A splitting lemma

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Abstract

In this paper, we study the relations between the numerical structure of the optimal solutions of a convex programming problem defined on the edge set of a simple graph and the stability number (i.e. the maximum size of a subset of pairwise non-adjacent vertices) of the graph. Our analysis shows that the stability number of every graph \( G \) can be decomposed in the sum of the stability number of a subgraph containing a perfect 2-matching (i.e. a system of vertex-disjoint odd-cycles and edges covering the vertex-set) plus a term computable in polynomial time. As a consequence, it is possible to bound from above and below the stability number in terms of the matching number of a subgraph having a perfect 2-matching and other quantities computable in polynomial time. Our results are closely related to those by Lorentzen [6], Balinsky and Spielberg [1], and Pulleyblank [8] on the linear relaxation of the vertex-cover problem. Moreover, The convex programming problem involved has important applications in information theory and extremal set theory where, as a graph capacity formula, has been used to answer some longstanding open questions (see [3] and [4]).

keywords. matching, 2-matching, stability number, packing, covering, entropy, graph capacity.

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1 Terminology and notation

Given any two positive reals $0 < p, q < 1$ we define the function

$$h(p, q) \triangleq (p + q) h\left(\frac{p}{p + q}\right),$$

where

$$h(x) = -x \log x - (1 - x) \log(1 - x), \quad (0 < x < 1)$$

is the binary entropy and (here and in the sequel) log's are to the base 2.

A stable set in a simple graph $G$ is a set of vertices that does not contain any edge. The size of a maximum stable set in $G$ is the stability number of $G$ and it is denoted by $\alpha(G)$. A set of vertices is a vertex cover of $G$ if each edge has at least one endpoint in the set. The minimum size of a vertex cover is the cover number of $G$ and it is denoted by $\tau(G)$. These two invariants are closely related by the Gallai identity:

$$\alpha(G) + \tau(G) = n \quad (n = |V(G)|).$$

The matching number of $G$, denoted by $\nu(G)$, is the maximum number of disjoint edges contained in the graph.

If $X$ is a set of vertices of $G$, we denote by $\bar{X} = V(G) \setminus X$ the complementary of $X$. Moreover $\Gamma X$ is the set of vertices of $\bar{X}$ adjacent to some vertex in $X$. Finally, if $X \subseteq V(G)$ and $F = (V(F), E(F))$ is a subgraph of $G$:

$$\bar{X}(F) = \bar{X} \cap V(F) \quad \text{and} \quad X(F) = X \cap V(F).$$

2 Introduction

In this paper we study the numerical structure of the optimal solutions of the following convex programming problem defined on the edge set of a graph. Let $G = (V(G), E(G))$ be a simple graph, $P$ a distribution of probability defined on $V(G)$ and set:

$$l(G, P) = \min_{(x, y) \in E(G)} h(P(x), P(y)).$$

We define the conjunctive capacity of $G$ as:

$$\Theta(G) = \max_P l(G, P). \quad (1)$$
Note that, being $h$ concave, problem (1) can be reduced to a convex programming problem. A distribution $P$ is $G$-balanced if it achieves the maximum in (1). Let us define the $t$-th power of $G$ as the graph $G^t = (V(G)^t, E(G^t))$ such that $\{(x_1, \ldots, x_t), (y_1, \ldots, y_t)\} \in E(G^t)$ if for every edge $e \in E(G)$ there exists a position $1 \leq i \leq t$ such that $\{x_i, y_i\} = e$.

In [3] and [4] the authors show that $\Theta(G)$ is the asymptotic exponent of the clique number (i.e. the size of the largest complete subgraph of $G$) of the powers of $G$:

$$\Theta(G) = \lim_{t \to \infty} \frac{1}{t} \log \omega(G^t).$$

This result has been used to answer a long-standing open question on the asymptotics of the maximum number of qualitatively independent partitions in the sense of Rényi [9].

We point out that in these papers the conjunctive capacity of graphs is considered as a particular case of the Sperner capacity of a family of directed graphs. For the applications in information theory see for example [3].

By considering the uniform distribution on the vertex set of the graph, one easily see that for every graph with $n$ vertices, $\Theta(G) \geq 2/n$. A 2-matching is a vector $x = (x_e : e \in E(G))$ with components 0, 1 or $1/2$, such that for every $x \in V(G)$ the sum of the weights to the edges incident in $x$ is at most 1. A 2-matching is maximum if the overall sum of the weights assigned to the edges is maximum. A 2-matching is perfect if every vertex has some incident edge with nonzero weight. It is easy to see that a graph has a perfect 2-matching if and only if it contains a system of vertex-disjoint odd cycles and edges covering the vertex-set (for more on 2-matchings and related problems see [7]).

