Randomness Extraction over Bilinear Character Sums

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Abstract

This work is based on the proposal of a deterministic randomness extractor of a random Diffie-Hellman element defined over two prime order multiplicative subgroups of a finite fields $F_p$, $G_1$ and $G_2$. We show that the least significant bits of a random element in $G_1 \ast G_2$, are indistinguishable from a uniform bit-string of the same length. One of the main application of this extractor is to replace the use of hash functions in pairing by the use of a good deterministic randomness extractor.

Keywords: Finite fields, elliptic curves, randomness extractor, key derivation, bilinear sums.

1 Introduction

The shared element after a Diffie-Hellmann exchange is $g^{ab} \in G$, where $G$ is a cyclic subgroup of a finite field. $g^{ab}$ is indistinguishable from any other element of $G$ under the decisional Diffie-Hellman (DDH) assumption [4]. This hypothesis argues that, given two distribution $(g^a, g^b, g^{ab})$ and $(g^a, g^b, g^c)$ there is no efficient algorithm that can distinguish them. However, the encryption key should be indistinguishable from a random bit string having a uniform distribution. So we could not directly use $g^{ab}$ as the encryption key. It is therefore of adequate arrangements to ensure the indistinguishability of the key such as hash functions, pseudo-random functions or random extractors.

Deterministic random extractor have been introduced in complexity theory by Trevisan and Vadhan [17]. Most of the work on deterministic extractors using exponential sums for their security proof work with simple exponential sums [5, 10, 11, 12, 13]. Here we introduce deterministic random extractors that extract a perfectly random bit string of an element derived from the combination of two separate source.

Related work

In 1998, Boneh et al. [5] show that calculate the $k$-most significant bits of a secret is also difficult as to calculate the common secret. The authors rely on Hidden Number Problem. Hastad et al. [14] propose random extractor based on the probabilistic Leftover Hash Lemma, capable of removing all of the entropy random source having sufficient min-entropy. This technique and its variants, however, requires the use of hash functions and perfect random.

The particularity of these extractors is that they belong to the random oracle model. Thus, indistinguishability can not be proven under the DDH assumption unless you add a random oracle. However, these are considered some limitations in practice.
In 2008, Fouque et al. [13] propose a simple extractor capable of extracting the $k$ least significant bits or the $k$ most significant bits of a strong random element issued to the Diffie-Hellman exchange on a sufficient big subgroup of $\mathbb{Z}_p$. They rely on exponential sums to bound the statistical distance between two variable.

In 2009, Chevalier et al. [10] also use exponential sums but bound the collision probability of bits extracted to prove the security of the extractor. They use the Vinogradov inequality to limit the incomplete character sums. They improve the results of Fouque by providing an extractor capable of extracting up to two times more bits. They also feature extractor on the group of points of an elliptic curve defined over a finite field. However, their work was limited to the finite prime fields.

In 2011, Ciss et al. [11] extend the work of Chevalier over finite non prime fields $\mathbb{F}_{p^n}$ and elliptic curves over $\mathbb{F}_{p^n}$ and more particularly on binary finite fields. They use the Winterhof inequality to limit the incomplete character sums.

All that previous work are based on the character model, using single character sums. We focus on the extraction of a random string of bits from a random element from multiple source in particular, two source.

**Our work**
We proposed a deterministic random extractor under the DDH assumption, which maps two multiplicative subgroup of a finite field to the set $\{0,1\}^k$, permitting to extract the $k$-least significant bits of a random element issued of the two subgroup. We use the double exponential sums to bound the collision probability and give a security proof of our extractor.

**Organization of work**
This work is organize as follow: In section 2, we recall some definition and results about randomness, character sums and bilinear character sums. In section 3, we present and analyze our randomness extractor. In section 4, we finish by giving some applications of our extractor.

## 2 Preliminaries

Measures of randomness  In this section, we introduce some definitions and results on the measurement parameters of randomness [13] and on character sums.

