Momentum and Uncertainty Relations in the Entropic Approach to Quantum Theory

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Abstract

In the Entropic Dynamics (ED) approach to quantum theory the particles have well-defined positions but since they follow non differentiable Brownian trajectories they cannot be assigned an instantaneous momentum. Nevertheless, four different notions of momentum can be usefully introduced. We derive relations among them and the corresponding uncertainty relations. The main conclusion is that momentum is a statistical concept: in ED the momenta are not properties of the particles; they are attributes of the probability distributions.

1 Introduction

In the Entropic Dynamics (ED) framework quantum theory is derived as an application of the method of maximum entropy [1]-[4]. The goal is to do for quantum mechanics what Jaynes did for statistical mechanics [5].

The basic assumption is that in addition to the particle of interest the world contains other variables whose entropy $S$ depends on the positions $x$ of the particles, $S = S(x)$. An important new feature is that the phase of the wave function also receives a statistical interpretation: the phase keeps track of the entropy $S(x)$ of those extra variables.

Entropic Dynamics differs from other information-based approaches to quantum theory in that the position observable assumes a privileged role: particles have well-defined, albeit unknown, positions. This opens the possibility of explaining all other observables in purely informational terms. In this paper our specific goal is to discuss momentum.

The notion of momentum has undergone a remarkable evolution from Descartes’ early imperfect notion of a scalar “quantity of motion” to Newton’s vectorial...
quantity of motion, then through Lagrange’s generalized momenta and Hamilton’s canonical momenta to the modern quantum version of momentum as the generator of infinitesimal translations. Each theory of motion demands its own concept of momentum. Our goal is to identify what concept, within the entropic framework, plays the role of momentum.

Since particles follow Brownian trajectories that are continuous but non-differentiable it is not possible to assign an instantaneous momentum to the particles. Nevertheless, four different notions of momentum can be usefully introduced. They are not associated to the particles but rather to their probability distributions: (1) the current momentum is associated to the velocity with which probabilities flow; (2) the drift momentum reflects flow along the entropy gradient; (3) the osmotic momentum is associated to the velocity with which probabilities diffuse; and (4) the familiar quantum momentum is the generator of infinitesimal translations. We find relations among these four momenta and the corresponding uncertainty relations. There is a formal similarity to analogous relations derived in the context of Nelson’s stochastic mechanics [6]–[10] and the Hall-Reginatto exact uncertainty formalism [11].

We show that in the entropic framework the current momentum can reasonably be called the momentum: its expected value agrees with that of the quantum momentum operator and in the classical limit it coincides with the classical momentum. The more important conclusion, however, is that in the entropic framework momentum is a statistical concept. In ED, unlike the standard interpretation of quantum mechanics, the positions of particles have definite values just as they would in classical physics. The price we pay for this feature is that particles do not have a momentum. More explicitly, momentum is not an attribute of the particles but of the probability distributions.

Entropic dynamics is not quite a “theory”; rather it is a framework for the construction of theories. In addition to familiar examples such as the standard quantum theory, its classical limit, and the usual dissipative Brownian motion it also includes less familiar examples. We briefly explore a new kind of model with unusual hybrid features [1]. The model resembles Brownian motion but there is no energy dissipation. It obeys both the classical Hamilton-Jacobi equation and also the usual uncertainty principle. It applies to the usual quantum regime where \( \hbar \) is not negligible but obeys a “classical” non-linear Schrödinger equation [12].

Section 2 is devoted to a brief review of entropic quantum dynamics—for details see [1]. Momentum and uncertainty relations are discussed in section 3, the hybrid model in section 4, and we conclude in section 5.

