PARALLEL*-RICCI TENSOR OF REAL HYPERSURFACES
IN $\mathbb{C}P^2$ AND $\mathbb{C}H^2$

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Abstract. In this paper the idea of studying real hypersurfaces in non-flat complex space forms, whose *-Ricci tensor satisfies geometric conditions is presented. More precisely, three dimensional real hypersurfaces in non-flat complex space forms with parallel *-Ricci tensor are studied. At the end of the paper ideas for further research on *-Ricci tensor are given.

1. INTRODUCTION

A complex space form is an $n$-dimensional Kaehler manifold of constant holomorphic sectional curvature $c$. A complete and simply connected complex space form is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n$ if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$,
- or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$.

The symbol $M_n(c)$ is used to denote the complex projective space $\mathbb{C}P^n$ and complex hyperbolic space $\mathbb{C}H^n$, when it is not necessary to distinguish them. Furthermore, since $c \neq 0$ in previous two cases the notion of non-flat complex space form refers to both them.

Let $M$ be a real hypersurface in a non-flat complex space form. An almost contact metric structure $(\varphi, \xi, \eta, g)$ is defined on $M$ induced from the Kaehler metric $G$ and the complex structure $J$ on $M_n(c)$. The structure vector field $\xi$ is called principal if $A\xi = \alpha\xi$, where $A$ is the shape operator of $M$ and $\alpha = \eta(A\xi)$ is a smooth function. A real hypersurface is called Hopf hypersurface, if $\xi$ is principal and $\alpha$ is called Hopf principal curvature.
The Ricci tensor $S$ of a Riemannian manifold is a tensor field of type (1,1) and is given by
\[ g(SX, Y) = \text{trace}\{Z \mapsto R(Z, X)Y\}. \]
If the Ricci tensor of a Riemannian manifold satisfies the relation
\[ S = \lambda g, \]
where $\lambda$ is a constant, then it is called Einstein.

Real hypersurfaces in non-flat complex space forms have been studied in terms of their Ricci tensor $S$, when it satisfies certain geometric conditions extensively. Different types of parallelism or invariance of the Ricci tensor are issues of great importance in the study of real hypersurfaces.

In [4] it was proved the non-existence of real hypersurfaces in non-flat complex space forms $M_n(c)$, $n \geq 3$ with parallel Ricci tensor, i.e. $(\nabla_X S)Y = 0$, for any $X$, $Y \in TM$. In [5] Kim extended the result of non-existence of real hypersurfaces with parallel Ricci tensor in case of three dimensional real hypersurfaces. Another type of parallelism which was studied is that of $\xi$-parallel Ricci tensor, i.e. $(\nabla_\xi S)Y = 0$ for any $Y \in TM$. More precisely in [6] Hopf hypersurfaces in non-flat complex space forms with constant mean curvature and $\xi$-parallel Ricci tensor were classified. More details on the study of Ricci tensor of real hypersurfaces are included in Section 6 of [7].

Motivated by Tachibana, who in [9] introduced the notion of $^\ast$-Ricci tensor on almost Hermitian manifolds, in [2] Hamada defined the $^\ast$-Ricci tensor of real hypersurfaces in non-flat complex space forms by
\[ g(S^\ast X, Y) = \frac{1}{2}(\text{trace}\{\varphi \circ R(X, \varphi Y)\}), \quad \text{for } X, Y \in TM. \]

The $^\ast$-Ricci tensor $S^\ast$ is a tensor field of type (1,1) defined on real hypersurfaces. Taking into account the work that so far has been done in the area of studying real hypersurfaces in non-flat complex space forms in terms of their tensor fields, the following issue raises naturally:

The study of real hypersurfaces in terms of their $^\ast$-Ricci tensor $S^\ast$, when it satisfies certain geometric conditions.

In this paper three dimensional real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ equipped with parallel $^\ast$-Ricci tensor are studied. Therefore, the following condition is satisfied
\[ (\nabla_X S^\ast)Y = 0, \quad X, Y \in TM. \quad (1.1) \]

More precisely the following Theorem is proved.

**Main Theorem.** There do not exist real hypersurfaces in $\mathbb{C}P^2$, whose $^\ast$-Ricci tensor is parallel. In $\mathbb{C}H^2$ only the geodesic hypersphere has parallel $^\ast$-Ricci tensor with $\coth(r) = 2$. 

