Random Bond Potts Model: the Test of the Replica Symmetry Breaking.

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ABSTRACT

Averaged spin-spin correlation function squared $<\sigma(0)\sigma(R)>^2$ is calculated for the ferromagnetic random bond Potts model. The technique being used is the renormalization group plus conformal field theory. The results are of the $\epsilon$ - expansion type fixed point calculation, $\epsilon$ being the deviation of the central charge (or the number of components) of the Potts model from the Ising model value. Calculations are done both for the replica symmetric and the replica symmetry broken fixed points. The results obtained allow for the numerical simulation tests to decide between the two different criticalities of the random bond Potts model.

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1 Introduction.

For the ferromagnetic random bond Potts model there exist two different fixed point solutions, one which is replica symmetric [1,2] and a second in which the replica symmetry is broken [3]. Both fixed points are of a universal nature and one or another is reached depending on the initial conditions for the coupling constant in the renormalization group (RG) equation. More specifically, if one starts with the model defined on a lattice, then the partition function shall be given by:

\[ Z(\beta) = \sum_{\{\sigma\}} \exp\{-\beta H[\sigma]\} \]  

(1.1)

where \( H[\sigma] \) is a nearest neighbor interaction hamiltonian for classical spins \( \{\sigma_x\} \):

\[ H[\sigma] = \sum_{x,\alpha} J_{x,\alpha} V(\sigma_x, \sigma_{x+\alpha}) \]  

(1.2)

Each spin \( \sigma_x \) is taking \( q \) values, in case of \( q \)-component Potts model; \( x \) stands for lattice sites; \( (x, \alpha) \) - for lattice bonds: in case of a square lattice \( \alpha = 1, 2 \); \( V(\sigma_x, \sigma_{x+\alpha}) \) is the spin-spin interaction potential: in case of the Potts model the usual choice would be \( V(\sigma, \sigma') \propto 1 - \delta_{\sigma, \sigma'} \). \( \{J_{x,\alpha}\} \) are the bond coupling constants. They are supposed to be ferromagnetic but taking random values, independently for each lattice bond, with some distribution, characterized by a width \( g_0 \), around a mean value \( J_0 \).

Weak disorder corresponds to a small value of \( g_0 : g_0 \ll J_0 \). In this case the model could be studied in the continuum limit, if \( \beta \) is close to a critical value \( \beta_c \), \( (\beta - \beta_c)/\beta_c = \tau \ll 1 \). In this limit the partition function could be given in the form:

\[ Z(\beta) = Tr \exp\{-H_0 - H_1\} \]  

(1.3)

\( Tr \) stands symbolically to represent the sum over the spin configurations, but in the context of the continuum limit theory. Its explicit realization is not important because we shall eventually be dealing with correlation functions, and these are defined unambiguously by the corresponding conformal field theory. \( H_0 \) stands for a hamiltonian, or a field theory action, of the conformal field theory corresponding to a given \( q \)-states Potts model, being defined on a perfect lattice with the spin-spin coupling constant \( J_0 \) for all
lattice bonds and taken at its critical point, $\beta = \beta_c$; $H_1$ represents a deviation from the critical point and it contains disorder. $H_1$ could be given in the following form:

$$H_1 = \int d^2x \tau(x) \varepsilon(x)$$  \hspace{1cm} (1.4)

where

$$\tau(x) \propto \beta J(x) - \beta_c J_0$$  \hspace{1cm} (1.5)

is the random temperature parameter of the continuum limit theory; $x$ takes values on the continuum $2D$ plane; $J_{x,\alpha}$ of the lattice is replaced by $J(x)$; $\varepsilon(x)$ is the energy operator of the Potts model replacing $V(\sigma_x, \sigma_{x+\alpha})$ on the lattice. As far as critical properties are concerned the reduced continuum limit form of the model, defined by the eqs. (1.3), (1.4), is sufficient. For simplicity we shall assume that $\tau(x)$ has a Gaussian distribution, for each $x$, with a width $g_0$, so that

$$\overline{\tau(x)} = \tau_0 = (\beta - \beta_c)/\beta_c$$  \hspace{1cm} (1.6)

$$\overline{(\tau(x) - \tau_0)(\tau(x') - \tau_0)} = g_0 \delta^{(2)}(x - x')$$  \hspace{1cm} (1.7)

The partition function (1.3) is of the form:

$$Z(\beta) = Tr \exp\{-H_0 - \int d^2x \tau(x) \varepsilon(x)\}$$  \hspace{1cm} (1.8)

To take the average over the disorder one introduces replicas, $n$ copies of the same model:

$$(Z(\beta))^n = Tr \exp\{-\sum_{a=1}^{n} H_0^{(a)} - \int d^2x \tau(x) \sum_{a=1}^{n} \varepsilon_a(x)\}$$  \hspace{1cm} (1.9)

and then one takes the average:

$$\overline{(Z(\beta))^n} = Tr \exp\{-\sum_{a=1}^{n} H_0^{(a)} - \tau_0 \int d^2x \sum_{a=1}^{n} \varepsilon_a(x) + g_0 \int d^2x \sum_{a \neq b}^{n} \varepsilon_a(x) \varepsilon_b(x)\}$$  \hspace{1cm} (1.10)

One arrives in this way at a homogeneous field theory of $n$ coupled models with the coupling action:

$$H_{\text{int}} = -g_0 \int d^2x \sum_{a \neq b}^{n} \varepsilon_a^{(x)} \varepsilon_b^{(x)}$$  \hspace{1cm} (1.11)
In $H_{\text{int}}$ the non-diagonal terms only, $a \neq b$, are being kept. The diagonal ones could be put back into $\sum_{a=1}^{n} H_{0}^{(a)}$. Moreover, in case of the Potts model the energy operator $\varepsilon(x)$ corresponds to the operator $\Phi_{1,2}(x)$ of the corresponding conformal theory. Its operator algebra is known

$$\Phi_{1,2}\Phi_{1,2} \rightarrow \Phi_{1,1} + \Phi_{1,3} \quad (1.12)$$

Here $\Phi_{1,1} = I$ is the identity operator, and the operator $\Phi_{1,3}(x)$ is irrelevant, $2\Delta_{1,3} > 2$.

So there will be no relevant diagonal subtractions from $\sum_{a,b} \varepsilon_{a}\varepsilon_{b}$, the diagonal terms can just be dropped. In our analysis of the random bond Potts model we shall take the replicated field theory form of it, equations (1.10), (1.11), as our starting point.

In the renormalization group analysis, with the interaction term given by the equation (1.11), the qualitatively different initial conditions for the renormalization group equation correspond to either taking the initial coupling constant as it is in eq.(1.11), $g(\xi = 0) = g_{0}$ ($\xi$ is the RG parameter), or to breaking the replica symmetry initially by putting $H_{\text{int}}$ into the form:

$$H_{\text{int}} = - \int d^{2}x \sum_{a \neq b} g_{ab} \varepsilon_{a}(x)\varepsilon_{b}(x) \quad (1.13)$$

This corresponds to assuming different couplings, initially, for different replicas.

Replica symmetric form of $g_{ab}$ is

$$g_{ab} = g_{0}, \quad \text{all} \; a \neq b \quad (1.14)$$

and $g_{ab} = 0$ for $a = b$. Taking $g_{ab}$ with different components corresponds to replica symmetry breaking perturbation. Relevance of this type of perturbation has been observed initially in [4] for the random XY model and in [5] for the standard $\varphi^{4}$ theory, with randomness. These two models move, under RG, into a strong coupling regime. The first fixed point with a replica symmetry breaking has been found in [3] for the 2D random-bond Potts model. For the moment this remains to be the only solution of this type.

To repeat it again, the random bond Potts model, i.e. the model with $H_{\text{int}}$ in (1.13), has two different fixed point solutions:

1) replica symmetric, if initially $g_{ab}$ of the form in eq.(1.14);
2) replica symmetry broken, if initially $g_{ab}$ is different from (1.14).

To check which one of the two solutions realizes for the original spin model defined on a lattice with random bonds, eqs.(1.1), (1.2), in the numerical simulation experiment in particular if not in the real one, we shall calculate in this paper one particular observable quantity for which the replica symmetry breaking effects are expected to be most pronounced. The calculations will be done for both RG fixed points to make the verification problem well defined.

The quantity in question is the averaged spin-spin correlation function squared

$$<\sigma(0)\sigma(R)>^2$$

(1.15)

The related quantity is the renormalization group amplitude

$$Z_{ab}(\xi)$$

(1.16)

for the replicated model operator

$$O_{ab}(x) = \sigma_a(x)\sigma_b(x)$$

(1.17)

Here the spins $\sigma_a$, $\sigma_b$, belong to different replicas, $a \neq b$, and are taken at the same point. The amplitude (1.16) is the spin-spin "overlap function" for our critical model, if one uses the terminology of the theory of spin glasses [6].

Further discussions will be postponed until the formal analytic solutions are obtained.

In the next section the analytic problem will be defined and the RG equation for the amplitude $Z_{ab}(\xi)$ will be obtained, up to second order in the coupling constant $g_{ab}(\xi)$. In Section 3 the amplitude $Z_{ab}$ and the correlation function $<\sigma(0)\sigma(R)>^2$ will be calculated. We shall also define the corresponding magnetization. In Section 4 the results will be analyzed, in particular as applied to the 3 and 4 component Potts models. We shall also consider in this section the distribution function for products of local magnetizations. This distribution is encoded in the spin-spin "overlap function", the amplitude $Z_{ab}$, in a way analogous to that for the corresponding object in the spin-glass theory. In section 5 results of numerical simulations will be presented for the 3 and 4 - component Potts
2 Definition of the problem and calculation of the 
RG equation for \( Z_{ab} \).

The RG equation for the coupling constant \( g_{ab} \) in the action \( H_{\text{int}} \), eq.(1.13), has been
derived and solved in [3], for the replica symmetry broken fixed point. The replica
symmetric solution has originally been found in [1]. We shall use those results but first
we shall concentrate on the RG equation for the amplitude \( Z_{ab}(\xi) \). This function occurs
in the calculation of the correlation function \( \langle \sigma(0)\sigma(R) \rangle \).

In terms of replicas the averaged spin-spin correlation function,
\[
\langle \sigma(0)\sigma(R) \rangle^2 = \lim_{n \to 0} \frac{1}{2n(n-1)} \sum_{a \neq b} \langle \sigma_a(0)\sigma_a(R)\sigma_b(0)\sigma_b(R) \rangle,
\]
can be represented in the following form (see Sec.3, eqs.(3.7)-(3.9)):
\[
\langle \sigma(0)\sigma(R) \rangle^2 = \lim_{n \to 0} \frac{1}{2n(n-1)} \sum_{a \neq b} \langle \sigma_a(0)\sigma_b(0) \sum_{c \neq d} \sigma_c(R)\sigma_d(R) \rangle \quad (2.1)
\]
Then the operator to be renormalized is:
\[
O_{ab}(x) = \sigma_a(x)\sigma_b(x), \quad a \neq b \quad (2.2)
\]
As we are going to use the perturbative RG, we have to consider the exponential of \( H_{\text{int}} \),
the way it would enter the partition function (1.10), in the presence of the operator \( O_{ab}(x) \):
\[
O_{ab}(x) \exp\{-H_{\text{int}}\} = \sigma_a(x)\sigma_b(x) \exp\{\int d^2y \sum_{a \neq b} g_{ab}\varepsilon_a(y)\varepsilon_b(y) \} \quad (2.3)
\]
We expand \( \exp\{-H_{\text{int}}\} \):
\[
O_{ab}(x)(1 - H_{\text{int}} + \frac{1}{2}(H_{\text{int}})^2 + \cdots) = O_{ab}(x) - O_{ab}(x)H_{\text{int}} + \frac{1}{2}O_{ab}(x)H_{\text{int}}^2 + \cdots \quad (2.4)
\]
and then we do all possible contractions and operator algebra, which lead to reproducing
the operator \( O_{ab}(x) \). Contractions in the case of our non-gaussian field theory amounts
to equalizing replica indices in all possible ways which occur under the summation over indices and then replacing the products of operators with the same index by correlation functions (or using directly the operator algebra) of the unperturbed theory. We remind that the unperturbed theory is a collection of independent and identical Potts models. Each Potts model, in the continuum limit, is a minimal conformal field theory with the energy operator \( \varepsilon(x) \) represented by the primary field \( \Phi_{1,2}(x) \), and with the central charge related to the number of components \( q \) of the original Potts model as defined on the lattice. We shall specify this relation later, see also [7].

Proceeding in this way, one gets in the first order:

\[
-O_{ab}(x)H_{\text{int}} = \sigma_a(x)\sigma_b(x) \int d^2 y \sum_{c \neq d} g_{cd} \varepsilon_c(y) \varepsilon_d(y) \\
\rightarrow 2g_{ab} \sigma_a(x)\sigma_b(x) \int_{1<|y-x|<a} d^2 y \frac{D^2}{|x-y|^{2\Delta_{\varepsilon}}} 
\]

We have contracted here the indices \( b \) and \( c \), and \( a \) and \( d \) (or \( b \) and \( d \), and \( a \) and \( c \), which amounts to a combinatorial factor of 2), and then we have used the operator algebra:

\[
\sigma(x)\varepsilon(y) = \frac{D}{|x-y|^{\Delta_{\varepsilon}}} \sigma(x) + \cdots 
\]

\( D \) stands for the operator algebra coefficient \( D_{\sigma,\varepsilon} \). For minimal conformal theories these coefficients have been calculated in [8].

Integration in (2.5) is around \( x \), over the distances ranging between the ”old” and the ”new” cut-offs. The old one we choose to be equal to 1 (like it would be on a lattice, when distances are measured in lattice spacings) and the new one \( a, a >> 1 \).

\( \varepsilon(x) \) is the primary field \( \Phi_{1,2} \), so one has:

\[
\Delta_{\varepsilon} = \Delta_{1,2} + \bar{\Delta}_{1,2} = 2\Delta_{1,2} 
\]

We use next the Kac formula for conformal dimensions of primary fields \( \Phi_{n',n}(x) \):

\[
\Delta_{n',n} = \frac{(\alpha_{n'} + \alpha_+ n)^2 - (\alpha_- + \alpha_- n)^2}{4} 
\]

The parameters \( \alpha_+, \alpha_- \) are related to the central charge of the Virasoro algebra:

\[
c = 1 - 24\alpha_0^2, \\
\alpha_{\pm} = \alpha_0 \pm \sqrt{1 + \alpha_1^2} 
\]
Notice that
\[ \alpha_+ \alpha_- = -1 \] (2.10)

We shall use in the following the parameter \( \alpha_+^2 \) to specify the models, instead of the central charge \( c \). For Ising model \( c = \frac{1}{2} \), \( \alpha_+^2 = \frac{4}{3} \). One gets conformal theories for the critical Potts models with the number of components \( q \) varying continuously from 2 to 4 if the central charge is taken to vary between 1/2 and 1, or \( \alpha_+^2 \) varying between 4/3 and 1 [7]. For the Ising model \( \Delta_1,2 = \frac{1}{2}, \Delta_\varepsilon = 1 \), and the perturbation \( H_{\text{int}} \) (1.13) is marginal, \( g_{ab} \) is dimensionless. This perturbation becomes slightly relevant if, following Ludwig [1], one takes
\[ \alpha_+^2 = \frac{4}{3} - \epsilon \] (2.11)

and one studies the Potts models by the \( \epsilon \)-expansion RG assuming \( \epsilon \) to be small.

