ON THE SUBCLASSES ASSOCIATED WITH THE BESSEL-STRUVE KERNEL FUNCTIONS

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Abstract. The article investigate the necessary and sufficient conditions for the normalized Bessel-struve kernel functions belonging to the classes $T_{\lambda}(\alpha)$, $L_{\lambda}(\alpha)$. Some linear operators involving the Bessel-Struve operator are also considered.

1. Introduction

In this article we will consider the class $H$ of all analytic functions defined in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $A$ is the subclass of $H$ consisting of function which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $U$. Thus each function $f \in A$ posses the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hfill (1)

We also consider the subclass $J$ of $H$ consisting of the function have the power series as

$$f(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n > 0.$$  \hfill (2)

The Hadamard product (or convolution) of two function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in H$ is defined as

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in D.$$  

Dixit and Pal [14] introduced the subclass $R_\tau(A, B)$ consisting the function $f \in H$ which satisfies the inequality

$$|f'(z) - 1|^{\tau - B[f'(z) - 1]} < 1,$$  

for $\tau \in \mathbb{C} \setminus \{0\}$ and $-1 \leq B < A \leq 1$. The class $R_\tau(A, B)$ is the generalization of many well known subclasses, for example, if $\tau = 1$ and $A = -B = \alpha \in [0, 1)$, then $R_1(A, B)$ is the subclass of $H$ which consist the functions satisfying the inequality $f'(z) - 1 < B$ which was studied by Padmanabhan [14], Caplinger and Causey [6] and many others. If the function $f$ of the form (1) belong to the class $R_\tau(A, B)$, then

$$|a_n| \leq \frac{(A - B)\tau}{n}, \quad n \in \mathbb{N}.$$  \hfill (3)

The bounds given in (3) is sharp.

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In this article we will consider two well-known subclass of $\mathcal{H}$ and denoted as $\mathcal{T}_\lambda(\alpha)$ and $\mathcal{L}_\lambda(\alpha)$. The function $f$ in the subclass $\mathcal{T}_\lambda(\alpha)$ satisfy the analytic criteria

$$\mathcal{T}_\lambda(\alpha) := \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} \right) > \alpha \right\},$$

while $f \in \mathcal{L}_\lambda(\alpha)$ have the analytic characterization as

$$\mathcal{L}_\lambda(\alpha) := \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{\lambda z^3 f'''(z) + (1 + 2\lambda)z^2 f''(z) + zf'(z)}{zf'(z) + \lambda z^2 f''(z)} \right) > \alpha \right\}.$$

Here $z \in \mathbb{D}$, $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$. For more details about this classes and its generalization see \cite{1, 3, 17} and references therein. The sufficient coefficient condition by which a function $f \in \mathcal{A}$ as defined in (1) belongs to the class $\mathcal{T}_\lambda(\alpha)$ is

$$\sum_{n=2}^{\infty} |n\lambda - \lambda + 1| (n-\alpha)|a_n| \leq 1 - \alpha,$$

and $f$ is in $\mathcal{L}_\lambda(\alpha)$ is

$$\sum_{n=2}^{\infty} n(n\lambda - \lambda + 1)(n-\alpha)|a_n| \leq 1 - \alpha.$$

The significance of the class $\mathcal{T}_\lambda(\alpha)$ and $\mathcal{L}_\lambda(\alpha)$ is that, it will reduce to some classical subclass of $\mathcal{H}$ for specific choice of $\lambda$. For example, if $\lambda = 0$, $\mathcal{T}_\lambda(\alpha) = \mathbb{S}^*(\alpha)$ is the class of all starlike functions of order $\alpha$ with respect to the origin and $\mathcal{L}_\lambda(\alpha) = \mathcal{C}(\alpha)$ is the class of all starlike functions of order $\alpha$.

Denote $\mathcal{T}_\lambda^*(\alpha) = \mathcal{T}_\lambda(\alpha) \cap \mathcal{J}$ and $\mathcal{L}_\lambda^*(\alpha) = \mathcal{L}_\lambda(\alpha) \cap \mathcal{J}$. Then (1) and (5) are respectively the necessary and sufficient condition for any $f \in \mathcal{J}$ is in $\mathcal{T}_\lambda^*(\alpha)$ and $\mathcal{L}_\lambda^*(\alpha)$.

