OKOUNKOV BODIES AND RESTRICTED VOLUMES ALONG
VERY GENERAL CURVES

SHIN-YAO JOW

Abstract. Given a big divisor $D$ on a normal complex projective variety $X$, we show that the restricted volume of $D$ along a very general complete-intersection curve $C \subset X$ can be read off from the Okounkov body of $D$ with respect to an admissible flag containing $C$. From this we deduce that if two big divisors $D_1$ and $D_2$ on $X$ have the same Okounkov body with respect to every admissible flag, then $D_1$ and $D_2$ are numerically equivalent.

Introduction

Motivated by earlier works of Okounkov [Oko96, Oko03], Lazarsfeld and Mustață gave an interesting construction in a recent paper [LM08], which associates a convex body $\Delta(D) \subset \mathbb{R}^d$ to any big divisor $D$ on a projective variety $X$ of dimension $d$. (Independent of [LM08], Kaveh and Khovanskii also came up with a similar construction around the same time: see [KK08, KK09].) This so called “Okounkov body” encodes many asymptotic invariants of the complete linear series $|mD|$ as $m$ goes to infinity. For example, the volume of $D$, which is the limit

$$\text{vol}_X(D) = \lim_{m \to \infty} \frac{h^0(X, mD)}{m^d/d!},$$

is equal to $d!$ times the Euclidean volume $\text{vol}_{\mathbb{R}^d}(\Delta(D))$ of $\Delta(D)$. This viewpoint renders transparent several basic properties about volumes of big divisors.

Let us recall now the construction of Okounkov bodies from [LM08]. The construction depends upon the choice of an admissible flag on $X$, which is by definition a flag

$$Y_{\bullet} : X = Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_{d-1} \supset Y_d = \{\text{pt}\}$$

of irreducible subvarieties of $X$, where $\text{codim}_X(Y_i) = i$ and each $Y_i$ is nonsingular at the point $Y_d$. The purpose of this flag is that it will determine a valuation $\nu_{Y_{\bullet}}$ which maps any nonzero invariant of the complete linear series $|mD|$ as $m$ goes to infinity. For example, the volume of $D$, which is the limit

$$\text{vol}_X(D) = \lim_{m \to \infty} \frac{h^0(X, mD)}{m^d/d!},$$

is equal to $d!$ times the Euclidean volume $\text{vol}_{\mathbb{R}^d}(\Delta(D))$ of $\Delta(D)$. This viewpoint renders transparent several basic properties about volumes of big divisors.

Let us recall now the construction of Okounkov bodies from [LM08]. The construction depends upon the choice of an admissible flag on $X$, which is by definition a flag

$$Y_{\bullet} : X = Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_{d-1} \supset Y_d = \{\text{pt}\}$$

of irreducible subvarieties of $X$, where $\text{codim}_X(Y_i) = i$ and each $Y_i$ is nonsingular at the point $Y_d$. The purpose of this flag is that it will determine a valuation $\nu_{Y_{\bullet}}$ which maps any nonzero invariant of the complete linear series $|mD|$ to a $d$-tuple of nonnegative integers

$$\nu_{Y_{\bullet}}(s) = (\nu_1(s), \ldots, \nu_d(s)) \in \mathbb{N}^d$$

defined as follows. Assuming that all the $Y_i$’s are smooth after replacing $X$ by an open subset, we can set to begin with

$$\nu_1(s) = \text{ord}_{Y_1}(s).$$
After choosing a local equation for $Y_1$ in $X$, $s$ determines a section 

$$s_1 \in H^0(X, mD - \nu_1(s)Y_1)$$

that does not vanish identically along $Y_1$, so after restriction we get a nonzero section 

$$s_1 \in H^0(Y_1, mD - \nu_1(s)Y_1).$$

Then we set 

$$\nu_2(s) = \text{ord}_{Y_2}(s_1),$$

and continue in this manner to define the remaining $\nu_i(s)$. Once we have the valuation $\nu_{Y^*}$, we can define 

$$\Gamma(D)_m := \text{Im}((H^0(X, mD) - \{0\}) \xrightarrow{\nu_{Y^*}} \mathbb{N}^d).$$

Then the Okounkov body of $D$ (with respect to the flag $Y^*$) is the compact convex set 

$$\Delta(D) = \Delta_{Y^*}(D) := \text{closed convex hull} \left( \bigcup_{m \geq 1} \frac{1}{m} \cdot \Gamma(D)_m \right) \subset \mathbb{R}^d.$$

Lazarsfeld and Mustaţă have shown in [LM08, Proposition 4.1] that Okounkov bodies are numerical in nature, i.e. if $D_1$ and $D_2$ are two numerically equivalent big divisors, then $\Delta_{Y^*}(D_1) = \Delta_{Y^*}(D_2)$ for every admissible flag $Y^*$. It is, however, not clear whether one can read off all numerical invariants of a given big divisor from its Okounkov bodies with respect to various flags. In this paper, we give an affirmative answer to this question when $X$ is normal:

**Theorem A.** Let $X$ be a normal complex projective variety of dimension $d$. If $D_1$ and $D_2$ are two big divisors on $X$ such that 

$$\Delta_{Y^*}(D_1) = \Delta_{Y^*}(D_2)$$

for every admissible flag $Y^*$ on $X$, then $D_1$ and $D_2$ are numerically equivalent.

We will derive Theorem A from the following Theorem B, which says that the restricted volume of $D$ to a very general complete-intersection curve can be read off from its Okounkov body:

**Theorem B.** Let $X$ be a normal complex projective variety of dimension $d$. Let $D$ be a big divisor on $X$, and let $A_1, \ldots, A_{d-1}$ be effective very ample divisors on $X$. If the $A_i$’s are very general, and $Y^*$ is an admissible flag such that 

$$Y_r = A_1 \cap \cdots \cap A_r, \quad \forall r \in \{1, \ldots, d-1\},$$

then the Euclidean volume (length) of 

$$\Delta_{Y^*}(D)|_{0}^{d-1} := \{ x \in \mathbb{R} \mid (0, \ldots, 0, x) \in \Delta_{Y^*}(D) \}$$

is equal to the restricted volume of $D$ to the curve $Y_{d-1}$, which is the limit 

$$\text{vol}_{X|Y_{d-1}}(D) = \lim_{m \to \infty} \frac{\dim(H^0(X, mD)|_{Y_{d-1}})}{m}.$$
See Theorem 3.4 for a full statement, including the precise general position condition we need on the very ample divisors $A_i$.

