ON SELF-APPROACHING AND INCREASING-CHORD DRAWINGS OF 3-CONNECTED PLANAR GRAPHS*

Martin Nöllenburg,† Roman Prutkin,‡ and Ignaz Rutter‡

Abstract. An st-path in a drawing of a graph is self-approaching if during the traversal of the corresponding curve from s to any point t' on the curve the distance to t' is non-increasing. A path is increasing-chord if it is self-approaching in both directions. A drawing is self-approaching (increasing-chord) if any pair of vertices is connected by a self-approaching (increasing-chord) path.

We study self-approaching and increasing-chord drawings of triangulations and 3-connected planar graphs. We show that in the Euclidean plane, triangulations admit increasing-chord drawings, and for planar 3-trees we can ensure planarity. We prove that strongly monotone (and thus increasing-chord) drawings of trees and binary cactuses require exponential resolution in the worst case, answering an open question by Kindermann et al. [14]. Moreover, we provide a binary cactus that does not admit a self-approaching drawing. Finally, we show that 3-connected planar graphs admit increasing-chord drawings in the hyperbolic plane and characterize the trees that admit such drawings.

1 Introduction

Finding paths between two vertices is one of the most fundamental tasks users want to solve when considering graph drawings [16], for example to find a connection in a schematic map of a public transport system. Empirical studies have shown that users perform better in path-finding tasks if the drawings exhibit a strong geodesic-path tendency [12, 21]. Not surprisingly, graph drawings in which a path with certain properties exists between every pair of vertices have become a popular research topic. Over the last years a number of different drawing conventions implementing the notion of strong geodesic-path tendency have been suggested, namely greedy drawings [22], (strongly) monotone drawings [2], and self-approaching and increasing-chord drawings [1]. Note that throughout this paper, all drawings are straight-line and vertices are mapped to distinct points.

The notion of greedy drawings came first and was introduced by Rao et al. [22]. Motivated by greedy routing schemes, e.g., for sensor networks, one seeks a drawing where for every pair of vertices s and t, there exists an st-path, along which the distance to t decreases in every vertex. This ensures that greedily sending a message to a vertex that is

* A preliminary version of this paper appeared at the 22nd International Symposium on Graph Drawing, Würzburg, Germany [19]
† Algorithms and Complexity Group, TU Wien, Vienna, Austria, noellenburg@ac.tuwien.ac.at
‡ Institute of Theoretical Informatics, Karlsruhe Institute of Technology, Karlsruhe, Germany, [roman.prutkin|ignaz.rutter]@kit.edu

JoCG 7(1), 47–69, 2016 47
closer to the destination guarantees delivery. Papadimitriou and Ratajczak conjectured that every 3-connected planar graph admits a greedy embedding into the Euclidean plane [20]. This conjecture has been proved independently by Leighton and Moitra [17] and Angelini et al. [5]. Kleinberg [15] showed that every connected graph has a greedy drawing in the hyperbolic plane. Eppstein and Goodrich [9] showed how to construct such an embedding, in which the coordinates of each vertex are represented using only $O(\log n)$ bits, and Goodrich and Strash [11] provided a corresponding succinct representation for greedy embeddings of 3-connected planar graphs in $\mathbb{R}^2$. Angelini et al. [3] showed that some graphs require exponential area for a greedy drawing in $\mathbb{R}^2$. Wang and He [25] used a custom distance metric to construct planar, convex and succinct greedy embeddings of 3-connected planar graphs using Schnyder realizers [24]. Nöllenburg and Prutkin [18] characterized trees admitting a Euclidean greedy embedding. However, a number of interesting questions remain open, e.g., whether every 3-connected planar graph admits a planar and convex Euclidean greedy embedding (strong Papadimitriou-Ratajczak conjecture [20]). Regarding planar greedy drawings of triangulations, the only known result is an existential proof and a heuristic construction by Dhandapani [8] based on face-weighted Schnyder drawings.

While getting closer to the destination in each step, a greedy path can make numerous turns and may even look like a spiral, which hardly matches the intuitive notion of geodesic-path tendency. To overcome this, Angelini et al. [2] introduced monotone drawings, where one requires that for every pair of vertices $s$ and $t$ there exists a monotone path, i.e., a path that is monotone with respect to some direction. Ideally, the monotonicity direction should be $st$. This property is called strong monotonicity. Angelini et al. showed that biconnected planar graphs admit monotone drawings [2] and that plane graphs admit monotone drawings with few bends [4]. Kindermann et al. [14] showed that every tree admits a strongly monotone drawing. The existence of strongly monotone planar drawings remains open, even for triangulations.

Both greedy and monotone paths may have arbitrarily large detour, i.e., the ratio between the path length and the distance of the endpoints can, in general, not be bounded by a constant. Motivated by this fact, Alamdari et al. [1] recently initiated the study of self-approaching graph drawings. Self-approaching curves, introduced by Icking [13], are curves, where for any point $t'$ on the curve, the distance to $t'$ is continuously non-increasing while traversing the curve from the start to $t'$. Equivalently, a curve is self-approaching if, for any three points $a$, $b$, $c$ in this order along the curve, it is $\text{dist}(a, c) \geq \text{dist}(b, c)$, where $\text{dist}()$ denotes the Euclidean distance. An even stricter requirement are so-called increasing-chord curves, which are curves that are self-approaching in both directions. The name is motivated by the characterization of such curves, which states that a curve has increasing chords if and only if for any four distinct points $a$, $b$, $c$, $d$ in that order, it is $\text{dist}(b, c) \leq \text{dist}(a, d)$. Self-approaching curves have detour at most 5.333 [13] and increasing-chord curves have detour at most 2.094 [23]. Alamdari et al. [1] studied the problem of recognizing whether a given graph drawing is self-approaching and gave a complete characterization of trees admitting self-approaching drawings. Furthermore, Alamdari et al. [1] and Frati et al. [7] investigated the problem of connecting given points to obtain an increasing-chord drawing.

We note that every increasing-chord drawing is self-approaching and strongly monotone [1]. The converse is not true. A self-approaching drawing is greedy, but not necessarily
monotone, and a greedy drawing is generally neither self-approaching nor monotone. For trees, the notions of self-approaching and increasing-chord drawing coincide since all paths are unique.

**Contribution.** We obtain the following results on constructing self-approaching or increasing-chord drawings.

1. We show that every triangulation has an increasing-chord drawing (answering an open question of Alamdari et al. [1]) and construct a binary cactus that does not admit a self-approaching drawing (Sect. 3). The latter is a notable difference to greedy drawings since both constructions of greedy drawings for 3-connected planar graphs [5, 17] essentially show that every binary cactus has a greedy drawing. We also prove that strongly monotone (and, thus, increasing-chord) drawings of trees and binary cactuses require exponential resolution in the worst case, answering an open question by Kindermann et al. [14].

2. We show how to construct plane increasing-chord drawings for planar 3-trees (a special class of triangulations) using Schnyder realizers (Sect. 4). To the best of our knowledge, this is the first construction for this graph class, even for greedy and strongly monotone plane drawings, which addresses an open question of Angelini et al. [2].

3. We show that, similar to the greedy case [15], the hyperbolic plane \( H^2 \) allows representing a broader class of graphs than \( \mathbb{R}^2 \) (Sect. 5). We prove that a tree has a self-approaching or increasing-chord drawing in \( H^2 \) if and only if it either has maximum degree 3 or is a subdivision of \( K_{1,4} \) (this is not the case in \( \mathbb{R}^2 \); see the characterization by Alamdari et al. [1]), implying that every 3-connected planar graph has an increasing-chord drawing. (Barnette proved [6] that 3-connected planar graphs can always be spanned by binary trees.) We also show how to construct planar increasing-chord drawings of binary cactuses in \( H^2 \).

2 Preliminaries

For points \( a, b, c, d \in \mathbb{R}^2 \), let \( \text{ray}(a, b) \) denote the ray with origin \( a \) and direction \( \overrightarrow{ab} \) and let \( \text{ray}(a, bc) \) denote the ray with origin \( a \) and direction \( \overrightarrow{bc} \). Let \( \text{dir}(ab) \) be the vector \( \overrightarrow{ab} \) normalized to unit length. Let \( \angle(ab, cd) \) denote the smaller angle formed by the two vectors \( \overrightarrow{ab} \) and \( \overrightarrow{cd} \), and let \( \angle abc \) denote the angle \( \angle(\overrightarrow{ba}, \overrightarrow{ca}) \). For vectors \( \overrightarrow{v_1}, \overrightarrow{v_2} \) with \( \text{dir}(\overrightarrow{v_2}) = R_\alpha \cdot \text{dir}(\overrightarrow{v_1}) \), \( \alpha \in [0, 2\pi) \), we write \( \angle_{\text{ccw}}(\overrightarrow{v_1}, \overrightarrow{v_2}) := \alpha \).

We reuse some notation from the work of Alamdari et al. [1]. For points \( p, q \in \mathbb{R}^2 \), \( p \neq q \), let \( l^+_pq \) denote the halfplane not containing \( p \) bounded by the line through \( q \) orthogonal to the segment \( pq \). A piecewise-smooth curve is self-approaching if and only if for each point \( a \) on the curve, the line perpendicular to the curve at \( a \) does not intersect the curve at a later point [13]. This leads to the following characterization of self-approaching paths.

**Fact 1** (Corollary 2 in [1]). Let \( \rho = (v_1,v_2,\ldots,v_k) \) be a directed path embedded in \( \mathbb{R}^2 \) with straight-line segments. Then, \( \rho \) is self-approaching if and only if for all \( 1 \leq i < j \leq k \), the point \( v_j \) lies in \( l^+_{v_i,v_{i+1}} \).
We shall denote the reverse of a path $\rho$ by $\rho^{-1}$. Let $\rho = (v_1, v_2, \ldots, v_k)$ be a self-approaching path. Define $\text{front}(\rho) = \bigcap_{i=1}^{k-1} l^+_{v_iv_{i+1}}$, see also Fig. 1a. Using Fact 1, we can decide whether a concatenation of two paths is self-approaching.

**Fact 2.** Let $\rho_1 = (v_1, \ldots, v_k)$ and $\rho_2 = (v_k, v_{k+1}, \ldots, v_m)$ be self-approaching paths. The path $\rho_1 \rho_2 := (v_1, \ldots, v_k, v_{k+1}, \ldots, v_m)$ is self-approaching if and only if $\rho_2 \subseteq \text{front}(\rho_1)$.

A path $\rho$ has increasing chords if for any points $a, b, c, d$ in this order along $\rho$, it is $\text{dist}(b, c) \leq \text{dist}(a, d)$. A path has increasing chords if and only if it is self-approaching in both directions. The following result can be obtained as a corollary of Lemma 3 in [13].

