Scattering of a particle on the $q$-deformed Euclidean space

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Abstract

We develop a formalism for the scattering of a particle on the $q$-deformed Euclidean space. We write down $q$-versions of the Lippmann-Schwinger equation. Their iterative solutions for a weak scattering potential lead us to $q$-versions of the Born series. With the expressions for the wave functions of the scattered particle, we can write down S-matrix elements. We show that these S-matrix elements satisfy unitarity conditions. Considerations about the interaction picture for a quantum system in the $q$-deformed Euclidean space and a discussion of a $q$-version of time-dependent perturbation theory conclude our studies.

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1 Introduction

The Euclidean space and the Minkowski space show special symmetries, also known as rotational symmetry, translational symmetry, or Lorentz symmetry. These space-time symmetries are mathematically described by Lie groups. Thus, the Euclidean space or the Minkowski space is a representation of a Lie group.

In the 1980s, it became possible to deform Lie groups and matrix groups together with their representation spaces within the category of Hopf algebras [1–5]. This insight led to the construction of $q$-deformed quantum groups and quantum spaces. The latter include the $q$-Euclidean space and the $q$-Minkowski space [6,7]. These $q$-deformed quantum spaces are noncommutative algebras, i.e., their coordinate generators no longer commute. Two non-commuting coordinates cannot be measured simultaneously with infinite precision because the measurement of one coordinate interferes with the measurement results for the other coordinates. Therefore, the accuracy of a position measurement should be limited by a "smallest length" [8–10] if space-time is described by a $q$-deformed quantum space.

It has been argued that the problem with ultraviolet divergences in quantum field theories does not arise in a discrete space-time since its smallest length should constrain the values of momentum [11]. With this in mind, it is appealing to formulate a quantum theory or quantum field theory on a $q$-deformed space. In Ref. [12] and Ref. [13], we have discussed $q$-versions of the Schrödinger equation for a free nonrelativistic particle and $q$-versions of the free Klein-Gordon equation on $q$-Euclidean space. We have solved these $q$-deformed wave equations by using $q$-deformed momentum eigenfunctions [14]. In this paper, we will study the scattering of a particle on $q$-deformed Euclidean space. Our considerations are analogous to the undeformed case [15] since the time evolution operator in $q$-deformed Euclidean space has the same structure as in the undeformed case [16].

Using the so-called star product formalism [17–20], we can develop a multivariable calculus on $q$-deformed Euclidean space [21,24]. In the appendix, we have summarized the basics of this $q$-deformed multivariable calculus as far as it is necessary for our considerations. Moreover, we need to know the solutions and the propagators for the wave equations describing a free particle in $q$-deformed Euclidean space. We have summarized these results from Ref. [12] and Ref. [13] in Chap. 2 of the present paper.

Using the propagators for the $q$-deformed wave equations of a free particle, we can write down $q$-versions of the Lippmann-Schwinger equation. Solving
these \(q\)-deformed Lippmann-Schwinger equations by iteration, we obtain perturbation expansions for the wave functions of a scattered particle, i.e., \(q\)-versions of the famous Born series (see Chap. 3). With the help of these expansions, we write down S-matrix elements for the scattering of a \(q\)-deformed particle (see Chap. 4.1, Chap. 4.2, and Chap. 4.4). These S-matrix elements again satisfy unitarity conditions. We show this in Chap. 4.3. Finally, we discuss how to formulate our considerations with the help of the interaction picture (see Chap. 5) or in the framework of time-dependent perturbation theory (see Chap. 6).

2 Propagators for a free particle

In this chapter, we summarize some results from Ref. [12] or Ref. [13]. As Hamiltonian operator for a free nonrelativistic particle with mass \(m\), we choose the following expression:

\[
H_0 = -\frac{1}{2} g_{AB} \partial^A \partial^B = -\frac{1}{2} \partial^A \partial_A.
\]

(1)

Since we have different actions of \(q\)-deformed partial derivatives [cf. Eq. (247) in App. B], we can write down the following \(q\)-versions of the Schrödinger equation for a free nonrelativistic particle [12, 16]:

\[
i \partial_t \triangleright \phi_R(x,t) = H_0 \triangleright \phi_R(x,t), \quad \phi_L(x,t) \triangleright \partial_t i = \phi_L(x,t) \triangleright H_0, \\
i \partial_t \triangleleft \phi^*_L(x,t) = H_0 \triangleleft \phi^*_L(x,t), \quad \phi^*_R(x,t) \triangleright \partial_t i = \phi^*_R(x,t) \triangleright H_0.
\]

(2)

We require that the solutions to the above Schrödinger equations behave as follows under quantum space conjugation:

\[
\phi_L(x,t) = \phi_R(x,t), \quad \phi^*_L(x,t) = \phi^*_R(x,t).
\]

(3)

This condition ensures that quantum space conjugation transforms the Schrödinger equations on the left side of Eq. (2) into the Schrödinger equations on the right side of Eq. (2) and vice versa.

For plane wave solutions, the Schrödinger equations above become

\[
i \partial_t \triangleright u_p(x,t) = H_0 \triangleright u_p(x,t) = -\frac{1}{2} \partial^A \partial_A \triangleright u_p(x,t)
= u_p(x,t) \oplus p^2 (2m)^{-1},
\]

(4)

\[
(u^*)_p(x,t) \triangleright \partial_t i = (u^*)_p(x,t) \triangleright H_0 = -\frac{1}{2} \partial^A \partial_A (2m)^{-1}
= (2m)^{-1} p^2 \oplus (u^*)_p.
\]

(5)

and

\[
u^p(x,t) \triangleright \partial_i i = v^p(x,t) \triangleright H_0 = -\frac{1}{2} v^p(x,t) \triangleright \partial^A \partial_A (2m)^{-1}
= (2m)^{-1} p^2 \oplus v^p(x,t),
\]

(6)

\[
i \partial_t \triangleright (u^*)_p(x,t) = H_0 \triangleright (u^*)_p(x,t) = -\frac{1}{2} \partial^A \partial_A \triangleright u_p(x,t)
= (u^*)_p(x,t) \oplus p^2 (2m)^{-1}.
\]

(7)
These plane wave solutions depend on $q$-deformed exponentials \[12\], i.e.

$$u_p(x,t) = \text{vol}^{-1/2} \exp_q(x|ip) \oplus e^{-it\varepsilon_p},$$

$$u^p(x,t) = e^{it\varepsilon_p} \exp_q(i^{-1}p|x) \text{vol}^{-1/2}, \quad (8)$$

and

$$(u^*)_p(x,t) = \text{vol}^{-1/2} e^{it\varepsilon_p} \oplus \exp'_q(ip|x),$$

$$(u^*)_p(x,t) = \text{vol}^{-1/2} \exp'_q(x|i^{-1}p) \oplus e^{-it\varepsilon_p}. \quad (9)$$

The time-dependent phase factor and the volume element are given by

$$e^{\pm it\varepsilon_p} = \exp(\pm it p^2(2m)^{-1}) = \sum_{k=0}^{\infty} \frac{1}{k!} [\pm it(2m)^{-1}]^k p^{2k} \quad (10)$$

and

$$\text{vol} = \int d_3q \int d_3x \exp_q(i^{-1}p|x). \quad (11)$$

Note that powers of $p^2(= g_{AB} p_A \oplus p_B)$ are calculated by using the star-product (cf. App. A):

$$p^{2k} = p^2 \oplus \ldots \oplus p^2 = \sum_{l=0}^{k} q^{-2l} (-\lambda_+)^{k-l} \binom{k}{l}_q (p_-)^{k-l} (p_3)^{2l} (p_+)^{k-l}. \quad (12)$$

The $q$-deformed binomial coefficients are defined in complete analogy to the undeformed case [also see Eq. (230) in App. A]:

$$\binom{n}{k}_q = \frac{[[n]]_q!}{[[n-k]]_q![[k]]_q!}. \quad (13)$$

The $q$-deformed plane wave solutions transform under quantum space conjugation as follows:

$$u^p(x,t) = u_p(x,t), \quad (u^*)^p(x,t) = (u^*)_p(x,t). \quad (14)$$

Moreover, they satisfy the orthonormality relations

$$\int d^3_q x (u^*)_p(x,t) \oplus u^p(x,t) = \text{vol}^{-1} \delta^3_q((\ominus \kappa^{-1} p) \oplus p'),$$

$$\int d^3_q x u^p(x,t) \oplus (u^*)_p(x,t) = \text{vol}^{-1} \delta^3_q(p \oplus (\ominus \kappa^{-1} p')), \quad (15)$$

and the completeness relations

$$\int d^3_q p u_p(x,t) \oplus (u^*_p)(y,t) = \text{vol}^{-1} \delta^3_q(x \oplus (\ominus \kappa^{-1} y)),$$

$$\int d^3_q p (u^*_p)^p(y,t) \oplus u^p(x,t) = \text{vol}^{-1} \delta^3_q((\ominus \kappa^{-1} y) \oplus x). \quad (16)$$
Note that the $q$-deformed delta function is defined as follows:

$$\delta^q_{\theta}(z) = \int d^3q q^{\theta} \exp(\frac{i}{q} |z|).$$  \hspace{1cm} (17)

The general solutions to the $q$-deformed Schrödinger equations for a free non-relativistic particle can be written as expansions in terms of $q$-deformed plane waves:

$$\phi_R(x, t) = \int d^3p u_p(x, t) \otimes c_p, \quad \phi^*_L(x, t) = \int d^3p (c^*)_p \otimes (u^*)_p(x, t),$$  

$$\phi_L(x, t) = \int d^3p c_p \otimes u^p(x, t), \quad \phi^*_R(x, t) = \int d^3p (u^*)_p(x, t) \otimes (c^*)_p. \hspace{1cm} (18)$$

With the $q$-deformed plane wave solutions, we can write down $q$-versions of the propagator for a free nonrelativistic particle \cite{12}. To this end, we need the expressions

$$K_R(x', t'; x, t) = \int d^3p u_p(x', t') \otimes (u^*)_p(x, t),$$  

$$K_L(x, t; x', t') = \int d^3p (u^*)_p(x, t) \otimes u^p(x', t'), \hspace{1cm} (19)$$

and

$$K^*_R(x', t'; x, t) = \int d^3p (u^*)_p(x', t') \otimes u^p(x, t),$$  

$$K^*_L(x, t; x', t') = \int d^3p u_p(x, t) \otimes (u^*)_p(x', t'). \hspace{1cm} (20)$$

The retarded propagators take on the following form:

$$(K_R)^+(x', t'; x, t) = \theta(t' - t) K_R(x', t'; x, t),$$

$$(K_L^+)^+(x, t; x', t') = \theta(t' - t) K_L^+(x, t; x', t'). \hspace{1cm} (21)$$

$$(K^*_R)^+(x', t'; x, t) = \theta(t' - t) K^*_R(x', t'; x, t),$$

$$(K^*_L)^+(x, t; x', t') = \theta(t' - t) K^*_L^+(x, t; x', t'). \hspace{1cm} (22)$$

Note that $\theta(t)$ stands for the Heaviside function:

$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{otherwise}. \end{cases} \hspace{1cm} (23)$$

We can also write down advanced propagators describing the propagation of a wave function into the past:

$$(K_R^-)^-(x', t'; x, t) = \theta(t - t') K_R(x', t'; x, t),$$

$$(K_L^-)^-(x, t; x', t') = \theta(t - t') K_L^-(x, t; x', t'). \hspace{1cm} (24)$$

$$(K^*_R^-)^-(x', t'; x, t) = \theta(t - t') K^*_R(x', t'; x, t),$$

$$(K^*_L^-)^-(x, t; x', t') = \theta(t - t') K^*_L(x, t; x', t'). \hspace{1cm} (25)$$
The conjugation properties of the $q$-deformed plane waves [cf. Eq. (14)] and the expressions in Eqs. (19) and (20) give

$$
(K_R)^{\pm}(x', t'; x, t) = (K_L)^{\pm}(x, t; x', t'),
$$
$$
(K_R^*)^{\pm}(x, t; x', t') = (K_L^*)^{\pm}(x', t'; x, t).
$$

(26)

The $q$-deformed propagators for a free nonrelativistic particle satisfy the equations [12]

$$(i\partial_t - H_0^0) \triangleright (K_R)^{\pm}(x', t'; x, t) = \pm i \text{vol}^{-1} \delta(t' - t) \delta^0_q(x' \oplus (\ominus \kappa^{-1}x)),
$$

$$(K_L)^{\pm}(x, t; x', t') \triangleright (\partial_t i - H_0^0) = \mp i \text{vol}^{-1} \delta(t - t') \delta^0_q((\ominus \kappa^{-1}x) \oplus x'),
$$

(27)

or

$$(i\partial_t - H_0^0) \triangleright (\psi_R)^{\pm}(x, t) = \pm i \text{vol}^{-1} \delta(t' - t) \delta^0_q((\ominus \kappa^{-1}x') \oplus x),
$$

$$(K_L^*)^{\pm}(x, t; x', t') \triangleright (\partial_t i - H_0^0) = \mp i \text{vol}^{-1} \delta(t - t') \delta^0_q(x \oplus (\ominus \kappa^{-1}x')).
$$

(28)

These equations enable us to obtain solutions to the following inhomogeneous wave equations:

$$(i\partial_t - H_0^0) \triangleright (\psi_R)^{\pm}(x, t) = \varrho(x, t),
$$

$$(\psi_L)^{\pm}(x, t) \triangleright (\partial_t i - H_0^0) = \varrho(x, t),
$$

(29)

$$(i\partial_t - H_0^0) \triangleright (\psi_R^*)^{\pm}(x, t) = \varrho(x, t),
$$

$$(\psi_L^*)^{\pm}(x, t) \triangleright (\partial_t i - H_0^0) = \varrho(x, t).
$$

(30)

The solutions to the equations above can be written in the following form [12]:

$$(\psi_R)^{\pm}(x, t) = \mp i \int dt' \int d^3x' (K_R)^{\pm}(x, t; x', t') \varrho(x', t'),
$$

$$(\psi_L)^{\pm}(x, t) = \pm i \int dt' \int d^3x' \varrho(x', t') \oplus (K_L)^{\pm}(x, t'; x, t),
$$

(31)

$$(\psi_R^*)^{\pm}(x, t) = \mp i \int dt' \int d^3x' (K_R^*)^{\pm}(x, t; x', t') \varrho(x', t'),
$$

$$(\psi_L^*)^{\pm}(x, t) = \pm i \int dt' \int d^3x' \varrho(x', t') \oplus (K_L^*)^{\pm}(x', t'; x, t).
$$

(32)

Alternatively, we can consider a $q$-version of the Klein-Gordon equation in dealing with a free particle on the $q$-deformed Euclidean space [13]:

$$
e^{-2} \partial_t^2 \triangleright \varphi_R - \nabla^2_q \triangleright \varphi_R + (mc)^2 \varphi_R = 0.
$$

(33)

Note that the $q$-deformed Laplace operator $\nabla^2_q$ depends on the metric of the three-dimensional $q$-deformed Euclidean space [see Eq. (215) in App. A]:

$$
\nabla^2_q = \partial^A \partial_A = g^{AB} \partial_B \partial_A.
$$

(34)
By conjugating Eq. (33), we obtain another $q$-version of the Klein-Gordon equation:

$$\varphi_L \triangleleft \partial_t^2 c^{-2} - \varphi_L \triangleright \nabla^2_q + \varphi_L (mc)^2 = 0.$$  \hspace{2cm} (35)

Accordingly, the wave function $\varphi_R$ transforms into $\varphi_L$ by conjugation:

$$\varphi_R = \varphi_L.$$  \hspace{2cm} (36)

There are two types of left-actions and two types of right-actions for $q$-deformed partial derivatives [see Eq. (241) and Eq. (244) in App. B]. Thus, we get further $q$-versions of the Klein-Gordon equation by applying the following substitutions in Eq. (33) or Eq. (35):

$$\triangleleft \leftrightarrow \triangleright, \quad \triangleright \leftrightarrow \triangleleft, \quad \varphi_R \leftrightarrow \varphi_R^*, \quad \varphi_L \leftrightarrow \varphi_L^*.$$  \hspace{2cm} (37)

