Anisotropy and universality in finite-size scaling: Critical Binder cumulant of a two-dimensional Ising model

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We reanalyze transfer-matrix and Monte Carlo results for the critical Binder cumulant \( U^* \) of an anisotropic two-dimensional Ising model on a square lattice in a square geometry with periodic boundary conditions. Spins are coupled between nearest-neighboring sites and between next-nearest-neighboring sites along one of the lattice diagonals. We find that \( U^* \) depends only on the asymptotic critical long-distance features of the anisotropy, irrespective of its realization through ferromagnetic or antiferromagnetic next-nearest-neighbor couplings. We modify an earlier renormalization-group calculation to obtain a quantitative description of the anisotropy dependence of \( U^* \). Our results support our recent claim towards the validity of universal finite-size scaling for critical phenomena in the presence of a weak anisotropy.

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Universality is a key concept in the theory of critical phenomena for bulk and confined systems; for reviews, see, for example, [1–3]. Within a given bulk universality class, critical exponents, certain critical amplitude ratios, and the critical behavior of thermodynamic functions are identical and independent of the specific microscopic realization. For instance, the bulk correlation length in the asymptotic critical domain, that is, for asymptotically small positive or negative \( t \equiv (T - T_c)/T_c \), is, for isotropic systems, described by

\[
\xi = \xi_{\pm,0} |t|^{-\nu}, \quad T \gtrsim T_c,
\]

where \( T_c \) is the bulk critical temperature, \( \nu > 0 \) is a universal critical exponent, \( \xi_{\pm,0} \) are nonuniversal critical amplitudes, and \( R_c \equiv \xi_{+0}/\xi_{-0} \) is a universal critical amplitude ratio.

Besides critical exponents, the thermodynamic functions describing the asymptotic critical domain are universal if one allows for adjusting of only two amplitudes. Consider, for example, for small \( t \) and small field \( h \) conjugate to the order parameter, the part \( f_{b,s} \) of the bulk free energy density that becomes singular at the critical point \( t, h = 0 \). \( f_{b,s} \) is asymptotically described by universal scaling functions \( W^\pm \) for \( t \gtrsim 0 \) according to [4]

\[
\beta f_{b,s}(t,h) = A_1 |t|^{2-\alpha} W^\pm (A_2 h|t|^{-\Delta}),
\]

with \( \beta \equiv 1/(8k_B T) \), with universal critical exponents \( \alpha \) and \( \Delta \), and where \( A_1 \) and \( A_2 \) are nonuniversal amplitudes, that is, they differ between different systems within the universality class under consideration.

Universality also extends to situations, where the system under consideration is confined on a length scale \( L \) that is large compared to all microscopic length scales, such as lattice constants. If the singular part of the free energy density of the system under consideration exhibits scaling, its asymptotic critical form may be written as [4]

\[
\beta f_0(t,h,L) = L^{-d} F(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}),
\]

where the nonuniversal constants \( C_1 \) and \( C_2 \) are universally related to the constants \( A_1 \) and \( A_2 \) in (2). For given shape and boundary conditions, the function \( F \) is universal.

Equations (1)–(3) were stated for isotropic critical systems, that is, where the bulk asymptotic near-critical correlation length is isotropic. Here we are interested in the fate of universal finite-size scaling in the presence of a weak anisotropy, that is, where the amplitudes \( \xi_{\pm,0} \) depend on the direction, but \( \nu \) does not (for strong anisotropies, where even \( \nu \) depends on the direction, see, for example, [5] and references therein). Representatives of spatially anisotropic systems in the context of critical phenomena are, for example, magnetic materials, alloys, superconductors [6], and solids with structural phase transitions [7, 8].

Most of the literature on critical phenomena has focused on the isotropic case. In Refs. [9–18], investigations were carried out on a number of specific anisotropic systems, but no general picture of universality for weakly anisotropic systems in restricted geometries was suggested. The more recent publications [19–23] have attempted to clarify the situation from a more general point of view. References [19–21] concluded that the dependence of some bulk amplitude ratios, scaling functions, and the critical Binder cumulant on the parameters describing the anisotropy indicates a violation of universality. In Ref. [22], it was suggested that universality should be defined only after relating the system under consideration to an isotropic system by means of a shear transformation (see [9] for such a suggestion in the context of the two-dimensional Ising model).

In contrast, the present author suggested in [23] that quantities that are universal in the isotropic case remain universal in the presence of a weak anisotropy, if their

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list of arguments is augmented by the parameters describing the asymptotic critical long-distance features of the anisotropy. Universality then implies that the quantity under consideration should exhibit no dependence on the particular microscopic realization of these features. Rather, it should depend universally on the anisotropy, that is, in a way that is identical for all members of the respective bulk universality class. Explicit support for this claim was provided in [23] through results for free energy scaling functions of the two-dimensional Ising model on infinite strips. These examples, however, exhibited the simplification that a shear transformation exactly related the anisotropic systems to their isotropic counterparts with identical boundary conditions and geometry. Accordingly, the scaling functions for the anisotropic case could be expressed in terms of the isotropic scaling functions, and the consequences of the anisotropy could be interpreted to be of a mere geometric nature. In this work, we consider a situation where this simplification is absent.