Theorem A will follow from Theorem B because the difference between $Y_{d-1} \cdot D$ and $\text{vol}_{X|Y_{d-1}}(D)$ can also be read off from the Okounkov bodies, and that very general complete-intersection curves are enough to span $N_1(X)$, the dual of the Néron-Severi space $N^1(X)$. Theorem B in turn is proved by introducing certain graded linear series $V_*(D; a)$ on $Y_{d-1}$ and studying its asymptotic behaviors; in particular, we will need a way to compute the volume of $V_*(D; a)$. This is accomplished by generalizing [ELMNP2, Theorem B] which computes the restricted volume by asymptotic intersection number. More precisely, recall from [LM08, Definition 2.5] that a graded linear series $W_\ast$ on a variety $X$ is said to satisfy condition (B) if $W_m \neq 0$ for all $m \gg 0$, and if for all sufficiently large $m$ the rational map $\phi_m: X \dasharrow \mathbb{P}(W_m)$ defined by $|W_m|$ is birational onto its image. Then we have

**Theorem C.** Let $X$ be a projective variety of dimension $d$, and let $W_\ast$ be a graded linear series on $X$ satisfying the condition (B) above. Fix a positive integer $m > 0$ sufficiently large so that the linear series $W_m$ defines a birational mapping of $X$, and denote by $B_m = \text{Bs}(W_m)$ the base locus of $W_m$. We define the moving self-intersection number $(W_m)^{[d]}$ of $W_m$ by choosing $d$ general divisors $D_1, \ldots, D_d \in |W_m|$ and setting

$$(W_m)^{[d]} := \#(D_1 \cap \cdots \cap D_d \cap (X - B_m)).$$

Then the volume of $W_\ast$, which is by definition

$$\text{vol}(W) := \lim_{m \to \infty} \frac{\dim(W_m)}{m^d/d!},$$

can be computed by the following asymptotic intersection number:

$$\text{vol}(W) = \lim_{m \to \infty} \frac{(W_m)^{[d]}}{m^d}. $$

This paper is organized into three sections. In Section 1 we collect some definitions and notations about graded linear series in general, and define the graded linear series $V_*(D; a)$ which will play a key role in the proof of Theorem B. Then we study the properties of $V_*(D; a)$ in Section 2 and use them to prove the theorems in Section 3.

**Acknowledgements.** The author would like to thank Robert Lazarsfeld and Mircea Mustaţă for valuable discussions and suggestions.

1. **Graded linear series**

In this section we first introduce some basic definitions and notations about graded linear series which we will need. Then we define a graded linear series $V_*(D; a)$, which will play a key role in the proof of Theorem B. We refer the readers to [Laz04, §2.4] for more details on graded linear series.
Definition 1.1. Let $X$ be an irreducible variety, and let $L$ be a line bundle on $X$. A graded linear series on $X$ associated to $L$ consists of a collection

$$V_\bullet = \{V_m\}_{m \in \mathbb{N}}$$

of finite dimensional vector subspaces $V_m \subset H^0(X, L^\otimes m)$, satisfying

$$V_k \cdot V_\ell \subset V_{k+\ell} \quad \text{for all } k, \ell \in \mathbb{N},$$

where $V_k \cdot V_\ell$ denotes the image of $V_k \otimes V_\ell$ under the homomorphism

$$H^0(X, L^\otimes k) \otimes H^0(X, L^\otimes \ell) \longrightarrow H^0(X, L^\otimes (k+\ell))$$
determined by multiplication. It is also required that $V_0$ contains all constant functions.

Notation 1.2. Let $X$ be a projective variety and let $L$ be a line bundle on $X$. We write $C_\bullet(X, L)$ to mean the complete graded linear series associated to $L$, namely

$$C_m(X, L) = H^0(X, L^\otimes m) \quad \text{for all } m \in \mathbb{N}.$$ 

If $D$ is a Cartier divisor on $X$, we will also write $C_\bullet(X, D)$ for $C_\bullet(X, \mathcal{O}_X(D))$.

Definition 1.3. Given two graded linear series $V_\bullet$ and $W_\bullet$, we define a morphism $f_\bullet: V_\bullet \to W_\bullet$ of graded linear series to be a collection of linear maps $f_m: V_m \to W_m$, $m \in \mathbb{N}$, such that

$$f_{k+\ell}(s_1 \otimes s_2) = f_k(s_1) \otimes f_\ell(s_2)$$

for all $s_1 \in V_k$, $s_2 \in V_\ell$ and all $k, \ell \in \mathbb{N}$. It is also required that $f_0$ preserves constant functions.

Example 1.4. Let $X$ be a projective variety and let $L$ and $M$ be two line bundles on $X$. If $V_\bullet$ is a graded linear series associated to $L$, and $s \in H^0(X, M)$, then the linear maps

$$V_m \to H^0(X, L^\otimes m \otimes M^\otimes m), \quad s' \mapsto s' \otimes s^\otimes m$$

for all $m \in \mathbb{N}$ form a morphism from $V_\bullet$ to $C_\bullet(X, L \otimes M)$. We will denote this morphism as $\mu(s)_\bullet: V_\bullet \to C_\bullet(X, L \otimes M)$.

Definition 1.5. Let $U_\bullet$, $V_\bullet$, and $W_\bullet$ be graded linear series.

(a) We say that $U_\bullet$ is a subseries of $V_\bullet$, denoted by $U_\bullet \subset V_\bullet$, if $U_m \subset V_m$ for all $m \in \mathbb{N}$.

