FUNCTORIAL CHARACTERIZATIONS OF MITTAG-LEFFLER MODULES

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ABSTRACT. We give some functorial characterizations of Mittag-Leffler modules and strict Mittag-Leffler modules.

1. Introduction

Let \( R \) be a commutative (associative with unit) ring. Let \( \mathcal{R} \) be the covariant functor from the category of commutative \( R \)-algebras to the category of rings defined by \( \mathcal{R}(S) := S \) for any commutative \( R \)-algebra \( S \). Let \( M \) be an \( R \)-module. Consider the functor of \( R \)-modules, \( \mathcal{M} \), defined by \( \mathcal{M}(S) := M \otimes_R S \), for any commutative \( R \)-algebra \( S \). \( \mathcal{M} \) is said to be the functor of quasi-coherent \( R \)-modules associated with \( M \). It is easy to prove that the category of \( R \)-modules is equivalent to the category of functors of quasi-coherent \( R \)-modules. Consider the dual functor \( \mathcal{M}^* := \text{Hom}_R(\mathcal{M}, \mathcal{R}) \) defined by \( \mathcal{M}^*(S) := \text{Hom}_S(M \otimes_R S, S) \). \( \mathcal{M}^* \) is called an \( R \)-module scheme. In general, the canonical morphism \( M \to \mathcal{M}^{**} \) is not an isomorphism, but, surprisingly, \( M = \mathcal{M}^{**} \) (see 2.14). This result has many applications in Algebraic Geometry (see [7]), for example the Cartier duality of commutative affine groups and commutative formal groups.

In [2], we proved that an \( R \)-module \( M \) is a projective module of finite type iff \( \mathcal{M} \) is a module scheme. In [9], we proved that \( M \) is a flat \( R \)-module iff \( \mathcal{M} \) is a direct limit of module schemes. It is also proved that \( M \) is a flat Mittag-Leffler module iff \( \mathcal{M} \) is the direct limit of its submodule schemes. In [10], we proved that \( M \) is a flat strict Mittag-Leffler module (see [4], for definition and properties) iff \( \mathcal{M} \) is the direct limit of its submodule schemes, \( \mathcal{M} = \lim_{\to} N_i^* \), and the morphisms \( \mathcal{M}^* \to N_i \) are epimorphisms.

The definition of a Mittag-Leffler module is slightly elaborated (see [11, Tag 0599]). Mittag-Leffler conditions were first introduced by Grothendieck in [5], and deeply studied by some authors, such as Raynaud and Gruson in [6]. Every module is a direct limit of finitely presented modules. Roughly speaking, we prove that a module \( M \) is a Mittag-Leffler module iff \( \mathcal{M} \) is a direct limit of finitely presented functors of submodules.

Let \( P \) be an \( R \)-module. \( P \) is a finitely presented module iff

\[
\text{Hom}_R(P, \lim_{\to} N_i) = \lim_{\to} \text{Hom}_R(P, N_i),
\]

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for any direct system \( \{N_i\} \) of \( R \)-modules (see [13]). We will say that a functor of \( R \)-modules, \( P \), is an FP-functor if

\[
\text{Hom}_R(P, \lim_{i} N_i) = \lim_{i} \text{Hom}_R(P, N_i)
\]

for every direct system of quasi-coherent modules \( \{N_i\} \). Module schemes are FP-functors. We prove the following theorem.

**Theorem 1.1.** \( M \) is an FP-functor of \( R \)-modules iff \( M^* \) is the cokernel of a morphism \( F: \oplus_i P_i^* \to \oplus_j Q_j^* \), where \( P_i, Q_j \) are finitely presented \( R \)-modules, for every \( i,j \).

Let \( \{N_i\} \) be the set of the finitely generated submodules of \( M \). Let \( \tilde{N}_i := \text{Im}[N_i \to M] \), for any \( N_i \). Then, \( M = \lim_{\to i} \tilde{N}_i \). We prove the following theorems.

**Theorem 1.2.** Let \( M \) be an \( R \)-module. The following statements are equivalent:

1. \( M \) is a Mittag-Leffler module.
2. \( \tilde{N}_i \) is an FP-functor, for any \( i \).
3. \( M \) is a direct limit of FP-functors of \( R \)-submodules.
4. The kernel of every morphism \( R^n \to M \) is isomorphic to a quotient of a module scheme.
5. The kernel of every morphism \( N^* \to M \) is isomorphic to a quotient of a module scheme, for any \( R \)-module \( N \).

**Theorem 1.3.** Let \( M \) be an \( R \)-module. The following statements are equivalent:

1. \( M \) is a strict Mittag-Leffler module.
2. \( \tilde{N}_i \) is an FP-functor and the natural morphism \( M^* \to \tilde{N}_i^* \) is an epimorphism, for any \( i \).
3. \( M \) is an \( R \)-module of some \( R \)-module \( \prod_r P_r \), where \( P_r \) is a finitely presented modules, for every \( r \).
4. The cokernel of every morphism \( M^* \to R^n \) is isomorphic to an \( R \)-submodule of a quasi-coherent module.
5. The cokernel of every morphism \( M^* \to N \) is isomorphic to an \( R \)-submodule of a quasi-coherent module, for any \( R \)-module \( N \).

2. Preliminaries

Let \( R \) be a commutative ring (associative with a unit). All the functors considered in this paper are covariant functors from the category of commutative \( R \)-algebras (always assumed to be associative with a unit) to the category of sets. A functor \( \mathcal{X} \) is said to be a functor of sets (resp. groups, rings, etc.) if \( \mathcal{X} \) is a functor from the category of commutative \( R \)-algebras to the category of sets (resp. groups, rings, etc.).

**Notation 2.1.** For simplicity, we shall sometimes use \( x \in \mathcal{X} \) to denote \( x \in \mathcal{X}(S) \). Given \( x \in \mathcal{X}(S) \) and a morphism of commutative \( R \)-algebras \( S \to S' \), we shall still denote by \( x \) its image by the morphism \( \mathcal{X}(S) \to \mathcal{X}(S') \).

An \( R \)-module \( M \) is a functor of abelian groups endowed with a morphism of functors

\[
R \times M \to M
\]
satisfying the module axioms (in other words, the morphism \( R \times M \to M \) yields an \( S \)-module structure on \( M(S) \) for any commutative \( R \)-algebra \( S \)). Let \( M \) and \( M' \) be two \( R \)-modules. A morphism of \( R \)-modules \( f: M \to M' \) is a morphism of functors such that the morphism \( f_S: M(S) \to M'(S) \) defined by \( f \) is a morphism of \( S \)-modules, for any commutative \( R \)-algebra \( S \). We shall denote by \( \text{Hom}_R(M, M') \) the family of all the morphisms of \( R \)-modules from \( M \) to \( M' \).

**Remark 2.2.** Direct limits, inverse limits of \( R \)-modules and kernels, cokernels, images, etc., of morphisms of \( R \)-modules are regarded in the category of \( R \)-modules.

One has

\[
\begin{align*}
(\text{Ker} f)(S) &= \text{Ker} f_S, \\
(\text{Coker} f)(S) &= \text{Coker} f_S, \\
(\text{Im} f)(S) &= \text{Im} f_S,
\end{align*}
\]

(\text{lim}_{i \in I} M_i)(S) = \lim_{i \in I} (M_i(S)), \quad (\text{lim}_{j \in J} M_j)(S) = \lim_{j \in J} (M_j(S)),

(where \( I \) is an upward directed set and \( J \) a downward directed set). \( M \otimes_R M' \) is defined by \((M \otimes_R M')(S) := M(S) \otimes_S M'(S)\), for any commutative \( R \)-algebra \( S \).

**Definition 2.3.** Given an \( R \)-module \( M \) and a commutative \( R \)-algebra \( S \), we shall denote by \( M|_S \) the restriction of \( M \) to the category of commutative \( R \)-algebras, i.e.,

\[
M|_S(S') := M(S'),
\]

for any commutative \( S \)-algebra \( S' \).

We shall denote by \( \mathbb{H}\text{om}_R(M, M')^{[1]} \) the \( R \)-module defined by

\[
\mathbb{H}\text{om}_R(M, M')(S) := \text{Hom}_S(M|_S, M'|_S).
\]

Obviously,

\[
(\mathbb{H}\text{om}_R(M, M'))|_S = \mathbb{H}\text{om}_S(M|_S, M'|_S).
\]

**Notation 2.4.** Let \( M \) be an \( R \)-module. We shall denote \( M^* = \text{Hom}_R(M, R) \).

**Proposition 2.5.** Let \( M \) and \( N \) be two \( R \)-modules. Then,

\[
\text{Hom}_R(M, N^*) = \text{Hom}_R(N, M^*), \quad f \mapsto \tilde{f},
\]

where \( \tilde{f} \) is defined as follows: \( \tilde{f}(n)(m) := f(m)(n) \), for any \( m \in M \) and \( n \in N \).

