Searching for a Shoreline

STEVEN R. FINCH AND LI-YAN ZHU

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Abstract. Logarithmic spirals are conjectured to be optimal escape paths from a half plane ocean. Assuming this, we find the rate of increase for both min-max and min-mean interpretations of “optimal”. For the one-dimensional analog, which we call logarithmic coils, our min-mean solution differs from a widely-cited published account.

A ship is lost in a dense fog at sea and must reach land as soon as possible. The captain knows that the shore is straight line, but has no information about its distance or direction. Equivalently, the sea is known to be a half plane, but the ship’s location and orientation relative to the boundary is unknown. Assuming its speed is constant, what is the best path for the ship to follow in its search for the shore?

The word “best” can be understood in several ways [1]. We start with minimizing the worst-case scenario (min-max); a relevant conjecture that the family of logarithmic spirals contains the minimal path remains open. Our small contribution is that of providing the computational details that underlie a proposition due to Baeza-Yates, Culberson & Rawlins [2, 3, 4]. We then adopt a different sense of “best” and determine the logarithmic spiral that minimizes the expected pathlength (min-mean), in which shoreline directions are assumed to be uniformly distributed. Except for the (admittedly large) theoretical gap regarding the optimality of logarithmic spirals, the calculations in this two-dimensional setting are straightforward.

We subsequently turn to the one-dimensional analog of the search problem. The shore is now simply a point on a line and the spirals here are necessarily self-intersecting. A large computer science literature on this problem exists. The solution of the min-max problem was first found by Beck & Newman [5]. Their approach to the min-mean problem, however, suffers from the assumption of a nonuniform target distribution (a certain scaling property, true in the two-dimensional setting, is less apparent here). We give our solution, which is distinct from theirs, and hope to initiate discussion on this issue.

The three-dimensional analog, for which shores are planes in space, would seem to be very difficult. We wonder if an appropriate extension of spiral has ever been examined in the past.

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0.1. Planar Setting: Min-Max. Let $\kappa > 0$. Three preliminary results are:

**Lemma 1.** The distance between the line $Ax + By + C = 0$ and the origin is $|C|/\sqrt{A^2 + B^2}$.

**Lemma 2.** The equation of a line tangent to the circle of radius $R$, center at the origin, is $r = R \sec(\theta - \omega)$, where $\omega$ corresponds to the point of tangency.

**Lemma 3.** The equation of a line tangent to the spiral $r = e^{\kappa \theta}$ is $y - e^{\kappa \theta} \sin(\theta) = m(x - e^{\kappa \theta} \cos(\theta))$, where $\theta$ corresponds to the point of tangency and the slope is given by

$$m = \frac{\kappa \sin(\theta) + \cos(\theta)}{\kappa \cos(\theta) - \sin(\theta)}.$$

**Proof of Lemma 1.** The unit vector $(A, B)/\sqrt{A^2 + B^2}$ is normal to the line $Ax + By + C = 0$, hence the point $(-CA, -CB)/(A^2 + B^2)$ on the line determines its distance from $(0, 0)$.

**Proof of Lemma 2.** In rectangular coordinates, the line is given by

$$y = R \sin(\omega) - \cot(\omega) (x - R \cos(\omega)).$$

In polar coordinates, therefore, we have

$$r \sin(\theta) = R \sin(\omega) - \cot(\omega) (r \cos(\theta) - R \cos(\omega))$$

and so

$$\frac{r}{R} = \frac{\sin(\omega) + \cot(\omega) \cos(\omega)}{\sin(\theta) + \cot(\omega) \cos(\theta)} = \sec(\theta - \omega).$$

**Proof of Lemma 3.** Clearly

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(e^{\kappa \theta} \sin(\theta))'}{(e^{\kappa \theta} \cos(\theta))'} = \frac{\kappa \sin(\theta) + \cos(\theta)}{\kappa \cos(\theta) - \sin(\theta)}.$$

