QUANTIZATIONS OF NILPOTENT ORBITS VS 1-DIMENSIONAL REPRESENTATIONS OF W-ALGEBRAS

IVAN LOSEV

ABSTRACT. Let \( g \) be a semisimple Lie algebra over an algebraically closed field \( \mathbb{K} \) of characteristic 0 and \( \mathcal{O} \) be a nilpotent orbit in \( g \). Then \( \mathcal{O} \) is a symplectic algebraic variety and one can ask whether it is possible to quantize \( \mathcal{O} \) (in an appropriate sense) and, if so, how to classify the quantizations. On the other hand, for the pair \((g, \mathcal{O})\) one can construct an associative algebra \( \mathcal{W} \) called a (finite) W-algebra. The goal of this paper is to clarify a relationship between quantizations of \( \mathcal{O} \) (and of its coverings) and 1-dimensional \( \mathcal{W} \)-modules. In the first approximation, our result is that there is a one-to-one correspondence between the two. The result is not new: it was discovered (in a different form) by Moeglin in the 80’s.

1. Introduction

We fix a base field \( \mathbb{K} \) which is assumed to be algebraically closed and of characteristic 0. Let \( g \) be a semisimple Lie algebra, \( G \) the corresponding simply connected algebraic group, and \( \mathcal{O} \subset g \cong g^\ast \) a nilpotent orbit. The variety \( \mathcal{O} \) is symplectic with respect to the Kostant-Kirillov form. So one can pose the problem of quantizing \( \mathcal{O} \).

The purpose of this paper is to relate two types of objects:
- Some special quantizations of \( \mathcal{O} \) and, more generally, of its \( G \)-equivariant coverings,
- 1-dimensional representations of a so called \( W \)-algebra \( \mathcal{W} \) constructed from \( g \) and \( \mathcal{O} \).

There are several (related but different) notions of quantizations. In this paper we are mostly going to deal with Deformation quantization in the algebro-geometric setting. So our quantization \( \mathcal{D} \) of \( \mathcal{O} \) will be a formal deformation of the structure sheaf \( \mathcal{O}_\mathcal{O} \) of Poisson algebras. We will require quantizations to be compatible with \( G \) and \( \mathbb{K}^\times \)-actions on \( \mathcal{O} \), the \( \mathbb{K}^\times \)-action on \( \mathcal{O} \) being induced from the usual action on \( g \) by multiplications by scalars. All necessary definitions will be given in Subsection 3.1. Apart from nilpotent orbits themselves we will also consider their \( G \)-equivariant coverings. For a \( G \)-equivariant covering \( X \) of \( \mathcal{O} \) we denote the set of isomorphism classes of quantizations (satisfying the conditions mentioned above) of \( \mathcal{O} \) by \( \mathcal{Q}(X) \).

Deformation quantization has been studied extensively starting from the seminal paper [BFFLS] mostly in the \( C^\infty \)-setting. Some work was done in the algebraic setting, see, e.g., [K], [BK], [Y]. In particular, in [BK] it was shown that a symplectic algebraic variety \( X \) can be quantized provided it satisfies some vanishing-like condition on the cohomology groups \( H^i(X, \mathcal{O}_X) \). Unfortunately, nilpotent orbits and their coverings are very far from satisfying these conditions.

Key words: Nilpotent orbits, Deformation quantization, Dixmier algebras, W-algebras, 1-dimensional modules.

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Supported by the NSF grant DMS-0900907.

Address: MIT, Dept. of Math., 77 Massachusetts Avenue, Cambridge MA02139, USA.

E-mail: ivanlosev@math.mit.edu.

The second object we are interested in is 1-dimensional modules over a W-algebra \( \mathcal{W} \). Here we give a very brief reminder on W-algebras, more details will be given in Subsection 4.1.

