Gravitational radiation from rotating monopole-string systems

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We study the gravitational radiation from a rotating monopole-antimonopole pair connected by a string. While at not too high frequencies the emitted gravitational spectrum is described asymptotically by \(P_n \propto n^{-1}\), the spectrum is exponentially suppressed in the high-frequency limit, \(P_n \propto \exp(-n/n_{cr})\). Below \(n_{cr}\), the emitted spectrum of gravitational waves is very similar to the case of an oscillating monopole pair connected by a string, and we argue therefore that the spectrum found holds approximately for any moving monopole-string system. As application, we discuss the stochastic gravitational wave background generated by monopole-antimonopole pairs connected by strings in the early Universe and gravitational wave bursts emitted at present by monopole-string networks. We confirm that advanced gravitational wave detectors have the potential to detect a signal for string tensions as small as \(G\mu \sim 10^{-13}\).

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I. INTRODUCTION

The formation of topological defects during symmetry-breaking phase transitions in the early Universe [1] is a generic prediction of Grand Unified Theories. Most prominent example is the breaking of a semi-simple group \(G \rightarrow H \otimes U(1)\) giving rise to monopoles. If in a second phase transition the \(U(1)\) is broken down to \(Z_N\),

\[ G \rightarrow H \otimes U(1) \rightarrow H \otimes Z_N, \tag{1} \]

each monopole gets attached to \(N\) strings. For \(N > 1\), a long-lived monopole-string network (“necklaces”) forms which may evolve towards the scaling regime with several long (infinite) strings and many closed string loops inside the Hubble horizon. The case \(N = 1\), i.e.

\[ G \rightarrow H \otimes U(1) \rightarrow H, \tag{2} \]

is rather different: the second phase transition results in single monopole-antimonopole pairs connected by one string. Typically, the lifetime of this system is much shorter than the Hubble time and it does not survive until present.

An important signature of both systems is the emitted gravitational radiation. Oscillations of monopole-antimonopole pairs connected by strings in the early Universe may form a stochastic gravitational wave background which is—as shown in Ref. [3]—detectable for advanced gravitational wave detectors like LIGO [3], VIRGO [4] and LISA [5]. For the case of string networks that survive until present, Damour and Vilenkin [6, 7] considered the gravitational radiation from cusps of a string network. They argued that the resulting Gravitational Wave Bursts (GWBs) may be detectable for string tensions as small as \(G\mu \sim 10^{-13}\).

In this work, we investigate the gravitational radiation from a simple rotating system consisting of a straight string and two monopoles attached to its ends. We assume that the (anti-) monopole charges are confined inside the string and thus the main energy losses of the topological defects is the emission of gravitational radiation. The solution to the equation of motion for such a system, dubbed “rotating rod”, was found in Ref. [8]. We derive the spectrum of gravitational waves emitted and simple asymptotic formulas for different frequency ranges. We find that the dimensionless parameter \(\mu R/m\), where \(\mu\) is the string tension, \(R\) the distance between monopoles and \(m\) the monopole mass, determines the main features of the emitted spectrum: For not too high mode numbers, \(n \ll n_{cr} = (\mu R/m)^{3/2}\), the gravitational wave spectrum can be approximated by \(P_n \approx 5.77G\mu^2/n\). In the high-frequency limit, \(n \gg n_{cr}\), the gravitational radiation is exponentially suppressed. For moderate mode numbers, \(n \ll n_{cr}\), in which most energy is radiated, the spectrum obtained is very similar to the one of an oscillating monopole-antimonopole pair connected by a string. We conclude therefore that the spectrum found applies also to the generic case of a superposition of a rotational and oscillatory motion of the monopole-string system.

Using the obtained gravitational radiation spectrum of the rotating rod solution we study how the presence of monopoles and antimonopoles in a string network influences the possibility to observe GWBs. We find that the presence of monopoles for any mass below \(M_{Pl}\) does not affect the possibility of GWBs detection.

For the case of a symmetry-breaking scheme with \(N = 1\), we estimate how the differences in the emission spectra between the rotating and oscillating monopole pair connected by string changes the predictions of Ref. [8] for the produced gravitational wave background. We find that the different high-frequency behavior of the two solutions only marginally influences the results.

The paper is organized as follows. In Sec. [9] we study the emission of gravitational radiation from a rotating rod in detail. Then we compare our results with the spec-
trum of another simple solution considered in Ref. \[2\]. In Sec. III we investigate the potential of gravitational wave detectors to observe GWBs from a string-monopole network or the gravitational wave background from $N = 1$ monopole-string systems. We summarize briefly our results in Sec. IV.

II. GRAVITATIONAL RADIATION

The equations of motion for two monopoles connected by a string of length $L$ can be solved analytically only in special cases. One case was dubbed “rotating rod” by the authors of Ref. \[2\] and is given by

\[
x(\sigma, t) = R \sin(\sigma/R) y(t/R),
\]

\[
x_i(t) = \pm (L/2) y(t/R),
\]

\[
\sigma_i = \pm R \arcsin(L/2R),
\]

where $x(\sigma, t)$ is the coordinate of the string at time $t$ and position $\sigma$, $x_i(t)$ are the positions of two point-like masses ($i = 1, 2$), $\sigma_i$ are the spatial parameters of the string at the positions of the two monopoles, $y(\theta) = (\cos \theta, \sin \theta)$, $L = [(1 + 4a^2 R^2)^{1/2} - 1]/a$ and $a = \mu/m$. The system \[2\] has the period $T = \pi R$.