**Theorem 4.** Of all lines tangent to the spiral $r = e^{\kappa \theta}$, there is exactly one that is tangent to the circle of radius $R$, center at the origin. Call this line $L$. The tangency point of $L$ with the spiral is

$$\theta_0 = \frac{1}{\kappa} \left( \ln(R) + \frac{1}{2} \ln(1 + \kappa^2) \right).$$

The tangency point of $L$ with the circle is

$$\omega_0 = \theta_0 - \arccos \left( \frac{1}{\sqrt{1 + \kappa^2}} \right) < \theta_0;$$
thus $L$ has equation $r = R \sec(\theta - \omega_0)$.

**Proof of Theorem 4.** Apply Lemma 1 with $A = m$, $B = -1$ and $C = e^{\kappa \theta}(\sin(\theta) - m \cos(\theta))$ to obtain $R^2$ as an expression in $\kappa$, $\theta$, $m$. Lemma 3 further gives $m$ as an expression in $\kappa$, $\theta$. We find that $R^2(1 + \kappa^2) = e^{2\kappa \theta}$ and hence the formula for $\theta_0$ is true. By Lemma 2, $e^{\kappa \theta_0} = R \sec(\theta_0 - \omega)$ and thus the formula for $\omega_0$ is true.

Think of a ship, starting at the origin and moving along the spiral. Its first contact point with $L$ is at $\theta_0$. We wish to compute its second contact point $\theta_1$. The reason is that, in the interval $\theta_0 < \theta < \theta_1$, the spiral intersects all other tangent lines to the circle of radius $R$. At $\theta = \theta_1$, repetition begins so we stop there: All possible shorelines at distance $R$ from the origin have at this point been found.

A closed-form expression for $\theta_1$ is not known, but it uniquely satisfies the equation

$$e^{\kappa \theta_1} = R \sec(\theta_1 - \omega_0), \quad \theta_0 < \theta_1 < \theta_0 + 2\pi.$$ 

Once we have $\theta_1$ for $R = 1$, we have it for all $R$ via the formula

$$\theta_1(R) = \frac{1}{\kappa} \ln(R) + \theta_1(1)$$

since

$$\omega_0(R) = \frac{1}{\kappa} \ln(R) + \omega_0(1),$$

so $\theta_1(R) - \omega_0(R) = \theta_1(1) - \omega_0(1)$ and thus

$$e^{\kappa \theta_1(R)} = Re^{\kappa \theta_1(1)} = R \sec(\theta_1(1) - \omega_0(1)) = R \sec(\theta_1(R) - \omega_0(R)).$$

Such scaling behavior is valuable here - we may consider $R = 1$ without loss of generality – but this property fails in Section 3.

**Lemma 5.** The arclength of the spiral $r = e^{\kappa \theta}$ up to $\Theta$ is

$$\sqrt{1 + \kappa^2} \int_{-\infty}^{\Theta} e^{\kappa \theta} d\theta = \frac{1}{\kappa} \sqrt{1 + \kappa^2} e^{\kappa \Theta}.$$

**Proof of Lemma 5.** From $dr = \kappa e^{\kappa \theta} d\theta$, we deduce that $ds^2 = r^2 d\theta^2 + dr^2 = e^{2\kappa \theta} d\theta^2 + \kappa^2 e^{2\kappa \theta} d\theta^2 = (1 + \kappa^2) e^{2\kappa \theta} d\theta^2$.

With the assumption that $R = 1$, the two-dimensional min-max problem reduces to minimizing $(\sqrt{1 + \kappa^2}/\kappa) e^{\kappa \theta_1}$ as a function of $\kappa$. While an explicit formula for $\theta_1$ in terms of $\kappa$ is unavailable, a purely numerical scheme suffices to give $\kappa = 0.2124695594...$ with arclength $13.8111351795...$. The latter is consistent with the
Figure 1: Two helpful pictures for the proof of Theorems 6 and 7.
estimate 13.81 reported in [2]; earlier estimates 0.22325 and 13.49 from [3, 4] arose when erroneously minimizing $e^{\kappa_1}/\kappa$.

