HOMOMORPHISM COMPLEXES AND K-CORES

GREG MALEN

ABSTRACT. We prove that the topological connectivity of a graph homomorphism complex \( \text{Hom}(G, K_m) \) is at least \( m - D(G) - 2 \), where \( D(G) = \max_{H \subseteq G} \delta(H) \). This is a strong generalization of a theorem of Ćukić and Kozlov, in which \( D(G) \) is replaced by the maximum degree \( \Delta(G) \). It also generalizes the graph theoretic bound for chromatic number, \( \chi(G) \leq \Delta(G) + 1 \), as \( \chi(G) = \min \{m : \text{Hom}(G, K_m) \neq \emptyset\} \). Furthermore, we use this result to examine homological phase transitions in the random polyhedral complexes \( \text{Hom}(G(n, p), K_m) \) when \( p = c/n \) for a fixed constant \( c > 0 \).

1. Introduction

The study of graph homomorphism complexes, \( \text{Hom}(G, H) \) for fixed graphs \( G \) and \( H \), grew out of a rising interest in finding topological obstructions to graph colorings, as in Lovász’s proof of the Kneser Conjecture [11]. Specifically, Lovász used the topological connectivity of the Neighborhood Complex, \( N(G) \), to provide a lower bound on the chromatic number of a Kneser graph. It turns out that \( N(G) \) is a specialization of a homomorphism complex, as it is homotopy equivalent to \( \text{Hom}(K_2, G) \) (see Proposition 4.2 in [2]). The structure of the underlying graphs has many quantifiable effects on the topology of these complexes. The 0-cells of \( \text{Hom}(G, H) \) are precisely the graph homomorphisms from \( G \to H \), and so the most straightforward such result is that the chromatic number of \( G \), \( \chi(G) \), is the minimum \( m \) for which \( \text{Hom}(G, K_m) \) is non-empty. For \( \Delta(G) \) the maximum degree of \( G \), Babson and Kozlov conjectured [2], and Ćukić and Kozlov later proved [6] that if \( \Delta(G) = d \), then \( \text{Hom}(G, K_m) \) is at least \( (m - d - 2) \)-connected. Note that when \( m = d + 1 \) this recovers the well known bound \( \chi(G) \leq \Delta(G) + 1 \).

In graph theory, there is an improvement of this bound using the minimum degree of induced subgraphs, namely that \( \chi(G) \leq \max_{H \subseteq G} \delta(H) + 1 \), for \( \delta(H) \) the minimum degree of \( H \). The value \( D(G) := \max_{H \subseteq G} \delta(H) \) is known as the degeneracy of \( G \), a graph property which is commonly utilized in computer science and the study of large networks. Using \( D(G) \) in place of \( \Delta(G) \), in Section 3 we prove a generalization of this graph theoretic result which will also specialize to the Ćukić–Kozlov Theorem. In Section 4 we focus on the case that \( H = K_3 \) and show, with only a few notable exceptions, that \( \text{Hom}(G, K_3) \) is disconnected if it is non-empty.

Furthermore, in the setting of random graphs the evolution of \( D(G) \) for a sparse Erdős–Rényi graph \( G \sim G(n, p) \) has been well studied. For \( p = c/n \) for a constant \( c > 0 \), Pittel, Spencer, and Wormald exhibited sharp thresholds for the appearance and size of subgraphs with minimum degree \( k \) [13]. In Section 5 we combine our results with their work to exhibit both phase transitions and lower bounds for the topological connectivity of the random polyhedral complexes \( \text{Hom}(G(n, c/n), K_m) \) for constant \( c > 0 \). When \( m = 3 \),
we are able to use the results from Section 4 to evaluate the limiting probability for the connectivity of $\text{Hom}(G(n, c/n), K_3)$ for all $c > 0$.

2. BACKGROUND AND DEFINITIONS

In the following, all graphs are undirected simple graphs, and $K_n$ is the complete graph on $n$ vertices. For a graph $G$ and a vertex $v \in V(G)$, when it is clear from context we may write simply that $v \in G$. In this section we define a graph homomorphism complex, or hom-complex, and various properties which will be used to acquire bounds on the topological connectivity of these complexes.

**Definition 2.1.** For fixed graphs $G$ and $H$, a graph homomorphism from $G \to H$ is an edge preserving map on the vertices, $f : V(G) \to V(H)$ such that $\{f(v), f(u)\} \in E(H)$ if $\{v, u\} \in E(G)$.

**Definition 2.2.** For fixed graphs $G$ and $H$, cells in $\text{Hom}(G, H)$ are functions $\eta : V(G) \to 2^{V(H)} \setminus \emptyset$ with the restriction that if $\{x, y\} \in E(G)$, then $\eta(x) \times \eta(y) \subseteq E(H)$. The dimension of $\eta$ is defined to be $\dim(\eta) = \sum_{v \in V(G)} (|\eta(v)| - 1)$, with ordering $\eta \subseteq \tau$ if $\eta(v) \subseteq \tau(v)$ for all $v \in V(G)$.

The set of 0-cells of $\text{Hom}(G, H)$ is then precisely the set of graph homomorphisms from $G \to H$, with higher dimensional cells formed over them as multihomomorphisms. Furthermore, every cell in a hom-complex is a product of simplices, and hom-complexes are entirely determined by their 1-skeletons. If $\eta$ is a product of simplices and its 1-skeleton is contained in $\text{Hom}(G, H)$, then $\eta \in \text{Hom}(G, H)$. The 1-skeleton of $\text{Hom}(G, H)$ can be thought of in the following manner. Two graph homomorphisms $\eta, \tau : G \to H$ are adjacent in $\text{Hom}(G, H)$ if and only if their images differ on exactly one vertex $v \in G$. So $\eta(v) \neq \tau(v)$, $\eta(u) = \tau(u)$ for all $u \in G \setminus \{v\}$, and the 1-cell joining $\eta$ and $\tau$ is the multihomomorphism $\sigma$ for which $\sigma(v) = \eta(v) \cup \tau(v)$ and $\sigma(u) = \eta(u) = \tau(u)$ for all $u \in G \setminus \{v\}$.

Since the property of being a homomorphism is independent on disjoint connected components of a graph, hom-complexes obey a product rule for disjoint unions. For any graphs $G, G'$ and $H$,

$$\text{Hom}(G \sqcup G', H) = \text{Hom}(G, H) \times \text{Hom}(G', H)$$

Thus, when convenient we can always restrict our attention to connected graphs. Here we also introduce the notion of a graph folding.

**Definition 2.3.** Denote the neighborhood of a vertex $v \in G$ by $N(v) = \{w \in G : w \sim v\}$. Let $v, u \in G$ be distinct vertices such that $N(v) \subseteq N(u)$. Then a fold of $G$ is a homomorphism from $G \to G \setminus \{v\}$ which sends $v \mapsto u$ and is otherwise the identity.

**Lemma 2.4.** (Babson, Kozlov; Proposition 5.1 in [2]). If $G$ and $H$ are graphs and $v, u \in G$ are distinct vertices such that $N(v) \subseteq N(u)$, then the folding $G \to G \setminus \{v\}$ which sends $v \mapsto u$ induces a homotopy equivalence $\text{Hom}(G \setminus \{v\}, H) \to \text{Hom}(G, H)$.

If $T$ is a tree, for example, then $T$ folds to a single edge, so $\text{Hom}(T, K_m) \approx \text{Hom}(K_2, K_m)$. And by Proposition 4.5 in [2], $\text{Hom}(K_n, K_m)$ is homotopy equivalent to a wedge of $(m-n)$-spheres, and in particular $\text{Hom}(K_2, K_m) \approx S^{m-2}$. So $\text{Hom}(T, K_m) \approx S^{m-2}$ for any tree $T$. For a thorough introduction to hom-complexes, see [2].
Notation 2.5. For a graph $G$ define the maximum degree to be $\Delta(G) := \max_{v \in V(G)} \{\deg(v)\}$, and the minimum degree to be $\delta(G) := \min_{v \in V(G)} \{\deg(v)\}$.

Definition 2.6. The $k$-core of a graph $G$ is the subgraph $c_k(G) \subseteq G$ obtained by the process of deleting vertices with degree less than $k$, along with all incident edges, one at a time until there are no vertices with degree less than $k$.

Regardless of the order in which vertices are deleted, this process always terminates in the unique induced subgraph $c_k(G) \subseteq G$ which is maximal over all subgraphs of $G$ which have minimum degree at least $k$. The $k$-core of a graph may be the empty graph, and the existence of a non-empty $k$-core is a monotone question, as $c_j(G) \subseteq c_l(G)$ whenever $j \leq l$. Here we are interested in the most highly connected non-trivial subgraph.

Definition 2.7. The degeneracy of $G$ is $D(G) := \max_{H \subseteq G} \delta(H)$, for $H$ an induced subgraph.

