AN ANALYSIS OF SPARSITY PRESERVING PIVOT STRATEGIES FOR DISCONTINUOUS GALERKIN METHODS APPLIED TO ACOUSTIC SCATTERING

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Abstract. In this paper we discuss and analyze the sparse structure of matrices associated to the interior penalty discontinuous Galerkin (IP-DG) method applied to the Helmholtz equation. It is well-known that LU-factorization causes fill-in and this paper discusses three pivoting strategies: approximate minimal degree (AMD), nested dissection, and reverse Cuthill-McKee, that can reduce fill-in associated to the LU-factorization. Numerical experiments are included that demonstrate the performance of these pivoting strategies. These experiments include both uniform and non-uniform mesh structures, the inclusion of a scattering boundary, and both piecewise linear and quadratic solution spaces.

Key words. Helmholtz equation, acoustic scattering, discontinuous Galerkin methods, sparse matrices, LU factorization, pivoting

1. Introduction. First, we will define the domain used to define the acoustic scattering model studied in this paper. Let $D_1$ and $D_2$ be two open and bounded sets in $\mathbb{R}^d$ where $d = 2, 3$ is the spatial dimension for the problem. Further, assume that $D_2 \subset D_1$ such that both $D_1$ and $D_2$ satisfy a star-shaped condition involving the same point. Define the domain $D = D_1 / D_2$ with boundary $\partial D$. We will decompose the boundary $\partial D$ into $\partial D = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \partial D_1$ and $\Gamma_2 = \partial D_2$.

Using the domain $D$ defined above we define the following acoustic scattering model:

\begin{align}
-\Delta u - k^2 u &= f, \quad \text{in } D, \\
\frac{\partial u}{\partial \nu} + ik u &= g, \quad \text{on } \Gamma_1, \\
u &= 0, \quad \text{on } \Gamma_2.
\end{align}

(1.1) is the acoustic Helmholtz equation and is used to model time-harmonic acoustic waves in some homogeneous medium. In this equation $u : D \rightarrow \mathbb{C}$ models the displacement of the medium caused by the acoustic wave. The domain $D$ represents the acoustic medium in which the wave is traveling. $k$ is the wave number parameter of the wave and is defined as $k = \omega / c$ where $\omega$ is the angular frequency of the wave and $c$ is the speed of the wave. $f, g : D \rightarrow \mathbb{C}$ are given source functions.

In the boundary condition (1.2) \( \nu \) refers to the unit outward normal vector to the domain $D_1$ and $i = \sqrt{-1}$ is the imaginary unit. Typically wave propagation models are posed on unbounded domains with far-field conditions dictating the behavior of the wave. These unbounded domains are computationally prohibitive. Thus, the above acoustic scattering model is defined on a bounded domain $D$ with boundary $\Gamma_1$ due to the truncation of an unbounded domain. To approximate an unbounded domain an absorbing boundary condition like (1.2) is used (1.3). (1.2) simulates the absorption of the acoustic wave into the boundary $\Gamma_1$.

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The open and bounded set $D_2$ represents a scattering object in the homogeneous media. Thus, (1.3) is a scattering boundary condition. In particular, the homogeneous Dirichlet condition (1.3) used in this paper models a sound-soft scattering object.

This paper is mainly concerned with the case of a large wave number $k$. It is well-known that in this case the Helmholtz differential operator is indefinite and this leads to difficulties in analyzing the Helmholtz PDE (1.1). Furthermore, numerical discretization techniques of the Helmholtz problem (1.1)–(1.3) lead to a non-Hermitian indefinite system of equations that must be solved to resolve the wave. This leads to subsequent difficulties in the numerical analysis of the discretization method as well as difficulties in solving the linear system.

Many discretization methods have been applied to the acoustic Helmholtz problem and analyzed. These include the finite element method (FEM) [3, 11, 12, 34, 37], the plane wave discontinuous Galerkin (PW-DG) method [22, 24, 35], and the ultra weak variational form (UWVF) [5, 6, 7, 25, 26, 29]. It is well-known that one must use a fine enough spatial mesh when defining the discretization in order to resolve the wave length $\ell = 2\pi/k$ in each coordinate direction. This leads to a minimum mesh constraint of the form $O(kh) = 1$ where $h$ is the mesh size parameter and represents the maximum diameter of each element of the partition of the spatial domain. In [27] the authors showed that this minimum constraint is necessary for the finite element method (FEM) applied to the 1-dimensional acoustic Helmholtz equation. Furthermore, in [27] it was shown that the $H^1$ error bound for the solution of the FEM applied to the 1-d acoustic Helmholtz equation contains a term of the order $k^3h^2$ called the pollution term. This pollution term causes an increase in the $H^1$ error as $k$ is increased when $h$ is chosen to satisfy the minimum mesh constraint $O(kh) = 1$. The increase in error under the minimum mesh constraint is called the pollution effect. It has been shown that the pollution effect is inherent in Helmholtz-type problems and leads to a loss of stability in standard discretization techniques [4, 11]. Due to this loss of stability, a strict mesh constraint of the form $O(k^2h) = 1$ (called the asymptotic mesh constraint) was required to obtain optimal and quasi-optimal error estimates for the acoustic Helmholtz problem [3, 11, 12].

