Generation of inequivalent generalized Bell bases

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The notion of equivalence of maximally entangled bases of bipartite $d$–dimensional Hilbert spaces $\mathcal{H}_d \otimes \mathcal{H}_d$ is introduced. An explicit method of inequivalent bases construction is presented.

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Maximally entangled states of two qubits - the so called Bell states [1] form a basic element of many quantum information protocols - e.g. quantum teleportation [2] and quantum dense coding [3] to mention only the most popular. Recently some protocols have been generalized to make use of $d$–dimensional quantum systems – qudits instead of qubits. e.g. [4, 5, 6, 7, 8]. In this context attention was paid to the $d$–dimensional generalization of Bell states. For any $d$ it is possible to construct $d^2$ mutually orthogonal maximally entangled states. These states form a basis of the Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$, which will be called maximally entangled basis (MEB). In a slight abuse of language, we use MEB to denote the plural form maximally entangled bases, as well. The expansion of the acronym will be clear from context. The construction of MEB is, of course, not unique and one can ask if there is any interesting classification of the possible MEB of a given space. In this paper we propose to classify the MEB with the use of the following notion of equivalence.

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Two MEB are equivalent if and only if (iff) there exists a bilocal unitary operation 
\[ U = U_1 \otimes U_2 \] 
which transforms all the states of the first MEB into the states of the second one. Let us denote the basis states of the two MEB as \(|\Psi^{(1)}_{jk}\rangle\) and \(|\Psi^{(2)}_{jk}\rangle\) 
\((j, k = 0, 1 \ldots, d-1)\). Then these two MEB are equivalent iff there exist permutations \(\pi_1\) and \(\pi_2\), phases \(\theta_{jk}\) and unitary operations \(U_1\) and \(U_2\) such that

\[ |\Psi^{(1)}_{jk}\rangle = e^{i\theta_{jk}}(U_1 \otimes U_2)|\Psi^{(2)}_{\pi_1(j)\pi_2(k)}\rangle \]  
\(1\)

Of course it is always possible to locally transform a given maximally entangled state into any other maximally entangled state. What we require here is that all states of one basis be transformed into the states of the second basis by the same unitary operation \(U_1 \otimes U_2\). We will present an explicit construction of inequivalent MEB of the Hilbert space \(H_d \otimes H_d\), provided that \(d\) is not prime.

Let us begin by restating some basic properties of the well known two-qubit Bell states

\[ |\Psi_{jk}^{\text{Bell}}\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{1} (-1)^{jm} |m\rangle |m \oplus k\rangle, \]  
\(2\)

where \(\oplus\) denotes addition modulo 2, while \(j, k = 0, 1\). These states can be generated from product states of two qubits by applying the CNOT gate (CNOT\(|j\rangle |k\rangle = |j\rangle |k \oplus j\rangle\)) as in the following equation

\[ |\Psi_{jk}^{\text{Bell}}\rangle = \text{CNOT}|\phi_j\rangle |\chi_k\rangle, \]  
\(3\)

provided that the states \(|\phi_j\rangle\) and \(|\chi_k\rangle\) are appropriately chosen, namely \(|\phi_j\rangle = 2^{-1/2} \sum_{m=0}^{1} (-1)^{jm} |m\rangle\) and \(|\chi_k\rangle = |k\rangle\). The Hadamard gate \(H\) \((H|j\rangle = 2^{-1/2} \sum_{k=0}^{1} (-1)^{jk}|k\rangle)\) transforms computational basis states \(|j\rangle\) into the states \(|\phi_j\rangle\), i.e. \(|\phi_j\rangle = H|j\rangle\). The Bell states clearly form an orthonormal basis \(\{|\Psi_{jk}^{\text{Bell}}\rangle\}\), since
they are generated from the orthonormal basis \{ | j \rangle | k \rangle \} via the unitary transformation CNOT(\( H \otimes 1 \)). In order to check if the states obtained in this manner are really maximally entangled it suffices (notice that whole system is in a pure state) to calculate the von Neuman entropy of the reduced density operator of the first qubit \( S(\rho^A_{jk}) = -\text{Tr}_A \rho^A_{jk} \log \rho^A_{jk} \), where \( \rho^A_{jk} = \text{Tr}_B(|\Psi_{jk}\rangle_A \langle \Psi_{jk}|) \) for all \( j, k \) (A and B denote the first and second qubit respectively). For maximally entangled states the von Neuman entropy takes the value \( S(\rho^A_{jk}) = 1 \). It is easy to show that \( \rho^A_{jk} = 2^{-1} \sum_{m=0}^{d-1} | m \rangle \langle m | \) and consequently \( S(\rho^A_{jk}) = 1 \) indeed. Thus, the Bell states form a MEB in the Hilbert space \( H_2 \otimes H_2 \).