The dimension of the energy operator is now given by:
\[ \Delta_\varepsilon = 2\Delta_{1,2} = \frac{(\alpha_- + 2\alpha_+)^2 - (\alpha_- + \alpha_+)^2}{2} = 1 - \frac{3}{2} \epsilon \] (2.12)

The first order correction in (2.5) takes the form:
\[
2g_{ab}\sigma_a(x)\sigma_b(x)D^2 \int_{|y|<a} \frac{d^2y}{|y|^{2-3\epsilon}} = 2g_{ab}\sigma_a(x)\sigma_b(x)D^22\pi \frac{1}{3\epsilon} a^{3\epsilon} \] (2.13)

In integrating over scales from 1 to \( a \) one gets a factor \( \frac{1}{\epsilon}(a^{3\epsilon} - 1) \). We have replaced it, in a standard way, by \( \frac{1}{\epsilon}a^{3\epsilon} \), assuming \( a^{3\epsilon} \gg 1 \).

Next, if one introduces the amplitude \( Z_{ab} \) and studies renormalization of the operator
\[ \tilde{O}_{ab}(x) = Z_{ab}\sigma_a(x)\sigma_b(x) \] (2.14)

then (2.13) will correspond to the first order correction to \( Z_{ab} \):
\[ \delta Z_{ab}^{(1)} = Z_{ab}4\pi D^2g_{ab}\frac{1}{3\epsilon} a^{3\epsilon} \] (2.15)

This correction, the corresponding RG equation, and the renormalized amplitude \( Z_{ab} \) has first been defined in [1], (for the replica symmetric case), as well as the corresponding amplitudes for higher moments of \( \sigma \): \( \tilde{O}_{ab \cdots d} = Z_{ab \cdots d}\sigma_a\sigma_b \cdots \sigma_d, a \neq b \neq \cdots \neq d \).
It has been shown in [3] that the replica symmetry breaking effects appear in the second order. One needs at least two orders, for all the quantities, to treat these effects. For this reason our analytic problem will be to define renormalization of \( Z_{ab} \) in two orders.

From eqs. (2.3), (2.4) one gets in the second order:

\[
O_{ab} \frac{1}{2}(H_{\text{int}})^2 = Z_{ab} \sigma_a(x) \sigma_b(x) \frac{1}{2} \int d^2y \sum_{c \neq d} g_{cd} \varepsilon_c(y) \varepsilon_d(y) \int d^2y' \sum_{e \neq f} g_{ef} \varepsilon_e(y') \varepsilon_f(y')
\]

\[
\rightarrow D_1^{(2)} + D_2^{(2)} + D_3^{(2)}
\]

We have defined as \( D_i^{(2)} \) the diagrams of the second order. The first diagram is of the form:

\[
D_1^{(2)} = 8 Z_{ab} \sigma_a(0) \sigma_b(0) \frac{1}{2} \sum_d g_{bd} g_{ad} \times \int d^2y \int d^2y' < \sigma(0) \varepsilon(y) \sigma(\infty) > < \sigma(0) \varepsilon(y') \sigma(\infty) > < \varepsilon(y) \varepsilon(y') >
\]

\[
= 4 Z_{ab} \sigma_a(0) \sigma_b(0) (g^2)_{ab} \int d^2y \int d^2y' \frac{D}{|y|^{\Delta_c}} \frac{D}{|y'|^{\Delta_c}} \frac{1}{|y - y'|^{2\Delta_c}}
\]

To simplify somewhat the expressions in the integrals we have put the operator \( \tilde{O}_{ab}(x) = Z_{ab} \sigma_a(x) \sigma_b(x) \) at \( x = 0 \).

To get \( D_1^{(2)} \) we have made equal in (2.16) \( b = c \) and \( a = e \) (we remind that \( a \neq b \), \( c \neq d \), \( e \neq f \) : diagonal elements of \( Z_{ab}, g_{cd}, g_{ef} \) are assumed to be zero). Next we have used the operator product expansion (2.6) for the products \( \sigma(0) \varepsilon(y) \) and \( \sigma(0) \varepsilon(y') \), keeping just the first term, the only relevant one, or, which is the same, we have calculated the projections of \( \sigma(0) \varepsilon(y) \) and of \( \sigma(0) \varepsilon(y') \) on the spin operator placed at infinity. In addition, we have assumed the replica coupling matrix \( g_{cd} \) to be symmetric, \( g_{cd} = g_{dc} \), as this is the case for the Parisi matrices and for the fixed point solution for \( g_{ab} \) in [3]. In this case \( \sum_d g_{bd} g_{da} = \sum_d g_{ad} g_{db} = (g^2)_{ab} \). Finally, the extra factor of 8 in the first line in (2.17) is due to combinatorics: there are 8 ways to make coupling of indices which give the diagram \( D_1^{(2)} \).

The calculation of the integral in (2.17) is straightforward, it is exposed in the Appendix A.1. One gets the following result:

\[
D_1^{(2)} = Z_{ab} \sigma_a(0) \sigma_b(0) 8\pi^2 D^2 (g^2)_{ab} (1 + \epsilon K) \frac{1}{9\epsilon^2} a^{6\epsilon} + O(1)
\]
where $K = 6 \log 2$. In (2.18) we have kept only the terms which are singular in $\epsilon, \sim \frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$, the ones which are relevant for the RG.

The next diagram is obtained from (2.16) by setting $b = c = e$ and, separately, $d = f$, plus all equivalents. This gives:

$$D_2^{(2)} \propto Z_{ab}\sigma_a(0)\sigma_b(0)(g^2)_{ab} \int d^2y \int d^2y' <\sigma(0)\varepsilon(y)\varepsilon(y')\sigma(\infty) > <\varepsilon(y)\varepsilon(y') >$$  \hspace{1cm} (2.19)

This diagram does not produce singularities in $\epsilon$, it is finite for $\epsilon \to 0$, see Appendix A.2. As a result, it does not contribute to the RG evolution of $Z_{ab}$.

Next diagram, $D_3^{(2)}$, is obtained by setting $b = c = e$ and $a = d = f$, plus all equivalents. This gives the following expression:

$$D_3^{(2)} = 4Z_{ab}\sigma_a(0)\sigma_b(0)\frac{1}{2}(g_{ab})^2 \int d^2y \int d^2y' (<\sigma(0)\varepsilon(y)\varepsilon(y')\sigma(\infty) >)^2$$  \hspace{1cm} (2.20)

Calculation of this integral, which is fairly complicated, is described in the Appendices A and B. Finally one gets the following result:

$$D_3^{(2)} = Z_{ab}\sigma_a(0)\sigma_b(0)(g_{ab})^2(8\pi^2D^4\frac{1}{9\epsilon^2} - \frac{\pi^2}{3}\frac{1}{6\epsilon})a^{6\epsilon}$$  \hspace{1cm} (2.21)

Putting (2.18) and (2.21), for $D_1^{(2)}$ and $D_3^{(2)}$, into (2.16), and dropping $D_2^{(2)}$, eq.(19), one derives the second order correction to the amplitude $Z_{ab}$:

$$\delta Z_{ab}^{(2)} = Z_{ab}\{8\pi^2D^2(1 + \epsilon K)(g^2)_{ab}\frac{1}{9\epsilon^2} + 8\pi^2D^4(g_{ab})^2a^{6\epsilon} - \frac{\pi^2}{3}(g_{ab})^2\frac{1}{6\epsilon}\}a^{6\epsilon}$$  \hspace{1cm} (2.22)

Together with $\delta Z_{ab}^{(1)}$ in (2.15) one obtains, up to second order:

$$\tilde{Z}_{ab} = Z_{ab} + \delta Z_{ab}^{(1)} + \delta Z_{ab}^{(2)}$$

$$= Z_{ab}\{1 + 4\pi D^2g_{ab}\frac{1}{3\epsilon}a^{3\epsilon} + 8\pi D^2(1 + \epsilon K)(g^2)_{ab}\frac{1}{9\epsilon^2}a^{6\epsilon}$$

$$+ 8\pi^2D^4(g_{ab})^2\frac{1}{9\epsilon^2}a^{6\epsilon} - \frac{\pi^2}{3}(g_{ab})^2\frac{1}{6\epsilon}\}a^{6\epsilon}\}$$  \hspace{1cm} (2.23)

Further in this Section we shall denote by tilde, like $\tilde{Z}_{ab}$ or $\tilde{g}_{ab}$, the corresponding renormalized quantities.
One obtains, in a standard way, the RG equation for $\tilde{Z}_{ab}$ by taking a derivative with respect to $\xi = \log a$:

$$\frac{d\tilde{Z}_{ab}}{d\xi} \equiv \frac{a d\tilde{Z}_{ab}}{da} = Z_{ab}\{4\pi D^2 g_{ab}a^{3\epsilon} + 8\pi^2 D^2(1 + \epsilon K)(g^2)_{ab}\frac{2}{3\epsilon}a^{6\epsilon}$$

$$+ 8\pi^2 D^4 (g_{ab})^2 \frac{2}{3\epsilon}a^{6\epsilon} - \frac{\pi^2}{3}(g_{ab})^2 a^{6\epsilon}\}$$

(2.24)

In the r.h.s. of this equation we have to replace $Z_{ab}$ and $g_{ab}$ by the corresponding quantities renormalized in the first order. One easily checks, by using the technique described above, and, in addition, in case of $g_{ab}$, a dilatation of coordinates, in order to return to the cut-off scale $a = 1$, that to the first order:

$$\tilde{g}_{ab} = a^{3\epsilon}(g_{ab} + 4\pi (g^2)_{ab}\frac{a^{3\epsilon}}{3\epsilon})$$

(2.25)

$$\tilde{Z}_{ab} = Z_{ab}(1 + 4\pi D^2 g_{ab}\frac{a^{3\epsilon}}{3\epsilon})$$

(2.26)

We have not been adding the dilatation term in case of renormalization of $Z_{ab}$ renormalization because, by its usual definition, we have put in this amplitude only the terms which are produced by interactions. The trivial scaling factor in the renormalization of the operator $O_{ab}(x)$ will be supplied later when we shall be calculating the correlation function.

Inversely, in the first order:

$$g_{ab} = a^{-3\epsilon}(\tilde{g}_{ab} - 4\pi D^2(\tilde{g}^2)_{ab}\frac{1}{3\epsilon})$$

(2.27)

$$Z_{ab} = \tilde{Z}_{ab}(1 - 4\pi D^2 \tilde{g}_{ab}\frac{1}{3\epsilon})$$

(2.28)

Substituting these expressions in the r.h.s. of (2.24), and keeping terms up to second order in $g_{ab}$, one obtains:

$$\frac{d\tilde{Z}_{ab}}{d\xi} = \tilde{Z}_{ab}\{4\pi D^2 \tilde{g}_{ab} + \frac{16\pi^2}{3}D^2 K(\tilde{g}^2)_{ab} - \frac{\pi^2}{3}(\tilde{g}_{ab})^2\}$$

(2.29)

This is the RG equation for the amplitude $\tilde{Z}_{ab}$ that we were aiming at.
It is easy to check that in the replica symmetric case:

\[ g_{ab} \rightarrow g \]  
\[ (2.30) \]

\[ Z_{ab} \sigma_a \sigma_b \rightarrow Z \sigma_a \sigma_b \]  
\[ (2.31) \]

– for all \( a, b \) \((a \neq b)\), the RG equation (2.29) takes the form:

\[
\frac{d\tilde{Z}}{d\xi} = \tilde{Z}\left\{4\pi D^2 \tilde{g} + \frac{16\pi^2}{3} D^2 K \tilde{g}^2 (n - 2) - \frac{\pi^2}{3} \tilde{g}^2 \right\}
\]  
\[ (2.32) \]

Here \( n \) is the number of replicas, to be put equal to 0 eventually.

In the next Section we shall derive solutions for both equations, (2.29) and (2.32), in order to have predictions for both fixed points, symmetric and non-symmetric one.

### 3 Solution of the RG equation for the amplitude \( Z_{ab} \).

#### Correlation function \( \langle \sigma(0) \sigma(R) \rangle^2 \).

We shall drop tildes in \( \tilde{g}_{ab} \) and \( \tilde{Z}_{ab} \) in the following, as all the quantities that we shall use will be the renormalized ones. Also, to simplify somewhat the equations, we shall change the normalization of \( g_{ab} \):

\[ g_{ab} \rightarrow \frac{g_{ab}}{4\pi} \]  
\[ (3.1) \]

And we shall substitute the value of the operator algebra constant:

\[ D = D(\epsilon) \equiv D^{a,\epsilon}_\sigma(\epsilon) = \frac{1}{2} + O(\epsilon^2) \]  
\[ (3.2) \]

It is well known that for the Ising model \( D^{\sigma}_\sigma = 1/2 \). One could check that the correction is \( \sim \epsilon^2 \), by using the formulas derived in [8]. For our present calculation the \( \epsilon^2 \) correction is irrelevant, so we can put the Ising model value \( D = 1/2 \).

The equation for \( Z_{ab} \) (2.29) takes the following form:

\[
\frac{dZ_{ab}(\xi)}{d\xi} = Z_{ab}(\xi)\gamma_{ab}(\xi)
\]  
\[ (3.3) \]

\[
\gamma_{ab}(\xi) = \frac{1}{4}g_{ab}(\xi) + \frac{K}{12}(g^2(\xi))_{ab} - \frac{1}{48}(g_{ab}(\xi))^2
\]  
\[ (3.4) \]
We remind that the constant $K = 6 \log 2$. For the replica symmetric case we shall have:

$$\frac{dZ(\xi)}{d\xi} = Z(\xi)\gamma(\xi) \quad (3.5)$$

$$\gamma(\xi) = \frac{1}{4} g(\xi) + \frac{K}{12} g^2(\xi)(n - 2) - \frac{1}{48} g^2(\xi) \quad (3.6)$$

We are interested in the correlation function $<\sigma(0)\sigma(R)>^2$ which is expressed through replicas by eq.(2.1). Assuming the RG evolution from the lattice cut-off up to the scale $\sim R$, one gets:

$$<\sigma(0)\sigma(R)>^2 \sim \lim_{n \to 0} \frac{1}{2n(n - 1)} \sum_{a \neq b} \sum_{c \neq d} Z_{ab}(\xi_R)Z_{cd}(\xi_R) \frac{1}{R^{4\Delta_\sigma(0)}}$$

$$\times <\sigma_a(0)\sigma_b(0)\sigma_c(1)\sigma_d(1)> \quad (3.7)$$

Here $\xi_R = \log R$; $\Delta_\sigma^{(0)}$ is the unperturbed dimension of the spin operator in the Potts model. The factor $1/R^{2\Delta_\sigma^{(0)}}$ could have been a part of $Z_{ab}(\xi)$. But, unlike for $g_{ab}$, we have included in $Z_{ab}$ only the scaling effects due to interactions. In this case the factor due to trivial scaling of the unperturbed operator $\sigma$ has to be added separately.

Correlation of spins at the cut-off distance $a \sim 1$ is that of the unperturbed decoupled replica models: it is $\sim 1$ if pairs of spins have same replica indices and 0 otherwise:

$$<\sigma_a(0)\sigma_b(0)\sigma_c(1)\sigma_d(1)> \sim \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \quad (3.8)$$

One get from (3.7):

$$<\sigma(0)\sigma(R)>^2 \sim \lim_{n \to 0} \frac{1}{n(n - 1)} \sum_{a \neq b} (Z_{ab}(\xi_R))^2 \frac{1}{R^{4\Delta_\sigma^{(0)}}} \quad (3.9)$$

The matrix $Z_{ab}(\xi)$ is assumed to be symmetric in its indices.

