ON THE PERIODIC NAVIER–STOKES EQUATION: AN ELEMENTARY APPROACH TO EXISTENCE AND SMOOTHNESS FOR ALL DIMENSIONS $n \geq 2$.

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Abstract. In this paper we study the periodic Navier–Stokes equation. From the periodic Navier–Stokes equation and the linear equation $\partial_t u = \nu \Delta u + P[v \nabla u]$ we derive the corresponding equations for the time dependent Fourier coefficients $a_k(t)$. We prove the existence of a smooth solution $u$ of the linear equation by a Montel space version of Arzelà–Ascoli. We gain bounds on the $a_k$’s of $u$ depending on $v$. These bounds provide the small time existence of a unique smooth solution of the Navier–Stokes equation. They prove that if all first derivatives of the solution are bounded for all times $t \in [0, T]$, then the solution is smooth for all $t \in [0, T]$. We prove that the Navier–Stokes equation with small initial data, e.g. when $\|u_0\|_A + \sqrt{n} \cdot (\|\partial_1 u_0\|_A + \cdots + \|\partial_n u_0\|_A) \leq \nu$, has a unique smooth solution for all times $t \geq 0$. All results hold for all dimensions $n \geq 2$.

CONTENTS

1. Introduction 1  
2. A simple one-dimensional initial value problem where $\nu\Delta$ removes a finite blowup time for any $\nu > 0$ 4  
3. Preliminaries: Periodic formulation with the Leray projection 6  
4. Solutions and Bounds of $\partial_t u = \nu \Delta u + P[v \nabla u]$ with $\nu \geq 0$ and simple $v$ 9  
5. Solutions and Bounds of $\partial_t u = \nu \Delta u + P[v \nabla u]$ with $\nu > 0$ 13  
6. Solutions and bounds of the Navier–Stokes equations 22  
7. Conclusion 25  
References 26  
Appendix A. Bessel function calculations of Theorem 4.4 27

1. INTRODUCTION

The dynamics of (incompressible) fluids on $\mathbb{R}^n$ or in the periodic case on $\mathbb{T}^n$ (with $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ and $n = 2, 3$) is described by the Euler ($\nu = 0$) and Navier–Stokes ($\nu > 0$) equations

\begin{align}
\partial_t u(x, t) &= \nu \Delta u(x, t) - u \cdot \nabla u(x, t) - \nabla p(x, t) + F(x, t) \\
\text{div } u(x, t) &= 0
\end{align}

with $x \in \mathbb{R}^n$ or $\mathbb{T}^n$, $t \geq t_0$ (w.l.o.g. $t_0 = 0$), and initial conditions

$u(x, t_0) = u_0(x)$. 

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The velocity field of the fluid, $u(x,t) = (u_1(x_1, \ldots, x_n, t), \ldots, u_n(x_1, \ldots, x_n, t))^T$ is the velocity field, $p(x,t)$ is the pressure, and $F(x,t) = (F_1(x_1, \ldots, x_n, t), \ldots, F_n(x_1, \ldots, x_n, t))^T$ are externally applied forces.

The Euler and Navier–Stokes equations belong to the most important partial differential equations in pure mathematics and applications. The literature about these equations is enormous, see e.g. [Lad63, MB02, BF13, LR16] and reference therein. Essentially the Navier–Stokes problem boils down to five aspects:

(A) **Local** (small time) *existence* of a smooth solution.
(B) **Regularity** (smoothness) from bounded first derivatives.
(C) **Global** (all time) *existence* of a smooth solution with small initial data.
(D) **Uniqueness** of a smooth solution.
(E) **Global existence** of a smooth solution.

The dynamic equation for a compressible or incompressible fluid was derived by Leonhard Euler [Eul57]. Claude-Louis Navier included viscosity [Nav27] and this equation has been rediscovered several times but remained controversial [LR16, p. 2]. George G. Stokes presented a more rigorous derivation [Sto49]. See e.g. [LR16, Lad03, Dar02] for more on the early history of these equations.

Local existence of a smooth solution (A) was proved by Carl W. Oseen [Ose11, Ose27]. In 1933 in his thesis [Ler33], Jean Leray completely solved the two-dimensional case, as Olga A. Ladyzhenskaya did put it, by a “happy circumstance [which] is valid only in the two-dimensional case and only for the Cauchy problem, for periodic boundary conditions, and for a certain special boundary condition” [Lad03, p. 271]. In 1934, Leray showed that Oseen’s local solution is unique (D), exists for all times if the initial data are sufficiently small (C), and for all initial data a turbulent (what nowadays is called a weak) solution for all times (weak E) exists [Ler34a, Ler34b] and therefore introducing the concept of weak solutions and what is now called the Sobolev space $H^1$. Starting in 1953, Ladyzhenskaya rigorously introduced and investigated weak solutions for partial differential equations [Lad53]. Eberhard Hopf in 1951 [Hop51] and Andrei A. Kiselev and Ladyzhenskaya in 1957 [KL57] used the Galerkin (Fadeo–Galerkin) method for these non-linear equations. In 1969, Ladyzhenskaya finally proved the non-uniqueness of weak (Hopf) solutions [Lad69]. In her review paper she summarizes the development as ‘seeking solutions of variational problems in spaces dictated by the functional rather than in spaces of smooth functions’ [Lad03, p. 253]. Since then the number of function (Sobolev) spaces used or even introduced to treat the Navier–Stokes equation exploded, see e.g. [LR16] for a large collection and discussion of such spaces and results. In 1984, J. Thomas Beale, Tosio Kato, and Andrew J. Majda proved the result “that the maximum norm of the vorticity $\omega(x,t) := \text{rot } u(x,t)$ controls the breakdown of smooth solutions of the 3-D-Euler equations” [BKM84]. Uniqueness (D) of a (local or global) smooth solution on $\mathbb{R}^n$ or $\mathbb{T}^n$, $n \geq 2$, is established by energy estimates, see e.g. [MB02, Section 3.1.1 and Prop. 3.1].

In summary, (A) to (E) are completely solved for the two-dimensional case (on $\mathbb{R}^2$ and $\mathbb{T}^2$) and (A) to (D) are solved for the $n$-dimensional case ($n \geq 3$, on $\mathbb{R}^n$ and $\mathbb{T}^n$). However, all these results and introduced function spaces did not provide a solution for (E) with $n \geq 3$, not even in the simplest case on $\mathbb{R}^3$ or $\mathbb{T}^3$ without external forces ($F = 0$). Finally, proving or disproving (by counterexample) that the Navier–Stokes equation on $\mathbb{R}^3$ or $\mathbb{T}^3$ with $F = 0$ has a smooth solution $u(x,t)$ with bounded energy

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \, d^n x < C \quad \text{resp.} \quad \int_{\mathbb{T}^n} |u(x,t)|^2 \, d^n x < C$$

for all $t \geq 0$ even became one of the seven millennium problems [Fef06].
Recently an example was given where the Euler equation breaks down in finite time if the initial condition \(u_0\) belongs only in the Hölder space \(C^{1,\alpha}(\mathbb{R}^3)\). In [EG19, EGM19] the initial value \(u_0\) shall be “physically reasonable”, i.e., \(u_0 \in \mathcal{S}(\mathbb{R}^3)\) a Schwartz function or \(u_0 \in C^\infty(\mathbb{R}^3)^3\) a smooth periodic function. In [Bra04, Prop. 3.1] Lorenzo Brandolese showed that the vorticity \(\omega(x,t) := \text{rot} u(x,t)\) of the (local) smooth solution of the Navier–Stokes equation remains a Schwartz function for small times. In [D19] we treated the question how a Schwartz function valued vorticity solution of the Euler or Navier–Stokes equation stops being a Schwartz function. The difference between other works (which work in or with function spaces like Sobolev spaces) is that we worked solely with the fact that the Schwartz function space \(\mathcal{S}(\mathbb{R}^n)\) and the space of smooth periodic functions \(C^\infty(\mathbb{T}^n)\) are complete Montel spaces. I.e., bounding of one norm on a Sobolev space exploded into bounding all (ininitely but countably many) semi-norms \(|x^\alpha \cdot \partial^\beta f(x)|_{\infty}\), \(\alpha, \beta \in \mathbb{N}_0^n\). But in Montel spaces the Heine–Borel property still holds, i.e., bounded sets are pre-compact. Therefore, also the Arzelà–Ascoli Theorem still holds in Montel spaces. We used the following version which will also be the only technical tool in this study. We state it here only for \(C^\infty(\mathbb{T}^n \times [0,T], \mathbb{R}^n)\) functions, see e.g. [D19, Lem. 2.1] for the Schwartz and complex smooth periodic function statement including the full proof, verbatim the same as the original proof for continuous functions, see e.g. [Yos68] pp. 85–86.

**Lemma 1.1** (\(C^\infty(\mathbb{T}^n, \mathbb{R}^n)\)-Valued Version of Arzelà–Ascoli). Let \(n, m \in \mathbb{N}, T > 0\), and \(\{f_N\}_{N \in \mathbb{N}}\) be a family of functions \(f_N(\cdot, t) \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)\) for all \(t \in [0,T]\). Assume that

i) \(\sup_{N \in \mathbb{N}, t \in [0,T]} \|\partial_\nu f_N(x, t)\|_{\infty} < \infty\) for all \(\nu \in \mathbb{N}_0^n\), and

ii) \(\{f_N\}_{N \in \mathbb{N}}\) is equi-continuous, i.e., for all \(\varepsilon > 0\) exists \(\delta = \delta(\varepsilon) > 0\) such that for all \(N \in \mathbb{N}\) we have

\[|t - s| < \delta \implies \|f_N(x, t) - f_N(x, s)\|_{\infty} \leq \varepsilon.\]

Then \(\{f_N\}_{N \in \mathbb{N}}\) is relatively compact in \(C^\infty(\mathbb{T}^n \times [0,T], \mathbb{R}^n)\).

With the Arzelà–Ascoli Theorem we proved the following.

**Theorem 1.2** ([D19, Thm. 2.5]). Let \(n, m \in \mathbb{N}\) be natural numbers, \(\nu := (\nu_1, \ldots, \nu_n)\) be a tuple of reals with \(\nu_i \geq 0\), \(\nu \cdot \Delta := \nu_1 \partial^2_{x_1} + \cdots + \nu_n \partial^2_{x_n}\), and \(g_0, h_{i,j} : \mathbb{T}^n \times [0, \infty) \to \mathbb{R}\) be periodic functions such that \(\|\partial^\alpha g_0(x, t)\|_{\infty} < \infty\) and \(\|\partial^\alpha h_{i,j}(x, t)\|_{\infty} < \infty\) for all \(k = 1, \ldots, n\), \(i, j = 1, \ldots, m\), \(\alpha \in \mathbb{N}_0^n\) and \(t \in [0, \infty)\). Let \(k \in C([0, \infty), C^\infty(\mathbb{T}^n)^m)\). Set \(g(x, t) := (g_1(x, t), \ldots, g_n(x, t))^T\) and \(h(x, t) := (h_{i,j}(x, t))_{i,j=1}^m\). Then for any \(f_0 \in C^\infty(\mathbb{T}^n)^m\) the initial value problem

\[
\begin{align*}
\partial_t f(x, t) &= (\nu \cdot \Delta) f(x, t) + (g(x, t) \cdot \nabla) f(x, t) + h(x, t) \cdot f(x, t) + k(x, t) & (2a) \\
f(x, 0) &= f_0(x) & (2b)
\end{align*}
\]

has a solution \(f(\cdot, t) \in C^\infty(\mathbb{T}^n)^m\) for all \(t \in [0, \infty)\).

The important part in this theorem is that it contains explicit bounds on the semi-norms \(\|\partial^\alpha f(x, t)\|_{\infty}\), \(\alpha \in \mathbb{N}_0^n\), which allowed for conditions when a vorticity solution \(\omega(x, t) = \text{rot} u(x, t)\) of the Euler or Navier–Stokes equations leaves \(C^\infty(\mathbb{T}^n)^n\). The drawback was that we had to go to the vorticity formulation of the Euler and Navier–Stokes equations since no \(\| \cdot \|_{\infty}\)-estimates of the Leray projector is known (to us) and no relations of the \(\| \cdot \|_{\infty}\)-norm of the vorticity \(\omega\) compared to \(u\) are known (to us).

In the present study we will not study the vorticity formulation of the Euler and the Navier–Stokes equations but the “original” Navier–Stokes equation \(\square\) without external forces \((F = 0)\). To remove the pressure component in \(\square\) we use the well-known Hodge decomposition of a vector field, or to be more precise the
Leray projection $P$ which gives the div-free part of a vector field. The Euler and Navier–Stokes equations are then (with $v = -u$) the following:

$$
\partial_t u(x, t) = \nu \cdot \Delta u(x, t) + P [v(x, t) \cdot \nabla u(x, t)] \\
(3a)
$$

$$
u(x, t) \in C^\infty(T^n, \mathbb{R}^n) \text{ with } \nabla v(x, t) = 0 \text{ for all } t \in [0, \infty), \nu \cdot \Delta := \nu_1 \partial^2_{x_1} + \cdots + \nu_n \partial^2_{x_n}. \\
(3b)
$$

The paper is structured as follows. In Section 2 we give a simple one-dimensional example of (3), i.e., without the Leray projector $P$, with given $u_0$ and $v$ where the unique smooth solution breaks down in finite time for $\nu = 0$ but exists globally for any $\nu > 0$. In Section 3 we give the preliminaries, i.e., basic facts about the Montel space $C^\infty(T^n)$, Fourier coefficients of functions $f \in C^\infty(T^n)$, and translate (3) into a system of ordinary differential equations (10) of the Fourier coefficients of $u$ and $v$ (Theorem 3.4). In Section 4 we give analytic (exact) solution of (3) for simple functions $v$. In Section 5 these simple solutions are glued together in a Trotter type fashion [Tro59] and existence of a smooth solution of (3) is ensured by the Arzelà–Ascoli Theorem (Lemma 1.1). We gain explicit bounds on $T$ of the form (1) and existence of a smooth solution breaks down in finite time for any $\nu > 0$, with given $u_0$ and $v$ where the unique smooth solution breaks down in finite time for $\nu = 0$ but exists globally for any $\nu > 0$. In Section 6 these bounds provide (A) local existence and (B) regularity of a unique solution from bounded first derivatives (Theorem 6.1), and (C) global existence of a unique smooth solution for all times with small initial data (Theorem 6.3). In Section 7 we give some conclusions.

In summary, the aim of this paper is to show that (A) to (C) for the periodic Navier–Stokes equation can be proved for all $n \geq 2$ in a simple and unified way using only the Arzelà–Ascoli Theorem.