Then $D(G)$ is the maximum $k$ such that $c_k(G)$ is non-empty. Given these definitions, we are now able to state the main theorem:

Theorem 2.8. For any graph $G$, $\text{Hom}(G, K_m)$ is at least $(m - D(G) - 2)$-connected.

And $D(G) \leq \Delta(G)$, hence this will imply the Ćukić–Kozlov Theorem. Our interest is primarily in applying this result to the case that $G$ is a sparse Erdős–Rényi random graph, where $D(G)$ has been studied extensively and is much smaller than $\Delta(G)$. It should also be noted that Engström gave a similar strengthening of the Ćukić–Kozlov Theorem in [8], replacing $\Delta(G)$ with a graph property which is independent of $D(G)$ and which may provide a better tool for studying the dense regime.

3. Proof of Theorem 2.8

To prove the theorem, we first introduce a result of Csorba which finds subcomplexes that are homotopy equivalent to $\text{Hom}(G, K_m)$ by removing independent sets of $G$.

Notation 3.1. Let $G$ and $H$ be graphs. Define

$$\text{Ind}(G) := \{S \subseteq V(G) : S \text{ is an independent set}\}$$

And for $I \in \text{Ind}(G)$ define

$$\text{Hom}_I(G, H) := \{\eta \in \text{Hom}(G \setminus I, H) : \text{there is } \overline{\eta} \in \text{Hom}(G, H) \text{ with } \overline{\eta}|_{G \setminus I} = \eta\}$$

So $\text{Hom}_I(G, H)$ is the subcomplex of $\text{Hom}(G \setminus I, H)$ comprised of all multihomomorphisms from $G \setminus I \to H$ which extend to multihomomorphisms from $G \to H$.

Theorem 3.2. (Csorba; Theorem 2.36 in [5]) For any graphs $G$ and $H$, and any $I \in \text{Ind}(G)$, $\text{Hom}(G, H)$ is homotopy equivalent to $\text{Hom}_I(G, H)$.

The proof of this is an application of both the Nerve Lemma and the Quillen Fiber Lemma, and a more thorough examination of this property is given by Schultz in [14]. Note that when $H = K_m$ and $I = \{v\}$ for any vertex $v \in G$, a multihomomorphism $\eta : G \setminus \{v\} \to K_m$
has an extension \(7 : G \to K_m\) as long as there is some vertex in \(K_m\) which is not in \(\eta(w)\) for any \(w \in N(v)\). Thus

\[
\text{Hom}_{\{v\}}(G, K_m) = \left\{ \eta \in \text{Hom}(G \setminus \{v\}, K_m) : \left| \bigcup_{w \in N(v)} \eta(w) \right| \leq m - 1 \right\}
\]

**Lemma 3.3.** Let \(G\) be a graph with a vertex \(v\) such that \(\deg(v) \leq k\). Then the \((m - k - 1)\)-skeleton of \(\text{Hom}(G \setminus \{v\}, K_m)\) is contained in \(\text{Hom}_{\{v\}}(G, K_m)\).

**Proof of Lemma 3.3.** Let \(\eta \in \text{Hom}(G \setminus \{v\}, K_m) \setminus \text{Hom}_{\{v\}}(G, K_m)\). Then

\[
m = \left| \bigcup_{w \in N(v)} \eta(w) \right| \leq \sum_{w \in G\{v\}} |\eta(w)|
\]

And

\[
dim(\eta) = \sum_{w \in G\{v\}} (|\eta(w)| - 1) = \sum_{w \in G\{v\}} |\eta(w)| - |N(v)| \geq m - k
\]

Hence if \(\eta \in \text{Hom}(G \setminus \{v\}, K_m)\) with \(\dim(\eta) \leq m - k - 1\), then \(\eta \in \text{Hom}_{\{v\}}(G, K_m)\). □

So if \(\text{Hom}(G \setminus \{v\}, K_n)\) is \((m - k - 2)\)-connected, then \(\text{Hom}_{\{v\}}(G, K_n)\) is \((m - k - 2)\)-connected. Combining this with Csorba’s theorem, we have the following corollary and the proof of Theorem 2.8:

**Corollary 3.4.** Let \(G\) be a graph with a vertex \(v\) such that \(\deg(v) \leq k\). If \(\text{Hom}(G \setminus \{v\}, K_n)\) is \((m - k - 2)\)-connected, then \(\text{Hom}(G, K_n)\) is \((m - k - 2)\)-connected.

**Proof of Theorem 2.8.** Let \(G\) be a graph with \(D(G) = k\). Then \(c_{k+1}(G)\) is the empty graph, and there is a sequence \(G = G_0, G_1 = G_{i-1} \setminus \{v_i\}\), with \(\deg(G_{i-1}\{v_i\}) \leq k\), terminating in \(G_{|V(G)|} = \varnothing\). So \(G_{|V(G)|-1}\) is a single vertex, and \(\text{Hom}(G_{|V(G)|-1}, K_m) = \Delta^m\), the \(m\)-simplex, which is contractible, and thus \((m - k - 2)\)-connected. Hence, by induction, \(\text{Hom}(G, K_m)\) is also \((m - k - 2)\)-connected. □

4. \(\text{Hom}(G, K_3)\)

When \(G\) is a graph such that \(\chi(G) \leq 3\), \(\text{Hom}(G, K_3)\) has a particularly nice structure. If \(G\) does not have an isolated vertex, then \(\text{Hom}(G, K_3)\) is a cubical complex, and Babson and Kozlov showed that it admits a metric with nonpositive curvature [2]. Notice that for a connected graph \(G\) with \(\chi(G) \leq 3\), the bound obtained by Theorem 2.8 when \(m = 3\) provides no new information. When \(D(G) = 1\), \(G\) is a tree and folds to a single edge, so \(\text{Hom}(G, K_3) \cong \text{Hom}(K_2, K_3) \cong S^1\). When \(D(G) = 2\), the bound merely confirms that \(\text{Hom}(G, K_3)\) is non-empty, and for \(D(G) > 2\) it gives no information at all. Here we show that \(\text{Hom}(G, K_3)\) is, in fact, disconnected for a large class of graphs \(G\) with \(\chi(G) \leq 3\).

**Theorem 4.1.** If \(G\) is a graph with \(\chi(G) = 3\), then \(\text{Hom}(G, K_3)\) is disconnected.

This result has been formulated previously in the language of statistical physics, where Glauber dynamics examines precisely the 1-skeleton of \(\text{Hom}(G(n, p), K_3)\). See, for example, the work of Cereceda, van den Heuvel and Johnson [3]. We provide a proof here in the context of work on hom-complexes of cycles done by Ćukić and Kozlov, and offer a
more general approach for lifting disconnected components via subgraphs in the following lemma.

**Lemma 4.2.** Let $G \subseteq G'$ and $H$ be graphs such that $\text{Hom}(G, H)$ is non-empty and disconnected. Let $\eta_1, \eta_2 \in \text{Hom}(G, H)$ be 0-cells, i.e. graph homomorphisms from $G \to H$, such that they are in distinct connected components of $\text{Hom}(G, H)$. If there are extensions $\eta_1^{'}, \eta_2^{' \in} \text{Hom}(G', H)$ with $\eta_i^{'|G} = \eta_i$ for $i \in \{1, 2\}$, then $\text{Hom}(G', H)$ is also disconnected.

**Proof of Lemma 4.2.** Let $\eta_1, \eta_2$ be graph homomorphisms from $G \to H$ such that they are in distinct connected components of $\text{Hom}(G, H)$, with extensions $\eta_1^{'}, \eta_2^{' \in} \text{Hom}(G', H)$. We may assume that $\dim(\eta_i) = 0$ for each $i$, since otherwise any 0-cells they contain would also be extensions of $\eta_i$. Suppose that $\eta_1^{'}, \eta_2^{' \in}$ are in the same connected component of $\text{Hom}(G', H)$. Then there is a path $\eta_1^{' = \eta_0^{' \sim \eta_1^{' \sim \cdots \sim \eta_l^{' \sim H G}}}$ in $\text{Hom}(G', H)$, with $\dim(\eta_j^{'}) = 0$ for all $j$. Let $\tau_j^{' = \eta_j^{'|G}}$, for $0 \leq j \leq l$. Then as functions on $V(G')$, for each $j$ there is one vertex $v_j^{' \in G'}$ for which $\tau_j^{'(v_j^{'})} = \tau_j^{'(v_j^{'})}$, and they agree on all other vertices. If $v_j^{' \in G' \setminus G}$, then for the restrictions $\tau_j^{' = \tau_j^{'(v_j^{'})} = \tau_j^{'(v_j^{'})}}$, but they agree on all other vertices, so $\tau_j^{' \sim \tau_j^{'1 \in G} \sim \text{Hom}(G, H)$, Hence the path in $\text{Hom}(G', H)$ projects onto a possibly shorter path from $\eta_1$ to $\eta_2$ in $\text{Hom}(G, H)$, which is a contradiction.

