A stochastic recursive optimal control problem under the G-expectation framework

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Abstract. In this paper, we study a stochastic recursive optimal control problem in which the objective functional is described by the solution of a backward stochastic differential equation driven by G-Brownian motion. Under standard assumptions, we establish the dynamic programming principle and the related Hamilton-Jacobi-Bellman (HJB) equation in the framework of G-expectation. Finally, we show that the value function is the viscosity solution of the obtained HJB equation.

Key words. G-expectation, backward stochastic differential equations, stochastic optimal control, dynamic programming principle, viscosity solution

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1 Introduction

It is well known that the nonlinear backward stochastic differential equation (BSDE) was first introduced by Pardoux and Peng [19]. Independently, Duffie and Epstein [5] presented a stochastic differential recursive utility which corresponds to the solution of a particular BSDE. Then the BSDE point of view gives a simple formulation of recursive utilities (see [6]).

Since then, the classical stochastic optimal control problem is generalized to a so called "stochastic recursive optimal control problem" in which the cost functional is described by the solution of BSDE. Peng [28] obtained the Hamilton–Jacobi–Bellman equation for this kind of problem and proved that the value function is its viscosity solution. In [29], Peng generalized his results and originally introduced the notion of stochastic backward semigroups which allows him to prove the dynamic programming principle in a very straightforward way. This backward semigroup approach is proved to be a useful tool for the stochastic optimal control problems. For instance, Wu and Yu [31] adopted this approach to study one kind of stochastic recursive optimal control problem with the cost functional described by the solution of a reflected BSDE. It is also introduced in the theory of stochastic differential games by Buckdahn and Li in [3]. We emphasize

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that Buckdahn et al. \[2\] obtained an existence result of the stochastic recursive optimal control problem.

Motivated by measuring risk and other financial problems with uncertainty, Peng \[22\] introduced the notion of sublinear expectation space, which is a generalization of probability space. As a typical case, Peng studied a fully nonlinear expectation, called $G$-expectation $\hat{\mathbb{E}}[\cdot]$ \(\text{(see } \[25\] \text{and the references therein)}\), and the corresponding time-conditional expectation $\hat{\mathbb{E}}_t[\cdot]$ on a space of random variables completed under the norm $\hat{\mathbb{E}}[|\cdot|^{1/p}]$. Under this $G$-expectation framework ($G$-framework for short) a new type of Brownian motion called $G$-Brownian motion was constructed. The stochastic calculus with respect to the $G$-Brownian motion has been established. The existence and uniqueness of solution of a SDE driven by $G$-Brownian motion can be proved in a way parallel to that in the classical SDE theory. But the solvability of BSDE driven by $G$-Brownian motion becomes a challenging problem. For a recent account and development of $G$-expectation theory and its applications we refer the reader to \[7, 8, 16, 20, 21, 26, 27, 32, 35, 36\].

Let us mention that there are other recent advances and their applications in stochastic calculus that do not require a probability space framework. Denis and Martini \[3\] developed quasi-sure stochastic analysis, but they did not have conditional expectation. This topic was further examined by Denis et al. \[4\] and Soner et al. \[33\]. It is worth noting that Soner et al. \[34\] have obtained a deep result of existence and uniqueness theorem for a new type of fully nonlinear BSDE, called $2$BSDE. Various stochastic control (game) problems are investigated in \[13, 17, 18, 30\] and the applications in finance are studied in \[14, 15\].

Recently Hu et al. studied the following BSDE driven by $G$-Brownian motion in \[11\] and \[10\]:

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).
\]

They proved that there exists a unique triple of processes $(Y, Z, K)$ within our $G$-framework which solves the above BSDE under a standard Lipschitz conditions on $f(s, y, z)$ and $g(s, y, z)$ in $(y, z)$. The decreasing $G$-martingale $K$ is aggregated and the solution is time consistent. Some important properties of the BSDE driven by $G$-Brownian motion such as comparison theorem and Girsanov transformation were given in \[10\].

In this paper, we study a stochastic recursive optimal control problem in which the objective functional is described by the solution of a BSDE driven by $G$-Brownian motion. In more details, the state equation is governed by the following controlled SDE driven by $G$-Brownian motion

\[
\begin{align*}
    &dX^{t,x,u}_s = b(s, X^{t,x,u}_s, u_s)ds + h_{ij}(s, X^{t,x,u}_s, u_s)d\langle B^i, B^j \rangle_s + \sigma(s, X^{t,x,u}_s, u_s)dB_s, \\
    &X^{t,x,u}_t = x.
\end{align*}
\]

The objective functional is introduced by the solution $Y^{t,x,u}_t$ of the following BSDE driven by $G$-Brownian motion at time $t$:

\[
\begin{align*}
    -dY^{t,x,u}_s &= f(s, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u_s)ds + g_{ij}(s, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u_s)d\langle B^i, B^j \rangle_s - Z^{t,x,u}_s dB_s - dK^{t,x,u}_s, \\
    Y^{t,x,u}_T &= \Phi(X^{t,x,u}_T), \quad s \in [t, T].
\end{align*}
\]
We define the value function of our stochastic recursive optimal control problem as follows:

\[ V(t, x) = \text{ess sup}_{u \in \mathcal{U}[t,T]} Y_t^{t,x,u}, \]

where the control set is in the G-framework.

It is well known that dynamic programming and related HJB equations is a powerful approach to solving optimal control problems (see [9], [37] and [28]). The objective of our paper is to establish the dynamic programming principle and investigate the value function in G-framework. The main result of this paper states that \( V \) is deterministic continuous viscosity solution of the following HJB equation

\[
\partial_t V(t, x) + \sup_{u \in U} H(t, x, V, \partial_x V, \partial^2_{xx} V, u) = 0
\]

\[ V(T, x) = \Phi(x), \quad x \in \mathbb{R}^n, \]

where

\[ H(t, x, v, A, u) = G(F(t, x, v, p, A, u)) + \langle p, b(t, x, u) \rangle + f(t, x, v, \sigma(x, u)p, u), \]

\[ F_{ij}(t, x, v, A, u) = \langle \sigma_i(t, x, u), \sigma_j(t, x, u) \rangle + 2\langle p, h_{ij}(t, x, u) \rangle + 2g_{ij}(t, x, v, \sigma(x, u)p, u), \]

\((t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_n \times U\) and \(\sigma_i\) is the \(i\)-th column of \(\sigma\).

Notice that under the G-framework, there is no reference probability measure. Thus our results generalizes the results in Peng [28] and [29] which was only considered in the Wiener space (corresponding to \(G\) is linear in our paper). Under a family of non-dominated probability measures, it is far from being trivial to prove that the value function \(V\) is wellposed and deterministic. Furthermore, the BSDE driven by G-Brownian motion contains the decreasing G-martingale \(K\), which is more difficult to deal with.

The paper is organized as follows. In section 2, we present some fundamental results on G-expectation theory and formulate our stochastic recursive optimal control problem. We establish the dynamic programming principle in section 3. In section 4, we first derive the HJB equation and prove that the value function is the viscosity solution of the obtained HJB equation.

## 2 Preliminaries

We review some basic notions and results of G-expectation, the related spaces of random variables and the backward stochastic differential equations driven by a G-Brownian motion. The readers may refer to [11, 22–25] for more details.

Let \(\Omega\) be a given set and let \(\mathcal{H}\) be a vector lattice of real valued functions defined on \(\Omega\), namely \(c \in \mathcal{H}\) for each constant \(c\) and \(|X| \in \mathcal{H}\) if \(X \in \mathcal{H}\). \(\mathcal{H}\) is considered as the space of random variables.

**Definition 2.1** A sublinear expectation \(\hat{E} : \mathcal{H} \to \mathbb{R}\) satisfying the following properties: for all \(X, Y \in \mathcal{H}\),

(i) **Monotonicity:** If \(X \geq Y\) then \(\hat{E}[X] \geq \hat{E}[Y]\);

(ii) **Constant preservation:** \(\hat{E}[c] = c\);

(iii) **Sub-additivity:** \(\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]\);

(iv) **Positive homogeneity:** \(\hat{E}[\lambda X] = \lambda \hat{E}[X]\) for each \(\lambda \geq 0\).