We now obtain trigonometric equations that serve to define the best spiral more precisely.

**Theorem 6.** The min-max logarithmic spiral has parameter $\kappa = \tan \alpha = 0.2124695594... = \ln(1.2367284662...)$ with arclength $\csc \alpha \sec \beta = 13.8111351795...$, where $\alpha, \beta$ satisfy the simultaneous equations

\[
\frac{1}{\tan \alpha} + \frac{1}{\tan \beta} = \frac{2\pi - \alpha - \beta}{\cos^2 \alpha}, \quad \frac{\cos \alpha}{\cos \beta} = e^{(2\pi - \alpha - \beta)\tan \alpha}.
\]

**Proof of Theorem 6.** Define angles $\alpha, \beta$ and lengths $u, v$ by

\[
\theta_0 = \alpha + \omega_0, \quad \theta_1 = (2\pi - \alpha - \beta) + \theta_0, \quad u = e^{\kappa \theta_0} = \sec \alpha, \quad v = e^{\kappa \theta_1} = \sec \beta.
\]

Differentiating with respect to $\alpha$, we obtain

\[
u' = \sec \alpha \tan \alpha = u \tan \alpha, \quad (1)\]

\[
u' = \beta' \sec \beta \tan \beta = \beta' v \tan \beta, \]

that is,

\[
\beta' = \frac{v'}{v} \cot \beta = v' \cos \beta \cot \beta. \quad (2)
\]

From $e^{\kappa \theta_1} = e^{\kappa \theta_0} e^{\kappa (2\pi - \alpha - \beta)}$, it follows that

\[
v = u e^{(2\pi - \alpha - \beta)\tan \alpha} \quad (3)
\]

hence

\[
v' = \left( (2\pi - \alpha - \beta) \sec^2 \alpha - (1 + \beta') \tan \alpha \right) v + u' \frac{v}{u} \quad (4)
\]

\[
= \left( -v' \tan \alpha \cos \beta \cot \beta + (2\pi - \alpha - \beta) \sec^2 \alpha \right) v
\]

by (1) and (2). Since the objective function

\[
e^{\kappa \theta_1} \sqrt{1 + \kappa^2} = v \csc \alpha
\]

is minimized when

\[
v' \csc \alpha - v \csc \alpha \cot \alpha = 0, \quad (5)
\]
we have the additional formula \( v' = v \cot \alpha \). Substituting this twice into (4) yields
\[
\cot \alpha = -v \cos \beta \cot \beta + (2\pi - \alpha - \beta) \sec^2 \alpha,
\]
thus
\[
\cot \alpha + \cot \beta = (2\pi - \alpha - \beta) \sec^2 \alpha.
\]
Also, (3) implies immediately that
\[
\sec \beta = \sec \alpha e^{(2\pi - \alpha - \beta) \tan \alpha}
\]
which we observe is true independent of (5).

