The incompressible quantum liquids (IQLs) [1] formed in a high magnetic field \( B \) by two-dimensional electrons filling various fractions \( \nu \) of different Landau levels (LL\(_n\), \( n = 0, 1, \ldots \)) have been the subject of extensive studies ever since the famous discovery of the fractional quantum Hall (FQH) effect [2]. The most recent storm of interest is motivated by the concept of “topological quantum computation” [3,4] employing non-Abelian statistics of the Moore–Read “pfaffian” wave function [5,6] believed to describe the FQH state in a half-filled LL. Other wave functions with different complexities of braiding statistics have also been proposed [7,8,9], elevating the convincing demonstration of non-Abelian statistics in a real physical system to the challenge of greatest current importance.

With the partially filled LL\(_0\) successfully described by the composite fermion (CF) theory [10] and with ordered electron phases favored in higher LLs, signatures of non-Abelian statistics are most strongly anticipated in the few known IQLs in LL\(_1\). Crucial recent experiments in this LL include confirmation of the quasiparticle (QP) charge of \( \frac{1}{3} e \) for the half-filled state at \( \nu = \frac{5}{2} \) [11] and careful measurements of the minute excitation gaps [12,13]. In theory, the most recent advances have been related to the breaking of particle-hole symmetry in the Moore–Read state [14], the role of layer width [15] and LL mixing [16,17] in real systems, and the nature of QPs [18,19].

Despite intensive studies, connection of the FQH states in LL\(_1\) (\( \nu = \frac{5}{2}, \frac{7}{3}, \frac{12}{5}, \) or \( \frac{11}{2} \)) to the few proposed wave functions is not conclusively established, and in some cases it is only tentatively assumed for the lack of other candidates. This is an urgent problem, as the connection of some of these wave functions to the particular conformal field theories is precisely what fuels the anticipation of non-Abelian statistics in nature. Another unresolved puzzle is the discrepancy [15] between the experimental and numerical gaps at \( \nu = \frac{5}{2} \), which undermines the understanding of this state in simple models assuming spin polarization or decoupling from the crystal lattice. On the other hand, a wealth of IQLs found in various systems (electrons or CFs at different fillings of different LLs, in layers of varied width \( w \)) invites a more general question of possible IQLs with arbitrary interactions \( V \).

In this Letter we demonstrate (numerically) that non-Abelian IQLs can only emerge in LL\(_1\) in the narrow range of filling factors \( \frac{5}{3} < \nu < \frac{3}{2} \). In this range, the known non-Abelian candidates at \( \nu = \frac{5}{2} \) (Moore–Read) and \( \frac{12}{5} \) (Read–Rezayi). Outside of it, the four-flux noninteracting composite fermion model is validated. The borderline \( \nu = \frac{5}{3} \) state is adiabatically connected to the Laughlin liquid, but its short-range correlations and charge excitations are different.

We study spin-polarized \( N \)-particle systems on a unit sphere with the magnetic monopole of strength \( 2Q(hc/e) \) inside [21]. In this geometry, LL\(_n\) is a shell of angular momentum \( \ell = Q + n \), and different \( N \)-body wave functions at the same \( \nu \) are distinguished by a ‘shift’ \( \gamma \) between LL degeneracy and \( \nu^{-1}N \). In contrast to previous exact diagonalization calculations, we do not confine ourselves to the systems with particular \( V \), but search for the universality classes \( (\nu,\gamma) \) of gapped ground states with arbitrary interactions, attainable in realistic systems (of electrons or CFs) with the suitable choice of \( n \) and \( w \).

We begin by recalling that many-body dynamics in a degenerate LL is determined by an interaction pseudopotential, defined [21] as the dependence of the pair energy \( V \) on the relative angular momentum \( m = 1, 3, \ldots \). Moreover, \( V_m \) induces particular correlations through its deviation from a straight line over any consecutive \( m \)'s [22]. Hence, it is known that low-energy spectra of \( V_m \) are reproduced by a suitable effective pseudopotential \( U_m \) with only a few (positive or negative) coefficients.

