QUASI INVARIANT CONHARMONIC TENSOR OF SPECIAL CLASSES OF LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLD

The authors classified a locally conformal almost cosymplectic manifold (LCACS-manifold) according to the conharmonic curvature tensor. In particular, they have determined the necessary conditions for a conharmonic curvature tensor on the LCACS-manifold of classes $CT_i$, $i = 1, 2, 3$ to be $\Phi$-quasi invariant. Moreover, it has been proved that any LCACS-manifold of the class $CT_1$ is conharmonically $\Phi$-paracontact.

Keywords: locally conformal almost cosymplectic manifold, conharmonic curvature tensor, $\Phi$-quasi invariant, conharmonically $\Phi$-paracontact.

DOI: 10.35634/vm200201

Introduction

The first classification of an almost contact metric structure has been introduced by Chinea and Marrero [9]. Like an almost Hermitian structure, another principle of classification of an almost contact metric structure for differential-geometric invariants of the second order depends on the properties of the Riemannian curvature tensor $R$. Volkova introduced analogues of these classes in contact geometry $CR_1, CR_2$ and $CR_3$ [19].

Our study focuses on the conharmonic tensor of the locally conformal almost cosymplectic manifold of three spacial classes $CT_i$, $i = 1, 2, 3$ which are related with the classes $CR_1, CR_2, CR_3$.

A harmonic function is a function whose Laplacian vanishes. It is known that the conformal transformation on the Riemannian manifold preserves the angle between two vectors. Generally, a harmonic function is not invariant. The condition to remain such a function invariant has been studied by Ishi [13]. Specifically, he introduced a conharmonic transformation that preserves the harmonicity of a certain function. On the other hand, Ghosh et al. [11] studied the $N(k)$-contact metric manifolds satisfying curvature conditions on the conharmonic tensor. Dwivedi and Kim [10] obtained certain necessary and sufficient conditions for the $K$-contact and Sasakian manifolds to be quasi conharmonically flat, $\xi$-conharmonically flat and $\Phi$-conharmonically flat. Furthermore, Asghari and Taleshian [4] studied the conharmonic curvature tensor on the Kenmotsu manifold. Chanyal and Uperti [8] proved that $\Phi$-conharmonically flat $(k, \mu)$-contact manifolds are $\eta$-Einstein manifolds. Taleshian et al. [18] considered $LP$-Sasakian manifolds admitting a conharmonic curvature tensor. Abood and Al-Hussaini [3] studied the geometrical properties of the conharmonic tensor of a locally conformal almost cosymplectic manifold. In particular, the authors established the necessary and sufficient conditions for the conharmonic tensor to be flat, the aforementioned manifold to be normal and an $\eta$-Einstein manifold.

§ 1. Preliminaries

The main purpose of this section is to construct an almost contact metric structure and a locally conformal almost cosymplectic manifold in the adjoined $G$-structure space.

**Definition 1** (see [5]). Let $M$ be $2n + 1$ dimensional smooth manifold, $\eta$ be differential 1-form called a contact form, $\xi$ be a vector field called a characteristic, $\Phi$ be an endomorphism of the module of the vector fields $X(M)$ called a structure endomorphism. The family of tensors $\{\eta, \xi, \Phi\}$ is called an almost contact structure if the following conditions hold.
1. \( \eta(\xi) = 1; \)
2. \( \Phi(\xi) = 0; \)
3. \( \eta \circ \Phi = 0; \)
4. \( \Phi^2 = -id + \eta \otimes \xi. \)

In addition, if there is a Riemannian metric \( g = \langle \cdot, \cdot \rangle \) on \( M \) such that
\[
\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in X(M),
\]
then the family of tensors \( \{\eta, \xi, \Phi, g\} \) is called an almost contact metric structure. In this case, the mentioned manifold \( M \) endowed with this structure is called an almost contact metric manifold.

**Definition 2** (see [16]). At each point \( p \in M^{2n+1} \), there is a frame in a complexification of a tangent space \( T^c_p(M) \) of the form \( (p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_\hat{1}, ..., \varepsilon_\hat{n}) \), where \( \varepsilon_a = \sqrt{2\pi}e_a, \varepsilon_\hat{a} = \sqrt{2\bar{\pi}}e_\hat{a}, \) \( \hat{a} = a + n, \) \( \varepsilon_0 = \xi_p, \) \( \pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi) \) and \( \bar{\pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi). \) The frame \( (p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_\hat{1}, ..., \varepsilon_\hat{n}) \) is called an \( A \)-frame.

A set of such frames defines \( G \)-structure on \( M \) with a structure group \( 1 \times U(n) \). This structure is called a \( G \)-adjoined structure.

