3-MANIFOLD INVARIANTS FROM COSETS

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Abstract. We construct unitary modular categories for a general class of coset conformal field theories based on our previous study of these theories in the algebraic quantum field theory framework using subfactor theory. We also consider the calculations of the corresponding 3-manifold invariants. It is shown that under certain index conditions the link invariants colored by the representations of coset factorize into the products of the the link invariants colored by the representations of the two groups in the coset. But the 3-manifold invariants do not behave so simply in general due to the nontrivial branching and selection rules of the coset. Examples in the parafermion cosets and diagonal cosets show that 3-manifold invariants of the coset may be finer than the products of the 3-manifold invariants associated with the two groups in the coset, and these two invariants do not seem to be simply related in some cases, for an example, in the cases when there are issues of “fixed point resolutions”. In the later case our framework provides a representation theory understanding of the underlying unitary modular categories which has not been obtained before.

Introduction

This is the fourth paper in a series of papers on algebraic coset conformal field theories (cf. [X1], [X2] and [X3]), devoted to the construction and calculation of 3-manifold invariants associated with a general class of coset conformal field theories.

The history of quantum invariants started from the striking discovery of Vaughan Jones of a new polynomial invariant of classical knots and links (cf. [J2]) by using his subfactor theory (cf. [J1]). The quantum 3-manifold invariants were predicted by E. Witten (cf. [Wi]) motivated by a quantum field theory interpretation of Jones polynomial. There are several mathematical approaches to the construction of quantum 3-manifold invariants. The original approach [RT] (also cf. [KM]) is based on theory of quantum groups at roots of unity, which has a subtle tensor product structure (cf. [A]) among a distinguished class of finite dimensional modules. The work of [TW1], [TW2] and [W2] relies on ideas from quantum groups and subfactors. There are also work (cf. [EK1], [EK2] and references therein) based on subfactors but there are no general calculations of the invariants. The approach we take in this paper (cf. [R] and [FRS]) is very different from that of [RT], [TW1], [TW2] and
[W2]. We still have a finite set of modules, but they are all infinite dimensional. The tensor product in this framework is very natural: they correspond to compositions of certain endomorphisms (cf. P. 533 of [C]). There are several analytical problems in this approach, one of the most difficult ones is the proof of finite index of certain inclusions. For the examples considered in this paper, this problem is settled by [Wآ], [X1] and [X7]. There are several advantages to this approach. The first is that our construction is automatically unitary\(^1\) since we are using von Neumann algebras. The second is that one of the axioms (non-degeneracy of $S$-matrix) of modular categories (cf. §1.1) which is most difficult to check in other approaches, follows from finiteness of certain index by [X3]. The third and perhaps the most important one is that our framework provides a mathematical understanding of some unitary modular categories which has not been obtained by other approaches (cf. §3.6). However the proof of the existence of invariants does not guarantee an explicit formula. One of the advantages in the approaches of [RT] (also cf. [KM] and [CM]) is that explicit formulas are given for any 3-manifold. The second goal of this paper is to obtain explicit formulas of 3-manifold invariants for a large class of cosets.

Now we’d like to explain why 3-manifold invariants from cosets may be interesting. Let $G$ be a simply connected compact Lie group and let $H \subset G$ be a Lie subgroup. Consider representations of loop group $L G$ (cf. §1.5) with positive energy at level $k^2$. Assume the coset $H \subset G_k$ verifies the conditions of Th. A in §1.7. Let $M$ be a closed oriented 3-manifold, and denote by $\tau_{G/H}(M)$ (resp. $\tau_{G}(M), \tau_{H}(M)$) the 3-manifold invariants associated to the coset $H \subset G_k$ (resp. groups $G, H$) whose existence is implied by Th. A. By Th. B in §2.3, it is natural to believe that $\tau_{G/H}(M)$ should contain the same information as $\tau_{G}(M)\tau_{H}(M)$, but the following example shows that this is not true.

Take $G = SU(2)_1 \times SU(2)_1$, and $H = SU(2)_2$ is the diagonal subgroup of $G$ at level 2. It is well known that the coset is the critical Ising model. By an easy adaption of the argument in §7 of [KM], it is shown in Prop. 3.6.3 that

$$\tau_{G/H}(M) = \sum_{\theta} \exp(-\frac{2\pi i \mu(M_\theta)}{16}),$$

where $\mu(M_\theta)$ is the $\mu$-invariant of $M$ with spin structure $\theta$, and the sum is over all spin structures (cf. §7 of [KM]). This formula can be compared to $\tau_{H}(M)$, which is $\tau_4(M)$ in the notation of [KM], and is given by

$$\tau_{H}(M) = \sum_{\theta} \exp(-\frac{6\pi i \mu(M_\theta)}{16}).$$

\(^1\)The unitarity or positivity question in the case of quantum groups at certain roots of unity is not trivial (cf. [X5] and [W3]).

\(^2\)When $G$ is the direct product of simple groups, $k$ is a multi-index, i.e., $k = (k_1, ..., k_n)$, where $k_i \in \mathbb{N}$ corresponding to the level of the $i$-th simple group. The level of $LH$ is determined by the Dynkin indices of $H \subset G$. To save some writing we write the coset as $H \subset G_k$. 

Now compare $\tau_{G/H}(M)$ with $\tau_G(M)\tau_H(M)$, which in the notation of [KM] is

$$\tau_3(M)^2\tau_4(M).$$

By (1) of Th. 6.3 in [KM], if $\tau_3(M) \neq 0$, then there exists an integer $\mu$ depending on $M$ such that

$$\mu(M_\theta) \equiv \mu \mod 4, \forall \theta,$$

and it follows that

$$\tau_G(M)\tau_H(M) = \tau_3(M)^2\exp\left(\frac{\mu \pi i}{2}\right)\tau_{G/H}(M).$$

Thus when $\tau_3(M) \neq 0$, $\tau_{G/H}(M)$ agrees with $\tau_G(M)\tau_H(M)$ up to a homotopy invariant $\tau_3(M)^2$ and a phase $\exp\left(\frac{\mu \pi i}{2}\right)$. However when $\tau_3(M) = 0$, $\tau_G(M)\tau_H(M)$ is always 0, but $\tau_{G/H}(M)$ may not be 0, for example when $M = \mathbb{R}P^3$ (Note that $\mathbb{R}P^3 = L(2,1)$).

The above example is a special case of Prop. 3.6.1. More examples in §3 show that $\tau_{G/H}(M)$ may be finer than $\tau_G(M)\tau_H(M)$, and these two invariants do not seem to be simply related when there are issues of “fixed point resolutions”, a problem resolved mathematically in [X2].

The calculation in §3 is based on Th. B of §2, which states that under certain index conditions the link invariants colored by the representations of coset factorize into the products of the the link invariants colored by the representations of the two groups in the coset. The proof is based on certain braiding properties first appeared in [X4] and further analyzed in [BE3]. These properties are also used in a very interesting recent paper [BEK].

The main new results of this paper are Th. A, Th. B and formulas (3.4.5), (3.5.2) and (3.6.1).

Here is a more detailed account of the paper. In §1.1 we recall the definitions of unitary modular category from [Tu]. In §1.2 to §1.5 we recall some basic definitions from algebraic conformal theories and results which will be used in this paper. These sections are included to set up notations and for the benefit of reader with less operator algebra background. In §1.6 we discuss the nondegeneracy condition of cosets based on [X3]. In §1.7 we prove (Th. A) that a general class of cosets naturally give rise to unitary modular categories by checking all the axioms listed in §1.1.

In §2.1 we recall some definitions and results from [X4] and provide a relation between [X4] and [BE1], [BE2]. In §2.2 the relative braidings introduced in [BE3] are incorporated into our formalism of cosets (Prop. 2.3.1) by using §2.1, and these properties are used to prove factorization of framed link invariants under certain index conditions (Th. B) in §2.3.

§3 are applications of the results in §1 and §2. In §3.1 we show that our invariants in the case of a type $A$ group is the same as those of [TW1], [W2] in the type $A$ case. The proof is based on certain properties of our invariants (lemma 1.7.4,
1.7.5) similar to the cabling idea introduced in [W2]. In §3.2 we give a different proof of the symmetry principle ([KM], [KT1]). In §3.3 we give a different proof of the result in [KT2] on the level-rank duality of type A invariants as an almost immediate application of Th. B. In §3.4 we calculate invariants associated with the simple current extensions of affine su(N) based on [BE2] and [BE3]. In §3.5 we consider the invariants associated with the parafermion cosets and observe that even in the simplest parafermion coset our invariants may not always be related to the product of invariants associated to the two groups in the coset. In §3.6 formula for the invariants associated with the diagonal cosets of type A are obtained. It is shown that for a special class of diagonal cosets known as coset W algebras the corresponding invariants are related to the product of invariants associated to the two groups in the coset by using the symmetry principle (Prop. 3.6.1 and Cor. 3.6.2). However there does not appear to be such a relation in general, especially when there are nontrivial fixed points under certain diagram automorphisms. In §3.7 a “Maverick” coset is considered which does not verify the conditions of Th. B, but we identify the corresponding invariants with that of a parafermion coset and a coset W algebra by using the ideas of lemma 1.7.5 and lemma 3.1.1. Finally in §3.8 a question is raised about the pertubative aspects of invariants from cosets.

§1. UNITARY MODULAR CATEGORIES FROM COSETS

§1.1 Unitary modular categories. Let us first recall the definitions of unitary modular tensor category from I.1, II.5 of [Tu].

A ribbon category is a monoidal category (assumed to be strict in the following) equipped with a braiding $c$, a twist $\theta$, and a compatible duality $(\ast, b, d)$ over a ground field $K$. Let us explain the conditions on these three ingredients. A braiding in a monoidal category consists of a family of isomorphisms

$$c = \{ c_{V,W} : V \otimes W \to W \otimes V \},$$

where $V, W$ run over all objects in the category, such that for any three objects $U, V, W$ we have

$$c_{U,V\otimes W} = (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W), c_{U\otimes V,W} = (c_{U,W} \otimes id_V \otimes/id_U \otimes c_{U,V})$$

The naturality of the isomorphisms of $c$ means that for any morphisms $f : V \to V', g : W \to W'$, we have

$$(g \otimes f)c_{V,W} = c_{V',W'}(f \otimes g)$$

The Yang-Baxter Equation (YBE), i.e.

$$(id_W \otimes c_{U,V})(c_{U,W} \otimes id_V)(id_U \otimes c_{V,W}) = (c_{V,W} \otimes id_U)(id_V \otimes c_{U,W})(c_{U,V} \otimes id_W)$$

is a consequence of (1.1) and (1.2) (cf. P. 19 of [Tu]).
The twist $\theta$ in a monoidal category with a braiding $c$ consists of a natural family of isomorphisms $\theta = \{\theta_V : V \to V\}$, where $V$ runs over all objects of the category, such that for any two objects $V, W$ we have

$$\theta_{V \otimes W} = c_{W,V} c_{V,W}(\theta_V \otimes \theta_W) \quad (1.1.3)$$

The naturality of $\theta$ means that for any morphisms $f : U \to V$, we have:

$$\theta_V f = f \theta_U \quad (1.1.4)$$

Assume that to each object $V$ of the category there are associated an object $V^*$ and two morphisms $b_V : 1 \to V \otimes V^*$, $d_V : V^* \otimes V \to 1$, where 1 denotes the unit object in $C$. This is called a duality if the following identities are satisfied:

$$(id_V \otimes d_V)(b_V \otimes id_{V^*}) = id_V, (d_V \otimes id_{V^*})(id_{V^*} \otimes b_V) = id_{V^*} \quad (1.1.5)$$

We need one axiom relating the duality morphisms $b_V, d_V$ with braiding and twist. We say that the duality is compatible with the braiding $c$ and the twist $\theta$ if for any object $V$ in the category we have:

$$ (\theta_V \otimes id_{V^*})b_V = (id_V \otimes \theta_{V^*})b_V \quad (1.1.6)$$

By a ribbon category, we mean a monoidal category equipped with a braiding $c$, a twist $\theta$, and a compatible duality $(*, b, d)$ which verifies (1.1) to (1.6). A ribbon category is called strict if its underlying monoidal category is strict, and is called abelian if its underlying monoidal category is abelian. For an endomorphism $f : V \to V$ in a ribbon category, define its trace $tr(f) \in K$ to be the following composition:

$$tr(f) = d_V c_{V,V^*}((\theta_V f) \otimes id_{V^*})b_V : 1 \to 1 \quad (1.1.7)$$

In general ribbon categories do not admit direct sums. It turns out that instead of decompositions of objects into direct sums one may decompose their identity endomorphisms. This leads to the following notion of domination. We say $V$ is dominated by $\{V_i\}_{i \in I}$ if the images of the pairings

$$\{(g, f) \to fg : Hom(V, V_i) \otimes_K Hom(V_i, V) \to End(V)\}_{i \in I}$$

additively generate $End(V)$.

A modular category is a pair consisting of an abelian ribbon category $C$ and a finite family $\{V_i\}_{i \in I}$ of simple objects (i.e., $End(V_i)$ is a free $K$ module of rank 1) in $C$ satisfying the following four axioms:

(1) (Normalization axiom). There exists $0 \in I$ such that $V_0 = 1$;
(2) (Duality axiom). For any \( i \in I \), there exists \( i^* \in I \) such that the object \( V_{i^*} \) is isomorphic to \((V_i)^*\);

(3) (Axiom of domination). All objects of \( C \) are dominated by the family \( \{V_i\}_{i \in I} \).

To formulate the next and last axiom we need some notation. For \( i, j \in I \), set (we add script Tu to distinguish this matrix from the \( S \) matrix defined in §1.5):

\[
S_{i,j}^{Tu} = \text{tr}(c_{V_j,V_i}c_{V_i,V_j}).
\]

(4) (Non-degeneracy axiom). The square matrix \( S = (S_{i,j})_{i,j \in I} \) is invertible over \( K \).

Let \( C \) be a strict monoidal category. A conjugation in \( C \) assigns to each morphism \( f : V \to W \) in \( C \) a morphism \( \bar{f} : W \to V \) so that the following identities hold:

\[
\bar{\bar{f}} = f, \quad \bar{f} \otimes g = \bar{\bar{f}} \otimes \bar{g}, \quad \bar{f} \bar{g} = \bar{g} \bar{f}.
\]

In the second formula \( f, g \) are arbitrary morphisms in \( C \) and in the third formula \( f, g \) are arbitrary composable morphisms in \( C \).

A hermitian ribbon category is a ribbon monoidal category \( C \) endowed with a conjugation \( f \to \bar{f} \) satisfying the following conditions: For any objects \( V, W \in C \), we have

\[
\bar{c}_{V,W} = (c_{V,W})^{-1},
\]

and for any object \( V \) of \( C \)

\[
\bar{\theta}_V = (\theta_V)^{-1}, \quad \bar{b}_V = d_V c_{V,V^*} (\theta_V \otimes \text{id}_{V^*}), \quad \bar{d}_V = (\text{id}_{V^*} \otimes \theta_V^{-1}) c_{V,V^*}^{-1} b_V \quad \text{(1.1.9)}
\]

A unitary modular category is a Hermitian modular category over \( C \) such that for any morphism \( f \) we have

\[
\text{tr}(f \bar{f}) \geq 0 \quad \text{(1.1.10)}
\]

Suppose \( C \) is a strict unitary modular category with a set of simple \( V_i, i \in I \) objects as above. When no confusion arises we will denote \( V_i \) simply by \( i \) and \( V_i^* \) by \( \bar{i} \).

We will use the following convention of [Tu], \( V^{+1} := V, V^{-1} := V^* \). A graphical calculus is introduced in I.1.6 of [Tu]. In this calculus, the dualities, braidings and twistings are represented by special diagrams called tangles. The braiding \( c_{V,W} \) is represented by a positive crossing and \( c_{V,W}^{-1} \) is represented by a negative crossing (One usually follows a right-hand rule when drawing these crossings, cf. figure 1.6 on Page 25 of [Tu]). A general morphism \( f : V \to W \) which is not a tangle is represented by a diagram called a coupon. If a morphism is represented by a diagram, the closure of this morphism is simply obtained by closing all the free ends of the diagram (cf. Page 43 of [Tu] for a pictorial representation). For more details on this graphical calculus including proofs using these diagrams see I.1.6 of [Tu].
Suppose \( L \) is a framed oriented link in 3-sphere with \( n \) components colored by \( i_1, \ldots, i_n \). We will denote the corresponding invariant as defined on P. 39 of [Tu] by \( L(i_1, \ldots, i_n) \in \mathbb{C} \). It is often convenient to represent \( L(i_1, \ldots, i_n) \) as the closure of a tangle \( T_L \) which consists of only braidings and twists. Applying further braidings if necessary, we will assume that \( T_L \in \text{End}(I_1 I_2 \ldots I_n) \), where \( I_j := (\iota_j) \epsilon_j \) with \( \epsilon_j = 1 \) (resp. \( \epsilon_j = -1 \)) if the \( j \)-th component of \( L \) is oriented anti-clockwise (resp. clockwise). Recall from the above that \( i^{-1} = \bar{i} \).

Define \( d(i) := \text{tr}(\text{id}_i) \). Define (cf. P. 76 and P. 115 of [Tu]) \( D_C > 0 \) (called rank of \( C \)) such that

\[
D_C^2 := \sum_i d(i)^2. \tag{1.1.11}
\]

Note since \( V_i \) is irreducible, the twist \( \theta_{V_i} \) must act on \( V_i \) as multiplication by a nonzero element \( \omega(i) \in \mathbb{C} \). Set (cf. Page 21 of [MW])

\[
k_C^3 := D_C^{-1} \sum_i \omega(i) d(i)^2,
\]

then

\[
D_C^{-1} \sum_i \omega(i)^{-1} d(i)^2 = k_C^{-3}. \tag{1.1.12}
\]

Given a closed oriented 3-manifold \( M_L \) obtained by surgery on a framed oriented link \( L \) (with \( n \) components) in the 3-sphere. The 3-manifold invariant \( \tau_C(M_L) \) is defined by the following formula

\[
\tau_C(M_L) := k_C^{3(b_+(L) - b_-(L))} D_C^{-n} \sum_{i_1 \in I, \ldots, i_n \in I} d(i_1) \ldots d(i_n) L(i_1, \ldots, i_n), \tag{1.1.13}
\]

where \( b_+(L) \) and \( b_-(L) \) are the numbers of positive and negative eigenvalues of the linking matrix of \( L \) (this matrix includes the framings on the diagonal). The normalization of the invariant (1.11) differs from that of [Tu] but agrees [MW]. It is normalized so that it is multiplicative under connected summing; in particular \( \tau_C(S^3) = 1 \).

For the constraints of unitarity on the invariants (1.1.13), see Page 224 of [Tu].

Representation theory of certain quantum groups at certain roots of unity provides examples of unitary modular category (cf. Chap. XII of [Tu] and [W3]). In such examples the most difficult axioms to check are the non-degeneracy axioms and positivity (1.10) (cf. [W2], [W3] and [MW]). In §1.7 we will show that algebraic coset conformal field theories as formulated in [X1], [X3] provides a new class of unitary modular categories.

§1.2 Sectors and Correspondences. Let us first recall some definitions from [L1] and [L2].

Let \( M, N \) be von Neumann algebras, that we always assume to have separable preduals, and \( \mathcal{H} \) a \( M - N \) correspondence, namely \( \mathcal{H} \) is a (separable) Hilbert space, where \( M \) acts on the left, \( N \) acts on the right and the actions are normal.
We denote by $x \xi y$, $x \in M$, $y \in N$, $\xi \in \mathcal{H}$ the relative actions.

The trivial $M - M$ correspondence is the Hilbert space $L^2(M)$ with the standard actions given by the modular theory

$$x \xi y = x J y^* J \xi, \quad x, y \in M, \quad \xi \in L^2(M),$$

where $J$ is the modular conjugation of $M$; the unitary correspondence is well defined modulo unitary equivalence.

If $\rho$ is a normal homomorphism of $M$ into $M$ we let $H_\rho$ be the Hilbert space $L^2(M)$ with actions:

$$x \cdot \xi \cdot y \equiv \rho(x) \cdot \xi \cdot y, \quad x, y \in M, \quad \xi \in L^2(M).$$

Denote by $\text{End}(M)$ the semigroup of the endomorphism of $M$ and $\text{Corr}(M)$ the set of all $M - M$ correspondences. The following proposition is proved in [L2] (Corollary 2.2 in [L2], also cf. [C]).

**Proposition 1.1.1.** Let $M$ be an infinite factor. There exists a bijection between the unitary equivalence classes of $\text{End}(M)$ and the unitary equivalence classes of $\text{Corr}(M)$, i.e., given $\rho, \rho' \in \text{End}(M)$, $H_\rho$ is unitarily equivalent to $H_{\rho'}$ iff there exists a unitary $u \in M$ with $\rho'(x) = u \rho(x) u^*$. 

Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in $M$ as in Proposition 1.1. We call sectors the elements of the semigroup $\text{Sect}(M)$; if $\rho \in \text{End}(M)$ we denote by $[\rho]$ its class in $\text{Sect}(M)$. By Proposition 2.2 $\text{Sect}(M)$ may be naturally identified with $\text{Corr}(M) \sim$ the quotient of $\text{Corr}(M)$ modulo unitary equivalence. It follows from [L1] and [L2] that $\text{Sect}(M)$, with $M$ a properly infinite (on Hilbert space $\mathcal{H}$) von Neumann algebra, is endowed with a natural involution $\theta \rightarrow \bar{\theta}$. The tensor product of correspondences correspond to the composition of sectors.

Suppose $\rho \in \text{End}(M)$ is given together with a normal faithful conditional expectation $\epsilon : M \rightarrow \rho(M)$. We define a number $d_\epsilon$ (possibly $\infty$) such that (cf. [PP]):

$$d_\epsilon^{-2} := \text{Max}\{\lambda \in [0, +\infty) | \epsilon(m_+) \geq \lambda m_+, \forall m_+ \in M_+\}$$

Now assume $\rho \in \text{End}(M)$ is given together with a normal faithful conditional expectation $\epsilon : M \rightarrow \rho(M)$, and assume $d_\epsilon < +\infty$. We define

$$d = \text{Min}_\epsilon \{d_\epsilon\}.$$ 

d is called the statistical dimension of $\rho$. It is clear from the definition that the statistical dimension of $\rho$ depends only on the unitary equivalence classes of $\rho$. We will denote the statistical dimension of $\rho$ by $d(\rho)$ (in [X4] it is denoted by $d_\rho$, here we choose $d(\rho)$ to avoid the confusion with the duality $d_V$ in (1.1.5)). $d(\rho)^2$ is called the minimal index of $\rho$.

The properties of the statistical dimension can be found in [L1], [L2], [L3] and [K].

Denote by $\text{Sect}_0(M)$ those elements of $\text{Sect}(M)$ with finite statistical dimensions. For $\lambda, \mu \in \text{Sect}_0(M)$, let $\text{Hom}(\lambda, \mu)$ denote the space of intertwiners from $\lambda$ to $\mu$, i.e.
\[ a \in \text{Hom}(\lambda, \mu) \text{ iff } a \lambda(x) = \mu(x)a \text{ for any } x \in M. \] \text{Hom}(\lambda, \mu) \text{ is a finite dimensional vector space and we use } \langle \lambda, \mu \rangle \text{ to denote the dimension of this space. } \langle \lambda, \mu \rangle \text{ depends only on } [\lambda] \text{ and } [\mu]. \text{ Moreover we have } \langle \nu \lambda, \mu \rangle = \langle \lambda, \bar{\nu} \mu \rangle, \langle \nu \lambda, \mu \rangle = \langle \lambda, \mu \lambda \rangle \text{ which follows from Frobenius duality (See [L2] or [Y]). We will also use the following notations: if } \mu \text{ is a subsector of } \lambda, \text{ we will write as } \mu \prec \lambda \text{ or } \lambda \succ \mu. \text{ Recall (cf. [L2]) for each } \rho \in \text{End}(M) \text{ and its conjugate } \bar{\rho} \text{ with finite minimal index, there exists } R_\rho \in \text{Hom}(id, \bar{\rho} \rho) \text{ and } \bar{R}_\rho \in \text{Hom}(id, \rho \bar{\rho}) \text{ such that}

\begin{align*}
R_\rho \rho(R_\rho) &= 1, \quad R_\rho \rho \rho\bar{R}_\rho(R_\rho) = 1 \\
||R_\rho|| = ||\bar{R}_\rho|| &= \sqrt{d_\rho} \tag{1.2.1}
\end{align*}

The choice of } R_\rho, \bar{R}_\rho \text{ above is unique up to a phase only if } \rho \text{ is irreducible. However the minimal left inverse } \phi_\rho \text{ of } \rho \text{ defined by}

\[ \phi_\rho(m) = R_\rho^* \bar{\rho}(m) R_\rho \]

is unique and depends only one \( \rho \).

Recall from [L3] that

\[ \phi_{\rho_1 \rho_2} = \phi_{\rho_2} \phi_{\rho_1}. \]

For any } f \in \text{End}(\rho), \text{ it is convenient to define an unnormalized trace as

\[ tr_\rho(f) := d(\rho) \phi_\rho(f), \]

\text{and this indeed satisfies tracial properties due to the following:

**Lemma 1.2.1.** If } f \in \text{Hom}(\rho, \mu), g \in \text{Hom}(\mu, \rho), \text{ then

\[ tr_\rho(gf) = tr_\mu(fg). \]

**Proof.** First assume that } \mu \text{ is irreducible. We can assume } \mu \text{ is a subsector of } \rho \text{ since otherwise } f = 0, g = 0 \text{ and the lemma holds trivially. Choose } h_i \in \text{Hom}(\mu, \rho) \text{ such that } h_i^* h_j = \delta_{i,j} id_\mu, \sum_i h_i h_i^* = id_\rho. \text{ Then } g = \sum_i h_i h_i^* g, f = \sum_i f h_i h_i^*. \text{ Note that } h_i^* g, f h_i \in \text{End}(\mu), \text{ and therefore are complex numbers since } \mu \text{ is assumed to be irreducible. Also note by [L2] we have } tr_\rho(h_i h_i^*) = d(\mu). \text{ We have}

\[ tr_\rho(gf) = tr_\rho(\sum_{i,j} h_i h_i^* g f h_j h_j^*) \\
= \sum_{i,j} h_i^* g f h_j tr_\rho(h_i h_j^*) \\
= \sum_i h_i^* g f h_i tr_\rho(h_i h_i^*) \\
= \sum_i h_i^* g f h_i d(\mu) \\
= \sum_{i,j} tr_\mu(f h_i h_i^* h_j h_j^* g) \\
= tr_\mu(fg). \]
If \( \mu \) is not irreducible, let \( \mu_i \) be a set of irreducible subsectors of \( \mu \), \( T_i \in \text{Hom}(\mu_i, \mu) \) such that \( T_i^* T_j = \delta_{i,j} \text{id} \), \( \sum_i T_i T_i^* = \text{id}_\mu. \) Note that \( T_i^* f \in \text{Hom}(\rho, \mu_i), g T_i \in \text{Hom}(\mu_i, \rho) \). By the previous argument we have

\[
\text{tr}_\rho(g T_i T_i^* f) = \text{tr}_{\mu_i}(T_i^* f g T_i) = d(\mu_i) T_i^* f g T_i.
\]

So

\[
\text{tr}_\rho(g) = \text{tr}_\rho\left( \sum_{i,j} g T_i T_i^* T_j T_j^* f \right)
= \sum_i \text{tr}_\rho(g T_i T_i^* f)
= d(\mu_i) T_i^* f g T_i
= \sum_{i,j} \text{tr}_\mu(T_i T_i^*) T_i T_j^* f g T_j
= \text{tr}_\mu(g f).
\]

\[\square\]

We will drop the subscript of tr in the following since it can always be recovered by tracing the domain of morphisms.

§1.3 Conformal precosheaves on \( S^1 \) and superselection structures. In this section we recall the basic properties enjoyed by the family of the von Neumann algebras associated with a conformal Quantum Field Theory on \( S^1 \). All the propositions in this section and §1.3 are proved in [GL1].

By an interval in this section we shall always mean an open connected subset \( I \) of \( S^1 \) such that \( I \) and the interior \( I' \) of its complement are non-empty. We shall denote by \( \mathcal{I} \) the set of intervals in \( S^1 \).

