THE VLASOV-POISSON EQUATION, THE MOEBIUS GEOMETRY AND THE CURVED n-BODY PROBLEM IN ONE NEGATIVE SPACE FORM

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Abstract. By using, the Vlasov-Poisson equation defined in one Riemannian region of $\mathbb{R}^k$ and a Dirac distribution function, we obtain the equations of motion of any curved $n$-body problem with a pairwise acting potential on such region for $k = 3n$. We apply this result for study the negatively curved $n$–body problem in one negative space form (the hyperbolic Klein’s half plane $\mathbb{H}^2_\mathbb{R}$) with the hyperbolic potential. Following the Klein’s geometric Erlangen program, with methods of Moebius geometry and using the Iwasawa decomposition of the Moebius isometric group $SL(2, \mathbb{R})$ via its representation in one Clifford Algebra, we give algebraic conditions for the existence of all the Moebius solutions (relative equilibria) of the problem. We show several families of these kind of solutions for $n = 2$ and $n = 3$. 

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1. Introduction

The relation between the modern geometry and the physical problems has been reflected most notably for the last two hundred years. The number of physical problems that require to be raised a geometric device is uncountable. Among them, it is the problem of generalizing the Newton gravitational equations for spaces with non-zero Gaussian curvature as Lobachevsky and Boljai tried in the early 19th century (see [3]). In recent years, the problem has been relived and have been published a series of papers where using suitable potentials that generalize the Newtonian one, have been obtained interesting solutions from the geometric point of view. However, the method for obtaining the equations of motion in each curved \( n \)-body problem has been particular for each case, and only for constant Gaussian curvature (see [3], [4], and [11]). In this paper, by using, the classical curved Vlasov-Poisson equation defined in one Riemannian region of \( \mathbb{R}^k \) and a Dirac distribution function defined on positions and velocities, we obtain the equations of motion of any curved \( n \)-body problem where the particles are moving in \( \mathbb{R}^3 \) and with a pairwise acting potential on such region, for \( k = 3n \). In other words, we obtain the equations of motion for any curved space with intrinsic coordinates. Such that equations compare the geometry of the space (geodesic curves) with the dynamic of the physical problem (curved gradient of the potential) as it is shown in the equation (6) of the Theorem 1.

We apply this result for the two dimensional constant negative curvature case, i.e. the motion of \( n \) point masses in a negative space form (space of constant Gaussian curvature (see [5])) moving under the influence of a gravitational potential that naturally extends Newton’s law. This work completes the earlier study of this problem done by Florin Diacu, Ernesto Pérez-Chavela and J. Guadalupe Reyes Victoria, [4]. For do this complementation, with the help of algebraic methods, we define formally the Killing vector fields in \( \mathbb{H}^2_R \) associated to the true conics motions, through the representation of its isometric group \( SL(2, \mathbb{R}) \) into a suitable Clifford Algebra.

For an interesting history of this negative \( n \)-body problem, the reader can be referenced to the seminal papers of Florin Diacu, et al [1], [2] and [3].

The paper is organized in the following way.

In section 2 we recall the curved Vlasov-Poisson equation (VP) in a general non-euclidian space and we obtain from it the equations of motion for a general mechanical system. We apply these for obtain here again the equation of motions for \( \mathbb{H}^2_R \) with a hyperbolic cotangent potential.

In section 3 following the Klein’s Erlangen program (see [10]), we do the representation of the Moebius group of isometries \( SL(2, \mathbb{R}) \) of \( \mathbb{H}^2_R \) into a four dimensional Clifford algebra. This allows us to factorize such that group, via the Iwasawa decomposition Theorem, and by one dimensional subgroups, in three different suitable ways, which define three different geometries: the elliptic, the parabolic or the hyperbolic (see [10]). There, we obtain from the Lie algebra \( sl(2, \mathbb{R}) \), when we project it via the exponential map onto the
Lie group $SL(2,\mathbb{R})$, the possible five Killing vector fields associated to all the one dimensional factors subgroups in such decomposition. These vector fields shall define the Moebius solutions (also called relative equilibria in the dynamic world) of the problem.

In section 4 by using the same decomposition, we proceed to find the Moebius relative equilibria of the problem in $H^2_R$, by the method of matching the gravitational field with each one of the Killing vector fields obtained in section 3 as we did before in [4], [11] and [12]. In subsection 4.1 we obtain functional algebraic conditions for the existence of Moebius hyperbolic normal relative equilibria. In subsection 4.2 we do the same for the parabolic nilpotent Moebius relative equilibria. We find the algebraic conditions for the Moebius elliptic relative equilibria in subsection 4.3, for the Moebius parabolic cyclic relative equilibria in subsection 4.4, and for the Moebius hyperbolic cyclic relative equilibria in subsection 4.5. We show examples in the cases $n = 2$ and $n = 3$ of the former orbits for elliptic and hyperbolic normal Moebius solutions. We confirm here, within our general context, a more general result than that obtained in [3], which states that parabolic nilpotent Moebius relative equilibria do not exist. We prove here that there are not, neither parabolic cyclic nor hyperbolic cyclic Moebius relative equilibria in this problem.

We note that the non existence of the aforementioned type of the true conic orbits (cycles parabolic and hyperbolic) in the curved spaces of non-zero curvature is not a surprise because in the parabolic euclidian geometry some geometrical features are richer (see [10]).

2. Vlasov-Poisson equation and equations of motions for the $n$-body problem

In an euclidian space $\mathbb{R}^k$ with coordinates $x = (x^1, x^2, \cdots , x^k)$, the Vlasov-Poisson equation or equation of self-consistent field has the form (see [7]),

\begin{equation}
\frac{\partial F}{\partial t} + \left\langle v, \frac{\partial F}{\partial x} \right\rangle + \left\langle f(F), \frac{\partial F}{\partial v} \right\rangle = 0,
\end{equation}

where $f$ is a functional of the distribution function $F$ of particles moving along the space $\mathbb{R}^n$ with velocities $v = (v^1, v^2, \cdots , v^k)$.

A simple kind of force $f$ for the Vlasov-Poisson equation (1) is given by

\begin{equation}
f = -\nabla \left( \int U(x,y) F(y,v,t) \, dv \, dy \right),
\end{equation}

where $U = U(x,y)$ defines an interactive pairwise potential.

For the non-euclidean space $\mathbb{R}^k$ endowed with the Riemannian metric $G = (g_{ij})$ we consider the curved Vlasov-Poisson equation (see [7]),

\begin{equation}
\frac{\partial F}{\partial t} + \left\langle v, \frac{\partial F}{\partial x} \right\rangle_G + \left\langle f(F), \frac{\partial F}{\partial v} \right\rangle_G = 0,
\end{equation}
where $<, >_G$ denotes the scalar product in the metric $G = (g_{ij})$. If $\{\Gamma^l_{ij}\}$ is the compatible connection associated to the metric $G = (g_{ij})$, the equations of the corresponding geodesics are

\[
\ddot{x}^l + \Gamma^l_{ij} \dot{x}^i \dot{x}^j = 0,
\]

and if we put $v^i = \dot{x}^i$, we see that equations (4) can be written as

\[
\dot{v}^l + \Gamma^l_{ij} v^i v^j = 0.
\]

For one system of $n$-particles moving along the positions $X^i(t)$ and velocities $V^i(t)$ in the curved three dimensional space $\mathbb{R}^3_G$ endowed with the Riemannian metric $G = (g_{ij})$, let $\mathbb{R}^3_G^n$ be the curved phase space endowed with the Riemannian metric obtained by the diagonal action of the $G$. We obtain the equations of motions for the system from the curved Vlasov-Poisson equations (3).