**Lemma 1** (see [15]). The component matrices of the tensors \( \Phi_p \) and \( g_p \) in \( A \)-frame have the following forms respectively:
\[
(\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},
\]
where \( I_n \) is the identity matrix of order \( n \).

**Definition 3** (see [5]). A skew-symmetric tensor \( \Omega(X, Y) = g(X, \Phi Y) \) is called a fundamental form of the \( \mathcal{AC} \)-structure.

**Definition 4** (see [12]). An almost contact metric structure \( S = (\eta, \xi, \Phi, g) \) is called an almost cosymplectic structure (\( \mathcal{AC} \)-structure) if
1. \( d\eta = 0 \);
2. \( d\Omega = 0 \).

**Definition 5** (see [17]). A conformal transformation of an \( \mathcal{AC} \)-structure \( S = (\eta, \xi, \Phi, g) \) on a manifold \( M \) is a mapping from \( S \) to an \( \mathcal{AC} \)-structure \( \tilde{S} = (\tilde{\eta}, \tilde{\xi}, \tilde{\Phi}, \tilde{g}) \) such that
\[
\tilde{\eta} = e^{-\sigma}\eta, \quad \tilde{\xi} = e^{\sigma}\xi, \quad \tilde{\Phi} = \Phi, \quad \tilde{g} = e^{-2\sigma}g,
\]
where \( \sigma \) is the determining function of the conformal transformation. If \( \sigma = \text{const} \), then the conformal transformation is said to be trivial.

**Definition 6** (see [17]). An \( \mathcal{AC} \)-structure \( S \) on a manifold \( M \) is said to be a locally conformal almost cosymplectic (\( \mathcal{LC\.ACS} \)-structure) if the restriction of \( S \) on a some neighborhood \( U \) of an arbitrary point \( p \in M \), admits a conformal transformation of an almost cosymplectic structure. This transformation is called a locally conformal. A manifold \( M \) equipped with a \( \mathcal{LC\.ACS} \)-structure is called a \( \mathcal{LC\.ACS} \)-manifold.
Lemma 2 (see [14]). In the adjoined G-structure space, the structure equations of $\mathcal{L}CA_{C_{S}}$-manifold have the following forms:

1. $d\omega^a = -\omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}_d \omega^d \wedge \omega^c + B^a_b \omega \wedge \omega^b$;
2. $d\omega_a = \omega^a_c \wedge \omega_b + B^{abc}_d \omega^b \wedge \omega^c + B^{abc}_d \omega^d \wedge \omega^c + B^a_b \omega \wedge \omega_b$;
3. $d\omega = C^i_j \omega \wedge \omega^b + C^b_i \omega \wedge \omega^b$;

where $A^{ab}_{[cd]} = -2\delta^{[c}_{[a} \delta^{d]}_{b]} + 2\sigma^{[a} \delta^{d]}_{b]} \sigma_{[c]} \right] \right] + \frac{1}{2} B^{dca}_{bcd}$. Here $B^{abc}_{a}$, $B^{ab}_{a}$, $B^{b}_{a}$, $B^{a}_{a}$, $C^{ab}_{a}$, $C^{b}_{a}$, $A^{ac}_{a}$, $A^{ab}_{a}$, $A^{c}_{a}$, $A^{b}_{a}$, $B^{abc}_{a}$, $B^{b}_{a}$, $D^{ab}_{a}$, $D^{ab}_{b}$ and $\sigma_{ij}$ are smooth functions in the adjoined G-structure space. The tensors $B^{abc}_{a}$, $B^{ab}_{a}$ are called the second and third structure tensors respectively.

Definition 7 (see [7]). A Ricci tensor is a tensor of type $(2, 0)$ which is a contracting of the Riemannian curvature tensor and defined by

$$r_{ij} = -R^k_{ijk}.$$

Lemma 3 (see [1]). In the adjoined G-structure space, the components of the Ricci tensor of $\mathcal{L}CA_{C_{S}}$-manifold are given by the following forms:

1. $r_{ab} = 2(-2A^{(ab)c}_{(a)b} - 4(|\sigma| B^c_{[a}B^b_{c]} + \sigma^c B_{[a}B^b_{c]} + \sigma_0 B_{[a}B^b_{c]} + 2\sigma_0 B_{ab} - D_{ab} - \sigma_{ab} \delta_{a} \delta_{b} + 2B_{ab} \sigma^h)$;
2. $r_{ab} = -4(\delta^{[a}_{[b} \sigma^{c]}_{c]} - \sigma^c \delta^{b}_{[a} \delta^{h}_{c]} - \sigma^a \delta^{b}_{[a} \sigma^b \{B^c_{[a}B^h_{b]}B^b_{c]} + B^a_{[a}B^b_{c]}B^b_{c]} + (B^c_{[a}B^h_{b]} - B^b_{b}B^a_{c]})) + A^{c}_{ac} \delta^{b}_{[a} \sigma^{a]}_{a} - 2n \sigma^b \delta^a_{a} - \sigma^a \sigma^b$;
3. $r_{a0} = -A^{c}_{ac} \sigma^a - \sigma^c B_{ac} + \sigma_0 \sigma^a + 2(\sigma_0 \delta^c_{[a]} + B^a_{[a}B^h_{c]} - 2\sigma^c \delta^c_{[a]}B_{[a]}B^a_{c]}$;
4. $r_{oo} = -2n(\sigma_0 + \sigma^c \sigma^c) - 2B_{ac} B^{ac} - 2(\sigma^c + \sigma^c \sigma^c) + 4\sigma^c \delta^c_{[a]} \sigma_{c}$.

The remaining components are conjugate to the above-mentioned components.

Definition 8 (see [2]). A $\mathcal{L}CA_{C_{S}}$-manifold is said to have $\Phi$-invariant Ricci tensor, if $\Phi \circ r = r \circ \Phi$.

Lemma 4 (see [2]). A $\mathcal{L}CA_{C_{S}}$-manifold has $\Phi$-invariant Ricci tensor if and only if, in the adjoined G-structure space, the following condition

$$r^b_{b} = r_{ab} = r^0_{0} = r_{a0} = 0$$

holds.

Definition 9 (see [6]). A pseudo-Riemannian manifold $M$ is known as an $\eta$-Einstein of type $(\alpha, \beta)$ if its Ricci tensor satisfies the following condition:

$$r = \alpha g + \beta \eta \otimes \eta,$$

where $\alpha$ and $\beta$ are suitable smooth functions. If $\beta = 0$, then $M$ is referred to as an Einstein manifold.
**Definition 10** (see [12]). Let $M$ be an $\mathcal{A}$-manifold of dimension $2n+1$. A tensor $T$ of type $(4,0)$ which is invariant under conharmonic transformation and defined by the form

$$T_{ijkl} = R_{ijkl} - \frac{1}{2n-1}(r_{ij}g_{jk} - r_{jk}g_{ij} + r_{jk}g_{il} - r_{ik}g_{jl})$$

is called a conharmonic curvature tensor.

In the work [1], we have calculated all possible components of the mentioned tensor which are listed in the next lemma.

**Lemma 5.** In the adjoined $G$-structure space, the components of the conharmonic curvature tensor of $\mathcal{LCAC}_S$-manifold are given by the following forms:

1. $T_{abcd} = 2(2B_{[cd]|ab]} - 2\sigma_{[a}B_{b]cd} + B_{a[b}B_{d]}b);$
2. $T_{abcd} = 2(A_{[abc]}^{c} + 4\sigma\sigma_{[c}B_{d]ab} - \sigma_{0}B_{[d}(\delta_{a}^{b})_{c}) - 4\frac{1}{2n-1}(r_{bc}\delta_{d}^{a} - r_{cd}\delta_{b}^{a});$
3. $T_{abcd} = A_{[abc]}^{d} + 4\sigma\sigma_{[a}B_{c]db} - 4B_{[a}(\delta_{b}^{d})_{c} + B_{b}^{d}B_{c}^{d} - \delta_{a}^{d}\delta_{b}^{d}\sigma_{0}^{2} + \frac{1}{2n-1}(r_{a}^{d}\delta_{b}^{c} + r_{c}^{d}\delta_{b}^{d});$
4. $T_{abcd} = 2(2\delta_{[c}^{a}\sigma_{d]}^{a} + 2B_{h}^{a}B_{h}^{c} - \delta_{[c}^{a}\delta_{d]}^{a}\sigma_{0}^{2} - 4\frac{1}{2n-1}(r_{h}^{a}\delta_{c}^{a} + r_{c}^{d}\delta_{b}^{d});$
5. $T_{abcd} = 2(\sigma_{0}\delta_{d]}^{a} + B_{a}^{b}B_{b}^{c} - 2\sigma\sigma_{[c}B_{d]a}) + \frac{1}{2n-1}(r_{a}^{d}\delta_{b}^{a} - r_{c}^{d}\delta_{b}^{a});$
6. $T_{abcd} = A_{[abc]}^{d} + \sigma_{0}B_{b}^{a} + \sigma_{b}^{a} + \frac{1}{2n-1}(r_{a}^{d}\delta_{b}^{a});$
7. $T_{abcd} = 2B_{c_{a}b}^{d} + 2B_{c}^{a}\sigma_{d};$
8. $T_{abcd} = -\delta_{b}^{a}\delta_{0}^{a} - B_{b}^{a}B_{b}^{a} - \delta_{b}^{a}\sigma_{0}^{a} + 2\sigma\sigma_{[a}B_{b]c} + \frac{1}{2n-1}(r_{a}^{d}\delta_{b}^{a} + r_{b}^{a});$
9. $T_{abcd} = 2\sigma_{b}B_{b}^{a} - D_{a}^{b} - \sigma_{b}^{a} - \sigma_{a}^{b} + 2B_{b}^{a}B_{b}^{c} + \frac{1}{2n-1}(r_{a}^{d});$

