STRUCTURING THE SET OF INCOMPRESSIBLE QUANTUM HALL FLUIDS

Jürg Fröhlich\textsuperscript{1}, Thomas Kerler\textsuperscript{2}, Urban M. Studer,\textsuperscript{3}† and Emmanuel Thiran\textsuperscript{1}

\textsuperscript{1} Institut für Theoretische Physik, ETH-Hönggerberg, 8093 Zürich, Switzerland
\textsuperscript{2} Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
\textsuperscript{3} Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven, 3001 Leuven, Belgium

Abstract: A classification of incompressible quantum Hall fluids in terms of integral lattices and arithmetical invariants thereof is proposed. This classification enables us to characterize the plateau values of the Hall conductivity $\sigma_H$ in the interval $(0, 1]$ (in units where $e^2/h = 1$) corresponding to “stable” incompressible quantum Hall fluids. A bijection, called shift map, between classes of stable incompressible quantum Hall fluids corresponding to plateaux of $\sigma_H$ in the intervals $\left[1/(2p + 1), 1/(2p - 1)\right)$ and $\left[1/(2q + 1), 1/(2q - 1)\right)$, respectively, is constructed, with $p, q = 1, 2, 3, (\ldots), p \neq q$.

Our theoretical results are carefully compared to experimental data, and various predictions and experimental implications of our theory are discussed.

\textsuperscript{†}Present address: Institut für Theoretische Physik, ETH-Hönggerberg, 8093 Zürich, Switzerland
Figure 1.1. Observed Hall fractions $\sigma_H = n_H/d_H$ in the interval $0 < \sigma_H < 1$, and their experimental status in single-layer quantum Hall systems.

Well established Hall fractions are indicated by “•”. These are fractions for which an $R_{xx}$-minimum and a plateau in $R_H$ have been clearly observed, and the quantization accuracy of $\sigma_H = 1/R_H$ is typically better than 0.5%. Fractions for which a minimum in $R_{xx}$ and typically an inflection in $R_H$ (i.e., a minimum in $dR_H/dB_\perp$, but no well developed plateau in $R_H$) have been observed are indicated by “○”. If there are only very weak experimental indications or controversial data for a given Hall fraction, the symbol “·” is used. Finally, “B/n-p” is appended to fractions at which a magnetic field ($B$) and/or density ($n$) driven phase transition has been observed.
1 Introduction and Summary of Contents

In this paper, we continue our theoretical analysis of the fractional quantum Hall (QH) effect [1]. We describe and explore a classification of incompressible quantum Hall fluids (or, for short, QH fluids) in terms of pairs of an integral lattice and a primitive vector in its dual and of arithmetical invariants of these data. Our classification is derived from a combination of basic physical principles and some phenomenologically well confirmed assumptions concerning QH fluids.

Our theoretical results neatly reproduce and structure experimental data (see Refs. [2] through [7]) for the fractional QH effect. They provide insight into the origin of the phase transitions (disappearance and reappearance of plateaux when, e.g., the in-plane component of the external magnetic field is varied, at a fixed value of the filling factor) that have been observed in single-layer [3, 8, 9, 10] and double-layer or wide-single-quantum-well [11] samples. We also describe the possible structures of QH fluids corresponding to Hall fractions corresponding to even denominators [12, 13].

The analysis presented in this paper continues the thread of arguments initiated in Refs. [14] through [19]. It is a companion of the results we have discussed in [20]. In [20], we have focussed our attention on an explicit and systematic classification of some specific classes of incompressible QH fluids. In the present paper, we derive general mathematical organizing principles that enable us to reproduce and interpret the set of experimental data on incompressible QH fluids and propose some new experiments.

The theoretical framework underlying our work has originally been inspired by Halperin’s analysis of edge currents [21] and is related to ideas of Read [22], Stone [23] and Wen and collaborators [24, 25, 26]. The mathematical data, a pair of an integral
lattice, $\Gamma$, and a primitive vector, $Q$, in its dual, $\Gamma^*$, called a “$QH$ lattice”, characterizing a universality class of (incompressible) QH fluids become apparent when one studies the large-scale, low-frequency properties, i.e., the physics in the scaling limit, of QH fluids. It is convenient to describe the physics of QH fluids in the scaling limit in terms of an effective field theory of their conserved current densities. For incompressible (dissipation-free) QH fluids, i.e., ones with vanishing longitudinal resistance, the effective field theory of conserved current densities is topological, and, because of the presence of an external magnetic field, it breaks parity and time-reversal invariance \cite{15,16,17}. It turns out that this effective field theory is an abelian Chern-Simons theory of the vector potentials of conserved current densities. Physical state vectors of this theory are labelled by the points of a lattice $\Gamma_{\text{phys}}$ containing the lattice $\Gamma$ and contained in (or equal to) the dual lattice $\Gamma^*$, and the vector $Q$ in $\Gamma^*$, called ”charge vector”, determines the electric charge of a state labelled by a point in $\Gamma_{\text{phys}}$.

A QH lattice generalizes – in a sense made precise in Sects. 2 and 3 – the data of an odd, positive integer, $m$, characterizing the celebrated Laughlin fluid \cite{27} with Hall fraction (dimensionless Hall conductivity) $\sigma_H = 1/m$, $m = 1, 3, 5, \ldots$. Localizable, finite-energy excitations above the ground state of a QH fluid have “charges” corresponding to points in $\Gamma_{\text{phys}}$. The scalar product of a point, $q$, in $\Gamma_{\text{phys}}$ with the charge vector $Q$ is the electric charge of the excitation described by $q$; the scalar product of $q$ with itself determines its statistical phase. Since the scalar product of any pair of vectors in $\Gamma^*$ is a rational number, and since $\Gamma_{\text{phys}}$ is contained in (or equal to) $\Gamma^*$, statistical phases are rational multiples of $\pi$, and localizable, finite-energy excitations of an (incompressible) QH fluid thus exhibit rational fractional (anyon) statistics. QH lattices thus encode fundamental quantum numbers of finite-energy excitations of (incompressible) QH fluids.

The basic assumptions underlying our theoretical framework are summarized in Sect. 2, and their explicit implementation in the form of an effective theory describing conserved current densities of QH fluids is reviewed in Sect. 3. The precise mathematical definition of a QH lattice is given in Sect. 2, where, moreover, the important notion of a (“primitive”) chiral QH lattice (CQHL) is introduced. CQHLs are the “basic building blocks” of QH lattices and form the main objects of study in the present paper. Physically, they correspond to QH fluids that are either electron-rich or hole-rich. Furthermore, finite, but macroscopic QH samples classified by chiral QH lattices exhibit conserved edge currents that circulate along their boundaries in only one chiral sense.

In order to efficiently organize the classification of CQHLs, $(\Gamma, Q)$, it is convenient to characterize them in terms of numerical invariants. Such invariants are introduced in Sects. 2 and 4. Among them, the following ones play a key role:

(i) the Hall fraction (or dimensionless Hall conductivity), $\sigma_H(\Gamma, Q)$, which is given by the squared length of the charge vector $Q$ and thus turns out to be a rational number;

(ii) the dimension, $N$, of the integral lattice $\Gamma$, which equals the number of inde-
pendent, conserved current densities of the corresponding QH fluid;

(iii) the discriminant, \( \Delta(\Gamma, \mathcal{Q}) \), of \( \Gamma \), i.e., the order of the abelian group formed by the (equivalence) classes of elements in \( \Gamma^* \) modulo \( \Gamma \), which is related to the denominator, \( d_H(\Gamma, \mathcal{Q}) \), of the Hall fraction \( \sigma_H(\Gamma, \mathcal{Q}) \); and

(iv) an invariant, denoted \( \ell_{\text{max}}(\Gamma, \mathcal{Q}) \), that, physically, corresponds to the smallest relative angular momentum of a certain pair of identical excitations (with the quantum numbers of the electron); (for the Laughlin fluid with \( \sigma_H = 1/m \), \( \ell_{\text{max}}(\Gamma, \mathcal{Q}) = m \)).

For CQHLs, the invariants \( \ell_{\text{max}} \) and \( \sigma_H \) are related by

\[
\ell_{\text{max}}(\Gamma, \mathcal{Q}) \geq \sigma_H^{-1}(\Gamma, \mathcal{Q}),
\]

which is a simple consequence of the Cauchy-Schwarz inequality; see Sect. 4.

In terms of the two invariants \( N \) and \( \ell_{\text{max}} \), we can formulate a phenomenological stability principle which we shall appeal to in our comparison between theory and experiment:

**Stability Principle.** A QH fluid described (in the scaling limit) by a CQHL \( (\Gamma, \mathcal{Q}) \) is the more stable, the smaller the value of the invariant \( \ell_{\text{max}}(\Gamma, \mathcal{Q}) \) and, given the value of \( \ell_{\text{max}}(\Gamma, \mathcal{Q}) \), the smaller the dimension \( N \) of \( \Gamma \).

(A measure for the stability of a QH fluid is, e.g., the width of the plateau of \( \sigma_H \), as a function of the filling factor, corresponding to that QH fluid.) A detailed discussion of this stability principle is contained in [20]. Here it serves as a “working hypotheses”. A consequence of our stability principle is that QH fluids at values of \( \sigma_H \) that have large denominators are unstable.

If we are given physically plausible upper bounds, \( N_* \) and \( \ell_* \), on the invariants \( N \) and \( \ell_{\text{max}} \) (see Sect.4) then the set of CQHLs satisfying these bounds and thus physically observable can be shown [20] to be finite. In other words, in any interval of Hall fractions \( \sigma_H \), there are infinitely many fractions that cannot be realized by a physically observable chiral QH fluid!

Inequality (1.1) leads to a natural decomposition of the interval \((0, 1]\) of Hall fractions \( \sigma_H \) into subintervals (or “windows”), \( \Sigma_p \), with \( 1/(2p+1) \leq \sigma_H < 1/(2p-1) \). These windows can be further divided into two halves, \( \Sigma_p = \Sigma_p^+ \cup \Sigma_p^- \), where

\[
\Sigma_p^+ := \left[ \frac{1}{2p+1}, \frac{1}{2p} \right), \quad \text{and} \quad \Sigma_p^- := \left[ \frac{1}{2p}, \frac{1}{2p-1} \right), \quad p = 1, 2, \ldots . \quad (1.2)
\]

By inequality (1.1), CQHLs with \( \sigma_H \in \Sigma_p \) must satisfy \( \ell_{\text{max}} \geq 2p+1 \), and hence, by our stability principle, the stability of QH fluids with \( \sigma_H \in \Sigma_p \) decreases, as \( p \) increases.

Before summarizing further theoretical results, we pause to reflect on the experimental data on Hall fractions in the interval \( 0 < \sigma_H \leq 1 \), that have been established in the literature on single-layer/component QH systems; see Refs. [2] through [10]. These
data are displayed in Fig. 1. Indications on the experimental status of the fractions are provided. In Fig. 1, we write $\sigma_H = n_H/d_H$ and display the data in a “$d_H$ versus $\sigma_H$” plot. Moreover, a grid delimiting the subwindows $\Sigma_p^\pm$ (see (1.2)) is superimposed on the figure. Note that, in the interval of Hall fractions $0 < \sigma_H \leq 1$, there are no experimental data from single-layer/component QH systems showing the characteristics of the QH effect at even-denominator fractions. Celebrated observations on even-denominator QH fluids have been reported in [12], where a single-layer QH fluid with $\sigma_H = 5/2$ is described, and in [13], where two-layer/component QH fluids with $\sigma_H = 1/2$ have been established.

Partly motivated by the experimental data on single-layer QH systems, as collected in Fig. 1, partly for theoretical reasons (Wigner lattice instability), we expect a realistic upper bound on $\ell_{\text{max}}$ to be given by

$$\ell_{\text{max}} \leq \ell_* = 7 \text{ (or 9)}.$$  \hfill (1.3)

Assuming the bound $\ell_* = 7$, we predict that there are no (incompressible) chiral QH fluids with $\sigma_H \in \Sigma_p^+$, for $p \geq 4$. Furthermore, we shall see that, in the subwindow $\Sigma_3^+ = [1/7, 1/6)$, the only physically realizable Hall fractions are the elements of the series $\sigma_H = N/(6N + 1)$, $N = 1, 2, \ldots$, and each such fraction is realized by a unique CQHL (of dimension $N$).

The situation in the “complementary” subwindow $\Sigma_3^- = [1/6, 1/5)$ is much more involved. It is a general consequence of our analysis that, from a “structural” point of view, the classification problems for QH lattices with Hall fractions $\sigma_H$ in the two complementary subwindows $\Sigma_p^+$ and $\Sigma_p^-$, $p = 1, 2, \ldots$, are strikingly different. This fact is reflected by the experimental data and will be illustrated by a “uniqueness theorem” proven in Sect. 4:

According to our stability principle, CQHLs $(\Gamma, Q)$ with a small value of the invariant $\ell_{\text{max}}(\Gamma, Q)$ (and not too high dimension $N$) describe stable QH fluids. If the Hall fraction $\sigma_H(\Gamma, Q)$ belongs to $\Sigma_p$ then, by (1.3), the minimal value of the invariant $\ell_{\text{max}}(\Gamma, Q)$ is given by $2p + 1$, $p = 1, 2, \ldots$, and (“primitive”) CQHLs realizing this value are called $L$-minimal; see Sect. 4. We shall see that all $L$-minimal CQHLs with $\sigma_H \in \Sigma_p^+$ can be enumerated explicitly. There is a unique $N$-dimensional CQHL with Hall fraction $\sigma_H = N/(2pN + 1)$, $N, p = 1, 2, \ldots$, and the edge states of the corresponding QH fluids carry a representation of the Kac-Moody algebra $\hat{\mathfrak{su}}(N)$ at level 1; (for information on Kac-Moody algebras, see [28]). For $p \leq 3$ and sufficiently small values of the dimension $N$ (stability principle!), the above fractions correspond to experimentally well verified plateaux of $\sigma_H$. It is interesting to note that, e.g., in $\Sigma_4^+ = [1/3, 1/2)$, there are no $L$-minimal CQHLs at the fractions $\sigma_H = 4/11$, and $5/13$; see Fig. 1 and Sect. 5 for implications of this fact.

The classification of QH lattices (with Hall fractions) in the subwindows, $\Sigma_p^-$, $p = 1, 2, \ldots$, is greatly facilitated by the existence of a family of maps $S_p$, $p = 1, 2, \ldots$, called “shift maps”. These maps relate CQHLs of equal dimension at shifted values
of the inverse Hall fraction, \( S_p : \sigma^{-1}_H \mapsto \sigma^{-1}_H + 2p \); see Sect. 4. Restricting their action to the class of \( L \)-minimal CQHLs, we shall see that they yield one-to-one correspondences between the sets of such lattices in the windows \( \Sigma_1 \) and \( \Sigma_{p+1} \). Hence, for the classification of \( L \)-minimal CQHLs in the subwindows \( \Sigma_p \), it suffices to classify all \( L \)-minimal CQHLs in the “fundamental domain” \( \Sigma^-_1 = [1/2, 1) \), where \( \ell_{\text{max}} = 3 \). However, in contrast, to the possibility of completely enumerating all \( L \)-minimal CQHLs in the complementary fundamental domain \( \Sigma^+_1 = [1/3, 1/2) \), the classification of \( L \)-minimal CQHLs in \( \Sigma^-_1 \) is much more involved. In fact, this classification has been one of the main objectives in our recent work [20] where all low-dimensional \( (N \leq 4) \) “indecomposable”, \( L \)-minimal CQHLs in \( \Sigma^-_1 \) have been classified, as well as all those \( L \)-minimal CQHLs in arbitrary dimension that exhibit “large symmetries” and thus are called “maximally symmetric”. (These latter CQHLs are natural generalizations of the \( su(N) \)-QH lattices in \( \Sigma^+_1 \) mentioned above.) The most relevant results of this classification are recapitulated in Sect. 5. Here, we only mention one striking feature thereof:

In general, one finds more than one \( L \)-minimal CQHL realizing a given Hall fraction \( \sigma_H \in \Sigma^-_p \), \( p = 1, 2, \ldots \), and the different CQHLs at a given fraction \( \sigma_H \) typically form interesting patterns of “QH lattice embeddings”. Physically, these embeddings find a natural interpretation in terms of possible phase transitions between “structurally distinct” QH fluids. A theory of such phase transitions, which follow a “symmetry breaking” logic, has been developed in [20, Sect. 7]. The most likely Hall fractions \( \sigma_H \) at which such structural phase transitions may occur are found to be \( 2/3, 3/5, 4/7, 5/7, 5/9, \) and \( 1/2 \); compare with Fig. 1!

