Curvature function renormalisation, topological phase transitions and multicriticality

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A recently proposed curvature renormalization group scheme for topological phase transitions defines a generic ‘curvature function’ as a function of the parameters of the theory and shows that topological phase transitions are signalled by the divergence of this function at certain parameters, called critical points, in analogy with usual phase transitions. A renormalization group procedure was also introduced as a way of flowing away from the critical point towards a fixed point, where an appropriately defined correlation function goes to zero and topological quantum numbers characterising the phase are easy to compute. In this paper, using an inversion symmetry broken model in one dimension as an example, we show that there are cases where the fixed point and the critical point appear to intersect, which turn out to be multi-critical points and focus on understanding its implications.

I. INTRODUCTION

The Landau order parameter paradigm describes continuous phase transitions with spontaneous symmetry breaking. Systems undergoing these transitions often possess a local order parameter which is present only in one of the phases. Close to the phase transition point or critical point, the system exhibits self-similarity or scale invariance, and these transitions can be studied using Kadanoff’s scaling theory.

In the last couple of decades, topological phase transitions have garnered a lot of attention. These transitions fall outside Landau’s paradigm and cannot be described by a local order parameter. They are tuned by varying the coupling parameters in the model. Even though different topological models are classified based on the dimensions and the symmetries of the model, these transitions do not involve spontaneous symmetry breaking like in Landau’s theory. The different topological phases are distinguished by a topological invariant and the discrete change of this invariant signals the transition between the phases.

In this context, a renormalisation group approach has been introduced where a scaling procedure, analogous to the Kadanoff’s scaling theory, for topological systems is derived. This is based on the following observation: The topological invariant in many cases is calculated by integrating a function, known as the curvature function, over the whole Brillouin zone. This function, which diverges at the transition holds the information about the topology of the band structure. The concept of scaling here is to change the curvature function in such a way that the topology does not change. This scaling procedure is compared to that of stretching a messy string to reveal the number and types of knots that it contains. The renormalization group flow is then chosen to reduce the divergence, without changing the topology, and to finally terminate at a fixed point. At this point the curvature remains unchanged under further renormalization. Since the scaling procedure acts on a curvature function, this method has been called the curvature renormalization group (CRG) approach.

The CRG approach has been used in studying topological systems such as the Su-Schrieffer-Heeger model and periodically driven systems. It has been extended to analyze topological phase transitions involving higher order band crossing and models with $Z_2$ topological invariants. A class of systems which are different from the above mentioned cases are the ones which are not exactly solvable, such as interacting systems. The CRG is also a useful technique for studying topological transitions in weakly interacting systems as shown in Ref. It has also been successfully applied to a strongly interacting fractional Chern insulator system.

The strength of the scaling technique lies in the fact that the fixed points and critical points of the theory hold all the information about the topological phases of the model, thus sparing us from scanning the whole parameter space to construct the phase diagram. The CRG procedure also shines light on the behaviour of the curvature near the transition points and provides an insight into the differences in properties of the curvature in different phases.

The idea is that there may be situations, notably in higher dimensional or interacting theories, where the direct study of the curvature function may be more feasible than integrating the curvature function to compute the topological invariant, which requires the knowledge of the curvature function at all points in the Brillouin zone. Even if the topological invariants can be computed, the study of the curvature function close to the phase transition may give us new information and could lead us to a new way of classifying topological phase transitions.

In this paper, we carry forward the analysis to models which are not inversion symmetric, where there exists critical lines at non high symmetry points, besides the critical lines at HSP. Although these critical lines at non high symmetric points cannot be discovered by a straightforward application of the CRG method, it is possible to analyze the system using the scaling of the curvature function and the divergence of the Wannier...
state correlation length. Using this approach, we find the following main results in this paper. We find that the fixed point line intersects a critical line, at a crossing point, which turns out to be a multi-critical point. In fact, a reasonable fraction of the fixed point line overlaps with the non high symmetry critical line in the topological region. So the crossing point also turns out to be a multi-critical point where three phases (two topological and one non-topological) meet. We also find that at all points on the critical line, except the multi-critical point, two topological phases meet and the curvature function diverges as we approach any point on the line, independent of the path of approach. However, precisely at the multi-critical point, we find that there is no divergence as both the numerator and denominator of curvature function vanish here. Finally, we find the Wannier state correlation function, related to the correlation length associated with the divergence of the curvature function\cite{18} has interesting behaviour close to the two HSP critical lines identified by the CRG procedure. Close to one of the critical lines, the envelope of the Wannier state correlation function decays exponentially, whereas close to the other critical line, the amplitude of the oscillations of the correlation function decays exponentially. We also note that the envelope of correlation function vanishes on the fixed point curve when the CRG equations around \( k_0 = 0 \) are studied and the amplitude of the oscillations of the correlation function vanish on the fixed point curve of the CRG equations around \( k = \pi \).