To find $<\sigma(0)\sigma(R)>^2$, it remains to define $Z_{ab}(\xi)$. We shall do it first for the replica symmetric fixed point, which is simpler.

RS. In this case $Z_{ab} = Z$, $g_{ab} = g$, $a \neq b$. Equation (3.9) takes the form:

$$<\sigma(0)\sigma(R)>^2 \sim (Z(\xi_R))^2 \frac{1}{R^{4\Delta_\sigma^{(0)}}} \quad (3.10)$$

$Z(\xi)$ could be defined from the equations (3.5), (3.6). We shall be deriving correlations for the model which is assumed to be already at the fixed point, and not for the crossover.
behavior when the fixed point is approached. (In any case, for the replica symmetry
broken case we have solution for $g_{ab}$ for the fixed point only.) In this case we have to
substitute in eq.(3.6) the fixed point value of $g$. This has been derived in [1], up to
second order in $\epsilon$, and reproduced, by a somewhat different technique, in [2]. In the
normalization that we have chosen at the start of this section, eq.(3.1), the fixed point
data is given by:

$$g_* = \frac{3}{2} \epsilon + \frac{9}{4} \epsilon^2 + O(\epsilon^3)$$  \hfill (3.11)

From eqs.(3.5), (3.6), for $n = 0$, we obtain:

$$\gamma_* = \frac{1}{4} g_* - \left( \frac{K}{6} + \frac{1}{48} \right) g_*^2 = \frac{3}{8} \epsilon - \left( \frac{9}{4} \log 2 - \frac{33}{64} \right) \epsilon^2 + O(\epsilon^3)$$  \hfill (3.12)

$$Z(\xi_R) \sim \exp \{ \gamma_* \xi_R \} = (R)^{\gamma_*}$$  \hfill (3.13)

We have substituted $K = 6 \log 2$. From (3.10):

$$< \sigma(0) \sigma(R) >^2 \sim \frac{1}{(R)^{2 \Delta'_\sigma}}$$  \hfill (3.14)

with

$$\Delta'_{\sigma^2} = 2 \Delta'_{\sigma^0} - \gamma_*$$  \hfill (3.15)

By $\Delta'_{\sigma^2}$ we have denoted the scaling dimension of the operator

$$O_{ab}(x) = \sigma_a(x) \sigma_b(x), \ a \neq b$$  \hfill (3.16)

at the replica symmetric fixed point.

RSB. In the replica symmetry broken case all the matrices are assumed to be in
Parisi block-diagonal form. One passes to the continuous dependence on matrix indices
by using the following rules [6]:

$$g_{ab} \rightarrow g(t)$$  \hfill (3.17)

$$(g^2)_{aa} \rightarrow - \int_0^1 dt g^2(t)$$  \hfill (3.18)

$$(g^2)_{ab} \rightarrow -2g(t)\bar{g} - \int_0^t du (g(t) - g(u))^2$$  \hfill (3.19)

with

$$\bar{g} = \int_0^1 dt g(t)$$  \hfill (3.20)
and similar expressions for the matrices $Z_{ab}$ and $\gamma_{ab}$. The continuous parameter $t$ replaces indices, the way described in [6]. It varies originally in the range from 1 to $n$, natural for the matrix indices, but after the limit $n \to 0$ is taken the interval of values of $t$ becomes [0,1]. The expressions (3.17)- (3.19) are somewhat special since we assume that diagonal elements of $g_{ab}$ (and of $Z_{ab}$, $\gamma_{ab}$) are zero. Note also that there is no summation over the index $a$ in (3.18), and that for the Parisi matrices all the diagonal elements of the matrix $g^2$ are equal, $a$- independent.

Using these rules one gets for $\langle \sigma(0)\sigma(R) \rangle^2$ from (3.9):

$$<\sigma(0)\sigma(R)>^2 \sim \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{a=1}^{n} (Z^2(\xi_R))_{aa} \frac{1}{R^4\Delta_y^{(0)}}$$

$$= \lim_{n \to 0} \frac{1}{n-1} (Z^2(\xi_R))_{aa} \frac{1}{(R)^4\Delta_y^{(0)}}$$

$$= \int_0^1 dt Z^2(\xi_R, t) \frac{1}{R^4\Delta_y^{(0)}}$$

(3.21)

The equations (3.3) and (3.4) for $Z_{ab}$ and $\gamma_{ab}$ take on the following forms:

$$\frac{dZ(\xi, t)}{d\xi} = Z(\xi, t)\gamma(t)$$

(3.22)

$$\gamma(t) = \frac{1}{4} g(t) + \frac{K}{12}(-2g(t)\bar{g} - \int_0^t du (g(t) - g(u))^2) - \frac{1}{48} g^2(t)$$

(3.23)

We have assumed here that the model is at the fixed point, so that $g_{ab}$ and $\gamma_{ab}$ are $\xi$ independent. $g(t)$ is a fixed point function of $t$ only. It has been found in [3] and it has the following form:

$$g(t) = \begin{cases} \frac{1}{3} t, & 0 < t < t_1 \\ g_1, & t_1 < t < 1 \end{cases}$$

(3.24)

$$g_1 = \frac{3}{2}\epsilon + \frac{9}{2}\epsilon^2 + O(\epsilon^3)$$

(3.25)

$$t_1 = 3g_1$$

(3.26)

We notice that $g(t)$ has a linearly growing piece for $0 < t < t_1$, which makes it different from the replica symmetric solution:

$$g_{r.s.}(t) = \text{const} = g_*$$

(3.27)
\( g_* \) is given by eq.(3.11). We observe that \( g_* \) is different in its \( \epsilon^2 \) term from the constant part of \( g(t) \), \( g_1 \), eq.(3.25).

We notice also that the form of \( g(t) \) in (3.24) corresponds to the full replica symmetry breaking, in the terminology of the spin-glass theory [6].

\( \gamma(t) \) is defined by the eq.(3.23). It could still be simplified. First, from (3.20), (3.23):

\[
\bar{g} = \int_0^1 dt g(t) = g_1 - \frac{3}{2} g_1^2
\]

(3.28)

Keeping the second term of \( \bar{g} \) in the product \( g(t)\bar{g} \) in (3.23) would mean having terms \( \sim g_1^3 \sim \epsilon^3 \) in this equation, which exceeds the accuracy of our calculations. Second, the integral term \( \int_0^t du (g(t) - g(u))^2 \) is \( \sim g_1^3 \sim \epsilon^3 \). So it can also be dropped. Then \( \gamma(t) \) takes the following form:

\[
\gamma(t) \simeq \left( \frac{1}{4} - \frac{K}{6} g_1 \right) g(t) - \frac{1}{48} g^2(t) = a_1 g(t) - a_2 g^2(t)
\]

(3.29)

where

\[
a_1 = \frac{1}{4} - g_1 \log 2 \approx \frac{1}{4} - \frac{3}{2} \epsilon \log 2
\]

(3.30)

\[
a_2 = \frac{1}{48}
\]

(3.31)

From eq.(3.22)

\[
Z(\xi, t) \sim \exp \{ \xi \gamma(t) \}
\]

(3.32)

For the correlation function one obtains from (3.21):

\[
<\sigma(0)\sigma(R)>^2 \sim \int_0^1 dt Z^2(\xi_R, t) \frac{1}{R^{4\Delta_0}}
\]

\[
= \int_0^1 dt \exp \{2\xi_R \gamma(t)\} \frac{1}{R^{4\Delta_0}}
\]

(3.33)

Here \( \xi_R = \log R \). \( 2\xi_R \gamma(t) \) is typically small, \( < 1 \). This is the case for example for the 3-component Potts model and \( R \sim 10^3 \), the biggest lattice sizes accessible for numerical simulation experiments. In this case it is reasonable to calculate the integral in (3.33) by expanding the exponent. In this way one obtains:

\[
R^{4\Delta_0} <\sigma(0)\sigma(R)>^2 \sim \int_0^1 dt (1 + 2\gamma(t)\xi_R + 2\gamma^2(t)\xi_R^2 + \cdots)
\]

\[
= 1 + 2\gamma \xi_R + 2\gamma^2 \xi_R^2 + \cdots
\]

(3.34)
Here
\[ \bar{\gamma} = \int_0^1 dt \gamma(t) \simeq a_1 \bar{g} - a_2 \bar{g}^2 \]  
(3.35)
\[ \gamma^2 = \int_0^1 dt \gamma^2(t) \simeq a_1^2 \bar{g}^2 - 2a_1a_2 \bar{g}^3 \]  
(3.36)

We remind that the precision of our calculations allows us to keep only the two first terms in \( \bar{\gamma} \) or in \( \gamma^2 \). For \( \bar{g}, \bar{g}^2, \bar{g}^3 \) one finds:
\[ \bar{g} \simeq g_1 - \frac{3}{2} g_1^2 \]  
(3.37)
\[ \bar{g}^2 \simeq g_1^2 - 2g_1^3 \]  
(3.38)
\[ \bar{g}^3 \simeq g_1^3 - \frac{9}{4} g_1^4 \]  
(3.39)

From (3.35), (3.36), (3.30)–(3.31), (3.37)–(3.39) one obtains:
\[ \bar{\gamma} \simeq \frac{1}{4} g_1 - (\log 2 + \frac{19}{48}) g_1^2 \]  
(3.40)
\[ \gamma^2 \simeq \frac{1}{16} g_1^2 - \frac{1}{2} \log 2 - \frac{13}{96} g_1^3 \]  
(3.41)

The expression for the correlation function (3.34) could be given, more conveniently, in the form:
\[ \log R^{4\Delta_s^{(0)}} < \sigma(0)\sigma(R) >^2 = \text{const} + 2\bar{\gamma} \xi_R + 2(\bar{\gamma}^2 - (\bar{\gamma})^2) \xi_R^2 + \cdots \]  
(3.42)

The term \( \sim \xi_R \) gives correction to the effective scaling dimension of the operator
\[ O_{ab}(x) = \sigma_a(x)\sigma_b(x), \ a \neq b \]  
(3.43)
\[ \Delta'' = 2\Delta^{(0)} - \bar{\gamma} \]  
(3.44)

The term \( \sim \xi_R^2 \) in (3.42) corresponds to the deviation from the usual scaling form of the correlation function. Its coefficient is given by:
\[ 2(\bar{\gamma}^2 - (\bar{\gamma})^2) \simeq \frac{1}{8} g_1^3 = \frac{27}{64} \epsilon^3 + O(\epsilon^4) \]  
(3.45)
In numerical simulation experiment it is more appropriate to measure the corresponding "magnetization", instead of the correlation function. The "magnetization" in the present case will be the expectation value of the operator \( \frac{1}{n} \sum_{a \neq b} O_{ab} \equiv \frac{1}{n} \sum_{a \neq b} \sigma_a \sigma_b \), i.e. \( \frac{1}{n} \langle \sum_{a \neq b} O_{ab} \rangle \). This quantity is to be measured for a finite lattice, of size \( L \times L \), at the critical point of an infinite lattice, and then the finite size scaling analysis is to be applied.

In the theory one obtains \( \langle \sum_{a \neq b} O_{ab} \rangle_L \), for a finite lattice of size \( L \), from the correlation function \( \langle \sum_{a \neq b} O_{ab}(0) \times \sum_{c \neq d} O_{cd}(R) \rangle \propto \langle \sigma(0) \sigma(R) \rangle^2 \) defined on an infinite lattice, by putting \( R = L \) and taking a square root. In this way one finds, in the RSB case:

\[
\log \{ L^2 \Delta_0^2 \frac{1}{n} \langle \sum_{a \neq b} O_{ab}(0) \rangle_L \} = \log \{ L^2 \Delta_0^2 \frac{1}{n} \sum_{a \neq b} \langle \sigma_a(0) \rangle_L \langle \sigma_b(0) \rangle_L \} = \text{const} + \gamma \xi + (\gamma_2 - \gamma_1) \xi^2 + \cdots \quad (3.46)
\]

The operator \( O_{ab}(x) = \sigma_a(x) \sigma_b(x) \), in the theory, is a product of local spins for different replicas. In numerical simulations \( O_{ab}(x) \) will be a local product of spins for two copies of the same random lattice, simulated with different initial conditions for spins. Two identical disordered lattices, different starting conditions, they realize in numerical experiment two replicas of the theory. We shall have first

\[
\langle O_{ab}(x) \rangle_L = \langle \sigma_a(x) \rangle_L \langle \sigma_b(x) \rangle_L \quad (3.47)
\]

for a given disorder, and then the average over the disorder, is to be taken. According to the replica theory of spin-glasses [6] in the situation when the replica symmetry is broken, the space of states is divided into many "valleys" corresponding to different ground states, and the summation over replica indices in the thermally averaged quantities corresponds to the summation over all these "valleys". Since different initial conditions, in general, correspond to different ground states, the summation over various samples (for averaging over the disorder) with different initial conditions must correspond to the summation over the indices \( a, b \) in the quantity (3.47).
In the RS case, one obtains from eqs.(3.14), (3.15):

\[
\log\left\{ L^{2\Delta_0} \sum_{a \neq b} < O_{ab}(0) >_L \right\} = \log\left\{ L^{2\Delta_0} \sum_{a \neq b} < \sigma_a(0) >_L < \sigma_b(0) >_L \right\} = \text{const}_2 + \gamma_\epsilon \xi_L \tag{3.48}
\]

4 Analyses of the results. Distributions.

For the purpose of numerical simulation tests we shall give here the numbers for the coefficients, for the case of the 3-component Potts model, \( \epsilon = \frac{2}{15} \). One obtains:

\[
\gamma_\epsilon \simeq \frac{3}{8} \epsilon - \frac{9}{4} \log 2 - \frac{33}{64} \epsilon^2 \simeq 0.050 - 0.019 = 0.031 \tag{4.1}
\]

for the RS case, eqs.(3.12), (3.48).

\[
\bar{\gamma} \simeq \frac{3}{8} \epsilon - \frac{9}{4} \log 2 - \frac{15}{64} \epsilon^2 \simeq 0.050 - 0.024 = 0.026 \tag{4.2}
\]

\[
\bar{\gamma}^2 - (\bar{\gamma})^2 \simeq \frac{27}{128} \epsilon^3 = 0.0005 \tag{4.3}
\]

for the RSB case, eqs.(3.40), (3.45), (3.46).

Characteristic numbers to look at are the values of the products \( \bar{\gamma} \xi_L \), \( (\bar{\gamma}^2 - (\bar{\gamma})^2) \xi_L^2 \) in eq.(3.46), for \( L \) maximal in numerical simulations, \( L = 10^3 \). One gets,

\[
\bar{\gamma} \xi_L = 0.18 \tag{4.4}
\]

\[
(\bar{\gamma}^2 - (\bar{\gamma})^2) \xi_L^2 = 0.02 \tag{4.5}
\]

This gives an estimate of \( \sim 10\% \) on the deviation from scaling, in case of the 3-component Potts model. This is only an estimate as we could not know in advance the accuracy of the \( \epsilon \)-expansion calculation. In particular, the scaling violation term (4.5) might still be smaller. For this reason we have also done simulations for the 4-component Potts model. The results will be presented in the next Section. In the theory, the \( \epsilon \)-expansion values for the coefficients do not make sense for the 4-component model (\( \epsilon = \frac{1}{2} \)). In particular, the second term in the eq.(3.40) for \( \bar{\gamma} \) becomes bigger than the first one, and \( \bar{\gamma} \) becomes negative. On the other hand the model itself should evolve continuously with
the number of components \( q \), up to 4 and further. This is because the phase transition of the Potts model with random bonds remains second order for \( q > 4 \) \[9,10,11\]. Then we would expect that the effect of the deviation from scaling, if present, should become more pronounced as \( q \) is increased. So it should be easier observable for the 4-component model, compared to the 3-component one. Going from \( q = 3 \) to \( q = 4 \) we go out of the perturbative region, where the \( \epsilon \) expansion is valid, but the qualitative effect of scaling violation should increase, if the model is at the RSB fixed point.

