AFFINE FRACTIONAL \( L^p \) SOBOLEV INEQUALITIES

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Abstract. Sharp affine fractional \( L^p \) Sobolev inequalities for functions on \( \mathbb{R}^n \) are established. The new inequalities are stronger than (and directly imply) the sharp fractional \( L^p \) Sobolev inequalities. They are fractional versions of the affine \( L^p \) Sobolev inequalities of Lutwak, Yang, and Zhang. In addition, affine fractional asymmetric \( L^p \) Sobolev inequalities are established.

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1. Introduction

Sharp fractional \( L^2 \) Sobolev inequalities are receiving increasing attention in the last decades. They are central in the study of solutions of equations involving the fractional Laplace operator \((-\Delta)^{1/2}\) which arises naturally in many non-local problems such as the stationary form of reaction-diffusion equations [9], the Signorini problem (and its equivalent formulation as the thin obstacle problem) [3], and the Dirichlet-to-Neumann operator of harmonic functions in the half space [29]. Also, the general operators \((-\Delta)^s\) for \( s \in (0,1) \) arise in stochastic theory, associated with symmetric Levy processes (see [29] and the references therein).

Let \( 0 < s < 1 \) and \( 1 \leq p < n/s \). The fractional \( L^p \) Sobolev inequalities state that

\[
\|f\|_{W^{s,p}(\mathbb{R}^n)}^p \leq \sigma_{n,p,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^p |x - y|^{n+ps} \, dx \, dy
\]

for \( f \in W^{s,p}(\mathbb{R}^n) \), the fractional \( L^p \) Sobolev space of functions \( f \in L^p(\mathbb{R}^n) \) with finite right side in (1) (see, for example, [27]). In general, the optimal constants \( \sigma_{n,p,s} \) and extremal functions are not known (see [6] for a conjecture). Equality is always attained in (1). For \( p = 1 \), the extremal functions of (1) are multiples of indicator functions of balls and the constants are explicitly known. The only further known case is \( p = 2 \), where the constants \( \sigma_{n,2,s} \) can be obtained by duality from Lieb’s sharp Hardy–Littlewood–Sobolev inequalities [18] (see, for example, [10]). The asymptotic behavior of \( \sigma_{n,p,s} \) as \( s \to 1^- \) was studied in [5]. Almgren and Lieb [11] and Frank and Seiringer [12] showed that the extremal functions of (1) are radially symmetric and of constant sign.

By a result of Bourgain, Brezis, and Mironescu [4],

\[
\lim_{s \to 1^-} p(1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^p |x - y|^{n+ps} \, dx \, dy = \alpha_{n,p} \int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx
\]

for \( f \in W^{1,p}(\mathbb{R}^n) \), the Sobolev space of \( L^p \) functions \( f \) with weak \( L^p \) gradient \( \nabla f \), where

\[
\alpha_{n,p} = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta \rangle|^p \, d\xi
\]
Theorem 1. Let $\eta \in S^{n-1}$. Here, integration on the unit sphere $S^{n-1}$ is with respect to the $(n-1)$-dimensional Hausdorff measure, $\omega_n$ is the volume of the $n$-dimensional unit ball and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$. For $p = 1$ and $p = 2$, this allows to deduce the sharp $L^p$ Sobolev inequalities from \cite{n1} by calculating the limit of $\sigma_{n,p,s}/(1 - s)$ as $s \to 1^-$.

Zhang \cite{z2} and Lutwak, Yang, and Zhang \cite{z3} obtained the following sharp affine $L^p$ Sobolev inequality that is significantly stronger than the classical $L^p$ Sobolev inequality:

\begin{equation}
\|f\|_{p_{n,p},p}^{n,p} \leq \sigma_{n,p} \frac{n\omega_n^{n+p}}{\alpha_{n,p}} |\Pi_p^{s} f|^{-\frac{p}{n}} \leq \sigma_{n,p} \int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx
\end{equation}

for $f \in W^{1,p}(\mathbb{R}^n)$ and $1 < p < n$, where the inequality between the first and third terms is the classical $L^p$ Sobolev inequality and the optimal constants $\sigma_{n,p}$ were determined by Aubin \cite{a2} and Talenti \cite{z4}. We have rewritten the explicit constant for the first inequality from \cite{z3} using \cite{t2}. Here $\Pi_p^{s} f$ is the $L^p$ polar projection body of $f$, a convex body associated to $f$ that was introduced with different notation in \cite{z3} (see Section 2.2), and $|\cdot|$ is the $n$-dimensional Lebesgue measure.

The main aim of this paper is to establish affine fractional $L^p$ Sobolev inequalities that are stronger than the Euclidean fractional $L^p$ Sobolev inequalities from \cite{n1} and are fractional counterparts of \cite{n3}. The case $p = 1$ was studied in \cite{h4}, so from now on we let $p > 1$.

**Theorem 1.** Let $0 < s < 1$ and $1 < p < n/s$. For $f \in W^{s,p}(\mathbb{R}^n)$,

\begin{align*}
\|f\|_{p_{n,p},p}^{n,p} & \leq \sigma_{n,p,s} \frac{n\omega_n^{n+p}}{\omega_{n,s,p}} \left( \frac{1}{n} \int_{S^{n-1}} \left( \int_0^\infty t^{p-1} \int_{\mathbb{R}^n} |f(x + t\xi) - f(x)|^p \, dx \, dt \right)^{\frac{p}{n}} \, d\xi \right)^{-\frac{p}{n}} \\
& \leq \sigma_{n,p,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p+s}} \, dx \, dy.
\end{align*}

There is equality in the first inequality if and only if $f = h_{s,p} \circ \phi$ for some $\phi \in GL(n)$, where $h_{s,p}$ is an extremal function of \cite{n1}. There is equality in the second inequality if $f$ is radially symmetric.

In order to prove Theorem 1 we introduce the $s$-fractional $L^p$ polar projection body $\Pi_p^{s,s} f$ associated to $f$, defined as the star-shaped set whose gauge function for $\xi \in S^{n-1}$ is

\[ \|\xi\|_{\Pi_p^{s,s} f}^{p_{n,p},s} = \left( \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x + t\xi) - f(x)|^p \, dx \, dt \right)^{\frac{p}{n}} \]

(see Section 3 for details). The affine fractional Sobolev inequality now can be written as

\begin{equation}
\|f\|_{p_{n,p},p}^{n,p} \leq \sigma_{n,p,s} \frac{n\omega_n^{n+p}}{\omega_{n,s,p}} |\Pi_p^{s,s} f|^{-\frac{p}{n}}.
\end{equation}

Since both sides of (4) are invariant under translations of $f$, and for volume-preserving linear transformations $\phi : \mathbb{R}^n \to \mathbb{R}^n$,

\[ \Pi_p^{s,s} (f \circ \phi^{-1}) = \phi \Pi_p^{s,s} f, \]
it follows that (4) is an affine inequality. In Theorem 10 we will show that
\[
\lim_{s \to 1^-} p(1-s)\|\Pi_p^{s, \ast} f\|^{\frac{p}{n}} = |\Pi_p f|^{\frac{p}{n}},
\]
which establishes the connection to the \(L^p\) polar projection bodies introduced by Lutwak, Yang and Zhang [24].

In Section 4 we introduce fractional asymmetric \(L^p\) polar projection bodies as fractional counterparts of the asymmetric \(L^p\) polar projection bodies of Haberl and Schuster [14], which in turn are functional versions of the asymmetric \(L^p\) polar projection bodies of convex bodies introduced in [19]. We obtain affine fractional asymmetric \(L^p\) Sobolev inequalities for non-negative functions that are stronger than the inequalities for the symmetric fractional \(L^p\) polar projection bodies.

In the proofs of the main results, we use anisotropic fractional Sobolev norms, which were introduced in [20, 21] and depend on a star-shaped set \(K \subset \mathbb{R}^n\). In Section 10 we discuss which choice of \(K\) (with given volume) gives the minimal fractional Sobolev norm and connect it to the corresponding quest for an optimal \(L^p\) Sobolev norm solved by Lutwak, Yang, and Zhang [25].

2. Preliminaries

We collect results on function spaces, Schwarz symmetrization, star-shaped sets, anisotropic Sobolev norms and \(L^p\) polar projection bodies, that will be used in the following.