0.2. Planar Setting: Min-Mean. On the basis of Section 1, the two-dimensional min-mean problem clearly reduces to minimizing the average arclength
\[
\frac{1}{2\pi} \sqrt{1 + \kappa^2} \int_{\omega_0}^{\omega_0 + 2\pi} e^{\kappa \theta} d\omega
\]
\[
= \frac{\sqrt{1 + \kappa^2}}{2\pi \kappa} \left[ \int_{\theta_0}^{\theta_1} e^{\kappa \theta} \left( 1 - \frac{\kappa}{\sqrt{e^{2\kappa \theta} - 1}} \right) d\theta + \int_{0}^{\theta_1} e^{\kappa \theta} \left( 1 + \frac{\kappa}{\sqrt{e^{2\kappa \theta} - 1}} \right) d\theta \right]
\]
\[
= \frac{\sqrt{1 + \kappa^2}}{2\pi \kappa} \left[ \frac{e^{\kappa \theta_1}}{\kappa} + \ln \left( e^{\kappa \theta_1} + \sqrt{e^{2\kappa \theta_1} - 1} \right) - \frac{e^{\kappa \theta_0}}{\kappa} + \ln \left( e^{\kappa \theta_0} + \sqrt{e^{2\kappa \theta_0} - 1} \right) \right]
\]
as a function of \( \kappa \), assuming \( R = 1 \). Define for convenience
\[
\Phi(\alpha, \beta) = (-2 \csc \alpha + \ln(\sec \alpha + \tan \alpha) + \ln(\sec \beta + \tan \beta)) (\cot \alpha + \cot \beta),
\]
\[
\Psi(\alpha, \beta) = (\alpha + \beta - 2\pi)(\sec \alpha \csc \beta + \csc \alpha \sec \beta) \sec \alpha,
\]
\[
\Xi(\alpha, \beta) = \sec \alpha - \cot \alpha \csc \beta + (\tan \alpha \cot \beta - \csc \alpha \csc \beta) \sec \alpha - (\cot^2 \alpha + \csc^2 \alpha) \sec \beta.
\]

Theorem 7. The min-mean logarithmic spiral has parameter \( \kappa = \tan \alpha = 0.3732051316... = \ln(1.4523822387...) \) with arclength
\[
\frac{1}{2\pi} (\ln(\sec \alpha + \tan \alpha) + \ln(\sec \beta + \tan \beta) - (\sec \alpha - \sec \beta) \cot \alpha) \csc \alpha = 7.0321857865...,\]
where \( \alpha, \beta \) satisfy the simultaneous equations
\[
\Phi(\alpha, \beta) + \Psi(\alpha, \beta) = \Xi(\alpha, \beta), \quad \frac{\cos \alpha}{\cos \beta} = e^{(2\pi - \alpha - \beta) \tan \alpha}.
\]
Proof of Theorem 7. By the observation at the end of the proof of Theorem 6, the truth of the second equation is independent of the objective function. It remains to derive the first equation. Define

$$w = (v - u) \cot \alpha + \ln \left( v + \sqrt{v^2 - 1} \right) + \ln \left( u + \sqrt{u^2 - 1} \right),$$

then

$$w' = (v' - u') \cot \alpha - (v - u) \csc^2 \alpha + \frac{v'}{\sqrt{v^2 - 1}} + \frac{u'}{\sqrt{u^2 - 1}}$$

$$= v' \cot \alpha - u - (v - u) \csc^2 \alpha + \frac{v'}{\sqrt{v^2 - 1}} + \frac{u \tan \alpha}{\sqrt{u^2 - 1}}$$

(6)

using (1). The objective function

$$\frac{\sqrt{1 + \kappa^2}}{2\pi\kappa} \left[ e^{\kappa \theta_1} + \ln \left( e^{\kappa \theta_1} + \sqrt{e^{2\kappa \theta_1} - 1} \right) - e^{\kappa \theta_0} + \ln \left( e^{\kappa \theta_0} + \sqrt{e^{2\kappa \theta_0} - 1} \right) \right] = \frac{w \csc \alpha}{2\pi}$$

is minimized when $w' = w \cot \alpha$, as with (5). Between (6) and this additional formula, we eliminate $w'$ and solve for $v'$ in terms of $\alpha$, $\beta$, $u$, $v$. Substituting the resulting expression for $v'$ into (4) gives an equation involving $\Phi$, $\Psi$ and $\Xi$.

0.3. Linear Setting: Min-Max. Let $\gamma > 1$. The one-dimensional analog of the logarithmic spiral $r = e^{\kappa \theta}$ we study here is

$$x = (-\gamma)^{[t]} (1 - (\gamma + 1)(t - \lfloor t \rfloor)).$$

For lack of standard phraseology, we call this a logarithmic coil. Local maximum points occur at $(x, t) = (\gamma^{2i}, 2i)$ where $i$ is an integer; local minimum points occur at $(x, t) = (-\gamma^{2i-1}, 2i - 1)$.