2. A SIMPLE ONE-DIMENSIONAL INITIAL VALUE PROBLEM WHERE $\nu \Delta$ REMOVES A FINITE BLOWUP TIME FOR ANY $\nu > 0$

In Theorem 1.2 it is essential that $g$ is a real function resp. the partial differential equation has the form (2). If this is not the case, then the existence of a smooth solution can no longer be ensured for all times $t \geq 0$.

Let $\nu \geq 0$. Let us look at the initial value problem

$$
\begin{align*}
\frac{\partial f_1(x, t)}{\partial t} - \partial_x f_2(x, t) &= \nu \left( \cos x \cdot \sin x \right) + \left( \cos x \cdot \sin x \right) \frac{\partial f_1(x, t)}{\partial x} + \left( \cos x \cdot \sin x \right) \frac{\partial f_2(x, t)}{\partial x} \\
(4a)
\end{align*}
$$

on $T \times [0, \tau^*)$. By the Euler formula $e^{ix} = \cos x + i \cdot \sin x$ the initial value problem (4) is equivalent to the complex-valued $(f = f_1 + i \cdot f_2)$ initial value problem

$$
\begin{align*}
\partial_t f(x, t) &= \nu \cdot \partial_x^2 f(x, t) + e^{ix} \cdot \partial_x f(x, t) \\
f(x, 0) &= e^{ix} \\
(5a)
\end{align*}
$$

on $T \times [0, \tau^*)$, $\tau^* > 0$. Both equations are not covered by Theorem 1.2. In fact, the following example shows that for $\nu = 0$ (4) resp. (5) have a unique smooth solution which breaks down in finite time (at $\tau^* = 1$).

Example 2.1. The initial value problem (5) with $\nu = 0$ has the unique solution

$$
\begin{align*}
f(x, t) &= \sum_{k \in \mathbb{N}} a_k(t) \cdot e^{ikx} \\
a_k(t) &= e^{ikx} \\
\end{align*}
$$

i.e., $f(\cdot, t) \in C^\infty(T, \mathbb{C})$ for all $t \in [0, 1)$ but it breaks down at $t^* = 1$. 

In fact, we have \( \|f\| \), i.e., the time-dependent coefficients \( a_k(t) \) fulfill for \( k = 1 \)
\[ \dot{a}_1(t) = 0 \quad \text{with} \quad a_1(0) = 1 \quad \Rightarrow \quad a_1(t) = 1, \]
and for \( k \geq 2 \)
\[ \dot{a}_k(t) = i \cdot (k - 1) \cdot a_{k-1}(t) \quad \text{with} \quad a_k(0) = 0 \quad \Rightarrow \quad a_k(t) = i^{k-1} \cdot t^{k-1}. \]
Hence, we have the solution
\[ f(x, t) = \sum_{k \in \mathbb{N}} a_k(t) \cdot e^{i \cdot k \cdot x}, \]
and therefore \( f(x, t) \in C^\infty([0, 2\pi], \mathbb{C}) \) for all \( t \in [0, 1) \) but breaks down at \( t^* = 1 \).
In fact, we have \( \|f(x, t)\|_A = \frac{1}{1 + t}. \]
So we see that \( 5 \) with \( \nu = 0 \) breaks down in finite time \( t^* = 1 \). But the following example reveals that with \( \nu > 0 \) the solution of \( 5 \) never breaks down \( (t^* = \infty) \).

**Example 2.2.** The initial value problem \( 4 \) with \( \nu > 0 \) has the unique solution
\[ f(x, t) = \sum_{k \in \mathbb{N}} a_k(t) \cdot e^{i \cdot k \cdot x}, \quad \text{with} \quad |a_k(t)| \leq \frac{2 \cdot e^{-\nu t}}{\nu^{k-1} \cdot (k+1)!}, \]
i.e., \( f(\cdot, t) \in C^\infty(\Omega, \mathbb{C}) \) for all \( t \in [0, \infty) \).

**Proof.** With the Ansatz \( 6 \) we get
\begin{align*}
\sum_{k \in \mathbb{N}} \dot{a}_k(t) \cdot e^{i \cdot k \cdot x} &= \sum_{k \in \mathbb{N}} \left[ -\nu \cdot k^2 \cdot a_k(t) \cdot e^{i \cdot k \cdot x} + i \cdot k \cdot a_k(t) \cdot e^{i \cdot (k+1) \cdot x} \right]
\end{align*}
and therefore the time-dependent coefficients \( a_k(t) \) fulfill for \( k = 1 \)
\[ \dot{a}_1(t) = -\nu \cdot a_1(t) \quad \text{with} \quad a_1(0) = 1 \quad \Rightarrow \quad a_1(t) = e^{-\nu t}, \]
and for \( k \geq 2 \) we have
\[ \dot{a}_k(t) = -\nu \cdot k^2 \cdot a_k(t) + i \cdot (k - 1) \cdot a_{k-1}(t) \quad \text{with} \quad a_k(0) = 0 \]
which provides the induction
\begin{align*}
a_k(t) &= i \cdot (k - 1) \cdot \int_0^t a_{k-1}(s) \cdot e^{\nu \cdot s^2 \cdot s} \cdot e^{-\nu \cdot k^2 \cdot t}, \quad \text{with} \quad a_k(0) = 0 \quad \text{for} \quad k \in \mathbb{N}, \quad \text{define} \quad \nu \in \mathbb{R}^+.
\end{align*}
Hence, we have \( a_1(t) \geq 0 \) and \( 7 \) also implies \( i^{-k+1} \cdot a_k(t) \geq 0 \) for all \( k \in \mathbb{N} \). For \( k \in \mathbb{N} \) define
\[ a_k^*(t) := \frac{2 \cdot i^{k-1}}{\nu^{k-1} \cdot (k+1)!} \cdot e^{-\nu \cdot t}. \]
Then
\[ a_k^*(t) \geq \frac{2 \cdot i^{k-1}}{\nu^{k-1} \cdot (k+1)!} \cdot e^{-\nu \cdot t} = a_1(t), \]
i.e.,
\[ i^{-k+1} \cdot a_k^*(t) \geq i^{-k+1} \cdot a_k^*(t) \geq 0 \quad \text{(8)} \]
holds for \( k = 1 \) and all \( t \geq 0 \). Assume \( 8 \) holds for a \( k \in \mathbb{N} \) and all \( t \geq 0 \). Then
\[ i^{-k} \cdot a_{k+1}(t) = i^{-k} \cdot i \cdot k \cdot \int_0^t a_k(s) \cdot e^{\nu \cdot (k+1) \cdot s^2} \cdot ds \cdot e^{-\nu \cdot (k+1) \cdot t} \]
\[ k \cdot \int_0^t \int_{i-k+1} \cdot a_k(s) \cdot e^{\nu (k+1)^2} \cdot ds \cdot e^{-\nu (k+1)^2} \cdot t \]
\[ \leq k \cdot \int_0^t \int_{i-k+1} \cdot a_k(s) \cdot e^{\nu (k+1)^2} \cdot ds \cdot e^{-\nu (k+1)^2} \cdot t \]
\[ = k \cdot \int_0^t \int_{i-k+1} \cdot \frac{2 \cdot i^{-k-1}}{\nu (k+1)!} \cdot e^{-\nu \cdot s} \cdot e^{\nu (k+1)^2} \cdot ds \cdot e^{-\nu (k+1)^2} \cdot t \]
\[ = \frac{2}{\nu (k+1)!} \cdot \int_0^t e^{\nu [(k+1)^2 - 1]} \cdot \left[ e^{\nu [(k+1)^2 - 1]} \cdot s \right] \right|_{s=0}^{t} \cdot e^{-\nu (k+1)^2} \cdot t \]
\[ = \frac{2}{(k+2)!} \cdot e^{-\nu t} \cdot e^{-\nu (k+1)^2} \cdot t \]
\[ \leq \frac{2}{(k+2)!} \cdot e^{-\nu t} = i^{-k} \cdot a_{k+1}(t), \]

i.e., (8) holds also for \( k + 1 \). Therefore, (8) holds for all \( k \in \mathbb{N} \) and all \( t \geq 0 \). Hence,

\[ |a_k(t)| \leq |a_k^*(t)| = \frac{2 \cdot e^{-\nu t}}{\nu (k+1)!} \]

and

\[ \|D_x f(\cdot, t)\|_A = \sum_{k \in \mathbb{N}} \|k! \cdot a_k(t)\| = \sum_{k \in \mathbb{N}} \|k! \cdot a_k^*(t)\| = 2 \cdot e^{-\nu t} \cdot \sum_{k \in \mathbb{N}} \frac{k! \cdot (\nu^{-1})^{k-1}}{(k+1)!} < \infty \]

hold for all \( k \in \mathbb{N} \) and \( t \geq 0 \). In summary, for every \( l \in \mathbb{N}_0 \) there exists a \( c_l > 0 \) dependent on \( \nu > 0 \) but independent on \( t \) such that

\[ \|D_x f(\cdot, t)\|_A \leq c_l \cdot e^{-\nu t} \]

and \( f(\cdot, t) \in C^\infty([0, 2\pi], C) \) for all \( t \geq 0 \). For \( l = 0 \) we have the explicit bound

\[ \|f(\cdot, t)\| \leq 2 \cdot e^{-\nu t} \cdot [\nu^2 \cdot \exp(\nu^{-1}) - \nu^2 - \nu] \]

for all \( t \geq 0 \). All \( \|D_x f(\cdot, t)\|_A \) decay exponentially in time with rate \(-\nu\). \( \square \)

3. Preliminaries: Periodic formulation with the Leray projection

\( C^\infty(\mathbb{T}^n) \) is a complete Montel space with the semi-norms \( \|D^\alpha f\|_\infty \), \( \alpha \in \mathbb{N}_0^n \). The crucial property we need is that a subset \( F \subset C^\infty(\mathbb{T}^n) \) is bounded iff there are constants \( C_\alpha > 0 \) for all \( \alpha \in \mathbb{N}_0^n \) such that \( \|D^\alpha f\|_\infty \leq C_\alpha \) for all \( f \in F \) and \( \alpha \in \mathbb{N}_0^n \). \( C^\infty(\mathbb{T}^n) \) has as a Montel space the Heine--Borel property, i.e., every bounded set \( F \) is pre-compact. Since it is complete, \( F \) has at least one accumulation point and the Arzelà–Ascoli Theorem (Lemma 1.1) holds. For more see e.g. [Arz] or [SW99].

Periodic functions \( f : \mathbb{T}^n \to \mathbb{C}^n \) can be uniquely written as \( \sum_{k \in \mathbb{Z}^n} a_k \cdot e^{i k \cdot x} \) with \( k \cdot x = k_1 x_1 + \cdots + k_n x_n \) and \( a_k \in \mathbb{C}^n \), \( n \in \mathbb{N} \). We define the absolute convergence norm \( \|f\|_A \) on \( C^\infty(\mathbb{T}^n, \mathbb{C}^n) \) by

\[ \|f\|_A := \sum_{k \in \mathbb{Z}^n} |a_k| \]

where \( |\cdot| \) is one of the equivalent norms on \( \mathbb{C}^n \), w.l.o.g. \( |\cdot| = |\cdot|_2 \) the Euclidean norm. We have \( \|f\|_\infty \leq \|f\|_A \) for a general periodic function \( f = \sum_{k \in \mathbb{Z}^n} a_k \cdot e^{i k \cdot x} \).
Hence, on $T^n$ Example 3.3 (Heat equation on $T^n$). That gives the well-known periodic solution of the heat equation. 

Proof. Let $\alpha \in \mathbb{N}_0$. We define $P_0$ and $P_k \in \mathbb{R}^{n \times n}$ for all $k \in \mathbb{Z}^n \setminus \{0\}$ by

$$P_0 := id \quad \text{and} \quad P_k := \begin{pmatrix} k^2 - k_1^2 & -k_1 k_2 & \cdots & -k_1 k_n \\ -k_1 k_2 & k^2 - k_2^2 & \cdots & -k_2 k_n \\ \vdots & \vdots & \ddots & \vdots \\ -k_1 k_n & -k_2 k_n & \cdots & k^2 - k_n^2 \end{pmatrix}.$$ 

$id$ is always the identity matrix. The formulas for $P_k$ can easily be remembered. The diagonal entries $(i, i)$ are $k^2 - k_i^2$ and the other entries $(i, j)$ with $i \neq j$ are $-k_i k_j$. In fact, for any $v \in \mathbb{C}^n$ and $k \in \mathbb{Z}^n \setminus \{0\}$ the matrix $P_k$ is a projection, i.e., the Gram-Schmidt orthogonalization with respect to $k$:

$$P_k v = v - \frac{(v, k)}{k^2} k. \tag{9}$$

The Euler and Navier–Stokes equations work with div-free vector fields. The Leray projection $P$ gives the div-free part (Hodge decomposition) of a vector field.

Lemma 3.2. Let $n \in \mathbb{N}$ and $f(x) = \sum_{k \in \mathbb{Z}^n} a_k \cdot e^{i \cdot k \cdot x} \in C^\infty(T^n, \mathbb{C})^n$. Then

(i) $\text{div } f(x) = 0$ if and only if $P_k a_k = a_k$ for all $k \in \mathbb{Z}^n$.

(ii) $P f(x) = \sum_{k \in \mathbb{Z}^n} P_k a_k \cdot e^{i \cdot k \cdot x}$.

Proof. (i): By linearity of div it is sufficient to show the equivalence for $f(x) = a_k \cdot e^{i \cdot k \cdot x}$, $k \in \mathbb{Z}^n \setminus \{0\}$. By the Gram-Schmidt formula we have

$$0 = \text{div } f(x) = i \cdot a_k \cdot k \cdot e^{i \cdot k \cdot x} \Leftrightarrow a_k \cdot k = 0 \Leftrightarrow P_k a_k = a_k.$$

(ii): Follows immediately from (i) and (9).

Note that $e^{i \cdot k \cdot x}$ is an eigenvector/-function of $\Delta$ (and all $\partial_j$) for each $k \in \mathbb{Z}^n$. That gives the well-known periodic solution of the heat equation.

