Natural linearization for corrosion identification

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Abstract. We consider the numerical solution of a nonlinear problem arising in nondestructive detection of a corrosion at a hidden surface of a conductor. The problem can be naturally linearized and reduced to an elliptic Cauchy problem. In this paper we describe and test a regularized reconstruction algorithm based on a regularization by discretization, where the discretization level is chosen in data driven way.

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1. Introduction

The aim of the corrosion detection is to determine a possible presence of a corrosion damage by performing current and voltage measurements on the boundary. This means to apply a current of a density \( g \) on an accessible portion of the boundary of the conductor and to measure the corresponding voltage potential \( u \) on the same portion.

This physical problem is modeled as follows. A regular bounded domain \( \Omega \) represents the region occupied by the electrostatic conductor which contains no electric sources, and this is modeled by the Laplace equation, so that the voltage potential \( u \) satisfies

\[
\Delta u = 0 \text{ in } \Omega. \tag{1}
\]

We assume that the boundary \( \partial \Omega \) of the conductor is decomposed into three open nonempty and disjoint portions \( \Gamma_c, \Gamma_{un}, \Gamma_D \). The portion \( \Gamma_c \) is accessible to the electrostatic conductor which contains no electric sources, whereas the portion \( \Gamma_{un} \), where the corrosion may take place, is out of reach. The remaining portion \( \Gamma_D \) is assumed to be grounded, which means that the voltage potential \( u \) vanishes there, i.e.

\[
u = 0 \text{ on } \Gamma_D. \tag{2}\]

We impose a current of a density \( g \) on \( \Gamma_c \) and measure the corresponding potential \( h \) upon the same portion of the boundary.

By the pair of boundary measurements

\[
\begin{align*}
u &= h &\text{on } \Gamma_c, \\
\frac{\partial u}{\partial \nu} &= g &\text{on } \Gamma_c
\end{align*} \tag{3}
\]
we want to recover a relationship between voltage $u$ and current density $\frac{\partial u}{\partial \nu}$ on the inaccessible boundary portion $\Gamma_{un}$ suspected to be corroded.

A simple model of corrosion appearance consists in finding a coefficient $\gamma = \gamma(x)$ in a linear boundary condition of the type

$$\frac{\partial u}{\partial \nu} = -\gamma u \quad \text{on} \quad \Gamma_{un},$$

(4)

where $\nu$ is the outward unit normal at the boundary, and $\gamma \geq 0$ is the so-called Robin coefficient. The study of such a problem has been developed by many authors, among them, let us mention Fasino, Inglese [13, 14], Kaup, Santosa, Vogelius [16, 17], Alessandrini, Del Piero, Rondi [1], Chaabane, Fellah, Jaoua, Leblond [7], and [20].

A more accurate treatment of corrosion requires a nonlinear relationship between voltage and current density on the corroded surface. A model of this kind is associated to the name of Butler and Volmer and postulates the boundary condition

$$\frac{\partial u}{\partial \nu} = \lambda(\exp(\alpha u) - \exp(-(1-\alpha)u)) \quad \text{on} \quad \Gamma_{un},$$

(5)

This nonlinear boundary value problem, has been recently discussed by Bryan, Kavian, Vogelius and Xu in [5, 18, 21]. The authors have examined the questions of the existence and the uniqueness of the solution of the problem with a given nonlinearity of the type (5). Namely, they have assumed to know the nonlinearity (5) by prescribing the coefficients $\lambda$ and $\alpha$ in suitable ranges, and they have discussed the existence and the uniqueness issues for the direct problem.

Motivated by these studies, in [2, 3, 19] Alessandrini and the third author have also considered a more general choice of the nonlinear profile, namely of the form

$$\frac{\partial u}{\partial \nu} = f(u) \quad \text{on} \quad \Gamma_{un},$$

(6)

and they have dealt with the inverse problem of a reconstruction of the nonlinear profile $f$ from boundary measurements at $\Gamma_c$, and proved a logarithmic stability estimate for this problem. They have also observed that the uniqueness of the solution of this inverse problem is, in fact, a consequence of the uniqueness of corresponding Cauchy problem. See also [10] for a related work.

