THE COMMON STRUCTURE FOR OBJECTS IN APERIODIC ORDER AND THE THEORY OF LOCAL MATCHING TOPOLOGY

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ABSTRACT. In aperiodic order, non-periodic but “ordered” objects such as tilings, Delone sets, functions and measures are investigated. In this article we depict the common structure of these objects by using the general framework of abstract pattern spaces. In particular, using the common structure we define local matching topology and uniform structure for objects such as tilings in quite a general space and a symmetry group. We prove Hausdorff property of the topology and the completeness of the uniform structure under a mild assumption. We also prove finite local complexity implies the compactness of the continuous hull and often the converse holds.

1. INTRODUCTION

Ever since quasicrystals were discovered, mathematical objects such as tilings, Delone (multi) sets, weighted Dirac combs and almost periodic functions have been investigated, especially on their diffraction nature and the connection with topology and the theory of dynamical systems. In this context the continuous hulls and the corresponding dynamical systems are important, where the former are the closures of their orbits and the latter are obtained by group actions on continuous hulls and geometric analogues for symbolic dynamics. The choice of topology is crucial here, since we want the continuous hulls to be compact. The simplest topology (and uniform structure) is the local matching topology (and local matching uniform structure). If the ambient space where the above objects live is $\mathbb{R}^d$, it is known that a condition called finite local complexity (FLC) assures that the continuous hulls are compact with respect to the local matching topology. In the proof for this claim, the fact that the local matching uniform structure is complete is tacitly used. For discrete subsets (which include Delone sets), the completeness of the local matching uniform structure is proved in [13] and [15]. (These papers deal with the cases where the ambient space is not necessarily Euclidean, but mathematically there is no necessity to restrict ourselves to the Euclidean case.) However there seems to be no analogous results for tilings, Delone multi sets, weighted Dirac combs and functions, if the ambient space in which these objects live is general. Since there are results on construction of non-periodic tilings in general spaces ([4],[10],[9]), it is worthwhile to prove such completeness. In this article we prove such completeness in full generality.

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The argument is based on a general framework to discuss those objects in a unified manner. The essential structure of those objects is cutting-off operation and group action of sliding objects. For example, if $D$ and $C$ are subsets of a set $X$, we can “cut off” $D$ by $C$ by considering $D \cap C$. If a group $\Gamma$ acts on $X$, we can “slide” $D$ by considering $\gamma D$, for each $\gamma \in \Gamma$. For each object, these structures have common properties and we can axiomatize them. A set with a cutting-off operation and a group action that satisfy those axioms are called abstract pattern spaces and the elements of abstract pattern spaces are called abstract patterns. Naturally the above objects such as tilings and Delone sets are abstract patterns.

The structure of cutting-off operation and group action with the axioms are common structure of objects such as tilings and Delone sets. The common structure is enough to define the local matching topology and uniform structure on abstract pattern spaces. Under a mild assumption, the topology is Hausdorff (Proposition 3.7) and metrizable (Corollary 3.8). We also prove that often on a subspace of abstract pattern space, the local matching uniform structure is complete (Theorem 3.19). This means that FLC of an abstract pattern implies the compactness of the continuous hull (Theorem 3.25).

In Section 2 we give an introduction of the theory of abstract pattern spaces, although we omit most of the proofs, which can be found in [11]. In Section 3 we define the local matching topology and uniform structure and investigate the properties, such as completeness and metrizability.

Setting 1. Here is the setting of this article. The symbol $X$ represents a proper metric space. The metric on $X$ is denoted by $\rho_X$. $\Gamma$ is a group and $\rho_\Gamma$ is a left-invariant proper metric on $\Gamma$. We assume $\Gamma$ acts on $X$ as isometries and the action is jointly continuous, that is, the map $\Gamma \times X \ni (\gamma, x) \mapsto \gamma x \in X$ is continuous, where the domain is endowed with the product topology. We take $x_0 \in X$ and use it as a reference point throughout the article.

Notation 1.1. For $x \in X$ and $r > 0$, we define the closed ball $B(x, r)$ via

$$B(x, r) = \{y \in X \mid \rho_X(x, y) \leq r\}.$$ 

Similarly for $\gamma \in \Gamma$ and $r > 0$ we set

$$B(\gamma, r) = \{\eta \in \Gamma \mid \rho_\Gamma(\gamma, \eta) \leq r\}.$$ 

The one-dimensional torus is denoted by $T$:

$$T = \{z \in \mathbb{C} \mid |z| = 1\}.$$ 

2. General theory of abstract pattern spaces

In this section we summarize the contents of [11] to introduce abstract pattern spaces discussed in Introduction.

Objects such as tilings (Example 2.4) and Delone sets (Example 2.8) admit the following structures, which play important roles explicitly or implicitly.
(1) They admit cutting-off operation. For example, if $T$ is a tiling in $\mathbb{R}^d$ and $C \subset \mathbb{R}^d$, we can “cut off” $T$ by $C$ by considering
\[
T \land C = \{ T \in T \mid T \subset C \}.
\]
By this operation we forget the behavior of $T$ outside $C$.

(2) Some of the objects “include” other objects. For patches this means the usual inclusion of two sets; for measures this means one measure is a restriction of another.

(3) They admit gluing operation. For example, suppose $\{P_i \mid i \in I\}$ is a family of patch such that if $i, j \in I, T \in P_i$ and $S \in P_j$, then either $S = T$ or $S \cap T = \emptyset$. Then we can “glue” $P_i$’s and obtain a patch $\bigcup_{i \in I} P_i$.

(4) There are “zero elements”, which contains nothing. For example, empty set is a patch that contains no tiles; zero function also contains no information. Such a zero element is often unique for each category of objects.

We will axiomatize the cutting-off operation should satisfy and a set with a cutting-off operation that obey the axiom is called abstract pattern space. The other structures in the list are captured by the cutting-off operation.

First in Subsection 2.1 we give the axiom and define abstract pattern spaces. In Subsection 2.2 we deal with the “inclusion” in the list. In Subsection 2.3 we discuss “gluing” operation in the list, by describing abstract pattern spaces in which we “often” glue abstract patterns. Finally in Subsection 2.4 we discuss “zero elements” in the list and give a sufficient conditions for it to be unique. In Subsection 2.5 we deal with abstract pattern spaces with $\Gamma$-action, which are called $\Gamma$-abstract pattern spaces.

2.1. Definition and examples of abstract pattern space.

Definition 2.1. The set of all closed subsets of $X$ is denoted by $\text{Cl}(X)$.

Definition 2.2. A non-empty set $\Pi$ equipped with a map
\[
\Pi \times \text{Cl}(X) \ni (P, C) \mapsto P \land C \in \Pi
\]
such that
(1) $(P \land C_1) \land C_2 = P \land (C_1 \cap C_2)$ for any $P \in \Pi$ and any $C_1, C_2 \in \text{Cl}(X)$, and
(2) for any $P \in \Pi$ there exists $C_P \in \text{Cl}(X)$ such that
\[
P \land C = P \iff C \supset C_P,
\]
for any $C \in \text{Cl}(X)$, is called an abstract pattern space over $X$. The map $[\Pi]$ is called the cutting-off operation of the abstract pattern space $\Pi$. The closed set $C_P$ that appears in 2. is unique. It is called the support of $P$ and is represented by $\text{supp} P$. Elements in $\Pi$ are called abstract patterns in $\Pi$.

The following lemma describes a relation between the support and the cutting-off operation.

Lemma 2.3. Let $\Pi$ be an abstract pattern space over $X$. For any $P \in \Pi$ and $C \in \text{Cl}(X)$, we have $\text{supp}(P \land C) \subset (\text{supp} P) \cap C$. 

Proof.

\[(P \land C) \land ((\text{supp } P) \cap C) = (P \land \text{supp } P) \land C = P \land C.\]

\[\square\]

We now give several examples of abstract pattern spaces.

**Example 2.4** (The space of patches in \(X\)). An open, nonempty and bounded subset of \(X\) is called a *tile* (in \(X\)). A set \(P\) of tiles such that if \(S, T \in P\), then either \(S = T\) or \(S \cap T = \emptyset\) is called a *patch* (in \(X\)). The set of all patches in \(X\) is denoted by \(\text{Patch}(X)\). For \(P \in \text{Patch}(X)\) and \(C \in \text{Cl}(X)\), set

\[P \land C = \{T \in P \mid T \subset C\}.\]

With this cutting-off operation \(\text{Patch}(X)\) becomes an abstract pattern space over \(X\). For \(P \in \text{Patch}(X)\), its support is

\[\text{supp } P = \bigcup_{T \in P} T.\]

Patches \(P\) with \(\text{supp } P = X\) are called *tilings*.

**Remark 2.5.** Usually tiles are defined to be (1) a compact set that is the closure of its interior [3], or in Euclidean case, (2) a polygonal subset of \(\mathbb{R}^d\) [14] or (3) a homeomorphic image of closed unit ball (for example, [1]). The advantage of our definition is that we can give punctures to tiles and we do not need to consider labels (Example 2.6).

