MEAN VALUE ONE OF PRIME-PAIR CONSTANTS

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Abstract. For \( k > 1, r \neq 0 \) and large \( x \), let \( \pi_{2r}^{k}(x) \) denote the number of prime pairs \((p, p^k + 2r)\) with \( p \leq x \). By the Bateman–Horn conjecture the function \( \pi_{2r}^{k}(x) \) should be asymptotic to \( (2/k)C_{2r}^{k}\text{li}_2(x) \), with certain specific constants \( C_{2r}^{k} \). Heuristic arguments lead to the conjecture that these constants have mean value one, just like the Hardy–Littlewood constants \( C_{2r} \) for prime pairs \((p, p + 2r)\). The conjecture is supported by extensive numerical work.

1. Introduction

In the following \( p \) runs through the primes. We are interested in the ‘Bateman–Horn constants’ associated with prime pairs \((p, p^k + 2r)\), where \( k \geq 2 \) and \( r \) runs over \( \mathbb{Z} \setminus 0 \).

For convenience we first state the general Bateman–Horn conjecture. It involves an \( m \)-tuple \( f = \{f_1, \ldots, f_m\} \) of polynomials \( f_j \) with integer coefficients and nonconstant ratios, and of positive degrees \( d_1, \ldots, d_m \). The conjecture involves the counting function

\[
\pi_f(x) = \#\{1 \leq n \leq x : f_1(n), \ldots, f_m(n) \text{ all prime}\}.
\]

Let

\[
N_f(p) = \#\{n, 1 \leq n \leq p : f_1(n) \cdots f_m(n) \equiv 0 \pmod{p}\}.
\]

Assuming that the polynomials \( f_j(n) \) are irreducible and that \( N_f(p) < p \) for every prime \( p \), Schinzel and Sierpinski [22] had conjectured that \( \pi_f(x) \rightarrow \infty \) as \( x \rightarrow \infty \). The corresponding quantitative conjecture is due to Bateman and Horn [1, 2]. Without making the above assumptions, let

\[
BH(f) = \frac{1}{d_1 \cdots d_m} \prod_{p} \left(1 - \frac{1}{p}\right)^{-m} \left(1 - \frac{N_f(p)}{p}\right).
\]
The product will converge, but $BH(f)$ may be zero; if one of the polynomials $f_j(n)$ can be factored, one may define $BH(f) = 0$. The conjecture now reads as follows; cf. also Schinzel [21], Davenport and Schinzel [6], and the recent survey paper by Hindry and Rivoal [11].

**Conjecture 1.1.** Let $\pi_f(x)$ be as in (1.1). Then

$$\pi_f(x) \sim BH(f) \lim_{x \to \infty} \int_2^x \frac{dt}{\log^m t},$$

in the sense that $\pi_f(x)/\lim_{x \to \infty} \sim BH(f)$ as $x \to \infty$.

One may verify that for prime pairs $(p, p+2r)$ (with $r \in \mathbb{N}$), this gives the classical conjecture of Hardy and Littlewood [10]:

$$\pi_{2r}(x) = \#\{p \leq x : p + 2r \text{ prime}\} \sim 2C_{2r}\text{li}_2(x),$$

where

$$C_2 = \prod_{p \text{ prime}, p > 2} \left\{1 - \frac{1}{(p-1)^2}\right\}, \quad C_{2r} = C_2 \prod_{p \nmid r, p > 2} \frac{p-1}{p-2}.$$

Turning to prime pairs $(p, p^k+2r)$, where $k \geq 2$ and $r \in \mathbb{Z}\setminus0$, we consider the pair of polynomials

$$f_{2r}(n) = f_{2r}^k(n) = \{n, n^k + 2r\}.$$

Adjusting our earlier notation, it is convenient to write

$$\pi_{k}(x) = \pi_{2r}(x) = \pi_{f_{2r}}(x) = \#\{p \leq x : p^k + 2r \text{ prime}\},$$

$$N_{2r}(p) = N_{2r}^k(p) = \#\{n, 1 \leq n \leq p : n(n^k + 2r) \equiv 0 \pmod{p}\}.$$

Suitable constants are defined by

$$C_{2r} = C_{2r}^k = \prod_{p > 2} \left(\frac{p}{p-1}\right)^2 \frac{p - N_{2r}(p)}{p};$$

we set $C_{2r}^k = 0$ if $n^k + 2r$ can be factored. The corresponding BH constants would be $(2/k)C_{2r}^k$. It is known that the Hardy–Littlewood constants $C_{2r} = C_{2r}^k$ for prime pairs $(p, p+2r)$ have average one; cf. Section 4. Our paper provides both heuristic and numerical support for an extension involving prime pairs $(p, p^k + 2r)$:
Metathorem 1.2. For any degree \( k \geq 2 \), the adjusted Bateman–Horn constants \( C_{2r}^k \) have mean value one:

\[
S_{\lambda}^k = \sum_{0 < |2r| \leq \lambda} C_{2r}^k \sim \lambda \quad \text{as} \quad \lambda \to \infty.
\]

It is convenient to introduce the following auxiliary functions \( g_q(n) \), where \( q \in \mathbb{Z} \setminus 0 \) will usually be taken equal to \( 2r \):

\[
\begin{align*}
g_q(n) &= g_q^k(n) = n^k + q, \\
\nu_q(p) &= \nu_q^k(p) = \# \{ n, 1 \leq n \leq p : g_q(n) \equiv 0 \pmod{p} \}, \\
\gamma_q &= \gamma_q^k = \prod_{p > 2} \frac{p - \nu_q(p)}{p}.
\end{align*}
\]