Bessel and Struve functions arise in many problems of applied mathematics and mathematical physics. The properties of these functions were studied by many researchers in the past years from many different point of views. Recently the Bessel-Struve kernel and the so-called Bessel-Struve intertwining operator have been the subject of some research from the point of view of the operator theory, see \cite{9,11} and the references therein. The Bessel-Struve kernel function $S_\nu$ is defined by the series

$$S_\nu(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu + 1) \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi n! \Gamma \left( \frac{n+1}{2} + \nu + 1 \right)}} z^n,$$

where $\nu > -1$. This function is a particular case when $\lambda_1 = 1$ of the unique solution $S_\nu(\lambda x)$ of the initial value problem

$$L_\nu u(z) = \lambda_1^2 u(z), \quad u(0) = 1, u'(0) = \frac{\lambda_1 \Gamma(\nu + 1)}{\sqrt{\pi \Gamma(\nu + \frac{3}{2})}},$$

where for $\nu > -1/2$ the expression $L_\nu$ stands for the Bessel-Struve operator defined by

$$L_\nu u(z) = \frac{d^2 u}{dz^2}(z) + \frac{2\nu + 1}{x} \left( \frac{du}{dz}(z) - \frac{du}{dz}(0) \right)$$

with an infinitely differentiable function $u$ on $\mathbb{R}$.

Motivated by the results connecting various subclasses with the Bessel functions \cite{4,12}, the Struve functions \cite{13,18}, the hypergeometric function and the Confluent hypergeometric functions \cite{7,16}, in Section 2 we obtain several conditions by which the normalized Bessel-Struve kernel functions $z S_\nu$ is the member of $\mathcal{T}_\lambda^*(\alpha)$ and $\mathcal{L}_\lambda^*(\alpha)$. Also determined the condition for which $z(2 - S_\nu(z)) \in \mathcal{T}_\lambda^*(\alpha)$ and $z(2 - S_\nu(z)) \in \mathcal{L}_\lambda^*(\alpha)$. 
2. Main Result

2.1. Inclusion properties of the Bessel-Struve functions. For convenience throughout in the sequel, we use the following notations

\[ z S_\nu(z) = z + \sum_{n=2}^{\infty} c_{n-1}(\nu) z^n, \quad (6) \]

and

\[ \Phi(z) = z(2 - S_\nu(z)) = z - \sum_{n=2}^{\infty} c_{n-1}(\nu) z^n, \quad (7) \]

where \( c_n(\nu) = \frac{\Gamma(\nu+1)\Gamma(\frac{n+1}{2})}{\sqrt{\pi n!}\Gamma(\frac{n+\nu+1}{2})} \).

Further from (6) a calculation yield

\[ z^2 S'_\nu(z) + z S_\nu(z) = z + \sum_{n=2}^{\infty} nc_{n-1}(\nu) z^n. \quad (8) \]

A differentiation of both side of (8) two times with respect to \( z \), gives

\[ z^3 S''_\nu(z) + 3z^2 S'_\nu(z) + z S_\nu(z) = z + \sum_{n=2}^{\infty} n^2 c_{n-1}(\nu) z^n, \quad (9) \]

and

\[ z^4 S'''_\nu(z) + 6z^3 S''_\nu(z) + 7z^2 S'_\nu(z) + z S_\nu = z + \sum_{n=2}^{\infty} n^3 c_{n-1}(\nu) z^n. \quad (10) \]

Now we will state and proof our main results.

**Theorem 2.1.** For \( \nu > -1/2 \) and \( 0 \leq \alpha, \lambda < 1 \), let

\[ \lambda S''_\nu(1) + (1 - \lambda \alpha) S'_\nu(1) + (1 - \alpha) S_\nu(1) \leq 2(1 - \alpha). \quad (11) \]

Then the normalized Bessel-Struve function \( z S_\nu(z) \in T_\lambda(\alpha) \).

**Proof.** Consider the identity

\[ z S_\nu(z) = z + \sum_{n=2}^{\infty} c_{n-1}(\nu) z^n. \]

Then by virtue of (11) it is enough to show that \( F(n, \lambda, \alpha) \leq 1 - \alpha \), where

\[ F(n, \lambda, \alpha) := \sum_{n=2}^{\infty} (n\lambda - \lambda + 1)(n - \alpha)c_{n-1}(\nu) \]

The right hand side expression of \( F(n, \lambda, \alpha) \) can be rewrite as

\[ F(n, \lambda, \alpha) = \lambda \sum_{n=2}^{\infty} n^2 c_{n-1}(\nu) + (1 - \lambda(1 + \alpha)) \sum_{n=2}^{\infty} nc_{n-1}(\nu) + \alpha(\lambda - 1) \sum_{n=2}^{\infty} c_{n-1}(\nu). \quad (12) \]
Now for $|z| = 1$, it is evident from (6)–(9) that

\[ S_\nu(1) = 1 + \sum_{n=2}^{\infty} z^n c_{n-1}(\nu), \]

\[ S_\nu'(1) + S_\nu(1) = 1 + \sum_{n=2}^{\infty} n c_{n-1}(\nu), \]

\[ S_\nu''(1) + 3S_\nu'(1) + S_\nu(1) = 1 + \sum_{n=2}^{\infty} n^2 z^n c_{n-1}(\nu). \]

