Dirichlet heat kernel for unimodal Lévy processes

K. Bogdan, T. Grzywny, M. Ryznar
Institute of Mathematics and Computer Sciences
Wrocław University of Technology, Poland

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Abstract

We estimate the heat kernel of the smooth open set for the isotropic unimodal pure-jump Lévy process with infinite Lévy measure and weakly scaling Lévy-Khintchine exponent.

1 Introduction and preliminaries

1.1 Motivation

Heat kernels provide direct access to properties of operators with Dirichlet conditions. For instance the Green function and the harmonic measure are expressed by the kernel, cf. (1.13), (1.15) below. We shall estimate the heat kernels of open sets $D \subset \mathbb{R}^d$ with $C^{1,1}$ smoothness of the boundary and nonlocal translation-invariant integro-differential operators satisfying the maximum principle and certain unimodality and scaling conditions. Such operators are commonly used to model nonlocal phenomena [30, 12, 39, 26, 29]. Put differently, we shall study the transition density $p_D(t, x, y)$ of jump-type unimodal Lévy processes $X$ killed upon leaving $D$ under scaling conditions at infinity for the Lévy-Khintchine exponent of $X$.

We recall that precise estimates for the heat kernel of the Laplacian (and the Brownian motion) were given for $C^{1,1}$ domains in 2002 by Zhang [49]. In 2006 Siudeja [44] gave upper bounds for the heat kernel of the fractional Laplacian (and the isotropic stable Lévy process) in convex sets. In 2010 Chen, Kim and Song [14] gave sharp (two-sided) explicit estimates for the heat kernel of the fractional Laplacian in bounded $C^{1,1}$ open sets. Gradual extensions were then obtained for generators of many subordinate Brownian motions satisfying scaling conditions [14, 15, 17, 18], and for processes with comparable Lévy measure [33]. We note that subordinate Brownian motions form a proper subset of unimodal Lévy processes; in this work we present a synthetic approach to sharp estimates of $p_D(t, x, y)$ for $C^{1,1}$ open sets $D$ and general unimodal Lévy processes with scaling.

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Rather precise but less explicit bounds of \( p_D(t, x, y) \) are also known to hold for Lipschitz sets in a number of situations. Such bounds were first obtained for the Laplacian in 2003 by Varopoulos \[17\]. In 2010 the present authors proved that the following factorization,

\[
p_D(t, x, y) \approx P^x(\tau_D > t)P^y(\tau_D > t)p(t, x, y),
\]

holds for the fractional Laplacian under a geometric condition on \( x, y \in D \) and \( t > 0 \) for every open \( D \subset \mathbb{R}^d \) \[7\] Theorem 2, see also \[6\] \[7\]. Here \( P^y(\tau_D > t) \) is the survival probability of the corresponding (isotropic stable Lévy) process \( X \), see \[1.12\], and \( p(t, x, y) = p_{\mathbb{R}^d}(t, x, y) \) is the (free) heat kernel for \( D = \mathbb{R}^d \). Needless to say, the Dirichlet condition prescribed on \( D^c \) for the functions in the domains of the generator reflects the killing of \( X \) when the process first leaves \( D \). This accounts for the role played in the study by the first exit time \( \tau_D \) of \( X \) from \( D \). The comparison \[1.1\] is uniform in time and space for cones, homogeneous Lipschitz domains and exterior \( C^{1,1} \) sets, cf. \[7\], \[19\]. For these sets, \[1.1\] is made rather explicit by approximating the survival probability with superharmonic functions of \( X \) \[7\].

The above Lipschitz setting of \[7\], namely the approximate factorization of the heat kernel and the estimates of the survival probability, are closely related to the so-called boundary Harnack inequality. The setting offers a structured approach to heat kernel estimates of non-local operators. It is also relevant in the Markovian context of \[17\], where \[1.1\] serves as an intermediate step leading to explicit estimates for \( C^{1,1} \) sets. We therefore owe the reader an explanation why we postpone the setting here and instead use an approach which is tailor-made for \( C^{1,1} \) sets. The main reason is better economy and clarity of the presentation when the boundary Harnack principle is replaced by explicit estimates of superharmonic functions, and these are now provided by the preparatory work \[8\]. The second main reason is that the boundary Harnack inequality puts additional constraints on the process \( X \), and these may be circumvented in the present development. For instance, the so-called truncated stable Lévy process is manageable by our approach but cannot be resolved by previous methods because the boundary Harnack inequality fails in this case, see Example 1 in Section 6.

To bound the heat kernel \( p_D(t, x, y) \) of the unimodal Lévy process \( X \) and the \( C^{1,1} \) set \( D \) we use the estimates of the free transition density \( p(t, x, y) \) from \[9\] and the estimates of superharmonic functions of \( X \) at the boundary of \( D \) from \[8\]. For bounded, exterior, and halfspace-like \( C^{1,1} \) open sets we obtain explicit approximate factorizations of \( p_D(t, x, y) \) similar to \[1.1\], along with bounds for survival probability. The results are given in Theorem 5.4 and Theorem 5.8 below. Our estimates are sharp, meaning that the ratio of the upper bound and the lower bound is less than a constant, and they are global, that is hold with a uniform constant for all \( t > 0 \) and \( x, y \in \mathbb{R}^d \). We focus on the transient case, leaving open some cases of recurrent unimodal Lévy processes on unbounded subsets of the real line (see \[7\] for a comprehensive study of the isotropic stable Lévy processes, including the recurrent case).

Recall that an exterior set is the complement of a bounded set, and a halfspace-like set is one included between two translates of a halfspace. We thus cover bounded and some unbounded \( C^{1,1} \) sets. Unbounded sets are especially challenging: the \( C^{1,1} \) condition does not specify their geometry at infinity, whereas the geometry strongly influences the asymptotics of the heat kernel. We note that the exterior \( C^{1,1} \) sets and the halfspace-like sets were studied for the fractional Laplacian in \[7\] and \[19\]. The case of the subordinate Brownian motions with global scalings is resolved in \[33\] for the halfspace, and \[6\], \[7\] handle the fractional Laplacian in cones. Our present estimates for the heat kernel of exterior sets in Theorem 5.4 are new even for the sum of two independent isotropic stable Lévy processes. Noteworthy, the comparability constants in the estimates do not change upon dilation of \( D \) if the scalings of the Lévy-Khintchine exponent of \( X \) are global, which is an added bonus of our approach. This is so for the ball.
and for general exterior open sets, see Corollary 4.6 and Corollary 5.6. In general we strive to control comparability constants because they may be important in scaling arguments and applications to more general Markov processes. In passing we also refer the reader to [25] for heat kernel estimates of unbounded domains for second-order elliptic differential operators.

Our estimates are generally expressed in terms of \( V \), the renewal function of the ladder-height process of one-dimensional projections of \( X \), but they could equivalently be expressed in terms of the more familiar Lévy-Khintchine exponent \( \psi \) of \( X \), see (1.6). Accordingly, we observe a wide range of power-like asymptotics of heat kernels. The derivative of \( V \) is the \( \text{éminence grise} \) of the present project, see also [8]. It is quite delicate to control \( V' \), but under a mild Harnack-type condition (H), \( V' \) only influences the comparability constants, not the structure of the estimates, thus allowing for the present generality of results.

Here is a summary of our main estimates. We denote by \( \delta_D \) the distance of \( x \in \mathbb{R}^d \) to \( D^c \). The following comparisons are meant to hold for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \), i.e. globally:

If the Lévy-Khintchine exponent \( \psi \) of the unimodal Lévy process \( X \) has lower and upper scalings and \( D \) is a bounded \( C^{1,1} \), open set, then

\[
p(t,x,y) \approx p_D(t/2,x,y) p(t \wedge t_0, x, y),
\]

and

\[
\mathbb{P}^x(\tau_D > t) \approx e^{-\lambda_1 t} \left( \frac{V(\delta_D(x))}{\sqrt{t \wedge t_0}} \wedge 1 \right),
\]

where \( t_0 = V^2(r_0) \), \( r_0 > 0 \) is sufficiently small and \( -\lambda_1 \) is the principal Dirichlet eigenvalue for \( D \) and the generator of the semigroup of \( X \). The result is proved in Theorem 4.5.

If \( \psi \) has global lower and upper scalings and \( D \) is a \( C^{1,1} \) halfspace-like open set, then

\[
p_D(t, x, y) \approx p_D(t/2, x, y) p(t, x, y),
\]

and

\[
\mathbb{P}^x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1.
\]

The estimates are proved in Theorem 5.8. The same approximate factorization of \( p_D \) holds under global lower and upper scalings of \( \psi \) if \( D \) is an exterior \( C^{1,1} \) open set in dimension \( d \geq 2 \), too, except that in that case we have

\[
\mathbb{P}^x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1.
\]

The result is given in Theorem 5.4, Proposition 5.5 and Corollary 5.6. In particular we have \( \mathbb{P}^x(\tau_D > t) \approx \mathbb{P}^x(\tau_D > t/2) \) in the above two cases of unbounded \( D \), hence the approximate factorizations in all the three cases above may be considered identical in bounded time. In fact, Theorem 2.1, Remark 2.2 and Theorem 3.3 below give estimates which essentially resolve the asymptotics of the heat kernels in bounded time and space for every \( C^{1,1} \) open set \( D \), regardless of the geometry of \( D \) at infinity, and they are at the heart of our development.

We note that estimates for the Green function can in principle be obtained by integrating the estimates of the heat kernel against time, cf. (1.13) below and [14, 33].

Here are comments on possible directions of further research: Other specific unbounded \( C^{1,1} \) open sets, e.g. the parabola-shaped domains [1] deserve some attention, as they may shed light on the generality of approximate factorizations of heat kernels. By a theorem of Courrègge, if smooth compactly supported functions are in the domain of the generator of a Markovian
semigroup on $\mathbb{R}^d$, then the generator is of Lévy type \[27\]. Therefore one should expect similar estimates of superharmonic functions and heat kernels of Lévy and Markov processes under two-sided unimodal bounds for the intensity of jumps, cf. \[17, 35\]. In Remark \[6.1\] at the end of the paper we give more details in the case of Lévy processes which are isotropic and almost unimodal. Lastly, rather optimal isotropic upper bounds of $p(t, x, y)$ for a class of strongly anisotropic Lévy-type operators were given in \[46\]. In the anisotropic setting there is little hope for explicit (two-sided) sharp bounds for $p(t, x, y)$, hence for $p_D(t, x, y)$, but integrable isotropic upper bounds for $p(t, x, y)$ and upper bounds for $p_D(t, x, y)$ at the boundary of $D$ would be of much interest.

The paper is composed as follows. In Section 1.2 we recall the sharp estimates of the free heat kernel from \[9\]. In Section 1.3 we present a general framework for estimating heat kernels of jump processes and we recall the estimates of \[8\] for the first exit time of unimodal Lévy processes from $C^{1,1}$ sets. The upper bounds for $p_D(t, x, y)$ are given in Section 2 and the lower bounds are given in Section 3. In particular we propose techniques based on structure inequalities (2.1) and (3.1), which make our proofs shorter even in comparison with the case of the isotropic stable Lévy process. We also obtain a number of auxiliary bounds, which may be interesting on their own. Our estimates are generally uniform in bounded time and space, and if global scaling conditions are satisfied or the set is bounded, then the estimates are uniform in the whole of time and space. In Section 4 we complement the results of Section 2 and Section 3 with some spectral theory to obtain for bounded $C^{1,1}$ sets sharp heat kernel estimates which are global in time and space. Since they are obtained rather easily, we invest further attention in unbounded sets, the exterior sets and the halfspace-like sets. Thus, Section 5 focuses on processes with global scaling in unbounded sets, and shows best the strengths of our approach. In Section 6 we discuss specific examples of unimodal Lévy processes, which can be resolved by our methods. We encourage the reader to inspect the examples when following the general theory.

1.2 Estimates for the free process

Below in the paper we consider the Euclidean space $\mathbb{R}^d$ of arbitrary dimension $d \in \mathbb{N}$. All the considered sets, functions and measures are tacitly assumed to be Borel.

We write $f(x) \approx g(x)$ and say $f$ and $g$ are comparable if $f, g \geq 0$ and there is a positive number $C \geq 1$, called comparability constant, such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for all $x$. We write $C = C(a, \ldots, z)$ to indicate that $C$ may be so chosen to depend only on $a, \ldots, z$. Later on in Remark \[2.5\] we also make a specific convention regarding the dependence of the constants on $\psi$. Enumerated capitalized constants $C_1, C_3, \ldots$ are meant to be fixed throughout the paper. Our main motivations for such book-keeping is to facilitate scaling arguments and applications to more general Markov processes having variable Lévy characteristics of a given type.

A (Borel) measure on $\mathbb{R}^d$ is called isotropic unimodal, in short: unimodal, if on $\mathbb{R}^d \setminus \{0\}$ it is absolutely continuous with respect to the Lebesgue measure and has a (finite) radial nonincreasing density function. Such measures may have an atom at the origin. A Lévy process $X = (X_t, t \geq 0)$ \[11\], is called isotropic unimodal, in short: unimodal, if all of its one-dimensional distributions $p_t(dx)$ are unimodal. We will consider jump-type processes $X$. To actually define $X$, recall that Lévy measure is any measure concentrated on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty.$$ 

Unimodal pure-jump Lévy processes are characterized in \[48\] by unimodal Lévy measures.
\[ \nu(dx) = \nu(x)dx = \nu(|x|)dx. \] After fixing \( \nu \) we denote
\[ \psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \langle \xi, x \rangle) \nu(dx), \quad \xi \in \mathbb{R}^d. \]

Unless explicitly stated otherwise, in what follows we assume that \( \nu \) is an infinite unimodal Lévy measure, and \( X \) is the (pure-jump unimodal) Lévy process in \( \mathbb{R}^d \) given by
\[ \mathbb{E} e^{i\langle \xi, X_t \rangle} = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p_t(dx) = e^{-t\psi(\xi)}. \]

The Lévy-Khintchine exponent \( \psi \) of \( X \) is then unbounded. Since \( \psi \) is a radial function, we shall write \( \psi(u) = \psi(x) \), if \( u = |x| \geq 0 \) and \( x \in \mathbb{R}^d \).

The Lévy process \( X_t \) has the same function \( \psi(u) \). Clearly, \( \psi(0) = 0 \) and \( \psi(u) > 0 \) for \( u > 0 \). We also note that for \( t > 0 \), \( p_t(dx) \) has no atom at 0. This is equivalent to infiniteness of \( \nu \) [41, Theorem 30.10]. In fact, for \( t > 0 \), \( p_t \) has density function \( p_t(x) \) continuous on \( \mathbb{R}^d \setminus \{0\} \) [36, Lemma 2.5]. Furthermore, if the following Hartman-Wintner condition holds,
\[ \lim_{|\xi| \to \infty} \frac{\psi(\xi)}{\ln |\xi|} = \infty, \] then by Fourier inversion for each \( t > 0 \), \( p_t(dx) \) has smooth density function \( p_t(x) \) with integrable derivatives of all orders on \( \mathbb{R}^d \) [34, Lemma 3.1]. In fact, unimodality yields the following characterization.

**Lemma 1.1.** The density function \( p_t(x) \) is bounded for every \( t > 0 \) if and only if (1.2) holds.

**Proof.** The necessity of (1.2) follows from [9, Proposition 2] and [34, Proposition 4.1]. \( \square \)

For \( r > 0 \) we define Pruitt’s function [40],
\[ h(r) = \int_{\mathbb{R}^d} \left( \frac{|z|^2}{r^2} \wedge 1 \right) \nu(dz). \] (1.3)

Note that \( 0 < h(r) < \infty \) and \( h \) is decreasing.

We also consider the renewal function \( V \) of the (properly normalized) ascending ladder-height process of \( X_t^{(1)} \). The ladder-height process is a subordinator with the Laplace exponent
\[ \kappa(\xi) = \exp \left\{ \frac{1}{\pi} \int_0^\infty \frac{\log \psi(\xi \zeta)}{1 + \zeta^2} d\zeta \right\}, \quad \xi \geq 0, \]
and \( V(x) \) is its potential measure of the half-line \( (-\infty, x) \). Silverstein studied \( V \) and \( V' \) as \( g \) and \( \psi \) in [43] (1.8) and Theorem 2. The Laplace transform of \( V \) is
\[ \int_0^\infty V(x)e^{-\xi x}dx = \frac{1}{\xi \kappa(\xi)}, \quad \xi > 0. \] (1.4)

For instance, \( V(x) = x^{\alpha/2} \) for \( x \geq 0 \), if \( \psi(\xi) = |\xi|^\alpha \) [43, Example 3.7]. The definition of \( V \) is rather implicit and properties of \( V \) are delicate. In particular the decay properties of \( V' \) are not yet fully understood. For a detailed discussion of \( V \) we refer the reader to [8] and [43]. We have \( V(x) = 0 \) for \( x \leq 0 \) and \( V(\infty) := \lim_{r \to \infty} V(r) = \infty \). Also, \( V \) is subadditive:
\[ V(x + y) \leq V(x) + V(y), \quad x, y \in \mathbb{R}. \] (1.5)
It is known that $V$ is absolutely continuous and harmonic on $(0, \infty)$ for $X^1_t$. Also $V'$ is a positive harmonic function for $X^1_t$ on $(0, \infty)$, hence $V$ is actually (strictly) increasing. For the so-called complete subordinate Brownian motions [12] $V'$ is monotone, in fact completely monotone, cf. [8 Lemma 7.5]. This property was crucial for the development in [16, 33], but in general it fails in the present setting cf. [8 Remark 9].

