Orthogonally $a$-Jensen mappings on $C^*$-modules

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Abstract. We investigate the representation of the so-called orthogonally $a$-Jensen mappings acting on $C^*$-modules. More precisely, let $\mathfrak{A}$ be a unital $C^*$-algebra with the unit 1, let $a \in \mathfrak{A}$ be fixed such that $a, 1 - a$ are invertible and let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be inner product $\mathfrak{A}$-modules. We prove that if there exist additive mappings $\varphi, \psi$ from $\mathcal{F}$ into $\mathcal{E}$ such that $\langle \varphi(y), \psi(z) \rangle = 0$ and $a \langle \varphi(y), \varphi(z) \rangle a^* = (1 - a) \langle \psi(y), \psi(z) \rangle (1 - a)^*$ for all $y, z \in \mathcal{F}$, then a mapping $f : \mathcal{E} \to \mathcal{G}$ is orthogonally $a$-Jensen if and only if it is of the form $f(x) = A(x) + B(x, x) + f(0)$ for $x \in \mathcal{K} := \varphi(\mathcal{F}) + \psi(\mathcal{F})$, where $A : \mathcal{E} \to \mathcal{G}$ is an $a$-additive mapping on $\mathcal{K}$ and $B$ is a symmetric $a$-biadditive orthogonality preserving mapping on $\mathcal{K} \times \mathcal{K}$. Some other related results are also presented.

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1. Introduction

Orthogonal functionals on an inner product space when the orthogonality is the ordinary one were considered by Pinsker [10]. Next Sundaresan [13] generalized the result of Pinsker to arbitrary Banach spaces equipped with the Birkhoff–James orthogonality. In recent decades, mappings satisfying a functional equation under some orthogonality conditions have been investigated by several mathematicians, who have presented many interesting results and applications, see, e.g., [1–4, 7, 9, 11].

Jensen [5] first studied functions satisfying the condition $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$. It is easy to see that every continuous Jensen function on $\mathbb{C}$ is affine in the sense that $f - f(0)$ is additive. The Jensen functional equation has been extensively studied from many points by many mathematicians, see, e.g., [8, 12] and the references therein.

Let us recall some definitions and introduce our notation. An inner product module over a $C^*$-algebra $\mathfrak{A}$ is a (right) $\mathfrak{A}$-module $\mathcal{E}$ equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle$, which is $\mathbb{C}$-linear and $\mathfrak{A}$-linear in the second variable and
has the properties $\langle x, y \rangle^* = \langle y, x \rangle$ as well as $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$. An inner product $\mathfrak{A}$-module $\mathcal{E}$ is called a Hilbert $\mathfrak{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

Although inner product $C^*$-modules generalize inner product spaces by allowing inner products to take values in an arbitrary $C^*$-algebra instead of the $C^*$-algebra of complex numbers, some fundamental properties of inner product spaces are no longer valid in inner product $C^*$-modules. For example, not each closed submodule of an inner product $C^*$-module is complemented. Therefore, when we are studying inner product $C^*$-modules, it is always of some interest to find conditions to obtain results analogous to those for inner product spaces. We refer the reader to [6] for more information on the theory of $C^*$-algebras and the structure of Hilbert $C^*$-modules.

Let $\mathcal{E}$ and $\mathcal{F}$ be two inner product $\mathfrak{A}$-modules. A morphism between inner product $\mathfrak{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ is a mapping $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ satisfying $\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{E}$. A mapping $t : \mathcal{E} \rightarrow \mathcal{F}$ is called adjointable if there exists a mapping $s : \mathcal{F} \rightarrow \mathcal{E}$ such that $\langle tx, y \rangle = \langle x, sy \rangle$ for all $x \in \mathcal{E}, y \in \mathcal{F}$. The unique mapping $s$ is denoted by $t^*$ and is called the adjoint of $t$. Furthermore, inner product $\mathfrak{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ are unitarily equivalent (and we write $\mathcal{E} \sim \mathcal{F}$) if there exists an adjointable mapping $u : \mathcal{E} \rightarrow \mathcal{F}$ such that $u^*u = id_\mathcal{E}$ and $uu^* = id_\mathcal{F}$. A closed submodule $\mathcal{G}$ of an inner product $\mathfrak{A}$-module $\mathcal{E}$ is said to be orthogonally complemented if $\mathcal{E} \oplus \mathcal{G}^\perp = \mathcal{E}$, where $\mathcal{G}^\perp = \{x \in \mathcal{E} : \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{G}\}$. A closed submodule $\mathcal{K}$ of an inner product $\mathfrak{A}$-module $\mathcal{E}$ is said to be fully complemented if $\mathcal{K}$ is orthogonally complemented and $\mathcal{K} \sim \mathcal{E}$. Note that the theory of inner product $C^*$-modules is quite different from that of inner product spaces. For example, not any closed submodule of an inner product $C^*$-module is complemented and there might exist bounded $\mathfrak{A}$-linear operators that are not adjointable.