The main attention of the present paper is on \( L \)-minimal, chiral QH lattices which we expect (stability principle) to describe the most stable physical QH fluids. In Sect. 5, we analyze to which extent experimental data actually support the physical relevance of the two concepts of \( L \)-minimality and chirality. Explicit Hall fractions are listed where new experimental data could lead to new theoretical insights. Readers only interested in experimental consequences of our analysis are invited to jump, after a short look at Sect. 2, directly to Sect. 5, where our theoretical findings are summarized, and their main phenomenological implications are discussed.

## 2 QH Fluids and QH lattices: Basic Concepts

Our analysis of the fractional quantum Hall effect is based on a precise notion of incompressible (dissipation-free) quantum Hall fluids (or, for short, QH fluids) [16, 17, 19] – see assumptions (A1) through (A4) below – from which their main features can be derived. Our mathematical characterization, in terms of “QH lattices”, of (universality classes of) QH fluids enables us to enumerate and classify QH fluids. In this section, we review the defining properties of QH lattices.

The fractional QH effect is observed in two-dimensional gases of electrons at temperatures \( T \approx 0 K \) subject to a very nearly constant magnetic field, \( B_c \), transversal
to the plane of the system. Two-dimensional gases of electrons can be realized as heterojunctures [1]. Let \( E_0 \) denote the ground-state energy of the system in a fixed magnetic field \( B_c \). If there is a mobility gap \( \delta \) strictly positive, uniformly in the size of the system, i.e., if there are no extended (conducting) states in the spectral subspace corresponding to the energy interval \([E_0, E_0 + \delta]\) then we say that the system is a QH fluid. More precisely, a QH fluid is characterized by the property that connected Green functions of the electric charge - and current densities have cluster decomposition properties stronger than those encountered in a system where the electric charge - and current density couple the ground state of the system to a Goldstone boson (a London superconductor). One can then show [16] that the longitudinal resistance, \( R_L \), of a QH fluid vanishes. This can also be used as a definition of an (incompressible) QH fluid. Let

\[
\nu := \frac{n}{(eB_c^\perp/h)}
\]  

(2.1)

be the filling factor of the system, where \( n \) denotes the electron density, \( e \) the elementary electric charge, \( B_c^\perp \) is the component of the constant magnetic field \( B_c \) that is perpendicular to the plane of the system, and \( h \) is Planck’s constant. The longitudinal resistance \( R_L \) is a complicated function of \( \nu \), and it is a difficult problem of many-body theory to predict where \( R_L \) vanishes as a function of \( \nu \); see [1]. We do not solve this problem in this paper. Instead, we show that if \( R_L \) vanishes then the Hall conductivity,

\[
\sigma_H = R_H^{-1},
\]  

(2.2)

necessarily belongs to a certain set of rational multiples of \( e^2/h \), and, given such a value of \( \sigma_H \), we can determine the possible types of quasi-particles, i.e., the different “Laughlin vortices” of the system, their electric charges and their statistical phases; see [25, 13, 23, 17, 19].

Next, we describe the basic assumptions and physical principles underlying our analysis of QH fluids.

**(A1)** The temperature \( T \) of the system is close to 0 K. For an (incompressible) QH fluid at \( T = 0 K \), the total electric charge is a good quantum number to label physical states of the system describing excitations above the ground state; see [19, 29]. The charge of the ground state of the system is normalized to be zero.

**(A2)** In the regime of very short wave vectors and low frequencies, the scaling limit, the total electric current density is the sum of \( N = 1, 2, 3, \ldots \) separately conserved \( u(1) \)-current densities, describing electron and/or hole transport in \( N \) separate “channels” distinguished by conserved quantum numbers. (For a finite, but macroscopic sample, this assumption implies that there are \( N \) separately conserved chiral \( u(1) \)-edge currents [21] circulating round the boundary of the system.) In our analysis,
we regard $N$ as a free parameter. Physically, $N$ turns out to depend on the filling factor $\nu$ and other parameters characterizing the system.

(A3) In our units where $\hbar = -e = 1$, the electric charge of an electron/hole is $+1/−1$. Any local excitation (quasi-particle) above the ground state of the system with integer total electric charge $q_{el}$ satisfies Fermi-Dirac statistics if $q_{el}$ is odd, and Bose-Einstein statistics if $q_{el}$ is even.

(A4) The quantum-mechanical state vector describing an arbitrary physical state of an (incompressible) QH fluid is single-valued in the position coordinates of all those (local) excitations that are composed of electrons and/or holes.

The basic contention advanced in \cite{16, 17, 19} is that if a QH fluid is interpreted as a two-dimensional system of electrons with vanishing longitudinal resistance $R_L$, satisfying assumptions (A1) through (A4) above, then, in the scaling limit, its quantum-mechanical description is completely coded into a quantum Hall lattice (QH lattice). A QH lattice $(\Gamma, Q)$ consists of an odd, integral lattice $\Gamma$ and an integer-valued, linear functional $Q$ on $\Gamma$. In general, the metric on $\Gamma$ need not be positive definite. The number of positive eigenvalues of the metric corresponds, physically, to the number of edge currents of one chirality, the number of negative eigenvalues corresponds to the number of edge currents of opposite chirality. In this paper, we present results on QH fluids with edge currents of just one chirality. These QH fluids correspond to QH lattices $(\Gamma, Q)$ where $\Gamma$ is a euclidian lattice, i.e., a lattice with a positive-definite metric. These special QH lattices are called chiral QH lattices (CQHLs).

Before explaining more precisely what a CQHL is, we note that a physical hypothesis expressing a "chiral factorization" property of QH fluids motivates our study of chiral QH lattices.

(A5) The fundamental charge carriers of a QH fluid are electrons and/or holes. We assume that, in the scaling limit, the dynamics of electron-rich subfluids of a QH fluid is independent of the dynamics of hole-rich subfluids, and, up to charge conjugation, the theoretical analysis of an electron-rich subfluid is identical to that of a hole-rich subfluid.

A discussion of the “working hypothesis” (A5), including some proposals for its experimental testing, is given in \cite{20}. There it is also shown that all “hierarchy fluids” of the Haldane-Halperin \cite{30} and Jain-Goldman \cite{31} scheme, respectively, satisfy our assumptions (A1–4), and the status of assumption (A5) is carefully analyzed for such fluids; (see especially Appendix E in \cite{20}).

Chiral Quantum Hall Lattices. Let $V$ be an $N$-dimensional real vector space equipped with a positive-definite inner product, $\langle \cdot, \cdot \rangle$. In $V$ we choose a basis $\{e_1, \ldots, e_N\}$ such that

$$K_{ij} = K_{ji} := \langle e_i, e_j \rangle \in \mathbb{Z} , \quad \text{for all } i, j = 1, \ldots, N ,$$

(2.3)
i.e., all matrix elements are integers. The basis \{e_1, \ldots, e_N\} generates an integral, euclidean (i.e., positive definite) lattice, \(\Gamma\), defined by

\[
\Gamma := \{ q = \sum_{i=1}^{N} q^i e_i \mid q^i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, N \} .
\] (2.4)

The matrix \(K\) in (2.3) is called the Gram matrix of the basis \{e_1, \ldots, e_N\} generating the lattice \(\Gamma\). Let \{\varepsilon^1, \ldots, \varepsilon^N\} be the basis of \(V\) dual to \{e_1, \ldots, e_N\}, i.e., the basis satisfying

\[
< \varepsilon^i, e_j > = \delta^i_j , \quad \text{for all } i, j = 1, \ldots, N .
\] (2.5)

Then,

\[
\varepsilon^i = \sum_{j=1}^{N} (K^{-1})^{ij} e_j ,
\] (2.6)

where \(K^{-1}\) is the inverse of the matrix \(K = (K_{ij})\) with \(K_{ij}\) given in (2.3), and \((K^{-1})^{ij} = < \varepsilon^i, \varepsilon^j >\). The basis \{\varepsilon^1, \ldots, \varepsilon^N\} generates the dual lattice, \(\Gamma^*\), defined by

\[
\Gamma^* := \{ n \in V \mid < n, q > \in \mathbb{Z}, \text{ for all } q \in \Gamma \}
\]

\[
= \{ n = \sum_{i=1}^{N} n_i \varepsilon^i \mid n_i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, N \} ,
\] (2.7)

and, by (2.6), \(\Gamma^*\) contains \(\Gamma\). Denoting by \(\Delta := \det K \in \mathbb{Z}\) the determinant of the matrix \(K = (K_{ij})\) with \(K_{ij}\) given in (2.3), and \((K^{-1})^{ij} = < \varepsilon^i, \varepsilon^j >\). The basis \{\varepsilon^1, \ldots, \varepsilon^N\} generates the dual lattice, \(\Gamma^*\), defined by

\[
\Gamma^* := \{ n \in V \mid < n, q > \in \mathbb{Z}, \text{ for all } q \in \Gamma \}
\]

\[
= \{ n = \sum_{i=1}^{N} n_i \varepsilon^i \mid n_i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, N \} ,
\] (2.7)

and, by (2.6), \(\Gamma^*\) contains \(\Gamma\). Denoting by \(\Delta := \det K \in \mathbb{Z}\) the determinant of the matrix \(K = (K_{ij})\) with \(K_{ij}\) given in (2.3), and \((K^{-1})^{ij} = < \varepsilon^i, \varepsilon^j >\). The basis \{\varepsilon^1, \ldots, \varepsilon^N\} generates the dual lattice, \(\Gamma^*\), defined by

\[
\Gamma^* := \{ n \in V \mid < n, q > \in \mathbb{Z}, \text{ for all } q \in \Gamma \}
\]

\[
= \{ n = \sum_{i=1}^{N} n_i \varepsilon^i \mid n_i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, N \} ,
\] (2.7)

and, by (2.6), \(\Gamma^*\) contains \(\Gamma\). Denoting by \(\Delta := \det K \in \mathbb{Z}\) the determinant of the matrix \(K = (K_{ij})\) with \(K_{ij}\) given in (2.3), and \((K^{-1})^{ij} = < \varepsilon^i, \varepsilon^j >\). The basis \{\varepsilon^1, \ldots, \varepsilon^N\} generates the dual lattice, \(\Gamma^*\), defined by

\[
\Gamma^* := \{ n \in V \mid < n, q > \in \mathbb{Z}, \text{ for all } q \in \Gamma \}
\]

\[
= \{ n = \sum_{i=1}^{N} n_i \varepsilon^i \mid n_i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, N \} ,
\] (2.7)

and, by (2.6), \(\Gamma^*\) contains \(\Gamma\). Denoting by \(\Delta := \det K \in \mathbb{Z}\) the determinant of the matrix \(K = (K_{ij})\) with \(K_{ij}\) given in (2.3), and \((K^{-1})^{ij} = < \varepsilon^i, \varepsilon^j >\). The basis \{\varepsilon^1, \ldots, \varepsilon^N\} generates the dual lattice, \(\Gamma^*\), defined by

\[
\Gamma^* := \{ n \in V \mid < n, q > \in \mathbb{Z}, \text{ for all } q \in \Gamma \}
\]

\[
= \{ n = \sum_{i=1}^{N} n_i \varepsilon^i \mid n_i \in \mathbb{Z}, \text{ for all } i = 1, \ldots, N \} ,
\] (2.7)
In geometrical terms, a vector \( \mathbf{n} \in \Gamma^* \) is primitive if and only if the line segment joining the origin to the point \( \mathbf{n} \) is free of any lattice point. In particular, one can always take a primitive vector as the first vector of a lattice basis.

A lattice \( \Gamma \) is said to be *decomposable* if it can be written as an orthogonal direct sum of sublattices,

\[
\Gamma = \bigoplus_{j=1}^{r} \Gamma_j , \quad \text{for some } r \geq 2 ,
\]

(2.9)

with the property that \( \langle \mathbf{q}^{(i)}, \mathbf{q}^{(j)} \rangle = 0 \), for arbitrary vectors \( \mathbf{q}^{(i)} \) in \( \Gamma_i \) and \( \mathbf{q}^{(j)} \) in \( \Gamma_j \), and for arbitrary \( i \neq j \). Otherwise, \( \Gamma \) is said to be *indecomposable*.

If \( m \) and \( n \) are two integers we shall write \( m \equiv n \mod p \) if and only if \( m - n \) is an integer multiple of \( p \).

We are now prepared to define what we mean by a *chiral QH lattice (CQHL)*:

**Definition.** A CQHL is a pair \( (\Gamma, \mathbf{Q}) \) where \( \Gamma \) is an odd, integral, euclidean lattice and \( \mathbf{Q} \) is a primitive vector in the dual lattice \( \Gamma^* \) with the property that

\[
\langle \mathbf{Q}, \mathbf{q} \rangle \equiv \langle \mathbf{q}, \mathbf{q} \rangle \mod 2 , \quad \text{for all } \mathbf{q} \in \Gamma .
\]

(2.10)

Let \( (\Gamma, \mathbf{Q}) \) be a CQHL for which \( \Gamma = \bigoplus_{j=1}^{r} \Gamma_j \) is decomposable. Then \( \Gamma^* = \bigoplus_{j=1}^{r} \Gamma_j^* \) is the associated decomposition of the dual lattice. We say that \( (\Gamma, \mathbf{Q}) \) is *proper* if, in the decomposition of \( \mathbf{Q} \)

\[
\mathbf{Q} = \sum_{j=1}^{r} \mathbf{Q}^{(j)} , \quad \text{with } \mathbf{Q}^{(j)} := \mathbf{Q}|_{\Gamma_j^*} \in \Gamma_j^* ,
\]

(2.11)

corresponding to the decomposition of \( \Gamma^* \), every \( \mathbf{Q}^{(j)} \) is *non-zero*.

Finally, we introduce the following notion which plays an important role in the sequel:

**Definition.** A CQHL is called *primitive* if, in the decomposition (2.11) of \( \mathbf{Q} \), the component \( \mathbf{Q}^{(j)} \) is a primitive vector in \( \Gamma_j^* \), for all \( j = 1, \ldots, r \).

We consider this primitivity property to be a natural requirement on *composite* CQHLs that correspond to physically observable QH fluids; see the discussion at the end of Sect. 3. We note that any *indecomposable* CQHL, i.e., one with \( r = 1 \) in (2.9) and (2.11), is proper and primitive.

### 3 A Dictionary Between the Physics of QH Fluids and the Mathematics of QH Lattices

In this section, we construct a precise dictionary between mathematical properties of QH lattices and physical properties of QH fluids. Such a dictionary has already been
presented in [14, 17, 19]. Here we just recall its main contents and significance. The starting point is the idea to describe the physics of a QH fluid in the scaling limit in terms of an effective field theory of its conserved current densities. Since a QH fluid has a strictly positive mobility gap $\delta$, the scaling limit of the effective theory of its conserved current densities must be a topological field theory [13, 16]. The presence of a non-zero external magnetic field transversal to the plane to which the electrons of a QH fluid are confined implies that the quantum dynamics of the system violates the symmetries of parity (reflection-in-lines) and time reversal. Thus the topological field theory will not be parity- and time-reversal invariant.