This way, we have

$$c^{-2} \partial_t^2 \varphi_R^* - \nabla^2_q \varphi_R^* + (mc)^2 \varphi_R = 0,$$

$$\varphi_L \triangleright \partial_t^2 c^{-2} - \varphi_L \triangleright \nabla^2_q + \varphi_L^* (mc)^2 = 0,$$  \hspace{2cm} (38)

where $\varphi_R^*$ transforms into $\varphi_L^*$ by conjugation:

$$\varphi_R^* = \varphi_L^*.$$  \hspace{2cm} (39)

There are plane wave solutions to our different $q$-versions of the Klein-Gordon equation. For example, the plane waves

$$\varphi_p(x, t) = \frac{c}{\sqrt{2}} u_p(x) \ast \exp(-itE_p) \ast E_p^{-1/2},$$

$$\varphi^P(x, t) = \frac{c}{\sqrt{2}} E_p^{-1/2} \ast \exp(itE_p) \ast u^P(x)$$  \hspace{2cm} (40)

are subject to

$$c^{-2} \partial_t^2 \varphi_p = \nabla^2_q \varphi_p - (mc)^2 \varphi_p,$$

$$\varphi^P \triangleright \partial_t^2 c^{-2} = \varphi^P \triangleright \nabla^2_q - \varphi^P (mc)^2.$$  \hspace{2cm} (41)

The time-dependent phase factors are given by

$$\exp(\pm itE_p) = \sum_{n=0}^{\infty} \frac{(\pm itE_p)^n}{n!}.$$  \hspace{2cm} (42)

Note that we have to calculate the powers of the energy of a $q$-deformed scalar particle by the following formula [also see Eq. (12)]:

$$E_p^\alpha = c^\alpha (p^2 + (mc)^2)^{\alpha/2} = c^\alpha \sum_{k=0}^{\infty} \binom{\alpha/2}{k} p^{2k}(mc)^{\alpha-2k}.$$  \hspace{2cm} (43)
Moreover, we have dual plane waves:

\[
(\varphi^*)_p(x, t) = \frac{c}{\sqrt{2}} E_p^{-1/2} \oplus \exp(i t E_p) \oplus (u^*)_p(x),
\]

\[
(\varphi^*)_p(x, t) = \frac{c}{\sqrt{2}} (u^*)_p(x) \oplus \exp(-i t E_p) \oplus E_p^{-1/2}.
\]  

(44)

They are solutions to the \(q\)-deformed Klein-Gordon equations given in Eq. \(38\):

\[
(\varphi^*)_p \circ \partial_t^2 e^{-2} - (\varphi^*)_p \circ \nabla_q^2 + (\varphi^*)_p (mc)^2 = 0,
\]

\[
c^{-2} \partial_t^2 \circ (\varphi^*)_p \circ \nabla_q^2 \circ (\varphi^*)_p + (mc)^2(\varphi^*)_p = 0.
\]  

(45)

We can write the general solutions to the \(q\)-deformed Klein-Gordon equations as expansions in terms of plane wave solutions, i.e.

\[
\varphi_R(x, t) = (\varphi_R)^[+] + (\varphi_R)^[-] = \sum_{\varepsilon = \pm} \int d^3 q \varphi_p(\varepsilon x, \varepsilon t) \oplus f^{|\varepsilon|}_p,
\]

\[
\varphi_L(x, t) = (\varphi_L)^[+] + (\varphi_L)^[-] = \sum_{\varepsilon = \pm} \int d^3 q f^{|\varepsilon|}_p \oplus \varphi_p(\varepsilon x, \varepsilon t),
\]  

(46)

and

\[
\varphi^*_L(x, t) = (\varphi^*_L)^[+] + (\varphi^*_L)^[-] = \sum_{\varepsilon = \pm} \int d^3 q h^{|\varepsilon|}_p \oplus (\varphi^*)_p(\varepsilon x, \varepsilon t),
\]

\[
\varphi^*_R(x, t) = (\varphi^*_R)^[+] + (\varphi^*_R)^[-] = \sum_{\varepsilon = \pm} \int d^3 q (\varphi^*)_p(\varepsilon x, \varepsilon t) \oplus h^{|\varepsilon|}_p,
\]  

(47)

The plane waves solutions to our \(q\)-deformed Klein-Gordon equations satisfy the orthogonality relations \(13\):

\[
\int c^{-2} d^3 q \varphi_p(x, \varepsilon t) \circ \partial_t \oplus \varphi_p'(x, \varepsilon' t)
+ ic^{-2} \int d^3 q \varphi_p(x, \varepsilon t) \circ \partial_t \circ \varphi_p'(x, \varepsilon' t)
= \varepsilon \delta_{x, \varepsilon'} \text{vol}^{-1} \delta_q(\bigcirc \kappa^{-1} \mathbf{p} \oplus \mathbf{p}')
\]  

(48)

and

\[
\int c^{-2} d^3 q \varphi^p(x, \varepsilon t) \circ \partial_t \text{vol}^{-1} \delta_q(\bigcirc \kappa^{-1} \mathbf{p}' \oplus \bigcirc \kappa^{-1} \mathbf{p}').
\]  

(49)

We need to formulate the propagators for the \(q\)-deformed Klein-Gordon equations so that the positive energy solution runs forward in time while the negative
energy solution runs backward in time. Propagators with these properties are given by

\[ \Delta_R(x', t'; x, t) = \theta(t' - t) \int d^3 p \, \varphi_p(x', t') \otimes (\varphi^*)_p(x, t) + \theta(t - t') \int d^3 p \, \varphi_p(x', -t') \otimes (\varphi^*)_p(x, -t) \quad (50) \]

and

\[ \Delta_L(x, t; x', t') = \theta(t' - t) \int d^3 p \, (\varphi^*)_p(x, t) \otimes \varphi^P(x', t') + \theta(t - t') \int d^3 p \, (\varphi^*)_p(x, -t) \otimes \varphi^P(x', -t') \quad (51) \]

We can also introduce so-called dual propagators, i.e.

\[ \Delta_R^*(x', t'; x, t) = \theta(t' - t) \int d^3 p \, (\varphi^*_p(x, t) \otimes \varphi^P(x', t')) + \theta(t - t') \int d^3 p \, (\varphi^*_p(x, -t) \otimes \varphi^P(x', -t')) \quad (52) \]

and

\[ \Delta_L^*(x, t; x', t') = \theta(t' - t) \int d^3 p \, \varphi_p(x, t) \otimes (\varphi^*_p(x', t')) + \theta(t - t') \int d^3 p \, \varphi_p(x, -t) \otimes (\varphi^*_p(x', -t')) \quad (53) \]

The propagators for the \(q\)-deformed Klein-Gordon equations transform into each other by conjugation:

\[ \Delta_R(x', t'; x, t) = \Delta_L(x, t; x', t'), \]
\[ \Delta_L^*(x', t'; x, t) = \Delta_R^*(x, t; x', t'). \quad (54) \]

The propagators for the \(q\)-deformed Klein-Gordon equations satisfy the equations

\[ ((mc)^2 + c^{-2} \partial_t^2 - \nabla^2_q) \triangleright \Delta_R(x, t; x', t') = -i \text{ vol}^{-1} \delta(t - t') \delta^3_q(x \oplus (\ominus \kappa^{-1}x')), \quad (55) \]

\[ \Delta_L(x', t'; x, t) \triangleright (\partial^2_t c^{-2} - \nabla^2_q + (mc)^2) = i \text{ vol}^{-1} \delta^3_q((\ominus \kappa^{-1}x') \oplus x) \delta(t' - t), \quad (56) \]

or

\[ \Delta_L^*(x', t'; x, t) \triangleright (\partial^2_t c^{-2} - \nabla^2_q + (mc)^2) = i \text{ vol}^{-1} \delta^3_q((\ominus \kappa^{-1}x') \oplus x) \delta(t' - t), \quad (57) \]

\[ ((mc)^2 + c^{-2} \partial_t^2 - \nabla^2_q) \triangleright \Delta_R^*(x, t; x', t') = -i \text{ vol}^{-1} \delta(t - t') \delta^3_q(x \oplus (\ominus \kappa^{-1}x')). \quad (58) \]
Using the above identities, we can show that the functions

\[
\phi_R(x, t) = \varphi_R(x, t) + i \int d^3x' dt' \Delta_R(x, t; x', t') \otimes \varrho(x', t'),
\]

\[
\phi_L(x, t) = \varphi_L(x, t) - i \int d^3x' dt' \varrho(x', t') \otimes \Delta_L(x', t'; x, t)
\]

are solutions to the following q-deformed inhomogenous Klein-Gordon equations:

\[
\begin{align*}
&c^{-2} \partial_t^2 \varphi_R - \nabla_q^2 \varphi_R + (mc)^2 \varphi_R = \varrho, \\
&\phi_L \triangleleft \partial_t^2 c^{-2} \phi_L - \nabla_q^2 \phi_L + \phi_L (mc)^2 = \varrho.
\end{align*}
\]

Note that \(\varphi_R\) and \(\varphi_L\) in Eq. (59) are solutions to the free q-deformed Klein-Gordon equations given in Eqs. (33) and (35). Similarly, the functions

\[
\begin{align*}
&\phi^*_R(x, t) = \varphi^*_R(x, t) + i \int d^3x' dt' \Delta^*_R(x, t; x', t') \otimes \varrho(x', t'), \\
&\phi^*_L(x, t) = \varphi^*_L(x, t) - i \int d^3x' dt' \varrho(x', t') \otimes \Delta^*_L(x', t'; x, t)
\end{align*}
\]

satisfy the following q-deformed versions of the inhomogenous Klein-Gordon equation:

\[
\begin{align*}
&c^{-2} \partial_t^2 \varphi^*_R - \nabla_q^2 \varphi^*_R + (mc)^2 \varphi^*_R = \varrho, \\
&\phi^*_L \triangleleft \partial_t^2 c^{-2} \phi^*_L - \nabla_q^2 \phi^*_L + \phi^*_L (mc)^2 = \varrho.
\end{align*}
\]

Once again, \(\varphi^*_R\) and \(\varphi^*_L\) are solutions to the free q-deformed Klein-Gordon equations given in Eq. (38).

### 3 Lippmann-Schwinger equations

The Hamiltonian operator of a scattering experiment is the sum of the Hamiltonian operator \(H_0\) of a free particle and a potential \(V\):

\[
H = H_0 + V(x).
\]

The potential \(V\) transforms as a scalar under rotations in the q-deformed Euclidean space. It also shows trivial braiding properties. Moreover, \(V\) has to be real so that no particles are absorbed:

\[
\overline{V(x)} = V(x).
\]

We assume \(V\) to vanish at infinity. With the Hamiltonian operator in Eq. (63), we get the q-deformed Schrödinger equations

\[
\begin{align*}
&i \partial_t \varphi_R(x, t) - H_0 \varphi_R(x, t) = V(x) \otimes \varphi_R(x, t), \\
&i \partial_t \varphi^*_R(x, t) - H_0 \varphi^*_R(x, t) = V(x) \otimes \varphi^*_R(x, t).
\end{align*}
\]
and

\[
\psi_L(x, t) \triangleq \partial_t \psi_L(x, t) - \psi_L(x, t) \triangleq H_0 = \psi_L(x, t) \otimes V(x),
\]
\[
\psi^*_L(x, t) \triangleq \partial_t i - \psi^*_L(x, t) \triangleq H_0 = \psi^*_L(x, t) \otimes V(x).
\] (66)

In a scattering experiment, a particle starts at \( t = -\infty \) as a free particle. After the interaction, the particle at \( t = \infty \) will emerge as a free particle again. For this reason, the solutions to the \( q \)-deformed Schrödinger equations above have to satisfy the following boundary conditions\(^1\):

\[
\lim_{t \rightarrow -\infty} \left[ (\psi_L/R)^\pm(x, t) - \phi_L/R(x, t) \right] = 0,
\]
\[
\lim_{t \rightarrow -\infty} \left[ (\psi^*_L/R)^\pm(x, t) - \phi^*_L/R(x, t) \right] = 0.
\] (67)

Recall that \( \phi_R, \phi_L, \phi^*_R, \) and \( \phi^*_L \) denote solutions to the \( q \)-deformed Schrödinger equations for a free particle [cf. Eq. (2) of Chap. 2]. From Chap. 2 we know how to solve \( q \)-versions of inhomogeneous Schrödinger equations by using Green’s functions. This method enables us to derive integral equations for the solutions of the Schrödinger equations in (65) and (66). By doing so, we get \( q \)-analogs of the Lippmann-Schwinger equations. Concretely, we have to replace the inhomogeneity \( \varrho \) in Eq. (31) or Eq. (32) of Chap. 2 with the expressions on the right-hand side of the Schrödinger equations in (65) and (66). Since we can add a solution of the free Schrödinger equation to a solution of an inhomogeneous Schrödinger equation, we finally obtain

\[
(\psi_R)^\pm(x, t) = \phi_R(x, t)
\]
\[
\pm i \int dt' \int d^3x'(K_R)^\pm(x, t; x', t') \otimes V(x') \otimes (\psi_R)^\pm(x', t'),
\]
\[
(\psi^*_R)^\pm(x, t) = \phi^*_R(x, t)
\]
\[
\pm i \int dt' \int d^3x'(K^*_R)^\pm(x, t; x', t') \otimes V(x') \otimes (\psi^*_R)^\pm(x', t'),
\] (68)

and

\[
(\psi_L)^\pm(x, t) = \phi_L(x, t)
\]
\[
\pm i \int dt' \int d^3x'(\psi_L)^\pm(x', t') \otimes V(x') \otimes (K_L)^\pm(x', t'; x, t),
\]
\[
(\psi^*_L)^\pm(x, t) = \phi^*_L(x, t)
\]
\[
\pm i \int dt' \int d^3x'(\psi^*_L)^\pm(x', t') \otimes V(x') \otimes (K^*_L)^\pm(x', t'; x, t).
\] (69)

The above \( q \)-deformed Lippmann-Schwinger equations are compatible with the

\(^1\)We have also taken into account that particles could go backwards in time.
boundary conditions in Eq. (67), as can be shown by using the identities:

\[
\lim_{t \to \pm \infty} (K_R)^\pm(x; x') = \lim_{t \to \pm \infty} \theta(t \mp t') K_R(x; x') = 0,
\]

\[
\lim_{t \to \pm \infty} (K_R^*)^\pm(x; x') = \lim_{t \to \pm \infty} \theta(t \mp t') K_R^*(x; x') = 0,
\]

and

\[
\lim_{t \to \pm \infty} (K_L)^\pm(x'; x) = \lim_{t \to \pm \infty} \theta(t \mp t') K_L(x'; x) = 0,
\]

\[
\lim_{t \to \pm \infty} (K_L^*)^\pm(x'; x) = \lim_{t \to \pm \infty} \theta(t \mp t') K_L^*(x'; x) = 0.
\]

We aim to solve the \(q\)-deformed Lippmann-Schwinger equations by iteration. To this end, we introduce new Green’s functions defined by the following equations:

\[
(\psi_R)^\pm(x) = \lim_{t' \to \pm \infty} \int d^3q x' (G_R)^\pm(x; x') \phi_R(x'),
\]

\[
(\psi_R^*)^\pm(x) = \lim_{t' \to \pm \infty} \int d^3q x' (G_R^*)^\pm(x; x') \phi_R(x'),
\]

\[
(\psi_L)^\pm(x) = \lim_{t' \to \pm \infty} \int d^3q x' \phi_L(x') \otimes (G_L)^\pm(x'; x),
\]

\[
(\psi_L^*)^\pm(x) = \lim_{t' \to \pm \infty} \int d^3q x' \phi_L^*(x') \otimes (G_L^*)^\pm(x'; x).
\]

Using these identities, we rewrite the right-hand side of each \(q\)-deformed Lippmann-Schwinger equation. We get, for example:

\[
(\psi_R)^\pm(x) = \lim_{t' \to \pm \infty} \int d^3q x' (K_R)^\pm(x; x') \phi_R(x')
\]

\[
\mp i \lim_{t' \to \pm \infty} \int d^3q x' \int dt'' \int d^3q x'' (K_R)^\pm(x; x'') \otimes V(x'') \otimes (G_R)^\pm(x''; x') \phi_R(x').
\]