(b) If $f_\bullet: V_\bullet \to W_\bullet$ is a morphism of graded linear series, then the image of $f_\bullet$, denoted by $\text{Im}(f_\bullet)$, is the subseries of $W_\bullet$ consisting of the images of $f_m$ for all $m \in \mathbb{N}$. If $U_\bullet$ is a subseries of $W_\bullet$, then the preimage of $U_\bullet$ under $f_\bullet$, denoted by $f_\bullet^{-1}(U_\bullet)$, is the subseries of $V_\bullet$ consisting of the preimages of $U_m$ under $f_m$ for all $m \in \mathbb{N}$.

(c) If $V_\bullet$ is a graded linear series on the variety $X$, and $Y$ is a subvariety of $X$, then the restriction of $V_\bullet$ to $Y$, denoted by $V_\bullet|_Y$, is the graded linear series on $Y$ obtained by restricting all of the sections in $V_m$ to $Y$ for all $m \in \mathbb{N}$. 
Definition 1.6. Let $W_*$ be a graded linear series on a variety $X$. The stable base locus of $W_*$, denoted by $B(W_*)$, is the (set-theoretic) intersection of the base loci $Bs(W_m)$ for all $m \geq 1$:

$$B(W_*) := \bigcap_{m \geq 1} Bs(W_m).$$

If $W_* = C_*(X, D)$, the complete graded linear series of a divisor $D$, then we will simply denote its stable base locus by $B(D)$.

It is a simple fact that for any graded linear series $W_*$, $Bs(W_m) = B(W_*)$ for all sufficiently large and divisible $m$ (the proof given in [Laz04, Proposition 2.1.21] for complete graded linear series can be used without change). Hence $B(D)$ makes sense even if $D$ is a $\mathbb{Q}$-divisor.

The stable base locus of a divisor $D$ does not depend only on the numerical equivalence class of $D$, so it is sometimes preferable to work instead with the augmented base locus $B_+(D)$, defined as

$$B_+(D) = B(D - A)$$

for any small ample $\mathbb{Q}$-divisor $A$, this being independent of $A$ provided that it is sufficiently small.

We will also need the following concept of asymptotic order of vanishing which was studied in [ELMNP1]:

**Definition 1.7.** Let $X$ be a projective variety, and let $E$ be a prime Weil divisor not contained in the singular locus of $X$. Given a graded linear series $W_*$ on $X$, we define

$$\text{ord}_E(W_m) := \min \{ \text{ord}_E(s) \mid s \in W_m \}.$$

If $W_m \neq 0$ for all $m \gg 0$, then we can define the asymptotic order of vanishing of $W_*$ along $E$ as

$$\text{ord}_E(W_*) := \lim_{m \to \infty} \frac{\text{ord}_E(W_m)}{m}.$$  

(cf. [ELMNP1] Definition 2.2) When $W_*$ is the complete graded linear series associated to a Cartier divisor $D$, we will simply write $\text{ord}_E(|mD|)$ for $\text{ord}_E(W_m)$, and $\text{ord}_E(||D||)$ for $\text{ord}_E(W_*)$.

Next we want to recall conditions (A)–(C) on graded linear series introduced in [LM08 §2.3], which are mild requirements needed for most major statements in that paper. We have already seen condition (B) in the Introduction.

**Definition 1.8.** Let $W_*$ be a graded linear series on an irreducible variety $X$ of dimension $d$.

(a) We say that $W_*$ satisfies condition (A) with respect to an admissible flag $Y_*$ if there is an integer $b > 0$ such that for every $0 \neq s \in W_m$,

$$\nu_i(s) \leq mb$$
for all $1 \leq i \leq d$.

(b) We say that $W_\bullet$ satisfies condition (B) if $W_m \neq 0$ for all $m \gg 0$, and if for all sufficiently large $m$ the rational map
\[
\phi_m : X \dasharrow \mathbb{P}(W_m)
\]
defined by $|W_m|$ is birational onto its image.

(c) Assume that $X$ is projective, and that $W_\bullet$ is a graded linear series associated to a big divisor $D$. We say that $W_\bullet$ satisfies condition (C) if:

- For every $m \gg 0$ there exists an effective divisor $F_m$ on $X$ such that the divisor $A_m := mD - F_m$ is ample; and
- For all sufficiently large $p$,
\[
H^0(X, \mathcal{O}_X(pmD - pF_m)) \subset W_{pm} \subset H^0(X, \mathcal{O}_X(pmD)),
\]
where the first inclusion is the natural one determined by $pF_m$.

In the proof of [LM08, Lemma 1.10], it was established that condition (A) holds automatically if $X$ is projective. Hence condition (A) is insignificant for us since we will be working with projective varieties. Also note that condition (C) implies condition (B).

We will now define a graded linear series which is particularly relevant to our study of Okounkov bodies. For the basic setting, let $X$ be an irreducible projective variety of dimension $d$, and let $A_1, \ldots, A_{d-1}$ be general effective very ample divisors on $X$. Then by Bertini’s theorem, for each $r \in \{1, \ldots, d-1\}$,
\[
Y_r := A_1 \cap \cdots \cap A_r
\]
is an irreducible subvariety of $X$ which is of codimension $r$ in $X$ and smooth away from the singular locus $X_{\text{sing}}$ of $X$. Hence if we let $Y_0 := X$ and let $Y_d$ be a point in $Y_{d-1} - X_{\text{sing}}$, then
\[
Y_\bullet : Y_0 \supset Y_1 \supset \cdots \supset Y_{d-1} \supset Y_d
\]
is an admissible flag on $X$. We let $\nu_\bullet = (\nu_1, \ldots, \nu_d)$ be the valuation determined by the flag $Y_\bullet$ as described in the Introduction.

