Scalarly Essentially Integrable Locally Convex Vector Valued Tensor Fields.

Stokes Theorem

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values in E provided with the usual locally convex topology, we let $G\text{Meas}(G).$ Let $L$ underlie $G.$ Let $H$ be a Hausdorff locally convex space over $K.$ We let $G_0$ denote the linear space over $R$ underlying $G,$ while let $G_R$ denote the Hausdorff locally convex space over $R$ underlying $G.$ Let $\mathcal{L}(G, H)$ be the $K$-linear space of continuous linear maps from $G$ into $H$ and $G' := \mathcal{L}(G, K)$ be the topological dual of $G,$ so $(G')' = \text{Mor}_K-\text{mod}(G', K)$ is the algebraic dual of $G'.$ Next if $W$ is an open set of $R^n$ with $n \in N^+,$ and $k \in Z_+ \cup \{+\infty\},$ then we let $\mathcal{E}^k(W, G)$ be the $K$-linear space of $\mathcal{E}^k$-maps in the sense of Bastiani. For every $(a, v) \in W \times R^n$ we let $D_v^{W, G}_a f$ denote the derivative of $f$ at $a$ in the direction $v,$ and let $D_v^{W, G}_a f : W \ni a \mapsto D_v^{W, G}_a f \in G.$

If $X$ is a topological space and $E$ is a Hausdorff locally convex space over $R,$ then we let $\mathcal{H}(X, E)$ be the $R$-linear space of compactly supported continuous maps defined on $X$ and with values in $E$ provided with the usual locally convex topology, we let $\mathcal{H}(X) := \mathcal{H}(X, R).$ Let $\text{Meas}(X, E)$ be the $R$-linear space of vectorial measures on $X$ with values in $E,$ namely the space

**Abstract.** This note is propaedeutic to the forthcoming work [2]; here we develop the terminology and results required by that paper. More specifically we introduce the concept of scalarly essentially integrable locally convex vector-valued tensor fields on a smooth manifold, generalize on them the usual operations, in case the manifold is oriented define the weak integral of scalarly essentially integrable locally convex vector-valued maximal forms and finally establish the extension of Stokes theorem for smooth locally convex vector-valued forms. This approach to the basic theory of scalarly essentially integrable and smooth locally convex vector-valued tensor fields seems to us to be new. Specifically are new (1) the definition of the space of scalarly essentially integrable locally convex vector-valued tensor fields as a $A(U)$-tensor product, although motivated by a result in the usual smooth and real-valued context; (2) the procedure of $A(U)$-linearizing $A(U)$-bilinear maps in order to extend the usual operations especially the wedge product; (3) the exploitation of the uniqueness decomposition of the $A(U)$-tensor product with a free module in order to define not only (a) the exterior differential of smooth locally convex vector-valued forms, but also (b) the weak integral of scalarly essentially integrable locally convex vector-valued maximal forms; (4) the use of the projective topological tensor product theory to define the wedge product.

**Notation 0.1.** If $A$ is a ring, then let $A-\text{mod}$ be the category of $A$-modules and $A$-linear maps. If $E$ is a $A$-module, then let $E'$ be its $A$-dual. Let $r, s \in Z_+$ and $E$ be a $A$-module, define $[E, r, s]$ to be such that $[E, 0, 0] := A^*,$ otherwise be the map on $[1, r + s]$ such that

$$i \in [1, r] \cap Z \Rightarrow [E, r, s], \ i \Rightarrow E',$$

$$j \in [1, s] \cap Z \Rightarrow [E, r, s]_{r+j} := E.$$

Let $\prod[E, 0, 0] := [E, 0, 0]$ and and let $\prod[E, r, s]$ be the $A$-module product $\prod_{i=1}^{r+s}[E, r, s].$ If $F$ is a $A$-module, then define $\mathcal{X}_0(E, F)$ be the $A$-module of $A$-multilinear maps from $\prod[E, r, s]$ into $F$ whose elements are called tensors on $E$ of type $(r, s)$ at values in $F.$ Set $\mathcal{X}_0(E) := \mathcal{X}_0(E, A)$ and identify $\mathcal{X}_0(E)$ with $A.$ Let $A\text{lt}^k(E)$ be the $A$-submodule of the alternating maps in $\mathcal{X}_0(E).$

Let $K = \{R, C\}$ and let $G$ be a Hausdorff locally convex space over $K.$ We let $G_0$ denote the linear space over $R$ underlying $G,$ while let $G_R$ denote the Hausdorff locally convex space over $R$ underlying $G.$ Let $\mathcal{L}(G, H)$ be the $K$-linear space of continuous linear maps from $G$ into $H$ and $G' := \mathcal{L}(G, K)$ be the topological dual of $G,$ so $(G')' = \text{Mor}_K-\text{mod}(G', K)$ is the algebraic dual of $G'.$ Next if $W$ is an open set of $R^n$ with $n \in N^+,$ and $k \in Z_+ \cup \{+\infty\},$ then we let $\mathcal{E}^k(W, G)$ be the $K$-linear space of $\mathcal{E}^k$-maps in the sense of Bastiani. For every $(a, v) \in W \times R^n$ we let $D_v^{W, G}_a f$ denote the derivative of $f$ at $a$ in the direction $v,$ and let $D_v^{W, G}_a f : W \ni a \mapsto D_v^{W, G}_a f \in G.$

If $X$ is a topological space and $E$ is a Hausdorff locally convex space over $R,$ then we let $\mathcal{H}(X, E)$ be the $R$-linear space of compactly supported continuous maps defined on $X$ and with values in $E$ provided with the usual locally convex topology, we let $\mathcal{H}(X) := \mathcal{H}(X, R).$ Let $\text{Meas}(X, E)$ be the $R$-linear space of vectorial measures on $X$ with values in $E,$ namely the space

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of $\mathbb{R}$-linear and continuous maps from $\mathcal{H}(X)$ into $E$ \cite[VI.18 Def. 1]{4}. Let $\text{Meas}(X)$ denote the set of measures on $X$ \cite[Def. 2, §1, n°3, Ch. 3]{4}. A map $\mu : X \to \mathbb{C}$ is scalarly essentially $\mu$-integrable or simply essentially $\mu$-integrable if $\mathcal{R} \circ \iota^{E}_{C} \circ \mu$ and $\mathcal{I} \circ \iota^{E}_{C} \circ \mu$ are essentially $\mu$-integrable where $\mathcal{R} \in \mathcal{L}(\mathbb{C}, \mathbb{R})$ and $\mathcal{I} \in \mathcal{L}(\mathbb{C}, \mathbb{R})$ are the real and imaginary part respectively. Given a Hausdorff locally convex space $G$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and a map $f : X \to G$, we say that $f$ is scalarly essentially $\mu$-integrable if $\mathcal{I} \circ f$ is essentially $\mu$-integrable for every $\psi \in G$. Moreover we say that the integral of $f$ belongs to $G$ if there exists a necessarily unique $s \in G$ such that $\psi(s) = \int \psi \circ f$ for every $\psi \in G$. In case we set $\int f := s$.

Let $M$ be a smooth manifold with or without boundary, $N = \dim M$ and $U$ be an open set of $M$. A chart and an atlas of $M$ are understood smooth. Let $\mathcal{A}(M)$ be the unital algebra of real valued smooth maps on $M$ and let $\mathbf{1}_{M}$ denote its unit. Let $\mathcal{A}_{e}(M, \mathbb{R})$ be the subalgebra of those $f \in \mathcal{A}(M)$ whose support is compact, while let $\mathcal{A}_{e}(M)$ denote the unital subalgebra $\mathcal{A}_{e}(M, \mathbb{R}) \cup \{\mathbf{1}_{M}\}$. If $G$ is a Hausdorff locally convex space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then let $\mathcal{A}(M, G)$ be the set of maps $f : M \to G$ such that $f \circ \iota^{M}_{U} \circ \phi^{-1} \in \mathcal{C}^{\infty}(\phi(U), G)$, for every chart $(U, \phi)$ of $M$. A standard argument proves that $f \in \mathcal{A}(M, G)$ is equivalent to state that for every $x \in M$ there exists a chart $(V, \beta)$ such that $V \ni x$ and $f \circ \iota^{M}_{V} \circ \beta^{-1} \in \mathcal{C}^{\infty}(\beta(V), G)$. As a result the usual gluing lemma via a covering of charts extends to $\mathcal{A}(M, G)$. Let $\mathcal{A}_{e}(U, G)$ be the subset of those maps in $\mathcal{A}(U, G)$ with compact support, $\mathcal{A}(U, G)$ and $\mathcal{A}_{e}(U, G)$ are clearly $\mathcal{A}(U)$-modules. If $N \neq 0$, then for every chart $(U, \phi)$ of $M$ and $i \in [1, N] \cap \mathbb{Z}$, let $\partial_{i}^{\phi} : \mathcal{A}(U, G) \to \mathcal{A}(U, G)$ be defined as in the case $G = \mathbb{R}$ with the exception of replacing the operator $D_{e}$ with $D_{e_i}^{(U, G)}$, where $\{e_{i}\}_{i=1}^{N}$ is the standard basis of $\mathbb{R}^{N}$.

Let $TM$ and $T^{*}M$ be the tangent and cotangent bundle of $M$ respectively. Let $V$ be a smooth vector bundle over $M$, then let $\Gamma_{0}(U, V)$, $\Gamma^{0}(U, V)$ and $\Gamma(U, V)$ be the $\mathcal{A}(U)$-module of sections, continuous sections and smooth sections respectively of the restriction at $U$ of $V$. If $r, s \in \mathbb{Z}_{+}$, let $\mathfrak{X}^{r}_{s}(T_{p}M)$; while if $k \in \mathbb{Z}_{+}$, then let $\text{Alt}^{k}(U, M) := \text{Alt}^{k}(\Gamma(U, M))$ and $\text{Alt}^{k}(TM)$ be the vector bundle over $M$ whose fiber at $p$ equals $\mathfrak{X}^{r}_{s}(T_{p}M)$; while if $k \in \mathbb{Z}_{+}$, then let $\text{Alt}^{k}(U, M) := \text{Alt}^{k}(\Gamma(U, M))$ and $\text{Alt}^{k}(TM)$ be the vector bundle over $M$ whose fiber at $p$ equals $\mathfrak{X}^{r}_{s}(T_{p}M)$. Set $\mathfrak{X}^{r}_{s}(U, M) := \bigoplus_{(r,s) \in \mathbb{Z}_{+} \times \mathbb{Z}_{+}} \mathfrak{X}^{r}_{s}(U, M)$ and $\mathfrak{X}^{r}_{s}(TM) := \bigoplus_{k \in \mathbb{Z}_{+}} \text{Alt}^{k}(TM)$. We set $\Omega^{r}(U, M) := \Gamma(U, \text{Alt}^{r}(TM))$ and $\Omega^{*}(U, M) := \bigoplus_{k \in \mathbb{Z}_{+}} \Omega^{k}(U, M)$. Clearly $\text{Alt}^{r}(U, M), \text{Alt}^{*}(TM)$ and $\Omega^{*}(U, M)$ equal the direct sum over $[1, N] \cap \mathbb{Z}$.

We shall denote by $r_{x}$ or simply $x$ the usual $\mathcal{A}(U)$-isomorphism from $\Gamma(U, \mathfrak{X}^{r}_{s}(TM))$ onto $\mathfrak{X}^{r}_{s}(U, M)$ and by $t_{x}$ or simply $t$ the inverse of $r$. By abuse of language we let us denote with the same symbol the restriction at $\text{Alt}^{r}(U, M)$ and at its range of $t$ and by $t^{*}$ its inverse. Given a chart $(U, \phi)$ of $M$, in order to keep the notation as light as possible we confine to let $dx^{0}_{\phi} \in \Gamma(U, T^{*}M)$ denote also $t(dx^{0}_{\phi}) \in \Gamma(U, TM)$. Moreover we let $\{(\partial_{i}^{\phi}t_{x})_{a} \mid a \in \Xi(\partial_{i}^{\phi}t_{x})\}$ and $\{e_{dx^{0}}(U) \mid U \in M(k, N, <)\}$ be the basis of $\mathfrak{X}^{r}_{s}(U, M)$ and $\text{Alt}^{k}(U, M)$ image via the isomorphism $t$ of the basis of $\Gamma(U, \mathfrak{X}^{r}_{s}(TM))$ and $\Gamma(U, \text{Alt}^{k}(TM))$ associated with the chart $(U, \phi)$ respectively.