\((\Omega, \mathcal{H}, \hat{E})\) is called a sublinear expectation space.
Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ respectively. We will denote by $C_{1,\text{Lip}}(\mathbb{R}^n)$ the space of real continuous functions defined on $\mathbb{R}^n$ such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where $k$ and $C$ depend only on $\varphi$.

**Definition 2.2** We call $X_1$ and $X_2$ identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if for all $\varphi \in C_{1,\text{Lip}}(\mathbb{R}^n)$,

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)].$$

**Definition 2.3** For given $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, random vectors $Y = (Y_1, \ldots, Y_n)$ and $X = (X_1, \ldots, X_m)$, $Y_i, X_i \in \mathcal{H}$. We call $Y$ is independent of $X$ under $\hat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{1,\text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$ we have

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\varphi(X)|_x = X].$$

**Definition 2.4** *(G-normal distribution)* For given $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $X = (X_1, \ldots, X_d)$. $X$ is called $G$-normally distributed if for each $a, b \geq 0$, we have

$$aX + b\hat{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of $X$, i.e., $\hat{X} \overset{d}{=} X$ and $\hat{X} \perp X$.

For each $\varphi \in C_{1,\text{Lip}}(\mathbb{R}^d)$, we define

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)], \quad (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Peng [25] proved that $X$ is $G$-normally distributed if and only if $u$ is the solution of the following $G$-heat equation:

$$\partial_t u - G(D^2_{xx} u) = 0, \quad u(0, x) = \varphi(x)$$

where $G$ denotes the function

$$G(A) := \frac{1}{2}\hat{\mathbb{E}}[(AX, X)]: \mathbb{S}_d \to \mathbb{R}.$$

The function $G(\cdot) : \mathbb{S}_d \to \mathbb{R}$ is a monotonic, sublinear mapping on $\mathbb{S}_d$, where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices. There exists a bounded and closed subset $\Gamma \subset \mathbb{R}^{d \times d}$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma^T A], \quad (2.1)$$

where $\mathbb{R}^{d \times d}$ denotes the collection of $d \times d$ matrices.

In this paper we only consider non-degenerate $G$-normal distribution, i.e., there exists some $\alpha^2 > 0$ such that $G(A) - G(B) \geq \alpha^2 \text{tr}[A - B]$ for any $A \succeq B$.

Let $\Omega = C_0([0, \infty); \mathbb{R}^d)$ be the space of real valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$ and let $B_t(\omega) = \omega_t$ be the canonical process. Set

$$L_{ip}(\Omega) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) : n \geq 1, t_1, \cdots, t_n \in [0, \infty), \varphi \in C_{1,\text{Lip}}(\mathbb{R}^{d \times n})\}.$$
Let \( \{ \xi_n : n \geq 1 \} \) be a sequence of identically distributed \( d \)-dimensional \( G \)-normally distributed random vectors in a sublinear expectation space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{E}})\) such that \( \xi_{i+1} \) is independent of \((\xi_1, \ldots, \xi_i)\) for every \( i \geq 1 \).

**Definition 2.5** For each \( X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \in L_{ip}(\Omega) \) with \( 0 \leq t_0 < \cdots < t_m \), the \( G \)-expectation of \( X \) is defined by

\[
\tilde{\mathbb{E}}[X] = \mathbb{E}[\varphi(\sqrt{t_1 - t_0} \xi_1, \ldots, \sqrt{t_m - t_{m-1}} \xi_m)].
\]

The conditional \( G \)-expectation \( \tilde{\mathbb{E}}_t \) of \( X \) with \( t = t_i \) is defined by

\[
\tilde{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] = \varphi(B_{t_i} - B_{t_{i-1}}, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}}),
\]
where

\[
\varphi(x_1, \ldots, x_i) = \mathbb{E}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})].
\]

\((\Omega, L_{ip}(\Omega), \tilde{\mathbb{E}})\) is called a \( G \)-expectation space. The corresponding canonical process \((B_t : t \geq 0)\) is called a \( G \)-Brownian motion.

We denote by \( L^p_G(\Omega) \) the completion of \( L_{ip}(\Omega) \) under the norm \( \|X\|_{p,G} = (\tilde{\mathbb{E}}[|X|^p])^{1/p} \) for \( p \geq 1 \). For each \( t \geq 0 \), \( \tilde{\mathbb{E}}[-] \) can be extended continuously to \( L^2_G(\Omega) \) under the norm \( \| \cdot \|_{1,G} \). For each fixed \( T > 0 \), set

\[
L_{ip}(\Omega_T) := \{ \varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0,T], \varphi \in C([-T,T]) \}.
\]

Obviously, \( L_{ip}(\Omega_T) \subset L_{ip}(\Omega) \), then we can similarly define \( L^p_G(\Omega_T) \) for \( p \geq 1 \).

**Definition 2.6** Let \( M^0_G(0,T) \) be the collection of processes in the following form: for a given partition \( \{t_0, \ldots, t_N\} = \pi_T \) of \([0,T]\),

\[
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{(t_j, t_{j+1}]}(t),
\]

where \( \xi_i \in L_{ip}(\Omega_{t_i}), i = 0, 1, 2, \ldots, N-1 \).

We denote by \( M^p_G(0,T) \) the completion of \( M^0_G(0,T) \) under the norm \( \| \eta \|_{M^p_G} = (\tilde{\mathbb{E}}[\int_0^T |\eta_t|^p ds])^{1/p} \) for \( p \geq 1 \).

**Theorem 2.7** \((\text{Proposition 1})\) There exists a family of weakly compact probability measures \( \mathcal{P} \) on \((\Omega, \mathcal{B}(\Omega))\) such that

\[
\tilde{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in L^1_G(\Omega).
\]

\( \mathcal{P} \) is called a set that represents \( \tilde{\mathbb{E}} \).

Let \( \{W_t\} \) be a classical \( d \)-dimensional Brownian motion on a probability space \((\Omega^0, \mathcal{F}^0, P^0)\) and let \( F^0 = \{F^0_t\} \) be the augmented filtration generated by \( W \). Set

\[
\mathcal{P}_M := \{ P_\theta : P_\theta = P^0 \circ (B^0_t)^{-1}, B^0_t = \int_0^t \theta_s dW_s, \theta \in L^2_{F^0}([0,T]; \Gamma) \},
\]

where \( L^2_{F^0}([0,T]; \Gamma) \) is the collection of \( F^0 \)-adapted square integrable measurable processes with values in \( \Gamma \). Set \( \mathcal{P} = \overline{\mathcal{P}_M} \) the closure of \( \mathcal{P}_M \) under the topology of weak convergence, then \( \mathcal{P} \) is weakly compact. \([\text{1}]\) proved that \( \mathcal{P} \) represents \( \tilde{\mathbb{E}} \) on \( L^2_G(\Omega_T) \).
Proposition 2.8 \( [4] \) Let \( \{P_n : n \geq 1\} \subset \mathcal{P} \) converge weakly to \( P \). Then for each \( \xi \in L^1_G(\Omega) \), we have \( E_{P_n}[\xi] \rightarrow E_P[\xi] \).

For this \( \mathcal{P} \), we define capacity
\[
c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).
\]
A set \( A \in \mathcal{B}(\Omega) \) is polar if \( c(A) = 0 \). A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables \( X \) and \( Y \) if \( X = Y \) q.s.. We set
\[
L^p(\Omega_T) := \{ X \in \mathcal{B}(\Omega_T) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \} \quad \text{for} \ p \geq 1.
\]
It is important to note that \( L^p_G(\Omega_T) \subset L^p(\Omega_T) \).

2.1 Forward and backward SDEs driven by \( G \)-Brownian motion

We first give the definition of admissible controls.

Definition 2.9 For each \( t \in [0, T] \), \( u \) is said to be an admissible control on \( [t, T] \), if it satisfies the following conditions:
\[
\begin{align*}
(i) & \quad u : [t, T] \times \Omega \rightarrow U \text{ where } U \text{ is a compact set of } \mathbb{R}^m; \\
(ii) & \quad u \in M^2_G(t, T; \mathbb{R}^m).
\end{align*}
\]

The set of admissible controls on \( [t, T] \) is denoted by \( \mathcal{U}[t, T] \).

