Effect of network topology on the ordering dynamics of voter models

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Abstract. We introduce and study the reverse voter model, a dynamics for spin variables similar to the well-known voter dynamics. The difference is in the way neighbors influence each other: once a node is selected and one among its neighbors chosen, the neighbor is made equal to the selected node, while in the usual voter dynamics the update goes in the opposite direction. The reverse voter dynamics is studied analytically, showing that on networks with degree distribution decaying as $k^{-v}$, the time to reach consensus is linear in the system size $N$ for all $v > 2$. The consensus time for link-update voter dynamics is computed as well. We verify the results numerically on a class of uncorrelated scale-free graphs.

INTRODUCTION

Ordering dynamics is a classical topic in statistical physics, actively investigated for over two decades [1]. The best known realization of an ordering process is the dynamics of a ferromagnet suddenly quenched from a high-temperature state to a temperature below the critical one: ordered domains are formed and coarsen in time until the equilibrium state at temperature $T$ is reached.

More recently, ordering processes have started to be investigated in the completely different context of social sciences: the spreading of rumors or beliefs, the dynamics of opinions, the diffusion of cultural traits are all phenomena with the common feature that, starting from a disordered initial configuration, the dynamics tends to increase the similarity of interacting agents, leading toward a more ordered state. Simple (often oversimplified) models for such phenomena are akin to models studied in traditional statistical physics, but from the point of view of social sciences it is more interesting to study their behavior on networks, rather than on regular lattices.

The voter model is one of the simplest ordering processes [2]. Each individual is fully specified by a single ‘spin’ variable that can assume only two states, say $-1$ and $1$. At each time step an individual is chosen at random and his variable is set equal to the variable of a randomly chosen nearest neighbor. This step is then iterated many times. There are two absorbing configurations for the dynamics: all variables equal to $-1$ or all equal to $1$. Once these consensus states are reached no further evolution is possible. When $N$, the number of agents in the system, is finite, the dynamics reaches sooner or later one of the two absorbing configurations. An interesting quantity is then $\tau(N)$, the mean time needed to arrive at full consensus. For regular lattices in $d$ dimensions, this time scales as $N^2$ in $d = 1$, $N \ln(N)$ in $d = 2$ and as $N$ for $d > 2$ [2, 3].
Some activity has been recently devoted to the study of the voter model on graphs. In particular, it has been shown that the time to reach consensus scales as $N$ on the Watts-Strogatz graph [4,5]. On the Barabasi-Albert (BA) scale-free network it is found numerically that $\tau$ grows with $N$ with an exponent close to 0.88 [6,7]. At odds with what occurs on regular lattices, voter dynamics does not conserve the average magnetization when the number of nearest neighbors (degree) changes from node to node [6,8]. In a very recent contribution [9], Sood and Redner have evaluated analytically the consensus time for a generic uncorrelated heterogeneous graph with a scale-free degree distribution $n_k \sim k^{-\nu}$. It turns out that $\tau(N)$ scales as $N$ for $\nu > 3$, it is a constant for $\nu < 2$, and it scales as $N^2/(\nu - 1)$ for $2 < \nu < 3$. The presence of a logarithmic correction for $\nu = 3$, $\tau(N) \sim N/\ln(N)$ accounts well for the observed exponent 0.88 on the BA network.

In the elementary step of the original voter model, the selected node A picks at random a nearest neighbor B and becomes equal to it. When one considers simple modifications of this dynamics there are two possibilities that immediately come to mind. The first, similar to the so called Ochrombel simplification of the Sznajd model [10], consists in reversing the direction of the update, so that it is the neighbor B that becomes equal to the selected node A. We call this dynamics ‘reverse’ voter model. The second is the ‘link-update’ voter model [6]: instead of a site, a link is randomly chosen and the update occurs in random direction along it. The link-update dynamics conserves the average magnetization for any degree distribution [6]. It is clear instead that the ‘reverse’ voter model violates this conservation.

Whenever the degree distribution is a delta function the three models perfectly coincide, but this is no longer true on a generic graph. This is evident for example on a star graph, where all links connect a hub with the other $N-1$ nodes. In a single elementary step, the probability that the hub is updated is $1/N$ for the usual voter model, $1/2$ for the link-update dynamics and $1 - 1/N$ for the reverse voter model.

In this paper we investigate for which topology the three types of voter dynamics give similar results and when, instead, the ordering processes are qualitatively different.

**ANALYTICAL RESULTS**

We consider the reverse voter dynamics on a generic heterogeneous uncorrelated graph with normalized degree distribution $n_k$. Let us define $\rho_k$, the fraction of up spins on nodes of degree $k$, and compute the probability $P(k; - \rightarrow +)$ that a spin down on a node of degree $k$ flips in an elementary update.