We shall denote by \( PSL(2, \mathbb{R}) \) the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle \( S^1 \) globally invariant. Denote by \( \mathbb{G} \) the universal covering group of \( PSL(2, \mathbb{R}) \). Notice that \( \mathbb{G} \) is a simple Lie group and has a natural action on the unit circle \( S^1 \).

Denote by \( R(\vartheta) \) the (lifting to \( \mathbb{G} \) of the) rotation by an angle \( \vartheta \).

A conformal precosheaf \( \mathcal{A} \) of von Neumann algebras on the intervals of \( S^1 \) is a map

\[
I \to \mathcal{A}(I)
\]

from \( \mathcal{I} \) to the von Neumann algebras on a Hilbert space \( \mathcal{H} \) that verifies the following property:

A. Isotony. If \( I_1, I_2 \) are intervals and \( I_1 \subset I_2 \), then

\[
\mathcal{A}(I_1) \subset \mathcal{A}(I_2).
\]
**B. Conformal invariance.** There is a unitary representation $U$ of $G$ (the universal covering group of $PSL(2, \mathbb{R})$) on $\mathcal{H}$ such that

$$U(g)A(I)U(g)^* = A(gI), \quad g \in G, \quad I \in \mathcal{I}.$$ 

**C. Positivity of the energy.** The generator of the rotation subgroup $U(R)(\cdot)$ is positive.

**D. Locality.** If $I_0, I$ are disjoint intervals then $A(I_0)$ and $A(I)$ commute.

**E. Existence of the vacuum.** There exists a unit vector $\Omega$ (vacuum vector) which is $U(G)$-invariant and cyclic for $\forall I \in \mathcal{I} A(I)$.

A conformal precosheaf is called irreducible if it also satisfies the following:

**F. Uniqueness of the vacuum (or irreducibility).** The only $U(G)$-invariant vectors are the scalar multiples of $\Omega$.

By Prop. 1.1 of [GL] a conformal precosheaf satisfies Haag duality, i.e., $A(I)' = A(I')$.

A conformal precosheaf $A$ is called strongly additive if it satisfies the following property: let $I_1, I_2$ be the connected components of the interval $I$ with one internal point removed, then

$$A(I) = A(I_1) \lor A(I_2),$$

and $A$ is called split if $A(I_1) \lor A(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $A(I_1) \otimes A(I_2)$ with $\bar{I}_1 \cap \bar{I}_2 = \emptyset$. According to [KLM], $A$ is called completely rational, shortly $\mu$-rational, if $A$ is strongly additive and the minimal index of the inclusion $A(E) \subset A(E')'$ is finite, where $E = I_1 \cup I_2, \bar{I}_1 \cap \bar{I}_2 = \emptyset$. In this case the index of $A(E) \subset A(E')'$ is called the $\mu$-index of $A$, usually denoted by $\mu_A$.

Let $\mathcal{A}$ be an irreducible conformal precosheaf of von Neumann algebras as defined above. Our next goal is to define covariant representation of $\mathcal{A}$ and the associated concepts which will be important later on. Again all the definitions and propositions are given in [GL] and we refer the reader to [GL] for further details.

A covariant representation $\pi$ of $\mathcal{A}$ is a family of representations $\pi_I$ of the von Neumann algebras $A(I), I \in \mathcal{I}$, on a Hilbert space $\mathcal{H}_\pi$ and a unitary representation $U_\pi$ of the covering group $G$ of $PSL(2, \mathbb{R})$, with positive energy, i.e. the generator of the rotation unitary subgroup has positive generator, such that the following properties hold:

$$I \supset \bar{I} \Rightarrow \pi_{\bar{I}}|_{A(I)} = \pi_I \quad \text{(isotony)}$$

$$adU_\pi(g) \cdot \pi_I = \pi_{gI} \cdot adU(g) \quad \text{(covariance)}.$$ 

A unitary equivalence class of representations of $\mathcal{A}$ is called a superselection sector.

Assuming $\mathcal{H}_\pi$ to be separable, the representations $\pi_I$ are normal because the $A(I)$’s are factors. Therefore for any given $I_0$, $\pi_{I_0}'$ is unitarily equivalent $id_{A(I_0')}$ because $A(I_0')$ is a type III factor. By identifying $\mathcal{H}_\pi$ and $\mathcal{H}$, we can thus assume
that \( \pi \) is localized in a given interval \( I_0 \in \mathcal{I} \), i.e. \( \pi_{I_0} = \text{id}_{\mathcal{A}(I_0)} \). By Haag duality we then have \( \pi_I(\mathcal{A}(I)) \subset \mathcal{A}(I) \) if \( I \supset I_0 \). In other words, given \( I_0 \in \mathcal{I} \) we can choose in the same sector of \( \pi \) a \textit{localized endomorphism} with localization support in \( I_0 \), namely a representation \( \rho \) equivalent to \( \pi \) such that

\[
I \in \mathcal{I}, I \supset I_0 \Rightarrow \rho_I \in \text{End} \mathcal{A}(I), \quad \rho_{I_0} = \text{id}_{I_0}.
\]

To capture the global point of view we may consider the \textit{universal algebra} \( \mathcal{C}^*(\mathcal{A}) \). Recall that \( \mathcal{C}^*(\mathcal{A}) \) is a \( \mathcal{C}^* \)-algebra canonically associated with the precosheaf \( \mathcal{A} \) (see [Fre]). \( \mathcal{C}^*(\mathcal{A}) \) has the following properties: there are injective embeddings \( \iota_I : \mathcal{A}(I) \to \mathcal{C}^*(\mathcal{A}) \) so that the local von Neumann algebras \( \mathcal{A}(I), I \in \mathcal{I} \), are identified with subalgebras of \( \mathcal{C}^*(\mathcal{A}) \) and generate all together a dense \( \ast \)-subalgebra of \( \mathcal{C}^*(\mathcal{A}) \), and every representation of the precosheaf \( \mathcal{A} \) factors through a representation of \( \mathcal{C}^*(\mathcal{A}) \). Conversely any representation of \( \mathcal{C}^*(\mathcal{A}) \) restricts to a representation of \( \mathcal{A} \).

The vacuum representation \( \pi_0 \) of \( \mathcal{C}^*(\mathcal{A}) \) corresponds to the identity representation of \( \mathcal{A} \) on \( \mathcal{H} \), thus \( \pi_0 \) acts identically on the local von Neumann algebras. We shall often drop the symbols \( \iota_I \) and \( \pi_0 \) when no confusion arises.

By the universality property, for each \( g \in \text{PSL}(2, \mathbb{R}) \) the isomorphism \( \text{adU}(g) : \mathcal{A}(I) \to \mathcal{A}(gI), I \in \mathcal{I} \) lifts to an automorphism \( \alpha_g \) of \( \mathcal{C}^*(\mathcal{A}) \). We shall lift the map \( g \to \alpha_g \) to a representation, still denoted by \( \alpha \), of the universal covering group \( \mathcal{G} \) of \( \text{PSL}(2, \mathbb{R}) \) by automorphisms of \( \mathcal{C}^*(\mathcal{A}) \).

The covariance property for an endomorphism \( \rho \) of \( \mathcal{C}^*(\mathcal{A}) \) localized in \( I_0 \) means that \( \alpha_g \cdot \rho \cdot \alpha_g^{-1} \) is

\[
\text{ad}z_\rho(g)^* \cdot \rho = \alpha_g \cdot \rho \cdot \alpha_g^{-1}, \quad g \in \mathcal{G}
\]

for a suitable unitary \( z_\rho(g) \in \mathcal{C}^*(\mathcal{A}) \). We define

\[
\rho_g = \alpha_g \cdot \rho \cdot \alpha_g^{-1}, \quad g \in \mathcal{G}.
\]

\( \rho_{g,J} \) is the restriction of \( \rho_g \) to \( \mathcal{A}(J) \). The map \( g \to z_\rho(g) \) can be chosen to be a localized \( \alpha \)-cocycle, i.e.

\[
z_\rho(g) \in \mathcal{A}(I_0 \cup gI_0) \quad \forall g \in \mathcal{G} : I_0 \cup gI_0 \in \mathcal{I}
\]

\[
z_\rho(gh) = z_\rho(g)\alpha_g(z_\rho(h)), \quad g, h \in \mathcal{G}.
\]

To compare with the result of [FG], let us define:

\[
\Gamma_\rho(g) = \pi_0(z_\rho(g)^*).
\]

This notation will be used in §1.4.

An endomorphism of \( \mathcal{C}^*(\mathcal{A}) \) localized in an interval \( I_0 \) is said to have \textit{finite index} if \( \rho_I(= \rho|_{\mathcal{A}(I)}) \) has finite index, \( I_0 \subset I \). The index is independent of \( I \) due to the following (See Prop. 2.1 of [GL1])
1.3.1 Proposition. Let \( \rho \) be an endomorphism localized in the interval \( I_0 \). Then the index \( \text{Ind}(\rho) := \text{Ind}(\rho_I) \), the minimal index of \( \rho_I \), does not depend on the interval \( I \supset I_0 \).

The following proposition is Prop. 2.2 of [GL1]:

1.3.2 Proposition. Let \( \rho \) be a covariant (not necessarily irreducible) endomorphism with finite index. Then the representation \( U_\rho \) described before is unique. In particular, any irreducible component of \( \rho \) is a covariant endomorphism.

By the above proposition the \textit{univalence} of an endomorphism \( \rho \) is well defined by
\[
\omega(\rho) = U_\rho(2\pi),
\]
since \( U_{\rho'}(g) := \pi_0(u)U_\rho(g)\pi_0(u)^* \), where \( \rho'(\cdot) := u\rho(\cdot)u^* \), \( u \in C^*(A) \), \( \omega(\rho) \) depends only on the superselection class of \( \rho \).

When \( \rho \) is irreducible, \( \omega(\rho) \) is a complex number of modulus one since by definition \( \omega(\rho) \) belongs to \( \pi(C^*(A))' \) and \( \rho \) is irreducible, and we have
\[
\omega(\rho) = e^{2\pi i \Delta_\rho}
\]
with \( \Delta_\rho \) the lowest weight of \( U_\rho \). \( \Delta_\rho \) is also referred to as \textit{conformal dimension}. Formulas for certain conformal dimensions may be found in (1.5.10).

Let \( \rho_1, \rho_2 \) be endomorphisms of an algebra \( \mathcal{B} \). Recall from §1.1 that their intertwiner space is defined by
\[
\text{Hom}(\rho_1, \rho_2) = \{ T \in \mathcal{B} : \rho_2(x)T = T\rho_1(x) , \; x \in \mathcal{B} \}.
\]
In case \( \mathcal{B} = C^*(A) \), \( \rho_i \) localized in the interval \( I_i \) and \( T \in (\rho_1, \rho_2) \), then \( \pi_0(T) \) is an intertwiner between the representations \( \pi_0 \cdot \rho_i \). If \( I \supset I_1 \cup I_2 \), then by Haag duality its embedding \( \iota_I \cdot \pi_0(T) \) is still an intertwiner in \( (\rho_1, \rho_2) \) and a local operator. We shall denote by \( (\rho_1, \rho_2)_I \) the space of such local intertwiners
\[
(\rho_1, \rho_2)_I = (\rho_1, \rho_2) \cap \mathcal{A}(I).
\]
If \( I_1 \) and \( I_2 \) are disjoint, we may cover \( I_1 \cup I_2 \) by an interval \( I \) in two ways: we adopt the convention that, unless otherwise specified, a local intertwiner is an element of \( (\rho_1, \rho_2)_I \) where \( I_2 \) follows \( I_1 \) inside \( I \) in the clockwise sense.

We now define the statistics. Given the endomorphism \( \rho \) of \( \mathcal{A} \) localized in \( I \in \mathcal{I} \), choose an equivalent endomorphism \( \rho_0 \) localized in an interval \( I_0 \in \mathcal{I} \) with \( \overline{I_0} \cap \overline{I} = \emptyset \) and let \( u \) be a local intertwiner in \( (\rho, \rho_0) \) as above, namely \( u \in (\rho, \rho_0)_I \) with \( I_0 \) following clockwise \( I \) inside \( \overline{I} \).

The \textit{statistics operator} \( \sigma := u^*\rho(u) = u^*\rho_I(u) \) belongs to \( (\rho^2_I, \rho_I^2) \). Recall that if \( \rho \) is an endomorphism of a \( C^* \)-algebra \( \mathcal{B} \), a \textit{left inverse} of \( \rho \) is a completely positive map \( \phi \) from \( \mathcal{B} \) to itself such that \( \phi \cdot \rho = \text{id} \).
It follows from Cor. 2.12 of [GL] that there exists a unique left inverse $\phi$ of $\rho$ and that the statistics parameter
\[
\lambda_\rho := \phi(\sigma)
\]
depends only on the sector of $\rho$.

By [GL] The statistical dimension $d(\rho)$ are given by
\[
d(\rho) = |\lambda_\rho|^{-1}
\]
and the statistics phase $\kappa_\rho$ are defined by
\[
\kappa_\rho = \frac{\lambda_\rho}{|\lambda_\rho|}.
\]

In [GL1], the following theorem is proved:

**Theorem 1.3.3 (Spin-Statistics Theorem).**

If $\rho$ is irreducible and has finite index, then $\kappa_\rho = \omega(\rho)$.

We shall need the following easy corollary.

**Corollary 1.3.4.** Assume that $\rho$ has finite index. Then $d(\rho)\lambda_\rho$ is a constant a times identity if and only if for any irreducible subsector $x$ of $\rho$, $\omega(x) = a$.

**Proof.** Ad (1). By Prop. 6.3 of [DHR] (also cf. [L1]) we have
\[
d(\rho)\lambda_\rho = \sum_x d_x \lambda_x E_x = \sum_x \kappa_x E_x,
\]
where the summation is over all different irreducible subsectors $x$ of $\rho$, and $E_x$ is the minimal abelian projection in $\text{End}(\rho)$ corresponding to $x$. By the spin-statistics theorem we have
\[
d(\rho)\lambda_\rho = \sum_x \omega(x) E_x,
\]
and (1) follows immediately.

\[\square\]

**1.4 Coherence equations.**

In this section, we assume $\Delta$ is a set of localized covairant endomorphism of $A$ with localization support in $I_0$. Let $h, g$ be elements of $G$. We assume $hI_0 \cap I_0 = \emptyset$, $gI_0 \cap I_0 = \emptyset$, $hI_0 \cap gI_0 = \emptyset$. Choose $J_1, J_2 \in \mathcal{I}$ such that $J_1 \cup J_2 \subsetneq S^1$, $J_1 \supset I_0 \cup g.I_0$, $J_2 \supset I_0 \cup h.I_0$, $J_1 \cap h.I_0 = \emptyset$, $J_2 \cap g.I_0 = \emptyset$ and $J_1 \cap J_2 = I_0$. We assume in $J_1$ (resp. $J_2$), $g.I_0$ (resp. $h.I_0$) lies a clockwise (resp. anti clockwise) from $I_0$. The proofs of all the results of this section can be found in §1.4 of [X4].
Lemma 1.4.1. For any \( J \supset J_1 \cup J_2, J \in \mathcal{I}, \gamma, \lambda \in \Delta \) and \( x \in \mathcal{A}(J) \), we have

1. \( \Gamma_\lambda(g) \in \mathcal{A}(J_1) \).
2. \( \Gamma_\lambda(g) \gamma J_1 (\Gamma_\lambda(g)) \gamma J \cdot \lambda J(x) = \lambda J \cdot \gamma J (x) \Gamma_\lambda(g) \gamma J_1 (\Gamma_\lambda(g)) \).
3. \( \Gamma_\lambda(g) \gamma J_1 (\Gamma_\lambda(g)) = \lambda J_2 (\Gamma_\lambda(h)^*) \Gamma_\gamma(h) \).
4. \( \Gamma_\gamma(g) \gamma J_1 (\Gamma_\lambda(g)) \in \mathcal{A}(I_0) \).

Because the property (1) of Lemma 1.4.1, \( \Gamma_\lambda(g) \gamma J_1 (\Gamma_\lambda(g)) \) is called the braiding operator.

We shall use \( \sigma_{\gamma \lambda} \) to denote \( \Gamma_\lambda(g) \gamma J_1 (\Gamma_\lambda(g)) \). \( \sigma_{\gamma \lambda} \) will be called the positive braiding operator between \( \gamma \) and \( \lambda \). We define

\[
\sigma_{\gamma \lambda} := \sigma_{\lambda \gamma}^*_\gamma
\]

and \( \sigma_{\lambda \gamma} \) will be called the negative braiding operator between \( \gamma \) and \( \lambda \). We are now ready to state the following equations. For simplicity we will drop the subscript \( I_0 \) and write \( \mu_{\lambda \gamma} \) as \( \mu \) for any \( \mu \in \Delta \) in the following.

Proposition 1.4.2. (1) Yang-Baxter-Equation (YBE)

\[
\sigma_{\mu \gamma} \mu (\sigma_{\lambda \gamma} \sigma_{\lambda \mu}) = \gamma(\sigma_{\lambda \mu}) \sigma_{\lambda \gamma} \lambda(\sigma_{\mu \gamma}).
\]

(2) Braiding-Fusion-Equation (BFE)

For any \( w \in \text{Hom}(\mu \gamma \delta) \)

\[
\begin{align*}
\sigma_{\lambda \delta \lambda}(w) &= w \mu(\sigma_{\lambda \gamma}) \sigma_{\lambda \mu} & (a) \\
\sigma_{\delta \lambda \mu}(w) &= \lambda(w) \sigma_{\mu \lambda}(\sigma_{\gamma \lambda}) & (b) \\
\sigma_{\delta \lambda \mu}(w) &= w \mu(\sigma_{\gamma \lambda}) \sigma_{\mu \lambda} & (c) \\
\sigma_{\delta \lambda \mu}(w) &= w \mu(\sigma_{\gamma \lambda}) \sigma_{\mu \lambda}^* & (d)
\end{align*}
\]

Suppose \( \xi_1 \in I_{\xi_1} \subset J_1, I_{\xi_1} \cap g I_1 \cap I_1 = \emptyset, \xi_2 \in I_{\xi_2} \subset J_2 \), and \( I_{\xi_2} \cap I_1 \cap h I_1 = \emptyset \). Here \( g, h, J_1, J_2 \) are defined as the beginning of this section.

It follows from (2) of Lemma 1.4.1 that:

\[
\gamma_{I_{\xi_1}} (\Gamma_\lambda(g)^* \Gamma_\lambda(g) \gamma_{I_{\xi_2}} (\Gamma_\lambda(g)) = \gamma_{J_2} (\Gamma_\lambda(h)^*) \Gamma_\lambda(h) = \sigma_{\lambda \gamma}.
\]

Hence \( \sigma_{\lambda \gamma} \sigma_{\gamma \lambda} = \gamma_{I_{\xi_1}} (\Gamma_\lambda(g)^*) \gamma_{I_{\xi_2}} (\Gamma_\lambda(g)) \).

\( \sigma_{\lambda \gamma} \sigma_{\gamma \lambda} \) is called the monodromy operator.

Let \( T_e : \delta \rightarrow \gamma \lambda \) be an intertwiner. Recall \( \omega(\rho) = U_\rho(2\pi) \) is the univalence of a covariant endomorphism. When \( \rho \) is irreducible, \( \omega(\rho) \) is a complex number. We have:

Proposition 1.4.3 (monodromy equation). If \( \omega(\delta), \omega(\gamma), \omega(\lambda) \) are complex numbers, then

\[
T_{e^*}^* \gamma_{I_{\xi_1}} (\Gamma_\lambda(g)^*) \gamma_{I_{\xi_2}} (\Gamma_\lambda(g)) T_e = T_{e^*}^* \sigma_{\lambda \gamma} \sigma_{\gamma \lambda} T_e = \frac{\omega(\delta)}{\omega(\lambda) \omega(\gamma)}.
\]
1.5 Genus 0 and 1 modular matrices. Next we will recall some of the results of [Reh] (also cf. [FRS]) and introduce notations.

Let \([\rho_i]\) denote the equivalence classes of irreducible superselection sectors in a finite set. Suppose this set is closed under conjugation and composition. We will denote the conjugate of \([\rho_i]\) by \([\rho_i]\) and identity sector by \([1]\) if no confusion arises, and let \(N_{ij}^k = \langle [\rho_i], [\rho_j], [\rho_k] \rangle\). The algebra generated by \([\rho_i]\)'s will be called fusion rule algebra and \(N_{ij}^k\) will be referred to as fusion coefficients. We will denote by \(\{T_e\}\) a basis of isometries in \(\text{Hom}(\rho_k, \rho_i \rho_j)\). Recall from §1.3 the univalence of \(\rho_i\) is denoted by \(\omega_{\rho_i}\). Let \(\phi_i\) be the unique minimal left inverse of \(\rho_i\), define:

\[
Y_{ij} := d_{\rho_i} d_{\rho_j} \phi_j (\sigma_{\rho_j, \rho_i} \sigma_{\rho_i, \rho_j})^* ,
\]

where \(\sigma_{\rho_j, \rho_i}\) is the unitary braiding operator defined in §1.4.

We list two properties of \(Y_{ij}\) (cf. (5.13), (5.14) of [Reh]) which will be used later:

\[
Y_{ij} = Y_{ji} = Y_{ij}^*, \omega(i) = \omega(i)
\]

\[
Y_{ij} = \sum_k N_{ij}^k \omega_i \omega_j \omega_k d_{\rho_k}
\]

Define \(D_- := \sum_i d_{\rho_i}^2 \omega_{\rho_i}^{-1}\). If the matrix \((Y_{ij})\) is invertible, by proposition on P.351 of [Reh] \(D_-\) satisfies \(|D_-|^2 = \sum_i d_{\rho_i}^2\). Suppose \(D_- = |D_-| \exp(i x), x \in \mathbb{R}\). Define matrices

\[
S := |D_-|^{-1} Y, T := CDiag(\omega_{\rho_i}),
\]

where \(C := \exp(i \frac{x}{2})\). Then these matrices satisfy the algebra:

\[
SS^\dagger = TT^\dagger = id,
\]

\[
TSTST = S,
\]

\[
S^2 = \hat{C}, T\hat{C} = \hat{T}T = T,
\]

where \(\hat{C}_{ij} = \delta_{ij}\) is the conjugation matrix. Moreover

\[
N_{ij}^k = \sum_m \frac{S_{im} S_{jm}^* S_{km}^*}{S_{1m}}.
\]

(1.5.7) is known as Verlinde formula.

We will refer the \(S, T\) matrices as defined in (1.5.3) as genus 0 modular matrices since they are constructed from the fusions rules, monodromies and minimal indices which can be thought as genus 0 data associated to a Conformal Field Theory (cf. [MS]).

Now let us consider an example which verifies (1.5.1) to (1.5.7) above. Let \(G = SU(N)\). We denote \(LG\) the group of smooth maps \(f : S^1 \mapsto G\) under pointwise multiplication. The diffeomorphism group of the circle \(\text{Diff}S^1\) is naturally a
subgroup of $\text{Aut}(LG)$ with the action given by reparametrization. In particular the group of rotations $\text{Rot}S^1 \simeq U(1)$ acts on $LG$. We will be interested in the projective unitary representation $\pi : LG \to U(H)$ that are both irreducible and have positive energy. This means that $\pi$ should extend to $LG \ltimes \text{Rot}S^1$ so that $H = \oplus_{n \geq 0} H(n)$, where the $H(n)$ are the eigenspace for the action of $\text{Rot}S^1$, i.e., $r_{\theta} \xi = \exp^{in\theta}$ for $\theta \in H(n)$ and $\dim H(n) < \infty$ with $H(0) \neq 0$. It follows from [PS] that for fixed level $k$ which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

$$P_{++}^h = \left\{ \lambda \in P \mid \lambda = \sum_{i=1,\ldots,N-1} \lambda_i \Lambda_i, \lambda_i \geq 1, \sum_{i=1,\ldots,n-1} \lambda_i < h \right\}$$

where $P$ is the weight lattice of $SU(N)$ and $\Lambda_i$ are the fundamental weights and $h = N + k$. Elements of $P_{++}^h$ are also denoted by $(\lambda_1, \ldots, \lambda_{N-1})$. We will use 1 to denote the representation corresponding to $(1, 1, \ldots, 1)$, referred to as vacuum representation. For $\lambda, \mu, \nu \in P^K_{++}$, define

$$N^\nu_{\lambda\mu} = \sum_{\delta \in P^K_{++}} \frac{S_{\lambda\delta} S_{\mu\delta} S_{\nu\delta}^*}{S_{1\delta}}$$

(1.5.8)

where $S_{\lambda\delta}$ is given by the Kac-Peterson formula:

$$S_{\lambda\delta} = c \sum_{w \in S_N} \varepsilon_w \exp(iw(\delta) \cdot \lambda 2\pi / n).$$

Here $\varepsilon_w = \det(w)$ and $c$ is a normalization constant fixed by the requirement that $(S_{\lambda\delta})$ is an orthonormal system. It is shown in [Kac] P.288 that $N^\nu_{\lambda\mu}$ are non-negative integers. Moreover, define $Gr(C_K)$ to be the ring whose basis are elements of $P^K_{++}$ with structure constants $N^\nu_{\lambda\mu}$. The natural involution $\ast$ on $P^K_{++}$ is defined by $\lambda \mapsto \lambda^* = \text{the conjugate of } \lambda$ as representation of $SU(N)$.

The irreducible positive energy representations of $LSU(N)$ at level $K$ give rise to an irreducible conformal precosheaf $A_G$ and its covariant representations (cf. P. 362 of [X1]). Let us recall how $A_G$ is defined. Let $\pi^0$ be the vacuum representation of $LG$ on Hilbert space $H^0$. Then $A_G(I) := \pi^0(L_I G)^\prime\prime$, where $\pi^0(L_I G)^\prime\prime$ denote the von Neumann algebra generated by all elements of the form $\pi^0(x), x \in LG, x = eonI^c$ with $e$ the identity element of $G$.

This conformal precosheaf is strongly additive, split (cf. [Wal]) and $\mu$-rational (cf. [X7]). The unitary equivalent classes of such representations are the superselection sectors. We will use $\lambda$ to denote such representations.

For $\lambda$ irreducible, the univalence $\omega_\lambda$ is given by an explicit formula. Let us first define

$$\Delta_\lambda = \frac{c_2(\lambda)}{K + N}$$

(1.5.10)
where \( c_2(\lambda) \) is the value of Casimir operator on representation of \( SU(N) \) labeled by dominant weight \( \lambda \) (cf. 1.4.1 of [KW]). \( \Delta_\lambda \) is usually called the conformal dimension.

We have \( \omega_\lambda = \exp(2\pi i \Delta_\lambda) \). Note that \( \omega_\lambda = \bar{\omega}_\lambda \).

Define the central charge (cf. 1.4.2 of [KW])

\[
C_G := \frac{K \dim(G)}{K + N}
\]

and \( T \) matrix as

\[
T = \text{diag}(\hat{\omega}_\lambda),
\]

where \( \hat{\omega}_\lambda = \omega_\lambda \exp\left(\frac{-2\pi i C_G}{24}\right) \). By Th.13.8 of [Kac] \( S \) matrix as defined in (1.5.9) and \( T \) matrix in (1.5.12) satisfy relation (1.5.4), (1.5.5) and (1.5.6). Since \( S,T \) matrix defined in (1.5.8) and (1.5.11) are related to the modular properties of characters which are related to Genus 1 data of CFT (cf. [MS]), we shall call them genus 1 modular matrices.

By Cor.1 in §34 of [Wa], The fusion ring generated by all \( \lambda \in P_{++}^K \) is isomorphic to \( Gr(C_K) \), with structure constants \( N^\nu_{\lambda \mu} \) as defined in (1.5.8). One may therefore ask what are the \( Y \) matrix (cf. (1.5.0)) in this case. By using (1.5.2) and the formula for \( N^\nu_{\lambda \mu} \), a simple calculation shows:

\[
Y_{\lambda \mu} = \frac{S_{\lambda \mu}}{S_{1 \mu}},
\]

and it follows that \( Y_{\lambda \mu} \) is nondegenerate, and \( S,T \) matrices as defined in (1.5.3) are indeed the same \( S,T \) matrix defined in (1.5.8) and (1.5.11), which is a surprising fact. This fact is referred to as genus 0 modular matrices coincide with genus 1 modular matrices.