In the rest of this paper, we use Einstein summation convention.

Let \( t \in [0, T], \ v > 0, \ \xi \in L^2_G(\Omega_T) \) and \( u \in \mathcal{U}[t, T] \). Consider the following forward and backward SDEs driven by \( G \)-Brownian motion:
\[
\begin{align*}
\frac{dX_t^{t,\xi,u}}{dt} &= b(s, X_s^{t,\xi,u}, u_s)ds + h_{ij}(s, X_s^{t,\xi,u}, u_s)d\langle B^i, B^j \rangle_s + \sigma(s, X_s^{t,\xi,u}, u_s)dB_s, \\
X_t^{t,\xi,u} &= \xi,
\end{align*}
\]
and
\[
\begin{align*}
-dY_s^{t,\xi,u} &= f(s, X_s^{t,\xi,u}, Y_s^{t,\xi,u}, Z_s^{t,\xi,u}, u_s)ds + g_{ij}(s, X_s^{t,\xi,u}, Y_s^{t,\xi,u}, Z_s^{t,\xi,u}, u_s)d\langle B^i, B^j \rangle_s - Z_s^{t,\xi,u}dB_s - dK_s^{t,\xi,u}, \\
Y_T^{t,\xi,u} &= \Phi(X_T^{t,\xi,u}), \quad s \in [t, T],
\end{align*}
\]
where
\[
\begin{align*}
b : [t, T] \times \mathbb{R}^n \times U &\rightarrow \mathbb{R}^n; \\
h_{ij} : [t, T] \times \mathbb{R}^n \times U &\rightarrow \mathbb{R}^n; \\
\sigma : [t, T] \times \mathbb{R}^n \times U &\rightarrow \mathbb{R}^{n \times d}; \\
f : [t, T] \times \mathbb{R}^n \times \mathbb{R}^d \times U &\rightarrow \mathbb{R}; \\
g_{ij} : [t, T] \times \mathbb{R}^n \times \mathbb{R}^d \times U &\rightarrow \mathbb{R}; \\
\Phi : \mathbb{R}^n &\rightarrow \mathbb{R}.
\end{align*}
\]
Denote
\[ S_G^0(0, T) = \{ h(t, B_{t_1}, \ldots, B_{t_n}) : t_1, \ldots, t_n \in [0, T], h \in C_b, Lip(\mathbb{R}^{n+1}) \}; \]
\[ S_G^2(0, T) = \text{the completion of } S_G^0(0, T) \text{ under the norm } \| \eta \|_{S_G^2} = \{ \mathbb{E}[\sup_{t \in [0, T]} |\eta(t)|^2] \}^{\frac{1}{2}}. \]

For given \( t, u \) and \( \xi \), \((X^{t, \xi, u}), (Y^{t, \xi, u}), (Z^{t, \xi, u}), (K^{t, \xi, u})\) are called solutions of the above forward and backward SDEs respectively if \((X^{t, \xi, u}) \in M^2_G(t, T; \mathbb{R}^n); (Y^{t, \xi, u}, Z^{t, \xi, u}) \in S_G^2(0, T) \times M^2_G(0, T); K^{t, \xi, u} \) is a decreasing \( G \)-martingale with \( K^1_{t, \xi, u} = 0, K^{t, \xi, u} \in L^2_G(\Omega_T) \); (2.2) and (2.3) are satisfied respectively.

Assume \( b, h_{ij}, \sigma, f, g_{ij}, \Phi \) are deterministic functions and satisfying the following conditions:

**Assumption 2.10** There exists a constant \( c > 0 \) such that
\[
|b(s, x^1, u) - b(s, x^2, v)| + |h_{ij}(s, x^1, u) - h_{ij}(s, x^2, v)| + |\sigma(s, x^1, u) - \sigma(s, x^2, v)| \\
\leq c(|x^1 - x^2| + |u - v|), \forall (s, x^1, u), (s, x^2, v) \in [t, T] \times \mathbb{R}^n \times U
\]

and \( b, h_{ij}, \sigma \) are continuous about \( t \).

**Assumption 2.11** There exists a constant \( c > 0 \) such that
\[
|f(s, x^1, y^1, z^1, u) - f(s, x^2, y^2, z^2, v)| \leq c(|x^1 - x^2| + |y^1 - y^2| + |z^1 - z^2| + |u - v|); \\
|g_{ij}(s, x^1, y^1, z^1, u) - g_{ij}(s, x^2, y^2, z^2, v)| \leq c(|x^1 - x^2| + |y^1 - y^2| + |z^1 - z^2| + |u - v|); \\
|\Phi(x^1) - \Phi(x^2)| \leq c|x^1 - x^2|,
\]

\( \forall (s, x^1, y^1, z^1, u), (s, x^2, y^2, z^2, v) \in [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \)

and \( f, g_{ij} \) are continuous about \( t \).

**Remark 2.12** Suppose Assumptions (2.10) and (2.11) hold. Then there exists a constant \( K > 0 \) such that
\[
|b(s, x, u)| + |h_{ij}(s, x, u)| + |\sigma(s, x, u)| \leq K(1 + |x|); \\
|f(s, x, 0, 0, u)| + |g_{ij}(s, x, 0, 0, u)| \leq c(1 + |x|); \\
|\Phi(x)| \leq K(1 + |x|), \forall (s, x, u) \in [t, T] \times \mathbb{R}^n \times U.
\]

We have the following theorems.

**Theorem 2.13** (2.2) Let Assumption 2.10 hold. Then there exists a unique adapted solution \( X \) for equation (2.2).

**Theorem 2.14** (2.3) Let Assumption 2.11 hold. Then there exists a unique adapted solution \((Y, Z, K)\) for equation (2.3).
2.2 Stochastic optimal control problem

The state equation of our stochastic optimal control problem is governed by the above forward SDE (2.2) and the objective functional is introduced by the solution of the BSDE (2.3) at time $t$. Let $\xi$ equals a constant $x \in \mathbb{R}^n$. When $u$ changes, $Y_t^{t,x,u}$ (the solution $Y^{t,x,u}$ at time $t$) also changes. In order to study the value function of our stochastic optimal control problem, we need to define the essential supremum of $\{Y_t^{t,x,u} \mid u \in \mathcal{U}[t,T]\}$.

**Definition 2.15** The essential supremum of $\{Y_t^{t,x,u} \mid u \in \mathcal{U}[t,T]\}$, denoted by $\text{ess sup}_u Y_t^{t,x,u}$, is a random variable $\zeta \in L^2_\mathcal{P}(\Omega)$ satisfying:

(i). $\forall u \in \mathcal{U}[t,T]$, $\zeta \geq Y_t^{t,x,u}$ q.s., and

(ii). if $\eta$ is a random variable satisfying $\eta \geq Y_t^{t,x,u}$ q.s. for any $u \in \mathcal{U}[t,T]$, then $\zeta \leq \eta$ q.s.

**Remark 2.16** It is easy to verify that $c(A) = 0$ if and only if $P(A) = 0$ for each $P \in \mathcal{P}$. Thus $\zeta \leq \eta$ q.s. is equivalent to $\zeta \leq \eta$ $P$-a.s. for each $P \in \mathcal{P}$.

**Proposition 2.17** Let $\zeta, \eta \in L^2_\mathcal{P}(\Omega)$. If $\zeta \leq \eta$ $P$-a.s. for each $P \in \mathcal{P}_M$, then $\zeta \leq \eta$ q.s.

**Proof.** It is easy to check that $(\zeta - \eta)^+ \in L^2_\mathcal{P}(\Omega)$. By Proposition 2.8 we obtain

$$\mathbb{E}[(\zeta - \eta)^+] = \sup_{P \in \mathcal{P}} E_P[(\zeta - \eta)^+] = \sup_{P \in \mathcal{P}_M} E_P[(\zeta - \eta)^+] = 0.$$ 

Thus $\zeta \leq \eta$ q.s.. $\blacksquare$

**Remark 2.18** From the above proposition, it is easy to deduce that $\zeta \leq \eta$ q.s. if and only if $\zeta \leq \eta$ $P$-a.s. for each $P \in \mathcal{P}_M$.

