CAPTURING LINKS IN SPATIAL COMPLETE GRAPHS

RYO NIKKUNI

Abstract. We say that a set of pairs of disjoint cycles $\Lambda(G)$ of a graph $G$ is linked if for any spatial embedding $f$ of $G$ there exists an element $\lambda$ of $\Lambda(G)$ such that the 2-component link $f(\lambda)$ is nonsplittable, and also say minimally linked if none of its proper subsets are linked. In this paper, (1) we show that the set of all pairs of disjoint cycles of $G$ is minimally linked if and only if $G$ is essentially same as a graph in the Petersen family, and (2) for any two integers $p, q \geq 3$, we exhibit a minimally linked set of Hamiltonian $(p, q)$-pairs of cycles of the complete graph $K_{p+q}$ with at most eighteen elements.

1. Introduction

Throughout this paper we work in the piecewise linear category. An embedding $f$ of a finite graph $G$ into the 3-sphere is called a spatial embedding of $G$, and $f(G)$ is called a spatial graph of $G$. We denote the set of all spatial embeddings of $G$ by $\text{SE}(G)$. Two spatial graphs $f(G)$ and $g(G)$ are said to be ambient isotopic and denoted by $f(G) \cong g(G)$ if there exists an orientation-preserving self-homeomorphism $\Phi$ of the 3-sphere such that $\Phi(f(G)) = g(G)$. We call a subgraph of $G$ homeomorphic to the circle a cycle of $G$, and a cycle containing exactly $p$ vertices a $p$-cycle. We denote the set of all pairs of disjoint cycles of $G$ by $\Gamma(2)(G)$, and the subset of $\Gamma(2)(G)$ consisting of all pairs of a $p$-cycle and a $q$-cycle by $\Gamma_{p,q}(G)$. We call an element of $\Gamma_{p,q}(G)$ a $(p, q)$-pair of cycles of $G$. For an element $\lambda$ of $\Gamma(2)(G)$ and an element $f$ of $\text{SE}(G)$, we call $f(\lambda)$ a constituent 2-component link of $f(G)$, and if $\lambda$ is a $(p, q)$-pair of cycles, then we also say that $f(\lambda)$ is of type $(p, q)$. In particular, if $\lambda$ contains all vertices of $G$, then we call $\lambda$ a Hamiltonian pair of cycles and $f(\lambda)$ a 2-component Hamiltonian link of $f(G)$.

A graph $G$ is said to be intrinsically linked if for any element $f$ in $\text{SE}(G)$ there exists an element $\lambda$ of $\Gamma(2)(G)$ such that $f(\lambda)$ is a nonsplittable 2-component link. Conway–Gordon and Sachs independently proved that $K_6$ is intrinsically linked [3], [8], where $K_n$ is the complete graph on $n$ vertices, that is the loopless graph consisting of $n$ vertices, a pair of whose distinct vertices is connected by exactly one edge. Actually they showed that every spatial graph $f(K_6)$ has a nonsplittable Hamiltonian link of type $(3, 3)$. In the case of $n \geq 7$, since $K_n$ contains a subgraph $H$ isomorphic to $K_6$, every spatial graph $f(K_n)$ also has a nonsplittable constituent 2-component link of type $(3, 3)$. On the other hand, for any integer $n \geq 6$, Vesnin–Litvintseva showed that every spatial graph $f(K_n)$ has a nonsplittable 2-component Hamiltonian link [9]. Moreover, the following is also known.

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Theorem 1.1. (Morishita–Nikkuni [5]) Let \( p, q \geq 3 \) be two integers. Then every spatial graph \( f(K_{p+q}) \) has a nonsplittable Hamiltonian link of type \((p, q)\).

As the number of vertices \( n \) increases, the number of the nonsplittable 2-component Hamiltonian links of \( f(K_n) \) also fairly increases and their behavior seems to be elusive, see [4], [1] for example. Our purpose in this paper is to ensure that we capture a nonsplittable Hamiltonian link of any possible type \((p, q)\) in a spatial complete graph on \( n \geq 6 \) vertices by the image of a smaller family of Hamiltonian \((p, q)\)-pairs of cycles. Let \( \Lambda(G) \) be a subset of \( \Gamma(2)(G) \). We say that \( \Lambda(G) \) is linked if for any element \( f \) in \( SE(G) \) there exists an element \( \lambda \) of \( \Lambda(G) \) such that the 2-component link \( f(\lambda) \) is nonsplittable. For example, Theorem 1.1 says that \( \Gamma(p, q)(K_{p+q}) \) is linked for any integers \( p, q \geq 3 \). Note that \( G \) is intrinsically linked if and only if there exists a linked subset \( \Lambda(G) \) of \( \Gamma(2)(G) \). Moreover, we say that a linked set \( \Lambda(G) \) is minimally linked if every proper subset of \( \Lambda(G) \) is not linked [6, §5]. By the definition of minimal linkedness, it is clear that every linked set of pairs of cycles includes a minimally linked subset. In addition, we also have the following, which properly represents the characteristics of the minimally linked set of pairs of cycles.

Proposition 1.2. Let \( \Lambda(G) \) be a subset of \( \Gamma(2)(G) \) that is linked. Then \( \Lambda(G) \) is minimally linked if and only if for any element \( \lambda \) in \( \Lambda(G) \) there exists an element \( f_\lambda \) of \( SE(G) \) such that \( f_\lambda(\lambda) \) is a nonsplittable link and \( f_\lambda(\lambda') \) is a split link for any element \( \lambda' \) in \( \Lambda(G) \setminus \{\lambda\} \).