Throughout the paper let $\mathfrak{A}$ be a unital $C^*$-algebra with the unit $1$ and let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be inner product $\mathfrak{A}$-modules. We fix an element $a \in \mathfrak{A}$ such that $a, 1-a$ are invertible. For instance, $a$ can be an element of $\mathfrak{A}$ satisfying $0 < a < 1$, where the order $c < d$ in $\mathfrak{A}$ means that $c, d$ are self-adjoint and the spectrum of $d-c$ is contained in $(m, \infty]$ for some positive number $m$. An additive mapping $A : \mathcal{E} \rightarrow \mathcal{G}$ is called $a$-additive if $A(ax) = aA(x)$ for all $x \in \mathcal{E}$. A biadditive mapping $B : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{G}$ is called $a$-biadditive if $B(ax, ax) = aB(x, x)$ and $B((1-a)x, (1-a)x) = (1-a)B(x, x)$ for all $x \in \mathcal{E}$. It is symmetric if $B(x, y) = B(y, x)$ for all $x, y \in \mathcal{E}$. Furthermore, $B$ is said to be orthogonality preserving if for all $x, y \in \mathcal{E}$,

$$\langle x, y \rangle = 0 \implies B(x, y) = 0.$$
A mapping $Q : E \rightarrow G$ is said to be quadratic if it satisfies the so-called quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \quad (x, y \in E).$$

Clearly any biadditive mapping is quadratic. A mapping $f : E \rightarrow G$ is called orthogonally $a$-Jensen if

$$\langle x, y \rangle = 0 \implies f(ax + (1-a)y) = af(x) + (1-a)f(y) \quad (x, y \in E). \quad (1.1)$$

In particular if $p \in (0,1)$, with $a = p1$ the mapping $f$ satisfying (1.1) is said to be orthogonally $p$-Jensen. Further if $p = \frac{1}{2}$ we say that $f$ is orthogonally Jensen.

In this paper, we investigate the representation of the so-called orthogonally $a$-Jensen mappings acting on inner product $C^\ast$-modules. More precisely, we prove that if there exist additive mappings $\varphi, \psi$ from $F$ into $E$ such that $\langle \varphi(y), \psi(z) \rangle = 0$ and $a\langle \varphi(y), \varphi(z) \rangle a^* = (1-a)\langle \psi(y), \psi(z) \rangle (1-a)^*$ for all $y,z \in F$, then a mapping $f : E \rightarrow G$ is orthogonally $a$-Jensen if and only if it is of the form $f(x) = A(x) + B(x,x) + f(0)$ for $x \in H := \varphi(F) + \psi(F)$, where $A : E \rightarrow G$ is an $a$-additive mapping on $H$ and $B$ is a symmetric $a$-biadditive orthogonality preserving mapping on $H \times H$. In addition, we show that if $F$ is a fully complemented submodule of $E$ and $f$ is orthogonally Jensen, then $f$ is of the form $f(x) = A(x) + f(0)$ for $x \in F$.

2. Main results

We start our work with the following lemmas. The first lemma follows immediately from (1.1).

**Lemma 2.1.** If $f : E \rightarrow F$ is orthogonally $a$-Jensen, then

(i) $af(a^{-1}x) + (1-a)f(0) = f(x)$
(ii) $af(0) + (1-a)f((1-a)^{-1}x) = f(x)$
(iii) $f(a^{-1}x) + a^{-1}(1-a)f(0) = a^{-1}f(x)$
(iv) $(1-a)^{-1}af(0) + f((1-a)^{-1}x) = (1-a)^{-1}f(x)$
(v) $(1-a)^{-1}af(x) + f(0) = (1-a)^{-1}f(ax)$
(vi) $f(0) + a^{-1}(1-a)f(x) = a^{-1}f((1-a)x)$

for every $x \in E$.

**Lemma 2.2.** Suppose that there exist additive mappings $\varphi, \psi : F \rightarrow E$ such that $\langle \varphi(z), \psi(w) \rangle = 0$ and $a\langle \varphi(z), \varphi(w) \rangle a^* = (1-a)\langle \psi(z), \psi(w) \rangle (1-a)^*$ for all $z,w \in F$. If $f : E \rightarrow G$ is orthogonally $a$-Jensen, then
\[af(\varphi(x) + \varphi(y)) + (1 - a)f(\psi(x) - \psi(y))\]
\[= a\left[f(\varphi(x)) + a^{-1}(1 - a)f(\psi(x)) - (1 - a)a^{-1}f(0)\right]
\[+ (1 - a)\left[(1 - a)^{-1}af(\varphi(y)) - (1 - a)^{-1}af(0) + f(\psi(-y))\right]\]

for every \(x, y \in \mathcal{F}\).

