Completeness and expressiveness for gs-monoidal categories

Andrea Corradini
Department of Computer Science, University of Pisa, Pisa, IT

Fabio Gadducci
Department of Computer Science, University of Pisa, Pisa, IT

Davide Trotta
Department of Computer Science, University of Pisa, Pisa, IT

Abstract
Formalised in the study of symmetric monoidal categories, string diagrams are a graphical syntax that has found applications in many areas of Computer Science. Our work aims at systematising and expanding what could be thought of as the core of this visual formalism for dealing with relations and partial functions. To this end, we identify gs-monoidal categories and their graphical representation as a convenient, minimal structure that is useful to formally express such notions.

More precisely, to show that such structures naturally arise, we prove that the Kleisli category of a strong commutative monad over a cartesian category is gs-monoidal. Then, we discuss how other categories providing a formalisation of “partial arrows”, such as p-categories and restriction categories, are related to gs-monoidal categories. This naturally introduces a pre-order enrichment on gs-monoidal categories, and an equivalence of arrows, called “gs-equivalence”: we conclude presenting a completeness result of this equivalence for models defined as lax functors to \( \text{Rel} \).

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1 Introduction

String diagrams are a graphical calculus for expressing operations in monoidal categories. The basic idea is to think of objects as “strings” or “wires” and of morphisms from one tensor product to another as “nodes” in which the source strings enter and the target strings exit. We refer to [35] for a presentation of various notions of monoidal categories and their associated string diagrams. Notable features of string diagrams are their flexibility and intuitive aspects, and as a visual language they have been found useful in practice as they greatly simplify the appearance of complex equations, the interrelations expressed being discernible at a glance. This makes such a formalism a powerful tool also for reasoning about the interaction of parallel systems, and justifies its widespread adoption in recent years.

The main purpose of this work is to identify a core graphical formalism that is adequate for expressing relations and partial maps, pursuing the goal of minimalism in the presentation. In particular, we identify gs-monoidal categories and their graphical representation as a suitable structure that allows to formally express such a core. Originally introduced in the context of term graphs and their rewriting, their study was pursued in a series of papers (see [10] [11] among others), including their use for the functorial semantics of relational and partial algebras [12]. As for cartesian bicategories, the categorical counterpart of string diagrams, gs-monoidal categories have been used as a suitable graphical calculus in different areas of system modelling (see [5] [13] among others).

Shortly, gs-monoidal categories are symmetric monoidal categories equipped with two families of arrows, a duplicator \( \nabla_A \) and a discharger \( !_A \) for each object \( A \), on which no form
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of naturality is required. Arrows of such categories can be thought of as abstract relations, and in fact, the paper argues precisely this point. Indeed, the presence of duplicators and dischargers allows to formally distinguish between arrows representing either total or functional relations, and this fact, amply discussed in the paper, suggests that gs-monoidal categories represent a core graphical formalism dealing with relations and partial functions.

The paper has the following roadmap. First, we show that the gs-monoidal structure naturally arises in several situations, in particular in every Kleisli category of strong commutative monads on cartesian categories. Kleisli categories have been largely employed in computer science as well as category theory. The leading example is the seminal work of Moggi [31], where stateful programs have an interpretation in terms of Kleisli categories of a strong monad. In this work we show that any Kleisli category of a strong commutative monad of a cartesian category is a gs-monoidal category. This result automatically provides a large class of examples of gs-monoidal categories, and at the same time shows how this structure is present in several well-known and adopted categories.

We then investigate the properties of gs-monoidal categories and we show how they relate to categories providing a formalisation of “partial arrows”, such as p-categories [33] and restriction categories [8]. More precisely, we show that in gs-monoidal categories we can define the notion of domain of an arrow, capturing the usual definition of domain of a relation. We use this notion to prove that suitable sub-categories of a gs-monoidal category are p-categories and restriction categories, and which sub-categories are (weakly) cartesian.

As a third step, we present a completeness-like result for the notion of gs-equivalence. We have seen that gs-monoidal categories have enough structure to formalise the notion of domain. In the specific case of Rel this notion coincides with the usual one for relations, but in general, domains of gs-monoidal categories do not satisfies all the algebraic properties of domains of relations. This is because the structure of a gs-monoidal category is less rich than the structure of Rel. In order to identify this kind of arrows, we consider preorder-enriched gs-monoidal categories. In this setting, we say that two arrows $f$ and $g$ are gs-equivalent when $f \leq g$ and $g \leq f$. Notice that gs-monoidal equivalent arrows are then identified by gs-functors $F: C \to Rel$ since $Rel$ is poset enriched. Then, we show that two arrows of a preorder-enriched gs-monoidal category $C$ are gs-monoidal equivalent if and only if every lax gs-monoidal lax functor $F: C \to Rel$ identifies such arrows in $Rel$.

Finally, recent years witnessed an increasing interest in the correspondence between string diagrams and various flavours of graphs, see [2] and the references therein. Concerning gs-monoidal categories, their connection with categories of cospans has been recently addressed in [17, 30], see also the references in the former and [10, 19] for early assessments. Employing the results developed in the previous sections, we explore and make precise the connection with gs-equivalence, providing a 2-categorical version of previous correspondence results.

The paper has the following structure. Section 2 recalls some basic properties of gs-monoidal categories. Section 3 presents a characterisation of Kleisli categories having the gs-monoidal structure. Section 4 studies the connection with p-categories and restriction categories. Section 5 introduces preorder-enriched gs-monoidal categories and the notion of gs-equivalence, further presenting for such notion a completeness result and a combinatorial characterisation involving categories of cospans and term graphs.

Related works. We already mentioned of the large interest in the connections between string diagrams and graphs, referring to [2] for a short review of such works. Usually, the focus has been on cartesian bicategories [6] (i.e., where the operators of gs-monoidal categories have duals, equipping them with a Frobenius structure), for their ability to represent systems with multiple inputs and outputs: see [3] for an introduction to their use in system modelling.
Less investigated has been the case of gs-monoidal categories, which in this paper we argue to be a core language for relations and partial maps. They were originally introduced in the context of the research on term graphs, which were in turn devised as an implementation tool for solving efficiently equational theories in algebraic data types. A categorical characterisation of term graphs and their rewriting was presented in a series of papers (among others), including their use for the functorial semantics of relational and partial algebras. The notion has surfaced in the literature a few times, see for a recollection [Remark. 2.2]. Most related to us are the works on the connection between Kleisli categories for affine monads and cospan categories with gs-monoidal categories. We improve on the former by generalising the connection to strong commutative monads and on the latter by establishing the notion of gs-equivalence, which allows for a 2-categorical extension of those works. We also present a completeness result for such an equivalence and we describe its connection with other categorical proposals for partial maps: both results are original to the best of our knowledge.

2 On gs-monoidal categories

Originally introduced in the context of term graph rewriting, the notion of gs-monoidal category has been developed in a series of papers: we recall the basic definitions, using the visual presentation adopted for string diagrams. We refer to a complete presentation of various notions of monoidal categories and their associated string diagrams.

Definition 2.1 (gs-monoidal category). A gs-monoidal category $C$ is a symmetric monoidal category, where we denote by $\otimes$ the tensor product and by $I$ the unit, such that every object $A$ of $C_0$ is equipped with morphisms $\nabla_A : A \to A \otimes A$ and $!_A : A \to I$, graphically

\[
\begin{align*}
\nabla_A & : A \to A \otimes A \\
!_A & : A \to I
\end{align*}
\]

such that they satisfy the ACI axioms

\[
\begin{align*}
\begin{array}{ccc} \\
\nabla_A & \approx & \nabla_A \\
!_A & \approx & !_A \\
\end{array}
\end{align*}
\]

and the co-monoid axioms

\[
\begin{align*}
\begin{array}{ccc} \\
A \otimes B & \approx & A \\
B \otimes A & \approx & B
\end{array}
\end{align*}
\]

Remark 2.2. Notice that in every gs-monoidal category $C$ we have that $\nabla_I = \text{id}_I$. This follows combining the second and the last ACI axiom, i.e. $!_I = \text{id}_I$ and $(!_I \otimes \text{id}_I) \nabla_I = \text{id}_I$.