In the rest of the paper, we derive and elaborate upon these results. The paper is organized in the following way: In Sec.\textbf{II} we describe the curvature renormalization procedure briefly. The model is introduced in Sec.\textbf{III} where we carry out some preliminary analysis on its topological properties. In Sec.\textbf{IV} we study the behavior of topological transitions in the model using curvature RG and explain the results. The conclusion and summary are described in Sec.\textbf{V}.

\section{Curvature RG}

In this section, we briefly review the curvature renormalization group method (CRG) introduced in Refs.\textbf{13}\textendash\textbf{16} and further explained in Refs.\textbf{18}\textendash\textbf{21}. The different topological phases in a system are distinguished by a topological invariant, which is calculated by integrating a function over the Brillouin zone. This integrand function, called the curvature function in the rest of the paper, can either be the Berry curvature, the Berry connection or the Pfaffian of an appropriate ‘sewing matrix’ as dictated by the dimensions and the underlying symmetry class of the system\cite{13}. The topological invariant \( C \) is then calculated by the following equation:

\begin{equation}
C = \int_{BZ} \frac{d^Dk}{(2\pi)^D} F(k, M).
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The deviation-reduction mechanism: For a given \( k_0 + \delta k \) for \( M \), one has to find \( M' \) for which \( F(k_0 + \delta k, M) = F(k_0, M'). \)
\end{figure}

Here \( F(k, M) \) is the curvature function and \( M = (M_1, M_2, \ldots, M_i, \ldots) \) is the set of all the coupling parameters in the theory.

Consider a point, \( M = M_c \) in the parameter space where the system undergoes a topological transition i.e. the topological number \( C \) changes. At \( M_c \), the bulk band gap closes usually at a high symmetry point (HSP), \( k_0 \), in the Brillouin zone resulting in a diverging curvature function. For a small perturbation \( \delta k \) near \( k_0 \), the CRG procedure can be summarized in the following equation:

\begin{equation}
F(k_0 + \delta k, M) = F(k_0, M').
\end{equation}

Given the curvature function (LHS) at \( k_0 + \delta k \) for parameters \( M \), one has to find a new \( M' \) which makes \( F(k_0, M') \) equal to \( F(k_0 + \delta k, M) \). As discussed in detail in Ref.\textbf{13} this procedure gradually reduces the divergence of the curvature function at \( k_0 \) as demonstrated in Fig.\textbf{1}. This is known as the deviation-reduction mechanism. Under the iterative application of Eq.\textbf{2} the parameters \( M \) flow away from the critical point \( M_c \) towards a fixed point \( M_0 \). When the flow stops at \( M_0 \), the curvature function has the form: \( F(k_0 + \delta k, M_0) = F(k_0, M_0) \).

The equation to track the flow of curvature function to its fixed point in the parameter space can be derived by expanding the RHS and LHS of Eq.\textbf{2} to leading order in \( \delta k \) and \( \delta M = M_i - M_{ic} \). This gives,

\begin{equation}
\frac{dM_i}{dl} = \frac{1}{2} \frac{\partial^2}{\partial k_j \partial M_i} F(k_i, M) |_{k = k_0},
\end{equation}

where \( dl = \delta k_j^2 \), \( \partial k_j \) is the partial derivative with respect to \( k_j \), and \( \partial M_i \) is the partial derivative with respect to \( M_i \). Note that we have used the fact that the curvature function is an even function near a gap-closing high symmetry point, i.e. close to \( k_0 \), \( F(k_0 + \delta k, M) = F(k_0 - \delta k, M) \). The RHS of the above equation diverges at the critical point \( M_{ic} \) and vanishes at the fixed point \( M_0 \).

The curvature function is expected to have a Lorentzian form (as shown in Fig.\textbf{1}) near the critical
where $\xi_{k_0}$, known as the correlation length, is the length scale associated with the divergence of the curvature function at the transition\cite{13}. This can also be identified with the decay length scale of the Wannier state correlation function (defined as the overlap between two Wannier states which are at a distance $R$ from each other) $\lambda_R$\cite{13}. In one dimensional systems, $\lambda_R$ scales as $\lambda_R \sim e^{-R/\xi_{k_0}}$.

