Acceleration, streamlines and potential flows in general relativity: analytical and numerical results.

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Analytical and numerical solutions for the integral curves of the velocity field (streamlines) of a steady-state flow of an ideal fluid with $p = \rho$ equation of state are presented. The streamlines associated with an accelerate black hole and a rigid sphere are studied in some detail, as well as, the velocity fields of a black hole and a rigid sphere in an external dipolar field (constant acceleration field). In the latter case the dipole field is produced by an axially symmetric halo or shell of matter. For each case the fluid density is studied using contour lines. We found that the presence of acceleration is detected by these contour lines. As far as we know this is the first time that the integral curves of the velocity field for accelerate objects and related spacetimes are studied in general relativity.

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I. INTRODUCTION

The study of potential flows in general relativity is relevant in the understanding of several phenomena of interest in relativistic astrophysics like: fluid flows at relativistic speeds in the presence of a neutron star [1,2], star clusters moving in gaseous media [1], flows near a cosmic string [1], accretion in binary star systems and supermassive black holes [3], and others [4–6]. Also, numerical solutions of the equations of general relativistic hydrodynamics can simulate, and model, gravitational collapse and the evolution of neutron stars [7].

Most of the articles in this area deal with the important case of fluid motion evolving in the spacetime associated with compact stars and black holes. The implementation of new background metrics brings some new challenges. First, metrics other than Schwarzschild and Kerr are not so well studied, sometimes a complete understanding of the physical meaning of the metric is missing. Also, the solutions of the fluid equations in a non trivial metric may be quite involved. In particular, the search for significant boundary values (or initial conditions) presents a non trivial problem. A simple and paradigmatic case of a flow is the stationary-zero-vorticity flow of a fluid with adiabatic stiff equation of state. In this case, for relativistic flows, the fluid equations admit analytical solutions for some particular metrics [1,3]. These solutions are used as test-beds for testing almost all the numerical hydrodynamic codes in the subject. Other potential flows in a non stationary background and different equation of state have been studied, see for instance [2]. Also, the solutions for potential flows permit to test new optimized codes in resolving nonlinear hyperbolic systems of conservation laws [6], see for a representative sample of numerical schemes [7].

In this work we extend the investigations about potential flows by studying the streamlines of a steady-state ideal fluid in the presence of an accelerated black hole and rigid sphere, and of a black hole and rigid sphere with a dipole field produced by an axially symmetric halo of matter. This shell like structures are useful in modeling many situations of interest in astrophysics, as for example, the Supernova 1987A [8–10], for other applications see [11]. We assume that the fluid is a test fluid, i.e., the metric does not evolve and it is given a priori. The state equation and the idea of a rigid star (rigid sphere) we use are idealized. However, they bring important results about the instability and behaviour of the fluid, and the difficulties involved in this kind of scenarios.

This work is divided as follows. In Sec. II we present the basic equations that describe potential flows. In Sec. III we summarize some aspects of the Weyl C-metric that represents the spacetime associated with an uniformly accelerated black hole. In particular, we present the metric in different systems of coordinates to facilitate the physical interpretation of the results. In the subsections III.A and III.B we study potential flows for an accelerated black hole and an accelerated rigid sphere, respectively. In the first case we use a perturbative approach and in the

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second we solve numerically the potential equation. For both cases, we also study the behavior of the fluid density. In Sec. IV we introduce a metric that represents a black hole in a dipolar field, this field is produced by an external halo of matter. In the subsections IV.A and IV.B, respectively, we study the streamlines of a fluid in the presence of a rigid sphere and black hole, both with halo. Also, in both cases we study the fluid density. Finally in Sec. V we summarized our results.

II. BASIC EQUATIONS

The starting point of this work is an ideal fluid whose energy-momentum tensor is given by $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$, where $p$ is the pressure, $\rho$ the total energy density and $U_\mu$ the four-velocity. The conservation equations, $T_{\mu\nu} = 0$, for this kind of fluid reduce to

$$\left(\rho + p\right)U_\mu^\nu + \rho_\mu U^\mu = 0, \quad (1)$$

and

$$\left(\rho + p\right)U^\nu U_{\mu,\nu} + \rho_{\mu} + p_{\mu}U^\nu U_\mu = 0, \quad (2)$$

which are respectively the conservation and Euler equation. (Our conventions are: $G = c = 1$. Metric with signature +2. Partial and covariant derivatives with respect to the coordinate $x^\mu$ denoted by $,\mu$ and $;\mu$, respectively.) For isentropic flows we have, $(\sigma/n)_\mu = 0$, where $\sigma$ is the entropy per unit volume and $n$ the baryon number density. In this case the equations of motion (2) take the form [12],

$$U^\nu \omega_{\mu\nu} = 0, \quad (3)$$

where $\omega_{\mu\nu}$ is the relativistic vorticity tensor defined as

$$\omega_{\mu\nu} = \left[ \left( \frac{\rho + p}{n} \right) U_\mu \right]_{;\nu} - \left[ \left( \frac{\rho + p}{n} \right) U_\nu \right]_{;\mu}, \quad (4)$$

