\textbf{\textit{\textsc{\textit{A}}}_{6}-INVARIANT CURVES OF GENERA 10 AND 19}

YUSUKE YOSHIDA

Abstract. We study smooth curves on which the alternating group \textit{\textsc{\textit{A}}}_{6} acts faithfully. Let \(\mathcal{V} \subset \text{PGL}(3, \mathbb{C})\) be the Valentiner group, which is isomorphic to \textit{\textsc{\textit{A}}}_{6}. We see that there are integral \(\mathcal{V}\)-invariant curves of degree 12 which have geometric genera 10 and 19. On the other hand, if \textit{\textsc{\textit{A}}}_{6} acts faithfully on a curve \(C\) of genus 10 or 19, then we give an explicit description of the extension \(k(C/\textit{\textsc{\textit{A}}}_{5}) \rightarrow k(C/\textit{\textsc{\textit{A}}}_{6})\) for any icosahedral subgroup \textit{\textsc{\textit{A}}}_{5}. Using this, we show the uniqueness of smooth projective curves of genera 10 and 19 whose automorphism groups contain \textit{\textsc{\textit{A}}}_{6}.

1. Introduction

Automorphism groups of algebraic curves have long been studied. Recently, Harui gave a classification of automorphism groups of smooth plane curves over \(\mathbb{C}\) (\cite{Harui}). In this classification, “primitive” subgroups of \(\text{PGL}(3, \mathbb{C})\) occupies an important part. They are either conjugate to the Valentiner group \(\mathcal{V} \cong \textit{\textsc{\textit{A}}}_{6}\), the icosahedral group \(I \cong \textit{\textsc{\textit{A}}}_{5}\), the Klein group \(K \cong \text{PSL}(2, \mathbb{F}_7)\), the Hessian group \(H_{216}\) or its subgroup \(H_{72}\) or \(H_{36}\). In \cite{Yoshida}, the author studied projective plane curves invariant under \(G = \mathcal{V}, I\) or \(K\), and determined all degrees of nonsingular (resp. integral) curves whose automorphism groups contain \(G\). For the Valentiner group \(\mathcal{V}\), the result is as follows.

\textbf{Theorem 1.1.} (\cite{Yoshida} Theorem 3.7, Theorem 4.4) Let \(d\) be a positive integer.

1. There is a nonsingular projective plane curve of degree \(d\) whose automorphism group is \(\mathcal{V}\) (on equivalently contains \(\mathcal{V}\)) if and only if \(d \equiv 0, 6\) or 12 mod 30.

2. There exists an integral projective plane curve of degree \(d\) invariant under \(\mathcal{V}\) if and only if \(d = 0, 6, \) or 12, \(d \neq 18\) and \(d \neq 24\).

The next step would be to study, for each degree, the existence of curves invariant under such a group of a given geometric genus. This is also related to the problem of finding birational plane models of \textit{\textsc{\textit{A}}}_{6}-invariant curves. For example, by Theorem 1.1, a smooth plane model of an \textit{\textsc{\textit{A}}}_{6}-invariant curve of genus \(g = \frac{1}{2}(d - 1)(d - 2)\) for \(d \equiv 0, 6\) or 12 mod 30 is given.

A smooth projective curve over \(\mathbb{C}\) can be also seen as a compact Riemann surface. Automorphism groups of compact Riemann surfaces have been studied with the help of the Fuchsian groups. According to Breuer’s book \cite{Breuer} Ch.5, \(\S19 - \S20\), the genera of compact Riemann surfaces whose automorphism groups have order 360 are 10, 19 and so on. We have a genus 10 curve whose automorphism group is isomorphic to \textit{\textsc{\textit{A}}}_{6} as a \(\mathcal{V}\)-invariant smooth plane curve of degree 6, but there is no smooth projective plane curve of genus 19. On the other hand, if we take singular curves into consideration, then it turns out that we can give a plane model invariant under \(\mathcal{V}\). In this paper, we find \(\mathcal{V}\)-invariant projective plane curves of degree 12 with genera 10 and 19 and show the birational uniqueness in each genus:
Theorem 1.2. Let $C$ be a smooth projective curve of genus $g$ on which the alternating group $\mathfrak{A}_6$ acts faithfully. For $g = 10$ and 19, $C$ is unique up to birational equivalence.

(1) If $g = 10$, then $C$ is isomorphic to the following:
- the smooth projective plane curve of degree 6 invariant under $\mathcal{V}$,
- the normalization of the unique integral $\mathcal{V}$-invariant projective plane curve of degree 12 with 45 nodes.

(2) If $g = 19$, then $C$ is isomorphic to the normalization of the unique integral $\mathcal{V}$-invariant projective plane curve of degree 12 with 36 nodes.

Remark 1.3. For each case of Theorem 1.2, we will give the specific form of $C$ in Section 2. We also give an explicit description of maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 6 whose Galois closure is the curve $C$.

To find such curves, we use the description of the $\tilde{\mathcal{V}}$-invariant ring $\mathbb{C}[x, y, z]^{\tilde{\mathcal{V}}}$ where $\tilde{\mathcal{V}}$ is a lift of $\mathcal{V}$ in $\text{SL}(3, \mathbb{C})$. It is classically known that this ring $\mathbb{C}[x, y, z]^{\tilde{\mathcal{V}}}$ is generated by explicitly given polynomials of degrees 6, 12, 30 and 45. The first polynomial defines a nonsingular curve of genus 10 invariant under $\mathcal{V}$. Let $\mathfrak{A}_{12}$ be the linear system of the $\mathcal{V}$-invariant curves of degree 12. We see that $\dim \mathfrak{A}_{12} = 1$, and its general elements are nonsingular by Bertini’s theorem. Thus $\mathfrak{A}_{12}$ has finitely many singular members. By calculations with the computer algebra system SINGULAR ([3]), we find 5 singular members and their singular points concretely. We can check that two of them are integral and have geometric genera 10 and 19.

Next, we show the birational uniqueness for each of genera $g = 10$ and 19. In each case, let $C$ be a curve of genus $g$ on which $\mathfrak{A}_6$ acts faithfully. Since the alternating group $\mathfrak{A}_5$ can be considered as a subgroup of $\mathfrak{A}_6$, we have the natural morphisms $\pi_{\mathfrak{A}_6} : C \to C/\mathfrak{A}_6$, $\pi_{\mathfrak{A}_5} : C \to C/\mathfrak{A}_5$ and $f : C/\mathfrak{A}_5 \to C/\mathfrak{A}_6$ with $f \circ \pi_{\mathfrak{A}_5} = \pi_{\mathfrak{A}_6}$. Using Riemann-Hurwitz theorem, we see that the quotient curves $C/\mathfrak{A}_6$ and $C/\mathfrak{A}_5$ are isomorphic to $\mathbb{P}^1$. Hence, $f$ can be considered as a rational function. If we consider the corresponding field extension $f^* : K \to M$, then $C$ corresponding the Galois closure $L$. By calculations using information on the ramification of $f$, we find a rational function $f$, unique up to projective equivalence, with $\text{Gal}(L/K) \cong \mathfrak{A}_6$.

The organization of this paper is as follows. In Section 2, we recall the description of the Valentiner group $\mathcal{V}$ and its invariant curves of degree 12. After some calculations with SINGULAR, we find the $\mathcal{V}$-invariant (projective plane) curves of degree 12 with genera 10 and 19. In Section 3, we consider a curve on which $\mathfrak{A}_6$ acts faithfully, and study the quotients $C/\mathfrak{A}_6$ and $C/\mathfrak{A}_5$. Using this, we show the uniqueness for the genus 10 curve invariant under $\mathfrak{A}_6$. In Section 4, we show the uniqueness for the genus 19 using similar arguments.

2. $\mathcal{V}$-invariant curves of degree 12

There is a subgroup $\mathcal{V}$ of $\text{PGL}(3, \mathbb{C})$, called the Valentiner group, which is isomorphic to the alternating group $\mathfrak{A}_6$. Up to conjugacy, $\mathcal{V}$ is generated by the equivalence classes of
\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & \rho^2 \\
0 & -\rho & 0
\end{pmatrix} \quad \text{and} \quad \frac{1}{2} \begin{pmatrix}
1 & \tau^{-1} & -\tau \\
\tau^{-1} & \tau & 1 \\
\tau & -1 & \tau^{-1}
\end{pmatrix},
\]
where $\rho = e^{\frac{2 \pi i}{5}}$ and $\tau = \frac{1 + \sqrt{5}}{2}$.

