SUPERCRITICAL ELLIPTIC PROBLEMS ON THE ROUND SPHERE AND NODAL SOLUTIONS TO THE YAMABE PROBLEM IN PROJECTIVE SPACES

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Abstract. Given an isoparametric function $f$ on the $n$-dimensional round sphere, we consider functions of the form $u = w \circ f$ to reduce the semilinear elliptic problem

$$-\Delta g_0 u + \lambda u = \lambda |u|^{p-1} u \quad \text{on } S^n$$

with $\lambda > 0$ and $1 < p$, into a singular ODE in $[0, \pi]$ of the form $w'' + h(r) \sin r w' + \ell^2 \left( |w|^{p-1} w - w \right) = 0$, where $h$ is a strictly decreasing function having exactly one zero in this interval and $\ell$ is a geometric constant. Using a double shooting method, together with a result for oscillating solutions to this kind of ODE, we obtain a sequence of sign-changing solutions to the first problem which are constant on the isoparametric hypersurfaces associated to $f$ and blowing-up at one or two of the focal submanifolds generating the isoparametric family. Our methods apply also when $p > \frac{n+2}{n-2}$, i.e., in the supercritical case. Moreover, using a reduction via harmonic morphisms, we prove existence and multiplicity of sign-changing solutions to the Yamabe problem on the complex and quaternionic space, having a finite disjoint union of isoparametric hypersurfaces as regular level sets.

1. Introduction. Let $(M, g)$ be a closed (compact without boundary) Riemannian manifold of dimension $n \geq 3$. We will consider the Yamabe type equations

$$-\Delta_g u + \lambda u = \mu |u|^{p-1} u \quad \text{on } M$$

where $\lambda \in C^\infty(M)$, $\mu \in \mathbb{R}$ and $p > 1$. In case $\lambda = R_g$ is the scalar curvature and $p = p_n := \frac{n+2}{n-2}$ is the critical Sobolev exponent, equation 1 is the well known Yamabe equation, widely studied in the last 50 years (see, for example, [1, 6, 15] and

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When \( p < \frac{n+2}{n-2} \), we will say that the equation 1 is subcritical and we will call it supercritical if \( p > \frac{n+2}{n-2} \). In the subcritical case, as the Sobolev embedding \( H^1(M, g) \hookrightarrow L^p(M, g) \) is compact, the existence of positive and sign-changing solutions can be obtained using standard variational methods [42, 43]. When \( M = \Omega \) is a bounded domain of \( \mathbb{R}^{n+1} \) with smooth boundary, there has been recent progress in handling supercritical exponent problems like 1. A fruitful approach consists in reducing the supercritical problem to a more general elliptic critical or subcritical problem, either by considering rotational symmetries or by means of maps preserving the Laplace operator or by a combination of both, see [17] and the references therein. In case of closed Riemannian manifolds, these reduction methods also apply and have been combined with the Lyapunov-Schmidt reduction method in order to obtain sequences of positive and sign-changing solutions to similar supercritical problems, such that they blow-up or concentrate at minimal submanifolds of \( M \) [16, 20, 24, 31, 38].

The main interest of this paper is to seek for sign-changing solutions (also called nodal solutions) to the problem

\[
-\Delta_{g_0} u + \lambda u = \lambda |u|^{p-1} u \quad \text{on } (S^n, g_0).
\]

(2)

when \( p \) is either subcritical or supercritical. Here \( g_0 \) denotes the round metric and we will assume from now on that \( \lambda > 0 \) is constant. When \( p = p_{cm} \), 2 is a renormalization of the Yamabe problem 1 and this kind of solutions have been studied in [13, 15, 18, 19, 36] and more recently in [21, 30, 40]. The slightly subcritical case has been studied in [41], where the authors obtained multiplicity of nodal solutions blowing-up at points, while the general subcritical case has been studied in [27] and in [7]. The method introduced in [27] allowed the authors to obtain more information about the qualitative behavior of the solutions, for they showed the existence of an infinite number of non constant positive solutions having prescribed level sets in terms of isoparametric hypersurfaces. This method has been further generalized to supercritical exponents in [5] to prove a similar result on general closed Riemannian manifolds, including the round sphere. Other results concerning the existence and concentration of positive solutions along minimal submanifolds for the supercritical and slightly supercritical can be found in [20, 31]. However, very little is known about the existence, multiplicity and blow-up of nodal solutions for the supercritical problem on the sphere 2. One of the few results known by the authors is given in [26], where the existence of at least one sign-changing solution to the supercritical problem was settled. In this direction we will follow and generalize the ideas introduced in [21] to obtain an infinite number of nodal solutions to the supercritical and subcritical problem 2, having as level and critical sets isoparametric hypersurfaces and its focal submanifolds.

To state our main result and to describe the method, we briefly recall some aspects of the theory of isoparametric functions and hypersurfaces. For the details, we refer the reader to [3, 11]. A smooth function \( f : (M, g) \to \mathbb{R} \) is isoparametric if there exist smooth functions \( a, b : \mathbb{R} \to \mathbb{R} \) such that

\[
|\nabla f|^2 = b(f) \quad \text{and} \quad \Delta f = a(f).
\]

(3)

The regular level sets of \( f \) are called isoparametric hypersurfaces.

The theory of isoparametric hypersurfaces in the round sphere \( (S^n, g_0) \) is very rich and it is a vast research topic. In this case, isoparametric hypersurfaces coincide with the hypersurfaces of constant principal curvatures. Its classification
began with E. Cartan [8] and it is still an open problem, see [12, 32, 33] and the references therein. Some major progresses in the theory were made by Cartan himself, H. F. M"{u}nzner [34, 35] and D. Ferus, H. Karcher and M"{u}nzner [22]. Given an isoparametric hypersurface, there exist a huge number of isoparametric functions having it as level hypersurface, for if \( f : S^n \to \mathbb{R} \) is isoparametric, \( \nu : \text{Im}(f) \to \mathbb{R} \) is monotone and \( \alpha \in \mathbb{R} \setminus \{0\} \), then \( \alpha(\nu \circ f) \) is again isoparametric. However, there are “canonical” isoparametric functions, which are obtained by restricting Cartan-M"{u}nzner polynomials to the sphere [11, Section 3.5]. They are well understood and have some nice properties. For instance, if \( f : S^n \to \mathbb{R} \) is obtained in this way, then \( \text{Im} f = [-1,1] \), the inverse image of a regular value is a connected isoparametric hypersurface, its only critical values are \( t = \pm 1 \) and the functions \( a \) and \( b \) defined in 3 can be written explicitly. To give these explicit expressions, let \( \ell \) be the number of distinct principal curvatures of the level sets of \( f \). M"{u}nzner showed that \( \ell \in \{1,2,3,4,6\} \) and if \( \ell \) is odd, then all the multiplicities of the principal curvatures are the same, while if \( \ell \) is even, there exist, at most, two different multiplicities \( m_- \) and \( m_+ \) with \( 1 \leq m_- , m_+ \leq n - 1 \), see [34]. With this notation, if \( \Delta g_0, f = a(f) \) and \( |\nabla f|_{g_0} = b(f) \), then

\[
a(t) = -\ell(n + \ell - 1)t + \frac{\ell^2(m_+ - m_-)}{2} \quad \text{and} \quad b(t) = -\ell^2 t^2 + \ell^2.
\]

The sets \( M_- := f^{-1}(-1) \) and \( M_+ := f^{-1}(1) \) are smooth submanifolds of \( S^n \) of dimension \( n_- = (n-1) - m_- \) and \( n_+ := (n-1) - m_+ \), called focal submanifolds, see [11, Section 2.4]. The main feature of these submanifolds is that every isoparametric hypersurface is a tube around \( M_- \) and \( M_+ \).

If \( u \) denotes a sign-changing smooth function defined on a Riemannian manifold, we define the nodal and the critical sets of \( u \) to be the sets \( \{u = 0\} \) and \( \{\nabla u = 0\} \), respectively. We state the main result of this paper, which generalizes Theorem 1.3 in [21].

**Theorem 1.1.** Let \( S \subset S^n \) be an isoparametric hypersurface and let \( n_- \leq n_+ \) be the dimensions of its corresponding focal submanifolds. Then, for any \( \lambda > 0 \), any \( k \in \mathbb{N} \) and any \( p \in (1, \frac{n-n_-+1}{n-n_-+2}) \), equation 2 admits a nodal solution \( u_k \) such that its nodal set has at least \( k \) connected components, each of them being an isoparametric hypersurface diffeomorphic to \( S \). The critical set of \( u_k \) consists in the focal submanifolds \( M_- \) and \( M_+ \) and, at least, \( k - 1 \) isoparametric hypersurfaces diffeomorphic to \( S \). Moreover, the solutions \( u_k \) satisfy

\[
\lim_{k \to \infty} |u_k(x)| = \infty,
\]

for every \( x \in M_- \) or for every \( x \in M_+ \).