**Lemma 1.** Let $\rho = (v_1, \ldots, v_k)$ be a path such that for any $i < j$, $i, j \in \{1, \ldots, k-1\}$, it is $\angle(v_i v_{i+1}, v_j v_{j+1}) \leq 90^\circ$. Then, $\rho$ has increasing chords.

**Proof.** For any $j > i$, $i, j \in \{1, \ldots, k-1\}$, it is $\angle(v_{j+1} v_i, v_{i+1} v_j) \leq 90^\circ$. Thus, the condition of the lemma also holds for $\rho^{-1}$, and by symmetry it is sufficient to prove that $\rho$ is self-approaching.

We claim that for each $i \in \{1, \ldots, k-1\}$ and each $j \in \{i+1, \ldots, k\}$, it is $v_j \in l^+_{v_i v_{i+1}}$. Once the claim is proved, it follows from Fact 1 that $\rho$ is self-approaching. For the proof of the claim let $i \in \{1, \ldots, k-1\}$ be arbitrary and fixed. It suffices to show that $v_{i+2}, \ldots, v_k \in l^+_{v_i v_{i+1}}$.

First consider $v_{i+2}$. By the condition of the lemma, it is $\angle(v_{i+2} v_i, v_{i+1} v_{i+2}) \leq 90^\circ$. Therefore, $v_{i+2} \in l^+_{v_i v_{i+1}}$. Now assume $v_j \in l^+_{v_i v_{i+1}}$ for some $j \in \{i+2, \ldots, k-1\}$. We show $v_j+1 \in l^+_{v_i v_{i+1}}$. Consider the halfplane $h \subseteq l^+_{v_i v_{i+1}}$ whose boundary is parallel to that of $l^+_{v_i v_{i+1}}$ and contains $v_j$. Since $\angle(v_i, v_{j+1}) \leq 90^\circ$, it is $v_j+1 \in h \subseteq l^+_{v_i v_{i+1}}$. 

Let $G = (V, E)$ be a connected graph. A separating $k$-set is a set of $k$ vertices whose removal disconnects the graph. A vertex forming a separating 1-set is called a cutvertex. A graph is $c$-connected if it does not admit a separating $k$-set with $k \leq c-1$; 2-connected graphs are also called biconnected. A connected graph is biconnected if and only if it does not contain a cutvertex. A block is a maximal biconnected subgraph. The block-cutvertex
tree (or BC-tree) $T_G$ of $G$ has a $B$-node for each block of $G$, a $C$-node for each cutvertex of $G$ and, for each block $v$ containing a cutvertex $v$, an edge between the corresponding $B$- and $C$-node. We associate $B$-nodes with their corresponding blocks and $C$-nodes with their corresponding cutvertices.

The following notation follows the work of Angelini et al. [5]. Let $T_G$ be rooted at some block $v$ containing a non-cutvertex (such a block $v$ always exists). For each block $\mu \neq v$, let $\pi(\mu)$ denote the parent block of $\mu$, i.e., the grandparent of $\mu$ in $T_G$. Let $\pi^2(\mu)$ denote the parent block of $\pi(\mu)$ and, generally, $\pi^{i+1}(\mu)$ the parent block of $\pi^i(\mu)$. Further, we define the root $r(\mu)$ of $\mu$ as the cutvertex contained in both $\mu$ and $\pi(\mu)$. Note that $r(\mu)$ is the parent of $\mu$ in $T_G$. In addition, for the root node $v$ of $T_G$, we define $r(v)$ to be some non-cutvertex of $v$. Let $\text{depth}_B(\mu)$ denote the number of $B$-nodes on the $v\mu$-path in $T_G$ minus 1, and let $\text{depth}_C(r(\mu)) = \text{depth}_B(\mu)$. If $\mu$ is a leaf of $T_G$, we call it a leaf block.

A cactus is a graph in which every edge is part of at most one cycle. Note that every cactus is outerplanar. In a binary cactus every cutvertex is part of exactly two blocks. For a binary cactus $G$ with a block $\mu$ containing a cutvertex $v$, let $G^v_\mu$ denote the maximal connected subgraph containing $v$ but no other vertex of $\mu$. We say that $G^v_\mu$ is a subcactus of $G$. Let $G$ be a binary cactus with a fixed root and let $v$ be a cutvertex of $G$. Then the block $\mu$ containing $v$ such that $v \neq r(\mu)$ is unique, and we write $G^v_v$ for $G^v_\mu$.

A triangulated cactus is a cactus together with additional edges, which make each of the cactus blocks internally triangulated. A triangular fan with vertices $V_1 = \{v_0, v_1, \ldots, v_k\}$ and root $v_0$ is a graph on $V_i$ with edges $v_iv_{i+1}$, $i = 1, \ldots, k - 1$, as well as $v_0v_i$, $i = 1, \ldots, k$. Let us consider a special kind of triangulated cactuses, each of whose blocks $\mu$ is a triangular fan with root $r(\mu)$. We call such a cactus downward-triangulated and every edge of a block $\mu$ incident to $r(\mu)$ a downward edge. Fig. 1b and 1c show a downward-triangulated binary cactus and the corresponding BC-tree.

Consider a fixed straight-line drawing of a cactus $G$ with root $r$. We define the set of upward directed edges $E_U(G) = \{r(\mu)v \mid \mu \text{ is a block of } G \text{ containing } v, v \neq r(\mu)\}$. Note that if $G$ is not triangulated, some edges in $E_U(G)$ might not be edges in $G$. We define the set of upward directions $U(G) = \{r(\mu)v \mid \mu \text{ is a block of } G \text{ containing } v, v \neq r(\mu)\}$ and the set of downward directions $D(G) = \{uv \mid v = r(\mu)\}$ of $G$. If $G$ is binary, then, for cutvertex $u$, let $U_u$ denote the upward directed edges of the subcactus rooted at $u$ or, formally, $U_u = E_U(G^u)$.

3 Graphs with Self-Approaching Drawings

A natural approach to construct (not necessarily plane) self-approaching drawings is to construct a self-approaching drawing of a spanning subgraph. For instance, to draw a graph $G$ containing a Hamiltonian path $H$ with increasing chords, we simply draw $H$ consecutively on a line. In this section, we consider 3-connected planar graphs and the special case of triangulations, which addresses an open question of Alamdari et al. [1]. These graphs are known to have a spanning binary cactus [5,17]. Angelini et al. [5] showed that every triangulation has a spanning downward-triangulated binary cactus.
We show that every downward-triangulated binary cactus has an increasing-chord drawing.

**Theorem 1.** Let $G$ be a downward-triangulated binary cactus. For any $0^\circ < \varepsilon < 90^\circ$, there exists an increasing-chord drawing $\Gamma_\varepsilon$ of $G$, such that for each vertex $v$ contained in some block $\mu$, $v \neq r(\mu)$, the angle formed by $r(\mu)v$ and $\vec{e}^2$ is less than $\frac{\varepsilon}{2}$.

**Proof.** Let $G$ be rooted at block $\nu$. As our base case, let $\nu = G$ be a triangular fan with vertices $v_0, v_1, \ldots, v_k$ and root $v_0 = r(\nu)$. We draw $v_0$ at the origin and distribute $v_1, \ldots, v_k$ on the unit circle, such that $\angle(\vec{e}^2, \overrightarrow{v_0v_1}) = k\alpha/2$ and $\angle(\overrightarrow{v_0v_i}, \overrightarrow{v_0v_{i+1}}) = \alpha$, $\alpha = \varepsilon/2k$; see Fig. 2a. By Lemma 1, path $(v_1, \ldots, v_k)$ has increasing chords.

Now let $G$ have multiple blocks. We draw the root block $\nu$, $v_0 = r(\nu)$, as in the previous case, but with $\alpha = \frac{\varepsilon}{4k}$. Then, for each $i = 1, \ldots, k$, we choose $\varepsilon' = \frac{\varepsilon}{4}$ and draw the subcactus $G_i = G_i^{v_0}$ rooted at $v_i$ inductively, such that the corresponding drawing $\Gamma_{i,\varepsilon'}$ is aligned at $\overrightarrow{v_0v_i}$ instead of $\overrightarrow{e_2}$; see Fig 2b. Note that $\varepsilon'$ is the angle of the cones (gray) containing $\Gamma_{i,\varepsilon'}$. Obviously, all downward edges of $G$ form angles less than $\frac{\varepsilon}{2}$ with $\vec{e}^2$.

We must be able to reach any $t$ in any $G_j$ from any $s$ in any $G_i$ via an increasing-chord path $\rho$. To achieve this, we make sure that no normal on a downward edge of $G_j$ crosses the drawing of $G_j$, $j \neq i$. Let $\Lambda_i$ be the cone with apex $v_i$ and angle $\varepsilon'$ aligned with $\overrightarrow{v_0v_i}$, $v_0 \notin \Lambda_i$ (gray regions in Fig. 2b). Let $s_i^1$ and $s_i^2$ be the left and right boundary rays of $\Lambda_i$ with respect to the direction of $\overrightarrow{v_0v_i}$, and $h_i^1$, $h_i^2$ the halfplanes with boundaries containing $v_i$ and orthogonal to $s_i^1$ and $s_i^2$ respectively, such that $v_0 \in h_i^1 \cap h_i^2$. For $i = 2, \ldots, k - 1$, define $\mathcal{O}_i = \Lambda_i \cap h_{i-1}^1 \cap h_{i+1}^1$ (thin blue quadrilateral in Fig. 2c). Let $\mathcal{O}_1 = \Lambda_1 \cap h_2^1$ and $\mathcal{O}_k = \Lambda_k \cap h_{k-1}^1$. For any $i, j = 1, \ldots, k$, $i \neq j$, it holds $\mathcal{O}_j \subseteq h_i^1 \cap h_i^2$. We now scale each drawing $\Gamma_{i,\varepsilon'}$ such that it is contained in $\mathcal{O}_i$. In particular, for any downward edge $uv$ in $\Gamma_{i,\varepsilon'}$, we have $\Gamma_{j,\varepsilon'} \subseteq \mathcal{O}_j \subseteq h_{uv}^+$ for $j \neq i$. We claim that the resulting drawing of $G$ is an increasing-chord
drawing.

Consider vertices $s, t$ of $G$. If $s$ and $t$ are contained in the same subgraph $G_i$, an increasing-chord $st$-path in $G_i$ exists by induction. If $s$ is in $G_i$ and $t$ is $v_0$, let $\rho_i$ be the $sv_i$-path in $G_i$ that uses only downward edges. By Lemma 1, path $\rho_i$ is increasing-chord and remains so after adding edge $v_iv_0$.