Example 3.3 (Heat equation on $T^n$). Let $u_0(x) = \sum_{k \in \mathbb{Z}^n} a_k \cdot e^{i \cdot k \cdot x} \in C^\infty(T^n, \mathbb{C})^n$, $n \in \mathbb{N}$. Then the initial value problem

$$\partial_t u(x, t) = \nu \cdot \Delta u(x, t)$$
$$u(x, 0) = u_0(x)$$

on $T^n$ has the unique solution

$$u(x, t) = \sum_{k \in \mathbb{Z}^n} a_k \cdot e^{-\nu \cdot k^2 \cdot t} \cdot e^{i \cdot k \cdot x}.$$
Proof. By linearity of the initial value problem it is sufficient to prove the statement for $u_0(x) = e^{ik_0x}$. With the Ansatz $u(x, t) = a_k(t) \cdot e^{ikx}$ with $a_k(0) = 1$ we get
\[ \hat{a}_k(t) \cdot e^{ikx} = \partial_t u(x, t) = \nu \cdot \Delta u(x, t) = -\nu \cdot k^2 \cdot a_k(t) \cdot e^{ikx}, \]
i.e., $\hat{a}_k(t) = -\nu \cdot k^2 \cdot a_k(t)$ with $a_k(0) = 1$ and therefore $a_k(t) = e^{-\nu \cdot k^2 \cdot t}$. \hfill $\square$

We get the coefficient formulation of the Euler and Navier–Stokes equations \cite{3}.

**Theorem 3.4.** Let $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}_{\geq 0}^n$ and $v(x, t) = \sum_{k \in \mathbb{Z}^n} b_k(t) \cdot e^{ikx}$ be a $C^\infty(\mathbb{T}^n)$-function for all $t \in [0, \infty)$. Then
\[ u(x, t) = \sum_{k \in \mathbb{Z}^n} a_k(t) \cdot e^{ikx} \]
solves \cite{3}, i.e.,
\[ \partial_t u(x, t) = \nu \cdot \Delta u(x, t) + \mathcal{P}[v(x, t) \cdot \nabla u(x, t)] \]
for all $t \in [0, \infty)$, if and only if the $a_k(t)$ solve
\[ \hat{a}_k(t) = -\nu \cdot k^2 \cdot a_k(t) + i \cdot \sum_{l \in \mathbb{Z}^n} \langle b_{k-l}(t), l \rangle \cdot \mathcal{P}_k a_l(t). \quad (10) \]
If additionally $\text{div} v(x, t) = 0$ for all $t \in [0, \infty)$, then we have
\[ \langle b_{k-l}(t), l \rangle = \langle b_{k-l}(t), k \rangle. \quad (11) \]

**Proof.** By inserting the Fourier expansions of $u$ and $v$ into \cite{3} we get
\[
\sum_{k \in \mathbb{Z}^n} \hat{a}_k(t) \cdot e^{ikx} = \partial_t u(x, t)
= \nu \cdot \Delta u(x, t) + \mathcal{P}[v(x, t) \cdot \nabla u(x, t)]
= \nu \cdot \Delta \sum_{k \in \mathbb{Z}^n} a_k(t) \cdot e^{ikx} + \mathcal{P} \sum_{k', l \in \mathbb{Z}^n} b_{k'}(t) \cdot e^{ik'x} \cdot \nabla a_l(t) \cdot e^{ilx}
= \sum_{k \in \mathbb{Z}^n} -(\nu_1 k_1^2 + \ldots + \nu_n k_n^2) \cdot a_k(t) \cdot e^{ikx}
+ \mathcal{P} \sum_{k', l \in \mathbb{Z}^n} i \cdot \langle b_{k'-l}(t), l \rangle \cdot a_l(t) \cdot e^{i(k+l)x}
= \sum_{k \in \mathbb{Z}^n} -(\nu_1 k_1^2 + \ldots + \nu_n k_n^2) \cdot a_k(t) \cdot e^{ikx}
+ \mathcal{P} \sum_{k', l \in \mathbb{Z}^n} i \cdot \langle b_{k'-l}(t), l \rangle \cdot a_l(t) \cdot e^{i(k+l)x}
= \sum_{k \in \mathbb{Z}^n} -(\nu_1 k_1^2 + \ldots + \nu_n k_n^2) \cdot a_k(t) \cdot e^{ikx}
+ \mathcal{P} \sum_{k, l \in \mathbb{Z}^n} i \cdot \langle b_{k-l}(t), l \rangle \cdot a_l(t) \cdot e^{ikx} \quad (\text{Lemma 3.2})
\]
which proves the statement by comparing the coefficients of each $e^{ikx}$. From
$\text{div} v(x, t) = 0$ and Lemma 3.2 we get $\langle b_{k-l}(t), k-l \rangle = 0$ which implies
\[ \langle b_{k-l}(t), l \rangle = \langle b_{k-l}(t), (k-l) + l \rangle = \langle b_{k-l}(t), k \rangle. \quad \square \]

In the following we will study the $a_k(t)$ in (10) when the $b_k(t)$ are given. If we set $b_k(t) = -a_k(t)$ then we have the Euler and Navier–Stokes equations on $\mathbb{T}^n$. 

4. Solutions and Bounds of $\partial_t u = \nu \Delta u + P[v \nabla u]$ with $\nu \geq 0$ and simple $v$.

The simplest $v(x,t)$ is $v(x,t) = b_1(t) \cdot e^{i(t \cdot x)}$ for some $l \in \mathbb{Z}^n$ and continuous function $b_1$. For these we can solve the initial value problem \(3\) for $\nu = 0$ explicitly.

**Proposition 4.1.** Let $n \in \mathbb{N}$ with $n \geq 2$, $k, l \in \mathbb{Z}^n$, $a_k(0) \in \mathbb{C}^n$, and $b_l(t) \in C([0, \infty), \mathbb{C}^n)$ resp. $C([0, \infty), \mathbb{R}^n)$ for $l = 0$ with $a_k \cdot k = b_l(t) \cdot l = 0$ for all $t \in [0, \infty)$. Then the initial value problem
\[
\partial_t u(x,t) = P[(b_l(t) \cdot e^{i(t \cdot x)}) \nabla u(x,t)]
\]
\[u(x,0) = a_k(0) \cdot e^{i_k x}
\]
has for $l = 0$ the solution
\[
u(x,t) = a_k(0) \cdot \exp \left( i \cdot \int_0^t \langle b_0(s), k \rangle \, ds \right) \cdot e^{i_k x}
\]
and for $l \neq 0$ the solution
\[
u(x,t) = \sum_{j \in \mathbb{N}_0} \frac{1}{j!} \left( i \cdot \int_0^t \langle b_j(s), k \rangle \, ds \right)^j \cdot P_{k+j} \cdots P_{k+1} a_k(0) \cdot e^{i(k+j) \cdot x}.
\]

**Proof.** By Theorem 3.4 for $l = 0$ we have $\dot{a}_k(t) = i \cdot \langle b_0(t), k \rangle \cdot a_k(t)$ and for $l \neq 0$ and $j \in \mathbb{N}_0$ the system solvable by induction
\[
\dot{a}_k(t) = 0 \quad \text{with } a_k(0) \in \mathbb{C}^n
\]
\[
\dot{a}_{k+(j+1)}(t) = i \cdot \langle b_1(t), k + jl \rangle \cdot P_{k+(j+1)} a_{k+j}(t) \quad \text{with } a_{k+(j+1)}(0) = 0.
\]

In the previous result the factors in the induction actually are $\langle b_l(t), k + jl \rangle$, i.e., we would have the system
\[
\dot{a}_k(t) = 0 \quad \text{with } a_k(0) \in \mathbb{C}^n
\]
\[
\dot{a}_{k+(j+1)}(t) = i \cdot \langle b_1(t), k + jl \rangle \cdot P_{k+(j+1)} a_{k+j}(t) \quad \text{with } a_{k+(j+1)}(0) = 0.
\]
This effect of apparently increasing factors actually appeared in Example 2.1 in dimension 1 and caused a finite break down time $t^* = 1$. But in dimension $n \geq 2$ the requirement $\text{div } v(x,t) = 0$ gives by Lemma 3.2
\[
\langle b_l(t), l \rangle = 0
\]
and therefore
\[
\langle b_l(t), k + jl \rangle = \langle b_l(t), k \rangle + j \cdot \langle b_1(t), l \rangle = \langle b_l(t), k \rangle,
\]
i.e., the factors remain constant (independent on the induction step $j$). Hence, the behavior of the solution changes from a geometric series with its singularity to an exponential series where we no longer have a singularity. We get the following bounds on $\|\partial^\alpha u(x,t)\|_A$ for $\alpha \in \mathbb{N}_0^n$.

**Corollary 4.2.** Let $u(x,t)$ be from Proposition 4.1. Then for $l = 0$ or $l \parallel k$ we have for all $\alpha \in \mathbb{N}_0^n$ and $t \in [0, \infty)$ the bounds
\[
\|\partial^\alpha u(x,t)\|_A = \|\partial^\alpha u_0\|_A.
\]

For $l \neq 0$ with $l \parallel k$ we have the bound
\[
\|u(x,t)\|_A = |a_k(0)| \cdot \exp \left( \int_0^t \langle b_j(s), k \rangle \, ds \right),
\]
for the first derivative ($|\alpha| = 1, j_1 = 1, \ldots, n$) we have the bounds
\[
\|\partial_{j_1} u(x,t)\|_A \leq |a_k(0)| \cdot \left( |b_{j_1}| + \left| \int_0^t \langle b_{j_1}(s), k \rangle \, ds \right| \right) \cdot \exp \left( \int_0^t \langle b_j(s), k \rangle \, ds \right)
\]
for the same $j_1$.
and for the second derivatives \( |\alpha| = 2, j_1, j_2 = 1, \ldots, n \) we have the bounds

\[
\|\partial_{j_1} \partial_{j_2} u(x,t)\|_A \leq |a_k(0)| \left( |k_{j_1} k_{j_2}| + (|k_{j_1} l_{j_2}| + |k_{j_2} l_{j_1}| + |l_{j_1} l_{j_2}|) \right) \int_0^t \langle b_l(s), k \rangle \, ds \\
+ |l_{j_1} l_{j_2}| \left( \int_0^t \langle b_l(s), k \rangle \, ds \right)^2 \cdot \exp \left( \left| \int_0^t \langle b_l(s), k \rangle \, ds \right| \right).
\]

**Proof.** For \( l = 0 \) and real \( b_0(t) \) we have the bounds

\[
\|\partial^\alpha u(x,t)\|_A = \left\| k^\alpha \cdot a_k(0) \cdot \exp \left( i \cdot \int_0^t \langle b_l(s), k \rangle \, ds \right) \right\|_A = \|\partial^\alpha u_0\|_A.
\]

For \( t \neq 0 \) with \( l \parallel k \) we get

\[
\|u(x,t)\|_A = \sum_{j \in \mathbb{N}_0} \frac{1}{j!} |\mathbf{P}_{k+j,l} \cdots \mathbf{P}_{k+l,a} b_0(0)| \cdot \left| i \cdot \int_0^t \langle b_l(s), k \rangle \, ds \right|^j \\
\leq |a_k(0)| \sum_{j \in \mathbb{N}_0} \frac{1}{j!} \left| \int_0^t \langle b_l(s), k \rangle \, ds \right|^j \\
= |a_k(0)| \exp \left( \left| \int_0^t \langle b_l(s), k \rangle \, ds \right| \right).
\]

For the first derivative \( \partial_{j_1} \) we get

\[
\|\partial_{j_1} u(x,t)\|_A = \sum_{j \in \mathbb{N}_0} \frac{|k_{j_1} + j | l_{j_1}|}{j!} \cdot |\mathbf{P}_{k+j,l} \cdots \mathbf{P}_{k+l,a} b_0(0)| \cdot \left| i \cdot \int_0^t \langle b_l(s), k \rangle \, ds \right|^j \\
\leq |a_k(0)| \sum_{j \in \mathbb{N}_0} \frac{|k_{j_1} + j | l_{j_1}|}{j!} \cdot \left| \int_0^t \langle b_l(s), k \rangle \, ds \right|^j \\
= |a_k(0)| \left| |k_{j_1}| + \int_0^t |l_{j_1} \cdot b_l(s), k \rangle \, ds \right| \exp \left( \left| \int_0^t \langle b_l(s), k \rangle \, ds \right| \right).
\]

For the second derivative we have

\[
\|\partial_{j_1} \partial_{j_2} u(x,t)\|_A \\
\leq |a_k(0)| \sum_{j \in \mathbb{N}_0} \frac{|k_{j_1} + j | l_{j_1}| + |k_{j_2} + j | l_{j_2}|}{j!} \cdot \left| \int_0^t \langle b_l(s), k \rangle \, ds \right|^j \\
\leq |a_k(0)| \sum_{j \in \mathbb{N}_0} \frac{|k_{j_1} k_{j_2}| + j |k_{j_1} l_{j_2}| + j |k_{j_2} l_{j_1}| + j^2 |l_{j_1} l_{j_2}|}{j!} \cdot \left| \int_0^t \langle b_l(s), k \rangle \, ds \right|^j \\
\leq |a_k(0)| \sum_{j \in \mathbb{N}_0} \left( |k_{j_1} k_{j_2}| + (|k_{j_1} l_{j_2}| + |k_{j_2} l_{j_1}| + |l_{j_1} l_{j_2}|) \cdot \left| \int_0^t \langle b_l(s), k \rangle \, ds \right| \\
+ |l_{j_1} l_{j_2}| \left( \int_0^t \langle b_l(s), k \rangle \, ds \right)^2 \cdot \exp \left( \left| \int_0^t \langle b_l(s), k \rangle \, ds \right| \right) \right).
\]

From the bounds of the first and second derivatives it is eminent that the bounds for a derivative of any degree \( |\alpha| \in \mathbb{N}_0 \) can easily be calculated in the same manner.

For \( l = 0 \) in Proposition 4.1 we can even include \( \nu \cdot \Delta \) from Example 3.3.

**Proposition 4.3.** Let \( n \in \mathbb{N} \), \( \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}_{\geq 0}^n \), and \( b_0(t) \in C([0, \infty), \mathbb{R}^n) \). Then the initial value problem

\[
\partial_t u(x,t) = \nu \cdot \Delta u(x,t) + \mathbf{P}[b_0(t) \cdot \nabla u(x,t)]
\]
\[ u(x,0) = \sum_{k \in \mathbb{Z}^n} a_k(0) \cdot e^{i k \cdot x} \]

with \( \text{div} u(x,0) = 0 \) has the unique solution

\[ u(x,t) = \sum_{k \in \mathbb{Z}^n} a_k(t) \cdot \exp \left( -\nu \cdot k^2 \cdot t + i \cdot \int_0^t \langle b_0(s), k \rangle \, ds \right) \cdot e^{i k \cdot x}. \]

**Proof.** By Theorem 3.4 the initial value problem is equivalent to the coefficient formulation (10). So from (10) with \( b_0 = 0 \) for all \( k \neq 0 \), \( b_0 \in C([0, \infty), \mathbb{R}) \) and since by Lemma 3.2 we have \( P_k a_k(0) = a_k(0) \) we get

\[ \dot{a}_k(t) = -\nu \cdot k^2 \cdot a_k(t) + i \cdot \langle b_0(t), k \rangle \cdot a_k(t) \]

for all \( k \in \mathbb{Z}^n \), i.e., we get the unique solutions

\[ a_k(t) = a_k(0) \cdot \exp \left( -\nu \cdot k^2 \cdot t + i \cdot \int_0^t \langle b_0(s), k \rangle \, ds \right). \]

\( \square \)

Note, that from Proposition 4.3 we see that \( b_0 \) only induces a complex rotation of the coefficients. That is the reason we introduced \( \| \cdot \|_{A,0} \), i.e., \( \| \cdot \|_A \) without the \( k = 0 \) coefficient.