If \( g_q(n) \) can be factored we set \( \gamma_q = 0 \). Observe that \( N_{2r}(p) = \nu_{2r}(p) + 1 \) except when \( p|2r \); in the latter case \( N_{2r}(p) = \nu_{2r}(p) \). Thus if \( \gamma_{2r} \neq 0 \), the ratio \( C_{2r}/\gamma_{2r} \) is given by an absolutely convergent product:

\[
C_{2r}/\gamma_{2r} = \prod_{p|2r, p > 2} \frac{p}{p - 1} \prod_{p|2r} \frac{p}{p - 1} \frac{p - \nu_{2r}(p) - 1}{p - \nu_{2r}(p)}.
\]

2. Prime pairs \((p, p^2 + 2r)\)

Consider the pair of polynomials

\[
f(n) = f_{2r}(n) = \{ n, g_{2r}(n) \} = \{ n, n^2 + 2r \} \quad (r \in \mathbb{Z} \setminus 0).
\]

The functions \( \pi_{2r}^2(x) = \pi_{f_{2r}}(x) \) will remain bounded as \( x \to \infty \) if \( g_{2r}(n) \) can be factored, or if \( r \equiv 1 \pmod{3} \); in the latter case \( n(n^2 + 2r) \) is always divisible by 3. For primes \( p \) the number of solutions of the quadratic congruence

\[
n^2 \equiv -2r \pmod{p}
\]

is given by

\[
\nu_{2r}(p) = 1 + \left( \frac{-2r}{p} \right),
\]

where \(( -2r/p) \) is the Legendre symbol. It follows that

\[
N_{2r}(p) = 2 + \left( \frac{-2r}{p} \right) \quad \text{for} \quad p \nmid 2r,
\]

while \( N_{2r}(p) = \nu_{2r}(p) = 1 \) if \( p|2r \). The values \( \chi(p) = ( -2r/p) \) generate a real Dirichlet character (different from the principal character) belonging to
\[ x \quad \pi_{-2}(x) \quad L_2(x) \quad \rho(x) \]

| \(x\) | \(\pi_{-2}(x)\) | \(L_2(x)\) | \(\rho(x)\) |
|------|----------------|--------------|----------|
| 10   | 4              |              |          |
| 10^2 | 13             |              |          |
| 10^3 | 52             |              |          |
| 10^4 | 259            | 274          | 0.945    |
| 10^5 | 1595           | 1600         | 0.997    |
| 10^6 | 10548          | 10567        | 0.998    |
| 10^7 | 74914          | 75275        | 0.995    |
| 10^8 | 563533         | 564200       | 0.999    |

**Table 1.** Counting prime pairs \((p, p^2 - 2)\)

some modulus \(m = m_{2r}\). The convergence of the products for \(\gamma_{2r}^2\) and \(C_{2r}^2\) now follows from the convergence of the series

\[
\sum_{p > 2} \frac{\nu_{2r}(p) - 1}{p} = \sum_{p > 2} \frac{\chi(p)}{p},
\]

which is a classical result for Dirichlet characters; cf. Landau [19].

**The special case** \(2r = -2\) or \(g(n) = n^2 - 2\). For \(p > 2\)

\[
\nu_{-2}(p) = 1 + \left(\frac{2}{p}\right) = \begin{cases} 2 & \text{if } p \equiv \pm 1 \pmod{8}, \\ 0 & \text{otherwise} \end{cases}
\]

In this case the values

\[
\chi(2) = 0 \quad \text{and} \quad \chi(p) = (-1)^{(p^2 - 1)/8} \quad \text{for } p > 2
\]

generate a character modulo 8.

A rough computation shows that here

\[(2.3) \quad BH(f) = C_{-2}^2 \approx 1.6916.
\]

We have also computed the counting function

\[(2.4) \quad \pi_{-2}^2(x) = \# \{ p \leq x : p^2 - 2 \text{ prime} \}\]

for \(x = 10, 10^2, \cdots, 10^8\). In Table 1 the number \(\pi_{-2}^2(x)\) is compared to rounded values

\[L_2(x) \text{ of } 1.6916 \log_2(x) = 1.6916 \int_2^x \frac{dt}{\log^2 t}.
\]
Table 2. Bateman–Horn constants for $k = 2$

| $2r$ | $\gamma_{2r}^2$ | $C_{2r}^2$ | $\gamma_{-2r}^2$ | $C_{-2r}^2$ |
|------|-----------------|------------|-----------------|------------|
|  2   | 0.71            | 1.85       | 1.692           |            |
|  4   | 1.37            | 1.107      | 0               | 0          |
|  6   | 0.71            | 0.806      | 1.04            | 1.270      |
|  8   | 0.71            | 1.85       | 1.692           |            |
| 10   | 1.08            | 1.194      | 0.67            | 0          |
| 12   | 1.12            | 1.522      | 1.38            | 1.976      |
| 14   | 0.42            | 0          | 1.15            | 1.070      |
| 16   | 1.37            | 1.107      | 0               | 0          |
| 18   | 1.43            | 2.048      | 1.23            | 1.692      |
| 20   | 0.53            | 0          | 1.77            | 2.131      |
| 22   | 1.77            | 1.872      | 0.60            | 0          |
| 24   | 0.71            | 0.806      | 1.04            | 1.270      |
| 26   | 0.37            | 0          | 1.17            | 1.007      |
| 28   | 1.97            | 2.220      | 0.78            | 0          |
| 30   | 0.87            | 1.532      | 0.86            | 1.450      |

The table includes some ratios

$$\rho(x) = \frac{\pi_{-2}^2(x)}{L_2(x)}.$$

These seem to converge to 1 rather quickly!