Finally, assume $t$ is in $G_j$ with $j \neq i$. Let $\rho_j$ be the $tv_j$-path in $G_j$ that uses only downward edges. Due to the choice of $\epsilon'$, $h_i^+ \cap h_i^+ \subseteq \text{front}(\rho_i)$ contains $v_1, \ldots, v_k$ in its interior. Consider the path $\rho' = (v_i, v_{i+1}, \ldots, v_j)$. It is self-approaching by Lemma 1; also, $\rho' \subseteq \text{front}(\rho_i)$ and $\rho_j \subseteq \text{front}(\rho')$. It also holds $\rho_j \subseteq \text{front}(\rho_i)$. Fact 2 lets us concatenate $\rho_i$, $\rho'$ and $\rho_j^{-1}$ to a self-approaching path. By a symmetric argument, it is also self-approaching in the opposite direction and, thus, is increasing-chord.

Since every triangulation has a spanning downward-triangulated binary cactus [5], this implies that planar triangulations admit increasing-chord drawings.

**Corollary 1.** Every planar triangulation admits an increasing-chord drawing.

### 3.2 Exponential worst case resolution

The construction for a spanning downward-triangulated binary cactus in Section 3.1 requires exponential area. In this section, we show that we cannot do better in the worst case even for strongly monotone drawings of downward-triangulated binary cactuses. Recall that increasing-chord drawings are strongly monotone.

For the following lemma, we want to point out the difference between a greedy $st$-path and a greedy drawing of a graph $G$, such that $G$ is a path. In a fixed drawing, a $st$-path $\rho = (v_0 = s, v_1, \ldots, v_k, v_{k+1} = t)$ is greedy (or distance-decreasing), if it is $|v_{i+1}| < |v_it|$ for every $i = 0, \ldots, k$. Note that for some $0 \leq i < j \leq k + 1, \{i, j\} \neq \{0, k + 1\}$, the $v_iv_j$-path $(v_i, v_{i+1}, \ldots, v_j)$ is not necessarily greedy; see Fig. 3a. On the other hand, for a graph $G$ which is a path $\rho = (v_0, v_1, \ldots, v_k, v_{k+1})$, a drawing $\Gamma$ is a greedy drawing of $G$ if every $v_iv_j$-path $(v_i, v_{i+1}, \ldots, v_j)$ and every $v_jv_i$-path $(v_j, v_{j-1}, \ldots, v_i)$ in $\Gamma$ is a greedy path for any $0 \leq i < j \leq k + 1$.

The following lemma describes directions of certain edges in a greedy or monotone drawing of a cactus.

**Lemma 2.** For a cactus $G = (V, E)$ and two vertices $s, t \in V$, consider the cutvertices $v_1, \ldots, v_k$ lying on every $st$-path in $G$ in this order. In any greedy drawing of $G$, connecting consecutive vertices in $(s, v_1, \ldots, v_k, t)$ would form a greedy drawing of the path $(s, v_1, \ldots, v_k, t)$. In any monotone drawing, connecting consecutive vertices in $(s, v_1, \ldots, v_k, t)$ would form a monotone drawing of the path $(s, v_1, \ldots, v_k, t)$. In both cases, $\text{ray}(v_1, s)$ and $\text{ray}(v_k, t)$ diverge.

**Proof.** Let $v_0 = s, v_{k+1} = t$. For $0 \leq i < j \leq k + 1$, any $v_iv_j$-path and any $v_jv_i$-path in $G$ contains vertices $v_i, v_{i+1}, \ldots, v_j$. Since a path in a greedy drawing of $G$ remains greedy after
Figure 3: (a) The st-path $(s, v_1, v_2, v_3, t)$ is a greedy path, but its $sv_2$ subpath is not. Thus, this drawing is not a greedy drawing of a path. (b) Proof of Lemma 2. (c) Proof of Lemma 3.

Figure 4: Family of binary cactuses $G_k$ requiring exponential area for any strongly monotone drawing. (a) Central cactus $G'$; (b) binary subcactus $C_k$ attached to each vertex of degree 1 of $G'$. In a strongly monotone drawing of $G_k$, it must hold: (c) $|u_2u_4| \leq |u_2v_2| \tan \varepsilon$; (d) $|u_4v_4| \leq |u_2u_4| \tan \varepsilon$.

replacing subpaths by shortcuts, the segments $sv_1, v_1v_2, \ldots, v_{k-1}v_k, v_kv_t$ form a greedy drawing. By Lemma 7 of Angelini et al. [3], ray($v_1, s$) and ray($v_k, t$) diverge; see Fig. 3b.

Analogously, a path remains monotone after replacing subpaths by shortcuts. Therefore, in a monotone drawing of $G$, segments $sv_1, v_1v_2, \ldots, v_{k-1}v_k, v_kv_t$ form a monotone drawing. Since a monotone path cannot make a turn of $180^\circ$ or more, ray($v_1, s$) and ray($v_k, t$) must diverge.

For the following lemma, consider a greedy or monotone drawing of a cactus $G$ with root $r$. Recall that for a cutvertex $u$, the set $U_u$ denotes the upward directed edges of the subcactus rooted at $u$, or, formally, $U_u = E_U(G^u)$. Then, the following property holds.

**Lemma 3.** In a monotone or greedy drawing of a cactus with root $r$, consider cutvertices $u, v \neq r$, such that the subcactuses $G^u$ and $G^v$ are disjoint. Then the edges in $U_u$ and in $U_v$ each form a single interval in the circular order induced by their joint set of directions.

**Proof.** Consider four pairs of vertices $x_i, y_i, i = 1, \ldots, 4$, such that $x_iy_i \in U_u$ for $i = 1, 2$ and $x_iy_i \in U_v$ for $i = 3, 4$. Note that, by the definition of $U_u$ and $U_v$, vertices $x_i$ and $x_j$ are cutvertices. For $i = 1, 2, j = 3, 4$, let $\rho_{ij}$ denote the vertex sequence $y_i, x_i, u, v, x_j, y_j$. Since $x_i, u, v, x_j$ are cutvertices, $\rho_{ij}$ is a subsequence of every $y_iy_j$-path. By Lemma 2, each such $\rho_{ij}$ forms a monotone or a greedy drawing of a path, respectively. Hence, $\rho_{ij}$ is non-crossing and cannot make a turn of $180^\circ$ or more. Additionally, by Lemma 2, rays
ray\((x_i, y_i)\) and ray\((x_j, y_j)\) must diverge. Finally, ray\((x_j, y_j)\) cannot cross \(\rho_{ij}\), since this would imply that \(\rho_{ij}\) has made a turn of \(180^\circ\) or more, and neither can ray\((x_i, y_i)\).

We define \(p = (u + v)/2\) and choose an arbitrary \(R > 0\), such that all paths \(\rho_{ij}\) are contained inside a circle \(C\) with center \(p\) and radius \(R\). Let \(p_i\) be the intersection of ray\((x_i, y_i)\) and \(C\). Assume \(p_1, p_3, p_2, p_4\) is the counterclockwise order of \(p_i\) on the boundary of \(C\); see Fig. 3c. Then, for some pair \(i, j, i \in \{1, 2\}, j \in \{3, 4\}\), there exists a crossing of ray\((x_i, y_i)\) or ray\((x_j, y_j)\) with \(\rho_{ij}\) or with each other; a contradiction. Therefore, \(p_1, p_2\) as well as \(p_3, p_4\) appear consecutively on the boundary of \(C\). Now let the radius \(R\) approach infinity. Then, \(\overline{pp_i}\) becomes parallel with \(\overline{x_3y_1}\). Therefore, the circular order \(\overline{x_1y_1}, \overline{x_3y_3}, \overline{x_2y_2}, \overline{x_4y_4}\) is not possible, and the statement follows.

Note that for trees, Angelini et al. [2] call this property \textit{slope disjointness}. Consider the following family of binary cactuses \(G_k\). Let \(G'\) be a rooted binary cactus with eleven vertices \(r_1, \ldots, r_{11}\) of degree 1 and its root \(r\) as the only vertex of degree 2; see Fig. 4a. Next, consider cactus \(C_k\) consisting of a chain of \(k + 1\) triangles and some additional degree-1 nodes as in Fig. 4b. We construct \(G_k\) by attaching a copy of \(C_k\) to each \(r_i\) in \(G'\). From now on, consider a strongly monotone drawing of \(G_k\).

Using Lemma 3 and the pigeonhole principle, we can show the following fact.

**Lemma 4.** For some \(r_i, i \in \{1, \ldots, 11\}\), each pair of directions in \(U_{r_i}\) forms an angle at most \(\varepsilon = 360^\circ/11\).

**Proof.** Consider the two cutvertices of the root block of \(G_k\); see Fig. 4a. By Lemma 3, vectors in \(U_{r_1} \cup \cdots \cup U_{r_{11}}\) appear in the following circular order: first the vectors in \(U_{r_1} \cup \cdots \cup U_{r_7}\), then the vectors in \(U_{r_8} \cup \cdots \cup U_{r_{11}}\). By applying the same argument to the child blocks repetitively, it follows that the vectors have the following circular order: first the vectors in \(U_{r_{\pi(3)}}\), \ldots, then the vectors in \(U_{r_{\pi(11)}}\) for some permutation \(\pi\). Therefore, for some \(i\), each pair of directions in \(U_{r_i}\) forms an angle at most \(\varepsilon = 360^\circ/11\).

Now consider a vertex \(r_i\) with the property of Lemma 4. Let the vertices of its subcactus be named as in Fig. 4b. Without loss of generality, we may assume that each vector in \(U_{r_i}\) forms an angle at most \(\varepsilon/2\) with the upward direction \(\vec{e_2}\). We show that certain directions have to be almost horizontal.

**Lemma 5.** For even \(i, j, 2 \leq j \leq i \leq 2k + 2\), consider vertices \(u_i, v_j\). Vector \(\overline{u_i, v_j}\) forms an angle at most \(\varepsilon/2\) with the horizontal axis.

**Proof.** Consider a strongly monotone \(u_i, v_j\)-path \(\rho\). Vertices \(u_i, u_{i-1}, v_{j-1}, v_j\) must appear on \(\rho\) in this order. It is \(\angle(\overline{u_{i-1}u_i}, \overline{v_{j-1}v_j}) \leq \varepsilon\). Furthermore, by the strong monotonicity of \(\rho\), it is \(\angle u_{i-1}u_i v_j, \angle u_{i-1}v_j u_i < 90^\circ\), as well as \(\angle v_{j-1}u_i v_j, \angle v_{j-1}v_j u_i < 90^\circ\).