2 Entropic Quantum Dynamics

For simplicity we discuss a single particle. The configuration space \( \mathcal{X} \) is a flat three dimensional space with the Euclidean metric, \( \gamma_{ab} = \delta_{ab}/\sigma^2 \). The full significance of the scale factor \( \sigma^2 \) only becomes apparent when discussing several particles with different masses [1].
In addition to the particle of interest the world contains other variables—we call them $y$. Not much needs to be known about them except that the unknown $y$ are described by a probability distribution $p(y|x)$ that depends on the position $x$ of the particle. The entropy of the $y$ variables is given by

$$S[p, q] = -\int dy \ p(y|x) \log \frac{p(y|x)}{q(y)} = S(x).$$

where $q(y)$ is some underlying measure which need not be specified further. Since $x$ enters as a parameter in $p(y|x)$ the entropy is a function of $x$:

$$S[p, q] = S(x).$$

The probability $P(x'|x)$ that the particle takes a short step from $x$ to a nearby point $x'$ is obtained using the method of maximum entropy subject to two constraints (plus normalization). The first constraint is that as $(x, y)$ changes to the new $(x', y')$ the uncertainty in the new $y'$ depends only on the new position $x'$, and not on any previous value, $x$, that is, $p(y|x)$ changes to the corresponding $p(y'|x')$. The second constraint reflects the physical fact that motion is continuous, that is, motion over large distances happens through the successive accumulation of many short steps, $\Delta x = x' - x$. We require that the expectation $\langle \Delta \ell^2 \rangle = \langle \gamma_{ab} \Delta x^a \Delta x^b \rangle$ be some small numerical value, which we take to be independent of $x$ in order to reflect the translational symmetry of the space $X$. The resulting transition probability from $x^a$ to $x'^a = x^a + \Delta x^a$ is

$$P(x'|x) \approx \frac{1}{Z(x)} \exp \left[ -\frac{\tau}{2\sigma^2 \Delta t} \delta_{ab} (\Delta x^a - \Delta \tau^a) (\Delta x^b - \Delta \tau^b) \right].$$

where $\tau$ is a constant that defines the units of the time interval $\Delta t$ [2]; $\Delta \tau^a$ is the drift velocity. Any displacement $\Delta x^a$ can be expressed as an expected drift plus a fluctuation $\Delta w^a$,

$$\Delta x^a = b^a (x) \Delta t + \Delta w^a,$$

where

$$\langle \Delta w^a \rangle = 0 \quad \text{and} \quad \langle \Delta w^a \Delta w^b \rangle = \frac{\sigma^2}{\tau} \Delta t \delta^{ab}. $$

Since the fluctuations are the order of $\Delta w \sim (\Delta t)^{1/2}$ the trajectory of the particle is continuous but not differentiable as in Brownian motion.

Standard methods show that the successive iteration of $P(x'|x)$ yields a probability distribution $\rho(x, t)$ that evolves according to Fokker-Planck equation

$$\partial_t \rho = -\partial_a (\rho v^a)$$

where $v^a$ is the current velocity defined by

$$v^a = b^a + u^a$$
where
\[ u^a = -\frac{\sigma^2}{\tau} \partial^a \log \rho^{1/2} , \] (8)
is the osmotic velocity. The drift velocity reflects motion up the entropy gradient, while the osmotic velocity reflects diffusion as can be seen when written as \( \rho u^a \propto \partial^a \rho \), which is Fick’s law of diffusion. The current velocity can also be written as
\[ v^a = \frac{\sigma^2}{\tau} \partial^a \phi \quad \text{with} \quad \phi(x,t) = S(x) - \log \rho^{1/2}(x,t) \] (9)

The dynamics described so far is pure diffusion. In quantum mechanics, the wave function has two degrees of freedom, the amplitude and the phase. So far we have only one degree of freedom and that is \( \rho(x,t) \). In order to promote \( \phi(x,t) \) to a dynamical degree of freedom we allow \( p_y(x) \) and \( S(x) \) to be functions of time, \( S = S(x,t) \). The time evolution of \( S(x,t) \) is determined by imposing yet another constraint, that a certain quantity — an “energy” — be conserved. Thus we impose that the diffusion be non-dissipative. To this end introduce an energy functional,
\[ E[\rho, S] = \int d^3 x \rho(x,t) \left( \frac{1}{2} m v^2 + \frac{1}{2} \mu u^2 + V(x) \right) \] (10)

where \( m \) and \( \mu \) are constants that will be called the mass and the osmotic mass respectively. Imposing that the energy be conserved for arbitrary initial choices of \( \rho \) and \( S \) leads to the quantum Hamilton-Jacobi equation,
\[ \eta \dot{\phi} + \frac{\eta^2}{2m} (\partial_a \phi)^2 + V - \frac{\mu \eta^2}{2m^2} \nabla^2 \rho^{1/2} \rho^{1/2} = 0 , \] (11)

where we have defined a new constant \( \eta \) so that \( \eta \equiv m \sigma^2 / \tau \).

