ELLIPTIC AND PARABOLIC EQUATIONS WITH FRACTIONAL DIFFUSION AND DYNAMIC BOUNDARY CONDITIONS

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Abstract. We investigate a class of semilinear parabolic and elliptic problems with fractional dynamic boundary conditions. We introduce two new operators, the so-called fractional Wentzell Laplacian and the fractional Steklov operator, which become essential in our study of these nonlinear problems. Besides giving a complete characterization of well-posedness and regularity of bounded solutions, we also establish the existence of finite-dimensional global attractors and also derive basic conditions for blow-up.

1. Introduction. Physical phenomena which requires the use of fractional operators is dubbed as anomalous diffusion in the sense that it exhibits deviations from normal diffusion, in that diffusion happens at either slower or faster scales [25]. From the probabilistic point of view and in analogy with the classical view of Brownian motion, the best way to understand anomalous diffusion is the random walk formalism: in the former particles make only small steps with finite probability such that the interaction between close neighbors is always short-ranged while in the latter particles are also allowed to take “arbitrarily” large steps (up to the system size) with a small finite probability for each such step, and so the interaction between particles is long-ranged. While normal diffusion is mostly ubiquitous to systems close to equilibrium, anomalous diffusion seems to be inherent in dynamical systems far from equilibrium (see [17, 25, 27, 29]).

In recent years the mathematical community has commenced an intense agenda on understanding fractional Laplacian operators in the context of various applications that show up in optimization problems, nonlinear stochastic dynamics, phase-transition and anomalous-diffusion phenomena, crystal dislocation dynamics and

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many others (see [8, 25] for a more complete list of references for potential applications). Many versions of the fractional Laplacian

\[-\Delta^s u(x) = \mathcal{N}_s \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy, \quad x \in F\]  

have been introduced and studied in connection with classical problems associated with the Laplace operator \(\Delta\), either on \(F = \mathbb{R}^N\) or on bounded (sufficiently smooth) domains \(F = \Omega \subset \mathbb{R}^N, \ N \geq 1\). The case where \(|x-y|^{N-2s}\) has been replaced by a general symmetric kernel \(K(x, y)\) has been recently studied in [6, 7, 24]. We refer the reader to [2, 6, 7, 16, 24, 28] for further literature which now appears to grow at a rapid rate but it is still full with many questions that remain to be answered. In the case of a bounded domain, by far the most predominant version of fractional Laplace operator in the literature is \((-\Delta_{\Omega})^s, \ s \in (0, 1)\), named the restricted fractional Laplacian in [28]; for this operator functions in \(\mathcal{F} = \Omega\) are extended by zero outside of \(\Omega\), i.e., \(u = 0\) on \(\mathbb{R}^N \setminus \Omega\) (in this case, one also simply refers to \((-\Delta_{\Omega})^s\) as the fractional Dirichlet Laplacian and plays a similar role as the classical Laplace operator \(\Delta\) with the Dirichlet boundary condition \(u_{|\partial\Omega} = 0\); note that \((-\Delta_{\Omega})^s\) should not be confused with a fractional power of the classical Dirichlet Laplacian, which has a spectral definition, and which is quite different). Besides, (inhomogeneous) (nonlocal) Neumann boundary conditions can be also defined for \((-\Delta)^s\) in (1.1) but as in the previous case they must be posed on a “fat” collar surrounding the bounded set \(\Omega\) (see [10, 21]) due to the nonlocal character of the fractional Laplacian. An example of such a collar is \(\mathbb{R}^N \setminus \Omega\) (cf., for instance, [9]). For such an example, the definition of Neumann boundary condition requires that the function \(u\) is defined on all \(\mathbb{R}^N\) (or the “fat” collar surrounding \(\Omega\)) and not only on \(\Omega\). From the view of applications, it seems quite impractical to deal with boundary conditions on the whole \(\mathbb{R}^N \setminus \Omega\), especially when one has to deal with nonlinear (and even inhomogeneous) boundary conditions. Moreover, an initial datum must be also specified in the whole of \(\mathbb{R}^N\), even though the evolution equation that governs the dynamics of the underlying physics is only posed in \(\Omega\). Thus, it is essential to find a definition of the fractional Laplacian that applies only to functions defined only on \(\Omega\) and for which one has Dirichlet and fractional Neumann-Robin boundary conditions on \(\partial\Omega\), similar to the classical case. The so-called regional fractional Laplacian \((-\Delta_{\Omega})^s, \ s \in (0, 1)\) has been first introduced in [2, 18, 19] and also studied in detail in [30, 31] (see Section 2). In this connection, a semilinear reaction-diffusion equation modelling long-range diffusion processes was also fully investigated in [16] when \(\Omega\) is a bounded Lipschitz domain. It is worth emphasizing that all these Neumann problems for the various definitions of the fractional Laplace operator recover the classical Neumann problems in the limit case as \(s \to 1\).

The purpose of this paper is to introduce dynamic boundary conditions for parabolic and elliptic equations which model anomalous diffusion processes in a bounded domain \(\Omega\) with Lipschitz boundary \(\partial\Omega\). Strictly speaking, in the case when the first definition (1.1) is employed one could define a dynamic equation

\[\partial_t u + \mathcal{N}_s u = 0\]  

on \((\mathbb{R}^N \setminus \Omega) \times (0, \infty)\) and then have it coupled to the evolution equation

\[\partial_t u + (-\Delta)^s u = 0\]
in $\Omega \times (0, \infty)$ by means of the nonlocal Neumann derivative $N_s$ given as in [9]. However, as we already mentioned earlier this approach may pose at least one dilemma in the sense that in practice it is not always easy to collect an initial datum $u_0(x) = u(t = 0, x)$ in the whole of $\mathbb{R}^N \setminus \Omega$. More precisely this has a lot to do with how we measure $u_0$ in $\mathbb{R}^N \setminus \Omega$ if we chose to work with the first definition (1.1); indeed, the problem of assigning values to $u_0$ in the whole of $\mathbb{R}^N \setminus \Omega$ is not only one of a theoretical nature but may be also of some practical relevance since the values of $u$ inside $\Omega$ are well-affected by values outside $\Omega$ due to the nonlocal nature of $(-\Delta)^s$. It turns out that this is not the case for the regional fractional Laplacian $(-\Delta)^s_\Omega$ (see Section 2, for a rigorous definition). It is the latter operator that leads to the correct framework on which one can define the so-called fractional dynamic boundary condition on the boundary $\partial \Omega$, and thus solve the previous conundrum contrary to what was stated in [9, Section 7, pg. 33]. Indeed in the formulation in which $(-\Delta)^s_\Omega$ replaces $(-\Delta)^s$ in (1.3) (and $N_s$ in (1.2) is replaced by the corresponding fractional Neumann derivative associated with $(-\Delta)^s_\Omega$, see Section 2.3), one requires knowledge of the datum $u(t=0)$ only on $\Omega \cup \partial \Omega$. But in fact in this paper we shall do even more: we will investigate a class of semilinear reaction-diffusion problems with long-range interaction, of the form

$$\begin{align*}
\frac{\partial u}{\partial t} + (-\Delta)^s_\Omega u + f(u) &= 0, & & \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial t} + C_s N^{2-2s} u + \delta (-\Delta)^l u + g(u) &= 0, & & \text{on } \partial \Omega \times (0, \infty), \\
u(0) &= u_0 \text{ in } \Omega, & & \text{on } \partial \Omega.
\end{align*}$$

(1.4)

Here, $f$ and $g$ are nonlinear functions, $N^{2-2s}$ is called the fractional normal derivative (see Section 2.3), $\delta \in \{0, 1\}$ and $(-\Delta)^l$, $l \in (0, 1)$ corresponds to a kind of fractional surface diffusion operator on $\partial \Omega$. One particular interest in dealing with (1.4) is also probabilistic in that the corresponding linear operator for (1.4), the so-called fractional Wentzell Laplacian $A_{W, \delta}^s$, bears a close relationship to a kind of stable (restricted) Lévy process in $\Omega = \Omega \cup \partial \Omega$ such that any “particle” jumps between $\Omega$ and its complement $\mathbb{R}^N \setminus \Omega$ are in fact suppressed (see [2], for a rigorous analysis involving censored stable processes and the regional fractional Laplacian $(-\Delta)^s_\Omega$ as “generators” of these processes in $\Omega$). In plain physical terms, we may think of (1.4) as describing a nonlinear flow process in a closed bounded domain $\Omega$ with forced action from the boundary $\partial \Omega$. In that case, the operator $A_{W, \delta}^s$ describes a particle jumping with intensity proportional to $|x-y|^{-N-2s}$ from a point $x \in \Omega$ to another $y \in \Omega$ such that when the particle hits the boundary at a point $z \in \partial \Omega$ it may “choose” (only if $\delta = 1$) to jump to another point $\tilde{z} \in \partial \Omega$ with a strength proportional to $|z-\tilde{z}|^{-(N-1+2l)}$, before it is “spewed” back into the region $\Omega$. When such a fractional dynamic boundary condition is satisfied on $\partial \Omega$ (see (1.4)), we shall always refer to $\partial \Omega$ as a nonlocally reacting surface.

We now give a brief outline of the paper. Section 2 is divided into five subsections: in Section 2.1 we introduce the relevant definitions of fractional order Sobolev spaces while in Section 2.2 we recall the definition for the regional fractional Laplacian $(-\Delta)^s_\Omega$ and some results associated with a Dirichlet problem. Then in Section 2.3 we recall the notion of fractional-order normal derivative $N^{2-2s}$ and a fundamental integration by parts formula (i.e., the corresponding Green identity in the fractional case). In Section 2.4, we define the fractional Wentzell operator associated with the corresponding linear problem for (1.4) and establish some crucial properties for it that will become important in the study of the nonlinear problem. The final Section 2.5 defines a new operator that we call the fractional Steklov operator. The latter
is important in our investigation of an “elliptic-parabolic” problem which is similar to (1.4) in that its first equation is merely replaced by
\[ (-\Delta)^s \Omega u = 0, \quad \text{in } \Omega \times (0, \infty). \]

A rigorous analysis for the full nonlinear problems is performed in Sections 3 and 4 for the parabolic system (1.4) and the elliptic-parabolic system (see (4.1), for instance), respectively. Here our first goal is to derive strong and sharp results in terms of existence, regularity and stability of bounded solutions using a combination of nonlinear semigroup and energy methods. Then our second goal is to also show the existence of finite-dimensional global attractors for such problems in a non-reflexive setting by working instead with $L^\infty$-like solutions. We recall that the space $L^\infty$ is the natural choice for parabolic problems since it actually carries a real physical meaning for most real-world applications; therefore, our preferred notion of solutions will typically be given in such spaces. In fact, it is this context in which our conditions on the nonlinearities become in fact optimal and sharp in the sense that global existence and regularity holds for such problems (see Sections 3, 4). Besides, when such conditions are not satisfied then blow-up of some solutions may indeed occur (see Section 5). Finally, besides the semigroup approach of Section 4.1 we shall exploit in Section 4.2 a perturbation method to derive global existence results for the elliptic-parabolic problem, by viewing the latter as a singular perturbation of a sequence of parabolic problems of the form (1.4). We note that this method may be of some independent interest in the treatment of other nonlinear systems involving such fractional dynamic boundary conditions.

2. Intermediate Results. In this section we introduce the function spaces needed to investigate our problem and we prove some intermediate results that will be used to obtain our main results.

2.1. Some preliminaries. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set. For $s \in (0, 1)$, we denote by
\[ W^{s,2}(\Omega) := \{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy < \infty \} \]
the fractional order Sobolev space endowed with the norm
\[ \|u\|_{W^{s,2}(\Omega)} := \left( \int_{\Omega} \|u\|^2 \, dx + \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}} \]
having defined
\[ C_{N,s} = \frac{2s \Gamma \left( \frac{N+2s}{2} \right)}{\pi^{\frac{N}{2}} \Gamma(1-s)} \]
where $\Gamma$ denotes the usual Gamma function. We let
\[ W_0^{s,2}(\Omega) = D(\Omega)^{W^{s,2}(\Omega)}. \]

By definition, $W_0^{s,2}(\Omega)$ is the smallest closed subspace of $W^{s,2}(\Omega)$ containing $D(\Omega)$.

\textbf{Theorem 2.1.} Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary. Then for every $0 < s \leq \frac{1}{2}$, the spaces $W^{s,2}(\Omega)$ and $W_0^{s,2}(\Omega)$ coincide with equivalent norms.
In view of Theorem 2.1, we have that if \(0 < s \leq \frac{1}{2}\) and \(\Omega\) has a Lipschitz continuous boundary, then every \(u \in W^{s,2}(\Omega)\) is zero \(\sigma\text{-a.e.}\) on \(\partial \Omega\), where \(\sigma\) denotes the usual Lebesgue surface measure on \(\partial \Omega\). Therefore, to talk about traces of functions in \(W^{s,2}(\Omega)\) that are not necessarily null on \(\partial \Omega\), it is not a restriction to assume that \(\frac{1}{2} < s < 1\).

We have the following result taken from [4, 8].

**Proposition 2.2.** Let \(\frac{1}{2} < s < 1\) and \(\Omega \subset \mathbb{R}^N\) a bounded open set with a Lipschitz continuous boundary \(\partial \Omega\). Then the following assertions hold.

(a) If \(N > 2s\), that is, \(N \geq 2\), then \(W^{s,2}(\Omega) \hookrightarrow L^2(\Omega)\) where \(2^* := \frac{2N}{N-2s} > 2\).

(b) If \(N \leq 2s\), that is, \(N = 1\), then \(W^{s,2}(\Omega) \hookrightarrow C(\overline{\Omega})\).

(c) The continuous injection \(W^{s,2}(\Omega) \hookrightarrow L^2(\Omega)\) is also compact.

(d) There exists a linear continuous trace operator

\[
\text{Tr}: W^{s,2}(\Omega) \to L^s(\partial \Omega),
\]

such that \(\text{Tr}(u) = u|_{\partial \Omega}\) for every \(u \in W^{s,2}(\Omega) \cap C(\overline{\Omega})\) with \(q = 2^* = \frac{2(N-1)}{N-2s}\) if \(N > 2s\) and \(q = \infty\) if \(N \leq 2s\). Moreover, the continuous embedding \(W^{s,2}(\Omega) \hookrightarrow L^2(\partial \Omega)\) is also compact.

The following proposition gives another equivalent norm for the space \(W^{s,2}(\Omega)\). We refer to [31, Theorem 2.3] for its proof.

**Proposition 2.3.** Let \(\Omega \subset \mathbb{R}^N\) be a bounded open set with a Lipschitz continuous boundary \(\partial \Omega\) and let \(\frac{1}{2} < s < 1\). Then there exists a constant \(C = C(\Omega, N, s) > 0\) such that for all \(u \in W^{s,2}(\Omega)\),

\[
\int_{\Omega} |u|^2 \, dx \leq C \left( \frac{CN_s}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\partial \Omega} |u|^2 \, d\sigma \right).
\]

It follows from (2.3) that

\[
\|u\|_{W^{s,2}(\Omega)} = \left( \frac{CN_s}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\partial \Omega} |u|^2 \, d\sigma \right)^{1/2}
\]

defines an equivalent norm on \(W^{s,2}(\Omega)\).

Next, assume that \(\Omega\) has a Lipschitz continuous boundary \(\partial \Omega\) and let \(l \in (0, 1)\). We denote by

\[
W^{l,2}(\partial \Omega) := \{ u \in L^2(\partial \Omega) : \int_{\partial \Omega} \int_{\partial \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N-1+2l}} \, d\sigma_x \, d\sigma_y < \infty \}
\]

the fractional order Sobolev space endowed with the norm

\[
\|u\|_{W^{l,2}(\partial \Omega)} := \left( \int_{\partial \Omega} |u|^2 d\sigma + \frac{CN_{l,1}}{2} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N-1+2l}} \, d\sigma_x \, d\sigma_y \right)^{1/2}.
\]

Comparing with Proposition 2.2, we have the following embedding.

**Proposition 2.4.** Let \(0 < l < 1\) and \(\Omega \subset \mathbb{R}^N\) a bounded open set with a Lipschitz continuous boundary \(\partial \Omega\). Then the following assertions hold.

(a) If \(N > 1 + 2l\), then \(W^{l,2}(\partial \Omega) \hookrightarrow L^2(\partial \Omega)\) with \(2^{*} := \frac{2(N-1)}{N-2l} > 2\).

(b) If \(N \leq 1 + 2l\), then \(W^{l,2}(\partial \Omega) \hookrightarrow L^\infty(\partial \Omega)\).

(c) The continuous injection \(W^{l,2}(\partial \Omega) \hookrightarrow L^2(\partial \Omega)\) is also compact.
For \( \frac{1}{2} < s < 1 \) and \( 0 < l < 1 \) and \( \delta \in \{0,1\} \), we endow the Banach space
\[
W^{s,\delta,2}(\Omega) := \{ U = (u,u|_{\partial\Omega}) : u \in W^{s,2}(\Omega), \, \delta u|_{\partial\Omega} \in W^{l,2}(\partial\Omega) \}
\]
with the norm
\[
\|U\|^{2}_{W^{s,\delta,2}(\Omega)} = \|u\|^{2}_{W^{s,2}(\Omega)} + \|\delta u\|^{2}_{W^{l,2}(\partial\Omega)},
\]
if \( \delta = 1 \), and if \( \delta = 0 \), we let
\[
W^{s,0,2}(\Omega) := \{ U = (u,u|_{\partial\Omega}) : u \in W^{s,2}(\Omega) \}
\]
and we endow it with the norm
\[
\|U\|^{2}_{W^{s,0,2}(\Omega)} = \|u\|^{2}_{W^{s,2}(\Omega)} + \|u\|^{2}_{W^{s-rac{1}{2},2}(\partial\Omega)}.
\]
It follows from (2.4) that
\[
\|U\|^{2}_{W^{s,\delta,2}(\Omega)} = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{2}}{|x-y|^{N-1+2s}} dxdy
\]
\[
+ \frac{C_{N-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^{2}}{|x-y|^{N-1+2l}} d\sigma_{x}d\sigma_{y} + \int_{\partial\Omega} |u|^{2}d\sigma
\]
defines an equivalent norm on \( W^{s,\delta,2}(\Omega) \) if \( \delta = 1 \) and
\[
\|U\|^{2}_{W^{s,0,2}(\Omega)} = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{2}}{|x-y|^{N-1+2s}} dxdy + \int_{\partial\Omega} |u|^{2}d\sigma
\]
defines an equivalent norm on \( W^{s,0,2}(\Omega) \) if \( \delta = 0 \).

For \( r,q \in [1,\infty] \) with \( 1 \leq r,q < \infty \) or \( r = q = \infty \) we endow the Banach space
\[
X^{r,q}(\Omega) := L^{r}(\Omega) \times L^{q}(\partial\Omega) = \{ (f,g) , f \in L^{r}(\Omega), g \in L^{q}(\partial\Omega) \}
\]
with the norm
\[
\|(f,g)\|_{X^{r,q}(\Omega)} := \|f\|_{r,\Omega} + \|g\|_{q,\partial\Omega} \quad \text{if} \quad r \neq q,
\]
\[
\|(f,g)\|_{X^{r,r}(\Omega,\Sigma)} := (\|f\|^{r}_{r,\Omega} + \|g\|^{r}_{r,\partial\Omega})^{\frac{1}{r}},
\]
if \( 1 \leq r,q < \infty \) and
\[
\|(f,g)\|_{X^{\infty,\infty}(\Omega)} := \max\{\|f\|_{\infty,\Omega} , \|g\|_{\infty,\partial\Omega}\}.
\]
We will simple write \( X^{r}(\Omega) := X^{r,r}(\Omega) \). If \( \Omega \) has a Lipschitz continuous boundary, we get from Proposition 2.2 and Proposition 2.4 that if \( N > 2s \), then if \( \delta = 1 \),
\[
W^{s,l,2}(\Omega) \hookrightarrow X^{r,q}(\Omega)
\]
with
\[
\begin{cases}
\forall \ r \in [1,2^{*}], \ 2^{*} := \frac{2N}{N-2s}, & \forall \ q \in [1,2^{*}], \ 2^{*} = \frac{2(N-1)}{N-2l} \quad \text{if} \quad l \geq s - \frac{1}{2}, \\
\forall \ r \in [1,2^{*}], \ & \forall \ q \in [1,2^{*}], \ 2^{*} := \frac{2(N-1)}{N-2s} \quad \text{if} \quad l \leq s - \frac{1}{2},
\end{cases}
\]
and
\[
W^{s,0,2}(\Omega) \hookrightarrow X^{r,q}(\Omega), \quad \forall \ r \in [1,2^{*}], \ \forall \ q \in [1,2^{*}], \ 2^{*} := \frac{2(N-1)}{N-2s}.
\]
For more information on fractional order Sobolev spaces we refer to [1, 8, 22, 20, 30] and their references.
2.2. The Dirichlet problem for the regional fractional Laplacian. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary $\partial \Omega$. Given a function $g \in C(\mathbb{R}^N)$, it is known that the Dirichlet problem for the fractional Laplacian

$$(-\Delta)^s u = 0 \; \text{in} \; \Omega, \; u = g \; \text{on} \; \partial \Omega,$$

is not well-posed due to the nonlocal character of $(-\Delta)^s$ (see e.g. [19] and the references therein). This follows from the fact that

$$(-\Delta)^s u(x) = \frac{C_{N,s}}{2} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy$$

for the regional fractional Laplacian, it has been shown in [19, Theorem 7.6 and Corollary 7.7] that for every $u \in C(\mathbb{R}^N)$, provided that the limit exists for all functions $u \in C(\mathbb{R}^N)$, it is known that the Dirichlet problem for the fractional Laplacian

$$(-\Delta)^s u = 0 \; \text{in} \; \Omega, \; u = g \; \text{on} \; \partial \Omega,$$

is not well-posed due to the nonlocal character of $(-\Delta)^s$ (see e.g. [19] and the references therein). This follows from the fact that

$$(-\Delta)^s u(x) = \frac{C_{N,s}}{2} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy + \frac{C_{N,s}}{2} \text{P.V.} \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy$$

provided that the limit exists. Hence, the function $u$ must coincide with $g$ on all of $\mathbb{R}^N \setminus \Omega$ in order to have a well-posed Dirichlet problem, that is, the following problem

$$(-\Delta)^s u = 0 \; \text{in} \; \Omega, \; u = g \; \text{on} \; \mathbb{R}^N \setminus \Omega,$$

is well-posed instead. The situation turns out to be quite different for the regional fractional Laplace operator $(-\Delta)^s_{\Omega}$, which is defined as follows:

$$(-\Delta)^s_{\Omega} u(x) = \frac{C_{N,s}}{2} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy$$

provided that the limit exists for all functions $u \in L^1_s(\Omega)$, where

$L^1_s(\Omega) := \{ u : \Omega \to \mathbb{R} \; \text{measurable}, \; \int_{\Omega} \frac{|u(x)|}{(1 + |x|)^{N+2s}} dx < \infty \}.$

For the regional fractional Laplacian, it has been shown in [19, Theorem 7.6 and Corollary 7.7] that for every $g \in C(\partial \Omega)$, there exists a unique function $u \in C(\overline{\Omega})$ solution of the Dirichlet problem

$$(-\Delta)^s_{\Omega} u = 0 \; \text{in} \; \Omega, \; u = g \; \text{on} \; \partial \Omega.$$

Let $W^{-s,2}(\Omega)$ denote the dual of the reflexive Banach space $W^{s,2}_0(\Omega)$ and $\langle \cdot, \cdot \rangle$ the duality map between the two spaces. We have the following result.

