EXPLICIT DIFFERENTIAL CHARACTERIZATION OF
PDE SYSTEMS POINTWISE EQUIVALENT TO $Y_{X_1 X_2} = 0$

$1 \leq j_1, j_2 \leq n \geq 2$.

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ABSTRACT. In Lie1883, as an early result, Sophus Lie established that a second order ordinary differential equation $y_{xx} = F(x, y, y_x)$ is equivalent, through an invertible point transformation $(x, y) \mapsto (X(x, y), Y(x, y))$, to the free particle equation $Y_{XX} = 0$ if and only if the right member $F$ is a degree three polynomial in $y_x$, namely there exist four functions $G, H, L$ and $M$ of $(x, y)$ such that $F$ can be written as

$$F(x, y, y_x) = G(x, y) + y_x \cdot H(x, y) + (y_x)^2 \cdot L(x, y) + (y_x)^3 \cdot M(x, y),$$

and furthermore, the four functions $G, H, L$ and $M$ satisfy two second order partial differential equations:

$$0 = -2G_{yy} + \frac{4}{3} H_{xy} - \frac{2}{3} L_{xx} + 2(GL)y - 2G_x M - 4G_x M + \frac{2}{3} HL_x - \frac{4}{3} HH_y,$$

$$0 = -\frac{2}{3} H_{yy} + \frac{4}{3} L_{xy} - 2M_{xx} + 2LM_y + 4G_y M - 2(HM)_x - \frac{2}{3} H_y L + \frac{4}{3} LL_x.$$

In M2004b, this theorem was generalized to systems of Newtonian particles with $m \geq 2$ degree of freedom, i.e. with one independent variable $x$ and $m \geq 2$ dependent variables $(y^1, y^2, \ldots, y^m)$. In this paper, we generalize S. Lie’s theorem to the case of several independent variables $(x^1, x^2, \ldots, x^n), n \geq 2$, and one dependent variable $y$. Strikingly, as in M2004a, the (complicated) differential system which corresponds to the above two second order partial differential equations is of first order. By means of computer programming, this phenomenon was discovered in BN2002, N2003 in the case $n = 2$. In Bi2003, M2003, the general case $n \geq 2$ was handled.

Table of contents
1. Introduction ................................................................. 1.
2. Completely integrable systems of second order ordinary differential equations .......... 9.
3. First and second auxiliary systems ..................................... 16.

§1. INTRODUCTION

This paper, a direct continuation of M2004a, provides a summarized proof of the following statement, labelled as Theorem 1.23 in the Introduction of M2004b. All our functions are assumed to be analytic.

Theorem 1.1. (n = 2: BN2002, N2003; n \geq 2: Bi2003, M2003) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $n \in \mathbb{N}$, suppose $n \geq 2$ and consider a system of completely integrable partial differential equations in $n$ independent variables $x = (x^1, \ldots, x^n) \in \mathbb{K}^n$ and in one dependent variable $y \in \mathbb{K}$ of the form:

$$y_{x_1 x_2} (x) = F^{j_1, j_2} (x, y(x), y_{x_1}(x), \ldots, y_{x^n}(x)), \quad j_1, j_2 = 1, \ldots, n,$$

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where $F_{j_1,j_2} = F_{j_2,j_1}$. Under a local change of coordinates $(x, y) \mapsto (X, Y) = (X(x, y), Y(x, y))$, this system (1.2) is equivalent to the simplest system $Y_{X^{j_1}, X^{j_2}} = 0$, $j_1, j_2 = 1, \ldots, n$, if and only if there exist arbitrary functions $G_{j_1,j_2}$, $H_{j_1,j_2}^k$, $L_{j_1}^k$ and $M_{j_1}^k$ of the variables $(x, y)$, for $1 \leq j_1, j_2, k_1 \leq n$, satisfying the two symmetry conditions $G_{j_1,j_2} = G_{j_2,j_1}$ and $H_{j_1,j_2}^k = H_{j_2,j_1}^k$, such that the equation (1.2) is of the specific cubic polynomial form:

\begin{equation}
F_{j_1,j_2} = G_{j_1,j_2} + \sum_{k_1=1}^n y_{x^{k_1}} \left( H_{j_1,j_2}^k + \frac{1}{2} y_{x^{j_1}} L_{j_2}^k + \frac{1}{2} y_{x^{j_2}} L_{j_1}^k + y_{x^{j_1}} y_{x^{j_2}} M_{j_1}^k \right),
\end{equation}

for $j_1, j_2 = 1, \ldots, n$.

We refer the reader to [M2004a] for an extensive introduction and for a more substantial bibliography. Applying É. Cartan’s equivalence algorithm (see [G1989] and [OL1995] for modern expositions), the general case $n \geq 2$ of the above theorem has also been established in [Ha1937], where the construction of a projective connection associated to a second order ordinary differential equation achieved in [Ca1924] was extended to several variables. Theorem 1.1 was re-discovered in [BN2002], in [N2003], in [Bi2003] and in [M2003], thanks to partial parametric computations of the Hachtroudi-Chern tensors in $n \geq 2$ variables, which were described in a non-parametric way in [Ch1975] (see §1.8 below).

It may seem quite paradoxical and counter-intuitive (or even false?) that every system (1.3), for arbitrary choices of functions $G_{j_1,j_2}$, $H_{j_1,j_2}^k$, $L_{j_1}^k$ and $M_{j_1}^k$, is automatically equivalent to $Y_{X^{j_1}, X^{j_2}} = 0$. However, a strong hidden assumption holds: that of complete integrability. Shortly, this crucial condition amounts to say that

\begin{equation}
D_{x^{j_3}} \left( F_{j_1,j_2} \right) = D_{x^{j_2}} \left( F_{j_1,j_3} \right),
\end{equation}

for all $j_1, j_2, j_3 = 1, \ldots, n$, where, for $j = 1, \ldots, n$, the $D_{x^j}$ are the total differentiation operators defined by

\begin{equation}
D_{x^j} := \frac{\partial}{\partial x^j} + y_{x^j} \frac{\partial}{\partial y} + \sum_{i=1}^n F_{j,i} \frac{\partial}{\partial y_{x^i}}.
\end{equation}

These conditions are non-void precisely when $n \geq 2$. More concretely, writing out (1.4) when the $F_{j_1,j_2}$ are of the specific cubic polynomial form (1.3), after some nontrivial manual computation, we obtain the complicated cubic differential polynomial in the variables $y_{x^k}$ labelled as equation (1.25) in [M2004a]. Equating to zero all the coefficients of this cubic polynomial, we obtain four families (I’), (II’), (III’), and (IV’), of first order partial differential equations satisfied by $G_{j_1,j_2}$, $H_{j_1,j_2}^k$, $L_{j_1}^k$ and $M_{j_1}^k$:

\begin{equation}
(\text{I’}) \quad \left\{ \begin{array}{l}
0 = G_{j_1,j_2,x^{j_3}} - G_{j_1,j_3,x^{j_2}} + \sum_{k_1=1}^n H_{j_1,j_2}^{k_1} G_{k_1,j_3} - \sum_{k_1=1}^n H_{j_1,j_3}^{k_1} G_{k_1,j_2}.
\end{array} \right.
\end{equation}
Differential Characterization of $Y_{X_1 \mid X_2} = 0, \ 1 \leq j_1, j_2 \leq n \geq 2$

\[
\begin{align*}
(II') \quad 0 &= \delta_{j_3}^{k_1} \ G_{j_1, j_2, y} - \delta_{j_2}^{k_1} \ G_{j_1, j_3, y} + H_{j_1, j_2, x_3}^{k_1} - H_{j_1, j_3, x_2}^{k_1} + \\
&+ \frac{1}{2} \ G_{j_1, j_3} \ L_{j_2}^{k_3} - \frac{1}{2} \ G_{j_1, j_2} \ L_{j_3}^{k_3} + \\
&+ \frac{1}{2} \delta_{j_1}^{k_1} \sum_{k_2=1}^{n} G_{j_2, j_3} \ L_{j_3}^{k_2} - 2 \delta_{j_3}^{k_1} \sum_{k_3=1}^{n} G_{j_2, j_3} \ L_{j_3}^{k_3} + \\
&+ \frac{1}{2} \delta_{j_2}^{k_2} \sum_{k_2=1}^{n} G_{j_2, j_3} \ L_{j_3}^{k_2} - 2 \delta_{j_3}^{k_2} \sum_{k_2=1}^{n} G_{j_2, j_3} \ L_{j_3}^{k_2} + \\
&+ \sum_{k_2=1}^{n} H_{j_2, j_3}^{k_1} H_{j_1, j_2}^{k_2} - \sum_{k_2=1}^{n} H_{j_2, j_3}^{k_1} H_{j_1, j_3}^{k_2},
\end{align*}
\]

\[
(III') \quad 0 &= \sum_{\sigma \in \mathfrak{S}_2} \left( \delta_{j_3}^{k_1} H_{j_1, j_2, y}^{k_1} - \delta_{j_2}^{k_1} H_{j_1, j_3, y}^{k_1} + \\
&+ \frac{1}{2} \delta_{j_3}^{k_2} L_{j_1, x_3}^{k_3} - \frac{1}{2} \delta_{j_3}^{k_3} L_{j_1, x_2}^{k_3} + \\
&+ \frac{1}{2} \delta_{j_1}^{k_2} L_{j_2, x_3}^{k_3} - \frac{1}{2} \delta_{j_1}^{k_3} L_{j_2, x_2}^{k_3} + \\
&+ \delta_{j_2}^{k_2} G_{j_1, j_3} M_{j_1, j_3}^{k_3} - \delta_{j_3}^{k_2} G_{j_1, j_2} M_{j_1, j_2}^{k_3} + \\
&+ \delta_{j_1}^{k_2} \ G_{j_1, j_3} M_{j_1, j_3}^{k_3} - \delta_{j_3}^{k_2} \ G_{j_1, j_2} M_{j_1, j_2}^{k_3} + \\
&+ \frac{1}{2} \delta_{j_1}^{k_1} \sum_{k_3=1}^{n} G_{j_2, j_3} M_{j_3, j_3}^{k_3} - \frac{1}{2} \delta_{j_3}^{k_1} \sum_{k_3=1}^{n} G_{j_2, j_3} M_{j_3, j_3}^{k_3} + \\
&+ \frac{1}{2} \delta_{j_3}^{k_1} \sum_{k_3=1}^{n} G_{j_2, j_3} M_{j_3, j_3}^{k_3} - \frac{1}{2} \delta_{j_3}^{k_1} \sum_{k_3=1}^{n} G_{j_2, j_3} M_{j_3, j_3}^{k_3} + \\
&+ \sum_{k_3=1}^{n} H_{j_1, j_2}^{k_3} L_{j_3}^{k_3} - \sum_{k_3=1}^{n} H_{j_1, j_3}^{k_3} L_{j_3}^{k_3} \right),
\]

\[
(IV') \quad 0 &= \sum_{\sigma \in \mathfrak{S}_3} \left( \frac{1}{2} \delta_{j_3}^{k_1} \delta_{j_1}^{k_2} \ L_{j_2, y}^{k_3} - \frac{1}{2} \delta_{j_2}^{k_1} \delta_{j_1}^{k_2} \ L_{j_3, y}^{k_3} + \\
&+ \delta_{j_2}^{k_1} M_{j_1, j_3}^{k_3} - \delta_{j_3}^{k_1} M_{j_1, j_2}^{k_3} + \\
&+ \delta_{j_1}^{k_1} M_{j_1, j_3}^{k_3} - \delta_{j_3}^{k_1} M_{j_1, j_2}^{k_3} + \\
&+ \frac{1}{4} \delta_{j_1}^{k_1} \sum_{k_3=1}^{n} L_{j_2}^{k_3} L_{j_3}^{k_3} - \frac{1}{4} \delta_{j_1}^{k_1} \sum_{k_3=1}^{n} L_{j_2}^{k_3} L_{j_3}^{k_3} \right),
\]