Note that the first expression on the right-hand side of the above equation is a consequence of the following identity:

\[
\phi_R(x, t) = \lim_{t' \to \pm \infty} \int d^3q x' (K_R)^\pm(x, t; x', t') \phi_R(x', t').
\]

Comparing Eq. (74) with the first identity in Eq. (72), we can read off a Lippmann-Schwinger equation for the Green’s function \((G_R)^\pm\) or \((G_R)^-\):

\[
(G_R)^\pm(x; x') = (K_R)^\pm(x; x')
\]

\[
\mp i \int dt'' \int d^3q x'' (K_R)^\pm(x; x'') \otimes V(x'') \otimes (G_R)^\pm(x''; x').
\]

\[\text{In the following, we use the notation } x = (x, t) \text{ or } x' = (x', t').\]
In the same way, we get:

\[(G_L^*)^\pm(x'; x) = (K_R^*)^\pm(x'; x)\]

\[\pm i \int \! dt'' \int d^3x'' (G_L^*)^\pm(x; x'') \otimes V(x'') \otimes (K_L^*)^\pm(x''; x'). \quad (77)\]

We obtain the Lippmann-Schwinger equations for the other Green’s functions in Eqs. (72) and (73) by applying the following substitutions:

\[G_R \rightarrow G_R^*, \quad K_R \rightarrow K_R^*,\]

\[G_L^* \rightarrow G_L, \quad K_L^* \rightarrow K_L. \quad (78)\]

The new Green’s functions satisfy Schrödinger equations for the Hamiltonian operator \(H\) with a \(q\)-deformed delta function as inhomogeneity, i. e.

\[(i \partial_t - H) \triangleright (G_R^*)^\pm(x; x') = \mp i \text{vol}^{-1} \delta(\pm t \mp t') \delta_q^3(\mathbf{x} \oplus (\ominus \kappa^{-1}\mathbf{x}')), \]

\[(i \partial_t - H) \triangleright (G_L^*)^\pm(x; x') = \mp i \text{vol}^{-1} \delta(\pm t \mp t') \delta_q^3((\ominus \kappa^{-1}\mathbf{x}) \oplus \mathbf{x'}), \quad (79)\]

and

\[(G_L^*)^\pm(x'; x) \triangleleft (\partial_t i - H) = \mp i \text{vol}^{-1} \delta(\pm t \mp t') \delta_q^3((\ominus \kappa^{-1}\mathbf{x'}) \oplus \mathbf{x}), \]

\[(G_L^*)^\pm(x'; x) \triangleleft (\partial_t i - H) = \mp i \text{vol}^{-1} \delta(\pm t \mp t') \delta_q^3(\mathbf{x'} \oplus (\ominus \kappa^{-1}\mathbf{x})). \quad (80)\]

By way of example, we show how to derive these identities. First, we apply the operator \(i \partial_t - H\) to the Lippmann-Schwinger equation for \((G_R^*)^\pm\) [see Eq. (76)]:

\[(i \partial_t - H) \triangleright (G_R^*)^\pm(x; x') =\]

\[= (i \partial_t - H_0) \triangleright (K_R^*)^\pm(x; x') - V(\mathbf{x}) \otimes (K_R^*)^\pm(x; x')\]

\[\mp i \int \! dt'' \int d^3x'' (i \partial_t - H_0) \triangleright (K_R^*)^\pm(x; x'') \otimes V(x'') \otimes (G_R^*)^\pm(x''; x')\]

\[\pm i \int \! dt'' \int d^3x'' V(x) \otimes (K_R^*)^\pm(x; x'') \otimes V(x'') \otimes (G_R^*)^\pm(x''; x'). \quad (81)\]

Next, we use the identities for the free Schrödinger propagator \((K_R^*)^\pm\) given in Eq. (27) of Chap. 2

\[(i \partial_t - H) \triangleright (G_R^*)^\pm(x; x') =\]

\[= \pm i \text{vol}^{-1} \delta(t - t') \delta_q^3(\mathbf{x} \oplus (\ominus \kappa^{-1}\mathbf{x}')) - V(\mathbf{x}) \otimes (K_R^*)^\pm(x; x')\]

\[\pm \text{vol}^{-1} \int \! dt'' \int d^3x'' (t - t'') \delta_q^3(\mathbf{x} \oplus (\ominus \kappa^{-1}\mathbf{x}'')) \oplus V(\mathbf{x}'') \otimes (G_R^*)^\pm(x''; x')\]

\[\pm i V(\mathbf{x}) \otimes \int \! dt'' \int d^3x'' (K_R^*)^\pm(x; x'') \otimes V(x'') \otimes (G_R^*)^\pm(x''; x')\]

\[= \pm i \text{vol}^{-1} \delta(t - t') \delta_q^3(\mathbf{x} \oplus (\ominus \kappa^{-1}\mathbf{x}')). \quad (82)\]

For a better understanding, we note the following: After calculating the integrals of the expressions with the delta functions, all terms depending on the interaction potential \(V\) cancel out due to Eq. (76).
The Green’s functions for the $q$-deformed Schrödinger equations with interaction satisfy the identities:

$$\theta(\pm t \mp t')(\psi_R)^\pm(x) = \int d^3 x' (G_R)^\pm(x; x') \oplus (\psi_R)^\pm(x'),$$

$$\theta(\pm t \mp t')(\psi_R^*)^\pm(x) = \int d^3 x' (G_R^*)^\pm(x; x') \oplus (\psi_R^*)^\pm(x'),$$

and

$$\theta(\pm t \mp t')(\psi_L)^\pm(x) = \int d^3 x' (G_L)^\pm(x; x) \oplus (\psi_L)^\pm(x'),$$

$$\theta(\pm t \mp t')(\psi_L^*)^\pm(x) = \int d^3 x' (G_L^*)^\pm(x; x) \oplus (\psi_L^*)^\pm(x').$$

Indeed, if we take the limit $t' \to \mp\infty$ on both sides of the above identities and consider the boundary conditions in Eq. (67), we regain the identities in Eq. (72) or Eq. (73).

We can also use Eqs. (79) and (80) to derive the formulas in Eq. (83) and Eq. (84). We show this by an example, applying the operator $i\partial_t - H$ to both sides of the first identity of Eq. (83):

$$\int d^3 x' (i\partial_t - H) \triangleright (G_R)^\pm(x; x') \oplus (\psi_R)^\pm(x') =$$

$$= (i\partial_t - H) \triangleright \theta(\pm t \mp t')(\psi_R)^\pm(x) = \pm i\delta(\pm t \mp t') (\psi_R)^\pm(x)$$

$$= \pm i\delta(\pm t \mp t') \text{vol}^{-1} \int d^3 x' \delta^3(x \oplus (\ominus \kappa^{-1}x')) \oplus (\psi_R)^\pm(x').$$

The result above implies the first identity in Eq. (79).

The propagators for the Schrödinger equations with interaction can be combined in the following way:

$$(G_R)^\pm(x, t; x', t') = \int d^3 x'' (G_R)^\pm(x, t; x'', t'') \oplus (G_R)^\pm(x'', t''; x, t'),$$

$$(G_L)^\pm(x', t'; x, t) = \int d^3 x'' (G_L)^\pm(x', t'; x'', t'') \oplus (G_L)^\pm(x'', t''; x, t).$$

For the intermediate point $t''$, it is $t' \leq t'' \leq t$ in the case of retarded propagators, and $t \leq t'' \leq t'$ in the case of advanced ones. We can prove the above rules by applying the identities in Eqs. (83) and (84) twice. We obtain further identities if we substitute $(G_R)^\pm$ by $(G_R^*)^\pm$ or $(G_L)^\pm$ by $(G_L^*)^\pm$.

We consider the case that a retarded propagator transforms a wave function from time $t'$ to a later time $t$. An advanced propagator pulls the new wave function from time $t$ back to the initial time $t'$. This way, we regain the original wave function:

$$\psi_R(x') = \int d^3 x (G_R)(x'; x) \oplus \psi_R(x)$$

$$= \int d^3 x'' \int d^3 x (G_R)(x', t'; x, t) \oplus (G_R)(x, t; x'', t') \oplus \psi_R(x'', t').$$
Comparing this result with the identity
\[ \psi_R(x', t') = \text{vol}^{-1} \int dq \, \delta_q^3(x' + (\mp \kappa^{-1} x'')) \odot \psi_R(x'', t'), \]  
we obtain for \( t' \leq t \):
\[ \int dq \, (G_R)^-(x', t'; x, t) \odot (G_R)^+(x, t; x'', t') = \text{vol}^{-1} \delta_q^3(x' + (\mp \kappa^{-1} x'')). \]  
In the same manner, we obtain for \( t' \geq t \):
\[ \int dq \, (G_R)^+(x', t'; x, t) \odot (G_R)^-(x, t; x'', t') = \text{vol}^{-1} \delta_q^3(x' + (\mp \kappa^{-1} x'')). \]

Similarly, arguments lead to
\[ \int dq \, (G_L^*)^\pm(x', t'; x, t) \odot (G_L^*)^\mp(x, t; x', t') = \text{vol}^{-1} \delta_q^3((\mp \kappa^{-1} x'') \odot x') \]  
with \( t' \leq t \) for the upper sign and \( t' \geq t \) for the lower sign. For the other \( q \)-versions of Green’s functions, we get corresponding relations if we substitute \( (G_R)^\pm \) by \( (G_R^*)^\pm \) or \( (G_L)^\pm \) by \( (G_L^*)^\pm \) in the above identities.

If the interaction potential \( V \) is sufficiently small, we can solve the Lippmann-Schwinger equations in (88) and (89) iteratively. To this end, we consider the solution of a free \( q \)-deformed Schrödinger equation as an approximate solution and plug it into the right-hand side of the corresponding \( q \)-deformed Lippmann-Schwinger equation. Doing so, we get a second approximate solution, which we again plug into the right-hand side of the \( q \)-deformed Lippmann-Schwinger equation. Repeating this procedure, we finally obtain \( q \)-versions of the Born series, i.e.
\[ (\psi_R^*)^\pm(x) = \phi_R(x) + i^{\mp 1} \int dt' \int dq \, \delta_q^3 x' (K_R)^\pm(x; x') \odot V(x') \odot \phi_R(x') \]
\[ + i^{\pm 2} \int dt' \int dq \, \delta_q^3 x' \int dt'' \int dq' \, \delta_q^3 x'' (K_R)^\pm(x; x') \odot V(x') \odot (K_R^*)^\mp(x''; x') \odot V(x''') \odot \phi_R(x''') + \ldots \]  
and
\[ (\psi_L^*)^\pm(x) = \phi_L^*(x) + i^{\pm 1} \int dt' \int dq \, \delta_q^3 x' \phi_L^*(x') \odot V(x') \odot (K_L^*)^\pm(x'; x) \]
\[ + i^{\pm 2} \int dt' \int dq \, \delta_q^3 x' \int dt'' \int dq' \, \delta_q^3 x'' \phi_L^*(x'') \odot V(x'') \odot (K_L^*)^\mp(x'''; x') \odot V(x''') \odot \phi_R(x''') + \ldots \]  
We get the expansions for the other \( q \)-versions of the Lippmann-Schwinger equation by applying the following substitutions to the above identities:
\[ \psi_R \rightarrow \psi_R^*, \quad \phi_R \rightarrow \phi_R^*, \quad K_R \rightarrow K_R^*, \]
\[ \psi_L^* \rightarrow \psi_L, \quad \phi_L^* \rightarrow \phi_L, \quad K_L^* \rightarrow K_L. \]
We can also solve the $q$-deformed Lippmann-Schwinger equations for Green’s functions by iteration. For example, if we plug in the free propagator $(K_R)^\pm$ on the right-hand side of Eq. (76) as a first approximation for the Green’s function $(G_R)^\pm$, we get a second approximation for $(G_R)^\pm$, which we plug in again on the right-hand side of Eq. (76), and so on. This way, we have

$$(G_R)^\pm(x; x') = (K_R)^\pm(x; x')$$

$$+ i^{\mp 1} \int dt'' \int d^3 q'' (K_R)^\pm(x; x'') \otimes V(x'') \otimes (K_R)^\pm(x''; x')$$

$$+ i^{\mp 2} \int dt'' \int d^3 q'' \int dt''' \int d^3 q'''(K_R)^\pm(x; x''') \otimes V(x''')$$

$$\otimes (K_R)^\pm(x'''; x'') \otimes V(x''') \otimes (K_R)^\pm(x''''; x') + \ldots$$

(95)

and

$$(G_L^*)^\pm(x'; x) = (K_L^*)^\pm(x'; x)$$

$$+ i^{\pm 1} \int dt'' \int d^3 q'' (K_L^*)^\pm(x'; x'') \otimes V(x'') \otimes (K_L^*)^\pm(x''; x')$$

$$+ i^{\pm 2} \int dt'' \int d^3 q'' \int dt''' \int d^3 q'''(K_L^*)^\pm(x'; x''') \otimes V(x''')$$

$$\otimes (K_L^*)^\pm(x'''; x'') \otimes V(x''') \otimes (K_L^*)^\pm(x''''; x') + \ldots$$

(96)

To obtain the corresponding expansions for the other $q$-versions of Green’s functions, we apply the substitutions of Eq. (78) to the above identities again. If we compare the expansions for $(G_R)^\pm$ and $(G_L^*)^\pm$ and those for $(G_L)^\pm$ and $(G_R^*)^\pm$, we find the following identifications:

$$(G_R)^\pm(x; x') = (G_L^*)^\mp(x; x'),$$

$$(G_L)^\pm(x; x') = (G_R^*)^\mp(x; x').$$

(97)

Next, we study how Green’s functions for the Schrödinger equations with interaction behave under conjugation. If we conjugate Eq. (76) and take into account the conjugation properties of the free propagators [cf. Eq. (20) in Chap. 2], we obtain

$$\overline{(G_R)^\pm(x; x')} = (K_L)^\pm(x'; x)$$

$$\pm i \int dt'' \int d^3 q'' \overline{(G_R)^\pm(x''; x)} \otimes V(x'') \otimes (K_L)^\pm(x''; x')$$

(98)

for an interaction potential with $\overline{V(x)} = V(x)$. This result implies:

$$\overline{(G_R)^\pm(x; x')} = (G_L)^\pm(x'; x).$$

(99)

In the same way, we get:

$$\overline{(G_R^*)^\pm(x; x')} = (G_L^*)^\pm(x'; x).$$

(100)
The $q$-deformed propagators for a free nonrelativistic particle can be expanded in terms of plane waves [also see Eqs. (19) and (20) in Chap. 2]. Such expansions also exist for the propagators of the $q$-deformed Schrödinger equations with interaction. To show this, we consider the expressions for the scattered waves in Eqs. (72) and (73). We replace the free solutions in the expressions for the scattered waves with the plane waves given in Eqs. (8) and (9) of Chap. 2. This way, we obtain

$$U^\pm_p(x, t) = \lim_{t' \to \mp \infty} \int d^3q \phi^R (x, t') \otimes \varphi_p (x', t'),$$  

$$ (U^*)^\pm_p(x, t) = \lim_{t' \to \mp \infty} \int d^3q \phi'R (x', t') \otimes (u^*)_p (x', t'; x, t), \quad (101)$$

or

$$U^\mp_p(x, t) = \lim_{t' \to \mp \infty} \int d^3q \phi' (x', t') \otimes (G^R_L)^\pm (x', t'; x, t)$$

$$ (U^*)^\mp_p(x, t) = \lim_{t' \to \mp \infty} \int d^3q \phi' (x', t') \otimes (u^*)_p (x', t'). \quad (102)$$

These expressions for the scattered waves satisfy the orthonormality relations

$$\int d^3q \phi^R (x, t) \otimes (U^*)^\pm_p (x, t) = \int d^3q \phi^R (x, t) \otimes \varphi_p (x, t)$$

$$= \text{vol}^{-1} \delta^3(\ominus \kappa^{-1} p' \oplus p), \quad (103)$$

or

$$\int d^3q \phi' (x, t) \otimes (U^*)^\pm_p (x, t) = \int d^3q \phi' (x, t) \otimes (u^*)_p (x, t)$$

$$= \text{vol}^{-1} \delta^3(\ominus \kappa^{-1} p' \oplus p). \quad (104)$$

Since the wave functions in Eqs. (101) and (102) arise from the scattering of plane waves, we can prove the above orthonormality relations with a calculation similar to that in Eq. (142) of the next chapter.

