The Allen-Cahn equation on the complete Riemannian manifolds of finite volume

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Abstract

The semi-linear, elliptic PDE \( AC_\varepsilon(u) := -\varepsilon^2 \Delta u + W'(u) = 0 \) is called the Allen-Cahn equation. In this article we will prove the existence of finite energy solution to the Allen-Cahn equation on certain complete, non-compact manifolds. More precisely, suppose \( M^{n+1} \) (with \( n+1 \geq 3 \)) is a complete Riemannian manifold of finite volume. Then there exists \( \varepsilon_0 > 0 \), depending on the ambient Riemannian metric, such that for all \( 0 < \varepsilon \leq \varepsilon_0 \), there exists \( u_\varepsilon : M \rightarrow (-1,1) \) satisfying \( AC_\varepsilon(u_\varepsilon) = 0 \) with the energy \( E_\varepsilon(u_\varepsilon) < \infty \) and the Morse index \( \text{Ind}(u_\varepsilon) \leq 1 \). Moreover, \( 0 < \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) < \infty \). Our result is motivated by the theorem of Chambers-Liokumovich [CL20] and Song [Son19], which says that \( M \) contains a complete minimal hypersurface \( \Sigma \) with \( 0 < \mathcal{H}^n(\Sigma) < \infty \). This theorem can be recovered from our result.

1 Introduction

Minimal hypersurfaces are the critical points of the area functional. By the combined works of Almgren [Alm65], Pitts [Pit81] and Schoen-Simon [SS81], every closed Riemannian manifold \( (M^{n+1},g) \), \( n+1 \geq 3 \), contains a closed minimal hypersurface, which is smooth and embedded outside a singular set of Hausdorff dimension \( \leq n - 7 \).

Recently, Almgren-Pitts min-max theory has been further extended and it has been discovered that minimal hypersurfaces exist in abundance. By the works of Marques-Neves [MN17] and Song [Son18], every closed Riemannian manifold \( (M^{n+1},g) \), \( 3 \leq n+1 \leq 7 \), contains infinitely many closed, minimal hypersurfaces. This was conjectured by Yau [Yau82]. In [IMN18], Irie, Marques and Neves proved that for a generic metric \( g \) on \( M \), the union of all closed, minimal hypersurfaces is dense in \( (M,g) \). This theorem was later quantified by Marques, Neves and Song in [MNS19] where they proved that for a generic metric there exists an equidistributed sequence of closed, minimal hypersurfaces in \( (M,g) \). Recently, Song and Zhou [SZ20] proved the generic scarring phenomena for minimal hypersurfaces, which can be interpreted as the opposite of the equidistribution phenomena. In [Zho20], Zhou proved that for a generic (bumpy) metric, the min-max minimal hypersurfaces have multiplicity one, which was conjectured by Marques and Neves. Using this theorem, Marques and Neves

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[MN16, MN18] proved that for a generic (bumpy) metric $g$, there exists a sequence of closed, two-sided minimal hypersurfaces $\{\Sigma_k\}_{k=1}^{\infty}$ in $(M, g)$ such that $\text{Ind}(\Sigma_k) = k$ and $\mathcal{H}^n(\Sigma_k) \sim k^{\frac{1}{n+1}}$. In higher dimensions, Li [Li19] proved the existence of infinitely many closed minimal hypersurfaces (with optimal regularity) for a generic set of metrics. While the arguments in [MN18], [MNS19] and [Li19] depend on the Weyl law for the volume spectrum, which was conjectured by Gromov [Gro03] and proved by Liokumovich, Marques and Neves [LMN18], the arguments in [Son18] and [SZ20] use the cylindrical Weyl law, which was proved by Song [Son18].

In the above mentioned theorems, the ambient manifolds are assumed to be closed. If $M$ is a complete non-compact manifold, Gromov [Gro14] proved that either $M$ contains a complete minimal hypersurface with finite area or every compact domain of $M$ admits a (possibly singular) strictly mean convex foliation. In [Mon16], Montezuma proved that a complete Riemannian manifold with a bounded, strictly mean concave domain contains a complete minimal hypersurfaces with finite area. The existence of minimal surfaces in hyperbolic 3-manifolds has been proved by Collin-Hauswirth-Mazet-Rosenberg [CHMR17], Huang-Wang [HW17] and Coskunuzer [Cos18]. In [CK18], Chodosh and Ketover proved the existence of minimal planes in asymptotically flat 3-manifolds. In [CL20], Chambers and Liokumovich proved that every complete Riemannian manifold with finite volume contains a complete minimal hypersurface with finite area. In [Son19], Song proved Yau’s conjecture on certain complete non-compact manifolds. Moreover, he also proved the local version of the above mentioned theorem of Gromov [Gro14], using which he gave alternative proofs of the above mentioned theorems of Montezuma [Mon16] and Chambers-Liokumovich [CL20].

In [Gua18], Guaraco introduced a new approach for the min-max construction of minimal hypersurfaces, which was further developed by Gaspar and Guaraco in [GG18]. This approach is based on the study of the limiting behaviour of solutions to the Allen-Cahn equation. The Allen-Cahn equation (with parameter $\varepsilon > 0$) is the following semi-linear, elliptic PDE

$$AC_{\varepsilon}(u) := -\varepsilon^2 \Delta u + W'(u) = 0 \quad (1.1)$$

where $W : \mathbb{R} \to [0, \infty)$ is a double well potential e.g. $W(t) = \frac{1}{4}(1-t^2)^2$. The solutions of this equation are precisely the critical points of the energy functional

$$E_{\varepsilon}(u) = \int_M \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}.$$

Informally speaking, as $\varepsilon \to 0$, the level sets of the solutions to (1.1) (with uniformly bounded energy) accumulate around a generalized minimal hypersurface (called a limit-interface). In particular, Modica [Mod87] and Sternberg [Ste88] proved that as $\varepsilon \to 0$, the energy minimizing solutions to (1.1) converge to a area minimizing hypersurface. For general solutions to (1.1), Hutchinson and Tonegawa [HT00] proved that the limit-interface is a stationary, integral varifold. Moreover, if the solutions are stable, by the works of Tonegawa [Ton05], Wickramasekera
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[Wic14] and Tonegawa-Wickramasekera [TW12], the limit-interface is a stable minimal hypersurface with optimal regularity. In [Gua18], Guaraco proved that the limit-interface has optimal regularity if the solutions have uniformly bounded Morse index. Furthermore, by a mountain-pass argument, he proved the existence of critical points of $E_\varepsilon$ (on a closed Riemannian manifold) with uniformly bounded energy and Morse index. In this way he obtained a new proof of the previously mentioned theorem of Almgren-Pitts-Schoen-Simon. In the case of surfaces (i.e. when the ambient dimension = 2), Mantoulidis [Man17] proved the regularity of the geodesic limit-interface for the solutions with uniformly bounded Morse index.

The index upper bound of the limit-interface was proved by Hiesmayr [Hie18] assuming the limit-interface is two-sided and by Gaspar [Gas20] in the general case. In [GG19], Gaspar and Guaraco proved the Weyl law for the phase transition spectrum and gave alternative proofs of the density [IMN18] and the equidistribution [MNS19] theorems. In [CM20], Chodosh and Mantoulidis proved the multiplicity one conjecture in the Allen-Cahn setting in dimension 3 and the upper semi-continuity of the Morse index when the limit-interface has multiplicity one. As a consequence, they proved that for a generic (bumpy) metric $g$ on a closed manifold $M^3$, there exists a sequence of closed, two-sided minimal surfaces $\{\Sigma_p\}_{p=1}^{\infty}$ in $(M^3, g)$ such that $\text{Ind}(\Sigma_p) = p$ and $\text{area}(\Sigma_p) \sim p^{4/3}$. In higher dimensions, the multiplicity one conjecture for the one parameter Allen-Cahn min-max has been proved by Bellettini [Bel20a,Bel20b]. In [GMN19] Guaraco, Marques and Neves proved that a strictly stable limit-interface must have multiplicity one.

In [BW20], Bellettini and Wickramasekera proved the existence of closed prescribed mean curvature (PMC) hypersurfaces in arbitrary closed Riemannian manifolds using the min-max solutions of the inhomogeneous Allen-Cahn equations. To prove the regularity of the Allen-Cahn PMC hypersurfaces, they used their earlier works [BW18,BW19] on the regularity and compactness theory of stable PMC hypersurfaces. Previously, Zhou and Zhu [ZZ19,ZZ20] developed a min-max theory for the construction of closed PMC hypersurfaces which is parallel to the Almgren-Pitts min-max theory. The estimates for the index and nullity of the Allen-Cahn PMC hypersurfaces have been proved by Mantoulidis [Man20].

The asymptotic behaviour of the critical points of the Ginzburg–Landau functional (which approximates the codimension-2 area functional) has been studied by Stern [Ste16,Ste17], Cheng [Che17] and Pigati-Stern [PS19]. In particular, in [PS19] Pigati and Stern proved the existence of a codimension-2 stationary, integral varifold in an arbitrary closed Riemannian manifold. This theorem was previously proved by Almgren [Alm65] using more complicated geometric measure theory approach.

If $\Sigma$ is a non-degenerate, separating, closed minimal hypersurface in a closed Riemannian manifold, Pacard and Ritoré [PR03] constructed solutions of the Allen-Cahn equation, for sufficiently small $\varepsilon > 0$, whose level sets converge to $\Sigma$. The uniqueness of these solutions has been proved by Guaraco, Marques and Neves [GMN19]. The construction of Pacard and
Ritoré has been extended by Caju and Gaspar [CG19] in the case when all the Jacobi fields of $\Sigma$ are induced by the ambient isometries. Assuming a positivity condition on the Ricci curvature of the ambient manifold, del Pino-Kowalczyk-Wei-Yang [dKWy10] constructed solutions of the Allen-Cahn equation whose energies concentrate on a non-degenerate, closed minimal hypersurface with multiplicity $> 1$.

In this article we will show the existence of finite energy min-max solution to the Allen-Cahn equation (for $\varepsilon$ sufficiently small) on complete Riemannian manifolds of finite volume. More precisely, we will prove the following theorem, which is motivated by the previously mentioned theorem of Chambers-Liokumovich [CL20] and Song [Son19].

**Theorem 1.1.** Let $M^{n+1}$ be a complete, Riemannian manifold, $n+1 \geq 3$, such that $\text{Vol}(M)$ is finite. Then there exists $\varepsilon_0 > 0$, depending on the ambient Riemannian metric, such that for all $0 < \varepsilon \leq \varepsilon_0$, there exists $u_\varepsilon : M \to (-1,1)$ satisfying $AC_\varepsilon(u_\varepsilon) = 0$, $\text{Ind}(u_\varepsilon) \leq 1$ and $E_\varepsilon(u_\varepsilon) < \infty$. Moreover, there exists a good set $U \subset M$ (see Section 2.4) such that

$$0 < \liminf_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon, U) \leq \limsup_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon) < \infty.$$  \hfill (1.2)

We will prove Theorem 1.1 by adapting the argument of Chambers-Liokumovich [CL20] in the Allen-Cahn setting. From Theorem 1.1, one can recover the above mentioned theorem of Chambers-Liokumovich [CL20] and Song [Son19].

**Theorem 1.2.** [CL20, Son19] Let $M^{n+1}$ be a complete, Riemannian manifold, $n + 1 \geq 3$, such that $\text{Vol}(M)$ is finite. Then there exists a complete minimal hypersurface $\Sigma \subset M$ such that $0 < H^n(\Sigma) < \infty$ and $\Sigma$ has optimal regularity, i.e. $\Sigma$ is smooth and embedded outside a singular set of Hausdorff dimension $\leq n - 7$.

As in [CL20] and [Son19], Theorem 1.1 and Theorem 1.2 continue to hold if the assumption $\text{Vol}(M) < \infty$ is replaced by the weaker assumption that there exists a sequence $\{U_i\}_{i=1}^\infty$, where each $U_i \subset M$ is a bounded open set with smooth boundary, such that $U_i \subset U_{i+1}$ for all $i \in \mathbb{N}$ and $\lim_{i \to \infty} H^n(\partial U_i) = 0$.

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2 Notation and Preliminaries

2.1 Notation

Here we summarize the notation which will be frequently used later.
• $\mathcal{H}^k$: the Hausdorff measure of dimension $k$.
• $B(p, r)$: the geodesic ball centered at $p$ with radius $r$.
• $d(\cdot, S)$: distance from a set $S$.
• $H^1(N)$: the Sobolev space $\{ u \in L^2(N) : \text{the distributional derivative } \nabla u \in L^2(N, TN) \}$.
• $e_\varepsilon(u) = \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}$.
• $E_\varepsilon(u) = \int_N e_\varepsilon(u)$, where $N$ is the ambient manifold.
• $E_\varepsilon(u, S) = \int_S e_\varepsilon(u)$, where $S$ is a measurable set.
• $AC_\varepsilon(u) = -\varepsilon^2 \Delta u + W'(u)$.
• $2\sigma$ = the energy of the 1-D solution to the Allen-Cahn equation (see (2.1), (4.3)).
• For two measurable functions $u$ and $v$, we say that $u \leq v$ (resp. $u \geq v$) if $u(x) \leq v(x)$ (resp. $u(x) \geq v(x)$) for a.e. $x$.

2.2 The Allen-Cahn equation and convergence of the phase interfaces

In this subsection we will briefly discuss about the Allen-Cahn equation and its connection with the minimal hypersurfaces. Suppose $\Omega^{n+1}$ is the interior of a compact Riemannian manifold. Let $W : \mathbb{R} \to [0, \infty)$ be a smooth, symmetric, double well potential. More precisely, $W$ has the following properties. $W$ is bounded; $W(-t) = W(t)$ for all $t \in \mathbb{R}$; $W$ has exactly three critical points $0, \pm 1$; $W(\pm 1) = 0$ and $W''(\pm 1) > 0$ i.e. $\pm 1$ are non-degenerate minima; $0$ is a local maximum. For $u \in H^1(\Omega)$, the $\varepsilon$-Allen-Cahn energy of $u$ is given by

$$E_\varepsilon(u) = \int_{\Omega} \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}.$$

As mentioned earlier,

$$AC_\varepsilon(u) := -\varepsilon^2 \Delta u + W'(u) = 0$$

if and only if $u$ is a critical point of $E_\varepsilon$.

Let $F : \mathbb{R} \to \mathbb{R}$ and the energy constant $\sigma$ be defined as follows.

$$F(t) = \int_0^t \sqrt{W(s)/2} \, ds; \quad \sigma = \int_{-1}^1 \sqrt{W(s)/2} \, ds \quad \text{so that} \quad F(\pm 1) = \pm \frac{\sigma}{2}. \quad (2.1)$$

For an $n$-rectifiable set $S \subset \Omega$, let $|S|$ denote the $n$-varifold defined by $S$. Given $u \in C^1(\Omega)$, we set $\tilde{u} = F \circ u$. The $n$-varifold associated to $u$ is defined by

$$V[u](A) = \frac{1}{\sigma} \int_{-\infty}^\infty |\{\tilde{u} = s\}|(A) \, ds,$$
for every Borel set $A \subset G_n \Omega$ (where $G_n \Omega$ denotes the Grassmannian bundle of unoriented $n$-dimensional hyperplanes on $\Omega$).

Building on the works of Hutchinson-Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa-Wickramasekera [TW12], Guaraco [Gua18] has proved the following theorem.

**Theorem 2.1** ([HT00, Ton05, TW12, Gua18]). Suppose $\Omega^{n+1}$, $n + 1 \geq 3$, is the interior of a compact Riemannian manifold. Let $\{u_i : \Omega \to (-1, 1)\}_{i=1}^{\infty}$ be a sequence of smooth functions such that

(i) $A C_{\epsilon_i}(u_i) = 0$ with $\epsilon_i \to 0$ as $i \to \infty$;

(ii) $\sup_{i \in \mathbb{N}} E_{\epsilon_i}(u_i) < \infty$ and $\sup_{i \in \mathbb{N}} \text{Ind}(u_i) < \infty$.