The potential flow solution of this equation ($\omega_{\mu\nu} = 0$) is

$$\left( \frac{\rho + p}{n} \right) U_\mu = \Phi_{,\mu}, \quad (5)$$

where $\Phi$ is a scalar field. From (5) and the equation of continuity for the baryon number density $n$, $(nU_\mu)_{;\mu} = 0$, we obtain the differential equation for the scalar field $\Phi$,

$$\left[ \left( \frac{n^2}{\rho + p} \right) \Phi_{,\nu} g^{\nu\mu} \right]_{;\mu} = 0. \quad (6)$$

The normalization condition, $U_\mu U^\mu = -1$, provides a relation between the pressure, the total energy density, and scalar field: $(\rho + p)/n = \sqrt{-g_{\mu\nu}} \Phi_{,\mu}$. Also, $p$ and $\rho$ are assumed to be related by a barotropic equation of state, $p = p(\rho)$. In general, Eq. (6) is a nonlinear equation for the scalar field $\Phi$. However it becomes linear if we assume that $p = \rho \propto n^2$, i.e., a stiff equation of state. Thus,

$$n U_\mu = \Phi_{,\mu}, \quad n^2 = -\Phi_{,\sigma} \Phi^{,\sigma}, \quad (7)$$

and $\Phi$ is a solution of

$$\Box \Phi \equiv (\sqrt{-g} g^{\mu\nu} \Phi_{,\mu})_{,\nu} = 0, \quad (8)$$

which is the usual wave equation for a scalar field. Note that for the stiff equation of state the sound velocity in the fluid is equal to the velocity of light, therefore the flow is always subsonic and the presence of shockwaves is excluded.
III. ACCELERATED BLACK HOLES AND RIGID SPHERES

The Weyl C-metric is the member of the static axially symmetric Weyl family of solution of the vacuum Einstein equations [13],

\[ ds^2 = \frac{1}{A^2(q+p)^2} \left[ -F(q)dt^2 + \frac{dp^2}{G(p)} + \frac{dq^2}{F(q)} + G(p)d\varphi^2 \right], \]

where the functions \( G(p) \) and \( F(q) \) are the cubic polynomials,

\[
\begin{align*}
F(q) &= -1 + q^2 - 2mAq^3, \\
G(p) &= 1 - p^2 - 2mAq^3.
\end{align*}
\]

The coordinates \( t, p, q \) and \( \varphi \) are dimensionless, \( A \) is a constant with dimension of \( L^{-1} \). The range of \( t \) is \((-\infty, +\infty)\), while the range of \( \varphi \) is \([0, 2\pi]\). The ranges of the coordinates \( p \) and \( q \) depend on the roots of \( F \) and \( G \). The constraint \( m^2A^2 < 1/27 \) is imposed to ensure the existence of three real roots in (10). This condition is not a physical one, it is only due to the choice of coordinates [14]. From the fact that the functions have three real roots, the C-metric allows a description of different spacetimes [14] depending on the interval considered for \((q, p)\) in this coordinates, we do not have the correct limit that corresponds to the Schwarzschild metric. To obtain this last metric when \( A \to 0 \) we make the coordinate transformation [15],

\[ r = \frac{1}{A(q+p)}, \quad t \to At. \] (11)

Now the metric (9) reads,

\[ ds^2 = -Hdt^2 + \frac{1}{H}dr^2 + \frac{2Ar^2}{H}drdp + r^2 \left( \frac{1}{F} + \frac{1}{G} \right) dp^2 + r^2Gd\varphi^2, \] (12)

where

\[
\begin{align*}
H &= A^2r^2F \\
&= -A^2r^2G(p - A^{-1}r^{-1}) \\
&= 1 - \frac{2m}{r} + 6Am + ArG,p - A^2r^2G(p).
\end{align*}
\]

(13)

When \( A \to 0 \) the metric (12) reduces to Schwarzschild form provided that the spherical angular coordinate \( \theta \) be related to \( p \) by \( G(p) = 1 - p^2 = \sin^2 \theta \). Furthermore, if the mass is zero in (12) ( \( m \to 0 \) ) the space becomes Euclidean with a special form of a flat space in an uniformly accelerated frame [15]. For these reasons the line element (12) represents a uniformly accelerating Schwarzschild-type particle. The coordinates \((t, r, p, \varphi)\) is a coordinate system rigidly fixed on the accelerating particle. With the transformation (11) we gain the correct Schwarzschild limit, but now the metric is not diagonal.