As for any periodic system, we can introduce the Fourier transform of the energy-momentum tensor $T^{\mu\nu}(x, t)$ as

\[
\hat{T}^{\mu\nu}(\omega_n, t) = \frac{1}{T} \int_0^T dt \int d^3x \ T^{\mu\nu}(x, t)e^{i(\omega_n t - k x)},
\]

where $\omega_n = 2\pi n/T$ and $1$ is an arbitrary unit vector. The following calculations are simplified in the corotating basis associated with the vector $\mathbf{1} \mathbf{3}$: In the new basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{v}, \mathbf{w}\}$, where $\mathbf{v}$ and $\mathbf{w}$ are unit vectors perpendicular to each other and to the vector $\mathbf{1}$, the gravitational radiation power at frequency $\omega_n$ per solid angle $d\Omega$ is given by

\[
\frac{dP_n}{d\Omega} = \frac{G\omega_n^2}{\pi} \left[ \tau_{pq}^\mu \tau_{pq}^\nu - \frac{1}{2} \tau_{qq}^\mu \tau_{pp}^\nu \right],
\]

where $\tau_{pq}^{\mu\nu}$ is the Fourier transform of $T^{\mu\nu}$ in the corotating basis. Note that the indices $p, q$ in Eq. \[5\] take only the values 2 and 3.

The energy-momentum $T_{str}^{\mu\nu}$ of a string is given by

\[
T_{str}^{\mu\nu} = \mu \int d\sigma \ (\dot{x}^\mu \dot{x}^\nu - x^\mu x^\nu) \delta^{(3)}(\mathbf{x} - x(\sigma, t))
\]

The Fourier components $\tau_{pq}$ for a string moving according to the system \[3\] can be expressed in the corotating basis as

\[
\tau_{pq}(\omega_n, t) = \frac{\mu R}{\pi} \int_{e^{-\sigma_1}}^{e^{\sigma_2}} d\sigma \int_0^{\pi R} dt e^{i\omega_n(t - k x)}
\]

\[
\times \left[ (\dot{x}_p \dot{x}_q) - (x'_p x'_q) \right].
\]

Introducing the new variables $\tau = t/R$ and $\zeta = \sigma/R$ results in

\[
\tau_{pq}^{\mathrm{str}} = -\mu R \int_{-\zeta_0}^{\zeta_0} d\zeta \cos^2 \zeta \chi_{pq} + \mu R \int_{-\zeta_0}^{\zeta_0} d\zeta \sin^2 \zeta \psi_{pq},
\]

where $\zeta_0 = \arcsin(L/2R)$ and

\[
\chi_{pq} = \frac{1}{2\pi} \int_0^{2\pi} d\tau e^{2in(\pi - \sin \zeta) n y(\tau)} (y(\tau)e_p)(y(\tau)e_q),
\]

\[
\psi_{pq} = \frac{1}{2\pi} \int_0^{2\pi} d\tau e^{2in(\pi - \sin \zeta) n y(\tau)} (\dot{y}(\tau)e_p)(\dot{y}(\tau)e_q).
\]

The energy-momentum tensor $T_{\mathrm{mon}}^{ij}$ of two point-like particles moving according to Eq. \[3\] is given by

\[
T_{\mathrm{mon}}^{ij}(t, x) = m \gamma_0 [\delta(x-x_1(t)) + \delta(x-x_2(t))] \hat{x}_i \hat{x}_j
\]

where $\gamma_0 = (1 - (L/2R)^2)^{-1/2}$ is the Lorentz factor of the point-like masses. The Fourier components of the energy-momentum tensor in the corotating basis follow as

\[
\tau_{pq}^{\mathrm{mon}} = \gamma_0 m \left( \frac{L}{2R} \right)^2 \frac{1}{2\pi} \int_0^{2\pi} d\tau \left[ e^{2in(\pi - L/2R) y(\tau)} + e^{2in(\pi + L/2R) y(\tau)} \right] (\dot{y}(\tau)e_p)(\dot{y}(\tau)e_q).
\]

Applying the relation

\[
\frac{\mu}{m} = 2 \gamma_0 \left( \frac{L}{2R} \right)^2 = \frac{\sin \zeta_0}{R \cos^2 \zeta_0},
\]

then gives

\[
\tau_{pq}^{\mathrm{mon}} = R \mu \sin \zeta_0 \cos \zeta_0 \left( \frac{1}{2\pi} \int_0^{2\pi} d\tau \left[ e^{2in(\pi - L/2R) y(\tau)} + e^{2in(\pi + L/2R) y(\tau)} \right] (\dot{y}(\tau)e_p)(\dot{y}(\tau)e_q),
\]