Therefore the extensions $\eta_1^{'}, \eta_2^{' \in}$ must be in different connected components of $\text{Hom}(G', H)$, which is thus disconnected. □

The subgraphs we examine to obtain disconnected components of $\text{Hom}(G, K_3)$ will be cycles. In [7], Ćukić and Kozlov fully characterized the homotopy type of $\text{Hom}(C_n, C_m)$ for all $n, m \in \mathbb{N}$. In particular, they showed that all 0-cells in a given connected component have the same number of what they call return points. For these complexes, let $V(C_n) = \{v_1, \ldots, v_n\}$ such that $v_i \sim v_{(i+1) \mod n}$ for all $1 \leq i \leq n$, and let $V(C_m) = \{1, \ldots, m\}$ such that $j \sim (j + 1) \mod m$ for all $1 \leq j \leq m$. For the purpose of defining the return number of a 0-cell $\eta$, we momentarily drop the set bracket notation and write $\eta(v_i) = j$.

**Definition 4.3.** A return point of a 0-cell $\eta \in \text{Hom}(C_n, C_m)$ is a vertex $v_i \in C_n$ such that $\eta(v_{i+1}) = \eta(v_i) \equiv -1 \mod m$. Then $r(\eta)$, the return number of $\eta$, is the number of $v_i \in C_n$ which are return points of $\eta$.

Note that since $\eta$ is a homomorphism, the quantity $\eta(v_{i+1}) - \eta(v_i)$ is always $\pm 1 \mod m$. It simply measures in which direction $C_n$ is wrapping around $C_m$ on the edge $\{v_i, v_{i+1}\}$.

**Lemma 4.4.** (Ćukić, Kozlov, Lemma 5.3 in [7]) If two 0-cells of $\text{Hom}(C_n, C_m)$ are in the same connected component, then they have the same return number.

For a 0-cell $\eta \in \text{Hom}(G, K_m)$, one may always obtain another 0-cell by swapping the inverse images of two vertices in $K_m$. When the target graph is a complete graph, we refer to the inverse image of a vertex in $K_m$ as a color class of $\eta$. When $m = 3$, we have $K_3 = C_3$ and we can track the effect that interchanging two color classes has on the return number.

**Notation 4.5.** For a 0-cell $\eta \in \text{Hom}(G, K_m)$, let $\eta_{[i, j]}$ denote the $(i, j)$-interchanging 0-cell obtained by defining $\eta_{[i, j]}^{-1}(i) = \eta^{-1}(j)$, $\eta_{[i, j]}^{-1}(j) = \eta^{-1}(i)$, and $\eta_{[i, j]}^{-1}(i) = \eta^{-1}(i)$ for all $i \not\in \{i, j\}$.

**Lemma 4.6.** Fix a pair $\{i, j\} \subset \{1, 2, 3\}$, $i \neq j$. Then for any 0-cell $\eta \in \text{Hom}(C_n, K_3)$, every $v_i \in C_n$ is a return point for exactly one of $\eta$ and $\eta_{[i, j]}$. Hence $r(\eta_{[i, j]}) = n - r(\eta)$. 
Proof of Lemma 4.6. Consider the edge \{v_i, v_{i+1}\}. Since the target graph is \(K_3\), we have that \(\{\eta(v_i), \eta(v_{i+1})\} \cap \{l, j\} \neq \emptyset\). If \(\{\eta(v_i), \eta(v_{i+1})\} = \{l, j\}\), then
\[
\eta[l,j](v_{i+1}) - \eta[l,j](v_i) = -(\eta(v_{i+1}) - \eta(v_i))
\]
Thus \(v_i\) is a return point of \(\eta\) if and only if it is not a return point of \(\eta_{[l,j]}\). Alternatively, if \(|\{\eta(v_i), \eta(v_{i+1})\} \cap \{l, j\}| = 1\), then one of \(v_i\) and \(v_{i+1}\) has a stationary image under an \((l, j)\)-interchange while the other does not. By fixing the image of one vertex and changing the other, this flips the direction that \(C_n\) is wrapping around \(K_3\) on the edge \(\{v_i, v_{i+1}\}\). So returns become non-returns and vice versa. Therefore each \(v_i \in C_n\) is a return point of exactly one of \(\eta\) and \(\eta_{[l,j]}\) for any fixed \(\{l, j\} \subset \{1, 2, 3\}\). \(\square\)

The proof of Theorem 4.1 will then proceed by finding an odd cycle in \(G\) and using a color class interchange to produce 0-cells in distinct connected components, which can be lifted from \(\text{Hom}(C_{2k+1}, K_3)\) to \(\text{Hom}(G, K_3)\).

Proof of Theorem 4.1 Since \(\chi(G) = 3\), \(\text{Hom}(G, K_3)\) is non-empty and \(G\) contains an odd cycle \(H = C_{2k+1}\) for some \(k \in \mathbb{N}\). Note that we do not require \(H\) to be an induced cycle. Let \(\eta \in \text{Hom}(G, K_3)\) be a 0-cell, and let \(\tau = \eta|_H\) be the induced homomorphism on \(H\). Fix a distinct pair \([l, j] \subset \{1, 2, 3\}\) and consider the \((l, j)\)-interchange \(\eta_{[l,j]}\). It is straightforward that \(\eta_{[l,j]}|_H = \tau_{[l,j]} \in \text{Hom}(H, K_3)\). Thus by Lemma 4.6, \(r(\tau_{[l,j]}) = 2k + 1 - r(\tau) \neq r(\tau)\). Hence, by lemma 4.4 they are in different connected components of \(\text{Hom}(H, K_3)\), and by Lemma 4.2 their extensions \(\eta\) and \(\eta_{[l,j]}\) are in different connected components of \(\text{Hom}(G, K_3)\), which is thus disconnected. \(\square\)

Bipartite graphs are a bit more complicated. We have seen that for any graph \(G\) which can be reduced to a single edge, \(\text{Hom}(G, K_3) \cong S^1\). When \(G\) is bipartite and does not fold to an edge, it must contain an even cycle \(C_{2k}\) for \(k \geq 3\). So by the same method one can start with \(\eta \in \text{Hom}(G, K_3)\) and take its restriction \(\eta|_{C_{2k}} = \tau \in \text{Hom}(C_{2k}, K_3)\). But if \(r(\tau) = k\), then interchanging two color classes no longer guarantees disjoint components. In particular, the existence of too many 4-cycles close to \(C_{2k}\) may force \(r(\tau) = k\). For example, for \(Q_3\) the 1-skeleton of the 3-cube, \(\text{Hom}(Q_3, K_3)\) is connected. Note that \(D(Q_3) = 3\), so the lower bound achieved by Theorem 2.8 does not provide any information. Restricting to the case that \(G\) omits the subgraphs in Figure 4.1 is sufficient to ensure that this does not happen, and that \(\text{Hom}(G, K_3)\) is disconnected.

\[
H_1=
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ }
\end{array}
\quad
H_2=
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ }
\end{array}
\]

Figure 4.1. If \(\text{Hom}(G, K_3)\) is connected, then \(G\) must contain \(H_1\) or \(H_2\).

Theorem 4.7. If \(G\) is a finite, bipartite, connected graph which does not fold to an edge and does not contain \(H_1\) or \(H_2\) as a subgraph, then \(\text{Hom}(G, K_3)\) is disconnected.

To show this requires a great deal of case-by-case structural analysis when a minimal even cycle in \(G\) has length 6, 8, or 10. The method of the proof will suggest a stronger, albeit more technical result.
Proof of Theorem 4.7. Let $G$ be a finite, bipartite, connected graph which does not fold to an edge and does not contain $H_1$ or $H_2$ as a subgraph. Since $G$ is finite, bipartite and does not fold to an edge, it must contain an even cycle of length greater than 4. Let $H = C_{2k}$, $k \geq 3$ be a fixed cycle in $G$ which has minimal length over all cycles in $G$ of length greater than 4. Label the vertices of $H$ as $v_1, \ldots, v_{2k}$ such that $v_i \sim v_{i+1} \mod 2k$ for all $i$. $H$ cannot have internal chords, as any such edge would create a cycle $C_i$ for $4 < i < 2k$, or in the case that $k = 3$ an antipodal chord would create a copy of $H_1$. For $2k \equiv i \mod 3$, take $\tau \in \text{Hom}(H, K_3)$ such that $\tau(v_j) = \{j \mod 3\}$ for $j \leq 2k - i$, with $\tau(v_{2k}) = \{2\}$ if $i = 1$, and $\tau(v_{2k-1}) = \{1\}$, $\tau(v_{2k}) = \{2\}$ if $i = 2$. Figure 4.2 depicts $\tau$ for $k \in \{3, 4, 5\}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure42.png}
\caption{Image of $\tau$ on $H = C_{2k}$}
\end{figure}

As defined, only the last two vertices of $C_{2k}$ can be return points of $\tau$. So $r(\tau) \leq 2$ in all cases, and hence is not equal to $k$ for any $k \geq 3$. Then for any $\{l, j\} \subset \{1, 2, 3\}$, $r(\tau_{l,j}) \neq r(\tau)$. Thus $\tau_{l,j}$ and $\tau$ are in distinct connected components of $\text{Hom}(H, K_3)$. So if we can construct an extension $\eta$ to all of $G$, then $\eta(l,j)$ will extend $\tau_{l,j}$ and hence $\eta$ will be in distinct connected component of $\text{Hom}(G, K_3)$.