**Proposition 2.5.** Let $\frac{1}{2} < s < 1$. Then for every $f \in W^{-s,2}(\Omega)$ and $g \in W^{s-\frac{1}{2},2}(\partial \Omega)$, there exists a unique $u \in W^{s,2}(\Omega)$ satisfying

$$(-\Delta)^s_{\Omega} u = f \; \text{in} \; \Omega, \; u = g \; \text{on} \; \partial \Omega. \quad (2.7)$$

Moreover, there exists a constant $C > 0$ such that

$$\|u\|_{W^{s,2}(\Omega)} \leq C \left( \|f\|_{W^{-s,2}(\Omega)} + \|g\|_{W^{s-\frac{1}{2},2}(\partial \Omega)} \right). \quad (2.8)$$
If in addition $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial \Omega)$, then $u \in L^\infty(\Omega)$ and there exists a constant $C > 0$ such that
\[ \|u\|_{L^\infty(\Omega)} \leq C \left( \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial \Omega)} \right). \] (2.9)

**Proof.** Let $\frac{1}{2} < s < 1$. It is straightforward to show that for every $f \in W^{-s,2}(\Omega)$, there is a unique $u \in W^{s,2}_0(\Omega)$ satisfying
\[ (-\Delta)^s u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \]
that is, for every $v \in W^{s,2}_0(\Omega)$,
\[ \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy = \langle f, v \rangle. \] (2.10)

Taking $v = u$ as a test function in (2.10) and noticing that
\[ |\langle f, v \rangle| \leq \|f\|_{W^{-s,2}(\Omega)} \|u\|_{W^{s,2}(\Omega)} \]
we get that
\[ \|u\|_{W^{s,2}(\Omega)} \leq \|f\|_{W^{-s,2}(\Omega)}. \] (2.11)

Next, let $g \in W^{s-1/2}(\partial \Omega)$ and $\varphi \in W^{s,2}(\Omega)$ such that $g = \varphi|_{\partial \Omega}$. Let $u \in W^{s,2}(\Omega)$ satisfy (2.7) and set $v = u - \varphi$. Then $v \in W^{s,2}_0(\Omega)$ is a (weak) solution of the Dirichlet problem
\[ (-\Delta)^s v = f - (-\Delta)^s \varphi \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega. \]
Moreover, the solution $v$ is unique and independent of $\varphi$. It follows from (2.11) that
\[ \|v\|_{W^{s,2}(\Omega)} \leq \|f\|_{W^{-s,2}(\Omega)} + \|(-\Delta)^s \varphi\|_{W^{s,2}(\Omega)} \leq \|f\|_{W^{-s,2}(\Omega)} + \|\varphi\|_{W^{s,2}_0(\Omega)}. \]

Therefore, we have the estimate
\[ \|u\|_{W^{s,2}(\Omega)} = \|v + \varphi\|_{W^{s,2}(\Omega)} \leq \|f\|_{W^{-s,2}(\Omega)} + 2\|\varphi\|_{W^{s,2}_0(\Omega)} \leq \|f\|_{W^{-s,2}(\Omega)} + 2\inf_{\varphi \in W^{s,2}(\Omega), \varphi|_{\partial \Omega} = g} \|\varphi\|_{W^{s,2}_0(\Omega)} \leq \|f\|_{W^{-s,2}(\Omega)} + 2\|g\|_{W^{s-1/2}(\partial \Omega)} \]
and we have shown (2.8). Now assume that $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial \Omega)$. We infer from [19, Theorem 7.5] that $u \in L^\infty(\Omega) \cap C(\Omega)$ and that it is given for every $x \in \Omega$ by
\[ u(x) = E_x \left( g(X_\tau) + \int_0^\tau f(X_s) \, ds \right), \] (2.12)
where $(X_t)$ is the Markov process generated by $(-\Delta)^s$, and $\tau := \inf\{t > 0 : X_t \notin \Omega\}$ is the first exit time from $\Omega$ of $(X_t)$. Note that here, $(X_t)$ is an $s$-stable process with purely jump sample paths, hence, $X_\tau$ spreads in the whole area $\mathbb{R}^N \setminus \Omega$. It follows from identity (2.12) that $u$ satisfies the estimate (2.9). The proof of the proposition is finished. \qed
2.3. The fractional normal derivative. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,1}$ with boundary $\partial \Omega$. Let $\frac{1}{2} < s < 1$ and the constant
\[ C_s := \frac{C_{1,s}}{2s(2s-1)} \int_0^\infty \frac{|t-1|^{1-2s} - (t \vee 1)^{1-2s}}{t^{2-2s}} \, dt, \tag{2.13} \]
where $C_{1,s}$ is given by (2.1) with $N = 1$. Let the constant $B_{N,s}$ be such that
\[ \frac{C_{1,s}}{C_{N,s}} B_{N,s} := \begin{cases} C_s & \text{if } N = 1 \\ \frac{2\pi}{1(N-1)} \int_0^{\pi} \cos^2(\theta) \sin^{N-2}(\theta) \, d\theta & \text{if } N \geq 2. \end{cases} \tag{2.14} \]
A simple calculation (see, e.g. [31]) shows that in fact $B_{N,s} = C_s$ and depends on $s$ only.

The following integration by parts formula has been recently obtained in [18, Theorem 3.3].

**Theorem 2.6.** Let $\frac{1}{2} < s < 1$ and $C^2_1(\Omega) := \{ u : u(x) = f(x)\rho(x)^{2s-1} + g(x), \forall x \in \Omega, \text{ for some } f, g \in C^2(\Omega) \}$, where $\rho(x) := \text{dist}(x,\partial\Omega)$, $x \in \Omega$. For $u \in C^2_1(\Omega)$ and $z \in \partial\Omega$, we define the fractional normal derivative $N^{2-2s}u$ of the function $u$ by
\[ N^{2-2s}u(z) := -\lim_{t \downarrow 0} \frac{d}{dt} u(z-t \cos(\theta), t \sin(\theta)) \big|_{t=0}, \quad z \in \partial\Omega, \tag{2.15} \]
where $\nu(z)$ denotes the outer normal vector to $\Omega$ at the point $z$. Then for every $u \in C^2_1(\Omega)$ and $v \in W^{s,2}(\Omega)$, one has $(-\Delta)^s_{\partial\Omega} u \in L^2(\Omega)$, $N^{2-2s} u \in L^2(\partial\Omega)$ and
\[ \int_\Omega v(-\Delta)^s_{\partial\Omega} u \, dx = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy - C_s \int_{\partial\Omega} v N^{2-2s} u \, d\sigma. \tag{2.16} \]

The above integration by parts formula has been recently generalized to the case of the fractional $p$-Laplace operator ($p \in (1, \infty)$) in [32].

Next, we introduce a weak formulation for the fractional normal derivative on non-smooth domains.

**Definition 2.7.** Let $\frac{1}{2} < s < 1$ and $\Omega \subset \mathbb{R}^N$ a bounded open set with Lipschitz continuous boundary $\partial\Omega$.

(a) Let $u \in W^{s,2}(\Omega)$. We say that $(-\Delta)^s_{\partial\Omega} u \in L^2(\Omega)$ if there exists $w \in L^2(\Omega)$ such that
\[ \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} wv \, dx \]
for all $v \in \mathcal{D}(\Omega)$ and hence, for all $v \in W^{s,2}_0(\Omega)$ by density. In that case we write $(-\Delta)^s_{\partial\Omega} u = w$.

(b) Let $u \in W^{s,2}(\Omega)$ such that $(-\Delta)^s_{\partial\Omega} u \in L^2(\Omega)$. We say that $u$ has a fractional normal derivative in $L^2(\partial\Omega)$ if there exists $g \in L^2(\partial\Omega)$ such that
\[ \int_\Omega v(-\Delta)^s_{\partial\Omega} u \, dx = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy \tag{2.17} \]
for all $v \in \mathcal{D}(\Omega)$ and hence, for all $v \in W^{s,2}_0(\Omega)$ by density. In that case we write $(-\Delta)^s_{\partial\Omega} u = v$.
for all \( v \in W^{s,2}(\Omega) \cap C(\overline{\Omega}) \), hence for all \( v \in W^{s,2}(\Omega) \) by density and in view of (2.2). In that case, the function \( g \) is uniquely determined by (2.17), we write \( C_s \mathcal{N}^{-2s}u = g \) and then call \( g \) the fractional normal derivative of \( u \).

**Remark 2.8.** It follows from Definition 2.7 that the Green's type formula

\[
\int_{\Omega} v(-\Delta)^{s}_{\Omega} u dx = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} dxdy \quad (2.18)
\]

holds for all \( v \in W^{s,2}(\Omega) \) whenever \( u \in W^{s,2}(\Omega) \), \((-\Delta)^{s}_{\Omega} u \in L^2(\Omega) \) and \( \mathcal{N}^{-2s}u \) exists in \( L^2(\partial\Omega) \). If \( \Omega \) is a bounded open set of class \( C^{1,1} \) and if \( u \in C^2_p(\Omega) \), then \( \mathcal{N}^{-2s}u \) coincides with the function given by (2.15). Moreover, \( \mathcal{N}^{-2s}u \in L^2(\partial\Omega) \) and \((-\Delta)^{s}_{\Omega} u \in L^2(\Omega) \) (see e.g. [18]).

2.4. The fractional Wentzell boundary conditions. Throughout the remainder of the paper we assume that \( \Omega \subset \mathbb{R}^N \) is a bounded open set with Lipschitz continuous boundary \( \partial\Omega \). Our goal in this subsection is to introduce the so-called fractional Wentzell operator on \( \overline{\Omega} \). Let \( 0 < l < 1 \) and let \((W^{l,2}(\partial\Omega))^*\) denote the dual of the reflexive Banach space \( W^{l,2}(\partial\Omega) \). Consider the operator \( B^l_T : W^{l,2}(\partial\Omega) \to (W^{l,2}(\partial\Omega))^* \) defined by

\[
\langle B^l_T u, v \rangle := \frac{C_{N-l,s}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N-1+2l}} d\sigma_x d\sigma_y, \quad (2.19)
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality map between \((W^{l,2}(\partial\Omega))^* \) and \( W^{l,2}(\partial\Omega) \). Note that \( B^l_T \) is a bounded surjective operator. Indeed, denote the right hand side of (2.19) by \( B_T(u, v) \) for \( u, v \in W^{l,2}(\partial\Omega) \). Clearly, \( B^l_T \) is a continuous bilinear form, that is,

\[
|B^l_T(u, v)| \leq \|u\|_{W^{l,2}(\partial\Omega)} \|v\|_{W^{l,2}(\partial\Omega)}.
\]

Hence, by the classical Lax-Milgram theorem, for every \( \varphi \in (W^{l,2}(\partial\Omega))^* \), there exists a function \( u \in W^{l,2}(\partial\Omega) \) such that \( B_T(u, v) = \langle \varphi, v \rangle \) for every \( v \in W^{l,2}(\partial\Omega) \). Thus, the operator \( B^l_T \) is surjective, bounded and \( \|B^l_T u\|_{(W^{l,2}(\partial\Omega))^*} \leq \|u\|_{W^{l,2}(\partial\Omega)} \) for every \( u \in W^{l,2}(\partial\Omega) \).

Next, we can define a fractional Laplace-Beltrami type operator as

\[
(-\Delta)^{l}_{\Omega} u = C_{N-l,s} P.V. \int_{\partial\Omega} \frac{u(x) - u(y)}{|x - y|^{N-1+2l}} d\sigma_y, \quad (2.20)
\]

and observe that

\[
\langle B^l_T u, v \rangle = \langle (-\Delta)^{l}_{\Omega} u, v \rangle, \quad \text{for all } u, v \in W^{l,2}(\partial\Omega).
\]

We consider \( L^2(\partial\Omega) \) as a subspace of \((W^{l,2}(\partial\Omega))^* \), that is, \( L^2(\partial\Omega) \hookrightarrow (W^{l,2}(\partial\Omega))^* \) by letting

\[
\langle f, v \rangle = \int_{\partial\Omega} f v d\sigma, \quad f \in L^2(\Omega), \ v \in W^{l,2}(\partial\Omega).
\]

We denote the part of the operator \( B^l_T \) in \( L^2(\partial\Omega) \) by \( B^l_T \), again, that is,

\[
D(B^l_T) = \{ u \in W^{l,2}(\partial\Omega), \ (-\Delta)^{l}_{\Omega} u \in L^2(\partial\Omega) \}, \quad B^l_T u = (-\Delta)^{l}_{\Omega} u.
\]

**Proposition 2.9.** We have that \( B^l_T \) is the linear self-adjoint operator on \( L^2(\partial\Omega) \) associated with the closed bilinear symmetric form with domain \( W^{l,2}(\partial\Omega) \),

\[
B_T(u, v) = \frac{C_{N-l,s}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N-1+2l}} d\sigma_x d\sigma_y,
\]
in the sense that for any \( v \in W^{1,2}(\partial \Omega) \),
\[
\begin{align*}
D(B^s_{1}) & := \{ u \in W^{1,2}(\partial \Omega) : \exists w \in L^2(\partial \Omega), \ B_T(u,v) = (w,v)_{L^2(\partial \Omega)} \} \\
B^s_{1} u & = w.
\end{align*}
\] (2.22)

**Proof.** Let \( D(B^s_{1}) \) and \( D \) be given by (2.21) and (2.22), respectively. Let \( u \in D(B^s_{1}) \subset W^{1,2}(\partial \Omega) \) and set \( w := (-\Delta)^s u \). Then \( w \in L^2(\partial \Omega) \) and \( B_T(u,v) = (w,v)_{L^2(\partial \Omega)} \) for every \( v \in W^{1,2}(\partial \Omega) \). Hence, \( u \in D \). Conversely, let \( u \in D \). Then there exists \( w \in L^2(\partial \Omega) \) such that \( B_T(u,v) = (w,v)_{L^2(\partial \Omega)} \) for all \( v \in W^{1,2}(\partial \Omega) \). Using (2.19) we find that \( w = (-\Delta)^s u \in L^2(\partial \Omega) \). Hence, \( D \subset D(B^s_{1}) \) and the proof is finished. Moreover, we have shown that for every \( u \in D(B^s_{1}) \) and \( v \in W^{s,1}(\partial \Omega) \),
\[
\langle B^s_{1} u, v \rangle = B_T(u,v).
\]

Next, let \( \frac{1}{2} < s < 1 \), \( 0 < l < 1 \), \( \delta \in \{0,1\} \) and define the bilinear symmetric form \( a^\delta \) on the product space \( X^2(\bar{\Omega}) = L^2(\Omega) \times L^2(\partial \Omega) \) with domain
\[
D(a^\delta) = W^{s,\delta l,2}(\bar{\Omega}),
\] (2.23)
and given for \( U := (u,u|_{\partial \Omega}), \Phi := (\varphi,\varphi|_{\partial \Omega}) \in W^{s,\delta l,2}(\bar{\Omega}) \) by
\[
a^\delta(U, \Phi) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy + \int_{\partial \Omega} \beta(x) u \varphi \, ds \quad \beta \Omega \] (2.24)
\[
+ \frac{\delta C_{N-1,l}}{2} \int_{\partial \Omega} \int_{\partial \Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N-1+2l}} \, ds \, dx \, dy.
\]
The function \( \beta \in L^{\infty}(\partial \Omega) \) satisfies \( \beta(x) \geq \beta_0 > 0 \) for \( \sigma \)-a.e. \( x \in \partial \Omega \) and for some constant \( \beta_0 \). Of course, this is generally not a restriction and with some minor modifications all the results below also hold with \( \beta \equiv 0 \).

**Proposition 2.10.** The bilinear symmetric form \( a^\delta \) with domain \( W^{s,\delta l,2}(\bar{\Omega}) \) is a Dirichlet form in the space \( X^2(\bar{\Omega}) \), that is, it is closed and submarkovian.

**Proof.** Let \( a^\delta \) with domain \( W^{s,\delta l,2}(\bar{\Omega}) \) be the bilinear symmetric form in \( X^2(\bar{\Omega}) \) defined in (2.24). First we show that the form \( a^\delta \) is closed in \( X^2(\bar{\Omega}) \). Indeed, let \( U_n = (u_n, u_n|_{\partial \Omega}) \in W^{s,\delta l,2}(\bar{\Omega}) \) be a sequence such that
\[
\lim_{n,m \to \infty} \left( a^\delta(U_n - U_m, U_n - U_m) + \| u_n - u_m \|^2_{L^2(\Omega)} + \| u_n - u_m \|^2_{L^2(\partial \Omega)} \right) = 0.
\] (2.25)
It follows from (2.25) that \( \lim_{n,m \to \infty} \| u_n - u_m \|_{W^{s,2}(\Omega)} = 0 \). This implies that \( u_n \) converges strongly to some function \( u \in W^{s,2}(\Omega) \). It also follows from (2.25) that \( u_n|_{\partial \Omega} \) is a Cauchy sequence in the Banach space \( W^{l,2}(\partial \Omega) \) if \( \delta = 1 \) (resp. in \( W^{s+\frac{1}{2},2}(\partial \Omega) \) if \( \delta = 0 \)); hence, it converges in \( W^{l,2}(\partial \Omega) \) (resp., in \( W^{s+\frac{1}{2},2}(\partial \Omega) \) if \( \delta = 0 \)) to some function \( v \). By uniqueness of the limit and the trace function, we have that \( v = u|_{\partial \Omega} \). Setting \( U = (u, u|_{\partial \Omega}) \), we have shown that
\[
\lim_{n \to \infty} a^\delta(U_n - U, U_n - U) + \| U_n - U \|^2_{X^2(\bar{\Omega})} = 0
\]
and this implies that the form \( a^\delta \) is closed in \( X^2(\bar{\Omega}) \).