In order to generalize the above procedure to the \( d \)-dimensional case we will look for the set \( \Phi = \{ | \phi_j \rangle, j = 0, 1, \ldots, d - 1 \} \) of \( d \)-dimensional states \( | \phi_j \rangle \) which can be used to construct MEB in the Hilbert space \( H_d \otimes H_d \) in the following way

\[
| \Psi_{jk}^{\text{MEB}} \rangle = \text{GCNOT} | \phi_j \rangle | k \rangle, \quad j, k = 0, 1, \ldots, d - 1,
\]

(4)

where the \( d \)-dimensional generalization of the CNOT gate is defined as \( \text{GCNOT} | j \rangle | k \rangle = | j \rangle | j \ominus_d k \rangle \), and \( \ominus_d \) denotes subtraction modulo \( d \). We follow the method suggested in [4] of generalizing the CNOT gate with the use of subtraction instead of addition which leads to the useful property of self–inverseness \( \text{GCNOT}^2 = 1 \).

The states \( | \Psi_{jk}^{\text{MEB}} \rangle \) will form a MEB iff the states \( | \phi_j \rangle \) fulfil the following two conditions. First, the states \( | \phi_j \rangle \) have to form an orthonormal basis of \( H_d \). Second, as was shown by Stenholm and Bardroff [5], the condition

\[
| \langle k | \phi_j \rangle | = \frac{1}{\sqrt{d}}
\]

(5)

must hold for all \( j, k \). This condition follows from the fact that reduced density operator of the single qudit (for the two qudit state \( | \Psi_{jk}^{\text{MEB}} \rangle \)) is \( \rho^A_{jk} = \ldots \)
\[ |\langle m|\phi_j\rangle|^2 \sum_{m=0}^{d-1} |m\rangle \langle m|. \]

Let us denote by \( V_\phi = \sum_j |\phi_j\rangle \langle j| \) a unitary operation which transforms states \(|j\rangle\) into \(|\phi_j\rangle\). From Eq.(5) one sees that all elements of the corresponding matrix \((V_\phi)_{jk} = \langle j|V_\phi|k\rangle\) must have equal moduli \(|(V_\phi)_{jk}| = d^{-1/2}\). Unitary matrices fulfilling these properties have been called Zeilinger matrices \([5, 9]\). Two commonly used operators belonging to this class are: the \(d\)-dimensional discrete Fourier transform

\[
\text{DFT}_d = \sum_{j,k} \omega^j_k |j\rangle \langle k|
\]

where \(\omega_d = e^{i\frac{2\pi}{d}}\) is the \(d\)-th complex root of unity, and in the case when the dimension \(d = 2^n\) is some integer power of 2, the generalized Hadamard operation \(H_n\) defined recursively as follows

\[
H_{n+1} = H_1 \otimes H_n, \quad H_1 = H.
\]

In the qubit case these operations are identical, i.e. \(\text{DFT}_2 = H\). This is not true, when \(d = 4\) where \(\text{DFT}_4 \neq H_2\) and moreover, these operations will be shown later to generate inequivalent MEB. We say that the Zeilinger operation \(V\) generates a MEB if the states of this basis are given by

\[
|\Psi_{jk}^{\text{MEB}}\rangle = \text{GCNOT}(V \otimes \mathbb{I})|j\rangle|k\rangle
\]