Our stochastic optimal control problem is: for given $x \in \mathbb{R}^n$, to find $u(\cdot) \in \mathcal{U}[t,T]$ so as to maximize the objective function $Y_t^{t,x,u}$.

The value function $V$ is defined to be

$$V(t,x) := \text{ess sup}_{u(\cdot) \in \mathcal{U}[t,T]} Y_t^{t,x,u}. \tag{2.4}$$

Next we prove that $V(t,x)$ exists and is deterministic, and then we show that it satisfies a kind of HJB equation.

For $x \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}[t,T]$ and $P \in \mathcal{P}_M$, we consider the following forward and backward equation:

$$dX_s^{t,x,u,P} = b(s, X_s^{t,x,u,P}, u_s)ds + h_{ij}(s, X_s^{t,x,u,P}, u_s)d(B^i_s, B^j_s) + \sigma(s, X_s^{t,x,u,P}, u_s)dB_s, \tag{2.5}$$

$$X_t^{t,x,u,P} = x, \quad P - a.s. \quad \text{and}$$

$$dY_s^{t,x,u,P} = -f(s, X_s^{t,x,u,P}, Y_s^{t,x,u,P}, Z_s^{t,x,u,P}, u_s)ds - g_{ij}(s, X_s^{t,x,u,P}, Y_s^{t,x,u,P}, Z_s^{t,x,u,P}, u_s)d(B^i_s, B^j_s) + Z_s^{t,x,u,P}dB_s, \tag{2.6}$$

$$Y_T^{t,x,u,P} = \Phi(X_T^{t,x,u,P}), \quad s \in [t,T], \quad P - a.s..$$
Remark 2.19 Note that under probability $P \in \mathcal{PM}$, the process $\{B_s\}_{t \leq s \leq T}$ in the equation (2.5) and (2.6) is generally not a standard Brownian Motion. But the martingale representation property still holds for $P$ (see [32] and [34]), thus there still exist unique solutions for (2.5) and (2.6).

By [23], we have

$$X_t^{t,x,u,P} = X_t^{t,x,u} \quad P - a.s.$$ 

Soner et al. [34] give the following representation for the solution $Y_t^{t,x,u}$ of (2.3):

$$Y_t^{t,x,u} = \underset{Q \in \mathcal{PM}(t,P)}{\text{ess sup}} P Y_t^{t,x,u,Q} \quad P - a.s.,$$

where $\text{ess sup}^P$ is the ess sup with respect to probability $P$ in the classical sense and

$$\mathcal{PM}(t,P) := \{Q : Q(A) = P(A), \forall A \in \mathcal{F}_t, Q \in \mathcal{PM}\}.$$

For each fix $P \in \mathcal{PM}$, the value function $V^P$ is defined to be

$$V^P(t,x) := \underset{u(\cdot) \in \mathcal{H}[t,T]}{\text{ess sup}} \underset{Q \in \mathcal{PM}(t,P)}{\text{ess sup}} P Y_t^{t,x,u,Q} \quad P - a.s. \quad (2.7)$$

Remark 2.20 If $V^P(t,x)$ is a deterministic function and independent of $P$, then by Remark 2.19 we have $V(t,x) = V^P(t,x)$.

### 3 Dynamic programming principle

For given initial data $(t,x)$, a positive real number $\delta \leq T - t$ and $\eta \in L^2_G(\Omega_{t+\delta})$, we define

$$\mathcal{G}^{t,x,u}_{t,t+\delta}[\eta] := Y_t^{t,x,u},$$

where $(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u})_{t \leq s \leq t+\delta}$ is the solution of the following forward and backward equations:

$$dX_t^{t,x,u} = b(s, X_s^{t,x,u}, u_s)ds + h_j(s, X_s^{t,x,u}, u_s)d(B^j, B^j)_s + \sigma(s, X_s^{t,x,u}, u_s)dB_s,$$

$$\begin{align*}
X_t^{t,x,u} &= x \\
\end{align*}$$

and

$$-dY_t^{t,x,u} = f(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + g_j(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)(B^j, B^j)_s - Z_s^{t,x,u}dB_s - dK_s^{t,x,u},$$

$$Y_{t+\delta}^{t,x,u} = \eta, \quad s \in [t, t + \delta]. \quad (3.1)$$

Note that $\mathcal{G}^{t,x,u}_{t,t+\delta}[\cdot]$ is a (backward) semigroup which was first introduced by Peng in [29].

Now we give some notations:

$$L_{lip}(\Omega^t_s) := \{\varphi(B_{t_1} - B_t, ..., B_{t_n} - B_t) : n \geq 1, t_1, ..., t_n \in [t, s], \varphi \in C_{t,s}Lip(\mathbb{R}^{d\times n})\};$$

$$M_{t}^{0,t}(t,T) := \{\eta_s = \sum_{j=0}^{N-1} \xi_j I_{[t_j,t_{j+1})}(s) : s \in [t,T], t = t_0 < \cdots < t_N = T, \xi_j \in L_{lip}(\Omega^t_s)\};$$

$$M_{t}^{2,t}(t,T) := \{\text{the completion of } M_{t}^{0,t}(t,T) \text{ under } \| \cdot \|_{M^{2,t}}\};$$

$$\mathcal{U}^t[t,T] := \{u \in M_{t}^{2,t}(t,T; \mathbb{R}^m) \text{ with values in } U\};$$

$$\mathcal{U}_0[t,T] := \{u = \sum_{i=1}^{m} 1_{A_i} u^i : m \in \mathbb{N}, u^i \in \mathcal{U}^t[t,T], \{A_i\}_{i=1,...,m} \text{ is a partition of } \Omega, \{A_i\} \in \mathcal{B}(\Omega)\}.$$
Our main result in this section is the following dynamic programming principle.

**Theorem 3.1** Let Assumptions 2.10 and 2.11 hold. Then for any \( t \leq T, x \in \mathbb{R}^n \), \( V(t,x) \) exists and is deterministic. Furthermore, for any \( s \in [t,T] \), we have

\[
V(t,x) = \underset{u(\cdot) \in U[t,s]}{\text{ess sup}} \ G_{t,s}^{u}[V(s, X_t^{s,x,u})]
\]

(3.2)

In order to prove Theorem 3.1, we need to study \( V^P(t,x) \) through equations (2.5) and (2.6). The following priori estimates are classical and we omit the proof (refer to (4)).

**Lemma 3.2** Under Assumptions 2.10 and 2.11 for any \( \xi_1, \xi_2 \in L^2_G(\Omega_t) \) and \( u,v \in U[t,T] \), there exists a constant \( C_2 \) such that

\[
E_P[\sup_{s \in [t,T]} |X_t^{t,\xi_1,u:P} - X_t^{t,\xi_2,u:P}|^2 | F_t] \leq C_2 | \xi_1 - \xi_2 |^2;
\]

\[
| Y_t^{t,\xi_1,u:P} - Y_t^{t,\xi_2,u:P}|^2 \leq C_2 | \xi_1 - \xi_2 |^2;
\]

\[
| Y_t^{t,\xi_1,u:P} - Y_t^{t,\xi_2,v:P}|^2 \leq C_2 E_P[ \int_t^T | u_s - v_s |^2 ds | F_t], \quad P - a.s..
\]

The following theorem shows that \( V^P(t,x) \) is deterministic and independent of \( P \in \mathcal{P}_M \).

**Theorem 3.3** Under Assumptions 2.10 and 2.11, we have

(i) For a fixed \( P \in \mathcal{P}_M \), \( V^P(t,x) \) is a deterministic function and

\[
V^P(t,x) = \underset{u(\cdot) \in U[t,T]}{\text{ess sup}} \underset{Q \in \mathcal{P}_M(t,P)}{\text{ess sup}} P Y^{t,x,u,Q}_t, \quad P - a.s.;
\]

(ii) For each \( u \in U[t,T] \), \( Y^{t,x,u}_t \) (the solution of (3.3) at time \( t \)) is a deterministic function. Furthermore,

\[
V(t,x) = \sup_{u(\cdot) \in U[t,T]} Y^{t,x,u}_t.
\]

**Proof:** (i) Without loss of generality, for (2.5) and (2.6), we only study the case \( n = d = 1 \) and \( h_{ij} = g_{ij} = 0 \).