Define $d(u, v)$ to be the minimal number of edges in a path connecting two vertices $u, v \in G$. Let $\sigma$ be a bipartition of $G$, viewed as a 0-cell of $\text{Hom}(G, K_3)$ with image contained in $\{1, 2\}$. We will let $\eta = \sigma$ on $G \setminus H$ and then make adjustments as necessary when the definition conflicts with $\tau$. Define the following sets:

$$
B_i := \{u \in G : \min \{d(u, v) : v \in H\} = i\}
$$

$$
A := \{u \in B_1 : \exists v \in N(u) \cap H \text{ with } \sigma(u) = \tau(v)\}
$$

$$
A_{i,j} := \{u \in A : |N(u) \cap H| = i, |N(u) \cap A| = j\}
$$

So $B_i$ is the set of points distance $i$ from $H$, and $A$ is the set of points in $G \setminus H$ on which we cannot define $\eta = \sigma$. To arrive at an appropriate definition for $\eta$ on the set $A$, we must first make some structural observations concerning the partitioning of $A$ into the subsets $A_{i,j}$.

Claim 4.8. For every $u \in B_1$, $|N(u) \cap H| \leq 2$. And if $|N(u) \cap H| = 2$, then $N(u) \cap H = \{v_i, v_{(i+2) \mod 2k}\}$ for some $i \leq 2k$.

Proof of Claim 4.8. Let $u \in B_1$. Then connecting $u$ to any two points in $H$ creates a cycle of length at most $k + 2 < 2k$, as in Figure 4.3 (a). Since $G$ is bipartite and $H$ is minimal with respect to having length at least 6, this is valid only if it creates a 4-cycle, with
$N(u) \cap H = \{v_i, v_{(i+2) \mod 2k}\}$ for some $i \leq 2k$. Similarly, any further points of attachment would need to be in a 4-cycle with each of the other two vertices in $N(u) \cap H$. But this is only possible if $H = C_6$, in which case it would create several copies of $H_1$, shown in Figure 4.3 (b). □

![Figure 4.3](image.png)

Figure 4.3. Obstructions to $|N(u) \cap H| > 2$

This imposes that $i \leq 2$ for $A_{i,j} \subseteq A$, while the following two claims will show that $j$ is at most 1, and that $j = 1$ only if $i = 1$.

**Claim 4.9.** Let $k \geq 4$, and let $u \in B_1$. Then $|N(u) \cap B_1| \leq 1$, and if $N(u) \cap B_1 \neq \emptyset$ then $|N(u) \cap H| = 1$.

**Proof of Claim.** Let $k \geq 4$, let $u \in B_1$, and let $w \in N(u) \cap B_1$, with $u \sim v_i$ and $w \sim v_j$. $G$ is bipartite, so $v_i \neq v_j$. If $v_i \neq v_j$, as in Figure 4.4 (a), there is a cycle of length $l$ with $4 < l \leq k + 3 < 2k$. So $v_i \sim v_j$. Suppose further that $|N(u) \cap H| = 2$. Then $N(u) \cap H = \{v_{(j-1) \mod 2k}, v_{(j+1) \mod 2k}\}$ since both must be adjacent to $v_j$, as in Figure 4.4 (b). But then the induced graph on $\{u, w, v_{(j-1) \mod 2k}, v_j, v_{(j+1) \mod 2k}\}$ is a copy of $H_2$. So if $N(u) \cap B_1 \neq \emptyset$ for $u \in B_1$, then $|N(u) \cap H| = 1$. Now suppose $|N(u) \cap B_1| > 1$. Then there is a vertex $x \in N(u) \cap B_1$ with $x \neq w$. As with $w$, the vertex in $N(x) \cap H$ must be adjacent to $v_i$. If $x \sim v_j$, as in Figure 4.4 (c), then $\{u, w, x, v_i, v_j\}$ is a copy of $H_2$, and if $x$ is connected to the other neighbor of $v_i$, as in Figure 4.4 (d), then it creates a copy of $H_1$. Thus $|N(u) \cap B_1| \leq 1$. □

![Figure 4.4](image.png)

Figure 4.4. Invalid structures in $G$ if Claim 4.9 is false.

For $k = 3$ a similar result holds when $B_1$ is replaced by the subset $A$:

**Claim 4.10.** Let $k = 3$, and let $u \in A$. Then $|N(u) \cap A| \leq 1$, and if $N(u) \cap A \neq \emptyset$ then $|N(u) \cap H| = 1$. 
Proof of Claim. Let \( u \in A \) and let \( w \in N(u) \cap A \). Recall that for \( 2k = 6 \), \( \tau(v_i) = \{i \mod 3\} \) for all \( i \). Then without loss of generality assume that \( \sigma(u) = \{1\} \) and \( u \sim v_1 \). So \( \sigma(w) = \{2\} \), and since \( w \in A \) it must be adjacent to at least one of \( v_3 \) and \( v_5 \). But \( G \) is bipartite and \( u \sim v_5 \) creates a 5-cycle, as indicated in Figure 4.5 (a). Thus \( w \sim v_2 \), and the same argument holds to show that \( u \sim v_1 \) if one first assumes that \( w \sim v_2 \). Furthermore, if \( w \sim v_4 \) or \( w \sim v_6 \) there is a copy of \( H_1 \) or \( H_2 \), respectively, depicted in Figure 4.5 (b), (c). By symmetry \( u \sim v_3, \ u \sim v_5 \). Hence, if \( u \in A \) with \( N(u) \cap A \neq \emptyset \), then \(|N(u) \cap H| = 1\). Now suppose there is \( x \in N(u) \cap A \) with \( x \neq w \). Then \( \tau(x) = \{2\} \), and by the preceding argument we have that \( x \sim v_2 \). But, as shown in Figure 4.5 (d), \( \{u, w, x, v_3, v_2\} \) is then a copy of \( H_2 \). Hence for \( u \in A \), \(|N(u) \cap A| \leq 1\). \( \square \)

![Figure 4.5. Invalid structures in G if Claim 4.10 is false.](image)

Then for all \( k \geq 3 \) we have that \( A = A_{1,0} \cup A_{2,0} \cup A_{1,1} \), and vertices in \( A_{1,1} \) which are adjacent to each other have adjacent attaching vertices in \( H \). When possible, we will define \( \eta(v) = \{3\} \) for vertices \( v \in A \). However, this not possible on all vertices in \( A_{1,1} \), nor for vertices in \( A_{2,0} \) which have a neighbor in \( H \) that is already assigned \( \{3\} \). So define

\[
\overline{A} = \{u \in A_{2,0} : (\tau(v) : v \in N(u) \cap H) = \{\sigma(u), \{3\}\})\cup \{u \in A_{1,1} : \sigma(u) = \{2\}\}
\]

Now define \( \eta : G \to K_3 \) by:

\[
\eta(u) = \begin{cases} 
\tau(u) & \text{if } u \in H \\
\{1\} & \text{if } u \in \overline{A} \cap A_{1,1}; \\
\{2\} & \text{if } u \in \overline{A} \cap A_{2,0} \text{ with } \sigma(u) = \{2\}; \\
\{3\} & \text{if } u \in A \setminus \overline{A}; \\
\sigma(v) & \text{if } u \in G \setminus (A \cup H) \text{ with } N(u) \cap \overline{A} \neq \emptyset \\
\end{cases}
\]

Notice that for \( u \in \overline{A}, \eta(u) = \{1, 2\} \setminus \sigma(u) = \sigma(w) \) for any \( w \in N(u) \). Hence the last qualifier assigns \( \{3\} \) to any neighbors of vertices in \( \overline{A} \) which need to be switched, and \( \eta = \sigma \) on all remaining vertices.