The usual labeled tilings (Example 2.6) are often MLD with tilings with open tiles (Example 2.4). It is the cutting-off operation that is essential and the definition of tiles is not essential.

**Example 2.6** (The space of labeled patches, [7], [8]). Let \(L\) be a set. An *\(L\)-labeled tile* is a pair \((T, l)\) of a compact subset \(T\) of \(X\) and \(l \in L\), such that \(T = \overline{T^o}\) (the closure of the interior). An *\(L\)-labeled patch* is a collection \(P\) of \(L\)-labeled tiles such that if \((T, l), (S, k) \in P\), then either \(T^o \cap S^o = \emptyset\), or \(S = T\) and \(l = k\). For an \(L\)-labeled patch \(P\), define the support of \(P\) via

\[\text{supp } P = \bigcup_{(T, l) \in P} T.\]

An \(L\)-labeled patch \(T\) with \(\text{supp } T = X\) is called an *\(L\)-labeled tiling*. Sometimes we suppress \(L\) and call such tilings labeled tilings.

For an \(L\)-labeled patch \(P\) and \(C \in \text{Cl}(X)\), define a cutting-off operation via

\[P \land C = \{(T, l) \in P \mid T \subset C\}.\]

The space \(\text{Patch}_L(X)\) of all \(L\)-labeled patches is a pattern space over \(X\) with this cutting-off operation.
Example 2.7 (The space of all locally finite subsets of $X$). Let $LF(X)$ be the set of all locally finite subsets of $X$; that is,

$$LF(X) = \{ D \subset X \mid \text{for all } x \in X \text{ and } r > 0, D \cap B(x, r) \text{ is finite} \}.$$

With the usual intersection $LF(X) \times Cl(X) \ni (D, C) \mapsto D \cap C \in LF(X)$ of two subsets of $X$, $LF(X)$ is an abstract pattern space over $X$. For any $D \in LF(X)$, its support is $D$ itself.

Example 2.8 (The space of all uniformly discrete subsets). We say, for $r > 0$, a subset $D$ of $X$ is $r$-uniformly discrete if $\rho_X(x, y) > r$ for any $x, y \in D$ with $x \neq y$. The set $UD_r(X)$ of all $r$-uniformly discrete subsets of $X$ is an abstract pattern space over $X$ by the usual intersection as a cutting-off operation. If $D$ is $r$-uniformly discrete for some $r > 0$, we say $D$ is uniformly discrete. The set $UD(X) = \bigcup_{r>0} UD_r(X)$ of all uniformly discrete subsets of $X$ is also an abstract pattern space over $X$.

Take a positive real number $R$. A subset $D$ of $X$ is $R$-relatively dense if whenever we take $x \in X$, the intersection $D \cap B(x, R)$ is non-empty. A subset of $X$ is relatively dense if it is $R$-relatively dense for some $R > 0$. The uniformly discrete and relatively dense subsets of $X$ are called Delone sets.

Example 2.9. With the usual intersection of two subsets of $X$ as a cutting-off operation, the set $2^X$ of all subsets of $X$ and $Cl(X)$ are abstract pattern spaces over $X$. For example, the union of all Ammann bars for a Penrose tilings is an abstract pattern.

Example 2.10 (The space of maps). Let $Y$ be a nonempty set. Take one element $y_0 \in Y$ and fix it. The abstract pattern space $Map(X, Y, y_0)$ is defined as follows: as a set the space is equal to $Map(X, Y)$ of all mappings from $X$ to $Y$; for $f \in Map(X, Y, y_0)$ and $C \in Cl(X)$, the cutting-off operation is defined by

$$(f \land C)(x) = \begin{cases} f(x) & \text{if } x \in C \\ y_0 & \text{if } x \notin C. \end{cases}$$

With this operation $Map(X, Y, y_0)$ is an abstract pattern space over $X$ and for $f \in Map(X, Y, y_0)$ its support is $\text{supp } f = \{ x \in X \mid f(x) \neq y_0 \}$.

Example 2.11 (The space of measures). Let $C_c(X)$ be the space of all continuous and complex-valued functions on $X$ which have compact supports. Its dual space $C_c(X)'$ with respect to the inductive limit topology consists of Radon charges, that is, the maps $\Phi: C_c(X) \rightarrow \mathbb{C}$ such that there is a unique positive Borel measure $m$ and a Borel measurable map $u: X \rightarrow \mathbb{T}$ such that

$$\Phi(\varphi) = \int_X \varphi u dm$$

for all $\varphi \in C_c(X)$. For such $\Phi$ and $C \in Cl(X)$ set

$$(\Phi \land C)(\varphi) = \int_C \varphi u dm$$
for each \( \varphi \in C_c(X) \). Then the new functional \( \Phi \wedge C \) is a Radon charge. With this operation \( C_c(X)' \times \text{Cl}(X) \ni (\Phi, C) \mapsto \Phi \wedge C \in C_c(X)' \), the space \( C_c(X)' \) becomes an abstract pattern space over \( X \).

Next we investigate abstract pattern subspaces. The relation between an abstract pattern space and its abstract pattern subspaces is similar to the one between a set with a group action and its invariant subsets.

**Definition 2.12.** Let \( \Pi \) be an abstract pattern space over \( X \). Suppose a non-empty subset \( \Pi' \) of \( \Pi \) satisfies the condition

\[
P \in \Pi' \text{ and } C \in \text{Cl}(X) \Rightarrow P \wedge C \in \Pi'.
\]

Then \( \Pi' \) is called an **abstract pattern subspace** of \( \Pi \).

**Remark 2.13.** If \( \Pi' \) is an abstract pattern subspace of an abstract pattern space \( \Pi \), then \( \Pi' \) is a abstract pattern space by restricting the cutting-off operation.

**Example 2.14.** \( \text{Cl}(X) \) is an abstract pattern subspace of \( 2^X \). \( \text{LF}(X) \) is an abstract pattern subspace of \( \text{Cl}(X) \) and \( \text{UD}_r(X) \) is an abstract pattern subspace of \( \text{UD}(X) \) for each \( r > 0 \). Since we assume the metrics we consider are proper, \( \text{UD}(X) \) is an abstract pattern subspace of \( \text{LF}(X) \).

Next we deal with a way to construct new abstract pattern space from old ones, namely, taking product.

**Lemma 2.15.** Let \( \Lambda \) be an index set and \( \Pi_\lambda, \lambda \in \Lambda \), is a family of abstract pattern spaces over \( X \). The direct product \( \prod \Pi_\lambda \) becomes an abstract pattern space over \( X \) by a cutting-off operation

\[
(\mathcal{P}_\lambda)_{\lambda \in \Lambda} \wedge C = (\mathcal{P}_\lambda \wedge C)_{\lambda \in \Lambda},
\]

for \( (\mathcal{P}_\lambda)_\lambda \in \prod \Pi_\lambda \) and \( C \in \text{Cl}(X) \). The support is given by \( \text{supp}(\mathcal{P}_\lambda)_\lambda = \bigcup_{\lambda} \text{supp} \mathcal{P}_\lambda \).

**Definition 2.16.** Under the same condition as in Lemma 2.15, we call \( \prod \Pi_\lambda \) the product abstract pattern space of \( (\Pi_\lambda)_\lambda \).

By taking product, we can construct the space of uniformly discrete multi sets, as follows.

**Example 2.17** (uniformly discrete multi set, [7]). Let \( I \) be a set and \( r > 0 \). Consider a space \( \text{UD}_r^I(X) \), defined via

\[
\text{UD}_r^I(X) = \{(D_i)_{i \in I} \in \prod_{i \in I} \text{UD}_r(X) \mid \bigcup_i D_i \in \text{UD}_r(X)\}.
\]

Elements of \( \text{UD}_r^I(X) \) are called \( r \)-**uniformly discrete multi sets**. Elements of \( \text{UD}^I(X) = \bigcup_{r > 0} \text{UD}_r^I(X) \) are called **uniformly discrete multi sets**. A uniformly discrete multi set \( (D_i)_i \in \text{UD}^I(X) \) is called a **Delone multi set** if each \( D_i \) is a Delone set and the union \( \bigcup_i D_i \) is a Delone set.
A cutting-off operation on \( \text{UD}^I(X) \) is defined by regarding it as a subspace of the product space \( \prod_{i \in I} \text{UD}(X) \), that is, via an equation
\[
(D_i)_{i \in I} \wedge C = (D_i \cap C)_{i \in I}.
\]
By this operation \( \text{UD}^I(X) \) and \( \text{UD}^I_r(X) \) are abstract pattern spaces over \( X \).

2.2. An order on abstract pattern spaces. Here we discuss an order on abstract pattern spaces. All the proofs are found in [11].