So far in Proposition 4.1 the coefficients

\[ \mathbb{P}_{k+l} \cdots \mathbb{P}_{k+l} a_k(0) \]

appeared and since all \( \mathbb{P}_k : \mathbb{C}^n \rightarrow \mathbb{C}^n \) are projections we used the bounds

\[ |\mathbb{P}_{k+l} \cdots \mathbb{P}_{k+l} a_k(0)| \leq |a_k(0)|, \]

of course in the \( l^2 \)-norm \( | \cdot | \) on \( \mathbb{C}^n \). In Proposition 4.3 the coefficients \( a_k \) do not interact with each other since \( v(x,t) = b_0(t) \) is constant in \( x \) (\( l = 0 \)), i.e., no shift \( e^{i l \cdot x} \cdot e^{i k \cdot x} = e^{i (l+k) \cdot x} \) appears. But if we have \( a_k(0) \perp l \) we get the apparently worst case \( \mathbb{P}_{k+l} \cdots \mathbb{P}_{k+l} a_k(0) = a_k(0) \) in Proposition 4.1. So we will look at this special case which can only appear non-trivially for \( n \geq 3 \). We find a solution for

\[ v(x,t) = b_l(t) \cdot e^{i l \cdot x} + b_{-l}(t) \cdot e^{-i l \cdot x} \in C^\infty(T^n, \mathbb{R})^n. \]

Since \( v(x,t) \) shall be real we always have \( b_l(t) = b_{-l}(t) \). It reveals the connection to the well-studied Bessel functions of the first kind of order \( j \in \mathbb{Z} \) [Wat66]:

\[ J_j(t) = \sum_{a \in \mathbb{N}_0} \frac{(-1)^a \cdot t^{2a+|j|}}{a! \cdot (a+|j|)! \cdot 2^{2a+|j|}}. \]

**Theorem 4.4.** Let \( n \in \mathbb{N} \), \( k, l \in \mathbb{Z}^n \) with \( l \neq 0 \), \( a_k(0) \in \mathbb{C}^n \) with \( a_k(0) \perp l \), and \( b_l, b_{-l} \in C([0, \infty), \mathbb{C})^n \) with \( b_l(t) \cdot l = b_{-l}(t) \cdot l = 0 \) and \( \overline{b_l(t)} = b_{-l}(t) \) for all \( t \in [0, \infty) \). Then the initial value problem

\[ \partial_t u(x,t) = \mathbb{P} \left[ (b_l(t) \cdot e^{i l \cdot x} + b_{-l}(t) \cdot e^{-i l \cdot x}) \nabla u(x,t) \right] \]

\[ u(x,0) = a_k(0) \cdot e^{i k \cdot x} \]

has with \( B_+(t) := i \cdot \int_0^t \langle b(s), k \rangle \, ds \neq 0 \) the unique \( C^\infty \) solution

\[ u(x,t) = \sum_{j \in \mathbb{Z}} a_k(0) \cdot \left( \frac{B_+(t)}{|B_+(t)|} \right)^j J_j(2 \cdot |B_+(t)|) \cdot e^{i (k+j) \cdot x} \]

where \( J_j \) is the Bessel function of the first kind of order \( j \) and for \( B_+(t) = 0 \) the unique solution is \( u(x,t) = a_k(0) \cdot e^{i k \cdot x} \).
Proof. Set \( \beta_+(t) := i \cdot (b(t), k) \) and \( \beta_-(t) := i \cdot (b_{-1}(t), k) \). Since \( a_k(0) \perp t \) we have
\[
P_{k+j}a_k(0) = a_k(0)
\]
for all \( j \in \mathbb{Z} \). By Theorem 3.4 the initial value problem is equivalent to
\[
\dot{a}_{k+j}(t) = \beta_+(t) \cdot a_{k+j-1}(t) + \beta_-(t) \cdot a_{k+j+1}(t).
\]
Therefore, \( a_k'(t) = 0 \) for all \( k' \neq k + j \) for all \( j \in \mathbb{Z} \). Defining the shifts \( S_+ \) and \( S_- \) by
\[
S_+(a_{k'})_{k' \in \mathbb{Z}} := (a_{k'+1})_{k' \in \mathbb{Z}} \quad \text{and} \quad S_-(a_{k'})_{k' \in \mathbb{Z}} := (a_{k'-1})_{k' \in \mathbb{Z}},
\]
i.e.,
\[
S_+S_- = S_-S_+ = \text{id},
\]
we can write (*) as
\[
(a_{k+j}(t))_{j \in \mathbb{Z}} = (\beta_+(t) \cdot S_+ + \beta_-(t) \cdot S_-) \cdot (a_{k+j}(t))_{j \in \mathbb{Z}}
\]
which has the (unique) solution
\[
(a_{k+j}(t))_{j \in \mathbb{Z}} = \exp \left( \int_0^t \beta_+(s) \, ds \cdot S_+ + \int_0^t \beta_-(s) \, ds \cdot S_- \right) \cdot (a_{k+j}(0))_{j \in \mathbb{Z}}
\]
with of course \( a_{k+j}(0) = a_k(0) \) for \( j = 0 \) and \( a_{k+j}(0) = 0 \) for all \( j \neq 0 \). Check by differentiation and use that \( S_+ \) and \( S_- \) commute by (**).

If \( \int_0^t \beta_+(s) \, ds = 0 \), then \( u(x, t) = a_k(0) \cdot e^{i k \cdot x} \). So assume \( \int_0^t \beta_+(s) \, ds \neq 0 \).

Let \( e_{k+j} \) be the vector with the unit matrix \( \text{id} \in \mathbb{C}^{n \times n} \) at position \( k + j \) and the zero matrix \( 0 \in \mathbb{C}^{n \times n} \) everywhere else. Then
\[
a_{k+j}(t) = \langle e_{k+j}, \exp \left( \int_0^t \beta_+(s) \, ds \cdot S_+ + \int_0^t \beta_-(s) \, ds \cdot S_- \right) e_k \rangle \cdot a_k(0)
\]
for all \( j \in \mathbb{Z} \). Hence, it is sufficient to calculate
\[
\langle e_{k+j}, \exp \left( \int_0^t \beta_+(s) \, ds \cdot S_+ + \int_0^t \beta_-(s) \, ds \cdot S_- \right) e_k \rangle \in \mathbb{C}^{n \times n}.
\]
Let \( j = 2 \alpha \in \mathbb{N}_0 \) be non-negative and even. For simplicity we use
\[
B_+(t) := \int_0^t \beta_+(s) \, ds = i \cdot \int_0^t (b_1(s), k) \, ds
\]
and
\[
B_-(t) := \int_0^t \beta_-(s) \, ds = i \cdot \int_0^t (b_{-1}(s), k) \, ds.
\]
Then since \( b_1(t) = b_{-1}(t) \) we have
\[
B_+(t) = -B_-(t) \quad \text{and} \quad B_+(t) \cdot B_-(t) = -|B_+(t)|^2 = -|B_-(t)|^2.
\]
Since \( S_+ \) and \( S_- \) commute by (**), we have
\[
\langle e_{k+2\alpha}, \exp(B_+(t) \cdot S_+ + B_-(t) \cdot S_-) e_k \rangle = \langle S_+^{2\alpha} e_k, \sum_{a=0}^{\infty} \frac{1}{a!} (B_+(t) \cdot S_+ + B_-(t) \cdot S_-)^a e_k \rangle
\]
\[
= \langle e_k, \sum_{a=0}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} B_+(t)^b \cdot B_-(t)^{a-b} \cdot S_+^b \cdot S_-^{a-b} e_k \rangle
\]
\[
= \langle e_k, \sum_{a=0}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \cdot B_+(t)^b \cdot B_-(t)^{a-b} \cdot S_-^{a-2b+2\alpha} e_k \rangle
\]
for every \( k \in \mathbb{N} \). Since \( \sum_{a=0}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \cdot B_+(t)^b \cdot B_-(t)^{a-b} \cdot S_-^{a-2b+2\alpha} e_k \) is bounded if \( \alpha \in \mathbb{N} \), the above series converges uniformly and is absolutely convergent on \( [0, T] \) for every \( T > 0 \). Therefore, the above expression is bounded on \( [0, T] \) for every \( T > 0 \).
since \( a - 2b + 2\alpha \) must be zero by the orthogonality of the \( e_k \)'s, \( a \) must be even and so we sum over \( 2a \)

\[
\left\langle e_k, \sum_{a=0}^{\infty} \frac{1}{(2a)!} \sum_{b=0}^{2a} \left( \frac{2a}{b} \right) \cdot B_+ (t)^b \cdot B_- (t)^{2a-2b+b} \cdot S_-^{2a-2b+2\alpha} e_k \right\rangle
\]

here again the orthogonality of the \( e_k \)'s implies \( 2a - 2b + 2\alpha = 0 \), i.e., \( b = a + \alpha \),

\[
\sum_{a=0}^{\infty} \frac{1}{(2a)!} \left( \frac{2a}{a + \alpha} \right) \cdot B_+ (t)^a \cdot B_- (t)^{-\alpha}
\]

so the summation over \( a \) begins at \( a = \alpha \), since otherwise \( \left( \frac{2a}{a + \alpha} \right) \) is zero,

\[
= \sum_{a=0}^{\infty} \frac{1}{(2a)!} \cdot \left( \frac{2a}{a + \alpha} \right) \cdot B_+ (t)^{a+\alpha} \cdot B_- (t)^{-\alpha}
\]

and shifting the summation back to start at \( a = 0 \)

\[
= \sum_{a=0}^{\infty} \frac{1}{a! \cdot (a + 2\alpha)!} \cdot B_+ (t)^{a+2\alpha} \cdot B_- (t)^{a}
\]

\[
= B_+ (t)^{2\alpha} \cdot \sum_{a=0}^{\infty} \frac{(-1)^a \cdot |B_+ (t)|^{2a}}{a! \cdot (a + 2\alpha)!}
\]

\[
= \left( \frac{B_+ (t)}{B_+ (t)} \right)^{2\alpha} \cdot \sum_{a=0}^{\infty} \frac{(-1)^a \cdot (2 \cdot |B_+ (t)|)^{2a+2\alpha}}{a! \cdot (a + 2\alpha)! \cdot 2^{2a+2\alpha}}
\]

\[
= \left( \frac{B_+ (t)}{B_+ (t)} \right)^{2\alpha} \cdot J_{2\alpha} (2 \cdot |B_+ (t)|).
\]

In the same way we calculate the cases \( k + (2\alpha + 1)l \), \( k - 2\alpha l \), and \( k - (2\alpha + 1)l \) for all \( \alpha \in \mathbb{N}_0 \), see Appendix A and we therefore get

\[
a_{k+j} (t) = a_k (0) \cdot \left( \frac{B_+ (t)}{|B_+ (t)|} \right)^j \cdot J_j (2 \cdot |B_+ (t)|)
\]

for all \( j \in \mathbb{Z} \).}

\[
J_0 (x)^2 + 2 \sum_{j=1}^{\infty} J_j (x)^2 = 1 \quad \text{Wat66 §2.5 (3)} \text{ gives } \| u(x,t) \|_{L^2 (\mathbb{T}^n)} = \| u_0 \|_{L^2 (\mathbb{T}^n)}.
\]

\[
\frac{z^2}{2} = \sum_{j \in \mathbb{R}} \varepsilon_j j^2 J_j (z)^2 \text{ with } \varepsilon_j = \begin{cases} 1 \text{ for } j = 1, \\ 2 \text{ for } j \geq 1 \end{cases} \quad \text{Wat66 §2.72 (3)} \text{ for } k_1 = 0 \text{ implies }
\]

\[
\| \partial_1 u(x,t) \|_{L^2 (\mathbb{T}^n)} = (2\pi)^n |a_k (0) \cdot l_1| \cdot \sqrt{\frac{z^2}{2}} + J_1 (z)^2 \leq (2\pi)^n \sqrt{3} \cdot |a_k (0) \cdot l_1 \cdot B_+ (t)|
\]

with \( z = 2 \cdot |B_+ (t)| \) and for \( k_1 \in \mathbb{Z} \) gives

\[
\| \partial_1 u(x,t) \|_{L^2 (\mathbb{T}^n)} \leq \| \partial_1 u_0 \|_{L^2 (\mathbb{T}^n)} + (2\pi)^n \sqrt{3} \cdot |a_k (0) \cdot l_1 \cdot B_+ (t)|.
\]

Despite the fact, that we have in Theorem 4.3 the case that the projections \( P_{k+j} \) disappear, i.e., the norm of \( a_k (0) \) is not reduced as in Proposition 4.1 another cancellation effect is clearly seen. In Proposition 4.1 we only have \( v(x,t) = b_l (t) \cdot e^{i l \cdot x} \) which resulted in a polynomial growth of the coefficients. But in Theorem 4.4 we have \( v(x,t) = b_l (t) \cdot e^{i l \cdot x} + b_{-l} (t) \cdot e^{-i l \cdot x} \) with \( b_l (t) = b_{-l} (t) \). So in Proposition 4.1 we only go in one direction \( l \) while in Theorem 4.4 we go forth \( l \) and back \(-l\) and sign cancellations appear.

In Figure 3 we show the Bessel functions \( J_0 \) (red), \( J_1 \) (blue), and \( J_2 \) (green). The reason that these sign cancellations appear are found in the coefficient formulation of the initial value problem in Theorem 3.4. In (10) the \( i \) in \( \sum_{l \in \mathbb{Z}} \langle b_{-l} (t), l \cdot 
\)
The Bessel functions (solid lines) $J_0$ (red), $J_1$ (blue), and $J_2$ (green) of the first kind of order 0, 1, and 2 compared to the corresponding absolute sequences (dashed with the same color).