By the definition of \( V^P(t,x) \),

\[
V^P(t,x) \geq \underset{u(\cdot) \in U[t,T]}{\text{ess sup}} \underset{Q \in \mathcal{P}_M(t,P)}{\text{ess sup}} P Y^{t,x,u,Q}_t.
\]

Analysis similar to that in Lemma 43 in [4] shows that \( U_0[t,T] \) is dense in \( U[t,T] \) under probability \( Q \). It yields that

\[
V^P(t,x) = \underset{u(\cdot) \in U[t,T]}{\text{ess sup}} \underset{Q \in \mathcal{P}_M(t,P)}{\text{ess sup}} P Y^{t,x,u,Q}_t
\]

(3.3)
Set \( u = \sum_{i=1}^{m} 1_{A_i} \), consider the following equation

\[
X_{t,x,u}^{t,x,u} = x + \int_{t}^{r} b(r, X_{r}^{t,x,u}, u_{r})dr + \int_{t}^{s} \sigma(r, X_{r}^{t,x,u}, u_{r})dB_{r},
\]

\[
Y_{t,x,u}^{t,x,u} = \Phi(X_{T}^{t,x,u}) + \int_{t}^{T} f(r, X_{r}^{t,x,u}, Y_{r}^{t,x,u}, Z_{r}^{t,x,u}, u_{r})dr - \int_{t}^{T} Z_{r}^{t,x,u}dB_{r}.
\]

Multiplying by \( I_{A_i} \), and adding the corresponding terms, we obtain

\[
\sum_{i=1}^{N} 1_{A_i} X_{t,x,u}^{t,x,u} = x + \sum_{i=1}^{N} 1_{A_i} \int_{t}^{r} b(r, X_{r}^{t,x,u}, u_{r})dr + \sum_{i=1}^{N} 1_{A_i} \int_{t}^{s} \sigma(r, X_{r}^{t,x,u}, u_{r})dB_{r},
\]

\[
\sum_{i=1}^{N} 1_{A_i} Y_{t,x,u}^{t,x,u} = \sum_{i=1}^{N} 1_{A_i} \Phi(X_{T}^{t,x,u}) - \sum_{i=1}^{N} 1_{A_i} \int_{t}^{T} Z_{r}^{t,x,u}dB_{r} + \sum_{i=1}^{N} \int_{t}^{T} f(r, X_{r}^{t,x,u}, Y_{r}^{t,x,u}, Z_{r}^{t,x,u}, u_{r})dr.
\]

Then

\[
\sum_{j=1}^{N} 1_{A_j} X_{t,x,u}^{t,x,u} = x + \int_{t}^{s} b(r, \sum_{j=1}^{N} 1_{A_j} X_{r}^{t,x,u}, \sum_{j=1}^{N} 1_{A_j} u_{r})dr + \int_{t}^{s} \sigma(r, \sum_{j=1}^{N} 1_{A_j} X_{r}^{t,x,u}, \sum_{j=1}^{N} 1_{A_j} u_{r})dB_{r},
\]

\[
\sum_{j=1}^{N} 1_{A_j} Y_{t,x,u}^{t,x,u} = \Phi(\sum_{j=1}^{N} 1_{A_j} X_{T}^{t,x,u}) - \int_{t}^{T} (\sum_{j=1}^{N} 1_{A_j} Z_{r}^{t,x,u}dB_{r} + \int_{t}^{T} f(r, \sum_{j=1}^{N} 1_{A_j} X_{r}^{t,x,u}, \sum_{j=1}^{N} 1_{A_j} Y_{r}^{t,x,u}, \sum_{j=1}^{N} 1_{A_j} Z_{r}^{t,x,u}, \sum_{j=1}^{N} 1_{A_j} u_{r})dr.
\]

By the uniqueness theorem of BSDE, we have

\[
Y_{t,x,u}^{t,x,u} = \sum_{i=1}^{N} 1_{A_i} Y_{t,x,u}^{t,x,u} \leq \sum_{i=1}^{N} 1_{A_i} \text{ess sup}_{u \in U}[t,T] \text{ess sup}_{Q \in P_M(t,P)} P Y_{t,x,u}^{t,x,u}.
\]

From this we get

\[
V^{P}(t,x) \leq \text{ess sup}_{u \in U}[t,T] \text{ess sup}_{Q \in P_M(t,P)} P Y_{t,x,u}^{t,x,u}, \quad P - a.s.,
\]

Thus

\[
V^{P}(t,x) = \text{ess sup}_{u \in U}[t,T] \text{ess sup}_{Q \in P_M(t,P)} P Y_{t,x,u}^{t,x,u}.
\]

(ii) For each \( u \in U[t,T] \), it is easy to check that \( Y_{t,x,u}^{t,x,u} \) is a deterministic function. Note that

\[
Y_{t,x,u}^{t,x,u} = \text{ess sup}_{Q \in P_M(t,P)} P Y_{t,x,u}^{t,x,u}, \quad P - a.s., \quad (3.4)
\]

Thus for each \( P \in P_M \), we obtain

\[
V^{P}(t,x) = \sup_{u \in U}[t,T] Y_{t,x,u}^{t,x,u} - a.s.,
\]

which implies that \( V^{P}(t,x) \) is a constant and independent of \( P \). By the definition of \( V(t,x) \), we deduce that \( V(t,x) \) is deterministic and

\[
V(t,x) = \sup_{u \in U}[t,T] Y_{t,x,u}^{t,x,u}.
\]

This completes the proof. \( \square \)

We have the following estimations of the continuity of value function \( V(t,x) \) with respect to \( x \).
Lemma 3.4 \( \forall t \in [0, T] \) and \( x, x' \in \mathbb{R}^n \), there exists a constant \( C_0 \) such that
(i) \( |V(t, x) - V(t, x')| \leq C_0 |x - x'| \);
(ii) \( |V(t, x)| \leq C_0 (1 + |x|) \).

Proof. By Lemma 3.2 we have
\[ |Y_t^{t,x,u,Q} - Y_t^{t,x',u,Q}| \leq C_0 |x - x'|, \quad P = \text{a.s.} \]

It is easy to verify that for any \( P \in \mathcal{P}_M \),
\[ |V(t, x) - V(t, x')| \leq \sup_{u(.) \in U[t,T]} \sup_{Q \in \mathcal{P}_M(t,P)} \esssup \ |Y_t^{t,x,u,Q} - Y_t^{t,x',u,Q}| \]
\[ \leq C_0 |x - x'|. \]

This completes the proof. \( \square \)

\( \forall s \geq t, \) define
\[ M_2^0(s, T) = \{ \eta = \sum_{i=0}^{N-1} \xi_i A_{[t_i, t_{i+1})}(t) : s = t_0 < \cdots < t_N = T, \xi_i \in L^2(\Omega_i) \}; \]
\[ M_2^0(s, T) = \{ \text{the completion of } M_2^0(s, T) \text{ under } ||\eta||_{M^2} := (\mathbb{E}[\int_0^T |\eta|^2 dt])^{1/2} \}; \]
\[ U[s, T] = \{ u : [s, T] \times \Omega \to U : u \in M_2^0(s, T; \mathbb{R}^n) \}. \]

Lemma 3.5 Suppose \( s \in [0, T] \) and \( \xi \in L^2_\mathbb{Q}(\Omega_s) \). Then we have
(i) for any \( v(.) \in U[s, T] \) and \( Q \in \mathcal{P}_M(s, P) \),
\[ V(s, \xi) \geq Y_s^{s,\xi,v,Q}, \quad P = \text{a.s.}; \quad (3.5) \]
(ii) for any \( \varepsilon > 0 \), there is an admissible control \( v'(.) \in U[s, T] \) and \( Q' \in \mathcal{P}_M(s, P) \) such that
\[ V(s, \xi) \leq Y_s^{s,\xi,v',Q'} + \varepsilon, \quad P = \text{a.s.}; \quad (3.6) \]
(iii)
\[ V(s, \xi) = \esssup_{v(.) \in U[s,T]} Y_s^{s,\xi,v}. \quad (3.7) \]

Proof. (i) Set
\[ \xi = \sum_{i=1}^N 1_{A_i} x_i \in L(\Omega_s), \]
where \( \{ A_i \}_{i=1,...,N} \) is a partition of \( \Omega \), \( A_i \in \mathcal{B}(\Omega_s) \) and \( x_i \in \mathbb{R}^n \).