To the best of our knowledge there are only a few papers devoted to a numerical study of parameter identification problem in nonlinear model of the corrosion detection. In [15] only the case of $\Omega = (0,1) \times (0,a), a \ll 1$, has been discussed. In [22] the method from [6] for solving intermediate Cauchy problem has been employed. This method requires a sufficiently smooth domain boundary $\partial \Omega$. Moreover, only its convergence for vanishing noise level has been proven.

At the same time, a rather general method for solving elliptic Cauchy problem has been proposed recently in [9]. In this method a Cauchy problem is regularized by discretization, and the discretization level is chosen in data driven way. Under rather mild assumption an order-optimality of the method has been proven in [9]. In the present paper our goal is to adjust the method [9] for solving parameter identification problem in nonlinear model of the corrosion detection (1) - (3), (6). We also present numerical experiments demonstrating the performance of the adjusted method.

2. Natural linearization

It is easy to realize that the reconstruction of a nonlinearity $f$ in (6) can be achieved in two steps. First, an elliptic Cauchy problem (1)-(3) can be reformulated as a linear operator equation with
a compact operator. Then, using an approximate solution of the problem (1)-(3) one is able to find its Dirichlet and Neumann traces at \( \Gamma_{un} \), and reconstruct \( f \) from (6) pointwise.

There are several ways to reduce the Cauchy problem to a linear operator equation. Here we follow the way leading to an equation with a compact operator for unknown Neumann trace \( s(x,y) \) of the solution \( u(x,y) \) at the part \( \Gamma_{un} \) of the boundary where no data was prescribed. Namely, we introduce a linear continuous operator \( A : L^2(\Gamma_{un}) \rightarrow L^2(\Gamma_c) \) that assigns to \( s \in L^2(\Gamma_{un}) \) a Dirichlet trace of the (weak) solution \( u_s \) of the mixed boundary value problem

\[
\begin{cases} 
\Delta u = 0, & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = s, & \text{on } \Gamma_{un} \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_c \\
u = 0 & \text{in } \Gamma_D
\end{cases}
\]

at \( \Gamma_c \), i.e. \( As = u_s|_{\Gamma_c} \).

Then we consider a function \( u_0 \) solving an auxiliary mixed boundary value problem

\[
\begin{cases} 
\Delta u = 0, & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_{un} \\
\frac{\partial u}{\partial \nu} = g, & \text{on } \Gamma_c \\
u = 0 & \text{in } \Gamma_D
\end{cases}
\]

If \( s \) in (7) is such that \( u_s|_{\Gamma_c} + u_0|_{\Gamma_c} = h \) then in view of uniqueness results the function \( u_s + u_0 \) solves the Cauchy problem (1)-(3). Therefore, unknown Neumann trace \( s \) can be found from the operator equation

\[ As = r, \quad r = h - u_0|_{\Gamma_c}. \]

If only noisy measurements \( g^\delta, h^\delta \) are available, then one can construct a noisy version of (9)

\[ As = r^\delta, \quad r^\delta = h^\delta - u_0^\delta|_{\Gamma_c}, \]

where \( u_0^\delta \) is the solution of (8) with \( g \) substituted for \( g^\delta \). As in [11], we assume that we are given noisy Cauchy data \( h^\delta, g^\delta \), or alternatively \( r^\delta \), such that

\[ ||r - r^\delta||_{L^2(\Gamma_c)} \leq \delta. \]

Note, that the existence of weak solutions with \( L^2 \)-traces on \( \Gamma_c \) can be guaranteed for auxiliary problems (7), (8) under rather mild assumptions, which are much weaker than the usual assumption about Sobolev regularity \( 1/2 \) and \( -1/2 \) for Dirichlet and Neumann data that are usually adopted in the literature. We refer to [9] for further discussions.

