Diffusion Limited Aggregation on a Cylinder

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Abstract

We consider the DLA process on a cylinder $G \times \mathbb{N}$. It is shown that this process “grows arms”, provided that the base graph $G$ has small enough mixing time. Specifically, if the mixing time of $G$ is at most $\log(2-\varepsilon) |G|$, the time it takes the cluster to reach the $m$-th layer of the cylinder is at most of order $m \cdot \frac{|G|}{\log \log |G|}$. In particular we get examples of infinite Cayley graphs of degree 5, for which the DLA cluster on these graphs has arbitrarily small density.

In addition, we provide an upper bound on the rate at which the “arms” grow. This bound is valid for a large class of base graphs $G$, including discrete tori of dimension at least 3.

It is also shown that for any base graph $G$, the density of the DLA process on a $G$-cylinder is related to the rate at which the arms of the cluster grow. This implies, that for any vertex transitive $G$, the density of DLA on a $G$-cylinder is bounded by $2/3$.

1 Introduction

Diffusion Limited Aggregation (DLA), is a growth model introduced by Witten and Sander ([12]). The process starts with a particle at the origin of $\mathbb{Z}^d$. At each time step,
a new particle starts a simple random walk on $\mathbb{Z}^d$ from infinity (far away). The particle is conditioned to hit the existing cluster (when $d \geq 3$). When the particle first hits the outer boundary of the cluster, it sticks and the next step starts, forming a growing family of clusters.

We consider a variant of this model, where the underlying graph of the process is a cylinder with base $G$, $G$ being some finite graph. A precise definition is given in Section 2.

This paper contains three main results:

The first, Theorem 2.1 states that if $G$ has small enough mixing time, then the time it takes the cluster to reach the $m$-th layer of the cylinder is $o(m \cdot |G|)$, were $|G|$ is the size of $G$. In fact, for a graph $G$ with mixing time at most $\log^{(2-\varepsilon)} |G|$ (for any constant $\varepsilon$), the time to reach the $m$-th layer is at most of order $m \cdot \frac{|G|}{\log \log |G|}$. This phenomenon is sometimes dubbed as “the aggregate grows arms”, i.e. grows faster than order $|G|$ particles per layer. The analogous phenomenon in the original DLA model on $\mathbb{Z}^d$ is considered a notoriously difficult open problem. In [6, 7, 8], Kesten provides upper bounds on the growth rate of the DLA aggregate in $\mathbb{Z}^d$. Eberz-Wagner [5] proved the existence of infinitely many holes in the two-dimensional DLA aggregate.

The second result concerns the density of the limit cluster, the union of all clusters obtained at some finite time. Theorem 4.2 shows that the expected rate at which the cluster grows bounds this density. This has two implications:

1. Theorem 4.6 states that for any vertex transitive graph $G$, the DLA process on the $G$-cylinder has density bounded by $2/3$. This includes the cases where $G$ is a $d$-dimensional Torus.

2. Theorem 4.8 shows that for $G$ with small enough mixing time, the density tends to 0 as the size of $G$ tends to infinity.

Finally, Theorem 5.1 is a lower bound on the expected time the cluster reaches the $m$-th layer, complementing the upper bound in Theorem 2.1. This lower bound implies that
the cluster cannot grow too fast, and in fact for many natural graphs it cannot grow faster than \(|G|^c\) for some universal \(0 < c < 1\). The lower bound holds for a wider range of graphs at the base of the cylinder than the upper bound (including \(d\)-dimensional tori for \(d \geq 3\)).

We remark that our estimates for the upper bound are crude, and simulations indicate that there is much room for improvement. In fact we believe the truth to be closer to the lower bound, see Conjecture 2.2. Proving Conjecture 2.2 will imply that for any family of graphs \(\{G_n\}\), the density of the DLA process on the \(G_n\)-cylinder tends to 0 as the size of \(G_n\) tends to infinity (see Conjecture 2.2).

For other very different variants of one dimensional DLA see [2, 9]. Another paper dealing with random-walk related questions on cylinders with varying bases is [13].

The rest of this paper is organized as follows:

First we introduce some notation. In Section 2 we define the process, and random variables associated with it. In Section 2.3 we state the first main result. Section 3 is devoted to proving Theorem 3.1, the main tool used to prove the main results. After the formulation of this theorem, a sketch of the key dichotomy idea is given, followed by a short discussion. In Section 4, we define the density of the DLA process on a cylinder. We also prove the theorems bounding the density in the above mentioned cases, Theorems 4.6 and 4.8. Finally, in Section 5 we prove the lower bound on the growth rate of the cluster, Theorem 5.1.

Let us note that the set up of DLA on a cylinder suggests another natural problem we are now pursuing. That is, how long does it take until the cluster clogs the cylinder? (This problem may be related to [3].)

Other possible directions for further research are presented in the last section, followed by an appendix which contains a few standard variants on some simple random walk results we need.
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1.1 Notation

Let $G$ be a graph. $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively. We use the notation $v \in G$ to denote $v \in V(G)$. For two vertices $u, v$ in $G$ we use the notation $u \sim v$ to denote that $u$ and $v$ are adjacent.

For a graph $G$, define the cylinder with base $G$, denoted $G \times \mathbb{N}$, by: The vertex set of $G \times \mathbb{N}$, is the set $V(G) \times \mathbb{N}$. The edge set is defined by the following relations: For all $u, v \in G$ and $m, k \in \mathbb{N}$, $(u, m) \sim (v, k)$ if and only if: either $m = k$ and $u \sim v$, or $|m - k| = 1$ and $u = v$. The cylinder with base $G$ is just placing infinitely many copies of $G$ one over the other, and connecting each vertex in a copy to its corresponding vertices in the adjacent copies.

By the simple random walk on a graph, we refer to the process where at each step the particle chooses a neighbor uniformly at random and moves to that neighbor. By the lazy random walk with holding probability $\alpha$, we mean a walk that with probability $\alpha$ stays at its current vertex, and with probability $1 - \alpha$ chooses a neighbor uniformly at random. By lazy random walk (without stating the holding probability) we refer to the walk that chooses uniformly at random from the set of neighbors and the current vertex. A lazy random walk is a simple random walk on the same graph with a self loop added at each vertex.

For simplicity, this paper will only deal with regular graphs; i.e. graphs such that all vertices are of the same degree.

We define the notion of the mixing time of a $d$-regular graph $G$: Let $\{g'_{t}\}_{t \geq 0}$ be a lazy random walk on $G$. The mixing time of $G$, is defined by

$$m = m(G) \overset{\text{def}}{=} \min \left\{ t > 0 \mid \forall u, v \in G \forall s \geq t , \Pr [g'_s = u \mid g'_0 = v] \geq \frac{1}{2|G|} \right\}. \quad (1.1)$$
This is a valid definition, since for all $v \in G$,
\[
\lim_{t \to \infty} \max_{u \in G} \left| \Pr \left[ g_t' = u \mid g_0' = v \right] - \frac{1}{|G|} \right| = 0.
\]
(This can be seen via Lemma [B.1].)

For a probability event $A$, we denote by $\overline{A}$ the complement of $A$.

## 2 Cylinder DLA

### 2.1 Definition

Fix a graph $G$. We define the $G$-Cylinder-DLA process:

Consider the graph $G \times \mathbb{N}$. Denote by $G_i$ the induced subgraph on the vertices $V(G) \times \{i\}$, for all $i \in \mathbb{N}$. We call $G_i$ the $i$-th layer of $G \times \mathbb{N}$.

The process is an increasing sequence, $\{A_t\}_{t=0}^{\infty}$, of connected subsets of $G \times \mathbb{N}$. We start with $A_0 = G_0$. Given $A_t$, define the set $A_{t+1}$ as follows:

Let $\partial A_t$ be the set of all vertices of $G \times \mathbb{N}$ that are not in $A_t$, but are adjacent to some vertex of $A_t$. That is,
\[
\partial A_t = \{u \in G \times \mathbb{N} \mid u \notin A_t, \exists v \in A_t : u \sim v\}.
\]

Let a particle perform a simple random walk on $G \times \mathbb{N}$ starting from infinity, and stop when the particle hits $\partial A_t$. Let $u$ be the vertex in $\partial A_t$ where the particle is stopped. Then, set $A_{t+1} = A_t \cup \{u\}$.

We find it convenient to use the following alternative (but equivalent) definition:

Let $M(t) = \min \{i \in \mathbb{N} \mid G_i \cap A_t = \emptyset\}$. That is, $M(t)$ is the lowest layer of $G \times \mathbb{N}$ that does not intersect the cluster $A_t$. Let $(g_{t+1}(i), \zeta_{t+1}(i)) \in G \times \mathbb{N}$, $i = 0, 1, 2, \ldots$, be a simple random walk on $G \times \mathbb{N}$, such that $g_{t+1}(0)$ is uniformly distributed in $G$, and $\zeta_{t+1}(0) = M(t)$. 

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Let $\kappa(t + 1)$ be the first time at which the walk is in $\partial A_t$. That is,

$$\kappa(t + 1) = \min \{ r \geq 0 \mid (g_{t+1}(r), \zeta_{t+1}(r)) \in \partial A_t \}.$$ 

Since the walk is recurrent, $\kappa(t + 1) < \infty$ with probability 1. Let $\kappa = \kappa(t + 1)$. Then, $(g_{t+1}(\kappa), \zeta_{t+1}(\kappa))$ is distributed on the set $\partial A_t$. Set $A_{t+1} = A_t \cup \{(g_{t+1}(\kappa), \zeta_{t+1}(\kappa))\}$.

This construction is equivalent to “starting from infinity”; a simple random walk starting at higher and higher layers, will take more and more steps before reaching the layer $M(t)$. Thus, as the starting layer tends to infinity, the distribution of the particle at the first time it hits the layer $M(t)$ is tending to uniform.

![Figure 1: G-Cylinder-DLA, where G is the cycle on 500 vertices. The number of particles is approximately 64,400.](image)

**2.2**

Let $\{A_t\}_{t=0}^{\infty}$ be a $G$-Cylinder-DLA process. $A_t$ is called the ($G$-Cylinder-DLA) cluster at time $t$. Define the following random variables:

For $A_t$, the $G$-Cylinder-DLA cluster at time $t$, define the load of the $i$-th layer at time $t$ by

$$L_t(i) = |A_t \cap G_i|.$$ 

$L_t(i)$ is the number of particles in the cluster at time $t$, on the $i$-th layer. Also define

$$L_t(\geq i) = \sum_{j \geq i} L_t(j), \quad \text{and} \quad L_t(> i) = \sum_{j > i} L_t(j).$$
\( L_t(\geq i) \) (respectively \( L_t(> i) \)) is the total load on layers \( \geq i \) (respectively \( > i \)). Note that

\[
L_t(\geq i) = \sum_{j=i}^{M(t)-1} L_t(j), \quad \text{and} \quad L_t(> i) = \sum_{j=i+1}^{M(t)-1} L_t(j).
\]

When subscripts become too small, we write \( L(t, i) \) instead of \( L_t(i) \) (and similarly for \( L(t, \geq i) \) and \( L(t, > i) \)).

Here are some properties of the Cylinder-DLA process, that we leave for the reader to verify. (This can help to get used to the notation.)

1. For all \( s > t \), \( A_t \subseteq A_s \).
2. For any \( i \in \mathbb{N} \), and \( s \geq t \), \( L_t(i) \leq L_s(i) \).
3. If \( L_t(i) = 0 \), then \( L_t(j) = 0 \) for all \( j \geq i \).
4. For all \( i \geq M(t) \), \( L_t(i) = 0 \). For all \( i < M(t) \), \( L_t(i) \geq 1 \).
5. For all \( t \),

\[
\sum_{i=0}^{\infty} L_t(i) = \sum_{i=0}^{M(t)-1} L_t(i).
\]

6. The following events are identical (for any \( t > 0 \)):

\[
\{L_t(\geq i) > L_{t-1}(\geq i)\} = \{L_t(\geq i) = 1 + L_{t-1}(\geq i)\} = \{\zeta_t(\kappa(t)) \geq i\}.
\]

### 2.3 \( G \)-Cylinder-DLA grows arms, for quickly mixing \( G \)

**Theorem 2.1.** Let \( 2 \leq d \in \mathbb{N} \). There exists \( n_0 = n_0(d) \), such that the following holds for all \( n > n_0 \):

Let \( G \) be a \( d \)-regular graph of size \( n \), and mixing time

\[
m(G) \leq \frac{\log^2(n)}{(\log \log(n))^5},
\]

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Let \( \{A_t\} \) be a \( G \)-Cylinder-DLA process. For \( m \in \mathbb{N} \), define

\[
T_m = \min \left\{ t \geq 0 \mid A_t \cap G_m \neq \emptyset \right\}.
\]

\( T_m \) is the time the cluster first reaches the layer \( m \).

Then, for all \( m \),

\[
\mathbb{E}[T_m] < \frac{4mn}{\log \log n}.
\]

The proof of Theorem 2.1 is via Theorem 3.1 below.

**Remark.** One may suggest that the reason Theorem 2.1 can be proved, is that we use for the base graph \( G \), graphs that are so highly connected that in some sense there is no geometry. We stress that the class of graphs that have \( \log^{(2-\epsilon)} |G| \) mixing time, is much larger than what is known as “expander graphs”. This class includes many natural families of graphs, including lamplighter graphs on tori of dimension 2 and above (see [10]).

We remark that Theorem 3.1 below is in some sense a “worst case” analysis. Thus, we believe that our results are not optimal. In fact, we conjecture that a stronger result than Theorem 2.1 should hold for any graph at the base of the cylinder:

**Conjecture 2.2.** Let \( \{G_n\} \) be a family of \( d \)-regular graphs such that \( \lim_{n \to \infty} |G_n| = \infty \). There exist \( 0 < \gamma < 1 \) and \( n_0 \) such that for all \( n > n_0 \) the following holds:

Set \( G = G_n \) and let \( \{A_t\} \) be a \( G \)-Cylinder-DLA process. For \( m \in \mathbb{N} \), define

\[
T_m = \min \left\{ t \geq 0 \mid A_t \cap G_m \neq \emptyset \right\}.
\]

\( T_m \) is the time the cluster first reaches the layer \( m \).