In the case of \( A \neq 0 \), the angular coordinates \( p \) and \( \theta \) are related by \( G(p) = \sin^2 \theta \). The mapping between them is [15],

\[
p = \begin{cases} 
-\frac{1}{6Am} \left[ 2\cos \left( \frac{\Theta(\theta)}{3} + \frac{4\pi}{3} \right) + 1 \right] & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\
-\frac{1}{6Am} \left[ 2\cos \left( \frac{\Theta(\theta)}{3} + \frac{2\pi}{3} \right) + 1 \right] & \text{for } \frac{\pi}{2} \leq \theta \leq \pi,
\end{cases}
\] (14)

where

\[
\cos \Theta(\theta) = 1 - 54A^2m^2\cos^2 \theta.
\] (15)

Due to the acceleration the black hole horizon deforms. The exact form of this horizon is [15],

\[
r_{Sch} = \begin{cases} 
-\frac{3m}{\cos \left( \frac{\Theta(\theta)}{3} + \frac{4\pi}{3} \right) + \cos \left( \frac{2\pi}{3} + \frac{\Theta(\theta)}{3} \right)} & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\
-\frac{3m}{\cos \left( \frac{\Theta(\theta)}{3} + \frac{2\pi}{3} \right) + \cos \left( \frac{\Theta(\theta)}{3} + \frac{4\pi}{3} \right)} & \text{for } \frac{\pi}{2} \leq \theta \leq \pi.
\end{cases}
\] (16)

The form of the Schwarzschild horizon will be used in a perturbation scheme to place an upper limit on the acceleration.
A. Potential flows and accelerated black holes

In order to study streamlines of the flow we need to know the field \( \Phi \) that determines the fluid velocity. The explicit form of the equation for \( \Phi \) can be obtained from (8) and (12). We find,

\[
\Box \Phi = -\frac{\mu}{H} \Phi_{,tt} + \left( [H + A^2 r^2 G] \Phi_{,r} + (G \Phi_{,p})_{,p} + \frac{1}{G} \Phi_{,\varphi \varphi} - (r^2 GA \Phi_{,r})_{,p} - (r^2 GA \Phi_{,p})_{,r} \right) = 0.
\]

We shall assume that the fluid is stationary and also independent of the variable \( \varphi \). In this case the time dependence of \( \Phi \) will be set by adding the term \(-at\) to the final solution, where \( a \) is a constant, this term clearly satisfies (17).

The arbitrary constant \( a \) is related to the 0th component of the four-velocity. With these assumptions, and setting \( m = 1 \), (17) takes the form

\[
\Box \Phi = \Box_{Sch} \Phi + A \Box_1 \Phi + A^2 \Box_2 \Phi = 0,
\]

with

\[
\Box_{Sch} \Phi = \left( [r^2 - 2r] \Phi_{,r} + (1 - p^2) \Phi_{,p} \right)_{,r} + \left( [1 - p^2] \Phi_{,p} \right)_{,p} = 0,
\]

\[
\Box_1 \Phi = \left( [6 pr^2 - 2 pr^3] \Phi_{,r} - (2p^3 \Phi_{,p})_{,p} - (r^2 [1 - p^2] \Phi_{,r})_{,p} - (r^2 [1 - p^2] \Phi_{,p})_{,r} \right),
\]

\[
\Box_2 \Phi = -(6 r^3 p^2 \Phi_{,r})_{,p} + (2r^2 p^3 \Phi_{,p})_{,r} + (2r^2 p^3 \Phi_{,p})_{,p}.
\]

Finding an analytical solution to Eq. (18) does not appear possible. We shall look for a meaningful approximate solution. For small enough \( A \), say \( A \leq 0.01 \), the Schwarzschild surface remains almost unaltered [cf. (16)]. Then we can approximate (18) as

\[
\Box \Phi = \Box_{Sch} \Phi + A \Box_1 \Phi = 0.
\]

By using the fact that \( A \approx 0 \) we may solve this equation by a perturbative method, but first we recall some earlier results. The solution for a black hole when \( A = 0 \) with the condition that at infinity the fluid velocity is constant and parallel to the \( z \)-axes of the inertial frame is known [3],

\[
\Phi_{PST} = -at - 2a \ln \left( 1 - \frac{2}{r} \right) + b(r - 1) \cos \theta,
\]