In addition to the above results on the correlation function squared and the associated magnetization, one could also look at the corresponding distributions. This implies that after the calculation of the thermodynamic expectation values of \(< \sigma_a(x) >\) and \(< \sigma_b(x) >\), for two identical lattices, different starting conditions, one changes disorder and performs the thermodynamic measurement again, and gets another values of \(< \sigma_a(x) >, < \sigma_b(x) >\). Having done these measurements many times, one constructs the distribution for the values of products of local magnetizations

\[
< O_{ab}(x) >_L = < \sigma_a(x) >_L < \sigma_b(x) >_L \quad (4.6)
\]

This is instead of summing up the values for products and calculating in this way the average over the disorder. [We observe that at the stage of calculating either the distribution or the average over the disorder of the products \( (4.6) \) one could use also the values of local products for different points \( x \) on the lattice, if they have been measured. But we stress again that local products have to be taken first, summation over \( x \) second.]

The distribution obtained in this way, the “overlap function” of local magnetizations, could be obtained in the theory from the RG result for the amplitude

\[
Z(\xi_L, t) \sim e^{\gamma(t)\xi_L} \quad (4.7)
\]

It is more convenient to study the log of this function:

\[
Q(t) = \log Z(\xi_L, t) = \text{const} + \gamma(t)\xi_L \equiv Q_0 + \gamma(t)\xi_L \quad (4.8)
\]

which corresponds to

\[
Q_{ab} = \log \{(L)^{2\Delta^{(0)}} < \sigma_a(x) >_L < \sigma_b(x) >_L \} \quad (4.9)
\]
We shall define
\[ q(t) = Q(t) - Q_0 \] (4.10)

In analogy with the theory of spin-glasses, to obtain the distribution of values of \( Q_{ab} \) one has to define the inverse function of \( Q(t) \), or of \( q(t) \), and calculate its derivative. One gets:

\[
q(t) = \gamma(t) \xi_L \] (4.11)

\[
\gamma(t) = \begin{cases} 
\frac{t}{12}, & 0 < t < t_1 (= 3g_1) \\
\gamma_1, & t_1 < t < 1 
\end{cases} \] (4.12)

\[
\gamma_1 = a_1g_1 - a_2g_1^2 = \frac{1}{4}g_1 - (\log 2 + \frac{1}{48})g_1^2 = \frac{3}{8} - \left(\frac{9}{4}\log 2 - \frac{69}{64}\right)\epsilon^2 \] (4.13)

We have used equations (3.29)–(3.31) for \( \gamma(t) \) and we have been keeping the accuracy of our calculations by dropping extra terms. In particular, in the interval \( 0 < t < t_1 \) we can keep linear terms only. This is because the interval itself is \( \sim \epsilon \) and under integration the linear terms become quadratic, \( \sim \epsilon^2 \). One has:

\[
q(t) = \begin{cases} 
\frac{t}{12} \xi_L, & 0 < t < t_1 \\
q_1, & t_1 < t < 1 
\end{cases} \] (4.14)

\[
q_1 = \gamma_1 \xi_L \] (4.15)

In the interval \( 0 < q < q_1 \) the inverse function is given by:

\[
t(q) = \frac{12}{\xi_L}q, \quad 0 < q < q_1 \] (4.16)

Its derivative:

\[
\frac{dt(q)}{dq} = \frac{12}{\xi_L}, \quad 0 \leq q < q_1 \] (4.17)

Finally one gets the following distribution function:

\[
N(q) = \frac{dt}{dq} = \begin{cases} 
\frac{12}{\xi_L} + (1 - 3g_1)\delta(q - q_1), & 0 < q \leq q_1 \\
0, & q > q_1 
\end{cases} \] (4.18)

Numerical measurement of the distribution \( N(q) \) might be complicated. For the product of magnetizations on finite lattices of size \( L \), eq.(4.9), the distribution \( N(q) \), will
actually be rounded by finite size effects. To have it more distinct one have to look for the limit of big $L$, but then the extra structure in $N(q)$, for $0 < q < q_1$, would become lower and lower, being of the height $\sim \frac{1}{\xi L}$.

The way out could be to look at the distribution of a products of correlation functions themselves:

$$Q_{ab}^{(2)}(R) = \log \{(R)^{4\Delta_{(0)}} < \sigma_a(0)\sigma_a(R) >_L < \sigma_b(0)\sigma_b(R) >_L \}$$

(4.19)

The index (2) of $Q_{ab}^{(2)}$ is meant to tell that we are looking at the overlap function of a two-point object. The above considered $Q_{ab}$ for magnetizations could have been noted $Q_{ab}^{(1)}$.

In the theory, the corresponding $Q^{(2)}(R, t)$ shall be given by $Z^2(\xi R, t)$:

$$Q^{(2)}(R, t) = \log Z^2(\xi R, t)$$

(4.20)

$\xi R = \log R$, and it is assumed that $1 \ll R \ll L$, $L$ being the size of the lattice. $R$ has to be big enough so that the continuum limit theory applies and the fixed point is reached. In an analogous way one finds in this case

$$N^{(2)}(q) = \frac{dt}{dq} = \begin{cases} \frac{6}{\xi R} + (1 - 3g_1)\delta(q - q_1), & 0 < q \leq q_1 \\ 0, & q > q_1 \end{cases}$$

(4.21)

Now, as we increase $L$, the profile of $N^{(2)}(q)$ will be sharpened while the hight of the extra structure $\sim \frac{1}{\xi R}$ remain unchanged.

5 Simulations.

In this section, we are going to present some results of numerical simulations that we performed in order to check the validity of our results. In particular, we want to try to find a way to choose between the two possibles scenarios, replica breaking or replica symmetry. The easiest thing that we can simulate, as explained earlier, is the scaling dimension of the square magnetization. Thus we have performed the following simulations: On a square lattice of size $L \times L$, we simulate two configurations ($\sigma^a$ and $\sigma^b$) of the $q$-state Potts
model with the same disorder, but with different initial conditions. Then we compute the product of the magnetization

\[ Q_{ab} = \frac{1}{L^2} \sum_{i=1,L^2} <\sigma_i^a><\sigma_i^b> \]  

(5.1)

Here \(<\sigma_i^a>\) means the thermal average of the local magnetization

\[ \sigma_i^a = \vec{\sigma}_i^a \cdot \vec{m}^a \]  

(5.2)

and \(\vec{m}^a\) is the total magnetization

\[ \vec{m}^a = \frac{1}{L^2} \sum_{i=1,L^2} \vec{\sigma}_i^a \]  

(5.3)

In practice, as we have checked numerically, it turns out that this quantity is the same as

\[ Q_{ab} = \frac{1}{L^2} \sum_{i=1,L^2} <\sigma_i^a \sigma_i^b> \]  

(5.4)

The Hamiltonian of the simulated model is given by

\[ H = -\sum_{\{i,j\}} J_{ij} (\delta_{\sigma_i^a,\sigma_j^a} + \delta_{\sigma_i^b,\sigma_j^b}) \]  

(5.5)

where the coupling constant between nearest neighbor spins takes the value

\[ J_{ij} = \begin{cases} J_0 & \text{with probability } p \\ J_1 & \text{with probability } 1 - p \end{cases} \]

Measurements were performed on a square lattice with helical boundary conditions. Without any lost of generality, we can consider the case where \(p = \frac{1}{2}\). Then the model is self-dual and thus the critical temperature is exactly known. It is given by the solution of the equation \([13]\)

\[ \frac{1 - e^{-\beta J_0}}{1 + (q-1)e^{-\beta J_0}} = e^{-\beta J_1}. \]  

(5.6)

The disorder that we choose to simulate is \(J_0 = 1, J_1 = \frac{1}{10}\). This disorder is in fact quit strong in order to avoid problems of cross-over \([4]\). Monte Carlo data were obtained by using the well known Wolff cluster algorithm \([15]\). Due to the strong disorder that we considered, we needed to have large statistics over the number of configurations of
disorder. Simulations were performed for lattice with size ranging from $L = 10$ to $L = 1000$. The number of configurations of disorder were 20000 for $L = 20 - 200$, 6000 for $L = 500$ and 1000 for $L = 1000$. For each of these configuration of disorder, measurements were taken over $t_1$ updates, after $t_0$ updates for thermalisation. The statistical error $\delta A$ of a quantity $A$ has two contributions, one from the thermal fluctuation, with a variance $\sigma_T$, and one from the disorder fluctuation, with a variance $\sigma_N$. Thus the statistical error is given by

$$ (\delta A)^2 = \frac{\sigma_N^2}{N} + \frac{\sigma_T^2}{N t_1 / \tau} \quad (5.7) $$

where $N$ is the number of configurations of disorder and $\tau$ is the autocorrelation time.

For the quantity that we measured, it turns out that the two variance are near equal and then

$$ (\delta A)^2 \simeq \frac{\sigma_N^2}{N} (1 + \frac{\tau}{t_1}) \quad (5.8) $$

Thus we just need to choose $t_1$ such that $t_1 \gg \tau$ and we can ignore the thermal fluctuations. Then the first step is to measure the autocorrelation time. For the 3-state Potts model, we measured $\tau \simeq 3$ for $L=10$ up to $\tau \simeq 25$ for $L = 1000$. Thus by choosing $t_0 = t_1 = 1000$, we can safely ignore the thermal fluctuations. For the 4-state Potts model, we measured $\tau \simeq 5$ for $L=10$ up to $\tau \simeq 50$ for $L = 1000$. Again, we choosed the parameters $t_0 = t_1 = 1000$ except for $L = 1000$ for which we took $t_0 = t_1 = 2000$ and this in order to be sure to have thermalized data.

Results of these measurements for the 3-state Potts model are displayed in Fig. 1.

In this figure, we plot $\ln(L^{2\Delta_r Q_{ab}})$ versus $\ln(L)$. Here, $\langle \cdots \rangle$ means the average over the disorder. As explained in section 3, we expected two possible behaviors for this quantity, either $\ln(L^{2\Delta_r Q_{ab}}) \simeq \text{const}_1 + \gamma_4 \ln(L) + \cdots$ in the RS scenario (see eq. (3.48)) or $\ln(L^{2\Delta_r Q_{ab}}) \simeq \text{const}_1 + \gamma \ln(L) + (\gamma^2 - \gamma^2) \ln^2(L) + \cdots$ in the RSB scenario (see eq. (3.46)). According to the RS scenario, we would expect a scaling behavior i.e. a linear (Log-Log) plot, which is in very good agreement with what we obtain. Moreover, we estimate that, for the RSB scenario, the deviation from such a linear behavior due to the $\ln^2(L)$ term should be of order 10% at $L = 1000$. We do not see such a deviation. Thus our numerical simulation of the 3-state Potts model does clearly favor the replica
Figure 1: Plot of $\ln(L^{2\Delta\sigma Q_{ab}})$ vs. $\ln(L)$ for the 3-state Potts model.
symmetry solution over the replica symmetry breaking one.

Performing a fit of the plot

\[ L^{2\Delta_*} Q_{ab} \simeq L^{\gamma_*} \]  

we obtain a value of \( \gamma_* = 0.04 \pm 0.002 \), which is reasonably close to the predicted value \( \gamma_* = 0.031 \) (see eq. (4.1)).

Fig. 2. corresponds to the same plot but for the 4-state Potts model. As explained in section 4, we do not expect that the \( \epsilon \)-expansion is still valid for \( q = 4 \), but we would still expect to see a deviation from scaling due to the RSB and we expect that this deviation is more pronounced as we increase \( q \). But again, we do not see a deviation from a scaling behavior, thus again not seeing any evidence in favor of the replica symmetry breaking scenario. Performing a fit of the plot with eq.(5.9), we do obtain here \( \gamma_* = 0.023 \pm 0.002 \).

6 Conclusions and Discussions.

We consider that the results of numerical simulations presented in the preceding Section support the RS fixed point critical behavior. Neither for the 3-component nor for the 4-component models could we detect the deviation from scaling, characteristic of the RSB fixed point. In case of the 3-component model, for which the \( \epsilon \)-expansion could be expected to be reasonably well defined, the value of the slope of the curve for the scaling function (3.48) agrees sufficiently well with the value of \( \gamma_* \) in eq.(4.1).

Still we would like to make a remark on possible physical significance of RSB fixed point, so that it would not appear totally formal.

The coupling constant \( g_{ab} \) in (1.13) is proportional to the overlap function of local energies, in a similar way as \( Z_{ab} \) is proportional to the overlap function of spins. Having \( g_{ab} \) different for different \( a, b \), as it is the case for the RSB fixed point, could be attributed to a multiplicity of ground states in the statistical model. One starts with different initial conditions for spins (one breaks initially equivalence of replicas in this way) and the model can end in different ground states. In turn, multiplicity of ground states could be
Figure 2: Plot of $\ln(L^{2\Delta r Q_{ab}})$ vs. $\ln(L)$ for the 4-state Potts model.
interpreted as a kind of localization phenomenon, in the configurational space of spins. Unlike in spin-glasses, in the present model with relatively weak disorder the localization would be due to fluctuations, which make disorder important at large distances. It is this spontaneous phenomenon that we had in mind behind the notion of RSB in the critical model with disorder.

We would also like to remark that one of the possibilities, why the RSB phenomena have not been detected in the particular models considered, could be due to the so called "marginal stability" of the RSB fixed point: it is well known that in the linear order stability analysis of the RSB fixed point (3.24) there exists the so called "zero-mode" with the zero eigenvalue. The detailed second order calculations of the stability of this fixed point shows that to enter the critical regime defined by the RSB fixed point (3.24), in general one needs to reach exponentially large spatial scales [16], which could be well beyond the lattice sizes of the present numerical tests.

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A Appendix
A.1 The integral of $D_1^{(2)}$, eq.(2.17).

$$I_1 = \int d^2y \int d^2y' \frac{1}{|y|^\Delta \epsilon} \frac{1}{|y'|^\Delta \epsilon} \frac{1}{|y-y'|^{2\Delta \epsilon}}$$  \hspace{1cm} (A.1)

Here $\Delta \epsilon = 1 - \frac{3}{2} \epsilon$. We change the variable $y' : y' = yt$. This gives:

$$I_1 = \int d^2y|y|^{-4\Delta \epsilon} \int d^2t|t|^{-\Delta \epsilon}|1-t|^{-2\Delta \epsilon}$$

$$= \int_{1<|y|<\alpha} d^2y|y|^{-2+6\epsilon} \int d^2t|t|^{2\alpha}|1-t|^{2\beta}$$

$$= 2\pi \frac{1}{6\epsilon} (a^{6\epsilon} - 1) \pi \Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(-1-\alpha-\beta)$$

$$\frac{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(2+\alpha+\beta)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(2+\alpha+\beta)}$$  \hspace{1cm} (A.2)

Here $\alpha = -\frac{\Delta \epsilon}{2} = -\frac{1}{2} + \frac{3}{4} \epsilon$, $\beta = -\Delta \epsilon = -1 + \frac{3}{2} \epsilon$, and we have used the known result:

$$\int d^2t|t|^{2\alpha}|1-t|^{2\beta} = \pi \Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(-1-\alpha-\beta)$$

$$\frac{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(2+\alpha+\beta)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(2+\alpha+\beta)}$$  \hspace{1cm} (A.3)

For its derivation see e.g.[12]. Putting the values of $\alpha, \beta$ and expanding in $\epsilon$ one obtains:

$$\frac{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(-1-\alpha-\beta)}{\Gamma(-\alpha)\Gamma(-\beta)\Gamma(2+\alpha+\beta)} \approx \frac{2}{3\epsilon} (1 + 3\epsilon (\psi(1) - \psi(\frac{1}{2})))$$

$$= \frac{2}{3\epsilon} (1 + \epsilon 6 \log 2)$$  \hspace{1cm} (A.4)

Finally one gets:

$$I_1 = 2\pi^2 (1 + \epsilon K) \frac{1}{9\epsilon^2} (a^{6\epsilon} - 1)$$  \hspace{1cm} (A.5)

$K = 6 \log 2$.