These systems (I’), (II’), (III’) and (IV’) should be distinguished from the systems (I), (II), (III) and (IV) of Theorem 1.7 in [M2004a], although they are quite similar. Here, the indices $j_1, j_2, j_3, k_1, k_2, k_3$ vary in $\{1, 2, \ldots, n\}$. By $\mathfrak{S}_2$ and by $\mathfrak{S}_3$, we denote the permutation group of $\{1, 2\}$ and of $\{1, 2, 3\}$. To facilitate hand- and Latex-writing, partial
interesting to achieve the computations of [Ch1975] in a parametric way, because at the
explicit hand computations. The trick in this lemma is to bypass the (often too long) para-
respect, the thesis [N2003] of Sylvain Neut is extremely interesting, since it implements
of variables are still undeveloped. Consequently, it is in-
with a general number of variables are still undeveloped. Consequently, it is in-
nowadays, to study an equivalence problem, one often achieves the non-parametric com-
functions of Newtonian particles, so that most steps of the proof will be summarized. In the
partial differential equations (I'), (II'), (III') and (IV') above.
for all $k_1, k_2, k_3 = 1, \ldots, n$.
In conclusion, the functions $G_{j_1, j_2}$, $H_{j_1, j_2}^{k_1}$, $L_{j_2}^{k_1}$, and $M^{k_1}$ in the statement of Theo-
1.8. Confirmation of Theorem 1.1 by the method of equivalence. Let us summarize
the system described in the abstract. The main technical part of the proof of Theorem 1.1 will
be to establish that this second order system is a consequence, by linear combinations and
for instance (1.25) in [M2004a] is equivalent to the annihilation of the following
symmetric sums of its coefficients:
\begin{equation}
\begin{aligned}
0 &= A, \\
0 &= B_{k_1}, \\
0 &= C_{k_1, k_2} + C_{k_2, k_1}, \\
0 &= D_{k_1, k_2, k_3} + D_{k_1, k_2, k_1} + D_{k_2, k_1, k_3} + D_{k_3, k_1, k_2} + D_{k_1, k_2, k_3}.
\end{aligned}
\end{equation}
(1.7)
Our proof of Theorem 1.1 is similar to the one provided in [M2004a], in the case of
systems of Newtonian particles, so that most steps of the proof will be summarized. In the end of
this paper, we will delineate a complicated system of second order partial differential equa-
tions satisfied by $G_{j_1, j_2}$, $H_{j_1, j_2}^{k_1}$, $L_{j_2}^{k_1}$ and $M^{k_1}$ which is the exact analog of
the system described in the abstract. The main technical part of the proof of Theorem 1.1 will
be to establish that this second order system is a consequence, by linear combinations and
by differentiations, of the first order system (I'), (II'), (III') and (IV'). Before proceeding
further, let us describe rapidly a second proof of Theorem 1.1, confirming its validity.

end, the vanishing of all the invariant tensors that one obtains in the final \( \{e\} \)-structure would show explicitly under which condition the system (1.2) is equivalent to the system \( Y_{X^{j_1} X^{j_2}} = 0 \), \( 1 \leq j_1, j_2 \leq n \geq 2 \).

To our knowledge, this problem is considered as open in the field of symmetries of Cauchy-Riemann submanifolds of \( \mathbb{C}^n \), a subfield of the mathematics area called Several Complex Variables. However, no specialist of the subfield seems to be aware of the old reference [Ha1937], a text only read, apparently, by S.-S. Chern, who was a student of É. Cartan contemporary to M. Hachtroudi.

Let us expose briefly how one obtains the final \( \{e\} \)-structure attached to the equivalence problem associated to the system (1.2). Since we have not been able to follow everything in the ancient text [Ha1937], we follow [Ch1975] with the only change that we do not introduce the imaginary number \( i = \sqrt{-1} \) in our structure equations and also, we make some minor changes of sign; in [Ch1975], the computations are conducted over \( \mathbb{K} = \mathbb{C} \) with the idea that they apply to Complex Analysis, but they do hold as well over \( \mathbb{K} = \mathbb{R} \), or even over an arbitrary commutative field of characteristic zero equipped with a valuation (in order to provide \( \mathbb{K}\)-analytic functions).

To begin with, consider the following family of \( 2n + 1 \) initial differential forms:

\[
\begin{align*}
\tilde{\omega} &:= dy - \sum_{\beta} y_{x\beta} \, dx^\beta, \\
\tilde{\omega}^\alpha &:= dx^\alpha, \\
\tilde{\omega}_\alpha &:= dy_{x^\alpha} - \sum_{\beta} F^{\alpha,\beta} \, dx^\beta.
\end{align*}
\]

Here, the Greek indices \( \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \) etc., run from 1 to \( n \). Sums \( \sum_{\alpha=1}^{n} \) are abbreviated as \( \sum_\alpha \). In the paragraph preceding the statement of Theorem 1.7 in [M2004a], we explain why we cannot use coherently the Einstein summation convention throughout the paper. For the same reason, we shall abandon this convention in the present paper. However, we would like to mention that in (1.9) above and in (1.10), (1.11), (1.12), (1.13), (1.14), and (1.15) below, the summation convention applies coherently, so that the reader may drop the sums if (s)he is used to.

Following [Ch1975], define the local Lie group consisting of \( (2n + 1) \times (2n + 1) \) matrices of the form

\[
g := \begin{pmatrix}
  u & 0 & 0 \\
  u^\alpha & u^\beta_\beta & 0 \\
  u_\alpha & 0 & u u'^\beta_\alpha
\end{pmatrix},
\]

where \( u^\alpha \) and \( u_\alpha \) are \( n \times 1 \) vectors close to the zero vector, where \( u^\beta_\beta \) is a \( n \times n \) invertible matrix close to the identity matrix, where \( u \) is a scalar close to 1 and where \( u'^\alpha_\beta \) denotes the inverse matrix of \( u^\beta_\beta \). With this group, define the initial \( G \)-structure, consisting of the following collection of \( (2n + 1) \) differential forms, called the lifted coframe:

\[
\begin{align*}
\omega &:= u \cdot \tilde{\omega}, \\
\omega^\alpha &:= u^\alpha \cdot \tilde{\omega} + \sum_\beta u^\beta_\beta \cdot \tilde{\omega}^\beta, \\
\omega_\alpha &:= u_\alpha \cdot \tilde{\omega} + \sum_\beta u u'^\beta_\alpha \cdot \tilde{\omega}^\beta.
\end{align*}
\]
These forms depend both on the “horizontal” variables \((x^i, y)\) and on the “vertical” (“group”, “fiber”) variables \(u, u^\alpha, u_\alpha\) and \(u^\beta_\alpha\). The introduction of this lifted coframe may be justified as follows.

It is the very first step of the method of equivalence to check that there exists a transformation \((x^i, y) \mapsto (x^i, \tilde{y})\) of the completely integrable system \(y^\alpha_{x^i} = F^{\alpha\beta}_{x^i}\) in the coordinates \((x^i, y)\) to another completely integrable system \(\tilde{y}^{\alpha}_{x^i} = \tilde{F}^{\alpha\beta}_{x^i}\) in other coordinates \((\tilde{x}^i, \tilde{y})\) if and only if there exist functions \(\tilde{u}, \tilde{u}^\alpha, \tilde{u}_\alpha, \tilde{u}^\beta_\alpha\) and \(\tilde{v}^\beta_\alpha\) of \((x^i, y)\), which depend on the functions \((\tilde{x}^i, \tilde{y})\), on their partial derivatives with respect to \((x^i, y)\), on the \(y^\beta_i\) and on the \(F^{\gamma\delta}\), such that the following matrix identity holds:

\[
\begin{pmatrix}
\omega^\alpha \\
\omega_\alpha \\
\omega^\beta_\alpha
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{u} & 0 & 0 \\
\tilde{u}^\alpha & \tilde{v}^\beta_\alpha & 0 \\
\tilde{u}_\alpha & 0 & \tilde{v}_\beta_\alpha
\end{pmatrix}
\begin{pmatrix}
\tilde{\omega} \\
\tilde{\omega}^\alpha \\
\tilde{\omega}^\beta_\alpha
\end{pmatrix},
\]

where the \((2n + 1)\) differential forms \(\left(\tilde{\omega}, \tilde{\omega}^\alpha, \tilde{\omega}^\beta_\alpha\right)\) are defined similarly as in (1.9), in the barred coordinates. Of course, it is understood that in the left hand side of this matrix identity, the variables \((\tilde{x}^i, \tilde{y})\) are replaced by their values with respect to the variables \((x^i, y)\) (in the language of modern differential geometry, one usually speaks of “pull-back”). By a more careful examination of the explicit expressions of the functions \(\tilde{u}, \tilde{u}^\alpha, \tilde{u}_\alpha, \tilde{u}^\beta_\alpha\) and \(\tilde{v}^\beta_\alpha\) in terms of the \(F^{\gamma\delta}\), one observes that \(\tilde{v}^\beta_\alpha = \tilde{u} \tilde{u}^\beta_\alpha\) (M2003). Thus, the collection of differential forms \((\omega, \omega^\alpha, \omega_\alpha)\) is defined modulo multiplication by a matrix of functions of \((x^i, y)\) which is of the specific form (1.10). Based on this preliminary observation, the general procedure of the equivalence method (G1989, OL1995) associates to the system (1.2) the lifted coframe (1.11).

Applying the exterior differential operator \(d\) to \(\omega\), to \(\omega^\alpha\) and to \(\omega_\alpha\) and absorbing the torsion, it is shown in Ch1975 that the initial structure equations may be written under the form

\[
\begin{align*}
d\omega &= \sum_\alpha \omega^\alpha \wedge \omega_\alpha + \omega \wedge \varphi, \\
d\omega^\alpha &= \sum_\beta \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha, \\
d\omega_\alpha &= \sum_\beta \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha,
\end{align*}
\]

where \(\varphi, \varphi^\beta_\alpha, \varphi^\alpha_\beta\) and \(\varphi_\alpha\) are modified Maurer-Cartan forms. We notice that there are no torsion coefficient to normalize. In fact, the choice of \(v^\alpha_\beta := u u^\alpha_\beta\) made in advance above corresponds to having achieved a first normalization implicitly.

Since the dimension of the Lie symmetry group of the system (1.2) is always finite and in fact bounded by \(n^2 + 4n + 3\) (Ha1937, CM1974, Ch1975, SO2001, GM2003), according to the general procedure of the method of equivalence (G1989, OL1995), one simply has to prolong the initial \(G\)-structure. One could also apply the Élie Cartan involutivity test to deduce that it is necessary to prolong.

