On the Geometry of Spacetime I:
baby steps in quantum ring theory

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Abstract

Vierbeins provide a bridge between the curved space of general relativity and the flat tangent space of special relativity. Both spaces should be causal and spin. We posit intertwining the two symmetries of spacetime bundles asymmetrically; disentangling the non-trivial \text{Id} between the base, curved space as a locally ringed space and its Zariski (co-)tangent space. This involves the introduction of a “two-sided vector space” as a section of the smooth, stratified diffeomorphism bundle of spacetime. A change of paradigm from the fiber bundle approach ensues where the bundle space takes an active role and the group actions are implemented through asymmetric “scalar multiplication” by elements of a skewed \text{K}-algebra on a free \text{K}-bimodule. Accordingly, the left action is augmented from that on the right algebraically by a left-sided ring endomorphism \text{via} a left \(\alpha\)-derivation as a non-central Ore extension of a Weyl algebra. Curiously, summoning the left \(\alpha\)-derivation in the context of spacetime symmetries may constitute the key to an asymmetric quantization of the theory. Furthermore, it is conjectured that causal and spin structure may be endowed upon the spacetime itself, independently of the tangent space structure.

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1 Introduction

One advantage of a runaway rather than a true cosmological “constant” is that, by analogy with a zero cosmological constant scenario first outlined by Dyson many years ago, in a runaway scenario life can possibly adapt and survive and develop forever by working at lower and lower temperatures and with longer time and length scales. A strict cosmological constant would bring this to a grim end by introducing effective length and time cutoffs.

Ed Witten, Dark Matter 2000, Marina del Rey, CA.

Nearly 140 years ago, Felix Klein was first to notice that certain invariant geometrical properties—namely angles, parallelisms, and cross-ratios—gradually lose invariance as the transformation group that embodies the geometry is enlarged. Such an insight led to his celebrated Erlangen Programme: the characterization of classical geometries on the basis of the transformation groups that preserve intrinsic geometrical invariants. In modern parlance, the latter group is presented as the group of automorphisms of the geometry thus embodying the notion of “symmetry”. Furthermore, the following hierarchy of symmetry groups: Euclidean ⊂ Affine ⊂ Projective, naturally led to the notion of homogeneous model geometry as a coset space: the quotient of a transitive Lie group epitomizing the geometry by the stabilizer sub-group of a point in the manifold. Implicit in this geometrical construction is that the manifold and the group action both be smooth. Important modern generalizations of the notion of Lie group structure include topological manifold with continuous action and algebraic variety with regular action under the Zariski topology.

Notably, a homogeneous model space need not admit Lie group structure unless the quotient is by a normal sub-group of the original group. Common knowledge holds that the product of equivalent classes in a coset space with group structure does not depend on the choice of “representative” group elements. Yet, without group structure no invariant notion of product exists. In a minimalist way, a group is a set with (two-sided) identity and a single invertible operation, i.e., the product operation. On the other hand, a ring is a set with identity and with two binary operations: addition and multiplication (the latter distributing over the former). Moreover, a ring is further endowed with additive monoid an abelian group already looking very much like a vector space. Within the scope of Lie theory, when the quotient in a homogeneous model geometry is not by a normal subgroup, the algebraic coset space is simply labeled a vector space as opposed to a Lie subalgebra.

Four dimensional Euclidian space, $R^4$, represents a canonical trivialization of what a physicist refers to as a vector space. In special relativity, $R^4$ is equipped with some minimal extra structure to enforce causality through a globally flat metric: $R^{1,3} = (R^{1+3}, \langle, \rangle)$. In general relativity, the world metric is derived from the flat metric through the introduction of vierbeins which possess one flat index and one world index. These are often called solder forms in analogy to the case when one interprets $R^4$ as an affine space with automorphism group $GL(4, R)$ and “reduces” this space to one with automorphism group the Lorentz group $SO(3,1)$. Group reduction is then embodied by the solder form which identifies the tangent space to the base manifold with the vector space of special relativity thereby breaking affine and scale invariance. However, this is clearly a misnomer for vierbeins, as introduced by Élie Cartan, are far richer geometrical objects than the solder forms introduced by his pupil, Charles Ehresmann. More formally, vierbeins are not exact external differentials; i.e. they are not 1-forms, but following common practice we will refer to them as solder “forms” although perhaps a better descriptor would be infinite forms or simply graviforms.

Algebraically (in the sense of Lie algebras) and for the compact case, a resolution of this issue in terms of a general infinitesimal coset space, $G'/H'$, depends on how the sub-algebra $H'$ is embedded in $G'$. In the particularly tame case of a reductive model geometry, the algebra is endowed with a canonical choice of infinitesimal symmetry perpendicular to the stabilizer subgroup $H$ so that “pure translations” (or transvections) may be generated by the $Ad(H)$-invariant complement of $H'$ in $G'$: $K' \supseteq [H', K']$, in the so-called reductive splitting of the Lie algebra. Such a canonical splitting leads to the notion of vector space as the algebraic quotient of $G'$ by $H'$ although this is not, groupwise, given a closed subgroup $H \subset G$, define the left coset (or quotient) space as the set of all equivalent classes: $G/H = \sum_{g_i \in g_{i-1}} [g_i]$ where $g_0 = e$ and $[g_i] = \{g_i h \mid h \in H\} \equiv g_i H$. If $H$ is normal in $G$, then $ghg^{-1} \in H$ so there $\exists h' \in H \mid gh = h'g$ and the product and inverse of equivalent classes do not depend on the choice of representative for the class: $[g][g'] = [gg']$ and $[g^{-1}] = [g]^{-1}$. Right cosets may be defined likewise: $G\backslash H = \sum_{g_i \in g_{i-1}} Hg_i$, as opposed to $g_i H$ for left cosets ($\forall h \in H$).

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1 Groupwise, given a closed subgroup $H \subset G$, define the left coset (or quotient) space as the set of all equivalent classes: $G/H = \sum_{g_i \in g_{i-1}} [g_i]$ where $g_0 = e$ and $[g_i] = \{g_i h \mid h \in H\} \equiv g_i H$. If $H$ is normal in $G$, then $ghg^{-1} \in H$ so there $\exists h' \in H \mid gh = h'g$ and the product and inverse of equivalent classes do not depend on the choice of representative for the class: $[g][g'] = [gg']$ and $[g^{-1}] = [g]^{-1}$. Right cosets may be defined likewise: $G\backslash H = \sum_{g_i \in g_{i-1}} Hg_i$, as opposed to $g_i H$ for left cosets ($\forall h \in H$).
Geometrically (and still for the compact case), this issue had led us to the rather abstract notion of a “stratified manifold” as a submersion of closed, totally geodesic submanifolds (= the fixed point set a given isometry) generated by distinct isotropy types in the orbit space of a given G-space. To the best of our knowledge, a working notion of vector space to support appropriate group representations of the field content of a given theory of gauge interactions with multiple stabilizer subgroups does not exist. In essence, this amounts to inducing a representation of a suitable vector space from the isotropy sub-groups of the field content (see Appendix C).

Physically, motivation for the above construction in flat spacetime comes from Wigner’s original classification scheme for unitary irreducible representations of the Poincaré group[10]. On an irreducible representation of the first Casimir operator of the Poincaré algebra, \( P^2 \), representations necessarily split into massless and massive states according to \( P^2 \geq 0 \). Choosing a non-zero momentum on the mass shell \( p^2 = m^2 \), the “little” group of the double cover of the Lorentz group, SL(2,C), is the subgroup which leaves \( p \mu \) invariant under \( M^1 p \mu M \), with \( M \in \text{SL}(2,\mathbb{C}) \). For positive energy states \( p_0 > 0 \), there are two stabilizer sub-groups: \( G_{m^2 > 0} = \text{SU}(2) \) and \( G_{m^2 = 0} = \text{Spin}(2) \otimes \mathbb{R}^2 \). Thus, simply labeling the irrep’s by the first Casimir element, the need for stratification of the vector space to support such representations becomes manifest. We posit that stratification causes the breaking of the affine symmetry for spacetime bundles.

On the other hand, Einstein’s gravity—when viewed as a gauge theory in the language of fiber bundles—requires for the local and global actions to belong in a broad sense to a reductive pair of symmetry groups. Curved spacetime is then to be interpreted as the coset space that results from the very large, local diffeomorphism symmetry acting on the left quotiented by the rigid, global Lorentz symmetry acting on the right of the fibration. Whereas the Lorentz group admits a double cover suitable for the representation of spinor fields as spinor bundles associated to the tangent bundle to spacetime, it is much less clear how such structure may be instilled upon the local symmetry of gravity: the diffeomorphisms of spacetime as a base manifold. In fact, even before attempting to address the issues of causal and spin structure rather little is formally well understood of the structure of this symmetry (as a group) beyond one dimension and interpretations surrogate to the notions of Lie theory for finite dimensional groups. Yet, a critical view of the standard reductive G-structure framework reveals clear pitfalls when attempting to address the full geometrical structure of this symmetry in 4D (see §2).

Broadly speaking, given a distinguished set of stabilizers for the particle content of the theory, it is highly desirable to build a stratified vector space “from the bottom up” by: i- noting the existence of a non-free group action induced by multiple maximally compact stabilizer subgroups, ii- selecting an algebraic ideal in a suitable ring and its associated module tailored for the treatment of gravity as a “gauge” theory in the jargon of fiber bundles and iii- replacing group morphisms by ring morphisms to implement the representation in a more general non-commutative algebro-geometric setting.

This is a ambitious agenda of which we only touch “the tip of the iceberg”. Tacitly assuming that the existence of distinct strata breaks the affine symmetry of spacetime, in this paper we build a mathematical framework to address ii- and iii- via the novel notion of a double-sided vector space as a conjecture for how the full bundle

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2 If an algebraic representation of the full group exists in terms of a finite set of infinitesimal generators and a well behaved exponential map, an (infinitesimal sub-) representation of the group quotient as a vector space relaxes the requirement that it be a Lie algebra since the isotropy group is infinitesimally represented by a Lie sub-algebra, not necessarily an ideal in \( G' \). The difference between these two may be tracked to the latter requiring “absorption” of coset generators (see, e.g., Sharpe 2.2.7. & 3.4.7.). Explicitly, with \( G' = \mathcal{H'} \oplus \mathcal{J}' \):

1. \([\mathcal{H'}, \mathcal{H'}] \subset \mathcal{H'}\), \( \mathcal{H'} \) is a sub-algebra of \( G' \).
2. \([\mathcal{H'}, \mathcal{J'}] \subset \mathcal{J'}\), a consequence of 1.
3. \([\mathcal{J'}, \mathcal{J'}] \subset \mathcal{H'}\), “absorption” of the quotient generators.

Reductive geometry brings about 1. and 2. while condition 3. is a special requirement leading to complete integrability (the in the sense of Frobenius theorem which we keyed as strongly involutive condition) and zero curvature[19]. Ideal rings in general are defined by closure and absorption; e.g., even integers under multiplication.
space of spacetime symmetries may be built. On the other hand, at this stage, we simply rely upon the notions of groupoids and stacks as a suggestion for how to address non-free group actions in the “amalgamate” of a stratified manifold (or quotient stack\cite{44} to be technical) leaving a more explicit construction of the stratified double-sided vector space for future work.