It is clear that this defines a homomorphism on \( A \cup H \), and on any edges between vertices \( u \in G \setminus (A \cup H) \) for which \( N(u) \cap \overline{A} = \emptyset \). Thus it suffices to check that this is consistent on edges with a vertex \( u \in G \setminus (A \cup H) \) for which \( N(u) \cap \overline{A} \neq \emptyset \). Note that by definition, any such \( u \) will have \( \eta(u) = \{3\} \). So to ensure that \( \eta(w) \neq \{3\} \) for any \( w \in N(u) \),
we must check that \( N(u) \cap (A \setminus \overline{A}) = \emptyset \), and for all \( w \in N(u) \) that \( N(w) \cap \overline{A} = \emptyset \). The cases \( k = 3 \) and \( k \geq 4 \) must again be handled separately.

**Claim 4.11.** Let \( k \geq 4 \), and let \( u \in G \setminus (A \cup H) \) with \( x \in N(u) \cap \overline{A} \). Then \( N(u) \cap (A \setminus \overline{A}) = \emptyset \), and if \( w \in N(u) \) then \( N(w) \cap \overline{A} = \emptyset \).

**Proof of Claim.** Let \( u \) and \( x \) be as in the claim. Since \( x \in A \subseteq B_1 \), it must be that \( u \in (B_1 \cup B_2) \setminus A \). Suppose first that \( u \in B_1 \). Then \( |N(u) \cap B_1| \leq 1 \) by Claim 4.9, and hence \( N(u) \cap A = \{x\} \). So \( N(u) \cap (A \setminus \overline{A}) = \emptyset \). Similarly, Claim 4.9 implies that \( N(x) \cap B_1 = \{u\} \) and that \( |N(x) \cap H| = 1 \), so \( x \in A_{1,0} \), which contradicts our assumption that \( x \in \overline{A} \subseteq A_{2,0} \cup A_{1,1} \). So \( u \notin B_1 \).

Suppose instead that \( u \in B_2 \), and thus \( N(u) \subseteq B_1 \cup B_2 \cup B_3 \). For \( w \in N(u) \cap B_3 \) it is immediate that \( w \notin A \) and that \( \{N(w) \cap \overline{A}\} \subseteq N(w) \cap B_1 = \emptyset \). Then assume that there is \( w \in N(u) \cap B_1 \), \( w \neq x \). Then \( \sigma(w) = \sigma(x) \) since \( G \) is bipartite, and thus for any attaching points in \( H \), \( v_i \sim x \), \( v_j \sim w \), it must be that \( i \) and \( j \) are either both odd or both even. Since \( x \in A_{2,0} \cup A_{1,1} \), we may assume that either \( i \neq j \) or \( x \in A_{1,1} \). Recall that an adjacent pair in \( A_{1,1} \) forms a square with its adjacent pair in \( H \). So if \( i = j \) and \( x \in A_{1,1} \) with \( \{z\} = N(x) \cap A_{1,1} \), then \( z \) is adjacent to one of the two vertices in \( N(v_i) \cap H \). As shown in Figure 4.6 (a), this creates a copy of \( H_1 \). So we may assume that \( i \neq j \). But then the short path between \( v_i \) and \( v_j \) in \( H \) along with \( \{u, w, x\} \) creates a cycle of length \( l \), \( 6 \leq l \leq k + 4 \).

![Figure 4.6](image_url)

If \( k \geq 5 \) this length is less than \( 2k \), as in Figure 4.6 (b), which contradicts the minimality of \( H \). If \( k = 4 \), this creates a cycle of length 8 only if \( j = (i + 4) \mod 8 \), as in Figure 4.6 (c). Recall that as defined for \( k = 4 \) (see Figure 4.2 (b)), \( \tau(v_i) \neq \tau(v_{(i+4) \mod 8}) \) for all \( i \). But \( \sigma(x) = \sigma(w) \), and therefore \( x \) and \( w \) cannot both be in \( A \). As this is true for any pair of vertices in \( N(u) \cap B_1 \), it must be that \( |N(u) \cap A| \leq 1 \), and hence by assumption \( N(u) \cap A = \{x\} \). Furthermore, for any \( w \in B_1 \setminus A \), if \( w \sim z \) with \( z \in A \), then by Claim 4.9 it must be that \( z \in A_{1,0} \). Thus \( N(w) \cap A = \emptyset \) for all \( w \in N(u) \cap B_1 \).

Now suppose that \( w \in N(u) \cap B_2 \). Note that \( w \notin A \subseteq B_1 \). Then let \( x = x_u \) and let \( x_w \in N(w) \cap B_1 \). Note that \( \sigma(x_u) \neq \sigma(x_w) \) since there is a path of length 3 from \( x_u \) to \( x_w \), so the connecting vertices for \( x_u \) and \( x_w \) in \( H \) must be distinct to avoid creating an odd cycle. Then the path between their connecting vertices in \( H \) along with \( x_u, u, w, \) and \( x_w \) creates a cycle of length \( l \), \( 6 \leq l \leq k + 5 \). For \( k > 5 \) such a cycle contradicts the minimality of \( H \).

Suppose \( k = 4 \). Then to avoid creating a cycle of length less than 8, this construction requires that the path in \( H \) between the connecting points of \( x_u \) and \( x_w \) is length 3. Suppose by way of contradiction that \( x_w \in \overline{A} \). So both \( x_u, x_w \in \overline{A} \subseteq A_{2,0} \cup A_{1,1} \). Without loss of
generality, assume that $x_u \in A_{2,0}$ with $N(x_u) \cap H = \{v_i, v_{(i+2) \mod 8}\}$ for some $1 \leq i \leq 8$. Then $v_{(i+5) \mod 8}$ is the only vertex in $H$ which is distance 3 from both $v_i$ and $v_{(i+2) \mod 8}$. So $x_w$ cannot also be in $A_{2,0}$, and thus $x_w \in A_{1,1}$ with $N(x_w) \cap H = \{v_{(i+5) \mod 8}\}$. By definition, $\sigma(v) = \{2\}$ for all $v \in \overline{A} \cap A_{1,1}$, so we have that $\sigma(x_u) = \{2\}$, and thus also $\tau(v_{(i+5) \mod 8}) = \{2\}$. For $\tau$ as defined, this forces $i \in \{3,5,8\}$. But now $\sigma(x_w) = \{1\}$, and so $x_u, x_w \in \overline{A} \cap A_{2,0}$ implies that $\{\tau(v_i), \tau(v_{(i+2) \mod 8})\} = \{\{1\}, \{3\}\}$, which is only possible if $i \in \{1,4\}$. Figure 4.7 (a) and (b) shows the arrangements if $i = 1$ and $i = 4$ respectively, with $\tau(v_6), \tau(v_1) \neq \{2\}$. Hence it must be that both $x_u, x_w \in A_{1,1}$. However, $\sigma(x_u) \neq \sigma(x_w)$, and as noted above $\sigma(v) = \{2\}$ for all $v \in \overline{A} \cap A_{1,1}$. So at most one of $x_u$ and $x_w$ can be in $\overline{A}$, and by assumption $x_u \in \overline{A}$. Thus $x_w \notin \overline{A}$ and $N(w) \cap \overline{A} = \emptyset$ for all $w \in N(u) \cap B_2$.

![Figure 4.7](https://example.com/figure4.7.png)

**Figure 4.7.** $x_u \in \overline{A} \cap A_{2,0}, x_w \in \overline{A} \cap A_{1,1}, \sigma(x_u) = \{1\}$ when $k = 4$.

Now suppose $k = 5$, and again assume that $x_w \in \overline{A}$. Here cycles created by adjoining $u, w, x_u,$ and $x_w$ will be of length 10 only if the connecting vertices in $H = C_{10}$ are distance 5 apart. Hence neither $x_u$ nor $x_w$ can be in $A_{2,0}$, and both must be in $A_{1,1}$. Thus, as in the case above for $k = 4$, $\sigma(x_u) \neq \sigma(x_w)$ implies that at most one of $x_u$ and $x_w$ can be in $\overline{A}$. Therefore $x_w \notin \overline{A}$ and $N(w) \cap \overline{A} = \emptyset$ for all $w \in N(u) \cap B_2$.

**Claim 4.12.** Let $k = 3$, and let $u \in G \setminus (A \cup H)$ with $x \in N(u) \cap \overline{A}$. Then $N(u) \cap (A \setminus \overline{A}) = \emptyset$, and if $w \in N(u)$ then $N(w) \setminus \overline{A} = \emptyset$.

