Entanglement Content of Quasi-Particle Excitations

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We investigate the quantum entanglement content of quasi-particle excitations in extended many-body systems. We show that such excitations give an additive contribution to the bi-partite von Neumann and Rényi entanglement entropies that takes a simple, universal form. It is largely independent of the momenta and masses of the excitations, and of the geometry, dimension and connectedness of the entanglement region. The result has a natural quantum information theoretic interpretation as the entanglement of a state where each quasi-particle is associated with two qubits representing their presence within and without the entanglement region, taking into account quantum (in)distinguishability. This applies to any excited state composed of finite numbers of quasi-particles with finite De Broglie wavelengths or finite intrinsic correlation length. This includes particle excitations in massive quantum field theory and gapped lattice systems, and certain highly excited states in conformal field theory and gapless models. We derive this result analytically in one-dimensional massive bosonic and fermionic free field theories and for simple setups in higher dimensions. We provide numerical evidence for the harmonic chain and the two-dimensional harmonic lattice in all regimes where the conditions above apply. Finally, we provide supporting calculations for integrable spin chain models and other interacting cases without particle production. Our results point to new possibilities for creating entangled states using many-body quantum systems.

Introduction.— Measures of entanglement, such as the entanglement entropy (EE) [1] and entanglement negativity [2–7], have attracted much attention in recent years, both theoretically [8–10] and experimentally [11, 12]. Quantum entanglement encodes correlations between degrees of freedom associated to independent factors of the Hilbert space, and as such, it separates quantum correlations from the particularities of observables. As a consequence, the entanglement in extended systems encodes, in a natural fashion, universal properties of the state. For instance, at criticality, the entanglement of ground states provides an efficient measure of universal properties of quantum phase transitions, such as the (effective) central change of the corresponding conformal field theory (CFT) and the primary operator content [13–22]. Near criticality, it is universally controlled by the masses of excitations [23–25]. In states that are highly excited, with finite energy densities, the entanglement gives rise to local thermalisation effects: at the heart of the eigenstate thermalisation hypothesis [26–30], the large entanglement between local degrees of freedom and the rest of the system effectively generates a Gibbs ensemble (or generalised Gibbs ensemble in integrable systems). The entanglement effects of a finite number of excitations are less known. Some results are available in critical systems: using the methods of Holzhey, Larsen and Wilczek [14], combining a geometric description with Riemann uniformization techniques in CFT it was shown in [31, 32] that certain excitations, with energies tending to zero in the large volume limit, correct the ground state entanglement by power laws in the ratio of length scales. Various few-particle states have also been studied in special cases of integrable spin chains [33–37].

In this paper we propose a universal formula, with a simple quantum information theoretic interpretation, for the entanglement content of states with quasi-particle excitations. We consider the von Neumann and Rényi EEs: these are measures of the amount of quantum entanglement, in pure quantum states, between the degrees of freedom associated to two sets of independent observables whose union is complete on the Hilbert space. We use the setup where the Hilbert space is factorised as \( \mathcal{H}_A \otimes \mathcal{H}_B \), according to two complementary spatial regions \( A \) and \( B \), of typical length scales (diameters) \( \ell_A \) and \( \ell_B \), respectively. The regions can be of generic geometry and connectedness. Let \( \xi \) be the correlation length, and \( \zeta = \max\{2\pi/|\vec{p}_i|\} \), where \( \vec{p}_i \) are the momenta, be the maximal particles’ De Broglie wavelength. We find exact formulae in the limit \( \min(\xi, \zeta) \ll \min(\ell_A, \ell_B) \), independent of the model studied, of the connectedness or shape of the entanglement region, and of the dimension. For instance, this condition includes the limit of large regions’ diameters \( \ell_A, \ell_B \to \infty \) in massive quantum field theory and gapped quantum lattice systems, where the correlation length \( \xi \) is finite. It also includes this same limit in certain states of conformal field theory and gapless models whose energies are finitely separated from that of the ground state, and which have well-defined particle content with finite momenta (finite De Broglie’s wavelengths). The results extend the “semiclassical” form discussed in the context of spin chains in [33]. They have a very natural qubit interpretation where qubits representing the particles are entangled according to the particles’ distribution in space, taking into account quantum
indistinguishability in the bosonic case. Quasi-particle excitations are ubiquitous in many-body quantum systems, and our results are expected to apply to a large family of states with well defined quasi-particle content. We give evidence by studying a variety of clear-cut cases of different dimensions.

\[
\Delta S^k_n (r) = \lim_{\lambda \to \infty} \Delta S^\Psi_\lambda (A, B).
\]

This is the contribution of the excitations to the entanglement, or “excess entanglement” as named in [31, 32].