Definition 2.3. A gs-monoidal functor (or gs-functor) $F : C \to C'$ is a symmetric monoidal functor such that $F(!_A) = !_F A$ and $F(\nabla_A) = \nabla_{F A}$.

We denote the category of gs-monoidal categories and gs-monoidal functors by GSM-Cat.
Definition 2.4. Let $C$ be a gs-monoidal category. Then, an arrow $\xymatrix{ A \ar@{.}[r] & B }$ is called $C$-total if $\xymatrix{ A \ar[r]^f & A }$ and $\xymatrix{ A \ar@{.}[r] & B }$ is $C$-functional if $\xymatrix{ A \ar[r]^f & A }$.

The sub-category of $C$ of $C$-functional arrows is denoted by $C\text{-Pfn}$, the sub-category of $C$-total arrows by $C\text{-Total}$, and the sub-category of $C$-total, $C$-functional arrows by $C\text{-Fn}$.

Example 2.5. The category $\text{Rel}$ of sets and relations with the operation $\times : \text{Rel} \times \text{Rel} \to \text{Rel}$ given by the direct product of sets is a gs-monoidal category. In particular, we have that $\text{Rel}$-functional arrows are precisely partial functions, $\text{Rel}$-total arrows are total relations, and $\text{Rel}$-total, $\text{Rel}$-functional arrows are functions.

Example 2.6. Every Markov category $M$ [15, 16] (see also [23]) is a gs-monoidal category. In fact, Markov categories are exactly gs-monoidal categories whose element $I$ is terminal. This is equivalent to say that in a Markov category $M$ every arrow is $M$-total, while $M$-functional arrows as in Definition 2.4 are precisely those called deterministic arrows there.

In the following proposition we summarise some useful properties of $C$-total and $C$-functional arrows of a gs-monoidal category.

Proposition 2.7. Let $C$ be a gs-monoidal category. Then
1. arrows $!_A$ and $\nabla_A$ are $C$-total and $C$-functional;
2. $C\text{-Pfn}$, $C\text{-Total}$ and $C\text{-Fn}$ are symmetric monoidal sub-categories of $C$.

Proof. See Appendix B.1

3 The gs-monoidal structure of Kleisli categories

In recent years, strong monads and Kleisli categories have been largely used to provide categorical models in several branches of computer science. The leading examples are Moggi’s works [31, 32] regarding an abstract approach to the notion of computation. We refer to [4, 28, 29] for a general introduction to the theory of monads, and to [27, 36] for more details. In Appendix A you can find a short recap with the main definitions.

Here we start by recalling the definition of Kleisli category for a given monad.

Definition 3.1. The Kleisli category $A_T$ of a monad $(T, \mu, \eta)$ on a category $A$ is defined as follows: the objects of $A_T$ are the same as those of $A$, and for every $A$ and $B$ in $A_T$ we define $A_T(A, B) = A(\mu_A T(B), A)$. To define the composition we consider $f \in A_T(A, B)$ and $g \in A_T(B, C)$. Then the composition $g * f$ is defined as $g * f = \mu_C T(g) f$.

In particular we can observe that $\eta_B * f := \mu_B T(\eta_B) f = f$ and $f * \eta_A := \mu_B T(f) \eta_A = \mu_B T(\eta_B) f = f$. Therefore, $\eta_A : A \to \mu_A T(A)$ is the identity of $A_T(A, A)$.

Example 3.2. The category of sets and relations $\text{Rel}$ is the Kleisli category of the powerset monad $P : \text{Set} \to \text{Set}$ sending a set $X$ into its powerset $P(X)$. 
Example 3.3. The Kleisli category of the distribution monad is the category of sets and stochastic maps. Recall that the distribution monad $\mathcal{P} : \textbf{Set} \to \textbf{Set}$ takes a set $X$ and returns the set $\mathcal{P}(X)$ of finitely supported probability measures on $X$.

Example 3.4. The category of measurable spaces and Markov processes is one of the key examples in Markov category: it coincides with the Kleisli category of the Giry monad $\mathcal{G} : \textbf{Meas} \to \textbf{Meas}$, where $\textbf{Meas}$ denotes the category of measurable spaces and measurable functions, see [20, 22].

Notation: to avoid confusion between arrows of $A$ and arrows of a Kleisli category $A_T$, we adopt the notation $f^k : A \to B$ for an arrow of $A_T$, i.e. to denote the arrow $f : A \to TB$. Given this, we can adopt the usual notation for the composition of arrows of $A_T$.

Given a monad $(T, \mu, \eta)$ on a symmetric monoidal category $A$, it is well-known that the Kleisli category $A_T$ inherits the symmetric monoidal structure precisely when the monad $T$ is strong and commutative, see Appendix A. This result is considered folklore, but it has been employed in several works and several situations.

Theorem 3.5 (Folklore). Let $(T, \mu, \eta)$ be a strong commutative monad on a symmetric monoidal category $A$. Then the Kleisli category $A_T$ is a symmetric monoidal category.

The previous theorem characterises the structure of Kleisli categories of strong commutative monads whose base categories are symmetric and monoidal.

However, when the base category is cartesian, the monoidal product induced on the Kleisli category is not cartesian in general. The leading example is the powerset monad $\mathcal{P} : \textbf{Set} \to \textbf{Set}$ on the category of sets and functions presented in Example 3.2. In this case, we have that the Kleisli category $\textbf{Set}_P$ is the category of relations $\textbf{Rel}$, and the categorical product of $\textbf{Set}$ induces a monoidal product on $\textbf{Rel}$, given by the direct products. However, the direct product of sets is not a categorical product in $\textbf{Rel}$, but just a monoidal product.

In the following theorem, we show that when the base category is cartesian and the monad is strong and commutative, the precise structure the Kleisli category inherits from the base category is that of gs-monoidal category.

Theorem 3.6. Let $(T, \mu, \eta)$ be a strong commutative monad on a cartesian category $A$. Then the Kleisli category $A_T$ is a gs-monoidal category, with $\nabla^k_A := \eta_A \times_A \nabla_A$ and $!^k_A := \eta_1 !_A$.

Proof. See Appendix B.2.

Remark 3.7. Observe that Theorem 3.6 can be generalised for any strong commutative monad on a gs-monoidal category. In particular, if $A$ is a gs-monoidal category, then the Kleisli category $A_T$ is a gs-monoidal category for every strong commutative monad $T : A \to A$. In fact, in the proof of Theorem 3.6, we just use the fact that in any cartesian category, the arrows $\nabla$ and $!$ satisfies the axioms of gs-monoidal categories. The proof depends on the fact that the monad is strong and commutative, and that $\nabla$ and $!$ satisfies the axioms of gs-monoidal categories in the base $A$.

Remark 3.8. Notice that an alternative choice for the morphism $\nabla^k_A$ in Theorem 3.6 could be $\nabla^k_A := c_{A,A} \nabla_X \eta_A$, where $c_{A,A} : TA \times TA \to T(A \times A)$ is the canonical arrow defined via the strong and commutative structure of the monad $T$, see Appendix A for all the details. However, it is straightforward to check that $c_{A,A} \nabla_X \eta_A = \eta_A !_A \nabla_A$.