**Proof.** \( \text{Hom}_R(M, N^*) = \text{Hom}_R(M \otimes_R N, R) = \text{Hom}_R(N, M^*) \).

**□**

**Proposition 2.6.** \([1] 1.15\)** Let \( M \) be an \( R \)-module, \( S \) a commutative \( R \)-algebra and \( N \) an \( S \)-module. Then,

\[
\text{Hom}_S(M|_S, N) = \text{Hom}_R(M, N).
\]

In particular,

\[
M^*(S) = \text{Hom}_R(M, S).
\]

\footnote{In this paper, we shall only consider well-defined functors \( \mathbb{H}\text{om}_R(M, M') \), that is to say, functors such that \( \text{Hom}_S(M|_S, M'|_S) \) is a set, for any \( S \).}
2.1. Quasi-coherent modules.

**Definition 2.7.** Let $M$ (resp. $N$, $V$, etc.) be an $R$-module. We shall denote by $\mathcal{M}$ (resp. $\mathcal{N}$, $\mathcal{V}$, etc.) the $R$-module defined by $\mathcal{M}(S) := M \otimes_R S$ (resp. $\mathcal{N}(S) := N \otimes_R S$, $\mathcal{V}(S) := V \otimes_R S$, etc.). $\mathcal{M}$ will be called the quasi-coherent $R$-module associated with $M$.

$\mathcal{M}|_S$ is the quasi-coherent $S$-module associated with $M \otimes_R S$. For any pair of $R$-modules $M$ and $N$, the quasi-coherent module associated with $M \otimes_R N$ is $\mathcal{M} \otimes_R \mathcal{N}$.

**Proposition 2.8.** \cite{1, 1.12] The functors

\[
\text{Category of } R\text{-modules} \rightarrow \text{Category of quasi-coherent } R\text{-modules}
\]

\[M \mapsto \mathcal{M}\]

\[\mathcal{M}(R) \mapsto M\]

establish an equivalence of categories. In particular,

\[\text{Hom}_R(\mathcal{M}, \mathcal{M}') = \text{Hom}_R(M, M').\]

Let $f: M \rightarrow N$ be a morphism of $R$-modules and $\hat{f}: \mathcal{M} \rightarrow \mathcal{N}$ the associated morphism of $R$-modules. Let $C = \text{Coker } f$, then $\text{Coker } \hat{f} = C$, which is a quasi-coherent module.

**Proposition 2.9.** \cite{1, 1.3] For every $R$-module $\mathbb{M}$ and every $R$-module $M$, it is satisfied that

\[\text{Hom}_R(\mathbb{M}, \mathbb{M}) = \text{Hom}_R(M, M(R)), f \mapsto f_R.\]

**Notation 2.10.** Let $\mathbb{M}$ be an $R$-module. We shall denote by $\mathbb{M}_{qc}$ the quasi-coherent module associated with the $R$-module $M(R)$, that is,

\[\mathbb{M}_{qc}(S) := M(R) \otimes_R S.\]

**Proposition 2.11.** For each $R$-module $\mathbb{M}$ one has the natural morphism

\[\mathbb{M}_{qc} \rightarrow \mathbb{M}, m \otimes s \mapsto s \cdot m,\]

for any $m \otimes s \in \mathbb{M}_{qc}(S) = M(R) \otimes_R S$, and a functorial equality

\[\text{Hom}_R(\mathcal{N}, \mathbb{M}_{qc}) = \text{Hom}_R(\mathcal{N}, \mathbb{M}),\]

for any quasi-coherent $R$-module $\mathcal{N}$.

**Proof.** Observe that $\text{Hom}_R(\mathcal{N}, \mathbb{M}) \cong \text{Hom}_R(N, M(R)) \cong \text{Hom}_R(\mathcal{N}, \mathbb{M}_{qc})$. \hfill \square

Obviously, an $R$-module $\mathbb{M}$ is a quasi-coherent module iff the natural morphism $\mathbb{M}_{qc} \rightarrow \mathbb{M}$ is an isomorphism.

**Theorem 2.12.** \cite{1, 1.8] Let $M$ and $M'$ be $R$-modules. Then,

\[\mathcal{M} \otimes_R \mathcal{M}' = \text{Hom}_R(\mathcal{M}, \mathcal{M}'), m \otimes m' \mapsto m \circ m',\]

where $m \circ m'(w) := w(m) \cdot m'$, for any $w \in \mathcal{M}$. 

**Note 2.13.** In particular, $\text{Hom}_R(\mathcal{M}, \mathcal{M}') = M \otimes_R M'$, and it is easy to prove that the morphism $f = \sum m_i \otimes m'_i \in \text{Hom}_R(\mathcal{M}, \mathcal{M}') = M \otimes_R M'$ factors through the quasi-coherent module associated with the submodule $\langle m'_i \rangle \subseteq M'$. 
If we make $M' = \mathcal{R}$ in the previous theorem, we obtain the following theorem.

**Theorem 2.14.** \(\text{[III, §1, 2.5] [1, 1.10]}\) Let $M$ be an $\mathcal{R}$-module. Then, $\mathcal{M} = M^{**}$.

**Definition 2.15.** Let $\mathcal{M}$ be an $\mathcal{R}$-module. We shall say that $\mathcal{M}^*$ is a dual functor. We shall say that an $\mathcal{R}$-module $\mathcal{M}$ is reflexive if $\mathcal{M} = \mathcal{M}^{**}$.

**Example 2.16.** Quasi-coherent modules are reflexive.

### 2.2. $\mathcal{R}$-module schemes.

**Definition 2.17.** Let $\mathcal{M}$ be an $\mathcal{R}$-module. $\mathcal{M}^*$ will be called the $\mathcal{R}$-module scheme associated with $\mathcal{M}$.

**Definition 2.18.** Let $\mathcal{N}$ be an $\mathcal{R}$-module. We shall denote by $\mathcal{N}_{sch}$ the $\mathcal{R}$-module scheme defined by $\mathcal{N}_{sch} := ((\mathcal{N}^*)_{qc})^*$.

**Proposition 2.19.** Let $\mathcal{N}$ be a functor of $\mathcal{R}$-modules. Then, we have a canonical morphism $\mathcal{N} \to \mathcal{N}_{sch}$ and

\[
\text{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{M}^*) = \text{Hom}_{\mathcal{R}}(\mathcal{N}_{sch}, \mathcal{M}^*), \quad \text{for any module scheme } \mathcal{M}^*.
\]

\[
\text{Hom}_{\mathcal{R}}(\mathcal{N}_{sch}, \mathcal{M}) = \mathcal{N}^*(R) \otimes_{\mathcal{R}} M, \quad \text{for any quasi-coherent module } \mathcal{M}.
\]

**Proof.** $\text{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{M}^*) = \text{Hom}_{\mathcal{R}}(\mathcal{M}, (\mathcal{N}^*)_{qc}) = \text{Hom}_{\mathcal{R}}(\mathcal{N}_{sch}, \mathcal{M}^*)$, and $\text{Hom}_{\mathcal{R}}(\mathcal{N}_{sch}, \mathcal{M}) = (\mathcal{N}^*)_{qc}(R) \otimes_{\mathcal{R}} M = \mathcal{N}^*(R) \otimes_{\mathcal{R}} M$.

Let $\{U_i\}_{i \in I}$ be an open covering of a scheme $X$. We shall say that the obvious morphism $Y = \coprod_{i \in I} U_i \to X$ is an open covering.

**Definition 2.20.** Let $\mathcal{F}$ be a functor of sets. $\mathcal{F}$ is said to be a sheaf in the Zariski topos if for any commutative $\mathcal{R}$-algebra $S$ and any open covering $\text{Spec} S_1 \to \text{Spec} S$, the sequence of morphisms

\[
\mathcal{F}(S) \longrightarrow \mathcal{F}(S_1) \longrightarrow \mathcal{F}(S_1 \otimes_{S} S_1)
\]

is exact.

**Example 2.21.** $\mathcal{M}$ is a sheaf in the Zariski topos.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be sheaves in the Zariski topos. If $f: \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of $\mathcal{R}$-modules, it is easy to check that $\text{Ker} f$ is a sheaf in the Zariski topos.

**Theorem 2.22.** \(\text{[9, 1.28]}\) Let $\{\mathcal{F}_i\}$ be a direct system of sheaves of $\mathcal{R}$-modules. Then, $\text{Hom}_{\mathcal{R}}(\mathcal{N}^*, \lim_{i} \mathcal{F}_i) = \lim_{i} \text{Hom}_{\mathcal{R}}(\mathcal{N}^*, \mathcal{F}_i)$.