Now, we summarize the remainder of Ch1975 very rapidly. After one prolongation, two normalizations and one supplementary prolongation, the final \(\{e\}\)-structure is of the
following form:

\[
\begin{align*}
    d\omega &= \sum_{\alpha} \omega^\alpha \wedge \omega_\alpha + \omega \wedge \varphi, \\
    d\omega^\alpha &= \sum_{\beta} \omega^\beta \wedge \varphi^\alpha_\beta + \omega \wedge \varphi^\alpha, \\
    d\omega_\alpha &= \sum_{\beta} \varphi^\beta_\alpha \wedge \omega_\beta + \omega_\alpha \wedge \varphi + \omega \wedge \varphi_\alpha, \\
    d\varphi &= \sum_{\alpha} \omega^\alpha \wedge \varphi_\alpha - \sum_{\alpha} \omega_\alpha \wedge \varphi^\alpha + \omega \wedge \psi, \\
    d\varphi^\beta_\alpha &= \sum_{\beta} \sum_{\gamma} S^{\alpha\beta}_{\gamma\rho} \cdot \omega^\rho \wedge \omega_\gamma + \sum_{\gamma} R^\alpha_{\beta\gamma} \cdot \omega \wedge \omega_\gamma + \sum_{\gamma} T^{\alpha\gamma}_{\beta} \cdot \omega \wedge \omega_\gamma + \sum_{\gamma} \varphi^\gamma_\beta \wedge \varphi_\gamma + \frac{1}{2} \delta^\gamma_\beta \cdot \omega \wedge \psi, \\
    d\varphi^\alpha &= \sum_{\beta} \sum_{\gamma} T^{\alpha\gamma}_{\beta} \cdot \omega^\beta \wedge \omega_\gamma + \frac{1}{2} \sum_{\beta} Q^\alpha_{\beta} \cdot \omega \wedge \omega_\beta + \sum_{\beta} \varphi^\beta_\alpha \wedge \varphi^\beta + \frac{1}{2} \omega^\alpha \wedge \psi, \\
    d\varphi_\alpha &= \sum_{\beta} \sum_{\gamma} R^{\gamma}_{\beta\alpha} \cdot \omega^\beta \wedge \omega_\gamma + \frac{1}{2} \sum_{\beta} P^\alpha_{\beta} \cdot \omega \wedge \omega_\beta + \sum_{\beta} \varphi^\beta_\alpha \wedge \varphi^\beta + \frac{1}{2} \omega_\alpha \wedge \psi, \\
    d\psi &= \sum_{\alpha} \sum_{\beta} Q^\beta_\alpha \cdot \omega^\alpha \wedge \omega_\beta + \sum_{\alpha} H^\alpha_\alpha \cdot \omega \wedge \omega^\alpha + \sum_{\alpha} K^\alpha_\alpha \cdot \omega \wedge \omega_\alpha + \varphi \wedge \psi + 2 \sum_{\alpha} \varphi^\alpha \wedge \varphi_\alpha.
\end{align*}
\]

These structure equations incorporate 8 families \(\omega, \omega^\alpha, \omega_\alpha, \varphi, \varphi^\alpha_\beta, \varphi^\alpha, \varphi_\alpha, \psi\) of differential forms, of total cardinality \(n^2 + 4n + 3\), together with 8 invariant tensors \(S^{\alpha\beta}_{\gamma\rho}, R^{\alpha}_{\beta\gamma}, T^{\alpha\gamma}_{\beta}, Q^\alpha_{\beta}, L^{\alpha\beta}, P_{\alpha\beta}, H_\alpha\) and \(K_\alpha\) having some specific index symmetries that we shall not use.

Applying the exterior differential operator \(d\) to these 8 families of structure equations (1.14), one verifies that the seven invariant tensors \(R^{\alpha}_{\beta\gamma}, T^{\alpha\gamma}_{\beta}, Q^\alpha_{\beta}, L^{\alpha\beta}, P_{\alpha\beta}, H_\alpha, K_\alpha\) are in fact functionally dependent on the fundamental tensors \(S^{\alpha\beta}_{\gamma\rho}\), namely they are certain coframe derivatives of the \(S^{\alpha\beta}_{\gamma\rho}\). In the (very similar) context of the equivalence problem associated with a Levi non-degenerate hypersurface of \(\mathbb{C}^{n+1}\), this computation was achieved by S.M. Webster in the Appendix of [CM1974]; in the precise context of the equivalence problem associated to the system (1.2), this computation is not achieved in [Ch1975], but see [BN2002], [B2003], [N2003] and [M2003] for details. For us, the precise nature of this functional dependence does not matter.

In fact, it is only in the computer science thesis [N2003] that the complete explicit parametric computation of the 8 families of differential forms \(\omega, \omega^\alpha, \omega_\alpha, \varphi, \varphi^\alpha_\beta, \varphi^\alpha, \varphi_\alpha, \psi\) together with the 8 invariant tensors \(S^{\alpha\beta}_{\gamma\rho}, R^{\alpha}_{\beta\gamma}, T^{\alpha\gamma}_{\beta}, Q^\alpha_{\beta}, L^{\alpha\beta}, P_{\alpha\beta}, H_\alpha\) and \(K_\alpha\) is achieved, in the case \(n = 2\) and with the help of Maple. Although the task is really of impressive size, the author of this paper has the project of achieving manually the general computation for \(n \geq 2\) variables ([M2005]). In [Ha1937], the parametric computations are achieved completely only in the case \(n = 2\). According to the author’s experience, it
appears that the tensorial formalism helps to shorten importantly the size of the electronic computations and that the case \( n \geq 2 \) is not much more difficult than the case \( n = 2 \), as one may already observe by reading [M2004a]. This is why the project [M2005] seems to be an accessible task. A similar project would be to compute in a parametric way the structure equations obtained in [F1995] for the equivalence problem associated to a system of Newtonian particles; the computations are quite similar, indeed, as well as the computations of this paper as quite similar to the computations of [M2004a]. This will be achieved in the future, if time permits.

At present, fortunately, there is a kind of “miracle”: the interesting tensor \( S_{\beta \rho}^{\alpha \sigma} \) appears essentially at the beginning of the computation of the final \( \{ e \} \)-structure. Thus, it is not necessary to go up to the end of the algorithm in order to deduce the final parametric expression of \( S_{\beta \rho}^{\alpha \sigma} \), and especially to characterize the maximally symmetric systems (1.2), i.e. those for which all the \( 8 \) tensors \( S_{\beta \rho}^{\alpha \sigma}, R_{\beta \gamma}^{\alpha}, T_{\beta}^{\alpha \gamma}, Q_{\beta}^{\alpha}, L_{\alpha \beta}, P_{\alpha \beta}, H_{\alpha} \) and \( K_{\alpha} \) vanish. In [M2003], we obtained:

\[
S_{\beta \rho}^{\alpha \sigma} = \delta_{\rho}^{\alpha} \left( \frac{1}{n + 2} \sum_{\gamma} \sum_{\delta} \sum_{\varepsilon} u^{-1} u_{\beta}^{\delta} u_{\varepsilon}^{\sigma} F_{y_{\varepsilon} y_{\gamma}}^{\gamma, \delta} \right) + \delta_{\beta}^{\alpha} \left( \frac{1}{n + 2} \sum_{\gamma} \sum_{\delta} \sum_{\varepsilon} u^{-1} u_{\rho}^{\delta} u_{\varepsilon}^{\sigma} F_{y_{\varepsilon} y_{\gamma}}^{\gamma, \delta} \right) + \delta_{\beta}^{\sigma} \left( \frac{1}{n + 2} \sum_{\gamma} \sum_{\delta} \sum_{\varepsilon} u^{-1} u_{\rho}^{\delta} u_{\varepsilon}^{\gamma} F_{y_{\varepsilon} y_{\gamma}}^{\gamma, \delta} \right) - (\delta_{\rho}^{\sigma} \delta_{\beta}^{\delta} + \delta_{\rho}^{\delta} \delta_{\beta}^{\sigma}) \cdot \left( \frac{1}{(n + 1)(n + 2)} \sum_{\gamma} \sum_{\delta} u^{-1} F_{y_{\varepsilon} y_{\gamma}}^{\gamma, \delta} \right) - \sum_{\gamma} \sum_{\delta} \sum_{\varepsilon} \sum_{\zeta} u^{-1} u_{\rho}^{\delta} u_{\varepsilon}^{\sigma} u_{\zeta}^{\gamma} F_{y_{\varepsilon} y_{\gamma}}^{\gamma, \delta}.
\]

It may be verified (cf. also [B2004]) that the general tensor \( S_{\beta \rho}^{\alpha \sigma} \) is a certain “tensorial rotation” of its value “at the identity”, namely its value when the matrix (1.10) is the identity; it suffices to put \( u := 1, u_{\rho}^{\delta} := \delta_{\rho}^{\delta} \) and \( u_{\beta}^{\delta} := \delta_{\beta}^{\delta} \) in (1.15) above. Next, it may be verified that the vanishing of the value of \( S_{\beta \rho}^{\alpha \sigma} \) at the identity (which is equivalent to the vanishing of the general \( S_{\beta \rho}^{\alpha \sigma} \)) provides a system of second linear second order partial differential equations involving only the partial derivatives \( F_{y_{\varepsilon} y_{\gamma}}^{\gamma, \delta} \). Finally, one verifies (B2004) that this last system is equivalent to the fact that the \( F_{j_{1} j_{2}}^{1} \) are of the specific cubic polynomial form (1.3). In conclusion, this provides a second (very summarized) proof of Theorem 1.1.

In the remainder of the paper, we shall not speak anymore of the equivalence method and we will focus on describing precisely how to establish Theorem 1.1 using S. Lie’s original techniques, as in [M2004a].

1.17. Acknowledgment. We are very grateful to Sylvain Neut and to Michel Petitot (LIFL, University of Lille 1), who implemented the Élie Cartan equivalence algorithm and who discovered the validity of Theorem 1.1 in the case \( n = 2 \), and also of Theorem 1.7 of [M2004a] in the case \( m = 2 \). Without these discoveries, we would not have
pushed our manual computations up to the very end of the proof of Theorem 1.1 (and of Theorem 1.7 in [M2004a]). We also acknowledge interesting exchanges about the method of equivalence with Camille Bièche, Sylvain Neut and Michel Petitot.

1.18. Closing remark. After Theorem 1.1 was established, namely after the references [BN2002], [N2003], [Bi2003] and [M2003] were completed, we discovered at the mathematical library of the École Normale Supérieure of Paris that Mohsen Hachtroudi, an Iranian student of Élie Cartan, also obtained a proof of Theorem 1.1 ([Ha1937], p. 53), sixty seven years ago. However, his computations follow the method of equivalence, in the spirit of his master, whereas we conduct ours in the spirit of Sophus Lie. Also, in [Ha1937], M. Hachtroudi only refers to the abbreviated formulation (1.4) of the compatibility conditions, and one does not find there the explicit and complete formulation of the systems (I'), (II'), (III') and (IV'). Finally, the combinatorial formulas that we provide in (2.10), in (2.14) in (2.15), in (2.22), in (2.23), in (2.31) and in (2.35) below seem to be new in the Lie theory of symmetries of partial differential equations (see also [M2004b]).

