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A Biconvex Form for Copulas

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Abstract: We study the integration of a copula with respect to the probability measure generated by another copula. To this end, we consider the map 
\[ [C, D] := \int_{[0,1]^d} C(u) \, dQ^D(u) \]
where \( C \) denotes the collection of all \( d \)-dimensional copulas and \( Q^D \) denotes the probability measures associated with the copula \( D \). Specifically, this is of interest since several measures of concordance such as Kendall’s tau, Spearman’s rho and Gini’s gamma can be expressed in terms of the map \([ , , , ]\). Quite generally, the map \([ , , , ]\) can be applied to construct and investigate measures of concordance.

Keywords: biconvex form, copulas, concordance order, group of transformations, measures of concordance

MSC: 62H05, 62H20

1 Introduction

In the present paper we study the map \([ , , , ] : C \times C \to \mathbb{R} \) given by
\[ [C, D] := \int_{[0,1]^d} C(u) \, dQ^D(u) \]
where \( C \) denotes the collection of all \( d \)-dimensional copulas and \( Q^D \) denotes the probability measures associated with the copula \( D \). A probability measure generated by a copula is said to be a copula measure. The map \([ , , , ]\) is linear with respect to convex combinations in both arguments and is therefore called a biconvex form.

For dimension \( d = 2 \), integration of a copula with respect to a copula measure has been frequently studied in connection with measures of concordance (see e.g. [8–12, 14]), with respect to transformations of copulas (see e.g. [3]), and also with regard to applications to the joint distribution of random vectors (see [7]). For \( d = 2 \) the biconvex form \([ , , , ]\) is symmetric, but it turns out that this property gets lost in higher dimensions.

Here we discuss the biconvex form \([ , , , ]\) for an arbitrary but fixed dimension \( d \geq 2 \). We show that the map \([ , , , ]\) is bounded by \( 0 \) from below and by \( 1/2 \) from above, and that it is monotone with regard to concordance order. We further study the biconvex form with respect to the group \( \Gamma \) of transformations on \( C \) discussed in [6] for the bivariate case and in [5] for the general case. The group \( \Gamma \) contains an involution \( \tau \) which transforms every copula into its survival copula, and it turns out that this transformation changes the arguments of the biconvex form. Moreover, we identify those transformations \( \gamma \in \Gamma \) which satisfy \([\gamma(C), \gamma(D)] = [C, D]\) for all \( C, D \in C \), and we consider a class of copulas \( D \) such that \([C, D] = 1/2^d\) holds for all \( C, D \in D \).

Several measures of concordance such as Kendall’s tau \( \kappa_\tau \) and Spearman’s rho \( \kappa_\rho \) can be expressed in terms of the biconvex form \([ , , , ]\). Namely, using the Fréchet–Hoeffding upper bound \( M \) and the product copula \( \Pi \)

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\[ \kappa_r(\mathcal{C}) = \frac{[C, C] - [I, I]}{[M, M] - [I, I]} \]

and

\[ \kappa_\rho(\mathcal{C}) = \frac{\left[ \frac{1}{\rho} C + \frac{1}{\rho} \tau(C), I \right] - [I, I]}{[M, M] - [I, I]} \]

(see [11]). However, the biconvex form \([\ldots, \cdot, \cdot]\) and its properties with regard to the transformations in \(\Gamma\) can not only be employed to construct but also to investigate measures of concordance (see e.g. [1, 15]).

This paper is organized as follows: We first recapitulate essential definitions and results concerning copulas, copula measures and the integration with respect to a copula measure (Section 2). In Section 3 we then only be employed to construct but also to investigate measures of concordance (see e.g. [1, 15]). However, the biconvex form \([\ldots, \cdot, \cdot]\) cannot be kept fix throughout this paper. For the sake of a concise definition of a copula we consider, for \(L \subseteq \{1, \ldots, d\}\), the map \(\eta_L : I^d \times I^d \to I^d\) given coordinatewise by

\[
(\eta_L(u, v))_l := \begin{cases} u_l & l \in \{1, \ldots, d\} \setminus L \\ v_l & l \in L \end{cases}
\]

and we put \(\eta_i := \eta_{\{i\}}\) for \(i \in \{1, \ldots, d\}\).

A copula is a function \(C : I^d \to I\) satisfying the following conditions:

(i) The inequality

\[ \sum_{L \subseteq \{1, \ldots, d\}} (-1)^{|L|} C(\eta_L(u, v)) \geq 0 \]

holds for all \(u, v \in I^d\) such that \(u \leq v\).

(ii) The identity \(C(\eta_i(u, 0)) = 0\) holds for all \(u \in I^d\) and all \(i \in \{1, \ldots, d\}\).

(iii) The identity \(C(\eta_i(1, u)) = u_i\) holds for all \(u \in I^d\) and all \(i \in \{1, \ldots, d\}\).

Note that, for \(u \leq v\), the family \(\{\eta_L(u, v)\}_{L \subseteq \{1, \ldots, d\}}\) consists of all vertices of the interval \([u, v]\). Thus, this definition of a copula is appropriate and in accordance with the literature; see [2, 12]. The collection \(\mathcal{C}\) of all copulas is convex.

### The Group \(\Gamma\)

A map \(\varphi : \mathcal{C} \to \mathcal{C}\) is said to be a transformation. We denote by \(\Phi\) the collection of all transformations and define the composition \(\circ : \Phi \times \Phi \to \Phi\) by letting \((\varphi_1 \circ \varphi_2)(C) := \varphi_1(\varphi_2(C))\). The transformation \(\iota \in \Phi\) given by \(\iota(C) := C\) is called the identity on \(\mathcal{C}\). Thus, \((\Phi, \circ)\) is a semigroup with neutral element \(\iota\). For the composition of \(n \in \mathbb{N}_0\) transformations \(\varphi_m \in \Phi, m \in \{1, \ldots, n\}\), we write

\[
\bigcirc_{m=1}^{n} \varphi_m := \begin{cases} \iota & n = 0 \\ \varphi_n \circ \bigcirc_{m=1}^{n-1} \varphi_m & \text{otherwise} \end{cases}
\]

and, for \(N = \{1, \ldots, n\}\) and a set of pairwise commuting \(\varphi_m \in \Phi, m \in N\), we put \(\bigcirc_{m \in N} \varphi_m := \bigcirc_{m=1}^{n} \varphi_m\).

