On the dynamical anomalies in numerical simulations of selfgravitating systems.

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Abstract

According to self-similarity hypothesis, the thermodynamic limit could be defined from the scaling laws for the system self-similarity by using the microcanonical ensemble. This analysis for selfgravitating systems yields the following thermodynamic limit: \( N \rightarrow \infty \) keeping constant \( E/N^2 \) and \( LN^{3/2} \), in which is ensured the extensivity of the Boltzmann entropy \( S_B = \ln W(E,N) \). It is shown how the consideration of this thermodynamic limit allows us to explain the origin of dynamical anomalies in numerical simulations of selfgravitating systems.

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I. INTRODUCTION

In the last years there is a special interest to perform a well-defined thermodynamical description for selfgravitating systems and many interesting results have been obtained. In ref. \[11\], de Vega and Sanchez have pointed out that a kind of thermodynamical limit of a self-gravitating system can be defined if one considers what they call the diluted limit: send the number of particles, \( N \), and the volume, \( V \), to infinity, keeping constant the ratio \( N/V^{1/3} \) instead of the density, \( N/V \). Recently, V. Laliena in ref. \[12\] showed that this thermodynamic limit suffers from the same problems as the usual thermodynamical limit and leads to divergent thermodynamical functions.

In the recent paper \[13\], we proposed a new thermodynamic limit for selfgravitating system which is obtained from the self-similarity properties \[14, 15, 16, 17, 18\] of this kind of systems:

\[ N \rightarrow \infty, \text{ keeping constant } \frac{E}{N^2} \text{ and } LN^{3/2}. \]  \( \text{(1)} \)

As already shown in ref. \[13\], the consideration of this thermodynamic limit allows us to perform a well-defined thermodynamical description for selfgravitating systems by using almost the same arguments used by the standard Thermodynamics for describing the extensive systems. The aim of this paper is to show that the dynamical anomalies in numerical simulations of selfgravitating systems disappear when the self-similarity properties of this kind of systems are taken into account.

II. WHAT IS THE SYSTEM SELF-SIMILARITY?

Self-similarity is a natural generalization of the extensivity of the traditional system, which should be taken into account in order to perform a well-defined thermodynamical description based on the microcanonical ensemble \[14, 15, 16\]. Basically, it can be obtained from the asymptotical scaling behavior of thermodynamical variables and the microcanonical phase-space volume \( W \) when the many particle limit \( N \rightarrow \infty \) is invoked.

There are many studies in long-range Hamiltonian systems where the validity of the thermodynamical description in the thermodynamic limit is intimately related to the scaling with \( N \) of the thermodynamic variables and potentials. The problems with the nonextensive nature of those systems are avoided using the physically unjustified Kac prescription \[19\] in which the coupling constants are scaled by some power of \( N \) in order to deal with an extensive total energy \( E \). Self-similarity can not be reduced to the Kac prescription, because the energy extensivity is not demanded in the self-similarity framework. Geometrical aspects of the probabilistic distribution function of the statistical ensembles allows us to define a well-defined thermodynamic formalism even in cases where the extensive nature of systems can not be ensured \[13, 14, 15\].

The type of self-similarity scaling laws determines which is the generalized Boltzmann entropy \[14\] which should be considered in the application in order to guarantee the ensemble equivalence of the microcanonical ensemble with some adequate generalization of the canonical ensemble \[14\]. Ordinarily, scaling laws are exponential, it means that the microcanonical accessible volume \( W \) has an exponential growing with \( N \) in the thermodynamic limit. There is not a priory reason for supposing that a Hamiltonian system should necessarily satisfy this kind of growing in the thermodynamic limit, although there is a huge world of Hamiltonian systems that do.
obey it. In ref. [16] we used the self-similarity hypothesis in order to find the necessary conditions for the validity of Tsallis’ Statistics [20] by considering these arguments and using a procedure inspired in the one used by Gross in deriving his microcanonical thermostatistical theory [21]. It was proved in that paper the nonuniqueness of a thermodynamic formalism based on the microcanonical ensemble. In this sense, self-similarity also differs from the Kac prescription.

