Derivation of the Linearity Principle of Intriligator-Leigh-Seiberg

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Utilizing the techniques recently developed for $\mathcal{N}=1$ super Yang-Mills theories by Dijkgraaf, Vafa and collaborators, we derive the linearity principle of Intriligator, Leigh and Seiberg, for the confinement phase of theories with semi-simple gauge groups and matter fields in a non-chiral representation that satisfies a further technical assumption.

**Introduction and results**

Recently, Dijkgraaf and Vafa proposed in Ref. 1) that the non-perturbative superpotential $W_{\text{eff}}$ of $\mathcal{N}=1$ super Yang-Mills theories is obtained through a perturbative calculation in the holomorphic matrix models. This triggered an avalanche of works\(^2\) checking and extending their proposal, and now we have two different, purely field-theoretic derivations of this superpotential.\(^3\),\(^4\)

Originally, the proposal was made for matter superfields in adjoint or bi-fundamental representations, but recent works\(^5\) extended this method to matter superfields in the fundamental representation.

In the method of Dijkgraaf and Vafa, one first introduces a bare superpotential $W_{\text{tree}} = \sum g_i O_i$, which leads the system to the confinement phase, and then the effective superpotential for the gaugino condensate $S$ is calculated perturbatively. Therefore, to check the proposal, the effective superpotential has to be calculated by some other methods. Usually this is done by combining two ingredients: one is the exact results, such as the Affleck-Dine-Seiberg superpotential or the Seiberg-Witten curves, which describe the system without $W_{\text{tree}}$, and the other is the linearity principle of Intriligator, Leigh and Seiberg.\(^6\),\(^7\) which governs the reaction of the system to the bare superpotential $W_{\text{tree}}$. For example, Ferrari showed in Ref. 8) that for the maximally confined phase of the $\mathcal{N}=1 U(N)$ super Yang-Mills theory with one adjoint matter superfield, the effective superpotential calculated with the Dijkgraaf-Vafa method is identical to that calculated with the linearity principle combined with the Seiberg-Witten solution. This suggests that the linearity principle can be derived generically from the Dijkgraaf-Vafa proposal.

Let us now briefly review the linearity principle. It states that when the effective superpotential $W_{\text{eff}}$ is written as a sum of two terms, $W_{\text{eff}} = W_{\text{n.p.}} + W_{\text{tree}}$, the non-perturbatively generated part, $W_{\text{n.p.}}$, is independent of $g_i$ and is a function of $\Lambda$ and $\langle O_i \rangle$ only. Here $\Lambda$ denotes the dynamical scale of the theory and the quantities $\langle O_i \rangle$ are the vacuum expectation values of gauge-invariant combinations of matter chiral superfields. Hence $W_{\text{eff}}$ is linear in the $g_i$. The authors of Refs. 6) and 7) checked this principle for several simple cases using symmetry and holomorphy, but for matter superfields in a more complicated representation, symmetry and holomor-
phy themselves are not strong enough to ensure the absence of further correction to \( W_{\text{eff}} \). In their paper, they proposed that this linearity holds in general, if composite operators are suitably defined. Anomalous global symmetries play a significant role in that analysis. An example is the \( R \) symmetry, which transforms matter fermions \( \psi \) and gauginos \( \lambda \) as \( \psi(x) \rightarrow e^{-i\varphi} \psi(x) \) and \( \lambda(x) \rightarrow e^{i\varphi} \lambda(x) \), respectively.

Now that we have derivations for the Dijkgraaf-Vafa proposal, we can turn the argument around and derive the linearity principle for a general class of theories, using the ideas in Refs. 3) and 4). We demonstrate the linearity of theories that satisfy the following three criteria.

First, we consider only the phase in which the gauge group is completely Higgsed or completely confined, so that the low energy gaugino condensate for each group is characterized by a single superfield \( S \).

Second, if the \( R \) charge of the dynamical scale \( \Lambda \) of the theory is non-zero, we further impose the following restriction:

Restriction A There are \( r \) basic gauge invariants, \( F_1, \cdots, F_r \), such that any of the gauge invariants \( O_i \) can be written as a polynomial in them and that no constraints \( P(\langle F_1 \rangle, \langle F_2 \rangle, \cdots, \langle F_r \rangle) = 0 \) independent of the coupling constants exist for \( F_1, \cdots, F_r \).