We have used \( U = [U_1, U_3, U_5] \) in the search for IQLs with (nearly) arbitrary interactions. Higher terms (causing long-range order) were ignored. On the other hand, inclusion of \( U_5 \) was needed for an accurate description of the two lowest (electron or CF) LLs known to host IQL states, while it still allowed for useful graphical representation of the ground state properties in an effectively two-dimensional space of (normalized) parameters \( U_m \).

We looked at various finite systems \( (N, 2\ell) \) in search of
the series of gapped ground states at $2\ell = \nu^{-1}N - \gamma$. A few maps of the ‘neutral’ gap $\Delta$ (gap to the first excited state in the same spectrum) appear in Fig. 1. The IQL candidates are the maxima in $\Delta$ repeating regularly for different $N$. Their location on the map must be compared with the actual pseudopotentials in different LLs, whose approximate positions (A)–(D) are indicated in frame (i).

Let us begin with a half-filled LL represented by $2\ell = 2N - 3$. The maximum near $U_{(B)} = [\frac{3}{2}, \frac{1}{2}, 0]$ found for each $N$ is relevant for the electrons in LL$_1$, whose actual Coulomb pseudopotential is almost linear between $m = 1$ and 5. The maximum coincides with a peak in similar maps (not shown) of the squared overlap of the ground state of $U$ with the exact Moore–Read state, and also with an essentially zero-level minimum of the triplet Hal- 

![FIG. 1: (color online) Ternary contour plots of the neutral excitation gaps $\Delta$ for $N$ fermions in a partially filled Landau level with shell angular momentum $\ell$, interacting via model pseudopotential $U_m$. In each plot, three corners of the big triangles correspond to $U_m = \delta_{m,\mu}$ with $\mu = 1, 3, 5$ marked in each corner. Points relevant for the actual interactions in different electron or CF Landau levels are marked on a triangle in frame (i). Different candidate incompressible states are indicated in frames (a)–(h), labeled by filling factors $\nu$ and shifts $\gamma$ assigned to each finite system $(N, 2\ell)$. For example, the Moore–Read state is marked as (B) in frames (a) and (b).](image)

![FIG. 2: (color online) Symmetric second-order differences $\delta^2 E$ of the ground-state energy per particle $E$ for $N = 10$ and 12 electrons in LL$_0$ and LL$_1$, as a function of shell angular momentum $\ell$, for layer widths $w = 0$ and 3$\lambda$. Candidate incompressible states are marked as $\Phi_{\nu,\gamma}$, where $\Phi = P, P$, L, J, B denote Moore–Read pfaffian, anti-pfaffian, Laughlin and Jain states, and the state of Ref. [3]. Squared overlaps between the states repeating in both LLs at $\nu \leq 4$ are indicated.](image)
particle $E$ on the LL degeneracy. The cusps are most pronounced in the plots of $\delta^2 E_{2\ell} = E_{2\ell+1} + E_{2\ell+1-2} - 2E_{2\ell}$. For an IQL, its (positive) value gives the QP gap $\Delta$ times the number of QPs created per flux quantum.

Besides complementing the maps of Fig. 1 in the identification of the universality classes of particular IQLs, Fig. 2 reveals a simple connection between statistics and filling factor in LL$_1$. At $\nu < \frac{1}{2}$ the same Laughlin/Jain IQLs occur in LL$_1$ and LL$_0$, with high overlaps and similar gaps. This similarity, earlier pointed out in Ref. [27], is caused by sufficiently high $V_1$ and similar $V_{m \geq 3}$ in both LLs. It validates the noninteracting CF model [11] with four flux quanta attached to each electron in LL$_1$.