The paper is organized as follows: In Section 2 preliminaries relations for real hypersurfaces in non-flat complex space forms are presented. In Section 3 the proof of Main Theorem is provided. Finally, in Section 4 ideas for further research on the above issue are included.

2. Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class $C^\infty$ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces $M$ are supposed to be without boundary.

Let $M$ be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure $J$ of constant holomorphic sectional curvature $c$. Let $N$ be a locally defined unit normal vector field on $M$ and $\xi = -JN$ the structure vector field of $M$.

For a vector field $X$ tangent to $M$ the following relation holds $$JX = \varphi X + \eta(X)N,$$
where $\varphi X$ and $\eta(X)N$ are the tangential and the normal component of $JX$ respectively.

The Riemannian connections $\nabla$ in $M_n(c)$ and $\nabla$ in $M$ are related for any vector fields $X, Y$ on $M$ by
$$\nabla_X Y = \nabla_X Y + g(AX, Y)N,$$
where $g$ is the Riemannian metric induced from the metric $G$.

The shape operator $A$ of the real hypersurface $M$ in $M_n(c)$ with respect to $N$ is given by $$\nabla_X N = -AX.$$

The real hypersurface $M$ has an almost contact metric structure $(\varphi, \xi, \eta, g)$ induced from the complex structure $J$ on $M_n(c)$, where $\varphi$ is the structure tensor and it is a tensor field of type (1,1). Moreover, $\eta$ is an 1-form on $M$ such that
$$g(\varphi X, Y) = G(JX, Y), \quad \eta(X) = g(X, \xi) = G(JX, N).$$

Furthermore, the following relations hold
$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0, \quad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y).$$

Since $J$ is complex structure implies $\nabla J = 0$. The last relation leads to

$$\nabla_X \xi = \varphi AX, \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi.$$
The ambient space $M_n(c)$ is of constant holomorphic sectional curvature $c$ and this results in the Gauss and Codazzi equations to be given respectively by

$$
R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X
-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z]
+g(AY, Z)AX - g(AX, Z)AY,
$$

\[(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi]\]

where $R$ denotes the Riemannian curvature tensor on $M$ and $X, Y, Z$ are any vector fields on $M$.

The tangent space $T_PM$ at every point $P \in M$ can be decomposed as

$$
T_PM = \text{span}\{\xi\} \oplus \mathbb{D},
$$

where $\mathbb{D} = \text{ker}\eta = \{X \in T_PM : \eta(X) = 0\}$ and is called holomorphic distribution. Due to the above decomposition the vector field $A\xi$ can be written

$$
A\xi = \alpha\xi + \beta U,\tag{2.3}
$$

where $\beta = \|\varphi \nabla\xi\|$ and $U = -\frac{1}{\beta}\varphi \nabla\xi \in \text{ker}(\eta)$ provided that $\beta \neq 0$.

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature $c$ following similar calculations to those in Theorem 2 in [3] and taking into account relation (2.2), it is proved that the *-Ricci tensor $S^*$ of $M$ is given by

$$
S^* = -\left[\frac{cn}{2}\varphi^2 + (\varphi A)^2\right].\tag{2.4}
$$

3. Proof of Main Theorem

Let $M$ be a non-Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, i.e. $M_2(c)$. Then the following relations hold on every non-Hopf three-dimensional real hypersurface in $M_2(c)$.

**Lemma 3.1.** Let $M$ be a real hypersurface in $M_2(c)$. Then the following relations hold on $M$

\begin{align}
(3.1) & \quad AU = \gamma U + \delta \varphi U + \beta \xi, \quad A\varphi U = \delta U + \mu \varphi U, \\
(3.2) & \quad \nabla_U \xi = -\delta U + \gamma \varphi U, \quad \nabla_{\varphi U} \xi = -\mu U + \delta \varphi U, \quad \nabla_\xi \varphi U = \beta \varphi U, \\
(3.3) & \quad \nabla_U \varphi U = \kappa_1 \varphi U + \delta \xi, \quad \nabla_{\varphi U} \varphi U = \kappa_2 \varphi U + \mu \xi, \quad \nabla_\xi U = \kappa_3 \varphi U, \\
(3.4) & \quad \nabla_U \varphi U = -\kappa_1 U - \gamma \xi, \quad \nabla_{\varphi U} \varphi U = -\kappa_2 U - \delta \xi, \quad \nabla_\xi \varphi U = -\kappa_3 U - \beta \xi,
\end{align}

where $\gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on $M$ and $\{U, \varphi U, \xi\}$ is an orthonormal basis of $M$. 
For the proof of the above Lemma see [8].