Proof. We have
\[
\langle \varphi(x) + a^{-1}(1 - a)\psi(x), (1 - a)^{-1}a\varphi(y) - \psi(y) \rangle
\]
\[= \langle \varphi(x), \varphi(y) \rangle(1 - a)^{-1}a^* - \langle \varphi(x), \psi(y) \rangle
\]
\[+ a^{-1}(1 - a)\langle \psi(x), \varphi(y) \rangle(1 - a)^{-1}a^* - a^{-1}(1 - a)\langle \psi(x), \psi(y) \rangle
\]
\[= \langle \varphi(x), \varphi(y) \rangle(1 - a)^{-1}a^* - a^{-1}(1 - a)\langle \psi(x), \psi(y) \rangle
\]
\[\text{(since } \langle \varphi(x), \psi(y) \rangle = 0 \text{ and } \langle \psi(x), \varphi(y) \rangle = \langle \varphi(y), \psi(x) \rangle^* = 0 )
\]
\[= \langle \varphi(x), \varphi(y) \rangle(1 - a)^{-1}a^* - a^{-1}(1 - a)[(1 - a)^{-1}a\langle \varphi(x), \varphi(y) \rangle((1 - a)^{-1}a)^*]
\]
\[= (1 - a)\langle \psi(x), \psi(y) \rangle(1 - a)^* = a\langle \varphi(x), \varphi(y) \rangle a^*
\]
\[= 0 \quad (2.1)
\]

for every \(x, y \in \mathcal{F}\). Therefore, we arrive at
\[af(\varphi(x) + \varphi(y)) + (1 - a)f(\psi(x) - \psi(y))
\]
\[= af(\varphi(x + y)) + (1 - a)f(\psi(x - y))
\]
\[= f(a\varphi(x + y) + (1 - a)\psi(x - y)
\]
\[\text{(since } \langle \varphi(x + y), \psi(x - y) \rangle = 0 \text{ and } f \text{ is orthogonally } a\text{-Jensen)}
\]
\[= f\left(a\varphi(x) + a^{-1}(1 - a)\psi(x) \right) + (1 - a)\left[(1 - a)^{-1}a\varphi(y) - \psi(y) \right]
\]
\[= af(\varphi(x) + a^{-1}(1 - a)\psi(x)) + (1 - a)f(1 - a)^{-1}a\varphi(y) - \psi(y)
\]
\[\text{(since } f \text{ is orthogonally } a\text{-Jensen and (2.1) holds)}
\]
\[= af(a^{-1}\varphi(x)) + (1 - a)[(1 - a)^{-1}a^{-1}(1 - a)\psi(x)]
\]
\[+ (1 - a)f\left(a[a^{-1}(1 - a)^{-1}a\varphi(y)] + (1 - a)\left[ - (1 - a)^{-1}\psi(y) \right]\right)
\]
\[= af\left(a^{-1}\varphi(x) \right) + (1 - a)f\left((1 - a)^{-1}a^{-1}(1 - a)\psi(x))\right]
\]
\[+ (1 - a)\left[af(a^{-1}(1 - a)^{-1}a\varphi(y)) + (1 - a)f\left( - (1 - a)^{-1}\psi(y) \right)\right]
\]
\[\text{(since } \langle a^{-1}\varphi(x), (1 - a)^{-1}a^{-1}(1 - a)\psi(x) \rangle = 0,
\]
\[\langle a^{-1}(1 - a)^{-1}a\varphi(y), -(1 - a)^{-1}\psi(y) \rangle = 0
\]
and $f$ is orthogonally $a$-Jensen

$$\begin{align*}
&= a \left[ f(\varphi(x)) - (1 - a)f(0) + f(a^{-1}(1 - a)\psi(x)) - af(0) \right] \\
&\quad + (1 - a) \left[ f((1 - a)^{-1}a\varphi(y)) - (1 - a)f(0) + f(-\psi(y)) - af(0) \right] \\
&\quad \left( \text{by Lemma 2.1 (i) and (ii)} \right) \\
&= a \left[ f(\varphi(x)) - (1 - a)f(0) + a^{-1}f((1 - a)\psi(x)) \right] \\
&\quad - a^{-1}(1 - a)f(0) - af(0) \\
&\quad + (1 - a) \left[ (1 - a)^{-1}f(a\varphi(y)) - (1 - a)^{-1}af(0) \right] \\
&\quad - (1 - a)f(0) + f(-\psi(y)) - af(0) \\
&\quad \left( \text{by Lemma 2.1 (iii) and (iv)} \right) \\
&= a \left[ f(\varphi(x)) - (1 - a)f(0) + f(0) + a^{-1}(1 - a)f(\psi(x)) \right] \\
&\quad - (1 - a)a^{-1}f(0) - af(0) \\
&\quad + (1 - a) \left[ (1 - a)^{-1}af(\varphi(y)) + f(0) \right] \\
&\quad - (1 - a)^{-1}af(0) - (1 - a)f(0) + f(\psi(-y)) - af(0) \\
&\quad \left( \text{by Lemma 2.1 (v) and (vi)} \right) .
\end{align*}$$

