POINT MODULES OF QUANTUM PROJECTIVE SPACES

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ABSTRACT. In this note we give an explicit description of the irreducible components of the reduced point varieties of quantum polynomial algebras.

Consider the quantum polynomial algebra on $k+1$-variables with quantum commutation relations

$$A_Q = \mathbb{C}(u_o, u_1, \ldots, u_k)/(u_iu_j - q_{ij}u_ju_i, 0 \leq i, j \leq k)$$

where the entries of the $k+1 \times k+1$ matrix $Q = (q_{ij})$ are all non-zero and satisfy $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$.

As $A_Q$ is a graded connected, iterated Ore-extension, it is Auslander-regular of dimension $k+1$ and we can consider the corresponding non-commutative projective space $\mathbb{P}_Q^n = \text{Proj}(A_Q)$ in the sense of [1].

Recall from [1] that a point module $P$ is a graded left $A_Q$-module which is cyclic (that is, generated by one element in degree 0), critical (implying that normalizing elements of $A_Q$ act on it either as zero or a non-zero divisor) with Hilbert-series $(1-t)^{-1}$.

Hence, a point module $P$ is necessarily of the form $A_Q/(A_Ql_1 + \ldots + A_Ql_k)$ where the $l_i$ are linearly independent degree one elements in $A_Q$, and hence determines a unique point $x_P = V(l_1, \ldots, l_k)$ in commutative $k$-dimensional projective space $\mathbb{P}^n = \text{Proj}(A_Q^*|)$.

In this note we will describe the reduced subvariety of $\mathbb{P}^k$, which is called the point-variety of $A_Q$

$$\text{pts}(A_Q) = \{x_P \in \mathbb{P}^k \mid P \text{ a point module of } A_Q\}.$$ 

We can approach this problem inductively.

**Proposition 1.** For each of the generators $u_i$ of $A_Q$ we have

$$\text{pts}(A_Q) = (\text{pts}(A_Q) \cap V(u_i^*)) \sqcup (\text{pts}(A_Q) \cap X(u_i^*))$$

and these pieces can be described as follows:

1. $\text{pts}(A_Q) \cap V(u_i^*) = \text{pts}(A_Q \cap X(u_i^*))$ where $\overline{Q}$ is the $k \times k$ matrix obtained from $Q$ after deleting the $i$-th row and column.

2. $\text{pts}(A_Q) \cap X(u_i^*)$ is the affine variety

$$\bigcap_{j \neq i} V((r_{jl} - 1)v_j^*v_i^*)$$

for the polynomial functions on $X(u_i^*)$: $v_j^* = u_j^*(u_i^*)^{-1}$ and with $r_{jl} = q_{ij}q_{jl}^{-1}$.

3. In particular, $\text{pts}(A_Q) = \mathbb{P}^k$ if and only if the rank of the matrix $Q$ is equal to one.
Proof. As \( u_i \) is normalizing in \( A_Q \) it acts either as zero on a point module \( P \) or as a non-zero divisor. The point modules on which \( u_i \) acts as zero are \( \text{pts}(A_Q) \cap \mathbb{V}(u_i^*) \), correspond to point modules of the quantum polynomial algebra \( A_Q/(u_i) \cong A_Q[1]/u_i \) and are contained in the projective subspace \( \mathbb{P}^{k-1} = \mathbb{V}(u_i^*) = \text{Proj}((A_Q[1])^*_1) \). This proves (1).

If \( u_i \) acts as a non-zero divisor on the point module \( P \), it extends to a graded module over the localization of \( A_Q \) at the multiplicative system of homogeneous elements \( \{1, u_i, u_i^2, \ldots \} \) and as this localization has an invertible degree one element it is a strongly graded algebra, see \( [3, \S 5] \), and hence is a skew Laurent-polynomial extension

\[
A_Q[u_i^{-1}] \cong (A_Q[u_i^{-1}])_0[u_i, u_i^{-1}, \sigma]
\]

where \( (A_Q[u_i^{-1}])_0 \) is the degree zero part of the localization and \( \sigma \) the automorphism on it induced by conjugation with \( u_i \).