Then, for all \( m \),

\[
\mathbb{E}[T_m] \leq m |G_n|^\gamma.
\]
3 The time to stick to a new layer

The following theorem states that under the assumption that $G$ has small enough mixing time, in the $G$-Cylinder-DLA process, the expected amount of particles until one sticks to the new layer, is substantially less than $|G|$.

Note that since for all $m$, we can write the telescopic sum

$$T_m = \sum_{\ell=1}^{m} (T_{\ell} - T_{\ell-1}),$$

Theorem 2.1 follows from Theorem 3.1 by linearity of expectation.

**Theorem 3.1.** Let $2 \leq d \in \mathbb{N}$. There exists $n_0 = n_0(d)$, such that the following holds for all $n > n_0$:

Let $G$ be a $d$-regular graph of size $n$, and mixing time

$$m(G) \leq \frac{\log^2(n)}{(\log \log(n))^5}.$$

Let $A_t$ be a $G$-Cylinder-DLA cluster at time $t$. Define

$$T = \min \{s > t \mid M(s) > M(t)\}.$$

$T$ is the first time that a particle sticks to the empty layer, $G_{M(t)}$. Then,

$$\mathbb{E}[T - t] < \frac{4n}{\log \log n}.$$

In order to prove Theorem 3.1, we need a few lemmas, stated and proved in this section. The proof of Theorem 3.1 is deferred to Section 3.4. The main idea of the proof is in the following proof sketch:

**Proof Sketch.** The cluster $A_t$ can be in two states: Either it is such that particles stick quickly to it; i.e. particles take few steps before sticking to the cluster. Or, the particles take many steps before sticking to the cluster.
In the first case, the particles take few steps before sticking. Thus, the particles cannot stick many layers below \( M(t) \), so they build up a heavy load on the layers near \( M(t) \). Each time a layer has a heavy load, there is better chance of the next particles to stick to the layers above it. So, in less than \( O\left(\frac{n}{\log \log n}\right) \) particles, there is a heavy load on the layer \( M(t) - 1 \), and the probability of sticking to the layer \( M(t) \) is now substantially greater than \( 1/n \). This case is dealt with in Lemma 3.5.

In the second case, the particles take many steps before sticking. Thus, they also make many long excursions above the layer \( M(t) \). Because the base of the cylinder, \( G \), has small enough mixing time, after each such excursion, there is a chance of at least \( 1/2n \) to stick to the layer \( M(t) \). This occurs many times, so the probability of sticking to the layer \( M(t) \) is much greater than \( 1/n \). This case is dealt with in Corollary 3.14.

The proof of Theorem 3.1 in Section 3.4 combines both cases, to show that in both cases, the expected time until a particle sticks to the new layer \( M(t) \), is substantially smaller than \( n \).

**Remark.** As stated above, the proof of Theorem 3.1 is in some sense a “worst-case” analysis. The first part, regarding the case where particles take few steps before sticking, is valid for any regular \( G \) (not only those with small mixing time). But in reality, simulations show that this is not what really happens. The particles do not build a series of higher and higher layers with large loads.

On the other hand, the second part, (where particles take many steps and thus return to the layer \( M(t) \) many times, thus increasing the probability of sticking to \( M(t) \)) is probably what does actually occur. In fact, we suspect that this is true not only for graphs with small mixing time, but for any graph at the base of the cylinder (see Conjecture 2.2).

**Remark.** It may be of use to note that Theorem 3.1 holds also if \( A_t \) is replaced with any subset of \( G \times N \) intersecting all layers up to \( M(t) \). In particular, given any cluster, not necessarily grown by a \( G \)-Cylinder-DLA process, the expected time until a particle sticks to the new layer is bounded by order \( \frac{|G|}{\log \log |G|} \).
3.1 A large load on a high layer

In this section, we show that if there is a high enough layer \((\geq M(t) - \frac{\log n}{4 \log \log n})\) with large load (at least \(\frac{n}{\log n}\)), then the expected time until a particle sticks to the new layer, \(M(t)\), is \(o(n)\).

**Lemma 3.2.** There exists \(n_0\), such that the following holds for all \(n > n_0\): Let \(G\) be a \(d\)-regular graph of size \(n\). Set

\[ \mu = \mu(n) = \left\lfloor \frac{\log(n)}{4 \log \log(n)} \right\rfloor, \quad \text{and} \quad \nu = \nu(n) = \log(n). \]

Let \(A_t\) be a \(G\)-Cylinder-DLA cluster at time \(t\). Let

\[ T = \min \left\{ s > t \mid M(s) > M(t) \right\}. \]

\(T\) is the first time that the cluster reaches the new layer.

Assume that there exists \(j \geq M(t) - \mu\) such that \(L_t(j) \geq \frac{n}{\nu}\). Then,

\[ \mathbb{E} [T - t] \leq \frac{n}{4 \log \log(n)}. \]

The main idea of the proof is as follows: If a layer \(j\) has load \(m\), then the probability to stick above layer \(j\) is at least \(m/n\). Thus, to get a layer \(i > j\) with load \(\frac{m}{\log(m)}\), we need \(o(n)\) particles. Thus, building higher and higher layers with high loads, we reach the empty layer in \(o(n)\) particles.

**Proof of Lemma 3.2.** The following proposition states that if there is a layer with load \(m\), then the probability of particles sticking above that layer is at least \(m/n\).

**Proposition 3.3.** Let \(G\) be a \(d\)-regular graph of size \(n\). Let \(A_t\) be a \(G\)-Cylinder-DLA cluster at time \(t\). Fix a layer \(j > 0\). Assume that \(L_t(j) \geq m\). For \(s > t\), let \(I_s\) be the indicator function of the event that the \(s\)-th particle sticks to a layer \(\geq j + 1\). That is,

\[ I_s = 1_{\{L(s, \geq j+1) > L(s-1, \geq j+1)\}}. \]
Then, for all \( s > t \),
\[
\Pr[I_s = 1 \mid I_r, \ t < r < s] \geq \frac{m}{n},
\]
for any values of \( I_r, \ t < r < s \).

**Proof.** Set \( s > t \). Condition on the values of \( I_r, \ t < r < s \). Let \( A_{s-1} \) be the cluster at time \( s - 1 \). Let \((g(\cdot), \zeta(\cdot)) = ((g_s(\cdot), \zeta_s(\cdot))\) be the walk of the \( s \)-th particle. Note that for any \( r \), if \((g(r), \zeta(r)) \in \partial A_{s-1} \cup A_{s-1} \), then \( \kappa(s) \leq r \). At time \( s - 1 \) the layer \( j \) has load \( L_{s-1}(j) \geq L_t(j) \geq m \). Thus,
\[
|'(\partial A_{s-1} \cup A_{s-1}) \cap G_{j+1}| \geq m.
\]
Let \( k \) be the first time the walk \((g(\cdot), \zeta(\cdot))\) hits the layer \( j + 1 \). Then, since the uniform distribution on \( G \) is the stationary distribution, \( g(k) \) is uniformly distributed in \( G_{j+1} \). Thus,
\[
\Pr[\kappa(s) \leq k \mid A_{s-1}] \geq \Pr[(g(k), \zeta(k)) \in \partial A_{s-1} \cup A_{s-1} \mid A_{s-1}] \geq \frac{m}{n}.
\]
Since for all \( 0 \leq r \leq k \) we have that \( \zeta(r) \geq j + 1 \), we get that
\[
\Pr[I_s = 1 \mid A_{s-1}] \geq \Pr[\kappa(s) \leq k \mid A_{s-1}] \geq \frac{m}{n}.
\]
Let \( \mathcal{A} \) be the set of all clusters \( A \subseteq G \times \mathbb{N} \) such that \( \Pr[A_{s-1} = A \mid I_r, \ t < r < s] > 0 \). Then we have,
\[
\Pr[I_s = 1 \mid I_r, \ t < r < s] \geq \sum_{A \in \mathcal{A}} \Pr[I_s = 1 \mid A_{s-1} = A] \Pr[A_{s-1} = A \mid I_r, \ t < r < s] \geq \frac{m}{n}.
\]

Assume there is a layer with load \( m \). Since each particle sticks above this layer with probability at least \( m/n \), the expected time until there are \( \ell \) new particles above this layer should be at most \( \ell \cdot (n/m) \). This is captured in the following proposition:
Proposition 3.4. Let $G$ be a $d$-regular graph of size $n$. Let $A_t$ be a $G$-Cylinder-DLA cluster at time $t$. Fix a layer $j > 0$. Assume that $L_t(j) \geq m$. For $\ell \in \mathbb{N}$, define
\[
S_\ell = \min \{ s \geq t \mid L_s(\geq j + 1) = \ell + L_t(\geq j + 1) \}. \tag{3.1}
\]
That is, $S_\ell$ is the first time that there are $\ell$ new particles in the layers $\geq j + 1$ (so $S_0 = t$).

Then,
\[
\mathbb{E} [S_\ell - t] = \mathbb{E} [S_\ell - S_0] \leq \frac{\ell n}{m}.
\]

Proof. By Proposition 3.3, for all $k \geq 1$, $S_k - S_{k-1}$ is dominated by a geometric random variable with mean $\leq \frac{n}{m}$. Thus,
\[
\mathbb{E} [S_\ell - t] = \sum_{k=1}^\ell \mathbb{E} [S_k - S_{k-1}] \leq \frac{\ell m}{n}.
\]
\[\square\]

With these two propositions, we continue with the proof of Lemma 3.2.

Set $M = M(t)$. Set $T_0 = t$. For $r \geq 0$, define inductively the following stopping times:
\[
T_r = \min \{ s \geq T_{r-1} \mid \exists i \geq j + r : L_s(i) \geq n\nu^{-(2r+1)} \}.
\]
That is, $T_r$ is the first time that there exists a “high enough” layer (higher than $j + r$), such that the load on that layer is “large enough” (larger than $n\nu^{-(2r+1)}$).

Consider time $T_\mu$. At this time, we have that there exists a layer $i \geq j + \mu \geq M$ such that $L_{T_\mu}(i) \geq n\nu^{-(2\mu+1)} \geq 1$. So $M(T_\mu) > M$ and $T \leq T_\mu$. Thus, we can write
\[
T - t = \sum_{r=1}^\mu (\min \{ T, T_r \} - \min \{ T, T_{r-1} \}).
\]

For all $r \geq 0$, set $\tau(r) = \min \{ T, T_r \}$.

Claim. For all $r > 0$,
\[
\mathbb{E} [\tau(r) - \tau(r-1)] \leq \frac{n}{\nu}.
\]
Proof. Fix \( r > 0 \). For \( \ell \in \mathbb{N} \), define

\[
S_\ell = \min \{ s \geq T_{r-1} \mid L_s(\geq j + r) = \ell + L_{T_{r-1}}(\geq j + r) \}.
\]

That is, \( S_\ell \) is the first time that there are \( \ell \) new particles in the layers \( \geq j + r \). So \( S_0 = T_{r-1} \). Let \( a = \mu[n\nu^{-2r+1}] \).

Case 1: \( T \leq T_{r-1} \). Then \( \tau(r) - \tau(r - 1) = 0 \leq S_a - S_0 \).

Case 2: \( T > T_{r-1} \) and \( T \leq S_a \). Then \( \tau(r) - \tau(r - 1) \leq T - T_{r-1} \leq S_a - S_0 \).

Case 3: \( T > T_{r-1} \) and \( T > S_a \). Note that if \( T > S_a \), then \( M(S_a) = M \). At time \( S_a \), there are at least \( a \) particles on the layers \( \geq j + r \). So, if \( T > S_a \) then

\[
a \leq \sum_{i=j+r}^{M(S_a)-1} L_{S_a}(i) = \sum_{i=j+r}^{M-1} L_{S_a}(i).
\]

So there exists some \( j + r \leq i \leq M - 1 \) such that \( L_{S_a}(i) \geq \frac{a}{M-(j+r)} \). Since \( j \geq M - \mu \), we have \( L_{S_a}(i) \geq \frac{a}{\mu} \geq n\nu^{-2r+1} \).

So we conclude that if \( T > S_a \) then \( T_r \leq S_a < T \). So \( T > S_a \) implies that

\[
\tau(r) - \tau(r - 1) = T_r - T_{r-1} \leq S_a - S_0.
\]

Thus, in all three cases, \( \tau(r) - \tau(r - 1) \leq S_a - S_0 \).

At time \( S_0 = T_{r-1} \), by the definition of \( T_{r-1} \), we have that for some \( i \geq j + r - 1 \), there is a load \( L_{S_0}(i) \geq n\nu^{-2r+1} \) (for \( r = 1 \) we can choose \( i = j \), and since \( T_0 = t \) we have by assumption that \( L_t(j) \geq \frac{n}{\nu} \)). By Proposition 3.4 with \( j = i, t = T_{r-1} = S_0 \) and \( m = n\nu^{-2r+1} \), we have that for large enough \( n \)

\[
\mathbb{E}[\tau(r) - \tau(r - 1)] \leq \mathbb{E}[S_a - S_0] \leq a\frac{n}{n\nu^{-2r+1}} \leq \frac{n}{\nu}.
\]

\( \square \)
Returning to the proof of Lemma 3.2 for all \( r > 0 \),

\[
E [\tau(r) - \tau(r-1)] \leq \frac{n}{\nu}.
\]

Thus, for large enough \( n \),

\[
E [T - t] = \sum_{r=1}^{\mu} E [\tau(r) - \tau(r-1)] \leq \mu \cdot \frac{n}{\nu} \leq \frac{n}{4 \log \log(n)}.
\]

\[\square\]

### 3.2 Particles take few steps

Recall that \( \kappa(s) \) is the number of steps the \( s \)-th particle takes until it sticks (to \( \partial A_{s-1} \)). In this section, we show that if \( \kappa(t + 1) \) is small, then all particles \( s > t \), have a good chance of sticking at high layers. Thus, a small amount of particles is needed to get a high layer with large load.