In what follows we let $K \in \{\mathbb{R}, \mathbb{C}\}$ and let $G, H, G_{1}$ and $H_{1}$ be Hausdorff locally convex spaces over $K$, and let $M$ be a finite dimensional smooth manifold $M$, with or without boundary, such
that \( N := \dim M \neq 0 \). Let \( \lambda \) be the Lebesgue measure on \( \mathbb{R}^N \) and for every open set \( A \) of \( \mathbb{R}^N \) we let \( \lambda_A \) be the restriction at \( A \) of \( \lambda \).

**Introduction** 0.2. Let us outline the main ideas underlying this note. We opt to avoid employing the concept of manifold modelled over locally convex spaces via the Bastiani differential calculus. Fortunately this is possible if we generalize to our context the well-known fact that given a finite dimensional vector bundle \( Z \) on \( M \), then \( \Gamma(Z \otimes \operatorname{Alt}^*(TM)) \) is \( \mathcal{A}(M) \)-isomorphic to \( \Gamma(Z) \otimes_{\mathcal{A}(M)} \Gamma(\operatorname{Alt}^*(TM)) \).

Therefore motivated by the above result, given a finite dimensional vector bundle \( V \) on \( M \) and an open set \( U \) of \( M \), we shall define the space of \( G \)-valued scalarly essentially \( \lambda \)-integrable sections of type \( V \) defined on \( U \), as the \( \mathcal{A}(U) \)-module

\[
\mathcal{U}^1_c(U, G, \lambda) \otimes_{\mathcal{A}(U)} \Gamma(U, V);
\]

where \( \mathcal{U}^1_c(U, G, \lambda) \) is the \( \mathcal{A}(U) \)-module of compactly supported scalarly essentially \( \lambda \)-integrable maps from \( U \) at values in \( G \) as defined in a natural way in Def. 1.1. Similar definition is given for \( G \)-valued smooth sections of type \( V \) defined on \( U \) by replacing \( \mathcal{U}^1_c(U, G, \lambda) \) with \( \mathcal{A}(U) \).

The advantages of employing the above definition are the following.

First it is well-known that for any (possibly noncommutative ring) \( A \), any \( A \)-module \( B \) and any free \( A \)-module \( C \) we have a unique decomposition of every element of the \( \mathbb{Z} \)-module \( B \otimes A C \) in terms of elements of \( B \) and elements of the basis of \( C \). In addition when \( U \) is the domain of a chart, then \( \Gamma(U, V) \) is a free \( \mathcal{A}(U) \)-finite dimensional module. As a result we obtain for instance Cor. 1.12 and Cor. 1.35. As a result any element of \( \mathcal{A}(U, G) \otimes_{\mathcal{A}(U)} \Omega^*(U, M) \) admits a unique decomposition which among other properties permits to define the exterior differential in a natural way and then to extend it in the usual manner see Def. 2.36 and Thm. 2.37. Furthermore the unique decomposition applied to any \( \mathbb{R} \)-valued scalarly essentially \( \lambda \)-integrable form over an open set of \( \mathbb{R}^N \), permits to define its integral Def. 2.45 that is the first step to define the weak integral.

Second all the standard operations over tensor fields can be extended to the \( G \)-valued setting just by \( \mathcal{A}(U) \)-linearization of \( \mathcal{A}(U) \)-bilinears. A paradigmatic example showing this procedure is the wedge product in Def. 2.22 provided a sequence of preliminary results, where an extra care must be implemented since the use in the definition of the projective topological tensor product of two Hausdorff locally convex spaces.

Third by pushing forward via any continuous functional on \( G \) the operation so obtained between \( G \)-valued sections we obtain the usual corresponding operation between \( \mathbb{R} \)-valued sections Prp. 1.25 and Prp. 2.25.

Fourth and most importantly by pushing forward via any continuous linear map \( \psi \) from \( G \) into \( H \) a \( G \)-valued scalarly essentially \( \lambda \)-integrable section \( \eta \) of type \( V \) defined on \( U \) we obtain a \( H \)-valued scalarly essentially \( \lambda \)-integrable section \( \psi_x(\eta) \) of type \( V \) defined on \( U \) Def. 1.24. This permits when \( H = \mathbb{K} \) to define in Def. 2.49 the weak integral of a \( G \)-valued smooth maximal form \( \eta \) as the map associating to any continuous functional \( \psi \) on \( G \) the integral of \( \psi_x(\eta) \), then as a result a vectorial measure on \( M \) with values in the real locally convex space \( \langle (G^*)^\prime, \sigma((G^*)^\prime, G^\prime) \rangle_\mathbb{R} \) is constructed in Thm. 2.52. Finally the
Stokes theorem Thm. 2.54 for a G-valued smooth \((N - 1)\)-form \(\theta\) results as a consequence of the usual Stokes theorem applied to \(\psi(x)\) for every \(\psi\) in the topological dual of \(G\).

1. G-Valued Integrable and Smooth Tensor Fields

**Definition 1.1** (G-Valued Scalarly Essentially Integrable Maps on \(M\)). Define \(\mathcal{L}^1(M, G, \lambda)\) to be the set of maps \(f : M \to G\) such that \(f \circ i_M \circ \phi^{-1}\) is scalarly essentially \(\lambda_{\phi(U')}\)-integrable, for every chart \((U, \phi)\) of \(M\). Let \(\mathcal{L}_c^1(M, G, \lambda)\) be the subset of the maps in \(\mathcal{L}^1(M, G, \lambda)\) with compact support.

**Remark 1.2.** The theorem of change of variable in multiple integrals along with a standard argument prove that \(f \in \mathcal{L}^1(M, G, \lambda)\) is equivalent to state that for every \(x \in M\) there exists a chart \((V, \beta)\) such that \(V \ni x\) and \(f \circ i_V \circ \beta^{-1}\) is scalarly essentially \(\lambda_{\beta(V')}\)-integrable. As a result the usual gluing lemma via a covering of charts extends to \(\mathcal{L}^1(M, G, \lambda)\).

Recall that \(A_c(M)\) is by definition the unital subalgebra of \(A(M)\) generated by the unit \(1_M\) and by the subalgebra \(A_c(M, \mathbb{R})\) of the maps in \(A(M)\) with compact support. Thus \(A_c(M) = A_c(M, \mathbb{R}) \cup \{1_M\}\).

**Lemma 1.3.** \(\mathcal{L}^1(M, G, \lambda)\) is a \(A_c\)-module and \(\mathcal{L}^1(M, G, \lambda)\) is a \(A(M)\)-module.

**Proof.** \(\mathcal{L}^1(M, G, \lambda)\) is a \(A_c\)-module since \(A_c(M, \mathbb{R}) \subseteq \mathcal{H}(M)\). Next let \(f \in \mathcal{L}^1(M, G, \lambda)\) and \(\psi : M \to \mathbb{R}\) be a smooth bump function for \(\text{supp}(f)\) supported in \(M\), then \(f = \psi f\) therefore for any \(g \in A(M)\) we have \(gf = g\psi f\), but \(g\psi \in A_c(M)\) and the second sentence of the statement follows by the first sentence of the statement above proven. \(\square\)

Until the end of this work we let \(U\) be an open set of \(M\).

**Definition 1.4.** Let \(\Gamma(c)(U, TM)\) be the \(A_c\)-module \(\Gamma(U, TM)\), define \(\Xi_s^c(U, M) = \Xi_s^c(\Gamma(c)(U, TM))\), set \(\Xi_s^c(M) = \Xi_s^c(M, M)^c\). Moreover define the \(A(U)\)-modules

\[
\Gamma(U, \Xi_s^c(TM))^c = \{ f \in \Gamma(U, \Xi_s^c(TM)) | \tau_{\mathbb{R}}(f) \in \Xi_s^c(U, M)^c \};
\]

\[
\Gamma_c(U, \Xi_s^c(TM)) = \{ f \in \Gamma(U, \Xi_s^c(TM)) | \text{supp}(f) \in \text{Cmp}(M) \};
\]

\[
\Xi_s^c(U, M)^c = \{ \zeta \in \Xi_s^c(U, M) | \text{supp}(\tau_{\mathbb{R}}(\zeta)) \in \text{Cmp}(M) \}.
\]

By construction \(\Xi_s^c(U, M)^c\) is a \(A_c(U)\)-module however we have also that

**Lemma 1.5.** Let \(r, s \in \mathbb{Z}_+\), thus \(\Xi_s^c(U, M)^c\) is a \(A(U)\)-module; \(\Gamma_c(U, \Xi_s^c(TM))\) is an \(A(U)\)-submodule of \(\Gamma(U, \Xi_s^c(TM))^c\), and then \(\Xi_s^c(U, M)^c\) is an \(A(U)\)-submodule of \(\Xi_s^c(U, M)^c\).

**Definition 1.6** (G-Valued Scalarly Essentially Integrable Tensor Fields). Let \(r, s \in \mathbb{Z}_+\), define the \(A(U)\)-module of \(G\)-valued scalarly essentially \(\lambda\)-integrable tensor fields on \(M\) defined on \(U\) of type \((r, s)\) to be

\[
\Xi_s^c(U, M; G, \lambda) := \mathcal{L}_c^1(U, G, \lambda) \otimes_{A(U)} \Xi_s^c(U, M).
\]
Define the $A(U)$-modules

$$\mathcal{I}'(U, M; G, \lambda) := \mathfrak{S}'(\Gamma(U, TM), \mathcal{L}_1(U, G, \lambda)).$$

and

$$\mathfrak{S}_s'(U, M; G, \lambda)^{\mathfrak{C}} := \mathcal{L}_1(U, G, \lambda) \otimes_{A(U)} \mathcal{S}_s'(U, M)^{\mathfrak{C}};$$

$$\mathfrak{S}_s'(U, M; G, \lambda)_c := \mathcal{L}_1(U, G, \lambda) \otimes_{A(U)} \mathcal{S}_s'(U, M)^{\mathfrak{C}}.$$  

Finally define

$$\mathfrak{S}_s'(U, M; G, \lambda) := \bigoplus_{(s,t) \in \mathbb{Z} \times \mathbb{Z}} \mathfrak{S}_s'(U, M; G, \lambda),$$

$$\mathcal{I}'(U, M; G, \lambda) := \bigoplus_{(s,t) \in \mathbb{Z} \times \mathbb{Z}} \mathcal{I}'(U, M; G, \lambda);$$

and

$$\mathfrak{S}_s'(U, M; G, \lambda)^{\mathfrak{C}} := \bigoplus_{(s,t) \in \mathbb{Z} \times \mathbb{Z}} \mathfrak{S}_s'(U, M; G, \lambda)^{\mathfrak{C}};$$

$$\mathfrak{S}_s'(U, M; G, \lambda)_c := \bigoplus_{(s,t) \in \mathbb{Z} \times \mathbb{Z}} \mathfrak{S}_s'(U, M; G, \lambda)_c.$$  

Remark 1.7. Clearly $\mathfrak{S}_s'(U, M; G, \lambda)^{\mathfrak{C}}$ is $A(U)$-isomorphic to a submodule of $\mathfrak{S}_s'(U, M; G, \lambda)$ and in what follows we shall identify these two modules. Similarly we identify $\mathfrak{S}_s'(U, M; G, \lambda)_c$ with a submodule of $\mathfrak{S}_s'(U, M; G, \lambda)$, in particular we have $\mathfrak{S}_s'(U, M; G, \lambda)_c \subseteq \mathfrak{S}_s'(U, M; G, \lambda)^{\mathfrak{C}}$.

Proposition 1.8. $\mathfrak{S}_s'(U, M; G, \lambda) = \mathfrak{S}_s'(U, M; G, \lambda)_c = \mathfrak{S}_s'(U, M; G, \lambda)^{\mathfrak{C}}$.

Proof. If $f \in \mathcal{L}_1(U, G, \lambda)$ and $T \in \mathfrak{S}_s'(U, M)$ and $\psi$ is a smooth bump function for $\text{supp}(f)$ supported in $U$, then $f = \psi f$, so $f \otimes T = (\psi f) \otimes T = f \otimes (\psi T)$. Thus the statement follows since Rmk. [1.7].