In [5] we show the following

**Theorem 1** For every simple graph $G$ without isolated vertices,

$$\Theta(G) = \frac{2}{n}, \quad (n = |V(G)|)$$

if and only if $G$ has a perfect 2-matching.

Note that an easy corollary is that if the uniform distribution is $G$-balanced then this it is also the unique optimal solution to (1). We recall also the following characterization for graph having a perfect 2-matching [10]:

3
Theorem T  

\( G \) has a perfect 2-matching if and only if for every stable set \( X \subseteq V(G) \):

\[
\frac{|\Gamma X|}{|X|} \geq 1.
\]

In the sequel \( G \) is a graph without isolated vertices. A vertex is \textit{critical} if its deletion strictly decreases the stability number of the graph. It is easy to see that a vertex is critical if and only if it belongs to every stable set of maximum size. If \( P \) is a probability distribution on \( V(G) \), the vertex \( x \) is \( P \)-\textit{critical} if, for some \( y \in \Gamma x \), \( P(x) < P(y) \). In the next section we prove the following

\textbf{Lemma 1 (Splitting Lemma)}  

For any graph \( G \) and \( G \)-balanced distribution \( P \),

1. All the \( P \)-critical vertices are critical,

2. If \( X \) is the set of the \( P \)-critical vertices, then the subgraph of \( G \):

\[
F = G - (X \cup \Gamma X)
\]

has a perfect 2-matching.

Note that an immediate consequence of the Splitting Lemma is the following, already known, result (for example, see [2]):

\textbf{Corollary 1}  

If \( G \) has no critical points then it has a perfect 2-matching.

\textbf{Proof:} By the hypothesis and the Splitting Lemma it follows that every \( G \)-balanced distribution has an empty set of \( P \)-critical vertices and \( G \) has a perfect 2-matching. \( \square \)

The number of the \( P \)-critical vertices is computable in polynomial time. So, it is interesting to investigate its relations with the stability number of \( G \):

\textbf{Theorem 2}  

Let \( G \) be a graph and \( X \) the set of \( P \)-critical vertices for a \( G \)-balanced distribution \( P \). Then

\[
\alpha(G) = |X| + \alpha(F),
\]

where \( F = G - (X \cup \Gamma X) \) has a perfect 2-matching.
So, the stability number of every graph can be expressed as the sum of the stability number of a graph with a perfect 2-matching plus some quantity computable in polynomial time. Now, the stability number of a graph with a perfect 2-matching can be bounded from above and below in terms of the matching number of the graph. Indeed, by observing that the set of vertices non covered by a maximal matching of $G$ is a stable set, one gets the general lower-bound:

$$\alpha(G) \geq n - 2\nu(G).$$

On the other hand, if $G$ has a perfect 2-matching and $X$ is any maximum stable set in $G$ then, by Theorem T

$$|\Gamma Y| \geq |Y|, \text{ for every } Y \subseteq X.$$ 

By Hall’s Theorem, $G$ contains a matching covering every vertex in $X$, and

$$\alpha(G) \leq \nu(G).$$

It follows

**Corollary 2** Let $G$ be a graph, $P$ a $G$-balanced distribution, $X$ the set of $P$-critical points, and $F = G - [X \cup \Gamma X]$, then:

$$|X| + |V(F)| - 2\nu(F) \leq \alpha(G) \leq |X| + \nu(F),$$

In particular

$$\nu(F) = \frac{|V(F)|}{3} \Rightarrow \alpha(G) = |X| + \nu(F).$$

**Remark:** Note that if a graph $F$ has a perfect 2-matching then

$$\nu(F) \geq \frac{|V(F)|}{3}.$$ 

The set $P$-critical points, $P$ balanced, plays a similar role of the set of vertices with weight zero in a minimum $2-$cover of $G$. A fractional vertex cover is any feasible solution $y = (y_u : u \in V(G))$, of the following dual of a linear programming problem