### 2.3. From the category of $\mathcal{R}$-algebras to the category of $\mathcal{R}$-modules.

**Notation 2.23.** Let $\mathcal{F} = \mathcal{N}^*$ be a dual functor of $\mathcal{R}$-modules. We can consider the following functor from the category of $\mathcal{R}$-modules to the category of $\mathcal{R}$-modules $\mathcal{F}(N) := \text{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{N}) \longrightarrow \text{Hom}_{\mathcal{R}}(\mathcal{N}^*, \mathcal{F})$, for any $\mathcal{R}$-module $N$. 
**Examples 2.24.** $\overline{\mathcal{M}}(N) = \text{Hom}_R(M, N)$. $\mathcal{M}(N) = M \otimes_R N$.

Observe that $\overline{\mathcal{F}}(S) = \mathcal{F}(S)$, for any commutative $R$-algebra $S$. Given an $R$-module $N$, consider the $R$-algebra $R \oplus N$, where $(r, n) \cdot (r', n') := (rr', rn' + r'n)$. It is easy to check that

$$\mathcal{F}(R \oplus N) = \mathcal{F}(R) \oplus \mathcal{F}(N),$$

and $\overline{\mathcal{F}}(N) = \text{Ker}(\mathcal{F}(R \oplus N) \to \mathcal{F}(R))$.

Let $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ be dual functors of $R$-modules. Then, $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ is an exact sequence of morphisms of $R$-modules iff $\overline{\mathcal{F}}(N) \to \overline{\mathcal{F}'}(N) \to \overline{\mathcal{F}''}(N)$ is an exact sequence of morphisms of $R$-modules, for any $R$-module $N$.

**Lemma 2.25.** The obvious morphism

$$\text{Hom}_R(\prod_{i \in I} R, N) \to \text{Hom}_R(\oplus_{i \in I} R, N), \; g \mapsto g|_{\oplus_{i \in I} R}$$

is injective.

**Proof.** Write $M = \oplus_{i \in I} R$. Then,

$$\text{Hom}_R(\prod_{i \in I} R, N) = \text{Hom}_R(\mathcal{M}^*, N) \cong M \otimes_R N = \oplus_{i \in I} N \subseteq \prod_{i \in I} N = \text{Hom}_R(\oplus_{i \in I} R, N).$$

\[\square\]

**Proposition 2.26.** Let $\{M_i\}_{i \in I}$ be a set of dual functors of $R$-modules and let $N$ be an $R$-module. Then,

$$\text{Hom}_R(\prod_{i \in I} M_i, N) = \oplus_{i \in I} \text{Hom}_R(M_i, N)$$

In particular, $(\prod_{i \in I} M_i)^* = \oplus_{i \in I} M_i^*$ and if $M_i$ is reflexive, for any $i$, then $\prod_{i \in I} M_i$ is reflexive.

**Proof.** If $f|_{\oplus_{i \in I} M_i} = 0$, then $f = 0$: Given $m = (m_i)_{i \in I} \in \prod_{i \in I} M_i$, define

$$g: \prod_{i \in I} R \to N, \; g((r_i)) := f((r_i \cdot m_i)).$$

Observe that $g|_{\oplus_{i \in I} R} = 0$, hence $g = 0$, by Lemma 2.25. Therefore, $f = 0$.

Obviously,

$$\oplus_{i \in I} \text{Hom}_R(M_i, N) \subseteq \text{Hom}_R(\prod_{i \in I} M_i, N).$$

Let $f \in \text{Hom}_R(\prod_{i \in I} M_i, N)$ and $J := \{i \in I: f_i := f|_{M_i} \neq 0\}$. For each $j \in J$, let $R_j$ be a commutative $R$-algebra and $m_j \in M_j(R_j)$ such that $0 \neq f_j(m_j) \in N \otimes_R R_j$. Let $S := \prod_{j \in J} R_j$. The obvious morphism of $R$-algebras $S \to R_i$ is surjective, and this morphism of $R$-modules has a section. Hence, the natural morphism $\pi_i: M_i(S) = \overline{M_i}(S) \to \overline{M_i}(R_i) = M_i(R_i)$ has a section of $R$-modules. Let $m'_i \in M_i(S)$ be such that $\pi_i(m'_i) = m_i$. The morphism of $S$-modules $g: \prod_{j \in J} S \to N \otimes_R S, \; g((s_j)) := f((s_j \cdot m'_j))$ satisfies that $g|S \neq 0$, for every factor $S \subseteq \prod_{j \in J} S$. Then, $\# J < \infty$ by Lemma 2.25.

Finally, define $h := \sum_{j \in J} f_j \in \oplus_{i \in I} \text{Hom}_R(M_i, N)$, then $f = h$.

\[\square\]

Let $\{F_i\}_{i \in I}$ be a set of reflexive functors, then $\oplus_{i \in I} F_i$ is a reflexive functor and $\overline{\oplus_{i \in I} F_i} = \oplus_{i \in I} \overline{F_i}$. 
3. Quasi-coherent modules associated with finitely presented modules

Let $M$ be an $R$-module. There exists an $R$-module $N$ such that $M = N^*$ iff $M$ is an $R$-module projective of finite type (see [2]). In other words, $M = M_{sch}$ iff $M$ is an $R$-module projective of finite type. $M = M_{sch}$ iff

$$M \otimes_R N' = \overline{M}(N') = \overline{M}_{sch}(N') = \text{Hom}_R(M^*, N')$$

for any $R$-module $N'$.

**Theorem 3.1.** The morphism $M^*_{qc} \to M^*$ is an epimorphism iff $M$ is a projective module of finite type.

**Proof.** $\Rightarrow$) The morphism $\overline{M^*_{qc}} \to \overline{M^*}$ is an epimorphism. Then, the morphism

$$M^* \otimes_R N \to \text{Hom}_R(M, N)$$

is surjective, for every $R$-module $N$. Let $N = M$. Then, there exist $w_i \in M^*$ and $m_i \in M$, $i = 1, \ldots, r$, such that $\sum_i w_i \otimes m_i \to Id$. Therefore, $\sum_i w_i(m)m_i = m$, for every $m \in M$. Let $f: M \to R^r$, $f(m) := (w_i(m))$ and $g: R^r \to M$, $g(a_i) := \sum_i a_i m_i$. Observe that $(g \circ f)(m) = g((w_i(m))) = \sum_i w_i(m)m_i = m$, that is, $g \circ f = Id$ and $M$ is a direct summand of $R^r$.

$\square$

**Corollary 3.2.** A morphism $N \to M^*$ is an epimorphism iff $M$ is a projective module of finite type and the morphism $N \to M^*$ is an epimorphism.

**Proof.** $\Rightarrow$) $N \to M^*$ factors through the morphism $M^*_{qc} \to M^*$, which is an epimorphism because $N \to M^*$ is an epimorphism. Then, $M$ is a projective module of finite type and the morphism $N \to M^*$ is an epimorphism.

$\Leftarrow$) The morphism $N \to M^*_{qc}$ is an epimorphism because $N \to M^*$ is an epimorphism. The morphism $M^*_{qc} \to M^*$ is an isomorphism because $M$ is a projective module of finite type. Then, $N \to M^*$ is an epimorphism.

$\square$

**Lemma 3.3.** Let $f: V_2 \to V_1$ be a morphism of $R$-modules between quasi-coherent modules. Then, $f$ is an epimorphism iff $f^*: V_1^* \to V_2^*$ is a monomorphism.

**Proof.** $\Leftarrow$) Coker $f$ is the quasi-coherent module associated with to Coker $f_R$, and $(\text{Coker } f)^* = \ker f^* = 0$. Then, Coker $f = (\text{Coker } f)^* = 0$.

$\square$

**Proposition 3.4.** If

$$0 \to V_2^* \xrightarrow{f} V_1^* \xrightarrow{g} M \to 0$$

is an exact sequence of morphisms of functors of $R$-modules, then $M$ is a projective module of finite type.

**Proof.** 1. $M$ is a finitely generated $R$-module, by Note 2.13

2. Given an $R$-module $N$, if we take $\text{Hom}_R(-, N)$ on the above exact sequence we obtain the exact sequence

$$0 \to \text{Hom}_R(M, N) \to V_2 \otimes N \to V_1 \otimes N \to 0$$

by Proposition 2.3 and Lemma 3.3

3. Consider an exact sequence of morphisms of $R$-modules

$$0 \to N_1 \to N_2 \to N_3 \to 0$$
We obtain the diagram

\[
\begin{array}{ccccccccc}
0 & \to & \text{Hom}_R(M, N_1) & \to & V_2 \otimes N_1 & \to & V_1 \otimes N_1 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \text{Hom}_R(M, N_2) & \to & V_2 \otimes N_2 & \to & V_1 \otimes N_2 & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \text{Hom}_R(M, N_3) & \to & V_2 \otimes N_3 & \to & V_1 \otimes N_3 & \to & 0 \\
\end{array}
\]

By the snake lemma, \(\text{Hom}_R(M, N_2) \to \text{Hom}_R(M, N_3)\) is surjective. Then, \(M\) is a projective module of finite type.