We show that the wave functions in Eq. (101) or Eq. (102) also form a complete system of functions. The general solutions to the $q$-deformed Schrödinger equations for a free nonrelativistic particle can be written as expansions in terms of $q$-deformed plane waves. For example, we have

$$\phi_R (x, t) = \int d^3q \phi_p (x, t) \otimes c_p. \quad (105)$$

Plugging this expansion into the first identity of Eq. (72) and taking into account the completeness of the plane wave solutions, we can establish the orthonormality relations as above. The completeness of the plane wave solutions is guaranteed by the orthonormality relations (101) and (102)

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\(^{3}\)See Eq. (18) in Chap. 2.
the definition of $U_p^{(±)}$ [see Eq. (101)], we find:

\[
(\psi_R^{±}(x)) = \lim_{t' \rightarrow \pm \infty} \int d^3x' (G_R)^{±}(x; x') \otimes \phi_R(x') = \\
= \lim_{t' \rightarrow \pm \infty} \int d^3x' (G_R)^{±}(x; x') \otimes \int d^3p u_p(x') \otimes c_p \\
= \int d^3p \lim_{t' \rightarrow \mp \infty} \int d^3x' (G_R)^{±}(x; x') \otimes u_p(x') \otimes c_p \\
= \int d^3p U_p^{±}(x) \otimes c_p.
\]

(106)

In the same way, we get the expansions

\[
(\psi_R^*(x))^{±}(x) = \int d^3p (U^*)_p^{±}(x) \otimes (c^*)^p,
\]

(107)

and

\[
(\psi_L^{±}(x, t)) = \int d^3p c^p \otimes U_p^{±}(x, t),
\]

\[
(\psi_L^*(x, t))^{±}(x, t) = \int d^3p (c^*)^p \otimes (U^*)_p^{±}(x, t)
\]

(108)

if we have

\[
\phi_R(x, t) = \int d^3p (u^*)^p(x, t) \otimes (c^*)^p,
\]

(109)

and

\[
\phi_L(x, t) = \int d^3p c^p \otimes u^p(x, t),
\]

\[
\phi_L^*(x, t) = \int d^3p (c^*)^p \otimes (u^*)^p(x, t).
\]

(110)

We calculate the coefficients in the above expansions using the orthonormality relations in Eqs. (103) and (104). For example, we have:

\[
\int d^3q (U^*)_p^{±}(x, t) \otimes (\psi_R)^{±}(x, t) = \int d^3q u_p' \int d^3x (U^*)_p^{±}(x, t) \otimes U_p^{±}(x, t) \otimes c_{p'} \\
= \int d^3q' \text{vol}^{-1} d^3q ((\otimes \kappa^{-1}) \otimes \text{p'}) \otimes c_{p'} \\
= c_p.
\]

(111)

By similar reasoning, we find

\[
\int d^3q (\psi_L^{±}(x, t)) \otimes (U^*)_p^{±}(x, t) = c_p.
\]

(112)
and
\[
\int \! d^3q \, U^p_{\pm}(\mathbf{x}, t) \otimes (\psi^*_R)^{\pm}(\mathbf{x}, t) = (c^*)_p,
\]
\[
\int \! d^3q \,(\psi^*_R)^{\pm}(\mathbf{x}, t) \otimes U^\pm_p(\mathbf{x}, t) = (c^*)_p. \tag{113}
\]

Using the scattered wave functions in Eq. (101) or Eq. (102), we can write down expressions for the Green’s functions to Schrödinger equations with an interaction. To this end, we plug the integral expression for \(c_p\) given in Eq. (111) into the expansion in Eq. (106):
\[
(\psi_R)^{\pm}(\mathbf{x}, t) = \int \! d^3p U^\pm_p(\mathbf{x}, t) \otimes c_p
= \int \! d^3p \int \! d^3p U^\pm_p(\mathbf{x}, t) \otimes (U^*)_p(\mathbf{x}', t') \otimes (\psi^*_R)^{\pm}(\mathbf{x}', t'). \tag{114}
\]
By comparing this result with the first identity in Eq. (83), we find
\[
(G_R)^{\pm}(\mathbf{x}, t; \mathbf{x}', t') = \int \! d^3q U^\pm_p(\mathbf{x}, t) \otimes (U^*)_p(\mathbf{x}', t'), \tag{115}
\]
where \(t > t'\) for \((G_R)^+\) and \(t < t'\) for \((G_R)^-\). By similar reasoning, we find
\[
(G^*_R)^{\pm}(\mathbf{x}, t; \mathbf{x}', t') = \int \! d^3q (U^*)_p(\mathbf{x}, t) \otimes U^\pm_p(\mathbf{x}', t'), \tag{116}
\]
and
\[
(G_L)^{\pm}(\mathbf{x}', t'; \mathbf{x}, t) = \int \! d^3q (U^*)_p(\mathbf{x}', t') \otimes U^\pm_p(\mathbf{x}, t),
\]
\[
(G^*_L)^{\pm}(\mathbf{x}', t'; \mathbf{x}, t) = \int \! d^3q U^\pm_p(\mathbf{x}', t') \otimes (U^*)_p(\mathbf{x}, t), \tag{117}
\]
where \(t > t'\) for the retarded Green’s functions and \(t < t'\) for the advanced Green’s functions.

### 4 Scattering matrices

#### 4.1 Definition

An initially free particle can change its state by scattering. The S-matrix relates the initial state and the final state of the particle undergoing a scattering process.

The wave function \((\psi_R)^{\pm}\) represents the state which emerges from a free state with wave function \(\phi_R\) by scattering. If the wave function \(\phi^*_L\) describes the free particle state after scattering, we have the following scattering amplitude \([15]\):
\[
(S_R)^{\pm}(\phi, \psi) = \lim_{t \to \pm \infty} \int \! d^3x \phi^*_L(x) \otimes (\psi_R)^{\pm}(x)
= \lim_{t \to \pm \infty} \lim_{t' \to \mp \infty} \int \! d^3x \int \! d^3x' \phi^*_L(x) \otimes (G_R)^{\pm}(x; x') \otimes \phi_R(x'). \tag{118}
\]
Note that the second identity follows from the first formula in Eq. (72) of the previous chapter. Eq. (118) shows that the elements of the S-matrix \((S_R)_{±}\) are identical to the matrix elements of the Green’s function \((G_R)_{±}\). The same applies to the other versions of Green’s functions, i.e.

\[
(S_L)_{±}(ψ, φ) = \lim_{t→±∞} \int d^3x (ψ_L)^{±}(x) ⊗ φ_R^*(x)
= \lim_{t→±∞, t’→±∞} \int d^3x \int d^3x’ ϕ_L(x’) ⊗ (G_L)_{±}^*(x’; x) ⊗ φ_R^*(x),
\]

(119)

\[
(S_R^*)_{±}(φ, ψ) = \lim_{t→±∞} \int d^3x ϕ_L(x) ⊗ (ψ_R^*)_{±}(x)
= \lim_{t→±∞, t’→±∞} \int d^3x \int d^3x’ ϕ_L(x) ⊗ (G_R^*)_{±}^*(x; x’) ⊗ φ_R^*(x’),
\]

(120)

and

\[
(S_L^*)_{±}(ψ, φ) = \lim_{t→±∞} \int d^3x (ψ_L^*)_{±}(x) ⊗ φ_R(x)
= \lim_{t→±∞, t’→±∞} \int d^3x \int d^3x’ ϕ_L^*(x’) ⊗ (G_L^*)_{±}(x’; x) ⊗ φ_R(x).
\]

(121)

If we compare the different versions of S-matrix elements and take into account Eq. (97) from the previous chapter, we can find the following identities:

\[
(S_R)_{±}(φ, ψ) = (S_L^*)_{±}(ψ, φ), \quad (S_L)_{±}(ψ, φ) = (S_R^*)_{±}(φ, ψ).
\]

(122)

Next, we study the conjugation properties of the S-matrix elements. Conjugating the expressions in Eqs. (118)-(121) and taking into account Eq. (3) of Chap. 2 as well as Eqs. (99) and (100) of the previous chapter, we find:

\[
(S_R^*)_{±}(φ, ψ) = (S_L)_{±}(ψ, φ), \quad (S_R)_{±}(φ, ψ) = (S_L^*)_{±}(ψ, φ).
\]

(123)

The conjugation properties of the free wave functions carry over to the scattered wave functions. Using the series expansions in Eqs. (92) and (93) of the previous chapter and the conjugation properties of the free propagators [see Eq. (26) of Chap. 2], we can verify the following identities:

\[
ψ_L(x, t) = ψ_R(x, t), \quad ψ_L^*(x, t) = ψ_R^*(x, t).
\]

(124)

### 4.2 Momentum representation

To get S-matrix elements in a basis of momentum eigenfunctions, we replace the solutions to the free Schrödinger equations with their expansions in terms of plane waves [cf. Eqs. (105), (109), and (110) of the previous chapter]. This
way, we have
\[
(S_R)^\pm(\phi, \psi) = \lim_{t \to \pm \infty} \lim_{t' \to \mp \infty} \int d^3x d^3x' \phi^*_L(x) \otimes (G_R)^\pm(x; x') \otimes \phi_R(x')
\]
\[
= \int d^3p d^3p' (c^*)_p \otimes (S_R)^\pm_{pp'} \otimes c_{p'},
\]
(125)

where
\[
(S_R)^\pm_{pp'} = \lim_{t' \to \mp \infty} \lim_{t \to \pm \infty} \int d^3x d^3x' (u^*)_p(x) \otimes (G_R)^\pm(x; x') \otimes u_{p'}(x').
\]
(126)

If we apply these considerations to the other versions of the S-matrix, we also get
\[
(S_L)^\pm(\psi, \phi) = \int d^3p' d^3p e^{p'} \otimes (S_L)^\pm_{pp'} \otimes (c^*)^p
\]
(127)

with
\[
(S_L)^\pm_{pp'} = \lim_{t' \to \mp \infty} \lim_{t \to \pm \infty} \int d^3x' d^3x u^{p'}(x') \otimes (G_L)^\pm(x'; x) \otimes (u^*)_p(x),
\]
(128)

and
\[
(S_R^*)^\pm(\phi, \psi) = \int d^3p' d^3p e^{p} \otimes (S_R^*)^\pm_{pp'} \otimes (c^*)^p',
\]
\[
(S_L^*)^\pm(\psi, \phi) = \int d^3p' d^3p (c^*)_p' \otimes (S_L^*)^\pm_{pp'} \otimes c_p
\]
(129)

with
\[
(S_R^*)^\pm_{pp'} = \lim_{t' \to \mp \infty} \lim_{t \to \pm \infty} \int d^3x d^3x' u^p(x) \otimes (G_R^*)^\pm(x; x') \otimes (u^*)_p'(x'),
\]
\[
(S_L^*)^\pm_{pp'} = \lim_{t' \to \mp \infty} \lim_{t \to \pm \infty} \int d^3x' d^3x (u^*)_p(x') \otimes (G_L^*)^\pm(x'; x) \otimes u_{p'}(x).
\]
(130)

Comparing the expressions for the different types of S-matrix elements and considering Eq. (97) of Chap. 3 we find the following identities:
\[
(S_R)^\pm_{pp'} = (S_L^*)^\mp_{pp'}, \quad (S_L)^\pm_{pp'} = (S_R^*)^\mp_{pp'}.
\]
(131)

The conjugation properties of momentum eigenfunctions and Green’s functions [see Eq. (13) of Chap. 2 and Eqs. (99) and (100) of Chap. 3] imply the following conjugation properties for the S-matrix elements in momentum space:
\[
(S_R)^\pm_{pp'} = (S_L^*)^\mp_{pp'}, \quad (S_L^*)^\pm_{pp'} = (S_R)^\mp_{pp'}.
\]
(132)

A look at Eq. (126) shows that the Green’s function \((G_R)^\pm\) determines the S-matrix \((S_R)^\pm\). If we represent \((G_R)^\pm\) by the Born series in Eq. (95)
of Chap. 3 we obtain the following perturbation expansion for the S-matrix 
\((S_R)^\pm\) in momentum space:

\[
(S_R)^\pm_{pp'} = \frac{\text{vol}^{-1}}{q^3}(\odot (\kappa^{-1}p) \odot p') + i\frac{1}{q^2} \int d^3x \int dt (u^*)_{p}(x) \otimes V(x) \otimes u_{p'}(x) \\
+ i\frac{2}{q} \int d^3x d^3x' \int dt dt' (u^*)_{p}(x) \otimes V(x) \otimes (K_R)^\pm(x;x') \otimes V(x') \otimes u_{p'}(x' + \ldots \quad (133)
\]

In deriving this formula, we have made use of the following identities:

\[u_{p}(x,t) = \lim_{t' \to \pm \infty} \int d^3x' (K_R)^\pm(x;x') \otimes u_{p}(x'),\]

\[(u^*)_{p}(x',t') = \lim_{t' \to \pm \infty} \int d^3x (u^*)_{p}(x) \otimes (K_R)^\pm(x;x'). \quad (134)\]

Moreover, we have taken into account the completeness of \(q\)-deformed plane waves [see Eq. (135) in Chap. 2]. There are similar expansions for the other \(q\)-versions of the S-matrix. For example, if we use the first identity in Eq. (131) together with \((K_R^L)^\pm = (K_R)^\pm\), we find:

\[
(S_L^\pm)_{pp'} = \frac{\text{vol}^{-1}}{q^3}(\odot (\kappa^{-1}p) \odot p') + i\frac{1}{q^2} \int d^3x \int dt (u^*)_{p}(x) \otimes V(x) \otimes u_{p'}(x) \\
+ i\frac{2}{q} \int d^3x d^3x' \int dt dt' (u^*)_{p}(x) \otimes V(x) \otimes (K_R^L)^\pm(x;x') \otimes V(x') \otimes u_{p'}(x' + \ldots \quad (135)
\]

### 4.3 Unitarity and conservation of probability

We clarify in which way the \(q\)-deformed S-matrices are unitary. Using the results in Eqs. (126) and (130), we can calculate the following product of two S-matrices:

\[
\int d^3p'' (S_L^\pm)_{pp'} \otimes (S_R)^\pm_{pp'} = \lim_{t',t'' \to \pm \infty} \int d^3p'' \int d^3x' d^3x'' (u^*)_{p'}(x') \otimes (G_L^\pm) (x';x'') \otimes u_{p'}(x'') \\
\otimes \int d^3x''' d^3x (u^*)_{p'}(x''') \otimes (G_R^\pm) (x''';x) \otimes u_{p}(x). \quad (136)
\]

With the identities

\[
\lim_{t',t'' \to \pm \infty} \int d^3p'' u_{p'}(x',t'') \otimes (u^*)_{p'}(x''',t''') = \int d^3p'' u_{p'}(x',t'') \otimes (u^*)_{p'}(x''',t''') = \frac{\text{vol}^{-1}}{q^3}(\odot (\kappa^{-1}x'''))\quad (137)
\]
we can carry out the integral with respect to \( x''' \) on the right-hand side of Eq. 136:

\[
\int d^3q'' (S_L^*_{p''} \circ (S_R^\pm)_{p'}) = \\
\quad = \lim_{\nu' \to \mp \infty} \lim_{\nu'' \to \mp \infty} \int d^3q' d^3q'' d^3q (u^*)_{p'} (x') \circ (G_R^*)_{p'}(x'') \circ (G_L^*)_{p'}(x'; x'') \circ u_p(x).
\]

Due to Eqs. 89 and 97 of Chap. 3, we have:

\[
\int d^3q'' (G_L^*)_{p''} (x'; x'') \circ (G_R^*), x''; x) = \text{vol}^{-1}\delta^3_q (x' + (\sigma \kappa^{-1} x)). \quad (139)
\]

If we apply this identity to the right-hand side of Eq. 138, we can also carry out the integral with respect to \( x' \):

\[
\int d^3q'' (S_L^*_{p''} \circ (S_R^\pm)_{p'}) = \\
\quad = \lim_{\nu' \to \mp \infty} \lim_{\nu'' \to \mp \infty} \int d^3q' d^3q'' d^3q (u^*)_{p'} (x') \circ \text{vol}^{-1}\delta^3_q (x' + (\sigma \kappa^{-1} x)) \circ u_p(x) \\
\quad = \lim_{\nu' \to \mp \infty} \int d^3q' (u^*)_{p'} (x', t) \circ u_p(x, t) = \text{vol}^{-1}\delta^3_q ((\sigma \kappa^{-1} p') \circ p). \quad (140)
\]

By similar reasoning, we obtain:

\[
\int d^3q'' (S_L^*_{p''} \circ (S_R^\pm)_{p'}) = \text{vol}^{-1}\delta^3_q (p' \circ (\sigma \kappa^{-1} p)). \quad (141)
\]

Since the \( q \)-deformed plane waves form a complete system of functions, we can interpret the identities in Eqs. 140 and 141 as \( q \)-versions of the unitarity condition of the \( S \)-matrix.