From the divergence and the associated behavior described above, a scaling form can be attributed to the curvature function near the critical point. Therefore one can write, $|F(k_0, M)| \sim |M - M_c|^{-\gamma}$. Similarly using Eq.\[4\] the scaling form of the correlation length is written as $\xi_i \sim |M - M_c|^{-\nu_i}$. The exponents $\gamma$ and $\nu_i$ are the critical exponents associated with this topological transition. In 1D systems they are simply related by the expression: $\gamma = \nu_i$.

So far, we have discussed a scenario known as the peak divergence scenario. The curvature function develops a peak at one of the high symmetry points as we approach the transition. There is another case known as the shell divergence scenario. Here, as we move towards the critical point in the parameter space, the curvature function peaks in the forms of a 1D dimensional shell around the HSP. The RG formalism in this case is discussed in Refs.\[19\] and \[23\].

As explained in this section, the CRG procedure is a very efficient method when the number of couplings is large. The invariant needs to be calculated only for a few points, which are from topologically different regions in this space.

III. THE KITAEV CHAIN WITH EXTENDED COUPLINGS

Here, we consider the 1D Kitaev spinless p-wave superconducting chain\cite{23} with both nearest and next-nearest coupling terms\cite{24,25}. The Hamiltonian is given by,

$$H = -t_1 \sum_{\langle ij \rangle} (c_i^\dagger c_j + h.c) - t_2 \sum_{\langle\langle ij \rangle\rangle} (c_i^\dagger c_j + h.c) - \lambda_1 \sum_{\langle ij \rangle} (c_i^\dagger c_j + h.c) - \lambda_2 \sum_{\langle\langle ij \rangle\rangle} (c_i^\dagger c_j + h.c) + g \sum_i (2c_i^\dagger c_i - 1).$$ (5)

where $g$ is the chemical potential, $t_{1,2}$ are the nearest(NN) and next-nearest-neighbour(2NN) hopping terms and $\lambda_{1,2}$ represent NN and NNN superconductor pairing terms respectively. We take the pairing amplitudes $\lambda_1$ and $\lambda_2$ to be real. This model was studied in detail in Ref.\cite{25} for $\lambda_1 = \lambda_2$ and $t_1 = t_2$ where the different topological phases of this system were analyzed. Here we remove this constraint to extend our parameter space. After Fourier transformation, the Hamiltonian can be written in the Boguliobov-de Gennes form in the basis $(c_k^\dagger, c_{-k})$ as

$$H(k) = d_2(k)\sigma_2 + d_3(k)\sigma_3,$$ (6)

where $d_2(k) = (-2\lambda_1 \sin k - 2\lambda_2 \sin 2k)$, $d_3(k) = (2g - 2t_1 \cos k - 2t_2 \cos 2k)$ and $\sigma_2,\sigma_3$ are the Pauli matrices. The energy eigenvalues are given by

$$E_k = \pm \sqrt{(\lambda_1 \sin k + \lambda_2 \sin 2k)^2 + (g - t_1 \cos k - t_2 \cos 2k)^2}.$$ (7)

$E_k$ has been scaled by a factor of 4. We can see from the above equation, that the gap closes when both the square terms vanish together for some $k$ i.e.,

$$\lambda_1 \sin k + \lambda_2 \sin 2k = 0, \quad g - t_1 \cos k - t_2 \cos 2k = 0.$$ (8)
The first equation here is trivially satisfied at the two HSPs $k_0 = 0$ and $k_0 = \pi$, for all $\lambda_1$ and $\lambda_2$. The second one reduces to, $g = t_2/|t_1|$ for $k_0 = 0$ (top sign) and $k_0 = \pi$ (bottom sign). Combining the two equations in [3] gives a set of gap closing points which are not at HSPs. These gap closing points are at momentum values, 

$$k = \frac{1}{2} \arccos \left[ \frac{t_1}{t_2} \left( \frac{\lambda}{2} + \frac{g}{\lambda} \right) \right],$$

(9)

in the Brillouin zone, with $\lambda = \lambda_1/\lambda_2$. These non-HSP gap closing points in the parameter space are given by the following equation:

$$g = t_2 \frac{\lambda^2 - 2}{2} - t_1 \frac{\lambda}{2}$$

(10)

with the constraint $-2 < \lambda < 2$. In the parameter space these lines forms the boundaries of different phases as shown in Fig.2.