### 2.1 Measures of randomness

**Definition 2.1. Guessing probability**
Let $\mathcal{X}$ be a set of cardinality $|\mathcal{X}|$ and $X$, an $\mathcal{X}$-valued random variable. The guessing probability $\gamma(X)$ of $X$ is given by:

$$\gamma(X) = \max\{P[X = v] : v \in \mathcal{X}\}$$

**Definition 2.2. Collision probability**
Let $\mathcal{X}$ be a finite set and $X$, an $\mathcal{X}$-valued random variable. The collision probability of $X$, denoted by $Col(X)$, is the probability

$$Col(X) = Pr[X = X'] = \sum_{x \in \mathcal{X}} Pr[X = x]^2$$
Definition 2.3. Statistical distance
Let $X$ be a finite set. If $X$ and $Y$ are $X$-valued random variables, then the statistical distance $SD(X,Y)$ between $X$ and $Y$ is defined as

$$SD(X,Y) = \frac{1}{2} \sum_{x \in X} |Pr[X = x] - Pr[Y = x]|$$

Let $U_X$ be a random variable uniformly distributed on $X$ and $\delta \leq 1$ a positive real number. Then a random variable $X$ on $X$ is said to be $\delta$-uniform if

$$SD(X,U_X) \leq \delta$$

Lemma 2.1. Relation between $SD$ and $Col(X)$
Let $X$ be a random variable over a finite set $X$ of size $|X|$ and $\Delta = SD(X,U_S)$ be the statistical distance between $X$ and $U_X$, where $U_X$ is a uniformly distributed random variable over $X$. Then,

$$Col(X) \geq 1 + 4\Delta^2 / |X|$$

Definition 2.4. Deterministic $(Y,\delta)$-extractor
Let $X$ and $Y$ be two finite sets. Let $Ext$ be a function $Ext : X \rightarrow Y$. We say that $Ext$ is a deterministic $(Y,\delta)$-extractor for $X$ if $Ext(U_X)$ is $\delta$-uniform on $Y$. That is

$$SD(Ext(U_X),U_Y) \leq \delta$$

Definition 2.5. Two-sources-extractor
Let $X$, $Y$ and $Z$ be finite sets. The function $F : X \times Y \rightarrow Z$ is a two-sources-extractor if the distribution $F(X,Y)$ is $\delta$-close to the uniform distribution $U_Z \in Z$ for every uniformly distributed random variables $X \in X$ and $Y \in Y$

2.2 Characters
Definition 2.6. Let $G$ be an abelian group. A character of $G$ is a homomorphism from $G \rightarrow \mathbb{C}^*$. A character is trivial if it is identically 1. We denote the trivial character by $X_0$ or $\psi_0$.

Definition 2.7. Let $\mathbb{F}_q$ be a given finite field. An additive character $\psi : \mathbb{F}_q^+ \rightarrow \mathbb{C}$ is a character with $\mathbb{F}_q$ considered as an additive group. A multiplicative character $\chi : \mathbb{F}_q^* \rightarrow \mathbb{C}$ is a character with $\mathbb{F}_q^* = \mathbb{F}_q - \{0\}$ considered as a multiplicative group. We extend $X$ to $\mathbb{F}_q$ by defining $X(0) = 1$ if $X$ is trivial, and $X(0) = 0$ otherwise. Note that the extended $X$ still preserves multiplication.

2.3 Exponential sums over finite fields
The main interests of exponential sums is that they allows to construct some caracteristic functions and in some cases we know good bounds for them. The use of these caracteristic functions can permit to evaluate the size of these sets.

We focus on certain character sums, those involving the character $e_p$ define as it follows.

Theorem 2.1. Multiplicative characters of $\mathbb{F}_q$
The multiplicative characters of $\mathbb{F}_q$ are given by:

$$\forall x \in \mathbb{F}_q, e_q(x) = e^{2\pi i x / q} \in \mathbb{C}^*$$

Theorem 2.2. Additive characters of $\mathbb{F}_q$
Suppose $q = p^r$ with $p$ prime. The additive characters of $\mathbb{F}_q$ are given by

$$\psi(x) = e_p(Tr(x))$$

where $Tr(x) = x + x^p + \ldots + x^{p^{r-1}}$ is the trace of $x$. 

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2.3.1 Single character sums

Let $p$ be a prime number, $G$ a multiplicative subgroup of $\mathbb{F}_p^*$. For all $a \in \mathbb{F}_p^*$, let introduce the following notation:

$$S(a, G) = \sum_{x \in G} e_p(ax)$$

Lemma 2.2. Let $p$ be a prime number, $G$ a multiplicative subgroup of $\mathbb{F}_p^*$.