In order to present an explicit construction of MEB, we introduce a function from the Cartesian product \(\mathcal{H}_d \times \mathcal{H}_d\) to \(\mathcal{H}_d\), which we will call vector multiplication. Multiplication of vectors \(|a\rangle = \sum_j a_j |j\rangle\) and \(|b\rangle = \sum_j b_j |j\rangle\) gives a vector \(|c\rangle = |a\rangle \circ |b\rangle\) such that \(|c\rangle = \sum_j c_j |j\rangle\), where \(c_j = \sqrt{da_j b_j}\). It is quite easy to prove that if the set \(\Phi = \{|\phi_j\rangle, j = 0, 1, \ldots, d - 1\}\) of \(d\) mutually orthogonal states \(|\phi_j\rangle\) together with the above defined multiplication form a group \(G = (\Phi, \circ)\), then these states can be used to construct a MEB in accordance with Eq.(4). The order of the group \(G\) is \(d\).
Therefore any element \( |\phi_j\rangle \) of group \( G \) must fulfil

\[
( |\phi_j\rangle )^d = |1_G\rangle ,
\]

where \( |1_G\rangle \) denotes the unit element of the group. Clearly, \( |1_G\rangle = d^{-1/2} \sum_j |j\rangle \). On the other hand, by the definition of the group operation \( ( |\phi_j\rangle )^d = \sum_k d(d-1)/2 \langle k|\phi_j\rangle^d|k\rangle \). Thus Eq.(9) leads to the identity \( d^{1/2}\langle k|\phi_j\rangle^d = 1 \) and consequently to \( \langle k|\phi_j\rangle = d^{-1/2}\omega_d^m \) for some integer \( m \). One can thus see that the condition of Eq.(5) is automatically fulfilled for the elements of the group \( G \). Let us now present an explicit construction of the elements \( |\phi_j\rangle \) of group \( G \). \( G \) is a finite Abelian group, so it is either a cyclic group \( G_d \) or a direct product of cyclic groups \( G_{d_1} \times G_{d_2} \times \ldots G_{d_r} \) (\( G_k \) denotes the cyclic group of order \( k \)). In the former case we can define inequivalent characters \( \chi^{(n)} \) \( (n = 0, 1, \ldots d - 1) \) of irreducible representations of the group \( G_d \) as \( \chi^{(n)}(|\phi_j\rangle) = \omega_d^{nj} \). The characters fulfil the orthogonality relation \[ \sum_{j=0}^{d-1} \chi^{(n)}(|\phi_j\rangle)\chi^{(m)}(|\phi_j\rangle)^* = d\delta_{nm} \]. Thus, if we take vectors \( |\phi_j\rangle \) of the form

\[
|\phi_j\rangle = d^{-1/2} \sum_{k=0}^{d-1} \chi^{(j)}(|\phi_k\rangle)|k\rangle ,
\]

they will form an orthonormal basis \( \langle \phi_j|\phi_k\rangle = \delta_{jk} \). It should be emphasized here that the structure of the group does not change under the permutation of the computational basis states. Thus, there exist \( d! \) bases given by \( |\phi_j^n\rangle = d^{-1/2} \sum_{j=0}^{d-1} \chi^{(j)}(|\phi_k\rangle)|\pi(k)\rangle \), where \( \pi \) is an arbitrary permutation. On the other hand, if \( G \) is a direct product of cyclic groups, Eq.(10) can be also used to obtain an alternative construction of vectors \( |\phi_j\rangle \) provided that we use the characters of the irreducible representations of group \( G \) nonisomorphic with \( G_d \). This is possible if the order \( d \) of the group can be expressed as a nontrivial product of integers \( d = d_1d_2\ldots d_r \). Any such decomposition of \( d \) leads to group \( G \) of the form
$G = G_{d_1} \times G_{d_2} \times \ldots \times G_{d_r}$. For each $G_{d_i}$ we define $d_i$ inequivalent characters of its irreducible representation $\chi_i^{n_i}(|\phi_j\rangle) = \omega_{d_i}^{n_i m_i}$ ($n_i, m_i = 0, 1, \ldots, d_i - 1$), where the one-to-one mapping between indices $j$ ($j = 0, 1, \ldots, d - 1$) and $\{m_i, i = 1, \ldots, r\}$ given by $j = \sum_{i=1}^{r} m_i \delta_i$, where $\delta_i = d(\Pi_{k=i} d_k)^{-1}$ is used. We can now obtain $d$ characters of the irreducible representations of group $G$ as a product of characters of groups $G_{d_i}$, i.e. $\chi^{(n)}(|\phi_j\rangle) = \Pi_{i=1}^{r} \chi_i^{(n_i)}(|\phi_j\rangle)$, where $n = \sum_{i=1}^{r} n_i D_i$ and $D_i = \Pi_{k=i+1} d_k$. In this way we obtain