Let $(T, \mu, \eta)$ be a strong commutative monad on a cartesian category $A$. We can define a symmetric monoidal functor $I_A : A \to A_T$ as $I_A(X) := X$ and $I_A(f) := \eta_B f$ for $f : A \to B$. 

Proof. See Appendix B.2.
Proposition 3.9. For every cartesian category \( \mathcal{A} \) and every strong commutative monad \( T: \mathcal{A} \to \mathcal{A} \) the functor \( I_\mathcal{A}: \mathcal{A} \to \mathcal{A}_T \) is a gs-functor.

Proof. We just need to check that \( I_\mathcal{A}(\nabla_\mathcal{A}) = \nabla_{\mathcal{A}_T} \) and \( I_\mathcal{A}(!_\mathcal{A}) = !_{\mathcal{A}_T} \), but both follow by the definitions of the action \( I_\mathcal{A} \) on the morphisms of \( \mathcal{A} \), \( \nabla_\mathcal{A} \) and \( !_\mathcal{A} \). In fact, \( I_\mathcal{A}(\nabla_\mathcal{A}) = \eta_{\mathcal{A} \times \mathcal{A}} \nabla_\mathcal{A} = \nabla_{\mathcal{A}_T} \) and \( I_\mathcal{A}(!_\mathcal{A}) = \eta_1 !_{\mathcal{A}} = !_{\mathcal{A}_T} \).

Corollary 3.10. Let \( (T, \mu, \eta) \) be a strong commutative monad on a cartesian category \( \mathcal{A} \). Then every cartesian functor \( F: \mathcal{C} \to \mathcal{A} \) lifts to a gs-functor from \( \mathcal{C} \) to \( \mathcal{A}_T \), given by the composition \( I_\mathcal{A}F: \mathcal{C} \to \mathcal{A}_T \).

Proof. It follows from the fact that \( F \) is in particular a gs-functor, and that \( I_\mathcal{A} \) is a gs-functor by Proposition 3.9, since gs-functors compose.

Remark 3.11. When \( \mathcal{A} \) is a cartesian category and \( (T, \mu, \eta) \) is a strong commutative monad, we have that 1 is terminal in \( \mathcal{A}_T \) if and only if \( T(1) = 1 \) in \( \mathcal{A} \). Therefore, when the monad \( T \) preserves the terminal object, we have that every arrow of \( \mathcal{A}_T \) is \( \mathcal{A}_T \)-total. As an example, consider the non-empty powerset monad, i.e., associating with a set \( X \) the family of its non-empty subsets \( P(X) \setminus \emptyset \): the arrows of the Kleisli categories are total relations, meaning that \( \forall a \exists b. aRb \). And indeed, we have that \( P(1) \setminus \emptyset = 1 \). Monads preserving the terminal arrows are called affine in the context of Markov categories, we refer to [15] for more details.

Remark 3.12. Notice that Theorem 3.6 and Corollary 3.10 can be considered a general version of [15, Cor. 3.2]. The difference is that in [15] the monad is also assumed to be affine (see [15]) in order to achieve totality.

Remark 3.13. As for functional relations, it suffices to consider the lifting monad, associating with a set \( X \) the pointed set \( X_\perp \).

Example 3.14. The category of measurable spaces and Markov processes of Example 3.4 is a gs-monoidal category. In fact, the Giry monad \( G: \text{Meas} \to \text{Meas} \) is commutative and strong with respect to the cartesian monoidal structure of \( \text{Meas} \) given by the direct product of sets. Therefore, by Theorem 3.6 we can conclude that the category \( \text{Meas}_G \), usually denoted by \( \text{Stoch} \) because its morphisms are stochastic maps, is a gs-monoidal category.

Example 3.15. The category of quasi-Borel spaces \( \text{QBS} \) is cartesian, and the monad \( P: \text{QBS} \to \text{QBS} \) of probability measures on the category of quasi-Borel spaces is strong and commutative, see [21] for all the details. Therefore, by Theorem 3.6 we can conclude that the category \( \text{QBS}_P \) is a gs-monoidal category.

4 On gs-monoidal categories and partiality

The main purpose of this section is to investigate the categorical structures of gs-categories and their subcategories. In particular, we are going to explore connections with restriction categories \([8]\), p-categories in the sense of \([33]\) and cartesian categories.

To achieve this goal, we start observing the gs-monoidal categories has enough structure to properly express the notion of domain of morphisms.

Definition 4.1. Let us consider a gs-monoidal category \( \mathcal{C} \) and an arrow \( f: A \to B \). We define the domain of \( f \) the arrow \( \text{dom}(f): A \to A \) given by \( \text{dom}(f) := (\text{id}_A \otimes !_B)f \nabla_A \), i.e.

\[
\begin{array}{ccc}
A & \rightarrow & A \\
| & & | \\
\downarrow & & \downarrow \\
B & \rightarrow & B
\end{array}
\]
The motivation behind the choice of the arrow above, i.e. $\text{dom}(f) = (\text{id}_A \otimes \triangleright B f) \triangleright A$, is that in the specific case of the category $\text{Rel}$ of sets and relations, for a given relation $R$ from $A$ to $B$, the arrow $\text{dom}(R) := (\text{id}_A \times \triangleright B R)(\text{id}_A \times R) \triangleright A$ is exactly the relation representing the domain of definition of $R$, i.e. $a \text{ dom}(R) a' \iff a = a'$ and $\exists b \in B.$ such that $a R b$.

As we will show, the arrow $\text{dom}(f)$ in a gs-monoidal category enjoys several algebraic properties of the usual notion of domain of a relation. A first property we expect to hold from a morphism that wants to abstract the notion of domain of a relation, is that the domain of a total arrow has to be the identity. This is exactly what we prove in the next proposition.

\textbf{Proposition 4.2.} Let $C$ be a gs-monoidal category. Then $f: A \to B$ is $C$-total if and only if $\text{dom}(f) = \text{id}_A$.

\textbf{Proof.} If $f$ is $C$-total, then $!_B f = !_A$, and hence $\text{dom}(f) = (\text{id}_A \otimes !_B f) \triangleright A = (\text{id}_A \otimes !_A) \triangleright A = \text{id}_A$ by the second axiom of gs-monoidal categories. Conversely, if $\text{dom}(f) = \text{id}_A$, then $\triangleright A \text{ dom}(f) = !_A$, hence $\triangleright A (\text{id}_A \otimes !_B f) \triangleright A = !_A$. But since $\triangleright A (\text{id}_A \otimes !_B f) \triangleright A = !_B f \triangleright !_A \otimes \text{id}_A \triangleright A = !_B f$, we can conclude that $!_B f = !_A$, i.e., that $f$ is $C$-total.

\textbf{Proposition 4.3.} Let $C$ be a gs-monoidal category. If $f: A \to B$ is $C$-functional, then $\text{dom}(f): A \to A$ is $C$-functional.

\textbf{Proof.} By Proposition 2.7 we have that $C$-functional arrows are closed under composition and tensor product. Since $!$ and $\triangleright$ are always $C$-functional arrows and $f$ is $C$-functional by hypothesis, we have that $\text{dom}(f) = (\text{id}_A \otimes !_B f) \triangleright A$ is $C$-functional.

\textbf{Remark 4.4.} In $\text{Rel}$ we have that every arrow $\text{dom}(f)$ satisfies $f \text{ dom}(f) = f$, i.e. graphically

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (0,-1) {$B$};
  \node (F) at (1,0) {$f$};
  \node (G) at (1,-1) {$f$};
  \draw[->] (A) -- (B);
  \draw[->] (A) -- (F);
  \draw[->] (F) -- (G);
  \draw[->] (G) -- (B);
\end{tikzpicture}
\end{center}

However, in an arbitrary gs-monoidal category this does not hold in general.