The Fourier transform of the total energy-momentum tensor $\tau_{\mathrm{total}}^{pq}$ is the sum of $\tau_{\mathrm{str}}$ and $\tau_{\mathrm{mon}}$. The final expression for $\tau_{\mathrm{total}}^{pq}$ is (see Appendix A for details of the calculation)

\[
\tau_{\mathrm{total}}^{pq} = \frac{\mu R}{\sin \theta} \times
\]

\[
\times \left[ \int_0^{\zeta_0} \frac{d\zeta}{R_{n}^{\mu\nu}(\sin \theta, \zeta)} + S_{n}^{\mu\nu}(\sin \theta, \zeta_0) \right],
\]

where

\[
R_{n}^{22}(x, z) = -\frac{2 \cos^2 \theta}{x} J_{2n}(2nx \sin z),
\]

\[
R_{n}^{23}(x, z) = -2i \cos \theta J_{2n}^*(2nx \sin z) \sin z,
\]

\[
R_{n}^{33}(x, z) = \frac{2}{x} J_{2n}(2nx \sin z) \left[ 1 + x^2 \cos(2z) \right],
\]
and

\[ S_{1}^{22}(x, z) = -\frac{2(1-x^2)}{x}J_{2n}(2nx \sin z) \frac{\cos z}{\sin z}, \]

\[ S_{1}^{23}(x, z) = -i \sqrt{1-x^2}J_{2n}(2nx \sin z) \cos z, \quad (15) \]

\[ S_{1}^{33}(x, z) = \frac{2}{x}J_{2n}(2nx \sin z) \cos \frac{z}{\sin z} [1-x^2 \sin^2 z]. \]

Let us now consider the limit of heavy monopoles or short, light strings, \( \zeta \ll 1 \). Keeping only terms up to first order in \( \zeta \), the integrals in Eq. (13) containing \( R_{ij}^{pq} \) can be neglected. This means that the contribution of the string to the total energy-momentum tensor is negligible. For \( S_{ij}^{pq} \) one finds

\[ S_{1}^{22} = S_{1}^{23} = S_{1}^{33} \approx \nu x \zeta, \]

\[ S_{1}^{pq} \approx 0, \quad l \geq 2. \quad (16) \]

Substituting Eq. (16) into Eq. (13), the components of the energy-momentum are in the limit \( \mu R/m \ll 1 \)

\[ \tau_{total}^{22} = -\frac{(\mu R)^2}{m} \cos^2 \theta, \]

\[ \tau_{total}^{23} = -\frac{i(\mu R)^2}{m} \cos \theta, \]

\[ \tau_{total}^{33} = \frac{(\mu R)^2}{m}. \quad (17) \]

This expression coincides with the one calculated using the quadrupole approximation for moving particles,

\[ \tau_{pq}^{ij} = -\frac{\omega^2}{2} \int d^3x (xe^p)(xe^q) T^{00}(\omega, x). \quad (18) \]

The opposite case of light particles, \( \mu R/m \gg 1 \) or \( \zeta_0 - \pi/2 \ll 1 \), is more complicated. For high frequencies, the radiation is highly concentrated in the plane of rotation of the system. For not too high mode numbers, \( 1 \ll n \ll n_{cr} \), where \( n_{cr} \) is given by

\[ n_{cr} = \left( \frac{\mu R}{m} \right)^{3/2}, \quad (19) \]

the radiation rate of the whole system behaves as if there are no monopoles at the ends of the string: Using the expression for the Lorentz-factor in the ultra-relativistic limit,

\[ \gamma_0 = \left( \frac{\mu R}{m} \right)^{1/2}, \quad (20) \]

the cut-off number \( n_{cr} \) can be rewritten as

\[ n_{cr} = \gamma_0^3. \quad (21) \]

In the range \( 1 \ll n \ll n_{cr} \), the gravitational radiation is well approximated by (see Appendix B)

\[ P_n = CG\mu^2 \frac{2n}{n}, \]

where the coefficient \( C \) is given by Eq. (B10); its numerical value is \( C \approx 5.77 \). For \( 1 \ll n \ll n_{cr} \), the gravitational radiation depends only on the string tension \( \mu \) and not on the monopole mass \( m \). In the absence of monopoles, \( m \to 0 \), the total radiated power diverges logarithmically. In this case, the back reaction of the gravitational radiation on the string should be taken into account. In contrast, the gravitational spectrum from cosmic strings with monopoles on their ends has a finite cut-off frequency and, thus, the total emitted gravitational radiation is finite.

For very high mode numbers, \( n \gg n_{cr} \), the radiated gravitational power depends on the ratio \( \mu R/m \) as well as on \( \mu \) and is given by (see Appendix B)

\[ P_n = G\mu^2 \sqrt{\frac{2}{\pi}} \left( \frac{m}{\mu R} \right)^{9/4} \sqrt{n} \exp \left[ -4n \left( \frac{m}{\mu R} \right)^{3/2} \right] \]

\[ = G\mu^2 \frac{2n}{n} \gamma_0^{3/2} e^{-4n\gamma_0^{3/2}}. \quad (23) \]

An example of the gravitational spectrum for \( \gamma_0 = 100 \) is shown in Fig. 2 together with the two asymptotic expressions, Eq. (22) and Eq. (23). For very small \( n \), the discrete spectrum of the radiated energy cannot be neglected and the low-frequency approximation underestimates the gravitational radiation slightly, while in the range \( 1 \ll n \ll \gamma_0^3 \) the approximation describes well the exact result. For \( n \gg \gamma_0^3 \), the high-frequencies limit approximates well Eq. (23).