Take the preimage $\tilde{\mathcal{V}}$ of $\mathcal{V}$ by the natural projection $\text{SL}(3, \mathbb{C}) \to \text{PGL}(3, \mathbb{C})$. By [2] Lemma 2.4, any $\tilde{\mathcal{V}}$-invariant projective plane curve is defined by a $\tilde{\mathcal{V}}$-invariant
homogeneous polynomial. In a suitable coordinate system, different from the one employed above, the polynomial

$$F(x, y, z) := 10x^3y^3 + 9x^5z + 9y^5z - 45x^2y^2z^2 - 135xyz^4 + 27z^6$$

degree 6 is invariant under \(\overline{\mathcal{V}}\). The curve defined by \(F(x, y, z)\) is called a Wiman’s curve [3]. (Wiman studied another series of curves of degree 6 whose automorphism groups are the symmetric group \(S_5\). These curves are often called “Wiman curves”. We remark that they are not isomorphic to \(V(F)\).) We define the homogeneous polynomial \(\Phi(x, y, z)\) of degree 12 as the Hessian of \(F\), i.e.,

$$\Phi(x, y, z) := \det H(F)(x, y, z) = \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{vmatrix}.$$  

Finally, the homogeneous polynomial \(\Psi(x, y, z)\) of degree 30 is defined as the “border Hessian” of \((F, \Phi)\), i.e.,

$$\Psi(x, y, z) := \begin{vmatrix} H(F) \\ \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} & 0 \end{vmatrix}.$$  

Let \(\mathfrak{d}_d\) be the linear system generated by \(\mathcal{V}\)-invariant curves of degree \(d\). If \(d\) is even, then any member of \(\mathfrak{d}_d\) is defined by a linear combination of \(F^2 \Phi \Psi^k\) with \(6i + 12j + 30k = d\). By [7] Theorem 3.7, a general element of \(\mathfrak{d}_d\) is nonsingular for \(d \equiv 0, 6\) or 12 mod 30.

We consider the lowest degree curve, i.e., the case of \(d = 6\). The only \(\mathcal{V}\)-invariant curve of degree 6 is \(C = V(F)\). Since \(C\) is nonsingular, it is a curve of genus 10 invariant under \(\mathcal{V}\).

Take \(d = 12\). Then any element of \(\mathfrak{d}_{12}\) is defined by a homogeneous polynomial \(aF(x, y, z)^2 + b\Phi(x, y, z)\) for a point \((a : b) \in \mathbb{P}^1\), and \(\dim \mathfrak{d}_{12} = 1\). It is known that \(V(\Phi)\) is a nonsingular curve. Thus the number of singular elements is finite. Let \(P = \frac{a}{b}\) and \(C_P\) the curve corresponding to the point \((a : b)\). Now, we will see that there are 5 special points \(P\) such that \(C_P\) is singular.

**Proposition 2.1.** The curve \(C_P\) is singular if and only if \(P = \infty, 20250, -10125, \xi\) or \(\overline{\xi}\) where \(\xi\) and \(\overline{\xi}\) are solutions to the quadratic equation

$$p^2 + 3375p + 7593750 = 0.$$

Except for \(P = \infty\), the curve \(C_P\) is reduced.

**Proof.** By definition, the curve \(C_\infty\) is defined by \(F(x, y, z)^2\). Thus it is nonreduced and singular. Assume that \(b \neq 0\). We look for all singular points of the curves \(C_P\) defined by \(PF(x, y, z)^2 + \Phi(x, y, z)\) for all \(P \in \mathbb{C}\).
We calculate the values of $P$ such that $C_P$ is singular using the computer algebra system SINGULAR ([3]). In the following, $F$ denotes $F$ and $\Phi$ denotes the polynomial $\Phi$, i.e., the determinant of the Hessian matrix of $F$. We let $G$ be the homogeneous polynomial $PF^2 + \Phi$.

**Code 1.** Setting

```plaintext
LIB "primdec.lib";
ring R = 0,(x,y,z),dp;
poly F = 10x3y3+9x5z+9y5z-45x2y2z2-135xyz4+27z6;
matrix HF = jacob(jacob(F));
// HF is the Hessian matrix of F.
poly Phi = det(HF);
ring S = 0,(x,y,z,P),dp;
map f = R,x,y,z;
poly G = P*(f(F)^2) + f(Phi);
```

To find singularities in that $z = 0$, let $J$ be the ideal generated by $G$ and its derivative of $G$ with respect to $x$, $y$ and $z$. We put $JG_0$ the ideal $J$ in $z = 0$, and check the radical ideal of $JG_0$.

```plaintext
ideal J = G,diff(G,x),diff(G,y),diff(G,z);
ideal JG0 = subst(J,z,0);
radical(JG0);
```

This code gives 3 generators of the radical of $JG_0$, and we obtain a system of equations

\[
\begin{align*}
    y(P + 10125) &= 0, \\
    x(P + 10125) &= 0, \\
    x^5 + y^5 &= 0.
\end{align*}
\]

Since $(x,y,z) \neq (0,0,0)$, we have $x \neq 0$ or $y \neq 0$, and $P = -10125$. Then 5 points $(x:y:0)$ with $x^5 + y^5 = 0$ satisfy the system of equations.

We consider the affine part $z \neq 0$, i.e., we look at the singular points on $V(PF^2 + \Phi) \subset \mathbb{C}^3$. Take the map $g: (x,y,z,P) \mapsto (x,y,1,P)$. Then $g(G)$ defines $V(PF^2 + \Phi)$. Let $JG1$ be the ideal generated by $g(G)$ and its derivatives with respect to $x$ and $y$. We eliminate $x$ and $y$ from $JG1$.

```plaintext
ring T = 0,(x,y,P),dp;
map g = S,x,y,1,P;
ideal JG1 = g(G),diff(g(G),x),diff(g(G),y);
ideal J = eliminate(radical(JG1),xy);
J;
factorize(J[1]);
```

This code returns the polynomial

\[ P^4 - 6750P^3 - 231609375P^2 - 76886718750P - 1556956054687500 \]

in $P$, which can be factored into

\[ (P - 20250)(P + 10125)(P^2 + 3375P + 7593750). \]

Furthermore, the following code returns the colength of the radical ideal $JG1$.

```plaintext
vdim(std(radical(JG1)));
```
This returns 196. Hence, for $z \neq 0$, the curve $C_P$ has singularities only at finitely many points.

Therefore, if $C_P$ has a singular point, then $P = 20250, -10125$ or a solution of $p^2 + 3375p + 7593750 = 0$. Since the number of singular points of $C_P$ is finite, $C_P$ is reduced. □

We will show that $C_{-10125}$ is a $V$-invariant curve of genus 10 and $C_{20250}$ is of genus 19. First, we consider irreducible components of $C_P$.

Notation 2.2. Let $C_n$ be the cyclic group of order $n$ and $D_n$ the dihedral group of order $2n$.

Table 1. The conjugacy classes of subgroups of $\mathfrak{A}_6$

| No. | isomorphic group | order | index |
|-----|-----------------|-------|-------|
| 1   | 1               | 1     | 360   |
| 2   | $C_2$           | 2     | 180   |
| 3   | $C_3$           | 3     | 120   |
| 4   | $C_3$           | 3     | 120   |
| 5   | $C_2^2$         | 4     | 90    |
| 6   | $C_2^2$         | 4     | 90    |
| 7   | $C_4$           | 4     | 90    |
| 8   | $C_5$           | 5     | 72    |
| 9   | $S_3$           | 6     | 60    |
| 10  | $S_3$           | 6     | 60    |
| 11  | $D_4$           | 8     | 45    |
| 12  | $C_3^2$         | 9     | 40    |
| 13  | $D_5$           | 10    | 36    |
| 14  | $A_4$           | 12    | 30    |
| 15  | $A_4$           | 12    | 30    |
| 16  | $C_3 \times S_3$| 18    | 20    |
| 17  | $S_4$           | 24    | 15    |
| 18  | $S_4$           | 24    | 15    |
| 19  | $C_5 \times C_4$| 36    | 10    |
| 20  | $A_5$           | 60    | 6     |
| 21  | $A_5$           | 60    | 6     |
| 22  | $A_6$           | 360   | 1     |

Lemma 2.3. Assume that $C_P$ is reducible and write $C_P = \sum_{i=1}^n D_i$ where $D_1, \cdots, D_n$ are integral curves in $\mathbb{P}^2$. Then one of the following holds:

(i) $n = 2$ and $D_1 = D_2 = V(F)$. Thus $P = \infty$ and $C_P = 2V(F)$ is nonreduced.

(ii) $n = 6$ and $\deg D_i = 2$ for any $i$. Then $C_P$ is reduced and reducible.

(iii) $n = 12$ and $\deg D_i = 1$ for any $i$. Then $C_P$ is reduced and reducible.

Proof. Let $\mathcal{O}$ be the set \{\(D_1, \cdots, D_n\)\} with $n \geq 2$. Since $C_P$ is invariant under $V$, $V$ acts on $\mathcal{O}$. Then

$$\frac{|V|}{|\text{Stab}(D_i)|} = |V \cdot D_i|$$

for any $i$ where $V \cdot D_i$ is the $V$-orbit of $D_i$ in $\mathcal{O}$. Since

$$|V \cdot D_i| \leq n \leq \sum_{i=1}^n \deg D_i = \deg C_P = 12,$$
we obtain $|\text{Stab}(D_i)| \geq 30$. If a subgroup $H$ of $V$ satisfies $|H| \geq 30$, then $H$ is $V$ or isomorphic to $\mathfrak{A}_5$ or $C_2^2 \times C_4$ by Table 11. Hence, any element $D_i$ is invariant under one of these groups.

Suppose that one of $D_i$ is invariant under $V$. Since $\deg D_i < 12$, $\deg D_i = 6$ and $D_i = V(F)$. Thus $aF^2 + b\Phi$ is divisible by $F$. Since $\Phi$ is not divisible by $F$, $C_P$ is defined by $aF^2$ with $a \neq 0$. Thus the statement (i) holds.