### 3. Prime pairs $(p, p^3 \pm 2r)$

For $r \in \mathbb{N}$ we now consider the pairs of polynomials

$$(3.1) \quad f(n) = f_{2r}(n) = \{n, g_{2r}(n)\} = \{n, n^3 \pm 2r\}.$$  

Modulo $p$ the number $\nu(p)$ of solutions of the cubic congruence

$$(3.2) \quad n^3 \equiv q \pmod{p} \quad (q \in \mathbb{Z} \setminus 0)$$

is equal to 1, 0 or 3, depending on $|q|$; cf. Ireland and Rosen [12]. If $p|q$ one has $\nu(p) = 1$; in the case $q = 2r$ and $p|2r$ also $N(p) = 1$. Ignoring such $p$ for the moment, the primes divide into three classes. The class of ‘1-primes’ is independent of $q$. It consists of $p = 3$ and the primes $p \equiv 2 \pmod{3}$, cf.
For these \( p \), roughly half of the primes, equation (3.2) always has precisely one solution \( n \).

The primes \( p \equiv 1 \pmod{3} \) are ‘unstable’. The convergence of the product for \( \gamma_q^3 \) in (1.11) shows that \( \nu(p) = 0 \) for roughly two thirds of these primes and \( \nu(p) = 3 \) for roughly one third of them. The classes of ‘0-primes’ and ‘3-primes’ depend on \( |q| \).

**Example 3.1.** The special case \( f(n) = \{n, n^3 \pm 2\} \). The 3-primes in the case \( 2r = \pm 2 \) were characterized by Euler and Gauss; cf. Cox [5]. They are the primes of the form \( p = a^2 + 27b^2 \), cf. Sloane [25] and additional references in Section 4:

\[ 31, 43, 109, 127, 157, 223, 229, 277, 283, 307, \ldots \]

The remaining primes \( p \equiv 1 \pmod{3} \) are 0-primes, cf. Sloane [24]:

\[ 7, 13, 19, 37, 61, 67, 73, 79, 97, 103, 139, 151, 163, 181, 193, 199, 211, 241, 271, 313, \ldots \]

Using the primes \( p \) up to large \( N \), formulas (1.11) and (1.12) give

\[
\gamma_2^3 \approx \prod_{p \leq N, \nu(p)=0} \frac{p}{p-1} \prod_{p \leq N, \nu(p)=3} \frac{p-3}{p-1} \approx 1.30,
\]

\[
C_2^3 = \gamma_2^3 \prod_{\nu(p)=1; p>2} \frac{p-2}{p-1} \prod_{\nu(p)=3} \frac{p-4}{p-3} \approx 0.89.
\]

**Example 3.2.** The special case \( f(n) = \{n, n^3 \pm 10\} \). Using the 3-primes \( p < 500 \) from Table 3 and paying special attention to \( p = 5 \) one finds

\[
\gamma_{10}^3 \approx \prod_{p<500, \nu(p)=0} \frac{p}{p-1} \prod_{p<500, \nu(p)=3} \frac{p-3}{p-1} \approx 1.34,
\]

\[
C_{10}^3 = \gamma_{10}^3 \cdot \frac{5}{4} \prod_{\nu(p)=1; p \neq 2,5} \frac{p-2}{p-1} \prod_{\nu(p)=3} \frac{p-4}{p-3} \approx 1.22.
\]
4. Average of constants $C_{2r}^k$

The constants $C_{2r} = C_{2r}^1$, associated with ordinary prime-pairs $(p, p+2r)$, with $r \in \mathbb{N}$, have mean value one:

**Proposition 4.1.** One has

$$S_m = \sum_{1 \leq r \leq m} C_{2r} \sim m \text{ as } m \to \infty. \quad (4.1)$$

An extension to the constants in the ‘prime $n$-tuple conjecture’ was given by Gallagher [8]. Strong estimates for the sums $S_m$ are due to Bombieri and Davenport [3], Montgomery [20], and Friedlander and Goldston [7]. Using singular series the latter showed that

$$S_m = m - (1/2) \log m + \mathcal{O}(\log^{2/3} m). \quad (4.2)$$

We sketch a simple proof of (4.1). By (1.6)

$$\frac{C_{2r}}{C_2} = \prod_{p | r, p > 2} \left(1 + \frac{1}{p - 2}\right).$$

Hence, numbering the primes $p > 2$ as $p_1, p_2, \ldots$, using the principle of inclusion-exclusion and letting $\lfloor \cdot \rfloor$ denote the integral-part function,

$$\frac{C_2 + \cdots + C_{2m}}{C_2} = m + \sum_{j} \left\lfloor \frac{m}{p_j} \right\rfloor \frac{1}{p_j - 2} + \sum_{j, k; j < k} \left\lfloor \frac{m}{p_j p_k} \right\rfloor \cdot \left\{ \left(1 + \frac{1}{p_j - 2}\right) \left(1 + \frac{1}{p_k - 2}\right) - \frac{1}{p_j - 2} - \frac{1}{p_k - 2} - 1 \right\} + \cdots.$$

Now simplify, divide by $m$ and let $m \to \infty$. Then by dominated convergence

$$\frac{C_2 + \cdots + C_{2m}}{mC_2} \to 1 + \sum \frac{1}{p_j(p_j - 2)} + \sum \frac{1}{p_j p_k(p_j - 2)(p_k - 2)} + \cdots = \prod \left(1 + \frac{1}{p_j(p_j - 2)}\right) = \prod \frac{(p_j - 1)^2}{p_j^2 - 2p_j} = \frac{1}{C_2}.$$

