Quantization of the $N = 2$ Supersymmetric KdV Hierarchy

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Abstract

We continue the study of the quantization of supersymmetric integrable KdV hierarchies. We consider the $N=2$ KdV model based on the $\mathfrak{sl}(2|1)$ affine algebra but with a new algebraic construction for the $L$-operator, different from the standard Drinfeld-Sokolov reduction. We construct the quantum monodromy matrix satisfying a special version of the reflection equation and show that in the classical limit, this object gives the monodromy matrix of $N=2$ supersymmetric KdV system. We also show that at both the classical and the quantum levels, the trace of the monodromy matrix (transfer matrix) is invariant under two supersymmetry transformations and the zero mode of the associated $U(1)$ current.

Keywords: superconformal field theory, quantum superalgebras, supersymmetric KdV equation, supersymmetric integrable systems, quantization.

1 Introduction

In [1], [3] we considered the quantization of the Drinfeld-Sokolov hierarchies associated with affine superalgebras along the lines of the bosonic approach introduced in [1] for the usual $sl(2)$ KdV model. Here, we extend this approach to nonstandard KdV hierarchies related to the $sl^{(1)}(m+1|m)$ superalgebras. We consider the $N=2$ KdV model with the underlying affine superalgebra $sl^{(1)}(2|1)$ in detail, but the generalization to the higher rank case is straightforward. The interest in such integrable models arises because the quantization procedure becomes more involved than for the standard Drinfeld-Sokolov hierarchies.

To begin quantizing this model, we first consider the corresponding classical theory (see Sec. 2). The classical version of the associated monodromy matrix is represented using the familiar $P$-exponential, but the exponent consists not...
only of generators corresponding to simple roots and their quadratic combinations but also of generators corresponding to more complicated composite roots (see Sec. 2). We prove that all these terms corresponding to composite roots (as in the “standard” case) disappear from the first iteration of the quantum generalization of the P-exponential. In addition, we show that this “quantum” P-exponential coincides with the reduced universal R-matrix of $\hat{sl}_q(2|1)$ quantum affine superalgebra (see Sec. 3 and 4). We also prove that supertraces of the quantum version of the monodromy matrix, the so-called transfer matrices (by an obvious analogy with the lattice case), commute with the supersymmetry generators. Hence, these generators can be included in the family of integrals of motion at both the classical and the quantum levels (see Sec. 5). In the last section, we also discuss the relation of N=2 supersymmetric KdV model to the topological theories and their integrable perturbations.

2 The $N=2$ SUSY KdV hierarchy in nonstandard form

The matrix L-operator of the N=2 SUSY KdV hierarchy has the explicit form [3]:

$$L_F = D - \begin{pmatrix} D\Phi_1 & 1 & 0 \\ \lambda & D\Phi_1 - D\Phi_2 & 1 \\ \lambda D(\Phi_1 - \Phi_2) & \lambda & -D\Phi_2 \end{pmatrix},$$

(1)

where $D = \partial_u + \theta \partial_u$, $u$ is a variable on a cylinder of circumference $2\pi$, $\theta$ is a Grassmann number, and $\Phi_i(u, \theta) = \phi_i(u) - \frac{1}{\sqrt{2}}\theta \xi_i(u)$ are the superfields with the Poisson brackets

$$\{\Phi_i(u_1, \theta_1), \Phi_i(u_2, \theta_2)\} = 0 \quad (i = 1, 2)$$

$$\{D_{u_1, \theta_1} \Phi_1(u_1, \theta_1), D_{u_2, \theta_2} \Phi_2(u_2, \theta_2)\} = -D_{u_1, \theta_1} (\delta(u_1 - u_2)(\theta_1 - \theta_2))$$

This is the odd L-operator related to the $sl^{(1)}(2|1)$ affine superalgebra (the elements in the representation column vector are graded top down in the order “even, odd, even”). There exists another (canonical) operator, also corresponding to the N=2 SUSY KdV model, related to the $sl^{(1)}(2|2)$ superalgebra [4].

This realization, which corresponds to a higher rank, can be quantized using the procedure outlined in [2], [3], which adds nothing new to the quantization procedure compared with [1]-[3]. Moreover, the representation theory of $sl^{(1)}(2|2)$ is quite complicated at both the classical and the quantum levels.

The form of the L-operator given above, corresponding to a lower rank, allows working with the much simpler representation theory of the $sl^{(1)}(2|1)$ superalgebra and provides very interesting features of the quantization formalism.