$\mathbb{P}_k a_l(t)$ has the affect that when in a small time step a small portion of $i \cdot a_k(0)$ is added to the $k+l$ position, then in the second small time step a portion of $i^2 \cdot a_k(0) = -a_k(0)$ is added to the original $k$ position and therefore the coefficient decreases. This is mathematically captured in Theorem 4.4. We have e.g.

$$J_0(2t) = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} \pm \ldots$$

In every second small time step the $k$th position ($j = 0$) gets something back with the sign $i^2 a = (-1)^a$.

In the following approximations and calculations of the bounds in Theorem 5.1 such sign cancellations can not be captured and we can only sum the absolute series. Compared to the Bessel functions we show the sums of the absolute summands

$$\sum_{a=0}^{\infty} \frac{|x|^{2a+j}}{a! \cdot (a + |j|)!}$$

for $j = 0$ (dashed red), 1 (dashed blue), and 2 (dashed green) in Figure 1. They grow much faster than the original Bessel functions and are unbounded.

Note, that the proof gives $J_1(t) = \langle e_l, \exp(i \cdot t(S_+ + S_+)/2) e_0 \rangle$ and therefore provides a simple proof of the addition theorem for Bessel functions of the first kind:

$$J_1(t_1 + t_2) = \langle e_l, \exp(i \cdot (t_1 + t_2)(S_+ + S_-)/2) e_0 \rangle$$

$$= \langle e_l, \exp(i \cdot t_1(S_+ + S_-)/2) \exp(i \cdot t_2(S_+ + S_-)/2) e_0 \rangle$$

$$= \langle \exp(i \cdot t_1(S_+ + S_-)/2) e_l, \exp(i \cdot t_2(S_+ + S_-)/2) e_0 \rangle$$

$$= \sum_{m \in \mathbb{Z}} \langle \exp(i \cdot t_1(S_+ + S_-)/2) e_l, e_m \rangle \cdot \langle e_m, \exp(i \cdot t_2(S_+ + S_-)/2) e_0 \rangle$$

$$= \sum_{m \in \mathbb{Z}} J_{-m}(t_1) \cdot J_m(t_2).$$

Since $S_+$ and $S_-$ commute we can extend Theorem 4.4 to linear combinations.
Corollary 4.5. Let \( n, M \in \mathbb{N} \) with \( n \geq 2 \), \( k, l \in \mathbb{Z}^n \) with \( l \neq 0 \), \( a_k(0) \in \mathbb{C}^n \) with \( a_k(0) \perp k, l \), and
\[
v(x, t) = \sum_{m=-M}^{N} b_{m \cdot x}(t) \cdot e^{i m \cdot x} \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)
\]
with \( \text{div} \, v(x, t) = 0 \) for all \( t \in [0, \infty) \). Then the initial value problem
\[
\partial_t u(x, t) = \mathcal{P}[v(x, t) \nabla u(x, t)]
\]
\[
u(x, 0) = a_k(0) \cdot e^{i k \cdot x}
\]
has the solution
\[
u(x, t) = a_k(0) \cdot e^{i k \cdot x} \cdot \prod_{m=0}^{M} \left( \sum_{j \in \mathbb{Z}} \left( B_m(t) \cdot |B_m(t)| \right)^j \cdot J_j \left( 2 \cdot |B_m(t)| \cdot e^{i j \cdot m \cdot x} \right) \right)
\]
where \( B_m(t) := \int_{0}^{t} |b_m(s), k| \, ds \).

Proof. Differentiating \( u(x, t) \) with respect to \( t \) by the product rule and using Theorem 4.4 gives the statement.

Since \( v \in C^\infty(\mathbb{T}^n, \mathbb{R}^n) \) the previous results even holds for \( M \to \infty \).

5. Solutions and Bounds of \( \partial_t u = \nu \Delta u + \mathcal{P}[v \nabla u] \) with \( \nu > 0 \)

We prove now the main theorem.

Theorem 5.1. Let \( n \in \mathbb{N} \) with \( n \geq 2 \), \( \nu > 0 \), \( v \in C^\infty(\mathbb{T}^n \times [0, \infty), \mathbb{R}^n) \) be a smooth periodic real function with \( \text{div} \, v(x, t) = 0 \), and \( u_0 \in C^\infty(\mathbb{T}^n, \mathbb{R}^n) \). Then
\[
\partial_t u(x, t) = \nu \Delta u(x, t) + \mathcal{P}[v(x, t) \nabla u(x, t)]
\]
\[
u(x, 0) = u_0(x)
\]
has a real smooth solution
\[
u : \mathbb{T}^n \times [0, \infty) \to \mathbb{R}^n \quad \nu(x, t) = \sum_{k \in \mathbb{Z}^n} a_k(t) \cdot e^{i k \cdot x}
\]
with \( a_0(t) = a_0(0) \) for all \( t \geq 0 \) and the bounds
\[
\|u(x, t)\|_{A,0} \leq \|u_0\|_{A,0} \cdot \exp \left( \frac{1}{4\nu} \int_{0}^{t} \|v(x, s)\|_{A,0}^2 \, ds \right), \tag{12}
\]
\[
\|u(x, t)\|_{A,1} \leq \|u_0\|_{A,1} \cdot \exp \left( \sqrt{n} \int_{0}^{t} \|v(x, s)\|_{A,1} \, ds + \frac{1}{4\nu} \int_{0}^{t} \|v(x, s)\|_{A,0} \, ds \right), \tag{13}
\]
and for all \( d \geq 2 \)
\[
\|u(x, t)\|_{A,d} \leq \exp \left( \frac{1}{4\nu} \int_{0}^{t} \|v(x, s)\|_{A,0}^2 \, ds + \sqrt{n} \cdot d \cdot \int_{0}^{t} \|v(x, s)\|_{A,1} \, ds \right) \tag{14}
\]
\[
\times \left( \|u_0\|_{A,d} + \sqrt{n} \cdot \sum_{j=2}^{d} \binom{d}{j} \cdot \int_{0}^{t} \|v(x, s)\|_{A,j-1} \, ds \sup_{s \in [0,t]} \|u(x, s)\|_{A,d+1-j} \right)
\]
for all \( t \in [0, \infty) \). If \( \|v(x, t)\|_{A,0} + \delta \leq \nu \) for all \( t \in [0, \tau] \) with \( \tau \geq 0 \) and \( \delta \geq 0 \), then
\[
\|u(x, t)\|_{A,0} \leq \|u_0\|_{A,0} \cdot e^{-\delta \cdot t} \tag{15}
\]
and
\[
\|u(x, t)\|_{A,1} \leq \|u_0\|_{A,1} \cdot \exp \left( \sqrt{n} \cdot \int_{0}^{t} \|v(x, s)\|_{A,1} \, ds - \delta \cdot t \right) \tag{16}
\]
for all \( t \in [0, \tau] \).
Proof. Let \( N \in \mathbb{N} \) and \( T > 0 \). For \([0,T]\) take a decomposition \( \mathcal{Z} = \{t_0 = 0, t_1, \ldots, t_N = T\} \) with \( t_0 < t_1 < \cdots < t_N \) and \( \Delta \mathcal{Z} := \max_{j=0,\ldots,N-1} |t_{j+1} - t_j| \).

Since \( v \) is \( C^\infty(\mathbb{T}^n, \mathbb{R}^n) \) we have

\[
v(x,t) = \sum_{l \in \mathbb{Z}^n} b_l(t) \cdot e^{i \cdot l \cdot x}
\]

where \( b_l \) are the Fourier coefficients of \( v \).

Let \( \{u_N\}_{N \in \mathbb{N}} \) be a family of functions \( u_N : \mathbb{T}^n \times [0,T] \rightarrow \mathbb{R} \) defined in the following way. Each function \( u_N \) is piece-wise on \([t_j, t_{j+1}]\), \( j = 0, \ldots, N-1 \), defined by its Fourier coefficients

\[
a_{k,N}(t) := a_{k,N}(t_j) \cdot e^{-\nu \cdot k^2 \cdot (t-t_j)} + i \cdot \sum_{l \in \mathbb{Z}^n \setminus \{0\}} \int_{t_j}^t \langle b_{k-l}(s), l \rangle \, ds \cdot P_{k,l,N}(t_j) \cdot e^{-\nu \cdot l^2 \cdot (t-t_j)} + i \cdot \sum_{l \in \mathbb{Z}^n} \langle b_{k-l}(t_j), k \rangle \cdot P_{k,l,N}(t_j).
\]  

(17)

for all \( t \in [t_j, t_{j+1}] \), i.e.,

\[
\partial_t a_{k,N}(t) = -\nu \cdot k^2 \cdot a_{k,N}(t_j) + i \cdot \sum_{l \in \mathbb{Z}^n} \langle b_{k-l}(t_j), k \rangle \cdot P_{k,l,N}(t_j).
\]  

(18)

From Proposition [4.1] we see that \( 17 \) is after reordering as in Theorem [3.4] the following: For each \( k \in \mathbb{Z}^n \) the component \( a_{k,N}(t_j) \cdot e^{i \cdot k \cdot x} \) of \( u \) at time \( t_j \) becomes at \( t \in [t_j, t_{j+1}] \)

\[
a_{k,N}(t_j) \cdot e^{-\nu \cdot k^2 \cdot (t-t_j)} \cdot e^{i \cdot k \cdot x} + i \cdot \sum_{l \in \mathbb{Z}^n \setminus \{0\}} \int_{t_j}^t \langle b_l(s), k \rangle \, ds \cdot P_{k+l,N}(t_j) \cdot e^{-\nu \cdot k^2 \cdot (t-t_j)} + i \cdot \sum_{l \in \mathbb{Z}^n} \langle b_l(t_j), k \rangle \cdot P_{k,l,N}(t_j).
\]

(19)

Hence, both \( 17 \) and \( 19 \) contain the same summands. If we sum in the following over the absolute values of the Fourier coefficients both formulations give the same sum, i.e., only a renumbering of the indices appears.

Since \( \text{div} \, v(x,t) = 0 \) we have \( \langle b_l(s), l \rangle = 0 \) by Lemma [3.2] and hence \( a_{0,N}(t) = a_0(0) \) for all \( t \in [0,T] \) and \( N \in \mathbb{N} \). Since \( b_0(t) \in \mathbb{R}^n \) for all \( t \in [0,T] \) the exponent \( i \cdot \sum_{l \in \mathbb{Z}^n \setminus \{0\}} \langle b_l(s), k \rangle \, ds \) induces only a complex rotation, i.e., the absolute values are unchanged. Since we only sum the absolute values, for notational simplicity we let \( b_0 = 0 \), i.e., we ignore this imaginary exponent in the following calculations.

In the following we show that for each \( d \in \mathbb{N}_0 \) there exists a \( C_d > 0 \) such that

\[
\sup_{\alpha \in \mathbb{N}_0^n, |\alpha|=d, N \in \mathbb{N}, t \in [0,T]} ||\partial^\alpha u_N(x,t)||_{\Lambda} \leq \sup_{t \in [0,T], N \in \mathbb{N}} ||u_N(x,t)||_{\Lambda,0} \leq \sup_{t \in [0,T], N \in \mathbb{N}} ||u_N(x,t)||_{\Lambda,d} \leq C_d
\]

(20)

to apply Lemma [1.1]. Since \( T > 0 \) is arbitrary, it is is sufficient to look at \( t = T = t_N \). For \( ||u_N(x,t)||_{\Lambda,0} \) we get from [19] directly

\[
||u_N(x,t)||_{\Lambda,0} = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_N)| \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N-t_{N-1})}
\]

\[
+ \sum_{k,l \in \mathbb{Z}^n \setminus \{0\}} \int_{t_{N-1}}^{t_N} |\langle b_l(s), k \rangle \, ds| \cdot |a_{k,N}(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N-t_{N-1})}
\]

(21)
\[
\begin{align*}
&= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_{N-1})| \cdot e^{-\nu \cdot k^2 (t_N - t_{N-1})} \cdot \left[ 1 + \sum_{t \in \mathbb{Z}^n \setminus \{0\}} \int_{t_{N-1}}^{t_N} |\beta_t(s, k)| \, ds \right] \\
&\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_{N-1})| \cdot \exp \left( |k| \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,0} \, ds - \nu k^2 (t_N - t_{N-1}) \right) \tag{(5)}
\end{align*}
\]

If \(\|v(x,s)\|_{A,0} + \delta \leq \nu\) for all \(s \in [0,T]\), then \(\|v(x,s)\|_{A,0} - \nu \cdot |k| \leq \|v(x,s)\|_{A,0} - \nu \leq -\delta\) and we get
\[
\sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_{N})| \cdot e^{-\delta (t_N - t_{N-1})}
\]
proving (15). Assume \(\|v(x,s)\|_{A,0} \not\leq \nu\), set \(K := \nu^{-1} \max_{s \in [t_{N-1}, t_N]} \|v(x,s)\|_{A,0}\), then for all \(k\) with \(|k| \geq K\) we have \(|k| \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,0} \, ds - \nu k^2 (t_N - t_{N-1}) \leq 0\) (contraction) and for \(0 < |k| < K\) we have the exponent possibly \(> 0\) (expansion) with maximum at \(K\). We divide the sum (5) into contraction and expansion parts and use the maximum for the expansions (keep this argument in mind, we will use it several times throughout the proof)
\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_{N-1})| \cdot e^{-\delta (t_N - t_{N-1})}
\]
\[
+ \exp \left( \frac{(t_N - t_{N-1})}{4\nu} \max_{s \in [t_{N-1}, t_N]} \|v(x,s)\|_{A,0}^2 \right) \cdot \sum_{k \neq 0, |k| < K} |a_k(t_{N-1})| \leq \exp \left( \frac{(t_N - t_{N-1})}{4\nu} \max_{s \in [t_{N-1}, t_N]} \|v(x,s)\|_{A,0}^2 \right) \cdot \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_{N-1})| \tag{\#}
\]
\[
\leq \exp \left( \frac{1}{4\nu} \sum_{j=1}^{N} (t_j - t_{j-1}) \max_{s \in [t_{j-1}, t_j]} \|v(x,s)\|_{A,0}^2 \right) \cdot \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |a_{k,N}(t_{N-1})| \quad \text{which goes for } \nu N \to \infty \text{ with } \Delta Z \to 0
\]
\[
\to \|u_0(x)\|_{A,0} \cdot \exp \left( \frac{1}{4\nu} \int_{0}^{T} \|v(x,s)\|_{A,0}^2 \, ds \right).
\]
Since the bound converges there exists a \(C_0 > 0\) such that \(\|u_N(x,t)\|_{A,0} \leq C_0\) for all \(t \in [0,T]\) and \(N \in \mathbb{N}\).