The spinless Boguliobov-de Gennes Hamiltonian $H(k)$ is time reversal (TR) invariant $TH(k)T^{-1} = H(-k)$, with $T$ being complex conjugation, and, in addition, due to the particle hole symmetry of the BdG Hamiltonian, it has the chiral symmetry $SH(k)S^{-1} = -H(k)$, where the chirality operator is given by $S = \sigma_1$. Consequently, a unitary transformation by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

brings the BdG Hamiltonian to the block-off diagonal form: $UH(k)U^{-1} = (2g - 2t_1 \cos k - 2t_2 \cos 2k)\sigma_1 + (-2\lambda_1 \sin k - 2\lambda_2 \sin 2k)\sigma_2$. This model exhibits non-trivial topological phases distinguished by the winding number $W$,

$$W = \frac{1}{V_B Z} \int_{BZ} d\theta_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{d\theta_k}{dk} \right) dk,$$

(11)

where $\theta_k$ is given by $\tan \theta_k = d_1(k)/d_2(k)$. The winding numbers of the different phases are also shown in Fig.2.

From Eq.(11) we read the curvature function to be,

$$F(k, M) = \frac{d^2 \theta_k}{dk^2} = \frac{d_2 \partial_k d_3 - d_3 \partial_k d_2}{d_3^2 + d_4^2}$$

(12)

where, $d_2 = -2\lambda_1 \sin k - 2\lambda_2 \sin 2k$, $d_3 = 2g - 2\cos k - 2t \cos 2k$, and $M = (g,t,\lambda_1,\lambda_2)$, with $t_1$ being set to one and $t_2 = t$. We plot the curvature function for $t = 0.9$ and $\lambda_2 = 0.85$ in Fig.3. For these parameter values, the gap closes for $g_c = 1.9$ at $k_0 = 0$. From the figure, it can be observed that the curvature function peaks as we approach $g_c$ and flips sign as we cross it.

In the next section, we use the curvature function in Eq.[12] to construct the flow equations of the model. The nature of these flow equations determine the different topological transitions in the model. We aim to achieve more than just to understand the topological phases in the Kitaev chain with NNN couplings. Using the relatively new CRG technique, we attempt to understand some general properties of the model by analyzing the flow equations and curvature function at different areas in the parameter space.

**IV. FLOW EQUATIONS, FIXED POINTS AND CRITICAL POINTS**

Using the RG procedure discussed in Sec.II we obtain the following flow equations for the four coupling parameters $\lambda$, $\lambda_2$, $g$, and $t$, around the high symmetry points $k_0 = 0$ and $k_0 = \pi$:

$$\frac{d\lambda}{dl} = -\frac{1}{2(g-t+1)^2} \alpha(g,t,\lambda,\lambda_2)$$

$$\frac{d\lambda_2}{dl} = \frac{t_1}{2(g-t+1)(\lambda^2 \pm 2)} \alpha(g,t,\lambda,\lambda_2)$$

$$\frac{dg}{dl} = \frac{1}{2(g-t+1)(\lambda^2 \pm 2)} \alpha(g,t,\lambda,\lambda_2)$$

$$\frac{dt}{dl} = \frac{1}{2(g-t+1)(\lambda^2 \pm 2)} \alpha(g,t,\lambda,\lambda_2)$$

(13)

where

$$\alpha(g,t,\lambda,\lambda_2) = (\lambda \pm 8)(g-t+1)^2 + 2\lambda_2^2(\lambda^2 \pm 2)^3 + 3(\lambda \pm 2)(g-t+1)(4t+1).$$

(14)

Here the upper sign is for $k_0 = 0$ and lower sign for $k_0 = \pi$. From the above set of equations, it is obvious that $dg/dl = -dt/dl$.

Using the above flow equations, we generate flow diagrams to analyze the fixed points and the critical points of this model. The flow diagrams, depicted in Fig.4a for $k_0 = 0$ and Fig.4b for $k_0 = \pi$, are in the $g$-$\lambda$ plane for fixed values of $t$ and $\lambda_2$. The blue dotted lines are the flow lines for different choices of initial conditions and the arrows denote the direction of flow. Throughout this paper we use $\lambda_2 = 0.85$ and $t = 0.9$. The qualitative features will remain same for other values of these parameters.