(21)

where \( a \) and \( b \) are constants. We shall consider for the unperturbed potential \( (A = 0) \) that the fluid is at rest at infinity. Following the procedure of [3] we find that the unperturbed potential is,

\[
\Phi_0 = -at - 2a \ln \left( 1 - \frac{2}{r} \right).
\]

(22)

This solution can also be found in a different way. Note that \( \Phi_{,r} \) diverges near the black hole horizon. This is a necessary condition to avoid the divergence of the particle density on the horizon [3]. Therefore, near the black hole horizon, we can neglect \( \Phi_{,p} \) when compared with \( \Phi_{,r} \). It is illustrative to consider (18) with \( A = 0 \)

\[
\Box_{Sch} \Phi = \left( [r^2 - 2r] \Phi_{,r} + (1 - p^2) \Phi_{,p} \right)_{,r} = 0.
\]

(23)

When near the black hole we neglect the terms containing derivatives with respect to \( p \) the equation reduces to an equation for \( r \) only. By considering that the temporal dependence is given by \(-at\) this equation has for solution (22). Hence, the above imposed condition is valid and moreover, in the case of fluid at rest at infinity, gives the exact solution.

Now we shall consider the perturbation,

\[
\Phi = \Phi_0 + A \Upsilon,
\]

(24)

where \( \Upsilon \) is a function of \( r \) and \( p \). From (20) we find,

\[
\Box_{Sch} \Upsilon = -\Box_1 \Phi_0.
\]

(25)
Thus,

$$([r^2 - 2r]Υ_{,r})_{,r} + [(1 - p^2)Υ_{,p})_{,p} = \frac{16ap(r - 3)}{(r - 2)^2}. \tag{26}$$

Again, to solve this equation we use the fact that near the black hole we can neglect the term containing derivatives of $p$. We find,

$$Υ = C_1 + \frac{1}{2}C_2 \ln \left(1 - \frac{2}{r}\right) - 4ap \left[\frac{2}{r - 2} + \ln \left(1 - \frac{2}{r}\right)\right] + 2\ln \left(\frac{r}{2}\right) \ln(r - 2) - [\ln(r - 2)]^2 + 2\text{PolyLog} \left(2, 1 - \frac{r}{2}\right), \tag{27}$$

with

$$\text{PolyLog}(n, z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \tag{28}$$

To obtain the range of validity of the perturbation $AY$, we compare it with the unperturbed function $Φ_0$. We consider that the perturbation is valid only when the absolute value of $AY$ is at least a 15% of the value of $Φ_0$. We can take values of the constants $C_1$ and $C_2$ to enlarge the range of the radial coordinate and still be in the required precision. For example, for $A ≈ 0.01$ and the $C_1$ and $C_2$ in the interval $(0, 2)$ the range of the radial coordinate is $r ≈ (2, 10)$. In Fig. 1, we show the flux lines for the case $A = 0$ (dotted lines) and $A ≈ 0.01$ (full lines). When $A ≈ 0.01$, the streamlines that were straight lines are now curved due to the black hole acceleration. These lines are similar to the corresponding lines for a fluid in the presence of a black hole, with the condition that the fluid velocity be constant at infinity $[1, 3]$.

Another significant quantity is the particle density of the fluid, $n^2 = -Φ_{,\mu}Φ_{,\mu}$. At first order in $A$ we obtain

$$n^2 = \frac{a^2}{1 + 6Ap - \frac{2}{3} - 2Apr} - \frac{16a^2}{(r - 2)r^3} + \frac{8aAC_2}{(r - 2)r^3} + \frac{32a^2pA}{(r - 2)^2r^3} \frac{r(r - 3) + 4(1 + (r - 2) \ln(r - 2))}{(r - 2)^2r^3}. \tag{29}$$

The particle density (29) is positive in the interval wherein the perturbation is valid. When $A \to 0$ we obtain the correct limit for a fluid at rest at infinity $[3]$. The fluid density contours are shown in Fig. 2. We see that near the black hole these lines are closed and more dense in the forward direction. These contour lines are also similar to the corresponding density lines of a fluid in the presence of a black hole, with the condition that the fluid velocity be constant at infinity $[3]$. Far from the black hole these lines are open, while in $[3]$ they are closed, this clearly shows the difference between a standing black hole in a moving fluid and an accelerated black hole moving in a fluid which is at rest at infinity. The qualitative aspects of the density contours do not vary when lowering the acceleration or varying the constant $C_2$. When $A \to 0$ we density contours are circles centered in the black hole.