We shall use $V$ and its inverse function $V^{-1}$ in the estimates of heat kernels. In fact, $V$ and $\psi$ may be used interchangeably because of the following lemma.

\textbf{Lemma 1.2.} The constants in the following comparisons depend only on the dimension,

$$h(r) \approx |V(r)|^{-2} \approx \psi(1/r), \quad r > 0. \quad (1.6)$$

\textit{Proof.} The constant in the first comparison depends only on $d$ and the second comparison is absolute; see [9 Proposition 2] and [8 Proof of Proposition 2.4]. \hfill \square

\textbf{Lemma 1.3.} There is a constant $C_1 = C_1(d)$ such that

$$p_t(x) \leq C_1 \frac{t}{|x|^d V^2(|x|)}, \quad t > 0, \ x \in \mathbb{R}^d \setminus \{0\}. \quad (1.7)$$

\textit{Proof.} By [9 Corollary 7 and Proposition 2], there is $C = C(d)$ such that

$$p_t(x) \leq C \frac{t \psi(1/|x|)}{|x|^d}, \quad t > 0, \ x \in \mathbb{R}^d \setminus \{0\}.$$ 

Replacing $\psi(1/|x|)$ with $1/V^2(|x|)$ and using Lemma 1.2, we get the present statement. \hfill \square

Clearly then, we also have $\nu(x) \leq C_1 |x|^{-d} V(|x|)^{-2}, \ x \in \mathbb{R}^d \setminus \{0\}$. It is rather natural to assume (relative) power-type asymptotics at infinity for the characteristic exponent $\psi$ of $X$. To this end we consider $\psi$ as a function on $(0, \infty)$. Let $\theta \in [0, \infty)$. We say that $\psi$ satisfies the weak lower scaling condition at infinity (WLSC) if there are numbers $\underline{\alpha} > 0$ and $\underline{c} \in (0, 1]$, such that

$$\psi(\lambda \theta) \geq \underline{c} \lambda^{\underline{\alpha}} \psi(\theta) \quad \text{for} \quad \lambda \geq 1, \ \theta > \underline{\theta}.$$ 

In short we write $\psi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ or $\psi \in \text{WLSC}$. If $\psi \in \text{WLSC}(\alpha, 0, 1)$, then we say that $\psi$ satisfies \textit{global} WLSC. Similarly, let $\overline{\theta} \in [0, \infty)$. The weak upper scaling condition at infinity (WUSC) means that there are numbers $\overline{\alpha} < 2$ and $\overline{c} \in [1, \infty)$ such that

$$\psi(\lambda \theta) \leq \overline{c} \lambda^{\overline{\alpha}} \psi(\theta) \quad \text{for} \quad \lambda \geq 1, \ \theta > \overline{\theta}.$$ 

In short, $\psi \in \text{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{c})$ or $\psi \in \text{WUSC}$. \textit{Global} WUSC means $\text{WUSC}(\overline{\alpha}, 0, \overline{c})$. The reader may find representative examples of characteristic exponents with scaling in Section 6 below.

We call $\underline{\alpha}, \underline{\theta}, \underline{c}, \overline{\alpha}, \overline{\theta}, \overline{c}$ the scaling characteristics of $\psi$. We emphasize that in our setting $\psi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c}) \cap \text{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{c})$ entails $0 < \underline{\alpha} \leq \overline{\alpha} < 2$. It may help to recall the connection of the weak scalings to the Matuszewska indices [3]. Namely, $\psi \in \text{WLSC}$ if and only if the lower Matuszewska index of $\psi$ is positive, and $\psi \in \text{WUSC}$ if and only if the upper Matuszewska index of $\psi$ is smaller than 2. Furthermore, $\psi$ satisfies global WLSC if and only if the lower Matuszewska indices of $\psi(\lambda)$ and $1/\psi(1/\lambda)$ are positive, and $\psi$ satisfies global WUSC if and only if the upper Matuszewska indices of $\psi(\lambda)$ and $1/\psi(1/\lambda)$ are smaller than 2. The connections are explained in [9 Remark 2 and Section 4]. In what follows we usually skip the word “weak” when referring to scaling.
Here are further remarks from [9]: We have $\psi \in \text{WLSC}(\alpha, \theta, \varrho)$ if and only if $\psi(\theta)/\theta^a$ is comparable to a nondecreasing function on $(\theta, \infty)$, and $\psi \in \text{WUSC}(\alpha, \theta, \varrho)$ if and only if $\psi(\theta)/\theta^a$ is comparable to a nonincreasing function on $(\theta, \infty)$. Scalings “at zero” may also be considered and are discussed in [9, Section 3]. Generally, the lower scaling for large arguments changes to upper scaling for small arguments by taking the reciprocal argument, as in the above discussion of Matuszewska indices for global scalings. We are thus led to the behavior of $V$ and its inverse function, $V^{-1}$, at zero, cf. (1.6). Namely, let $\psi \in \text{WLSC}(\alpha, \theta, \varrho)$ and $K(\theta) = [V(1/\theta)]^{-2}$, $\theta > 0$. By the proof of Lemma 1.2 there is an absolute constant $C \geq 1$ such that for all $V$, the range of scalings of $\psi$ is comparable to a nonincreasing function on $(\theta, \infty)$. Consequently, the lower bound in Lemma 1.5 implies the lower and upper scalings of $\psi$.

**Theorem 26**, which shows the importance of the scaling conditions in the study of unimodal Lévy processes. The next result elaborates on (1.7) when scaling is assumed.

**Proof.** We replace $\psi$ with $V$ and use Lemma 1.2 to reformulate [9, Theorem 21].

To clarify, the estimates in Lemma 1.5 hold for all $x \in \mathbb{R}^d$ and $t > 0$ if $\theta = 0$. Further, (1.10) holds and only if $t > V^2(|x|)$.

It is convenient to assume $\theta = \overline{\theta} = \theta$ in Lemma 1.5 and it entails no essential loss of generality because we can take $\theta = \max\{\theta, \overline{\theta}\}$ or extend the range of the scalings by using Remark 1.4. Conversely, the lower bound in Lemma 1.5 implies the lower and upper scalings of $\psi$, see [9, Theorem 26], which shows the importance of the scaling conditions in the study of unimodal Lévy processes. The next result is a variant of [9, Proposition 19].
Lemma 1.6. If \( \psi \in \text{WLSC}(\alpha, \theta, \underline{c}) \), \( r > 0 \) and \( 0 < t \leq rV^2(1/\underline{c}) \), then
\[
c_2e^{-c_1r} \left[ V^{-1}\left(\sqrt{t/r}\right) \right]^d \leq p_t(0) \leq c_3 \left( 1 + (v^d)^{-1-d/\underline{c}} \right) \left[ V^{-1}\left(\sqrt{t/r}\right) \right]^d,
\]
where \( c_1 \) is an absolute constant, \( c_2 = c_2(d) \) and \( c_3 = c_3(d, \underline{c}) \).

Proof. Note that (1.2) holds and we have
\[
p_t(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\psi(t, \xi)} \, d\xi,
\]
where \( \omega_d = 2\pi^{d/2}/\Gamma(d/2) \) is the surface measure of the unit sphere in \( \mathbb{R}^d \). Since \( \Psi^{-1}(s) = (V^{-1}(1/\sqrt{s}))^{-1} \), the lower bound in (1.11) obtains. If \( 0 < t \leq rV^2(1/\underline{c}) = r/\Psi(\underline{c}) \), then \( tc_1^{-1} \leq (rc_1^{-1})/\Psi(\underline{c}) \), and [9, Lemma 16 and Remark 6] with \( \epsilon = r c_1^{-1} \) and \( tc_1^{-1} \) instead of \( t \), yields
\[
p_t(0) \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{tc_1^{-1}\psi(t, \xi)} \, d\xi \leq c_4 \left( 1 + (\underline{c}r)^{-1-d/\underline{c}} \right) \left( \Psi^{-1}(r/t) \right)^d.
\]
\[\square\]

Remark 1.7. Under the assumptions of Lemma 1.6, if \( 0 < t \leq CV^2(1/\underline{c}) \), then \( p_t(0) \geq c p_t/2(0) \) with constant \( c = c(X, C) \). This follows from (1.11), (1.9), and Remark 1.4.

Definition 1. We say that condition \((H)\) holds if for every \( r > 0 \) there is \( H_r \geq 1 \) such that
\[
V(z) - V(y) \leq H_r V'(x)(z - y) \quad \text{whenever} \quad 0 < x \leq y \leq z \leq 5x \leq 5r.
\]
We say that \((H^*)\) holds if \( H_\infty = \sup_{r > 0} H_r < \infty \).

We consider \((H)\) and \((H^*)\) as variants of Harnack inequality because \((H)\) is implied by the following property of \( V' \):
\[
\sup_{y \in [x, 5x], \ x \in r} V'(y) \leq H_r \inf_{y \in [x, 5x], \ x \in r} V'(y), \quad r > 0.
\]
Both the above conditions control relative growth of \( V \). If \((H)\) holds, then we may and do chose \( H_r \) nondecreasing in \( r \). By [3, Section 7.1], in each of the following cases, \((H)\) holds:

1. \( X \) is a subordinate Brownian motion governed by a special \([22]\) subordinator. (In this case \( V \) is concave so \((H^*)\) holds with \( H_\infty = 1 \).)
2. \( d \geq 3 \) and \( \psi \) satisfies WLSC. (If \( d \geq 3 \) and \( \psi \in \text{WLSC}(\underline{c}, 0, \underline{c}) \), then \((H^*)\) holds.)
3. \( d \geq 1 \) and \( \psi \) satisfies WLSC and WUSC. (If \( d \geq 1 \) and \( \psi \in \text{WLSC}(\underline{c}, 0, \underline{c}) \cap \text{WUSC}(\pi, 0, \underline{c}) \), then \((H^*)\) holds.)

We do not know any \( V \) failing \((H)\), nor a proof that \((H)\) always holds in our setting, which would be interesting to know. Below approximate factorizations of heat kernels are proved under Case 3, from whence \((H)\) follows for all dimensions \( d = 1, 2, \ldots \). However, many auxiliary results of independent interest hold under weaker assumptions, see, e.g., Remark 2.7 below.
1.3 Dirichlet condition

Recall that $d \in \mathbb{N}$. We let $B(x,r) = \{y \in \mathbb{R}^d : |y - x| < r\}$, the open ball with center at $x \in \mathbb{R}^d$ and radius $r > 0$, and $B_r = B(0,r)$. Recall that by $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ we denote the surface measure of $\partial B_1$, the unit sphere in $\mathbb{R}^d$. We also let $\overline{B(x,r)} = (B(x,r))^c = \{y \in \mathbb{R}^d : |y - x| > r\}$ and $\overline{B_r} = B(0,r)$. For $a \in \mathbb{R}$, we consider the upper halfspace $\mathbb{H}_a = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > a\}$. All other halfspaces are obtained by rotations. The ball, the complement of the ball and the halfspace represent three distinctly different geometries at infinity which are in focus in this paper.

We consider nonempty open set $D \subset \mathbb{R}^d$, its diameter $\text{diam}(D) = \sup\{|y - x| : x, y \in D\}$, and the distance to its complement:

$$\delta_D(x) = \text{dist}(x, D^c), \quad x \in \mathbb{R}^d.$$ 

We say that $D$ satisfies the inner ball condition at scale $r$ if $r > 0$ and for every $Q \in \partial D$ there is ball $B(x',r) \subset D$ such that $Q \in \partial B(x',r)$. We say $D$ satisfies the outer ball condition at scale $r$ if $r > 0$ and for every $Q \in \partial D$ there is ball $B(x'',r) \subset D^c$ such that $Q \in \partial B(x'',r)$.

We say that $D$ is of class $C^{1,1}$ at scale $r$, if $D$ satisfies the inner and outer ball conditions at the scale $r$. We call $B(x',r)$ and $B(x'',r)$ above the inner and outer balls for $D$ at $Q$, respectively. Estimates of potential-theoretic objects for $C^{1,1}$ sets $D$ often rely on the inclusion $B(x',r) \subset D \subset \overline{B(x'',r)}$ and on explicit calculations for its extreme sides. If $D$ is $C^{1,1}$ at some positive but unspecified scale (hence also at all smaller scales), then we simply say $D$ is $C^{1,1}$. We refer the reader to [10, Lemma 1] for more delicate aspects of geometry of $C^{1,1}$ sets.

We are interested in the behavior of the unimodal Lévy process $X$ as it approaches the complement of the open set $D$. We shall use the usual Markovian notation: for $x \in \mathbb{R}^d$ we write $\mathbb{E}^x$ and $\mathbb{P}^x$ for the expectation and distribution of $x + X$, but we use the same symbol $X$ for the resulting process [41, Chapter 8]. We shall also alternatively write $p_t(y-x) = p(t,x,y)$. We define the time of the first exit of $X$ from open set $D \subset \mathbb{R}^d$:

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$ 

The transition density of the process $X$ killed upon the first exit from $D$ is defined by

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, y) ; \tau_D < t], \quad t > 0, x, y \in \mathbb{R}^d,$$

see [20]. We call $p_D$ the heat kernel of $X$ on $D$. The definition is rather implicit, but tractable. For instance, the reader may check that $y \mapsto p_B(t,0,y)$ is a radial function for all $r, t > 0$. It is well known that $p_D$ satisfies the Chapman-Kolmogorov equations, which yields the following simple connection of the heat kernel and the survival probability.

**Lemma 1.8.** For all $t > 0$ and $x, y \in \mathbb{R}^d$, we have $p_D(t,x,y) \leq p_{t/2}(0) \mathbb{P}^x(\tau_D > t/2)$ and

$$p_D(t,x,y) \leq p_{t/2}(0) \mathbb{P}^x(\tau_D > \frac{t}{4}) \mathbb{P}^y(\tau_D > \frac{t}{4}).$$

**Proof.** The estimates obtain as follows,

$$p_D(t,x,y) = \int p_D\left(\frac{t}{2}, x, z\right) p_D\left(\frac{t}{2}, z, y\right) dz$$

$$\leq \sup_{w,y \in \mathbb{R}^d} p_D\left(\frac{t}{2}, w, y\right) \int p_D\left(\frac{t}{2}, x, z\right) dz \leq p_{t/2}(0) \mathbb{P}^x(\tau_D > \frac{t}{2}),$$

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\[ p_D(t,x,y) = \int \int p_D \left( \frac{t}{4}, x, z \right) p_D \left( \frac{t}{2}, z, w \right) p_D \left( \frac{t}{4}, w, y \right) dz dw \]
\[ \leq \sup_{u, v \in \mathbb{R}^d} p_D \left( \frac{t}{2}, u, v \right) \int p_D \left( \frac{t}{2}, x, z \right) dz \int p_D \left( \frac{t}{2}, w, y \right) dw \]
\[ \leq p(t/2)(0) \mathbb{P}^x \left( \tau_D > \frac{t}{4} \right) \mathbb{P}^y \left( \tau_D > \frac{t}{4} \right). \]

**Remark 1.9.** If \( \mathbb{P}^x(\tau_D = 0) = 1 \), then \( p_D(t, x, y) = 0 \) for all \( y \in \mathbb{R}^d, t > 0 \). The assumption holds for all \( x \in D^c \) less a polar set because \( X \) is symmetric and has transition density function, see [37, VI.4.10, VI.4.6, II.3.3]. If \( D \) is a \( C^{1,1} \) open set, then the assumption holds for all \( x \in D^c \) by familiar arguments of radial symmetry and Blumenthal’s zero-one law, see [20, the proof of Proposition 1.2].

The survival probability may be expressed via \( p_D \):
\[ \mathbb{P}^x(\tau_D > t) = \int_{\mathbb{R}^d} p_D(t,x,y)dy, \quad t > 0, \ x \in \mathbb{R}^d, \] (1.12)
and the Green function of \( D \) for \( X \) is defined as
\[ G_D(x,y) = \int_0^\infty p_D(t,x,y)dt, \quad x, y \in \mathbb{R}^d. \] (1.13)

The expected exit time is
\[ \mathbb{E}^x_{\tau_D} = \int_0^\infty \mathbb{P}^x(\tau_D > t)dt = \int_0^\infty \int_{\mathbb{R}^d} p_D(t,x,y)dy dt = \int_{\mathbb{R}^d} G_D(x,y)dy, \quad x \in \mathbb{R}^d. \]

If \( x \in D \), then the \( \mathbb{P}^x \)-distribution of \( (\tau_D, X_{\tau_D-}, X_{\tau_D}) \) restricted to the event \( \{ X_{\tau_D-} \neq X_{\tau_D} \} \) is given by the following density function \( 28 \):
\[ (0, \infty) \times D \times D^c \ni (s, u, z) \mapsto \nu(z-u)p_D(s,x,u). \] (1.14)

Integrating against \( ds, du \) and/or \( dz \) gives marginal distributions. For instance, if \( x \in D \), then
\[ \mathbb{P}^x(\tau_D \in dz) = \left( \int_D G_D(x,u)\nu(z-u)du \right) dz, \] (1.15)
on \( (D)^c \) or even on \( D^c \) if \( \mathbb{P}^x(\tau_D = \partial D) = 0 \). Such identities resulting from (1.14) are called Ikeda-Watanabe formulae. They enjoy intuitive interpretations in terms of “occupation time measures” \( p_D(s,x,u)duds \) and \( G_D(x,u)du \) and “intensity of jumps” \( \nu(z-u)dz \), cf. [5, p.17].