The notion of domain in a gs-monoidal category allows us to state the precise link between gs-monoidal and restriction categories. For an introduction to the latter we refer to [8].

\textbf{Theorem 4.5.} Let $C$ be a gs-monoidal category. The sub-category $C$-$\text{Pfn}$ of $C$-functional arrows is a restriction category, with the restriction structure given by $\text{dom}(\cdot)$, i.e., it satisfies the following conditions for any arrow $f: A \to B$

1. $f \circ \text{dom}(f) = f$

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (0,-1) {$B$};
  \node (F) at (1,0) {$f$};
  \node (G) at (1,-1) {$f$};
  \draw[->] (A) -- (B);
  \draw[->] (A) -- (F);
  \draw[->] (F) -- (G);
  \draw[->] (G) -- (B);
\end{tikzpicture}
\end{center}

2. $\text{dom}(f) \circ \text{dom}(g) = \text{dom}(g) \circ \text{dom}(f)$ whenever the source of $g$ is the same of $f$

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (0,-1) {$B$};
  \node (C) at (0,-2) {$C$};
  \node (F) at (1,0) {$f$};
  \node (G) at (1,-1) {$f$};
  \node (H) at (1,-2) {$f$};
  \draw[->] (A) -- (B);
  \draw[->] (A) -- (C);
  \draw[->] (B) -- (C);
  \draw[->] (A) -- (F);
  \draw[->] (B) -- (F);
  \draw[->] (C) -- (F);
  \draw[->] (A) -- (G);
  \draw[->] (B) -- (G);
  \draw[->] (C) -- (G);
  \draw[->] (A) -- (H);
  \draw[->] (B) -- (H);
  \draw[->] (C) -- (H);
\end{tikzpicture}
\end{center}

3. $\text{dom}(g \circ \text{dom}(f)) = \text{dom}(g) \circ \text{dom}(f)$ whenever the source of $g$ is the same of $f$

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (0,-1) {$B$};
  \node (C) at (0,-2) {$C$};
  \node (F) at (1,0) {$f$};
  \node (G) at (1,-1) {$f$};
  \node (H) at (1,-2) {$f$};
  \draw[->] (A) -- (B);
  \draw[->] (A) -- (C);
  \draw[->] (B) -- (C);
  \draw[->] (A) -- (F);
  \draw[->] (B) -- (F);
  \draw[->] (C) -- (F);
  \draw[->] (A) -- (G);
  \draw[->] (B) -- (G);
  \draw[->] (C) -- (G);
  \draw[->] (A) -- (H);
  \draw[->] (B) -- (H);
  \draw[->] (C) -- (H);
\end{tikzpicture}
\end{center}
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4. dom(g) ∘ f = f ∘ dom(g ∘ f) whenever the source of g is equal to the target of f

\[ \text{Diagram: } \]

\[ A \xrightarrow{f} B \xrightarrow{g} C \]

\[ A \xrightarrow{\text{id}_A \otimes !_B} A \otimes B \xrightarrow{\text{id}_A \otimes !_B} A \]

\[ C \xrightarrow{(f \otimes g) \nabla} B \]

Proof. See Appendix B.3.

The rest of this section is devoted to analyse what structure the tensor product of a gs-monoidal category induces in the three subcategories \( C\text{-Pfn} \), \( C\text{-Total} \) and \( C\text{-Fn} \). We start by considering the sub-category \( C\text{-Pfn} \) of \( C\)-functional arrows.

Recall that the notion of \( p\)-category was introduced by Rosolini and Robinson in [33] to properly present categories of partial maps. Such a notion can be considered an improvement of the notion of \( \text{dominical categories} \) introduced by Di Paola and Heller [14]. A \( p\)-category is essentially a category \( C \) endowed with a bifunctor \( \times : C \times C \to C \), called product, a natural transformation \( \nabla : \text{Id}_C \to \text{Id}_C \times \text{Id}_C \) which is called the diagonal and two families of natural transformations \( \{ p(-) : Y \times (-) \to (-) | Y \in \text{ob}(C) \} \) and \( \{ q_X(-) : X \times (-) \to (-) | X \in \text{ob}(C) \} \), called projections, satisfying certain axioms (see [33]).

In the following theorem we show that the tensor product of gs-monoidal categories induces the structure of \( p\)-category in the sub-category of \( C\)-functional arrows.

\[ \text{Theorem 4.6. Let us consider a gs-monoidal category } C. \text{ Then the monoidal operation } \otimes \text{, the natural transformation } \nabla \text{ and the projections given by arrows of the form } \text{id} \otimes ! \text{ induce the structure of } p\text{-category in the sub-category } C\text{-Pfn}. \]

Proof. See Appendix B.4.

If we consider the sub-category of \( C\)-total arrows, the family of morphisms of the form \( \text{id} \otimes ! \) does not provide the structure of \( p\)-category as in Theorem 4.6 since \( \nabla \) may not be natural for \( C\)-total arrows, but it induces the structure of weak cartesian products in \( C\text{-Total} \).

\[ \text{Theorem 4.7. Let us consider a gs-monoidal category } C. \text{ Then the monoidal operation } \otimes \text{ induces a weak cartesian product in the sub-category } C\text{-Total}. \]

Proof. See Appendix B.5.

\[ \text{Remark 4.8. In Theorem 4.7 we have seen that if we consider two } C\text{-total arrows } f : C \to A \text{ and } g : C \to B \text{ then } A \otimes B \text{ is a weak-product with projections } \text{id}_A \otimes !_B \text{ and } !_A \otimes \text{id}_B. \text{ This occur because we have that the arrow } (f \otimes g) \nabla : C \to A \otimes B \text{ satisfies the universal properties of cartesian products, but it is not necessarily unique. A counterexample to the uniqueness of this arrows is provided by the case of the category of sets and total relations. In this case it is direct to check that we can construct another total relation } h : C \to A \otimes B \text{ such that the following diagram commutes} \]

\[ \text{Diagram: } \]

\[ A \xrightarrow{\text{id}_A \otimes !_B} A \otimes B \xrightarrow{\text{id}_A \otimes !_B} A \]

\[ C \xrightarrow{(f \otimes g) \nabla} B \]

\[ f \]

\[ g \]

In order to obtain cartesian products we have to consider arrows that are not only \( C\)-total, as in Theorem 4.7 but also \( C\)-functional.
Theorem 4.9. Let us consider a gs-monoidal category $\mathcal{C}$. Then the monoidal operation $\otimes$ induces a cartesian product in the sub-category $\mathcal{C}$-$\text{Fn}$.

Proof. See Appendix B.6.

5 Preorder-enriched gs-monoidal categories

We have seen that gs-monoidal categories enjoy the main features of the category of sets and relations with respect to total and functional arrows. We introduced the notion of domain and proved in Theorem 4.5 that the sub-category $\mathcal{C}$-$\text{Pfn}$ is a restriction category.

However, as we have already observed in Remark 4.4, we cannot prove in general that this notion of domain satisfies the same properties as in $\text{Rel}$ when we consider an arbitrary arrow of a gs-monoidal category. This is clearly due to the fact that $\text{Rel}$ is much richer of structures that an arbitrary gs-monoidal category, so the next step is finding the right structure of $\text{Rel}$ we have to abstract to be able to speak of domain properly.

Therefore, what we would like to have in an arbitrary gs-monoidal category, is a relation identifying, for example, the arrow $f$ with $f \text{ dom}(f)$. For this purpose we have to consider the structure of 2-category for gs-monoidal categories.