Lemma 1.9. Assume $\mathcal{K} = \mathbb{C}$, thus $\mathcal{L}_1(U, G, \lambda) = \mathcal{L}_1(U, G_{\mathbb{R}}, \lambda)$, in particular $\mathfrak{S}_s'(U, M; G_{\mathbb{R}}, \lambda) = \mathfrak{S}_s'(U, M; G, \lambda)$.

Proof. $\mathcal{L}_1(U, G_{\mathbb{R}}, \lambda) \subseteq \mathcal{L}_1(U, G, \lambda)$ since for every $\psi \in \mathcal{L}(G, \mathbb{C})$ we have $\mathfrak{K} \circ \mathfrak{I}_{G_{\mathbb{R}}} \circ \psi \circ \mathfrak{I}_{G_{\mathbb{R}}} \in \mathcal{L}(G_{\mathbb{R}}, \mathbb{R})$ and $\mathfrak{H} \circ \mathfrak{I}_{G_{\mathbb{R}}} \circ \psi \circ \mathfrak{I}_{G_{\mathbb{R}}} \in \mathcal{L}(G_{\mathbb{R}}, \mathbb{R})$. Next according to what stated immediately after [1.11] we have that

$$\forall \phi \in \mathcal{L}(G_{\mathbb{R}}, \mathbb{R}))(\exists \psi \in \mathcal{L}(G, \mathbb{C}))(\phi = \mathfrak{H} \circ \mathfrak{I}_{G_{\mathbb{R}}} \circ \psi \circ \mathfrak{I}_{G_{\mathbb{R}}});$$

from which we deduce that $\mathcal{L}_1(U, G_{\mathbb{R}}, \lambda) \subseteq \mathcal{L}_1(U, G_{\mathbb{R}}, \lambda)$.  

Proposition 1.10. $\mathfrak{S}_s'(U, M; G, \lambda)$ is isomorphic to $\mathcal{L}_1(U, G, \lambda) \otimes_{A(U)} \mathfrak{S}_s'(U, M)$; while $\mathfrak{S}_s'(U, M; G, \lambda)^{\mathfrak{C}}$ is isomorphic to $\mathcal{L}_1(U, G, \lambda) \otimes_{A(U)} \mathfrak{S}_s'(U, M)^{\mathfrak{C}}$ as well $\mathfrak{S}_s'(U, M; G, \lambda)_c$ is isomorphic to $\mathcal{L}_1(U, G, \lambda) \otimes_{A(U)} \mathfrak{S}_s'(U, M)_c$ in the category $A(U) - \text{mod}$.

Proof. Since [1.] II.61 Prp. 7] there exist (canonical) $\mathbb{Z}$-linear isomorphisms, which are clearly a $A(U) - \text{mod}$ isomorphisms by the definition of the module structure of the tensor product of modules over a commutative ring.  

$\Box$
We shall identify the above isomorphic modules.

**Proposition 1.11.**

\[ \exists! \Phi \in \text{Mor}_{\mathcal{A}(U) - \text{mod}} \left( \mathfrak{T}_c(U, M; G, \lambda), l_c'(U, M; G, \lambda) \right) \left( \forall (r, s) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \right) \]

\[ \left( \forall f \in \mathfrak{U}_c(U, G, \lambda) \left( \forall T \in \mathfrak{T}_c(U, M) \right) \left( \forall (\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \in \prod \left[ \Gamma(\mathcal{U}, TM), r, s \right] \right) \]

\[ \left( \Phi(f \otimes T)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) = T(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \cdot f \right). \]

**Proof.** By the universal property of the tensor product over a commutative ring applied to the $\mathcal{A}(U)$-bilinear map $* : \mathfrak{U}_c(U, G, \lambda) \times \mathfrak{T}_c(U, M) \to l_c'(U, M; G, \lambda)$, $(f, T) \mapsto f * T$, defined by $(f * T)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) = T(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \cdot f$. \hfill \Box

**Corollary 1.12 (Unique Decomposition of $G$-Valued Integrable Tensor fields at $M$ defined on a Chart).** Let $(U, \phi)$ be a chart of $M$, $r, s \in \mathbb{Z}_+$ and $T \in \mathfrak{T}_c(U, M; G, \lambda)$, thus

\[ \exists! f : \mathfrak{X}(b^{r,s,\phi}) \to \mathfrak{U}_c(U, G, \lambda) \left( T = \sum_{\alpha \in \Xi(b^{r,s,\phi})} f_\alpha \otimes (\otimes (b^{r,s,\phi}))_\alpha \right). \]

**Proof.** $\{(\otimes (b^{r,s,\phi}))_\alpha \mid \alpha \in \Xi(b^{r,s,\phi})\}$ is a basis of $\mathfrak{T}_c(U, M)$, thus the statement follows since [11 II.62 Cor.1]. \hfill \Box

**Definition 1.13 (Bar Operators on Integrable Tensor Fields).** Define the $\mathcal{A}(U)$-module

\[ \Gamma(U, \mathfrak{T}_c(TM); G, \lambda) : = \mathfrak{U}_c(U, G, \lambda) \otimes_{\mathcal{A}(U)} \Gamma(U, \mathfrak{T}_c(TM)). \]

Define

\[ t_C \in \text{Mor}_{\mathcal{A}(U) - \text{mod}} \left( \mathfrak{S}_c(U, M; G, \lambda), \Gamma(U, \mathfrak{T}_c(TM); G, \lambda) \right), \]

\[ t_C := \text{Id}_{\mathfrak{U}_c(U, G, \lambda)} \otimes t_R; \]

and

\[ r_C \in \text{Mor}_{\mathcal{A}(U) - \text{mod}} \left( \Gamma(U, \mathfrak{T}_c(TM); G, \lambda), \mathfrak{S}_c(U, M; G, \lambda) \right), \]

\[ r_C := \text{Id}_{\mathfrak{U}_c(U, G, \lambda)} \otimes r_R. \]

**Proposition 1.14.** $t_C$ and $r_C$ are isomorphisms one the inverse of the other in the category $\mathcal{A}(U) - \text{mod}$. \hfill \Box

**Remark 1.15.** Since Rmk. [12] the gluing lemma via a covering of charts extends to $\Gamma(U, \mathfrak{T}_c(TM); G, \lambda)$. \hfill \Box

**Proposition 1.16.**

\[ \exists! s_C \in \text{Mor}_{\mathcal{A}(U) - \text{mod}} \left( \Gamma(U, \mathfrak{T}_c(TM); G, \lambda), \Gamma_0(U, G_R \otimes_R \mathfrak{T}_c(TM)) \right), \]

such that

\[ (\forall f \in \mathfrak{U}_c(U, G, \lambda)) \left( \forall \beta \in \Gamma(U, \mathfrak{T}_c(TM))) \left( s_C(f \otimes \beta) = (U \ni p \mapsto f(p) \otimes \beta(p)) \right) . \]
Proof. The map \((f, \beta) \mapsto (U \ni p \mapsto f(p) \otimes \beta(p))\) is \(\mathcal{A}(U)\)-bilinear thus the statement follows by the universal property of the tensor product of modules over a commutative ring.

\[\square\]

Now we are able to define the support as follows

**Definition 1.17 (Support).** Define

\[
\supp : \mathfrak{X}(U, M; G, \lambda) \rightarrow \Cmp(M);
\]

\[
\supp(\theta) := \supp((s_G \circ t_G)(\theta)).
\]

**Convention 1.18.** We let \(r, t\) and \(s\) denote \(r_G, t_G\) and \(s_G\) respectively whenever it does not cause confusion.

We will employ the next result in order to construct in Prp. [2.52] a vectorial measure

**Proposition 1.19.** There exists a unique \(\mathcal{A}(U)\)-bilinear map \((g, \theta) \mapsto g \cdot \theta\) from \(\mathcal{H}(U, \mathbb{K}) \times \mathfrak{X}(U, M; G, \lambda)\) into \(\mathfrak{X}(U, M; G, \lambda)\) such that for every \(g \in \mathcal{H}(U, \mathbb{K})\) and every \(f \in \mathfrak{U}_1(U, M; G, \lambda)\) and \(T \in \mathfrak{X}(U, M)\) we have \(g \cdot (f \otimes T) = (gf) \otimes T\).

Proof. Let \(g \in \mathcal{H}(U, \mathbb{K})\), thus the map \((f, T) \mapsto (gf) \otimes T\) is \(\mathcal{A}(U)\)-bilinear since the \(\mathcal{A}(U)\)-module structure of \(\mathfrak{X}(U, M; G, \lambda)\), then by the universal property there exists a unique \(\mathcal{A}(U)\)-linear endomorphism \(t(g)\) of \(\mathfrak{X}(U, M; G, \lambda)\) such that \(t(g)(f \otimes T) = (gf) \otimes T\).

Next by the uniqueness characterization present in the universal property we deduce that \(t\) is a \(\mathcal{A}(U)\)-linear map from \(\mathcal{H}(U, \mathbb{K})\) into the \(\mathcal{A}(U)\)-module of \(\mathcal{A}(U)\)-endomorphisms of \(\mathfrak{X}(U, M; G, \lambda)\). Thus the statement follows since the isomorphism in [1II.74 Prp. 1(6)] and by the universal property of the tensor product.

\[\square\]

**Definition 1.20.** Let \(N\) be a differential manifold, \(W\) be an open set of \(N\) and \(F \in \mathfrak{C}^\infty(W, U)\) be a diffeomorphism. Define

\[
F^* : \mathfrak{U}_1(U, G, \lambda) \rightarrow \mathfrak{U}_1(W, G, \lambda),
\]

\[
f \mapsto f \circ F;
\]

well-set since the theorem of change of variable in multiple integrals.

Since \(F^*\) is \(\mathbb{R}\)-linear we can give the following

**Definition 1.21 (Pullback of Integrable Tensors of type \((0, s)\)).** Let \(N\) be a differential manifold, \(W\) be an open set of \(N\) and \(F \in \mathfrak{C}^\infty(W, U)\) be a diffeomorphism. Define

\[
(1.2) \quad \mathfrak{F} \in \text{Mor}_{\mathbb{R}_{-\text{mod}}} (\mathfrak{X}(U, M; G, \lambda), \mathfrak{X}(W, N; G, \lambda))
\]

\[
\mathfrak{F} := F^* \otimes F^*;
\]

and

\[
(1.3) \quad \mathfrak{F} \in \text{Mor}_{\mathbb{R}_{-\text{mod}}} (\Gamma(U, \mathfrak{X}(TM); G, \lambda), \Gamma(W, \mathfrak{X}(TN); G, \lambda));
\]

\[
\mathfrak{F} := F^* \otimes F^*.
\]
Next we prepare for the definition of pushforward.

**Definition 1.22.** Let $\psi \in \mathcal{L}(G, H)$, thus define

\[ \psi_\star : \mathcal{U}_c^1(U, G, \lambda) \ni f \mapsto \psi \circ f \in \mathcal{U}_c^1(U, H, \lambda). \]

Well-set definition since $\psi$ is linear and continuous. Clearly we have

**Lemma 1.23.** Let $\psi \in \mathcal{L}(G, H)$, thus $\psi_\star \in \text{Mor}_{A(U)-mod}(\mathcal{U}_c^1(U, G, \lambda), \mathcal{U}_c^1(U, H, \lambda))$.

The above result permits to give the following

**Definition 1.24 (Pushforward of $G$-Valued Integrable Tensors).** Let $\psi \in \mathcal{L}(G, H)$, define

\[ \psi_\star \in \text{Mor}_{A(U)-mod}(\mathfrak{S}_c^0(U, M; G, \lambda), \mathfrak{S}_c^0(U, M; H, \lambda)); \]

\[ \psi_\star := \psi_\star \otimes \text{Id}_{\mathfrak{S}_c^0(U, M)}. \]

Then easily we find that

**Proposition 1.25 (Pushforward Commutes with All the Above Operators).** Let $N$ be a differential manifold, $W$ be an open set of $N$, and $F \in C^\infty(W, U)$ be a diffeomorphism. If $\psi \in \mathcal{L}(G, H)$, then $\psi_\star \circ t = t \circ \psi_\star$, $\psi_\star \circ r = r \circ \psi_\star$, and $\psi_\star \circ F = F \circ \psi_\star$; and that

**Proposition 1.26.** Let $N$ be a differential manifold, $W$ be an open set of $N$, and $F \in C^\infty(W, U)$ be a diffeomorphism. Thus for every $h \in A(U)$ and every $\theta \in \mathfrak{S}_c^0(U, M; G, \lambda)$ we have $\dot{F}(h\theta) = (F^*h)\dot{F}(\theta)$.

