Risk-return arguments applied to options with trading costs

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Abstract

We study the problem of option pricing and hedging strategies within the framework of risk-return arguments. An economic agent is described by a utility function that depends on profit (an expected value) and risk (a variance). In the ideal case without transaction costs the optimal strategy for any given agent is found as the explicit solution of a constrained optimization problem. Transaction costs are taken into account on a perturbative way. A rational option price, in a world with only these agents, is then determined by considering the points of view of the buyer and the writer of the option. Price and strategy are determined to first order in the transaction costs.

1 INTRODUCTION

Options are financial contracts made out between two economic agents called the writer and the buyer. The content of such a contract is that it gives the buyer the optional right to buy from the writer a unit of some commodity at some time in the future at a determined price. Options differ mainly as to the type of underlying commodity (stock, stock indices, foreign currency, etc.), if the expiry time is fixed or may be chosen by the option buyer (European or American), and on the form of the pay-out function. Option pricing theory is generally regarded as one of the corner-stones of modern mathematical finance, for standard text-book treatments, see [8, 7, 11]. The outcomes of these theories are normative prescriptions of option prices and hedging strategies, the latter being portfolios of the underlying commodity to be held in conjunction with the option. The theory has generally been developed for the class of complete markets, the two main examples being the log-Brownian continuous-time model of Black and Scholes[3], and the dichotomic discrete-time model of Cox, Ross and Rubinstein[6].

Option pricing in incomplete markets have been much less well developed, partly because the Black-Scholes and Cox-Ross-Rubinstein theories already yield quite reasonable estimates of observed market prices, partly because there is no agreed-upon procedure to find the price in this more general situation. In economic terms this is is expressed as the price being contingent on individual investor attitudes.

The perhaps simplest assumption about investor attitudes is that they can be described only by their preferred trade-offs of risk versus return on investment. If the risk is measured by the standard deviation, then we are effectively looking at the option problem in the spirit of the Markowitz portfolio theory[13]. Even though this is quite a simplification, there remains (at least) one parameter describing the risk-aversiveness of an individual investor, which appears as the parameter $\lambda$ below. To have predictive power we must be able to say something about a price on the market without assuming, say, that all investors can be described by the same parameter $\lambda$. One recalls that in the
Capital Asset Pricing Model one is able to derive a relation between returns and (diversified) risks on the market\(^{[1]}\). The particular relation (the slope of the Capital Market Line) depends on the set of investment opportunities and investor attitudes, but once it has been established, it holds for all investors. In short, one would like to get out something similar from a mean-variance approach to option pricing.

One way to proceed is to postulate that the hedging problem is solved by minimizing risk unconditionally. This approach has been put forward by several authors, notably by Schweizer\(^{[14]}\) and Bouchaud & Sornette \(^{[5]}\). In the mean-variance language it directly corresponds to all investors being very risk-averse, characterized by very large values of \(\lambda\). If all investors share this attitude, and trade with one another, then the expected profit from each trade has to be zero, and this argument fixes the price.

When presented this way the risk-minimizing prescription has problems: Why all operators have to be very risk-averse? And if they are very risk-averse, and make on the average zero net profit on an option trade, then why would they bother to engage in it? However, it turns out that the special role of risk-minimizing hedging can be derived from a mean-variance approach in a different manner\(^{[1]}\). The main objective of this present paper is to present this result, including transaction costs in the calculations.

In any reasonable approach no economic agent should be prepared to sell an option more cheaply than the price he would be prepared to pay for it. There may however be some normative agent who is prepared to buy and sell at one and the same price. If that is the case, and all other agents buy at lower and sell at higher prices, then this price is in fact a possible market price. This is what happens in the mean-variance approach without transaction costs. It is more appealing than the straight-forward risk-minimizing procedure is that the normative agent is neither infinitely risk-averse nor infinitely risk-willing, but something in between. The case of agents still more risk-willing requires a discussion outside the scope of this paper, see \(^{[1]}\), but does not change the argument. The price proposed by the normative agent is such, that if he used the risk-minimizing strategy his expected profit would be zero. His price therefore agrees with that of the risk-minimizing prescription of Schweizer and Bouchaud-Sornette. Ultimately this is a consequence of the variance being a quadratic functional of the strategy, and it would not hold if we adopt another measure of risk. The strategy actually used by the normative agent is however not the risk-minimizing strategy, It can best be described as the the risk-minimizing strategy plus a component of pure investment in stock, unrelated to the option. In this way we can rederive the price of the risk-minimizing approach, but we do not imply that any rational agent actually uses that strategy, except as part of a larger portfolio.