From this it follows that

$$af(\varphi(x) + \varphi(y)) + (1 - a)f(\psi(x) - \psi(y))$$

$$= a \left[ f(\varphi(x)) + a^{-1}(1 - a)f(\psi(x)) - (1 - a)a^{-1}f(0) \right] \\
+ (1 - a) \left[ (1 - a)^{-1}af(\varphi(y)) - (1 - a)^{-1}af(0) + f(\psi(-y)) \right]$$

and the lemma is proved. \hfill \Box

**Remark 2.3.** The condition that additive mappings $\varphi, \psi$ satisfying $\langle \varphi(x), \psi(y) \rangle = 0$ and $a\langle \varphi(x), \varphi(y) \rangle a^* = (1 - a)\langle \psi(x), \psi(y) \rangle (1 - a)^*$ is not restrictive. In fact, there are non-trivial concrete examples of additive mappings satisfying this condition. A non-trivial example can be given in $l^2$ by $a = 1 - p$ with $p \in (0, 1)$ and

$$\left\{ \begin{array}{l}
\varphi, \psi : l^2 \longrightarrow l^2 \\
\varphi(\{a_n\}) = (\frac{1}{1-p}a_1, 0, \frac{1}{1-p}a_2, 0, \frac{1}{1-p}a_3, 0, \cdots) \\
\psi(\{a_n\}) = (0, \frac{1}{p}a_1, 0, \frac{1}{p}a_2, 0, \frac{1}{p}a_3, 0, \cdots).
\end{array} \right.$$
One can easily observe that \( \langle \varphi(\{a_n\}), \psi(\{b_n\}) \rangle = 0 \) and
\[
a \langle \varphi(\{a_n\}), \varphi(\{b_n\}) \rangle a^* = (1 - a) \langle \psi(\{a_n\}), \psi(\{b_n\}) \rangle (1 - a)^* = \sum_{n=1}^{\infty} a_n b_n.
\]

The following auxiliary results are needed in our investigation.

**Proposition 2.4.** Suppose that there exist additive mappings \( \varphi, \psi : \mathcal{F} \to \mathcal{E} \) such that \( \langle \varphi(z), \psi(w) \rangle = 0 \) and \( a \langle \varphi(z), \varphi(w) \rangle a^* = (1 - a) \langle \psi(z), \psi(w) \rangle (1 - a)^* \) for all \( z, w \in \mathcal{F} \). If \( f : \mathcal{E} \to \mathcal{F} \) is an odd orthogonally \( a \)-Jensen mapping, then \( f \) is additive on \( \mathcal{K} := \varphi(\mathcal{F}) + \psi(\mathcal{F}) \).

**Proof.** Since \( f \) is odd \( f(0) = 0 \). Thus for every \( x, y \in \mathcal{F} \), by Lemma 2.2 we conclude that
\[
a f(\varphi(x) + \varphi(y)) + (1 - a) f(\psi(x) - \psi(y))
= a f(\varphi(x)) + (1 - a) f(\psi(x)) + a f(\varphi(y)) + (1 - a) f(-\psi(y)).
\]
(2.2)
Switching \( x \) and \( y \) in (2.2) we obtain
\[
a f(\varphi(y) + \varphi(x)) + (1 - a) f(\psi(y) - \psi(x))
= a f(\varphi(y)) + (1 - a) f(\psi(y)) + a f(\varphi(x)) + (1 - a) f(-\psi(x)).
\]
(2.3)
Add (2.2) and (2.3) and use the fact that \( f \) is odd to get
\[
2 a f(\varphi(x) + \varphi(y)) = 2 a f(\varphi(x)) + 2 a f(\varphi(y)),
\]
or equivalently,
\[
f(\varphi(x) + \varphi(y)) = f(\varphi(x)) + f(\varphi(y)).
\]
Hence \( f \) is additive on \( \varphi(\mathcal{F}) \). Similarly \( f \) is additive on \( \psi(\mathcal{F}) \). Now for every \( z_1, z_2 \in \mathcal{K} \) there exist \( x_1, x_2, y_1, y_2 \in \mathcal{F} \) such that
\[
z_1 = \varphi(x_1) + \psi(y_1) \quad \text{and} \quad z_2 = \varphi(x_2) + \psi(y_2).
\]
We have
\[
f(z_1 + z_2) = f(\varphi(x_1 + x_2) + \psi(1 + y_2))
= f(a a^{-1} \varphi(x_1 + x_2) + (1 - a)(1 - a)^{-1} \psi(y_1 + y_2))
= a f(a^{-1} \varphi(x_1 + x_2)) + (1 - a) f((1 - a)^{-1} \psi(y_1 + y_2))
\]
(since \( \langle a^{-1} \varphi(x_1 + x_2), (1 - a)^{-1} \psi(y_1 + y_2) \rangle = 0 \)
and \( f \) is orthogonally \( a \)-Jensen)
\[
= f(\varphi(x_1 + x_2)) + f(\psi(y_1 + y_2)) \quad \text{(by Lemma 2.1 (i) and (ii))}
= f(\varphi(x_1)) + f(\varphi(x_2)) + f(\psi(y_1)) + f(\psi(y_2))
\]
(by the additivity of \( f \) on \( \varphi(\mathcal{F}) \) and \( \psi(\mathcal{F}) \))
\[
= a f(a^{-1} \varphi(x_1)) + (1 - a) f((1 - a)^{-1} \psi(y_1))
+ a f(a^{-1} \varphi(x_2)) + (1 - a) f((1 - a)^{-1} \psi(y_2))
\]