For this flip to occur it is necessary that the spin on the selected node (A) is up (and this happens with probability $\rho = \sum_k n_k \rho_k$) and that the neighbor chosen (B) has degree $k$ and is down. The degree distribution of the neighbors of a random node in an uncorrelated network is $kn_k/\mu_1$, where $\mu_1 = \sum_k k^2 n_k$. The probability that a node of degree $k$ is down is $1 - \rho_k$, hence

$$P(k; - \rightarrow +) = \rho \frac{kn_k}{\mu_1} (1 - \rho_k).$$

Similarly $P(k; + \rightarrow -) = (1 - \rho) \frac{kn_k}{\mu_1} \rho_k$. 


Using these expressions one can apply the analytical treatment of Ref. [9] to compute the consensus time $\tau(N)$. The idea is to write down a recursion formula for $\tau$ as a function of the $\rho_k$. Expanding to second order one obtains

$$\sum_k \frac{k}{\mu_1} (\rho_k - \rho) \partial_k \tau - \frac{1}{2N} \sum_k \frac{k}{\mu_1 n_k} (\rho + \rho_k - 2\rho \rho_k) \partial^2_k \tau = 1,$$

(2)

where $\partial_k = \partial / \partial \rho_k$.

To analyze the role of the different terms in Eq. (2) we have to consider the dynamics of the densities $\rho_k$. In an elementary update, the number $N_{kp}$ of up spins of degree $k$, changes by an amount $dN_{kp} = P(k; - \rightarrow +) - P(k; + \rightarrow -)$, so that

$$\dot{\rho}_k = dN_{kp} N_k dt = \frac{1}{n_k N dt} \left[ P(k; - \rightarrow +) - P(k; + \rightarrow -) \right] = \frac{k}{\mu_1} (\rho - \rho_k).$$

(3)

The equation of motion for the magnetization $\rho$ is

$$\dot{\rho} = \rho - \frac{1}{\mu_1} \sum_k n_k \rho_k.$$  

(4)

Starting from their initial values, the densities $\rho_k$ rapidly converge to a stationary value

$$\rho_k = \rho_s \quad \forall k.$$  

(5)

This stationary value can be related to the quantity $R = \mu_1 \sum_k \frac{n_k}{k} \rho_k$ that is conserved by the dynamics

$$R = \mu_1 \rho_s \sum_k \frac{n_k}{k} \rho_k = \frac{\rho_s}{C},$$

(6)

where $C = 1/(\mu_1 \mu_{-1})$.

It is now possible to rewrite Eq. (2) in the stationary state. Using Eqs. (5) and (6) the first term is seen to vanish; one is left with

$$R(1 - CR) \partial^2_R \tau = -N.$$  

(7)

The integration of Eq. (7), with boundary conditions $\tau(R = 0) = 0$ and $\tau(R = 1/C) = 0$, yields

$$\tau(N) = -N \left[ R \ln(CR) + \left( \frac{1}{C} - R \right) \ln(1 - CR) \right].$$  

(8)

For an uncorrelated initial configuration, i.e. $\rho_k(0) = \rho(0) \quad \forall k$,

$$\tau(N) = -N \mu_1 \mu_{-1} \left[ \rho(0) \ln \rho(0) + (1 - \rho(0)) \ln(1 - \rho(0)) \right].$$  

(9)

Since the prefactor is finite the time to consensus diverges linearly with $N \rightarrow \infty$ for all $\nu > 2$. For $\nu = 2$ a logarithmic correction is present: $\tau(N) \propto N \ln(N)$, while $\tau(N) \sim N^{1/(\nu - 1)}$ for $1 < \nu < 2$. 


Applying the same analytical approach, the consensus time for link-update dynamics \[6\] can be computed as well. The probability of flipping a down spin on a node of degree \(k\) has the form
\[
P(k; -\rightarrow +) = \omega \frac{\ln k}{\mu_1} (1 - \rho_k),
\]
where
\[
\omega = \sum_k n_k k \rho_k.
\]
The consensus time turns out to be
\[
\tau(N) = -N[\ln(1 - \rho) + \rho \ln \rho].
\]  
(10)

Also in this case the time to consensus depends linearly on the system size \(N\). Remarkably, there is no dependence at all on the network parameters.

**NUMERICAL RESULTS**

In order to validate the analytical results we have performed numerical simulations of the reverse voter dynamics on two classes of scale-free graphs.

In Fig. 1 we present the consensus time for reverse voter dynamics on a growing network with redirection \[11\] for several values of the exponent \(v\) of the scale-free degree distribution.