$$\chi^{(n)}(|\phi_j\rangle) = \omega_d^{n \cdot j}, \quad (11)$$

where we introduce the product $\vec{n} \cdot \vec{j} = d \sum_{i=1}^{r} \frac{n_i m_i}{d_i}$ dependent on the decomposition $d = d_1 d_2 \ldots d_r$. Eqs. (10) and (11) lead to construction of nonisomorphic groups of vectors $|\phi_j\rangle$ for each decomposition of $d$ with the use of operator $V$ ($V|j\rangle = |\phi_j\rangle$) given by

$$V = d^{-\frac{d-1}{2}} \sum_{j,k=0}^{d-1} \omega_d^{\vec{n} \cdot \vec{j}} |k\rangle \langle j|. \quad (12)$$

The operators DFT$_4$ and $H_2$, mentioned earlier, are two examples of operators constructed in this way for the two possible decompositions of $d = 4$, namely $4 = 4$ and $4 = 2 \times 2$.

Let us now consider two operators and given by Eq. (12) with the use of different decomposition of $d$ and two MEB generated by these operators

$$|\Psi_{jk}^{(1)}\rangle = \text{GCNOT}(V_1 \otimes \mathbb{1})|j\rangle|k\rangle, \quad (13a)$$

$$|\Psi_{jk}^{(2)}\rangle = \text{GCNOT}(V_2 \otimes \mathbb{1})|j\rangle|k\rangle. \quad (13b)$$

These equations lead to the following transformation

$$|\Psi_{jk}^{(1)}\rangle = \text{GCNOT}(V_1 P_1^{-1} V_2^{-1} \otimes P_2^{-1}) \text{GCNOT}\Psi_{\pi(j)\pi(k)}^{(2)} \rangle \quad (14)$$
where \( P_1 \) and \( P_2 \) are permutation operators corresponding to permutations \( \pi_1 \) and \( \pi_2 \). Thus two MEB given by Eq.(13) are equivalent (see Eq.(11)) iff there exist local operators \( U_1 \) and \( U_2 \) such that

\[
U_1 \otimes U_2 = \text{GCNOT}(V_1 P_1^{-1} V_2^{-1} \otimes P_2^{-1})\text{GCNOT}.
\]  

(15)

This is possible iff the operator \( V_1 P_1^{-1} V_2^{-1} \) is the product of the permutation operator \( P \) and unitary diagonal operator \( D \), i.e. \( V_1 P_1^{-1} V_2^{-1} = PD \), which leads to the following condition of equivalence of two MEB \( |\Psi_{jk}^{(1)}\rangle \) and \( |\Psi_{jk}^{(2)}\rangle \)

\[
P^{-1} V_1 P_1^{-1} = DV_2.
\]

(16)

Let us notice that due to group structure of the columns of matrix \( V_1 \), one of these columns must be of the form

\[
\begin{pmatrix}
  d^{-\frac{1}{2}} \\
  d^{-\frac{1}{2}} \\
  \vdots \\
  d^{-\frac{1}{2}}
\end{pmatrix}
\]

(17)

Obviously one of the columns of matrix \( V_1 P_1^{-1} V_2^{-1} \) must also have this form. This restricts the possible form of diagonal matrix \( D \), namely \( D_{jj} = (V_2)_{jk}^* \) must hold for some \( k \). On the other hand, the group structure guarantees that there exist \( k' \) such that \( (V_2)_{jk}^* = (V_2)_{jk'} \). It means that the action of \( D \) on \( V_2 \) is equivalent to performing vector multiplication \( \circ \) of the columns of \( V_2 \) by one of these columns, i.e. equivalent to permutation of the columns of \( V_2 \). This leads to the final conclusion that two operators \( V_1 \) and \( V_2 \) given by Eq.(12) generate MEB, which are equivalent iff they are equal to each other up to permutations of columns and rows. It follows that two operators \( V_1 \) and \( V_2 \) given by Eq.(12) with the use of different decomposition of \( d \)
generate inequivalent MEB. As an example we can take MEB generated by DFT$_4$ and $H_2$ and conclude that these MEB are inequivalent.

In conclusion, we have introduced the notion of equivalence of maximally entangled bases of qudits. In accordance with this notion, we have presented an explicit construction of inequivalent maximally entangled bases from group-theoretic concepts.

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