Thus \( f \) is additive on \( \mathcal{K} \).

**Proposition 2.5.** Suppose that there exist additive mappings \( \varphi, \psi : \mathcal{F} \to \mathcal{E} \) such that \( \langle \varphi(z), \psi(w) \rangle = 0 \) and \( a \langle \varphi(z), \varphi(w) \rangle a^* = (1 - a) \langle \psi(z), \psi(w) \rangle (1 - a)^* \) for all \( z, w \in \mathcal{F} \). If \( f : \mathcal{E} \to \mathcal{F} \) is an even orthogonally \( a \)-Jensen mapping such that \( f(0) = 0 \), then \( f \) is quadratic on \( \mathcal{K} := \varphi(\mathcal{F}) + \psi(\mathcal{F}) \).

**Proof.** Since \( f \) is even and \( f(0) = 0 \), putting \( x = y \) in Lemma 2.2 we infer that
\[
af(2\varphi(x)) = 2af(\varphi(x)) + 2(1 - a)f(\psi(x)) \quad (x \in \mathcal{F}).
\]
Similarly, we have
\[
(1 - a)f(2\psi(x)) = 2af(\varphi(x)) + 2(1 - a)f(\psi(x)) \quad (x \in \mathcal{F}).
\]
Therefore, we conclude that
\[
af(2\varphi(x)) = (1 - a)f(2\psi(x)) \quad (x \in \mathcal{F}). \quad (2.4)
\]
If we put \( \frac{x}{2} \) instead of \( x \) in (2.4) we get
\[
af(\varphi(x)) = (1 - a)f(\psi(x)) \quad (x \in \mathcal{F}). \quad (2.5)
\]
Now for every \( x, y \in \mathcal{F} \) we have
\[
f(\varphi(x) + \varphi(y)) + f(\varphi(x) - \varphi(y))
= f(\varphi(x) + \varphi(y)) + a^{-1}(1 - a)f(\psi(x) - \psi(y)) \quad \text{(by (2.5))}
= f(\varphi(x)) + a^{-1}(1 - a)f(\psi(x))
+ f(\varphi(y)) + a^{-1}(1 - a)f(\psi(y)) \quad \text{(by Lemma 2.2)}
= f(\varphi(x)) + f(\varphi(x)) + f(\varphi(y)) + f(\varphi(y)) \quad \text{(by (2.5))}
= 2f(\varphi(x)) + 2f(\varphi(y)).
\]
Thus
\[
f(\varphi(x) + \varphi(y)) + f(\varphi(x) - \varphi(y)) = 2f(\varphi(x)) + 2f(\varphi(y)).
\]
So \( f \) is quadratic on \( \varphi(\mathcal{F}) \). Similarly \( f \) is quadratic on \( \psi(\mathcal{F}) \).
By the same reasoning as in the last part of Proposition 2.4 we conclude that \( f \) is quadratic on \( \mathcal{K} \). \( \square \)

We are now in a position to establish the main result. If \( A : \mathcal{E} \rightarrow \mathcal{G} \) is \( a \)-additive and \( B : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{G} \) is \( a \)-biadditive orthogonality preserving, then the mapping \( f : \mathcal{E} \rightarrow \mathcal{G} \) defined by

\[
f(x) = A(x) + B(x, x) + f(0) \quad (x \in \mathcal{E})
\]

is an orthogonally \( a \)-Jensen mapping. Namely, if \( \langle x, y \rangle = 0 \) then

\[
\begin{align*}
f(ax + (1 - a)y) & = A(ax + (1 - a)y) + B(ax + (1 - a)y, ax + (1 - a)y) + f(0) \\
& = aA(x) + aB(x, x) + af(0) \\
& \quad + (1 - a)A(y) + (1 - a)B(y, y) + (1 - a)f(0) \\
& = af(x) + (1 - a)f(x).
\end{align*}
\]

The following theorem is a kind of converse of the previous discussion.