---

**Proposition 3.5.** Let \(M\) be an \(R\)-module. \(M\) is a finitely presented \(R\)-module iff there exists an exact sequence of functors of \(R\)-modules

\[
\begin{array}{ccccccc}
V^*_1 & \xrightarrow{i} & V^*_2 & \xrightarrow{\pi} & M & \to & 0 \\
\end{array}
\]

**Proof.** \(\Leftarrow\) 1. \(M\) is a finitely generated \(R\)-module, by Note 2.13.

2. Let \(f: L = R^n \to M\) be an epimorphism, \(K := \text{Ker} f \subseteq L\), and \(\tilde{f}: L \to M\) and \(i': K \to L\) the associated morphisms. There exists a morphism \(g: V^*_2 \to L\) such that \(\tilde{f} \circ g = \pi\), because \(\text{Hom}_R(V^*_2, L) \cong V^*_2 \otimes_R L \to V^*_2 \otimes_R M\) is surjective. We can suppose that \(g\) is an epimorphism, replacing \(V^*_2\) by \(V^*_2\) and \(L\). Consider the exact sequences of morphisms

\[
\begin{array}{ccccccc}
K & \to & L & \to & M & \to & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Hom}_R(V^*_1, K) & \xrightarrow{\cong} & \text{Hom}_R(V^*_1, L) & \xrightarrow{\cong} & \text{Hom}_R(V^*_1, M) \\
V_1 \otimes_R K & \xrightarrow{\cong} & V_1 \otimes_R L & \xrightarrow{\cong} & V_1 \otimes_R M & \to & 0 \\
\end{array}
\]

There exists a morphism \(g': V^*_1 \to K\) such that \(i' \circ g' = g \circ i\). The morphism \(g'_R\) is surjective because \(g_R\) is surjective. Then, \(g'\) is an epimorphism and \(K\) is a finitely generated module, by Note 2.13. Hence, \(M\) is a finitely presented module.

\(\square\)

**Proposition 3.6.** [11, Tag 058L] If \(0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{\pi} M_3 \to 0\) is an exact sequence of functors of \(R\)-modules and \(M_3\) is a finitely presented module, then this exact sequence splits.

**Proof.** Let \(V^*_1 \xrightarrow{i} V^*_2 \xrightarrow{\pi} M_3 \to 0\) be an exact sequence of \(R\)-modules and let \(V^*_0 := \text{Ker} i\) \((V_0 := \text{Coker}[V_2 \to V_1])\). Let \(i_0: V^*_0 \to V^*_1\) the inclusion morphism. Then,

\[
0 \to V^*_0 \xrightarrow{i_0} V^*_1 \xrightarrow{i} V^*_2 \xrightarrow{\pi} M_3 \to 0
\]
is an exact sequence, and if we take $\text{Hom}_R(\mathcal{M}, \mathcal{N})$, for any quasi-coherent $\mathcal{R}$-module $\mathcal{N}$, the sequence

$$
\begin{array}{c}
0 \longrightarrow \text{Hom}_R(\mathcal{M}_3, \mathcal{N}) \longrightarrow \text{Hom}_R(\mathcal{V}_2^*, \mathcal{N}) \longrightarrow \text{Hom}_R(\mathcal{V}_1^*, \mathcal{N}) \longrightarrow \text{Hom}_R(\mathcal{V}_0^*, \mathcal{N}) \longrightarrow 0
\end{array}
$$

(\ast)

is exact.

Let $f: \mathcal{V}_2^* \to \mathcal{M}_2$ be a morphism such that $\pi' \circ f = \pi$. Let $g: \mathcal{V}_1^* \to \mathcal{M}_1$ be the morphism such that the diagram

$$
\begin{array}{c}
0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{V}_2^* \longrightarrow \mathcal{V}_1^* \longrightarrow \mathcal{V}_0^* \longrightarrow 0
\end{array}
$$

is commutative. Observe that $i' \circ g \circ i_0 = f \circ i \circ i_0 = 0$, then $g \circ i_0 = 0$. By the exact sequence (\ast), there exists a morphism $f': \mathcal{V}_2^* \to \mathcal{M}_1$ such that $g = f' \circ i$. Therefore, $f - i' \circ f'$ is zero over $\mathcal{V}_1^*$. Then, there exists a morphism $s: \mathcal{M}_3 \to \mathcal{M}_2$ such that $f - i' \circ f' = s \circ \pi$. Then, $\pi = \pi' \circ f = \pi' \circ (f - i' \circ f') = \pi' \circ s \circ \pi$ and $\pi' \circ s = \text{Id}$.

\[\Box\]

**Proposition 3.7.** Let $P$ be a finitely presented $R$-module, $M \to P$ an epimorphism and $f: \mathcal{M} \to \mathcal{P}$ the associated morphism. Consider the exact sequence of morphisms of $R$-modules

$$
\begin{array}{c}
0 \to \text{Ker} f \to \mathcal{M} \to \mathcal{P} \to 0
\end{array}
$$

Then, the sequence of morphisms of $R$-modules

$$
\begin{array}{c}
0 \to \mathcal{P}^* \to \mathcal{M}^* \to (\text{Ker} f)^* \to 0
\end{array}
$$

is exact. More generally, the sequence of morphisms of $R$-modules

$$
\begin{array}{c}
0 \to \text{Hom}_R(\mathcal{P}, \mathcal{N}) \to \text{Hom}_R(\mathcal{M}, \mathcal{N}) \to \text{Hom}_R(\text{Ker} f, \mathcal{N}) \to 0
\end{array}
$$

is exact, for any $R$-module $N$.

Finally, $\text{Ker} f$ is a reflexive functor of $R$-modules.

**Proof.** We have to prove that every morphism $g: \text{Ker} f \to \mathcal{N}$ lifts to a morphism $\mathcal{M} \to \mathcal{N}$. It is equivalent to prove that the exact sequence of morphisms of $R$-modules

$$
\begin{array}{c}
0 \to \mathcal{N} \to \mathcal{N} \oplus \mathcal{M} \to \mathcal{P} \to 0
\end{array}
$$

splits. $\mathcal{N} \oplus \mathcal{M}$ is the quasicoherent $\mathcal{R}$-module associated with the cokernel of the morphism $\text{Ker} f_R \to N \oplus M$, $k \mapsto (k, -g_R(k))$. By 3.6 this exact sequence splits.

Finally, the sequence of morphisms of $R$-modules

$$
\begin{array}{c}
0 \to (\text{Ker} f)^{**} \to \mathcal{M}^{**} \to \mathcal{P}^{**} = \mathcal{P}
\end{array}
$$

is exact, then $(\text{Ker} f)^{**} = \text{Ker} f$.

\[\Box\]
4. FP-functors

Definition 4.1. We shall say that a functor of $R$-modules $\mathcal{M}$ is an FP-functor if
\[
\Hom_R(\mathcal{M}, \lim_{i} \mathcal{N}_i) = \lim_{i} \Hom_R(\mathcal{M}, \mathcal{N}_i)
\]
for every direct system of quasi-coherent modules $\{\mathcal{N}_i\}$.

Example 4.2. Module schemes are FP-functors:
\[
\Hom_R(\mathcal{M}^*, \lim_{i} \mathcal{N}_i) = \lim_{i} \Hom_R(\mathcal{M}^*, \mathcal{N}_i).
\]

Theorem 4.3. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be FP-functors of $R$-modules and $f: \mathcal{F}_1 \to \mathcal{F}_2$ a morphism of $R$-modules. Then, $\text{Coker } f$ is an FP-functor of $R$-modules.

Proposition 4.4. Let $P$ be a finitely presented module and $\{\mathcal{M}_i\}$ a direct system of $R$-modules. Then,
\[
\Hom_R(P, \lim_{i} \mathcal{M}_i) = \lim_{i} \Hom_R(P, \mathcal{M}_i).
\]
In particular, $P$ is an FP-functor.

Proof. By $\text{(2.10)}$, $\Hom_R(P, \lim_{i} \mathcal{M}_i) = \lim_{i} \Hom_R(P, \mathcal{M}_i(R)) = \lim_{i} \Hom_R(P, \mathcal{M}_i)$.

Proposition 4.5. Let $M$ be an $R$-module. $M$ is a finitely presented module iff $M$ is an FP-functor of $R$-modules.

Proof. $\Leftarrow$) Any $R$-module is a direct limit of finitely presented modules. Write $M = \lim_{i} P_i$, where $P_i$ is a finitely presented module, for any $i$. Then, $\text{Id}: M \to M$ factors through a morphism $f_i: M \to P_i$, for some $i$. Then, $M$ is a direct summand of $P_i$, and it is finitely presented.