Third, we also take the gauge group to be simple for the sake of brevity. Extension to the general case of semi-simple groups should be immediate.

It should be noted that many theories satisfy these criteria. As an example, let us recall the \( \text{Sp}(N) \) super Yang-Mills with \( 2N_f \) fundamentals \( Q_i \). The basic gauge invariants are \( T_{ij} = Q_i Q_j \). For \( N_f < N + 1 \), there are no constraints. For \( N_f = N + 1 \), there is a dynamical constraint, \( \text{Pf} T_{ij} = \Lambda^{2N_f} \), but in this case the \( R \) charge of \( \Lambda \) is zero. For \( N_f > N + 1 \), there are classical constraints analogous to that above, but they are all lifted dynamically. Therefore they all meet the above three criteria. The behavior of the \( SU(N) \) super Yang-Mills theory with fundamentals is similar.

Proof of the linearity is carried out in the following two steps. Firstly, we show that \( W_{\text{n.p.}} \) is equal to \( (N_c - N_f) S \), where \( N_c \) is the dual Coxeter number of the gauge group and \( N_f \) is the index of anomaly of the representation of the matter fields. Secondly, under the restriction A, we derive the result that \( S \) is independent of the coupling constants \( g_i \) in \( W_{\text{tree}} \) when expressed as a function of \( \Lambda \) and the basic gauge invariants \( \langle F_i \rangle \). Combining these two results, the linearity principle follows for the theories considered in this paper.

We follow the conventions used in Ref. 3).

Determination of \( W_{\text{n.p.}} \) Let us consider an \( \mathcal{N} = 1 \) super Yang-Mills system with a simple gauge group and matter fields in some non-chiral representation, so that the Feynman rules given in Ref. 3) can be used.

According to Ref. 3), with the bare superpotential \( W_{\text{tree}} \) introduced, the effective superpotential \( W_{\text{eff}} \) for the gaugino condensate is calculated by introducing the vector superfield as an external background and integrating out the matter superfields. We only use a few essential features, namely that i) the propagator for each superfield is the inverse of its mass, ii) the vertices come from terms in \( W_{\text{tree}} \) that are of at least cubic order, and iii) each loop integral introduces two factors of \( W_{\alpha} \) whose lowest component is the gaugino. \( S \) is found by extremizing \( W_{\text{eff}} \).
Another important feature is that, for pure super Yang-Mills theories with a simple gauge group, the chiral ring is generated by $S = \text{tr} W_\alpha W^\alpha/(32\pi^2)$, as noted in Ref. 4). Hence, after integrating out all the matter superfields, any possible contraction of $2n$ of the $W_\alpha$ becomes $S^n$ times some numerical constant. Therefore, any $n$ loop diagram is accompanied by the factor $S^n$.

Let us write $W_{\text{tree}} = \sum m_i Q_i + \sum g_j O_j$, where $m_i$ is the mass of the $i$th superfield, $Q_i$ is the corresponding quadratic gauge invariant, the $O_j$ are gauge invariants that are of at least cubic order in matter superfields, and the $g_j$ are their coupling constants. For the $SU(N)$ super Yang-Mills theory with an adjoint matter field $\Phi$, for example, we can introduce $\text{tr} \Phi^n$ and $(\text{tr} \Phi^n)^m$ as gauge invariants. We can also introduce baryonic invariants if there are sufficiently many fundamental matter fields. Let us also denote by $n_i$ the anomaly index* of the representation of the $i$th matter superfield, plus that of its conjugate if the representation is not real. For example, for the gauge group $SU(N)$, $n_i = N$ for an adjoint matter and $n_i = 1$ for a pair of fundamental and anti-fundamental superfields.

Once we have $W_{\text{eff}}$, we can calculate the vacuum expectation values for various operators $O_i$ by just differentiating $W_{\text{eff}}$ with respect to $g_i$, because
\[
\langle O_i \rangle = \frac{\partial}{\partial g_i} W_{\text{eff}}(S, g_j),
\]
where we have used the extremization condition $\partial W_{\text{eff}}/\partial S = 0$.