For moderate values of \( n_{cr} \), the total gravitational radiation power \( P \) can be calculated by summing up the radiation power at different modes \( P_n \), while for larger values of \( n_{cr} \), a semi-analytical formula for \( P \) is useful. With \( n_{cr} \propto \gamma_0^3 \), we obtain as total gravitational power

\[ P \approx G\mu^2 (3C \ln \gamma_0 + B). \quad (24) \]

The numerical coefficient \( B \approx 6.8 \) accounts for the deviation of the low-frequency limit Eq. (22) from \( P_n \) for \( n \to 1 \). In Fig. 2, the total gravitational radiation \( P \) (solid line) and the approximation (dashed line) is shown as a function of \( \ln \gamma_0 \).

At the end of this section, we want to compare our results for the gravitational radiation from a rotating monopole-string system with those for an oscillating monopole-string system derived in Ref. 2. There are two main difference between the rotating and the oscillating system: Firstly, the Lorentz factor of the monopoles depends differently on the dimensionless quantity \( \mu R/m \). While \( \gamma_0 = (\mu R/m)^{1/2} \) for the rotating rod, \( \gamma_0 = \mu R/m \) for the oscillating solution. Secondly, the critical mode number \( n_{cr} \) is for the oscillating solution given by \( n_{cr} = \gamma_0^3 \sim (\mu R/m)^2 \) compared to \( n_{cr} = \gamma_0^3 \sim (\mu R/m)^{3/2} \) for the rotating rod.

For modes below the critical number \( n_{cr} \) of both solutions, the spectrum of emitted gravitational waves is surprisingly similar: \( P_n/(G\mu^2)^3 \approx 4/n \) for the oscillating solution and \( P_n/(G\mu^2)^3 \approx 5.77/n \) for the rotating rod.
solution. The difference becomes more pronounced only for frequencies above $n_{cr}$. While the high-frequency limit of the rotating rod solution is well-behaved and modes with $n \gg n_{cr}$ are exponentially suppressed, the emitted power behaves as $P_n \propto n^{-2}$ for the oscillating solution. This $1/n^2$ behavior is caused by the discontinuity of the monopole acceleration and is probably unphysical. In the generic case of a superposition of an oscillating and rotating monopole-string system we expect therefore an exponential decay of the radiated power similar to the Eq. (23). We estimate therefore the emitted gravitational radiation in the generic case as

$$\frac{P_n}{G\mu^2} = \begin{cases} 5/n, & n \lesssim n_{cr}, \\ 0, & n \gtrsim n_{cr} \end{cases}, \quad (25)$$

where $n_{cr}$ is given by (19). The total gravitational power in the generic case follows as

$$P \sim 7G\mu^2 \ln \left(\frac{\mu R}{m}\right). \quad (26)$$

III. DETECTION OF THE GRAVITATIONAL WAVE SIGNAL

In this section we study the possibility to detect the gravitational radiation from oscillating and rotating monopole-string systems. In the first part, we consider the gravitational radiation from a network of monopoles and antimonopoles connected by $N$ strings. The evolution of such a network can be analyzed in two limiting cases. If monopoles are light enough, then the cosmic string network does not feel the presence of monopoles. Thus we obtain the “normal” scaling solution for the network evolution where the typical length $\xi$ of closed strings is

$$\xi \sim \Gamma G\mu t. \quad (27)$$

Here, $t$ is the cosmological time and the dimensionless coefficient $\Gamma \sim 50$ determines the gravitational radiation from strings. In the opposite limit, we may assume that the gravitational radiation from monopoles is the dominant energy loss mechanism. Then we find, following Ref. [11], another scaling solution for the typical distance $d$ between monopoles with

$$d \sim \tilde{\Gamma} G\mu t, \quad (28)$$

where $\gamma$ is the typical Lorentz-factor of the monopoles and $\Gamma \sim 10\ln\gamma$. Apart from an numerically not too important factor $5/\ln\gamma$, the two limits give the same characteristic length $\xi$ for the network [13].

The presence of monopoles (and antimonopoles) on the strings changes the GWBs from cusps of cosmic strings. Massive particles living on the string affect the cusp formation which happens only in the case of “ordinary” Nambu-Goto strings. Adding monopoles makes strings heavier and smoothes-out cusps, similar to the presence of superconducting currents on the string. The consequence of smoothed cusps is a cut-off in the emitted

FIG. 1: A log-log plot of the gravitational spectrum $P_n/G\mu^2$ for $\gamma = 100$ (solid line) is shown. The approximation is plotted by dashed line and high frequency approximation is plotted by dotted line.