**Definition 1.9.** Let $X$ and $Y_\bullet$ be as in the preceding paragraph, and let $D$ be a big Cartier divisor on $X$. Given an $r$-tuple of nonnegative integers $a = (a_1, \ldots, a_r)$ where $r \in \{0, \ldots, d-1\}$, the space of sections
\[
\{ s \in H^0(X, mD) \mid \nu_i(s) \geq ma_i, \ i = 1, \ldots, r \} \subset C_m(X, D), \ m \in \mathbb{N}
\]
form a subseries of $C_\bullet(X, D)$, and one can define a morphism from this subseries to $C_\bullet(Y_r, D - a_1A_1 - \cdots - a_rA_r)$ using an iterated restrictions process similar to the one we saw in the Introduction when we defined the valuation $\nu_\bullet$. We then set
\[
V_\bullet(D; a) = V_\bullet(D; a_1, \ldots, a_r) \subset C_\bullet(Y_r, D - a_1A_1 - \cdots - a_rA_r)\
\]
to be the image of this morphism.

More pedantically, \( V_\bullet(D; a) \) can be defined inductively in the following way. If \( r = 0 \), then we simply set

\[
V_\bullet(D; 0) := C_\bullet(Y_0, D).
\]

Assume that \( r > 0 \) and \( V_\bullet(D; a_1, \ldots, a_{r-1}) \) has been defined. To define \( V_\bullet(D; a_1, \ldots, a_r) \), we first pick a section \( s_r \in H^0(Y_{r-1}, \mathcal{O}_{Y_{r-1}}(A_r)) \) such that \( \text{div}(s_r) = Y_r \), and let

\[
\mu(s_r^{\otimes a_r})_* : C_\bullet(Y_{r-1}, D - a_1A_1 - \cdots - a_rA_r) \to C_\bullet(Y_{r-1}, D - a_1A_1 - \cdots - a_{r-1}A_{r-1})
\]

be the morphism as defined in Example 1.4. Then we define

\[
V_\bullet(D; a_1, \ldots, a_r) := \left( \mu(s_r^{\otimes a_r})_*(-1)(V_\bullet(D; a_1, \ldots, a_{r-1})) \right)_{Y_r}.
\]

It is obvious that the definition does not depend on the choice of \( s_r \). It will be convenient to also define \( V_\bullet(D; a) \) when \( a_1, \ldots, a_r \) are nonnegative rational numbers in the following way: let \( \ell \) be the smallest positive integer such that \( \ell a_i \in \mathbb{N} \) for all \( i \), then set

\[
V_m(D; a_1, \ldots, a_r) := \begin{cases} V_k(\ell D; \ell a_1, \ldots, \ell a_r), & \text{if } m = \ell k \text{ for some } k \in \mathbb{N}; \\ 0, & \text{otherwise}. \end{cases}
\]

Although in general this is not a graded linear series in the sense of Definition 1.9, for all of our purposes we are essentially dealing with the graded linear series \( V_\bullet(\ell D; \ell a) \), so no problem will occur.

2. Properties of \( V_\bullet(D, a) \)

In this section we will first explain the motivation behind the construction of \( V_\bullet(D; a) \) in Definition 1.9. Then we will show that \( V_\bullet(D; a) \) contains the restricted complete linear series \( C_\bullet(X, D - a_1A_1 - \cdots - a_rA_r)|_{Y_r} \). A consequence of this is that \( V_\bullet(D; a) \) satisfies the condition (C) given in [LM08, Definition 2.9] as long as \( |a| := \max\{|a_1|, \ldots, |a_r|\} \) is sufficiently small, which in turn allows us to compute its volume. We will end with giving a lower bound to the base locus of \( V_m(D; a) \) and its order of vanishing there. These ingredients will all go into the proof of Theorem B in Section 3.

As one might already notice, the construction of the graded linear series \( V_\bullet(D; a) \) is closely related to the construction of the Okounkov body of \( D \). Recall from the Introduction that given a \( d \)-dimensional projective variety \( X \) and an admissible flag \( Y_\bullet \) on \( X \), we get a valuation \( \nu_{Y_\bullet} \) which sends a nonzero global section of any line bundle on \( X \) to a \( d \)-tuple of integers. This allows us to define the graded semigroup of a graded linear series \( W_\bullet \) on \( X \) ([LM08, Definition 1.15]):

\[
\Gamma_{Y_\bullet}(W_\bullet) := \{ (\nu_{Y_\bullet}(s), m) \mid 0 \neq s \in W_m, m \geq 0 \} \subset \mathbb{N}^d \times \mathbb{N}.
\]
For any graded semigroup $\Gamma \subset \mathbb{N}^d \times \mathbb{N}$, a closed convex cone $\Sigma(\Gamma) \subset \mathbb{R}^d \times \mathbb{R}$ and a closed convex body $\Delta(\Gamma) \subset \mathbb{R}^d$ can be constructed:

$$\Sigma(\Gamma) := \text{the closed convex cone spanned by } \Gamma;$$

$$\Delta(\Gamma) := \{ x \in \mathbb{R}^d \mid (x, 1) \in \Sigma(\Gamma) \}.$$

Using these notations, the Okounkov body of a big divisor $D$ on $X$ is

$$\Delta(Y \cdot (D)) = \Delta_Y(C_\bullet(X, D)).$$

We will subsequently abbreviate $\Gamma_Y(C_\bullet(X, D))$ as $\Gamma_Y(D)$.