**Definition 2.18.** Let \( \Pi \) be an abstract pattern space over \( X \). We define a relation \( \geq \) on \( \Pi \) as follows: for each \( P, Q \in \Pi \), we set
\[
P \geq Q \text{ if } P \wedge \text{supp } Q = Q.
\]

**Lemma 2.19.**
(1) If \( P \geq Q \), then \( \text{supp } P \supset \text{supp } Q \).
(2) The relation \( \geq \) is an order on \( \Pi \).

**Lemma 2.20.**
(1) If \( P \in \Pi \) and \( C \in \text{Cl}(X) \), then \( P \geq P \wedge C \).
(2) If \( P, Q \in \Pi \), \( C \in \text{Cl}(X) \) and \( P \geq Q \), then \( P \wedge C \geq Q \wedge C \).

The supremum of \( \Xi \subset \Pi \) with respect to this order \( \geq \) describes “the union” of \( \Xi \) that is obtained by “gluing” elements of \( \Xi \). We will discuss this gluing operation in the next subsection.

**Definition 2.21.** Let \( \Xi \) be a subset of an abstract pattern space \( \Pi \). If the supremum of \( \Xi \) with respect to the order \( \geq \) defined in Definition 2.18 exists in \( \Pi \), it is denoted by \( \bigvee \Xi \).

The following lemma describes a relation between \( \bigvee \) and supports.

**Lemma 2.22.** If a subset \( \Xi \subset \Pi \) admits the supremum \( \bigvee \Xi \), then \( \text{supp } \bigvee \Xi = \bigcup_{P \in \Xi} \text{supp } P \).

**Remark 2.23.** It is not necessarily true that any element \( P_0 \) in \( \Pi \) that majorizes \( \Xi \) and \( \text{supp } P_0 = \bigcup_{P \in \Xi} \text{supp } P \) is the supremum of \( \Xi \) ([11]).

2.3. Glueable abstract pattern spaces. In this subsection \( \Pi \) is an abstract pattern space over \( X \).

Often we want to “glue” abstract patterns to obtain a larger abstract pattern. For example, suppose \( \Xi \) is a collection of patches such that if \( P, Q \in \Xi \), \( S \in P \) and \( T \in Q \), then we have either \( S = T \) or \( S \cap T = \emptyset \). Then we can “glue” patches in \( \Xi \), that is, we can take the union \( \bigcup_{P \in \Xi} P \), which is also a patch. Abstract pattern spaces in which we can often “glue” abstract patterns are called glueable abstract pattern spaces (Definition 2.27). This gluing operation is essential when we construct the limit of a Cauchy sequence (Theorem 3.19).

We first introduce notions that are used to define “glueable” abstract pattern spaces, where we can often “glue” abstract patterns.

**Definition 2.24.**
(1) Two abstract patterns \( P, Q \in \Pi \) are said to be compatible if there is \( R \in \Pi \) such that \( R \geq P \) and \( R \geq Q \).
(2) A subset $\Xi \subset \Pi$ is said to be pairwise compatible if any two elements $P, Q \in \Pi$ are compatible.

(3) A subset $\Xi \subset \Pi$ is said to be locally finite if for any $x \in X$ and $r > 0$, the set $\Xi \land B(x,r)$, which is defined via

$$\Xi \land B(x,r) = \{P \land B(x,r) | P \in \Xi\},$$

is finite.

Remark 2.25. We will see for many examples of abstract pattern spaces $\Pi$, a locally finite and pairwise compatible $\Xi \subset \Pi$ admits the supremum. We have to assume being pairwise compatible because if $\Xi \subset \Pi$ admits the supremum, any $P, Q \in \Xi$ are compatible (we can use the supremum for the role of $\mathcal{R}$ above). We have to assume local finiteness because without this there is a counterexample that do not admits the supremum (see [11]).

We use the following lemma to define glueable abstract pattern spaces.

Lemma 2.26. Let $\Xi$ be a subset of $\Pi$ and take $C \in \text{Cl}(X)$. Then the following hold.

1. If $\Xi$ is locally finite, then so is $\Xi \land C$, which is defined via

$$\Xi \land C = \{P \land C | P \in \Xi\}.$$

2. If $\Xi$ is pairwise compatible, then so is $\Xi \land C$.

Definition 2.27. $\Pi$ is said to be glueable if the following two conditions hold:

1. If $\Xi \subset \Pi$ is both locally finite and pairwise compatible, then there is the supremum $\bigvee \Xi$ for $\Xi$.

2. If $\Xi \subset \Pi$ is both locally finite and pairwise compatible, then for any $C \in \text{Cl}(X)$,

$$\bigvee (\Xi \land C) = (\bigvee \Xi) \land C.$$

Remark 2.28. By Lemma 2.26, for $\Xi \subset \Pi$ which is locally finite and pairwise compatible and $C \in \text{Cl}(X)$ the left-hand side of the equation (2) makes sense.

We finish this subsection with examples. The details are found in [11].

Example 2.29. Consider $\Pi = \text{Patch}(X)$ (Example 2.4). In this abstract pattern space, for two elements $P, Q \in \text{Patch}(X)$, the following statements hold:

1. $P \supseteq Q \iff P \supset Q$.

2. $P$ and $Q$ are compatible if and only if for any $T \in P$ and $S \in Q$, either $S = T$ or $S \land T = \emptyset$ holds.

If $\Xi \subset \text{Patch}(X)$ is pairwise compatible, then $\mathcal{P}_\Xi = \bigcup_{P \in \Xi} P$ is a patch, which is the supremum of $\Xi$. If $C \in \text{Cl}(X)$, then

$$\bigvee \Xi \land C = \bigvee_{P \in \Xi} \bigvee \Xi \land C = \bigvee \Xi \land C = \bigvee \Xi \land C.$$ 

Patch$(X)$ is glueable.
Example 2.30. For the abstract pattern space \(2^X\) in Example 2.49 two elements \(A, B \in 2^X\) are compatible if and only if 
\[
\overline{A} \cap B \subset A \quad \text{and} \quad A \cap \overline{B} \subset B.
\]

Suppose \(\Xi \subset 2^X\) is locally finite and pairwise compatible. Note that \(\bigcup_{A \in \Xi} \overline{A} = \bigcup_{A \in \Xi} A\). Set \(A_\Xi = \bigcup_{A \in \Xi} A\). For each \(A \in \Xi\), \(A_\Xi \cap \overline{A} = \bigcup_{B \in \Xi} (B \cap \overline{A}) = A\); \(A_\Xi\) is a majorant of \(\Xi\). If \(B\) is also a majorant for \(\Xi\), then 
\[
B \cap \overline{A_\Xi} = B \cap \left( \bigcup_{A \in \Xi} \overline{A} \right) = \bigcup_{A \in \Xi} (B \cap \overline{A}) = \bigcup_{A \in \Xi} A = A_\Xi,
\]
and so \(B \geq A_\Xi\). It turns out that \(A_\Xi\) is the supremum for \(\Xi\). Moreover, if \(C \in \text{Cl}(X)\), then \(A_\Xi \cap C = \bigcup_{A \in \Xi} (A \cap C) = \bigvee(\Xi \cap C)\). Thus \(2^X\) is a glueable space.

Remark 2.31. Let \(\Pi_0\) be a glueable abstract pattern space and \(\Pi_1 \subset \Pi_0\) an abstract pattern subspace. For any subset \(\Xi \subset \Pi_1\), if it is pairwise compatible in \(\Pi_1\), then it is pairwise compatible in \(\Pi_0\). Moreover, whether a set is locally finite or not is independent of the ambient abstract pattern space in which the set is included. For a subset \(\Xi \subset \Pi_1\) which is locally finite and pairwise compatible in \(\Pi_1\), since \(\Pi_0\) is glueable, there is the supremum \(\bigvee \Xi\) in \(\Pi_0\). If this supremum in \(\Pi_0\) is always included in \(\Pi_1\), then \(\Pi_1\) is glueable.

By this remark it is easy to see the abstract pattern spaces \(\text{Cl}(X)\) (Example 2.9), \(\text{LF}(X)\) (Example 2.7), and \(\text{UD}_r(X)\) (Example 2.8) \(r\) is an arbitrary positive number) are glueable.

However, \(\text{UD}(X)\) (Example 2.8) is not necessarily glueable. For example, set \(X = \mathbb{R}\). Set \(\mathcal{P}_n = \{n, n + \frac{1}{n}\}\) for each integer \(n \neq 0\). Each \(\mathcal{P}_n\) is in \(\text{UD}(\mathbb{R})\), \(\Xi = \{\mathcal{P}_n \mid n \neq 0\}\) is locally finite and pairwise compatible, but it does not admit the supremum.

Lemma 2.32. Let \(I\) be a non-empty set and \(r\) be a positive real number. The abstract pattern space \(\text{UD}_r^I(X)\) (Example 2.17) is glueable.