**Theorem 2.6.** Let \( a \) be an element of a unital \( C^* \)-algebra \( \mathfrak{A} \) such that \( a, 1 - a \) are invertible and let \( \mathcal{E}, \mathcal{F}, \mathcal{G} \) be inner product \( \mathfrak{A} \)-modules. Suppose that there exist additive mappings \( \varphi, \psi : \mathcal{F} \rightarrow \mathcal{E} \) such that \( \langle \varphi(z), \psi(w) \rangle = 0 \) and \( a\langle \varphi(z), \psi(w) \rangle a^* = (1 - a)\langle \varphi(z), \psi(w) \rangle (1 - a)^* \) for all \( z, w \in \mathcal{F} \). Let \( \mathcal{K} := \varphi(\mathcal{F}) + \psi(\mathcal{F}) \). If \( f : \mathcal{E} \rightarrow \mathcal{G} \) is an orthogonally \( a \)-Jensen mapping, then there exist unique mappings \( A : \mathcal{E} \rightarrow \mathcal{G} \) and \( B : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{G} \) such that \( A \) is \( a \)-additive on \( \mathcal{K} \), \( B \) is symmetric \( a \)-biadditive orthogonality preserving on \( \mathcal{K} \times \mathcal{K} \) and

\[
f(x) = A(x) + B(x, x) + f(0) \quad (x \in \mathcal{K}).
\]

Moreover, \( B(x, y) = \frac{1}{8} \left( f(x + y) + f(-x - y) - f(x - y) - f(-x + y) \right) \) and \( A(x) = \frac{1}{2} \left( f(x) - f(-x) \right) \) for every \( x, y \in \mathcal{E} \).

**Proof.** By passing to \( f - f(0) \), if necessary, we may assume that \( f(0) = 0 \). We decompose \( f \) into its even and odd parts by

\[
f_o(x) = \frac{1}{2} \left( f(x) - f(-x) \right) \quad \text{and} \quad f_e(x) = \frac{1}{2} \left( f(x) + f(-x) \right)
\]

for all \( x \in \mathcal{E} \). Set \( A(x) := f_o(x) \) and \( B(x, y) := \frac{1}{4} \left( f_e(x + y) - f_e(x - y) \right) \). It is easy to show that \( f_o \) is odd orthogonally \( a \)-Jensen. So by Proposition 2.4, \( f_o \) is additive on \( \mathcal{K} \). Also, by Lemma 2.1 (v), we have

\[
A(ax) = f_o(ax) = af_o(x) + (1 - a)f_o(0) = aA(x) \quad (x \in \mathcal{K}).
\]

Hence \( A \) is \( a \)-additive on \( \mathcal{K} \). Furthermore, \( f_e \) is even orthogonally \( a \)-Jensen. Since \( f_e(0) = f(0) - f_o(0) = 0 \), \( f_e \) is quadratic on \( \mathcal{K} \) by Proposition 2.5. Thus \( f_e(2x) = 4f_e(x) \) and so,

\[
B(x, x) = \frac{1}{4} \left( f_e(2x) - f_e(0) \right) = f_e(x).
\]
This implies
\[ f(x) = f_o(x) + f_e(x) = A(x) + B(x, x) \quad (x \in \mathcal{K}). \]

For each \( x, y, z \in \mathcal{E} \) we have
\[
B(x + y, 2z) = \frac{1}{4} \left( f_e(x + y + 2z) - f_e(x + y - 2z) \right) \\
= \frac{1}{4} \left( f_e((x + z) + (y + z)) + f_e((x + z) - (y + z)) \right. \\
- f_e((x - z) + (y - z)) - f_e((x - z) - (y - z)) \right) \\
= \frac{1}{4} \left( 2f_e(x + z) + 2f_e(y + z) - 2f_e(x - z) - 2f_e(y - z) \right) \\
= \frac{1}{2} \left( f_e(x + z) - f_e(x - z) + f_e(y + z) - f_e(y - z) \right) \\
= 2B(x, z) + 2B(y, z). \tag{2.6}
\]

In particular, by choosing \( y = 0 \), we get
\[
B(x, 2z) = 2B(x, z) + 2B(0, z) \\
= 2B(x, z) + \frac{1}{4} \left( f(x) + f(-x) - f(x) - f(-x) \right) \\
= 2B(x, z). \tag{2.7}
\]

If we replace \( x \) by \( x + y \) in (2.7), then by (2.6) we obtain
\[
B(x + y, z) = \frac{1}{2} B(x + y, 2z) = \frac{1}{2} [2B(x, z) + 2B(y, z)] = B(x, z) + B(y, z),
\]
and similarly \( B(x, y + z) = B(x, y) + B(x, z) \). Therefore \( B \) is biadditive on \( \mathcal{K} \times \mathcal{K} \). Also, by Lemma 2.1 (vi), we have
\[
B(ax, ax) = f_e(ax) = af_e(x) + (1 - a)f_e(0) = aB(x, x) \quad (x \in \mathcal{K})
\]
and analogously
\[
B((1 - a)x, (1 - a)x) = (1 - a)B(x, x) \quad (x \in \mathcal{K}).
\]