In [17, 18] an interior penalty discontinuous Galerkin (IP-DG) method was developed and analyzed for the acoustic Helmholtz problem (1.1)–(1.3). This IP-DG method made use of purely imaginary penalty parameters and penalization of the jumps of function values, normal derivatives, and tangential derivatives. In [17, 18] this IP-DG method was shown to be unconditionally stable. In particular, stability estimates were obtained in both the asymptotic and pre-asymptotic mesh regime. Also, sub-optimal error estimates were proven in the pre-asymptotic mesh regime which improve to optimal order error estimates in the asymptotic mesh regime. Numerical experiments in [17] show that when $k$ is large the IP-DG method outperforms the standard FEM in the number of degrees of freedom required to attain a given accuracy. For these reasons, we will focus on this unconditionally stable IP-DG method in this paper.

To resolve the solution of the acoustic Helmholtz problem (1.1)–(1.3) with $k$ large using standard discretization techniques, one must solve a large non-Hermitian, indefinite, and ill-conditioned system of linear equations. In [15] it was shown that standard iterative solvers applied to the acoustic Helmholtz problem do not perform well. In fact, many do not converge. Thus, to resolve the solution of the acoustic Helmholtz problem a direct linear solver is usually employed. In [16] an $LU$ direct solver was used to obtain an efficient and accurate discretization method for the
acoustic Helmholtz problem in random media.

In this paper, we will study the $LU$ decomposition of the system of equations obtained from the IP-DG method applied to the acoustic Helmholtz problem \( (1.1) - (1.3) \). The IP-DG method leads to a linear system defined with a sparse global matrix $A$. It is well-known that the $LU$ decomposition leads to “fill-in” of the sparse system and thus requires more CPU memory and leads to a loss of efficiency \[20, 21\]. To reduce fill-in, one can use pivoting prior to the $LU$ decomposition. This paper will study three popular sparsity preserving pivoting strategies applied to the linear system obtained from the IP-DG method \[17, 18\]: (1) minimum degree pivoting, (2) nested dissection, and (3) bandwidth/profile reduction. We will use numerical experiments to compare the fill-in produced by the $LU$ decomposition after using these pivoting strategies.

The paper is organized as follows. Section 2 will detail the IP-DG method used in this paper. Section 3 will discuss the sparse structure of the global matrix defined by the IP-DG method and detail the three sparsity preserving pivoting methods that we will study in this paper. Section 4 will be used to discuss multiple numerical experiments that were designed to study pivoting to preserve the sparsity of the global matrix obtained from the IP-DG method. Section 5 will be used to summarize the results and offer conclusions.

2. Interior penalty discontinuous Galerkin method for acoustic scattering. The goal of this section is to introduce the reader to the interior penalty discontinuous Galerkin (IP-DG) method developed and analyzed in \[17, 18\]. For more in-depth information regarding this discretization we encourage the reader to see those papers. In order to define the IP-DG method we will need to introduce standard notation used to define DG methods. For more understanding of discontinuous Galerkin (DG) methods we refer the reader to \[36\].

2.1. DG Notation. This subsection will be used to introduce the notation used to define the IP-DG method studied in this paper. Throughout this paper we will use the standard complex-valued $L^2$-space norm and inner product notation. In particular, for $\Omega \subset D$ and $\Sigma \subset \partial \Omega$ we let $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_\Sigma$ denote the complex $L^2$-inner product on $\Omega$ and $\Sigma$, respectively. That is,

\[
(u, v)_\Omega = \int_\Omega u^* v \, dx \quad \text{and} \quad \langle u, v \rangle_\Sigma = \int_\Sigma u^* v \, dS.
\]

To discretize the PDE problem \( (1.1) - (1.3) \) using the discontinuous Galerkin method, let $T_h$ be a shape regular partition of the domain $D$ parameterized by the size parameter $h$. Typically $T_h$ is a triangulation of a domain in $\mathbb{R}^2$ and $h$ specifies the maximum diameter of a triangle $K \in T_h$. Example triangulations used in later numerical experiments are given throughout Section 4 (see Figures 4.1, 4.6, 4.11, and 4.16).

One key characteristic of discontinuous Galerkin methods is the use of an energy space $H^1(T_h) = \prod_{K \in T_h} H^1(K)$ to derive the weak formulation. This function space includes functions that are discontinuous at edge/face boundaries of the partition $T_h$. To deal with these discontinuities we will need to define special notation that is standard in discontinuous Galerkin method. We begin by defining the following sets
of edges/faces:

\[ \mathcal{E}_h^I := \text{set of all interior edges/faces of } \mathcal{T}_h, \]
\[ \mathcal{E}_h^R := \text{set of all edges/faces of } \mathcal{T}_h \text{ on } \Gamma_1, \]
\[ \mathcal{E}_h^D := \text{set of all edges/faces of } \mathcal{T}_h \text{ on } \Gamma_2, \]
\[ \mathcal{E}_h := \mathcal{E}_h^I \cup \mathcal{E}_h^D. \]