Given a point $X > 0$, the distance $\delta$ that the ship travels to reach $X$ is

$$\delta = X + 2 \sum_{j=-\infty}^{2i-1} \gamma^j = X + \frac{2\gamma^{2i+1}}{\gamma - 1}$$

where $\gamma^{2i} < X \leq \gamma^{2i+2}$, that is,

$$i = \left\lfloor \frac{\ln(X)}{2 \ln(\gamma)} - 1 \right\rfloor = \left\lfloor \frac{1}{2} \left\lfloor \frac{\ln(X)}{\ln(\gamma)} - 1 \right\rfloor \right\rfloor.$$

Given a point $X < 0$, the corresponding distance $\delta$ is

$$\delta = -X + 2 \sum_{j=-\infty}^{2i} \gamma^j = -X + \frac{2\gamma^{2i+1}}{\gamma - 1}$$
where $\gamma^{2i-1} < -X \leq \gamma^{2i+1}$, that is,

$$i = \left\lfloor \frac{\ln(-X) - 1}{2 \ln(\gamma)} \right\rfloor = \left\lfloor \frac{1}{2} \left\lfloor \frac{\ln(-X)}{\ln(\gamma)} - 1 \right\rfloor \right\rfloor.$$

Scaling as observed in Section 1 no longer works here: for the points ±1, we have

$$\delta(1) = 1 + \frac{2}{\gamma - 1}, \quad \delta(-1) = 1 + \frac{2\gamma}{\gamma - 1},$$

which are not easily related to $\delta(X)$ and $\delta(-X)$. An analysis of $\delta(X)$ and $\delta(-X)$, which possess sizeable jump discontinuities at $\gamma^{2i}$ and $-\gamma^{2i-1}$, would seem to require different tools than before. Computer scientists traditionally normalize by $|X|$; see Figure 2 for a sample result. For more on the following theorem, see [5, 6, 7, 8, 9, 10, 11, 12].

**Theorem 8.** The min-max logarithmic coil has parameter $\gamma = 2$ with worst-case ratio $\delta/|X| = 9$.

**Proof of Theorem 8.** For simplicity, we examine only positive $X$. If $X = \gamma^{2k+\varepsilon}$ for some small $\varepsilon > 0$, then $i = k$ and $\delta/X \rightarrow (2\gamma^2 + \gamma - 1)/(\gamma - 1)$ as $\varepsilon \rightarrow 0^+$. Calculus gives that $\gamma = 2$ is the critical point, which yields in turn the least maximum value $\delta/|X| = 9$.

### 0.4. Linear Setting: Min-Mean

The use of $\delta/|X|$ in defining the min-max coil in Section 3 seems fairly natural; the formulation behind a min-mean coil, however,
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Figure 3: Graph of \( I(X) \) for \( \gamma = 2 \).

requires some careful thought. Consider the integral

\[
I(X) = \frac{1}{2X} \int_{-X}^{X} \frac{\delta(x)}{|x|} \, dx
\]

whose graph appears in Figure 3. (The minimum and maximum values are \( 1 + 6 \ln(2) = 5.1588... \) and \( 1 + 12/e = 5.4145... \) when \( \gamma = 2 \).) We wish to determine \( \gamma \) for which \( I \), a kind of normalized average, is minimal. Since \( I \) itself is a periodic function of \( X \) (although smoother than \( \delta/|X| \)), the word “minimal” can be used only loosely.