In $2 + 1$ space-time dimensions, the continuity equation

$$
\frac{\partial}{\partial t} j^0 + \vec{\nabla} \cdot \vec{j} = 0 ,
$$

(3.1)

obeyed by a conserved current density $j^\mu = (j^0, \vec{j})$, implies that $j^\mu$ is the curl of a vector potential, i.e., $j^\mu = \varepsilon^{\mu\nu\lambda} \partial_\nu b_\lambda$, for a vector potential $b_\lambda = (b_0, \vec{b})$ which is unique up to the gradient of a scalar field (gauge invariance!).

Let us consider a QH fluid which, in the scaling limit, has $N$ independent, conserved current densities $j^\mu_1, \ldots, j^\mu_N$, the electric current density, $J^\mu_\ell$, being a linear combination, $J^\mu_\ell = \sum_{i=1}^N Q_i j^\mu_i$, of these current densities. When formulated in terms of the vector potentials $b^\lambda_1, \ldots, b^\lambda_N$ of these conserved current densities, the topological field theory describing the physics of the QH fluid in the scaling limit can only be a pure, abelian Chern-Simons theory in the fields $b^\lambda_1, \ldots, b^\lambda_N$, as follows from the circumstance that it must violate parity - and time-reversal invariance. This has been shown in refs. [14, 16, 17, 19]. The bulk action is given by

$$
S(b) = \frac{1}{4\pi} \int C_{ij} b^i_\mu \partial_\mu b^j_\lambda \varepsilon^{\mu\nu\lambda} dtd^2x =: \frac{1}{4\pi} \int \langle b_\mu, \partial_\nu b_\lambda \rangle \varepsilon^{\mu\nu\lambda} dtd^2x ,
$$

(3.2)

where $C_{ij} = C_{ji}$ is some non-degenerate quadratic form (metric) on $\mathbb{R}^N$.

Physical states of a pure, abelian Chern-Simons theory describe static $N$-tuples of “charges” localized in bounded disks of space. Each $N$-tuple, $(q^1, \ldots, q^N)$, of “charges” localized in some disk $D$ of space is an $N$-tuple of eigenvalues of the operators

$$
\int_D j^{0i}(\vec{x}, t) d^2x = \oint_{\partial D} \vec{b}^i(\vec{x}, t) \cdot d\vec{x} , \quad i = 1, \ldots, N ,
$$

(3.3)

acting on a corresponding physical state of the theory. The equations of motion of pure, abelian Chern-Simons theory, with the currents $j^{\mu i}$ minimally coupled to $N$ external gauge fields $a_\mu$, $i = 1, \ldots, N$, read

$$
\varepsilon_{\mu\rho\sigma} j^{\rho i} = \partial_\mu b^i_\nu - \partial_\nu b^i_\mu = (C^{-1})^{ij} (\partial_\mu a_{\nu j} - \partial_\nu a_{\mu j}) =: (C^{-1})^{ij} f_{\mu\nu j} ,
$$

(3.4)
These equations imply that, to an $N$-tuple of “charges” $(q^1, \ldots, q^N)$, there corresponds an $N$-tuple of “fluxes”, $(\varphi_1, \ldots, \varphi_N)$, with $\varphi_j = \int_{D} f_{1j} \, d^2x$ related to $q^i$ through

$$q^i = (C^{-1})^{ij} \varphi_j, \quad i = 1, \ldots, N.$$ (3.5)

Consider a state of Chern-Simons theory describing two excitations with identical “charges” $(q^1, \ldots, q^N)$ localized in disjoint, congruent disks of space, $D_1$ and $D_2$. From the theory of the Aharonov-Bohm effect we know that if the positions of the two disks are exchanged adiabatically along counter-clockwise oriented paths, the state vector only changes by a phase factor, $\exp i \pi \theta$, given by

$$\exp i \pi \theta = \exp i \pi <q, q> ,$$ (3.6)

where

$$\theta = \sum_{i=1}^{N} \varphi_i q^i = \sum_{i=1}^{N} (C^{-1})^{ij} \varphi_i \varphi_j = \sum_{i=1}^{N} C_{ij} q^i q^j =: <q, q> .$$ (3.7)

It is well known that $\exp i \pi <q, q>$ has the meaning of a statistical phase of the excitation corresponding to the “charges” $(q^1, \ldots, q^N)$. The $N$-dimensional vector $q$ introduced here is equivalently defined through its components $(q^1, \ldots, q^N)$, which are “charges”, or, through (3.4), in terms of its dual (w.r.t. the metric $C_{ij}$) components $(\varphi_1, \ldots, \varphi_N)$, the “fluxes”. An excitation $q$ with $<q, q> \equiv 1 \mod 2$ is a fermion, while if $<q, q> \equiv 0 \mod 2$ it represents a boson.

Next, we consider a state describing two excitations with two different charge vectors $q_1(1)$ and $q_2(2)$ localized in disjoint disks $D_1$ and $D_2$. Imagine that $D_2$ is transported around $D_1$ adiabatically, along a counter-clockwise oriented loop. Then the theory of the Aharonov-Bohm effect teaches us that the state vector only changes by a phase factor given by

$$\exp 2i \pi <q_1(1), q_2(2)> ,$$ (3.8)

where

$$<q_1(1), q_2(2)> := \sum_{i=1}^{N} C_{ij} q_1^{i} q_2^{j} = \sum_{i=1}^{N} \varphi_1^{(1)i} q_2^{i} = \sum_{i=1}^{N} (C^{-1})^{ij} \varphi_1^{(1)i} \varphi_2^{(2)j}$$ (3.9)

Since the “charges”, $q^i$, of physical states—i.e., the eigenvalues of the operators $\int_{D} j^{0i}(\vec{x}, t) \, d^2x$, $i = 1, \ldots, N$, on physical states—are additive quantum numbers, the set of vectors $q$ of physical excitations of a QH fluid form a lattice, $\Gamma_{\text{phys}}$, whose dimension can be taken to be $N$. Expressing the electric current density $J_{cl}^{\mu}$ as a linear combination, $J_{cl}^{\mu} = \sum_{i=1}^{N} Q_i j^{\mu i}$, of the current densities $j^{\mu 1}, \ldots, j^{\mu N}$, the total electric
charge (in units of $-e$), $q_{el}(\mathbf{q})$, of an excitation of the system with charge vector $\mathbf{q}$ localized in some disk $D$ is found to be given by

$$q_{el}(\mathbf{q}) = \sum_{i=1}^{N} Q_i q^i =: < \mathbf{Q}, \mathbf{q} > ,$$

(3.10)

where the electric charge assignments $Q_i$ of each current have been collected in an $N$-dimensional vector $\mathbf{Q}$ henceforth referred to as the “charge vector”. Note that, according to the above definition, the $Q_i$ are the dual components of the charge vector $\mathbf{Q}$. The electric charge $q_{el}(\mathbf{q})$ is an eigenvalue of the electric charge operator

$$\int_{D} J_{el}^0(\vec{x}, t) \, d^2x = \int_{D} \left( \sum_{i=1}^{N} Q_i j^{0i}(\vec{x}, t) \right) \, d^2x .$$

(3.11)

We define $\Gamma$ as the set of all physical excitations $\mathbf{q}$ with integral electric charge, i.e.,

$$\Gamma := \{ \mathbf{q} \in \Gamma_{phys} \mid < \mathbf{Q}, \mathbf{q} > = q_{el}(\mathbf{q}) \in \mathbb{Z} \} .$$

(3.12)

Clearly $\Gamma$ is a sublattice of $\Gamma_{phys}$. If $\Gamma$ has at least one excitation $\mathbf{q}$ of electric charge $q_{el}(\mathbf{q}) = 1$ then the dimension of $\Gamma$ is equal to $N$. Assumption (A3) entails that arbitrary vectors $\mathbf{q}_{(i)}$ in $\Gamma$ must have integer statistical phases $< \mathbf{q}_{(i)}, \mathbf{q}_{(i)} >$ related to their electric charges by the congruence $q_{el}(\mathbf{q}_{(i)}) \equiv < \mathbf{q}_{(i)}, \mathbf{q}_{(i)} > \mod 2$. Writing the scalar product $< \mathbf{q}_{(1)}, \mathbf{q}_{(2)} >$ between two arbitrary charge vectors in $\Gamma$ as

$$< \mathbf{q}_{(1)}, \mathbf{q}_{(2)} > = \frac{1}{2} \left( < \mathbf{q}_{(1)} + \mathbf{q}_{(2)}, \mathbf{q}_{(1)} + \mathbf{q}_{(2)} > - < \mathbf{q}_{(1)}, \mathbf{q}_{(1)} > - < \mathbf{q}_{(2)}, \mathbf{q}_{(2)} > \right) ,$$

we see that this is always an integer and conclude that $\Gamma$ is an integral lattice. Since there is at least one electron-like excitation $\mathbf{q}$ in $\Gamma$ with $q_{el}(\mathbf{q}) = 1$, (A3) also constrains the lattice $\Gamma$ to be odd because $< \mathbf{q}, \mathbf{q} > \equiv q_{el}(\mathbf{q}) \equiv 1 \mod 2$. For an arbitrary lattice basis $\{ \mathbf{q}_{(1)}, \ldots, \mathbf{q}_{(N)} \}$ of $\Gamma$, all the scalar products $Q_i = < \mathbf{Q}, \mathbf{q}_{(i)} >$, $i = 1, \ldots, N$, are integers, and hence the vector $\mathbf{Q}$ belongs to the dual, $\Gamma^*$, of $\Gamma$ (see (2.7)). The charge vector $\mathbf{Q}$ must necessarily also be primitive—i.e., $\gcd(Q_1, \ldots, Q_N) = 1$—in order for an excitation of electric charge 1 to exist.

Every vector $\mathbf{q}$ of the sublattice $\Gamma$ of $\Gamma_{phys}$ can now be consistently interpreted as a physical state describing a (multi-)electron and/or (multi-)hole configuration (depending on the integral value of the electric charge) excited from the ground state of the system. Note that, starting from an electron-like excitation $\mathbf{q}_{(1)}$ in $\Gamma$, one can form a full basis for $\Gamma$ consisting of electron vectors, $\mathbf{q}_{(i)}$, with $q_{el}(\mathbf{q}_{(i)}) = 1$, for $i = 1, \ldots, N$. We call such a basis a “symmetric” lattice basis.

The next step consists in finding restrictions on the lattice $\Gamma_{phys}$ by making use of our last assumption (A4). Consider, for each $i = 1, \ldots, N$, a physical state of
the system describing an excitation with “charges” \( q' \) localized in a disk \( D_1 \) and an excitation corresponding to an electron with vector \( q_{(i)} \) localized in a disk \( D_2 \) disjoint from \( D_1 \). Then we derive from assumption \((A4)\) and \((3.8)\) that

\[
\exp 2\pi i < q', q_{(i)} > = 1 , \tag{3.13}
\]

i.e., \( < q', q_{(i)} > \) must be an integer, for all \( i = 1, \ldots, N \). From this, it follows that \( q' \) belongs to the dual lattice \( \Gamma^* \). Thus, vectors \( q' \) of arbitrary physical excitations of a QH fluid described by the lattice \( \Gamma \) and the charge vector \( Q \) must all belong to the lattice \( \Gamma^* \) dual to \( \Gamma \). We then have the following inclusions between the various lattices introduced so far:

\[
\Gamma^* \supseteq \Gamma_{\text{phys}} \supseteq \Gamma . \tag{3.14}
\]

To discover how the data \((\Gamma, Q)\) predict the value of the Hall conductivity \( \sigma_H \), we consider the effect of turning on a perturbing external magnetic field \( B = B_{\text{total}} - B_c \) localized in a small disk \( D \) inside the system and creating a total magnetic flux \( \Phi \), with

\[
\Phi = \int_D B(\vec{x}) \, d^2x , \tag{3.15}
\]

where \( B \) is the component of \( B \) perpendicular to the plane of the system. Since the total electric current density \( J_{el}^\mu \) is given by \( J_{el}^\mu = \sum_{i=1}^N Q_i j^{\mu i} \), of the current densities \( j^{\mu 1}, \ldots, j^{\mu N} \), the “flux” \( \varphi_i \) created by \( B \) is given by

\[
\varphi_i = Q_i \Phi , \tag{3.16}
\]

for \( i = 1, \ldots, N \). This follows from the way in which, in Chern-Simons theory, the currents, \( j^{\mu i} \), are coupled to an external gauge field. Let \( A_\mu = (A_0, \vec{A}) \) denote the electromagnetic vector potential of the external magnetic field \( B \). Then, from the fact that the total electric current \( J_{el}^\mu \) couples to the vector potential \( A_\mu \) through the term \( J_{el}^\mu A_\mu \), it follows that the currents \( j^{\mu i} \) are minimally coupled to the field \( Q_i A_\mu \). Thus the gauge fields \( a_{\mu i} \) appearing on the r.h.s. of \((3.4)\) are given by \( a_{\mu i} = Q_i A_\mu \) which implies \((3.16)\); see [16, 17, 19]. Since the charges \( q^i = \int_D j^0(\vec{x}, t) \, d^2x = \oint_{\partial D} \vec{b}(\vec{x}, t) \cdot d\vec{x} \), corresponding to the fluxes \( \varphi_k \), are equal to \( (C^{-1})^{ik} \varphi_k \), by the equations of motion \((3.4)\) of Chern-Simons theory, we conclude that the total excess electric charge created by the excess magnetic flux \( \Phi \) is given by

\[
q_{el}(\Phi) = \sum_{i=1}^N Q_i q^i = \sum_{i,k=1}^N Q_i (C^{-1})^{ik} \varphi_k
\]
\[
= \left( \sum_{i,k=1}^{N} Q_i (C^{-1})^{ik} Q_k \right) \Phi =: < Q, Q > \Phi .
\]  

(3.17)

Comparing this equation to the equations describing the electrodynamics of a QH fluid which have been described in \[14, 17\], in particular to the equation

\[
J^0_{el} = \sigma_H B ,
\]

(3.18)

we find that the coefficient of \(\Phi\) on the r.h.s. of (3.17) is the Hall conductivity, \(\sigma_H\), of the QH fluid. Thus, we arrive at the fundamental equation

\[
\sigma_H = < Q, Q > .
\]

(3.19)

Since, as shown above, \(Q\) belongs to the dual lattice \(\Gamma^*\), we have that

\[
Q = \sum_{i=1}^{N} Q_i \epsilon^i
\]

with \(Q_i \in \mathbb{Z}\), for \(i = 1, \ldots, N\), where \(\{\epsilon^1, \ldots, \epsilon^N\}\) is the basis of \(\Gamma^*\) dual to a basis \(\{e_1, \ldots, e_N\}\) of the integral lattice \(\Gamma\), with Gram matrix \(K_{ij} = < e_i, e_j > \in \mathbb{Z}\). We could choose, for example, a symmetric basis of electron-like excitations, i.e., \(e_i = q_{(i)}\), for all \(i = 1, \ldots, N\). In this situation, the electric charge requirements fix the coefficients \(Q_i = 1\), for all \(i = 1, \ldots, N\), since we must have \(q_{el}(e_i) = < Q, e_i > = Q_i = 1\). But any other choice of basis is admissible as well. The all important consequence of \(Q\) being a vector in the dual of \(\Gamma\) is that its squared length \(< Q, Q >\) is a rational number. To see this, we return to the discussion below (2.7). We have that \(< \epsilon^i, \epsilon^j > = (K^{-1})^{ij} = (\tilde{K})^{ij}/\Delta\), where \(\Delta = \text{det} K\) (the lattice discriminant) is an integer, and \((\tilde{K})^{ij}\) are integers. It follows that

\[
<Q, Q> = \sum_{i,j=1}^{N} Q_i Q_j < \epsilon^i, \epsilon^j > = \left( \sum_{i,j=1}^{N} Q_i (\tilde{K})^{ij} Q_j \right) \Delta^{-1}
\]

(3.20)

is a rational number whose denominator is a divisor of \(\Delta\); (in general, there will be non-trivial common divisors of \(\Delta\) and of the numerator of the expression on the r.h.s. of (3.20)). We conclude that the Hall conductivity \(\sigma_H\) of a QH fluid satisfying assumptions (A1) through (A4) is necessarily a rational number.