Here the numbers \( \frac{n-n_-+2}{n-n_-+2} \geq p_n \) are just the critical Sobolev exponents in dimensions \( n - n_- \). Our Theorem improves the existence result stated by Henry in [26], giving an infinite number of distinct solutions instead of one. It also extends the multiplicity result in [21] to the subcritical and supercritical exponents. However, this last result gives a better description of the nodal set of the solutions. We strongly believe that a refinement of our methods may give a prescribed number of connected components for the nodal sets of the sign-changing solutions to problem 1, as in Theorem 1.2 in [21].

The last assumption of Theorem 1.1 says that the sequence \( (u_k) \) is not compact with the \( C^0 \) topology and that the blow-up occurs on one of the focal submanifolds, which are minimal submanifolds of the sphere [11]. Other noncompactness
phenomenon of the same nature appears in the solutions to the critical Yamabe problem obtained in \[18\], where the blow-up occurs at a single point. However, it was recently proved by Premoselli and Vétois that this sequence of solutions is uniformly bounded from below, but not from above \[40\]. This does not holds true in general, as we state next.

**Corollary 1.** Let \( S \) be an isoparametric hypersurface with focal submanifolds \( M_- \) and \( M_+ \) satisfying \( \dim M_+ > 0 \). Then there exists a sequence \((u_k)\) of sign-changing solutions to the Yamabe problem on the sphere

\[- \Delta_{S^n} u + \frac{n(n-2)}{4} u = \frac{n(n-2)}{4} |u|^{p_n-1} u \quad \text{on} \ S^n, \tag{5}\]

satisfying that

\[ \lim_{k \to \infty} u_k(x) = \infty \quad \text{for every} \ x \in M_- \quad \text{and} \quad \lim_{k \to \infty} u_k(x) = -\infty \quad \text{for every} \ x \in M_+. \]

As another consequence of Theorem 1.1, we obtain a multiplicity result for the Yamabe problem on projective spaces.

**Corollary 2.** Let \((M,g)\) be the complex projective space \( \mathbb{C}P^m \) or the quaternionic projective space \( \mathbb{H}P^m \) endowed with their canonical metric. Then, if \( j = 2 \) in case of \( \mathbb{C}P^m \) and \( j = 4 \) in case of \( \mathbb{H}P^m \), for every \( k \in \mathbb{N} \), the Yamabe equation

\[- \frac{4(jm-1)}{jm-2} \Delta_g v + R_g v = |v|^{p_m-1} v \quad \text{on} \ (M,g), \tag{6}\]

admits a sequence of sign-changing solutions \((u_k)\) such that the regular level sets of \( u_k \) consist of isoparametric hypersurfaces in \((M,g)\) and

\[ \lim_{k \to \infty} \max_{x \in M} |u_k(x)| = \infty. \tag{7}\]

We describe briefly the method we shall use in order to prove Theorem 1.1. The details will be given in Section 2 and Section 3.

Let \( f: S^n \to \mathbb{R} \) be an isoparametric function obtained as the restriction of a Cartan-Münzner polynomial, and let \( \ell, m_- \) and \( m_+ \) be the number of principal curvatures and the multiplicities associated to the isoparametric hypersurfaces that \( f \) defines, as it was explained before. Then, it is easy to see that \( z: [-1,1] \to \mathbb{R} \) is a solution to the problem

\[ b(t)z'' + a(t)z' + \lambda [z|^{p-1} z - z] = 0 \quad \text{on} \ [-1,1], \tag{8}\]

with \( a(t) := -\ell(n+\ell-1)t + \frac{\ell^2(m_-m_+)}{2} \) and \( b(t) := -\ell^2 t^2 + \ell^2 \), if and only if \( u = z \circ f \) is a solution to the problem 2 (Cf. \[21\]). Therefore, if \( u = z \circ f \) is a solution to 2, its regular level sets and the set of its critical points are conformed by isoparametric hypersurfaces and focal submanifolds. We can simplify equation 8 even more by considering the new variable \( w(r) = z(\cos r) \), and, in this way, solving 8 is equivalent to solving the singular ODE

\[ w'' + \frac{b(r)}{\sin r} w' + \frac{\lambda}{\ell^2} (|w|^{p-1} w - w) = 0 \quad \text{on} \ [0,\pi], \tag{9}\]

where \( b(r) = \frac{n-1}{r} \cos r - \frac{m_-m_+}{2} = \frac{m_-+m_+}{\ell} \cos r - \frac{m_-m_+}{2} \).

Observe that the natural boundary conditions associated to this problem are given by \( w'(0) = w'(\pi) = 0 \). Theorem 1.1 will be a consequence of the following one.
Theorem 1.2. For any \( p \in \left( 1, \frac{n-n+2}{n-2} \right) \) and any \( k \in \mathbb{N} \), the equation 9 with boundary conditions \( u'(0) = u'(\pi) = 0 \) admits a sign changing solution \( w_k \) having at least \( k \) isolated zeroes in \( [0, \pi] \) and at least \( k + 1 \) isolated critical points.

We will prove this theorem in Section 3.

The function \( h \) appearing in Equation 9 has a unique zero \( a_0 \in (0, \pi) \). To prove Theorem 1.2 we will use the double shooting method developed in [21], which consists in considering the solutions \( w_d, \tilde{w}_c \) of Equation 9 with initial conditions \( w_d'(0) = \tilde{w}_c'(\pi) = 0 \), \( w_d(0) = d, \tilde{w}_c(\pi) = c \) and consider the maps \( I(d) = (w_d(a_0), w'_d(a_0)) \) and \( J(c) = (\tilde{w}_c(a_0), \tilde{w}'_c(a_0)) \). If \( I(c) = J(d) \), then \( w_d = \tilde{w}_c \) is a solution of Equation 9 with \( w_d'(0) = w'_d(\pi) = 0 \), as one can readily see. To understand the intersections of the curves \( I, J \) one needs information of the functions \( w_d, \tilde{w}_c \). In the next section we will prove that, for large \( l \) and \( c \), these functions have many zeroes close to 0 and \( \pi \) (respectively) and then, generalizing an argument based on a Pohozaev-type identity and presented in [9], we will prove that \( |I(d)| , |J(c)| \to \infty \) as \( c,d \to \infty \).

These two results will allow us to conclude that the curves \( I \) and \( J \) behave as spirals rotating in opposite directions and from this we will obtain the intersections needed to solve the double shooting problem.

2. Double shooting and the proof of Theorem 1.2. We now develop the double shooting method used to prove Theorem 1.2. First, observe that the function \( h \) defined in 9 satisfies \( h(0) = m_-, h(\pi) = -m_+ \), it is strictly decreasing, has a unique zero \( a_0 \in (0, \pi) \) and \( h(r) > 0 \) in \([0, a_0)\), while \( h(r) < 0 \) in \((a_0, \pi]\). Moreover, the function \( \tilde{h}(r) := -h(\pi - r) = \frac{m_- - m_+}{2} \cos r + \frac{m_+ - m_-}{2} \) has the same properties with \( m_- \) and \( m_+ \) interchanged and a unique zero at \( \pi - a_0 \). To handle both singularities in 9 at the same time, the strategy is to shoot solutions from each of them and expect that, for some suitable initial and final conditions, the solutions coincide.

That is, we consider the initial value problem

\[
\begin{cases}
w_i'(r) + \frac{h(r)}{\sin^2 r} w_i^2(r) + \frac{A}{r^2} (|w_i(r)|^{p-1} w_i - w_i) = 0 & \text{in } [0, a_0], \\
w_i(0) = d, \quad w_i'(0) = 0,
\end{cases}
\]

(10)

and the “final” value problem

\[
\begin{cases}
w_f'(r) + \frac{\tilde{h}(r)}{\sin^2 r} w_f^2(r) + \frac{A}{r^2} (|w_f(r)|^{p-1} w_f - w_f) = 0 & \text{in } [a_0, \pi], \\
w_f(\pi) = c, \quad w_f'(\pi) = 0,
\end{cases}
\]

(11)

looking for initial and final conditions \( d \) and \( c \) such that \( w_i(a_0, d) = w_f(a_0, c) \) and \( w'_i(a_0, d) = w'_f(a_0, c) \). Hence, by uniqueness of the solution, we would have a well defined solution to problem 9 given by \( w(r) = w_i(r, d) \) if \( r \in [0, a_0] \) and \( w(r) = w_f(r, c) \) if \( r \in [a_0, \pi] \). To construct the solutions with an arbitrarily large number of zeroes, we will need to use that the number of zeroes before and after \( a_0 \) goes to infinity as \( |d|, |c| \to \infty \).

Actually, problem 11 can be written as an initial condition problem having the form of 10. Indeed, if we consider the function \( \tilde{h}(r) \) defined above, then \( w_f \) solves 11 if and only if \( \omega(r) = w_f(\pi - r) \) solves the initial value problem

\[
\begin{cases}
\omega''(r) + \frac{\tilde{h}(r)}{\sin^2 r} \omega'(r) + \frac{A}{r^2} (|\omega(r)|^{p-1} \omega - \omega) = 0 & \text{in } [0, \pi - a_0],
\end{cases}
\]

\[
\omega(0) = c, \quad \omega'(0) = 0,
\]

(12)

So, in order to understand problem 11 it is enough to consider problem 10.