For the derivatives \((|\alpha| \geq 1)\) it is sufficient to bound \(\|u_N(x,t)\|_{A,|\alpha|}\). For simplicity we drop the index \(N\) in \(a_{k,N}\) and only write \(a_k\) (but keep in mind that these \(a_k\)'s still depend on \(N\), see (17) and (19)).

For the first derivatives \(d = |\alpha| = 1\) we get from (19)
\[
\|u_N(x,t_N)\|_{A,1} = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_{\infty} \cdot |a_k(t_N)| \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_{\infty} \cdot |a_{k,N}(t_{N-1})| \cdot e^{-\nu k^2 (t_N - t_{N-1})}
\]
\[
+ \sum_{k,l \in \mathbb{Z}^n \setminus \{0\}} |k + l|_\infty \int_{t_{N-1}}^{t_N} |\langle b_l(s), k \rangle| \, ds \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}
\]

\[
= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}, \quad \left( |k|_\infty + \sum_{l \in \mathbb{Z}^n \setminus \{0\}} |k + l|_\infty \int_{t_{N-1}}^{t_N} |\langle b_l(s), k \rangle| \, ds \right)
\]

\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}, \quad \left( 1 + \sum_{l \in \mathbb{Z}^n \setminus \{0\}} \int_{t_{N-1}}^{t_N} |\langle b_l(s), k \rangle| \, ds \right)
\]

\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_N)| \cdot \exp \left( \sqrt{n} \cdot \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,1} \, ds + \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,0} \, ds \cdot |k| - \nu \cdot k^2 \cdot (t_N - t_{N-1}) \right)
\]

\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_N)| \cdot \exp \left( \sqrt{n} \cdot \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,1} \, ds \right)
\]

\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_N)| \cdot \exp \left( \sqrt{n} \cdot \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,1} \, ds - \delta \cdot (t_N - t_{N-1}) \right)
\]

\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_0)| \cdot \exp \left( \sqrt{n} \cdot \int_{t_0}^{t} \|v(x,s)\|_{A,1} \, ds - \delta \cdot (t_N - t_0) \right)
\]

\[
= \|u_0\|_{A,1} \cdot \exp \left( \sqrt{n} \cdot \int_{t_0}^{t} \|v(x,s)\|_{A,1} \, ds - \delta \cdot T \right) =: C_1
\]

which proves (10). If \(\|v(x,t)\|_{A,0} + \delta \leq \nu\), then equivalently to the previous case of \(\|u_N(x,t)\|_{A,0}\) we divide the sum (9) into a contraction part (for \(|k| \geq K\)) and an expansion part (for \(|k| < K\)) with \(K := \nu^{-1} \cdot \max_{s \in [t_{N-1},T]} \|v(x,s)\|_{A,0}\) and in the expansion part its maximum is at \(K/2\)

\[
\leq \sum_{k:|k| \geq K} |k|_\infty \cdot |a_k(t_{N-1})| \cdot \exp \left( \sqrt{n} \cdot \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,1} \, ds \right)
\]

\[
+ \sum_{k:0 < |k| < K} |k|_\infty \cdot |a_k(t_{N-1})| \cdot \exp \left( \sqrt{n} \cdot \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,1} \, ds \right)
\]

\[
+ \frac{t_N - t_{N-1}}{4\nu} \cdot \max_{s \in [t_{N-1},T]} \|v(x,s)\|_{A,0}^2
\]

\[
\leq \exp \left( \sqrt{n} \cdot \int_{t_{N-1}}^{t_N} \|v(x,s)\|_{A,1} \, ds + \frac{t_N - t_{N-1}}{4\nu} \cdot \max_{s \in [t_{N-1},T]} \|v(x,s)\|_{A,0}^2 \right)
\]
which goes for \( N \to \infty \) with \( \Delta Z \to 0 \) to
\[
\to \|u_0\|_{A,1} \cdot \exp\left(\sqrt{n} \cdot \int_0^T \|v(x, s)\|_{A,1} \, ds + \frac{1}{4\nu} \cdot \int_0^T \|v(x, s)\|_{A,0}^2 \, ds\right)
\]
and proves (13). Since the limit converges the sequence is bounded, i.e., it exists a \( C_1 > 0 \) such that
\[
\|\partial_j u_N(x, t)\|_{A, j} \leq \|u_N(x, t)\|_{A,1} \leq C_1
\]
for all \( j = 1, \ldots, n \), \( t \in [0, T] \), and \( N \in \mathbb{N} \).
We have so far proved that for \( d = 0 \) and \( 1 \) there are constants \( C_d \) with
\[
\|u_N(x, t)\|_{A, d} \leq C_d
\]
for all \( t \in [0, T] \) and \( N \in \mathbb{N} \). We will now prove (14) and the existence of such constants \( C_d \) for all \( d \geq 2 \) by induction on \( d \).

(13) is (14) for \( d = 1 \) since then the sum \( \sum_{d=2}^d \) is empty. So for \( d = 1 \) (14) is true. So assume for \( d \in \mathbb{N} \) there are constants \( C_j \) with \( \|u_N(x, t)\|_{A, j} \leq C_j \) for all \( j = 0, \ldots, d-1 \), \( t \in [0, T] \) and \( N \in \mathbb{N} \). We show then also \( C_d \) exists.

From (19) we get
\[
\|u_N(x, t_N)\|_{A, \hat{d}} = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_N)|
\]
\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}
\]
\[
+ \sum_{k, l \in \mathbb{Z}^n \setminus \{0\}} |k + l|_\infty \cdot \int_{t_{N-1}}^{t_N} |b_l(s, k)| \, ds \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}
\]
\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}
\]
\[
+ \sum_{k, l \in \mathbb{Z}^n \setminus \{0\}} |k + l|_\infty \cdot \int_{t_{N-1}}^{t_N} |b_l(s, k)| \, ds \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}
\]
\[
+ \sum_{k, l \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |l|_\infty \cdot \int_{t_{N-1}}^{t_N} |b_l(s, k)| \, ds \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}
\]
\[
+ \sum_{k, l \in \mathbb{Z}^n \setminus \{0\}} \sum_{j=2}^d \left( |k|_\infty \cdot |l|_\infty \cdot \int_{t_{N-1}}^{t_N} |b_l(s, k)| \, ds \cdot |a_k(t_{N-1})| \right) 
\]
\[
\times e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})}
\]
\[
\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|_\infty \cdot |a_k(t_{N-1})| \cdot e^{-\nu \cdot k^2 \cdot (t_N - t_{N-1})} \cdot \left( 1 + |k| \cdot \int_{t_{N-1}}^{t_N} \|v(x, s)\|_{A,0} \, ds \right)
\]
\begin{align*}
&\vdash + \sqrt{n} \cdot d \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A_1} \, ds \\
&+ \sqrt{n} \cdot \sum_{j=2}^{d} \left( \frac{d}{j} \right) \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A_j} \, ds \cdot \| u_N(t_{N-1}) \|_{A,d+1-j} \\
&\leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^d \cdot |a_k(t_{N-1})| \cdot \exp \left( |k| \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A,0} \, ds - \nu \cdot k^2 \cdot (t_N - t_{N-1}) \right) \\
&+ \sqrt{n} \cdot \sum_{j=2}^{d} \left( \frac{d}{j} \right) \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A_j} \, ds \cdot \sup_{s \in [0, T]} \| u_N(x, s) \|_{A,d+1-j} \\
&\leq \exp \left( \frac{t_N - t_{N-1}}{4\nu} \max_{s \in [t_{N-1}, t_N]} \|v(x, s)\|_{A^2,0} + \sqrt{n} \cdot d \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A_1} \, ds \right) \\
&\times \sum_{k \in \mathbb{Z}^n} |k|^d \cdot |a_k(t_{N-1})| \\
&+ \sqrt{n} \cdot \sum_{j=2}^{d} \left( \frac{d}{j} \right) \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A_j} \, ds \cdot \sup_{s \in [0, T]} \| u_N(x, s) \|_{A,d+1-j} \\
&\vdash \\
&\leq \exp \left( \sum_{j=1}^{N} \frac{t_j - t_{j-1}}{4\nu} \max_{s \in [t_{j-1}, t_j]} \|v(x, s)\|_{A^2,0} + \sqrt{n} \cdot d \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A_1} \, ds \right) \\
&\times \left( \|u_0\|_{A,d} + \sqrt{n} \cdot \sum_{j=2}^{d} \left( \frac{d}{j} \right) \cdot \int_{T_{-1}}^{t_N} \|v(x, s)\|_{A_1} \, ds \right) \cdot \sup_{s \in [0, T]} \| u_N(x, s) \|_{A,d+1-j} \\
&\vdash \exp \left( \frac{1}{4\nu} \int_{0}^{T} \|v(x, s)\|_{A^2,0} \, ds + \sqrt{n} \cdot d \cdot \int_{0}^{T} \|v(x, s)\|_{A_1} \, ds \right) \\
&\times \left( \|u_0\|_{A,d} + \sqrt{n} \cdot \sum_{j=2}^{d} \left( \frac{d}{j} \right) \cdot \int_{0}^{T} \|v(x, s)\|_{A_j} \, ds \cdot \sup_{s \in [0, T]} \| u_N(x, s) \|_{A,d+1-j} \right)
\end{align*}

which proves (14) for \(d\). Since the limit exists the sequence is bounded, i.e., it exists a constant \(C_d\) such that (\%) holds. Hence, by induction (14) and (\%) hold for all \(d \in \mathbb{N}\) with \(d \geq 2\).

We apply now Lemma 1.1. We just proved
\[
\max_{t \in \mathbb{N}^n: |t| = d} \| \partial^t u_N(x, t) \|_{\infty} \leq \sup_{s \in [0, T]} \| u_N(x, s) \|_{A,d} \leq C_d < \infty
\]
for all \(d \in \mathbb{N}\), i.e., condition (i) of Lemma 1.1 is fulfilled for the family \(\{u_N\}_{N \in \mathbb{N}}\).

Since all \(u_N\) are piece-wise differentiable in \(t\) with (18) and all derivatives of \(\partial^t u_N\) are bounded, the family \(\{u_N\}_{N \in \mathbb{N}}\) is Lipschitz in \(t\) with a Lipschitz constant \(L\) independent on \(N \in \mathbb{N}\) and \(t \in [0, T]\), i.e., \(\{u_N\}_{N \in \mathbb{N}}\) is equi-continuous and condition (ii) of Lemma 1.1 is fulfilled. Hence, by Lemma 1.1 the family \(\{u_N\}_{N \in \mathbb{N}}\) is relatively compact and there exists an accumulation point \(u \in C^0(T^n \times [0, T], \mathbb{R}^n)\).
and a subsequence \( \{N_j\}_{j \in \mathbb{N}} \subseteq \mathbb{N} \) such that

\[
\| \partial^\alpha u_{N_j}(x, t) - \partial^\alpha u(x, t) \|_\infty \xrightarrow{j \to \infty} 0
\]

for all \( \alpha \in \mathbb{N}_0^n \). By \( (18) \) \( u \) is a smooth solution of the initial value problem with the desired bounds for all \( t \in [0, T] \). Since \( a_{0,N}(t) = a_0(0) \) for all \( t \in [0, T] \) is constant, so is \( a_0(t) \) of \( u \).

Since \( T > 0 \) was arbitrary, we can set \( T = 1 \) and from \( [0, \infty) = \bigcup_{j \in \mathbb{N}} [j, j + 1] \) it follows that \( u \) exists for all times \( t \in [0, \infty) \). \( \square \)

**Remark 5.2.** We showed Theorem 5.1 only for \( \nu \Delta = \nu (\partial_1^2 + \cdots + \partial_n^2) \) for notational simplicity. But of course it is clear it also holds for \( \nu \cdot \Delta = \nu_1 \partial_1^2 + \cdots + \nu_n \partial_n^2 \). Just use \( \nu_1 \cdot a_1^2 + \cdots + \nu_n \cdot a_n^2 \geq \min\{\nu_1, \ldots, \nu_n\} \cdot (k_1^2 + \cdots + k_n^2) \).

The construction of the approximate solutions \( u_N \) in \( (17) \) and \( (19) \) can be seen as gluing together the solutions (linear in \( t \)) from Proposition 4.1 and Proposition 4.3 in the Trotter type fashion \([Tro59]\) where convergence is ensured by the Montel space version of Arzelà–Ascoli (Lemma 1.1). At first it might seem impractical to replace a Sobolev space with one norm by a Montel space with infinitely many semi-norms. But as seen from Theorem 5.1 in case of \( (3) \) only the semi-norms \( \| \cdot \|_{A,0} \) and \( \| \cdot \|_{A,1} \) have to be bounded separately by long calculations. But for \( \| \cdot \|_{A,d} \) with \( d \geq 2 \) we can use induction.

In the following we will apply the bounds from Theorem 5.1 to the Navier–Stokes equation by setting \( u = -u \). Similar to our previous study \([DiJ9]\) we find that only the semi-norms \( \| \cdot \|_{A,0} \) and \( \| \cdot \|_{A,1} \) see themselves through the exponential function.

This will imply that we can not exclude a finite blow-up time for all initial data and can only ensure local existence (A) in Theorem 6.1. But all other semi-norms \( C_d = \| \cdot \|_{A,d} \) with \( d \geq 2 \) see themselves only linearly

\[
C_d(t) \leq B_d(t) + \tilde{B}_d(t) \cdot \int_0^t C_d(s) \, ds
\]

where \( B_d(t) \) and \( \tilde{B}_d(t) \) depend only on \( C_0, \ldots, C_{d-1} \). Therefore, as long as the lower semi-norms are bounded, higher semi-norms do not develop singularities which ensures regularity (B) in Theorem 6.2.

\( (17) \) and \( (19) \) were carefully chosen. At first we applied \( \nu \Delta + b_0(t) \cdot \nabla \) from Proposition 4.3 to get \( a_k(t_j) \cdot e^{-\nu k^2 (t_{j+1} - t_j) + i \int_{t_j}^{t_{j+1}} (b_0(s), k) \, ds} \) and then \( \mathcal{F}[\exp 0 \mathcal{F}[(v - b_0) \nabla \cdot] \mathcal{F} \cdot] \) is applied so that in the rearranging of the double sum \( \sum_{k,t} E \in \mathbb{Z} \mathcal{\backslash} \{0\} \) all new \( a_k \)'s have the same \( e^{-\nu k^2 (t_{j+1} - t_j) + i \int_{t_j}^{t_{j+1}} (b_0(s), k) \, ds} \) factor. We have to emphasize here that we had to treat the \( b_0 \)-term of \( v \) differently. We have seen in Proposition 4.3 that only a complex rotation is induced by \( b_0 \) and this disappears taking absolute values. On the other hand, when we look at the solution \( u \), the \( a_0 \)-term is always constant. And to smuggle in the \( -\delta \cdot t \) in \( (15) \) and \( (16) \) we needed to remove the \( a_0 \)-term since it is not affected by \( \nu \Delta \), see Example 3.3. Our proof of global existence from small initial data (C) in Theorem 6.3 will rely on this \( -\delta \cdot t \). This is the reason why we introduced \( \| \cdot \|_{A,0} = \| \cdot \|_{A} - a_0 \).