The remaining components are conjugate to the above-mentioned components or can be obtained by the property of symmetry for $T$ or identically equal to zero.

The next lemma gives analogues to the Gray’s identities in the adjoined $G$-structure space.

**Lemma 6** (see [19]). An $\mathcal{A}$-manifold is called a manifold of class

1. $CR_{1}$ if and only if, $R_{abcd} = R_{abcd} = R_{abcd} = 0$;
2. $CR_{2}$ if and only if, $R_{abcd} = R_{abcd} = 0$;
3. $CR_{3}$ if and only if, $R_{abcd} = 0$.

§ 2. The main results

In this section, we introduce analogues to the Gray’s identities for the conharmonic curvature tensor of a $\mathcal{LCAC}_S$-manifold.

**Definition 11.** In the adjoined $G$-structure space, a $\mathcal{LCAC}_S$-manifold is called a manifold of class

1. $CT_{1}$ if and only if, $T_{abcd} = T_{abcd} = T_{abcd} = 0$;
2. $CT_{2}$ if and only if, $T_{abcd} = T_{abcd} = 0$;
3. $CT_3$ if and only if, $T_{abcd} = 0$.

**Definition 12.** The conharmonic tensor is called $\Phi$-quasi invariant, if

$$\langle T(\Phi X, \Phi Y)X, \Phi Y \rangle - \langle T(\Phi^2 X, \Phi^2 Y)X, \Phi Y \rangle = \langle T(\Phi X, \Phi Y)X, \Phi Y \rangle$$

holds for each $X, Y, Z \in X(M)$.

**Definition 13.** A $\mathcal{LCAC}_S$-manifold is called a conharmonically $\Phi$-paracontact, if its conharmonic tensor satisfies the identity

$$\langle T(\Phi^2 X, \Phi^2 Y)X, \Phi^2 W \rangle = \langle T(\Phi^2 X, \Phi^2 Y)X, \Phi^2 W \rangle - \langle T(\Phi X, \Phi^2 Y)X, \Phi^2 W \rangle - \langle T(\Phi X, \Phi^2 Y)X, \Phi^2 W \rangle,$$

where $X, Y, Z \in X(M)$.

**Theorem 1.** A conharmonic tensor on $\mathcal{LCAC}_S$-manifold of class $CT_i$, $i = 1, 2, 3$, is $\Phi$-quasi invariant.

**Proof.** Consider the component $T_{\hat{a}ab} = 0$. Write this equation in $A$-frame, we have

$$\langle T(\hat{a}_a, \hat{b}_b)e\hat{a}, e\hat{b} \rangle = 0.$$

Then we get

$$\langle T(\hat{a}X, \sigma Y)X, \sigma Y \rangle = 0, \quad X, Y \in X(M)$$

We compensate for the value of the projections $\sigma$ and $\sigma$ and, using the linear property of the tensor, we can rewrite the previous expression in the form

$$\{ \langle T(X, Y)X, Y \rangle + \langle T(\Phi X, \Phi Y)X, Y \rangle + \langle T(\Phi X, Y)X, \Phi Y \rangle + \langle T(\Phi X, Y)X, \Phi Y \rangle - \langle T(\Phi X, \Phi Y)X, \Phi Y \rangle - \langle T(\Phi X, \Phi Y)X, \Phi Y \rangle \} +$$

$$i\{ \langle T(X, Y)X, Y \rangle + \langle T(\Phi X, Y)X, Y \rangle + \langle T(\Phi X, Y)X, Y \rangle + \langle T(\Phi X, Y)X, Y \rangle +$$

$$\langle T(\Phi X, Y)X, Y \rangle - \langle T(\Phi X, \Phi Y)X, Y \rangle - \langle T(\Phi X, \Phi Y)X, Y \rangle - \langle T(\Phi X, \Phi Y)X, Y \rangle \} = 0.$$

Note that the real part equals to zero, and regarding the symmetrical properties of the conharmonic curvature tensor, we have

$$\langle T(X, Y)X, Y \rangle + \langle T(\Phi X, Y)X, Y \rangle - \langle T(\Phi X, \Phi Y)X, Y \rangle - \langle T(\Phi X, \Phi Y)X, Y \rangle = 0. \quad (1)$$

Then by substituting $X \to -\Phi^2 X; Y \to -\Phi^2 Y$ in equation (1) and using the fact $\Phi(\hat{X}) = = -\Phi(X)$, we obtain the assertion of the theorem.