Next, we show that the form \( a^\delta \) is submarkovian. Indeed, let \( \epsilon > 0 \) and \( \phi_\epsilon \in C^\infty(\mathbb{R}) \) be such that
\[
\begin{align*}
\phi_\epsilon(t) &= t, \quad \forall t \in [0,1], \quad -\epsilon \leq \phi_\epsilon(t) \leq 1 + \epsilon, \quad \forall t \in \mathbb{R}, \\
0 &\leq \phi_\epsilon(t_1) - \phi_\epsilon(t_2) \leq t_1 - t_2 \quad \text{whenever} \quad t_2 < t_1.
\end{align*}
\] (2.26)
An example of such a function $\phi_x$ is contained in [12, Exercise 1.2.1, pg. 8]. We notice that it follows from (2.26) that
\[
0 \leq \phi'_x(t) \leq 1, \quad |\phi_x(t_1) - \phi_x(t_2)| \leq |t_1 - t_2| \quad \text{and} \quad |\phi_x(t)| \leq |t|.
\] (2.27)

Let $U := (u, u|_{\partial \Omega}) \in W^{s,2}(\Omega)$. It follows from (2.27) that $\phi_x(u) \in W^{s,2}(\Omega)$ and
\[
\int_{\Omega} \int_{\Omega} \frac{|\phi_x(u(x)) - \phi_x(u(y))|^2}{|x-y|^{N+2s}} \, dx \, dy \leq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.
\] (2.28)

It also follows from (2.27) that $\phi_x(u|_{\partial \Omega}) \in W^{1,2}(\partial \Omega)$ and
\[
\int_{\partial \Omega} \int_{\partial \Omega} \frac{|\phi_x(u(x)) - \phi_x(u(y))|^2}{|x-y|^{N-1+2l}} \, d\sigma(x) \, d\sigma(y) + \int_{\partial \Omega} \beta(x)|\phi_x(u)|^2 \, d\sigma \leq \int_{\partial \Omega} \int_{\partial \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N-1+2l}} \, d\sigma(x) \, d\sigma(y) + \int_{\partial \Omega} \beta(x)|u|^2 \, d\sigma.
\] (2.29)

Hence, $\Phi_x(U) := (\phi_x(u), \phi_x(u)|_{\partial \Omega}) \in W^{s,2}(\Omega)$. Moreover, these estimates also imply that
\[
\mathbf{a}^\Omega_\delta(\Phi_x(U), \Phi_x(U)) \leq \mathbf{a}^\Omega_\delta(U, U).
\] (2.30)

By [12, pp. 5-6 Formula (1.1.6)], the estimate (2.30) implies that the form $\mathbf{a}^\Omega_\delta$ is submarkovian on $X^2(\Omega)$. Therefore, $\mathbf{a}^\Omega_\delta$ with domain $W^{s,2}(\Omega)$ is a Dirichlet form on $X^2(\Omega)$. The proof is finished. \(\square\)

Let $(-\Delta)^s_{W,\delta}$ be the linear self-adjoint operator on $X^2(\Omega)$ associated with the form $\mathbf{a}^\Omega_\delta$ in the sense that for any $\Phi \in W^{s,2}(\Omega)$,
\[
\left\{ \begin{array}{l}
D((-\Delta)^s_{W,\delta}) := \{ U \in W^{s,2}(\Omega), \; \exists \; W \in X^2(\Omega), \; \mathbf{a}^\Omega_\delta(U, \Phi) = (W, \Phi)_{X^2(\Omega)} \}, \\
(-\Delta)^s_{W,\delta} U = W.
\end{array} \right.
\] (2.31)

We also have the following characterization of the operator $(-\Delta)^s_{W,\delta}$.

**Proposition 2.11.** Let $(-\Delta)^s_{W,\delta}$ be the operator defined in (2.31) where $s \in (1/2, 1)$, $l \in (0, 1)$ and $\delta \in (0, 1)$. Then the domain $D((-\Delta)^s_{W,\delta})$ of the operator $(-\Delta)^s_{W,\delta}$ consists of functions $U = (u, u|_{\partial \Omega}) \in W^{s,2}(\Omega)$, such that
\[
(-\Delta)^s_{\Omega} u \in L^2(\Omega), \; \delta(-\Delta)^s_{\partial} u \in L^2(\partial \Omega),
\]
$\mathcal{N}^{2-2s} u$ exists in $L^2(\partial \Omega)$ and
\[
C_s \mathcal{N}^{2-2s} u + \delta(-\Delta)^s_{\partial} u + \beta(u|_{\partial \Omega}) \in L^2(\partial \Omega).
\]
For $U \in D((-\Delta)^s_{W,\delta})$, the operator $(-\Delta)^s_{W,\delta}$ has the following matrix representation:
\[
(-\Delta)^s_{W,\delta} U = \begin{pmatrix}
0 \\
C_s \mathcal{N}^{2-2s} \delta(-\Delta)^s_{\partial} + \beta \end{pmatrix} \begin{pmatrix}
\mathbf{a}^\Omega_\delta \\
\mathbf{a}^\Omega_\delta
\end{pmatrix},
\]
\[
\begin{pmatrix}
u \\
u|_{\partial \Omega}
\end{pmatrix}.
\]

**Proof.** Let $(-\Delta)^s_{W,\delta}$ be the closed linear self-adjoint operator on $X^2(\Omega)$ defined in (2.31). Set
\[
D := \{ U = (u, u|_{\partial \Omega}) \in W^{s,2}(\Omega), \; (-\Delta)^s_{\Omega} u \in L^2(\Omega), \; \delta(-\Delta)^s_{\partial} u \in L^2(\partial \Omega), \; \mathcal{N}^{2-2s} u \text{ exists in } L^2(\partial \Omega), \;
\]
\[
C_s \mathcal{N}^{2-2s} u + \delta(-\Delta)^s_{\partial} u + \beta(u|_{\partial \Omega}) \in L^2(\partial \Omega).
\]
and $C_s \mathcal{N}^{2-2s} u + \delta(-\Delta)^s_{\partial} u + \beta(u|_{\partial \Omega}) \in L^2(\partial \Omega)$
and let $D((-\Delta)^s_{W,\delta})$ be given by (2.31). Let $U = (u, u_{\partial\Omega}) \in D((-\Delta)^s_{W,\delta})$. Then by definition, there exists $W = (w_1, w_2) \in X^2(\Omega)$ such that for every $\varphi \in C^1(\Omega)$, we have

$$
\int_{\Omega} w_1 \varphi dx + \int_{\partial\Omega} w_2 \varphi d\sigma = C_{N,s} \int_{\Omega} \frac{(u(x) - u(y))}{|x - y|^{N+2s}} (\varphi(x) - \varphi(y)) dx dy + \int_{\partial\Omega} \beta(x) w_2 \varphi d\sigma
$$

(2.32)

Choosing the support of $\varphi$, suitably integrating (2.32) by parts, using (2.33), Definition 2.7 and (2.20), we infer

$$
\int_{\Omega} w_1 \varphi dx + \int_{\partial\Omega} w_2 \varphi d\sigma
= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))}{|x - y|^{N+2s}} (\varphi(x) - \varphi(y)) dxdy + \int_{\partial\Omega} \beta(x) w_2 \varphi d\sigma + \delta \int_{\partial\Omega} \frac{|\delta \varphi(x)|^2}{|x - y|^{N-1+2l}} d\sigma d\sigma_y.
$$

In particular we get from (2.32) that for every $\varphi \in D(\Omega)$, we have

$$
\int_{\Omega} w_1 \varphi dx = C_{N,s} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))}{|x - y|^{N+2s}} (\varphi(x) - \varphi(y)) dx dy.
$$

(2.33)

It follows from (2.33) and Definition 2.7 that

$$
(-\Delta)^s_{\Omega} u = w_1 \in L^2(\Omega).
$$

Choosing the support of $\varphi$ suitably, integrating (2.32) by parts, using (2.33), Definition 2.7 and (2.20) we get that $N^{2-2s} u$ exists in $L^2(\partial\Omega)$ and

$$
\int_{\partial\Omega} w_2 \varphi d\sigma = C_s \int_{\Omega} N^{2-2s} u \varphi d\sigma + \int_{\partial\Omega} \beta(x) u \varphi d\sigma + \frac{\delta C_{N-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))}{|x - y|^{N-1+2l}} (\varphi(x) - \varphi(y)) d\sigma_x d\sigma_y
$$

$$
= \int_{\partial\Omega} \left( C_s N^{2-2s} u + \beta(x) u \right) \varphi d\sigma + \delta \delta \mathcal{L}_{(\Omega)}^2 u, \varphi \right). W^{1,2}(\partial\Omega), W^{1,2}(\partial\Omega)
$$

From this identity we deduce that $\delta (-\Delta)^s_{\Omega} u \in L^2(\partial\Omega)$ and

$$
C_s N^{2-2s} u + \beta(x) u_{\partial\Omega} = w_2 \in L^2(\partial\Omega).
$$

Hence, $U \in D$ and we have shown that $D((-\Delta)^s_{W,\delta}) \subset D$. Conversely, let $U \in D$ and set $w_1 := (-\Delta)^s_{\Omega} u$ and $w_2 := C_s N^{2-2s} u + \beta(x) u_{\partial\Omega} + \delta (-\Delta)^s_{\Omega} u_{\partial\Omega})$. Then $W = (w_1, w_2) \in X^2(\Omega)$. Let $\varphi \in C^1(\Omega)$ and set $\Phi := (\varphi, \varphi_{\partial\Omega})$. Integrating by parts using the generalized Green type identity (2.18) and (2.20), we infer

$$
\int_{\Omega} w_1 \varphi dx + \int_{\partial\Omega} w_2 \varphi d\sigma
= \int_{\Omega} (-\Delta)^s_{\Omega} u \varphi dx + \int_{\partial\Omega} w_2 \varphi d\sigma
$$

$$
= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))}{|x - y|^{N+2s}} (\varphi(x) - \varphi(y)) dxdy
$$

$$
- \int_{\partial\Omega} \frac{N^{2-2s}}{2} \varphi d\sigma d\sigma y
= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))}{|x - y|^{N+2s}} (\varphi(x) - \varphi(y)) dxdy
$$

$$
+ \frac{\delta C_{N-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))}{|x - y|^{N-1+2l}} (\varphi(x) - \varphi(y)) d\sigma_x d\sigma_y
$$

$$
+ \int_{\partial\Omega} \beta(x) u \varphi d\sigma
= \Phi^a(U, \Phi).
$$
We have shown that $D \subset D((-\Delta)^s_{W,\delta})$ and the proof is finished. \hfill \Box

We call $(-\Delta)^s_{W,\delta}$, the realization of the regional fractional Laplace operator with the fractional Wentzell-Robin type boundary conditions.

**Theorem 2.12.** Let $A^s_{W,\delta} := (-\Delta)^s_{W,\delta}$ be the operator defined in (2.31). Then the following assertions hold.

(a) The operator $-A^s_{W,\delta}$ generates a submarkovian semigroup $(e^{-tA^s_{W,\delta}})_{t \geq 0}$ on $X^2(\Omega)$. The semigroup can be extended to contraction semigroups on $X^p(\Omega)$ for every $p \in [1,\infty]$, and each semigroup is strongly continuous if $p \in [1,\infty)$ and bounded analytic if $p \in (1,\infty)$.

(b) The operator $A^s_{W,\delta}$ has a compact resolvent, and hence has a discrete spectrum. The spectrum of $A^s_{W,\delta}$ is an increasing sequence of real numbers $0 < \lambda^s_{W,1} \leq \lambda^s_{W,2} \leq \cdots \leq \lambda^s_{W,n} \leq \cdots$, that converges to $+\infty$. Let $N > 2s$. Then the semigroup $(e^{-tA^s_{W,\delta}})_{t \geq 0}$ is ultracontractive in the sense that it maps $X^2(\Omega)$ into $X^{\infty}(\Omega)$ and each semigroup on $X^p(\Omega)$ is compact for every $p \in [1,\infty]$. More precisely, we have that there is a constant $C > 0$ such that

$$
\|e^{-tA^s_{W,\delta}}f\|_{L(X^p(\Omega),X^{\infty}(\Omega))} \leq Ct^{-\frac{2}{q}}(1+\frac{2}{q-2})e^{-\lambda^s_{W,l}(\frac{1}{2}-\frac{1}{2})},
$$

(2.34)

for all $t > 0$ and $1 \leq p \leq q \leq \infty$, where

$$
\begin{cases}
\gamma = \frac{n}{2} > 2 & \text{if } l \geq s - \frac{n}{N} > s - \frac{1}{2}, \\
\gamma = \frac{n-1}{2} > 2 & \text{if } s - \frac{1}{2} \leq l \leq s - \frac{n}{N}, \\
\gamma = \frac{2(N-l)}{2s-1} & \text{if } l \leq s - \frac{1}{2},
\end{cases}
$$

(2.35)

in the case $\delta = 1$ and

$$
\gamma = \frac{2(N-1)}{2s-1},
$$

(2.36)

in the case $\delta = 0$.

(c) If $U_n$ is an eigenfunction associated with $\lambda_{s,n}$, then $U_n \in D(A^s_{W,\delta}) \cap X^{\infty}(\Omega)$.

(d) Let $N > 2s$. Assume $N < 4s$ if $l \geq s - \frac{n}{N} > s - \frac{1}{2}$, or $N > 1 + 4l$ if $s - \frac{1}{2} \leq l < s - \frac{n}{N}$, or $N < 4s - 1$ if $l \leq s - \frac{1}{2}$. Then the embedding $D(A^s_{W,1}) \subset X^{\infty}(\Omega)$ is continuous. If $N < 4s - 1$, then $D(A^s_{W,0}) \subset X^{\infty}(\Omega)$.

(e) Let $N > 2s$ and $q \in (2,\infty)$. Assume that $N < \frac{4s}{q-2}$ if $l \geq s - \frac{n}{N} > s - \frac{1}{2}$, or $N > 1 + \frac{4q}{q-2}$ if $s - \frac{1}{2} \leq l \leq s - \frac{n}{N}$, or $N < 1 + \frac{4q}{q-2}$ if $l \leq s - \frac{1}{2}$. Then the embedding $D(A^s_{W,1}) \subset X^q(\Omega)$ is continuous. If $N < 1 + \frac{2q}{2q-1}$, then $D(A^s_{W,0}) \subset X^q(\Omega)$.

**Proof.** Let $A^s_{W,\delta} := (-\Delta)^s_{W,\delta}$ be the operator defined in (2.31).

(a) We have shown in Proposition 2.10 that $a^s_{U_0}$ is a Dirichlet form on $X^2(\Omega)$. Hence, by [12, Theorem 1.4.1] the operator $-A^s_{W,\delta}$ generates a submarkovian semigroup $(e^{-tA^s_{W,\delta}})_{t \geq 0}$ on $X^2(\Omega)$ which is also analytic. It follows from [5, Theorem 1.4.1] that the semigroup can be extended to contraction semigroups on $X^p(\Omega)$ for every $p \in [1,\infty]$, and each semigroup is strongly continuous if $p \in [1,\infty)$ and bounded analytic if $p \in (1,\infty)$.

(b) Now exploiting Propositions 2.2 and 2.4 (see also (2.5) and (2.6)), we ascertain the existence of a constant $C > 0$ such that for every $U = (u, u|_{\partial\Omega}) \in W^{s,0,l}(\Omega)$,

$$
\|U\|_{X^q(\Omega)}^2 \leq C a^s_{U}(U,U)
$$

(2.37)
The assertion (d).

is ultracontractive, that is, it maps

\[ X \quad \gamma < \frac{N}{s} > s - \frac{1}{2}, \]

This completes the proof of part (c).

Then, for every

\[ X \quad \text{tracontractivity estimate from } \mathcal{W}_{s,\delta}(\Omega) \rightarrow X^{2}(\Omega) \]

and some abstract results on ultracontractivity of semigroups contained in [26, Lemma 6.1 and Lemma 6.5].

Lemma 6.1 and Lemma 6.5.

(c) Let \( U_n \in D(A_{W,\delta}) \subset \mathcal{W}_{s,\delta}(\Omega) \) be an eigenfunction associated with \( \lambda_{s,n}^W \).

Then, for every \( V \in \mathcal{W}_{s,\delta}(\Omega) \), \( a_{\delta}(U_n, V) = \lambda_{s,n}(U_n, V)_{X^{2}(\Omega)} \) which translates to

\[ A_{W,\delta}U_n = \lambda_{s,n}^W U_n. \]

Let \( \lambda > 0 \) be a real number. Since \( \lambda \in \rho(-A_{W,\delta}) \), we have that

\[ \lambda I + A_{W,\delta} \]

is invertible. From \( A_{W,\delta}U_n = \lambda_{s,n}^W U_n \) we have that

\[ U_n = (\lambda I + A_{W,\delta})^{-1}(\lambda_{s,n}^W U_n) = (\lambda_{s,n}^W + \lambda)(\lambda I + A_{W,\delta})^{-1}(U_n). \]

Since for every \( F \in X^{2}(\Omega) \) and \( \lambda > 0 \),

\[ (\lambda I + A_{W,\delta})^{-1}F = \int_0^\infty e^{-\lambda}e^{-\lambda I}e^{-\lambda I}F dt, \]

it follows from (2.34) that there exists a constant \( M > 0 \) such that

\[ \|U_n\|_{X^{\infty}(\Omega)} \leq M(\lambda_{s,n}^W + \lambda)\|U_n\|_{X^{2}(\Omega)}. \]

This completes the proof of part (c).

(d) We notice that since \( I + A_{W,\delta}^s \) is invertible we have that \( \|(I + A_{W,\delta}^s)^{-1}F\|_{D(A_{W,\delta})} \), for \( F \in X^{2}(\Omega) \), defines an equivalent norm on \( D(A_{W,\delta}) \). Using (2.34) with \( p = 2 \), \( q = \infty \), for \( t \in (0, 1) \) and the contractivity of \( e^{-\lambda I} \) for \( t > 1 \), for \( U \in D(A_{W,\delta}) \), we deduce

\[ \|U\|_{X^{\infty}(\Omega)} \leq C \|U\|_{D(A_{W,\delta})} \int_0^1 t^{-\frac{1}{2}} dt + C \|U\|_{D(A_{W,\delta})} \int_1^\infty e^{-t} dt. \]

The first integral is finite if and only if \( \gamma < 4 \). Using the expression of \( \gamma \) given in (2.35) and(2.36), we get the condition given on dimension \( N \) in the hypothesis of the assertion (d).
(e) As in part (d), if } \alpha > 0 \text{, then } \| (\alpha I + A_{W,\delta})^{-1} F \|_{D(A_{W,\delta})}, \text{ for } F \in X^2(\overline{\Omega}), \text{ defines an equivalent norm on } D(A_{W,\delta}). \text{ Using } (2.34) \text{ with } p = 2 \text{ and } q \in (2, \infty) \text{ and the contractivity of } e^{-tA_{W,\delta}} \text{ for } t > 1, \text{ for } U \in D(A_{W,\delta}), \text{ we deduce once again that}

\| U \|_{X^p(\overline{\Omega})} \leq C \| U \|_{D(A_{W,\delta})} \int_0^1 t^{-\frac{q}{2}(1-\frac{\delta}{2})} dt + C \| U \|_{D(A_{W,\delta})} \int_1^\infty e^{-\alpha t} dt < \infty

provided that the first integral is finite, i.e., } \gamma < \frac{4q}{q-2}. \text{ Using the expression of } \gamma \text{ in (2.35) and (2.36), we get the condition given on dimension } N \text{ in the hypothesis of the assertion (e). The proof of the theorem is finished.}

Next, for } \frac{1}{2} < s < 1, \text{ } 0 < l < 1, \text{ and } \delta \in \{0,1\}, \text{ we consider the elliptic boundary value problem}

\begin{align*}
(-\Delta)^2 u &= f \quad \text{in } \Omega, \\
C_\alpha N^{2-2s} u + \beta(x) u + \delta (-\Delta)^l u &= g \quad \text{on } \partial\Omega,
\end{align*}

where } (f,g) \in (W^{s,\delta,2}(\Omega))^*. \text{ is given.}

**Definition 2.13.** A function } u \text{ is said to be a weak solution of (2.40) if } U := (u,u|_{\partial\Omega}) \in W^{s,\delta,2}(\Omega) \text{ and for every } V = (v,v|_{\partial\Omega}) \in W^{s,\delta,2}(\Omega),

\begin{align*}
& a^s_\Omega(U,V) = (f,v)_{(W^{s-2}(\Omega))^*,W^{s-2}(\Omega)} + (g,v)_{(W^{s,2}(\Omega))^*,W^{s,2}(\Omega)} \\
& \text{if } \delta = 1, \text{ and } \\
& a^s_\Omega(U,V) = (f,v)_{(W^{s-2}(\Omega))^*,W^{s,2}(\Omega)} + (g,v)_{(W^{s-\frac{1}{2}}(\partial\Omega))^*,W^{s-\frac{1}{2}}(\partial\Omega)}
\end{align*}

if } \delta = 0, \text{ where we recall that } a^s_\Omega \text{ is given in (2.24). We will simply say that } U \text{ is a weak solution of (2.40).}

We have the following result as a consequence of Theorem 2.12.

**Theorem 2.14.** Let } \frac{1}{2} < s < 1, \text{ } 0 < l < 1 \text{ and } \delta \in \{0,1\}. \text{ Then the following assertions hold.}

(a) For every } (f,g) \in (W^{s,\delta,2}(\Omega))^*, \text{ the problem (2.40) has a unique weak solution.}

(b) Let } N > 2s. \text{ If } (f,g) \in X^p(\overline{\Omega}) \text{ with}

\begin{align*}
& p > \frac{N}{2s} \quad \text{if } l > s - \frac{s}{N}, \\
& p > \frac{N-1}{2s-1} \quad \text{if } s - \frac{1}{2} \leq l \leq s - \frac{s}{N}, \\
& p > \frac{N-1}{2s-1} \quad \text{if } l \leq s - \frac{1}{2},
\end{align*}

in the case } \delta = 1 \text{ and }

\begin{equation}
\frac{N}{2s-1} \geq p > \frac{N-1}{2s-1}
\end{equation}

in the case } \delta = 0, \text{ then the weak solution } U := (u,u|_{\partial\Omega}) \in X^\infty(\overline{\Omega}) \text{ and there is a constant } C > 0 \text{ such that}

\begin{equation}
\| U \|_{X^\infty(\overline{\Omega})} \leq C \|(f,g)\|_{X^p(\overline{\Omega})}.
\end{equation}

**Proof.** Let } \frac{1}{2} < s < 1, \text{ } 0 < l < 1 \text{ and } \delta \in \{0,1\}.

(a) We have already shown that } a^s_\Omega \text{ with domain } W^{s,\delta,2}(\Omega) \text{ is a bilinear, closed and continuous form. It also follows from Theorem 2.12, part (b), that } a^s_\Omega \text{ is coercive; more precisely there exists a constant } C > 0 \text{ such that}

\begin{equation}
a^s_\Omega(U,U) \geq \| U \|_{W^{s,\delta,2}(\Omega)}^2.
\end{equation}
for every $U \in \mathcal{W}^{s,\delta,2}(\Omega)$. By the Lax-Milgram theorem, this implies that the system (2.40) has a unique weak solution for every given $(f,g) \in (\mathcal{W}^{s,\delta,2}(\Omega))^\ast$.

