Effect of asynchronicity on the universal behaviour of coupled map lattices

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Abstract

We investigate the spatiotemporal dynamics of coupled circle map lattices, evolving under synchronous (parallel) updating on one hand and asynchronous (random) updating rules on the other. Synchronous evolution of extended spatiotemporal systems, such as coupled circle map lattices, commonly yields multiple co-existing attractors, giving rise to phenomena strongly dependent on the initial lattice. By marked contrast numerical evidence here strongly indicates that asynchronous evolution eliminates most of the attractor states arising from special sets of initial conditions in synchronous systems, and tends to yield more global attractors. Thus the phenomenology arising from asynchronous evolution is more generic and robust in that it is obtained from many different classes of initial states. Further we show that in parameter regions where both asynchronous and synchronous evolution yield spatio-temporal intermittency, asynchronicity leads to better scaling behaviour.
It is well-known that spatially extended systems undergoing temporal evolution show the presence of a large number of multiple co-existing attractors [1]. In the case of coupled map lattices, which constitute simple models of spatially extended systems [2, 3] evolving under synchronous updates, it has been seen in numerous examples that system attractors show strong sensitivity to initial conditions even in parameter regimes where the evolving dynamics is spatio-temporally periodic in nature [4]. This multiattractor property has significant consequences for problems like control and synchronisation [5] in physical, chemical, biological and engineering contexts. The stability of such attractors to perturbations and noise have also been studied in the case of globally coupled systems [3].

A typical CML consists of dynamical elements on a lattice which interact with suitably chosen sets of other elements, and evolve via synchronous or parallel updates wherein the dynamical elements at each lattice site are updated simultaneously. However, there have been several attempts to study CML-s which evolve via asynchronous evolution, that is, one in which the updates at lattice sites are not concurrent, but sequential instead. The study of asynchronous updates is considered to be interesting for several reasons. Notably, neurophysiological systems like neurons and neuron groups evolve asynchronously, and lattice dynamical models of such phenomena must of course employ asynchronous updating schemes. Therefore it is of importance to investigate the effects of asynchronicity in prototype models [6-13]. These effects can be quite significant, e.g. it has been argued that asynchronous updates can alter the universality class of spatio-temporal intermittency [8].

In this paper, we observe that asynchronicity can have a very important physical effect. Systems which show the existence of multiple co-existing attractors which depend on the state of the initial lattice when evolved via synchronous updates, evolve to the same global attractor from many kinds of initial states under asynchronous evolution. Thus asynchronicity wipes out multiple co-existing attractors and leads to a generic global attractor which is robust to the evolution of different classes of initial conditions.

We demonstrate this result for the following specific example. The space on which our CML dynamics occurs is a discrete 1-dimensional chain, with periodic boundary conditions. The sites are denoted by integers $i$, $i = 1, \ldots, N$, where $N$ is the linear size of the lattice. On each site is defined a continuous state variable denoted by $x_n(i)$, which corresponds to the physical variable of interest, with $n$ denoting the discrete time. Here the local on-site

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map is chosen to be the sine circle map, a map which has generated much research interest:

\[ f(x) = x + \omega - \frac{K}{2\pi} \sin(2\pi x) \]

where \(0 \leq x \leq 1\). The parameter \(K\) indicates the strength of the nonlinearity, and is chosen to be 1 here.

These local maps are coupled through their nearest neighbours, and the coupling form is the discretized Laplacian form (i.e. future coupled) [2], with the strength of coupling given by \(\epsilon\). Now extensive results have been obtained for the standard parallel time evolution of lattices of circle maps, i.e. the scheme in which all individual maps of the lattice are iterated forward simultaneously [3]. This synchronous evolution implies the dynamics:

\[ x_{n+1}(i) = (1 - \epsilon) f(x_n(i)) + \frac{\epsilon}{2} \{ f(x_n(i - 1)) + f(x_n(i + 1)) \} \quad \text{mod} \ 1 \quad (1) \]

It is known [2, 4] that varying \(\epsilon\) and \(\omega\), under the usual parallel updating, yields various dynamical phases, such as: the synchronized fixed point, i.e. spatial period 1 temporal period 1 (S1T1), spatio-temporally periodic solutions e.g. spatial period 2 temporal period 1 (S2T1) and spatial period 2 temporal period 2 (S2T2), spatial intermittency, spatiotemporal intermittency etc. For parallel updating an important feature of the spatiotemporal dynamics is the existence of multiple co-existing attractors. This implies that the resultant attractor is very strongly dependent on the spatiotemporal features of the initial lattice. Thus synchronous evolution can yield many different behaviours under different initial preparations even for identical parameter values. Detailed phase diagrams of the behaviour of this CML under synchronous evolution have been obtained [4, 14].