The paper is organized as follows: in section \(^{2}\) we set up the problem and rederive the results on mean-variance optimal portfolios of \(^{[1]}\). In section \(^{3}\) which contains the hard new results of the paper, we introduce transaction costs, and see how they modify the price and the strategy. In section \(^{4}\), included for completeness, we look at the buyer’s and the seller’s side of the market, and summarize the results.

## 2 RISK-RETURN WITHOUT TRANSACTION COSTS

We now specialize the discussion to a European Call Option contracted at time 0 to expire at time \(T\), with strike price \(S_c\). Time is taken discrete in units of \(\tau\), such that \(T = N\tau\). For convenience we refer to the underlying commodity as stock, which comes in units of shares. The price of the share is assumed to generated by a multiplicative random processes, such that \(S_{i+1} = u_i S_i\), where the \(u_i\)'s are independent and identically distributed random variables with finite variance. An important parameter is \(\mu\), the expected excess return on the share compared to a risk-less investment. Let us assume that a risk-less investment increases in value from 1 to \(r\) over one discrete trading period,
\[
\mu = < (u_i - r) > 
\]  
(1)

In realistic market models one would expect \( \mu \) to be larger than zero. The incremental wealth of an option writer that sells the option for \( C \) and uses the strategy to keep \( \phi_i(S_i) \) shares against the option at time \( t = i \tau \), if the realized share price is \( S_t \), is, in the absence of trading costs,

\[
\Delta W = C r^N + \sum_{i=0}^{N-1} (u_i - r) r^{N-i-1} S_i \phi_i(S_i) - \{ S_T - S_c \}^+ 
\]  
(2)

We follow here the notation of [3], to which we refer for a discussion and motivation of (2), see also [1, 2].

The gain and the risk are the expected value and the variance of the incremental wealth, with the price substracted, i.e.

\[
M[\phi] = < \Delta W - C r^N > 
\]  
(3)

\[
R[\phi] = < (\Delta W - C r^N)^2 > - \left( < \Delta W - C r^N > \right)^2 
\]  
(4)

The profit, the expected value of \( \Delta W \) in (2), is equal to the gain plus \( C r^N \).

\( M \) is a linear and \( R \) is a quadratic functional of \( \phi \). If we perform the minimization of \( R \) with \( M \) constrained to the value \( m \), it is clear that this \( R \) must be a quadratic polynomial in \( m \). To get the explicit coefficients of that polynomial we introduce the vector-valued set of functions given by

\[
F_i(S_i) = < \{ S_T - S_c \}^+ (u_i - r) >_{S_i}, 
\]  
(5)

the auxiliary variable

\[
\tilde{\psi}_i(S_i) = P(S_i|S_0) r^{N-i-1} S_i \phi_i(S_i), 
\]  
(6)

and the matrix

\[
K_{ij}(S_i, S_j) = \frac{P(S_i|S_0) P(S_j|S_0)}{P(S_i|S_0) P(S_j|S_0)} < (u_i - r)(u_j - r) >_{S_i, S_j} - \mu^2, 
\]  
(7)

where, naturally, \( P(S_i|S_0) \) is the conditional probability that the share price equals \( S_i \) at time \( i \), given that it was initially \( S_0 \). Likewise \( P(S_i, S_j|S_0) \) is the joint probability that the price equals \( S_i \) at time \( i \) and \( S_j \) at time \( j \), conditioned by having been initially \( S_0 \). The diagonal elements of \( K \) can be written a little more simply as

\[
K_{ii}(S_i, S_i) = \frac{1}{P(S_i|S_0)} < (u_i - r)^2 >_{S_i} - \mu^2. 
\]  
(8)