We now introduce two elementary transformations: For \(i, j \in \{1, \ldots, d\}\) with \(i \neq j\) we define the map \(\pi_{i,j} : \mathcal{C} \to \mathcal{C}\) by letting

\[ (\pi_{i,j}(C))(u) := C(\eta_{\{i,j\}}(u, u_i e_i + u_i e_i)) \]
and, for $k \in \{1, \ldots, d\}$, the map $\nu_k : \mathbb{C} \to \mathbb{C}$ by letting
\[
(v_k(C))(u) := C(\eta_k(u, 1)) - C(\eta_k(u, 1 - u))
\]
$\pi_{i,j}$ is called a transposition, and $\nu_k$ is called a partial reflection. Both, $\pi_{i,j}$ and $\nu_k$, are involutions.

Then there exists a smallest subgroup $(\Gamma, \circ)$ of $\Phi$ containing all transpositions and all partial reflections. A transformation is called a permutation if it can be expressed as a finite composition of transpositions, and a transformation is called a reflection if it can be expressed as a finite composition of partial reflections. We denote by $I^\pi$ the set of all permutations, and by $I^\nu$ the set of all reflections. Then $I^\pi$ and $I^\nu$ are subgroups of $\Gamma$ whereas only $I^\nu$ is commutative, and every transformation in $\Gamma$ can be expressed as a composition of a permutation and a reflection. Due to its particular interest we emphasize the reflection $\tau := O_{k=1}^d \nu_k$, an involution called total reflection. We set $I^\tau := \{1, \tau\}$ and
\[
I^{\pi, \tau} := \{\gamma \in \Gamma \mid \gamma = \pi \circ \varphi \text{ for some } \pi \in I^\pi \text{ and some } \varphi \in I^\tau\}
\]
Then $I^\tau$ is the center of $\Gamma$, and $I^{\pi, \tau}$ is a subgroup of $\Gamma$. The total reflection $\tau$ transforms every copula into its survival copula.

A copula $C$ is called invariant with respect to a subgroup $\Lambda$ of $\Gamma$ (or $\Lambda$-invariant) if it satisfies $\gamma(C) = C$ for every $\gamma \in \Lambda$.

The group $\Gamma$ is a representation of the hyperoctahedral group with $d! \cdot 2^d$ elements. Note that the hyperoctahedral group has another representation which is geometric and quite popular; see [15–17]: Consider the collection of all vector–valued functions from $I^d$ into $I^d$ equipped with the composition $\circ$ and the identity $i$. Then there is a smallest group $(\tilde{\Gamma}, \circ)$ containing the vector–valued functions $\tilde{\eta}_{i,j} : I^d \to I^d$, $\tilde{\nu}_k : I^d \to I^d$ and $\tilde{\tau} : I^d \to I^d$ with $i, j, k \in \{1, \ldots, d\}$ and $i \neq j$, given by
\[
\begin{align*}
\tilde{\eta}_{i,j}(u) &= \eta_{i,j}(u, u_j u_i + u_i u_j) \\
\tilde{\nu}_k(u) &= \eta_k(u, 1 - u) \\
\tilde{\tau}(u) &= 1 - u
\end{align*}
\]
Note that every $\tilde{\gamma} \in \tilde{\Gamma}$ can be expressed as a finite composition of $\tilde{\eta}_{i,j}$ and $\tilde{\nu}_k$ with $i, j, k \in \{1, \ldots, d\}$ and $i \neq j$. Since $\tilde{\eta}_{i,j}$ and $\tilde{\nu}_k$ are continuous we hence obtain that every $\tilde{\gamma} \in \tilde{\Gamma}$ is continuous as well. For the composition of $n \in \mathbb{N}_0$ functions $\tilde{\gamma}_m \in \tilde{\Gamma}$, $m \in \{1, \ldots, n\}$, we write
\[
\tilde{\gamma}_m := \bigcirc_{m=1}^{n} \tilde{\gamma}_m := \begin{cases} 
\tilde{\gamma}_m & n = 0 \\
\tilde{\gamma}_m \circ \bigcirc_{m=1}^{n-1} \tilde{\gamma}_m & \text{otherwise}
\end{cases}
\]
and, for $N = \{1, \ldots, n\}$ and a set of pairwise commuting $\tilde{\gamma}_m \in \tilde{\Gamma}$, $m \in N$, we put $\bigcirc_{m \in N} \tilde{\gamma}_m := \bigcirc_{m=1}^{n} \tilde{\gamma}_m$. We have $\tilde{\tau} := \bigcirc_{k=1}^{d} \tilde{\nu}_k$.

The groups $\Gamma$ and $\tilde{\Gamma}$ are related to each other by an isomorphism $T : (\Gamma, \circ) \to (\tilde{\Gamma}, \circ)$ satisfying $T(\pi_{i,j}) = \tilde{\eta}_{i,j}$ and $T(\nu_k) = \tilde{\nu}_k$ for all $i, j, k \in \{1, \ldots, d\}$ such that $i \neq j$ and hence $T(\tau) = \tilde{\tau}$. For a detailed discussion of the groups $(\Gamma, \circ)$ and $(\tilde{\Gamma}, \circ)$, see [5].