Proof. First we show the ‘if’ part. By the assumption, \( \Lambda(G) \) is linked and \( \Lambda(G) \setminus \{\lambda\} \) is not linked for any element \( \lambda \) in \( \Lambda(G) \). Thus \( \Lambda(G) \) is minimally linked. Next we show the ‘only if’ part. Assume that \( \Lambda(G) \) is minimally linked. Then for any element \( \lambda \) in \( \Lambda(G) \), the set \( \Lambda(G) \setminus \{\lambda\} \) is not linked. Thus there exists an element \( f_\lambda \) of \( SE(G) \) such that \( f_\lambda(\lambda') \) is a split link for any element \( \lambda' \) in \( \Lambda(G) \setminus \{\lambda\} \). Since \( \Lambda(G) \) is linked, the link \( f_\lambda(\lambda) \) must be nonsplittable.

Our first result in this paper is to reveal the relationship between the minimality of linked set of pairs of cycles and the minor-minimality of the intrinsic linkedness. An edge contraction on a graph is an operation that contracts an edge \( e \) of the graph which is not a loop to a new vertex \( v \) as illustrated in Fig. 1.1. The reverse operation of an edge contraction is called a vertex splitting. We say that an edge contraction (resp. vertex splitting) is topologically trivial if it does not change the topological type of the graph, or equivalently, the degree of either terminal vertices of \( e \) is 2. A graph \( H \) is called a minor of a graph \( G \) if there exists a subgraph \( G' \) of \( G \) such that \( H \) is obtained from \( G' \) by finite number of edge contractions. In particular, \( H \) is called a proper minor of \( G \) if \( H \neq G \). An intrinsically linked graph \( G \) is said

![Figure 1.1. Edge contraction](image_url)

to be minor-minimal if every proper minor of \( G \) is not intrinsically linked. It is well-known that every intrinsically linked graph has a minor-minimal intrinsically linked graph as a minor, and all minor-minimal intrinsically linked graphs are only
graphs in the *Petersen family* that is a family of exactly seven graphs as illustrated in Fig. 1.2 [7]. Then the following says that for an intrinsically linked graph $G$, the minimality of $\Gamma^{(2)}(G)$ and the minor-minimality of $G$ are equivalent.

**Theorem 1.3.** Let $G$ be a graph with no vertices of degree 0 or 1. Then $\Gamma^{(2)}(G)$ is minimally linked if and only if $G$ is homeomorphic to one of the graphs in the Petersen family.

![Figure 1.2. Petersen family $K_6$, $Q_7$, $Q_8$, $P_7$, $P_8$, $P_9$, $P_{10}$](image)

Our second result in this paper is to explicitly give a minimally linked subset of $\Gamma_{p,q}(K_{p+q})$ for any $p,q \geq 3$. To accomplish this, we give a minimally linked set of Hamiltonian pairs of cycles for some specific graphs as follows. Let $G$ be one of the graphs $G_8, G_9$ and $G_{10}$ as illustrated in Fig. 1.3. We denote an edge of $G$ connecting two vertices $i$ and $j$ by $ij$, and a $p$-cycle $i_1i_2\cup i_2i_3\cup \cdots \cup i_pi_1$ of $G$ by $[i_1 \ i_2 \ \cdots \ \ i_p]$. Then we define the subset $\Lambda(G)$ of $\Gamma^{(2)}(G)$ as follows. For $G = G_8$, we define $\Lambda(G_8)$ by the proper subset of $\Gamma_{5,3}(G_8)$ consisting of the following twelve Hamiltonian $(5,3)$-pairs of cycles

\[
[1 \ 8 \ 7 \ 2 \ 3] \cup [4 \ 5 \ 6], \ [1 \ 2 \ 8 \ 7 \ 3] \cup [4 \ 5 \ 6], \ [1 \ 2 \ 3 \ 8 \ 7], \ [1 \ 2 \ 3 \ 8 \ 7] \cup [4 \ 5 \ 6], \\
[1 \ 8 \ 7 \ 2 \ 4] \cup [3 \ 5 \ 6], \ [1 \ 8 \ 7 \ 2 \ 5] \cup [4 \ 3 \ 6], \ [1 \ 8 \ 7 \ 2 \ 6] \cup [4 \ 5 \ 3], \\
[6 \ 2 \ 8 \ 7 \ 3] \cup [4 \ 5 \ 1], \ [5 \ 2 \ 8 \ 7 \ 3] \cup [4 \ 1 \ 6], \ [4 \ 2 \ 8 \ 7 \ 3] \cup [1 \ 5 \ 6], \\
[1 \ 4 \ 3 \ 8 \ 7] \cup [2 \ 5 \ 6], \ [1 \ 5 \ 3 \ 8 \ 7] \cup [4 \ 2 \ 6], \ [1 \ 6 \ 3 \ 8 \ 7] \cup [4 \ 5 \ 2].
\]
For $G = G_9$, we define $\Lambda(G_9)$ by the proper subset of $\Gamma_{5,4}(G_9)$ consisting of the following nine Hamiltonian $(5, 4)$-pairs of cycles

\[
\begin{align*}
[1 \ 8 \ 7 \ 2 \ 6] &\cup [4 \ 9 \ 5 \ 3], \\
[1 \ 8 \ 7 \ 2 \ 4] &\cup [3 \ 5 \ 9 \ 6], \\
[1 \ 8 \ 7 \ 2 \ 5] &\cup [4 \ 3 \ 6 \ 9], \\
[6 \ 2 \ 8 \ 7 \ 3] &\cup [4 \ 9 \ 5 \ 1], \\
[4 \ 2 \ 8 \ 7 \ 3] &\cup [1 \ 5 \ 9 \ 6], \\
[5 \ 2 \ 8 \ 7 \ 3] &\cup [4 \ 1 \ 6 \ 9], \\
[1 \ 6 \ 3 \ 8 \ 7] &\cup [4 \ 9 \ 5 \ 2], \\
[1 \ 4 \ 3 \ 8 \ 7] &\cup [2 \ 5 \ 9 \ 6], \\
[1 \ 5 \ 3 \ 8 \ 7] &\cup [4 \ 2 \ 6 \ 9].
\end{align*}
\]