The analysis just completed shows that, in the scaling limit, the physics of a QH fluid is described by a pair \((\Gamma, Q)\) of an integral, odd lattice \(\Gamma\) and a primitive vector \(Q\) in the dual lattice, with \(< Q, q > \equiv < q, q > \mod 2\), for all vectors \(q\) in \(\Gamma\), i.e., by data that we have termed QH lattice in Sect. 2.

In the following, we specialize to chiral QH lattices (CQHLs) which are QH lattices where the lattice \(\Gamma\) is euclidean. They describe (universality classes of) QH
fluids with edge currents of definite chirality, the basic charge carriers being, say, electrons. The theory of an electron-rich QH fluid differs from that of a hole-rich QH fluid only in relative minus signs in all equations relating charges to fluxes and involving the electric charge of the basic charge carriers. One simply reverses the sign of the charge vector \( Q \) in involving the electric charge of the basic charge carriers. The theory of an electron-rich QH fluid only in relative minus signs in all equations relating charges to fluxes and electrons. The theory of an electron-rich QH fluid differs from that of a hole-rich QH fluid.

In such QH fluids the subsystems described by \( \Gamma_e \) and \( \Gamma_h \) are independent of each other; (in particular, left- and right moving edge excitations are independent of each other). The mathematical properties of \( \Gamma_e \) and \( \Gamma_h \) are analogous. It is therefore sufficient to study the properties of \( \Gamma_e \), say, and we shall omit the subscript “e” henceforth. Note that this “factorized” situation implements assumption \( (\text{A5}) \) of Sect. 2.

As in (2.9) and (2.11), let \( \Gamma = \bigoplus_{j=1}^{r} \Gamma_j \) be the decomposition of the lattice \( \Gamma \) into an orthogonal direct sum of indecomposable sublattices \( \Gamma_j \), and let \( Q = \sum_{j=1}^{r} Q^{(j)} \), with \( Q^{(j)} \in \Gamma_j^* \), be the associated decomposition of the vector \( Q \) of electric charges. Then, by (3.19),

\[
\sigma_H = \langle Q, Q \rangle = \sum_{j=1}^{r} \langle Q^{(j)}, Q^{(j)} \rangle = \sum_{j=1}^{r} \sigma_{H}^{(j)} \tag{3.22}
\]

is the corresponding decomposition of the Hall conductivity (or Hall fraction) as a sum of Hall fractions of subfluids described by the pairs \( (\Gamma_j, Q^{(j)}) \), \( j = 1, \ldots, r \). Let us imagine that, for some \( j \), \( Q^{(j)} = 0 \). Then \( \sigma_{H}^{(j)} = 0 \), and the subfluid corresponding to \( (\Gamma_j, Q^{(j)}) \) does not have any interesting electric properties. For the purpose of describing electric properties of QH fluids and classifying the possible values of the Hall fraction \( \sigma_H \) of QH fluids, subfluids described by \( (\Gamma_j, Q^{(j)}) \), with \( Q^{(j)} = 0 \), can therefore be discarded. Thus, we may assume henceforth that \( Q^{(j)} \neq 0 \), for all \( j = 1, \ldots, r \), i.e., we may limit our analysis to proper CQHLS; see (2.11). (However, if there are QH fluids with spin-charge separation, subfluids \( (\Gamma_j, Q^{(j)}) \), with \( Q^{(j)} = 0 \), will appear, and spin currents could receive contributions from such neutral subfluids; see [17].)

Next, suppose that, for some \( j \) with \( 1 \leq j \leq r \), \( Q^{(j)} \) is an integer multiple of a primitive vector \( Q^{(j)}_h \in \Gamma_j^* \), i.e., \( Q^{(j)} = \nu_j Q^{(j)}_h \), with \( \nu_j \geq 2 \); see (2.8). Then, for any \( q \in \Gamma_j \), \( \langle Q^{(j)}, q \rangle \) is an integer multiple of \( \nu_j \). This would mean that the electric charge of an arbitrary quasi-particle of the subfluid described by \( (\Gamma_j, Q^{(j)}) \) would be an integer multiple of \( \nu_j \) (in units where \( -e = 1 \)), i.e., only bound states of electrons and holes of electric charge \( n\nu_j \), \( n \in \mathbb{Z} \), would appear as quasi-particles of such a subfluid. There appear to exist QH fluids where this situation arises (e.g. “hierarchy
QH fluids”, see Appendix E of [20], or films of superfluid He$^3$, see [17]). For simplicity, we shall, however, assume henceforth that $\nu_j = 1$, i.e., that $Q^{(j)}$ is a primitive vector in $\Gamma^*_j$, for each $j = 1, \ldots, r$. This means that we limit our analysis of CQHLs to what we call primitive CQHLs; see the definition after (2.11).

4 Basic Invariants and Elementary Mathematical Properties of Chiral QH Lattices

Let $(\Gamma, Q)$ be a chiral QH lattice (CQHL), i.e., a QH lattice describing (a universality class of) QH fluids with edge currents of a definite chirality. In this section, we describe elementary mathematical properties of $(\Gamma, Q)$. This is conveniently done in terms of invariants of $(\Gamma, Q)$. Numerical invariants of a CQHL, $(\Gamma, Q)$, are numbers which only depend on its intrinsic properties. They are independent of the choice of a basis in $\Gamma$ and of a “reshuffling” of electric charge assignments corresponding to a transformation of $Q$ by an orthogonal symmetry of $\Gamma$.

The two most elementary invariants of an integral lattice $\Gamma$ are its dimension $N$ and its discriminant $\Delta = |\Gamma^*/\Gamma|$; see Sect. 2.

What do we know about the values of $N$ and $\Delta$ occurring for a CQHL that describes a real QH fluid with impurities?

The honest answer is: not much! Let $E_e$ denote the average energy per electron in the ground state of a QH fluid. Let $n_e \equiv n$ denote the electron density, $n_I$ the density of impurities, and $E_I$ the average potential energy corresponding to a single impurity. Then we can form the dimensionless quantity

$$\alpha = E_e / (E_I \cdot n_I / n_e) ,$$

and, at some fixed value of the filling factor $\nu$, the maximal value, $N_*$, of $N$ is an increasing function of $\alpha$; ($N_* \propto \alpha$).

One may argue that the density and strength of impurities and the Wigner-lattice instability yield upper bounds on the discriminant $\Delta$ of a CQHL $(\Gamma, Q)$.

What, as physicists, we are longing for are invariants, $J$, of CQHLs with the property that if $(\Gamma, Q)$ is a CQHL corresponding to a real QH fluid the values of its invariants $J = J(\Gamma, Q)$ can either be constrained by experimental data or by safe theoretical arguments. Such invariants are, for example, the “relative-angular-momentum invariants”, $\ell_{\text{min}}$ and $\ell_{\text{max}}$, described in [13]. We first give a mathematical definition of these invariants and then explain what physical quantities they correspond to.

Let $(\Gamma, Q)$ be a primitive CQHL. Since $Q$ is a primitive vector of $\Gamma^*$, there is a basis, $\{q_1, \ldots, q_N\}$, of $\Gamma$ such that

$$q_{el}(q_i) = <Q, q_i> = 1 , \quad \text{for all } i = 1, \ldots, N .$$
The set of all such bases of $\Gamma$ is denoted by $\mathcal{B}_Q$. We define

$$L_{\text{min}}(\Gamma, Q) := \min_{q \in \Gamma, <q, q> = 1} <q, q> ,$$

(4.2)

and

$$L_{\text{max}}(\Gamma, Q) := \min_{\{q_1, \ldots, q_N\} \in \mathcal{B}_Q} \left( \max_{1 \leq i \leq N} <q_i, q_i> \right).$$

(4.3)

By (2.11), $L_{\text{min}}(\Gamma, Q)$ and $L_{\text{max}}(\Gamma, Q)$ are odd, positive integers, and $L_{\text{min}}(\Gamma, Q) \leq L_{\text{max}}(\Gamma, Q)$.

Suppose that $(\Gamma, Q)$ is decomposable (see (2.9)) and let

$$(\Gamma, Q) = \bigoplus_{j=1}^r (\Gamma_j, Q^{(j)})$$

be the decomposition of $(\Gamma, Q)$ into indecomposable CQHLs, $(\Gamma_j, Q^{(j)})$, $j = 1, \ldots, r$. Note that, since $(\Gamma, Q)$ has been assumed to be primitive (see Sect. 2, following (2.11)), each vector $Q^{(j)}$ is a (non-zero) primitive vector of the lattice $\Gamma_j^*$ dual to $\Gamma_j$, and every sublattice $\Gamma_j$ is odd, so that $(\Gamma_j, Q^{(j)})$ is an indecomposable CQHL. We define the relative-angular-momentum invariants, $\ell_{\text{min}}$ and $\ell_{\text{max}}$, by

$$\ell_{\text{min}}(\Gamma, Q) := \min_{1 \leq j \leq r} L_{\text{min}}(\Gamma_j, Q^{(j)}) \geq L_{\text{min}}(\Gamma, Q),$$

(4.4)

and

$$\ell_{\text{max}}(\Gamma, Q) := \max_{1 \leq j \leq r} L_{\text{max}}(\Gamma_j, Q^{(j)}) \geq L_{\text{max}}(\Gamma, Q).$$

(4.5)

We pause to explain the physical meaning of the invariants $L_{\text{min}}$ and $L_{\text{max}}$. For this purpose, we consider a state of the system where two quasi-particles, with quantum numbers corresponding to vectors $q_1$ and $q_2$ in $\Gamma_{\text{phys}}$ and localized near two points $\vec{x}_1 \neq \vec{x}_2$ in the plane of the system, are created from the ground state. From Chern-Simons theory and its relation to chiral conformal field theory it is known that the physical state vector, $\Psi = \Psi(\vec{x}_1, q_1; \vec{x}_2, q_2)$, is then given by

$$\Psi(\vec{x}_1, q_1; \vec{x}_2, q_2) = (z_1 - z_2)^{<q_1, q_2>} \Phi(\vec{x}_1, q_1; \vec{x}_2, q_2),$$

(4.6)

where $z = x + iy$ is the complex number corresponding to a point $\vec{x} = (x, y)$ in the plane of the system, and where $\Phi(\vec{x}_1, q_1; \vec{x}_2, q_2)$ is single-valued in $\vec{x}_1$ and $\vec{x}_2$. Let $L_z$ denote the component of the relative-angular-momentum operator along the axis perpendicular to the plane of the system. Then (4.4) implies that

$$L_z \Psi = <q_1, q_2> \Psi + (z_1 - z_2)^{<q_1, q_2>} L_z \Phi.$$
Since $\Phi$ is single-valued in $\vec{x}_1$ and $\vec{x}_2$, it follows from (4.7) that the possible eigenvalues of $L_z$ corresponding to states describing two quasi-particles with vectors $\vec{q}_1$ and $\vec{q}_2$ are given by

$$<\vec{q}_1, \vec{q}_2> + m, \quad \text{with} \quad m \in \mathbb{Z}.$$  

If the two quasi-particles are electrons with quantum numbers $\vec{q}_1 = \vec{q}_2 = \vec{q}$, where $\vec{q}$ is a point of the lattice $\Gamma$ with $q_{el}(\vec{q}) = <\vec{Q}, \vec{q}> = 1$, then, in states of low energy, $m$ is non-negative, and it follows that the relative angular momentum, $L$, of the state vector $\Psi(\vec{x}_1, \vec{q}; \vec{x}_2, \vec{q})$ is at least as large as $<\vec{q}, \vec{q}>$, i.e.,

$$L \geq <\vec{q}, \vec{q}>.$$  \hspace{1cm} (4.8)

Thus, by (4.2), $L_{min}(\Gamma, Q)$ is the smallest possible relative angular momentum of a state describing two electrons excited from the ground state in a QH fluid corresponding to the CQHL $(\Gamma, Q)$. For a QH fluid describing a system of non-interacting electrons filling the lowest Landau level, the Hall conductivity $\sigma_H$ is unity, and $L_{min} = L_{max} = 1$. For the basic Laughlin fluid [27], we have that

$$\sigma_H = \frac{1}{3}, \quad \text{and} \quad L_{min} = L_{max} = 3.$$  

We shall prove below that, for an arbitrary QH fluid described by a CQHL $(\Gamma, Q)$,

$$L_{max}(\Gamma, Q) \geq L_{min}(\Gamma, Q) \geq <\vec{Q}, \vec{Q}>^{-1} = \sigma_H^{-1}(\Gamma, Q).$$  \hspace{1cm} (4.9)

Theoretical arguments and numerical simulations [34] suggest that the relative-angular-momentum invariant $L_{max}$ obeys a universal upper bound

$$L_{max} \leq \ell^*_s,$$  \hspace{1cm} (4.10)

with $\ell^*_s = 7$ or $9$. This can also be understood as follows: If $L_{max}$ were larger than $7$ or $9$, say, the density of electrons in the ground state of a QH fluid corresponding to such a value of the relative-angular-momentum invariant would be so small that the system could lower its energy if the electrons formed a $\textit{Wigner lattice}$ [35]. But a Wigner lattice is not an incompressible state.

By (4.9), the bound (4.10) implies a lower bound on the Hall fraction $\sigma_H$ of real QH fluids:

$$\sigma_H \geq \frac{1}{\ell^*_s} = \frac{1}{9} \text{ or } \frac{1}{7}.$$  \hspace{1cm} (4.11)
This lower bound is in agreement with experimental data; see Fig. 1.1.

Using Hadamard’s inequality for the determinant of the lattice Gram matrix (see [19, 36]), it is easy to prove that the discriminant $\Delta(\Gamma, Q) = |\Gamma^*/\Gamma|$ of a CQHL $(\Gamma, Q)$ corresponding to a QH fluid is bounded by

$$\Delta(\Gamma, Q) \leq L_{max}(\Gamma, Q)^N,$$  \hspace{1cm} (4.12)

where $N$ is the dimension of $\Gamma$ which can be bounded, in principle, in terms of the quantity $\alpha$ defined in (4.1), i.e., which remains finite for systems with a positive density of impurities of finite strength. Then (4.12) shows that the discriminant $\Delta$ can only take a finite (but possibly large) set of values. As discussed in [19], this implies that the problem of classifying all CQHL corresponding to physically realizable QH fluids is a finite problem.

Equipped with the invariants $N$, $\Delta$, $\ell_{min}$, and $\ell_{max}$ of CQHLs, we can begin to classify such lattices.