In the special case when \( \mu \) is equal to zero \( K \) is diagonal. The gain and the risk can now be written as ordinary scalar products involving \( \tilde{\psi} \), \( \tilde{\psi} \) and \( F \):

\[
M = \mathcal{B} + \mu (\tilde{\psi} \cdot 1) 
\]  
(9)

\[
R = R_c - 2 (\mathcal{F} \cdot \tilde{\psi}) + 2 \mu \mathcal{B} (\tilde{\psi} \cdot 1) + (\tilde{\psi} \cdot K \tilde{\psi}) 
\]  
(10)

where the notation \( \mathcal{B} \) stands for the average \( < \{ S_T - S_c \}^+ > \), and \( R_c \) the corresponding variance, \( < \{ S_T - S_c \}^+ > - < \{ S_T - S_c \}^+ > \) \( R_c \) \( \mu \) the minimum risk at given gain \( m \) is

\[
R[m] = \rho + \nu (m + \mathcal{A})^2 
\]  
(11)

with the following identification of the coefficients:

\[
\rho = R_c - (\mathcal{F} \cdot K^{-1} \mathcal{F}) + 2 \mathcal{B} \mu (1 - K^{-1} 1) - \mathcal{B}^2 \mu^2 (1 - K^{-1} 1) 
\]  
(12)

\[
\mathcal{A} = \mathcal{B} - \mu (1 - K^{-1} \mathcal{F}) + \mathcal{B} \mu^2 (1 - K^{-1} 1) 
\]  
(13)

\[
\nu = \frac{1}{\mu^2 (1 - K^{-1} 1)} 
\]  
(14)
The optimal strategy is given by
\[ \tilde{\psi}_i(S; m) = (K^{-1}F)_i(S) + \frac{m + B - \mu(1 \cdot K^{-1}F)}{\mu(1 \cdot K^{-1}1)}(K^{-1}1)_i(S). \] (15)

The value of \( m \) such that the lowest risk is attained is \( m = -A \). The risk-minimizing strategy is thus
\[ \tilde{\psi}_i(S; -A) = (K^{-1}F)_i(S) - \mu B - \mu(1 \cdot K^{-1}1)_i(S). \] (16)

The expected profit when trading with the risk-minimizing strategy is \( Cr^N - A \). If we adjust the price \( (C) \) such that the expected profit is zero, we have
\[ C = r^N A \] (17)
where \( A \) is given in (13). Equations (16) and (17) summarize the risk-minimizing approach to option pricing and hedging without transaction costs.

3 RISK-RETURN WITH TRANSACTION COSTS

We now assume that trading costs are present in the form
\[ \Delta W_{\text{costs}} = -\gamma F[\phi] \] (18)
where \( \gamma \) is a small parameter and \( F \) is a positive functional of the strategy \( \phi \). For instance, \( F \) could include proportional trading costs when changing the portfolio from \( \phi_i \) to \( \phi_{i+1} \).

We will assume that agents try to maximize a utility function of the following kind
\[ U[\phi, \gamma; \lambda] = M[\phi, \gamma] + Cr^N - \lambda \sqrt{R[\phi, \gamma]} \] (19)
where \( \lambda \) is a parameter and \( M[\phi, \gamma] \) and \( R[\phi, \gamma] \) denote the gain and the risk in the presence of transaction costs, i.e., the expected value and the variance of
\[ \Delta W - Cr^N = \sum_{i=0}^{N-1} (u_i - r)r^{N-i-1}S_i\phi_i(S_i) - \{S_T - S_c\}^+ - \gamma F[\phi] \] (20)

The gain and the risk are expanded in powers of \( \gamma \):
\[ M[\phi, \gamma] = M_0[\phi] - \gamma M_1[\phi] \] (21)
\[ R[\phi, \gamma] = R_0[\phi] + \gamma R_1[\phi] + \gamma^2 R_2[\phi] \] (22)
\[ \gamma M_1[\phi], \gamma^2 R_2[\phi] \]
where \( M_0[\phi] \) is identical to (3) and \( \gamma M_1[\phi] \), equal to \( \gamma <F[\phi]> \), are the expected trading costs using \( \phi \). We look for strategies that are expandable in power series in \( \gamma \)
\[ \phi = \phi_0 + \gamma \phi_1 + \ldots \] (24)

To maximize the utility (19) we first wish to minimize the risk at constant gain. We minimize
\[ Q[\phi, \gamma] = R[\phi, \gamma] - 2q (M[\phi, \gamma] - m) \] (25)
by varying \( \phi \) and \( q \). The first step gives
\[ \phi(q) = \phi_0(q) - \gamma (\delta^2 R_0[\phi] |_{\phi_0})^{-1} \left[ \frac{\delta R_1[\phi]}{\delta \phi} |_{\phi_0} + 2q \frac{\delta M_1[\phi]}{\delta \phi} |_{\phi_0} \right] + O(\gamma^2) \] (26)
By varying with respect to \( q \) we have

\[
q(m) = q_0(m) - \gamma \frac{-M_1[\phi_0(q_0)] + (\frac{\delta M_0[\phi]}{\delta \phi}|_{\phi=q_0}) \cdot \phi_1(q_0(m))}{(\frac{\delta M_0[\phi]}{\delta \phi}|_{\phi=q_0}) \cdot (\frac{\delta \phi_0(q)}{\delta q}|_{q=q_0(m)})} + O(\gamma^2)
\] (27)