**Lemma 3.5.** There exists \( n_0 \) such that the following holds for all \( n > n_0 \): Let \( G \) be a \( d \)-regular graph of size \( n \). Set

\[
\mu = \mu(n) = \left\lfloor \frac{\log(n)}{4 \log \log(n)} \right\rfloor, \quad \text{and} \quad \nu = \nu(n) = \log(n).
\]

Let \( A_t \) be a \( G \)-Cylinder-DLA cluster at time \( t \). Let

\[
T = \min \{ s > t \mid M(s) > M(t) \}.
\]

\( T \) is the first time that the cluster reaches the new layer.

Assume that \( \Pr \left[ \kappa(t + 1) \leq \frac{\mu^2}{4} \right] \geq \frac{1}{4} \). Then,

\[
E [T - t] \leq \frac{5n}{2 \log \log(n)}.
\]

**Proof.** In the following two propositions, we use the fact that with probability at least \( 1/4 \), the particle takes a small amount of steps to stick.
Proposition 3.6. Let $G$ be a $d$-regular graph. Let $A_t$ be a $G$-Cylinder-DLA cluster at time $t$, and consider the $(t+1)$-th particle. Let $y \in \mathbb{N}$ and assume that $\Pr [\kappa (t+1) \leq y^2/4] \geq \frac{1}{4}$. Then,

$$
\Pr \left[ \min_{0 \leq r \leq \kappa (t+1)} \zeta_{t+1}(r) \geq M(t) - y \right] \geq \frac{1}{8}.
$$

That is, with probability at least $1/8$, the particle sticks without ever going below the layer $M(t) - y$.

Proof. Set $x = \lfloor \frac{y^2}{4} \rfloor$. Note that

$$
\Pr [\kappa (t+1) \leq x] \geq \frac{1}{4}.
$$

Let $(g(\cdot), \zeta(\cdot)) = (g_{t+1}(\cdot), \zeta_{t+1}(\cdot))$ be the walk the $(t+1)$-th particle takes. That is, $g(0)$ is uniformly distributed in $G$, and $\zeta(0) = M(t)$. Let $\kappa = \kappa (t+1)$ be the first time the walk hits $\partial A_t$.

Note that

$$
\left\{ \min_{0 \leq r \leq x} \zeta(r) \geq M(t) - y \right\} \cap \{ \kappa \leq x \} \text{ implies } \left\{ \min_{0 \leq r \leq \kappa} \zeta(r) \geq M(t) - y \right\}.
$$

The walk $\zeta(0), \ldots, \zeta(x)$, is an $x$-step lazy random walk, with holding probability $1 - \alpha = \frac{d}{d+2}$. By Lemma A.6, we have that

$$
\Pr \left[ \min_{0 \leq r \leq x} \zeta(r) \geq M(t) - y \right] \geq \Pr \left[ \max_{1 \leq r \leq x} |\zeta(r) - M(t)| < \sqrt{8ax} \right] \geq 1 - \frac{1}{8}.
$$

Thus,

$$
\Pr \left[ \min_{0 \leq r \leq \kappa} \zeta(r) \geq M(t) - y \right] \geq \Pr \left[ \min_{0 \leq r \leq x} \zeta(r) \geq M(t) - y \right] - \Pr [\kappa > x] \geq \frac{1}{8},
$$

(where we have used the inequality $\Pr [A \cap B] \geq \Pr [A] - \Pr [\overline{B}]$, valid for any events $A, B$).

\[\square\]
Proposition 3.7. Let $G$ be a $d$-regular graph. Let $A_t$ be a $G$-Cylinder-DLA cluster at time $t$. Let $y \in \mathbb{N}$ and assume that

$$\Pr[\kappa(t + 1) \leq y^2/4] \geq \frac{1}{4}.$$ 

For $s > t$, define $H(s) = \zeta_s(\kappa(s))$; i.e. $H(s)$ is the height of the layer at which the $s$-th particle sticks. Then, for all $s > t$,

$$\Pr[H(s) \geq M(t) - y \mid H(r), \ t < r < s] \geq \frac{1}{8},$$

for any values of $H(r)$, $t < r < s$.

**Proof.** Let $s > t$. Let $(g(\cdot), \zeta(\cdot)) = (g_s(\cdot), \zeta_s(\cdot))$ be the walk the $s$-th particle takes. That is, $g(0)$ is uniformly distributed in $G$, and $\zeta(0) = M(s) \geq M(t)$. Set $k = \min \{ r \geq 0 \mid \zeta(r) = M(t) \}$, and set $k' = \min \{ r \geq k \mid (g(r), \zeta(r)) \in \partial A_t \}$. $k$ is the first time the $s$-th particle is at the layer $M(t)$ (this can be time 0, e.g. if $M(s) = M(t)$). $k'$ is the first time after $k$ that the particle hits the outer boundary of the cluster $A_t$. Since $A_t \subseteq A_{s-1}$, we have that $\kappa(s) \leq k'$. So,

$$\Pr[H(s) \geq M(t) - y \mid A_{s-1}] \geq \Pr\left[ \min_{0 \leq r \leq \kappa(s)} \zeta(r) \geq M(t) - y \mid A_{s-1} \right] \geq \Pr\left[ \min_{0 \leq r \leq k'} \zeta(r) \geq M(t) - y \mid A_{s-1} \right] \geq \Pr\left[ \min_{0 \leq r \leq k' - k} \zeta(k + r) \geq M(t) - y \mid A_{s-1} \right],$$

the last inequality following from the fact that for all $r < k$, by definition, $\zeta(r) \geq M(t) \geq M(t) - y$.

Since the uniform distribution is the stationary distribution on $G$, $g(k)$ is uniformly distributed in $G$. Thus, the walk $(g(k + r), \zeta(k + r))$ has the same distribution as the walk $(g_{t+1}(r), \zeta_{t+1}(r))$, and $k' - k$ has the same distribution as $\kappa(t+1)$. Using Proposition 3.6 we now conclude

$$\Pr[H(s) \geq M(t) - y \mid A_{s-1}] \geq \Pr\left[ \min_{0 \leq r \leq \kappa(t+1)} \zeta_{t+1}(r) \geq M(t) - y \right] \geq \frac{1}{8}.$$
Averaging over all $A \subset G \times \mathbb{N}$ such that $\Pr [A_{s-1} = A \mid H(r), \ t < r < s] > 0$, we get that
\[
\Pr [H(s) \geq M(t) - y \mid H(r), \ t < r < s] \geq \frac{1}{8}.
\]

Proposition 3.8. Let $G$ be a $d$-regular graph. Let $A_t$ be a $G$-Cylinder-DLA cluster at time $t$. Let $y \in \mathbb{N}$ and assume that
\[
\Pr [\kappa(t + 1) \leq y^2/4] \geq \frac{1}{4}.
\]
For $\ell \in \mathbb{N}$, define
\[
S_\ell = \min \left\{ s \geq t \mid L_s(\geq M(t) - y) = \ell + L_t(\geq M(t) - y) \right\}.
\]
That is, $S_\ell$ is the first time that there are $\ell$ new particles in the layers $\geq M(t) - y$ (so $S_0 = t$).

Then,
\[
\mathbb{E} [S_\ell - t] \leq 8\ell.
\]

Proof. The proof is similar to the proof of Proposition 3.4.

By Proposition 3.7, regardless of the previous particles, each particle $s > t$ has probability at least $1/8$ to stick to a layer $\geq M(t) - y$. Thus, the expected time until there are $\ell$ particles above this layer is bounded by $8\ell$.

We now put everything together to prove Lemma 3.5. We show that if $\kappa(t+1)$ is small, then after a small amount of particles there is a high layer with large load. Thus, after another small amount of particles, the cluster reaches the new layer $M(t)$.

Set $M = M(t)$. Let
\[
T' = \min \left\{ s \geq t \mid \exists j \geq M - \mu : L_s(j) \geq \frac{n}{\nu} \right\}.
\]
For $\ell \in \mathbb{N}$, define
\[
S_\ell = \min \{ s \geq t \mid L_s(\geq M - \mu) = \ell + L_t(\geq M - \mu) \}.
\]

Consider the time $S_a$ for $a = \mu \lceil n \nu \rceil$. Consider the case where $T > S_a$. Then $M(S_a) = M$.

At time $S_a$, there are at least $a$ particles in the layers $\geq M - \mu$, so
\[
a \leq \sum_{i=M-\mu}^{M(S_a)-1} L_{S_a}(i) = \sum_{i=M-\mu}^{M-1} L_{S_a}(i).
\]

Thus, there exists $M - \mu \leq j \leq M - 1$ such that $L_{S_a}(j) \geq \frac{a}{\mu} \geq \frac{n}{\nu}$. So $T' \leq S_a$.

We conclude that if $T > S_a$ then $T' \leq S_a$. In other words, we have shown that $\min \{ T, T' \} \leq S_a$. Hence, because it was assumed that $\Pr[\kappa(t + 1) \leq \mu^2/4] \geq \frac{1}{4}$, by Proposition 3.8
\[
E[\min \{ T, T' \} - t] \leq E[S_a - t] \leq 8a \leq \frac{2n}{\log \log(n)} + \frac{2\log(n)}{\log \log(n)}.
\]

Define the event
\[
B = \left\{ \exists \ j \geq M(T') - \mu : L_{T'}(j) \geq \frac{n}{\nu} \right\}.
\]

By Lemma 3.2, we have that for large enough $n$,
\[
E[1_B (T - T')] \leq \frac{n}{4 \log \log n}.
\]

We have that $T = \min \{ T, T' \} + 1_{\{ T < T' \}} (T - T')$. Now, at time $T'$, we have a layer $j \geq M - \mu$ such that $L_{T'}(j) \geq \frac{n}{\nu}$. If $j < M(T') - \mu$ then $M(T') > M$, and $T \leq T'$. So the event $\{ T' < T \}$ implies the event $B$. Thus, for large enough $n$,
\[
E[T - t] \leq E[\min \{ T, T' \} - t] + E[1_B (T - T')] \leq \frac{5n}{2 \log \log(n)}.
\]

$\square$
3.3 Particles take many steps

In the previous section, we analyzed what happens when \( \kappa(t+1) \) is “small”. This section is concerned with the case where \( \kappa(t+1) \) is “large”. The main goal of this section is proving Lemma 3.11 and Corollary 3.14. These are essential ingredients in the proof of Theorem 3.1.

We begin with two technical lemmas:

**Lemma 3.9.** Let \( G \) be a graph. Let \( G' \) be the graph obtained from \( G \) by adding a self loop at each vertex. That is,

\[
V(G') = V(G), \quad \text{and} \quad E(G') = E(G) \cup \{ \{v, v\} \mid v \in G \}.
\]

Let \( V = V(G \times \mathbb{N}) = V(G' \times \mathbb{N}) \). Consider the \( G \)-Cylinder-DLA and \( G' \)-Cylinder-DLA processes. Let \( P_t(A, v) = \Pr [A_t = A \cup \{v\} \mid A_{t-1} = A] \) where \( \{A_t\} \) is a \( G \)-Cylinder-DLA process. Let \( P'_t(A, v) = \Pr [A_t = A \cup \{v\} \mid A_{t-1} = A] \) where \( \{A_t\} \) is a \( G' \)-Cylinder-DLA process.

Then, for all \( A \subseteq V, v \in V, \) and all \( t > 0 \),

\[
P_t(A, v) = P'_t(A, v).
\]

**Proof.** Assume that \( A_t = A \). We can couple the walk of the \( (t+1) \)-th particle in both processes to hit the same vertex, as follows:

Denote by \( L \) the set of self loops added to \( G \) to form \( G' \). Let \( \{(g(r), \zeta(r)) \mid r \geq 0\} \) be the walk of the \( (t+1) \)-th particle, in the \( G' \)-Cylinder-DLA process. Define \( \Gamma \) to be the set of all \( r > 0 \) such that the step from \((g(r-1), \zeta(r-1))\) to \((g(r), \zeta(r))\) does not traverse one of the self loops in \( L \). For the \( G \)-Cylinder-DLA process, let the \( (t+1) \)-th particle take the path \( \{(g(r), \zeta(r)) \mid r \in \Gamma \cup \{0\}\} \). This path has the correct marginal distribution, as it is a simple random walk on \( G \times \mathbb{N} \). Note that both paths hit \( \partial A_t \) at the same vertex, since traversing a self loop does not move the particle to a new vertex. \( \square \)
Remark. If $G$ already has self loops, then by adding a self loop at each vertex, we mean adding a new self loop, treated as different from the original loop. This only adds technical complications, so we will not go into this issue. The reader can treat all graphs as not having self loops, though the results carry out to graphs with self loops as well.

The important consequence of Lemma 3.9 is that the Cylinder-DLA process does not change if we let the particles perform a lazy random walk on $G \times \mathbb{N}$. This is needed to avoid technical complications that arise from parity issues in bi-partite graphs. The following technical lemma is used to bypass this issue.

Recall our definition of the mixing time of a $d$-regular graph $G$: Let $\{g_t\}_{t \geq 0}$ be a lazy symmetric random walk on $G$. The mixing time of $G$, is defined by

$$m = m(G) \overset{\text{def}}{=} \min \left\{ t > 0 \mid \forall u, v \in G \forall s \geq t, \Pr [g_s' = u \mid g_0 = v] \geq \frac{1}{2|G|} \right\}.$$

**Lemma 3.10.** Let $G$ be a $d$-regular graph, and let $G'$ be the graph obtained from $G$ by adding a self loop at each vertex, as in Lemma 3.9. Let $\{g_t\}_{t \geq 0}$ be a simple random walk on $G'$. Then, for all $t \geq m(G)$, and all $u, v \in G$,

$$\Pr [g_t = u \mid g_0 = v] \geq \frac{1}{2|G|}.$$

**Proof.** This is immediate from the definition of $m(G)$, and the fact that $\{g_t\}_{t \geq 0}$ is distributed as a lazy symmetric random walk on $G$. \qed

This completes the two technical lemmas we require. Next we introduce some notation.