The paper is structured as follows. In §2 the non-linear realization of spacetime symmetries is reviewed within the framework of induced representations. This construction is then cast in the light of the diffeomorphism symmetry of the base space by inspecting adjacent diffeomorphism quotients according to their filtration by degree as operators on the ideal of vanishing Dirac “functions”. This sets the stage for the interpretation of the very large left symmetry of spacetime as a derivation. Along the way, Cartan’s construction of the frame bundle as a flat holomorph is reviewed. In particular, this involves defining tautological and fundamental 1-forms while peripherally touching upon the categorification of the full construct. In §3.1 takes a more formal look at the algebraic construction of a holomorph and at the standard way of defining the tangent bundle as a fibered product with the principal frame bundle. Causal structure is then instilled upon the base manifold through a reductive bundle morphism. This section ends with the well known algorithm for the reconstruction of the bundle space from transition maps on the base via equivariant maps. §3.2 constitutes the core of the paper. It begins by stating the fact that the existence of a principal bundle is a necessary condition for the definition of parallel transport when the topology of the fiber is non-compact. A brief primer on crossed modules and their connection to not necessarily abelian group extensions and semidirect products is then given. As it turns out, the non-linear transformation law for a section into a curved spacetime bundle naturally suggests itself for interpretation as a derivation on a double-sided vector space. We elaborate on this idea while dwelling on whether the germ of the “point” on the spacetime should be the vector space or the fiber. While the construction of a cross-module (a.k.a. a 2-group) is convoluted because these two groups are not necessarily naturally embedded into each other; we conclude that the embedding for the curved spacetime are in fact physically natural and that the generator of the spacetime “bundle” is the symmetric, associative, unital enveloping algebra of the (non-compact) double cover of the Lorentz group; so much for a choice of algebra generators. As a byproduct of the introduction of a doubled-sided vector space, the notion of a left $\alpha$-derivation now takes center stage. The deep meaning of this notion is that commutators in the Weyl algebra pick up a ring endomorphism upon commutating elements of the symmetric algebra from left to right. Furthermore, because higher Weyl algebras may be defined inductively, this construction may in principle encompass the full differential structure of the solder form to arbitrary degree. We close with a powerful conjecture: Summoning the left $\alpha$-derivation in the context of spacetime symmetries may constitute the key to a quantization of the theory; so much for the poetry, let us proceed.

2 Induced Representations of Spacetime Symmetries

We want to cast light on the interpretation of curved spacetime as a local, inhomogeneous coset space in the so-called non-linear realization of the local symmetry of gravity. This vague statement will find its formal motivation in the main body of the paper. In its full glory, the non-linear realization of the full diffeomorphism group of spacetime would involve inducing a “representation” of Diff M ($\equiv D_M$) from a maximally compact subgroup of SO(1,3) ($\equiv L$) under the implicit assumption that such representation could be defined via adjunction: Invoking Frobenius reciprocity so that the Induced and Restriction maps constitute a pair of adjoint functors ($\text{Ind}, \text{Res}$). Alas, the full diffeomorphism group of the fiber to spacetime is too complex to fit into a simple Lie theoretic framework (see, e.g., \cite{49} pg. 234). In fact, one could argue that $D_M$ must be envisioned in an affine, coordinate-free manner\cite{11}; with the breaking of affine symmetry bearing coordinates and frames\cite{12} in the linear representation space of the tangent bundle plus the particle content of the theory on the base “manifold”.

A more pedestrian approach leaves out affine and $C^\infty$ issues by linearly representing $D_M^\infty$ as a first order Jacobian via a finite dimensional quotient with group structure:

$$D_M^\infty \cong D_M^1 / D_M^2 \equiv \text{Aut}^1 M,$$  \hspace{1cm} (1)
where $\mathcal{D}_M^n$ is defined via algebraic geometry as the subgroup of the “differential little group” of spacetime, $\mathcal{D}_M^1$, that left acts as Id on the quotient of ideals of smooth “Dirac functions” vanishing up to order (n-1) in partial derivatives at x:

$$\mathcal{D}_M^1|_x \supset \mathcal{D}_M^n \cdot \left( \frac{I^n(M)}{I^n_x(M)} \right)_x = \text{Id}_x.$$

Note that by quotienting an ideal this states that higher order terms remain invariant by the action of $\mathcal{D}_M^n$ at $x$. Furthermore, in the linear representation $M$ equals the quotient of $\mathcal{D}_M$ (no super-script ⇒ linear representation) by its “differential little group”, $\mathcal{D}_M^1$:

$$M = \mathcal{D}_M/\mathcal{D}_M^1 \quad \text{or} \quad \mathcal{D}_M = \mathcal{D}_M^1 \otimes M.$$

(2)

In fact, the ring of differential operators $\mathcal{D}_M^i$’s forms a filtration by degree:

$$\mathcal{D}_M \supset \mathcal{D}_M^1 \supset \mathcal{D}_M^2 \supset \mathcal{D}_M^3 \ldots$$

with corresponding “higher order automorphism groups” as quotients:

$$\text{Aut}_x^i M = \mathcal{D}_M^i/\mathcal{D}_M^{i+1},$$

(3)

while $M \equiv \text{Aut}_x^0 M$. More formally, elements of the quotient $\mathcal{D}_M^n/\mathcal{D}_M^{n-1}$ are homogeneous differential operators of degree $n$ that may be identified with elements of the quotient $(\mathcal{U}\mathcal{L})^n/(\mathcal{U}\mathcal{L})^{n-1}$ of the (symmetric, associative, unital) graded universal enveloping algebra in left invariant vector fields of a suitable algebra. This Id assigns to the left invariant basis elements of a Lie algebra the role of first order differential operators on the “group manifold” via left invariance on the operators as well. Thus, elements of the quotient $(\mathcal{U}\mathcal{L})^n/(\mathcal{U}\mathcal{L})^{n-1}$ are homogeneous polynomials of degree $n$ in the left invariant vector fields. This is the content of the Poincaré-Birkhoff-Witt Theorem.

Although the full diffeomorphism symmetry of the spacetime encompasses the $C^\infty$ structure to all orders, the $\mathcal{D}_M^n$ quotients (as homogeneous polynomials of degree $n$) may be trivialized, order by order, to arbitrarily high order by choosing a suitable matrix representation as a linear automorphism group. Accordingly, $\text{Aut}_x^1 M$ may be thought of as a matrix group representation resulting from an infinitesimal coordinate change: $x^\mu \rightarrow x'^\mu - \xi^\mu$ (evidently this requires the presence of a fundamental length scale as in the absence of such there is no notion of “infinitesimal”; Ref. Professor Witten’s quote in the introduction). Tautologically, $\text{Aut}_x^1 M$ is identified with the automorphism group of Euclidean space, $\text{GL}(4, \mathbb{R})$, although clearly a more proper choice must enlarge the geometric data to enforce local causality, for instance (but not uniquely) via a locally flat metric: $\mathbb{R}^{1,3} = (\mathbb{R}^{1+3}, \langle , \rangle)$. However, a big problem does arise if one takes the non-compact $\mathcal{L} \simeq \mathcal{D}_M$ as the stabilizer subgroup for then left and right measures differ on the group manifold and a proper choice of automorphism group is not self-evident. On the other hand, since we advocate a vacuous picture of empty spacetime, it is tempting to adopt the maximally compact subgroup of $\mathcal{L}$ for massive states, $\text{SU}(2)$, to induce appropriate representations of the spacetime. Yet again, we are also strong advocates of local enforcement of causal structure. We propose an potential exit to such a conundrum in 132.

Note that inducing a representation of a “standard” vector space leads to a flat holomorph: $\mathcal{L} \otimes \text{Aut}(\mathcal{L})$; the semidirect product of $\mathcal{L}$ with its (inner) automorphism group $\text{Aut}(\mathcal{L})$. Indeed, in empty spacetimes the non-linear realization of its Lorentz subgroup from its isotropic embedding of the latter in the

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3 Let’s relate this construction to Klein’s original proposal. Associated to a general not necessarily effective (infinitesimal) Klein pair $(\mathcal{G}', \mathcal{H'})$ with non-empty kernel $\mathcal{N}'$ (⟩ largest ideal of $\mathcal{G}'$ contained in $\mathcal{H}'$); the “difference of Ad$(\mathcal{H})$ with the identity” relates the normalizer of $\mathcal{H}$ in $\mathcal{G}$ to such a kernel: $\mathcal{N} = \{ h \in \mathcal{H} \mid \text{Ad}(h)v - v \in \mathcal{N}' \}$ for all $v \in \mathcal{G}'$ (Sharpe 4.3.2). Moreover, all the normal, closed subgroups of $\mathcal{H}$ in $\mathcal{G}$ may be found recursively through an algorithm that kills the original kernel (Sharpe 4.4.1). $\mathcal{I}^n_x(M)$ in the above Eq. has the interpretation of the ideal of functions on a manifold with spin structure (gammas may be switched from the Dirac operator to the ideal) that vanish to all orders in an infinite polynomial ring in differential operators (as indeterminates) with variable coefficients. This structure can be simplified by assuming that the homogeneous differential operators of degree $n$ are homogeneous polynomials of the same degree in left invariant vector fields with constant coefficients.
former: $L\text{SO}(1,3) \equiv \text{SO}(1,3) \otimes \mathbb{R}^{1,3}$. This is an extension of the Lorentz group by a vector space representation of $\mathcal{L}$: $\mathbb{R}^{1,3} \simeq L\text{SO}(1,3)/\text{SO}(1,3)$, the normal, abelian group of (constrained) translations with Minkowski signature. Trivially, transition maps take values on the abelian group of translations and the corresponding equivalence class of objects invariant under such maps defines the points of the manifold: Minkowski space as an abelian, homogenous coset space. This all makes sense since a non-linear realization of the full Diffeomorphism group should not be expected to be strongly involutive (in the sense of Frobenius): abelian commutators for the generators of the coset generically lead to empty, globally symmetric spacetime solutions (see remarks in Footnote 2). This in turn implies the non-existence of holonomic bases for the “world” manifold: $[\partial_\mu, \partial_\nu] = \emptyset$.

Given a closed, invariant subgroup: $\mathcal{N} \subset \mathcal{G}$, much of the notion of homogeneous model geometry is succinctly embodied by the following short exact sequence:

$$1 \to \mathcal{N} \to \mathcal{G} \to \mathcal{C} \to 1,$$

which not incidentally parallels the construction of a (non-necessarily principal) fiber bundle $(E, F, \pi, M)_K$ with structure group $\mathcal{H}$ over base $M$: $\mathcal{H} \hookrightarrow E \twoheadrightarrow M$. In particular, this implies that the first map $\hookrightarrow$ is injective (algebraically one to one with possibly non-empty cokernel (although then $\mathcal{N}$ would not be normal nor “invariant”, see [3.2]) while the second map $\twoheadrightarrow$ is surjective (algebraically onto with empty cokernel and full use of the codomain) and unique although the definition is more broad in a categorical sense.

One of Élie Cartan’s most profound insights into algebraic constructions of differential geometry was to advance Klein’s notion of homogeneous model space to develop the more abstract machinery of fiber bundles. Yet, the colossal utility afforded by a full understanding of these notions was not readily forthcoming. In fact, nearly thirty years transpired before Cartan’s pupil, Charles Ehresmann, partially unravelled the physical practicality of this elegant construct by loosely detaching the identification of the coset space with the physical spacetime of General Relativity. In this paper we are chiefly concerned with the re-identification of the coset space with the base manifold as a physical spacetime: affine, inhomogeneous, non-compact, causal, and spin.