Choose an element \( e \in \mathfrak{O} \) and an \( \mathfrak{sl}_2 \)-triple \( (e, h, f) \). Recall the Slodowy slice \( S := e + \ker \text{ad}(f) \). The algebra \( \mathbb{K}[S] \) of regular functions on \( S \) has a natural grading called the Kazhdan grading. A W-algebra is a particularly nice filtered algebra \( \mathcal{W} \) with \( \text{gr} \mathcal{W} = \mathbb{K}[S] \). One of its features which will be important for us is that the group \( Q := Z_G(e, h, f) \) acts on \( \mathcal{W} \) by automorphisms. Let \( \mathfrak{J}(\mathcal{W}) \) denote the set of two-sided ideals of codimension 1 in \( \mathcal{W} \), this set is naturally identified with the set of isomorphism classes of 1-dimensional \( \mathcal{W} \)-modules). The group \( Q \) acts naturally \( \mathfrak{J}(\mathcal{W}) \). It turns out that this action descends to the component group \( C(e) := Q/Q^0 \).

The main result of this paper is the following theorem.

**Theorem 1.1.** Let \( \tilde{\mathfrak{O}} \) be a \( G \)-equivariant covering of \( \mathfrak{O} \). Pick a point \( x \in \tilde{\mathfrak{O}} \) lying over \( e \) and set \( \Gamma = G_x/(G_x)^0 \). Then \( \mathcal{Q}(\tilde{\mathfrak{O}}) \) is in bijection with the set \( \mathfrak{J}(\mathcal{W})^\Gamma \) of \( \Gamma \)-fixed points in \( \mathfrak{J}(\mathcal{W}) \).

One-dimensional \( \mathcal{W} \)-modules have been studied extensively during the last few years see [GRU], [Lo1], [Lo3], [Pr2], [Pr3]. We will recall these results in Subsection 4.4.

It turns out that Theorem 1.1 is not essentially new: a closely related result was obtained by Mœglin in [Mo]. She used a different language to speak about quantizations, namely, the language of Dixmier algebras. Also she dealt with so called Whittaker modeles of primitive ideals in \( U(\mathfrak{g}) \) rather than with 1-dimensional \( \mathcal{W} \)-modules. We will explain a relation of Mœglin’s work to ours in Subsection 5.3. Our proof is different from Mœglin’s and (at least, from the author point of view) easier and more natural. It is based on the construction of the jet bundle of a quantization and also on the decomposition theorem from [Lo1]. The latter is a very basic result about the relationship between the universal enveloping algebras and W-algebras.

This paper is organized as follows. We list the notation used in the paper in Section 2. In Section 3 we provide some preliminary results on Deformation quantization. In Section 4 we recall some known results and constructions related to W-algebras. Finally, in Section 5 we prove the main theorem as well as some other related results. In the beginning of each section its content is described in more detail.

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## 2. Notation

\begin{align*}
\hat{\otimes} & \quad \text{the completed tensor product of complete topological vector spaces/ modules.} \\
\text{Aut}(Y) & \quad \text{the automorphism group of an object } Y. \\
\text{Der}(A) & \quad \text{the Lie algebra of derivations of an algebra } A. \\
G_x & \quad \text{the stabilizer of } x \text{ in } G. \\
\text{gr } A & \quad \text{the associated graded vector space of a filtered vector space } A. \\
H^i_{DR}(X) & \quad \text{the } i\text{-th De Rham cohomology of a smooth algebraic variety } X. \\
\mathfrak{J}(A) & \quad \text{the set of all (two-sided) ideals of an algebra } A. \\
M_{\mathfrak{g}-\text{fin}} & \quad \text{the finite part of a } \mathfrak{g}\text{-module } M. \\
N_G(H) & \quad \text{the normalizer of a subgroup } H \text{ in a group } G. 
\end{align*}
Let us explain what we mean by this. Let $G \cdot \cdot \cdot$ be a group acting on $X$. Then $X$ acts on $\mathcal{G}$. For $\mathcal{G} = \mathcal{O}_X$, the structure sheaf of an algebraic variety $X$, the quantization is given by the Rees construction for $\mathcal{O}_X$.