1. If $a = 0$, \( \sum_{x=0}^{p-1} e_p(ax) = p \)
2. For all $a \in \mathbb{F}_p^*$, \( \sum_{x=0}^{p-1} e_p(ax) = 0 \)
3. For all $x_0 \in G$ and all $a \in \mathbb{F}_p^*$, $S(ax_0, G) = S(a, G)$

Proof. Follows [21], pp69-70

Theorem 2.3. Polya-Vinogradov bound

Let $p$ be a prime number, $G$ a multiplicative subgroup of $\mathbb{F}_p^*$. For all $a \in \mathbb{F}_p^*$:

$$\left| \sum_{x \in G} e_p(ax) \right| \leq \sqrt{p}$$

Proof. See [21] for the proof

Theorem 2.4. Winterhof bound

Let $V$ be an additive subgroup of $\mathbb{F}_{p^n}$ and let $\psi$ be an additive caracter of $\mathbb{F}_{p^n}$. Then

$$\sum_{a \in \mathbb{F}_{p^n}} \left| \sum_{x \in V} \psi(ax) \right| \leq p^n$$

Proof. See [20] for the proof

2.3.2 Bilinear character sums

Let $p$ be a prime number, $G, H$ be two multiplicative subgroups of $\mathbb{F}_p^*$. For all $a \in \mathbb{F}_p^*$, let introduce the following notation:

$$S(a, (G, H)) = \sum_{x \in G} \sum_{y \in H} e_p(ax y)$$

Lemma 2.3. Let $p$ be prime and, $G$ and $H$ two subsets of $\mathbb{F}_p^*$. Then

$$\max_{(n,p)=1} \left| \sum_{x \in G} \sum_{y \in H} (e_p(n xy)) \right| \leq (p|G||H|)^{1/2}$$

Proof. See [6] [19]
Lemma 2.4. For any subsets $G$, $H$ of $\mathbb{F}_p^*$, and for any complex coefficients $\alpha_x, \beta_y$ with $|\alpha_x| \leq 1$, $|\beta_y| \leq 1$, the following bound holds

$$|\sum_{x \in G} \sum_{y \in H} \alpha_x \beta_y \psi(xy)| \leq (p^n|G||H|)^{\frac{1}{2}}$$

2.4 Exponential sums over points of elliptic curves

2.4.1 Elliptic curves

Let $E$ be an elliptic curve over $\mathbb{F}_p$, $p \geq 3$ defined by an affine Weierstrass equation of the form

$$y^2 = x^3 + ax + b$$

(1)

with coefficients $a, b \in \mathbb{F}_p$. It is known that the set $E(\mathbb{F}_p)$ of $\mathbb{F}_p$-rational points of $E$, with the point at infinity $\mathcal{O}$ as the neutral element, forms an abelian group. The group law operation is denoted by $\oplus$. Every point $P \neq \mathcal{O} \in E(\mathbb{F}_p)$ is denote by $P = (x(P), y(P))$. Given an integer $n$ and a point $P \in E(\mathbb{F}_p)$, we write $nP$ for the sum of $n$ copies of $P$

$$nP = P \oplus P \oplus \ldots \oplus P, \, n \text{ copies.}$$

2.4.2 Bilinear sums over additive character

Given two subsets $P, Q$ of $E(\mathbb{F}_p)$, and arbitrary complex functions $\sigma, v$ supported on $P$ and $Q$ we consider the bilinear sums of additive characters.

$$V_{\sigma, v}(\psi, P, Q) = \sum_{P \in P} \sum_{Q \in Q} \sigma(P)v(Q)\psi(x(P \oplus Q))$$

Lemma 2.5. Let $E$ be an elliptic curve defined over $\mathbb{F}_q$ where $q = p^n$, with $n \geq 1$ and let

$$\sum_{P \in P} |\sigma(P)|^2 \leq R \text{ and } \sum_{Q \in Q} |v(Q)|^2 \leq T$$

Then, uniformly over all nontrivial additive character $\psi$ of $\mathbb{F}_q$

$$|V_{\sigma, v}(\psi, P, Q)| << \sqrt{qRT}$$

Proof. See [1].

3 Randomness extractor

3.1 Randomness extractor in finite fields

We propose and prove the security of a simple deterministic randomness extractor for two sub-

group $G_1$ and $G_2$ of $\mathbb{F}_q^*$ where $q = p^n$, with $p$ prime and $n \geq 1$. The main theorem of this section states that the $k$-least significant bits of a random element in $(G_1, G_2)$ are close to a truly random group-element in $\{0,1\}^k$. Our approach is from the model based on character sums.
3.1.1 Randomness extraction in $\mathbb{F}_p$

Let $\mathbb{F}_p$ be a finite prime field such that $|p| = m$.

Let $G_1$ and $G_2$ be two multiplicative subgroups of $\mathbb{F}_p^*$ of order $q_1$ (resp. $q_2$), with $|q_1| = l_1$, $|q_2| = l_2$.

Let $U_{G_1}$ (resp. $U_{G_2}$) be a random variable uniformly distributed on $G_1$ (resp. $G_2$), and $k$ a positive integer less than $m$.