An elegant proof of (4.1) was proposed by Tenenbaum [26]: apply the Wiener–Ikehara theorem to the Dirichlet series $\sum a(r) r^{-s}$, where $a(r)$ is the multiplicative function $C_{2r}/C_2$. One finds that the subsequences $\{C_{2hr}\}$ of $\{C_{2r}\}$ have mean value $\prod_{p | h, p > 2} p/(p - 1)$; cf. Montgomery [20], Lemma 17.4. It is plausible that more generally, the following is true:
Table 3. The ‘3-primes’ below 500

\[
\begin{array}{c|c}
q & \text{Corresponding 3-primes } p < 500 \\
\hline
2, 4, 8, 16 & 31, 43, 109, 127, 157, 223, 229, 277, 283, 307, 397, 433, 439, 457, 499 \\
3, 9, 24 & 61, 67, 73, 103, 151, 193, 271, 307, 367, 439, 499 \\
6 & 7, 37, 139, 163, 181, 241, 307, 313, 337, 349, 379, 409, 421, 439, 499 \\
10 & 37, 73, 79, 103, 127, 139, 271, 331, 349, 421, 457, 463 \\
12, 18 & 13, 19, 79, 97, 199, 211, 307, 331, 373, 439, 463, 487, 499 \\
14 & 13, 37, 79, 103, 139, 157, 193, 223, 379, 397, 409, 439 \\
20 & 7, 19, 61, 97, 127, 151, 193, 373, 421, 457 \\
22 & 7, 43, 67, 73, 79, 97, 103, 163, 181, 229, 331, 373, 457 \\
\end{array}
\]

Conjecture 4.2. The subsequences of \( \{C_{2r}\} \) that correspond to arithmetic subsequences of the index sequence \( \{2r\} \) all have a mean value.

For example, since \( \{C_{4r}\} \) has mean value one, so does the complementary subsequence \( \{C_{4r-2}\} \). The sequence \( \{C_{6r}\} \) has mean value 3/2, and the sequences \( \{C_{6r-2}\} \) and \( \{C_{6r-4}\} \) should both have mean value 3/4.

The speculative manuscript \[15\] suggests an extension of Proposition 4.1 to the case \( k \geq 2 \) given by Metatheorem 1.2. Machinery for theoretical approach to the metatheorem is developed in Sections 5–8.

The constants \( \gamma_2^q \) can be computed by using the Legendre symbol. Table 2 shows that the average of these constants with \( 1 \leq |r| \leq 15 \) is about 0.98.

When \( k = 3 \) the computations are more laborious. In order to obtain a reasonable approximation to \( \gamma_3^q \) and \( \gamma_3^q \), one has to know the corresponding 3-primes (and hence the 0-primes) up to a suitable level. Given \( q \) we restrict ourselves to primes \( p \equiv 1 \pmod{3} \) that do not divide \( q \). For which \( p \) do the congruences

\[
(4.3) \quad n^3 \equiv \pm q \pmod{p}
\]
have a solution \( n \)? Factorization mod \( p \) of \( n^3 - q \) for \( |n| \leq 100 \) will reveal all the 3-primes \( p < 200 \) and a good many beyond that. To test additional candidates \( p \equiv 1 \pmod{3} \) one may use the following criterion. For given \( q \), the congruences (4.3) have a solution (hence three solutions) if and only if

\[
q^{(p-1)/3} \equiv 1 \pmod{p};
\]

cf. Ireland and Rosen [12], Propositions 7.1.2 and 9.3.3. Table 3 lists the 3-primes \( p < 500 \) for a number of values \( q \).

For \( q = 2 \) and \( q = 3 \) the constants \( \gamma_3^q \) were computed by Bateman and Horn [2], and also by Davenport and Schinzel [6]; the latter constructed absolutely convergent products.

For \( n^3 \pm 4, n^3 \pm 8 \) and \( n^3 \pm 16 \) the 3-primes are the same as for \( n^3 \pm 2 \) in Example 3.1. Indeed, if \( n_1^3 \equiv 2 \pmod{p} \) with \( p > 2 \) and \( n_2 \equiv n_1^2 \), then \( n_2^3 \equiv 4 \). Conversely, if \( n_3^3 \equiv 4 \) and \( n_4 \equiv n_3^2/2 \pmod{p} \), then \( n_4^3 \equiv 2 \).

For \( n^3 \pm 9 \) and \( n^3 \pm 24 \) the 3-primes are the same as for \( n^3 - 3 \). By the work of Dedekind, the latter are the primes \( p \) for which \( 4p = a^2 + 243b^2 \). For \( n^3 \pm 18 \) the 3-primes are the same as for \( n^3 \pm 12 \).

Corresponding constants \( \gamma_3^q \) and \( C_3^2 \) are given in Table 4. Note that \( \gamma_1^3 = 0 \) and \( \gamma_8^3 = C_8^3 = 0 \) because the corresponding polynomials can be factored. The average of the constants \( C_3^2 \) for \( 1 \leq r \leq 12 \) is about 0.93.

We found 527 prime pairs \((p, p^3 + 2)\) and 556 prime pairs \((p, p^3 - 2)\) with \( p < 10^5 \). With our imprecise constant \( C_2^3 \approx 0.87 \), the Bateman–Horn
conjecture would give the approximate value
\[(2/3) \cdot 0.87 \log_2(10^5) \approx 550.\]

5. Auxiliary functions

For \(k \geq 2\) and \(r \in \mathbb{Z} \setminus 0\) we again consider the pair of polynomials \(f_{2r}(n) = \{n, n^k + 2r\}\). In addition to the counting function
\[
\pi_{2r}(x) = \pi_{2r}^{k}(x) = \pi_{f_{2r}}(x) = \#\{p \leq x : p^k + 2r \text{ prime}\}
\]
we need the function
\[
\theta_{2r}(x) = \theta_{2r}^{k}(x) = \sum_{p \leq x; p^k + 2r \text{ prime}} \log^2 p.
\]
Integration by parts will show that for the present case, the Bateman–Horn Conjecture is equivalent to the asymptotic relation
\[
\theta_{2r}(x) \sim BH(f_{2r}) x \quad \text{as} \quad x \to \infty.
\]
Incidentally, a sieving argument would give \(\theta_{2r}(x) = O(x)\); cf. Bateman and Horn [2], Halberstam and Richert [9], Hindry and Rivoal [11].