**Theorem 2.** A conharmonic curvature tensor on $\mathcal{LCAC}_S$-manifold of class $CR_3$ with $\Phi$-invariant Ricci tensor is $\Phi$-quasi invariant.

**Proof.** Since the $\mathcal{LCAC}_S$-manifold $M$ has $\Phi$-invariant Ricci tensor, then $CT_3$ and $CR_3$ are coincide. Regarding the Theorem 1, we get that the conharmonic curvature tensor is $\Phi$-quasi invariant.

**Theorem 3.** Any $\mathcal{LCAC}_S$-manifold $M$ of class $CT_1$ is conharmonically $\Phi$-paracontact.
Proof. On the $G$-adjoined structure space, we have $T_{abcd} = 0$. In A-frame, we get

$$\langle T(\varepsilon_a, \varepsilon_b) \varepsilon_c, \varepsilon_d \rangle = 0.$$  

Consequently, we have

$$\langle T(\sigma X, \sigma Y) \sigma X, \sigma Y \rangle = 0, \quad X, Y, Z, W \in X(M).$$  

We make up for the value of the projections $\sigma$ and $\bar{\sigma}$ and, using the linear property of the tensor, we can rewrite this expression in the form

$$\langle T(X, Y)Z, W \rangle - \langle T(\Phi X, \Phi Y)Z, W \rangle - \langle T(X, Y)\Phi Z, \Phi W \rangle + \langle T(\Phi X, Y)Z, \Phi W \rangle - \langle T(\Phi X, Y)\Phi Z, \Phi W \rangle = 0. \quad (2)$$  

According to the Definition 11, we have $T_{abcd} = 0$. Using the same technique, consequently we obtain

$$\langle T(X, Y)Z, W \rangle + \langle T(\Phi X, Y)\Phi Z, \Phi W \rangle - \langle T(X, Y)\Phi Z, \Phi W \rangle - \langle T(\Phi X, Y)Z, \Phi W \rangle - \langle T(\Phi X, \Phi Y)\Phi Z, \Phi W \rangle = 0. \quad (3)$$  

From the combination of the equations (2) and (3), it follows that

$$\langle T(X, Y)Z, W \rangle = \langle T(\Phi X, Y)\Phi Z, \Phi W \rangle - \langle T(\Phi X, Y)\Phi Z, W \rangle - \langle T(\Phi X, Y)Z, \Phi W \rangle. \quad (4)$$  

Then by substituting $X \to -\Phi^2 X; Y \to -\Phi^2 Y; Z \to -\Phi^2 Z, W \to -\Phi^2 W$ in the equation (4) and using the fact $\Phi^3(X) = -\Phi(X)$, we get the result. □

Theorem 4. Let $M$ be $\mathcal{LCAC}_S$-manifold of class $CR_2$, then the first structure tensor is parallel in the first canonical connection.

Proof. Suppose that $M$ is $\mathcal{LCAC}_S$-manifold of the class $CR_2$. So we have

$$2((B_{cabd} - B_{dabc}) - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b}) = 0. \quad (5)$$  

Symmetrizing (5) by the indices $(c, d)$ and then by the indices $(c, b)$, we get

$$B_{cabd} = 0.$$

Regarding the fundamental theorem of tensor analysis, we have

$$\nabla B_{abc} = dB_{abc} + B_{dbc}\omega^d_a + B_{abc}\omega^d_b + B_{abd}\omega^d_c = B_{abcd}\omega^d.$$

So, we get

$$\nabla B_{abc} = B_{abcd}\omega^d.$$  

It follows that

$$\nabla B_{abc} = 0.$$  

Therefore, the tensor $B_{abc}$ is parallel. □

Theorem 5. Let $M$ be $\mathcal{LCAC}_S$-manifold of class $CR_1$, then its second and third structure tensors identically vanish.
Let $M$ be $\mathcal{LCAC}_S$-manifold of the class $CR_1$. According to the component of Riemannian curvature tensor, we have

$$2(2B[c|ab|d] - 2\sigma[aB_{b|ed}B_{a|c}B_{d|b}]) = 0. \quad (6)$$

Symmetrizing and then antisymmetrizing (6) by the indices $(a, b)$, we deduce

$$B_{ac}B_{db} - B_{ad}B_{cb} = 0. \quad (7)$$

Antisymmetrizing (7) by the indices $(a, d)$, we get

$$B_{ac}B_{db} = 0. \quad (8)$$

Contracting (8) by the indices $(a, d)$ and then by $(c, b)$, we get $B_{ac}^2 = 0$, then

$$B_{ac} = 0.$$ 

From the condition $R_{abcd} = 0$, we obtain

$$B^{abc} = 0.$$ 

In the rest of this section, we establish a relation between the classes $CT_1$ and $CR_1$.