**Copula Measure**

A probability measure $Q : \mathcal{B}(I^d) \to I$ is said to be a copula measure if the identity $Q[\times_{i=1}^d B_i] = \Lambda[B_i]$ holds for every $j \in \{1, \ldots, d\}$ and every collection of Borel sets $\{B_i\}_{i \in \{1, \ldots, d\}} \subseteq \mathcal{B}(I)$ such that $B_i = 1$ for all $i \neq j$; here $\mathcal{B}(I^d) := \bigotimes_{i=1}^d \mathcal{B}(I)$ denotes the Borel $\sigma$–field on $I^d$, and $\Lambda$ denotes the Lebesgue measure on $\mathcal{B}(I)$. We denote by $\Omega$ the collection of all copula measures. Then $\Omega$ is convex. Since the functions in $\tilde{\Gamma}$ are continuous and hence measurable, they can be used to transform copula measures, and it turns out that $\Omega$ is stable under the transformations of the group $\tilde{\Gamma}$; see [5, Theorem 7.3.]. The next result is a refinement of the correspondence
Theorem relating distribution functions on $\mathbb{R}^d$ to probability measures on $\mathcal{B}(\mathbb{R}^d)$; see [5, Theorem 7.2]:

**2.1 Theorem.** There exists a one–to–one correspondence $S : \mathcal{C} \to \mathcal{Q}$. Moreover, every corresponding pair of a copula $C \in \mathcal{C}$ and a copula measure $Q \in \mathcal{Q}$ satisfies $C(u) = Q([0, u])$ for all $u \in \mathbb{R}^d$.

Moreover, we have the following representation including the isomorphism $T$ and the one–to–one correspondence $S$; Theorem 2.2 follows from the measure extension theorem:

**2.2 Theorem.** For every $C \in \mathcal{C}$ and every $\gamma \in \Gamma$ we have $S(\gamma(C)) = (S(C))_{T(\gamma)}$.

The previous result can be represented by the following commuting diagram:

```
\begin{array}{c}
\mathcal{C} \\
\downarrow \gamma \\
\mathcal{C} \\
\end{array}
\quad \begin{array}{c}
\xrightarrow{S} \\
\downarrow T(\gamma) \\
\xrightarrow{S} \\
\end{array}
\quad \begin{array}{c}
\mathcal{Q} \\
\end{array}
```

For the ease of notation we write $Q^C := S(C)$. As a consequence of Theorem 2.2 we can formulate the substitution rule for copula measures:

**2.3 Lemma.** For every $C \in \mathcal{C}$ and every $\gamma \in \Gamma$ the identity

$$\int_{\mathbb{R}^d} f(u) \, dQ^{\gamma(C)}(u) = \int_{\mathbb{R}^d} (f \circ T(\gamma))(u) \, dQ^C(u)$$

holds for all positive measurable functions $f : \mathbb{R}^d \to \mathbb{R}$.

We finally discuss some useful results concerning the integration of a positive measurable function with respect to a copula measure. Lemma 2.4 is due to Fubini’s Theorem and the fact that every copula measure has uniform margins.

**2.4 Lemma.** For every $C \in \mathcal{C}$ the identity

$$\int_{\mathbb{R}^d} u_i \, dQ^C(u) = \frac{1}{2}$$

holds for all $i \in \{1, \ldots, d\}$.

**2.5 Examples.**

1. The copula measure of $M$ satisfies $Q^M(\{u \in \mathbb{R}^d \mid u_i = u_j \text{ for all } i, j \in \{1, \ldots, d\}\}) = 1$; see [13, Example 2.2]. Moreover, the identity

$$\int_{\mathbb{R}^d} f(u) \, dQ^M(u) = \int_{\mathbb{R}^d} f(u_1) \, dQ^M(u)$$

holds for all positive measurable functions $f : \mathbb{R}^d \to \mathbb{R}$.

2. The copula measure of $\Pi$ satisfies $Q^\Pi = \lambda^d$ where $\lambda^d$ denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$; see [13, Example 2.2].

### 3 A Biconvex Form for Copulas

In the present section we study the integration of a copula with respect to a copula measure. To this end, we introduce a biconvex form on $\mathcal{C} \times \mathcal{C}$ which we investigate with regard to bounds, symmetry, convergence and order.
Note that a copula is a positive measurable function. We define the map \([\cdot, \cdot] : \mathcal{C} \times \mathcal{C} \to \mathbb{R}\) by letting

\[
[C, D] := \int_{I^d} C(u) \, dQ^D(u)
\]

The map \([\cdot, \cdot]\) is linear with respect to convex combinations in both arguments and is therefore called a biconvex form. For the copulas \(M\) and \(\Pi\) we have the following results:

### 3.1 Example

The copulas \(M\) and \(\Pi\) satisfy

\[
[M, M] = \frac{1}{2}, \quad [M, \Pi] = \frac{1}{d+1} = [\Pi, M] \quad \text{and} \quad \Pi, \Pi = \frac{1}{2d}
\]

Thus, the value \([M, M]\) does not depend on the dimension of \(M\).

The biconvex form \([\cdot, \cdot]\) is positive:

### 3.2 Lemma

The inequality \(0 \leq [C, D]\) holds for all \(C, D \in \mathcal{C}\).

Considering the identity \([M, \Pi] = [\Pi, M]\) stated in Example 3.1, one may suppose that the biconvex form \([\cdot, \cdot]\) is symmetric. This is actually true for \(d = 2\); see [9, p.196]. More precisely, we have the following result:

### 3.3 Theorem

The biconvex form \([\cdot, \cdot]\) is symmetric if, and only if, \(d = 2\).