The following lemma is instrumental in estimating the heat kernel \( p_D \). This present statement was preceded by [37, Theorem 4.2], [44, Lemma 3.2], [14, Lemma 2.2] and [7, Lemma 2].

**Lemma 1.10.** Consider disjoint open sets \( D_1, D_3 \subset D \). Let \( D_2 = D \setminus (D_1 \cup D_3) \). If \( x \in D_1, y \in D_3 \) and \( t > 0 \), then
\[ p_D(t,x,y) \leq p^x(X_{\tau_D} \in D_2) \sup_{s \leq t, z \in D_2} p(s,z,y) + (t \wedge \mathbb{E}^x_{\tau_D}) \sup_{u \in D_1, z \in D_3} \nu(z-u), \]
\[ p_D(t,x,y) \leq p^x(X_{\tau_D} \in D_2) \sup_{s \leq t, z \in D_2} p_D(s,z,y) + \sup_{u \in D_1, z \in D_3} \nu(z-u) \times \]
\[ \times \left( p^x(\tau_D > t/2) \int_0^{t/2} \mathbb{P}^y(\tau_D > s)ds + \mathbb{P}^y(\tau_D > t/2) \int_0^{t/2} \mathbb{P}^x(\tau_D > s)ds \right), \]
\[ p_D(t,x,y) \geq t \mathbb{P}^x(\tau_D > t) \mathbb{P}^y(\tau_D > t) \inf_{u \in D_1, z \in D_3} \nu(z-u). \]
Proof. By the strong Markov property,
\[ p_D(t, x, y) = \mathbb{E}^x[p_D(t - \tau_{D_1}, X_{\tau_{D_1}}, y), \tau_{D_1} < t]. \]

By Remark 1.9, this equals
\[ \mathbb{E}^x[p_D(t - \tau_{D_1}, X_{\tau_{D_1}}, y), \tau_{D_1} < t, X_{\tau_{D_1}} \in D_2] + \mathbb{E}^x[p_D(t - \tau_{D_1}, X_{\tau_{D_1}}, y), \tau_{D_1} < t, X_{\tau_{D_1}} \in D_3] = I + II. \]

Since \( D_3 \subset \mathbb{D}_1 \), by (1.14) the distribution of \((\tau_{D_1}, X_{\tau_{D_1}})\) at \( s > 0 \) and \( z \in D_3 \), is given by the density function
\[ f^x(s, z) = \int_{D_1} p_{D_1}(s, x, u) \nu(z - u) du. \]

Let \( m = \inf_{u \in D_1, z \in D_3} \nu(z - u) \). For \( z \in D_3 \) we have \( f^x(s, z) \geq m \mathbb{P}^x(\tau_{D_1} > s) \), and
\[
II = \int_0^t \int_{D_3} p_D(t - s, z, y) f^x(s, z) dz ds \\
\geq m \mathbb{P}^x(\tau_{D_1} > t) \int_0^t \int_{D_3} p_{D_3}(t - s, z, y) dz ds = m \mathbb{P}^x(\tau_{D_1} > t) \int_0^t \mathbb{P}^y(\tau_{D_3} > s) ds,
\]
hence the lower bound. For the upper bounds we let \( M = \sup_{u \in D_1, z \in D_3} \nu(z - u) \), obtaining
\[
II \leq M \int_0^t \int_{D_3} p_D(t - s, z, y) \mathbb{P}^x(\tau_{D_1} > s) dz ds \\
\leq M \int_0^t \mathbb{P}^x(\tau_{D_1} > s) \mathbb{P}^y(\tau_D > t - s) ds \\
\leq M \left( \mathbb{P}^x(\tau_D > t/2) \int_{t/2}^t \mathbb{P}^y(\tau_D > s) ds + \mathbb{P}^y(\tau_D > t/2) \int_0^{t/2} \mathbb{P}^x(\tau_{D_1} > s) ds \right). \tag{1.16}
\]

This, (1.16), and the inequality \( I \leq \mathbb{P}^x(X_{\tau_{D_1}} \in D_2) \sup_{s < t, z \in D_2} p_D(s, z, y) \), finish the proof. \( \square \)

Similar arguments provide the following relationship, which will be useful later on.

Lemma 1.11. For all \( t > 0 \) and \( y \in \mathbb{R}^d \),
\[ p_t(y) \geq 4^{-d} t \nu(y) \left( \mathbb{P}^0(\tau_{B_{|y|}/2} > t) \right)^2. \]

Proof. We use the notation from the previous lemma. Let \( y \neq 0 \), \( D_1 = B(0, |y|/2) \), \( D_2 = B(y, |y|/2) \) and \( D = D_1 \cup D_2 \). Let \( F = B(y/2, |y|/2) \). Observe that for \( u \in D_1 \cap F \) and \( z \in D_3 \cap F \) we have \( |z - u| \leq |y| \), hence by geometric considerations,
\[ f^0(s, z) = \int_{D_1} p_{D_1}(s, 0, u) \nu(z - u) du \geq \nu(y) \int_{D_1 \cap F} p_{D_1}(s, 0, u) du \geq 2^{-d} \nu(y) \int_{D_1} p_{D_1}(s, 0, u) du = 2^{-d} \nu(y) \mathbb{P}^0(\tau_{D_1} > s), \]
and
\[
p(t, 0, y) \geq \int_0^t \int_{D_3 \cap F} f^0(s, z) p(t - s, z, y) dz ds \\
\geq 2^{-d} \nu(y) \mathbb{P}^0(\tau_{D_1} > t) \int_0^t \int_{D_3 \cap F} p(t - s, z, y) dz ds \\
\geq 4^{-d} \nu(y) \mathbb{P}^0(\tau_{D_1} > t) \int_0^t \int_{D_3} p_D(t - s, z, y) dz ds \\
= 4^{-d} \nu(y) \mathbb{P}^0(\tau_{D_1} > t) \int_0^t \mathbb{P}^y(\tau_{D_3} > s) ds \\
\geq 4^{-d} \nu(y) \mathbb{P}^0(\tau_{D_1} > t) \left( \mathbb{P}^0(\tau_{D_1} > t) \right)^2. \]

\( \square \)
Lemma 1.12. $C_2 = C_2(d)$, $C_3 = C_3(d)$ and $C_4 = C_4(d)$ exist such that for all $t, r > 0$ and $|x| \leq r/2$,

\[
\begin{align*}
\mathbb{P}^x (|X_{\tau_D}| \geq r) &\leq C_2 \frac{E^x \tau_D}{V^2(r)}, & (1.17) \\
\mathbb{P}^x (\tau_{B_r} \leq t) &\leq C_3 \frac{t}{V^2(r)}, & (1.18) \\
\mathbb{P}^x (\tau_{B_r} > C_4 V^2(r)) &\geq 1/2. & (1.19)
\end{align*}
\]

Proof. The result combines Lemma 2.7 and Corollary 2.8 of [8].

Corollary 1.13. There is $c = c(d) > 0$ such that if $y \in \mathbb{R}^d \setminus \{0\}$ and $0 < t < c/\psi(1/|y|)$, then $p_t(y) \geq 4^{-d-1} t \nu(y)$.

Proof. The result follows from Lemma [1.11] and (1.19), by Lemma [1.2] and subadditivity of $V$.

Following [8], for $r > 0$ we define

\[
\mathcal{I}(r) = \inf_{0 < \rho \leq r/2} \nu(B_r \setminus B_\rho)V^2(\rho) \quad \text{and} \quad \mathcal{J}(r) = \inf_{0 < \rho \leq r} \nu(B_\rho)V^2(\rho).
\]

(1.20)

The quantities are meant to simplify notation in arguments leading from Ikeda-Watanabe formulas to estimates of the survival probability from below, where $\mathcal{I}$ is used, and to estimates of the expected exit time from above, where $\mathcal{J}$ is used. Note that by Lemma [1.2] $h(r)V^2(r) \approx 1$. Below we strive for lower bounds for $\mathcal{I}$ and $\mathcal{J}$. Such bounds can be interpreted as comparability of a part of the integral defining $h$ with the whole, cf. (1.3), and certainly, $\mathcal{I}$ and $\mathcal{J}$ describe the size of the Lévy measure in comparison to $V^{-2}$ and $h$. Additional information on $\mathcal{I}$ is given in Lemma [3.2] below.

The following result is taken from [8 Proposition 6.1].

Lemma 1.14. Let (H) hold. There are $C_5 = C_5(d) < 1$ and $C_6 = C_6(d)$ such that for $r > 0$,

\[
\mathbb{P}^x (\tau_{B_r} > t) \geq C_6 \frac{\mathcal{I}(r)}{H_r} \left( \frac{V(\delta_{B_r}(x))}{\sqrt{t}} \wedge 1 \right), \quad 0 < t \leq C_5 V^2(r), \quad x \in \mathbb{R}^d.
\]

In the next result we slightly extend [8 Remark 8], to include processes with local scalings.

Lemma 1.15. If $D$ is $C^{1,1}$ at scale $r$, $\nu(r) > 0$, and $\psi \in \text{WLSC} \cap \text{WUSC}$, then

\[
\mathbb{P}^x (\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1, \quad 0 < t \leq C_5 V^2(r), \quad x \in \mathbb{R}^d.
\]

The comparison depends only on $X$ and $r$. If the scalings are global, then the comparison depends only on $d$ and the scaling characteristics of $\psi$.

Proof. Let $x \in D$. If $\delta_D(x) \geq r/2$, then there is a ball $B \subset D$ with radius $r$ such that $\delta_B(x) \geq r/2$. By Lemma [1.14] and subadditivity of $V$ we obtain

\[
\mathbb{P}^x (\tau_D > t) \geq \mathbb{P}^x (\tau_B > t) \geq c_2 \left( \frac{V(r/2)}{\sqrt{C_5 V(r)}} \wedge 1 \right) > c_2/2,
\]
Thus, $\mathbb{P}^x(\tau_D > t) \approx 1 \approx \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1$. Here $c_2 > 0$ depends only on $X$ and $r$ (on $d$ and $\psi$ if global scalings hold), as follows from [8, Proposition 5.2(ii) and Lemma 7.3]. Namely, [8, Proposition 5.2(ii)] yields $\inf_{0 < s < R} \mathcal{I}(s) > 0$ for some $0 < R < r$. Since $\mathcal{I}(r) \geq \mathcal{I}(R) \wedge [\nu(r)|B_r \setminus B_{r/2}|\sqrt{V^2(R/2)}]$, we obtain $\mathcal{I}(r) > 0$. On the other hand, [8, Lemma 7.3] yields $\mathcal{H}$ hence $H_r < \infty$, as needed. In the case of global scalings, [8, Lemma 7.3] and the proof of Proposition 5.2(ii) show that $c_2$ only depends on $d$ and the parameters $\alpha$, $\zeta$, $\pi$ and $\mathcal{C}$ of the scalings (by [9, Theorem 26] we then automatically have $\nu(r) > 0$). If local scalings are only assumed, then [8, Proposition 5.2 and Lemma 7.3] yield $c_2 = c_2(d, \psi)$ if $r$ is small (the notion of smallness depends on the characteristics of the scalings).

If $\delta_D(x) < r/2$ and $S \subset \partial D$ is such that $\delta_D(x) = |x - S|$, then there are balls $B$ and $B'$ with radii $r$, tangent at $S$ and such that $B \subset D \subset B'$. Since $\delta_D(x) = \delta_B(x) = \delta_{\overline{B'}}(x)$, by Lemma [11] and [8, Lemma 6.2] we get the claim, see also [8, Proposition 5.2 and Lemma 7.3].

The above lemmas largely resolve the asymptotics of the survival probability in $C^{1,1}$ open sets in small time. Estimates of the survival probability for large time depend on specific geometry of $D$ at infinity and shall be studied later on in this paper.

The following result relates survival probabilities to the scenario of $X$ evading the complement of $D$ by going towards the center of the set.

**Lemma 1.16.** Let $0 < r \leq 1$, $x \in B_1$ and $\delta_B(x) < r/6$. Denote $x_0 = x/|x|$, $x_1 = x_0(1 - r/2)$ and $F_x = B(x_0, r/4) \cap B_1$. There is a constant $c = c(d)$ such that

$$
\int_{B(x_1, r/12)} p_{B_1}(t, x, v) dv \geq c t \nu(r) r^d \mathbb{P}(\tau_{F_x} > t) \mathbb{P}(\tau_{B_{r/12}} > t), \quad t > 0. \tag{1.21}
$$

**Proof.** We use Lemma [11] with $D = B_1$, $D_1 = F_x$, $D_3 = B(x_1, r/6)$. For $v \in B(x_1, r/12)$,

$$
p_{B_1}(t, x, v) \geq t \mathbb{P}(\tau_{D_1} > t) \mathbb{P}(\tau_{D_3} > t) \inf_{w \in D_1, z \in D_3} \nu(z - w)
$$

$$
\geq t \nu(r) \mathbb{P}(\tau_{D_1} > t) \mathbb{P}(\tau_{B_{r/12}} > t).
$$

Integrating against $v \in B(x_1, r/12)$ we obtain (1.21) with $c = \omega_d(12)^{-d/d}$. □

**Corollary 1.17.** Assume that $\mathcal{H}$ holds, $0 < r \leq 1$ and $x \in B_1$. Let $x_1 = x$, if $\delta_B(x) \geq r/6$, otherwise let $x_1 = x(1 - r/2)/|x|$. There are constants $C_7 = C_7(d)$ and $C_8 = C_8(d)$ such that if $0 < t \leq C_7 V^2(r)$, then

$$
\int_{B(x_1, r/12)} p_{B_1}(t, x, v) dv \geq C_8 \frac{\mathcal{I}(r/8)}{H_1} t \nu(r) r^d \left( \frac{V(\delta_B(x))}{\sqrt{t}} \wedge 1 \right).
$$

**Proof.** Let $C_7 = C_4/(12)^2 \wedge C_5/8^2$ and $0 < t \leq C_7 V^2(r)$. If $0 < \delta_B(x) < r/6$, then by Lemma [11] and Lemma [12]

$$
\int_{B(x_1, r/12)} p_{B_1}(t, x, v) dv \geq c t \nu(r) r^d \mathbb{P}(\tau_{B(x_1, r/8)/|x|, r/8} > t).
$$

By Lemma [11] we get the result, since $V(\delta_B(x) \wedge r/8) \geq V(\delta_B(x) \wedge r/4)/2 = V(\delta_B(x))/2$. If $\delta_B(x) \geq r/6$, then by (1.19),

$$
\int_{B(x_1, r/12)} p_{B_1}(t, x, v) dv \geq \int_{B(x, r/12)} p_{B(x, r/12)}(t, x, v) dv = \mathbb{P}(\tau_{B_{r/12}} > t) \geq 1/2.
$$

By [9, (16)] and (1.7), $t \nu(r) r^d \leq c(d)$ for $t \leq C_7 V^2(r)$. This ends the proof. □

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Lemma 1.18. Assume that (H) holds. Let $R > 0$ and $D = \overline{B}_R$. Let $0 < r < R$, $x \in D$, $0 < \delta_D(x) \leq r/2$, $x_0 = xR/|x|$ and $D_1 = B(x_0, r) \cap D$. Then

$$
\mathbb{E}^x \tau_{D_1} \leq C_9 \frac{H_R}{\mathcal{J}(R)^2} V(\delta_D(x)) V(r).
$$

(1.22)

Furthermore, $C_9H_R/\mathcal{J}(R)^2 \geq 1/(2C_2)$ and $C_2 \geq 1/2$.

Proof. (1.22) was proved in [8], see Corollary 4.5 ibid., but we need to justify the statement about the constants. Let $|x| = 5R/4$ and $D_1 = B(4x/5, R/2) \cap \overset{\circ}{B}_R$. Since $B(x, R/4) \subset D_1$, by (1.17) and [8] Corollary 4.1 we obtain

$$
C_9^{-1}V^2(R/4) \leq \mathbb{E}^x \tau_{D_1} \leq C_9H_R\mathcal{J}(R)^{-2}V(R/4)V(R/2).
$$

This and subadditivity of $V$ imply

$$
C_2C_9H_R\mathcal{J}(R)^{-2} \geq \frac{V(R/4)}{V(R/2)} \geq \frac{1}{2}.
$$

Due to (1.17) and [8] Lemma 2.3, $C_2 \geq 1/2$. \hfill \square

2 Upper bound

In this section we shall study consequences of the following structure assumption:

$$
p_t(x) \leq tF(|x|), \quad t > 0, \ x \in \mathbb{R}^d \setminus \{0\},
$$

(2.1)

where $F$ is a nonnegative nonincreasing function on $(0, \infty)$. We shall use (2.1) to estimate the heat kernel $p_D(t, x, y)$ of $C^{1,1}$ sets $D \subset \mathbb{R}^d$ for $t > 0$ and $x, y \in \mathbb{R}^d$. In view of Lemma 1.10 we may think of $F(r) = C_1/[d^dV^2(r)]$ here (the method however seems to generalize beyond the context of the present paper). We note that $p_D(t, x, y) = 0$ if $x \in D^c$ or $y \in D^c$, cf. Remark 1.3 so without much mention in what follows we only consider $x, y \in D$ and $D \neq \emptyset$. We start with the following upper bound, which elaborates Lemma 1.10 for the complement of the ball.