In what follows, we will consider the more general equation.
Proof. First observe that if \( w \) is isolated in \([0, A]\), Notice that equations 10 and 12 are special cases of the former by taking 
\[ \mu = \frac{A}{\pi}, H(r) = \frac{h(r)r}{\sin \pi r} \] in \([0, A]\) with \( A < a_0 \) and 
\[ H(r) = \frac{h(r)r}{\sin \pi r} \] in \([0, A]\) with \( A < \pi - a_0 \). Observe that now we are just dealing with a single singularity at \( r = 0 \).

A standard contraction map argument (Cf. [21, 28]) yields the existence and uniqueness, in the whole interval \([0, A]\), of the solutions to equation 13 with initial conditions \( w(0) = d \in \mathbb{R} \) and \( w'(0) = 0 \), depending continuously on \( d \). For \( d > 0 \), let \( w_d := w(., d) \) be the solution with initial values \( w_d(0) = d \) and \( w'_d(0) = 0 \). To assure the existence of an arbitrarily large number of zeroes, we use the following result, proven in [21, 25].

**Theorem 2.1.** Suppose that \( H(0) > 0, p > 1 \) and that the following inequality

\[ \frac{H(0) + 1}{2} < \frac{p + 1}{p - 1} \]  

holds true. Then, for any \( \varepsilon > 0 \) and any positive integer \( k \) there exists \( D_k > 0 \) so that the solution \( w_d \) of 13 has at least \( k \) zeroes in \((0, \varepsilon)\) for any \( d \geq D_k \).

In case of equations 10 and 12, we have that \( H(0) = h(0) = m_- \) and \( H(0) = \tilde{h}(0) = m_+ \) respectively. Taking \( 1 \leq m_- \leq m_+ \leq n - 1 \), and recalling that \( n_+ = (n - 1) - m_- \) and \( n_+ = (n - 1) - m_+ \), inequality

\[ 1 < p < \frac{n - n_- + 2}{n - n_+ - 2} = \frac{m_+ + 3}{m_+ - 1} \leq \frac{m_- + 3}{m_- - 1} = \frac{n - n_- + 2}{n - n_+ - 2} \]  

implies the validity of inequality 14 for both equations 10 and 12.

Since \( p_n \leq \frac{m_+ + 3}{m_+ - 1} \) is true for every \( 1 \leq m_+ \leq n - 1 \), we may guarantee the existence of an arbitrary large number of zeroes for equations 10 and 12 when \( p < p_n \), corresponding to the subcritical Yamabe problem, or when \( p_n < p < \frac{m_+ + 3}{m_- - 1} \), corresponding to the supercritical one. Also observe that inequality 15 is true when \( p = p_n \) and \( 1 \leq m_-, m_+ < n - 1 \), a fact used in [21] to assure the existence of a prescribed number of zeroes to equation 9 in the critical case. For the rest of this section, we will suppose that \( p > 1 \) satisfies inequality 15.

Even if the number of zeroes is arbitrarily large, it can not be infinite as we next show.

**Lemma 2.2.** For \( d_* > 0, d_* \neq 1 \) fixed, the zeroes and critical points of \( w_{d_*} \) are isolated in \([0, A]\).

**Proof.** First observe that if \( r_0 \in [0, A] \) is a zero of \( w_{d_*} \), then uniqueness of the solution implies that \( w'_{d_*}(r_0) \neq 0 \). Therefore \( w_{d_*} \) is monotone in a neighborhood of \( r_0 \) and it is an isolated zero. Now, to see that the critical points are isolated, suppose that \( w'_{d_*}(r_0) = 0 \) for some \( r_0 \in [0, A] \). As \( d \neq 0, 1 \), by uniqueness of the solutions, \( w_{d_*}(r_0) \neq 0 \) and this together with equation 13 implies that \( w''_{d_*}(r_0) \neq 0 \). Without loss of generality, suppose \( w''_{d_*} > 0 \). By continuity, there exists \( \varepsilon > 0 \) such that \( w''_{d_*}(r) > 0 \) in \((r_0 - \varepsilon, r_0 + \varepsilon) \cap [0, A] \). Hence \( w'_{d_*} \) is monotone in \((r_0 - \varepsilon, r_0 + \varepsilon) \cap [0, A] \) and, therefore, \( w'_{d_*}(r) \neq 0 \) in \((r_0 - \varepsilon, r_0 + \varepsilon) \cap [0, A] \). \( \{r_0\} \) and \( r_0 \) is an isolated critical point. \( \square \)
For \( \varepsilon < A \), take \( D_k > 0 \) as given in Theorem 2.1. Then, for \( d \geq D_k > 0 \), \( w_d \) has at least \( k \) zeroes before \( A \). Denote them by \( r_1(d) < r_2(d) < \ldots < r_k(d) \). The following statement holds true.

**Lemma 2.3.** For each \( j = 1, \ldots, k \),

\[
\lim_{d \to \infty} r_j(d) = 0
\]

**Proof.** Let

\[
z_d(r) := d^{-\frac{2m}{p-1}} w_d^{\frac{2}{p-1}} \left( \frac{r}{d^{\sqrt{\lambda}}} \right).
\]

For any \( K > 0 \), the functions \( z_d \) converge \( C^1 \)-uniformly on \([0, K]\) to the unique solution of the limit Cauchy problem

\[
\begin{cases}
v''(r) + \frac{H(0)}{v(0)} v'(r) + |v(r)|^{p-1} v = 0 & \text{in } [0, \infty), \\
v(0) = 1, \quad v'(0) = 0
\end{cases}
\]

and as \( p > 1 \) satisfies inequality 15, \( v \) has an infinite number of isolated zeroes in \((0, \infty)\) (see Lemma 3.2 and Theorem 3.3 in [21] and Proposition 3.6 in [25]).

Now observe that

\[
w_d(r) = \frac{dz}{dz} \left( \sqrt{\lambda d^{-\frac{2m}{p-1}} r} \right) \quad \text{and} \quad w'_d(r) = \sqrt{\lambda d^{-\frac{2m}{p-1}}} z_d^{-1} \left( \sqrt{\lambda d^{-\frac{2m}{p-1}} r} \right).
\]

Therefore \( r \) is a zero of \( w_d \) if and only if \( \sqrt{\lambda d^{-\frac{2m}{p-1}}} r \) is a zero of \( z_d^{-1} \). For each \( j = 1, \ldots, k \), denote by \( r_j(d) \) the \( j \)-th zero of \( z_d^{-1} \) and by \( a_j \) the \( j \)-th zero of the solution \( v_0 \) to the limit problem 16. Then \( r_j(d) = \lambda d^{-\frac{2m}{p-1}} r_j(d) \) and the \( C^1 \) uniform convergence of \( z_d \) to \( v_0 \) implies that \( r_j(d) \to a_j \) as \( d \to \infty \) for each \( j = 1, \ldots, k \). Hence \( r_j(d) \to 0 \) as \( d \to \infty \).

Now we focus on equation 10 in order to do a phase plane analysis. Let \( a_0 \) be the unique zero of \( h \) in \((0, \pi)\). Let \( w_d \) be the solution of 10 with initial conditions \( w_d(0) = d, w'_d(0) = 0 \).

Consider the curve \( I : \mathbb{R} \to \mathbb{R}^2 \) given by \( I(d) = (w_d(a_0), w'_d(a_0)) \). Note that \( I(0) = (0, 0), I(-d) = -I(d) \) and \( I(d) \neq (0, 0) \) if \( d \neq 0 \). It is then easy to see that we have a well defined continuous function \( \theta : (0, \infty) \to \mathbb{R} \) such that \( \theta(1) = 0 \) and \( \theta(d) \) gives an angle between \( I(d) \) and the positive \( x \)-axis for any \( d > 0 \). Note that, in a similar way, there is a unique continuous function \( \vartheta : (-\infty, 0) \to \mathbb{R} \) such that \( \vartheta(-1) = -\pi \) and \( \vartheta(d) \) gives an angle between \( I(d) \) and the positive \( x \)-axis. Thus, we have that for any \( d > 0 \) and \( \vartheta(-d) = \vartheta(d) - \pi \). Also notice that \( w_d(a_0) = 0 \) if and only if \( \theta(d) = -\frac{\pi}{2} - k\pi \) for some integer \( k \).

Next we proceed to define a second curve in the phase space, corresponding to the solutions to problem 11 in \([0, \pi]\) with condition \( w'(\pi) = 0 \). Let \( \hat{h}(r) = -h(\pi - r) = \frac{m-m+}{2} \cos r + \frac{m-m-}{2} \) and consider the problem 12. If \( \omega \) is a solution to this problem, then \( \bar{w}(r) := \omega(\pi - r) \) solves the “final” conditions problem 11. For \( c \in \mathbb{R} \), denote by \( \bar{w}_c \) the solution to the problem 11 and define the map \( J(c) := (\bar{w}_c(a_0), \bar{w}'_c(a_0)) \).