§2. Completely integrable systems of second order ordinary differential equations

2.1. Prolongation of a point transformation to the second order jet space. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $n \in \mathbb{N}$, suppose $n \geq 2$, let $x = (x^1, \ldots, x^n) \in \mathbb{K}^n$ and let $y \in \mathbb{K}$. According to the main assumption of Theorem 1.1, we have to consider a local $\mathbb{K}$-analytic diffeomorphism of the form

$$\tag{2.2} (x^{j_1}, y) \mapsto (X^j(x^{j_1}, y), Y(x^{j_1}, y)), $$

which transforms the system (1.2) to the system $Y_{X^{j_1}, X^{j_2}} = 0, 1 \leq j_1, j_2 \leq n$. Without loss of generality, we shall assume that this transformation is close to the identity. To obtain the precise expression (2.35) of the transformed system (1.2), we have to prolong the above diffeomorphism to the second order jet space. We introduce the coordinates $(x^j, y, y_x, y_{xx}, x^{j_1}, y_{x^{j_1}}, y_{x^{j_1}x^{j_2}})$ on the second order jet space. Let

$$\tag{2.3} D_k := \frac{\partial}{\partial x^k} + y_k \frac{\partial}{\partial y} + \sum_{l=1}^{n} y_{x^k x^l} \frac{\partial}{\partial y_{x^l}},$$

be the $k$-th total differentiation operator. According to [BK1989], for the first order partial derivatives, one has the (implicit, compact) expression:

$$\tag{2.4} \begin{pmatrix} Y_{X^1} \\ \vdots \\ Y_{X^n} \end{pmatrix} = \begin{pmatrix} D_1 X^1 & \cdots & D_1 X^n \\ \vdots & \ddots & \vdots \\ D_n X^1 & \cdots & D_n X^n \end{pmatrix}^{-1} \begin{pmatrix} D_1 Y \\ \vdots \\ D_n Y \end{pmatrix},$$

where $(\cdot)^{-1}$ denotes the inverse matrix, which exists, since the transformation (2.2) is close to the identity. For the second order partial derivatives, again according to [BK1989], one has the (implicit, compact) expressions:

$$\tag{2.5} \begin{pmatrix} Y_{X^{j_1} X^1} \\ \vdots \\ Y_{X^{j_1} X^n} \end{pmatrix} = \begin{pmatrix} D_1 X^1 & \cdots & D_1 X^n \\ \vdots & \ddots & \vdots \\ D_n X^1 & \cdots & D_n X^n \end{pmatrix}^{-1} \begin{pmatrix} D_1 Y_{X^j} \\ \vdots \\ D_n Y_{X^{j_1}} \end{pmatrix}.$$
for \( j = 1, \ldots, n \). Let \( DX \) denote the matrix \( (D_iX)^{1 \leq i \leq n} \), where \( i \) is the index of lines and \( j \) the index of columns, let \( Y_X \) denote the column matrix \( (Y^i_X)^{1 \leq i \leq n} \) and let \( DY \) be the column matrix \( (D_iY)^{1 \leq i \leq n} \).

By inspecting (2.5) above, we see that the equivalence between (i), (ii) and (iii) just below is obvious:

**Lemma 2.6.** The following conditions are equivalent:

(i) the differential equations \( Y^{1 \leq i \leq n} = 0 \) hold for \( 1 \leq j, k \leq n \);

(ii) the matrix equations \( D_k(Y^{X}_X) = 0 \) hold for \( 1 \leq k \leq n \);

(iii) the matrix equations \( DX \cdot D_k(Y^{X}_X) = 0 \) hold for \( 1 \leq k \leq n \);

(iv) the matrix equations \( 0 = D_k(DX) \cdot Y^{X}_X - D_k(DY) \) hold for \( 1 \leq k \leq n \).

Formally, in the sequel, it will be more convenient to achieve the explicit computations starting from condition (iv), since no matrix inversion at all is involved in it.

**Proof.** Indeed, applying the total differentiation operator \( D_k \) to the matrix equation (2.4) written under the equivalent form \( 0 = DX \cdot Y^{X}_X - DY \), we get:

\[
0 = D_k(DX) \cdot Y^{X}_X + DX \cdot D_x(Y^{X}_X) - D_k(DY),
\]

so that the equivalence between (iii) and (iv) is now clear. \( \square \)

### 2.8. An explicit formula in the case \( n = 2 \).

Thus, we can start to develop explicitly the matrix equations

\[
0 = D_k(DX) \cdot Y^{X}_X - D_k(DY).
\]

In it, some huge formal expressions are hidden behind the symbol \( D_k \). Proceeding inductively, we start by examining the case \( n = 2 \) thoroughly. By direct computations which require to be clever, we reconstitute some \( 3 \times 3 \) determinants in the four (in fact three) developed equations (2.9). After some work, the first equation is:

\[
0 = y_{z^1} \cdot \begin{vmatrix} X^1_{x^1} & X^1_{x^2} & X^1_{y} \\ X^2_{x^1} & X^2_{x^2} & X^2_{y} \\ Y_{x^1} & Y_{x^2} & Y_{y} \end{vmatrix} + y_{z^2} \cdot \begin{vmatrix} X^1_{x^1} & X^1_{x^2} & X^1_{y} \\ X^2_{x^1} & X^2_{x^2} & X^2_{y} \\ Y_{x^1} & Y_{x^2} & Y_{y} \end{vmatrix}
\]

\[
+ y_{z^1} \cdot \begin{vmatrix} 2 & X^1_{x^1} & X^1_{x^2} & X^1_{y} \\ X^2_{x^1} & X^2_{x^2} & X^2_{y} \\ Y_{x^1} & Y_{x^2} & Y_{y} \end{vmatrix} + \begin{vmatrix} X^1_{x^1} & X^1_{x^2} & X^1_{y} \\ X^2_{x^1} & X^2_{x^2} & X^2_{y} \\ Y_{x^1} & Y_{x^2} & Y_{y} \end{vmatrix}
\]

\[
+ y_{z^2} \cdot \begin{vmatrix} - & X^1_{x^1} & X^1_{x^2} & X^1_{y} \\ X^2_{x^1} & X^2_{x^2} & X^2_{y} \\ Y_{x^1} & Y_{x^2} & Y_{y} \end{vmatrix} + \begin{vmatrix} X^1_{x^1} & X^1_{x^2} & X^1_{y} \\ X^2_{x^1} & X^2_{x^2} & X^2_{y} \\ Y_{x^1} & Y_{x^2} & Y_{y} \end{vmatrix}
\]
where the coefficients $\text{[BK1989]}$, there exists a unique prolongation
do not involve the Kronecker symbol), we obtained the following perfect formulas:

\[
\begin{align*}
+ y_{x^1} y_{x^2} \cdot & \left\{ \begin{array}{c}
X^1_{x^1 x^2} X^1_{x^1 y} + X^1_{x^2 x^2} X^1_{x^2 y} - 2 X^1_{x^1 y} X^1_{x^2 y} + Y_{x^1} Y_{x^2} Y_{y} \\
Y_{x^1} Y_{x^2} Y_{y}
\end{array} \right\} \\
+ y_{x^1} y_{x^2} \cdot & \left\{ -2 \begin{array}{c}
X^1_{x^1 y} X^1_{x^2 y} X^1_{y}
X^2_{x^1 y} X^2_{x^2 y} X^2_{y}
Y_{x^1} Y_{x^2} Y_{y}
\end{array} \right\} \\
+ y_{x^1} y_{x^2} \cdot & \left\{ - \begin{array}{c}
X^1_{y} X^2_{s^2 y} X^1_{x^1 y}
X^2_{y} X^2_{s^2 y} X^2_{x^1 y}
Y_{y}
Y_{x^1} Y_{x^2} Y_{y}
\end{array} \right\} \\
+ y_{x^1} y_{x^2} \cdot & \left\{ - \begin{array}{c}
X^1_{y} X^2_{y} X^1_{x^1 y}
X^2_{y} X^2_{y} X^2_{x^1 y}
Y_{y}
Y_{x^1} Y_{x^2} Y_{y}
\end{array} \right\} \\
\end{align*}
\]

This formula and the two next (2.22), (2.23) have been checked by Sylvain Neut and
Michel Petitot with the help of Maple.

2.11. Comparison with the coefficients of the second prolongation of a vector field.
At present, it is useful to make an illuminating digression which will help us to devise
what is the general form of the development of the equations (2.9). Consider an arbitrary
vector field of the form

\[
(2.12) \quad L := \sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k} + Y \frac{\partial}{\partial y},
\]

where the coefficients $X^k$ and $Y$ are functions of $(x^1, y)$. According to [OL1986, BK1989],
there exists a unique prolongation $L^{(2)}$ of this vector field to the second order jet space, of the form

\[
(2.13) \quad L^{(2)} := L + \sum_{j_1=1}^{n} Y_{j_1} \frac{\partial}{\partial y_{x^1}} + \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} Y_{j_1, j_2} \frac{\partial}{\partial y_{x^1 x^2}},
\]

where the coefficients $Y_{j_1}, Y_{j_1, j_2}$ may be computed by means of some inductive devices
explained in [OL1986, BK1989]. In [GM2003] (see also [Su2001] for formulas which
do not involve the Kronecker symbol), we obtained the following perfect formulas:

\[
(2.14) \quad \begin{cases}
Y_{j_1, j_2} = Y_{x^1 x^2} + \sum_{k_1=1}^{n} y_{x^1} \cdot \left\{ \delta_{j_1}^{k_1} Y_{x^1 y} + \delta_{j_2}^{k_2} Y_{x^2 y} - X^k_{x^1 x^2} \right\} \\
+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} y_{x^1} y_{x^2} \cdot \left\{ \delta_{j_1, j_2}^{k_1, k_2} Y_{x^1 y} - \delta_{j_1}^{k_1} X^k_{x^1 y} - \delta_{j_2}^{k_2} X^k_{x^2 y} \right\} \\
+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} y_{x^1} y_{x^2} y_{x^3} \cdot \left\{ -\delta_{j_1, j_2}^{k_1, k_2} X^k_{y} \right\},
\end{cases}
\]

for $j_1, j_2 = 1, \ldots, n$. The expression of $Y_{j_1}$ does not matter for us here. Specifying this
formula to the the case $n = 2$ and taking account of the symmetry $Y_{1,2} = Y_{2,1}$ we get
the following three second order coefficients:

\[
\begin{aligned}
Y_{1,1} &= y_{x^1} + y_{x^1} \cdot \left\{ 2 \left( \frac{x_{x^1}^2}{x_{y}} - \frac{x_{x^1}^1}{x_{y}} \right) + y_{x^2} \cdot \left\{ -\frac{x_{x^2}^2}{x_{y}} \right\} + y_{x^1} \cdot (y_{x^1} - 2 \frac{x_{x^1}^1}{x_{y}}) + y_{x^1} \cdot \left\{ -\frac{x_{x^1}^1}{x_{y}} \right\} + y_{x^1} \cdot \frac{x_{x^1}^1}{x_{y}} \right\}, \\
y_{1,2} &= y_{x^1} + y_{x^1} \cdot \left\{ \frac{x_{x^1}^2}{y} - \frac{x_{x^1}^1}{y} \right\} + y_{x^2} \cdot \left\{ y_{x^1} - \frac{x_{x^1}^1}{y} \right\} + y_{x^1} \cdot \left\{ -\frac{x_{x^1}^1}{y} \right\} + y_{x^1} \cdot \frac{x_{x^1}^1}{y} \right\}, \\
y_{2,2} &= y_{x^1} + y_{x^1} \cdot \left\{ -\frac{x_{x^1}^1}{y} \right\} + y_{x^2} \cdot \left\{ 2 \frac{x_{x^1}^2}{y} - \frac{x_{x^1}^1}{y} \right\} + y_{x^2} \cdot \left\{ -\frac{x_{x^1}^1}{y} \right\} + y_{x^2} \cdot \frac{x_{x^1}^1}{y} \right\}.
\end{aligned}
\tag{2.15}
\]

We would like to mention that the computation of \(Y_{1,1}, Y_{1,2}, 1 \leq j_1, j_2 \leq 2\), above is easier than the verification of (2.10). Based on the three formulas (2.15), we claim that we can guess the second and the third equations, which would be obtained by developing and by simplifying (2.9), namely with \(y_{x^1} x^2, 1 \leq j_1, j_2 \leq 2\). In (2.10). Our dictionary to translate from the first formula (2.15) to (2.10) may be described as follows.