**Proof.** Consider \(d = 2\) and \(C, D \in \mathcal{C}\). Then \(Q^D\) satisfies \(0 \leq Q^D[[u, 1] \setminus (u, 1)] \leq Q^D[[u_1, 1] \setminus (u_1, 1) \times 1] = \lambda[[u_1, 1] \setminus (u_1, 1)] = 0\) for all \(u \in \mathbb{I}^2\), and Fubini’s theorem together with Lemma 2.4 yields

\[
[C, D] = \int_{I^2} C(v) \, dQ^D(v)
= \int_{I^2} \int_{I^2} X_{[0,v]}(u) \, dQ^C(u) \, dQ^D(v)
= \int_{I^2} \int_{I^2} X_{[u,1]}(v) \, dQ^D(v) \, dQ^C(u)
= \int_{I^2} Q^D[[u, 1]] \, dQ^C(u)
= \int_{I^2} Q^D[[u, 1]] \, dQ^C(u)
= \int_{I^2} (Q^D[[0, 1]] - Q^D[[0, u_1] \times 1] - Q^D[1 \times [0, u_2]] + Q^D[[0, u]]) \, dQ^C(u)
= \int_{I^2} (D(1, 1) - D(u_1, 1) - D(1, u_2) + D(u)) \, dQ^C(u)
= \int_{I^2} (1 - u_1 - u_2) \, dQ^C(u) + [D, C]
= [D, C]
\]

Therefore, the biconvex form \([\cdot, \cdot]\) is symmetric. The converse follows from Examples 3.4 below.

The next examples assert that, for any \(d \geq 3\), the map \([\cdot, \cdot]\) fails to be symmetric.
3.4 Examples.

1. Consider \( d \in 2\mathbb{N} + 1 \). Then the function \( E : I^d \to \mathbb{R} \) given by
   \[
   E(u) := II(u) + \prod_{i=1}^{d} u_i(1 - u_i)
   \]
   is a copula (see [1, Example 4.5]) and satisfies
   \[
   [E, II] = \frac{1}{2^d} + \frac{1}{6d} \neq \frac{1}{2^d} + \frac{1}{6d} = [II, E]
   \]

2. Consider \( d \in 2\mathbb{N} + 2 \). Then, for \( j \in \{1, \ldots, d\} \), the function \( E : I^d \to \mathbb{R} \) given by
   \[
   E(u) := II(u) + u_j \prod_{i=1, i\neq j}^{d} u_i(1 - u_i)
   \]
   is a copula (see [5, Proof of Theorem 3.1]) and satisfies
   \[
   [E, II] = \frac{1}{2^d} + \frac{3}{6d} \neq \frac{1}{2^d} + \frac{3}{6d} = [II, E]
   \]

The following result deals with convergence of the biconvex form \([\, , \,]\); it immediately follows from the dominated convergence theorem:

3.5 Theorem. For every \( D \in \mathcal{C} \) and every sequence of copulas \( \{C_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C} \) that converges uniformly to a copula \( C \in \mathcal{C} \), we have
   \[
   \lim_{n \to \infty} [C_n, D] = [C, D].
   \]

Now, we study the biconvex form \([\, , \,]\) with regard to the pointwise order relation \( \leq \) on \( \mathcal{C} \), and distinguish between the first and second argument of \([\, , \,]\):

3.6 Theorem.

1. Each of the maps \( C \mapsto [C, D], D \in \mathcal{C} \), is monotonically increasing with regard to the pointwise order relation.
2. Each of the maps \( D \mapsto [C, D], C \in \mathcal{C} \), is monotonically increasing with regard to the pointwise order relation if, and only if, \( d = 2 \).

Proof. Assertion (1) is evident. Now, if \( d = 2 \), from (1) and the symmetry of the biconvex form \([\, , \,]\), we have that the map \( D \mapsto [C, D] \) is monotonically increasing with regard to the pointwise order relation. The converse follows from Examples 3.7 below. \( \square \)

The next examples assert that, for any \( d \geq 3 \), the map \( D \mapsto [II, D] \) fails to be monotonically increasing with regard to the pointwise order relation:

3.7 Examples.

1. Consider \( d \in 2\mathbb{N} + 1 \). Then the copula \( E \) discussed in Example 3.4 (1) satisfies \( II \leq E \), but
   \[
   [II, II] = \frac{1}{2^d} > \frac{1}{2^d} + \frac{1}{6d} = [II, E]
   \]
2. Consider \( d \in 2\mathbb{N} + 2 \). Then the copula \( E \) discussed in Example 3.4 (2) satisfies \( II \leq E \), but
   \[
   [II, II] = \frac{1}{2^d} > \frac{1}{2^d} + \frac{3}{6d} = [II, E]
   \]

We conclude this section with the following result, whose proof for \( d = 2 \) is due to [4].

3.8 Theorem. Let \( D_1, D_2 \in \mathcal{C} \). Then, \( [C, D_1] = [C, D_2] \) for all \( C \in \mathcal{C} \) if, and only if, \( D_1 = D_2 \).
Proof. Assume that \([C, D_1] = [C, D_2]\) for all \(C \in \mathcal{C}\). First, consider \(v \in \mathbb{I}^d\) with \(v_i = 0\) for some \(i \in \{1, \ldots, d\}\). Then property (ii) of a copula implies \(D_1(v) = 0 = D_2(v)\).