The solutions to the zeroth order equations are the same as those discussed above in section 2.

\[
q_0(m) = -\nu(m + A) \tilde{\psi}_i(S_i; q) = (K^{-1}F)_i(S_i) - \mu(B + q)(K^{-1}1)_i(S_i)
\] (28)

where \( \nu \) and \( A \) are given in (14) and (13), and \( \tilde{\psi} \) is identical to (13), only expressed as a function of the Lagrange multiplier \( q \). As a function of \( m \) the optimal strategy may be written

\[
\phi(m) = \phi_0(q_0(m)) + \gamma \phi_1(q_0(m) + \gamma q_1(m) + \ldots) + \gamma \phi_1(q_0(m) + \gamma q_1(m) + \ldots) + \ldots
\]

which may be simplified to

\[
\phi_0(q_0(m)) + \gamma \frac{M_1[\phi_0(q_0(m))] }{(\frac{\delta M_0[\phi]}{\delta \phi}|_{\phi=q_0(m)}) \cdot (\frac{\delta \phi_0(q)}{\delta q}|_{q=q_0(m)}) + O(\gamma^2)
\] (30)

The minimal risk as a function of trading gain can now be expressed as

\[
R[m] = R_0[\phi_0(q_0(m)) + \gamma q_1(m) + \ldots] + \gamma \phi_1(q_0(m) + \gamma q_1(m) + \ldots) + \ldots
\]

which can be expanded into

\[
R[m] = R_0[\phi_0(q_0(m))] + \gamma \left( \frac{dR_0[m]}{dm} \cdot M_1[\phi_0(q_0(m))] + R_1[\phi_0(q_0(m))] \right) + O(\gamma^2)
\] (32)

Since the combination of \( \phi_1(q_0(m)) \) and \( \frac{\delta \phi_0(q)}{\delta q} \cdot q_1(m) \) simplifies we have, in fact, a much more compact expression for the risk as a function of \( m \), expanded up to first order in \( \gamma \), namely

\[
R[m] = R_0[\phi_0(q_0(m))] + \gamma \left( \frac{dR_0[m]}{dm} \cdot M_1[\phi_0(q_0(m))] + R_1[\phi_0(q_0(m))] \right) + O(\gamma^2)
\] (33)

The interpretation of (33) is straightforward. The expected trading costs, using the zeroth order trading strategy, appropriate for a level of gain equal to \( m \), is \( \gamma M_1[\phi_0(q_0(m))] \). When we use this strategy we actually realise a gain of \( m - \gamma M_1[\phi_0(q_0(m))] \). To reach a level of \( m \), compensating for trading losses, we must use a strategy that in the absence of trading costs would give \( m + \gamma M_1[\phi_0(q_0(m))] \). In doing so we run up the extra risk, to first order, of \( \gamma \frac{dR_0[m]}{dm} \cdot M_1[\phi_0(q_0(m))] \).

Since the risk is now expressed as a function of gain and the trading cost parameter \( \gamma \), we can write the utility as

\[
U[m, \gamma; \lambda] = m + Cr^N - \lambda \sqrt{R[m, \gamma]}
\]

(34)

At given risk-aversiveness parameter \( \lambda \) we seek to maximize the utility by varying \( m \). Let us assume that the maximum is obtained at a value of \( m[\lambda, \gamma] \). We then observe that a price acceptable to the writer must be such that the resulting utility is non-negative. If the option writer does not sell an option and performs no operations in the market, then his incremented wealth is identically zero, which carries zero utility. We hence have

\[
C[\gamma; \lambda] = r^{-N} \left( -m[\lambda, \gamma] + \lambda \sqrt{R[m[\lambda, \gamma], \gamma]} \right)
\]

(35)
where we understand that this is the lowest price that this option writer is prepared to ask for.