In contrast, at $\nu > \frac{1}{2}$ the Jain sequence of LL$_0$ is replaced in LL$_1$ by a different set of IQLs, including the pfaffian and anti-pfaffian pair at $\nu = \frac{1}{4}$ and, apparently, a different $\nu = \frac{5}{2}$ state with $\gamma = 2$ (Fig. 2 shows no signatures of the parafermion $\nu = \frac{5}{2}$ series with $\gamma = -2$ [2]). The breakdown of the noninteracting two-flux CF model in LL$_1$ opens, exclusively at $\frac{1}{4} < \nu < \frac{3}{4}$, a possibility for other IQLs, including several suggested more exotic states with various non-Abelian statistics of their QPs.

For the borderline $\nu = \frac{1}{3}$ state, separating the Abelian from (possibly) non-Abelian states, our calculation reinforces the theory of Ref. [27]. This state has a moderate overlap with the Laughlin ground state of LL$_0$, despite falling into the same class of $\gamma = 3$. It also has a smaller gap than the other Laughlin/Jain states in LL$_0$ or LL$_1$.

Having narrowed the search for non-Abelian IQLs in LL$_1$ to the range of $\frac{1}{7} < \nu < \frac{2}{3}$, let us now decide more conclusively between the competing candidate wave functions for the particular filling factors. For $\nu = \frac{1}{3}$, the issues of the interplay of pfaffian and anti-pfaffian states [15] and of spin polarization [28] were already discussed.

In Fig. 3(a) we compare the ground-state energies per particle $E$ of three known candidates for $\nu = \frac{7}{2}$ (we used $N \leq 16$ and $w = 0$; results for $w > 0$ are similar; energies $E$ are rescaled by $\sqrt{2}\nu/N$ to ensure equal units $e^2/\lambda$ for each $\gamma$). In the extrapolation we only used the open squares, discarding one state aliased with the anti-pfaffian and one apparently suffering from the small size.

Although Fig. 3(a) suggests that the ground state at $\nu = \frac{5}{2}$ might have $\gamma = -2$ (despite the lack of significant gaps in Figs. 1 and 2), the extrapolated energies of all three competing states are indecisively close. Hence, comparison of their susceptibility to LL mixing is crucial. We estimated the appropriate energy correction $dE$ by including in the diagonalization additional states involving a single cyclotron excitation [17]. For each $\gamma$, the value of $dE$ was calculated for $N \leq 10$ (it was also checked that $dE$ is less $N$-dependent than $E$). The assumed Coulomb-to-cyclotron energy ratio $\beta \equiv (e^2/\lambda)/(\hbar \omega_c) = 1.56$ corresponds to $B = 2.6$ T. Clearly, in addition to having the lowest $E$, the $\gamma = -2$ parafermion state also has the largest $|dE|$, and therefore it is convincingly predicted to define the universality class of the $\nu = \frac{7}{2}$ ground state in LL$_1$. This is important because this state is the only known candidate IQL [4] whose braiding rules are sufficiently complex to allow quantum computation.

Fig. 3(a) presents also the energies for $\nu = \frac{4}{3}$. Here, the $\gamma = 7$ series is firmly ruled out in favor of the $\gamma = 3$ states (adiabatically connected to the Laughlin state).

In Fig. 3(b) we plot the extrapolated neutral gaps $\Delta(w)$ for several IQLs in LL$_1$. Unlike in LL$_0$ and in agreement with the experiments [11,12], the gaps at $\nu = \frac{1}{2}$, $\frac{5}{2}$, $\frac{7}{3}$ are all similar, and the gap at $\nu = \frac{7}{2}$ is much smaller (not shown). Understanding that $\Delta$ includes the QE–QH attraction, it is nevertheless noteworthy that incorporating an earlier estimate of gap reduction due to LL mixing at $\nu = \frac{7}{2}$ gives a surprisingly good agreement with experiments in the high mobility limit: for Refs. [11,12], $w/\lambda = 2.7$ and $\beta = 1.1$, yielding the 35% reduction and $\Delta = 0.006 e^2/\lambda$; for Ref. [13], $w/\lambda = 2.5$ and $\beta = 1.7$, yielding the 45% reduction and $\Delta = 0.005 e^2/\lambda$.