(b) Let $N > 2s$. Let $F := (f, g) \in \mathcal{X}^p(\Omega)$ with $p$ satisfying (2.41) or (2.42). It follows from the embedding (2.5) or (2.6) that $\mathcal{X}^p(\Omega) \hookrightarrow (\mathcal{W}^{s,\delta,2}(\Omega))^\ast$ with

$$
\begin{cases}
    r' = \frac{2^*}{2^* - 1} = \frac{2N}{N + 2s} \\
    q' := \frac{2^*}{2^* - 1} = \frac{2(N - 1)}{N - 2 + 2s},
\end{cases}
$$

in the case $\delta = 1$ and

$$
\begin{cases}
    r' = \frac{2^*}{2^* - 1} = \frac{2N}{N + 2s} \\
    q' := \frac{2^*}{2^* - 1} = \frac{2(N - 1)}{N - 2 + 2s},
\end{cases}
$$

in the case $\delta = 0$. Since $N > 2s$ and $N - 1 > 2l$, we have that in the case $\delta = 1$,

$$
\begin{cases}
    p > \frac{N}{2^*} > \frac{2N}{N + 2s} = r' \\
    p > \frac{N - 1}{2^* - 1} > \frac{2(N - 1)}{N - 2 + 2s} = q',
\end{cases}
$$

and in the case $\delta = 0$,

$$
p > \frac{N - 1}{2s - 1} > \frac{2N}{N + 2s} = r'.
$$

We have shown that $F := (f, g) \in (\mathcal{W}^{s,\delta,2}(\Omega))^\ast$. Let now $U$ be the unique weak solution of (2.40). Since 0 is not an eigenvalue of $A_{W,\delta}^s$, then it is invertible and this shows that $U = (A_{W,\delta}^s)^{-1}F$. Let $\gamma$ be as in (2.38) and (2.39). Then $1 - \frac{\gamma}{2p} > 0$ and therefore $2p - \gamma > 0$. This together with the fact that $\lambda_{W,s}^1 > 0$ and the estimate (2.34) with $q = \infty$ give

$$
|U| = \left| \int_0^\infty e^{-tA_{W,\delta}}F dt \right| \leq \int_0^\infty \|e^{-tA_{W,\delta}}F\|_{X^{\infty}(\Omega)}dt
\leq C \left( \int_1^\infty t^{-\frac{\gamma}{2p}} e^{-\frac{\lambda_{W,s}^1}{2p}t}dt \right) ||F||_{X^p(\Omega)}
\leq C \left( \int_1^\infty e^{-\frac{\lambda_{W,s}^1}{2p}t}dt \right) ||F||_{X^p(\Omega)}
= C \left( \frac{1}{\lambda_{W,s}^1} + \frac{2p}{2p - \gamma} \right) ||F||_{X^p(\Omega)};
$$

This implies that there is a constant $C > 0$ such that (2.43) holds. The proof is finished.

2.5. The fractional Steklov operator. Throughout the remainder of the article, we denote by $(-\Delta)^s_{\Omega,D}$ the realization of the operator $(-\Delta)^s_{\Omega}$ with the Dirichlet boundary condition in $L^2(\Omega)$. That is, $(-\Delta)^s_{\Omega,D}$ is the linear self-adjoint operator associated with the bilinear symmetric closed form $b_D$ with domain $W_{0,s}^{s,2}(\Omega)$ and given by

$$
b_D(u,v) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx dy.
$$

It is well-known (see, e.g., [31]) that $(-\Delta)^s_{\Omega,D}$ hat a compact resolvent. We denote its discrete spectrum by $\sigma((-\Delta)^s_{\Omega,D})$ which consists of eigenvalues satisfying $0 <
\[\lambda_{s,1} < \lambda_{s,2} \cdots \lambda_{s,n} \cdots\] and \(\lim_{n \to \infty} \lambda_{s,n}^D = \infty\). The following Poincaré inequality holds: for every \(u \in W^s_0(\Omega)\), we have

\[
\lambda_{s,1}^D \int_\Omega |u|^2 \, dx \leq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dxdy. \tag{2.44}
\]

Next, let \(\frac{1}{2} < s < 1\) and \(\lambda \in \mathbb{R} \setminus \sigma((-\Delta)^s_{\Omega,D})\). By [31, Lemma 3.1], the following decomposition holds:

\[
W^{s,2}(\Omega) = W^{s,2}_0(\Omega) \oplus \mathcal{H}^{s,\lambda}(\Omega), \tag{2.45}
\]

where \(\mathcal{H}^{s,\lambda}(\Omega) := \{u \in W^{s,2}(\Omega) : (-\Delta)^s_{\Omega} u = \lambda u\}\) and by \((-\Delta)^s_{\Omega} u = \lambda u\) we mean that

\[
\frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy = \lambda \int_\Omega uvdx \tag{2.46}
\]

for all \(v \in D(\Omega)\), and hence, for all \(v \in W^{s,2}_0(\Omega)\) by density. Recall that

\[
W^{s,2}(\Omega) = \{u_{\partial \Omega} : u \in W^{s,2}(\Omega)\}.
\]

It follows from (2.45) that the trace operator restricted to \(\mathcal{H}^{s,\lambda}(\Omega)\), i.e., the mapping \(\mathcal{H}^{s,\lambda}(\Omega) \ni u \mapsto u_{\partial \Omega} \in W^{s,2}(\partial \Omega)\) is linear and bijective. Letting \(\|u\|_{W^{s-\frac{1}{2},2}(\partial \Omega)} := \|u\|_{\mathcal{H}^{s,\lambda}(\Omega)}\), then \(W^{s-\frac{1}{2},2}(\partial \Omega)\) becomes a Hilbert space. By the closed graph theorem, different choice of \(\lambda \in \mathbb{R} \setminus \sigma((-\Delta)^s_{\Omega,s})\) leads to an equivalent norm on \(W^{s-\frac{1}{2},2}(\partial \Omega)\). We define the bilinear symmetric form \(F_\lambda\) with domain \(W^{s-\frac{1}{2},2}(\partial \Omega)\) by

\[
F_\lambda(\varphi, \psi) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy - \lambda \int_\Omega uvdx \tag{2.47}
\]

where \(u, v \in \mathcal{H}^{s,\lambda}(\Omega)\) with \(\varphi = u_{\partial \Omega}\) and \(\psi = v_{\partial \Omega}\).

The following result is taken from [31, Proposition 3.1].

**Proposition 2.15.** The form \(F_\lambda\) is continuous and elliptic. Let \(D_{s,\lambda}\) be the linear self-adjoint operator on \(L^2(\partial \Omega)\) associated with \(F_\lambda\) in the sense that, for any \(\phi \in W^{s-\frac{1}{2},2}(\partial \Omega)\),

\[
\begin{cases}
D(D_{s,\lambda}) := \{\varphi \in W^{s-\frac{1}{2},2}(\partial \Omega), \exists \psi \in L^2(\partial \Omega), F_\lambda(\varphi, \psi) = (\psi, \phi)_{L^2(\partial \Omega)}\} \\
D_{s,\lambda} \varphi := \psi.
\end{cases}
\]

Then

\[
\begin{cases}
D(D_{s,\lambda}) = \{\varphi \in W^{s-\frac{1}{2},2}(\partial \Omega) : \exists u \in W^{s,2}(\Omega), u_{\partial \Omega} = \varphi, (-\Delta)^s_{\Omega} u = \lambda u, \text{ and } N^{2-2s}u \text{ exists in } L^2(\partial \Omega)\}, \\
D_{s,\lambda} \varphi = C_s N^{2-2s} u,
\end{cases}
\tag{2.48}
\]

where \(N^{2-2s}u\) is to be understood in the sense of Definition 2.7 and \(C_s\) is the constant given in (2.13).

The operator \(D_{s,\lambda}\) is called the fractional Dirichlet-to-Neumann map.

**Remark 2.16.** Let \(\frac{1}{2} < s < 1\), \(0 < l < 1\) and \(\delta \in \{0, 1\}\). Let \(u \in W^{s,2}(\Omega)\) be such that \((-\Delta)^s_{\Omega} u \in L^2(\Omega)\) in the sense of Definition 2.7 and assume that \(u_{\partial \Omega} = \varphi \in W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{l,2}(\partial \Omega)\). Throughout the remainder of the paper, we will say that \(N^{2-2s}u + \delta(-\Delta)^s_{\Omega}(u_{\partial \Omega})\) exists in \(L^2(\partial \Omega)\) if there is a function
\[ g \in L^2(\partial\Omega) \] such that for every \( \psi \in W^{s-\frac{1}{2}}(\partial\Omega) \cap W^{l,2}(\partial\Omega) \), \( \psi = v|_{\partial\Omega} \) where \( v \in W^{s,2}(\Omega) \), the following identity holds:

\[
\begin{align*}
&\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy - \int_{\Omega} v(-\Delta)^s_\Omega u \, dx \\
&+ \frac{\delta C_{N-1,l}}{2} \int_{\Omega} \int_{\partial\Omega} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N-1 + 2l}} \, ds_x \, ds_y = \int_{\partial\Omega} g \psi \, ds.
\end{align*}
\] (2.49)

In that case the function \( g \) is uniquely determined by the identity (2.49) and the function \( g \) is given by \( g = C_s N^{2-2s}u + \delta(-\Delta)^s \psi \).

We shall now define a perturbation of the operator \( D_{s,\lambda} \) and refer to the new operator as the \textit{fractional Steklov operator}. To this end, for \( s \in (1/2, 1), l \in (0, 1), \delta \in \{0, 1\} \) and \( \lambda \in \mathbb{R} \setminus \sigma((-\Delta)^s_{\Omega, l, \lambda}) \), we consider the bilinear symmetric form \( \mathbb{F}^s_{\lambda} \) with domain \( D(\mathbb{F}^s_{\lambda}) = W^{s-\frac{1}{2},2}(\partial\Omega) \cap W^{l,2}(\partial\Omega) \) if \( \delta = 1 \) and \( D(\mathbb{F}^s_{\lambda}) = W^{s-\frac{1}{2},2}(\partial\Omega) \) if \( \delta = 0 \) and given for \( u, v \in D(\mathbb{F}^s_{\lambda}) \) by

\[
\begin{align*}
\mathbb{F}^s_{\lambda}(\varphi, \psi) &= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy - \lambda \int_{\Omega} uv \, dx \\
&+ \frac{\delta C_{N-1,l}}{2} \int_{\Omega} \int_{\partial\Omega} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N-1 + 2l}} \, ds_x \, ds_y \\
&+ \int_{\partial\Omega} \beta(x) \varphi \psi \, ds
\end{align*}
\] (2.50)

where \( u, v \in \mathcal{H}^{s,\lambda}(\Omega) \) with \( \varphi = u|_{\partial\Omega} \) and \( \psi = v|_{\partial\Omega} \).

We have the following result.

**Proposition 2.17.** The bilinear symmetric form \( \mathbb{F}^s_{\lambda} \) is continuous and elliptic. Let \( \mathbb{B}^s_{\lambda} \) be the linear and self-adjoint operator on \( L^2(\partial\Omega) \) associated with \( \mathbb{F}^s_{\lambda} \). Then the domain \( D(\mathbb{B}^s_{\lambda}) \) of the operator consists of functions \( \varphi \in W^{s-\frac{1}{2},2}(\partial\Omega) \cap W^{l,2}(\partial\Omega) \) such that there exists \( u \in W^{s,2}(\Omega), u|_{\partial\Omega} = \varphi, (-\Delta)^s_{\Omega,u} = \lambda u, \) and \( N^{2-2s}u + (-\Delta)^l \varphi \) exists in \( L^2(\partial\Omega) \), and

\[ \mathbb{B}^s_{\lambda} \varphi = D_{s,\lambda} \varphi + \delta(-\Delta)^s_{\Omega} \varphi = C_s N^{2-2s}u + \delta(-\Delta)^l \varphi + \beta(x) \psi. \] (2.51)

Moreover, the following assertions hold.

(a) The operator \( \mathbb{B}^s_{\lambda} \) has a compact resolvent and \(-\mathbb{B}^s_{\lambda}\) generates a strongly continuous semigroup \( (e^{-t\mathbb{B}^s_{\lambda}})_t \geq 0 \) on \( L^2(\partial\Omega) \).

(b) If \( \lambda < \lambda^0_{s,\lambda} \), then the semigroup \( (e^{-t\mathbb{B}^s_{\lambda}})_t \geq 0 \) is positive.

(c) If \( \lambda \leq 0 \), then we have that the semigroup \( (e^{-t\mathbb{B}^s_{\lambda}})_t \geq 0 \) is submarkovian, that is, \( (e^{-t\mathbb{B}^s_{\lambda}})_t \geq 0 \) is positive and

\[ 0 \leq e^{-t\mathbb{B}^s_{\lambda}} \varphi \leq 1, \quad \forall t \geq 0 \] whenever \( \varphi \in L^2(\partial\Omega), 0 \leq \varphi \leq 1. \)

(d) Let \( \lambda \leq 0 \) and \( N > 2s \). Let \( \mu_{s,\lambda}(\lambda) \) be the first eigenvalue of \( \mathbb{B}^s_{\lambda} \). Then for all \( 1 \leq p \leq q \leq \infty \) and \( t > 0 \), the operator \( e^{-t\mathbb{B}^s_{\lambda}} \) is bounded from \( L^p(\partial\Omega) \) into \( L^q(\partial\Omega) \). More precisely, there is a constant \( C > 0 \) such that

\[
\| e^{-t\mathbb{B}^s_{\lambda}} \|_{L^p(\partial\Omega), L^q(\partial\Omega)} \leq C(t \wedge 1)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\mu_{s,\lambda}(\lambda)(\frac{1}{p} - \frac{1}{q}) t},
\] (2.52)

for all \( t > 0 \) and \( p, q \in [1, \infty] \) with \( p \leq q \), where

\[ \gamma = \frac{N - 1}{l} > 2 \quad \text{if} \quad l \geq s - \frac{1}{2} \quad \text{and} \quad \gamma = \frac{2(N - 1)}{2s - 1} > 2 \quad \text{if} \quad l \leq s - \frac{1}{2}, \]
in the case \( \delta = 1 \) and
\[
\tilde{\gamma} = \frac{2(N - 1)}{2s - 1} > 2
\]
in the case \( \delta = 0 \).

Proof. We give the proof of the proposition for the case \( \delta = 1 \). The case \( \delta = 0 \) follows similarly (see [31]). Let \( \delta = 1 \) and \( \varphi, \psi \in W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega) \). First observe that
\[
\mathcal{F}_\lambda(\varphi, \psi) = \mathcal{F}_\lambda(\varphi, \psi) + \mathcal{B}_1(\varphi, \psi)
\]
where
\[
\mathcal{B}_1(\varphi, \psi) := \frac{C_{N-1,1}}{2} \int_{\partial \Omega} \int_{\partial \Omega} \frac{(\varphi(y) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N-1+2l}} d\sigma_x d\sigma_y + \int_{\partial \Omega} \beta(x) \varphi \psi d\sigma,
\]
and \( \mathcal{F}_\lambda \) is given by (2.47). By Proposition 2.15, \( \mathcal{F}_\lambda \) is continuous, that is,
\[
|\mathcal{F}_\lambda(\varphi, \psi)| \leq C\|\varphi\|_{W^{s-\frac{1}{2},2}(\partial \Omega)} \|\psi\|_{W^{s-\frac{1}{2},2}(\partial \Omega)}
\]
\[
\leq C\|\varphi\|_{W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)} \|\psi\|_{W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)},
\]
(2.53)
for some constant \( C > 0 \), and is elliptic in the sense that there are two constants \( \omega_1 \in \mathbb{R} \) and \( C_1 > 0 \) such that
\[
\mathcal{F}_\lambda(\varphi, \varphi) + \omega_1 \int_{\partial \Omega} |\varphi|^2 \ d\sigma \geq C_1 \|\varphi\|^2_{W^{s-\frac{1}{2},2}(\partial \Omega)}.
\]
(2.54)
It is easy to see that there is a constant \( C_2 > 0 \) such that
\[
|\mathcal{B}_1(\varphi, \psi)| \leq C\|\varphi\|_{W^{s,2}(\partial \Omega)} \|\psi\|_{W^{s,2}(\partial \Omega)}
\]
\[
\leq C\|\varphi\|_{W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)} \|\psi\|_{W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)},
\]
(2.55)
Moreover,
\[
\mathcal{B}_1(\varphi, \varphi) \geq \min\{1, \beta_0\} \|\varphi\|^2_{W^{s,2}(\partial \Omega)}.
\]
(2.56)
Combining (2.53) and (2.55) we find that
\[
|\mathcal{F}_\lambda(\varphi, \psi)| \leq C\|\varphi\|_{W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)} \|\psi\|_{W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)},
\]
Hence, \( \mathcal{F}_\lambda \) is continuous. The estimate (2.54) together with (2.56) implies that there exists a constant \( \omega \in \mathbb{R} \) such that
\[
\mathcal{F}_\lambda(\varphi, \varphi) + \omega \int_{\partial \Omega} |\varphi|^2 \ d\sigma \geq C_1 \|\varphi\|^2_{W^{s-\frac{1}{2},2}(\partial \Omega)} + \min\{1, \beta_0\} \|\varphi\|^2_{W^{s,2}(\partial \Omega)}.
\]
We have shown that \( \mathcal{F}_\lambda \) is also elliptic. Next, let \( \mathcal{B}_1 \) with domain \( D_\lambda^1 \) be the linear and self-adjoint operator on \( L^2(\partial \Omega) \) associated with \( \mathcal{F}_\lambda \) in the sense that
\[
\begin{cases}
D_\lambda^1 = \{ \varphi \in W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega) : \exists w \in L^2(\partial \Omega), \\
\mathcal{F}_\lambda(\varphi, \psi) = (w, \psi)_{L^2(\partial \Omega)} \forall \psi \in W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega) \}
\end{cases}
\]
\[
\mathcal{B}_1^\lambda \varphi = w.
\]
We show that it coincides with the one defined in (2.51). Indeed, let \( \varphi \in D_\lambda^1 \) and \( u \in H^{s,\lambda}(\Omega) \) such that \( \varphi = u|_{\partial \Omega} \). Then by definition, there is \( w \in L^2(\partial \Omega) \) such that
\[
\mathcal{F}_\lambda(\varphi, \psi) = (w, \psi)_{L^2(\partial \Omega)} \forall \psi \in W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega).
\]
for every $\psi \in W^{s-\frac{1}{2}}(\partial \Omega) \cap W^{l,2}(\partial \Omega)$, we have

$$F^{s}_{\lambda}(\varphi, \psi) := \frac{C_{N,s}}{2} \int \int \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy - \lambda \int wdx - \int \int \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N-1+2l}} d\sigma_x d\sigma_y$$

(2.57)

$$+ \int \beta(x) \varphi \psi d\sigma$$

$$= \int w \psi d\sigma,$$

where $v = \psi|_{\partial \Omega}$ with $v \in H^{s,\lambda}(\Omega)$. We mention that for $u \in H^{s,\lambda}(\Omega)$ and $g \in L^2(\partial \Omega)$, we have that for all $v \in W^{s,2}(\Omega)$,

$$\frac{C_{N,s}}{2} \int \int \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy - \lambda \int wdx = \int \int \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N-1+2l}} d\sigma_x d\sigma_y = 0.$$
Using (2.59) and Definition 2.7 we have
\[
\lambda < \lambda
\]
Since
\[
By [5, Theorem 1.3.2], (2.64) implies that the semigroup \((\varphi_t)\) is positive.
\]
Combining (2.60), (2.61), (2.62) and (2.63) we get that
\[
\mathcal{F}_\lambda(\varphi^+, \varphi^-) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} dx dy - \lambda \int_\Omega u_1 u_2 dx
\]
Using (2.59) and Definition 2.7 we have
\[
\mathcal{F}_\lambda(\varphi^+, \varphi^-) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} dx dy - \lambda \int_\Omega u_1 u_2 dx
\]
\[
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_0(x) - u_0(y))}{|x - y|^{N+2s}} dx dy
\]
\[
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_0(x) - u_0(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} dx dy
\]
\[
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} dx dy
\]
\[
- \lambda \int_\Omega (u_1 + u_0)(u_2 + u_0) dx + \lambda \int_\Omega u_1 u_0 dx + \lambda \int_\Omega u_0 u_2 dx + \lambda \int_\Omega |u_0|^2 dx
\]
\[
= \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u^+(x) - u^-(y))(u^+(x) - u^-(y))}{|x - y|^{N+2s}} dx dy - \lambda \int_\Omega u^+ u^- dx
\]
\[
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} dx dy + \lambda \int_\Omega |u_0|^2 dx
\]
\[
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} dx dy + \lambda \int_\Omega |u_0|^2 dx.
\]
Since \(u^+(x)u^-(y) + u^-(x)u^+(y) \geq 0\) for almost every \((x, y) \in \Omega \times \Omega\), we have that
\[
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x - y|^{N+2s}} dx dy \leq 0.
\]
Since \(\lambda < \lambda_{1,s}^\varphi\) and \(u_0 \in W^{s,2}_0(\Omega)\), it follows from the Poincaré inequality (2.44) that
\[
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} dx dy + \lambda \int_\Omega |u_0|^2 dx \leq 0.
\]
A simple calculation (see, e.g., [30, Lemma 2.6]) shows that
\[
B_1(\varphi^+, \varphi^-) \leq 0.
\]
Combining (2.60), (2.61), (2.62) and (2.63) we get that
\[
\mathcal{F}_\lambda(\varphi^+, \varphi^-) = \mathcal{F}_\lambda(\varphi^+, \varphi^-) + B_1(\varphi^+, \varphi^-) \leq 0.
\]
By [5, Theorem 1.3.2], (2.64) implies that the semigroup \((e^{-t\varphi_1})_{t \geq 0}\) is positive.