Definition 5.1. A preorder-enriched gs-monoidal category $\mathcal{C}$ is a gs-monoidal category such that $\mathcal{C}$ is a preorder-enriched category and for every arrow $f: A \to B$ we have that the following inequalities are satisfied

\[
\begin{array}{c}
A \leq A \\
B \leq B
\end{array}
\]

Hence, we ask that the families $\nabla_A$ and $!_A$ form lax-natural transformations. Notice that the notion of preorder-enriched gs-monoidal category may remind that of cartesian structure on a bicategory presented in [6, Def. 1.2]. The main difference is that in Definition 5.1 we do not require the existence of right adjoints for $\nabla$ and $!$.

It is direct to check that we can generalise the notion of gs-monoidal functor introduced in Definition 2.3 in the context of preorder-enriched gs-monoidal categories, by requiring the preservation of the pre-order.

Remark 5.2. Recall that $\text{Rel}$ has a natural 2-categorical structure when we consider as 2-cells the set-theoretic inclusions. In particular, for every relation $R: A \to B$ we trivially have the 2-cells discussed in Definition 5.1.

Definition 5.3. Let $\mathcal{C}$ be a preorder-enriched gs-monoidal category, and let us consider two arrows $f, g: A \to B$. We say that $f$ and $g$ are gs-equivalent if there exists a 2-cell $f \leq g$ and a 2-cell $g \leq f$. We will use the notation $f =_{gs} g$ when $f$ and $g$ are gs-equivalent.

Exploiting this definition we can prove that, even if $f \text{ dom}(f)$ is different from $f$ in general, these two arrows are gs-equivalent in any preorder-enriched gs-monoidal category.

Proposition 5.4. In any preorder-enriched gs-monoidal category, for every arrow $f: A \to B$ we have that $\text{dom}(f) \leq \text{id}_A$ and $f \text{ dom}(f) =_{gs} f$, graphically

\[
\begin{array}{c}
A \\
B
\end{array}
\]

Proof. See Appendix B.7.
5.1 Completeness for gs-equivalence

The main goal of this section is to provide a completeness result for the notion of gs-equivalence. In particular, we show that for any preorder-enriched gs-monoidal category \( \mathcal{C} \) two arrows \( f, g : A \to B \) are gs-monoidal equivalent if and only if for any lax gs-monoidal lax functor \( F : \mathcal{C} \to \text{Rel} \) we have that \( F(f) = F(g) \). Notice that with lax gs-monoidal lax functors we mean a lax monoidal lax functor \( F : \mathcal{C} \to \mathcal{A} \) such that \( \nabla F_A \leq F(\nabla A) \) and \( !F_A \leq F(!A) \) for every object \( A \) of \( \mathcal{C} \).

For this purpose, let us consider a preorder-enriched gs-monoidal category \( \mathcal{C} \). We define a lax gs-monoidal lax functor \( H_C : \mathcal{C} \to \text{Rel} \) as follows

- for every object \( A \) of \( \mathcal{C} \), \( H(A) := \mathcal{C}/A \) (the set of morphisms of \( \mathcal{C} \) whose codomain is \( A \));
- for every arrow \( f : A \to B \), \( H(f) : H(A) \to H(B) \) is \( h \circ f \) if \( g \leq h \).

**Proposition 5.5.** For every preorder-enriched gs-monoidal category \( \mathcal{C} \), \( H_C : \mathcal{C} \to \text{Rel} \) is a lax gs-monoidal lax functor.

**Proof.** See Appendix B.8.

The lax functor \( H_C : \mathcal{C} \to \text{Rel} \) is used to provide a completeness result for gs-equivalence.

**Theorem 5.6.** Let \( \mathcal{C} \) be a preorder-enriched gs-monoidal category. Two arrows of \( \mathcal{C} \) are gs-monoidal equivalent if and only if for every lax gs-monoidal lax functor \( F : \mathcal{C} \to \text{Rel} \) we have \( F(f) = F(g) \).

**Proof.** Let us consider two arrows \( f : A \to B \) and \( g : A \to B \). If \( g \) and \( f \) are gs-equivalent, then \( F(g) = F(f) \) because a lax gs-functor preserves the order and \( \text{Rel} \) is poset-enriched.

Now suppose that \( F(g) = F(f) \) for every lax gs-functor \( F : \mathcal{C} \to \text{Rel} \). In particular we have that \( H_C(f) = H_C(g) \) where \( H_C : \mathcal{C} \to \text{Rel} \) is the canonical lax gs-monoidal lax functor defined in the previous proposition. Now we show \( H_C(f) = H_C(g) \) implies \( g \) and \( f \) are gs-equivalent. Notice when \( H_C(f) = H_C(g) \), in particular, we have that \( \text{id}_A H_C(f) g \) implies that \( \text{id}_A H_C(g) f \). This means that \( g \leq f \). Similarly, we have that \( \text{id}_A H_C(g) f \) implies that \( \text{id}_A H_C(f) g \), i.e. \( f \leq g \). Therefore we can conclude that \( f \) and \( g \) are gs-equivalent.

5.2 A combinatorial characterisation of gs-equivalence

There has been an increasing interest in the precise correspondence between string diagrams and various flavours of graphs, see \cite{22} and the references therein. Concerning gs-monoidal categories, their connection with suitable categories of cospans has been recently addressed in \cite{17, 30}, see also the references in the former and \cite{10, 19} for early assessments. In this section we want to explore and make precise the connection with gs-equivalence.

5.2.1 Some remarks on cospans

Let us consider a category \( \mathcal{C} \) that admits finite coproducts, binary pushouts when one leg is regular mono and such that regular monos are stable under pushouts and composition. For example, these conditions are satisfied by every coregular category \cite{34}. Following \cite{1}, we can construct a bicategory of cospans of \( \mathcal{C} \), denoted \textbf{CoSpan}(\( \mathcal{C} \)): its objects are the same of \( \mathcal{C} \), while an arrow from \( X \) to \( Y \) is a pair of arrows \( X \to A \leftarrow Y \) in \( \mathcal{C} \) where \( X \to A \) is a regular mono. A 2-cell \( \alpha : (X \to A \leftarrow Y) \Rightarrow (X \to B \leftarrow Y) \) is an arrow \( \alpha : A \to B \) in \( \mathcal{C} \) such that...
the diagram below commutes

\[ X \xrightarrow{\alpha} A \xleftarrow{f} B \xleftarrow{g} Y \]

Cospans \(X \to A \xleftarrow{f} Y\) and \(X \to B \xleftarrow{g} Y\) are isomorphic if there exists a 2-cell as above, where \(\alpha : A \to B\) is an isomorphism. For every object \(X\) in \(\mathcal{C}\), the identity cospan is \(X \xleftarrow{id_X} X \xrightarrow{id_X} X\). The composition of \(X \xleftarrow{f} Y \xrightarrow{g} Z\) is \(X \xleftarrow{f, g} Y \xrightarrow{g} Z\), obtained by taking the pushout of \(f\) and \(g\), noting that regular monos are preserved and composed by hypothesis. We will consider the 2-categorical version \(\text{CoSpan}(\mathcal{C})\) of such bicategory, obtained by simply identifying isomorphic cospans, and its pre-order enriched version \(\text{PCoSpan}(\mathcal{C})\).