Theorem 2.1. Let (H) hold, $R > 0$ and $D = \overline{B}_R$. There is $C = C(d)$ such that if (2.1) is true with nonincreasing function $F \geq 0$ on $(0, \infty)$, $0 < t \leq V^2(|x - y|)$ and $x, y \in D$, then

$$
p_D(t, x, y) \leq C\frac{H_R^2}{\mathcal{J}(R)^4} \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) tF(|x - y|/9).
$$

(2.2)

Proof. Let $t_0 = t \wedge V^2(R)$ and $x, y \in \overline{B}_R$. We choose $r > 0$ so that $V(12r) = \sqrt{t_0}$. In particular, $r \leq R/12$. If $\delta_D(x) \wedge \delta_D(y) \geq r/3$, then (2.2) is verified as follows. Since $\delta_D(x) \geq r/3$, by subadditivity of $V$ we have $V(\delta_D(x))/\sqrt{t_0} \geq 1/36$. Thus,

$$
p_D(t, x, y) \leq 36 \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) p(t, x, y) \leq 36 \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) tF(|x - y|/3).
$$

(2.3)

By Lemma 1.18 we have $H_R^2/\mathcal{J}(R)^4 \geq 1/(4C_2^2C_9^2)$. Hence for $\delta_D(x) \wedge \delta_D(y) \geq r/3$ we have

$$
p_D(t, x, y) \leq 4 \cdot 36C_2^2C_9^2 \frac{H_R^2}{\mathcal{J}(R)^4} \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) tF(|x - y|/9).
$$

(2.4)
We now may and do assume that $0 < \delta_D(x) < r/3$, hence $V(\delta_D(x))/\sqrt{t_0} < 1$. At first, we also assume that $V^2(3|x-y|) \geq t$, in particular $|x-y| \geq 4r$. We define

$$x_0 = Rx/|x|, \quad D_1 = B(x_0, r) \cap D, \quad D_3 = B(x, 2|x-y|/3) \cap D.$$ 

Note that $|z-y| \geq |x-y|/3$ if $z \in D_2 = D \setminus (D_1 \cup D_3)$. Radial monotonicity of $p_t$ implies

$$\sup_{s<t, z \in D_2} p(s, z, y) \leq tF(|x-y|/3).$$

If $u \in D_1, z \in D_3$, then $|z-u| \geq 2|x-y|/3 - |x-x_0| - |x_0-u| > |x-y|/3$. Hence,

$$\sup_{u \in D_1, z \in D_3} \nu(z-u) \leq \nu((x-y)/3) \leq F(|x-y|/3).$$

By Lemma [1.10]

$$p_D(t, x, y) \leq (t \mathbb{P}^x(X_{\tau_{D_1}} \in D_2) + \mathbb{E}^x_{\tau_{D_1}}) F(|x-y|/3).$$

By (1.17) and subadditivity of $V$,

$$\mathbb{P}^x(X_{\tau_{D_1}} \in D_2) \leq c_2 \frac{\mathbb{E}^x_{\tau_{D_1}}}{V^2(r)} \leq 144c_2 \frac{\mathbb{E}^x_{\tau_{D_1}}}{t_0}. \quad (2.4)$$

By Lemma [1.18]

$$\mathbb{E}^x_{\tau_{D_1}} \leq c_0 \frac{H_R}{\mathcal{J}(R)^2} V(r) V(\delta_D(x)) \leq c_0 \frac{H_R}{\mathcal{J}(R)^2} \sqrt{t_0} V(\delta_D(x)). \quad (2.5)$$

We let $c_1 = (144c_2 + 1)c_0$ and obtain

$$p_D(t, x, y) \leq c_1 \frac{H_R}{\mathcal{J}(R)^2} \frac{V(\delta_D(x))}{\sqrt{t_0}} tF(|x-y|/3). \quad (2.6)$$

Combining (2.3), (2.6) and Lemma [1.18] we see that

$$p_D(t, u, v) \leq c_2 \frac{H_R}{\mathcal{J}(R)^2} \left( \frac{V(\delta_D(u))}{\sqrt{t_0}} \wedge 1 \right) tF(|u-v|/3),$$

where $u, v \in D$, $V^2(3|u-v|) \geq t$ and $c_2 = c_1 \vee (72c_2 c_0)$. By symmetry,

$$p_D(t, u, v) \leq C^* \left( \frac{V(\delta_D(v))}{\sqrt{t_0}} \wedge 1 \right) tF(|u-v|/3),$$

where $C^* = c_2 H_R \mathcal{J}(R)^{-2}$.

We observe that $s \leq V^2(3|z-y|)$ if $s \leq t \leq V^2(|x-y|)$ and $z \in D_2$. By previous estimate,

$$\sup_{s<t, z \in D_2} p_D(s, z, y) \leq C^* \left( \frac{V(\delta_D(y))}{\sqrt{t_0}} \wedge 1 \right) tF(|x-y|/9).$$

Applying Lemma [1.10] and the estimate $\sup_{u \in D_1, z \in D_1} \nu(z-u) \leq \nu((x-y)/3)$, we obtain

$$p_D(t, x, y) \leq \mathbb{P}^x(X_{\tau_{D_1}} \in D_2) \sup_{s<t, z \in D_2} p_D(s, z, y) + \left( t \wedge \mathbb{E}^x_{\tau_{D_1}} \right) \mathbb{P}^y(\tau_D > t/2) + \mathbb{P}^x(\tau_{D_1} > t/2) \int_0^{t/2} \mathbb{P}^y(\tau_D > s) ds \nu((x-y)/3)$$

$$= I_1 + I_2.$$
Combining (2.4) and (2.5) we prove
\[ \mathbb{P}^x(X_{\tau_{D_1}} \in D_2) \leq C^* \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right). \]

Therefore,
\[ I_1 \leq (C^*)^2 \left( \frac{V(\delta_D(y))}{\sqrt{t_0}} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) t F(|x - y|/9). \]

By (2.5) we obtain
\[ t \wedge \mathbb{E}^x \tau_{D_1} \leq \frac{C^*}{73} \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) t. \]

From [S] Lemma 6.2 and its proof and Lemma L.18 it is clear that
\[ \int_0^{t/2} \mathbb{P}^y(\tau_D > s) ds \leq \frac{6}{146} C^* \left( \frac{V(\delta_D(y))}{\sqrt{t_0}} \wedge 1 \right) t \]
and
\[ \mathbb{P}^x(\tau_{D_1} > t/2) \leq \mathbb{P}^y(\tau_D > t/2) \leq \frac{3}{73} C^* \left( \frac{V(\delta_D(y))}{\sqrt{t_0}} \wedge 1 \right). \]

The estimates imply that
\[ I_2 \leq (C^*)^2 \left( \frac{V(\delta_D(y))}{\sqrt{t_0}} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) t \nu((x - y)/3). \]

Finally, for \( t \leq V^2(|x - y|) \) we have
\[ p_D(t, x, y) \leq 2(C^*)^2 \left( \frac{V(\delta_D(y))}{\sqrt{t_0}} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t_0}} \wedge 1 \right) t F(|x - y|/9). \]

\[ \square \]

Remark 2.2. With cosmetic adjustments, the proof also works for \( D = \left( B(Q_1, R) \cup B(Q_2, R) \right)^c \), where \( Q_1, Q_2 \in \mathbb{R}^d \). Then by domain monotonicity of heat kernels, the conclusion of Theorem 2.1 holds for every open set \( D \) having the outer ball property at scale \( R \).

Here is an analogue of Theorem 2.1 for convex sets. Noteworthy, we do not assume \((H)\) (or scalings) here.

**Theorem 2.3.** Suppose that \( p(t, x) \leq t F(|x|), \ t > 0, \ x \neq 0, \) with nonincreasing \( F \geq 0 \). Let \( D \) be open and convex. There is \( C = C(d) \) such that if \( x, y \in D \) and \( 0 < t \leq V^2(|x - y|) \), then
\[ p_D(t, x, y) \leq C \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{t}} \wedge 1 \right) (p_{t/2}(0) \wedge t F(|x - y|/9)). \]

**Proof.** By convexity of \( D \) and [S] (2.21) there is an absolute constant \( c \) such that
\[ \mathbb{P}^x(\tau_D > t) \leq c \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1 \right). \] (2.7)

Hence, by Lemma L.8 and subadditivity of \( V \),
\[ p_D(t, x, y) \leq 4c^2 \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{t}} \wedge 1 \right) p_{t/2}(0). \]
This provides the first part of the conclusion. The full conclusion follows by the proof of Theorem 2.2 with some modifications. We fix \( x_0 \) such that \( \delta_D(x) = |x - x_0| \) and define \( D_1, D_2 \) and \( D_3 \) exactly in the same way as in the proof of Theorem 2.1. To validate all the arguments we need to consider the case \( \tau_{D_1} \). Note that (2.25) provides a desired estimate for \( \mathbb{P}^\tau(\tau_D > t) \) and \( \mathbb{E}^\tau \tau_{D_1} \). With these estimates at hand, we may replace the constant \( H_R \) used in the proof of Theorem 2.1 (see, e.g., (2.20)) by a constant depending only on \( d \).

As we already indicated, the above two theorems apply to every pure-jump unimodal Lévy process with infinite Lévy measure: by Lemma 1.3 we can take \( F(r) = C_1/[r^dV^2(r)] \), to obtain the following consequences of Theorem 2.1.

**Corollary 2.4.** Let \( D \) be an open set satisfying the outer ball condition at a scale \( R \). There is a constant \( C = C(d) \) such that for all \( x, y \in D \) and \( t \leq V^2(|x - y|) \),

\[
p_D(t, x, y) \leq C \frac{H_R^2}{J(R)^2} \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) \left( p_{t/2}(0) \wedge \frac{t}{V^2(|x - y|)|x - y|^d} \right),
\]

provided (H) holds. If, additionally, \( \psi \in \text{WLSC}(\alpha, \theta, c) \), then for all \( t > 0 \),

\[
p_D(t, x, y) \leq C \frac{H_R^2}{c^{1+d/\alpha} J(R)^2} \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) \left( p_{t/2}(0) \wedge \frac{t}{V^2(|x - y|)|x - y|^d} \right),
\]

provided \( |x - y| < 1/\theta \). Here \( C = C(d, \alpha) \).

**Proof.** Let \( x \in D \) and \( S \in \partial D \) such that \( \delta_D(x) = |x - S| \). Since \( D \) satisfies the outer ball condition, there is a ball \( B \) of radius \( R \) such that \( B \subset D^c \) and \( S \in \overline{B} \) and \( \delta_D(x) = \overline{\delta_D}^B(x) \). By [17, Lemma 6.2],

\[
\mathbb{P}^x \left( \tau_D > \frac{t}{4} \right) \leq \mathbb{P}^x \left( \tau_B^D > \frac{t}{4} \right) \leq c \frac{H_R}{J(R)^2} \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right),
\]

with \( c = c(d) \). Hence, the first bound in the statement is a simple consequence of Lemma 1.8, Theorem 2.1 and Remark 2.2. To prove the second one we only need to consider the case \( t \geq V^2(|x - y|) \). Let \( t_0 = V^2(|x - y|) \). Since \( |x - y| \leq 1/\theta \), we have \( t_0 \leq V^2(1/\theta) \). Applying Lemma 1.8 we obtain

\[
p_{t/2}(0) \leq p_{t_0/2}(0) \leq C \frac{1}{c^{1+d/\alpha}} \left[ V^{-1}(t_0) \right]^{-d} = C \frac{1}{c^{1+d/\alpha}} \frac{1}{|x - y|^d} \leq C \frac{1}{c^{1+d/\alpha}} \frac{t}{V^2(|x - y|)|x - y|^d},
\]

with \( C = C(d, \alpha) \). This ends the proof due to Lemma 1.8.

Regarding the assumptions of Corollary 2.4 we recall that (H) holds automatically if \( \psi \in \text{WLSC} \) and \( d \geq 3 \).

**Remark 2.5.** In what follows, when we write \( \psi \in \text{WLSC} \cap \text{WUSC} \) and \( C = C(\psi, \ldots) \), we mean \( \psi \in \text{WLSC}(\alpha, \theta, c) \cap \text{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{c}) \) and \( C = C(\alpha, \theta, c, \overline{\alpha}, \overline{\theta}, \overline{c}, \ldots) \). Here is a simplifying convention.

**Theorem 2.6.** Let \( R > 0 \) and let \( D \) be an open set satisfying the outer ball condition at scale \( R \). Suppose that global \( \text{WLSC} \) and \( \text{WUSC} \) hold for \( \psi \). Then there is a constant \( C = C(d, \psi) \) such that for all \( t > 0 \) and \( x, y \in D \),

\[
p_D(t, x, y) \leq C \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) p(t, x, y).
\]

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Proof. Due to [3] Corollary 24] \( p_t(0) \approx p_{t/2}(0) \) and by Lemma 1.6 we have \( p_t(0) \approx [V^{-1}(\sqrt{t})]^{-d} \) with comparability constants depending only on \( d \) and \( \psi \). By Lemma 1.3 \( p(t, x, y) \approx p_{t/2}(0) \wedge [t|x - y|^{-d}/V^2(|x - y|)] \) with comparability constants depending only on \( d \) and \( \psi \).

By global WLSC and WUSC for \( \psi \), we have \( \inf_{R>0} J(R) > 0 \) (see [8, Proposition 5.2]) and \( (H^*) \) (see [8] Lemma 7.2 and Lemma 7.3), hence \( H_R/J(R)^2 \leq C = C(d, \psi) \).

Therefore the claim is an obvious consequence of the second bound of Corollary 2.4. \( \square \)

Remark 2.7. \((H)\) may usually be circumvented in the (exceptional) dimension \( d = 1 \), cf. [8, Proposition 2.6, Corollary 4.7]. This may be of interest for the upper bounds of the survival probability if \( \psi \) satisfies WLSC but not WUSC.

The following is a simple corollary to Theorem 2.3. We skip the proof, since it repeats the arguments used to prove Corollary 2.4.

**Corollary 2.8.** Let \( D \) be open and convex. If \( \psi \in WLSC(\alpha, \theta, \omega) \), \( t > 0 \), \(|x - y| < 1/\theta \) and \( x, y \in D \), then there is a constant \( C = C(d, \alpha) \) such that

\[
p_D(t, x, y) \leq \frac{C}{\xi^{2(1+\delta)/2\xi+1}} \left( \frac{V(\delta_D(x))}{t} \right)^{\wedge 1} \left( \frac{V(\delta_D(y))}{t} \right)^{\wedge 1} \left( p_{t/2}(0) \wedge \frac{t}{V^2(|x - y|)|x - y|^d} \right),
\]

and if \( \theta = 0 \), then there is a constant \( C = C(d, \psi) \) such that for all \( t > 0 \) and \( x, y \in D \),

\[
p_D(t, x, y) \leq C \left( \frac{V(\delta_D(x))}{t} \right)^{\wedge 1} \left( \frac{V(\delta_D(y))}{t} \right)^{\wedge 1} \left( p_t(0) \wedge \frac{t}{V^2(|x - y|)|x - y|^d} \right).
\]

### 3 Lower bound

By Lemma 1.3 \( p_t(x) \leq C_1 t/|V^2(|x|)|x|^d \). We shall often assume the following partial converse.

**Condition \( G_R \):** We say \( G_R \) holds if \( R > 0 \) and there is \( C_1^* \in [1, \infty) \) such that

\[
\frac{t}{V^2(|x|)|x|^d} \leq C_1^* p_t(x), \quad 0 < t \leq V^2(|x|), \quad |x| \leq R.
\] (3.1)

The condition is merely for notational convenience since it has the following characterization.

**Lemma 3.1.** Let \( 0 < R < \infty \). \( G_R \) holds if and only if \( \psi \in WLSC \cap WUSC \) and \( \nu(R^{-}) > 0 \).