The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the space of global sections of a sheaf $\mathcal{F}$ on $X$. We say that a quantization $\mathcal{D}$ of $\mathcal{O}_X$ is a deformation quantization if $\mathcal{D}$ is a filtered algebra of smooth functions on $X$.

In Subsection 3.1 we recall the notion of a deformation quantization of a symplectic algebraic variety. Also we will define $G$-equivariant and of a graded quantizations and also quantum comoment maps. We finish the subsection recalling some facts about the Fedosov quantization of affine varieties.

In Subsection 3.2 we will recall the notion of jet bundle of a quantization that we learned from [BK]. Then axiomatizing its properties we introduce the notion of a quantum jet bundle, which plays a crucial in the proof of Theorem 1.1.

Finally, in Subsection 3.3 we will prove a result on the existence and the uniqueness of a quantum comoment map which are used in the proof of Theorem 1.1.

3.1. Generalities. Let $X$ be a smooth algebraic variety equipped with a symplectic form $\omega$. Let $\{\cdot, \cdot\}_\omega$ denote the Poisson bracket on the structure sheaf $\mathcal{O}_X$ induced by $\omega$.

Suppose we are given a sheaf $\mathcal{D}$ of $\mathbb{K}[[\hbar]]$-algebras on $X$ together with an isomorphism $\mathcal{D}/\hbar\mathcal{D} \to \mathcal{O}_X$. We suppose that $\mathcal{D}$ is flat over $\mathbb{K}[[\hbar]]$ and is complete and separated in the $\hbar$-adic topology and such that an isomorphism $\theta : \mathcal{D}/\hbar\mathcal{D} \to \mathcal{O}_X$ is fixed. Further, we assume that $[\tilde{a}, \tilde{b}]$ is divisible by $\hbar^2$ for any local sections $\tilde{a}, \tilde{b}$ of $\mathcal{D}$. Then the identification $\mathcal{O}_X \cong \mathcal{D}/\hbar\mathcal{D}$ gives rise to a Poisson bracket on $\mathcal{D}$. Namely, fix $x \in X$ and pick two local sections $a, b$ of $\mathcal{O}_X$ on a neighborhood $U$ of $x$. Shrinking $U$ if necessary, we may assume that $a, b$ can be lifted to sections $\tilde{a}, \tilde{b}$ of $\mathcal{D}$. Then we set $\{a, b\}_D := \frac{1}{\hbar}[\tilde{a}, \tilde{b}] \mod \hbar$. We say that $\mathcal{D}$ (or, more precisely, the pair $(\mathcal{D}, \theta)$) is a quantization of the symplectic variety $X$ if $\{\cdot, \cdot\}_\omega = \{\cdot, \cdot\}_D$.

Usually the definition of a quantization is given in a slightly different way, one requires $[\tilde{a}, \tilde{b}]$ to be divisible by $\hbar$. Given the quantization in that sense we can get a quantization in our sense by replacing $\hbar$ with $\sqrt{\hbar}$. So our notion of quantization is (slightly) more restrictive. We choose our convention because it makes passing from filtered $\mathbb{K}$-algebras to graded $\mathbb{K}[[\hbar]]$-algebras by using the Rees construction more convenient. Peculiarly, we need to use the same convention while working with W-algebras, see [Lo1], [Lo2].

We assume that $X$ comes equipped with two group actions. First, let an algebraic group $G$ act on $X$ by symplectomorphisms. Second, suppose that the one-dimensional torus $\mathbb{K}^\times$ acts on $X$ commuting with the $G$-action such that $t \cdot \omega = t^2 \omega$ for all $t \in \mathbb{K}^\times$.

In this paper we are going to consider graded Hamiltonian $G$-equivariant quantizations. Let us explain what we mean by this. We say that a quantization $\mathcal{D}$ of $X$ is $G$-equivariant if the action of $G$ on $\mathcal{O}_X$ lifts to a $G$-action on $\mathcal{D}$ by algebra automorphisms such that $h$ is $G$-invariant. In particular, the $G$-action on $\mathcal{D}$ gives rise to a Lie algebra homomorphism $\mathfrak{g} \to \text{Der}(\mathcal{D})$, the image of $\xi \in \mathfrak{g}$ under this homomorphism by $\xi_\mathcal{D}$.