The fusion coefficient for \( N^\mu_{\nu \lambda} \) takes a particular simple form when \( \nu := 2\Lambda_1 + \Lambda_2 + ... + \Lambda_{N-1} \) corresponds to the defining representation of \( SU(N) \). Represent \( \lambda \) by \( (\lambda_1, ..., \lambda_{N-1}) \), we have

\[
\nu \lambda = (\lambda_1 + 1, \lambda_2, ..., \lambda_{N-1}) + (\lambda_1 - 1, \lambda_2 + 1, ..., \lambda_{N-1}) + ... + (\lambda_1, \lambda_2, ..., \lambda_{N-2} - 1, \lambda_{N-1} + 1)
\]

(1.5.13)

where on the righthand side if a weight is not in \( P_{++}^h \) it is defined to be 0.

**Lemma 1.5.1.** Denote by \( \nu := 2\Lambda_1 + \Lambda_2 + ... + \Lambda_{N-1}, a_i = \Lambda_1 + ... + 2\Lambda_i + ... + \Lambda_{N-1}, s_i = (i + 1)\Lambda_1 + \Lambda_2 + ... + \Lambda_{N-1}, i = 1, ..., N - 1. \) Then:

1. \([a_i]\) appears once and only once in \([\nu]\). All other subsectors except \([a_i]\) and \([s_i]\) appear with multiplicity at least 2.
2. If \( A \) is a map from the set \( P_{++}^h \) to itself, and

\[
S_A(\lambda)A(\mu) = S_{\lambda \mu}, \forall \lambda, \mu \in P_{++}^h,
\]
and $A(a_i) = a_i, i = 1, \ldots, N - 1$, then $A = id$.

Proof. Ad (1). When $k = 1$ the fusion ring is isomorphic to $Z_N$ and (1) is trivial. Let us assume that $k \geq 2$. Let us prove a slightly stronger statement by induction on $i$. The statement is that for any $i = 1, \ldots, N - 1$, (1) is true and in addition the subsectors of $[a_{i-1}v] := [a_i] + [b_i]$, appear $([a_0] = [id])$ in $[v^i]$, and if $[s_i]$ appears in $[v^i]$, then the subsectors of $[s_{i-1}v] := [s_i] + [c_i] \ ([s_0] = [id])$ appear in $[v^i]$. When $i = 1$ the statement is trivial. Assume it is proved for $1 \leq i \leq N - 2$, let us prove it for $i + 1$. Note that denote by $[a_{i}v] := [a_{i+1}] + [b_{i+1}]$, then $[b_{i}v] \succ b_{i+1}$ by (1.5.13), and similarly if $[s_{i+1}]$ appears in $[v^{i+1}]$, then $[s_{i}v] := [s_{i+1}] + [c_{i+1}]$, and $[c_{i}v] \succ c_{i+1}$. It is then easy to see by induction that the stronger statement as above is true for $i + 1$. By induction (1) is proved.

Ad(2): By assumption and (1.5.8), we have $N^A(\nu)_{A(\lambda)A(\mu)} = N^{\nu\mu}_{\lambda\mu}$. It follows that $a \to A(a)$ is a ring endomorphism of $Gr(C_k)$. Since $a_i, i = 1, 2, \ldots, N - 1$ are generators of $Gr(C_k)$ by §34 of [Wa] and $A(a_i) = a_i, i = 1, \ldots, N - 1$ by our assumption, it follows that $A = id$.

\[\square\]

1.6 Rationality and non-degeneracy of cosets. Let $G$ be a simply connected compact Lie group and let $H \subset G$ be a Lie subgroup. Let $\pi^i$ be an irreducible representations of $LG$ with positive energy at level $k$ as in the introduction. Suppose when restricting to $LH$, $H^i$ decomposes as:

$$H^i = \sum_{\alpha} H_{i,\alpha} \otimes H_{\alpha},$$

and $\pi_{\alpha}$ are irreducible representations of $LH$ on Hilbert space $H_{\alpha}$. The set of $(i, \alpha)$ which appears in the above decompositions will be denoted by $exp$.

Let $A_{G/H}$ be the Vacuum Sector of the coset $G/H$ as defined on Page 5 of [X4]. Let us recall how $A_{G/H}$ is defined. Let $\pi^0$ be the vacuum representation of $LG$ on Hilbert space $H^0$. Then $A_{G/H}(I)$ is naturally isomorphic to $\pi^0(L_1H)' \cap \pi^0(L_1G)'$, where for any algebra $A \subset B(H^0)$, $A'$ is defined to be its commutant. The coset $H \subset G_k$ is called cofinite if the inclusion

$$(\pi^0(L_1H)')' \cup (\pi^0(L_1H) \cap \pi^0(L_1G)') \subset \pi^0(L_1G)'$$

has finite index and the square root of its minimal index will be denoted by $d(G/H)$ (cf. §3 of [X1]). All the cosets considered in this paper are cofinite.

The decompositions above naturally give rise to a class of covariant representations of $A_{G/H}$, denoted by $\pi_{i,\alpha}$ or simply $(i, \alpha)$ on Hilbert space $H_{i,\alpha}$. Note that on $H_{i,\alpha}$, the rotation group acts as $e^{i\theta} \to e^{i(\Delta_i - \Delta_{\alpha})}$ by the nature of coset construction (cf. [GKO]), and hence for any irreducible sector of $(i, \alpha)$, its univalence is

$$e^{2\pi i(\Delta_i - \Delta_{\alpha})} = \omega(i)\omega(\alpha)^{-1},$$
hence the sector \((i, \alpha)\) has a uniform univalence with univalence \(\omega(i)\omega(\alpha)^{-1}\) (cf. definition in §1.7 before Cor. 1.7.3).

In [X4], certain rationality results (cf. Th. 4.2) are proved for a class of coset \(H \subset G\). A stronger rationality condition, \(\mu\)-rational is defined in §1.3. We will give a formula of \(\mu\)-index for the coset in (1.6.2). We will see in the following \(\mu\)-index is the square of the rank of certain unitary modular category.

We have:

**Lemma 1.6.** Suppose \(H \subset G_k\) is cofinite and the conformal precosheaf associated with \(H\) and \(G\) are \(\mu\)-rational. Then the coset conformal precosheaf has finite \(\mu\)-index. Moreover its \(\mu\)-index (denoted by \(\mu_{G/H}\)) is given by

\[
\mu_{G/H} = \frac{d(G/H)^4 \mu_G}{\mu_H},
\]

(1.6.1)

where \(d(G/H)\) is the cofinite statistical dimension defined above, and \(\mu_G, \mu_H\) are \(\mu\)-index of the conformal precosheaf associated with \(G\) and \(H\) respectively.

This lemma is proved in [X3].

Throughout this section and the rest of the paper, we assume that the vacuum sectors associated with \(H,G\) are \(\mu\)-rational , and \(H \subset G\) is cofinite (cf. §3 of [X1]). By the above lemma and Cor. 9 of [KLM] [KLM], the set of irreducible superselection sectors of the coset is finite, and by Cor. 3.2 of [X3], every such irreducible sector appears as an irreducible subsector of some \((j, \beta) \in \exp\). Moreover the \(Y\)-matrix as defined in §2.1 is non-degenerate by Prop. 2.4 of [X3]. We will see in the next section that this is the non-degeneracy condition of certain unitary modular category.

Let us give a formula for \(\mu_{G/H}\) as promised. It follows by Prop. 2.4 of [X3] that

\[
\mu_{G/H} = \sum_x d(x)^2,
\]

(1.6.2)

where the summation is over all the different irreducible sectors \(x\) which appears as irreducible sectors of some \((j, \beta) \in \exp\).

### 1.7 Unitary modular category from cosets.

Fix a proper interval \(I_0 \subset S^1\), and let \(\mathcal{A}\) be the conformal precosheaf on \(S^1\). Define \(M := \mathcal{A}(I_0)\). Suppose a set of covariant representations \(\rho_i, i \in I\) of \(\mathcal{A}\), localized on \(I_0\) are given (\(\rho_i\) can be thought as an element in \(\text{End}(M)\)). We will assume that there is an involution \(i \rightarrow \bar{i}\) on \(I\) and there is a distinguished element \(0 \in I\) so that

\[
[\rho_i] = [\bar{\rho_i}], \rho_0 = \text{id}, \bar{0} = 0.
\]

(1.7.0)

Recall from §1.2.1 that for each \(\rho_i \in \text{End}\) and its conjugate \(\bar{\rho_i}\) there exists \(R_{\rho_i} \in \text{Hom}(\text{id}, \rho_i \rho_i)\) and \(\bar{R}_{\rho_i} \in \text{Hom}(\text{id}, \rho_i \rho_i)\) such that

\[
\bar{R}_{\rho_i}^*\rho_i(R_{\rho_i}) = 1, R_{\rho_i}^*\rho_i(\bar{R}_{\rho_i}) = 1
\]
and \(||R_{\rho_i}|| = ||\bar{R}_{\rho_i}|| = \sqrt{d_{\rho_i}}\). The choice of the pair \((R_{\rho_i}, \bar{R}_{\rho_i})\) is not unique even up to a phase unless \(\rho_i\) is irreducible, but we fix a choice for each \(i \in I\). We will see later that the invariants we are interested in are independent of such choices (cf. (4) of Lemma 1.7.4.). We will also choose the pairs \((R_{\rho_i}, \bar{R}_{\rho_i})\) so that

\[
R_{\rho_i} = \bar{R}_{\rho_i}, \quad \bar{R}_{\rho_i} = R_{\rho_i}.
\]

Recall the minimal left inverse \(\phi_{\rho_i}\) of \(\rho_i\) is defined by

\[
\phi_{\rho_i}(m) = R^*_{\rho_i} \rho_i \bar{\rho}_i(m) \bar{R}_{\rho_i}.
\]

Let us start to define a category denoted by \(C(A)\) from \(\rho_i, i \in I\) subject to (1.7.0). The objects of \(C(A)\) are defined to be any finite compositions of \(\rho_i, i \in I\), considered as elements in \(\text{End}(M)\). The homomorphisms are defined as intertwiners as in §1.2. The tensor product of two objects \(V, W \in C(A)\) are defined by

\[
V \otimes W := VW
\]

where \(VW(m) := V(W(m)), \forall m \in M\), i.e., \(VW\) stands for the composition of endomorphism \(V\) with endomorphism \(W\). We have omitted the usual \(\cdot\) when no confusion arises. Now let \(f \in \text{Hom}(V_1, V_2), g \in \text{Hom}(W_1, W_2)\), then

\[
f \otimes g := fV_1(g).
\]

Note this definition makes sense since \(V_1 \in \text{End}(M), g \in M\). It is then easy to see that \(C(A)\) is a strict abelian monoidal category. The braidings \(c_{V,W}, \forall \lambda, \mu \in C(A)\) are defined to be \(\sigma_{VW}\) as in the beginning of §1.4. Note that (1.1.1), (1.1.2) follows immediately from Prop. 1.4.2.

Next we define duality. For each \(\rho_i\), we define

\[
b_{\rho_i} := \bar{R}_{\rho_i}, d_{\rho_i} := R^*_{\rho_i}.
\]

To save some writing we shall use \(i\) to denote \(\rho_i\) in the following. So \(\rho_{i_1}\rho_{i_2}\ldots\rho_{i_n}\) can be written as \(i_1i_2\ldots i_n\). For any objects \(i_1i_2\ldots i_n\) we define

\[
(i_1i_2\ldots i_n)^* := \bar{i}_n\ldots \bar{i}_2\bar{i}_1,
\]

and define

\[
b_{i_1i_2\ldots i_n} := i_1i_2\ldots i_{n-1}(b_{i_n})\ldots i_1(b_{i_2})b_{i_1},
\]

and

\[
d_{i_1i_2\ldots i_n} := \bar{i}_n\bar{i}_{n-1}\ldots \bar{i}_2(d_{i_1})\bar{i}_n\bar{i}_{n-1}\ldots \bar{i}_3(d_{i_2})\ldots d_{i_n}.
\]

For each object \(i_1i_2\ldots i_n \in C(A)\) and \(f \in \text{End}(i_1i_2\ldots i_n)\) we define unnormalized trace of \(f\) as in §1.2:

\[
tr(f) := d(i_n)\ldots d(i_1)\phi_n(\phi_{n-1}(\ldots (\phi_1(f))\ldots)) \tag{1.7.1}
\]
where $\phi_k$ is the unique minimal left inverse of $\rho_{i_k}, k = 1, \ldots, n$, and $d(i_k)$ is the statistical dimension of $\rho_{i_k}, k = 1, \ldots, n$. Note that (1.7.1) can be also written as

$$tr(f) := d_{i_n} \ldots d_{i_1} \phi_{i_1 \ldots i_n}(f)$$

where $\phi_{i_1 \ldots i_n}$ is the unique minimal left inverse of $i_1 \ldots i_n$ since by §1.2 $\phi_{i_1 \ldots i_n} = \phi_n \phi_{n-1} \ldots \phi_1$.

Next let us define twist:

$$\theta_{i_1 \ldots i_n} := d_{i_n} \ldots d_{i_1} \phi_{i_1 \ldots i_n}(c_{i_1 \ldots i_n,i_1 \ldots i_n}).$$

Denote by $\tilde{\theta}_V := V(b_V^\ast)c_{V,V}V(b_V)$.

Finally note that for any morphisms $f : U \to V$ in $C(A)$, $f$ is an element of von Neumann algebra $M$. The conjugation of $f$ is defined to be $f^* : V \to U$.

Let us first note the following lemma which follows immediately from the definitions:

**Lemma 1.7.1.** For the category $C(A)$ defined above, we have:

1. (1.1.1) and (1.1.2) are true;
2. (1.1.8) is true, i.e.,
   $$\theta_V^* = \theta_V^{-1};$$
3. (1.1.5) is true.

**Proof.** Ad (1): (1.1), (1.2) follows immediately from Prop. 1.4.2 of §1.4; (2) follows from Page 244 of [L1]. (3) follows from definitions and (1.2.1).

□

**Lemma 1.7.2.** (1) (1.1.3) is true (note by our definition $\theta_V \otimes \theta_W := V(\theta_W)$), i.e.,

$$\theta_{VW} = c_{W,V}c_{V,W}\theta_VV(\theta_W);$$

(2)

$$\tilde{\theta}_{VW} = c_{W,V}c_{V,W}\tilde{\theta}_VV(\tilde{\theta}_W);$$

(3) $\theta_V = \tilde{\theta}_V$ if and only if $\theta_{\rho_i} = \tilde{\theta}_{\rho_i};$

(4) $\theta_V = \tilde{\theta}_V$ if and only if (1.1.6) is true;

(5) $\theta_{\rho_i} = \tilde{\theta}_{\rho_i} = c_{i}id_{\rho_i}$ if and only if for any irreducible subsector $x$ of $\rho_i$, the univalence of $x$ is equal to $c_i$.

**Proof.** Ad (1): By Prop. 1.4.2 $c_{VW,VW} = V(c_{V,W})V^2(c_{W,V})c_{V,V}V(c_{V,W})$, and so

$$d(V)\phi_V(c_{VW,VW}) = c_{V,W}V(c_{V,W})\theta_Vc_{W,V}.$$

By using Braiding-Fusion-Equation of Prop. 1.4.2 and the fact that $c_{1,V} = id$ we have

$$\theta_Vc_{W,V} = c_{W,V}\theta_V, \theta_Wc_{V,W} = c_{V,W}V(\theta_W),$$
and by Yang-Baxter equation
\[ c_{V,W}V(c_{W,V})c_{W,V} = W(c_{W,V})c_{W,W}W(c_{V,W}) \]
and so
\[ \theta_{V,W} = d(V)d(W)\phi_W \phi_V (c_{V,W,V}) \]
\[ = d(W)\phi_W (c_{V,W}V(c_{W,V})\theta_V c_{W,V}) \]
\[ = d(W)\phi_W (c_{V,W}V(c_{W,V})W(\theta_V)) \]
\[ = d(W)\phi_W (W(c_{W,V})c_{W,W}W(c_{V,W})W(\theta_V)) \]
\[ = c_{W,V} \theta_W c_{V,W} \theta_V \]
\[ = c_{W,V}c_{V,W}V(\theta_W)\theta_V \]
\[ = c_{W,V}c_{V,W}V(\theta_W) \theta_V \]
which is (1). (2) is proved in similar way. (3) follows from (1), (2) and the construction of \( C(A) \). §1.2. Ad (4): By BFE of Prop. 1.4.2 and (1.1.5)
\[ \tilde{\theta}_V = V(b^*_V) V(c_{V,V^*}^{-1}) b_V \]
\[ = V(b^*_V) V(c_{V,V^*}^{-1}) b_V V^*(b_V V(d_V)) \]
\[ = V(b^*_V) V(c_{V,V^*}^{-1}) V V^*(b_V V(d_V)) b_V \]
\[ = V(d_V) V(b^*_V) V^2(c_{V^*,V^*}) V(b_V) b_V \]
\[ = V(d_V) V(\theta_{V^*}) b_V. \]
Hence \( \tilde{\theta}_V = \theta_V \) is equivalent to
\[ V(d_V) V(\theta_{V^*}) b_V = \theta_V, \]
and multiplied both sides on the right by \( b_V \), using
\[ V(d_V) V(\theta_{V^*}) b_V b_V = V(d_V) V(\theta_{V^*}) V V^*(b_V) b_V = V(d_V) V V^*(b_V) V(\theta_{V^*}) b_V \]
and (1.1.5) we get
\[ V(\theta_{V^*}) b_V = \theta_V b_V, \]
which is (1.1.6). The other direction is similar.

Ad (5): Suppose \( \theta_{\rho_i} = \tilde{\theta}_{\rho_i} = c_i d_{\rho_i} \). By Cor. 1.3.2 the univalence of \( x \) must be \( c_i \). Now suppose the univalence of any irreducible subsector \( x \) of \( \rho_i \) is the same constant \( c_i \). By Cor. 1.3.2, \( \theta_{\rho_i} = c_i d_{\rho_i} \), so we just have to show that \( \tilde{\theta}_{\rho_i} = \theta_{\rho_i} \) to finish the proof. By (4) this is equivalent to show that
\[ \theta_{\rho_i} = \theta_{\rho_i} = c_i, \]
and by Cor. 1.3.2 again it is sufficient to show that for any irreducible subsector \( y \) of \( \rho_i = \rho_i \), the univalence of \( y \) is \( c_i \), which follows from the third equation in (1.5.1).

\[ \square \]

For convenience we shall say that \( \rho_i \) has a uniform univalence if for any irreducible subsector \( x \) of \( \rho_i \), the univalence of \( x \) is equal to a constant \( c_i \) which depends only on \( \rho_i \), and in this case we will define \( \exp(2\pi i \Delta_{\rho_i}) := c_i \). Note that any irreducible endomorphism has a uniform univalence.

**Corollary 1.7.3.** (1): If each \( \rho_i, i \in I \) has a uniform valence, then the trace defined in (1.7.1) agrees with (1.1.7), and the category \( \mathcal{C}(\mathcal{A}) \) is an abelian unitary ribbon category;

(2): If the set \( \{\rho_i, i \in I\} \) contains a finite family \( \{V_j, j \in J\} \) of simple objects satisfying the four defining axioms of modular category in §1.1, then \( \mathcal{C}(\mathcal{A}) \) is a unitary modular category.

**Proof.** Ad (1): Let us first prove the second equation in (1.1.9), i.e.,

\[ b^*_V = d^*_V c_{V,V} \cdot \theta_V. \]

By (1.1.5) it is sufficient to show that

\[ d^*_V c_{V,V} \cdot \theta_V V(d^*_V) = 1. \]

Note \( d^*_V c_{V,V} \cdot \theta_V V(d^*_V) = d^*_V c_{V,V} \cdot V(d^*_V) \theta_V \), and by Prop. 1.4.2 we have

\[ d^*_V c_{V,V} \cdot V(d^*_V) = \theta^*_V, \]

and so

\[ d^*_V c_{V,V} \cdot (\theta_V) V(d^*_V) = \theta^*_V \theta_V = 1. \]

To prove the third equation in (1.1.9) \( d^*_V = V^* (\theta_V^{-1}) c^{-1}_{V^*,V} b_V \), by (1.1.5) it is sufficient to show that

\[ b^*_V V(V^* (\theta_V^{-1}) c^{-1}_{V^*,V} b_V) = 1, \]

but

\[ b^*_V V(V^* (\theta_V^{-1}) c^{-1}_{V^*,V} b_V) = \theta_V^{-1} b^*_V V(c^{-1}_{V^*,V}) V(b_V), \]

\[ b^*_V V(c^{-1}_{V^*,V}) V(b_V) = V(b^*_V) c_{V,V} V(b_V), \]

hence it is enough to show that

\[ V(b^*_V) c_{V,V} V(b_V) = \theta_V, \]

i.e., \( \tilde{\theta}_V = \theta_V \), which is implied by (3), (5) of lemma 1.7.2 and our assumptions.
Now let us show that (1.7.1) agrees with (1.1.7). We need to check for any \( f : V \to V \),
\[ d_V c_{V,V} \cdot \theta_V f b_V = d_V V^*(f) d^*_V, \]
and by (1.1.9) just proved, this is equivalent to
\[ b^*_V f b_V = d_V V^*(f) d^*_V. \]
Note by BFE and the last equation of (1.1.9) we have:
\[ b^*_V f b_V = b^*_V c_{V^*,V} V^*(f) c^{-1}_{V^*,V} b_V = d_V V^*(\theta^{-1}_V f \theta_V)) d^*_V, \]
but the last expression is equal to \( d_V V^*(f) d^*_V \) by the tracial property lemma 1.2.1.

Finally we check (1.1.4), i.e.,
\[ d_V V^*(c_{V,V}) d^*_V f = f d_U V^*(c_{U,U}) d^*_U \]
for any \( f : U \to V, U, V \in C(A) \). By using the naturality of the braiding (1.1.1), (1.1.2),
\[ d_V c_{V,V} d^*_V f = d_V V^*(f) V^*(c_{U,V}) d^*_V, \]
\[ f d_U c_{U,U} d^*_U = d_U U^*(c_{V,U}) U^*(f) d^*_U. \]
Let \( f_1 : V^* \to U^* \) be a homomorphism such that
\[ U^*(f) d^*_U = f_1 d^*_V. \]
Note by (1.1.4) the above defines \( f_1 \) uniquely. In fact
\[ f_1 = U^*(b^*_U) U^*(f) d^*_U. \]
Using \( f_1 \) we have
\[ d_U U^*(c_{V,U}) U^*(f) d^*_U = d_U U^*(c_{V,U}) f_1 d^*_V = d_U f_1 V^*(c_{V,U}) d^*_V, \]
hence to prove (1.1.4) we just have to show
\[ d_V V^*(f) = f_1 f, \]
or equivalently by using (1.1.5)
\[ U^*(b^*_U) U^*(f) d^*_U = d_V V^*(f) V^*(b_U). \]

Denote by
\[ A := U^*(b^*_U) U^*(f) d^*_U, B := d_V V^*(f) V^*(b_U). \]
Using the fact that (1.7.1) agrees with (1.1.7) which is proved above, we have:
\[ tr(AA^*) = tr(ff^*) = tr(BB^*), \]
and \( tr(AB^*) = tr(BA^*) = tr(f^*f) \). But by lemma 1.2.1 we have \( tr(ff^*) = tr(f^*f) \), so
\[
tr((A - B)(A - B)^*) = 0,
\]
and since \( tr \) is faithful, we have
\[
A = B,
\]
which completes the proof of (1.1.4).

Finally (1.10) follows from the definition.

So we have verified (1.1.1) to (1.1.6) and (1.1.9), (1.1.10) for \( C(A) \), which shows that \( C(A) \) is an abelian unitary ribbon category.

(2) follows from (1) and definitions.

□

Now suppose \( A \) is \( A_{G/H} \), the conformal precosheaf associated the coset \( H \subset G_k \) as defined in §1.6. We will denote \( C(A_{G/H}) \) simply by \( C(G/H) \) in the following. Let us make a specific choice of \( \rho_i, i \in I \). We will choose \( \rho_i, i \in I \) to be all the irreducible sectors of any \( (j, \beta) \in \text{exp} \) and also all \( (j, \beta) \in \text{exp} \). Note by (2.3.1) if \( (j, \beta) \in \text{exp} \), then \( (\tilde{j}, \tilde{\beta}) \in \text{exp} \) and \( [(j, \beta)] = [(\tilde{j}, \tilde{\beta})] \). Hence such a set verifies (1.7.0). Notice that for any \( (j, \beta) \in \text{exp} \), it has a uniform univalence which is equal to \( \text{exp}(2\pi i(\Delta_j - \Delta_\beta)) \) by §1.6. The reader may wonder why we include these (possibly reducible) sectors as our building blocks. The reason is that it is convenient to include these objects for the computations of invariants, cf. §3.4 and §3.6 below.

**Theorem A.** If a coset \( H \subset G_k \) verifies the conditions of §1.6, i.e., \( H \subset G_k \) is cofinite, and the conformal precosheaf associated with \( H \) and \( G \) are \( \mu \)-rational, then the category \( C(G/H) \) constructed from the choices of \( \rho_i, i \in I \) above is a unitary modular category.

**Proof.** The category \( C(G/H) \) is a unitary ribbon category by Cor. 1.7.3, so we just need to check axioms (1)-(4) in §1.1. Denote by \( x, x \in X \) the finite set of simple objects in \( C(G/H) \) obtained as irreducible subsectors of all \( (j, \beta) \in \text{exp} \). Denote the identity sector in \( X \) by 0 as in axiom (1). This set obviously satisfies axiom (1), and it satisfies axioms (2) and (3) since all irreducible sectors of the conformal precosheaf \( A_{G/H} \) appear in \( X \) by Cor. 3.2 of [X3]. To prove axiom (4), note that by definition (1.5.0)
\[
Y_{xy} = S^{u}_{xy},
\]
and since the matrix \( (Y_{xy}) \) is nondegenerate by §1.6, it follows that axiom (4) holds.

□

We conjecture that all the cosets \( H \subset G_k \) satisfy the conditions of Theorem A. Infinite series of cosets which satisfy the conditions of Theorem A will be discussed in §3. For more such infinite series, see §3.1 of [X3] and §3.2 of [X2].

Note that if we take \( H_1 = e \) to be a trivial group of \( G \), then \( H_1 \subset G_k \) is automatically cofinite, in fact the conformal precosheaf associated with \( H_1 = \{e\} \subset G \) is simply the conformal precosheaf associated with \( G \). So under the conditions
of Th. A, we have a unitary modular category associated \( G \), denoted by \( C(G) \). Similarly we have unitary modular category associated \( H \), denoted by \( C(H) \). Much of the rest of this paper focus on the relations between \( C(G/H) \) and the categories \( C(G) \) and \( C(H) \).

Let us calculate \( D_{C(G/H),k}^3 \), as defined for any modular category in (1.11) and (1.12). We will write \( D_{C(G/H),k}^3 \) simply as \( D_{G/H}^3 \) in the following.

\[
D_{G/H}^2 = \mu_{G/H}
\]

and similarly \( D_G^2 = \mu_G \), \( D_H^2 = \mu_H \), and by lemma 2.2 we have the following formula for \( D_{G/H} \):

\[
D_{G/H} = \frac{D_G d(G/H)^2}{D_H}
\]

(1.7.2)

By the remark after the proof of Prop. 3.1 in [X3] we have

\[
k_{G/H}^3 = \frac{k_G^3}{k_H^3}
\]

(1.7.3)

where \( k_G^3 \) (resp. \( k_H^3 \)) is \( k_{C(G)}^3 \) (resp. \( k_{C(H)}^3 \)) as defined in (1.12) for modular category \( C(G) \) (resp. \( C(H) \)).

To calculate \( D_G, k_G^3 \) (or \( D_H, k_H^3 \)), let us assume that \( G = SU(N) \) (or \( H = SU(k) \)) even though the following argument works for any \( G \) which has the property that its genus 0 modular matrix agrees with genus 1 modular matrix (cf. §1.5). By comparing (1.11), (1.5.0) and (1.5.9) we immediately have the following formula:

\[
D_G = \frac{1}{S_{11}}
\]

(1.7.4)

and comparing (1.12), (1.5.3) and (1.5.12) we have the following:

\[
k_G^3 = \exp(\frac{\pi i C_G}{4})
\]

(1.7.5)

where \( C_G \) the central charge as defined in (1.5.11).