We are therefore to maximize the expression \( m - \lambda \sqrt{R[m, \gamma]} \), and we do so by expanding

\[
m[\lambda, \gamma] = m_0[\lambda] + \gamma m_1[\lambda] + \ldots
\]

and all the preceding expansions, which together yield

\[
U[m, \gamma; \lambda] - r^N C = \left( m_0[\lambda] - \lambda \sqrt{R_0[m_0[\lambda]]} \right)
+ \gamma \left( m_1[\lambda] - \frac{1}{2} \frac{1}{\sqrt{R_0[m_0[\lambda]]}} \left( \frac{dR_0[m]}{dm} \right) m_1[\lambda] + R_1[\phi_0(m_0[\lambda])] \right)
+ \frac{dR_0[m]}{dm} \left( M_1[\phi_0(m_0[\lambda])] \right) + \mathcal{O}(\gamma^2)
\]

(37)

All zeroth order calculations can be done rather explicitly since we have \( R_0[m] = \rho + \nu(m + A)^2 \), with all the coefficients known. Hence

\[
m_0[\lambda] = -A + \sqrt{\frac{\rho}{\nu(\lambda^2 \nu - 1)}}
\]

(38)

\[
C[\gamma = 0; \lambda] = r^{-N} \left( A + \sqrt{\frac{\rho(\lambda^2 \nu - 1)}{\nu}} \right)
\]

(39)

with \( \rho, A \) and \( \nu \) given in (12), (13) and (14). The corresponding (zeroth order) optimal strategy is

\[
\tilde{\psi}_i(S_i; m_0[\lambda]) = \left( (K^{-1}F)_i(S_i) - \mu B(K^{-1}1)_i(S_i) \right) + \sqrt{\frac{\rho \nu}{\lambda^2 \nu - 1}} (K^{-1}1)_i(S_i)
\]

(40)

This is again equivalent to (13) and (28), but expressed this time as a function of the risk-aversiveness parameter \( \lambda \). The structure is that of a \( \lambda \)-dependent correction to the risk-minimizing strategy (16).

When \( \lambda \) is large the correction is small. When \( \lambda \) diminishes, such that the combination \( \lambda^2 \nu \) tends to zero, the correction is large. The case of \( \lambda^2 \nu < 1 \) is one that can be treated by comparing with pure stock investment [1]. An operator using utility function (19) would then invest an unlimited amount in stock. In other words, when \( \lambda^2 \nu - 1 \) is negative, the formulation of the problem using (19) gives unreasonable results, both for stock and options. If, however, the utility function is modified by adding a quantity \( -\frac{1}{2W_0} R[\phi] \), then the stock investor only invests an amount proportional to \( W_0 \).

The option price can then be fixed by comparing the utilities of option trading and pure stock investment, in a similar way as the option price has here been fixed by comparing option trading and doing nothing. The general structure of the solution will again be the risk-minimizing strategy (13) and the price in the risk-minimizing approach, with corrections which will now depend on both \( \lambda \) and \( W_0 \), see [1].

For the rest of this paper we will assume that \( \lambda^2 \nu \) is greater than one.

Coming back to (37) we see that the structure of \( R_0 \) also facilitates the optimization to next order in \( \gamma \), since

\[
\frac{1}{2} \lambda \frac{1}{\sqrt{R_0[m_0[\lambda]]}} \left( \frac{dR_0[m]}{dm} \right) \bigg|_{m = m_0[\lambda]} = 1
\]

(41)

We therefore have up to first order in \( \gamma \)

\[
U[m, \gamma; \lambda] - r^N C = \left( m_0[\lambda] - \lambda \sqrt{R_0[m_0[\lambda]]} \right) - \gamma \left( M_1[\phi_0(m_0[\lambda])] + \frac{1}{2} \sqrt{\frac{\lambda^2 \nu - 1}{\rho \nu}} R_1[\phi_0(m_0[\lambda])] \right) + \mathcal{O}(\gamma^2)
\]

(42)
which does not depend on \( m_1[\lambda] \). Hence we do not need to compute \( m_1[\lambda] \). The price, fixed by the requirement that the maximal utility is zero, is finally

\[
C[\gamma; \lambda] = r^{-N} \left( A + \sqrt{\frac{\rho(\lambda^2\nu - 1)}{\nu}} \right) + \\
\gamma r^{-N} \left( M_1[\phi_0[\lambda]] + \frac{1}{2} \sqrt{\frac{\lambda^2\nu - 1}{\rho\nu}} R_1[\phi_0(m_0[\lambda])] \right) + O(\gamma^2) \tag{43}
\]

Using (16), (28), (30) and (40) we can also express the optimal strategy up to first order in \( \gamma \) as the risk-minimizing strategy and a correction proportional to \( (K^{-1}1)_i(S_i) \).