Let $G$ be a graph. Let $(g(0), \zeta(0)), (g(1), \zeta(1)), \ldots$, be a simple random walk on $G \times \mathbb{N}$. For two times $r_1 < r_2$ denote

$$r_1 \to r_2 \overset{\text{def}}{=} \{(g(r_1), \zeta(r_1)), (g(r_1 + 1), \zeta(r_1 + 1)), \ldots, (g(r_2), \zeta(r_2))\}.$$  

$r_1 \to r_2$ is the path the walk takes between times $r_1$ and $r_2$. Define

$$L = \{ r > 0 \mid \zeta(r) = \zeta(0) \}.$$  

and assume that $L = \{\ell_1 < \ell_2 < \cdots \}$. $L$ is the set of times at which the walk visits the original layer. For $i \geq 1$ define $\rho_i \defeq \ell_{i-1} \to \ell_i$, where $\ell_0 = 0$. We call $\rho_i$ an excursion. For $i \geq 1$ and $\alpha \in \mathbb{R}$, we say that $\rho_i = \ell_{i-1} \to \ell_i$ is a positive $\alpha$-long excursion if the following conditions hold:

1. $\zeta(\ell_{i-1} + 1) = \zeta(0) + 1$; i.e. the excursion is on the positive side of the origin of the walk.

2. The walk takes at least $\alpha$ steps in $G$ during the excursion; that is,

$$\sum_{r=\ell_{i-1}+1}^{\ell_i} 1_{\{\zeta(r) = \zeta(r-1)\}} \geq \alpha.$$ 

We stress that ‘$\alpha$-long’ refers to the number of steps in $G$, not the total length of the excursion.

**Lemma 3.11.** Let $2 \leq d \in \mathbb{N}$. There exist $c = c(d) > 0$ and $C = C(d) > 0$ such that for any $x \geq 1$ the following holds:

Let $G$ be a $d$-regular graph of size $|G| = n$ and mixing time $m(G)$. Let $A_t$ be a $G$-Cylinder-DLA cluster at time $t$. Recall that $M(t)$ is the lowest empty layer at time $t$, and that $\kappa(t+1)$ is the number of steps the $(t+1)$-th particle takes before it sticks.

Then,

$$\Pr [ M(t+1) > M(t) \mid A_t ] > c \frac{x}{n \sqrt{m(G)}} \cdot \left( \frac{1}{2} - \Pr [ \kappa(t+1) \leq C x^2 ] \right).$$

**Proof.** Let $G$ be a $d$-regular graph. Let $(g(0), \zeta(0)), (g(1), \zeta(1)), \ldots$, be a simple random walk on $G \times \mathbb{N}$. Let $\{\rho_i = \ell_{i-1} \to \ell_i \mid i \geq 1\}$ be the excursions of the walk.

First, we need to calculate the probability of a positive $\alpha$-long excursion.

**Proposition 3.12.** For all $i \geq 1$ and any $2 \leq \alpha \in \mathbb{R}$, the probability that $\rho_i$ is a positive $\alpha$-long excursion is greater than $\frac{1}{12(d+2)\sqrt{\alpha}}$. 

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Proof. Because of the Markov property, and the fact that \( \zeta(\ell_i) = \zeta(0) \) for all \( i \), we get that \( \{\rho_i \mid i \geq 1\} \) are independent and identically distributed. Thus, it suffices to prove the proposition for \( \rho = \rho_1 \).

Fix \( 2 \leq \alpha \in \mathbb{R} \). Set \( m = \zeta(0) + 1 \). So, the probability that \( \rho \) is a positive \( \alpha \)-long excursion is equal to

\[
\Pr \left[ \zeta(1) = m, \sum_{r=1}^{\ell_1} 1_{\{\zeta(r) = \zeta(r-1)\}} \geq \alpha \right]
\]

\[
= \Pr[\zeta(1) = m] \Pr \left[ \sum_{r=2}^{\ell_1} 1_{\{\zeta(r) = \zeta(r-1)\}} \geq \alpha \mid \zeta(1) = m \right], \tag{3.2}
\]

(we use the fact that \( \zeta(1) = m \neq \zeta(0) \)). Define

\[
\Gamma = \{r > 1 \mid \zeta(r) = \zeta(r-1)\}, \quad \text{and}
\]

\[
Z = \{r > 1 \mid \zeta(r) \neq \zeta(r-1)\}.
\]

\( \Gamma \) (respectively, \( Z \)) is the set of times at which the walk moves in \( G \) (respectively, \( N \)). Let \( g_1 = g(1) \) and let \( g_2, g_3, \ldots \), be the walk \( \{g(r) \mid r \in \Gamma\} \). So \( g_1, g_2, \ldots \), is distributed as a simple random walk on \( G \), starting at \( g(1) \). Let \( \zeta_1 = \zeta(1) \) and let \( \zeta_2, \zeta_3, \ldots \), be the walk \( \{\zeta(r) \mid r \in Z\} \). So \( \zeta_1, \zeta_2, \ldots \), is distributed as a simple random walk on \( N \), starting at \( \zeta(1) \).

Set \( \alpha' = 8 \left\lceil \frac{\alpha}{2} \right\rceil \). For \( r > 1 \), let \( I_r = 1_{\{\zeta(r) = \zeta(r-1)\}} \). Set

\[
\gamma = \sum_{r=2}^{\alpha'+1} I_r.
\]

\( \gamma \) is the sum of \( \alpha' \) independent, identically distributed Bernoulli random variables, with mean \( \frac{\alpha'}{d+2} \geq \frac{1}{2} \). Using the Chernoff bound (see e.g. Appendix A in [1]),

\[
\Pr \left[ \gamma < \frac{\alpha'}{4} \right] \leq 2 \exp \left( -\frac{\alpha'}{8} \right) \leq \frac{2}{e} < \frac{3}{4}.
\]
\(\gamma\) is independent of \(\zeta(1)\), so

\[
\Pr[\gamma \geq \alpha \mid \zeta(1) = m] \geq \Pr[\gamma \geq \frac{\alpha'}{4} \mid \zeta(1) = m] > \frac{1}{4}.
\] (3.3)

Consider the walk \(\zeta_1, \zeta_2, \ldots, \zeta_{\alpha'+1}\), conditioned on the event \(\zeta(1) = m\). Define the event

\[B = \{\zeta_2 \geq m, \ zeta_3 \geq m, \ldots, \ zeta_{\alpha'+1} \geq m\}.
\]

Conditioned on \(\zeta(1) = m\), the walk \(\zeta_1, \zeta_2, \ldots, \zeta_{\alpha'+1}\) is a simple random walk on \(\mathbb{N}\) starting at \(\zeta_1 = m\). Using Corollary A.2,

\[
\Pr[B \mid \zeta(1) = m] \geq \Pr[\zeta_2 \geq m, \ldots, \zeta_{\alpha'+1} \geq m \mid \zeta(1) = m] = 2^{-\alpha'} \left(\frac{\alpha'}{\alpha'/2}\right).
\]

A careful application of Stirling’s approximation gives

\[
\Pr[B \mid \zeta(1) = m] > \frac{1}{3\sqrt{\alpha}},
\] (3.4)

for all \(\alpha \geq 2\).

Since \(\zeta(\ell_1) = \zeta(0) = m - 1\), we have that, conditioned on \(\zeta(1) = m\), the event \(B\) implies the event \(\{\alpha' + 1 \leq \ell_1\}\). Thus,

\[
\sum_{r=2}^{\ell_1} \mathbf{1}_{\{\zeta(r) = \zeta(r-1)\}} \geq \mathbf{1}_{\{B\}} \sum_{r=2}^{\alpha'+1} \mathbf{1}_{\{\zeta(r) = \zeta(r-1)\}} = \mathbf{1}_{\{B\}} \gamma.
\]

Now, \(\gamma\) is independent of the event \(B\), so, using (3.3) and (3.4),

\[
\Pr\left[\sum_{r=2}^{\ell_1} \mathbf{1}_{\{\zeta(r) = \zeta(r-1)\}} \geq \alpha \mid \zeta(1) = m\right] \geq \Pr\left[B, \gamma \geq \alpha \mid \zeta(1) = m\right] = \Pr[B \mid \zeta(1) = m] \Pr[\gamma \geq \alpha \mid \zeta(1) = m] > \frac{1}{12 \sqrt{\alpha}}.
\]

Plugging this into (3.2), we have that the probability that \(\rho\) is a positive \(\alpha\)-long excursion is greater than \(\frac{1}{12(d+2)\sqrt{\alpha}}\). \(\square\)
The next proposition bounds from below the probability of sticking to the layer $M(t)$ at each excursion.

**Proposition 3.13.** For all $i \geq 1$,

$$
\Pr \left[ \kappa(t + 1) = \ell_i \mid \kappa(t) > \ell_{i-1} \right] \geq \frac{c}{n \sqrt{m(G)}},
$$

where $c = c(d) > 0$ is a constant that depends only on $d$.

**Proof.** It suffices to prove that for any $u, v \in G$

$$
\Pr \left[ g(\ell_i) = u \mid g(\ell_{i-1}) = v \right] \geq \frac{c}{n \sqrt{m(G)}},
$$

for some constant $c = c(d) > 0$, depending only on $d$.

Let $G'$ be the graph obtained from $G$ by adding a self loop at each vertex. By Lemma 3.9 we can assume that $(g(\cdot), \zeta(\cdot))$ is a walk on $G' \times \mathbb{N}$.

Let $\Gamma = \{\ell_{i-1} < r \leq \ell_i \mid \zeta(r) = \zeta(r-1)\}$, and let $\gamma = |\Gamma|$. Let $\{h_r \mid 0 \leq r \leq \gamma\}$ be the walk $\{g(\ell_{i-1} + r) \mid r \in \Gamma \cup \{0\}\}$. $h_0, h_1, \ldots, h_\gamma$ is the walk measured only when moving in the $G$-coordinate. Note that conditioned on $\Gamma$, the walk $h_0, h_1, \ldots, h_\gamma$ has the distribution of a lazy random walk on $G$. Thus by Lemma 3.10 we have that for any $u, v \in G$,

$$
\Pr \left[ h_\gamma = u \mid \gamma \geq m(G), h_0 = v \right] \geq \frac{1}{2n}.
$$

Note that if $\rho_i$ is a $m(G)$-long excursion then $\gamma \geq m(G)$. Thus, for any $u, v \in G$, using Proposition 3.12

$$
\Pr \left[ g(\ell_i) = u \mid g(\ell_{i-1}) = v \right] \\
\geq \Pr \left[ h_\gamma = u \mid h_0 = v, \gamma \geq m(G) \right] \Pr \left[ \gamma \geq m(G) \mid h_0 = v \right] \\
\geq \frac{1}{2n} \cdot \frac{c}{\sqrt{m(G)}}.
$$

$\square$
Back to the proof of Lemma 3.11: Note that the events \( \{ \kappa(t+1) = \ell_i \}_{i=0}^{\infty} \) are pairwise disjoint, and that for every \( i \geq 0 \), we have \( \{ \kappa(t+1) = \ell_i \} \subset \{ M(t+1) > M(t) \} \). Thus, using Proposition 3.13 we now have for any \( x \geq 1 \),

\[
\Pr[M(t+1) > M(t)] \geq \sum_{i=1}^{\infty} \Pr[\kappa(t+1) = \ell_i] \\
= \sum_{i=1}^{\infty} \Pr[\kappa(t+1) = \ell_i \mid \kappa(t+1) > \ell_{i-1}] \Pr[\kappa(t+1) > \ell_{i-1}] \\
\geq x \Pr[\kappa(t+1) > \ell_x] \cdot \frac{c}{n\sqrt{m(G)}}, \tag{3.5}
\]

for a constant \( c = c(d) > 0 \) depending only on \( d \).