As is obvious from Eq.[13] the RHS of all equations diverges at $g = t + 1$ for $k_0 = 0$ and at $g = t - 1$ for $k_0 = \pi$. These are the vertical lines at $g = 1.9$ (red) in Fig.4a and at $g = -0.1$ (brown) in Fig.4b respectively. These critical lines for $k_0 = 0$ and $k_0 = \pi$ forms the boundary between the trivial and topological regions.

Note however that it is not easy to generalise the CRG procedure to non-high-symmetry $k$ points, and also that the RG flows around the high-symmetry points, do not
yield the non-HSP critical line denoted by the black dot-dash curve in Fig.4b.

Under the RG flow, the parameters $g$ and $\lambda$ flow away from the critical lines and towards the stable fixed points. By studying the RG flow diagrams, we note that the flows are towards the points $g \to \pm \infty$, $\lambda = \pm 2$ or they fall on the curved magenta line or curved cyan line in Figs.4a and 4b. In fact, the fixed point curve is obtained by setting the right side of the flow equation to zero, i.e. \( \alpha(g, t, \lambda, \lambda_2) = 0 \). The profile of the curvature function \( F(k, M_0) \) as a function of $k$ is flat (close to $k = 0$ or $k = \pi$) on the fixed point curve since it remains invariant under further renormalisation. Note, however, that parts of the fixed point curve are also unstable fixed points with the parameters flowing away from the curve. Surprisingly, these unstable fixed points in the $k = 0$ and the $k = \pi$ flow diagrams have considerable overlap with the non-HSP critical line, where, in fact, the curvature function diverges at a non-high symmetry $k$ point. This surprising overlap is shown in Fig.4c. We will say more about this later in the discussions section.

From Fig.4a (relevant for $k_0 = 0$), one can observe that the $\lambda = -2$ line is also a special one. This line, along with the critical line divides the diagram into four quadrants. To understand this, let us examine the curvature function at $k_0 = 0$. It has the form \( F(k_0 = 0, M) \sim (\lambda + 2)/(g - 1.9) \) for the parameters used in this paper. This function goes to zero throughout the $\lambda = -2$ line. When $\lambda = -2$ and $g = 1.9$, the curvature function is indeterminate. We will analyze this point (which we call the multi-critical point) further later in this section. The direction of the flow lines are in the opposite direction above and below the $\lambda = -2$ line. Consider a region in the flow diagram where the flow lines are almost horizontal. From Eq.12 one can write \( \frac{d\lambda}{dg} = -\frac{\lambda + 2}{g - 1.9} \sim (\lambda + 2) \).

In the second quadrant, when $\lambda \lesssim -2$, we get \( d\lambda \propto dg \) which implies that $\lambda$ decreases as $g$ flows to smaller values (more negative). Similarly, if we were in the third quadrant, i.e. where $\lambda \lesssim -2$, the above equation takes the form \( d\lambda \propto -dg \) which illustrates the decrease in $\lambda$ with increasing $g$. A similar analysis can be carried out in the other quadrant too which demonstrate the change in the flow direction on crossing $\lambda = -2$ line.

Using Fig.4a, we analyze the fixed point curve and critical line for $k_0 = 0$. Here, one can observe that the fixed point curve crosses itself and the critical line near $\lambda = -2.0$. The nature of the curvature function at the crossing point (the multi-critical point) is indeterminate as explained earlier. Similar behavior exists for the $k = \pi$ fixed point in Fig.4b too.

For further analysis, we plot the critical line and the fixed points lines for $k_0 = 0$ along with non-HSP critical lines in Fig.5a. The fixed point curve is divided in to three portions and plotted in three different colors. The red dotted line is the critical line for $k_0 = 0$ and the black dotted line denotes the non-HSP critical line. The orange line is part of the fixed point curve which falls on the non-HSP critical line in places. In Fig.5b we plot the curvature function for three points in the green portion of the fixed point curve, which falls in the topological phase. As expected, the curve forms a plateau around $k_0 = 0$ since the RG equations in Eq.13 vanish - $\alpha(g, t, \lambda, \lambda_2) = 0$. As we move closer to the critical point, there appears to be a peak in the curvature at $k_0 = 0$, which is contrary to the definition of fixed point curve. But a closer analysis of the peak given in the inset of Fig.5b reveals the plateau even for points very close to critical point.
The behaviour of the curvature function in the non-topological phase which has the blue branch of the fixed point curve in Fig. 5a is shown in Fig. 5c. Here we plot the curvature function along the line L which passes through the crossing point \((g, \lambda) = (1.9, -2)\). The curvature function plotted for different points from \(L\) all have the same peak value at \(k_0 = 0\) instead of showing a divergence (as is seen at other points on the critical line). This can be explained in the following way: the curvature function at \(k_0 = 0\) has the form \(F(k_0 = 0, M) = \frac{0.85(\lambda + 2)}{g - 1.9}\). This expression, rewritten slightly would give the equation of a straight line passing through \((1.9, -2)\) with a slope, \(m = F(0, M) / 0.85\). Since the slope is constant throughout the line, any point from a line \(L\) will give the same peak value of curvature function at \(k_0 = 0\). This tells that the limiting value of \(F(k, M)\) at the crossing point (singular point) is \(0.85 \times m\), indicating that limit depends on the choice of path. Instead of \(L\), if we had chosen another line, the (slope-dependent) limit would be different, but all points on that line would also give the same peak value of the curvature function at \(k_0 = 0\).