Consider the strip \(S = \mathbb{R}^2 \setminus (l_{u_i}^+ \cup l_{v_j}^+)\); see Fig. 5a. From the above observation on the angles, it follows \(u_{i-1}, v_{j-1} \in S\). Line segment \(u_i v_j\) divides \(S\) into two parts. Assume \(u_{i-1}\) and \(v_{j-1}\) are in different parts. But then, the angle \(\angle(\overline{u_{i-1}u_i}, \overline{v_{j-1}v_j})\) is at least \(90^\circ\), a contradiction. Thus, \(u_{i-1}\) and \(v_{j-1}\) are in the same part, and, since \(\overline{u_{i-1}u_i}, \overline{v_{j-1}v_j}\) point upwards, vertices \(u_{i-1}\) and \(v_{j-1}\) are below the line through the segment \(u_i v_j\).
Let \( p \) be the intersection of the lines through \( u_{i-1}u_i \) and \( v_{j-1}v_j \); see Fig. 5b. Point \( p \) also lies below the line through \( u_i, v_j \) and \( p \). It is \( \angle pu_iv_j = \angle (\overline{u_{i-1}u_i}, \overline{v_{j-1}v_j}) \leq \varepsilon \). Furthermore, it is \( \angle pu_iv_j = \angle u_{i-1}u_iv_j < 90^\circ \), and \( \angle pu_iv_i = \angle v_{j-1}v_ju_i < 90^\circ \). Therefore, \( \angle u_{i-1}u_iv_j, \angle v_{j-1}v_ju_i \in (90^\circ - \varepsilon, 90^\circ) \).

Assume it is \( \angle (\overline{u_{i-1}u_i}, \overline{e_1}) \geq \varepsilon/2 \), and let \( \overline{u_{i-1}u_i} \) point upwards and to the left. The other cases are analogous. Then, since \( \angle u_{i-1}u_iv_j \in (90^\circ - \varepsilon, 90^\circ) \), edge \( u_{i-1}u_i \) must point upwards and to the right, and it must be \( \angle (\overline{u_{i-1}u_i}, \overline{e_2}) < \varepsilon/2 \), a contradiction to the above assumption on the directions of the upward edges. Therefore, the statement follows.

The following lemma essentially shows that \( G_k \) requires exponential resolution.

**Lemma 6.** For \( i = 2, 4, \ldots, 2k \), it holds \( |u_{i+2}v_{i+2}| \leq (\tan \varepsilon)^2 |u_iv_i| \).

**Proof.** For brevity, let \( i = 2 \). First, we show that \( |u_2v_2| \) is significantly larger than \( |u_2u_4| \); see Fig. 4c. By Lemma 5, it is \( \angle u_2u_4v_2 \in (90^\circ - \varepsilon, 90^\circ + \varepsilon) \). Therefore, \( \sin \angle u_2v_2u_4 \geq \sin(90^\circ - \varepsilon) \).

\[
\frac{|u_2u_4|}{|u_2v_2|} = \frac{\sin \angle u_2v_2u_4}{\sin \angle u_2u_4v_2} \leq \frac{\sin \varepsilon}{\sin(90^\circ - \varepsilon)} = \tan \varepsilon.
\]

Next, we show that \( |u_2u_4| \) is significantly larger than \( |u_4v_4| \); see Fig. 4d. By Lemma 5, it is \( \angle u_2v_4u_4 \in (90^\circ - \varepsilon, 90^\circ + \varepsilon) \). Therefore, \( \sin \angle u_2v_4u_4 \geq \sin(90^\circ - \varepsilon) \).

\[
\frac{|u_4v_4|}{|u_2u_4|} = \frac{\sin \angle u_2v_4u_4}{\sin \angle u_2v_4u_4} \leq \frac{\sin \varepsilon}{\sin(90^\circ - \varepsilon)} = \tan \varepsilon.
\]

Thus, \( |u_4v_4| \leq |u_2u_4| \tan \varepsilon \leq |u_2v_2| (\tan \varepsilon)^2 \).

As a consequence of Lemma 6 we get \( |u_{2k+2}v_{2k+2}| \leq |u_2v_2| (\tan \varepsilon)^{2k} \). It is \( (\tan \varepsilon)^2 < 0.414 \). Since cactus \( G_k \) contains \( n = \Theta(1) + 44k \) vertices, the following exponential lower bound holds for the resolution of strongly monotone drawings.

**Theorem 2.** There exists an infinite family of binary cactuses with \( n \) vertices that require resolution \( \Omega(2^{\frac{n}{2}}) \) for any strongly monotone drawing.
Using this result, we can construct a family of trees requiring exponential area for any strongly monotone drawing. Consider the binary spanning tree $T_k$ of $G_k$ created by removing the thick green edges in Fig. 4a and 4b. Obviously, by Theorem 2 it requires resolution $\Omega(2^{\frac{n}{44}})$ for any strongly monotone drawing. This answers an open question by Kindermann et al. [14]. Replacing degree-2 vertices by shortcuts and applying a more careful analysis lets us prove the following result.

**Theorem 3.** There exists an infinite family of binary trees with $n$ vertices that require resolution $\Omega(2^{\frac{n}{22}})$ for any strongly monotone drawing.

Observe that exponential worst-case resolution of strongly monotone drawings of binary cactuses is a stronger result than the corresponding statement for trees. A strongly monotone drawing of a binary cactus does not necessarily induce a strongly monotone drawing of any of its spanning trees.

### 3.3 Non-triangulated cactuses

The construction for an increasing-chord drawing from Section 3.1 fails if the blocks are not triangular fans since we now cannot just use downward edges to reach the common ancestor block. Consider the family of rooted binary cactuses $G_n = (V_n, E_n)$ defined as follows. Graph $G_0$ is a single 4-cycle, where an arbitrary vertex is designated as the root. For $n \geq 1$, consider two disjoint copies of $G_{n-1}$ with roots $a_0$ and $c_0$. We create $G_n$ by adding new vertices $r_0$ and $b_0$ both adjacent to $a_0$ and $c_0$; see Fig. 6a. For the new block $\nu$ containing $r_0, a_0, b_0, c_0$, we set $r(\nu) = r_0$. We select $r_0$ as the root of $G_n$ and $\nu$ as its root block. For a block $\mu_i$ with root $r_i$, let $a_i, b_i, c_i$ be its remaining vertices, such that $b_i r_i \notin E_n$. For a given drawing, due to the symmetry of $G_n$, we can rename the vertices $a_i$ and $c_i$ such that $\angle_{ccw}(r_i c_i, r_i a_i) \leq 180^\circ$. If a fixed block $\mu_i$ is considered, we refer to $a_i, b_i, c_i$ as $a, b, c$ for brevity. We now prove the following negative result.

**Theorem 4.** For $n \geq 10$, $G_n$ has no self-approaching drawing.

The outline of the proof is as follows. We show that every self-approaching drawing $\Gamma$ of $G_{10}$ contains a self-approaching drawing of $G_3$ such that for each block $\mu$ of this $G_3$, the angle at $r = r(\mu)$ is very small, angles at $a$ and $c$ are 90° or slightly larger (Lemma 8) and such that sides $ra$ and $rc$ have almost the same length which is significantly greater than dist($a, c$) (Lemma 10). In addition, the following properties hold for this $G_3$.

1. If $\mu_i$ is contained in the subcactus rooted at $c_j$, each self-approaching $b_i a_j$-path uses edge $b_i a_i$, and analogously for the symmetric case; see Lemma 9.

2. Each block is drawn significantly smaller than its parent block; see Lemma 11(i).

3. If the descendants of block $\mu$ form subcactuses $G_k$ with $k \geq 2$ on both sides, the parent block of $\mu$ must be drawn smaller than $\mu$; see Lemma 11(ii).

Obviously, the second and third conditions are contradictory. Note that every block has to be self-approaching. However, it might be non-convex and even non-planar.
Observation 1. In a self-approaching drawing of a polygon $P$, no two non-consecutive angles can be both less than $90^\circ$.

Proof. If $P$ is a triangle, it is trivially self-approaching. Let now $v_1, v_2, v_3, v_4$ be pairwise distinct vertices appearing in this circular order around the boundary of $P$. Let the angles at both $v_2$ and $v_4$ be less than $90^\circ$. However, a self-approaching $v_1v_3$-path must use either $v_2$ or $v_4$, a contradiction. □

The following lemmas will be used to show that the drawings of certain blocks must be relatively thin, i.e., their downward edges have similar directions.

Lemma 7. Every self-approaching drawing of $G_{10}$ contains a cutvertex $\overline{r}$, such that each pair of directions in $U_{\overline{r}}$ form an angle at most $\varepsilon = 22.5^\circ$.

Proof. Denote by $r_j$, $j = 1, \ldots, 16$, the cutvertices with $\text{depth}_C(r_j) = 4$. By an argument similar to the one in the proof of Lemma 4, the edges in $U_{r_j}$ appear in the following circular order by their directions: first the edges in $U_{r_\pi(1)}$, then the edges in $U_{r_\pi(2)}$, ..., then the edges in $U_{r_\pi(16)}$ for some permutation $\pi$. Therefore, by the pigeonhole principle, the statement holds for some $j \in \{1, \ldots, 16\}$ and $\overline{r} = r_j$. □

Let $\overline{r}$ be a cutvertex in the fixed drawing at depth$_C(\overline{r}) = 4$ with the property shown in Lemma 7. Then, $G^\overline{r}$ is isomorphic to $G_6$. From now on, we only consider non-leaf blocks $\mu_i$ and vertices $r_i, a_i, b_i, c_i$ in $G^\overline{r}$. We shall sometimes name the points $a$ instead of $a_i$ etc. for convenience. We assume $\angle(\overrightarrow{e_2}, \overrightarrow{r a}), \angle(\overrightarrow{e_2}, \overrightarrow{r c}) \leq \varepsilon/2$. The following lemma is proved using basic trigonometric arguments.

Lemma 8. It holds:

(i) $\angle abc \geq 90^\circ$;

(ii) $G^a \subseteq l_{ba}^+, G^c \subseteq l_{bc}^+$;

(iii) $\angle bar \leq 90^\circ + \varepsilon$, $\angle ber \leq 90^\circ + \varepsilon$.

(iv) For vertices $u$ in $G^a$, $v$ in $G^c$ of degree 4 it is $\angle(\overrightarrow{uv}, \overrightarrow{e_1}) \leq \varepsilon/2$. 

---

Figure 6: (a) cactuses $G_n$; (b) Lemma 8(iii); (c),(d): Lemma 8(iv).
Proof. (i) It is $\angle \text{arc} \leq \varepsilon$. Thus, by Observation 1, $\angle \text{abc} \geq 90^\circ$.