III. THE SELF-SIMILARITY SCALING LAWS

In this section is recalled how can be obtained the thermodynamic limit (1) by considering the system self-similarity properties for the following tridimensional self-gravitating Hamiltonian system:

\[ H = T + U = \sum_{k=1}^{N} \frac{1}{2m} p_k^2 + \sum_{j>k=1}^{N} \phi(r_{jk}) . \]  

(2)

Here \( r_{jk} = |r_j - r_k| \) is the distance between the \( j \)-th and \( k \)-th particles, being \( \phi(r) \) the self-gravitating potential which behaves as Newtonian interaction when \( r \to \infty \):

\[ \phi(r) \sim 1/r. \]  

(3)

It is easy to show that the microcanonical phase-space accessible volume \( W \) is given by:

\[ W = \frac{1}{N! \Gamma \left( \frac{3N}{2} \right)} \int_V \prod_{k=1}^{N} d^3 r_k \left[ E - U \right]^{\frac{3N-1}{2}} \]  

(4)

where \( V \) is the tridimensional volume in which the system is enclosed. Since \( \phi(r) \) describes a long-range interaction, the potential energy \( U \) for \( N \) large could be estimated as follows:

\[ U \sim N^2 L^{-1} , \]  

(5)

where \( L \) is the characteristic linear dimension of the system. The factor \( N^2 \) takes into account the contribution of all couples of particles, while the factor \( L^{-1} \) appears as consequence of the behavior of the potential \( \phi(r) \) for \( r \) large. The total energy \( E \) will exhibit an identical behavior:

\[ E \sim N^2 L^{-1} . \]  

(6)

Thus, when \( N \) is sent to infinity, the microcanonical phase-space accessible volume \( W \) depends on \( N \) and \( L \) as follows:

\[ W \sim \frac{1}{N! \Gamma \left( \frac{3N}{2} \right)} \left[ N^2 L^{-1} \right]^{\frac{3N}{2}} L^{3N} \sim \left[ LN^{\frac{1}{2}} \right]^{\frac{3N}{2}} . \]  

(7)

Thus, when the thermodynamic limit (1) is taken into account, the Boltzmann entropy is extensive:

\[ S_B = \ln W \sim N . \]  

(8)

There are some connections suggesting the validity of the thermodynamic limit (1). It was proposed in ref. [18], an alternative model for the classic isothermal model of Antonov [3] which uses an energetic prescription instead of a box renormalization in order to avoid the long-range singularity of the \( N \)-body self-gravitating systems. In that paper, a classical gas of identical particles with mass \( m \) was considered. Taking into account the rest energy of the particles, the nonrelativistic limit is valid when the absolute value of the mechanical energy of the system is much smaller than its rest energy:

\[ \left| \epsilon_0 (\Phi_0) N^{\frac{2}{3}} \right| \ll mc^2 N , \]  

(9)

where \( \epsilon_0 \) is the characteristic energy of this model:

\[ \epsilon_0 = \frac{2\pi G^2 m^5}{\hbar^2} . \]

However, this condition can not be satisfied for an arbitrary number of particles. In fact, when \( N \) tends to \( N_0 \):

\[ N_0 = \left( \frac{2\pi mc^2}{\epsilon_0} \right)^{\frac{2}{3}} = \left( \frac{\hbar c}{G} \right)^{\frac{2}{3}} \frac{1}{m^2} , \]  

(10)

which corresponds to a characteristic mass \( M_0 \):

\[ M_0 = N_0 m = \left( \frac{\hbar c}{G} \right)^{\frac{2}{3}} \frac{1}{m^2} , \]  

(11)

the model loses its validity. Everybody can recognize the fundamental constant of the stellar systems, which has much to do with the stability conditions of the stars (see in ref. [22]). Note that this constant appears as a consequence of assuming the energy \( N \)-dependence (1), so that, it could not have been obtained if another thermodynamic limit had been adopted. A consequent analysis of these massive systems should be performed taking into account the relativistic effects.

Another link is found in the well-known white dwarfs model based on the consideration of the Thomas-Fermi method to describe the state equation of the degenerate nonrelativistic electronic gas, whose pressure supports the hydrostatic equilibrium of the star:

\[ \Delta \phi = -4\pi G \rho , \]  

(12)
where $\phi$ is the Newtonian potential, being $\rho$ the mass density:

$$\rho = \mu m_c^2 \frac{2^\frac{4}{3} m_c^3 (\phi_S - \phi)^{\frac{2}{3}}}{3\pi^2 h^3} \quad (13)$$

where $\mu$ is the number of nucleons per electron, $m$ and $m_c$ are the nucleon and electron masses, being $\phi_S$ the potential at the star surface. From this model it can be easily derived the $N$-dependence of total energy and linear dimension of the system given in (1) by using a simple dimensional analysis. This coincidence is not casual: for the nonrelativistic particles the thermodynamic limit only depends on the dimension of the physical space.