A.2 The integral of $D_2^{(2)}$, eq.(2.19).

$$I_2 = \int d^2y \int d^2y' <\sigma(0)\varepsilon(y)\varepsilon(y')\sigma(\infty)> <\varepsilon(y)\varepsilon(y')>$$  \hspace{1cm} (A.6)

This integral has already appeared in the calculations in the paper [2]. It is finite, the limit of $\epsilon \to 0$. This could also be seen without calculations, by using the operator algebra. In fact, when $y \to 0$ one has:

$$\sigma(0)\varepsilon(y) = \frac{D}{|y|^\Delta \epsilon} \sigma(0) + \cdots$$  \hspace{1cm} (A.7)

As $\Delta \epsilon \approx 1$, the integration over $y$ around 0 is finite.
When \( y, y' \to 0, \ y' \gg y \), one has:
\[
\sigma(0) \varepsilon(y) \varepsilon(y') = \frac{D^2}{|y|^{\Delta_e} |y'|^{\Delta_e}} \sigma(0) + \cdots
\]  
(A.8)
Again, the integration around 0 is finite.

Finally, for the configuration of \( y' \to y \) one obtains:
\[
\varepsilon(y) \varepsilon(y') = \frac{1}{|y-y'|^{2\Delta_e}} (1 + \cdots + |y-y'|^4 \bar{T}(\bar{y}) + \cdots) + \cdots
\]  
(A.9)
\( T(y), \bar{T}(\bar{y}) \) are components of the energy-momentum operator. In the OPE (A.9) the term \( \sim \bar{T} \bar{T} \) is the first after 1 which contributes when integrated in (A.6). Some extra terms, which vanish under integration, like \( (y-y')^2 T(y) \) or \( (\bar{y}-\bar{y}')^2 \bar{T}(\bar{y}) \) have been dropped. Putting \( \Delta_e \approx 1 \), one gets
\[
< \sigma(0) \varepsilon(y) \varepsilon(y') \sigma(\infty) > < \varepsilon(y) \varepsilon(y') > = < \sigma(0) \sigma(\infty) > \frac{1}{|y-y'|^4} (1 + |y-y'|^4 < \sigma(0) T(y) \bar{T}(\bar{y}) \sigma(\infty) > + \cdots) + \cdots
\]  
(A.10)
The first term, \( \sim 1/|y-y'|^4 \), leads to quadratic divergence in the integral (A.6). It is automatically subtracted in the analytic \( \epsilon \)-expansion calculation. It corresponds, in fact, to the constant shift renormalization of the action of the theory which is irrelevant. The next term is finite, when \( y' \to y \). So the integral (A.6) will be finite.

A.3 The integral of \( D_3^{(2)} \), eq.(2.21).
\[
I_3 = \int d^2 y' \int d^2 y (< \sigma(0) \varepsilon(y) \varepsilon(y') \sigma(\infty) >)^2
\]  
(A.11)
The variable \( y' \) could be scaled out by using invariance of correlation functions w.r.t. global dilatation. In general, when a set of operators \( O_1, O_2, \cdots, O_n \) is projected on the operator \( O_{n+1} \), placed at infinity, one has:
\[
< O_1(x_1) O_2(x_2) \cdots O_n(x_n) O_{n+1}(\infty) >
= \lambda^{\Delta_1+\Delta_2+\cdots+\Delta_n-\Delta_{n+1}} < O_1(\lambda x_1) O_2(\lambda x_2) \cdots O_n(\lambda x_n) O_{n+1}(\infty) >
\]  
(A.12)
where \( \lambda \) is a dilatation parameter. For \( < \sigma(0) \varepsilon(y) \varepsilon(y') \sigma(\infty) > \) this gives:
\[
< \sigma(0) \varepsilon(y) \varepsilon(y') \sigma(\infty) > = \lambda^{2\Delta_e} < \sigma(0) \varepsilon(\lambda y) \varepsilon(\lambda y') \sigma(\infty) >
\]  
(A.13)
Putting $\lambda = 1/y'$, one obtains:

$$<\sigma(0)\varepsilon(y)\varepsilon(y')\sigma(\infty)> = |y|^{-2\Delta} <\sigma(0)\varepsilon(\frac{y}{y'})\varepsilon(1)\sigma(\infty)>$$  \hspace{1cm} (A.14)

Using this relation in the integral (A.11) and changing the variable $y, y' = y'y$, one obtains:

$$I_3 = \int d^2 y'|y'|^{-2-4\Delta} \int d^2 y(\varepsilon(0)\varepsilon(\bar{y})\varepsilon(1)\sigma(\infty) >)^2$$  \hspace{1cm} (A.15)

Putting $\Delta_\varepsilon = 1 - \frac{3}{2}\varepsilon$ and integration $y'$ between the cut-offs $1, a$, one gets:

$$I_3 = \int_{1<|y'|<a} d^2 y'|y|^{-2+6\varepsilon} \bar{I}_3 = 2\pi \frac{1}{6\varepsilon}(a^{6\varepsilon} - 1)\bar{I}_3$$  \hspace{1cm} (A.16)

$$\bar{I}_3 = \int d^2 y(\varepsilon(0)\varepsilon(\bar{y})\varepsilon(1)\sigma(\infty) >)^2$$  \hspace{1cm} (A.17)

In the Coulomb gas representation the correlation function $<\sigma\varepsilon\varepsilon\sigma>$ could be presented in the following form:

$$<\sigma(0)\varepsilon(y)\varepsilon(1)\sigma(\infty)>
= N \int d^2 u <V_{\bar{\alpha}_{\sigma}}(0)V_{\alpha_{\varepsilon}}(y)V_{\alpha_{\varepsilon}}(1)V_{\alpha_{\sigma}}(u)V_{\alpha_{\sigma}}(\infty)>
= N \int d^2 u |y|^{4\alpha_{\sigma}\alpha_{\varepsilon}}|y - 1|^{4\alpha_{\varepsilon}}|u|^{4\alpha_{\sigma}\alpha_{\varepsilon}}|y - u|^{4\alpha_{\sigma}\alpha_{\varepsilon}}|1 - u|^{4\alpha_{\sigma}\alpha_{\varepsilon}}$$  \hspace{1cm} (A.18)

Here $N$ is the normalization constant. It can be fixed by using the operator algebra:

$$r \rightarrow 0, \quad \sigma(0)\sigma(r) = \frac{1}{|r|^{2\Delta}} + \cdots$$  \hspace{1cm} (A.19)

$$r \rightarrow 0, \quad V_{\bar{\alpha}_{\sigma}}(0)V_{\alpha_{\varepsilon}}(r) = \frac{1}{|r|^{2\Delta}}V_{2\alpha_{\varepsilon}}(0) + \cdots$$  \hspace{1cm} (A.20)

For the correlation function

$$<\sigma(0)\sigma(r)\varepsilon(y)\varepsilon(y')>
= N \int d^2 u <V_{\bar{\alpha}_{\sigma}}(0)V_{\alpha_{\varepsilon}}(r)V_{\alpha_{\varepsilon}}(y)V_{\alpha_{\varepsilon}}(y')V_{\alpha_{\sigma}}(u)>$$  \hspace{1cm} (A.21)

This gives, in the limit of $r \rightarrow 0$:

$$\frac{1}{|r|^{2\Delta}} <\varepsilon(y)\varepsilon(y')> = N \frac{1}{|r|^{2\Delta}} \int d^2 u <V_{2\alpha_{\varepsilon}}(0)V_{\alpha_{\varepsilon}}(y)V_{\alpha_{\varepsilon}}(y')V_{\alpha_{\sigma}}(u)>$$  \hspace{1cm} (A.22)

$$\frac{1}{|y - y'|^{2\Delta}} = N |y|^{8\alpha_{\sigma}\alpha_{\varepsilon}}|y'|^{8\alpha_{\sigma}\alpha_{\varepsilon}}|y - y'|^{4\alpha_{\varepsilon}}
\times \int d^2 u |u|^{8\alpha_{\sigma}\alpha_{\varepsilon}}|y - u|^{4\alpha_{\sigma}\alpha_{\varepsilon}}|y - y'|^{4\alpha_{\sigma}\alpha_{\varepsilon}}$$  \hspace{1cm} (A.23)
Changing the variable $u$ in the integral:

$$u = \frac{yy'}{y-y'} \times \frac{1}{\tilde{u} + \frac{y'}{y-y'}}$$

(A.24)

one obtains, after simple algebra:

$$\frac{1}{|y-y'|^{2\Delta_c}} = N \frac{1}{|y-y'|^{2\Delta_c}} \int d^2\tilde{u} |\tilde{u}|^{-2\alpha_+^2} |\tilde{u} - 1|^{-2\alpha_-^2}$$

(A.25)

Here $2\Delta_c = 4\Delta_1 - 8\alpha_2\alpha_0$. Using $\alpha_2 = -\alpha_-/2$, $2\alpha_0 = \alpha_+ + \alpha_-$, $\alpha_+\alpha_- = 1$, one gets also $2\Delta_c = -2 + 3\alpha_+^2$. From (A.25):

$$N = \left( \int d^2u |u|^{-2\alpha_+^2} |u - 1|^{-2\alpha_-^2} \right)^{-1}$$

(A.26)

Here $\alpha_+^2 = \frac{4}{3} - \epsilon$. One obtains:

$$N = \pi^{-1} \frac{\Gamma^2(\alpha_+^2)\Gamma(2 - 2\alpha_+^2)}{\Gamma^2(1 - \alpha_+^2)\Gamma(-1 + 2\alpha_+^2)} = \pi^{-1} \frac{\Gamma^2(\frac{4}{3} - \epsilon)\Gamma(-\frac{2}{3} + 2\epsilon)}{\Gamma^2(-\frac{1}{3} + \epsilon)\Gamma(\frac{2}{3} - 2\epsilon)}$$

(A.27)

We have used again the integral (A.3). In our further calculations the normalization constant $N$ will multiply expressions with singularities $\sim 1/\epsilon$, but not $1/\epsilon^2$. This implies that we don’t need to keep the $\epsilon$ correction of $N$. For $\epsilon = 0$, and using the usual properties of $\Gamma$-functions, one finds from (A.27):

$$N = -\frac{2\Gamma^3(-\frac{2}{3})}{9\pi \Gamma^3(-\frac{1}{3})} = -\frac{2}{9 \sin(\pi(-\frac{2}{3}))} \frac{\Gamma^2(-\frac{2}{3})}{\Gamma^2(-\frac{1}{3})\Gamma(\frac{2}{3} - 2\epsilon)}$$

(A.28)

or

$$N = -\frac{2}{\sqrt{3}} \gamma^{-2}, \quad \gamma = \frac{\Gamma^2(-\frac{1}{3})}{\Gamma(-\frac{2}{3})}$$

(A.29)

Returning to the integral $\tilde{I}_3$ in (A.17) and using the expression (A.18) for the correlation function $<\sigma\varepsilon\varepsilon\sigma>$, one gets:

$$\tilde{I}_3 = N^2 I$$

(A.30)

$$I = \int d^2y \int d^2u_1 \int d^2u_2 |y|^{4\alpha_2\alpha_+} |y - 1|^{4\alpha_-^2} |u_1|^{4\alpha_2\alpha_+} |u_1 - 1|^{4\alpha_-^2} |u_1 - y|^{4\alpha_+\alpha_-}$$

$$\times |u_2|^{4\alpha_2\alpha_+} |u_2 - 1|^{4\alpha_+\alpha_-} |u_2 - y|^{4\alpha_+\alpha_-}$$

(A.31)
The integral $I$ is calculated in the Appendix B, with the result:

$$I = -\frac{\pi}{16} \gamma^4 + O(\epsilon) \quad (A.32)$$

$\gamma$ is defined in (A.29). For $\tilde{I}_3$ one gets:

$$\tilde{I}_3 = N^2 I = -\frac{\pi}{12} + O(\epsilon) \quad (A.33)$$

and then for the integral $I_3$, (A.11), one obtains from (A.16):

$$I_3 = \cdots - \frac{\pi^2}{6} \frac{1}{6\epsilon} (a^{6\epsilon} - 1) + \cdots \quad (A.34)$$

This expression is in fact incomplete. It has to be corrected. There is an extra term in it, $\sim 1/\epsilon^2$, which is missed in the analytic technique of calculating the integral $I$ in the Appendix B. It could be recovered by using the original expression for the integral $I_3$ in the eq. (A.11) and the operator algebra. Configuration which is responsible for the leading singularity in $I_3$ is either

$$1 \ll |y| \ll |y'|, \quad y, y' \to 0 \quad (A.35)$$

or

$$1 \ll |y'| \ll |y|, \quad y, y' \to 0 \quad (A.36)$$

For (A.35) one has, by operator algebra:

$$\sigma(0)\varepsilon(y)\varepsilon(y') = \frac{D}{|y|^{\Delta_\epsilon}} \sigma(0)\varepsilon(y') + \cdots$$

$$= \frac{D}{|y|^{\Delta_\epsilon}} \frac{D}{|y'|^{\Delta_\epsilon}} \sigma(0) + \cdots \quad (A.37)$$

$D = D_{\sigma\varepsilon}^{\sigma\varepsilon}$ is the OA coefficient. For the integral $I_3$, this gives:

$$\int d^2y' \int d^2y (<\sigma(0)\varepsilon(y)\varepsilon(y')\sigma(\infty)>)^2$$

$$= \int_{1<|y'|<a} 2\pi |y'| |y'| d|y'| \int_{1<|y|<|y'|} 2\pi |y| |d|y| |y|^{4} \frac{D^4}{|y|^{2\Delta_\epsilon}|y'|^{2\Delta_\epsilon}} + \cdots$$

$$= 4\pi^2 D^4 \int_{1<|y'|<a} d|y'||y'|^{-1+3\epsilon} \int_{1<|y|<|y'|} d|y||y|^{-1+3\epsilon} + \cdots$$

$$= 4\pi^2 D^4 \int_{1<|y'|<a} d|y'|^{-1+3\epsilon} \frac{1}{3\epsilon} |y'|^{3\epsilon} + \cdots$$

$$= 2\pi^2 D^2 \frac{1}{9\epsilon^2} a^{6\epsilon} + \cdots \quad (A.38)$$
We have used: $< \sigma(0) \sigma(\infty) > = 1$, $2 \Delta_\varepsilon = 2 - 3 \varepsilon$, and we have kept the leading term when integrating: $|y'|^{3\varepsilon} - 1 \approx |y'|^{3\varepsilon}$, $a^{6\varepsilon} - 1 \approx a^{6\varepsilon}$.