For any \( e \in \mathcal{E}_h^I \) there exists two cells \( K, K' \in \mathcal{T}_h \) such that \( e = K \cap K' \). For such an edge/face \( e = K \cap K' \), let \( \nu_e \) be the unit normal vector pointing out of \( K \) if \( K \) has larger global label and pointing out of \( K' \) if \( K' \) has larger global label. Also for an edge/face \( e = K \cap K' \) we define the jump \([v]\) and average \(\{v\}\) operators of a function \( v \in H^1(\mathcal{T}_h) \) in the following way:

\[ [v] := \begin{cases} v|_K - v|_{K'}, & \text{if the global label of } K \text{ is larger,} \\ v|_{K'} - v|_K, & \text{if the global label of } K' \text{ is larger,} \end{cases} \]
\[ \{v\} := \frac{1}{2}(v|_K + v|_{K'}). \]

For \( e \in \mathcal{E}_h^R \cup \mathcal{E}_h^D \) we define \( \nu_e \) to be the unit outward normal vector on \( \partial D \) and \([v] = \{v\} = v|_e\). For \( e \in \mathcal{E}_h^I \cup \mathcal{E}_h^R \cup \mathcal{E}_h^D \), let \( \{\tau_j\}_{j=1}^{d-1} \) be an independent set of unit tangential vectors to \( e \) and \( h_e = \text{diam}(e) \).

To define the IP-DG method for (1.1)–(1.3), we define the finite dimensional solution space \( V_h^p := \prod_{K \in \mathcal{T}_h} P_h^p(K) \), where \( p \) is a positive integer. In other words \( V_h^p \) is the set of piecewise polynomials of degree \( p \) over the partition \( \mathcal{T}_h \). In numerical experiments later in this paper we will focus on the cases of \( p = 1, 2 \), i.e. solutions that are piecewise linear or quadratic over the partition \( \mathcal{T}_h \). Over the space \( V_h^p \times V_h^p \) we define the following sesquilinear form from [17] [18]:

\begin{align}
(2.2) \quad a_h(u, v) := b_h(u, v) - k^2(u, v)_D + ik(u, v) \gamma_1, \\
&\quad + i(J_0(u, v) + J_1(u, v) + L_1(u, v)) \quad \forall u, v \in V_h^p,
\end{align}

where

\[ b_h(u, v) := \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e \in \mathcal{E}_h^R} \left( \left\langle \frac{\partial u}{\partial \nu_e}, [v] \right\rangle_e + \left\langle [u], \frac{\partial v}{\partial \nu_e} \right\rangle_e \right), \]

\[ J_0(u, v) := \sum_{e \in \mathcal{E}_h^D} \gamma_0 e \frac{\gamma_0}{h_e} \left\langle [u], [v] \right\rangle_e, \]

\[ J_1(u, v) := \sum_{e \in \mathcal{E}_h^I} \gamma_1 e h_e \left\langle \left[ \frac{\partial u}{\partial \nu_e} \right], \left[ \frac{\partial v}{\partial \nu_e} \right] \right\rangle_e, \]

\[ L_1(u, v) := \sum_{e \in \mathcal{E}_h^D} \sum_{j=1}^{d-1} \beta_{1, e} \frac{\beta_{1, e}}{h_e} \left\langle \left[ \frac{\partial u}{\partial \tau_{e, j}} \right], \left[ \frac{\partial v}{\partial \tau_{e, j}} \right] \right\rangle_e. \]

The IP-DG method developed and analyzed in [17] [18] and used in this paper is defined as: Find \( u \in V_h^p \) such that

\begin{align}
(2.3) \quad a_h(u, v) = (f, v)_D + (g, v) \gamma_1, \quad \forall v \in V_h^p,
\end{align}
The terms $J_0, J_1, L_1$ are referred to in the DG literature as penalty terms with $\gamma_{0,e}, \gamma_{1,e}, \beta_{1,e}$ called penalty parameters. This IP-DG method is characterized by penalization of jump discontinuities in the function values, normal derivatives, and tangential derivatives at cell boundaries in the partition $T_h$. The penalty parameters $\gamma_{0,e}, \gamma_{1,e}, \beta_{1,e}$ can be complex values with positive real part, but it has been shown that the imaginary part of these parameters do not improve performance [17]. Thus, we will assume the penalty parameters $\gamma_{0,e}, \gamma_{1,e}, \beta_{1,e}$ are positive real numbers. With this in mind, the penalty terms in (2.3) characterize purely imaginary penalization which is a unique characteristic of this method.

The key feature of this method is the unconditional stability of the method. In [17, 18], stability estimates were proven for any positive parameters $k, h, \gamma_{0,e}, \gamma_{1,e}, \beta_{1,e}$. In contrast, for standard discretization methods such as the finite element method, stability estimates are only proven in the asymptotic mesh regime $O(k^2 h) = 1$. The unconditional stability of the IP-DG method in (2.3) leads to unconditional unique solvability of the IP-DG problem (2.3). The IP-DG method is also proven to be optimally convergent in the asymptotic mesh regime and sub-optimally convergent in the preasymptotic mesh regime [17, 18].