Define the central charge \( C_{G/H} \) of the coset by

\[
C_{G/H} := C_G - C_H
\]

(1.7.6)

It follows from (1.7.3) and (1.7.5) that

\[
k_{G/H}^3 = \exp(\frac{\pi i C_{G/H}}{4})
\]

(1.7.7)

if \( H, G \) are of type A.
The following two lemmas will be useful in the calculation of framed oriented link invariants from \( C(G/H) \). Given a framed oriented link \( L \) with \( n \) components, colored by objects \( i_1, ..., i_n \in C(G/H) \). The corresponding link invariants as given by Th. 1.2.5 of [Tu] will be denoted by \( L(i_1, ..., i_n) \). As in §1.1, suppose \( L(i_1, ..., i_n) \) is represented by the closure of a tangle \( T_L \in \text{End}(I_1 I_2 ... I_n) \), where \( I_m := (i_m^m)^k \), \( k \in \mathbb{N}, m = 1, ..., n \) and \( \epsilon_m = 1 \) (resp. \( \epsilon_m = -1 \)) if the \( m \)-th component of \( L \) is oriented anti-clockwise (resp. clockwise). Let \( j_k(l), 1 \leq l \leq f(k) \) be a set of subsets of \( i_k \), \( T_k(l) \in \text{Hom}(j_k(l), i_k) \) such that

\[
T_k(l)^*T_k(l') = \delta_{l,l'}, \sum_l T_k(l)T_k(l)^* = id_{i_k}.
\]

Let \( p_k(l) := T_k(l)T_k(l)^* \). Then we have the following:

**Lemma 1.7.4.** With the notations introduced above, we have

1. \( L(j_1(l_1), ... j_n(l_n)) = tr(p_1(l_1)I_1(p_2(l_2)) ... I_1 I_2 I_{n-1}(p_n(l_n))T_L) \);

2. \( \sum_{j_1(l_1), ..., j_n(l_n)} L(j_1(l_1), ... j_n(l_n)) = L(i_1, ..., i_n) \);

3. If \( j_1(l_1), ..., j_n(l_n) \in C(G/H) \) such that \( [j_m(l_m)] = [i_m], m = 1, 2, ..., n \), then

\[
L(j_1(l_1), ... j_n(l_n)) = L(i_1, ..., i_n);
\]

4. \( L(i_1, ..., i_n) \) are independent of the choices of \( (R_{\rho_i}, R'_{\rho_i})_{i \in I} \) in the construction of \( C(G/H) \).

**Proof.** Ad (1): The idea is to insert \( T_k(l_k)^*T_k(l_k) = id_{j_k(l_k)} \) on the \( k \)-th string of \( T_L \) colored by \( j_k(l_k) \). When we pull the coupon represented by \( T_k(l_k)^* \) clockwise around the \( k \)-th component of \( L \) and approaches \( T_k(l_k) \) from below, we obtain the righthand side of (1) by the property of ribbon category. (2) follows from (1). (3) follows from (1) since \( p_m(l_m) = id_{i_m} \). (4) follows since the tangle \( T_L \) consists of braiding and twistings only, and the closure of \( T_L \) are given by minimal conditional expectations which are independent of the choices of \( (R_{\rho_i}, R'_{\rho_i})_{i \in I} \) (cf. §1.2).

\[ \square \]

Now suppose given \( C(G/H) \). Let \( v \) be an object of \( C(G/H) \). We say that \( v \) is a *generator* of \( C(G/H) \) if there exists a finite subset \( K \subseteq \mathbb{N} \) such that any irreducible object of \( C(G/H) \), as a sector appears as a subsector of \( [v^m], m \in K \). Notice that there is a natural inclusion \( \text{End}(v^k) \subset \text{End}(v^l) \) if \( l \geq k \). Now suppose \( C(G/H) \) (resp. \( C(G_1/H_1) \)) are given with generators \( v \) (resp. \( v_1 \)). We say that \( C(G/H) \) is *compatible* with \( C(G_1/H_1) \) if:

1. \( v \) and \( v_1 \) are of the uniform univalence;
(c2): there exists a sequence of trace-preserving *-isomorphisms \( \psi_k : \text{End}(v^k) \to \text{End}(v_1^k), \forall k \geq 1 \) such that

\[
\psi_k(f) = \psi_1(f) \quad \text{if } k \geq 1, \psi_{k+1}(v(f)) = v_1(\psi_k(f)) \quad \forall f \in \text{End}(v^k);
\]

\[
\psi_2(c_{v,v}) = \psi_2(c_{v_1,v_1});
\]

(c3): there exists a finite subset \( K \subset \mathbb{N} \) such that the irreducible objects of \( C(G/H) \) (resp. \( C(G_1/H_1) \)), as sectors are in one-to-one correspondence with the irreducible subsectors of \( [v^m], m \in K \) (resp. \( [v_1^m], m \in K \)), i.e.,

\[
\langle [v^m], [v'^m] \rangle = 0, \quad \text{if } m \neq m', m, m' \in K;
\]

\[
\langle [v_1^m], [v_1'^m] \rangle = 0, \quad \text{if } m \neq m', m, m' \in K,
\]

and the set of irreducible subsectors of \( [v^m], m \in K \) (resp. \( [v_1^m], m \in K \)) are the same as the set of irreducible objects of \( C(G/H) \) (resp. \( C(G_1/H_1) \)) as sectors.

Note that by definition the univalence of \( v \) is \( \frac{\text{tr}(c_{v,v})}{\text{tr}(id_v)} \), and by (c2)

\[
\text{tr}(c_{v,v}) = \text{tr}(c_{v_1,v_1}), \text{tr}(id_v) = \text{tr}(id_{v_1}),
\]

so if \( C(G/H) \) and \( C(G_1/H_1) \) are compatible then \( v \) and \( v_1 \) have the same univalence. We can therefore add to (c1) that \( v \) and \( v_1 \) have the same univalence.

Let \( p_i \in \text{End}(v_i^m), i \in I, m_i \in K \) be the set of minimal projections corresponding to the set of irreducible objects (denoted by \( c(p_i) \)) of \( C(G/H) \). Let \( q_i := \psi_{m_i}(p_i), m_i \in K \) and \( c(q_i) \) be the simple objects of \( C(G_1/H_1) \).

**Lemma 1.7.5.** If \( C(G/H) \) and \( C(G_1/H_1) \) are compatible, then with the notations above we have:

1. \( d(c(p_i)) = d(c(q_i)), \omega(c(p_i)) = \omega(c(q_i)) \);
2. \( D_{G/H} = D_{G_1/H_1}, k_{G/H} = k_{G_1/H_1} \);
3. \( \tau_{G/H}(M_L) = \tau_{G_1/H_1}(M_L) \).

**Proof.** Ad (1): Note \( d(c(p_i)) = \text{tr}(c(p_i)), d(c(q_i)) = \text{tr}(c(q_i)) = \text{tr}(\psi_{m_i}(c(p_i))) \), so \( d(c(p_i)) = d(c(q_i)) \) since \( \psi_{m_i} \) is trace-preserving. By definition and naturality of the twist (cf. (1.1.4)) \( d(c(p_i))\omega(c(p_i)) = \text{tr}(\theta_{c(p_i)}) = \text{tr}(\theta_{v_i^m_p_i}) \) and similarly \( d(c(q_i))\omega(c(q_i)) = \text{tr}(\theta_{v_i^m_q_i}), \) so we just have to show \( \psi_{m_i}(\theta_{v_i^m}) = \theta_{v_i^m} \). By (1.1.3) \( \theta_{v_i^m} \) (resp. \( \theta_{v_i^m} \)) is a product of \( v_j^i(c_{v,v}), j \in \mathbb{N} \) (resp. \( v_j^i(c_{v_1,v_1}), j \in \mathbb{N} \)), their inverses, and the phase \( \omega(v_j^i) \) (resp. \( \omega(v_j^1) \)), and it follows from (c1), (c2) and the remark after them that \( \psi_{m_i}(\theta_{v_i^m}) = \theta_{v_i^m} \) and (1) is proved.

(2) follows from (1) and definitions.

Ad (3): Let \( L \) be any framed oriented link with \( n \) component in \( S^3 \), since \( \tau_{G/H}(M_L) \) (resp. \( \tau_{G_1/H_1}(M_L) \)) is independent of the orientations of \( L \), we can assume that \( L \) is represented by the closure of a tangle \( T_L \) with all the bands oriented downward. By (2), to prove (3) it is enough to show that for such \( L \)

\[
L(c(p_1), ..., c(p_n)) = L(c(q_1), ..., c(q_n)).
\]
By (1) of lemma 1.7.4, with \( i_1 = v^{m_1}, ..., i_n = v^{m_n} \),

\[
L(c(p_1), ..., c(p_n)) = tr(p_1 v^{k_1 m_1}(p_2) ... v^{k_1 m_1 + ... + k_{n-1} m_{n-1}}(p_n) T_L^c),
\]

where \( L^c \) is the cabling of \( L \) (cf. [W2]) with cabling vector \( c = (m_1, m_2, ..., m_n) \). By using the compatibility conditions

\[
tr(p_1 v^{k_1 m_1}(p_2) ... v^{k_1 m_1 + ... + k_{n-1} m_{n-1}}(p_n) T_L)
= tr(\psi_{k_1 m_1 + ... + k_{n-1} m_{n-1}}(p_1 v^{k_1 m_1}(p_2) ... v^{k_1 m_1 + ... + k_{n-1} m_{n-1}}(p_n))) \times
\psi_{k_1 m_1 + ... + k_{n-1} m_{n-1}}(T_L^c),
\]

note

\[
\psi_{k_1 m_1 + ... + k_{n-1} m_{n-1}}(v^{k_1 m_1 + ... + k_{n-1} m_{n-1}}(p_s))
= \psi_{k_1 m_1 + ... + k_{n-1} m_{n-1} + m_s}(v^{k_1 m_1 + ... + k_{n-1} m_{n-1}}(p_s))
= v_1(\psi_{k_1 m_1 + ... + k_{n-1} m_{n-1} + m_s-1}(v^{k_1 m_1 + ... + k_{n-1} m_{n-1} - 1}(p_s)))
= ... = v_1^{k_1 m_1 + ... + k_{n-1} m_{n-1} - 1}(\psi_{m_s}(p_s))
\]

by repeatedly using the compatibility conditions, and so we have

\[
L(c(p_1), ..., c(p_n)) = tr(q_1 v^{k_1 m_1}(q_2) ... v^{k_1 m_1 + ... + k_{n-1} m_{n-1}}(q_n) \psi_{k_1 m_1 + ... + k_{n-1} m_{n-1}}(T_L^c)).
\]

To finish the proof we just have to show that

\[
\psi_{k_1 m_1 + ... + k_{n-1} m_{n-1}}(T_L^c(v, v, ..., v)) = T_L^c(v_1, v_1, ..., v_1).
\]

Note that \( T_L^c \) consists of only twistings and braidings. So \( T_L^c(v, v, ..., v) \) up to a phase factor determined by the univalence of \( v \), is equal to products of \( v_j(c_{v, v}), j \in \mathbb{N} \) and their inverses. It follows from (c1) (c2) and the remark after them that

\[
\psi_{nm}(T_L^c(v, v, ..., v)) = T_L^c(v_1, v_1, ..., v_1),
\]

and the proof is complete.

\[\square\]

\section{Factorization of link invariants}

The goal of this section is to calculate link invariants colored by \((j, \beta) \in \exp\) under the condition that \( d((j, \beta)) = d(j)d(\beta) \). In [X1], [X2] the sectors of \([j, \beta] \) are studied by using the idea of [X4]. But to calculate link invariants one needs to analyze the braiding properties of the system in [X4]. This is essentially done in [BE3], and ideas are also contained in the proof of lemma 3.3 on Page 384 of [X4]. We will use the notations of [X1] and [X4], and set up a dictionary in the following between our notations and that of [BE1] to [BE3] so the similarity of the argument is clear.
2.1 Preliminaries. In this section we sketch some of the results of [LR] which will be used in this paper. For the details of all the proofs, we refer the reader to [LR]. We have changed some of the notations in [LR] since they have been used to denote different objects in this paper.

Let \( N \subset M \) be an inclusion of type III von Neumann algebras on a Hilbert space \( H \). Let \( \phi \in H \) be a joint cyclic and separating vector (which always exists if \( N \) is type III and \( H \) is separable). Let \( j_N = Ad_{J_N} \) and \( j_M = Ad_{J_M} \) be the modular conjugations w.r.t. \( \phi \) and the respective algebra. Then

\[ \alpha = j_N j_M |_M \in \text{End}(M) \]

maps \( M \) into a subalgebra of \( N \). We call \( \alpha \) the canonical endomorphism associated with the subfactor, and denote by

\[ N_1 := j_N j_M(M) \subset N, M \subset M_1 := j_M j_N(N) \]

the canonical extension resp. restriction. \( \phi \) is again joint cyclic and separating for the new inclusions above, giving rise to new canonical endomorphisms \( \beta = j_{N_1} j_N \in \text{End}(N) \) and \( \alpha_1 = j_{M_1} j_{M_1} \in \text{End}(M_1) \).

We have the following formula for canonical endomorphism: (cf. Prop.2.9 of [LR]):

**Proposition 2.1.1.** Let \( N \subset M \) be an inclusion of properly infinite factors, \( \epsilon : M \to N \) a faithful normal conditional expectation, and \( e_N \in N' \) the associated Jones projection. The canonical endomorphism \( \alpha : M \to N \) is given by

\[ \alpha = \Psi^{-1} \cdot \Phi \]

where

\[ \Phi(m) = U m U^*(m \in M) \]

is the isomorphism of \( M \) into \( Ne_N \) implemented by an isometry \( U \in M_1 = \langle M, e_N \rangle \) with \( UU^* = e_N \), and \( \Phi \) is the isomorphism of \( N \) with \( Ne_N \) given by

\[ \Phi(n) = ne_N(n \in N). \]

Every canonical endomorphism of \( M \) into \( N \) arises in this way.

**Definition:** A net of von Neumann algebras over a partially ordered set \( J \) is an assignment \( \mathcal{M} : i \to M_i \) of von Neumann algebras on a Hilbert space to \( i \in J \) which preserves the order relation, i.e., \( M_i \subset M_k \) if \( i \leq k \). A net of subfactors consists of two nets \( \mathcal{N} \) and \( \mathcal{M} \) such that for every \( i \in J \), \( N_i \subset M_i \) is an inclusion of subfactors. We simply write \( \mathcal{N} \subset \mathcal{M} \). The net \( \mathcal{M} \) is called standard if there is a vector \( \Omega \in H \) which is cyclic and separating for every \( M_i \). The net of subfactors \( \mathcal{N} \subset \mathcal{M} \) is called standard if \( \mathcal{M} \) is standard and \( \mathcal{N} \) is standard on a subspace \( H_0 \subset H \) with the same cyclic and separating vector \( \Omega \in H_0 \). For a net of subfactors
$\mathcal{N} \subset \mathcal{M}$, let $\epsilon$ be a consistent assignment $i \rightarrow \epsilon_i$ of normal conditional expectations. Consistency means that $\epsilon_i = \epsilon_k|_{M_i}$ whenever $i \leq k$. Then we call $\epsilon$ a normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. $\epsilon$ is called standard, if it preserves the vector state $\omega = (\Omega, \cdot)\Omega$.

If the index set $J$ is directed, i.e., for $j, k \in J$ there is $m \in J$ with $j, k \leq m$, we associate with a net $\mathcal{M}$ of von Neumann algebras the inductive limit $C^*$ algebra $(\bigcup_{i \in J} M_i)^{-}$ and denote it by the same symbol $\mathcal{M}$. Then we have (cf. Cor.3.3 of [LR]):

**Proposition 2.1.2.** Let $\mathcal{N} \subset \mathcal{M}$ be a directed standard net of subfactors (w.r.t. the vector $\Omega \in H$) over a directed set $J$, and $\epsilon$ a standard conditional expectation. For every $i \in J$ there is an endomorphism $\alpha^i$ of the $C^*$ algebra $\mathcal{M}$ into $\mathcal{N}$ such that $\alpha|_{M_j}$ is a canonical endomorphism of $M_j$ into $N_j$ whenever $i \leq j$. Furthermore, $\alpha^i$ acts trivially on $M_i \cap \mathcal{N}$. As $i \in J$ varies to $k$, the corresponding endomorphisms $\alpha^i$ and $\alpha^k$ are inner equivalent by a unitary in $N_i$ whenever $i, k \leq l$.

If the index set $J$ is directed, by Cor.4.2 of [LR] the index is constant in a directed standard net of subfactors with a standard conditional expectation. We have (cf. Cor.4.3 of [LR]):

**Proposition 2.1.3.** Let $\mathcal{N} \subset \mathcal{M}$ be a directed standard net of subfactors (w.r.t. the vector $\Omega \in H$) over a directed set $J$, and $\epsilon$ a standard conditional expectation. If the index $d^2 = \text{Ind}(\epsilon)$ is finite, then for any $i \in J$, there is an isomorphic intertwiner $v_1 : \text{id} \rightarrow \alpha$ in $M_i$ which satisfies the following identity with the isometric intertwiner $w_0 : \text{id} \rightarrow \beta$ ($\beta := \alpha|_{\mathcal{N}}$) in $N_i$:

$$w_0^*v_1 = d^{-1}\text{id} = w_0^*\alpha(v_1).$$

$\alpha$ is given on $\mathcal{M}$ by the formula

$$\alpha(m) = d^2\epsilon(v_1mv_1^*)(m \in \mathcal{M}).$$

Furthermore, every element in $\mathcal{M}$ is of the form $nv_1$ with $n \in \mathcal{N}$, namely

$$m = d^2\epsilon(mv_1^*)v_1 = d^2v_1^*\epsilon(v_1m).$$

Now assume that two conformal precosheaf (cf. §1.3) $N(I), M(I)$ are given, and use $H^0$ to denote the Hilbert space of the vacuum representation of $M(I)$, and $\Omega \in H^0$ is the vacuum vector (uniquely determined up to a nonzero constant). Assume that there is a covariant representation $\pi^0$ of $N(I)$ on $H^0$ such that $\pi^0(N(I)) \subset M(I)$, and moreover, fix any proper interval of $S^1$, the net $\pi^0(N(I)) \subset M(I)$ for any $I \subset J \subset S^1$ is a directed standard net of subfactors w.r.t. $\Omega$. An immediate corollary of the above proposition is the following:

**Lemma 2.1.4.** Suppose the net $\pi^0(N(I)) \subset M(I)$ has finite index. Then if $N(I)$ is strongly additive, $M(I)$ is also strongly additive.

**Proof.** We need to show that if $I_1, I_2$ are the connected components of an interval $I$ with one internal point removed, then $M(I) \subset M(I_1) \lor M(I_2)$. 

By the above proposition we can choose isometry $v_1 \in \pi^0(N(I_1))$ such that any element of $M(I)$ can be written as $nv_1$ for some $n \in \pi^0(N(I))$. Since $N(I)$ is strongly additive by our assumption, $\pi^0(N(I)) = \pi^0(N(I_1)) \cup \pi^0(N(I_2)) \subset M(I_1) \cup M(I_2)$, it follows that

$$M(I) \subset M(I_1) \cup M(I_2)$$

and the proof is complete.

For the rest of this section, we assume that a net $\pi^0(N(I)) \subset M(I)$ as described before lemma 2.1.4 with finite index is given. By Prop. 2.1.1 and Prop. 2.1.3 we have a canonical endomorphism $\alpha^{I_0} : M(I) \to N(I)$ for any $I \supset I_0$. Recall $\beta^{I_0} := \alpha^{I_0}|N(I)$. For simplicity, we shall drop the labels and write $\alpha, \beta$ simply as $\alpha, \beta$ in the following. Define $M := N(I_0)$.

Define $U(\gamma, I) : H^0 \to H_0$ to be a unitary operator which commutes with the action of $N(I)$, where $H_0$ denotes the Hilbert space of the vacuum representation $\pi_0$ of $N(I)$. Notice that such an operator always exists since $\pi^0|N(I)$ is equivalent to that of $\pi_0$. We shall think of $U(\gamma, I)$ as an element of $B(H^0)$ by identifying $H_0$ as a subspace of $H^0$. Define: $\phi_I : B(H^0) \to B(H_0)$ by

$$\phi_I(m) = U(\gamma, I)mU(\gamma, I)^*, m \in B(H^0)$$

As on Page 368 of [X4] $\alpha$ can be chosen to be:

$$\alpha(m) = \phi_I^{-1}(\phi_{I_0}(m)). \quad (2.1.1)$$

For any $a \in N(I)$, define:

$$\gamma_I(a) := \phi_I\beta\phi_I^{-1}(a).$$

Notice since $\phi_I^{-1}(N(I)) = \pi^0(N(I))$, $\gamma$ is well defined. By using (2.1), we have:

$$\gamma_I(a) = \phi_{I_0}(\phi_I^{-1}(a)) \quad (2.1.2)$$

which is precisely the definition of localized (localized on $I_0$) covariant endomorphism associated with the representation $H^0$ of $M(I)$.

We shall denote $\gamma_{I_0}$ for the fixed interval $I_0$ by $\gamma$. Notice $\gamma \in \text{End}(M)$.

Since $\beta|_{N(I_0)}$ is a canonical endomorphism, we can find $\rho_1 \in \text{End}(N(I_0))$ such that:

$$\beta|_{N(I_0)} = \rho_1\tilde{\rho}_1$$

with $\tilde{\rho}_1(N(I_0)) \subset N(I_0)$ conjugate to $N(I_0) \subset M(I_0)$. Moreover,

$$\rho_1(N(I_0)) = \alpha(M(I_0))$$

. Recall $M := N(I_0)$. Define $\rho \in \text{End}(M)$ by:

$$\rho := \phi_{I_0}\rho_1\phi_{I_0}^{-1}$$
Anticipating the dictionary below, we denote by $j_{BE}$ the localized covariant endomorphism (localized on $I_0$) of $M(I)$ on $H_{j_{BE}}$. Let us recall how $j_{BE}$ is defined. Let $U(j, I) : H_{j_{BE}} \to H^0$ be a unitary map which commutes with the action of $L_1 G$. Define: $\psi_I : B(H_{j_{BE}}) \to B(H^0)$ by $\psi_I(x) = U(j, I)xU(j, I)^*$. Then $j_{BE}$ is given by:

$$j_{BE}(m) = \psi_I(\psi_I^{-1}(m)) \quad (\forall m \in M(I)).$$

Let $\gamma_j$ be the reducible representation of $N(I)$ on $H_{j_{BE}}$. Then the localized endomorphism, denoted by the same $\gamma_j$, is given by:

$$\gamma_j = \phi_{I_0}^{c}\psi_I^{c}\psi_I^{-1}\phi_I^{-1}.$$ 

Define $\sigma_j \in \text{End}(M)$ by:

$$\sigma_j = \rho^{-1}\phi_{I_0}^{c}j_{BE}\phi_{I_0}^{-1}\rho$$ \hspace{1cm} (2.1.3)

Notice by the definition of $\rho_1$, $\rho_1(N(I_0)) = \alpha(M(I_0)) = \phi_{I_0}^{-1}\phi_{I_0}(M(I_0))$, and it follows that $\phi_{I_0}^{-1}\rho(M) = M(I_0)$, so $\sigma_j$ as above is well defined. Note that our $\sigma_j$ is $\sigma_j'$ in [X4] and our $j_{BE}$ is $\sigma_j$ in [X4].

For each covariant endomorphism $\lambda$ (localized on $I_0$) of $N(I)$ with finite index, let $\sigma_{\gamma\lambda} : \gamma \lambda \to \lambda \gamma$ (resp. $\sigma_{\bar{\gamma}\lambda} : \gamma \lambda \to \lambda \gamma$), be the positive (resp. negative) braiding operator as defined in §1.4. Denote by $\lambda_\sigma \in \text{End}(M)$ which is defined by

$$\lambda_\sigma(x) := \text{Ad}(\sigma_{\gamma\lambda}^*)\lambda(x) = \sigma_{\gamma\lambda}^*\lambda(x)\sigma_{\gamma\lambda}$$

for any $x \in M$. By (1) of Th. 3.1 of [X4], $\lambda_\sigma\rho(m) \in \rho(M)$

(similarly $\lambda_{\bar{\sigma}}\rho(m) \in \rho(M)$), and hence the following definition makes sense.

**Definition 2.1.1.** $a_\lambda$ is an endomorphism of $M$ defined by:

$$a_\lambda(m) := \rho^{-1}(\lambda_\sigma\rho(m)), \quad \bar{a}_\lambda(m) := \rho^{-1}(\lambda_{\bar{\sigma}}\rho(m)), \quad \forall m \in M.$$

The endomorphisms $a_\lambda$ are called braided endomorphisms in [X4] due to its braiding properties (cf. (2) of Cor. 3.4 of [X4]), and enjoy a remarkable set of properties (cf. §3 of [X4]). Motivated by [X4], these properties are also studied in a slightly different context in [BE1] and [BE2]. We will set up a dictionary between our notations here and that of [BE1] and [BE2]. In the following the notations from [BE1] and [BE2] will be given a subscript BE. The formulas are:

$$\gamma = \phi_{I_0}^{BE}\theta_{BE}\phi_{I_0}^{-1}, \quad \alpha = \gamma_{BE},$$

$$\lambda = \phi_{I_0}\lambda_{BE}\phi_{I_0}^{-1}, \quad \sigma_{\lambda\gamma} = \phi_{I_0}(\epsilon(\lambda_{BE}, \theta_{BE})), $$

$$\bar{\rho}(\sigma_{\lambda\gamma}) = \rho^{-1}\phi_{I_0}(\epsilon(\lambda_{BE}, \theta_{BE}))$$ \hspace{1cm} (2.1.4)

where is last formula is obtained by the following:

$$\rho^{-1}\phi_{I_0}(\epsilon(\lambda_{BE}, \theta_{BE})) = \rho^{-1}\gamma(\sigma_{\lambda\gamma}) = \bar{\rho}(\sigma_{\lambda\gamma})$$
which follows from (2.1.3) and $\rho \bar{\rho} = \gamma$. Now we are ready to set up a dictionary between $a_\lambda \in \text{End}(M)$ in definition 2.1.1 and $\alpha_\lambda^-$ as in Definition 3.3 and 3.5 of [BE1]. By using definitions above we have:

$$a_\lambda = \rho^{-1} \phi_0 \alpha_{\lambda BE} \phi_0^{-1} \rho$$

$$\tilde{a}_\lambda = \rho^{-1} \phi_0 \alpha_{\lambda BE} \phi_0^{-1} \rho$$ (2.1.5)

In fact

$$\rho^{-1} \phi_0 \alpha_{\lambda BE} \phi_0^{-1} \rho = \rho^{-1} \phi_0 \lambda \lambda_{BE} \phi_0^{-1} \rho$$

$$= \rho^{-1} \lambda \lambda_{BE} \phi_0^{-1} \rho$$

$$= \rho^{-1} \lambda \lambda_{BE} \phi_0^{-1} \rho$$

$$= \rho^{-1} \lambda \lambda_{BE} \phi_0^{-1} \rho$$

$$= \rho^{-1} \lambda \lambda_{BE} \phi_0^{-1} \rho$$

$$= \rho^{-1} \lambda \lambda_{BE} \phi_0^{-1} \rho$$

The second equation in (2.1.5) follows similarly. Notice the similarity of between (2.1.5) and (2.1.3). Formulas (2.1.4) and (2.1.5) above will be referred to as our dictionary between the notations of [X4] and that of [BE1] to [BE3].

§ 2.2 Relative braidings from [BE3]. We preserve the setup of § 2.1.1.

**Lemma 2.2.1.** Let $p, q \in \text{Hom}(a_{\mu_1}, a_\mu)$. Then

$$\bar{\rho}(\sigma_\mu \lambda)p \bar{\rho}(\sigma^*_{\mu_1 \lambda}) = a_\lambda(p), \bar{\rho}(\sigma_\mu \lambda)\tilde{a}_\lambda(q)\bar{\rho}(\sigma^*_{\mu_1 \lambda}) = q$$

**Proof.** The proof follows from line 9 (count from the top) on Page 385 of [X4] (In the diagram drawing below line 9 one needs to change $\mu$ to $\mu_1$). This proof can also be translated into the proof of Lemma 3.25 of [BE3] by using our dictionary (2.1.4) and (2.1.5).