$\Rightarrow$) It is an immediate consequence of Proposition 4.4.

Recall Notation $\text{(2.23)}$.

Proposition 4.6. $\mathcal{M}$ is an FP-functor iff $\mathcal{M}^*$ commutes with direct limits of $R$-modules.

Proof. It is an immediate consequence of the equality
\[
\Hom_R(\mathcal{M}, \lim_{i} \mathcal{N}_i) = \mathcal{M}^*(\lim_{i} \mathcal{N}_i).
\]

Proposition 4.7. Let $M$ be an $R$-module and let $N \subseteq M$ be an $R$-submodule. Let $\tilde{N}$ be the image of the obvious morphism $N \to M$. $\tilde{N}$ is an FP-functor iff there exist finitely presented modules $P$ and $P'$ and an exact sequence of morphisms of $R$-modules
\[
0 \to \tilde{N} \to P \to P' \to 0
\]
(and in particular, $N$ is isomorphic to a finitely generated submodule of $P$).
Proof. \(\Rightarrow\) Write \(M = \lim \limits_{\to} P_i\), where \(\{P_i\}\) is a direct system of finitely presented \(R\)-modules. Then, \(\mathcal{M} = \lim \limits_{\to} \mathcal{P}_i\) and for some \(i\) the morphism \(\tilde{N} \to \mathcal{M}\) factors through an injective morphism \(\tilde{N} \to \mathcal{P}_i\). Consider the composite morphism \(\mathcal{N} \to \tilde{N} \to \mathcal{P}_i\). 

\[\text{Coker}[\tilde{N} \to \mathcal{P}_i] = \text{Coker}[\mathcal{N} \to \mathcal{P}_i] =: Q\] is quasi-coherent. Then we have the exact sequence of morphisms 

\[0 \to \tilde{N} \to \mathcal{P}_i \to Q \to 0\] 

By 4.3, \(Q\) is an FP-functor. By 4.5, \(Q\) is a finitely presented \(R\)-module. We are done. 

\(\Leftarrow\) By 3.7, the sequence of morphisms of \(R\)-modules 

\[0 \to P'^* \to P^* \to (\text{Ker } f)^* \to 0\] 

is exact. \(P'^*\) and \(P^*\) commute with direct limits of \(R\)-modules, therefore \((\text{Ker } f)^*\) commutes with direct limits of \(R\)-modules. Hence, \(\text{Ker } f\) is an FP-functor by Proposition 4.6. 

\(\square\)

**Proposition 4.8.** Let \(P\) be a finitely presented module and \(f: \mathcal{M} \to \mathcal{P}\) an epimorphism. Then, \(\text{Ker } f\) is an FP-functor iff \(M\) is a finitely presented module.

**Proof.** \(\Rightarrow\) By Proposition 3.7, the rows of the diagram

\[
\begin{array}{cccccc}
0 & \to & \text{Hom}_R(P, \lim \limits_{\to} N_i) & \to & \text{Hom}_R(M, \lim \limits_{\to} N_i) & \to & \text{Hom}_R(\text{Ker } f, \lim \limits_{\to} N_i) & \to & 0 \\
\text{Hom}_R(\mathcal{P}, \lim \limits_{\to} N_i) & & \text{Hom}_R(\mathcal{M}, \lim \limits_{\to} N_i) & & \text{Hom}_R(\text{Ker } f, \lim \limits_{\to} N_i) & & 0 \\
\end{array}
\]

are exact, then \(\text{Hom}_R(\mathcal{M}, \lim \limits_{\to} N_i) = \lim \limits_{\to} \text{Hom}_R(\mathcal{M}, N_i)\). Then, \(\mathcal{M}\) is an FP-functor and \(M\) is a finitely presented \(R\)-module, by Proposition 4.6. 

\(\Leftarrow\) It is an immediate consequence of Proposition 4.7. \(\square\)

**Theorem 4.9.** \(\mathbb{M}\) is an FP-functor of \(R\)-modules iff \(\mathbb{M}^*\) is the cokernel of a morphism \(F: \oplus_i P_i^* \to \oplus_j Q_j^*\), where \(P_i, Q_j\) are finitely presented \(R\)-modules, for every \(i, j\).

**Proof.** \(\Leftarrow\) \(\oplus_i P_i^* = \text{Coker } F\) commutes with direct limits and \(\mathbb{M}\) is a FP-functor, by 4.6.

\(\Rightarrow\) Choose a set \(A\) of representatives of the isomorphism classes of finitely presented \(R\)-modules. Let \(B\) be the set of the pairs \((P^*, g)\), where \(P \in A\) and \(g \in \text{Hom}_R(P^*, \mathbb{M}^*)\). The obvious morphism 

\[G: \oplus_{(P^*, g) \in B} P^* \to \mathbb{M}^*\]
is an epimorphism: Let $Q$ be a finitely presented module and $g^* \in \mathbb{M}^*(Q) = \text{Hom}_R(Q^*, M^*)$. Obviously, through the morphism $g^*: Q^* \to M^*$, $\text{Id}_Q$ is mapped to $g^*$. Hence, the morphism

$$\bigoplus_{(P^*, g) \in B} P^*(Q) \to \mathbb{M}^*(Q)$$

is surjective. Every module is a direct limit of finitely presented $R$-modules. Let $K$ be a field. In [1, 2.2], it has been proved that reflexive FP-functors of $K$-modules are module schemes.

**Lemma 4.13.** Let $P$ and $P'$ be finitely presented modules and $f: P \to P'$ an epimorphism. Then,

1. $\text{Hom}_R(\bigoplus_i N_i^*, \text{Ker } f) = \text{Hom}_R((\bigoplus_i N_i^*)_{\text{sch}}, \text{Ker } f)$.
2. $\text{Ker } f_R$ is a finitely generated $R$-module and the morphism $(\text{Ker } f)_Q \to \text{Ker } f$ is an epimorphism. Let $L$ be a finite free module and let $\pi: L \to \text{Ker } f$ an epimorphism. There exists an exact sequence of morphisms of $R$-modules

$$V^* \to L \xrightarrow{\pi} \text{Ker } f \to 0$$

**Proof.** (1) Given a finitely presented $R$-module, $Q$, we have that

$$\text{Hom}_R(\bigoplus_i N_i^*, Q) = \bigoplus_i \text{Hom}_R(N_i^*, Q) \xrightarrow{\text{Id}} \prod(N_i \otimes_R Q) = (\prod N_i) \otimes_R Q \xrightarrow{\text{Id}} \text{Hom}_R((\bigoplus_i N_i^*)_{\text{sch}}, Q).$$

Write $N = \bigoplus_i N_i^*$. From the commutative diagram of exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}_R(N, \text{Ker } f) & \longrightarrow & \text{Hom}_R(N, P) & \longrightarrow & \text{Hom}_R(N, P') \\
& & \| & \| & \| & \\
0 & \longrightarrow & \text{Hom}_R(N_{\text{sch}}, \text{Ker } f) & \longrightarrow & \text{Hom}_R(N_{\text{sch}}, P) & \longrightarrow & \text{Hom}_R(N_{\text{sch}}, P')
\end{array}
$$

- Proposition 4.11. If $P$ is an FP-functor of $R$-modules, then $P|_S$ is an FP-functor of $S$-modules, for any commutative $R$-algebra.

**Proof.** Let $\{N_i\}$ be a direct system of $S$-modules. Then,

$$\text{Hom}_S(P|_S, \text{lim } N_i) \xrightarrow{\text{Id}} \text{Hom}_R(P, \text{lim } N_i) = \lim \text{Hom}_R(P, N_i) \xrightarrow{\text{Id}} \text{Hom}_S(P|_S, N_i) \xrightarrow{\text{Id}}$$

**Corollary 4.10.** Let $K$ be a field and $M$ an $K$-module. $M$ is an FP-functor iff $M^*$ is quasi-coherent.

- Proposition 4.12. Let $P$ be an FP-functor. Then,

$$\text{Hom}_R(P, \text{lim } N_i) = \lim \text{Hom}_R(P, N_i).$$

**Corollary 4.12.** Let $P$ be an FP-functor. Then,

$$\text{Hom}_R(P, \text{lim } N_i) = \lim \text{Hom}_R(P, N_i).$$

**Lemma 4.13.** Let $P$ and $P'$ be finitely presented modules and $f: P \to P'$ an epimorphism. Then,

1. $\text{Hom}_R(\bigoplus_i N_i^*, \text{Ker } f) = \text{Hom}_R((\bigoplus_i N_i^*)_{\text{sch}}, \text{Ker } f)$.
2. $\text{Ker } f_R$ is a finitely generated $R$-module and the morphism $(\text{Ker } f)_Q \to \text{Ker } f$ is an epimorphism. Let $L$ be a finite free module and let $\pi: L \to \text{Ker } f$ an epimorphism. There exists an exact sequence of morphisms of $R$-modules

$$V^* \to L \xrightarrow{\pi} \text{Ker } f \to 0$$

**Proof.** (1) Given a finitely presented $R$-module, $Q$, we have that

$$\text{Hom}_R(\bigoplus_i N_i^*, Q) = \bigoplus_i \text{Hom}_R(N_i^*, Q) \xrightarrow{\text{Id}} \prod(N_i \otimes_R Q) = (\prod N_i) \otimes_R Q \xrightarrow{\text{Id}} \text{Hom}_R((\bigoplus_i N_i^*)_{\text{sch}}, Q).$$