FIG. 2: the total gravitational radiation $P$ (solid line) and the approximation (dashed line) is shown as a function of $\ln\gamma_0$. 
gravitational radiation \[12\]. Therefore we assume that the presence of monopoles on the strings leads to a cut-off given by Eq. (19),

\[
\omega_{cr} \sim \frac{\gamma_0^3}{\xi}. \tag{29}
\]

In Ref. [8], Damour and Vilenkin argued that the gravitational bursts from ordinary string cusps can be detected by the gravitational wave detectors like LIGO, VIRGO and especially LISA. How is this analysis changed by the presence of monopoles on the strings? The main difference of the radiation of a string-monopole network from the radiation of an ordinary string network is the existence of a cut-off frequency, such that the signal from cusps with frequencies \(\omega > \omega_{cr}\) is highly suppressed.

The cut-off frequency of the arriving signal is

\[
\omega_{cr} = \frac{\gamma_0}{(\Gamma G \mu_0)}^{-1} \left\{ \begin{array}{l}
(1+z)^{1/2}, \\
(1+z)(1+z_{eq})^{-1/2},
\end{array} \right. \quad z \lesssim z_{eq}, \quad \frac{1}{1+z_{eq}} \lesssim z \lesssim z_{eq}, \tag{30}
\]

where at redshift \(z_{eq}\) matter and radiation densities are equal.

The typical length of loops decreases with \(z\) and consequently the cut-off frequency increases with red-shift. Using Eq. (27) and \(R \sim \xi\), the behavior of the Lorentz-factor as function of the redshift follows as

\[
\gamma_0 = (\Gamma G \mu)^{1/2} \left( \frac{\mu_0}{m} \right)^{1/2} \times \left\{ \begin{array}{l}
(1+z)^{-3/4}, \\
(1+z_{eq})^{-1/4},
\end{array} \right. \quad z \lesssim z_{eq}, \quad \frac{1}{1+z_{eq}} \lesssim z \lesssim z_{eq}. \tag{31}
\]

Hence, the cut-off frequency is given by

\[
\omega_{cr} = (\Gamma G \mu)^{1/2} \left( \frac{\mu_0}{m} \right)^{3/2} t_0^{-1} \times \left\{ \begin{array}{l}
(1+z)^{-7/4}, \\
(1+z_{eq})^{1/4}(1+z)^{-2},
\end{array} \right. \quad z \lesssim z_{eq}, \quad \frac{1}{1+z_{eq}} \lesssim z \lesssim z_{eq}. \tag{32}
\]

The analysis given by Damour and Vilenkin [8] leads to the following result: for a given frequency \(\omega\) and for a given string parameter \(\mu\) (or, strictly speaking, for a given parameter \(\Gamma G \mu\)) the amplitude of incoming signals \(h\) from cusps in the universe decrease with \(z\), but the number of such signals per unit time \(N(\mu, \omega)\) increases with \(z\). Thus, we should find the compromise between the rate of signals and their amplitude. For a given \(N\) one can find the minimal amplitude of incoming signals \(h(\mu, N, \omega)\) and corresponding maximal red-shift \(z_{m}(\mu, N, \omega)\).

How does the existence of the cut-off frequency \(\omega_{cr}\) modify these arguments? The presence of monopoles may effectively lower the amplitude of the signal from cusp. Let us estimate the minimal mass of monopoles that may affect the amplitude \(h\). There are three regimes of behavior of \(z_{m}\) in dependence of the function \(y(\mu, N, \omega)\),

\[
y(\mu, N, \omega) = 10^{-2}(N/c) t_0^{5/3} (\Gamma G \mu)^{8/3} \omega^{2/3}, \tag{33}
\]

where \(c\) is the the average number of cusps per loop period (usually it is taken that \(c \sim 0.1 - 1\)). The maximal redshift \(z_{m}(\mu)\) can be expressed as

\[
z_{m}(\mu) = \left\{ \begin{array}{l}
y^{1/3}, \quad y \leq 1, \\
y^{6/11}, \quad 1 \leq y \leq y_{eq}, \\
(y_{eq} y)^{3/11}, \quad y_{eq} \leq y, \end{array} \right. \tag{34}
\]

where \(y_{eq} = z_{11/6}^{11/6}\). Inserting (34) in (30) we can find from the equation \(\omega_{cr} = \omega\) the critical value \(\mu_0/m\) for which the monopoles suppress the amplitude of the gravitational signal from the network,

\[
\frac{\mu_0}{m} = \left( \frac{\omega t_0}{(\Gamma G \mu)} \right)^{2/3} \times \left\{ \begin{array}{l}
1, \quad y \leq 1, \\
\frac{11}{2}, \quad 1 \leq y \leq y_{eq}, \\
\frac{11}{4}, \quad y_{eq} \leq y. \end{array} \right. \tag{35}
\]

Let us fix the rate of observable GWBs \(N/c \sim 1 \text{ yr}^{-1}\), the frequency \(\omega \sim 10^{16}\) (preferable for detecting GWBs with LIGO/VIRGO) and find \(\mu_0/m\) as a function of \(G \mu\). From (35) we obtain

\[
\frac{\mu_0}{m} = \left\{ \begin{array}{l}
10^{12} (G \mu)^{-1/3}, \quad G \mu \leq 10^{-10}, \\
10^{29} (G \mu)^{15/11}, \quad 10^{-10} \leq G \mu \leq 10^{-7}, \\
10^{24} (G \mu)^{7/11}, \quad G \mu \geq 10^{-7}. \end{array} \right. \tag{36}
\]

For instance, for strings with \(G \mu \sim 10^{-13}\) we have \(\mu_0/m \sim 10^{16}\), while for grand unified theory strings with \(G \mu \sim 10^{-9}\) we find \(\mu_0/m \sim 10^{20}\). Thus the monopoles should have masses well beyond the Planck mass, which, of course, can not be realized in nature. Therefore we conclude that the presence of monopoles on the string network does not affect the observation of gravitational wave burst from a cosmic string network.