**Theorem 6.** Suppose that $M$ is $\mathcal{LCAC}_S$-manifold of class $CT_1$ with $\Phi$-invariant Ricci tensor, then $M$ is a manifold of class $CR_1$ if and only if $M$ has flat Ricci tensor.

Proof. Suppose that $M$ has flat Ricci tensor. Since $M$ is a $\mathcal{LCAC}_S$-manifold of class $CT_1$, then according to the Definition 11, we have

$$T_{abcd} = T_{\hat{a}\hat{b}\hat{c}\hat{d}} = T_{\hat{a}\hat{d}\hat{c}\hat{b}} = 0.$$ 

The condition $T_{abcd} = 0$, implics that $R_{abcd} = 0$; while the condition $T_{\hat{a}\hat{b}\hat{c}\hat{d}} = 0$ with $\Phi$-invariant Ricci tensor, implies that $R_{\hat{a}\hat{b}\hat{c}\hat{d}} = 0$. In addition, $T_{\hat{a}\hat{d}\hat{c}\hat{b}} = 0$ with flat Ricci tensor, indicates $R_{\hat{a}\hat{d}\hat{c}\hat{b}} = 0$. Then $M$ is a manifold of class $CR_1$. Conversely, according to the condition $T_{\hat{a}\hat{b}\hat{c}\hat{d}} = 0$, we have

$$2(2\delta_{[c|a}\sigma_{d]} + 2B^{hab}B_{hdc} - \delta_{[c|a}\delta_{d]}\sigma_{b]}^2) - \frac{4}{2n - 1}(r_{[c|a}\delta_{d]}^b) = 0. \quad (9)$$

Since $M$ is a manifold of class $CR_1$, we get

$$- \frac{4}{2n - 1}(r_{[c|a}\delta_{d]}^b) = 0. \quad (9)$$

Taking the contraction operation for (9) by the indices $(b, d)$, we deduce

$$r^{a}_{c\delta_b} - r^{a}_{b\delta_c} - r^{b}_{c\delta_a} + r^{b}_{b\delta_c} = 0.$$ 

Consequentaly, we obtain

$$(n - 2)r^{a}_{c\delta} + r^{b}_{b\delta_c} = 0. \quad (10)$$

Taking the symmetrization and antisymmetrization operations for (10) by the indices $(b, a)$, we conclude

$$r^{a}_{c\delta} = 0.$$ 

(11)
Making use of the Theorems 3 and 5, we have
\[ -4\left(\delta_{[c}^a\sigma_{d]}^c - \sigma_{[c}\delta_{h]}^c\sigma^{[h}\delta_{c]}^c - \frac{1}{2}\sigma_{[a}\delta_{h]}^c\sigma_{h]}^c + A_{ac}^{ch} - \delta_{c}^0\sigma_{00} - 2n\delta_{b}^0\sigma_{0}^2 - \sigma_{b}^0 - \sigma_{b}^a\sigma_{a} = 0. \] (12)
Symmetrizing (12) by the indices \((c, h)\), we obtain
\[ -\sigma_{[c}\delta_{h]}^c\sigma^{[h}\delta_{c]}^c = 0. \] (13)
Contracting (13) by the indices \((a, c)\), we get
\[ \sigma_{a}^\delta\delta_{a}^\delta\sigma_{b}^a\sigma_{b}^a - \sigma_{a}^\delta\delta_{b}^a\sigma_{b}^a\delta_{a}^\delta - \sigma_{h}^\delta\delta_{a}^\delta\delta_{a}^\delta + \sigma_{h}^\delta\delta_{a}^\delta\delta_{a}^\delta = 0. \]
Hence
\[ -n(n - 1)(\sigma_{h}\sigma_{h}^\delta) = 0. \] (14)
Or equivalently,
\[ \sigma_{h}\sigma_{h}^\delta = 0 \Leftrightarrow \sum_{h}\sigma_{h}|^2 = 0 \Leftrightarrow \sigma_{h} = 0. \] (15)
Once again, by using the equation (12), we have
\[ A_{ac}^{ch} - \delta_{c}^0\sigma_{00} - 2n\delta_{b}^0\sigma_{0}^2 = 0. \] (16)
By virtue of the Theorems 2, 5 and equation (14), the tensors \(A_{ac}^{ae} = A_{ac}^{ae} = 0\) become symmetric regarding the lower and upper indices. Antisymmetrizing (15) by the indices \((c, b)\), we obtain
\[ -\delta_{b}^0\sigma_{00} - 2n\delta_{b}^0\sigma_{0}^2 = 0. \] (17)
Contracting (16) by the indices \((a, b)\), consequently, we get
\[ r_{00} = 0. \] (18)
According to the equations (11), (17) and \(\Phi\)-invariant Ricci property, we get that \(M\) has flat Ricci tensor.