Then there exists a stationary, integral varifold $V$ in $\Omega$ such that possibly after passing to a subsequence, $V[u_i] \to V$ in the sense of varifolds. Moreover, $\text{spt}(V)$ is a minimal hypersurface with optimal regularity in $\Omega$. Furthermore, if $\|V\|$ denotes the Radon measure associated to $V$, then

$$\frac{1}{2\sigma} \left( \epsilon_i \frac{|\nabla u_i|^2}{2} + \frac{W(u_i)}{\epsilon_i} \right) d\text{Vol}_\Omega \to \|V\|,$$

in the sense of Radon measures.

The proof of the regularity of the limit-interface depends on the regularity theory of stable, minimal hypersurfaces, developed by Wickramasekera [Wic14]. In the ambient dimension $n + 1 = 3$, the regularity of the limit-interface can also be obtained from the curvature estimates of Chodosh and Mantoulidis [CM20].

We also state here the theorem proved by Smith [Smi16] about the generic finiteness of the number of solutions to the Allen-Cahn equation on a closed manifold. This theorem will be used to prove the Morse index upper bound in Theorem 1.1.

Let $\mathcal{M}$ be the space of all smooth Riemannian metrics on $N$, endowed with the $C^\infty$ topology. For $\varepsilon > 0$ and $\gamma \in \mathcal{M}$, we define

$$Z_{\varepsilon, \gamma} = \{ u \in C^\infty(N) : -\varepsilon^2 \Delta_{\gamma} u + W'(u) = 0 \}.$$

**Theorem 2.2.** [Smi16, Theorem 1.1 (2)] There exists a generic set $\tilde{\mathcal{M}} \subset \mathcal{M}$ such that if $\gamma \in \tilde{\mathcal{M}}$ and $\varepsilon^{-1} \notin \text{Spec}(-\Delta_{\gamma})$ (here we are using the convention that $\text{Spec}(-\Delta_{\gamma}) \subset [0, \infty)$), then $Z_{\varepsilon, \gamma}$ is finite.
2.3 Min-max theorem on the Hilbert space

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ be a $C^2$ functional. Suppose $B_0, B_1$ are closed subsets of $\mathcal{H}$. We define

$$\mathcal{F} = \{ \zeta : [0, 1] \to \mathcal{H} : \zeta \text{ is continuous}, \zeta(0) \in B_0, \zeta(1) \in B_1 \}$$

and

$$c = \inf_{\zeta \in \mathcal{F}} \sup_{t \in [0, 1]} \mathcal{E}(\zeta(t)).$$

A sequence $\{\zeta_i\}_{i=1}^\infty \subset \mathcal{F}$ is called a minimizing sequence if

$$\lim_{i \to \infty} \sup_{t \in [0, 1]} \mathcal{E}(\zeta_i(t)) = c.$$

For a minimizing sequence $\{\zeta_i\} \subset \mathcal{F}$, let $K(\{\zeta_i\})$ denote the set of all $v \in \mathcal{H}$ for which there exist sequences $\{i_j\} \subset \{i\}$ and $\{t_j\} \subset [0, 1]$ such that

$$v = \lim_{j \to \infty} \zeta_{i_j}(t_j).$$

**Definition 2.3.** Given a minimizing sequence $\{\zeta_i\}$ in $\mathcal{F}$, we say that $\mathcal{E}$ satisfies the Palais-Smale condition along $\{\zeta_i\}$ if every sequence $\{v_i\}$, satisfying the conditions

$$\lim_{i \to \infty} \mathcal{E}'(v_i) = 0 \quad \text{and} \quad \lim_{i \to \infty} d(v_i, \zeta_i([0, 1])) = 0,$$

has a convergent subsequence.

**Definition 2.4.** [Gho91, Section 3, page 53] Let

$$K_c = \{ v \in \mathcal{H} : \mathcal{E}'(v) = 0, \mathcal{E}(v) = c \}.$$

A compact subset $\mathcal{C}$ of $K_c$ is called an isolated critical set for $\mathcal{E}$ in $K_c$ if there exists an open set $\mathcal{U} \subset \mathcal{H}$ such that $\mathcal{C} \subset \mathcal{U}$ and

$$K_c \cap \mathcal{U} = \mathcal{C}.$$

The following min-max theorem, which was proved by Ghoussoub [Gho91] in a much more general setting, will be used to prove Theorem 1.1.

**Theorem 2.5.** [Gho91] (a) Let $L \subset \mathcal{H}$ be a closed set such that the following conditions are satisfied:

(a1) $L \cap (B_0 \cup B_1) = \emptyset$;

(a2) for all $\zeta \in \mathcal{F}$, $L \cap \zeta([0, 1]) \neq \emptyset$;
\( \inf_{v \in L} \mathcal{E}(v) \geq c. \)

Suppose \( \mathcal{E} \) satisfies the Palais-Smale condition along a minimizing sequence \( \{ \zeta_i \}_{i=1}^{\infty} \). Then

\[ K_c \cap L \cap K(\{ \zeta_i \}) \neq \emptyset. \]

(b) In addition to the assumptions stated in part (a), let us also assume that:

(b1) \( K_c \cap L \) is an isolated critical set for \( \mathcal{E} \) in \( K_c \);

(b2) \( \mathcal{E}'' \) is Fredholm on \( K_c \).

Then there exists

\[ v \in K_c \cap L \cap K(\{ \zeta_i \}) \] such that \( m(v) \leq 1, \]

where \( m(v) \) is the Morse index of the critical point \( v \), i.e. \( m(v) \) is equal to the index of the bilinear form \( \mathcal{E}'' \big|_v \).

Remark 2.6. In the definition of \( \mathcal{F} \) in (2.2) and in Theorem 2.5, we have assumed that \( B_0 \) and \( B_1 \) are closed subsets of \( \mathcal{H} \). This is slightly different from the hypothesis made in [Gho91, Theorem (1.bis) and Theorem (4)], where \( B_0 \) and \( B_1 \) are assumed to be singleton sets. However this does not affect the proof of Theorem 2.5 in [Gho91] for the following reason (see [Gho91, Remark (3) in page 32 and Remark (11) in page 60]). If \( \zeta \in \mathcal{F} \) (as defined in (2.2)) and \( \zeta' : [0, 1] \to \mathcal{H} \) is another map satisfying \( \zeta'(0) = \zeta(0) \) and \( \zeta'(1) = \zeta(1) \), then \( \zeta' \in \mathcal{F} \) as well.

2.4 The notion of the good set

In this subsection we will recall the definition of the good set from [CL20, Section 2.2]. Let \( N \) be a complete Riemannian manifold and \( \Omega \subset N \) be a bounded open set with smooth boundary \( \partial \Omega \). \( \mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2) \) denotes the space of \((n+1)\)-dimensional mod 2 flat chains in \( \Omega \); \( \mathbf{Z}_{\text{rel}}(\Omega, \partial \Omega; \mathbb{Z}_2) \) denotes the space of \( n \)-dimensional mod 2 relative flat cycles in \( \Omega \) and \( \partial : \mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2) \to \mathbf{Z}_{\text{rel}}(\Omega, \partial \Omega; \mathbb{Z}_2) \) is the boundary map. Both the spaces \( \mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2) \) and \( \mathbf{Z}_{\text{rel}}(\Omega, \partial \Omega; \mathbb{Z}_2) \) are assumed to be equipped with the flat topology. (We refer to [LMN18, Section 2] and [LZ16, Section 3] for more details about these spaces.) Let \( \mathcal{F} \) be the set of all continuous maps \( \Gamma : [0, 1] \to \mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2) \) such that \( \Gamma(0) = \emptyset \) and \( \Gamma(1) = \Omega \). The (relative) width of \( \Omega \), denoted by \( \mathbb{W}(\Omega) \), is defined as follows [Gro88, Gut09, LMN18, CL20, LZ16].

\[
\mathbb{W}(\Omega) = \inf_{\Gamma \in \mathcal{F}} \sup_{t \in [0,1]} M(\partial \Gamma(t)).
\]

(2.3)

\( \Omega \) is called a good set if

\[
\mathbb{W}(\Omega) > 4\mathcal{H}^n(\partial \Omega).
\]

(2.4)
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3 Nested families in the Sobolev space $H^1(N)$

The notion of the nested family of open sets played an important role in the proof of the main theorem of Chambers-Liokumovich in CL20. In this section, we will deal with the notion of the nested family in the function space $H^1(N)$. Throughout this section, $N$ will be assumed to be a closed Riemannian manifold (of dimension $n+1$) and $\varepsilon > 0$. We begin with the following lemma.

**Lemma 3.1.** Let $u, v \in H^1(N)$ such that $v \geq u$ and $|u|, |v| \leq 1$. Suppose

$$S = \{ w \in H^1(N) : v \geq w \geq u \}.$$

Then there exists $w^* \in S$ such that

$$E_\varepsilon(w^*) = \inf \{ E_\varepsilon(w) : w \in S \}.$$

**Proof.** Let

$$\alpha = \inf \{ E_\varepsilon(w) : w \in S \}$$

and $\{w_i\}_{i=1}^\infty \subset S$ be such that

$$\int_N \varepsilon \frac{|\nabla w_i|^2}{2} + \frac{W(w_i)}{\varepsilon} \leq \alpha + \frac{1}{i}. \quad (3.1)$$

Since $|u|, |v| \leq 1$,

$$|w_i| \leq 1 \quad \forall \ i \in \mathbb{N}. \quad (3.2)$$

(3.1) and (3.2) imply that $\{w_i\}_{i=1}^\infty$ is a bounded sequence in $H^1(N)$. Therefore, by Rellich’s compactness theorem, there exist $w^* \in H^1(N)$ and a subsequence $\{w_{i_k}\}_{k=1}^\infty$ such that

$$w_{i_k} \to w^* \text{ strongly in } L^2(N) \text{ and pointwise a.e.} \quad (3.3)$$

and

$$\nabla w_{i_k} \to \nabla w^* \text{ weakly in } L^2(N). \quad (3.4)$$

(3.3) implies that $w^* \in S$. (3.3), (3.4) and (3.1) together imply that $E_\varepsilon(w^*) \leq \alpha$ and hence $E_\varepsilon(w^*) = \alpha$ (as $w^* \in S$). \qed

**Definition 3.2.** A continuous map $u : [a, b] \to H^1(N)$ is called *nested* if $u(t) \geq u(s)$ whenever $t \leq s$.
3.1 Truncation and concatenation of the nested maps

The following Lemmas 3.3 and 3.4 are the Allen-Cahn counterparts of [CL20, Lemma 5.1] and [CL20, Proposition 6.3], respectively.

**Lemma 3.3.** (a) Let \( u : [0, 1] \to H^1(N) \) be nested. Suppose \( v \in H^1(N) \) has the following properties:

- \( v \geq u(1) \);
- for any \( v' \in H^1(N) \) with \( v \geq v' \geq u(1) \), we have \( E_\varepsilon(v) \leq E_\varepsilon(v') \).

Then there exists \( \tilde{u} : [0, 1] \to H^1(N) \) such that

(i) \( \tilde{u} \) is nested;
(ii) \( \tilde{u}(0) \geq u(0) \) and \( \tilde{u}(1) = v \); moreover if \( u(0) \geq v \), one can choose \( \tilde{u}(0) = u(0) \);
(iii) \( E_\varepsilon(\tilde{u}(t)) \leq E_\varepsilon(u(t)) \) for all \( t \in [0, 1] \);
(iv) if \( \|v\|_{L^\infty(N)} \leq 1 \) and \( \sup_{t \in [0,1]} \|u(t)\|_{L^\infty(N)} \leq 1 \), then \( \sup_{t \in [0,1]} \|\tilde{u}(t)\|_{L^\infty(N)} \leq 1 \) as well.

(b) Let \( u : [0, 1] \to H^1(N) \) be nested. Suppose \( v \in H^1(N) \) has the following properties:

- \( u(0) \geq v \);
- for any \( v' \in H^1(N) \) with \( u(0) \geq v' \geq v \), we have \( E_\varepsilon(v) \leq E_\varepsilon(v') \).

Then there exists \( \tilde{u} : [0, 1] \to H^1(N) \) such that

(i) \( \tilde{u} \) is nested;
(ii) \( \tilde{u}(0) = v \) and \( \tilde{u}(1) \leq u(1) \); moreover if \( u(1) \leq v \), one can choose \( \tilde{u}(1) = u(1) \);
(iii) \( E_\varepsilon(\tilde{u}(t)) \leq E_\varepsilon(u(t)) \) for all \( t \in [0, 1] \);
(iv) if \( \|v\|_{L^\infty(N)} \leq 1 \) and \( \sup_{t \in [0,1]} \|u(t)\|_{L^\infty(N)} \leq 1 \), then \( \sup_{t \in [0,1]} \|\tilde{u}(t)\|_{L^\infty(N)} \leq 1 \) as well.

**Proof.** To prove part (a), we define

\[
\tilde{u}(t) = \max\{u(t), v\}.
\]

Items (i) and (ii) follow from the assumptions that \( u \) is nested and \( v \geq u(1) \), respectively. Item (iv) follows from the definition of \( \tilde{u} \). To prove (iii), we consider

\[
u'(t) = \min\{u(t), v\}.
\]

Then \( v \geq u'(t) \geq u(1) \) for all \( t \in [0, 1] \). By our hypothesis,

\[
E_\varepsilon(v) \leq E_\varepsilon(u'(t)) = E_\varepsilon(u(t), \{u(t) < v\}) + E_\varepsilon(v, \{u(t) \geq v\}).
\]
Hence
\[ E_\varepsilon(v, \{u(t) < v\}) \leq E_\varepsilon(u(t), \{u(t) < v\}). \] (3.5)
Therefore, using (3.5),
\[ E_\varepsilon(\tilde{u}(t)) = E_\varepsilon(u(t), \{u(t) \geq v\}) + E_\varepsilon(v, \{u(t) < v\}) \leq E_\varepsilon(u(t)). \]

Part (b) can also be proved in a similar way by defining
\[ \tilde{u}(t) = \min\{u(t), v\} \]
and using the fact that
\[ u(0) \geq \max\{u(t), v\} \geq v. \]

\[ \square \]

**Lemma 3.4.** Let \( u_1, u_2 : [0, 1] \to H^1(N) \) be such that
\begin{itemize}
  \item \( u_1, u_2 \) are nested;
  \item \( \sup_{t \in [0,1]} \|u_i(t)\|_{L^\infty(N)} \leq 1 \) for \( i = 1, 2 \);
  \item \( u_2(0) \geq u_1(1) \);
  \item \( \sup_{t \in [0,1]} E_\varepsilon(u_i(t)) \leq A \) for \( i = 1, 2 \).
\end{itemize}
Then there exists \( \tilde{u} : [0, 1] \to H^1(N) \) such that
\begin{enumerate}
  \item \( \tilde{u} \) is nested;
  \item \( \sup_{t \in [0,1]} \|\tilde{u}(t)\|_{L^\infty(N)} \leq 1 \);
  \item \( \tilde{u}(0) \geq u_1(0) \) and \( \tilde{u}(1) \leq u_2(1) \);
  \item \( \sup_{t \in [0,1]} E_\varepsilon(\tilde{u}(t)) \leq A \).
\end{enumerate}