**Corollary 1.27.** Assume $\mathbb{K} = \mathbb{C}$. Let $N$ be a differential manifold, $W$ be an open set of $N$ and $F \in C^\infty(W, U)$ be a diffeomorphism. If $\{G_j\}_{j \in J}$ is a family of real locally convex spaces and $G$ is such that $G_\mathbb{R} = \prod_{j \in J} G_j$ provided with the product topology. Thus for every $j \in J$ we have that

\[ \text{Pr}_j^t \circ t = t \circ \text{Pr}_j^t, \text{Pr}_j^t \circ r = r \circ \text{Pr}_j^t, \text{and Pr}_j^t \circ F = F \circ \text{Pr}_j^t. \]

**Proof.** $\text{Pr}_j^t \in \mathcal{L}(G_\mathbb{R}, G)$ and the product topology is locally convex as a particular case of what stated in [3, II.5]. Thus the statement is well-set and it follows since $\text{Prp. 1.25}$ applied to $\mathbb{K} = \mathbb{R}$, to $G$ replaced by $G_\mathbb{R}$ and to $\psi$ replaced by $\text{Pr}_j^t$. \hfill \Box

**Definition 1.28 (G-Valued Smooth Tensor fields at $M$ defined on $U$).** Let $r, s \in \mathbb{Z}_+$, define the $A(U)$-module of $G$-valued differential tensor fields at $M$ defined on $U$ of type $(r, s)$ to be

\[ \mathfrak{T}_c^r(U, M; G) := A(U, G) \otimes_{A(U)} \mathfrak{S}_c^r(U, M). \]

Next we define the $A(U)$-module

\[ T_c^r(U, M; G) := \mathfrak{T}_c^r(\Gamma(U, TM), A(U; G)). \]
Finally define the \( A(U) \)-modules
\[
\mathcal{I}_s'(U, M; G) := \bigoplus_{(r,s) \in \mathbb{Z}_+ \times \mathbb{Z}_+} \mathcal{I}_s'(U, M; G);
\]
\[
\mathsf{T}_s'(U, M; G) := \bigoplus_{(r,s) \in \mathbb{Z}_+ \times \mathbb{Z}_+} \mathsf{T}_s'(U, M; G).
\]

**Definition 1.29.** Let \( r, s \in \mathbb{Z}_+ \), define the \( A(U) \)-modules
\[
\mathcal{I}'(U, M; G)_{[r]} := A_c(U, G) \otimes_{A(U)} \mathcal{I}_s'(U, M);
\]
\[
\mathcal{I}_s'(U, M; G)_r := A(U, G) \otimes_{A(U)} \mathcal{I}_s'(U, M);
\]
\[
\mathcal{I}_s'(U, M; G)^c := A(U, G) \otimes_{A(U)} \mathcal{I}_s'(U, M)^c.
\]

**Remark 1.30.** Clearly \( \mathcal{I}_s'(U, M; G)^c \) is \( A(U) \)-isomorphic to a submodule of \( \mathcal{I}_s'(U, M; G) \) and in what follows we shall identify these two modules. Similarly we identify \( \mathcal{I}_s'(U, M; G)^c \) (respectively \( \mathcal{I}_s'(U, M; G)_{[r]} \)) with a submodule of \( \mathcal{I}_s(U, M; G) \), in particular we have \( \mathcal{I}_s'(U, M; G)^c \subset \mathcal{I}_s(U, M; G)^c \).

**Proposition 1.31.** Let \( r, s \in \mathbb{Z}_+ \), thus \( \mathcal{I}_s'(U, M; G)_{[r]} = \mathcal{I}_s'(U, M; G)^c \).

**Proof.** If \( f \in A_c(U, G) \) and \( T \in \mathcal{I}_s(U, M) \) and \( \psi \) is a smooth bump function for \( \text{supp}(f) \) supported in \( U \), then \( f \psi f \), so \( f \otimes T = (f \psi f) \otimes T = f \otimes (\psi T) \). If \( g \in A(U, G) \) and \( S \in \mathcal{I}_s(U, M) \) and \( \psi \) is a smooth bump function for \( \text{supp}(S) \) supported in \( U \), then \( S = \psi S \), thus \( g \otimes S = g \otimes (\psi S) = (\psi g) \otimes S \). Thus the statement follows since Rmk. 1.30.

**Remark 1.32.** \( \mathcal{I}_s'(U, M; G) = \mathcal{I}_s'(U, M; G_R) \) since \( A(U, G) = A(U, G_R) \).

**Proposition 1.33.** \( \mathcal{I}_s(U, M; G) \) is isomorphic to \( A(U, G) \otimes_{A(U)} \mathcal{I}_s(U, M) \) in the category \( A(U) \) – mod.

**Proof.** Since \( \mathbb{I} II.61 \text{Prp. 7} \) there exists a (canonical) \( \mathbb{Z} \)-linear isomorphism, which is clearly a \( A(U) \) – mod isomorphism by the definition of the module structure of the tensor product of modules over a commutative ring.

**Proposition 1.34.**
\[
(1.4) \quad \left( \exists! \Psi \in \text{Mor}_{A(U) - \text{mod}} \left( \mathcal{I}_s'(U, M; G), \mathcal{T}_s'(U, M; G) \right) \right) \quad (V(r, s) \in \mathbb{Z}_+ \times \mathbb{Z}_+)
\]
\[
(V f \in A(U, G))(\forall T \in \mathcal{I}_s(U, M))(\forall (\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \in \prod_r [\Gamma(U, TM), r, s])
\]
\[
\left( \Psi(f \otimes T)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) = T(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \cdot f \right).
\]

**Proof.** By the universal property of the tensor product over a commutative ring applied to the \( A(U) \)-bilinear map \( \star : A(U, G) \times \mathcal{I}_s(U, M) \to \mathcal{T}_s(U, M; G) \), \( (f, T) \mapsto f \star T \), defined by \( (f \star T)(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) = T(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \cdot f \).

The following result justifies the choice of the above definition.
Corollary 1.35 (Unique Decomposition of G-Valued Smooth Tensor fields at M defined on a Chart). Let \((U, \phi)\) be a chart of \(M\), \(r, s \in \mathbb{Z}_+\) and \(T \in \mathfrak{X}_s^r(U, M; G)\), thus

\[
\left( \exists f : \mathfrak{X}(b^{r,s,\phi}) \rightarrow \mathcal{A}(U, G) \right) \left( T = \sum_{\alpha \in \mathfrak{X}(b^{r,s,\phi})} f_\alpha \otimes (\otimes(b^{r,s,\phi})_\alpha) \right).
\]

Proof. \(\{\otimes(b^{r,s,\phi})_\alpha| \alpha \in \mathfrak{X}(b^{r,s,\phi})\}\) is a basis of \(\mathfrak{X}_s^r(U, M)\), thus the statement follows since \([1, II.62\text{ Cor.1}].\)

Definition 1.36 (Bar Operators on Smooth Tensor Fields). Define the \(\mathcal{A}(U)\)-module

\[
\Gamma(U, \mathfrak{X}^\bullet(TM); G) := \mathcal{A}(U, G) \otimes_{\mathcal{A}(U)} \Gamma(U, \mathfrak{X}^\bullet(TM)).
\]

Define with abuse of language the following maps

\[
\begin{align*}
t_G & \in \text{Mor}_{\mathcal{A}(U)-\text{mod}}(\mathfrak{X}(U, M; G), \Gamma(U, \mathfrak{X}^\bullet(TM); G)), \\
t_G & := \text{Id}_{\mathcal{A}(U, G)} \otimes t_R,
\end{align*}
\]

and

\[
\begin{align*}
r_G & \in \text{Mor}_{\mathcal{A}(U)-\text{mod}}(\Gamma(U, \mathfrak{X}^\bullet(TM); G), \mathfrak{X}(U, M; G)), \\
r_G & := \text{Id}_{\mathcal{A}(U, G)} \otimes r_R.
\end{align*}
\]

Proposition 1.37. \(t_G\) and \(r_G\) are isomorphisms one the inverse of the other in the category \(\mathcal{A}(U) - \text{mod}\).

Proof. Since \(t_R\) and \(r_R\) are isomorphisms one the inverse of the other in the category \(\mathcal{A}(U) - \text{mod}\).

Remark 1.38. The gluing lemma via a covering of charts extends to \(\Gamma(U, \mathfrak{X}^\bullet(TM); G)\), since it extends for maps in \(\mathcal{A}(U, G)\).

We shall use convention [1.18] also for the above defined maps.

Definition 1.39. Let \(N\) be a differential manifold, \(W\) be an open set of \(N\) and \(F \in \mathcal{C}^\infty(W, U)\). Define

\[
F^* : \mathcal{A}(U, G) \rightarrow \mathcal{A}(W, G), \\
f \mapsto f \circ F.
\]

Since \(F^*\) is \(\mathbb{R}\)-linear we can give the following

Definition 1.40 (Pullback of Smooth Tensor of type \((0, s)\)). Let \(N\) be a differential manifold, \(W\) be an open set of \(N\) and \(F \in \mathcal{C}^\infty(W, U)\). Define

\[
(1.5) \\
\begin{align*}
\times F & \in \text{Mor}_{\mathbb{R}-\text{mod}}(\mathfrak{X}_0^r(U, M; G), \mathfrak{X}_0^s(W, N; G)) \\
\times F & := F^* \otimes F;
\end{align*}
\]
and

\[ \bar{F} \in \text{Mor}_{\mathbb{R}}(\Gamma(U, \mathcal{F}_0^0(TM)); G), \Gamma(W, \mathcal{F}_0^0(TN)); G); \]

(1.6)

\[ \bar{F} := F^* \otimes F^*. \]

**Definition 1.41.** Let \( \psi \in \mathcal{L}(G, H) \), thus define by abuse of language \( \psi_* : \mathcal{A}(U, G) \ni f \mapsto \psi \circ f \in \mathcal{A}(U, H) \).

Well-set definition since \( \psi \) is linear and continuous. Clearly we have

**Lemma 1.42.** Let \( \psi \in \mathcal{L}(G, H) \), thus \( \psi_* \in \text{Mor}_{\mathcal{A}(U)}(\mathcal{A}(U, G), \mathcal{A}(U, H)) \).

The above result permits to give the following

**Definition 1.43 (Pushforward of \( G \)-Valued Smooth Tensors).** Let \( \psi \in \mathcal{L}(G, H) \), define

\[ \psi_x \in \text{Mor}_{\mathcal{A}(U)}(\mathcal{F}_x^0(U, M; G), \mathcal{F}_x^0(U, M; H)); \]

\[ \psi_x := \psi_* \otimes \text{Id}_{\mathcal{F}_x^0(U, M)}. \]

Then easily we find that

**Proposition 1.44 (Pushforward Commutes with All the Above Operators).** Let \( N \) be a differential manifold, \( W \) be an open set of \( N \), and \( F \in \mathcal{C}^\infty(W, U) \). If \( \psi \in \mathcal{L}(G, H) \), then

\[ \psi_x \circ t = t \circ \psi_x, \quad \psi_x \circ r = r \circ \psi_x, \quad \text{and} \quad \psi_x \circ \bar{F} = \bar{F} \circ \psi_x; \]

and that

**Proposition 1.45.** Let \( N \) be a differential manifold, \( W \) be an open set of \( N \), and \( F \in \mathcal{C}^\infty(W, U) \). Thus for every \( h \in \mathcal{A}(U) \) and every \( \theta \in \mathcal{F}_x^0(U, M; G) \) we have

\[ \bar{F}(h\theta) = (F^*h)\bar{F}(\theta). \]

**Corollary 1.46.** Assume \( \mathbb{K} = \mathbb{C} \). Let \( N \) be a differential manifold, \( W \) be an open set of \( N \) and \( F \in \mathcal{C}^\infty(W, U) \). If \( \{G_j\}_{j \in J} \) is a family of real locally convex spaces and \( G \) is such that \( G_R = \prod_{j \in J} G_j \) provided with the product topology. Thus for every \( j \in J \) we have that

\[ \text{Pr}_x^j \circ t = t \circ \text{Pr}_x^j, \quad \text{Pr}_x^j \circ r = r \circ \text{Pr}_x^j, \quad \text{and} \quad \text{Pr}_x^j \circ \bar{F} = \bar{F} \circ \text{Pr}_x^j. \]