Clearly, if \(\mathcal{C}\) admits monoidal products, so does the associated bicategory of cospans. We can then provide another natural source of gs-monoidal categories.

\[ \text{Proposition 5.7.} \text{ Let } \mathcal{C} \text{ be a category that admits finite coproducts, pushouts when one leg is regular mono and such that regular monos are stable under pushouts and composition. Then } \text{PCoSpan}(\mathcal{C}) \text{ is a pre-order enriched gs-monoidal category, with } \nabla_A := (A \xleftarrow{\nabla_{1}} A \xrightarrow{\nabla_{2}} A + A) \text{ and } !_A := (A \nrightarrow A \xleftarrow{0} 0). \]

\[ \text{Proof.} \text{ See Appendix B.9} \]

### 5.2.2 Term graphs and gs-equivalence

Let us now move to consider term graphs, adapting the presheaf presentation for graphs adopted in [2]. A term graph consists of a set of nodes \(G\) and for each \(k \in \mathbb{N}\) a (possibly empty) set of edges \(G_k\) with \(k\) (ordered) sources and 1 target. That is, for each \(1 \leq i \leq k\) we have the \(i\)th source map \(s_i^k : G_k \to G\), and the target map \(t_k : G_k \to G\). Differently from graphs, it must hold that the target maps are jointly mono, that is, the disjoint union of all \(t_k\)'s is mono. The arrows are term graph homomorphisms: functions \(G_i \to H_j\) and \(G_k \to H_k\) for each \(k\) such that they respect the source and target maps in the obvious way.

\[ \text{Definition 5.8.} \text{ Let } \mathbf{I} \text{ be the category whose objects are natural numbers } k \in \mathbb{N} \text{ together with one extra object } \ast \text{ such that for each } k \in \mathbb{N}, \text{ there are } k+1 \text{ arrows from } k \text{ to } \ast. \text{ The category of finite term graphs } \mathcal{G} \text{ is the sub-category of the functor category } \mathbf{FinSet}^\mathbf{I} \text{ whose objects are the functors } F \text{ such that } \bigsqcup F(t_k) \text{ is mono.} \]

The category \(\mathcal{G}\) admits finite coproducts (given by disjoint unions), and is quasi-adhesive [7], thus it has pushouts if one of the legs is regular mono, and regular monos are stable under pushouts and composition [20]. More precisely, a mono \(f : F \to G\) is regular if nodes in \(F_t\) that are not in the image of \(\bigsqcup F(t_k)\) are not in the image of \(\bigsqcup f(F(t_k))\), that is, in the jargon of the term graph community, if empty nodes are mapped into empty nodes.

We will further restrict to term graphs that are acyclic, with the obvious meaning. Moreover, we will consider their typed version, that is, given a term graph \(G\) we will focus on the slice category \(\mathcal{G} \downarrow G\), which is also quasi-adhesive. Now, given a term graph \(G\), one can build a pre-order enriched gs-monoidal category \(\text{GS}(G)\) with natural numbers as objects by viewing each edge with \(k\) sources as an arrow \(k \to 1\) and adding \(\nabla\)'s and \(!\)'s and the relevant 2-cells. We can then close this section by a 2-categorical version of the correspondence between free gs-monoidal categories and cospans over term graphs.
Proposition 5.9. Let \( f, g \) be arrows of \( GS(G) \), and \([f], [g] \) their image in \( PCoSpan(TG \downarrow G) \). Then \( f \) and \( g \) are gs-equivalent if and only if \([f] \) and \([g] \) are so.

Proof. See Appendix B.10.

6 Conclusions and future works

Our work presented a preliminary investigation on gs-monoidal categories. Arisen in the area of term graph rewriting, this paper shows how such categories fit in the current interest for the use of graphical formalisms in system description, as a core language of string diagrams. More precisely, we argue that they represent the minimal structure that is needed to discuss about relations and partial maps. This is done by introducing the notion of domain for arrows in a gs-monoidal category, and discussing the relationship with other categorical models of partiality. Moreover, the characterisation we propose via Kleisli categories (for strong commutative monads on cartesian categories) shows that the gs-monoidal structure naturally arises in several situations, thus providing a large number of potential case studies. Also the presentation via cospans (in particular, in the category of term graphs) provide additional examples, systematising a series of results that appeared in the graph transformation literature and putting them in a firm ground as part of the string diagram formalism.

On a technical side, future work will keep the focus on the one-side on the completeness result, in order to strengthen it via a proper functor instead of a lax one, following the preliminary ideas in [19]. On the other side, it will try to establish a firm connection between the presentations using Kleisli and cospan categories, also addressing the cyclic case, and discussing the whole 2-categorical structure of \( CoSpan(C) \) for a presentation of graph rewriting and of inequational deduction for relations, investigated in a set-theoretical flavour in [11] [13]. Concerning the case studies arising from such characterisations, it will be put at work the preorder machinery developed for gs-monoidal categories.

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A strong commutative monads

A strength and a costrength for a monad on a monoidal category are structures relating the monad with the tensor product of the category at least in one direction. A monad equipped with a strength is called a strong monad.

Originally, the notion of strong monad was introduced by Kock in [25, 24] as an alternative description of enriched monads. Strong monads have been later successfully employed in computer science, playing a fundamental in Moggi's theory of notions of computation [31, 32].

We recall these notions in the following definitions.

▶ Definition A.1. A strong monad \((T, \mu, \eta)\) over a symmetric monoidal category \((A, \otimes, I, \alpha, r, l, \gamma)\) is a monad together with a natural transformation \(t_{A,B}: A \otimes TB \to T(A \otimes B)\) called strength, such that

\[
\begin{align*}
I \otimes TA & \xrightarrow{t_{I,A}} TA \\
& \downarrow T(r_A) \\
& T(I \otimes A)
\end{align*}
\]

\[
\begin{align*}
(A \otimes B) \otimes TC & \xrightarrow{t_{(A \otimes B),C}} T((A \otimes B) \otimes C) \\
& \downarrow \alpha_{A,B,TC} \\
A \otimes (B \otimes TC) & \xrightarrow{id_A \otimes T_{B,C}} A \otimes T(B \otimes C) \\
& \downarrow t_{A,B \otimes C} \\
& T(A \otimes (B \otimes C))
\end{align*}
\]

\[
\begin{align*}
A \otimes B & \xrightarrow{id \otimes \eta_B} A \otimes TB \\
& \downarrow \eta_{A \otimes B} \\
T(A \otimes B) & \xrightarrow{t_{A,B}} T(A \otimes B)
\end{align*}
\]

\[
\begin{align*}
A \otimes T^2 B & \xrightarrow{t_A \cdot T_B} T(A \otimes TB) \\
& \downarrow \mu_{A \otimes B} \\
A \otimes TB & \xrightarrow{t_{A,B}} T(A \otimes B)
\end{align*}
\]

commute for every object \(A, B\) and \(C\) of \(A\).

▶ Remark A.2. Notice that for every strong monad on a symmetric monoidal category \((A, \otimes, I, \alpha, r, l, \gamma)\) we can define a costrength

\[t'_{A,B}: TA \otimes B \to T(A \otimes B)\]

as

\[t'_{A,B} := T(\gamma_{B,A})t_{B,A}T_{\gamma_{A,TB}}.\]
Definition A.3. A strong monad \((T, \mu, \eta)\) on a symmetric monoidal category \((A, \otimes, I, \alpha, r, l, \gamma)\) said to be commutative when the diagram
\[
\begin{array}{ccc}
TA \otimes TB & \xrightarrow{t_{A,B}} & T(TA \otimes B) \\
\downarrow{t_{A,TB}} & & \downarrow{\mu_{A\otimes B}} \\
T(A \otimes TB) & \xrightarrow{T\eta_{A\otimes B}} & T^2(A \otimes B)
\end{array}
\]
commutes for every object \(A\) and \(B\). Let us denote the previous arrow by \(c_{A,B} : TA \otimes TB \rightarrow T(A \otimes B)\).

Example A.4. The list monad \(T_{\text{list}} : \text{Set} \rightarrow \text{Set}\) is strong and commutative. Given two sets \(X\) and \(Y\), the strength is given by the function assigning to an element \(x, [y_1, \ldots, y_m]\) of \(X \times T_{\text{list}}(Y)\) the element \([x, y_1, \ldots, (x, y_m)]\) of \(T_{\text{list}}(X \times Y)\). The previous example is just a particular case of a more general result. We refer to [25, 24] for more details.