Finally, we established an application by identifying the necessary and sufficient condition for the locally conformal almost contact manifold to be \(\eta\)-Einstein manifold.

**Theorem 7.** Suppose that \(M\) is \(LCAC_{S}\)-manifold of class \(CT_1\) with \(\Phi\)-invariant Ricci tensor. Then the necessary and sufficient condition for \(M\) to be \(\eta\)-Einstein manifold is \(\sigma_{d}^a = E\delta_{d}^a\), where \(E = \frac{2(1-n)\sigma_{0}^2}{n-2} - \frac{\alpha}{2n-1}\).

**Proof.** Suppose that \(M\) is \(LCAC_{S}\)-manifold of class \(CT_1\). According to the Definition 11 and Lemma 5, we have
\[ 2(2\delta_{[c}^a\sigma_{d]}^a - \delta_{[c}^a\delta_{d]}^a\sigma_{0}^2) - \frac{4}{2n-1}(r_{[d}^a\delta_{c]}^a) = 0. \] (19)
Contracting (19) by indices \((b, c)\), we get
\[ (n - 2)\sigma_{d}^b + \delta_{d}^a\sigma_{b}^a - 2(1 - n)\sigma_{0}^2\delta_{d}^a - \frac{1}{2n-1}((n - 2)\sigma_{d}^b + \delta_{d}^a\sigma_{b}^a) = 0. \]
Then, we deduce

\[ \sigma^a_d = \frac{2(1 - n)\sigma^0_d\delta^a_d}{n - 2} - \frac{r^a_d}{2n - 1}. \] (19)

Let \( M \) be \( \eta \)-Einstein manifold. We use the Definition9, so the equation (19) becomes

\[ \sigma^a_d = E\delta^a_d. \] (20)

Conversely, from the equations (19) and (20), we have

\[ r^a_d = \alpha\delta^a_d \]

where \( \alpha \) is the cosmological constant. Also

\[ r_{00} = \alpha + \beta \]

where \( \alpha = -2n(\sigma_{00} + \sigma^2_0) - 2B_{hc}B^{hc} - 2(\sigma^c_c + \sigma^c_c\sigma_c) + 4\sigma^c_c\delta^h_c\sigma_h - \alpha \)

According to \( \Phi \)-invariant of Ricci tensor, we obtain that \( M \) is \( \eta \)-Einstein manifold. \( \square \)

REFERENCES

1. Abood H. M., Al-Hussaini F. H. Locally conformal almost cosymplectic manifold of \( \Phi \)-holomorphic sectional conharmonic curvature tensor, *European Journal of Pure and Applied Mathematics*, 2018, vol. 11, no. 3, pp. 671–681. https://doi.org/10.29020/nybg.ejpam.v11i3.3261

2. Abood H. M., Al-Hussaini F. H. Constant curvature of a locally conformal almost cosymplectic manifold, *AIP Conference Proceedings*, 2019, vol. 2086, issue 1, 030003. https://doi.org/10.1063/1.5095088

3. Abood H. M., Al-Hussaini F. H. On the conharmonic curvature tensor of a locally conformal almost cosymplectic manifold, *Communications of the Korean Mathematical Society*, 2020, vol. 35, no. 1, pp. 269–278. https://doi.org/10.4134/CKMS.c190003

4. Asghari N., Taleshian A. On the conharmonic curvature tensor of Kenmotsu manifolds, *Thai Journal of Mathematics*, 2014, vol. 12, no. 3, pp. 525–536. http://thaijmath.in.cmu.ac.th/index.php/thaijmath/article/view/410

5. Blair D. E. The theory of quasi-Sasakian structures, *Journal of Differential Geometry*, 1967, vol. 1, no. 3–4, pp. 331–345. https://doi.org/10.4310/jdg/1214428097

6. Blair D. E. *Riemannian geometry of contact and symplectic manifolds*, Boston, MA: Birkhäuser, 2010. https://doi.org/10.1007/978-0-8176-4959-3

7. Cartan E. *Riemannian geometry in an orthogonal frame*, Singapore: World Scientific, 2001.

8. Chanyal S. K., Upreti J. Conharmonic curvature tensor on \((\kappa, \mu)\)-contact metric manifold, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)*, 2016, vol. 2, issue F2, pp. 681–694. https://www.math.uaic.ro/~annalsmath/new/?page_id=351

9. Chinea D., Marrero J. C. Classification of almost contact metric structures, *Revue Roumaine de Mathématiques Pures et Appliquées*, 1992, vol. 37, no. 3, pp. 199–211.