For $G = G_{10}$, we define $\Lambda(G_{10})$ by the proper subset of $\Gamma_{5,5}(G_{10})$ consisting of the following eighteen Hamiltonian $(5, 5)$-pairs of cycles

\[
\begin{align*}
[1 \ 8 \ 7 \ 2 \ 3] &\cup [4 \ 10 \ 9 \ 5 \ 6], \\
[1 \ 8 \ 7 \ 2 \ 3] &\cup [4 \ 5 \ 10 \ 9 \ 6], \\
[1 \ 2 \ 8 \ 7 \ 3] &\cup [4 \ 10 \ 9 \ 5 \ 6], \\
[1 \ 2 \ 8 \ 7 \ 3] &\cup [4 \ 5 \ 10 \ 9 \ 6], \\
[1 \ 2 \ 3 \ 8 \ 7] &\cup [4 \ 10 \ 9 \ 5 \ 6], \\
[1 \ 2 \ 3 \ 8 \ 7] &\cup [4 \ 5 \ 10 \ 9 \ 6], \\
[1 \ 8 \ 7 \ 2 \ 6] &\cup [4 \ 10 \ 9 \ 5 \ 3], \\
[1 \ 8 \ 7 \ 2 \ 4] &\cup [3 \ 5 \ 10 \ 9 \ 6], \\
[1 \ 8 \ 7 \ 2 \ 5] &\cup [4 \ 3 \ 6 \ 10 \ 9], \\
[6 \ 2 \ 8 \ 7 \ 3] &\cup [4 \ 10 \ 9 \ 5 \ 1], \\
[4 \ 2 \ 8 \ 7 \ 3] &\cup [1 \ 5 \ 10 \ 9 \ 6], \\
[5 \ 2 \ 8 \ 7 \ 3] &\cup [4 \ 1 \ 6 \ 10 \ 9], \\
[1 \ 6 \ 3 \ 8 \ 7] &\cup [4 \ 10 \ 9 \ 5 \ 2], \\
[1 \ 4 \ 3 \ 8 \ 7] &\cup [2 \ 5 \ 10 \ 9 \ 6], \\
[1 \ 5 \ 3 \ 8 \ 7] &\cup [4 \ 2 \ 6 \ 10 \ 9].
\end{align*}
\]

Theorem 1.4. Let $G$ be one of the graphs $G_8, G_9$ and $G_{10}$. Then $\Lambda(G)$ is minimally linked.

By using Theorem 1.4, we have the following for the complete graph $K_n$ on $n \geq 7$ vertices.

**Theorem 1.5.**

1. For any integer $p \geq 5$, there exists a minimally linked subset of $\Gamma_{p,3}(K_{p+3})$ with exactly twelve elements.
2. For any integer $p \geq 3$, there exists a minimally linked subset of $\Gamma_{p,4}(K_{p+4})$ with exactly nine elements.
3. For any two integers $p, q \geq 5$, there exists a minimally linked subset of $\Gamma_{p,q}(K_{p+q})$ with exactly eighteen elements.

The proof is constructive, namely a minimally linked set of Hamiltonian pairs of cycles is explicitly given in any case. Therefore, for any two integers $p, q \geq 3$, by lying in ambush on at most eighteen specific Hamiltonian $(p, q)$-pairs of cycles, we can capture a nonsplittable Hamiltonian link of type $(p, q)$ for any spatial graph of $K_{p+q}$. We prove Theorem 1.3 in Section 2 and Theorem 1.4 and Theorem 1.5 in Section 3.
2. Proof of Theorem 1.3

We have already known that the set of all pairs of disjoint cycles of a graph in the Petersen family is linked [3]. First, we show their minimal linkedness.

**Lemma 2.1.** Let $G$ be a graph in the Petersen family. Then $\Gamma_G(2)$ is minimally linked.

In the following, an automorphism of $G$ means a self-homeomorphism of $G$, and we may identify an automorphism of $G$ with a permutation of degree $n$, where $n$ is the number of vertices of $G$.

**Proof of Lemma 2.1.** Let $h$ be a spatial embedding of $G$ as illustrated in Fig. 1.2. Then we can check that each of the spatial graphs contains a Hopf link in the thick line part as exactly one nonsplittable link in the images of all elements in $\Gamma_G(2)$.

In the case of $K_6$, for any element $\lambda$ in $\Gamma_G(2)(K_6) = \Gamma_G(3,3)(K_6)$, there exists an automorphism $\sigma$ of $K_6$ such that $\sigma$ sends $\lambda$ to $[1\ 3\ 5\ ] \cup [2\ 4\ 6\ ]$. Then $h \circ \sigma(\lambda)$ is a Hopf link and $h \circ \sigma(\lambda')$ is a split link for any element $\lambda'$ in $\Gamma_G(2)(K_6) \setminus \{\lambda\}$. Thus by Proposition 1.2, $\Gamma_G(2)(K_6)$ is minimally linked.