**Definition 3.1.** Extractor $f_k$ on $\mathbb{F}_p$

The extractor $f_k$ is defined as a function

$$f_k : G_1 \times G_2 \to \{0,1\}^k$$

$$(x_1, x_2) \mapsto \text{lsb}_k(x_1x_2)$$

The following lemma shows that $f_k$ is a good randomness extractor.

**Lemma 3.1.** Let $p$ be an $m$-bits prime, $G_1$ and $G_2$ be two multiplicative subgroups of $\mathbb{F}_p^*$ of order $q_1$ (resp. $q_2$), we denote $|q_1| = l_1$ and $|q_2| = l_2$.

Let $U_{G_1}$ (resp. $U_{G_2}$) be a random variable uniformly distributed on $G_1$ (resp. $G_2$), and $k$ a positive integer less than $m$.

Let $U_k$ be a random variable uniformly distributed on $\{0,1\}^k$.

If $\Delta = \text{SD}(f_k(U_{G_1}, U_{G_2}), U_k)$ then

$$2\Delta \leq \sqrt{\frac{2^k}{p} + \frac{2^k M(\log_2(p))^{\frac{3}{2}}}{q_1 q_2}} = 2^{ \frac{k + m \log_2(p) - (1 + \log_2(2))}{2} }$$

**Proof.** Since $f_k(x_1, x_2) = \text{lsb}_k(x_1x_2)$, this means $x_1x_2 = 2^k a + b$ or $x'_1x'_2 = 2^k a' + b'$ where $0 \leq a, a' \leq 2^{m-k}$ et $0 \leq b, b' \leq 2^k - 1$.

Thus $x_1x_2 - x'_1x'_2 = 2^k(a - a') + (b - b')$. If $\text{lsb}_k(x_1x_2)$ and $\text{lsb}_k(x'_1x'_2)$ coincide then $x_1x_2 - x'_1x'_2 = 2^k(a - a')$.

Let $u = a - a'$ thus $0 \leq u \leq 2^{m-k}$.

Let us define $K = 2^k$, $u_0 = \text{msb}_{m-k}(p - 1)$.

If $w = 2^m w_m + \ldots + 2^1 w_1 + 2^0 w_0$, $z = 2^m z_m + \ldots + 2^1 z_1 + 2^0 z_0$, and $z < w$ then $\text{msb}_k(z) < \text{msb}_k(w)$.

Since $0 \leq a, a' \leq p - 1$ therefore $u \leq u_0$.

We introduce the following notation,

$$S(a, (G_1, G_2)) = \sum_{x_1 \in G_1} \sum_{x_2 \in G_2} e_p(ax_1x_2)$$
We construct the characteristic function, \(1((x_1, x_2), (x'_1, x'_2), u) = \frac{1}{p} \sum_{a=0}^{p-1} e_p(a(x_1x_2 - x'_1x'_2 - Ku))\), by properties (1) and (2) of Lemma 22.

which is equal to 1 if \(x_1x_2 - x'_1x'_2 = Ku \mod (p)\) and 0 otherwise. Therefore, we can evaluate \(\text{Col}(f_k(U_{G_1}, U_{G_2}))\) where \(U_{G_1}\) (resp. \(U_{G_2}\)) is uniformly distributed in \(G_1\) (resp. in \(G_2\)):

\[
\text{Col}(f_k(U_{G_1}, U_{G_2})) = \frac{1}{(q_1q_2)^2} \sum_{(x_1, x_2) \in (G_1, G_2)^2} \sum_{u=0}^{u_0} \sum_{a=0}^{a_0} e_p(a(x_1x_2 - x'_1x'_2 - Ku))
\]

Then we manipulate the sums, separate some terms \((a = 0)\) and obtain:

For \(a = 0\),
\[
\text{Col}(f_k(U_{G_1}, U_{G_2})) = \frac{1}{(q_1q_2)^2} \sum_{a=0}^{p-1} \sum_{(x_1, x_2) \in (G_1, G_2)^2} \sum_{u=0}^{u_0} e_p(a(x_1x_2 - x'_1x'_2 - Ku))
\]

For \(a \in \mathbb{F}_p^*\),
\[
\text{Col}(f_k(U_{G_1}, U_{G_2})) = \frac{1}{(q_1q_2)^2} \sum_{a=0}^{p-1} \sum_{(x_1, x_2) \in (G_1, G_2)^2} \sum_{u=0}^{u_0} e_p(a(x_1x_2 - x'_1x'_2 - Ku))
\]