Now let us define relative braiding as first introduced in §3.3 of [BE3]. Let $\tilde{\beta}, \delta \in \text{End}(M)$ be subsectors of $\tilde{a}_\lambda$ and $a_\mu$. By lemma 3.3 of [X4], $[\tilde{\beta}]$ and $[\delta]$ commute. Denote by $\epsilon_r(\tilde{\beta}, \delta)$ given by:

$$\epsilon_r(\tilde{\beta}, \delta) := s^* a_\mu (t^*) \rho(\sigma_\mu \lambda) \tilde{a}_\lambda(s)t \in \text{Hom}(\beta \delta, \delta \beta)$$

$$\epsilon_r(\delta, \tilde{\beta}) := \epsilon_r(\tilde{\beta}, \delta)^{-1},$$ (2.2.1)

with isometries $t \in \text{Hom}(\tilde{\beta}, \tilde{a}_\lambda)$ and $s \in \text{Hom}(\delta, a_\mu)$. Note that the first formula above is exactly (10) of [BE3] using (2.1.4), (2.1.5). Also note from (2.2.1) we have

$$\epsilon_r(\tilde{a}_\lambda, a_\mu) = \bar{\rho}(\sigma_\mu \lambda), \epsilon_r(a_\lambda, \tilde{a}_\mu) = \bar{\rho}(\lambda_{\mu})$$ (2.2.2)
Lemma 2.2.2. The operator $\epsilon_r(\beta, \delta)$ defined above does not depend on $\lambda, \mu$ and the isometries $s, t$ in the sense that, if there are isometries $x \in \text{Hom}(\beta, \tilde{a}_\nu)$ and $y \in \text{Hom}(\delta, \tilde{a}_{\delta_1})$, then

$$\epsilon_r(\beta, \delta) = s^* a_{\delta_1} (t^*) \tilde{\rho}(\sigma_{\nu, \lambda_1}) \tilde{a}_\nu(y)x$$

Proof. The proof follows from Lemma 2.2.1 as in the proof of Lemma 3.11 of [BE3] by using the dictionary (2.1.4) and (2.1.5).

The following two lemmas are Prop. 3.12 and Prop. 3.15 of [BE3], translated into our notations by using the dictionary (2.1.4) and (2.1.5).

Lemma 2.2.3. The system of unitaries of Eq. (6) provides a relative braiding between representative endomorphisms of subsectors of $\tilde{a}_\lambda$ and $a_\mu$ in the sense that, if $\beta, \delta, \omega, \xi$ are subsectors of $[\tilde{a}_\lambda], [a_\mu], [\tilde{a}_\nu], [a_{\delta_1}]$, respectively, then we have initial conditions

$$\epsilon_r(id_M, \delta) = \epsilon_r(\beta, id_M) = 1,$$

compositions rules

$$\epsilon_r(\beta \omega, \delta) = \epsilon_r(\beta, \delta) \beta(\epsilon_r(\omega, \delta)), \epsilon_r(\beta, \delta \xi) = \delta(\epsilon_r(\beta, \xi)) \epsilon_r(\beta, \delta),$$

and naturality

$$\delta(q_+) \epsilon_r(\beta, \delta) = \epsilon_r(\omega, \delta) q_+, q_- \epsilon_r(\beta, \delta) = \epsilon_r(\beta, \xi) \beta(q_-)$$

whenever $q_+ \in \text{Hom}(\beta, \omega)$ and $q_- \in \text{Hom}(\delta, \xi)$.

Lemma 2.2.4. $\rho^{-1} \phi_{\tilde{\sigma}_i}(\sigma_{BEjBE}) = \epsilon_r(\tilde{\sigma}_i, \sigma_j)$, where $\tilde{\sigma}_i$ means that the relative braiding $\epsilon_r(\tilde{\sigma}_i, \sigma_j)$ is defined so that its first argument is considered to be a subsector of some $\tilde{a}_\lambda$ (and hence the second argument is considered to be a subsector of some $a_\mu$), and $\sigma_{BEjBE}$ is the braiding operator as defined in §1.4 for the conformal precosheaf associated with $\{M(I), \forall I\}$.

§2.3 Factorization of coset link invariants. Let $H \subset G_k$ be an inclusion which satisfies the assumptions of §1.6. Let $A_{G/H}(I)$ be the conformal precosheaf of the coset as described in §1.6. Recall from §4.2 of [X4] that $N(I) := A_{G/H}(I) \otimes \pi_0(L_I H)''$ and $M(I) := \pi^0(L_I G)''$ verifies all the assumptions of §2.1 and §2.2 so we can apply the results of §2.1 and §2.2. The sectors $\pi_{i,\alpha}$ of $A_{G/H}(I)$ are obtained in the decompositions of $\pi^i$ of $LG$ with respect to subgroup $LH$, and we will denote the set of such $(i, \alpha)$ by $\exp$. We will use the following tensor notation

Tensor Notation. Let $\rho \in \text{End}(A_{G/H}(I) \otimes \pi_0(L_I H)'')$. We will denote $\rho$ by $\rho_1 \otimes \rho_2$ if

$$\rho(x \otimes 1) = \rho_1(x) \otimes 1, \forall x \in A_{G/H}(I), \rho(1 \otimes y) = 1 \otimes \rho_2(y), \forall y \in \pi_0(L_I H)'',$$
where \( \rho_1 \in \text{End}(A_{G/H}(I)), \rho_2 \in \text{End}(\pi_0(L_1H)''). \)

So \((i, \alpha) \otimes \beta\) will be a covariant endomorphism of \(N(I) = A_{G/H}(I) \otimes \pi_0(L_1H)''\) obtained from the covariant endomorphisms \((i, \alpha)\) and \(\beta\) of \(A_{G/H}(I)\) and \(\pi_0(L_1H)''\) respectively. Note we have

\[
\omega((i, \alpha) \otimes \beta) = \omega((i, \alpha))\omega(\beta).
\]

For simplicity we shall denote \((i, \alpha) \otimes 1\) (resp. \(1 \otimes \beta\)) as \((i, \alpha)\) (resp. \(\beta\)) where \(1\) stands for identity automorphism.

As in §2 fix a proper open interval \(I_0 \subset S^1\). For the rest of this section we will fix a choice a finite set of irreducible endomorphisms of \(\mathcal{A}(I_0)\) which appear as irreducible subsectors of all \((i, \alpha) \in \exp\) localized on \(I_0\) and fix a choice of all irreducible endomorphisms (denoted by \(\beta\)) of \(\pi_0(L_nH)''\) obtained from the irreducible covariant representations of the conformal precosheaf associated with \(H\) localized on \(I_0\). Also as in §2.1 fix a choice of \(\sigma_i, \gamma\) and \(\rho, \tilde{\rho}\) such that \(\rho\tilde{\rho} = \gamma\).

Note \([\sigma_i a_{(i, \alpha)}]\) is a subsector of \([\sigma_i a_{\tilde{\alpha}}]\) by (2) of Prop. 4.2 of [X1]. In fact the proof of (2) of Prop. 4.2 of [X1] also shows that if \((i, \alpha) \in \exp\), then

\[
[(i, \alpha)] = [(\tilde{i}, \tilde{\alpha})]
\]

and it follows that \((\tilde{i}, \tilde{\alpha}) \in \exp\). Let us give a proof of (2.3.1) following the proof of (2) of Prop. 4.2 of [X1]. Assume that \([(i, \alpha)] = \sum_j m_j [x_j]\), where the sum is finite, \(m_j \in \mathbb{N}\), and \(x_j\) is irreducible. So \([(i, \alpha)] = \sum_j m_j [\tilde{x}_j]\). Since

\[
\langle a_{x_j \otimes 1}, a_{1 \otimes \tilde{\alpha}} \sigma_i \rangle = \langle x_j, (i, \alpha) \rangle = m_j
\]

by the proof of (2) of Prop. 4.2 of [X1], and \([a_\lambda] = [\tilde{\alpha}_\lambda]\) by (2) of Cor. 3.5 of [X4], \([\tilde{\sigma}_i] = [\sigma_i]\) by definition, we have

\[
\langle a_{\tilde{x}_j \otimes 1}, a_{1 \otimes \alpha} \sigma_i \rangle = \langle \tilde{x}_j, (\tilde{i}, \tilde{\alpha}) \rangle = m_j
\]

and it follows that \([(\tilde{i}, \tilde{\alpha}] \succ [i, \alpha])\) for any \((i, \alpha) \in \exp\). This shows that \((\tilde{i}, \tilde{\alpha}) \in \exp\) if \((i, \alpha) \in \exp\). Replacing \(i, \alpha\) by \(\tilde{i}, \tilde{\alpha}\) we get \([(i, \alpha)] \succ [(\tilde{i}, \tilde{\alpha})]\), and so \([(\tilde{i}, \tilde{\alpha})]\) \succ \([(i, \alpha)]\), and this together with \([(\tilde{i}, \tilde{\alpha})]\) \succ \([(i, \alpha)]\) proves (2.3.1).

**Proposition 2.3.1.** Suppose \(u_{i, \alpha} \in \text{Hom}(a_{(i, \alpha)}, \sigma_i a_{\tilde{\alpha}})\), \(u_{j, \beta} \in \text{Hom}(a_{(j, \beta)}, \sigma_j a_{\tilde{\beta}})\), then: (1)

\[
\epsilon_r(\tilde{\alpha}_\lambda, \sigma_j a_{\tilde{\beta}})a_{\tilde{\alpha}}(u_{j, \beta}) = u_{j, \beta} \epsilon_r(\tilde{\alpha}_\lambda, a_{(j, \beta)});
\]

\[
\epsilon_r(\alpha_{\tilde{\alpha}}, \tilde{\sigma}_j a_{\tilde{\beta}})a_{\tilde{\alpha}}(u_{j, \beta}) = u_{j, \beta} \epsilon_r(a_{\tilde{\alpha}}, a_{(j, \beta)});
\]

\[
\epsilon_r(\tilde{\sigma}_i a_{\tilde{\alpha}}, a_{(j, \beta)})u_{i, \alpha} = a_{(j, \beta)}(u_{i, \alpha}) \epsilon_r(\tilde{\alpha}(i, \alpha), a_{(j, \beta)});
\]
(2) 
\[ \epsilon_r(\tilde{a}_\alpha, a_{(j, \beta)}) = \epsilon_r(a_\alpha, \tilde{a}_{(j, \beta)}) ; \]

(3) Define 
\[ c(\sigma_i a_\bar{\alpha}, \sigma_j a_{\bar{\beta}}) := \epsilon_r(\bar{\sigma}_i, \sigma_j a_\beta)\sigma_i(\epsilon_r(a_\alpha, \bar{\sigma}_j \bar{a}_{\bar{\beta}})) . \]

Then 
\[ c(\sigma_i a_\bar{\alpha}, \sigma_j a_{\bar{\beta}})\sigma_i a_\bar{\alpha}(u_{j, \beta})u_{i, \alpha} = u_{j, \beta}a_{(j, \beta)}(u_{i, \alpha})\epsilon_r(\tilde{a}_{(i, \bar{\alpha})}, a_{(j, \beta)}) . \]

Proof. Ad (1): The first equation follows from the naturality of relative braiding (cf. Lemma 2.2.3). The second equation also follows from the naturality of relative braiding and the fact that 
\[ \text{Hom}(a_{(j, \beta)}, \sigma_j a_{\bar{\beta}}) = \text{Hom}(a_{(j, \beta)}\bar{\rho}, \sigma_j a_{\bar{\beta}}\bar{\rho}) = \text{Hom}(\bar{a}_{(j, \beta)}\bar{\rho}, \sigma_j \bar{a}_{\bar{\beta}}\bar{\rho}) = \text{Hom}(\bar{a}_{(j, \beta)}, \sigma_j \bar{a}_{\bar{\beta}}), \]
where we have used Cor. 3.2 and (1) of Th. 3.3 in [X4]. The third equation follows similarly.

Ad(2): By monodromy equation (cf. Prop. 1.4.3)
\[ \sigma_{\bar{\alpha}, (j, \beta)}\sigma_{(j, \alpha), \bar{\alpha}} = \exp 2\pi i(\Delta_{(j, \beta)} \otimes \bar{\alpha} - \Delta_{\bar{\alpha}} - \Delta_{(j, \beta)}) = 1 , \]

since 
\[ \Delta_{(j, \beta)} \otimes \bar{\alpha} = \Delta_{\bar{\alpha}} + \Delta_{(j, \beta)} . \]

It follows that 
\[ \sigma_{\bar{\alpha}, (j, \beta)} = \bar{\sigma}_{(j, \beta)} , \]
and by applying \( \bar{\rho} \) to both sides and use definition (2.2.1) we obtain (2).

(3) We have 
\[ \epsilon_r(a_\alpha, \bar{\sigma}_j \bar{a}_{\bar{\beta}})a_\alpha(u_{j, \beta}) = u_{j, \beta}\epsilon_r(a_\alpha, \bar{a}_{j, \beta}) \]
\[ = u_{j, \beta}\epsilon_r(\tilde{a}_\alpha, a_{j, \beta}) \]
\[ = \epsilon_r(\tilde{a}_\alpha, \sigma_j a_{\bar{\beta}})\tilde{a}_\alpha(u_{j, \beta}) \]

where we have used (2) in the second = and (1) in the the first and last =. It follows that 
\[ c(\sigma_i a_\bar{\alpha}, \sigma_j a_{\bar{\beta}})\sigma_i a_\bar{\alpha}(a_{j, \beta}) = \epsilon_r(\bar{\sigma}_i, \sigma_j a_\beta)\sigma_i(\epsilon_r(a_\alpha, \bar{\sigma}_j \bar{a}_{\bar{\beta}})a_\alpha(u_{j, \beta})) \]
\[ = \epsilon_r(\bar{\sigma}_i, \sigma_j a_\beta)\sigma_i(\epsilon_r(\tilde{a}_\alpha, \sigma_j a_{\bar{\beta}})\tilde{a}_\alpha(u_{j, \beta})) \]
\[ = \epsilon_r(\bar{\sigma}_i, \sigma_j a_\beta)\sigma_i(\epsilon_r(\tilde{a}_\alpha, \sigma_j a_{\bar{\beta}})\tilde{a}_\alpha(u_{j, \beta})) \]
\[ = \epsilon_r(\bar{\sigma}_i \tilde{a}_\alpha, \sigma_j a_{\bar{\beta}})\tilde{a}_\alpha(u_{j, \beta}) \]
\[ = u_{j, \beta}\epsilon_r(\bar{\sigma}_i \tilde{a}_\alpha, a_{(j, \beta)}) . \]
where we have used the equation derived above in the second = and (1) and lemma 2.2.3 in the rest of equalities (note by the convention of lemma 2.2.4, $\tilde{\sigma}_i = \sigma_i$). The proof is then complete by using the third equation in (1).

\[ \square \]

Note that in (3) of the above proposition, the operator $c(\sigma_i a_{\tilde{\alpha}}, \sigma_j a_\beta)$ involve relative braidings between $\sigma_i$ and $a_\beta$, $\sigma_j$ and $a_{\tilde{\alpha}}$ as defined in §2.2, where $\sigma_i$, $\sigma_j$ are considered as subsectors of some $\tilde{\alpha}_\mu, \tilde{\alpha}_\nu$. Since

$$\epsilon_r(\tilde{\sigma}_i, a_\beta)\epsilon_r(a_\beta, \tilde{\sigma}_i) = 1, \epsilon_r(\tilde{\sigma}_j, a_{\tilde{\alpha}})\epsilon_r(a_{\tilde{\alpha}}, \tilde{\sigma}_j) = 1$$

by definition, we can think that the braidings between $\sigma_i$ and $a_\beta$, $\sigma_j$ and $a_{\tilde{\alpha}}$ are “trivial”. Also note the braidings between $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$. The next lemma shows that $\tilde{\alpha}_\mu, \tilde{\alpha}_\nu$ is important in the proof of Th. B below.

Now let us focus on the cases that $[\sigma_i a_{\tilde{\alpha}}] = [a_{(i,\alpha)}]$ which holds for a general class of cosets including diagonal cosets of type A (cf. Th. 4.3 of [X4]). By (2) of Prop. 4.2 of [X1], this is equivalent to $d(i, \alpha) = d(i)d(\alpha)$. Fix a choice of unitaries $u_{i,\alpha} \in Hom(a_{(i,\alpha)}, \sigma_i a_{\tilde{\alpha}})$ for each $(i, \alpha) \in exp$. The relations between the braidings among $a_{(i,\alpha)}$ and the braidings among $\sigma_i a_{\tilde{\alpha}}$ is given by (3) of Prop. 2.3.1. Now we need to relate the dualities. Denote by $r_{i,\alpha} := \rho^{-1}\phi_{f_0}(R_{iB}) \in Hom(id, \sigma_i a_{\tilde{\alpha}})$, $\tilde{r}_{i,\alpha} := \rho^{-1}\phi_{f_0}(R_{i\tilde{B}}) \in Hom(id, \sigma_i \tilde{\sigma}_i)$, and $r_{a_{\tilde{\alpha}}} = \tilde{\rho}(R_{a_{\tilde{\alpha}}}) \in Hom(id, a_{\tilde{\alpha}} a_{\alpha})$, $\tilde{r}_{a_{\tilde{\alpha}}} = \tilde{\rho}(\tilde{R}_{a_{\tilde{\alpha}}}) \in Hom(id, a_{\tilde{\alpha}} a_{\alpha})$, where $R$ and $\tilde{R}$ are defined as in (1.2.1). Define:

$$r_{i,\alpha} = a_{(i,\alpha)}(u_{i,\alpha}^* u_{i,\alpha}^* \epsilon_r(a_{(i,\alpha)}, \tilde{\sigma}_i) a_{(i,\alpha)}(r_{\tilde{\alpha}}) r_{a_{\tilde{\alpha}}},$$

and

$$\tilde{r}_{i,\alpha} = a_{(i,\alpha)}(u_{i,\alpha}^* u_{i,\alpha}^* \epsilon_r(a_{(i,\alpha)}, \tilde{\sigma}_i) \sigma_i a_{\tilde{\alpha}}(r_{\tilde{\alpha}}) \tilde{r}_{i}.$$  

Note by definition and Prop. 4.2 of [X1]

$$r_{i,\alpha} \in Hom(id, a_{(i,\tilde{\alpha})} a_{(i,\alpha)}) = Hom(\tilde{\rho}, \tilde{\rho}(\tilde{i}, \tilde{\alpha})(\tilde{i}, \alpha))$$

$$= \tilde{\rho}(Hom(1, (\tilde{i}, \tilde{\alpha})(\tilde{i}, \alpha)))$$

and similarly $\tilde{r}_{i,\alpha} \in \tilde{\rho}(Hom(1, (\tilde{i}, \alpha)(\tilde{i}, \tilde{\alpha})))$, we may assume that

$$r_{i,\alpha} = \tilde{\rho}(R_{i,\alpha}), \tilde{r}_{i,\alpha} = \tilde{\rho}(\tilde{R}_{i,\alpha}).$$

The next lemma shows that $R_{i,\alpha}, \tilde{R}_{i,\alpha}$ indeed satisfy the duality conditions (thus justifying the notations).

**Lemma 2.3.2.** (1):

$$a_{\tilde{\alpha}}(u_{i,\alpha}) r_{i,\alpha} = u_{i,\tilde{\alpha}}^* \epsilon_r(a_{\tilde{\alpha}}, \tilde{\sigma}_i) a_{\tilde{\alpha}}(r_{\tilde{\alpha}}) r_{a_{\tilde{\alpha}},}$$
\[
a_{i,\alpha}(u_{i,\alpha})\tilde{r}_{i,\alpha} = u_{i,\alpha}^*\epsilon_r(a_\alpha, \sigma_i)r_{i,\alpha}\tilde{r}_i.
\]

(2):
\[
\tilde{R}_{i,\alpha}^*(i, \alpha)(R_{i,\alpha}) = 1_{(i,\alpha)}, R_{i,\alpha}^*(\tilde{r}, \tilde{\alpha})(\tilde{R}_{i,\alpha}) = 1_{(\tilde{i}\tilde{\alpha})};
\]
\[
||R_{i,\alpha}|| = ||\tilde{R}_{i,\alpha}|| = \sqrt{d(i,\alpha)};
\]

(3)
\[
u_{i,\alpha}\tilde{\rho}(\theta_{(i,\alpha)}) = \rho^{-1}\phi_{I_c^0}(\theta_{iBE})\rho(\theta^{-1}_{\tilde{\alpha}})u_{i,\alpha},
\]
where \(\theta\)'s are the twistings as constructed in \(\S 1.7\).

**Proof.** (1) follows from definitions.

Ad (2): Since braidings among \(\sigma_i\) and \(a_\alpha\) are relative braiding and hence
\[
\epsilon_r(\tilde{\sigma}_i, a_\alpha)\epsilon_r(a_\alpha, \sigma_i) = 1,
\]
in another words, the braiding among \(\sigma_i\) and \(a_\alpha\) are trivial, by using the naturality of the relative braiding (cf. lemma 2.2.3) we have
\[
\tilde{r}_{i,\alpha}^*a_{(i,\alpha)}(r_{i,\alpha}) = u_{i,\alpha}^*\tilde{r}_{\sigma_i}^*\sigma_i(r_{\sigma_i})\sigma_i(\tilde{r}_{\alpha}^*a_{\alpha}(r_{\alpha}))u_{i,\alpha} = id_{a_{(i,\alpha)}},
\]
and this implies the first equation in (2) by definitions. The second equation is proved similarly. The last equation of (2) follows from the definitions and \(d((i,\alpha)) = d(i)d(\alpha)\).

(3) follows since both sides are equal to \(u_{i,\alpha}\exp(2\pi i(\Delta_i - \Delta_\alpha))\).

\(\square\)

Note by (2) of lemma 2.3.2 we can choose our pairs \((R_{i,\alpha}, \tilde{R}_{i,\alpha})\) as in lemma 2.3.2 in our construction of \(\mathcal{C}(G/H)\) in \(\S 1.7\). Note that a different choices of the pairs \((R_{i,\alpha}, \tilde{R}_{i,\alpha})\) will not change the value of link invariants by (4) of lemma 1.7.4.

Denote by \(L((i_1,\alpha_1), ..., (i_n,\alpha_n))\) the link invariant of an oriented framed link \(L\) with \(n\) components colored by \((i_1,\alpha_1), ..., (i_n,\alpha_n)\) \(\in \exp\). Since
\[
L((i_1,\alpha_1), ..., (i_n,\alpha_n)) = \tilde{\rho}(L((i_1,\alpha_1), ..., (i_n,\alpha_n)))
\]
and \(\tilde{\rho}\) is an endomorphism, we can think of
\[
L((i_1,\alpha_1), ..., (i_n,\alpha_n))
\]
as a link whose \(k\)-th component is colored by \(a_{i_k,\alpha_k}, k = 1, ..., n\), and replacing the operators corresponding to the tangles in \(L\) by their image under \(\tilde{\rho}\). We can do the same with links colored by \(\alpha\)'s. For the link \(L\) whose \(k\)-th component is colored by \(i_k\), using the isomorphism \(\rho^{-1}\phi_{I_c^0}\) we can think of \(L(i_1, ..., i_n)\) as a link whose \(k\)-th component is colored by \(\sigma_{i_k}, k = 1, ..., n\), and replacing the operators corresponding to the tangles in \(L\) by their image under \(\rho^{-1}\phi_{I_c^0}\).
Theorem B. Assume a coset $H \subset G_k$ verifies the conditions of Theorem A. Let $L$ be an oriented framed link in three sphere with $n$ components. If $d((i_k, \alpha_k)) = d(i_k)d(\alpha_k), (i_k, \alpha_k) \in \exp, k = 1, \ldots, n$, then

$$L((i_1, \alpha_1), \ldots, (i_n, \alpha_n)) = L(i_1, \ldots, i_n)\overline{L(\alpha_1, \ldots, \alpha_n)}.$$  

Proof. It is sufficient to show

$$\tilde{\rho}(L((i_1, \alpha_1), \ldots, (i_n, \alpha_n))) = \rho^{-1}\phi_{c_0}(L(i_1, \ldots, i_n))\tilde{\rho}(\overline{L(\alpha_1, \ldots, \alpha_n)}).$$

By the remark before the theorem, we can think of each component of $L_k$ as colored by $a_{(i_k, \alpha_k)}$. Insert coupons $u^*_{i_k,i}u_{i_k} = 1$ on $L_k$. By (3) of Prop. 2.3.1 and Lemma 2.3.2, we can pull the coupon $u_{i,k}$ away from $u^*_{i,k}$ around the component $L_k$, and when $u_{i,k}$ returns close to $u^*_{i,k}$ from the other direction, using $u_{i,k}u^*_{i,k} = 1$, we get a link with components colored by $\sigma_{i,k}a_{\bar{\alpha}_k}$. We can effectively think of $L_k$ consists of two parallel parts with one part colored by $\sigma_{i,k}$ and the other part colored by $a_{\bar{\alpha}_k}$. By (3) of Prop. 2.3.1, the braidings among $\sigma_{i,k}$ and $a_{\bar{\alpha}_k}$ are trivial, and by using the naturality of the relative braiding we can move all the strings colored by $\sigma_{i,k}$’s away from the link colored by $a_{\bar{\alpha}_k}$. The result is two links $L_1$, $L_2$, where $L_1$ colored by $\sigma_{i,k}$’s, and $L_1$ is identical to $L$, and $L_2$ is colored by $a_{\bar{\alpha}_k}$’s, but $L_2$ is identical to $L$ except all the overcrossing and the undercrossings of $L$ have been exchanged by (3) of Prop. 2.3.1. Note $L_1(\sigma_{i_1}, \ldots, \sigma_{i_n}) = \rho^{-1}\phi_{c_0}(L_1(i_1, \ldots, i_n)) = L(i_1, \ldots, i_n)$, since $L_1(i_1, \ldots, i_n) = L(i_1, \ldots, i_n) \in \mathbb{C}$. Similarly $L_2(a_{\bar{\alpha}_1}, \ldots, a_{\bar{\alpha}_n}) = L_2(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$, and by Cor. 2.8.1 of [Tu]

$$L_2(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$$

is the same as $L'_2(\alpha_1, \ldots, \alpha_n)$ where $L'_2$ is obtained from $L_2$ by reversing the direction of each component. Hence $L'_2$ is exactly the negation of $L$ defined on Page 110 of [Tu]. It follows by Lemma 5.1.3 of [Tu] that

$$L'_2(\alpha_1, \ldots, \alpha_n) = \overline{L(\alpha_1, \ldots, \alpha_n)}$$

since $C(H)$ is a unitary modular category by Th. A. So we have

$$L((i_1, \alpha_1), \ldots, (i_n, \alpha_n))) = L(i_1, \ldots, i_n)\overline{L(\alpha_1, \ldots, \alpha_n)}$$

and the proof is complete.

\[\square\]

3. Applications

§3.1 $G = SU(N)_k, H = e$. In this section we show that the framed link invariants from the category $C(G)$ with $G = SU(N)_k$ is the same as the link invariants in [W3]. This imply that $\tau_G(M)$ in this case is the same as the 3-manifold invariants
of [W3], [TW1] and [RT]. The proof is based on lemma 3.1.1 which slightly improves (3) of Th. 3.8 of [X4].

Let us recall the representations \( \pi_{\lambda}^{(N,N+K)} \) as defined on Page 368 of [W1]. This a \( C^* \) representation of Hecke algebra \( H(n,q) \) which is an algebra with generators \( 1, g_i, i = 1,...,n-1 \) and the following relations

\[
\begin{align*}
g_{i}g_{i+1} & = g_{i+1}g_{i}g_{i+1} \\
g_{i}g_{j} & = g_{j}g_{i}, \text{if } |j - i| \geq 2 \\
g_{i}^{2} & = (q - 1)g_{i} + q.
\end{align*}
\]

The minimal projections \( q_{\lambda} \) of \( \pi_{\lambda}^{(N,N+K)}(H_n(q)) \) are labeled by \((N,N+k)\) Young diagrams \([x_1,...,x_N]\). The correspondence between \( q_{\lambda} \) and irreducible representations of \( LSU(N)_k \) is given by (cf. P.267 of [W2])

\[
\lambda = [x_1,...,x_N] \rightarrow \mu = (x_1 - x_N + 1,...,x_{N-1} - x_N + 1) \tag{3.1.1}
\]

Denote by \( v \) the vector representation of \( LG \). Consider the sequences of algebras

\[
End(v^n) \subset End(v^{n+1}),
\]

where the \( \subset \) is the natural inclusion. The minimal projection corresponding to \( \lambda \prec f^n \) will be denoted by \( p_{\lambda} \).