Hence, \( B \) is \( a \)-biadditive on \( \mathcal{K} \times \mathcal{K} \). Further, for each \( x, y \in \mathcal{K} \), it follows from \( \langle x, y \rangle = 0 \) that
\[
B(x, y) = \frac{1}{2} (B(x, y) + B(y, x)) \\
= \frac{1}{2} \left( B(x + y, x + y) - B(x, x) - B(y, y) \right) \\
= \frac{1}{2} \left( f(x + y) - A(x + y) \right) - \frac{1}{2} \left( f(x) - A(x) \right) - \frac{1}{2} \left( f(y) - A(y) \right) \\
= \frac{1}{2} \left( f(x + y) - f(x) - f(y) \right) \quad \left( A \text{ is additive on } \mathcal{K} \right) \\
= \frac{1}{2} \left( f(aa^{-1}x + (1 - a)(1 - a)^{-1}y) - f(x) - f(y) \right)
\]
= \frac{1}{2} (af(a^{-1}x) + (1-a)f((1-a)^{-1}y)) - \frac{1}{2} f(x) - \frac{1}{2} f(y)

\left(\text{since } \langle a^{-1}x, (1-a)^{-1}y \rangle = 0 \text{ and } f \text{ is orthogonally } a\text{-Jensen}\right)

= \frac{1}{2} f(x) + \frac{1}{2} f(y) - \frac{1}{2} f(x) - \frac{1}{2} f(y) \quad \left(\text{by Lemma 2.1 (i) and (ii)}\right)

= 0.

Also, since \( f_e \) is even, \( B \) is symmetric. Thus \( B \) is symmetric \( a \)-biadditive orthogonality preserving on \( \mathcal{K} \times \mathcal{K} \). Finally suppose \( f(x) = A_1(x) + B_1(x, x) + f(0) = A_2(x) + B_2(x, x) + f(0) \) for any \( x \) for the specified kind of mappings \( A \) and \( B \). Hence, \( A_1(x) - A_2(x) = B_1(x, x) - B_2(x, x) \) for any \( x \). However, the left part is an odd mapping, and the right part is an even mapping. So both of these terms are equal to zero for any \( x \). Thus we conclude that \( A \) and \( B \) are uniquely determined by \( f \).

\begin{proof}
\end{proof}

Remark 2.7. The \( a \)-additive mappings \( \varphi, \psi \) from \( \mathcal{F} \) to \( \mathcal{E} \) need not to be injective. Also, the linear span of their ranges need not coincide with \( \mathcal{E} \). So, \( a \) and \( 1-a \) might be assumed merely to admit generalized inverses inside the \( C^* \)-algebra \( \mathfrak{A} \). By the requested equality \( a \langle \varphi(z), \varphi(w) \rangle a^* = (1-a) \langle \psi(z), \psi(w) \rangle (1-a)^* \) for any \( z, w \in \mathcal{F} \) the range projections of \( a \) and \( 1-a \) in the bidual von Neumann algebra \( \mathfrak{A}^{**} \) of \( \mathfrak{A} \) have to coincide. The domain projections of \( a \) and \( 1-a \) in the bidual von Neumann algebra \( \mathfrak{A}^{**} \) of \( \mathfrak{A} \) have to majorize the support projection the subset \( \langle \varphi(\mathcal{F}), \varphi(\mathcal{F}) \rangle \) and of the subset \( \langle \psi(\mathcal{F}), \psi(\mathcal{F}) \rangle \) in \( \mathfrak{A}^{**} \), respectively. For \( a \) and \( 1-a \) admitting generalized inverses in \( \mathfrak{A} \) the domain and the range projections belong to \( \mathfrak{A} \subseteq \mathfrak{A}^{**} \). However, they might not belong to the center of \( \mathfrak{A} \), so the Hilbert \( C^* \)-modules \( \mathcal{F} \) and \( \mathcal{E} \) cannot be reduced appropriately compatible with their module structure, in general.

In the following result we obtain the representation of orthogonally \( p \)-Jensen mappings in inner product modules.