3. Sparsity preserving pivot strategies. It is well-known that the IP-DG method (2.3) is equivalent to solving a system of linear equations of the form

$$ Ac = b. $$

(3.1)

In this section, we will discuss the sparse properties of this linear system as well as sparsity preserving pivoting strategies that can be used to reduce fill-in of the $LU$ decomposition.

First, we note that the IP-DG function space $V^p_h$ is a finite dimensional subspace of $H^1(T_h)$, thus there exists a basis $\{\phi_j\}_{j=1}^N$ of $V^p_h$. Recall, that $V^1_h$ is the space of piecewise linear polynomials over the partition $T_h$. A typical basis function used in IP-DG methods for $\Omega \subset \mathbb{R}^2$ takes the form

$$ \phi_j = \begin{cases} \text{ax + by + c} & \text{in } K^*, \\ 0 & \text{in } K \neq K^*, \end{cases} $$

(3.2)

where $K^*$ is a single cell in the partition $T_h$. In other words, a typical basis function is a polynomial on a single cell that is 0 on all other cells of $T_h$.

Given a basis $\{\phi_j\}_{j=1}^N$ of $V^p_h$, the linear system in (3.1) is defined by $A \in \mathbb{C}^{N \times N}$, $c, b \in \mathbb{C}^N$ where

$$ [A]_{i,j} = a_h(\phi_j, \phi_i), \quad b_i = (f, \phi_i)_D + (g, \phi_i)_{\Gamma_1}, $$

and the IP-DG solution $u \in V^p_h$ is defined as

$$ u = \sum_{i=1}^N c_i \phi_i. $$

Thus, for a basis comprised of functions $\phi_i$ with support on a single cell $K \in T_h$, 

$$ [A]_{i,j} = a_h(\phi_j, \phi_i) \neq 0, $$

if and only if $\phi_i$ and $\phi_j$ are basis functions associated to the same cell or neighboring cells of the partition $T_h$. Thus, the matrix $A$ is a sparse matrix. Also, since
\[ a_h(\phi_j, \phi_i) = a_h(\phi_i, \phi_j) \] for real-valued basis functions in (2.2), then \( A \) is symmetric. The distribution of nonzero element in \( A \) depends on the enumeration of the partition \( \mathcal{T}_h \) as well as the basis functions used.

As stated earlier in the paper, we will focus on an \( LU \) factorization as the method for solving the system in (3.1). It is well known that the \( LU \) factorization causes fill-in of sparse systems. To mitigate this fill-in, we will use different permutation matrices \( P \) such that the permuted matrix \( PAP^T \) has an \( LU \) factorization with less fill-in. In this paper, we will test three sparsity preserving pivoting strategies for the system (3.1). These are minimum degree pivoting (MDP), nested dissection (ND), and bandwidth/profile reducing pivoting. These methods were developed for the Cholesky factorization of a symmetric positive definite matrix, but can also be applied to the \( LU \) factorization of a more general square matrix. [10] discusses these pivoting strategies and others in depth.

MDP was developed as a sparsity preserving pivoting strategy in calculating a right-looking sparse Cholesky factorization. In particular, MDP is a greedy algorithm with the goal of choosing the sparsest pivot row and column in each step of the Cholesky factorization. MDP was first introduced by Markowitz in [30]. In this paper we use the approximate minimal degree (AMD) pivoting algorithm developed by Amestoy, Davis, and Duff [1, 2]. For numerical experiments discussed in Section 4, we made use of Matlab’s \texttt{amd} function [31] to obtain the AMD permutation matrix \( P \).

The nested dissection (ND) permutation was developed by George [19]. The ND permutation was designed specifically to reduce fill-in due to Cholesky factorization applied to linear systems generated by the finite element method. ND might be a good choice for the matrices discussed in this paper due to the similarities between the matrix generated by the finite element method and the IP-DG method. Duff, Erisman, and Reid [13] made an early comparison of MDP and ND permutations. For numerical experiments discussed in Section 4, we made use of Matlab’s \texttt{dissect} function [32] to obtain the ND permutation matrix \( P \).

The last class of pivoting strategies that we consider is one in which \( P \) is generated such that the bandwidth/profile of \( PAP^T \) is decreased, thus reduces the fill-in from the \( LU \) decomposition. One of the first pivoting techniques developed for bandwidth/profile reduction was developed by Cuthill and McKee in [9]. Later Liu and Sherman introduced the reverse Cuthill McKee (RCM) method [28]. RCM reverses the Cuthill-McKee ordering which can further reduce the profile of the matrix. Chan and George developed an efficient RCM algorithm in [8]. For numerical experiments discussed in Section 4, we made use of Matlab’s \texttt{symrcm} function [33] to obtain the RCM permutation matrix \( P \).