$$\begin{cases}
\min \ 1y \\
y_u + y_v \geq 1 \ \forall \{u, v\} \in E(G) \\
y \geq 0
\end{cases} \quad (2)$$
An optimal solution is a minimum fractional cover. A 2-cover of $G$ is a fractional cover whose components are 0, 1 or 1/2. A 2-cover is basic if the graph induced in $G$ by the set of vertices with weight 1 is not bipartite. Lorentzen [6] and independently Balinsky and Spielberg [1] proved that the set of vertices of the feasible region of problem (2) coincides with the set of the basic 2-covers of the graph. It is possible to prove that the uniform 2-cover (i.e. the assignment of weight constantly equal to 1/2) is an optimal solution to the minimum fractional cover problem if and only if $G$ has a perfect 2-matching. Nevertheless, this does not mean that the uniform fractional cover is the unique optimal solution. For example consider a complete bipartite graph with color classes of same size. Having this graph a perfect matching, the uniform fractional cover is optimal. But another optimal solution is the one having value 0 on a color class and value 1 on the complementary class. This simple example shows that the analogous of the Splitting lemma does not hold for the set of vertices having weight 0 in an optimal fractional cover of $G$ (Pulleyblank in [8] prove that the uniform fractional cover is the unique optimal solution if and only if for every vertex $v \in V(G)$ the graph $G - \{v\}$ has a perfect 2-matching).

In the next section we give a proof of the Splitting Lemma.

3 Proof of the Splitting Lemma

In [5] we proved the following three lemmas. In all the statement $G$ has no isolated vertices. In the first lemma, a line cover of $G$ is a set of lines collectively incident with each point of $G$:

**Lemma 2** [5] Let $G = (V(G), E(G))$ be a simple graph and $P$ a $G$-balanced distribution, then:

$$\mathcal{L}(P) = \{\{x, y\} \in E(G) : h(P(x), P(y)) = \Theta(G)\}$$

is a line cover of $G$.

Now, set

$$e(P) = \{x \in V(G) : P(x) = P(y) \text{ for any } y \in \Gamma X\},$$

and let us denote by $m(P)$ the set of $P$-critical vertices in $G$
Lemma 3  [5] Let $G = (V(G), E(G))$ be a simple and $P$ a $G$-balanced distribution. Then $m(P)$ is a stable set in $G$ and for every maximal stable $S \supseteq m(P)$, $S \setminus m(P)$ is a maximal stable in the subgraph induced in $G$ by $e(P)$.

Let $S$ be a maximal stable set of $G$. Every distribution $P$ such that $m(P) \subseteq S$ is called centered on $S$. The family of all the distributions centered on $S$ will be denoted by $Cr(S)$. Note that the uniform distribution is centered on every maximal stable set of $G$.

Lemma 4  [5] Let $G = (V(G), E(G))$ be a simple graph without isolated vertices and $P$ a $G$-balanced distribution centered on $S$. Then for every connected component $F = (V(F), E(F))$ of the graph $(V(G), L(P))$ there exist two reals $q_F \leq p_F$ such that

$$P(v) = \begin{cases} p_F & \text{if } v \in V(F) \cap \bar{S} \\ q_F & \text{if } v \in V(F) \cap S. \end{cases} \quad (3)$$

Now, it is interesting to consider our maxmin problem for probability distributions that assume at most two different values on the vertex set of a graph. In particular if $q$ and $p$ are these two values with $q \leq p$, by Lemmas 3 and 4 there must exist a maximal stable set $S$ in $G$ such that $P(v) = q$ if $v \in S$ and $P(v) = p$ otherwise. In particular, for those graphs $G$ for which there exists a two valued balanced distribution $P$ we obtain the exact solution of (1). Let $G$ be a graph and $S$ a maximal stable set of $G$ with $|S| = \alpha$. We write $|\bar{S}| = \tau$. Then the maxmin problem for a two-valued distribution can be defined as:

$$\phi(w, \alpha, \tau) = \max_{(q,p) \in D_{w,\alpha,\tau}} h(p, q) \quad (4)$$

$$D_{w,\alpha,\tau} = \{(q,p) \in (0,1]^2 \mid q \leq p \text{ and } q\alpha + p\tau = w\}, \quad (5)$$

where $w = 1$ and $\alpha$, $\tau$ are positive constants. In the following proofs we will consider the general setting where $0 < w \leq 1$.

It will be convenient to rewrite the above, setting

$$t = \frac{p}{q} = \frac{w - q\alpha}{q\tau} \quad (6)$$
and define
\[
\phi(w, \alpha, \tau) = \max_{t \geq 1} z(t, w, \alpha, \tau). \tag{7}
\]
where
\[
z(t, w, \alpha, \tau) = h \left( \frac{wt}{t\tau + \alpha}, \frac{w}{t\tau + \alpha} \right). \tag{8}
\]

Now, we formulate the two main properties of \(h\) that will be used in the sequel (proof in Appendix A).