First, we prove the bound (4.9) on $\sigma_{H}^{-1}$ which holds for general CQHLs, not necessarily proper or primitive. The proof is an easy application of the Cauchy-Schwarz inequality,

$$\frac{<n, q>^2}{<n, n>} \leq <q, q>,$$

for arbitrary vectors $q$ and $n \neq 0$ in a vector space $V$. Setting $n = Q \in \Gamma^* \subset V$ and choosing $q$ to lie in $\Gamma$ we conclude that

$$<q, q> \geq <Q, q>^2 <Q, Q>^{-1} \geq <Q, q>^2 \sigma_{H}^{-1}(\Gamma, Q).$$ \hspace{1cm} (4.13)

Since, for any non-zero vector $Q$ in $\Gamma^*$, and any vector $q$ in $\Gamma$ with $q_{el}(q) = <Q, q> \neq 0$, we have that $<Q, q>^2 \geq 1$, (4.13) implies that

$$<q, q> \geq \sigma_{H}^{-1}(\Gamma, Q),$$ \hspace{1cm} (4.14)

for an arbitrary vector $q$ in $\Gamma$, with $<Q, q> \neq 0$. Recalling the definition (4.2) of the invariant $L_{min}(\Gamma, Q)$, we find that the bound (4.14) implies (4.9). Moreover, by (4.4) and (4.5), we have that

$$\ell_{max}(\Gamma, Q) \geq \ell_{min}(\Gamma, Q) \geq \sigma_{H}^{-1}(\Gamma, Q).$$ \hspace{1cm} (4.15)
In the following, we focus on the classification of CQHLs \((\Gamma, Q)\) with
\[
\sigma_\mathcal{H}(\Gamma, Q) = \langle Q, Q \rangle \leq 1 ;
\]
corresponding to a partially or fully filled lowest Landau level. We divide the half-open interval \((0, 1]\) into a sequence of subintervals, or “windows”, \(\Sigma_p\), where
\[
\Sigma_p := \{ \sigma_\mathcal{H} | \frac{1}{2p+1} \leq \sigma_\mathcal{H} < \frac{1}{2p-1} \} ,
\]
for \(p = 1, 2, \ldots\), and \(\Sigma_0 := \{ \sigma_\mathcal{H} = 1 \} \).

In each such window, we attempt to classify all those CQHLs which are \(L\)-minimal in the sense of the following definition.

**Definition.** A CQHL \((\Gamma, Q)\) with \(\sigma_\mathcal{H}(\Gamma, Q) = \langle Q, Q \rangle \in \Sigma_p\), \(p = 1, 2, \ldots\), is \(L\)-minimal if and only if it is primitive and
\[
\ell_{\text{max}}(\Gamma, Q) = 2p + 1 .
\]

Note that \(2p + 1\) is the smallest possible value of the invariant \(\ell_{\text{max}}\) that is compatible with the bound (4.15) when \(\sigma_\mathcal{H}\) belongs to the window \(\Sigma_p\). It then follows from (4.15) and (4.18) that, for an \(L\)-minimal CQHL \((\Gamma, Q)\) with \(\sigma_\mathcal{H} \in \Sigma_p\)
\[
\ell_{\text{max}}(\Gamma, Q) = \ell_{\text{min}}(\Gamma, Q) = L_{\text{max}}(\Gamma, Q) = L_{\text{min}}(\Gamma, Q) = 2p + 1 ,
\]
i.e., all relative-angular-momentum invariants take the smallest possible value compatible with the value of \(\sigma_\mathcal{H}\).

**Proposition.** Any proper CQHL \((\Gamma, Q)\) with \(\sigma_\mathcal{H}(\Gamma, Q) \in \Sigma_p\) and \(L_{\text{max}}(\Gamma, Q) = 2p + 1\), \(p = 1, 2, \ldots\), is a primitive CQHL with \(\ell_{\text{max}}(\Gamma, Q) = L_{\text{max}}(\Gamma, Q) = 2p + 1\), i.e., it is \(L\)-minimal. Moreover, if \(\sigma_\mathcal{H}(\Gamma, Q) < 2/3\), it is indecomposable.

**Proof.** The proposition is obviously true if the lattice \(\Gamma\) is indecomposable, since then it is necessarily primitive and, by definition (4.5), \(\ell_{\text{max}}(\Gamma, Q) = L_{\text{max}}(\Gamma, Q)\).

If the lattice is decomposable, i.e.,
\[
\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_r , \quad \text{and} \quad \Gamma^* = \Gamma_1^* \oplus \cdots \oplus \Gamma_r^* ,
\]
for some \(r > 1\), the vector \(Q \in \Gamma^*\) has a non-vanishing projection, \(Q |_{\Gamma_i^*}\), along each orthogonal summand \(\Gamma_i^*\) of the dual lattice \(\Gamma^*\), \(i = 1, \ldots, r\), since \((\Gamma, Q)\) is proper by
hypothesis (see (2.11)). The projections $Q |_{\Gamma^*_I}$, however, are not necessarily primitive vectors, i.e., there are strictly positive integers $q_i = \gcd(Q |_{\Gamma^*_I})$ and primitive vectors $Q^{(i)}$ in $\Gamma^*_I$ such that

$$Q = q_1 Q^{(1)} + \cdots + q_r Q^{(r)} ,$$

(4.21)

with $\gcd(q_1, \ldots, q_r) = 1$, since $Q$ is primitive.

By hypothesis, we have that $\sigma_H \in \Sigma_p$ and $L_{\max} = 2p + 1$; hence (4.9) gives

$$L_{\max}(\Gamma, Q) = L_{\min}(\Gamma, Q) = 2p + 1 ,$$

which implies that there is a basis for $\Gamma$ made of vectors $e$ satisfying

$$<e, e> = 2p + 1 ,$$

(4.22)

and, for the Hall fraction $\sigma_H$, the decomposition (4.21) implies

$$\sigma_H(\Gamma, Q) = <Q, Q> = \sum_{i=1}^{r} q_i^2 \sigma^{(i)} ,$$

with

$$\sigma^{(i)} := <Q^{(i)}, Q^{(i)}> (> 0) , \quad i = 1, \ldots, r .$$

(4.25)

In order to prove the first part of the proposition ($(\Gamma, Q)$ is primitive) we have to show that all $q_i$ in (4.21) equal unity.

Since the electric charge of any one of the above basis vectors $e$ equals unity, $q_{el}(e) = <Q, e> = 1$, at least one of its projections, say $e^{(I)} \in \Gamma_I$, must carry an odd charge, i.e., $<Q, e^{(I)}> = q_I <Q^{(I)}, e^{(I)}> > 0$ is an odd integer. Hence $q_I$ and $<Q^{(I)}, e^{(I)}> > 0$ are both odd, and $<e^{(I)}, e^{(I)}> > 0$ must be odd, too, by (2.10). Thus, we have found a vector $e^{(I)}$ in the euclidean lattice $\Gamma_I \subset \Gamma$ whose squared length satisfies

$$<e^{(I)}, e^{(I)}> \leq <e, e> = 2p + 1 ,$$

(4.26)
(equality holding only if \(e^{(I)} = e\)), and whose charge \(q_{\mu}(e^{(I)}) = \langle Q^{(I)}, e^{(I)} \rangle\) is non-vanishing and odd.

Next, given the decomposition (4.25) and the assumption that \(\sigma_H \in \Sigma_p\), the value of \(\sigma^{(I)} = \langle Q^{(I)}, Q^{(I)} \rangle\) is bounded, and we have
\[
\sigma^{-1}_{(I)} > q_I^2 (2p - 1) ,
\]
the inequality being strict because \(r > 1\) and no summand \(q_i^2 \sigma_{(i)}\) is vanishing in (4.25). Inequality (4.27) and the Cauchy-Schwarz inequality (4.13) for the pair \((\Gamma_I, Q^{(I)})\) then imply
\[
\langle e^{(I)}, e^{(I)} \rangle \geq \langle Q^{(I)}, e^{(I)} \rangle^2 \sigma^{-1}_{(I)} > q_I^2 (2p - 1) .
\]
This inequality, however, contradicts inequality (4.26) unless
\[
q_I = 1 , \quad \text{and} \quad e^{(I)} = e .
\]
Hence the basis vector \(e\) necessarily lies entirely in one orthogonal summand of the decomposition (4.20); and since there is a basis of \(\Gamma\) of such vectors \(e\), we conclude that in (4.21)
\[
q_i = 1 , \quad \text{for all} \quad i = 1, \ldots, r .
\]
This proves that \((\Gamma, Q)\) is primitive. Furthermore, for the relative-angular-momentum invariant \(\ell_{\text{max}}(\Gamma, Q)\) (see (4.5)), the above reasoning implies that
\[
\ell_{\text{max}}(\Gamma, Q) = L_{\text{max}}(\Gamma, Q) = 2p + 1 ,
\]
i.e., the CQHL \((\Gamma, Q)\) is \(L\)-minimal.

We can now easily prove the second part of the proposition about the indecomposability of \(L\)-minimal CQHLs with \(\sigma_H < 2/3\). Let \((\Gamma, Q)\) be such a lattice with \(\sigma_H \in \Sigma_p\), \(p = 1, 2, \ldots\). Assuming \(r > 1\) in (4.20), we have proven that, for each summand \(\Gamma^{(j)}\) of \(\Gamma\), \(j = 1, \ldots, r\), there is a basis, \(\{e_1^{(j)}, \ldots, e_s^{(j)}\}\), of minimal-length vectors with charge 1, i.e.,
\[
\langle e_k^{(j)}, e_k^{(j)} \rangle = 2p + 1 , \quad \text{and} \quad \langle Q^{(j)}, e_k^{(j)} \rangle = 1 , \quad k_j = 1, \ldots, s_j . \quad (4.28)
\]
Moreover, for the Hall fraction \(\sigma_H\), we have that
\[
\frac{1}{2p+1} \leq \sigma_{H}(\Gamma, Q) = \sigma_{(1)} + \cdots + \sigma_{(r)} < \min \left\{ \frac{1}{2p-1}, \frac{2}{3} \right\}.
\]

Then, by the assumption that \( r > 1 \), at least one component, say, \((\Gamma_{I}, Q^{(I)})\) satisfies
\[
\sigma_{(I)} = <Q^{(I)}, Q^{(I)}> < \min \left\{ \frac{1}{2} \frac{1}{2p-1}, \frac{1}{3} \right\}, \quad \text{(4.29)}
\]
since \( \sigma_{(j)} \neq 0 \), for all \( j = 1, \ldots, r \). Applying again the Cauchy-Schwarz inequality \((\ref{4.13})\) to the pair \((\Gamma_{I}, Q^{(I)})\), we find, with \((\ref{4.29})\), that
\[
<e_{ki}^{(I)}, e_{kj}^{(I)}> \geq \sigma_{(I)}^{-1} > \max \left\{ 4p - 2, 3 \right\}, \quad \text{(4.30)}
\]
which contradicts the equality in \((\ref{4.28})\). Hence, for an \( L \)-minimal CQHL \((\Gamma, Q)\), \( \sigma_{H}(\Gamma, Q) < 2/3 \) implies \( r = 1 \), which means that the lattice \( \Gamma \) is indecomposable. \(\blacksquare\)

Note that the above bound for the value of \( \sigma_{H} \) of indecomposable \( L \)-minimal CQHLs is optimal: It is very easy to construct a composite \( L \)-minimal CQHL at the threshold value \( \sigma_{H} = 2/3 \). E.g., one takes the direct sum, \( \Gamma = \Gamma_{1} \oplus \Gamma_{2} \), and \( Q = Q^{(1)} + Q^{(2)} \), of two one-dimensional CQHLs with \( \sigma_{(1)} = \sigma_{(2)} = 1/3 \), i.e., \( \Gamma_{i} = \sqrt{3} \mathbb{Z} \) is generated by \( e^{(i)} \) with squared length \( <e^{(i)}, e^{(i)}> = 3 \), and \( Q^{(i)} = e^{(i)} = \frac{1}{3} e^{(i)} \) is the dual basis vector in \( \Gamma_{i}^{*}, i = 1, 2; \sigma_{H} = \sigma_{(1)} + \sigma_{(2)} = 2/3 \), and the lattice \( \Gamma \) is primitive and \( L \)-minimal. The two one-dimensional CQHLs \((\Gamma_{i}, Q^{(i)}), i = 1, 2, \) correspond to the basic Laughlin fluid \([27]\) at \( \sigma_{H} = 1/3 \).

In order to state some powerful general classification results on \( L \)-minimal CQHLs with \( \sigma_{H} < 1 \) (Thms. 4.1 and 4.2 below), we introduce a family of maps \( S_{p}, p = 1, 2, \ldots, \) between CQHLs of equal dimension. As we will see shortly, acting with \( S_{p} \) on a CQHL \((\Gamma, Q)\) simply shifts the inverse Hall fraction \( \sigma_{H}^{-1}(\Gamma, Q) \) and the relative-angular-momentum invariants \( L_{\text{max}}(\Gamma, Q) \) and \( L_{\text{min}}(\Gamma, Q) \) by a common even integer \( 2p \).

**Definition.** For any positive integer \( p \), the shift map \( S_{p} \) is a map between proper CQHLs of equal dimensions, \( S_{p} : (\Gamma, Q) \mapsto (\Gamma', Q') \). From an arbitrary basis \( \{f_{1}, \ldots, f_{N}\} \) of \((\Gamma, Q)\), a basis \( \{f'_{1}, \ldots, f'_{N}\} \) and a charge vector \( Q' \) of the image \((\Gamma', Q')\) is constructed such that
\[
<f'_{i}, f'_{j}> = <f_{i}, f_{j}> + 2p <f_{i}, Q><Q, f_{j}> , \quad \text{(4.31)}
\]
and
\[
<Q', f'_{j}> = <Q, f_{j}> . \quad \text{(4.32)}
\]

To see that this definition makes sense, i.e., that \((\Gamma', Q')\) is a well-defined proper CQHL, take any so-called “normal” basis \( \{q, e_{2}, \ldots, e_{N}\} \) of \( \Gamma \) which is characterized
by \( \langle Q, q \rangle = 1 \) and \( \langle Q, e_i \rangle = 0 \), for \( i = 2, \ldots, N \). In such a basis, the map \( S_p \) “shifts” the squared length of the vector \( q \) by the even integer \( 2p \),

\[
\langle q', q' \rangle = \langle q, q \rangle + 2p ,
\]

and the neutral sublattice,

\[
\Gamma_0(\Gamma, Q) := \{ v \in \Gamma \mid \langle Q, v \rangle = 0 \} ,
\]

(4.33)

(generated by the vectors \( e_i \), \( i = 2, \ldots, N \)) is left unchanged, i.e.,

\[
\Gamma_0(\Gamma', Q') = \Gamma_0(\Gamma, Q) .
\]

(4.34)

Given (4.34), and defining \( \gamma(\Gamma, Q) := \det(K |_{\Gamma_0}) = \tilde{K}_{11} (> 0) \), where \( \tilde{K} \) is the cofactor matrix of the Gram matrix \( K \) of the lattice \( \Gamma \) which is associated to the normal basis above, one finds that

\[
\gamma(\Gamma', Q') = \gamma(\Gamma, Q) .
\]

(4.35)

Furthermore, computing the lattice discriminant \( \Delta(\Gamma', Q') := \det(K') \) of \( \Gamma' \) generated by \( \{ q', e_2, \ldots, e_N \} \), we find that

\[
\Delta(\Gamma', Q') = \Delta(\Gamma, Q) + 2p \gamma(\Gamma, Q) .
\]

(4.36)

Eqs. (4.35) and (4.36) show that \( \Gamma' \) is a euclidean lattice which, by (4.31), is odd. Primitivity of \( Q' \) results from that of \( Q \) through (4.32). Moreover, since any improper orthogonal component of a CQHL necessarily lies in its neutral sublattice (see (2.11)), the image \( (\Gamma', Q') \) is a proper CQHL if \( (\Gamma, Q) \) is proper.