The normalizations of the wave functions do not change by scattering due to the unitarity of the \( S \)-matrix. We can show this fact by the following calculation:

\[
\int d^3q' \phi_L^*(x, t) \circ \phi_R(x, t) = \\
\quad = \lim_{\nu', \nu'' \to \mp \infty} \int d^3q'' d^3q' d^3q \phi_L^*(x'') \circ (G_L^*)_{p'}(x'') \circ (G_R^*)_{p'}(x'; x') \circ \phi_R(x') \\
\quad = \text{vol}^{-1}\lim_{\nu', \nu'' \to \mp \infty} \int d^3q'' d^3q' \phi_L^*(x'') \circ \delta^3_q (x'' + (\sigma \kappa^{-1} x')) \circ \phi_R(x') \\
\quad = \lim_{\nu' \to \mp \infty} \int d^3q' \phi_L^*(x', t') \circ \phi_R(x', t') \\
\quad = \int d^3q' \phi_L^*(x, t) \circ \phi_R(x, t). \quad (142)
\]

First, we have applied the identities in Eqs. 72 and 73 of Chap. 3. The next step follows from Eq. 139. Due to the \( q \)-deformed delta function, we could
carry out the integral with respect to $x''$. Moreover, we have identified the time variables $t'$ and $t''$ since they approach the same limit. The last identity holds since the normalization of the wave functions for a free particle is time-independent [16]. We can verify the following identity by similar reasoning:

$$
\int d^3_\gamma x (\psi_L)^\pm(x, t) \otimes (\psi_R^*)^\pm(x, t) = \int d^3_\gamma x \phi_L(x, t) \otimes \phi_R^*(x, t).
$$

(143)

4.4 Scattering of a scalar particle

We can also employ our reasonings in the case of a $q$-deformed scalar particle. To this end, we extend $q$-versions of the free Klein-Gordon equation [see Eqs. (33), (35), and (38) in Chap. 2] by an interaction term that depends on a real scattering potential $V$, i.e.

$$
c^{-2} \partial_t^2 \phi_R - \nabla^2_q \phi_R + (mc_0)^2 \phi_R = -V \otimes \phi_R,
$$

$$
c^{-2} \partial_t^2 \phi_R^* - \nabla^2_q \phi_R^* + (mc_0)^2 \phi_R^* = -V \otimes \phi_R^*,
$$

(144)

or

$$
\phi_L \otimes \partial_t^2 c^{-2} - \phi_L \otimes \nabla^2_q + \phi_L (mc_0)^2 = -\phi_L \otimes V,
$$

$$
\phi_L^* \otimes \partial_t^2 c^{-2} - \phi_L^* \otimes \nabla^2_q + \phi_L^* (mc_0)^2 = -\phi_L^* \otimes V.
$$

(145)

We can get Lippmann-Schwinger equations for the solutions to the above $q$-deformed Klein-Gordon equations by using the propagators for the $q$-versions of the free Klein-Gordon equation. For example, if we replace the inhomogeneity $\rho$ in Eq. (59) of Chap. 2 with the interaction term $-V \otimes \phi_R$ or $-\phi_L \otimes V$, we find

$$
\phi_R(x) = \varphi_R(x) - i \int d^3_\gamma x' dt' \Delta_R(x; x') \otimes V(x') \otimes \phi_R(x'),
$$

$$
\phi_L(x) = \varphi_L(x) + i \int d^3_\gamma x' dt' \phi_L(x') \otimes V(x') \otimes \Delta_L(x'; x).
$$

(146)

By similar reasoning, we get:

$$
\phi_R^*(x) = \varphi_R^*(x) - i \int d^3_\gamma x' dt' \Delta_R^*(x; x') \otimes V(x') \otimes \phi_R^*(x'),
$$

$$
\phi_L^*(x) = \varphi_L^*(x) + i \int d^3_\gamma x' dt' \phi_L^*(x') \otimes V(x') \otimes \Delta_L^*(x'; x).
$$

(147)

Next, we introduce S-matrix elements for the scattering of a $q$-deformed scalar particle. Similarly to the definitions in Eqs. (118)-(121) of Chap. 4.1, we denote the solutions to the $q$-versions of the free Klein-Gordon equations by $\varphi_L$ or $\varphi_R$. 

24
have

\[(S_R)^\pm(\varphi, \phi) = \lim_{t \to \pm\infty} ic^{-2} \int d^3x \left[ \left( \phi^*_L \right)^\pm(x) \triangleleft \partial_t \triangleright \phi_R(x) + \left( \varphi^*_R \right)^\pm(x) \triangleright \partial_t \triangleleft \phi_R(x) \right], \quad (148)\]

\[(S_L)^\pm(\phi, \varphi) = \lim_{t \to \pm\infty} ic^{-2} \int d^3x \left[ \phi_L(x) \triangleleft \partial_t \triangleright \varphi^*_R(x) + \phi_L(x) \triangleright \partial_t \triangleleft \varphi^*_R(x) \right], \quad (149)\]

and

\[(S^*_R)^\pm(\varphi, \phi) = \lim_{t \to \pm\infty} ic^{-2} \int d^3x \left[ \left( \phi^*_L \right)^\pm(x) \triangleleft \partial_t \triangleright \varphi^*_R(x) + \left( \varphi^*_R \right)^\pm(x) \triangleright \partial_t \triangleleft \varphi^*_R(x) \right], \quad (150)\]

\[(S^*_L)^\pm(\phi, \varphi) = \lim_{t \to \pm\infty} ic^{-2} \int d^3x \left[ \phi^*_L(x) \triangleleft \partial_t \triangleright \phi_R(x) + \phi^*_L(x) \triangleright \partial_t \triangleleft \phi_R(x) \right]. \quad (151)\]

The plus or minus sign indicates whether we consider particle states with positive or negative energy after scattering [15].

We consider a situation in which a free scalar particle starts with momentum \(p\). After it has taken part in an interaction, the particle will reappear as a free particle with momentum \(p'\). For this reason, the wave function for the final free particle is given by [see Eq. (44) in Chap. 2]:

\[(\varphi^*_L)^\pm(x, t) = (\varphi^*_{p'})^{\pm}(x, t) = \frac{c}{\sqrt{2}} E_{p'}^{-1/2} \circ \exp(it E_p) \circ (u^*)_p(x). \quad (152)\]

Since the wave function \(\phi_R(x, t)\) for the interacting particle evolves from a state with positive energy and momentum \(p\) in the distant past, it holds

\[\lim_{t \to -\infty} [\phi_R(x, t) - \varphi^{[\pm]}_{p}(x, t)] = 0 \quad (153)\]

with [see Eq. (40) in Chap. 2]

\[\varphi^{[\pm]}_{p}(x, t) = \frac{c}{\sqrt{2}} u_p(x) \circ E_{p'}^{-1/2} \circ \exp(-it E_p). \quad (154)\]

The wave function \(\phi_R(x, t)\) is a solution to the first Klein-Gordon equation in (144) and satisfies the boundary condition in Eq. (153). For this reason, \(\phi_R(x, t)\) has to be a solution to the first Lippmann-Schwinger equation in Eq. (146):

\[\phi_R(x, t) = \varphi^{[\pm]}_{p}(x, t) - i \int d^3x' dt' \Delta_R(x; x') \circ V(x') \circ \phi_R(x'). \quad (155)\]
If we plug the expressions of Eqs. (152) and (155) into the first formula of Eq. (159), we finally obtain:

\[
\begin{align*}
(S_R)^+_{p'p} &= \lim_{t \to +\infty} \text{ic}^{-2} \int d^3x \left[ (\varphi^+)^{[+]}_{p'} \circ \partial_t \circ \varphi^+_{p} + (\varphi^+)^{[+]}_{p'} \circ \partial_t \circ \varphi^+_{p} \right] \\
&+ \lim_{t \to +\infty} \text{c}^{-2} \int d^3x \int d^3x' dt' \left[ (\varphi^+)^{[+]}_{p'}(x) \circ \partial_t \circ \Delta_R(x; x') \circ V(x') \circ \phi_R(x') \\
&+ (\varphi^+)^{[+]}_{p'}(x) \circ \partial_t \circ \Delta_R(x; x') \circ V(x') \circ \phi_R(x') \right].
\end{align*}
\]

(156)

The propagator \(\Delta_R\) is subject to the following identity [13]:

\[
\begin{align*}
\text{ic}^{-2} \int d^3x \varphi^+_L(x, t) &\circ \partial_t \circ \Delta_R(x, t; x', t') \\
&+ \text{ic}^{-2} \int d^3x \varphi^+_L(x, t) \circ \partial_t \circ \Delta_R(x, t; x', t') = \\
&= \theta(t - t') (\varphi^+_L)^{[+]}(x', t') - \theta(t - t) (\varphi^+_L)^{[-]}(x', t').
\end{align*}
\]

(157)

For this reason, we have:

\[
\begin{align*}
\text{ic}^{-2} \int d^3x \left[ (\varphi^+)^{[+]}_{p'}(x) \circ \partial_t \circ \Delta_R(x; x') \circ V(x') \circ \phi_R(x') \\
&+ \text{ic}^{-2} \int d^3x \left( (\varphi^+)^{[+]}_{p'}(x) \circ \partial_t \circ \Delta_R(x; x') \circ V(x') \circ \phi_R(x') \right) \right] = \\
&= \theta(t - t') (\varphi^+)^{[+]}_{p'}(x') \circ V(x') \circ \phi_R(x').
\end{align*}
\]

(158)

Using this result and the orthogonality relation in Eq. (18) of Chap. 2, we can write the S-matrix in Eq. (159) as follows [13]:

\[
(S_R)^+_{p'p} = \text{vol}^{-1} \delta^3((\otimes p') \oplus p) - i \int d^3x' dt' (\varphi^+)^{[+]}_{p'}(x') \circ V(x') \circ \phi_R(x').
\]

(159)

There are no limits in the above formula since the orthogonality relations are time-independent. Moreover, the following identity holds:

\[
\lim_{t \to +\infty} \theta(t - t') = 1.
\]

(160)

We can solve the Lippmann-Schwinger equation for the wave function \(\phi_R(x)\) by iteration [see Eq. (155)]:

\[
\phi_R(x) = \varphi^+_{p'}(x) - i \int d^3x' dt' \Delta_R(x; x') \circ V(x') \circ \varphi^+_{p'}(x') \\
+ (-i)^2 \int d^3x' dt' \int d^3x'' dt'' \Delta_R(x; x') \circ V(x') \circ \Delta_R(x'; x'') \circ V(x'') \circ \varphi^+_{p'}(x'') + \ldots
\]

(161)
By substituting this result into the right-hand side of Eq. (159), we get:

\[
(S_R)_p^+ = \text{vol}^{-1} \delta_0^3((\otimes p') \oplus p) - i \int d^3 x' dt' (\varphi^+_p(x')) V(x') \otimes \varphi^+_p(x') \\
- \int d^3 x' dt' \int d^3 x'' dt'' (\varphi^+_p(x')) \otimes V(x') \\
\oplus \Delta_R(x', x'') \oplus V(x'') \otimes \varphi^+_p(x'') + \ldots
\]  

(162)

We can modify the above considerations to apply them to antiparticle scattering, pair production, and annihilation.

5 Interaction picture

In Ref. [16], we have discussed two formalisms of quantum dynamics on the q-deformed Euclidean space: the Schrödinger picture and the Heisenberg picture. Whenever the Hamiltonian operator of a quantum system splits into a time-independent part \(H_0\) and a time-dependent interaction \(V(t)\), the interaction picture is often more helpful [25]. The interaction picture also applies to scattering processes in the q-deformed Euclidean space.

We can assume that a particle before scattering is non-interacting. Moreover, we can expand its wave function in terms of q-deformed momentum eigenfunctions [14]. The corresponding expansion coefficients determine the probability for the particle being in a particular momentum eigenstate. After the interaction, we can again expand the wave function in terms of q-deformed momentum eigenfunctions. The expansion coefficients will have changed due to the scattering.

For the wave functions describing the scattered q-deformed particle in the interaction picture, we have the expansions:[3]

\[
\Psi_R(x, t) = \int d^3 p u_p(x) \otimes C_p(t),
\]

\[
\Psi_L(x, t) = \int d^3 p C_p(t) \otimes u_p(x),
\]  

(163)

and

\[
\Psi^*_R(x, t) = \int d^3 p (C^*)_p(t) \otimes (u^*)_p(x),
\]

\[
\Psi^*_L(x, t) = \int d^3 p (u^*)_p(x) \otimes (C^*)_p(t)
\]  

(164)

with

\[
u_p(x) = u_p(x, t = 0), \quad (u^*)_p(x) = (u^*)_p(x, t = 0),
\]

\[
u^p(x) = u^p(x, t = 0), \quad (u^*)_p(x) = (u^*)_p(x, t = 0).
\]  

(165)

We write wave functions of the interaction picture in upper case Greek letters.
The potential $V(t)$ alone determines the time dependence of the expansion coefficients $C_p(t)$, $C^p(t)$, $(C^*)_p(t)$, and $(C^*)^p(t)$.

We can regain the wave functions of the Schrödinger picture from the expansions in Eq. (163) or Eq. (164). To this end, we only need to replace the $q$-deformed momentum eigenfunctions with the corresponding plane wave solutions of the $q$-deformed free Schrödinger equations, i.e.

$$
\psi^*_R(x, t) = \exp(-itH_0) \triangleright \Psi^*_R(x, t) = \int d^3p u_p(x, t) \otimes C_p(t),
$$

$$
\psi^*_L(x, t) = \exp(-itH_0) \triangleright \Psi^*_L(x, t) = \int d^3p (u^*)_p(x, t) \otimes (C^*)_p(t),
$$

and

$$
\psi^*_L(x, t) = C^*_L(x, t) \triangleleft \exp(itH_0) = \int d^3p (C^*)_p(t) \otimes (u^*)_p(x, t),
$$

$$
\psi^*_L(x, t) = L(x, t) \triangleleft \exp(itH_0) = \int d^3p C^p(t) \otimes u^p(x, t).
$$

The wave functions of the Schrödinger picture are solutions to Schrödinger equations with the complete Hamiltonian operator $H$:

$$
i \partial_t \triangleright \psi_R(x, t) = H \triangleright \psi_R(x, t), \quad \psi^*_L(x, t) \triangleleft i \partial_t = \psi^*_L(x, t) \triangleleft H,
$$

$$
i \partial_t \triangleright \psi^*_R(x, t) = H \triangleright \psi^*_R(x, t), \quad \psi^*_L(x, t) \triangleleft i \partial_t = \psi^*_L(x, t) \triangleleft H.
$$

The wave functions of the interaction picture, however, satisfy the equations

$$
i \partial_t \triangleright \Psi_R(x, t) = V_I \triangleright \Psi_R(x, t), \quad \Psi^*_L(x, t) \triangleleft i \partial_t = \Psi^*_L(x, t) \triangleleft V_I,
$$

$$
i \partial_t \triangleright \Psi^*_R(x, t) = V_I \triangleright \Psi^*_R(x, t), \quad \Psi^*_L(x, t) \triangleleft i \partial_t = \Psi^*_L(x, t) \triangleleft V_I,
$$

where $V_I$ is understood to be the potential in the interaction picture:

$$
V_I = \exp(itH_0) V \exp(-itH_0).
$$

We can prove the identities in Eq. (169) in the following way:

$$
i \partial_t \triangleright \Psi_R(x, t) = i \partial_t \triangleright [\exp(itH_0) \triangleright \psi_R(x, t)]
$$

$$
= -H_0 \exp(itH_0) \triangleright \psi_R(x, t) + \exp(itH_0) H \triangleright \psi_R(x, t)
$$

$$
= -H_0 \triangleright \Psi_R(x, t) + \exp(itH_0) H \exp(-itH_0) \triangleright \Psi_R(x, t)
$$

$$
= \exp(itH_0) V \exp(-itH_0) \triangleright \Psi_R(x, t).
$$

First, we have expressed the wave function of the interaction picture by that of the Schrödinger picture. After applying the product rule for the time derivative, we use the Schrödinger equation for $\psi_R$ [cf. Eq. (168)]. In the penultimate step, we switch from the Schrödinger picture to the interaction picture. The final step follows from the decomposition $H = H_0 + V$.  