In the end of this section, let us consider the gravitational radiation background from hybrid topological defects, formed in the sequence of phase transitions with \(N = 1, 2\). The evolution of single monopole-antimonopole pairs connected by one string is very different from the one considered above. This system has a lifetime much less than the Hubble time, i.e. does not survive till now. The efficient emission of gravitational waves depends crucially when the phase transition producing monopoles happens. If monopoles are formed after inflation and later get connected by strings (e.g. as in the Langacker-Pi model [8]), the monopole-string system loses most of its energy due to friction with the surrounding plasma and a negligible gravitational wave background results. However, in another possible scenario proposed in Ref. [8] traces of topological defects might be left in the form of gravitational waves. In this scenario, monopoles are formed during inflation, but are not swept away completely. The evolution of the monopole-string system is even during the period of large friction similar to the evolution of “ordinary” strings without monopoles, because of the large monopoles separation. Therefore, the monopole-antimonopole pairs connected by strings survive the period of large damping forces. After that,
the string-monopole systems begin to move relativistically and radiate most of their energy into gravitational waves. Martin and Vilenkin estimated the gravitational wave background from such hybrid defects using for the calculations the gravitational spectrum of the oscillating solution. The results of Ref. do not change strongly for a generic moving monopole-string system, because of two reasons: (i) at low frequencies the spectra are almost the same except an unimportant factor 1.4 and (ii) the different dependence of the cut-off frequency on the parameter $\mu R/m$ enters in the formula for the total power emitted only logarithmically. Thus the conclusion of Ref. that the gravitational wave background might be observable by advanced interferometers does not change.

IV. CONCLUSION

We have studied the radiation of gravitational waves by a “rotating rod”, i.e. a rotating monopole-antimonopole pair connected by a cosmic string. We have found that the dimensionless parameter $\mu R/m$ determines the main features of the emitted spectrum: For not too high mode numbers, $n \ll n_c = (\mu R/m)^{3/2}$, the gravitational wave spectrum can be approximated by $P_n \approx 5.77 G \mu^2/n$, while the gravitational radiation is exponentially suppressed for $n \gg n_c$. The total gravitational radiation from the rotating rod is given by $P \approx G \mu^2 (17.3 \ln \gamma + 6.8)$.

The total power emitted and also the spectrum at not too high frequencies found by us agrees approximately with the spectrum of an oscillating monopole-antimonopole pair connected by a string from Ref. Only in the high frequency limit differences appear: While the gravitational radiation is as expected on physical grounds exponentially suppressed for the rotating solution, it decays only as $1/n^2$ for the oscillating solution. Therefore, we have concluded that the spectrum found by us applies approximately also to the generic case of a superposition of a rotational and oscillatory motion of the monopole-string system. Moreover, studies about the potential of interferometers to observe gravitational wave emitted by topological defect networks that were based on the oscillating solution are valid also for more general motions of the monopole-string systems. We have confirmed that advanced gravitational wave detectors like LIGO, VIRGO and LISA have the potential to detect a signal topological hybrid defects for string tensions as small as $G \mu \sim 10^{-13}$.

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APPENDIX A: ENERGY-MOMENTUM TENSOR

In this Appendix, we present the details of our calculation of the energy-momentum tensor of a rotating rod. The considered system is axially symmetric and, therefore, it is sufficient to calculate the radiation only in the plane perpendicular to the plane of rotation. For definiteness, let us choose the following corotating basis,

\[ l = \{\sin \theta, 0, \cos \theta\}, \]
\[ v = \{\cos \theta, 0, -\sin \theta\}. \]  

\[ \text{(A1)} \]
\( w = \{0, 1, 0\} \).

Substituting (A1) into (8), one obtains

\[
\begin{align*}
\chi^{22} &= \frac{1}{2} \cos^2 \theta \left[ J_{2n-1}^{'}(2nr) - J_{2n+1}^{'}(2nr) \right], \\
\chi^{23} &= i \frac{1}{2} \cos \theta \left[ J_{2n-1}^{'}(2nr) + J_{2n-1}(2nr) \right], \\
\chi^{33} &= -\frac{1}{2} \left[ J_{2n-1}^{'}(2nr) - J_{2n+1}(2nr) \right] - J_{2n}(2nr),
\end{align*}
\]