B. Potential flows and accelerated rigid spheres

In the case of rigid spheres moving in a fluid, the particle density is no longer divergent on the sphere surface nor $Φ_{,r}$ is divergent. Then, even near the surface of the sphere, we cannot neglect terms containing derivatives of $p$ in the equation for $Φ$. To find a numerical solution we need to solve the partial differential Eq. (8) with mixed boundary conditions, Neumann and Dirichlet. In this case it is more convenient to work with the metric (9) which is diagonal and later perform the coordinate transformation (11). In this metric Eq. (8) reads,

$$-\frac{Φ_{,tt}}{F(q + p)^2} + \left[\frac{G}{(q + p)^2}Φ_{,p}\right]_{,p} + \left[\frac{F}{(q + p)^2}Φ_{,q}\right]_{,q} + \frac{Φ_{,ϕϕ}}{G(q + p)^2} = 0. \tag{30}$$

From the condition that the fluid is stationary and that $Φ$ is independent of $ϕ$ we get,

$$\left[\frac{G}{(q + p)^2}Φ_{,p}\right]_{,p} + \left[\frac{F}{(q + p)^2}Φ_{,q}\right]_{,q} = 0. \tag{31}$$
The boundary conditions are: i) Zero normal component of the fluid velocity on the sphere surface, and ii) Faraway from the sphere we set the fluid velocity to be constant and parallel to the acceleration. The condition i) do not describe the typical flow of gas around a star because the surface of the star is not hard but gaseous, except in special astrophysical situations; but describes an idealized strong-field star.

The code used to solve (31) is a finite difference multigrid method [17] with second order precision. The numerical multigrid is evenly spaced in \( r \), but not in \( \theta \). We find that the convergence is better when we increase the number of points in \( \theta \) rather than in \( r \). To solve (31) we employ four domains (grids) limited by spherical shells of radius greater than 2.5 (Schwarzschild radius equal 2). For the first grid we have \( r \in [2.5, 3.9] \) and we use 79300 points; the respective numbers for the other three grids are [3.9, 5.3], 39700, [5.3, 11.1], 19900, and [11.1, 25.5], 5000. An iterative method in which at every step each point of the grid is calculated from the values of the four nearest neighbors is employed. The error is estimated with the expression \( |\Phi_{\text{new}}(x_i) - \Phi_{\text{old}}(x_i)| \), in which \( \Phi_{\text{old}} \) is the old value of \( \Phi \) at the point \( (x_i) \) and \( \Phi_{\text{new}} \) is the new calculated value at the same point. The program will stop when the sum of all errors of the grid points reach some pre-establish value, say \( \sum_{x_i} |\Phi_{\text{new}}(x_i) - \Phi_{\text{old}}(x_i)| \leq \text{Error} \). The code was tested with the exact solution for a rigid sphere in a fluid flow [1].

In Fig. 3, the streamlines of a fluid disturbed by a moving rigid sphere with constant acceleration \( A = 0.01 \) are depicted. We see that the streamlines concentrate in the frontal part and separate when passing the rigid sphere. This may be due to the deformation that suffer the killing horizons in an accelerated frame [15]. The qualitative aspects of the density contours for this case are similar to the black hole case.

IV. POTENTIAL FLOWS IN BLACK HOLES AND RIGID SPHERES WITH DIPOLAR HALO

To incorporate the dipolar field into the Schwarzschild metric we consider the static axially symmetric Weyl metric,

\[
d s^2 = -e^{2\nu} d t^2 + e^{2(\gamma - \nu)} (d z^2 + d \rho^2) + e^{-2\nu} \rho^2 d \varphi^2,
\]

where \( \nu \) and \( \gamma \) are functions of \( \rho \) and \( z \) only and satisfy the conditions [18],

\[
\nu_{,\rho \rho} + \frac{1}{\rho} \nu_{,\rho} + \nu_{,zz} = 0,
\]

\[
d \gamma = \rho \left[ (\nu_{,\rho})^2 - (\nu_{,z})^2 \right] d \rho + 2 \rho \nu_{,\rho} \nu_{,z} d z.
\]

The first equation is the usual Laplace’s equation in cylindrical coordinates, and the second (once \( \nu \) is known) gives \( \gamma \) as a quadrature.

For a multipolar expansion of \( \nu \), the spherical coordinates \(( r, \theta, \varphi) \) or the prolate spherical coordinates \(( u, v, \varphi) \) are more adequate than the cylindrical ones. The relation between these coordinates are

\[
u = \frac{1}{2m} \left[ \sqrt{\rho^2 + (z + m)^2} + \sqrt{\rho^2 + (z - m)^2} \right],
\]

\[
v = \frac{1}{2m} \left[ \sqrt{\rho^2 + (z + m)^2} - \sqrt{\rho^2 + (z - m)^2} \right],
\]

\[
\varphi = \varphi.
\]

\[
\rho = m \sqrt{(u^2 - 1)(1 - v^2)}
\]

\[
= \sqrt{r(r - 2m)} \sin \theta, \quad r \geq 2m,
\]

\[
z = muv
\]