**Proof of Claim.** Let vertices $u$ and $x$ be as in the claim, and let $v_i \in N(x) \cap H$ be a vertex such that $\sigma(x) = \tau(v_i)$, which exists by the assumption that $x \in A$. Then since $G$ is bipartite, it must be that $\{v \in H : d(v, u) = 2\} \subseteq \{v_j, v_{(j+2) \mod 6}, v_{(j+4) \mod 6}\}$. Recall that $\tau$ is defined such that for each $j \in K_3$ the set $\tau^{-1}(\{j\})$ contains one vertex with an odd index and one vertex with an even index (see Figure 4.2 (a)). So $\{v_j, v_{(j+2) \mod 6}, v_{(j+4) \mod 6}\} \cap \tau^{-1}(\{v_i\}) = \emptyset$. Hence $w \sim v_i$ for every $w \in N(u) \cap A$. Then if $|N(u) \cap A| \geq 3$, this would create a copy of $H_2$, depicted in Figure 4.8 (a). Thus $|N(u) \cap A| \leq 2$.
Suppose that \(|N(u) \cap A| = 2\). If there was an adjacency from the set \(\{v_{(i+2)} \mod 6, v_{(i+4)} \mod 6\}\) to \(N(u) \cap A\), it would create a copy of \(H_1\), as in Figure 4.8 (b). So \(N(u) \cap A_{2,0} = \emptyset\). And adjacent pairs in \(A_{1,1}\) form squares with their connecting pair in \(H\). So if \(w \in N(u) \cap A_{1,1}\), then there is \(w' \in N(w) \cap A_{1,1}\) such that, without loss of generality, \(w' \sim v_{i+1} \mod 6\). But, as depicted in Figure 4.8 (c), this creates a copy of \(H_1\). Thus it cannot be that both \(|N(u) \cap A| > 1\) and \(N(u) \cap (A_{2,0} \cup A_{1,1}) \neq \emptyset\). Since \(x \in \overline{A} \subseteq A_{2,0} \cup A_{1,1}\) by assumption, we have that \(N(u) \cap A = \{x\}\), and so \(N(u) \cap (A \setminus \overline{A}) = \emptyset\).

Now it suffices to check for all \(w \in N(u) \setminus (H \cup \{x\})\) that \(N(w) \cap \overline{A} = \emptyset\). Suppose that \(u \in B_1\). Then \(N(u) \cap H \subseteq \{v_{(i+1)} \mod 6, v_{(i+3)} \mod 6, v_{(i+5)} \mod 6\}\) since \(x \sim v_i\). Further suppose that \(x \in A_{2,0}\), so without loss of generality \(N(x) \cap H = \{v_i, v_{(i+2)} \mod 6\}\). Then all three possible adjacencies for \(u\) in \(H\) are shown in Figure 4.9, with \(u \sim v_{(i+1)} \mod 6\) creating a copy of \(H_2\), while \(v \sim v_{(i+3)} \mod 6\) or \(v \sim v_{(i+5)} \mod 6\) each creates a copy of \(H_1\). Hence \(x \notin A_{2,0}\).

So \(x \in \overline{A} \cap A_{1,1}\) which implies that \(\sigma(x) = \{2\}\), and so \(\sigma(u) = \{1\}\). Then without loss of generality let \(i = 2\). Note \(x \sim v_2\) implies \(u\) must be connected to a vertex in \(H\) which has odd index. But \(u \notin A\), so \(u \sim v_1\). Let \(w = N(x) \cap A_{1,1}\), so it must be that \(\sigma(w) = \{1\}\) and \(w \sim v_1\).

Then \(u \sim v_3\) would create a copy of \(H_1\), shown in Figure 4.10 (a), and so \(u \sim v_5\). Now consider a vertex \(z \in N(u) \setminus (H \cup \{x\})\), and suppose there exists \(v_{\overline{z}} \in N(z) \cap \overline{A}\). Then \(d(u, v_{\overline{z}}) = 2\), so \(\sigma(v_{\overline{z}}) = \sigma(u) = \{1\}\). And by definition, we have that \(\{v \in \overline{A} : \sigma(v) = \{1\}\} \subseteq A_{2,0}\), so \(v_{\overline{z}} \in A_{2,0}\) and the two vertices in \(N(v_{\overline{z}}) \cap H\) have images \(\{1\}\) and \(\{3\}\) under \(\sigma\). Since \(\sigma(v_{\overline{z}}) = \sigma(u)\), vertices in \(N(v_{\overline{z}}) \cap H\) must have odd index. Thus \(N(v_{\overline{z}}) \cap H = \{v_1, v_3\}\). As shown in Figure 4.10 (b), \(\{w, x, v_2, v_1, v_2, v_3\}\) is then a copy of \(H_1\). So if \(u \in B_1\) there cannot be any vertex \(z \in N(u)\) with \(N(z) \cap \overline{A} \neq \emptyset\).
Now assume that $u \in B_2$. Let $x = x_u$, and suppose that there is a vertex $w \in N(u)$ which has a vertex $x_w \in N(w) \cap \overline{A}$. Then there is a path of length 3 from $x_u$ to $x_w$, so $x_u \neq x_w$ and also $\sigma(x_u) \neq \sigma(x_w)$. Then they cannot both be in $A_{1,1}$, since $\sigma(v) = [2]$ for all $v \in \overline{A} \cap A_{1,1}$. Without loss of generality, let $x_u \in A_{2,0}$. Suppose that $x_u \in A_{1,1}$ and let $\{z\} = N(x_u) \cap A_{1,1}$. Then $\sigma(x_u) = [2]$ and without loss of generality we have that $x_u \sim v_2$ and $z \sim v_1$. Now $x_u$ must be adjacent to two vertices in $H$ with odd indices, one of which has image $[3]$ under $\tau$. And $\sigma(x_u) = [1]$, so $N(x_u) \cap H = \{v_1, v_3\}$. This is depicted in Figure 4.11 (a), where $\{v_1, v_2, v_3, x_u, x_w, z\}$ forms a copy of $H_1$. Suppose instead that $x_u \in A_{2,0}$. Then each $x_w$ and $x_u$ must be adjacent to a vertex whose image under $\tau$ is $[3]$. Without loss of generality assume that $x_w \sim v_3$ and $x_u \sim v_6$, and let $x_u \sim v_j$ and $x_w \sim v_l$ be the other two adjacencies. Then $j$ must be odd and $l$ must be even, and $\tau(v_j) = \sigma(x_u) \neq \sigma(x_w) = \tau(v_l)$. So either $j = 1$ and $l = 2$, or $j = 5$ and $l = 4$. By symmetry we may assume that $x_u \sim v_1$ and $x_w \sim v_2$, as in Figure 4.11 (b). But then $\{v_1, v_2, v_3, x_u, x_w\}$ forms a copy of $H_1$. Thus $N(w) \cap \overline{A} = \emptyset$ for all $w \in N(u)$. □

It follows immediately from Claims 4.11 and 4.12 that $\eta : G \to K_3$ is a homomorphism for all $k \geq 3$. And $\eta$ extends $\tau$, so $\eta_{[i,j]}$ extends $\tau_{[i,j]}$. Thus by Lemma 4.2, $\eta$ and $\eta_{[i,j]}$ are in distinct connected components of $\text{Hom}(G, K_3)$, which is therefore disconnected. □

The details of this proof suggests that it is only necessary to require that there be some cycle $C_{2k} \subset G$, $k \geq 3$, such that $H' \cap C_{2k} = \emptyset$ for any subgraph $H' \subset G$ with $H' \in \{H_1, H_2\}$. But even this leaves out many bipartite $G$ for which $\text{Hom}(G, K_3)$ is disconnected. For example, consider the graph $G'$ which is a circular ladder with six rungs. So $G'$ is...
two disjoint copies of $C_6$ with respective vertices labeled consecutively as $v_1, \ldots, v_6$, and $w_1, \ldots, w_6$, and additional edges $v_i \sim w_i$ for each $i$. Note that $G'$ is bipartite, does not admit any folds, and every edge is contained in at least two copies of $H_1$. Define $\eta(v_i) = [i \mod 3]$ and $\eta(w_i) = [(i + 1) \mod 3]$ for all $i$. Then let $\tau$ be the restriction of $\eta$ to the induced cycle on $[v_1, \ldots, v_6]$. So $r(\tau) = 0$, and as before $\eta$ and $\eta_{[l,j]}$ are in distinct connected components of $\text{Hom}(G', K_3)$ for any fixed pair $[l, j] \subset \{1, 2, 3\}$.

In the contrarian case of $Q_3$, again every edge is contained in multiple copies of $H_1$. The real issue, however, is that every 6-cycle in $Q_3$ is contained in either an induced copy of $H_1$ or an induced copy of $Q_3 \setminus \{v\}$ for some $v \in Q_3$. Both of these graphs fold to an edge, so $\text{Hom}(H_1, K_3)$ and $\text{Hom}(Q_3 \setminus \{v\}, K_3)$ are each connected. But then any distinct 0-cells $\tau_1, \tau_2 \in \text{Hom}(C_6, K_3)$ which extend to all of $Q_3$ must first extend to either $H_1$ or $Q_3 \setminus \{v\}$, where their extensions are necessarily in a single connected component.