Example A.5. Any monad on \(\text{Set}\) is strong and commutative. The strength can be defined as the usual strength of the list monad.

B Omitted proofs

B.1 Proof of Proposition 2.7

1. Every arrow \(!_A\) is \(C\)-total because \(!_I = \text{id}_I\) by Definition 2.1. Moreover \(!_A\) is \(C\)-functional because \((!_A \otimes !_A)\nabla_A = (!_A \otimes \text{id}_I)(\text{id}_A \otimes !_A)\nabla_A\) and, applying the axioms of Definition 2.1, we have that \((!_A \otimes !_A)\nabla_A = (\text{id}_A \otimes !_A) = !_A = \nabla_I!_A\). Every arrow \(\nabla_A\) is \(C\)-total because \(!_A \otimes !_A \nabla_A = (\text{id}_A \otimes !_A)(\text{id}_A \otimes !_A)\nabla_A = !_A\), and it is \(C\)-functional by the first co-monoid axiom combined with the second axiom of \(\nabla_A\).

The proofs of 2 are straightforward.

B.2 Proof of Theorem 3.6

By Theorem 3.5, we have that \(A_T\) is a symmetric monoidal category. Now, let us consider an object \(A\) of the Kleisli category \(A_T\). We define the arrow \(\nabla^A_A : A \rightarrow A \times A\) of \(A_T\) as the arrow
\[
\begin{array}{ccc}
A & \xrightarrow{\nabla_A} & A \times A \\
& \xrightarrow{\eta_{A\times A}} & T(A \times A)
\end{array}
\]
of \(A\). Similarly, we define the arrow \(!^k_A : A \rightarrow 1\) of \(A_T\) as the arrow
\[
\begin{array}{ccc}
A & \xrightarrow{!_A} & 1 \\
& \xrightarrow{\eta} & T1
\end{array}
\]
Since cartesian categories are a particular case of gs-monoidal categories and since \(\nabla^k\) and \(!^k\) are defined in terms of the structural arrows \(\nabla\) and \(!\) of the base plus the unit of the monad, just combining the naturality of the \(\eta\) with the axioms validated by \(\nabla\) and \(!\) in \(C\), we can conclude that \(A_T\) is a gs-monoidal category. One can also directly check that the \(\nabla^k\) and \(!^k\) satisfy the axioms of gs-monoidal categories. In the following, we provide a formal verification of this.
1. We prove that for every object $A$ of $\mathcal{A}$, $(id_A^k \otimes t_A^k)\eta^k_A = id_A$. By definition of arrows (recall that $(id_A^k \otimes t_A^k) = c_{A,1}(\eta_A \times \eta_1)$) and composition in Kleisli categories, we have to prove that

$$(id_A^k \otimes t_A^k)\eta^k_A := \mu_A T(c_{A,1}(\eta_A \times \eta_1))\eta_{A \times A} \eta_A = \eta_A.$$  

First, notice that $(\eta_A \times \eta_1) = (\eta_A \times \eta_1)(id_A \otimes !_A)$. Since $\eta$ is a natural transformation, we have that

$$(id_A^k \otimes t_A^k)\eta^k_A = \mu_A T(c_{A,1}(\eta_A \times \eta_1))\eta_A(id_A \otimes !_A) \eta_A$$

and since $(id_A \otimes !_A)\eta_A = id_A$, we have

$$(id_A^k \otimes t_A^k)\eta^k_A = \mu_A T(c_{A,1}(\eta_A \times \eta_1))\eta_A.$$  

Now, since $c_{A,1} = \mu_A T(t_A'_{1,1})$, and $(\eta_A \times \eta_1) = (id_A \times \eta_1)(\eta_A)$, we have that

$c_{A,1}(\eta_A \times \eta_1) = \mu_A T(t_A'_{1,1})t_A(\eta_A \times \eta_1)(\eta_A) = \mu_A T(t_A'_{1,1})\eta_{1A}\eta_A$  

Now notice that $t_A'_{1,1} = id_A$, hence

$$\mu_A T(t_A'_{1,1}) = \mu_A \eta_{1A}\eta_A = \eta_A$$

and then

$$(id_A^k \otimes t_A^k)\eta^k_A = \mu_A T(c_{A,1}(\eta_A \times \eta_1))\eta_A = \mu_A T(\eta_A)\eta_A = \eta_A.$$  

Therefore we can conclude that $(id_A^k \otimes t_A^k)\eta^k_A = id_A$.

2. We prove that $t_A^1 = id_A$. This follows from the fact that $t_A^1 = \nabla !_1$ and $!_1 = id_A$.

3. We prove that $\gamma_A^n = \nabla^n$. By definition, we have that

$$\gamma_A^n = \mu_{A \times A} T(\eta_{A \times A}) T(\gamma_{A,A}) \eta_{A \times A} \nabla_A.$$  

Since $\mu_{A \times A} T(\eta_{A \times A}) = id_{A \times A}$ and $\eta$ is a natural transformation we have that

$$\mu_{A \times A} T(\eta_{A \times A}) T(\gamma_{A,A}) \eta_{A \times A} \nabla_A = \eta_{A \times A} \gamma_{A,A} \nabla_A = \eta_{A,A} \nabla_A$$

and since $\eta_{A,A} \nabla_A = \nabla^n_A$, we can conclude.

4. Similarly one can check that $(id_A^k \otimes \nabla^k_A)\nabla^k_A = (\nabla_A \otimes id_A)\nabla_A$ and the co-monoid axioms hold.

### B.3 Proof of Theorem 4.5

1. Since $f$ is $C$-functional and applying the second axiom of gs-monoidal categories we have that the followings hold

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\Lambda} \\
A
\end{array}
\end{array}
\begin{array}{c}
\xrightarrow{B}
\end{array}
- \begin{array}{c}
\xrightarrow{A}
\end{array}
\begin{array}{c}
B
\end{array}
- \begin{array}{c}
\xrightarrow{A}
\end{array}
\begin{array}{c}
C
\end{array}
\]

(2) Employing the associativity axiom of $\nabla$ we have the chain of equalities

\[
\begin{array}{c}
\begin{array}{c}
\xrightarrow{A}
\end{array}
\begin{array}{c}
\xrightarrow{C}
\end{array}
\begin{array}{c}
\xrightarrow{B}
\end{array}
\end{array}
- \begin{array}{c}
\xrightarrow{A}
\end{array}
\begin{array}{c}
\xrightarrow{B}
\end{array}
\begin{array}{c}
\xrightarrow{C}
\end{array}
- \begin{array}{c}
\xrightarrow{A}
\end{array}
\begin{array}{c}
\xrightarrow{B}
\end{array}
\begin{array}{c}
\xrightarrow{C}
\end{array}
\end{array}
\]
(3) This point follows directly from the associativity of $\nabla$.

(4) Since $f$ is assumed to be $C$-functional we have that

$$A \otimes B \xrightarrow{C} A \otimes B$$

and from this we can conclude that

$$A \otimes B \xrightarrow{C} A \otimes B$$

B.4 Proof of Theorem 4.6

Since every arrow of $C\text{-}\text{Pfn}$ is $C$-functional, $\nabla$ induces a natural transformation in this sub-category. Moreover, note that since a gs-monoidal category is symmetric we need just to define the projection $\text{id} \otimes !$: the other one is defined by applying the symmetry. Finally, it is immediate to observe that the axioms of p-categories follow from those of gs-monoidal ones.