In an entirely similar way, \( J(1) = (1, 0), J(0) = (0, 0), J(c) \neq (0, 0) \) if \( c \neq 0 \) and \( J(-c) = -J(c) \). So, there is a well defined angle \( \vartheta : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) such that \( \vartheta(1) = 0 \) and

\[
J(c) = (|J(c)| \cos(\vartheta(c)), |J(c)| \sin(\vartheta(c))).
\]
Note that $\theta$ and $\vartheta$ run in opposite directions. If $n(d)$ denotes the number of zeroes of $w_d$ before $a_0$ and $N(c)$ the number of zeroes of $\tilde{w}_c$ after $a_0$, then the angles $\theta$ and $\vartheta$ are related with these numbers by the following formulas, proved in [21],

$$
n(d) = -\left\lceil \left( \theta(d) - \frac{\pi}{2} \right) / \pi \right\rceil - 1 \quad \text{and} \quad N(c) = -\left\lceil \left( -\vartheta(c) - \frac{\pi}{2} \right) / \pi \right\rceil - 1, \quad (17)
$$

where, as usual for $x \in \mathbb{R}$, $[x]$ denotes the maximum integer such that $[x] \leq x$. As a consequence of these formulas and Theorem 2.1, we have the following limits

$$
\lim_{d \to \infty} \theta(d) = -\infty \quad \text{and} \quad \lim_{|c| \to \infty} \vartheta(c) = \infty. \quad (18)
$$

We will show that the curves $I$ and $J$ behave like spirals turning in opposite directions. The above limits show that the spirals actually turn. Next we see that the spirals are not bounded. If $w_d$ is a solution to the initial value problem 10 and $w_c$ is a solution to 11, define $\rho(r,d) := \sqrt{w_d^2(r) + (w_d')^2(r)}$ and $\varrho(r,c) = \sqrt{w_c^2(r) + (w_c')^2(r)}$.

**Lemma 2.4.** We have that

$$
\lim_{d \to \infty} \rho(r,d) = \infty, \quad \text{and} \quad \lim_{|c| \to \infty} \varrho(r,c) = \infty,
$$

uniformly in $[0,a_0]$ and in $[a_0,\pi]$, respectively.

The proof of this Lemma is technical and we postpone it to the Appendix A.

Consider the functions $\rho(d) := \rho(d,a_0)$, $\varrho(c) := \varrho(c,a_0)$, $\theta(d) := \theta(d,a_0)$ and $\vartheta(c) := \vartheta(c,a_0)$.

Now, define in the radius-argument plane the curves $R : [1,\infty) \to \mathbb{R} \times \mathbb{R}_{>0}$ and $S : [1,\infty) \to \mathbb{R} \times \mathbb{R}_{>0}$ given by

$$
R(d) := (\theta(d),\rho(d)), \quad \text{and} \quad S(c) := (\vartheta(c),\varrho(c))
$$

From uniqueness of the solutions to problems 10 and 11, the curves $R$ and $S$ are simple and they intersect at the point $(0,1)$.

For each $i,j \in \mathbb{N}$, define

$$
d_i := \max \{ d : \theta(d) = -i\pi \} \quad \text{and} \quad c_j := \max \{ c : \vartheta(c) = j\pi \},
$$

and

$$
\hat{d}_i := \min \{ d : \theta(d) = -i\pi \} \quad \text{and} \quad \hat{c}_j := \min \{ c : \vartheta(c) = j\pi \},
$$

These numbers are well defined by 18 and they form unbounded sequences by the same limit. Observe that $\hat{d}_i$ and $d_i$ are the first and last time that the curve $R$ hits the line $\theta = -i\pi$, respectively, while $\hat{c}_j$ and $c_j$ correspond to the first and last time that $S$ hits the line $\vartheta = j\pi$. It was also shown in [21] that for any $c,d > 0$, $\theta(d) < \pi/2$ and $\vartheta(c) > -\pi/2$. So, it follows that the curve $R$ is completely contained in $(-\infty,\pi/2) \times \mathbb{R}_{>0}$, while $S$ is contained in $(-\pi/2,\infty) \times \mathbb{R}_{>0}$, and that $R$ restricted to $[d_1,\infty)$ and $S$ restricted to $[c_1,\infty)$ do not intersect.

We can now prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $k \in \mathbb{N}$ and for each $i,j \in \mathbb{N}$, set $x_i := \rho(d_i)$, $\hat{x}_i := \rho(\hat{d}_i)$, $y_j := \varrho(c_j)$ and $\hat{y}_j := \varrho(\hat{c}_j)$. By Lemma 2.4, these sequences are unbounded. Therefore, we can find $i,j > k$ and $\alpha > j$ such that $y_j < \min \{ x_i, \hat{x}_i, x_{\alpha+1-j} \} < \hat{y}_\alpha$.

Observe that the curves $R$ and $S - ((\alpha+j)\pi,0)$ restricted to the intervals $[d_i,\hat{d}_{\alpha+i-j}]$ and $[c_j,\hat{c}_\alpha]$, respectively, are both contained in $\mathcal{A}_k := [-(\alpha+i-j)\pi,-i\pi] \times [y_j,\hat{y}_\alpha]$. As $\hat{x}_{\alpha+i-j} > y_j$ and $x_i < \hat{y}_\alpha$ and as $R$ restricted to $[d_j,\hat{d}_{\alpha+j-i}]$ intersects $\mathcal{A}_k$ only
at the points \((-\alpha + j - i)\pi, \tilde{x}_{\alpha + j - i}\) and \((-j\pi, x_j)\), the intermediate value Theorem implies that the curve \(R\) must intersect \(S - ((\alpha + j)\pi, 0)\). Let \(d_R > 1\) and \(c_S > 1\) be the points such that \(R(d_R) = S(c_S) - ((\alpha + j)\pi, 0)\). Using the formulas 17, we can argue as in the proof of Theorem 1.2 in [21] to conclude that \(w_{d_R} = \tilde{w}_{c_S}\) is a solution to the problem 9 with exactly \(\alpha + j > k\) zeroes and, since \(w_{d_R}'(0) = w_{d_R}'(\pi) = 0\), it has, at least, \(k + 1\) critical points by Rolle’s Theorem. The fact that the zeroes and that the critical points are isolated follows from Lemma 2.2.

This theorem implies 1.1 as follows.

Proof of Theorem 1.1. Associated to \(S\), there is a Cartan-Münzner polynomial such that its restriction to the sphere is an isoparametric function \(f : S^n \to [-1, 1]\). Then equation 2 can be reduced into equation 9. By Theorem 1.2, this equation admits a sign-changing solution \(w_k\) having at least \(k\) isolated zeroes and at least \(k + 1\) isolated critical points in \([0, \pi]\). Therefore, \(u_k = w_k(\arccos f)\) is a solution to the problem 2 having as regular level sets a disjoint union of connected isoparametric hypersurfaces diffeomorphic to \(S\). As \(w_k\) has at least \(k\) isolated zeroes, then the nodal set \(w_k^{-1}(0)\) has at least \(k\) connected components, all of them being isoparametric hypersurfaces diffeomorphic to \(S\). As \(w_k\) has at least \(k - 1\) isolated critical points in \((0, \pi)\) and as \(w_k'(0) = 0 = w_k'(\pi)\), the critical set of \(u_k\) consists in, at least, \(k - 1\) connected isoparametric submanifolds diffeomorphic to \(S\), together with the focal submanifolds \(M_- = f^{-1}(-1)\) and \(M_+ = f^{-1}(1)\). Finally, by construction of \(w_k\), we have that \(w_k(0) \to \infty\) or \(|w_k(\pi)| \to \infty\) as \(k \to \infty\), for the number of zeroes increases as the initial or final conditions increase. Suppose, without loss of generality, that \(w_k(0) \to \infty\). In this case, as \(\arccos(f(x)) = 0\) for all \(x \in M_-\) we conclude the limit 4 for every \(x \in M_+\).

Even if Theorem 1.1 holds for subcritical, critical and supercritical values of \(p > 1\), the methods developed here do not allow us to prove the existence of solutions such that the final value \(w_d(\pi) = e\) is negative. However, in case of the critical exponent \(p_n\), we can use the refined version of this theorem given in [21] to construct a sequence of solutions which is not uniformly bounded from below.

Proof of Corollary 1. As the focal manifolds that generate \(S\) have positive dimension, the number of distinct principal curvatures \(\ell\) must be bigger that 1. Let \(f\) be the Cartan-Münzner polynomial associated to \(S\). Then, for any \(k \in \mathbb{N}\), Theorem 1.2 in [21] gives a solution to the Yamabe problem on the sphere 5 having the form \(u_k = \arccos(f(w_k))\), where \(w_k\) is a solution to the problem 9, with \(p = p_n\), having exactly \(k\) zeroes in \([0, \pi]\). Lemma 2.4 together with Lemma 4.6 in [21] imply that the sequences \((x_i)\) and \((y_j)\) are both increasing and unbounded. In this way, the situation in Lemma 4.7 in [21] can not happen and the number of zeroes before and after \(a_0\) must increase as the initial and final conditions, \(w_k(0)\) and \(|w_k(\pi)|\) respectively, increase. For \(k\) odd, by Lemma 2.2 and its proof, necessarily \(w_k(0) > 0\) and \(w_k(\pi) < 0\) and \(w_k\) can not have an infinite number of zeroes before and after \(a_0\). Therefore there exists a subsequence \((w_{k_j})\), with each \(k_j\) odd, such that \(w_{k_j}(0) \to \infty\) and \(w_{k_j}(\pi) \to -\infty\) as \(j \to \infty\). Since \(\arccos(f(x)) = 0\) for every \(x \in M_+\) and \(\arccos(f(x)) = \pi\) for every \(x \in M_-\), the sequence \((u_{k_j})\) satisfies the desired limits as \(j \to \infty\).