4. Numerical experiments. In this section, we present a number of numerical experiments with the intent to compare the performance of approximate minimal degree (AMD), nested dissection (ND), and reverse Cuthill-McKee (RCM) pivoting strategies. In particular, we will observe the percent of fill-in when an \( LU \) factorization is used without these pivoting strategies and after these pivoting strategies have been used. The results will be presented using tables that specify the total number of nonzero entries compared to the total number of entries, as well as a table that gives the percent of fill-in. In addition to these tables, plots will be used to demonstrate the sparsity structure of the matrix \( A \) before and after pivoting as well as the \( LU \) decomposition before and after pivoting. These plots were generated using Matlab’s \texttt{spy} function. To save space, the plots of the \( LU \) factorization show the combined \( LU \)
factorization defined as \( L + U - I \) where \( I \) is the identity matrix of the same size as \( A \) and \( A = LU \).

In all experiments in this section we let \( D_1 \) be the unit square at the origin \( D_1 = [-0.5, 0.5] \times [-0.5, 0.5] \) and \( k = 5 \). Since our intent is only to study the matrix \( A \) defined in (3.1) and its \( LU \) factorization it is unnecessary to specify the functions \( f, g \) in (1.1) and (1.2). Since the structure of the matrix \( A \) depends on the structure of the mesh \( T_h \) and the function space \( V^p_h \) used in the IP-DG method (2.3), our experiments are conducted on using different mesh structures and function spaces. Following [17], we chose the following mesh parameters for each experiment

\[
\gamma_{1,e} = 0.1, \quad \gamma_{0,e} = (k^2 h_e)^{2/3}, \quad \beta_{1,e} = 1.
\]

To produce the mesh \( T_h \) and matrix \( A \) associated to the IP-DG method, FreeFEM++ [23] was used. All matrix analysis was done using Matlab’s built-in functions. All experiments were conducted on the same Mac computer with a 2 GHz Intel Core i7 processor and 8 GB 1600 MHz DDR3 RAM.

### 4.1. Numerical experiment 1

In this first set of experiments the scattering portion of the domain was chosen to be \( D_2 = \emptyset \). Thus, \( D = D_1 \) and \( \Gamma_2 = \emptyset \). To define the mesh \( T_h \) we used a uniform triangulation of \( D \) with \( n = 5, 10, 15, 20 \) intervals along each side of \( D \). The mesh size parameter is then defined as \( h = 1/n \). Examples of the mesh used in this experiment are displayed in Figure 4.1. In this set of experiments the IP-DG function space \( V^1_h \) was used, i.e. the set of piecewise linear functions across the triangulation \( T_h \).

![Figure 4.1. Mesh used in experiment 1 with \( n = 5 \) (left) and \( n = 10 \) (right).](image)

Figure 4.2 demonstrates the sparse structure of the matrix \( A \) along with its combined \( LU \) decomposition for \( n = 5 \). At the bottom of each figure the number of nonzero entries is given as \( nz \). This figure demonstrates the fill-in associated to the \( LU \) factorization. Figures 4.3–4.5 show the sparsity structure of the permutation \( PAP^T \) produced along with the \( LU \) factorization after permutation using AMD, ND, and RCM, respectively. From these images it is clear that the pivoting strategies give different sparsity structure and varying results with respect to fill-in reduction. Recall that RCM is designed to reduce bandwidth/profile of the matrix \( A \). We can see this bandwidth/profile reduction clearly in Figure 4.5.

Tables 4.1 and 4.2 summarize the results of experiment 1. In particular, Table 4.1 gives the number of total entries of the matrix \( A \), the number of nonzero entries of \( A \), the number of nonzero entries in the \( LU \) factorization of \( A \), and the number of nonzero
entries of $LU$ factorization of $PAP^T$ after AMD, ND, and RCM pivoting. Table 4.2 presents the same data from Table 4.1 as percentage of fill-in. From Table 4.2, it is clear that $LU$ factorization produces a large amount of fill-in. The percent of fill-in
decreases as the mesh is refined. It is also clear that each pivoting strategy decreases the percent of fill-in which leads to improved performance. In this experiment both AMD and ND pivoting decreases fill-in better than RCM pivoting. For a coarse mesh $n = 5$ AMD performs slightly better than ND, but as the mesh is refined ND performs slightly better than AMD. In fact, for a fine mesh using $n = 20$, ND reduced fill-in by a factor of 2.3. In contrast, for the same mesh, AMD reduces fill-in by a factor of 1.9 and RCM reduces fill-in by a factor of 1.5.

| $n$ | Total Entries | Nonzero Entries | LU   |
|-----|---------------|----------------|------|
| 5   | 22500         | 1620           | 4680 |
| 10  | $3.60 \times 10^6$ | 6840           | 36810|
| 15  | $1.82 \times 10^6$ | 15660          | $1.23 \times 10^6$ |
| 20  | $5.76 \times 10^6$ | 28000          | $2.91 \times 10^6$ |

| $n$ | AMD | ND  | RCM |
|-----|-----|-----|-----|
| 5   | 2970 | 3258 | 3600 |
| 10  | 20427 | 20508 | 26550 |
| 15  | 69567 | 61647 | 87804 |
| 20  | $1.53 \times 10^6$ | $1.33 \times 10^6$ | $2.04 \times 10^6$ |

Table 4.1 (Above) Total entries, number of nonzero entries before finding the LU decomposition, and the number of nonzero entries in the LU decomposition. (Below) Number of nonzero entries of the LU decomposition after applying AMD, ND, and RCM pivoting.