**Proof.** Let
\[ S = \{v \in H^1(N) : u_2(0) \geq v \geq u_1(1)\}. \]
By Lemma 3.1, there exists \( v^* \in S \) such that
\[ E_\varepsilon(v^*) = \inf\{E_\varepsilon(v) : v \in S\}. \]
We note that \( v^* \geq v' \geq u_1(1) \) implies that \( v' \in S \) and hence \( E_\varepsilon(v^*) \leq E_\varepsilon(v') \). Therefore, by Lemma 3.3, part (a), there exists a nested map \( \tilde{u}_1 : [0, 1] \to H^1(N) \) such that
\[ \tilde{u}_1(0) \geq u_1(0), \tilde{u}_1(1) = v^*, \sup_{t \in [0,1]} \|\tilde{u}_1(t)\|_{L^\infty(N)} \leq 1 \text{ and } \sup_{t \in [0,1]} E_\varepsilon(\tilde{u}_1(t)) \leq A. \]
Similarly, \( u_2(0) \geq v' \geq v^* \) implies that \( v' \in \mathcal{S} \) and hence \( E_\varepsilon(v^*) \leq E_\varepsilon(v') \). Therefore, by Lemma 3.3, part (b), there exists a nested map \( \tilde{u}_2 : [0, 1] \to H^1(N) \) such that

\[
\tilde{u}_2(0) = v^*, \quad \tilde{u}_2(1) \leq u_2(1), \quad \sup_{t \in [0, 1]} \|\tilde{u}_2(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\tilde{u}_2(t)) \leq A.
\]

Finally, we define \( \tilde{u} : [0, 1] \to H^1(N) \) by

\[
\tilde{u}(t) = \begin{cases} 
\tilde{u}_1(2t) & \text{if } t \in [0, 1/2]; \\
\tilde{u}_2(2t - 1) & \text{if } t \in [1/2, 1].
\end{cases}
\]

\( \square \)

### 3.2 Approximation by nested maps

In [CL20, Proposition 6.1], Chambers and Liokumovich proved that if \( \{\Omega_t\}_{t \in [0, 1]} \) is a family of open sets and \( \kappa > 0 \), there exists a nested family of open sets \( \{\tilde{\Omega}_t\}_{t \in [0, 1]} \) such that \( \tilde{\Omega}_0 \subset \Omega_0 \), \( \tilde{\Omega}_1 \supset \Omega_1 \) and

\[
\sup_{t \in [0, 1]} \mathcal{H}^n(\partial \tilde{\Omega}_t) \leq \sup_{t \in [0, 1]} \mathcal{H}^n(\partial \Omega_t) + \kappa.
\]

The following Proposition 3.5 is the Allen-Cahn analogue of this theorem.

**Proposition 3.5.** Let \( \phi : [0, 1] \to H^1(N) \) be a continuous map such that \( |\phi(t)| \leq 1 \) for all \( t \in [0, 1] \) and \( \kappa > 0 \). Then there exists a nested map \( \psi : [0, 1] \to H^1(N) \) such that \( \psi(0) \geq \phi(0) \), \( \psi(1) \leq \phi(1) \), \( |\psi(t)| \leq 1 \) for all \( t \in [0, 1] \) and

\[
\sup_{t \in [0, 1]} E_\varepsilon(\psi(t)) \leq \sup_{t \in [0, 1]} E_\varepsilon(\phi(t)) + \kappa.
\]

Before we prove Proposition 3.5, we need to prove few lemmas.

**Lemma 3.6.** Let \( \varepsilon > 0 \) and \( w : [0, 1] \to H^1(N) \) be a continuous map. Then, for all \( \delta > 0 \), there exists \( r > 0 \) such that

\[
E_\varepsilon(w(t), B(p, r)) \leq \delta,
\]

for all \( t \in [0, 1] \) and \( p \in N \).

**Proof.** We assume by contradiction that there exist \( \delta > 0 \) and sequences \( \{t_i\}_{i=1}^\infty \subset [0, 1] \) and \( \{p_i\}_{i=1}^\infty \subset N \) such that

\[
E_\varepsilon(w(t_i), B(p_i, \varepsilon^{-1})) > \delta. \tag{3.6}
\]

Without loss of generality, we can assume that \( t_i \to t_0 \) and \( p_i \to p_0 \). Then, for all \( m \in \mathbb{N} \), if \( i \) is sufficiently large, \( B(p_i, \varepsilon^{-1}) \subset B(p_0, m^{-1}) \). Therefore, by (3.6), for all \( m \in \mathbb{N} \),

\[
E_\varepsilon(w(t_0), B(p_0, m^{-1})) = \lim_{i \to \infty} E_\varepsilon(w(t_i), B(p_0, m^{-1})) \geq \delta.
\]
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This contradicts the fact that

$$\lim_{m \to \infty} E_\varepsilon(w(t_0), B(p_0, m^{-1})) = 0.$$ 

\[ \square \]

**Lemma 3.7.** Let $u_0, u_1 \in L^\infty(N)$ with $u_0 \geq u_1$ and $|u_0|, |u_1| \leq 1$. Suppose $h : \mathbb{R} \to [-1, 1]$ is a piecewise $C^1$ function so that $h' \in L^\infty(\mathbb{R})$ and $e_\varepsilon(h)$ is compactly supported inside the compact interval $[-a, a]$. For $p \in N$ and $r > 0$, let $b^r : N \to \mathbb{R}$ be defined by $b^r(x) = h(d_p(x) - r)$, where $d_p(x) = d(x, p)$. Then, for all $0 < s_1 < s_2$ and $\varepsilon > 0$,

$$\int_{s_1}^{s_2} E_\varepsilon (b^r, \{u_0 > b^r > u_1\}) \, dr \leq \int_{B(p, s_2 + a)} \int_{\{u_0(x) > h > u_1(x)\}} e_\varepsilon(h)(t) \, dt \, d\mathcal{H}^{n+1}(x).$$

**Proof.** $d_p : N \to \mathbb{R}$ is a Lipschitz continuous function with $|\nabla d_p| = 1$, $\mathcal{H}^{n+1}$-a.e. As a consequence,

$$\mathcal{H}^{n+1}(\{d_p = s\}) = 0 \quad \forall \, s \geq 0,$$

(3.7)

since otherwise $\nabla d_p = 0$ on a set of positive $\mathcal{H}^{n+1}$-measure. For all $r > 0$, $b^r$ is Lipschitz continuous; by [GT01, Theorem 7.8] and (3.7),

$$\nabla b^r(x) = h'(d_p(x) - r)\nabla d_p(x) \text{ for } \mathcal{H}^{n+1}$-a.e. \, x \in N.$$

Therefore,

$$\int_{s_1}^{s_2} E_\varepsilon (b^r, \{u_0 > b^r > u_1\}) \, dr$$

$$= \int_{s_1}^{s_2} \int_{\{u_0 > b^r > u_1\}} \left[ \frac{\varepsilon}{2} h'(d_p(x) - r)^2 + \frac{1}{\varepsilon} W(h(d_p(x) - r)) \right] \, d\mathcal{H}^{n+1}(x) \, dr$$

$$= \int_{s_1}^{s_2} \int_{-a}^{a} \left[ \frac{\varepsilon}{2} h'(t)^2 + \frac{1}{\varepsilon} W(h(t)) \right] \mathcal{H}^n(\{d_p - r = t\} \cap \{u_0 > b^r > u_1\}) \, dt \, dr. \quad (3.8)$$

In the last step we have used the co-area formula. It follows from the definition of $b^r$ that for fixed $r$ and $t$,

$$\{d_p - r = t\} \cap \{u_0 > b^r > u_1\} = \{d_p = r + t\} \cap \{u_0 > h(t) > u_1\}.$$
Hence, by Fubini’s theorem and the co-area formula, \((3.8)\) implies that
\[
\int_{s_1}^{s_2} E_\varepsilon(b^r, \{ u_0 > b^r > u_1 \}) \, dr \\
= \int_{s_1}^{s_2} \int_{-a}^{a} \left[ \frac{\varepsilon}{2} h(t)^2 + \frac{1}{\varepsilon} W(h(t)) \right] \mathcal{H}^n(\{ d_p = r + t \} \cap \{ u_0 > h(t) > u_1 \}) \, dt \, dr \\
\leq \int_{-a}^{a} e_\varepsilon(h(t)) \mathcal{H}^{n+1}(B(p,s_2 + a) \cap \{ u_0 > h(t) > u_1 \}) \, dt \\
= \int_{B(p,s_2+a) \cap \{ u_0(x) > h > u_1(x) \}} e_\varepsilon(h(t)) \, dt \, d\mathcal{H}^{n+1}(x).
\]

For \(\rho > 0\), let \(h^\rho : \mathbb{R} \to [-1,1]\) be defined by
\[
h^\rho(t) = \begin{cases} 
\frac{1}{\rho} & \text{if } |t| \leq \rho; \\
1 & \text{if } t \geq \rho; \\
-1 & \text{if } t \leq -\rho.
\end{cases}
\]

Setting \(h = h^\rho\) in Lemma 3.7 and using the notation \(b^\rho,r(x) = h^\rho(d_p(x) - r)\), one obtains
\[
\int_{s_1}^{s_2} E_\varepsilon(b^\rho,r, \{ u_0 > b^\rho,r > u_1 \}) \, dr \leq C(\varepsilon, \rho) \| u_0 - u_1 \|_{L^1(N)}, \tag{3.10}
\]
where \(u_0, u_1\) are as in Lemma 3.7 and
\[
C(\varepsilon, \rho) = \frac{\varepsilon}{2\rho^2} + \frac{1}{\varepsilon} \| W \|_{L^\infty([-1,1])}. \tag{3.11}
\]

**Lemma 3.8.** Let \(\varepsilon, \delta > 0\). Suppose \(u_0, u_1 \in H^1(N) \cap L^\infty(N)\) such that \(u_0 \geq u_1; |u_0|, |u_1| \leq 1\) and
\[
\int_N |e_\varepsilon(u_0) - e_\varepsilon(u_1)| \leq \delta. \tag{3.12}
\]
We fix a nested map \(w : [0,1] \to H^1(N)\) such that \(w(0) \equiv 1, w(1) \equiv -1\) and \(|w(t)| \leq 1\) for all \(t \in [0,1]\). Let \(R > 0\) be such that for all \(p \in N\),
\[
E_\varepsilon(u_0, B(p,4R)) \leq \delta, \quad E_\varepsilon(u_1, B(p,4R)) \leq \delta \quad \text{and} \quad \sup_{t \in [0,1]} E_\varepsilon(w(t), B(p,4R)) \leq \delta. \tag{3.13}
\]
Suppose \(N\) can be covered by \(I\) balls of radius \(R\). Then, using the notation of \((3.11)\),
\[
\| u_0 - u_1 \|_{L^1(N)} \leq \frac{\delta R}{C(\varepsilon, R)I} \tag{3.14}
\]
The Allen-Cahn equation on the complete Riemannian manifolds of finite volume implies that there exists a nested map \( u : [0, 1] \to H^1(N) \) such that \( u(0) = u_0, u(1) = u_1, |u(t)| \leq 1 \) for all \( t \in [0, 1] \) and
\[
\sup_{t \in [0, 1]} E_\epsilon(u(t)) \leq \min\{E_\epsilon(u_0), E_\epsilon(u_1)\} + 9\delta.
\]

**Proof.** Let
\[
N = \bigcup_{i=1}^I B(p_i, R). \quad (3.15)
\]
For \( r > 0 \) and \( i = 1, 2, \ldots, I \), let \( b_i^r : N \to [-1, 1] \) be defined by (using the notation as in (3.9))
\[
b_i^r(x) = h^R(d_{p_i}(x) - r).
\]
We inductively define a sequence \( \{v_i\}_{i=0}^I \) with
\[
v_0 \geq v_1 \geq \cdots \geq v_I
\]
as follows. Set \( v_0 = u_0 \). Let us assume that \( v_k \) has been defined for \( 1 \leq k \leq i - 1 \) and \( u_0 = v_0 \geq v_1 \geq \cdots \geq v_{i-1} \geq u_1 \), which implies that
\[
\|v_i - 1 - u_1\|_{L^1(N)} \leq \|u_0 - u_1\|_{L^1(N)}. \quad (3.16)
\]
Therefore, using (3.10), (3.16) and (3.14),
\[
\int_{2R}^{3R} E_\epsilon(b_i^r, \{v_i - 1 > b_i^r \geq u_1\}) \, dr \leq \frac{\delta R}{I}.
\]
So there exists \( r_i \in (2R, 3R) \) such that
\[
E_\epsilon(b_i^r, \{v_i - 1 > b_i^r \geq u_1\}) \leq \frac{\delta}{I}. \quad (3.17)
\]
We define
\[
v_i = \min\{v_i - 1, \max\{u_1, b_i^r\}\} = \begin{cases} v_i - 1 & \text{on } \{b_i^r \geq v_i - 1\}; \\
b_i^r & \text{on } \{v_i - 1 > b_i^r \geq u_1\}; \\
u_1 & \text{on } \{u_1 \geq b_i^r\}. \end{cases} \quad (3.18)
\]
Then \( v_{i-1} \geq v_i \geq u_1 \).

Using the definition of \( v_i \) in (3.18), one can prove by induction that for each \( 1 \leq i \leq I \), there exist pairwise disjoint, \( H^{n+1} \)-measurable sets \( G_i^0, G_i^1, \{G_{k,i}\}_{k=1}^i \) with
\[
N = G_i^0 \cup G_i^1 \cup \left( \bigcup_{k=1}^{i} G_{k,i} \right) \quad (3.19)
\]
such that the following conditions are satisfied.
(i) 
\[ v_i = \begin{cases} 
  u_0 & \text{on } G_i^0; \\
  u_1 & \text{on } G_i^1; \\
  b_k^{r_k} & \text{on } G_{k,i}, \quad 1 \leq k \leq i.
\end{cases} \] (3.20)

(ii) 
\[ G_{i,i} = \{v_{i-1} > b_i^{r_i} > u_1\}. \] (3.21)

(iii) For \(1 < i \leq I\) and \(1 \leq k \leq i - 1\),
\[ G_{i-1}^0 \supset G_i^0; \quad G_{i-1}^1 \subset G_i^1; \quad G_{k,i-1} \supset G_{k,i}. \] (3.22)

(iv) 
\[ \bigcup_{k=1}^{i} B(p_k, R) \subset G_i^1 \] (3.23)

(For this item one needs to use the fact that \(b_k^{r_k} \equiv -1\) on \(B(p_k, R)\).)

(3.23) implies that (by (3.15)) \(v_I = u_1\). Moreover,

\[ E_\e(v_i) = E_\e(u_0, G_i^0) + E_\e(u_1, G_i^1) + \sum_{k=1}^{i} E_\e(b_k^{r_k}, G_{k,i}) \] (by (3.19), (3.20))

\[ \leq \min\{E_\e(u_0, N), E_\e(u_1, N)\} + \int_N |e_\e(u_0) - e_\e(u_1)| + \sum_{k=1}^{i} E_\e(b_k^{r_k}, G_{k,k}) \] (by (3.22))

\[ \leq \min\{E_\e(u_0), E_\e(u_1)\} + 2\delta \] (by (3.12), (3.21), (3.17)). (3.24)

Similarly, using (3.19) and (3.20), for any \(p \in N\),

\[ E_\e(v_i, B(p, 4R)) \]

\[ = E_\e(u_0, G_i^0 \cap B(p, 4R)) + E_\e(u_1, G_i^1 \cap B(p, 4R)) + \sum_{k=1}^{i} E_\e(b_k^{r_k}, G_{k,i} \cap B(p, 4R)) \]

\[ \leq 3\delta \] (by (3.13), (3.22), (3.21), (3.17)). (3.25)

For each \(1 \leq i \leq I\), we define \(\beta_i : [0, 1] \to H^1(N)\) by

\[ \beta_i(t) = \min\{v_{i-1}, \max\{v_i, w(t)\}\} = \begin{cases} 
  v_{i-1} & \text{on } \{w(t) \geq v_{i-1}\}; \\
  w(t) & \text{on } \{v_{i-1} > w(t) > v_i\}; \\
  v_i & \text{on } \{v_i \geq w(t)\}.
\end{cases} \]

Here \(w : [0, 1] \to H^1(N)\) is as stated in the Lemma 3.8. Since \(w\) is nested, \(\beta_i\) is also nested.