2.1. Function spaces. For \(p \geq 1\) and measurable \(f : \mathbb{R}^n \to \mathbb{R}\), let
\[
\|f\|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.
\]
We set \(\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\}\) for \(t \in \mathbb{R}\) and use similar notation for level sets, etc. We say that \(f\) is non-zero, if \(\{f \neq 0\}\) has positive measure, and we identify functions that are equal up to a set of measure zero. For \(p \geq 1\), let
\[
L^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} : f \text{ is measurable}, \|f\|_p < \infty \right\}.
\]
Here and below, when we use measurability and related notions, we refer to the \(n\)-dimensional Lebesgue measure on \(\mathbb{R}^n\).

For \(0 < s < 1\) and \(p \geq 1\), we define the fractional Sobolev space \(W^{s,p}(\mathbb{R}^n)\) as
\[
W^{s,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} \, dx \, dy < \infty \right\}.
\]
For \(p \geq 1\), we set
\[
W^{1,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : |\nabla f| \in L^p(\mathbb{R}^n) \right\},
\]
where \(\nabla f\) is the weak gradient of \(f\).

2.2. Symmetrization. For a set \(E \subset \mathbb{R}^n\), the indicator function \(1_E\) is defined by \(1_E(x) = 1\) for \(x \in E\) and \(1_E(x) = 0\) otherwise. Let \(E \subseteq \mathbb{R}^n\) be a Borel set of finite measure. The Schwarz symmetrization of \(E\), denoted by \(E^{\ast}\), is the closed centered Euclidean ball with same volume as \(E\).

Let \(f : \mathbb{R}^n \to \mathbb{R}\) be a non-negative measurable function with super-level sets \(\{f \geq t\}\) of finite measure. The layer cake formula states that
\[
f(x) = \int_0^\infty 1_{\{f \geq t\}}(x) \, dt.
for almost every $x \in \mathbb{R}^n$ and allows us to recover the function from its super-level sets. The Schwarz symmetrals of $f$, denoted by $f^*$, is defined by

$$f^*(x) = \int_0^\infty 1_{\{f \geq t\}}(x) \, dt$$

for $x \in \mathbb{R}^n$. Hence, $f^*$ is determined by the properties of being radially symmetric, decreasing and having super-level sets of the same measure as those of $f$. Note that $f^*$ is also called the symmetric decreasing rearrangement of $f$.

The proofs of our results make use of the Riesz rearrangement inequality, which is stated in full generality, for example, in [7].

**Theorem 2** (Riesz’s rearrangement inequality). For $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative, measurable functions with super-level sets of finite measure,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) k(x - y) g(y) \, dx \, dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^*(x) k^*(x - y) g^*(y) \, dx \, dy.$$

We will use the characterization of equality cases of the Riesz rearrangement inequality due to Burchard [8].

**Theorem 3** (Burchard). Let $A, B$ and $C$ be sets of finite positive measure in $\mathbb{R}^n$ and denote by $\alpha, \beta$ and $\gamma$ the radii of their Schwarz symmetrals $A^*, B^*$ and $C^*$. For $|\alpha - \beta| < \gamma < \alpha + \beta$, there is equality in

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_A(y) 1_B(x - y) 1_C(x) \, dx \, dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{A^*}(y) 1_{B^*}(x - y) 1_{C^*}(x) \, dx \, dy$$

if and only if, up to sets of measure zero,

$$A = a + \alpha D, B = b + \beta D, C = c + \gamma D,$$

where $D$ is a centered ellipsoid, and $a, b$ and $c = a + b$ are vectors in $\mathbb{R}^n$.

### 2.3. Star-shaped sets and star bodies

A set $K \subseteq \mathbb{R}^n$ is star-shaped (with respect to the origin), if the interval $[0, x] \subset K$ for every $x \in K$. The gauge function $\| \cdot \|_K : \mathbb{R}^n \to [0, \infty]$ of a star-shaped set is defined as

$$\|x\|_K = \inf \{ \lambda > 0 : x \in \lambda K \},$$

and the radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to [0, \infty]$ as

$$\rho_K(x) = \|x\|_K^{-1} = \sup \{ \lambda \geq 0 : \lambda x \in K \}.$$

The $n$-dimensional Lebesgue measure or volume of a star-shaped set $K$ in $\mathbb{R}^n$ with measurable radial function is given by

$$|K| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K(\xi)^n \, d\xi.$$

We call a star-shaped set $K \subset \mathbb{R}^n$ a star body if its radial function is strictly positive and continuous in $\mathbb{R}^n \setminus \{0\}$. On the set of star bodies, the $q$-radial sum for $q \neq 0$ of $K, L \subset \mathbb{R}^n$ is defined by

$$\rho^q(K + q L, \xi) = \rho^q(K, \xi) + \rho^q(L, \xi)$$

for $\xi \in \mathbb{S}^{n-1}$ (cf. [28, Section 9.3]). The dual Brunn–Minkowski inequality (cf. [28 (9.41)]) states that for star bodies $K, L \subset \mathbb{R}^n$ and $q > 0,$

$$|K + q L|^{-q/n} \geq |K|^{-q/n} + |L|^{-q/n},$$

with equality precisely if $K$ and $L$ are dilates, that is, there is $\lambda > 0$ such that $K = \lambda L.$
Let \( \alpha \in \mathbb{R} \setminus \{0, n\} \). For star-shaped sets \( K, L \subseteq \mathbb{R}^n \) with measurable volume functions, the dual mixed volume is defined as
\[
\tilde{V}_\alpha(K, L) = \frac{1}{n} \int_{\mathbb{R}^{n-1}} \rho_K(\xi)^{n-\alpha} \rho_L(\xi)^\alpha \, d\xi.
\]
Notice that
\[
\tilde{V}_\alpha(K, K) = |K|
\]
and that
\[
\tilde{V}_\alpha(K, L_1 +_a L_2) = \tilde{V}_\alpha(K, L_1) + \tilde{V}_\alpha(K, L_2)
\]
for star-shaped sets \( K, L_1, L_2 \subseteq \mathbb{R}^n \) with measurable radial functions.

Equality holds if and only if \( K \) and \( L \) are dilates, where we say that star-shaped sets \( K \) and \( L \) are dilates if \( \rho_K = \lambda \rho_L \) almost everywhere on \( \mathbb{S}^{n-1} \) for some \( \lambda > 0 \).

The definition of dual mixed volume for star bodies is due to Lutwak [22], where also the dual mixed volume inequality is derived from Hölder’s inequality (also see [28, Section 9.3] or [13, B.29]).

### 2.4. Anisotropic fractional Sobolev norms

Let \( 0 < s < 1 \) and \( p \geq 1 \). For \( K \subseteq \mathbb{R}^n \) a star body and \( f \in W^{s,p}(\mathbb{R}^n) \), the anisotropic fractional \( L^p \) Sobolev norm of \( f \) with respect to \( K \) is
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_K^{n+ps}} \, dx \, dy.
\]

It was introduced in [21] for \( K \) a convex body (also, see [20]). For \( K = B^n \), the Euclidean unit ball, we obtain the classical \( s \)-fractional \( L^p \) Sobolev norm of \( f \). The limit as \( s \to 1^- \) was determined in [4] in the Euclidean case and in [21] in the anisotropic case.

We will also consider the following asymmetric versions of (7),
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|_K^{n+ps}} \, dy \, dx, \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|_K^{n+ps}} \, dx \, dy,
\]
where \( a_+ = \max\{a, 0\} \) and \( a_- = \max\{-a, 0\} \) for \( a \in \mathbb{R} \). The limits as \( s \to 1^- \) were determined in [20].

### 2.5. \( L^p \) polar projection bodies

For \( p \geq 1 \) and \( f \in W^{1,p}(\mathbb{R}^n) \), the \( L^p \) polar projection body is defined as the star body with gauge function given by
\[
\|\xi\|_{\Pi^*_K f}^p = \int_{\mathbb{R}^n} |(\nabla f(x), \xi)|^p \, dx
\]
for \( \xi \in \mathbb{S}^{n-1} \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product. It is the polar body of a convex body. The definition is due to Lutwak, Yang, and Zhang [24]. For a convex body \( K \subseteq \mathbb{R}^n \), they defined the \( L^p \) polar projection body (with a different normalization) in [23] by
\[
\|\xi\|_{\Pi^*_K}^p = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta \rangle|^p \, dS_\mu(K, \eta),
\]
where \( S_\mu(K, \cdot) \) is the \( L^p \) surface area measure of \( K \) (for the definition of \( L^p \) surface area measures, see, for example, [28, Section 9.1]).
Hence, for \( f \in W^{1,p}(\mathbb{R}^n) \), the asymmetric \( L^p \) polar projection bodies of \( f \) are defined as the star bodies with gauge function given by

\[
\|\xi\|_{\Pi^{p,\pm}_f}^p = \int_{\mathbb{R}^n} (\nabla f(x), \xi)^p dx
\]

for \( \xi \in \mathbb{S}^{n-1} \).