In the case of $G = Q_7, Q_8, P_7, \text{or } \Gamma_G(4)(G)$ if $G = Q_8$, there exists an automorphism $\sigma$ of $G$ generated by $(1\ 2\ 3)$ and $(4\ 5\ 6)$ such that $\sigma$ sends $\lambda$ to $[1\ 3\ 5\ ] \cup [2\ 4\ 6\ ]$ if $G = Q_7, [1\ 5\ 2\ 6\ ] \cup [7\ 3\ 4\ ]$ if $G = P_7$ and $[1\ 7\ 3\ 5\ ] \cup [2\ 4\ 8\ 6\ ]$ if $G = Q_8$. Then $h \circ \sigma(\lambda)$ is a Hopf link and $h \circ \sigma(\lambda')$ is a split link for any element $\lambda'$ in $\Gamma_G(2)(G) \setminus \{\lambda\}$. Thus by Proposition 1.2, $\Gamma_G(2)(G)$ is minimally linked.

In the case of $P_5$, note that $\Gamma_G(2)(P_5) = \Gamma_G(3,5)(P_5) \cup \Gamma_G(4,4)(P_5)$. For any element $\lambda$ in $\Gamma_G(5,3)(P_5)$, there exists an automorphism $\sigma$ of $P_5$ generated by $(2\ 3)$ and $(4\ 5)$ such that $\sigma$ sends $\lambda$ to $[1\ 5\ 2\ 6\ 8\ ] \cup [7\ 3\ 4\ ]$. Then $h \circ \sigma(\lambda)$ is a Hopf link and $h \circ \sigma(\lambda')$ is a split link for any element $\lambda'$ in $\Gamma_G(2)(P_5) \setminus \{\lambda\}$. On the other hand, for any element $\mu$ in $\Gamma_G(4,4)(P_5)$, there exists an automorphism $\tau$ of $P_5$ generated by $(2\ 3)$, $(4\ 5)$ and $(1\ 6)(3\ 4)(5\ 2\ 5)$ such that $\tau$ sends $\mu$ to $[8\ 1\ 5\ 7\ ] \cup [4\ 2\ 6\ 3\ ]$. Let $g$ be an element of $\text{SE}(P_6)$ obtained from $h$ by a single crossing change between $h(1-5$ and $h(34)$ and ambient isotopies such that $g([1\ 5\ 2\ 6\ 8\ ] \cup [7\ 3\ 4\ ])$ is a trivial link. Then we have that $g([8\ 1\ 5\ 7\ ] \cup [4\ 2\ 6\ 3\ ])$ has nonzero linking number, namely this is nonsplittable. Since no other element in $\Gamma_G(2)(P_5)$ contains $1-5$ and $34$ in the different components, $g(\lambda')$ is a split link for any element $\lambda'$ in $\Gamma_G(2)(P_5) \setminus \{[8\ 1\ 5\ 7\ ] \cup [4\ 2\ 6\ 3\ ]\}$. Then $g \circ \tau(\mu)$ is a nonsplittable link and $g \circ \tau(\mu')$ is a split link for any element $\mu'$ in $\Gamma_G(2)(P_5) \setminus \{\mu\}$. Thus by Proposition 1.2, $\Gamma_G(2)(P_5)$ is minimally linked.

In the case of $P_8$, note that $\Gamma_G(2)(P_8) = \Gamma_G(5,4)(P_8) \cup \{\mu\}$, where $\mu = [1\ 8\ 6\ 3\ 9\ 5\ ] \cup [2\ 7\ 4\ ]$. For any element $\lambda$ in $\Gamma_G(5,4)(P_8)$, there exists an automorphism $\sigma$ of $P_8$ generated by $(1\ 6\ 9)(3\ 5\ 2\ 7\ 4)$ and $(4\ 7\ 2)(1\ 8\ 6\ 3\ 9\ 5)$ such that $\sigma$ sends $\lambda$ to $[1\ 8\ 6\ 3\ 4\ ] \cup [5\ 2\ 7\ 9\ ]$. Then $h \circ \sigma(\lambda)$ is a Hopf link and $h \circ \sigma(\lambda')$ is a split link for any element $\lambda'$ in $\Gamma_G(2)(P_8) \setminus \{\lambda\}$. On the other hand, let $g$ be an element of $\text{SE}(P_8)$ obtained from $h$ by a single crossing change between $h(27$ and $h(56)$ and ambient isotopies such that $g([1\ 8\ 6\ 3\ 4\ ] \cup [5\ 2\ 7\ 9\ ])$ is a trivial link. Then we have that $g(\mu)$ has nonzero linking number, namely this is nonsplittable. Since no other element in $\Gamma_G(2)(P_8)$ contains $27$ and $56$ in the different components, $g(\mu')$ is a split link for any element $\mu'$ in $\Gamma_G(2)(P_8) \setminus \{\mu\}$. Thus by Proposition 1.2, $\Gamma_G(2)(P_8)$ is minimally linked.

In the case of $P_{10}$, for any element $\lambda$ in $\Gamma_G(2)(P_{10}) = \Gamma_G(5,5)(P_{10})$, there exists an automorphism $\sigma$ of $P_{10}$ generated by $(1\ 6\ 9)(8\ 3\ 5)(2\ 7\ 4)$ and $(4\ 7\ 2)(1\ 8\ 6\ 3\ 9\ 5)$
such that $\sigma$ sends $\lambda$ to $[1 8 6 3 4] \cup [5 2 10 7 9]$. Then $h \circ \sigma(\lambda)$ is a Hopf link and $h \circ \sigma(\lambda')$ is a split link for any element $\lambda'$ in $\Gamma^{(2)}(P_{10}) \setminus \{\lambda\}$. Thus by Proposition 1.2 $\Gamma^{(2)}(P_{10})$ is minimally linked.