As before, we will set \( m = 1 \).
Using the transformations (36), Eqs. (33) and (34) can be written in terms of and as

\[
\left[(u^2 - 1)v_\nu\right]_u + \left[(1 - v^2)v_\nu\right]_v = 0,
\]

\[
\gamma, u = u^2 - 1 \left[ u(u^2 - 1)(v_\nu)^2 - u(1 - v^2)(v_\nu)^2 - 2v(u^2 - 1)v_\nu v_\nu \right],
\]

\[
\gamma, v = \frac{u^2 - 1}{u^2 - v^2} \left[ v(u^2 - 1)(v_\nu)^2 - v(1 - v^2)(v_\nu)^2 + 2u(1 - v^2)v_\nu v_\nu \right].
\]

The authors in [11] solved these equations using an external multipolar expansion up to octopoles using a Legendre expansion with their corresponding terms increasing with the distance in the intermediate vacuum between the core and the shell. In this work we are only interested in the dipolar case. Hence, the functions and reduce to

\[
2\nu = 2\nu_0 + \kappa \ln \left( \frac{u - 1}{u + 1} \right) + 2Duv,
\]

\[
2\gamma = 2\gamma_0 + \kappa^2 \ln \left( \frac{u^2 - 1}{u^2 - v^2} \right) + 4\kappa Dv - D^2 \left[ u^2(1 - v^2) + v^2 \right],
\]

where is the value of the dipolar field produced by an axially symmetric halo (or shell) of matter, and is a constant. The Schwarzschild solution is recovered with and . The constants and can be used to rule out conical singularities and to ensure analyticity of the metric at the horizon [11]; here we have made them zero. From the metric (32) in coordinates and , and (38) and (39), we find

\[
ds^2 = - \left( \frac{u - 1}{u + 1} \right) e^{2Duv} dt^2 + \left( u + 1 \right)^2 e^{2Dv(2-u) - D^2[u^2(1-v^2) + v^2]} \left[ \frac{du^2}{u^2 - 1} + \frac{dv^2}{1 - v^2} \right] + (u + 1)^2(1 - v^2)e^{-2Duv} d\varphi^2.
\]

This metric represents a monopolar core \((\kappa = 1)\) in the presence of a external dipolar field \((D)\) that is associated to a distant shell or halo of matter.

### A. Potential flow of a rigid sphere in an external dipolar field

For a stationary fluid that does not depend on \(\varphi\) the scalar equation (8) reduces to

\[
\left[(u^2 - 1)\Phi, u\right]_u + \left[(1 - v^2)\Phi, v\right]_v = 0.
\]

This is exactly the same equation that we solved in Sec. III.A, but in a different system of coordinates. Note that all the Weyl solutions have the same differential equation (41) for \(\Phi\). The presence of the dipolar field will be taken into account in the boundary conditions. On the surface of the rigid sphere the boundary condition is the usual one for the Euler’s equation, i.e., that the normal component of the fluid velocity vanish on the surface. For large values of \(u\) the boundary condition deserves more attention. Since, the time dependence of \(\Phi\) is \(-at\), we have for the density,

\[
n^2 = a^2 \left( \frac{u + 1}{u - 1} \right) e^{-2Duv} - \frac{e^{2Dv(u-2)+D^2[u^2(1-v^2)+v^2]}}{(u + 1)^2} \left[ (\Phi, u)^2(u^2 - 1) + (\Phi, v)^2(1 - v^2) \right].
\]

Due to the presence of the dipolar field \(D\), at some value of \(u\) greater than a certain \(u_0\), the first term in (42) will always be smaller than the others two. Then we will have \(n^2 < 0\). This is not allowed; we say that the fluid is no longer stationary, i.e., the assumed time dependence is not right for \(u > u_0\). If \(D\) is large enough, say \(D \approx 1\), the fluid will never be stationary. The first term also depends on \(a^2\). By varying the value of \(a\) we can enlarge or decrease the domain where the fluid is stationary, also for \(D \rightarrow \infty\) we can set \(a\) large enough to keep the fluid stationary. For example, for \(a = 1.25\), we must have \(D \approx 0.001\) to obtain a relative large domain wherein the fluid remains stationary. Test particles moving in the metric (40) can have chaotic behaviour [11]. We believe that the lost of the stationary character of the fluid for \(u > u_0\) may be another manifestation of the same type of instability. This point is under active consideration by the authors.