Proof. Let \(p_i : \text{UD}_r^I(X) \to \text{UD}_r(X)\) be the projection to \(i\)-th element. For two \(D, E \in \text{UD}_r^I(X)\), if they are compatible, there is \(F \in \text{UD}_r^I(X)\) such that \(F \geq D\) and \(F \geq E\). By the definition of cutting-off operation, we have \(p_i(F) \cap \text{supp } D = p_i(D)\) and \(p_j(F) \cap \text{supp } E = p_j(E)\) for each \(i, j \in I\), which means if \(x \in p_i(D)\) and \(y \in p_j(E)\), then we have either \(x = y\) or \(\rho_X(x, y) \geq r\). We also have \(p_i(D) \cap \text{supp } E \subset p_i(F) \cap \text{supp } E = p_i(E)\) for each \(i\).

Let \(\Xi\) be a pairwise compatible subset of \(\text{UD}_r^I(X)\). For each \(i\) set \(D_i = \bigcup_{E \in \Xi} p_i(E)\). By the above observation, the tuple \(D = (D_i)_{i \in I}\) is an element of \(\text{UD}_r^I(X)\). The above observation also implies that for each \(E \in \Xi\), we have \(D_i \cap \text{supp } E = p_i(E)\), which means \(D \geq E\). We see \(D\) is a majorant of \(\Xi\), and since \(\text{supp } D = \bigcup_{E \in \Xi} \text{supp } E\), we see \(D\) is the supremum. \(\square\)

We mention the following proposition without proving it.

Proposition 2.33. If \(Y\) is a non-empty set and \(y_0 \in Y\), the abstract pattern space \(\text{Map}(X, Y, y_0)\) (Example 2.10) is glueable.
2.4. **Zero Element and Its Uniqueness.** In this subsection we discuss zero elements, which is on the list of structures at the beginning of this section.

**Definition 2.34.** Let $\Pi$ be an abstract pattern space over $X$. An element $P \in \Pi$ such that $\text{supp}(P) = \emptyset$ is called a zero element of $\Pi$. If there is only one zero element in $\Pi$, it is denoted by $0$.

**Remark 2.35.** Zero elements always exist. In fact, take an arbitrary element $P \in \Pi$. Then by Lemma 2.23 $\text{supp}(P \land \emptyset) = \emptyset$ and so $P \land \emptyset$ is a zero element.

**Lemma 2.36.** If $\Pi$ is a glueable abstract pattern space over $X$, there is only one zero element in $\Pi$.

*Proof.* The subset $\emptyset$ of $\Pi$ is locally finite and pairwise compatible. Set $P = \bigvee \emptyset$. By Lemma 2.22 $P$ is a zero element. If $Q$ is a zero element, then since $Q$ is a majorant for $\emptyset$, we see $Q \geq P$. We have $Q = Q \land \emptyset = P$. □

**Lemma 2.37.** Let $\Pi$ be a glueable abstract pattern space over $X$. Take a locally finite and pairwise compatible subset $\Xi$ of $\Pi$. Then $\bigvee \Xi \cup \{0\}$ exists and $\bigvee \Xi \cup \{0\} = \bigvee \Xi$.

2.5. **$\Gamma$-abstract pattern spaces over $X$, or abstract pattern spaces over $(X, \Gamma)$**.

Here we incorporate group actions to the theory of abstract pattern spaces. First we define abstract pattern spaces over $(X, \Gamma)$, or $\Gamma$-abstract pattern spaces over $X$. We require there is an action of the group $\Gamma$ on such an abstract pattern space and the cutting-off operation is equivariant.

In this subsection, $\Pi$ is an abstract pattern space over $X$.

**Definition 2.38.** Suppose there is a group action $\Gamma \curvearrowright \Pi$ such that for each $P \in \Pi, C \in \text{Cl}(X)$ and $\gamma \in \Gamma$, we have $(\gamma P) \land (\gamma C) = \gamma(P \land C)$, that is, the cutting-off operation is equivariant. Then we say $\Pi$ is a $\Gamma$-abstract pattern space over $X$ or a abstract pattern space over $(X, \Gamma)$. For an abstract pattern space $\Pi$ over $(X, \Gamma)$, its nonempty subset $\Sigma$ such that $P \in \Sigma$ and $\gamma \in \Gamma$ imply $\gamma P \in \Sigma$ is called a subshift of $\Pi$.

We first investigate the relation between the group action and the two construction of abstract pattern spaces, taking subspace and taking product.

**Lemma 2.39.** Let $\Pi$ be an abstract pattern space over $(X, \Gamma)$. Suppose $\Pi'$ is an abstract pattern subspace of $\Pi$. If $\Pi'$ is closed under the $\Gamma$-action, then $\Pi'$ is an abstract pattern space over $(X, \Gamma)$.

**Lemma 2.40.** Let $\Lambda$ be a set and $(\Pi_{\lambda})_{\lambda \in \Lambda}$ be a family of abstract pattern spaces over $(X, \Gamma)$. Then $\Gamma$ acts on the product space $\prod_{\lambda} \Pi_{\lambda}$ by $\gamma(P_{\lambda})_{\lambda} = (\gamma P_{\lambda})_{\lambda}$ and by this action $\prod_{\lambda} \Pi_{\lambda}$ is an abstract pattern space over $(X, \Gamma)$.

**Definition 2.41.** The abstract pattern space $\prod \Pi_{\lambda}$ is called the product $\Gamma$-abstract pattern space.

We now list several examples of $\Gamma$-abstract pattern spaces.
Example 2.42. For $\mathcal{P} \in \text{Patch}(X)$ and $\gamma \in \Gamma$, set $\gamma \mathcal{P} = \{ \gamma T \mid T \in \mathcal{P} \}$. This defines an action of $\Gamma$ on $\text{Patch}(X)$ and makes $\text{Patch}(X)$ an abstract pattern space over $(X, \Gamma)$.

For an $L$-labeled tile $(T, l)$ and $\gamma \in \Gamma$, set $\gamma(T, l) = (\gamma T, l)$. This defines an action of $\Gamma$ on $\text{Patch}_L(X)$ (Example 2.6), which makes $\text{Patch}_L(X)$ an abstract pattern space.

Example 2.43. $2^X$ (Example 2.9) is an abstract pattern space over $(X, \Gamma)$, by the action $\Gamma \rhd 2^X$ inherited from the action $\Gamma \rhd X$. By Lemma 2.39, the spaces $\text{LF}(X)$ (Example 2.7), $\text{Cl}(X)$ (Example 2.9), $\text{UD}(X)$ and $\text{UD}_r(X)$ (Example 2.8, $r > 0$) are all abstract pattern spaces over $(X, \Gamma)$.

The abstract pattern space $\text{UD}^I(X)$ (Example 2.17) is also an $\Gamma$-abstract pattern space.

Example 2.44. Take a non-empty set $Y$, an element $y_0 \in Y$ and an action $\phi : \Gamma \rhd Y$ that fixes $y_0$. As was mentioned before (Example 2.10), $\text{Map}(X, Y, y_0)$ is an abstract pattern space over $X$. Define an action of $\Gamma$ on $\text{Map}(X, Y, y_0)$ by

$$ (\gamma f)(x) = \phi(\gamma)(f(\gamma^{-1}x)). $$

By this group action $\text{Map}(X, Y, y_0)$ is $\Gamma$-abstract pattern space. This $\Gamma$-abstract pattern space is denoted by $\text{Map}^\phi(X, Y, y_0)$. If $\phi$ sends every group element to the identity, we denote the corresponding space by $\text{Map}(X, Y, y_0)$. This group action is essential when we study pattern-equivariant functions, since if $P$ is an abstract pattern, a function $f \in \text{Map}(X, Y, y_0)$ is $P$-equivariant if and only if $f$ is locally derivable from $P$ (11).

Example 2.45. The dual space $C_c(X)'$ is an abstract pattern space over $X$ (Example 2.11). For $\varphi \in C_c(X)'$ and $\gamma \in \Gamma$, set $(\gamma \varphi)(x) = \varphi(\gamma^{-1}x)$. For $\Phi \in C_c(X)'$ and $\gamma \in \Gamma$, set $\gamma \Phi(\varphi) = \Phi(\gamma^{-1} \varphi)$. Then $C_c(X)'$ is an abstract pattern space over $(X, \Gamma)$.

Let $r$ be a positive real number. The space $WDC_r(X)$ of all weighted Dirac combs $\sum_{x \in D} w(x) \delta_x$, where $D$ is an $r$-uniformly discrete subset of $X$, $w : D \to \mathbb{C} \setminus \{0\}$ and $\delta_x$ is the Dirac measure on $x$, is an abstract pattern space over $(X, \Gamma)$, which is a subshift of $C_c(X)'$.

We mention two examples of subshifts.

Example 2.46. The set $\text{Del}(X)$ of all Delone sets in $X$ is a subshift of $\text{UD}(X)$ (Example 2.8).

Example 2.47. The space of all tilings is a subshift of $\text{Patch}(X)$ (Example 2.4).

In the previous subsection, we defined glueable abstract pattern spaces. Here, we define corresponding notions for $\Gamma$-abstract pattern spaces and subshifts.