The algebra \( (A_Q[u_i^{-1}])_0 \) is generated by the \( k \) elements \( v_j = u_ju_i^{-1} \) and as we have the commuting relations \( u_ju_i^{-1} = q_{ij}u_i^{-1}u_j \) we have

\[
v_{j\ell}v_l = q_{ij}q_{il}^{-1}u_ju_lu_i^{-1} = q_{ij}q_{jl}u_i^{-1}u_ju_lu_i^{-1} = q_{ij}q_{jl}q_{il}^{-1}u_iu_lu_ju_i^{-1} = q_{ij}q_{jl}q_{il}^{-1}v_lv_j
\]

Therefore, \( (A_Q[u_i^{-1}])_0 \) is again a quantum polynomial algebra of the form \( A_R \) where \( R = (r_{jl})_{j,l} \) is the \( k \times k \) matrix with entries

\[
r_{jl} = q_{ij}q_{jl}q_{il}^{-1}
\]

Because \( (A_Q[u_i^{-1}])_0 \) is strongly graded, the localization \( P[u_i^{-1}] \) (and hence the point module \( P \)) is fully determined by the one-dimensional representation \( P[u_i^{-1}]_0 \) of \( (A_Q[u_i^{-1}])_0 \), see \([3, \S 5.3]\) or \([1, \text{Proposition 7.5}]\).

One-dimensional representations of \( A_R \) correspond to points \( (a_j) \in \mathbb{A}^k \) (via the association \( v_j \mapsto a_j \) for all \( j \neq i \)) satisfying all the defining relations of \( A_R \), that is, they must satisfy the relations

\[
(1 - r_{jl})a_ja_l = 0
\]

which proves (2).

As for (3), observe that \( \text{pts}(A_Q) \cap \mathbb{X}(u_i^*) = \mathbb{A}^k \) if and only if all the \( r_{jl} = 1 \). This in turn is equivalent, by the definition of the \( r_{jl} \) to

\[
\forall j, l \neq i : q_{jl} = q_{il}^{-1}
\]

but then, any \( 2 \times 2 \) minor of \( Q \) has determinant zero as

\[
\begin{bmatrix}
q_{ju} & q_{jv} \\
q_{lu} & q_{lv}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

and the same applies to minors involving the \( i \)-th row or column of \( Q \). Hence, \( Q \) is of rank one (and so is \( \mathbb{Q} \)) finishing the proof.

This result also allows us to describe the irreducible components of the point-varieties of quantum polynomial algebras directly. Take a quantum polynomial algebra on \( n+1 \) variables

\[
A_M = \mathbb{C}(x_0, x_1, \ldots, x_n)/(x_ix_j - m_{ij}x_jx_i, \ 0 \leq i, j \leq n)
\]

Consider the points \( \delta_i = [\delta_{i0} : \ldots : \delta_{in}] \) in \( \mathbb{P}^n = \text{Proj}((A_M)^*_1) \). For any \( k+1 \)-tuple \((i_0, i_1, \ldots, i_k)\) with \( 0 \leq i_0 < i_1 < \ldots < i_k \leq n \) consider the \( k \)-dimensional projective
subspace $\mathbb{P}(i_0, \ldots, i_k) \subset \mathbb{P}^n$ spanned by the points $\delta_{i_j}$. Also denote the $k+1 \times k+1$ minor of $M$ by

$$M(i_0, \ldots, i_k) = \begin{bmatrix}
1 & m_{i_0 i_1} & \cdots & m_{i_0 i_k} \\
m_{i_1 i_0} & 1 & \cdots & m_{i_1 i_k} \\
\vdots & \vdots & \ddots & \vdots \\
m_{i_k i_0} & m_{i_k i_1} & \cdots & 1
\end{bmatrix}$$