Here we focus on randomly updated CMLs, i.e. a CML where the elements of the lattice do not update simultaneously, but update one after another in random sequence. Time now is measured in units of one complete updating sweep of the lattice (in random order), i.e. after one unit of time, all sites in the lattice have been updated. This random updating scheme should help us test the implications of asynchronicity in extended systems, and help us gauge the degree of robustness of various physical features emerging under conventional synchronous evolution.

We study the different \(\epsilon\) and \(\omega\) regions of this coupled circle map system, under three classes of initial lattices: (i) spatial period 2 initial condition; (ii)
spatial period 2 initial lattice, with a kink; and (iii) random initial lattice. We observe the spatiotemporal dynamics under parallel updating on one hand and completely asynchronous updating on the other, and find out how this “phase diagram” changes under asynchronicity, for the different classes of initial states mentioned above.

**Case I:** In the region with very high coupling parameter $\epsilon$ and low $\omega$, for instance around $\epsilon \sim 0.97$, $\omega \sim 0.02$, parallel updating yields spatial period 2, temporal period 2 behaviour when evolved from spatial period 2 initial lattices. But for the case of the initial periodic lattice having a kink, and for the case of random initial lattices, spatio-temporal intermittency is obtained. (See [4] for a detailed phase diagram).

By contrast, under asynchronous updating in this parameter regime one obtains spatiotemporal fixed points for all initial conditions: spatial period 2, spatial period 2 with a kink, and random initial lattices. This fixed point is at $\theta = f(\theta) \sim \omega$ for low $\omega$. (See Fig. 1.) Note that this is in agreement with earlier observations that under strong coupling asynchronicity has the effect of regularising the system [12].

Thus, under randomness in updating rules, spatiotemporal dynamics is no longer sensitive to initial conditions. The spatiotemporal fixed point is now the global attractor of the system.

**Case II:** In the region around $\epsilon \sim 0.8$, $\omega \sim 0.2$, asynchronicity yields spatiotemporal chaos from all initial conditions. By contrast, parallel updating yields spatiotemporal periodicity when evolved from spatial period 2 initial lattices (with or without kink), and yields spatiotemporal intermittency when evolved from random initial lattices (see Fig. 2).

Also note here, that while asynchronicity had the effect of regularising the system under strong coupling (as seen above in Case I), under weaker coupling asynchronicity has the effect of inducing spatiotemporal disorder. In this example for instance, we see that parallel updating gives rise to spatiotemporal periodicity for certain classes of initial states, while asynchronous evolution always leads to spatiotemporal chaos.

**Case III:** Around the region $\epsilon \sim 0.8$, $\omega \sim 0.08$, asynchronicity yields spatiotemporal intermittency for all initial conditions. In contrast, parallel updates yield exact spatiotemporal periodicity for period 2 initial lattices,
and spatiotemporal intermittency for random lattices.

Interestingly, when both asynchronous and synchronous evolution yield spatiotemporal intermittency from random initial lattices, the scaling exponents obtained from the probability distribution of laminar lengths in space and time are quite distinct for the two cases. The intermittent dynamics arising from asynchronous evolution exhibits scaling in both time and space, with the spatial and temporal scaling exponents being approximately the same ($\phi \sim 3$). Synchronous evolution on the other hand leads only to temporal scaling, and not good spatial scaling (see Fig. 3).

We note that similar phenomena are observed for coupled logistic map lattices as well, i.e. a CML with the local on-site map being $f(x) = rx(1-x)$, with $r = 4$. See figures 6 and 7 for two examples, one for high coupling strength $\epsilon = 0.9$ and the other for low coupling strength $\epsilon = 0.1$. Clearly here too asynchronous evolution is insensitive to differences in initial conditions, whereas multiple attractors co-exist for the case of the traditional synchronous evolution.

In summary then, we have investigated the spatiotemporal dynamics of coupled map lattices evolving under asynchronous updating rules. We have shown that asynchronicity has a pronounced effect on the spatiotemporal dynamics: it yields a more global attractor where multiple attractors co-existed for the case of parallel updating. In that sense asynchronous updating yields more generic and robust phenomena in extended systems. Conversely, introducing some degree of asynchronicity in an extended system may help lead the system to its most generic attractor from any initial condition, however special. The nature of the generic attractor i.e. whether regular or disordered, appears to depend on the strength of the coupling, as well as the synchronicity /asynchronicity. In the case of spatio-temporal intermittency, asynchronicity leads to better spatial scaling. Thus, synchronous and asynchronous updates can lead to different universality classes of spatio-temporal behaviour for systems which are otherwise identical in all respects. Therefore asynchronicity constitutes a relevant perturbation in the evolution of extended systems, and can perhaps be used to direct spatially extended systems to desired global attractors. We hope our observations will be useful towards the understanding the diversity of phenomena which can be seen in spatially extended systems and for controlling their behaviour.
References

[1] J.C. Sommerer and E. Ott, Nature (London) 365 136 (1993).