**Proof.** \( \text{Pr}_x^j \in \mathcal{L}(G_{R_j}, G_j) \) and the product topology is locally convex as a particular case of what stated in [3 II.5]. Thus the statement is well-set and it follows since Prop. 1.44 applied to \( \mathbb{K} = \mathbb{R} \), to \( G \) replaced by \( G_R \) and to \( \psi \) replaced by \( \text{Pr}_x^j \).

\[ \square \]

2. \textit{G-Valued Integrable and Smooth Forms}

**Definition 2.1 (G-valued Scalarly Essentially Integrable Forms at M defined on U).** For every \( k \in \mathbb{Z}_+ \) define the \( \mathcal{A}(U) \)-module of \( G \)-valued scalarly essentially \( \lambda \)-integrable \( k \)-forms at \( M \) defined on \( U \) as follows

\[ \text{Alt}^k(U, M; G, \lambda) := \mathcal{L}_x^1(U, G, \lambda) \otimes_{\mathcal{A}(U)} \text{Alt}^k(U, M). \]
Next define the $\mathcal{A}(U)$-module
\[
\text{Alt}^\bullet(U, M; G, \lambda) := \bigoplus_{k \in \mathbb{Z}} \text{Alt}^k(U, M; G, \lambda).
\]

Set $\text{Alt}^k(M; G, \lambda) := \text{Alt}^k(M, M; G, \lambda)$ and $\text{Alt}^\bullet(M; G, \lambda) := \text{Alt}^\bullet(M, M; G, \lambda)$. Finally define the $\mathcal{A}(U)$-modules
\[
\text{Alt}^\bullet_0(U, M; G, \lambda) := \mathcal{H}(U, G_R \otimes_{\mathcal{A}(U)} \text{Alt}^\bullet(U, M);
\]
and
\[
\Omega^\bullet(U, M; G, \lambda) := \mathcal{L}^1_c(U, G, \lambda) \otimes_{\mathcal{A}(U)} \Omega^\bullet(U, M).
\]

Clearly $\text{Alt}^\bullet_0(U, M; G, \lambda)$ is isomorphic to a $\mathcal{A}(U)$-submodule of $\text{Alt}^\bullet(U, M; G, \lambda)$ and this is isomorphic to a $\mathcal{A}(U)$-submodule of $\mathcal{Z}^\bullet_0(U, M; G, \lambda)$. In what follows we shall identify these isomorphic modules.

**Proposition 2.2.** $\text{Alt}^\bullet(U, M; G, \lambda)$ is isomorphic to $\mathcal{L}^1_c(U, G, \lambda) \otimes_{\mathcal{A}(U)} \text{Alt}^\bullet(U, M)$ in the category $\mathcal{A}(U) - \text{mod}$.

**Proof.** Since [II] II.61 Prp. 7 there exists a canonical $\mathbb{Z}$-linear isomorphism that is clearly a $\mathcal{A}(U) - \text{mod}$ isomorphism by the definition of the module structure of the tensor product of modules over a commutative ring. \hfill \Box

**Remark 2.3.** $\text{Alt}^\bullet(U, M; G, \lambda) = \mathcal{L}^1_c(U, G, \lambda) \otimes_{\mathcal{A}(U)} \text{Alt}^\bullet(U, M)$ since Prp. [I.8] where we employ the convention described in Rmk. [I.7]

**Corollary 2.4 (Unique Decomposition of $G$-Valued Scalarly Essentially Integrable Forms).** Let $(\phi, U)$ be a chart of $M$ and $\theta \in \text{Alt}^k(U, M; G, \lambda)$, thus
\[
\left( \exists ! f : M(k, N, <) \rightarrow \mathcal{L}^1_c(U, G, \lambda) \right) \left( \theta = \sum_{i \in M(k, N, <)} f_i \otimes \mathcal{E}_{d\phi^i}(I) \right).
\]

**Proof.** $\{\mathcal{E}_{d\phi^i}(I) \mid I \in M(k, N, <)\}$ is a basis of $\text{Alt}^k(U, M)$, thus the statement follows since [II] II.62 Cor.1. \hfill \Box

**Definition 2.5.** Define by abuse of language
\[
t_G \in \text{Mor}_{\mathcal{A}(U) - \text{mod}} \left( \text{Alt}^\bullet(U, M; G, \lambda), \Omega^\bullet(U, M; G, \lambda) \right),
\]
be the restriction of $t_G$ defined in Def. [II.13] and let
\[
t_G \in \text{Mor}_{\mathcal{A}(U) - \text{mod}} \left( \Omega^\bullet(U, M; G, \lambda), \text{Alt}^\bullet(U, M; G, \lambda) \right);
\]
be the restriction of $t_G$ defined in Def. [II.13]

**Proposition 2.6.** $t_G$ and $t_G$ are isomorphisms one the inverse of the other in the category $\mathcal{A}(U) - \text{mod}$.

**Proof.** Since Prp. [I.14]

We shall use convention [I.18] also for the above defined maps.
Definition 2.7. Let $N$ be a differential manifold, $W$ be an open set of $N$ and $F \in \mathcal{C}^\infty(W, U)$ be a diffeomorphism. Define by abuse of language the $\mathbb{R}$-linear map

$$ \tilde{F} : \text{Alt}^\bullet(U, M; G, \lambda) \to \text{Alt}^\bullet(W, N; G, \lambda); $$

as the restriction of the map defined in (1.2). Similarly define by abuse of language the $\mathbb{R}$-linear map

$$ \tilde{F} : \Omega^\bullet(U, M; G, \lambda) \to \Omega^\bullet(W, N; G, \lambda); $$

as the restriction of the map defined in (1.3).

Easily we see that

Theorem 2.8 (Pushforward Commutes with All the Above Operators). Let $N$ be a differential manifold, $W$ be an open set of $N$, $F \in \mathcal{C}^\infty(W, U)$ be a diffeomorphism. If $\psi \in \mathcal{L}(G, H)$, then $\psi_x \circ t = t \circ \psi_x$, $\psi_x \circ r = r \circ \psi_x$ and $\psi_x \circ \tilde{F} = \tilde{F} \circ \psi_x$.

Definition 2.9 (G-valued Smooth Forms at $M$ defined on $U$). For every $k \in \mathbb{Z}_+$ define the $\mathcal{A}(U)$-module of $G$-valued differential $k$-forms at $M$ defined on $U$ as follows

$$ \text{Alt}^k(U, M; G) := \mathcal{A}(U, G) \otimes_{\mathcal{A}(U)} \text{Alt}^k(U, M); $$

and define the $\mathcal{A}(U)$-module of $G$-valued differential forms at $M$ defined on $U$ as follows

$$ \text{Alt}^\bullet(U, M; G) := \bigoplus_{k \in \mathbb{Z}_+} \text{Alt}^k(U, M; G), $$

set $\text{Alt}^k(M; G) := \text{Alt}^k(M, M; G)$ and $\text{Alt}^\bullet(M; G) := \text{Alt}^\bullet(M, M; G)$. Similarly

$$ \text{Alt}^k_c(U, M; G) := \mathcal{A}_c(U, G) \otimes_{\mathcal{A}(U)} \text{Alt}^k(U, M). $$

and define the $\mathcal{A}(U)$-module of $G$-valued differential forms at $M$ defined on $U$ and with compact support as follows

$$ \text{Alt}^\bullet_c(U, M; G) := \bigoplus_{k \in \mathbb{Z}_+} \text{Alt}^k(U, M; G), $$

set $\text{Alt}^k_c(M; G) := \text{Alt}^k_c(M, M; G)$ and $\text{Alt}^\bullet_c(M; G) := \text{Alt}^\bullet_c(M, M; G)$.

Proposition 2.10. $\text{Alt}^\bullet(U, M; G)$ is isomorphic to $\mathcal{A}(U, G) \otimes_{\mathcal{A}(U)} \text{Alt}^\bullet(U, M)$ and $\text{Alt}^\bullet_c(U, M; G)$ is isomorphic to $\mathcal{A}_c(U, G) \otimes_{\mathcal{A}(U)} \text{Alt}^\bullet(U, M)$ in the category $\mathcal{A}(U) - \text{mod}$.

Proof. Since [11] II.61 Prp. 7] there exists a canonical $\mathbb{Z}$-linear isomorphism that is clearly a $\mathcal{A}(U) - \text{mod}$ isomorphism by the definition of the module structure of the tensor product of modules over a commutative ring.

Remark 2.11. $\text{Alt}^\bullet_c(U, M; G) = \mathcal{A}(U, G) \otimes_{\mathcal{A}(U)} \text{Alt}^\bullet_c(U, M)$ since Prp. [1.31] where we used the convention described in Rmk. [1.30]
Corollary 2.12 (Unique Decomposition of \( G \)-Valued Smooth Forms). Let \((\phi, U)\) be a chart of \( M \), \( \theta \in \text{Alt}^k(U, M; G) \) and \( \eta \in \text{Alt}^\ell(U, M; G) \) thus
\[
\left( \exists ! f : M(k, N, \langle \rangle) \to \mathcal{A}(U, G) \right) \left( \exists \sum_{i \in M(k, N, \langle \rangle)} f_i \otimes \varepsilon_{\phi^0}(I) \right);
\]
and
\[
\left( \exists ! g : M(k, N, \langle \rangle) \to \mathcal{A}_c(U, G) \right) \left( \exists \sum_{i \in M(k, N, \langle \rangle)} g_i \otimes \varepsilon_{\phi^0}(I) \right).
\]

Proof. \( \{ \varepsilon_{\phi^0}(I) | I \in M(k, N, \langle \rangle) \} \) is a basis of \( \text{Alt}^k(U, M) \), thus the statement follows since \[1\] II.62 Cor.1.

Definition 2.13. By abuse of language define
\[
t_G \in \text{Mor}_{\mathcal{A}(U) - \text{mod}}\left( \text{Alt}^*(U, M; G), \Omega^*(U, M; G) \right),
\]
be the restriction of the map \( t_G \) defined in Def. 2.14. Similarly by abuse of language let
\[
r_G \in \text{Mor}_{\mathcal{A}(U) - \text{mod}}\left( \Omega^*(U, M; G), \text{Alt}^*(U, M; G) \right),
\]
be the restriction of the map \( r_G \) defined in Def. 2.14.

Proposition 2.14. \( t_G \) and \( r_G \) defined in Def. 2.13 are isomorphisms one the inverse of the other in the category \( \mathcal{A}(U) - \text{mod} \).

Proof. Since Prp. 1.37 also for the above defined maps.

Definition 2.15. Let \( N \) be a differential manifold, \( W \) be an open set of \( N \) and \( F \in \mathcal{C}_\infty(W, U) \). By abuse of language let
\[
\check{\mathcal{F}} : \text{Alt}^*U, M; G) \to \text{Alt}^*(W, N; G)
\]
be the restriction of the map defined in (1.5), and let
\[
\check{\mathcal{F}} : \Omega^*U, M; G) \to \Omega^*(W, N; G)
\]
be the restriction of the map defined in (1.6).

Next we start the sequence of results required to define the wedge product in Def. 2.22.