Denote by \( A = \cup_n End(v^n) \). Define

\[
h_i = q^{\frac{N+k}{2N}}v^{i-1}(c_{v,v}), i = 1,2,\ldots,
\]

where \( q := \exp(\frac{2\pi i}{N+k}) \). Define a normalized trace \( Tr \) on \( A \) by

\[
Tr(f) = \frac{1}{d(v)^n}tr(f), \forall f \in End(v^n)
\]

where \( tr \) is defined as (1.7.1).

**Lemma 3.1.1.** (1) The trace as defined above is a Markov trace on \( A \), i.e.,

\[
Tr(fh_n) = Tr(f)z, \forall f \in End(v^n), \text{ where } z = \frac{q-1}{1-q^{-N}}.
\]

(2) \( h_i^2 = (q - 1)h_i + q, \forall i \);

(3) \( 1, h_1,...,h_{n-1} \) generates \( End(v^n) \);

(4) The map

\[
\psi_n : End(v^n) \rightarrow \pi_{\lambda}^{(N,N+K)}(H_n(q))
\]

such that \( \psi_n(h_i) = \pi_{\lambda}^{(N,N+K)}(g_i), i = 1,2,...,n-1 \) gives a trace-preserving * isomorphism between the algebras \( End(v^n) \) and \( \pi_{\lambda}^{(N,N+K)}(H_n(q)) \);
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(5) Let $p_{2\Lambda_i}$ be the minimal projection in $\text{End}(v^i)$ corresponding to $2\Lambda_i$, then

$$\psi_i(p_{2\Lambda_i})$$

is a minimal projection in $\pi^{(N,N+K)}(H_i(q))$ corresponding to $2\Lambda_i$ via (3.1.1).

**Proof.** Ad (1): Since $C(G)$ is a ribbon category, we have

$$tr(fh_n) = tr(f)q^{\frac{N+1}{2N}}\omega(v), \forall f \in \text{End}(v^n),$$

where $\omega(v) = \exp(\frac{2\pi i (N^2-1)}{2N})$. (1) now follows from $d(v) = \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}}$ and the definitions.

Ad (2): It is sufficient to show (2) for $i = 1$. Note by (1.5.13)

$$[v^2] = [3\Lambda_1] + [2\Lambda_2].$$

Let $T_s \in \text{Hom}(3\Lambda_1,v^2), T_a \in \text{Hom}(2\Lambda_2,v^2)$ be two isometries, uniquely determined up to a phase, and $p_s := T_s T_s^*, p_a := T_a T_a^*$ (s here stands for symmetric and a stands for antisymmetric). Suppose $h = q^{\frac{N+1}{2}}c_{v,v} = xp_a + yp_s$. It is enough to show that $x = -1, y = q$. By Prop. 1.4.3 we have

$$c_{v,v}^2 p_a = \omega(2\Lambda_2)\omega(v)^{-2} = q^{\frac{N-1}{2}}, c_{v,v}^2 p_s = \omega(3\Lambda_1)\omega(v)^{-2} = q^{\frac{N-1}{N}},$$

and so

$$x^2 = 1, y^2 = q^2.$$  

On the other hand by Spin-Statistical theorem (Th. 1.3.3)

$$tr(h) = d(v)\omega(v),$$

and since $tr(h) = xd(2\Lambda_2) + yd(3\Lambda_1)$, we get the following equation

$$xd(2\Lambda_2) + yd(3\Lambda_1) = d(v)\omega(v),$$

and a little algebra using

$$d(2\Lambda_2) = \frac{[N]_q[N-1]_q}{[2]_q}, d(3\Lambda_1) = \frac{[N]_q[N+1]_q}{[2]_q},$$

where $[m]_q := \frac{q^{m/2}-q^{-m/2}}{q^{1/2}-q^{-1/2}}$, gives

$$(x + 1)[N - 1]_q + (y - q)[N + 1]_q = 0.$$  

It is then easy to rule out the three cases $x = 1, y = \pm q; x = -1, y = -q$. and we must have $x = -1, y = q.$
Denote by $B_n$ the subalgebra of $\text{End}(v^n)$ generated by $1, h_1, \ldots, h_{n-1}$. From (1), (2) and (b) of Th. 3.6 of [W1] the map $\psi_n : B_n \to \pi^{(N,N+K)}(H_n(q))$ defined by $\psi_n(h_i) = \pi^{(N,N+K)}(g_i), i = 1, 2, \ldots, n - 1$ is a trace-preserving $*$ isomorphism. So to prove (4) it is enough to show (3), i.e., $B_n = \text{End}(v^n)$. By Th. 3.6 of [W1], the simple ideals of $B_n$ are given by $(N, K + N)$ diagrams with $n$ boxes, which are in one-to-one correspondence with the irreducible descendants of $v^n$. The inclusion matrix of $B_n \subset B_{n+1}$, given by (2.14) on Page 369 of [W1], is precisely the same as the inclusion matrix of $\text{End}(v^n) \subset \text{End}(v^{n+1})$ by (1.5.13). Since $B_1 = \text{End}(v) \equiv \mathbb{C}$, this proves $B_n = \text{End}(v^n)$ and completes the proof of (3) and (4).

Ad (5): (5) is obviously true for $i = 1$, so let us assume $i > 1$. First we claim that $h_m p_{2\Lambda_i} = -p_{2\Lambda_i}, k = m, \ldots, i - 1$. Note since $2\Lambda_i$ appears in $v^i$ once and only once, and by (1) of lemma 1.5.1, the only other possible subsector of $v^i$ with multiplicity 1 is $(i + 1)\Lambda_1$. By the fusion rule (1.5.13) we can choose co-isometries $T_a^* \in \text{Hom}(v^2, 2\Lambda_2), T^* : \text{Hom}(2\Lambda_2 v^{i-2}, 2\Lambda_i)$ such that

$$p_{2\Lambda_i} = T_a T^* T_a^*;$$

and it follows from (2) that $h_1 p_{2\Lambda_i} = -p_{2\Lambda_i}$. Hence $\psi_i(p_{2\Lambda_i})$ is minimal projection of $\pi^{(N,N+k)}(H_i(q))$ corresponding to the one dimensional simple ideal with the property

$$\pi^{(N,N+k)}(h_m) \psi_i(p_{2\Lambda_i}) = -\psi_i(p_{2\Lambda_i}), m = 1, 2, \ldots, i - 1.$$

Similarly by using (2) one can show that if $(i + 1)\Lambda_1$ appear in $v^i$ and $p_{(i+1)\Lambda_1}$ is the corresponding minimal projection, then

$$\pi^{(N,N+k)}(h_m) \psi_i(p_{(i+1)\Lambda_1}) = q\psi_i(p_{(i+1)\Lambda_1}), m = 1, 2, \ldots, i - 1.$$

Note that there are at most two one dimensional simple ideals in $\pi^{(N,N+k)}(H_i(q))$ by (1) of lemma 1.5.1, it is then easy to see that $\psi_i(p_{2\Lambda_i})$ corresponds to $q^i\Lambda$ where $\Lambda = [1^i]$ (it is one of the trivial $*$ representations defined on p. 370 of [W1]) . Note via (3.1.1) $[1^i]$ corresponds to $2\Lambda_i$, so (5) is proved.

\[\square\]

(5) of Lemma 3.1.1 may seem to be redundant. However note that the minimal projections $p_{\Lambda}$ and $q_{\Lambda}$ as described before the lemma are all related to the irreducible representations of $\text{LSU}(N)_k$, but it is not obvious that $\psi(p_{\Lambda})$ should correspond to $\Lambda$ via (3.1.1). One must keep in mind that our construction of invariants is different from that of [W2] even though it will be shown in Prop. 3.1.2 that they lead to the same invariants, and (5) of lemma 3.1.1 is used in the proof.

Now assume that $L$ is an oriented framed link in $S^3$ with $n$ components colored by $\lambda_1, \ldots, \lambda_n$, and $M_L$ a 3-manifold obtained by surgery on $L$. Denote by $L^Q(\lambda_1, \ldots, \lambda_n)$ (resp. $\tau^Q(M_L)$) the link invariant (resp. the 3-manifold invariant) defined by using the quantum group associated with $\text{SU}(N)$ with the deformation parameter $q = \exp(\frac{2\pi i}{N+k})$ (cf. [RT], [TW]), and let $L^c(\lambda_1, \ldots, \lambda_n)$ (resp. $\tau^c(M)$) be the link invariant (resp. the 3-manifold invariant) via cabling as (*) on p. 247 of [W2].
Proposition 3.1.2. (1) 

\[ L(\lambda_1, ..., \lambda_n) = L^c(\lambda_1, ..., \lambda_n) = L^Q(\lambda_1, ..., \lambda_n); \]

(2) 

\[ \tau(M) = \tau^c(M) = \tau^Q(M) \]

for any closed oriented 3-manifold \( M \). In fact it is also true if \( M \) is replaced by a pair \( (M, \Omega) \) where \( \Omega \) is a colored ribbon graph in \( M \) as defined on Page 82 of [Tu].

Proof. 

\[ L^c(\lambda_1, ..., \lambda_n) = L^Q(\lambda_1, ..., \lambda_n) \]

follows from Cor. 3.3.3 of [W2], so we just have to show 

\[ L(\lambda_1, ..., \lambda_n) = L^c(\lambda_1, ..., \lambda_n). \]

Since both sides transform in the same way under the change of orientation of a component of \( L \) (the invariant is unchanged if one change the orientation of a component and change the color of that component by its dual), we can assume that \( L \) is represented as the closure of a tangle \( T_L \) as in the proof of (3) of Lemma 1.7.5. Now let \( p_{\lambda_i} \) be the minimal projection in \( \text{End}(v^{m_i}) \) corresponding to \( \lambda_i \), and let \( \psi_{m_i}(p_{\lambda_i}) \) be a minimal projection in \( \pi^{(N,k+N)}(H_i(q)) \) where \( \psi_{m_i} \) is defined as in (4) of Lemma 3.1.1. Suppose that \( \psi_{m_i}(p_{\lambda_i}) \) corresponds to \( a(\lambda_i) \) which is a representation of \( \text{LSU}(N)_k \) via (3.1.1). By using (4) of Lemma 3.1.1, the proof of (3) of Lemma 1.7.5 now applies verbatim, and we have 

\[ L(\lambda_1, ..., \lambda_n) = L^c(a(\lambda_1), ..., a(\lambda_n)). \]

To finish the proof we just have to show that \( a = \text{id} \). Note by (5) of lemma 3.1.1 \( a(\Lambda_i) = \Lambda_i, i = 1, ..., N - 1 \). On the other hand since 

\[ L(\lambda_1, ..., \lambda_n) = L^c(a(\lambda_1), ..., a(\lambda_n)) = L^Q(a(\lambda_1), ..., a(\lambda_n)), \]

we must have 

\[ S_{\lambda,\mu} = S_{a(\lambda),a(\mu)}; \]

since the elements of \( S \) matrices are special cases of colored link invariants where the link is oriented downward, and by §3.5 of [TW] the \( S \) matrix computed from type \( A \) quantum group at \( q = \exp(\frac{2\pi i}{N+k}) \) agrees with the \( S \) matrix defined at the end of §1.5. So we have \( S_{\lambda,\mu} = S_{a(\lambda)a(\mu)} \), and \( a(\Lambda_i) = \Lambda_i, i = 1, ..., N - 1 \). By lemma 1.5.1 \( a = \text{id} \) and the proof of (1) is complete. (2) now follows immediately from definitions.

\( \square \)
§3.2 Symmetry principle. The symmetry principle in the case of \( N = 2 \) first appeared in [MK] and later generalized to \( N > 2 \) case in [KT]. The proof in [KT] uses the subtle property of \( R \) matrix at roots of unity. Here we give a proof of the symmetry principle of our invariants \( L(i_1, ..., i_n) \), and by Prop. 3.1.1 this also gives a slightly different proof of the results of [KT1]. This result will be used in §3.6.

Note that
\[
\Delta_{\sigma(\lambda)} - \Delta_\lambda - \frac{((N - 1)k - 2\tau(\lambda))}{2N} \in \mathbb{Z}, \tag{3.2.1}
\]
which follows from a simple computation using definitions. It follows from (3.2.1) that
\[
\Delta_{\sigma(\lambda)} - \Delta_\lambda - \Delta_{\sigma(1)} + \frac{\tau(\lambda)}{N} \in \mathbb{Z}, \tag{3.2.2}
\]
\[
\Delta_{\sigma^{-1}(\lambda)} - \Delta_\lambda - \Delta_{\sigma^{-1}(1)} - \frac{\tau(\lambda)}{N} \in \mathbb{Z}, \tag{3.2.3}
\]
Also note from the definitions that
\[
\tau(\lambda) + \tau(\bar{\lambda}) \equiv 0 \mod(N) \tag{3.2.4}
\]
For an oriented framed link \( L \) with \( n \) components \( L_i, i = 1, 2, ..., n \) the linking matrix is given by \( (L_i \cdot L_j) \) where \( L_i \cdot L_i \) is defined to be the framing.

Lemma 3.2.1. Let \( L \) be an oriented framed knot. Then
\[
L(\sigma(\lambda)) = \exp(2\pi i(\Delta_{\sigma(\lambda)} - \Delta_\lambda) L \cdot L) L(\lambda)
\]

Proof. First assume that \( L \) is presented as the closure of a counter-clockwise braid, i.e., \( L \) is the closure of a tangle \( T_L \) with all bands oriented downwards, and we will also assume that \( T_L \) consists of only twists and braidings. Choose the blackboard framing so that \( L \cdot L = b_+ + t_+ - b_- - t_- \), where \( b_+ \) (resp. \( b_- \)) are the number of positive (resp. negative) crossings in \( T_L \), and \( t_+ \) (resp. \( t_- \)) are the number of positive (resp. negative) twistings in \( T_L \).

Since \([\sigma(\lambda)] = [\sigma(1)\lambda]\), by (3) of Lemma 1.7.4, we can replace the color \( \sigma(\lambda) \) by \( \sigma(1)\lambda \), and by Cor. I.2.8.3 of [Tu], we can obtain the same invariant \( L(\sigma(\lambda)) \) by cutting the band \( L \) along its core and coloring two newly emerging annuli denoted by \( L_1, L_2 \), with color \( \lambda \) and \( \sigma(1) \) respectively. Now for each positive crossing or twisting of \( L \) we can change the positive crossing of \( \sigma(1) \) and \( \lambda \) to negative crossings of \( \sigma(1) \) and \( \lambda \) provided we multiply the resulting expression by
\[
\exp(2\pi i(\Delta_{\sigma(\lambda)} - \Delta_\lambda - \Delta_{\sigma(1)})),
\]
by Prop. 1.4.3. Similarly for each negative crossing or twisting of \( L \) we can change the negative crossing of \( \lambda \) and \( \sigma(1) \) to positive crossings of \( \sigma(1) \) and \( \lambda \) provided we multiply the resulting expression by
\[
\exp(-2\pi i(\Delta_{\sigma(\lambda)} - \Delta_\lambda - \Delta_{\sigma(1)})),
\]
by Prop. 1.4.3. Afterwards it is easy to see that we can then pull $L_1$ and $L_2$ apart, and $L(\sigma(\lambda))$ is equal to
\[
\exp(2\pi i (\Delta_{\sigma(\lambda)} - \Delta_{\lambda} - \Delta_{\sigma(1)})L \cdot L) L(\sigma(1)) L(\lambda).
\]
Note that for each positive crossing of $L$ colored by $\sigma(0)$, the corresponding braiding operator is a scalar multiple of identity since $\text{End}(\sigma(1))^2 \equiv \mathbb{C}$, and this scalar is equal to $\omega(\sigma(1)) = \exp(2\pi i \Delta(\sigma(1)))$ by Th. 1.3.3 since $d(\sigma(1)) = 1$. Similarly for each negative crossing of $L$ colored by $\sigma(1)$, the corresponding braiding operator is a scalar multiple of identity and this scalar is equal to $\omega(\sigma(1))^{-1} = \exp(-2\pi i \Delta_{\sigma(1)})$.

So
\[
L(\sigma(1)) = \exp(2\pi i \Delta_{\sigma(1)} L \cdot L).
\]
It follows that
\[
L(\sigma(\lambda)) = \exp(2\pi i (\Delta_{\sigma(\lambda)} - \Delta_{\lambda}) L \cdot L) L(\lambda)
\]
when $L$ is oriented as the closure of counter-clockwise braids. Now suppose $L$ is oriented as the closure of clockwise braids. Denote by $L^\text{op}$ the knot obtained from $L$ by changing its direction. From the above proof we have
\[
L^\text{op}(\bar{\lambda}) = \exp(2\pi i (\Delta_{\bar{\lambda}} - \Delta_{\sigma^{-1}(\bar{\lambda})}) L^\text{op} \cdot L^\text{op}) L^\text{op}(\sigma^{-1}(\bar{\lambda})),
\]
and it follows that
\[
L(\sigma(\lambda)) = \exp(2\pi i (\Delta_{\sigma(\lambda)} - \Delta_{\lambda}) L \cdot L) L(\lambda)
\]
by using $\sigma(\bar{\lambda}) = \sigma^{-1}(\bar{\lambda})$, $\Delta_{\lambda} = \Delta_{\bar{\lambda}}$ and $L^\text{op} \cdot L^\text{op} = L \cdot L$.

\textbf{Proposition 3.2.2 (Symmetry principle).}

Suppose $L$ has $k$ components $L_1, \ldots, L_k$. Then
\[
L(\lambda_1, \ldots, \sigma(\lambda_i), \ldots, \lambda_k) = L(\lambda_1, \ldots, \lambda_k) \times \exp(2\pi i ((\Delta_{\sigma(\lambda_i)} - \Delta_{\lambda_i})L_i \cdot L_i + \sum_{j \neq i}(\Delta_{\sigma(\lambda_j)} - \Delta_{\lambda_j} - \Delta_{\sigma(1)})L_j \cdot L_i))
\]

\textbf{Proof.} Without loss of generality we assume that $i = 1$. As in the proof of Lemma 3.2.1 we can obtain the same invariant by cutting the band $L_1$ along its core and coloring two newly emerging annuli denoted by $N_2, N_1$, with color $\lambda$ and $\sigma(1)$ respectively. By using Lemma 3.2.1, $L(\sigma(\lambda), \lambda, \ldots, \lambda_n)$, up to multiplication by $\exp(2\pi i (\Delta_{\sigma(\lambda_1)} - \Delta_{\lambda_1})L_1 \cdot L_1)$ is the same as the invariant of a link $L'$ where $N_1$ colored by $\sigma(1)$ is unknotted and split from $N_2$ colored by $\lambda_1$, and the rest of $L'$ is the same as $L$. Now suppose that $N_1$ and $L_j, j = 2, \ldots, k$ are all oriented anti-clockwise. The crossings of $N_1$ and $L_j$ occur in right or left handed pairs which algebraically sum to $L_1 \cdot L_j$. For each righthanded pair of crossings of $N_1$ and $L_j$,
the associated operator is a scalar multiple of identity since $\text{End}(\sigma(1)\lambda_j) \equiv \mathbb{C}$, and the scalar is
\[
\exp(2\pi i(\Delta_{\sigma(\lambda_j)} - \Delta_{\lambda_j} - \Delta_{\sigma(1)}))
\]
by Prop. 1.4.3, and similarly for each lefthanded pair of crossings of $N_1$ and $L_j$, the associated operator is a scalar multiple of identity and the scalar is
\[
\exp(-2\pi i(\Delta_{\sigma(\lambda_j)} - \Delta_{\lambda_j} - \Delta_{\sigma(1)})),
\]
so all told we have shown the proposition in the case that all $L_i$ are oriented anticlockwise. To finish the proof we just have to show the phase factor of the equation in the proposition remain the same under changing the orientations of components $L_j$ and in the same time changing the color $\lambda_j$ to its dual $\bar{\lambda}_j$ when $j > 1$, and changing the orientation of $L_1$ and in the same time changing the color $\lambda_1$ to $\bar{\lambda}_1$, $\sigma$ to $\sigma^{-1}$. For $j > 1$ this is equivalent to
\[
(\Delta_{\sigma(\lambda_j)} - \Delta_{\lambda_j} - \Delta_{\sigma(1)}) + (\Delta_{\sigma(\bar{\lambda}_j)} - \Delta_{\bar{\lambda}_j} - \Delta_{\sigma(1)}) \in \mathbb{Z},
\]
which follows from immediately from (3.2.2), (3.2.3) and (3.2.4). For $j = 1$ this is equivalent to
\[
(\Delta_{\sigma^{-1}(\lambda_j)} - \Delta_{\lambda_j} - \Delta_{\sigma^{-1}(1)}) + (\Delta_{\sigma(\lambda_j)} - \Delta_{\lambda_j} - \Delta_{\sigma(1)}) \in \mathbb{Z},
\]
which also follows from immediately from (3.2.2), (3.2.3), (3.2.4) and
\[
\sigma(\lambda) = \sigma^{-1}(\bar{\lambda}), \Delta_{\lambda} = \Delta_{\bar{\lambda}}, L_1^{op} \cdot L_1^{op} = L \cdot L.
\]

\[\square\]

§ 3.3 Level-rank duality. An almost immediate consequence of Th. B is a result of [KT] relating two three manifold invariants. Our proof below is different from [KT]. Let us first prepare some notations. The conformal inclusion $SU(n)_m \times SU(m)_n \subset SU(nm)_1$ has been used in [X1] (also. cf. [X7] and [X8]) and the branchings rules are given in [ABI]. Let $\pi^0$ be the vacuum representation of $LSU(nm)$ on Hilbert space $H^0$. The decomposition of $\pi^0$ under $L(SU(m) \times SU(n))$ is known, see, e.g. [ABI]. To describe such a decomposition, let us prepare some notation. We shall use $\check{S}$ to denote the $S$-matrices of $SU(m)$, (see §2.2), and $\check{S}$ to denote the $S$-matrices of $SU(n)$. The level $n$ (resp. $m$) weight of $LSU(m)$ (resp. $LSU(n)$) will be denoted by $\hat{\lambda}$ (resp. $\check{\lambda}$).

We start by describing $\check{P}_n$ (resp. $\check{P}_n^m$), i.e. the highest weights of level $n$ of $LSU(m)$ (resp. level $m$ of $LSU(n)$). $\check{P}_n^m$ is the set of weights
\[
\hat{\lambda} = \check{k}_0 \check{\lambda}_0 + \check{k}_1 \check{\lambda}_1 + \cdots + \check{k}_{m-1} \check{\lambda}_{m-1}
\]
where $\tilde{k}_i$ are non-negative integers such that

$$\sum_{i=0}^{m-1} \tilde{k}_i = n$$

and $\tilde{\lambda}_i = \tilde{\lambda}_0 + \tilde{\omega}_i$, $1 \leq i \leq m - 1$, where $\tilde{\omega}_i$ are the fundamental weights of $SU(m)$.

Instead of $\tilde{\lambda}$ it will be more convenient to use

$$\dot{\lambda} + \dot{\rho} = \sum_{i=0}^{m-1} k_i \dot{\lambda}_i$$

with $k_i = \tilde{k}_i + 1$ and $\sum_{i=0}^{m-1} k_i = m + n$ as in the notation of §1.5. Due to the cyclic symmetry of the extended Dykin diagram of $SU(m)$, the group $\mathbb{Z}_m$ acts on $\dot{P}_n^m$ by

$$\dot{\lambda}_i \to \dot{\lambda}_{(i+\sigma)} \mod m, \quad \sigma \in \mathbb{Z}_m.$$ 

Let $\Omega_{m,n} = \dot{P}_n^m / \mathbb{Z}_m$. Then there is a natural bijection between $\Omega_{m,n}$ and $\Omega_{n,m}$ (see §2 of [ABI]).

We shall parameterize the bijection by a map

$$\beta : \dot{P}_n^m \to \dot{P}_m^m$$

as follows. Set

$$r_j = \sum_{i=j}^{m} k_i, \quad 1 \leq j \leq m$$

where $k_m \equiv k_0$. The sequence $(r_1, \ldots, r_m)$ is decreasing, $m + n = r_1 > r_2 > \cdots > r_m \geq 1$. Take the complementary sequence $(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n)$ in $\{1, 2, \ldots, m + n\}$ with $\bar{r}_1 > \bar{r}_2 > \cdots > \bar{r}_n$. Put

$$S_j = m + n + \bar{r}_n - \bar{r}_{n-j+1}, \quad 1 \leq j \leq n.$$ 

Then $m + n = s_1 > s_2 > \cdots > s_n \geq 1$. The map $\beta$ is defined by

$$(r_1, \ldots, r_m) \to (s_1, \ldots, s_n).$$

The following lemma summarizes what we will . For the proof, see Lemma 3, 4 of [ABI].
Lemma 3.3.1. (1) Let $\hat{Q}$ be the root lattice of $SU(m)$, $\hat{\Lambda}$, $0 \leq i \leq m - 1$ its fundamental weights and $\hat{Q}_0 = (\hat{Q} + \hat{\Lambda}_0) \cap \hat{P}_+^n$. Then for each $\hat{\lambda} \in \hat{Q}_0$, there exists a unique $\check{\lambda} \in \check{P}_m$ with $\check{\lambda} = \sigma \beta(\hat{\lambda})$ for some $\sigma \in \mathbb{Z}_n$ such that $H_{\check{\lambda}} \otimes H_{\check{\lambda}}$ appears once and only once in $H^0$. Moreover, $H^0$, as representations of $L(SU(m) \times SU(n))$, is a direct sum of all such $H_{\check{\lambda}} \otimes H_{\check{\lambda}}$.

(2) $\sum_{\hat{\lambda} \in \hat{Q}_0} \hat{S}_{\hat{\lambda}1} = \frac{1}{m}$.

(3) $\check{S}_{\check{\lambda}1} = (\frac{n}{m})^{\frac{1}{2}} \check{S}_{\sigma \beta(\hat{\lambda})1}$.

We shall denote the bijection as in (1) of lemma 3.3.1 by $\hat{\lambda} \in \hat{Q}_0 \rightarrow \check{\lambda} = F(\hat{\lambda}) \in \check{Q}_0$.

Consider the coset $H = SU(m)_n \subset G = SU(nm)_1$. By using the above decompositions, it is proved (cf. Lemma 3.2 of [X1]) that the coset conformal precosheaf is the conformal precosheaf of $H_1 = SU(n)_m$, and it follows by definition that $F(\hat{\lambda})$ is isomorphic with $(1, \check{\lambda})$ as representations of the conformal precosheaf of $H_1 = SU(n)_m$, where $1$ denotes the identity sector of $G$, $\check{\lambda} \in \check{Q}_0$. By Th. B we have

$$L(F(\hat{\lambda}_1), ..., F(\hat{\lambda}_n)) = L(\check{\lambda}_1, ..., \check{\lambda}_n)L(1, ..., 1).$$

But $L(1, ..., 1) = 1$ so we have

$$L(F(\hat{\lambda}_1), ..., F(\hat{\lambda}_n)) = L(\check{\lambda}_1, ..., \check{\lambda}_n).$$

When $m$ and $n$ are relatively prime, denote by

$$\tau_{SU(n)_m/\mathbb{Z}_n}(M)$$

and

$$\tau_{SU(m)_n/\mathbb{Z}_m}(M)$$

the three manifold invariants associated with $SU(n)_m/\mathbb{Z}_n$ and $SU(m)_n/\mathbb{Z}_m$ respectively as defined on Page 226 of [MW] (our notation $H_m/\mathbb{Z}_n$ corresponds to their $PSU(n)$ at level $m$) or as on Page 260 of [KT2].

Since $C_H + C_{H_1} = mn - 1$,

$$\sum_{\check{\lambda} \in \check{Q}_0} d(\check{\lambda})^2 = \sum_{\hat{\lambda} \in \hat{Q}_0} \frac{\check{S}_{\check{\lambda}1}^2}{\check{S}_{11}^2} = \frac{1}{m} \frac{1}{\check{S}_{11}^2} = \frac{1}{n} \check{S}_{11}^2 = \sum_{\hat{\lambda} \in \hat{Q}_0} d(\hat{\lambda})^2$$

by lemma 3.3.1, it follows by above definitions that we have proved the following proposition:
Proposition 3.3.2. If $m$ and $n$ are relatively prime, then

$$\tau_{SU(n)/Z_n}(M) = \tau_{SU(m)/Z_m}(M)$$

for any closed oriented 3-manifold.