B.5 Proof of Theorem 4.7

Let us consider two objects $A$ and $B$ of $C$. We claim that $A \otimes B$ is the weak-product of $A$ and $B$ in $C\text{-}\text{Total}$, with projections $\text{id}_A \otimes !_B : A \otimes B \to A$ and $!_A \otimes \text{id}_B : A \otimes B \to B$. First notice that by Proposition 2.7 these projections are $C$-total arrows, since they are monoidal products of $C$-total and $C$-functional morphisms.

Now let us consider two $C$-total arrows $f : C \to A$ and $g : C \to B$. We claim that the following diagram commutes

$$A \xrightarrow{\text{id}_A \otimes !_B} A \otimes B \xrightarrow{!_A \otimes \text{id}_B} B$$

In particular, let us consider the left triangle. In this case $(\text{id}_A \otimes !_B)(f \otimes g)\nabla_C = (f \otimes !_B g)\nabla_C$, and since $g$ is total, then $!_B g = !_C$, and hence $(\text{id}_A \otimes !_B)(f \otimes g)\nabla_C = (f \otimes !_C)\nabla_C = f(\text{id}_C \otimes !_C)\nabla_C = f$. Similarly one can check that $g = (!_A \otimes \text{id}_B)(f \otimes g)\nabla_C$.

B.6 Proof of Theorem 4.9

Let us consider two objects $A$ and $B$ of $C$. We claim that $A \otimes B$ is the cartesian product of $A$ and $B$ in $C\text{-}\text{Fn}$, with projections $\text{id}_A \otimes !_B : A \otimes B \to A$ and $!_A \otimes \text{id}_B : A \otimes B \to B$. First, by Theorem 4.7 we have that this is a weak-product in $C\text{-}\text{Total}$, and in particular, since $C$-total, $C$-functional arrows compose by Proposition 2.7 we have that $A \otimes B$ is a weak-product in $C\text{-}\text{Fn}$. In particular, given two $C$-total, $C$-functional arrows $f : C \to A$ and $g : C \to B$, the arrow $(f \otimes g)\nabla_C$ is $C$-total and $C$-functional and it commutes with projections correctly. We have to prove that it is unique. Now suppose there exists another $C$-total, $C$-functional arrow $h : C \to A \otimes B$ such that $(\text{id}_A \otimes !_B)h = f$ and $(!_A \otimes \text{id}_B)h = g$. Then we have that

$$(f \otimes g)\nabla_C = ((\text{id}_A \otimes !_B)h \otimes (!_A \otimes \text{id}_B)h)\nabla_C = ((\text{id}_A \otimes !_B) \otimes (!_A \otimes \text{id}_B))(h \otimes h)\nabla_C$$
and then, since \( h \) is \( \mathcal{C} \)-functional, we have that
\[
(f \otimes g)\nabla_C = ((\text{id}_A \otimes !_B) \otimes (!_A \otimes \text{id}_B))(h \otimes h)\nabla_C = ((\text{id}_A \otimes !_B) \otimes (!_A \otimes \text{id}_B))\nabla_{A \otimes B} h = h.
\]

### B.7 Proof of Proposition 5.4

1. By definition of preorder-enriched gs-monoidal category we have a 2-cell \( !_B f \leq !_A \), and then we have a 2-cell \( \text{id}_A \otimes !_B f \leq \text{id}_A \otimes !_A \), and hence there is 2-cell \( (\text{id}_A \otimes !_B f)\nabla_A = \text{dom}(f) \leq (\text{id}_A \otimes !_A)\nabla_A = \text{id}_A \).

2. The existence of the 2-cell \( f \text{dom}(f) \leq f \text{dom}(f) \) follows from point (1). Then, we can construct a 2-cell \( f \leq f \text{dom}(f) \) by considering the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f \circ \text{id}_B} & B \\
\nabla_A & \xrightarrow{f \otimes \text{id}_B} & B \otimes B & \xrightarrow{\text{id}_B \otimes !_B} & B \\
A \otimes A & \xrightarrow{\text{id}_A \otimes f} & A \otimes B & \xrightarrow{\text{id}_A \otimes !_B} & A.
\end{array}
\]

The second and the third square commute, while in the first one we only have the 2-cell provided by the lax-natural transformation \( \nabla \). Since \( (\text{id}_B \otimes !_B)\nabla_B = \text{id}_B \) by the second axiom of the definition of gs-monoidal categories, we obtain precisely the 2-cell \( f \leq f \text{dom}(f) \).

### B.8 Proof of Proposition 5.5

First we show that \( H_C : \mathcal{C} \rightarrow \text{Rel} \) is a lax functor between preorder-enriched categories. Let us consider two arrows \( f : A \rightarrow B \) and \( g : B \rightarrow C \) of \( \mathcal{C} \). It is direct to check \( H_C(g) H_C(f) \leq H_C(g f) \) because, by definition of composition of relations, we have that \( e H_C(g) H_C(f) d \) if and only if there exists an arrow \( m \) such that \( m H_C(g) d \) and \( e H_C(f) m \). This means that \( d \leq g m \) and \( m \leq f e \), but by transitivity, this implies that \( d \leq g f e \), and hence that \( e H_C(g f) d \), i.e. \( H_C(g) H_C(f) \leq H_C(g f) \). Similarly, one can check that \( H_C(\text{id}_A) \leq \text{id}_C(A) \).

Notice that the coherence axioms are trivially satisfied since \( \mathcal{C} \) is preorder-enriched.

Moreover, when \( m \leq l \) in \( \mathcal{C}(A, B) \) we have \( H_C(m) \leq H_C(l) \) because if \( h H(m) g \), i.e. \( h \leq m g \) then we can conclude that \( h \leq m g \leq l g \), i.e. \( h H(l) g \). Therefore \( H : \mathcal{C} \rightarrow \text{Rel} \) is a well-defined lax functor.

Now we consider the lax monoidal structure. We can define a natural transformation \( \iota_{A, B} : H_C(A) \times H_C(B) \rightarrow H_C(A \otimes B) \) sending \( (m, n) \mapsto m \otimes n \) for all \( A \) and \( B \) objects of \( \mathcal{C} \) and a trivial morphism \( \epsilon : 1 \rightarrow H_C(I) \). Again we have that these data satisfy the coherence axioms of lax monoidal lax functor.

Finally, for every object \( A \) of \( \mathcal{C} \) we have that \( \nabla_{H_C(A)} \leq H_C(\nabla_A) \) and that \( !_{H_C(A)} \leq H_C(!_A) \).

### B.9 Proof of Proposition 5.7

The proof is constructive. Without the restriction of the left-leg to be regular mono, it is folklore that the monoidal operator induced by binary coproducts is lifted, while its bicategorical structure (and in fact, its 2-categorical one) appears already in the original paper by Bénabou. The fact that it is gs-monoidal is also known when the left-leg is mono, and it is easily shown by a direct proof. Thus, it boils down to show that for the arrows \( \nabla \)’s and \( ! \)’s the left-leg is a regular mono, and that such property is preserved by composition and finite products: the former is assumed, while the latter is easily shown by direct inspection.
B.10 Proof of Proposition 5.9 (sketch)

The equivalence between the free gs-monoidal category on a signature $G$ and the category of discrete cospans of graphs, namely, where the interfaces contains only nodes, has been recently discussed in the context of the string diagram literature in [17, 30]; see in particular [17, Section 3] and [10, 19] for early assessments appearing in the term graph literature. The correspondence instead boils down to the existence of a logical system for inequational deduction over relations and its correspondence with term graph rewriting presented in [13]. More explicitly, the fact that $f$ and $g$ are gs-equivalent in the free gs-monoidal category immediately implies that $[f]$ and $[g]$ are so. Vice versa, morphisms between term graphs have an epi-mono factorisation: each epi can be decomposed into a series of steps merging two edges with the same label, and each mono can be decomposed into a series of steps adding a single edge, thus precisely mimicking the inequalities presented in Definition 5.1.