Eqs. (6) and (11) can be combined into a single complex equation by using \( \Psi = \rho^{1/2} e^{i\phi} \)
\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi + \frac{\eta^2}{2m} \left( 1 - \frac{\mu}{m} \right) \nabla^2 (\Psi \Psi^*)^{1/2} \Psi . \] (12)

In [1] we showed that it is possible to change units and rescale \( \eta \rightarrow \kappa \eta \) and \( \tau \rightarrow \kappa \tau \) by some constant \( \kappa \), while simultaneously introducing \( \kappa \phi_{\text{new}} = \phi \) and \( \mu_{\text{new}} = \kappa^2 \mu \). This means that there are essentially two possibilities: all theories with \( \mu > 0 \) are physically equivalent in that they can be rescaled/regraded to a theory with \( \mu_{\text{new}} = m \). Setting \( \kappa \eta = \hbar \) results in the Schrödinger equation,
\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi . \] (13)
The other possibility occurs for \( \mu = 0 \) which allows no regraduation and leads to a non-linear Schrödinger equation,
\[ i\hbar \dot{\Psi} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi + \frac{\hbar^2}{2m} \nabla^2 (\Psi \Psi^*)^{1/2} \Psi . \] (14)

This case will be further discussed in section 4. This concludes our brief review.
3 Momentum in Entropic Dynamics

During the transition from classical to quantum mechanics a central problem was to identify some concept that would correspond to the classical momentum in some appropriate limit. We face an analogous (but easier) problem: our goal is to identify what concept, within the entropic framework, may reasonably be called momentum.

Since the particle follows a Brownian non-differentiable trajectory it is clear that the classical momentum $m \frac{d\vec{x}}{dt}$ along the trajectory cannot be defined. The obvious momentum candidates correspond to the various velocities available to us. Thus, we define the drift, osmotic, and current momenta,

\[
\vec{p}_d = m \vec{\dot{b}} = \hbar \vec{\nabla} S ,
\]

\[
\vec{p}_o = m \vec{\dot{u}} = -\hbar \vec{\nabla} \log \rho^{1/2} ,
\]

\[
\vec{p}_c = m \vec{\dot{v}} = \hbar \vec{\nabla} \phi ,
\]

where $\phi$ is given in eq.(9). The fourth notion of momentum that one can introduce in ED is the differential operator that generates infinitesimal translations—it coincides, of course, with the standard quantum momentum $\vec{p}_q = -i\hbar \vec{\nabla}$. Notice that the three momenta $\vec{p}_d$, $\vec{p}_o$, and $\vec{p}_c$ are local functions of $\vec{x}$ and this makes them conceptually very different from the momentum operator $\vec{p}_q$. To explore their differences and similarities we calculate the first and second moments.

Expected Values

The important theorem here is the vanishing expectation of the osmotic momentum. Using (16) and since $\rho$ vanishes at infinity,

\[
\langle p^a_o \rangle = -\hbar \int d^3 x \rho \partial^a \log \rho^{1/2} = -\frac{\hbar}{2} \int d^3 x \partial^a \rho = 0 .
\]

Since $p_c = p_d + p_o$ the immediate consequence is that $\langle p^a_c \rangle = \langle p^a_d \rangle$.

To study the connection to the quantum mechanical momentum we calculate

\[
\langle p^a_q \rangle = \int d^3 x \Psi \frac{\hbar}{i} \partial^a \Psi .
\]

Using $\Psi = \rho^{1/2} e^{i(S - \log \rho^{1/2})}$ and (18) one gets

\[
\langle p^a_q \rangle = -i\hbar \int d^3 x \rho \left( \partial^a \log \rho^{1/2} + i \partial^a S - i \partial^a \log \rho^{1/2} \right) = \hbar \langle \partial^a S \rangle .
\]

Therefore

\[
\langle \vec{p}_q \rangle = \langle \vec{p}_c \rangle = \langle \vec{p}_d \rangle ,
\]

the expectations of quantum momentum, current momentum and drift momentum coincide.
Uncertainty Relations