In Proposition 4.4 and Corollary 4.2 we have seen that the solution fulfills

\[
\| u(x, t) \|_{A,0} = |a_k(0)| \cdot \exp \left( \int_0^t (b_1(s), k) \, ds \right).
\]

\( \lambda k \to \infty \) as \( \lambda \to \infty \) affects the summation in the proof for \( \| u_N(x, t) \|_{A,0} \) (and all of the following calculations for \( \| u_N(x, t) \|_{A,d}, d \geq 1 \)). But since \( \nu > 0 \) we have

\[
\sum_{k \in \mathbb{Z} \mathcal{\backslash} \{0\}} \left| a_k(N(t_N - j)) \right| \cdot \exp \left( |k| \int_{t_N - j}^{t_N} \| v(x, s) \|_{A,0} \, ds - \nu k^2 (t_N - t_N - j) \right), \quad (8)
\]
While ($\$) would blow up immediately if $\nu = 0$, for $\nu > 0$ we have a quadratic equation in $|k|$ in the exponent with negative leading coefficient. Using the elementary fact, that this quadratic equation

$$ (t_N - t_{N-1}) \cdot |k| \cdot \left( \max_{s \in [t_{N-1}, t_N]} \|v(x, s)\|_{A,0} - \nu \cdot |k| \right) $$

is $\leq 0$ for all $k$ with $|k| \geq K = \nu^{-1} \cdot \max_{s \in [t_{N-1}, t_N]} \|v(x, s)\|_{A,0}$ and positive only for finitely many $k$’s with $0 < |k| < K$ with maximum at $K/2$ enabled us to split the sum ($\$)$ into a contraction and an expansion sum in (#)

$$(\#) \leq \sum_{k:|k|\geq K} |a_{k,N}(t_{N-1})| + \exp \left( \frac{(t_N - t_{N-1})}{4} \max_{s \in [t_{N-1}, t_N]} \|v(x, s)\|_{A,0}^2 \right) \cdot \sum_{k:0 < |k| < K} |a_k(t_{N-1})|.$$  

Here lies the reason why we get for different $\nu$’s depending on $\max_s \|v(x, s)\|_{A,0}$ different bounds. In the worst case the second expansion sum is non-empty and we use the exp-factor $\geq 1$ but independent on $k$ also for the first (contraction) sum. But if $\nu$ is large enough compared to $\max_s \|v(x, s)\|_{A,0}$ we know that the second (expansion) sum is empty and we get the second bounds (15) and (16) without the $\int_0^t \|v(x, s)\|_{A,0}^2$ ds in the exponent. We can even keep some $\delta \geq 0$ to get $-\delta \cdot t$ because we treated the $a_0$- and $b_0$-Fourier coefficients differently.

6. Solutions and bounds of the Navier–Stokes equations

We will now apply Theorem 5.1 and its bounds to the Navier–Stokes equation, i.e., we will set $v = -u$ and hence $b_k = -a_k$. We use time-delayed solutions to approximate the solutions. Let $\epsilon > 0$. Set $v(x, t) = u_0(x)$ for all $t \in [0, \epsilon]$. Then from Theorem 5.1 we get a smooth solution $u^{(\epsilon)}(x, t)$ for all $t \in [0, \epsilon]$. Then set $v(x, t) := u^{(\epsilon)}(x, t - \epsilon)$ for all $t \in [\epsilon, 2\epsilon]$ and by induction we get a solution $u^{(\epsilon)}$ for all $t \in [0, \infty)$. We call this a time-delayed solution. For more on time-delayed solutions see e.g. [CR01], [Var08], or [dD19].

As discussed in the introduction uniqueness of smooth solutions (D) was first proved by Leray [Ler34a, Ler34b]. In [MB02, Prop. 3.1] this is very efficiently proved by the energy method and holds on $\mathbb{R}^n$ and $\mathbb{T}^n$ for all $n \in \mathbb{N}$. We will therefore use this result.

The local existence (A) with explicit minimal times is ensured by the following.

Theorem 6.1. Let $n \in \mathbb{N}$, $n \geq 2$, $\nu > 0$ and $u_0 \in C^{\infty}(\mathbb{T}^n, \mathbb{R}^n)$. The Navier–Stokes equation has a unique smooth solution $u(x, t) \in C^{\infty}(\mathbb{T}^n, \mathbb{R}^n)$ for all $t \in [0, T^*)$ with

$$ T^* \geq \begin{cases} \infty, & \text{if } \|u_0\|_{A,1} = 0, \\ \frac{2\nu}{\|
abla u_0\|_{A,1}^2} - \frac{\|
abla u_0\|_{A,0}^2}{2\nu \|
abla u_0\|_{A,1}^2} + \frac{3\nu}{\|
abla u_0\|_{A,0}^2} & \text{if } \|u_0\|_{A,0}^2 \geq 4\sqrt{n} \cdot \nu \cdot \|u_0\|_{A,1}^2, \\ \geq 0. & \text{else} \end{cases} $$

Proof. For $\|u_0\|_{A,1} = 0$ we have that $u_0(x)$ is constant and hence $u(x, t) = u_0(x)$ is a smooth solution for all times $t \geq 0$. Therefore, we can assume w.l.o.g. $\|u_0\|_{A,1} > 0$.

For $\epsilon \in (0, 1]$ let $u^{(\epsilon)}$ be a time-delayed solution from Theorem 5.1. We want to apply Lemma 1.1 to the family $\{u^{(\epsilon)}\}_{\epsilon \in (0, 1]}$. Hence, we want to determine the interval $I \subseteq [0, \infty)$ where we have

$$ C_d(t) := \lim_{\epsilon \to 0} \sup_{\epsilon \in [0, 1]} \|u^{(\epsilon)}(x, t)\|_{A,d} < \infty $$

for $t \in I$ and all $d \in \mathbb{N}_0$. 
From Theorem 5.1[12], we find for \( e \to 0 \)

\[
C_0(t) \leq \|u_0\|_{A,0} \cdot \exp \left( \frac{1}{4\nu} \int_0^t C_0(s)^2 \, ds \right).
\]

In the worst case we have equality. Differentiating this equality gives

\[
f_0(t) = f_0(0) \cdot \exp \left( \frac{1}{4\nu} \int_0^t f_0(s)^2 \, ds \right)
\]

\[
\Rightarrow C_0(t) \leq f_0(t) = \frac{1}{4\nu} \cdot f_0^3(t)
\]

(20)

with singularity at \( T_0 := 2\nu\|u_0\|_{A,0}^{-2} > 0 \). For \( d = 1 \) we find from the bound \[13\]

\[
f_1(t) = f_1(0) \cdot \exp \left( \sqrt{n} \cdot \int_0^t f_1(s) \, ds + \frac{1}{4\nu} \int_0^t f_0(s)^2 \, ds \right)
\]

\[
\Rightarrow C_1(t) \leq f_1(t) = \frac{1}{\gamma \cdot 4\nu \cdot \|u_0\|_{A,0}^2 - 2t + \sqrt{n} \cdot (4\nu \cdot \|u_0\|_{A,0}^2 - 2t)}
\]

\[
= \frac{1}{\sqrt{c - 2t} \cdot (\gamma + \sqrt{n} \cdot \sqrt{c - 2t})}
\]

(21)

with \( \gamma := \frac{\|u_0\|_{A,0}^2}{2\sqrt{2\nu \|u_0\|_{A,0}^2}} \) and \( c = 4\nu \cdot \|u_0\|_{A,0}^2 \).

If \( \gamma \geq 0 \), then \( f_1 \) (and \( f_0 \)) exists on \([0,T_0]\) with a singularity at \( T_0 \).

If \( \gamma \in (-\sqrt{n} \cdot c,0) \), then \( \gamma + \sqrt{n} \cdot \sqrt{c - 2t} \geq 0 \) for all \( t \in [0,T_1] \) with \( T_1 := \frac{\gamma^{-2} - 2}{2\nu} = 2\nu\|u_0\|_{A,0}^2 \), and \( f_1 \) has a singularity at \( T_1 \in (0,T_0) \). \( f_0 \) and \( f_1 \) exist both on \([0,T_1]\). Which proves the third case.

Let \([0,\tau] \subset [0,T] \) where \( T \) is one of the three cases \( \infty, T_0, \) or \( T_1 \) we just calculated. Then \( f_0(t) \) and \( f_1(t) \) are continuous on \([0,\tau]\) and therefore bounded. \( C_d(t) \) fulfills the bounds \[14\] for all \( d \geq 2 \). But \[14\] has the structure

\[
C_d(t) \leq B_d(t) + \tilde{B}_d(t) \cdot \int_0^t C_d(s) \, ds
\]

(22a)

where \( B_d \) and \( \tilde{B}_d \) are continuous non-decreasing functions depending only on \( C_0, \ldots, C_{d-1} \). Hence,

\[
C_d(t) \leq B_d(\tau) + \tilde{B}_d(\tau) \cdot \int_0^\tau C_d(s) \, ds
\]

(22b)

with equality in the worst cast. Differentiating the worst cast gives

\[
f_d'(t) = \tilde{B}_d(\tau) \cdot f_d(t) \quad \Rightarrow \quad C_d(t) \leq f_d(t) = B_d(\tau) \cdot \exp \left( \tilde{B}_d(\tau) \cdot t \right),
\]

(22c)

i.e., all \( C_d \) are bounded functions on \([0,\tau] \subset [0,T] \).

Since \( C_d(t) < \infty \) for all \( d \in \mathbb{N} \) and \( t \in [0,\tau] \) the family \( \{ u^{(e)} \}_{e \in [0,1]} \) fulfills condition (i) of Lemma 1.1. But all \( u^{(e)} \) fulfill \[3\] with \( v(x,t) = u^{(e)}(x,t - \varepsilon) \), i.e.,

\[
\sup_{s \in [0,\tau], e \in (0,1]} \| \partial_t u^{(e)}(x,s) \|_{\infty} < \infty,
\]

and condition (ii) of Lemma 1.1 is fulfilled. Therefore, the family \( \{ u^{(e)} \}_{e \in (0,1]} \) is relatively compact and has an accumulation point \( u \in C^\infty(\mathbb{T}^n \times [0,\tau], \mathbb{R}^n) \). Let \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1] \) with \( \varepsilon_j \to 0 \) as \( j \to \infty \). Then

\[
\| \partial^n u^{(\varepsilon_j)}(x,t) - \partial^n u(x,t) \|_{\infty} \xrightarrow{j \to \infty} 0.
\]
and \( u(x,t) \) is a smooth solution of the Navier–Stokes equation for all \( t \in [0,\tau] \).

But since \([0,\tau] \subset [0,T]\) was arbitrary let \((\tau_j)_{j \in \mathbb{N}}\) with \( \tau_1 = 0 \) be an increasing sequence with \( \tau_j \to T \) as \( j \to \infty \). Then \([0,T] = \bigcup_{j \in \mathbb{N}} [\tau_j,\tau_{j+1}] \) and a solution \( u \) on \( \bigcup_{j=1}^{\infty} [\tau_j,\tau_{j+1}] \) can be extended to a solution \( u \) on \( \bigcup_{j=1}^{\infty} [\tau_j,\tau_{j+1}] \). Hence, there exists a smooth solution \( u \) for all \( t \in [0,T) \). By [MB02, Prop. 3.1] it is unique. □

The smoothness is solely determined by the first derivatives \((B)\) as we see in the following results.

**Theorem 6.2.** Let \( \nu > 0 \) and \( n \in \mathbb{N} \) with \( n \geq 2 \). Assume the solution of the Navier–Stokes equation with initial value \( u_0 \in C^\infty (\mathbb{T}^n, \mathbb{R}^n) \) has the priori bounds
\[
\| \partial_j u(x,t) \|_A < C < \infty
\]
for all \( j = 1, \ldots , n \) and \( t \in [0,\infty) \) or \([0,T], T > 0\). Then there exists a unique smooth solution of the Navier–Stokes equation for all \( t \in [0,\infty) \) resp. \([0,T]\).

**Proof.** It is sufficient to prove the statement for \([0,\infty)\). Let \( u \) be the unique smooth solution of the Navier–Stokes equation for small times \( t \in [0,\tau) \) from Theorem 6.1 where \( \tau \) is one of the three cases in Theorem 6.1. By the boundedness of the first derivatives we get
\[
\| u(x,t) \|_{A,0}, \| u(x,t) \|_{A,1} \leq \| \partial_j u(x,t) \|_A + \cdots + \| \partial_n u(x,t) \|_A \leq n \cdot C.
\]
But since \( \tau \) only depends on \( \| u_0 \|_{A,0} \) and \( \| u_0 \|_{A,1} \) and these do not exceed \( n \cdot C \) on \([0,\tau/2]\), \( u \) exists also on \([\tau/2,\tau]\), \([\tau,3\tau/2]\), etc. Hence, there exists a unique smooth solution \( u \) for all \( t \in [0,\infty) \). □

For small initial data \( u_0 \) with respect to \( \nu \) we get that a smooth solution exists for all times \((C)\), i.e., the Navier–Stokes equation never breaks down.

**Theorem 6.3.** Let \( \nu > 0 \), \( n \in \mathbb{N} \) with \( n \geq 2 \), and \( u_0 \in C^\infty (\mathbb{T}^n, \mathbb{R}^n) \) with
\[
\| u_0 \|_{A,0} + \sqrt{n} \cdot \| u_0 \|_{A,1} \leq \nu.
\](23)

Then the Navier–Stokes equation has a unique smooth solution for all times.