To build the scalar L-operator related to the matrix operator, we consider the linear problem $L\Psi = 0$ and express the second and third elements in the vector $\Psi$ in terms of the first (top) one. The linear equation for this element is

$$((D + D\Phi_1)(D - D(\Phi_1 - \Phi_2))(D - D\Phi_2) + 2\lambda D)\Psi_1 = 0.$$  

(3)
We thus obtain the scalar linear L-operator as

\[ L = D^3 + (V + 2\lambda)D + U, \]  

where the Miura map is specified by the relations

\[ V = -D\Phi_2 D\Phi_1 + \partial \Phi_2 - \partial \Phi_1 = \frac{i}{\sqrt{2}}(\alpha^+ - \alpha^-) + V, \]

\[ U = -\partial \Phi_2 D\Phi_1 - D\partial \Phi_1 = \theta U + \frac{i}{\sqrt{2}}\alpha^+, \]  

where the Miura map is specified by the relations

\[ \alpha^+ = \xi_2 \phi'_1 + \xi'_2, \quad V = \phi'_2 - \phi'_1 + \frac{1}{2}\xi_2 \xi_1 \]

and \( U, V, \) and \( \alpha^\pm \) satisfy the N=2 superconformal algebra under the Poisson brackets:

\[
\begin{align*}
\{ U(u), \alpha^+(v) \} &= -\alpha'^+(u)\delta(u-v) - 2\alpha^+(u)\delta'(u-v), \\
\{ U(u), \alpha^-(v) \} &= -\alpha'^-(u)\delta(u-v), \\
\{ \alpha^+(u), \alpha^-(v) \} &= -2U(u)\delta(u-v) - 2V(u)\delta'(u-v) - 2\delta''(u-v), \\
\{ V(u), \alpha^+(v) \} &= -\alpha'^+(u)\delta(u-v), \\
\{ V(u), \alpha^-(v) \} &= \alpha'-u)\delta(u-v), \\
\{ U(u), V(v) \} &= -V(u)\delta'(u-v), \\
\{ V(u), V(v) \} &= -2\delta'(u-v), \\
\{ U(u), U(v) \} &= -U'(u)\delta(u-v) - 2U(u)\delta'(u-v). 
\end{align*}
\]

We now rewrite this L-operator in the algebraic form as

\[ \mathcal{L}_F = D - (h_{\alpha_1} D\Phi_1 + h_{\alpha_2} D\Phi_2 + e_{\alpha_1} + e_{\alpha_2} + [e_{\alpha_2}, e_{\alpha_0}]) + [e_{\alpha_0}, e_{\alpha_1}] + D(\Phi_1 - \Phi_2) e_{\alpha_0}, \]  

where \( h_{\alpha_i} \) and \( e_{\alpha_i} \) are the generators of the upper Borel algebra of \( sl^{(1)}(2|1) \) with the commutation relations

\[
\begin{align*}
[e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{i,j}h_{\alpha_i} \quad (i = 0, 1, 2), \\
[h_{\alpha_j}, e_{\pm\alpha_i}] &= \pm e_{\pm\alpha_i} \quad (i = 1, 2, i \neq j), \\
h_{\alpha_1}, e_{\pm\alpha_0} &= \mp e_{\pm\alpha_0} \quad (i = 1, 2), \\
h_{\alpha_0}, e_{\pm\alpha_1} &= \mp e_{\pm\alpha_1} \quad (i = 1, 2), \\
h_{\alpha_1}, e_{\pm\alpha_0} &= 0 \quad (i = 1, 2), \\
h_{\alpha_0}, e_{\pm\alpha_1} &= 0 \quad (k = 1, 2), \\
[ad_{\pm\alpha_i}, e_{\pm\alpha_j}] &= 0 \quad (k = 1, 2), \\
\end{align*}
\]

where the generators \( e_{\pm\alpha_i} \) are odd for \( i = 1, 2 \) and even for \( i = 0 \) and \([,] \) denotes the supercommutator.

The symmetrized Cartan matrix \( b_{ij} = (\alpha_i, \alpha_j) \) corresponding to the given affine superalgebra is

\[
\begin{pmatrix}
0 & 1 & -1 \\
1 & 0 & -1 \\
-1 & -1 & 2
\end{pmatrix}.
\]
In this paper, we use only the evaluation representations (for which the central charge of the corresponding affine algebra is equal to zero).