**Theorem 1.** The solution curves of one system of $n$ point particles in $\mathbb{R}^3$ with masses $m_1, m_2, m_3, \ldots, m_n$ moving under the influence of the potential $U = U(x) = U(x_1, x_2, \ldots, x^k)$ with $k = 3n$, along the positions $x = x(t)$ and velocities $v = v(t)$ in the curved space $\mathbb{R}^3_G$ satisfy the system of $n$ second order differential equations, for $i = 1, \ldots, n$ and $s = 1, 2, 3$.

\[
\ddot{x}^i_s + \Gamma^i_{lj} \dot{x}^l_s \dot{x}^j_s = -g^{ik} \frac{\partial U}{\partial x^k},
\]

where $G^{-1} = (g^{ik})$ is the inverse matrix for the Riemannian metric $G$.

**Remark 1.** We observe that the left hand side of equation (6), when is equaled to zero, defines the equations of the geodesic curves associate to the metric $G = (g_{ij})$, whereas the left hand side corresponds to the curved (via the metric) gradient of the potential $U$. In other words, we are comparing geometry versus dynamics, as in the classical way.

**Remark 2.** In order of proving the Theorem, since the metric in $\mathbb{R}^3_G^n$ is diagonal, then we can consider, for convenience, each position vector $X^i(t) = (X^i_1(t), X^i_2(t), X^i_3(t))$ and its velocity vector $V^i(t) = (V^i_1(t), V^i_2(t), V^i_3(t))$ as the single real coordinates $X^i(t)$ and $V^i(t)$, and the same for the coordinates of $x$ and $v$ of $\mathbb{R}^k$, missing the upper index $s$ in equation (6) along all the proof.

**Proof.** We consider for the force function (2), the particular case when the distribution function of particles moving under the influence of the potential $U = U(x) = U(x_1, x_2, \ldots, x^k)$ is,

\[
F(t, v, x) = \sum_{i=1}^{n} m_i \delta(v - V^i(t)) \delta(x - X^i(t)),
\]

where $\delta = \delta(x)$ is the ordinary Dirac-function, and $X^i(t), V^i(t)$ are time dependent vector functions which locate the position and velocity of the $i$-th particle.
We denote by $\nabla^g_z$ the gradient respect to the variable $z$ in the metric $G$, and by $\frac{D}{dt}$ to the corresponding covariant derivative. Then the following chain of results hold.

For a fixed $i = 1, 2, 3, \cdots$ we have, by using the properties of the Dirac function in the force that,

$$f(t, x) = -\nabla^g_x \int U(x, y) F(t, v, y) dy \, dv$$

$$= -\nabla^g_x \int U(X^i, y) \sum_{j=1}^n m_j \delta(v - V^j(t)) \delta(y - Y^j(t)) \, dy \, dv$$

$$= -\sum_{j=1}^n m_j \nabla^g_x \int U(X^i, y) \delta(v - V^j(t)) \delta(y - Y^j(t)) \, dy \, dv$$

$$= -\sum_{j=1}^n m_j \nabla^g_x U(X^i, X^j),$$

(8)

when we have used $x = X^i(t)$, $v = V^i(t)$ and the corresponding integral has value 1.

By a straight forward computation we obtain,

$$\frac{\partial F}{\partial t} = \sum_{i=1}^n m_i \left\langle \nabla^g_v \delta(v - V^i(t), -\frac{DV^i}{dt}) \delta(x - X^i(t)) \right\rangle_G$$

$$+ \sum_{i=1}^n m_i \left\langle \nabla^g_x \delta(x - X^i(t)), -\dot{X}^i \right\rangle_G \delta(v - V^i(t))$$

$$= \sum_{i=1}^n m_i \left[ \left\langle \nabla^g_v \delta(v - V^i(t), -\frac{DV^i}{dt}) \right\rangle_G + \left\langle \nabla^g_x \delta(x - X^i(t)), -\dot{X}^i \right\rangle_G \right],$$

(9)

and we have used again that $x = X^i(t)$, $v = V^i(t)$.

On the other hand,

$$\left\langle v, \frac{\partial F}{\partial x} \right\rangle_G = \left\langle v, \sum_{i=1}^n m_i \delta(v - V^i(t)) \nabla^g_x \delta(x - X^i(t)) \right\rangle_G$$

$$= \sum_{i=1}^n m_i \left\langle v, \nabla^g_x \delta(x - X^i(t)) \right\rangle_G,$$

(10)

for the case when $v = V^i(t)$. 

Finally, we have that,
\[
\left\langle f(F), \frac{\partial F}{\partial v} \right\rangle_G = \left\langle -\sum_{j=1}^{n} m_j \nabla_g^U(X^i, X^j), \sum_{i=1}^{n} m_i \delta(x - X^i(t)) \nabla_g^\delta(v - V^i(t)) \right\rangle_G \\
= \sum_{i=1}^{n} m_i \left\langle -\sum_{j=1}^{n} m_j \nabla_g^U(X^i, X^j), \nabla_g^\delta(v - V^i(t)) \right\rangle_G \\
\tag{11}
\]
for the case when \( x = X^i(t) \).

Therefore when we substitute and factorize \( \sum_{i=1}^{n} m_i \), from equations (9), (10) and (11), a sufficient condition for equation (3) holds is that, the following equality holds also for all \( i = 1, 2, \ldots, n \),
\[
0 = \left\langle \nabla^\delta_v(v - V^i(t)), -\frac{D V^i}{d t} \right\rangle_G + \left\langle \nabla^\delta(x - X^i(t)), -\dot{X}^i \right\rangle_G \\
+ \left\langle v, \nabla^\delta_g(x - X^i(t)) \right\rangle_G + \left\langle -\sum_{j=1}^{n} m_j \nabla^g_g^U(X^i, X^j), \nabla^\delta_v(v - V^i(t)) \right\rangle_G . \\
\tag{12}
\]

By comparing the first and the last terms in the right hand side of equation (12) and the second and third ones, if we put \( v^i = \dot{X}^i \), then, for such number vanishes it is necessary that
\[
\frac{D V^i}{d t} = -\sum_{j=1}^{n} m_j \nabla^g_g^U(X^i, X^j) . \\
\tag{13}
\]

Since
\[
\frac{D V^i}{d t} = \frac{d \dot{X}^i}{d t} = \dot{V}^i + \Gamma^i_{lj} V^l \dot{V}^j = \ddot{x}^i + \Gamma^i_{lj} \dot{x}^l \dot{x}^j ,
\]
and
\[
\sum_{j=1}^{n} m_j \nabla^g_g^U(X^i, X^j) = m_j g^{ij} \frac{\partial U}{\partial x^j}
\]
the claim result follows, and ends the proof. \( \square \)

**Remark 3.** We observe that if \( g_{ij} = \delta_{ij} \) is the euclidian metric, then \( \Gamma^i_{lj} \equiv 0 \) and we obtain the classical Newtonian \( n \)-body problem.