Definition 2.48. Assume $\Pi$ is an abstract pattern space over $(X, \Gamma)$. We say $\Pi$ is a glueable abstract pattern space over $(X, \Gamma)$ if it is a glueable abstract pattern space over $X$. For a glueable abstract pattern space $\Pi$, its subshift $\Sigma$ is said to be supremum-closed if for any pairwise compatible and locally finite $\Xi \subset \Sigma$, we have $\bigvee \Xi \in \Sigma$.

Remark 2.49. In the definition of supremum-closed subshifts, the supremum $\bigvee \Xi$ exists by the definition of glueable abstract pattern space.
3. The definition and properties of local matching topology

**Notation 3.1.** Let $\text{Cpt}(X)$ be the set of all compact subsets of $X$ and $\mathcal{V}$ the set of all compact neighborhoods of $e \in \Gamma$ (the identity element of $\Gamma$).

In this section $\Pi$ is an abstract pattern space over $(X, \Gamma)$.

In this section we define and investigate the local matching topology on $\Pi$. We use the theory of uniform structure to define them. (For the theory of uniform structures, see [5].) The uniform structure will be metrizable (Corollary 3.8), but the description of a metric is not simple when $\Gamma$ is non-commutative, and this is why we prefer uniform structures. With respect to this uniform structure, two abstract patterns $\mathcal{P}$ and $\mathcal{Q}$ in $\Pi$ are “close” when they match in a “large region” after sliding $\mathcal{Q}$ by a “small” $\gamma \in \Gamma$. This is analogous to the product topology of the space $A^Z$, where $A$ is a finite set; in fact we can show that the relative topology of the local matching topology on a space of maps $\text{Map}(Z, A \cup \{\ast\})$ ($\ast$ is a point outside $A$) coincides with the product topology.

Here is the plan of this section. In Subsection 3.1 we define the local matching uniform structure and topology. In Subsection 3.2 we give a sufficient condition for the local matching topology on a subshift to be Hausdorff, and prove many examples of subshifts satisfies this condition. In Subsection 3.3 we give a mild condition that assures that the local matching uniform structure is complete, and show if the action $\Gamma \rtimes X$ is proper, the usual definition of FLC implies the compactness of the continuous hull (the closure of the orbit).

### 3.1. The definition of local matching topology

We first define the following notation, which will be used to define a uniform structure.

**Definition 3.2.** For $K \in \text{Cpt}(X)$ and $V \in \mathcal{V}$, set

$$U_{K,V} = \{(P, Q) \in \Pi \times \Pi \mid \text{there is } \gamma \in V \text{ such that } P \wedge K = (\gamma Q) \wedge K\}.$$

We first remark the following lemma, the proofs of which are easy.

**Lemma 3.3.** If $K_1 \subset K_2$ and $V_2 \subset V_1$, then $U_{K_2,V_2} \subset U_{K_1,V_1}$.

We define a uniform structure by constructing a filter basis on $\Pi \times \Pi$ that satisfies the axiom of fundamental system of entourages ([5] II.2,§1.1]).

**Lemma 3.4.** The set

$$\{U_{K,V} \mid K \in \text{Cpt}(X), V \in \mathcal{V}\}$$

satisfies the axiom of fundamental system of entourages.

**Proof.** (1) For any $K \in \text{Cpt}(X)$, $V \in \mathcal{V}$, we show

$$\{(P, P) \mid P \in \Pi\} \subset U_{K,V}.$$

This is clear since if $P \in \Pi$, we have $P \wedge K = P \wedge K$.

(2) For any $K$ and $V$, we show there are $K'$ and $V'$ such that

$$U_{K',V'} \subset U_{K,V}^{-1} = \{(Q, P) \mid (P, Q) \in U_{K,V}\}.$$
Take \((P, Q) \in U_{V^{-1}K,V^{-1}}\). There is \(\gamma \in V\) such that
\[
P \land V^{-1}K = (\gamma^{-1}Q) \land V^{-1}K.
\]
Multiplying by \(\gamma\) both sides we have
\[
(\gamma P) \land \gamma V^{-1}K = Q \land \gamma V^{-1}K,
\]
and so
\[
(\gamma P) \land K = (\gamma P) \land \gamma V^{-1}K \land K
= Q \land \gamma V^{-1}K \land K
= Q \land K.
\]
We have \((Q, P) \in U_{K,V}\) and so \(U_{V^{-1}K,V^{-1}} \subset U_{K,V}\).

(3) For each \(K_1, K_2 \in \text{Cpt}(X)\) and \(V_1, V_2 \in \mathcal{V}\), we show there are \(K_3 \in \text{Cpt}(X)\) and \(V_3 \in \mathcal{V}\) such that
\[
U_{K_3,V_3} \subset U_{K_1,V_1} \cap U_{K_2,V_2}.
\]
By Lemma 3.3 for \(K_1, K_2 \in \text{Cpt}(X)\) and \(V_1, V_2 \in \mathcal{V}\), we have
\[
U_{K_1 \cup K_2,V_1 \cap V_2} \subset U_{K_1,V_1} \cap U_{K_2,V_2}.
\]

(4) Take \(K \in \text{Cpt}(X)\) and \(V \in \mathcal{V}\) arbitrarily. We show there are \(K'\) and \(V'\) such that
\[
U_{K',V'}^2 = \{(P, R) \mid \text{there is } Q \text{ with } (P, Q), (Q, R) \in U_{K',V'}\} \subset U_{K,V}.
\]
Set \(K_1 = (V^{-1}K) \cup K\) and take \(V_1 \in \mathcal{V}\) such that \(V_1 \cup V \subset V\). Note that \(V_1 \subset V\). If \((P_1, P_2), (P_2, P_3) \in U_{K_1,V_1}\), then there are \(\gamma_1\) and \(\gamma_2\) in \(V_1\) such that \(P_1 \land K_1 = (\gamma_1 P_2) \land K_1\) and \(P_2 \land K_1 = (\gamma_2 P_3) \land K_1\). We have
\[
(\gamma_1 \gamma_2 P_3) \land K = \gamma_1 ((\gamma_2 P_3) \land K_1) \land K
= \gamma_1 (P_2 \land K_1) \land K
= (\gamma_1 P_2) \land K
= ((\gamma_1 P_2) \land K_1) \land K
= (P_1 \land K_1) \land K
= P_1 \land K.
\]
Thus \((P_1, P_3) \in U_{K,V}\). We have proved \(U_{K_1,V_1}^2 \subset U_{K,V}\). \(\Box\)

**Definition 3.5.** Let \(\mathcal{U}\) be the set of all entourages generated by (3). That is, \(\mathcal{U}\) is the set of all subsets \(U\) of \(\Pi \times \Pi\) such that there are \(K \in \text{Cpt}(X)\) and \(V \in \mathcal{V}\) with \(U \supset U_{K,V}\).
The uniform structure defined by \(\mathcal{U}\) is called the local matching uniform structure. The topology defined by this uniform structure, that is, the topology in which the set
\[
\{U_{K,V}(P) \mid K \in \text{Cpt}(X), V \in \mathcal{V}\}
\]
is a fundamental system of neighborhoods for $\mathcal{P} \in \Pi$, is called the local matching topology. Here,

$$U_{K,V}(\mathcal{P}) = \{ \mathcal{Q} \in \Pi \mid (\mathcal{P}, \mathcal{Q}) \in U_{K,V} \}.$$  

3.2. **Hausdorff property of local matching topology.** Next we give a sufficient condition for the local matching topology to be Hausdorff in Proposition 3.7.

**Definition 3.6.** Suppose $\Pi$ admits a unique zero element $0$. An abstract pattern $\mathcal{P} \in \Pi$ is called an atom if $\text{supp} \mathcal{P}$ is compact and $\mathcal{Q} \in \Pi$ and $\mathcal{Q} \subseteq \mathcal{P} \Rightarrow \mathcal{P} = \mathcal{Q} \text{ or } \mathcal{Q} = 0$.

For $\mathcal{P} \in \Pi$ set

$$A(\mathcal{P}) = \{ \mathcal{Q} : \text{atom } | \mathcal{Q} \subseteq \mathcal{P} \}.$$  

A subset $\Sigma \subset \Pi$ is said to be atomistic if for any $\mathcal{P} \in \Pi$ we have $\mathcal{P} = \bigvee A(\mathcal{P})$.

A subset $\Sigma \subset \Pi$ is said to have limit inclusion property if the following condition is satisfied:

- for any $\mathcal{P} \in \Sigma$ and an atom $\mathcal{Q} \in \Pi$, if for any $V \in \mathcal{V}$ there is $\gamma_V \in V$ such that $\gamma_V \mathcal{Q} \subseteq \mathcal{P}$, we have $\mathcal{Q} \subseteq \mathcal{P}$.

**Proposition 3.7.** Suppose $\Pi$ admits a unique zero element $0$. Let $\Sigma$ be a nonempty subset of $\Pi$ which is atomistic and has limit inclusion property. Then the local matching topology on $\Sigma$ is Hausdorff.