The outer boundary condition is still missing. With all these fluid instabilities it is not clear which is the right boundary condition. The previous condition, that the fluid velocity to be constant far from the sphere, is not a valid
condition in this case because the fluid is accelerating and is in this region where the instabilities appear. To find some characteristics values of \( \Phi \) to be used as an outer boundary condition we will integrate \( \Phi, u \) along a line of constant \( v \) from the surface of the sphere (black hole) to the stationary limit of the fluid. From Eq. (7) we get

\[
\Phi, u \propto \frac{g_{uu} U^u}{g_t U^t}.
\]  

(43)

The proportionality constant is of no importance, because \( U_{\mu} \) in (7) is invariant under a rescaling of \( \Phi \). To find the \( t \) component of the four-velocity, \( U^t \), we will assume – without further justification – that the fluid particle follows the usual geodesic equation of motion for a test particle, i.e., the Euler-Lagrange equation,

\[
\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{x}^{\mu}} \right) - \frac{\partial L}{\partial x^{\mu}} = 0,
\]

(44)

where \( \dot{x}^{\mu} \) represents total derivative of \( x^{\mu} \) with respect to the parameter \( s \) and 

\[
L(\dot{x}^{\lambda}, x^{\lambda}) = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}.
\]

Since the coordinate \( t \) is cyclic, we find,

\[
\dot{t} = U^t = -k \left( \frac{u + 1}{u - 1} \right) e^{-2Duv},
\]

(45)

where the constant \( k \) is the energy function of the fluid particle. The component \( U^u \) along the lines of constant \( v \) can be computed from the metric (40),

\[
U^u = \dot{u} = \sqrt{\left( \frac{u - 1}{u + 1} \right) e^{2Duv} + k^2 e^{-2Dv + \frac{D^2}{4}(v^2 - u^2)}}
\]

(46)

Therefore, \( \Phi \) along the lines of constant \( v \) can be written as

\[
\Phi = \int \left( \frac{u + 1}{u - 1} \right) e^{2Dv(1-u) - \frac{D^2}{4}(v^2 - u^2)} \sqrt{\left( \frac{u - 1}{u + 1} \right) e^{2Duv} + k^2} \, du,
\]

(47)

where a multiplicative constant was set equal to one. We numerically integrate (47) to obtain the outer boundary condition at a certain value \( u \) near the limit of stability. We use the code described in Sec. III.B to solve (41). For different values of the constant \( k \) the qualitative features of the streamlines and density contours do not change. In Fig. 4 the streamlines of a fluid in the presence of a rigid sphere of radius 2.5, Schwarzschild radius equal 2, and an external dipolar moment field with \( D = 0.001 \) are shown. We also set \( k = 1 \) in this case. The values of \( \Phi \) used for boundary condition and all the ones computed satisfy the condition \( n^2 > 0 \). We see little difference in the flows with \( D = 0.001 \) and \( D = 0 \). In Fig. 5 the density contours are plotted for the same values of the parameters used in Fig 4, sphere of radius 2.5 and \( D = 0.001 \). We see the same qualitative features discussed in Sec. II.A in the context of the C-metric. In this case the fluid is accelerating something that we should expect since an external field (in this case the external dipolar field) exert a force on the fluid.

**B. Potential flows for black holes with dipolar halo**

Like in the precedent case we solve Eq. (41), but to avoid the singularity at the black hole horizon, we use the tortoise radial coordinate [19],

\[
r^* = r + 2 \ln \left( \frac{r}{2} - 1 \right) = (u + 1) + 2 \ln \left( \frac{u - 1}{2} \right),
\]

(48)

(49)

Now, Eq. (41) reads,

\[
\left( \frac{u + 1}{u - 1} \right) [(u + 1)^2 \Phi, r^*], r^* + [(1 - v^2) \Phi, v] = 0,
\]

(50)

and the particle density takes the form,
\[ n^2 = a_1^2 \left( \frac{u + 1}{u - 1} \right) e^{-2Duv} - e^{2Dv(u-2) + D^2[u^2(1-v^2)+v^2]} \left[ \frac{u + 1}{u - 1} (\Phi, r^*)^2 + \frac{1 - u^2}{(u + 1)^2} (\Phi, v^*)^2 \right]. \]  

(51)

From a computational viewpoint the difference between a rigid sphere and a black hole laid in the inner boundary condition. For a rigid sphere the condition is zero normal velocity on its surface, and for a black hole is finite particle density on the horizon. This last condition requires that \( \Phi, r^* \) be limited in such a way that cancels the singular term in \( \Phi, v^* \). In the general time dependent case this condition reads [2, 6],

\[ \Phi, r^* = \left( 1 - \frac{2}{r} \right) \Phi, r = \Phi, t + a_1(t)(r - 2) + a_2(t)(r - 2)^2 + \cdots, \]  