Consider weakly anisotropic critical bulk systems in $d$ dimensions. Their asymptotic long-distance correlations are described by a correlation length ellipsoid that may be represented by a symmetric positive definite matrix $\Xi$. This is diagonalized by a rotation matrix $R$, so that

$$R \Xi R^{-1} = \text{diag}(\xi_1^2, \xi_2^2, \ldots, \xi_d^2).$$

(4)

The $\xi_i$ are the asymptotic near-critical bulk correlation lengths along the principal axes of the ellipsoid. The scale-free matrix [23]

$$\tilde{A} \equiv (\text{det} \, \Xi)^{-1/d} \Xi$$

(5)

is normalized to have $\text{det} \, \tilde{A} = 1$ and describes the shape and orientation of the ellipsoid. Since $\nu$ is unique by assumption, $\tilde{A}$ is independent of $t$ within the asymptotic critical domain. It may be parametrized by $d - 1$ correlation length ratios and $d(d-1)/2$ rotation angles. According to [23], universal quantities may receive an additional dependence on $\tilde{A}$, but are independent of its particular microscopic realization. This implies that the functions $W^\pm(x)$ and $F(y, z)$ on the right-hand sides of Eqs. (2) and (3), respectively, have to be replaced by universal functions $W^\pm(x, \tilde{A})$ and $F(y, z, \tilde{A})$, which describe the isotropic situation as a special case $\tilde{A} = 1$.

The use of a normalized matrix $\tilde{A}$ to describe the anisotropy of a model was suggested in [19–21]. However, while $\tilde{A}$ from (5) is defined through the physical correlation lengths in the asymptotic critical domain, explicit versions of $\tilde{A}$ in [19–21] were obtained by expanding the Hamiltonian under consideration in small wave numbers $k$ through order $k^2$. While, for standard $\varphi^4$ field theory, these definitions coincide due to an exact mapping between anisotropic and isotropic bulk Hamiltonians [19], this procedure in general does not, for lattice models, lead to the same $\tilde{A}$ as defined through the physical correlation lengths. This will be demonstrated explicitly for the two-dimensional Ising model below. The matrix $\tilde{A}$ used in [19–21] is thus not universally related to the matrix defined in (5). This explains why the author of [20, 21] arrived at the conclusion that the anisotropy effects exhibit “a kind of restricted universality” that only holds for a subclass of systems within a given universality class. Here we give support to our claim from [23] that the use of the matrix $\tilde{A}$ as defined in (5) leads to unrestricted universality in the traditional sense.

For $d = 2$-dimensional models, the ellipsoid reduces to an ellipse [24], whose shape may be characterized by the ratio $r \equiv \xi_\perp/\xi_\parallel$ of the smallest and largest correlation lengths (thus $0 < r \leq 1$) and an angle $\theta$ describing its orientation (we select the convention $-\pi/2 < \theta \leq \pi/2$). If $\theta$ is chosen to be the inclination of the direction of the largest correlation length with respect to the “1” direction, see Fig. 1(a), the explicit form of $\tilde{A}$ for $d = 2$ is [23]

$$\tilde{A}_2 = \begin{pmatrix} r \sin^2 \theta + r^{-1} \cos^2 \theta & \frac{1}{2}(r^{-1} - r) \sin 2\theta \\ \frac{1}{2}(r^{-1} - r) \sin 2\theta & r \cos^2 \theta + r^{-1} \sin^2 \theta \end{pmatrix}.$$ 

(6)

Therefore, universal quantities may receive an additional dependence on the variables $r$ and $\theta$ as compared to the isotropic case.

Consider a two-dimensional Ising model in a square $L \times L$ geometry on a square lattice with lattice constant $a$ and ferromagnetic couplings $J > 0$ of neighboring spins and couplings $J_d$ of next-nearest-neighboring spins in the direction of only one of the diagonals, see Fig. 1(b). With $L = Na$, the Hamiltonian of this model reads

$$H = - \sum_{m,n=1}^{N} s_{m,n} \left[ J(s_{m+1,n} + s_{m,n+1}) + J_d s_{m+1,n+1} \right],$$

(7)

with identifications of coordinates $N + 1$ and 1 in both principal lattice directions (i.e., periodic boundary conditions) and where $s_{m,n} = \pm 1$ is the Ising spin at the site.
We only consider couplings $J_d > -J$ that allow for a ferromagnetic bulk phase transition at a temperature $T_c > 0$, given by [25, 26]

$$\sinh^2(2\beta_c J) + 2\sinh(2\beta_c J) \sinh(2\beta_c J_d) = 1. \quad (8)$$

We choose $\theta$ to be the inclination of the direction of the largest correlation length with respect to one of the lattice axes. Using results for the general triangular lattice [9, 23], we obtain, at the critical point,

$$r = [\sinh(2\beta_c J)]^{1/2}, \quad \theta = \mp \pi/4 \quad \text{for} \quad J_d \geq 0. \quad (9)$$