3. Fractional \( L^p \) Polar Projection Bodies

Let \( 0 < s < 1 \) and \( 1 < p < n/s \). For \( f \in W^{s,p}(\mathbb{R}^n) \), define the \( s \)-fractional \( L^p \) polar projection body \( \Pi^{p,s}_f \) as the star-shaped set given by the gauge function

\[
\|\xi\|^{p,s}_{\Pi^{p,s}_f} = \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} |f(x + t\xi) - f(x)|^p dx \, dt
\]

for \( \xi \in \mathbb{R}^n \). Note that \( \| \cdot \|^{p,s}_{\Pi^{p,s}_f} \) is a one-homogeneous function on \( \mathbb{R}^n \).

Let \( K \subset \mathbb{R}^n \) be a star body. The following simple calculation turns out to be useful. For \( f \in W^{s,p}(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+ps}_K} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y + z) - f(y)|^p}{\|z\|^{n+ps}_K} \, dz \, dy
\]

\[
= \int_{\mathbb{S}^{n-1}} \int_0^\infty \|t\xi\|^{-n-ps}_K \int_{\mathbb{R}^n} |f(y + t\xi) - f(y)|^p t^{n-1} dy \, dt \, d\xi
\]

\[
= \int_{\mathbb{S}^{n-1}} \int_0^\infty \|\xi\|^{-n-ps}_K t^{-ps-n} \|f(\cdot + t\xi) - f\|^p_{L^p} t^{n-1} dt \, d\xi
\]

\[
= \int_{\mathbb{S}^{n-1}} \rho_K (\xi)^{n+ps} \int_0^\infty t^{-ps-1} \|f(\cdot + t\xi) - f\|^p_{L^p} dt \, d\xi
\]

\[
= \int_{\mathbb{S}^{n-1}} \rho_K (\xi)^{n+ps} \rho^{p,s}_{\Pi^{p,s}_f}(\xi)^{-ps} d\xi.
\]

Hence,

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+ps}_K} \, dx \, dy = n \bar{V}_{-ps}(K, \Pi^{p,s}_f)
\]

in this case.

Next, we establish basic properties of fractional \( L^p \) polar projection bodies.

**Proposition 4.** For non-zero \( f \in W^{s,p}(\mathbb{R}^n) \), the set \( \Pi^{p,s}_f \) is an origin-symmetric star body with the origin in its interior. Moreover, there is \( c > 0 \) depending only on \( f \) and \( p \) such that \( \Pi^{p,s}_f \subseteq cB^n \) for every \( s \in (0, 1) \).

**Proof.** First, note that since for \( \xi \in \mathbb{R}^n \) and \( t > 0 \),

\[
\int_{\mathbb{R}^n} |f(x - t\xi) - f(x)|^p dx = \int_{\mathbb{R}^n} |f(x) - f(x + t\xi)|^p dx,
\]

the set \( \Pi^{p,s}_f \) is origin-symmetric.
Next, we show that $\Pi_{p}^{s,a}f$ is bounded. We take $r > 1$ large enough so that $\|f\|_{L^{p}(B^{n})} \geq \frac{2}{3} \|f\|_{p}$ and easily see that for $t > 2r$,

$$\|f(\cdot + t\xi) - f(\cdot)\|_{p} \geq \|f(\cdot + t\xi) - f(\cdot)\|_{L^{p}(B^{n} - t\xi)}$$

$$= \|f(\cdot) - f(\cdot - t\xi)\|_{L^{p}(B^{n})}$$

$$\geq \|f\|_{L^{p}(B^{n})} - \|f(\cdot - t\xi)\|_{L^{p}(B^{n})}$$

$$\geq \frac{2}{3} \|f\|_{p} - \frac{1}{3} \|f\|_{p}.$$  

Hence,

$$\int_{0}^{\infty} t^{-ps-1} \int_{\mathbb{R}^{n}} |f(x + t\xi) - f(x)|^{p} \, dx \, dt \geq \frac{\|f\|_{p}^{p}}{3^{p}} \int_{r}^{\infty} t^{-ps-1} \, dt \geq \frac{\|f\|_{p}^{p}}{3^{p}} \frac{r^{-ps}}{ps} \geq c,$$

which implies that $\Pi_{p}^{s,a}f \subseteq c B^{n}$ for $c > 0$ independent of $s$.

Now, we show that $\Pi_{p}^{s,a}f$ has the origin in its interior. First observe that for $\xi, \eta \in \mathbb{R}^{n}$, by the triangle inequality and a change of variables,

$$\|\xi + \eta\|_{\Pi_{p}^{s,a}f}^{ps}$$

$$= \int_{0}^{\infty} t^{-ps-1} \|f(\cdot + t\xi + t\eta) - f(\cdot)\|_{p}^{p} \, dt$$

$$\leq \int_{0}^{\infty} t^{-ps-1} (\|f(\cdot + t\xi + t\eta) - f(\cdot + t\xi)\|_{p} + \|f(\cdot + t\xi) - f(\cdot)\|_{p})^{p} \, dt$$

$$\leq \int_{0}^{\infty} t^{-ps-1} 2^{ps-1} (\|f(\cdot + t\eta) - f(\cdot)\|_{p}^{p} + \|f(\cdot + t\xi) - f(\cdot)\|_{p}^{p}) \, dt$$

$$= 2^{ps-1} \|\xi\|_{\Pi_{p}^{s,a}f}^{ps} + 2^{ps-1} \|\eta\|_{\Pi_{p}^{s,a}f}^{ps}.$$  

Using the relation (11) with $K = B^{n}$, we get

$$\int_{S^{n-1}} \xi \|\xi\|_{\Pi_{p}^{s,a}f}^{ps} \, d\xi = \frac{1}{n} \int_{\mathbb{R}} \int_{\mathbb{R}^{n}} \frac{|f(x) - f(y)|^{p}}{|x - y|^{n+ps}} \, dx \, dy,$$

which is finite since $f \in W^{s,p}(\mathbb{R}^{n})$. We choose $r > 0$ large enough so that the set $A = \{\xi \in S^{n-1} : \|\xi\|_{\Pi_{p}^{s,a}f}^{s} < r\}$ has positive $(n - 1)$-dimensional Hausdorff measure and contains a basis $\{\xi_{1}, \ldots, \xi_{n}\} \subseteq A$ of $\mathbb{R}^{n}$. Applying (if necessary) a linear transformation to $\Pi_{p}^{s,a}f$, we may assume without loss of generality that $\xi_{i} = e_{i}$ are the canonical basis vectors. For every $x \in \mathbb{R}^{n}$, writing $x = \sum x_{i}e_{i}$ and using (11), we get

$$\|x\|_{\Pi_{p}^{s,a}f} \leq \left(2^{n(p-1)} \sum_{i=1}^{n} |x_{i}|^{ps} \|e_{i}\|_{\Pi_{p}^{s,a}f}^{ps} \right)^{\frac{1}{ps}} \leq d |x|,$$

where $d > 0$ is independent of $x$. This shows that $\Pi_{p}^{s,a}f$ has the origin as interior point.
Finally, we show that \( \| \cdot \|_{\Pi_p^{s,f}} \) is continuous. For \( \xi, \eta \in \mathbb{R}^n \), by the triangle inequality and (12), we have

\[
\| \xi + \eta \|_{\Pi_p^{s,f}}^{p_s} = \int_0^\infty t^{1-ps} \| f(\cdot + t\xi + t\eta) - f(\cdot) \|_p dt \\
\leq \int_0^\infty t^{1-ps} \left( \| f(\cdot + t\eta) - f(\cdot) \|_p + \| f(\cdot + t\xi) - f(\cdot) \|_p \right)^p dt \\
\leq (1 + |\eta|^{\frac{s}{p'}})^{p-1} \int_0^\infty t^{1-ps} \left( \frac{\| f(\cdot + t\eta) - f(\cdot) \|_p}{|\eta|^{\frac{s}{p'}}} + \| f(\cdot + t\xi) - f(\cdot) \|_p \right) dt \\
= (1 + |\eta|^{\frac{s}{p'}})^{p-1} (|\eta|^{\frac{s}{p'}} \| \Pi_p^{s,f} \| + \| \Pi_p^{s,f} \|_{\Pi_p^{s,f}}) \\
\leq (1 + |\eta|^{\frac{s}{p'}})^{p-1} (d|\eta|^{\frac{s}{p'}} + \| \Pi_p^{s,f} \|_{\Pi_p^{s,f}}),
\]

where we used the inequality \( a + b \leq (1 + r^{p/(p-1)})^{(p-1)/p} (r^{-1}a + b)^{1/p} \) for \( a, b, r > 0 \), which is a consequence of Hölder’s inequality.