Today, Yang-Mills gauge theories are robustly formulated within the versatile language of fiber bundles: Demanding Lie-group structure of the “internal” fiber space but with tacit trivialization of the geometry of the base manifold. Accordingly, the coset space is identified pointwise homeomorphically with the linear, flat tangent space to spacetime. $TM_x \simeq \mathbb{R}^{1,3}$; $x \in M$, whereas postulating the existence of a frame bundle $\mathcal{F}M$ supported on a fiducial vector space $V$ as an Aut($V$)-principal bundle allows for an impromptu inclusion of the “external” symmetries of General Relativity in a linear representation space. The tangent bundle to spacetime then is associated to $\mathcal{F}M$, as a pull back bundle (a fibration!) and this whole construction descends on spacetime proper through bundle reduction and soldering with the base manifold. Algebraically, the “egg” here is the germ of a point in each stratum of the spacetime while the chicken is the frame bundle as an associated, principal Aut($V$)-bundle thereby identifying a choice of finite-dimensional vector space with the horizontal distribution or co-normal sheaf. Differentially speaking, frame bundles are “mathematically natural” and one element of the Lie group of k-jets of origin preserving diffeomorphisms $[9, 22]$. As a corollary to natural bundles have finite order $[7].$
the structure group of a natural bundle can always be reduced to $O(n)$ (in the absence of causal structure); while however that of smooth fiber bundle (with the full $\text{Diff} F$ in lieu of a matrix representation) may not.

The advent of the Ehresmann connection as a choice of horizontal submanifold to the entire tangent space of a principal $\mathcal{H}$-bundle, $\mathcal{H} \hookrightarrow E \rightarrow M$, furnishes a flexible technology generically suitable to address the internal symmetries of particle physics where loosely speaking “spontaneous symmetry breaking” corresponds to doing physics in a coset space; i.e., modulo the stability group of a point in a coordinatized manifold. Note that such a principal $\mathcal{H}$-bundle may be identified with a larger symmetry, $\mathcal{G}$, with base space the coset $\mathcal{G}/\mathcal{H}$ as a trivial, abelian Lie group and fiber space $\mathcal{H}$; yet, without a globally defined $\mathcal{G}$-action on the entire bundle space since an internal symmetry cannot generate the spacetime. Instead, a standard fiber bundle relies on the construction of the bundle space as a central group extension of the fiber by the “Lie algebra” of $V$ as an abelian normal subgroup by demanding that all Lie brackets in the vector space generators vanish. (see [3.1]). This is the starting point in Ehresmann’s generalization to Cartan’s approach.

Obstructions to a physically viable framework appropriate for inclusion of external symmetries lie in a geometric interpretation of the notion of spontaneous symmetry breaking [26, 23]. For internal symmetries, spontaneous symmetry breaking corresponds to a map from a principal $\mathcal{G}$-bundle over $M$ into a finite dimensional, unstratified vector space, $V$, when the image of this map belongs to a single, “stable” sub-orbit of $\mathcal{G}$ in $V$ as the putative support of the $\mathcal{G}$-representation. In effect, such a map is a restriction of the orbits of $\mathcal{G}$ to a sub-space of the full vector space $W \subset V$ and thus represents a stratification of $V$ in the formal sense of Appendix [C] The stability group is identified with $\mathcal{H} \subset \mathcal{G}$ while the sub-orbit is generated by the algebraic generators of the coset $\mathcal{G}/\mathcal{H}'$. Such a map intertwines the right action of $\mathcal{H}$ on the vertical tangent space with a suitable representation of $\mathcal{H}$ on a linear algebraic structure locally isomorphic to $\mathcal{H}'$ as a canonical choice of “vertical space” or “normal sheaf” [E].

If the algebra of the larger group is identified with the full vector space, the intertwining operator encodes a trivial restriction map to the normal space while the the geometric data that defines the horizontal distribution as a co-normal space is non-trivially attributed to the kernel of such a map. Furthermore, this is done pointwise throughout the spacetime. It follows that if the spacetime amalgamates from distinct strata the intertwining operator is indeed a very interesting object! The attentive reading will recognize this simply as the definition of a connection as a “choice” of horizontal space. Geometrically, spontaneous symmetry breaking is a restriction map on the full vector space while the Higgs field is given by the pullback of such an intertwiner, a zero-form $\eta$, by a local section $\sigma$ of the principal $\mathcal{G}$-bundle [26]: $N \xrightarrow{\sigma} \mathcal{G} \xrightarrow{\eta} W$ where $N \subset M$ and $W \subset V$. In homological terms, this basically means that the sequence splits on the right. The counterpart to this construct for spacetime symmetries is given at the end of §3.1.

The space of intertwining operators, $\mathcal{J}(V,W)$, between two vector spaces $V$ and $W$ is the fundamental object of interest in the theory of group representations [50] effectively defining equivalence of vector spaces and “naturality” of linear maps between them as equivariance in the sense of Eqs [5] and [13] below. Such mathematical notion of naturality constitutes a key step in the categorification of Lie theoretic notions; particularly in the abstraction of algebraic structures aiming to free such structures from representation issues; e.g., thinking of representable functors as “geometric spaces” (although useful functors such as the tangent space functor are usually selected to be representable by construction in algebraic geometry through the crafty use of Yoneda’s Lemma: basically objects in a category may be recovered from equivaraint the maps into it (up to unique isomorphism, of course; see Appendix [A]).

For finite dimensional Lie groups, the general framework was worked out long ago by Élie through his proposition of a flat $\mathcal{G}'$-valued Cartan connection [28] with empty kernel: $\alpha$. Postulating the existence of a frame bundle supported on a standard vector space, i.e., a principal bundle with structure group $\text{Aut}(V) = GL(V) \subset \mathcal{G}$, prompts the Cartan connection to split: $\alpha = \beta + \gamma$, into an $\text{End}(V)$-valued Ehresmann connection $\beta$: $T\mathcal{G} \rightarrow \text{gl}(V)$ and a $V$-valued tautological 1-form $\gamma$: $T\mathcal{G} \rightarrow V$, with the latter mediating an “infra-natural”

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6 In the formal sense of category theory, identifying the quotient with the co-normal submanifold via the co-normal sheaf is more (mathematically) “natural” than the canonical choice of normal sheaf as the “internal space” generated by the stability group. This is a non-issue for torsion-less modules (i.e., “reflexive” modules Vakil (2012) 23.2.15) but it does restrict the support of the module as the set of prime ideals.
identification between the support of the representation and the tangent space to the base manifold as a local homogeneous quotient:

\[ TM \simeq V \times \mathcal{G}/\text{GL}(V) \simeq V \times V; \]

but not yet with the physical spacetime of General Relativity as base manifold. That \textbf{Id} must still address two contentious issues: descent (e.g., gluing of affine spaces, holonomy, torsors, zero sections, group reduction of the underlying bundle space, etc) and naturalness of the \textbf{Id} between base and double dual vector spaces. The latter issue may have confronted Élie Cartan and Alexander Grothendieck views on advancing the beauty of mathematics perhaps at the cost of undermining the pursuit of truth in physics. Classically, Cartan’s view may seem to be more physically rooted; yet, the flexibility and prowess afforded by Grothendieck’s program may be required for a better understanding of a complete theory of Quantum Gravity.

Notwithstanding the unquestionable success of YMGT in presenting a unified picture of nature away from curved spacetime, at a fundamental level the standard construction represents a rather incomplete picture. On the right hand, the transformation properties of a section into the diffeomorphism bundle of spacetime symmetries, Eq [21] below, require for the global and local actions to belong to a “reductive pair” of symmetry groups. Lorentz transformations on the right of the section are rigid, spacetime independent, causal structure enforcing transformations whose algebra generates the fiber in the standard sense of Lie groups. On the left hand, the identity component of the automorphism group of Euclidean space, \( \text{GL}^+(4,\mathbb{R}) \), is simply connected and does not admit a double cover\(^{11}\) so the alleged local symmetry preludes straightforward representations of finite-dimensional spinor fields\(^{25, 24}\). The former objection clearly represents an obstruction to the standard construction of an associated frame bundle as a “principal” bundle already at the classical level whereas at the quantum level the latter objection may represent an even earlier obstruction to the assembling of the tangent bundle \textit{pointwise} from all of the tangent spaces at each point of the base in the presence of localized spinor and matter fields. These seemingly disconnected issues are in fact intertwined and disentangling their relationship lays at the core of our enterprise. We will advocate a picture of quantum gravity where the notion of “empty space” is vacuous: There is no gravity (nor space or time) without stable massive and spin states; indeed, how else could one define inertial observers? Thus, to understand quantum gravity—least as a “gauge” theory—one must consider stable matter and spinor fields embedded in spacetime proper through the amalgamate of a stratified manifold.

Moreover, a better understanding of the tangent space to physical spacetime as a representable functor is essential to incorporate the two symmetries of gravity in a unified framework. As the attentive reader may have anticipated, we will closely scrutinize the putative “choice” of vector space as the support of the representation of spacetime symmetries in \( \S \overline{3.2} \). In this vein, an algebro-geometric construction of spacetime requires that the tangent space be a fibered product of schemes and that its representable functor be “made out” of Zariski sheaves (see, e.g., Vakil 2012\(^{59}\): 10.1.6).

3 The Geometry of the Solder Form: algebraic vs differential

Given a generic fiber bundle \((E, F, \pi, M)\) with the standard identifications, the kernel of the derivative map,

\[ d\pi_p : TE_p \rightarrow TM_{\pi(p)}; \]

at a point \( p = (x, h) \) in the fiber bundle is a vertical subspace tangent to the fiber at \( p \). The total tangent space of the bundle may thus be written as a Whitney sum:

\[ TE_p = TE^V_p \oplus TE^H_p \cong \ker[d\pi_p] \oplus TM_{\pi(p)}, \]

where the identification of \( TE^V_p \) with \( \ker[d\pi_p] \) is \textit{canonical} while the identification \( TE^H_p \cong TM_{\pi(p)} \) is \textit{degenerate}.

A working definition of an Ehresmann connection corresponds to a choice of horizontal subspace \( TE^H_p \) as follows. First, note that for a \textit{principal} \( \mathcal{H} \)-bundle with Lie algebra \( \mathcal{H}' \), the sub-space \( TE^H_p \) is invariant under the
right action of an element of the $\mathcal{H}$ group (with $R_{h'}(p) = (ph') \equiv (x, hh')$, $p \in E$):
$$R_{h*} : TE^H|_p \to TE^H|_{ph} \cong TE^H|_p$$
so that right translation corresponds to “orbital motion” along the fiber only. When the symmetry of the fiber is non-compact as a group, Ehresmann connections are only defined for principal bundles: the existence of a right action on the full bundle space makes it possible to parallel transport vector fields and to lift horizontal curves to the bundle space in a self-consistent manner [49]; see [3.2]. In the case of an associated non-principal bundle, e.g., in a stratified manifold, the choice of horizontal space must be modified accordingly.

Next, construct the annihilator of horizontal vector fields $\Theta : TE \to TE^V$ as an $\mathcal{H}'$-valued one-form while identifying $TE^V$ with $\mathcal{H}'$. It follows that the subspace spanned by vectors $u$ such that $\Theta(u) = 0$ defines the horizontal subspace:
$$TE^H = \text{Ann } \Theta \equiv \{ u \in TE | \Theta(u) = 0 \}. \quad (4)$$

Invariance of this space under the right action of $\mathcal{H}$ generically imposes the requirement of “equivariance” on $\Theta$:
$$\Theta \circ R_{h*} = \text{Ad}_{h'}(h^{-1}) \circ \Theta. \quad (5)$$
Written in composition form makes manifest the intertwining property of vector spaces that makes $\Theta$ an “equivariant one-form”; i.e., $\Theta \circ R_{h*} \equiv R_{h*}^t \Theta$. More concretely, one says that $\Theta$ intertwines the right action of $\mathcal{H}$ on the vertical tangent bundle with the Adjoint representation of $\mathcal{H}$ on its own Lie algebra.