Due to the symmetry of the $L \times L$ geometry, no dependence of universal critical quantities on the sign of $\theta$ in (9) is possible. Thus, compared to the isotropic case, universal quantities may receive an additional dependence only on $r$. Note that for general $r < 1$, a shear transformation to an isotropic system causes the periodicity of the boundary conditions to be no longer in mutually perpendicular directions and therefore no transformation to an isotropic system with identical geometry and boundary conditions to the original system is possible. Thus the consequences of the anisotropy cannot be interpreted to be of a mere geometric nature.

Here we test anisotropic universality for the critical Binder cumulant. The Binder cumulant [27] is a measure of the order parameter distribution. It may be used to locate the phase transition for a given model from the consequences of the anisotropy cannot be interpreted to be universal critical quantities on the sign of $\theta$.

We conclude by discussing the use of the renormalization group (RG) to describe the anisotropy dependence of $U^*$. In [21], the RG in three dimensions was used to describe the anisotropy dependence of $U^*$ in two dimensions. A two-dimensional anisotropy matrix $A_2(s) = (1 - s^2)^{-1/2} (\alpha \beta^T)$ with $s$ from above was incorporated into a block-diagonal $3 \times 3$ matrix $A_3(s)$ to estimate the $s$ dependence of $U^*$. The result provided a good approximation to the available data from [17, 31] for $J_d \geq 0$. However, later Monte Carlo data for $J_d < 0$ were not in agreement with the RG results, unless $J_d$ is small [18].

The reason for this discrepancy is easy to identify. The matrix $A_2(s)$ does not provide the correct description of the asymptotic shape of the correlation length ellipse in contrast to

$$\tilde{A}_2(r) = \frac{1}{2} \left( \begin{array}{cc} r + r^{-1} & \mp (r^{-1} - r) \\ \mp (r^{-1} - r) & r + r^{-1} \end{array} \right) \quad (12)$$

obtained by setting $\theta = \mp \pi/4$ in (6) for $J_d \geq 0$. While the deviation is small for positive $J_d$, it becomes large for negative $J_d$, unless $J_d$ is small. Using $\tilde{A}_2(r)$ instead, a natural choice for the appropriate three-dimensional
matrix is

\[ \tilde{A}_3(r) = \begin{pmatrix} A_2(r) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]  

(13)

which localizes the anisotropy within the first two dimensions. Plugging our choice of \( \tilde{A}_3(r) \) into the RG result for \( U_{\text{RG}}^*(r) \equiv U(0, \tilde{A}) \), provided in Eq. (7.8) of [21], and following Ref. [18] in normalizing the resulting function \( U_{\text{RG}}^*(r) \) by plotting \( U_{\text{RG}}^*(r)/U_{\text{RG}}^*(1) \) with \( U^*(1) \) from (11a), we obtain the solid line in Fig. 2, valid for both signs in (12) and thus for both ferromagnetic and antiferromagnetic \( J_d \). We conclude that the three-dimensional RG calculation of [21] results in a good description of the relative anisotropy dependence of the two-dimensional critical Binder cumulant, provided the anisotropy is described by the appropriate matrix \( \tilde{A}_2(r) \) from (12) and \( \tilde{A}_3(r) \) is naturally embedded into \( \tilde{A}_3(r) \) as in (13).

In summary, we considered the critical Binder cumulant \( U^* \) of a two-dimensional Ising model on a square lattice in a square geometry with periodic boundary conditions and nearest-neighbor couplings. Additional couplings on one of the lattice diagonals allowed us to adjust a variable diagonal anisotropy of the correlation lengths in the asymptotic critical domain. We found that \( U^* \) depends universally on its argument \( r \), defined as the ratio of the smallest and largest bulk correlation length, independently of the particular realization of \( r \) through ferromagnetic or antiferromagnetic diagonal couplings. Observing the true physical asymptotic long-distance behavior of the anisotropy, represented by the asymptotic critical correlation length ellipse, we repeated Dohm’s renormalization group calculation and found good quantitative agreement of the relative anisotropy dependence of \( U^* \) with precision transfer-matrix and Monte Carlo data. Our results support the validity of universal finite-size scaling for critical phenomena in the presence of a weak anisotropy. The ideas presented here and in [23] may be used to analyze other quantities in weakly anisotropic systems, for example, those (besides the Binder cumulant) for which nonuniversality was claimed in [19–21].

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[32] We show the Monte Carlo data provided to us by W. Selke and L.N. Shchur. For \( s = 0.6 \) and \( s = 0.6666 \), corresponding to \( \ln(1/r) \approx 0.710 \) and \( \ln(1/r) \approx 0.839 \), respectively, their error bars are somewhat larger than those plotted in Fig. 4 of [18].