**Property 1** \(h(., .) \in C^{(1)}(0, 1]^2\) is a symmetric and strictly increasing function of its arguments.

**Property 2** For fixed \(w, \alpha, \tau\) the function \(z(., w, \alpha, \tau)\) has a unique absolute point of maximum \(t(w, \alpha, \tau) \in [1, +\infty)\). If \(t(w, \alpha, \tau) > 1\) then it is also the unique stationary point of \(z(., w, \alpha, \tau)\) and if \(t(w, \alpha, \tau) = 1\) then \(z\) is a strictly decreasing function for \(t > 1\).

**Claim 1** For any balanced distribution \(P\), we have
\[
e(P) = V(G) \setminus [m(P) \cup \Gamma m(P)]. \tag{9}
\]

**Proof:** By Property 1, it is clear that \(e(P) \cap m(P) = \emptyset\). Suppose that for an \(x \in m(P)\), \(e(P) \cap \Gamma x \neq \emptyset\) and fix \(y \in e(P) \cap \Gamma x\). Now, let \(F\) and \(F'\) be the connected components in \((V(G), \mathcal{L}(P))\) containing \(x\) and \(y\) respectively. By \(x \in m(P)\) and \(y \in e(P)\) \(F \neq F'\) and \(\{x, y\} \notin \mathcal{L}(P)\). Hence:
\[
p(F') \geq P(x) = q(F') > q(F) = p(F) = P(y),
\]
and by using Property 1 and Lemma 4 one gets a contradiction with
\[
h(p(F), q(F)) = h(p(F'), q(F')).
\]
Therefore
\[
e(P) \subseteq V(G) \setminus [m(P) \cup \Gamma m(P)].
\]

For the converse, suppose \(x \notin e(P) \cup m(P)\). Then for any \(\{x, y\} \in \mathcal{L}(P)\), \(P(x) > P(y)\) and \(x \in \Gamma m(P)\). \qed
Now, we prove item 2 in the Splitting Lemma. We will use the following

\[ t(w, \alpha, \tau) = 1 \text{ iff } \alpha \leq \tau. \tag{10} \]

(see Appendix A)

**Proof of 2 in Lemma 1:** By (9) it suffices to show that, for every \( G \)-balanced distribution \( P \), the subgraph induced in \( G \) by \( e(P) \) has a perfect 2-matching. We have

\[ e(P) = \bigcup_{F : q(F) = p(F)} V(F) \]

where the union ranges into the family of the components \( F \) of \( (V(G), \mathcal{L}(P)) \) such that \( q(F) = p(F) \). We show that if \( q(F) = p(F) \) then \( F \) has a perfect 2-matching. By Tutte’s Theorem we must prove that for every stable set \( Y \) in \( F \),

\[ |\Gamma FY| \geq |Y| \quad (\Gamma FY = V(F) \cap \Gamma Y). \]

Suppose the contrary and let us fix

\[ t = t(w, |Y|, |\Gamma FY|), \quad (|\Gamma FY| < |Y|) \]

where

\[ w = P(Y \cup \Gamma FY) = (|Y| + |\Gamma FY|)q \text{ and } q = q(F) = p(F). \]

Note that by \( |\Gamma FY| < |Y| \) and (10) \( t > 1 \).

We replace \( P \) with a new probability distribution \( P' \) where \( \mathcal{L}(P') \) is not a line cover but

\[ l(G, P') \geq l(G, P). \]

By Lemma 2 it follows that \( P \) cannot be \( G \)-balanced. Fix

\[ R = |\Gamma FY||Y|^{-1} \]

and

\[ \epsilon = \min \left\{ \frac{q(t - 1)}{Rt + 1}, R^{-1} \min_{F' \in \mathcal{C}} [q - q(F')], \min_{F' \in \mathcal{C}} [p(F') - q] \right\}, \quad \nu = \epsilon R, \]

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where $\mathcal{C}$ is the family of the components $F'$ of $(V(G), \mathcal{L}(P))$ such that $q(F') \neq q$. Note that $\epsilon > 0$ (in particular, by Property 1, $p(F') = q$ implies $q(F') = q$).

Define $P'$ as:

$$P'(v) = \begin{cases} q + \epsilon & \text{if } v \in \Gamma F Y \\ q - \nu & \text{if } v \in Y \end{cases}$$

and $P'(v) = P(v)$ if $v \not\in Y \cup \Gamma F Y$. $P'$ is a probability distribution. Indeed, from $\epsilon \leq q(t - 1)(Rt + 1)^{-1}$ it follows

$$q - \nu = q - \epsilon R \geq \frac{q(R + 1)}{Rt + 1} > 0,$$

and $\epsilon$ and $\nu$ are fixed so as to leave the total amount of probability of $Y \cup \Gamma F Y$ unchanged. We prove that the global minimum does not decrease, that is $l(G, P') \geq l(G, P)$.