(ii) Let $t$ be a vertex of $G^c$. Since $\angle \text{arc} \leq \varepsilon < 90^\circ$, any self-approaching at-path must contain $bc$. Thus, $t \in \overleftrightarrow{bc}$, and the claim for $G^c$ and, similarly, for $G^a$ follows.

(iii) Consider block $\mu'$ containing $d' \neq a$, $r(\mu') = a$; see Fig. 6b. Then, $\angle(\bar{r}d', dd') \leq \varepsilon$. By (ii), it is $\bar{b}a \geq 90^\circ$. If $\angle \text{bar} > 90^\circ + \varepsilon$, it is $\angle(\bar{r}d', dd') > \varepsilon$, a contradiction. The same argument applies for $\angle \text{bc}r$.

(iv) Since $u, v$ have degree 4, they are roots of some blocks. Let $u_1$ be a neighbor of $u$ in $G^u$ and $v_1$ a neighbor of $v$ in $G^v$ maximizing $\angle u_1uv$ and $\angle v_1vu$; see Fig. 6c. By considering self-approaching $u_1v$ and $v_1u$-paths, it follows $\angle u_1uv = \angle v_1vu \geq 90^\circ$. Also, $\angle \text{ray}(u,v)$ and $\angle \text{ray}(v_1,v)$ converge by Lemma 2. Let $p$ be their intersection. Then, $\angle upv \leq \varepsilon$ and $\angle pvu, \angle pvu \leq 90^\circ$. It is $\angle(\bar{p}u, \bar{p}v) \leq \varepsilon/2$ and $\angle(\bar{p}v, \bar{v}c) \leq \varepsilon/2$. Therefore, if $\bar{v}u$ points upward, it forms an angle at most $\varepsilon/2$ with the horizontal direction. If $\bar{v}u$ points downward, by symmetric arguments, $\bar{v}u$ forms an angle at most $\varepsilon/2$ with the horizontal direction. The same holds for $\bar{a}r, \bar{a}r, \bar{a}c$.

It remains to show that $u$ is “to the left” of $v$. Since it is $\angle \text{ccw}(\bar{r}a, \bar{c}a) < 180^\circ$ and $\angle \text{cw}(\bar{r}a, \bar{a}c) \leq \varepsilon$, it is $\angle(\bar{a}r, \bar{a}c) \leq \varepsilon/2$. Consider the two vertically aligned cones with apices $a$ and $c$ and angle $\varepsilon$ (gray in Fig. 6d). Vertex $u$ must be in the cone of $a$, and vertex $v$ in the cone of $c$. If $u$ is not in the cone of $c$ and, at the same time, $v$ not in the cone of $a$, then $v$ is to the right of $u$. In this case, it is $\angle(\bar{v}u, \bar{v}c) \leq \varepsilon/2$, and we are done.

Now assume $\angle(\bar{v}u, \bar{v}c) \leq \varepsilon/2$. Then, by the above argument, $u$ is in the cone of $c$ or $v$ in the cone of $a$ (without loss of generality, $u$ is in the cone of $a$). Thus, $u$ must be in the dark gray area in Fig. 6d). This contradicts the fact that $\bar{a}r$ forms an angle of at most $\varepsilon/2$ with the horizontal direction. \hfill $\square$

We can now describe block angles at $a_i, c_i$ more precisely and characterize certain self-approaching paths in $G^a$. We show that a self-approaching path from $b_i$ downwards and to the left, i.e., to an ancestor block $\mu_j$ of $\mu_i$, such that $\mu_j$ is in $G^a$, must use $a_i$. Similarly, a self-approaching path downwards and to the right must use $c_i$. Since for several ancestor blocks of $\mu_i$ the roots lie on both of these two kinds of paths, we can bound the area containing them and show that it is relatively small. This implies that the ancestor blocks are small as well, providing a contradiction.

Lemma 9. Consider non-leaf blocks $\mu_0, \mu_1, \mu_2$, such that $r(\mu_1) = a_0$ and $\mu_2$ in $G^a_1$; see Fig. 7a.

(i) It is $\angle r_2a_2b_2, \angle r_2c_2b_2 \in [90^\circ, 90^\circ + \varepsilon]$, $b_2$ lies to the right of $\text{ray}(r_2, a_2)$ and to the left of $\text{ray}(r_2, c_2)$.

(ii) Each self-approaching $b_2a_0$-path uses $a_2$; each self-approaching $b_2c_1$-path uses $c_2$.

Proof. (i) Assume $\angle r_2a_2b_2 < 90^\circ$. Then, all self-approaching $b_2a_0$ and $b_2c_1$-paths must use $c_2$. By Lemma 8(iv), the lines through $a_0c_2$ and $c_2c_1$ are “almost horizontal”, i.e., $\angle(\bar{a}c_2, \bar{e}_1), \angle(\bar{c}_2c_1, \bar{e}_1) \leq \varepsilon/2$. Since $r_2c_2$ is “almost vertical”, $r_2$ must lie below these lines and it is $\angle a_0c_2r_2, \angle c_1c_2r_2 \in [90^\circ - \varepsilon, 90^\circ + \varepsilon]$, see Fig. 7b.
Therefore, \( \angle r_2c_2b_2 \geq 90^\circ \). Furthermore, since every self-approaching \( b_2a_0 \)-path must use \( c_2 \), it is \( \angle a_0c_2b_2 \geq 90^\circ \). Therefore, \( b_2 \) cannot lie inside the counterclockwise angle between \( c_2a_0 \) and \( c_2r_2 \), since it is \( \angle_{ccw}(c_2a_0,c_2r_2) \leq 90^\circ + \varepsilon < \angle r_2c_2b_2 + \angle a_0c_2b_2 \). Thus, \( b_2 \) is above \( a_0c_2 \), and it is \( \angle r_2c_2b_2 = \angle a_0c_2r_2 + \angle a_0c_2b_2 \geq (90^\circ - \varepsilon) + 90^\circ = 180^\circ - \varepsilon \). Since \( \varepsilon < 22.5^\circ \), this contradicts Lemma 8(iii).

Now let \( b_2 \) lie to the right of \( r_2, c_2 \). Recall that every self-approaching \( b_2c_1 \)-path must use \( c_2 \) so it is \( \angle c_1c_2b_2 \geq 90^\circ \). Therefore, \( b_2 \) cannot lie inside the counterclockwise angle between \( c_2r_2 \) and \( c_2c_1 \), since it is \( \angle_{ccw}(c_2r_2,c_2c_1) \leq 90^\circ + \varepsilon < \angle r_2c_2b_2 + \angle b_2c_2c_1 \). Thus, \( b_2 \) is above \( c_2c_1 \), and it is \( \angle r_2c_2b_2 = \angle c_1c_2r_2 + \angle c_1c_2b_2 \geq (90^\circ - \varepsilon) + 90^\circ = 180^\circ - \varepsilon \). Again, since \( \varepsilon < 22.5^\circ \), this contradicts Lemma 8(iii). It follows \( \angle r_2a_2b_2 \geq 90^\circ \).

Analogously, we prove \( \angle r_2c_2b_2 \geq 90^\circ \). Thus, by Lemma 8(iii), \( \angle r_2a_2b_2, \angle r_2c_2b_2 \in [90^\circ,90^\circ + \varepsilon] \). Since \( \angle a_2b_2c_2 \geq 90^\circ \) by Lemma 8(i), \( b_2 \) lies to the right of \( r_2, a_2 \) and to the left of \( r_2, c_2 \). (If \( b_2 \) lies to the left of both rays, it is \( \angle a_2b_2c_2 = \angle (a_2b_2, c_2b_2) \leq 2\varepsilon < 90^\circ \).)

(ii) Similarly, if a self-approaching \( b_2a_0 \)-path uses \( c_2 \) instead of \( a_2 \), it is \( \angle r_2c_2b_2 \geq 180^\circ - \varepsilon \). The last part follows analogously.

The next lemma allows us to show that certain blocks are drawn smaller than their ancestors.

**Lemma 10.** It holds:

(i) \[ \frac{|ra|}{|rc|}, \frac{|rc|}{|ra|} \geq \cos \varepsilon; \]

(ii) \[ \frac{|ac|}{|ra|}, \frac{|ac|}{|rc|} \leq \tan \varepsilon; \]

(iii) The distance from \( a \) to the line through \( rc \) is at least \( |ac| \cos \varepsilon \).

(iv) Consider block \( \mu \) containing \( a, b, c, r \), vertex \( u \neq a \) in \( G^a \) and \( v \neq c \) in \( G^c \), \( \deg(u) = \deg(v) = 4 \). Then, \[ \frac{|au|}{|ac|} \leq \tan \varepsilon, \frac{|cv|}{|ac|} \leq \tan \varepsilon, \text{ and } |uv| \leq (1 + 2\tan \varepsilon)|ac|. \]

**Proof.** (i) Due to symmetry, we show only one part. By Lemma 8(iv), \( \angle acr \in [90^\circ - \varepsilon, 90^\circ + \varepsilon] \). Therefore,

\[ \frac{|ra|}{|rc|} = \frac{\sin \angle acr}{\sin \angle rac} \geq \frac{\sin(90^\circ - \varepsilon)}{1} = \cos \varepsilon. \]
(ii) It is $\angle arc \leq \varepsilon$. Therefore,

$$\frac{|ac|}{r} = \frac{\sin \angle arc}{\sin \angle acr} \leq \frac{\sin(\varepsilon)}{\sin(90^\circ - \varepsilon)} = \tan \varepsilon.$$ 

(iii) Let $d$ be the point on the line through $rc$ minimizing $|ad|$. Since $\angle acr \in [90^\circ - \varepsilon, 90^\circ + \varepsilon]$, it is $\angle(ac, ad) \leq \varepsilon$. Thus, $|ad| \geq |ac| \cos \varepsilon$.

(iv) By Lemma 8(iv), $\angle acu \leq \varepsilon$ and $\angle auc \in [90^\circ - \varepsilon, 90^\circ + \varepsilon]$. Thus,

$$\frac{|au|}{|ac|} = \frac{\sin \angle acu}{\sin \angle auc} \leq \frac{\sin \varepsilon}{\sin(90^\circ - \varepsilon)} = \tan \varepsilon.$$  

Similarly, $|vc| \leq |ac| \tan \varepsilon$. Thus, it is $|uv| \leq |ua| + |ac| + |cv| \leq (1 + 2 \tan \varepsilon)|ac|$.