The statement of Theorem B involves the intersection of $\Delta(Y \cdot (D))$ with the last coordinate axis, so it is natural to study the following more general intersections:

**Notation 2.1.** Given a graded semigroup $\Gamma \subset \mathbb{N}^d \times \mathbb{N}$, and an $r$-tuple of nonnegative rational numbers $a = (a_1, \ldots, a_r)$ where $r \leq d$, we denote by $\Gamma|_a \subset \mathbb{N}^{d-r} \times \mathbb{N}$ the graded semigroup

$$\Gamma|_a := \{ (\nu_{r+1}, \ldots, \nu_d, m) \in \mathbb{N}^{d-r} \times \mathbb{N} \mid (a_1 m, \ldots, a_r m, \nu_{r+1}, \ldots, \nu_d, m) \in \Gamma \}.$$

Similarly for a subset $S \subset \mathbb{R}^d$, we denote by $S|_a \subset \mathbb{R}^{d-r}$ the subset

$$S|_a := \{ (\nu_{r+1}, \ldots, \nu_d) \in \mathbb{R}^{d-r} \mid (a_1, \ldots, a_r, \nu_{r+1}, \ldots, \nu_d) \in S \}.$$

**Remark 2.2.** Note that in general we have $\Delta(\Gamma|_a) \subset \Delta(\Gamma)|_a$. If $\Delta(\Gamma)|_a$ meets the interior of $\Delta(\Gamma)$, then $\Delta(\Gamma|_a) = \Delta(\Gamma)|_a$ by [LM08, Proposition A.1].

To compute the Euclidean volume of $\Delta_Y(D)|_{0^{d-1}}$ in Theorem B, our plan is to study the Euclidean volume of $\Delta_Y(D)|_a$ and let $a$ goes to $0^{d-1}$. Since

$$\Delta_Y(D)|_a = \Delta(\Gamma_Y(D)|_a)$$

by Remark 2.2, it is thus desirable to realize the semigroup $\Gamma_Y(D)|_a$ as the semigroup of some graded linear series. This is precisely what motivates the definition of $V_\bullet(D; a)$.

**Lemma 2.3.** Let $X$, $Y_\bullet$, $D$, and $a$ be as in Definition 1.9. Then

$$\Gamma_{Y_\bullet|_{Y_\bullet}}(V_\bullet(D; a)) = \Gamma_{Y_\bullet}(D)|_a,$$

where $Y_\bullet|_{Y_\bullet}$ is the admissible flag $Y_r \supset Y_{r+1} \supset \cdots \supset Y_d$ on $Y_r$.

**Proof.** $V_\bullet(D; a)$ is defined in the way that makes this true. \qed

Another important property of $V_\bullet(D; a)$ is that it satisfies the condition (C) in Definition 1.8 when $|a|$ is sufficiently small.

**Lemma 2.4.** The graded linear series $V_\bullet(D; a_1, \ldots, a_r)$ satisfies

$$C_\bullet(X, D - a_1 A_1 - \cdots - a_r A_r)|_{Y_\bullet} \subset V_\bullet(D; a_1, \ldots, a_r) \subset C_\bullet(Y_\bullet, D - a_1 A_1 - \cdots - a_r A_r).$$
(In the case when \( a_1, \ldots, a_r \) are rational numbers and \( \ell \) is the smallest positive integer such that \( \ell a_i \in \mathbb{N} \) for all \( i \), we interpret \( C_\bullet(X, D - a_1 A_1 - \cdots - a_r A_r) \) in the following manner:

\[
C_m(X, D - a_1 A_1 - \cdots - a_r A_r) := \begin{cases} 
C_k(X, \ell D - \ell a_1 A_1 - \cdots - \ell a_r A_r), & \text{if } m = \ell k; \\
0, & \text{otherwise.}
\end{cases}
\]

And similarly for \( C_\bullet(Y_r, D - a_1 A_1 - \cdots - a_r A_r) \).)

**Proof.** It suffices to prove the case when \( a_1, \ldots, a_r \) are integers. The containment on the right follows from the definition. We show the containment on the left by induction on \( r \). The case \( r = 0 \) is trivial, so we assume that \( r > 0 \) and

\[
C_\bullet(X, D - a_1 A_1 - \cdots - a_{r-1} A_{r-1})|_{Y_{r-1}} \subset V_\bullet(D; a_1, \ldots, a_{r-1}).
\]

Let \( s \) be an arbitrary element in \( C_m(X, D - a_1 A_1 - \cdots - a_r A_r) \), \( m \in \mathbb{N} \). Let \( s_r \in H^0(X, \mathcal{O}_X(A_r)) \) be a section whose divisor is \( A_r \). Then

\[
\mu(s_r^{|\cdot|_{Y_{r-1}}})(s|_{Y_{r-1}}) = (s \otimes s_r^{|\cdot|_{Y_{r-1}}})|_{Y_{r-1}} \in C_m(X, D - a_1 A_1 - \cdots - a_{r-1} A_{r-1})|_{Y_{r-1}} 
\subset V_m(D; a_1, \ldots, a_{r-1}),
\]

hence \( s|_{Y_r} \in V_m(D; a_1, \ldots, a_r) \) by definition. \( \square \)

**Corollary 2.5.** The graded linear series \( V_\bullet(D; a_1, \ldots, a_r) \) on \( Y_r \) satisfies condition (C) if \( Y_r \not\subseteq B_+(D) \) and \(|a| \) is sufficiently small. (Strictly speaking this is an abuse of terminology: as remarked toward the end of Definition 1.9, what actually satisfies condition (C) is \( V_\bullet(\ell D; \ell a) \), but this will not cause any trouble for us.)

**Proof.** This is because under these assumptions the restricted complete linear series contained in \( V_\bullet(D; a) \) already satisfies condition (C) by [LM08, Lemma 2.16]. \( \square \)

Next we want to give a lower bound to the base locus of \( V_m(D; a) \) and its order of vanishing there.

