Hardy Inequalities for Fractional $(k,a)$-Generalized Harmonic Oscillators

We define $a$-deformed Laguerre operators $L_{a,\alpha}$ and $a$-deformed Laguerre holomorphic semigroups on $L^2((0,\infty),d\mu_{a,\alpha})$. Then we give a spherical harmonic expansion, which reduces to the Bochner-type identity when taking the boundary value $z = \pi i/2$, of the $(k,a)$-generalized Laguerre semigroup introduced by Ben Saïd, Kobayashi and Ørsted. We prove a Hardy inequality for fractional powers of the $a$-deformed Dunkl harmonic oscillator $\triangle_{k,a} := |x|^{2-a} \triangle_k - |x|^a$ using this expansion. When $a = 2$, the fractional Hardy inequality reduces to that of Dunkl-Hermite operators given by Ciaurri, Roncal and Thangavelu. The operators $L_{a,\alpha}$ also give a tangible characterization of the radial part of the $(k,a)$-generalized Laguerre semigroup on each $k$-spherical component $H_{k,a}^m(\mathbb{R}^N)$ for

$$
\lambda_{k,a,m} := \frac{2m + 2\langle k \rangle + N - 2}{a} \geq -\frac{1}{2}
$$

defined via a decomposition of the unitary representation.

Keywords: Spherical harmonic expansion of $(k,a)$-generalized Laguerre semigroup, $a$-deformed Laguerre operators, fractional Hardy inequality, $(k,a)$-generalized harmonic oscillator.

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