**Correlation length and critical exponents**

As discussed in Sec 11, the curvature function exhibits scaling behavior near the topological transition given by \(F(k_0, M) \sim |M - M_0|^{-\gamma}\). From the Lorentzian form in Eq. 4, the width of the curvature function goes to zero in this limit, resulting in a scaling form \(\xi_{k_0} \sim |M - M_0|^{-\nu}\). The exponents \(\nu\) and \(\gamma\) are the critical exponents characterizing this transition.

The curvature function at the high symmetry points \(k_0 = 0, \pi\) is given by \(F(k_0, M) = \pm \lambda_2(\lambda + 2)/(g - t + 1)\). Here the upper sign is for \(k_0 = 0\) and the lower sign is for \(k_0 = \pi\). Near the critical points, i.e. \(g \to t \pm 1\), \(F(k_0, M)\) diverges as,

\[
F(k_0, M) \sim \frac{1}{|g - t + 1|} \quad (15)
\]

making \(\gamma = 1\). To calculate the exponent \(\nu\), we expand the full curvature function around \(k_0 = 0, \pi\) and bring it to the form in Eq. 4. The form of \(\xi_{k_0}\) thus extracted is given by,

\[
\xi_{k_0}(M) = \frac{\alpha(g, t, \lambda, \lambda_2)}{2(g - t + 1)^2(\lambda + 2)} \sim \frac{1}{g - t + 1} \quad (16)
\]

giving \(\nu = 1\).

In the rest of this section, we will discuss the physical significance of correlation length as a length scale to determine the correlation between Wannier states. In the case of one dimensional systems, the curvature function is given by the Berry connection \(F(k, M) = \sum_n \langle u_{nk}|i\partial_k|u_{nk}\rangle\), where \(n\) in all occupied bands, the Fourier transform of which gives the charge polarization correlation function \(\langle \xi_{k_0} (M) \rangle\) between Wannier states at a distance \(R\) apart:

\[
\lambda_R = \int_{BZ} \frac{dk}{2\pi} e^{ikR} F(k, M) = \int_{BZ} \frac{dk}{2\pi} e^{ikR} \sum_n \langle u_{nk}|i\partial_k|u_{nk}\rangle = \sum_n \langle Rn|00n\rangle \quad (17)
\]

We have two bands in our model and only the lower band \(n = 1\) is occupied. Therefore we have \(\lambda_R = \langle Rn|00n\rangle\), which is a measure of overlap between Wannier states at 0 and \(R\). The zeroth component \(\lambda_0\) is the charge polarization, which is the topological invariant. Since Wannier state \(\langle R|n\rangle = W_n(r - R)\) is a localized function with center at \(R\), the quantity \(\langle Rn|00n\rangle\) is expected to decay with \(R\) to zero.

The behavior of \(\lambda_R\) in both topological and trivial phases are depicted in Fig. 6 for \(\lambda = 10\). In Fig. 6a we consider the effect of the \(k_0 = 0\) (\(g_t = 1.9\)) critical line on the correlation function. We consider three points in the parameter space, \(g = 2.0, 5.0\) in the trivial phase and \(g = 1.8\) in the topological phase. For \(g = 5\), which is far from the critical line, we note that the envelope of the correlation function decays as \(\lambda_R \sim e^{-R/\xi_{k_0}}\), as expected.
from the discussion in Sec[11] The non-decaying behavior of the envelope of $\lambda_R$ for both $g = 1.8$ and $g = 2.0$ can be explained using the argument that correlation length, $\xi_{k_0}$ needs to diverge near critical points. This means that the decay will be extremely slow close to the critical line. So the length scale $\xi_{k_0=0}$ governs the exponential decay. However, the graphs also exhibit oscillatory behaviour, which comes from the remaining part of $F(k, M)$ far from $k_0 = 0$, which still plays a role, since we integrate over the whole Brillouin zone in computing $\lambda_R$ in Eq[17]. It turns out, as we shall show below, that the amplitude of the oscillations are governed by the only other length scale we have in this model, which is $\xi_{k=\pi}$ coming from $k$ close to $\pm \pi$. We shall show this shortly.