Since the Hall fraction \( \sigma_H \) can be written as

\[
\sigma_H(\Gamma, Q) = \langle Q, Q \rangle = (K^{-1})_{11} = \frac{\tilde{K}_{11}}{\Delta(\Gamma, Q)} = \frac{\gamma(\Gamma, Q)}{\Delta(\Gamma, Q)} ,
\]

(4.37)

the change in \( \sigma_H \), when acting with the shift map \( S_p \), is given by

\[
\sigma_H^{-1}(\Gamma', Q') = \sigma_H^{-1}(\Gamma, Q) + 2p .
\]

(4.38)

(Here the reader may recognize the “D-operation” (or “first move”) of the Jain-Goldman hierarchy scheme \[31, 37\]). Note that (4.38) implies that all CQHLs \( (\Gamma', Q') \) obtained through the action of shift maps \( S_p \), \( p = 1, 2, \ldots \), necessarily have
\[ \sigma_H (\Gamma', Q') < \frac{1}{2p} \]  \hspace{1cm} (4.39) 

In (4.37), the quantities \( \gamma \) and \( \Delta \) are obviously not necessarily coprime integers. Thus, defining by \( l(\Gamma, Q) := \gcd(\Delta, \gamma) \) the “level” of a CQHL \((\Gamma, Q)\) (see [19]), and denoting by \( n_H \) and \( d_H \) the numerator and denominator, respectively, of the Hall fraction, i.e., \( \sigma_H = n_H / d_H \), one has \( n_H = \gamma / l \), and \( d_H = \Delta / l \). Applying the shift map \( S_p \), we find from (4.35) and (4.36) that \( l(\Gamma', Q') = \gcd(\Delta', \gamma') = \gcd(\Delta + 2p \gamma, \gamma) = \gcd(\Delta, \gamma) = l(\Gamma, Q) \), and hence

\[ n_H (\Gamma', Q') = n_H (\Gamma, Q) , \quad \text{and} \quad d_H (\Gamma', Q') = d_H (\Gamma, Q) + 2p n_H (\Gamma, Q) . \]  \hspace{1cm} (4.40)

Starting again from a normal basis, the change of the relative-angular-momentum invariants \( L_{\min} \) and \( L_{\max} \), defined in (4.2) and (4.3), is given by simply shifting them by \( 2p \),

\[ L_{\min} (\Gamma', Q') = L_{\min} (\Gamma, Q) + 2p , \quad \text{and} \quad L_{\max} (\Gamma', Q') = L_{\max} (\Gamma, Q) + 2p . \]  \hspace{1cm} (4.41)

We emphasize that this simple transformation rule does not hold, in general, for the invariants \( \ell_{\min} \) and \( \ell_{\max} \) defined in (4.4) and (4.5). This fact points to a basic pitfall one has to avoid when using the shift maps \( S_p \) in the classification of CQHLs: the maps \( S_p \) do not necessarily preserve the decomposability properties of CQHLs. Moreover, they do not, in general, preserve the primitivity property that we have required for physically relevant composite CQHLs (see the end of Sect. 2); a CQHL that is not primitive can have as its image under \( S_p \) a primitive CQHL.

We illustrate this situation by a pair of CQHLs in two dimensions for which we give the corresponding Gram matrices, \( K_{ij}^{(\ell)} = \langle q_i^{(\ell)}, q_j^{(\ell)} \rangle \), in symmetric bases, \( \{ q_1^{(\ell)}, q_2^{(\ell)} \} \), where \( \langle Q^{(\ell)}, q_i^{(\ell)} \rangle = 1 \), for \( i = 1, 2 \):

\[ K = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \]  \hspace{1cm} \text{at} \quad \sigma_H = 3 \quad \overset{S_1}{\rightarrow} \quad K' = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \]  \hspace{1cm} \text{at} \quad \sigma_H = \frac{3}{7} . \]  \hspace{1cm} (4.42)

One then easily checks that the preimage lattice described by \( K \) is decomposable according to \( \Gamma = \Gamma_1 \oplus \Gamma_2 \simeq \mathbb{Z} \oplus \sqrt{2} \mathbb{Z} \), where the first summand is generated by \( e^{(1)} \) with \( \langle e^{(1)}, e^{(1)} \rangle = 1 \), and the second one by \( e^{(2)} \) with \( \langle e^{(2)}, e^{(2)} \rangle = 2 \). Moreover, the decomposition of the charge vector \( Q \) corresponding to \( \Gamma^* = \Gamma_1^* \oplus \Gamma_2^* \) reads \( Q = \varepsilon^{(1)} + 2 \varepsilon^{(2)} \), where the dual basis in \( \Gamma^* \) is given by \( \varepsilon^{(1)} = e^{(1)} \) and \( \varepsilon^{(2)} = e^{(2)}/2 \). Thus, for the restriction of \( Q \) to the second summand, \( Q \mid_{\Gamma_2^*} = 2 \varepsilon^{(2)} \), we find that \( \gcd(Q \mid_{\Gamma_2}) = \gcd(2 \varepsilon^{(2)}, e^{(2)}) = 2 \), and the composite CQHL \((\Gamma, Q)\) is not primitive.
Physically, the second summand in the decomposition of \((\Gamma, Q)\) corresponds to a QH subfluid at \(\sigma_H = 2\) that consists of a bosonic charge-2 condensate in which there are no elementary electrons. Finally, the image CQHL \((\Gamma', Q')\) specified by \(K'\) in (4.42) can be shown to be indecomposable and hence primitive.

From the discussion above it follows that the shift maps \(S_p, p = 1, 2, \ldots\), are injective on the set of proper CQHLs and that they are surjective onto the subsets of proper CQHLs with \(\sigma_H < 1/(2p)\). Hence they establish interesting bijections between subsets of proper CQHLs at corresponding “shifted” values of the inverse Hall fraction \(\sigma_H^{-1}\) (see (4.38)) and of the relative-angular-momentum invariants \(L_{\text{min}}\) and \(L_{\text{max}}\) (see (4.41)).

Although such bijections do not hold, in general, on the physically relevant restricted subset of primitive CQHLs, they hold again on the more restricted subset of \(L_{\text{-minimal}}\) CQHLs (see (4.18)), and two powerful, general classification results can be established there.

We define the following classes of \(L_{\text{-minimal}}\) CQHLs with Hall fractions \(\sigma_H < 1\):

\[
\mathcal{H}_p := \{ (\Gamma, Q) \mid \sigma_H(\Gamma, Q) \in \Sigma_p \text{ and } L_{\text{-minimal}}, \text{i.e., } (\Gamma, Q) \text{ is primitive and } \ell_{\text{min}}(\Gamma, Q) = \ell_{\text{max}}(\Gamma, Q) = 2p + 1 \}, \quad p = 1, 2, \ldots ,
\]  

where the windows \(\Sigma_p\) of Hall fractions in the interval \((0, 1]\) have been defined in (4.17).

Recalling (i) the proposition stating that the set of \(L_{\text{-minimal}}\) CQHLs coincides with the set of all proper CQHLs with minimal value of \(L_{\text{max}}\) consistent with the value of the Hall fraction \(\sigma_H\) (see (4.9)), and (ii) the “shifting” properties of the invariants \(L_{\text{max}}\) and \(\sigma_H\) under the shift maps \(S_p, p = 1, 2, \ldots\) (see (4.41) and (4.38)), we are led to the following structural result.

**Bijection Theorem.** The sets \(\mathcal{H}_p, p = 2, 3, \ldots\), of \(L_{\text{-minimal}}\) CQHLs with \(\sigma_H \in \Sigma_p\) are in one-to-one correspondence with the set \(\mathcal{H}_1\). The corresponding bijections are realized by the shift maps \(S_{p-1} : \mathcal{H}_1 \rightarrow \mathcal{H}_p\).

Our second result goes much beyond this bijection theorem: namely, on half of each window \(\Sigma_p\), the class \(\mathcal{H}_p, p = 1, 2, \ldots\), can be determined completely. Let each window \(\Sigma_p\) can be split into two subwindows by its mid value \(1/(2p)\), i.e., we define

\[
\Sigma^+_p := \{ \sigma_H \mid \frac{1}{2p+1} \leq \sigma_H < \frac{1}{2p} \}, \quad (4.44)
\]

and

\[
\Sigma^-_p := \{ \sigma_H \mid \frac{1}{2p} \leq \sigma_H < \frac{1}{2p-1} \}, \quad p = 1, 2, \ldots . \quad (4.45)
\]

**Uniqueness Theorem.** The sets \(\mathcal{H}^+_p \subset \mathcal{H}_p\) of all \(L_{\text{-minimal}}\) CQHLs with \(\sigma_H \in \Sigma^+_p, p = 1, 2, \ldots\), coincide with the infinite series \((N = 1, 2, \ldots)\) of indecomposable,
$N$-dimensional, maximally symmetric CQHLs with $SU(N)$-symmetry of $N$-ality 1, meaning that the elementary charge-1 fermions (electrons) described by these CQHLs transform under the fundamental representation of $SU(N)$. The corresponding Hall fractions are

$$\sigma_h = \frac{N}{2pN + 1}, \quad p, N = 1, 2, \ldots \; . \; \; (4.46)$$

(In the notation of [19, 18, 20],

$$\mathcal{H}_p^+ = \{ (2p + 1)^{A_{N-1}} \mid N = 1, 2, \ldots \}, \quad \text{for} \quad p = 1, 2, \ldots \; . \; \; (4.47)$$

Before proving this theorem, we make a few remarks. First, note that each Hall fraction $\sigma_h = N/(2pN + 1)$, $p, N = 1, 2, \ldots$, that appears in (4.46) is realized by a unique element of these $su(N)$-series of $L$-minimal CQHLs.

Second, since the level $l$ of all the CQHLs given in the uniqueness theorem equals unity, there holds a charge-statistics relation for the quasi-particle excitations of the corresponding QH fluids; see [19] and [20].

Third, the above $su(N)$-series of $L$-minimal CQHLs have already been discussed in [15] and, following the approach of [22], they can be shown to describe the same states that have been proposed by the Haldane-Halperin hierarchy scheme [30] as well as by Jain’s scheme [37, 31] at the corresponding Hall fractions $\sigma_h$. More details on these equivalences are given in [20], in particular, in Appendix E. Furthermore, it has been shown in [15] (see also [17, 19]) that the excitations (quasi-particles) of the low-energy spectrum of the corresponding QH fluids carry a representation of the Kac-Moody algebra $\hat{su}(N)$ at level 1.

We now turn to the proof of the uniqueness theorem.

**Proof.** Let $(\Gamma', Q')$ be any CQHL in $\mathcal{H}_p^+$. Since it is $L$-minimal, we have that $\ell_{\max}(\Gamma', Q') = L_{\max}(\Gamma', Q') = 2p + 1$. Moreover, since $(\Gamma', Q')$ is proper and has $\sigma_h < 1/(2p)$, we can act with the inverse shift map $S_p^{-1} = S_{-p}$ on it and get as preimage a proper CQHL, $(\Gamma, Q)$, of equal dimension with $L_{\max}(\Gamma, Q) = 1$. This implies that there is a lattice basis, $\{e_1, \ldots, e_N\}$, of $\Gamma$ consisting of charge-1 vectors, $<Q, e_i> = 1$, for all $i = 1, \ldots, N$, with unit squared length, $<e_i, e_i> = 1$, for all $i = 1, \ldots, N$. Thus, the lattice $\Gamma$ of the preimage has to be the $N$-dimensional unit hypercubic lattice,

$$\Gamma \simeq \mathbb{Z}^N \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \; , \; \; (4.48)$$

where each $e_i$ generates a one-dimensional unit lattice $\Gamma_i \simeq \mathbb{Z}$, $i = 1, \ldots, N$.

The decomposition of the charge vector $Q$ corresponding to (4.48) can be written in the general form

$$Q = q_1 Q^{(1)} + \cdots + q_N Q^{(N)} \; ,$$

28
where none of the \( q_i := \gcd(Q_i) \), \( i = 1, \ldots, N \), vanishes by the condition that \( \Gamma' \), and hence \( \Gamma \), be proper, and each \( Q^{(i)} \) is a primitive vector in \( \Gamma_i \simeq \mathbb{Z}^* \simeq \mathbb{Z} \) (i.e., \( Q^{(i)} = \pm e^{(i)} \), \( i = 1, \ldots, N \); see also (1.21)). Next, we show that \( Q \) is actually completely fixed by the mere existence of a symmetric basis \( \{e_1, \ldots, e_N\} \) for \( \Gamma \). The fact that

\[ 1 = q_{el}(e_i) = \langle Q, e_i \rangle = q_i \langle Q^{(i)}, e_i \rangle , \quad \text{for all} \ i = 1, \ldots, N , \]

implies that \( q_i = 1 \) and \( Q^{(i)} = e_i \), for each \( i = 1, \ldots, N \). Hence

\[ Q = Q^{(i)} + \cdots + Q^{(N)} = e_1 + \cdots + e_N , \quad (4.49) \]

and

\[ \sigma_h(\Gamma, Q) = \langle Q, Q \rangle = 1 + \cdots + 1 = N . \quad (4.50) \]

Thus, by Eqs. (4.48)–(4.50), we conclude that the preimage \((\Gamma, Q)\) of the \( L \)-minimal CQHL \((\Gamma', Q')\) is completely fixed by its dimension \( N = 1, 2, \ldots \). Applying the shift map \( S_p \), \( p = 1, 2, \ldots \), to \((\Gamma, Q)\), we obtain the desired result of a unique \( N \)-dimensional CQHL \((\Gamma', Q')\) in \( H^+_p \) with Hall fraction \( \sigma_h = N/(2pN + 1) \), \( N, p = 1, 2, \ldots \). More explicitly, in a symmetric basis, \( \{f'_1, \ldots, f'_N\} \), with \( \langle Q', f'_i \rangle = 1 \), for \( i = 1, \ldots, N \), the Gram matrix \( K' \) of \( \Gamma' \) reads

\[ K'_{ij} := \langle f'_i, f'_j \rangle = \delta_{ij} + 2p . \quad (4.51) \]

Following the transformation steps given in [13], we can finally make the presence of the (global) \( SU(N) \)-symmetry exhibited by \((\Gamma', Q')\) and encoded in (4.51) more transparent. Choosing to a suitable normal basis \( \{q'_1, e'_2, \ldots, e'_N\} \), where \( \langle Q', q'_1 \rangle = 1 \), and \( \langle Q', e'_i \rangle = 0 \), for \( i = 2, \ldots, N \), the associated Gram matrix \( K'_n \) (which is equivalent to \( K' \) in (4.51)) reads

\[
K'_n = \begin{pmatrix}
2p + 1 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 \\
0 & 0 & -1 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2
\end{pmatrix} , \quad (4.52)
\]

The lower-right, \((N-1)\)-dimensional block is recognized to be the Cartan matrix of \( A_{N-1} \simeq su(N) \), the Lie algebra of the (global) symmetry group \( SU(N) \), i.e., the
neutral sublattice $\Gamma_0(\Gamma',Q')$ of the $L$-minimal CQHL $(\Gamma',Q')$ in $\mathcal{H}_p^+$ is isomorphic to the root lattice of $su(N)$. ■

5 Phenomenological Implications

The purpose of this section is to explore the phenomenological implications of the theoretical results presented in Sect. 4 and to confront them with experimental data. We have focused our attention on chiral QH lattices with Hall fractions $\sigma_H \leq 1$. We recall that such lattices describe QH fluids that have only electrons (or holes) as fundamental charge carriers and whose edge currents are of definite chirality. We emphasize that, when restricting considerations to CQHLs, we do not make use of the idea of “charge conjugation”, $\sigma_H = 1 - \sigma'_H$, in the discussion of QH fluids with $1/2 < \sigma_H < 1$. In this interval, charge conjugation is usually invoked in other approaches to the fractional QH effect; see [30, 37, 31]. There is no difficulty, within our framework, to go beyond the chirality assumption and to consider the general classification problem of mixed-chirality QH lattices including, e.g., those corresponding to the QH fluids proposed by the charge-conjugation picture; see [20]. The general problem, however, is exorbitantly involved, and conclusions lack the simplicity and transparency of the results obtained when restricting the analysis to the class of CQHLs. Experimentally, it would be interesting to test the chirality assumption by direct edge-current measurements, e.g., of the type reported in [38]; (which, by the way, are compatible with a purely chiral structure of the QH fluid at $\sigma_H = 2/3$ that has been studied there).