Suppose that $D_i$ is not $V$-invariant for any $i$. Since $\text{Stab}(D_i) \cong \mathfrak{A}_5$ or $C_2^2 \times C_4$, we have $|V \cdot D_i| = 6$ or 10 for each $i$. Thus we see

$$\sum_{i=1}^{n} \deg D_i = \sum_{\text{Stab}(D_i) \cong C_2^2 \times C_4} \deg D_i + \sum_{\text{Stab}(D_i) \cong \mathfrak{A}_5} \deg D_i$$

$$= 10k + 6l$$

where $k$ and $l$ are nonnegative integers. Since the left hand side is equal to 12, the only possible pair is $(k, l) = (0, 2)$. Then $\text{Stab}(D_i) = \mathfrak{A}_5$ for any $i$, and $n = 6$ or 12. Hence, the statement (ii) or (iii) holds. Then $C_P$ is not equal to $C_{\infty}$. By Proposition 2.1, $C_P$ is reduced. \hfill \qed

**Remark 2.4.** We can show that the curves $C_5$ and $C_7$ are reducible and are unions of six conics, and they are the only curves $C_P$ satisfying the condition (ii). Furthermore, there is no curve $C_P$ which satisfies the condition (iii).

**Lemma 2.5.** Let $O \subset \mathbb{P}^2$ be a $V$-orbit. Then $|O| \geq 30$.

**Proof.** We consider the stabilizer of a point. Let $H$ be a subgroup of $V$ whose index is less than 30 (i.e., groups No. 16–22 in Table 11). Then $n := [V : H] = 1, 6, 10, 15$ or 20.

Assume that $H$ fixes a point. By the representation theory of finite groups, $H$ also fixes a line in $\mathbb{P}^2$. (See Remark 2.6.) Let $C'$ be a union of $n$ lines invariant under $V$. For $n = 1$, $C'$ is a line. Otherwise, $C'$ is a reducible or nonreduced curve of degree $n$ invariant under $V$. Since the only $V$-invariant curve of degree 6 is $V(F)$ and it is irreducible, we see $C' \neq V(F)$, thus $n \neq 6$. On the other hand, the degree of a $V$-invariant curve is divisible by 6 or is greater than 45. Hence, $n$ is neither 1, 10, 15 nor 20. Therefore, there is no $V$-orbit whose size is less than 30. \hfill \qed

**Remark 2.6.** There is a $H$-invariant subspace $V$ of dimension 1 in $C^3$. Since $H$ is finite, the (orthogonal) compliment subspace $L$ of $V$ is invariant under $H$ and dimension 2. The subspace $L$ corresponds to a line.

We look at the singularities of $C_{-10125}$ and $C_{20250}$.

**Proposition 2.7.** The curves $C_{-10125}$ and $C_{20250}$ satisfy the following.

1. The singular locus of $C_{20250}$ is a $V$-orbit $O_{36}$ of order 36.
2. The singular locus of $C_{-10125}$ is a $V$-orbit $O_{45}$ of order 45.
3. $C_{-10125}$ and $C_{20250}$ are nodal.
4. $C_{-10125}$ and $C_{20250}$ are irreducible.

**Proof.** We prove the claims (1), (2) and (3) by calculations with SINGULAR. We use Code [H] in the proof of Proposition 2.1. First, suppose that $P = 20250$. Then no singular point of $C_{20250}$ is contained in $V(z)$. Thus we may assume that $z = 1$. Let $f$ be the ring homomorphism $S \to T$ defined as $z \mapsto 1$ and $P \mapsto 20250$ where $T$ is the polynomial ring $\mathbb{Q}[x,y]$. Then the affine curve $C_{20250} - V(z)$ embedded in $\mathbb{A}^2$ is defined by $f(G)$. Take the ideal $\mathfrak{jfG}$ generated by $f(G)$ and its derivatives with respect to $x$ and $y$. We can calculate the dimension of the quotient ring of $T$ modulo the ideal generated by the initial terms of the standard Gröbner basis of $\mathfrak{jfG}$ with Code [H] and the following code.
The code returns 36, i.e., the colength of \( jfG \) is 36. Since the set \( \text{Sing}(C_{20250}) \) is a \( V \)-orbit, if the multiplicity of \( jfG \) was greater than 1 at a singular point, then the number of the singular points is at most 18. This contradicts Lemma 2.5. Therefore, all singularities are nodes.

Next, suppose that \( P = -10125 \). By the proof of Proposition 2.1, \( C_{-10125} \) has 5 singular points on \( V(z) \). Now, we check the singular points of \( C_{-10125} \) on \( P^2 - V(z) \). Then the affine curve \( C_{-10125} - V(z) \) embedded to \( \mathbb{A}^2 \) is defined by \( g(G) \). Take the ideal \( jgG \) generated by \( g(G) \) and its derivatives with respect to \( x \) and \( y \). We can calculate the dimension of the quotient ring of \( T \) modulo the ideal generated by the initial terms of the standard Gröbner basis of \( jgG \) with the following code.

The code returns 40 as the colength of \( jgG \). There are also 5 singular points of \( C_{-10125} \) on \( z = 0 \) by the proof of Proposition 2.1. Since \( \text{Sing}(C_{20250}) \) has at most \( 45 \) points, it is a \( V \)-orbit by Lemma 2.5. If the multiplicity of \( jgG \) was greater than \( 1 \) at a singular point, then the number of the singular points were at most \( \frac{45}{2} \), and this is a contradiction. Therefore, all singularities are nodes and \( C_{-10125} \) has 45 nodes.

Finally, we show that \( C_{20250} \) and \( C_{-10125} \) are irreducible. We assume that they are reducible and derive a contradiction. Let \( C \) be \( C_{20250} \) or \( C_{-10125} \) and we write \( C = D_1 \sqcup \cdots \sqcup D_n \) where \( D_1, \cdots, D_n \) are pairwise distinct integral curves. Since \( C \) is reduced and nodal, by Bézout’s theorem, we see

\[
\sum_{i<j} \deg D_i \deg D_j = \# \text{Sing} \ C.
\]

The left hand side is 60 or 66 by Lemma 2.3 (ii) and (iii). However, \( \# \text{Sing} C_{20250} = 36 \) and \( \# \text{Sing} C_{-10125} = 45 \). This is a contradiction. □

For an integral curve of degree 12 with \( n \) nodes, we calculate that its genus is \((55 - n) \) by the genus-degree formula. To summarize, we have the following.

**Corollary 2.8.**

1. \( C_{-10125} \) is an integral, nodal and \( A_6 \)-invariant projective plane curve of genus 10.
2. \( C_{20250} \) is an integral, nodal and \( A_6 \)-invariant projective plane curve of genus 19.

3. **The \( A_6 \)-invariant curve of genus 10**

Let \( C \) be a curve on which the alternating group \( A_6 \) acts. The group \( A_6 \) has 2 conjugacy classes of subgroups which are isomorphic to \( A_5 \). For \( G = A_6 \) or \( A_5 \), we consider the natural morphism \( \pi_G : C \to C/G \). Then \( \pi_G \) corresponds to
the field extension $k(C)/k(C/G)$. We can take the morphism $f : C/\mathfrak{A}_5 \to C/\mathfrak{A}_6$ corresponding to the field extension $k(C/\mathfrak{A}_5)/k(C/\mathfrak{A}_6)$.

\[
\begin{array}{ccc}
C & \xrightarrow{\pi_{\mathfrak{A}_5}} & C/\mathfrak{A}_5 \\
\downarrow f & & \downarrow \pi_{\mathfrak{A}_6} \\
C/\mathfrak{A}_6 & & k(C/\mathfrak{A}_5) \\
k(C) & \xrightarrow{\pi_{\mathfrak{A}_6}} & k(C/\mathfrak{A}_6)
\end{array}
\]

The following lemma is an elementary consequence of Galois theory.

**Lemma 3.1.**

1. If $L$ is the Galois closure of $k(C/\mathfrak{A}_5)/k(C/\mathfrak{A}_6)$, then $L = k(C)$.
2. Take an subgroup $\mathfrak{A}'_5$ of $\mathfrak{A}_6$ isomorphic to $\mathfrak{A}_5$. Then $\mathfrak{A}'_5$ is conjugate to $\mathfrak{A}_5$ as a subgroup of $\mathfrak{A}_6$ if and only if there exists an isomorphism $k(C/\mathfrak{A}'_5) \to k(C/\mathfrak{A}_5)$ over $k(C/\mathfrak{A}_6)$.

**Proof.**

1. Since $\mathfrak{A}_6$ is simple, the only normal subgroup of $\mathfrak{A}_6$ contained in $\mathfrak{A}_5$ is the trivial group 1.
2. If the permutation $\sigma \in \mathfrak{A}_6$ satisfies $\sigma^{-1} \mathfrak{A}_5 \sigma = \mathfrak{A}'_5$, then it corresponds an isomorphism $k(C/\mathfrak{A}'_5) \to k(C/\mathfrak{A}_5)$ over $k(C/\mathfrak{A}_6)$.

Conversely, let $\varphi$ be an isomorphism $k(C/\mathfrak{A}'_5) \to k(C/\mathfrak{A}_5)$ over $k(C/\mathfrak{A}_6)$. For any isomorphism $\tau : k(C) \to k(C)$ over $k(C/\mathfrak{A}_5)$, $\varphi^{-1} \circ \tau \circ \varphi$ is an isomorphism $k(C) \to k(C)$ over $k(C/\mathfrak{A}_5)$. Therefore, $\mathfrak{A}'_5$ is conjugate to $\mathfrak{A}_5$ as a subgroup of $\mathfrak{A}_6$. □

To study the curve $C$, we first describe the map $f : C/\mathfrak{A}_5 \to C/\mathfrak{A}_6$. We give the ramification indices at the ramification points of $\pi_G$.