From now on, let $\mu_0$ be the root block of $G^\rho$ and $\mu_1$, $\mu_2$, $\mu_3$ its descendants such that $r(\mu_1) = c_0$, $r(\mu_2) = a_1$, $r(\mu_3) \in \{a_2, c_2\}$; see Fig. 7c. Light gray blocks are the subject of Lemma 11(i), which shows that several ancestor roots lie inside a cone with a small angle. Dark gray blocks are the subject of Lemma 11(ii), which considers the intersection of the cones corresponding to a pair of sibling blocks and shows that some of their ancestor roots lie inside a narrow strip; see Fig. 8a for a sketch.

**Lemma 11.** Let $\mu$ be a block in $G^c_2$ with vertices $a$, $b$, $c$, $r(\mu)$.

(i) Let $\mu$ have depth 5 in $G^\rho$. Then, the cone $l^+_{ba} \cap l^+_{bc}$ contains $r(\mu)$, $r(\pi(\mu))$, $r(\pi^2(\mu))$ and $r(\pi^3(\mu))$.

(ii) Let $\mu$ have depth 4 in $G^c$. There exist $u$ in $G^a$ and $v$ in $G^c$ of degree 4 and a strip $S$ containing $r(\mu)$, $r(\pi(\mu))$, $r(\pi^2(\mu)) = r(\mu_2)$, such that $u$ and $v$ lie on the different boundaries of $S$, and it holds: $|uv| \leq (1 + 2 \tan \varepsilon)(\tan \varepsilon) \min\{|r(\mu)a|, |r(\mu)c|\}$.

**Proof.** (i) Consider a self-approaching $bb_0$-path $\rho_0$ and a self-approaching $bb_1$-path $\rho_1$. By Lemma 9(ii) applied to $\mu$, $ba$ is the first edge of $\rho_0$ and $be$ is the first edge of $\rho_1$. Since the cutvertices $r(\mu)$, $r(\pi(\mu))$, $r(\pi^2(\mu))$, $r(\pi^3(\mu))$ are on both $\rho_0$ and $\rho_1$, the statement holds.

(ii) Consider blocks $\mu_1$, $\mu_2$, such that $r(\mu_1) = a$ and $r(\mu_2) = c$. By (i), $r(\mu)$, $r(\pi(\mu))$, $r(\pi^2(\mu))$ are in $\Lambda := l^+_{b_0a_0} \cap l^+_{b_0c_0} \cap l^+_{b_0a_1} \cap l^+_{b_0c_1}$. Let $\vec{a_1}$ be the vector $b_0c_1$ rotated by $90^\circ$ clockwise and $\vec{a_1}'$ be the vector $b_0a_1$ rotated by $90^\circ$ counterclockwise. Note that by Lemma 8(ii), $G^b_1$, $G^a_0$ lie in $l^+_{b_0c_1}$, $l^+_{b_0a_1}$ respectively. Therefore, $\text{ray}(c_1, \vec{a_1})$ and $\text{ray}(a_1, \vec{a_1}')$ (green resp. blue arrows in Fig. 8a) converge, since the converse would contradict Lemma 2. Let $p$ be their intersection. Due to the chosen directions, $r(\mu)$, $r(\pi(\mu))$, $r(\pi^2(\mu))$ are below both $c_1$ and $a_1$. Therefore, $r(\mu)$, $r(\pi(\mu))$, $r(\pi^2(\mu))$ are contained in the triangle $c_1a_1p$, which lies inside a strip $S$ of width at most $|c_1a_1|$, whose respective boundaries contain $c_1$ and $a_1$. By Lemma 10(iv) and (ii), $|c_1a_1| \leq (1 + 2 \tan \varepsilon)|ac| \leq (1 + 2 \tan \varepsilon)(\tan \varepsilon) \min\{|r(\mu)a|, |r(\mu)c|\}$.

Again, we consider two siblings and the intersection of their corresponding strips, which forms a small diamond containing the root of the ancestor block; see Fig. 8b, 8c.
Lemma 12. Consider block $\mu_3$ containing $r = r(\mu_3), a, b, c,$ and let $r_\pi := r(\pi(\mu_3))$. It holds:  
(i) $|r'r_\pi| \leq \frac{(1+2\tan \varepsilon)(\tan \varepsilon)^2}{\cos \varepsilon}(|ra| + |rc|)$; (ii) $|ra|, |rc| \leq |rr_\pi|((\tan \varepsilon)^2$.

Proof. (i) Define $d = (1 + 2\tan \varepsilon)(\tan \varepsilon)^2|ac|$. Then, by Lemma 10(ii) and (iv) and Lemma 11(ii), vertices $a, r$ and $r_\pi$ are contained in a strip $s_1$ (green in Fig. 8b) of width at most $d$. Additionally, both boundaries of $s_1$ contain vertices of $G^a$ (red dots), which lie in $l_{ba}^+$ and, by Lemma 9(i), to the left of ray$(r,a)$. Thus, the downward direction along $s_1$ is counterclockwise compared to $\overrightarrow{ar}$. (Otherwise, the green strip could not contain $a$.) Similarly, vertices $c, r$ and $r_\pi$ are contained in a strip $s_2$ (blue) of width at most $d$, and both boundaries of $s_2$ contain vertices of $G^c$, which lie to the right of ray$(r,c)$. Thus, the downward direction along $s_2$ is clockwise compared to $\overrightarrow{cr}$; see Fig. 8b.

Let us find an upper bound for the diameter of the parallelogram $s_1 \cap s_2$. In the critical case, the right side of $s_1$ touches $ra$, the left side of $s_2$ touches $rc$, and the width of both strips is $d$; see Fig. 8c. Let $a'$ (resp. $c'$) be the intersection of the right (resp. left) sides of $s_1$ and $s_2$, and $r'$ the intersection of the left side of $s_1$ and right side of $s_2$. Let $d_a$ be the distance from $a$ to the line through $rc$ and $d_c$ the distance from $c$ to the line through $ra$. By Lemma 10(iii), it is $d_a, d_c \geq |ac| \cos \varepsilon$. Moreover, it holds: $\frac{|ra'|}{|ra|} = \frac{d}{d_a}$ and $\frac{|rc'|}{|rc|} = \frac{d}{d_c}$.

Therefore, $|ra'| \leq \frac{d|ra|}{|ac| \cos \varepsilon}$, $|rc'| \leq \frac{d|rc|}{|ac| \cos \varepsilon}$ and

$$|rr'| \leq |ra'| + |rc'| \leq \frac{(1+2\tan \varepsilon)(\tan \varepsilon)^2}{\cos \varepsilon}(|ra| + |rc|).$$

Since $\angle a'r'c' \leq \varepsilon$, $rr'$ is the diameter, thus, $|rr_\pi| \leq |rr'|$.

(ii) Let $a_\pi$ and $c_\pi$ be the two neighbors of $r_\pi$ in the block $\mu_2 = \pi(\mu_3)$. It is $r \in \{a_\pi, c_\pi\}$. Assume $r = c_\pi$ as in Fig. 8b. By Lemma 10(iv), it is $\frac{|ac_\pi|}{|ra_\pi|} \leq \tan \varepsilon$. By Lemma 10(ii), it is $\frac{|a_\pi c_\pi|}{|ra_\pi c_\pi|} \leq \tan \varepsilon$. It follows: $|ra| = |ac_\pi| \leq |r_\pi c_\pi|((\tan \varepsilon)^2 = |rr_\pi|((\tan \varepsilon)^2$. Analogously, $|rc| \leq |rr_\pi|((\tan \varepsilon)^2$. □

For $\varepsilon \leq 22.5^\circ$, the two claims of Lemma 12 contradict each other. This concludes the proof of Theorem 4.
4 Planar Increasing-Chord Drawings of 3-Trees

In this section, we show how to construct planar increasing-chord drawings of planar 3-trees. We make use of Schnyder labelings [24] and drawings of triangulations based on them. For a plane triangulation \( G = (V, E) \) with external vertices \( r, g, b \), its Schnyder labeling is an orientation and partition of the interior edges into three trees \( T_r, T_g, T_b \) (called red, green and blue tree), such that for each internal vertex \( v \), its incident edges appear in the following clockwise order: exactly one outgoing red, an arbitrary number of incoming blue, exactly one outgoing green, an arbitrary number of incoming red, exactly one outgoing blue, an arbitrary number of incoming green. Each of the three outer vertices \( r, g, b \) serves as the root of the tree in the same color and all its incident interior edges are incoming in the respective color. For \( v \in V \), let \( R_v^r \) (the red region of \( v \)) denote the region bounded by the \( vg \)-path in \( T_g \), the \( vb \)-path in \( T_b \) and the edge \( gb \). Let \( |R_v^r| \) denote the number of the interior faces in \( R_v^r \). The green and blue regions \( R_v^g, R_v^b \) are defined analogously. Assigning \( v \) the coordinates \((|R_v^r|, |R_v^g|, |R_v^b|) \in \mathbb{R}^3 \) results in a plane straight-line drawing of \( G \) in the plane \( \{ x = (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = f - 1 \} \) called Schnyder drawing. Here, \( f \) denotes the number of faces of \( G \). For a thorough introduction to this topic, see the book of Felsner [10].

For \( \alpha, \beta \in [0^\circ, 360^\circ] \), let \([\alpha, \beta] \) denote the corresponding counterclockwise cone of directions. We consider drawings satisfying the following constraints.

**Definition 1.** Let \( G = (V, E) \) be a plane triangulated graph with a Schnyder labeling. For \( 0^\circ \leq \alpha \leq 60^\circ \), we call an arbitrary planar straight-line drawing of \( G \) \( \alpha \)-Schnyder if for each internal vertex \( v \in V \), its outgoing red edge has direction in \( [90^\circ - \frac{\alpha}{2}, 90^\circ + \frac{\alpha}{2}] \), blue in \( [210^\circ - \frac{\alpha}{2}, 210^\circ + \frac{\alpha}{2}] \) and green in \( [330^\circ - \frac{\alpha}{2}, 330^\circ + \frac{\alpha}{2}] \) (see Fig. 9a).

According to Definition 1, classical Schnyder drawings are \( 60^\circ \)-Schnyder; see, e.g., Lemma 4 in [8]. The next lemma shows an interesting connection between \( \alpha \)-Schnyder and increasing-chord drawings.

**Lemma 13.** For any \( \alpha \leq 30^\circ \), \( \alpha \)-Schnyder drawings are increasing-chord drawings.

**Proof.** Let \( G = (V, E) \) be a plane triangulation with a given Schnyder labeling and \( \Gamma \) a corresponding \( 30^\circ \)-Schnyder drawing. Let \( r, g, b \) be the red, green and blue external vertex, respectively, and \( T_r, T_g, T_b \) the directed trees of the corresponding color.