To write a monodromy matrix, we must consider the equivalent bosonic L-operator. Expressing the linear problem associated with operator \( \Psi \) as

\[
L \Psi = (\partial_\theta + \theta \partial_u + N_1 + \theta N_0)(\Psi_0 + \theta \Psi_1),
\]

we can easily rewrite it as a bosonic linear problem for \( \Psi_0 \):

\[
L_B \Psi_0 \equiv (\partial_u + N_1^2 + N_0)\Psi_0 = 0
\]

with \( \Psi_1 = -N_1 \Psi_0 \), where

\[
N_1 = \frac{i}{\sqrt{2}} \xi_1 h_{\alpha_1} + \frac{i}{\sqrt{2}} \xi_2 h_{\alpha_2} - e_{\alpha_1} - e_{\alpha_2} - [e_{\alpha_2}, e_{\alpha_0}] - [e_{\alpha_0}, e_{\alpha_1}],
\]

\[
N_0 = -\phi_1' h_{\alpha_1} - \phi_2' h_{\alpha_2} - (\phi_1' - \phi_2') e_{\alpha_0}.
\]

Hence,

\[
L_B = \partial_u + \left( \frac{i}{\sqrt{2}} \xi_1 h_{\alpha_1} + \frac{i}{\sqrt{2}} \xi_2 h_{\alpha_2} - [e_{\alpha_2}, e_{\alpha_0}] - [e_{\alpha_0}, e_{\alpha_1}] - e_{\alpha_1} - e_{\alpha_2} + \frac{i}{\sqrt{2}} (\xi_1 - \xi_2) e_{\alpha_0} \right)
\]

\[
- e_{\alpha_1} - e_{\alpha_2} + \frac{i}{\sqrt{2}} (\xi_1 - \xi_2) e_{\alpha_0} + (\phi_1' - \phi_2') e_{\alpha_0} - \phi_1' h_{\alpha_1} - \phi_2' h_{\alpha_2}. \quad (10)
\]

Considering the associated linear problem, we can write the solution as

\[
\chi(u) = e^{\sum_{i=1,2} \phi_i(u) h_{\alpha_i}} \exp \int_0^u \left( \sum_{k=0,1,2} W_{\alpha_k}(u') e_{\alpha_k} + K(u') \right) \quad (11)
\]

where

\[
W_{\alpha_j} = \int d\theta e^{-\phi_j}, \quad j = 1, 2, \quad W_{\alpha_0} = \int d\theta (D\Phi_1 - D\Phi_2) e^{\Phi_1 + \Phi_2}
\]

\[
K(u) = -\frac{i}{\sqrt{2}} \xi_2 [e_{\alpha_2}, e_{\alpha_0}] e^{\phi_2} - \frac{i}{\sqrt{2}} \xi_1 [e_{\alpha_0}, e_{\alpha_1}] e^{\phi_1} - [e_{\alpha_0}, e_{\alpha_1}] e^{-\phi_1 + \phi_2} - [e_{\alpha_0}, [e_{\alpha_2}, e_{\alpha_0}]] e^{\phi_1 + \phi_2} - [e_{\alpha_2}, [e_{\alpha_0}, e_{\alpha_1}]] - [e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_0}]]. \quad (12)
\]

We can then define the monodromy matrix on the interval \([0, 2\pi]\) with the quasiperiodic boundary conditions

\[
\phi_i(u + 2\pi) = \phi_i(u) + 2\pi i p_i, \quad \xi_i(u + 2\pi) = \pm \xi_i(u)
\]

as

\[
M = e^{\sum_{i=1,2} 2\pi i p_i h_{\alpha_i}} \exp \int_0^{2\pi} du \left( \sum_{k=0,1,2} W_{\alpha_k}(u) e_{\alpha_k} + K(u) \right). \quad (13)
\]

The reason for separating the integrand into the \( K \) term and the covariant part expressed in terms of superfields is that in the quantum case (see below), the
noncovariant $K$ term disappears from the expression for the quantum monodromy matrix.

As in the case of the standard KdV models, we define the auxiliary $L$-operators

$$L = e^{-\sum_{i=1,2} \frac{i\pi p_i}{h_a} M_i},$$

which satisfy the quadratic r-matrix relation \[\text{[7]}\]:

$$\{L(\lambda), \otimes L(\mu)\} = [r(\lambda \mu^{-1}), L(\lambda) \otimes L(\mu)],$$

where we restore the dependence on the spectral parameters $\lambda$ and $\mu$ corresponding to some fixed evaluation representations. Here, $r(\lambda \mu^{-1})$ is the classical trigonometric r-matrix associated with $sl^{(1)}(2|1)$ \[\text{[8]}\]. As usual, this yields a relation leading to the classical integrability,

$$\{t(\lambda), t(\mu)\} = 0,$$

where $t(\lambda) = \text{str} M(\lambda)$ and the supertrace is taken in one of the $sl^{(1)}(2|1)$ representations.