**Notation 3.2.** For the morphism $\pi_G : C \to C/G$, let $P_1, \ldots, P_s \in C/G$ be the branched points and $r_i$ the ramification index over $P_i$. We may assume that $r_1 \leq \cdots \leq r_s$. Then we often write $(r_1, \cdots, r_s)$ as a tuple.

In general, the following proposition is a result of elementary calculations.

**Proposition 3.3.** Let $L := \frac{2g - 2}{|G|}$ for $g \geq 2$, $g'$ be the genus of $C/G$ and $s$ the number of branched points of $\pi_G$.

1. If $L < 1$, then one of the following holds:
   - (i) $g' = 1$, $s = 1$ and $r_1 = \frac{1}{1 - L}$.
   - (ii) $g' = 0$ and $3 \leq s \leq 5$.

   In particular, if $L < \frac{1}{2}$, then $g' = 0$ and $s = 3$ or 4.

2. If $L < \frac{1}{6}$, then $g' = 0$, $s = 3$ and the triple of the ramification indices is one of the following:
   - $(2, r_2, r_3)$ where $3 \leq r_2 \leq 5$ and $r_3 = \frac{1}{2 - L - \frac{1}{r_2}}$.
   - $(3, 3, 4)$. Then $L = \frac{1}{12}$.
   - $(3, 3, 5)$. Then $L = \frac{2}{15}$.
Proof. Note that $L > 0$ by the assumption $g \geq 2$. By Riemann-Hurwitz formula, we have

(I) \[ L = 2g' - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right). \]

(1) Assume $L < 1$. If $s = 0$, then $L = 2g' - 2$ is a positive integer, a contradiction. Thus $s > 0$. We determine $g'$ and $s$. By the equality (I), we see that

(II) \[ L - (2g' - 2) = \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right) = s - \sum_{i=1}^{s} \frac{1}{r_i}. \]

Since

(III) \[ \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right) \geq \frac{s}{2} \geq \frac{1}{2}, \]

we obtain $L \geq 2g' - \frac{3}{2}$. If $L < 1$, then $g' = 0$ or 1. If $L < \frac{1}{2}$, then $g' = 0$.

On the other hand,

(IV) \[ s - \sum_{i=1}^{s} \frac{1}{r_i} < s. \]

Hence, by the equality (II), the inequality (III) and (IV), we have

$L - (2g' - 2) < s \leq 2(L - (2g' - 2))$.

If $g' = 1$, then $L < s \leq 2L$. Since $L > 0$ and $L < 1$, we see $0 < s < 2$. Then $s = 1$, and $L = 1 - \frac{1}{r_1}$, i.e., (i) holds. If $g' = 0$, then $L + 2 < s \leq 2(L + 2)$. From $L < 1$, we see $2 < s < 6$, and (ii) holds. For $L < \frac{1}{2}$, we obtain $2 < s < 5$, i.e., $s = 3$ or 4.

(2) Assume that $L < \frac{1}{6}$. By the statement (1), $g' = 0$ and $s = 3$ or 4. We consider the ramification indices $(r_1, \ldots, r_s)$. By the equality (I), we have

(V) \[ \sum_{i=1}^{s} \frac{1}{r_i} = (s - 2) - L. \]

Suppose that $s = 3$. If $r_1 \geq 4$, then $\sum_{i=1}^{3} \frac{1}{r_i} \leq \frac{3}{4}$ since $r_i \leq r_1$ for any $i$ by the assumption of Notation 3.2. However, the right hand side of (V) is greater than $\frac{5}{6}$.

Thus $r_1 = 2$ or 3. Since $\frac{1}{r_2} + \frac{1}{r_3} = 1 - \frac{1}{r_1} - L$ and $r_2 \leq r_3$, we see $\frac{2}{r_2} \geq 1 - \frac{1}{r_1} - L$, and

$r_1 \leq r_2 \leq \frac{2}{1 - \frac{1}{r_1} - L}$.

Take $r_1 = 2$. If $r_2 = 2$, then $1 + \frac{1}{r_3} > 1 > 1 - L$, which contradicts the equality (V). Then $r_3 \geq 3$. Since $\frac{2}{1 - \frac{1}{r_1} - L} = \frac{2}{\frac{2}{3} - L} < 6$, we obtain $3 \leq r_2 \leq 5$. On the other hand, take $r_1 = 3$. Then $\frac{2}{1 - \frac{1}{r_1} - L} = \frac{2}{\frac{2}{3} - L} < 4$. Thus $r_2 = 3$. Since $\frac{1}{r_3} = \frac{1}{3} - L$ and $0 < L < \frac{1}{6}$, we see $r_3 = 4$ or 5.
Suppose that $s = 4$. By $0 < L < \frac{1}{6}$ and $\sum_{i=1}^{4} \frac{1}{r_i} = 2 - L$, we see $\frac{11}{6} < \sum_{i=1}^{4} \frac{1}{r_i} < 2$.

Since $(r_1, r_2, r_3, r_4) = (2, 2, 4, 2)$ does not satisfy the inequality, we have $\sum_{i=1}^{4} \frac{1}{r_i} \leq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$. However, this is a contradiction to the former inequality. Hence, there is no 4-tuple satisfying the equality (V) for $s = 4$. □

This proposition implies the following lemma.

**Lemma 3.4.** Let $C$ be a smooth curve of genus 10 with a faithful $\mathfrak{A}_6$-action.

1. $C/\mathfrak{A}_6$ is rational, and the ramification indices at the branched points of $\pi_{\mathfrak{A}_6}$ are $(2, 4, 5)$.
2. $C/\mathfrak{A}_5$ is rational, and the ramification indices at the branched points of $\pi_{\mathfrak{A}_5}$ are $(4, 4, 5)$ or $(2, 2, 2, 5)$.

**Proof.** (1) We see

$$L = 2 \cdot 10 - 2 - \frac{1}{|\mathfrak{A}_6|} = \frac{1}{20} < \frac{1}{6}.$$  

Hence, by Proposition 3.3 (2), the genus of $C/\mathfrak{A}_6$ is 0, i.e., $C/\mathfrak{A}_6$ is rational, and the number $s$ of branched points of $\pi_{\mathfrak{A}_6}$ is 3. Since $L \neq \frac{1}{12}$ and $L \neq \frac{2}{15}$, the ramification indices are $(2, r_2, r_3)$ with $3 \leq r_2 \leq 5$ and $r_3 = \frac{1}{20} - \frac{1}{r_2}$.

Since $r_3$ is an integer and $r_2 \leq r_3$, we have $r_2 = 4$ and $r_3 = 5$. Thus the ramification indices at the branched points of $\pi_{\mathfrak{A}_6}$ are $(2, 4, 5)$.

(2) We see

$$L = 2 \cdot 10 - 2 - \frac{3}{|\mathfrak{A}_5|} = \frac{3}{10} < \frac{1}{2}.$$  

By Proposition 3.3 (1), the genus of $C/\mathfrak{A}_5$ is 0, i.e., $C/\mathfrak{A}_5$ is rational, and the number $s$ of branched points of $\pi_{\mathfrak{A}_5}$ is 3 or 4. On the other hand, since we have a morphism $f : C/\mathfrak{A}_5 \rightarrow C/\mathfrak{A}_6$ with $\pi_{\mathfrak{A}_6} = f \circ \pi_{\mathfrak{A}_5}$, the ramification indices $r_i$ are divisors of 2, 4 or 5, i.e., $r_i = 2, 4$ or 5. We recall the equation (V) in the proof of Proposition 3.3:

$$\sum_{i=1}^{s} \frac{1}{r_i} = (s - 2) - \frac{3}{10}.$$  

Suppose that $s = 3$. Then

$$\sum_{i=1}^{3} \frac{1}{r_i} = \frac{7}{10}.$$  

If $r_1 = 2$, then $r_2 = 2, 4$ or 5 and we have $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} > \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$. This is a contradiction. Assume that $r_1 = 4$. Then $\frac{1}{r_2} + \frac{1}{r_3} = \frac{9}{20}$, and $\frac{20}{9} < r_2 \leq \frac{40}{9}$. Thus $(r_1, r_2, r_3) = (4, 4, 5)$. Assume that $r_1 = 5$. Then $r_1 = r_2 = r_3 = 5$, but then

$$\sum_{i=1}^{3} \frac{1}{r_i} = \frac{3}{5} \neq \frac{7}{10}.$$  

Suppose that $s = 4$. Then

$$\sum_{i=1}^{4} \frac{1}{r_i} = \frac{17}{10}.$$
If \( r_2 > 2 \), then \( r_2 \geq 4 \) and
\[
\sum_{i=1}^{4} \frac{1}{r_i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} < \frac{17}{10}.
\]
Thus \( r_1 = r_2 = 2 \).

We obtain \( \frac{1}{r_3} + \frac{1}{r_4} = \frac{7}{10} \), and \( \frac{10}{7} < r_3 \leq \frac{20}{7} \). Hence, \( r_3 = 2 \), and we see \( r_4 = 5 \).