Next, we will apply the equations (6) to obtain the equations of motion of the curved \( n \)-body problem in upper half complex plane with coordinate \( w \),
\[
\mathbb{H}^2_R = \{ (w, \bar{w}) \in \mathbb{C} | \text{Im}(w) > 0 \}.
\]
endowed with the conformal Riemannian metric
\[
- ds^2 = \frac{4R^2}{(w - \bar{w})^2} dw d\bar{w} \\
\tag{14}
\]
The equations of the geodesic curves are given by (see [5]),

\[ \ddot{w} - \frac{2\dot{w}^2}{w - \bar{w}} = 0, \]

and they are either half circles orthogonal to the real axis \((y = 0)\) or half lines perpendicular to it.

The space \(\mathbb{H}_R^2\) endowed with the metric [14] is called the Klein upper half plane model of hyperbolic geometry.

We define the acting hyperbolic cotangent potential (see [3] and [4]) in the coordinates \((w, \bar{w})\), given by

\[ V_R(w, \bar{w}) = \frac{1}{R} \sum_{1 \leq k < j \leq n} \frac{m_km_j(w_k + w)(\bar{w}_j + w_j) - 2(|w_k|^2 + |w_j|^2)}{[\Theta(k,j)(w, \bar{w})]^{1/2}}, \]

where

\[ \Theta(k,j)(w, \bar{w}) = [(\bar{w}_k + w_k)(\bar{w}_j + w_j) - 2(|w_k|^2 + |w_j|^2)]^2 - (\bar{w}_k - w_k)^2(\bar{w}_j - w_j)^2 \]

defines the singular set of the problem.

Another main result of this section is the following one, which follows from a direct substitution of equations [15] and [16] together with the conformal factor of the metric [14] in Theorem 1.

**Theorem 2.** The solution curves of the system defining the negatively curved \(n\)-body problem in the negative space form \(\mathbb{H}_R^2\) with acting hyperbolic potential [16], satisfy the system of \(n\) second order differential equations, \(k = 1, \ldots, n\),

\[ \ddot{w}_k - \frac{2\dot{w}_k^2}{w_k - \bar{w}_k} = \frac{2}{\mu(w_k, \bar{w}_k)} \frac{\partial V_R}{\partial \dot{w}_k} = -\frac{(w_k - \bar{w}_k)^2}{2R^2} \frac{\partial V_R}{\partial \dot{w}_k} = -\frac{2(w_k - \bar{w}_k)^3}{R} \sum_{j=1}^{n} m_j(\bar{w}_j(w_k - w_j)(\bar{w}_j - w_k)[\Theta(k,j)(w, \bar{w})]^{3/2}}. \]

(18)

As we have pointed in Remark 1, the left hand side of equation (18) defines the equations of the geodesic curves in \(\mathbb{H}_R^2\), whereas the left hand side corresponds to the curved gradient of the potential \(V_R\).

### 3. Invariants of the Moebius Geometry

In this section we obtain the geometric invariants (conic curves) of the Moebius geometry of \(\mathbb{H}_R^2\) according to the Klein’s Erlangen Program and with the methodology of representing the isometry Moebius group into a four dimensional Clifford algebra as in [10]. These invariants will define five Killing vector fields in the hyperbolic half plane.

Let

\( SL(2, \mathbb{R}) = \{ A \in GL(2, \mathbb{R}) \mid \det A = 1 \}, \)
be the special linear real 2-dimensional group, which is a 3-dimensional simply connected, smooth real manifold. It is well known (see [6]) that the group of proper isometries of $\mathbb{H}^2_R$ is the projective quotient group $\text{SL}(2, \mathbb{R})/\{\pm I\}$. Every class

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})/\{\pm I\}$$

has also associated a unique M"obius transformation $f_A : \mathbb{H}^2_R \to \mathbb{H}^2_R$, where

$$f_A(z) = \frac{az + b}{cz + d},$$

which satisfies that $f_{-A}(z) = f_A(z)$.

We give the representations of the $\text{SL}(2, \mathbb{R})$ group in a Clifford algebra with two generators, knowing that there are three different Clifford algebras $\text{Cl}(e)$, $\text{Cl}(p)$, $\text{Cl}(h)$, corresponding to the elliptic, parabolic and hyperbolic cases.

A Clifford algebra $\text{Cl}()$ is a four dimensional linear real space spanned by $\{1, e_0, e_1, e_0e_1\}$ with the non-commutative product defined by, (see [10] for more details)

\[
e_0^2 = -1, \quad e_1^2 = \sigma = \begin{cases} -1 & \text{for } \text{Cl}(e), \\
0 & \text{for } \text{Cl}(p), \\
1 & \text{for } \text{Cl}(h) \end{cases} \quad e_0e_1 = -e_1e_0.
\]

(19)

For the space $\mathbb{R}^2 = \{ue_0 + ve_1 \mid u, v \in \mathbb{R}\}$, we denote by $\mathbb{R}^e$, $\mathbb{R}^p$ or $\mathbb{R}^h$ to the corresponding Clifford algebra $\mathbb{R}^\sigma$. Therefore, an isomorphic representation of the group $\text{SL}(2, \mathbb{R})$ with the same product is obtained if we replace the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} a & be_0 \\ -ce_0 & d \end{pmatrix}$, which give us the advantage of defining the Moebius transformation $\mathbb{R}^\sigma \to \mathbb{R}^\sigma$, given by (see [10] for more details)

\[
\begin{pmatrix} a & be_0 \\ -ce_0 & d \end{pmatrix} : \quad ue_0 + ve_1 \to \frac{a(ue_0 + ve_1) + be_0}{-c_0(ue_0 + ve_1) + d}.
\]

In this representation, we can factorize by the Iwasawa decomposition Theorem (see [8], [9], [10]) to $\text{SL}(2, \mathbb{R}) = ANK$, where each matrix factorizes,

\[
(20) \quad \begin{pmatrix} a & be_0 \\ -ce_0 & d \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & \nu e_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & e_0 \sin \phi \\ e_0 \sin \phi & \cos \phi \end{pmatrix},
\]

with

$$\alpha = \sqrt{c^2 + d^2}, \quad \nu = ac + bd, \quad \phi = \arctan \frac{c}{d},$$

where the one dimensional subgroup $A$ is the normalizer of the nilpotent one dimensional subgroup $N$, and $K$ is a maximal compact subgroup of $\text{SL}(2, \mathbb{R})$. 