Similarly the correlation functions for parameter points separated by the $k_0 = \pi$ ($g_c = -0.1$) critical line are plotted in Fig[6a]. We consider $g = 0$ in topological phase (close to the critical line) and $g = -5$ in the trivial phase (faraway from the critical line). Here, we note that the amplitude of the oscillations damp out very quickly and displays the exponential decay as predicted in Sec[11] for $g = -5$. But being in the vicinity of the $k_0 = \pi$ critical line, the amplitude of oscillations for $g = 0$ exhibits non-decaying behavior (divergent correlation length dependence). So it is the amplitude of the oscillations which is governed by the length scale $\xi_{\pi}$. Here, again, besides the behaviour of the amplitude of the oscillations, there is an overall decay of the envelope, which decays to zero at the same rate for both the curves. As explained in the previous paragraph, this is because the remaining part of the $F(k, M)$ which is far from $k = \pi$ also plays a role, since we integrate over the whole Brillouin zone, and so the length scale $\xi_0$ also plays a role.

The critical behavior close to the $k_0 = 0$ and the $k_0 = \pi$ critical lines demonstrated in Fig[6b] can be understood in the following way: Near the $k_0$ critical line, the total curvature function can be approximated by the Lorentzian function centered at $k_0 = 0$. Therefore the correlation function can be written in this limit given by,

$$\lambda_R \sim \int_{BZ} dk e^{ikR} F(0, M) \frac{1}{1 + \xi_0^2 k^2} \sim F(0, M) e^{-|R|/\xi_0}. \quad (18)$$

Near the critical point $g_c = 1.9$, $\xi_0 \to \infty$ this function would not decay to zero. Similarly, for a point near the $k_0 = \pm \pi$ critical line, the curvature function can approximated as a sum of the two Lorentzian peaked around $\pm \pi$. In this case, the correlation function can be written as,

$$\lambda_R \sim \int_{BZ} dk e^{ikR} \frac{F(\pi, M)}{1 + \xi_{\pi}^2 (k - \pi)^2} + \frac{F(-\pi, M)}{1 + \xi_{-\pi}^2 (k + \pi)^2} \sim F(\pi, M) \cos(\pi R) e^{-|R|/\xi_{\pi}}. \quad (19)$$

Note that $\xi_{\pi} = \xi_{-\pi}$ and $F(\pi, M) = F(-\pi, M)$. The $\cos(\pi R)$ in the last equation resulted from the sum of the two Lorentzian caused the oscillatory behavior of $\lambda_R$ in Fig[6b].

In general, we expect $F(k, M)$ to be a complicated function and have universal behaviour only close to the critical points. Near a critical point $k_0$, the curvature function is highly peaked, and we expect the correlation function $\lambda_R = \int dk e^{ikR} F(k, M)$ to have the largest contribution from $k_0$, where the curvature function is well approximated by a Lorentzian. However, in simple models with just two HSP, $k_0 = 0$ and $k_0 = \pm \pi$, the behaviour of the curves in Fig[6b] appears to be well-approximated by assuming that $F(k, M)$ is a sum of the Lorentzians at $k_0 = 0$ and $k_0 = \pm \pi$. So using Eqs[18] and [19] we are able to obtain