Thus (12) reduce to

\[ F(n, \lambda, \alpha) = \lambda S_\nu''(1) + (1 - \lambda \alpha + 2\lambda)S_\nu'(1) + (1 - \alpha)(S_\nu(1) - 1). \]

and is bounded above by $1 - \alpha$ if (11) holds. Thus the proof is completed. □

**Corollary 2.1.** The normalized Bessel-Struve kernel function is starlike of order $\alpha \in [0, 1]$ with respect to origin if $S_\nu'(1) + (1 - \alpha)S_\nu(1) \leq 2(1 - \alpha)$.

**Theorem 2.2.** For $\nu > -1/2$ and $0 \leq \alpha, \lambda < 1$, suppose that

\[ \lambda S_\nu''(1) + (5\lambda + 1 - \lambda \alpha)S_\nu'(1) + (4\lambda - 2\lambda \alpha - \alpha + 3)S_\nu(1) + (1 - \alpha)S_\nu(1) \leq 2(1 - \alpha). \]

Then the normalized Bessel-Struve function $zS_\nu \in \mathcal{L}_\lambda(\alpha)$.

**Proof.** By virtue of (5), it suffices to prove that $G(n, \lambda, \alpha) \leq 1 - \alpha$, where

\[ G(n, \lambda, \alpha) = \sum_{n=2}^{\infty} n(n\lambda - \lambda + 1)(n - \alpha)c_{n-1}(\nu) \]

\[ = \lambda \sum_{n=2}^{\infty} n^3 c_{n-1}(\nu) + (1 - \lambda(1 + \alpha)) \sum_{n=2}^{\infty} n^2 c_{n-1}(\nu) + \alpha(\lambda - 1) \sum_{n=2}^{\infty} n c_{n-1}(\nu). \]

For $|z| = 1$ and using (8)–(10), it follows that

\[ G(n, \lambda, \alpha) = \lambda(S_\nu''(1) + 7S_\nu''(1) + 6S_\nu'(1) + S_\nu(1) - 1) \]

\[ + (1 - \lambda(1 + \alpha))(S_\nu''(1) + 3S_\nu'(1) + S_\nu(1) - 1) + \alpha(\lambda - 1)(S_\nu'(1) + S_\nu(1) - 1) \]

\[ = \lambda S_\nu''(1) + (5\lambda + 1 - \lambda \alpha)S_\nu'(1) + \lambda(\lambda - 1)S_\nu(1) + (1 - \alpha)(S_{\gamma,\beta}(1) - 1) \]

which is bounded above by $1 - \alpha$ if (13) holds, and hence the conclusion. □

**Corollary 2.2.** The normalized Bessel-Struve kernel function is convex of of order $\alpha \in [0, 1]$ if

\[ S_\nu''(1) + (3 - \alpha)S_\nu'(1) + (1 - \alpha)S_\nu(1) \leq 2(1 - \alpha). \]

**Remark 2.1.** The condition (11) is necessary and sufficient for $z(2 - S_\nu(z)) \in \mathcal{T}^*(\alpha)$, while $z(2 - S_\nu(z)) \in \mathcal{L}^*(\alpha)$ if and only if the condition (13) hold.

2.2. **Operators involving the Bessel-Struve functions and its inclusion properties.** In this section we considered the linear operator $J_\nu : A \to A$ defined by

\[ J_\nu f(z) = z S_\nu(z) * f(z) = z + \sum_{n=2}^{\infty} c_{n-1}(\nu)a_n z^n, \]

In the next result we will study the action of the Bessel-struve operator on the class $\mathcal{R}^+(A, B)$. 