From the main part of the figure it is clear that, while the asymptotic behavior is, as expected, proportional to the system size, sizeable corrections occur for smaller values of \(N\). This might be caused by the residual preasymptotic dependence of \(\mu_1\) and \(\mu_{-1}\) on \(N\), for the relatively small values of \(N\) considered. In the inset we have therefore plotted \(\tau(N)/(N\mu_1 \mu_{-1})\) and compared to the value \(\ln(2) = 0.6931\ldots\), predicted by Eq. (9). The agreement is not satisfactory. This suggests that correlations present in the network have an important influence on the ordering process; they do not alter the asymptotic linear dependence on \(N\) but affect rather strongly the prefactor.
We have then considered the evolution of the reverse voter model on another class of graphs, recently introduced by Catanzaro et al. [12]. Such graphs are built according to the usual algorithm for the configuration model, with the additional prescription that no node can have a degree larger than $N^{1/2}$. This modification has the goal of avoiding the correlations of the node connectivities present in the usual configuration model.

It is evident from the results, displayed in Fig. 2, that even on this class of networks one does not see a clean linear dependence on $N$ for the system sizes considered here. The effective exponent for small $N$ is larger than one; a crossover to the expected asymptotic behavior is seen for larger values of $N$. The plot of $\tau(N)/(N\mu_1\mu_{-1})$, presented in the inset, indicates that the bending in the main plot actually comes from the residual $N$-dependence of $\mu_1$ and $\mu_{-1}$. The quantity $\tau(N)/(N\mu_1\mu_{-1})$ goes asymptotically to a constant value that, as predicted from Eq. (9), does not depend on the exponent $\nu$. The only feature of Fig. 2 not in agreement with the theory is the value of the constant, that is actually of the order of two times the expected value, ln(2).

On the uncorrelated scale-free graphs we have also studied numerically the ordering of the link-update voter model. Although Eq. (10) predicts the consensus time to be completely independent from $\nu$ we find that the curves do not superimpose (Fig. 3). A simple linear fit yields exponents ranging from 0.96 to 1.02, but a more careful inspection indicates that the curves are slightly bent.

Further insight is provided by the inset, where the consensus time divided by the system size $N$ is plotted as a function of $N$. A transient regime, for small $N$, with non constant $\tau(N)/N$ is followed by plateau, indicating that the asymptotic linear regime is reached. At odds with Eq. (10), the height of the plateau depends on the value of $\nu$, ranging from approximately one to two times the expected value ln(2).

Hence we find that also in this case the analytical agreement correctly captures the linear behavior independent from $\nu$, while it fails for what concerns the prefactor. We
FIGURE 3. Main: Double logarithmic plot of the consensus time $\tau(N)$ as a function of $N$ for the link-update voter dynamics on the uncorrelated scale-free networks of Ref. [12] with minimum degree 3, for $\rho_i(0) = \rho(0) = 1/2$. Inset: The same data divided by $N$. The thick line is the result predicted by Eq. (10), $\ln(2)$.

do not have a clear understanding of the origin of this disagreement. One possible reason could be the presence of residual correlations in the modified configuration model that we have used. However, the mismatch occurs also for values of $\nu$ so large that correlations are surely negligible. It is more likely that the different prefactors are an effect of the term containing the first derivative in Eq. (2), that we have neglected in the analytical treatment, but that may play a role in the initial stage of the ordering process.

CONCLUSIONS

We have studied analytically and numerically the ordering behavior of the reverse and the link-update voter models, finding that the time to reach consensus grows linearly with the system size for scale-free networks with $\nu > 2$.

It is interesting to notice that, for the reverse voter model, the time to reach consensus gets larger as $\nu$ is decreased, since $\mu_1$ grows. This can be qualitatively understood as follows: in the reverse voter dynamics it is very easy that a hub changes its state, being chosen as the neighbor of the selected node. In the usual voter dynamics, instead, the presence of a hub favors order, since its many neighbors tend to become equal to it. Correspondingly, the time to consensus in the normal voter dynamics is reduced as $\nu$ is decreased [9].

As mentioned above, the three types of voter model are perfectly equal when the degree is the same for all nodes. This can be seen also from Eqs. (9) and (10). For $\nu \to \infty$, $\mu_i = (\mu_1)^i$ and in all cases (including the usual voter dynamics), the consensus time is given by Eq. (10). For finite values of $\nu$, instead, the usual voter dynamics leads to consensus most quickly, and the reverse dynamics is the slowest (Fig. 4).

All the results presented here were obtained under the assumption that the dynamics
FIGURE 4. Main: Plot of the consensus time $\tau$ as a function of $\nu$ for the three types of voter dynamics on the uncorrelated scale-free networks of Ref. [12] with minimum degree 3, for $\rho_k(0) = \rho(0) = 1/2$. The upper curves are for $N = 3000$, the lower curves are for $N = 300$.

takes place on an uncorrelated network. An interesting issue for future work is the investigation of voter dynamics on correlated graphs.

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