Equivariance in this sense may be compared with the left invariance of a Maurer-Cartan 1-form valued on the Lie algebra: $L^* h \omega = \omega$. In fact, a local “pointwise” pull back of $\Theta$ (from $E$) to the fiber, $\pi^{-1}x$, yields the left-invariant (fundamental) one-form on $\mathcal{H}$ [19],
$$\omega = \iota^*_x \Theta,$$
where $\iota_x : h \mapsto \pi^{-1}x$, is a right equivariant, left invariant map into the fiber space. I.e., $\iota_x$ is the fiber part of a local trivialization, a “diffeomorphism”, over an open neighborhood of $x \in M : \iota : U \times \mathcal{H} \to \pi^{-1}(U)$, consistent with right translation spanning the fiber space: $\iota_x(hg) = \iota_x(h)g$ and invariant by the left composite diffeomorphism, $(\iota_x \circ L_P)^* \Theta = \iota^*_x \Theta$, induced by a change of trivialization on overlaps. Such a left invariance of the vertical space “naturally” suggests an algebraic identification of $TE^V$ with $\mathcal{H}'$ (a.k.a. the intertwining property).

3.1 Anatomy of a Tautology:
from tautological to solder form via reduced bundle homomorphism

$\Theta$—a.k.a. the fundamental 1-form on $E$—is in one to one correspondence with the Ehresmann connection on a principal $\mathcal{H}$-bundle. Notably, while $\Theta$ is valued on a Lie algebra which is isomorphic to the fiber space and is tangent to the fibers in the sense that it vanishes on horizontal vector fields; in the particular case of a frame bundle the complementary notion of a tautological 1-form, $\Lambda$, is a horizontal one-form that vanishes on vectors along the fiber direction,
$$\text{Ann } \Lambda = \{ u \in TE | \Lambda(u) = 0 \} \equiv TE^V \quad (6)$$
(compare with Eq [4]), while taking values on the quotient of a larger algebra by the algebra of the fiber: the associative, unital algebra of endomorphisms of the support of the representation, gl($V$). Purportedly, such a “large” group is endowed a priori with algebraic generators for the quotient plus additional generators for the fiber as an Ad-invariant sub-module of the principal bundle. Let us elaborate on these issues.

---

\textsuperscript{7} Such quotients tend to be better-behaved than sub-objects (note that this is a categorical distinction in the SSE) for coherent sheaves which generalize the notion of vector bundle (14.1.9, 17.7, 14.5 Module-like constructions & 3.5). In algebraic geometry, vector bundles do not conform an abelian category so one must restrict the category to (quasi)coherent sheaves: the category such that maps between locally-free sheaves have well behaved cokernels as $\mathcal{O}_X$-modules.
At a fundamental level, Cartan’s construction involves the existence of a principal \(\text{Aut}(V)\)-Frame bundle. By construction, such a principal bundle, \((E = G, F = \text{Aut}(V), \pi, M)_\text{\textsuperscript{\text{r}}},\) is the holomorph ensuing from the central, algebraic group extension of the vector space by inner automorphisms (since the extension is trivial, this is insensitive to the order of the arguments in the Ext-functor; i.e, ref. Eq[17], below). This, in turn, leads to the vector space playing the role of ideal for the bundle itself. More formally, this holomorph results from taking V to be a necessarily abelian right \(L_\text{\textsuperscript{r}}\)-module\(^8\) which is at the same time a Lie algebra \(L_\text{\textsuperscript{r}}\) such that the module mapping \(x \rightarrow x\ell\) is the inner derivation in V defined by inner adjoint action (see, e.g., [29] pg. 17-18) on implicit left invariant vector fields (following Jacobson[29], the reader is encouraged to read the equations from right to left; such practice in ambidexterity will come in handy when building hermitian operators in [42]):

Define:

\[
(x\overleftarrow{D})y + x(y\overleftarrow{D}) = (xy)\overleftarrow{D}
\]

\[
±xya + yxa - axy = (xy)\overleftarrow{D}_a
\]

\[
.(x\overleftarrow{D}_a)y + x(y\overleftarrow{D}_a) =
\]

Next, map: \(x \mapsto x\ell\), under: \(0 = [xy]\), while assuming: \((x\ell)y + x(y\ell) = xy\ell\), so that

\[
x\ell y - y\ell x = [xy]\ell
\]

\[
x(y\ell) + (x\ell)y - y(x\ell) - (y\ell)x =
\]

\[
.([x\ell], y) + [x, (y\ell)] =
\]

For later reference, we now include a slight generalization of the “plain vanilla” derivation above. Simply assuming the following “right-\(a\)-derivation” rule: \((x\ell)(y_\alpha) + x(y\ell) = xy\ell\), which has the informal meaning that commuting the \(\ell\) past the \(y\) from right to left makes it “pick up an \(a\) on the right”, we get:

\[
x\ell y - y\ell x = [xy]\ell
\]

\[
x(y\ell) + (x\ell)(y_\alpha) - y(x\ell) - (y\ell)(x_\alpha) =
\]

\[
.([x\ell], y_\alpha) + [(x\ell), y_\alpha] =
\]

Note that commuting \(x\ell\)’s and \(y\ell\)’s do not pick up \(\alpha\)’s because \(\ell\) is “doing” the derivation. This leads to the following “right-\(\alpha\) commutation rule”

\[
xy - yx_\alpha = [x, y_\alpha]
\]

Back to the algebraic construction of the frame bundle (and to reading Eq’s from left to right!), this is given by the split extension of the derivation algebra of V by V obeying the following short exact sequence:

\[
0 \rightarrow V \rightarrow V \rightarrow \mathcal{D}(V) \rightarrow \mathcal{D}(V) \rightarrow 0,
\]

In fact, \(V\) represents a two sided ideal of the Lie algebra embodied by the holomorph (essentially two-sided absorption by generators of the coset; ref. footnote[2]). Recall that such an ideal leads to a normal subgroup via the \(\exp\ell\) map near 0 when such a map is well defined; i.e., modulo infinite dimensional group structure. Furthermore, conditioning the vector space to be abelian as a sub-group restricts the extension to inner automorphisms of \(V, \text{Inn}(V)\) since the derivation algebra is, by definition, the Lie algebra of linear transformations; i.e., with trivial kernel as the center \(Z\) of the full space and empty co-kernel (\(\equiv \text{Out}(G)\)) by the \(\text{Ad}(s)\) map:

\[
1 \rightarrow Z(G) \hookrightarrow V \xrightarrow{\text{Ad}(s)} \text{Aut}(V) \twoheadrightarrow \text{Out}(V) \rightarrow 1,
\]

The ambiguity between trivial Lie algebra, abelian group and vector space is prevalent in the literature. We aim to resolve this issue algebraically, below. Meanwhile, note that the algebraic statements above are robust while those referring to the group as a vector space are not.

\(^8\) Recall that a right \(\ell\)-module for an associative algebra \(U\) is a vector space \(M\) over a field \(\psi\) together with a binary operation \(\Psi : M \times U \rightarrow M\) which is right and left distributive with respect to addition, commutative and associative with respect to “scalar” product \(\beta \in \psi\) and where a multiplicative unit exists on the right.
Group-wise, relaxing the abelian condition and enabling the “vector space” to be normal as a sub-group leads to non-empty outer automorphism group (see below). Juxtaposing this with the extension of a non-trivial stabilizer sub-group, \( \mathcal{H} \), via adjunction, a frame bundle is a trivialized construction of a flat holomorph since the fundamental building block is a “standard” vector space which does not possess a rich algebraic structure: it only need be additive abelian as a right \( \mathcal{L}_n \)-module.

The above construction corresponds to a central group extension of \( \mathcal{H} = \text{Aut}(V) \) by \( \mathcal{N} = V: \mathcal{G} = \mathcal{N} \otimes \mathcal{H}; \) or equivalently, to a surjective group homomorphism

\[
t : \mathcal{N} \to \mathcal{H} \mid \text{Ker}(t) = \text{Id}_\mathcal{N},
\]

with trivial kernel at the center of \( \mathcal{N} \). A generalization of this construction with non-trivial kernel leads to a non-trivial co-kernel as the outer automorphism group in Eq [12]. This is related to cross-modules and Lie 2-groups [12]. Yet, any notion of “non-central extension” by a subgroup is non-existent, at least not in the language of groups. In either case, the interpretation of \( \mathcal{N} \) as a “vector space” runs into trouble because of its non-abelian nature but it is this case that may be of interest to us in the interpretation of spacetime as a coset space.

The tangent space to the homogeneous model geometry is identified with an associated (pullback or inverse image) vector bundle (note that under the equivalence relation: \([p,v] \sim [ph,h^{-1}v]\) where \( p \in E, \ h \in \mathcal{H} \) and \( v \in \mathcal{G}'/\mathcal{H}' \) the dimensions of the tangent bundle on the r.h.s. match twice the dimension of the (algebraic) group quotient on the l.h.s. as expected; i.e., \( \dim(E) \) is reduced to \( \dim(\mathcal{H}'/\mathcal{G}') \) under “\( \sim \)”).

\[
\mathcal{T}[\mathcal{G}/\mathcal{H}] \simeq E \times_\mathcal{H} [\mathcal{G}'/\mathcal{H}'],
\]

but this Id is not canonical; it is determined only up to the Adjoint action of \( \mathcal{H} \) on the algebraic coset, \( \text{Ad}(\mathcal{G}'/\mathcal{H}')(h) \), as a vector space, not as a Lie algebra which ensues from the right invariance of the horizontal distribution (see [19] pg. 163). Thus, this tangent bundle is not a principal bundle but an associated vector bundle under a suitable equivalence relation with “target object” \( \mathcal{H} = \text{Aut}(V) \) in the fibered product sense.

Furthermore, in Cartan’s picture, the tautological one form is an equivariant operator intertwining the right action of \( \mathcal{H} \) on the horizontal distribution of the frame bundle with the fundamental representation of the stabilizer on the algebraic quotient:

\[
\Lambda \circ R_h = L_{h^{-1}} \circ \Lambda. \quad (13)
\]

Evidently, such a decomposition assumes algebraic structure (not nesses. group) of the underlying quotient as a vector space, not necessarily as a Lie sub-algebra; see Footnote [2]. Moreover, consistent with the definition of \( \Theta \) as annihilator of horizontal vector fields, this decomposition implicitly requires right invariance of the vector fields that span the horizontal distribution: \( \pi R_h = \pi \). Thus, we may equivalently write \( \Lambda \circ R_h = \text{Ad}_{h^{-1}} \circ \Lambda \).