Now, consider \(v \in (0, 1]\). Then there exists some \(\epsilon \in (0, 1)\) such that \(M_\epsilon := [\epsilon 1, v - \epsilon 1] \neq \emptyset\). For every \(L \subseteq \{1, \ldots, d\}\), we put \(M_{L, v, \epsilon} := (\eta_L(0, v - \epsilon 1), \eta_L(\epsilon 1, v))\) and define the function \(C_{v, \epsilon} : \mathbb{I}^d \to \mathbb{R}\) given by

\[
C_{v, \epsilon}(u) := II(u) + \sum_{L \subseteq \{1, \ldots, d\}} (-1)^{|L|} \lambda^d([0, u] \cap M_{L, v, \epsilon})
\]

Then \(C_{v, \epsilon}\) satisfies \(C_{v, \epsilon}(u) = II(u) + \epsilon^d\) for all \(u \in M_\epsilon\) and \(C_{v, \epsilon}(u) = II(u)\) for all \(u \in \mathbb{I}^d \setminus (0, v)\). We prove that \(C_{v, \epsilon}\) is a copula: For every \(u, v \in \mathbb{I}^d\) such that \(u \leq v\) we obtain

\[
\sum_{K \subseteq \{1, \ldots, d\}} (-1)^{|K|} C_{v, \epsilon}(\eta_K(u, v))
\]

\[
= \sum_{K \subseteq \{1, \ldots, d\}} (-1)^{|K|} II(\eta_K(u, v)) + \sum_{K \subseteq \{1, \ldots, d\}} (-1)^{|K|} \sum_{L \subseteq \{1, \ldots, d\}} (-1)^{|L|} \lambda^d([0, \eta_K(u, v)] \cap M_{L, v, \epsilon})
\]

\[
= \prod_{i=1}^{d} (v_i - u_i) + \sum_{K \subseteq \{1, \ldots, d\}} (-1)^{|K|} \sum_{L \subseteq \{1, \ldots, d\}} (-1)^{|L|} \lambda^d([0, \eta_K(u, v)] \cap M_{L, v, \epsilon})
\]

\[
= \prod_{i=1}^{d} (v_i - u_i) + \sum_{L \subseteq \{1, \ldots, d\}} (-1)^{|L|} \lambda^d([u, v] \cap M_{L, v, \epsilon})
\]

\[
\geq 0
\]

which proves condition (i); conditions (ii) and (iii) are straightforward. Thus, \(C_{v, \epsilon}\) is a copula.

Now, choose some \(m \in \mathbb{N}\) with \(m \geq 1/\epsilon\). For all \(n \geq m\), we then have \(M_{1/n} \neq \emptyset\) and \(C_{v, 1/n} \in \mathcal{C}\). We further define the function \(f_{v, n} : \mathbb{I}^d \to \mathbb{R}\), \(n \geq m\), given by

\[
f_{v, n} := \frac{C_{v, 1/n} - II}{1/n^d}
\]

Then \(f_{v, n}\) is continuous and hence measurable, and satisfies

\[
f_{v, n}(u) = \begin{cases} 
0 & u \in \mathbb{I}^d \setminus (0, v) \\
1 & u \in M_{1/n} \\
0 & u \in (0, 1) \text{ otherwise}
\end{cases}
\]

as well as \(f_{v, n}^{-1}((0)) = \mathbb{I}^d \setminus (0, v)\) for all \(n \geq m\). Moreover, the sequence \(f_{v, n} \mid_{\mathbb{I}^d}\) is monotonically increasing with \(\sup_{n \geq m} f_{v, n} = \chi(0, v)\). Since every copula measure \(Q\) satisfies \(0 \leq Q([0, u] \setminus (0, u)) \leq Q([0, u_1] \setminus (0, u_1) \times I^{d-1}) = A([0, u_1] \setminus (0, u_1)) = 0\) for all \(u \in I^d\), the monotone convergence theorem yields

\[
D_1(v) = Q^{B_1}([0, v])
\]

\[
= Q^{B_1}([0, v])
\]

\[
= \int_{I^d} \chi(0, v)(u) \, dQ^B(u)
\]

\[
= \int_{I^d} \sup_{n \geq m} f_{v, n}(u) \, dQ^B(u)
\]

\[
= \sup_{n \geq m} \int_{I^d} f_{v, n}(u) \, dQ^B(u)
\]

\[
= \sup_{n \geq m} n^d \int_{I^d} C_{v, 1/n}(u) - II(u) \, dQ^B(u)
\]

\[
= n^d \int_{I^d} ([C_{v, 1/n}, D_1] - [II, D_1])
\]
\[
\begin{align*}
= \sup_{n \geq m} \left( [C_{v,1/n}, D_2] - [H, D_2] \right) \\
= D_2(v)
\end{align*}
\]

which concludes the proof. \(\square\)

**Remark.** For \(d = 2\), the symmetry discussed in Theorem 3.3 implies results analogous to those obtained in Theorems 3.5 and 3.8 for the other argument of \([., .]\).

### 4 The Biconvex Form and \(\tau\)

In the present section we first show that the total reflection \(\tau\) changes the arguments of the biconvex form \([., .]\) (Lemma 4.1). This is of interest since it provides a useful tool in order to prove that certain properties which are valid for one of the arguments of the map \([., .]\) are also valid for the other argument. By applying this tool the results for the biconvex form \([., .]\) discussed in Section 3 can be extended or completed.

Lemma 4.1 is essentially due to [1, Lemma 3.1]:

**4.1 Lemma.** The identity \([C, D] = [\tau(D), \tau(C)]\) holds for all \(C, D \in \mathbb{C}\).

**Proof.** Theorem 2.2 first implies

\[
((\tau(D)) \circ T(\tau))(u) = (\tau(D))(1 - u)
= Q^{r(D)}[[0, 1 - u]]
= (Q^{D})_{T(\tau)}([T(\tau)]^{-1}([u, 1])]
= (Q^{D})_{T(\tau)\circ T(\tau)}([u, 1]]
= Q^{D}[[u, 1]]
\]

for all \(D \in \mathbb{C}\) and all \(u \in I^d\). Lemma 2.3 together with Fubini’s theorem then yield

\[
[\tau(D), \tau(C)] = \int_{I^d} (\tau(D))(u) \, dQ^{r(C)}(u)
= \int_{I^d} ((\tau(D)) \circ T(\tau))(u) \, dQ^C(u)
= \int_{I^d} Q^{D}[[u, 1]] \, dQ^C(u)
= \int_{I^d} \int_{I^d} \chi_{[u,1]}(v) \, dQ^D(v) \, dQ^C(u)
= \int_{I^d} \int_{I^d} \chi_{[0,v]}(u) \, dQ^C(u) \, dQ^D(v)
= \int_{I^d} C(v) \, dQ^D(v)
= [C, D]
\]

for all \(C, D \in \mathbb{C}\). This proves the assertion. \(\square\)

We are now able to identify the upper bound of the biconvex form \([., .]\); Theorem 4.2 hence completes Lemma 3.2:
4.2 Theorem. The inequality
\[ 0 \leq [C, D] \leq \frac{1}{2} \]
holds for all \( C, D \in \mathcal{C} \). In particular, both bounds are attainable.