Write $N = \bigoplus_i N_i^*$. From the commutative diagram of exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}_R(N, \text{Ker } f) & \longrightarrow & \text{Hom}_R(N, P) & \longrightarrow & \text{Hom}_R(N, P') \\
& & \| & \| & \| & \\
0 & \longrightarrow & \text{Hom}_R(N_{\text{sch}}, \text{Ker } f) & \longrightarrow & \text{Hom}_R(N_{\text{sch}}, P) & \longrightarrow & \text{Hom}_R(N_{\text{sch}}, P')
\end{array}
$$
we obtain that $\text{Hom}_R(\oplus_i N^*_i, \text{Ker } f) = \text{Hom}_R(\oplus_i N^*_i)_{sch}, \text{Ker } f)$.

(2) By 3.7 the sequence of morphisms of $R$-modules

$$0 \to P^* \to P^* \to (\text{Ker } f)^* \to 0$$

is exact. By 4.3 $(\text{Ker } f)^*$ is an FP-functor.

Ker $f_R$ is a finitely generated $R$-module and the morphism $(\text{Ker } f)_{qc} \to \text{Ker } f$ is an epimorphism. Hence, there exist a finite free module $L$ and an epimorphism $\pi: L \to \text{Ker } f$. Ker $\pi$ is equal to the quasi-coherent functor of modules associated with $\text{Coker } \pi$, which is a finitely presented module. By 4.7, Im $\tilde{\gamma}$ is an FP-functor. Then,

$$\text{Im} \tilde{\gamma} = \text{Im} \tilde{\gamma}$$

is an FP-functor.

5. MITTAG-LEFFLER MODULES

**Definition 5.1.** [11] Tag 0599 Let $M$ an $R$-module. $M$ is said to be a Mittag-Leffler module if for every finite free $R$-module $F$ and morphism of $R$-modules $f: F \to M$, there exists a finitely presented $R$-module $Q$ and a morphism of $R$-modules $\gamma: F \to Q$ such that Ker $[F \otimes_R N \to M \otimes_R N] = \text{Ker } [F \otimes_R N \to Q \otimes_R N]$ for every $R$-module $N$, that is, the kernel of the associated morphism $\tilde{f}: F \to M$ is equal to the kernel of the associated morphism $\tilde{\gamma}: F \to Q$, that is, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{f} & \mathcal{M} \\
\downarrow \tilde{\gamma} & & \downarrow \tilde{\gamma} \\
\text{Im } \tilde{\gamma} = \text{Im } \tilde{\gamma} & & Q
\end{array}
\]

**Theorem 5.2.** Let $M$ be an $R$-module. $M$ is a Mittag-Leffler module iff for every finitely generated $R$-submodule $N \subseteq M$ the image of the morphism $N \to M$ is an FP-functor of $R$-modules.

**Proof.** $\Rightarrow$) Let $N \subseteq M$ be a finitely generated submodule. Let $F$ be a finite free module and an epimorphism $F \to N$. Let $f$ be the composite morphism $F \to N \to M$. Let $Q$ be a finitely presented module and $\gamma: F \to Q$ a morphism of $R$-modules, such that the images of the associated morphisms $\tilde{f}: F \to M$ and $\tilde{\gamma}: F \to Q$ are equal. Observe that Coker $\tilde{\gamma}$ is equal to the quasi-coherent functor of modules associated with Coker $g$, which is a finitely presented module. By 4.7, Im $\tilde{\gamma}$ is an FP-functor. Then,

$$\text{Im}[N \to M] = \text{Im } \tilde{f} = \text{Im } \tilde{\gamma}$$

is an FP-functor.
\[ \text{\(\Leftarrow\)} \] Let \( F \) be a finite free module, \( f : F \to M \) a morphism of \( R \)-modules and \( N := \text{Im} f \subseteq M \). Consider the associated morphism \( \tilde{f} : F \to M \). Obviously \( \text{Im} \tilde{f} \) is equal to the image of the morphism \( N \to M \). Then, \( \text{Im} \tilde{f} \) is an FP-functor.

Write \( M = \varprojlim P_i \), where \( P_i \) is a finitely presented \( R \)-module, for every \( i \). Then, \( \text{Hom}_R(\text{Im} \tilde{f}, M) = \text{Hom}_R(\text{Im} \tilde{f}, \varprojlim P_i) = \varprojlim \text{Hom}_R(\text{Im} \tilde{f}, P_i) \)

Hence, there exist an \( i \) and a morphism \( \text{Im} \tilde{f} \to P_i \) such that the composite morphism \( \text{Im} f \to P_i \to M \) is the inclusion morphism. Hence, the morphism \( \text{Im} \tilde{f} \to P_i \) is a monomorphism and we have the commutative diagram

\[ \begin{array}{ccc}
   & & M \\
   & \nearrow \searrow & \\
F & \rightarrow & \text{Im} \tilde{f} \\
   & \searrow \nearrow & \\
   & & P_i \\
\end{array} \]

Hence, \( M \) is a Mittag-Leffler module.

\[ \text{\(\Rightarrow\)} \] Let \( \mathcal{M} \) be an \( R \)-module. \( \mathcal{M} \) is a Mittag-Leffler module iff \( \mathcal{M} \) is a direct limit of FP-functors of \( R \)-submodules.

**Proof.** \( \Leftarrow \) Let \( \mathcal{M} \) be the direct limit of a direct system of FP-functors of submodules \( \{P_j\} \). Let \( L = \mathbb{Z}^n \) be a finite free module. A morphism \( f : L \to \mathcal{M} = \varprojlim P_i \), factors through a morphism \( g : L \to P_j \), by Proposition 4.4. Any \( R \)-module is a direct limit of finitely presented modules, write \( M = \varprojlim P_i \), where \( P_i \) is a finitely presented \( R \)-module, for every \( i \). Then, each canonical morphism \( P_j \to \mathcal{M} = \varprojlim P_i \), factors through a morphism \( P_j \to P_i \) for some \( i \). Hence, we have a commutative diagram

\[ \begin{array}{ccc}
   L & \rightarrow & \text{Im} f \\
   \downarrow & & \downarrow \\
   P_j & \rightarrow & P_i \\
\end{array} \]

Then, \( M \) is a Mittag-Leffler module.

\( \Rightarrow \) Let \( \{M_i\} \) be the set of all the finitely generated submodules of \( M \). Let \( \tilde{M}_i \) be the image of the morphism \( M_i \to \mathcal{M} \). By Theorem 5.2 \( \tilde{M}_i \) is an FP-functor for any \( i \). We have the morphisms \( M_i \to \tilde{M}_i \to \mathcal{M} \). Taking direct limits we have

\[ \mathcal{M} = \varprojlim M_i \overset{\text{Id}}{\longrightarrow} \varprojlim \tilde{M}_i \overset{\text{Id}}{\longrightarrow} \mathcal{M} \]

Then, \( \varprojlim \tilde{M}_i = \mathcal{M} \).
Corollary 5.4.  \[\text{[11 Tag 059H]}\] Let \(M\) be an \(R\)-module. \(M\) is a Mittag-Leffler module if and only if the natural morphism \(M \otimes_R \prod_{i \in I} Q_i \to \prod_{i \in I}(M \otimes_R Q_i)\) is injective, for every set of \(R\)-modules \(\{Q_i\}_{i \in I}\).

Proof. \(\Rightarrow\) Let \(\{M_j\}\) be the set of the submodules of \(M\) and \(\tilde{M}_j\) the image of the morphism \(M_j \to M\). Then, \(M = \text{lim}_j \tilde{M}_j\). Then,

\[
\begin{align*}
M & \otimes_R \prod_{i} Q_i \cong \text{Hom}_R((\oplus_i Q_i^*)_{\text{sch}}, M) \\
& \xrightarrow{\text{[2.19]} \text{Hom}_R((\oplus_i Q_i^*)_{\text{sch}}, \tilde{M}_j)} \\
& \xrightarrow{\text{[2.22]} \text{Hom}_R((\oplus_i Q_i^*)_{\text{sch}}, \tilde{M}_j)} \\
& \text{lim}_j \text{Hom}_R((\oplus_i Q_i^*)_{\text{sch}}, \tilde{M}_j) = \text{lim}_j \text{Hom}_R(Q_i^*, \tilde{M}_j) = \text{Hom}_R(Q_i^*, M) \\
& = \prod_i (M \otimes_R Q_i)
\end{align*}
\]

\(\Leftarrow\) Let \(L\) be a finitely generated \(R\)-module and \(L \to M\) a morphism of \(R\)-modules. By Theorem 5.2 we have to prove that the image of the associated morphism \(f : L \to M\) is an FP-functor. Let \(L'\) be a free \(R\)-module and \(L' \to L\) an epimorphism. Obviously, the image of the composite morphism \(L' \to L \to M\) is equal to the image of the morphism \(L \to M\). Then, we can suppose that \(L\) is a finite free module.