**Case 1:** Edges $\{x, y\}$ such that one endpoint $x$ belongs to $\Gamma F Y$. If $y \not\in Y$ then

$$P'(y) \geq P(y) \text{ and } P'(x) > P(x) \Rightarrow$$

$$\Rightarrow h(P'(x), P'(y)) > h(P(x), P(y)) \geq l(G, P).$$

If $y \in Y$ note that by

$$\epsilon \leq \frac{q(t - 1)}{(Rt + 1)},$$

one has

$$1 < \frac{q + \epsilon}{q - \nu} \leq t.$$

By Property 2, setting $\alpha = |Y|$ and $\tau = |\Gamma F Y|:

$$h(P(x), P(y)) = h(q, q) = z(1, w, \alpha, \tau) < h(P'(x), P'(y)) \leq z(t, w, \alpha, \tau).$$

**Case 2:** $x \in Y$ and $y \not\in \Gamma F Y$. Clearly, it follows that $y$ belongs to a component $F' \neq F$. Note that $F' \not\in \mathcal{C}$, otherwise by definition of $\mathcal{C}$, $q(F') = q$ would imply $p(F') = q$, and $F = F'$. In addition by $x \in e(P)$ it follows $y \not\in m(P)$. Hence $P(y) = p(F')$, by

$$\epsilon \leq \min\{R^{-1}[q - q(F')], [p(F') - q]\}$$

one has

$$q(F') \leq q - \nu \leq q + \epsilon \leq p(F').$$
and:
\[ h(P'(x), P'(y)) = h(q - \nu, p(F')) \geq h(q(F'), p(F')) = l(G, P). \]
Now, note that (Case 1) no nodes in $\Gamma F Y$ are endpoints of edges in $L(P')$ and so $P'$ is not $G$-balanced. \hfill \Box

Now, we prove item 1 in Lemma 1. For an arbitrary maximal stable set $X$ such that $P \in Cr(X)$, let us introduce the following relation between the components of the graph $(V(G), L(P))$:
\[ F \prec F' \iff F \neq F' \text{ and } \exists \{x, y\} \in E(G) : x \in X(F), y \in V(F'), \tag{12} \]
For the transitive closure $\prec$ of $\prec$ we prove

Claim 2
\[ F \not\prec F' \Rightarrow q(F') < q(F) \leq p(F) < p(F'). \]

Proof: Let $F = F_1 \prec F_2 \prec \ldots \prec F_m = F'$ be any chain of relations $\prec$. We show
\[ q_m < \ldots < q_2 < q_1 \leq p_1 < p_2 < \ldots < p_m \]
where $q(F_i) = q_i$ and $p(F_i) = p_i$. By definition of $\prec$ there exist $m - 1$ edges $(u_i, v_{i+1})$ such that for each $1 \leq i < m$
\[ u_i \in X(F_i) \text{ and } v_{i+1} \in \bar{X}(F_{i+1}) \]
and
\[ h(P(u_i), P(v_{i+1})) = h(q_i, p_{i+1}) > l(G, P) = h(q_i, p_i), \]
where the strict inequality follows from $(u_i, v_{i+1}) \notin L(P)$. By Property 1 it follows the claim. \hfill \Box

Observation 2: Note that by Claim 2 if $q(F) = p(F)$ and $P \in Cr(X)$, then
\[ X(F) = X \cap \Gamma \bar{X}(F), \]
or else there would exist a component $F' \prec F$.

The proof of the following property of $h$ can be found in Appendix A
**Property 3** If $\alpha/\tau > 1$ then for any $w > 0$

$$\frac{\alpha}{\tau} < \frac{\alpha'}{\tau'} \iff t(w, \tau, \alpha) < t(w, \tau', \alpha').$$

Note also that $t$ is independent by $w$. That is, for any $\alpha, \tau, w$ and $w'$ (see Appendix A):

$$t(w, \tau, \alpha) = t(w', \tau, \alpha) = t(\tau, \alpha) \quad (13)$$

**Lemma 5** If $P$ is a $G$-balanced distribution centered on a stable set $X$ and $F$ is a connected component of the graph $(V(G), \mathcal{L}(P))$, then for any $U \subseteq X(F)$

$$t(|X(F) \cap \Gamma U|, |U|) \geq \frac{p(F)}{q(F)}.$$  

**Proof:** Suppose that the above inequality is false for $U \subseteq X(F)$. As in the proof of item 2, we replace $P$ with a new probability distribution $P'$ where $\mathcal{L}(P')$ is not a line cover of $G$ and $l(G, P') \geq l(G, P)$.