In the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $\text{SL}(2, \mathbb{R})$ we consider the following suitable set of Killing vector fields,

$$\left\{ X_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & e_0 \\ -e_0 & 0 \end{pmatrix} \right\}.$$

If we consider also the exponential map of matrices,

$$\exp : \mathfrak{sl}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R}),$$

applied to the one-parameter additive subgroups (straight lines) \{tX_1\}, \{tX_2\}, and \{tX_3\}, we obtain the following one-dimensional factor subgroups of the Lie group $\text{SL}(2, \mathbb{R})$.

1. The isometric normal homothetic subgroup in $A$,

$$\exp(tX_1) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

which defines the one-parameter family of acting Möbius transformations

$$f_1(w, t) = e^t w,$$

and associated to the differential equation

$$\dot{w} = w.$$

The flow of corresponding vector field is a set of straight lines arising in the origin of coordinates and is shown in Figure 1.

![Figure 1. The vector field $\dot{w} = w$.](image)

2. The isometric nilpotent shift subgroup in $N$,

$$\exp(tX_2) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

which defines the one-parameter family of acting Möbius transformations

$$f_2(w, t) = w + t,$$
and associated to the differential equation

\[ \dot{w} = 1. \tag{24} \]

The flow of corresponding vector field is a set of horizontal parallel straight lines and it is shown in Figure 2.

\begin{center}
\includegraphics{figure2.png}
\end{center}

**Figure 2.** The vector field $\dot{w} = 1$.

3. The isometric rotation subgroup in $K$,

\[ \exp(tX_3) = \begin{pmatrix} \cos t & e_0 \sin t \\ e_0 \sin t & \cos t \end{pmatrix}, \]

which defines the one-parameter family of acting Möbius transformations

\[ f_3(w, t) = \frac{(\cos t) w + e_0 \sin t}{(e_0 \sin t) w + \cos t}, \tag{25} \]

and associated to the differential equation

\[ \dot{w} = e_0(1 - w^2). \tag{26} \]

We obtain three different Killing vector fields associated to $\mathbb{R}^e$, $\mathbb{R}^p$ or $\mathbb{R}^h$ for each corresponding Clifford algebra.

a. The flow of corresponding vector field in $\mathbb{R}^e$ is a set of coaxal circles with focus (in the sense of the Möbius geometry, see [10]) in the point $w = i$ and it is shown in Figure 3.

b. The flow of corresponding vector field in $\mathbb{R}^p$ is a set of vertical parabolas with horizontal directrices, and it is shown in Figure 4.

c. The flow of corresponding vector field in $\mathbb{R}^h$ is a set of vertical hyperbolas with asymptotes parallel to the diagonal, and it is shown in Figure 5.
Figure 3. The elliptic vector field $\dot{w} = \epsilon_0 (1 - w^2)$ for $\epsilon_1^2 = -1$.

Figure 4. The parabolic vector field $\dot{w} = \epsilon_0 (1 - w^2)$ for $\epsilon_1^2 = 0$.

Figure 5. The hyperbolic vector field $\dot{w} = \epsilon_0 (1 - w^2)$ for $\epsilon_1^2 = 1$. 
4. Moebius solutions of the curved n-body problem in $\mathbb{H}^2_R$

In this section we shall obtain the whole set of the so called Moebius solutions or relative equilibria solutions of the mechanical system (18). For this, we will use the method of matching the gravitational hyperbolic cotangent field with each one of the Killing vector fields associated to the above one-dimensional subgroups as in [4] and [12].

**Definition 1.** An Moebius solution or relative equilibrium for the negatively curved $n$–body problem in the Klein half plane $\mathbb{H}^2_R$, is a solution $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))$ of the equations of motion (18) which is invariant under any one dimensional subgroup $A(t)$ of the group $SL(2, \mathbb{R})$. In other words, the function obtained by the action denoted by $z(t) = A(t)w(t)$ is also a solution of (18).

Since the basic one-dimensional parametric subgroups (22), (24) and (26) generate under the composition of functions all one-dimensional parametric subgroups $\{A(t)\} \subset SL(2, \mathbb{R})/\{\pm I\}$, we shall show in the following sections the corresponding Moebius motions associated to such subgroups.

4.1. Hyperbolic normal Moebius solutions. In this subsection we state conditions for the existence of normal hyperbolic (homothetic) Moebius solutions (relative equilibria) for the negatively curved problem in the Klein’s upper half plane model $\mathbb{H}^2_R$. These results were obtained in Diacu et al. [4].

**Theorem 3.** Consider $n$ point particles with masses $m_1, \ldots, m_n > 0$, $n \geq 2$, moving in $\mathbb{H}^2_R$. A necessary and sufficient condition for the function $w = (w_1, \ldots, w_n)$ to be a solution of system (18) and, at the same time, a relative equilibrium associated to the Killing vector field $X_1$ defined by equation (21) is that, for every $k = 1, \ldots, n$, the coordinates satisfy the algebraic conditions

$$R(w_k + \bar{w}_k) w_k = \sum_{j=1\atop j \neq k}^n \frac{m_j(w_j - \bar{w}_j)^2(w_k - \bar{w}_j)(\bar{w}_j - w_k)}{[\Theta_{(k,j)}(w, \bar{w})]^{3/2}}.$$  

**Proof.** Solutions of the aforementioned type the bodies move along straight half lines converging to the origin of the coordinate system and must therefore satisfy equation (22), which, when we derive and substitute in equation (18), allows equation (27).

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**Definition 2.** We will call hyperbolic normal Moebius solutions (relative equilibria) to these solutions of system (18) in $\mathbb{H}^2_R$ that satisfy equations (27).

We state here a description of the hyperbolic relative equilibria for 2 and 3 interacting particles in $\mathbb{H}^2_R$, the proof of these results can be seen in [4], and we omit them here.

We state first the existence of hyperbolic normal Moebius (relative equilibria) when one body is moving along a non-geodesic curve and the other
along another non-geodesic curve, both equidistant from a given geodesic, and when the 2 non-geodesic curves satisfy a certain relationship that depends on the values of the masses.

Recall that the slope $\beta$ of any straight line in the complex plane is defined by the formula

$$i\beta = \frac{w - \bar{w}}{w + \bar{w}}.$$ 

Without loss of generality, we choose the initial conditions such that the heights satisfy $w_1(0) - \bar{w}_1(0) = 2iv_1$ and $w_2(0) - \bar{w}_2(0) = 2iv_2$. Then, in terms of the slopes $\beta_1$ and $\beta_2$ of the straight lines, equation (27) becomes

$$m_1 \beta_2 v_2 (|w_1|^2 - w_2 w_1) = -m_2 \beta_1 v_1 (|w_2|^2 - w_2 w_1).$$

We have the following result.