Consider vertices \( s, t \in V \). First, note that monochromatic directed paths in \( \Gamma \) have increasing chords by Lemma 1. Assume \( s \) and \( t \) are not connected by such a path. Then, they are both internal and \( s \) is contained in one of the regions \( R_s^r, R_s^g, R_s^b \). Without loss of generality, we assume \( s \in R_s^r \). The \( sr \)-path in \( T_r \) crosses the boundary of \( R_s^r \), and we assume without loss of generality that it crosses the blue boundary of \( R_s^b \) in \( u \neq t \); see Fig. 9b. The other cases are symmetric.

Let \( \rho_r \) be the \( su \)-path in \( T_r \) and \( \rho_b \) the \( tu \)-path in \( T_b \); see Fig. 9c. On the one hand, the direction of a line orthogonal to a segment of \( \rho_r \) is in \( [345^\circ, 15^\circ] \cup [165^\circ, 195^\circ] \). On the other hand, \( \rho_b \) is contained in a cone \( [15^\circ, 45^\circ] \) with apex \( u \). Thus, \( \rho_b^{-1} \subseteq \text{front}(\rho_r) \), and \( \rho_r \cdot \rho_b^{-1} \) is self-approaching by Fact 2. By a symmetric argument it is also self-approaching in the other direction, and hence has increasing chords. \( \square \)

---

\( \text{JoCG 7(1), 47–69, 2016} \)
Planar 3-trees are the graphs obtained from a triangle by repeatedly choosing a (triangular) face $f$, inserting a new vertex $v$ into $f$, and connecting $v$ to each vertex of $f$.

**Lemma 14.** Planar 3-trees have $\alpha$-Schnyder drawings for any $0^\circ < \alpha \leq 60^\circ$.

**Proof.** We describe a recursive construction of an $\alpha$-Schnyder drawing of a planar 3-tree. We use the pattern in Fig. 9a consisting of three cones with angle $0^\circ < \alpha \leq 60^\circ$ to maintain the following invariant.

*For each inner face $f$, the pattern can be centered at a point $p$ in the interior of $f$, such that every cone of the pattern contains one vertex of $f$ in its interior.*

We start with an equilateral triangle. Obviously, the invariant holds for the single inner face $f$ by choosing $p$ to be the barycenter of $f$.

Assume the invariant holds for each inner face of the drawing created so far. We prove that the invariant can be maintained after adding a new vertex. Consider an inner face $f$ with corners $x,y,z$. We move the pattern from Fig. 9a, such that its center lies in the interior point $p$ of $f$ from the invariant. Without loss of generality, let $x$ be in the red cone of the pattern, $y$ in the blue cone and $z$ in the green. We insert the new vertex $v$ at point $p$ and connect $v$ to $x,y,z$. We make the edge $vx$ outgoing red, $vy$ outgoing blue and $vz$ outgoing green.

We now show that the invariant holds for the three newly created faces $f_1 = xyv$, $f_2 = yzv$ and $f_3 = zxv$. Consider $f_1$ first. If we place the pattern at $v$, by the invariant for face $f$, one cone of the pattern contains $x$ and another contains $y$ in its interior; see Fig. 9d. It is now possible to move the pattern inside the triangle $xyv$ slightly, such that $v$ is in the interior of the third cone; see Fig. 9e. This proves the invariant for $f_1$, and the proof for $f_2$ and $f_3$ is analogous.

Lemmas 13 and 14 provide a constructive proof for the following theorem.

**Theorem 5.** Every planar 3-tree has a planar increasing-chord drawing.

## 5 Self-Approaching Drawings in the Hyperbolic Plane

Kleinberg [15] showed that every tree can be drawn greedily in the hyperbolic plane $\mathbb{H}^2$. This is not the case in $\mathbb{R}^2$. Thus, $\mathbb{H}^2$ is more powerful than $\mathbb{R}^2$ in this regard. Since
self-approaching drawings are closely related to greedy drawings, it is natural to investigate the existence of self-approaching drawings in $\mathbb{H}^2$.

We shall use the Poincaré disk model for $\mathbb{H}^2$, in which $\mathbb{H}^2$ is represented by the unit disk $D = \{ x \in \mathbb{R}^2 : |x| < 1 \}$ and the geodesics are represented by arcs of circles orthogonal to the boundary of $D$. We consider a drawing of a graph in $\mathbb{H}^2$ straight-line, if the edges are drawn as arcs of such circles. For an introduction to the Poincaré disk model, see, for example, Kleinberg [15] and the references therein.

First, let us consider a tree $T = (V, E)$. A drawing of $T$ in $\mathbb{R}^2$ is self-approaching if and only if no normal on an edge of $T$ in any point crosses another edge [1]. The same condition holds in $\mathbb{H}^2$.

**Lemma 15.** A straight-line drawing $\Gamma$ of a tree $T$ in $\mathbb{H}^2$ is self-approaching if and only if no normal on an edge of $T$ crosses $\Gamma$ in another point.

**Proof.** The proof is similar to the Euclidean case. We present it for the sake of completeness. First, let $\Gamma$ be a self-approaching drawing, for which the condition of the lemma is violated. Without loss of generality, let $\rho = (s, u, \ldots, t)$ be the $st$-path in $T$, such that the normal on $su$ in a point $r$ crosses $\rho$ in another point. Due to the piecewise linearity of $\rho$, we may assume $r$ to be in the interior of $su$. Let $H_+ = \{ p = (p_x, p_y) \in D \mid p_y > 0 \}$ and $H_- = \{ p = (p_x, p_y) \in D \mid p_y < 0 \}$ the top and bottom hemispheres of $D$. For $p_1, p_2 \in D$, let $d(p_1, p_2)$ denote the hyperbolic distance between $p_1$ and $p_2$, i.e., the hyperbolic length of the corresponding geodesics. We recall the following basic fact whose proof is given, e.g., by Kleinberg [15].

**Claim 1.** Let $0 < y < 1$, $p_- = (0, -y)$, $p_+ = (0, y)$. Then, for each $p \in H_-$, it is $d(p, p_-) < d(p, p_+)$. 

Due to isometries, we can assume that $r$ is in the origin of $D$, $su$ is vertical, $s \in H_-$, $u \in H_+$. Let $a \in H_-$, $b \in H_+$ be two points on $su$, such that $|ar| = |rb|$. Since the normal on $su$ in $r$ crosses $\rho$, there must exist a point $c$ on $\rho$, $c \in H_-$, such that $a, b, c$ are on $\rho$ in this order. However, it is $d(a, c) < d(b, c)$, a contradiction to $\rho$ being self-approaching.

Let $\Gamma$ be a drawing of $T$, for which the condition holds. Let $a, b, c$ be three consecutive points on a path $\rho$ in $\Gamma$. First, assume $a, b$ lie on the same arc of $\Gamma$. We apply an isometry to $\Gamma$, such that $ab$ is vertical, $a \in H_-$, $b \in H_+$, and $a, b$ are equidistant from the origin $o$. The normal to $\rho$ in $o$ is the equator. Thus, $c \notin H_-$, and $d(b, c) \leq d(a, c)$. By applying this argument iteratively, this inequality also holds if $a, b$ lie on different arcs.

According to the characterization by Alamdari et al. [1], some binary trees have no self-approaching drawings in $\mathbb{R}^2$. We show that this is no longer the case in $\mathbb{H}^2$.

**Theorem 6.** Let $T = (V, E)$ be a tree, such that each node of $T$ has degree either 1 or 3. Then, $T$ has a self-approaching drawing in $\mathbb{H}^2$, in which every arc has the same hyperbolic length and every pair of incident arcs forms an angle of $120^\circ$.

**Proof.** For convenience, we subdivide each edge of $T$ once. We shall show that both pieces are collinear in the resulting drawing $\Gamma$ and have the same hyperbolic length.
First, consider a regular hexagon $\odot = p_0 p_1 p_2 p_3 p_4 p_5$ centered at the origin $o$ of $D$; see Fig. 10a. In $H^2$, it can have angles smaller than 120°. We choose them to be 90° (any angle between 0° and 90° would work). Next, we draw a $K_{1,3}$ with center $v_0$ in $o$ and the leaves $v_1, v_2, v_3$ in the middle of the arcs $p_0 p_1, p_2 p_3, p_4 p_5$ respectively.

For each such building block of the drawing consisting of a $K_{1,3}$ inside a regular hexagon with 90° angles, we add its copy mirrored at an arc of the hexagon containing a leaf node of the tree constructed so far. For example, in the first iteration, we add three copies of $\odot$ mirrored at $p_0 p_1$, $p_2 p_3$ and $p_4 p_5$, respectively, and the corresponding inscribed $K_{1,3}$ subtrees. The construction after two iterations is shown in Fig. 10b. This process can be continued infinitely to construct a drawing $\Gamma_\infty$ of the infinite binary tree. However, we stop after we have completed $\Gamma$ for the tree $T$.

We now show that $\Gamma_\infty$ (and thus also $\Gamma$) has the desired properties. Due to isometries and Lemma 15, it suffices to consider edge $e = v_0 v_1$ and show that a normal on $e$ does not cross $\Gamma_\infty$ in another point. To see this, consider Fig. 10a. Due to the choice of the angles of $\odot$, all the other hexagonal tiles of $\Gamma_\infty$ are contained in one of the three blue quadrangular regions $\square_i := l^+_{v_0 v_1} \setminus (l^+_{v_i p_{2i-1}} \cup l^+_{v_i p_{2i-2}})$, $i = 1, 2, 3$. Thus, the regions $l^+_{v_1 p_1}$ and $l^+_{v_1 p_0}$ (gray) contain no point of $\Gamma_\infty$. Therefore, since each normal on $v_0 v_1$ is contained in the “slab” $D \setminus (l^+_{v_0 v_1} \cup l^+_{v_1 v_0})$ bounded by the diameter through $p_2, p_3$ and the line through $p_0, p_1$ (dashed) and is parallel to both of these lines, it contains no other point of $\Gamma_\infty$.

We note that our proof is similar in spirit to the one by Kleinberg [15], who also used tilings of $H^2$ to prove that any tree has a greedy drawing in $H^2$.

As in the Euclidean case, it can be easily shown that if a tree $T$ contains a node $v$ of degree 4, it has a self-approaching drawing in $H^2$ if and only if $T$ is a subdivision of $K_{1,4}$ (apply an isometry, such that $v$ is in the origin of $D$). This completely characterizes the trees admitting a self-approaching drawing in $H^2$. Further, it is known that every binary cactus and, therefore, every 3-connected planar graph has a binary spanning tree [5, 17].

**Corollary 2.** (i) A tree $T$ has an increasing-chord drawing in $H^2$ if and only if $T$ either has maximum degree 3 or is a subdivision of $K_{1,4}$.