Lemma 2.16. Assume that there exist \( \mathbb{K} \)-linear subspaces \( X \) of \( G^* \) and \( Y \) of \( H^* \) such that the topology on \( G \) and \( H \) are \( \sigma(G, X) \) and \( \sigma(H, Y) \) respectively. Thus the following
\[
\mathcal{L}_c^1(U, G, \lambda) \times \mathcal{A}(U, H) \to \mathcal{L}_c^1(U, G \otimes H, \lambda),
\]
\[
(f, g) \mapsto (x \mapsto f(x) \otimes g(x));
\]
is a well-defined \( \mathcal{A}(U) \)-bilinear map.
Proof. Since the topological dual of a Hausdorff topological linear space is $\mathcal{K}$-isomorphic to the topological dual of its completion, we deduce by [6, Prp.2 pg. 30] that $(G \widehat{\otimes} H)'$ is $\mathcal{K}$-isomorphic via the universal property to the space of bilinear continuous $\mathcal{K}$-forms on $G \times H$. Therefore given any continuous bilinear $\mathcal{K}$-form $b$ on $G \times H$ we have $\hat{b} \in (G \widehat{\otimes} H)'$, where $\hat{b}$ is the continuous extension at $G \widehat{\otimes} H$ of the linearization of $b$ via the universal property, any element of $(G \widehat{\otimes} H)'$ arises uniquely in this way, and finally there exist $a > 0$, $\psi \in X = G'$ and $\phi \in Y = H'$ such that for every $(u,v) \in G \times H$ we have $|\hat{b}(u \otimes v)| \leq a|\psi(u)||\phi(v)|$. Then the map in the statement is well-defined since $\psi \times (f) \in \hat{\Omega}_c^1(U, \mathcal{K}, \lambda)$ for every $f \in \hat{\Omega}_c^1(U,G,\lambda)$, $\phi \times (g) \in \mathcal{A}(U, \mathcal{K})$ for every $g \in \mathcal{A}(U, G)$ and by Prp. [1.8] applied to $r = s = 0$. The $\mathcal{A}(U)$-bilinearity is trivially true. □

Lemma 2.16 permits to give the following

**Definition 2.17.** Assume that there exist $\mathcal{K}$-linear subspaces $X$ of $G^*$ and $Y$ of $H^*$ such that the topology on $G$ and $H$ are $\sigma(G, X)$ and $\sigma(H, Y)$ respectively. Define

$$\tau \in \text{Mor}_{\mathcal{A}(U)-\text{mod}} \left( \hat{\Omega}_c^1(U,G,\lambda) \otimes_{\mathcal{A}(U)} \mathcal{A}(U, H), \hat{\Omega}_c^1(U, G \widehat{\otimes} H, \lambda) \right)$$

such that

$$\tau(f \otimes g) = (x \mapsto f(x) \otimes g(x)).$$

**Definition 2.18.** Assume that there exist $\mathcal{K}$-linear subspaces $X$ of $G^*$ and $Y$ of $H^*$ such that the topology on $G$ and $H$ are $\sigma(G, X)$ and $\sigma(H, Y)$ respectively. Let $k, l \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(U, M)$ and $g \in \mathcal{A}(U, H)$. Define

$$\wedge_{g, \omega, 1}^l : \hat{\Omega}_c^1(U,G,\lambda) \times \text{Alt}^l(U, M) \to \text{Alt}^{k+l}(U, M; G \widehat{\otimes} H, \lambda),$$

$$(f, \zeta) \mapsto \tau(f \otimes g) \otimes (\zeta \wedge \omega),$$

and

$$\wedge_{g, \omega, 2}^l : \hat{\Omega}_c^1(U,G,\lambda) \times \text{Alt}^l(U, M) \to \text{Alt}^{k+l}(U, M; G \widehat{\otimes} H, \lambda),$$

$$(f, \zeta) \mapsto \tau(f \otimes g) \otimes (\omega \wedge \zeta).$$

**Proposition 2.19.** Assume that there exist $\mathcal{K}$-linear subspaces $X$ of $G^*$ and $Y$ of $H^*$ such that the topology on $G$ and $H$ are $\sigma(G, X)$ and $\sigma(H, Y)$ respectively. Let $k, l \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(U, M)$ and $g \in \mathcal{A}(U, H)$. Thus $\wedge_{g, \omega, 2}^l = (-1)^{kl} \wedge_{g, \omega, 1}^l$ and $\wedge_{g, \omega, i}^l$ is $\mathcal{A}(U)$-bilinear for every $i \in \{1, 2\}$.

Proof. The wedge product in $\text{Alt}^* (U, M)$ is $\mathcal{A}(U)$-bilinear, thus the statement follows since Def. 2.17 and the $\mathcal{A}(U)$-module structure of $\text{Alt}^{k+l}(U, M; G \widehat{\otimes} H, \lambda)$.

The above result permits the following

**Definition 2.20.** Assume that there exist $\mathcal{K}$-linear subspaces $X$ of $G^*$ and $Y$ of $H^*$ such that the topology on $G$ and $H$ are $\sigma(G, X)$ and $\sigma(H, Y)$ respectively. Let $k, l \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(U, M)$ and $g \in \mathcal{A}(U, H)$. For every $i \in \{1, 2\}$ define $\overline{\wedge}_{g, \omega, i}^l$ as the unique

$$\overline{\wedge}_{g, \omega, i}^l \in \text{Mor}_{\mathcal{A}(U)-\text{mod}} \left( \text{Alt}^l(U, M; G, \lambda), \text{Alt}^{k+l}(U, M; G \widehat{\otimes} H, \lambda) \right).$$
such that 
\[(\forall f \in \mathcal{U}_1^1(U, G, \lambda))(\forall \zeta \in \text{Alt}^l(U, M))(\wedge_{g, \alpha, l}^1(f \otimes \zeta) = \wedge_{g, \alpha, l}^1(f, \zeta)).\]

Easily we see that

**Lemma 2.21.** Assume that there exist \(K\)-linear subspaces \(X\) of \(G^*\) and \(Y\) of \(H^*\) such that the topology on \(G\) and \(H\) are \(\sigma(G, X)\) and \(\sigma(H, Y)\) respectively. Let \(k, l \in \mathbb{Z}_+\), Thus the map \((g, \omega) \mapsto \wedge_{g, \alpha, l}^1\) is \(\mathcal{A}(U)\)-bilinear. In particular there exists a unique
\[
\wedge_{1}^k \in \text{Mor}_{\mathcal{A}(U)\text{-mod}} \left( \text{Aut}^k(U, M; H), \text{Mor}_{\mathcal{A}(U)\text{-mod}} \left( \text{Alt}^l(U, M; G, \lambda), \text{Alt}^{k+l}(U, M; G \overset{\wedge}{\otimes} H, \lambda) \right) \right)
\]
such that
\[(\forall g \in \mathcal{A}(U, H))(\forall \omega \in \text{Aut}^k(U, M))(\wedge_{1}^k(g \otimes \omega) = \wedge_{1}^k).\]

**Definition 2.22 (The Wedge Products of G-Valued Integrable Forms).** Assume that there exist \(K\)-linear subspaces \(X\) of \(G^*\) and \(Y\) of \(H^*\) such that the topology on \(G\) and \(H\) are \(\sigma(G, X)\) and \(\sigma(H, Y)\) respectively. Let \(k, l \in \mathbb{Z}_+\), define
\[
\wedge_{1}^k : \text{Alt}^l(U, M; G, \lambda) \times \text{Alt}^l(U, M; H) \to \text{Alt}^{k+l}(U, M; G \overset{\wedge}{\otimes} H, \lambda);
\]
\[
(\theta, \varepsilon) \mapsto \wedge_{1}^k(\varepsilon)(\theta),
\]
and
\[
\wedge_{2}^k : \text{Alt}^l(U, M; H) \times \text{Alt}^l(U, M; G, \lambda) \to \text{Alt}^{k+l}(U, M; G \overset{\wedge}{\otimes} H, \lambda);
\]
\[
(\varepsilon, \theta) \mapsto \wedge_{2}^k(\varepsilon)(\theta).
\]

Next define
\[
\wedge_{1} : \text{Alt}^*(U, M; G, \lambda) \times \text{Alt}^*(U, M; H) \to \text{Alt}^*(U, M; G \overset{\wedge}{\otimes} H, \lambda);
\]
\[
(\theta, \varepsilon) \mapsto \wedge_{1}(\varepsilon, \theta),
\]
and
\[
\wedge_{2} : \text{Alt}^*(U, M; H) \times \text{Alt}^*(U, M; G, \lambda) \to \text{Alt}^*(U, M; G \overset{\wedge}{\otimes} H, \lambda);
\]
\[
(\varepsilon, \theta) \mapsto \wedge_{2}(\varepsilon, \theta).
\]

\(\wedge \) will be also denoted by \(\wedge\).

**Remark 2.23.** \((f \otimes \zeta) \wedge_{1}(g \otimes \omega) = \tau(f \otimes g) \otimes (\zeta \wedge \omega)\) and \((g \otimes \omega) \wedge_{2}(f \otimes \zeta) = \tau(f \otimes g) \otimes (\omega \wedge \zeta)\).

**Corollary 2.24 (The Wedge Products are \(\mathcal{A}(U)\)-Bilinear).** \(\wedge_{1}^i\) in Def. 2.22 is \(\mathcal{A}(U)\)-bilinear for every \(i \in \{1, 2\}\).

**Proof.** \(\wedge_{1}^{k+l}\) is \(\mathcal{A}(U)\)-bilinear for every \(k, l \in \mathbb{Z}_+\) and \(i \in \{1, 2\}\) as a consequence of Lemma 2.21, then the statement follows. \(\Box\)

**Proposition 2.25 (Pushforward Commutes with Wedge).** Assume that there exist \(K\)-linear subspaces \(X\) of \(G^*\) and \(Y\) of \(H^*\) such that the topology on \(G\) and \(H\) are \(\sigma(G, X)\) and \(\sigma(H, Y)\) respectively. Similarly assume that there exist \(K\)-linear subspaces \(X_1\) of \(G_1\) and \(Y_1\) of \(H_1\) such that the topology on \(G_1\) and \(H_1\) are \(\sigma(G_1, X_1)\) and \(\sigma(H_1, Y_1)\) respectively. Let \(\theta \in \text{Alt}^*(U, M; G, \lambda), \varepsilon \in \text{Alt}^*(U, M; H)\). If \(\psi \in \mathcal{L}(G, G_1)\), and \(\phi \in \mathcal{L}(H, H_1)\), then \((\psi \otimes \phi)_*(\theta \wedge \varepsilon) = \psi_*(\theta) \wedge \phi_*(\varepsilon)\).
Proof. The statement is well-set since \( \psi \otimes \phi \in \mathcal{L}(G \otimes H, G_1 \otimes H_1) \) by [6 pg.37], then the statement is trivially true. \( \square \)

**Corollary 2.26.** Assume \( \mathbb{K} = \mathbb{C} \). Let \( N \) be a differential manifold, \( W \) be an open set of \( N \), \( F \in C^\infty(W, U) \) be a diffeomorphism, \( \eta \in \text{Alt}^\bullet(U, M; G, \lambda) \) and \( \varepsilon \in \text{Alt}^\bullet(U, M; H) \). If \( \{G_j\}_{j \in J} \) is a family of real locally convex spaces and \( G \) is such that \( G_R = \prod_{j \in J} G_j \) provided with the product topology and if \( \{H_k\}_{k \in K} \) is a family of real locally convex spaces and \( H \) is such that \( H_R = \prod_{k \in K} H_k \) provided with the product topology, then for every \( j \in J \) we have that \( (\text{Pr}_G^j)_x \circ t_G = t_G \circ (\text{Pr}_G^j)_x \), \( (\text{Pr}_G^j)_x \circ t_G = t_G \circ (\text{Pr}_G^j)_x \), \( (\text{Pr}_G^j)_x \circ F = F \circ (\text{Pr}_G^j)_x \), moreover for every \( k \in K \) we have that

\[
\left( (\text{Pr}_G^j) \otimes (\text{Pr}_H^k) \right)(\eta \wedge \varepsilon) = (\text{Pr}_G^j)(\eta) \wedge (\text{Pr}_H^k)(\varepsilon).
\]

Proof. \( \eta \in \text{Alt}^\bullet(U, M; G_R, \lambda) \) since Lemma 1.9 while \( \text{Pr}_G^j \in \mathcal{L}(G_R, G_j) \) and the product topology is locally convex as a particular case of what stated in [3 II.5]. Thus the statement is well-set and it follows since Thm. 2.8 and Prp. 2.25 applied to \( \mathbb{K} = \mathbb{R} \), to \( G \) replaced by \( G_R \) and to \( \psi \) replaced by \( \text{Pr}_G^j \in \mathcal{L}(G_R, G_j) \) and to \( H \) replaced by \( H_R \) and to \( \phi \) replaced by \( \text{Pr}_H^k \in \mathcal{L}(H_R, H_k) \). \( \square \)

Next we start to define the wedge product for \( G \)-valued smooth forms.

**Lemma 2.27.** The following

\[
\mathcal{A}(U, G) \times \mathcal{A}(U, H) \to \mathcal{A}(U, G \otimes H),
\]

\[
(f, g) \mapsto (x \mapsto f(x) \otimes g(x));
\]

is a well-defined \( \mathcal{A}(U) \)-bilinear map.