### 3. Regularization by projection

The equation (10) is regularized by projection. Let \( P = P_n \) be the orthogonal projector from \( L^2(\Gamma_{un}) \) onto \( n \)-dimensional subspace \( U_n \subset L^2(\Gamma_{un}) \). We seek for the minimal norm solution \( s_n^\delta \in U_n \) meeting

\[ P_nA^*AP_n s = P_nA^* r^\delta, \]

where \( A^* : L^2(\Gamma_c) \rightarrow L^2(\Gamma_{un}) \) is the adjoint of \( A \).
Seeking for an approximate solution \( s^\delta_n \) of (10) in the subspace \( U_n \) one needs to fix a linearly independent system \( \{ \Phi_i^n \}_{i=1}^n \) forming a basis in \( U_n \). Then \( s^\delta_n \) can be represented in the form of a linear combination

\[
s^\delta_n = \sum_{i=1}^n \gamma_i \Phi_i^n
\]

(13)
of the fixed system \( \{ \Phi_i^n \} \) with a coefficient vector \( \gamma = \{ \gamma_i \}_{i=1}^n \).

Plugging the representation (13) into (12) and keeping in mind that \( P_n \Phi_i^n = \Phi_i^n, i = 1, 2, \ldots, n \), and \( \langle \Phi_i^n, A^*A \Phi_j^n \rangle = \langle A \Phi_i^n, A \Phi_j^n \rangle \), we obtain the following system of linear algebraic equations for the vector \( \gamma \)

\[
M \gamma = Y_\delta,
\]

(14)

where matrix and right-hand side vector are given as

\[
M := \left( \langle A \Phi_i^n, A \Phi_j^n \rangle \right)_{i,j=1,\ldots,n},
\]

\[
Y_\delta := \left( \langle A \Phi_i^n, r^\delta \rangle \right)_{i=1,\ldots,n}.
\]

(15)

The error \( \| s - s^\delta_n \|_{L_2(\Gamma_{un})} \) is analyzed under the following assumptions:

- Convergence for noise free data: The solution \( s_0^n \) of (12) for noise free data converges to the solution \( s \) of (9) as \( n \to \infty \). It means that there exists an increasing continuous function \( \psi(\lambda) = \psi(A,s; \lambda) \) such that \( \psi(0) = 0 \), and

\[
\| s - s_0^n \|_{L_2(\Gamma_{un})} \leq \psi(n^{-1}).
\]

(16)

- Noise propagation error: In discrete approximations \( s^\delta_n \) a noise propagates with an exponential rate such that

\[
e^{an\delta} \leq \| s^\delta_n - s_0^n \|_{L_2(\Gamma_{un})} \leq e^{aqn \delta}, \quad n = 1, 2, \ldots
\]

(17)

for some unknown \( a > 0 \). The quantity \( q > 1 \) in (17) is a priori chosen design parameter which reflects the magnitude of a gap in our knowledge of the problem. In [9] we argue that this assumption reflects a typical noise propagation for severely ill-posed problems. Note that a similar observation has been made in [15] (see the formula (2.10) there).

4. **Balancing principle for the choice of \( n \)**

In this section we describe an adaptive procedure for choosing a discretization level \( n \) without knowledge of \( a \) in (17) and \( \psi \) in (16). This procedure can be seen as successive testing of the hypotheses that the power \( aq \) in a robustness estimation (17) is not larger than some \( b_j \in \{ b_i = b_0 q^i, i = 0, 1, \ldots, M \} \), where \( q > 1 \) is the same as in (17). For each of these hypotheses one can choose \( n = n(b_j) \) using the strategy from [4]:

\[
n(b_j) = \min \left\{ n : \| s^\delta_n - s^\delta_m \| \leq 4\delta e^{b_j m}, \quad m = N, N-1, \ldots, n+1 \right\},
\]

where \( N \) is so large that

\[
N > \max \left\{ \frac{4 \ln 2}{b_0(q-1)}, q^2 \left( \frac{1}{b_0} \ln \left( \frac{6}{\delta} \right) + 1 \right) \right\}.
\]

Then it is easy to verify that the sequence \( \{ n(b_j) \} \) is non-increasing, and it turns out that one can detect whether the hypothesis \( aq \leq b_j \) is untrue. The following statements can be proven along the lines of [9] and presented here for the sake of completeness.
Let
\[ n(b_{j-1}) > \tau(N, q, b_0, \delta) := \frac{N}{q^2} - \frac{1}{b_0} \ln \left( \frac{3e^{N(q-1)b_0}}{e^{N(q-1)b_0} - 4} \right). \]

Let
\[ n_\tau = \max \{ n(b_j) : n(b_j) \leq \tau(N, q, b_0, \delta) \}. \]

If \( n_\tau = n(b_i) \), then we take \( n_+ := n(b_{i+2}) \) as the final discretization level.