### 3 Quantum R-matrix and the Cartan-Weyl basis for $sl_q^{(1)}(2|1)$

The quantum algebra $sl_q^{(1)}(2|1)$ has the commutation relations

\begin{align*}
[e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{i,j}[h_{\alpha_i}] && (i = 0, 1, 2), \quad [h_{\alpha_i}, e_{\pm \alpha_j}] = \pm e_{\pm \alpha_j} \quad (i = 1, 2, i \neq j) \\
\overline{[h_{\alpha_i}, e_{\pm \alpha_0}] &= \mp e_{\pm \alpha_0} \quad (i = 1, 2), \quad [h_{\alpha_0}, e_{\pm \alpha_i}] = \mp e_{\pm \alpha_i} \quad (i = 1, 2) \\
[h_{\alpha_i}, e_{\pm \alpha_j}] &= 0 \quad (i = 1, 2), \quad [h_{\alpha_0}, e_{\pm \alpha_0}] = \pm 2 e_{\pm \alpha_0}, \\
[e_{\pm \beta_i}, e_{\pm \beta_j}, e_{\pm \beta_j}] &= 0, \\
\overline{e_{\pm \beta_i}} &= 0 \quad (k = 1, 2).
\end{align*}

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ and the quantum supercommutator is defined as

$$[e_{\alpha_i}, e_{\beta_j}]_q = e_{\alpha_i} e_{\beta_j} - (-1)^{p(\alpha)p(\beta)} q^{(\alpha, \beta)} e_{\beta_i} e_{\alpha_j}.$$

The universal quantum R-matrix for quantum affine superalgebras is \[\text{[9]}\]

$$R = K \overline{R} = K( \prod_{\alpha \in \Delta^+} R_{\alpha}),$$

where $\overline{R}$ is a reduced R-matrix and $R_{\alpha}$ are defined by the formulas

$$R_{\alpha} = \exp_{q_{\alpha^{-1}}}((-1)^{p(\alpha)}(q - q^{-1})(a(\alpha))^{-1}(e_{\alpha} \otimes e_{-\alpha}))$$

for real roots and

$$R_{\alpha \delta} = \exp((q - q^{-1}) \sum_{i,j} c_{ij}(n) e_{\alpha i}^{(i)} \otimes e_{-\alpha j}^{(j)})$$

for quantum roots.
for pure imaginary roots. Here, $\Delta_+$ is the reduced positive root system (the bosonic roots equal to twice a fermionic root are excluded) and $q$-exponential is defined as usual,

$$\exp_q(x) = \sum_{n=0}^{\infty} x^n/(n)_q!, \quad (n)_q = (q^n - 1)/(q - 1).$$

The generators corresponding to composite roots and the ordering in (18) are defined in accordance with the construction of the Cartan-Weyl basis (see, e.g., [13]). The Cartan factor $K$ in the case of the $sl_q(1|2)$ is equal to $K = q^{h_{\alpha_1} \otimes h_{\alpha_2} + h_{\alpha_2} \otimes h_{\alpha_1}}$. The coefficients $a(\alpha), c_{ij}(n)$ and $d_{ij}(n)$ are defined using relations

$$[e_\gamma, e_{\gamma}^-] = a(\gamma)(k_\gamma - k_\gamma^{-1})/(q - q^{-1}),$$
$$[e_{n\delta}^{(i)}, e_{n\delta}^{(j)}] = d_{ij}(n)(q^{nh_{\delta}} - q^{-nh_{\delta}})/(q - q^{-1})$$

and $c_{ij}(n)$ is the matrix inverse to $d_{ij}(n)$. The first few generators corresponding to composite roots are constructed using the above procedure as

$$e_{\alpha_1 + \alpha_2} = [e_{\alpha_1}, e_{\alpha_2}]_{q^{-1}}$$
$$e_{\delta - \alpha_1} \equiv e_{\alpha_0 + \alpha_2} = [e_{\alpha_0}, e_{\alpha_2}]_{q^{-1}}, \quad e_{\delta - \alpha_2} \equiv e_{\alpha_0 + \alpha_1} = [e_{\alpha_1}, e_{\alpha_0}]_{q^{-1}}$$
$$e_{\delta}^{(1)} = [[e_{\alpha_0}, e_{\alpha_2}]_{q^{-1}}, e_{\alpha_1}], \quad e_{\delta}^{(2)} = [[e_{\alpha_1}, e_{\alpha_0}]_{q^{-1}}, e_{\alpha_2}]$$
$$e_{2\delta - \alpha_1 - \alpha_2} = [e_{\delta - \alpha_2}, e_{\delta - \alpha_1}]_{q^{-1}}$$
$$e_{-\alpha_1 - \alpha_2} = [e_{-\alpha_2}, e_{-\alpha_1}]_{q^{-1}}$$
$$e_{-\delta + \alpha_1} \equiv e_{-\alpha_0 - \alpha_2} = [e_{-\alpha_2}, e_{-\alpha_0}]_{q^{-1}}, \quad e_{-\delta + \alpha_2} \equiv e_{-\alpha_0 - \alpha_1} = [e_{-\alpha_0}, e_{-\alpha_1}]_{q^{-1}}$$
$$e_{\delta}^{(1)} = [[e_{-\alpha_2}, e_{-\alpha_0}]_{q^{-1}}, e_{-\alpha_1}], \quad e_{\delta}^{(2)} = [[e_{-\alpha_0}, e_{-\alpha_1}]_{q^{-1}}, e_{-\alpha_2}]$$
$$e_{-2\delta + \alpha_1 + \alpha_2} = [e_{-\delta + \alpha_1}, e_{-\delta + \alpha_2}]_{q^{-1}}.$$