For any \( v(.) \in U[s, T], Q \in \mathcal{P}_M(s, P) \), we have
\[ Y_s^{s,\xi,v,Q} = \sum_{i=1}^N 1_{A_i} Y_s^{s,x_i,v,Q} \leq \sum_{i=1}^N 1_{A_i} V(s, x_i) = V(s, \sum_{i=1}^N 1_{A_i} x_i) = V(s, \xi). \]

For the general case, note that \( V \) is continuous in \( x \) and \( Y_s^{s,\xi,v,Q} \) is continuous in \( \xi \). We can choose a sequence of simple random variables \( \{ \xi^i \} \) \( (i = 1, 2, \ldots) \) which converges to \( \xi \). Using similar techniques in Lemma 3.2 and 3.4 we have

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\[ E_P \left| Y_{s}^{s,\xi,v,Q} - Y_{s}^{s,\xi',v,Q} \right|^2 \to 0, \quad E_P \left| V(s,\xi) - V(s,\xi') \right|^2 \to 0. \]

Then (3.5) holds.

(ii) For \( \xi \in L^2_G(\Omega_s) \), we can construct a random variable
\[
\eta = \sum_{i=1}^{\infty} 1_{A_i} x^i \in L^2(\Omega_s),
\]
such that
\[
| \eta - \xi | \leq \frac{\varepsilon}{3C}
\]
where \( C := \max\{C_0, C_2\} \).

By Lemma 3.2 and 3.4, for any \( v(\cdot) \in U[\cdot, T] \),
\[
| Y_{s}^{s,\eta,v,Q} - Y_{s}^{s,\xi,v,Q} | \leq \frac{\varepsilon}{3}, \quad | V(s,\eta) - V(s,\xi) | \leq \frac{\varepsilon}{3}.
\]
For every \( x^i \), we can choose an admissible control \( v^i(\cdot) \in U[\cdot, T] \) and \( Q^i \) such that
\[
V(s, x^i) \leq Y_{s}^{s,x^i,v^i,Q^i} + \frac{\varepsilon}{3}, \quad P - a.s.
\]
Denote
\[
v(\cdot) := \sum_{i=1}^{\infty} 1_{A_i} v^i(\cdot) \in U[\cdot, T],
\]
\[
Q'(A) := \sum_{i=1}^{\infty} Q^i(A \cap A_i), \quad \forall A \in \mathcal{B}(\Omega_T).
\]
We have
\[
Y_{s}^{s,\xi,v,Q} \geq -| Y_{s}^{s,\eta,v,Q} - Y_{s}^{s,\xi,v,Q} | + Y_{s}^{s,\eta,v,Q}
\]
\[
\geq -\frac{\varepsilon}{3} + \sum_{i=1}^{\infty} 1_{A_i} Y_{s}^{s,x^i,v^i,Q^i}
\]
\[
\geq -\frac{\varepsilon}{3} + \sum_{i=1}^{\infty} 1_{A_i} (V(s,x^i) - \frac{\varepsilon}{3})
\]
\[
= -\frac{2\varepsilon}{3} + \sum_{i=1}^{\infty} 1_{A_i} V(s,x^i)
\]
\[
= -\frac{2\varepsilon}{3} + V(s,\eta)
\]
\[
\geq -\varepsilon + V(s,\xi), \quad P - a.s.
\]

(iii) By (3.5) and (3.6), it is easy to prove (3.7).

The proof is completed. \( \square \)

Define the (backward) semigroup
\[
G_{t,s}^{t,x,u,P}[\eta] = Y_{t}^{t,x,u,P}.
\]
where \( \eta \in L^{2+\varepsilon}_G(\Omega_s) \) and \((X^{t,x,u,P}_r, Y^{t,x,u,P}_r, Z^{t,x,u,P}_r)_{t \leq r \leq s}\) is the solution of the following forward-backward system:

\[
dX^{t,x,u,P}_r = b(s, X^{t,x,u,P}_r, u_s)ds + h_{ij}(s, X^{t,x,u,P}_r, u_s)d(B^i, B^j)_s + \sigma(s, X^{t,x,u,P}_r, u_s)dB_s,
\]

\[
X^{t,x,u,P}_t = x
\]

and

\[
-dY^{t,x,u,P}_r = f(X^{t,x,u,P}_r, Y^{t,x,u,P}_r, Z^{t,x,u,P}_r, u_r)dr - Z^{t,x,u,P}_r dB_r + g_{ij}(X^{t,x,u,P}_r, Y^{t,x,u,P}_r, Z^{t,x,u,P}_r, u_r)d(B^i, B^j)_r,
\]

\[
Y^{t,x,u,P}_s = \eta, \quad r \in [t, s], \quad P \text{-a.s.}
\]

It is obvious that for \( Q \in \mathcal{P}_M(t, P) \)

\[
G^{t,x,u,Q}_t \left[ \Phi(X^{t,x,u}_T) \right] = G^{t,x,u,Q}_t \left[ Y^{t,x,u}_s \right].
\]

Now we give the proof of Theorem 3.3.

**Proof.** By Theorem 3.3, for each fixed \( P \in \mathcal{P}_M \), we have

\[
V(t, x) = \sup_{u(.) \in \mathcal{U}[t, T]} \sup_{Q \in \mathcal{P}_M(t, P)} P \left[ G^{t,x,u,Q}_T \left[ \Phi(X^{t,x,u}_T) \right] \right]
\]

\[
= \sup_{u(.) \in \mathcal{U}[t, T]} \sup_{Q \in \mathcal{P}_M(t, P)} P \left[ G^{t,x,u,Q}_s \left[ Y^{t,x,u}_s \right] \right]
\]

\[
= \sup_{u(.) \in \mathcal{U}[t, s]} \sup_{Q \in \mathcal{P}_M(t, P)} P \left[ G^{t,x,u,Q}_s \left[ Y^{t,x,u}_s \right] \right], \quad P \text{-a.s.}
\]

By Lemma 3.3 and the comparison theorem of BSDE, we have

\[
V(t, x) \leq \sup_{u(.) \in \mathcal{U}[t, s]} \sup_{Q \in \mathcal{P}_M(t, P)} P \left[ G^{t,x,u,Q}_s \left[ V(s, X^{t,x,u}_s) \right] \right], \quad P \text{-a.s.}
\]

On the other hand, for each fixed \( u(.) \in \mathcal{U}[t, s], \forall \varepsilon > 0 \), by Lemma 3.3 there exist \( \bar{u}(.), \bar{\bar{u}}(.) \in \mathcal{U}[s, T] \) and \( \bar{Q} \in \mathcal{P}_M(s, Q) \) such that

\[
V(s, X^{t,x,u}_s) \leq Y^{s, X^{t,x,u}_s, \bar{u}, \bar{\bar{u}}, \bar{Q}} \leq V(t, x),
\]

where

\[
\bar{u}_s = 1_{\{t \leq r \leq s\}} u_r + 1_{\{s < r \leq T\}} \bar{u}_r.
\]

By the above inequality and the comparison theorem, we have

\[
Y^{s, X^{t,x,u}_s, \bar{u}, \bar{\bar{u}}, \bar{Q}} \geq V(s, X^{t,x,u}_s) - \varepsilon,
\]

\[
V(t, x) \geq G^{t,x,u,Q}_s \left[ Y^{s, X^{t,x,u}_s, \bar{u}, \bar{\bar{u}}, \bar{Q}} \right] \geq G^{t,x,u,Q}_s \left[ V(s, X^{t,x,u}_s) - \varepsilon \right].
\]