It follows from (3.18) that \(|v_k| \leq 1\) for all \(0 \leq k \leq I\). Hence \(|\beta_i(t)| \leq 1\) for all \(t \in [0, 1]\).
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Moreover, \( w(0) \equiv 1 \) (resp. \( w(1) \equiv -1 \)) implies that \( \beta_i(0) = v_{i-1} \) (resp. \( \beta_i(1) = v_i \)). Using the fact that \( b_i^n \equiv 1 \) on \( N \setminus B(p_i, 4R) \), it also follows from (3.18) that \( v_{i-1} = v_i \) on \( N \setminus B(p_i, 4R) \); hence for all \( t \in [0, 1] \),

\[
\beta_i(t) = v_{i-1} = v_i \quad \text{on} \quad N \setminus B(p_i, 4R).
\]

Therefore, for all \( t \in [0, 1] \),

\[
E_\varepsilon(\beta_i(t)) \leq E_\varepsilon(v_i, N) + E_\varepsilon(v_{i-1}, B(p_i, 4R)) + E_\varepsilon(v_i, B(p_i, 4R)) + E_\varepsilon(w(t), B(p_i, 4R)) \\
\leq \min\{E_\varepsilon(u_0), E_\varepsilon(u_1)\} + 9\delta \quad \text{(by (3.24), (3.25), (3.13))}.
\]

Finally we obtain the required map \( u : [0, 1] \to H^1(N) \) by concatenating all the maps \( \beta_i \), \( i = 1, 2, \ldots, I \). \hfill \Box

**Proof of Proposition 3.5.** We fix a nested map \( w_0 : [0, 1] \to H^1(N) \) such that \( w_0(0) \equiv 1 \), \( w_0(1) \equiv -1 \) and \( |w_0(t)| \leq 1 \) for all \( t \in [0, 1] \). (For instance, one can define \( w_0(t) \) to be equal to the constant function \( 1 - 2t \).) Let \( \delta_0 = \kappa/9 \) and

\[
A_0 = \sup_{t \in [0, 1]} E_\varepsilon(\phi(t)).
\]

By Lemma 3.6, there exists \( R_0 > 0 \) such that for all \( p \in N \) and \( t \in [0, 1] \),

\[
E_\varepsilon(\phi(t), B(p, 4R_0)) \leq \frac{\delta_0}{2} \quad \text{and} \quad E_\varepsilon(w_0(t), B(p, 4R_0)) \leq \delta_0. \tag{3.26}
\]

Suppose \( N \) can be covered by \( I_0 \) balls of radius \( R_0 \). One can choose \( m \in \mathbb{N} \) such that \( |t_1 - t_2| \leq 1/m \) implies

\[
\int_N |e_\varepsilon(\phi(t_1)) - e_\varepsilon(\phi(t_2))| \leq \frac{\delta_0}{2} \tag{3.27}
\]

and

\[
\|\phi(t_1) - \phi(t_2)\|_{L^1(N)} \leq \frac{\delta_0 R_0}{C(\varepsilon, R_0) I_0}. \tag{3.28}
\]

We define the sequence \( \{\hat{\phi}_i\}_{i=0}^{2m} \) by setting \( \hat{\phi}_{2k} = \phi(k/m) \) and

\[
\hat{\phi}_{2k+1} = \min\{\hat{\phi}_{2k}, \hat{\phi}_{2k+2}\} = \begin{cases} \hat{\phi}_{2k} & \text{on} \ \{\hat{\phi}_{2k} \leq \hat{\phi}_{2k+2}\} ; \\ \hat{\phi}_{2k+2} & \text{on} \ \{\hat{\phi}_{2k} > \hat{\phi}_{2k+2}\}. \end{cases}
\]

Hence, (3.27) implies that for \( 0 \leq k \leq m - 1 \),

\[
\int_N |e_\varepsilon(\hat{\phi}_{2k}) - e_\varepsilon(\hat{\phi}_{2k+1})| = \int_{\{\hat{\phi}_{2k} > \hat{\phi}_{2k+2}\}} |e_\varepsilon(\hat{\phi}_{2k}) - e_\varepsilon(\hat{\phi}_{2k+2})| \leq \frac{\delta_0}{2}. \tag{3.29}
\]

Similarly, (3.28) implies that for \( 0 \leq k \leq m - 1 \),

\[
\|\hat{\phi}_{2k} - \hat{\phi}_{2k+1}\|_{L^1(N)} \leq \frac{\delta_0 R_0}{C(\varepsilon, R_0) I_0}.
\]
By (3.26) and (3.29), for all $0 \leq i \leq 2m$ and $p \in N$, 
\[
E_\varepsilon(\hat{\phi}_i, B(p, 4R_0)) \leq \delta_0.
\]

By Lemma 3.8, for $0 \leq i \leq m - 1$, there exists a nested map \( \gamma_i : [0, 1] \to H^1(N) \) such that \( \gamma_i(0) = \hat{\phi}_2i, \gamma_i(1) = \hat{\phi}_{2i+1} \), \( |\gamma_i(t)| \leq 1 \) for all \( t \in [0, 1] \) and 
\[
\sup_{t \in [0, 1]} E_\varepsilon(\gamma_i(t)) \leq E_\varepsilon(\hat{\phi}_{2i}) + \kappa \leq A_0 + \kappa.
\]

Therefore, 
\[
\gamma_i(1) = \hat{\phi}_{2i+1} \leq \hat{\phi}_{2i+2} = \gamma_{i+1}(0).
\]

One obtains the map \( \psi \) in Proposition 3.5 from the maps \( \{\gamma_i\}_{i=0}^{m-1} \) by repeatedly applying Lemma 3.4. More precisely, setting \( u_1 = \gamma_0 \) and \( u_2 = \gamma_1 \) in Lemma 3.4, we get a nested map \( \bar{\gamma}_1 : [0, 1] \to H^1(N) \) such that 
\[
\bar{\gamma}_1(0) \geq \hat{\phi}_0, \quad \bar{\gamma}_1(1) \leq \hat{\phi}_3, \quad \sup_{t \in [0, 1]} \|\bar{\gamma}_1(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\bar{\gamma}_1(t)) \leq A_0 + \kappa.
\]

Let us assume that there exists a nested map \( \bar{\gamma}_i : [0, 1] \to H^1(N), 1 \leq i < m - 1 \), such that 
\[
\bar{\gamma}_i(0) \geq \hat{\phi}_0, \quad \bar{\gamma}_i(1) \leq \hat{\phi}_{2i+1} \leq \hat{\phi}_{2i+2} = \gamma_{i+1}(0), \quad \sup_{t \in [0, 1]} \|\bar{\gamma}_i(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\bar{\gamma}_i(t)) \leq A_0 + \kappa.
\]

Then choosing \( u_1 = \bar{\gamma}_i \) and \( u_2 = \gamma_{i+1} \) in Lemma 3.4, one gets a nested map \( \bar{\gamma}_{i+1} : [0, 1] \to H^1(N) \) such that 
\[
\bar{\gamma}_{i+1}(0) \geq \hat{\phi}_0, \quad \bar{\gamma}_{i+1}(1) \leq \hat{\phi}_{2i+3}, \quad \sup_{t \in [0, 1]} \|\bar{\gamma}_{i+1}(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\bar{\gamma}_{i+1}(t)) \leq A_0 + \kappa.
\]

The map \( \psi \) in Proposition 3.5 is obtained by setting \( \psi = \bar{\gamma}_{m-1} \). \qed

4 A deformation lemma

The following Lemma 4.1 is motivated by [CL20, Lemma 7.1 (3)]. To prove this lemma, we adapt the argument of Chambers and Liokumovich [CL20, Proof of Lemma 7.1] in the Allen-Cahn setting.

Lemma 4.1. Let \( N \) be a closed Riemannian manifold and \( \Omega \subset N \) be an open set with smooth boundary \( \partial \Omega \). Suppose \( f : N \to [1/3, \infty) \) is a Morse function so that in the interval \([1/3, 2/3]\), \( f \) has no critical value which is a non-global local maxima or minima;
\[
\min_N f = 1/3; \quad \max_N f > 1; \quad \Omega \subset f^{-1}([1/3, 2/3]).
\]
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We set
\[ \tilde{\Omega} = f^{-1}([1/3, 1]). \]

Then, for all \( \eta > 0, \) there exist \( \varepsilon_1, \tilde{\eta} > 0, \) depending on \( \eta, \Omega, \tilde{\Omega}, f|_{\tilde{\Omega}}, \) such that the following two conditions are satisfied.

(i) If \( 0 < \varepsilon \leq \varepsilon_1 \) and \( u_0 \in H^1(N) \) satisfies \( |u_0| \leq 1, \|1 - u_0\|_{L^1(\tilde{\Omega})} \leq \tilde{\eta}, \) then there exists \( u : [0, 1] \to H^1(N) \) such that \( u(0) = u_0, u(1)|_{\tilde{\Omega}} \equiv 1 \) and
\[
\sup_{t \in [0,1]} E_\varepsilon(u(t)) \leq E_\varepsilon(u_0) + 2\sigma H^n(\partial \Omega) + \eta.
\]

(ii) If \( 0 < \varepsilon \leq \varepsilon_1 \) and \( u_0 \in H^1(N) \) satisfies \( |u_0| \leq 1, \|1 + u_0\|_{L^1(\tilde{\Omega})} \leq \tilde{\eta}, \) then there exists \( u : [0, 1] \to H^1(N) \) such that \( u(0) = u_0, u(1)|_{\tilde{\Omega}} \equiv -1 \) and
\[
\sup_{t \in [0,1]} E_\varepsilon(u(t)) \leq E_\varepsilon(u_0) + 2\sigma H^n(\partial \Omega) + \eta.
\]

Proof. Let \( q : \mathbb{R} \to \mathbb{R} \) be the unique solution of the following ODE.
\[
\varphi'(t) = \sqrt{2W(\varphi(t))}; \quad \varphi(0) = 0. \tag{4.1}
\]

For all \( t \in \mathbb{R}, -1 < q(t) < 1 \) and
\[
as t \to \pm \infty, (q(t) \mp 1) \text{ converges to zero exponentially fast.} \tag{4.2}
\]

\( q_\varepsilon(t) = q(t/\varepsilon) \) is a solution of the one dimensional Allen-Cahn equation
\[
\varepsilon^2 \varphi''(t) = W'(\varphi(t))
\]

with finite total energy:
\[
\int_{-\infty}^{\infty} \left[ \frac{\varepsilon}{2} \left( q_\varepsilon'(t) \right)^2 + \frac{1}{\varepsilon} W(q_\varepsilon(t)) \right] dt = 2\sigma. \tag{4.3}
\]

For \( \varepsilon > 0, \) we define Lipschitz continuous function
\[
\hat{q}_\varepsilon(t) = \begin{cases} 
q_\varepsilon(t) & \text{if } |t| \leq \sqrt{\varepsilon}; \\
q_\varepsilon(\sqrt{\varepsilon}) + \left( \frac{1}{\sqrt{\varepsilon}} - 1 \right) (1 - q_\varepsilon(\sqrt{\varepsilon})) & \text{if } \sqrt{\varepsilon} \leq t \leq 2\sqrt{\varepsilon}; \\
1 & \text{if } t \geq 2\sqrt{\varepsilon}; \\
q_\varepsilon(-\sqrt{\varepsilon}) + \left( \frac{1}{\sqrt{\varepsilon}} + 1 \right) (1 + q_\varepsilon(-\sqrt{\varepsilon})) & \text{if } -2\sqrt{\varepsilon} \leq t \leq -\sqrt{\varepsilon}; \\
-1 & \text{if } t \leq -2\sqrt{\varepsilon}.
\end{cases} \tag{4.4}
\]

For \( t \in \mathbb{R} \) and \( x \in N, \) we set
\[
d_1^d(x) = d_{\partial \Omega}(x) - t, \quad \text{where} \quad d_{\partial \Omega}(x) = \begin{cases} 
-d(x, \partial \Omega) & \text{if } x \in \Omega; \\
d(x, \partial \Omega) & \text{if } x \notin \Omega.
\end{cases} \tag{4.5}
\]
For \( t \in [1/3, 1] \) and \( x \in N \), we set
\[
d_t^2(x) = \begin{cases} 
-d(x, f^{-1}(t)) & \text{if } f(x) \leq t; \\
d(x, f^{-1}(t)) & \text{if } f(x) \geq t.
\end{cases}
\]

Following [Gua18, Section 7 and Section 9], we define the continuous maps \( w_{1,\varepsilon} : \mathbb{R} \to H^1(N) \) and \( w_{2,\varepsilon} : [0, 1] \to H^1(N) \) by
\[
w_{1,\varepsilon}(t) = \hat{q}_\varepsilon \circ d_1^t; \quad (4.6)
\]
\[
w_{2,\varepsilon}(t) = \begin{cases} 
\hat{q}_\varepsilon \circ d_2^t & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}; \\
1 - 3t(1 - w_{2,\varepsilon}(1/3)) & \text{if } 0 \leq t \leq \frac{1}{3}; \\
-1 + 3(1 - t)(1 + w_{2,\varepsilon}(2/3)) & \text{if } \frac{2}{3} \leq t \leq 1.
\end{cases} \quad (4.7)
\]

Since in the interval \([1/3, 2/3]\), \( f \) has no critical value which is a non-global local maxima or minima, \( t \mapsto f^{-1}(t) \) is continuous on \([1/3, 2/3]\) in the Hausdorff topology. This implies that \( w_{2,\varepsilon} \) is continuous (see [Gua18, Proposition 9.2]).

Let us fix \( \eta > 0 \). From the argument in [Gua18, Section 9], it follows that there exist \( \varepsilon', t_0 > 0 \), depending on \( \eta, \partial \Omega \) and \( \tilde{\Omega} \), such that if \( 0 < \varepsilon \leq \varepsilon' \) and \( |t| \leq 2t_0 \) then
\[
E_\varepsilon(w_{1,\varepsilon}(t)) \leq 2\sigma H^n(\partial \Omega) + \frac{\eta}{2}. \quad (4.8)
\]

By the “no concentration of mass” property ([MN17, Lemma 5.2]), there exists \( 0 < R < t_0/5 \), depending on \( \eta, \Omega, \tilde{\Omega} \) and \( f|_{\tilde{\Omega}} \), such that
\[
H^n(f^{-1}(t) \cap B(p, 5R)) \leq \frac{1}{2\sigma} \frac{\eta}{3},
\]
for all \( t \in [1/3, 2/3] \) and \( B(p, 5R) \subset \Omega \). Moreover, \( w_{2,\varepsilon}(1/3) > 0 \) on \( N \) and \( w_{2,\varepsilon}(2/3) < 0 \) on \( \Omega \). As a consequence, by the results in [Gua18, Section 9], there exists \( 0 < \varepsilon'' \leq R^2/4 \), depending on \( \Omega, \tilde{\Omega} \) and \( f|_{\tilde{\Omega}} \), such that
\[
E_\varepsilon(w_{2,\varepsilon}(t), B(p, 4R)) \leq \frac{\eta}{2}, \quad (4.9)
\]
for all \( t \in [0, 1] \) and \( B(p, 5R) \subset \Omega \). We define \( \varepsilon_1 = \min\{\varepsilon', \varepsilon''\} \). By our definitions of \( R \) and \( \varepsilon'' \),
\[
2\sqrt{\varepsilon_1} \leq R < \frac{t_0}{5}. \quad (4.10)
\]

Let us fix \( \varepsilon \in (0, \varepsilon_1] \). Using the notation of (4.5), let
\[
\Omega_r = \{ x \in N : d_{\partial \Omega}(x) \leq r \}. \quad (4.11)
\]

For \( r > 0 \) and for a fixed \( p \in \Omega_{-t_0} \), we define \( \omega_\varepsilon^r : N \to \mathbb{R} \) by
\[
\omega_\varepsilon^r(x) = \hat{q}_\varepsilon(r - d_p(x)),
\]

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where \( d_p(x) = d(x, p) \).