$$\lambda_R = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikR} F(k, M)$$

$$\approx w_0 \cos(\pi R) \exp \left[ -\frac{|R|}{\xi_{\pi}(M)} \right]$$

$$+ w_{\pi} \exp \left[ -\frac{|R|}{\xi_{0}(M)} \right]. \quad (20)$$
where the prefactors $w_{k_0} = F(k_0, M)/\xi_{k_0}(M)$ for $k_0 = 0$ and $k_0 = \pi$ are slowly varying functions of $M$ because the topological invariant $C = \lambda_0 \approx w_0 + w_\pi$. We have extended the limit of integration because the Lorentzians in Eqs.18 and 19 decay very fast beyond the range $(-\pi, \pi)$, so there is a negligible contribution coming from outside the BZ. The right hand side depends only on the two length scale $\xi_0(M)$ and $\xi_\pi(M)$, where $\xi_{k_0}(M)$ is given by Eq.10. Note that the expression of $\lambda_R$ in Eq.20 is universal in our model. Using the expression in Eq.20 both the figures in Fig.6 can be explained easily. The envelope of the correlation function is given by the second term and the BZ. The right hand side depends only on the two $\xi_0$ and $\xi_\pi$, where $\xi_0$ diverges close to the critical line $g_c = 1.9$ and hence the envelope decays very quickly for the points close to the critical line in Fig.6b. The oscillation in $\lambda_R$ is due to the presence of the $\cos(\pi R)$ in the first term. The amplitude of the oscillation is controlled by the correlation length $\xi_c$. On the other hand, $\xi_c$ diverges close to the critical line $g_c = -0.1$ and hence the amplitude of oscillation in $\lambda_R$, close to $g_c = -0.1$, decays very slowly in Fig.6b and the decay of the envelope is governed by $\xi_{k_0=0}$. So in both the figures, the exponential decay of the envelope is governed by $\xi_{k_0=0}$ and the exponential decay of the amplitude of the oscillations by the scale $\xi_{k_0=\pi}$.

At the fixed points, since $\alpha = 0$, as seen from Eq.16 the length scale $\xi_{k_0}$ vanishes. We find that the amplitude of the oscillations in Fig.6b disappears, when parameters are taken from the fixed point curve (cyan curve in Fig.4) of $k_0 = \pi$ and then $\lambda_R$ decays monotonically. Similarly, when we take parameters from the fixed points of $k_0 = 0$ curve (the magenta curve in Fig.4), we find that the envelope of $\lambda_R$ vanishes and $\lambda_R$ oscillates around the $\lambda_R = 0$ axis. Once again, this is evidence that the envelope and the amplitude of oscillations are governed by the two independent critical lines of the model.

V. DISCUSSION AND CONCLUSIONS

In this paper, we analyzed the topological transitions of the 1D Kitaev superconducting model with next nearest neighbour couplings using the curvature RG technique. Our model falls in the "peak divergence" scenario where the curvature function peaks and subsequently diverges at the gap closing point in BZ. The RG procedure has to be done individually for each gap closing point, and therefore this formalism is best suited to models with isolated gap closing points - e.g. for models with inversion symmetry. In order to capture the limits of the formalism, we considered a model where the gap closes at an array of non-HSPs in addition to $k_0 = 0$ and $k_0 = \pi$ in the BZ. These non-HSP gap closing points are depicted as black dotted lines in Figs.2 and 15.

We calculated the flow equations for both $k_0 = 0$ and $k_0 = \pi$. Upon calculating the flow equations for both HSPs, we found that the unstable parts of both fixed point lines overlapped with the non-HSP critical line. The overlap was found to be always with the repulsive part of the fixed point line (where the parameters flowed away from the fixed point line) as demonstrated in Fig.4c where the overlap has been shown with the fixed point lines of both the $k_0 = 0$ and $k_0 = \pi$ flows. Although the region of overlap changes with the parameters and the overlap is not exact, we believe that it is because the RG equations are perturbative in the parameters. However, we are able to obtain a large fraction of the non-HSP critical points of the model by studying just the first order perturbative RG equations around the two HSP. We believe that this is an indication that a perturbative RG study of the unstable or repulsive fixed points around the HSP of the model is sufficient to obtain all information about the non high symmetry critical points as well, because they can be obtained from the flow diagrams via flows to unstable fixed points. This can prove to be a very useful technique in models where it is not possible to obtain all the critical lines exactly.

We have also calculated the critical exponents for both the transitions and found $\nu = \gamma$ as is expected in 1D systems. The critical behaviour is apparent in the non-decaying nature of the envelope and amplitude of the correlation functions near the critical lines as demonstrated in Fig.6.

Going towards the future, we expect further generalisations to models in different dimensions and in different symmetry classes. A recent review tries to give a unified picture of topological phase transitions in a variety of static and periodically driven systems, and in both weakly and strongly interacting systems with the aim of classifying these transitions using standard concepts of critical exponents and universality classes. Since multicriticality is also a standard concept in phase transitions, it would also be of interest to extend our study of multicriticality in topological phase transitions to other models in different dimensions and symmetry classes.

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