For the conjecture in the form (5.3) we introduce the Dirichlet series
\[
D_{2r}(s) = D_{2r}^{k}(s) = \sum_{p^k + 2r \text{ prime}} \frac{\log^2 p}{p^s} \quad (s = \sigma + i\tau, \sigma > 1).
\]
By a two-way Wiener–Ikehara theorem for Dirichlet series with positive coefficients, relation (5.3) is true if and only if the difference
\[
G_{2r}(s) = D_{2r}(s) - \frac{BH(f_{2r})}{s - 1}
\]
has ‘good’ boundary behavior as \(\sigma \searrow 1\). That is, \(G_{2r}(\sigma + i\tau)\) should tend to a distribution \(G_{2r}(1 + i\tau)\) which is locally equal to a pseudofunction. By a pseudofunction we mean the distributional Fourier transform of a bounded function which tends to zero at infinity; it cannot have poles. A pseudofunction may be characterized as a tempered distribution which is locally given by Fourier series whose coefficients tend to zero; see [13]. In particular \(D_{2r}(s)\) itself would have to show pole-type behavior, with residue \(BH(f_{2r})\), for angular approach of \(s\) to 1 from the right; there should be no other poles on the line \(\{\sigma = 1\}\).

In the following we have to use repeated complex integrals related to those in [14].
6. Complex integral for a sieving function

Our integrals involve sufficiently smooth even sieving functions $E^\lambda(\nu) = E(\nu/\lambda)$ depending on a parameter $\lambda > 0$. The basic functions $E(\nu)$ have $E(0) = 1$ and support $[-1, 1]$; it is required that $E, E'$ and $E''$ be absolutely continuous, with $E'''$ of bounded variation. One may for example take

$$E^\lambda(\nu) = \frac{3}{4\pi} \int_0^\infty \frac{\sin(\lambda t/4)}{\lambda^3(t/4)^4} \cos \nu t \, dt$$

for $|\nu| \leq \lambda/2$,

$$2(1 - |\nu|/\lambda)^3$$

for $\lambda/2 \leq |\nu| \leq \lambda$,

$$0$$

for $|\nu| \geq \lambda$.

(6.1)

An important role is played by a Mellin transform associated with the Fourier transform of the kernel $E^\lambda(\nu) = E(\nu/\lambda)$. For $0 < x = \Re z < 1$

$$M^\lambda(z) \overset{\text{def}}{=} \frac{1}{\pi} \int_0^\infty \hat{E}^\lambda(t) t^{-z} \, dt = \frac{2}{\pi} \int_0^\infty t^{-z} \, dt \int_0^\lambda E^\lambda(\nu) \cos \nu t \, d\nu$$

$$= \frac{2}{\pi} \int_0^\lambda E(\nu/\lambda) \, d\nu \int_0^\infty (\cos \nu t) t^{-z} \, dt$$

$$= \frac{2 \lambda^z}{\pi} \Gamma(1 - z) \sin(\pi z/2) \int_0^\lambda E(\nu/\lambda) \nu^{z-1} \, d\nu$$

$$= \frac{2 \lambda^z}{\pi} \Gamma(1 - z) \sin(\pi z/2) \int_0^1 E(\nu) \nu^{z-1} \, d\nu$$

$$= \frac{2 \lambda^z}{\pi} \Gamma(-z - 3) \sin(\pi z/2) \int_0^{1+} \nu^{z+3} \, dE'''(\nu).$$

(6.2)

The Mellin transform extends to a meromorphic function for $x > -3$ with simple poles at the points $z = 1, 3, \cdots$. The residue of the pole at $z = 1$ is $-2(\lambda/\pi)A^E$ with $A^E = \int_0^1 E(\nu) \, d\nu$, and $M^\lambda(0) = 1$. Setting $z = x + iy$ (and later $w = u + iv$), the standard order estimates

$$\Gamma(z) \ll |y|^{-1/2} e^{-\pi|y|/2}, \quad \sin(\pi z/2) \ll e^{\pi|y|/2}$$

(6.3)

for $|x| \leq C$ and $|y| \geq 1$ imply the useful majorization

$$M^\lambda(z) \ll \lambda^x(|y| + 1)^{-x-7/2} \quad \text{for} \quad -3 < x \leq C, \ |y| \geq 1.$$
Repeated complex integral for $E^\lambda(\alpha - \beta)$. We write $L(c)$ for the ‘vertical line’ $\{x = c\}$; the factor $1/(2\pi i)$ in complex integrals will be omitted. Thus

$$\int_{L(c)} f(z)dz \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(z)dz.$$ 

Since it is important for us to have absolutely convergent integrals, we often have to replace a line $L(c)$ by a path $L(c, B) = L(c_1, c_2, B)$ with suitable $c_1 < c_2$ and $B > 0$:

![Diagram of path L(c, B)](image)