Proof. Lemma 3.2, the fact that \( C \leq M \) for all \( C \in \mathcal{C} \) together with Theorem 3.6, Lemma 4.1, the fact that \( \tau(M) = M \) and Example 3.1 yield \( 0 \leq [C, D] \leq [M, D] = [\tau(D), \tau(M)] = [\tau(D), M] \leq [M, M] = 1/2 \) for all \( C, D \in \mathcal{C} \). That the lower bound is attainable follows from Example 5.8 below.

The following result deals with convergence of the biconvex form \([\ , \ ]\) and extends Theorem 3.5:

4.3 Theorem. For every sequence of copulas \( \{C_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C} \) that converges uniformly to a copula \( C \in \mathcal{C} \), and every sequence of copulas \( \{D_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C} \) that converges uniformly to a copula \( D \in \mathcal{C} \), we have \( \lim_{n \to \infty} [C_n, D_n] = [C, D] \).

Proof. Consider \( \varepsilon \in (0, \infty) \). Since \( \{C_n\}_{n \in \mathbb{N}} \) converges uniformly to \( C \), there exists some \( n_1 \in \mathbb{N} \) such that, for every \( n \geq n_1 \), we have \( |C_n(u) - C(u)| < \frac{\varepsilon}{2} \) for all \( u \in \mathbb{R}^d \). Moreover, since \( \{D_n\}_{n \in \mathbb{N}} \) converges uniformly to \( D \), it follows that \( |\tau(D_n)(u) - \tau(D)(u)| < \frac{\varepsilon}{2} \) for all \( u \in \mathbb{R}^d \). Thus, for every \( n \geq n_2 \), Lemma 4.1 yields
\[
|\tau(D_n)(u) - \tau(D)(u)| < \frac{\varepsilon}{2} \quad \text{for all } u \in \mathbb{R}^d.
\]

This proves the assertion.

Now, we study the biconvex form \([\ , \ ]\) with regard to the concordance order \( \preceq_C \) on \( \mathcal{C} \) \((C \preceq_D D \text{ if } C \subseteq D \text{ and } \tau(C) \leq \tau(D))\), and we distinguish between the first and second argument of \([\ , \ ]\):

4.4 Theorem.
1. Each of the maps \( C \mapsto [C, D], D \in \mathcal{C} \), is monotonically increasing with regard to the concordance order relation.
2. Each of the maps \( D \mapsto [C, D], C \in \mathcal{C} \), is monotonically increasing with regard to the concordance order relation.

Proof. Assertion (1) is evident and (2) follows from (1) and Lemma 4.1.

Remark. Note that, for \( d = 2 \), pointwise order is the same as concordance order. Thus, each of the maps \( D \mapsto [C, D], C \in \mathcal{C} \), is monotonically increasing as stated in Theorems 3.6 and 4.4.

We conclude this section by completing Theorem 3.8:

4.5 Theorem. Let \( C_1, C_2, D_1, D_2 \in \mathcal{C} \).
1. \([C_1, D] = [C_2, D] \text{ for all } D \in \mathcal{C} \text{ if, and only if, } C_1 = C_2\).
2. \([C, D_1] = [C, D_2] \text{ for all } C \in \mathcal{C} \text{ if, and only if, } D_1 = D_2\).
5 The Biconvex Form and $\Gamma$

In the present section we investigate the biconvex form $[\cdot, \cdot]$ with respect to the transformations in $\Gamma$. We first identify those transformations $\gamma \in \Gamma$ which satisfy $[\gamma(C), \gamma(D)] = [C, D]$ for all $C, D \in \mathcal{C}$. Finally, we consider a class of copulas $\mathcal{D}$ such that $[C, D] = 1/2^d$ holds for all $C, D \in \mathcal{D}$.

We first present a representation for $[\gamma_1(C), \gamma_2(D)]$ with $\gamma_1, \gamma_2 \in \Gamma$:

**5.1 Lemma.** The identity

$$[\gamma_1(C), \gamma_2(D)] = \int_{\mathbb{I}^d} (\gamma_1(C) \circ T(\gamma_2))(u) \ dQ^B(u)$$

holds for all $\gamma_1, \gamma_2 \in \Gamma$ and all $C, D \in \mathcal{C}$.

**Proof.** The result is immediate from Lemma 2.3. □

We now identify transformations in $\Gamma$ which preserve the value of the biconvex form $[\cdot, \cdot]$. Theorem 5.2 emphasizes the particular role of the transformations which belong to the subgroup $I^{n,r}$. The first implication in Theorem 5.2 (2) is essentially due to [3, Lemma 2.3].

**5.2 Theorem.** Let $\gamma \in \Gamma$.

1. $[\gamma(C), \gamma(D)] = [C, D]$ for all $C \in \mathcal{C}$ if, and only if, $\gamma \in I^{n,r}$.
2. Let $d = 2$. Then, $[\gamma(C), \gamma(D)] = [C, D]$ for all $C, D \in \mathcal{C}$ if, and only if, $\gamma \in I^{n,r}$.
3. Let $d \geq 3$. Then, $[\gamma(C), \gamma(D)] = [C, D]$ for all $C, D \in \mathcal{C}$ if, and only if, $\gamma \in I^r$.