We inject the result of (*) then,
\[
\text{Col}(f_k(U_{G_1}, U_{G_2})) = \frac{u_0 + 1}{p} + \frac{1}{(q_1q_2)^2} \sum_{a=1}^{p-1} |S(a, (G_1, G_2))|^2 \sum_{u=0}^{u_0} e_p(-Ku)
\]

We have
\[
\sum_{a=1}^{p-1} \sum_{u=0}^{u_0} e_p(-Ku)
\]

\[
= \sum_{a=1}^{u_0} e_p(-au), \text{ it comes from a change of variable } (a' = Ka = 2^k a \mod (p), \text{ with } \gcd(2, p) = 1).
\]

\[
= \sum_{a=1}^{p-1} \frac{1 - e_p(-a(u_0 + 1))}{1 - e_p(-a)}, \text{ considere the fact that } [0, u_0] \text{ is an interval, the sum is the geometric sum.}
\]

\[
= \sum_{a=1}^{p-1} \sin\left(\frac{\pi a (u_0 + 1)}{p}\right) \sin\left(\frac{\pi a}{p}\right) = \sum_{a=1}^{p-1} \sum_{a=1}^{p-1} \sin\left(\frac{\pi a}{p}\right) \sin\left(\frac{\pi a}{p}\right)
\]

\[
\leq 2 \sum_{a=1}^{p-1} \frac{1}{\sin\left(\frac{\pi a}{p}\right)} \leq 2 \sum_{a=1}^{p-1} \frac{1}{\left|\frac{p}{a}\right|} \leq p \log_2(p)
\]
Therefore
\[
\text{Col}(f_k(U_{G_1}, U_{G_2})) \leq \frac{u_0 + 1}{p} + \frac{1}{(q_1q_2)^2p} |S(a_i(G_1, G_2))|^2p \log_2(p)
\]
\[
\leq \frac{u_0 + 1}{p} + \frac{1}{(q_1q_2)^2p} (pq_1q_2p \log_2(p)) , \text{ by Lemma [2.3]}
\]
\[
\leq \frac{1}{p} + \frac{p \log_2(p)}{q_1q_2}
\]

We now use the Lemma 2.4 which gives a relation between the statistical distance $\Delta$ of $f_k(U_{G_1}, U_{G_2})$ with the uniform distribution and the collision probability:
\[
\text{Col}(f_k(U_{G_1}, U_{G_2})) = \frac{1+4\Delta^2}{2^n}.
\]
The previous upper bound, combined with some manipulations, gives:
\[
2\Delta \leq \sqrt{2k \cdot \text{Col}(f_k(U_{G_1}, U_{G_2})) - 1} \leq \sqrt{\frac{2k}{p}} + \sqrt{\frac{2kp(\log_2(p))}{q_1q_2}} \leq 2^{\frac{k+m+\log_2(m)+(l_1+l_2)}{2}}
\]

### 3.1.2 Randomness extraction in $\mathbb{F}_{p^n}$

Consider the finite field $\mathbb{F}_{p^n}$, where $p$ is prime and $n$ is a positive integer greater than 1. $\mathbb{F}_{p^n}$ is a $n$-dimensional vector space over $\mathbb{F}_p$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$. That means, every element $x$ and $y$ in $\mathbb{F}_{p^n}$ can be represented in the form
\[
x = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n, \text{ and } x' = x'_1\alpha_1 + x'_2\alpha_2 + \ldots + x'_n\alpha_n, \text{ where } x_i (\text{resp. } x'_i) \in \mathbb{F}_{p^n}.
\]
Let $G_1$ and $G_2$ be two multiplicative subgroups of $\mathbb{F}_{p^n}^*$ of order $q_1$ (resp. $q_2$), we denote $|q_1| = l_1$, $|q_2| = l_2$.
Let $U_{G_1}$ (resp. $U_{G_2}$) be a random variable uniformly distributed on $G_1$ (resp. $G_2$), and $k$ a positive integer less than $n$.

**Definition 3.2.** *Extractor $F_k$ on $\mathbb{F}_{p^n}$*

The extractor $F_k$ is defined as a function
\[
F_k : G_1 \times G_2 \rightarrow \{0, 1\}^k
\]
\[
(x, x') \rightarrow (x_1x'_1, x_2x'_2, \ldots, x_kx'_k)
\]

The following lemma shows that $F_k$ is a good randomness extractor.