(ii) Every binary cactus and, therefore, every 3-connected planar graph has an increasing-chord drawing in $H^2$. 

---

**Figure 10:** Constructing increasing-chord drawings of binary trees and cactuses in $H^2$. 

(a) (b) (c)
Again, note that this is not the case for binary cactuses in $\mathbb{R}^2$; see the example in Theorem 4. We use the above construction to produce planar self-approaching drawings of binary cactuses in $\mathbb{H}^2$. We show how to choose a spanning tree and angles at vertices of degree 2, such that non-tree edges can be added without introducing crossings; see Fig. 10c for a sketch.

**Corollary 3.** Every binary cactus has a planar increasing-chord drawing in $\mathbb{H}^2$.

**Proof.** Without loss of generality, let $G$ be a binary cactus rooted at block $\nu$ such that each block $\mu$ of $G$ is either a single edge or a cycle. For each block $\mu$ forming a cycle $r(\mu) = v_0, v_1, \ldots, v_k, v_0$, we remove edge $v_0v_k$, thus obtaining a binary tree $T$. We embed it similarly to the proof of Theorem 6 such that additionally the counterclockwise angle $\angle v_{j-1}v_jv_{j+1} = 120^\circ$ for $j = 1, \ldots, k - 1$. Obviously, $T$ is drawn in a planar way since for each edge $e$ of $T$, each half of $e$ is drawn inside its hexagon.

It remains to show that for each $\mu$, adding arc $v_0v_k$ introduces no crossings. For each $j = 1, \ldots, k - 1$, we can apply an isometry to the drawing, such that $v_j$ is in the origin and $v_jv_{j+1}$ points upwards; see Fig. 10c. According to the construction of $T$, subcactus $G^0_\mu$ (maximal subcactus of $G$ containing $v_0$ and no other vertex of $\mu$) lies in the green region contained in $l_1^+v_0$ and $G^k_\mu$ in the blue region contained in $l_{k-1}^+v_k$. Since $v_0 \notin l_{k-1}^{-1}v_k$ and $v_k \notin l_1^{-1}v_0$, arc $v_0v_k$ crosses neither $G^0_\mu$ nor $G^k_\mu$. Furthermore, $v_0$ and $v_k$ lie inside the $120^\circ$ cone $\Lambda_j$ formed by ray($v_j, v_{j+1}$) and ray($v_j, v_{j-1}$). Thus, $v_0v_k$ does not cross $v_{j-1}v_j$, $v_jv_{j+1}$. Since subcactus $G^0_\mu$ is in $\mathbb{H}^2 \setminus \Lambda_j$ (it lies in the red area in Fig. 10c), it is not crossed by $v_0v_k$ either.

6 Conclusion

We have studied the problem of constructing self-approaching and increasing-chord drawings of 3-connected planar graphs and triangulations in the Euclidean and hyperbolic plane. Due to the fact that every such graph has a spanning binary cactus, and in the case of a triangulation even one that has a special type of triangulation (downward-triangulation), self-approaching and increasing-chord drawings of binary cactuses played an important role.

We showed that, in the Euclidean plane, downward-triangulated binary cactuses admit planar increasing-chord drawings, and that the condition of being downward-triangulated is essential as there exist binary cactuses that do not admit a (not necessarily planar) self-approaching drawing. Naturally, these results imply the existence of non-planar increasing-chord drawings of triangulations. It remains open whether every 3-connected planar graph has a self-approaching or increasing-chord drawing. If this is the case, according to our example in Theorem 4, the construction must be significantly different from both known proofs [5, 17] of the weak Papadimitriou-Ratajczak conjecture [20] (you cannot just take an arbitrary spanning binary cactus) and would prove a stronger statement.

For planar 3-trees, which are special triangulations, we introduced $\alpha$-Schnyder drawings, which have increasing chords for $\alpha \leq 30^\circ$, to show the existence of planar increasing-chord drawings. It is an open question whether this method works for further
classes of triangulations. Which triangulations admit $\alpha$-Schnyder drawings for arbitrarily small values of $\alpha$ or for $\alpha = 30^\circ$?

Finally, we studied drawings in the hyperbolic plane. Here we gave a complete characterization of the trees that admit an increasing-chord drawing (which then is planar) and used it to show the existence of non-planar increasing-chord drawings of 3-connected planar graphs. For binary cactuses even a planar increasing-chord drawing exists.

It is worth noting that all self-approaching drawings we constructed are actually increasing-chord drawings. Is there a class of graphs that admits a self-approaching drawing but no increasing-chord drawing?

References

[1] Alamdari, S., Chan, T.M., Grant, E., Lubiw, A., Pathak, V.: Self-approaching graphs. In: W. Didimo, M. Patrignani (eds.) Graph Drawing (GD’12), LNCS, vol. 7704, pp. 260–271. Springer (2013). doi:10.1007/978-3-642-36763-2_23

[2] Angelini, P., Colasante, E., Di Battista, G., Frati, F., Patrignani, M.: Monotone drawings of graphs. J. Graph Algorithms Appl. 16(1), 5–35 (2012). doi:10.7155/jgaa.00249

[3] Angelini, P., Di Battista, G., Frati, F.: Succinct greedy drawings do not always exist. Networks 59(3), 267–274 (2012). doi:10.1002/net.21449

[4] Angelini, P., Didimo, W., Kobourov, S., Mchedlidze, T., Roselli, V., Symvonis, A., Wismath, S.: Monotone drawings of graphs with fixed embedding. Algorithmica 71(2), 233–257 (2013). doi:10.1007/s00453-013-9790-3

[5] Angelini, P., Frati, F., Grilli, L.: An algorithm to construct greedy drawings of triangulations. J. Graph Algorithms Appl. 14(1), 19–51 (2010). doi:10.7155/jgaa.00197

[6] Barnette, D.: Trees in polyhedral graphs. Canad. J. Math. 18, 731–736 (1966)

[7] Dehkordi, H.R., Frati, F., Gudmundsson, J.: Increasing-chord graphs on point sets. J. Graph Algorithms Appl. 19(2), 761–778 (2015). doi:10.7155/jgaa.00348

[8] Dhandapani, R.: Greedy drawings of triangulations. Discrete Comput. Geom. 43(2), 375–392 (2010). doi:10.1007/s00454-009-9235-6

[9] Eppstein, D., Goodrich, M.T.: Succinct greedy geometric routing using hyperbolic geometry. IEEE Trans. Computers 60(11), 1571–1580 (2011). doi:10.1109/TC.2010.257

[10] Felsner, S.: Geometric Graphs and Arrangements, chap. 2, pp. 17–42. Advanced Lectures in Mathematics. Vieweg+Teubner Verlag (2004). doi:10.1007/978-3-322-80303-0_2

[11] Goodrich, M.T., Strash, D.: Succinct greedy geometric routing in the Euclidean plane. In: Y. Dong, D.Z. Du, O. Ibarra (eds.) Algorithms and Computation (ISAAC’09), LNCS, vol. 5878, pp. 781–791. Springer (2009). doi:10.1007/978-3-642-10631-6_79

[12] Huang, W., Eades, P., Hong, S.H.: A graph reading behavior: Geodesic-path tendency. In: IEEE Pacific Visualization Symposium (PacificVis’09), pp. 137–144 (2009). doi:10.1109/PACIFICVIS.2009.4906848

[13] Icking, C., Klein, R., Langetepe, E.: Self-approaching curves. Math. Proc. Camb. Phil. Soc. 125, 441–453 (1999). doi:10.1017/S0305004198003016
[14] Kindermann, P., Schulz, A., Spoerhase, J., Wolff, A.: On monotone drawings of trees. In: C. Duncan, A. Symvonis (eds.) Graph Drawing (GD’14), LNCS, vol. 8871, pp. 488–500. Springer (2014). doi:10.1007/978-3-662-45803-7_41

[15] Kleinberg, R.: Geographic routing using hyperbolic space. In: Computer Communications (INFOCOM’07), pp. 1902–1909 (2007). doi:10.1109/INFCOM.2007.221

[16] Lee, B., Plaisant, C., Parr, C.S., Fekete, J.D., Henry, N.: Task taxonomy for graph visualization. In: AVI Workshop on Beyond Time and Errors: Novel Evaluation Methods for Information Visualization (BELIV’06), pp. 1–5. ACM (2006). doi:10.1145/1168149.1168168

[17] Leighton, T., Moitra, A.: Some results on greedy embeddings in metric spaces. Discrete Comput. Geom. 44(3), 686–705 (2010). doi:10.1007/s00454-009-9227-6

[18] Nöllenburg, M., Prutkin, R.: Euclidean greedy drawings of trees. In: H. Bodlaender, G. Italiano (eds.) European Symposium on Algorithms (ESA’13), LNCS, vol. 8125, pp. 767–778. Springer (2013). doi:10.1007/978-3-642-40540-4_65

[19] Nöllenburg, M., Prutkin, R., Rutter, I.: On self-approaching and increasing-chord drawings of 3-connected planar graphs. In: C.A. Duncan, A. Symvonis (eds.) Graph Drawing (GD’14), LNCS, vol. 8871, pp. 476–487. Springer Berlin Heidelberg (2014). doi:10.1007/978-3-662-45803-7_40

[20] Papadimitriou, C.H., Ratajczak, D.: On a conjecture related to geometric routing. Theoret. Comput. Sci. 344(1), 3–14 (2005). doi:10.1016/j.tcs.2005.06.022

[21] Purchase, H.C., Hamer, J., Nöllenburg, M., Kobourov, S.G.: On the usability of Lombardi graph drawings. In: W. Didimo, M. Patrignani (eds.) Graph Drawing (GD’12), LNCS, vol. 7704, pp. 451–462. Springer (2013). doi:10.1007/978-3-642-36763-2_40

[22] Rao, A., Ratnasamy, S., Papadimitriou, C., Shenker, S., Stoica, I.: Geographic routing without location information. In: Mobile Computing and Networking (MobiCom’03), pp. 96–108. ACM (2003). doi:10.1145/938985.938996

[23] Rote, G.: Curves with increasing chords. Math. Proc. Camb. Phil. Soc. 115, 1–12 (1994). doi:10.1017/S0305004100071875

[24] Schnyder, W.: Embedding planar graphs on the grid. In: ACM-SIAM Symposium on Discrete Algorithms (SODA’90), pp. 138–148. SIAM (1990)

[25] Wang, J.J., He, X.: Succinct strictly convex greedy drawing of 3-connected plane graphs. Theoret. Comput. Sci. 532, 80–90 (2014). doi:10.1016/j.tcs.2013.05.024