We obtain

\[
\| \xi + \eta \|_{\Pi_p^{s,f}}^{p_s} \leq (1 + |\eta|^{\frac{s}{p'}})^{p-1} (d|\eta|^{\frac{s}{p'}} + \| \Pi_p^{s,f} \|_{\Pi_p^{s,f}}).
\]

Applying inequality (13) to the vectors \( \xi + \eta \) and \( -\eta \), we get

\[
\| \xi + \eta \|_{\Pi_p^{s,f}}^{p_s} = \| \xi - \eta \|_{\Pi_p^{s,f}}^{p_s} \leq (1 + |\eta|^{\frac{s}{p'}})^{p-1} (d|\eta|^{\frac{s}{p'}} + \| \Pi_p^{s,f} \|_{\Pi_p^{s,f}}),
\]

which implies

\[
\| \xi + \eta \|_{\Pi_p^{s,f}}^{p_s} \geq (1 + |\eta|^{\frac{s}{p'}})^{p-1} \| \xi \|_{\Pi_p^{s,f}}^{p_s} - d|\eta|^{\frac{s}{p'}}.
\]

The continuity of \( \| \cdot \|_{\Pi_p^{s,f}} \) now follows from (13) and (14). \( \square \)

### 4. Fractional Asymmetric \( L^p \) Polar Projection Bodies

Let \( 0 < s < 1 \) and \( 1 < p < n/s \). For \( f \in W^{s,p}(\mathbb{R}^n) \), define the asymmetric \( s \)-fractional \( L^p \) polar projection bodies \( \Pi_{p,s}^{*,f} \) and \( \Pi_{p,s}^{*,f} \) as the star-shaped sets given by the gauge functions

\[
\| \xi \|_{\Pi_{p,s}^{*,f}}^{p_s} = \int_0^\infty t^{-ps-1} \int_{\mathbb{R}^n} (f(x + t\xi) - f(x))^p dx dt
\]

for \( \xi \in \mathbb{R}^n \). We have \( \Pi_{p,s}^{*,f} \) and \( \Pi_{p,s}^{*,f} \) defined for \( f \) and \( \Pi_{p,s}^{*,f} \) for \( f \). Note that, as in the symmetric case, \( \| \cdot \|_{\Pi_{p,s}^{*,f}} \) is a one-homogeneous function on \( \mathbb{R}^n \). Also note that

\[
\| \xi \|_{\Pi_{p,s}^{*,f}}^{p_s} = \| \xi \|_{\Pi_{p,s}^{*,f}}^{p_s} + \| \xi \|_{\Pi_{p,s}^{*,f}}^{p_s}
\]

for \( \xi \in \mathbb{R}^n \).

Let \( K \subset \mathbb{R}^n \) be a star body and \( f \in W^{s,p}(\mathbb{R}^n) \). As in (10), we obtain that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{|x - y|^{n + ps}} dx dy = n \tilde{V}_{-ps}(K, \Pi_{p,s}^{*,f} f).
\]

In the following proposition, we derive the basic properties of fractional asymmetric \( L^p \) polar projection bodies.
Proposition 5. For non-zero $f \in W^{s,p}(\mathbb{R}^n)$, the set $\Pi^s_{p,+}f$ is a star body with the origin in its interior. Moreover, there is $c > 0$ depending only on $f$ and $p$ such that $\Pi^s_{p,+}f \subseteq cB^n$ for every $s \in (0,1)$.

Proof. Since the functions $(a)^{p}_{s}$ and $(a)^{p}_{s}$ are convex, the inequalities $(a + b)^{p}_{s} \geq (a)^{p}_{s} + p(a)^{p-1}_{s}b$ and $(a + b)^{p}_{s} \geq (a)^{p}_{s} + p(a)^{p-1}_{s}b$ hold for $a, b \in \mathbb{R}$.

If $\int_{\mathbb{R}^n} (f(x))^p dx > 0$, take $\varepsilon > 0$ so small that $\varepsilon + p\varepsilon^{1/p} \|f\|^{p-1}_p \leq \frac{1}{2} \int_{\mathbb{R}^n} (f(x))^p dx$, and take $r > 0$ so large that $\int_{\mathbb{R}^n \setminus rB^n} |f(x)|^p dx < \varepsilon$. For $z \in \mathbb{R}^n \setminus 2rB^n$, we obtain by Hölder's inequality that

$$\int_{rB^n} (f(x) - f(x + z))^p dx$$

$$\geq \int_{rB^n} (f(x))^p - p(f(x))^{p-1}_x f(x + z) dx$$

$$\geq \int_{rB^n} (f(x))^p dx - p\left(\int_{rB^n} (f(x))^p dx\right)^{p-1} \left(\int_{rB^n} |f(x + z)|^p dx\right)^\frac{1}{p}$$

$$\geq \int_{rB^n} (f(x))^p dx - p\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{p-1} \left(\int_{\mathbb{R}^n \setminus rB^n} |f(x)|^p dx\right)^\frac{1}{p}$$

$$\geq \int_{\mathbb{R}^n} (f(x))^p dx - \varepsilon - p \|f\|^{p-1}_p \varepsilon^\frac{1}{p}$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^n} (f(x))^p dx.$$ 

In case $\int_{\mathbb{R}^n} (f(x))^p dx = 0$ the previous inequality holds trivially for any $r > 0$.

By an analogous calculation, and eventually increasing the value of $r$, we obtain that

$$\int_{rB^n - z} (f(x) - f(x + z))^p dx = \int_{rB^n} (f(x) - f(x + z))^p dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^n} (f(x))^p dx.$$ 

It follows that $\int_{\mathbb{R}^n} (f(x) - f(x + z))^p dx \geq \frac{1}{2} \|f\|^{p}_p$ for every $z \in \mathbb{R}^n \setminus 2rB^n$ with $r > 0$ depending only on $f$. Finally,

$$\|\xi\|^{ps}_{\Pi^s_{p,+}} \geq \int_{2r}^{\infty} t^{-1-qs} \int_{rB^n} (f(x) - f(x + z))^p dx dt$$

$$\geq \int_{2r}^{\infty} t^{-1-qs} \int_{\mathbb{R}^n} |f(x)|^p dx$$

$$\geq \frac{(2r)^{-p}}{ps} \int_{\mathbb{R}^n} |f(x)|^p dx$$

$$\geq \frac{(2r)^{-p}}{2p} \|f\|^{p}_p.$$ 

Note that $\Pi^s_{p,+}f \subseteq \Pi^s_{p,+}f$. Hence, it follows from Proposition 3 that $\Pi^s_{p,+}f$ contains the origin in its interior, that is, there is $d > 0$ such that

$$\|x\|_{\Pi^s_{p,+}} \leq d |x|$$ 

for every $x \in \mathbb{R}^n$. 

Finally, we show that \( \| \cdot \|_{\Pi^{p,s}_{\varepsilon}} \) is continuous. Observe that the inequality 
\[ (a + b)_{+}^{p} \leq (a_{+} + b_{+})^{p} \] 
holds for any \( a, b \in \mathbb{R} \). Hence, for \( \xi, \eta \in \mathbb{R}^{n} \), we obtain that

\[
\int_{\mathbb{R}^{n}} (f(x + t\xi + t\eta) - f(x))^{p}_{+} \, dx \\
= \int_{\mathbb{R}^{n}} (f(x + t\xi + t\eta) - f(x + t\xi) + f(x + t\xi) - f(x))^{p}_{+} \, dx \\
\leq \int_{\mathbb{R}^{n}} ((f(x + t\xi + t\eta) - f(x + t\xi))_{+} + (f(x + t\xi) - f(x))_{+})^{p} \, dx \\
\leq \int_{\mathbb{R}^{n}} (1 + |\eta|^{\frac{p}{p-1}})^{p-1} \left( \frac{\|f(x + t\xi + t\eta) - f(x + t\xi)\|^{p}_{\Pi^{p,s}_{\varepsilon}}}{|\eta|^{\frac{p}{p-1}}} + \|f(x + t\xi) - f(x)\|^{p}_{\Pi^{p,s}_{\varepsilon}} \right) \, dx \\
\leq (1 + |\eta|^{\frac{p}{p-1}})^{p-1} \left( \|f(\cdot + t\eta) - f(\cdot)\|_{\Pi^{p,s}_{\varepsilon}}^{p} + \|f(\cdot + t\xi) - f(\cdot)\|_{\Pi^{p,s}_{\varepsilon}}^{p} \right),
\]

where we used the inequality \( a + b \leq (1 + r^{p/(p-1)})^{(p-1)/p}(r^{-1}a + b)^{1/p} \) for \( a, b, r > 0 \), which is a consequence of Hölder’s inequality. Thus, integrating and using (17), we obtain

\[
\tag{18} \|\xi + \eta\|_{\Pi^{p,s}_{\varepsilon}}^{p} \leq (1 + |\eta|^{\frac{p}{p-1}})^{p-1}(d|\eta|^{\frac{p}{p-1}} + \|\xi\|_{\Pi^{p,s}_{\varepsilon}}^{p}).
\]