By (4.10), \( B(p, 3R + 2\sqrt{\varepsilon_1}) \subset \Omega \). Therefore, it follows from Lemma 3.7 that for \( u \in L^\infty(N) \) with \( |u| \leq 1 \),

\[
\int_{2R}^{3R} E_\varepsilon(\omega_\varepsilon^r, \{1 > \omega_\varepsilon^r > u\}) \, dr \\
\leq \int_{\Omega} \int_{u(x)}^{1} e_\varepsilon(\hat{q}_\varepsilon)(t) \, dt \, dH^{n+1}(x) \quad \text{(since \( \hat{q}_\varepsilon \) is an odd function)}
\]

\[
= \int_{\Omega} \int_{u(x)}^{1} \left[ \frac{\varepsilon}{2} \hat{q}_\varepsilon'(q_\varepsilon^{-1}(s)) + \frac{1}{\varepsilon} \frac{W(s)}{\hat{q}_\varepsilon'(q_\varepsilon^{-1}(s))} \right] \, ds \, dH^{n+1}(x). \tag{4.12}
\]

In (4.12), \( \hat{q}_\varepsilon \) is thought of as a bijective map from \([-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}]\) to \([-1, 1]\). We claim that there exists \( C_0 = C_0(W, \varepsilon_1) > 0 \) such that the \( L^\infty([-1, 1]) \) norm of the integrand in (4.12) is bounded by \( C_0 \). (In particular, \( C_0 \) does not depend on \( \varepsilon \).) Indeed, the integrand is a non-negative, even function. If \( 0 \leq t < \sqrt{\varepsilon} \), then

\[
\frac{\varepsilon}{2} \hat{q}_\varepsilon'(t) + \frac{1}{\varepsilon} \frac{W(\hat{q}_\varepsilon(t))}{\hat{q}_\varepsilon'(t)}
\]

\[
= \frac{1}{2} q'(t/\varepsilon) + \frac{W(q(t/\varepsilon))}{q'(t/\varepsilon)}
\]

\[
= \sqrt{2W(q(t/\varepsilon))} \quad \text{(by (4.1))}
\]

\[
\leq \sqrt{2}\|W\|_{L^\infty([-1, 1])}. \tag{4.13}
\]

If \( \sqrt{\varepsilon} < t < 2\sqrt{\varepsilon} \), using the fact that

\[
\sup_{t \in [-1, 1]} \frac{W(t)}{(1 - t)^2} = C_1 < \infty,
\]

we obtain

\[
\frac{\varepsilon}{2} \hat{q}_\varepsilon'(t) + \frac{1}{\varepsilon} \frac{W(\hat{q}_\varepsilon(t))}{\hat{q}_\varepsilon'(t)}
\]

\[
\leq \frac{\sqrt{\varepsilon}}{2} \left(1 - q \left(\varepsilon^{-1/2}\right)\right) + \frac{1}{\sqrt{\varepsilon}} \frac{W(q(\varepsilon^{-1/2}))}{(1 - q(\varepsilon^{-1/2}))}
\]

\[
\leq \left( \frac{\sqrt{\varepsilon}}{2} + \frac{C_1}{\sqrt{\varepsilon}} \right) \left(1 - q \left(\varepsilon^{-1/2}\right)\right). \tag{4.14}
\]

By (4.2), the expression in (4.14) is bounded by some constant \( C_2 = C_2(\varepsilon_1, C_1) \). Thus our claim follows from (4.13) and (4.14). (4.12), together with the claim, implies that

\[
\int_{2R}^{3R} E_\varepsilon(\omega_\varepsilon^r, \{1 > \omega_\varepsilon^r > u\}) \, dr \leq C_0\|1 - u\|_{L^1(\Omega)}. \tag{4.15}
\]
We choose a covering
\[
\Omega_{-t_0} = \bigcup_{i=1}^{I} B(p_i, R); \tag{4.16}
\]
each \( p_i \in \Omega_{-t_0} \) so that (by (4.10)) \( B(p_i, 5R) \subset \Omega \). To prove part (i) of Lemma 4.1, we set
\[
\tilde{\eta} = \frac{\eta R}{2C_0 I}, \tag{4.17}
\]
where \( C_0 \) is as in the above claim. Let \( u_0 \) be as in the statement of Lemma 4.1, part (i). We inductively define a sequence \( \{v_i\}_{i=0}^{I} \),
\[
-1 \leq v_0 \leq v_1 \leq \cdots \leq v_I \leq 1,
\]
as follows. Set \( v_0 = u_0 \). Suppose \( v_k \) has been defined for \( 0 \leq k \leq i-1 \) so that
\[
u_0 = v_0 \leq v_1 \leq \cdots \leq v_{i-1} \leq 1;
\]
hence
\[
\|1 - v_{i-1}\|_{L^1(\Omega)} \leq \|1 - u_0\|_{L^1(\Omega)} \leq \tilde{\eta}. \tag{4.18}
\]
Let \( \omega_{i, \varepsilon}^r(x) = \hat{q}_\varepsilon(r - d_{p_i}(x)) \). By (4.15), (4.18) and (4.17), there exists \( r_i \in (2R, 3R) \) such that
\[
E_\varepsilon \left( \omega_{i, \varepsilon}^r, \{v_{i-1} < \omega_{i, \varepsilon}^r < 1\} \right) \leq \frac{\eta}{2I}. \tag{4.19}
\]
We define
\[
v_i = \max \{v_{i-1}, \omega_{i, \varepsilon}^r\} = \begin{cases}
v_{i-1} & \text{on } \{v_{i-1} \geq \omega_{i, \varepsilon}^r\}; \\
\omega_{i, \varepsilon}^r & \text{on } \{v_{i-1} < \omega_{i, \varepsilon}^r\}.
\end{cases} \tag{4.20}
\]
It follows from (4.20) that \( v_{i-1} \leq v_i \leq 1 \). Moreover, since \( \omega_{i, \varepsilon}^r \equiv -1 \) on \( N \setminus B(p_i, 4R) \),
\[
v_i = v_{i-1} \text{ on } N \setminus B(p_i, 4R). \tag{4.21}
\]
Thus we have obtained the sequence \( \{v_i\}_{i=0}^{I} \) with \( v_0 = u_0 \) and we set \( v_I = \bar{u} \). As \( \omega_{i, \varepsilon}^r \equiv 1 \) on \( B(p_i, R) \), using (4.20) and (4.16), one can prove by induction that
\[
\bar{u} \big|_{\Omega_{-t_0}} \equiv 1. \tag{4.22}
\]
By (4.20) and (4.19),
\[
E_\varepsilon(v_i, \{v_i \neq v_{i-1}\}) = E_\varepsilon \left( \omega_{i, \varepsilon}^r, \{v_{i-1} < \omega_{i, \varepsilon}^r < 1\} \right) \leq \frac{\eta}{2I}. \tag{4.23}
\]
Thus
\[
E_\varepsilon(v_i) = E_\varepsilon(v_{i-1}) + E_\varepsilon(v_i, \{v_i \neq v_{i-1}\}) \leq E_\varepsilon(v_{i-1}) + \frac{\eta}{2I}
\]
\[
\implies E_\varepsilon(v_i) \leq E_\varepsilon(u_0) + \frac{\eta i}{2I} \quad \forall \ 0 \leq i \leq I. \tag{4.24}
\]
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For $1 \leq i \leq I$, let $\beta_i : [0, 1] \to H^1(N)$ be defined by

$$\beta_i(t) = \max\{v_{i-1}, \min\{v_i, -w_{2,\varepsilon}(t)\}\} = \begin{cases} v_{i-1} & \text{on } \{-w_{2,\varepsilon}(t) \leq v_{i-1}\}; \\ -w_{2,\varepsilon}(t) & \text{on } \{v_{i-1} < -w_{2,\varepsilon}(t) < v_i\}; \\ v_i & \text{on } \{v_i \leq -w_{2,\varepsilon}(t)\}. \end{cases} \tag{4.25}$$

Since $w_{2,\varepsilon}(0) \equiv 1$ and $w_{2,\varepsilon}(1) \equiv -1$, $\beta_i(0) = v_{i-1}$ and $\beta_i(1) = v_i$. Moreover, by (4.21),

$$\beta_i(t) = v_{i-1} = v_i \text{ on } N \setminus B(p_i, 4R).$$

Therefore, using (4.25), (4.24), (4.23) and (4.9), one obtains

$$E_\varepsilon(\beta_i(t)) \leq E_\varepsilon(v_{i-1}) + E_\varepsilon(v_i, \{v_i \neq v_{i-1}\}) + E_\varepsilon(-w_{2,\varepsilon}(t), B(p_i, 4R)) \leq E_\varepsilon(u_0) + \frac{\eta(i-1)}{2I} + \frac{\eta}{2I} + \frac{\eta}{2} \leq E_\varepsilon(u_0) + \eta.$$ 

Concatenating all the $\beta_i$’s we get a map $\beta' : [0, 1] \to H^1(N)$ such that $\beta'(0) = u_0$, $\beta'(1) = \bar{u} = v_I$ and

$$\sup_{t \in [0,1]} E_\varepsilon(\beta'(t)) \leq E_\varepsilon(u_0) + \eta.$$

Let $\beta'' : [-2t_0, t_0] \to H^1(N)$ be defined by

$$\beta''(t) = \max\{\bar{u}, -w_{1,\varepsilon}(t)\}.$$ 

Then $\beta''(-2t_0) = \bar{u}$ as $\bar{u} \equiv 1$ on $\Omega_{-t_0}$ ((4.22)) and $w_{1,\varepsilon}(-2t_0) \equiv 1$ on $N \setminus \Omega_{-t_0}$ (by (4.10)). Moreover, $\beta''(t_0)|_\Omega \equiv 1$ as by (4.10), $w_{1,\varepsilon}(t_0)|_\Omega \equiv -1$. Using (4.8) and (4.24), we conclude that for all $t \in [-2t_0, t_0]$,

$$E_\varepsilon(\beta''(t)) \leq E_\varepsilon(\bar{u}) + E_\varepsilon(-w_{1,\varepsilon}(t)) \leq E_\varepsilon(u_0) + 2\sigma\mathcal{H}^n(\partial\Omega) + \eta.$$ 

Finally, the map $u$ in Lemma 4.1 part (i) is obtained by concatenating $\beta'$ and $\beta''$. This finishes the proof of part (i) of Lemma 4.1; part (ii) of the lemma can be deduced from part (i) by replacing $u_0$ by $-u_0$. \hfill \Box

The next lemma is motivated by the properties of the isoperimetric profile of a compact Riemannian manifold (see [CL20, Lemma 7.1 (2)]).

**Lemma 4.2.** Let $\Omega$ be a compact Riemannian manifold (not necessarily closed). For all $\eta_1 > 0$, there exist $\varepsilon_2$, $\eta_2 > 0$, depending on $\Omega$ and $\eta_1$, such that the following holds. If $0 < \varepsilon \leq \varepsilon_2$ and $u \in H^1(\Omega)$ satisfies $\|u\|_{L^\infty(\Omega)} \leq 1$ and $\min\{\|1 - u\|_{L^1(\Omega)}, \|1 + u\|_{L^1(\Omega)}\} > \eta_1$, then $E_\varepsilon(u, \Omega) > \eta_2$.
Proof. We assume by contradiction that there exist sequences \( \{u_i\}_{i=1}^{\infty} \subset H^1(\Omega) \) with \( |u_i| \leq 1 \) for all \( i \) and \( \{\alpha_i\}_{i=1}^{\infty} \subset (0, \infty) \) with \( \alpha_i \to 0 \) such that
\[
\min\{\|1 - u_i\|_{L^1(\Omega)}, \|1 + u_i\|_{L^1(\Omega)}\} > \eta_i \quad \forall \ i \in \mathbb{N},
\] (4.26)
and
\[
E_{\alpha_i}(u_i, \Omega) \to 0. \tag{4.27}
\]
Let \( F : [-1, 1] \to [-\sigma/2, \sigma/2] \) be as defined in (2.1) and \( v_i = F \circ u_i \). As argued in [HT00, Section 2.1], for all \( i \), \( |v_i| \leq \sigma/2 \) and
\[
\int_{\Omega} |\nabla v_i| \leq \frac{1}{2} E_{\alpha_i}(u_i, \Omega). \tag{4.28}
\]
Therefore, there exists a subsequence \( \{v_{i_k}\} \subset \{v_i\} \) and \( v_\infty \in BV(\Omega) \) such that
\[
v_{i_k} \to v_\infty \text{ in } L^1(\Omega) \text{ and pointwise a.e.}
\]
and (using (4.27) and (4.28))
\[
\int_{\Omega} |Dv_\infty| \leq \liminf_{k \to \infty} \int_{\Omega} |\nabla v_{i_k}| = 0.
\]
Thus \( v_\infty \) is a constant function. Denoting \( u_\infty = F^{-1}(v_\infty) \), by the dominated convergence theorem,
\[
u_{i_k} \to u_\infty \text{ pointwise a.e. and in } L^1(\Omega).
\]
Moreover, by (4.27),
\[
\int_{\Omega} W(u_\infty) = \lim_{k \to \infty} \int_{\Omega} W(u_{i_k}) = 0.
\]
Since \( u_\infty \) is a constant function, either \( u_\infty \equiv 1 \) or \( u_\infty \equiv -1 \). However, this contradicts the assumption (4.26). \( \square \)

5 Proof of Theorem 1.1

Let \( N \) be a closed Riemannian manifold and \( \Omega \subset N \) be an open set with smooth boundary \( \partial \Omega \). For \( \varepsilon > 0 \), the \( \varepsilon \)-Allen-Cahn width of \( \Omega \), which we denote by \( \lambda_\varepsilon(\Omega) \), is defined as follows [Gua18]. Let \( \mathcal{A} \) be the set of all continuous maps \( \zeta : [0, 1] \to H^1(\Omega) \) such that \( \zeta(0) \equiv 1 \) and \( \zeta(1) \equiv -1 \). Then
\[
\lambda_\varepsilon(\Omega) = \inf_{\zeta \in \mathcal{A}} \sup_{t \in [0,1]} E_\varepsilon(\zeta(t), \Omega).
\]
It follows from [Gua18, Section 8] that
\[
\mathbb{W}(\Omega) \leq \frac{1}{2\sigma} \liminf_{\varepsilon \to 0^+} \lambda_\varepsilon(\Omega), \tag{5.1}
\]
where \( \mathbb{W}(\Omega) \) is as defined in (2.3). Motivated by [CL20, Section 2.2], we also make the following definition.

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Definition 5.1. Let $B$ be the set of all continuous maps $\zeta : [0,1] \to H^1(N)$ such that $\zeta(0)_{|\Omega} \equiv 1$ and $\zeta(1)_{|\Omega} \equiv -1$. For $\varepsilon > 0$, we define

$$\tilde{\lambda}_\varepsilon(\Omega) = \inf_{\zeta \in B} \sup_{t \in [0,1]} E_\varepsilon(\zeta(t), N).$$

Given $\tilde{\zeta} \in B$ one can define $\zeta \in A$ by $\zeta(t) = \tilde{\zeta}(t)_{|\Omega}$; hence, for all $\varepsilon > 0$,

$$\lambda_\varepsilon(\Omega) \leq \tilde{\lambda}_\varepsilon(\Omega).$$

The following Proposition 5.2 is the Allen-Cahn analogue of [CL20, Proposition 2.1].