Set

$$t = t(|X(F) \cap \Gamma U|, |U|).$$

By hypothesis

$$t < \frac{p(F)}{q(F)}.$$  

Fix

$$R = \frac{|U|}{|X(F) \cap \Gamma U|} \text{ and } L = \{F' : F' \prec F\}$$

and the two real numbers

$$\epsilon = \min \left\{ \frac{p(F) - q(F)t}{Rt + 1}, R^{-1} \min_{F' \in L} [q(F') - q(F)], \min_{F' \in L} [p(F) - p(F')] \right\},$$

$$\nu = \epsilon R,$$

by Claim 2 and $p(F) > q(F)t$, $\epsilon > 0$. Define $P'$ as:

$$P'(v) = \begin{cases} p(F) - \epsilon & \text{if } v \in U \\ q(F) + \nu & \text{if } v \in X(F) \cap \Gamma U \\ P(v) & \text{if } v \notin U \cup (X(F) \cap \Gamma U) \end{cases} \quad (14)$$

and $P'(v) = P(v)$ if $v \notin U \cup (X(F) \cap \Gamma U)$. 

$P'$ is a probability distribution. Indeed:

\[ p(F) - \epsilon \geq \frac{t(q(F) + Rp(F))}{Rt + 1} > 0, \]

and $\nu$ is chosen so as to leave unchanged the total amount of probability of $U \cup (X(F) \cap \Gamma U)$. Also note that by $t \geq 1$,

\[ p(F) - \epsilon \geq q(F) + \nu, \]

that is

\[ \epsilon \leq \frac{p(F) - q(F)t}{Rt + 1} \leq \frac{p(F) - q(F)}{R + 1}. \]

Now, we show that $l(G, P') \geq l(G, P)$.

**Case 1:** Edges $\{x, y\}$ such that $x \in X(F) \cap \Gamma U$. If $y \notin U$ then $P'(y) \geq P(y)$ and $P'(x) > P(x)$ make this case trivial. If $y \in U$, by

\[ \epsilon \leq \frac{(p(F) - q(F))t}{Rt + 1}, \]

one has

\[ \frac{p(F)}{q(F)} > \frac{p(F) - \epsilon}{q(F) + \nu} \geq t. \]

Now, set

\[ \alpha = |X(F) \cap \Gamma U|, \quad \tau = |U| \]

and

\[ P(F) = \sum_{v \in V(F)} P(v), \]

By Property 2, and (13) one has

\[ h(P(x), P(y)) = h(p(F), q(F)) = z\left(\frac{p(F)}{q(F)}, P(F), \alpha, \tau\right) \]

\[ < h(P'(x), P'(y)) \leq z(t, P(F), \alpha, \tau). \]

**Case 2:** $x \in U$ and $y \notin X(F) \cap \Gamma U$. Let $y \in V(F')$. If $y \in X(F')$ suppose $p(F') \geq p(F) - \epsilon$, then

\[ h(P'(x), P'(y)) = h(p(F) - \epsilon, p(F')) \geq h(p(F) - \epsilon, p(F) - \epsilon) \geq \]

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\[ \geq h(p(F) - \epsilon, q(F) + \nu) > h(p(F), q(F)) = l(G, P). \]

The case \( p(F') < p(F) - \epsilon \) can be evaluated in a similar way.

Finally, if \( y \in X(F') \) then by hypothesis \( F' \neq F \) and by \( U \subseteq \bar{X}(F), F' \prec F \). Being

\[ \epsilon \leq \min \left\{ R^{-1}[q(F') - q(F)], [p(F) - p(F')] \right\}, \]

it follows

\[ q(F) + \nu \leq q(F') \leq p(F') \leq p(F) - \epsilon. \]

That is

\[ h(P'(x), P'(y)) = h(p(F) - \epsilon, q(F')) \geq \]
\[ \geq h(p(F) - \epsilon, q(F) + \nu) > h(p(F), q(F)) = l(G, P). \]

Now, note that the value of \( h(.,.,.) \) strictly increases over all the edges with at least one point in \( X(F) \cap \Gamma U \) (Case 1). Hence, unless \( U = \bar{X} \) (in this case one should have directly \( t = p(F)/q(F) \)) it follows that \( \mathcal{L}(P') \) is not a line cover of \( G \) which proves the statement.