Now, recall that the effective superpotential $W_{\text{eff}}$ is a sum of three terms:
\[
W_{\text{eff}} = W_{\text{VY}} + W_{\text{one loop}} + W_{\text{higher}},
\]
where $W_{\text{VY}}$ is the Veneziano-Yankielowicz term $N_c S(1 - \log(S/\Lambda^2))$, $W_{\text{one loop}} = \sum n_i S \log m_i/\Lambda$ is the one-loop contribution without any vertex insertion, and $W_{\text{higher}} = \sum D$ is the sum of diagrams with some vertex insertions (see Ref. 9 for a more detailed explanation). The non-perturbative part of the superpotential $W_{\text{n.p.}}$ is defined as
\[
W_{\text{n.p.}} = W_{\text{eff}} - \langle W_{\text{tree}} \rangle.
\]

Let us note that the number of vertices $V$, the number of propagators $E$, and the number of loops $L$ satisfy $V - E + L = 1$ and that $V$, $E$ and $L$ of each diagram can be counted by
\[
\sum g_j \frac{\partial}{\partial g_j}, \quad \sum m_i^{-1} \frac{\partial}{\partial m_i} = -\sum m_i \frac{\partial}{\partial m_i}, \quad \text{and} \quad S \frac{\partial}{\partial S},
\]
respectively. Hence, the value of $D$, with the convention that we denote a diagram and its value by the same letter, satisfies
\[
\left(1 - S \frac{\partial}{\partial S} \right) D = \left(\sum m_i \frac{\partial}{\partial m_i} + \sum g_j \frac{\partial}{\partial g_j} \right) D,
\]

* The index of anomaly $T(r)$ of the representation $r$ is defined by the formula $\text{tr}_r(t^a t^b) = T(r) \delta^{ab}$, where the operators $t^a$ are the generators of the gauge group normalized so that for the standard $SU(2)$ subgroup they become equal to the Pauli sigma matrices multiplied by $1/2$. 

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but the right-hand side of this expression is simply the contribution of the diagram 
$D$ to $\langle W_{\text{tree}} \rangle$. This implies that

$$
\langle W_{\text{tree}} \rangle = \left( \sum m_i \frac{\partial}{\partial m_i} + \sum g_j \frac{\partial}{\partial g_j} \right) (W_{\text{one loop}} + W_{\text{higher}})
$$

$$
= \left( \sum m_i \frac{\partial W_{\text{one loop}}}{\partial m_i} \right) + \left( 1 - S \frac{\partial}{\partial S} \right) W_{\text{higher}}. \tag{6}
$$

Combining this with the equation $\partial W_{\text{eff}} / \partial S = 0$, we can derive

$$
W_{\text{n.p.}} = W_{\text{eff}} - \langle W_{\text{tree}} \rangle
$$

$$
= W_{\text{VY}} + \left( 1 - \sum m_i \frac{\partial}{\partial m_i} \right) W_{\text{one loop}} + S \frac{\partial}{\partial S} W_{\text{higher}}
$$

$$
= \left( 1 - S \frac{\partial}{\partial S} \right) W_{\text{VY}} + \left( 1 - S \frac{\partial}{\partial S} - \sum m_i \frac{\partial}{\partial m_i} \right) W_{\text{one loop}}
$$

$$
= \left( N_c - \sum n_i \right) S. \tag{7}
$$

As a simple application of this, it is easily shown that there is no non-perturbative superpotential generated for the $SU(N_c)$ super Yang-Mills theory with $N_f = N_c$ pairs of fundamental flavors.

**Independence of $S$ from coupling constants**

In the following, we assume that the theory under consideration satisfies the restriction $A$. We do not need to distinguish quadratic operators from operators of higher than quadratic order, and therefore we collectively denote them as $O_i$ and write $W_{\text{tree}} = \sum \lambda_i O_i$. We also take $W_{\text{eff}}$ as a function of $A$, $S$ and the coupling constants $\lambda_i$. As a convention, we take $O_i = F_i$ for $i = 1, 2, \cdots, r$.

In this section, we show that the gaugino condensate $S$ can be written as a function of $A$ and $\langle F_i \rangle$ without explicit dependence on $\lambda_i$. In other words, we show that as long as the quantities $\langle F_i \rangle$ are left invariant, $S$ does not change when the $\lambda_i$ are varied.

To demonstrate the above claim, first let us recall that $S$ is determined by extremizing $W_{\text{eff}}$, so that

$$
- \frac{\partial^2 W_{\text{eff}}}{\partial S \partial S} \delta S = \sum_i \delta \lambda_i \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial S}. \tag{8}
$$

(Note that $-\frac{\partial^2 W_{\text{eff}}}{\partial S^2}$ contains a term $N_c / S$ from the Veneziano-Yankielowicz contribution, and therefore generally non-zero.)