4 MARKET EQUILIBRIUM

Rationalizations of market prices are economically meaningful if there are both buyers and sellers. We have so far exposed the point of view of the writer of the option. In the context of mean-variance arguments the analysis for the buyer is however very similar. If, in fact, an option buyer would use a strategy \( \phi^b \), then, in the absence of trading costs, his incremental wealth would be equal in size but opposite in sign of that of an option writer using \( -\phi^b \). From this follows immediately the strategy and the price appropriate for an option buyer described by a risk-aversiveness parameter \( \lambda \). The bid-ask spread of agents is thus, in the absence of trading costs,

\[
C^{\text{bid/ask}}[\gamma = 0; \lambda] = r^{-N} \left( A \pm \sqrt{\frac{\rho(\lambda^2\nu - 1)}{\nu}} \right) \tag{44}
\]

The only agents prepared to buy and sell are those with a value of \( \lambda \) such that \( \lambda^2\nu - 1 \) vanishes, and the price they offer is \( r^{-N}A \), which we recognize by (17) to be the same as that from the risk-minimizing procedure.

The trading costs break the symmetry between buyer and writer. All things being equal, the buyer will be prepared to pay a little less, and the writer will ask a little more. A gap opens between the smallest ask price and the largest bid price. Strictly speaking, we do not find a market price in the presence of trading costs. When \( \lambda^2\nu \) tends to one from above, the zeroth order strategy, \( \phi_0(m[\lambda]) \), diverges. Hence the average trading costs become large. It is convenient to introduce the notation

\[
\phi^\text{risk} = \frac{1}{P(S_i|S_0)r^N_{N-i}S_i} \left( (K^{-1}F)_i(S_i) - \mu B(K^{-1}1)_i(S_i) \right) \tag{45}
\]

\[
\phi^\text{gain} = \frac{1}{P(S_i|S_0)r^N_{N-i}S_i} (K^{-1}1)_i(S_i) \tag{46}
\]

such that equation (44) can be written for the variable \( \phi \) as

\[
\phi_0(\lambda) = \phi^\text{risk} + \sqrt{\frac{\rho\nu}{\lambda^2\nu - 1}} \phi^\text{gain} \tag{47}
\]

The first order correction to the strategy can by (28) and (30) be written

\[
\phi_1(\lambda) = \mu \nu M_1[\phi_0(\lambda)] \phi^\text{gain} \tag{48}
\]

When \( \lambda^2\nu \) is close to one the trading costs are dominated by the component proportional to \( \phi^\text{gain} \). The only way in which the dominating contribution in this case could come from \( \phi^\text{risk} \) would be if \( \phi^\text{gain} \) were actually a buy-and-hold strategy. It is fairly straight-forward to see that this is not so. It suffices to look at the case when \( \mu \) is equal to zero, the matrix \( K \) diagonal, and use (8).
When $\lambda^2\nu$ is larger, $\phi^{\text{gain}}$ and $\phi^{\text{risk}}$ contribute each to the trading costs. We thus have

$$M_1[\phi_0(\lambda)] = \sqrt{\frac{\rho\nu}{\lambda^2\nu - 1}} M_1[\phi^{\text{gain}}] + M_1[\phi^{\text{risk}}]$$  \hspace{1cm} (49)$$

We can also address in a similar manner the first order correction to the risk that enters in (43). When $\lambda^2\nu$ is close to one we should have

$$\sqrt{\frac{\lambda^2\nu - 1}{\rho\nu}} R_1[\phi_0(\lambda)] \sim \sqrt{\frac{\rho\nu}{\lambda^2\nu - 1}} R_1[\phi^{\text{gain}}]$$ \hspace{1cm} (50)$$

which depends in a similar way as the first term in (49) on $\lambda$. After a possible redefinition of $M_1[\phi^{\text{gain}}]$ to incorporate the incremental risk $R_1$ we can therefore write the bid/ask prices as

$$C_{\text{bid/ask}}[\gamma; \lambda] = r^{-N} \left( A \pm \sqrt{\frac{\rho(\lambda^2\nu - 1)}{\nu}} + \gamma M_1[\phi^{\text{gain}}] \sqrt{\frac{\rho\nu}{\lambda^2\nu - 1}} + \gamma M_1[\phi^{\text{risk}}] \right)$$ \hspace{1cm} (51)$$