**Proof.** We first show that every permutation preserves the value of the biconvex form $[\cdot, \cdot]$. Lemma 5.1 implies

$$[\pi_{i,i+1}(C), \pi_{i,i+1}(D)] = \int_{\mathbb{I}^d} (\pi_{i,i+1}(C) \circ T(\pi_{i,i+1}))(u) \ dQ^B(u)$$

for all $i \in \{1, \ldots, d-1\}$ and all $C, D \in \mathcal{C}$. The assertion then follows from the fact that $\{\pi_{i,i+1}, i \in \{1, \ldots, d-1\}\}$ generates $I^r$.

We further show that none of the transformations $\gamma \in \Gamma \setminus I^{n,r}$ satisfies $[\gamma(C), \gamma(D)] = [C, D]$ for all $C, D \in \mathcal{C}$. To this end, consider $\gamma \in \Gamma \setminus I^{n,r}$. Then there exist unique $\pi \in I^r$ and $\nu \in I^r \setminus I^r$ such that $\gamma = \nu \circ \pi$ (see [5, Lemma 3.7]), and the copula $M$ satisfies $\gamma(M) = (\nu \circ \pi)(M) = \nu(M)$ (see [5, Example 4.2(1)]). Example 5.8 below and Example 3.1 then yield $[\gamma(M), \gamma(M)] = [\nu(M), \nu(M)] = 0 \neq 1/2 = [M, M]$.

We now prove (1). It remains to show that every transformation $\gamma \in I^{n,r} \setminus I^r$ satisfies $[\gamma(C), \gamma(C)] = [C, C]$ for all $C \in \mathcal{C}$. To this end, consider $\gamma \in I^{n,r} \setminus I^r$. Then there exists a unique $\pi \in I^r$ such that $\gamma = \pi \circ \tau$ (see [5, Lemma 3.7]). Lemma 4.1 then yields $[\gamma(C), \gamma(C)] = [\pi(\tau(C)), \pi(\tau(C))] = [\tau(C), \tau(C)] = [C, C]$ for all $C \in \mathcal{C}$.

We further prove (2). It remains to show that every transformation $\gamma \in I^{n,r} \setminus I^r$ satisfies $[\gamma(C), \gamma(D)] = [C, D]$ for all $C, D \in \mathcal{C}$. To this end, consider $d = 2$ and $\gamma \in I^{n,r} \setminus I^r$. Then there exists a unique $\pi \in I^r$ such that $\gamma = \pi \circ \tau$ (see [5, Lemma 3.7]). Lemma 4.1 and Theorem 3.3 then yield $[\gamma(C), \gamma(D)] = [\pi(\tau(C)), \pi(\tau(D))] = [\tau(C), \tau(D)] = [C, D]$ for all $C, D \in \mathcal{C}$. 

**Proof.** Assertion (2) was proven in Theorem 3.8 and (1) follows from (2) and Lemma 4.1. □
We finally prove (3). It remains to show that none of the transformations $\gamma \in \Gamma^{a,r} \setminus \Gamma^{a}$ satisfies $[\gamma(C), \gamma(D)] = [C, D]$ for all $C, D \in \mathcal{C}$. This result will be discussed in Examples 5.3 below. □

The next examples assert that, for any $d \geq 3$, the transformations $\gamma \in \Gamma^{a,r} \setminus \Gamma^{a}$ fail to satisfy

$$[\gamma(C), \gamma(D)] = [C, D]$$

for all $C, D \in \mathcal{C}$:

5.3 Examples. Consider $\gamma \in \Gamma^{a,r} \setminus \Gamma^{a}$. Then there exists a unique $\pi \in \Gamma^{a}$ such that $\gamma = \pi \circ \tau$ (see [5, Lemma 3.7]).

1. Consider $d \in 2\mathbb{N} + 1$. Then the copula $E$ discussed in Example 3.4 (1) satisfies

$$[\gamma(E), \gamma(\Pi)] = [\tau(E), \tau(\Pi)] = [\Pi, E] = \frac{1}{2^d} + (-1)^{d} \frac{1}{6^{d}} \neq \frac{1}{2^d} + \frac{1}{6^{d}} = [E, \Pi]$$

2. Consider $d \in 2\mathbb{N} + 2$. Then the copula $E$ discussed in Example 3.4 (2) satisfies

$$[\gamma(E), \gamma(\Pi)] = [\tau(E), \tau(\Pi)] = [\Pi, E] = \frac{1}{2^d} + (-1)^{d-1} \frac{3}{6^{d}} \neq \frac{1}{2^d} + \frac{3}{6^{d}} = [E, \Pi]$$

Thus, the transformations in $\Gamma \setminus \Gamma^{a,r}$ do not preserve the value of the biconvex form $[\cdot, \cdot]$.

We further investigate the map $[\cdot, \cdot]$ with respect to reflections. Lemma 5.4 provides a result for partial reflections:

5.4 Lemma. The identities

$$[C, D] + [v_k(C), v_k(D)] = \int_{I^d} C(\eta_k(u, 1)) \ dQ^D(u) = [C, v_k(D)] + [v_k(C), D]$$

hold for all $k \in \{1, \ldots, d\}$ and all $C, D \in \mathcal{C}$.