(52)

where \( a_1, a_2, \ldots \) are functions of time coordinate. Hence, the following equation is valid near \( r = 2 \)

\[ \frac{\partial}{\partial r^*} \left[ \frac{\Phi, r^* - \Phi, t}{r - 2} \right] = 0. \]  

(53)

With the assumption that \( \Phi, t = -a \) this condition is also valid in our case and it will be taken as the inner boundary condition. In (52) we cannot set \( \Phi, r^* = \Phi, t \) because we can have \( n^2 \leq 0 \). The outer condition for this problem is found integrating Eq. (47) as before. The numerical code employed is the same of Sec. IV.A with the implementation of the inner boundary condition (53) and the change of \( r \) or \( u \) to the tortoise coordinate \( r^* \). The program was tested with the exact solution for a black hole in a moving fluid [3]. The computed values of \( \Phi \) on the black hole horizon have less than 1% of error when compared with the exact solution. Again, outside the horizon the precision is better.

In Fig. 6 the streamlines of a perfect fluid in the presence of a black hole in an external dipolar moment field with \( D = 0 \) are plotted. Like the rigid sphere case the streamlines show little difference with respect to the case \( D = 0 \). In Fig. 7 we present the density contours for the same values of the parameters used in the previous figure. Once more we see the same pattern encountered in the C-metric case. This shows that the effect of acceleration appears always in the density contours rather than in the fluid streamlines. In the last four figures we see that the streamlines are practically the same compared to the respective case with \( D = 0 \), but the density contours are quite different.

V. CONCLUSIONS

We can summarized the results of this work as follows: in the C-metric and dipolar field cases we found that, for a stationary fluid, the difference between the accelerate and non accelerate cases lies on the form of the density contours and not, as one may think, in the shape of the streamlines. We assumed in both cases that the acceleration is near zero. In the dipolar field case the potential flow becomes very unstable and the presence of chaos may appear [11].

It is important to note that the method presented here is valid for a stiff equation of state and the incorporation of a new barotropic equation of state for the fluid is not easily applied. In that case we need a different approach for solving the difference equation (6) because it is nonlinear and it is necessary to impose initial conditions to the enthalpy, this may help or not in the convergence of the method. A negative value of the enthalpy could indicate a nonstationary regime for the fluid, see for instance [2]. In the different scenarios of this work the search for suitable boundary conditions to be treated, with \( A \approx 0 \), have been a constant problem, we think that this difficulties would increase with a new barotropic equation due to the condition for the enthalpy. Future applications of this work could be the study of fluids with another equation of state to model a more realistic situation, the study of chaos in the presence of instabilities, and the study of fluids in non stationary states. All this applications are currently under investigation by the authors.

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FIG. 1. Analytical results for the streamlines of a fluid in the presence of an accelerating black hole in the direction of the positive $z$-axis. The solid lines represent the case with $A \neq 0$ and the dotted lines $A = 0$. The black hole has radius $r \approx 2$ (Schwarzschild radius equal 2), this approximation is due to the deformation of the Schwarzschild horizon. The axes are defined as $X = r \sin \theta$ and $Z = \cdots$

FIG. 2. Analytical results of the fluid density contours of an accelerating black hole. The black hole radius and the meaning of the axes are defined as in Fig. 1.
FIG. 3. Numerical results for the fluid streamlines in the presence of a rigid sphere accelerating in the direction of the positive z-axis. The sphere radius is $r \approx 2.5$ (Schwarzschild radius equal 2), this approximation is due to the deformation of the Schwarzschild horizon.

FIG. 4. Numerical results for the fluid streamlines when an external dipolar field with $D = 0.001$ is present and a rigid sphere of radius $r = 2.5$ (Schwarzschild radius equal 2) is placed as an obstacle. The axes are defined as $X = r \sin \theta$ and $Z = r \cos \theta$, with $r = u + 1$ and $\cos \theta = v$. 
FIG. 5. Numerical results for the fluid density contours when an external dipolar field with $D = 0.001$ is present and a rigid sphere of radius $r = 2.5$ (Schwarzschild radius equal 2) is placed as an obstacle. The axes are defined as in Fig. 4.

FIG. 6. Numerical results for the streamlines when an external dipolar field of value $D = 0.001$ is present and a black hole is placed as an obstacle. The black hole has radius $r = 2$ (Schwarzschild radius equal 2). The meaning of axes are the same of Fig. 4.
FIG. 7. Numerical results for the fluid density contours when an external dipolar field of value $D = 0.001$ is present and a black hole is placed as an obstacle. The black hole has radius $r = 2$ (Schwarzschild radius equal 2). The meaning of the axes are the same of Fig. 4.