Note by (2) of Prop. 3.1.2 the above proposition gives a different proof of the main result Th. 4.2.7 of [KT2] which is based on symmetries of certain Boltzmann weights.

§ 3.4 Simple current extensions of $\hat{su}(N)$.

Recall that the finite set of irreducible representations of LSU($N$) at level $k$ is given by

$$P^h_{++} = \left\{ \lambda \in P \mid \lambda = \sum_{i=1}^{N-1} \lambda_i \Lambda_i, \lambda_i \geq 1, \sum_{i=1}^{N-1} \lambda_i < h \right\}$$

where $P$ is the weight lattice of $SU(N)$ and $\Lambda_i$ are the fundamental weights and $h = N + k$. Recall that this set admits a $\mathbb{Z}_N$ automorphism generated by

$$\sigma_1 : \lambda = (\lambda_1, \lambda_2, ..., \lambda_{N-1}) \rightarrow \sigma_1(\lambda) = (h - \sum_{j=1}^{N-1} \lambda_j, \lambda_1, ..., \lambda_{N-2}).$$

Note $\tau(\lambda) \equiv \sum_i (\lambda_i - 1) \mod(N)$ and $Q$ is the root lattice of $\hat{SL}(N)$ (cf. §1.3 of [KW]). Also note that $\lambda \in Q$ iff $\frac{1}{N} \tau(\lambda) \in \mathbb{Z}$.

Suppose $N = mq$, with $m, q$ positive integers. Assume level $k$ satisfies:

$$kq \in 2m\mathbb{Z} \text{ if } N \in 2\mathbb{Z}; \quad kq \in m\mathbb{Z} \text{ if } N \notin 2\mathbb{Z} + 1,$$

then there is an extension by the simple current $\sigma^q(1)$, realized as standard net of subfactors (cf. §2.1) as in Prop. 6.4 of [BE3]. The relation between condition (3.4.1) and orbifold subfactors in the string algebra framework was noticed in [X5].

Let $G_1 := SU(N)/\mathbb{Z}_m$. We shall denote the above nets of extensions as described in Prop. 6.4 of [BE3] simply by $\hat{G} \subset \hat{G}_1$. We use $\hat{G}_1$ since it is likely that the net as described in Prop. 6.4 of [BE3] are related to the representations of loop group $LG_1$ which is not connected. The representation theory of such loop groups should be very close to that of [PS] but has not been developed yet. Note that the index of the inclusion $\hat{G} \subset \hat{G}_1$ is $m$ (cf. §6 of [BE3]). Now we can apply the formulas in §2.1 to this inclusion. Note that this is not a coset construction, but we shall see that Cor. 1.7.3 applies to the theory determined by $\hat{G}_1$, and we can use the inclusion $\hat{G} \subset \hat{G}_1$ to calculate the 3-manifold invariants of $G_1$ in terms of $G$. 
Lemma 3.4.1. (1) The conformal precosheaf $\hat{G}_1$ is strongly additive and $\mu$-rational. Its $\mu$-index, denoted by $\mu_{G_1}$, is given by

$$\mu_{G_1} = \frac{\mu_G}{m^2}.$$ 

(2) Let $X$ be the finite set of all irreducible sectors of $\hat{G}_1$, and let $D_- := \sum_x d_x^2\omega_x^{-1}$. Then

$$D_- = \sqrt{\mu_{G_1}} \exp\left(-\frac{\pi i C_G}{4}\right),$$

where $C_G$ is the central charge of $\text{LSU}(N)$ at level $k$.

Proof. Ad (1): Since $\hat{G} \subset \hat{G}_1$ has index $m$, $\hat{G}_1$ is strongly additive by lemma 2.1.4. The vacuum representation of $\hat{G}_1$, denoted by $\pi_0$, decomposes into a finite number of irreducible representations of $\hat{G}$, and the same argument showing that $\hat{G}$ is split, using the asymptotics of the generator of the rotation proved in Th. B of [KW] (cf. [BAF] or P. 21 of [X7]) now applies verbatim, and shows that $\hat{G}_1$ is split. Note

$$\pi^0(\hat{G}(E))^\prime \subset \pi^0(\hat{G}_1(E))^\prime \subset \pi^0(\hat{G}_1(E^c))^\prime \subset \pi^0(\hat{G}(E^c))^\prime,$$

where $E \subset S^1$ is union of two open intervals $I_1, I_2 \subset S^1$ such that the intersection of their closure in $S^1$ is empty. Since $\pi^0(\hat{G}(E))^\prime \subset \pi^0(\hat{G}(E^c))^\prime$ has finite index (cf. Th. 3.5 of [X7]), it follows that

$$\pi^0(\hat{G}_1(E))^\prime \subset \pi^0(\hat{G}_1(E^c))^\prime$$

also has finite index, and so $\hat{G}_1$ has finite $\mu$ index. Now applying Prop. 21 of [KLM] gives the formula of the $\mu$-index.

Ad(2): By (1) and Th. 30 of [KLM] the set $X$ of all irreducible sectors of $\hat{G}_1$ is finite. Apply the formalism of §1.5 to this set $X$. By Cor. 32 of [KLM], the $Y$-matrix as defined (1.5.0) is non-degenerate, and

$$D_- = \sqrt{\mu_{G_1}} \check{C}^3$$

for some $\check{C} \in \mathbb{C}$ with $|\check{C}| = 1$. By the remark after the proof of Prop. 3.1 of [X3] (note in this case $\check{C}^3$ there is 1 since the coset $\hat{G} \subset \hat{G}_1$ is trivial and $C^3$ there is our $k_{G}^{-3}$), we have

$$k_{G}^{-3} = \check{C}^3,$$

and this finishes the proof of (2) by (1.7.5).
Lemma 3.4.2. (1) \[
\langle a_\lambda, a_\mu \rangle = \sum_{0 \leq j \leq m-1} \delta(\lambda, \sigma^j(\mu));
\]
(2) \[
\langle a_\lambda, \tilde{a}_\mu \rangle = \delta^m(\tau(\lambda)) \sum_{0 \leq j \leq m-1} \delta(\lambda, \sigma^j(\mu))
\]
where \(\delta^m(\tau(\lambda))\) is defined to be 1 if \(\tau(\lambda) \equiv 0 \mod (m)\) and 0 otherwise.

Proof. Both equations follow from similar equations in Th. 6.9 of [BE3], using the dictionary in §2.1.

Denote the set \(\lambda, \tau(\lambda) \equiv 0 \mod (m)\) by \(P_m\). Since \(\tau(\sigma^q) = qk \equiv 0 \mod (m)\) by (3.4.1), \(P_m\) admits a \(\mathbb{Z}_m\) action generated by \(\sigma^q\), and decomposed into disjoint union of orbits, denoted by \(O_1, \ldots, O_p\). Consider an orbit \(O_s\). Suppose \(O_s = \{\lambda_s, \sigma^q(\lambda_s), \ldots \sigma^f(s)(\lambda_s)\}\). Denote irreducible sectors of \(a_\lambda, \lambda \in O_s\) by \(\sigma_t(O_s)\), \(1 \leq t \leq g(O_s)\). By (2) of the above lemma
\[
\langle \sigma_t(O_s), \sigma_{t'}(O_{s'}) \rangle = 0
\]
if \(s \neq s'\). Note that these \(\sigma_t(O_s)\) are in one-to-one correspondence with some of the irreducible sectors of the extended net \(\hat{G}_1\), and with a slightly abuse of notations we will use \(\sigma_t(O_s)\) to denote the irreducible sector of \(\hat{G}_1\). We will see in the following that they are in fact all of the irreducible sectors of the extended net \(\hat{G}_1\). We will (it is always possible ) choose an involution of our labels \(s \rightarrow \bar{s}\) such that
\[
[a_\lambda s] = [a_{\lambda \bar{s}}],
\]
and an involution of our labels \(t \rightarrow \bar{t}\) such that
\[
[\sigma_t(O_s)] = [\sigma_t(O_{\bar{s}})].
\]

Now consider the covariant irreducible representation \(\pi^{\sigma_t(O_s)}\) of \(\hat{G}_1\). Consider the following analogue of Jones’ s basic construction:
\[
\pi^{\sigma_t(O_s)}(\hat{G}(I))'' \subset \pi^{\sigma_t(O_s)}(\hat{G}_1(I))'' \subset \pi^{\sigma_t(O_s)}(\hat{G}_1(I^e))'' \subset \pi^{\sigma_t(O_s)}(\hat{G}(I^e))''.
\]
Note that the minimal index of
\[
\pi^{\sigma_t(O_s)}(\hat{G}_1(I))'' \subset \pi^{\sigma_t(O_s)}(\hat{G}_1(I^e))''
\]
is \(d(\sigma_t(O_s))^2\), and the minimal index of
\[
\pi^{\sigma_t(O_s)}(\hat{G}(I))'' \subset \pi^{\sigma_t(O_s)}(\hat{G}_1(I))''
\]
\[
\pi_{\sigma_t(O_s)}(\hat{G}_1(I))'' \subset \pi_{\sigma_t(O_s)}(\hat{G}_1(I^c))''
\]
are the same and equal to \(m\). When restricting to \(\hat{G}\), \(\pi_{\sigma_t(O_s)}\) decomposes into a finite number of irreducible representations, and by our choices, each of such irreducible representation belong to the orbit \(O_s\), and the multiplicity of any such \(\sigma_{jq}(\lambda_s)\) is given by
\[
\langle \sigma_t(O_s), a_{\sigma_{jq}(\lambda_s)} \rangle = \langle \sigma_t(O_s), a_{\lambda_s} \rangle
\]
by (2) of lemma 3.4.2. By the additivity of statistical dimensions (cf. [L3]) the minimal index of the inclusion
\[
\pi_{\sigma_t(O_s)}(\hat{G}(I))'' \subset \pi_{\sigma_t(O_s)}(\hat{G}(I^c))''
\]
is
\[
f(s)^2 d(\lambda_s)^2 (\langle \sigma_t(O_s), a_{\lambda_s} \rangle)^2.
\]
On the other hand by using the multiplicative properties of statistical dimensions (cf. [L3]) the minimal index of
\[
\pi_{\sigma_t(O_s)}(\hat{G}(I))'' \subset \pi_{\sigma_t(O_s)}(\hat{G}(I^c))''
\]
is \(m^2 d(\sigma_t(O_s))^2\), so we have:
\[
d(\sigma_t(O_s)) = \frac{f(s)}{m} d(\lambda_s) \langle \sigma_t(O_s), a_{\lambda_s} \rangle
\]  
(3.4.2)

**Lemma 3.4.3.** The irreducible sectors \(\sigma_t(O_s), 1 \leq t \leq g(O_s)\) above are all the irreducible sectors of \(\hat{G}_1\).

**Proof.** It is sufficient to show that
\[
\sum_{O_s, 1 \leq t \leq g(O_s)} d(\sigma_t(O_s))^2 = \mu_{G_1}
\]
by Th. 30 of [KLM]. By (3.4.2) we have
\[
\sum_{O_s, 1 \leq t \leq g(O_s)} d(\sigma_t(O_s))^2 = \sum_{O_s, 1 \leq t \leq g(O_s)} \frac{f(s)^2}{m^2} d(\lambda_s)^2 \langle \sigma_t(O_s), a_{\lambda_s} \rangle^2
\]
\[
= \sum_{O_s} \frac{f(s)^2}{m^2} d(\lambda_s)^2 \sum_{1 \leq t \leq g(O_s)} \langle \sigma_t(O_s), a_{\lambda_s} \rangle^2
\]
\[
= \sum_{O_s} \frac{f(s)^2}{m^2} d(\lambda_s)^2 \langle a_{\lambda_s}, a_{\lambda_s} \rangle
\]
\[
= \sum_{O_s} \frac{f(s)}{m} d(\lambda_s)^2
\]
\[
= \frac{1}{m} \sum_{\lambda \in P_m} d(\lambda)^2
\]
where we used (cf. (2) of lemma 3.4.2)

\[ \sum_{1 \leq t \leq g(O_s)} \langle \sigma_t(O_s), a_{\lambda_s} \rangle = \langle a_{\lambda_s}, a_{\lambda_s} \rangle = \frac{m}{f(s)}. \]

Let us show

\[ \sum_{\lambda \in P_m} d(\lambda)^2 = \frac{1}{m} \sum_{\lambda \in P_{++)}} d(\lambda)^2. \]

By definitions

\[
\sum_{\lambda \in P_m} d(\lambda)^2 = \sum_{\lambda : \tau(\lambda) \equiv \emptyset \text{mod}(m)} d(\lambda)^2 \\
= \sum_{1 \leq i \leq q} \sum_{\lambda : \tau(\lambda) \equiv mi \text{mod}(N)} d(\lambda)^2 \\
= \sum_{1 \leq i \leq q} \sum_{\lambda : \lambda \equiv \lambda_i \text{mod}(Q)} d(\lambda)^2 \\
= \sum_{1 \leq i \leq q} \frac{1}{N^i} \mu_G \\
= \frac{1}{m^i} \mu_G
\]

where for each \(1 \leq i \leq q\) we choose a \(\lambda_i\) such that \(\tau(\lambda_i) \equiv mi \text{mod}(N)\), and in the sixth = we have used Cor. 2.7 of [KW]. It follows that

\[ \sum_{O_s, 1 \leq t \leq g(O_s)} d(\sigma_t(O_s))^2 = \frac{1}{m^2} \mu_G, \]

and by (1) of lemma 3.4.1 the proof is complete.

\[ \square \]

Note that combine the above lemma and Cor. 32 of [KLM] gives a different proof of Th. 6.12 of [BE3] (also cf. [EK3]).

Now let us construct a unitary modular category \(C(G_1)\) as in §1.7. We choose the set of objects \(\rho_i, i \in I\) to be \(\sigma_t(O_s), 1 \leq t \leq g(O_s)\) and \(\sigma(O_s)\) such that

\[ [\sigma(O_s)] = \sum_{1 \leq t \leq g(O_s)} [\langle \sigma_t(O_s), a_{\lambda_s} \rangle [\sigma_t(O_s)]]. \]

Note that since \(\sigma_t(O_s)\) has univalence \(\omega(\lambda_s)\) independent of \(t\), so \(\sigma(O_s)\) is of uniform valence with univalence \(\omega(\lambda_s)\). By (1) of Cor. 1.7.3 \(C(G_1)\) is an abelian unitary ribbon category. Now using lemma 3.4.1, lemma 3.4.3 and Cor. 32 of [KLM] we conclude that \(C(G_1)\) is a unitary modular category.
Note that $[\sigma(O_s)] = [a_{\lambda_s}]$ by our definition. Choose unitaries
$$u(O_s) \in Hom(a_{\lambda_s}, \sigma(O_s)).$$
Define
$$r(O_s) := \sigma(O_s)(u^*(O_s))u^*(O_s)\bar{\rho}(R_{\lambda_s}), \bar{r}(O_s) := \sigma(O_s)(u^*(O_s))u^*(O_s)\bar{\rho}(\bar{R}_{\lambda_s}),$$
then we have the following analogue of (3) of Prop. 2.3.1 and Lemma 2.3.2:

**Lemma 3.4.4.** (1)
$$u^*(O_t)\sigma(O_t)(u^*(O_s))\sigma(O_s)\sigma(O_t) = \bar{\rho}\sigma_{\lambda_s, \lambda_t}u^*(O_t)\sigma(O_t)(u^*(O_s))$$
where $\sigma_{xy}$ is the braiding operator between $x$ and $y$ as defined in §1.4;
(2)
$$\bar{r}(O_s)^*\sigma(O_s)(r(O_s)) = id_{\sigma(O_s)}, r(O_s)^*\sigma(O_s)(\bar{r}(O_s)) = id_{\sigma(O_s)};$$
(3)
$$u^*(O_s)\theta_{\sigma(O_s)} = \bar{\rho}(\theta_{\lambda_s})u^*(O_s);$$
(4)
$$L(O_{s_1}, ..., O_{s_n}) = L(\lambda_{s_1}, ..., \lambda_{s_n}).$$

**Proof.** (1) follows from lemma 2.4. (2) follows from definitions. (3) follows since $\omega(\sigma(O_s)) = \omega(\lambda_s)$.

To prove (4), insert coupons
$$u(O_{s_k})^*u(O_{s_k}) = 1,$$
on the $k$-th component of $L, k = 1, ..., n$ as in the beginning of the proof of Th. B. By (1), (2) and (3) we can pull the coupon $u_{i,k}$ away from $u_{i,k}^*$ around the component $L_k$, and when $u_{i,k}$ returns close to $u_{i,k}^*$ from the other direction, using $u_{i,k}u_{i,k}^* = 1$, we get a link identical to $L$, but with the $k$-th components colored by $a_{\lambda_{s_k}}, k = 1, ..., n$. Since
$$L(a_{\lambda_{s_1}}, ..., a_{\lambda_{s_n}}) = \bar{\rho}(L(\lambda_{s_1}, ..., \lambda_{s_n})) = L(\lambda_{s_1}, ..., \lambda_{s_n}),$$
the proof of (4) is complete.

Note
$$\sum_{1 \leq t \leq g(O_s)} d(\sigma_t(O_s))L(\sigma_t(O_s), ...) = \frac{1}{m} \sum_{1 \leq t \leq g(O_s)} f(s)(\sigma_t(O_s), a_{\lambda_s})d(\lambda_s)L(\sigma_t(O_s), ...)
= \frac{1}{m} f(s)d(\lambda_s)L(\sigma(O_s), ...),$$  (3.4.3)
where in the second $=$ we have used (2) of Lemma 1.7.4 and the colors indicated by ... are kept fixed in the summation. So we have

$$
\sum_{1 \leq t_j \leq g(O_{s_j})} d(\sigma_{t_1}(O_{s_1})) \cdots d(\sigma_{t_n}(O_{s_n}))L(\sigma_{t_1}(O_{s_1}), \sigma_{t_2}(O_{s_2}), \ldots, \sigma_{t_n}(O_{s_n}))
$$

$$= \frac{1}{m^n} f(s_1) f(s_2) \cdots f(s_n) d(\lambda_{s_1}) \cdots d(\lambda_{s_n})L(\sigma(O_{s_1}), \ldots, \sigma(O_{s_n}))
$$

$$= \frac{1}{m^n} f(s_1) f(s_2) \cdots f(s_n) d(\lambda_{s_1}) \cdots d(\lambda_{s_n})L(\lambda_{s_1}, \ldots, \lambda_{s_n})
$$

$$= \frac{1}{m^n} \sum_{\lambda_j \in O_{s_j}, j = 1, \ldots, n} d(\lambda_1) \cdots d(\lambda_n)L(\lambda_1, \ldots, \lambda_n) \quad (3.4.4)
$$

where in the second and the third $=$ we have used (4) of Lemma 3.4.4. Apply the above to the closed 3-manifold invariants associated with $C(G_1)$ we have

$$\tau_{G_1}(ML) = k_{G_1}^{3(b_-(L) - b_+(L))} \frac{1}{D_{G_1}^{b_{t_1}}(O_{s_1}), k = 1, \ldots, n} d(\sigma_{t_1}(O_{s_1})) \cdots d(\sigma_{t_n}(O_{s_n})) \times L(\sigma_{t_1}(O_{s_1}), \ldots, \sigma_{t_n}(O_{s_n}))
$$

$$= k_{G_1}^{3(b_-(L) - b_+(L))} \frac{1}{D_{G_1}^{b_{t_1}}} \frac{1}{m^n} \sum_{s_j, \lambda_j \in O_{s_j}, j = 1, \ldots, n} d(\lambda_1) \cdots d(\lambda_n)L(\lambda_1, \ldots, \lambda_n)
$$

$$= k_{G}^{3(b_-(L) - b_+(L))} \frac{1}{D_{G}^{b_p}} \sum_{\lambda_j \in P_m, \ldots, \lambda_n \in P_m} d(\lambda_1) \cdots d(\lambda_n)L(\lambda_1, \ldots, \lambda_n)
$$

$$= k_{G}^{3(b_-(L) - b_+(L))} \frac{1}{D_{G}^{b_p}} \sum_{\lambda_j \in P_m, \tau(\lambda_j) \equiv 0 \text{mod}(m), j = 1, \ldots, n} d(\lambda_1) \cdots d(\lambda_n)L(\lambda_1, \ldots, \lambda_n), \quad (3.4.5)
$$

where we used lemma 3.4.1 that $k_{G_1}^{3} = k_{G}^{3}$, $D_G = m D_{G_1}$, and $D_G$ and $k_{G}^{3}$ are given by (1.7.4) and (1.7.5) respectively.

Let us consider a special case of (3.4.5) when $N = 2$, $m = 2$. Choose $k = 4k_1$, $k_1 \in \mathbb{N}$ so that (3.4.1) is satisfied. Note that there are $k_1 + 2$ simple objects in $C(G_1)$, 2 of them coming from the 2 irreducible subsectors of $a_{k_1}$ by (2) of lemma 3.4.2, where as in §3.5, we use half integers to denote the irreducible representations of $SU(2)$. It is easy to see that our invariants $\tau_{G_1}$ in this case is exactly the invariants discussed on Page 234 of [MW] when $l = k + 2$ is even but not divisible by 4. What is new here is a description of a unitary modular tensor category (consequently a 3 dimensional Topological Quantum Field Theory in the sense of Chap. 4 of [Tu]) $C(G_1)$ underlying this invariant.

**3.5 Parafermion cosets.** Consider the coset $H \subset G_m$ with $G = SU(n)$ and $H$ is the Cartan subgroup of $G$, a $n - 1$ dimensional torus. This coset is an example
of parafermion cosets (cf. [DJ]). It is proved in §4.4 of [X1] that this coset verifies all the assumptions of Th. 1, and hence we have a 3 manifold invariant $\tau_{G/H}$. Also by §4.4 of [X1] the assumption of Th. B is also satisfied in this case, hence we can effectively calculate these invariants. We will do this for the case of $G = SU(2)$ in the following.

We will label the representations of $LSU(2)$ at level $k$ by $i$ such that $0 \leq 2i \in \mathbb{Z} \leq k$. The representations of the subalgebra $LU(1)$ at level $2k$ is labeled by $0 \leq \alpha \in \mathbb{Z} \leq 2k - 1$. The fusion ring of $LU(1)$ at level $2k$ is isomorphic to the group ring of $\mathbb{Z}_{2k}$ and the conformal dimension of $\alpha$ is given by $\Delta_\alpha = \exp(2\pi i \frac{\alpha^2}{4k})$. The central charge of this coset (cf. (1.7.6)) is given by $C_{G/H} = 2k - 2 + \frac{2}{k+2}$.

Note $(i, \alpha) \in \text{exp}$ iff $i + \frac{\alpha}{2} \in \mathbb{Z}$. By (2.6.12) of [KW], we know that $(i, \alpha) \succ (0, 0)$ iff $(i, \alpha) = (0, 0)$ or $(k/2, k)$. On the other hand since $[a_{(2k/2)}] = [\sigma_{k/2} a_k]$ has statistical dimension 1, it follows that

$$[(k/2, k)] = [(0, 0)].$$

This leads to the only selection rule that

$$[(i, \alpha)] = [(k/2 - i, \alpha + k)].$$

(3.5.1)

It is then easy to calculate the sum of the index of all different sectors $[(i, \alpha)]$, and the result is $\frac{1}{2} k \mu_G$. But by Lemma 2.2 of [X3], we have

$$\mu_{G/H} = \frac{d(G/H)^4 \mu_G}{\mu_H},$$

and since $d(G/H)^4 = k^2, \mu_H \geq 2k, \frac{1}{2} k \mu_G \leq \mu_{G/H}$, we must have

$$\mu_H = 2k, \mu_{G/H} = \frac{1}{2} k \mu_G,$$

and by [KLM] this shows that $(i, \alpha) \in \text{exp}$ subject to (1) are all the irreducible sectors of the coset $G/H$.

By Th. B, we have

$$L((i_1, \alpha_1), \ldots, (i_n, \alpha_n)) = L(i_1, \ldots, i_n) L(\overline{\alpha_1}, \ldots, \overline{\alpha_n}).$$

Note that the fusion ring generated by $\alpha$’s is isomorphic to the group ring of $\mathbb{Z}_{2k}$, and easier argument than that in the proof of Prop. 3.2.2 shows that (also cf. [MOO])

$$L(\alpha_1, \ldots, \alpha_n) = \exp\left(\frac{2\pi i}{4k} \alpha L \cdot \alpha L\right),$$

where $\alpha L := \sum_i \alpha_i L_i$, and

$$\alpha L \cdot \alpha L := \sum_{1 \leq i, j \leq n} \alpha_i \alpha_j L_i \cdot L_j.$$
\(L(i_1, ..., i_n)\) is the same as the framed link invariants in [KM] by Prop. 3.1.2. The three manifold invariant is given by
\[
\tau_{G/H}(M_L) = k_G^{3(b_-(L) - b_+(L))} D_{G/H}^{-n} \sum_{i_j + \frac{\alpha_j}{2} \in \mathbb{Z}, j=1, ..., n} d(i_1)...d(i_n) \times \\
L(i_1, ..., i_n) \exp(-\frac{2\pi i}{4k} \alpha L \cdot \alpha L) = k_G^{3(b_-(L) - b_+(L))} (D_GD_H)^{-n} \sum_{i_j + \frac{\alpha_j}{2} \in \mathbb{Z}, j=1, ..., n} d(i_1)...d(i_n) \times \\
L(i_1, ..., i_n) \exp(-\frac{2\pi i}{4k} \alpha L \cdot \alpha L),
\]
(3.5.2)
where \(D_G = \frac{1}{\sqrt{k+1 \sin(\frac{\pi}{k+2})}}, D_H = \sqrt{2k}, \) and \(k_{G/H} = \exp(2\pi i \frac{2k-2}{k+2}),\) which follows from (1.7.3) and (1.7.4). The \(2^{-n}\) factor appears in the first = due to (3.5.1). Note that the summation above is over all those representation labels given at the beginning of this section subject to the indicated constraints.

When \(k = 2k_1 + 1\) is odd, we can choose a unique representative of \((i, \alpha)\) in the equivalent class obtained by relation (3.5.1) such that \(2i\) is even, and in this case \(\alpha\) is also even since \(i + \frac{\alpha}{2} \in \mathbb{Z}\). So
\[
\tau_{G/H}(M_L) = k_G^{3(b_-(L) - b_+(L))} D_{G/H}^{-n} \sum_{i_j \in \mathbb{Z}, j=1, ..., n} d(i_1)...d(i_n) L(i_1, ..., i_n) \times \\
\sum_{\alpha_j \in \mathbb{Z}, j=1, ..., n} \exp(-\frac{2\pi i}{4k} \alpha L \cdot \alpha L) = 2^{n/2} k_G^{3(b_-(L) - b_+(L))} D_G^{-n} \sum_{i_j \in \mathbb{Z}, j=1, ..., n} d(i_1)...d(i_n) L(i_1, ..., i_n) \\
\times 2^{n/2} k_H^{3(b_-(L) - b_+(L))} D_H^{-n} \sum_{\alpha_j \in \mathbb{Z}, j=1, ..., n} \exp(-\frac{2\pi i}{4k} \alpha L \cdot \alpha L),
\]
note that
\[
2^{n/2} k_G^{3(b_-(L) - b_+(L))} D_G^{-n} \sum_{i_1 \in \mathbb{Z}, j=1, ..., n} d(i_1)...d(i_n) L(i_1, ..., i_n)
\]
is \(\tau_{SO(3)}(M)\) (cf. [MW] or [KT2]), and
\[
2^{n/2} k_H^{3(b_-(L) - b_+(L))} D_H^{-n} \sum_{\alpha_j \in \mathbb{Z}, j=1, ..., n} \exp(-\frac{2\pi i}{4k} \alpha L \cdot \alpha L)
\]
is also a 3 manifold invariant as defined on P. 545 of [MOO] with \(N = k\), and we denote it by \(\tau_{U(1)/\mathbb{Z}_2}\). Hence
\[
\tau_{G/H}(M) = \tau_{SO(3)}(M) \tau_{U(1)/\mathbb{Z}_2}(M).
\]
(3.5.3)
Note that \( \tau_{U(1)/\mathbb{Z}_2} \) depends only on the linking matrix, and it follows from Page 521 of [KM] that it is homotopy invariant, determined in fact by the first Betti number of \( M_L \) and the linking pairing on \( \text{Tor}H_1(M_L) \). In fact \( \tau_{U(1)/\mathbb{Z}_2} \) is expressed in terms of classical invariants in [MOO].