**Theorem 4.1** Under assumptions above, if the function \( \psi \) increases with a polynomial rate, then
\[ \| s - s_{n_+}^\delta \| \leq c\psi(n_{opt}^{-1}) \leq c\psi(\ln^{-1} \frac{1}{\delta}). \]

where \( c \) does not depend on \( \delta \), and \( n_{opt} \) is the solution of the equation \( \psi(n) = \delta e^{a_n} \), which balances unknown approximation error bound (16) with unknown noise propagation rate (17).

5. Numerical examples

Using the approximate reconstruction \( s_{n_+}^\delta \), we can find an approximate voltage potential in \( \Omega \) as the solution \( u_{n_+}^\delta \) of the mixed boundary value problem (1), (2) completed by the boundary conditions
\[ \frac{\partial u}{\partial \nu} = g^\delta \text{ in } \Gamma_c, \quad \frac{\partial u}{\partial \nu} = s_{n_+}^\delta \text{ in } \Gamma_{un}. \]

Now the searched-for nonlinearity \( f \) can be reconstructed from (6), where \( u \) and \( \frac{\partial u}{\partial \nu} \) are substituted for \( u_{n_+}^\delta \mid_{\Gamma_{un}} \) and \( s_{n_+}^\delta \) respectively. It gives an equation
\[ f(u_{n_+}^\delta(x, y)) = s_{n_+}^\delta(x, y), \quad (x, y) \in \Gamma_{un}. \tag{18} \]

In numerical tests below a regularized approximation \( s_{n_+}^\delta \) is so accurate that \( u_{n_+}^\delta \mid_{\Gamma_{un}} \) preserves a monotony of the exact trace. Then \( f \) can be reconstructed point-wise directly from (18). If \( u_{n_+}^\delta \mid_{\Gamma_{un}} \) is a non-monotone then (18) can be regularized in the same way as in [12, 8].

In our tests the basis \( \{ \Phi^\delta_n \} \) in the space \( U_n \) consists of the traces at \( \Gamma_{un} \) of two-dimensional polynomials \( x, y, x^2, xy, y^2, \ldots, x^{k-1}, x^{k-2}y, \ldots, y^{k-1}, k = 1, 2, \ldots, n = k(k + 1)/2 \).

Then to calculate the matrix \( M \) in (15) one needs to find \( A\Phi^\delta_i, i = 1, 2, \ldots, n \). Note that \( A\Phi^\delta_i \) is the Dirichlet trace at \( \Gamma_c \) of the solution of (7), where \( s \) is substituted for \( \Phi^\delta_i \). To find such solutions one can use Matlab PDE toolbox with FEM method, where the domain \( \Omega \) is partitioned into triangles, and an approximate solution is constructed as a corresponding sum of piece-wise linear finite elements. The auxiliary problem (8) is solved in the same way.

The inner products \( \langle A\Phi^\delta_i, A\Phi^\delta_j \rangle, \langle A\Phi^\delta_i, r^\delta \rangle \) in \( L^2(\Gamma_c) \) are computed approximately by trapezoidal rule using as knots on \( \Gamma_c \) the apexes of triangles forming \( \Omega \)-triangulation. Moreover, calculating \( \langle A\Phi^\delta_i, r^\delta \rangle \) we add to corresponding values of \( h \) and \( u_0 \) the values of independent zero mean random variables uniformly distributed on \( [-\delta, \delta] \). In this way noisy data are simulated.

In the considered case it means that a mixed boundary value problem (7) should be solved \( n \) times for \( s = \Phi^\delta_i, i = 1, 2, \ldots, n \). Usually in practice it is enough to take \( n \) much less than 100 (see, for example, numerical test presented below). At the same time, some other methods for solving elliptic Cauchy problems may require hundreds or even thousands calls of the direct solver for similar mixed boundary value problems (see, for example, numerical experiments reported in [11] for Mann-Maz’ya iterations).