We start by stating a couple of definitions and an inequality. The variance of a quantity $A$ is

$$\text{Var}A = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2,$$  \hspace{1cm} (22)

and its covariance with $B$ is

$$\text{Cov} (A, B) = \langle (A - \langle A \rangle)(B - \langle B \rangle) \rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle.$$ \hspace{1cm} (23)

The general form of uncertainty relation to be used below follows from the Schwarz inequality,

$$\text{Var}A \text{Var}B = \langle (A - \langle A \rangle)^2 \rangle \langle (B - \langle B \rangle)^2 \rangle \geq |\langle (A - \langle A \rangle)(B - \langle B \rangle) \rangle|^2 = \text{Cov}^2 (A, B).$$ \hspace{1cm} (24)

Next we apply these notions to the various momenta. An analogous calculation in the context of stochastic mechanics is given in [9].

Osmotic Momentum

For simplicity we consider the one-dimensional case. The generalization is immediate. Eq. (24) is

$$\text{Var} \text{Var} \geq \text{Cov}^2.$$ \hspace{1cm} (25)

Using (16) and (18) we have

$$\text{Cov} (x, p_o) = \langle xp_o \rangle - \langle x \rangle \langle p_o \rangle = \langle xp_o \rangle \hspace{1cm} (26)$$

Therefore

$$\text{Var} x \text{Var} p_o \geq \left( \frac{\hbar}{2} \right)^2 \text{ or } \Delta x \Delta p_o \geq \frac{\hbar}{2}. \hspace{1cm} (27)$$

which coincides with the Heisenberg uncertainty relation.

Drift Momentum

The uncertainty relation is

$$\text{Var} x \text{Var} p_d \geq \text{Cov}^2 (x, p_d) \hspace{1cm} (28)$$

Consider

$$\text{Cov} (x, p_d) = \langle xp_d \rangle - \langle x \rangle \langle p_d \rangle \hspace{1cm} (29)$$
The integrands involve $\rho$ and $\partial S$ which can be chosen independently. We can choose as narrow a probability distribution as we like, for example $\rho \rightarrow \delta(x - x_0)$, which trivially leads to $\text{Cov}(x, p_d) = 0$. Therefore, the uncertainty relation for drift momentum is

$$(\text{Var } x) (\text{Var } p_d) \geq \text{Cov}^2(x, p_d) \geq 0 \quad \text{or} \quad \Delta x \Delta p_d \geq 0.$$  \hspace{1cm} (30)

**Quantum Momentum: the Schrödinger and the Heisenberg Uncertainty Relations**

There appears to be no useful insight to be found from the uncertainty relation for current momentum. It is nevertheless true that

$$(\text{Var } x) (\text{Var } p_c) \geq \text{Cov}^2(x, p_c)$$  \hspace{1cm} (31)

Since $p_c = p_d + p_o$, we have

$$\text{Cov} (x, p_c) = \text{Cov} (x, p_d) + \text{Cov} (x, p_o)$$  \hspace{1cm} (32)

Let us now focus our attention on the quantum momentum. Using $\Psi = \rho^{1/2} e^{i\phi}$, (16) and (17) we have, after an integration by parts,

$$\langle p_q^2 \rangle = \int dx \Psi^* \left( \frac{\hbar}{i} \partial \right)^2 \Psi = \langle p_c^2 \rangle + \langle p_o^2 \rangle.$$  \hspace{1cm} (33)

Together with eqs. (18) and (21) this leads to

$$\text{Var } p_q = \langle p_q^2 \rangle - \langle p_q \rangle^2 = \text{Var } p_c + \text{Var } p_o.$$  \hspace{1cm} (34)

Combining inequalities (27) and (31) gives,

$$(\text{Var } x) (\text{Var } p_q) \geq \text{Cov}^2(x, p_c) + \left( \frac{\hbar}{2} \right)^2.$$  \hspace{1cm} (35)

Finally, a straightforward calculation gives

$$\text{Cov} (x, p_q) = \frac{1}{2} \langle xp_q + p_q x \rangle - \langle x \rangle \langle p_q \rangle = \text{Cov} (x, p_c).$$  \hspace{1cm} (36)