Adding to (A.34) the $1/\varepsilon^2$ piece in (A.36), multiplied by two because of two equivalent configurations, (A.35) and (A.36), one finally obtains:

$$I_3 = (4 \pi^2 D^4 \frac{1}{9 \varepsilon^2} - \frac{\pi^2}{6 \varepsilon}) a^{6\varepsilon} + O(1) \quad (A.39)$$

The leading singularity, $\sim 1/\varepsilon^2$, is missed in the calculation of the integral $I$ in the Appendix B for the following reason. When $I_3$ is put in the form of (A.16), (A.17), the factor $1/\varepsilon$ is already in front, $1/\varepsilon^2$ singularity of $I_3$ would be produced by $1/\varepsilon$ singularity of $\tilde{I}_3$, or $I$, eq.(A.31). Instead we find the result (A.32) for $I$, $I \sim 1$. In the integral $\tilde{I}_3$, (A.17), or in $I$, (A.31), the integration over $y$ is performed over the whole infinite plane. This is correct to define the finite piece of the integral, but the singularity $1/\varepsilon$ get cancelled in this way.

In fact, let us consider the integral $\tilde{I}_3$, (A.17). There are two configurations which lead to $1/\varepsilon$ singularity in the integral (A.17):

$$0 < |y| \ll 1 \quad (A.40)$$

and

$$1 \ll |y| < \infty \quad (A.41)$$

For the first one we use:

$$\sigma(0) \varepsilon(y) = \frac{D}{|y|^{\Delta_\varepsilon}} \sigma(0) + \cdots \quad (A.42)$$

and we get:

$$< \sigma(0) \varepsilon(y) \varepsilon(1) \sigma(\infty) > = \frac{D}{|y|^{\Delta_\varepsilon}} < \sigma(0) \varepsilon(1) \sigma(\infty) > = \frac{D^2}{|y|^{\Delta_\varepsilon}} \quad (A.43)$$

We have used $< \sigma(0) \varepsilon(1) \sigma(\infty) > = D$. For the second configuration, (A.41), we use:

$$\sigma(0) \varepsilon(1) = D \sigma(0) + \cdots \quad (A.44)$$

and we get:

$$< \sigma(0) \varepsilon(y) \varepsilon(1) \sigma(\infty) > = < \sigma(0) \varepsilon(1) \varepsilon(y) \sigma(\infty) >$$
In the last line we have used the scaling properties of the function \( < \sigma(0) \varepsilon(y) \sigma(\infty) > \).

In the integral \( \tilde{I}_3 \), (A.17), the above two configurations will provide the following contribution:

\[
\tilde{I}_3 = \int_{0 < |y| \ll 1} d^2 y \left( \frac{D^4}{|y|^{2\Delta_\epsilon}} \right) + \int_{1 < |y| < \infty} d^2 y \frac{D^4}{|y|^{2\Delta_\epsilon}} + \cdots
\]

\[
= 2\pi \int_{0 < |y| \ll 1} d|y||y|^{-1+3\epsilon} + 2\pi \int_{1 < |y| < \infty} d|y||y|^{-1+3\epsilon} + \cdots \quad (A.46)
\]

Next we change the variable in the second integral: \(|y| \rightarrow 1/|y|\). Then we obtain:

\[
\tilde{I}_3 = 2\pi \int_{0 < |y| \ll 1} d|y||y|^{-1+3\epsilon} + 2\pi \int_{0 < |y| \ll 1} d|y||y|^{-1-3\epsilon} + \cdots \quad (A.47)
\]

If the first integral is \( \sim 2\pi/3\epsilon \) (by extending the integration up to 1), then, by analytic continuation in \( \epsilon \), the second integral should be defined to be \( \sim -2\pi/3\epsilon \). In the result the two contributions \( \sim 1/\epsilon \) cancel one another. This is what is happening with the \( 1/\epsilon \) terms in the calculation of the integral \( I \) in the Appendix B. The calculation there is all based on the analytic continuation technique.

**B Appendix**

The integral \( I \), eq.(A.31). The integral is of the following general form:

\[
I = \int d^2 t \int d^2 x \int d^2 y |t|^{2\alpha'}|t-1|^{2\beta'}
\]

\[
\times |x|^{2\alpha}|x-1|^{2\beta}|x-t|^{2\rho}|y|^{2\alpha}|y-1|^{2\beta}|y-t|^{2\rho} \quad (B.1)
\]

We shall calculate it for the values of exponents \( \alpha', \beta', \alpha, \beta, \rho \) which will be specified below. By the standard technique, see e.g.[12], this integral could be factorized into the following sum of products of contour integrals:

\[
I = -\{ j_1^{(+)} [s(\beta')s^2(\beta)j_1^{(-)} + s(\beta')s(\beta)s(\beta+\rho)j_2^{(-)} + s(\beta')s^2(\beta+\rho)j_3^{(-)}] \}
\]
\[ +j_1^{(+)}[s(\beta' + \rho)s^2(\beta)j_1^{(-)} + \frac{1}{2}(s(\beta')s^2(\beta) + s(\beta' + \rho)s(\beta)s(\beta + \rho))j_2^{(-)} \]
\[ + s(\beta')s(\beta + \rho)j_3^{(-)}] \]
\[ +j_3^{(+)}[s(\beta' + 2\rho)s^2(\beta)j_1^{(-)} + s(\beta' + \rho)s^2(\beta)j_2^{(-)} + s(\beta')s^2(\beta)j_3^{(-)}] \} \quad \text{(B.2)} \]

Here \( s(\beta) \equiv \sin \pi \beta \), etc. The contour integrals are defined in the following way:

\[ j_1^{(+)} = \int_0^1 dt \int_t^1 dx \int_0^t dy(t)^{\alpha'}(1 - t)^{\beta'} \]
\[ \times (x)^{\alpha}(1 - x)^{\beta}(t - x)^{\rho}(y)^{\alpha}(1 - y)^{\beta}(t - y)^{\rho} \quad \text{(B.3)} \]
\[ j_2^{(+)} = 2 \int_0^1 dt \int_1^t dx \int_0^t dy(\cdots) \quad \text{(B.4)} \]
\[ j_3^{(+)} = \int_0^1 dt \int_1^t dx \int_1^t dy(\cdots) \quad \text{(B.5)} \]

The symbol (\( \cdots \)) in (B.4), (B.5) stands for the same expression as in (B.3) except that the variables are put in the order corresponding to the order of integration, so that the differences of variables are always positive. E.g. the factors \((t - x)^\rho, (t - y)^\rho\) in (B.3) will be in the form \((x - t)^\rho, (x - t)^\rho\) in the integral (B.5).

\[ j_1^{(-)} = \int_1^\infty dt \int_1^t dx \int_1^t dy(t)^{\alpha'}(t - 1)^{\beta'} \]
\[ \times (x)^{\alpha}(x - 1)^{\beta}(t - x)^{\rho}(y)^{\alpha}(y - 1)^{\beta}(t - y)^{\rho} \quad \text{(B.6)} \]
\[ j_2^{(-)} = 2 \int_1^\infty dt \int_1^\infty dx \int_1^t dy(\cdots) \quad \text{(B.7)} \]
\[ j_3^{(-)} = \int_1^\infty dt \int_1^\infty dx \int_1^t dy(\cdots) \quad \text{(B.8)} \]

The integrals \( j_1^{(-)}, j_2^{(-)}, j_3^{(-)} \) can be put in the same form as the integrals \( j_1^{(+)}, j_2^{(+)}, j_3^{(+)} \) under the change of variables \( t \to 1/t, \ x \to 1/x, \ y \to 1/y \). One obtains:

\[ j_1^{(-)} = \int_0^1 dt \int_t^1 dx \int_0^1 dy(t)^{\tilde{\alpha'}}(1 - t)^{\tilde{\beta'}} \]
\[ \times (x)^{\tilde{\alpha}}(1 - x)^{\tilde{\beta}}(x - t)^{\rho}(y)^{\tilde{\alpha}}(1 - y)^{\tilde{\beta}}(y - t)^{\rho} \quad \text{(B.9)} \]
\[ j_2^{(-)} = 2 \int_0^1 dt \int_0^t dx \int_0^1 dy(\cdots) \quad \text{(B.10)} \]
\[ j_3^{(-)} = \int_0^1 dt \int_0^t dx \int_0^t dy(\cdots) \quad \text{(B.11)} \]
Here
\[ \tilde{\alpha}' = -2 - \alpha' - \beta' - 2\rho, \quad \tilde{\alpha} = -2 - \alpha - \beta - \rho \]  
(B.12)

Originally the values of the exponents are given by:
\[ \alpha' = 4\tilde{\alpha}_\sigma \alpha_\varepsilon = \frac{1}{3} - \frac{\varepsilon}{2} \]  
(B.13)
\[ \beta' = 4\alpha_\varepsilon^2 = \frac{4}{3} + \varepsilon \]  
(B.14)
\[ \alpha = 2\tilde{\alpha}_\sigma \alpha_+ = -\frac{1}{3} + \frac{\varepsilon}{2} \]  
(B.15)
\[ \beta = 2\alpha_\varepsilon \alpha_+ = -\frac{4}{3} - \varepsilon \]  
(B.16)
\[ \rho = 2\alpha_\varepsilon \alpha_+ = -\frac{4}{3} - \varepsilon \]  
(B.17)

Here \( \tilde{\alpha}_\sigma, \alpha_\varepsilon \) correspond to the Coulomb gas operators
\[ V_{\tilde{\alpha}_\sigma}(x) =: \exp\{i\tilde{\alpha}_\sigma \varphi(x)\} ; \quad V_{\alpha_\varepsilon}(x) =: \exp\{i\alpha_\varepsilon \varphi(x)\} ; \]
which represent the operators of spin \( \sigma(x) \) and energy \( \varepsilon(x) \); \( \alpha_+ \) corresponds to the screening operator. In particular:
\[ \alpha_\varepsilon = \alpha_{1,2} = -\frac{\alpha_+}{2} \]  
(B.18)

For the spin operator, if \( \alpha_+^2 \) is put in the form:
\[ \alpha_+^2 = \frac{2p}{2p - 1} = \frac{4}{3} + \varepsilon \]  
(B.19)
(which corresponds to the Kac table of the size \( (2p - 1) \times (2p - 2) \)) then the Potts model spin operator is
\[ \sigma \sim V_{p,p-1}, \overline{V_{p,p-1}} \]  
(B.20)

\( V_{p,p-1}(x) \) is the conjugate Coulomb gas operator (with respect to \( V_{p,p-1}(x) \)), which we are actually using:
\[ V_{\overline{p,p-1}}(x) =: \exp\{i\alpha_{p,p-1} \varphi(x)\} ; \quad \alpha_{p,p-1} \equiv \tilde{\alpha}_\sigma = \frac{1}{2} + \frac{p}{2} \alpha_- + \frac{p}{2} \alpha_+ \]  
(B.21)
\[ \alpha_{p,p-1} \equiv \tilde{\alpha}_\sigma = \frac{1}{2} + \frac{p}{2} \alpha_- + \frac{p}{2} \alpha_+ \]  
(B.22)

One gets the values of exponents in (B.13)-(B.17), using in addition the relation between \( p \) and \( \alpha_+^2 \):
\[ p(\alpha_+^2 - 1) = \frac{\alpha_+^2}{2} \]  
(B.23)
which follows from (B.19), and the usual relation $\alpha_+ \alpha_- = -1$ for the Coulomb gas parameters.

We have found that the calculation of the integral $I$ is simpler for the values of exponents $\alpha', \beta', \alpha, \beta, \rho$ obtained after the transformation of variables:

$$
t \to 1 - \frac{1}{t}, \quad x \to 1 - \frac{1}{x}, \quad y \to 1 - \frac{1}{y}
$$

(B.24)

One gets then the following values:

$$
\alpha' = \frac{4}{3} + \epsilon, \quad \beta' = -1 + \frac{3}{2} \epsilon
$$

(B.25)

$$
\alpha = -\frac{4}{3} - \epsilon, \quad \beta = 1 + \frac{3}{2} \epsilon
$$

(B.26)

$$
\rho = -\frac{4}{3} - \epsilon
$$

(B.27)

The exponents $\tilde{\alpha}', \tilde{\alpha}$ of the contour integrals $j_1^{(-)}, j_2^{(-)}, j_3^{(-)}$ eqs. (B.9)-(B.11), are obtained from (B.12):

$$
\tilde{\alpha}' = \frac{1}{3} - \frac{\epsilon}{2}
$$

(B.28)

$$
\tilde{\alpha} = -\frac{1}{3} + \frac{\epsilon}{2}
$$

(B.29)

We shall give next some details on the calculation of the contour integrals $j_1^{(+)}, j_2^{(+)}, j_3^{(+)}$, which enter into the decomposition of the integral $I$, eq.(B.2). The calculation of the integrals $j_3^{(-)}, j_2^{(-)}, j_1^{(-)}$ is respectively analogous.

$$
j_1^{(+)} = \int_0^1 dt \ t^{\alpha'} (1 - t)^{\beta'} \int_0^t dx \ x^{\alpha} (1 - x)^{\beta (t - x)^\rho} \times \int_0^t dy \ y^{\alpha} (1 - y)^{\beta (t - y)^\rho}
$$

(B.30)

We change variables: $x = \tilde{x}t$, $y = \tilde{y}t$ and next we drop the tildes of the new variables $\tilde{x}$, $\tilde{y}$.

$$
j_1^{(+)} = \int_0^1 dt \ t^{\alpha' + 2\alpha + 2\rho} (1 - t)^{\beta'} \int_0^1 dx \ x^{\alpha} (1 - x)^{\rho} (1 - xt)^{\beta} \times \int_0^1 dy \ y^{\alpha} (1 - y)^{\rho} (1 - yt)^{\beta}
$$

(B.31)
We expand next the factors $(1 - xt)^\beta$, $(1 - yt)^\beta$:

$$(1 - xt)^\beta = \sum_{k=0}^{\infty} \frac{(-\beta)_k}{k!} (xt)^k$$

(B.32)

Here $(-\beta)_k = (-\beta)(-\beta + 1) \cdots (-\beta + k - 1)$. One obtains:

$$j_1^{(+)} = \sum_{k_1} \sum_{k_2} \frac{(-\beta)_{k_1} (-\beta)_{k_2}}{k_1! k_2!} \int_0^1 dt \ t^{2 + \alpha' + 2\alpha + 2\rho + k_1 + k_2} (1 - t)^{\beta'}$$

$$\times \int_0^1 dx \ x^{\alpha + k_1} (1 - x)^{\rho} \int_0^1 dy \ y^{\alpha + k_2} (1 - y)^{\rho}$$

(B.33)

Next we use the integral

$$\int_0^1 dt \ t^a (1 - t)^b = \frac{\Gamma(1 + a)\Gamma(1 + b)}{\Gamma(2 + a + b)}$$

(B.34)

to obtain:

$$j_1^{(+)} = \sum_{k_1} \sum_{k_2} \frac{(-\beta)_{k_1} (-\beta)_{k_2}}{k_1! k_2!} \frac{\Gamma(3 + \alpha' + 2\alpha + 2\rho + k_1 + k_2)\Gamma(1 + \beta')}{\Gamma(4 + \alpha' + \beta' + 2\alpha + 2\rho + k_1 + k_2)}$$

$$\times \frac{\Gamma(1 + \alpha + k_1)\Gamma(1 + \rho)}{\Gamma(2 + \alpha + \rho + k_1)} \frac{\Gamma(1 + \alpha + k_2)\Gamma(1 + \rho)}{\Gamma(2 + \alpha + \rho + k_2)}$$

(B.35)