Since for any \( C > 0 \),

\[
\Pr[\kappa(t+1) > \ell_x] \geq \Pr[\ell_x \leq Cx^2] - \Pr[\kappa(t+1) \leq Cx^2],
\]

we are left with proving that there exists \( C > 0 \) such that \( \Pr[\ell_x \leq Cx^2] \geq \frac{1}{2} \) for any \( x \geq 1 \). Note that \( \ell_x > Cx^2 \) implies that the number of times the walk \( \zeta(\cdot) \) visits the layer \( M(t) \) up to time \( \lceil Cx^2 \rceil \) is less than \( x \). Thus by Lemma A.5, there exists \( C = C(d) > 0 \) such that

\[
\Pr[\ell_x > Cx^2] \leq \frac{1}{2}.
\]

\( \square \)

**Corollary 3.14.** Let \( 2 \leq d \in \mathbb{N} \). There exist \( n_0 = n_0(d) \) such that the following holds for all \( n > n_0 \):

Let \( G \) be a \( d \)-regular graph of size \( n \), and mixing time

\[
m(G) \leq \frac{\log^2(n)}{(\log \log(n))^5}.
\]

Consider the \( G \)-Cylinder-DLA process. Let \( A_t \) be a \( G \)-Cylinder-DLA cluster at time \( t \). Set

\[
\mu = \mu(n) = \left\lfloor \frac{\log(n)}{4 \log \log(n)} \right\rfloor.
\]
Assume that $\Pr[\kappa(t + 1) \leq \mu^2/4] < \frac{1}{4}$. Then,

$$\Pr[M(t + 1) > M(t)] > \frac{\log \log(n)}{n}.$$ 

Proof. Let $C$ and $c$ be as in Lemma 3.11. We can choose $x \geq \frac{\log(n)}{(\log \log(n))^{3/2}}$ such that $Cx^2 \leq \mu^2/4$ and $\frac{cx}{\sqrt{m(G)}} \geq \log \log(n)$ for large enough $n$. Plugging this into Lemma 3.11 we get

$$\Pr[M(t + 1) > M(t)] \geq \frac{\log \log(n)}{n}.$$

$\square$

### 3.4 Proof of Theorem 3.1

For convenience, we restate the Theorem:

**Theorem 3.1.** Let $2 \leq d \in \mathbb{N}$. There exists $n_0 = n_0(d)$, such that the following holds for all $n > n_0$:

Let $G$ be a $d$-regular graph of size $n$, and mixing time

$$m(G) \leq \frac{\log^2(n)}{(\log \log(n))^5}.$$

Let $A_t$ be a $G$-Cylinder-DLA cluster at time $t$. Define

$$T = \min \{ s > t \mid M(s) > M(t) \}.$$ 

$T$ is the first time that a particle sticks to the empty layer, $M(t)$. Then,

$$\mathbb{E}[T - t] \leq \frac{4n}{\log \log n}.$$

Proof. Set $M = M(t)$ and

$$\mu = \mu(n) = \bigg\lfloor \frac{\log(n)}{4 \log \log(n)} \bigg\rfloor, \quad \text{and} \quad \nu = \nu(n) = \log(n).$$
For $s \geq t$, define

$$\alpha(s) = \Pr \left[ \kappa(s+1) \leq \mu^2/4 \mid A_s \right],$$

(which is random variable that is a function of $A_s$). Define

$$\tau = \min \left\{ s \geq t \mid \alpha(s) \geq \frac{1}{4} \right\}.$$

Fix $s > t$, and $t + 1 \leq r \leq s$. By Corollary 3.14 there exists $n_0 = n_0(d)$ such that for all $n > n_0$,

$$\Pr \left[ M(r) = M(t) , \alpha(r) < \frac{1}{4} \mid \forall t + 1 \leq q \leq r - 1 M(q) = M(t) , \alpha(q) < \frac{1}{4} \right]$$

$$\leq \Pr \left[ M(r) = M(t) \mid \forall t + 1 \leq q \leq r - 1 M(q) = M(t) , \alpha(q) < \frac{1}{4} \right]$$

$$\leq 1 - \frac{\log \log(n)}{n}.$$

Thus, for all $s > t$,

$$\Pr \left[ \min \{ T, \tau \} > s \right] = \Pr \left[ T > s , \tau > s \right]$$

$$\leq \Pr \left[ \forall t + 1 \leq r \leq s M(r) = M(t) , \alpha(r) < \frac{1}{4} \right]$$

$$= \prod_{r=t+1}^{s} \Pr \left[ M(r) = M(t) , \alpha(r) < \frac{1}{4} \mid \forall t + 1 \leq q \leq r - 1 M(q) = M(t) , \alpha(q) < \frac{1}{4} \right]$$

$$\leq \left( 1 - \frac{\log \log(n)}{n} \right)^{s-t}.$$

Since, $\Pr \left[ \min \{ T, \tau \} > t \right] \leq 1$, we get that

$$\mathbb{E} \left[ \min \{ T, \tau \} - t \right] \leq \frac{n}{\log \log(n)}.$$

Define

$$T' = \begin{cases} 
\min \{ s > 0 \mid M(\tau + s) > M(\tau) \} & \tau < \infty \\
0 & \tau = \infty.
\end{cases}$$
Then we have $T \leq \min\{T, \tau\} + T'$. If $\tau = \infty$ then $E[T' | \tau = \infty] = 0$. Assume that $\tau < \infty$. Then, at time $\tau$, we have that $\Pr[\kappa(\tau + 1) \leq \mu^2/4 | A_{\tau}] \geq \frac{1}{4}$. So, using Lemma 3.5

$$E[T' | \tau < \infty] \leq \frac{5n}{2\log \log(n)},$$

and consequently,

$$E[T'] < \frac{3n}{\log \log(n)}.$$

Thus, we conclude that

$$E[T - t] \leq E[\min\{T, \tau\} - t] + E[T'] < \frac{4n}{\log \log(n)}.$$

\qed

4 Density

4.1 Definitions and Notation

**Definition 4.1.** Fix a graph $G$, and let $\{A_t\}$ be a $G$-Cylinder-DLA process. Define the \textit{cluster at infinity} by

$$A_\infty = \bigcup_{t=0}^{\infty} A_t.$$

For $m \in \mathbb{N}$, define

$$D(m) = \frac{1}{mn} \sum_{i=1}^{m} |A_\infty \cap G_i|.$$

$D(m)$ is the fractional amount of particles in the finite cylinder $G \times \{1, \ldots, m\}$.

Define the \textit{density at infinity} by

$$D = D_\infty = \lim_{m \to \infty} D(m). \quad (4.1)$$
Using standard arguments from ergodic theory it can be shown that the limit in (4.1) exists, and is constant almost surely. Since \(D(m)\) are bounded random variables, we get by dominated convergence (see e.g. Chapter 9 in [4]):

\[
D = \mathbb{E}[D] = \lim_{m \to \infty} \mathbb{E}[D(m)].
\]

Recall the random times:

\[
T_m = \min \{ t \geq 0 \mid A_t \cap G_m \neq \emptyset \}.
\]

\(T_m\) is the time the cluster first reaches the layer \(m\).

**Theorem 4.2.** Let \(G\) be a \(d\)-regular graph of size \(n\), and let \(\{A_t\}\) be a \(G\)-Cylinder-DLA process. Let \(D = D_\infty\) be the density at infinity, and for all \(m\) let

\[
T_m = \min \{ t \geq 0 \mid A_t \cap G_m \neq \emptyset \}.
\]

Then,

\[
D = \lim_{m \to \infty} \frac{1}{mn} \mathbb{E}[T_m].
\]

Theorem [4.2] relates the density at infinity to the average growth rate. Theorem [4.2] is proved via the following propositions. The proof of the theorem is in Section 4.3.

### 4.2

The main objective of this section is Proposition [4.5]. This proposition is the main observation in proving Theorem [4.2].

First we require some notation: For a \(G\)-Cylinder-DLA process \(\{A_t\}\), recall \(L_t(i) = |A_t \cap G_i|\), the load of the \(i\)-th layer at time \(t\). Define the load of the \(i\)-th layer at infinity:

\[
L(i) = L_\infty(i) \overset{\text{def}}{=} |A_\infty \cap G_i|.
\]
Define:

\[ L_t(\leq i) \overset{\text{def}}{=} \sum_{j=1}^{i} L_t(j). \]

\[ L(\leq i) = L_\infty(\leq i) \overset{\text{def}}{=} \sum_{j=1}^{i} L(j). \]

For \( 0 \leq t \leq \infty \), \( L_t(\leq i) \) is the total load of all layers below \( i \), including \( i \) but not including the 0-layer. (When indices become too small we write \( L(t, \leq i) \) instead of \( L_t(\leq i) \).)

Also define \( H(t) = \zeta_t(\kappa(t)) \). That is, \( H(t) \) is the layer at which the \( t \)-th particle sticks (the height of the \( t \)-th particle).

The following proposition bounds the probability that a particle sticks to a “low” layer.

**Proposition 4.3.** Fix \( m < m' \in \mathbb{N} \). Let \( G \) be a \( d \)-regular graph of size \( n \), with spectral gap \( 1 - \lambda \) (i.e., \( \lambda \) is the second eigenvalue of the transition matrix of \( G \)). Consider the \( G \)-Cylinder-DLA process. Let \( t > T_m' \) and let \( A_{t-1} \) be the \( G \)-Cylinder-DLA cluster at time \( t - 1 \). Then,

\[ \Pr[H(t) \leq m \mid A_{t-1}] < 3 \exp\left(-\frac{1 - \lambda}{8n}(m' - m)\right). \]

**Proof.** Let \( t > T_m' \). Let \( M = M(t-1) \) and \( \varphi = m' - m \). Note that

\[ M - 1 - m \geq M(T_{m'}) - 1 - m = m' - m = \varphi. \]

Let \( (g(\cdot), \zeta(\cdot)) = (g_t(\cdot), \zeta_t(\cdot)) \) be the walk of the \( t \)-th particle. So \( \zeta(0) = M \) and \( g(0) \) is uniformly distributed in \( G \). Let \( k \) be the first step at which the walk is at the layer \( m \). That is, \( k = \min \{ r > 0 \mid \zeta(r) = m \} \). Let \( \kappa = \kappa(t) \) be the step at which the particle sticks to the cluster.

Note that the event \( \{ H(t) \leq m \} \) implies the event \( \{ \kappa \geq k \} \). Moreover, \( \{ \kappa \geq k \} \) implies the event

\[ \{ g(k - i) \not\in A_{t-1} \cap G_{\zeta(k-i)} : i = 1, 2, \ldots, \varphi \}. \]
Also, for all $1 \leq i \leq \varphi$ we have that $|A_{t-1} \cap G_{\zeta(k-i)}| \geq 1$ (because $\zeta(k-i) \leq m+i \leq M-1$).

Define

$$\Gamma = \{k - \varphi \leq r < k \mid \zeta(r) = \zeta(r-1)\},$$

and assume that

$$\Gamma = \{r_1 < r_2 < \cdots < r_s\}$$

(note that $s = |\Gamma|$ is a random variable). For $1 \leq i \leq s$ let $g_i = g(r_i)$. So $g_1, g_2, \ldots, g_s$ is distributed as an $s$-step simple random walk on $G$, starting from a uniformly chosen vertex.

For all $1 \leq i \leq s$ define $C_i = A_{t-1} \cap G_{\zeta(r_i)}$. Thus, the event $\{H(t) \leq m\}$ implies the event $\{g_i \notin C_i, i = 1, 2, \ldots, s\}$.

By Lemma B.4 we have that

$$\Pr\left[\left. g_i \notin C_i, i = 1, 2, \ldots, s \right| C_1, C_2, \ldots, C_s, s \geq \frac{1}{4}\varphi \right] \leq \exp\left(\frac{-1 - \lambda}{2n} \sum_{i=1}^{s} |C_i|\right) \leq \exp\left(\frac{-1 - \lambda}{8n}\varphi\right).$$

Hence,

$$\Pr\left[\left. H(t) \leq m \right| A_{t-1}\right] \leq \Pr\left[\left. g_i \notin C_i, i = 1, 2, \ldots, s \right| C_1, C_2, \ldots, C_s, s \geq \frac{1}{4}\varphi \right] \leq \Pr\left[ s < \frac{1}{4}\varphi \right] + \exp\left(\frac{-1 - \lambda}{8n}\varphi\right).$$

Note that

$$s = \sum_{i=1}^{\varphi} 1_{\{\zeta(k-i) = \zeta(k-i-1)\}}.$$ 

That is, $s$ is the sum of independent identically distributed Bernoulli random variables, with mean $\frac{d}{d+2} \geq \frac{1}{2}$. Thus, using the Chernoff bound (see e.g. Appendix A in [1]),

$$\Pr\left[ s < \frac{1}{4}\varphi \right] < 2 \exp\left(\frac{-\varphi}{8}\right).$$
Thus,

$$\Pr[H(t) \leq m \mid A_{t-1}] < 3 \exp \left(-\frac{1 - \lambda}{8n} \varphi \right).$$

Consider the following event in the $G$-Cylinder-DLA process: Given a cluster $A_t$, the next $|G|$ particles appear in exactly the right order so that they completely fill up the layer $M(t)$. (There is always such an order; e.g., consider a spanning tree of $G$ rooted at a vertex in $\partial A_t$.) Thus, an impassible “wall” is created. Specifically, we are interested in the event that $L_{t+n}(M(t)) = n$. The following proposition bounds from below the probability of this event.

**Proposition 4.4.** Let $G$ be a $d$-regular graph of size $n$. Consider the $G$-Cylinder-DLA process. Let $A_t$ be the $G$-Cylinder-DLA cluster at time $t$. Then,

$$\Pr[L_{t+n}(M(t)) = n \mid A_t] \geq (d + 2)^{-(n-1)} n^{-n}.$$  

**Proof.** Consider the following event $W$: The $(t+1)$-th particle appears at a vertex in $G_{M(t)}$ that is in $\partial A_t$. Since there is at least one such vertex, this happens with probability at least $1/n$. For $i = 2, \ldots, n$, the $(t+i)$-th particle appears at the layer $M(t) + 1$, and moves to a vertex in $G_{M(t)}$ that is in $\partial A_{t+i-1}$. Since there is at least one such vertex, the probability of this is at least $n^{-1}(d+2)^{-1}$, for each $i = 2, \ldots, n$.