Begin with the Jacobian determinant

\[
\begin{vmatrix}
X_{x^1}^1 & X_{x^1}^2 & X_y^1 \\
X_{x^2}^1 & X_{x^2}^2 & X_y^2 \\
Y_{x^1} & Y_{x^2} & Y_y
\end{vmatrix}
\tag{2.16}
\]

of the change of coordinates (2.2). Since this change of coordinates is close to the identity, we may consider that the following Jacobian matrix approximation holds:

\[
\begin{pmatrix}
X_{x^1}^1 \\
X_{x^2}^1 \\
Y_{x^1}
\end{pmatrix}
\approx
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\].

The Jacobian matrix has three columns. There are six possible second order derivatives with respect to the variables \((x^1, x^2, y)\), namely

\[
(\cdot)_{x^1 x^1}, \ (\cdot)_{x^1 x^2}, \ (\cdot)_{x^2 x^2}, \ (\cdot)_{x^1 y}, \ (\cdot)_{x^2 y}, \ (\cdot)_{y y}.
\tag{2.18}
\]

In the Jacobian determinant (2.16), by replacing any one of the three columns of first order derivatives with a column of second order derivatives, we obtain exactly 3 \times 6 = 18 possible determinants. For instance, by replacing the third column by the second order derivative \((\cdot)_{x^1 y}\), or the first column by the second order derivative \((\cdot)_{x^1 x^2}\), we get:

\[
\begin{vmatrix}
X_{x^1}^1 & X_{x^1}^2 & X_{x^2}^1 \\
X_{x^2}^1 & X_{x^2}^2 & X_{x^2}^1 \\
Y_{x^1} & Y_{x^2} & Y_{x^1}
\end{vmatrix}
\quad \text{or} \quad
\begin{vmatrix}
X_{x^1}^1 & X_{x^1}^2 & X_{x^2}^1 \\
X_{x^2} & X_{x^2}^2 & X_{x^2}^1 \\
Y_{x^1} & Y_{x^2} & Y_{x^1}
\end{vmatrix}
\tag{2.19}
\]

We recover the two determinants appearing in the second line of (2.10). On the other hand, according to the approximation (2.17), these two determinants are essentially equal to

\[
\begin{vmatrix}
1 & 0 & X_{x^1 y}^1 \\
0 & 1 & X_{x^2 y}^1 \\
0 & 0 & Y_{x^1 y}^1
\end{vmatrix}
= Y_{x^1 y}^1
\quad \text{or to} \quad
\begin{vmatrix}
X_{x^1}^1 & 0 & 0 \\
X_{x^2}^1 & 1 & 0 \\
Y_{x^1} & 0 & 1
\end{vmatrix}
= X_{x^1}^1.
\tag{2.20}
\]
Consequently, in the second line of (2.10), up to a change to calligraphic letters, we recover the coefficient

\[
0 = y_{x_1 x^2} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right| + y_{x_1} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right| + y_{x_1 x_2} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right|
\]

(2.21)

\[
2 \mathcal{Y}_{x_1 y} - \mathcal{X}_{x_1 x_1}^1
\]

of \(y_{x_1}\) in the expression of \(Y_{1,1}\) in (2.15). In conclusion, we have discovered how to pass symbolically from the first equation (2.15) to the equation (2.10) and conversely.

Translating the second equation (2.15), we deduce, without any further computation, that the second equation which would be obtained by developing (2.9) in length, is:

\[
0 = y_{x_1 x^2} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right| + y_{x_1} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right| + y_{x_1 x_2} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right|
\]

Using the third equation (2.15), we also deduce, without any further computation, that the third equation which would be obtained by developing (2.9) in length, is:

\[
0 = y_{x_2 x^2} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right| + y_{x_1} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right| + y_{x_1 x_2} \cdot \left| \begin{array}{ccc}
X_{x_1}^1 & X_{x_2}^1 & X_{x_1 x_2}^1 \\
Y_{x_1} & Y_{x_2} & Y_{x_1 x_2}
\end{array} \right|
\]

(2.23)
we simply mean which column of first order derivatives is replaced by a column of second
determinants:
\[
\begin{pmatrix}
\Delta(x^2|x^1|y) := \begin{vmatrix}
X_{x^1 x^2} & X_{x^2 x^1} & X_{x^2 y} \\
Y_{x^1 x^2} & Y_{x^2 x^1} & Y_{x^2 y}
\end{vmatrix}
\end{pmatrix}
\]

Here, in the notation \( \Delta(x^1|x^2|y) \), the three spaces between the two vertical lines \( | \) refer
to the three columns of the Jacobian determinant, and the terms \( x^1, x^2, y \) in \( (x^1|x^2|y) \)
designate the partial derivatives appearing in each column. Accordingly, in the following
two examples of modified Jacobian determinants:
\[
\begin{align*}
\Delta(x^1|x^2|y) &:= \begin{vmatrix}
X_{x^1 x^2} & X_{x^2 x^1} & X_{x^2 y} \\
Y_{x^1 x^2} & Y_{x^2 x^1} & Y_{x^2 y}
\end{vmatrix} \\
\Delta(x^1|x^2|y) &:= \begin{vmatrix}
X_{x^1 x^2} & X_{x^2 x^1} & X_{x^2 y} \\
Y_{x^1 x^2} & Y_{x^2 x^1} & Y_{x^2 y}
\end{vmatrix}
\end{align*}
\]

we simply mean which column of first order derivatives is replaced by a column of second
order derivatives in the original Jacobian determinant.

As there are 6 possible second order derivatives \((\cdot)_{x^1 x^2}, (\cdot)_{x^1 x^2}, (\cdot)_{x^1 x^2}, (\cdot)_{x^2 y}, (\cdot)_{x^2 y}, (\cdot)_{x^2 y} \)
and \((\cdot)_{y y}\) together with 3 columns, we obtain \( 3 \times 6 = 18 \) possible modified Jacobian
determinants:
\[
\begin{align*}
\Delta(x^1|x^2|y) &\Delta(x^2|x^1|y) \Delta(x^1|x^2|y) \\
\Delta(x^1|x^2|y) &\Delta(x^2|x^1|y) \Delta(x^1|x^2|y) \\
\Delta(x^1|x^2|y) &\Delta(x^2|x^1|y) \Delta(x^1|x^2|y) \\
\Delta(x^1|x^2|y) &\Delta(x^2|x^1|y) \Delta(x^1|x^2|y) \\
\Delta(x^1|x^2|y) &\Delta(x^2|x^1|y) \Delta(x^1|x^2|y) \\
\Delta(x^1|x^2|y) &\Delta(x^2|x^1|y) \Delta(x^1|x^2|y)
\end{align*}
\]

Next, we observe that if we want to solve with respect to \( y_{x^1 x^2} \) in (2.10), with respect
to \( y_{x^1 x^2} \) in (2.22) and with respect to \( y_{x^2 x^2} \) in (2.23), we have to divide by the Jacobian
Consequently, we introduce 18 new square functions as follows:

\[
\begin{align*}
\Box^1_{x^1} & := \Delta(x^1|x^2|^2) / \Delta(x|x^2|^2) \\
\Box^1_{x^2} & := \Delta(x^1|x^2|^2) / \Delta(x|x^2|^2) \\
\Box^2 \quad & \\
\Box^2 & := \Delta(x^1|x^2|^2) / \Delta(x|x^2|^2) \\
\Box^3 & := \Delta(x^1|x^2|^2) / \Delta(x|x^2|^2) \\
\Box^3 & := \Delta(x^1|x^2|^2) / \Delta(x|x^2|^2) \\
\Box^3 & := \Delta(x^1|x^2|^2) / \Delta(x|x^2|^2) \\
\Box^3 & := \Delta(x^1|x^2|^2) / \Delta(x|x^2|^2)
\end{align*}
\]

Thanks to these notations, we can rewrite the three equations (2.10), (2.22) and (2.23) in a more compact style.

**Lemma 2.29.** A completely integrable system of three second order partial differential equations

\[
\begin{align*}
y_{x^1} & := F^{1,1}(x^1, x^2, y(x), y_{x^1}(x), y_{x^2}(x)) \\
y_{x^2} & := F^{1,2}(x^1, x^2, y(x), y_{x^1}(x), y_{x^2}(x)) \\
y_{x^3} & := F^{2,2}(x^1, x^2, y(x), y_{x^1}(x), y_{x^2}(x))
\end{align*}
\]

is equivalent to the simplest system $Y_{X^1:x^1} = 0, Y_{X^2:x^2} = 0, Y_{X^2:x^2} = 0$, if and only if there exist local \( \mathbb{C} \)-analytic functions $X^1, X^2, Y$ such that it may be written under the specific form:

\[
\begin{align*}
y_{x^1} & = -\Box^1_{x^1} + y_{x^1} \cdot (-2 \Box^2_{x^1,y} + \Box^1_{x^1,y}) + y_{x^2} \cdot (\Box^2_{x^1,y}) + \\
y_{x^2} & = -\Box^2_{x^2} + y_{x^1} \cdot (-\Box^3_{x^2,y} + \Box^1_{x^2,y}) + y_{x^2} \cdot (2 \Box^2_{x^1,y}) + \\
y_{x^3} & = -\Box^3_{x^3} + y_{x^1} \cdot (\Box^2_{x^1,y}) + y_{x^2} \cdot (\Box^2_{x^1,y}) + \end{align*}
\]

2.32. **General formulas.** The formal dictionary between the original determinant formulas (2.10), (2.22), (2.23), between the coefficients (2.15) of the second order prolongation of a vector field and between the new square formulas (2.31) above is evident. Consequently, *without any computation*, just by translating the family of formulas (2.14), we may deduce the exact formulation of the desired generalization of Lemma 2.29 above.

**Lemma 2.33.** A completely integrable system of second order partial differential equations of the form

\[
y_{x^1} \cdot y_{x^1} + y_{x^2} \cdot y_{x^2} + (x, y(x), y_{x^1}(x), \ldots, y_{x^n}(x)), \quad j_1, j_2 = 1, \ldots, n,
\]
is equivalent to the simplest system \( Y_{x_{j_1} x_{j_2}} = 0, j_1, j_2 = 1, \ldots, n \), if and only if there exist local \( \mathbb{K} \)-analytic functions \( X^1, Y \) such that it may be written under the specific form:

\[
\begin{aligned}
    y_{x_{j_1} x_{j_2}} &= -\Box^{n+1}_{x_{j_1} x_{j_2}} + \sum_{k_1 = 1}^{n} y_{x_{k_1}} \cdot \left( \Box^{k_1}_{x_{j_1} x_{j_2}} - \delta^{k_1}_{j_1} \Box^{n+1}_{x_{j_2} y} - \delta^{k_1}_{j_2} \Box^{n+1}_{x_{j_1} y} \right) \\
    &\quad + y_{x_{j_1}} \cdot \left( \Box^{k_1}_{x_{j_2} y} - \frac{1}{2} \delta^{k_1}_{j_2} \Box^{n+1}_{y y} \right) + y_{x_{j_2}} \cdot \left( \Box^{k_1}_{x_{j_1} y} - \frac{1}{2} \delta^{k_1}_{j_1} \Box^{n+1}_{y y} \right) \\
    &\quad + y_{x_{j_1}} y_{x_{j_2}} \cdot \Box^{k_1}_{y y}.
\end{aligned}
\]

Of course, to define the square functions in the context of \( n \geq 2 \) independent variables \((x^1, x^2, \ldots, x^n)\), we introduce the Jacobian determinant

\[
\Delta(x^1 | x^2 | \cdots | x^n | y) := \begin{vmatrix}
    X^1_{x^1} & \cdots & X^1_{x^n} & X^1_{y} \\
    \vdots & \cdots & \vdots & \vdots \\
    X^n_{x^1} & \cdots & X^n_{x^n} & X^n_{y} \\
    Y_{x^1} & \cdots & Y_{x^n} & Y_{y}
\end{vmatrix},
\]

together with its modifications

\[
\Delta \left( x^1 | \cdots | k_1 x^{j_1} x^{j_2} | \cdots | y \right),
\]

in which the \( k_1 \)-th column of partial first order derivatives \( |k_1 x^{k_1}| \) is replaced by the column \( |k_1 x^{j_1} x^{j_2}| \) of partial derivatives. Here, the indices \( k_1, j_1, j_2 \) satisfy \( 1 \leq k_1, j_1, j_2 \leq n + 1 \), with the convention that we adopt the notational equivalence

\[
x^{n+1} \equiv y.
\]

This convention will be convenient to write some of our general formulas in the sequel.