3. Submersions with minimal fibers and the proof of Corollary 2. We shall study Riemannian submersions of the form \(\pi : (\mathbb{S}^n, g_0) \to (M^m, g)\), where \((M, g)\) is
Lemma 3.1. Let $\pi : (N,h) \to (M,g)$ be a Riemannian submersion with minimal fibers. If a smooth function $f : M \to \mathbb{R}$ is isoparametric, then $\tilde{f} := f \circ \pi : N \to \mathbb{R}$ is isoparametric. Conversely, if $\pi : N \to \mathbb{R}$ is isoparametric and $f(x) = f(y)$ whenever $\pi(x) = \pi(y)$, then there exists a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $f = \tilde{f} \circ \pi$ and $\tilde{f}$ is isoparametric.

Proof. Suppose $\tilde{f} : M \to \mathbb{R}$ is isoparametric with $|\nabla \tilde{f}|^2_g = \tilde{a}(\tilde{f})$ and $\Delta \tilde{f} = \tilde{b}(\tilde{f})$. On the one hand, as $\pi : N \to M$ is a harmonic morphism with dilation $\varphi \equiv 1$ (see [2]), we have that

$$
\Delta_h f = \Delta_h (\tilde{f} \circ \pi) = (\Delta_{\tilde{g}} \tilde{f}) \circ \pi = \tilde{b}(\tilde{f} \circ \pi) = \tilde{b}(f).
$$

Next, observe that

$$
\nabla \tilde{f} = \pi_* \nabla f
$$

because $\pi : N \to M$ is a Riemannian submersion.

Therefore,

$$
|\nabla f(x)|^2_h = h(\nabla f(x), \nabla f(x)) = g(\pi_* \nabla f(x), \pi_* \nabla f(x))
$$

$$
= g(\nabla \tilde{f}(\pi(x)), \nabla \tilde{f}(\pi(x))) = |\nabla \tilde{f}(\pi(x))|^2_g
$$

$$
= a(\tilde{f}(\pi(x))) = \tilde{a}(f(x)),
$$

and $f$ is an isoparametric function.

Now suppose that $f$ is an isoparametric function such that $f(x) = f(y)$ if $\pi(x) = \pi(y)$. Let $a, b : \mathbb{R} \to \mathbb{R}$ be such that $|\nabla f|^2_h = a(f)$ and $\Delta_h f = b(f)$. As $\pi : N \to M$ is a submersion, it is a quotient map [29] and the function $f$ passes to the quotient as a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $f = \tilde{f} \circ \pi$. Then, as before

$$
a \circ \tilde{f} \circ \pi(x) = a(f(x)) = |\nabla f(x)|^2_h = |\nabla \tilde{f}(\pi(x))|^2_g
$$

$$
= a(\tilde{f}(\pi(x))) = \tilde{a}(f(x)),
$$

and

$$
b \circ \tilde{f} \circ \pi = b(f) = \Delta_h f = \Delta_h (\tilde{f} \circ \pi) = (\Delta_{\tilde{g}} \tilde{f}) \circ \pi.
$$

Since $\pi$ is surjective, it has a right inverse and we conclude that $|\nabla \tilde{f}|^2_g = a(\tilde{f})$ and $\Delta_{\tilde{g}} \tilde{f} = b(\tilde{f})$. \qed

Let $(M, g)$ denote $(\mathbb{C}P^m, g_0)$ or $(\mathbb{H}P^m, g_0)$, the complex and quaternionic spaces with their corresponding canonical metrics. Both of them are Einstein manifolds with constant positive scalar curvature (see [4, Theorem 14.39] and Sections 8.1 and 8.3 in [37]). We will denote it by $R_g = \Lambda_m > 0$. Recall that $\mathbb{C}P^m = S^{2m+1}/S^1$ and that $\mathbb{H}^m = S^{4m+3}/SU(2)$, where $S^1$ is the circle group and $SU(2)$ is the group of unit quaternions. We need the following lemma.
Lemma 3.2. We have the following:

1. For each $m \geq 3$, there exists an $S^1$-invariant Cartan-Münzner polynomial $f : S^{2m+1} \to \mathbb{R}$ such that the associated focal manifolds $M_-$ and $M_+$ satisfy $\dim M_\pm \geq 2$.

2. For each $m \geq 3$ there exists an $SU(2)$-invariant Cartan-Münzner polynomial $f : S^{4m+3} \to \mathbb{R}$ such that the associated focal manifolds $M_-$ and $M_+$ satisfy $\dim M_\pm \geq 4$.

Proof. We begin with the proof of 1. In this case, we consider $S^{2m+1} \subset \mathbb{R}^{2m+2}$. As $m \geq 3$, we can write $m = 2 + k$ with $k \in \mathbb{N} \setminus \{0\}$ and we can decompose $\mathbb{R}^{2m+2} \equiv \mathbb{R}^{2\alpha} \times \mathbb{R}^{2\beta} \equiv \mathbb{C}^\alpha \times \mathbb{C}^\beta$ where $\alpha = \beta = 2$ if $k = 1$ and $\alpha = k$, $\beta = 3$ if $k \geq 2$. Hence, we have an action of $S^1$ by isometries given as $\zeta(x, y) = (\zeta x, \zeta y)$, where $\zeta \in S^1 \subset \mathbb{C}$ and $x, y \in \mathbb{C}$. We can then consider the degree two Cartan-Münzner polynomial $f : \mathbb{R}^{2m+2} \equiv \mathbb{C}^\alpha \times \mathbb{C}^\beta \to \mathbb{R}$, $f(x, y) = |x|^2 - |y|^2$, which is clearly $S^1$-invariant. The restriction of $f$ to $S^{2m+1}$ is an isoparametric function with focal submanifolds $M_- = S^{2\alpha-1} \times \{0\}$ and $M_+ = \{0\} \times S^{2\beta-1}$, see [11]. If $k = 1$, $\dim M_\pm = 3$ while if $k \geq 2$, $\dim M_- = 2k - 1 \geq 3$ and $\dim M_+ = 5$, and the lemma follows.

Now we prove 2 in a similar way. For each $m \geq 4$, write $m = 4 + k$ with $k \in \mathbb{N} \cup \{0\}$. Hence $S^{4m+3} \subset \mathbb{R}^{4m+4} \equiv \mathbb{H}^\alpha \times \mathbb{H}^\beta \equiv \mathbb{H}^\alpha \times \mathbb{H}^\beta$, where $\alpha = \beta = 4$ if $m = 3$ and $\alpha = k + 2$ and $\beta = 3$ if $m \geq 4$. Hence we have a natural action of $SU(2)$ on $\mathbb{R}^{4m+4}$ given by $\zeta(x, y) = (\zeta x, \zeta y) = (\zeta x_1, \ldots, \zeta x_\alpha, \zeta y_1, \ldots, \zeta y_\beta)$ where $\zeta \in SU(2)$ and $x_i, y_j \in \mathbb{H}$. Since $|q| = |\zeta q| = |q|$ for each $q \in \mathbb{H}$ and $\zeta \in SU(2)$, we have that the Cartan-Münzner polynomial $f : \mathbb{R}^{2m+2} \equiv \mathbb{H}^\alpha \times \mathbb{H}^\beta \times \mathbb{R}^{2m+2} \to \mathbb{R}$ given by $f(x, y) = |x|^2 - |y|^2$ is $SU(2)$-invariant. As before, the corresponding focal submanifolds are given by $M_- = S^{4\alpha-1} \times \{0\}$ and $M_+ = \{0\} \times S^{4\beta-1}$. Thus, $\dim M_\pm = 7$ if $m = 3$, while $\dim M_- = 4(k + 2) - 1 \geq 11$ and $\dim M_+ = 11$ for every $m \geq 4$, where we conclude the lemma in this case.

Proof of Corollary 2. We analyze the case of the quaternionic projective space, being the case for $\mathbb{C}P^m$ completely analogous. First we write 6 in the form 19,

$$-\Delta_{g_m} v + \lambda_m u = \mu_m |u|^{p_{4m-1}} \quad \text{on } \mathbb{H}P^m,$$

where $\lambda_m := \frac{\lambda_m (4m - 2)}{16m - 4}$ and $\mu_m := \frac{4m - 2}{16m - 4}$. Let $\pi : S^{4m+3} \to \mathbb{H}P^m$ be the natural projection. This map is a Riemannian submersion with minimal fibers and, so, it is a harmonic morphism [2].