Proof. The bilinear \( \otimes : G \times H \to G \otimes H \) is continuous as a result of [6 Prp.2 pg. 30], thus the statement follows since any continuous bilinear map is smooth w.r.t. the Bastiani differential calculus. \( \square \)

Lemma 2.27 permits to give the following

**Definition 2.28.** Define by abuse of language

\[
\tau \in \text{Mor}_{\mathcal{A}(U)_{\text{mod}}}(\mathcal{A}(U, G) \otimes \mathcal{A}(U, H), \mathcal{A}(U, G \otimes H));
\]

such that

\[
\tau(f \otimes g) = (x \mapsto f(x) \otimes g(x)).
\]

**Definition 2.29.** Let \( k, l \in \mathbb{Z}_+ \), \( \omega \in \text{Alt}^k(U, M) \) and \( g \in \mathcal{A}(U, H) \), define

\[
\Lambda_{\omega, 1}^l : \mathcal{A}(U, G) \times \text{Alt}^l(U, M) \to \text{Alt}^{k+l}(U, M; G \otimes H),
\]

\[
(f, \zeta) \mapsto \tau(f \otimes g) \otimes (\zeta \wedge \omega),
\]

and

\[
\Lambda_{\omega, 2}^l : \mathcal{A}(U, G) \times \text{Alt}^l(U, M) \to \text{Alt}^{k+l}(U, M; G \otimes H),
\]

\[
(f, \zeta) \mapsto \tau(f \otimes g) \otimes (\omega \wedge \zeta).
\]
Proposition 2.30. Let $k, l \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(U, M)$ and $g \in A(U, H)$. Thus $\Lambda^l_{\omega, 2} = (-1)^{k+l} \Lambda^l_{\omega, 1}$ and $\Lambda^l_{\omega, 1}$ is $A(U)$-bilinear for every $i \in \{1, 2\}$.

Proof. The wedge product in $\text{Alt}^*(U, M)$ is $A(U)$-bilinear, thus the statement follows since Def. 2.28 and the $A(U)$-module structure of $\text{Alt}^{k+l}(U, M; G\hat{\otimes}H)$.

The above result permits the following

Definition 2.31. Let $k, l \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(U, M)$ and $g \in A(U, H)$. For every $i \in \{1, 2\}$ define $\Lambda^l_{\omega, i}$ as the unique

$$\Lambda^l_{\omega, i} \in \text{Mor}_{A(U)-\text{mod}} \left( \text{Alt}^l(U, M; G), \text{Alt}^{k+l}(U, M; G\hat{\otimes}H) \right),$$

such that

$$(\forall f \in A(U, G))(\forall \zeta \in \text{Alt}^l(U, M)) \left( \Lambda^l_{\omega, i} \left( f \otimes \zeta \right) = \Lambda^l_{\omega, i}(f, \zeta) \right).$$

Easily we see that

Lemma 2.32. Let $k, l \in \mathbb{Z}_+$. Thus the map $(g, \omega) \mapsto \Lambda^l_{\omega, i}$ is $A(U)$-bilinear. In particular there exists a unique

$$\Lambda^l_i \in \text{Mor}_{A(U)-\text{mod}} \left( \text{Aut}^k(U, M; H), \text{Mor}_{A(U)-\text{mod}} \left( \text{Alt}^l(U, M; G), \text{Alt}^{k+l}(U, M; G\hat{\otimes}H) \right) \right),$$

such that

$$(\forall g \in A(U, H))(\forall \omega \in \text{Aut}^k(U, M)) \left( \Lambda^l_i(g \otimes \omega) = \Lambda^l_{\omega, i} \right).$$

Definition 2.33 (The Wedge Products of $G$-Valued Smooth Forms). Let $k, l \in \mathbb{Z}_+$, define

$$\Lambda^k_1 : \text{Alt}^k(U, M; G, \lambda) \times \text{Alt}^l(U, M; H) \to \text{Alt}^{k+l}(U, M; G\hat{\otimes}H);$$

$$\left( \theta, \epsilon \right) \mapsto \Lambda^l_1(\epsilon)(\theta),$$

and

$$\Lambda^k_2 : \text{Alt}^k(U, M; H) \times \text{Alt}^l(U, M; G) \to \text{Alt}^{k+l}(U, M; G\hat{\otimes}H),$$

$$\left( \epsilon, \theta \right) \mapsto \Lambda^l_2(\epsilon)(\theta).$$

Next define

$$\Lambda : \text{Alt}^*(U, M; G) \times \text{Alt}^*(U, M; H) \to \text{Alt}^*(U, M; G\hat{\otimes}H);$$

$$\left( \theta, \epsilon \right) \mapsto \Lambda^{\text{ord}(\epsilon),\text{ord}(\theta)} \left( \theta, \epsilon \right),$$

and

$$\Lambda : \text{Alt}^*(U, M; H) \times \text{Alt}^*(U, M; G) \to \text{Alt}^*(U, M; G\hat{\otimes}H),$$

$$\left( \epsilon, \theta \right) \mapsto \Lambda^{\text{ord}(\epsilon),\text{ord}(\theta)} \left( \epsilon, \theta \right).$$

$\Lambda_1$ will be also denoted by $\Lambda$.

Remark 2.34. $(f \otimes \zeta) \Lambda_1(g \otimes \omega) = \tau(f \otimes g) \otimes (\zeta \wedge \omega)$ and $(g \otimes \omega) \Lambda_2(f \otimes \zeta) = \tau(f \otimes g) \otimes (\omega \wedge \zeta)$. 


Corollary 2.35 (The Wedge Products are \( A(U) \)-Bilinear). \( \wedge_i \) in Def. 2.33 is \( A(U) \)-bilinear for every \( i \in \{1, 2\} \).

Proof. \( \wedge_{kl} \) is \( A(U) \)-bilinear for every \( k, l \in \mathbb{Z}_+ \) and \( i \in \{1, 2\} \) as a consequence of Lemma 2.32, then the statement follows. \( \square \)

Next we shall use Cor. 2.12 to define the differential of \( G \)-valued differential forms defined on a chart of \( M \).

Definition 2.36 (Differential of a \( G \)-valued differential form on a chart). Let \( (U, \phi) \) be a chart of \( M \), define for every \( i \in \{1, N\} \cap \mathbb{Z} \) the following map

\[
A(U, G) \times \text{Alt}^i(U, M) \to \text{Alt}^i(U, M; G)
\]

\[
(f, \zeta) \mapsto \partial_{i}^{\phi,G}(f) \otimes (dx_i^\phi \wedge \zeta).
\]

The above map is \( \mathbb{R} \)-bilinear since the \( A(U) \)-module structure of \( \text{Alt}^i(U, M; G) \), since Cor. 2.35 and since \( \partial^{\phi,G}_i \) is \( \mathbb{R} \)-linear. \( \square \) Therefore by the universal property of the tensor product

\[
\exists \delta_i \in \text{Mor}_{\mathbb{R} \text{-mod}}(\text{Alt}^i(U, M; G), \text{Alt}^i(U, M; G)) ,
\]

such that

\[
(\forall f \in A(U, G))(\forall \zeta \in \text{Alt}^i(U, M))(\delta_i(f \otimes \zeta) = \partial_{i}^{\phi,G}(f) \otimes (dx_i^\phi \wedge \zeta)).
\]

Therefore we are legitimate to define \( d : \text{Alt}^i(U, M; G) \to \text{Alt}^i(U, M; G) \) such that for every \( \theta \in \text{Alt}^i(U, M; G) \)

\[
d\theta := \sum_{i \in M(\text{ord} \theta, N, <)} \sum_{j=1}^N \delta_i(f_i \otimes \mathcal{E}_{dx^\phi}(I));
\]

where \( f : M(\text{ord} \theta, N, <) \to A(U, G) \) is the unique map in the decomposition of \( \theta \) established in Cor. 2.12

Theorem 2.37 (Differential of a \( G \)-Valued Smooth Form). Let \( \{U_i\}_{a \in D} \) be a collection of domains of charts of \( M \) which are subsets of \( U \) covering \( U \). Thus there exists a unique \( d : \text{Alt}^i(U, M; G) \to \text{Alt}^i(U, M; G) \) called the exterior \( G \)-differentiation such that for all \( k \in \mathbb{Z}_+ \) we have \( d : \text{Alt}^k(U, M; G) \to \text{Alt}^{k+1}(U, M; G) \) and

\[
(\forall \theta \in \text{Alt}^i(U, M; G))(\forall \alpha \in D)((i^U_{i \uparrow a})^\times \circ d)(d \circ (i^M_{M \uparrow i \downarrow a})^\times)(\theta).
\]

Proof. Since the gluing lemma via charts that is legitimate by Rmk. 1.38 where the compatibility is ensured by the uniqueness of the decomposition established in Cor. 2.12 and by the fact that \( (i^M_{M \uparrow i \downarrow a})^\times = (i^U_{i \uparrow a \cap U^i_{i \downarrow a}})^\times \circ (i^M_{U^i_{i \downarrow a} \cap U^i_{i \downarrow a}})^\times = (i^U_{U^i_{i \downarrow a}})^\times \circ (i^M_{U^i_{i \downarrow a}})^\times \).

Remark 2.38. Since Thm. 2.37 and Prp. 2.14 we can define \( d \) on \( \Omega^i(U, M; G) \).

\[\text{but not } A(U) \text{-linear so the linearization of the above bilinear map is w.r.t. } \mathbb{R} \text{ rather than } A(U), \text{ however this does not affect the goal for which this map has been introduced, namely to legitimate the definition of } d \text{ as below.}\]
Definition 2.39. Let $\psi \in \mathcal{L}(G, H)$, define by abuse of language

$$\psi_* : \mathcal{A}(U,G) \ni f \mapsto \psi \circ f \in \mathcal{A}(U,H).$$

Well-set definition since $\psi$ is linear and continuous. Clearly we have

Lemma 2.40. Let $\psi \in \mathcal{L}(G, H)$, thus $\psi_* \in \text{Mor}_{\mathcal{A}(U)\text{-mod}}(\mathcal{A}(U,G), \mathcal{A}(U,H))$. If in addition $(U, \phi)$ is a chart of $M$, then

$$\partial^\phi_i \circ \psi_* = \psi_* \circ \partial_i^\phi G.$$  \hfill (2.1)

The above result permits to give the following

Definition 2.41. Let $\psi \in \mathcal{L}(G, H)$, define by abuse of language

$$\psi_x \in \text{Mor}_{\mathcal{A}(U)\text{-mod}}(\text{Alt}^*(U,M,G), \text{Alt}^*(U,M;H));$$

$$\psi_x := \psi_* \otimes \text{Id}_{\text{Alt}^*(U,M)},$$

and the same symbol denotes also

$$\psi_x \in \text{Mor}_{\mathcal{A}(U)\text{-mod}}(\Omega^*(U,M,G), \Omega^*(U,M;H));$$

$$\psi_x := \psi_* \otimes \text{Id}_{\Omega^*(U,M)}.$$  

Theorem 2.42 (Pushforward Commutes with All the Above Operators). Let $N$ be a differential manifold, $W$ be an open set of $N$, $F \in \mathcal{C}^\infty(W,U)$, $\eta \in \text{Alt}^*(U,M;G)$ and $\varepsilon \in \text{Alt}^*(U,M;H)$. If $\psi \in \mathcal{L}(G, G_1)$, and $\phi \in \mathcal{L}(H_1,H_1)$, then $\psi_x \circ \iota = \iota \circ \psi_x$, $\psi_x \circ r = r \circ \psi_x$, $\psi_x \circ F = F \circ \psi_x$, $\psi_x \circ d = d \circ \psi_x$ and $(\psi \otimes \phi)_x(\eta \wedge \varepsilon) = \psi_x(\eta) \wedge \phi_x(\varepsilon)$.  