Since the event $W$ implies that $L_{t+n}(M(t)) = n$, we have

$$\Pr[L_{t+n}(M(t)) = n \mid A_t] \geq (d + 2)^{-(n-1)} n^{-n}.$$  

\hfill \Box

**Proposition 4.5.** Fix $m \in \mathbb{N}$. Let $\varphi = \varphi(m)$ be a positive integer, and let $m' = m + \varphi$. Let $G$ be a $d$-regular graph of size $n$, with spectral gap $1 - \lambda$. Consider the $G$-Cylinder-DLA process. Let $X(m)$ be the event that there exists $t > T_{m'}$ such that $H(t) \leq m$. That
is, $X(m)$ is the event that a particle sticks to a layer $\leq m$ after the cluster has reached the layer $m'$. Then,

$$\Pr[X(m)] \leq (d + 2)^{n-1}(n + 1)n^n \cdot 3 \exp\left(-\frac{1 - \lambda}{8n} \varphi\right).$$

Proof. Fix $m \in \mathbb{N}$ and let $m' = m + \varphi(m)$. Let $t > T_{m'}$. For $i \in \mathbb{N}$ define the events

$$W(t + i) = \{L_{t+i+n}(M(t + i)) = n\}, \quad \text{and} \quad B(t + i) = \{H(t + i) \leq m\}.$$

Set

$$F(t + i) = \overline{B(t + i)} \cap \overline{W(t + i)}.$$

By Proposition 4.3 we have that for all $i \geq n + 1$,

$$\Pr[B(t + i) \mid \forall 0 \leq j \leq i - (n + 1) \ F(t + j)] \leq 3 \exp\left(-\frac{1 - \lambda}{8n} \varphi(m)\right).$$

By Proposition 4.4 we have that for all $i \geq n + 1$,

$$\Pr[F(t + i) \mid \forall 0 \leq j \leq i - (n + 1) \ F(t + j)] \leq \Pr[\overline{W(t + i)} \mid \forall 0 \leq j \leq i - (n + 1) \ F(t + j)] \leq 1 - (d + 2)^{-(n-1)n^n}.$$

Thus, for all $i \geq n + 1$,

$$\Pr[B(t + i), F(t + i - 1), \ldots, F(t)] \leq \Pr[B(t + i) \mid \forall 0 \leq j \leq i - (n + 1) \ F(t + j)] \prod_{\ell=1}^{[i/(n+1)]} \Pr[F(t + i - \ell(n+1)) \mid \forall 0 \leq j \leq i - (\ell + 1)(n+1) \ F(t + j)] \leq p(m)(1 - q)^{[i/(n+1)]},$$

where

$$p(m) = 3 \exp\left(-\frac{1 - \lambda}{8n} \varphi(m)\right) \quad \text{and} \quad q = (d + 2)^{-(n-1)n^n}.$$
Note that for all $t > T_m'$, the event $W(t)$ implies that $B(t + i)$ for all $i \geq 0$ (since the first $n$ particles must stick to the layer $M(t) > m$, and after time $t + n$ no particle can pass the layer $M(t) > m$). Thus, setting $t = T_m' + 1$, the event $X(m)$ implies that there exists $i \geq 0$ such that

$$B(t + i) \cap \bigcap_{j=0}^{i-1} F(t + j)$$

occurs (i.e. take the first $i$ for which $B(t + i)$ occurs). So,

$$\Pr [X(m)] \leq \sum_{i=0}^{\infty} \Pr [B(t + i), F(t + i - 1), \ldots, F(t)]$$

$$\leq \sum_{i=0}^{\infty} p(m)(1 - q)^{\lfloor i/(n+1) \rfloor}$$

$$= \sum_{\ell=0}^{\infty} (n + 1)p(m)(1 - q)^{\ell} = (n + 1)p(m)\frac{1}{q}.$$  

$\square$

4.3 Proof of Theorem 4.2

We restate the theorem:

**Theorem (4.2).** Let $G$ be a $d$-regular graph of size $n$, and let $\{A_t\}$ be a $G$-Cylinder-DLA process. Let $D = D_\infty$ be the density at infinity, and for all $m$ let

$$T_m = \min \{ t \geq 0 \mid A_t \cap G_m \neq \emptyset \}.$$  

Then,

$$D = \lim_{m \to \infty} \frac{1}{mn} \mathbb{E} [T_m].$$

**Proof.** Let $\varphi : \mathbb{N} \to \mathbb{N}$ be any function such that

$$\lim_{m \to \infty} \varphi(m) = \infty \quad \text{and} \quad \lim_{m \to \infty} \frac{\varphi(m)}{m} = 0.$$  

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For $m \in \mathbb{N}$ let $m' = m + \varphi(m)$.

Recall that $H(t) = \zeta_t(\kappa(t))$ is the height of the layer at which the $t$-th particle sticks.

For $m \in \mathbb{N}$ let $X(m)$ be the event that there exists $t > T_{m'}$ such that $H(t) \leq m$. Then, for all $\ell$, we have that

$$
\{L_\infty(\leq m) > \ell\} \text{ implies } \{L(T_{m'}, \leq m) > \ell\} \cup X(m).
$$

This is because if $L(T_{m'}, \leq m) \leq \ell$, then at least one more particle is needed to stick at a layer $\leq m$ after time $T_{m'}$, in order for $L_\infty(\leq m) > \ell$ to hold.

Thus, since $L_\infty(\leq m) \leq mn$, using Proposition 4.5

$$
\mathbb{E}[L_\infty(\leq m)] = \sum_{\ell=0}^{\infty} \Pr[L_\infty(\leq m) > \ell] = \sum_{\ell=0}^{mn-1} \Pr[L_\infty(\leq m) > \ell] \leq \sum_{\ell=0}^{\infty} \Pr[L(T_{m'}, \leq m) > \ell] + \sum_{\ell=0}^{mn-1} \Pr[X(m)] = \mathbb{E}[L(T_{m'}, \leq m)] + mn \cdot p(m) \frac{n+1}{q},
$$

for

$$
p(m) = 3 \exp\left(-\frac{1 - \lambda}{8n} \varphi(m)\right) \quad \text{and} \quad q = (d + 2)^{-(n-1)n^{-n}}.
$$

Note that $L_t(\leq m) \leq t$ for all $t$, so

$$
\mathbb{E}[L(T_{m'}, \leq m)] \leq \mathbb{E}[T_{m'}].
$$

Also,

$$
\lim_{m \to \infty} p(m) \frac{n+1}{q} = 0.
$$
So,

\[
E[D] = \lim_{m \to \infty} E[D(m)] = \lim_{m \to \infty} \frac{1}{mn} E[L_\infty(\leq m)] \\
\leq \lim_{m \to \infty} \frac{1}{mn} E[T_{m'}] + \lim_{m \to \infty} p(m) \frac{n+1}{q} \\
= \lim_{m \to \infty} \frac{1}{mn} E[T_{m'}].
\]

Since for all \(k' > k\), \(E[T_{k'} - T_k] \leq n(k' - k)\), we have that

\[
E[T_{m'}] = E[T_m] + E[T_{m'} - T_m] \leq E[T_m] + \varphi(m)n.
\]

Thus,

\[
D = E[D] \leq \lim_{m \to \infty} \frac{1}{mn} E[T_{m}] + \lim_{m \to \infty} \frac{\varphi(m)}{m} = \lim_{m \to \infty} \frac{1}{mn} E[T_{m}]. \tag{4.2}
\]

Note that for all \(m\),

\[
T_m = L(T_m, \leq m) \leq L_\infty(\leq m),
\]

so \(E[T_m] \leq E[L_\infty(\leq m)]\). Thus,

\[
\lim_{m \to \infty} \frac{1}{mn} E[T_m] \leq \lim_{m \to \infty} \frac{1}{mn} E[L_\infty(\leq m)] = E[D] = D. \tag{4.3}
\]

\((4.2)\) and \((4.3)\) together give equality. \(\square\)

### 4.4 Density of Cylinder-DLA with transitive base

In this section we assume that \(G\) is vertex transitive; i.e. for any \(u, v \in G\) there exists an automorphism (of graphs) \(\varphi_{uv} : G \to G\) such that \(\varphi(u) = v\).

**Theorem 4.6.** Let \(G\) be a vertex transitive graph. Let \(\{A_t\}\) be the \(G\)-Cylinder-DLA process. Let \(D = D_\infty\) be the density at infinity. Then,

\[D \leq \frac{2}{3}.\]
The key to proving Theorem 4.6 is Lemma 4.7 below. The proof of the Theorem follows the proof of the Lemma.

**Lemma 4.7.** Let $G$ be a vertex transitive graph. Let $A_{t-1}$ be a $G$-Cylinder-DLA cluster at time $t - 1$. Then,

$$\Pr [M(t) > M(t - 1) \mid A_{t-1}] \geq \frac{2d + 2}{(d + 2)n}.$$  

**Proof.** Recall $(g_t(\cdot), \zeta_t(\cdot))$ is the walk of the $t$-th particle, so $g_t(0)$ is uniformly distributed in $G$, and $\zeta_t(0) = M(t - 1)$.

Define $\Xi(t - 1)$ to be the newest particle in the top layer of the cluster $A_{t-1}$. That is, if $A_{t-1} \cap G(M(t-1) - 1) = \{v_1, \ldots, v_{\ell}\}$, then $\Xi(t - 1)$ is the vertex $v_i$ that is the last vertex to join the cluster.

Note that $(\Xi(t - 1), M(t - 1)) \in \partial A_{t-1}$.

Because the graph $G$ is vertex transitive, we get that $\Xi(t - 1)$ is uniformly distributed in $G$. Moreover, $\Xi(t - 1)$ depends only on the clusters $A_{t-1}, \ldots, A_0$, and is independent of the walk $(g_t(\cdot), \zeta_t(\cdot))$.

Define $S(t)$ to be the set of vertices in $G$ that the walk $(g_t(\cdot), \zeta_t(\cdot))$ visits before leaving the layer $M(t - 1)$. That is:

$$\tau = \min \{r > 0 \mid \zeta_t(r) \neq M(t - 1)\},$$

$\tau$ is the first step the $t$-th particle is not in the layer $M(t - 1)$.

$$S(t) = \{v \in G \mid \exists 0 \leq r \leq \tau - 1 : g_t(r) = v\}.$$  

**Claim.** For all $t > 1$,

$$\Pr [\Xi(t - 1) \in S(t)] = \frac{1}{n} \mathbb{E} [|S(t)|].$$
Proof. For any \( u \in G \),
\[
Pr \left[ \Xi(t-1) = u \mid S(t) \right] = \frac{1}{n}.
\]
Consequently,
\[
Pr \left[ \Xi(t-1) \in S(t) \right] = \sum_{s} \sum_{v \in S} Pr \left[ \Xi(t-1) = v \mid S(t) = S \right] \Pr \left[ S(t) = S \right]
\]
\[
= \frac{1}{n} \sum_{s} |S| \Pr \left[ S(t) = S \right] = \frac{1}{n} \mathbb{E} \left[ |S(t)| \right].
\]
\[
\Box
\]
Recall that \( d \) is the degree of \( G \).

Claim. For all \( t > 0 \),
\[
\mathbb{E} \left[ |S(t)| \right] \geq \frac{2d + 2}{d + 2}.
\]

Proof. Let \( R(k) \) denote the range of a \( k \)-step random walk on \( G \). Then, for \( s \geq 2 \),
\[
Pr \left[ |S(t)| = s \right] = \sum_{k=s-1}^{\infty} \Pr \left[ R(k) = s \right] \left( \frac{d}{d+2} \right)^{k} \cdot \frac{2}{d+2}.
\]
For \( s = 1 \),
\[
Pr \left[ |S(t)| = 1 \right] = \frac{2}{d+2}.
\]
Thus,
\[
\mathbb{E} \left[ |S(t)| \right] = \frac{2}{d+2} + \sum_{k \geq 2} \sum_{s \geq 1} \frac{2}{d+2} \cdot \sum_{k=1}^{k+1} s \Pr \left[ R(k) = s \right] \left( \frac{d}{d+2} \right)^{k}
\]
\[
= \frac{2}{d+2} \cdot \sum_{k \geq 0} \mathbb{E} \left[ R(k) \right] \left( \frac{d}{d+2} \right)^{k}
\]
\[
= \frac{2}{d+2} \cdot \mathbb{E} \left[ R(0) \right] \left( \frac{d}{d+2} \right)^{k}
\]
\[
= \frac{2}{d+2} \cdot \mathbb{E} \left[ R(k) \right] \left( \frac{d}{d+2} \right)^{k} \quad (4.4)
\]
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Substitute in (4.4) the naive bound $R(k) \geq 2$ for all $k \geq 1$:

$$
E[|S(t)|] \geq \frac{2}{d+2} \cdot \left[1 + 2 \sum_{k=1}^{\infty} \left(\frac{d}{d+2}\right)^k \right] = \frac{2}{d+2} \cdot \left[1 + 2 \frac{d}{d+2} \frac{d+2}{2} \right] = \frac{2d+2}{d+2}.
$$

Thus, using the claim, we have that for all $t > 1$,

$$
\Pr[\Xi(t-1) \in S(t)] \geq \frac{2d+2}{(d+2)n}.
$$

The lemma now follows from the fact that the event $\{\Xi(t-1) \in S(t)\}$ implies $\{S(t) \cap \partial A_{t-1} \neq \emptyset\}$. So $\{\Xi(t-1) \in S(t)\}$ implies that the $t$-th particle sticks to the layer $M(t-1)$.  

\textbf{Proof of Theorem 4.6} For $m \in \mathbb{N}$ recall

$$
T_m = \min \left\{t \geq 0 \mid A_t \cap G_m \neq \emptyset \right\}.
$$

By Lemma 4.7 for all $m$,

$$
E[T_m] \leq m \frac{(d+2)n}{2d+2}.
$$

Thus,

$$
\frac{1}{mn} E[T_m] \leq \frac{d+2}{2d+2}.
$$

Plugging this into Theorem 4.2 we have

$$
D \leq \frac{d+2}{2d+2} \leq \frac{2}{3}.
$$
4.5 Density of Cylinder-DLA with quickly mixing base

In this section we combine two main results: For a family of graphs \( \{G_n\} \) with small mixing time, we show that since the \( G \)-Cylinder-DLA process grows arms, the densities at infinity tend to 0 as \( n \) tends to infinity. Formally:

**Theorem 4.8.** Let \( 2 \leq d \in \mathbb{N} \). Let \( \{G_n\} \) be a family of \( d \)-regular graphs such that \( \lim_{n \to \infty} |G_n| = \infty \), and for all \( n \),

\[
m(G_n) \leq \frac{\log^2 |G_n|}{(\log \log |G_n|)^5}.
\]

For all \( n \) let \( D(n) \) be the density at infinity of the \( G_n \)-Cylinder-DLA process. Then,

\[
\lim_{n \to \infty} D(n) = 0.
\]

**Proof.** There exists \( n_0 = n_0(d) \) such that the following holds for all \( n > n_0 \):

Set \( G = G_n \) and consider \( \{A_t\} \), a \( G \)-Cylinder-DLA process. By Theorem 2.1, for all \( m \),

\[
\mathbb{E}[T_m] < mn \frac{4}{\log \log n}.
\]

Thus, using Theorem 4.2

\[
D(n) \leq \frac{4}{\log \log n},
\]

for all \( n > n_0 \). Thus,

\[
\lim_{n \to \infty} D(n) = 0.
\]

\( \Box \)

5 Lower bound on the growth rate

In this section we prove a lower bound on the expected growth rate of the \( G \)-Cylinder-DLA cluster, provided that the spectral gap is at least \( |G|^{-2/3} \). This regime of the spectral gap includes graphs with small mixing time as in Theorem 2.1 and many more natural families of graphs such as discrete cubes and tori of dimension at least 3.
Theorem 5.1. Let $2 \leq d \in \mathbb{N}$. There exists $n_0 = n_0(d)$, such that the following holds for all $n > n_0$:

Let $G$ be a $d$-regular graph such that

$$|G| = n \quad \text{and} \quad 1 - \lambda \geq n^{-2/3},$$

where $1 - \lambda$ is the spectral gap of $G$. Consider $\{A_t\}$, a $G$-Cylinder-DLA process. For $m \in \mathbb{N}$, define

$$T_m = \min \{t \geq 0 \mid A_t \cap G_m \neq \emptyset\}.$$ 

$T_m$ is the time the cluster first reaches the layer $m$.