Therefore, $v$ is a solution to the Yamabe problem 6 if and only if $u = \left[ \frac{\mu_m}{\lambda_m} \right]^{\frac{1}{p_{4m-1}}} v \circ \pi$ is a solution to the supercritical problem on the sphere

$$-\Delta_{g_m} u + \lambda_m u = \lambda_m |u|^{p_{4m-1}} \quad \text{on } S^{4m+3}. \quad (21)$$

By Lemma 3.2, there exists an $SU(2)$-invariant isoparametric function $f : S^{4m+3} \to \mathbb{R}$, which is the restriction of a Cartan-Münzner polynomial and such that its corresponding focal submanifold have dimension at least 4. Hence, if we take $\kappa = \min\{\dim M_-, \dim M_+\} \geq 4$, then

$$p_{4m} < \frac{(4m + 3 - \kappa) + 2}{(4m + 3 - \kappa) - 2}$$

and the supercritical problem 21 admits a sequence of nodal solutions $(u_k)$ of the form $u_k = z_k \circ f$ where $z_k$ is a solution to the problem 8 with at least $k$ zeroes. As $f$ is $SU(2)$-invariant, there exists $\hat{f} : \mathbb{H}P^m \to \mathbb{R}$ such that $f = \hat{f} \circ \pi$. Therefore
where the isoparametric family satisfy $1 \leq \phi_m$ is a solution to 21, implying that $v_k := \left[ \frac{\nu_k}{\nu_m} \right]^{-\frac{1}{r_m}} z_k \circ \hat{f}$ is a solution to the Yamabe problem 6. Moreover, by Lemma 3.1, $\hat{f}$ is an isoparametric function and, thus, $v_k$ has isoparametric hypersurfaces in $\mathbb{H}P^m$ as level sets. Finally, the blow-up of the sequence $(v_k)$ is a consequence of the limit 4 for the sequence $(u_k)$ at $M_-$ or at $M_+$. \qed

Appendix A. Energy analysis and proof of Lemma 2.4. In this appendix we prove Lemma 2.4 for the solutions to the initial value problem 10. An entirely similar argument will hold for the problem 12 instead of 11. As in Section 2, we will suppose in what follows that the multiplicities of the principal curvatures of the isoparametric family satisfy $1 \leq m_- \leq m_+$ and that $p$ satisfies inequality 15. To simplify the notation, we write equation 9 as

$$w'' + \frac{h(r)}{\sin r} w' + g(w) = 0 \quad \text{in } [0, \pi], \quad (22)$$

where $g(t) := \lambda \frac{1}{r^p} (|t|^{p-1} t - t)$. For the initial conditions $w(0) = d$ and $w'(0) = 0$, let $w_d$ be the unique solution to problem 22 on $[0, a_0]$. Define the energy function

$$E(r, d) := \frac{(w_d'(r))^2}{2} + G(w_d(r)), \quad \text{where}$$

$$G(t) := \int_0^t g(s) ds = \frac{\lambda}{r^p} \left( \frac{|t|^{p+1}}{p+1} - \frac{t^2}{2} \right).$$

Observe that $E$ is nonincreasing on $r \in [0, a_0]$, for

$$E'(r, d) = -\frac{h(r)}{\sin r} (w_d'(r))^2.$$

The aim of this appendix is to prove that $E(a_0, d) \to \infty$ uniformly in $[0, a_0]$ as $d \to \infty$, for this will immediately imply Lemma 2.4. Since the proof is long and technical, we first sketch it in few lines, following the ideas given in [9] and [10].

First, in Lemma A.1 we prove the existence of the value $r_0(d)$ for which $w_d(r_0) = \kappa d$ and $d \geq w_d(r) \geq \kappa d$ for a suitable $\kappa \in (0, 1)$ and for every $r \in [0, r_0]$. Since we will show that $r_0(d) \to 0$, we need to prove the existence of fixed $T > 0$, independent of $d$, such that $E(r, d) \to \infty$ uniformly in $[0, T]$ as $d \to \infty$. To see this, we establish a version of the Pohozaev identity [39] for equation 22, generalizing the ones given in [9, 10]. This identity together with the properties of $r_0$ and inequality 15 will imply the existence of such $T > 0$. Finally, we prove that $E(r, d) \geq e^{-2T} E(T, d) + C$ for every $r \in [T, a_0]$, where $C$ is a constant independent of $d$ and $r$. The last inequality implies the desired uniform limit.

We begin with the existence of $r_0$.

Lemma A.1. For each $\kappa \in (0, 1)$, there exists $\bar{\kappa} = \bar{\kappa}(\kappa) \geq 1$ such that for every $d \geq \bar{D}_1$, $\kappa d \geq 1$ and there is $r_0 = r_0(d) \in (0, \pi)$ such that

$$w_d(r_0) = \kappa d \quad \kappa d \leq w_d(r) \leq d, \quad \text{for every } r \in [0, r_0] \quad \text{and} \quad \lim_{d \to \infty} r_0(d) = 0. \quad (23)$$

Proof. As inequality 15 implies 14, Theorem 2.1 gives the existence of $\bar{D}_1 \geq 1$ such that $w_d$ has a zero in $[0, a_0]$. Take $\bar{D}_1$ so that $\kappa d \geq 1$ for every $d \geq \bar{D}_1$ and for each $d \geq \bar{D}_1$, denote by $r_1(d)$ the first zero of $w_d$. Then $w_d$ is strictly decreasing in $[0, r_1]$ and $r_1$ depends continuously on $d$. Therefore, there exists $r_0(d) \in (0, r_1(d))$
such that \( k d = w_d(r) \leq w_d(r) \leq w_d(0) = d \) for every \( r \in [0, r_0] \). By Lemma 2.3 we conclude that \( r_0(d) \to 0 \) as \( d \to \infty \).

Recall \( \ell \) denotes the number of distinct principal curvatures of the isoparametric hypersurfaces associated to the isoparametric function \( f \) on the sphere, and let \( 1 \leq m_- \leq m_+ \leq n - 1 \) be the (possibly equal) multiplicities of the principal curvatures. In this situation, \( \frac{n-1}{\ell} = \frac{m_- + m_+}{2} \) and an integrating factor for equation 22 is given by

\[
q(r) := (\sin r)^{n-1}(\tan r/2)^{\frac{m_+ - m_-}{2}} = 2^{\frac{m_- + m_+}{2}}(\sin r/2)^{m_+ - (\cos r/2)^{m_+}}.
\]

Therefore, equation 22 can be written in divergence form as

\[
(qw')' + q(r)g(w) = 0 \quad \text{in } [0, \pi] \tag{24}
\]

and \( q \) satisfies

\[
\frac{q'(r)}{q(r)} = \frac{h(r)}{\sin r} \tag{25}
\]

Define \( \zeta : [0, a_0] \to \mathbb{R} \) by

\[
\zeta(r) := \begin{cases} 
q(r) \int_r^{a_0} q^{-1}(s) ds, & \text{if } r \neq 0, \\
0, & \text{if } r = 0.
\end{cases}
\]

Observe this function is continuous at \( r = 0 \), for

\[
\lim_{r \to 0} \zeta(r) = 0 \tag{26}
\]

by L'Hôpital's rule and 25. Notice also that

\[
\zeta'(r) = \frac{h(r)}{\sin r} \zeta(r) - 1. \tag{27}
\]

We next derive a useful Pohozaev-like identity.

**Lemma A.2.** If \( w_d \) is the solution to 22 in \( [0, a_0] \) with initial conditions \( w_d(0) = d \) and \( w_d'(0) = 0 \), then

\[
P(r, d) := qw_dw_d' + 2q\zeta E(r, d) = \int_0^r q \left\{ G(w_d)\zeta \left[ \frac{h(s)}{\sin s} - 2 \right] - g(w_d)w_d \right\} ds. \tag{28}
\]

**Proof.** On the one hand, multiplying equation 22 by \( qw_d \), integrating from 0 to \( r \leq a_0 \) and integrating by parts we obtain

\[
qw_dw_d - \int_0^r q(w_d')^2 ds + \int_0^r qg(w_d)w_d = 0 \tag{29}
\]

because \( w_d'(0) = 0 \) and \( q' = \frac{h(s)}{\sin s} \).

On the other hand, multiplying equation 22 by \( q\zeta w_d' \), integrating from 0 to \( r \), using integration by parts and 27 we have that

\[
q\zeta(w_d')^2 + \int_0^r q(w_d')^2 ds + 2 \int_0^r q\zeta g(w_d)w_d ds = 0. \tag{30}
\]

But also, integration by parts yields

\[
\int_0^r q\zeta g(w_d)w_d' ds = q\zeta G(w_d) - \int_0^r qG(w_d)\zeta \left[ \frac{2h(s)}{\sin s} - 1 \right] ds
\]

Thus, adding 29 and 30, and using the above equality, identity 28 follows. \( \square \)
Observe that the derivative of $w_d$ does not appear in the right hand side of the identity, while the energy appears explicitly on the left hand side. Observe also that if $r \in [0, a_0]$ is such that $P(r, d) \rightarrow \infty$ as $d \rightarrow \infty$, then also $E(r, d) \rightarrow \infty$ as $d \rightarrow \infty$. In this direction, we state and prove the following Lemma.