As we promised to only summarize the proof of Theorem 1.1 in this paper, we will not reproduce the proof of Lemma 2.37 from [M2003]. Involving linear algebra considerations, this proof is similar to (and in fact slightly simpler than) the proof of the analogous Lemma 3.32 in [M2004].

§3. First and Second Auxiliary System

3.1. Functions \( G_{j_1, j_2}, H^{k_1}_{j_1, j_2}, L^{k_1}_{j_1}, \text{ and } M^{k_1} \). To discover the four families of functions appearing in the statement of Theorem 1.1, by comparing (2.35) and (1.3), it suffices (of course) to set:

\[
\begin{aligned}
    G_{j_1, j_2} &:= -\Box^{n+1}_{x_{j_1} x_{j_2}}, \\
    H^{k_1}_{j_1, j_2} &:= \Box^{k_1}_{x_{j_1} x_{j_2}} - \delta^{k_1}_{j_1} \Box^{n+1}_{x_{j_2} y} - \delta^{k_1}_{j_2} \Box^{n+1}_{x_{j_1} y}, \\
    L^{k_1}_{j_1} &:= 2 \Box^{k_1}_{x_{j_1} y} - \delta^{k_1}_{j_1} \Box^{n+1}_{y y}, \\
    M^{k_1} &:= \Box^{k_1}_{y y}.
\end{aligned}
\]

Consequently, we have shown the “only if” part of Theorem 1.1, which is the easiest implication.

To establish the “if” part, by far the most difficult implication, the very main lemma can be stated as follows.
Lemma 3.3. The partial differential relations (I'), (II'), (III') and (IV') which express in length the compatibility conditions for the system (1.3) are necessary and sufficient for the existence of functions $X^i, Y$ of $(x^i, y)$ satisfying the second order nonlinear system of partial differential equations (3.2) above.

Indeed, the collection of equations (3.2) is a system of partial differential equations with unknowns $X^i, Y$, by virtue of the definition of the square functions.

3.4. First auxiliary system. To proceed further, we observe that there are $(m + 1)$ more square functions than functions $G_{j_1,j_2}^1, H_{j_1,j_2}^k, L_{j_1}^k,$ and $M_{j_1}^k$. Indeed, a simple counting yields:

\[
\begin{align*}
\#\{\Box_{x^i,y}^{k_1}\} &= \frac{n^2(n + 1)}{2}, \\
\#\{\Box_{x^i}^{k_1}\} &= \frac{n^2}{2}, \\
\#\{\Box_{y}^{n+1}\} &= n, \\
\#\{\Box_{x^i y}^{n+1}\} &= n, \\
\#\{\Box_{y}^{n+1}\} &= \frac{n(n + 1)}{2}, \\
\#\{\Box_{x^i}^{n+1}\} &= 1, \\
\#\{G_{j_1}^{j_2}\} &= \frac{n(n + 1)}{2}, \\
\#\{H_{j_1}^{j_2}X^i\} &= \frac{n^2(n + 1)}{2}, \\
\#\{L_{j_1}^{k_1}\} &= n^2, \\
\#\{M_{j_1}^{k_1}\} &= n.
\end{align*}
\]

Here, the indices $j_1, j_2, k_1$ satisfy $1 \leq j_1, j_2, k_1 \leq n$. Similarly as in [M2004a], to transform the system (3.2) in a true complete system, let us introduce functions $\Pi_{j_1,j_2}^{k_1}$ of $(x^i,y)$, where $1 \leq j_1, j_2, k_1 \leq n + 1$, which satisfy the symmetry $\Pi_{j_1,j_2}^{k_1} = \Pi_{j_1,j_2}^{k_1}$, and let us introduce the following first auxiliary system:

\[
\begin{align*}
\Box_{x^i y}^{k_1} &= \Pi_{j_1,j_2}^{k_1}, \\
\Box_{x^i}^{k_1} &= \Pi_{j_1,j_2}^{k_1}, \\
\Box_{y}^{n+1} &= \Pi_{j_1,j_2}^{k_1}, \\
\Box_{x^i}^{n+1} &= \Pi_{j_1,j_2}^{k_1}, \\
\Box_{y}^{n+1} &= \Pi_{j_1,j_2}^{k_1}.
\end{align*}
\]

It is complete. The necessary and sufficient conditions for the existence of solutions $X^i, Y$ follow by cross differentiations.

Lemma 3.8. For all $j_1, j_2, j_3, k_1 = 1, 2, \ldots, n + 1$, we have the cross differentiation relations

\[
\Box_{x^i x^j}^{k_1} - \Box_{x^j x^i}^{k_1} = -\sum_{k_3=1}^{n+1} \Box_{x^i,x^j,x^k_3}^{k_3} - \sum_{k_3=1}^{n+1} \Box_{x^i,x^k_3}^{k_3} \Box_{x^j}^{k_3}.
\]

The proof of this lemma is exactly the same as the proof of Lemma 3.40 in [M2004a].

As a direct consequence, we deduce that a necessary and sufficient condition for the existence of solutions $\Pi_{j_1,j_2}^{k_1}$ to the first auxiliary system is that they satisfy the following compatibility partial differential relations:

\[
\frac{\partial \Pi_{j_1,j_2}^{k_1}}{\partial x^i} - \frac{\partial \Pi_{j_1,j_2}^{k_1}}{\partial x^j} = -\sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \cdot \Pi_{j_2,j_3}^{k_1} + \sum_{k_2=1}^{n} \Pi_{j_1,j_3}^{k_2} \cdot \Pi_{j_2,j_2}^{k_1},
\]

for all $j_1, j_2, j_3, k_1 = 1, \ldots, n + 1$.

We shall have to specify this system in length according to the splitting $\{1, 2, \ldots, n\}$ and $\{n + 1\}$ of the indices of coordinates. We obtain six families of equations equivalent...
to (3.10) just above:

\begin{equation}
(\Pi_{j_1,j_2}^{n+1})_{x_{j_3}} - (\Pi_{j_1,j_2}^{n+1})_{x_{j_3}} = - \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{j_3,k_2}^{n+1} - \Pi_{j_1,j_2}^{n+1} \Pi_{j_3,n+1}^{n+1} + \\
+ \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{j_3,k_2}^{n+1} + \Pi_{j_1,j_3}^{n+1} \Pi_{j_2,n+1}^{n+1},
\end{equation}

\begin{equation}
(\Pi_{j_1,n+1}^{n+1})_y - (\Pi_{j_1,n+1}^{n+1})_y = - \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{n+1,j_2}^{n+1} - \Pi_{j_1,j_2}^{n+1} \Pi_{n+1,n+1}^{n+1} + \\
+ \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{n+1,j_2}^{n+1} + \Pi_{j_1,n+1}^{n+1} \Pi_{j_2,n+1}^{n+1},
\end{equation}

\begin{equation}
(\Pi_{j_1,j_3}^{n+1})_{x_{j_1}} - (\Pi_{j_1,j_3}^{n+1})_{x_{j_1}} = - \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{j_3,k_2}^{n+1} - \Pi_{j_1,j_2}^{n+1} \Pi_{j_3,n+1}^{n+1} + \\
+ \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{j_3,k_2}^{n+1} + \Pi_{j_1,j_3}^{n+1} \Pi_{j_2,n+1}^{n+1},
\end{equation}

\begin{equation}
(\Pi_{j_1,n+1}^{n+1})_y - (\Pi_{j_1,n+1}^{n+1})_y = - \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{n+1,j_2}^{n+1} - \Pi_{j_1,j_2}^{n+1} \Pi_{n+1,n+1}^{n+1} + \\
+ \sum_{k_2=1}^{n} \Pi_{j_1,j_2}^{k_2} \Pi_{n+1,j_2}^{n+1} + \Pi_{j_1,n+1}^{n+1} \Pi_{j_2,n+1}^{n+1},
\end{equation}

where the indices $j_1, j_2, j_3, k_1$ vary in the set $\{1, 2, 1, \ldots , n\}$.

3.12. Principal unknowns. As there are $(m + 1)$ more square (or Pi) functions than the functions $G_{j_1,j_2}, H_{j_1,j_2}^{k_1}, L_{j_1}^{k_1}$ and $M^{k_1}$, we cannot invert directly the linear system (3.2). To quasi-inverse it, we choose the $(m + 1)$ specific square functions

\begin{equation}
\Theta^1 := \Box_{x_{j_1}x_{j_1}}, \quad \Theta^2 := \Box_{x_{j_2}x_{j_2}}, \ldots , \Theta^{n+1} := \Box_{x_{j_{n+1}}x_{j_{n+1}}},
\end{equation}
calling them principal unknowns, and we get the quasi-inversion:

\[(3.14)\]

\[
\begin{align*}
\Pi_{j_1,j_2}^k &= \frac{\delta^{k_1}_{j_1} \cdot \delta^{k_2}_{j_2}}{\Pi_{j_1,j_2}} = H_{j_1,j_2}^{k_1} \cdot H_{j_2,j_1}^{k_2} - \frac{1}{2} \delta^{k_2}_{j_2} \cdot H_{j_1,j_1}^{k_1} \cdot \Theta_{j_2} + \frac{1}{2} \delta^{k_1}_{j_1} \cdot \Theta_{j_1}, \\
\Pi_{j_1,n+1}^{k_1} &= \frac{\delta^{k_1}_{j_1}}{\Pi_{j_1,n+1}^1} = \frac{1}{2} L_{j_1}^{k_1} + \frac{1}{2} \delta^{k_1}_{j_1} \cdot \Theta_{n+1}, \\
\Pi_{n+1,n+1}^{k_1} &= \frac{\delta^{k_1}_{j_1}}{M^{k_1}}, \\
\Pi_{j_1,n+1}^{n+1} &= \frac{\delta^{n+1}_{j_1}}{\Pi_{j_1,n+1}^1} = -G_{j_1,j_2}, \\
\Pi_{j_1,n+1}^{n+1} &= \frac{\delta^{n+1}_{j_1}}{\Pi_{j_1,n+1}^1} = -\frac{1}{2} H_{j_1,j_1}^{n+1} + \frac{1}{2} \Theta_{j_1}.
\end{align*}
\]

### 3.15. Second auxiliary system.

Replacing the five families of functions \(\Pi_{j_1,j_2}^{k_1}, \Pi_{j_1,n+1}^{k_1}, \Pi_{n+1,n+1}^{k_1}, \Pi_{j_1,j_2}^{n+1}, \Pi_{j_1,n+1}^{n+1}\) by their values obtained in (3.14) just above together with the principal unknowns

\[(3.16)\]

\[
\begin{align*}
\Pi_{j_1,j_1}^{j_1} &= \Theta_{j_1}, \\
\Pi_{n+1,n+1}^{n+1} &= \Theta_{n+1},
\end{align*}
\]

in the six equations (3.11)_1, (3.11)_2, (3.11)_3, (3.11)_4, (3.11)_5 and (3.11)_6, after hard computations that we will not reproduce here, we obtain six families of equations. From now on, we abbreviate every sum \(\sum_{k=1}^{n} \) as \(\sum_{k_1}\).