Therefore, the tuple of ramification indices is \((2, 2, 2, 5)\).

By Lemma 3.4, \( f \) is a rational surjection \( \mathbb{P}^1 \to \mathbb{P}^1 \), and we can write it as a rational function.

**Lemma 3.5.** Let \( C \) be a smooth curve of genus 10 with a faithful \( A_6 \)-action. Take the coordinate on \( C/\mathbb{A}_6 \sim \mathbb{P}^1 \) for which \( 1 \in \mathbb{P}^1 \) is the branched point of index 2, \( 0 \in \mathbb{P}^1 \) is the branched point of index 4 and \( \infty \in \mathbb{P}^1 \) is the branched point of index 5. Then the following holds.

1. If the ramification indices at the branched points of \( \pi_{\mathbb{A}_5} \) are \((4, 4, 5)\), then we can write
\[
f(w) = \frac{w^4(w^2 + 4w + 20)}{256(w - 1)}
\]
in a suitable coordinate on \( C/\mathbb{A}_5 \).

2. If the ramification indices at the branched points of \( \pi_{\mathbb{A}_5} \) are \((2, 2, 2, 5)\), then we can write
\[
f(w) = \frac{\lambda_0 w^4((256\lambda_0 - 25)w - 50)^2}{2500(w - 1)}
\]
in a suitable coordinate on \( C/\mathbb{A}_5 \) where \( \lambda_0 \) is a solution of the quadratic equation
\[
442368\Lambda^2 - 72000\Lambda + 3125 = 0.
\]

**Remark 3.6.** The possibility of the form (1) is later excluded.

**Proof.** Let \( Q_1^{(r)}, Q_2^{(r)}, \cdots \) denote the branch points of \( \pi_{\mathbb{A}_5} \) of index \( r \). For each \( r \) and \( i \), the ramification index of \( \pi_{\mathbb{A}_5} \) at \( f(Q_1^{(r)}) \) is divisible by \( r \).

1. The branch points of \( \pi_{\mathbb{A}_5} \) are \( Q_1^{(4)}, Q_2^{(4)} \) and \( Q_1^{(5)} \). Since the only ramification index of \( \pi_{\mathbb{A}_5} \) divisible by 4 (resp. 5) is 4 (resp. 5), the map \( f \) sends \( Q_1^{(4)} \) and \( Q_2^{(4)} \) to 0 and \( Q_1^{(5)} \) to \( \infty \). Since \( \deg f = [k(C/\mathbb{A}_5) : k(C/\mathbb{A}_6)] = [\mathbb{A}_6 : \mathbb{A}_5] = 6 \), there are 1 ramified point of \( f \) of index 4 over \( 0 \in \mathbb{P}^1 \), 1 ramified point of index 5 over \( \infty \in \mathbb{P}^1 \) and 3 ramified points of index 2 over \( 1 \in \mathbb{P}^1 \). By taking a suitable projective transformation of \( C/\mathbb{A}_5 \sim \mathbb{P}^1 \), we may assume that \( f^{-1}(0) = \{0, Q_1^{(4)}, Q_2^{(4)}\} \) and \( f^{-1}(\infty) = \{\infty, Q_1^{(5)} = 1\} \). Then we can write
\[
f(w) = \frac{w^4(\kappa w^2 + \lambda w + \mu)}{w - 1}
\]
for some \( \kappa, \lambda, \mu \in \mathbb{C} \). On the other hand, the numerator of
\[
f(w) - 1 = \frac{\kappa w^6 + \lambda w^5 + \mu w^4 - w + 1}{w - 1}
\]
decomposes as \((\alpha w^3 + \beta w^2 + \gamma w + \delta)^2\) for \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) since \( f^{-1}(1) \) consists of 3 ramified points of index 2. By equating the corresponding coefficients of powers of
We see $\delta = \pm 1$ by the first equality, $\gamma = \mp \frac{1}{2}$ by the second equality, $\beta = \mp \frac{1}{8}$ by the third equality and $\alpha = \mp \frac{1}{16}$ by the fourth equality. Therefore, by the 5-th to 7-th equalities,

$$(\kappa, \lambda, \mu) = \left( \frac{1}{256}, \frac{1}{64}, \frac{5}{64} \right),$$

and $f(w)$ is given.

(2) The branch points of $\pi_{A_5}$ are $Q^{(2)}_1$, $Q^{(2)}_2$, $Q^{(2)}_3$ and $Q^{(5)}_1$. The map $f$ sends $Q^{(5)}_1$ to $\infty$ and $Q^{(2)}_i$ to 0 or 1 for each $i$. Thus $f$ has 1 ramified point of index 5 over $\infty$ and $Q^{(2)}_1$, $Q^{(2)}_2$ and $Q^{(2)}_3$ satisfy one of the following:

(i) All points are sent to 0.
(ii) 2 points are sent to 0 and 1 point is sent to 1.
(iii) 1 point is sent to 0 and 2 points are sent to 1.
(iv) All points are sent to 1.

If $Q^{(2)}_1$ is sent to 1, then it is an unramified point for $f$. Since any point which is sent to 1 and different from $Q^{(2)}_1$ is a ramified point of index 2 for $f$ and $\deg f = 6$ is even, (ii) and (iv) are impossible.

Assume the case (i). Then there are 3 ramified points $Q^{(2)}_1$, $Q^{(2)}_2$, $Q^{(2)}_3$ for $f$ of index 2 over 0 and 3 ramified points of index 2 over 1. By taking a suitable projective transformation on $C/\mathfrak{A}_5 \cong \mathbb{P}^1$, we may assume that $f^{-1}(0) = \{Q^{(2)}_1 = 0, Q^{(2)}_2, Q^{(2)}_3\}$ and $f^{-1}(\infty) = \{\infty, Q^{(5)}_1 = 1\}$. Then we can write

$$f(w) = \frac{w^2(\kappa w^2 + \lambda w + \mu)^2}{w - 1}$$

for some $\kappa, \lambda, \mu \in \mathbb{C}$. On the other hand, since $f^{-1}(1)$ consists 3 ramified point $Q^{(2)}_1$, $Q^{(2)}_2$, $Q^{(2)}_3$ of index 2, the numerator of

$$f(w) - 1 = \frac{\kappa^2 w^6 + 2\kappa \lambda w^5 + (2\kappa \mu + \lambda^2)w^4 + 2\lambda \mu w^3 + \mu^2 w^2 - w + 1}{w - 1}$$

decomposes as $(\alpha w^3 + \beta w^2 + \gamma w + \delta)^2$ for some $\alpha \neq 0, \beta, \gamma, \delta \in \mathbb{C}$. Then we may assume that $\alpha = \kappa$. We obtain

$$\begin{cases}
    1 & = \delta^2, \\
    -1 & = 2\gamma \delta, \\
    \mu^2 & = 2\beta \delta + \gamma^2, \\
    2\lambda \mu & = 2\alpha \delta + 2\beta \gamma, \\
    2\alpha \mu + \lambda^2 & = 2\alpha \gamma + \beta^2, \\
    2\alpha \lambda & = 2\alpha \beta,
\end{cases}$$
by equating the corresponding coefficients. Since we see $\delta = \pm 1$, $\gamma = \mp \frac{1}{2}$ and $\beta = \lambda$ by the first, second and sixth equations,

\[
\begin{align*}
\mu^2 &= \pm 2\beta + \frac{1}{4}, \\
2\beta\mu &= \pm 2\alpha \mp \beta, \\
2\alpha\mu + \beta^2 &= \mp \alpha + \beta^2.
\end{align*}
\]

By the last equation, we see $\alpha(2\mu \pm 1) = 0$. Since $\alpha \neq 0$, we have $\mu = \mp \frac{1}{2}$.

Thus $\beta = 0$ by the first equation, and $\alpha = 0$ by the second equation. This is a contradiction.

Therefore, we have (iii). Apart from $Q^{(2)}_0$, there is 1 ramified point of index 4 over 0 and there are 2 ramified points of index 2 over 1. By taking a suitable projective transformation on $C/\mathbb{A}_5 \cong \mathbb{P}^1$, we may assume that $f^{-1}(0) = \{0, Q^{(2)}_1\}$ and $f^{-1}(\infty) = \{\infty, Q^{(5)}_1\}$. Then we can write

\[
 f(w) = \frac{w^4(\kappa w + \lambda)^2}{w - 1}
\]

for $\kappa, \lambda \in \mathbb{C}$. On the other hand, since $f^{-1}(1)$ consists of $Q^{(2)}_2, Q^{(2)}_3$ and 2 ramified points of index 2, the numerator of

\[
 f(w) - 1 = \frac{\kappa^2 w^6 + 2\kappa\lambda w^5 + \lambda^2 w^4 - w + 1}{w - 1}
\]

decomposes as $(w^2 + \alpha w + \beta)^2(\gamma w^2 + \delta w + \epsilon)$ for some $\alpha \neq 0$, $\beta, \gamma, \delta, \epsilon \in \mathbb{C}$. By equating the corresponding coefficients for powers of $w$,

\[
\begin{align*}
1 &= \beta^2 \epsilon, \\
-1 &= 2\alpha\beta \epsilon + \beta^2 \delta, \\
0 &= (\alpha^2 + 2\beta) \epsilon + 2\alpha\beta \delta + \beta^2 \gamma, \\
0 &= 2\alpha \epsilon + (\alpha^2 + 2\beta) \delta + 2\alpha \beta \gamma, \\
\lambda^2 &= \epsilon + 2\alpha \delta + (\alpha^2 + 2\beta) \gamma, \\
2\kappa \lambda &= \delta + 2\alpha \gamma, \\
\kappa^2 &= \gamma.
\end{align*}
\]

We solve this system of equations with SINGULAR. We put $a = \alpha$, $b = \beta$, $c = \gamma$, $d = \delta$, $e = \epsilon$, $k = \kappa$ and $l = \lambda$. Let $I$ be the ideal corresponding to (*) We check the elimination ideal $\mathcal{E}$ of $I$.

```plaintext
ring R = 0,(a,b,c,d,e,k,l),dp;
poly p0 = b2e - 1;
poly p1 = 2abe + b2d + 1;
poly p2 = a2e + 2be + 2abd + b2c;
poly p3 = 2ae + a2d + 2bd + 2abc;
poly p4 = e + 2ad + a2c + 2bc - 12;
poly p5 = d + 2ac - 2kl;
poly p6 = c - k2;
ideal I = p0,p1,p2,p3,p4,p5,p6;
//Eliminate a,b,c,d and e from I.
//Take E to be this elimination ideal.
ideal E = eliminate(I,abcde);
E;
```

The code returns generators of $\mathcal{E}$.  