One may now elegantly re-assemble fundamental and tautological one forms into a single Cartan connection over the vector space (not a group):

\[
\Xi = \Theta + \Lambda,
\]

A more categorical, algebro-geometric construction of such a (co)tangent bundle as a fibered product implicitly involves an underlying “structure morphism” from the vector space \( V \), as an \( \mathcal{R} \)-module, to an \( \mathcal{R} \)-scheme \( S_\mathcal{G}: V \to S_\mathcal{G} \), where \( \mathcal{R} \) is a field or a ring or a scheme in increasing degrees of generality (see 7.3.4 of Vakil; also, 10.3.: fibers of morphisms and pulling back families). The fibered (pullback or inverse image) product then occurs over such a scheme locally:

\[
\mathcal{T}^*M \simeq V \times_S [\mathcal{G}/\text{GL}(V)],
\]

and this in turn implies yet another formal map from the group quotient to the structure sheaf in the Zariski topology:

\[
\mathcal{G}/\text{GL}(V) \to S.
\]

Pullback or inverse image diagram refers to the commutative diagram in 2.3.6. of Vakil illustrating the notion of universality up to “unique isomorphism”. Pullback sheaf is reserved for quasi-coherent sheaves while inverse image sheaf is left adjoint to pushforward for general sheaves (3.6.1.). For classical differentiable manifolds, given a continuous map of manifolds: \( f: X \to Y \), one can always pullback the sheaf of (differentiable) functions but the notion of “function” is more subtle in the category of schemes. Given a morphism of schemes as topological spaces \( f: X \to Y \), this carries a pullback map of structure sheaves as a morphisms of locally ringed spaces \( f^*: \mathcal{O}_Y \to \mathcal{O}_X \); see also Definition 7.3.1. We shall say a bit more about this structure morphism in a future publication.
which is manifestly equivariant (in the sense of Eq [5]) under the action of \( \mathcal{H} = \text{Aut}(V) \) \[19\]. Note that the interpretation of \( \Lambda \) here as an equivariant operator is much more subtle than before. Indeed, this makes implicit use of the fact that the Cartan connection is an absolute parallelism of the frame bundle as a central group extension.

Consider now a not necessarily closed subgroup of \( \mathcal{H} \supset \mathcal{L} \), and build the following fibered product:
\[
E \times_{\mathcal{H}} [\mathcal{H}/\mathcal{L}] \rightarrow E/\mathcal{L}.
\]
Standard lore \[51\] posits that the obstruction to a reduction of the structure group from \( \mathcal{H} \) to \( \mathcal{L} \) lies in the admission of sections into this associated bundle with putative fiber the equivalent class \([\mathcal{H}/\mathcal{L}]\) invariant under left translation by \( a^{-1} \in \mathcal{H} \) (corresponding to group action by the fundamental representation). Furthermore, the “canonical” \( \text{Id} \) with \( E/\mathcal{L} \) through the map: \((u, a\xi_0) \rightarrow u \in E/\mathcal{L}\), purposefully forgets the origin of the coset space \( \xi_0 \in \mathcal{H}/\mathcal{L} \) (this is an example of a “forgetful functor” mapping elements of the associated bundle to (affine) elements of \( E/\mathcal{L} \)). Thus, in this picture causal structure ensues locally if the associated bundle with reduced Lorentz group structure, \( \mathcal{L} \), admits a local section but a “gravitational field” is not yet present since the holomorph was flat to begin with. This is the first manifestation of the dual character of the solder form; it is an entity that lives on both, the tangent space; i.e., in the flat holomorph, and on the base, curved spacetime. Clearly the latter is endowed with a much richer geometrical structure as an infinite dimensional Folk space.

In fiber bundle language, the pullback of the tautological 1-form along an \( \mathcal{L} \)-bundle morphism \[20\],
\[
f^\psi: Q\mathcal{L} \rightarrow E_{\text{Aut}(V)},
\]
yields the solder form on a causal, geodesic and holonomic bundle:
\[
f^\psi \Lambda = \Upsilon,
\]
thereby reducing the structure group of the tangent/frame bundle from \( \text{Aut}(V) \) to \( \mathcal{O}(V) \), the orthogonal group of the putative vector space which supports the representation.

This so-called bundle reduction presumes that the fiber algebra of the frame bundle encompasses the algebra of the \( Q \)-fiber as an \( \text{Ad-} \mathcal{H} \)-invariant sub-module of \( gl(V) \), so that the frame bundle is formally endowed with a reductive \( G \)-structure. Such a reductive model geometry anchors the frame bundle to the tangent bundle of the base manifold: locally \( \text{Id} \)'ing the algebraic quotient pointwise, homeomorphically with the support of the representation as a physical spacetime, i.e., \( D'/L' \simeq R^{1,3} \). The corresponding curtailment from tautological to solder form is the enactment of spontaneous symmetry breaking in the context of spacetime symmetries.

On the other hand, it is well known that one can reconstruct the bundle from transition maps as follows. Given a partition of unity on the base manifold as a topological space and charts subordinate to the open cover indexed by such a partition,
\[
\psi_i: U_i \times H \rightarrow \pi^{-1}(U_i),
\]
each such diffeomorphism as inclusion into the fiber bundle must satisfy right equivalence on the fibers. Thus, \( \psi_i(u_i, h_i) = \psi_i(u_i, e)h_i \) so that a given chart corresponds to a local section \( \sigma \) over \( U_i \) : \( \psi(u_i, h) = \sigma(u)h \). On overlaps, two sections from different trivializations must agree on the fiber
\[
\psi_i \rightarrow \pi^{-1}(U_i) = \sigma_i(u_{ij}) \quad h_i = \sigma_j(u_{ji}) \quad h_j = \pi^{-1}(U_j) \quad \leftrightarrow \quad \psi_j
\]
so that \( \sigma_i \) and \( \sigma_j \) are related by a smooth map, \( k: U_{ij} \rightarrow \mathcal{H} \), acting on the right of sections: \( \sigma_j = \sigma_i k \) or
\[
k = \sigma_i^{-1} \sigma_j|_{u_{ij}}.
\]
Thus, composition of sections on overlaps is in one to one correspondence with the transition functions and one can reconstruct the bundle from transition maps as previously advertised. The crucial key in this result is the existence of “equivariant structure” on the underlying bundle (or gerbe \[43, 44, 46\]). This is a powerful statement in the context of category theory since, according to Yoneda’s Lemma, the object in the category may be recovered by knowing the maps into it (up to unique isomorphism, of course).
3.2 Anatomy of a Tautology II: a geometric re-tooling of the solder form

Recall that when the symmetry of the fiber in non-compact, the notion of parallel transport is only defined for principal bundles\cite{59}. On the other hand, we concluded in the last section that the bundle may be reconstructed from transition maps which in turn are surrogate to the holonomies of the base manifold. The latter must take into account manifold stratification and also include the monodromies that result from “integrating over the boundary” of non-contractible (i.e., disconnected pieces on the manifold), embedded stable massive and spin states. In this paper, we address the former issue but leave the latter for a future publication.

Assuming simply that the “large” group $D$ has $L$ as a closed, normal subgroup, the following principal fiber bundle may be readily constructed (with its corresponding short exact sequence):

$$\mathcal{D}, \mathcal{L}, \pi, Q \equiv \mathcal{D}/\mathcal{L}; \quad 1 \rightarrow \mathcal{L} \hookrightarrow \mathcal{D} \rightarrow Q \rightarrow 1,$$

as a (necessarily central) group extension of the quotient $Q$ by $L$ ($= \text{Ext}(Q, \mathcal{L})$):

$$\mathcal{D} = Q \otimes _{\mathcal{L} \rightarrow \text{Aut}(Q)} \mathcal{L} \quad (17)$$

where the standard group morphism implicit in the semi-direct product\cite{52}: $\mathcal{L} \rightarrow \text{Aut}(Q)$, is exhibited and furthermore particular emphasis is placed on the implicit map\cite{53}

$$\alpha : Q \rightarrow \text{Aut}(\mathcal{L}), \quad (18)$$

as a section (not necessarily a group homomorphism) enforcing part of the consistency conditions for a Lie 2-group $Q$\cite{54}. Notably, the short exact sequence splits only if kernel of the map

$$t : \mathcal{L} \rightarrow Q \quad (19)$$

is central \cite{58}.

We can inequivalently reverse this construction leading to a more standard picture of the semidirect product as an extension of $L$ by an abelian group $U$ (which we coined as the “vector space” in Eq \[11\]: an algebraic extension via a derivation algebra):

$$1 \rightarrow U \hookrightarrow \mathcal{L} \otimes_{U \rightarrow \text{Aut}(\mathcal{L})} U \rightarrow \mathcal{L} \rightarrow 1, \quad (20)$$

but, as noted, this is not equivalent to the normal, non-abelian group extension of the quotient “group”.

In fact, the fundamental difference resides in the kernel of $t$-map, Eq \[19\].

Now, the solder form may be pictorially represented by its “non-linear transformation law”\cite{57, 35, 12}:

$$\mathcal{D} \sigma \mathcal{L}^{-1} = \sigma' \quad (21)$$

where $\sigma(\xi) \equiv \exp(\xi I_f)$ is a section of the fiber bundle $D$ with fiber $L$ over the coset $D/L$ and $D$ acts on $\sigma$ locally from the left\cite{10} while $L$ acts on $\sigma$ globally from the right\cite{11}. The crucial issue here is that left action by

\[10\] We thank Juan Maldacena for this key observation on the melding of the local and global symmetry groups for gravity while highlighting the fact that because $D$ is induced from $L$, both of these enforce local causality: the former at the microscopic level and the latter at the macroscopic level.

\[11\] Note that local and global roles here are in fact exchanged from what is encountered in the standard literature on non-linear realizations of spacetime symmetries. Given a principal H-bundle $G(H,G/H)$ with (large) group structure (i.e., a holomorph), “coordinates” in the full bundle space are related to the section through right multiplication by an [arbitrary group] element of the fiber\cite{57}. A full bundle diffeomorphism is implemented through left translation by an element of the group, $g$, embodied by the bundle as a holomorph. Such a translation moves both the original base point and the coordinates on the fiber. Standard lore poses that to “compensate”\cite{14, 15, 16} for the motion along the fiber, right multiplication on the translated section by a fiber element $h$ is required. Thus, base point translation gets “dumped” into “spacetime dependencies” for the translated fiber and coset elements as follows: $\sigma(\xi) \rightarrow g_0\sigma(\xi')$ and $\sigma(\xi') \rightarrow \sigma'(\xi')h(\xi', g)$ where the prime refers to the translated coset representative. Thus, identifying coset representatives $\xi$ with base point coordinates, $H$ can be seen as the local group for its dependence on $\xi$. On the other hand, looking at elements of $D'$ as the differential generators of the spacetime naturally leads to its interpretation as the (infinitesimal) local group whereas $L'$ can be seen as generating rigid global transformation that enforce (local) causality. This is the interpretation that we will give to the meaning of Eq\[21\].
$\mathcal{D}$ does not preserve the fibers pointwise on the base; instead it induces a \textit{diffeomorphism of the entire bundle} and in particular a change in the algebraic coset representatives, $\xi$, that are endowed with the transformation properties of gravitational fields $^{12}$\textsuperscript{13}. This is in essence what defines our \textit{geometric re-tooling} of the solder form: a section into the very large “diffeomorphism space” of spacetime embodied by a \textit{non-central extension} of the quotient space as a \{quasi-\}group, $\mathcal{Q}$, by the Lorentz group $\mathcal{L}$; c.f., Eq [17]. On the other hand, we are strong advocates of interpreting the fiber as the “egg” that generates the entire bundle space in some suitable fashion. Thus, once we setup the construct below in the fashion devised by Elie Cartan for a “principal” frame bundle, it will be reversed and the bundle space will be induced from the algebra of the fiber for comparison and to try to make ends meet. Recall that whereas we have a fair understanding of the structure of $\mathcal{L}$ as a non-compact Lie group, the structure of the quotient $\mathcal{Q}$ and of $\mathcal{D}$ as “groups” is uncertain at best and open to debate. This seems to preclude a well behaved embedding into a \textit{standard} fiber bundle construction; unless, that is, one calls upon groupoids and non-commutative algebra.