**Lemma 3.2.** Let $p$ be a $m$-bits prime. Let $G_1$ and $G_2$ be two multiplicative subgroups of $\mathbb{F}_{p^n}^*$ of order $q_1$ (resp. $q_2$), we denote $|q_1| = l_1$, $|q_2| = l_2$.
Let $U_{G_1}$ (resp. $U_{G_2}$) be a random variable uniformly distributed on $G_1$ (resp. $G_2$), and $k$ a positive integer less than $m$. Let $U_k$ be a random variable uniformly distributed on $\{0, 1\}^k$.
If $\Delta = SD(F_k(U_{G_1}, U_{G_2}), U_k)$ then
\[
\Delta \leq \sqrt{\frac{p^n+k-2}{q_1q_2}} = 2^{\frac{k+n(m-(l_1+l_2+2}}{2}}
\]
Proof. Let \((x, x'), (y, z) \in (G_1, G_2)^2\)
Let us introduce the notation
\[
T(a, (G_1, G_2)) = \sum_{x \in G_1} \sum_{x' \in G_2} \psi(axx')
\]
Let us define the following sets
\[
R = \{x_{k+1}x_{k+1} \alpha_{k+1} + x_{k+2}x_{k+2} \alpha_{k+2} \ldots + x_nx_n \alpha_n\}, \text{ a subgroup of } \mathbb{F}_{p^n}
\]
\[
C = \{(x, x'), (y, z) \in (G_1, G_2)^2 / \exists r \in R, xx' - yz = r\}
\]
\[
|C| = \frac{1}{p^n} \sum_{x \in G_1, x' \in G_2} \sum_{y \in G_1, z \in G_2} \sum_{r \in R} \sum_{a \in \mathbb{F}_{p^n}} \psi(a(xx' - yz - r))
\]
we can evaluate the collision probability:
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) = \frac{|C|}{|G_1 \times G_2|^2}
\]
\[
= \frac{1}{(q_1 q_2)^2 p^n} \sum_{(x, x') \in (G_1, G_2)^2} \sum_{(y, z) \in (G_1, G_2)^2} \sum_{r \in R} \sum_{a \in \mathbb{F}_{p^n}} \psi(a(xx' - yz - r))
\]
Then we manipulate the sums, separate some terms \((a = 0)\) and obtain:
For \(a = 0\)
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) = \frac{1}{(q_1 q_2)^2 p^n} \sum_{a \in \mathbb{F}_{p^n}} \sum_{(x, x') \in (G_1, G_2)^2} \sum_{(y, z) \in (G_1, G_2)^2} \psi(0) = \frac{1}{p^k}
\]
For \(a \in \mathbb{F}_{p^n}\)
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) = \frac{1}{(q_1 q_2)^2 p^n} \sum_{a \in \mathbb{F}_{p^n}} \sum_{(x, x') \in (G_1, G_2)^2} \sum_{(y, z) \in (G_1, G_2)^2} \psi(-ayz) \sum_{r \in R} \psi(-ar)
\]
Then for all \(a \in \mathbb{F}_{p^n}\)
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) = \frac{1}{p^k} + \frac{1}{(q_1 q_2)^2 p^n} \sum_{a \in \mathbb{F}_{p^n}} \sum_{(x, x') \in (G_1, G_2)^2} \sum_{(y, z) \in (G_1, G_2)^2} \psi(-ayz) \sum_{r \in R} \psi(-ar)
\]
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) = \frac{1}{p^k} + \frac{1}{(q_1 q_2)^2 p^n} \sum_{a \in \mathbb{F}_{p^n}} |T(a, (G_1, G_2))|^2 \sum_{r \in R} \psi(-ar)
\]
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) \leq \frac{1}{p^k} + \frac{p^n (q_1 q_2)^n}{(q_1 q_2)^2 p^n}, \text{ by Lemma 2.4 and Theorem 2.4}
\]
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) \leq \frac{1}{p^k} + \frac{p^n}{(q_1 q_2)^2}
\]
We now use the Lemma 2.1 which gives a relation between the statistical distance \(\Delta\) of \(F_k(U_{G_1}, U_{G_2})\) with the uniform distribution \(U_k\) and the collision probability:
\[
\text{Col}(F_k(U_{G_1}, U_{G_2})) = \frac{1 + 4 \Delta^2}{2}
\]
\[
2 \Delta \leq \sqrt{2^{2k}.\text{Col}(F_k(U_{G_1}, U_{G_2}))} - 1
\]
\[
\Delta \leq \sqrt{\frac{p^{n+k}}{4 q_1 q_2} - 1} \leq \sqrt{\frac{2^{2+k}}{q_1 q_2}}
\]
\[ \Delta \leq \sqrt{\frac{p^{n+k-2}}{q_1q_2}} \]