General Structuring Results

- The inequality $\ell_{max} \geq \sigma_H^{-1}$. The first important result of our analysis is that, for an arbitrary CQHL $(\Gamma, Q)$, the associated invariants $\sigma_H(\Gamma, Q)$ and $\ell_{max}(\Gamma, Q)$ satisfy the fundamental inequality $\ell_{max}(\Gamma, Q) \geq \sigma_H^{-1}(\Gamma, Q)$; see (4.4) and (4.15). This result implies a first organizing principle for CQHLs with $0 < \sigma_H \leq 1$. It offers a natural splitting of the interval $(0, 1]$ into successive windows, $\Sigma_p$, defined by $1/(2p+1) \leq \sigma_H < 1/(2p-1)$, $p = 1, 2, \ldots$. The relative-angular-momentum invariant $\ell_{max}$ characterizing an arbitrary CQHL with $\sigma_H \in \Sigma_p$ is thus greater or equal to $2p + 1$, $p = 1, 2, \ldots$. Adopting the heuristic stability principle of Sect. 1, CQHLs in successive windows $\Sigma_p$, for increasing values of $p$, are expected to describe QH fluids of decreasing stability.

Combining the fundamental inequality above with a (universal) physical upper bound $\ell_*$ on the relative-angular-momentum invariant $\ell_{max}$, we conclude that, physically, no chiral QH fluid can form with a Hall conductivity $\sigma_H < 1/\ell_*$. As mentioned in Sect. 4 (see (4.10)), theoretical and numerical arguments as well as the present-day experimental data (see Fig. 1) suggest that $\ell_* = 7$. In the sequel, we impose this bound.

Thus, for the classification of physically relevant CQHLs, we have to consider only the first three windows, $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$, because only there CQHLs with $\ell_{max} \leq 7$ can be found. In particular, in the third window $\Sigma_3$, any physically relevant CQHL
(Γ, Q) must have \(\ell_{\text{max}}(\Gamma, Q) = 7\) meaning that \((\Gamma, Q)\) has to be \(L\)-minimal, i.e., all relative-angular-momentum invariants take their smallest possible value; see (4.19).

- **Uniqueness theorem.** The uniqueness theorem of the previous section classifies all \(L\)-minimal CQHLs in the left halves, \(\Sigma_p^+\), of the windows \(\Sigma_p\), i.e., in the subintervals \(1/(2p+1) \leq \sigma_h < 1/(2p)\), \(p = 1, 2, \ldots\). A unique, \(N\)-dimensional, \(L\)-minimal CQHL is found at every Hall fraction \(\sigma_h = N/(2pN + 1)\), \(p, N = 1, 2, \ldots\). Given this uniqueness theorem and the bound \(\ell_\ast = 7\), we find the following remarkable result: The classification problem of physically relevant CQHLs can be completely solved in the small subwindow \(\Sigma_3^+ = [1/7, 1/6)\). In \(\Sigma_3^+\), the only fractions at which such lattices can be found are \(\sigma_h = N/(6N + 1)\), \(N = 1, 2, \ldots\), and the electrons of the corresponding unique, chiral QH fluids carry an \(SU(N)\)-symmetry.

In the introduction we have mentioned that any set of CQHLs satisfying upper bounds on their invariants \(\ell_{\text{max}}\) and \(N\) is finite. Here it is interesting to note that, independently of any upper bound, \(N_\ast\), on the dimension \(N\), every closed subinterval in \(\Sigma_3^+\) contains only a finite number of physically relevant \((\ell_{\text{max}} \leq 7)\) CQHLs. Presently we do not know to which extent this is also true in other (sub)windows. The relevance of this remark derives from the fact that it is rather difficult to find satisfactory bounds \(N_\ast\) on the dimension \(N\) of physically relevant QH lattices; see the discussion at the beginning of Sect. 4. Thus, by-passing the need for a bound on the dimension would be progress.

So far, the only indication of a QH fluid in \(\Sigma_3^+\) has been found at \(\sigma_h = 1/7\), corresponding to the first member of this series. It coincides with the Laughlin fluid at \(m = 7\); see Sect. 1. Further probing (although difficult experimentally) of the subwindow \(\Sigma_3^+\) would clearly be interesting!

- **\(L\)-minimality.** We wish to comment on the assumption of \(L\)-minimality that we require when trying to classify CQHLs that correspond to stable physical QH fluids. As discussed above, this assumption is strictly justified only in the window \(\Sigma_3\). In general, it is an implementation of the heuristic stability principle described in Sect. 1 stating that the lower the values of \(\ell_{\text{max}}\) and \(N\) of a CQHL, the more stable the corresponding QH fluid. Thus, the justification of the assumption of \(L\)-minimality in the other two windows, \(\Sigma_2\) and \(\Sigma_1\), is directly connected to the validity of this stability principle. Thanks to the precise predictions it yields, experimental tests can be proposed to settle its validity. Such tests are discussed in great detail in [20].

In order to formulate such tests, however, one has to go beyond the assumption of \(L\)-minimality in classifying CQHLs, which requires much more work and can be achieved, in full generality, only for small values of the bounds \(N_\ast\) and \(\ell_\ast\). In [20], we have fully classified all CQHLs with upper bounds on \(\ell_{\text{max}}\) and \(N\) given by \((\ell_\ast, N_\ast) = (7, 2)\), \((\ell_\ast, N_\ast) = (5, 3)\), and \((\ell_\ast, N_\ast) = (3, 4)\). We emphasize that the problem is bound to be very complicated, since it involves as an input the knowledge of the complete classification of integral lattices with given bounds on the lattice discriminant, and this is a very intricate mathematical problem (unsolved, in general, for euclidean lattices, as needed here). With more patience and computing skill one might extend the above
results slightly (at most, however, up to cases where $N = 6$, or 7) because, in low dimensions, complete lists of lattices can be computed \cite{39, 40}; see \cite{20}. Partial results (for small discriminants) derived from the lattice classification in \cite{41} have already been given in Table 2 of \cite{19}. Furthermore, we note that there is a natural subclass of CQHLs which generalize the ones of the uniqueness theorem. They exhibit “large symmetries” and, for that reason, have been called “maximally symmetric”. In \cite{20}, we have classified all $L$-minimal, maximally symmetric CQHLs with $\sigma_H \leq 1$, independent of any upper bound on the dimension $N$.

The upshot of the analysis in \cite{20} is that the assumption of $L$-minimality for physically relevant CQHLs is well supported by the presently available experimental data. In the sequel, we adopt it as our basic working hypothesis.

- **Bijection theorem.** The second main theorem that we have proven in the previous section, the bijection theorem, asserts that there are one-to-one correspondences between the sets of $L$-minimal CQHLs in the different windows $\Sigma_p$, $p = 1, 2, \ldots$. Although this theorem does not classify CQHLs, it is a powerful structuring device for $L$-minimal CQHLs with $\sigma_H \leq 1$. In particular, it reduces the classification of $L$-minimal CQHLs with Hall fractions in the entire interval $0 < \sigma_H < 1$ to the classification of such lattices in the “fundamental domain” $\Sigma_1 = [1/3, 1]$!

**Theoretical Implications versus Experimental Data**

In the remaining part of this section, we shall discuss explicit consequences of the uniqueness and bijection theorem, complement them with classification results given in \cite{20}, and confront our conclusions with experimental data, as given in Fig. 1.

- **The subwindows $\Sigma_p^+$.** Adopting the hypothesis of $L$-minimality, the uniqueness theorem tells us that, in the subwindows $\Sigma_p^+$, no (chiral) QH fluids can be found with Hall conductivities $\sigma_H \neq N/(2pN + 1)$, $N, p = 1, 2, \ldots$. Taking a look at Fig. 1, remarkable agreement between this theoretical prediction and experimental data is found: QH fluids have been observed at $N/(2N + 1)$, $N = 1, \ldots, 9$, in $\Sigma_1^+$; at $N/(4N + 1)$, $N = 1, 2$, and 3, in $\Sigma_2^+$; and, as already mentioned, at just one value of $N/(6N + 1)$, namely $N = 1$, in $\Sigma_3^+$. The reader has probably recognized these Hall fractions as the ones of the “basic Jain states” \cite{37}. We repeat that, following the arguments in \cite{22} and \cite{14}, one can show \cite{20} that, at the above fractions, the proposals of the hierarchy schemes \cite{30, 31} and of our $L$-minimal-CQHL scheme coincide. The additional insight our approach offers is that all these proposals have a unique status as $L$-minimal, chiral QH fluids!

A closer inspection of Fig. 1 shows that, in the subwindows $\Sigma_p^+$, there seems to be only one fraction, $\sigma_H = 4/11$, at which a weak signal of a QH fluid has been reported, and which does not belong to the set of fractions described by the uniqueness theorem. The corresponding experimental data (reported only once) are somewhat controversial; see \cite{32}. Theoretically, a QH fluid at $\sigma_H = 4/11$ is predicted by the Haldane-Halperin \cite{30} and the Jain-Goldman \cite{31} hierarchy scheme at low(!) “level” 2 and 3, respectively. These two proposals can be shown \cite{20} to belong to the
same universality class of QH fluids described by a non-$L$-minimal, two-dimensional (primitive) CQHL which, in some sense, provides the “simplest” example of a non-$L$-minimal CQHL. This fraction marks thus an interesting plateau value where further experiments might challenge the hierarchy schemes and/or our working hypothesis of $L$-minimality.

It has been emphasized in the literature (see [42]) that the absence in the data of Fig. 1 of a QH fluid at $\sigma_H = 5/13$ is quite remarkable. Indeed, this fraction is conspicuous by its absence from the list of observed Hall fractions (in single-layer systems) with denominator $d_H = 13$, which are $\sigma_H = 3/13, 4/13, 6/13, 7/13, 8/13, 9/13$. Theoretically, the Haldane-Halperin [30] and the Jain-Goldman [31] hierarchy scheme predict a QH fluid with $\sigma_H = 5/13$ at low(!) “level” 3 and 2, respectively. These two proposals correspond to a non-chiral QH lattice. We note that, in addition, there is an (inequivalent) chiral, but non-$L$-minimal QH lattice in three dimensions with $\sigma_H = 5/13$; see [20]. This fraction is thus another interesting plateau value where the hierarchy schemes and/or the $L$-minimality assumption can be tested further.

By a similar reasoning process, in the first subwindow $\Sigma_1^+$, all fractions in the open intervals $N/(2N + 1) < \sigma_H < (N + 1)/(2N + 3), N = 1, 2, \ldots$, are interesting plateau values for testing the $L$-minimality assumption. To be explicit, we do not expect stable QH fluids to form at $\sigma_H = 4/11(!), 5/13(!), 6/17, 7/19, 8/21, \ldots$ in $(1/3, 2/5)$, and at $\sigma_H = 7/17, 7/19, \ldots$ in $(2/5, 3/7)$. Note that, with the help of the shift maps $S_1$ and $S_2$ (see (4.38)), these predictions can be translated into predictions in the subwindows $\Sigma_2^+$ and $\Sigma_3^+$.

- The subwindows $\Sigma_p^-$, $p = 1, 2, \ldots$ . In the “complementary” subwindows, $\Sigma_p^-$, defined by $1/(2p) \leq \sigma_H < 1/(2p - 1), p = 1, 2, \ldots$, we do not have a complete classification of $L$-minimal CQHLs. Nevertheless, we can make interesting observations for these subwindows by exploiting the bijection theorem of the previous section and the (partial) classification results given in [24].

First, we note that the experimental data in $\Sigma_1^- = [1/2, 1)$ (see Fig. 1) can hardly be interpreted as a complete “mirror image” of the data in the interval $(0, 1/2]$, as one would expect if charge conjugation were at work in general. Second, comparing, the data in the two complementary subwindows $\Sigma_1^-$ and $\Sigma_1^+$, we find, besides the prominent series of fractions $\sigma_H = n/(2n - 1), n = 2, \ldots, 9$, “mirroring” the unique fractions in $\Sigma_1^+$ (i.e., $\sigma_H = 1 - \sigma_H'$), data points at $\sigma_H = 4/5, 5/7, 7/11, 8/11, 8/13, 9/13$, and possibly at 10/17. This is a first experimental indication that the sets of QH fluids appearing in the complementary subwindows $\Sigma_p^+$ and $\Sigma_p^-$, $p = 1, 2, \ldots$, are “structurally distinct”.

We may ask to which extent the experimental data in Fig. 1 also support the one-to-one correspondences predicted by the bijection theorem between QH fluids in the different subwindows $\Sigma_p^-$. We can act “formally” with the shift maps $S_{p-1}, p = 2$ and 3, of the bijection theorem on the fractions $\sigma_H$ given in $\Sigma_1^-$ of Fig. 1, e.g., $S_{p-1} : n/(2n - 1) \mapsto n/(2pn - 1)$; see (4.38). The resulting fractions $\sigma_H$ that we obtain in the two subwindows $\Sigma_2^-$ and $\Sigma_3^-$ are fully consistent with the experimental data given
in Fig. 1. Experimentally observed are the fractions \( \sigma_n = n/(4n - 1) \), \( n = 2, 3, 4 \), and \( \sigma_n = 4/13 \) (very weakly) in \( \Sigma^-_2 = [1/4, 1/3] \), and only one fraction in \( \Sigma^-_3 = [1/6, 1/5] \), namely \( \sigma_n = n/(6n - 1) \), with \( n = 2 \).

If the QH fluids appearing in \( \Sigma^-_1 \) were to correspond to \( L \)-minimal CQHLs then, by the logic of the bijection theorem, we would predict the formation of (chiral) QH fluids at \( \sigma_n = 4/13 \), 5/17, 5/19, \ldots in \( \Sigma^-_2 \), and at \( \sigma_n = 3/17, 4/21, \ldots \) in \( \Sigma^-_3 \). These are thus interesting plateau-values for experimentation.

What do we know explicitly about \( L \)-minimal CQHLs in the subwindows \( \Sigma^-_p \)? As mentioned above, the analysis presented in [20] contains, in particular, a complete classification of all low-dimensional \( (N \leq N_* = 4) \) and of all maximally symmetric, \( L \)-minimal CQHLs with \( \sigma_n \leq 1 \).

- **Summary of results presented in [20] for the fundamental subdomain \( \Sigma^-_1 \).** The upshot of the analysis given in [20] is that, in \( \Sigma^-_1 \), natural proposals for QH fluids at the fractions of the series \( \sigma_n = n/(2n - 1), \) \( n = 2, 3, \ldots \), are provided by the charge-conjugation picture, meaning that the corresponding QH fluids are composite. They consist of an electron-rich subfluid with a partial Hall fraction \( \sigma_{(1)} = 1 \), and of a hole-rich subfluid corresponding to an \( L \)-minimal CQHL of the \( su(N) \)-series in \( \Sigma^+_1 \) with partial Hall fraction \( \sigma_{(2)} = -N/(2N + 1) \), where \( N = n - 1 \). This is, however, not the full story! As we have already mentioned in the introduction, it is a structural property of the subwindows \( \Sigma^-_p \) that, at a given Hall fraction \( \sigma_n \), one typically finds more than one \( L \)-minimal CQHL realizing that fraction. (We recall that this is much in contrast to the unique realization of the fractions \( \sigma_n = N/(2N + 1) \) in the complementary subwindow \( \Sigma^+_1 \).)