We find that for a wide variety of quantum systems, the results depend only on the proportion \( r \) of the system's volume occupied by the entanglement region, and are largely independent of the momenta of the quasi-particles. Suppose the state is formed of \( k \) particles of equal momenta. Denoting \( \Delta S^\Psi_n (r) = \Delta S^k_n (r) \), we find

\[
\Delta S^k_n (r) = \frac{\log \sum_{q=0}^k f_q^k (r) n}{1 - n}, \quad \Delta S^1_n (r) = - \sum_{q=0}^k f_q^k (r) \log f_q^k (r)
\]

with \( f_q^k (r) = \binom{k}{q} r^q (1-r)^{k-q} \). For a state composed of \( k \) particles divided into groups of \( k_i \) particles of equal momenta \( \tilde{p}_i \), with \( i = 1, 2, \ldots \) and \( \sum_i k_i = k \), we denote \( \Delta S^\Psi_n (r) = \Delta S^{k_1, k_2, \ldots} (r) \) and have

\[
\Delta S^{k_1, k_2, \ldots} (r) = \sum_i \Delta S^{k_i}_n (r).
\]

In particular, for \( k \) particles of distinct momenta the result is \( k \) times that for a single particle, which is

\[
\Delta S^1_n (r) = \frac{\log(r^n + (1-r)^n)}{1 - n},
\]

\[
\Delta S^1_n (r) = -r \log r - (1-r) \log (1-r).
\]

We observe that in all cases, the entanglement is maximal at \( r = 1/2 \). For \( k \) distinct-momenta particles, the maximum is \( k \log 2 \), while when some particles have coinciding momenta, the maximal value is smaller. Interestingly, single-copy entropies present non-analytic features. For distinct momenta, we have

\[
\Delta S^1_n (r) = \begin{cases} -\log(1-r) & 0 \leq r < \frac{1}{2} \\ -\log r & \frac{1}{2} \leq r \leq 1 \end{cases}
\]

Again, the result is just multiplied by \( k \) for a state consisting of \( k \) distinct-momentum particles. For equal momenta it is a function which is non-differentiable at \( k \) points in the interval \( r \in (0, 1) \) (generalizing (6)). The positions of these cusps are given by the values

\[
r = \frac{1 + q}{1 + k} \quad \text{for} \quad q = 0, \ldots, k - 1,
\]

and the single copy entropy is given by

\[
\Delta S^\infty_n (r) = -\log f_q^k (r) \quad \text{for} \quad \frac{q}{1 + k} \leq r < \frac{1 + q}{1 + k}
\]

and \( q = 0, \ldots, k \). See FIG. 1 for a numerical evaluation.

The results take their full meaning under a quantum information theoretic interpretation that combines a “semiclassical” picture of particles with quantum indistinguishability. Consider a bi-partite Hilbert space \( \mathcal{H} = \mathcal{H}_{\text{int}} \otimes \mathcal{H}_{\text{ext}} \). Each factor \( \mathcal{H}_{\text{int}} \simeq \mathcal{H}_{\text{ext}} \) is a tensor product \( \otimes_k \mathcal{H}^{k_i} \) of Hilbert spaces \( \mathcal{H}^{k_i} \simeq \mathbb{C}^{k_i+1} \) for \( k_i \) indistinguishable qubits, with, as above, \( \sum_i k_i = k \). We associate \( \mathcal{H}_{\text{int}} \) with the interior of the region \( A \) and \( \mathcal{H}_{\text{ext}} \) with its exterior, and we identify the qubit state 1 with...
the presence of a particle and 0 with its absence. We construct the state $|\Psi_{q_b}\rangle \in \mathcal{H}$ under the picture according to which equal-momenta particles are indistinguishable, and a particle can lie anywhere in the full volume of the system with flat probability. That is, any given particle has probability $r$ of lying within $A$, and $1-r$ of lying outside of it. We make a linear combination of qubit states following this picture, with coefficients that are square roots of the total probability of a given qubit configuration, taking proper care of (in)distinguishability. For instance, for a single particle,

$$|\Psi_{q_b}\rangle = \sqrt{r} |1\rangle \otimes |0\rangle + \sqrt{1-r} |0\rangle \otimes |1\rangle$$

as either the particle is in the region, with probability $r$, or outside of it, with probability $1-r$. If two particles of coinciding momenta are present, then we have

$$|\Psi_{q_b}\rangle = \sqrt{r^2} |2\rangle \otimes |0\rangle + \sqrt{2r(1-r)} |1\rangle \otimes |1\rangle + \sqrt{(1-r^2)} |0\rangle \otimes |2\rangle$$

as either the two particles are in the region, with probability $r^2$, or one is in the region and one outside of it (no matter which one), with probability $2r(1-r)$, or both are outside the region, with probability $(1-r)^2$. For two particles of different momenta,

$$|\Psi_{q_b}\rangle = \sqrt{r^2} |11\rangle \otimes |00\rangle + \sqrt{(1-r^2)} |00\rangle \otimes |11\rangle + \sqrt{(1-r)} ( |10\rangle \otimes |01\rangle + |01\rangle \otimes |10\rangle)$$

as either both are in the region, with probability $r^2$, or one is in the region and one outside of it (no matter which one), with probability $2r(1-r)$, or both are outside the region, with probability $(1-r)^2$. The quantity $\Delta S^\Psi_n(r)$ is the vacuum state. Both $|0\rangle_n$ and the state $|\Psi\rangle_n$ have the structure

$$|\Psi\rangle_n = |\Psi\rangle^1 \otimes |\Psi\rangle^2 \otimes \cdots \otimes |\Psi\rangle^n.$$

Here $|\Psi\rangle^i \simeq |\Psi\rangle$ is the $i$-particle excited state of interest, implemented in the $i$th copy.