Assuming \( \gamma^{2i} < X \leq \gamma^{2i+2} \) and \(-\infty < p < i\), we have

\[
\int_{\gamma^{2p}}^{\gamma^{2p+2}} \frac{1}{x} \left( x + \frac{2\gamma^{2p+2}}{\gamma - 1} \right) \, dx = f(p, \gamma, \gamma^{2p+2}), \quad \int_{\gamma^{2i}}^{X} \frac{1}{x} \left( x + \frac{2\gamma^{2i+2}}{\gamma - 1} \right) \, dx = f(i, \gamma, X)
\]

where

\[
f(i, \gamma, X) = (X - \gamma^{2i}) + \frac{2\gamma^{2i+2}}{\gamma - 1} (\ln(X) - 2i \ln(\gamma)) .
\]

Assuming \(-\gamma^{2j+1} \leq X < -\gamma^{2j-1} \) and \(-\infty < q < j\), we have

\[
\int_{-\gamma^{2q+1}}^{-\gamma^{2q-1}} \frac{1}{x} \left( -x + \frac{2\gamma^{2q+1}}{\gamma - 1} \right) \, dx = g(q, \gamma, -\gamma^{2q+1}), \quad \int_{X}^{-\gamma^{2j-1}} \frac{1}{x} \left( -x + \frac{2\gamma^{2j+1}}{\gamma - 1} \right) \, dx = g(j, \gamma, X)
\]
where
\[ g(j, \gamma, X) = (X + \gamma^{2j-1}) - \frac{2\gamma^{2j+1}}{\gamma - 1} \left( \ln(-X) - (2j - 1) \ln(\gamma) \right). \]

Clearly
\[ I(X) = \frac{1}{2X} \left( \sum_{p=-\infty}^{i-1} f(p, \gamma, \gamma^{2p+2}) + f(i, \gamma, X) - \sum_{q=-\infty}^{j-1} g(q, \gamma, -\gamma^{2q+1}) - g(j, \gamma, -X) \right) \]
and, if \([\ln(X)/\ln(\gamma) - 1]\) is even, then \(i = j\). Upon summation, it can be proved that the minimum and maximum values of \(I(X)\) are, respectively,
\[ 1 + \frac{\gamma(\gamma + 1)}{(\gamma - 1)^2} \ln(\gamma), \quad 1 + \frac{1}{e \gamma} \gamma^{\gamma/(\gamma - 1)}. \]
The former quantity is least when \(\gamma = 5.7041372673\ldots\); the latter quantity is least when \(\gamma = 3.2232549401\ldots\). The corresponding mean ratios are 4.0089813375\ldots and 4.8131558458\ldots. These values together constitute our solution to the min-mean problem.

An alternative approach is due to Beck & Newman [5, 7, 8, 13, 14]. It uses a single random variable \(H\), assumed to be uniformly distributed on the interval \([0, 2)\), to sample different logarithmic coils with rate of increase \(\gamma\). For simplicity, take \(X > 0\). Then
\[ E(\delta(X)) = X + 2E \left( \sum_{j=-\infty}^{2i+1} \gamma^{j+H} \mid \gamma^{2i+H} < X \leq \gamma^{2i+2+H} \right) \]
\[ = X + \frac{2\gamma^2}{\gamma - 1} E \left( \gamma^{2i+H} \mid \frac{X}{\gamma^2} \leq \gamma^{2i+H} < X \right) \]
\[ = X + \frac{2\gamma^2}{\gamma - 1} E \left( X \gamma^{H-2} \right) \]
\[ = X + \frac{2\gamma^2}{\gamma - 1} \int_{0}^{2} X \gamma^{h-2} \frac{1}{2} dh = X \left( 1 + \frac{\gamma + 1}{\ln(\gamma)} \right) \]
and this is least when \(\gamma = 3.591121476669\ldots = 1/W(1/e)\), where \(W\) denotes Lambert’s function. A search strategy as such is called a mixed strategy (in game theory) or a random strategy (in computer science). Note, however, that a uniform distribution on \(H\) does not imply a uniform distribution on \(X\). It is not clear to us whether the optimal mixed strategy (with \(\gamma = 1/W(1/e)\)) is necessarily preferable to our deterministic strategy discussed earlier.
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