**Theorem 4.** Necessary and sufficient conditions for the existence of a hyperbolic normal relative equilibrium in $H^2_R$ as a solution of system (18) with $n = 2$ are that one particle moves along a non-geodesic half line, while the other particle moves along another non-geodesic half line, both half lines converging to the origin of the coordinates system, such that the supporting lines have slopes of opposite signs that satisfy the relationship

$$\frac{m_1}{m_2} = -\frac{\beta_2 v_2}{\beta_1 v_1},$$

and that at every time instant there is a geodesic half circle centered at the origin of the coordinate system on which both particles are located.

For the case of 3 bodies in the Klein upper half plane, $H^2_R$, with masses $m_1, m_2, m_3 > 0$ we have the following result (see [4]).

**Theorem 5.** Consider 3 point particles of masses $m_1, m_2, m_3 > 0$ moving in $H^2_R$. Assume that $m_1$ and $m_3$ move along non-geodesic half lines emerging from the origin of the coordinate system at angles $\theta_1$ and $\theta_3$, respectively, and that $m_2$ moves along the geodesic vertical half line. Moreover, at every time instant, there is a geodesic half circle on which all 3 bodies are located, and the motion of the particles is given by the function $w = (w_1, w_2, w_3)$. Then $w$ is a hyperbolic relative equilibrium that is a solution of system (18) with $n = 3$ if and only if $\theta_1 = -\theta_3$ and $m_1 = m_3$, with $\theta_1, \theta_3 \in (-\pi/2, 0) \cup (0, \pi/2)$.

4.2. **Parabolic nilpotent Moebius solutions.** In this subsection we show the Moebius solutions associated to the subgroup generated by the Killing vector field $X_2$ and which defines the one-parametric family of acting Möbius transformations (24) in the upper half plane $H^2_R$. These orbits correspond to parabolic relative equilibria, and we will show that they do not exist in $H^2_R$. We state the following result obtained in Diacu et al. [4].

**Theorem 6.** Consider $n \geq 2$ point particles of masses $m_1, \ldots, m_n > 0$ moving in $H^2_R$. Then a necessary and sufficient condition for the function
$w = (w_1, \ldots, w_n)$ to be a solution of system (18) that is a relative equilibrium associated to the Killing vector field $X_2$ is that the coordinate functions satisfy the equations

$$\frac{R}{4(w_k - \bar{w}_k)^3} = \sum_{j=1}^{n} \frac{m_j(\bar{w}_j - w_j)^2(w_k - w_j)(\bar{w}_j - w_k)}{[\Theta(k,j)(w, \bar{w})]^{3/2}},$$

**Proof.** We saw that relative equilibria associated with the Killing vector field $X_2$ must satisfy equations (24), which when we derive and substitute in equations (18) allows us to equations (30). This completes the proof. □

**Definition 3.** We will call parabolic nilpotent Moebius solution (relative equilibria) the solutions of system (18) in $\mathbb{H}_R^2$ that satisfy equations (30).

We have the following result (see [4]).

**Theorem 7.** In the curved $n$-body problem with negative curvature there are no parabolic nilpotent Moebius relative equilibria.

4.3. Elliptic cyclic Moebius solutions. In this subsection we shall study the Moebius solutions of (18) corresponding to the action of the Killing vector field $X_3$ associated to the differential equation (26) for the case when $e_2^2 = -1$. It is not hard to see (Kisil [10]) that in this case such that equation becomes into the one,

$$\dot{w} = 1 + w^2,$$

and then we obtain the condition for a solution of equation (18) to be a Moebius invariant under such that Killing vector field.

**Theorem 8.** Consider $n$ point particles with masses $m_1, \ldots, m_n > 0$, $n \geq 2$, moving in $\mathbb{H}_R^2$. A necessary and sufficient condition for the function $w = (w_1, \ldots, w_n)$ to be a Moebius solution of system (18), that is, a relative equilibrium associated to the Killing vector field $X_3$ defined by equation (31) is that for all $k = 1, \ldots, n$, the following algebraic functional equations are satisfied

$$R(1 + w_k^2)(1 + |w_k|^2) = \sum_{j=1}^{n} \frac{m_j(\bar{w}_j - w_j)^2(w_k - w_j)(\bar{w}_j - w_k)}{[\Theta(k,j)(w, \bar{w})]^{3/2}}.$$

**Proof.** Result follows when from the equations $\dot{w}_k = 1 + w_k^2$, $k = 1, \ldots, n$, we differentiate and substitute in the equations of motion (18). This completes the proof. □

**Definition 4.** We call elliptic cyclic Moebius solutions (relative equilibria) to the solutions of system (18) in $\mathbb{H}_R^2$ that satisfy the conditions (32).

The circle passing through the points $i\alpha$ and $\frac{i}{\alpha}$ (with $1 < \alpha$) is centered at the point $\frac{i}{2} \left( \alpha + \frac{1}{\alpha} \right)$ and has radius $r = \frac{\alpha^2 + 1}{2\alpha}$. Its equation is given
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by (see [10])

\[ 2|z|^2 + i(z - \bar{z}) \left( \alpha + \frac{1}{\alpha} \right) + 2 = 0. \]

The following results are obtained from Theorems 3, 4 and 5 in [4] via the Moebius linear fractional transformation $z : \mathbb{H}^2_R \to \mathbb{D}^2_R$ defined by

\[ z = z(w) = \frac{-Rw + iR^2}{w + iR}, \]

where $\mathbb{D}^2_R$ is the Poincaré disk of radius $R$ with center at the origin of coordinates in the complex plane, endowed with the conformal metric

\[ ds^2 = \frac{4R^4}{(R^2 - |z|^2)^2} dz d\bar{z}. \]

We state the following result, which characterizes the elliptic Moebius relative equilibria of the curved 2-body problem in the hyperbolic plane $\mathbb{H}^2_R$ (see [4]).

**Theorem 9. (Two bodies)** Consider 2 point particles of masses $m_1, m_2 > 0$ moving in the half plane $\mathbb{H}^2_R$ along circles (33) whose center is sited on the imaginary axis. Then a function $w = (w_1, w_2)$ is an elliptic Moebius relative equilibrium of system (18) with $n = 2$, if and only if for every circle centered at $i \left( \frac{\alpha}{2} + \frac{1}{\alpha} \right)$ of radius $\frac{\alpha^2 + 1}{2\alpha}$ along which $m_1$ moves, there is a unique circle centered at $i \left( \frac{\beta}{2} + \frac{1}{\beta} \right)$ of radius $\frac{\beta^2 + 1}{2\beta}$, along which $m_2$ moves, such that, at every time instant, $m_1$ and $m_2$ are sitted on some the same geodesic of $\mathbb{H}^2_R$ but in opposite sides of the isometric circles. Moreover,

1. if $m_2 > m_1 > 0$ and $\alpha$ are given, then $\beta < \alpha$;
2. if $m_1 = m_2 > 0$ and $\alpha$ are given, then $\beta = \alpha$;
3. if $m_1 > m_2 > 0$ and $\alpha$ are given, then $\beta > \alpha$.

In the case of 3 particles in the half plane $\mathbb{H}^2_R$, we state the main results obtained in [4] for the Eulerian and Lagrangian cases.