Proposition 5.2. Let $N$ be a closed Riemannian manifold and $\Omega \subset N$ be a good set (as defined in Section 2.4). Suppose $f : N \to [1/3, \infty)$ is a Morse function so that in the interval $[1/3, 2/3]$ $f$ has no critical value which is a non-global local maxima or minima;

$$\min_N f = 1/3; \quad \max_N f > 1; \quad \Omega \subset \subset f^{-1}([1/3, 2/3]); \quad 1 \text{ is a regular value of } f.$$

We set

$$\tilde{\Omega} = f^{-1}([1/3, 1]).$$

Then there exist $\varepsilon^*, \eta^* > 0$, depending on $\Omega$, $\tilde{\Omega}$ and $f|_{\tilde{\Omega}}$, such that for all $0 < \varepsilon \leq \varepsilon^*$ the following condition is satisfied. For every $\zeta \in B$, there exists $t^0 \in [0,1]$ such that

$$E_\varepsilon(\zeta(t^0)) \geq \tilde{\lambda}_\varepsilon(\Omega) \quad \text{and} \quad E_\varepsilon(\zeta(t^0), \Omega) \geq \eta^*.$$

Remark 5.3. The constants $\varepsilon^*$ and $\eta^*$ in the above Proposition 5.2 depend on the ambient Riemannian metric restricted to $\tilde{\Omega}$. (By our hypothesis, $\tilde{\Omega}$ is smooth.) Let us fix a Riemannian metric $g_0$ on $\tilde{\Omega}$. If $g'$ is an arbitrary Riemannian metric on $N$, from the proofs of Proposition 5.2, Lemmas 4.1 and 4.2, it follows that there exists $g > 0$, depending on $g_0$ and $g'|_{\tilde{\Omega}}$, such that the following holds. One can choose $\varepsilon^*$ and $\eta^*$ in Proposition 5.2 in such a way that the proposition holds for all Riemannian metrics $g''$ on $N$ satisfying

$$\|g'|_{\tilde{\Omega}} - g''|_{\tilde{\Omega}}\|_{C^2(\Omega, g_0)} < \rho.$$

Proof of Proposition 5.2. The proof will be presented in four parts.

Part 1. Let

$$\tau = \frac{\sigma}{2} H^n(\partial \Omega).$$

We set $\eta = \tau$ in Lemma 4.1 and choose $\varepsilon_1^* > 0$ and

$$0 < \tau_1 < H^{n+1}(\Omega)$$

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so that Lemma 4.1 holds for \( \eta = \tau, \varepsilon_1 = \varepsilon_1^* \) and \( \tilde{\eta} = 3\tau_1 \). Next, we set \( \eta_1 = \tau_1 \) in Lemma 4.2 and choose \( \varepsilon_2^* > 0 \) and

\[
0 < \tau_2 \leq \frac{\sigma}{12} \mathcal{H}^n(\partial \Omega)
\]  

(5.6)

so that Lemma 4.2 holds for \( \eta_1 = \tau_1, \varepsilon_2 = \varepsilon_2^* \) and \( \eta_2 = \tau_2 \). Let us define \( \eta^* = \tau_2 \). We also define \( \varepsilon^* \) to be a positive real number so that the following conditions are satisfied.

- \( \varepsilon^* \leq \min\{\varepsilon_1^*, \varepsilon_2^*\} \).
- Let

\[
w_\varepsilon = w_{1,\varepsilon}(-2\sqrt{\varepsilon}),
\]

where \( w_{1,\varepsilon} \) is as defined in (4.6). Then, for all \( 0 < \varepsilon \leq \varepsilon^* \),

\[
E_\varepsilon(w_\varepsilon) \leq 2\sigma \mathcal{H}^n(\partial \Omega) + \tau_2.
\]

(5.8)

(For this item one needs to use (4.8).)

- For all \( 0 < \varepsilon \leq \varepsilon^* \), using the notation of (4.11),

\[
\mathcal{H}^{n+1}\left(\Omega \setminus \Omega_{-4\sqrt{\varepsilon}}\right) \leq \frac{\tau_1}{2}.
\]

(5.9)

- For all \( 0 < \varepsilon \leq \varepsilon^* \),

\[
\frac{1}{2\sigma} \lambda_\varepsilon(\Omega) > \frac{7}{2} \mathcal{H}^n(\partial \Omega).
\]

(5.10)

(For this item one needs (5.1) and the hypothesis (2.4) that \( \Omega \) is a good set.)

We will show that Proposition 5.2 holds for the above choices of \( \varepsilon^* \) and \( \eta^* \). Let us assume by contradiction that there exist \( \alpha \in (0, \varepsilon^*] \) and \( h \in \mathcal{B} \) such that

for \( t \in [0, 1] \), if \( E_\alpha(h(t)) \geq \tilde{\lambda}_\alpha(\Omega) \) then \( E_\alpha(h(t), \Omega) < \eta^* = \tau_2 \).

(5.11)

Without loss of generality, we can assume that

\[
|h(t)| \leq 1 \quad \forall t \in [0, 1].
\]

(5.12)

Indeed, if

\[
\tilde{h}(t) = \min\{1, \max\{-1, h(t)\}\},
\]

then for all \( t \in [0, 1] \), \( |\tilde{h}(t)| \leq 1 \) and \( e_\varepsilon(\tilde{h}(t)) \leq e_\varepsilon(h(t)) \). Therefore,

\[
E_\alpha(\tilde{h}(t)) \geq \tilde{\lambda}_\alpha(\Omega) \implies E_\alpha(h(t)) \geq \tilde{\lambda}_\alpha(\Omega) \implies \eta^* > E_\alpha(h(t), \Omega) \geq E_\alpha(\tilde{h}(t), \Omega).
\]

To prove Proposition 5.2, we will show that the existence of such \( h \in \mathcal{B} \) and \( \alpha \in (0, \varepsilon^*] \) imply there exists \( \gamma \in \mathcal{B} \) satisfying \( \sup_{t \in [0, 1]} E_\alpha(\gamma(t)) < \tilde{\lambda}_\alpha(\Omega) \).

**Part 2.** Let \( h \) and \( \alpha \) be as defined above in (5.11) and (5.12).
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**Lemma 5.4.** There exist $0 < a < b < 1$ such that the following conditions are satisfied.

- $E_\alpha(h(a), \Omega) = \tau_2 = E_\alpha(h(b), \Omega)$.
- $\|1 - h(a)\|_{L^1(\Omega)} \leq \tau_1$ and $\|1 + h(b)\|_{L^1(\Omega)} \leq \tau_1$.
- $E_\alpha(h(t), \Omega) \geq \tau_2$ for all $t \in [a, b]$.

**Proof.** Let

$$S_1 = \left\{ t \in [0, 1] : E_\alpha(h(t), \Omega) \leq \tau_2 \text{ and } \|1 - h(t)\|_{L^1(\Omega)} \leq \tau_1 \right\};$$

$$S_2 = \left\{ t \in [0, 1] : E_\alpha(h(t), \Omega) \leq \tau_2 \text{ and } \|1 + h(t)\|_{L^1(\Omega)} \leq \tau_1 \right\}.$$ 

By (5.5),

$$\nexists t \in [0, 1] \text{ such that } \max \left\{ \|1 - h(t)\|_{L^1(\Omega)}, \|1 + h(t)\|_{L^1(\Omega)} \right\} \leq \tau_1. \quad (5.13)$$

(5.13), together with the choices of $\tau_1$, $\tau_2$ and Lemma 4.2, implies that

$$S_1 \cap S_2 = \emptyset; \quad S_1 \cup S_2 = \{ t \in [0, 1] : E_\alpha(h(t), \Omega) \leq \tau_2 \}.$$ 

(5.14) $S_1$ and $S_2$ are closed subsets of $[0, 1]$. Since $h \in \mathcal{B}$, $0 \in S_1$ and $1 \in S_2$. Let

$$a = \max S_1, \quad b = \min (S_2 \cap [a, 1]).$$

(5.14) implies that $\|1 + h(a)\|_{L^1(\Omega)} > \tau_1$. Suppose $E_\alpha(h(a), \Omega) < \tau_2$. By continuity, there exists $a' > a$ such that

$$E_\alpha(h(a'), \Omega) < \tau_2 \quad \text{and} \quad \|1 + h(a')\|_{L^1(\Omega)} > \tau_1,$$

which implies (by (5.14)) $a' \in S_1$. This contradicts the definition of $a$; hence $E_\alpha(h(a), \Omega) = \tau_2$. A similar argument shows that $E_\alpha(h(b), \Omega) = \tau_2$ as well. Suppose there exists $t' \in (a, b)$ such that $E_\alpha(h(t'), \Omega) < \tau_2$. Then by (5.14), $t' \in S_1 \cup S_2$, which contradicts the definitions of $a$ and $b$. This finishes the proof of the lemma. □

**Part 3.** By (5.11) and Lemma 5.4, there exists $\delta > 0$ such that

$$\sup_{t \in [a, b]} E_\alpha(h(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \delta.$$ 

By Proposition 3.5, there exists a nested map $\tilde{h} : [0, 1] \to H^1(N)$ such that $\tilde{h}(0) \geq h(a)$, $\tilde{h}(1) \leq h(b)$, $|\tilde{h}(t)| \leq 1$ for all $t \in [0, 1]$ and

$$\sup_{t \in [0, 1]} E_\alpha(\tilde{h}(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \frac{\delta}{2}, \quad (5.15)$$

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We recall from (5.7) that \( w_\varepsilon = w_{1,\varepsilon}(-2\sqrt{\varepsilon}) \); hence using the notation of (4.11),

\[
w_\varepsilon = \begin{cases} 
-1 & \text{on } \Omega_{-4\sqrt{\varepsilon}}; \\
1 & \text{on } (N \setminus \Omega).
\end{cases}
\] (5.16)

**Lemma 5.5.** Let \( T : H^1(N) \to H^1(N) \) be defined by

\[
T(u) = \min \{-w_\alpha, \max \{w_\alpha, u\}\}.
\]

If \( |u| \leq 1 \), then denoting \( \hat{u} = T(u) \), we have \( |\hat{u}| \leq 1 \);

\[
E_\alpha(\hat{u}) \leq E_\alpha(w_\alpha) + E_\alpha(u_{-w_\alpha \geq u \geq w_\alpha}) \leq E_\alpha(w_\alpha) + E_\alpha(u, \Omega); 
\] (5.17)

\[
\|1 - \hat{u}\|_{L^1(\Omega)} \leq \|1 - u\|_{L^1(\Omega)} + 2H^{n+1} \left( \Omega \setminus \Omega_{-4\sqrt{\varepsilon}} \right); 
\] (5.18)

\[
\|1 + \hat{u}\|_{L^1(\Omega)} \leq \|1 + u\|_{L^1(\Omega)} + 2H^{n+1} \left( \Omega \setminus \Omega_{-4\sqrt{\varepsilon}} \right). 
\] (5.19)

*Proof.*

\[
\hat{u} = \begin{cases} 
u & \text{on } \{-w_\alpha \geq u \geq w_\alpha\}; \\
\pm w_\alpha & \text{otherwise}. 
\end{cases}
\] (5.20)

Therefore, \( |\hat{u}| \leq 1 \). Moreover,

\[
\{-w_\alpha \geq u \geq w_\alpha\} \subset \{w_\alpha \leq 0\} \subset \Omega \text{ (by (5.16)).} 
\] (5.21)

Combining (5.20) and (5.21), one gets (5.17). It follows from (5.16) and (5.20) that

\[
\hat{u} = u \text{ on } \Omega_{-4\sqrt{\varepsilon}}. 
\] (5.22)

Moreover,

\[
0 \leq 1 \pm w_\alpha \leq 2. 
\] (5.23)

Equations (5.18) and (5.19) both follow from (5.20), (5.22) and (5.23).

Let us define

\[
\ell = \min \{h(a), -h(b)\}; 
\] (5.24)

so

\[-\ell = \max \{-h(a), h(b)\}. 
\]

Moreover,

\[
0 \leq 1 - \ell \leq (1 - h(a)) + (1 + h(b)). 
\] (5.25)
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**Lemma 5.6.** Let $\tilde{h}$ be as in (5.15). There exists $t^* \in [0,1]$ such that

$$E_\alpha \left( \tilde{h}(t^*), \{w_\alpha \leq \tilde{h}(t^*) \leq -w_\alpha \} \cup \{-\ell \leq \tilde{h}(t^*) \leq \ell\} \right) \leq E_\alpha(w_\alpha) + 2\tau_2. \quad (5.26)$$

Here, for $S \subset N$, $S^c = (N \setminus S)$.

**Proof.** Let $h' : [0,1] \to H^1(N)$ be defined by

$$h'(t) = \min\{\ell, \max\{-\ell, \tilde{h}(t)\}\} = \begin{cases} \tilde{h}(t) & \text{on } \{\ell \geq \tilde{h}(t) \geq -\ell\}; \\ \pm \ell & \text{otherwise}. \end{cases} \quad (5.27)$$

Since

$$\ell \leq h(a) \leq \tilde{h}(0) \leq \max\{-\ell, \tilde{h}(0)\} \quad \text{and} \quad -\ell \geq h(b) \geq \tilde{h}(1),$$

we have

$$h'(0) = \ell \quad \text{and} \quad h'(1) = \min\{\ell, -\ell\}. \quad (5.28)$$

Let $h'' : [0,1] \to H^1(N)$ be defined by $h''(t) = T(h'(t))$, where $T$ is as in Lemma 5.5. By Lemma 5.5, (5.28), (5.8) and Lemma 5.4,

$$E_\alpha(h''(0)) \leq E_\alpha(w_\alpha) + E_\alpha(\ell, \Omega) \leq 2\sigma \mathcal{H}^n(\partial \Omega) + 3\tau_2;$$
$$E_\alpha(h''(1)) \leq E_\alpha(w_\alpha) + E_\alpha(\ell, \Omega) \leq 2\sigma \mathcal{H}^n(\partial \Omega) + 3\tau_2.$$

Moreover, by Lemma 5.5, (5.25), Lemma 5.4 and (5.9),

$$\|1 - h''(0)\|_{L^1(\Omega)} \leq \|1 - \ell\|_{L^1(\Omega)} + 2\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\alpha}}) \leq 3\tau_1;$$
$$\|1 + h''(1)\|_{L^1(\Omega)} \leq \|1 + \ell\|_{L^1(\Omega)} + 2\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\alpha}}) \leq 3\tau_1.$$

Therefore, by our choices of $\tau_1$, $\tau_2$, $\tau$ and Lemma 4.1, there exists a continuous map $\beta_0 : [0,1] \to H^1(N)$ such that $\beta_0(0) = h''(0)$, $\beta_0(1)|_\Omega \equiv 1$ and

$$\sup_{t \in [0,1]} E_\alpha(\beta_0(t)) \leq 4\sigma \mathcal{H}^n(\partial \Omega) + 3\tau_2 + \tau \leq 5\sigma \mathcal{H}^n(\partial \Omega) \quad \text{(by (5.4) and (5.6))}. \quad (5.29)$$

Similarly, there exists a continuous map $\beta_1 : [0,1] \to H^1(N)$ such that $\beta_1(0) = h''(1)$, $\beta_1(1)|_\Omega \equiv -1$ and

$$\sup_{t \in [0,1]} E_\alpha(\beta_1(t)) \leq 4\sigma \mathcal{H}^n(\partial \Omega) + 3\tau_2 + \tau \leq 5\sigma \mathcal{H}^n(\partial \Omega). \quad (5.30)$$

Let us define $\beta : [0,1] \to H^1(N)$ by

$$\beta(t) = \begin{cases} \beta_0(1 - 3t) & \text{if } 0 \leq t \leq 1/3; \\ h''(3t - 1) & \text{if } 1/3 \leq t \leq 2/3; \\ \beta_1(3t - 2) & \text{if } 2/3 \leq t \leq 1. \end{cases}$$
Since $\beta \in \mathcal{B}$, there exists $t^* \in [0, 1]$ such that

$$E_\alpha(\beta(t^*)) \geq \tilde{\lambda}_\alpha(\Omega) \geq \lambda_\alpha(\Omega) > 7\sigma \mathcal{H}^d(\partial\Omega) \text{ (using (5.2), (5.3) and (5.10)).}$$

Therefore, by (5.29) and (5.30), there exists $t^* \in [0, 1]$ such that

$$E_\alpha(h''(t^*)) \geq \tilde{\lambda}_\alpha(\Omega). \quad (5.31)$$

However, by Lemma 5.5, (5.27) and Lemma 5.4, for all $t \in [0, 1]$,

$$E_\alpha(h''(t)) \leq E_\alpha(w_\alpha) + E_\alpha(h'(t), \{w_\alpha \leq h'(t) \leq -w_\alpha\})$$

$$\leq E_\alpha(w_\alpha) + E_\alpha(\tilde{h}(t), \{w_\alpha \leq \tilde{h}(t) \leq -w_\alpha\} \cap \{-\ell \leq \tilde{h}(t) \leq \ell\}) + E_\alpha(\ell, \Omega)$$

$$\leq E_\alpha(w_\alpha) + E_\alpha(\tilde{h}(t), \{w_\alpha \leq \tilde{h}(t) \leq -w_\alpha\} \cap \{-\ell \leq \tilde{h}(t) \leq \ell\}) + 2\tau_2. \quad (5.32)$$

Further, by (5.15),

$$E_\alpha(h(t^*)) < \tilde{\lambda}_\alpha(\Omega). \quad (5.33)$$

Combining (5.31), (5.32) and (5.33), one obtains (5.26).