Consider the dual morphism \(f^* : M^* \to L^*\). Coker \(f^*\) is an FP-functor, by 4.3, and \(\text{Ker} f = (\text{Coker} f^*)^*\). By Theorem 4.3 there exists an epimorphism \(\pi : \oplus_i Q_i^* \to \text{Ker} f\). Consider the commutative diagram

\[
\begin{array}{ccc}
\oplus_i Q_i^* & \xrightarrow{\text{[2.19]} \text{Hom}_R((\oplus_i Q_i^*)_{\text{sch}}, M)} & L \\
& \text{Im} f & \text{M} \\
& \text{(\oplus_i Q_i^*)_{sch}} & \text{[(\oplus_i Q_i^*)_{sch} \to L]} = \text{Im} f
\end{array}
\]

\(\text{Hom}_R((\oplus_i Q_i^*)_{\text{sch}}, M) \cong M \otimes_R \prod_i Q_i \to \text{lim}_i (M \otimes_R Q_i) = \text{Hom}_R(\oplus_i Q_i^*, M)\). Then, the morphism \((\oplus_i Q_i^*)_{\text{sch}} \to M\) is zero and

\[
\text{Coker}[(\oplus_i Q_i^*)_{\text{sch}} \to L] = \text{Im} f
\]

\((\oplus_i Q_i^*)_{\text{sch}}\) and \(L\) are FP-functors, then \(\text{Im} f\) is an FP-functor by 4.3.

\[\square\]

Lemma 5.5. Let \(M\) be a Mittag-Leffler module, and \(N_1 \subseteq N_2 \subseteq M\) two finitely generated submodules. Let \(\tilde{N}_1 \subseteq \tilde{N}_2\) be the images of the morphisms \(N_1, N_2 \to M\). The dual morphism \(\tilde{N}_2^* \to \tilde{N}_1^*\) is an epimorphism.

Proof. \(M\) is equal to a direct limit of finitely presented modules, \(M = \text{lim}_i P_i\). The morphism \(\tilde{N}_2 \to M\) factors through a morphism \(\tilde{N}_2 \to P_i\), which is a monomorphism. \(\text{Coker}[\tilde{N}_2 \to P_i] = \text{Coker}[N_2 \to P_i]\), which is the quasi-coherent module associated with \(P' = \text{Coker}[N_2 \to P_i]\). We have the exact sequence

\[
0 \to \tilde{N}_2 \to P_i \to P' \to 0
\]

\[\square\]
By Proposition 3.7, the morphism $P_i^* \to \tilde{N}_2^*$ is an epimorphism. Consider the composite morphism $\tilde{N}_1 \to \tilde{N}_2 \to P_i$. Coker[$\tilde{N}_1 \to P_i$] = Coker[$N_1 \to P_i$], which is the quasi-coherent module associated with $Q' = \text{Coker}[\tilde{N}_1 \to P_i]$. Again by Proposition 3.7, the morphism $P_i^* \to \tilde{N}_1^*$ is an epimorphism. Then, the morphism $\tilde{N}_2^* \to \tilde{N}_1^*$ is an epimorphism. 

Let $\{F_i\}$ be a direct system of finitely presented modules, so that $M = \lim_{\to} F_i$. $M$ is a strict Mittag-Leffler module iff for every commutative $R$-algebra $S$ and index $i$ there exists $j \geq i$ such that $\text{Im}[M^*(S) \to F_i^*(S)] = \text{Im}[F_j^*(S) \to F_i^*(S)]$ (see [9 II 2.3.2]).

**Theorem 5.6.** Let $M$ be an $R$-module. $M$ is a strict Mittag-Leffler module iff for every finitely generated submodule $N \subseteq M$ the image $\tilde{N}$ of the associated morphism $N \to M$ is an FP-functor and the morphism $M^* \to \tilde{N}^*$ is an epimorphism.

**Proof.** Let $\{F_i\}$ be a direct system of finitely presented $R$-modules, so that $M = \lim_{\to} F_i$. Let $\{N_k\}$ be the set of the finitely generated submodules of $M$, and let $\tilde{N}_k$ be the image of the morphism $N_k \to M$. Then, $M = \lim_{\to} \tilde{N}_k$.

$\Rightarrow$) Let $S$ be a commutative $R$-algebra. We have to prove that the morphism $M^*(S) \to \tilde{N}^*(S)$ is an epimorphism. The morphism $\tilde{N} \to M$ factors through some morphism $F_i \to M$, because $\tilde{N}$ is an FP-functor ($M$ is a Mittag-Leffler module). There exists $j \geq i$ such that $\text{Im}[M^*(S) \to F_i^*(S)] = \text{Im}[F_j^*(S) \to F_i^*(S)]$. $M = \lim_{\to} \tilde{N}_k$. The morphism $F_j \to M$ factors through some morphism $\tilde{N}_k \to M$, by Proposition 4.4. We have the morphisms

$$M^*(S) \to \tilde{N}_k^*(S) \to F_j^*(S) \to F_i^*(S) \to \tilde{N}^*(S)$$

Then,

$$\text{Im}[M^*(S) \to F_i^*(S)] \subseteq \text{Im}[\tilde{N}_k^*(S) \to F_j^*(S)] \subseteq \text{Im}[F_j^*(S) \to F_i^*(S)]$$

$$= \text{Im}[M^*(S) \to F_i^*(S)]$$

Hence, $\text{Im}[\tilde{N}_k^*(S) \to F_i^*(S)] = \text{Im}[M^*(S) \to F_i^*(S)]$ and $\text{Im}[M^*(S) \to \tilde{N}^*(S)] = \text{Im}[\tilde{N}_k^*(S) \to \tilde{N}^*(S)]$.

$\Leftarrow$) Consider a commutative $R$-algebra $S$ and an index $i$. The morphism $F_i \to M$ factors through some morphism $\tilde{N}_k \to M$, by Proposition 4.4. The morphism $\tilde{N}_k \to M$ factors through some morphism $F_j \to M$ (because $\tilde{N}_k$ is an FP-functor). We have the morphisms

$$M^*(S) \to F_j^*(S) \to \tilde{N}_k^*(S) \to F_i^*(S)$$

From the commutative diagram

$$\begin{array}{ccc}
\text{Im}[M^*(S) \to F_i^*(S)] & \xrightarrow{\cong} & \text{Im}[\tilde{N}_k^*(S) \to F_i^*(S)] \\
\text{Im}[F_j^*(S) \to F_i^*(S)] & & \text{Im}[F_j^*(S) \to F_i^*(S)]
\end{array}$$
we have that Im[\mathcal{M}^*(S) \to \mathcal{F}^*_j(S) = \text{Im}[\mathcal{F}^*_j(S) \to \mathcal{F}^*_i(S)].

\textbf{Corollary 5.7.} \( M \) is a strict Mittag-Leffler module iff \( M \) is an \( R \)-submodule of some \( R \)-module \( \prod_r P_r \), where \( P_r \) is a finitely presented \( R \)-module, for every \( r \).

\textit{Proof.} \( \Leftarrow \) Let \( P \) be a finitely presented \( R \)-module. Then,

\[ \text{Hom}_R(\oplus_i N_i^*, P) \overset{\text{def}}{=} \prod_i (N_i \otimes P) = (\prod_i N_i) \otimes P \overset{\text{def}}{=} \text{Hom}_R((\oplus_i N_i^*)_{sch}, P) \]

Hence, \( \text{Hom}_R(\oplus_i N_i^*, \prod_r P_r) = \text{Hom}_R((\oplus_i N_i^*)_{sch}, \prod_r P_r) \). Consider a monomorphism \( \mathcal{M} \hookrightarrow \prod_r P_r \) and the commutative diagram,

\[ \begin{array}{ccc}
\prod_i (N_i \otimes M) & \rightarrow & \text{Hom}_R(\oplus_i N_i^*, \mathcal{M}) \\
\downarrow & & \downarrow \\
(\prod_i N_i) \otimes M & \rightarrow & \text{Hom}_R((\oplus_i N_i^*)_{sch}, \mathcal{M}) \\
\end{array} \]

Then, \( \prod_i (N_i \otimes M) \to (\prod_i N_i) \otimes M \) is injective. By \( \text{5.4} \) \( M \) is a strict Mittag-Leffler module.