Direct inspection of Eq [21] suggests that the section could be built algebraically \textit{via} non-commutative ring theory\textsuperscript{30} as a “double-sided vector space”\textsuperscript{31} or possibly as a \textit{Frobenius bimodule}\textsuperscript{32}—for which the right and left “scalar” actions differ in the prescribed manner—left action augmented from that on the right by a \textit{left-sided ring endomorphism} $\alpha$—while being subject to the interpretation of the coset $\mathcal{D}/\mathcal{L} \equiv \mathcal{Q}$ as a spacetime. Thus, following the algorithm for the algebraic building of the frame bundle as a holomorph $^{11}$, we take $\mathcal{Q}$ as a (necess. abelian) right $\mathcal{Q}_{\alpha}$-module which is at the same time a \textit{skew (non-Lie)} K-algebra $\mathcal{Q}_{\alpha}',$ such that the module mapping $x \mapsto \varphi x$ is a \textit{left} $\alpha$-derivation (\(\varphi\) now moving from left to right) in an \textit{associative} algebra $\mathcal{A}$:

$$\alpha \triangledown (a b) = \alpha \triangledown (a) b + \alpha (a) \alpha \triangledown (b),$$

where $a, b \in \mathcal{A}$, $\alpha = \text{End} \mathcal{A}$, so that

$$\alpha \triangledown a = \alpha \triangledown (a) + \alpha (a) \alpha \triangledown$$

(22)
as a \textit{non-central} ring generalization of a Weyl algebra\textsuperscript{4}. The definition of the algebra is given by David Patrick\textsuperscript{31}: A skewed (or twisted) K-algebra is a ring A, together with an injective ring homomorphism: $K \rightarrow A,$ where elements of K may be interpreted as the scalars of A; yet, K is not constrained to be central in A so the scalars have different left and right actions on A. Such an algebra results from the free polynomial ring in a single indeterminate cut out by the ideal generated by the left $\alpha$-derivation\textsuperscript{31}:

$$A[x; \alpha, \alpha \triangledown] = \frac{A(x)}{(x a - \alpha (a) x - \alpha \triangledown (a))}$$

(23)

where $a$ ranges over all elements of $A$. This is a left-Ore extension in a single variable. Quite remarkably, the mere fact that $\alpha \triangledown$ is a derivation $^{29}$, enables the existence of an algebraic (not necess. flat) holomorph in the sense of Eq $^{11}$, above.

We may now assemble the bundle algebraically over the non-commutative ring via the following short exact sequence:

$$0 \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}' \oplus \alpha \triangledown (\mathcal{Q}') \rightarrow \alpha \triangledown (\mathcal{Q}') \rightarrow 0.$$ 

(24)

Yet, this obviates the fact that inductively speaking (in the sense of representation theory) in principle the egg here is a maximally compact (sub-)group of $\mathcal{L}$, i.e., $\text{SU}(2)$ and not the vector space, or rather, the \textit{non-commutative} ring module that results in the case of the algebraic spacetime embodied in Eq $^{24}$.

In this spirit, let us reverse this construction and start with the algebra of the fiber instead: $\mathcal{L}'$. Thus, we take $\mathcal{L}'$ as a \textit{right} $\mathcal{L}_{\alpha}$-module which is at the same time a \textit{skew} K-algebra $\mathcal{L}_{\alpha}'$ such that the module mapping $x \mapsto \ell x$ is a \textit{left} $\alpha$-derivation in an \textit{associative} algebra $\mathcal{A}$; e.g., Eq [22]. The ensuing algebraic sequences should

\textsuperscript{12}In fact, the quotient of \textit{analytic} diffeomorphisms by $\mathcal{L}$ yields \textit{spacetime} coordinates, the vierbein and an “infinite tower of ‘generalized’ connections” as \textit{dynamical} gravitational fields in the spacetime proper\textsuperscript{13}. This complements our analysis here on the (algebraic and non-commutative) geometry of the solder form (meaning there is much more than analyticity in $\text{Diff} \, M$).
be compared side by side with the group sequences for the crossed-modules embedded in the semidirect products, Eqs[17, 20], while re-interpreting the “vector space” as a quotient $Q$ with group structure:

\[ 0 \to L' \hookrightarrow L' \oplus \alpha \overrightarrow{D}(L') \to \alpha \overrightarrow{D}(L') \to 0 \] (25)

\[ 1 \to L \hookrightarrow Q \otimes_{L \to \text{Aut}(Q)} L \to Q \to 1 \] (26)

Yet again, inducing a representation from the non-compact $L$ brushes over the need, in the sense of Frobenius, for a maximally compact (sub-)group of $L$, e.g. SU(2), to properly induce the representation despite the fact that imposing compactness may wipe out causal structure locally (see below). Globally, on the other hand, a spacetime foliation via groupoids may help alleviate this issue.

Recall now that the subscripts in the semidirect products represent implicit structure morphisms into a base object (i.e., a ring scheme, Ref. footnote [9]) so that an immediate $\text{Id}$ seems trivial:

\[ \text{Aut}(L) \simeq \alpha \overrightarrow{D}(Q') \quad \wedge \quad \text{Aut}(Q) \simeq \alpha \overrightarrow{D}(L'); \]

however, this is formally incorrect since Eqs[24 & 25] represent extensions of algebraic rings while Eqs[20 & 26] represent group extensions (of a quotient and a stabilizer respectively). A quick fix to such an oversight starts with replacing automorphisms of the group by endomorphisms of the algebra:

\[ \text{End}(L') \simeq \alpha \overrightarrow{D}(Q') \quad \wedge \quad \text{End}(Q') \simeq \alpha \overrightarrow{D}(L'); \] (27)

however, this is again formally incorrect since a priori neither the groups nor the algebras are trivially embedded into each other. Instead, we will adhere to the construction of a crossed-module to understand more precise relations.

With this aim in mind, recall the following three maps: the implicit structure morphism in the standard definition of the semi-direct product, call it $\beta$; the $\alpha$-map as a section (not necess. a group homomorphism) and the $t$-map, respectively: Eq[17]: $\beta : L \to \text{Aut}(Q)$, Eq[18]: $\alpha : Q \to \text{Aut}(L)$, Eq[19]: $t : L \to Q$. Remarkably, the $\alpha$-map becomes a group homomorphism if one considers outer automorphisms of a normal (not necess. abelian) group:

\[ \alpha : Q \to \text{Out}(L). \]

A key fact is that the standard ad-map is an inner derivation[29]. Thus, by enlarging the algebra of the fiber with generators for the coset this derivation can be extended by “outer endomorphisms” of the fiber, $\alpha_{\text{Out}}$, as follows:

\[ \varphi\ell - \alpha \ell\varphi h = \alpha^{\text{out}}_{\alpha}\overrightarrow{D}(\ell) \equiv \alpha^{\text{out}}_{\alpha}\overrightarrow{D}(\ell) + \alpha^{\text{out}}_{\alpha}(\ell) \alpha^{\text{out}}_{\alpha}\overrightarrow{D}, \] (28)

where $\varphi \in Q'$ and $\ell \in L'$. Algebraically, we end up with the following short exact sequence:

\[ 0 \to Q' \hookrightarrow Q' \oplus \alpha^{\text{out}}_{\alpha}\overrightarrow{D}(L') \to \alpha^{\text{out}}_{\alpha}\overrightarrow{D}(L') \to 0, \]

along with the implied group extension:

\[ 1 \to Q \hookrightarrow L \otimes_{Q \to \text{Out}(L)} Q \to L \to 1. \]

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13 The standard reference for non-abelian group extensions and their higher order cohomology groups (up to $H^3$) is the book by Charles Weibel[53], in particular §6.6; a very nice and concise review may be found at Patrick Morandi’s webpage: [http://sierra.nmsu.edu/morandi/notes/GroupExtensions.pdf](http://sierra.nmsu.edu/morandi/notes/GroupExtensions.pdf). A more technical yet delightful reference and one that includes infinite dimensional groups is given by Karl-Hermann Neeb[58].
However, such a construct still does not enable non-central extensions of a normal subgroup $L$ by the double sided quotient $Q$. This is why we need to adhere to non-commutative ring theory.

Furthermore, invoking cross-modules is not truly necessary since for spacetime symmetries the embedding are all in fact physically natural: the germ of a fermionic stratum in the spacetime bundle is given by SU(2) which generates the double cover of the Lorentz group $SL(2,C)$ via ’t Hooft symbols. By the same token, the image of a section belonging to the the sheaf of algebraic generators at the stack of a “point” is the set of germs over some open neighborhood of the identity which enforce causal structure and encompass stable massive and spin states. Following the lines developed by Robert Gerosch in “Domains of Dependence” where causally evolving volume measures[6, 7] bound the local spacetime; we will aim to produce the explicit forms for causality enforcing generators in a future publication. Curiously, when spacetime symmetries are involved, the Goldstone bosons are all necessarily massive and moreover some are spin.

Let’s re-write Eq[22] with explicit reference to the endomorphisms in the associative algebra $\alpha \to D\alpha = \alpha \to D(a) + \text{End}(a)\, \alpha \to D; \ a \subset \mathcal{L}^\prime$, where the “hat” alludes to the universal enveloping algebra. It is well known that for a division ring $D$; e.g., a field, defining $\mathcal{M}$ as the ring of $n \times n$ matrices over $D$, the endomorphism ring $\text{End}(\mathcal{M}V)$ viewed as a ring of right operators on its unique left simple module $V$ is isomorphic to the original division ring $D$ (Ref. Lam[30], Theorem 3.3.3.). However, Ore extensions constitute instances of simple rings that are not matrix rings over a division ring.

In naive terms, enlarging the algebraic structure from an algebra to a group requires appropriate use of a suitable exp$_x$ map near the origin when such a map is well defined. In principle, this yields a one to one correspondence between endomorphisms of the (not necessarily associative) algebra and automorphisms of the group. However, in general going from the algebra to the group is not so straightforward. The data attached to an algebra is closest to that of a module: both live over a ring as two sets with operations. The algebra is endowed with multiplication while the module is enlarged by an additive operation. The data attached to a group is closest to that of a ring. The group is the simplest: one set with identity and a single invertible operation (combined, these two axioms allow for the existence of an inverse). A ring is a set with identity and with two operations: addition and multiplication (the latter distributing over the former). Moreover, a ring is further endowed with additive monoid an abelian group.

Our proposal leads to an algorithm on how to build the bundle space of spacetime symmetries from the causal generators of the Lorentz group (or from the non-causal generators of the maximally compact little group of the Lorentz group for massive states if it need be). However, important issues remain; in particular, local enforcement of causality and endowment of spin structure on the curved spacetime remain wide open issues outside the scope of the present note. Further, the author makes suggestion for the explicit forms for the curved spacetime generators at the end of § 4 but leaves a more concrete representation of such objects for a future publication.

We now proceed to identify the ring endomorphisms, $\alpha$, in Eq[22] with the higher automorphism “groups” of Eq[3]. Then, the skewed $K$-algebra encodes the full differential geometry of the solder form as follows:

- Generate a ring of differential operators naturally filtered by degree via the left $\alpha$-derivation to provide a “total ordering” of the operators to be used in the Poincaré-Birkhoff-Witt Theorem.
- Invoke the PBW Theorem to identify the quotients $D^\alpha_n / D^\alpha_{n-1}$ with homogenous polynomials of degree $n$ in left-invariant vector fields generated by the Lorentz algebra via the (commutative) grading of its universal envelope. This is a trivial statement on the particle content of the spacetime.
- Assign to the (non-commutative) invariant basis elements of the free polynomial ring the role of first order differential operators on the group “manifold” by demanding left invariance of the differential operators as well.
It follows that the full diffeomorphism symmetry of the spacetime—encompassing the $\mathcal{C}^\infty$ structure to all orders—is encoded in the equivalent classes parsed by the free polynomial ring generated by the Lorentz algebra cut out by the “ideal”—in the sense of (non-commutative) symmetric algebras—generated by the left $\alpha$-derivation. This defines the skewed K-algebra. The automorphism “groups” of Eq\[8\] are in fact endomorphism elements of the free polynomial ring for $\mathcal{L}'$. Investigation of the structure of such endomorphisms appears as a fertile ground for future research.