\[ zS_\nu(z) * f(z) = z + \sum_{n=2}^{\infty} c_{n-1}(\nu)a_n z^n, \]
It is evident from the hypothesis \(14\) that
\[
G(n, \lambda, \alpha) \leq (A - B)|\tau| \sum_{n=2}^{\infty} (n\lambda - \lambda + 1)(n - \alpha)c_{n-1}(\nu)
\]
\[
= (A - B)|\tau| \left( \lambda \sum_{n=2}^{\infty} n^2c_{n-1}(\nu) + (1 - \lambda(1 + \alpha)) \sum_{n=2}^{\infty} nc_{n-1}(\nu) + \alpha(\lambda - 1) \sum_{n=2}^{\infty} c_{n-1}(\nu) \right)
\]
Using \((8)\) and \((9)\), the right hand side of the above inequality reduce to
\[
G(n, \lambda, \alpha) = (A - B)|\tau| \{\lambda(S''_\nu(1) + 3S'_\nu(1) + S_\nu(1) - 1) + (1 - \lambda(1 + \alpha))(S'_\nu(1) + S_\nu(1) - 1) + \alpha(\lambda - 1)(S_\nu(1) - 1)\}
\]
It is evident from the hypothesis \((14)\) that \(G(n, \lambda, \alpha)\) is bounded above by \(1 - \alpha\) and hence the conclusion.

**Theorem 2.3.** Suppose that \(\nu > -\frac{1}{2}\) and \(f \in \mathcal{R}^\tau(A, B)\). For \(\alpha, \lambda \in (0, 1]\), if
\[
(A - B)|\tau| \{\lambda S''_\nu(1) + (1 - \lambda\alpha + 2\lambda)S'_\nu(1) + (1 - \alpha)(S_\nu(1) - 1) \leq 1 - \alpha,
\]
then \(J_\nu(f) \in \mathcal{L}_\lambda(\alpha)\).

**Proof.** Let \(f\) be of the form \((1)\) belong to the class \(\mathcal{R}^\tau(A, B)\). By virtue of the inequality \((3)\), it suffices to show that \(G(n, \lambda, \alpha) \leq 1 - \alpha\).

It follows from \((3)\) that the coefficient \(a_n\) for each \(f \in \mathcal{R}^\tau(A, B)\) satisfy the inequality \(|a_n| \leq (A - B)|\tau|/n\), and hence
\[
G(n, \lambda, \alpha) \leq (A - B)|\tau| \sum_{n=2}^{\infty} (n\lambda - \lambda + 1)(n - \alpha)c_{n-1}(\nu)
\]
\[
= (A - B)|\tau| \left( \lambda \sum_{n=2}^{\infty} n^2c_{n-1}(\nu) + (1 - \lambda(1 + \alpha)) \sum_{n=2}^{\infty} nc_{n-1}(\nu)
\]
\[
+ \alpha(\lambda - 1) \sum_{n=2}^{\infty} c_{n-1}(\nu) \right)
\]
Using \((8)\) and \((9)\), the right hand side of the above inequality reduce to
\[
G(n, \lambda, \alpha) = (A - B)|\tau| \{\lambda S''_\nu(1) + (2\lambda - \lambda\alpha + 1)S'_\nu(1) + (1 - \alpha)(S_\nu(1) - 1)\}
\]
It is evident from the hypothesis \((14)\) that \(G(n, \lambda, \alpha)\) is bounded above by \(1 - \alpha\) and hence the conclusion.

**Theorem 2.4.** For \(\nu > -\frac{1}{2}\), let \(Q_\nu(z) := \int_0^z (2 - S_\nu(t))dt\). Then for \(\alpha, \lambda \in (0, 1]\), the operator \(Q_\nu \in \mathcal{L}_\lambda(\alpha)\) if and only if
\[
\lambda S''_\nu(1) + (2\lambda - \lambda\alpha + 1)S'_\nu(1) + (1 - \alpha)S_\nu(1) \leq 2(1 - \alpha).
\]

**Proof.** Note that
\[
Q_\nu(z) = z - \sum_{n=2}^{\infty} c_{n-1}(\nu) \frac{z^n}{n}.
\]
It is enough to show that
\[
\sum_{n=2}^{\infty} n(n\lambda - \lambda + 1)(n - \alpha) \left( \frac{c_{n-1}(\nu)}{n} \right) \leq 1 - \alpha.
\]
Since
\[
\sum_{n=2}^{\infty} n(n\lambda - \lambda + 1)(n - \alpha) \left( \frac{c_{n-1}(\nu)}{n} \right) = \sum_{n=2}^{\infty} (n\lambda - \lambda + 1)(n - \alpha)c_{n-1}(\nu),
\]
proceeding as the proof of Theorem \(2.1\) we get
\[
\sum_{n=2}^{\infty} n(n\lambda - \lambda + 1)(n - \alpha) \left( \frac{c_{n-1}(\nu)}{n} \right) = \lambda S''_\nu(1) + (2\lambda - \lambda\alpha + 1)S'_\nu(1) + (1 - \alpha)S_\nu(1),
\]
which is bounded above by \(1 - \alpha\) if and only if \((15)\).
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