Corollary 2.8. Let \( p \in (0, 1) \) be rational. Suppose that there exist additive mappings \( \varphi, \psi : \mathcal{F} \rightarrow \mathcal{E} \) such that \( \langle \varphi(z), \psi(w) \rangle = 0 \) and \( (1-p)^2 \langle \varphi(z), \varphi(w) \rangle = p^2 \langle \psi(z), \psi(w) \rangle \) for all \( z, w \in \mathcal{F} \). If \( f : \mathcal{E} \rightarrow \mathcal{G} \) is orthogonally \( p \)-Jensen, then there exists a unique mapping \( A : \mathcal{E} \rightarrow \mathcal{G} \) such that \( A \) is additive on \( \mathcal{K} := \varphi(\mathcal{F}) + \psi(\mathcal{F}) \) and

\[ f(x) = A(x) + f(0) \quad (x \in \mathcal{K}). \]

Proof. By Theorem 2.6, there exist unique mappings \( A : \mathcal{E} \rightarrow \mathcal{G} \) and \( B : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{G} \) such that \( A \) is additive on \( \mathcal{K} \), \( B \) is symmetric biadditive orthogonality preserving on \( \mathcal{K} \times \mathcal{K} \) and

\[ f(x) = A(x) + B(x, x) + f(0) \quad (x \in \mathcal{K}). \]

We have

\[ A(x) + B(x, x) + f(0) = f(x) \]
\[(1 - p)f(0) + pf\left(\frac{1}{p}x\right) \quad \text{(by Lemma 2.1)}
\]
\[(1 - p)f(0) + p\left(\frac{1}{p}A_{-x} + B_{-x, -x} + f(0)\right) \quad \text{(by Theorem 2.6)}
\]
\[= A(x) + \frac{1}{p}B(x, x) + f(0), \quad \text{(by the } \mathbb{Q}\text{-linearity of } A \text{ and the } \mathbb{Q}\text{-bilinearity of } B)\]

for any \(x \in \mathcal{X}\). Consequently, \((1 - \frac{1}{p})B(x, x) = 0\) and therefore, \(B(x, x) = 0\) for all \(x \in \mathcal{X}\). Thus \(B = 0\) on \(\mathcal{X}\) and \(f(x) = A(x) + f(0)\) on \(\mathcal{X}\).

Note that if \(A(x) = 0\) for some \(x \in \mathcal{X}\) then by the \(\mathbb{Q}\)-linearity of \(A\) we reach \(A(qx) = 0\) for all rational numbers \(q\). So, \(f(qx) = A(qx) + f(0) = f(0)\) for all rational numbers \(q\).

**Corollary 2.9.** Let \(\mathcal{F}\) be a submodule of \(\mathcal{E}\), and \(\varphi : \mathcal{F} \longrightarrow \mathcal{E}\) be a morphism such that \(\varphi(\mathcal{F}) \subseteq \mathcal{F}^\perp\). If \(f : \mathcal{E} \longrightarrow \mathcal{G}\) is orthogonally Jensen, then there exists a unique mapping \(A : \mathcal{E} \longrightarrow \mathcal{G}\) such that \(A\) is additive on \(\mathcal{K} := \mathcal{F} \oplus \varphi(\mathcal{F})\) and

\[f(x) = A(x) + f(0) \quad (x \in \mathcal{X}).\]

**Proof.** Let \(id : \mathcal{F} \longrightarrow \mathcal{F}\) be the identity mapping. Since \(\varphi\) is a morphism and \(\varphi(\mathcal{F}) \subseteq \mathcal{F}^\perp\), for every \(x, y \in \mathcal{F}\) we obtain \(\langle \varphi(x), \varphi(y) \rangle = \langle id(x), id(y) \rangle\) and \(\langle \varphi(x), id(y) \rangle = 0\). It remains to apply Corollary 2.8 for \(p = \frac{1}{2}\). \(\square\)

**Corollary 2.10.** Let \(\mathcal{F}\) be a fully complemented submodule of \(\mathcal{E}\). If \(f : \mathcal{E} \longrightarrow \mathcal{G}\) is orthogonally Jensen, then there exists a unique mapping \(A : \mathcal{E} \longrightarrow \mathcal{G}\) such that \(A\) is additive on \(\mathcal{F}\) and \(f(x) = A(x) + f(0)\) for every \(x \in \mathcal{F}\).

**Proof.** Since \(\mathcal{F}\) is a fully complemented submodule of \(\mathcal{E}\), \(\mathcal{F} \oplus \mathcal{F}^\perp = \mathcal{E}\) and \(\mathcal{F}^\perp \sim \mathcal{E}\). So, there exists an adjointable mapping \(\phi : \mathcal{E} \longrightarrow \mathcal{F}^\perp\) such that \(\phi^* \phi = id_\mathcal{E}\). We have \(\langle \phi(x), \phi(y) \rangle = \langle \phi^* \phi(x), y \rangle = \langle x, y \rangle\) for all \(x, y \in \mathcal{E}\). Let us define \(\varphi := \phi|_\mathcal{F}\). Then \(\varphi(\mathcal{F}) \subseteq \mathcal{F}^\perp\) and \(\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle\) for all \(x, y \in \mathcal{F}\). Thus \(\varphi : \mathcal{F} \longrightarrow \mathcal{E}\) is a morphism such that \(\varphi(\mathcal{F}) \subseteq \mathcal{F}^\perp\). Now the statement follows from Corollary 2.9. \(\square\)

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