By Lemma 3.2 there exists a constant \( C_0 \) such that
\[ V(t, x) \geq G_{t,s}^{t,x,u,Q}[V(s, X_s^{t,x,u})] - C_0\varepsilon. \]

From this we get

\[ V(t, x) \geq \text{ess sup}_{u(\cdot) \in U[t,s]} \text{ess sup}_{Q \in \mathcal{P}_M(t,P)} \ P \ G_{t,s}^{t,x,u,Q}[V(s, X_s^{t,x,u})] - C_0\varepsilon. \]

Thus by letting \( \varepsilon \downarrow 0 \), we obtain

\[ V(t, x) = \text{ess sup}_{u(\cdot) \in U[t,s]} \text{ess sup}_{Q \in \mathcal{P}_M(t,P)} \ P \ G_{t,s}^{t,x,u,Q}[V(s, X_s^{t,x,u})], \quad P - a.s.. \]

Similar to the proof of Theorem 3.3, we can get

\[ V(t, x) = \text{ess sup}_{u(\cdot) \in U[t,s]} \text{ess sup}_{Q \in \mathcal{P}_M(t,P)} \ P \ G_{t,s}^{t,x,u,Q}[V(s, X_s^{t,x,u})], \quad P - a.s.. \]

Note that

\[ \text{ess sup}_{Q \in \mathcal{P}_M(t,P)} P G_{t,s}^{t,x,u,Q}[V(s, X_s^{t,x,u})] = G_{t,s}^{t,x,u}[V(s, X_s^{t,x,u})], \quad P - a.s.. \]

We have

\[ V(t, x) = \text{ess sup}_{u(\cdot) \in U[t,s]} G_{t,s}^{t,x,u}[V(s, X_s^{t,x,u})] \]

This completes the proof. \( \Box \)

The following lemma show the continuity of \( V \) about \( t \).

**Lemma 3.6** The value function \( V \) is \( \frac{1}{2} \) Hölder continuous in \( t \).

**Proof.** Set \( (t, x) \in \mathbb{R}^n \times [0, T] \) and \( \delta > 0 \). By dynamic programming principle, \( \forall \varepsilon > 0 \), there exist \( u(\cdot) \in U \) such that

\[ G_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] + \varepsilon \geq V(t, x) \geq G_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})]. \quad (3.8) \]

We first show that there exists \( C > 0 \) such that \( V(t+\delta, x) - V(t, x) \leq C\delta^{\frac{1}{2}} \). Similarly, we can prove

\[ V(t+\delta, x) - V(t, x) \geq -C\delta^{\frac{1}{2}}. \]

By equation (3.8), we have

\[ V(t+\delta, x) - V(t, x) \leq I_1^\delta + I_2^\delta, \quad (3.9) \]

where

\[ I_1^\delta = G_{t,t+\delta}^{t,x,u}[V(t+\delta, x)] - G_{t,t+\delta}^{t,x,u}[V(t, x)], \]

\[ I_2^\delta = V(t+\delta, x) - G_{t,t+\delta}^{t,x,u}[V(t+\delta, x)]. \]
By Lemma 3.4, note that $V$ is 1-H"older continuous in $x$. We have
\[ |I_1^\delta| \leq \left| C\hat{E} \left| V(t + \delta, x) - V(t + \delta, X_{t+\delta}^{t,x,u}) \right| \right|^2 \leq \left| C\hat{E} \left| X_{t+\delta}^{t,x,u} - x \right|^2 \right|^2. \]
Then by $\hat{E} |X_{t+\delta}^{t,x,u} - x|^2 \leq C\delta$ (C will change line by line),
\[ |I_1^\delta| \leq C\delta. \]

According to the definition of $G_{t,t+\delta}$, $I_2^\delta$ can be rewritten as
\[
I_2^\delta = V(t + \delta, x) - \hat{E}[V(t + \delta, x) + \int_t^{t+\delta} f(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds \\
+ \int_t^{t+\delta} g_{ij}(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)d(B^i, B^j)_s].
\]
It yields that
\[
|I_2^\delta| \leq \delta^\frac{1}{2} \left\{ E \left[ \int_t^{t+\delta} \left| f(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s) \right|^2 ds \right]^\frac{1}{2} \\
+ E \left[ \int_t^{t+\delta} \left| g_{ij}(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s) \right|^2 d(B^i, B^j)_s \right]^\frac{1}{2} \right\} \\
\leq C\delta^\frac{1}{2}.
\]

Thus, we have
\[ V(t + \delta, x) - V(t, x) \leq C\delta^\frac{1}{2}. \]
This completes the proof. \(\square\)

4 The viscosity solution of HJB equation

The following theorem gives the relationship between the value function $V$ and the second-order partial differential equation (4.1).

**Theorem 4.1** Let Assumptions 2.10 and 2.11 hold. $V$ is the value function defined by (2.7). Then $V$ is a viscosity solution of the following second-order partial differential equation:
\[
\partial_t V(t, x) + \sup_{u \in U} H(t, x, V, \partial_x V, \partial_{xx} V, u) = 0, \quad (4.1)
\]
where
\[
H(t, x, v, p, A, u) = G(F(t, x, v, p, A, u)) + \langle p, b(t, x, u) \rangle + f(t, x, v, \sigma(t, x, u)p, u),
\]
\[
F_{ij}(t, x, v, p, A, u) = \langle A \sigma_i(t, x, u), \sigma_j(t, x, u) \rangle + 2\langle p, h_{ij}(t, x, u) \rangle + 2g_{ij}(t, x, v, \sigma(t, x, u)p, u),
\]
$(t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_n \times U$, $\sigma_i$ is the $i$-th column of $\sigma$, $G$ is defined by equation (2.7).
For simplicity, we only consider the case $h_{ij} = g_{ij} = 0$.
Suppose $\varphi \in C^{2,3}_{b,loc}([t, T] \times \mathbb{R}^n)$. Define

$$F_1(s, x, y, z, u) = \partial_s \varphi(s, x) + \langle b(s, x, u), \partial_x \varphi(s, x) \rangle + f(s, x, y + \varphi(s, x), z + \partial_x \varphi(s, x)\sigma(s, x, u), u),$$

$$F_2^{ij}(s, x, u) = \frac{1}{2} \langle \partial^2_{xx} \varphi(s, x) \sigma_i(s, x, u), \sigma_j(s, x, u) \rangle.$$  \hfill (4.2)

Consider the following G-BSDE: \forall $t \in [t, T]$, we have

$$Y^{1,u}_s = \int_t^s \partial_s F_1(r, X^{t:x,u}_r, Y^{1,u}_r, Z^{1,u}_r, u_r)dr + \int_t^s F_2^{ij}(r, X^{t:x,u}_r, u_r)d(B^i, B^j)_r$$

$$- \int_t^s Z^{1,u}_r dB_r - (K^{1,u}_{t+\delta} - K^{1,u}_s),$$

\hfill (4.3)

and

$$Y^u_s = \varphi(t + \delta, X^{t:x,u}_t) + \int_t^s f(r, X^{t:x,u}_r, Y^u_r, Z^u_r, u_r)dr - \int_s^t Z^u_r dB_r - (K^u_{t+\delta} - K^u_s).$$ \hfill (4.4)

**Lemma 4.2** \forall $s \in [t, t + \delta]$, we have

$$Y^{1,u}_s = Y^u_s - \varphi(s, X^{t:x,u}_s).$$ \hfill (4.5)

**Proof.** Applying Itô’s formula to $\varphi(s, X^{t:x,u}_s)$, we have

$$d(Y^u_s - \varphi(s, X^{t:x,u}_s)) = dY^{1,u}_s.$$

Since $Y^{1,u}_t - \varphi(t + \delta, X^{t:x,u}_t) = Y^{1,u}_{t+\delta} = 0$, we obtain

$$Y^{1,u}_s = Y^u_s - \varphi(s, X^{t:x,u}_s), \quad \forall s \in [t, t + \delta].$$

The proof is completed. \hfill \Box

Consider the G-BSDE: \forall $s \in [t, t + \delta]$,

$$Y^{2,u}_s = \int_t^s F_1(r, x, Y^{2,u}_r, Z^{2,u}_r, u_r)dr + \int_t^s F_2^{ij}(r, x, u_r)d(B^i, B^j)_r - \int_s^t Z^{2,u}_r dB_r - (K^{2,u}_{t+\delta} - K^{2,u}_s).$$ \hfill (4.6)

We have the following estimation.