**Proof.** For one implication we assume that \( \psi \in WLSC \cap WUSC \) and \( \nu(R^{-}) > 0 \). By Lemma 1.5 there is \( r = r(d, \psi) > 0 \) such that \( G_r \) holds. We may and do assume that \( R > r \). Let \( r \leq |x| \leq R \). By Lemma 1.11 and continuity of \( p_t \), for \( 0 < t \leq V^2(R) \) we have

\[
p_t(R) \geq 4^{-d} t \nu(R^{-})[P_0(\tau_{B_R/4} > V^2(R))]/V^2(R^2) = ct.
\]

By radial monotonicity of \( p_t \),

\[
p_t(x) \geq p_t(R) \geq cV^2(r) r^d \frac{t}{V^2(|x|)|x|^d},
\]
as needed. For the converse implication, we note that \( G_R \) and [3] Theorem 26 imply scalings of \( \psi \) with \( \theta = R^{-1} \). Since \( p_t(x)/t \rightarrow \nu(x) \) vaguely on \( \mathbb{R}^d \setminus \{0\} \), \( G_R \) and monotonicity of \( \nu \) yield \( \nu(R^{-}) \geq C_1^*^{-1}/[V^2(R)|R^d|] > 0 \), which ends the proof. \( \square \)
Thus in many cases, if $G_R$ holds for some value $R$ and $C_1^*$, then it holds for every $R \in (0, \infty)$ with $C_1^*$ depending on $R$. This is so, e.g., for every subordinate Brownian motion, due to Lemma 3.1 and positivity of $\nu$. It may also happen that (3.1) is true for some $R$, but it fails for larger values of $R$. This is the case for the truncated Lévy process, whose Lévy measure is supported by a bounded set (see Section 5). For clarity, $G_\infty$ is equivalent to global scaling conditions on $\psi$ [9 Theorem 26]. Notice also that due to [9 Theorem 26] and [8] Lemmas 7.2 and 7.3, $G_R$ implies (H). Furthermore, if we replace $X$ by $X/R$, then by (1.4), $V(x)$ is replaced by $V(Rx)$, and if we subsequently replace $x$ by $Rx$, then we equivalently obtain $G_1$ for $X/R$.

Before stating the next result we recall that $\mathcal{I}$ is defined in (1.20).

**Lemma 3.2.** There is $c = c(d)$ such that if $G_R$ holds, then

$$\inf_{r \in R} \mathcal{I}(r) \geq c/C_1^*.$$  

**Proof.** Let $0 < r \leq R$. Note that (3.1) implies

$$\nu(x) \geq \frac{1}{C_1^* V^2(|x|)|x|^d} |x| < R.$$  

(3.2)

For $\rho \leq r/2$ we obtain

$$V^2(\rho) \nu(B_r \setminus B_\rho) \geq V^2(\rho) \int_{B_{2r} \setminus B_\rho} \frac{dx}{C_1^* V^2(|x|)|x|^d} \geq \frac{1}{4} \int_{B_{2r} \setminus B_\rho} \frac{dx}{C_1^* |x|^d} = \frac{c}{C_1^*},$$

which completes the proof. \hfill \Box

We now give the lower bound for the heat kernel for union of two balls of the same radius.

**Theorem 3.3.** Let $R > 0$ and $\psi \in \text{WLSC}(\underline{\alpha}, R^{-1}, \underline{\lambda})$. Assume that $G_R$ is satisfied. Let $D = B(z_1, R) \cup B(z_2, R)$. There exist $c = c(d) < 1, c_1 = c_1(d, \underline{\alpha})$ such that

$$p_D(t, x, y) \geq \frac{c_1 c^{1+d/\underline{\alpha}}}{H_R^2(C_1^*)^{d/\underline{\alpha}}} \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{t}} \wedge 1 \right) (p_{t/2}(0) \wedge [t \nu(2|x-y|\wedge \text{diam}(D)))],$$

provided $0 < t \leq cV^2(R)/C_1^*, x, y \in D, \delta_D(x) = \delta_{B(z_1, R)}(x)$ and $\delta_D(y) = \delta_{B(z_2, R)}(y)$.  

**Proof.** By the discussion at the beginning of the section we may and do assume that $R = 1$. We may also assume that $z_1 = 0$. Let

$$c^* = \min \{(12^d + 43^d + 32C_3 C_1^* - 1, C_5/36, C_7)\},$$

where $C_5 < 1, C_7$ are from Lemma 1.14 and Corollary 1.17 and $C_1^*$ is from (3.1). Let $0 < t \leq (c^*/9)V^2(1) \leq c^*V^2(1/3)$. Let $0 < r \leq 1/3$ be such that

$$t = c^*V^2(r), \quad \text{or} \quad r = V^{-1}(\sqrt{t/c^*}).$$

Let $x \in D$. If $\delta_D(x) < r/6$, then we let $x_0 = x/|x|, x_1 = x_0(1 - r/2)$ and $r_2 = r/2$, otherwise we let $x_1 = x$ and $r_2 = \delta_D(x)$. Denote $D_x = B(x_1, r_x)$. Similarly, we let $D_y = B(y_1, r_y)$, where

$$y_1 = y_0(1 - r/2) + z_2 \text{ if } \delta_D(y) < r/6 \text{ and } r_y = r/2, \text{ with } y_0 = (y - z_2)/|y - z_2|, \text{ and we let } y_1 = y, r_y = \delta_D(y) \text{ otherwise.}$$
CASE I. We first assume that $|x - y| > 2r$. For $u \in D_x$ and $v \in D_y$ we have $|u - v| \leq |u - x| + |x - y| + |y - v| \leq |x - y| + 2r \leq 2|x - y|$. We next use Lemma 1.10 with $D_1 = D_x$ and $D_3 = D_y$, and obtain

$$p_D(t, x, y) \geq \mathbb{P}^\tau(\tau_{D_x} > t)\mathbb{P}^\nu(\tau_{D_y} > t) t \inf_{u \in D_1, v \in D_3} \nu(u - v) \geq \mathbb{P}^\tau(\tau_{D_x} > t)\mathbb{P}^\nu(\tau_{D_y} > t) t \nu(2|x - y| \wedge \text{diam}(D)).$$

By subadditivity of $V$ we have $t = c^*V^2(r) \leq C_3V^2(r)/36 \leq C_5V^2(r/6) \leq C_5V^2(r_x)$. By Lemma 1.14 and 3.2,

$$\mathbb{P}^\tau(\tau_{D_1} > t) \geq C_6\frac{c_1}{H_1C_1^t} \left( \frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) = C_6\frac{c_1}{H_1C_1^t} \left( \frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right),$$

where $c_1 = c_1(d)$. Hence,

$$p_D(t, x, y) \geq c_2(H_1C_1^t)^{-2} \left( \frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) t \nu(2|x - y| \wedge \text{diam}(D)),$$

where $c_2 = c_2(d)$. Since $C_1^t \geq 1$ and $\epsilon \leq 1$, we have a complete proof in this case.

CASE II. $x, y \in D : |x - y| \leq 2r$. We define $\tilde{D}_x = B(x, r/12)$. Since $t = c_0V^2(r) \leq C_7V^2(r)$, by Corollary 1.17, Lemma 3.2 and (3.2) we have,

$$\int_{\tilde{D}_x} p_D(t/3, x, v)dv \geq C_8\frac{c_3}{H_1C_1^t} c^*V^2(r)\nu(r)r^d \left( \frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \geq \frac{c_4}{H_1(C_1^t)^3} \left( \frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right),$$

where $c_4 = c_4(d)$. A similar inequality obtains for $\int_{\tilde{D}_y} p_D(t/3, y, v)dv$.

Let $u \in \tilde{D}_x$ and $v \in \tilde{D}_y$. We claim that there is $c_5 = c_5(d, \alpha)$ such that

$$p_D(t/3, u, v) \geq c_5 \frac{e^{1+d/\alpha}}{(C_1^t)^3+d/\alpha} p_t(0).$$

Indeed, we have $|u - v| \leq 3r$. Our aim is to estimate $\mathbb{E}^u p(t/3 - \tau_D, X_{\tau_D}, v)$. Since $|z - v| \geq r/12$ for all $z \in D^c$, by (1.7) and subadditivity of $V$ we obtain

$$\mathbb{E}^u p(t/3 - \tau_D, X_{\tau_D}, v) \leq 12dC_1 \frac{t}{V^2(r/12)r^d} \mathbb{P}^u(\tau_D \leq t/3) \leq 12d^2C_1 \frac{t}{V^2(r)r^d} \mathbb{P}^0(\tau_{B_{r/12}} \leq t/3) \leq \frac{12d^4C_3C_1}{r^d} \left( \frac{t}{V^2(r)} \right)^2,$$

where the last step uses (1.18). Next, since $t \leq V^2(3r)$ and $r \leq 1/3$, by (3.1) we have

$$p(t/3, u, v) \geq \frac{t}{3C_1^tV^2(3r)(3r)^d} \geq \frac{t}{3d^3C_1^tV^2(r)r^d}.$$

Recall that $\psi \in \text{WLSC}(\alpha, 1, \ell)$, $t = c^*V^2(r) \leq c^*V^2(1)$ and $c^*C_1^tC_3C_112d^43^{d+3} \leq 1/2$. Thus, $\mathbb{E}^\psi p(t/3 - \tau_D, X_{\tau_D}, v)$

$$\geq c^* \frac{1}{3d^3C_1^t r^d} - (c^*)^2 \frac{12d^4C_3C_1}{r^d} \geq c^* \frac{1}{2C_1^t 3^{d+3} r^d} = \frac{c^*}{2C_1^t 3^{d+3}} (V^{-1}(\sqrt{t}/c^*))^{-d},$$

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and by (1.11),

\[ \left( V^{-1} \left( \sqrt{t/c} \right) \right)^{-d} \geq c_6 (c^* t)^{1+d/\alpha} p_t(0), \]

with \( c_6 = c_6(d, \alpha) \). Since \( c^* C_1^* \) is a positive constant depending only on \( d \), we obtain (3.5). By (3.5) and (3.4),

\[
p_D(t, x, y) \geq \int_{\tilde{B}_x} \int_{\tilde{B}_y} p_D(t/3, x, u) p_D(t/3, u, v) p_D(t/3, v, y) du dv
\]

\[
\geq c_5 \frac{c^{1+d/\alpha}}{(C_1^*)^{3+d/\alpha}} p_t(0) \int_{\tilde{B}_x} p_D(t/3, x, u) du \int_{\tilde{B}_y} p_D(t/3, v, y) dv
\]

\[
\geq c_7 \frac{c^{1+d/\alpha}}{H_1^2(C_1^*)^{3+d/\alpha}} \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1 \right) p_t(0) \left( \frac{V(\delta_D(y))}{\sqrt{t}} \wedge 1 \right),
\]

with \( c_7 = c_7(d, \alpha) \). The proof is complete, cf. Remark 1.7.

The end of the above proof shows a major strategy for \( X \) to connect \( x \) and \( y \) and survive in \( D \) time \( t \): evade \( D^c \) by going from \( x \) and \( y \) towards the center of \( D \) and connect then.

**Remark 3.4.** Suppose that global WLSC and WUSC hold for \( \psi \). Then (3.1) holds for all \( t \) and \( x \) such that \( 0 < t \leq V^2(|x|) \) with the constant \( C_1^* \) depending only on \( d \) and \( \psi \). Furthermore, (H*) holds. It follows that the constant in the lower bound in Theorem 3.3 may be so chosen to depend only on \( d \) and \( \psi \).

**Corollary 3.5.** Suppose that global WLSC and WUSC hold for \( \psi \). Constants \( c^* = c^*(d, \psi) \) and \( C^* = C^*(d, \psi) \) exist such that for every open \( D \) with inner ball condition at scale \( R \),

\[
p_D(t, x, y) \geq C^* \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{t}} \wedge 1 \right) p(t, x, y),
\]

if \( 0 < t \leq c^* V^2(R) \). If \( D = \mathbb{H}_0 \) or \( D = \overline{B}_R \), then the estimate is true for all \( t > 0 \).

**Proof.** This easily follows from domain monotonicity by using a union of two balls of radius \( R \) instead of \( D \) and applying Theorem 3.3 along with Remark 3.4 and [9, Corollary 23].

The following improvement of [33, Theorem 5.10] stems from Corollary 3.5 and Theorem 2.6.

**Corollary 3.6.** If global WLSC and WUSC hold for \( \psi \), then for \( D = \mathbb{H}_0 \) and \( x, y \in D \) and \( t > 0 \),

\[
p_D(t, x, y) \approx \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{t}} \wedge 1 \right) p(t, x, y),
\]

with comparability constant depending only on \( d \) and the scaling characteristics of \( \psi \).

Below we give sharp heat kernel estimates for other classes of \( C^{1,1} \) sets.
4 Global estimates for bounded $C^{1,1}$ sets

In this section we provide sharp explicit estimates of the heat kernel of bounded $C^{1,1}$ open sets. To this end we combine spectral properties of the heat kernel $p_D(t, x, y)$ for large time with the finite-time estimates obtained in Sections 2 and 3. Our discussion of spectral properties of $p_D$ closely follows that in [15, the proof of Theorem 1.1] but we additionally provide explicit control of comparability constants, which is delicate in the intermediate region between small and large times. For instance under global scaling conditions on $\psi$ we give an estimate of the heat kernel of the ball of arbitrary radius, uniform enough to reproduce optimal estimates of the heat kernel of a halfspace.

Let $D$ be an open bounded set. In the remainder of the section we assume that $p_t(0)$ is finite for every $t > 0$, cf. Lemma 1.1. Then the semigroup of integral operators on $L^2(D)$ with kernels $p_D(t, x, y) \leq p_t(0)$ is compact, in fact Hilbert-Schmidt. General theory yields eigenvalues $0 < \lambda_1 < \lambda_2 \leq \ldots$ and orthonormal basis of eigenfunctions $\phi_1 \geq 0, \phi_2, \phi_3, \ldots$:

$$\phi_k(x) = e^{\lambda_k t} \int p_D(t, x, z) \phi_k(z) dz.$$  

**Lemma 4.1.** Let $f \in L^2(D)$. Then

$$e^{-\lambda_1 t} \left( \int f(w) \phi_1(w) dw \right)^2 \leq \int \int f(w) f(z) p_D(t, z, w) dz dw \leq e^{-\lambda_1 t} \int f^2(w) dw.$$

**Proof.** The result obtains from the identities

$$\int f^2(w) dw = \sum_k \left( \int f(w) \phi_k(w) dw \right)^2,$$

$$\int \int f(w) f(z) p_D(t, z, w) dz dw = \sum_k e^{-\lambda_k t} \left( \int f(w) \phi_k(w) dw \right)^2.$$

The following general bound is an easy consequence of Lemma 4.1.

**Lemma 4.2.** Let $t_0 > 0$. For $t \geq t_0$ and $x, y \in D$,

$$p_D(t, x, y) \leq |D| p_{t_0/4}(0)^2 \mathbb{P}^x \left( \tau_D > \frac{t_0}{4} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{4} \right) e^{\lambda_1 t_0} e^{-\lambda_1 t}.$$

**Proof.** Let $t_0 > 0$. We use Lemma 1.8 and Lemma 4.1 with $f \equiv I_D$. Then for $t > t_0$,

$$p_D(t, x, y) = \int \int p_D \left( \frac{t_0}{2}, x, z \right) p_D \left( t - t_0, z, w \right) p_D \left( \frac{t_0}{2}, w, y \right) dz dw$$

$$\leq \left[ p_{t_0/4}(0) \right]^2 \mathbb{P}^x \left( \tau_D > \frac{t_0}{4} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{4} \right) \int \int p_D \left( t - t_0, z, w \right) dz dw$$

$$\leq |D| p_{t_0/4}(0)^2 \mathbb{P}^x \left( \tau_D > \frac{t_0}{4} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{4} \right) e^{\lambda_1 t_0} e^{-\lambda_1 t}.$$


We now discuss the corresponding lower bound.

**Lemma 4.3.** If \( t_0 > 0 \) and \( c_\ast > 0 \) are such that

\[
p_D \left( \frac{t_0}{2}, x, z \right) \geq c_\ast \mathbb{P}^x \left( \tau_D > \frac{t_0}{2} \right) \mathbb{P}^z \left( \tau_D > \frac{t_0}{2} \right), \quad x, z \in D, \tag{4.1}
\]

then for \( t \geq t_0 \) and \( x, y \in D \),

\[
p_D(t, x, y) \geq \left( \frac{c_\ast}{\sqrt{|D|} p_{t_0/2}(0)} \right)^2 e^{-\lambda_1 t_0} \mathbb{P}^x \left( \tau_D > \frac{t_0}{2} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{2} \right) e^{-\lambda_1 t}.
\]

**Proof.** Since \( \lambda_1 \) is the eigenvalue corresponding to \( \phi_1 \),

\[
\phi_1(x) = e^{2\lambda_1 s} \int p_D(2s, x, z) \phi_1(z) dz.
\]

From Lemma 1.8 \( p_D(2s, x, z) \leq \mathbb{P}^x(\tau_D > s) p_s(0) \), and by Schwartz inequality,

\[
\phi_1(x) \leq e^{2\lambda_1 s} p_s(0) \mathbb{P}^x(\tau_D > s) \int \phi_1(z) dz \leq \sqrt{|D|} e^{2\lambda_1 s} \mathbb{P}^x(\tau_D > s) p_s(0).
\]

Taking \( s = t_0/2 \), we obtain

\[
\phi_1(x) \leq \sqrt{|D|} p_{t_0/2}(0) e^{\lambda_1 t_0} \int \phi_1(z) \mathbb{P}^z \left( \tau_D > \frac{t_0}{2} \right) dz.
\]

which in turn yields

\[
1 = \int \phi_1^2(z) dz \leq \sqrt{|D|} p_{t_0/2}(0) e^{\lambda_1 t_0} \int \phi_1(z) \mathbb{P}^z \left( \tau_D > \frac{t_0}{2} \right) dz. \tag{4.2}
\]

Let \( t > t_0 \). By (4.1), Lemma 4.1 with \( f(z) = \mathbb{P}^z(\tau_D > t_0/2) \) and (4.2) we have

\[
p_D(t, x, y) = \int \int p_D \left( \frac{t_0}{2}, x, z \right) p_D \left( t - t_0, z, w \right) p_D \left( \frac{t_0}{2}, w, y \right) dz dw
\]
\[
\geq c_\ast^2 \mathbb{P}^x \left( \tau_D > \frac{t_0}{2} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{2} \right)
\]
\[
\times \int \int \mathbb{P}^z \left( \tau_D > \frac{t_0}{2} \right) \mathbb{P}^w \left( \tau_D > \frac{t_0}{2} \right) p_D \left( t - t_0, z, w \right) dz dw
\]
\[
\geq c_\ast^2 \mathbb{P}^x \left( \tau_D > \frac{t_0}{2} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{2} \right) e^{-\lambda_1 (t-t_0)} \left( \int \mathbb{P}^w \left( \tau_D > \frac{t_0}{2} \right) \phi_1(w) dw \right)^2
\]
\[
\geq \left( \frac{c_\ast}{\sqrt{|D|} p_{t_0/2}(0)} \right)^2 e^{-\lambda_1 t_0} \mathbb{P}^x \left( \tau_D > \frac{t_0}{2} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{2} \right) e^{-\lambda_1 t}.
\]

Lemma 4.2 and 4.3 indicate the asymptotics of the killed semigroup for large times.