**Proof.** Take $\mathcal{P}, \mathcal{Q} \in \Sigma$ and suppose $(\mathcal{P}, \mathcal{Q}) \in U_{K,V}$ for any $K \in \text{Cpt}(X)$ and $V \in \mathcal{V}$. We show $\mathcal{P} = \mathcal{Q}$. Take $R \in A(\mathcal{P})$. Set $K = \text{supp} R$. For any $V \in \mathcal{V}$ there is $\gamma_V \in V$ such that $\gamma_V^{-1} \mathcal{Q} \subseteq \mathcal{P}$.

This implies that

$$\gamma_V \mathcal{R} \subseteq \mathcal{Q},$$

and so by limit inclusion property, we have

$$\mathcal{R} \subseteq \mathcal{Q}.$$  

Since $\Sigma$ is atomistic, we have $\mathcal{P} \subseteq \mathcal{Q}$. The converse is proved in the same way and we have $\mathcal{Q} \subseteq \mathcal{P}$, and so $\mathcal{P} = \mathcal{Q}$. □

**Corollary 3.8.** Under the same assumption on $\Pi$ and $\Sigma$, the local matching topology on $\Sigma$ is metrizable.

**Proof.** This follows from Proposition 3.7, [6, IX, §2] and the fact that $\{U_{K_n,V_n} \mid n = 1,2,\ldots \}$, where $K_n = B(x_0, R_n), V_n = B(e, r_n), R_n \not\to \infty$ and $r_n \not\to 0$, forms a fundamental system of entourages. □

We then give examples of sets of abstract patterns on which the local matching topology is Hausdorff, by checking the conditions in Proposition 3.7.
Lemma 3.9. Let $Y$ be a non-empty topological space and $y_0$ be an element of $Y$. Take a group homomorphism $\phi : \Gamma \to \text{Homeo}(Y)$ which is continuous with respect to the compact-open topology and such that $\phi(\gamma)y_0 = y_0$ for each $\gamma \in \Gamma$. Then $C_b(X,Y,y_0) = \{ f \in \text{Map}_\phi(X,Y,y_0) \mid \text{continuous and bounded} \}$ is atomistic and has limit inclusion property as a subset of the abstract pattern space $\text{Map}_\phi(X,Y,y_0)$ (Definition 2.44).

Proof. For each $x \in X$ and $y \in Y \setminus \{y_0\}$, the function defined by

$$
\varphi^y_x(x') = \begin{cases} 
y & \text{if } x = x' \\
y_0 & \text{if } x \neq x'
\end{cases}
$$

is an atom of $\text{Map}_\phi(X,Y,y_0)$. Any atom of $\text{Map}_\phi(X,Y,y_0)$ is of this form. For $f \in C_b(X,Y,y_0)$, we have

$$
A(f) = \{ \varphi^f_x \mid x \in X \text{ and } f(x) \neq y_0 \}.
$$

We see $f = \bigvee A(f)$. We have proved that $C_b(X,Y,y_0)$ is atomistic.

Next we show $C_b(X,Y,y_0)$ has limit inclusion property. Take any $x \in X$ and $y \in Y \setminus \{y_0\}$, and assume that for any $V \in \mathcal{V}$ there is $\gamma_V \in \mathcal{V}$ such that $\gamma_V \varphi^y_x \leq f$. Since $\text{supp} \gamma_V \varphi^y_x = \{ \gamma_V x \}$, we have

$$
f(\gamma_V x) = (\gamma_V \varphi^y_x)(\gamma_V x) = \phi(\gamma_V)(\varphi^y_x(x)) = \phi(\gamma_V)(y).
$$

Since $f$ is continuous and the action $\Gamma \curvearrowright X$ is continuous,

$$
f(x) = \lim_V f(\gamma_V x) = \lim_V \phi(\gamma_V)(y) = y,
$$

and so $f \geq \varphi^y_x$. We have shown $C_b(X,Y,y_0)$ has limit inclusion property. \hfill \Box

Corollary 3.10. The relative topology of the local matching topology on $C_b(X,Y,y_0)$ is Hausdorff.

Proof. Clear by Proposition 3.7 and Lemma 3.9 \hfill \Box

As the following lemma shows, the local matching topology on $\text{Map}_\phi(X,Y,y_0)$ is not necessarily Hausdorff:

Lemma 3.11. On $\text{Map}(\mathbb{R}, \mathbb{C}, 0)$, the local matching topology is not Hausdorff.

Proof. Take characteristic functions $f = 1_Q$ and $g = 1_{Q+a}$, where $a$ is any irrational number. Then $(f, g)$ belongs to any entourage. \hfill \Box

We then prove on Patch$(X)$ (Example 2.4, Example 2.42), the local matching topology is Hausdorff.

Lemma 3.12. The abstract pattern space Patch$(X)$ over $(X, \Gamma)$ is atomistic and has limit inclusion property.

Proof. Let $T$ be a tile. Then $\{T\}$ is an atom. Any atom in Patch$(X)$ is of this form. For any patch $P \in \text{Patch}(X)$, we have

$$
A(P) = \{ \{T\} \mid T \in P \},
$$
and so \( \mathcal{P} = \bigcup A(\mathcal{P}) = \bigvee A(\mathcal{P}) \). We have shown that \( \text{Patch}(X) \) is atomistic.

To prove \( \text{Patch}(X) \) satisfies limit inclusion property, take \( \mathcal{P} \in \text{Patch}(X) \) and a tile \( T \), and assume for any \( V \in \mathcal{V} \) there is \( \gamma_V \in V \) such that \( \gamma_V \{ T \} \subseteq \mathcal{P} \), that is, \( \gamma_V T \in \mathcal{P} \). We show \( T \in \mathcal{P} \). There is \( V_0 \in \mathcal{V} \) such that if \( V_1, V_2 \in \mathcal{V} \) and \( V_j \subseteq V_0 \) for each \( j \), then \( \gamma_{V_1} T \cap \gamma_{V_2} T \neq \emptyset \). Since \( \gamma_{V_j} T \) is in a patch \( \mathcal{P} \) for each \( j \), we see \( \gamma_{V_1} T = \gamma_{V_2} T \). Now it suffices to show that \( T = \gamma_{V_0} T \) since \( \gamma_{V_0} T \in \mathcal{P} \). If \( x \in T \), then if \( V_1 \in \mathcal{V} \) is small enough we have \( V_1 \subseteq V_0 \) and \( \gamma_{V_1}^{-1} x \in T \). Since \( \gamma_{V_1} T = \gamma_{V_0} T \), we see \( x \in \gamma_{V_0} T \). Conversely, if \( x \in \gamma_{V_0} T \), then if \( V_1 \in \mathcal{V} \) is small enough \( \gamma_{V_1} x \in \gamma_{V_0} T = \gamma_{V_1} T \), and so \( x \in T \). We have shown \( T = \gamma_{V_0} T \).

**Corollary 3.13.** The local matching topology on \( \text{Patch}(X) \) is Hausdorff.

**Proof.** Clear by Proposition 3.7 and Lemma 3.12.

By a similar argument we obtain the following:

**Lemma 3.14.** The abstract pattern space \( \text{Patch}_L(X) \) (Example 2.42), where \( L \) is a non-empty set, is atomistic and has limit inclusion property.

**Corollary 3.15.** The local matching topology on \( \text{Patch}_L(X) \) is Hausdorff.

Finally we directly prove on the following product abstract pattern space, the local matching topology is Hausdorff.

**Lemma 3.16.** On the product \( \prod_{i \in I} \text{Cl}(X) \), where \( I \) is a non-empty set, \( \text{Cl}(X) \) is given in Example 2.43, and the structure of \( \Gamma \)-abstract pattern space is given by Lemma 2.17 and Lemma 2.40, the local matching topology is Hausdorff.

**Proof.** Take two elements \( D = (D_i)_{i \in I} \) and \( E = (E_i)_{i \in I} \) from \( \prod_{i \in I} \text{Cl}(X) \) and assume \( (D, E) \in \mathcal{U}_{K, V} \) for any \( K \in \text{Cpt}(X) \) and \( V \in \mathcal{V} \). We will show \( D = E \).

For each \( i \in I \), \( x \in D_i \) and \( V \in \mathcal{V} \), there is \( \gamma_V \in V \) such that \( D \cap \{ x \} = (\gamma_V E) \cap \{ x \} \). This implies \( \{ x \} = D_i \cap \{ x \} = (\gamma_V E_i) \cap \{ x \} \), and so \( \gamma_V^{-1} x \in E_i \). Since the action \( \Gamma \curvearrowright X \) is continuous and \( E_i \) is closed, we see \( x \in E_i \) and \( D_i \subseteq E_i \). By symmetry \( D_i = E_i \). Since \( i \) is arbitrary, we have \( D = E \).

**Corollary 3.17.** The local matching topologies on \( \text{Cl}(X), \text{LF}(X), \text{UD}_r(X), \text{UD}(X) \) and \( \text{UD}^t(X) \) are Hausdorff.