**Figure 1.** The path $L(c_1, c_2, B)$

(6.5) $L(c, B) = \left\{ \begin{array}{l}
\text{the half-line } \{x = c_1, -\infty < y \leq -B\} \\
+ \text{the segment } \{c_1 \leq x \leq c_2, y = -B\} \\
+ \text{the segment } \{x = c_2, -B \leq y \leq B\} \\
+ \text{the segment } \{c_2 \geq x \geq c_1, y = B\} \\
+ \text{the half-line } \{x = c_1, B \leq y < \infty\};
\end{array} \right.$

cf. Figure [1]. Thus, for example,

$$\cos \alpha = \int_{L(c,B)} \Gamma(z)\alpha^{-z} \cos(\pi z/2)dz \quad (\alpha > 0),$$

with absolute convergence if $c_1 < -1/2$ and $c_2 > 0$. Similarly for $\sin \alpha$. For the combination

$$\cos(\alpha - \beta)t = \cos \alpha t \cos \beta t + \sin \alpha t \sin \beta t$$
with $\alpha, \beta, t > 0$, one can now write down an absolutely convergent repeated integral. In [14] it was combined with (6.2) to obtain a repeated complex integral for the sieving function $E^\lambda(\alpha - \beta)$ in which $\alpha$ and $\beta$ occur separately. Taking $-3 < c_1 + c'_1 < 0$, $c_2, c'_2 > 0$ and $c_2 + c'_2 < 1$ one has

$$E^\lambda(\alpha - \beta) = \int_{L(c,B)} \Gamma(z)\alpha^z dz \int_{L(c',B)} \Gamma(w)\beta^w \cdot M^\lambda(z + w) \cos\{\pi(z - w)/2\} dw.$$  

(6.6)

To verify the absolute convergence of the repeated integral one would substitute $z = x + iy, w = u + iv$ and then use the inequalities (6.3), (6.4) and $\cos\{\pi(z - w)/2\} \ll e^{\pi(|y| + |v|)/2}$, together with a simple lemma:

**Lemma 6.1.** For real constants $a, b, c$, the function

$$\phi(y, v) = (|y| + 1)^{-a}(|v| + 1)^{-b}(|y + v| + 1)^{-c}$$

is integrable over $\mathbb{R}^2$ if and only if $a + b > 1$, $a + c > 1$, $b + c > 1$ and $a + b + c > 2$.

7. Repeated integral for a function $T_k^\lambda(s)$

Taking $k \geq 2$ and using paths specified below, we consider integrals

$$T_k^\lambda(s) = \int_{L(c,B)} \Gamma(z - s)\zeta'(kz) \zeta(kz) dz \int_{L(c',B)} \Gamma(w - s)\zeta'(w) \zeta(w) \cdot M^\lambda(z + w - 2s) \cos\{\pi(z - w)/2\} dw.$$  

(7.1)

The case $k = 1$ was used in [14] to study prime pairs $(p, p + 2r)$. Proceeding in a similar way, we use the Dirichlet series

$$\frac{\zeta'(Z)}{\zeta(Z)} = - \sum \frac{\Lambda(n)}{n^Z} = - \sum (\log p) \left( \frac{1}{p^Z} + \frac{1}{p^{2Z}} + \cdots \right)$$

and formula (6.6) to obtain the (as yet formal) expansion

$$T_k^\lambda(s) = \sum_{h, j} \Lambda(h)\Lambda(j)h^{-ks}j^{-s}E^\lambda(hk - j)$$

$$= \sum_{0 \leq |d| \leq \lambda} \sum_h \Lambda(h)\Lambda(hk - d)h^{-ks}(hk - d)^{-s}E^\lambda(d)$$

$$= \sum_{0 \leq |2r| \leq \lambda} \sum_h \Lambda(h)\Lambda(hk - 2r)h^{-2ks}E^\lambda(2r) + H^\lambda_1(ks),$$
where $H^\lambda_1(Z)$ is holomorphic for $X > 0$. Indeed, $(h^k - d)^{-s}$ may be approximated by $h^{-ks}$, and for odd numbers $d$, the product $\Lambda(h)\Lambda(h^k - d)$ can be $\neq 0$ only if $h = 2$ or $h^k - d = 2$. The expansion can be used to define $T^\lambda_k(s)$ as a holomorphic function for $\sigma > 1/(2k)$.

Having even $d = 2r$, the principal contributions to the expansion come from the cases where $h$ is a prime $p$, and either $h^k - 2r$ is the prime power $p^k$ (if $r = 0$), or a prime $q$ (if $r \neq 0$). In the latter case $\Lambda(h^k - 2r) = \log q = \log(p^k - 2r)$ is well-approximated by $k \log p$. Using (5.4) one thus finds that

$$T^\lambda_k(s) = D_0(2ks) + k \sum_{0 < 2r \leq \lambda} E(2r/\lambda)D_2v(2ks) + H^\lambda_2(ks),$$

where $D_0(Z) = \sum_p (\log^2 p)/p^Z$ and $H^\lambda_2(Z)$ is holomorphic for $X > 1/3$.

Comparison of $D_0(Z)$ with $(d/dz) \sum \Lambda(n)/n^Z$ shows that $D_0(Z)$ is holomorphic for $X > 1/4$, except for purely quadratic poles at $Z = 1$, $Z = 1/2$ and the complex zeros $Z = \rho$ of $\zeta(Z)$.