In what follows, we interchangeably write \( \lambda_1(D) = \lambda_1 \). Here is a sharp bound for the first eigenvalue in terms of \( V \).
Proposition 4.4. Let \( D \) be an open bounded set containing a ball of radius \( r \). Then
\[
\frac{1}{8} \left( \frac{r}{\text{diam } D} \right)^2 \leq \lambda_1(D) V^2(r) \leq c \left( \frac{\text{diam } D}{r} \right)^{d/2},
\]
where \( c = c(d) \).

Proof. The following bound is proved in [2, Proposition 2.1]:
\[
\frac{1}{\sup_{x} \mathbb{E}^x \tau_D} \leq \lambda_1(D) \leq \frac{\int_{D} \mathbb{E}^x \tau_D dx}{\int_{D} (\mathbb{E}^x \tau_D)^2 dx}.
\]

By the Cauchy-Schwarz inequality,
\[
\int_{D} \mathbb{E}^x \tau_D dx \leq \|D\|^{1/2} \left( \int_{D} (\mathbb{E}^x \tau_D)^2 dx \right)^{1/2}.
\]

Let \( B(x_0, r) \subset D \). By [8, Lemma 2.3], \( \sup_{x} \mathbb{E}^x \tau_D \leq 2V^2(\text{diam } D) \) and by (1.17), \( \inf_{x \in B(x_0, r/2)} \mathbb{E}^x \tau_D \geq V^2(r) / C_2 \). Hence,
\[
\lambda_1(D) \leq C_2 \frac{1}{V^2(r)} \sqrt{\frac{|D|}{|B(x_0, r/2)|}} \leq C_2 \frac{2^{d/2}}{V^2(r)} \left( \frac{\text{diam } D}{r} \right)^{d/2}
\]
and
\[
\lambda_1(D) \geq \frac{1}{2V^2(\text{diam } D)} = \frac{1}{2V^2(r)} \frac{V^2(r)}{V^2(\text{diam } D)} \geq \frac{1}{8V^2(r)} \left( \frac{r}{\text{diam } D} \right)^2,
\]
where in the last step we used subadditivity of \( V \).

Here is the main result of this section (cf. Remark 2.5 for our notational conventions).

Theorem 4.5. Let \( \psi \in \text{WLSC} \cap \text{WUSC} \). There is \( r_0 = r_0(d, \psi) > 0 \) such that if \( 0 < r < r_0 \) and open \( D \subset \mathbb{R}^d \) is bounded and \( C^{1,1} \) at scale \( r \), and \( \nu(\text{diam } D) > 0 \), then for all \( x, y \in \mathbb{R}^d, t > 0 \),
\[
p_D(t, x, y) \approx \mathbb{P}^x(\tau_D > t/2) \mathbb{P}^y(\tau_D > t/2) p \left( t \wedge V^2(r), x, y \right)
\]
and
\[
\mathbb{P}^x(\tau_D > t) \approx e^{-\lambda_1 t} \left( \frac{V(\delta_D(x))}{t \wedge V(r) \wedge 1} \right).
\]

If the scalings are global, then we may take \( r_0 = \infty \) and comparability constants depending only on \( d, \text{diam } D/r \) and scaling characteristics of \( \psi \).

Proof. Define \( t_0 = V^2(r) \), and for \( x \in \mathbb{R}^d, t > 0, s > 0 \),
\[
\tilde{p}_t(x) = p_{t/2}(0) \wedge \frac{t}{V^2(|x|)|x|^d},
\]
\[
\tilde{p}_t(x) = p_{t/2}(0) \wedge [t \nu([2|x| \wedge \text{diam}(D)]),
\]
\[
\phi(t, x, s) = \frac{V(\delta_D(x))}{t \wedge V(s) \wedge 1},
\]
\[
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\]
with the convention that \( \hat{p}_t(0) = p_{t/2}(0) = \hat{p}_t(0) \). Clearly, \( \hat{p}_t(x) \) and \( \hat{p}_t(x) \) are nonincreasing functions of \(|x|\), \( \phi(t, x, s) \) is nonincreasing in \( t \),

\[
\hat{p}_t(x)/\hat{p}_t(x) = p_{t/2}(0)/[\sqrt{t/V^2(|x|)|x|^d}] \wedge [\nu(|2x| \wedge \text{diam}(D))/[1/V^2(|x|)|x|^d]],
\]

\[
\hat{p}_t(x)/\hat{p}_t(y) \geq \nu(|\text{diam}(D)|)/\nu(|2y| \wedge \text{diam}(D)), \quad \text{and } k\hat{p}_t(x) \geq \hat{p}_{kt}(x) \text{ for } k \geq 1. \]

Since \( \psi \in \text{WLSC} \cap \text{WUSC} \), by Lemma 1.5 and Lemma 1.6,

\[
c^*\hat{p}_t(x) \leq p_t(x) \leq C_1\hat{p}_t(x), \quad (4.5)
\]

where \( c^* = c^*(d, \psi) \leq 1 \), \( r_0 = r_0(d, \psi) \), \(|x| < r_0 \), \( 0 < t < t_1 \) and \( t_1 = V^2(r_0) \). The upper bound in (4.5) even holds for all \( c \) where \( C^* = c(d)/c^* \). By letting \( t \to 0 \) in (4.5), we get for \(|x| < r_0\),

\[
\frac{c^*}{V^2(|x||x|^d)} \leq \nu(x) \leq \frac{C_1}{V^2(|x||x|^d)}. \quad (4.6)
\]

If \( r_0 \leq \text{diam } D \), then we can extend the lower bound in (4.5) and (4.6) to \( r_0 \leq |x| \leq \text{diam } D \) and \( 0 < t \leq t_0 \) due to Lemma 3.1. For \( s = 2|x| \wedge \text{diam } D \), by (4.6) and subadditivity of \( V \),

\[
\nu(s) \geq \frac{c^*}{V^2(s|s|^d)} \geq \frac{c^*}{V^2(2|x|)|2x|^d} \geq \frac{c^*}{2^{d+2} V^2(|x||x|^d)}. \]

By the definitions of \( \hat{p} \) and \( \hat{p} \), for \( x, y \in D \) and \( 0 < t \leq t_0 \) we have

\[
\hat{p}_t(x - y) \geq \frac{c^*}{2^{d+2} \hat{p}_t(x - y)} \geq \frac{c^*}{2^{d+2} C_1} p(t, x, y). \quad (4.7)
\]

From now on we let \( t > 0 \) and \( x, y \in D \) (additional restrictions are indicated as we proceed). We may assume that \( \psi \in \text{WLSC}(\alpha, \theta, \zeta) \) with \( \theta = 1/r \), cf. Remark 1.4. By Corollary 2.4,

\[
p_D(t, x, y) \leq c_1\phi(t, x, r)\phi(t, y, r)\hat{p}_t(x - y), \quad t \leq t_0. \quad (4.8)
\]

where \( c_1 = cH^2\mathcal{J}(r)^{-4}c^{-1-d/\alpha} \) and \( c = c(d, \alpha) \). To facilitate justification of the last statement of the theorem, all enumerated constants are fixed throughout the proof. Let \( \lambda_1 = \lambda_1(D) \). By [8 Lemma 6.2] there is \( c_2 = c_2(d, r, \psi) \) such that \( \mathbb{P}^x(\tau_D > t/4) \leq \sqrt{c_2\phi(t, x, r). \text{ By Lemma 1.2}} \)

for \( t \geq t_0 \) we have

\[
p_D(t, x, y) \leq |D| p_{t_0/4}^2(0) \mathbb{P}^x \left( \tau_D > \frac{t_0}{4} \right) \mathbb{P}^y \left( \tau_D > \frac{t_0}{4} \right) e^{\lambda_1 t_0} e^{-\lambda_1 t}
\]

\[
\leq c_2 |D| p_{t_0/4}^2(0) \phi(t, 0, x, r)\phi(t, 0, y, r)e^{\lambda_1 t_0} e^{-\lambda_1 t}
\]

\[
\leq c_2 |D| p_{t_0/4}^2(0) e^{\lambda_1 t_0} \phi(t, 0, x, r)\phi(t, 0, y, r)p(t, 0, x, y)e^{-\lambda_1 t}. \quad (4.9)
\]

Combining (4.8) and (4.9) with (4.5) we get

\[
p_D(t, x, y) \leq c_3\phi(t, x, r)\phi(t, y, r)p(t \wedge t_0, x, y)e^{-\lambda_1 t}, \quad t > 0, \quad (4.10)
\]

where \( c_3 = (c_2 |D| p_{t_0/4}^2(0)/p_{t_0/4}(\text{diam } D) + c_1/c^*) e^{\lambda_1 t_0} \).
We now give a similar lower bound. By Theorem 3.3 and domain monotonicity of heat kernels, there is $c = c(d, \psi) < 1$ such that for $t \leq t_2 := ct_0,$

$$p_D(t, x, y) \geq c_4 \phi(t, x, r)\phi(t, y, r)\hat{p}_t(x - y),$$

(4.11)

with $c_4 = c_0d^{1+d/2}H_r^{-2}(C_1)^{-9-d/2}.$ Since $\phi(t, x, r) \geq P^x(t_D > t)/\sqrt{c_2},$ we have

$$p_D(t, x, y) \geq c_5 P^x(t_D > t)P^y(t_D > t)\hat{p}_t(x - y), \quad t \leq t_2.$$

where $c_5 = c_4/c_2.$ In particular,

$$p_D(t_2/2, x, y) \geq c_5\hat{p}_{t_2/2}(\text{diam } D)P^x(t_D > t_2/2)P^y(t_D > t_2/2).$$

By Lemma 1.14 there is $c_6 = c_6(d, r, \psi)$ such that $P^z(t_D > t_2/2) \geq \sqrt{c_6} \phi(t_2, z, r),$ $z \in D.$ By Lemma 4.3 for $t \geq t_2$ we have

$$p_D(t, x, y) \geq \left(\frac{c_5\hat{p}_{t_2/2}(\text{diam } D)}{\sqrt{|D|\hat{p}_{t_2/2}(0)}}\right)^2 e^{-\lambda t_2} P^x(t_D > t_2/2)P^y(t_D > t_2/2) e^{-\lambda t}$$

$$\geq c_6 \left(\frac{c_5\hat{p}_{t_2/2}(\text{diam } D)}{\sqrt{|D|\hat{p}_{t_2/2}(0)}}\right)^2 e^{-\lambda t_2} \phi(t_2, x, r)\phi(t_2, y, r)e^{-\lambda t}.$$

By the above-mentioned monotonicity properties of $\hat{p}_t$ for $t \geq t_2$ we have

$$p_D(t, x, y) \geq c_7 \hat{p}_{t_2}(x - y)\phi(t_2, x, r)\phi(t_2, y, r)e^{-\lambda t},$$

(4.12)

where $c_7 = c_6c^2e^{-\lambda t_2}\hat{p}^2_{t_2/2}(\text{diam } D)/\left[\hat{p}_{t_2/2}(0)|D|\hat{p}_{t_2}(0)\right].$ Combining (4.11) and (4.12) we get

$$p_D(t, x, y) \geq (c_4 + c_7)\hat{p}_{t_2}(x - y)\phi(t \land t_2, x, r)\phi(t \land t_2, y, r)e^{-\lambda t} \quad (t > 0).$$

(4.13)

Since $\hat{p}_{t_2}(x - y) \geq (t_2/t_0)\hat{p}_{t_0}(x - y),$ by the aforementioned monotonicity of $\phi$ and (4.7)

$$p_D(t, x, y) \geq c_8\hat{p}_{t_0}(x - y)\phi(t \land t_0, x, r)\phi(t \land t_0, y, r)e^{-\lambda t}$$

$$\geq c_8c_9\phi(t \land t_0, x, y)\phi(t, x, r)\phi(t, y, r)e^{-\lambda t},$$

(4.14)

where $c_8 = (t_2/t_0)(c_4 + c_7),$ and $c_9 = c^*/(2d+2C_1) > 0.$ Combining (4.10) with (4.14) we get

$$p_D(t, x, y) \approx \phi(t, x, r)\phi(t, y, r)p(t \land t_0, x, y)e^{-\lambda t}.$$  

(4.15)

Now we prove (4.3). The upper bound: $P^t(t_D > t) \leq c_3e^{-\lambda t} \phi(t, x, r)$ is an easy consequence of (4.15) since $\phi(t, y, r) \leq 1$ and $\int_{\mathbb{R}^d} p(t \land t_0, x, y)dy = 1.$ Lemma 1.15 implies the lower bound for $t \leq t_2,$ cf. 3.3. If $t > t_2$ then, by (4.13) and monotonicity of $\phi,$

$$P^t(t_D > t) \geq (c_4 + c_7)e^{-\lambda t}\phi(t_2, x, r)\hat{p}_{t_2}(\text{diam } D)\int_D \phi(t_2, y, r)dy \geq c_{10}e^{-\lambda t}\phi(t_2, x, r),$$

where $c_{10} = (c_4 + c_7)\hat{p}_{t_2}(\text{diam } D)\int_D \phi(t_2, y, r)dy.$

We now assume that $\psi$ satisfies global scaling conditions. Then $t_1 = r_0 = \infty.$ We shall investigate the dependence of the constants $c_1 - c_{10}$ on $r$ and $\text{diam } D.$ The constants $c_1, c_2, c_4 - c_6,$ $c_9$ depend only on the dimension and $\psi$ (through scaling characteristics), but not on $r$ or $\text{diam } D.$ This is due to [8 Proposition 5.2, Lemma 7.2 and Lemma 7.3] and [24]
Proposition 3.5], which imply that quantities \( J(s) \), \( \mathcal{I}(s) \) and \( H_s \) are uniformly bounded in \( s \in (0, \infty) \) from below and above by two positive constants. Furthermore, \( c_8 \) depends only on \( c_7 \), \( d \) and the scaling characteristics. Therefore we only need to inspect \( c_3 \), \( c_7 \) and \( c_{10} \).

We claim that \( c_3 \leq c_3^* = c_3^*(\text{diam } D/r, d, \psi) < \infty \), \( c_7 \geq c_7^* = c_7^*(\text{diam } D/r, d, \psi) > 0 \) and \( c_{10} \geq c_{10}^* = c_{10}^*(\text{diam } D/r, d, \psi) > 0 \). The remaining comparisons in this proof depend only on \( \psi \) and \( d \). We have \( t_0/4 \approx t_2/2 \approx V^2(r) \), then \( p_{t_0/4}(0) \approx p_{t_2/2}(0) = \hat{p}_{t_2}(0) \approx r^{-d} \). Furthermore,

\[
\hat{p}_{t_0}(\text{diam } D) \approx \hat{p}_{t_2}(\text{diam } D) \approx \hat{p}_{t_2}(\text{diam } D) \approx p_{t_0}(\text{diam } D) \approx \frac{V^2(r)}{V^2(\text{diam } D)(\text{diam } D)^d},
\]

and

\[
\frac{|D| p_{t_0/4}(0)^2}{\hat{p}_{t_0}(\text{diam } D)} \approx \frac{|D|(\text{diam } D)^d V^2(\text{diam } D)}{V^2(r) r^{-2d}} \leq \frac{c(\text{diam } D)^{2d} V^2(\text{diam } D)}{V^2(r) r^{-2d}} \leq c \left( \frac{\text{diam } D}{r} \right)^{2d+2},
\]

where in the last step we used subadditivity (1.5) of \( V \), and \( c = c(d) \). By the same arguments,

\[
\left( \frac{\hat{p}_{t_2/2}(\text{diam } D)}{p_{t_2/2}(0) \sqrt{|D| \hat{p}_{t_2}(0)}} \right)^2 \geq c^{-1} \left( \frac{r}{\text{diam } D} \right)^{3d+4}.
\]

By Proposition 4.4 we have

\[
\lambda_1 t_2 \leq \lambda_1 t_0 = \lambda_1 V^2(r) \leq c \left( \frac{\text{diam } D}{r} \right)^{d/2},
\]

where \( c = c(d) \). Furthermore,

\[
\int_D \phi(t_0, y, r) dy \geq \int_{B_{r/2}} \left( \frac{V(r/2)}{V(r)} \wedge 1 \right) dy \geq |B_{r/2}|/2,
\]

hence

\[
\hat{p}_{t_2}(\text{diam } D) \int_D \phi(t_0, y, r) dy \geq c \frac{V^2(r)r^d}{V^2(\text{diam } D)(\text{diam } D)^d} \geq c \left( \frac{r}{\text{diam } D} \right)^{d+2}.
\]

The above arguments indeed show that if the global scaling conditions hold, then the constants \( c_1-c_{10} \) depend only through \( \text{diam } D/r, d \) and the scaling characteristics of \( \psi \).