**Proof.** Clear by Lemma 3.16 since these spaces are included in \( \text{Cl}(X) \) or \( \prod_{i \in I} \text{Cl}(X) \).

### 3.3. The completeness of local matching topology, FLC and the compactness of the continuous hull

Next we prove that under a mild condition the local matching uniform structure on a subshift is complete.

In this subsection \( \Pi \) is a glueable abstract pattern space over \((X, \Gamma)\).

**Lemma 3.18.** For each \( n = 1, 2, \ldots \) take \( \gamma_n \in \Gamma \) such that \( \rho_{\Gamma}(e, \gamma_n) < \frac{1}{2^n + 1} \). Then the following hold:

1. \( \rho_{\Gamma}(\gamma_{n} \gamma_{m-1} \cdots \gamma_{m}, e) < \frac{1}{m} \) for each \( n \geq m \geq 1 \).
2. For any \( m \geq 1 \) the sequence \( (\gamma_{n} \gamma_{n-1} \cdots \gamma_{m})_{n \geq m} \) is a Cauchy sequence.
Proof. 1. We have
\[
\rho(\gamma_n \cdots \gamma_m, e) \leq \sum_{k=m}^{n-1} \rho(\gamma_n \cdots \gamma_k, \gamma_n \cdots \gamma_{k+1}) + \rho(\gamma_n, e) \\
= \sum_{k=m}^{n} \rho(e, \gamma_k) \\
< \sum \frac{1}{2^{k+1}} \\
< \frac{1}{2^m}.
\]

2. For any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that if \( \gamma, \eta, \zeta \in B(e, 1) \) and \( \rho(\gamma, \eta) < \delta \), then \( \rho(\gamma\zeta, \eta\zeta) < \varepsilon \). This follows from the fact that \( B(e, 1) \) is compact and so the multiplication \( B(e, 1) \times B(e, 1) \ni (\gamma, \eta) \mapsto \gamma\eta \in \Gamma \) is uniformly continuous. If \( n > k \geq m \) and \( k \) is large enough, by 1.,
\[
\rho(\gamma_n \cdots \gamma_{k+1}, e) < \delta.
\]

By the definition of \( \delta \), we have
\[
\rho(\gamma_n \cdots \gamma_m, \gamma_k \cdots \gamma_m) < \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, we see the sequence is Cauchy. \( \square \)

**Theorem 3.19.** Let \( \Sigma \) be a supremum-closed subshift of \( \Pi \) on which the local matching topology is Hausdorff. Then the local matching uniform structure on \( \Sigma \) is complete.

**Proof.** By [6, IX, §2], the local matching uniform structure on \( \Sigma \) is metrizable (see also Corollary [3.8]). It suffices to show that any Cauchy sequences in \( \Sigma \) converge.

Let \( (\mathcal{P}_n)_n \) be a Cauchy sequence in \( \Sigma \). Set \( K_n = B(x_0, n) \) and \( V_n = B(e, \frac{1}{2^{n+1}}) \subset \Gamma \) for each \( n = 1, 2, \ldots \). Since it suffices to show a subsequence of \( (\mathcal{P}_n) \) converges, we may assume that \( (\mathcal{P}_k, \mathcal{P}_l) \in \mathcal{U}_{K_n, V_n} \) for any \( n > 0 \) and \( k, l \geq n \). For each \( n > 0 \) there is \( \gamma_n \in V_n \) such that
\[
(\gamma_n \mathcal{P}_n) \cap K_n = \mathcal{P}_{n+1} \cap K_n.
\]

By Lemma 3.18, since \( \Gamma \) is complete, there is a limit
\[
\xi_n = \lim_{m \to \infty} \gamma_m \gamma_{m-1} \cdots \gamma_n \in B(e, \frac{1}{2^n})
\]
for each \( n > 0 \). Note that \( \xi_n = \xi_{n+1}\gamma_n \) for each \( n \).

Since the group action is continuous, we can take \( n_0 \in \mathbb{N} \) such that if \( \gamma \in B(e, \frac{1}{2^{n_0}}) \), then \( \gamma x_0 \in B(x_0, 1) \). If \( n, m \geq n_0 \) and \( n < m \), then since
\[
\xi_{m+1}K_m = B(\xi_{m+1}x_0, m) \supset B(x_0, m-1) \supset B(x_0, n) = K_n,
\]

the sequence is Cauchy. \( \square \)
we have

\[(\xi_m P_m) \land K_n = (\xi_{m+1}(\gamma_m P_m) \land K_m) \land K_n = (\xi_{m+1}(P_{m+1} \land K_m)) \land K_n = (\xi_{m+1}P_{m+1}) \land K_n.\]

By induction we have

\[(\xi_m P_m) \land K_n = (\xi_{n+1}P_{n+1}) \land K_n\]

for each \(n, m > n_0\) with \(m > n\). This means that

\[(\xi_{n+1}P_{n+1}) \land K_n \leq (\xi_{n+2}P_{n+2}) \land K_{n+1}\]

for any \(n > n_0\).

Set

\[Q_k = \bigvee \{ (\xi_{n+1}P_{n+1}) \land K_n \mid n > k \}\]

for each \(k > n_0\). We need to show that such a supremum exists. To this objective it suffices to show that \(\Xi_k = \{ (\xi_{n+1}P_{n+1}) \land K_n \mid n > k \}\) is locally finite and pairwise compatible. By (4), we have

\[(\xi_{m+1}P_{m+1}) \land K_m \land K_n = (\xi_{n+1}P_{n+1}) \land K_n\]

for any \(n, m\) with \(k < n < m\), and so \(\Xi_k\) is pairwise compatible. To prove \(\Xi\) is locally finite, take a closed ball \(B\). For any sufficiently large \(n\), we have \(K_n \supset B\), and so if \(m\) is larger than this \(n\) we have by (4)

\[(\xi_{m+1}P_{m+1}) \land K_m \land B = (\xi_{n+1}P_{n+1}) \land K_n \land B,\]

and so \(\Xi \land B\) is finite. Since \(B\) was arbitrary, \(\Xi\) is locally finite. Thus \(Q_k\) is well-defined and is in \(\Sigma\) since \(\Sigma\) is supremum-closed.

By \(\Xi_1 \supset \Xi_k\), we have \(Q_1 \geq Q_k\) for each \(k\). On the other hand, by (5) \(Q_k \geq (\xi_{n+1}P_{n+1}) \land K_n\) for any \(n\) and so \(Q_k \geq Q_1\); we have shown \(Q_1 = Q_k\) for any \(k > 0\).

Finally \(Q_1\) is the limit of \((P_n)\), since for each \(k > n_0\), (4) implies that

\[Q_1 \land K_k = \bigvee \{ (\xi_{n+1}P_{n+1}) \land K_n \mid n > k \} = (\xi_{k+1}P_{k+1}) \land K_k\]

and so \(P_{k+1} \in \mathcal{U}_{K_{k+1}V_{k+1}}(Q_1)\). (Note that \(\mathcal{U}_{K_{k+1}V_{k+1}} \subset \mathcal{U}_{K_{k}V_{k}}\).)

\[\square\]

**Corollary 3.20.** On \(C_b(X, Y)\) (Lemma 3.9), Patch\((X)\), Patch\(_L\)(\(X\)) (Example 2.42), UD\(_I\)\((X)\) (Example 2.44), Cl\((X)\), LF\((X)\), UD\((X)\) (Example 2.46), and WDC\(_r\)(\(X\)) (Example 2.48), the local matching uniform structures are complete.

**Remark 3.21.** This theorem is similar to Proposition 2.1 in [13] and Theorem 3.10 in [15]. The former proves the completeness of discrete and closed subsets of \(\sigma\)-compact locally compact abelian group, which is included in the latter. The latter proves the completeness of the space of discrete subsets of a \(\sigma\)-compact space, which is more general than Theorem
in that the author does not assume the existence of metrics in the space and the group, but less general in that it only deals with discrete sets, rather than any abstract patterns.

Finally we define FLC and the continuous hull for abstract patterns and prove that FLC implies the compactness of the continuous hull, if the action \( \Gamma \acts X \) is proper.

**Definition 3.22.** Take \( P \in \Pi \). The continuous hull \( X_P \) of \( P \) is defined by
\[
X_P = \{ \gamma P | \gamma \in \Gamma \},
\]
where the closure is taken with respect to the local matching topology.

**Definition 3.23.** An abstract pattern \( P \in \Pi \) is said to have finite local complexity (FLC) if whenever we take a compact \( K \subset X \), the set
\[
\{ (\gamma P) \wedge K | \gamma \in \Gamma \}
\]
is finite modulo the group action \( \Gamma \acts \Pi \).

In what follows, \( \Sigma \) is a supremum-closed, atomistic subshift of \( \Pi \) with limit inclusion property.

We will use the following lemma to prove the compactness of a continuous hull.