**Lemma A.3.** There exists $T > 0$ small enough and fixed such that

$$
\lim_{d \rightarrow \infty} P(r, d) = \infty \quad (31)
$$

uniformly in $[r_0, T]$.

The proof of this lemma is long and technical, and will take the following four pages. As we will continue with the argument after its proof with Lemma A.4, the reader may skip it in a first reading.

**Proof.** The proof uses strongly that inequality 15 holds true. For the reader convenience, we divide it into three steps.

We begin with some estimates of $r_0(d)$ in terms of the initial condition $d$.

**Step 1.** Let $\kappa \in (0, 1)$ and let $\hat{D}_1 := \hat{D}_1(\kappa)$ as in Lemma A.1. Then, for $d \geq \hat{D}_1$, there exist $\kappa_1, \kappa_2, \kappa_3 > 0$ independent of $d$ such that

$$
\kappa_1 e^{d \cdot \frac{d}{g(d)}} \leq \cos r_0/2 \quad \text{and} \quad \kappa_2 \left[ \frac{d}{g(d)} \right] ^{\frac{1}{2}} \leq \sin r_0/2 \leq \kappa_3 \sqrt{\frac{d}{g(d)}}. \quad (32)
$$

**Proof of Step 1.** First observe that the integrating factor $q$ satisfies, for any $R \in (0, \pi)$, the following estimates

$$
\frac{2^{m+1} - 2}{m - 1} (\cos R/2)^{m+1} - (\sin R/2)^{m+1} \leq \int_0^R q(r) dr \leq \frac{2^{m+1} - 2}{m - 1} (\sin R/2)^{m+1}.
$$

Now, let $\kappa \in (0, 1)$ and $\hat{D}_1$ be as in the hypotheses of Lemma A.1. As $1 \leq \kappa d \leq w_d(r) \leq d$ in $[0, r_0]$ when $d \geq \hat{D}_1$, and as $g$ is nondecreasing in $[1, \infty)$, then $0 \geq -g(\kappa d) \geq -g(w_d) \geq -g(d)$ in $[0, r_0]$. Now, integrating equation 24 from 0 to $r < r_0$ and recalling that $w'_d(0) = 0$, we get

$$
\frac{2^{m+1} - 2}{m - 1} (\sin r/2)^{m} - (\cos r/2)^{m+1} w'_d(r) = q(r)w'_d(r) = qw'_d|_0^r = \int_0^r (qw'_d)' ds = - \int_0^r qg(w_d) \leq -g(\kappa d) \int_0^r q(s) ds \leq -\frac{2^{m+1} - 2}{m - 1} g(\kappa d)(\cos r/2)^{m+1} - (\sin r/2)^{m+1}.
$$

Hence

$$
w'_d \leq -\frac{g(\kappa d)}{m - 1} (\cos r/2)^{-1} \sin r/2.
$$
Integrating over \([0, r_0]\) we obtain
\[
(\kappa - 1)d = w_d(r_0) - w_d(0) \\
\leq - \frac{g(\kappa d)}{m_- + 1} \int_0^{r_0} (\cos r/2)^{-1} \sin r/2 \, dr \\
= 2 \frac{g(\kappa d)}{m_- + 1} \int_1^{\cos r_0/2} x^{-1} \, dx \\
= 2 \frac{g(\kappa d)}{m_- + 1} \ln \cos r_0/2.
\]
Therefore
\[
\kappa_1 e^{\frac{d}{m_- + 1}} \leq \cos r_0/2, \quad \text{with} \quad 0 < \kappa_1 := e^{\frac{(\kappa - 1)(m_- + 1)}{2}} < 1
\]
Similarly we obtain the second estimate: Integrating equation 24 from 0 to \(r < r_0\) we have that
\[
2 \frac{m_- + m_+}{m_- + 1} (\sin r/2)^m w'_d(r) \geq q(r) w'_d(r) \\
= - \int_0^r qg(w_d) \, ds \\
\geq - g(d) \int_0^r q(s) \, ds \\
\geq - 2 \frac{m_- + m_+}{m_- + 1} (\sin r/2)^m + 1.
\]
So \(w'_d \geq - \frac{g(d)}{m_- + 1} \sin r/2\). Integrating over \([0, r_0]\) we get that
\[
(\kappa - 1)d = w_d(r_0) - w_d(0) \geq - \frac{g(d)}{m_- + 1} \int_0^{r_0} \sin r/2 \, dr = \frac{2g(d)}{m_- + 1} [\cos r_0/2 - 1].
\]
Noticing that \(0 \leq \cos r/2 \leq 1\) in \([0, \pi]\) implies \(\sin^2 r/2 = 1 - \cos^2 r/2 \geq 1 - \cos r/2\), we write
\[
(1 - \kappa)(m_- + 1) \frac{d}{g(d)} \leq 1 - \cos r_0/2 \leq \sin^2 r_0/2,
\]
where we conclude that
\[
\kappa_2 \left[ \frac{d}{g(d)} \right]^{1/2} \leq \sin r_0/2, \quad \text{with} \quad \kappa_2 := \left[ \frac{(1 - \kappa)(m_- + 1)}{2} \right]^{1/2} > 0.
\]
Next, for the third one, since \(-(\cos r/2)^{m_+ + 1} \geq -(\cos r/2)^{m_- - 1}\), we get in the same fashion that
\[
2 \frac{m_- + m_+}{m_- + 1} (\cos r/2)^m w'_d(r) \\
= - \int_0^r qg(w_d) \leq - g(\kappa d) \int_0^r q(s) \, ds \\
\leq - 2 \frac{m_- + m_+}{m_- + 1} g(\kappa d)(\cos r/2)^{m_- - 1}(\sin r/2)^{m_+ + 1} \\
\leq - 2 \frac{m_- + m_+}{m_- + 1} g(\kappa d)(\cos r/2)^{m_+ + 1}(\sin r/2)^{m_- + 1}.
\]
So
\[
w'_d(r) \leq - \frac{g(\kappa d)}{m_- + 1} \cos r/2 \sin r/2.
\]
Integrating from 0 to $r_0$ we obtain

$$\int_0^{r_0} w'_d(s)ds \leq -\frac{g(\kappa d)}{m_- + 1} \int_0^{r_0} \cos s/2 \sin s/2 \, ds = -\frac{g(\kappa d)}{m_- + 1} \sin^2 r_0/2.$$ 

Hence

$$\kappa^2 \frac{d}{g(\kappa d)} \geq \sin^2 r_0/2, \quad \text{with} \quad \kappa^2 := (m_- + 1)(1 - \kappa) > 0.$$

\[\square\]

**Step 2.** There exist $\kappa \in (0, 1)$ and $\theta > 0$ such that

$$\lim_{d \to \infty} [\theta G(\kappa d) - g(d)\lambda] \left[ \frac{d}{g(d)} \right]^{\frac{m_- + 1}{2}} = \infty. \quad (34)$$

**Proof of Step 2.** As $\lim_{r \to 0} \zeta(r) = 0$ and $\lim_{r \to 0} h(r) = m_-$, if we write $\sin r = 2\sin \frac{r}{2} \cos \frac{r}{2}$ and use L'Hôpital's rule, we then have that

$$\lim_{r \to 0} \zeta \left[ \frac{h(s)}{\sin s} - 2 \right] = 4m_- \lim_{r \to 0} \frac{g \int_0^r g^{-1}(s)ds}{\sin s} = 2m_- \lim_{r \to 0} \frac{\int_r^\infty (\sin s/2)^{m_- - 1}(\cos s/2)^{1 - m_-} ds}{(\cos r/2)^{1 - m_-}} = 2m_- \lim_{r \to 0} \frac{1 - m_-}{2 \cos^2 r/2 - \frac{1 - m_-}{2} \sin^2 r/2} = \frac{4m_-}{m_- - 1}.$$

Inequality 15 yields that $0 < \frac{4m_-}{m_- - 1} - (p + 1)$. Therefore, we can fix $0 < \theta := \frac{4m_-}{m_- - 1} - \varepsilon$ and $\kappa := (1 - \delta)^{1/(p+1)} \in (0, 1)$ with $\varepsilon, \delta > 0$ small enough so that

$$\frac{\theta - (p + 1)}{p + 1} > \frac{\theta \kappa^{p+1} - (p + 1)}{p + 1} > 0.$$

Hence, the functions $\theta G(\kappa t) - tg(t) = \lambda \left[ \frac{\theta \kappa^{p+1} - (p + 1)}{p + 1} \right] |t|^{p+1} - \frac{\theta \kappa^{2} - 2}{2} |t|^2$ and $G(t) - tg(t)$ are bounded from below. Using this, inequality 14 and that $1 < p$, we have that

$$\lim_{d \to \infty} [\theta G(\kappa d) - g(d)\lambda] \left[ \frac{d}{g(d)} \right]^{\frac{m_- + 1}{2}} = \lim_{d \to \infty} \lambda^{\frac{1 - m_-}{2}} \left[ |d|^{p-1} - \frac{(\theta \kappa^{p+1} - (p + 1)}{p + 1} |d|^2 \right]^{\frac{m_- + 1}{2}} = \lim_{d \to \infty} \lambda^{\frac{1 - m_-}{2}} \left[ |d|^{p-1} - \frac{(\theta \kappa^{p+1} - (p + 1)}{p + 1} |d|^2 \right]^{\frac{m_- + 1}{2}} = \infty. \quad (36)$$

\[\square\]
Step 3. There exists $T > 0$ such that
\[
\lim_{d \to \infty} P(r, d) = \infty
\]
uniformly in $[r_0, T]$.