4 Construction of the Quantum Monodromy Matrix

In this section, we construct the quantum version of the monodromy matrix introduced in Sec. 2. We show that matrix $[13]$ is reproduced in the classical limit.

We first consider the quantum versions of the free-field exponentials (vertex operators):

$$W_{\alpha_i} = \int d\theta : e^{-\phi_i} : \equiv \frac{i}{\sqrt{2}} \xi_i : e^{-\phi_i} : \quad (i = 1, 2),$$
$$W_{\alpha_0} = \int d\theta : (D\Phi_1 - D\Phi_2)e^{\phi_1 + \phi_2} : = : e^{\phi_1 + \phi_2}(\phi_1' - \phi_2' + \xi_1 \xi_2) : .$$

6
We can express the superfields as \( \Phi_1 = \frac{i \Phi_+ + \Phi_-}{\sqrt{2}} \) and \( \Phi_2 = \frac{i \Phi_+ - \Phi_-}{\sqrt{2}} \), where

\[
\phi_\pm(u) = i Q^\pm + i P^\pm u + \sum_n \frac{a^\pm_n}{n} e^{inu}, \quad \xi_\pm(u) = i^{-1/2} \sum_n \xi^\pm_n e^{-inu},
\]

\[
\left[ Q^\pm, P^\pm \right] = i \beta^2, \quad \left[ a^\pm_n, a^\pm_m \right] = \beta^2 n \delta_{n+m,0}, \quad \{ \xi^\pm_n, \xi^\pm_m \} = \beta^2 \delta_{n+m,0}. \tag{23}
\]

and the normal ordering in (22) is defined as

\[
e^{\phi_\pm(u)} = \exp \left( c \sum_{n=1}^\infty \frac{a^\pm_n}{n} e^{inu} \right) \exp \left( ci(Q^\pm + P^\pm u) \right) \exp \left( -c \sum_{n=1}^\infty \frac{a^\pm_n}{n} e^{-inu} \right).
\]

Here, the \( a^\pm_n \) operators with negative \( n \) are placed to the left and those with positive \( n \), to the right.

Vertex operators (22) integrated from \( u_1 \) to \( u_2 \) satisfy the quantum Serre and “nonstandard” Serre relations for the lower Borel subalgebra with \( q = e^{i\pi \beta^2/2} \). Proving this nontrivial because the usual proof of the Serre relations given in [10] for the bosonic case, based on transforming the product of the integrated vertex operators to ordered integrals, is inapplicable here because of the singularities generated by the fermion fields in the corresponding operator product expansions. But there is another way to prove it, relying on the standard conformal field theory technique of contour integration and analytic continuation of operator product expansions of nonlocal vertex operators [11]. This proof is also applicable to a quantum affine superalgebra and the corresponding vertex operators because this method allows isolating the divergences in each product of vertex operators and then canceling them in the standard and “nonstandard” Serre relations. This will be considered elsewhere for a general quantum affine superalgebra.

Because the operators \( (q - q^{-1})^{-1} \int_{u_1}^{u_2} W_\alpha \) satisfy the Serre relations, we can represent the lower Borel subalgebra using the correspondence

\[
e_{-\alpha_i} \rightarrow (q - q^{-1})^{-1} \int_{u_1}^{u_2} W_\alpha.
\]

It was shown in [12] that the corresponding reduced R-matrix \( \tilde{R} \), denoted by \( \tilde{L}^{(q)}(u_2, u_1) \) here, has the P-exponential property, satisfying the functional relation

\[
\tilde{L}^{(q)}(u_3, u_1) = \tilde{L}^{(q)}(u_3, u_2) \tilde{L}^{(q)}(u_2, u_1). \tag{24}
\]