First, in one dimension, $A$ is a union of segments, and $T(A, B)$ becomes a product of branch-point twist fields [23] on the boundary points of these segments. Let us consider the case $A = [0, \ell]$ in a periodic system of length $L$. Then $T(A, B) = \tilde{T}(0)\tilde{T}(\ell)$, where $\tilde{T}$ is the branch point twist field and $\tilde{T}$ is its hermitian conjugate. In expression (13) one may then expand in a basis $\{|\Phi\rangle\}$ of quasi-particles,

$$n\langle\Psi|\tilde{T}(0)\tilde{T}(\ell)|\Psi\rangle_n = \sum_\Phi e^{-ip_\Phi\ell} \langle n|\langle\Psi|\tilde{T}(0)|\Phi\rangle^2$$

where $P_\Phi$ are the momentum eigenvalues (in finite volume, they are quantised, and the set of states is discrete). The evaluation of (13) using (15) is in principle feasible in integrable 1+1-dimensional QFT, but this present a number of challenges. Although matrix elements of branch-point twist fields in infinite volume are known [23, 46, 47], they cannot be used in order to evaluate the limit $L \rightarrow \infty$ in (13): divergencies occur whenever momenta of intermediate particles in $|\Phi\rangle$ coincide with those in $|\Psi\rangle_n$. One must first evaluate finite-volume matrix elements, re-sum the series (15), and take the limit. Finite-volume matrix elements of generic fields are related [48, 49] to their infinite-volume counterpart up to exponentially decaying terms in $L$, but for twist fields the theory has not been developed yet. We have solved these problems for the massive free real boson and the massive free Majorana fermion. By performing the summation over intermediate states at large $L$, noting that the so-called “kinematic singularities” of infinite-volume matrix elements provide the leading contribution, we have derived the full results (3) and (4). The details are technical, and presented in a separate paper [50].

Second, we performed a numerical evaluation of the quantity $\Delta S^\Psi_n(r)$ using wave functional methods in the harmonic chain and the two-dimensional harmonic lattice, see FIG. 2 and the supplementary material (SM). In the finite-volume Klein-Gordon theory, the vacuum wave functional takes the Gaussian form

$$\langle\varphi|0\rangle \propto \exp \left[ -\frac{1}{2} \int_{C \times C} d^d x d^d y K(\tilde{x} - \tilde{y}) \varphi(\tilde{x}) \varphi(\tilde{y}) \right]$$

where $K(\tilde{x}) = \sum_\ell \text{Vol}_d (C)^{-1} E_\ell e^{i\tilde{x}\cdot\tilde{p}}$. Excited state wave functionals have extra polynomial-functional factors, obtained by applying the operator

$$A^\dagger (\tilde{p}) = \int_{C} d^d x e^{i\tilde{p}\cdot\tilde{x}} (E_\tilde{p} \varphi(\tilde{x}) - i \varphi(\tilde{x})) \sqrt{2E_\tilde{p} \text{Vol}_d (C)}, \quad [A_\tilde{p}, A^\dagger_\tilde{q}] = \delta_{\tilde{p}, \tilde{q}},$$

with the representation of the canonical momentum $\varphi(\tilde{x}) = -i\delta/\delta \varphi(\tilde{x})$ satisfying $[\varphi(\tilde{x}), \varphi(\tilde{y})] = i\delta(\tilde{x} - \tilde{y})$. Implementing the permutation $T(A, B)$ on the space of field configurations, the ratio (13) becomes a Gaussian average of polynomial functionals of the fields. With finite lattice spacing $\Delta x$ the dispersion relation is $E_\tilde{p}^2 = $
accuracy, and have analysed regimes where both $r$-ric: regions
connected and disconnected regions. We concentrate
ter regimes are seen to agree with our predictions, for
are small, finding even greater accuracy. We have also
analysis extends to other quasi-one-dimensional
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QFT is also interesting: QFT locality is formally based on the vanishing of space-like commutation relations, not on particles, yet our results show how quantum entanglement clearly “sees” localised particles. This suggests that entanglement entropy could be used as a diagnostic tool for determining if excitations are of quasi-particle type. The relation (12) suggests that quasi-particle excitations in extended systems of any dimension can be used to create simple entangled states with controllable entanglement, where the control parameter is the region-to-system volume ratio $r$. It would be interesting to investigate the possible applications of such a result in the area of quantum information.

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