5. $\text{Hom}(G(n, p), K_m)$

The topological connectivity of random hom-complexes has been studied previously under the guise of the Neighborhood Complex of $G(n, p)$, the Erdős–Rényi model for a random graph. $G(n, p)$ is the probability space of graphs on $n$ vertices where each edge is inserted independently with probability $p = p(n)$. Kahle [9] showed that the connectivity of $\mathcal{N}(G(n, p))$ is concentrated between $1/2$ and $2/3$ of the expected value of the largest clique in $G(n, p)$, and also obtained asymptotic bounds on the number of dimensions with non-trivial homology. As we discussed earlier, $\mathcal{N}(G(n, p))$ is homotopy equivalent to $\text{Hom}(K_2, G(n, p))$. Here we take the opposite perspective and consider the random polyhedral complex $\text{Hom}(G(n, p), K_m)$.

The major benefit of utilizing $D(G)$ in a generalization of the Ćukić-Kozlov Theorem is that $k$-cores have been well-studied in random graphs models. We say that $G(n, p)$ has property $\mathcal{P}$ with high probability if $\lim_{n \to \infty} \text{Pr}[G(n, p) \in \mathcal{P}] = 1$. A property $\mathcal{P}$ is said to have a sharp threshold $\hat{p} = \hat{p}(n)$ if for all $\epsilon > 0$

$$\lim_{n \to \infty} \text{Pr}[G(n, p) \in \mathcal{P}] = \begin{cases} 0 & \text{if } p \leq (1 - \epsilon)\hat{p}, \\ 1 & \text{if } p \geq (1 + \epsilon)\hat{p}. \end{cases}$$

When $p = c/n$ for constant $c > 0$, Pittel, Spencer and Wormald [13] showed that the existence of a $k$-core in $G(n, p)$ has a sharp threshold $c = c_k$ for $k \geq 3$, and that asymptotically $c_k = k + \sqrt{k \log k} + O(\log k)$. Approximate values for small $k$ are known, such as $c_3 \approx 3.35$, $c_4 \approx 5.14$, $c_5 \approx 6.81$. When $k = 2$, the existence of cycles has a one-sided sharp threshold at $c = 1$. For $c > 1$, indeed $\text{Pr}[D(G(n, c/n)) \geq 2] \to 1$. However, for all $0 < c < 1$ there is a constant $0 < f(c) < 1$ for which $\text{Pr}[D(G(n, c/n)) \geq 2] \to f(c)$. To simplify this issue, define $c_k$ for all $k \geq 2$ by

$$c_k := \sup \left\{ c > 0 : \lim_{n \to \infty} \text{Pr}[D(G(n, c/n) \geq k) = 0]\right\}$$

In particular $c_2 = 0$, and for $k \geq 3$ these are precisely the sharp thresholds mentioned above.

By applying Theorem 2.8 to these thresholds, we then immediately obtain lower bounds on the topological connectivity of $\text{Hom}(G(n, p), K_3)$. For notational convenience, define $M(n, c, m) := \text{conn} [\text{Hom}(G(n, c/n), K_m)]$.

Theorem 5.1. If $k \geq 2$ and $p = c/n$ with $c < c_{k+1}$, then for all $m \geq 3$

$$\lim_{n \to \infty} \text{Pr}[M(n, c, m) \geq m - k - 2] = 1$$
Theorem 2.8 does not require that the input graph be connected, and in fact $G(n, p)$ will be disconnected when $p = c/n$. Let the disjoint connected components of $G(n, p)$ be $G_0, G_1, \ldots, G_t$, ordered from most vertices to least. For $c > 1$, with high probability $G_0$ is a giant component containing more than half the vertices, and $G_i$ is either an isolated vertex, a tree, or a unicyclic graph for $i \geq 1$. So

$$\text{Hom}(G(n, c/n), K_m) = \text{Hom}\left( \bigcup_{i=0}^{t} G_i, K_m \right) = \prod_{i=0}^{t} \text{Hom}(G_i, K_m)$$

And thus

$$M(n, c, m) = \min_{0 \leq i \leq t} \{ \text{conn}[\text{Hom}(G_i, K_m)]\}$$

For $G_i = \{v_i\}$, $\text{Hom}(G_i, K_m) = \Delta^2$, which is contractible. If a finite connected graph $G_i$ is a tree, then it folds to a single edge and $\text{Hom}(G_i, K_m) \approx \text{Hom}(K_2, K_m) \approx S^{m-2}$. Finally, if $G_i$ is a finite unicyclic graph, then it folds to its cycle $C_n$. If $n = 4$, then $C_4$ folds further to $K_2$ and $\text{Hom}(G_i, K_m) \approx S^{m-2}$. Thus, with high probability we have

$$\text{Hom}(G(n, c/n), K_m) = \left( \Delta^2 \right)^{t_1} \times \left( S^{m-2} \right)^{t_2} \times \text{Hom}(G_0, K_m) \times \prod_{j=1}^{n} \text{Hom}(C_{n_j}, K_m)$$

where $t_1$ is the number of isolated vertices, $t_2$ is the number of trees and unicyclic components whose cycle is $C_{t_1}$, $t_3 = t - (t_1 + t_2)$, and $n_j \neq 4$ for all $j$.

When $n = 3$, $\text{Hom}(C_n, K_m) = \text{Hom}(K_3, K_m)$, which is a wedge of $(m-3)$-spheres \cite{2}. Kozlov also computed the homology of $\text{Hom}(C_n, K_m)$ for all $n \geq 5$, $m \geq 4$ in \cite{10}. In particular, for all $m \geq 4$:

$$\text{conn}[\text{Hom}(C_{2r+1}, K_m)] = m - 4 \quad \text{for } r \geq 1$$

$$\text{conn}[\text{Hom}(C_{2r}, K_m)] = m - 3 \quad \text{for } r \geq 2$$

Note that the cases $n = 3$ and $n = 4$, which yield spheres, are consistent with these values. And the threshold for all small components to be isolated vertices is $p = \frac{\log n}{4m}$, so for $c < c_{k+1}$, $k \geq 2$, with high probability there is some $i \geq 1$ such that

$$\text{conn}[\text{Hom}(G_i, K_m)] \in (m - 3, m - 4)$$

This provides an upper bound for $M(n, c, m)$, and so when $m \geq 4$ we improve Theorem 5.1 to the following:

**Theorem 5.2.** If $k \geq 2$ and $p = c/n$ with $c < c_{k+1}$, then for all $m \geq 4$

$$\lim_{n \to \infty} \Pr[m - k - 2 \leq M(n, c, m) \leq m - 3] = 1$$

For fixed $m$, the lower bound decreases as $c$ gets bigger, and the gap between upper and lower bounds becomes worse. We expect that $\text{conn}[\text{Hom}(G_0, K_m)]$ should decrease as $D(G_0)$ increases and the giant component becomes more highly connected. However, the lower bound in Theorem 2.8 is not tight in general. For example, the complete bipartite graph $K_{ii}$ folds to an edge, so $\text{Hom}(K_{ii}, K_m) \approx S^{m-2}$. But $D(K_{ii}) = \min(i, j)$, so Theorem 2.8 yields $\text{conn}[\text{Hom}(K_{ii}, K_m)] \geq m - \min(i, j) - 2$. If $i, j \geq 2$, then this bound is not tight, and for large $i, j$ it can be arbitrarily bad.

To sharpen these results, we turn to examining the chromatic number of $G(n, c/n)$. Similar to the thresholds for the appearance of $k$-cores, Achlioptas and Friedgut \cite{11} showed that there is a sharp threshold sequence $d_k(n)$ such that for any $\varepsilon > 0$,

$$c < (1 - \varepsilon)d_k(n) \iff \lim_{n \to \infty} \Pr[\chi(G(n, c/n)) \leq k] = 1$$
\[ c > (1 + \epsilon) \frac{d_k(n)}{m} \Rightarrow \lim_{n \to \infty} \Pr[\chi(G(n, c/n)) \geq k + 1] = 1 \]

The convergence of the \( d_k(n) \) remains an open problem, but Coja-Oghlan and Vilenchik [4] improved the lower bound on \( \liminf_{n \to \infty} d_k(n) \) to within a small constant of the upper bound for \( \limsup_{n \to \infty} d_k(n) \). In particular, for \( o_k(1) \) a term which goes to 0 as \( k \) becomes large,

\[
d_k^- := \liminf_{n \to \infty} d_k(n) \geq 2k \log k - \log k - 2 \log 2 - o_k(1)
\]

\[
d_k^+ := \limsup_{n \to \infty} d_k(n) \leq 2k \log k - \log k - 1 + o_k(1)
\]

Then as \( k \) increases, the difference \( |d_k^+ - d_k^-| \) approaches \( 2 \log 2 - 1 \approx 0.39 \). And in the context of hom-complexes it is immediate that with high probability

\[
\text{Hom}(G(n, c/n), K_m) = \emptyset \quad \text{for} \ c > d_m^+ \quad \text{and} \ \text{Hom}(G(n, c/n), K_m) \neq \emptyset \quad \text{for} \ c < d_m^-
\]

For \( m = 3 \), we can then evaluate the connectivity of \( \text{Hom}(G(n, c/n), K_3) \) for all \( c > 0 \), excluding the gap between \( d_3^- \) and \( d_3^+ \).