[2] K. Kaneko, ed. Theory and Applications of Coupled Map Lattices, (Wiley, 1993); J. Crutchfield and K. Kaneko, in Directions in Chaos, edited by Hao Bai-Lin (1987, World Scientific, Singapore) and references therein.

[3] K. Kaneko, Phys. Rev. Lett., 78, 2736 (1997).

[4] G.R. Pradhan, N. Chatterjee and N. Gupte, To appear Phys. Rev. E.

[5] S. Sinha and N. Gupte, Phys. Rev. E 64 R015203 (2001).

[6] For instance, for neural networks, an independent choice of update times for the various elements of the system is believed to provide a closer approximation to reality (Ref: J. Hertz, A. Krogh and R.G. Palmer, Introduction to the Theory of Neural Computation (Addison-Wesley, reading, 1991).

[7] P. Marcq, H. Chate and P. Manneville, Phys. Rev. Lett., 77 (1996) 4003; Phys. Rev. E 55 (1997) 2606.

[8] J. Rolf, T. Bohr and M.H. Jensen, Phys. Rev. E 57 (1998) R2503.

[9] There are cases where synchronous updates can lead to complicated spatial structures in cellular automata (Ref: M. Nowak and R.M. May, Nature 359 (1992) 826) which disappear under asynchronous updating (B.A. Huberman and N. Glance, Proc. Natl. Acad. Sci. USA 90 (1993) 7716) and are therefore considered unphysical in the absence of an external clock.

[10] In the context of MonteCarlo simulations the importance of different kinds of updating has been appreciated. For instance, R.H. Swendsen and J.-S. Wang, Phys. Rev. Letts, 58 (1987) 86, studied the differences induced by varied asynchronous updating schemes in MonteCarlo simulations of equilibrium systems.

[11] E.D. Lumer and G. Nicolis, Physica D 71 (1994) 440.
[12] M. Mehta and S. Sinha, *Chaos* 10 (2000) 350.

[13] G. Abramson and D. Zanette, Phys. Rev. E 58 (1998) 4454; S. Sinha, Int. J. Bif. and Chaos 12 (2002).

[14] G.R. Pradhan and N. Gupte, Int. J. Bifurcations and Chaos, 11, 2501, (2001).
Figure 1: Area of $\varepsilon$-$\omega$ parameter space occupied by the spatiotemporal fixed point for the case of parallel updating (top panel) and completely asynchronous updating (bottom panel), for different initial conditions: (a) period 2 initial lattice (b) period 2 with a kink and (c) random initial conditions. Here lattice size $N = 100$. 
Parallel Updating  

Asynchronous Updating
Figure 2: Space-time density plots showing the evolution of a circle map lattice of size $N = 100$ over 100 iterations via parallel updating and via completely asynchronous updating. Each slice parallel to the $x$ axis displays a snapshot of the spatial profile at some instant of time. Here $\epsilon = 0.8, \omega = 0.2$.

The lattices are evolved from (top to bottom) period 2 initial lattice, period 2 lattice with a kink at $i = 37$ and random initial lattice.
Figure 3: Probability $P$ of occurrence of (a) temporal laminar regions and (b) spatial laminar regions vs length of laminar regions $l$ (on a log-log plot), for the case of asynchronous updating (solid squares) and parallel updating (solid triangles). The solid lines are the best fit straight lines, with slope equal to $-3.0$ for asynchronous updating and $-2.4$ for parallel updating in (a), and with slope $-3.2$ for asynchronous updating in (b). These results are for circle map lattices of size $N = 1000$, with $\epsilon = 0.8, \omega = 0.08$, evolved from random initial conditions. Sample size is $\sim 10^4$. 
Parallel Updating

Asynchronous Updating
Figure 4: Space-time density plots showing the evolution of a logistic map lattice of size $N = 100$ over 100 iterations via parallel updating and via completely asynchronous updating. Each slice parallel to the $x$ axis displays a snapshot of the spatial profile at some instant of time. Here $\epsilon = 0.9$. The lattices are evolved from (top to bottom) uniform (spatial period 1) initial lattice, spatial period 2 initial lattice and random initial lattice.
Parallel Updating    Asynchronous Updating
Figure 5: Space-time density plots showing the evolution of a logistic map lattice of size $N = 100$ over 100 iterations via parallel updating and via completely asynchronous updating. Each slice parallel to the $x$ axis displays a snap shot of the spatial profile at some instant of time. Here $\epsilon = 0.1$. The lattices are evolved from (top to bottom) uniform (spatial period 1) initial lattice, spatial period 2 initial lattice and random initial lattice.