By using repeatedly the recurrence relation for the $\Gamma$-function, $\Gamma(z + 1) = z\Gamma(z)$, one can put (B.35) into the following form:

$$j_1^{(+)} = \gamma_1^{(+)} \times S_1^{(+)}$$

(B.36)

$$\gamma_1^{(+)} = \frac{\Gamma(3 + \alpha' + 2\alpha + 2\rho)\Gamma(1 + \beta')}{\Gamma(4 + \alpha' + \beta' + 2\alpha + 2\rho)} \left( \frac{\Gamma(1 + \alpha)\Gamma(1 + \rho)}{\Gamma(2 + \alpha + \rho)} \right)^2$$

(B.37)

$$S_1^{(+)} = \sum_{k_1} \sum_{k_2} \frac{(-\beta)_{k_1} (-\beta)_{k_2}}{k_1! k_2!} \frac{(3 + \alpha' + 2\alpha + 2\rho)_{k_1 + k_2}}{(4 + \alpha' + \beta' + 2\alpha + 2\rho)_{k_1 + k_2}}$$

$$\times \frac{(1 + \alpha)_{k_1}}{(2 + \alpha + \rho)_{k_1}} \frac{(1 + \alpha)_{k_2}}{(2 + \alpha + \rho)_{k_2}}$$

(B.38)

Substituting the values of exponents (B.25)-(B.27) one obtains:

$$\gamma_1^{(+)} = \frac{\Gamma(-1 - 3\epsilon)\Gamma(\frac{\epsilon}{2})}{\Gamma(-1 - \frac{3}{2}\epsilon)} \left( \frac{\Gamma(-\frac{1}{3} - \epsilon)\Gamma(-\frac{1}{3} - \epsilon)}{\Gamma(-\frac{2}{3} - 2\epsilon)} \right)^2$$

(B.39)

$$S_1^{(+)} = \sum_{k_1} \sum_{k_2} \frac{(-1 - \frac{3}{2}\epsilon)_{k_1}}{k_1!} \frac{(-1 - \frac{3}{2}\epsilon)_{k_2}}{k_2!} \frac{(-\frac{1}{3} - \epsilon)_{k_1}}{(-\frac{2}{3} - 2\epsilon)_{k_1}} \frac{(-\frac{1}{3} - \epsilon)_{k_2}}{(-\frac{2}{3} - 2\epsilon)_{k_2}} \frac{(-1 - 3\epsilon)_{k_1 + k_2}}{(-1 - \frac{3}{2}\epsilon)_{k_1 + k_2}}$$

(B.40)
We shall do calculations by expanding in $\epsilon$. For the renormalization group equation we need to keep the first two terms only.

For $\gamma_1^{(+)}$ in (B.39) a simple calculation gives:

$$\gamma_1^{(+)} = \frac{\gamma^2}{3\epsilon}(1 - \frac{3}{2}\epsilon - 4\epsilon\kappa) + O(\epsilon) \tag{B.41}$$

We have defined here:

$$\gamma = \frac{(\Gamma(-\frac{1}{3}))^2}{\Gamma(-\frac{2}{3})} \tag{B.42}$$

$$\kappa = \psi(-\frac{1}{3}) - \psi(-\frac{2}{3}) = \frac{\pi}{\sqrt{3}} + \frac{3}{2} \tag{B.43}$$

$\psi(z)$ is the standard $\psi$-function:

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) \tag{B.44}$$

The calculation of $S_1^{(+)}$ is more involved. First we check the convergence of the series in (B.40). In general one has:

$$\frac{(a)_k}{(b)_k} \approx (k)^{a-b}, \quad k \gg 1 \tag{B.45}$$

Using this asymptotic form one gets the following estimate for large $k_1, k_2$ in (B.40):

$$S_1^{(+)} \sim \sum_{k_1} \sum_{k_2} (k_1)^{-\frac{2}{3} - \frac{2}{3} - \frac{2}{3}} (k_2)^{-\frac{2}{3} - \frac{2}{3} - \frac{2}{3}} (k_1 + k_2)^{-\frac{3}{2}} \tag{B.46}$$

(We observe that $k! = (1)_k$). So the series converge and we can do safely its $\epsilon$-expansion.

Using the specific values of the parameters in the series (B.40) one can develop it in the following way:

$$S_1^{(t)} \approx (0, 0) + 2(0, 1) + (1, 1) + 2 \sum_{k_2=2}^{\infty} (\cdots) + 2 \sum_{k_2=2}^{\infty} (\cdots) \tag{B.47}$$

(0,0) stand for the term of the series $k_1 = 0, k_2 = 0$, etc. We have dropped the part of the series corresponding to $(k_1 = 2, \cdots \infty, k_2 = 2, \cdots, \infty)$ since it is $\sim \epsilon^2$. We are keeping only terms $\sim 1$ and $\sim \epsilon$ in $S_1^{(+)}$. Explicitly one gets:

$$S_1^{(+)} \approx 1 + 2 \frac{(-1 - \frac{3}{2}\epsilon)}{1 - \frac{1}{2}} \frac{(-\frac{1}{2} - \epsilon)}{\frac{1}{3} - \frac{1}{2} - \epsilon} \frac{(-1 - 3\epsilon)}{(-\frac{2}{3} - 2\epsilon)} \tag{B.48}$$
To simplify further, we use the following relations:

\[(a)_{k+2} = a(a+1)(a+2)k,\]

\[(a)_{k+3} = a(a+1)(a+2)(a+3)k\]  \hspace{1cm} (B.49)

In particular:

\[(-1 - \frac{3}{2} \epsilon)_{k+2} = (-1 - \frac{3}{2} \epsilon)(-\frac{3}{2} \epsilon)(1 - \frac{3}{2} \epsilon)k \approx \frac{3}{2} \epsilon(1)k = \frac{3}{2} \epsilon k!\]  \hspace{1cm} (B.50)

\[(-\frac{1}{3} - \epsilon)_{k+2} \approx (-\frac{1}{3})_{k+2} = (-\frac{1}{3})(\frac{2}{3})(\frac{5}{3})k\]  \hspace{1cm} (B.51)

etc. Using these simple rules, after some algebra one gets \(S_1^{(+)}\) in the following form:

\[S_1^{(+)} = \frac{1}{2} + \epsilon(3s - \frac{3}{4}) + O(\epsilon^2)\]  \hspace{1cm} (B.52)

where

\[s = \sum_{k=0}^{\infty} \frac{k!}{(k+2)!} \frac{(\frac{5}{3})_k}{(\frac{4}{3})_k}\]  \hspace{1cm} (B.53)

It remains to calculate this sum. We shall use the following general results:

\[\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\]  \hspace{1cm} (B.54)

\[\sum_{k=0}^{\infty} \frac{k!}{(k+2)!} \frac{(b)_k}{(c)_k} = \frac{c-1}{b-1} \left(\frac{c-1}{c-2} + \frac{c-b}{b-2} \tilde{\kappa}\right)\]  \hspace{1cm} (B.55)

Here

\[\tilde{\kappa} = \psi(c-b) - \psi(c-2)\]  \hspace{1cm} (B.56)

Eq. (B.55) could be derived by using the sum (B.54), which is standard, and some rather simple algebra.

From (B.55) and (B.53) one gets in a straightforward way:

\[s = -\frac{1}{4} + \frac{\kappa}{2}\]  \hspace{1cm} (B.57)
The constant $\kappa$ is defined in (B.43). By eq.(B.52)

$$S_{1(+)} \approx \frac{1}{2} + \epsilon(3s - \frac{3}{4}) = \frac{1}{2} - \frac{3}{2} \epsilon + \frac{3}{2} \epsilon \kappa$$ (B.58)

Returning still back to eqs. (B.36), (B.41) one obtains:

$$j_{1(+)} = \gamma_{1(+)} S_{1(+)} = \frac{\gamma^2}{6\epsilon} (1 - \frac{9}{2} \epsilon - \kappa \epsilon) + O(\epsilon)$$ (B.59)

The integral $j_{3(-)}$, eq.(B.11), is of the same form as $j_{1(+)}$ with only the exponents $\alpha', \alpha$ replaced by $\tilde{\alpha}', \tilde{\alpha}$, eqs.(B.28), (B.29). The calculation follows the same lines, with the result:

$$j_{3(-)} = \gamma^2 (-\frac{7}{4} + \frac{\kappa}{2}) + O(\epsilon)$$ (B.60)

For calculation of $j_{2(+)}$ it is useful to use the following linear relation of the integrals

$$u_2 = -\frac{s(\alpha)}{s(\alpha + \rho)} u_1 - \frac{s(\alpha + \beta + \rho)}{s(\alpha + \rho)} \tilde{u}_2$$ (B.61)

Here we have redefined, for the purpose of this particular calculation only,

$$u_2 = \frac{1}{2} j_{2(+)} , \quad u_1 = j_{1(+)}$$ (B.62)

and $\tilde{u}_2$ is a new integral:

$$\tilde{u}_2 = \int_0^1 dt \ t^\alpha (1 - t)^{\beta'} \int_0^t dy \ y^\alpha (1 - y)^\beta (t - y) \int_1^\infty dx \ x^\alpha (x - 1)^\beta (x - t)^\rho$$ (B.63)

$s(\alpha)$ in (B.61) is $\sin \pi \alpha$. The relation (B.61) is obtained in a standard way by doing transformation of the contours of integration [12].

Putting the problem this way, to calculate $j_{2(+)}$ we have to calculate the integral $\tilde{u}_2$, (B.63). Because of specific values of the exponents the calculation of the integral $\tilde{u}_2$ is simpler, as compared to the direct calculation of $j_{2(+)}$.

We do first the change of variables in (B.63): $x \rightarrow 1/x, \ y \rightarrow ty$. One gets:

$$\tilde{u}_2 = \int_0^1 dt \ t^{1+\alpha' + \alpha + \rho} (1 - t)^{\beta'} \int_0^t dy \ y^{\alpha} (1 - y)^\rho (1 - ty)^\beta$$

$$\times \int_0^1 dx \ x^{\tilde{\alpha}} (1 - x)^\beta (1 - xt)^\rho$$ (B.64)
\( \tilde{\alpha} \) was defined in (B.12), (B.29). Next we expand the factors \((1 - ty)^\beta, (1 - xt)^\rho \) and proceed like we did for the integral \( j_1^{(+)} \), starting with eq.(B.31). We get \( \tilde{u}_2 \) in the following form:

\[
\tilde{u}_2 = \tilde{\gamma}_2 \tilde{S}_2 \tag{B.65}
\]

\[
\tilde{\gamma}_2 = \frac{\Gamma(2 + \alpha' + \alpha + \rho) \Gamma(1 + \beta') \Gamma(1 + \alpha) \Gamma(1 + \rho) \Gamma(1 + \tilde{\alpha}) \Gamma(1 + \beta)}{\Gamma(3 + \alpha' + \beta' + \alpha + \rho) \Gamma(2 + \alpha + \rho) \Gamma(2 + \tilde{\alpha} + \beta)} \tag{B.66}
\]

\[
\tilde{S}_2 = \sum_k \sum_l \frac{(-\beta)_k (-\rho)_l}{k!} \frac{(1 + \alpha)_k (1 + \tilde{\alpha})_l (2 + \alpha' + \alpha + \rho)_{k+l}}{(2 + \alpha + \rho)_k (2 + \tilde{\alpha} + \beta)_l (3 + \alpha' + \beta' + \alpha + \rho)_{k+l}} \tag{B.67}
\]

Substituting the values of the parameters, eqs.(B.25)-(B.29), one obtains:

\[
\tilde{\gamma}_2 = \frac{\Gamma(2 + \epsilon) \Gamma(3 + \epsilon) \Gamma(2 - \epsilon)}{\Gamma(2 + 2\epsilon) \Gamma(-\epsilon)} \tag{B.68}
\]

\[
\tilde{S}_2 = \sum_k \sum_l \frac{(-1 - \frac{3}{2}\epsilon)_k}{k!} \frac{(-\frac{1}{3} - \epsilon)_l}{l!} \frac{\frac{4}{3} + \epsilon)_l}{\left(-\frac{2}{3} - 2\epsilon\right)_k \left(\frac{8}{3} + 2\epsilon\right)_l} \frac{(\frac{2}{3} - \epsilon)_l}{(\frac{2}{3} + \frac{3}{2})_l} \tag{B.69}
\]

For \( \tilde{\gamma}_2 \) one gets:

\[
\tilde{\gamma}_2 = \frac{3}{5\epsilon} \gamma(1 + \frac{15}{2} - \frac{6}{5}) - 2\epsilon \kappa - 3\epsilon \psi(-\frac{1}{3}) + 3\epsilon \psi(1) + O(\epsilon) \tag{B.70}
\]

For \( \tilde{S}_2 \) one gets asymptotically, for large \( k, l \):

\[
\tilde{S}_2 \sim \sum_k \sum_l (k)^{-\frac{3}{4} - \frac{1}{2}l} (2 - \frac{1}{2}l)^{-\frac{3}{4} - \frac{1}{2}(k + l)} \tag{B.71}
\]

the series is convergent. In order to develop in \( \epsilon \), \( \tilde{S}_2 \) can be decomposed in the following way:

\[
\tilde{S}_2 = (0, l) + (1, l) + (k + 2, l) \tag{B.72}
\]

or explicitly:

\[
\tilde{S}_2 \approx \sum_{l=0}^\infty \frac{(-\frac{1}{3} - \epsilon)_l \frac{2}{3} + \epsilon)_l \frac{2}{3} - \epsilon)_l}{l! \left(-\frac{2}{3} + 2\epsilon\right)_l \left(\frac{2}{3} + \frac{3}{2}\right)_l} \frac{(\frac{4}{3} + \epsilon)_l \frac{2}{3} + \epsilon)_l \frac{2}{3} - \epsilon)_l}{l! \left(\frac{2}{3} + 2\epsilon\right)_l \left(\frac{2}{3} + \frac{3}{2}\right)_l} \tag{B.73}
\]

In the last term we have put \( \epsilon = 0 \) because of the factor \( -\frac{3}{2}\epsilon \) in front. Finally one gets:

\[
\tilde{S}_2 \approx s_1 - \frac{1}{2}(1 - \frac{3}{4}\epsilon)s_2 + \frac{3}{2}\epsilon s_3 s_4 \tag{B.74}
\]
with

\[ s_1 = \sum_{l=0}^{\infty} \frac{\left(\frac{4}{3} + \epsilon\right)_l (\frac{5}{3} - \epsilon)_l}{l! (\frac{8}{3} + 2\epsilon)_l} \]  
(B.75)

\[ s_2 = \sum_{l=0}^{\infty} \frac{\left(\frac{4}{3} + \epsilon\right)_l (\frac{5}{3} + \frac{\eta}{2})_l (\frac{5}{3} - \epsilon)_l}{l! (\frac{8}{3} + 2\epsilon)_l (\frac{5}{3} + \frac{\eta}{2})_l} \]  
(B.76)

\[ s_3 = \sum_{k=0}^{\infty} \frac{k!}{(k+2)!} (\frac{5}{3})_k \]  
(B.77)

\[ s_4 = \sum_{l=0}^{\infty} \frac{\left(\frac{4}{3}\right)_l (\frac{2}{3})_l}{l! (\frac{5}{3})_l} \]  
(B.78)

By using the results for sums in (B.54), (B.55) the calculation of \( s_1, s_3, s_4 \) is straightforward. One obtains:

\[ s_1 = -\frac{5}{9} \left( 1 + \left( \frac{6}{5} - \frac{27}{2} \right) \epsilon + \kappa \epsilon + 3\epsilon \psi\left( -\frac{1}{3} \right) - 3\epsilon \psi(1) \right) + O(\epsilon^2) \]  
(B.79)

\[ s_3 = -\frac{1}{4} + \frac{\kappa}{2} \]  
(B.80)

\[ s_4 = -\frac{5}{9} \gamma \]  
(B.81)