Let $H$ be a minor of a graph $G$, namely there exists a subgraph $G'$ of $G$ and edges $e_1, e_2, \ldots, e_m$ of $G'$ such that $H$ is obtained from $G'$ by edge contractions along $e_1, e_2, \ldots, e_m$. Then by composing the injective map from $\Gamma^{(2)}(H)$ to $\Gamma^{(2)}(G')$ induced from vertex splittings on $H$ and the inclusion from $\Gamma^{(2)}(G')$ to $\Gamma^{(2)}(G)$, we obtain the natural injective map

$$\Psi^{(2)}_{H,G} : \Gamma^{(2)}(H) \to \Gamma^{(2)}(G).$$

On the other hand, for an element $f$ of $\text{SE}(G)$, the element $\psi(f)$ of $\text{SE}(H)$ is obtained from $f(G')$ by contracting spatial edges $f(e_i)$ ($i = 1, 2, \ldots, m$). This correspondence from $f$ to $\psi(f)$ defines a surjective map

$$\psi : \text{SE}(G) \to \text{SE}(H).$$

Note that we can easily see the following.

**Proposition 2.2.** Let $f$ be an element of $\text{SE}(G)$. Then for any element $\lambda$ in $\Gamma^{(2)}(H)$, 2-component links $\psi(f)(\lambda)$ and $f(\Psi^{(2)}_{H,G}(\lambda))$ are ambient isotopic.

Then the next lemma says that a minimally linked set of pairs of cycles for a minor $H$ of a graph $G$ naturally induces a minimally linked set for $G$.

**Lemma 2.3.** Let $\Lambda(H)$ be a subset of $\Gamma^{(2)}(H)$ and $\Lambda(G)$ the image of $\Lambda(H)$ by $\Psi^{(2)}_{H,G}$. If $\Lambda(H)$ is (minimally) linked then $\Lambda(G)$ is also (minimally) linked.

**Proof.** Let $f$ be an element of $\text{SE}(G)$. Then by the assumption that $\Lambda(H)$ is linked and Proposition 2.2 there exists an element $\lambda_0$ of $\Lambda(H)$ such that $f(\Psi^{(2)}_{H,G}(\lambda_0)) \cong \psi(f)(\lambda_0)$ is a nonsplittable link. Thus $\Lambda(G) = \Psi^{(2)}_{H,G}(\Lambda(H))$ is linked. Suppose that $\Lambda(H)$ is minimally linked. Then for any element $\Psi^{(2)}_{H,G}(\lambda)$ in $\Lambda(G)$, there exists an element $\bar{f}_{\lambda}$ of $\text{SE}(H)$ such that $\bar{f}_{\lambda}(\lambda)$ is a nonsplittable link and $\bar{f}_{\lambda}(\lambda')$ is a split link for any element $\lambda'$ in $\Lambda(H) \setminus \{\lambda\}$. Then for an element $\bar{f}_{\lambda}$ of $\psi^{-1}(\bar{f}_{\lambda})$, by Proposition 2.2 we have that $f_{\lambda}(\Psi^{(2)}_{H,G}(\lambda)) \cong \bar{f}_{\lambda}(\lambda)$ is a nonsplittable link and $f_{\lambda}(\Psi^{(2)}_{H,G}(\lambda')) \cong \bar{f}_{\lambda}(\lambda')$ is a split link for $\lambda' \neq \lambda$. Since $\Psi^{(2)}_{H,G}$ is injective, we have $\Psi^{(2)}_{H,G}(\lambda') \neq \Psi^{(2)}_{H,G}(\lambda)$. Thus $\Lambda(G) = \Psi^{(2)}_{H,G}(\Lambda(H))$ is minimally linked.

A cut-edge of a graph is an edge of the graph whose deletion increases the number of the connected components. Then the next lemma says that if $\Gamma^{(2)}(G)$ is minimally linked then $G$ is essentially '2-edge-connected'.

**Lemma 2.4.** Let $G$ be a graph with no vertices of degree 0 or 1. If $\Gamma^{(2)}(G)$ is minimally linked then $G$ is connected and does not have a cut-edge.

**Proof.** First, assume that there exists a disconnected graph $G$ such that $\Gamma^{(2)}(G)$ is minimally linked. Let $G_1, G_2, \ldots, G_m$ be the connected components of $G$ ($m \geq 2$). Then for a pair of cycles $\lambda = \gamma_1 \cup \gamma_2$ where $\gamma_i$ is a cycle of $G_i$ ($i = 1, 2$), by Proposition 1.2 there exists an element $f_{\lambda}$ of $\text{SE}(G)$ such that $f_{\lambda}(\lambda)$ is a nonsplittable link and $f_{\lambda}(\lambda')$ is a split link for any element $\lambda'$ in $\Gamma^{(2)}(G) \setminus \{\lambda\}$. This implies that $\Gamma^{(2)}(G_i)$ is not linked for any $i = 1, 2, \ldots, m$. Then we can see that there
exists an element \( g \) of \( \text{SE}(G) \) such that \( g(G) \) does not contain a nonsplittable 2-component link. This is a contradiction. Hence, if \( \Gamma^{(2)}(G) \) is minimally linked then \( G \) is connected.