Let $M$ be a real hypersurface in $M_2(c)$, i.e. $\mathbb{C}P^2$ or $\mathbb{C}H^2$, whose $*$-Ricci tensor satisfies relation (1.1), which is more analytically written

\begin{equation}
\nabla_X (S^*Y) = S^*(\nabla_X Y), \quad X, Y \in TM.
\end{equation}

We consider the open subset $\mathcal{N}$ of $M$ such that

$\mathcal{N} = \{ P \in M : \beta \neq 0, \text{ in a neighborhood of } P \}$. In what follows we work on the open subset $\mathcal{N}$.

On $\mathcal{N}$ relation (2.3) and relations (3.1)-(3.4) of Lemma 3.1 hold. So relation (2.4) for $X \in \{U, \varphi U, \xi\}$ taking into account $n = 2$ and relations (2.3) and (3.1) yields

\begin{equation}
\begin{aligned}
S^*\xi &= \beta \mu U - \beta \delta \varphi U, \\
S^*U &= (c + \gamma \mu - \delta^2)U \quad \text{and} \\
S^*\varphi U &= (c + \gamma \mu - \delta^2)\varphi U.
\end{aligned}
\end{equation}

The inner product of relation (3.5) for $X = Y = \xi$ with $\xi$ due to the first and the third of (3.6), the first of (2.1) for $X = \xi$ and the third of relations (3.3) and (3.4) implies

\begin{equation}
\delta = 0.
\end{equation}

Moreover, the inner product of relation (3.5) for $X = \varphi U$ and $Y = \xi$ with $\xi$ because of (3.7), the first of (2.1) for $X = \varphi U$, the first and the second of (3.6) and the second of (3.3) results in

\begin{equation}
\mu = 0.
\end{equation}

Finally, the inner product of relation (3.5) for $X = \xi$ and $Y = \varphi U$ with $\xi$ taking into account $\mu = \delta = 0$, the first and the third of (3.6) and the last relation of (3.4) leads to

\begin{equation}
c = 0,
\end{equation}

which is a contradiction. So the open subset $\mathcal{N}$ is empty and we lead to the following Proposition.

**Proposition 3.2.** Every real hypersurface in $M_2(c)$ whose $*$-Ricci tensor is parallel, is a Hopf hypersurface.

Since $M$ is a Hopf hypersurface, the structure vector field $\xi$ is an eigenvector of the shape operator, i.e. $A\xi = \alpha \xi$. Due to Theorem 2.1 in [7] $\alpha$ is constant. We consider a point $P \in M$ and choose a unit principal vector field $W \in D$ at $P$, such that $AW = \lambda W$ and $A\varphi W = \nu \varphi W$. Then $\{W, \varphi W, \xi\}$ is a local orthonormal basis and the following relation holds (Corollary 2.3 [7])

\begin{equation}
\lambda \nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}.
\end{equation}
The first of relation (2.1) and relation (2.4) for $X \in \{W, \varphi W, \xi\}$ because of $A\xi = \alpha \xi$, $AW = \lambda W$ and $A\varphi W = \nu \varphi W$ implies respectively
\[
\nabla_W \xi = \lambda \varphi W \quad \text{and} \quad \nabla_{\varphi W} \xi = -\nu W
\]
(3.9)
\[
S^* \xi = 0, \quad S^* W = (c + \lambda \nu)W \quad \text{and} \quad S^* \varphi W = (c + \lambda \nu)\varphi W.
\]
(3.10)
Relation (3.5) for $X = W$ and $Y = \xi$ because of the first of (3.9) and the first and third relation of (3.10) yields
\[
\lambda (c + \lambda \nu) = 0.
\]
Suppose that $(c + \lambda \nu) \neq 0$ then the above relation results in $\lambda = 0$. Moreover, relation (3.5) for $X = \varphi W$ and $Y = \xi$ because of the second of (3.9) and the first and second relation of (3.10) yields
\[
\nu = 0.
\]
Substitution of $\lambda = \nu = 0$ in (3.8) results in $c = 0$, which is a contradiction. So relation $c = -\lambda \nu$ holds. The last one implies $\lambda \nu \neq 0$ since $c \neq 0$.