With these notations, we deduce from Proposition 1:

**Theorem 2.** The reduced point-variety of the quantum polynomial algebra $A_M$ is equal to

$$\text{pts}(A_M) = \bigcup_{\text{rk}(M(i_0, i_1, \ldots, i_k)) = 1} \mathbb{P}(i_0, i_1, \ldots, i_k) \subset \mathbb{P}^n$$

As a consequence, the union of all lines $\cup_{i,j} \mathbb{P}(i,j)$ is always contained in $\text{pts}(A_M)$ and will be equal to it for generic $M$.

**Proof.** As $\mathbb{P}(i_0, \ldots, i_k) = \text{Proj}((A_Q)_n)$ with $A_Q = A_M/(x_{j_1}, \ldots, x_{j_{n-k}})$ where $\{0, 1, \ldots, n+1\} = \{i_0, i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\}$, the description of $\text{pts}(A_M)$ follows from Proposition 1.

As for the generic statement, consider the matrix $M$ with $m_{ij} = -1$ if $i \neq j$. Clearly, as any $3 \times 3$ minor

$$M(i, j, k) = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{bmatrix}$$

has rank 3, we have that $\text{pts}(A_M) = \cup_{i,j} \mathbb{P}(i,j)$. Alternatively, one can use the results of [2]. In this case, the algebra $A_M$ is a Clifford algebra over $\mathbb{C}[x_0^2, \ldots, x_n^2]$ with associated quadratic form $D = \text{diag}(x_0^2, \ldots, x_n^2)$. In [2] it was shown that for a Clifford algebra, the point-variety is a double cover $\text{pts}(A_M) \longrightarrow \mathbb{V}(\text{minor}(3, D)) \subset \mathbb{P}^n_{[x_0^2, \ldots, x_n^2]}$, ramified over $\mathbb{V}(\text{minor}(2, D))$. For the given $M$ one easily checks that $\mathbb{V}(\text{minor}(3, D))$ is one-dimensional, hence so is $\text{pts}(A_M)$. \hfill $\square$

We leave the combinatorial problem of determining which subvarieties of $\mathbb{P}^n$ can actually arise as a suggestion for further research. Not all unions as above can occur.

**Example 3.** In $\mathbb{P}^3$ only two of the $\mathbb{P}^2$ (out of four possible) can arise in a proper subvariety $\text{pts}(A_M) \subset \mathbb{P}^3$. For example, take

$$M = \begin{bmatrix}
1 & a & b & x \\
a^{-1} & 1 & a^{-1}b & c \\
b^{-1} & ab^{-1} & 1 & bca^{-1} \\
x^{-1} & c^{-1} & ba^{-1}c^{-1} & 1
\end{bmatrix}$$

then, for generic $x$ we have

$$\text{pts}(A_M) = \mathbb{P}(0, 1, 2) \cup \mathbb{P}(1, 2, 3) \cup \mathbb{P}(0, 3)$$

but once we want to include another $\mathbb{P}^2$, for example, $\mathbb{P}(0, 1, 3)$ we need the relation $x = ac$ in which case $M$ becomes of rank one, whence $\text{pts}(A_M) = \mathbb{P}^3$. 
References

[1] Michael Artin, John Tate and Michel Van den Bergh, *Modules over regular algebras of dimension 3*, Inventiones mathematicae **106**, 1, 335-388, Springer (1991)

[2] Lieven Le Bruyn, *Central singularities of quantum spaces*, Journal of Algebra **177**, 1, 142-153, Elsevier (1995)

[3] C. Nastasescu and F. Van Oystaeyen, *Graded Ring Theory*, North Holland Publ. Co. (1982)

[4] user2013, *Point modules of quantum projective space* $P^n$ Mathoverflow question, October 11 (2012)

[mathoverflow.net/questions/109347/point-modules-of-quantum-projective-space-mathbbpn]

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