(c) Let \(\lambda \leq 0\). It follows from part (b) that \((e^{-t\varphi_1})_{t \geq 0}\) is positive. Let then \(0 \leq \varphi \in W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)\). Then \(\varphi \wedge 1 \in W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)\). In fact, let \(0 \leq u \in W^{s,2}(\Omega)\) such that \(|u|_{\partial \Omega} = \varphi\). It follows from [30, Lemma 2.7] that \(u \wedge 1 \in W^{s,2}(\Omega)\) and
\[
\|u \wedge 1\|^2_{W^{s,2}(\Omega)} \leq \|u\|^2_{W^{s,2}(\Omega)}.
\]
Moreover, \((u \land 1)|_{\partial \Omega} = \varphi \land 1\). Now, let \(0 \leq \varphi \in W^{s-\frac{1}{2}, 2}(\partial \Omega) \cap W^{l, 2}(\partial \Omega), u|_{\partial \Omega} = \varphi\) where \(0 \leq u \in \mathcal{H}^{s+\lambda}(\Omega)\). We write \(u \land 1 = u_0 + u_1 \in W^{s+\lambda}(\Omega) \oplus \mathcal{H}^{s+\lambda}(\Omega)\). Since 
\(u_1 \in \mathcal{H}^{s+\lambda}(\Omega)\) and \(u_0 \in W^{l, 2}(\Omega)\), we have that 
\[
\frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_0(x) - u_0(y))}{|x - y|^{N+2s}} \, dx \, dy = \lambda \int_\Omega u_1 u_0 \, dx. \quad (2.66)
\]
Using (2.66), Definition 2.7, the fact that \(\lambda \leq 0 < \lambda^D_{1,s}\), the Poincaré inequality (2.44) and (2.65), we get 
\[
\mathcal{F}_\lambda(\varphi \land 1, \varphi \land 1) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_1(x) - u_1(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_\Omega |u_1|^2 \, dx \quad (2.67)
\]
\[
= \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|(u_1 + u_0)(x) - (u_1 + u_0)(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_\Omega |u_1 + u_0|^2 \, dx \\
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_0(x) - u_0(y))}{|x - y|^{N+2s}} \, dx \, dy \\
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + 2\lambda \int_\Omega u_0 u_1 \, dx + \lambda \int_\Omega |u_0|^2 \, dx \\
= \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|(u \land 1)(x) - (u \land 1)(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_\Omega |u \land 1|^2 \, dx \\
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \lambda \int_\Omega |u_0|^2 \, dx \\
\leq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|(u \land 1)(x) - (u \land 1)(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_\Omega |u \land 1|^2 \, dx \\
- \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \lambda^D_{1,s} \int_\Omega |u_0|^2 \, dx \\
\leq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_\Omega |u|^2 \, dx = \mathcal{F}_\lambda(\varphi, \varphi). 
\]
Proceeding as the proof of (2.65) in [30, Lemma 2.7] we get that 
\[
\mathcal{B}_1(\varphi \land 1, \varphi \land 1) \leq \mathcal{B}_1(\varphi, \varphi). \quad (2.68)
\]
Combining (2.67) and (2.68) we obtain that 
\[
\mathcal{F}^1_\lambda(\varphi \land 1, \varphi \land 1) = \mathcal{F}_\lambda(\varphi \land 1, \varphi \land 1) + \mathcal{B}_1(\varphi \land 1, \varphi \land 1) \leq \mathcal{F}_\lambda(\varphi, \varphi). \quad (2.69)
\]
By [5, Theorem 1.3.3], the estimate (2.69) implies that the semigroup \((e^{-t\mathcal{F}^1_\lambda})_{t \geq 0}\) is submarkovian. This completes the proof of part (c).

(d) Let \(\lambda \leq 0\) and \(N > 2s\). Note that 
\[
W^{s-\frac{1}{2}, 2}(\partial \Omega) \cap W^{l, 2}(\partial \Omega) = \begin{cases} 
W^{l, 2}(\partial \Omega) \hookrightarrow L^2(\partial \Omega) & \text{if } l \geq s - \frac{1}{2} \\
W^{s-\frac{1}{2}, 2}(\partial \Omega) \hookrightarrow L^2(\partial \Omega) & \text{if } l \leq s - \frac{1}{2}.
\end{cases} \quad (2.70)
\]
It follows from (2.70) that there exists a constant \(C > 0\) such that for every \(\varphi \in D(\mathcal{F}^1_\lambda),\) 
\[
\|\varphi\|_{\mathcal{F}^1_\lambda(\partial \Omega)}^2 \leq C\mathcal{F}^1_\lambda(\varphi, \varphi),
\]
with 
\[
q = 2^* = \frac{2(N - 1)}{N - 1 - 2l} = \frac{2\tilde{\gamma}}{\tilde{\gamma} - 2} > 2, \quad \tilde{\gamma} = \frac{N - 1}{l} \quad \text{if } l \geq s - \frac{1}{2},
\]
and

\[ q = 2s = \frac{2(N - 1)}{N - 2s} = \frac{2\gamma}{\gamma - 2} > 2, \quad \gamma = \frac{2(N - 1)}{2s - 1} \quad \text{if} \quad l \leq s - \frac{1}{2}. \]

The estimate (2.71) implies that the semigroup \((e^{-t\mathcal{B}^1_{\cdot}^\gamma})_{t \geq 0}\) is ultracontractive in the sense that it maps \(L^2(\partial\Omega)\) into \(L^\infty(\partial\Omega)\). Proceeding as in [31] we get the estimate (2.52) by exploiting (2.71), the ultracontractivity, the submarkovian property, that is, for every \(p \in [1, \infty] \) and \(t \geq 0\),

\[ \|e^{-t\mathcal{B}^1_{\cdot}^\gamma}\|_{L^p(\partial\Omega)} \leq 1, \]

and some abstract results on ultracontractivity of semigroups contained in [26, Lemma 6.1 and Lemma 6.5]. The proof of the theorem is finished. \(\square\)

3. Parabolic equations with fractional dynamic boundary conditions. In this section, we shall consider a semilinear parabolic equation associated with the fractional Wentzell operator \(A^1_{W,\delta}\) of the form

\[
\begin{aligned}
&\partial_t u + (-\Delta)_{\Omega}^s u + f(u) = 0, \\
&\partial_t u + C_s \mathcal{N}^{2-2s} u + \beta(x) u + \delta(-\Delta)_{\Gamma}^l u + g(u) = 0, \\
&u(0) = u_0 \in \Omega, \quad u(0) = v_0 \text{ on } \partial\Omega.
\end{aligned}
\]

where we recall that \(s \in (1/2, 1), \ l \in (0, 1)\) and \(\delta \in \{0, 1\}\). As usual, no additional connection between \(v_0\) and \(u_0\) is required at this point.

In what follows strong solutions to problem (3.1) are defined via nonlinear semigroup theory for bounded initial data and satisfy the differential equations almost everywhere in \(t > 0\). We first introduce the rigorous notion of (global) weak solutions to the problem (3.1) as in the classical case for the semilinear parabolic equation with classical diffusion \(\Delta\) and the corresponding dynamic boundary conditions. For the sake of simplicity of notation, the symbol \(\langle \cdot, \cdot \rangle\) stands for the duality pairing between Banach space \(X\) and its dual \(X^*\). The classical notion of a weak energy solution in the space \(X^2(\Omega)\) is given by the following.

**Definition 3.1.** Let \(p, q > 1\). The function \(U = (u, u|_{\partial\Omega})\) is said to be a weak solution of (3.1) if, for a.e. \(t \in (0, T)\), for any \(T > 0\), the following properties are valid:

- **Regularity:**

\[
\begin{aligned}
U &\in L^\infty((0, T); X^2(\Omega)) \cap X^{p,q}((0, T) \times \Omega) \cap L^2((0, T); \mathcal{W}^{s,\delta,l,2}(\Omega)), \\
\partial_t U &\in L^p((0, T); (\mathcal{W}^{s,\delta,l,2}(\Omega))^*) \oplus X^{p',q'}((0, T) \times \Omega), \\
f(U) &\in L^p((0, T) \times \Omega), \quad g(U) \in L^q((0, T) \times \partial\Omega),
\end{aligned}
\]

where \(p' = p/(p - 1)\) and \(q' = q/(q - 1)\).

- **Variational identity:** for the weak solutions the following equality

\[
\langle \partial_t U(t), \xi \rangle + a^\Omega_{\Omega}(U(t), \xi) + \langle f(u(t)), \xi \rangle + \langle g(u(t)), \xi \rangle = 0
\]

(3.3) holds for all \(\xi \in \mathcal{W}^{s,\delta,l,2}(\Omega) \cap X^{p,q}(\Omega)\), a.e. \(t \in (0, T)\). Finally, we have, in the space \(X^2(\Omega)\), \(U(0) = (u_0, v_0)\) almost everywhere.
• Energy identity: weak solutions satisfy the following identity
\[
\frac{1}{2} \| \bar{U} (t) \|^2_{\mathcal{X}^2(\Omega)} + \int_0^t a^\delta_\Omega (U (\tau), U (\tau)) \, d\tau
+ \int_0^t [\langle f (u (\tau)) , u (\tau) \rangle + \langle g (u (\tau)), u (\tau) \rangle] \, d\tau
= \frac{1}{2} \| U (0) \|^2_{\mathcal{X}^2(\Omega)}.
\]

**Remark 3.2.** Note that by (3.2), \( U \in C_w ([0, T]; \mathcal{X}^2(\Omega)) \), that is, the space of all \( \mathcal{X}^2(\Omega) \)-valued weakly continuous functions on the interval \([0, T] \). Therefore the initial value \( U (0) = (u_0, v_0) \) is meaningful when \((u_0, v_0) \in \mathcal{X}^2(\Omega) \). As usual, \( v_0 \) is some function in \( L^2 (\partial \Omega) \) and there need not be any connection between \( v_0 \) and the boundary trace of \( u_0 \), which may not exist for \( u_0 \in L^2 (\Omega) \).

Finally, our notion of (global) strong solution is as follows.

**Definition 3.3.** Let \((u_0, v_0) \in \mathcal{X}^\infty (\Omega) \) be given. A weak solution \( U \) is “strong” if, in addition, it fulfills the regularity properties:
\[
U \in W^{1, \infty} ([\eta, T]; \mathcal{X}^2(\Omega)) \cap C ([0, T]; \mathcal{X}^\infty (\Omega)),
\]
such that \( U \in L^\infty ([\eta, T]; D (A^\lambda_{W, \delta})) \), for any \( T > \eta > 0 \).

This section consists of two main parts. At first we will establish the existence and uniqueness of a (local) strong solution on a finite time interval using the theory of monotone operators exploited in [15] and [16]. Then exploiting a Moser iteration argument we show that the local solution is actually a global solution. In the second part, we establish the existence of a compact (finite dimensional) global attractor for problem (3.1).

We first state a Poincaré-type inequality associated with the quadratic form \( a^\delta_\Omega \) as defined by (2.24). As in [15, 16] it is crucial in the proof of the existence of strong solutions to semilinear parabolic equations with fractional diffusion and dynamic boundary conditions.

**Lemma 3.4.** Let \( s \in (1/2, 1), l \in (0, 1), \delta \in \{0, 1\} \) and let \( \Omega \subset \mathbb{R}^N \) be a domain with Lipschitz continuous boundary \( \partial \Omega \). Then for all \( \varepsilon \in (0, 1) \) there is \( \zeta > 0 \) such that
\[
\| U \|^2_{\mathcal{X}^2(\Omega)} \leq \varepsilon a^\delta_\Omega (U, U) + \varepsilon^{-\zeta} \| U \|^2_{\mathcal{X}^2(\Omega)}, \quad \text{for all } U \in \mathcal{W}^{s, \delta l, 2} (\Omega).
\]

**Proof.** The proof is similar to that of [16, Lemma 3.3] and hence omitted. \( \square \)

Now we state the first main theorem of this section.

**Theorem 3.5.** Assume that the functions \( f, g \in C_{loc}^1 (\mathbb{R}) \) satisfy for all \( \tau \in \mathbb{R} \),
\[
f (\tau) \tau \geq -c_f \tau^2 - C_f, \quad g (\tau) \tau \geq -c_g \tau^2 - C_g,
\]
for some constants \( c_f, c_g \in \mathbb{R}, C_f, C_g > 0 \). Furthermore, we assume that the first eigenvalue \( \lambda_{W, \delta, 1} \) for the eigenvalue problem
\[
A^\lambda_{W, \delta} \left( u \left|_{\partial \Omega} \right. \right) - \left( \frac{c_f u}{c_g u} \left|_{\partial \Omega} \right. \right) = \lambda \left( u \left|_{\partial \Omega} \right. \right),
\]
is positive, that is, \( \lambda_{W, \delta, 1} > 0 \). Then the following assertions hold.

(a) For every \((u_0, v_0) \in \mathcal{X}^\infty (\Omega) \), there exists a unique strong solution \( U = (u, u|_{\partial \Omega}) \) of (3.1) in the sense of Definition 3.3.
\( \|U_1(t) - U_2(t)\|_{X^2(\Omega)}^2 + \int_0^t \mathcal{A}_\Omega^\delta (U_1(\tau) - U_2(\tau), U_1(\tau) - U_2(\tau)) \, d\tau \leq C e^{Lt} \|U_1(0) - U_2(0)\|_{X^2(\Omega)}^2, \)

for \( t \geq 0 \), for some constants \( C, L > 0 \).

**Proof.** To prove the first part (a), one can proceed exactly as in the proof of [16, Theorem 3.7] with some minor modifications. In this proof, \( c > 0 \) will denote a constant that is independent of \( t, T_{\text{max}}, m, k \) and initial data, which only depends on the other structural parameters of the problem. Such a constant may vary even from line to line. Moreover, we shall denote by \( Q_\tau(m) \) a monotone nondecreasing function in \( m \) of order \( \tau \), for some nonnegative constant \( \tau \), independent of \( m \). More precisely, \( Q_\tau(m) \sim cm^\tau \) as \( m \to +\infty \). We now give some brief details. Indeed, let \((u_0, v_0) \in X^\infty(\Omega) \subset X^2(\Omega) = D(A^s_{W,\delta} X^2(\Omega)) \). By Theorem 2.12 we know that 

\(-A^s_{W,\delta} \sim \partial \Phi^s_{\Omega}, \quad \text{i.e., } A^s_{W,\delta} \text{ equals the subdifferential of the proper convex and lower-semicontinuous functional } \Phi^s_{\Omega} \text{ defined on } X^2(\Omega) \) by

\[
\Phi^s_{\Omega}(U) = \begin{cases} \frac{1}{2} A^s_{\Omega}(U, U) & \text{if } U \in D(A^s_{\Omega}), \\ +\infty & \text{if } U \in X^2(\Omega) \setminus D(A^s_{\Omega}), \end{cases}
\]

and 

\(-A^s_{W,\delta} \) generates a strongly continuous (linear) semigroup \( \{e^{-tA^s_{W,\delta}}\}_{t \geq 0} \) of contraction operators on \( X^2(\Omega) \). Finally, \( e^{-tA^s_{W,\delta}} \) is non-expansive on \( X^\infty(\Omega) \) and \( A^s_{W,\delta} \) is strongly accretive on \( X^2(\Omega) \). As usual, we can rewrite (3.1) in functional form as

\[ \partial_t U = -A^s_{W,\delta} U - F(U), \]

where we have set \( F(U) = (f(u), g(u|_{\partial\Omega})) \). The existence of locally bounded solution can now be sought as the fixed point of the following map

\[ \Sigma(U)(t) = e^{-tA^s_{W,\delta} U_0} - \int_0^t e^{-(t-\tau)A^s_{W,\delta}} F(U(\tau)) \, d\tau, \quad t \in [0, T^*]. \]

in the Banach space

\[ X_{T^*, R^*} = \left\{ U \in C([0, T^*]; X^\infty(\Omega)) : \|U(t)\|_{X^\infty(\Omega)} \leq R^* \right\}, \]

for some \( 0 < T^* \leq T \) and \( R^* > 0 \) (see, e.g., [16, Theorem 3.7]). This local solution can certainly be (uniquely) extended on a right maximal time interval \([0, T_{\text{max}}]\), with \( T_{\text{max}} > 0 \) depending on \( \|U_0\|_{X^\infty(\Omega)} \), such that, either \( T_{\text{max}} = \infty \) or \( T_{\text{max}} < \infty \), in which case \( \lim_{T \to T_{\text{max}}} \|U(t)\|_{X^\infty(\Omega)} = \infty \). The regularity of the “fixed-point” solution in Definition 3.3 can be ascertained by exploiting some regularity results for nonhomogeneous equations (see again [16, Theorem 3.7, Step 1]).

We briefly explain why we can take \( T_{\text{max}} = \infty \) because of condition (3.6). To this end, let \( m \geq 1 \) and consider the function \( E_m : (0, \infty) \to [0, \infty) \) defined by \( E_m(t) := \|U(t)\|_{X^{m+1}(\Omega)}^{m+1} \). First, notice that \( E_m \) is well-defined on \((0, T_{\text{max}}]\) because \( U \) is bounded in \( \Omega \times (0, T_{\text{max}}] \). Since \( U \) is a strong solution on \((0, T_{\text{max}}] \), the function \( E_m(t) \) is also differentiable for a.e. \( t \in (0, T_{\text{max}}] \). For strong solutions and \( t \in (0, T_{\text{max}}] \), integration by parts yields the following standard energy identity

\[
\frac{1}{2} \frac{d}{dt} E_1(t) + a^\delta_{\Omega}(U(t), U(t)) + \int_\Omega f(u(t)) u(t) \, dx + \int_{\partial\Omega} g(u(t)) u(t) \, d\sigma = 0. \]
Assumptions (3.6)-(3.7) together with the application of Gronwall’s inequality gives the following estimate for \( t \in (0, T_{\text{max}}) \),
\[
\|U(t)\|^2_{L^2(\Omega)} + \int_0^t a_\delta^\alpha (U(\tau), U(\tau)) \, d\tau \leq C \|U_0\|^2_{L^2(\Omega)} e^{-\lambda W, \epsilon_1 t} + C,
\]
for some \( C = C (f,g) > 0 \), depending on the constants from (3.6) and \(|\Omega|, \sigma(\partial \Omega)|

Next, owing to Lemma 3.4, we have
\[
\alpha
\]
and
\[
\text{standard estimates, for every } \epsilon > 0,
\]
\[
Q_1 (m_k + 1) \int_\Omega |u|^{1+m_k} \, dx \leq \varepsilon a_\Omega^\delta (|U(t)|^{\frac{m_k+1}{2}}, |U(t)|^{\frac{m_k+1}{2}})
\]
\[
+ Q_{\alpha_1} (m_k + 1) \left( \int_\Omega |u|^{1+m_k-1} \, dx \right)^2,
\]
for some \( \alpha_i = \alpha_i (\varepsilon, \epsilon) > 0 \) independent of \( k \), such that
\[
\frac{d}{dt} E_{m_k} (t) + \varepsilon a_\Omega^\delta (|U(t)|^{\frac{m_k+1}{2}}, |U(t)|^{\frac{m_k+1}{2}})
\]
\[
\leq Q_\gamma (m_k + 1) \left( \left( \int_\Omega |u|^{1+m_k-1} \, dx \right)^2 + \left( \int_{\partial \Omega} |u|^{1+m_k-1} \, d\sigma \right)^2 \right),
\]
for some positive constant \( \gamma > 0 \) independent of \( k \), for a sufficiently small \( 0 < \epsilon \leq \frac{1}{2} \).