The following tells that, when one particle is fixed in the imaginary axis of $\mathbb{H}^2_R$ as the center of one circle, there is just one class of Eulerian elliptic Moebius relative equilibria, namely orbits for which the geodesic distance from the fixed body to the 2 rotating bodies is the same, and consequently those masses must be equal (see [4]).

**Theorem 10. (Eulerian motions)** Consider 3 point particles of masses $m_1, m_2, m_3 > 0$, two of which are moving in one isometric circle of the Half plane $\mathbb{H}^2_R$, whose center is sited along the imaginary axis of the coordinate system. Take a function $w = (w_1, w_2, z_3)$ that describes the positions of the particles, with $w_1(t) = i \left( \frac{\alpha}{2} + \frac{1}{\alpha} \right)$ for all $t$. Then the triplet $w$ is an Eulerian elliptic Moebius relative equilibrium of system (18) with $n = 3,$
if and only if, \( m_2 \) and \( m_3 \) are at the opposite sides of the same uniformly rotating diameter of the isometric circle of radius \( \alpha^2 + \frac{1}{2\alpha} \) in \( \mathbb{H}^2_R \), centered at \( \frac{i}{2}(\alpha + \frac{1}{\alpha}) \) and on the same geodesic, with \( m_1 = m_2 \).

We will next show Lagrangian elliptic relative equilibria in \( \mathbb{H}^2_R \) (see [4]).

**Theorem 11.** (Lagrangian motions) Assume that 3 point particles of equal masses move along a circle of radius \( \alpha^2 + \frac{1}{2\alpha} \) centered at the point \( \frac{i}{2}(\alpha + \frac{1}{\alpha}) \) on the imaginary axis of the half plane \( \mathbb{H}^2_R \). Then a necessary and sufficient condition for the existence of an elliptic Moebius relative equilibrium is that the particles form an equilateral geodesic triangle for all time \( t \).

### 4.4. Parabolic cyclic Moebius solutions

In this subsection we shall study the Moebius solutions of (18) corresponding to the action of the Killing vector field \( X_3 \) associated to the differential equation (26) for the case when \( e^2_1 = 0 \). It is not hard to see (Kisil [10]) that in this case such that equation becomes into the one (see [10]),

\[
\dot{w} = 1 + w^2 - \frac{(w - \bar{w})^2}{4},
\]

and then we obtain the condition for a solution of equation (18) to be a Moebius invariant under such that Killing vector field.

**Theorem 12.** Consider \( n \) point particles with masses \( m_1, \ldots, m_n > 0 \), \( n \geq 2 \), moving in \( \mathbb{H}^2_R \). A necessary and sufficient condition for the function \( w = (w_1, \ldots, w_n) \) to be a Moebius solution of system (18), that is, a relative equilibrium associated to the Killing vector field \( X_3 \) defined by equation (34), is that for all \( k = 1, \ldots, n \), the following algebraic functional equations are satisfied

\[
\frac{R((w_k - \bar{w}_k)^2(8 - w_k^2 + 6|w_k|^2 + 3\bar{w}_k^2) - 16(1 + w_k^2)(1 + |w_k|^2))}{16(w_k - \bar{w}_k)^4} = \sum_{\substack{j=1 \atop j \neq k}}^n \frac{m_j(w_j - w_j)^2(w_k - w_j)(\bar{w}_j - w_k)}{[\Theta_{(k,j)}(w, \bar{w})]^{3/2}}.
\]

**Proof.** Result follows when from the equations \( \dot{w}_k = 1 + w_k^2 - \frac{(w_k - \bar{w}_k)^2}{4} \), for \( k = 1, \ldots, n \), we differentiate them and substitute in the equations of motion (18). This completes the proof. \( \square \)

**Definition 5.** We call parabolic cyclic Moebius solutions (relative equilibria) to the solutions of system (18) that satisfy the conditions (35).
We observe that equation (35) is so hard to solve, and instead of trying this, we shall find explicitly the parabolic cyclic flow of (34) and we will propose these type of solutions as in [2].

If we use real coordinates \((u, v)\) for \(w = u + iv\), the complex differential equation (34) can be written (see [10]) as the system of real differential equations

\[
\begin{align*}
\dot{u} &= 1 + u^2, \\
\dot{v} &= 2uv,
\end{align*}
\]

(36)

which when is integrated for the initial conditions \((u(0) = \alpha, v(0) = \beta)\) has the isometric parametrization

\[
\begin{align*}
u(t) &= \frac{\alpha + \tan t}{1 - \alpha \tan t}, \\
v(t) &= \frac{\beta \sec^2 t}{(1 - \alpha \tan t)^2}.
\end{align*}
\]

(37)

In this way, the isometric parametrization of the solution of equation (34) with initial condition \(w(0) = \alpha + i\beta\) is

\[
w(t) = \frac{\alpha + \tan t}{1 - \alpha \tan t} + i \frac{\beta \sec^2 t}{(1 - \alpha \tan t)^2}.
\]

(38)

Let \(s = \tan t\) be one isometric diffeomorphic variable. Then \(\sec^2 t = 1 + s^2\), and the parametrization (38) becomes

\[
w(s) = \frac{\alpha + s}{1 - \alpha s} + i \frac{\beta (1 + s^2)}{(1 - \alpha s)^2}.
\]

(39)

Since

\[
\begin{align*}
\frac{ds}{dt} &= 1 + s^2, \\
\frac{d^2s}{dt^2} &= 2(1 + s^2)s,
\end{align*}
\]

(40)

and \(w' = \frac{dw}{ds}\), then

\[
\begin{align*}
\dot{w} &= \frac{dw}{ds} \frac{ds}{dt} = (1 + s^2)w', \\
\ddot{w} &= w'' \left(\frac{dw}{ds}\right)^2 + w' \frac{d^2s}{dt^2} = w''(1 + s^2)^2 + 2s(1 + s^2)w'.
\end{align*}
\]

(41)
If we substitute equations (39), (40) and (41) into equation (35), the respective real parts become

\[ R(\alpha_k + s)(1 + \alpha_k^2)(1 - \alpha_k s)^2 \]
\[ \frac{64\beta_k^2(1 + s^2)^5}{(1 + \alpha_k^2)(1 - \alpha_k s)^2} \]
\[ = \sum_{j \neq k} m_j \beta_j^2 \]
\[ [\Theta(k,j)(\alpha, \beta)]^{3/2}(1 - \alpha_j s)^4 \left( \frac{\alpha_j + s}{1 - \alpha_j s} - \frac{\alpha_k + s}{1 - \alpha_k s} \right), \]
\[ (42) \]