**Lemma 2.6.** Let \( X, Y_r, D, \) and \( a \) be as in Definition 1.9. If \( Y_i \) intersects \( Y_{i-1} \cap B(D) \) properly in \( Y_{i-1} \) for all \( i \in \{1, \ldots, r\} \), then

\[
B(V_\bullet(D; a)) \supset (Y_r \cap B(D)).
\]

Moreover, if \( E \) is a prime Weil divisor of \( X \) contained in \( B(D) \), and if \( F \) is an irreducible component of \( Y_r \cap E \) such that \( F \) is not contained in the singular locus of \( X \), then

\[
\text{ord}_F(V_m(D; a)) \geq \text{ord}_E(|mD|), \quad \forall m \in \mathbb{N}.
\]

**Proof.** We will proceed by induction on \( r \). The case \( r = 0 \) is trivial. Assuming that \( r > 0 \), then by definition any section

\[
s \in V_m(D; a_1, \ldots, a_r) \subset H^0(Y_r, m(D - a_1 A_1 - \cdots - a_r A_r))
\]
is the restriction to $Y_r$ of some section
\[ s' \in H^0(Y_{r-1}, m(D - a_1A_1 - \cdots - a_rA_r)) \]
such that
\[ s' \otimes s_r^{\otimes m} \in V_m(D; a_1, \ldots, a_{r-1}) \subset H^0(Y_{r-1}, m(D - a_1A_1 - \cdots - a_{r-1}A_{r-1})) , \]
where $s_r \in H^0(Y_{r-1}, A_r)$ is a section whose divisor is $Y_r$. By the induction hypothesis, $s' \otimes s_r^{\otimes m}$ vanishes on $Y_{r-1} \cap B(D)$, thus $s'$ vanishes on $Y_{r-1} \cap B(D)$ since $Y_r$ intersects $Y_{r-1} \cap B(D)$ properly. Hence $s = s'|_{Y_r}$ vanishes on $Y_r \cap B(D)$, proving that $Y_r \cap B(D)$ is contained in the stable base locus of $V_*(D; a_1, \ldots, a_r)$. Let $F' \subset Y_{r-1}$ be an irreducible component of $Y_{r-1} \cap E$ containing $F$. Then
\[ \text{ord}_{F'}(s' \otimes s_r^{\otimes m}) \geq \text{ord}_E(|mD|) \]
by the induction hypothesis, hence
\[ \text{ord}_{F'}(s') \geq \text{ord}_E(|mD|) \]
since $Y_r$ does not contain $F'$. Therefore
\[ \text{ord}_F(s) = \text{ord}_F(s'|_{Y_r}) \geq \text{ord}_{F'}(s') \geq \text{ord}_E(|mD|) . \]

\[ \square \]

3. Proof of theorems

We start with the proof of Theorem C:

\textit{Proof of Theorem C.} Since $W_*$ satisfies condition (B), it must belong to a big divisor $L$. Fix a positive integer $m > 0$ sufficiently large so that the linear series $W_m$ defines a birational mapping of $X$. Let $\pi_m : X_m \to X$ be a resolution of the base ideal $b_m := b(W_m)$. Then we have a decomposition
\[ \pi_m^*|W_m| = |M_m| + F_m , \]
where $F_m$ is the fixed divisor (i.e. $b_mO_{X_m} = O_{X_m}(-F_m)$), and
\[ M_m \subset H^0(X_m, \pi_m^*L - F_m) \]
is a base-point-free linear series. Let $M_{m,*}$ be the graded linear series on $X_m$ associated to $\pi_m^*L - F_m$ given by
\[ M_{m,k} := \text{Im} \left( S^k(M_m) \to H^0(X_m, k(\pi_m^*L - F_m)) \right) . \]
By Fujita’s approximation theorem [LM08, Theorem 3.5], for any $\epsilon > 0$, there exists an integer $m_0 = m_0(\epsilon)$ such that if $m \geq m_0$, then
\[ \text{vol}(W_*) - \epsilon \leq \frac{1}{md} \cdot \lim_{k \to \infty} \frac{\dim M_{m,k}}{k^d/d!} \leq \text{vol}(W_*) . \]
Since we assume that \( m \) is sufficiently large so that the morphism defined by the linear series \( M_m \) maps \( X_m \) birationally onto its image \( X'_m \) in some \( \mathbb{P}^N \), we have

\[
\lim_{k \to \infty} \frac{\dim M_{m,k}}{k^d/d!} = (\mathcal{O}_{\mathbb{P}^N}(1)|_{X'_m})^d = (\pi^*_m L - F_m)^d = (W_m)^d.
\]

Hence the desired conclusion follows.

Before we go on to prove Theorem B, recall that in its statement the very ample divisors \( A_i \)'s are required to be very general. The precise requirement we will need is the assumptions on the flag \( Y \) in Lemma 3.2 below.

**Definition 3.1.** Let \( Z \) be a variety, and let \( Y \subset Z \) be a prime divisor. Let \( C \subset Z \) be a closed algebraic subset, and denote by \( C_1, \ldots, C_n \) the irreducible components of \( C \). We say that \( Y \) intersects \( C \) very properly in \( Z \) if for every \( I \subset \{1, \ldots, n\} \) such that \( \bigcap_{i \in I} C_i \neq \emptyset \), \( Y \) does not contain any irreducible component of \( \bigcap_{i \in I} C_i \).

**Lemma 3.2.** Let \( X, Y, D \) be as in Definition 1.9, and assume that \( X \) is normal. Let \( E_1, \ldots, E_n \) be all of the irreducible \((d-1)\)-dimensional components of \( B(D) \). Assume that \( Y_i \) intersects \( Y_{i-1} \cap B(D) \) very properly in \( Y_{i-1} \) for all \( i \in \{1, \ldots, d\} \). Then

\[
B(C_m(X, D)|_{Y_{d-1}}) = (Y_{d-1} \cap B(D)) = \prod_{i=1}^n (Y_{d-1} \cap E_i).
\]

Moreover, let \( m \in \mathbb{N} \) be sufficiently large and divisible so that \( Bs(|mD|) = B(D) \). If, in addition to the above assumptions, the curve \( Y_{d-1} \) intersects each of the \( E_i \)'s transversally at smooth points of \( X \), and none of these intersection points lies in an embedded component of the base scheme of \( |mD| \), then

\[
\text{ord}_p(C_m(X, D)|_{Y_{d-1}}) = \text{ord}_{E_i}(|mD|)
\]

for every \( p \in Y_{d-1} \cap E_i, \ i \in \{1, \ldots, n\} \).