Applying inequality (18) to the vectors \( \xi + \eta \) and \( -\eta \), we get

\[
\|\xi\|_{\Pi^{p,s}_{\varepsilon}}^{p} = \|\xi + \eta - \eta\|_{\Pi^{p,s}_{\varepsilon}}^{p} \leq (1 + | - \eta|^{\frac{p}{p-1}})^{p-1}(d|\eta|^{\frac{p}{p-1}} + \|\xi + \eta\|_{\Pi^{p,s}_{\varepsilon}}^{p}).
\]

which implies

\[
\tag{19} \|\xi + \eta\|_{\Pi^{p,s}_{\varepsilon}}^{p} \geq (1 + |\eta|^{\frac{p}{p-1}})^{-(p-1)}\|\xi\|_{\Pi^{p,s}_{\varepsilon}}^{p} - d|\eta|^{\frac{p}{p-1}}.
\]

The continuity of \( \| \cdot \|_{\Pi^{p,s}_{\varepsilon}} \) now follows from (18) and (19).

\[ \square \]

5. The Limit of Fractional \( L^{p} \) Polar Projection Bodies

We establish the limiting behavior of \( s \)-fractional \( L^{p} \) polar projection bodies for 
\( 1 < p < n/s \) as \( s \to 1^{-} \) in the symmetric and asymmetric case. For \( p = 1 \), a corresponding result was proved in [16].

Let \( 0 < s < 1 \) and \( 1 < p < n/s \). Set \( p' = p/(p-1) \). We say that \( f_{k} \to f \) weakly in \( L^{p}(\mathbb{R}^{n}) \) if

\[
\int_{\mathbb{R}^{n}} f_{k}(x)g(x) \, dx \to \int_{\mathbb{R}^{n}} f(x)g(x) \, dx
\]

for every \( g \in L^{q'}(\mathbb{R}^{n}) \) as \( k \to \infty \). Set \( B_{p',+} = \{ g \in L^{p'}(\mathbb{R}^{n}) : g \geq 0, \|g\|_{p'} \leq 1 \} \).
We require the following lemmas.

**Lemma 6.** The following statements hold.

1. For \( f \in L^p(\mathbb{R}^n) \),
   \[
   \|f_+\|_p = \sup_{g \in B_{p',+}} \int_{\mathbb{R}^n} f(x)g(x) \, dx.
   \]

2. Let \( f_k, f \in L^p(\mathbb{R}^n) \). If \( f_k \to f \) weakly in \( L^p(\mathbb{R}^n) \) as \( k \to \infty \), then
   \[
   \liminf_{k \to \infty} \|f_k+\|_p \geq \|f_+\|_p.
   \]

3. Assume \( f_k \) is a bounded sequence in \( L^p(\mathbb{R}^n) \). If
   \[
   \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k(x)g(x) \, dx = \int_{\mathbb{R}^n} f(x)g(x) \, dx
   \]
   for every \( g \) in a dense subset \( D \subseteq L^{p'}(\mathbb{R}^n) \), then \( f_k \to f \) weakly in \( L^p(\mathbb{R}^n) \) as \( k \to \infty \).

**Proof.** First we prove (1). Let \( g \in B_{p',+} \) and write \( f = f_+ - f_- \). Since \( f_- \) and \( g \) are non-negative, it follows from Hölder’s inequality that
   \[
   \int_{\mathbb{R}^n} f(x)g(x) \, dx \leq \int_{\mathbb{R}^n} f_+(x)g(x) \, dx \leq \|f_+\|_p.
   \]
   For the opposite inequality, take \( g = \|f_+\|_p^{-p'/p'} f_+^{p'/p'} \) and notice that \( g \in B_{p',+} \) and
   \[
   \int_{\mathbb{R}^n} f(x)g(x) \, dx = \|f_+\|_p^{-\frac{p}{p'}} \int_{\mathbb{R}^n} f(x)f_+(x)^{\frac{p}{p'}} \, dx \leq \|f_+\|_p^{-\frac{p}{p'}} \int_{\mathbb{R}^n} f_+(x)^{p} \, dx = \|f_+\|_p.
   \]
   Next we prove (2). Fix \( k_0 \) and \( g_0 \in B_{p',+} \). By (1), we have
   \[
   \int_{\mathbb{R}^n} f_{k_0}(x)g_0(x) \, dx \leq \sup_{g \in B_{p',+}} \int_{\mathbb{R}^n} f_{k_0}(x)g(x) \, dx = \|(f_{k_0})_+\|_p.
   \]
   Since this inequality holds for every \( k_0 \),
   \[
   \int_{\mathbb{R}^n} f(x)g_0(x) \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k(x)g_0(x) \, dx \leq \liminf_{k \to \infty} \|(f_k)_+\|_p.
   \]
   Thus, by (1),
   \[
   \|f_+\|_p = \sup_{g \in B_{p',+}} \int_{\mathbb{R}^n} f(x)g(x) \, dx \leq \liminf_{k \to \infty} \|(f_k)_+\|_p.
   \]
   Finally, we prove (3). Take \( c \geq \max\{\|f_k\|_p, \|f\|_p\} \). Let \( \varepsilon > 0 \) and \( g \in L^{p'}(\mathbb{R}^n) \). Take \( h \in D \) such that \( \|g - h\|_{p'} < \varepsilon/(2c) \). Then
   \[
   \left| \int_{\mathbb{R}^n} f_k(x)g(x) \, dx - \int_{\mathbb{R}^n} f(x)g(x) \, dx \right|
   \leq \left| \int_{\mathbb{R}^n} f_k(x)(g(x) - h(x)) \, dx \right| + \left| \int_{\mathbb{R}^n} f_k(x)h(x) \, dx - \int_{\mathbb{R}^n} f(x)h(x) \, dx \right|
   \leq c\varepsilon/(2c) + \left| \int_{\mathbb{R}^n} f_k(x)h(x) \, dx - \int_{\mathbb{R}^n} f(x)h(x) \, dx \right| + c\varepsilon/(2c)
   \]
   and the statement follows. \( \square \)
Lemma 7. For $f \in W^{1,p}(\mathbb{R}^n)$ and fixed $\xi \in S^{n-1}$,
\[
\lim_{t \to 0} \left\| \left( \frac{f(\cdot + t\xi) - f(\cdot)}{t} \right) \right\|^p = \int_{\mathbb{R}^n} \langle \nabla f(x), \xi \rangle^p \, dx.
\]

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Write $\text{div}_x$ for the divergence taken with respect to the variable $x$. Using integration by parts, we obtain for $\xi \in S^{n-1}$ and $t > 0$,
\[
\int_{\mathbb{R}^n} g(x) \frac{f(x + t\xi) - f(x)}{t} \, dx = \int_{\mathbb{R}^n} f(x) \frac{g(x - t\xi) - g(x)}{t} \, dx
\]
\[
= - \int_{\mathbb{R}^n} f(x) \int_0^1 \langle \nabla g(x - rt\xi), \xi \rangle \, dr \, dx
\]
\[
= - \int_{\mathbb{R}^n} f(x) \text{div}_x \left( \int_0^1 g(x - rt\xi) \, dr \right) \xi \, dx
\]
\[
= \int_{\mathbb{R}^n} \left( \int_0^1 g(x - rt\xi) \, dr \right) \langle \nabla f(x), \xi \rangle \, dx.
\]

By Minkowski’s integral inequality $\| \int_0^1 g(\cdot - rt\xi) \, dr \|_{p'} \leq \| g \|_{p'}$, and we deduce
\[
\left\| \left( \frac{f(\cdot + t\xi) - f(\cdot)}{t} \right) \right\|^p \leq \| \langle \nabla f(\cdot), \xi \rangle \|_p < \infty.
\]

Hence, $\frac{f(\cdot + \xi) - f(\cdot)}{t}$ is uniformly bounded in $L^p(\mathbb{R}^n)$ on $(0, \infty)$.