Therefore,

$$(\text{Var } x) (\text{Var } p_q) \geq \text{Cov}^2(x, p_q) + \left( \frac{\hbar}{2} \right)^2,$$  \hspace{1cm} (37)

which is a version of the quantum uncertainty relation originally proposed by Schrödinger [9]. Since $\text{Cov}^2(x, p_q) \geq 0$ the somewhat weaker Heisenberg uncertainty relation follows immediately,

$$(\text{Var } x) (\text{Var } p_q) \geq \left( \frac{\hbar}{2} \right)^2 \quad \text{or} \quad \Delta x \Delta p_q \geq \frac{\hbar}{2}.$$  \hspace{1cm} (38)
4  A Hybrid Theory

Non-dissipative ED is defined by the Fokker-Planck equation (6) and the quantum Hamilton-Jacobi eq. (11). Here we focus on the special case with $\mu = 0$. Setting $\hbar = \eta$ and $S_{HJ} = \hbar \phi$ in eq. (11) gives

$$\dot{S}_{HJ} + \frac{1}{2m} \left( \vec{\nabla} S_{HJ} \right)^2 + V = 0 ,$$

which is classical Hamilton-Jacobi equation.

One might be tempted to interpret the $\mu = 0$ model as a classical ensemble dynamics but this is wrong. To see this it is useful to contrast $\mu = 0$ with the usual classical limit defined by $\hbar/m \to 0$. As $\hbar \to 0$ with $S_{HJ}$, $m$, and $\mu$ fixed, the current and the osmotic momenta, given in (17) and (16), become

$$\vec{p}_c = m \vec{v} = \vec{\nabla} S_{HJ} \quad \text{and} \quad \vec{p}_o = m \vec{u} = 0 ,$$

and $S_{HJ}$ satisfies the classical eq. (39). Furthermore, according to eq. (3) the particle is expected to move along the entropy gradient, while eq. (5),

$$\langle \Delta w^a \rangle = 0 \quad \text{and} \quad \langle \Delta w^a \Delta w^b \rangle = \frac{\hbar}{m} \Delta t \delta^{ab} \to 0 ,$$

shows that the fluctuations about the expected trajectory vanish. Therefore, in the limit $\hbar \to 0$ ED reproduces classical mechanics with classical trajectories following the entropy gradient. The same conclusion is obtained for fixed $\hbar$ provided the mass $m$ is sufficiently large.

The limit $\mu \to 0$ is very different! Here $\hbar$ and $m$ are fixed and $\hbar/m$ need not be small. This situation is also ruled by the classical Hamilton-Jacobi equation (39), but the osmotic momentum does not vanish,

$$\vec{p}_c = m \vec{v} = \vec{\nabla} S_{HJ} \quad \text{and} \quad \vec{p}_o = m \vec{u} = -\hbar \vec{\nabla} \log \rho^{1/2} .$$

The expected trajectory lies along a classical path but now the fluctuations $\Delta w^a$ about the classical trajectory, eq. (11), no longer vanish.

All the considerations about momentum described in the previous section apply to this $\mu = 0$ model. In particular, the momentum operator $\vec{p}_q = -i\hbar \vec{\nabla}$ can be introduced—for exactly the same reasons that one would introduce it in quantum theory—as a generator of translations, and this means that the $\mu = 0$ model obeys uncertainty relations identical to quantum theory. And yet, this is not quantum theory: the corresponding Schrödinger equation, eq. (14), is nonlinear and therefore there is no superposition principle.

5  Conclusion

We have explored the notion of momentum in entropic quantum dynamics. We find that the current momentum can reasonably be called the momentum because its expected value agrees with that of the quantum momentum operator.
and in the classical limit it coincides with the classical momentum. We have derived uncertainty relations within the entropic framework. A new insight is the reason the Heisenberg relation arises which is traced to the peculiar form of the osmotic velocity—it is essentially a diffusion effect described by Fick’s law.

The main conclusion is that in the entropic framework momentum is a statistical concept. In ED, unlike the standard interpretation of quantum mechanics, the positions of particles have definite values just as they would in classical physics. The price we pay for this feature is that particles do not have a momentum. More explicitly, momentum is not an attribute of the particles but of the probability distributions.

Finally, we explored ED for $\mu = 0$ which yields what we believe to be an altogether new kind of theory, neither classical nor quantum—we call it a hybrid theory. Whether it can usefully describe any actual physical system remains to be seen.

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