**Proof.** This is obvious.

**Corollary 3.3.** If the very ample divisors \( A_1, \ldots, A_{d-1} \) are very general so that Lemma 3.2 holds for all \( m \) such that \( Bs(|mD|) = B(D) \), then

\[
\text{vol}_{X|Y_{d-1}}(D) = Y_{d-1} \cdot D - \sum_{i=1}^n \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_{E_i} \|D\|.
\]
Proof. Let $m$ be sufficiently large and divisible so that $\text{Bs}(|mD|) = B(D)$. By Theorem C and Lemma 3.2,

$$\frac{\text{vol}_{X|Y_{d-1}}(D)}{Y_{d-1} \cdot (mD) - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_p(C_m(X, D)|_{Y_{d-1}}) m} = Y_{d-1} \cdot D - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \lim_{m \to \infty} \text{ord}_{E_i}(|mD|) m = Y_{d-1} \cdot D - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_{E_i} D.$$ 

We will now prove Theorem B by proving the following more precise statement:

**Theorem 3.4.** Let $X$, $Y_\bullet$, and $D$ be as in Definition 1.9. Assume that $Y_r \not\subseteq B_+(D)$ for all $r \in \{0, \ldots, d-1\}$.

(a) If $D$ is ample, then for any $r \in \{0, \ldots, d-1\},$

$$\text{vol}_{R^{d-r}}(\Delta_{Y_\bullet}(D)|_{0^r}) = \frac{\text{vol}_{X|Y_r}(D)}{(d-r)!} = \frac{Y_r \cdot D^{d-r}}{(d-r)!},$$

where $0^r$ denotes $(0, \ldots, 0)$.

(b) If $D$ is big but not necessarily ample, and assume that $X$ is normal and that the very ample divisors $A_1, \ldots, A_{d-1}$ are very general so that Lemma 3.2 and Corollary 2.3 hold, then the first equality in (a) still holds when $r = d-1$, i.e.

$$\text{vol}_{R^{1}}(\Delta_{Y_\bullet}(D)|_{0^{d-1}}) = \frac{\text{vol}_{X|Y_{d-1}}(D)}{(d-r)!}.$$ 

Proof. Since $Y_{d-1} \not\subseteq B_+(D)$, we have $\text{vol}_{X|Y_{d-1}}(D) > 0$. Hence

$$\text{vol}_{R^{1}}(\Delta_{Y_\bullet}(D)|_{0^{d-1}}) = \text{vol}_{R^{1}}(\Delta(\Gamma_{Y_r}(D))|_{0^{d-1}}) \geq \text{vol}_{R^{1}}(\Delta(\Gamma_{Y_r}(D)|_{0^{d-1}})) = \frac{\text{vol}_{X|Y_{d-1}}(D) > 0}. $$

This implies we can find $a \in \mathbb{Q}_{+}^r$ with arbitrarily small norm $|a|$ such that $\Delta_{Y_\bullet}(D)|_{a}$ meets the interior of $\Delta_{Y_r}(D)$. Hence by [LM08, Proposition A.1],

$$\Delta_{Y_\bullet}(D)|_{a} = \Delta(\Gamma_{Y_r}(D))|_{a} = \Delta(\Gamma_{Y_r}(V_r(D; a))) = \Delta(\Gamma_{Y_r}|_{Y_r}(V_r(D; a)))$$,

where the last equality follows from Lemma 2.3. By Corollary 2.3 $V_r(D; a)$ satisfies condition (C) as long as $|a|$ is sufficiently small, hence we can calculate the Euclidean volume of $\Delta(\Gamma_{Y_r}|_{Y_r}(V_r(D; a))) = \Delta_{Y_r}(V_r(D; a))$ by [LM08, Theorem 2.13]:

$$\text{vol}_{R^{d-r}}(\Delta_{Y_r}|_{Y_r}(V_r(D; a))) = \frac{\text{vol}(V_r(D; a))}{(d-r)!}. $$
Combining the above equalities gives us

\[(*) \quad \text{vol}_{R^{d-r}}(\Delta_{Y_\bullet}(D)|_a) = \frac{\text{vol}(V_\bullet(D; a))}{(d-r)!}.\]

If \( D \) is ample, then \( D - \sum_{i=1}^{r} a_i A_i \) is also ample as long as \( |a| \) is sufficiently small, and hence

\[C_m(X, D - \sum_{i=1}^{r} a_i A_i) \big|_{V_r} = C_m(Y_r, D - \sum_{i=1}^{r} a_i A_i)\]

for all sufficiently large \( m \). So by Lemma [2.4],

\[\text{vol}(V_\bullet(D; a)) = \text{vol}(C_\bullet(Y_r, D - \sum_{i=1}^{r} a_i A_i)) = Y_r \cdot (D - \sum_{i=1}^{r} a_i A_i)^{d-r}.\]

Substituting this back into \((*)\), we get

\[\text{vol}_{R^{d-r}}(\Delta_{Y_\bullet}(D)|_a) = \frac{Y_r \cdot (D - \sum_{i=1}^{r} a_i A_i)^{d-r}}{(d-r)!},\]

and letting \( a \) goes to \( 0^r \) proves part (a).

Now suppose that \( D \) is big but not necessarily ample, and \( r = d - 1 \). Then as long as \( |a| \) is sufficiently small, \( D - \sum_{i=1}^{d-1} a_i A_i \) is also big, and \( V_\bullet(D; a) \) satisfies condition (C) by Corollary [2.5]. Let \( E_1, \ldots, E_n \) be all of the irreducible \((d-1)\)-dimensional components of \( B(D) \). By Theorem C, Lemma [2.6] and Lemma [3.2].