But in the supersymmetric case involving fermionic operators, the associated singularities in the operator products do not allow writing \( \tilde{L}^{(q)}(u_3, u_1) \) in the standard manner in terms of ordered integrals. We therefore call it the “quantum” P-exponential. In our case it can be written as

\[
\tilde{L}^{(q)}(u_2, u_1) = Pexp(q) \int_{u_1}^{u_2} du \left( \sum_{k=0,1,2} W_{\alpha_k}(u) e_{\alpha_k} \right). \tag{25}
\]
It can be shown that the operators $e^{\sum_{i=1,2} \pi ip_i h_a}, \bar{L}^{(q)}(2\pi, 0) = L^{(q)}$ satisfy the RTT-relation [13]:

$$R(\lambda \mu^{-1}) \left( L^{(q)}(\lambda) \otimes I \right) \left( I \otimes L^{(q)}(\mu) \right) = \left( I \otimes L^{(q)}(\mu) \right) \left( L^{(q)}(\lambda) \otimes I \right) R(\lambda \mu^{-1}),$$

(26)

where the dependence on $\lambda$ and $\mu$ means that we consider $L^{(q)}$-operators in the appropriate evaluation representation of $sl_q(1|2)$. We also note that

$$M^{(q)} = e^{\sum_{i=1,2} \pi p_i h_a}, \bar{L}^{(q)}$$

satisfies the reflection equation [14]

$$\tilde{R}_{12}(\lambda \mu^{-1}) M_1^{(q)}(\lambda) F_{12}^{-1} M_2^{(q)}(\mu) = M_2^{(q)}(\mu) F_{12}^{-1} M_1^{(q)}(\lambda) R_{12}(\lambda \mu^{-1}),$$

(27)

where $F = K^{-1}$, the inverse Cartan factor from the universal R-matrix,

$$\tilde{R}_{12}(\lambda \mu^{-1}) = F_{12}^{-1} R_{12}(\lambda \mu^{-1}) F_{12}$$

and labels 1 and 2 indicate the position of factors in the tensor product. This leads to the quantum integrability relation

$$[t^{(q)}(\lambda), t^{(q)}(\mu)] = 0,$$

(28)

where $t^{(q)}(\lambda) = \text{str} M^{(q)}(\lambda)$.

We next show that $L^{(q)}$ and $M^{(q)}$ pass into the respective auxiliary $L$-matrix and monodromy matrix in the classical limit as $q \to 1$. We also note that the quantum universal R-matrix, as usual, tends to the classical $r$-matrix in the limit as $q \to 1$, and the classical limit of the $L^{(q)}$-operator therefore gives a realization of the classical $r$-matrix via the classical counterparts of the corresponding vertex operators.

To find the classical limit, we decompose $\bar{L}^{(q)}$ as [2]:

$$\bar{L}^{(q)}(2\pi, 0) = \lim_{N \to \infty} \prod_{m=1}^N \bar{L}^{(q)}(x_m, x_{m-1}),$$

(29)

where we divide the interval $[0, 2\pi]$ into infinitesimal intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \epsilon = 2\pi/N$. We next find the terms that can contribute to $\bar{L}^{(q)}(x_m, x_{m-1})$ in the first order in $\epsilon$. For this, we need the operator product expansions of vertex operators. Nontrivial terms occur in the expansions

$$\xi_1(u) \xi_2(u') = \frac{i\beta^2}{(iu - iu')} + \sum_{k=0}^{\infty} c_k(u)(iu - iu')^k,$$

$$: e^{a\phi_1(u)} : : e^{b\phi_2(u')} := (iu - iu')^{-\frac{\alpha h}{2}} (e^{(a\phi_1(u)) + b\phi_2(u)} : +$$
\[
\sum_{k=1}^{\infty} d_k(u)(iu - iu')^k,
\]
\[
\phi_1(u) : e^{b_\phi(u')} := -ib\beta^2 : e^{b_\phi(u)} : \frac{2}{2(iu - iu')} + \sum_{k=0}^{\infty} f_k(u)(iu - iu')^k,
\]
\[
\phi_2(u) : e^{b_\phi(u')} := -ib\beta^2 : e^{b_\phi(u)} : \frac{2}{2(iu - iu')} + \sum_{k=0}^{\infty} f_k(u)(iu - iu')^k. \quad (30)
\]

In the cases considered in [2], only two types of terms contribute to $\tilde{L}^{(q)}(x_{m-1}, x_m)$ in the order $\epsilon$ as $q \to 1$. Terms of the first type are operators of the first order in $W_{\alpha_i}$, and terms of the second type are the operators quadratic in $W_{\alpha_i}$, which contribute in the order $\epsilon^1\beta^2$ by virtue of operator product expansion. These second-type contributions correspond to the composite roots that are equal to the sum of two simple roots.