\( \Rightarrow \) Write \( M = \lim_i P_i \), where \( P_i \) is a finitely presented \( R \)-module, for any \( i \).

Consider the natural morphism \( P_i \to \mathcal{M} \) and let \( \tilde{P}_i \) the image of this morphism. The morphism \( \tilde{P}_i \hookrightarrow \mathcal{M} \), factors through a morphism \( \tilde{P}_i \to \mathcal{P}_{f(i)} \), for some \( f(i) > i \), because \( \tilde{P} \) is an FP-functor, by the hypothesis. \( \mathcal{M}^* \to \tilde{P}_i^* \) is an epimorphism, by the hypothesis again. Hence, \( \mathcal{M}^* \to \tilde{P}_i^* \) is an epimorphism. Then, there exists a morphism \( s_{f(i)}: \mathcal{M} \to \mathcal{P}_{f(i)} \) such that the diagram

\[ \begin{array}{ccc}
\tilde{P}_i & \rightarrow & \mathcal{M} \\
\downarrow & \searrow & \downarrow \\
\mathcal{P}_{f(i)} & \rightarrow & \mathcal{M} \\
\end{array} \]

is commutative. Now it is easy to check that the morphism \( \mathcal{M} \to \prod_i \mathcal{P}_{f(i)} \), \( m \mapsto (s_{f(i)}(m)) \) is a monomorphism.

\[ \square \]

\textbf{Theorem 5.8.} Let \( M \) be an \( R \)-module. The following statements are equivalent

1. \( M \) is a Mittag-Leffler module.
2. The kernel of every morphism \( \mathcal{R}^n \to \mathcal{M} \) is isomorphic to a quotient of a module scheme.
3. The kernel of every morphism \( \mathcal{N}^* \to \mathcal{M} \) is isomorphic to a quotient of a module scheme, for any \( R \)-module \( N \).

\textit{Proof.} \( 1 \Rightarrow 2 \) Let \( g: \mathcal{R}^n \to \mathcal{M} \) be a morphism of \( R \)-modules. By Theorem \( \text{5.6} \) \( \text{Im} g \) is an FP-functor. By Proposition \( \text{4.4} \) \( \text{Im} g \) is the kernel of an epimorphism \( \mathcal{P} \to \mathcal{P}' \), where \( P \) and \( P' \) are finitely presented \( R \)-modules. By Lemma \( \text{4.13} \) there exists an exact sequence of morphisms of \( R \)-modules

\[ \mathcal{V}^* \to \mathcal{R}^n \to \text{Im} g \to 0 \]

Then, \( \text{Ker} g = \text{Ker}[\mathcal{R}^n \to \text{Im} g] \) is a quotient of \( \mathcal{V}^* \).
\((2) \Rightarrow (3)\) Let \(L\) be a free module and \(g: \mathcal{L}^* \to \mathcal{M}\) be a morphism of \(\mathcal{R}\text{-}\)modules. Consider the dual morphism \(g^*: \mathcal{M}^* \to \mathcal{L}\). Observe that \(g^*\) factors through a finite free direct summand \(\mathcal{L}'\) of \(\mathcal{L}\). Then, \(g\) factors through \(\mathcal{L}'^*\) and

\[
\text{Ker } g \cong (\mathcal{L}/\mathcal{L}')^* \times \text{Ker}[\mathcal{L}'^* \to \mathcal{M}].
\]

Hence, \(\text{Ker } g\) is a quotient of a module scheme.

Let \(N\) be an \(\mathcal{R}\text{-}\)module and let \(f: N^* \to \mathcal{M}\) be a morphism of \(\mathcal{R}\text{-}\)module. Let \(L\) be a free \(\mathcal{R}\text{-}\)module, \(j: L \to N\) be an epimorphism and \(N' := \text{Ker } j\). The morphism \(\text{Hom}_\mathcal{R}(\mathcal{L}^*, \mathcal{M}) = M \otimes L \to M \otimes N = \text{Hom}_\mathcal{R}(N^*, \mathcal{M})\) is surjective. Hence, there exists a morphism \(g: \mathcal{L}^* \to \mathcal{M}\)

\[
\begin{array}{ccc}
\mathcal{L}^* & \xrightarrow{g} & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{L}^* & \xrightarrow{f} & \mathcal{M}
\end{array}
\]

is commutative. Let \(\pi: \mathcal{V}^* \to \text{Ker } g\) be an epimorphism and let \(h\) be the composite morphism \(\text{Ker } g \to \mathcal{L}^* \to \mathcal{N}^*\). Observe that \(\text{Ker } f = \text{Ker } h\). Let \(W^* = \text{Ker}(h \circ \pi)\). The morphism \(W^* = \text{Ker}(h \circ \pi) \xrightarrow{\sim} \text{Ker } f = \text{Ker } f\) is an epimorphism, because \(\pi\) is an epimorphism.

\((3) \Rightarrow (1)\) Let \(f: \mathcal{R}^n \to \mathcal{M}\) a morphism of \(\mathcal{R}\text{-}\)module. There exists an epimorphism \(\pi: \mathcal{V}^* \to \text{Ker } f\). Then, \(\text{Im } f = \text{Coker } \pi\) is an FP-functor by 4.3. Then, \(\text{Mittag-Leffler module by } 5.2\)

\[
\boxed{\text{Proposition 5.9. Let } \mathcal{M} \text{ be an } \mathcal{R}\text{-}\text{module. The following statements are equivalent}}
\]

\((1)\) \(\mathcal{M}\) is a strict Mittag-Leffler module.

\((2)\) The cokernel of every morphism \(\mathcal{M}^* \to \mathcal{R}^n\) is isomorphic to an \(\mathcal{R}\text{-}\text{submodule of a quasi-coherent module.}\)

\((3)\) The cokernel of every morphism \(\mathcal{M}^* \to N\) is isomorphic to an \(\mathcal{R}\text{-}\text{submodule of a quasi-coherent module, for any } \mathcal{R}\text{-}\text{module } N.}\)

\[
\text{Proof. } (1) \Rightarrow (2)\) Let \(f: \mathcal{M}^* \to \mathcal{R}^n\) a morphism. Consider the dual morphism \(f^*: \mathcal{R}^n \to \mathcal{M}\). The morphism \(\mathcal{M}^* \to (\text{Im } f^*)^*\) is an epimorphism, by 5.6

\[
\text{There exists an exact sequence of morphisms}
\]

\[
\mathcal{V}^* \to \mathcal{R}^n \to \text{Im } f^* \to 0
\]

by 5.8 Then, \(0 \to (\text{Im } f^*)^* \to \mathcal{R}^n \to \mathcal{V}\) is exact and \(\text{Coker } f = \text{Coker}[\text{Im } f^* \to \mathcal{R}^n]\) is an \(\mathcal{R}\text{-}\text{submodule of } \mathcal{V}\).

\((2) \Rightarrow (3)\) Dually, proceed as (2) \(\Rightarrow (3)\) in 5.8

\((3) \Rightarrow (2)\) It is obvious.

\((2) \Rightarrow (1)\) Let \(\phi: \mathcal{P} \to \mathcal{P}'\) be a morphism of \(\mathcal{R}\text{-}\text{modules. It is easy to check that}

\[
(\text{Coker } \phi)^* = \text{Ker } \phi^*.
\]

Let \(f: \mathcal{R}^n \to \mathcal{M}\) a morphism of \(\mathcal{R}\text{-}\text{modules. Consider the dual morphism } f^*: \mathcal{M}^* \to \mathcal{R}^n\). \(\text{Coker } f^*\) is an FP-functor, by 4.3. By the hypothesis, there exists a monomorphism \(i: \text{Coker } f^* \to \mathcal{V}\). By 4.7 there exist finitely presented \(\mathcal{R}\text{-}\text{modules and an exact sequence of morphisms}

\[
0 \to \text{Coker } f^* \to \mathcal{P} \to \mathcal{P}' \to 0
\]
By [3.7] the natural morphism $P^* \to (\text{Coker } f^*)^* = \text{Ker } f$ is an epimorphism. Let $h$ be the composite morphism $P^* \to \text{Ker } f \to R^*$. Then, $\text{Im } f = \text{Coker } h$, which is an FP-functor by [4.3] and $\text{Ker } h^* = \text{Im } f^*$. Observe that

$$(\text{Im } f)^* = (\text{Coker } h)^* = \text{Ker } h^* = \text{Im } f^*$$

Hence, the morphism $M^* \to (\text{Im } f)^*$ is an epimorphism and $M$ is a strict Mittag-Leffler by [5.6]. □

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