4 A Conjecture on Quantization: baby steps in quantum ring theory

On the basis of the given arguments, we now fancy the following conjecture: Quantization of the curved spacetime involves a slight generalization of the canonical commutator brackets in flat spacetime where elements of the free polynomial ring in the Lorentz algebra generators as non-commuting indeterminates undergo an $\alpha$-mutation upon commuting elements from left to right.

The deep meaning of the conjecture is that the flat spacetime symbols $x$ & $p$ in the canonical commutator relation:

$$[p, x] = -i\hbar,$$

must be replaced by the algebro-geometric objects $\wp$ & $\ell$ as elements of the free polynomial ring generated by the Lorentz algebra. When one of these objects, say $\ell$, is simply considered as an (asymmetric) inner derivation; e.g., following the procedure in Eqs\[8 & 9\] but from left to right, we obtain a left $\alpha$-skewed version of the canonical commutation relations:

$$\wp\ell - \alpha \ell\wp = -i\bar{\hbar},$$

which has the informal interpretation that the $\ell$ element undergoes an $\alpha$-mutation upon commuting $\wp$ through $\ell$ from left to right. The $\alpha$-mutation is in fact a ring endomorphism and a suitable framework to develop this notions further in that of a quantum ring theory.

As a quick check for consistency, recall that

$$[p, x] = [x, p] = -[p, x]$$

and look at the anti-hermicity relation for the skewed bracket:

$$\left(\frac{1}{i} \alpha [\wp, \ell]_c\right)^\dagger = -\hbar$$

$$i\{(\wp\ell)^\dagger - (\alpha \ell\wp)^\dagger\} =$$

$$i\{\ell^\dagger \wp^\dagger - \wp^\dagger (\alpha \ell)^\dagger\} =$$

$$i\{\ell\wp - \wp\ell_\alpha\} = i[\ell, \wp]_\alpha.$$  \hspace{1cm} (30)

Thus, the skewed counterpart to Eq\[29\] is

$$\alpha [\wp, \ell]^\dagger = [\ell, \wp]_\alpha.$$

Evidently $\alpha \ell = \ell_\alpha = \text{End}(\ell)$, but switching $\alpha$ from left to right upon taking the hermitian operation in the context of the above relation is a lot more meaningful. This has the elegant interpretation that the operators $\wp$ and $\ell$ are hermitian but the skewed derivation switches “preference” from left to right $\alpha$-derivation upon the hermitian operation. Since the construction of the double-sided vector space in insensitive to which action is right or left as long as the “large” differential action is derived from the rigid one, this scheme is seen to be self-consistent!

The immediate question that arises is: what does this mean for the rest of quantum mechanics as formulated in flat space? Let’s take a quick look at the second order fundamental operators in the elements of the free
polynomial ring (these have the standard interpretation as angular momentum and Lorentz boost operators but for the more obscure nature of the $\varphi$ and $\ell$ symbols as elements of the free polynomial ring):

$$L_{\mu\nu} = -i(\ell_\mu \varphi_\nu - \varphi_\mu \ell_\nu).$$

These are unaffected by the skewed derivation algebra since they involve different index combinations. Moreover, as is well known, their commutator algebras involve the flat metric and these too are unaffected by the skew Weyl algebra. These considerations suggest that these operators constitute a good set of generators for the rest of the spacetime via the free polynomial ring that is built from them. So much for the digression.

5 Outlook

We have proposed a reformulation of the standard fiber bundle approach to address spacetime symmetries where the bundle space plays an active role in encoding the geometric data. A novel notion going by the name of a double-sided vector space proves to be a handy tool pulled out of non-commutative algebraic geometry to represent sections into spacetime fiber bundles. In the mathematical literature, this is motivated by the study of the non-commutative analogs of symmetric algebras\cite{31} and their relationship to noncommutative ruled surfaces\cite{67}. This is highly technical field yet unexplored by the physical community. In particular, we have completely omitted allusion to the issue of admissibility of symmetric algebras for a given two-sided vector space\cite{31}. Yet, the conclusions of this note are independent of such admission.

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A Category theory, naturality and intertwining operators

A (small) category is a pair of related sets: (objects, morphisms) where the morphisms act on the objects. A large category eases a requirement on the morphisms: these need only constitute a class, not necessarily a set. A functor between two categories, with source A and target B, is the following data. A map

$$F : \text{obj}(A) \rightarrow \text{obj}(B)$$

and related morphisms in B,

$$F(m) : F(a_i) \rightarrow F(a_j),$$

induced by arbitrary morphisms in A,

$$m : a_i \rightarrow a_j.$$  

Composition of morphisms is transitive,

$$l : x \rightarrow y \text{ and } m : y \rightarrow z \text{, } m \circ l : x \rightarrow z$$

and associative:

$$(l \circ m) \circ n = l \circ (m \circ n).$$

Most importantly, for each object in the category, $$a_i$$, there exist an identity morphism

$$\text{id}_{a_i} : a_i \rightarrow a_i.$$  

Naturality of a “transformation” $$\eta : F \rightarrow G$$ between a pair of functors $$F,G : \text{obj}(A) \rightarrow \text{obj}(B)$$ is stated in terms of the commutativity of functor induced morphisms $$F(m)$$ and $$G(m)$$ with “intertwined” morphisms (transformations) of functorial objects $$\eta_{a_k} : F(a_k) \rightarrow G(a_k)$$ on the target space:

$$\eta_{a_j} \circ F(m) = G(m) \circ \eta_{a_i}.$$  

This may be conveniently stated through the commutativity of the diagram implied by this algebraic relation.

In particular, when $$F$$ and $$G$$ belong to the category of finite dimensional (smooth, topological or algebraic) vector spaces, natural transformations are interpreted as the intertwining operators between the group (groupoid) representations embodied by $$F$$ and $$G$$ as (affine) vector spaces, or torsors. Within a Lie theoretic framework, the category of (finite) vector spaces is interpreted as a category with a single object, with invertible morphisms as group elements and with Lie algebra elements as generators of the vector space.

Note that this is reminiscent of the equivalence relation for Čech 1-cocycle defined for an open cover $$\mathcal{U} = \{U_\alpha\}$$:

$$g,h \in \mathcal{G}(U_{\alpha \beta})$$ are equivalent iff $$\exists f \in \mathcal{G}(U_\alpha)$$ such that

$$f_\alpha g_{\alpha \beta} = h_{\alpha \beta} f_\beta.$$  

The set of equivalent classes yields the first cohomology class on the open cover with coefficients in $$\mathcal{G}$$, $$\Sigma_i [g_i] = H^1(\mathcal{U}, \mathcal{G})$$, and the colimit over the refinement of covers turns the first Čech cohomology class over the base manifold: $$\check{H}^1(X, \mathcal{G}) = \lim_{\leftarrow \mathcal{U}} H^1(\mathcal{U}, \mathcal{G})$$. For completeness, we quote here the very important theorem from Moerdijk notes on Stacks and Gerbes (2002: Theorem 1.3):

There is a bijective correspondence between isomorphism classes of $$\mathcal{G}$$-torsors and cohomology classes in $$\check{H}^1(X, \mathcal{G})$$.  

So much for the digression.

Given objects A and B in a category $$\mathcal{C}$$ as above, **Yoneda’s Lemma** is the statement that a new, fully faithful functor can be built from $$\mathcal{C}$$ to its “functor category” where objects are contravariant functors $$\mathcal{C} \rightarrow \text{Sets}$$ and the morphisms are natural transformations of such functors (Vakil2012). More explicitly, given a third object
in the same category $C \in \mathcal{C}$, this induces a set of morphisms from $C$ into $A$: $\text{Mor}(C, A)$, so that given a single morphism, $f : B \to C$, from a different object $B$ into $C$ a map of sets in induced:

$$\text{Mor}(C, A) \to \text{Mor}(B, A)$$

by composition

$$B \xrightarrow{f} C \to A$$

where the second map is given. Now, given two distinct objects $A, A' \in \mathcal{C}$ and another set of morphisms

$$j_C : \text{Mor}(C, A) \to \text{Mor}(C, A')$$

that commute with the first map of sets, the $j_C$’s are induced from a unique morphism $g : A \to A'$ for all the $C \in \mathcal{C}$! Furthermore, if all the $j_C$’s are bijections, $g$ is necessarily an isomorphism. Lastly, there is a functor $h^A : \mathcal{C} \to \text{Sets}$ sending $B \in \mathcal{C}$ to $\text{Mor}(A, B)$ and $f : B_1 \to B_2$ to $\text{Mor}(A, B_1) \to \text{Mor}(A, B_2)$. This is given by

$$[g : A \to B_1] \mapsto [f \circ g : A \to B_1 \to B_2].$$

Yoneda’s Lemma is the statement that there is a bijection between the natural transformations $h^A \to h^B$ of covariant functors $\mathcal{C} \to \text{Sets}$ and the morphisms $B \to A$. 
B Left and right actions in representation theory

By definition, a (finite) representation of a (finite) group $\mathcal{H}$ on a (finite) vector space $V$ is a homomorphism of $\mathcal{H}$ to the group of automorphisms of $V$, $\rho: \mathcal{H} \to \text{Aut}(V)$. The assignment of a continuous map from a simply connected Lie group $\mathcal{H}$:

$$\rho: \mathcal{H} \to \mathcal{I},$$
i.e. a Lie group homomorphism, is uniquely determined by its differential at the identity element $e$:

$$d\rho_e: T_e\mathcal{H} \to T_e\mathcal{I}$$

More formally, one says that any smooth map from $\mathcal{H}$ is uniquely determined by its germ at $e$.

As a homomorphism of Lie groups, left conjugation, $\text{int}_g(h) = ghg^{-1}$ (with $g, h \in \mathcal{H}$), preserves the action of the group $\mathcal{H}$ on itself by inner automorphism. The differential of this map yields automorphisms of the tangent space (expanding $h$ as a 1-parameter subgroup $h = h(t) \cong e + At$):

$$\text{Ad}_g \equiv d(\text{int}_g)_e : \mathcal{H} \to \text{Aut}(T_e\mathcal{H}); \quad g, e \in \mathcal{H}. $$

$$d(\text{int}_g) : A \mapsto gAg^{-1}; \quad g \in \mathcal{H}, \quad A \in \mathcal{H}'.$$  

As a map from $\mathcal{H}$ to $\text{Aut}(T_e\mathcal{H})$ this is by definition a representation of $\mathcal{H}$ in its own tangent space; it constitutes a bookkeeping devise to track the “difference” between left and right translations by elements of $\mathcal{H}$ on elements of $\mathcal{H}'$. In fact, the composition law for homomorphisms as $C^\infty$ maps: $\rho(gh) = \rho(g) \cdot \rho(h)$, also permits either left or right translation as group operation in lieu of conjugation; these maps respect the action of the group $\mathcal{H}$ on itself by diffeomorphism. Nevertheless, automorphisms are favored over diffeomorphisms as differential structure preserving morphisms because these enforce the presence of a fixed point, namely the choice of origin, “$e$”, to contain the representation locally. One then exploits the continuous structure of the group at the identity to build the representation. The main obstruction to a locally Diffeomorphic construction is the absence of generic fixed points.