Then, for all $m$,

$$\mathbb{E}[T_m] > Cmn^{1/20},$$

where $C$ is some constant that depends only on $d$.

Proof. Fix $t > 0$, and let $A_{t-1}$ be the $G$-Cylinder-DLA cluster at time $t - 1$.

Claim. There exists a constant $C = C(d)$ (that depends on $d$) such that for all $t > 0$,

$$\Pr \left[ M(t) > M(t-1) \mid A_{t-1} \right] < \frac{C |\partial A_{t-1} \cap G_{M(t-1)}|}{n^{1/10}}.$$ 

Proof. Let $(g(\cdot), \zeta(\cdot)) = (g_t(\cdot), \zeta_t(\cdot))$ be the walk of the $t$-th particle. Set

$$L = \{ r > 0 \mid \zeta(r) = \zeta(0) \} = \{ \ell_1 < \ell_2 < \cdots \},$$

and let $\rho_i = \ell_{i-1} \to \ell_i$ be the excursions of the walk. For $2 \leq \alpha \in \mathbb{R}$, let $p(\alpha)$ be the probability that an excursion is a negative $\alpha$-long excursion; that is $p(\alpha)$ is the probability that

$$\zeta(\ell_{i-1} + 1) = \zeta(0) - 1 \quad \text{and} \quad \sum_{r=\ell_{i-1}+1}^{\ell_i} 1_{\{\zeta(r) = \zeta(r-1)\}} \geq \alpha.$$

(This is independent of $i$.) By symmetry and Proposition 3.12, we have that $p(\alpha) > (1/c(d))\alpha^{-1/2}$, where $c(d) = 12(d + 2)$.
Fix $2 \leq \alpha \in \mathbb{R}$, and set $p = p(\alpha)$.

For an integer $k \in \mathbb{N}$, let $N(k)$ denote the number of negative $\alpha$-long excursions out of the first $k$ excursions. So $N(k) = \sum_{i=1}^{k} I_i(\alpha)$, where $I_i(\alpha)$ is the indicator of the event that $\rho_i$ is a negative $\alpha$-long excursion. Since $\{I_i(\alpha)\}$ are independent, we have by Chebychev's inequality that
\[
\Pr \left[ N(k) \leq \frac{1}{2} pk \right] \leq \frac{4}{pk} < 4c(d) \frac{\sqrt{\alpha}}{k}.
\]

Let $Z = Z(k)$ be the number of times up to $\ell_k$ the walk moves in $G$ while on the negative side of $\zeta(0)$; i.e.,
\[
Z(k) = \sum_{r=1}^{\ell_k} 1_{\{\zeta(r) = \zeta(r-1)\}} 1_{\{\zeta(r) < \zeta(0)\}}.
\]

We have that $Z \geq \alpha \cdot N(k)$ (since each negative $\alpha$-long excursion contributes at least $\alpha$ to the sum). Thus,
\[
\Pr \left[ Z \leq \frac{\sqrt{\alpha k}}{2c(d)} \right] \leq \Pr \left[ Z \leq \frac{\alpha}{2} pk \right] < 4c(d) \frac{\sqrt{\alpha}}{k}.
\]

Set $A = \partial A_{t-1} \cap G_{M(t-1)}$. Set $B = \partial A_{t-1} \setminus A$. That is, $B$ is the set $\partial A_{t-1}$ with the highest layer removed.

For all $r \geq 0$ let $C_r = B \cap G_{\zeta(r)}$. Note that if $\zeta(r) < \zeta(0)$ then $|C_r| \geq 1$ (because any layer below $M(t-1)$ contains at least one particle). Define a simple random walk on $G$ by $h_0 = g(0)$ and
\[
\{h_1, h_2, \ldots, h_s\} = \{g(r) \mid \zeta(r) = \zeta(r-1), 1 \leq r \leq \ell_k\}.
\]

Let $F = F(\ell_k)$ be the event that the particle does not hit the set $B$ up to time $\ell_k$. That is,
\[
F = \{\forall 0 \leq r \leq \ell_k \ (g(r), \zeta(r)) \not\in B\}.
\]
Conditioned on a specific path \( \zeta(0), \zeta(1), \ldots, \zeta(\ell_k) \), and on \( A_{t-1} \), we have that \( h_0, h_1, \ldots, h_s \) is distributed as a simple random walk on \( G \). Using Lemma B.4 that

\[
\Pr \left[ F \mid \zeta(0), \ldots, \zeta(\ell_k), A_{t-1} \right] \leq \Pr \left[ \forall \ 0 \leq r \leq s \ h_r \notin C_r \mid \zeta(0), \ldots, \zeta(\ell_k) \right] \leq \exp \left( - \frac{1 - \lambda}{2n} \sum_{r=1}^{s} C_r \right) \leq \exp \left( - \frac{1 - \lambda}{2n} Z \right).
\]

Thus, averaging over all possible paths \( \zeta(0), \zeta(1), \ldots, \zeta(\ell_k) \), we have that

\[
\Pr [F \mid A_{t-1}] < \Pr [Z \leq \frac{\alpha}{2} pk] + \exp \left( - \frac{1 - \lambda}{4n} \alpha pk \right) < 4c(d) \sqrt{\frac{\alpha}{k}} + \exp \left( - \frac{1 - \lambda}{4c(d)n} \sqrt{\alpha k} \right).
\]

Note that the event \( \{\kappa(t) > \ell_k\} \) implies the event \( F \), so we have that

\[
\Pr [\kappa(t) > \ell_k \mid A_{t-1}] < 4c(d) \sqrt{\frac{\alpha}{k}} + \exp \left( - \frac{1 - \lambda}{4c(d)n} \sqrt{\alpha k} \right).
\]

On the other hand, consider the times \( \ell_0, \ell_1, \ldots, \ell_k \). Since \( \partial A_{t-1} \cap G_{\zeta(0)} = \partial A_{t-1} \cap G_{M(t-1)} = A \), we have by a union bound,

\[
\Pr \left[ \exists \ x \in A, \exists \ 0 \leq i \leq k : (g(\ell_i), \zeta(\ell_i)) = x \mid A_{t-1} \right] \leq \frac{|A| (k+1)}{n}.
\]

Now, the event \( \{M(t) > M(t-1)\} \) implies that there exists \( i \geq 0 \) such that the particle does not stick to \( \partial A_{t-1} \) before time \( \ell_i \), and \( (g(\ell_i), \zeta(\ell_i)) = x \) for some \( x \in A \). Thus, we have for all \( 2 \leq \alpha \in \mathbb{R} \) and all \( k \in \mathbb{N} \),

\[
\Pr [M(t) > M(t-1) \mid A_{t-1}] \leq \Pr [\kappa(t) > \ell_k \mid A_{t-1}] + \Pr \left[ \exists \ x \in A, \exists \ 0 \leq i \leq k : (g(\ell_i), \zeta(\ell_i)) = x \mid A_{t-1} \right] < 4c(d) \sqrt{\frac{\alpha}{k}} + \exp \left( - \frac{1 - \lambda}{4c(d)n} \sqrt{\alpha k} \right) + \frac{|A| (k+1)}{n}.
\]
Set $\varepsilon = 1/10$, $k = n^{1-\varepsilon}$, $\alpha = n^{2-4\varepsilon}$. Then, if $1 - \lambda \geq \frac{1}{n^{\varepsilon}}$, we have that for large enough $n$ (depending on $d$),

\[
\Pr \left[ M(t) > M(t-1) \mid A_{t-1} \right] < \frac{C |A|}{n^{\varepsilon}},
\]

for some constant $C = C(d)$.  

Back to the proof of Theorem 5.1: Fix $m > 0$, and consider the time $T_m$. Note that for all $1 \leq j \leq n$,

\[
| \partial A_{T_m+j-1} \cap G_{M(T_m+j-1)} | \leq j,
\]

(because at most $j$ particles could have stuck to the layer $M(T_m+j-1) - 1$ by time $T_m+j-1$). Thus, for all $1 \leq j \leq n$ we have that for $C$ and $\varepsilon$ as above

\[
\Pr \left[ M(T_m+j) = M(T_m+j-1) \mid A_{T_m+j-1} \right] > 1 - \frac{Cj}{n^{\varepsilon}}.
\]

This implies that for $\lambda < \frac{\varepsilon}{C}$,

\[
\Pr \left[ T_{m+1} - T_m > \lambda \right] > \prod_{j=1}^{\lambda} \left( 1 - \frac{Cj}{n^{\varepsilon}} \right) \geq \left( 1 - \frac{C\lambda}{n^{\varepsilon}} \right)^{\lambda},
\]

and so, there exists a constant $C'$ (depending on $C$) such that for $\lambda = \lceil n^{\varepsilon/2} \rceil$,

\[
\mathbb{E} \left[ T_{m+1} - T_m \right] > \lambda \Pr \left[ T_{m+1} - T_m > \lambda \right] > C' n^{\varepsilon/2}.
\]

Hence, we get that for all $m \geq 2$,

\[
\mathbb{E} [T_m] > \frac{C'}{2} mn^{\varepsilon/2}.
\]

For completeness, we state the immediate Corollary of Theorems 5.1 and 4.2.
Corollary 5.2. Let $2 \leq d \in \mathbb{N}$. There exists $n_0 = n_0(d)$, such that the following holds for all $n > n_0$:

Let $G$ be a $d$-regular graph such that

$$|G| = n \quad \text{and} \quad 1 - \lambda \geq n^{-2/3},$$

where $1 - \lambda$ is the spectral gap of $G$. Consider $\{A_t\}$, a $G$-Cylinder-DLA process. Let $D_\infty$ be the density at infinity. Then, for some constant $C$ that depends only on $d$,

$$D_\infty \geq \frac{C}{n^{19/20}}.$$

6 Further research directions

The results and methods in this paper raise a few natural questions:

1. Let $G$ be a $d$-regular graph. Let $H$ be obtained from $G$ by only adding edges to $G$, so that $V(H) = V(G)$ and $H$ is $(d + 1)$-regular. Is there monotonicity in the expected speed of the cluster on the Cylinder-DLA processes with base $G$ and with base $H$. That is, let $T^G_m$, respectively $T^H_m$, be the first time the cluster reaches the layer $m$ in the $G$-Cylinder-DLA, respectively $H$-Cylinder-DLA, process. Is it true that $\mathbb{E} [T^G_m] \geq \mathbb{E} [T^H_m]$ for all $m$?

2. Consider a $G$-Cylinder-DLA process, started with $A_0 = \{x_0\}$ for a specific vertex $x_0 \in G$. Let $\tau$ be the mixing time of a simple random walk on $G$ (i.e. the time it takes for a simple random walk to come close in total-variation distance to the stationary distribution). For $m > 0$, let $x_m \in G$ be the vertex in $G$ that is the first vertex in the layer $m$ that a particle sticks to. In our notation above $x_m = v$ such that $A_{T_m} \cap G_m = \{(v, m)\}$. How long does it take for the distribution of $x_m$ to be close to the uniform distribution? Does there exist a constant $c$ such that $x_{c\tau}$ is close to being uniformly distributed on $G$?
3. Directed G-Cylinder-DLA: Consider a model of G-Cylinder-DLA where particles cannot move to layers above, only to layers below or in their current layer. Is the density of directed G-Cylinder-DLA always greater than undirected? Are there graphs G for which these quantities are of the same order? Are there graphs for which the ratio between the density of undirected G-Cylinder-DLA and directed G-Cylinder-DLA goes to 0 as the size of G goes to infinity?

The model of directed G-Cylinder-DLA can also be generalized to a model where particles move up with probability \( \frac{\alpha}{d+2} \) and down with probability \( \frac{2-\alpha}{d+2} \) (and to a neighbor in the current layer with probability \( \frac{1}{d+2} \)), for some \( \alpha < 1 \). Thus, there is a drift down. The same questions can be asked of this model.

We remark that some of our results still hold in directed G-Cylinder-DLA. Mainly, Lemma 3.5 (that states that if the particle takes a small amount of steps to stick, then the expected time to reach the new layer is small,) still holds with the assumption that \( \Pr [\kappa(t+1) \leq \mu] \geq \frac{1}{4} \).

4. The G-Cylinder-DLA process, is of course not a stationary process (since \( A_t \subset A_{t+1} \) for all t). But, each time a “wall” is built (i.e. \( L_{t+n}(M(t)) = n \), see Proposition 4.4), we start the cluster again, independently of the cluster below the wall. If we identify clusters that are the same above walls, we get a stationary Markov chain on clusters. Our analysis throughout this paper in some sense evades this stationary distribution. It would be interesting if some properties of the cluster generated under the stationary distribution could be worked out. Perhaps, calculating properties of the “typical cluster” could help improve the results of this paper (e.g., reduce the spectral gap required to grow arms).