**Lemma 4.3** We have

$$\left| \sup_{u(\cdot) \in U[t,T]} Y^{1,u}_t - \sup_{u(\cdot) \in U[t,T]} Y^{2,u}_t \right| \leq C\delta^{3/2},$$ \hfill (4.7)

where $C$ is a positive constant independent of $u(\cdot)$.  

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Proof. By Proposition 3.9 in [10], we have for any fixed \( u(\cdot) \in \mathcal{U}[t, T] \) and \( p > 2 \)

\[
|Y_{t}^{1,u} - Y_{t}^{2,u}|^2 \leq \mathbb{E}\left[ \sup_{s \in [t, t+\delta]} |Y_{s}^{1,u} - Y_{s}^{2,u}|^2 \right] 
\leq C\left\{ \mathbb{E}\left[ \sup_{s \in [t, t+\delta]} \left| \hat{F}_{r,s}(\int_{s}^{t+\delta} \tilde{F}_{r,dr}) \right|^{2/p} \right] + \mathbb{E}\left[ \sup_{s \in [t, t+\delta]} \left| \hat{F}_{r,s}(\int_{s}^{t+\delta} \tilde{F}_{r,dr}) \right| \right] \right\},
\]

where \( \hat{F}_{r} = |F_{1}(r, X_{r}^{t,x,u}, Y_{r}^{2,u}, Z_{r}^{2,u}, u_{r}) - F_{1}(r, x, Y_{r}^{2,u}, Z_{r}^{2,u}, u_{r})| + \sum_{i,j=1}^{d} |F_{2}^{ij}(r, X_{r}^{t,x,u}, Y_{r}^{2,u}, Z_{r}^{2,u}, u_{r}) - F_{2}^{ij}(r, x, Y_{r}^{2,u}, Z_{r}^{2,u}, u_{r})| \). It is easy to verify that

\[
\hat{F}_{r} \leq C_{1}(|X_{r}^{t,x,u} - x| + |X_{t}^{t,x,u} - x|^{2}),
\]

where \( C_{1} \) is independent of \( u(\cdot) \). By standard estimates of G-SDE, we can obtain that for any \( p' \geq 2 \)

\[
\mathbb{E}\left[ \sup_{r \in [t, t+\delta]} |X_{r}^{t,x,u} - x|^{p'} \right] \leq C_{2}(1 + |x|^{p'})^{p'/2},
\]

where \( C_{2} \) is independent of \( u(\cdot) \). Then by Theorem 2.13 in [10] we can deduce that \( |Y_{t}^{1,u} - Y_{t}^{2,u}| \leq C\delta^{3/2} \), where \( C \) is independent of \( u(\cdot) \). Thus

\[
\text{ess sup}_{u(\cdot) \in \mathcal{U}[t, T]} Y_{t}^{1,u} - \text{ess sup}_{u(\cdot) \in \mathcal{U}[t, T]} Y_{t}^{2,u} \leq \text{ess sup}_{u(\cdot) \in \mathcal{U}[t, T]} |Y_{t}^{1,u} - Y_{t}^{2,u}| \leq C\delta^{3/2}.
\]

This completes the proof. \( \square \)

Now we compute \( \text{ess sup}_{u(\cdot) \in \mathcal{U}[t, t+\delta]} Y_{t}^{2,u} \).

Lemma 4.4 We have

\[
\text{ess sup}_{u(\cdot) \in \mathcal{U}[t, t+\delta]} Y_{t}^{2,u} = Y_{t}^{0},
\]

where \( Y_{t}^{0} \) is the solution of the following ordinary differential equation

\[
-dY_{s}^{0} = F_{0}(s, x, Y_{s}^{0}, 0)ds, \quad Y_{t+t+\delta}^{0} = 0, \quad s \in [t, t+\delta]
\]

and

\[F_{0}(s, x, y, z) := \sup_{u \in \mathcal{U}} [F_{1}(s, x, y, z, u) + 2G(F_{2}(s, x, u))].\]

Proof. By Theorem 3.7 of [11], we have

\[Y_{s}^{2,u} \leq Y_{s}^{0}, \quad s \in [t, t+\delta],\]

where \((Y_{t}^{0}, Z_{t}^{0}, K_{t}^{0})\) is the solution of the following G-BSDE:

\[Y_{s}^{0} = \int_{s}^{t+\delta} F_{0}(r, x, Y_{r}^{0}, Z_{r}^{0})dr - \int_{s}^{t+\delta} Z_{r}^{0}dB_{r} - (K_{s+\delta}^{0} - K_{s}^{0}) \quad s \in [t, t+\delta].\]

Since \( F_{1} \) and \( G(F_{2}) \) are deterministic functions, we obtain that \( Z_{s}^{0} = 0, K_{s}^{0} = 0 \) and \( Y_{s}^{0} \) is the solution of equation (4.8).

We denote the class of all deterministic controls in \( \mathcal{U}[t, t+\delta] \) by \( \mathcal{U}_{1} \). Then, for every \( u(\cdot) \in \mathcal{U}_{1}, Y_{t}^{2,u} \) is the solution of the following ordinary differential equation:
\[-dY_2^{2,u} = [F_1(s, x, Y_2^{2,u}, 0, u_s) + 2G(F_2(s, x, u_s))]ds, \quad s \in [t, t + \delta],
\]
\[Y_2^{2,u}_{t+\delta} = 0.\]

It is easy to check that
\[Y_t^0 = \text{ess sup}_{u(\cdot) \in \mathcal{U}[t, t+\delta]} Y_t^{2,u}.\]

This completes the proof. \(\Box\)

Finally we give the proof of Theorem 4.1.

**Proof:** By Lemma 3.4, 3.6, \(V\) is a continuous functions on \([0, T] \times \mathbb{R}^n\). We first prove that \(V\) is the subsolution of (4.1).

Given \(t \leq T\) and \(x \in \mathbb{R}^n\), suppose \(\varphi \in C^{2,3}_{b,Lip}([0, T] \times \mathbb{R}^n)\) such that \(\varphi(t, x) = V(t, x)\) and \(\varphi \geq V\) on \([0, T] \times \mathbb{R}^n\). By Theorem 3.1, we have
\[V(t, x) = \text{ess sup}_{u(\cdot) \in \mathcal{U}[t, t+\delta]} \{G_{t,t+\delta}[\varphi(t + \delta, X_{t+\delta}^{t,x,u})] - \varphi(t, x)\}.\]

So
\[\text{ess sup}_{u(\cdot) \in \mathcal{U}[t, t+\delta]} \{G_{t,t+\delta}[\varphi(t + \delta, X_{t+\delta}^{t,x,u})] - \varphi(t, x)\} \geq 0.\]

By (4.5), we have
\[\text{ess sup}_{u(\cdot) \in \mathcal{U}[t, t+\delta]} Y_{t,u}^{1,u} \geq 0.\]

By (4.7) and Lemma 4.4, we get
\[\text{ess sup}_{u(\cdot) \in \mathcal{U}[t, t+\delta]} Y_{t,u}^{2,u} \geq -C\delta^{3/2}\]
and
\[Y_t^0 \geq -C\delta^{3/2}.\]

Thus,
\[-C\delta^{3/2} \leq \delta^{-1}Y_t^0 = \delta^{-1}\int_t^{t+\delta} F_0(r, x, Y_r^0, 0)dr.\]

Letting \(\delta \to 0\), we get \(F_0(t, x, 0, 0) = \sup_{u \in \mathcal{U}}(F_1(t, x, y, z, u) + G(F_2(t, x, u))) \geq 0\), which implies that \(V\) is a subsolution of (4.1). Using the same method, we can prove \(V\) is the supersolution of (4.1).

This completes the proof. \(\Box\)

**Attachment**

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