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To get solutions to the equations in Eq. (169), we introduce time-evolution operators defined by
\[
\Psi_R(x, t) = (\mathcal{U}_R)_I(t, t_0) \Psi_R(x, t_0),
\]
\[
\Psi^*_L(x, t) = (\mathcal{U}^*_L)_I(t, t_0) \Theta \Psi^*_R(x, t_0),
\]
(172)

or
\[
\Psi^*_L(x, t) = \Psi^*_L(x, t_0) \circ (\mathcal{U}^*_L)_I(t_0, t),
\]
\[
\Psi_L(x, t) = \Psi_L(x, t_0) \bowtie (\mathcal{U}_L)_I(t_0, t).
\]
(173)

Due to these definitions and the identities in Eq. (169), the time-evolution operators satisfy the equations
\[
i \partial_t \triangleright (\mathcal{U}_R)_I(t, t_0) = V_I(t)(\mathcal{U}_R)_I(t, t_0),
\]
\[
i \partial_t \bowtie (\mathcal{U}^*_R)_I(t, t_0) = V_I(t)(\mathcal{U}^*_R)_I(t, t_0),
\]
(174)

or
\[
(\mathcal{U}_L)_I(t_0, t) \ll (-\partial_t) \ll (\mathcal{U}_L)_I(t_0, t) = V_I(t),
\]
\[
(\mathcal{U}_L)_I(t_0, t) \ll \partial_t \ll (\mathcal{U}_L)_I(t_0, t) = V_I(t).
\]
(175)

We require that the solutions to the equations in Eq. (172) or Eq. (173) be subject to the following conditions:
\[
(\mathcal{U}_R)_I(t, t) = (\mathcal{U}^*_R)_I(t, t) = (\mathcal{U}_L)_I(t, t) = (\mathcal{U}^*_L)_I(t, t) = 1.
\]
(176)

In this case, the differential equations in Eq. (174) or Eq. (175) are equivalent to the integral equations
\[
(\mathcal{U}_R)_I(t, t_0) = 1 - i \int_{t_0}^{t} dt' V_I(t') (\mathcal{U}_R)_I(t', t_0),
\]
\[
(\mathcal{U}^*_R)_I(t, t_0) = 1 - i \int_{t_0}^{t} dt' V_I(t') (\mathcal{U}^*_R)_I(t', t_0),
\]
(177)

or
\[
(\mathcal{U}_L)_I(t_0, t) = 1 + i \int_{t_0}^{t} dt' (\mathcal{U}^*_L)_I(t_0, t') V_I(t'),
\]
\[
(\mathcal{U}_L)_I(t_0, t) = 1 + i \int_{t_0}^{t} dt' (\mathcal{U}_L)_I(t_0, t') V_I(t').
\]
(178)

Solving these integral equations by iteration, we obtain *q-versions of the Dyson series* i.e.
\[
(\mathcal{U}_R)_I(t, t_0) = 1 + \sum_{n=1}^{\infty} i^{-n} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n V_I(t_1) \cdots V_I(t_n),
\]
\[
(\mathcal{U}^*_R)_I(t, t_0) = 1 + \sum_{n=1}^{\infty} i^{-n} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n V_I(t_1) \cdots V_I(t_n),
\]
(179)
or

\begin{align*}
(\mathcal{U}_L)(t_0, t) &= 1 + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n V_I(t_n) \ldots V_I(t_1), \\
(\mathcal{U}_R)(t_0, t) &= 1 + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n V_I(t_n) \ldots V_I(t_1). (180)
\end{align*}

Note that the product of the potentials is time-ordered. For this reason, we can write the above expansions as follows [15]:

\begin{align*}
(\mathcal{U}_L)(t_0, t) &= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n \hat{T}^+ [V_I(t_1) \ldots V_I(t_n)], \\
(\mathcal{U}_R)(t_0, t) &= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n \hat{T}^- [V_I(t_n) \ldots V_I(t_1)]. (181)
\end{align*}

The time-ordered products are defined by

\begin{align*}
\hat{T}^+ [V_I(t_1) \ldots V_I(t_n)] &= V_I(t_{i_1}) V_I(t_{i_2}) \ldots V_I(t_{i_n}), \\
\hat{T}^- [V_I(t_1) \ldots V_I(t_n)] &= V_I(t_{i_n}) V_I(t_{i_{n-1}}) \ldots V_I(t_{i_1}). (182)
\end{align*}

if \( \{i_1, i_2, \ldots, i_n\} \) denotes a permutation of the natural numbers from 1 to \( n \), such that \( t_{i_1} \geq t_{i_2} \geq \ldots \geq t_{i_n} \).

The \( q \)-deformed momentum eigenfunctions in Eq. (165) form a complete and orthogonal system [cf. Eq. (15) and Eq. (16) in Chap. 2]. The coefficients for the expansions in Eqs. (163) and (164) can be calculated by

\begin{align*}
C_p(t) &= \int d^3x (u^*_p(x) \otimes \Psi_R(x, t)), \\
C^p(t) &= \int d^3x \Psi_L(x, t) \otimes (u^*)^p(x), \quad (183)
\end{align*}

and

\begin{align*}
(C^*_p)(t) &= \int d^3x \Psi^*_L(x, t) \otimes u^*_p(x), \\
(C^*_p)(t) &= \int d^3x u^p(x) \otimes \Psi^*_R(x, t). \quad (184)
\end{align*}

These coefficients determine the probability for a particle to be in a particular momentum eigenstate after its interaction. In a scattering experiment, they are related to the S-matrix.

Remember that only the potential \( V \) determines the time evolution of the wave function in the interaction picture. For this reason, the wave function of a
free particle does not depend on time in the interaction picture. It follows from Eqs. (166) and (167) that the wave function of a free particle at time $t = 0$ in the Schrödinger picture is the same as in the interaction picture:

$$
\Phi_R(x) = \phi_R(x, t = 0), \quad \Phi_R^*(x) = \phi_R^*(x, t = 0),
\Phi_L(x) = \phi_L(x, t = 0), \quad \Phi_L^*(x) = \phi_L^*(x, t = 0).
$$

(185)

In a scattering experiment, a particle is free in the distant past or distant future. For this reason, the wave functions for a scattered particle in the interaction picture satisfy the conditions

$$
\lim_{t \to \pm \infty} (\Psi_R)^\pm(x, t) = \Phi_R(x),
\lim_{t \to \pm \infty} (\Psi_L)^\pm(x, t) = \Phi_L(x),
$$

(186)

and

$$
\lim_{t \to \pm \infty} (\Psi_R^*)^\pm(x, t) = \Phi_R^*(x),
\lim_{t \to \pm \infty} (\Psi_L^*)^\pm(x, t) = \Phi_L^*(x).
$$

(187)

If we take into account Eq. (172) or Eq. (173), the above conditions imply

$$
(\Psi_R)^+(x, t) = (\mathbb{U}_R)_{I}(t, -\infty) \triangleright \Phi_R(x),
(\Psi_R^*)^+(x, t) = (\mathbb{U}_R^*)_{I}(t, -\infty) \triangleright \Phi_R^*(x),
$$

(188)

or

$$
(\Psi_L)^+(x, t) = \Phi_L(x) \triangleright (\mathbb{U}_L)_{I}(-\infty, t),
(\Psi_L^*)^+(x, t) = \Phi_L^*(x) \triangleright (\mathbb{U}_L^*)_{I}(-\infty, t).
$$

(189)

Thus, the matrix representations of the time evolution operator in the interaction picture determine the S-matrices for a scattering experiment [26], i.e.

$$
(S_R)^+(\Phi, \Psi) = \lim_{t \to \infty} \int d^3x \Phi_R^*(x) \otimes (\Psi_R)^+(x, t)
= \int d^3x \Phi_R^*(x) \otimes (\mathbb{U}_R)_{I}(\infty, -\infty) \triangleright \Phi_R(x),
$$

(190)

$$
(S_L)^+(\Phi, \Psi) = \lim_{t \to \infty} \int d^3x (\Psi_L)^+(x, t) \otimes \Phi_R^*(x)
= \int d^3x \Phi_L(x) \triangleright (\mathbb{U}_L)_{I}(-\infty, \infty) \otimes \Phi_R^*(x),
$$

(191)
and
\[
(S^+_R)(\Phi, \Psi) = \lim_{t \to \infty} \int d^3q x \Phi_L(x) \otimes (\Phi^+_R)(x, t)
\]
\[
= \int d^3q x \Phi_L(x) \otimes (U^*_R)_{t}(\infty, -\infty) \odot \Phi^+_R(x),
\]
(192)
\[
(S^+_L)(\Phi, \Psi) = \lim_{t \to \infty} \int d^3q x (\Psi^+_L)(x, t) \otimes \Phi_R(x)
\]
\[
= \int d^3q x \Phi^+_L(x) \odot (U^*_L)_{t}(-\infty, \infty) \odot \Phi_L(x).
\]
(193)

6 Time-dependent perturbation theory

We describe how a time-dependent perturbation causes transitions between the \(q\)-deformed momentum eigenstates of a nonrelativistic particle. For this purpose, we assume that the particle moves in the presence of a weak potential \(V(x, t)\). The corresponding Schrödinger equations read as [cf. Eqs. (65) and (66) in Chap. 3]
\[
i \partial_t \triangleright \psi_R(x, t) - H_0 \triangleright \psi_R(x, t) = V(x, t) \otimes \psi_R(x, t),
\]
\[
i \partial_t \triangleright \psi^*_R(x, t) - H_0 \triangleright \psi^*_R(x, t) = V(x, t) \otimes \psi^*_R(x, t),
\]
(194)
or
\[
\psi_L(x, t) \triangleright \partial_t i - \psi_L(x, t) \triangleright H_0 = \psi_L(x, t) \otimes V(x, t),
\]
\[
\psi^*_L(x, t) \triangleright \partial_t i - \psi^*_L(x, t) \triangleright H_0 = \psi^*_L(x, t) \otimes V(x, t).
\]
(195)

We can expand their solutions in terms of \(q\)-deformed plane waves as the latter form a complete and orthogonal system [cf. Eq. (15) and Eq. (16) in Chap. 2]. Concretely, we have
\[
\psi_R(x, t) = \int d^3p u_p(x, t) \otimes c_p(t),
\]
\[
\psi_L(x, t) = \int d^3p c^p(t) \otimes u^p(x, t),
\]
(196)
or
\[
\psi^*_R(x, t) = \int d^3p (u^*)^p(x, t) \otimes (c^*)^p(t),
\]
\[
\psi^*_L(x, t) = \int d^3p (c^*)^p(x, t) \otimes (u^*)^p(x, t).
\]
(197)

We plug these expansions into the Schrödinger equations (194) or (195). If we take into account the identities
\[
i \partial_t \triangleright u_p(x, t) = H_0 \triangleright u_p(x, t),
\]
\[
u^p(x, t) \triangleright \partial_t i = u^p(x, t) \triangleright H_0,
\]
(198)
Due to Eqs. (201), (203), and (204), the expansion coefficients satisfy the integral

\[ i \partial_t \triangleright (u^*)_p(x, t) = H_0 \triangleright (u^*)_p(x, t), \]

\[ (u^*)_p(x, t) \triangleleft \partial_t i = (u^*)_p(x, t) \triangleleft H_0, \tag{199} \]

we obtain, for example:

\[ \int d^3q [u_p(x, t) \odot i \partial_t \triangleright c_p(t) - V(x, t) \odot u_p(x, t) \odot c_p(t)] = 0. \tag{200} \]

We multiply this equation by \((u^*)_p'(x, t)\) from the left and integrate over all space. Due to the orthogonality of \(q\)-deformed plane waves, we get

\[ i \partial_t \triangleright c_{p'}(t) = \int d^3q e^{i\tau p'} \odot V_{p'p}(t) \odot e^{-i\tau p} \odot c_p(t) \tag{201} \]

with

\[ V_{p'p}(t) = \int d^3q x (u^*)_p(x) \odot V(x, t) \odot u_p(x). \tag{202} \]

Similarly, we can show

\[ c_{p'}(t) \triangleright \partial_t i = \int d^3q p c_p(t) \odot e^{i\tau p} \odot V_{pp'}(t) \odot e^{-i\tau p'} \tag{203} \]

and

\[ i \partial_t \triangleright (c^*)_{p'}(t) = \int d^3q e^{i\tau p'} \odot V_{p'p}(t) \odot e^{-i\tau p} \odot (c^*)_p(t), \]

\[ (c^*)_p(t) \triangleright \partial_t i = \int d^3q p (c^*)_p(t) \odot e^{i\tau p} \odot V_{pp'}(t) \odot e^{-i\tau p'} \tag{204} \]

with

\[ V_{p'p}(t) = \int d^3q x u^p(x) \odot V(x, t) \odot (u^*)_p(x). \tag{205} \]

Due to Eqs. (201), (203), and (204), the expansion coefficients satisfy the integral equations

\[ c_{p'}(t) = c_{p'}(0) + i \int_0^t \! d\tau \int d^3q (V_i)_{p'p}(\tau) \odot c_p(\tau), \]

\[ c_{p'}(t) = c_{p'}(0) + i \int_0^t \! d\tau \int d^3q c_p(\tau) \odot (V_i)^{pp'}(\tau), \tag{206} \]

and

\[ (c^*)_{p'}(t) = (c^*)_{p'}(0) + i \int_0^t \! d\tau \int d^3q (V_i)^{p'p}(\tau) \odot (c^*)_p(\tau), \]

\[ (c^*)_{p'}(t) = (c^*)_{p'}(0) + i \int_0^t \! d\tau \int d^3q (c^*)_p(\tau) \odot (V_i)^{pp'}(\tau). \tag{207} \]
with
\[(V_I)^p_0(t) = e^{i \varepsilon_p} \circ V_{p_0}(t) \circ e^{-i \varepsilon_p},\]
\[(V_I)^p_0(t) = e^{i \varepsilon_p} \circ V_{p_0}(t) \circ e^{-i \varepsilon_p}.\] (208)

Solving the above integral equations by iteration, we obtain the perturbation expansions
\[c_p(t) = c_p(0) + \sum_{n=1}^{\infty} i^{-n} \int_0^t \int_0^{t_1} \cdots \int_0^{t_2} \cdots \int_0 \dd t_n \cdots \int_0 \dd t_1 \int_0 \dd p_n \cdots \int_0 \dd p_1 \(V_I)^p_{p_n}(t_n)\]
\[\oplus \cdots \oplus (V_I)^p_{p_2}(t_1) \oplus c_{p_1}(0),\] (209)
\[c^p(t) = c^p(0) + \sum_{n=1}^{\infty} i^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_2} \cdots \int_0 \dd t_n \cdots \int_0 \dd t_1 \int_0 \dd p_n \cdots \int_0 \dd p_1 c^p_{p_n}(0)\]
\[\oplus (V_I)^p_{p_{n-1}}(t_n) \oplus \cdots \oplus (V_I)^p_{p_2}(t_1) \oplus (V_I)^p_{p_1}(0),\] (210)
and
\[(c^*)^p(t) = (c^*)^p(0) + \sum_{n=1}^{\infty} i^{-n} \int_0^t \int_0^{t_1} \cdots \int_0^{t_2} \cdots \int_0 \dd t_n \cdots \int_0 \dd t_1 \int_0 \dd p_n \cdots \int_0 \dd p_1 (V_I)^p_{p_n}(t_n)\]
\[\oplus \cdots \oplus (V_I)^p_{p_{n-1}}(t_n) \oplus \cdots \oplus (V_I)^p_{p_2}(t_1) \oplus (c^*)^p_{p_1}(0),\] (211)
\[(c^*)_p(t) = (c^*)_p(0) + \sum_{n=1}^{\infty} i^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_2} \cdots \int_0 \dd t_n \cdots \int_0 \dd t_1 \int_0 \dd p_n \cdots \int_0 \dd p_1 (c^*)_p_{p_n}(t_n)\]
\[\oplus (V_I)^p_{p_{n-1}}(t_n) \oplus \cdots \oplus (V_I)^p_{p_2}(t_1) \oplus (c^*)_p_{p_1}(0).\] (212)