Next, owing to Lemma 3.4, we have
\[
\varepsilon a_\Omega^\delta (|U|^{\frac{m_k+1}{2}}, |U|^{\frac{m_k+1}{2}}) \geq E_{m_k} (t) - \epsilon^{-\zeta} (E_{m_k-1})^2,
\]
for some \( \zeta > 0 \) independent of \( U, k \). We can now combine (3.17) with (3.16) to deduce
\[
\frac{d}{dt} E_{m_k} (t) + E_{m_k} (t) \leq Q_\gamma \left( 2^k \right) M_{k-1}^2,
\]
for \( t \in (0, T_{\text{max}}) \). Integrating (3.18) over \( (0, t) \), we infer from the Gronwall inequality (see, e.g., [3, Lemma 1.2.4]) that there exists yet another constant \( c > 0 \), independent of \( k \), such that
\[
M_k \leq \max \left\{ E_{m_k} (0), c 2^{k\gamma} M_{k-1}^2 \right\}, \text{ for all } k \geq 2.
\]
Iterating this inequality (see, e.g., [3, Lemma 9.3.1]), we finally obtain the estimate
\[
sup_{t \in (0,T_{\text{max}})} \|U(t)\|_{X^\infty(\Omega)} \leq e \max \left\{ \|U_0\|_{X^\infty(\Omega)}, \sup_{t \in (0,T_{\text{max}})} \|U(t)\|_{X^2(\Omega)} \right\}, \tag{3.20}
\]
which together with the energy estimate (3.11) gives \(T_{\text{max}} = +\infty\) so that a strong solution is in fact global. Having proved that the solution \(U\) is globally (and uniformly in \(T\)) bounded on \([0,T]\) for any \(T > 0\), estimate (3.8) can be proved in a standard way. This completes the proof of the theorem. \(\Box\)

**Remark 3.6.** If both \(c_f, c_g < 0\) in (3.6), then automatically \(\lambda_{\Omega, 1} > 0\). Therefore, our assumption \(\lambda_{W, 1} > 0\) is more general since it may also allow other cases in which either \(c_f > 0, c_g < 0\), \(c_f < 0, c_g > 0\) or \(c_f, c_g > 0\). Furthermore, it is worth emphasizing that even without the assumption (3.7), a global bound in \(C\left([0,T] ; X^\infty(\Omega)\right)\) (for any \(T > 0\)) can still be derived with any constants \(c_f, c_g > 0\) in (3.6), with the exception that in this case the bound is no longer uniform with respect to time and grows like \(e^{CT}\), for some \(C > 0\). The condition on the first eigenvalue of (3.7) turns out to be crucial for the attractor theory (see below).

We note that under the assumptions of Theorem 3.5, the parabolic system (3.1) defines a (nonlinear) continuous semigroup
\[
S_\delta (t) : X^\infty(\Omega) \to X^\infty(\Omega), \quad S_\delta (t) U_0 = U(t), \tag{3.21}
\]
where \(U\) is the (unique) strong solution in the sense of Definition 3.3.

As usual, the existence of a weak energy solution in the sense of Definition 3.1 follows from classical arguments. We shall only state this result without proof but refer the reader to [13, 16] for further details.

**Theorem 3.7.** Assume that the functions \(f, g \in C^1_{\text{loc}}(\mathbb{R})\) obey the rule that \(f(\tau) \tau \sim C_f |\tau|^p\), \(g(\tau) \tau \sim C_g |\tau|^q\) as \(|\tau| \to \infty\), for some \(p, q > 1\) and some \(C_f, C_g > 0\). Then, for any initial data \(U_0 \in X^2(\Omega)\), there exists at least one (globally-defined) weak solution \(U \in C\left([0,T] ; X^2(\Omega)\right)\) in the sense of Definition 3.1. Moreover, the map \(t \mapsto \|U(t)\|_{X^2(\Omega)}^2\) is absolutely continuous on \([0,T]\) and \(U\) satisfies the energy identity (3.4).

In the final part of this section, we focus on the long-term analysis for our problem (3.1). We proceed to investigate its asymptotic properties using the notion of an exponential attractor. We begin with the following.

**Definition 3.8.** A set \(G_\delta\) is an **exponential attractor** of the semigroup \(S_\delta(t)\) on \(X^\infty(\Omega)\) associated with (3.1) if

- \(G_\delta\) is compact in \(X^2(\Omega)\) and bounded in \(X^\infty(\Omega) \cap D(A_{W, \delta})\);
- \(G_\delta\) is positively invariant, that is, \(S_\delta(t)G_\delta \subseteq G_\delta \forall t \geq 0\);
- \(G_\delta\) attracts the images of all bounded subsets of \(X^\infty(\Omega)\) at an exponential rate, namely, there exists two constants \(\rho > 0, C > 0\) such that
  \[
  \text{dist}_{X^\infty(\Omega)}(S_\delta(t)B, G_\delta) \leq Ce^{-\rho t}, \quad \text{for all } t \geq 0,
  \]
  for every bounded subset \(B\) of \(X^\infty(\Omega)\). Here, \(\text{dist}_H\) denotes the standard Hausdorff semidistance between sets in a Banach space \(H\);
- \(G_\delta\) has finite fractal dimension in \(X^\infty(\Omega)\).

The next result gives the existence of such an attractor.
Theorem 3.9. Let the assumptions of Theorem 3.5 be satisfied. Then \((S_{\delta}(t), X^\infty(\Omega))\) has an exponential attractor \(\mathcal{G}_{\delta}\) in the sense of Definition 3.8.

Since the exponential attractor always contains the global attractor, as a consequence of Theorem 3.9 we immediately have the following.

**Theorem 3.10.** The semigroup \(S_{\delta}(t)\) associated with the parabolic problem (3.1) possesses a global attractor \(\mathcal{A}_{\delta}\), bounded in \(X^\infty(\Omega)\cap D(A^{1}_{W,\delta})\), compact in \(X^2(\Omega)\) and of finite fractal dimension in the \(X^\infty(\Omega)\)-topology. This attractor is generated by all complete bounded trajectories of (3.1), that is, \(\mathcal{A}_{\delta} = \mathcal{K}_{\delta} = 0\), where \(\mathcal{K}_{\delta}\) is the set of all strong solutions \(U = (u, u|_{\partial\Omega})\) which are defined for all \(t \in \mathbb{R}_+\) and bounded in the \(X^\infty(\Omega) \cap D(A^{1}_{W,\delta})\)-norm.

Our construction of an exponential attractor is based on the following abstract result [11, Proposition 4.1].

**Proposition 3.11.** Let \(\mathcal{H}, \mathcal{V}, \mathcal{V}_1\) be Banach spaces such that the embedding \(\mathcal{V}_1 \hookrightarrow \mathcal{V}\) is compact. Let \(\mathcal{B}\) be a closed bounded subset of \(\mathcal{H}\) and let \(S : \mathcal{B} \rightarrow \mathcal{B}\) be a map. Assume also that there exists a uniformly Lipschitz continuous map \(T : \mathcal{B} \rightarrow \mathcal{V}_1\), i.e.,

\[
\|Tb_1 - Tb_2\|_{\mathcal{V}_1} \leq L \|b_1 - b_2\|_{\mathcal{H}}, \quad \forall b_1, b_2 \in \mathcal{B},
\]

for some \(L \geq 0\), such that

\[
\|Sb_1 - Sb_2\|_{\mathcal{H}} \leq \gamma \|b_1 - b_2\|_{\mathcal{H}} + K \|Tb_1 - Tb_2\|_{\mathcal{V}_1}, \quad \forall b_1, b_2 \in \mathcal{B},
\]

for some constant \(0 \leq \gamma < \frac{1}{2}\) and \(K \geq 0\). Then, there exists a (discrete) exponential attractor \(\mathcal{M}_d \subset \mathcal{B}\) of the semigroup \(\{S(n) = S^n, n \in \mathbb{Z}_+\}\) with discrete time in the phase space \(\mathcal{H}\), which satisfies the following properties:

- semi-invariance: \(S(\mathcal{M}_d) \subset \mathcal{M}_d\);
- compactness: \(\mathcal{M}_d\) is compact in \(\mathcal{H}\);
- exponential attraction: \(\text{dist}_{\mathcal{H}}(S^n\mathcal{B}, \mathcal{M}_d) \leq Ce^{-\alpha n}\), for all \(n \in \mathbb{N}\) and for some \(\alpha > 0\) and \(C \geq 0\), where \(\text{dist}_{\mathcal{H}}\) denotes the standard Hausdorff semidistance between sets in \(\mathcal{H}\);
- finite-dimensionality: \(\mathcal{M}_d\) has finite fractal dimension in \(\mathcal{H}\).

**Remark 3.12.** The constants \(C\) and \(\alpha\), and the fractal dimension of \(\mathcal{M}_d\) can be explicitly expressed in terms of \(L, K, \gamma, \|\|_{\mathcal{H}}\) (and hence, in terms of the Sobolev-Poincaré constants involved in the previous Poincaré inequalities) and Kolmogorov’s \(\kappa\)-entropy of the compact embedding \(\mathcal{V}_1 \hookrightarrow \mathcal{V}\), for some \(\kappa = \kappa(L, K, \gamma)\). We recall that the Kolmogorov \(\kappa\)-entropy of the compact embedding \(\mathcal{V}_1 \hookrightarrow \mathcal{V}\) is the logarithm of the minimum number of balls of radius \(\kappa\) in \(\mathcal{V}\) necessary to cover the unit ball of \(\mathcal{V}_1\).

We will prove the main theorem by carrying first a sequence of dissipative estimates for the strong solution and then applying Proposition 3.11 to our situation at the end.

**Lemma 3.13.** Under the assumptions of Theorem 3.5, there exists a sufficiently large radius \(R > 0\) independent of time and the initial data, such that the ball

\[
\mathcal{B}_{\delta} \overset{\text{def}}{=} \left\{ U \in \mathcal{X}_{\delta} = W^{s,\delta,2}(\Omega) \cap X^\infty(\Omega) : \|U\|_{\mathcal{X}_{\delta}} \leq R \right\},
\]

is an absorbing set for \(S_{\delta}(t)\) in \(X^\infty(\Omega)\). More precisely, for any bounded set \(B \subset X^\infty(\Omega)\), there exists a time \(t_* = t_*(B) > 0\) such that \(S_{\delta}(t) B \subset \mathcal{B}_{\delta}\), for all \(t \geq t_*\).
Lemma 3.15. Let the assumptions of Theorem 3.5 hold, and let \( U_1 = (u_1, u_1|_{\partial \Omega}) \) and \( U_2 = (u_2, u_1|_{\partial \Omega}) \) be two strong solutions of (3.1) such that \( U_1(0) \in B_\delta \). Then

Proof. We claim that the existence of an absorbing set in the topology of \( X^\infty(\overline{\Omega}) \) is a consequence of the following recursive inequality for \( E_m(t) \):

\[
\sup_{t \geq t_{k-1}} E_m(t) \leq C \left( \frac{q^k}{k!} \right)^{s} \left( \sup_{\sigma \geq t_k} E_{m_{k-1}}(\sigma) \right)^2, \quad \text{for all } k \geq 1, \quad (3.24)
\]

where the sequence \( \{t_k\}_{k \in \mathbb{N}} \) is defined recursively \( t_k = t_{k-1} - \mu / 2^k, \ k \geq 1, \ t_0 = \tau' \).

Here we recall that \( C = C(\mu) > 0 \) and \( \mu > 0 \) are independent of \( k \) and that \( C(\mu) \) is uniformly bounded in \( \mu \) if \( \mu \geq 1 \). We recall that \( (3.24) \) is in fact a generic property that follows directly from (3.18) (see, e.g., [16, 13] and the references therein). Iterating in (3.24) with respect to \( k \geq 1 \), we obtain

\[
\sup_{t \geq t_{k-1}} E_m(t) \leq C \left( \frac{q^k}{k!} \right)^{s} \left( \sup_{\sigma \geq t_k} E_{m_{k-1}}(\sigma) \right)^2 \left\| U(\sigma) \right\|_{X^2(\overline{\Omega})}^{2k}. \quad (3.25)
\]

Taking the \( 2^k \)-th root on both sides of (3.25) and letting \( k \to +\infty \) allows us to obtain the claim in view of the \( X^2(\overline{\Omega}) \)-estimate (3.11) and the fact that the series in (3.25) are convergent. Indeed, the existence of an absorbing ball in the topology of \( X^2(\overline{\Omega}) \) for the semigroup \( S_\delta(\tau) \) together with (3.25) gives an absorbing ball for \( S_\delta(t) \) in the space \( X^\infty(\overline{\Omega}) \). In order to get the existence of a bounded absorbing set in \( D(a_\delta^1) = \mathcal{W}^{s,\delta,2}(\overline{\Omega}) \) we argue as follows. Testing (3.3) with \( \xi = (\partial_t u(t), \partial_t u(t)|_{\partial \Omega}) \) (note that such a test function is allowed by the regularity of the strong solution) we find

\[
\frac{d}{dt} \left( a_\Omega^\delta (U(t), U(t)) + 2 \left( \overline{f}(u(t)), 1 \right)_{L^2(\Omega)} + 2 \left( \overline{g}(u(t)), 1 \right)_{L^2(\partial \Omega)} \right) = -2 \left\| \partial_t u(t) \right\|_{L^2(\Omega)}^2 - 2 \left\| \partial_t u(t) \right\|_{L^2(\partial \Omega)}^2,
\]

for all \( t \geq t_1 \), where \( t_1 = t_1(\|B\|_{X^\infty(\overline{\Omega})}) > 0 \) is such that \( S_\delta(t) B \subset B(0, R_\delta) \) for all \( t \geq t_1 \). \( B(0, R_\delta) \) denotes a \( X^\infty(\overline{\Omega}) \)-ball of radius \( R_\delta > 0 \), centered at \( 0 \). Here and below, \( \overline{f} \) and \( \overline{g} \) denote the primitives of \( f \) and \( g \), respectively, such that \( \overline{f}(0) = \overline{g}(0) = 0 \). The application of the uniform Gronwall’s lemma together with (3.11) and the existence of an absorbing set for \( S_\delta(t) \) in the space \( X^\infty(\overline{\Omega}) \) yields the existence of a time \( t_* = t_*(B) \geq 1 \) (\( B \) is any bounded set of initial data contained in \( X^\infty(\overline{\Omega}) \)) such that

\[
\sup_{t \geq t_*} \left( \int_{t}^{t+1} \left\| \partial_t U(\tau) \right\|_{X^2(\overline{\Omega})}^2 d\tau + a_\Omega^\delta (U(t), U(t)) \right) \leq C, \quad (3.27)
\]

for some constant \( C > 0 \) independent of time and the initial data. This final estimate implies the existence of a bounded absorbing set in \( \mathcal{W}^{s,\delta,2}(\overline{\Omega}) \) and the final claim follows.

Remark 3.14. In view of Lemma 3.13, we can also exploit the regularity results from [16, Theorems 3.5-3.6] to show in fact that \( U \in L^\infty((t_*, \infty); D(A_{\mathcal{W},\delta})) \).

Next we carry some estimates for the difference of any two strong solutions, estimates which will become crucial in the final proof.

Lemma 3.15. Let the assumptions of Theorem 3.5 hold, and let \( U_1 = (u_1, u_1|_{\partial \Omega}) \) and \( U_2 = (u_2, u_1|_{\partial \Omega}) \) be two strong solutions of (3.1) such that \( U_1(0) \in B_\delta \). Then
the following estimates are valid:
\[
\|U_1(t) - U_2(t)\|_{X^2(\Omega)}^2 \leq M \|U_1(0) - U_2(0)\|_{X^2(\Omega)}^2 e^{-\omega t} + K \|U_1(t) - U_2(t)\|_{L^2([0,T];X^2(\Omega))}^2,
\]
and
\[
\| \partial_t U_1 - \partial_t U_2 \|_{L^2([0,T];X^2(\Omega))}^2 + \int_0^T \left( a_{ij}^1(U_1(\tau) - U_2(\tau), U_1(\tau) - U_2(\tau)) \right) d\tau \\
\leq C e^{LT} \|U_1(0) - U_2(0)\|_{X^2(\Omega)}^2,
\]
for some \( \omega, L > 0, M, K, C > 0, \) all independent of \( t \) and \( U_i \).

**Proof.** Recall that the injection \( D(\mathbf{a}_{ij}^1) = \mathbb{W}^{s,sl,2}_{r}(\Omega) \hookrightarrow X^2(\Omega) \) is compact and continuous. Owing to Lemma 3.13, we also have
\[
\sup_{t \geq 0} \left( \|U_i(t)\|_{X^\infty(\Omega)} + a_{ij}^1(U_i(t), U_i(t)) \right) \leq C = C \left( \|U_i(0)\|_{B_{\delta}} \right), \quad i = 1, 2. \tag{3.30}
\]
Setting \( U = U_1 - U_2 \), in light of Definition 3.1 the identity (3.3) holds for all \( \xi \in D(\mathbf{a}_{ij}^1) \), and for a.e. \( t \in (0, T) \). Choosing \( \xi = U(t) \) into (3.3) and owing to the uniform bound (3.30), we deduce
\[
\frac{d}{dt} \|U(t)\|_{X^2(\Omega)}^2 + 2a_{ij}^1(U(t), U(t)) \leq C_{f,g} \left( \|U_1(0)\|_{B_{\delta}} \right) \|U(t)\|_{X^2(\Omega)}^2
\]
for some constant \( C_{f,g} > 0 \) which depends only on \( f, g \) and on the constant from (3.30). Integrating the foregoing inequality in time entails the desired estimate (3.28) owing to the Gronwall inequality and the fact that \( \|\cdot\|_{X^2(\Omega)} \leq C \|\cdot\|_{D(\mathbf{a}_{ij}^1)} \), for some \( C > 0 \). Finally, we observe that for any test function \( \xi \in D(\mathbf{a}_{ij}^1) \) there holds
\[
\langle \partial_t U(t), \xi \rangle = -a_{ij}^1(U(t), \xi) - (F(U_1(t)) - F(U_2(t)), \xi) \leq C \|U(t)\|_{D(\mathbf{a}_{ij}^1)} \|\xi\|_{D(\mathbf{a}_{ij}^1)},
\]
since \( f, g \in C^1_{loc}(\mathbb{R}) \), owing to (3.30). This estimate together with (3.8) gives the desired control on the time derivative in (3.29).

The last ingredient we need is the uniform Hölder continuity of the time map \( t \mapsto S_t(\cdot) U_0 \) in the \( X^\infty(\Omega) \)-norm, namely,

**Lemma 3.16.** Let the assumptions of Theorem 3.5 be satisfied. Consider \( U(t) = S_t(\cdot) U_0 \) with \( U_0 \in B_{\delta} \). Then the following estimate holds:
\[
\|U(t) - U(\tau)\|_{X^\infty(\Omega)} \leq C |t - \tau|^{s}, \quad \text{for all } t, \tau \in (0, T], \tag{3.31}
\]
where \( s < 1, C > 0 \) are independent of \( t, s, U \).

**Proof.** Exploiting the bound (3.30), by comparison in (3.3), we have as in the proof of Lemma 3.15 that
\[
\int_0^T \|\partial_t U(t)\|_{L^2([0,T];X^2(\Omega))}^2 dt \leq C_T,
\]
for any \( T > 0 \). This estimate entails the inequality
\[
\|U(t) - U(\tau)\|_{L^2([0,T];X^2(\Omega))} \leq C_T |t - \tau|^{s}, \quad \text{for all } t, \tau \in [0, T]. \tag{3.32}
\]
By a duality argument, (3.32) and the uniform bound (3.30) further yield
\[
\|U(t) - U(\tau)\|_{X^2(\Omega)} \leq C_T |t - \tau|^{s}, \quad \text{for all } t, \tau \in [0, T]. \tag{3.33}
\]
Inequality (3.31) is a consequence of (3.33) and the \( X^2 \to X^\infty \) smoothing property (3.25). More precisely, the derivation of the \( X^2 \to X^\infty \) continuous dependance
estimate for the difference $U(t) - U(\tau)$ of any two strong solutions $U(t), U(\tau)$ is actually reduced to the same iteration procedure leading to (3.25) (cf. the proof of Lemma 3.13). The proof is completed. \hfill \Box

We can now finish the proof of Theorem 3.9, using the abstract scheme of Proposition 3.11.