Therefore with some manipulations, we obtain the expected result:
\[ \Delta \leq \sqrt{\frac{p^{n+k-2}}{q_1q_2}} = 2^\frac{k + n + \log_q(n) - (l_1 + l_2 + 2)}{2} \]

\section*{3.2 Randomness extraction in elliptic curves}

\subsection*{3.2.1 Randomness extractor in \( \mathcal{E}(\mathbb{F}_p) \)}

\textbf{Definition 3.3.} Let \( p \) be a prime greater than 5. Let \( \mathcal{E} \) be an elliptic curve over the finite field \( \mathbb{F}_p \) and let \( P, Q \) be two subgroups of \( \mathcal{E}(\mathbb{F}_p) \). Let denote \( |P| = q_1 \) and \( |Q| = q_2 \). Then is define the function

\[ \text{extrac}_k : \mathcal{P} \times \mathcal{Q} \rightarrow \{0,1\}^k \]

\[ (P, Q) \mapsto \text{lsb}_k(x(P), x(Q)) \]

\textbf{Lemma 3.3.} We now show an equivalent of Lemma 3.1

Let \( \mathcal{E} \) be an elliptic curve over the finite field \( \mathbb{F}_p \) and let \( \mathcal{P}, \mathcal{Q} \) be two subgroups of \( \mathcal{E}(\mathbb{F}_p) \). Let denote \( |\mathcal{P}| = q_1 \) and \( |\mathcal{Q}| = q_2 \). Let \( U_\mathcal{P} \) and \( U_\mathcal{Q} \) be two random variables uniformly distributed in \( \mathcal{P} \) and \( \mathcal{Q} \) respectively. Let \( U_k \) be the uniform distribution in \( \{0,1\}^k \). Then

\[ \Delta(\text{extrac}_k(U_\mathcal{P}, U_\mathcal{Q}), U_k) \ll \sqrt{\frac{2^{k-2}p \log_2(p)}{q_1q_2}} = 2^\frac{k + n + \log_q(n) - (l_1 + l_2 + 2)}{2} \]

\textbf{Proof.} Let us define \( K = 2^k \), \( u_0 = \text{msb}_{m-k}(p-1) \)

Define the characteristic function

\[ 1((P, Q), (A, B), u) = \frac{1}{p} \sum_{\psi \in \Psi} \psi(x(P)x(Q) - x(A)x(B) - Ku) \]

which is equal to 1 if \( \psi = \psi_0 \) and to 0, otherwise.

Let us compute the collision probablity

\[ \text{Col(\text{extrac}_k(U_\mathcal{P}, U_\mathcal{Q}))} = \frac{1}{(q_1q_2)^2p} \sum_{\psi \in \Psi} \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{A \in \mathcal{P}} \sum_{B \in \mathcal{Q}} \sum_{u \leq u_0} \sum_{\psi} (\psi(x(P)x(Q) - x(A)x(B) - Ku)) \]

Then we manipulate the sums, separate some terms \( (\psi = \psi_0) \) and obtain:

For \( (\psi = \psi_0) \),

\[ \text{Col(\text{extrac}_k(U_\mathcal{P}, U_\mathcal{Q}))} = \frac{1}{(q_1q_2)^2p} \sum_{\psi = \psi_0} \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{A \in \mathcal{P}} \sum_{B \in \mathcal{Q}} \sum_{u \leq u_0} \psi_0(0) = \frac{1}{(q_1q_2)^2p} \sum_{\psi = \psi_0} \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{A \in \mathcal{P}} \sum_{B \in \mathcal{Q}} \sum_{u \leq u_0} \psi_0(0) = \frac{1}{p} \]