Assuming Riemann’s Hypothesis (RH) for simplicity, one may in (7.1) take $c_1 = (1/4) + \eta$, $c_2 = (1/2) + \eta$ and $c'_1 = (1/2) + \eta$, $c'_2 = 1 + \eta$ with small $\eta > 0$. Varying $\eta$, the integral will represent $T^\lambda_k(s)$ as a holomorphic function for $3/8 < \sigma < 1/2$ and $|\tau| < B$. Indeed, for given $s$ and small $\eta$ there will be no singular points on the paths. Absolute convergence (locally uniform in $s$) may be verified with the aid of Lemma 6.1, using (6.3), (6.4) and the fact that $(\zeta'/\zeta)(Z)$ grows at most logarithmically in $Y$ on vertical lines $\{X = d\}$ with $d \neq 1/2$; cf. Titchmarsh [27]. Thus on the remote parts of the paths, the integrand is majorized by

$$C(\lambda)|y|^{c_1 - \sigma - 1/2}(\log |y|) |v|^{c'_1 - \sigma - 1/2}(\log |v|)(|y + v| + 1)^{-c_1 - c'_1 + 2\sigma - 7/2}.$$

Similar estimates will enable us to move the paths of integration; cf. [14].

Starting with $s$ and the paths in (7.1) as above, we now move the $w$-path $L(c', B)$ across the poles at the points $w = 1$, $\rho$ and $s$ to the path $L(d, B)$ with $d_1 = -1/2$ and $d_2 = 0$. Then the residue theorem gives

$$T^\lambda_k(s) = \int_{L(c, B)} \cdots \int_{L(d, B)} \cdots dw + U^\lambda_k(s) + T^\lambda_k(s),$$

say, where

$$U^\lambda_k(s) = \int_{L(c, B)} \Gamma(z - s) J(z, s) dz,$$
with

\[ J(z, s) = -\Gamma(1 - s)M^\lambda(z + 1 - 2s) \cos\{\pi(z - 1)/2\} \]

(7.5)

\[ + \sum_\rho \Gamma(\rho - s)M^\lambda(z + \rho - 2s) \cos\{\pi(z - \rho)/2\} \]

\[ + \frac{\zeta'(s)}{\zeta(s)} M^\lambda(z - s) \cos\{\pi(z - s)/2\} \]

Observe that the apparent poles of \( J(z, s) \) at the points \( s = 1 \) and \( s = \rho \) cancel out. For given \( s \) with \( 3/8 < \sigma < 1/2 \) and \( |\tau| < B \), and for suitably small \( \eta \), the function \( J(z, s) \) is holomorphic in \( z \) on and between the paths \( L(c, B) \) and \( L(d, B) \). We now move the path \( L(c, B) \) in (7.4) to \( L(d, B) \).

Picking up residues at \( z = s, 1/k \) and the zeros \( \rho'/k \) of \( \zeta(kz) \), one finds that

\[ U^\lambda_k(s) = \int_{L(d, B)} \Gamma(z - s) \frac{\zeta'(kz)}{\zeta(kz)} J(z, s) dz + V^\lambda_k(s) \]

(7.6)

\[ = U^\lambda_k(s) + V^\lambda_k(s), \]

say, where

\[ V^\lambda_k(s) = \frac{\zeta'(ks)}{\zeta(kz)} J(s, s) - (1/k)\Gamma\{(1/k) - s\}J(1/k, s) \]

(7.7)

\[ + \sum_{\rho'} (1/k)\Gamma\{(\rho'/k) - s\}J(\rho'/k, s). \]

The single integral for \( U^\lambda_k(s) \) in (7.6) defines a holomorphic function for \( 0 < \sigma < 1/2 \). Varying \( B \), the same is true for the repeated integral defining \( T^\lambda_k(s) \) in (7.3). For verification one may use Lemma 6.1 or an analog for the integral of a sum; the number of points \( \rho \) with \( n - 1 < \text{Im} \rho \leq n \) is \( \mathcal{O}(\log n) \).

We know that \( T^\lambda_k(s) \) is holomorphic in the strip

\[ S = \{1/(2k) < \sigma < 1/2\}. \]

For our work we have to know the behavior of \( T^\lambda_k(s) \) near the boundary line \( \{\sigma = 1/(2k)\} \). This is determined by the sum \( V^\lambda_k(s) \) in (7.7), which in view of (7.5) splits into nine separate terms. All but one of these clearly represent meromorphic functions in the closed strip \( \overline{S} \). The exception is the function
defined by the double series arising from the third term in (7.7):

$$\Sigma_k^\lambda(s) \overset{\text{def}}{=} \sum_{\rho, \rho'} (1/k) \Gamma(\rho - s) \Gamma\{(\rho'/k) - s\} \cdot M^\lambda(\rho - 2s + \rho'/k) \cos\{\pi(\rho - \rho'/k)/2\}.$$ (7.8)

By a discrete analog of Lemma 6.1 the double series is absolutely convergent for $$(1 + 1/k)/4 < \sigma < 1/2$$. The analysis below will show that the sum has an analytic continuation [also denoted $$\Sigma_k^\lambda(s)$$] to $$\mathcal{S}$$; see (8.4).

8. Behavior of $$T_k^\lambda(s)$$ near the line \(\{\sigma = 1/(2k)\}\)

We start with the second term of $$V_k^\lambda(s)$$ in (7.7). By (7.5) the factor $$J(1/k, s)$$ is holomorphic in $$\mathcal{S}$$, except for a simple pole at $$s = 1/(2k)$$ due to the pole of $$M^\lambda(Z)$$ for $$Z = 1$$ with residue $$-2(\lambda/\pi)A^E$$; cf. (6.2). Also taking into account the other factor $$-(1/k)\Gamma\{(1/k) - s\}$$, a short calculation gives the principal part of the pole at $$s = 1/(2k)$$ as

$$\frac{(1/k)A^E \lambda}{s - 1/(2k)}, \quad \text{where} \quad A^E = \int_0^1 E(\nu)d\nu.$$ (8.1)