In this paper, we show that there are contributions of the composite roots equal to the sum of three and even four simple roots, ensuring the desired terms in the classical expression [12], [13]. We first consider the quadratic terms corresponding to the negative roots $-\alpha_1 - \alpha_2$, $-\delta + \alpha_2$ and $-\delta + \alpha_1$. The commutation relations between vertex operators on a circle are such that for $u > u'$,

\[
W_{\alpha_i}(u)W_{\alpha_j}(u') = q^{-1}W_{\alpha_j}(u')W_{\alpha_i}(u) \quad u > u' \quad (i, j = 1, 2 \quad i \neq j)
\]
\[
W_{\alpha_i}(u)W_{\alpha_0}(u') = qW_{\alpha_0}(u')W_{\alpha_i}(u) \quad u > u' \quad (i = 1, 2)
\]
\[
W_{\alpha_0}(u)W_{\alpha_i}(u') = qW_{\alpha_i}(u')W_{\alpha_0}(u), \quad i = 1, 2. \quad (31)
\]

This allows writing the generators corresponding to the negative composite roots $-\delta + \alpha_2$ and $-\delta + \alpha_1$ as

\[
[e_{-\alpha_0}, e_{-\alpha_1}]_q = \frac{1}{q - q^{-1}} \int_{x_{m-1}}^{x_m} du W_{\alpha_1}(u) \int_{x_{m-1}}^{u} du' W_{\alpha_0}(u'),
\]
\[
[e_{-\alpha_2}, e_{-\alpha_0}]_q = \frac{1}{q - q^{-1}} \int_{x_{m-1}}^{x_m} du W_{\alpha_0}(u) \int_{x_{m-1}}^{u} du' W_{\alpha_2}(u'). \quad (32)
\]

The exponents of the corresponding q-exponentials in quantum R-matrix [15] are equal to

\[
\int_{x_{m-1}}^{x_m} du W_{\alpha_1}(u) \int_{x_{m-1}}^{u} du' W_{\alpha_0}(u') [e_{\alpha_1}, e_{\alpha_0}]_q^{-1},
\]
\[
\int_{x_{m-1}}^{x_m} du W_{\alpha_0}(u) \int_{x_{m-1}}^{u} du' W_{\alpha_2}(u') [e_{\alpha_0}, e_{\alpha_2}]_q^{-1}. \quad (33)
\]

In the classical limit ($\beta^2 \to 0$), their contribution calculated using the corresponding operator product expansions is given by

\[
\int_{x_{m-1}}^{x_m} du (-\frac{i}{\sqrt{2}} \xi_2 e^{\phi_2}[e_{\alpha_2}, e_{\alpha_0}]) - \frac{i}{\sqrt{2}} \xi_1 e^{\phi_1}[e_{\alpha_0}, e_{\alpha_1}] \quad \text{ (34)}
\]
(similarly to \(1\)). The contribution of the root \(-\alpha_1 - \alpha_2\) is

\[
\int_{x_{m-1}}^{x_m} du (-e^{-\phi_1 - \phi_2}[e_{\alpha_1}, e_{\alpha_2}]).
\]

(35)

In this case, the commutator of the integrated vertex operators cannot be rewritten in terms of ordered integrals as above. We use the fact that the integrated vertex operators must be radially ordered, i.e., the product \(e_{-\alpha_1}e_{-\alpha_2}\), for example, must be written as

\[
\int_{x_{m-1}}^{x_m} du W_{\alpha_1}(u - i0) \int_{x_{m-1}}^{x_m} du' W_{\alpha_2}(u' + i0),
\]

and recall the well known relation

\[
\frac{1}{x + i0} - \frac{1}{x - i0} = -2i\pi\delta(x).
\]

(36)

Calculations performed as a generalization of the results in \([2]\) lead to (35). At the “quadratic” level, we therefore have complete agreement with the classical expression.