\[ \square \]

**Proof of 1 in Lemma 1:** Let \( \mathcal{I}(.) \) be the family of all the *maximum* stable sets in a graph. Note that 1 is equivalent to the following equality

\[ \mathcal{I}(G) = \{ Z : Z = m(P) \cup A, A \in \mathcal{I}(F) \}. \]  \hspace{1cm} (15)

Let us consider any maximum stable set \( Z \) in \( G \). Then, if we show that for every \( G \)-balanced distribution \( P, P \in \mathcal{C}r(Z) \) this would imply (15). Indeed, by definition of \( \mathcal{C}r(Z) \),

\[ Z = A \cup m(P), \]

where \( A \) is a maximal stable set in the subgraph induced in \( G \) by \( e(P) \). By Claim 1, if \( S \) is any stable set in such a subgraph then the set \( Z' = S \cup m(P) \) is a stable set in \( G \). Hence,

\[ |Z'| = |S| + |m(P)| \leq |Z| = |A| + |m(P)| \Rightarrow A \in \mathcal{I}(F). \]

Vice versa, once again by Claim 1, if \( S \in \mathcal{I}(F) \) then \( S \cup m(P) \) is a maximal stable set in \( G \) and by \( S \) maximum in \( F \)

\[ |S| + |m(P)| \geq |A| + |m(P)|. \]
On the other hand, we supposed \( Z \) maximum that is \( S \cup m(P) \in \mathcal{I}(G) \).

Suppose \( P \in Cr(Y) \). If \( Z = Y \) we have finished. Let \( F \) be any connected component of the graph \((V(G), \mathcal{L}(P))\), and set

\[
\Delta_Z(F) = V(F) \cap (Z \setminus Y).
\]

Let us fix

\[
C = \{F : \Delta_Z(F) \neq \emptyset\}.
\]

Being \( \mathcal{L}(P) \) a line cover, \( \{\Delta_Z(F) : F \in C\} \) is a partition of \( Z \setminus Y \). Further, if \( F \in C \) then

\[
\Delta_Y(F) = V(F) \cap (Y \setminus Z) \neq \emptyset.
\]

Indeed, if \( x \in \Delta_Z(F) \) then \( x \in \bar{Y}(F) \) and, being \( \mathcal{L}(P) \) a line cover of \( G \), \( Y(F) \cap \Gamma x \neq \emptyset \). Now, by \( x \in Z \setminus Y \) it follows

\[
Y(F) \cap \Gamma x \subseteq Y \setminus Z.
\]

So, being

\[
|Z \setminus Y| = \left| \bigcup_{F \in C} \Delta_Z(F) \right| = \sum_{F \in C} |\Delta_Z(F)|
\]

and

\[
|Y \setminus Z| \geq \left| \bigcup_{F \in C} \Delta_Y(F) \right| = \sum_{F \in C} |\Delta_Y(F)|,
\]

we have

\[
\min_{F \in C} \frac{|\Delta_Y(F)|}{|\Delta_Z(F)|} \leq \frac{\sum_{F \in C} |\Delta_Y(F)|}{\sum_{F \in C} |\Delta_Z(F)|} \leq \frac{|Y \setminus Z|}{|Z \setminus Y|} \leq 1.
\]

Therefore, we can fix any \( C \in \mathcal{C} \) such that

\[
|\Delta_Y(C)| \leq |\Delta_Z(C)|.
\]

By \( \Delta_Z(C) \subseteq Z \setminus Y \) it follows \( Y \cap \Gamma [\Delta_Z(C)] \subseteq Y \setminus Z \) and in particular

\[
Y(C) \cap \Gamma [\Delta_Z(C)] \subseteq \Delta_Y(C).
\]

Hence

\[
\frac{|Y(C) \cap \Gamma [\Delta_Z(C)]|}{|\Delta_Z(C)|} \leq \frac{|\Delta_Y(C)|}{|\Delta_Z(C)|} \leq 1. \tag{16}
\]
In accordance with Lemma 5 and (10)

\[ \frac{p(C)}{q(C)} \leq t(|Y(C) \cap \Gamma[\Delta_Z(C)]|, |\Delta_Z(C)|) = 1. \]

So \( p(C) = q(C) \) and \( \Delta_Z(C) \subseteq e(P) \). Moreover, by Observation 2 it follows that \( Y(C) = Y \cap \Gamma Y(C) \) and then

\[ Y \cap \Gamma[\Delta_Z(C)] = Y(C) \cap \Gamma[\Delta_Z(C)] \subseteq e(P). \quad (17) \]