Firstly:

\[(3.17)\]

\[
0 = G_{j_1,j_2,x^j_3} - G_{j_2,j_3,x^j_3} + \sum_{k_1} G_{j_3,k_1} H_{j_1,j_2}^{k_1} - \sum_{k_1} G_{j_2,k_1} H_{j_1,j_3}^{k_1}.
\]

This is (I’) of Theorem 1.1. Just above and below, we plainly underline the monomials involving a first order derivative. Secondly:

\[
\begin{align*}
\Theta_{j_2}^{j_1} &= -2 G_{j_1,j_2,y} + H_{j_1,j_1,x^j_2} + \\
&\quad + \sum_{k_1} G_{j_2,k_1} L_{j_1}^{k_1} + \frac{1}{2} H_{j_1,j_1}^{k_1} H_{j_2,j_2}^{k_2} - \sum_{k_1} H_{j_1,j_2}^{k_1} H_{j_1,j_1}^{k_1} - \\
&\quad - G_{j_1,j_2} \Theta_{n+1} - \frac{1}{2} H_{j_1,j_1}^{j_1} \Theta_{j_2} - \frac{1}{2} H_{j_2,j_2}^{j_1} \Theta_{j_1} + \sum_{k_1} H_{j_1,j_2}^{k_1} \Theta_{j_1} + \\
&\quad + \frac{1}{2} \Theta_{j_1} \Theta_{j_2}.
\end{align*}
\]

Thirdly:

\[
\begin{align*}
-\Theta_{x^j_1}^{n+1} + \frac{1}{2} \Theta_{j_1}^{j_1} &= \frac{1}{2} H_{j_1,j_1,y} + \\
&\quad - \sum_{k_1} G_{j_1,k_1} M^{k_1} + \frac{1}{4} \sum_{k_1} H_{k_1,k_1}^{k_1} L_{j_1}^{k_1} + \\
&\quad + \frac{1}{4} H_{j_1,j_1}^{j_1} \Theta_{n+1} - \frac{1}{4} \sum_{k_1} L_{j_1}^{k_1} \Theta_{k_1} - \frac{1}{4} \Theta_{j_1} \Theta_{n+1}.
\end{align*}
\]
\[ \begin{align*}
\frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_2}^{j_3} x^{j_3} & - \frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_2}^{j_3} x^{j_3} + \frac{1}{2} \delta_{j_2}^{k_1} \Theta_{j_1}^{j_3} x^{j_3} - \frac{1}{2} \delta_{j_3}^{k_1} \Theta_{j_2}^{j_3} x^{j_3} = \\
&= -H_{j_1,j_2,x,j_3}^{k_1} + H_{j_1,j_3,x,j_2}^{k_1} - \frac{1}{2} \delta_{j_1}^{k_1} H_{j_1,j_2,x,j_3}^{j_3} + \frac{1}{2} \delta_{j_2}^{k_1} H_{j_2,j_1,x,j_3}^{j_3} - \\
&\quad - \frac{1}{2} \delta_{j_3}^{k_1} H_{j_3,j_1,x,j_2}^{j_2} + \frac{1}{2} \delta_{j_2}^{k_1} H_{j_2,j_1,x,j_3}^{j_3} - \\
&\quad - \frac{1}{2} G_{j_1,j_2} L_{j_3}^{k_1} + \frac{1}{2} G_{j_1,j_3} L_{j_2}^{k_1} - \frac{1}{4} \delta_{j_1}^{k_1} H_{j_1,j_1}^{j_2,j_3} + \frac{1}{4} \delta_{j_2}^{k_1} H_{j_2,j_1}^{j_3,j_1} - \\
&\quad - \frac{1}{4} \delta_{j_3}^{k_1} H_{j_3,j_1}^{j_2,j_2} - \frac{1}{2} \delta_{j_1}^{k_1} H_{j_1,j_2}^{k_2,k_2} - \frac{1}{2} \delta_{j_2}^{k_1} H_{j_2,j_1}^{k_2,k_2} - \\
&\quad - \frac{1}{4} \delta_{j_3}^{k_1} H_{j_3,j_1}^{k_2,k_2} + \frac{1}{2} \delta_{j_2}^{k_1} H_{j_2,j_1}^{k_2,k_2} - \\
&\quad - \frac{1}{2} \delta_{j_2}^{k_1} G_{j_1,j_3} \Theta_{j_1}^{j_1} + \frac{1}{4} \delta_{j_2}^{k_1} G_{j_1,j_3} \Theta_{j_1}^{j_1} - \\
&\quad - \frac{1}{4} \delta_{j_2}^{k_1} H_{j_1,j_1}^{j_2,j_3} - \frac{1}{2} \delta_{j_2}^{k_1} H_{j_2,j_1}^{j_3,j_1} + \frac{1}{4} \delta_{j_2}^{k_1} H_{j_2,j_2}^{j_3,j_3} - \\
&\quad - \frac{1}{4} \delta_{j_3}^{k_1} \sum_{k_1} H_{j_1,j_2}^{k_2} \Theta_{j_2}^{k_2} + \frac{1}{2} \delta_{j_2}^{k_1} \sum_{k_1} H_{j_1,j_3}^{k_2} \Theta_{j_3}^{k_2} - \\
&\quad - \frac{1}{4} \delta_{j_3}^{k_1} \Theta_{j_2}^{j_2} + \frac{1}{4} \delta_{j_2}^{k_1} \Theta_{j_1}^{j_1} \Theta_{j_3}^{j_3}.
\end{align*} \]

Fourthly:
\[ \begin{align*}
\frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_2}^{j_3} y + \frac{1}{2} \delta_{j_2}^{k_1} \Theta_{j_1}^{j_3} y - \frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_2}^{j_3} x^{j_3} = \\
&= G_{j_1,j_2} M^{k_1} + \frac{1}{2} \sum_{k_2} H_{j_1,k_2}^{k_2} L_{j_1}^{k_2} - \frac{1}{2} \sum_{k_2} H_{j_1,j_2}^{k_2} L_{j_2}^{k_2} - \frac{1}{4} \delta_{j_1}^{k_1} \sum_{k_2} H_{j_2,k_2}^{k_2} L_{j_1}^{k_2} - \\
&\quad - \frac{1}{4} \delta_{j_2}^{k_1} H_{j_1,j_1}^{j_2,j_3} \Theta_{j_1}^{j_1} + \frac{1}{4} \delta_{j_1}^{k_1} \sum_{k_2} L_{j_1}^{k_2} \Theta_{j_2}^{k_2} + \frac{1}{4} \delta_{j_2}^{k_1} \Theta_{j_1}^{j_1} \Theta_{j_3}^{j_3}.
\end{align*} \]

Fifthly:
\[ \begin{align*}
\delta_{j_1}^{k_1} \Theta_{j_2}^{j_3} y^{n+1} = -f_{j_1,j_2}^{k_1} + 2 M_{j_1,j_2}^{k_1} + \\
&+ \sum_{k_2} H_{j_1,k_2}^{k_2} M_{k_2}^{k_2} - \delta_{j_1}^{k_1} \sum_{k_2} H_{k_2,k_2}^{k_2} M_{k_2}^{k_2} - \frac{1}{2} \sum_{k_2} L_{j_1}^{k_2} L_{j_1}^{k_2} + \\
&+ \delta_{j_1}^{k_1} M_{k_2}^{k_2} \Theta_{j_2}^{j_3} + \frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_1}^{j_1} \Theta_{j_2}^{j_2} + \frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_2}^{j_3}.n+1.
\end{align*} \]

Sixthly:
\[ \begin{align*}
\delta_{j_1}^{k_1} \Theta_{j_2}^{j_3} x^{j_3} = -f_{j_1,j_2}^{k_1} + 2 M_{j_1,j_2}^{k_1} + \\
&+ \sum_{k_2} H_{j_1,k_2}^{k_2} M_{k_2}^{k_2} - \delta_{j_1}^{k_1} \sum_{k_2} H_{k_2,k_2}^{k_2} M_{k_2}^{k_2} - \frac{1}{2} \sum_{k_2} L_{j_1}^{k_2} L_{j_1}^{k_2} + \\
&+ \delta_{j_1}^{k_1} M_{k_2}^{k_2} \Theta_{j_2}^{j_3} + \frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_1}^{j_1} \Theta_{j_2}^{j_2} + \frac{1}{2} \delta_{j_1}^{k_1} \Theta_{j_2}^{j_3}.n+1.
\end{align*} \]

3.23. Solving $\Theta_{x^{j_1}}^{j_2}$, $\Theta_{y^{j_1}}^{j_2}$, $\Theta_{x^{j_1}}^{n+1}$ and $\Theta_{y^{n+1}}$. From the six families of equations (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22), we can solve $\Theta_{x^{j_1}}^{j_2}$, $\Theta_{y^{j_1}}^{j_2}$, $\Theta_{x^{j_1}}^{n+1}$ and $\Theta_{y^{n+1}}$. Not mentioning the (hard) intermediate computations, we obtain firstly:
\[ \begin{align*}
\Theta_{x^{j_1}}^{j_2} = -G_{j_1,j_2} H_{j_1,j_1,x^{j_2}}^{j_2} + \sum_{l} G_{j_2,l} L_{j_1}^{l} + \frac{1}{2} H_{j_1,j_1}^{j_2,j_2} + \sum_{l} H_{j_1,j_2}^{l} - \\
&- G_{j_1,j_2} \Theta_{x^{j_2}}^{n+1} - \frac{1}{2} H_{j_1,j_1}^{j_2} \Theta_{j_2}^{j_2} - \frac{1}{2} H_{j_2,j_2}^{j_2} \Theta_{j_1}^{j_1} + \sum_{l} H_{j_1,j_2}^{l} \Theta_{j_2}^{j_2} + \frac{1}{2} \Theta_{j_1}^{j_1} \Theta_{j_2}^{j_2}.
\end{align*} \]
Secondly:
(3.25)
\[ \Theta_y^{j_i} = \frac{1}{3} H_{j_1,j_2,y}^{j_i} + \frac{2}{3} L_{j_1,x}^{j_i} + \frac{4}{3} G_{j_1,j_2} M^{j_i} + \frac{2}{3} \sum_l G_{j_1,l} M^l - \frac{1}{2} \sum_l H_{l,l}^j L_{j_1}^l + \]
+ \frac{2}{3} \sum_l H_{j_1,l}^j L_{j_1}^l - \frac{2}{3} \sum_l H_{j_1,l}^j L_{j_1}^l - \frac{1}{2} H_{j_1,j_2}^j \Theta^{n+1} + \frac{1}{2} \sum_l L_{j_1}^l \Theta^l + \]
+ \frac{1}{2} \Theta^{j_i} \Theta^{n+1}.