**Part 4.** For $r \in \mathbb{R}$, let $r^+ = \max\{r, 0\}$; $r^- = \min\{r, 0\}$. The maps $\Phi, \Psi, \Theta : H^1(N) \times H^1(N) \times H^1(N) \to H^1(N)$ are defined as follows [Dey20, Equation (3.70)].

$$(u_0, u_1, w) = \min\{\max\{u_0, w\}, \max\{u_1, w\}\};$$

$$\Psi(u_0, u_1, w) = \max\{\min\{u_0, w\}, \min\{u_1, -w\}\};$$

$$\Theta(u_0, u_1, w) = \Phi(u_0, u_1, w)^+ + \Psi(u_0, u_1, w)^-. $$

**Lemma 5.7.** Let $u_0, u_1, w \in H^1(N)$ such that $|u_0|, |u_1|, |w| \leq 1$; $\phi = \Phi(u_0, u_1, w)$, $\psi = \Psi(u_0, u_1, w)$, $\theta = \Theta(u_0, u_1, w)$.

(i) If $w(x) = 1$, then $\theta(x) = u_0(x)$; if $w(x) = -1$, then $\theta(x) = u_1(x)$.

(ii) If $u_0(x) = u_1(x)$, then $\theta(x) = u_0(x) = u_1(x)$.

(iii) For all $x \in N$, either $\theta(x) = \phi(x)$ or $\theta(x) = \psi(x)$; hence $\theta(x) \in \{u_0(x), u_1(x), w(x), -w(x)\}$.

(iv) For $\varepsilon > 0$ and $S \subset N$,

$$E_\varepsilon(\theta, S) \leq E_\varepsilon(w, S) + E_\varepsilon(u_0, S \cap \{u_0 > -w\} \cup \{u_0 < w\})$$

$$+ E_\varepsilon(u_1, S \cap \{u_1 > w\} \cup \{u_1 < -w\}).$$

(v) If $v_0, v_1 \in H^1(N)$ such that $v_0 \geq u_0, u_1 \geq v_1$, then $v_0 \geq \theta \geq v_1$. 

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Proof. For the proofs of items (i) – (iii), we refer to [Dey20, Proof of Proposition 3.12]. (iv) follows from (iii) and the definitions of $\Phi$ and $\Psi$. To prove item (v), we note the following. If $r_1, r_1', r_2, r_2' \in \mathbb{R}$ such that $r_1 \geq r_2$ and $r_1' \geq r_2'$, then $\max\{r_1, r_1'\} \geq \max\{r_2, r_2'\}$ and $\min\{r_1, r_1'\} \geq \min\{r_2, r_2'\}$. In particular, if we set $r_1' = r_2' = s$, then $\max\{r_1, s\} \geq \max\{r_2, s\}$ and $\min\{r_1, s\} \geq \min\{r_2, s\}$. Hence $v_0 \geq u_0, u_1 \geq v_1$ implies

$$\begin{align*}
\Phi(v_0, u_0, w) &\geq \Phi(u_0, u_1, w) \geq \Phi(v_1, v_1, w), \\
\Psi(v_0, v_0, w) &\geq \Psi(u_0, u_1, w) \geq \Psi(v_1, v_1, w).
\end{align*}$$

Therefore, using item (ii), we obtain

$$v_0 \geq \Theta(u_0, u_1, w) \geq v_1.$$  

Let $t^*$ be as in Lemma 5.6 and $\ell$ be as defined in (5.24). We define

$$\begin{align*}
\ell_0 &= \max\{\tilde{h}(t^*), \ell\}; \quad \ell_1 = \min\{\tilde{h}(t^*), -\ell\}; \\
\alpha \tilde{h}_0^* &= \Theta(\tilde{h}(t^*), \ell_0, w_\alpha); \quad \alpha \tilde{h}_1^* = \Theta(\tilde{h}(t^*), \ell_1, w_\alpha).
\end{align*}$$

Using the fact that

$$\ell_0 = \begin{cases} 
\ell & \text{on } \{\tilde{h}(t^*) \leq \ell\} \\
\tilde{h}(t^*) & \text{on } \{\tilde{h}(t^*) > \ell\},
\end{cases}$$

and Lemma 5.7 (ii), (iv), we obtain

$$E_\alpha(\alpha \tilde{h}_0^*) = E_\alpha(\Theta(\tilde{h}(t^*), \ell, w_\alpha), \{\tilde{h}(t^*) \leq \ell\}) + E_\alpha(\tilde{h}(t^*), \{\tilde{h}(t^*) > \ell\})$$

$$\leq E_\alpha(w_\alpha) + E_\alpha(\ell, \{\ell > w_\alpha\} \cup \{\ell < -w_\alpha\})$$

$$+ E_\alpha(\tilde{h}(t^*), \{\tilde{h}(t^*) > \ell\} \cup \{\tilde{h}(t^*) > -w_\alpha\} \cup \{\tilde{h}(t^*) < w_\alpha\})$$

$$\leq 2E_\alpha(w_\alpha) + 4\tau_2. \quad (5.34)$$

In the last step we have used Lemma 5.4, Lemma 5.6 and the fact that

$$\{\ell > w_\alpha\} \cup \{\ell < -w_\alpha\} \subset \{w_\alpha < 1\} \subset \Omega \text{ (by (5.16))}.$$  

By a similar argument,

$$E_\alpha(\alpha \tilde{h}_1^*) \leq 2E_\alpha(w_\alpha) + 4\tau_2. \quad (5.35)$$

By Lemma 5.7 (i) and (5.16), $\alpha \tilde{h}_0^* = \ell_0$ on $\Omega_{-4\sqrt{\alpha}}$. Further, $1 \geq \ell_0 \geq \ell$ and by Lemma 5.7 (v), $|\alpha \tilde{h}_0^*| \leq 1$. Hence,

$$\|1 - \alpha \tilde{h}_0^*\|_{L^1(\Omega)} \leq \|1 - \ell_0\|_{L^1(\Omega_{-4\sqrt{\alpha}})} + 2H^{n+1}\left(\Omega \setminus \Omega_{-4\sqrt{\alpha}}\right)$$

$$\leq \|1 - \ell\|_{L^1(\Omega)} + \tau_1 \text{ (by (5.9))}$$

$$\leq 3\tau_1 \text{ (by Lemma 5.4).} \quad (5.36)$$

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Similarly, one can show that
\[ \|1 + h_1^*\|_{L^1(\Omega)} \leq 3\tau_1. \]  
(5.37)

Using Lemma 5.7 (v) and the definitions of \( \tilde{h} \) and \( \ell \), we obtain
\[ \tilde{h}(0) \geq h_0^* \geq \tilde{h}(t^*); \quad \tilde{h}(t^*) \geq h_1^* \geq \tilde{h}(1). \]  
(5.38)

By Lemma 3.1, there exists \( \tilde{h}(0) \geq h_0^* \geq h_0^\ast \) such that
\[
E_\alpha(h_0^*) = \inf \{ E_\alpha(u) : \tilde{h}(0) \geq u \geq h_0^* \} \\
\leq E_\alpha(h_0^\ast) \\
\leq 2E_\alpha(w_\alpha) + 4\tau_2 \text{ (by (5.34))} \\
\leq 4\sigma H^n(\partial\Omega) + 6\tau_2 \text{ (by (5.8))}.
\]  
(5.39)

Moreover, (5.36) and \( 1 \geq h_0^* \geq h_0^\ast \) imply that
\[ \|1 - h_0^\ast\|_{L^1(\Omega)} \leq \|1 - h_0^*\|_{L^1(\Omega)} \leq 3\tau_1. \]  
(5.40)

Similarly, by Lemma 3.1, there exists \( h_1^* \geq h_1^\ast \geq \tilde{h}(1) \) such that
\[
E_\alpha(h_1^*) = \inf \{ E_\alpha(u) : h_1^* \geq u \geq \tilde{h}(1) \} \\
\leq E_\alpha(h_1^\ast) \\
\leq 2E_\alpha(w_\alpha) + 4\tau_2 \text{ (by (5.35))} \\
\leq 4\sigma H^n(\partial\Omega) + 6\tau_2 \text{ (by (5.8))}.
\]  
(5.41)

Moreover, (5.37) and \( h_1^* \geq h_1^\ast \geq -1 \) imply that
\[ \|1 + h_1^\ast\|_{L^1(\Omega)} \leq \|1 + h_1^*\|_{L^1(\Omega)} \leq 3\tau_1. \]  
(5.42)

Setting \( u = \tilde{h} \) and \( v = h_1^* \) in Lemma 3.3 (a), we conclude that there exists a nested map \( \gamma_1 : [0,1] \to H^1(N) \) such that \( \gamma_1(0) = \tilde{h}(0), \gamma_1(1) = h_1^* \) ((5.38) implies that \( \tilde{h}(0) \geq h_1^* \)) and
\[
\sup_{t \in [0,1]} E_\alpha(\gamma_1(t)) \leq \sup_{t \in [0,1]} E_\alpha(\tilde{h}(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \frac{\delta}{2} \text{ (by (5.15)).}
\]

Next, setting \( u = \gamma_1 \) and \( v = h_0^\ast \) in Lemma 3.3 (b), we obtain another nested map \( \gamma_2 : [0,1] \to H^1(N) \) such that \( \gamma_2(0) = h_0^\ast, \gamma_2(1) = h_1^\ast \) ((5.38) implies that \( h_0^\ast \geq h_1^\ast \)) and
\[
\sup_{t \in [0,1]} E_\alpha(\gamma_2(t)) \leq \sup_{t \in [0,1]} E_\alpha(\gamma_1(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \frac{\delta}{2}. \]  
(5.43)
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By the definitions of $\tau_1$, $\tau$ and Lemma 4.1, (5.39) and (5.40) imply that there exists $\tilde{\gamma}_0 : [0, 1] \rightarrow H^1(N)$ such that $\tilde{\gamma}_0(0) = h_0^\bullet$, $\tilde{\gamma}_0(1)|_\Omega \equiv 1$ and

$$\sup_{t \in [0,1]} E_\alpha(\tilde{\gamma}_0(t)) \leq 6\sigma H^n(\partial \Omega) + 6\tau_2 + \tau \leq 7\sigma H^n(\partial \Omega) \text{ (by (5.4) and (5.6))}. \tag{5.44}$$

Similarly (5.41) and (5.42) imply that there exists $\tilde{\gamma}_1 : [0, 1] \rightarrow H^1(N)$ such that $\tilde{\gamma}_1(0) = h_1^\bullet$, $\tilde{\gamma}_1(1)|_\Omega \equiv -1$ and

$$\sup_{t \in [0,1]} E_\alpha(\tilde{\gamma}_1(t)) \leq 6\sigma H^n(\partial \Omega) + 6\tau_2 + \tau \leq 7\sigma H^n(\partial \Omega) \text{ (by (5.4) and (5.6))}. \tag{5.45}$$

Let $\gamma : [0, 1] \rightarrow H^1(N)$ be defined by

$$\gamma(t) = \begin{cases} 
\tilde{\gamma}_0(1 - 3t) & \text{if } 0 \leq t \leq 1/3; \\
\gamma_2(3t - 1) & \text{if } 1/3 \leq t \leq 2/3; \\
\tilde{\gamma}_1(3t - 2) & \text{if } 2/3 \leq t \leq 1. 
\end{cases}$$

Then $\gamma \in \mathcal{B}$; (5.43), (5.44), (5.45), (5.10) and (5.3) imply that

$$\sup_{t \in [0,1]} E_\alpha(\gamma(t)) < \tilde{\lambda}_\alpha(\Omega),$$

which contradicts the definition of $\tilde{\lambda}_\alpha(\Omega)$ ((5.2)). This finishes the proof of Proposition 5.2. \hfill $\square$

**Theorem 5.8.** Let $(N^{n+1}, g)$, $n + 1 \geq 3$, be a closed Riemannian manifold and $\Omega \subset N$ be a good set. Suppose $\varepsilon^*$ and $\eta^*$ are as in Proposition 5.2 and Remark 5.3 (where we set $g' = g$ in Remark 5.3). Then for all $0 < \varepsilon \leq \varepsilon^*$, there exists $\partial_\varepsilon : N \rightarrow (-1, 1)$ satisfying $AC_\varepsilon(\partial_\varepsilon) = 0$, $\text{Ind}(\partial_\varepsilon) \leq 1$, $E_\varepsilon(\partial_\varepsilon) = \tilde{\lambda}_\varepsilon(\Omega)$ and $E_\varepsilon(\partial_\varepsilon, \Omega) \geq \eta^*$.