Now, set

\[ K = (Y \setminus \Gamma[\Delta_Z(C)]) \cup \Delta_Z(C). \]

Note that by (16) and (17), \( |K| \geq |Y| \) and it is easy to check that \( K \) is a stable set in \( G \). By

\[ \Delta_Z(C) \cup (Y \cap \Gamma[\Delta_Z(C)]) \subseteq e(P), \]

and \( m(P) \subseteq Y \), it follows \( m(P) \subseteq K \). If \( R_K \) is any maximal stable set in the subgraph induced by \( e(P) \) in \( G \) containing \( K \setminus m(P) \) then by Claim 1

\[ Y_1 = R_K \cup m(P) \]

is a stable set in \( G \). We have \( |Y_1| \geq |K| \geq |Y| \) and \( P \in Cr(Y_1) \). In addition

\[ |Z \setminus Y_1| \leq |Z \setminus K| = |Z \setminus Y| - |\Delta_Z(C)| < |Z \setminus Y|, \]

and

\[ |Z \setminus Y_1| = 0 \text{ implies } Y_1 = Z. \]

Iteratively applying the above procedure, we find a sequence of maximal stable sets:
\( Y = Y_0, Y_1, \ldots \) such that \( P \in Cr(Y_i) \) and \( |Z \setminus Y_i| \) strictly decreases with \( i \geq 0 \). Hence, for some \( m > 0 \) we get \( Y_m = Z \) and the statement. \( \square \)

4 Appendix A: basic properties of \( \bar{h} \)

We prove the three main properties of \( \bar{h} \). Property 1 is easy to verify.
1. **Property 2**, (13), (10) We have:

\[
z(t, w, \alpha, \tau) = \frac{w}{t\tau + \alpha} \left[ \log(t + 1) + \log\left(\frac{1}{t} + 1\right) \right]
\]  

(18)

and:

\[
\frac{dz}{dt} = \frac{w}{(t\tau + \alpha)^2} \left[ \alpha \log\left(\frac{1}{t} + 1\right) - \tau \log(t + 1) \right].
\]

Hence \( t(w, \alpha, \tau) \) is independent by \( w \) and it follows (13). Now, if \( \alpha \leq \tau \) the point of maximum of \( z \) is \( t(\tau, \alpha) = 1 \). Otherwise \( t(\tau, \alpha) \) is the unique number greater than 1 that is a root of:

\[
\rho(t) = (t + 1)^{\alpha-\tau} - t^\alpha.
\]

This proves Property 2 and (10). We note that (10) can be proved for any function verifying Properties 1 and 2 (the proof is not trivial).

2. **Property 3**: Remember that if \( \alpha > \tau \), \( t = t(\alpha, \tau) \) is the unique root greater than 1 of

\[
\rho(t) = (t + 1)^{\alpha-\tau} - t^\alpha.
\]

Hence

\[
\frac{\tau}{\alpha} = 1 - \frac{\log t}{\log(t + 1)}
\]

and it is sufficient to note that the right hand side is a strictly decreasing function on the semi-interval \( t \geq 1 \).

**References**

[1] M.L. Balinski and K. Spielberg. Methods for integer programming: algebraic, combinatorial and enumerative. In J. Aronofsky, editor, *Progress in operation research*, volume III, pages 195–292, Wiley, New York, 1969.

[2] C. Berge. *Graphes*. Gaulhier-Villars, Paris, 1973.

[3] G. Cohen, J. Körner, and G. Simonyi. Zero error capacities and very different sequences. In R.M. Capocelli, editor, *Sequences: combinatorics, compression security and transmission*, pages 144–155. Springer-Verlag, 1990.
[4] L. Gargano, J. Körner, and U. Vaccaro. Sperner capacities. *Graphs and combinatorics*, 9:31–46, 1993.

[5] G. Greco. Capacities of graphs and 2-matchings. *Discrete Mathematics*, 186:135–143, 1998.

[6] L.C. Lorentzen. *Notes on covering of arcs by nodes in an undirected graph*. Technical report, 1966.

[7] L. Lovász and M.D. Plummer. *Matching Theory*. North-Holland, New-York, 1986.

[8] W.R. Pulleyblank. Minimum node covers and 2-bicritical graphs. *Mathematical programming*, 17:91–103, 1979.

[9] A. Rényi. *Probability theory*. North-Holland, Amsterdam/New-York, 1970.

[10] W.T. Tutte. The 1-factors in oriented graphs. *Proceedings american mathematical society*, 22:107–111, 1947.