whereas the imaginary parts are

\[ - R(1 + \alpha_k^2)(1 + \alpha_k^2)(1 - \alpha_k s)^2 + 2](1 - \alpha_k s)^2 \]
\[ \frac{64\beta_k^2(1 + s^2)^4}{(1 + \alpha_k^2)(1 - \alpha_k s)^2} \]
\[ = \sum_{j \neq k} m_j \beta_j^2 \]
\[ [\Theta(k,j)(\alpha, \beta)]^{3/2}(1 - \alpha_j s)^4 \times \]
\[ \times \left[ \left( \frac{\alpha_j + s}{1 - \alpha_j s} - \frac{\alpha_k + s}{1 - \alpha_k s} \right)^2 - (1 + s^2) \left( \frac{\beta_j^2}{(1 - \alpha_j s)^4} - \frac{\beta_k^2}{(1 - \alpha_k s)^4} \right) \right] \]
\[ (43) \]

where

\[ \Theta(k,j)(\alpha, \beta) \]
\[ = \left[ 4 \left( \frac{\alpha_j + s}{1 - \alpha_j s} \right) \left( \frac{\alpha_k + s}{1 - \alpha_k s} \right) - 2 \left( \left( \frac{\alpha_j + s}{1 - \alpha_j s} \right)^2 + \left( \frac{\alpha_k + s}{1 - \alpha_k s} \right)^2 + \frac{\beta_j^2(1 + s^2)^2}{(1 - \alpha_j s)^4} + \frac{\beta_k^2(1 + s^2)^2}{(1 - \alpha_k s)^4} \right) \right]^2 \]
\[ - \frac{16\beta_k^2\beta_j^2(1 + s^2)^4}{(1 - \alpha_k s)^4(1 - \alpha_j s)^4} \]
\[ (44) \]
is the corresponding evaluation of the singular part (17) in the 2n-dimensional parametric vector \( (\alpha, \beta) \).

We obtain the following result.

**Theorem 13.** There are not parabolic cyclic Moebius solutions (relative equilibria) in the negatively curved problem in \( \mathbb{H}^2_R \).

**Proof.** Consider \( n \) point particles with masses \( m_1, \ldots, m_n > 0, \) \( n \geq 2, \) moving in \( \mathbb{H}^2_R \) under the influence of the hyperbolic cotangent potential and satisfying the equations (42) and (43). Since the parabolic flow carries all these solutions to the right positive half plane as \( t \) goes to infinity, without loss of generality we can suppose that the \( k \)-th particle is one of the last reaching the imaginary axis. If we take these configuration as initial conditions, then \( \alpha_k = 0 \) and \( \alpha_j \geq 0 \) for \( s = 0 \). Therefore, for these initial
conditions equation (42) becomes

\[ 0 = \sum_{j \neq k} \frac{m_j \beta_j^2}{[\Theta(k,j)(\alpha, \beta)]^{3/2}} \alpha_j, \]

which implies necessarily that \( \alpha_j = 0 \) for all \( j = 1, 2, \ldots, n \).

This is, if one particle reaches the imaginary axis for some time, then the whole set of particles are sited also on the imaginary axis for that time. This implies that we can suppose, in the early, that all the particles are sited along the imaginary axis, and the isometric parametrizations are \( w_j = s + i(1 + s^2)\beta_j \) and \( w_k = s + i(1 + s^2)\beta_k \) respectively. If this is the case, then, for \( s = 0 \) and \( \alpha_j = 0 \) in the imaginary parts (43), we have that

\[ R_{64} \beta_k^2 = -\sum_{j \neq k} \frac{m_j \beta_j^2}{[\Theta(k,j)(\beta)]^{3/2}} (\beta_j^2 - \beta_k^2), \]

where

\[ \Theta(k,j)(\beta) = 4[\beta_j^2 - \beta_k^2]^2. \]

If we re-numerate the particles such that the \( k \)-th particle be sited in the bottom, this is \( \beta_k \leq \beta_j \) for all \( j \), then the left hand side of equation (46) is positive whereas the right hand side is negative. This contradiction proves the claim and ends the proof.

4.5. **Hyperbolic cyclic Moebius solutions.** Now, we shall study the Moebius solutions of (18) corresponding to the action of the Killing vector field \( X_3 \) associated to the differential equation (26) for the case when \( e_j^2 = 1 \). It is not hard to see (Kisil [10]) that in this case such that equation becomes into the one (see [10]),

\[ \dot{w} = 1 + w^2 - \frac{(w - \bar{w})^2}{2}. \]

and then we obtain the condition for a solution of equation (18) to be a Moebius invariant under such Killing vector field.

**Theorem 14.** Consider \( n \) point particles with masses \( m_1, \ldots, m_n > 0 \), \( n \geq 2 \), moving in \( \mathbb{H}_R^2 \). A necessary and sufficient condition for the function \( w = (w_1, \ldots, w_n) \) to be a Moebius solution of system (18), that is, a relative equilibrium associated to the Killing vector field \( X_3 \) defined by equation (48), is that for all \( k = 1, \ldots, n \), the following algebraic functional equations are satisfied

\[ -\frac{w_k^2 + 10|w_k|^2 + 2w_k|w_k|^2 + 2|w_k|^2\bar{w}_k^2 - 2\bar{w}_k^4 + 2w_k^4 - 3\bar{w}_k^2 + 4}{2(w_k - \bar{w}_k)} = \sum_{j=1}^{n} \frac{m_j(\bar{w}_j - w_j)^2(w_k - w_j)(\bar{w}_j - w_k)}{[\Theta(k,j)(w, \bar{w})]^{3/2}}. \]
Proof. Result follows when from the equations \( \dot{w}_k = 1 + w_k^2 - \frac{(w_k - \bar{w}_k)^2}{2} \), for \( k = 1, \ldots, n \), we differentiate and substitute in the equations of motion (18). This remark completes the proof. \( \square \)

Definition 6. We call hyperbolic cyclic Moebius solutions (relative equilibria) the solutions of system (18) that satisfy the conditions (49).

As in the parabolic case, we observe that equation (49) is also hard of solving, and again as there, instead of trying this, we shall find explicitly the hyperbolic flow of (48) and we will propose these type of solutions.

In this case, for the real coordinates \((u, v)\) in \( w = u + iv \), the complex differential equation (48) can be written (see [10]) as the system of real differential equations

\[
\begin{align*}
\dot{u} &= 1 + u^2 + v^2, \\
\dot{v} &= 2uv,
\end{align*}
\]

(50)

which when is integrated for the initial conditions

\[
\begin{align*}
u(0) + v(0) &= \alpha, \\
u(0) - v(0) &= \beta,
\end{align*}
\]

(51)

has the isometric parametrization

\[
\begin{align*}
u(t) &= \frac{1}{2} \left[ \frac{\tan t + \alpha}{1 - \alpha \tan t} + \frac{\tan t + \beta}{1 - \beta \tan t} \right], \\
v(t) &= \frac{1}{2} \left[ \frac{\tan t + \alpha}{1 - \alpha \tan t} - \frac{\tan t + \beta}{1 - \beta \tan t} \right].
\end{align*}
\]

(52)