**Proof.** To prove this theorem, we apply Theorem 2.5 to the functional $E_\varepsilon : H^1(N) \rightarrow \mathbb{R}$, $0 < \varepsilon \leq \varepsilon^*$. In Theorem 2.5, we set

$$B_0 = \{ u \in H^1(N) : u|_{\Omega} \equiv 1 \}; \quad B_1 = \{ u \in H^1(N) : u|_{\Omega} \equiv -1 \};$$

$\mathcal{F} = \mathcal{B}$ so that $c = \tilde{\lambda}_\varepsilon(\Omega)$ and

$$L = \{ u \in H^1(N) : E_\varepsilon(u) \geq \tilde{\lambda}_\varepsilon(\Omega) \} \cap \{ u \in H^1(N) : E_\varepsilon(u, \Omega) \geq \eta^* \}. \tag{5.46}$$

Since $u|_{\Omega} \equiv 1$ or $u|_{\Omega} \equiv -1$ imply that $E_\varepsilon(u, \Omega) = 0$, the condition (a1) of Theorem 2.5 is satisfied. By Proposition 5.2, the condition (a2) is also satisfied. It follows from (5.46) that (a3) is satisfied as well. Following [Gua18, Section 4], if $\{h_i\}_{i=1}^\infty$ is an arbitrary minimizing sequence for $E_\varepsilon$ in $\mathcal{B}$, we define $\{h_i\}_{i=1}^\infty \subset \mathcal{B}$ by

$$h_i(t) = \min\{1, \max\{-1, h_i(t)\}\}.$$
Then

\[-1 \leq h_i(t) \leq 1 \quad \forall \, i \in \mathbb{N}, \, t \in [0, 1]\]

and \(E_\varepsilon(h_i) \leq E_\varepsilon(\tilde{h}_i)\). Hence \(\{h_i\}\) is again a minimizing sequence. By [Gua18, Proposition 4.4], \(E_\varepsilon\) satisfies the Palais-Smale condition along \(\{h_i\}\). Therefore, by Theorem 2.5, part (a), there exists \(\vartheta_\varepsilon \in K(\{h_i\})\) such that

\[AC_\varepsilon(\vartheta_\varepsilon) = 0, \quad E_\varepsilon(\vartheta_\varepsilon) = \tilde{\lambda}_\varepsilon(\Omega), \quad E_\varepsilon(\vartheta_\varepsilon, \Omega) \geq \eta^*.

Moreover, by (5.47), \(|\vartheta_\varepsilon| \leq 1\); hence by the strong maximum principle \(|\vartheta_\varepsilon| < 1\). In addition, if the ambient metric \(g \in \mathcal{M}\) (where \(\mathcal{M}\) is as defined in Theorem 2.2) and \(\varepsilon^{-1} \notin \text{Spec}(-\Delta_g)\), then \(Z_{\varepsilon, g}\) is finite. In that case, the condition (b1) of Theorem 2.5, part (b) is satisfied and one can ensure that \(\vartheta_\varepsilon\) satisfies \(\text{Ind}(\vartheta_\varepsilon) \leq 1\).

To get the Morse index upper bound for arbitrary metric \(g\) and \(\varepsilon \in (0, \varepsilon^*]\), we use an approximation argument. Since, by Theorem 2.2, \(\mathcal{M}\) is a generic subset of \(\mathcal{M}\), it is possible to choose \(\{g_i\}_{i=1}^\infty \subset \mathcal{M}\) such that \(g_i\) converges to \(g\) smoothly. Let \(\{\epsilon_i\}_{i=1}^\infty\) be a sequence in \((0, \varepsilon^*]\) such that \(\epsilon_i^{-1} \notin \text{Spec}(-\Delta_g)\) and \(\epsilon_i \to \varepsilon\). Since the width \(\mathcal{W}(\Omega)\) depends continuously on the ambient metric [IMN18, Lemma 2.1], \(\Omega\) is a good set with respect to \(g_i\) if \(\epsilon_i\) is sufficiently large. Therefore, by the above discussion, Theorem 2.2, Theorem 2.5, Proposition 5.2 and Remark 5.3 imply that for all \(i\) sufficiently large, there exists \(\vartheta^i : N \to (-1, 1)\) such that

\[AC_{\epsilon_i, g_i}(\vartheta^i) = 0; \quad \text{Ind}_{g_i}(\vartheta^i) \leq 1; \quad E_{\epsilon_i, g_i}(\vartheta^i) = \tilde{\lambda}_{\epsilon_i, g_i}(\Omega); \quad E_{\epsilon_i, g_i}(\vartheta^i, \Omega) \geq \eta^*.

(5.48)

In this equation, the subscript \(g_i\) indicates that these quantities are computed with respect to the metric \(g_i\). Since \(|\vartheta^i| < 1\), by the elliptic regularity and the Arzela-Ascoli theorem, there exists \(\vartheta^\infty : N \to [-1, 1]\) such that up to a subsequence \(\vartheta^i\) converges to \(\vartheta^\infty\) in \(C^2(N)\). Using (5.48) and the fact that the min-max quantity \(\tilde{\lambda}_\varepsilon(\Omega)\) depends continuously on the ambient metric [GG19, Lemma 5.4], we obtain

\[AC_{\varepsilon, g}(\vartheta^\infty) = 0; \quad \text{Ind}_g(\vartheta^\infty) \leq 1; \quad E_{\varepsilon, g}(\vartheta^\infty) = \tilde{\lambda}_{\varepsilon, g}(\Omega); \quad E_{\varepsilon, g}(\vartheta^\infty, \Omega) \geq \eta^*.

Furthermore, by the strong maximum principle, \(|\vartheta^\infty| < 1\). This finishes the proof of Theorem 5.8.

Proof of Theorem 1.1. As proved in [CL20, Section 8.1], \(\text{Vol}(M) < \infty\) implies that there exists a sequence \(\{U_i\}_{i=1}^\infty\), where each \(U_i \subset M\) is a bounded open set with smooth boundary, such that \(U_i \subset U_{i+1}\) for all \(i \in \mathbb{N}\) and \(\lim_{i \to \infty} \mathcal{H}^n(\partial U_i) = 0\). As a consequence, there exists \(i_0 \in \mathbb{N}\) such that \(U_{i_0}\) is a good set. For simplicity, let us denote \(U_{i_0}\) by \(U\).

The following proposition was proved in [Mon16, Section 12.2].

Proposition 5.9. [Mon16, Section 12.2] Let \(M'\) be a complete Riemannian manifold and \(f' : M' \to [a_0, \infty)\) be a proper Morse function. Suppose \(a_1\) is a regular value of \(f'\) and define

\[\mathcal{R} = \{x \in M' : f'(x) \leq a_1\}.

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Then there exists a closed Riemannian manifold $N'$ and a Morse function $f'' : N' \to [a_0, \infty)$ such that $N'$ contains an isometric copy of $\mathcal{R}$, $f''$ coincides with $f'$ on $\mathcal{R}$ and $f'' > a_1$ on $N' \setminus \mathcal{R}$.

Coming back to the proof of Theorem 1.1, we choose a proper Morse function $f_1 : M \to [0, \infty)$ with $\min_M f_1 = 0$. Let $t_1$ be a regular value of $f_1$ so that $U \subset \subset f_1^{-1}([0, t_1/2])$. Furthermore, by suitably modifying $f_1$, we can assume that in the interval $[0, t_1/2]$, $f_1$ has no critical value which is a non-global local maxima or minima. Let

$$f_2 = \frac{1}{3} + \frac{2f_1}{3t_1}.$$ 

Then $f_2 : M \to [1/3, \infty)$, $U \subset \subset f_2^{-1}([1/3, 2/3])$ and we set $\bar{U} = f_2^{-1}([1/3, 1])$.

We choose an increasing sequence $\{s_i\}_{i=1}^\infty$ such that $s_1 \geq 1$, each $s_i$ is a regular value of $f_2$ and $s_i \to \infty$ as $i \to \infty$. Let

$$Q_i = \{x \in M : f_2(x) < s_i\}, \quad \bar{Q}_i = \{x \in M : f_2(x) \leq s_i\}.$$ 

By Proposition 5.9, there exists a sequence $\{N_i\}_{i=1}^\infty$ of closed Riemannian manifolds such that $N_i$ contains an isometric copy of $\bar{Q}_i$. Moreover, for every $i$, there exists a Morse function $\bar{f}_i : N_i \to [1/3, \infty)$ such that $\bar{f}_i$ coincides with $f_2$ on $\bar{Q}_i$ and $\bar{f}_i > s_i$ on $N_i \setminus \bar{Q}_i$. In particular, $N_i$ contains isometric copies of $U$ and $\bar{U}$; suppose $\bar{U}_i$ denotes the isometric copy of $U$ in $N_i$. Setting $N = N_i$ and $f = \bar{f}_i$ in Proposition 5.2 and Remark 5.3 and using Theorem 5.8, we obtain $\varepsilon_0, \eta_0 > 0$, which depend only on $U, \bar{U}$, the ambient metric on $M$ restricted to $\bar{U}$ and $f_2|_{\bar{U}}$ such that the following holds.\footnote{In particular, $\varepsilon_0$ and $\eta_0$ do not depend on $i$.} For all $0 < \varepsilon \leq \varepsilon_0$, there exists $\vartheta_{\varepsilon,i} : N_i \to (-1, 1)$ satisfying

$$AC_\varepsilon(\vartheta_{\varepsilon,i}) = 0; \quad \text{Ind}(\vartheta_{\varepsilon,i}) \leq 1; \quad E_\varepsilon(\vartheta_{\varepsilon,i}) = \bar{\lambda}_\varepsilon(\bar{U}_i); \quad E_\varepsilon(\vartheta_{\varepsilon,i}, \bar{U}_i) \geq \eta_0.$$

Let $b \in [1/3, \infty)$ such that

$$\{x \in M : d(x, U) \leq 2\sqrt{\varepsilon_0}\} \subset f_2^{-1}([1/3, b]).$$

One can modify $f_2$ on $f_2^{-1}([1/3, b])$ and define another proper Morse function $f_3 : M \to [1/3, \infty)$ so that

$$\{x \in M : d(x, U) \leq 2\sqrt{\varepsilon_0}\} \subset f_3^{-1}([1/3, b]) \quad (5.49)$$

and in the interval $[1/3, b]$, $f_3$ has no critical value which is a non-global local maxima or minima. For $1/3 \leq t \leq b$, let $d^t : M \to \mathbb{R}$ be defined by

$$d^t(x) = \begin{cases} -d(x, f_3^{-1}(t)) & \text{if } f_3(x) \leq t; \\ d(x, f_3^{-1}(t)) & \text{if } f_3(x) \geq t. \end{cases}$$
\[ U \subset \{ d^b \leq -2\sqrt{\varepsilon_0} \}. \]  

(5.50)

We choose \( i_1 \in \mathbb{N} \) so that

\[ \{ d^b \leq 2\sqrt{\varepsilon_0} \} \subset Q_{i_1}. \]  

(5.51)

For \( 0 < \varepsilon \leq \varepsilon_0 \), let \( \zeta_\varepsilon : [0, b] \to H^1(Q_{i_1}) \) be defined by

\[
\zeta_\varepsilon(t) = \begin{cases} 
\hat{q}_\varepsilon \circ d^t & \text{if } \frac{1}{3} \leq t \leq b \\
1 - 3t(1 - \zeta_\varepsilon(1/3)) & \text{if } 0 \leq t \leq \frac{1}{3},
\end{cases}
\]

where \( \hat{q}_\varepsilon \) is as defined in (4.4). \( \zeta_\varepsilon \) is continuous (see the discussion after equation (4.7)), \( \zeta_\varepsilon(0) \equiv 1 \) and (5.50) implies that \( \zeta_\varepsilon(b)\mid_U \equiv -1 \). Further, by [Gua18, Section 9],

\[ \Lambda_\varepsilon := \sup_{0 \leq t \leq b} E_\varepsilon(\zeta_\varepsilon(t)) \]

satisfies

\[ \limsup_{\varepsilon \to 0^+} \Lambda_\varepsilon \leq 2\sigma \sup_{1/3 \leq t \leq b} \mathcal{H}^n \left( f_3^{-1}(t) \right). \]  

(5.52)

For each \( i \geq i_1, N_i \) contains an isometric copy of \( Q_{i_1} \). Hence (5.51) implies that \( \zeta_\varepsilon \) canonically defines a continuous map \( \zeta_\varepsilon,i : [0, b] \to H^1(N_i) \) (\( \zeta_\varepsilon,i(t) \equiv 1 \) on \( N_i \setminus Q_{i_1} \) for all \( t \in [0, b] \)) so that \( \zeta_\varepsilon(0) \equiv 1 \) and \( \zeta_\varepsilon(b)\mid_{U_i} \equiv -1 \). Thus

\[ \hat{\lambda}_\varepsilon(U_i) \leq \Lambda_\varepsilon \quad \forall \ i \geq i_1. \]  

(5.53)

Restricting \( \vartheta_{\varepsilon,i} \) to the isometric copy of \( Q_i \) contained in \( N_i \), one gets \( u_{\varepsilon,i} : Q_i \to (-1, 1) \) such that

\[ AC_\varepsilon(u_{\varepsilon,i}) = 0; \quad \text{Ind}(u_{\varepsilon,i}) \leq 1; \quad E_\varepsilon(u_{\varepsilon,i}, Q_i) \leq \hat{\lambda}_\varepsilon(U_i); \quad E_\varepsilon(u_{\varepsilon,i}, U) \geq \eta_0. \]  

(5.54)

For each fixed \( j \in \mathbb{N}, \) by the elliptic estimates, \( \| u_{\varepsilon,i} \|_{C^{2,\alpha}(Q_j)} \) is uniformly bounded for all \( i > j \) (since \( |u_{\varepsilon,j}| < 1 \) for all \( i \in \mathbb{N} \)). Using a diagonal argument and the Arzela-Ascoli theorem, we conclude that there exists \( u_\varepsilon : M \to [-1, 1] \) such that a subsequence \( \{ u_{\varepsilon,i_k} \} \) converges to \( u_\varepsilon \) in \( C^2_{\text{loc}}(M) \). Hence, using (5.54) and (5.53),

\[ AC_\varepsilon(u_\varepsilon) = 0; \quad \text{Ind}(u_\varepsilon) \leq 1; \quad E_\varepsilon(u_\varepsilon) \leq \Lambda_\varepsilon; \quad E_\varepsilon(u_\varepsilon, U) \geq \eta_0. \]

By the strong maximum principle, \( |u_\varepsilon| < 1 \). Further, as mentioned in (5.52), \( \limsup_{\varepsilon \to 0^+} \Lambda_\varepsilon < \infty \). This completes the proof of Theorem 1.1.
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**Proof of Theorem 1.2 using Theorem 1.1.** Let

\[ \Omega_1 \subset \cdots \subset \Omega_i \subset \Omega_{i+1} \subset \cdots \]

be an exhaustion of \( M \) by bounded open subsets with smooth boundaries and \( U \subset \subset \Omega_1 \). Using Theorem 2.1 and a diagonal argument, there exist a sequence \( \{ \epsilon_i \}_{i=1}^\infty \) converging to 0 and a stationary, integral varifold \( V_k \) in \( \Omega_k \) (for each \( k \in \mathbb{N} \)) such that the following conditions are satisfied.

- \( V \left[ u_{\epsilon_i} | \Omega_k \right] \rightarrow V_k \) \hspace{1cm} (5.55)
  in the sense of varifolds.
- \( \text{spt}(V_k) \) is a minimal hypersurface with optimal regularity in \( \Omega_k \).
- \[ \|V_k\| (\text{Clos}(U)) \geq \frac{1}{2\sigma} \liminf_{i \to \infty} E_{\epsilon_i} (u_{\epsilon_i}, U) . \] \hspace{1cm} (5.56)
- \[ \|V_k\| (\Omega_k) \leq \frac{1}{2\sigma} \limsup_{i \to \infty} E_{\epsilon_i} (u_{\epsilon_i}, \Omega_k) \leq \frac{1}{2\sigma} \limsup_{i \to \infty} E_{\epsilon_i} (u_{\epsilon_i}) . \] \hspace{1cm} (5.57)

By (5.55), \( V_i \cap \Omega_j = V_j \) if \( i > j \). Therefore, there exists a stationary, integral varifold \( V \) in \( M \) such that \( V \cap \Omega_i = V_i \) and \( \text{spt}(V) \) is a minimal hypersurface with optimal regularity. Further, (1.2), (5.56) and (5.57) imply

\[ 0 < \|V\| (\text{Clos}(U)) \leq \|V\| (M) < \infty. \]

\[ \square \]

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