Here is a simple consequence of Theorem 4.5.

**Corollary 4.6.** If \( D = B_R, \lambda_1(R) = \lambda_1(D) \) and \( \psi \in \text{WLSC}(\underline{a}, 0, \underline{c}) \cap \text{WUSC}(\overline{a}, 0, \overline{c}) \), then

\[
p_D(t, x, y) \approx e^{-\lambda_1(R)t} \left( \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{t} \wedge V(R)} \wedge 1 \right) p(t \wedge V^2(R), x, y)
\]

\[
\approx \mathbb{P}^x \left( \tau_D > \frac{t}{2} \right) \mathbb{P}^y \left( \tau_D > \frac{t}{2} \right) p(t \wedge V^2(R), x, y),
\]

for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \). The comparability constants depend only on \( d \) and the scaling characteristics of \( \psi \).

Such uniform estimates should be useful in approximation and scaling arguments, especially that \( p_D \) is monotone in \( D \).
Corollary 4.7. Under the assumptions of Corollary 4.6, let $\phi_1^R$ be the (positive) eigenfunction corresponding to $\lambda_1(R)$. There is $c = c(d, \psi)$ such that

$$c^{-1} \frac{V(\delta_D(x))}{R^{d/2}V(R)} \leq \phi_1^R(x) \leq c \frac{V(\delta_D(x))}{R^{d/2}V(R)}, \quad x \in \mathbb{R}^d.$$  

Proof. By Corollary 4.6 for $t \geq V(R)$, we obtain

$$p_D(t, x, x) \approx e^{-\lambda_1(R)t} \left( \frac{V(\delta_D(x))}{V(R)} \right)^2 p_{V^2(R)}(0).$$

By Lemma 1.6,

$$p_{V^2(R)}(0) \approx R^{-d}.$$  

Since $\nu$ is radial and infinite, by [22, Theorem 3.1], the semigroup $P_t^D$ is intrinsically ultracontractive. Hence, by [21, Theorem 4.2.5]

$$\lim_{t \to \infty} \frac{p_D(t, x, x)}{e^{-\lambda_1(R)t}(\phi_1^R(x))^2} = 1, \quad x \in D,$$

which gives the claimed result.  

5 Unbounded sets

Throughout this section we assume that global WLSC and WUSC hold for $\psi$. Due to [8, Proposition 5.2, Lemma 7.2 and Lemma 7.3] and [21, Proposition 3.5] the quantities $\mathcal{J}(r)$, $\mathcal{I}(r)$ and $H_r$ employed above now depend only on the dimension and scaling characteristics of the Lévy-Khintchine exponent $\psi$. Denote $L(r) = \nu(B_r)$, $r > 0$, the tail of the Lévy measure. Our first unbounded set is the complement of a ball.

Proposition 5.1. Let $\psi \in \text{WLSC}(\alpha, 0, \underline{c}) \cap \text{WUSC}(\alpha, 0, \overline{C})$. There is $C_{10} = C_{10}(d, \psi)$, such that for all $R > 0$ and $t \geq V^2(R)$,

$$\mathbb{P}^x(T_{\overline{c} R} \in dt)/dt \leq C_{10}p_{t/2}(0) \frac{R^d}{V^2(R)}, \quad x \in \mathbb{R}^d.$$  

Proof. By (1.14) and symmetry of $\nu$, the Radon-Nikodym derivative satisfies

$$\mathbb{P}^x(T_{\overline{c} R} \in dt)/dt = \int_{B_R^c} \nu_{\overline{c} R}(t, x, y) \nu(B(y, R)) dy.$$  

Since $\nu$ is radially decreasing, for $|y| > R$,

$$\nu(B(y, R)) \leq \min\{L(|y| - R), \frac{\omega_d}{d} \nu(|y| - R)) R^d\}.$$  

By Lemma 1.8 and [8, Theorem 6.3], there exists $c_1 = c_1(d, \psi)$ such that, for $t \geq V^2(R)$,

$$p_{\overline{c} R}(t, x, y) \leq c_1 p_{t/2}(0) \left(1 \wedge \frac{V(|y| - R)}{V(R)}\right), \quad x, y \in \mathbb{R}^d.$$  

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Therefore,
\[
\mathbb{P}^x(\tau^{B_R}_y \in dt) / dt \leq c_1 p_{t/2}(0) \int_{B_R} \left( 1 \wedge \frac{V(|y| - R)}{V(R)} \right) \nu(B(y, R)) \, dy
\leq c_1 p_{t/2}(0) \left( \int_{R < |y| < 2R} \frac{V(|y| - R)}{V(R)} L(|y| - R) \, dy + \frac{\omega_d R^d}{d} \int_{|y| > 2R} \nu(|y| - R) \, dy \right)
\leq \omega_d 2^{d-1} c_1 p_{t/2}(0) \left( \frac{R^{d-1}}{V(R)} \int_0^R V(\rho) L(\rho) \, d\rho + R^d L(R) \right)
\leq c_2 p_{t/2}(0) \frac{R^d}{V^2(R)},
\]
where in the last line we used Lemma \[2\] and \[3\] Proof of Proposition 3.4] to get
\[
\int_0^R V(\rho) L(\rho) \, d\rho \leq c \frac{R}{V(R)}.
\]
In fact, \(c_2 = c_1 c(d)\).

The assumption \(d > \alpha\) in the next result secures the transience of the underlying unimodal Lévy process \(X\) \[\gamma\] Corollary 37.6]. The proof below asserts in relative terms that hitting the ball \(B_R\) is unlikely for \(X\) when the points \(x\) and \(y\) are far away from the ball.

**Proposition 5.2.** Let \(\psi \in \text{WLSC}(\alpha, 0, \varrho) \cap \text{WUSC}(\sigma, 0, \varphi)\) and \(d > \alpha\). There is \(C_{11} = C_{11}(d, \psi)\) such that if \(R > 0\), \(|x|, |y| \geq C_{11} R\), and \(t > 0\), then
\[
P^{t,y} (t, x, y) \geq \frac{1}{2} p(t, x, y).
\]

**Proof.** Assume that \(|y| \geq |x| \geq 2R\). Let \(y^*\) be a projection of \(y\) onto the boundary of \(B_R\). Let \(f(t, x, y) = E^x[p(t - \tau^{B_R}_y, y^*, y), \tau^{B_R}_y < t]\). Since \(p_t(\cdot)\) is radially decreasing,
\[
p^{t,y} (t, x, y) = p(t, x, y) - E^x[p(t - \tau^{B_R}_y, X^{\tau^{B_R}_y}_y, y), \tau^{B_R}_y < t] \geq p(t, x, y) - f(t, x, y).
\]
Clearly,
\[
f(t, x, y) \leq \sup_{s \leq t} p(s, y^*, y) \mathbb{P}^x(\tau^{B_R}_y < \infty).
\]
Observe that \(|y - y^*| \geq R \vee \frac{|x-y|}{4}\). Indeed
\[
|x - y| \leq |x| + |y| \leq 2|y| \leq 4(|y| - R) = 4|y - y^*|.
\]
Due to Lemma \[1,3\] and radial monotonicity of \(p_t\),
\[
\sup_{s \leq t} p(s, y^*, y) \leq c_1 \frac{t}{V^2 \left( R \vee \frac{|x-y|}{4} \right)} \left( R \vee \frac{|x-y|}{4} \right)^d.
\]
This, Lemma \[1,5\] radial monotonicity of \(p_t\) and \[9\] Corollary 24] give, for \(t \leq 2V^2 \left( R \vee \frac{|x-y|}{4} \right)\),
\[
\sup_{s \leq t} p(s, y^*, y) \leq c_2 p_t \left( R \vee \frac{|x-y|}{4} \right) \leq c_3 p(t, x, y),
\]
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and so by [8 Proposition 5.8],

$$f(t, x, y) \leq c_4 p(t, x, y) \frac{R^d V^2(|x|)}{V^2(R)|x|^d}. \quad (5.1)$$

Let $t \geq 2V^2 \left( R \vee \frac{|x-d|}{4} \right)$. By Proposition 5.1 and [23] Theorem 3 and Section 4 for $d \leq 2$,

$$I := E[x][p(t - \tau_{\overline{B}_R}, y, y^*), t/2 \leq \tau_{\overline{B}_R} < t] \leq C_{10} p_{t/2}(0) \frac{R^d}{V^2(R)} \int_{t/2}^t p(t - s, y, y^*) ds$$

$$\leq C_{10} p_{t/2}(0) \frac{R^d}{V^2(R)} U(y - y^*) \leq c_5 p_{t/2}(0) \frac{R^d}{V^2(R)} V^2(|y| - R),$$

where $U(y) = \int_0^\infty p_t(y) dt \approx V^2(|y|)/|y|^d$ is the potential kernel of $X \ [23]$. By [8 Proposition 5.8] and monotonicity of $s \mapsto p_s(0)$,

$$II := E[x][p(t - \tau_{\overline{B}_R}, y^*, y), \tau_{\overline{B}_R} < t/2] \leq c_6 p_{t/2}(0) \frac{R^d V^2(|x|)}{V^2(R)|x|^d}. \quad (5.2)$$

Since $t \geq V^2(|x - y|)/8$, by Lemma 1.5, $p_{t/2}(0) \leq c_7 p(t, x, y)$. Therefore,

$$f(t, x, y) = I + II \leq c_8 p(t, x, y) \frac{R^d}{V^2(R)} \left( \frac{V^2(|x|)}{|x|^d} + \frac{V^2(|y| - R)}{(|y| - R)^d} \right). \quad (5.2)$$

Finally, combining (5.1) with (5.2), for all $t > 0$ we have

$$f(t, x, y) \leq (c_8 + c_4) p(t, x, y) \frac{R^d}{V^2(R)} \left( \frac{V^2(|x|)}{|x|^d} + \frac{V^2(|y| - R)}{(|y| - R)^d} \right).$$

By global WUSC, for all $t > 0$ and $|x|, |y| \geq 2R$,

$$f(t, x, y) \leq c_9 p(t, x, y) \left( \left( \frac{R}{|x|} \right)^{d-\bar{\eta}} + \left( \frac{R}{|y| - R} \right)^{d-\bar{\eta}} \right).$$

Therefore there exists a constant $c_{10}$ such that for $|x|, |y| \geq c_{10} R$ we have

$$|p_{\overline{B}_R}(t, x, y) - p(t, x, y)| \geq \frac{1}{2} p(t, x, y). \quad \square$$

**Lemma 5.3.** Let $d \geq 1$ and $\psi$ satisfy global WLSC and WUSC. There is a constant $C = C(d, \psi)$ such that if $\lambda > 1$ and $|x - z| \leq \lambda R$, then

$$C^{-1} \lambda^{-2-d} \leq \frac{p_{V^2(R)}(x)}{p_{V^2(R)}(z)} \leq C \lambda^{2+d}.$$  

**Proof.** Assume that $|x - z| \leq \lambda R$. By symmetry it is enough to prove the upper bound. By scaling and Lemma 1.5 we have, for $x \in \mathbb{R}^d$,

$$p_{V^2(R)}(x) \approx \min \left\{ R^{-d}, \frac{V^2(R)}{V^2(|x|)|x|^d} \right\}. \quad (5.3)$$
If $|z| \geq 2\lambda R$, then $|z| \leq 2|x|$. Hence, by radial monotonicity and \cite[Corollary 24]{9},
\[ p_t(x) \leq p_t(z/2) \leq c_1 p_t(z), \quad t > 0. \]
For $|z| < 2\lambda R$, again by radial monotonicity,
\[ p_t(z) \geq p_t(2\lambda R). \]
This, subadditivity of $V$, and (5.3) complete the proof:
\[ \frac{p_{V^2(R)}(z)}{p_{V^2(R)}(0)} \leq \frac{p_{V^2(R)}(2\lambda R)}{p_{V^2(R)}(0)} \leq c_2 \lambda^{2+d}. \]

The next theorem may be considered as the main result of this section.

**Theorem 5.4.** Let $\psi \in WLSC(\overline{\alpha}, 0, \overline{\beta}) \cap WUSC(\overline{\beta}, 0, \overline{\alpha})$ and $d > \overline{\alpha}$. Let $D$ be a $C^{1,1}$ at scale $R_1$ and $D^\delta \subset \overline{B}_{R_2}$. Constants $c_* = c_*(d, \psi), c^* = c^*(d, \psi)$ exist such that for all $x, y \in \mathbb{R}^d, t > 0,$
\[ c_* \left( \frac{R_1}{R_2} \right)^{4+2d} \left( \frac{V(\delta_D(x))}{\sqrt{\lambda}} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{\lambda}} \wedge 1 \right) p(t, x, y) \leq p_D(t, x, y) \]
and
\[ p_D(t, x, y) \leq c^* \left( \frac{V(\delta_D(x))}{\sqrt{\lambda}} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{\sqrt{\lambda}} \wedge 1 \right) p(t, x, y). \]

**Proof.** We only deal with the lower bound since the upper bound follows from Theorem 2.6. Assume that $|x| \leq |y|$ and denote $l(x, y) = \left( \frac{V(\delta_D(x))}{V(R_1)} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{V(R_1)} \wedge 1 \right)$. By Corollary 5.3, it is enough to consider $t > t_0 = c^* V^2(R_1)$, where $c^*$ is the constant from that corollary. By Proposition 5.2, for $|x|, |y| \geq C_{11} R_2$, we have
\[ p_D(t, x, y) \geq p_{\overline{B}_{R_2}}(t, x, y) \geq \frac{1}{2} p(t, x, y). \]
In the remaining part of the proof, we closely follow the ideas of \cite[Theorem 1.3]{13}, where a similar result is proved for the isotropic stable Lévy processes. Let $|x| < C_{11} R_2$. Fix $v \in \mathbb{R}^d$ with $|v| = 1$, such that $\langle x, v \rangle \geq 0$ and $\langle y, v \rangle \geq 0$ if $d \geq 2$, and $v = 2y/|y|$ if $d = 1$. Define $x_0 = x + C_{11} R_2 v$ and $y_0 = y + C_{11} R_2 v$. Then $|x_0|, |y_0| \geq C_{11} R_2$. By Lemma 5.3 and \cite[Corollary 24]{9}, there exists $c_1 = c_1(d, \psi)$ such that, for all $z \in \mathbb{R}^d$,
\[ p(t_0/2, x, z) \geq c_1 \left( \frac{R_1}{R_2} \right)^{2+2d} p(t_0/2, x_0, z). \]
Hence, by Corollary 3.5 and Theorem 2.6, there exists $c_2 = c_2(d, \psi)$, such that for $z, x \in D$,
\[ p_D(t_0/2, x, z) \geq c_2 \left( \frac{R_1}{R_2} \right)^{2+2d} \left( \frac{V(\delta_D(x))}{V(R_1)} \wedge 1 \right) p_D(t_0/2, x_0, z). \]
This and the semigroup property imply
\[ p_D(t, x, y) = \int \int p_D(t_0/2, x, z) p_D(t - t_0, z, w) p_D(t_0/2, w, y) \, dz \, dw \]
\[ \geq c_2^2 \left( \frac{R_1}{R_2} \right)^{4+2d} l(x, y) \int \int p_D(t_0/2, x_0, y_0) p_D(t - t_0, z, w) p_D(t_0/2, w, y_0) \, dz \, dw \]
\[ = c_2^2 \left( \frac{R_1}{R_2} \right)^{4+2d} l(x, y) p_D(t, x_0, y_0) \geq \frac{c_2^2}{2} \left( \frac{R_1}{R_2} \right)^{4+2d} l(x, y) p(t, x, y), \]
where in the last step we used (5.4). Since $p(t, x, y_0) = p(t, x, y)$, we obtain the conclusion. \qed
We recall that the assumption $d > \alpha$ above yields the transience of the process $X$. We note that the results for recurrent unimodal Lévy processes in dimension 1 should be quite different: for exterior domains in the case of recurrent the isotropic stable Lévy processes we refer to [7].