Next, assume that \( G \) has a cut-edge \( e \). Let \( G_1 \) and \( G_2 \) be the connected components of \( G \setminus e \). Then for a pair of cycles \( \lambda = \gamma_1 \cup \gamma_2 \) where \( \gamma_i \) is a cycle of \( G_i \) (\( i = 1, 2 \)), by Proposition 1.2 there exists an element \( f_\lambda \) of \( \text{SE}(G) \) such that \( f_\lambda(\lambda) \) is a nonsplittable link and \( f_\lambda(\lambda') \) is a split link for any element \( \lambda' \) in \( \Gamma^{(2)}(G) \setminus \{\lambda\} \). This implies that \( \Gamma^{(2)}(G_i) \) is not linked for any \( i = 1, 2 \). Then we also can see that there exists an element \( g \) of \( \text{SE}(G) \) such that \( g(G) \) does not contain a nonsplittable 2-component link. This is a contradiction. \( \square \)

Now let us prove Theorem 1.3. In the following, a subdivision \( H' \) of a graph \( H \) is a graph obtained from \( H \) by subdividing some edges by a finite number of vertices, or equivalently, \( H' \) is obtained from \( H \) by finite number of vertex splittings which are topologically trivial. Thus a graph \( H \) is a minor of its subdivision \( H' \).

Proof of Theorem 1.3. First we show the ‘if’ part. Let \( G \) be a graph homeomorphic to a graph \( P \) in the Petersen family. Then \( G \) is a subdivision of \( P \) and the map \( \Psi^{(2)}_{P,G} : \Gamma^{(2)}(P) \to \Gamma^{(2)}(G) \) is bijective. Thus by Lemma 2.1 and Lemma 2.3 we have that \( \Gamma^{(2)}(G) = \Psi^{(2)}_{P,G}(\Gamma^{(2)}(P)) \) is minimally linked.

Next we show the ‘only if’ part. Suppose that \( \Gamma^{(2)}(G) \) is minimally linked. Since \( G \) is intrinsically linked, \( G \) has a minor \( P \) in the Petersen family, namely there exists a subgraph \( G' \) of \( G \) such that \( P \) is obtained from \( G' \) by some edge contractions. Here, \( G' \) can be taken as having no vertices of degree 0 or 1. Let \( \Psi^{(2)}_{P,G} : \Gamma^{(2)}(P) \to \Gamma^{(2)}(G) \) be the natural injective map. Then by Lemma 2.1 and Lemma 2.3 we have that \( \Psi^{(2)}_{P,G}(\Gamma^{(2)}(P)) \) is linked. Since \( \Gamma^{(2)}(G) \) is minimally linked, \( \Psi^{(2)}_{P,G}(\Gamma^{(2)}(P)) \) must coincide with \( \Gamma^{(2)}(G) \). This implies that the map \( \Psi^{(2)}_{P,G} \) is bijective. Let \( P' \) be a graph obtained from \( P \) by a single vertex splitting at a vertex \( v \). If the degree of \( v \) is greater than or equal to 4 and this vertex splitting is not topologically trivial, then it can be seen that \( \sharp \Gamma^{(2)}(P') > \sharp \Gamma^{(2)}(P) \) by checking each of the graphs in the Petersen family one by one. This implies that \( G' \) is a subdivision of \( P \). Suppose that \( G \neq G' \). Then we divide our situation into the following two cases. First we consider the case that \( G' \) is a spanning subgraph of \( G \). Let \( e \) be an edge of \( G \) not contained in \( G' \). Note that for any pair of vertices \( u, v \) of \( P \) (possibly \( u = v \)), there exist a shortest path \( \rho \) between \( u \) and \( v \) and a cycle \( c \) of \( P \) such that \( u \) and \( v \) are disjoint. Therefore, if both of the terminal vertices of \( e \) are the original vertices \( u, v \) of \( P \) in the subdivision \( G' \), then there exist a path \( \rho' \) between \( u \) and \( v \) and a cycle \( c' \) of \( G' \) such that \( e \cup \rho' \) and \( c' \) are disjoint pair of cycles in \( G' \) not contained in \( G' \). Thus we have \( \sharp \Gamma^{(2)}(P') > \sharp \Gamma^{(2)}(G') \). and this is a contradiction. If either of the terminal vertices of \( e \) is not a vertex of \( P \) in \( G' \), then we can take another subdivision \( G'' \) of \( P \) so that both of the terminal vertices of \( e \) are the vertices of \( P \) in \( G'' \). Thus, similar to the previous argument, we find that there is a contradiction. Next we consider the case that \( G' \) is not a spanning subgraph of \( G \). Let \( F' \) be a connected component of a subgraph of \( G \) induced by the vertices of \( G \) which are not contained in \( G' \). Note that by Lemma 2.4 \( G \) does not have a cut-edge. Thus there exist at least two edges connecting \( G' \) and \( F' \). Hence, there exists a path of \( G \) connecting the vertices of \( G' \) which is edge-disjoint with \( G' \). Then by the same argument as in the previous case, we have
\[ \# \Gamma(2)(P) = \# \Gamma(2)(G') < \# \Gamma(2)(G), \] and this is a contradiction. From the above we have \( G = G' \), namely \( G \) is homeomorphic to \( P \).

3. Proof of Theorem 1.4 and Theorem 1.5

For a graph \( G = G_8, G_9, G_{10} \), we first show the linkedness of \( \Lambda(G) \).

**Lemma 3.1.** Let \( G \) be one of the graphs \( G_8, G_9 \) and \( G_{10} \). Then \( \Lambda(G) \) is linked.

**Proof.** The proof is given in exactly the same way as Conway–Gordon theorem for \( K_6 \) \cite{3}. Let \( f \) be a spatial embedding of \( G \). In the following we show

\[ \sum_{\lambda \in \Lambda(G)} \text{lk}(f(\lambda)) \equiv 1 \pmod{2}, \]

where \( \text{lk} \) denotes the linking number in the 3-sphere. This implies that there exists an element \( \lambda \) in \( \Lambda(G) \) such that \( f(\lambda) \) has odd linking number, namely \( f(\lambda) \) is nonsplittable. We define \( \varsigma(f) \in \mathbb{Z}_2 \) by the modulo two reduction of \( \sum_{\lambda \in \Lambda(G)} \text{lk}(f(\lambda)) \).