**Proof of Theorem 3.9.** First, we construct the exponential attractor $\mathcal{E}_\delta$ of the discrete map $S_\delta(T^*)$ on $\mathcal{B}_\delta$ (the above constructed absorbing ball in $X_\delta$), for a sufficiently large $T^*$. Indeed, let $B_{1,\delta} = \cup_{t \geq T^*} S_\delta(t) \mathcal{B}_\delta$, where $\| \cdot \|_{X_2}$ denotes the closure in the space $X^2(\Omega)$ and then set $\mathbb{B}_\delta := S_\delta(1) B_{1,\delta}$. Thus, $\mathbb{B}_\delta$ is a semigroup invarient closed but also compact (for the $X^2$-metric) on $X^\infty(\Omega)$ and $S_\delta(T^*) : \mathbb{B}_\delta \rightarrow \mathbb{B}_\delta$, provided that $T^*$ is large enough. Then, we apply Proposition 3.11 on the set $\mathbb{B}_\delta$ with $H = X^2(\Omega)$ and $S = S_\delta(T^*)$, with $T^* > 0$ large enough so that $Me^{-\omega T^*} < \frac{1}{2}$ (see (3.28)). Besides, letting

$$V_1 = L^2([0,T^*]; D(a^\delta_{1,0})) \cap W^{1,2}([0,T^*]; (D(a^\delta_{1,0}))^\ast),$$

$$V = L^2([0,T^*]; X^2(\Omega)),$$

we have that $V_1 \hookrightarrow V$ is compact owing to the fact that $D(a^\delta_{1,0}) \hookrightarrow X^2(\Omega)$ is compact. Secondly, define $T: \mathbb{B}_\delta \rightarrow V_1$ to be the solving operator for (3.1) on the time interval $[0,T^*]$ such that $TU_0 := U \in V_1$, with $U(0) = U_0 \in \mathbb{B}_\delta$. Due to Lemma 3.15 and (3.29), we have the global Lipschitz continuity (3.22) of $T$ from $\mathbb{B}_\delta$ to $V_1$, and (3.28) gives us the basic estimate (3.23) for the map $S = S_\delta(T^*)$. Therefore, the assumptions of Proposition 3.11 are verified and, consequently, the map $S = S_\delta(T^*)$ possesses an exponential attractor $\mathcal{E}_\delta$ on $\mathbb{B}_\delta$. In order to construct the (continuous) exponential attractor $\mathcal{G}_\delta$ for the semigroup $S_\delta(t)$ with continuous time, we note that this semigroup is Lipschitz continuous with respect to the initial data in the topology of $X^2(\Omega)$ (in fact it is also Lipschitz continuous with respect to the metric topology of $X^\infty(\Omega)$, owing to the $X^2 \rightarrow X^\infty$ smoothing property). Moreover, by Lemma 3.16 the map $(t,U_0) \mapsto S_\delta(t)U_0$ is also uniformly Hölder continuous on $[0,T^*] \times \mathbb{B}_\delta$, where $\mathbb{B}_\delta$ is endowed with the metric topology of $X^\infty(\Omega)$. Hence, the desired exponential attractor $\mathcal{G}_\delta$ for the continuous semigroup $S_\delta(t)$ can be obtained by the standard formula

$$\mathcal{G}_\delta = \bigcup_{t \in [0,T^*]} S_\delta(t) M_\delta. \tag{3.34}$$

Finally, the finite-dimensionality of $\mathcal{G}_\delta$ in $X^\infty(\Omega)$ follows from the finite-dimensionality of $\mathcal{E}_\delta$ in $X^2(\Omega)$ and the $X^2 \rightarrow X^\infty$ smoothing property. The remaining properties of $\mathcal{G}_\delta$ are also immediate. Theorem 3.9 is now proved. \hfill \Box

4. Elliptic equations with fractional dynamic boundary conditions. In this section, we wish to investigate a simple prototype of initial-boundary value problem, of the form

$$\begin{cases}
(-\Delta)^{\alpha/2}_\Omega u = 0, & \text{in } \Omega \times (0,\infty), \\
\partial_t u + C_s \nabla^2 u + \beta(x) u + \delta B^1_t u + g(u) = 0, & \text{on } \partial \Omega \times (0,\infty), \\
u|_{t=0} = v_0, & \text{on } \partial \Omega.
\end{cases} \tag{4.1}$$

In what follows we shall exploit two different basic approaches based on semigroup and perturbation methods to derive a well-posed theory and long-term results for problem (4.1).
4.1. **A semigroup approach.** Our goal is to argue as in Section 3 for the parabolic problem (3.1) by employing instead the results stated in Proposition 2.17 for the fractional Steklov operator associated with (4.1). Indeed, letting \( \lambda = 0 \) we observe that (4.1) can be recast as a semilinear parabolic equation on \( \partial \Omega \) for the fractional Steklov operator \( \mathbb{B}_0^\delta \) as follows:

\[
\partial_t u + \mathbb{B}_0^\delta u + g(u) = 0, \quad \text{on } \partial \Omega \times (0, \infty), \quad u|_{t=0} = v_0 \quad \text{on } \partial \Omega.
\]

(4.2)

For the latter problem, we shall argue exactly as in the proof of Theorem 3.5 for the parabolic problem. For notion of (global) strong solution for (4.2) we shall use the following.

**Definition 4.1.** Let \( v_0 \in L^\infty(\partial \Omega) \) be given. A strong solution \( u \) of (4.2) fulfills the regularity properties

\[
u \in W^{1,\infty}([\eta, T]; L^2(\partial \Omega)) \cap C([0, T]; L^\infty(\partial \Omega)),
\]

\[
u \in L^2(0, T; D(\mathbb{F}_0^\delta)) \cap W^{1,2}(0, T; (D(\mathbb{F}_0^\delta))^*)
\]

such that \( u \in L^\infty([\eta, T]; D(\mathbb{F}_0^\delta)) \), for any \( T > \eta > 0 \). The equation (4.2) is satisfied in the following sense:

\[
\langle \partial_t u(t), v \rangle + \mathbb{F}_0^\delta(u(t), v) + \langle g(u(t)), v \rangle = 0, \quad \text{a.e. on } (0, T),
\]

for all \( v \in D(\mathbb{F}_0^\delta) = W^{s-\frac{1}{2}}(\partial \Omega) \cap W^{1,2}(\partial \Omega) \).

The corresponding Poincaré-Young inequality for the quadratic form \( \mathbb{F}_0^\delta \), as defined in (2.50) is as follows.

**Lemma 4.2.** Let \( s \in (1/2, 1) \), \( l \in (0, 1) \), \( \delta \in \{0, 1\} \) and let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with Lipschitz continuous boundary \( \partial \Omega \). Then for all \( \epsilon \in (0, 1) \) there is \( \zeta > 0 \) such that

\[
\|u\|_{L^2(\partial \Omega)}^2 \leq \epsilon \mathbb{F}_0^\delta(u|_{\partial \Omega}, u|_{\partial \Omega}) + \epsilon^{-\zeta} \|u\|_{L^1(\partial \Omega)}^2, \quad \text{for all } U = (u, u|_{\partial \Omega}) \in W^{s,\delta,2}(\Omega).
\]

**Proof.** The proof follows by a contradiction argument (see [16, Lemma 3.3] for further details).

The corresponding result on global existence of strong solutions is the following.

**Theorem 4.3.** Assume that \( g \in C^1_{loc}(\mathbb{R}) \) satisfies for all \( \tau \in \mathbb{R}, \)

\[
g(\tau) \tau \geq -c_g \tau^2 - C_g,
\]

(4.3)

for some constants \( c_g \in \mathbb{R}, C_g > 0 \). Furthermore, we assume that the first eigenvalue for the eigenvalue problem \( \mathbb{B}_0^\delta u|_{\partial \Omega} - c_g u|_{\partial \Omega} = \lambda u|_{\partial \Omega} \) satisfies \( \lambda_1 = \lambda_1(\mathbb{B}_0^\delta, c_g) > 0 \). Then the following assertions hold.

(a) For every \( v_0 \in L^\infty(\partial \Omega) \), there exists a unique strong solution \( u \) of (4.1) in the sense of Definition 4.1.

(b) Finally, for any two strong solutions \( u_1, u_2 \) the following estimate holds:

\[
\|u_1(t) - u_2(t)\|_{L^2(\partial \Omega)}^2 + \int_0^t \mathbb{F}_0^\delta(u_1(\tau) - u_2(\tau), u_1(\tau) - u_2(\tau)) \, d\tau \leq C e^{L t} \|u_1(0) - u_2(0)\|_{L^2(\partial \Omega)}^2,
\]

(4.4)

for \( t \geq 0 \), for some constants \( C, L > 0 \).
Proof. As in the proof of Theorem 3.5, owing to Proposition 2.17, the existence of a locally bounded (generalized) solution can be sought as a fixed point of the map
\[ \Xi(u)(t) = e^{-t\partial_0^\delta}u_0 - \int_0^t e^{-(t-\tau)\partial_0^\delta}g(u(\tau)) \, d\tau, \quad t \in [0,T^*]. \]
in the Banach space
\[ \mathcal{Y}_{T^*,R^*} = \left\{ u \in C([0,T^*];L^\infty(\partial\Omega)) : \|u(t)\|_{L^\infty(\partial\Omega)} \leq R^* \right\}, \]
for some \( 0 < T^* \leq T \) and \( R^* > 0 \) (see, e.g., [16, Theorem 3.7]). Of course, the local solution can be (uniquely) extended by continuity on a right maximal time interval \( [0,T_{\text{max}}] \), with \( T_{\text{max}} > 0 \) depending on \( \|U_0\|_{\mathcal{X}^S(\Pi)} \). To conclude that the solution \( u \) belongs to the class in Definition 4.1, we argue as in [16] by further setting \( \theta(t) := -g(u(t)) \), for \( u \in C([0,T_{\text{max}}];L^\infty(\partial\Omega)) \) and notice that \( u \) is also the “generalized” solution of
\[ \partial_t u + \partial_0^\delta u = \theta(t), \quad t \in [0,T_{\text{max}}], \tag{4.5} \]
such that \( u(0) = u_0 \in L^\infty(\partial\Omega) \subseteq L^2(\partial\Omega) = D(\partial_0^\delta) \). We can then employ [16, Theorem 3.5] to observe that the “generalized” solution \( u \) has the additional regularity \( \partial_t u \in L^2(\eta,T_{\text{max}});L^2(\partial\Omega) \), for every \( \eta > 0 \), such that together with \( g \in C^1_{\text{loc}}(\mathbb{R}) \) and \( u \in C([0,T_{\text{max}}];L^\infty(\partial\Omega)) \) it yields
\[ \theta \in W^{1,2}(\eta,T_{\text{max}});L^2(\partial\Omega) \cap L^\infty((\eta,T_{\text{max}});L^2(\partial\Omega)). \tag{4.6} \]
Thus, we can apply [16, Theorem 3.6] to deduce that
\[ u \in L^\infty((\eta,T_{\text{max}});D(\partial_0^\delta)) \cap W^{1,\infty}(\eta,T_{\text{max}});L^2(\partial\Omega), \tag{4.7} \]
such that the solution \( u \) is Lipschitz continuous on \( [\eta,T_{\text{max}}] \), for every \( \eta > 0 \). Thus, we have obtained a locally-defined strong solution in the sense of Definition 3.3.

Next, we can exploit a Moser iteration argument to prove that one can take \( T_{\text{max}} = \infty \). As usual, this iteration will be performed for the \( L^{m+1}(\partial\Omega) \)-norm of the solution \( u \) but first we notice that on \( (0,T_{\text{max}}) \), integration by parts in (4.2) yields
\[ \frac{d}{dt} \|u(t)\|_{L^2(\partial\Omega)}^2 + 2\mathcal{E}_0^\delta(u(t),u(t)) + 2\int_{\partial\Omega} g(u(t))u(t) \, d\sigma = 0. \tag{4.8} \]
The application of Gronwall’s inequality together with the assumptions of the theorem give
\[ \|u(t)\|_{L^2(\partial\Omega)}^2 + \int_0^t \mathcal{E}_0^\delta(u(\tau),u(\tau)) \, d\tau \leq C \|u_0\|_{L^2(\partial\Omega)}^2 e^{-\lambda_1 t} + C, \tag{4.9} \]
for some \( C = C(g) > 0 \), depending on the constants from (4.3) and \( \sigma(\partial\Omega) \). Furthermore, one can check that \( E_m(t) \) satisfies a local recursive relation which can be used to perform an iterative argument exactly as in [16, Theorem 3.7, Step 3]. We have, owing to [16, Lemma 3.4] the following inequality:
\[ \frac{d}{dt} \|u(t)\|_{L^{m+1}(\partial\Omega)}^{m+1} + \frac{4m}{m+1} \mathcal{E}_0^\delta(|u(t)|^{\frac{m+1}{2}},|u(t)|^{\frac{m+1}{2}}) \leq Q_1(m+1) \left( \|u(t)\|_{L^{m+1}(\partial\Omega)}^m + \|u(t)\|_{L^{m+1}(\partial\Omega)}^m \right). \tag{4.10} \]
We emphasize that (4.10) is the same inequality as that of (3.12) if we set \( E_m(t) := \|u(t)\|_{L^{m+1}(\partial\Omega)} \) in (4.10), owing to the fact that \( \mathcal{E}_0^\delta \) and \( a_0^\delta \) coincide as quadratic forms (compare (2.24) and (2.50)). Thus, one can still establish the estimate (3.15).
and the analogue of inequality (3.19) for the elliptic problem (4.1) by using instead Lemma 4.2. The iteration argument yields as usual the estimate
\[
\sup_{t \in [0,T_{\text{max}}]} \|u(t)\|_{L^\infty(\partial\Omega)} \leq \max \left\{ \|u_0\|_{L^\infty(\partial\Omega)}, \sup_{t \in [0,T_{\text{max}}]} \|u(t)\|_{L^2(\partial\Omega)} \right\}.
\]
Once we recall (4.9), the foregoing estimate easily allows us to reach the claim, i.e., the function \( u \in C([0,T]; L^\infty(\partial\Omega)) \), for any \( T > 0 \).

To prove the second part, we set \( u = u_1 - u_2 \) and observe that the strong solution \( u \) satisfies
\[
\partial_t u + \mathbb{L}_0^\partial u + g(u_1) - g(u_2) = 0, \text{ a.e. on } \partial\Omega \times [\eta,T],
\]
for any \( \eta > 0 \), with \( u(\eta) = u_1(\eta) - u_2(\eta) \in L^\infty(\partial\Omega) \). Testing this equation with \( u \) in \( L^2(\partial\Omega) \), we find
\[
\frac{d}{dt} \|u(t)\|_{L^2(\partial\Omega)}^2 + 2\mathbb{L}_0^\partial(u(t),u(t)) \leq L \|u(t)\|_{L^2(\partial\Omega)}^2,
\]
for some constant \( L > 0 \) depending only on \( g \) and the \( C([0,T]; L^\infty(\partial\Omega)) \)-norm of \( u_i \). Integrating this inequality over \((\eta,t)\), for every \( t \geq \eta \), we infer from the Gronwall inequality that
\[
\|u(t)\|_{L^2(\partial\Omega)}^2 \leq \|u(\eta)\|_{L^2(\partial\Omega)}^2 e^{Lt-\eta}.
\]
Letting \( \eta \to 0^+ \) and recalling that \( u \) is also continuous, gives
\[
\|u(t)\|_{L^2(\partial\Omega)}^2 \leq \|u(0)\|_{L^2(\partial\Omega)}^2 e^{Lt}, \text{ for any } t \geq 0.
\]
Integrating (4.11) over \((0,t)\) and exploiting once more the previous inequality, we easily arrive at the stability estimate (4.4). The proof is finished. \( \square \)

Under the full force of Theorem 4.3, the elliptic-parabolic system (4.1) defines a (nonlinear) continuous semigroup
\[
T_\delta(t) : L^\infty(\partial\Omega) \to L^\infty(\partial\Omega), \ T_\delta(t)u_0 = u(t),
\]
where \( u \) is the (unique) strong solution in the sense of Definition 4.1.

The last results concern the long-term behavior for the problem (4.1) in terms of finite dimensional global and exponential attractors.

**Theorem 4.4.** Let the assumptions of Theorem 4.3 be satisfied. Then the semigroup \((T_\delta(t), L^\infty(\partial\Omega))\) has an exponential attractor \( \mathcal{E}_\delta \) in the following sense:

(a) \( \mathcal{E}_\delta \) is compact in \( L^2(\partial\Omega) \) and bounded in \( L^\infty(\partial\Omega) \cap D(\mathbb{L}_0^\partial) \);

(b) \( \mathcal{E}_\delta \) is positively invariant, that is, \( T_\delta(t)\mathcal{E}_\delta \subseteq \mathcal{E}_\delta, \forall t \geq 0; \)

(c) \( \mathcal{E}_\delta \) attracts the images of all bounded subsets of \( L^\infty(\partial\Omega) \) at an exponential rate, namely, there exists two constants \( \rho > 0, C > 0 \) such that
\[
\text{dist}_{L^\infty(\partial\Omega)} (T_\delta(t)B, \mathcal{E}_\delta) \leq Ce^{-\rho t}, \text{ for all } t \geq 0,
\]
for every bounded subset \( B \) of \( L^\infty(\partial\Omega) \).

(d) \( \mathcal{E}_\delta \) has finite fractal dimension in \( L^\infty(\partial\Omega) \).

**Proof.** The proof follows the same scheme of the proof of Theorem 3.9 and requires only minor inessential modifications in the statements of Lemmas 3.13, 3.15 and 3.16. We leave the tedious details for the interested reader to check. \( \square \)

**Corollary 4.5.** The semigroup \( T_\delta(t) \) possesses a global attractor \( \mathbb{A}_\delta \), bounded in \( L^\infty(\partial\Omega) \cap D(\mathbb{L}_0^\partial) \), compact in \( L^2(\partial\Omega) \) and of finite fractal dimension in the \( L^\infty(\partial\Omega) \)-topology.
4.2. A perturbation method. In this subsection, inspired by [14] we wish to consider a different approach to handle the well-posedness of our elliptic-parabolic system, of that in which (4.1) can be viewed as a singular perturbation of a sequence of fully parabolic problems, of the form

\[
\begin{cases}
\varepsilon \partial_t u + (-\Delta)^s u = 0, & \text{in } \Omega \times (0, \infty), \\
\partial_t u + C_s N^{2-2s} u + \beta(x) u + \delta B_t^1 u + g(u) = 0, & \text{on } \partial \Omega \times (0, \infty), \\
u|_{t=0} = u_0 \text{ in } \Omega, \quad u|_{t=0} = v_0 \text{ on } \partial \Omega,
\end{cases}
\]  

(4.12)

where \( \varepsilon \in (0, 1) \) is a given relaxation parameter. Indeed, if we formally set \( \varepsilon = 0 \) in the first equation of (4.12), then we can easily deduce (4.1).

It turns out that (4.12) possesses a unique strong solution due to Theorem 3.5.

**Theorem 4.6.** Assume that

\[ g(\tau) \tau \geq -c_0 \tau^2 - C_g, \text{ for all } \tau \in \mathbb{R}. \]  

(4.13)

Then for any initial datum \( U_0 = (u_0, v_0) \in X^\infty(\Omega) \cap W^{s, \delta, 2}(\Omega) \) and every \( \varepsilon \in (0, 1) \), (4.12) possesses a unique strong solution in the sense of Definition 3.3. Moreover, this solution has the additional regularity properties:

\[ U \in C([0, T]; W^{s, \delta, 2}(\Omega)) \cap W^{1, 2}((0, T); X^{2}(\Omega)) \cap L^2((0, T); D(A_{W, \delta}^1)). \]  

(4.14)

In this case, the equations of (3.1) are satisfied a.e. in \( \Omega \times (0, T) \) and a.e. on \( \partial \Omega \times (0, T) \), respectively.

**Proof.** Without loss of generality, we take \( \varepsilon = 1 \). First, we note that we no longer make an assumption on the sign of \( c_0 \in \mathbb{R} \) as far as the first eigenvalue \( \lambda_{W, \delta, 1} \) of (3.7) is concerned. That is, in general it may happen that \( \lambda_{W, \delta, 1} \) could also be negative. However, this is not a restriction since estimate (3.11) still holds in that case effectively providing the bound \( U \in C([0, T]; X^{2}(\Omega)) \), for any \( T > 0 \). This bound together with (3.20) yields the desired global bound of \( U \) in \( C([0, T]; X^{\infty}(\Omega)) \) assuming that \( U_0 \in X^{\infty}(\Omega) \). If additionally \( U_0 = (u_0, v_0) \in W^{s, \delta, 2}(\Omega) \), we gain some additional control of the solution \( U \) over \([0, T]\) from the identity (3.26) (where of course, \( f \equiv 0 \)). In particular, we have

\[
\frac{d}{dt} a^1_t(U(t), U(t)) + 2 \| \partial_t u(t) \|_{X^2(\Omega)}^2 = 2 (g(u(t)), \partial_t u(t))_{L^2(\partial \Omega)} \leq \| \partial_t u(t) \|_{L^2(\partial \Omega)}^2 + L(\| u(t) \|_{L^\infty(\partial \Omega)}),
\]  

(4.15)

for some constant \( L > 0 \) depending only on \( g \) and the bound of \( u \in C([0, T]; L^\infty(\partial \Omega)) \). Integrating this inequality over \((0, t)\), for any \( t \in (0, T) \), gives

\[ U \in L^\infty((0, T); W^{s, \delta, 2}(\Omega)) \text{ and } U \in W^{1, 2}((0, T); X^{2}(\Omega)). \]

In particular, \( U \in C_w((0, T); W^{s, \delta, 2}(\Omega)) \). Next, we observe that comparison in the functional equation \( \partial_t U + A^1_{W, \delta} U + F(U) = 0 \), having set \( F(U) = (0, g(u|_{\partial \Omega})) \), also yields \( U \in L^2((0, T); D(A^1_{W, \delta})) \) owing to \( F(U) \in C([0, T]; X^2(\Omega)) \). Moreover, from this latter regularity and (4.15) we can infer that the map \( t \mapsto a^1_t(U(t), U(t)) \) is also absolutely continuous on \((0, T)\) for any \( T > 0 \), and this fact together with the weak continuity of \( U \) in \( W^{s, \delta, 2}(\Omega) \) yields the first part of (4.14). \(\square\)

Our goal then is to prove the existence of at least one strong solution to (4.1) in the sense of Definition 4.1 by passing to the limit as \( \varepsilon \to 0 \) in the parabolic system (4.12). However, this is not so straightforward since when we collapse (4.12) into (4.1) by taking \( \varepsilon = 0 \), we lose the information on the initial datum \( u_0 \) in \( \Omega \). Indeed,
(4.1) requires only knowledge of the initial value of $u(t)|_{\partial \Omega}$ at the initial time $t = 0$. It is here where the linear elliptic problem (2.7) for the fractional regional Laplacian with inhomogeneous Dirichlet data enters into the picture. By virtue of Proposition 2.5, for a given $g \in L^\infty(\partial \Omega) \cap W^{s-\frac{1}{2},2}(\Omega)$ the system

$$
\begin{align*}
(-\Delta)^s u &= 0, \quad \text{in } \Omega, \\
\left. u \right|_{\partial \Omega} &= g,
\end{align*}
$$

(4.16)

has a unique weak solution $u \in L^\infty(\Omega) \cap W^{s,2}(\Omega)$. We denote the corresponding solving operator as the map

$$
O : L^\infty(\partial \Omega) \cap W^{s-\frac{1}{2},2}(\partial \Omega) \to L^\infty(\Omega) \cap W^{s,2}(\Omega), \quad u = O(g).
$$

(4.17)

We now set $Z_0 := W^{s-\frac{1}{2},2}(\partial \Omega)$ and $Z_1 = W^{s-\frac{1}{2},2}(\partial \Omega) \cap W^{1,2}(\partial \Omega)$. Our main result in this subsection is the following.