4.2. Numerical experiment 2. Similar to the experiments in Section 4.1 in this section we will let $D_2 = \emptyset$ and use the IP-DG function space $V^1_h$. The main difference between the experiments in this section and Section 4.1 is the use of a less structured triangular mesh $T_h$. In this section we use a mesh $T_h$ defined using $n = 5, 10, 15, 20$ intervals along the vertical sides of $D$ and $10n$ intervals along the horizontal sides of $D$. Figure 4.6 shows examples of meshes used in this section. The structure of the mesh will determine the structure of the matrix $A$ in (2.1). Thus, we expect to see different results in this experiment. Similar to Section 4.1 the IP-DG function space $V^1_h$ was used in this set of experiments.
Table 4.2

| n  | Nonzero Entries | LU   | AMD  | ND   | RCM  |
|----|----------------|------|------|------|------|
| 5  | 7.2 %          | 20.8 % | 13.2 % | 14.5 % | 16 % |
| 10 | 1.9 %          | 10.2 % | 5.7 %  | 5.7 %  | 7.4 % |
| 15 | 0.86 %         | 6.8 %  | 3.8 %  | 3.4 %  | 4.8 % |
| 20 | 0.49 %         | 5.1 %  | 2.7 %  | 2.3 %  | 3.5 % |

Percentage of nonzero entries before finding the LU decomposition, in the LU decomposition, and in the LU decomposition after applying AMD, ND, and RCM pivoting.

Figure 4.6. Mesh used in experiment 2 with $n = 5$ (left) and $n = 10$ (right).

Similar to Section 4.1, we plotted the sparsity structure of the matrix $A$ and its combined $LU$ factorization for $n = 5$. We also plotted the matrix $PAP^T$ after AMD, ND, and RCM pivoting along with the $LU$ factorization of each of these pivoted matrices. These plots are given in Figure 4.7–4.10. Also, similar to Section 4.1, the data for the number of nonzero entries for the matrices in this set of experiments are summarized in Tables 4.3 and 4.4. From these tables we see that all three pivoting strategies reduce the fill-in associated to $LU$ factorization in this set of experiments. Also, as the mesh parameter $n$ increases, the pivoting strategies perform better. In this experiment, RCM reduces fill-in the least and ND reduces fill-in the most. For instance, for $n = 20$, RCM reduces fill-in by a factor of 1.99 and ND reduces fill-in by a factor of 5.36.
**Figure 4.7.** (Left) sparsity structure of the global matrix $A$ produced by the IP-DG method with $n = 5$. (Right) sparsity structure of the combined LU decomposition of $A$.

**Figure 4.8.** (Left) sparsity structure of the global matrix after AMD pivoting. (Right) sparsity structure of the combined LU decomposition the global matrix after AMD pivoting.

**Figure 4.9.** (Left) sparsity structure of the global matrix after ND pivoting. (Right) sparsity structure of the combined LU decomposition the global matrix after ND pivoting.
Figure 4.10. (Left) sparsity structure of the global matrix after RCM pivoting. (Right) sparsity structure of the combined LU decomposition the global matrix after RCM pivoting.

| n  | Total Entries | Nonzero Entries | LU       |
|----|---------------|----------------|----------|
| 5  | $1.71 \times 10^9$ | 14706         | 93387    |
| 10 | $9.91 \times 10^7$ | $1.17 \times 10^7$ | $4.52 \times 10^6$ |
| 15 | $1.24 \times 10^8$ | $1.31 \times 10^7$ | $3.24 \times 10^6$ |
| 20 | $1.46 \times 10^9$ | $4.55 \times 10^7$ | $5.88 \times 10^6$ |

| n  | AMD          | ND         | RCM       |
|----|--------------|------------|-----------|
| 5  | 45609        | 45138      | 84786     |
| 10 | $1.41 \times 10^9$ | $9.71 \times 10^7$ | $3.33 \times 10^6$ |
| 15 | $1.43 \times 10^8$ | $1.24 \times 10^8$ | $2.79 \times 10^7$ |
| 20 | $1.97 \times 10^7$ | $1.10 \times 10^7$ | $2.95 \times 10^7$ |

(Above) Total entries, number of nonzero entries before finding the LU decomposition, and the number of nonzero entries in the LU decomposition. (Below) Number of nonzero entries of the LU decomposition after applying AMD, ND, and RCM pivoting.

| n  | Nonzero Entries | LU | AMD | ND | RCM |
|----|----------------|----|-----|----|-----|
| 5  | 0.86 %         | 5.5 % | 2.7 % | 2.6 % | 4.96 % |
| 10 | 0.12 %         | 4.6 % | 1.4 % | 0.98 % | 3.4 % |
| 15 | 0.11 %         | 2.6 % | 1.2 % | 1.0 % | 2.2 % |
| 20 | 0.031 %        | 4.02 % | 1.3 % | 0.75 % | 2.02 % |