There is also a pole at $$s = 1/k$$, but it is cancelled by a pole of the first term in $$V_k^\lambda(s)$$. That term involves $$J(s, s)$$, which by (7.3) is holomorphic in $$\mathcal{S}$$, and $$(\zeta'/\zeta)(ks)$$, which besides $$s = 1/k$$ has poles at the points $$\rho'/k$$ on the line $$\sigma = 1/(2k)$$. The latter have principal parts

$$\frac{(1/k)J(\rho'/k, \rho'/k)}{s - \rho'/k}.$$ (8.2)

The third term of $$V_k^\lambda(s)$$ involves an infinite series of products. The factors $$J(\rho'/k, s)$$ are holomorphic in $$\mathcal{S}$$, but the factors $$(1/k)\Gamma\{(\rho'/k) - s\}$$ introduce poles at the points $$s = \rho'/k$$. The poles in the products have principal parts

$$\frac{-(1/k)J(\rho'/k, \rho'/k)}{s - \rho'/k},$$ (8.3)

hence these poles cancel those given by (8.2). The final term of $$J(\rho'/k, s)$$ leads to the function $$\Sigma_k^\lambda(s)$$ defined by the double series in (7.8).
Summary 8.1. Assume RH. Combination of (7.2) and the subsequent results in Sections 7 and 8 shows that in the strip $\mathcal{S} = \{1/(2k) < \sigma < 1/2\}$,

$$T^\lambda_k(s) = D_0(2ks) + k \sum_{0 < |2r| \leq \lambda} E(2r/\lambda)D_{2r}(2ks) + H^\lambda_2(ks)$$

(8.4)

$$= \frac{(1/k)AE_\lambda}{s - 1/(2k)} + \Sigma^\lambda_k(s) + H^\lambda_3(s),$$

where $H^\lambda_2(ks)$ and $H^\lambda_3(s)$ are holomorphic for $1/(2k) \leq \sigma < 1/2$.

We now focus on the difference $\Sigma^\lambda_k(s) - D_0(2ks)$, which by (8.4) can be considered as a holomorphic function in $\mathcal{S}$. How does it behave as $s$ approaches the line $\{\sigma = 1/(2k)\}$? The function $D_0(2ks)$ has a purely quadratic pole at $s = 1/(2k)$; see (7.2). By sieving, the functions $D_{2r}(2ks)$ cannot have a pole at $s = 1/(2k)$ of higher order than the first; cf. Section 5, hence $\Sigma^\lambda_k(s)$ must cancel the quadratic pole of $D_0(2ks)$. On the basis of the Bateman–Horn conjecture in (5.3) it is plausible that the functions $D_{2r}(2ks)$ do have first-order poles at $s = 1/(2k)$, with respective residues $BH(f_{2r})(2k)$; cf. (5.5).

Assuming (5.3), what can we say about the residue of $\Sigma^\lambda_k(s) - D_0(2ks)$ for $s \downarrow 1/(2k)$? By (8.4) and (8.1) it will be equal to

$$R^\lambda_k(\lambda) = k \sum_{0 < |2r| \leq \lambda} E(2r/\lambda)BH(f_{2r})(2k)/(2k) - (\lambda/k) \int_0^1 E(\nu) d\nu.$$  

(8.5)

Now it is plausible that this residue is $o(\lambda)$ as $\lambda \to \infty$. Indeed, $\lambda$ occurs in the terms of $\Sigma^\lambda_k(s)$ only as a factor $\lambda^{\nu - 2s + \rho'/k}$. Cf. the case of $T^\lambda_1(s)$ and the sum $\Sigma^\lambda_1(s)$ in [14], where one dealt with ordinary prime pairs $(p, p + 2r)$, so that $BH(f_{2r}) = 2C_{2r}$ and it is known that $R^1_1(\lambda) = o(\lambda)$; see (4.1). By analogy assuming $R^\lambda_k(\lambda) = o(\lambda)$, and letting $E(\nu) \leq 1$ approach the constant function 1 on $[0, 1]$, it follows from (8.5) that

$$\sum_{0 < |r| \leq \lambda/2} BH(f_{2r})/2 \sim \lambda/k \quad \text{as} \quad \lambda \to \infty.$$  

(8.6)

Hence by (1.3) or (1.9), the numbers $C^\lambda_{2r} = (k/2)BH(f_{2r})$ should have mean value 1, as asserted in Metathemorem 1.2.

Remark 8.2. By more refined treatment of $T^\lambda_k(s)$ the conclusion can be obtained without RH; cf. the analysis of $T^\lambda_1(s)$ in [14].
9. The Bateman–Horn constants $\gamma_q^k$

Heuristics based on the relevant Bateman–Horn conjectures suggest that the constants $\gamma_q^k$ also have mean value one. This is supported by numerical evidence; a rough computation gives the average of $\gamma_q^2$ for $1 \leq q \leq 20$ as 0.99 and for $-20 \leq q \leq -1$ as 1.03. There are corresponding results for even $q$; cf. Tables 2 and 4.

For the study of $\gamma_q^k$ one may introduce a related Dirichlet series. By (1.11),

$$
\gamma_q^k = \lim_{s \to 1} \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1} \prod_p \left( 1 - \frac{\nu_q^k(p)}{p^s} \right) = \lim_{s \to 1} \zeta(s) G_q^k(s),
$$

say. Kurokawa [17], cf. [18], has studied the general product

$$
Z(s, f) = \zeta^m(s) \prod_p \{ 1 - N_f(p)p^{-s} \},
$$

which is related to the product for $BH(f)$ in (1.3). See also Conrad [4].

The mean value one of prime $n$-tuple constants plays a role in recent work of Kowalski [16].

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