f} \) for the case (ii), the dynamics seems to be different from our case; in particular, the invariance of \( U_r \) does not hold if \( |\lambda| < 1 \).

6. A generalization to holomorphic germs

Until now we have dealt with a holomorphic germ of the form \( f(z, w) = (p(z), q(z, w)) \) such that \( p(z) = z^\delta + a_{\delta+1}z^{\delta+1} + \cdots \) and
\[
q(z, w) = bw^\gamma w^d + \sum b_jw^{n_j}w^{m_j},
\]
where \( b \neq 0, \gamma \leq n_j, \) and \( d < m_j \) if \( \gamma = n_j \). Since the origin is a super-attracting fixed point, \( \delta \geq 2, \gamma + d \geq 2 \) and \( n_j + m_j \geq 2 \). In this section we perturb \( p \) to a holomorphic germ \( \tilde{p} \) in \( z \) and \( w \) such that \( \tilde{p}(z, w) = a(w)z^\delta + a_{\delta+1}(w)z^{\delta+1} + \cdots \), where \( a(0) = 1 \). In other words,
\[
\tilde{p}(z, w) = z^\delta + \sum a_iw^{n_i}w^{m_i},
\]
where \( n_i \geq \delta \), and \( m_i \geq 1 \) if \( n_i = \delta \). Let \( \tilde{f}(z, w) = (\tilde{p}(z, w), q(z, w)) \).
Let us explain that we need not to change the definition of $\alpha$. For $|c| = |z/w^d|$, it follows that

$$\frac{|c^{n_i}w^{m_i}|}{|z|^{\alpha}} = \frac{|(cW^d)^{n_i}w^{m_i}|}{(cW^d)^{\alpha}} = \frac{|c^{n_i}w^{am_i}+m_i|}{c^{\delta W^{d\delta}}} = |c|^{n_i-\delta}|w|^{am_i+m_i-a\delta}.$$ 

The condition $an_i + m_i \geq a\delta$ ensures that $\tilde{p}(z,w) \sim z^{\delta}$ on $U_\alpha$ as $r$ tends to 0. However, this condition is already included in the condition $a \geq 0$ in the definition of $\alpha$, since it is equivalent to $a \geq -m_i/(n_i - \delta)$ if $n_i > \delta$. We remark that the interval $I_\tilde{f}$ may differ whether we add the condition above. Consequently, we have the following lemma without changing the definition of $\alpha$.

**Lemma 6.** Let $\alpha$ be well-defined. If $d \geq 2$ or if $d = 1$ and $\alpha < (\delta - 1)/\gamma$, then $\tilde{f}(z,w) \sim (z^{\delta}, b\gamma w^d)$ on $U_\alpha$ as $r \to 0$, and $\tilde{f}(U_\alpha) \subset U_r$ for small $r$.

This lemma induces the existence of the limit of the compositions of $f_i^{\sim n}$ and $\tilde{f}^n$ as previous cases, where $f_i^{\sim n}(z,w) = (z^{\delta}, b\gamma w^d)$.

**Theorem 4.** Let $\alpha$ be well-defined. If $d \geq 2$ or if $d = 1$ and $\alpha < (\delta - 1)/\gamma$, then there is a biholomorphic map $\phi$ defined on $U_\alpha$, with $\phi \sim id$ on $U_\alpha$ as $r \to 0$, that conjugates $\tilde{f}$ to $(z, w) \to (z^{\delta}, b\gamma w^d)$.

The proof of the existence of $\phi$ is similar to the skew product case. The difficult part of the proof is the biholomorphicity of $\phi$; the rest of the paper is used to prove it. Since $\phi$ is clearly biholomorphic if $\alpha = 0$, we may assume that $\alpha > 0$ hereafter.

Let us state the idea of the proof. As in Section 4, we prove that the lift $\Phi$ of $\phi$ is biholomorphic, which implies the biholomorphicity of $\phi$ because $\Phi \sim id$. For the skew product case, we applied Rouché’s theorem to $\Phi$ restricted to a vertical line in order to show that $\Phi_Z$ is conformal, where $\Phi = (\Phi_1, \Phi_Z)$. Since we may assume that $\Phi_1$ is conformal, this implies that $\Phi$ is biholomorphi‌c. On the other hand, for the map $\tilde{f}$, we apply Rouché’s theorem to $\Phi$ restricted to a line, which may not be vertical, as follows. Let $\Phi$ be well-defined and holomorphic on $V$, and take a sufficiently small region $V'$ in $V$. Let $w_1$ and $w_2$ be two points in $V'$ such that $\Phi(w_1) = \Phi(w_2)$. Applying Rouché’s theorem to $\Phi$ restricted to the intersection of $V$ and the line $L$ passing through $w_1$ and $w_2$, we can show that $w_1 = w_2$.