Percentage of nonzero entries before finding the LU decomposition, in the LU decomposition, and in the LU decomposition after applying AMD, ND, and RCM pivoting.
4.3. Numerical experiment 3. In the third set of numerical experiments presented in this paper, our goal was to observe the performance of sparsity preserving pivoting on a domain with a scattering boundary. With this goal in mind, a circular scattering domain

\[(4.1) \quad D_2 = \{ (x, y) \mid x^2 + y^2 \leq 1/5 \}, \]

was included. In this experiment we used a quasi-uniform triangular mesh \(\mathcal{T}_h\) generated by using \(n = 5, 10, 15, 20\) intervals along each side of \(\Gamma_1 = \partial D_1\) and along \(\Gamma_2 = \partial D_2\). Examples of the mesh generated by \(n = 5, 10\) are given in Figure 4.11. Similar to Sections 4.1 and 4.2 the IP-DG function space \(V^1_h\) was used in this set of experiments.

Figure 4.11. Mesh used in experiment 3 with \(n = 5\) (left) and \(n = 10\) (right).

Figure 4.12 shows the sparsity structure of the matrix \(A\) defined by (3.1) along with its combined \(LU\) factorization. Similarly, Figures 4.13–4.15 give the sparsity structure of \(P A P^T\) after AMD, ND, and RCM pivoting along with the combined \(LU\) factorization of each matrix for \(n = 5\). The number of nonzero entries for \(A\) and the combined \(LU\) factorization of \(A\) and \(P A P^T\) after AMD, ND, and RCM are summarized in Table 4.5. Similarly, the percent of non-zero entries for the combined \(LU\) factorization of these matrices is given in Table 4.6.

Figure 4.12. (Left) sparsity structure of the global matrix \(A\) produced by the IP-DG method with \(n = 5\). (Right) sparsity structure of the combined \(LU\) decomposition of \(A\).
The results of this experiment and summarized in Table 4.6 are similar to the results using the uniform mesh in Section 4.1. For instance, Table 4.6 indicates that AMD and ND reduce fill-in at roughly the same rate, with AMD performing slightly
better for the coarse mesh $n = 5$ and ND performing slightly better for the fine mesh $n = 20$. Similar to experiments in Section 4.1 and 4.2 all three pivoting strategies reduce fill-in associated to $LU$ factorization and as $n$ increases the rate at which pivoting reduces fill-in increases. For the fine mesh given by $n = 20$, RCM reduces fill-in the least with a reduction factor of 1.66 and ND reduces fill-in the most with a reduction factor of 2.58.

**4.4. Numerical experiment 4.** Similar to Section 4.3 this section studies the IP-DG method with a circular scattering domain $D_2$ define by (4.1). In this section we chose to use a non-uniform triangular mesh $T_h$. The triangular mesh $T_h$ was generated using $n = 5, 10, 15, 20$ intervals along each side of $\Gamma_1 = \partial D_1$ and $10n$ intervals along $\Gamma_2 = \partial D_2$. Thus, $T_h$ is coarse along the boundary $\Gamma_1$ and fine along the boundary $\Gamma_2$. Example meshes for $n = 5, 10$ are given in Figure 4.16. Similar to Sections 4.1–4.3 the IP-DG function space $V^n_1$ was used in this set of experiments.

The sparsity structure of the matrix $A$ defined in (3.1) along with its combined $LU$ factorization are given in Figure 4.17. This is followed up with the sparsity structure of the pivoted matrix $PAP^T$ and the combined $LU$ factorization of this matrix for the AMD, ND, and RCM pivot strategies in Figure 4.18–4.20. Tables 4.7 and 4.8 are used to summarize the data from the experiments in this section. In particular, Table 4.7 give the number of nonzero entries in the matrix $A$, its $LU$ factorization, and the $LU$ factorization of the matrix $PAP^T$ after AMD, ND, and RCM pivoting has been applied. Table 4.8 presents this nonzero data as a percent of the total number of entries in the matrix.
In contrast to experiments in Sections 4.1–4.3, the RCM method does not perform well in the experiments in this section. In fact, RCM increases the fill-in for the cases of $n = 5, 10, 15$. RCM only reduces fill-in in the fine mesh case with $n = 20$, but even then the reduction is only by a factor of 1.13.

Similar to our previous experiments in Sections 4.1–4.3, the AMD method reduces fill-in at a greater rate than ND for a coarse mesh characterized by $n = 5$. As the mesh is refined ND reduces fill-in at a rate greater than AMD. For instance, when $n = 20$ ND reduces fill-in at a rate of 2.36 while AMD reduces fill-in at a rate of 1.33.

**Figure 4.17.** (Left) sparsity structure of the global matrix $A$ produced by the IP-DG method with $n = 5$. (Right) sparsity structure of the combined LU decomposition of $A$.