**Proof.** For \( \varepsilon \in (0,1) \) let \( u^{(\varepsilon)} \) be a time-delayed solution from Theorem 5.1.

Since \( \| u_0 \|_{A,0} + \delta \leq \nu \) with \( \delta = \sqrt{n} \cdot \| u_0 \|_{A,1} \) we have (15) which implies
\[
\| u^{(\varepsilon)}(x,t) \|_{A,0} \leq \| u_0 \|_{A,0} \cdot e^{-\delta \cdot t} \leq \| u_0 \|_{A,0}
\]
for all \( t \in [0,\infty) \) and \( \varepsilon \in (0,1) \). By (16) we have
\[
\| u^{(\varepsilon)}(x,t) \|_{A,1} \leq \| u_0 \|_{A,1}
\]
for all \( t \in [0,\infty) \) and from (22) we find that there exist continuous non-decreasing functions \( C_d : [0,\infty) \to [0,\infty) \) such that \( \| u^{(\varepsilon)}(x,t) \|_{A,d} \leq C_d(t) \) for all \( d \in \mathbb{N}_0 \).

Hence, for \( t \in [0,1] \) the family of time-delayed solutions \( \{ u^{(\varepsilon)} \}_{\varepsilon \in (0,1]} \) fulfills all conditions in Lemma 5.1 and an accumulation point \( u \) solves the Navier–Stokes equation for all \( t \in [0,1] \). But \( u \) fulfills \((*)\) and \((***)\) and by induction the Navier–Stokes equation has a smooth solution for all times. □

(23) holds when the more restrictive conditions
\[
\| u_0 \|_{A,0} + \sqrt{n} \cdot (\| \partial_1 u_0 \|_A + \cdots + \| \partial_n u_0 \|_A) \leq \nu
\]
or
\[
\| u_0 \|_A + \sqrt{n} \cdot (\| \partial_1 u_0 \|_A + \cdots + \| \partial_n u_0 \|_A) \leq \nu
\]
hold.
7. Conclusion

We started the investigation of the periodic Navier–Stokes equation by expressing the Leray projection to the div-free part of a vector field as the Gram–Schmidt orthogonalization and found the simple fact that the $l^2$-norm of the Fourier coefficients is of course non-increasing under this projection, i.e., $\|Pf\|_A \leq \|f\|_A$ which not necessarily holds for $\|\cdot\|_\infty$. By expanding $u$ and the given $v$ in terms of the Fourier coefficients $a_k$ resp. $b_l$ we gained the system of ordinary differential equations (10) of the $a_k$’s in Theorem 3.4 which is equivalent to the Navier–Stokes equation (1) and (3). Hence, we needed only to follow the $a_k$’s in time. This was done in any dimension $n \geq 2$. Compared to our previous study [dD19] where we had to incorporate the Leray projection by going to the vorticity formulation of the Navier–Stokes equation here in the periodic case we have simple access to the Leray projection.

Simple solutions of (3) with given $v$ were calculated in Section 4 and insight into the behavior and growth of the Fourier coefficients were presented. We showed in Theorem 4.4 that certain (the worst) cases lead to Bessel functions. These cases can only appear for $n \geq 3$. That in these cases the $a_k(t)$ are multiples of Bessel functions shows that cancellations of the coefficients appear. While a bound which only treats the absolute values grows exponentially fast, the analytic $a_k$’s as Bessel functions remain bounded and even oscillate, see Figure 1. Here we found the first indication that the later bounds in Theorem 5.1 are probably not optimal.

With the bounds from Theorem 5.1 we treated the Navier–Stokes equation. The essential calculations are for how long the $\|u(x, t)\|_{A,d}$’s remain finite. We see that the local existence (A) in Theorem 6.1 and the regularity (B) in Theorem 6.2 of solutions of the Navier–Stokes equation reduces to the elementary calculations (20), (21), and (22). If $\nu$ is large enough compared to $u_0$, e.g. when

$$\|u_0\|_A + \sqrt{n} \cdot (\|\partial_1 u_0\|_A + \cdots + \|\partial_n u_0\|_A) \leq \nu,$$

then we even find in Theorem 6.3 that a global unique smooth solution of the Navier–Stokes equation exists since we kept $-\delta \cdot t$ in (15) and (16).

We also want to emphasize that the blow up in $f_0$ is a pole of order $1/2$ and for $f_1$ a pole of order 1. But from the examples in Section 2 we have seen that $\nu \Delta$ is able to remove a pole of order 1 in $f_0$ and a pole of order 2 in $f_1$.

Charles L. Fefferman stated at the end of the Navier–Stokes millennium problem description [Fef06]: “Since we don’t even know whether these solutions [of the Navier–Stokes equation] exist, our understanding is at a very primitive level. Standard methods from PDE appear inadequate to settle the problem. Instead, we probably need some deep, new ideas.” But our method (Montel space version of Arzelà–Ascoli applied to the Fourier coefficients) with elementary calculations (which do not exceed the level of undergraduate mathematics) immediately answers three (A to C) out of the five main questions (A to E) of the Navier–Stokes equation (at least in the periodic case but here for all dimensions $n \geq 2$). For $n = 2$ the (periodic) Navier–Stokes problem was solved completely by Leray [Ler33] and therefore the summations in Theorem 5.1 remain finite. In future it is important to understand how these summations for $n = 2$ remain finite and if this can be carried over to higher dimensions, especially since cases as in Theorem 4.4 can only appear for $n \geq 3$. Our approach is simple and still has a lot of techniques which can be incorporated. But it also shows that probably the most elementary and oldest technique and idea in mathematics could provide an answer (not only for the Navier–Stokes but for other non-linear partial differential equations as well):

Direct calculations.

We used this idea a lot!
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THE PERIODIC NAVIER–STOKES EQUATION: EXISTENCE AND SMOOTHNESS

We retain the notations in the proof of Theorem 4.4. We calculate

\[
\langle \varepsilon_{k+1}\rangle, \exp(B_+(t) \cdot S_+ + B_-(t) \cdot S_-) \varepsilon_k \rangle \in C^{n \times n}
\]

for the remaining cases \( j = 2\alpha + 1, -2\alpha, \) and \(-2\alpha - 1\) with \( \alpha \in \mathbb{N}_0 \). We assume \( B_+(t) \neq 0 \) for all three cases, otherwise (\#) is zero for each case.

Let \( j = 2\alpha + 1 \) with \( \alpha \in \mathbb{N}_0 \). Then

\[
\langle \varepsilon_{k+(2\alpha+1)}\rangle, \exp(B_+(t) \cdot S_+ + B_-(t) \cdot S_-) \varepsilon_k \rangle = \langle S_+^{2\alpha+1}\varepsilon_k, \sum_{a=0}^{\infty} \frac{1}{a!}(B_+(t) \cdot S_+ + B_-(t) \cdot S_-)^{a}\varepsilon_k \rangle
\]

\[
= \langle \varepsilon_k, S_-^{2\alpha+1} \sum_{a=0}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \cdot B_+(t)^b \cdot B_-(t)^{a-b} \cdot S_+^{b} \cdot S_-^{a-b}\varepsilon_k \rangle
\]

\[
= \langle \varepsilon_k, \sum_{a=0}^{\infty} \frac{1}{a!} \sum_{b=0}^{a} \binom{a}{b} \cdot B_+(t)^b \cdot B_-(t)^{a-b} \cdot S_-^{2\alpha+2b+2\alpha+1}\varepsilon_k \rangle
\]

hence, \( a - 2b + 2\alpha + 1 \) must be zero, i.e., we sum only over the odd \( a \)'s,

\[
= \langle \varepsilon_k, \sum_{a=0}^{\infty} \frac{1}{(2a+1)!} \sum_{b=0}^{2a+1} \binom{2a+1}{b} \cdot B_+(t)^{2a+1-b} \cdot B_-(t)^{2a+1-b} \cdot S_-^{2a-2b+2\alpha+2}\varepsilon_k \rangle
\]

hence, \( 2a - 2b + 2\alpha + 2 \) must be zero, i.e., \( b = a + \alpha + 1 \),

\[
= \sum_{a=0}^{\infty} \frac{1}{(2a+1)!} \cdot \binom{2a+1}{a+\alpha+1} \cdot B_+(t)^{a+\alpha+1} \cdot B_-(t)^{a-\alpha}
\]

the summation can start at \( a = \alpha \), otherwise \( \binom{2a+1}{a+\alpha+1} \) is zero,

\[
= \sum_{a=0}^{\infty} \frac{1}{(2a+1)!} \cdot \binom{2a+1}{a+\alpha+1} \cdot B_+(t)^{a+\alpha+1} \cdot B_-(t)^{a-\alpha}
\]

shifting \( a \) back to start at \( a = 0 \) gives

\[
= \sum_{a=0}^{\infty} \frac{1}{(2a+2\alpha+1)!} \cdot \binom{2a+2\alpha+1}{a+2\alpha+1} \cdot B_+(t)^{a+2\alpha+1} \cdot B_-(t)^{a}
\]

\[
= \sum_{a=0}^{\infty} \frac{B_+(t)^{a+2\alpha+1} \cdot B_-(t)^{a}}{(a+2\alpha+1)! \cdot a!}
\]

\[
= B_+(t)^{2\alpha+1} \sum_{a=0}^{\infty} \frac{(-1)^a \cdot |B_+(t)|^{2a}}{(a+2\alpha+1)! \cdot a!}
\]
Hence, the statement is true for $a$ shift now 2 hence $a$ 28 THE PERIODIC NAVIER–STOKES EQUATION: EXISTENCE AND SMOOTHNESS summation can start at $a$ hence the statement is also true for $j = 2\alpha + 1$ with $\alpha \in \mathbb{N}_0$. Finally, let $j = -2\alpha$ with $\alpha \in \mathbb{N}$. Then

$$
\langle e_k, \exp(2\alpha \cdot (S_+ + B_-(t) \cdot S_-)) e_k \rangle = \langle S^{2\alpha}_+ e_k, \sum_{\alpha = 0}^{\infty} \frac{1}{(2\alpha)!} \sum_{b=0}^{\infty} \binom{2\alpha}{b} \cdot B_+(t)^{2\alpha-b} \cdot B_-(t)^b \cdot S_+^{\alpha-b} \cdot S_-^b e_k \rangle
$$

ow 2a - 2b + 2a must be zero, i.e., $b = a + \alpha$

$$
\sum_{\alpha = 0}^{\infty} \frac{1}{(2\alpha)!} \cdot \left( \binom{2\alpha}{a + \alpha} \cdot B_+(t)^{a-\alpha} \cdot B_-(t)^{a+\alpha} \right)
$$

summation can start at $a = \alpha$, since otherwise $\binom{2\alpha}{a+\alpha}$ is zero,

$$
\sum_{\alpha = \alpha}^{\infty} \frac{1}{(2\alpha)!} \cdot \left( \binom{2\alpha}{a + \alpha} \cdot B_+(t)^{a-\alpha} \cdot B_-(t)^{a+\alpha} \right)
$$

shift $a$ to start at $a = 0$ again

$$
\sum_{\alpha = 0}^{\infty} \frac{B_+(t)^{\alpha} \cdot B_-(t)^{\alpha+2\alpha}}{a! \cdot (a + 2\alpha)!}
$$

$$
\frac{(-1)^\alpha}{B_+(t)^\alpha} \sum_{\alpha = 0}^{\infty} \frac{(-1)^\alpha \cdot (2 \cdot B_+(t))^{2\alpha+2\alpha}}{a! \cdot (a + 2\alpha)! \cdot 2^{2\alpha+2\alpha}}
$$

$$
\left( \frac{B_+(t)}{|B_+(t)|} \right)^{-2\alpha} \cdot J_{2\alpha}(2 \cdot |B_+(t)|).
$$

Hence, the statement is true for $j = -2\alpha$ with $\alpha \in \mathbb{N}$. Finally, let $j = -2\alpha - 1$ with $\alpha \in \mathbb{N}_0$. Then

$$
\langle e_{k-(2\alpha+1)} \cdot \exp(2\alpha \cdot (S_+ + B_-(t) \cdot S_-)) e_k \rangle = \langle S^{2\alpha+1}_+ e_k, \sum_{\alpha = 0}^{\infty} \frac{1}{(2\alpha)!} \sum_{b=0}^{\infty} \binom{2\alpha+1}{b} \cdot B_+(t)^{a-\alpha} \cdot B_-(t)^b \cdot S_+^{\alpha-b} \cdot S_-^b e_k \rangle
$$


\[ = (e_k, \infty \sum_{a=0}^{\infty} \frac{1}{a!} \cdot \sum_{b=0}^{a} \binom{a}{b} \cdot B_+(t)^{a-b} \cdot B_-(t)^{a} \cdot \sqrt{\frac{a-2b+2\alpha+1}{e_k}}) \]

hence \( a - 2b + 2\alpha + 1 \) must be zero, i.e., summation can be done over all odd \( a \)'s

\[ = (e_k, \infty \sum_{a=0}^{\infty} \frac{1}{(2a + 1)!} \cdot \sum_{b=0}^{2a+1} \binom{2a+1}{b} B_+(t)^{2a+1-b} \cdot B_-(t)^{b} \cdot \sqrt{\frac{2a-2b+2\alpha+2}{e_k}}) \]

and therefore \( 2a - 2b + 2\alpha + 2 \) must be zero, i.e., \( b = a + \alpha + 1 \),

\[ = \infty \sum_{a=\alpha}^{\infty} \frac{1}{(2a + 1)!} \left( \frac{2a + 1}{a + \alpha + 1} \right) \cdot B_+(t)^{a-\alpha} \cdot B_-(t)^{a+\alpha+1} \]

and the summation can start at \( a = \alpha \), since otherwise \( \left( \frac{2a+1}{a+\alpha+1} \right) \) is zero,

\[ = \infty \sum_{a=\alpha}^{\infty} \frac{1}{(2a + 1)!} \left( \frac{2a + 1}{a + \alpha + 1} \right) \cdot B_+(t)^{a-\alpha} \cdot B_-(t)^{a+\alpha+1} \]

shifting \( a \) to start summation at \( a = 0 \) again we get

\[ = \infty \sum_{a=0}^{\infty} \frac{B_+(t)^{a} \cdot B_-(t)^{a+2\alpha+1}}{a! \cdot (a + 2\alpha + 1)!} \]

\[ = \left( \frac{B_+(t)}{|B_+(t)|} \right)^{-2\alpha-1} \frac{(-1)^a \cdot (2 \cdot B_+(t)^2)^{2a+2\alpha+1}}{a! \cdot (a + 2\alpha + 1)! \cdot 2^{2a+2\alpha+1}} \]

\[ = \left( \frac{B_+(t)}{|B_+(t)|} \right)^{-2\alpha-1} \cdot J_{-2\alpha-1}(2 \cdot |B_+(t)|). \]

So finally also the last case \( j = -2\alpha - 1 \) with \( \alpha \in \mathbb{N}_0 \) is proved. \( \square \)