A Star-products

The three-dimensional $q$-deformed Euclidean space $\mathbb{R}^3_q$ has the generators $X^+$, $X^3$, and $X^-$, subject to the following commutation relations [27]:
\[X^3 X^+ = q^2 X^+ X^3,\]
\[X^3 X^- = q^{-2} X^- X^3,\]
\[X^- X^+ = X^+ X^- + (q - q^{-1}) X^3 X^3.\] (213)

We can extend the algebra of $\mathbb{R}^3_q$ by a time element $X^0$, which commutes with the generators $X^+$, $X^3$, and $X^-$ [16]:
\[X^0 X^A = X^A X^0, \quad A \in \{+, 3, -\}.\] (214)

In the following, we refer to the algebra spanned by the generators $X^i$ with $i \in \{0, +, 3, -\}$ as $\mathbb{R}^{3,t}_q$. 
There is a $q$-analog of the three-dimensional Euclidean metric $g^{AB}$ with its inverse $g_{AB}$ \textsuperscript{(27)} (rows and columns are arranged in the order $+,,3,-$):

$$
\begin{pmatrix}
0 & 0 & -q \\
0 & 1 & 0 \\
-q^{-1} & 0 & 0
\end{pmatrix},
$$

(215)

We can use the $q$-deformed metric to raise and lower indices:

$$
X_A = g_{AB} X^B, \quad X^A = g^{AB} X_B.
$$

(216)

The algebra $\mathbb{R}^3_q,t$ has a semilinear, involutive, and anti-multiplicative mapping, which we call quantum space conjugation. If we indicate conjugate elements of a quantum space by a bar \textsuperscript{6} we can write the properties of quantum space conjugation as follows ($\alpha, \beta \in \mathbb{C}$ and $u, v \in \mathbb{R}^3_q,t$):

$$
\overline{\alpha u + \beta v} = \overline{\alpha} \overline{u} + \overline{\beta} \overline{v}, \quad \overline{u} = u, \quad \overline{uv} = \overline{v} \overline{u}.
$$

(217)

The conjugation on the algebra $\mathbb{R}^3_q,t$ is compatible with the commutation relations in Eqs. \textsuperscript{(213)} and \textsuperscript{(214)} if the following applies \textsuperscript{[16]}:

$$
\overline{X_A} = X_A = g_{AB} X^B, \quad \overline{X^0} = X_0.
$$

(218)

We can only prove a physical theory if it predicts measurement results. The problem, however, is: How can we associate the elements of the non-commutative space $\mathbb{R}^3_q,t$ with real numbers? One solution to this problem is to introduce a vector space isomorphism between the noncommutative algebra $\mathbb{R}^3_q,t$ and a corresponding commutative coordinate algebra $\mathbb{C}[x^+, x^3, x^-, t]$.

We can write each element $F \in \mathbb{R}^3_q,t$ uniquely as a finite or infinite linear combination of monomials with a given normal ordering (Poincaré-Birkhoff-Witt property):

$$
F = \sum_{n_+ \ldots n_0} a_{n_+ \ldots n_0} (x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-}(x^0)^{n_0}, \quad a_{n_+ \ldots n_0} \in \mathbb{C}.
$$

(219)

For this reason, we can define a vector space isomorphism

$$
\mathcal{W} : \mathbb{C}[x^+, x^3, x^-, t] \rightarrow \mathbb{R}^3_q,t
$$

(220)

with

$$
\mathcal{W}((x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-}(x^0)^{n_0}) = (x^+)^{n_+}(x^3)^{n_3}(x^-)^{n_-}(x^0)^{n_0}.
$$

(221)

In general, we have

$$
\mathbb{C}[x^+, x^3, x^-, t] \ni f \mapsto F \in \mathbb{R}^3_q,t,
$$

(222)

\textsuperscript{6}A bar over a complex number indicates complex conjugation.
where

\[
    f = \sum_{n_+, \ldots, n_0} a_{n_+, \ldots, n_0} (x^+)^{n_+} (x^3)^{n_3} (x^-)^{n_-} t^{n_0},
\]

\[
    F = \sum_{n_+, \ldots, n_0} a_{n_+, \ldots, n_0} (X^+)^{n_+} (X^3)^{n_3} (X^-)^{n_-} (X^0)^{n_0}.
\]  (223)

The vector space isomorphism \( W \) is nothing else than the Moyal-Weyl mapping, which gives an operator \( F \) to a complex valued function \( f \) [17][20].

We can extend this vector space isomorphism to an algebra isomorphism if we introduce a new product on the commutative coordinate algebra. This so-called star-product symbolized by \( \ast \) satisfies the following homomorphism condition:

\[
    W(f \ast g) = W(f) \cdot W(g).
\]  (224)

Since the Moyal-Weyl mapping is invertible, we can write the star-product as follows:

\[
    f \ast g = W^{-1}(W(f) \cdot W(g)).
\]  (225)

To get explicit formulas for calculating star-products, we first have to write a noncommutative product of two normal-ordered monomials as a linear combination of normal-ordered monomials again (see Ref. [21] for details):

\[
    (X^+)^{n_+} \ldots (X^0)^{n_0} \ast (X^+)^{m_+} \ldots (X^0)^{m_0} = \sum_{k=0}^\infty B_{\frac{n}{k}}^{\frac{m}{k}} (X^+)^{k_+} \ldots (X^0)^{k_0}.
\]  (226)

We achieve this by using the commutation relations for the noncommutative coordinates [cf. Eq. (213)]. From the concrete form of the expansion in Eq. (226), we can finally read off a formula to calculate the star-product of two power series in commutative space-time coordinates (\( \lambda = q - q^{-1} \))\(^7\)

\[
    f(x, t) \ast g(x, t) = \sum_{k=0}^\infty \lambda^k \frac{(x^3)^{2k}}{[k]_q!} q^{2(\hat{n}_{A+} + \hat{n}_{A-})} D_{q^k, x} f(x, t) D_{q^k, x'} g(x', t) \bigg|_{x' \to x}.
\]  (227)

The expression above depends on the operators

\[
    \hat{n}_A = x^A \frac{\partial}{\partial x^A}
\]  (228)

and the so-called Jackson derivatives [28]:

\[
    D_{q^k, x} f = \frac{f(q^k x) - f(x)}{q^k x - x}.
\]  (229)

The q-numbers are given by

\[
    [[a]]_q = \frac{1 - q^a}{1 - q},
\]  (230)

\(^7\)The argument \( x \) indicates a dependence on the spatial coordinates \( x^+, x^3 \), and \( x^- \).
and the $q$-factorials are defined in complete analogy to the undeformed case:

$$[n]_q! = [1]_q[2]_q \cdots [n-1]_q[n]_q, \quad [0]_q! = 1. \quad (231)$$

The algebra isomorphism $W^{-1}$ enables us to carry over the conjugation for the quantum space algebra $\mathbb{R}^3_q$ to the commutative coordinate algebra $\mathbb{C}[x^+, x^3, x^-, t]$. In other words, the mapping $W^{-1}$ is a $\ast$-algebra homomorphism:

$$W(\overline{f}) = \overline{W(f)} \quad \iff \quad f = W^{-1}(W(\overline{f})). \quad (232)$$

This relationship implies the following property for the star-product:

$$f \ast g = g \ast f. \quad (233)$$

With $\overline{f}$, we designate the power series obtained from $f$ by quantum space conjugation. It follows from Eq. (218) and Eq. (232) that $\overline{f}$ takes the following form (if $\overline{a}_{n_+, n_3, n_-, n_0}$ stands for the complex conjugate of $a_{n_+, n_3, n_-, n_0}$) [16, 24]:

$$\overline{f(x,t)} = \sum_{n} \overline{a}_{n_+, n_3, n_-, n_0} (-q x^-)^{n_+} (x^3)^{n_3} (-q^{-1} x^+)^{n_-} t^{n_0}$$

$$= \sum_{n} (-q)^{n_-} (-q)^{n_+} \overline{a}_{n_-, n_3, n_+, n_0} (x^+)^{n_+} (x^3)^{n_3} (x^-)^{n_-} t^{n_0}$$

$$= \overline{f(x,t)}. \quad (234)$$

### B Partial derivatives and integrals

There are partial derivatives for $q$-deformed space-time coordinates [29, 30]. These partial derivatives again form a quantum space with the same algebraic structure as that of the $q$-deformed space-time coordinates. Thus, the $q$-deformed partial derivatives $\partial_i$ satisfy the same commutation relations as the covariant coordinate generators $X_i$:

$$\partial_0 \partial_+ = \partial_+ \partial_0, \quad \partial_0 \partial_- = \partial_- \partial_0, \quad \partial_0 \partial_3 = \partial_3 \partial_0,$$

$$\partial_+ \partial_3 = q^2 \partial_3 \partial_+, \quad \partial_3 \partial_- = q^2 \partial_- \partial_3,$$

$$\partial_+ \partial_- - \partial_- \partial_+ = \lambda \partial_3 \partial_3. \quad (235)$$

The commutation relations above are invariant under conjugation if the derivatives show the following conjugation properties\footnote{The indices of partial derivatives are raised and lowered in the same way as those of coordinates [see Eq. (216) in App. A].}

$$\overline{\partial_A} = -\partial^A = -g^{AB} \partial_B, \quad \overline{\partial_0} = -\partial^0 = -\partial_0. \quad (236)$$

There are two ways of commuting $q$-deformed partial derivatives with $q$-deformed space-time coordinates. One is given by the following $q$-deformed Leibniz
rules \[16,29,30\]:
\[
\partial_B X^A = \delta^A_B + q^4 \hat{R}^{AC}_{BD} X^D \partial_C,
\]
\[
\partial_A X^0 = X^0 \partial_A,
\]
\[
\partial_0 X^A = X^A \partial_0,
\]
\[
\partial_0 X^0 = 1 + X^0 \partial_0.
\]
(237)

\(\hat{R}^{AC}_{BD}\) denotes the vector representation of the R-matrix for the three-dimensional q-deformed Euclidean space \[31\]. Introducing \(\hat{\partial}_A = q^6 \partial_A\) and \(\hat{\partial}_0 = \partial_0\), we can write the Leibniz rules for the second differential calculus in the following form:
\[
\hat{\partial}_B X^A = \delta^A_B + q^{-4} (\hat{R}^{-1})^{AC}_{BD} X^D \hat{\partial}_C,
\]
\[
\hat{\partial}_A X^0 = X^0 \hat{\partial}_A,
\]
\[
\hat{\partial}_0 X^A = X^A \hat{\partial}_0,
\]
\[
\hat{\partial}_0 X^0 = 1 + X^0 \hat{\partial}_0.
\]
(238)

Using the Leibniz rules in Eq. (237) or Eq. (238), we can calculate how partial derivatives act on normal-ordered monomials of noncommutative coordinates. We can carry over these actions to commutative coordinate monomials with the help of the Moyal-Weyl mapping:
\[
\partial^i \triangleright f(x, t) = W^{-1} (\partial^i \triangleright W(f(x, t))).
\]
(239)

Since the Moyal-Weyl mapping is linear, we can apply the action above to space-time functions that can be written as a power series:
\[
\partial^i \triangleright f(x, t) = W^{-1} (\partial^i \triangleright W(f(x, t))).
\]
(240)

If we use the ordering given in Eq. (221), the Leibniz rules in Eq. (237) lead to the following operator representations \[22\]:
\[
\partial_+ \triangleright f(x, t) = D_{q^4, x^+} f(x, t),
\]
\[
\partial_3 \triangleright f(x, t) = D_{q^2, x^3} f(q^2 x^+, x^3, x^-, t),
\]
\[
\partial^- \triangleright f(x, t) = D_{q^4, x^+} f(x^+, q^2 x^3, x^-, t) + \lambda x^+ D_{q^4, x^3}^2 f(x, t).
\]
(241)

The derivative \(\partial_0\), however, is represented on the commutative space-time algebra by an ordinary partial derivative:
\[
\partial_0 \triangleright f(x, t) = \partial_t \triangleright f(x, t) = \frac{\partial f(x, t)}{\partial t}.
\]
(242)

Using the Leibniz rules in Eq. (238), we get operator representations for the partial derivatives \(\hat{\partial}_i\). The Leibniz rules in Eq. (237) transform into those in Eq. (238) by the following substitutions:
\[
q \rightarrow q^{-1}, \quad X^- \rightarrow X^+, \quad X^+ \rightarrow X^-,
\]
\[
\partial^+ \rightarrow \hat{\partial}^-, \quad \partial^- \rightarrow \hat{\partial}^+, \quad \partial^3 \rightarrow \hat{\partial}^3, \quad \partial^0 \rightarrow \hat{\partial}^0.
\]
(243)
For this reason, we obtain the operator representations of the partial derivatives \( \bar{\partial}_A \) from those of the partial derivatives \( \partial A \) [cf. Eq. (241)] if we replace \( q \) with \( q^{-1} \) and exchange the indices + and −:

\[
\begin{align*}
\hat{\partial}_- \triangleright f(x,t) &= D_{q^{-1}x^-} f(x,t), \\
\hat{\partial}_3 \triangleright f(x,t) &= D_{q^{-2}x^-} f(q^{-2}x^-, x^3, x^+, t), \\
\hat{\partial}_+ \triangleright f(x,t) &= D_{q^{-1}x^-} f(q^{-1}x^-, q^{-2}x^3, x^+, t) - \lambda x^- D_{q^{-2}x^-} f(x,t). \tag{244}
\end{align*}
\]

Once again, \( \hat{\partial}_0 \) is represented on the commutative space-time algebra by an ordinary partial derivative:

\[
\hat{\partial}_0 \triangleright f(x,t) = \partial_t \triangleright f(x,t) = \frac{\partial f(x,t)}{\partial t}. \tag{245}
\]

Due to the substitutions given in Eq. (243), the actions in Eqs. (244) and (245) refer to normal-ordered monomials different from those in Eq. (221):

\[
\hat{W} \left( t^0 x^+ n_0 (x^3)^n_3 (x^-)^n_- \right) = (X^0)^{n_0} (X^-)^{n_-} (X^3)^{n_3} (X^+)^{n_+}. \tag{246}
\]

We should not forget that we can also commute \( q \)-deformed partial derivatives from the right side of a normal-ordered monomial to the left side by using the Leibniz rules. This way, we get so-called right representations of partial derivatives, for which we write \( f \triangleright \hat{\partial}^i \) or \( f \triangleright \hat{\partial}^i \). Note that the operation of conjugation transforms left actions of partial derivatives into right actions and vice versa [22]:

\[
\begin{align*}
\partial_i \triangleright f &= -f \triangleright \partial_i, \\
\bar{\partial}_i \triangleright f &= -f \triangleright \bar{\partial}_i.
\end{align*}
\]

In general, the operator representations in Eqs. (241) and (244) consist of two terms, which we call \( \partial^A_{\text{cla}} \) and \( \partial^A_{\text{cor}} \):

\[
\partial^A \triangleright F = (\partial^A_{\text{cla}} + \partial^A_{\text{cor}}) \triangleright F. \tag{248}
\]

In the undeformed limit \( q \to 1 \), \( \partial^A_{\text{cla}} \) becomes an ordinary partial derivative, and \( \partial^A_{\text{cor}} \) disappears. We get a solution to the difference equation \( \partial^A \triangleright F = f \) with given \( f \) by using the following formula [23]:

\[
F = (\partial^A_{\text{cla}})^{-1} \triangleright f = (\partial^A_{\text{cla}} + \partial^A_{\text{cor}})^{-1} \triangleright f = \sum_{k=0}^{\infty} \left[ -(\partial^A_{\text{cla}})^{-1} \partial^A_{\text{cor}} \right]^k (\partial^A_{\text{cla}})^{-1} \triangleright f. \tag{249}
\]

Applying the above formula to the operator representations in Eq. (241), we get

\[
\begin{align*}
(\partial_-)^{-1} \triangleright f(x,t) &= D_{q^{-1}x^-}^{-1} f(x,t), \\
(\partial_3)^{-1} \triangleright f(x,t) &= D_{q^{-2}x^3}^{-1} f(q^{-2}x^3, x^-, x^+, t), \tag{250}
\end{align*}
\]
\[
(\partial_-)^{-1} \triangleright f(x, t) = \sum_{k=0}^{\infty} q^{2k(k+1)} \left( -\lambda x^+ D_{q^2 x^+}^{-1} - D_{q^2 x^3}^2 \right)^k D_{q^2 x^-}^{-1} f(x^+, q^{-2(k+1)} x^3, x^-, t). \tag{251}
\]

Note that \( D_{q^2 x}^{-1} \) stands for a Jackson integral with \( x \) being the variable of integration \[32\]. The explicit form of this Jackson integral depends on its limits of integration and the value for the deformation parameter \( q \). If \( x > 0 \) and \( q > 1 \), for example, the following applies:

\[
\int_0^x dq z f(z) = (q - 1)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x). \tag{252}
\]

The integral for the time coordinate is an ordinary integral since \( \partial_0 \) acts on the commutative space-time algebra like an ordinary partial derivative \[cf. \ Eq. (242)\]:

\[
(\partial_0)^{-1} \triangleright f(x, t) = \int dt f(x, t). \tag{253}
\]

The above considerations also apply to the partial derivatives with a hat. However, we can obtain the representations of \( \hat{\partial}_i \) from those of the derivatives \( \partial_i \) if we replace \( q \) with \( q^{-1} \) and exchange the indices \(+\) and \(-\). Applying these substitutions to the expressions in Eqs. (250) and (251), we immediately get the corresponding results for the partial derivatives \( \hat{\partial}_i \).