**Theorem 4.7.** Assume (4.13) and recall that $\delta \in \{0,1\}$. Then, for any initial datum satisfying $v_0 \in Z_3 \cap L^\infty(\partial \Omega)$, the nonlinear elliptic system (4.1) possesses a unique strong solution in the sense of Definition 4.1 with the additional properties

$$
\begin{align*}
\{ u \in C([0,T]; Z_\delta \cap L^\infty(\partial \Omega)), & \quad U \in L^\infty((0,T); W^{s,\delta,2}(\Omega)), \\
\partial_t u \in L^2((0,T) \times \partial \Omega), & \quad U \in L^2((0,T); D(A_{W,\delta}^s)).
\}
\end{align*}
$$

(4.18)

**Proof.** Step 1. We reconstruct an initial datum $u_0$ for (4.12) from the given $v_0 \in L^\infty(\partial \Omega) \cap Z_\delta$, $\delta \in \{0,1\}$. It is reconstructed by $u_0 = O(v_0)$, see (4.17). Observe that owing to Proposition 2.5 we have

$$
\| (u_0,v_0) \|_{W^{s,\delta,2}(\Omega)} \leq C(\| v_0 \|_{Z_\delta}) \quad \text{and} \quad \| (u_0,v_0) \|_{X^\infty(\Omega)} \leq C(\| v_0 \|_{L^\infty(\partial \Omega)}),
$$

(4.19)

for some constant $C > 0$ which is independent of $\varepsilon > 0$ and the initial datum $v_0$. Therefore, for each initial datum $U_0 = (u_0,v_0) \in X^\infty(\Omega) \cap W^{s,\delta,2}(\Omega)$ constructed this way we can infer from Theorem 4.6 that there exists a unique strong solution $u_\varepsilon(t) = (u_\varepsilon,v_\varepsilon|_{\partial \Omega})$, $t \in (0,T)$, to the parabolic problem (4.12) for any $T > 0$.

**Step 2.** We aim to provide sufficiently strong estimates for $U_\varepsilon$ that are also uniform in $\varepsilon \in (0,1]$. We claim that by virtue of assumption (4.13), the following estimates hold:

$$
sup_{t \in (0,T)} \| u_\varepsilon(t) \|_{L^\infty(\partial \Omega)} \leq C(\| v_0 \|_{L^\infty(\partial \Omega)}),
$$

(4.20)

and

$$
sup_{t \in (0,T)} a_{W,\delta}^s(U_\varepsilon, U_\varepsilon(t)) + \int_0^T \left( \| \partial_t u_\varepsilon(\tau) \|_{L^2(\partial \Omega)}^2 + \varepsilon \| \partial_t u_\varepsilon(\tau) \|_{L^2(\Omega)}^2 \right) d\tau \leq C(\| v_0 \|_{L^\infty(\partial \Omega) \cap Z_\delta}),
$$

(4.21)

for some function $C > 0$ which is independent of $\varepsilon$ and the initial data. To show (4.20), we slightly modify the argument in the proof of Theorem 3.5 by emphasizing the explicit independence of the various constants in $\varepsilon > 0$. For each $\varepsilon > 0$, we naturally define

$$
E_{k,\varepsilon}(t) := \varepsilon \int_{\Omega} |u_\varepsilon(t)|^{m_k+1} dx + \int_{\partial \Omega} |u_\varepsilon(t)|^{m_k+1} d\sigma,
$$

where
where the sequence \( \{ m_k = 2^k - 1 \}_{k \in \mathbb{N}} \). We observe that in this case the assumption (4.13) yields the following analogue of inequality (3.12) for problem (4.12):

\[
\frac{d}{dt} E_{k,\varepsilon} (t) + \frac{4m_k}{m_k + 1} \mathcal{M} (t) \left( |U_{\varepsilon} (t)|^{\frac{m_k + 1}{2}}, |U_{\varepsilon} (t)|^{\frac{m_k + 1}{2}} \right)
\]

\[
\leq Q_1 (m_k + 1) \left( \int_{\partial \Omega} |u_{\varepsilon} (t)|^{m_k + 1} \, d\sigma \right) + \left( \int_{\partial \Omega} |u_{\varepsilon} (t)|^{m_k + 1} \, d\sigma \right)^{\frac{m_k}{m_k + 1}}.
\]

As usual, with the same definition (3.13) for \( M_{k,\varepsilon} = \sup_{t \in (0, T)} E_{k,\varepsilon} (t) \), we still have estimate (3.15) for some sufficiently small \( \tilde{\varepsilon} > 0 \) (independent of \( \varepsilon > 0 \)) and therefore,

\[
\frac{d}{dt} E_{k,\varepsilon} (t) + \frac{\varepsilon}{2} \mathcal{F}_0^k (|U_{\varepsilon} (t)|^{\frac{m_k + 1}{2}}, |U_{\varepsilon} (t)|^{\frac{m_k + 1}{2}})
\]

\[
\leq Q_\gamma (m_k + 1) \left( \int_{\partial \Omega} |u|^ {1 + m_k - 1} \, d\sigma \right)^2,
\]

for some positive constant \( \gamma > 0 \) independent of \( k, \varepsilon \), for a sufficiently small \( 0 < \varepsilon \leq \tilde{\varepsilon} \) (here, we have also exploited the fact that \( \mathcal{F}_0^k \) and \( \mathcal{A}_1^k \) coincide as quadratic forms). Thus, arguing in the same fashion as in the proof of Theorem 3.5, by employing Lemma 4.2 and the Gronwall’s inequality, we eventually arrive at the estimate:

\[
M_{k,\varepsilon} \leq \max \{ E_{k,\varepsilon} (0), c 2^{k\gamma} M_{k-1,\varepsilon}^2 \}, \quad \text{for all } k \geq 2,
\]

for some constant \( c > 0 \), independent of \( k \) and \( \varepsilon > 0 \). On the other hand, freezing \( \varepsilon \in (0, 1] \) and noting that

\[
( E_{k,\varepsilon} (0) )^{1/2k} \leq C (\| v_0 \|_{L^\infty (\Omega)} + \| v_0 \|_{L^\infty (\partial \Omega)} ) \leq C (\| v_0 \|_{L^\infty (\partial \Omega)} )
\]

owing to (4.19), we obtain by iterating in (4.23), taking the \( 1/2^k \)-root and passing to the limit as \( k \to \infty \) that

\[
\sup_{t \in (0, T)} \| u (t) \|_{L^\infty (\partial \Omega)} \leq \lim_{k \to \infty} \sup_{t \in (0, T)} E_{k,\varepsilon} (t)^{1/2k} \leq c \max \left\{ \| v_0 \|_{L^\infty (\partial \Omega)}, \sup_{t \in (0, T)} (E_{1,\varepsilon} (t)^{1/2} \right\}.
\]

In order to show that \( E_{1,\varepsilon} \in L^\infty (0, T) \) uniformly in \( \varepsilon > 0 \) we test both equations of (4.12) with \( u_{\varepsilon} \) itself and get

\[
\frac{d}{dt} \left( \varepsilon \| u_{\varepsilon} (t) \|_{L^2 (\Omega)}^2 + \| u_{\varepsilon} (t) \|_{L^2 (\partial \Omega)}^2 \right) + 2 \mathcal{A}_1^k (|U_{\varepsilon} (t)|^{\frac{m_k + 1}{2}}, |U_{\varepsilon} (t)|^{\frac{m_k + 1}{2}})
\]

\[
= 2 (g (u_{\varepsilon} (t)), u_{\varepsilon} (t))_{L^2 (\partial \Omega)}.
\]

Assumption (4.13) allows us to deduce

\[
\frac{d}{dt} \left( \varepsilon \| u_{\varepsilon} (t) \|_{L^2 (\Omega)}^2 + \| u_{\varepsilon} (t) \|_{L^2 (\partial \Omega)}^2 \right) \leq C \left( \| u_{\varepsilon} (t) \|_{L^2 (\partial \Omega)}^2 + 1 \right),
\]

for some \( C > 0 \) independent of \( \varepsilon \) and \( t \), as well as the initial data. Thus, by Gronwall’s inequality,

\[
E_{1,\varepsilon} (t) = \varepsilon \| u_{\varepsilon} (t) \|_{L^2 (\Omega)}^2 + \| u_{\varepsilon} (t) \|_{L^2 (\partial \Omega)}^2 \leq (tC + E_{1,\varepsilon} (0)) e^{Ct}, \quad t \in (0, T).
\]

Owing once more to (4.19), we have \( E_{1,\varepsilon} (0) \leq C (\| v_0 \|_{L^\infty (\partial \Omega)} ) \) uniformly in \( \varepsilon \in (0, 1] \). The desired inequality (4.20) follows then by combining this estimate with (4.24). The proof of (4.21) is standard especially now that we have the first uniform
compactness theorem we have

\[ E_\varepsilon (t) := a_0^{\delta} (U_\varepsilon (t), U_\varepsilon (t)) + 2 (g (u_\varepsilon (t)), 1)_{L^1 (\partial \Omega)} + C_\varepsilon , \]

we obtain

\[ 2 \varepsilon \| \partial_t u_\varepsilon \|_{L^2 (\Omega)} + 2 \| \partial_t u_\varepsilon \|_{L^2 (\partial \Omega)} + \partial_t E_\varepsilon (t) = 0, \text{ a.e. } t \in (0, T) . \quad (4.27) \]

Here the constant \( C_\varepsilon > 0 \) is taken large enough independent of \( \varepsilon > 0 \), depending only on the initial data \( v_0 \), in order to ensure that \( E_\varepsilon (t) \) is nonnegative (recall that \( g (t) \) is uniformly bounded for \( |\tau| \leq r \) and by \( (4.20) \)). Furthermore, one can easily check

\[
\| U_\varepsilon (t) \|_{W^{2,1;2} (\Omega)} - L (\| v_0 \|_{L^\infty (\partial \Omega)}) + C_\varepsilon \leq E_\varepsilon (t)
\]

\[
\leq \| U_\varepsilon (t) \|_{W^{2,1;2} (\Omega)} + L (\| v_0 \|_{L^\infty (\partial \Omega)}),
\]

for some positive function \( L \) independent of \( \varepsilon \). Integrating \((4.27)\) over time with \( t \in (0, T) \), then exploiting \((4.20)\) together with the fact that \( E_{1, \varepsilon} \in L^\infty (0, T) \) and \((4.19)\) yields the desired bound in \((4.21)\).

We now exploit the preceding results in order to derive additional uniform estimates for the solutions of \((4.12)\) as \( \varepsilon \to 0 \). First, a comparison in \((4.12)\) for every \( \varepsilon \leq 1 \) shows that

\[
\int_0^T \| A_{W, \delta} U_\varepsilon (\tau) \|^2_{X^2 (\Omega)} d\tau \leq C (\| v_0 \|_{L^\infty (\partial \Omega)}),
\]

on account of estimates \((4.20)-(4.21)\). This implies in particular that \(-\Delta u_\varepsilon \in L^2 ((0, T); L^2 (\Omega))\) uniformly with respect to \( \varepsilon > 0 \). Finally, owing to the Green identity \((2.18)\), \((4.28)\) and \((4.21)\), we obtain that also \( C_s N^{2-2s} u_\varepsilon \in L^2 ((0, T); L^2 (\partial \Omega)) \) and \( \delta (-\Delta) u_\varepsilon \in L^2 ((0, T); L^2 (\partial \Omega)) \), both uniformly with respect to \( \varepsilon > 0 \).

**Step 3.** We are now ready to pass to the limit, as \( \varepsilon \to 0 \), in the parabolic problem \((4.12)\), using the uniform estimates \((4.28)\) and \((4.20)-(4.21)\). Indeed, on account of these uniform inequalities, we can find \( u \) such that, up to subsequences,

\[
\begin{align*}
  u_\varepsilon & \to u && \text{weakly-* in } L^\infty ((0, T); L^\infty (\partial \Omega)) , \\
  U_\varepsilon & \to U && \text{weakly-* in } L^\infty ((0, T); W^{2,1;2} (\Omega)) , \\
  \partial_t u_\varepsilon & \to \partial_t u && \text{weakly in } L^2 ((0, T) \times \partial \Omega) , \\
  \varepsilon \partial_t u_\varepsilon & \to 0 && \text{strongly in } L^2 ((0, T) \times \Omega) , \\
  (-\Delta)^{\delta}_0 u_\varepsilon & \to (-\Delta)^{\delta}_0 u && \text{weakly in } L^2 ((0, T); L^2 (\Omega)) , \\
  C_s N^{2-2s} u_\varepsilon & \to C_s N^{2-2s} u && \text{weakly in } L^2 ((0, T); L^2 (\partial \Omega)) , \\
  \delta (-\Delta)^{\varepsilon}_1 u_\varepsilon & \to \delta (-\Delta)^{\varepsilon}_1 u && \text{weakly in } L^2 ((0, T); L^2 (\partial \Omega)) .
\end{align*}
\]

Note that the second and third convergences of \((4.29)\) imply that \( u \in C ([0, T]; L^2 (\partial \Omega)) \) such that \( u (0) = v_0 \) a.e. on \( \partial \Omega \). Also by the classical Aubin-Lions-Simon compactness theorem we have

\[
u_\varepsilon \to u \quad \text{strongly in } L^2 ((0, T); L^2 (\partial \Omega)) , \quad (4.30)
\]

which is enough to pass to the limit in the nonlinear boundary term. More precisely, using the fact that \( g \in C^1 \), we have

\[
g (u_\varepsilon) \to g (u) \quad \text{strongly in } L^2 ((0, T); L^2 (\partial \Omega)) , \quad (4.31)
\]
5. Blow-up results. The goal in this section is to check that our assumptions on \( f, g \) in the previous sections, which ascertain the global well-posedness of strong solutions of both problems (3.1) and (4.1), respectively, are in fact optimal. We observe for instance that assumption (3.6) implies that if \( f \) and \( g \) are sources with a bad sign at infinity then they can only be of at most linear growth at infinity (i.e., if \( f(\tau) \sim -c_f|\tau|^{p-2}\tau, \ g(\tau) \sim -c_g|\tau|^{q-2}\tau \), as \( |\tau| \to \infty \) for some \( p, q > 2 \) then we necessarily must have \( c_f < 0, c_g < 0 \), exactly as in the classical case (see, however, Remark 5.4 below). Indeed, following the well-known concavity method of Levine and Payne we can easily show that as soon as both \( f \) and \( g \) have superlinear growth and a bad sign at infinity, then blow up in finite time of some strong solutions for (3.1) will occur. We establish a similar result for the elliptic-parabolic system (4.1).

We first recall the following local result which is a consequence of results proven in the previous sections.

**Proposition 5.1.** Let \( f, g \in C^1_{\text{loc}}(\mathbb{R}) \) and assume that \( U_0 \in W^{s,\delta,2}(\bar{\Omega}) \cap X^\infty(\bar{\Omega}) \), \( s \in (1/2, 1) \), \( l \in (0, 1) \) and \( \delta \in \{0, 1\} \). Then (3.1) possesses a unique maximal (strong) solution in the sense of Definition 3.3 such that also

\[
U \in C([0, T_{\max}); W^{s,\delta,2}(\bar{\Omega})) \cap W^{1,2}(0, T_{\max}) \cap L^2(0, T_{\max}); D(A^s_{W,\delta}).
\]

**Theorem 5.2.** Let \( U \) be a solution of (3.1) in the sense of Proposition 5.1 and recall that \( \beta \in L^\infty(\partial \Omega) \) with \( \beta \geq 0 \). Let \( h \) be a continuous, convex and positive function on \((0, \infty)\) such that either one of the following conditions holds:

(a) We have \(-f(\tau) \geq -g(\tau) + \|\beta\|_\infty \tau \geq h(\tau) > 0\), for all \( \tau \geq \tau_0 \) (for some \( \tau_0 > 0 \)) and

\[
\int_{\tau_0}^{\infty} \frac{1}{h(\tau)} d\tau < \infty. \tag{5.1}
\]

(b) We have \(-g(\tau) + \|\beta\|_\infty \tau \geq -f(\tau) \geq h(\tau) > 0\), for all \( \tau \geq \tau_0 \) (for some \( \tau_0 > 0 \)) and (5.1) holds.

In either case (a) or (b), the solution \( U \) must blow-up in finite time.

**Remark 5.3.** Note that the function \( h(\tau) = c_h|\tau|^{r-2}\tau, \ c_h > 0 \), for some \( r > 2 \), obeys (5.1).

**Proof.** Our proof is inspired from \([23]\). We prove the claim in the case (a), the other is quite similar. Let \( U = (u, u|_{\partial \Omega}) \) be a solution of (3.1) and consider the function

\[
V(t) := \frac{1}{\mu(\bar{\Omega})} \left( \int_\Omega U(t) \, d\mu \right) = \frac{1}{\mu(\bar{\Omega})} \left( \int_\Omega u(t) \, dx + \int_{\partial \Omega} u(t) \, d\sigma \right),
\]

thanks to (4.30), the first convergence of (4.29) and estimate (4.20). By means of the above convergence properties (4.29), (4.31), we can now pass to the limit in both equations of (4.12) to deduce a function \( U = (u, u|_{\partial \Omega}) \) which solves the elliptic-parabolic system (4.1). Moreover, due to the arbitrariness of \( T > 0 \), passing to limit as \( \varepsilon \to 0 \) in (4.28) and (4.20)-(4.21), and recalling (4.29), we also deduce that the limit (strong) solution \( U \) satisfies these inequalities with a constant \( C > 0 \) independent of \( \varepsilon > 0 \). The continuity of \( u \) in \( C([0, T]; Z_\delta \cap L^\infty(\partial \Omega)) \) is proved exactly in the same fashion as in the proof of Theorem 4.6. Finally, the uniqueness of the strong solution to the elliptic-parabolic problem (4.1) follows from (4.4). The proof is finished.
for \( t > 0 \), where we have set \( \mu (\Omega) = |\Omega| + \sigma (\partial \Omega) \). Integrating both equations of (3.1) over \( \Omega \) and \( \partial \Omega \), respectively and adding the corresponding identities, we find

\[
V (t) = -\frac{1}{\mu (\Omega)} a(x) \delta (\Omega) (U(t), 1) - \frac{1}{\mu (\Omega)} \left( \int_{\Omega} f (u(t)) \, dx + \int_{\partial \Omega} g (u(t)) \, d\sigma \right).
\]

The Green identity (2.18) together with the definition (2.24) gives

\[
V (t) = -\frac{1}{\mu (\Omega)} \left( \int_{\Omega} f (u(t)) \, dx + \int_{\partial \Omega} g (u(t)) + \beta (x) u(t) \, d\sigma \right),
\]

for \( t > 0 \) and for as long the strong solution \( U \) exists. From (a) and the Jensen’s inequality, we obtain

\[
V (t) \geq \frac{1}{\mu (\Omega)} \left( \int_{\Omega} h (U(t)) \, d\mu \right) \geq h (V (t)), \quad t > 0
\]

for as along the strong solution exists. Since \( h > 0 \) this implies as usual that

\[
\int_{V (0)}^{V (t)} \frac{1}{h (x)} \, dx \geq t;
\]

hence, by (5.1) it follows that

\[
t \leq \int_{V (0)}^{\infty} \frac{1}{h (x)} \, dx < \infty \tag{5.2}
\]

and therefore if \( U (t) \) is a global solution for problem (3.1), (5.2) leads to a contradiction since the left-hand side of (5.2) goes to infinity as time goes to infinity. The proof is finished.

**Remark 5.4.** Note that this result does not say anything in the case when \( g \) is dissipative and of good sign at infinity, but \( f \) still has a bad sign at infinity. Indeed, for the standard reaction-diffusion system associated with the classical diffusion \( \Delta \), blow-up in finite time of some strong solutions actually holds (see [13, Section 3]). On the other hand, it was also proven in [13] for the same standard system that global boundedness of solutions still holds provided that \( g \) is of supercritical growth and of bad sign at infinity such that \( f \) is a “good” dissipative source whose role is to absorb the “bad” boundary reaction. Unfortunately, such questions remain open for the system (3.1).

The corresponding result for the elliptic-parabolic system (4.1) is proved in the same fashion.

**Theorem 5.5.** Let \( u \) be a maximal solution on \((0, T_{\max})\) of (4.1) in the sense of Definition 4.1, satisfying also (4.18) on \((0, T_{\max})\), for some \( v_0 \in Z_{\delta} \cap L^{\infty} (\partial \Omega) \). Let \( h \) be a continuous, convex and positive function on \((0, \infty)\) such that

\[
-g (\tau) + ||\beta||_{\infty} \tau \geq h (\tau) > 0, \quad \text{for all } \tau \geq \tau_0 \text{ (for some } \tau_0 > 0) \text{ and (5.1) holds. Then the solution } u \text{ must blow-up in finite time.}
\]

Combining Theorem 5.5 with Theorem 4.6 yields the following.

**Theorem 5.6.** Assume that \( g (s) \sim c_g |s|^{r-2} s \text{ as } |s| \to \infty \text{ for some } c_g \in \mathbb{R} \text{ and } r > 1 \). Then for all \( v_0 \in Z_{\delta} \cap L^{\infty} (\partial \Omega) \), problem (4.1) has a unique global strong solution if and only if either \( c_g \geq 0 \) or \( r \leq 2 \).
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