By Lemma 6(3),
\[
\lim_{t \to 0} \int_{\mathbb{R}^n} g(x) \frac{f(x + t\xi) - f(x)}{t} \, dx = \int_{\mathbb{R}^n} g(x) \langle \nabla f(x), \xi \rangle \, dx
\]
for every $g \in L^p(\mathbb{R}^n)$. Hence, $\frac{f(\cdot + t\xi) - f(\cdot)}{t}$ converges weakly to $\langle \nabla f(\cdot), \xi \rangle$ as $t \to 0$.

By Lemma 6(2),
\[
\liminf_{t \to 0} \left\| \left( \frac{f(\cdot + t\xi) - f(\cdot)}{t} \right) \right\|_p \geq \| \langle \nabla f(\cdot), \xi \rangle \|_p.
\]

For the opposite inequality we recall that for any $g \in B_{p',+}$, the function $x \mapsto \int_0^1 g(x - rt\xi) \, dr$ is in $B_{p',+}$ as well. Hence,
\[
\int_{\mathbb{R}^n} g(x) \frac{f(x + t\xi) - f(x)}{t} \, dx = \int_{\mathbb{R}^n} \left( \int_0^1 g(x - rt\xi) \, dr \right) \langle \nabla f(x), \xi \rangle \, dx
\]
\[
\leq \| \langle \nabla f(x), \xi \rangle \|_p.
\]

Again by Lemma 6(1),
\[
\left\| \left( \frac{f(\cdot + t\xi) - f(\cdot)}{t} \right) \right\|_p \leq \| \langle \nabla f(\cdot), \xi \rangle \|_p
\]
for each $t > 0$. \hfill \Box

The following result is Lemma 4 in [10].

Lemma 8. If $\varphi : [0, \infty) \to [0, \infty)$ be a measurable function with $\lim_{t \to 0^+} \varphi(t) = \varphi(0)$ and such that $\int_0^\infty t^{-s_0} \varphi(t) \, dt < \infty$ for some $s_0 \in (0, 1)$, then
\[
\lim_{s \to 1^-} (1 - s) \int_0^\infty t^{-s} \varphi(t) \, dt = \varphi(0).
\]
We are now able to prove the main result of this section.

**Theorem 9.** Let \( f \in W^{1,p}(\mathbb{R}^n) \). For \( \xi \in S^{n-1} \),
\[
\lim_{s \to 1^-} (p(1-s))^{\frac{1}{p}} \| \Pi_{p,s}^* f \|_{L^p} = \| \xi \|_{L^p}.
\]
Moreover,
\[
\lim_{s \to 1^-} p(1-s) | \Pi_{p,s}^* f |^{-\frac{1}{p}} = | \Pi_p^* f |^{-\frac{1}{p}},
\]
and
\[
\lim_{s \to 1^-} p(1-s) \tilde{V}_{p,s}(K, \Pi_{p,s}^*) = \tilde{V}_p(K, \Pi_p^*)
\]
for every star body \( K \subset \mathbb{R}^n \).

**Proof.** Define \( \varphi : [0, \infty) \to [0, \infty) \) by
\[
\varphi(t) = \left\| \left( \frac{f(\cdot + t\xi) - f(\cdot)}{t} \right)_+ \right\|_p^p,
\]
and note that \( \varphi(t) \leq \left( \frac{2\|f\|_p}{t} \right)^p \) for \( t > 0 \). By Lemma 8 and Lemma 7,
\[
\lim_{s \to 1^-} p(1-s) \int_0^\infty t^{p(1-s)-1} \left\| \left( \frac{f(\cdot + t\xi) - f(\cdot)}{t} \right)_+ \right\|_p^p dt = \int_{\mathbb{R}^n} \langle \nabla f(x), \xi \rangle_p^p dx.
\]
By Proposition 4, we can use the dominated convergence theorem to obtain
\[
\lim_{s \to 1^-} n |(p(1-s))^{-\frac{1}{p}} \Pi_{p,s}^* f| = \lim_{s \to 1^-} \int_{S^{n-1}} \left( p(1-s) \int_0^\infty t^{p(1-s)-1} \left\| \left( \frac{f(\cdot + t\xi) - f(\cdot)}{t} \right)_+ \right\|_p^p dt \right)^{-\frac{1}{p}} d\xi
\]
\[
= \int_{S^{n-1}} \left( \int_{\mathbb{R}^n} \langle \nabla f(x), \xi \rangle_p^p dx \right)^{-\frac{1}{p}} d\xi
\]
\[
= n | \Pi_p^* f |,
\]
and
\[
\lim_{s \to 1^-} n p(1-s) \tilde{V}_{p,s}(K, \Pi_{p,s}^*) = \lim_{s \to 1^-} p(1-s) \int_{S^{n-1}} \left\| \xi \right\|_K^{n+p^s} \| \xi \|_{L^p}^{p_n} d\xi
\]
\[
= \int_{S^{n-1}} \| \xi \|_K^{n} \| \xi \|_{L^p}^{p_n} d\xi
\]
\[
= n \tilde{V}_p(K, \Pi_p^*)
\]
which completes the proof of the theorem. \( \square \)

The following result is an immediate consequence of Theorem 9 and (15).

**Theorem 10.** Let \( f \in W^{1,p}(\mathbb{R}^n) \). For \( \xi \in S^{n-1} \),
\[
\lim_{s \to 1^-} (p(1-s))^{\frac{1}{p}} \| \Pi_{p,s}^* f \|_{L^p} = \| \xi \|_{L^p}.
\]
Moreover,
\[
\lim_{s \to 1^-} p(1-s) | \Pi_{p,s}^* f |^{-\frac{1}{p}} = | \Pi_p^* f |^{-\frac{1}{p}},
\]
and
\[
\lim_{s \to 1^-} p(1-s) \tilde{V}_{p,s}(K, \Pi_{p,s}^*) = \tilde{V}_p(K, \Pi_p^*)
\]
for every star body \( K \subset \mathbb{R}^n \).
6. Anisotropic Fractional Pólya–Szegő Inequalities

We will establish anisotropic Pólya–Szegő inequalities for fractional $L^p$ Sobolev norms and their asymmetric counterparts.

**Theorem 11.** If $f \in L^p(\mathbb{R}^n)$ is non-negative and $K \subset \mathbb{R}^n$ a star body, then

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|_K^{n+ps}} \, dx \, dy \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f^*(x) - f^*(y))^p}{\|x - y\|_K^{n+ps}} \, dx \, dy.
\]

Equality holds for non-zero $f \in W_s^p(\mathbb{R}^n)$ if and only if $K$ is a centered ellipsoid and $f$ is a translate of $f^* \circ \phi$ for some $\phi \in SL(n)$.

**Proof.** Writing

\[
\|z\|_K^{n-ps} = \int_0^\infty k_t(z) \, dt
\]

where $k_t(z) = 1_{t^{-1/(n+ps)}K}(z)$, we obtain

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|_K^{n+ps}} \, dx \, dy = \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - f(y))^p k_t(x - y) \, dx \, dy \, dt.
\]

Note that

\[
(f(x) - f(y))^p = p \int_0^\infty (f(x) - r)^{p-1} 1_{(f < r)}(y) \, dr.
\]

Hence, for $t > 0$, it follows from Fubini’s theorem that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - f(y))^p k_t(x - y) \, dx \, dy
\]

\[
= p \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)^{p-1} k_t(x - y) 1_{(f < r)}(y) \, dx \, dy \, dr
\]

\[
= p \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)^{p-1} k_t(x - y) (1 - 1_{(f \geq r)}(y)) \, dx \, dy \, dr.
\]

Let $r, t > 0$. Note that $\int_{\mathbb{R}^n} (f(x) - r)^{p-1} \, dx < \infty$ and that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)^{p-1} k_t(x - y) (1 - 1_{(f \geq r)}(y)) \, dx \, dy
\]

\[
= p \|k_t\|_1 \int_{\mathbb{R}^n} (f(x) - r)^{p-1} \, dx - p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)^{p-1} k_t(x - y) 1_{(f \geq r)}(y) \, dx \, dy.
\]

The first term is finite since $\{f > r\}$ has finite measure, $f \in L^{n/(n-p)}(\mathbb{R}^n)$ and $\frac{np}{n-p} > p - 1$. Clearly the first term is invariant under Schwarz symmetrization. For the second term, by the Riesz rearrangement inequality, Theorem 2 we have