Proof. The proof for the operators $\iota$, $r$, $F$ and $\wedge$ is trivial, where the statement concerning $\wedge$ is well-set since $\psi \otimes \phi \in \mathcal{L}(G \otimes H, G_1 \otimes H_1)$ by [6] pg.37. The proof for the operator $d$ follows by Def. 2.36, (2.1), by what right now said and by Thm. 2.37. \hfill \Box

Corollary 2.43. Assume $\mathbb{K} = \mathbb{C}$. Let $N$ be a differential manifold, $W$ be an open set of $N$, $F \in \mathcal{C}^\infty(W,U)$, $\eta \in \text{Alt}^*(U,M;G)$ and $\varepsilon \in \text{Alt}^*(U,M;H)$. If $\{G_j\}_{j \in J}$ is a family of real locally convex spaces and $G$ is such that $G_R = \prod_{j \in J} G_j$ provided with the product topology, and if $\{H_k\}_{k \in K}$ is a family of real locally convex spaces and $H$ is such that $H_R = \prod_{k \in K} H_k$ provided with the product topology; then for every $j \in J$ we have that $(\text{Pr}_G^j)_x \circ \iota = \iota \circ (\text{Pr}_G^j)_x$, $(\text{Pr}_G^j)_x \circ r = r \circ (\text{Pr}_G^j)_x$, $(\text{Pr}_G^j)_x \circ F = F \circ (\text{Pr}_G^j)_x$, moreover for every $k \in K$ we have that

$$((\text{Pr}_G)_x \otimes (\text{Pr}_H)_x)(\eta \wedge \varepsilon) = (\text{Pr}_G)_x(\eta) \wedge (\text{Pr}_H)_x(\varepsilon).$$

Proof. Since Thm. 2.42. \hfill \Box

Corollary 2.44 (Properties of the $G$-differential). Let $d$ the operator uniquely determined in Thm. 2.37. Thus

(1) $d$ is $\mathbb{R}$–linear;
(2) For all \( \omega \in \text{Alt}^\bullet(U, M) \) and \( \eta \in \text{Alt}^\bullet(U, M; G) \)
\[
d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\text{ord}(\omega)} \omega \wedge d\eta;
\]

(3) \( d \circ d = 0 \);

(4) for all \( \psi \in G' \), \( f \in \mathcal{A}(U, G) \) and \( X \in \Gamma(U, M) \) we have
\[
\left((\mathcal{R} \circ i^{\mathcal{R}}_K \circ \psi \circ i^{\mathcal{R}}_{G^r}) \times \circ d\right)(i^{\mathcal{R}}_G \circ f)(X) = X\left((\mathcal{R} \circ i^{\mathcal{R}}_K \circ \psi \circ f)\right),
\]
where in case \( \mathcal{R} = \mathbb{R} \) in the above equality \( \mathcal{R} \) has to be understood \( \text{Id}_{\mathbb{R}} \).

Moreover let \( N \) be a manifold, \( U' \) be an open set of \( N \) and \( F \in \mathcal{C}^\infty(U, U') \). Thus the following equality of operators defined on \( \text{Alt}^\bullet(U', N; G) \) holds true
\[
d \circ F = F \circ d.
\]

**Proof.** \((G^r)\)' separates the points of \( G^r, \) thus the statement follows by Rmk. \[1.32\] by Thm. \[2.42\] applied for \( K = \mathbb{R}, G \) replaced by \( G^r \) and for \( H = \mathbb{R} \), and by the fact that the statement is true for the special case of real valued smooth forms. \( \square \)

Now the unique decomposition established in Cor. \[2.4\] permits to define the integral of a maximal \( \mathbb{R} \)-valued essentially integrable form defined on an open set of \( \mathbb{R}^N \) as in the standard case

**Definition 2.45.** Let \( V \) be an open set of \( \mathbb{R}^N \), and for every \( \omega \in \text{Alt}^N(V, \mathbb{R}^N; \mathbb{R}, \lambda) \) let \( f_\omega \) be the unique map in \( \mathcal{C}^1(V, \mathbb{R}^N; \mathbb{R}, \lambda) \) such that \( \omega = f_\omega \wedge \bigwedge_{i=1}^N (i^{\mathbb{R}}_V)^* (dx_i) \) via the decomposition established in Cor. \[2.4\] Define the map
\[
\text{Alt}^n(V, \mathbb{R}^N; \mathbb{R}, \lambda) \ni \omega \mapsto \int f_\omega d\lambda_V \in \mathbb{R}.
\]

**Definition 2.46.** Let \( M \) be oriented and \( (U, \phi) \) be an oriented chart of \( M \). Define \( \gamma_\phi \in \{1, -1\} \) such that \( \gamma_\phi = 1 \) if \( (U, \phi) \) is positively oriented, otherwise \( \gamma_\phi = -1 \).

Def. \[2.45\] and the concept of support as introduced in Def. \[1.17\] permit to give the following definition as in the standard case

**Definition 2.47.** Let \( M \) be oriented, \( \omega \in \text{Alt}^N(M; \mathbb{R}, \lambda) \), \( \{(U_a, \phi_a)\}_{a \in D} \) be a finite family of oriented charts of \( M \) such that \( \{U_a\}_{a \in D} \) is a covering of \( \text{supp}(\omega) \) moreover by setting \( D^+ = D \cup \{\hat{\alpha}\} \) and \( U_\hat{\alpha} = \bigcap_M \text{supp}(\omega) \), let \( \{\psi_a\}_{a \in D^+} \) be a smooth partition of unity subordinate to \( \{U_a\}_{a \in D^+} \). Define
\[
\int \omega := \sum_{a \in D} \gamma_{\phi_a} \int (i^M_{U_a} \circ \phi_a^{-1})^\times (\psi_a \omega).
\]

Standard arguments as for instance \[5, 13.1.9\] permit to show that the above definition does not depend by the choice of the covering and of the partition of unity subordinate to it. Now \( \text{Alt}^N(M; \mathbb{R}, \lambda) = \text{Alt}^N(M; \mathbb{R}^r, \lambda) \) since Lemma \[1.9\] while \( \mathcal{R}, \mathcal{S} \in \mathcal{L}(\mathcal{C}_R, \mathbb{R}) \) therefore Def. \[2.47\] allows us to provide the following
Definition 2.48. Let $M$ be oriented, define

\[
\text{Alt}^N(M; \mathbb{K}, \lambda) \ni \beta \mapsto \int \beta := \int \Re_x(\beta) + i \int \Im_x(\beta) \in \mathbb{K}, \text{ if } \mathbb{K} = \mathbb{C};
\]
\[
\text{Alt}^N(M; \mathbb{K}, \lambda) \ni \omega \mapsto \int \omega \in \mathbb{K}, \text{ if } \mathbb{K} = \mathbb{R}.
\]

Definition 2.49 (Weak Integral of $G$-Valued Scalarly $\lambda$-Integrable Maximal Forms). Let $M$ be oriented and $\eta \in \text{Alt}^N(M; G, \lambda)$. Define $\int \eta \in (G')^*$ such that

\[
\int \eta : G' \ni \psi \mapsto \int \psi_x(\eta) \in \mathbb{K},
\]

called the weak integral of $\eta$. We say that $\int \eta$ belongs to $G$ or that $\int \eta \in G$ iff there exists a necessarily unique element $s \in G$ such that $\psi(s) = \int \psi_x(\eta)$ for every $\psi \in G'$, in such a case and whenever there is no confusion we let $\int \eta$ denote also the element $s$.

Clearly $\int$ is a $\mathbb{R}$-linear operator by considering the $\mathbb{R}$-module underlying the $A(U)$-module $\text{Alt}^N(M; G, \lambda)$. By recalling Def. 2.45, since $\mathbb{R}^N$ is locally compact, since the Lebesgue measure on $\mathbb{R}^N$ is a measure, and since the weak integral of any compactly supported continuous $G$-valued map against any measure belongs to $G$ as established in [4, III.38 Cor. 2].

Definition 2.51. Define $G^* := \langle (G')^*, o((G')^*, G') \rangle_{\mathbb{R}}$

Now we can state the following

Theorem 2.52 (Vectorial Measure Associated with an Integrable $G$-Valued Form). Let $M$ be oriented, thus there exists a unique map

\[
m \in \text{Mor}_{\mathbb{R} - \text{mod}} \left( \text{Alt}^N(M; G, \lambda), \text{Meas}(M, G^*) \right);
\]

such that

\[
(\forall \eta \in \text{Alt}^N(M; G, \lambda))(\forall g \in \mathcal{H}(M)) \left\{ m_{\eta}(g) = \int g \cdot \eta \right\};
\]

where $(\cdot)$ is the $A(M)$-bilinear map constructed in Prp. 1.19.
Proof. \( m \) is \( \mathbb{R} \)-linear since it is so the weak integral and since \((-)\) is \( A(M) \)-bilinear. Next let \( E \) denote \( \langle (G'), \sigma((G'), G') \rangle \) so \( G^* = E_\mathbb{R} \) and for every \( \psi \in G \) let \( b_\psi : (G')^* \rightarrow \mathbb{K}, z \mapsto z(\psi) \), thus

\[
(2.3) \quad E' = \{ b_\psi \}_{\psi \in G'}.
\]

Let \( g \in \mathcal{H}(M) \) and \( \psi \in G \) thus \( \int \psi_x(g \cdot \eta) = \int g\psi_x(\eta) \) so

\[
(2.4) \quad b_\psi \circ m_\eta \in \text{Meas}(M, \mathbb{K});
\]

in particular \( b_\psi \circ m_\eta \) is continuous. Therefore \( m_\eta : \mathcal{H}(M) \rightarrow E \) is continuous by \( (2.3) \), by \( (2.4) \), since the definition of weak topologies and since \([2 \text{ I.12 Prp. 4}]\). Hence the statement follows since the topology on \( G^* \) is the topology on \( E \).

\[\square\]

**Corollary 2.53.** Let \( M \) be oriented, \( \eta \in \text{Alt}^N(M; G, \lambda), \{(U_\alpha, \phi_\alpha)\}_{\alpha \in D} \) be a finite family of oriented charts of \( M \) such that \( \{U_\alpha\}_{\alpha \in D} \) is a covering of \( \text{supp}(\eta) \) moreover by setting \( D^\prime = D \cup \{\dagger\} \) and \( U_\dagger = \bigcap M \text{supp}(\eta) \), let \( \{\psi_\alpha\}_{\alpha \in D^\prime} \) be a smooth partition of unity subordinate to \( \{U_\alpha\}_{\alpha \in D^\prime} \). Thus

\[
\int \eta = \sum_{\alpha \in D} \int \psi_\alpha \cdot \eta.\]

Proof. Since \( D \) is finite we can define in \( A(M) \) the map \( g = \sum_{\alpha \in D} \psi_\alpha, \) in particular \( g \in \mathcal{H}(M) \), while \( g \circ i^M_{\text{supp}(\eta)} = 1_{\text{supp}(\eta)} \) since \( \psi_\dagger \circ i^M_{\text{supp}(\eta)} = 0_{\text{supp}(\eta)} \) and since \( g + \psi_\dagger = 1_M \) by definition of partition of unity. Therefore \( \eta = g \cdot \eta, \) then \( \int \eta = m_\eta(g) = \sum_{\alpha \in D} m_\eta(\psi_\alpha) \)

where the second equality follows since Thm. \[2.52\]

\[\square\]

Finally we can establish the following

**Theorem 2.54 (Stokes Theorem for G-Valued Smooth Forms).** Let \( M \) be oriented and with boundary and \( \theta \in \text{Alt}^{N-1}_c(M; G), \) thus

\[
\int d\theta = \int (i^M_{\partial M})^\times(\theta);\]

furthermore if \( G \) is quasi-complete, then the above integrals belong to \( G \). Here if \( \partial M = \emptyset \), then the right-hand side of the equality has to be understood equal to \( 0 \).

Proof. The statement is well set since \( \text{Alt}^*_c(M; G) \) is isomorphic to a submodule of \( \text{Alt}^*(M; G, \lambda) \). Let \( \psi \in G ', \) thus \( \psi_x(d\theta) = d(\psi_x(\theta)) \) and \( \psi_x(i^M_{\partial M})^\times \theta = (i^M_{\partial M})^\times \psi_x \theta \) since Thm. \[2.42\]. Henceforth the equality follows by \( (2.2) \), by Stokes theorem, and in case \( \mathbb{K} = \mathbb{C} \) also by \( \text{Alt}_c^*(M; C, \lambda) = \text{Alt}_c^*(M; C_R, \lambda) \) since Rmk. \[1.32\] and by Cor. \[2.43\] applied to the projectors \( \mathbb{R}, \mathbb{I} \in \mathbb{V}(C_R, \mathbb{R}) \). The last sentence of the statement follows since Prp. \[2.50\]

\[\square\]
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