Note that the mod 2 linking number of a 2-component link does not change under crossing changes on the same component, and changes under a single crossing change between different components. So \( \varsigma(f) \) does not change under crossing changes on the same edge, and between adjacent edges. On the other hand, it can be checked that for any disjoint edges \( e \) and \( e' \) of \( G \), there exist even number of pairs of cycles in \( \Lambda(G) \) containing both \( e \) and \( e' \) in each of the components separately. So \( \varsigma(f) \) does not change under crossing changes between disjoint edges. Therefore \( \varsigma(f) \) does not change under any crossing change on \( f(G) \). Since any two spatial embeddings of a graph are transformed into each other by crossing changes and ambient isotopies, this implies that the value \( \varsigma(f) \) does not depend on the choice of \( f \). Let \( h \) be a spatial embedding of \( G \) as illustrated in Fig. \cite{13}. Then we can check that each of the spatial graphs contains a Hopf link in the thin line part as exactly one nonsplittable link in the images of all elements in \( \Lambda(G) \). Thus we have \( \varsigma(f) = \varsigma(h) = 1 \).

**Proof of Theorem 1.4.** In the case of \( G_8 \), for any element \( \lambda \) in \( \Lambda(G_8) \) which does not contain \([4 5 6]\) as one of the components, there exists an automorphism \( \sigma \) of \( G_8 \) generated by \((1 2 3)\) and \((4 5 6)\) such that \( \sigma \) sends \( \lambda \) to \([1 5 3 8 7] \cup [4 2 6]\). Let \( h \) be an element of \( \text{SE}(G_8) \) as illustrated in Fig. \cite{13}. Then \( h \circ \sigma(\lambda) \) is a Hopf link and \( h \circ \sigma(\lambda') \) is a split link for any element \( \lambda' \) in \( \Lambda(G_8) \) \( \setminus \{ \lambda \} \). On the other hand, for any element \( \mu \) in \( \Lambda(G_8) \) containing \([4 5 6]\) as one of the components, there exists an automorphism \( \tau \) of \( G_8 \) generated by \((1 2 3)\) such that \( \tau \) sends \( \mu \) to \([1 2 3 8 7] \cup [4 5 6]\).

Let \( g \) be an element of \( \text{SE}(G_8) \) obtained from \( h \) by a single crossing change between \( h([3 8]) \) and \( h([4 6]) \) and ambient isotopies such that \( g([1 5 3 8 7] \cup [4 2 6]) \) is a trivial link. Then we have that \( g([1 2 3 8 7] \cup [4 5 6]) \) has nonzero linking number, namely this is nonsplittable. Since no other element in \( \Lambda(G_8) \) contains both \([3 8]\) and \([4 6]\), \( g(\lambda') \) is a split link for any element \( \lambda' \) in \( \Lambda(G_8) \) \( \setminus \{ \lambda \} \). Then \( g \circ \tau(\mu) \) is a nonsplittable link and \( g \circ \tau(\mu') \) is a split link for any element \( \mu' \) in \( \Lambda(G_8) \) \( \setminus \{ \mu \} \).

Thus by Proposition \cite{12}, \( \Lambda(G_8) \) is minimally linked.

In the case of \( G_9 \), for any element \( \lambda \) in \( \Lambda(G_9) \) there exists an automorphism \( \sigma \) of \( G_9 \) generated by \((1 2 3)\) and \((4 5 6)\) such that \( \sigma \) sends \( \lambda \) to \([1 5 3 8 7] \cup [4 2 6 9]\). Let \( h \) be an element of \( \text{SE}(G_9) \) as illustrated in Fig. \cite{13}. Then \( h \circ \sigma(\lambda) \) is a Hopf link and \( h \circ \sigma(\lambda') \) is a split link for any element \( \lambda' \) in \( \Lambda(G_9) \) \( \setminus \{ \lambda \} \). Thus by Proposition \cite{12}, \( \Lambda(G_9) \) is minimally linked.
In the case of $G_{10}$, for any element $\lambda$ in $\Lambda(G_{10})$ where not all of the vertices 1, 2 and 3 (equivalently, 4, 5 and 6) are included in the same component, there exists an automorphism $\sigma$ of $G_8$ generated by (1 2 3) and (4 5 6) such that $\sigma$ sends $\lambda$ to $[1 5 3 8 7] \cup [4 2 6 10 9]$. Let $h$ be an element of $\text{SE}(G_{10})$ as illustrated in Fig. 13. Then $h \circ \sigma(\lambda)$ is a Hopf link and $h \circ \sigma(\lambda')$ is a split link for any element $\lambda'$ in $\Lambda(G_{10}) \setminus \{\lambda\}$. On the other hand, for any element $\mu$ in $\Lambda(G_{10})$ where all of the vertices 1, 2 and 3 (equivalently, 4, 5 and 6) are included in the same component, there exists an automorphism $\tau$ of $G_{10}$ generated by (1 2 3) and (4 5 6) such that $\tau$ sends $\mu$ to $[1 2 3 8 7] \cup [4 5 6 10 9]$. Let $g$ be an element of $\text{SE}(G_{10})$ obtained from $h$ by a single crossing change between $h(6 \, 10)$ and $h(38)$ and ambient isotopies such that $g([1 5 3 8 7] \cup [4 2 6 10 9])$ is a trivial link. Then we have that $g([1 2 3 8 7] \cup [4 5 6 10 9])$ has nonzero linking number, namely this is nonsplittable. Since no other element in $\Lambda(G_{10})$ contains both $6 \, 10$ and $38$, $g(\lambda')$ is a split link for any element $\lambda'$ in $\Lambda(G_{10}) \setminus \{[1 2 3 8 7] \cup [4 5 6 10 9]\}$. Then $g \circ \tau(\mu)$ is a nonsplittable link and $g \circ \tau(\mu')$ is a split link for any element $\mu'$ in $\Lambda(G_{10}) \setminus \{\mu\}$. Thus by Proposition 1.2, $\Lambda(G_{10})$ is minimally linked.