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(x) - r)^{p-1} k_t(x - y) 1_{(f \geq r)}(y) \, dx \, dy
\]

\[
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f^*(x) - r)^{p-1} k_t^*(x - y) 1_{(f^* \geq r)}(y) \, dx \, dy
\]

for $r, t > 0$. Note that

\[
(f(x) - r)^{p-1} = (p - 1) \int_0^\infty (\tilde{r} - r)^{p-2} 1_{(f \geq \tilde{r})}(x) \, d\tilde{r}
\]
and that the corresponding equation holds for $f^*$. Hence, if there is equality in (21), then, for $(\tilde{r}, r, t) \in (0, \infty)^3 \setminus M$ with $|M| = 0$, we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{f \geq \tilde{r}\}}(x) 1_{t^{-1/(n+p)}K}(x - y) 1_{\{f \geq r\}}(y) \, dx \, dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{f^* \geq \tilde{r}\}}(x) 1_{t^{-1/(n+p)}K^*}(x - y) 1_{\{f^* \geq r\}}(y) \, dx \, dy.
\]

For almost every $(\tilde{r}, r, t) \in (0, \infty)^2$, we have $(\tilde{r}, r, t) \in (0, \infty)^3 \setminus M$ for almost every $t > 0$. For such $(\tilde{r}, r)$ with $\tilde{r} \leq r$ and $t > 0$ sufficiently large, the assumptions of Theorem 3 are fulfilled and therefore there are a centered ellipsoid $D$ and $a, b \in \mathbb{R}^n$ (depending on $(\tilde{r}, r)$) such that
\[
\{f \geq \tilde{r}\} = a + aD, \quad t_{-1/(n+p)}K = b + \beta D, \quad \{f \geq r\} = c + \gamma D
\]
where $c = a + b$. Since $K = t_{-1/(n+p)}b + (|K|/|D|)^{1/n}D$, the centered ellipsoid $D$ does not depend on $(\tilde{r}, r, t)$ and also $a, c$ do not depend on $t$. It follows that $b = 0$ and that $K$ is a multiple of $D$. Hence, $a = c$ is a constant vector which concludes the proof. □

The following result is a variation of [17, Theorem 3.1].

**Theorem 12.** If $f \in L^p(\mathbb{R}^n)$ is non-negative and $K \subset \mathbb{R}^n$ a star body, then
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+ps}} \, dx \, dy \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f^*(x) - f^*(y)|^p}{\|x - y\|^{n+ps}} \, dx \, dy.
\]
Equality holds for non-zero $f \in W^{s,p}(\mathbb{R}^n)$ if and only if $K$ is a centered ellipsoid and $f$ is a translate of $f^* \circ \phi$ for some $\phi \in \text{SL}(n)$.

**Proof.** Since
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|^{n+ps}_K} \, dx \, dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{\|x - y\|^{n+ps}_{-K}} \, dx \, dy,
\]
the result follows from Theorem 11 for $K$ and $-K$. □

### 7. Affine Fractional Pólya–Szegő Inequalities

We establish affine Pólya–Szegő inequalities for fractional asymmetric and symmetric $L^p$ polar projection bodies.

**Theorem 13.** If $f \in W^{s,p}(\mathbb{R}^n)$ is non-negative, then
\[
|\Pi_{p, f}|^{-ps/n} \geq |\Pi_{p, f^*}|^{-ps/n}.
\]
Equality holds if and only if $f$ is a translate of $f^* \circ \phi$ for some $\phi \in \text{SL}(n)$.

**Proof.** By Theorem 11 (16) and the dual mixed volume inequality, we obtain for $K \subset \mathbb{R}^n$ a star body that
\[
\tilde{V}_{ps}(K, \Pi_{p, f}) \geq \tilde{V}_{ps}(K^*, \Pi_{p, f^*}) \geq |K^*|^{(n+ps)/n} |\Pi_{p, f^*}|^{-ps/n} = |K|^{(n+ps)/n} |\Pi_{p, f^*}|^{-ps/n}.
\]
Setting $K = \Pi_{p, f}$, we see that
\[
|\Pi_{p, f}| = \tilde{V}_{ps}(\Pi_{p, f}, \Pi_{p, f}) \geq |\Pi_{p, f}|^{(n+ps)/n} |\Pi_{p, f^*}|^{-ps/n},
\]
and that the corresponding equation holds for $f^*$. Hence, if there is equality in (21), then, for $(\tilde{r}, r, t) \in (0, \infty)^3 \setminus M$ with $|M| = 0$, we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{f \geq \tilde{r}\}}(x) 1_{t^{-1/(n+p)}K}(x - y) 1_{\{f \geq r\}}(y) \, dx \, dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{f^* \geq \tilde{r}\}}(x) 1_{t^{-1/(n+p)}K^*}(x - y) 1_{\{f^* \geq r\}}(y) \, dx \, dy.
\]
which completes the proof of the inequality. By Theorem 14, there is equality in (22) if and only if \( f \) is a translate of \( f^* \circ \phi \) for some \( \phi \in \text{SL}(n) \).

The following result is obtained in the same way as Theorem 13 by replacing Theorem 11 with Theorem 12.

**Theorem 14.** If \( f \in L^p(\mathbb{R}^n) \) is non-negative, then

\[
|\Pi_p^s f|^{-ps/n} \geq |\Pi_{p,s} f^*|^{-ps/n}.
\]

Equality holds for \( f \in W^{s,p}(\mathbb{R}^n) \) if and only if \( f \) is a translate of \( f^* \circ \phi \) for some \( \phi \in \text{SL}(n) \).

We remark that by Theorem 10 we obtain from Theorem 14 in the limit as \( s \to 1^- \) that

\[
|\Pi_p f|^{-p/n} \geq |\Pi_{p} f^*|^{-p/n},
\]

which is equivalent to the Pólya–Szegő inequality for \( L^p \) projection bodies by Cianchi, Lutwak, Yang, and Zhang [11, Theorem 2.1]. Similarly, by Theorem 9 we obtain from Theorem 13 in the limit as \( s \to 1^- \) that

\[
|\Pi_{p,s} f|^{-p/n} \geq |\Pi_{p,s} f^*|^{-p/n},
\]

which is equivalent to the Pólya–Szegő inequality for asymmetric \( L^p \) projection bodies by Haberl, Schuster and Xiao [15, Theorem 1].

8. **Affine Fractional Asymmetric \( L^p \) Sobolev Inequalities**

We establish the following affine fractional asymmetric \( L^p \) Sobolev inequalities and show that they are stronger than Theorem 1.

**Theorem 15.** Let \( 0 < s < 1 \) and \( 1 < p < n/s \). For non-negative \( f \in W^{s,p}(\mathbb{R}^n) \),

\[
\|f\|_{n,p}^{p_n,p_s} \leq 2 \sigma_{n,p,s} n\omega_n^{n+ps} |\Pi_{p} f|^{-p/n} \leq 2\sigma_{n,p,s} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{|x - y|^{n+ps}} \, dx \, dy.
\]

There is equality in the first inequality if and only if \( f = h_{s,p} \circ \phi \) for some \( \phi \in \text{GL}(n) \) where \( h_{s,p} \) is an extremal function of (1). There is equality in the second inequality if \( f \) is radially symmetric.