We can now find the optimal $\lambda$ as a function of $\gamma$ by minimizing the bid-ask spread. The minimum is attained at $(\lambda^*)^2 = \frac{1}{\nu} + \gamma M_1[\phi^{\text{gain}}]$ and leads to

$$C_{\text{bid/ask}}[\gamma; \lambda^*] = r^{-N} \left( A \pm 2\sqrt{\rho\gamma M_1[\phi^{\text{gain}}] + \gamma M_1[\phi^{\text{risk}}]} \right)$$ \hspace{1cm} (52)$$

The two components in the bid-ask spread are the expected trading costs using the risk-minimizing strategy, $\gamma M_1[\phi^{\text{risk}}]$, and a term proportional to $\sqrt{\rho}$, the square root of the minimal residual risk. This is also the form used in [3]. The new piece in (52) is that the proportionality factor of $\sqrt{\rho}$ is not a free parameter, but in principle computable. Note that $\phi^{\text{gain}}$ does not involve the pay-out function of the option. Hence $\gamma M_1[\phi^{\text{gain}}]$ takes the same value for different options. The optimal strategies are approximately

$$\phi^{\text{writer/buyer}}(\lambda^*) = \pm \phi^{\text{risk}} + \sqrt{\frac{\rho}{\gamma M_1[\phi^{\text{gain}}]}} \phi^{\text{gain}}$$ \hspace{1cm} (53)$$

In a mean-variance world with transaction costs and homogeneous expectations there is a minimum gap between the smallest ask and the largest bid price. We should therefore, in that world, not expect to see any trading in options at all. In the real world all agents do not of course hold identical expectations. In addition, there is no reason to expect that all agents are necessarily well described by their preferred trade-offs of risk vs. return, measured by a variance and an expected value. In somewhat extreme parameter ranges mean-variance option pricing lead to paradoxical results [16, 10, 2], also discussed in the finance literature quite some time ago [9].

It is therefore not a problem per se that we find a gap between bid and ask prices. The price $A$ from (52) and (13) is an estimate of a market price, and it is in the end an empirical question to decide how useful that estimate is. If the gap as predicted by (52) is sufficiently small compared to inhomogeneous expectations and all other externalities that tend to move and push apart bids and asks, then there is no conceptual difference between testing $A$ or the Black-Scholes price against observed market data. For tests of the risk-minimizing prescriptions against market data, chiefly for the spacial case $\mu = 0$, see [3] [4].

The analysis of this paper has been performed in perturbation theory in $\gamma$. This assumes that the trading costs are relatively small. It is however perfectly natural to consider the case where the minimal risk $\rho$ is very small, but trading with the strategy $\phi^{\text{risk}}$ leads to very large costs. For instance, suppose that risk is minimized by rehedging very often, and one pays some amount for every trade, an example considered in [3]. It then seems clear that the best strategy is probably not very close to the optimal strategy without transaction costs, and the perturbation cannot really be applied.
considered small. We may however get information also on this case by imagining that the space of possible strategies may be varied (for instance, by trading more or less often). The minimal risk and the trading costs will then both change with the class considered. In minimizing the bid-ask spread there is a trade-off between minimizing minimal risk (by trading more often) and minimizing trading costs (by trading less often). One expects that the best trade-off is obtained when minimal risk and trading costs are of the same order.

To conclude, the mean-variance approach to option pricing leads to the same price as the risk-minimizing prescription of [5] and [14]. The optimal strategy is, in the absence of transaction costs, equal to the risk-minimizing strategy plus another strategy more related to direct investment in stock. Transaction costs lead to a gap between bid and ask prices. By minimizing this gap one arrives at a specific level of risk-taking appropriate to a given level of transaction costs. All these calculations can be done perturbatively around the case of no trading friction, and therefore assume that transaction costs using the optimal (zero-cost) strategy are small. The case when the transaction costs using the optimal (zero-cost) strategy are actually large can be done in a somewhat more heuristic manner by varying the class of strategies considered. This leads to the estimate that the smallest bid/ask spreads are obtained when trading costs and residual risk are about equal.

The results of the mean-variance approach therefore finally agree in surprising detail with those of the risk-minimizing prescription.

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