Let $\lambda \neq \nu$ then $\lambda = -\frac{\xi}{\nu}$. Substitution of the last one in (3.8) leads to
\[
2\alpha \nu^2 + 5\epsilon \nu - 2\alpha c = 0.
\]
(3.11)
In case of $\mathbb{C}P^2$ we have that $c = 4$ and from equation (3.11) there is always a solution for $\nu$. So $\nu$ is constant and $\lambda$ will be also constant. Therefore, the real hypersurface has three distinct constant eigenvalues. The latter results in $M$ being a real hypersurface of type ($B$), i.e. a tube of radius $r$ over complex quadric. Substitution of the eigenvalues of type ($B$) in $\lambda \nu = -c$ leads to a contradiction. So no real hypersurface in $\mathbb{C}P^2$ has parallel $^*$-Ricci tensor (eigenvalues can be found in [7]).

In case of $\mathbb{C}H^2$ we have that $c = -4$ and from equation (3.11) there is a solution for $\nu$ if $0 \leq \alpha^2 \leq \frac{25}{4}$. If $\alpha = 0$ equation (3.11) implies $c\nu = 0$, which is impossible. So there is a solution for $\nu$ if $0 < \alpha^2 \leq \frac{25}{4}$ and $\nu$ will be constant. The latter results in that $\lambda$ is also constant and so the real hypersurface is of type ($B$), i.e. a tube of radius $r$ around totally geodesic $RH^2$. Substitution of the eigenvalues of type ($B$) in $\lambda \nu = -c$ leads to a contradiction and this completes the proof of our Main Theorem (eigenvalues can be found in [1]).

In case $\lambda = \nu$ then $c + \lambda^2 = 0$, which results in $c < 0$. So $M$ is locally congruent to a real hypersurface of type ($A$) in $\mathbb{C}H^2$. In this case only the geodesic hypersphere satisfies the above relation and we obtain $\coth(r) = 2$ and the $^*$-Ricci tensor vanishes identically.

4. DISCUSSION-OPEN PROBLEMS

In this paper three dimensional real hypersurfaces in non-flat complex space forms with parallel $^*$-Ricci tensor are studied and the non-existence of them is proved. Therefore, a question which raises in a natural way is
Are there real hypersurfaces in non-flat complex space forms of dimension greater than three with parallel *-Ricci tensor?

Generally, the next step in the study of real hypersurfaces in non-flat complex space forms is to study them when a tensor field $P$ type (1,1) of them satisfies other types of parallelism such as the $\mathbb{D}$-parallelism or $\xi$-parallelism. The first one implies that $P$ is parallel in the direction of any vector field $X$ orthogonal to $\xi$, i.e. $(\nabla_X P)Y = 0$, for any $X \in \mathbb{D}$, and the second one implies that $P$ is parallel in the direction of the structure vector $\xi$, i.e. $(\nabla_{\xi} P)Y = 0$. So the questions which should be answered are the following

Are there real hypersurfaces in non-flat complex space forms whose *-Ricci tensor satisfies the condition of $\mathbb{D}$-parallelism or $\xi$-parallelism?

Finally, other types of parallelism play important role in the study of real hypersurfaces is that of semi-parallelism and pseudo-parallelism. A tensor field $P$ of type $(1,s)$ is said to be semi-parallel if it satisfies $R \cdot P = 0$, where $R$ is the Riemannian curvature tensor and acts as a derivation on $P$. Moreover, $P$ is said to be pseudo-parallel if there exists a function $L$ such that $R(X,Y) \cdot P = L\{(X \wedge Y) \cdot P\}$, where $(X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y$. So the questions are:

Are there real hypersurfaces in non-flat complex space forms with semi-parallel or pseudo-parallel *-Ricci tensor?

The importance of answering the above question lays in the fact that the class of real hypersurfaces with parallel *-Ricci tensor is included in the class of real hypersurfaces with semi-parallel *-Ricci tensor. Furthermore, the last one is included in the class of real hypersurfaces with pseudo-parallel *-Ricci tensor.

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