To apply the balancing principle presented in the previous section one should choose design parameters \( N, q, b_0 \). One possible way to specify these parameters is the application of the balancing principle to a simple problem with a known solution. In this case one can try several
parameter values, and the values corresponding to the smallest error can be used in applications of the balancing principle to problems with unknown solutions.

To illustrate this approach we consider at first the problem (1) - (3), (6) with \( \Omega = [0, \pi] \times [0, 1] \)
\( \Gamma_c = [0, \pi] \times \{0\}, \Gamma_{un} = [0, \pi] \times \{1\}, \Gamma_D = \{0\} \times [0, 1] \cup \{\pi\} \times [0, 1], u(x, y) = -\sin(x) \exp(-y), \)
\( g(x) = -\sin(x), f(t) = -t. \) Noise level is \( 4 \cdot 10^{-4}. \)

Numerical experiments with this calibrating problem suggest the values \( N = 66, \quad q = 1.3, \quad b_0 = 0.084. \) For \( \{b_j = 0.084 \times (1.3)^j, j = 0, 1, \ldots, 20\} \) the threshold \( \tau(N, q, b_0, \delta) = 9.08. \)

The balancing principle selects \( \{n(b_j)\}_{j=1}^{20} \) as 66, 66, 66, 66, 52, 39, 21, 11, 7, 3, 3, 1, 1, 1, 1, 1, 1, 1. In this case \( n_r := n(b_9) = 7 \) and \( n_+ := n(b_{11}) = 3. \) The results of a reconstruction are displayed in Figure 1 and 2.

In the second example we apply the balancing principle calibrated as above to the problem (1) - (3), (6) with the domain \( \Omega \) displayed in Figure 3, where

\[
\begin{align*}
\Gamma_{un} &= \{(x, y) \in \partial B(0, 1), \ x \geq 0, y \geq 0\}, \\
\Gamma_c &= \Gamma_{c,1} \cup \Gamma_{c,2} \cup \Gamma_{c,3}
\end{align*}
\]

with

\[
\begin{align*}
\Gamma_{c,1} &= \{(x, y) \in \partial B(0, 1), \ x \leq 0, y \geq 0\}, \\
\Gamma_{c,2} &= \{(x, y), \ 3y + 4x + 4 = 0, -1 \leq x \leq 1/2\}, \\
\Gamma_{c,3} &= \{(x, y), \ x = 1, -2 \leq y \leq 0\},
\end{align*}
\]

and

\( \Gamma_D = \{(x, y), \ y = -2, \frac{1}{2} \leq x \leq 1\}. \)

In (3) synthetic data \( h, g \) are chosen in such a way that the exact solution is given as

\[
u(x, y) = \frac{y + 2}{(y + 2)^2 + x^2}, \quad (x, y) \in \Omega.
\]

Then it is easy to check that on \( \Gamma_{un} \)

\[
\frac{\partial u}{\partial \nu} = -\frac{5y + 4}{(4y + 5)^2}, \quad u = \frac{y + 2}{4y + 5},
\]
and these two functions are related by the equation (6), where
\[ f(t) = 4t^2 - \frac{11}{3}t + \frac{2}{3}, \quad \frac{1}{3} \leq t \leq \frac{2}{5}. \]

We simulate a noise with the level \( \delta = 10^{-5} \). The balancing principle selects \( \{n(b_j)\}_{j=1}^{20} = 66, 66, 65, 59, 59, 28, 25, 14, 10, 8, 5, 3, 3, 2, 2, 1, 1, 1, 1 \). Then \( n_r := n(b_{10}) = 8 \), and the final discretization level \( n_+ := n(b_{12}) = 3 \). The results of a reconstruction are displayed in Figure 4 and 5.

As one can see from Figure 1 and 4, the method for solving Cauchy problem proposed in [9] gives an accurate approximation of Neumann traces required for solving the equation (6). When such traces are reconstructed, the application of the method from [12, 8] equipped with the balancing principle allows a recovery of unknown nonlinear profile of \( f \) in (6) with good quality. It can be seen in Figures 2 and 5.

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