For example, at the finite series of fractions \( \sigma_n = n/(2n - 1), \) \( n = 2, \ldots, 7 \), one finds maximally symmetric, \( L \)-minimal CQHLs in dimensions \( N = 10 - n \) which are based on the root lattices of the exceptional Lie algebras \( E_{9-n} \), similarly to the way in which the \( su(N) \)-QH lattices in \( \Sigma^+_1 \) are based on the root lattices of \( su(N) \); see the end of Sect. 4. While the last two members of this finite series, \( \sigma_n = 6/11 \) and \( 7/13 \), are realized by unique, low-dimensional \( (N = 4 \text{ and } 3, \text{ respectively}) \), \( L \)-minimal CQHLs, the higher dimensional members of this “E-series” of CQHLs contain several \( L \)-minimal, chiral QH sublattices of lower dimensions. All these sublattices (although not necessarily maximally symmetric) represent possible proposals for QH fluids at the corresponding Hall fractions \( \sigma_n \). They arise naturally from symmetry breaking patterns existing for exceptional Lie groups.

Physically, QH lattice embeddings can describe phase transitions at a given Hall fraction between “structurally different” QH fluids related by symmetry breaking. For example, in [20, Appendix D], we have found 13 and 5 QH sublattices embedded into the QH lattice of the E-series at \( \sigma_n = 2/3 (E_7) \) and \( 3/5 (E_6) \), respectively. The composite \( L \)-minimal CQHL at \( \sigma_n = 2/3 \) which consists of two Laughlin subfluids with partial Hall fraction \( \sigma_{(i)} = 1/3, \) \( i = 1, 2 \), is the lowest dimensional \( (N = 2) \) QH sublattice of the \( E_7 \)-QH lattice at \( \sigma_n = 2/3 \). All the other QH sublattices at \( \sigma_n = 2/3 \) are indecomposable. (Recall that we have proven in Sect. 4 that all \( L \)-minimal CQHLs
with \( \sigma_H < 2/3 \) are indecomposable.) In \( \Sigma^- \), complex embedding patterns of \( L \)-minimal CQHLs are found at the fractions \( \sigma_H = 4/7, 5/7, 5/9, \) and \( 1/2 \). These fractions are interesting in the light of the data in Fig. 1, where phase transitions are indicated at \( \sigma_H = 2/3, 3/5, \) and possibly at \( 5/7 \), driven by an added in-plain component of the external magnetic field (see [8, 10, 3]), and at \( \sigma_H = 2/3 \), driven by changing the density of charge carriers in the system (see [1]); see also the data reported in [11] on phase transitions in wide-single-quantum-well systems.

(At this point, we remark that by the uniqueness of the \( L \)-minimal CQHLs in the subwindows \( \Sigma_p^+ \) and our heuristic stability principle, we do not expect structural phase transitions there. Thus, what about a possible indication of a magnetic field driven phase transition at \( \sigma_H = 2/5 \)? As a matter of fact, in [17] (see also [20]), we have argued that \( \sigma_H = 2/5 \) is the most likely plateau value where we may expect a phase transition from a spin-polarized to a spin-singlet QH fluid. While, structurally, the two phases are described by one and the same \( L \)-minimal CQHL, the phase transition corresponds to a change from an internal \( SU(2) \)-symmetry to a spatial \( SU(2) \) spin-symmetry.)

We complete our short review of results derived in [20], see Fig. 1.2) by mentioning that to all data points in \( \Sigma_1^- \) (including the fraction \( \sigma_H = 1/2 \)) one can associate at least one \( L \)-minimal CQHL that is either generic (without special symmetry properties) and low-dimensional (\( N \leq 4 \)), maximally symmetric with dimension \( N \leq 9 \) (based on the root lattice of a simple or semi-simple Lie algebra), or charge-conjugated to an \( su(N) \)-lattice in \( \Sigma_1^+ \). Within these three subclasses of CQHLs, predictions of new QH fluids are made at \( \sigma_H = 6/7, 10/13, 10/17(!), 13/17, 10/19, 12/19, 14/19, \ldots \), and at the even-denominator fractions \( \sigma_H = 3/4 \) and \( 5/8 \). The CQHLs that yield even-denominator Hall fractions have a structure that can naturally be interpreted as describing double-layer QH systems. Furthermore, staying within these three subclasses, we do not expect stable(!) QH fluids to form at \( \sigma_H = 9/11, 11/17, 14/17, 13/19, \) and \( 15/19 \) in \( \Sigma_1 \), where we have omitted fractions with \( d_H \geq 21 \) and within the “domain of attraction” of the most stable Laughlin fluid at \( \sigma_H = 1 \). None of these fractions has been observed experimentally! These predictions are rather different from those of the standard hierarchy schemes [30, 31]; for further discussions, see [20].

- **Concluding remarks.** By the bijection theorem, all the statements about \( L \)-minimal chiral QH lattices in \( \Sigma_1^- = [1/2, 1] \) have their precise analogues in the shifted subwindows \( \Sigma_p^- = [1/(2p), 1/(2p - 1)] \), for \( p = 2, 3, \ldots \). For example, interpreting the phase transitions observed at \( \sigma_H = 2/3, 3/5, \) and possibly \( 5/7 \) in \( \Sigma_1^- \) as structural phase transitions, we predict analogous transitions at the Hall fractions \( \sigma_H = 2/7, 3/11, (5/17) \) in \( \Sigma_2^- \), and at \( \sigma_H = 2/11, 3/17, (5/27) \) in \( \Sigma_3^- \).

We remark that when acting with the shift maps \( S_p \) on the composite, mixed-chirality QH lattices at \( \sigma_H = n/(2n - 1), \) \( n = 2, 3, \ldots \), corresponding to the charge-conjugation picture, we obtain non-euclidean QH lattices which are not primitive, where primitivity has been defined at the end of Sect. 2. Thus, the corresponding images do not figure in the present paper which is restricted to primitive CQHLs.
A discussion of non-primitive QH lattices and the corresponding QH fluids has been given in [20, Appendix E]. In particular, in [20], we have described composite, non-euclidean (but “factorized”) QH lattices corresponding to the hierarchy fluids in the windows $\Sigma^-_p$, $p = 1, 2, 3$.

For the complementary subwindows $\Sigma^+_p = [1/(2p + 1), 1/(2p))$, the one-to-one correspondences between the different sets of $L$-minimal CQHLs are implicit in the classification result of the uniqueness theorem discussed above.

These results may suffice to convince the reader that there is a significant structural asymmetry between the sets of QH fluids with Hall conductivities in the two complementary subwindows $\Sigma^+_p$ and $\Sigma^-_p$, for a given $p = 1, 2, \ldots$, while there is a structural similarity between all the sets of QH fluids with conductivities in the “+”-windows $\Sigma^+_p$ and all the ones with conductivities in the “−”-windows $\Sigma^-_p$, for different values of $p$.

We conclude by noting that, after more then ten years since its discovery [1], the fractional QH effect in the interval $0 < \sigma_H \leq 1$ is still an interesting field of experimental and theoretical research. In the present paper and in [20], we have argued that the QH-lattice approach provides an efficient instrument for describing universal properties of QH fluids. In our analysis of physically relevant QH lattices we have assumed chirality and $L$-minimality as basic properties. New and refined experimental data in the neighbourhood of the various plateau-values discussed above would either support or question these hypotheses, and hence could lead to further progress in the understanding of this fascinating effect.

Acknowledgements

We wish to thank our colleagues Yosi Avron, Rudolf Morf, Duncan Haldane, and Paul Wiegmann for helpful discussions and encouragement. We also thank Mansour Shayegan for sending us a copy of [1] prior to publication. T.K. acknowledges partial support from the NSF (grant DMS-9305715), and U.M.S. acknowledges support from the Onderzoeksfonds K.U. Leuven (grant OT/92/9).

References

[1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980); D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. B 48, 1559 (1982); for reviews, see, e.g.: R.E. Prange and S.M. Gervin, eds., The Quantum Hall Effect, Second Edition, Graduate Texts in Contemporary Physics (Springer, New York, 1990);
T. Chakraborty and P. Pietiläinen, The Fractional Quantum Hall Effect: Properties of an Incompressible Quantum Fluid, Springer Series in Solid State Science 85 (Springer, Berlin, 1988);
M. Stone, ed., *Quantum Hall Effect* (World Scientific, Singapore, 1992).

[2] D.C. Tsui, Physica B **164**, 59 (1990).

[3] T. Sajoto, Y.W. Suen, L.W. Engel, M.B. Santos, and M. Shayegan, Phys. Rev. B **41**, 8449 (1990).

[4] H.L. Stormer, Physica B **177**, 401 (1992).

[5] V.J. Goldman and M. Shayegan, Surf. Sci. **229**, 10 (1990).

[6] R.R. Du, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer, and K.W. West, Phys. Rev. Lett. **70**, 2944 (1993).

[7] W. Kang, H.L. Stormer, L.N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **71**, 3850 (1993).

[8] R.G. Clark, S.R. Haynes, J.V. Branch, A.M. Suckling, P.A. Wright, P.M.W. Oswald, J.J. Harris, and C.T. Foxon, Surf. Sci. **229**, 25 (1990).

[9] J.P. Eisenstein, H.L. Stormer, L.N. Pfeiffer, and K.W. West, Phys. Rev. B **41**, 7910 (1990).

[10] L.W. Engel, S.W. Hwang, T. Sajoto, D.C. Tsui, and M. Shayegan, Phys. Rev. B **45**, 3418 (1992).

[11] Y.W. Suen, H.C. Manoharan, X. Ying, M.B. Santos and M. Shayegan, Surf. Sci. **305**, 13 (1994).

[12] R.L. Willett, J.P. Eisenstein, H.L. Stormer, D.C. Tsui, A.C. Gossard, and J.H. English, Phys. Rev. Lett. **59**, 1776 (1987); the first observations of fractional QH fluids with \( \sigma_H > 2 \) are given in this reference;

J.P. Eisenstein, R.L. Willett, H.L. Stormer, D.C. Tsui, A.C. Gossard, and J.H. English, Phys. Rev. Lett. **61**, 997 (1988);

J.P. Eisenstein, R.L. Willett, H.L. Stormer, L.N. Pfeiffer, and K.W. West, Surf. Sci. **229**, 31 (1990).

[13] Y.W. Suen, L.W. Engel, M.B. Santos, M. Shayegan, and D.C. Tsui, Phys. Rev. Lett. **68**, 1379 (1992);

J.P. Eisenstein, G.S. Boebinger, L.N. Pfeiffer, K.W. West, and Song He, Phys. Rev. Lett. **68**, 1383 (1992).

[14] J. Fröhlich and T. Kerler, Nucl. Phys. B **354**, 369 (1991).

[15] J. Fröhlich and A. Zee, Nucl. Phys. B **364**, 517 (1991).

[16] J. Fröhlich and U.M. Studer, Commun. Math. Phys. **148**, 553 (1992).
[17] J. Fröhlich and U.M. Studer, Rev. Mod. Phys 65, 733 (1993); see also:
J. Fröhlich and U.M. Studer, “Incompressible Quantum Fluids, Gauge-Invariance, and Current Algebra”, in Proc. of New Symmetry Principles in Quantum Field Theory, Cargèse, 1991, J. Fröhlich et al., eds., (Plenum Press, New York, 1992).

[18] J. Fröhlich, U.M. Studer and E. Thiran, “An ADE-O Classification of Minimal Incompressible Quantum Hall Fluids”, in Proc. of On Three Levels, Leuven, 1993, M. Fannes et al., eds., (Plenum Press, New York, 1994); see cond-mat/9406009.

[19] J. Fröhlich and E. Thiran, J. Stat. Phys. 76, 209 (1994).

[20] J. Fröhlich, U.M. Studer and E. Thiran, “A Classification of Quantum Hall Fluids”, preprint KUL-TF-94/35; see cond-mat/9503113.

[21] B.I. Halperin, Phys. Rev. B 25, 2185 (1982).

[22] N. Read, Phys. Rev. Lett. 65, 1502 (1990).

[23] M. Stone, Int. J. Mod. Phys. B 5, 509 (1991); Ann. Phys. (N.Y.) 207, 38 (1991).

[24] X.G. Wen, Phys. Rev. B 40, 7387 (1989); Phys. Rev. Lett. 64, 2206 (1990); Phys. Rev. B 41, 12838 (1990); Phys. Rev. B 43, 11025 (1991); Phys. Rev. Lett. 66, 802 (1991); Int. J. Mod. Phys. B 6, 1711 (1992).

[25] B. Block and X.G. Wen, Phys. Rev. B 42, 8133, and 8145 (1990).

[26] X.G. Wen and A. Zee, Phys. Rev. B 46, 2290 (1992).

[27] R.B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983); Phys. Rev. B 27, 3383 (1983).

[28] P. Goddard and D. Olive, Int. J. Mod. Phys. A1, 303 (1986).

[29] J. Fröhlich, R. Götschmann, and P.A. Marchetti, “Bosonization of Fermi Systems in Arbitrary Dimension in Terms of Gauge Forms”, preprint DFPD 94/TH/36.

[30] F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983); B.I. Halperin, ibid. 52, 1583 (1984).

[31] J.K. Jain and V.J. Goldman, Phys. Rev. B 45, 1255 (1992).

[32] Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959).

[33] J. M. Leinaas and J. Myrheim, Nuovo Cimento 37 B, 1 (1977); F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982); 49, 957 (1982); for a review see:
J. Fröhlich, “Quantum Statistics and Locality”, in Proc. of The Gibbs Symposium, Yale University, 1989, D.G. Caldi and G.D. Mostow, eds., (Am. Math. Soc., Providence, RI, 1990).

[34] K. Esfarjani and S.T. Chui, Phys. Rev. B 42, 10758 (1990);
U. Brockstieger and G. Meissner, Verhandlungen DPG (VI), vol. 26, 11042 (1991);
G. Meissner, “Goldstone Mode of a Two-dimensional Electron Solid in High Magnetic Fields”, in From Phase Transitions to Chaos: Topics in Modern Statistical Physics, G. Györgyi et al., eds., (World Scientific, Singapore, 1992);
G. Meissner, Physica B 184, 66 (1993).

[35] P.K. Lam and S.M. Girvin, Phys. Rev. B 30, 473 (1984);
D. Levesque, J.J. Weiss, and A.H. MacDonald, Phys. Rev. B 30, 1056 (1984);
B. McCombe and A. Nurmikko, eds., Electronic Properties of Two-Dimensional Systems (North-Holland, New York, 1994).

[36] C.L. Siegel, Lectures on the Geometry of Numbers, Springer-Verlag (Berlin, Heidelberg, 1989).

[37] J.K. Jain, Phys. Rev. Lett. 63, 199 (1989); Phys. Rev. B 41, 7653 (1990); Adv. Phys. 41, 105 (1992).

[38] R.C. Ashoori, H.L. Stormer, L.N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. B 45, 3894 (1992).

[39] J.H. Conway and N.J.A. Sloane, Sphere-packings, lattices and groups, Springer (New York, 1988).

[40] L.E. Dickson, Studies in the theory of numbers, University of Chicago Press (Chicago, 1930), reprinted by Chelsea Publishing Company (New York, 1957).

[41] J.H. Conway and N.J.A. Sloane, Proc. R. Soc. Lond. A 418, 17 (1988); and references therein.

[42] C.-R. Hu, Int. J. Mod. Phys. B 5, 1739 (1991).