Second, the change in the vacuum expectation values induced by a change in the coupling constants is given by

$$
\delta \langle F_j \rangle = \delta \frac{\partial W_{\text{eff}}}{\partial \lambda_j} = \delta S \frac{\partial^2 W_{\text{eff}}}{\partial S \partial \lambda_j} + \sum_i \delta \lambda_i \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial \lambda_j} = \sum_i \delta \lambda_i G_{ij}, \tag{9}
$$

where

$$
G_{ij} = \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial \lambda_j} - \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial S} \frac{\partial^2 W_{\text{eff}}}{\partial S \partial \lambda_j} / \frac{\partial^2 W_{\text{eff}}}{\partial S \partial S}. \tag{10}
$$
It should be noted that if some dynamical constraints emerge, the rank of $G_{ij}$ is less than $r$, the number of basic gauge invariants. Therefore, the restriction $A$ ensures that the rank of $G_{ij}$ is equal to $r$.

Let us regard the $\langle O_i \rangle = \partial W_{\text{eff}} / \partial \lambda_i$ as the expectation value of $O_i$ in the static background of the gaugino condensate $S$. Then, using the factorization of gauge invariants $\delta \langle O_i \rangle = \sum_k \langle \partial O_i / \partial F_k \rangle \delta \langle F_k \rangle$, we can rewrite $\delta S$ as

$$-\frac{\partial^2 W_{\text{eff}}}{\partial S \partial S} \delta S = \sum_i \delta \lambda_i \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial S} = \sum_i \delta \lambda_i \frac{\partial \langle O_i \rangle}{\partial S} = \sum_{k=1}^r \left( \sum_i \delta \lambda_i \left( \frac{\partial O_i}{\partial F_k} \right) \right) \frac{\partial F_k}{\partial S} \quad (11)$$

and $\delta \langle F_j \rangle$ as

$$\delta \langle F_j \rangle = \sum_i \delta \lambda_i \left( \frac{\partial \langle O_i \rangle}{\partial \lambda_j} - \frac{\partial \langle O_i \rangle}{\partial \lambda_j} \frac{\partial^2 W_{\text{eff}}}{\partial S \partial \lambda_j} / \partial S \partial S \right)$$

$$= \sum_{k=1}^r \left( \sum_i \delta \lambda_i \left( \frac{\partial O_i}{\partial F_k} \right) \right) \left( \frac{\partial \langle F_k \rangle}{\partial \lambda_j} - \frac{\partial \langle F_k \rangle}{\partial \lambda_j} \frac{\partial^2 W_{\text{eff}}}{\partial S \partial \lambda_j} / \partial S \partial S \right)$$

$$= \sum_{k=1}^r \left( \sum_i \delta \lambda_i \left( \frac{\partial O_i}{\partial F_k} \right) \right) G_{kj} \quad (12)$$

As noted above, $G_{kj}$ is invertible if the indices $k$ and $j$ are restricted to the range $1, 2, \cdots, r$. Thus, multiplying (12) by $G^{-1}_{jk}$, we obtain

$$\sum_i \delta \lambda_i \left( \frac{\partial O_i}{\partial F_k} \right) = 0 \quad (13)$$

for $k = 1, 2, \cdots, r$, so that $\delta S = 0$ follows.

**Conclusion**

Let us combine the results obtained to this point. First, if the $R$ charge of $\Lambda$ is zero, $N_c - N_f$ is also zero. Hence, the non-perturbatively generated superpotential satisfies $W_{\text{eff}} = (N_c - N_f)S = 0$. Second, if the $R$ charge of $\Lambda$ is non-vanishing, we can use the result that $S$ is independent of the coupling constants, because we assume that the theory satisfies the restriction $A$. Therefore $W_{\text{eff}} = (N_c - N_f)S$ does not depend explicitly on the coupling constants, while it is a function of $\Lambda$ and $\langle F_i \rangle$. This is the result that we hoped to derive.

The extension of the present result to a wider class of theories would be interesting and is worth studying. Extending the analysis to the case of chiral matter content seems to be much more difficult, because recent developments have been mainly focused on theories with matter content in a non-chiral representation.

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