EI \[1\]=3456kl +1152l^2 -125

EI \[2\]=54k^2 -9kl + l^2

EI \[3\]=256l^3 +50k -25l

By this third line, the pair \((κ,λ)\) satisfies 50κ = −λ(256λ^2 - 25). Thus we can write

\[ f(w) = \frac{λ^2w^4((256λ^2 - 25)w - 50)^2}{2500(w - 1)} \]

Finally, we eliminate \(k\) from EI to give the value of \(λ^2\).

The line returns

442368λ^4 - 72000λ^2 + 3125.

This means that λ^2 is a root of 442368Λ^2 - 72000Λ + 3125. Therefore, we have \(f(w)\) as in the statement (2).

Now we exclude the possibility of Lemma 3.5 (1) by considering the Galois closure of the corresponding extension \(M/K\).

Lemma 3.7. Let \(φ\) be a surjective morphism \(\mathbb{P}^1 → \mathbb{P}^1\) given by

\[ φ(w) = \frac{w^4(w^2 + 4w + 20)}{256(w - 1)} \]

\(M/K\) the corresponding field extension and \(L\) its Galois closure. Then \(\text{Gal}(L/K) \neq \mathfrak{S}_6\).

Proof. Since the fields \(M\) and \(K\) correspond to \(\mathbb{P}^1\), we may assume that \(M = \mathbb{C}(w)\) and \(K = \mathbb{C}(t)\) where \(w\) and \(t\) are transcendental over \(\mathbb{C}\). Then the pull back \(φ^*\) is an injective homomorphism \(\mathbb{C}(t) → \mathbb{C}(w)\) given by \(φ(w) = t\). Thus we obtain

\[ p(w) := w^6 + 4w^5 + 20w^4 + (-256t)w + (256t) = 0. \]

Suppose that the factorization of \(p(w)\) over \(L\) is \((w - φ_1)\cdots(w - φ_6)\). The Galois group \(\text{Gal}(L/K)\) is a subgroup of the permutation group of \(φ_1,\cdots, φ_6\). We will give an intermediate field \(F\) of \(L/K\) which has degree 2 over \(K\). We can check that the discriminant \(Δ\) of \(p(w)\) is

\[ Δ = R(p, \frac{∂p}{∂w}) \]

\[
\begin{vmatrix}
1 & 4 & 20 & 0 & 0 & -256t & 256t & 0 & 0 & 0 & 0 \\
0 & 1 & 4 & 20 & 0 & 0 & -256t & 256t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 20 & 0 & 0 & -256t & 256t \\
6 & 20 & 80 & 0 & 0 & 0 & -256t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 20 & 80 & 0 & 0 & -256t \\
\end{vmatrix}
\]

\[ = -87960930222080000t^3 + 263882790666240000t^4 \\
-263882790666240000t^5 + 87960930222080000t^6. \]

Let \(ψ := \prod_{i<j}(φ_i - φ_j) ∈ L\), and then \(ψ^2 = Δ\). Since the order of \(Δ\) in \(t\) is 3, which is odd, we have \(ψ \not∈ K\), and \(F := K(ψ)\) is an extension field of degree 2 over \(K\). Therefore, the Galois group \(\text{Gal}(L/F)\) is a subgroup of \(\text{Gal}(L/K)\) of index 2. Since \(\mathfrak{S}_6\) has no subgroup of index 2, \(\text{Gal}(L/K)\) is not isomorphic to \(\mathfrak{S}_6\). □
Theorem 3.8. Let $C$ be a smooth projective curve of genus 10 with a faithful $A_6$-action. Then the morphism $f : C/A_5 \to C/A_6$ can be written as

$$f(w) = \frac{\lambda_0 w^4((256\lambda_0 - 25)w - 50)^2}{2500(w - 1)}$$

in a suitable coordinate on $C/A_5$ and $C/A_6$ where $\lambda_0$ is a solution of the quadratic equation $442368\Lambda^2 - 72000\Lambda + 3125 = 0$.

In particular, an $A_6$-invariant curve of genus 10 is unique up to isomorphism.

Proof. By Lemma 3.5 and Lemma 3.7, for non-conjugate icosahedral subgroups $A_5$ and $A'_5$, the corresponding morphism $f : C/A_5 \to \mathbb{P}^1$ and $f' : C/A'_5 \to \mathbb{P}^1$ are written in the above form. (Note that the base $\mathbb{P}^1$'s can be identified since the branched points are same.) Since $f$ and $f'$ are not conjugate by Lemma 3.1(2), they have to correspond to the 2 choice for $\lambda_0$. Thus the Galois closure gives $C$ for any of $\lambda_0$. □

4. The $A_6$-Invariant Curve of Genus 19

In this section, let $C$ be an $A_6$-invariant curve of genus 19. (Note that Lemma 3.1 and Proposition 3.3 also hold in the case of genus 19.) Fix an icosahedral group $A_5 < A_6$. Recall that $\pi_G$ is the natural morphism $C \to C/G$ for $G = A_6$ or $A_5$.

Lemma 4.1. Let $C$ be a smooth curve of genus 19 with a faithful $A_6$-action.

(1) $C/A_6$ is rational, and the ramification indices at the branched points of $\pi_{A_6}$ are $(2, 5, 5)$. 
(2) $C/A_5$ is rational, and the ramification indices at the branched points of $\pi_{A_5}$ are $(2, 2, 5, 5)$.

Proof. (1) We see

$$L = \frac{2 \cdot 19 - 2}{|A_6|} = \frac{1}{10} < \frac{1}{6},$$

and $L \neq \frac{1}{12}$ and $L \neq \frac{2}{15}$. By the Proposition 3.3 (2), $C/A_6$ is rational and the ramification indices is $(2, r_2, r_3)$ with $3 \leq r_2 \leq 5$ and

$$r_3 = \frac{\frac{2}{5} - \frac{1}{r_2}}{r_2}.$$

Since $r_3$ is an integer and $r_2 \leq r_3$, we have $(r_2, r_3) = (3, 15)$ or $(5, 5)$. However, $A_6$ has no subgroup of order 15. Thus the ramification indices of $\pi_{A_6}$ are $(2, 5, 5)$.

(2) We see

$$L = \frac{2 \cdot 19 - 2}{|A_5|} = \frac{3}{5} < 1.$$

By the Proposition 3.3 (1), the genus $g$ of $C/A_5$ and the number $s$ of the branched points of $A_5$ satisfy one of the following:

(i) $g = 1$ and $s = 1$, 
(ii) $g = 0$ and $3 \leq s \leq 5$.

If the condition (i) holds, then the ramification index is

$$r_1 = \frac{1}{1 - \frac{3}{5}} = \frac{5}{2}.$$
However, this is not an integer. Hence, we have (ii) and \( C/A \) is rational. We recall the equation (V) in the proof of the Proposition 3.3:

\[
\sum_{i=1}^{s} \frac{1}{r_i} = (s - 2) - \frac{3}{5}.
\]

On the other hand, since we have a morphism \( f : C/A \to C/A \) with \( \pi_{A} = f \circ \pi_{A} \), the ramification indices \( r_i \) are divisors of 2 or 5, i.e., \( r_i = 2 \) or 5. Thus there is a positive integer \( k \) such that

\[
\sum_{i=1}^{s} \frac{1}{r_i} = \frac{1}{2}k + \frac{1}{5}(s - k).
\]

Since this is equal to \((s - 2) - \frac{3}{5}\), we see \( k = \frac{1}{3}(8s - 26) \). If \( s = 3 \) or 5, then \( k \) is not an integer. Therefore, \( s = 4 \) and \((r_1, r_2, r_3, r_4) = (2, 2, 5, 5)\).

By Lemma 4.1 (1) and (2), \( C/A \) and \( C/A \) are rational. Thus the corresponding morphism \( f : C/A \to C/A \) can be identified with a surjective rational map \( P^1 \to P^1 \).