and

\[
\begin{align*}
\psi^{22} &= -\frac{1}{2} \cos^2 \theta \times \\
&\times \left[ \frac{2n-1}{2nr} J_{2n-1}(2nr) + \frac{2n+1}{2nr} J_{2n+1}(2nr) \right], \\
\psi^{33} &= \frac{1}{2} \left[ \frac{2n-1}{2nr} J_{2n-1}(2nr) + \frac{2n+1}{2nr} J_{2n+1}(2nr) \right] - J_{2n}(2nr),
\end{align*}
\]

where \( J_n(z) \) is \( n \)-th Bessel function and \( r = |\sin \zeta \sin \theta| \). Inserting (A2) and (A3) into (8), we find as components of the energy-momentum tensor of the string

\[
\begin{align*}
\tau^{22}_{\text{str}} &= -2\mu R \cos^2 \frac{\theta}{\sin^2 \theta} \\
&\times \left[ \int_0^{\phi_0} J_{2n}(2nr) d\zeta - \frac{\sin \theta}{2n} J_{2n}^{'}(2nr_0) \cos \zeta_0 \right], \\
\tau^{23}_{\text{str}} &= -\mu R \frac{\cos \theta}{\sin \theta} \\
&\times \left[ \int_0^{\phi_0} \left[ J_{2n-1}(2nr) - J_{2n+1}(2nr) \right] d\zeta \\
&- \frac{1}{n \sin \theta \sin \zeta_0} J_{2n}(2nr_0) \cos \zeta_0 \right], \\
\tau^{33}_{\text{str}} &= 2\mu R \frac{1}{\sin \theta} \\
&\times \left[ \int_0^{\phi_0} \left[ J_{2n}(2nr) + \sin^2 \theta \cos(2\zeta) J_{2n}(2nr) \right] d\zeta \\
&- \frac{\sin \theta}{2n} J_{2n}^{'}(2nr_0) \cos \zeta_0 \right].
\end{align*}
\]

Let us turn now to the calculation of the energy-momentum tensor of the monopoles. Using the fact that the expression (A2) is very similar to the expression for \( \psi^{pq} \) in (8), one easily obtains

\[
\begin{align*}
\tau^{22}_{\text{mon}} &= -\mu R \cos \zeta_0 \frac{\cos^2 \theta}{\sin \theta} \\
&\times \left[ \frac{2n-1}{2l} J_{2n-1}(2nr_0) + \frac{2n+1}{2n} J_{2n+1}(2nr_0) \right], \\
\tau^{23}_{\text{mon}} &= -\mu R \frac{\cos \zeta_0}{\sin \theta} \times \left[ \frac{2n-1}{2n} J_{2n-1}(2nr_0) - \frac{2n+1}{2n} J_{2n+1}(2nr_0) \right], \\
\tau^{33}_{\text{mon}} &= \mu R \cos \zeta_0 \frac{1}{\sin \theta} \times \left[ \frac{2n-1}{2n} J_{2n-1}(2nr_0) + \frac{2n+1}{2n} J_{2n+1}(2nr_0) \right] \\
&- \mu R \sin(2\zeta_0) J_{2n}(2nr_0).
\end{align*}
\]

Summing then (A4) and (A5), we finally obtain the Fourier components of energy-momentum of the whole system.

**APPENDIX B: HIGH-FREQUENCY BEHAVIOR**

Throughout this Appendix we assume that the monopoles are very light, i.e. \( R\mu/m \gg 1 \). The qualitative behavior of the radiation emitted by a rotating string at high frequencies can be derived using the following asymptotic relations for Bessel functions of large order (cf. Ref. [14]),

\[
J_n(nx) \propto \begin{cases}
 n^{-1/3}, & n \ll n_{cr}, \\
 \exp(-n/n_{cr}), & n \gg n_{cr},
\end{cases}
\]

and

\[
J_n'(nx) \propto \begin{cases}
 n^{-2/3}, & n \ll n_{cr}, \\
 \exp(-n/n_{cr}), & n \gg n_{cr},
\end{cases}
\]

where \( n_{cr} \) is the cut-off frequency given by

\[
n_{cr} \simeq 3 \left( \frac{\mu R}{m} \right)^{3/2} \times \frac{3}{(\pi/2 - \zeta_0)^3}.
\]

Using the relations (B1) and (B2), from (B4) follows that the radiated power \( P_n \) behaves for \( 1 \ll n \ll n_{cr} \) as

\[
P_n \propto n^{-1},
\]

which was pointed by Martin and Vilenkin [2]. If there are no point-like masses on the ends of a string, then the total radiated power diverges because of presents of string points, permanently moving with the speed of light. The presence of particles of any non-zero mass at the ends of string points removes such a divergence. For \( n \gg n_{cr} \), the radiated gravitational power falls exponentially (as can be seen from (B1) and (B2)), thus making the sum over all modes finite.