**Theorem 5.3.** For \( 1 \leq c < d_3^- \), \( \text{Hom}(G(n, c/n), K_3) \) is disconnected with high probability.

**Proof of Theorem 5.3.** By the definition of \( d_3^- \) and the existence of odd cycles with high probability for \( c \geq 1 \), \( \chi(G(n, c/n)) = 3 \). Then by Theorem 4.6 \( \text{Hom}(G(n, c/n), K_3) \) is disconnected with high probability. \( \square \)

When \( 0 < c < 1 \), all connected components are isolated vertices, trees, or unicyclic graphs with high probability. If there is a component which contains edges and does not fold to a single edge, then \( \text{Hom}(G(n, c/n)) \) will be disconnected. But if every connected component of \( G(n, c/n) \) is an isolated vertex or folds to an edge, then \( \text{Hom}(G(n, c/n), K_3) \) is connected. In the latter case, specifically

\[
\text{Hom}(G(n, c/n), K_3) = (\Delta^2)^{t_1} \times T^{t_2}
\]

where \( t_1 \) = number of isolated vertices, and \( t_2 \) = the number of connected components which contain at least one edge. And \( t_2 > 0 \) with high probability for \( p = c/n \), so the complex will not be contractible, and \( M(n, c, 3) = 0 \).

**Theorem 5.4.** For \( 0 < c < 1 \), \( c' = \frac{1}{2} \log(1 - c) + \frac{c}{2} + \frac{c^2}{4} + \frac{c^4}{8} \)

\[
\lim_{n \to \infty} \Pr[M(n, c, 3) = -1] = 1 - e^{c'} \quad \text{and} \quad \lim_{n \to \infty} \Pr[M(n, c, 3) = 0] = e^{c'}
\]

**Proof of Theorem 5.4.** If \( G \) is a connected unicyclic graph whose cycle is \( C_l \) for \( l \neq 4 \), then \( G \) folds to \( C_l \) and \( \text{Hom}(G, K_3) \cong \text{Hom}(C_l, K_3) \), which is disconnected. So if \( G(n, c/n) \) does not contain any cycle \( C_l \) for \( l \neq 4 \), then all components of \( G(n, c/n) \) will be isolated vertices or will fold to an edge. For a fixed \( l \), the number of \( l \)-cycles in \( G(n, c/n) \) approaches a limiting Poisson distribution with mean \( \frac{c^3}{l} \), and so

\[
\Pr[C_l \notin G(n, c/n) \text{ for all } l \neq 4] \to \exp \left( -\frac{c^3}{6} + \sum_{l=5}^{\infty} \frac{c^l}{2l} \right)
\]

For \( c < 1 \), the sum in the exponent converges to \( c' = \frac{1}{2} \log(1 - c) + \frac{c}{2} + \frac{c^2}{4} + \frac{c^4}{8} \). Thus

\[
\Pr[M(c, n, 3) = 0] \to e^{c'} \text{ and } \Pr[M(c, n, 3) = -1] \to 1 - e^{c'} \quad \square
\]
6. Further Questions

The biggest question left unanswered is that of bounding \( \text{conn}[\text{Hom}(G_0, K_m)] \) from above. We expect that \( \text{conn}[\text{Hom}(G_0, K_m)] \) remains close to \( m - k - 2 \) when this bound makes sense, that \( M(n, c, m) = \text{conn}[\text{Hom}(G_0, K_m)] \), and that \( M(n, c, m) \) is a non-increasing function for fixed \( m \), with high probability. However, exhibiting non-trivial homology classes in \( \text{Hom}(G, H) \) is difficult in general. In Section 4 we showed that disconnected components can be lifted via subgraphs, which can be viewed as lifting non-trivial 0-cycles, but there is no known analogous result for lifting higher dimensional non-trivial cycles.

We also expect that \( M(n, c, m) \) increases monotonically when \( c \) is fixed and \( m \) is increasing, but this is not true in general for a fixed input graph. For instance, consider again our favorite counter example, \( Q_3 \). We noted in Section 4 that \( \text{conn}[\text{Hom}(Q_3, K_3)] \geq 0 \). And by Theorem 2.8, \( \text{conn}[\text{Hom}(Q_3, K_3)] \geq 0 \). But for \( m = 4 \), there are colorings of \( Q_3 \) in which each color class is a pair of antipodal corners. In such a coloring, every vertex is adjacent to one vertex from each of the other three color classes, so it represents an isolated 0-cell in \( \text{Hom}(Q_3, K_3) \). Thus \( \text{conn}[\text{Hom}(Q_3, K_3)] = -1 \). In this example with \( Q_3 \), the problem seems to occur when \( m \leq D(G) \), and it is plausible that \( \text{conn}[\text{Hom}(G, K_m)] \) is either increasing or non-decreasing for \( m \geq D(G) + 1 \), if \( G \) is a fixed graph which cannot be reduced via folds. This, however, remains an open question.

In the random setting, considering \( m \leq D(G) \) presents a serious roadblock to evaluating \( M(n, c, m) \) for all pairs of \( c \) and \( m \). Since the \( d_k^c \) grow much faster than the \( c_k \), there are intervals where \( D(G(n, c/n)) \) is much larger than \( \gamma(G(n, c/n)) \). For example, Molloy [12] pointed out that \( c_{40} \approx 52.23 \) while \( d_{40} > 53.88 \). So if \( c = 53 \), with high probability \( D(G(n, c/n)) = 40 \), but \( m \) can be as small as 19 before \( \text{Hom}(G(n, c/n), K_m) \) becomes empty. When \( m \leq D(G) \) the lower bound from Theorem 2.8 provides no information, and new methods are required.

A different direction altogether would be to indulge in a closer examination of precise numerical estimates on the Betti numbers and Euler characteristic of \( \text{Hom}(G(n, c/n), K_m) \). Ćukić and Kozlov’s [7] work on cycles in hom-complexes makes the case that \( m = 3 \) a tantalizingly tractable place to start this type of investigation.

References

[1] Dimitris Achlioptas and Ehud Friedgut. A sharp threshold for \( k \)-colorability. Random Structures Algorithms, 14(1):63–70, 1999.
[2] Eric Babson and Dmitry N. Kozlov. Complexes of graph homomorphisms. Israel J. Math., 152:285–312, 2006.
[3] Luis Cerdeira, Jan van den Heuvel, and Matthew Johnson. Mixing 3-colourings in bipartite graphs. European J. Combin., 30(7):1593–1606, 2009.
[4] Amin Coja-Oghlan and Dan Vilenchik. Chasing the \( k \)-colorability threshold. In 2013 IEEE 54th Annual Symposium on Foundations of Computer Science—FOCS 2013, pages 380–389. IEEE Computer Soc., Los Alamitos, CA, 2013.
[5] Peter Csorba. Non-Tidy Spaces and Graph Colorings. PhD thesis, ETH Zürich, 2005.
[6] Sonja Lj. Ćukić and Dmitry N. Kozlov. Higher connectivity of graph coloring complexes. Int. Math. Res. Not., (25):1543–1562, 2005.
[7] Sonja Lj. Ćukić and Dmitry N. Kozlov. The homotopy type of complexes of graph homomorphisms between cycles. Discrete Comput. Geom., 36(2):313–329, 2006.
[8] Alexander Engström. A short proof of a conjecture on the connectivity of graph coloring complexes. Proc. Amer. Math. Soc., 134(12):3703–3705 (electronic), 2006.
[9] Matthew Kahle. The neighborhood complex of a random graph. J. Combin. Theory Ser. A, 114(2):380–387, 2007.
[10] Dmitry N. Kozlov. Cohomology of colorings of cycles. *Amer. J. Math.*, 130(3):829–857, 2008.

[11] L. Lovász. Knösen’s conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[12] Michael Molloy. A gap between the appearances of a $k$-core and a $(k + 1)$-chromatic graph. *Random Structures Algorithms*, 8(2):159–160, 1996.

[13] Boris Pittel, Joel Spencer, and Nicholas Wormald. Sudden emergence of a giant $k$-core in a random graph. *J. Combin. Theory Ser. B*, 67(1):111–151, 1996.

[14] Carsten Schultz. Small models of graph colouring manifolds and the Stiefel manifolds $\text{Hom}(C_5, K_n)$. *J. Combin. Theory Ser. A*, 115(1):84–104, 2008.

Department of Mathematics, The Ohio State University