Proof of Theorem 1.5

(1) Let $p \geq 5$ be an integer. Then $K_{p+3}$ contains a spanning subgraph $G'_s$ which is a subdivision of $G_s$ obtained by subdividing the edge $78$ by $p - 5$ vertices. Since $G_s$ is a minor of $K_{p+3}$, by Theorem 1.4 and Lemma 2.3, $\Lambda(K_{p+3}) = \Psi_{G_s,K_{p+3}}(\Lambda(G_s))$ is minimally linked. Since any element in $\Lambda(G_s)$ is a Hamiltonian $(5, 3)$-pair of cycles where the edge $78$ is included in the 5-cycle component, $\Lambda(K_{p+3})$ is a subset of $\Gamma_{p,3}(K_{p+3})$ consisting of exactly twelve elements.

(2) Let $p \geq 3$ be an integer. In the case of $p = 3$, $K_7$ contains a spanning subgraph $P'_7$ that is equal to $P_7$. Since $P'_7$ is a minor of $K_7$, by Lemma 2.1 and Lemma 2.3, $\Lambda(K_7) = \Psi_{P'_7,K_7}(\Lambda(P_7))$ is minimally linked. In the case of $p = 4$, $K_8$ contains a spanning subgraph $Q'_8$ that is equal to $Q_8$. Since $Q_8$ is a minor of $K_8$, by Lemma 2.1 and Lemma 2.3, $\Lambda(K_8) = \Psi_{Q'_8,K_8}(\Lambda(Q_8))$ is minimally linked. Note that $\sharp \Lambda(K_7) = \sharp \Gamma_{3,4}(P_7) = 9$ and $\sharp \Lambda(K_8) = \sharp \Gamma_{4,8}(Q_8) = 9$. In the case of $p \geq 5$, $K_{p+4}$ contains a spanning subgraph $G'_s$ which is a subdivision of $G_s$ obtained by subdividing the edge $78$ by $p - 5$ vertices. Then in the same way as (1), we can obtain a minimally linked subset $\Lambda(K_{p+4}) = \Psi_{G'_s,K_{p+4}}(\Lambda(G_s))$ of $\Gamma_{p,4}(K_{p+4})$ consisting of exactly nine elements.

(3) Let $p, q \geq 5$ be two integers. Then $K_{p+q}$ contains a spanning subgraph $G'_{10}$ which is a subdivision of $G_{10}$ obtained by subdividing the edge $78$ by $p - 5$ vertices and $9 \, 10$ by $q - 5$ vertices. Since $G_{10}$ is a minor of $K_{p+q}$, by Theorem 1.4 and Lemma 2.3, $\Lambda(K_{p+q}) = \Psi_{G'_{10},K_{p+q}}(\Lambda(G_{10}))$ is minimally linked. Since any element in $\Lambda(G_{10})$ is a Hamiltonian $(5, 5)$-pair of cycles containing both $78$ and $9 \, 10$ in each of the components separately, $\Lambda(K_{p+q})$ is a subset of $\Gamma_{p,5}(K_{p+q})$ consisting of exactly eighteen elements.

Remark 3.2. Note that $K_{10}$ contains a spanning subgraph $P'_{10}$ that is equal to $P_{10}$. Since $P_{10}$ is a minor of $K_{10}$, by Lemma 2.1 and Lemma 2.3, $\Lambda'(K_{10}) = \Psi_{P_{10},K_{10}}(\Lambda(P_{10}))$ is minimally linked. Since $\sharp \Lambda'(K_{10}) = 6$, the minimally linked subset $\Lambda'(K_{10})$ of $\Gamma_{5,5}(K_{10})$ found in Theorem 1.5 (3) does not realize the minimum number of elements in a minimally linked set of Hamiltonian $(5, 5)$-pairs of cycles. It is still open to determine the minimum and maximum numbers of elements in a minimally linked set of Hamiltonian $(p, q)$-pairs of cycles for $K_n$. 


Remark 3.3. For a graph $G$, let us denote the set of all cycles of $G$ by $\Gamma(G)$, and the set of all $p$-cycles of $G$ by $\Gamma_p(G)$. A subset $\Gamma$ of $\Gamma(G)$ is said to be knotted if for any element $f$ in $\mathrm{SE}(G)$ there exists an element $\gamma$ of $\Gamma$ such that the knot $f(\gamma)$ is nontrivial, and a knotted set $\Gamma$ is said to be minimally knotted if every proper subset of $\Gamma$ is not knotted. Note that $G$ is said to be intrinsically knotted if there exists a knotted subset $\Gamma$ of $\Gamma(G)$. It is known that $\Gamma_7(K_7)$ is minimally knotted \cite{3}, and for $n \geq 8$, $\Gamma_n(K_n)$ is also knotted \cite{2}, but an example of a minimally knotted subset of $\Gamma_n(K_n)$ has not been known yet \cite[Problem 5.3]{6}. See \cite[§5]{6} for related open problems.

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