\[ = \frac{u_0 + 1}{p} \]
For \((\psi \neq \psi_0)\),
\[
\text{Col(extrac}_k(U_P, U_Q)) = \frac{1}{(q_1 q_2)^2 p} \sum_{\psi \neq \psi_0} \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{A \in \mathcal{A}} \sum_{B \in \mathcal{B}} \sum_{u \leq u_0} \psi(x(P)x(Q) - x(A)x(B) - Ku)
\]
Then
\[
\text{Col(extrac}_k(U_P, U_Q)) = \frac{u_0 + 1}{p} + \frac{1}{(q_1 q_2)^2 p} \sum_{\psi \neq \psi_0} \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{A \in \mathcal{A}} \sum_{B \in \mathcal{B}} \sum_{u \leq u_0} \psi(x(P)x(Q) - x(A)x(B) - Ku)
\]
\[
= \frac{u_0 + 1}{p} + \frac{1}{(q_1 q_2)^2 p} \sum_{\psi \neq \psi_0} \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{u \leq u_0} \sum_{\psi(-x(A)x(B))} \psi(-Ku)
\]
\[
\leq \frac{1}{p} \left( \frac{1}{(q_1 q_2)^2 p} \right) \sum_{\psi \neq \psi_0} q_1 q_2 p \sum_{u \leq u_0} \psi(-Ku), \text{ by Lemma 2.5}
\]
\[
\leq \frac{1}{p} \left( \frac{1}{(q_1 q_2)^2 p} \right) p q_1 q_2 \log_2(p), \text{ since it is shown that } \sum_{\psi \neq \psi_0} \sum_{u \leq u_0} \psi(-Ku) \leq p \log_2(p)
\]
\[
\leq \frac{1}{p} \left( \frac{1}{(q_1 q_2)^2 p} \right) p \log_2(p)
\]
We now use the Lemma 2.1
\[
2\Delta(extrac}_k(U_P, U_Q), U_k) \ll \sqrt{\frac{2^k \cdot \text{Col}(F_k(U_{G_1}, U_{G_2})) - 1}{q_1 q_2}}
\]
\[
2\Delta(extrac}_k(U_P, U_Q), U_k) \ll \sqrt{\frac{2^k \left( \frac{1}{p} + \frac{1}{(q_1 q_2)^2 p} \right) \log_2(p) - 1}{q_1 q_2}}
\]
Therefore with some manipulations,
\[
\Delta(extrac}_k(U_P, U_Q), U_k) \ll \sqrt{\frac{2^{k-2} p \log_2(p)}{q_1 q_2}} = 2^{k+n+\log_2(n)-(t_1+t_2+2)}
\]
\[
\square
\]

3.2.2 Randomness extractor in \(E(\mathbb{F}_{p^n})\)

**Definition 3.4.** Let \(p\) be a prime, \(p > 5\). Let \(E\) be an elliptic curve over the finite field \(\mathbb{F}_{p^n}\). Let \(\mathcal{P}, \mathcal{Q}\) be two subgroups of \(E(\mathbb{F}_{p^n})\). Let denote \(|\mathcal{P}| = q_1\) and \(|\mathcal{Q}| = q_2\). Then is define the function

\[
\text{Extrac}_k : \mathcal{P} \times \mathcal{Q} \rightarrow \{0, 1\}^k
\]

\[
(P, Q) \mapsto \text{lsbk}(x(P).x(Q))
\]

Where \(x(P).x(Q) = t_1 \alpha_1 + t_2 \alpha_2 + t_3 \alpha_3 + \ldots + t_n \alpha_n\)

**Lemma 3.4.** Let \(E\) be an elliptic curve over the finite field \(\mathbb{F}_{p^n}\) and let \(\mathcal{P}, \mathcal{Q}\) be two subgroups of \(E(\mathbb{F}_{p^n})\). Let denote \(|\mathcal{P}| = q_1\) and \(|\mathcal{Q}| = q_2\). Let \(U_P\) and \(U_Q\) be two random variables uniformly distributed in \(\mathcal{P}\) and \(\mathcal{Q}\) respectively. Let \(U_k\) be the uniform distribution in \(\{0, 1\}^k\). Then

\[
\Delta(\text{Extrac}_k(U_P, U_Q), U_k) \ll \sqrt{\frac{2^{n+k}}{q_1 q_2}} = 2^{k+n+\log_2(n)-(t_1+t_2+2)}
\]

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Proof. Using Lemma 2.5 and Theorem 2.4, the sketch of the proof is the same as those of Lemma 3.2.

4 Application

The first most well-known and use tools for the extraction phase of a key exchange protocol in order to create a secure channel are hash function. Hash functions are the most often adopted solution because of their flexibility and efficiency. However, they have a significant drawback. That is, the validity of this technique holds in the random oracle model only. Definitely the truncation of the bit-string of the random element is the most efficient randomness extractor, since it is deterministic and does not require any computation. The interest of studying randomness extraction has several cryptographic applications specially the randomness extraction from a point of elliptic curve. Some of these various applications are find as we have already said in key derivation function, key exchange protocols[12], design cryptographically secure pseudorandom number generator[16].

Today the trend is towards cryptography identification and pairing on elliptic and hyperelliptic curves are widely used in this field, especially for key exchange between three entities and for authentication. Nevertheless, we find that the tools used in most of the protocols based on the pairing, in this case for authentication using hash functions in the extraction phase. The extractor on two sources would be good candidates to replace these functions. They are not only deterministic but also offer the possibility of increasing the randomness considering either one but two sources.

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