10. Dwivedi M. K., Kim J.-S. On conharmonic curvature tensor in \( K \)-contact and Sasakian manifolds, *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, 2011, vol. 34, no. 1, pp. 171–180.

11. Ghosh S., De U. C., Taleshian A. Conharmonic curvature tensor on \( N(K) \)-contact metric manifolds, *ISRN Geometry*, 2011, vol. 2011, article ID 423798, 11 p. https://doi.org/10.5402/2011/423798

12. Goldberg S. I., Yano K. Integrability of almost cosymplectic structures, *Pacific Journal of Mathematics*, 1969, vol. 31, no. 2, pp. 373–382. https://doi.org/10.2140/pjm.1969.31.373

13. Ishii Y. On conharmonic transformation, *Tensor: New Ser.*, 1957, vol. 7, pp. 73–80. https://zbmath.org/?q=an:0079.15702
14. Kharitonova S. V. On the geometry of locally conformal almost cosymplectic manifolds, Mathematical Notes, 2009, vol. 86, no. 1, pp. 121–131. https://doi.org/10.1134/S0001434609070116

15. Kirichenko V. F. Differential’no-geometricheskie struktury na mnogoobraziyakh (Differential-geometric structures on manifolds), Odessa: Pechatnyi Dom, 2013.

16. Kirichenko V. F., Rustanov A. R. Differential geometry of quasi Sasakian manifolds, Sbornik: Mathematics, 2002, vol. 193, no. 8, pp. 1173–1201. https://doi.org/10.1070/SM2002v193n08ABEH000675

17. Olszak Z. Locally conformal almost cosymplectic manifolds, Colloquium Mathematicum, 1989, vol. 57, no. 1, pp. 73–87. https://doi.org/10.4064/cm-57-1-73-87

18. Taleshian A., Prakasha D. G., Vikas K., Asghari N. On the conharmonic curvature tensor of $LP$-Sasakian manifolds, Palestine Journal of Mathematics, 2016, vol. 5, no. 1, pp. 177–184.

19. Volkova E. S. Curvature identities of normal manifolds of Killing type, Mathematical Notes, 1997, vol. 62, no. 3, pp. 296–305. https://doi.org/10.1007/BF02360870

Received 12.02.2020

Al-Hussaini Farah Hassan, University of Basrah, Basrah, Iraq.
E-mail: farahhussaini14@yahoo.com

Abood Habeeb Mtashar, PhD, Professor, University of Basrah, Basrah, Iraq.
E-mail: iraqsafwan2006@gmail.com

Citation: F. H. Al-Hussaini, H. M. Abood. Quasi invariant conharmonic tensor of special classes of locally conformal almost cosymplectic manifold, Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp’yuternye Nauki, 2020, vol. 30, issue 2, pp. 147–157.
Ф. Х. Аль-Хусейни, Х. М. Абуд
Квазиинвариантный конгармонический тензор специальных классов локально конформного почти косимплектического многообразия

Ключевые слова: локально конформное почти косимплектическое многообразие, тензор конгармонической кривизны, Ф-квазиинвариант, конгармонический Ф-параконтакт.

УДК 514.7
DOI: 10.35634/vm200201

В работе описывается классификация локально конформного почти косимплектического многообразия (LC\(_{AC}\)S-многообразия) в соответствии с тензором конгармонической кривизны. В частности, были получены необходимые условия Ф-инвариантности тензора конгармонической кривизны на LC\(_{AC}\)S-многообразии классов CT\(_i\), \(i = 1, 2, 3\). Кроме того, доказано, что любое LC\(_{AC}\)S-многообразие класса CT\(_1\) оказывается конгармоничным и Ф-параконтактным.

Поступила в редакцию 12.02.2020

Аль-Хусейни Фара Хасан, университет Басры, Ирак, г. Басра.
E-mail: farahalhussaini14@yahoo.com

Абуд Хабиб Мгашар, PhD, профессор, университет Басры, Ирак, г. Басра.
E-mail: iraqsafwan2006@gmail.com

Цитирование: Ф. Х. Аль-Хусейни, Х. М. Абуд. Квазиинвариантный конгармонический тензор специальных классов локально конформного почти косимплектического многообразия // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2020. Т. 30. Вып. 2. С. 147–157.