**Lemma 3.24.** Assume the action \( \Gamma \acts X \) is proper. Take \( P_1, P_2, \ldots \) from \( \Sigma \). Assume \( \{ P_n \wedge K | n = 1,2,\ldots \} \) is finite modulo \( \Gamma \acts \Pi \), for any \( K \in \text{Cpt}(X) \). Then for any compact \( K \subset X \) and \( V \in \mathcal{V} \), there is a subsequence \( (P_{n_j})_{j=1,2,\ldots} \) of \( (P_n)_n \) such that
\[
(P_{n_j}, P_{n_k}) \in U_{K,V}
\]
for any \( j \) and \( k \).

**Proof.** Take \( K \in \text{Cpt}(X) \) and \( V \in \mathcal{V} \), and we will prove we can take a subsequence of \( (P_n)_n \) with the above property.

Take \( R > 0 \) large enough so that \( B(x_0, R) \supset K \) hold.

Set \( K' = B(x_0, R + 1) \). By the second condition in the statement of this lemma,
\[
\{ P_n \wedge K' | n = 1,2,\ldots \}
\]
is finite modulo \( \Gamma \)-action. We can take an increasing map \( \sigma : \mathbb{N} \to \mathbb{N} \) and elements \( \eta_1, \eta_2, \ldots \in \Gamma \) such that
\[
P_{\sigma(1)} \wedge K' = \eta_n(P_{\sigma(n)} \wedge K')
\]
for each \( n = 1,2,\ldots \).

We may assume \( P_{\sigma(1)} \wedge K' \neq 0 \) (0 is the zero element in \( \Pi \), which is unique by Lemma 2.36), since if it is 0, \( P_{\sigma(n)} \wedge K' = P_{\sigma(m)} \wedge K' \) for each \( n \) and \( m \), which means \( (P_{\sigma(n)}, P_{\sigma(m)}) \in U_{K,V} \). In the case where it is not 0, the set \( \text{supp} P_{\sigma(n)} \wedge K' \) is non-empty and included in \( K' \) by Lemma 2.3. Since the action \( \Gamma \acts X \) is proper, the set \( \{ \eta_1, \eta_2,\ldots \} \) is relatively compact. We can take an increasing map \( \tau : \mathbb{N} \to \mathbb{N} \) such that
\begin{itemize}
  \item \( \rho_X(\eta_{\tau(n)}^{-1}\eta_{\tau(m)}^{-1})x_0, x_0 < 1 \), and
  \item \( \eta_{\tau(n)}^{-1}\eta_{\tau(m)} \in V \)
\end{itemize}
for each \( n, m \in \mathbb{N} \).

If \( n \) and \( m \) are natural numbers, by (6) we have

\[
\eta_{r(n)}(P_{\sigma(n)} \land K') = P_{\sigma(1)} \land K' = \eta_{r(m)}(P_{\sigma(m)} \land K').
\]

By multiplying both sides by \( \eta_{r(m)}^{-1} \) and using the fact that the cutting-off operation is equivariant (Definition 2.38), we obtain

\[
(\eta_{r(m)}^{-1}\eta_{r(n)}P_{\sigma(n)}) \land (\eta_{r(m)}^{-1}\eta_{r(n)}K') = P_{\sigma(m)} \land K'.
\]

Since \( \eta_{r(m)}^{-1}\eta_{r(n)}K' = B(\eta_{r(m)}^{-1}\eta_{r(n)}x_0, R + 1) \supset B(x_0, R) \supset K \) by the definition of \( \tau \), we have, by cutting off both sides of (7) by \( K \),

\[
(\eta_{r(m)}^{-1}\eta_{r(n)}P_{\sigma(n)}) \land K = P_{\sigma(m)} \land K,
\]

which means \((P_{\sigma(m)}), P_{\sigma(n)}) \in U_{K,V} \) by the definition of \( \tau \). \( \square \)

The following diagonalization argument is well-known, compare for example [12].

**Theorem 3.25.** Assume the action \( \Gamma \bowtie X \) is proper. Take \( \mathcal{P} \in \Pi \) and assume it has FLC. Then the continuous hull \( X_{\mathcal{P}} \) is compact.

**Proof.** Since on \( \Sigma \) the local matching topology is complete (Theorem 3.19), is suffices to show that \( \{\gamma \mathcal{P} \mid \gamma \in \Gamma \} \) is totally bounded. To this aim we take a sequence \((\gamma_n \mathcal{P})\) from this set and prove there is a Cauchy subsequence.

Take a sequence of compact sets \( K_1, K_2, \ldots \in \text{Cpt}(X) \) and a sequence \( V_1, V_2, \ldots \) such that \( \{U_{K_n, V_n} \mid n = 1, 2, \ldots \} \) is a fundamental system of entourages and \( U_{K_n, V_n}^2 \subset U_{K_{n+1}, V_{n+1}} \) for each \( n = 2, 3, \ldots \). For example set \( K_n = B(x_0, R_n) \) and \( V_n = B(\epsilon, r_n) \), where \( (R_n) \) is rapidly increasing and \( (r_n) \) is rapidly decreasing. Then we have, for each \( n, m \) with \( n < m \),

\[
U_{K_n, V_n} U_{K_{m-1}, V_{m-1}} \cdots U_{K_{n+1}, V_{n+1}} \subset U_{K_n, V_n}.
\]

By Lemma 3.24 we can take a subsequence \((P_{n}^{(1)})_{n=1,2,\ldots} \) of \((\gamma_n \mathcal{P})\) such that \((P_{n}^{(1)}) \in U_{K_n, V_n} \) for any \( n \) and \( m \). We can further take a subsequence \((P_{n}^{(2)})_{n} \) of \((\gamma_n \mathcal{P})_{n} \) such that \((P_{n}^{(2)}) \in U_{K_n, V_n} \) for each \( n \). We proceed in this way to obtain a sequences \((P_{n}^{(k)})_{n} \), \( k = 1, 2, \ldots \). Set \( Q_n = P_{n}^{(n)} \). Then by (8), the sequence \((Q_n)_{n} \) is a Cauchy subsequence of \((\gamma_n \mathcal{P})_{n} \). \( \square \)

FLC of an abstract pattern is not necessarily inherited by local derivability, which was defined in [2] and [11]. For example, \( \mathbb{Z} \) has FLC, but the function \( \mathbb{R} \ni t \mapsto \sin(2\pi t) \in \mathbb{R} \) does not have FLC, although the latter is locally derivable from the former. However, the fact that the continuous hull is compact is often inherited, since if \( \mathcal{Q} \) is locally derivable from \( \mathcal{P} \) and \( \mathcal{Q} \) lies in a set \( \Sigma \) on which the local matching topology is complete, the map

\[
\{\gamma \mathcal{P} \mid \gamma \in \Gamma \} \ni \gamma \mathcal{P} \mapsto \gamma \mathcal{Q} \in \{\gamma \mathcal{Q} \mid \gamma \in \Gamma \}
\]

is uniformly continuous and extended to \( X_{\mathcal{P}} \to X_{\mathcal{Q}} \).

With an additional assumption, we have the following:
Theorem 3.26. Take $P \in \Sigma$ and assume the set $A(P \wedge K)$ is finite for each $K \in \text{Cpt}(X)$. If $X_P$ is compact, then the abstract pattern $P$ has FLC.

Proof. Assume for some $K \in \text{Cpt}(X)$ the set

$$\{ (\gamma P) \wedge K \mid \gamma \in \Gamma \}$$

is infinite up to $\Gamma$-action. Then there are $\gamma_1, \gamma_2, \ldots \in \Gamma$ such that if $n \neq m$, $(\gamma_n P) \wedge K$ and $(\gamma_m P) \wedge K$ are not equivalent with respect to $\Gamma$-action. Since $X_P$ is compact, by passing to a subsequence if necessary, we may assume there is $Q \in X_P$ such that $\gamma_n P \to Q$ as $n \to \infty$.

Take a compact neighborhood $V$ of $e \in \Gamma$ and a compact $K' \subset X$ such that if $\xi, \zeta \in V$ and $x \in K$, then $\xi^{-1}\zeta x \in K'$. By passing to a subsequence if necessary, we may assume for each $n$ there is $\eta_n \in V$ such that $$(\eta_n \gamma_n P) \wedge K' = Q \wedge K'.$$

If $R \in A((\gamma_n P) \wedge K)$, then $\eta_n R \in A((\eta_n \gamma_n P) \wedge K') \subset A(\eta_1 \gamma_1 P)$. We have $\eta_n R \in A((\eta_1 \gamma_1 P) \wedge K')$. Since the latter is finite, there are distinct $n$ and $m$ such that

$$\eta_n A((\gamma_n P) \wedge K) = \eta_m A((\gamma_m P) \wedge K),$$

which implies

$$\eta_n ((\gamma_n P) \wedge K) = \eta_m ((\gamma_m P) \wedge K).$$

This contradicts the assumption at the beginning and we see $P$ has FLC. \hfill $\square$

This means that for Delone sets and tilings with finitely many tile types up to $\Gamma$-action, FLC is equivalent to compactness of the continuous hull, which is well-known for the case of $X = \mathbb{R}^d$.

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