Let us calculate now the high-frequency behavior of the gravitational radiation more precisely. For not very high frequencies, \( 1 \ll n \ll n_{cr} \), the contribution from monopoles to the gravitational radiation may be neglected. In this case we can simply set \( \zeta_0 = \pi/2 \) in the expression for the components of the energy-momentum tensor, Eq. (13). Then \( S_{pq}^{\alpha} = 0 \) and, moreover, the integrals in Eq. (13) can be integrated in closed form. Using...
the properties of integrals involving Bessel functions, we obtain

\[
\tau_{\text{total}}^{22} = -\pi \mu R \frac{\cos^2 \theta}{\sin^2 \theta} J_n^2(n \sin \theta),
\]

\[
\tau_{\text{total}}^{23} = -i \pi \mu R \frac{\cos \theta}{\sin \theta} J_n(n \sin \theta) J'_n(n \sin \theta),
\]

\[
\tau_{\text{total}}^{33} = \pi \mu R J'_n(n \sin \theta).
\]

Substituting these expressions into Eq. (5) results in

\[
\frac{dP_n}{d\Omega} = 2\pi G \mu^2 n^2 \left[ \frac{\cos^4 \theta}{\sin^2 \theta} J_n^4(n \sin \theta) \right]
+ 6 \frac{\cos \theta}{\sin \theta} J_n^2(n \sin \theta) J_n^2(n \sin \theta) + J_n^4(n \sin \theta),
\]

and, introducing the variable \( x = \sin \theta \),

\[
P_n = 8\pi^2 G \mu^2 n^2 \int_0^1 \frac{d x}{\sqrt{1 - x^2}} \left[ \frac{(1 - x^2)^2}{x^4} J_n^4(nx) \right]
+ \frac{6 - x^2}{x^2} J_n^2(nx) J_n^2(nx) + J_n^4(nx).
\]

This expression coincides with the result of Martin and Vilenkin for a rotating string without monopoles \[2\]. To calculate the integral in (B7) we use further the following approximations for Bessel functions and their derivatives,

\[
J_n(nx) \simeq \left( \frac{2}{n} \right)^{1/3} \operatorname{Ai} \left[ 2^{1/3} n^{2/3} (1 - x) \right],
\]

\[
J'_n(nx) \simeq -\left( \frac{2}{n} \right)^{2/3} \operatorname{Ai}' \left[ 2^{1/3} n^{2/3} (1 - x) \right],
\]

which is valid for \( n \gg 1 \). Here, \( \operatorname{Ai}(z) \) is the Airy function.

Substituting (B8) into (B7) and then changing the variables to \( u = 2^{1/3} n^{2/3} (1 - x) \), we obtain after some algebra

\[
P_n \simeq \frac{C G \mu^2}{n},
\]

where the numerical factor \( C \) is given by

\[
C = 32\pi^2 \int_0^\infty \frac{d u}{\sqrt{u}} \left[ u^2 \operatorname{Ai}^4(u) \right] + 6u \operatorname{Ai}^2(u) \operatorname{Ai}^2(u) + \operatorname{Ai}^4(u).
\]

(In the last integral we changed the limit \( 2^{1/3} n^{2/3} \) to \( \infty \).) The numerical value of \( C \) is \( C \simeq 5.77 \).

For \( n \gg n_{\text{cr}} \) the Airy functions in (B8) may be approximated by exponentials resulting in the following approximated expressions for Bessel function and its derivative

\[
J_{2n}(2nz) \simeq \frac{1}{2\sqrt{\pi n}} e^{-2n(1-z^2)^{3/2}/3},
\]

\[
J'_{2n}(2nz) \simeq \frac{(1 - z^2)^{1/4}}{2\sqrt{\pi n}} e^{-2n(1-z^2)^{3/2}/3}.
\]

Substituting (B11) into (B8), one finds that the \( S \) terms are enhanced by the factor \( n(1 - x^2 \sin^2 \zeta_0)^{1/2} \gg 1 \) compared to the \( R \) terms. Therefore, the \( R \) terms can be neglected. Further, we introduce as a new variable \( y = 2n(1 - x) \cos \zeta_0 \) and notice that the components of the energy-momentum tensor (13) are proportional to the same exponential, but with prefactors proportional to \( n^{-3/2}, n^{-1} \) and \( n^{-1/2} \) for \( \tau_{22}, \tau_{23}, \tau_{33} \), respectively. Thus the leading term in the expression for the radiated power is the \( \tau_{33} \) term. Keeping only this component of the energy-momentum tensor, expanding the argument of the exponent around \( y = 0 \) and taking the other slower varying terms as being constant, we obtain

\[
P_n \simeq G \mu^2 \frac{\cos^{9/2} \zeta_0}{\pi} \sqrt{\frac{2n}{\pi}} \times
\]

\[
\times e^{-(4/3)n \cos^2 \zeta_0} \int_0^{4n \cos \zeta_0} \frac{y^{-1/2}}{y} dy e^{-y}
\]

\[
\simeq G \mu^2 \left( \frac{m}{\mu R} \right)^9 \sqrt{\frac{2n}{\pi}} \times
\]

\[
\times e^{-(4/3)n(m/\mu R)^{3/2}} \int_0^{8n} y^{-1/2} dy e^{-y}
\]

\[
= G \mu^2 \sqrt{\frac{2}{\pi}} \left( \frac{m}{\mu R} \right)^9 \sqrt{n} \exp \left[ -\frac{1}{3} \left( \frac{m}{\mu R} \right)^{3/2} \right].
\]

As expected, the radiation rate is exponentially suppressed for \( n \to \infty \).