According to the exposed above, during the many particle limit, $N \to \infty$, the selfgravitating Hamiltonian systems (2) exhibit self-similarity under the following scaling laws:

$$N_o \to N (\alpha) = \alpha N_o,
E_o \to E (\alpha) = \alpha^\frac{2}{3} E_o,
L_o \to L (\alpha) = \alpha^{-\frac{1}{3}} L_o$$

where the functional $F [W, \alpha]$ defines an exponential scaling laws:

$$F [W, \alpha] = \exp [\alpha \ln W], \quad (15)$$

being

$$W (\alpha) = W [E (\alpha), N (\alpha), L (\alpha)]. \quad (16)$$

As already mentioned in our previous works [14, 15, 16, 17], a system exhibits self-similarity under certain scaling transformations of its fundamental macroscopic observables when the functional $F [W, \alpha]$ obeys to the general condition of self-similarity [14]:

$$F [F [x, \alpha_1], \alpha_2] = F [x, \alpha_1 \alpha_2]. \quad (17)$$

Only in this case, the scaling transformations constitute a group of transformations compatible with the system macroscopic properties, whose ordering information is contained in the microcanonical accessible phase space volume $W$. We suppose that when the thermodynamic limit is carried out by using these special scaling laws:

- all those system self-similarity properties are protected, even though, its dynamic characteristics,
- as well as under certain conditions, the microcanonical ensemble could be equivalent to certain generalization of canonical ensemble.

Contrary, we guess that the non consideration of the self-similarity scaling laws should provoke a trivial ensemble inequivalence, as well as others anomalous behaviors [17]. According to our viewpoint, the Boltzmann’s definition of entropy is applicable to this system, which allows us to classify the selfgravitating model as a pseudoeextensive system [15].

IV. DISCUSSIONS AND CONCLUSIONS

Many investigators do not pay so much attention to the analysis of the system self-similarity when it is performed its macroscopic description. However, the non or bad consideration of the system self-similarity scaling laws leads in many cases to the trivial nonequivalence of the statistics ensembles as well as other anomalous behaviors, such as the reduction of the mixing and slow relaxation regime towards the equilibrium.

As an example of the above affirmation are the anomalies presented in the dynamical study of the selfgravitating systems performed by Cerruti-Sola & Pettini in the ref. [23]. In that paper, they observed a weakening of the system chaotic behavior with the increasing of the particles number $N$. It is very well-known the consequence of this fact on the ergodicity of the system: the chaotic dynamics provides the mixing property in the phase space necessary for obtaining the equilibrium. In that example the chaoticity time grows with $N$, and therefore, the systems could expend so much time to arrive to the equilibrium.

In a recent paper [24], Taruya and Sakagami carried out a $N$-body numerical experiment of the same situation as investigated in classic papers [2, 3]. That is, they confined the $N$ particles interacting via Newton gravity in a spherical adiabatic wall, which reverses the radial components of the velocity if the particle reaches the wall. The typical timescales appearing in this system are the free-fall time, $T_{ff} = (G\rho)^{-1/2}$, and the global relaxation time driven by the two-body encounter, $T_{relax} = (0.1 N/ \ln N) T_{ff}$ [23]. Since they considered the density $\rho$ as $N$-independent quantity, the relaxation time $T_{relax}$ diverges when the thermodynamic limit is invoked.

These facts can be interpreted as the non-commutativity of the thermodynamic limit ($N \to \infty$) with the infinite time limit ($t \to \infty$): when the first is performed before the second, the systems will not relax to the microcanonical ensemble. However, our analysis allows us to understand the origin of all these anomalous behaviors.

It was considered in the study [23] that the energy is scaled proportional to $N$ during the realization of the thermodynamic limit, which does not take into account the system self-similarity. It is very easy to see that the instability exponent $\lambda_H$ versus energy dependence (Figure 4 of the ref. [23]) is corrected when the self-similarity scaling law for the energy is considered: its proportionality to $N^{\frac{2}{3}}$. The weakening of chaoticity disappears when self-similarity is taken into account. Anyway, in spite of they assumed a different scaling law for the energy, they obtained the correct dependency of the instability exponent $\lambda_H$ with the energy per particle: the power law $\epsilon^{\frac{2}{3}}$.

The dynamical anomalies in the Taruya and Sakagami analysis are also explained by the non consideration of self-similarity. When the thermodynamic limit (13) is taken into consideration, system volume scales as $\sim 1/N$,
and therefore, the characteristic density $\rho$ scales as $\sim N^2$. Thus, $T_{ff} \sim 1/N$, and hence, $T_{relax} \sim 1/\ln N \to 0$. So that, the origin of the dynamical anomalies is a trivial consequence of the consideration of a thermodynamic limit incompatible with the system self-similarity properties of the self-gravitating systems.

There are some other examples of dynamical anomalies which at first glance could lead to the inapplicability of the microcanonical or the canonical ensemble (see for example in refs. [27, 28, 29, 30, 31, 32, 33, 34]). However, it could be proved that all these dynamical anomalies can be explained by the same arguments exposed in this paper.

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