The following proposition may be proved in a similar way as [8, Theorem 6.3], where the result is proved for the isotropic stable Lévy processes. Let $\psi \in WLSC(\alpha, 0, \mathcal{C}) \cap WUSC(\alpha, 0, \overline{\mathcal{C}})$ and $d > \alpha$. Let $D$ be a $C^{1,1}$ at scale $R_1$ and $D^c \subset \overline{B_{R_2}}$. Then there are constants $c_\ast = c_\ast (d, \psi)$, $c^* = c^*(d, \psi)$ such that for all $x, y, \in \mathbb{R}^d$ and $t > 0$,

$$c_\ast \left( \frac{R_1}{R_2} \right)^2 \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge V(R_1) \wedge 1 \right) \leqslant \mathbb{P}^x(\tau_D > t) \leqslant c^* \left( \frac{V(\delta_D(x))}{\sqrt{t}} \wedge V(R_1) \wedge 1 \right).$$

One can also prove sharp estimates of $\mathbb{P}^x(\tau_D > t)$ above by integrating the estimates in Theorem 5.4 but it results with a suboptimal dependence of comparability constants on $R_1/R_2$.

The following corollary is an immediate consequence of Theorem 5.4 and Proposition 5.5.

**Corollary 5.6.** Let $\psi \in WLSC(\alpha, 0, \mathcal{C}) \cap WUSC(\alpha, 0, \overline{\mathcal{C}})$ and $d > \alpha$. Let $D$ be a $C^{1,1}$ at scale $R_1$ and such that $D^c \subset \overline{B_{R_2}}$. For all $x, y, \in \mathbb{R}^d$ and $t > 0$ we have

$$p_D(t, x, y) \approx \mathbb{P}^x(\tau_D > t)\mathbb{P}^y(\tau_D > t)p(t, x, y),$$

with comparability constant $C = C(d, \psi, R_2/R_1)$.

The next lemma is helpful to handle halfspace-like $C^{1,1}$ sets.

**Lemma 5.7.** Let $t_0 > 0$, $r_0 = V^{-1}(\sqrt{t_0})$. Then, for $r > 0$, $\lambda \geqslant 1$, $t > t_0$,

$$\frac{1}{\lambda + 2} \left( \frac{V(r)}{\sqrt{t_0}} \wedge 1 \right) \left( \frac{V(r + \lambda r_0)}{\sqrt{t}} \wedge 1 \right) \leqslant \left( \frac{V(r)}{\sqrt{t}} \wedge 1 \right) \left( \frac{V(r + \lambda r_0)}{\sqrt{t}} \wedge 1 \right).$$

**Proof.** By subadditivity and monotonicity of $V$ we have

$$V(r_0 \vee r) \leqslant V(r + \lambda r_0) \leqslant (\lambda + 2)V(r_0 \vee r).$$

Considering cases $r \leqslant r_0$ and $r > r_0$, this observation easily leads to the conclusion. \(\square\)

Here is our main result for halfspace-like $C^{1,1}$ sets. Recall that $\mathbb{H}_a$ is defined in Section 1.2.

**Theorem 5.8.** Let $\psi$ satisfy global WLSC and WUSC, $D$ be $C^{1,1}$ at scale $R$ and $\mathbb{H}_a \subset D \subset \mathbb{H}_b$. Then for all $x, y, \in \mathbb{R}^d$ and $t > 0$,

$$p_D(t, x, y) \approx \mathbb{P}^x(\tau_D > t)\mathbb{P}^y(\tau_D > t)p(t, x, y) \quad \text{and} \quad \mathbb{P}^x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1,$$

and constants in the comparisons may be so chosen to depend only on $d$, $\psi$, $a - b$ and $R$.

**Proof.** Without loss of generality we may and do assume that $a > b = 0$. Let $x, y, \in D$. Due to Corollary 3.5 and Theorem 2.6 it remains to prove the comparisons for $t > t_0 = c^*V^2(R)$, where $c^*$ is the constant from Corollary 3.5. Our arguments below are similar to those proving [19, Theorem 1.2], where the result is proved for the isotropic stable Lévy processes.
\[ r_0 = V^{-1}(\sqrt{t_0}), \lambda = 1 + a/r_0, x_0 = x + \lambda r_0 e_d \text{ and } y_0 = y + \lambda r_0 e_d, \text{ where } e_d = (0, \ldots, 0, 1). \]

By Lemma \[5.3\] and \[9\] Corollary \[24\] the following comparison depends only on \(d, \psi\) and \(\lambda\):

\[ p(t_0/2, x, z) \approx p(t_0/2, x_0, z), \quad x, z \in \mathbb{R}^d. \]

Since \(\delta_D(x_0) \geq \delta_{\mathbb{H}_a}(x_0) \geq r_0\), we have \(V(\delta_D(x_0)) \geq \sqrt{t_0}\). By Corollary \[3.5\] and Theorem \[2.6\]

\[ p_D(t_0/2, x, z) \approx \left(1 \land \frac{V(\delta_D(x))}{\sqrt{t_0}}\right) p_D(t_0/2, x_0, z), \quad x, z \in \mathbb{R}^d, \quad (5.5) \]

where the comparability constant depends on dimension \(\psi\), \(a\) and \(R\). We denote \(l(x, y) = \left(1 \land \frac{V(\delta_D(x))}{\sqrt{t_0}}\right) \left(1 \land \frac{V(\delta_D(x))}{\sqrt{t_0}}\right)\). By \(5.5\) and the semigroup property,

\[
\begin{align*}
p_D(t, x, y) & \approx l(x, y) \int_D p_D(t_0/2, x_0, z) p_D(t - t_0, z, w) p_D(t_0/2, w, y_0) dz dw \\
& = l(x, y) p_D(t, x_0, y_0).
\end{align*}
\]

We have

\[ \delta_D(x) + r_0 \leq \delta_{\mathbb{H}_a}(x_0) \leq \delta_{\mathbb{H}_0}(x_0) \leq \delta_D(x) + 2\lambda r_0, \]

hence, by Lemma \[5.7\]

\[
\left(1 \land \frac{V(\delta_{\mathbb{H}_a}(x_0))}{\sqrt{t}}\right) \left(1 \land \frac{V(\delta_D(x))}{\sqrt{t_0}}\right) \leq (2 + 2\lambda) \left(1 \land \frac{V(\delta_D(x))}{\sqrt{t}}\right)
\]

and

\[
\left(1 \land \frac{V(\delta_{\mathbb{H}_a}(x_0))}{\sqrt{t}}\right) \left(1 \land \frac{V(\delta_D(x))}{\sqrt{t_0}}\right) \geq \left(1 \land \frac{V(\delta_D(x))}{\sqrt{t}}\right). \quad (5.7)
\]

The last two estimates also hold if \(x_0, x\) are replaced by \(y_0, y\). We note that

\[ p_{\mathbb{H}_a}(t, x_0, y_0) \leq p_D(t, x_0, y_0) \leq p_{\mathbb{H}_0}(t, x_0, y_0), \]

and \(p(t, x_0 - y_0) = p(t, x - y)\). Also, \(\delta_D(x_0) \approx \delta_{\mathbb{H}_a}(x_0) \approx \delta_{\mathbb{H}_0}(x_0)\) because \(\delta_{\mathbb{H}_0}(x_0) \leq \delta_D(x_0) = a + \delta_{\mathbb{E}_a}(x_0) \leq \lambda \delta_{\mathbb{H}_a}(x_0)\). From Corollary \[3.6\] subadditivity of \(V\), and \(5.6\) and \(5.7\) (along with their variants for \(y_0\) and \(y\)), we obtain

\[ p_D(t, x, y) \approx \left(\frac{V(\delta_D(x))}{\sqrt{t}} \land 1\right) \left(\frac{V(\delta_D(y))}{\sqrt{t}} \land 1\right) p(t, x, y). \]

This gives a sharp approximate factorization of \(p_D\). One consequence is that

\[ \mathbb{P}^x(\tau_D > t) \leq c \left(\frac{V(\delta_D(x))}{\sqrt{t}} \land 1\right) \int_D p(t, x, y) dy \leq c \left(\frac{V(\delta_D(x))}{\sqrt{t}} \land 1\right). \]

By Lemma \[1.19\] a matching lower bound holds for \(0 < t \leq t_0\). If \(t > t_0\), then by \(5.5\) and the semigroup property,

\[ p_D(t, x, y) \approx \left(\frac{V(\delta_D(x))}{\sqrt{t_0}} \land 1\right) p_D(t, x_0, y), \]

cf. the proof \[5.6\]. We integrate the comparison against \( y \) and use \[38\] Theorem 3.1, to get

\[
\mathbb{P}^x(\tau_D > t) \approx \left(\frac{V(\delta_D(x))}{\sqrt{t_0}} \land 1\right) \mathbb{P}^{x_0}(\tau_D > t) \geq \left(\frac{V(\delta_D(x))}{\sqrt{t_0}} \land 1\right) \mathbb{P}^{x_0}(\tau_{\mathbb{H}_a} > t)
\]

\[ \approx \left(\frac{V(\delta_D(x))}{\sqrt{t_0}} \land 1\right) \left(1 \land \frac{V(\delta_{\mathbb{H}_a}(x_0))}{\sqrt{t}}\right). \]

We end the proof by using \(5.7\).
6 Examples

In a recent work [17], Chen, Kim and Song provide estimates of Dirichlet heat kernels for a class of pure-jump Markov processes with intensity of jumps comparable to that of a complete subordinate Brownian motion with scaling. In fact the assumptions of [17] imply the (scale invariant) boundary Harnack inequality, which leads to the “Lipschitz setting” mentioned in the Section 1.1, and allows to handle the so-called $\kappa$-fat sets (see [8] for the case of the isotropic stable Lévy processes). In this sense [17] is a culmination of the line of research presented in [11, 16, 15, 17].

Therefore in the examples below we focus on processes which are not covered by [17]. Namely, in the first three examples the (scale invariant) boundary Harnack inequality is not known or simply fails for some $C^{1,1}$ sets, but our method provides satisfactory estimates. Our last two examples are more straightforward, and the reader may find others in [9] and [8].

Example 1. Let $\nu(x) = \log^\beta(1 + |x|^{-1})|x|^{-d-\alpha}1_{B_r}(x)$, $\alpha \in (0, 2)$, $\beta \geq 0$. It is known that the scale invariant boundary Harnack inequality fails for some $C^{1,1}$ sets for the corresponding truncated Lévy process [31], but the characteristic exponent $\psi$ satisfies the desired scaling conditions. Indeed, it is easy to verify that $h(r) \approx r^{-2} \wedge \log^\beta(2 + r^{-1})r^{-\alpha}$. Then, by Lemma 1.2 $\psi(x) \approx h(|x|^{-1}) \approx |x|^2 \wedge \log^\beta(2 + |x|)|x|^\alpha$ and $\psi \in \mbox{WUSC}(\alpha + \epsilon, 1, C_x) \cap \mbox{WLSC}(\alpha, 1, \emptyset)$, if $\alpha < \epsilon + \alpha < 2$. Our results apply to this case, e.g. for $0 < r < 1/2$, $0 < t < r^\alpha/|\log r|^{\beta}$ and $x,y \in B_r$, we have

$$p_{B_r}(t, x, y) \approx \left(1 \wedge \frac{(r - |x|^\alpha)}{t \log^\beta \frac{1}{r - |x|}}\right)^{1/2} \left(1 \wedge \frac{(r - |y|^\alpha)}{t \log^\beta \frac{1}{r - |y|}}\right)^{1/2} \left[t \log^\beta \frac{1}{t} \wedge \frac{t \log^\beta \frac{1}{|x - y|}}{|x - y|^{d+\alpha}}\right],$$

and the comparability constant depends only on $d$ and $\psi$.

Example 2. Let $T$ be a subordinator with Lévy density $\mu(r) = r^{-1-\alpha/2}1_{(0,1)}(r)$, $\alpha \in (0, 2)$, and $X$ be a subordinate Brownian motion governed by $T$. Then $\psi(x) \approx |x|^2 \wedge |x|^\alpha$ and $\nu(x) \approx e^{-|x|^2/4} |x|^{-2}$, $|x| \geq 1$. This $\psi$ satisfies WUSC and WUSC with $\alpha = \pi = \alpha$, but the scale invariant boundary Harnack inequality does not hold (see [11 Example 5.14]).

Example 3. Let $\phi$ be a complete Bernstein function [42] and $\phi(| \cdot |^2) \in \mbox{WUSC}(\pi, 0, \emptyset) \cap \mbox{WLSC}(\alpha, 0, \emptyset)$. If $\psi(x) = |x|^2 + \phi(|x|^2)$ then, by [38] Proposition 4.5 and Theorem 4.4] the renewal function $V$ of the ascending ladder-height process is a Bernstein function and $\psi_1(x) = V(|x|^2) \in \mbox{WUSC}(\pi, 1, C_t) \cap \mbox{WLSC}(\alpha, 1, \emptyset)$ is the characteristic exponent of a subordinate Brownian motion. For this process it is not clear if the boundary Harnack inequality holds. In particular, it is not clear how to construct a complete subordinate Brownian motion with comparable Lévy measure. Nevertheless, our approach applies because of scaling and isotropy.

In the next two examples we assume global scaling conditions and we focus on estimates for exterior $C^{1,1}$ sets for the full range of time and space. To the best of our knowledge such estimates were known only for the isotropic stable Lévy process. Even the estimate from the next example seems to be new.

Example 4. Let $0 < \alpha_1 \leq \alpha_2 < 2$, $d > \alpha_2$ and $\psi(x) = |x|^{\alpha_1} + |x|^{\alpha_2}$. Then $\psi \in \mbox{WLSC}(\alpha_1, 0, 1) \cap \mbox{WUSC}(\alpha_2, 0, 1)$. In particular, by Lemma 1.5

$$p_t(x) \approx (t^{-1/\alpha_1} + t^{-1/\alpha_2})^d \wedge \frac{t(|x|^{-\alpha_1} + |x|^{-\alpha_2})}{|x|^d}, \quad t > 0, \ x \in \mathbb{R}^d,$$

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and by Corollary 5.6
\[ p_{\beta}(t, x, y) \approx \left( 1 \wedge \frac{(|x| - r)^{\alpha_1}}{t \wedge \theta} \wedge \frac{(|y| - r)^{\alpha_2}}{t \wedge \theta} \right)^{1/2} \left( 1 \wedge \frac{(|y| - r)^{\alpha_1}}{t \wedge \theta} \wedge \frac{(|y| - r)^{\alpha_2}}{t \wedge \theta} \right)^{1/2} p(t, x, y), \]
where \( r > 0, t > 0, x, y \in B \) and the comparability constant depends only on \( d, \alpha_1 \) and \( \alpha_2 \).

**Example 5.** If \( f \in \text{WLSC}(\theta, 0, C) \cap \text{WUSC}(\theta, 0, C) \) is nonincreasing and \( \nu(|x|) = f(1/|x|)/|x|^d \), then \( \psi \) has both global scalings [9] Proposition 28. E.g. we may let \( \alpha_1, \alpha_2 \in (0, 2) \) and
\[
\begin{align*}
f(r) &= (r^{-1} \log(r + 1/r))^{\alpha_1}, & \text{or} \\
f(r) &= (r \log(r + 1/r))^{-\alpha_1}, & \text{or} \\
f(r) &= \begin{cases} r^{-\alpha_1} & \text{if } 0 < r \leq 1, \\
r^{-\alpha_2}/2 & \text{if } r > 1.
\end{cases}
\end{align*}
\]
In particular we do not need continuity of \( \nu \), which was assumed in [17]. The results from Section 5 give estimates which are uniform in the whole of time and space for exterior \( C_{1,1} \) sets and halfspace-like sets. For clarity, the global scaling conditions imply the scale boundary Harnack inequality ([32] and [9, Corollary 27]), so short time estimates would follow from [17] and [8], if we also assumed continuity of the Lévy density.

We finally suggest a possible generalization of our estimates which relaxes the assumption of monotonicity of the Lévy density.

**Remark 6.1.** We can work with more general isotropic pure-jump Lévy processes. Assume that the Lévy measure of \( X \) is absolutely continuous and its density function satisfies \( \nu(x) = \nu_0(|x|) \approx f(1/|x|)/|x|^d \), where \( f \) is nonincreasing and satisfies WLSC(\( \theta, \theta \)) and WUSC (\( \theta, \theta \)). Then by [9] Proposition 28, \( \psi(x) \approx f(x) \) for \( |x| \geq \theta \), hence \( \psi \) satisfies (local) WLSC and WUSC. By [13] for \( \theta > 0 \) and [18] for \( \theta = 0 \) we get estimates for the heat kernel. This implies that \( x \rightarrow p_t(x) \) is radial and almost decreasing locally in time and space (for all \( t > 0 \) and \( x \in \mathbb{R}^d \), if \( \theta = 0 \)). Moreover, the scale invariant Harnack inequality holds [13] [18]. This allows to repeat the arguments in Section 4, Section 5 and Section 6 to obtain analogous estimates for bounded \( C_{1,1} \) open sets if \( \theta > 0 \) and for all considered \( C_{1,1} \) sets if \( \theta = 0 \).

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