It should be emphasized that the Adjoint representation is a mixing bilinear gadget adjoining elements of the group with those on its algebra. Since in order to exploit the continuous structure of the group at the identity one desires a condition purely on the differential of the homomorphism map, $d\rho_e$, it is necessary go one step further and take the differential of the Adjoint map, $d\text{Ad} \equiv ad$, to get the adjoint representation acting on elements of the Lie algebra alone. In the context of finite dimensional Lie Groups where there is a well defined exponential map relating elements of the algebra to elements of the group the procedure is a straightforward; but the absence of such a map in the infinite dimensional case, i.e. for the diffeomorphism group, requires a more sophisticated (and uncertain) procedure.

For finite dimensional groups, the map $ad$ is realized as map from the tangent space into endomorphisms of the same:

$$ad = d\text{Ad}_e : (T_e\mathcal{H}) \to \text{End}(T_e\mathcal{H}).$$

$$ad(A) = \lim_{t \to 0} \frac{d(\text{Ad}_e(t))}{dt} = [A, \cdot]$$

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14 A homomorphism is just a $C^\infty$ map that respects the composition law: $\rho(gh) = \rho(g) \cdot \rho(h)$ plus possibly some algebraic or geometric structure.

15 How could one reach the latter if $\mathcal{J}$ were non–simply-connected?

16 If $U \subset \mathcal{H}$ is any open neighborhood of $e$, then $U$ generates $\mathcal{H}$. The differential structure of a variety may be readily generalized by introducing the notions of sheaves and stalks. Whereas group representations yield “the group element at the translated point” (e.g. the value of a continuous function at an adjacent point) in terms of the differential map, by contrast the image of a section belonging to a sheaf $\mathcal{F}$ at the stalk $\mathcal{F}_p$ are sets of germs over some open set containing $p$ and $e$. The value of a section at a “point” is thus undefined.

17 At the level of unbroken symmetries, we will see that another representation arises naturally: the fundamental representation (built from the highest weights or roots in the adjoint representation?) which corresponds to right-left (or left-right!) diffeomorphisms. The ambiguity arising in the mod'ing of the stabilizer...

18 Just what are these in the tangent space?
where \( g(t) = e^{tA} \) is a one parameter subgroup starting at the identity \( e \). One then has \( dp_e(\text{ad}(x)(y)) = \text{ad}(dp_e(x)) \rho_e(y) \). Note that in both cases, the tangent spaces must be interpreted as vector spaces and not as Lie algebras per se. Also, automorphisms constitute a dense open subset of endomorphisms since the latter allow for non-invertible maps into the target space. Diffeomorphisms, on the other hand, add the extra requirement of differentiability of both the map and its inverse. This is a stronger requirement than continuity which yields homeomorphisms of manifolds. For further explanation including the implicit commutative diagrams involved in these “natural” maps, the reader is referred to the beautiful exposition by Fulton and Harris[35].

### C Strata, Slices and Associated Bundles

We proceed to the formal definition of a stratified manifold following the expositions by Borel et al. [21] and Palais and Terng[8].

Geometrically, the issue of vector spaces as quotients by not necessarily normal subgroups leads to the following rather formal construction. Consider a \( G \)-manifold, \( M \), where \( G \) acts on \( M \) from the right. The orbit space is defined by the coset \( \tilde{M} = M/G \) and this orbit map is a smooth fiber bundle: \( M/G \times G \). Given a closed, not invariant/normal sub-group \( H \subset G \), one defines an \( H \)-slice, \( S \), in \( M \) as an \( H \)-invariant subset of \( M \) such that, given a local section in the coset \( \sigma: G/H \rightarrow G \mid \sigma(H) = e \), the map \( \mathcal{J}: U_H \times S \rightarrow M \) given by \( \mathcal{J}(u, s) = \sigma(u) \cdot s \) is a local homeomorphism of \( U_H \times S \) into an open set in \( M \) (note that in this map \( G \) is left acting). Here \( U_H \) is an open neighborhood of \( H \) in \( G/H \) (recall that such a neighborhood generates the representation via its germ at the identity of \( G/H \); Appendix [11]; i.e., \( [g] = gH \) is “close” to \( eH \) so that points in the coset \( G/H \) are in one to one correspondence with points of the slice \( S \) in \( M \).

A slice at a point \( x \in M \) is implicitly understood to be a \( G_x \)-slice where \( G_x \) is the isotropy or stabilizer subgroup of \( x \). If such a slice exists, then there is an open neighborhood of \( x \) where the isotropy groups of each point are conjugate to that of \( x \): \( G_y = gG_xg^{-1} \) for some \( g \in G \) and \( y \in U_x \). This statement makes use of the local section as: \( G_{\sigma(u)}s = \sigma(u)Gx\sigma(u)^{-1} \). Note that an \( H \)-slice (not to be confused with an \( S \)-representation) is the equivalent of a co-normal neighborhood of \( H \) in \( G \) passing through \( e \): transverse to \( H \) and of dimension complementary to \( H \) in \( G \). Thus, the action is neither transitive nor free. We choose to act on the right of \( M \) in order to make a distinction explicit in the construction of the associated bundle below.

Next, recall the definition of the normalizer of \( H \) in \( G \) as the maximal subgroup of \( G \) where \( H \) is normal. For finite dimensional Lie groups with a well defined exponential map, such group can always be found (Ref. footnote [3] & [2]). A key fact ([21] pg. 157) is that the fixed point set of a closed subgroup \( H \) in the \( H \)-invariant subspace of orbits of type \( G/H \), \( M(H) \), is an \( N_H \)-slice and under the action of \( N_H \) it is a principal \( N_H/H \)-bundle over the restricted orbit space \( M/H \). Further, taken as a space with right translation as group action, \( G/H \), is a principal \( N_H/H \)-bundle over \( G/N_H \). Furthermore, the action of \( G \) commutes with that of \( N_H/H \) so these may be combined into a single \( G \times N_H/H \)-action for the \( G/H \)-space.
References

[1] Klein, F., “A Comparative Review of Recent Researches in Geometry”, [http://arXiv.org/pdf/0807.3161v1.pdf]
[2] Eguchi, T., Gilkey, P., & Hanson, A. “Gravitation, Gauge Theories and Differential Geometry”, Phys. Rep. 66, #6, (1980) 213-393
[3] Wise, D., “Symmetric Space Cartan Connections and Gravity in Three and Four Dimensions”, Sym. Int. & Geo Met & App (SIGMA) 5, 80 (2009)
[4] Sternberg, S. & Ungar, T., “Classical and pre-quantized mechanics without Lagrangians or Hamiltonians”, Hadronic Journal, 1, 33 (1978).
[5] Sternberg, S. “Lie Algebras”, (2004) PDF found at [http://www.math.harvard.edu/~shlomo/]
[6] Gerosch, R., “Spinor Structure of Space-Times in General Relativity. I.”, J. of Math. Phys. 9, 1739 (1968).
[7] Gerosch, R., “Domains of Dependence”, J. of Math. Phys. 11, 437 (1970).
[8] Palais, R. & Terng, C., “Critical Point Theory and Submanifold Geometry”, Lecture Notes in Mathematics 1353, Springer-Verlag (1988).
[9] Palais, R. & Terng, C., “Natural Bundles have Finite Order”, Topology 16, 271 (1977).
[10] Wigner, E. P. (1939), “On unitary representations of the inhomogeneous Lorentz group”, Annals of Mathematics 40 (1): 149204.
[11] Gornwald, F., ”Metric-Affine Gauge Theory of Gravity I: Fundamental Structure and Field Equations”, Int. J., Mod., Phys., D 6, 263, (1997) (arXiv:gr-qc/9702034)
[12] Kirsch, I., "A Higgs Mechanism for Gravity", Phys. Rev. D72 (2005) 024001 (arXiv:hep-th/0503024)
[13] Boulanger, N & Kirsch, I., “A Higgs Mechanism for Gravity. Part II: Higher Spin Connections”, Phys. Rev. D73 (2006) 124023 (arXiv:hep-th/0602225)
[14] Colleman, S., Wess, J. & Zumino, B. “Structure of Phenomenological Lagrangians. I.” Phys. Rev. 177, #5, 2239.
[15] Weinberg, S. “Quantum Theory of Fields V2: Modern Applications”, Cambridge University Press, 1995.
[16] Lamberti, N. & West, P. “Duality Groups, Automorphic Forms and Higher Derivative Corrections Phys.Rev.D 75:066002 (2007)
[17] Frankel, T., “The Geometry of Physics: An Introduction”, Cambridge University Press (1997).
[18] Fatibene, L. & Francaviglia, M., “Natural and Gauge Natural Formalism for Classical Field Theories: a geometric perspective including spinors and gauge theories”, Kluver Academic Publishers, Dordrecht/Boston/London (2003).
[19] Sharpe, R.W., “Differential Geometry: Cartan’s Generalization of Klein’s Erlangen Program”. Graduated Texts in Mathematics #166, Springer-Verlag (1997).
[20] Sternberg, S., “On the Interaction of Spin and Torsion II. The Principle of General Covariance”, Ann. of Phys. 162, 85 (1985).
[21] Borel, A., Bredon, G., Floyd, E., Montgomery, D. & Palais, R., “Seminar on Transformation Groups”, Annals of Mathematical Studies 46, Princeton University Press (1960).
[22] Epstein, D. & Thurston, W. (1979), “Transformation Groups and Natural Bundles” Proc. London Math. Soc.(3) 38, 219.
[23] Rapoport, D., & Sternberg, S., “On the interaction of Spin and Torsion” Ann. of Phys. 158, 447 (1984).
[24] Heyl, F.W., McCrea, J.D., Mielke, E.W., Ne’eman, Y. “Metric-affine gauge theory of gravity: field equations, Noether indentities, world spinors, and the braking of dilation invariance” Phys. Rep. 258 1-171 (1995)
[25] Ne’eman, Y. “Gravitational interaction of hadrons: Band-spinor representations of GL(n,R)” Proc. Natl. Acad.Sci. USA V74, No.10, 4157-4159 (1977)
[26] Trautman, A., “The Geometry of Gauge Fields” Czech. J. of Phys. 29, 107 (1979).
[27] Ehresmann, C., “Les connexions infinitésimales dans un space fibré différentiable”, Colloque de Topologie, Bruxelles (1950) pag 29-55
[28] Cartan, E., “Sur les variétés à connexion affine et la théorie de la relativité généralisée”, ann. Ec. Norm. 40 (1923), 325-412; ibid. 41 (1924), 1-25; ibid. 42 (1925), 205-241
[29] Jacobson, N., “Lie Algebras”, Interscience Tracts in Pure and Applied Mathematics No. 10, John Wiley & Sons, New York • London (1962).
[30] Lam, T.Y., “A First Course in Noncommutative Rings” Graduate Texts in Mathematics # 131, Springer (1990).
[31] Patrick, D., “Noncommutative Symmetric Algebras of Two-Sided Vector Spaces” J. of Alg. 233, 16-36 (2000).
[32] Pappacena, C.J., “Frobenius bimodules between noncommutative spaces” J. of Alg. 275, 675-731 (2004).
[33] Alvarez-Gaumé, L. & Ginsparg, P. “The structure of Gauge and Gravitational Anomalies” Ann. of Phys. 161, 423 (1985).
[34] Borisov, A.B. & Ogievetskii I., “Theory of Dynamical Affine and Conformal Symmetries as the theory of the Gravitational Field” Theor. Math. Phys. 21, 1179 (1975) [Teor. Mat. Fiz 21, 329 (1974)]