Let us now examine more closely the crossover $\nu = \frac{1}{3}$ ground states in LL$_1$. For $N \leq 12$, their squared overlaps with the Laughlin state are merely $\sim 50\%$. This reflects quite different correlations (e.g., for $N = 12$ and $w = 3\lambda$), the pair amplitudes at $m = 1, 3, 5, 7$ are $0.02, 0.13, 0.17, 0.10$ in LL$_1$, as opposed to $0.00, 0.18, 0.13, 0.09$ in LL$_0$. The magneto-roton band is also absent in the spectra of LL$_1$. The low-energy states resembling Laughlin QEs and QHs are found at $2\ell = 3N - 3 \pm 1$, but they are not generally the lowest states in their spectra. For example, the ground state at $2\ell = 3N - 4$ has angular momentum $L = 2, \frac{5}{3}, \frac{1}{3}, 0$ for $N = 8, \ldots, 12$ (instead of $L = \frac{1}{2}N$).

Although we have checked that the eigenstates adiabatically connected to the Laughlin QPs become the low-
est states under bias, their joint energy $\tilde{\Delta} = \varepsilon_{\text{QE}} + \varepsilon_{\text{QH}}$ is unreasonably high. Fig. 4(a,b) shows the size extrapolation of QE and QH energies $\varepsilon$, and width dependences of the extrapolated $\varepsilon$'s. Remarkably, $\varepsilon_{\text{QH}} > 0$ requires finite width $w > 3\lambda$. Negative QH energies in narrow layers are also found in other states in LL1 identified in Fig. 4(b). For instance, $\varepsilon = \frac{1}{2}(E_{2\nu+1} - E_{2\nu})$ plus the size corrections [21] are plotted in Fig. 4(c,d) for $\nu = \frac{1}{2}$.

Note that the gap $\Delta$ at $\nu = \frac{1}{2}$ is not immediately weakened in wider wells (in contrast to Ref. [20] where only data for $N = 10$ and 14 was used for $w > 0$). The initial gap enhancement in wider wells is even more pronounced with the inclusion of LL mixing (here, the corrections $dE$ were calculated for $N \leq 14$ at $2\ell = 2N - 3$ and $2N - 3 \pm 1$). While the reasonable match between the experimental gaps and $\Delta$ (rather than $\tilde{\Delta}$) at $\nu = \frac{1}{2}$ remains a mystery, the difference between the latter two is well understood as the QE–QH attraction [20]. However, the difference between $\Delta$ and $\tilde{\Delta}$ at $\nu = \frac{1}{3}$ is too large to be explained in this way. Instead, Laughlin QEs are unlikely to be the elementary negative carriers in LL1.

In conclusion, we have found the range of filling factors $\frac{1}{3} < \nu < \frac{2}{3}$ in LL1 in which the emergence of non-Abelian statistics is possible. Inside this range, our calculations including finite layer width and LL mixing demonstrate that the spin-polarized ground states at $\nu = \frac{1}{2}$ and $\frac{2}{3}$ are described by the “pfaffian” and “parafermion” wave functions (both non-Abelian). Outside the Jain states of noninteracting CFs repeat in both lowest LLs, precluding more exotic phases. The borderline $\nu = \frac{1}{3}$ state is adiabatically connected to the Laughlin liquid but has a smaller gap and distinct excitations.

The author thanks Gunnar Möller, Nigel Cooper, and Guillaume Gervais for many insightful comments, and acknowledges support from EU under the Marie Curie Intra-European Grant No. PIEF-GA-2008-221701 and from the Polish MNiSW under grant N202-071-32/1513.