Proof. Lemma 5.1 first implies

$$[C, v_k(D)] + [v_k(C), D] = \int_{I^d} (C \circ T(v_k))(u) \ dQ^D(u) + \int_{I^d} (v_k(C))(u) \ dQ^D(u)$$

$$= \int_{I^d} C(\eta_k(u, 1 - u)) \ dQ^D(u) + \int_{I^d} C(\eta_k(u, 1)) - C(\eta_k(u, 1 - u)) \ dQ^D(u)$$

$$= \int_{I^d} C(\eta_k(u, 1)) \ dQ^D(u)$$

for all $C, D \in \mathcal{C}$. This identity together with Lemma 2.3 further yields

$$[C, D] + [v_k(C), v_k(D)] = [C, v_k(v_k(D))] + [v_k(C), v_k(D)]$$

$$= \int_{I^d} C(\eta_k(u, 1)) \ dQ^{v_k(D)}(u)$$

$$= \int_{I^d} C(\eta_k(T(v_k))(u), 1) \ dQ^D(u)$$

$$= \int_{I^d} C(\eta_k(\eta_k(u, 1 - u), 1)) \ dQ^D(u)$$

$$= \int_{I^d} C(\eta_k(u, 1)) \ dQ^D(u)$$

$$= [C, v_k(D)] + [v_k(C), D]$$
for all $C, D \in \mathcal{C}$. This proves the assertion. □

The next result is a refinement of Lemma 5.4 for $d = 2$, and it completes Theorem 5.2 (2); implication (a) to (c) in Corollary 5.5 is essentially due to [3, Lemma 2.3]:

**5.5 Corollary.** Let $d = 2$ and $\gamma \in \Gamma$. Then the following assertions are equivalent:

(a) $\gamma \in \Gamma \setminus \Gamma^{\pi, T}$.
(b) $[C, C] + [\gamma(C), \gamma(C)] = \frac{1}{2}$ for all $C \in \mathcal{C}$.
(c) $[C, D] + [\gamma(C), \gamma(D)] = \frac{1}{2}$ for all $C, D \in \mathcal{C}$.

**Proof.** Assume first that (a) holds. We prove (c). To this end, consider $\gamma \in \Gamma \setminus \Gamma^{\pi, T}$. Then assertion (c) follows from Lemma 5.4, Lemma 2.4 and Theorem 5.2 (2). Implication (c) to (b) is evident. We finally prove that (b) implies (a). To this end, consider $\gamma \in \Gamma^{\pi, T}$. The fact that $\gamma(M) = M$ (see [5, Example 4.2(1)]) and Example 3.1 then yield $[M, M] + [\gamma(M), \gamma(M)] = 2 [M, M] = 1 \neq \frac{1}{2}$. This proves the assertion. □

We proceed with a generalization of Lemma 5.4:

**5.6 Corollary.** The identities

$$
\sum_{L \subseteq K} \left[ \left( \bigcirc_{l \in L} v_l \right)(C), \left( \bigcirc_{l \in L} v_l \right)(D) \right] = \int_{I^d} C(q^L(u, 1)) \, dQ^D(u) = \sum_{L \subseteq K} \left[ \left( \bigcirc_{l \in L} v_l \right)(C), \left( \bigcirc_{l \in L \setminus K} v_l \right)(D) \right]
$$

hold for all $K \subseteq \{1, \ldots, d\}$ and all $C, D \in \mathcal{C}$. In particular, we have

$$
\sum_{v \in \Gamma^T} [v(C), v(D)] = 1
$$

for all $C, D \in \mathcal{C}$.

**Proof.** The first identities follow from Lemma 5.4 via induction. Moreover, we have

$$
\sum_{v \in \Gamma^T} [v(C), v(D)] = \sum_{L \subseteq \{1, \ldots, d\}} \left[ \left( \bigcirc_{l \in L} v_l \right)(C), \left( \bigcirc_{l \in L} v_l \right)(D) \right] = \int_{I^d} 1 \, dQ^D(u) = 1
$$

for all $C, D \in \mathcal{C}$. □

Finally, we discuss the biconvex form $[\cdot, \cdot]$ with regard to $\Gamma^\nu$–invariant arguments; Theorem 5.7 is immediate from Corollary 5.6 and the fact that $|\Gamma^\nu| = 2^d$:

**5.7 Theorem.** The identity

$$
[C, D] = \frac{1}{2^d}
$$

holds for all $\Gamma^\nu$–invariant copulas $C, D \in \mathcal{C}$.

Thus, for the class of $\Gamma^\nu$–invariant copulas, Theorem 5.7 allows us to solve the biconvex form $[\cdot, \cdot]$ without integrating. Note that the $\Gamma^\nu$–invariance of a copula is quite simple to verify due to the fact that the set $\{v_k, k \in \{1, \ldots, d\}\}$ generates $\Gamma^\nu$.

We conclude this section with an example required in Theorems 4.2 and 5.2:

**5.8 Example.** For every transformation $v \in \Gamma^\nu \setminus \Gamma^T$ the copula $M$ satisfies

$$
[v(M), v(M)] = 0
$$

Indeed, consider $v \in \Gamma^\nu \setminus \Gamma^T$ such that $v = \bigcirc_{k \in K} v_k$ for some $K \subseteq \{1, \ldots, d\}$ with $1 \leq |K| \leq d - 1$. Then $T(v) = \bigcirc_{k \in K} T(v_k) = q_k(u, 1 - u)$ and Lemma 5.1, [5, Theorem 4.1, Example 2.5 (1) and Lemma 2.4 yield

$$
[v(M), v(M)] = \int_{I^d} (v(M) \circ T(v))(u) \, dQ^M(u)
$$
\[
\int (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} M \left( \eta_L (\eta_K (u, 1 - u), 1 - \eta_K (u, 1 - u)), 1) \right) \, dQ^M(u)
= \int (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} M \left( \eta_L (\eta_K (u, 1), 1) \right) \, dQ^M(u)
= (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} \int_{\mu} M(\eta_L (u, 1)) \, dQ^M(u)
= (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} \int_{\mu} M(\eta_L (u_1 1, 1)) \, dQ^M(u)
= (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} \int_{\mu} u_1 \, dQ^M(u)
= (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} \frac{1}{2}
= 0
\]

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