5. As stated in the introduction, DLA on a cylinder suggests studying the problem of “clogging”. That is, run a G-Cylinder-DLA process for some graph G. Let T be the (random) time at which the cluster clogs the cylinder. That is, T is the first time at which there exists a layer such that no particle can pass this layer;
i.e.,

\[ T = \min \{ t > 0 \mid \exists m > 1 : \Pr[H(t) \leq m] = 0 \} . \]

Provide bounds on \( E[T] \). How is \( T \) distributed?

## A Random Walks on \( \mathbb{Z} \)

We collect some facts about a simple random walk on \( \mathbb{Z} \), \( S(n) \), starting at \( S(0) = 0 \).

The following is Theorem 9.1 of [11]:

**Lemma A.1.** Let

\[ \rho(1) = \min \{ i \geq 0 \mid S(i) = 1 \} . \]

Then, for all \( n \),

\[ \Pr[\rho(1) > 2n] = 2^{-2n} \binom{2n}{n} . \]

**Corollary A.2.** For all \( n \),

\[ \Pr[\forall 1 \leq i \leq 2n, \ S(i) \geq 0] = 2^{-2n} \binom{2n}{n} . \]

**Proof.** Let

\[ \tau = \min \{ i \geq 0 \mid S(i) = -1 \} . \]

By symmetry, \( \tau \) has the same distribution as \( \rho(1) \) above. Thus, for all \( n \),

\[ \Pr[\forall 1 \leq i \leq 2n, \ S(i) \geq 0] = \Pr[\tau > 2n] = \Pr[\rho(1) > 2n] = 2^{-2n} \binom{2n}{n} . \]

\qed

The following is Theorem 9.3 of [11]:
Lemma A.3. Let $L(n)$ be the number of times the walk has visited 0, i.e.

$$L(n) = \left| \{ 1 \leq i \leq n \mid S(i) = 0 \} \right|. $$

Then for $m \leq n$,

$$\Pr [L(2n) < m] = 2^{-2n} \sum_{j=0}^{m-1} 2^j \binom{2n-j}{n}. \hspace{1cm} \text{(A.1)}$$

Corollary A.4. For $L(n)$ as above, and $m \leq n/2$,

$$\Pr [L(n) < m] < \frac{m}{\sqrt{n} - 2m}. $$

Proof. This is a careful application of Stirling’s approximation to (A.1). \hfill \Box

Lemma A.5. Let $S(\cdot)$ be a lazy random walk on $\mathbb{Z}$, starting at $S(0) = 0$, with holding probability $1 - \alpha$. That is,

$$S(n) = \sum_{i=1}^{n} x_i, $$

where $x_i$ are i.i.d., such that $\Pr [x_i = 0] = 1 - \alpha$, and

$$\Pr [x_i = 1] = \Pr [x_i = -1] = \frac{\alpha}{2}. $$

Let $L(n)$ be the number of times the walk visits 0 up to time $n$. That is,

$$L(n) = \left| \{ 1 \leq i \leq n \mid S(i) = 0 \} \right|. $$

Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \alpha) > 0$ such that for all $n \geq 1$,

$$\Pr \left[ L \left( \lfloor Cn^2 \rfloor \right) < n \right] \leq \varepsilon.$$

Proof. Let $m(n)$ be the number of times the walk moves in the first $n$ steps. Then, $m(n) = \sum_{i=1}^{n} r_i$, where $r_i$ are i.i.d. Bernoulli random variables of mean $\alpha$. By the Chernoff bound (see e.g. Appendix A in [1]),

$$\Pr \left[ m(n) \leq \frac{\alpha}{2} n \right] \leq \Pr \left[ |m(n) - \alpha n| \geq \frac{\alpha}{2} n \right] < 2 \exp \left( -\frac{\alpha^2}{2} n \right).$$

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Conditioned on \( m(n) \), the walk is a \( m(n) \)-step simple random walk. Thus, for \( 2k \leq m \), by Corollary A.4,
\[
\Pr \left[ L(n) < k \mid m(n) = m \right] \leq \frac{k}{\sqrt{m - 2k}}.
\]
Let \( C > \frac{4}{\alpha} \) and set \( j = \lceil Cn^2 \rceil \). If \( i \geq \frac{\alpha j}{2} \) then \( 2n \leq i \). Thus,
\[
\Pr \left[ L(j) < n \right] \leq \Pr \left[ m(j) \leq \frac{\alpha}{2} j \right] + \sum_{i \geq (\alpha j)/2} \Pr \left[ L(j) < n, m(j) = i \right]
\leq \exp \left( -\frac{\alpha^2}{2} Cn^2 \right) + \frac{\sqrt{2n}}{\alpha Cn^2 - 4n}.
\]
For large enough \( C \) this is less than \( \varepsilon \). \( \Box \)

**Lemma A.6.** Let \( S(\cdot) \) be a lazy random walk on \( \mathbb{Z} \), starting at \( S(0) = 0 \), with holding probability \( 1 - \alpha \). That is,
\[
S(n) = \sum_{i=1}^{n} x_i,
\]
where \( x_i \) are i.i.d., such that \( \Pr \left[ x_i = 0 \right] = 1 - \alpha \), and
\[
\Pr \left[ x_i = 1 \right] = \Pr \left[ x_i = -1 \right] = \frac{\alpha}{2}.
\]
Let \( m \geq 1 \). Then, for all \( \beta > 0 \),
\[
\Pr \left[ \max_{1 \leq i \leq m} |S(i)| < \sqrt{\beta am} \right] \geq 1 - \frac{1}{\beta}.
\]
**Proof.** The assertion is equivalent to
\[
\Pr \left[ \max_{1 \leq i \leq m} |S(i)| \geq \sqrt{\beta am} \right] \leq \frac{1}{\beta}.
\]
But this follows immediately from the Kolmogorov inequality, since \( S(i) \) is the sum of i.i.d. random variables, and \( \text{Var} \left[ S(m) \right] = \alpha m \). \( \Box \)
B Random walks on finite graphs

In this section we recall some properties of a simple random walk on a finite graph. Given a finite $d$-regular graph $G$ we define two matrices, whose columns and rows are indexed by the vertices of the graph. The adjacency matrix of $G$ is the matrix $A(u, v) = 1_{\{u \sim v\}}$ for all $u, v \in G$. The transition matrix of $G$ is the matrix $P = \frac{1}{d}A$. It is well known that the eigenvalues of $P$ are all real. Further, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|G|}$ are the eigenvalues of $P$, then $\lambda_1 = 1$, and if $G$ is not bi-partite $|\lambda_i| < 1$ for $1 < i \leq |G|$. We denote by $\lambda = \max_{i > 1} |\lambda_i|$. $\lambda$ is called the second eigenvalue of $G$, and $1 - \lambda$ is called the spectral gap.

The following lemma is standard in the theory of random walks on graphs, and in fact stronger statements can be proved. We omit the proof (see \cite{1}).

Lemma B.1. Let $G$ be a non-bi-partite $d$-regular graph. Let $\lambda$ be the second eigenvalue of $G$. Let $\mu(i), i \in G$ be any distribution on the vertices of $G$. Let $x_0, x_1, \ldots, x_t$ be a random walk on $G$, such that $x_0$ is distributed like $\mu$. Then, for any $j \in G$,

$$\left| \Pr [x_t = j] - \frac{1}{n} \right| \leq \lambda^t.$$

We now prove that the spectral gap of a graph, measures how close the random walk on the graph is to independent sampling of the vertices. This is a slight generalization of results from Chapter 9 of \cite{1}, and the proof is similar.

In what follows $G$ is a $d$-regular graph of size $n$. $A$ is its adjacency matrix. $\lambda$ is the second eigenvalue of the transition matrix. Thus, $d$ is the largest eigenvalue of $A$, and all other eigenvalues are at most $d\lambda$.

Let $C \subseteq V(G)$ of size $|C| = cn$. Define the matrix

$$Q_C(i, j) = \begin{cases} A(i, j) & \text{if } j \in C, \\ 0 & \text{otherwise.} \end{cases}$$
For two vectors we use the usual inner product $\langle x, y \rangle = \sum_i x(i) y(i)$, and norm $\|x\|^2 = \langle x, x \rangle$.

Claim B.2.

$$\|Q_C\| \leq \sqrt{cd^2 + (1-c)d^2\lambda^2}$$

Proof. Let $x$ be any vector, and let $\tilde{x}$ be the vector defined by

$$\tilde{x}(i) = \begin{cases} x(i) & \text{if } i \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Q_C x = Q_C \tilde{x} = A\tilde{x}$. Also, note that

$$\|x\|^2 = \sum_i x(i)^2 \geq \sum_{i \in C} x(i)^2 = \|\tilde{x}\|^2.$$ 

Thus,

$$\|Q_C\|^2 = \max_{x \neq 0} \frac{\langle Q_C x, Q_C x \rangle}{\langle x, x \rangle} \leq \max_{\tilde{x} \neq 0} \frac{\langle A\tilde{x}, A\tilde{x} \rangle}{\langle \tilde{x}, \tilde{x} \rangle}.$$ 

So it is enough to prove that for all $x$ such that $\|x\| = 1$ and such that $x(i) = 0$ for all $i \not\in C$, that $\langle Ax, Ax \rangle \leq cd^2 + (1-c)d^2\lambda^2$. Let $x$ be a vector such that $x(i) = 0$ for all $i \not\in C$, and assume that $\|x\| = 1$. Let $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ be the eigenvalues of $A$, and let $v_1, \ldots, v_n$ be the orthonormal basis of eigenvectors of $A$, corresponding to these eigenvalues. We have that $v_1 = n^{-1/2} e$ where $e$ is the all-ones vector. Decompose $x$,

$$x = \sum_{s=1}^n \alpha_s v_s.$$ 

So, by the Cauchy-Schwartz inequality,

$$\alpha_1 = \langle x, v_1 \rangle = \sum_{i \in C} x(i) \frac{1}{\sqrt{n}} \leq \sqrt{\sum_{i \in C} x(i)^2} \cdot \sqrt{\sum_{i \in C} \frac{1}{n}} = \sqrt{c}.$$ 

Note that $\sum_s \alpha_s^2 = \|x\| = 1$. Thus,

$$\langle Ax, Ax \rangle = \sum_{s=1}^n \gamma_s^2 \alpha_s^2 \leq d^2 \alpha_1^2 + (1 - \alpha_1^2)(d\lambda)^2 \leq cd^2 + (1 - c)(d\lambda)^2. \quad \square$$
Claim B.3. Let $C_1, C_2, \ldots, C_t$ be subsets of $V(G)$ such that $|C_s| = c_s n$ for all $s$. Let $\Lambda$ be the number of paths $x_0, x_1, \ldots, x_t$ in $G$ such that $x_s \in C_s$ for all $s \geq 1$. Then,

$$\Lambda \leq n \prod_{s=1}^{t} \sqrt{c_s d^2 + (1 - c_s) d^2 \lambda^2}.$$  

Proof. For $1 \leq s \leq t$, let $Q_s = Q_{C_s}$. Let $Q = Q_1 Q_2 \cdots Q_t$. We claim that

$$Q(i, j) \text{ is the number of paths } i = x_0, x_1, \ldots, x_t = j \text{ such that } x_s \in C_s \text{ for all } s \geq 1.$$  

(B.1)

This is proven by induction on $t$. For $t = 1$, $Q = Q_1$. So $Q(i, j) = 1$ iff $j \in C_1$ and $i \sim j$, and $Q(i, j) = 0$ otherwise. Assume (B.1) for $t - 1$. Let $Q' = Q_1 Q_2 \cdots Q_{t-1}$. Then by the induction hypothesis, $Q'(i, k)$ is the number of paths $i = x_0, x_1, \ldots, x_{t-1} = k$ such that $x_s \in C_s$ for all $1 \leq s \leq t - 1$. Thus,

$$Q(i, j) = (Q' Q_t)(i, j) = \sum_{k} Q'(i, k) Q_t(k, j)$$

is the required quantity.

Thus, if $e$ is the all-ones vector, using (B.1) and claim B.2 we get that

$$\Lambda = \sum_{i,j} Q(i, j) = \langle Q e, e \rangle \leq \langle e, e \rangle \|Q\| \leq n \prod_{s=1}^{t} \|Q_s\| \leq n \prod_{s=1}^{t} \sqrt{c_s d^2 + (1 - c_s) d^2 \lambda^2}$$

\[\Box\]

Lemma B.4. Let $x_0, x_1, \ldots, x_t$ be a random walk on $G$ starting at a uniformly chosen vertex. Let $C_1, C_2, \ldots, C_t$ be subsets of $V(G)$ such that $|C_s| = c_s n$ for all $s$. Let $E$ be the event that $x_s \in C_s$ for all $s \geq 1$. Set $c = \sum_{s} (1 - c_s)$. Then,

$$\Pr[E] \leq \exp\left(-\frac{c}{2}(1 - \lambda)\right).$$
Proof. The total number of possible paths is \( nd^t \). Thus, by claim B.3,

\[
\Pr[E] = \frac{\Lambda}{nd^t} \leq \prod_{s=1}^{t} \sqrt{c_s + (1-c_s)\lambda^2}
\]

\[
\leq \prod_{s=1}^{t} \sqrt{1 - (1-c_s) + (1-c_s)\lambda^2} \leq \prod_{s=1}^{t} \exp \left( -\frac{(1-c_s)}{2}(1 - \lambda^2) \right)
\]

\[
= \exp \left( -\sum_{s=1}^{t} \frac{(1-c_s)}{2}(1 - \lambda^2) \right) < \exp \left( -\frac{c}{2}(1 - \lambda) \right).
\]

\[\square\]

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