Thirdly:
(3.26)
\[ \Theta^{n+1} = \frac{1}{3} H_{j_1,j_2,y}^{j_i} + \frac{2}{3} L_{j_1,x}^{j_i} + \frac{2}{3} G_{j_1,j_2} M^{j_i} + \frac{4}{3} \sum_l G_{j_1,l} M^l - \frac{1}{2} \sum_l H_{l,l}^j L_{j_1}^l + \]
+ \frac{1}{3} \sum_l H_{j_1,l}^j L_{j_1}^l - \frac{1}{3} \sum_l H_{j_1,l}^j L_{j_1}^l - \frac{1}{2} H_{j_1,j_2}^j \Theta^{n+1} + \frac{1}{2} \sum_l L_{j_1}^l \Theta^l + \]
+ \frac{1}{2} \Theta^{j_i} \Theta^{n+1}.

Fourthly:
(3.27)
\[ \Theta_y^{n+1} = \frac{2}{3} H_{j_1,j_2,y}^{j_i} + 2 M_{j_1}^{j_i} + 2 \sum_l H_{j_1,l}^j M^l - \sum_l H_{l,l}^j M^l - \frac{1}{2} \sum_l L_{j_1}^l L_{j_1}^l + \]
+ \sum_l M^l \Theta^l + \frac{1}{2} \Theta^{n+1} \Theta^{n+1}.

These four families of partial differential equations constitute the second auxiliary system.

By replacing these solutions in the three remaining families of equations (3.20), (3.21) and (3.22), we obtain supplementary equations (which we do not copy) that are direct consequences of (I’), (II’), (III’), (IV’).

To complete the proof of the main Lemma 3.3 above, it suffices now to establish the first implication of the following list, since the other three have been already established.

- Some given functions \( G_{j_1,j_2} \), \( H_{j_1,j_2}^{k_1} \), \( L_{j_1}^{k_1} \), \( M^{k_1} \) of \((x^{j_1}, y)\) satisfy the four families of partial differential equations (I’), (II’), (III’), (IV’) of Theorem 1.1.

- There exist functions \( \Theta^{j_1}, \Theta^{n+1} \) satisfying the second auxiliary system (3.24), (3.25), (3.26) and (3.27).

- These solution functions \( \Theta^{j_1}, \Theta^{n+1} \) satisfy the six families of partial differential equations (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22).

- There exist functions \( \Pi_{j_1,j_2}^{k_1} \) of \((x^{j_1}, y)\), \( 1 \leq j_1, j_2, k_1 \leq m + 1 \), satisfying the first auxiliary system (3.7) of partial differential equations.

- There exist functions \( X^{l_2} \), \( Y \) of \((x^{l_1}, y)\) transforming the system \( y_{x^{j_1},x^{j_2}} = F^{j_1,j_2}(x^{l_1}, y_{x^{l_2}}), j_1, j_2 = 1, \ldots, n, \) to the simplest system \( Y_{x^{j_1},x^{j_2}} = 0, j_1, j_2 = 1, \ldots, n. \)
3.28. Compatibility conditions for the second auxiliary system. We notice that the second auxiliary system is also a complete system. Thus, to establish the first above implication, it suffices to show that the four families of compatibility conditions:

\[
\begin{align*}
0 &= \left( \Theta_j^{x_1x_3} \right)_{x_3} \Theta_j^{x_3} - \left( \Theta_j^{x_3} \right)_{x_3} \Theta_j^{x_1} , \\
0 &= \left( \Theta_j^{x_1y} \right)_{y} - \left( \Theta_j^{x_1} \right)_{y} , \\
0 &= \left( \Theta_j^{n+1} \right)_{x_1} - \left( \Theta_j^{x_1} \right)_{x_1} , \\
0 &= \left( \Theta_j^{n+1} \right)_{y} - \left( \Theta_j^{y} \right)_{y} ,
\end{align*}
\]

are a consequence of (I'), (I''), (III'), (IV').

For instance, in (3.29)1, replacing \( \Theta^{x_1} \) by its expression (3.24), differentiating it with respect to \( x^{y} \), replacing \( \Theta^{x_1} \) by its expression (3.24), differentiating it with respect to \( x^{y} \) and substracting, we get:

\[
0 = -2 G_{j_1,j_2,yx_3} + 2 G_{j_1,j_3,yx_3} + H_{j_1,j_2,x_2j_3} - H_{j_1,j_3,x_2j_3} +
\]

\[
\begin{align*}
+ & \frac{1}{2} \Theta_j^{x_1x_3} \Theta_j^{x_1} + \frac{1}{2} \Theta_j^{x_2x_3} \Theta_j^{x_2} - \frac{1}{2} \Theta_j^{x_3} \Theta_j^{x_1} + \frac{1}{2} \Theta_j^{x_3} \Theta_j^{x_2} - \\
- & \frac{1}{2} H_{j_1,j_2,x_3} \Theta_j^{x_2} - \frac{1}{2} H_{j_1,j_1,x_3} \Theta_j^{x_3} + \frac{1}{2} H_{j_1,j_2,x_3} \Theta_j^{x_3} + \frac{1}{2} H_{j_1,j_1,x_3} \Theta_j^{x_3} - \\
- & \frac{1}{2} H_{j_2,j_2,x_3} \Theta_j^{x_3} - \frac{1}{2} H_{j_2,j_2,x_3} \Theta_j^{x_3} + \frac{1}{2} H_{j_2,j_2,x_3} \Theta_j^{x_3} + \frac{1}{2} H_{j_2,j_2,x_3} \Theta_j^{x_3} - \\
- G_{j_1,j_2,x_3} \Theta^{n+1} - G_{j_1,j_2,x_3} \Theta^{n+1} + G_{j_1,j_3} \Theta^{n+1} + G_{j_1,j_3} \Theta^{n+1} +
\end{align*}
\]

\[
+ \sum_l H_{j_1,j_2,x_3} \Theta^l + \sum_l H_{j_1,j_2,x_3} \Theta^l - \sum_l H_{j_1,j_3,x_2} \Theta^l - \sum_l H_{j_1,j_3,x_2} \Theta^l +
\]

\[
+ \frac{1}{2} H_{j_1,j_1,x_3} H_{j_3,j_2} - \frac{1}{2} H_{j_1,j_1,x_3} H_{j_3,j_2} - \frac{1}{2} H_{j_1,j_1,x_3} H_{j_3,j_2} - \frac{1}{2} H_{j_1,j_1,x_3} H_{j_3,j_2} -
\]

\[
- \sum_l H_{j_1,j_2,x_3} H_{j_1,l} - \sum_l H_{j_1,j_2,x_3} H_{j_1,l} + \sum_l H_{j_1,j_3,x_2} H_{j_1,l} + \sum_l H_{j_1,j_3,x_2} H_{j_1,l} +
\]

\[
+ \sum_l G_{j_1,l,x_3} L_{j_1} + \sum_l G_{j_2,l,x_2} L_{j_1} - \sum_l G_{j_3,l,x_2} L_{j_1} - \sum_l G_{j_3,l,x_2} L_{j_1},
\]

Next, replacing the twelve first order partial derivatives underlined just above:

\[
\begin{align*}
\Theta_j^{x_1x_3}, & \quad \Theta_j^{x_2x_3}, & \quad \Theta_j^{x_1x_3}, & \quad \Theta_j^{x_2x_3}, & \quad \Theta_j^{x_3} , & \quad \Theta_j^{x_3} , & \quad \Theta_j^{x_3} , \\
\Theta_j^{x_1x_3}, & \quad \Theta_j^{x_1x_3}, & \quad \Theta_j^{x_1x_3}, & \quad \Theta_j^{x_1x_3}, & \quad \Theta_j^{x_1x_3}, & \quad \Theta_j^{x_1x_3}, & \quad \Theta_j^{x_1x_3}, \n\end{align*}
\]
by their values issued from \((3.24), (3.26)\) and adapting the summation indices, we get the explicit developed form of the first family of compatibility conditions \((3.29)_1^1\):

\[
0 = -2 G_{j_1, j_2, x_1 x_2 y} + 2 G_{j_1, j_3, x_1 x_2 y} - \\
\sum_l G_{j_3, l, x_1 x_2} L^l_{j_1} + \sum_l G_{j_2, l, x_1 x_3} L^l_{j_1} - G_{j_1, j_2, y} H^j_{j_3, j_3} + G_{j_1, j_3, y} H^j_{j_2, j_2} - \\
- 2 \sum_l G_{j_1, j_3} H^j_{j_1, j_2} + 2 \sum_l G_{j_1, j_2} H^j_{j_1, j_3} - \sum_l H^j_{j_1, j_2, x_1 x_3} H^l_{1, l} + \sum_l H^j_{j_1, j_3, x_1 x_2} H^l_{1, l} - \\
- \frac{2}{3} H^j_{j_2, j_2, y} G_{j_1, j_3} + \frac{2}{3} H^j_{j_3, j_3, y} G_{j_1, j_2} - \frac{2}{3} L^j_{j_3, x_1 x_2} G_{j_1, j_2} + \frac{2}{3} L^j_{j_2, x_1 x_2} G_{j_1, j_3} - \\
\sum_l L^l_{1, x_1 x_2} G_{j_3, l} + \sum_l L^l_{1, x_1 x_3} G_{j_2, l} - \\
- \frac{2}{3} G_{j_1, j_2} G_{j_3, j_3} M^j + \frac{2}{3} G_{j_1, j_3} G_{j_2, j_2} M^j - \frac{4}{3} \sum_l G_{j_1, j_2} G_{j_3, l} M^l + \\
+ \frac{4}{3} \sum_l G_{j_1, j_3} G_{j_2, l} M^l - \frac{1}{2} \sum_l G_{j_3, l} H^j_{j_1, j_1} L^l_{j_2} + \frac{1}{2} \sum_l G_{j_2, l} H^j_{j_3, j_3} L^l_{j_1} - \\
- \frac{1}{2} \sum_l G_{j_3, l} H^j_{j_2, j_2} L^l_{j_1} + \frac{1}{2} \sum_l G_{j_2, l} H^j_{j_3, j_3} L^l_{j_1} - \frac{1}{2} \sum_l G_{j_3, l} H^j_{j_1, j_1} L^l_{j_2} + \\
+ \frac{1}{2} \sum_l G_{j_1, j_2} H^j_{j_1, j_2} L^l_{j_3} + \frac{1}{3} \sum_l G_{j_1, j_2} H^j_{j_2, j_2} L^l_{j_1} + \frac{1}{3} \sum_l G_{j_1, j_3} H^j_{j_2, j_2} L^l_{j_1} - \\
- \frac{1}{3} G_{j_1, j_3} H^j_{j_2, j_2} L^j_{j_2} + \frac{1}{3} G_{j_1, j_2} H^j_{j_3, j_3} L^j_{j_2} - \\
- \sum_l \sum_p G_{j_2, p} H^j_{j_1, j_2} L^p_{l} + \sum_p G_{j_3, p} H^j_{j_1, j_3} L^p_{l} - \\
- \sum_l \sum_p H^j_{j_1, j_2} H^p_{j_1, j_2} H^j_{j_3, j_3} H^p_{j_3, j_3}.
\]

**Lemma 3.33.** \([\text{M2003}, \text{M2004}]\) *This first family of compatibility conditions for the second auxiliary system obtained by developing \((3.29)_1^1\) in length, together with the three remaining families obtained by developing \((3.29)_2, (3.29)_3, (3.29)_4\) in length, are consequences, by linear combinations and by differentiations, of \((\Gamma), (\Pi'), (\Pi''), (\Pi'''), (\Pi''''), \text{ of Theorem 1.1}.*

The proof of Theorem 1.1 is complete.
24 JOËL MERKER

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