**Lemma 4.2.** Let \( C \) be a smooth curve of genus 19 with a faithful \( A \) -action. Take the coordinate on \( C/A \) for which 1 is the branched point of index 2 and 0 and \( \infty \) are two branched points of index 5. In a suitable coordinate, the following holds:

1. \( f(w) = \frac{w^5(w + 4)}{64(w - 1)} \),
2. \( f(w) = \frac{w^5(-783\lambda_0 + 64)w + 972\lambda_0}{972(w - 1)} \), where \( \lambda_0 \) is a solution of the quadratic equation \( 11664\Lambda^2 - 1647\Lambda + 64 = 0 \).

**Remark 4.3.** Similar to Section 3, the possibility of the form (1) is later excluded.

**Proof.** Let \( Q_1^{(r)}, Q_2^{(r)}, \cdots \) denote the branch points of \( \pi_{A} \) of index \( r \). We note that the ramification index of \( \pi_{A} \) at \( f(Q_1^{(r)}) \) is divisible by \( r \) for each \( r \) and \( i \). The branch points of \( \pi_{A} \) are \( Q_1^{(2)}, Q_2^{(2)}, Q_1^{(5)} \) and \( Q_2^{(5)} \) by Lemma 4.1 and the map \( f \) sends \( Q_1^{(2)} \) to 1 for any \( i \) and \( Q_1^{(5)} \) to 0 or \( \infty \) for each \( i \).

Assume that both \( Q_1^{(5)} \) and \( Q_2^{(5)} \) are sent to 0 (resp. \( \infty \)). They are unramified points for \( f \). Thus \( f^{-1}(0) \) (resp. \( f^{-1}(\infty) \)) has another point \( Q \). Since the ramification index at \( Q \) is equal to the ramification index of 0 (resp. \( \infty \)), i.e., \( 5 \), we see \( \deg f^*(0) \geq 7 \) (resp. \( \deg f^*(\infty) \geq 7 \)). However, \( \deg f = \left[ A_6 : A_5 \right] = 6 \). This is a contradiction.

By taking a suitable projective transformation of \( P^1 \), we may assume that \( f \) sends \( Q_1^{(5)} = 0 \to 0 \), \( Q_2^{(5)} = \infty \to 0 \) and 1 to \( \infty \). Then we can write

\[
f(w) = \frac{w^5(\kappa w + \lambda)}{w - 1} \]

where \( \kappa, \lambda \in \mathbb{C} \) and \( \kappa \neq 0 \). On the other hand, since there are 2 ramified points of \( f \) of index 2 at 1, the numerator of

\[
f(w) - 1 = \frac{\kappa w^6 + \lambda w^5 - w + 1}{w - 1}
\]
decomposes as \((w^2 + \alpha w + \beta)^2(\kappa w^2 + \gamma w + \delta)\) for \(\alpha \neq 0, \beta, \gamma, \delta \in \mathbb{C}\). By equating the corresponding coefficients for \(w\),

\[
\begin{align*}
1 &= \beta^2 \delta, \\
-1 &= 2\alpha\beta\delta + \beta^2 \gamma, \\
0 &= (\alpha^2 + 2\beta)\delta + 2\alpha\beta\gamma + \beta^2 \kappa, \\
0 &= 2\alpha\delta + (\alpha^2 + 2\beta)\gamma + 2\alpha\beta\kappa, \\
0 &= \delta + 2\alpha\gamma + (\alpha^2 + 2\beta)\kappa, \\
\lambda &= \gamma + 2\alpha\kappa.
\end{align*}
\]

We use SINGULAR to solve this system of equations. We put \(a = \alpha, b = \beta, c = \gamma, d = \delta, k = \kappa\) and \(l = \lambda\). Let \(I\) be the ideal corresponding to \((*)\). We check the elimination ideal \(EI\) of \(I\).

```plaintext
ring R = 0,(a,b,c,d,k,l),dp;
poly p0 = b2d - 1;
poly p1 = 2 abd + b2c + 1;
poly p2 = a2d + 2bd + 2abc + b2k;
poly p3 = 2ad + a2c + 2bc + 2abk;
poly p4 = d + 2ac + a2k + 2bk;
poly p5 = c + 2ak - 1;
ideal I = p0,p1,p2,p3,p4,p5;

// Eliminate a,b,c and d from I.
// Take EI to be this elimination ideal.
ideal EI = eliminate(I,abcd);
EI;
```

This code returns generators of \(EI\).

\[
\begin{align*}
EI[1] &= 216 l^2 - 954 k - 799 l + 64 \\
EI[2] &= 46656 kl - 864 l^2 + 166896 k + 136801 l - 11200 \\
EI[3] &= 3024 k^2 + 2448 kl - l^2 - 200 k
\end{align*}
\]

Its first generator gives the equation

\((**\)) \quad 216\lambda^2 - 799\lambda + 64 = 954\kappa.

Furthermore, eliminate \(k\) from \(EI\).

```plaintext
ideal J = eliminate(EI,k);
J;
factorize(J[1]);
```

The code returns the polynomial of \(\lambda\)

\[
186624\lambda^3 - 38016\lambda^2 + 2671\lambda - 64,
\]

and it is factored into \((16\lambda - 1)(11664\lambda^2 - 1647\lambda + 64)\). Therefore, \(\lambda = \frac{1}{16}\) or a solution of \(11664\lambda^2 - 1647\lambda + 64 = 0\). By the equality \((**\)) if \(\lambda = \frac{1}{16}\), then \(\kappa = \frac{1}{64}\). Otherwise, we see

\[
\kappa = \frac{1}{954} (\frac{1537}{2}\lambda + \frac{1696}{27}) = \frac{-783\lambda + 64}{972}.
\]

\[\square\]

**Remark 4.4.** For the rational function \(f(w) = \frac{w^5(\kappa w + \lambda)}{w - 1}\), we calculate

\[
f\left(\frac{-\lambda}{\kappa w}\right) = \left(\frac{\lambda}{\kappa w}\right)^5 \left(\frac{\lambda^2}{\kappa w} + \frac{\lambda}{\kappa w}\right) = \frac{\lambda^6(w - 1)}{\kappa^4 w^3(\kappa w + \lambda)} = \frac{\lambda^6}{\kappa^4} \frac{1}{f(w)}.
\]
Since \( \frac{\lambda^3}{\kappa^2} = -1 \) for \((\kappa, \lambda) = \left( \frac{-783\lambda_0 + 64}{972}, \lambda_0 \right)\), we have \( f \left( -\frac{\lambda}{\kappa w} \right) = \frac{1}{f(w)} \).

Now we exclude the possibility of Lemma 4.2 (1) by considering the Galois closure of the corresponding extension \( M/K \).

**Lemma 4.5.** Let \( \varphi \) be a surjective morphism \( \mathbb{P}^1 \to \mathbb{P}^1 \) given by

\[
\varphi(w) = \frac{w^5(w+4)}{64(w-1)}. 
\]

\( M/K \) the corresponding field extension and \( L \) its Galois closure. Then \( \text{Gal}(L/K) \not\cong \mathfrak{A}_6 \).

**Remark 4.6.** Since the discriminant of \( p(w) \) is square, we can not prove this in the same way as in Lemma 4.7. Here we use the *resolvent* \( f_{15}(x) \) used in [5, Section 3].

Take a polynomial \( \omega_0 \in K[p_1, \cdots, p_6] \). Under the action of \( S_6 \) on \( p_1, \cdots, p_6 \), let \( G \) be the stabilizer subgroup \( \text{Stab}(\omega_0) \). Let \( \Omega \) be the orbit of \( \omega_0 \) under the action \( S_6 \). Then the polynomial \( f(x) = \prod_{\omega \in \Omega} (x - \omega) \) is called a resolvent for \( G \). For example, \( f_2(x) = x^2 - \Delta \) where \( \Delta \) is the discriminant is a resolvent for \( \mathfrak{A}_6 \).

**Proof.** Since the fields \( M \) and \( K \) correspond to \( \mathbb{P}^1 \), we can write \( M = \mathbb{C}(w) \) and \( K = \mathbb{C}(t) \) where \( w \) and \( t \) are transcendental over \( \mathbb{C} \). The pull back \( \varphi^* \) is an injective homomorphism \( \mathbb{C}(t) \to \mathbb{C}(w) \) given by \( t = \varphi(w) \). Thus we see

\[
p(w) := w^6 + 4w^5 - 64tw + 64t = 0.
\]

Assume that the factorization of \( p(w) \) over \( L \) is \((w - p_1) \cdots (w - p_6)\). Let \( S_i \) be the elementary symmetric polynomial of degree \( i \) for \( p_1, \cdots, p_6 \). Then their values are

\[
\begin{align*}
S_1 &= -4, \\
S_2 &= 0, \\
S_3 &= 0, \\
S_4 &= 0, \\
S_5 &= 64t, \\
S_6 &= 64t.
\end{align*}
\]

The Galois group \( \text{Gal}(L/K) \) is a subgroup of the permutation group of \( p_1, \cdots, p_6 \), which we identify with \( S_6 \).