We now consider the contribution corresponding to the composite roots \(e^{(i)}_{\beta}\) and \(2\delta - \alpha_1 - \alpha_2\). We first consider the purely imaginary roots,

\[
e^{(1)}_{\beta} = [e_{-\alpha_2}, e_{-\alpha_0}]_q, \quad e^{(2)}_{\beta} = [e_{-\alpha_0}, e_{-\alpha_1}]_q, \quad e^{(3)}_{\beta} = [e_{-\alpha_0}, e_{-\alpha_1}, e_{\alpha_2}].
\]

Finding their contribution in the classical limit requires calculating the contribution to \([e_{\alpha_2}, e_{\alpha_0}]_q\) and \([e_{\alpha_0}, e_{\alpha_1}]_q\) of the terms of the order \(\epsilon^{1+\beta^2}\), proportional to \(\int d\theta d\phi\) and \(\int dud\phi\). We then consider their supercommutators with \(e_{\alpha_1}\) and \(e_{\alpha_2}\). Rewriting these supercommutators in terms of ordered integrals as above and following the calculations in \([1]\), we find that the contribution of the associated exponent of q-exponential is given by

\[
\int_{x_{m-1}}^{x_m} du (-[e_{\alpha_2}, [e_{\alpha_0}, e_{\alpha_1}]] - [e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_0}]]).
\]

(37)

The generator \(e^{2\delta - \alpha_1 - \alpha_2}\) is expressed similarly to the q-commutator of \([e_{\alpha_2}, e_{\alpha_0}]_q\) and \([e_{\alpha_0}, e_{\alpha_1}]_q\). Taking the terms of the order \(\epsilon^{1+\beta^2}\) into account and using the formula (36) as explained above, we obtain the order-\(\epsilon\) contribution to the classical expression

\[
\int_{x_{m-1}}^{x_m} du (-[[e_{\alpha_0}, e_{\alpha_1}], [e_{\alpha_2}, e_{\alpha_0}]]e^{\phi_1 + \phi_2}).
\]

(38)

Gathering all the terms, we thus find that the first iteration of the \(\tilde{L}\)-operator in the classical limit is given by

\[
\lim_{q \to 1} \mathbf{L}^{(q)}(x_m, x_{m-1}) = 1 + \int_{x_{m-1}}^{x_m} du \left( \sum_{k=0,1,2} W_{\alpha_k}(u)e_{\alpha_k} - \frac{i}{\sqrt{2}}\xi_2[e_{\alpha_2}, e_{\alpha_0}]e^{\phi_2}
\]

\[
- \frac{i}{\sqrt{2}}\xi_1[e_{\alpha_0}, e_{\alpha_1}]e^{\phi_1} - [e_{\alpha_1}, e_{\alpha_2}]e^{-\phi_1 - \phi_2} - [[e_{\alpha_0}, e_{\alpha_1}], [e_{\alpha_2}, e_{\alpha_0}]]e^{\phi_1 + \phi_2}
\]

\[
[[e_{\alpha_2}, [e_{\alpha_0}, e_{\alpha_1}]] - [e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_0}]]] + O(\epsilon^2).
\]

(39)
Collecting all the infinitesimal \( \mathbf{L} \) operators and multiplying them by the appropriate Cartan factor, we obtain the sought classical expressions for the \( \mathbf{L} \)-operators and the monodromy matrix.

5 Conclusions

The construction of the monodromy matrix given in the previous section is based on the method proposed in \([\text{1] - [4], [12}\). In other words, we have shown that the quantum version of the auxiliary \( \mathbf{L} \)-matrix coincides with the universal R-matrix with the lower Borel algebra represented by the appropriate vertex operators. This construction also allows showing that the supersymmetry generators commute with the supertrace of the monodromy matrix and can therefore be included in the series of integrals of motion. Indeed, the commutators of the supersymmetry generators

\[
G_0^+ = \beta^{-2} \sqrt{2} i^{-1/2} \int_0^{2\pi} du \phi_1'(u) \xi_2(u), \quad G_0^- = \beta^{-2} \sqrt{2} i^{-1/2} \int_0^{2\pi} du \phi_2'(u) \xi_1(u)
\]

with vertex operators reduce to total derivatives; the argument then repeats Sec. 4 in \([3]\), where it was shown (following \([12]\)) that in the case of the standard Drinfeld-Sokolov hierarchies, the supersymmetry generator commutes with the trace of the monodromy matrix if the system of simple roots is purely fermionic (the oddness condition for the system of simple roots then guarantees that the commutators of supersymmetry generators with the corresponding vertex operators are equal to total derivatives). It can be shown similarly that the transfer matrices commute with the zero mode of the U(1) current of the \( N = 2 \) superconformal algebra (the quantum version of the \( V \) field in \([6]\)).

This result has important consequences. If we make the twist transformation \([15]\) in the underlying \( N=2 \) superconformal algebra, then we find that one of the generators \( G_0^\pm \) becomes a BRST operator. That is, the transfer-matrices become BRST exact, providing an infinite series of pairwise commuting “physical” integrals of motion (of zero ghost number, as follows from the commutation relations with the zero mode of U(1) current). This allows studying two-dimensional topological models and their integrable perturbations using the methods of integrable theories, for example, the well-known quantum inverse scattering method \([13]\).

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