Proof of Step 3. For $\kappa$ and $\theta$ as above, consider $\hat{\mathcal{D}}_1 \geq 1$ as in Lemma A.1. By limit 35, we can choose $T \in (0, a_0)$ small enough so that
\[
\zeta \left[ \frac{4h(s)}{\sin s} - 2 \right] \geq \theta, \quad r \in [0, T].
\]
By Step 1, $r_0(d) \to 0$ when $d \to \infty$ and we can choose $\hat{\mathcal{D}}_2 \geq \hat{\mathcal{D}}_1$ such that $r_0(d) < T$ for every $d \geq \hat{\mathcal{D}}_2$. Since for every $d \geq \hat{\mathcal{D}}_2$, we have that $1 \leq \kappa d \leq w_d(r) \leq d$ for every $r \in [0, r_0]$, and since the functions $G(t)$ and $tg(t)$ are nondecreasing when $t \geq 1$, it follows that $G(w_d) \geq G(\kappa d) \geq 0$ and that $-g(w_d)w_d \geq -g(d)d$. Hence
\[
G(w_d)\zeta \left[ \frac{4h(r)}{\sin r} - 2 \right] - g(w_d)w_d \geq \theta G(\kappa d) - g(d)d > 0,
\]
for every $r \in [0, r_0]$ and every $d \geq \hat{\mathcal{D}}_3$, where $\hat{\mathcal{D}}_3 \geq \hat{\mathcal{D}}_3$ is such that $G(\kappa d) - g(d)d > 0$ for every $d \geq \hat{\mathcal{D}}_3$.

First we prove the following limit
\[
\lim_{d \to \infty} P(r_0, d) = \infty
\]
Indeed, since $d \geq \hat{\mathcal{D}}_3$, the estimates 33 and the ones obtained in Step 1 yield that
\[
\begin{align*}
P(r_0, d) &= \int_0^{r_0} q \left\{ G(w_d)\zeta \left[ \frac{4h(s)}{\sin s} - 2 \right] - g(w_d) \right\} ds \\
&\geq [\theta G(\kappa d) - g(d)d] \int_0^{r_0} qds \\
&\geq 2^{\frac{m_+ - m_1}{m_- + 1}} \kappa_1^{-\frac{m_1 - 1}{m_- + 1}} \kappa_2^{-\frac{m_1 + 1}{m_- + 1}} \frac{1}{\pi^{\frac{m_1 + 1}{m_- + 1}}} \frac{1}{\pi^{\frac{m_1 + 2}{m_- + 1}}} [\theta G(\kappa d) - g(d)d] e^{(m_1 - 1) \frac{d}{g(d)}} \left[ \frac{1}{g(d)} \right]^{m_1 + 1}
\end{align*}
\]
and since $e^{(m_1 - 1) \frac{d}{g(d)}} \to 1$ as $d \to \infty$, 34 implies $P(r_0, d) \to \infty$ as $d \to \infty$ as we wanted.

Next, as $\theta G(\kappa d) - g(d)d$ is bounded from below, there exists $M < 0$ such that $\theta G(\kappa d) - dg(d) \geq M$ for every $d \in \mathbb{R}$. Then, for every $r \in [0, T]$ we have that
\[
\begin{align*}
P(r, d) &= P(r_0, d) + \int_{r_0}^{r} q \left\{ G(w_d)\zeta \left[ \frac{4h(s)}{\sin s} - 2 \right] - g(w_d) \right\} ds \\
&\geq P(r_0, d) + [\theta G(\kappa d) - g(d)d] \int_{r_0}^{r} q(s)ds \\
&\geq P(r_0, d) + 2^{\frac{m_+ - m_1}{m_- + 1}} \frac{M}{m_- + 1} \left[ (\sin r_0/2)^{m_- + 1} - (\sin r_0/2)^{m_- + 1} \right] \\
&\geq P(r_0, d) + 2^{\frac{m_+ - m_1}{m_- + 1}} \frac{M}{m_- + 1}
\end{align*}
\]
and as the last constant does not depend on \( r \in [r_0, T] \) and \( d \geq \hat{D}_1 \), it follows that
\[
\lim_{d \to \infty} P(r, d) = \infty \quad \text{uniformly on} \quad [r_0, T]
\]
as we wanted to prove.

With the proof of Step 3 we conclude the proof of Lemma A.3.

We can now prove the uniform convergence of the energy function.

**Lemma A.4.**
\[
\lim_{d \to \infty} E(r, d) = \infty, \quad \text{uniformly in} \quad [0, a_0].
\]

**Proof.** Take \( T \) as in the previous lemma. Then clearly the limit 31 implies that
\[
\lim_{d \to \infty} E(r, d) = \infty \quad \text{uniformly in} \quad [r_0, T].
\]
Now we show that \( E(r, d) \) also converges uniformly in \([0, r_0]\) and in \([T, a_0]\) as \( d \to \infty \).

Let \( \kappa \in (0, 1) \) be as in Step 2 of the proof of Lemma A.3 and consider \( \hat{D}_1(\kappa) > 1 \)
as in Lemma A.1. Then, for every \( d \geq \hat{D}_1(\kappa) \) and every \( r \in [0, r_0] \), we have that
\[
1 \leq \kappa d \leq w_d(r) \leq d,
\]
which implies that \( G(w_d(r)) \geq G(\kappa d) \). Since \( G(\kappa d) \to \infty \) as \( d \to \infty \) and since
\[
E(r, d) = \frac{(w_d')^2}{2} + G(w_d) \geq G(w_d) \geq G(\kappa d), \quad \text{for every} \quad r \in [0, r_0],
\]
we conclude that \( E(r, d) \to \infty \) as \( d \to \infty \) uniformly on \([0, r_0]\).

Finally, define \( \hat{E}(r, d) = E(r, d) - K \), where \( K < 0 \) is a lower bound for \( G(d) \). As \( h(r) \geq 0 \) in \([T, a_0]\), being \( a_0 \) the unique zero of this function, by continuity we have that \( \tau := \max_{r \in [T, a_0]} \frac{h(r)}{\sin \tau} > 0 \). Hence
\[
\hat{E}'(r, d) = E'(r, d) = \frac{h(r)(w_d')^2}{\sin r} \geq -2\tau \frac{(w_d')^2}{2} + 2\tau K - 2\tau K
\]
\[
\geq -2\tau \frac{(w_d')^2}{2} + 2\tau K - 2\tau G(w_d) = -2\tau \left[ \frac{(w_d')^2}{2} + G(w_d) - K \right]
\]
\[
= -2\tau \hat{E}(r, d).
\]
Integration on \([T, r]\) yields \( \hat{E}(r, d) \geq e^{-2\tau T \hat{E}(T, d)} \) for every \( r \in [T, a_0] \). Since \( E'(T, d) \to \infty \) as \( d \to \infty \), we get that \( E(r, d) \to \infty \) uniformly in \([T, a_0]\) as \( d \to \infty \) and the lemma follows.

**Proof of Lemma 2.4.** If \( \lim_{d \to 0} \rho(r, d) = \infty \) is not true, then \( w_d \) and \( w_d' \) are bounded as \( d \to \infty \), contradicting 37.

Now, if \( w_c(r) \) is a solution to 22 with initial conditions \( w(\pi) = c \) and \( w'(\pi) = 0 \), then, as it was mentioned in Section 2, the function \( \omega_c(r) := w_c(\pi - r) \) is a solution to the equivalent problem 12. As \( h \) and \( \bar{h} \) have the same properties interchanging \( m_- \) and \( m_+ \) and taking \( \bar{a}_0 := \pi - a_0 \) instead of \( a_0 \), Lemmas A.1-A.4 hold true for the energy \( \hat{E}(r, c) := \frac{(w_c'(r))^2}{2} + G(\omega_c(r)), \ r \in [0, \bar{a}_0] \), because of inequality 15. Therefore \( \lim_{c \to \infty} \hat{E}(r, c) = \infty \) uniformly in \([0, \bar{a}_0]\) and \( |(\omega_c(r), \omega_c'(r))| \) \to \infty uniformly in \([0, \bar{a}_0]\) as \( |c| \to \infty \), concluding the proof of the lemma.

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