Let \( \omega_0 = p_1p_2 + p_3p_4 + p_5p_6 \). Then the orbit \( \Omega = \mathfrak{S}_6 \cdot \omega_0 \) is of order 15, and \( \mathfrak{A}_6 \) acts transitively on \( \Omega \). To see this, note that there is a permutation \( \sigma \in \mathfrak{S}_6 \) with \( \sigma \cdot \omega_0 = \omega \) for any \( \omega \in \Omega \). If \( \sigma \) is even, then \( \sigma \in \mathfrak{A}_6 \). Suppose that \( \sigma \) is odd. Since \((12) \cdot \omega_0 = \omega_0 \), we obtain \( (\sigma(12)) \cdot \omega_0 = \sigma \cdot ((12) \cdot \omega_0) = \sigma \cdot \omega_0 = \omega \). Thus \( \sigma(12) \) is an even permutation which sends \( \omega_0 \) to \( \omega \).

Assume that \( \text{Gal}(L/K) = \mathfrak{A}_6 \). We consider the polynomial of degree 15

\[
f_{15}(x) = \prod_{\omega \in \Omega} (x - \omega(p_1, \cdots, p_6)) = \sum_{i=0}^{15} a_i x^{15-i} \in (\mathbb{C}(t))[x]
\]

which decomposes over an intermediate field \( N := K(\{\omega(p_1, \cdots, p_6) \mid \omega \in \Omega\}) \) of \( L/K \). Then, since \( \mathfrak{A}_6 \) acts transitively on \( \Omega \), it also acts transitively on the roots of \( f_{15}(x) \). The polynomial \( f_{15}(x) \) must be a power of an irreducible polynomial over \( K \).

We calculate \( f_{15}(x) \) from the coefficients of \( p(w) \) with SINGULAR. We put \( x(1), \cdots, x(6) \) to be variables corresponding to \( p_1, \cdots, p_6 \). First, we give the set \( \Omega \) as \( \text{Omg} \) with the following code.
LIB "sets.lib";
LIB "ellipticcovers.lib";

ring R1 = 0,r(1..6),dp;

list S6 = permute(list(1,2,3,4,5,6));  // S6 is the list of permutations of 1, 2, 3, 4, 5 and 6.

poly omg0 = r(1)*r(2)+r(3)*r(4)+r(5)*r(6);
Set Omg = list();
for (int i = 1;i<=720;i=i+1)
{
  map sigma = R1,r(S6[i][1]),r(S6[i][2]),r(S6[i][3]),r(S6[i][4]),
  r(S6[i][5]),r(S6[i][6]);
  Omg = addElement(Omg, sigma(omg0));
}

// Put Omglist be a list of elements in Omg.
list Omglist = Omg.elements;

Next, we calculate the polynomial \( f_{15}(x) \). We take \( S[i] \) to be \( S_i \) and \( I \) be the ideal generated by the polynomials corresponding to \( \ast \) and \( f_{15}(x) \). Then we take \( EI \) to be the ideal obtained by eliminating \( r(1), \cdots, r(6) \) from \( I \).

LIB "poly.lib";
ring R2 = 0,(r(1..6),t,x),dp;
map phi = R1,r(1),r(2),r(3),r(4),r(5),r(6);

c = 0;
for(int j=1;j<=15;j=j+1)
{
  f15 = f15*(x - phi(Omglist)[j]);
}

ideal S = elemSymmId(6);
ideal I = S[1]+4,S[2],S[3],S[4],S[5]-64t,S[6]-64t,f15;
ideal EI = eliminate(1,r(1)*r(2)*r(3)*r(4)*r(5)*r(6));

EI;
factorize(EI[1]);
The code returns the polynomial of degree 15 and its factorization
\[ f_{15}(x) = x^{15} - 4480t x^{12} - 86016t^2x^{10} - 20480t^3x^{11} + 583680t^4x^9 - 49152t^5x^8 + 125829120t^6x^7 + 47513600t^8x^5 - 5922357248t^9x^5 - 2495610880t^{10}x^6 + 199229440t^2x^7 - 93952409600t^4x^4 - 23395827712t^5x^4 + 671088640t^6x^6 + 1417339207680t^7x^3 - 187904819200t^8x^3 - 104018739200t^9x^3 + 805306368t^2x^5 + 219902325552t^6 + 15375982919680t^8x^4 - 10485760t^3x^2 + 4915200t^2x^3 - 16777216t^4 - 115343360t^5x^3 + 5242880t^2x^2 - 213909504t^3 + 83886080t^2x + 268435456t^2. \]

Therefore, the polynomial \( f_{15}(x) \) is not a power of an irreducible polynomial over \( K \), and \( \text{Gal}(L/K) \neq \mathfrak{A}_6 \). \( \square \)

**Theorem 4.7.** Let \( C \) be a smooth curve of genus 19 with a faithful \( \mathfrak{A}_6 \)-action. Then \( C \) is isomorphic to the Galois closure curve of the morphism \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) defined by
\[ f(w) = \frac{w^5((-783\lambda_0 + 64)w + 972\lambda_0)}{972(w - 1)} \]
where \( \lambda_0 \) is a solution of the quadratic equation \( 11664\Lambda^2 - 1647\Lambda + 64 = 0 \).

The two solutions give rise to isomorphic Galois closure curves. In particular, a smooth \( \mathfrak{A}_6 \)-invariant curve of genus 19 is unique up to isomorphism.

**Proof.** The first statement follows from Lemma 4.2 and Lemma 4.5. To show that the two value of \( \lambda_0 \) give rise to isomorphic Galois closure curves, we proceed as follows. Let us fix a curve \( C \) and an action of \( \mathfrak{A}_6 \) on \( C \), and take \( \mathfrak{A}_5 \) and \( \mathfrak{A}_5' \) to be non-conjugate icosahedral subgroups. It suffices to show that they correspond to different values of \( \lambda_0 \).

Assume to the contrary that they correspond to the same value of \( \lambda_0 \) and write \( f : C/\mathfrak{A}_5 \to C/\mathfrak{A}_6 \) and \( f' : C/\mathfrak{A}_5' \to C/\mathfrak{A}_6 \) for the natural morphism. Then there are two isomorphisms \( \alpha : C/\mathfrak{A}_6 \to \mathbb{P}^1 \) and \( \beta : C/\mathfrak{A}_5 \to \mathbb{P}^1 \) with \( f = \alpha^{-1} \circ f_{\lambda_0} \circ \beta \) where \( f_{\lambda_0} \) is rational function define as \( f_{\lambda_0}(w) = \frac{w^5((-783\lambda_0 + 64)w + 972\lambda_0)}{972(w - 1)} \).

Then \( \alpha \) sends each 2 branched points of the ramification index 5 to 0 and \( \infty \) and the branching point of the ramification index 2 to 1 \( \in \mathbb{P}^1 \). Similarly, there are two isomorphisms \( \alpha' : C/\mathfrak{A}_6 \to \mathbb{P}^1 \) and \( \beta' : C/\mathfrak{A}_5 \to \mathbb{P}^1 \) with \( f' = \alpha'^{-1} \circ f_{\lambda_0} \circ \beta' \). Then the projective transformation \( \iota := \alpha \circ (\alpha')^{-1} \) on \( \mathbb{P}^1 \) permutes 0 and \( \infty \) and fixes 1 \( \in \mathbb{P}^1 \). Hence, \( \iota(z) = z \), or \( \frac{1}{z} \). We claim that there exists a projective transformation \( \eta \) on \( \mathbb{P}^1 \) with \( f_{\lambda_0} = \iota^{-1} \circ f_{\lambda_0} \circ \eta \). If \( \iota = \text{id}_{\mathbb{P}^1} \), then we take \( \eta = \text{id}_{\mathbb{P}^1} \). On the other hand, if \( \iota(z) = \frac{1}{z} \), noting that we have \( f_{\lambda_0} \left( \frac{972\lambda_0}{(783\lambda_0 - 64)} \right) = \frac{1}{f_{\lambda_0}(w)} \) by Remark.
then we take \( \eta(w) = \frac{972\lambda_0}{(783\lambda_0 - 64)w} \).

Then we see that \( \beta'^{-1} \circ \eta \circ \beta \) is an isomorphism \( C/\mathcal{A}_5 \to C/\mathcal{A}'_5 \) and \( f' \circ (\beta'^{-1} \circ \eta \circ \beta) = f \), i.e., there is an isomorphism \( k(C/\mathcal{A}'_5) \to k(C/\mathcal{A}_5) \) over \( k(C/\mathcal{A}_6) \). By Lemma 3.1(2), the subgroup \( \mathcal{A}'_5 \) is conjugate to \( \mathcal{A}_5 \), which is a contradiction. Therefore, the morphism \( f \) corresponds to \( \lambda_0 \) or \( \lambda_0 \) and the morphism \( f' \) corresponds to the other one. \( \square \)

References

[1] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series 280, Cambridge University Press, Cambridge, 2000.

[2] A. Broughton, T. Shaska and A. Wootton, *On automorphisms of algebraic curves*, Contemp. Math. 724 (2019), Algebraic curves and their applications, 175–212.

[3] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönenmann, H.: *SINGULAR 4-2-1* — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2021).

[4] T. Harui, *Automorphism groups of smooth plane curves*, Kodai Math. J. 42 (2019), no. 2, 308–331.

[5] T. R. Hagedorn, *General formulas for solving solvable sextic equations*, Journal of Algebra 233 (2000), no. 2, 704–757.

[6] A. Wiman, *Ueber eine einfache Gruppe von 360 ebenen Collineationen*, Mathematische Annalen 45 (1895), 531–556.

[7] Y. Yoshida, *Projective plane curves whose automorphism groups are simple and primitive*, Kodai Math. J. 44 (2021), no. 2, 334–368.