However when \( k \) is even, it is not clear if \( \tau_{G/H} \) can be factorized into the products of other invariants. One may try to express \( \tau_G(M)\overline{\tau}_H(M) \) in terms of \( \tau_{G/H} \), but this does not seem to be possible for general \( M \). To illustrate this let us consider the case that \( k = 2k_1 \) with \( k_1 \) odd, and \( L \) has only one component with framing \( n = 2n_1 \), and \( n_1 \) is odd. Consider the transformation \( \alpha \to \alpha + k_1 \) which changes the parity of \( \alpha \), and since

\[
\exp\left(\frac{2\pi i}{4k}(\alpha + k_1)^2n\right) = \exp\left(\frac{2\pi i}{4k}\alpha^2n\right)(-1)^{\alpha n_1} \exp\left(\frac{\pi i k_1 n_1}{2}\right),
\]

we have

\[
\sum_{\alpha \in 2\mathbb{Z}} \exp\left(\frac{2\pi i}{4k}\alpha^2n\right) = -\exp\left(\frac{\pi i k_1 n_1}{2}\right) \sum_{\alpha \in 2\mathbb{Z}+1} \exp\left(\frac{2\pi i}{4k}\alpha^2n\right).
\]

Denote the Gauss sum

\[
\sum_{0 \leq \alpha \leq 2k-1} \exp\left(\frac{2\pi i}{4k}\alpha^2n\right) = \sum_{0 \leq \alpha \leq 4k_1-1} \exp\left(\frac{2\pi i}{4k_1}\alpha^2n_1\right)
\]

by \( G(n_1, 4k_1) \), note that \( G(n_1, 4k_1) \neq 0 \) (cf. [Lang]). From the above we have

\[
\sum_{\alpha \in 2\mathbb{Z}} \exp\left(\frac{2\pi i}{4k}\alpha^2n\right) = \frac{1}{(1 - \exp(-\frac{\pi i k_1 n_1}{2}))} G(n_1, 4k_1),
\]

and

\[
\sum_{\alpha \in 2\mathbb{Z}+1} \exp\left(\frac{2\pi i}{4k}\alpha^2n\right) = \frac{1}{(1 - \exp(\frac{\pi i k_1 n_1}{2}))} G(n_1, 4k_1).
\]

So

\[
\tau_{G/H}(M_L) = \frac{1}{2} k^{3(b_-(L) - b_+(L))} D_{G/H}^{-1}(\sum_{j,2j \in \mathbb{Z}} L(j) - \sum_{j,2j \in \mathbb{Z}+1} L(j) \exp(\frac{\pi i k_1 n_1}{2})) \times \frac{G(n_1, 4k_1)}{(1 - \exp(\frac{\pi i k_1 n_1}{2}))}.
\]

Compared to

\[
\tau_G(M_L)\overline{\tau}_H(M_L) = k^{3(b_-(L) - b_+(L))} D_G^{-1} D_H^{-1}(\sum_{j,2j \in \mathbb{Z}} L(j) + \sum_{j,2j \in \mathbb{Z}+1} L(j))\overline{G}(n_1, 4k_1),
\]
expressing $\tau_G\bar{\tau}_H$ in terms of $\tau_{G/H}$ would imply a relation between

$$\sum_{j,2j\in\mathbb{Z}} L(j) - \sum_{j,2j\in\mathbb{Z}+1} L(j) \exp\left(\frac{\pi ik_1n_1}{2}\right)$$

and

$$\sum_{j,2j\in\mathbb{Z}} L(j) + \sum_{j,2j\in\mathbb{Z}+1} L(j),$$

which seems unlikely since when $k$ is even the symmetry $j \to k/2 - j$ preserves the parity of $2j$, and does not relate the above two sums, so the symmetry principle as on Page 513 of [KM] (also cf. Prop. 3.2.2) does not help. Also since in this case $k = 2k_1$ is not divisible by 4, the $SO(3)$ invariants as discussed in the end of §3.4 does not exist. It is not clear if $\tau_{G/H}$ is related to any of the previous known invariants in a simple way.

§3.6 Diagonal cosets of type A. We consider the coset

$$H := SU(N)_{m',m''} \subset G := SU(N)_{m'} \times SU(N)_{m''},$$

where the embedding $H \subset G$ is diagonal. We use $i$ (resp. $\alpha$) to denote the irreducible positive energy representations of $LG$ (resp. $LH$). To compare our notations with that §2.7 of [KW], note that our $i$ is $(\Lambda', \Lambda'')$ of [KW], and our $\alpha$ is $\Lambda$ of [KW]. We will identify $i = (\Lambda', \Lambda'')$ and $\alpha = \Lambda$ where $\Lambda', \Lambda''$, $\Lambda$ are the weights of $SL(N)$ at levels $m', m'', m' + m''$ respectively. Suppose

$$i = (\Lambda_1', \Lambda_1''), j = (\Lambda_2', \Lambda_2''), k = (\Lambda_3', \Lambda_3''), \alpha = \Lambda_1, \beta = \Lambda_2, \delta = \Lambda_3.$$
so

\[ \mu_{G/H} = \frac{\mu_H \mu_G}{N^2}. \]

A direct way of calculating the sum of index of all irreducible sectors of \((i, \alpha) \in \text{exp}\) can be found in (2) of Th. 2.3 of [X2]. From (1) of Th. 2.3 we have

\[ k_{G/H}^3 = (k_G/k_H)^3, \]

and this also follows from (1.7.3).

To calculate \(\tau_{G/H}\) we need to prepare some further notations from [X3]. We define a vector space \(W\) over \(\mathbb{C}\) whose orthonormal basis are denoted by \(i \otimes \alpha\) with \(i = (\Lambda', \Lambda''), \alpha = \Lambda\). \(W\) is also a commutative ring with structure constants given by \(N^k_{ij}N^d_{\alpha \beta}\). Let \(V\) be the vector space over \(\mathbb{C}\) whose basis are given by the irreducible components of \(\sigma_i a_1 \otimes \bar{\alpha}\) (cf. §4.3 of [X4]). Then \(V = V_0 \oplus V_1\), where \(V_0\) is a subspace of \(V\) whose basis are given by the irreducible components of \(\sigma_i a_1 \otimes \bar{\alpha}\) with \((i, \alpha) \in \text{exp}\), and \(V_1\) is the orthogonal complement of \(V_0\) in \(V\). The composition of sectors gives \(V\) a ring structure. By (1) of theorem 4.3 of [X4], the irreducible subrepresentations of \((i, \alpha)\) of the coset are in one-to-one correspondence with the basis of \(V_0\) and this map is a ring isomorphism by (1) of Prop.4.2 of [X4], and we will identify the irreducible subrepresentations of \((i, \alpha)\) of the coset with the basis of \(V_0\) in the following when no confusion arises. Note that \(V_0\) is a subring of \(V\) and \(V_0. V_1 \subset V_1\).

Define a linear map \(P : W \to V\) such that \(P(i \otimes \alpha) = \sigma_i a_1 \otimes \bar{\alpha}\). By Th. 4.3 of [X4]

\[ P(i \otimes \alpha) = \sigma_i a_1 \otimes \bar{\alpha} = P(i' \otimes \bar{\alpha}') = \sigma_{i'} a_1 \otimes \bar{\alpha}', \]

iff \(\sigma^s(i) = i', \sigma^s(\alpha) = \alpha'\) for some \(s \in \mathbb{Z}\). Also \(\langle P(i \otimes \alpha), P(j \otimes \beta) \rangle = 0\) if \(P(i \otimes \alpha) \neq P(j \otimes \beta)\) by (*) of §4.3 of [X4]. Note \(P(\sigma(1) \otimes \sigma(1)) = 1\) and \(P\) is a ring homomorphism from \(W\) to \(V\). Define \(W_0 := P^{-1}(V_0), W_1 := P^{-1}(V_1)\), then \(W = W_0 \oplus W_1\) since \(\text{exp}\) is \(\sigma\) invariant. Note that \(i \otimes \alpha \in W_0\) iff \(i - \alpha \in Q\). Define the action of \(Z_N\) on \(W\) as \(\sigma(i \otimes \alpha) = \sigma(i) \otimes \sigma(\alpha)\).

Assume \(\sigma^s(i \otimes \alpha) = i \otimes \alpha\) for some \(i \otimes \alpha \in W_0\), and \(0 < s \leq N\) is the least positive integer with this property. Let \(t = \frac{N}{s}\). By equation (*) on Page 30 of [X4] we have:

\[ \langle P(i \otimes \alpha), P(i \otimes \alpha) \rangle = t. \]

In particular when \(t > 1\), i.e., when the subgroup of \(Z_N\) which fix \((i \otimes \alpha)\) is nontrivial, the sector \(P(i \otimes \alpha)\) is not irreducible. The question of decomposing \(P(i \otimes \alpha)\) when \(t > 1\) into irreducible pieces is known as Fixed point resolutions which is discussed in [LVW], [FSS1] [FSS2]. The problem is answered in a mathematical framework in [X2]. The result is (cf. Page 10 of [X2]) that there exists \(c_1, ..., c_t \in V_0\) such that

\[ P(i \otimes \alpha) = \sum_{1 \leq k \leq t} c_k, \quad d(c_k) = \frac{1}{t} d(i) d(\alpha), k = 1, ..., t. \]
Note that if \( P^{-1}(P(i \otimes \alpha)) = \{i_1 \otimes \alpha_1, ..., i_s \otimes \alpha_s\} \), then \( st = N \).

Note we identify the covariant representations of the coset with the basis of \( P(W_0) = V_0 \). The univalence of \( A := P(i \otimes \alpha), i \otimes \alpha \in W_0 \) are given by: \( \omega_A = \exp(2\pi i(\Delta_i - \Delta_\alpha)) \), where \( \Delta_i, \Delta_\alpha \) are the conformal dimensions (cf. §1.5, and if \( i = (\Lambda', \Lambda''), \Delta_i := \Delta_{\Lambda'} + \Delta_{\Lambda''}. \) Note if \( A \succ a \), then \( \omega_a = \omega_A. \)

Under the action of \( \sigma, W_0 \) decomposes into disjoint orbits denoted by \( O_p := \{(i_p \otimes \alpha_p), \sigma((i_p \otimes \alpha_p)), ..., \sigma^{f(O_p)}((i_p \otimes \alpha_p))\}. \) Note that all the elements of \( O_p \) are mapped to the same element \( P(i_p \otimes \alpha_p) \) by \( P. \) Notice that

\[
\langle P(i_p \otimes \alpha_p), P(i_p \otimes \alpha_p) \rangle = \frac{N}{f(O_p)},
\]

and so \( P(i_p \otimes \alpha_p) \) decomposes into \( g(O_p) := \frac{N}{f(O_p)} \) irreducible sectors denoted by \( x_1(O_p), ..., x_{g(O_p)}(O_p) \) by (3.6.0). To summarize, we have to each orbit \( O_p \) associate irreducible sectors denoted by \( x_1(O_p), ..., x_{g(O_p)}(O_p) \), and

\[
\langle x_j(O_p), x_i(O_p') \rangle = 0
\]

if \( O_p \neq O_{p'}. \) Moreover the statistical dimension \( d(x_j(O_p)) = \frac{f(p)}{N}d(i_p)d(\alpha_p) \) by (3.6.0). Use Th. B the calculation of \( \tau_{G/H} \) is now rather similar to the calculation of \( \tau_{G_2} \), and in fact essentially the same argument as (3.4.3) and (3.4.4) with some change of notations gives the following

\[
\tau_{G/H}(M_L) = \left(\frac{k_G}{k_H}\right)^{3(b_-(L) - b_+(L))}(D_GD_H)^{-3} \sum_{j \neq j \in Q, 1 \leq j \leq n} L(i_1, ..., i_n) \overline{L(\alpha_1, ..., \alpha_n)}
\]

(3.6.1)

where \( k_G, k_H \) are given by (1.7.5), \( D_G, D_H \) are given by (1.7.4), \( Q \) is the root lattice (note \( \lambda \in Q \) iff \( \frac{\tau(\lambda)}{N} \in \mathbb{Z} \)), and

\[
L(i_1, ..., i_n) = L(\Lambda(1)', ..., \Lambda(n)')L(\Lambda(1)'', ..., \Lambda(n)'')
\]

if \( i_j = (\Lambda(j)', \Lambda(j)''), j = 1, ..., n. \)

Let us first consider a special case of (3.6.1) when \( m'' = 1 \). This coset is called Coset \( W_N \) algebras with critical parameters in [X1] due to its close relations with \( W \) algebras (cf. [FKW] and [BBSS]).

**Proposition 3.6.1.** Suppose \( m'' = 1. \) Then there exists a function \( F \) which only depends on the linking matrix of \( L \) such that

\[
\tau_G(M_L) \tau_H(M_L) = F\tau_{G/H}(M_L).
\]

**Proof.** To save some writing only in this proof we shall denote write \( i_j = (s_j, t_j) \) and \( I_j := (s_j, t_j; \alpha_j), j = 1, 2, ..., n. \) The link invariants \( L(i_1, ..., i_n)L(\alpha_1, ..., \alpha_n) \) will be denoted by \( L(I_1, I_2, ..., I_n). \) Define \( \tau(I_j) := \tau(s_j) + \tau(t_j) - \tau(\alpha_j), d(I_j) =
Let \( d(s_j)d(t_j)d(\alpha_j), \sigma(I_j) := (\sigma(s_j), \sigma(t_j); \sigma(\alpha_j)), \) and \( \sigma'(I_j) := (s_j, \sigma(t_j); \alpha_j), j = 1, 2, ..., n. \) By symmetry principle Prop. 3.2.2

\[
L(I_1, ..., \sigma(I_i), ..., I_n) = \exp\left(-2\pi i \sum_{j \neq i} \frac{\tau(I_j)}{N} L_j \cdot L_i + 2\pi i \frac{\tau(I_i)}{N} L_i \cdot L_i\right) L(I_1, I_2, ..., I_n) \tag{3.6.2}
\]

\[
L(I_1, ..., \sigma'(I_i), ..., I_n) = \exp\left(2\pi i \frac{N - 1}{2N} \tau(t_i)L_i \cdot L_i - 2\pi i \sum_{j \neq i} \frac{\tau(t_j)}{N} L_j \cdot L_i\right) \tag{3.6.3}
\]

Consider the following summation

\[
\sum_{\tau(I_i) \equiv x_i \text{mod}(N)} d(I_1) \ldots d(I_n) L(I_1, ..., I_n). \tag{3.6.4}
\]

Since \( \tau(\sigma(I_i)) - \tau(I_i) \equiv 0 \text{mod}(N), \) the set which is summed over above is invariant under the action of \( \sigma. \) By using (3.6.2) it is easy to see that if

\[
\sum_{j \neq i} -x_j L_j \cdot L_i + x_i L_i \cdot L_i
\]

is not divisible by \( N \) for some \( i, \) then the summation in (3.6.4) is 0. Let us assume that

\[
\sum_{j \neq i} -x_j L_j \cdot L_i + x_i L_i \cdot L_i \equiv 0 \text{mod}(N), \forall i \tag{3.6.5}
\]

By using (3.6.2) repeatedly it is easy to see that under (3.6.5) any term \( L(I_1, ..., I_n) \) in the summation (3.6.4) has the property that

\[
L(\sigma^{y_1}(I_1), ..., \sigma^{y_n}(I_n)) = L(I_1, ..., I_n), \forall y_1, ..., y_n.
\]

Using this we have:

\[
\sum_{\tau(I_i) \equiv x_i \text{mod}(N)} L(I_1, ..., I_n) = N^n \sum_{t_i = \sigma^{x_i}(1), \tau(I_i) \equiv x_i \text{mod}(N)} L(I_1, ..., I_n)
\]

\[
= N^n \sum_{t_i = 1, \tau(I_i) \equiv 0 \text{mod}(N)} L(\sigma^{x_1}(I_1), ..., \sigma^{x_n}(I_n))
\]

\[
= N^n f(x_i, L) \sum_{t_i = 1, \tau(I_i) \equiv 0 \text{mod}(N)} L(I_1, ..., I_n),
\]

where \( f(x_i, L) \) depends only on the linking matrix of \( L \) and numbers \( \{x_i\} \) and in the last step we have repeatedly used (3.6.3). Also keep in mind that the action of
σ on the label \( t_i \) is transitive and \( \tau(\sigma^{x_i}(1)) = i \). When \( \{x_i\} \) does not verify (3.6.5), we define \( f(x_i, L) = 0 \).

So we have

\[
\tau_G(M_L)\bar{\tau}_H(M_L) = \left(\frac{k_G}{k_H}\right)^{3(b_-(L) - b_+(L))} D_G^{-n} D_H^{-n} \sum_{I_i} d(I_1)...d(I_n) L(I_1, \ldots, I_n)
\]

\[
= \left(\frac{k_G}{k_H}\right)^{3(b_-(L) - b_+(L))} D_G^{-n} N^{-n} \times \sum_{0 \leq x_i \leq N-1} \sum_{\tau(I_i) \equiv x_i \bmod(N)} d(I_1)...d(I_n) L(I_1, \ldots, I_n)
\]

\[
= \left(\frac{k_G}{k_H}\right)^{3(b_-(L) - b_+(L))} D_G^{-n} N^{-n} \sum_{0 \leq x_i \leq N-1} N^n f(x_i, L) \times \sum_{t_i = 1, \tau(I_i) \equiv 0 \bmod(N)} d(I_1)...d(I_n) L(I_1, \ldots, I_n)
\]

\[
= \sum_{0 \leq x_i \leq N-1} f(x_i, L) \tau_{G/H}(M_L).
\]

The proposition now follows by choosing

\[
F = \sum_{0 \leq x_i \leq N-1} f(x_i, L).
\]

\[\square\]

**Corollary 3.6.2.** Under the condition of the previous proposition, there exists a 3-manifold invariant \( F_1 \) such that

\[
\tau_G(M_L)\bar{\tau}_H(M_L) = F_1(M_L)\tau_{G/H}(M_L).
\]

**Proof.** By the previous proposition we can define

\[
F_1(M_L) := \frac{\tau_G(M_L)\bar{\tau}_H(M_L)}{\tau_{G/H}(M_L)},
\]

if \( \tau_{G/H}(M_L) \neq 0 \), and

\[
F_1(M_L) := 0,
\]

if \( \tau_{G/H}(M_L) = 0 \). \( F_1 \) is a 3-manifold invariant since \( \tau_G(M_L), \bar{\tau}_H(M_L) \) and \( \tau_{G/H}(M_L) \) are 3-manifold invariants.

\[\square\]

It seems to be possible to choose \( F_1 \) in Cor. 3.6.2 to be a homotopy invariant, but the above proof does not show this.
Let us consider the simplest case $H = SU(2)_2 \subset G = SU(2)_1 \times SU(2)_1$. This coset has central charge $C_{G/H} = 1/2$ and corresponds to the critical Ising model. There are three simple objects in $C(G/H)$, denoted by $1 := (0, 0; 0), x := (0, 1/2; 1/2), y := (0, 0; 1)$, with univalence $\omega(x) = \exp(2\pi i \frac{1}{16})$. As sectors we have $[x^2] = [1] + [y], [xy] = [x], [y^2] = 1$. Note that $C(G/H)$ has the same number of simple objects with identical fusion rules as $C(SU(2)_2)$, but with one different univalence $\omega(1/2) = \exp(2\pi i \frac{3}{16})$. We can evaluate $\tau_{G/H}$ by using the formula (3.6.1) but instead let us take the following short cut by using the similarity with $C(SU(2)_2)$.

Consider the sequence of algebras

$$Hom(x^n, x^n) \subset Hom(x^{n+1}, x^{n+1})...$$

By a similar argument as in the proof of lemma 3.1.1 (2) we have If

$$h_n := \exp\left(\frac{\pi i}{8}x^{n-1}(c_{x,x})\right),$$

then

$$h_n^2 = (q - 1)h_n + q,$$

with $q = \exp\left(\frac{2\pi i}{4}\right)$. The Bratteli diagram of the algebras

$$Hom(x^n, x^n) \subset Hom(x^{n+1}, x^{n+1})...$$

is determined by the fusion rules and it is then easy to see that this sequence of algebras are a special case of the sequence of algebras analyzed by Jones in §5.2 of [J1] (with $\tau = 1/2$). Now an almost identical argument as in the proof of Cor. 4.11 of [KM] shows that

$$L(x, x, ..., x) = \sqrt{2} \exp\left(\frac{2\pi i}{16}L \cdot L\right)\tilde{V}_L,$$

where $\tilde{V}_L$ is the modified Jones polynomial as in Cor. 4.11 of [KM]. Now the proof of Th. 7.1 of [KM] applies verbatim, with the constant $c = \exp(-\frac{6\pi i}{16})$ on page 524 of [KM] replaced by $\exp(-\frac{2\pi i}{16})$. The result is a formula for $\tau_{G/H}(M)$ as in Th. 7.1 of [KM], with their $c = \exp(-\frac{6\pi i}{16})$ replaced by $\exp(-\frac{2\pi i}{16})$. Let us record this result in the following proposition:

**Proposition 3.6.3.** Let $M$ be a closed, oriented 3-manifold, $H = SU(2)_4 \subset G = SU(2)_2 \times SU(2)_2$. Then

$$\tau_{G/H}(M) = \sum_{\theta} c^\mu(M_\theta),$$

where $c' = \exp\left(\frac{2\pi i}{16}\right)$, and $\mu(M_\theta)$ is the $\mu$-invariant of the spin structure $\theta$ on $M$ and the sum is taken over all spin structures.

It is also interesting to compare the above coset with the simplest parafermion coset $H_1 := U(1)_4 \subset G_1 := SU(2)_2$. This coset has central charge $1/2$ with three
simple objects. In the notations of §3.5 these three simple objects are given by 
\[ x_1 := (1/2, 1), y_1 := (1, 0), 1 := (0, 0) \] with fusion rules
\[ [x_1^2] = [1] + [y_1], [x_1 y_1] = [x_1], [y_1^2] = [1], \]
and the conformal dimension (modulo integers) of \( x_1 \) is \( \frac{1}{16} \). Now exactly the same argument as above shows that the coset \( H_1 := U(1)_4 \subset G_1 := SU(2)_2 \) gives the same 3-manifold invariant as in Prop. 3.6.3.

In general there is no clear relation between \( \tau_G(M_L)\tilde{\tau}_H(M_L) \) and \( \tau_{G/H}(M_L) \) as in Prop. 3.6.1 by inspecting (3.6.1) and using the symmetry principle. This is especially true in the case when the action of \( \sigma \) has fixed points on \( \exp \). For example the coset \( SU(2)_{2(k+l)} \subset SU(2)_{2k} \times SU(2)_{2l} \) has a fixed point \( (k/2, l/2; \frac{k+l}{2}) \). Any of the diagram automorphisms of the three groups \( SU(2)_{2(k+l)}, SU(2)_{2k} \) and \( SU(2)_{2l} \) preserves the parity of the representation labels. One runs into a problem similar to the example at the end of §3.5 if \( \tau_G(M_L)\tilde{\tau}_H(M_L) \) can be expressed in terms of \( \tau_{G/H}(M_L) \).

The simplest case when there is a fixed point under the action of \( \sigma \) is the coset \( H = SU(2)_4 \subset G = SU(2)_2 \times SU(2)_2 \). In this case there is a unique fixed point \( (1/2, 1/2, 1) \). The corresponding unitary modular category \( C(G/H) \) has 13 simple objects: 11 from the 11 orbits of \( \mathbb{Z}_2 \) action on the set \( (i_1, i_2, \alpha) \) with \( i_1 + i_2 - \alpha \in \mathbb{Z} \), where each of this orbit contains two elements, and 2 from the fixed point \( (1/2, 1/2, 1) \) by (3.6.0). Note from (3.6.1) that in this case the three manifold invariants are expressed in terms of Jones polynomial at sixth and fourth roots of unity via cabling (cf. [KM]), and since Jones polynomial at sixth and fourth roots of unity can be expressed in terms of classical topological invariants, it is an interesting question to see if one can do the same with our \( \tau_{G/H} \) along the lines of [KMZ].

§3.7 A “Maverick” Coset. Let us consider a “Maverick” Coset \( H = SU(2)_8 \subset G = SU(3)_2 \) (cf. [DJ]) considered at the end of [X1]. This coset is also considered in [X2] as the first counter-example to a hypothesis of Kac-Wakimoto (cf. [KW], [X8]) which is not a conformal inclusion. This coset verifies Th. A, so there is a unitary modular category \( C(G/H) \) as constructed in §1.7. However as shown in [X1] this example does not verify the condition of Th. B, so we need to find a way to calculate \( \tau_{G/H} \).

As shown in [X1], \( C(G/H) \) has 6 simple objects, denoted by the following: 
\[ 1 = (00, 0), x = (00, 4), y = (10, 2), \bar{y} = (01, 2), z = (10, 4), \bar{z} = (01, 4) \]. The notation for representation is slightly different from §1.5. Here the representation of \( \hat{su}(3)_2 \) is denoted by two integers \( (\lambda_1, \lambda_2), \lambda_1 + \lambda_2 \leq 2 \) such that the vacuum representation is given by \( (00) \) and the representation of \( \hat{su}(2)_8 \) is labelled by an integer \( k, 0 \leq k \leq 8 \) such that 0 denotes the vacuum representation.

The nontrivial fusion rules are
\[ [x^2] = [1] + [x], [y \bar{y}] = [1] + [x], [z^3] = [1], [y] = [xz]. \]

The central charge of this coset is 0.8, which is the same as the diagonal coset \( H_1 = SU(3)_2 \subset G_1 = SU(3)_1 \times SU(3)_1 \). Furthermore the diagonal coset \( H_1 = \)
$SU(3)_2 \subset G_1 = SU(3)_1 \times SU(3)_1$ also has 6 irreducible sectors labeled in the following

$$1 = (00,00;00), x_1 = (00,00;11), y_1 = (00,01;01), \bar{y} = (00,10;10),$$
$$z_1 = (00,20;20), \bar{z} = (00,02;20).$$

where we use the same notations for representations of $SU(3)$ as above. It is an easy exercise to show that the map $1 \rightarrow 1, [x] \rightarrow [x_1], [y] \rightarrow [y_1], [z] \rightarrow [z_1]$ is a ring isomorphism, and

$$\omega(y) = \omega(y_1) = \exp(2\pi i \frac{1}{15}).$$

Now notice that $y$ (resp. $y_1$) is a generating element in $C(G/H)$ (resp. $C(G_1/H_1)$), and by a simpler argument as in lemma 3.1.1 $Hom(y^n, y^n)$ (resp. $Hom(y_1^n, y_1^n)$) is isomorphic to Wenzl’s representation (cf. [W1]) $\pi^{(3,5)}(H_n(q))$ with $q = \exp(\frac{2\pi i}{6})$. Hence $C(G/H)$ and $C(G_1/H_1)$ are compatible as defined before lemma 1.7.5. By Lemma 1.7.5 we have shown

$$\tau_{G/H}(M) = \tau_{G_1/H_1}(M).$$

Finally note that the parafermion coset (cf. §3.5) $U(1)_6 \subset SU(2)_3$ also has central charge 0.8 and 6 simple objects. A simple calculation shows that the element $(1/2,1)$ is a generating element with univalence $\exp(2\pi i \frac{1}{15})$. Now similar argument as above shows that the modular category associated with the coset $U(1)_6 \subset SU(2)_3$ is compatible with $C(G/H)$ or $C(G_1/H_1)$ above, and so by Lemma 1.7.5 give the same 3-manifold invariants. We can therefore use (3.5.3) to calculate the three manifold invariant associated with the coset $H = SU(2)_8 \subset G = SU(3)_2$.

For more such “Maverick” Cosets, see [DJ2] and [FSS2]. It will be interesting if one can calculate the corresponding 3-manifold invariants by a similar identification as above.

### 3.8 A Question.

An interesting question is to investigate the perturbative aspects of $\tau_{G/H}$ similar to the case of $\tau_G$ (cf. [Ro], [Gar], [O]). Note that 3-d Chern-Simons action and 2-d gauged WZW action exist for the coset (cf. [MS], [KS]), it remains to see if one can draw any conclusion on the calculations of $\tau_{G/H}$ similar to that of [Ro].

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