\[
\text{vol}(V_\bullet(D; a)) = \lim_{m \to \infty} \frac{1}{m} \left( Y_{d-1} \cdot m(D - \sum_{i=1}^{d-1} a_i A_i) - \sum_{p \in Bs(V_m(D; a))} \text{ord}_p(V_m(D; a)) \right)
\]

\[
\leq \lim_{m \to \infty} \frac{1}{m} \left( Y_{d-1} \cdot m(D - \sum_{i=1}^{d-1} a_i A_i) - \sum_{p \in Y_{d-1} \cap B(D)} \text{ord}_p(V_m(D; a)) \right)
\]

\[
= \lim_{m \to \infty} \frac{1}{m} \left( Y_{d-1} \cdot m(D - \sum_{i=1}^{d-1} a_i A_i) - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_p(V_m(D; a)) \right)
\]

\[
\leq \lim_{m \to \infty} \frac{1}{m} \left( Y_{d-1} \cdot m(D - \sum_{i=1}^{d-1} a_i A_i) - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_{E_i}(|mD|) \right)
\]

\[
= Y_{d-1} \cdot (D - \sum_{i=1}^{d-1} a_i A_i) - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_{E_i}(|D|).
\]
Substituting this back into (3), we get
\[ \text{vol}_{R^1}(\Delta_{Y^*}(D)|_a) \leq Y_{d-1} \cdot (D - \sum_{i=1}^{d-1} a_i A_i) - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_{E_i}(\|D\|), \]
and letting \( a \) goes to \( 0^{d-1} \) gives
\[ \text{vol}_{R^1}(\Delta_{Y^*}(D)|_{0^{d-1}}) \leq Y_{d-1} \cdot D - \sum_{i=1}^{n} \sum_{p \in Y_{d-1} \cap E_i} \text{ord}_{E_i}(\|D\|) = \text{vol}_{X|Y_{d-1}}(D) \]
by Corollary 3.3. Since we saw that \( \text{vol}_{R^1}(\Delta_{Y^*}(D)|_{0^{d-1}}) \geq \text{vol}_{X|Y_{d-1}}(D) \) in the beginning of the proof, part (b) is thus established. \( \Box \)

Finally to prove Theorem A, we need the following lemma:

**Lemma 3.5.** Let \( X \) be a smooth projective variety of dimension \( d \), and let \( Y \subset X \) be a transversal complete intersection of \( (d - 2) \) very ample divisors. If \( D_1, \ldots, D_\rho \) are ample divisors on \( X \) whose numerical classes form a basis of \( N^1(X)_{\mathbb{Q}} \), then the curve classes
\[ \{ C_i := Y \cdot D_i \mid i = 1, \ldots, \rho \} \]
form a basis of \( N_1(X)_{\mathbb{Q}} \).

**Proof.** By the Lefschetz hyperplane theorem, the numerical classes of \( D_1|_Y, \ldots, D_\rho|_Y \) are linearly independent in \( N^1(Y)_{\mathbb{Q}} \), hence by the Hodge index theorem for surfaces, the intersection matrix
\[ (D_i|_Y \cdot D_j|_Y) = (C_i \cdot D_j) \]
is nondegenerate. \( \Box \)

**Proof of Theorem A.** We first prove the theorem assuming that \( X \) is smooth. We start with an observation that for any admissible flag \( Y^* \) on \( X \) and any big divisor \( D \) on \( X \), the asymptotic order of vanishing \( \text{ord}_{Y^*}(\|D\|) \) equals the minimum of the projection of \( \Delta_{Y^*}(D) \) to the first coordinate axis. From this we see that for every prime divisor \( E \) on \( X \),
\[ \text{ord}_E(\|D_1\|) = \text{ord}_E(\|D_2\|), \]
since we can always extend \( E \) into an admissible flag. By Lemma 3.5 we can choose \( \rho \) admissible flags \( Y^*_1, \ldots, Y^*_\rho \), such that each one is sufficiently general to make Corollary 3.3 and Theorem 3.4 (b) hold, and the numerical classes of the curves \( Y^*_{d-1}, \ldots, Y^*_d \) form a basis of \( N_1(X)_{\mathbb{Q}} \). It then follows that \( Y^*_i \cdot D_1 = Y^*_d \cdot D_2 \) for all \( i \in \{1, \ldots, \rho\} \), hence \( D_1 \) and \( D_2 \) are numerically equivalent.

When \( X \) is just normal but not smooth, we let \( \pi: X' \to X \) be a resolution of singularities which is an isomorphism over \( X - X_{\text{sing}} \). Then
\[ H^0(X, D) = H^0(X', \pi^*D) \]
for any big divisor $D$ on $X$, and therefore if $Y'$ is an admissible flag on $X'$ such that $\pi(Y'_d) \notin X_{\text{sing}}$, then

$$\Delta_{Y'}(\pi^*D) = \Delta_{\pi(Y')} (D).$$

It thus follows from the previous paragraph on the smooth case that $\pi^*D_1$ and $\pi^*D_2$ are numerically equivalent, and hence so are $D_1$ and $D_2$. \hfill \Box

References

[ELMNP1] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, *Asymptotic invariants of base loci*, Ann. Inst. Fourier 56, 6 (2006), 1701–1734.

[ELMNP2] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, *Restricted volumes and base loci of linear series*, Amer. J. Math. 131 (2009), no. 3, 607–651.

[KK08] Kiumars Kaveh and Askold Khovanskii, *Convex bodies and algebraic equations on affine varieties*, preprint, arXiv:0804.4095

[KK09] Kiumars Kaveh and Askold Khovanskii, *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, preprint, arXiv:0904.3350

[Laz04] Robert Lazarsfeld, *Positivity in Algebraic Geometry I–II*, Ergeb. Math. Grenzgeb., vols. 48–49, Berlin: Springer, 2004.

[LM08] Robert Lazarsfeld and Mircea Mustaţă, *Convex bodies associated to linear series*, to appear in Ann. Sci. École Norm. Sup., arXiv:0805.4559

[Oko96] Andrei Okounkov, *Brunn-Minkowski inequality for multiplicities*, Invent. Math. 125 (1996), 405–411.

[Oko03] Andrei Okounkov, *Why would multiplicities be log-concave?*, The orbit method in geometry and physics, Progress in Mathematics vol. 213, Boston, MA: Birkhäuser Boston, 2003, pp. 329–347.