**Proof.** By Theorem 13

\[
|\Pi_{p,s} f|^{-ps/n} \geq |\Pi_{p,s} f^*|^{-ps/n},
\]

with equality if \( f \) is a translate of \( f^* \circ \phi \) for some \( \phi \in \text{SL}(n) \). Since \( f^* \) is radially symmetric, \( \Pi_{p,s} f^* = \Pi_{p,s} f^* \) is a ball. Hence, it follows from (16) that

\[
2n\omega_n^\frac{n+ps}{n} |\Pi_{p,s} f^*|^{-\frac{ps}{n}} = 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f^*(x) - f^*(y))^p}{|x - y|^{n+ps}} \, dx \, dy
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f^*(x) - f^*(y)|^p}{|x - y|^{n+ps}} \, dx \, dy.
\]

The fractional Sobolev inequality (11) shows that

\[
\sigma_{n,p,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f^*(x) - f^*(y)|^p}{|x - y|^{n+ps}} \, dx \, dy \geq \|f^*\|_{p,n}^{p_n,p_s}.
\]

Combining these inequalities and their equality cases, we complete the proof of the first inequality of the theorem.
For the second inequality, we set \( K = B^n \) in (10) and apply the dual mixed volume inequality (3) to obtain
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^p}{|x - y|^{n+ps}} \, dx \, dy = n \lambda_{-ps}(B^n, \Pi^{s,\ast}_p f) \geq n \omega_n \frac{n+ps}{n} |\Pi^{s,\ast}_p f|^{-\frac{ps}{n}}.
\]
There is equality precisely if \( \Pi^{s,\ast}_p f \) is a ball, which is the case for radially symmetric functions. \( \square \)

Note that it follows from the definition of fractional symmetric and asymmetric \( L^p \) polar projection bodies that
\[
\Pi^{s,\ast}_p f = \Pi^{s,\ast}_{p_+} f + \Pi^{s,\ast}_{p_-} f.
\]
We use the dual Brunn–Minkowski inequality (5) and obtain that
\[
|\Pi^{s,\ast}_p f|^{-\frac{ps}{n}} \geq |\Pi^{s,\ast}_{p_+} f|^{-\frac{ps}{n}} + |\Pi^{s,\ast}_{p_-} f|^{-\frac{ps}{n}},
\]
with equality precisely if the star bodies \( \Pi^{s,\ast}_{p_+} f \) and \( \Pi^{s,\ast}_{p_-} f \) are dilates. Thus, it follows that for non-negative \( f \), Theorem 15 implies Theorem 1 and it is, in general, substantially stronger than Theorem 1. Of course, they coincide for even functions.

9. **Affine Fractional \( L^p \) Sobolev Inequalities: Proof of Theorem 1**

For non-negative \( f \), the first inequality in Theorem 1 follows from Theorem 15 as mentioned before. For general \( f \) and \( x, y \in \mathbb{R}^n \), we use
\[
|f(x) - f(y)| \geq |f(x) - |f(y)||,
\]
where equality holds if and only if \( f(x) \) and \( f(y) \) are both non-negative or non-positive. We obtain
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} \, dx \, dy \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - |f(y)||^p}{|x - y|^{n+ps}} \, dx \, dy,
\]
with equality if and only if \( f \) has constant sign for almost every \( x, y \in \mathbb{R}^n \). Using the result for \( |f| \), we obtain the first inequality of the theorem and its equality case.

For the second inequality, we set \( K = B^n \) in (10) and apply the dual mixed volume inequality (3) as in the proof of Theorem 15.

10. **Optimal Fractional \( L^p \) Sobolev Bodies**

The following important question was asked by Lutvak, Yang and Zhang [25] for a given \( f \in W^{1,p}(\mathbb{R}^n) \) and \( 1 \leq p < n \): For which origin-symmetric convex bodies \( K \subset \mathbb{R}^n \) is
\[
(23) \quad \inf \left\{ \int_{\mathbb{R}^n} \|
abla f(x)\|^p_{L^p} \, dx : K \text{ origin-symmetric convex body, } |K| = \omega_n \right\}
\]
attained? An optimal \( L^p \) Sobolev body of \( f \) is a convex body where the infimum is attained.

Lutvak, Yang and Zhang [25] showed that the infimum in (23) is attained (up to normalization) at the unique origin-symmetric convex body \( \langle f \rangle_p \) in \( \mathbb{R}^n \) such that
\[
(24) \quad \int_{\mathbb{R}^{n-1}} g(\xi) \, dS_p(\langle f \rangle_p, \xi) = \int_{\mathbb{R}^n} g(\nabla f(x)) \, dx
\]
for every even \( g \in C(\mathbb{R}^n) \) that is positively homogeneous of degree \( p \), where \( S_p(K, \cdot) \) is the \( L_p \) surface area measure of \( K \). Setting \( g = \| \cdot \|_{K^*} \), they obtain from the \( L^p \) Minkowski inequality that

\[
\frac{1}{n} \int_{\mathbb{R}^n} \| \nabla f(x) \|_{K^*}^p \, dx = V_p((f)_p, K) \geq \| (f)_p \|_{(n-p)/n} |K|^{p/n},
\]

with equality precisely if \( K \) and \( (f)_p \) are homothetic (see \cite{28} Section 9.1) for the definition of the \( L_p \) mixed volume \( V_p(\cdot, \cdot) \) and the \( L^p \) Minkowski inequality. Hence, they obtain from their solution to their functional version \cite{24} of the \( L^p \) Minkowski problem that \( (f)_p \) is the optimal \( L^p \) Sobolev body associated to \( f \). Tuo Wang \cite{31} obtained corresponding results for \( f \in BV(\mathbb{R}^n) \) and \( p = 1 \).

Let \( 0 < s < 1 \) and \( 1 < p < n/s \). The results by Lutwak, Yang and Zhang \cite{25} suggest the following question for a given \( f \in W^{s,p}(\mathbb{R}^n) \): For which star bodies \( L \subset \mathbb{R}^n \) is

\[
\inf \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_L^{n+ps}} \, dx \, dy : L \text{ star body}, |L| = \omega_n \right\}
\]

attained? An optimal \( s \)-fractional \( L^p \) Sobolev body of \( f \) is a star body where the infimum is attained.

By \cite{10} and the dual mixed volume inequality \cite{4},

\[
\frac{1}{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_L^{n+ps}} \, dx \, dy = \tilde{V}_{-ps}(L, \Pi_p^{s,s} f) \geq |L|^{(n+ps)/n} \| \Pi_p^{s,s} f \|_{-ps}/n,
\]

and there is equality precisely if \( L \) is a dilate of \( \Pi_p^{s,s} f \). Hence, \( \Pi_p^{s,s} f \) is the unique optimal \( s \)-fractional \( L^p \) Sobolev body associated to \( f \).

To understand how the solutions to \cite{23} and \cite{26} are related, we use the following result: For \( f \in W^{1,p}(\mathbb{R}^n) \) and \( L \subset \mathbb{R}^n \) a star body,

\[
\lim_{s \to 1^{-}} p(1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_L^{n+ps}} \, dx \, dy = \int_{\mathbb{R}^n} \| \nabla f(x) \|_{Z_p L} \, dx,
\]

where the convex body \( Z_p K \), defined for \( \xi \in \mathbb{S}^{n-1} \) by

\[
h_{Z_p L}(\xi)^p = \int_{\mathbb{S}^{n-1}} |\langle \xi, \eta \rangle|^p \rho_L(\eta)^{n+p} \, d\eta,
\]

is a multiple of the \( L^p \) centroid body of \( L \). This can be proved as in \cite{21}, where the corresponding result was established for a convex body \( L \) (with a different normalization of \( Z_p L \)). It also follows from Theorem \cite{10}. Indeed, by \cite{10} and \cite{20},

\[
\lim_{s \to 1^{-}} p(1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_L^{n+ps}} \, dx \, dy = \tilde{V}_{-p}(L, \Pi_p^s f).
\]

Using that

\[
\Pi_p^s f = \Pi_p^s (f)_p
\]

for \( f \in W^{1,p}(\mathbb{R}^n) \), which follows from \cite{24} by setting \( g = |\cdot, \eta|^p \) for \( \eta \in \mathbb{S}^{n-1} \) and using \cite{8} and \cite{4} (cf. \cite{25}), and that

\[
V_p(K, Z_p L) = \tilde{V}_{-p}(L, \Pi_p^s K)
\]

for \( K \) a convex body and \( L \) a star body, a well-known relation that follows from Fubini’s theorem, we now obtain \cite{27} from the first equation in \cite{25}.
Using (27), we obtain from (26) in the limit as $s \to 1^−$ for a given $f \in W^{1,p}(\mathbb{R}^n)$, the following question: For which star bodies $L \subset \mathbb{R}^n$ is
\begin{equation}
\inf \left\{ \int_{\mathbb{R}^n} \|\nabla f(x)\|\|_Z^p L \, dx : L \text{ star body}, |L| = \omega_n \right\}
\end{equation}
attained? By (25) and the dual mixed volume inequality (6), we have
\begin{align*}
\frac{1}{n} \int_{\mathbb{R}^n} \|\nabla f(x)\|\|_Z^p L \, dx = V_p(\langle f \rangle_p, Z_p L) = \bar{V}_p(L, \Pi_p^* f) \geq |L|^{(n+p)/n} |\Pi_p^* f|^{-p/n},
\end{align*}
with equality precisely if $L$ and $\Pi_p^* f$ are dilates, where we have used (28) and (29).

From Theorem 10, we obtain that a suitably scaled sequence of optimal $s$-fractional Sobolev bodies converges to a multiple of the optimal body for (30) as $s \to 1^−$.

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