The point is taking a smaller region $V'$ in $V$ such that $L \cap (V - V')$ has a suitable width for any line $L$ intersecting $V'$, as in Section 4. Recall that

$$V = \left\{ \frac{\text{Re} Z}{\alpha} - \frac{\log r_1}{\alpha} < \text{Re} W < \log r_2 \right\},$$
and let $|\Phi - id| < \varepsilon$. Then the following region is what we need:

$$V' = \left\{ \frac{\text{Re}Z}{\alpha} - \frac{\log r_1}{\alpha} + \frac{1 + \alpha}{\alpha} \cdot 2\varepsilon < \text{Re}W < \log r_2 - 2\varepsilon \right\}.$$ 

Let us illustrate where the constant $(1 + \alpha) / \alpha$ comes from. First, consider everything in $\mathbb{R}^2$. Let $L = \{y = mx\}$, $V = \{y > x/\alpha\}$ and $V' = \{y > x/\alpha + R \cdot 2\varepsilon\}$ for a constant $R$, where $(x, y) \in \mathbb{R}^2$ and $m \in \mathbb{R}$. If $|m| \geq 1$, then we take the projection $\pi_2$ to the second coordinate, and require that the length of the interval $\pi_2(L \cap (V - V'))$ in $\mathbb{R}$ is greater than or equal to $2\varepsilon$. It is enough to consider the case $m = -1$, since the length takes the minimum for this case. By an elementary calculation in terms of two right-angled triangles, it follows that, if $R = 1 + 1 / \alpha$, then the length coincides with $2\varepsilon$. If $|m| \leq 1$, then we take the projection $\pi_1$ to the first coordinate. By the same argument, it follows that, if $R = 1 + 1 / \alpha$, then the length of $\pi_1(L \cap (V - V'))$ is greater than or equal to $2\varepsilon$. This sketch works for complex setting as well:

**Lemma 7.** Let $L$ be a line $\{W = mZ + n\}$ which intersects $V'$. Then

$$\text{dist}(\pi_1^{-1}(L \cap V'), \partial \pi_1^{-1}(L \cap V)) \geq 2\varepsilon \text{ if } |m| \leq 1,$$

and

$$\text{dist}(\pi_2^{-1}(L \cap V'), \partial \pi_2^{-1}(L \cap V)) \geq 2\varepsilon \text{ if } |m| \geq 1,$$

where $\pi_1$ and $\pi_2$ are the projections to Z and W coordinates, respectively.

**Proof.** Let $n = 0$ for simplicity. We only prove the case $|m| \geq 1$. Note that

$$\pi_2^{-1}(L \cap V') = H \cap \left\{ \text{Re}W < \frac{1}{\alpha} \frac{W}{m} - \frac{\log r_1}{\alpha} + \frac{1 + \alpha}{\alpha} \cdot 2\varepsilon \right\}$$

$$= H \cap \{ \text{Re}((\alpha - 1/m)W) < -\log r_1 + (1 + \alpha)2\varepsilon \},$$

where $H = \{ \text{Re}W < \log r_2 - 2\varepsilon \}$. It is enough to show that $\text{dist}(l_0, l_{\varepsilon}) \geq 2\varepsilon$, where $l_0 : \{ \text{Re}((\alpha - 1/m)W) = 0 \}$ and $l_{\varepsilon} : \{ \text{Re}((\alpha - 1/m)W) = (1 + \alpha)2\varepsilon \}$. Actually, we have

$$\text{dist}(l_0, l_{\varepsilon}) = \frac{(1 + \alpha)2\varepsilon}{|\alpha - 1/m|} \geq 2\varepsilon \text{ since } |\alpha - \frac{1}{m}| \leq \alpha + \frac{1}{|m|} \leq \alpha + 1.$$ 

Now we are ready to prove the biholomorphicity of $\Phi$.

**Proposition 5.** The map $\Phi$ is biholomorphic on $V'$.

**Proof.** Let $\Phi(w_1) = \Phi(w_2)$ for points $w_1$ and $w_2$ in $V'$. Let $L$ be the line passing through $w_1$ and $w_2$. It is enough to consider the case $L =$
\{W = mZ + n\}. Define \( \tilde{\Phi}_1 \) and \( \tilde{\Phi}_2 \), where 
\[ u(Z) = (Z, mZ + n) \quad \text{and} \quad v(W) = (W/m, W + n) \] 

It then follows from Lemma 7 that \( w_1 = w_2 \), by applying Rouché’s theorem to \( \tilde{\Phi}_1 \) or \( \tilde{\Phi}_2 \) if \( |m| \leq 1 \) or \( |m| \geq 1 \) as in Proposition 1 in Section 4. 

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Daido University, Nagoya 457-8530, Japan
E-mail address: k-ueno@daido-it.ac.jp