### 4.5. Numerical experiment 5

In the final set of experiments of this paper, we use the same domain and mesh as Section 4.1. See Figure 4.1 for examples of the computational mesh $T_h$ used in this experiment. In contrast to the experiments in Section 4.1, this section makes use of the finite dimensional function space $V_h^2$ in (2.3), i.e., the IP-DG method used in this section involves piecewise quadratic solutions across the triangulation $T_h$. This change in the approximation space changes the structure of the global matrix in (3.1) and further leads to different results with regard to pivoting and fill-in.

Figure 4.21 shows the sparse structure of the matrix $A$ without pivoting and the sparse structure of the combined $LU$ factorization of $A$ when $n = 5$. Figures 4.22–4.24 display the sparse structure of the pivoted matrix $PAP^T$ along with its combined $LU$
factorization for the AMD, ND, and RCM pivoting strategies when $n = 5$. From these figures we see differences in the fill-in produced in this section compared to those in Section 4.1. In particular, we see that the amount of fill-in from $LU$ factorization...
Table 4.7

| n  | Total Entries | Nonzero Entries | LU     |
|----|---------------|----------------|--------|
| 5  | $6.46 \times 10^9$ | 9018           | 49113  |
| 10 | $8.50 \times 10^9$ | 33732          | $3.04 \times 10^9$ |
| 15 | $4.45 \times 10^9$ | 78174          | $1.08 \times 10^9$ |
| 20 | $1.44 \times 10^9$ | $1.42 \times 10^9$ | $3.69 \times 10^9$ |

Table 4.8

| n  | Nonzero Entries | AMD  | ND   | RCM   |
|----|----------------|------|------|-------|
| 5  | 1.4 %          | 7.6 %| 4.3 %| 4.7 % |
| 10 | 0.4 %          | 3.6 %| 2.3 %| 2.2 % |
| 15 | 0.18 %         | 2.4 %| 1.96 %| 1.3 % |
| 20 | 0.098 %        | 2.6 %| 1.96 %| 1.1 % |

Percentage of nonzero entries before finding the LU decomposition, in the LU decomposition, and in the LU decomposition after applying AMD, ND, and RCM pivoting.

Increased in this set of experiments. Tables 4.9 and 4.10 summarize the results of the experiments using $n = 5, 10, 15, 20$ intervals along each side of the domain to generate the mesh $T_h$.

Surprisingly, unlike previous experiments the RCM method out-performs both AMD and ND for all experiments in this section. For a coarse mesh characterized by $n = 5$ all three methods reduce fill-in with a reduction rate of 1.26 by AMD, 1.58 by ND, and 1.674 by RCM. The performance of AMD and ND deteriorates as the mesh is
refined in this set of experiments. In fact, fill-in increases in the cases of $n = 10, 15, 20$ when using AMD and ND pivoting prior to the LU factorization. RCM continues to reduce fill-in when the mesh is refined, but for $n = 20$ the rate of reduction is
Table 4.9

| n  | Total Entries | Nonzero Entries | LU     |
|----|---------------|----------------|--------|
| 5  | 90000         | 6480           | 24275  |
| 10 | 1.44 × 10^6  | 27360          | 2.00 × 10^6 |
| 15 | 7.29 × 10^6  | 62640          | 7.09 × 10^6 |
| 20 | 2.30 × 10^7  | 1.12 × 10^7   | 1.25 × 10^6 |

(Above) Total entries, number of nonzero entries before finding the LU decomposition, and the number of nonzero entries in the LU decomposition after applying AMD, ND, and RCM pivoting.

Table 4.10

| n  | Nonzero Entries | LU     | AMD  | ND  | RCM |
|----|----------------|--------|------|-----|-----|
| 5  | 7.2 %          | 26.97 %| 21.4 %| 17.1 %| 16.1 %|
| 10 | 1.9 %          | 13.9 %| 22.7 %| 15.1 %| 8.1 %|
| 15 | 0.86 %         | 9.7 %  | 15.9 %| 12.8 %| 5.8 %|
| 20 | 0.49 %         | 5.4 %  | 13.1 %| 7.6 %  | 4.3 %|

Percentage of nonzero entries before finding the LU decomposition, in the LU decomposition, and in the LU decomposition after applying AMD, ND, and RCM pivoting.

decreased to 1.26.

5. Conclusions. The results discussed in Section 4 demonstrate that pivoting strategies are effective in reducing the fill-in due to LU factorization applied to the coefficient matrix generated by the IP-DG method applied to the Helmholtz problem (1.1)–(1.3). For experiments using a piecewise linear solution space $V_1^h$, AMD and ND reduced fill-in by similar amounts with AMD usually performing better than ND for coarse meshes and ND usually performing better than AMD for fine meshes (c.f. Sections 4.1, 4.3, and 4.4). In Section 4.2 ND performs better than AMD in both coarse and fine meshes. For a piecewise linear solution space $V_1^h$ we observe that RCM does reduce fill-in, but not by as large of a factor as AMD and ND (c.f. Sections 4.1, 4.4). Surprisingly, for a piecewise quadratic solution space $V_2^h$, RCM reduces fill-in by a larger factor than AMD and ND. In fact, Section 4.5 demonstrates that AMD and ND increase fill-in for the solution space $V_2^h$ when using a fine mesh.

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