By successively applying the integral operators given in Eqs. (250) and (251), we can explain an integration over all space \[23, 24\]:

\[
\int_{-\infty}^{+\infty} d^3 q x f(x^+, x^3, x^-) = (\partial_-)^{-1} \big|_{-\infty}^{+\infty} (\partial_0)^{-1} \big|_{-\infty}^{+\infty} (\partial_+)^{-1} \big|_{-\infty}^{+\infty} \triangleright f. \tag{254}
\]

On the right-hand side of the above relation, the different integral operators can be simplified to Jackson integrals \[23, 33\]:

\[
\int_{-\infty}^{+\infty} d^3 q x f(x) = D_{q^2 x^+}^{-1} D_{q^2 x^3}^{-1} D_{q^2 x^-}^{-1} f(x). \tag{255}
\]

Note that the Jackson integrals in the formula above refer to a smaller \( q \)-lattice. Using such a smaller \( q \)-lattice ensures that our integral over all space is a scalar with trivial braiding properties \[34\]. Finally, we mention that the \( q \)-integral over all space behaves as follows under quantum space conjugation:

\[
\int_{-\infty}^{+\infty} d^3_q x \triangleright \int_{-\infty}^{+\infty} d^3_q x \bigtriangledown. \tag{256}
\]

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\section{Exponentials and translations}

A \(q\)-deformed exponential is an eigenfunction of each partial derivative of a given \(q\)-deformed quantum space \cite{35,37}. In the following, we consider \(q\)-deformed exponentials that are eigenfunctions for left-actions or right-actions of partial derivatives:

\[
i^{-1} \partial^A \triangleright \exp_q(\mathbf{x}|\mathbf{p}) = \exp_q(\mathbf{x}|\mathbf{p}) \otimes p^A,
\]

\[
\exp_q(i^{-1} \mathbf{p}|\mathbf{x}) \triangleleft \partial^A i^{-1} = p^A \otimes \exp_q(i^{-1} \mathbf{p}|\mathbf{x}).
\] (257)

The \(q\)-exponentials are uniquely defined by the above eigenvalue equations and the following normalization conditions:

\[
\exp_q(\mathbf{x}|\mathbf{p})|_{\mathbf{x}=0} = \exp_q(\mathbf{x}|\mathbf{p})|_{\mathbf{p}=0} = 1,
\]

\[
\exp_q(i^{-1} \mathbf{p}|\mathbf{x})|_{\mathbf{x}=0} = \exp_q(i^{-1} \mathbf{p}|\mathbf{x})|_{\mathbf{p}=0} = 1.
\] (258)

Using the operator representation in Eq. (241), we have found the following expressions for the \(q\)-exponentials of the three-dimensional Euclidean quantum space \cite{37}:

\[
\exp_q(\mathbf{x}|\mathbf{p}) = \sum_{n=0}^{\infty} \frac{(x^+)^n(x^3)^3(x^-)^n(-ip_3)^n(i\mathbf{p})^n}{[[n_+]]_{q^4}[[n_3]]_{q^2}[[n_-]]_{q^4}},
\]

\[
\exp_q(i^{-1} \mathbf{p}|\mathbf{x}) = \sum_{n=0}^{\infty} \frac{(i^{-1} p^+)^n(i^{-1} p^3)^3(i^{-1} p^-)^n(-x^-)^n(x_3)^3(x_+)^n}{[[n_+]]_{q^4}[[n_3]]_{q^2}[[n_-]]_{q^4}}.
\] (259)

If we substitute \(q\) with \(q^{-1}\) in both expressions of Eq. (259), we get two more \(q\)-exponentials, which we designate \(\overline{\exp}_q(\mathbf{x}|\mathbf{p})\) and \(\overline{\exp}_q(i^{-1} \mathbf{p}|\mathbf{x})\). We obtain the eigenvalue equations and normalization conditions of these two \(q\)-exponentials by applying the following substitutions to Eqs. (257) and (258):

\[
\exp_q \rightarrow \overline{\exp}_q, \quad \triangleright \rightarrow \bar{\triangleright}, \quad \bar{\triangleright} \rightarrow \triangleleft, \quad \partial^A \rightarrow \bar{\partial}^A.
\] (260)

We can use \(q\)-exponentials to generate \(q\)-translations. If we replace the momentum coordinates in the expressions for \(q\)-exponentials with derivatives, it applies \cite{24,35,38}:

\[
\exp_q(x|\partial_y) \triangleright g(y) = g(x \bar{\oplus} y),
\]

\[
\overline{\exp}_q(x|\partial_y) \triangleright g(y) = g(x \oplus y),
\] (261)

and

\[
g(y) \triangleleft \exp_q(-\partial_y|x) = g(y \bar{\oplus} x),
\]

\[
g(y) \triangleleft \overline{\exp}_q(-\bar{\partial}_y|x) = g(y \oplus x).
\] (262)
In the case of the three-dimensional $q$-deformed Euclidean space, for example, we can get the following formula for calculating $q$-translations [39]:

$$f(x \oplus y) = \sum_{i_+ = 0}^{\infty} \sum_{i_3 = 0}^{\infty} \sum_{i_4 = 0}^{\infty} \sum_{k = 0}^{\infty} \left(-q^{-1} \lambda \lambda_+\right)^k (x^-)^{i_4} (x^3)^{i_3-k} (y^-)^k \left( [2k]_{q^{-2}} [2]_{q^{-2}} \right)^k \left( [2l]_{q^{-2}} [2]_{q^{-2}} \right)^l \left( [2]_{q^{-2}} [2]_{q^{-2}} \right)^m \times \left( D_{q^{-4}}^1 q^{-4}, y^+ \right)^k \left( D_{q^{-4}}^1 q^{-2}, y^+ \right)^l \left( D_{q^{-4}}^1 q^{-2}, y^+ \right)^m \left( D_{q^{-4}}^1 q^{-2}, y^+ \right)^n \left( q^{2(k-l)} y^-, q^{-2(k-l)} y^3 \right).$$ (263)

In analogy to the undeformed case, $q$-exponentials satisfy addition theorems [24, 35, 36]. Concretely, we have:

$$\exp_q(x \oplus y |ip) = \exp_q(x| \exp_q(y|ip) \oplus ip),$$
$$\exp_q(ix|p \oplus p') = \exp_q(x \oplus \exp_q(x|ip)|ip'), (264)$$

and

$$\overline{\exp}_q(x \oplus y |ip) = \overline{\exp}_q(x| \overline{\exp}_q(y|ip) \oplus ip),$$
$$\overline{\exp}_q(ix|p \oplus p') = \overline{\exp}_q(x \oplus \overline{\exp}_q(x|ip)|ip'). (265)$$

Note that we obtain further addition theorems from the above identities by substituting position coordinates with momentum coordinates and vice versa. For a better understanding of the meaning of the two addition theorems in Eq. (264), we have given their graphic representation in Fig. 1.

The $q$-deformed quantum spaces we have considered so far are so-called braided Hopf algebras [40]. From this point of view, the two versions of $q$-translations are nothing else but realizations of two braided co-products $\Delta$ and $\bar{\Delta}$ on the corresponding commutative coordinate algebras [24]:

$$f(x \oplus y) = (W^{-1} \otimes W^{-1}) \circ \Delta(W(f)),$$
$$f(x \oplus y) = (W^{-1} \otimes W^{-1}) \circ \bar{\Delta}(W(f)). (266)$$
The braided Hopf algebras also have braided antipodes $\mathcal{S}$ and $\bar{\mathcal{S}}$, which can be realized on the corresponding commutative coordinate algebras as well:

$$f(\oplus x) = (W^{-1} \circ \mathcal{S})(W(f)),$$
$$f(\bar{\oplus} x) = (W^{-1} \circ \bar{\mathcal{S}})(W(f)). \quad (267)$$

In the following, we refer to the operations in Eq. (267) as $q$-inversions. In the case of the $q$-deformed Euclidean space, for example, we found the following operator representation for $q$-inversions [39]:

$$\hat{U}^{-1} f(\oplus x) = \sum_{i=0}^{\infty} (-q \lambda \lambda^*)^i \left( \frac{(x^+, x^-)^i}{[2i]_q^{-2i+2}} \right) q^{-2 \tilde{n}_+ (\tilde{n}_+ + \tilde{n}_3) - 2 \tilde{n}_- (\tilde{n}_- + \tilde{n}_3) - \tilde{n}_3 \tilde{n}_3}$$

$$\times D_{q^{-2i}, x^3} f(-q^{2-4i} x^-, -q^{1-2i} x^3, -q^{2-4i} x^+). \quad (268)$$

Note that the operators $\hat{U}$ and $\hat{U}^{-1}$ act on a commutative function $f(x^+, x^3, x^-)$ as follows:

$$\hat{U} f = \sum_{k=0}^{\infty} (-\lambda)^k \left( \frac{x^3)^k}{[k]_q^{-4k}} \right) q^{-2 \tilde{n}_3 (\tilde{n}_+ + \tilde{n}_-) + k} D_{q^{-2i}, x^+}^k D_{q^{-2i}, x^-}^k f,$$
$$\hat{U}^{-1} f = \sum_{k=0}^{\infty} \lambda^k \left( \frac{x^3)^k}{[k]_q^{4k}} \right) q^{2 \tilde{n}_3 (\tilde{n}_+ + \tilde{n}_-) + k} D_{q^4, x^+}^k D_{q^4, x^-}^k f. \quad (269)$$

The braided co-products and braided antipodes satisfy the axioms (also see Ref. [40]),

$$m \circ (\mathcal{S} \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes \mathcal{S}) \circ \Delta = \xi,$$
$$m \circ (\bar{\mathcal{S}} \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes \bar{\mathcal{S}}) \circ \Delta = \bar{\xi}, \quad (270)$$

and

$$(\text{id} \otimes \mathcal{S}) \circ \Delta = \text{id} = (\mathcal{S} \otimes \text{id}) \circ \Delta,$$
$$(\text{id} \otimes \bar{\mathcal{S}}) \circ \Delta = \text{id} = (\bar{\mathcal{S}} \otimes \text{id}) \circ \Delta. \quad (271)$$

In the identities above, we denote the operation of multiplication on the braided Hopf algebra by $m$. The co-units $\xi, \bar{\xi}$ of the two braided Hopf structures are both linear mappings that vanish on the coordinate generators:

$$\xi(X^i) = \bar{\xi}(X^i) = 0. \quad (272)$$

For this reason, we can realize the co-units $\xi$ and $\bar{\xi}$ on a commutative coordinate algebra as follows:

$$\xi(W(f)) = \bar{\xi}(W(f)) = f(x)|_{x=0} = f(0). \quad (273)$$

We can also translate the Hopf algebra axioms in Eqs. (270) and (271) into corresponding rules for $q$-translations and $q$-inversions [14], i.e.

$$f((\oplus x) \oplus x) = f(x \oplus (\oplus x)) = f(0),$$
$$f((\bar{\oplus} x) \bar{\oplus} x) = f(x \bar{\oplus} (\bar{\oplus} x)) = f(0), \quad (274)$$

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Figure 2: Invertibility of $q$-exponentials.

and

$$f(x \oplus y)|_{y=0} = f(x) = f(y \oplus x)|_{y=0},$$
$$f(x \ominus y)|_{y=0} = f(x) = f(y \oplus x)|_{y=0}. \quad (275)$$

Using $q$-inversions, we are able to introduce inverse $q$-exponentials:

$$\exp_q(\tilde{x} | ip) = \exp_q(ip | \tilde{x}). \quad (276)$$

Due to the addition theorems and the normalization conditions of our $q$-exponentials, the following applies:

$$\exp_q(ip \otimes \exp_q(\tilde{x} | ip) \odot p) = \exp_q(x \odot (\tilde{x} | ip) = \exp_q(x | ip)|_{x=0} = 1. \quad (277)$$

For a better understanding of these identities, we have again given their graphic representation in Fig. 2. The conjugate $q$-exponentials $\overline{\exp_q}$ are subject to similar rules, which we obtain from the above identities by using the following substitutions:

$$\exp_q \rightarrow \overline{\exp_q}, \quad \odot \rightarrow \odot, \quad \ominus \rightarrow \ominus. \quad (278)$$

Finally, we describe another way of obtaining $q$-exponentials. For this purpose, we exchange the two tensor factors of a $q$-exponential using the inverse of the so-called universal R-matrix (also see the graphic representation in Fig. 3):

$$\exp_q^*(ip | x) = \tau \circ ([R^{-1}_{[2]} \otimes R^{-1}_{[1]}] \triangleright \exp_q(ip \otimes p)],$$
$$\exp_q^*(x | ip) = \tau \circ [(R^{-1}_{[2]} \otimes R^{-1}_{[1]}) \triangleright \exp_q(\ominus \otimes p | ix)]. \quad (279)$$

In the expressions above, $\tau$ denotes the ordinary twist operator. It can be shown that the new $q$-exponentials satisfy the following eigenvalue equations (see Fig. 3):

$$\exp_q^*(ip | x) \triangleleft \partial^A = ip^A \odot \exp_q^*(ip | x),$$
$$\partial^A \triangleright \exp_q^*(x | i^{-1} p) = \exp_q^*(x | i^{-1} p) \odot ip^A. \quad (280)$$

You find some explanations of this sort of graphical calculations in Ref. [41].
Similar considerations apply to the conjugate $q$-exponentials. To this end, we only need to modify Eqs. (279) and (280) by performing the following substitutions:

$$
\exp^*_q \rightarrow \exp_q, \quad \mathcal{R}_{[2]}^{-1} \otimes \mathcal{R}_{[1]}^{-1} \rightarrow \mathcal{R}_{[1]} \otimes \mathcal{R}_{[2]}, \quad \odot \rightarrow \bar{\odot}, \\
\bar{s} \rightarrow \bar{v}, \quad \bar{\alpha} \rightarrow \bar{\alpha}, \quad \partial^A \rightarrow \hat{\partial}^A.
$$

(281)

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