For the sum \( s_2 \), we observe at first that

\[ \frac{(\frac{2}{3} + \frac{\eta}{2})_l}{(\frac{2}{3} + \frac{\eta}{2})_l} = \frac{\frac{2}{3} + \frac{\eta}{2} + l}{\frac{2}{3} + \frac{\eta}{2} + l} \]  
(B.82)

So one gets:

\[ s_2 = \left( \frac{2}{3} + \frac{\epsilon}{2} \right) \sum_{l=0}^{\infty} \frac{\left(\frac{4}{3} + \epsilon\right)_l (\frac{5}{3} - \epsilon)_l}{l! (\frac{8}{3} + 2\epsilon)_l (\frac{5}{3} + \frac{\eta}{2} + l)} \]  
(B.83)

\[ = \frac{2}{3} + \frac{\epsilon}{2} \sum_{l=0}^{\infty} \frac{\left(\frac{4}{3} + \epsilon\right)_l (\frac{5}{3} - \epsilon)_l}{l! (\frac{8}{3} + 2\epsilon)_l (\frac{5}{3} + \frac{\eta}{2} + l)} \]  
(B.84)

Calculation of the first sum is straightforward. One obtains:

\[ \sum_l \frac{\left(\frac{4}{3} + \epsilon\right)_l (\frac{5}{3} - \epsilon)_l}{l! (\frac{8}{3} + 2\epsilon)_l} \approx -\frac{5}{9} \gamma \left( 1 + \frac{6}{5} \epsilon - \frac{27}{2} \epsilon + \epsilon \kappa + 3\epsilon \psi\left( -\frac{1}{3} \right) - 3\epsilon \psi(1) \right) \]  
(B.84)
The second sum could be calculated by decomposing

$$\frac{1}{(\frac{2}{3} + l)(\frac{2}{3} + l)^2} = \frac{1}{(\frac{2}{3} + l)^2} - \frac{1}{\frac{2}{3} + l} + \frac{1}{\frac{2}{3} + l}$$  \hspace{1cm} (B.85)$$

and using the formulas:

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} \frac{1}{b + k} = \frac{\Gamma(1 - a)\Gamma(b)}{\Gamma(1 - a + b)} \hspace{1cm} (B.86)$$

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} \frac{1}{(b + k)^2} = \frac{\Gamma(1 - a)\Gamma(b)}{\Gamma(1 - a + b)}(\psi(1 - a + b) - \psi(b)) \hspace{1cm} (B.87)$$

One gets in this way:

$$\sum_{l} \frac{(\frac{4}{3})l}{l!} \frac{1}{(\frac{2}{3} + l)(\frac{2}{3} + l)^2} = \gamma \frac{5}{2} \left( \frac{1}{2} - \kappa \right)$$  \hspace{1cm} (B.88)$$

From eqs.(B.83), (B.84), (B.88) one obtains:

$$s_2 = -\frac{5}{9}\gamma(1 + \frac{6}{5}\epsilon - \frac{15}{2}\epsilon - \frac{\kappa}{2}\epsilon + 3\epsilon\psi(-\frac{1}{3}) - 3\epsilon\psi_1)$$  \hspace{1cm} (B.89)$$

Putting together the results for \(s_1, s_2, s_3, s_4\) into eq.(B.74) for \(\tilde{S}_2\) one gets:

$$\tilde{S}_2 = -\frac{5}{18}\gamma(1 + \frac{6}{5}\epsilon - \frac{39}{2}\epsilon + 4\kappa\epsilon + 3\epsilon\psi(-\frac{1}{3}) - 3\epsilon\psi(1)) + O(\epsilon^2)$$  \hspace{1cm} (B.90)$$

Returning back to the eqs.(B.65), (B.68) one derives:

$$\tilde{u}_2 = -\frac{\gamma^2}{6\epsilon}(1 - 7\epsilon + 2\kappa\epsilon) + O(\epsilon)$$  \hspace{1cm} (B.91)$$

Going still back to the relation (B.61), putting the values of the parameters and substituting the value of \(u_1\) obtained earlier, one finally derives:

$$j_2^{(+)} \equiv 2u_2 = \frac{\gamma^2}{6}(\frac{1}{2} - 3\kappa) + O(\epsilon)$$  \hspace{1cm} (B.92)$$

The integral \(j_2^{(-)}\), eq.(B.10), is of the same form as \(j_2^{(+)}\) with only the exponents \(\alpha', \alpha\) replaced by \(\tilde{\alpha}', \tilde{\alpha}\), eqs.(B.28), (B.29). The calculation follows the same line. There is one additional detail in the calculation. In the decomposition of the sum \(\tilde{S}_2\), corresponding to the integral \(j_2^{(-)}\), one gets, in particular, the sums:

$$s_1 = \sum_{l} \frac{(\frac{4}{3} + \epsilon)_l}{l!} \frac{(-\frac{1}{3} - \epsilon)_l(\frac{2}{3} - \epsilon)_l}{(\frac{4}{3} + \frac{6}{2}l)(\frac{4}{3} + \frac{4}{2}l)}$$  \hspace{1cm} (B.93)$$
\[
\sum_{l} \frac{1}{l!} \left( \frac{4}{3} + \epsilon \right)_l \left( -\frac{1}{3} - \epsilon \right)_l \left( \frac{5}{3} - \epsilon \right)_l \left( \frac{5}{3} + \frac{\epsilon}{2} \right)_l
\]

The sum \( s_1 \) is transformed as follows:

\[
s_1 = (-\frac{1}{3} - \epsilon) \left( \frac{2}{3} + \frac{\epsilon}{2} \right) \sum_{l} \frac{1}{l!} \left( \frac{4}{3} + \epsilon \right)_l \left( \frac{5}{3} - \epsilon \right)_l \frac{1}{(-\frac{1}{3} - \epsilon + l)(\frac{5}{3} + \frac{\epsilon}{2} + l)}
\]

and next relation is used:

\[
\left( \frac{\frac{2}{3} - \epsilon}{\frac{5}{3} + \frac{\epsilon}{2}} \right)_l = \frac{\frac{2}{3} - 2\epsilon}{\frac{5}{3} + \epsilon} + O(\epsilon^2)
\]

So, up to linear order in \( \epsilon \), one obtains \( s_1 \) in the form:

\[
s_1 \simeq (-\frac{1}{3} - \epsilon) \left( \frac{2}{3} + \frac{\epsilon}{2} \right) \sum_{l} \frac{1}{l!} \left( \frac{4}{3} + \epsilon \right)_l \left( \frac{5}{3} - 2\epsilon \right)_l \frac{1}{(-\frac{1}{3} - \epsilon + l)(\frac{5}{3} + \frac{\epsilon}{2} + l)}
\]

In the rest of the calculation one proceeds in the way analogous to the calculation of the sum

\( \sum s_2 \) for the integral \( j_2^{(+)} \), eq.(B.83). The sum \( s_2 \), of the integral \( j_2^{(-)} \), eq.(B.94), is dealt with in a similar manner. Finally one obtains the following result for \( j_2^{(-)} \):

\[
j_2^{(-)} = -\frac{\gamma^2}{3\epsilon} (1 - \frac{3}{2} \epsilon - \kappa \epsilon) + O(\epsilon)
\]

\( j_3^{(+)} , j_1^{(-)} \).

We redefine: \( j_3^{(+)} \equiv u_3 \), and we shall use the following linear relation between the integrals:

\[
u_3 = A u_1 + 2AB \tilde{u}_2 + B^2 u_1^*
\]

Here \( u_1 = j_1^{(+)} \), \( \tilde{u}_2 \) is defined in (B.63), \( u_1^* \) is a new integral:

\[
u_1^* = \int_0^1 dt \ t^\alpha (1 - t)^\beta \int_1^\infty dx \ x^\alpha(x - 1)^\beta(x - t)^\rho \int_1^\infty dy \ y^\alpha(y - 1)^\beta(y - t)^\rho
\]

A and B in (B.99) are coefficients:

\[
A = \frac{-s(\alpha)}{s(\alpha + \rho)}, \ B = \frac{-s(\alpha + \beta + \rho)}{s(\alpha + \rho)}
\]

The relation (B.99) is obtained by transformations of the contours of integration [12], similar to the relation (B.61)
The integrals $u_1, \tilde{u}_2$ had already been calculated. To define $u_3(j_3^{(+*)})$ we have to define $u_1^*$, which turns out to be easier.

We change the variables: $x \rightarrow 1/x$, $y \rightarrow 1/y$. This gives:

$$u_1^* = \int_0^1 dt \ t^{\alpha'}(1-t)^{\beta'} \int_0^1 dx \ x^{\tilde{\alpha}}(1-x)^{\beta}(1-xt)^{\rho} \int_0^1 dy \ y^{\tilde{\alpha}}(1-y)^{\beta}(1-yt)^{\rho} \quad (B.102)$$

$\tilde{\alpha} = -2 - \alpha - \beta - \rho$. Next we expand the factors $(1-xt)^{\rho}, (1-yt)^{\rho}$. This leads to the following form of $u_1^*$:

$$u_1^* = \gamma_1^* S_1^* \quad (B.103)$$

$$\gamma_1^* = \frac{\Gamma(1+\alpha')\Gamma(1+\beta')}{\Gamma(2+\alpha'+\beta')} \frac{\Gamma(1+\tilde{\alpha})\Gamma(1+\beta)}{\Gamma(2+\tilde{\alpha}+\beta)}$$

$$= \frac{\Gamma(\frac{7}{3}+\epsilon)\Gamma(\frac{3}{2}+\epsilon)}{\Gamma(\frac{5}{3}+\epsilon)} \frac{\Gamma(\frac{2}{3}+\epsilon)\Gamma(2+\frac{3}{2}+\epsilon)}{\Gamma(\frac{2}{3}+2\epsilon)}$$

$$S_1^* = \sum_k \sum_l \left( -\frac{\rho}{k!} \frac{1}{l!} \frac{(1+\tilde{\alpha})_k}{(2+\tilde{\alpha}+\beta)_k} \frac{(1+\tilde{\alpha})}{(2+\tilde{\alpha}+\beta)_l} \frac{(1+\alpha')_{k+l}}{(2+\alpha'+\beta')_{k+l}} \right)$$

$$= \sum_k \sum_l \left( \frac{\frac{4}{3}+\epsilon}{k!} \frac{\frac{5}{3}+\epsilon}{l!} \frac{\frac{2}{3}+\epsilon}{(\frac{8}{3}+2\epsilon)_k} \frac{\frac{7}{3}+\epsilon}{(\frac{8}{3}+2\epsilon)_l} \right)$$

$$= \frac{5}{9} \gamma^2 + O(\epsilon) \quad (B.105)$$

One easily checks convergence of this sum.

It turns out that the contour integral $j_3^{(+*)}$, like also $j_1^{(-*)}$, enters into the decomposition of the integral $I$, eq.(B.2), with a coefficients $\sim \epsilon^2$. This immediately follows from eq.(B.2) after the substitution of the values of the exponents $\beta' = -1+\frac{3}{2}\epsilon$, $\beta = 1+\frac{3}{2}\epsilon$, $\rho = -\frac{4}{3} - \epsilon$ eqs.(B.25)-(B.27). As a result, it is sufficient to define the leading $\sim 1/\epsilon$ parts of the integrals $j_3^{(+*)}, j_1^{(-*)}$.

One finds from (B.104):

$$\gamma_1^* = \frac{2}{3\epsilon} \left( \frac{9}{10} \right)^2 + O(1) \quad (B.106)$$

and from (B.105):

$$S_1^* \simeq \left( \sum_k \frac{(\frac{4}{3})_k(\frac{2}{3})_k}{k!(\frac{8}{3})_k} \right)^2 = \left( \frac{\Gamma(\frac{4}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})\Gamma(2)} \right)^2 = \left( \frac{5}{9} \right)^2 \gamma^2 + O(\epsilon) \quad (B.107)$$

This gives:

$$u_1^* = \gamma_1^* S_1^* = \frac{\gamma^2}{6\epsilon} + O(1) \quad (B.108)$$
Finally from (B.99), one gets:

\[ u_3 \equiv j_3^{(+)} = \frac{\gamma^2}{6\epsilon} - 2\frac{\gamma^2}{6\epsilon} + \frac{\gamma^2}{6\epsilon} + O(1) \]  

or

\[ j_3^{(+)} = O(1) \]  

This implies that \( j_3^{(+)} \) will not contribute to \( I \) (to the finite part of \( I \), to be more precise).

Similar calculation for the integral \( j_1^{(-)} \) gives:

\[ j_1^{(-)} = -\frac{\gamma^2}{6\epsilon} + O(1) \]  

We come back now to the eq.(B.2). Substituting the values of the exponents in the coefficients, developing the coefficients in \( \epsilon \) and keeping the leading terms only, one obtains:

\[ I \approx -\{j_1^{(+)}[-\pi^2 \epsilon j_1^{(-)}] - \pi^2 \epsilon^2 \frac{\sqrt{3}}{2} j_1^{(-)} - \pi \epsilon \frac{2}{3} j_3^{(-)}\]  

\[ + j_2^{(+)}[-\pi^2 \epsilon^2 \frac{\sqrt{3}}{2} j_1^{(-)} - \frac{1}{2} \pi \epsilon \frac{3}{4} j_2^{(-)} - \pi^2 \epsilon^2 \frac{\sqrt{3}}{2} j_3^{(-)} \]  

\[ + j_3^{(+)}[\pi^2 \epsilon^2 \frac{\sqrt{3}}{2} j_1^{(-)} - \pi^2 \epsilon^2 \frac{\sqrt{3}}{2} j_2^{(-)} - \pi^3 \epsilon^3 j_3^{(-)}]\} \]

(B.112)

Here we have noted \( \frac{3}{2} \epsilon = \epsilon \). The leading terms of the contour integrals are the following:

\[ j_1^{(+)} \approx \frac{\gamma^2}{6\epsilon}, \quad j_2^{(+)} \approx \frac{\gamma^2}{6}(\frac{1}{2} - 3\kappa), \quad j_3^{(+)} \approx 1 \]  

(B.113)

\[ j_1^{(-)} \approx -\frac{\gamma^2}{6\epsilon}, \quad j_2^{(-)} \approx -\frac{\gamma^2}{3\epsilon}, \quad j_3^{(-)} \approx \gamma^2(-\frac{7}{4} + \kappa) \]  

(B.114)

Taking them into account, the expression (B.112) for \( I \) can further be reduced:

\[ I \approx j_1^{(+)}(\pi^2 \epsilon^2 \frac{\sqrt{3}}{2} j_2^{(-)} + \pi \epsilon \frac{3}{4} j_3^{(-)}) + j_2^{(+)}\frac{1}{2} \pi \epsilon \frac{3}{4} j_2^{(-)} \]

(B.115)

It turns out finally that we need to know only the leading terms of the integrals. Still, the leading terms for \( j_2^{(+)} \) and \( j_3^{(-)} \) are finite terms. So we had to calculate them, the singular \( \sim \frac{1}{\epsilon} \) terms for all the contour integrals would not be sufficient.

Substituting the values (B.113), (B.114) of the integrals, replacing back \( \epsilon \) by \( \frac{3}{2} \epsilon \) and using \( \kappa = \frac{\pi}{\sqrt{3}} + \frac{3}{2} \), one finds after some simple algebra:

\[ I = -\frac{\pi}{16} \gamma^4 + O(\epsilon) \]  

(B.116)
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