In this way, by using equations (51), the isometric parametrization of the solution of equation (48) with initial condition \( w(0) = u(0) + iv(0) \) is

\[
\begin{align*}
(53) \quad w(t) &= \frac{1}{2} \left[ \frac{\tan t + \alpha}{1 - \alpha \tan t} + \frac{\tan t + \beta}{1 - \beta \tan t} \right] + i \left[ \frac{\tan t + \alpha}{1 - \alpha \tan t} - \frac{\tan t + \beta}{1 - \beta \tan t} \right].
\end{align*}
\]

Let \( s = \tan t \) be again the isometric diffeomorphic variable. Then the parametrization (53) becomes

\[
\begin{align*}
(54) \quad w(s) &= \frac{1}{2} \left[ \frac{s + \alpha}{1 - \alpha s} + \frac{s + \beta}{1 - \beta s} \right] + i \left( \frac{s + \alpha}{1 - \alpha s} - \frac{s + \beta}{1 - \beta s} \right).
\end{align*}
\]
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For the arbitrary index \( l = 1, 2, \ldots, n \), we define the equalities,

\[
A_l = A_l(s) = \frac{1}{2} \frac{\alpha_l + s}{1 - \alpha_l s}, \\
B_l = B_l(s) = \frac{1}{2} \frac{\beta_l + s}{1 - \beta_l s}, \\
C_l = C_l(s) = A_l + B_l, \\
D_l = D_l(s) = A_l - B_l,
\]

(55)

and if we substitute equations (54), (40) and (41) together the equations (55) into equation (49), the respective real parts for \( l = k \) become

\[
(1 + s^2)^2 \left( \frac{\alpha_k (1 + \alpha_k^2)}{(1 - \alpha_k s)^3} + \frac{\beta_k (1 + \beta_k^2)}{(1 - \beta_k s)^3} \right) \\
+ s(1 + s^2) \left( \frac{1 + \alpha_k^2}{(1 - \alpha_k s)^2} - \frac{\beta_k (1 + \beta_k^2)}{(1 - \beta_k s)^2} \right) \\
- 4 \left[ \frac{(\alpha_k + \beta) + 2(1 - (\alpha_k + \beta))s - (\alpha_k + \beta)s^2}{(1 + s^2)(1 - \alpha_k s)(1 - \beta_k s)} \right] \left[ \frac{1 + \alpha_k^2}{(1 - \alpha_k s)^2} + \frac{1 + \beta_k^2}{(1 - \beta_k s)^2} \right] \\
= \frac{4(1 + s^2)^3(\alpha_k - \beta_k)^3}{R} \sum_{j \neq k} \left( \frac{\alpha_j + s}{1 - \alpha_j s} - \frac{\beta_j + s}{1 - \beta_j s} \right)^2 m_j D_k(C_j - C_k) \\
\[\Theta_{(k,j)}(\alpha, \beta)]^{3/2}. \\
\]

(56)

On the other hand, the corresponding imaginary parts are

\[
(1 + s^2)^2 \left( \frac{\alpha_k (1 + \alpha_k^2)}{(1 - \alpha_k s)^3} - \frac{\beta_k (1 + \beta_k^2)}{(1 - \beta_k s)^3} \right) \\
+ s(1 + s^2) \left( \frac{1 + \alpha_k^2}{(1 - \alpha_k s)^2} + \frac{\beta_k (1 + \beta_k^2)}{(1 - \beta_k s)^2} \right) \\
+ \frac{8(1 + \alpha_k^2)(1 + \beta_k^2)}{(\alpha_k - \beta_k)(1 + s^2)(1 - \alpha_k s)(1 - \beta_k s)} \\
= -\frac{2(1 + s^2)^3(\alpha_k - \beta_k)^3}{R} \sum_{j \neq k} \left( \frac{\alpha_j + s}{1 - \alpha_j s} - \frac{\beta_j + s}{1 - \beta_j s} \right)^2 \times \\
\times m_j [D_k^2 - D_j^2 - (C_j - C_k)^2] \\
\[\Theta_{(k,j)}(\alpha, \beta)]^{3/2}. \\
\]

(57)

Here, \( \Theta_{(k,j)}(\alpha, \beta) \) is the corresponding evaluation of the singular function (17) in the \( 2n \)-parametric vector \( (\alpha, \beta) \).

We obtain the following result.

**Theorem 15.** There are not hyperbolic cyclic Moebius solutions (relative equilibria) in the negatively curved problem in \( \mathbb{H}^2_R \).
Proof. The proof follows the same method as for Theorem \[\text{[15]}\] Regardless we give it here. Consider again \(n\) point particles with masses \(m_1, \ldots, m_n > 0, n \geq 2\), moving in \(\mathbb{H}_R^2\) under the influence of the hyperbolic cotangent potential and satisfying the equations (56) and (57). Since the hyperbolic flow also carries all these solutions to the right positive half plane as \(t\) goes to infinity, without loss of generality we can suppose that the \(k\)-th particle is one of the last reaching the imaginary axis, this is, \(u_k(0) = 0\). If we take these configuration as initial conditions, then for \(s = 0\) we have the values
\[
A_k(0) = \frac{\alpha_k}{2},
B_k(0) = \frac{\beta_k}{2},
C_k(0) = \frac{\alpha_k + \beta_k}{2} = u_k(0) = 0,
D_k(0) = \frac{\alpha_k - \beta_k}{2} = v_k(0) = -\beta_k > 0,
\]
which implies that \(0 = C_k \leq C_j(0) = A_j(0) + B_j(0)\) for all \(k \neq j\).

Therefore, for these initial conditions equation (56) become
\[
0 = \sum_{j \neq k} \left( \frac{\alpha_j + s}{1 - \alpha_j s} - \frac{\beta_j + s}{1 - \beta_j s} \right)^2 \frac{m_j}{[\Theta_{(k,j)}(\alpha, \beta)]^{3/2}} (-\beta_k) C_j,
\]
which implies that \(C_j(0) = A_j(0) + B_j(0) = u_j(0) = 0\) for all \(j \neq k\), and therefore, if the \(k\)-particle reaches the imaginary axis for some time, the whole set of particles also must reach the imaginary axis for this same time. This also implies that \(\alpha_j = -\beta_j > 0\) for all \(j = 1, 2, \ldots, n\).

On the other hand, if we consider again for \(s = 0\) the initial conditions \(w_k(0) = i\beta_k, w_j(0) = i\beta_j\), such that \(\beta_j \leq \beta_k\), then \(C_j = C_k = 0\) and \(D_j \leq D_k\). If we substitute in the imaginary parts (57), then it follows that
\[
(\alpha_k - \beta_k)(1 + \beta_k^2) + \frac{8(1 + \alpha_k^2)(1 + \beta_k^2)}{(\alpha_k - \beta_k)} = -\frac{2(\alpha_k - \beta_k)^3}{R} \sum_{j \neq k} (\alpha_j - \beta_j)^2 \frac{m_j[D_k^2 - D_j^2]}{[\Theta_{(k,j)}(\alpha, \beta)]^{3/2}},
\]
which is a contradiction, because the left hand side of this last equation is positive whereas that the right hand side is negative.

This contradiction proves the claim and ends the proof. \(\square\)

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