Now let us explain what we mean by a Hamiltonian quantization. Recall that a morphism $\mu : X \to \mathfrak{g}^*$ is called a moment map if $\mu$ is $G$-equivariant and has the following property:
for \( \xi \in \mathfrak{g} \) and a local section \( f \) of \( \mathcal{O}_X \) the equality \( \{ \mu^* (\xi), \cdot \} = \xi_X \) holds (here \( \xi_X \) is the derivation of \( \mathcal{O}_X \) induced by the \( G \)-action).

Suppose that \( X \) is equipped with a moment map \( \mu : X \to \mathfrak{g}^* \). A \( G \)-equivariant quantization \( \mathcal{D} \) of \( X \) is called Hamiltonian if it is equipped with a \( G \)-equivariant homomorphism \( \varphi_h : \mathfrak{g} \to \Gamma (X, \mathcal{D}) \) (a quantum comoment map) satisfying \( \frac{1}{\hbar} [\varphi (\xi), \cdot] = \xi_D \) for all \( \xi \in \mathfrak{g} \) and coinciding with \( \mu^* \) modulo \( \hbar \).

In the next section we will see that under some conditions on \( X \) and on \( G \) any \( G \)-equivariant quantization \( \mathcal{D} \) of \( X \) a quantum comoment map exists and is unique.

Now let us define graded Hamiltonian \( G \)-equivariant quantizations.

We say that a \( G \)-equivariant Hamiltonian quantization \( \mathcal{D} \) of \( X \) is graded if \( \mathcal{D} \) is equipped with a \( \mathbb{K}^\times \)-action by algebra automorphisms satisfying the following conditions.

1. \( t, h = th \).
2. The \( \mathbb{K}^\times \)-action commutes with the \( G \)-action.
3. \( t[\varphi_h (\xi)] = t^2 \varphi_h (\xi) \) for all \( t \in \mathbb{K}^\times, \xi \in \mathfrak{g} \).
4. The isomorphism \( \theta : \mathcal{D} / h \mathcal{D} \xrightarrow{\sim} \mathcal{O}_X \) is \( \mathbb{K}^\times \)-equivariant.

Two graded Hamiltonian \( G \)-equivariant quantizations \( (\mathcal{D}_1, \theta_1), (\mathcal{D}_2, \theta_2) \) with quantum comoment maps \( \varphi^1_h, \varphi^2_h \) of \( X \) are said to be isomorphic if there is a \( G \times \mathbb{K}^\times \)-equivariant isomorphism \( \psi : \mathcal{D}_1 \to \mathcal{D}_2 \) of sheaves of \( \mathbb{K}[h] \)-algebras such that \( \theta_1 = \theta_2 \circ \psi, \varphi^1_h = \varphi^2_h \circ \psi \). All graded Hamiltonian \( G \)-equivariant quantizations of \( X \) form a category (in fact, a groupoid).

Until a further notice \( G \) denotes a simply connected semisimple algebraic group. We identify \( \mathfrak{g} \) with \( \mathfrak{g}^* \) using the Killing form. Let us provide a class of examples of algebraic varieties equipped with the structures described above. A variety \( X \) we are going to consider will be either \( \mathbb{O} \) or, more generally, some \( G \)-equivariant covering of \( \mathbb{O} \).

Let \( \eta : X \to \mathbb{O} \) be the projection. Then \( \omega := \eta^* \omega_{KK} \), where \( \omega_{KK} \) stands for the Kostant-Kirillov form on \( \mathbb{O} \), is a symplectic form on \( X \). A natural \( G \)-action on \( X \) preserves \( \omega \). The composition of \( \eta \) with the inclusion \( \mathbb{O} \hookrightarrow \mathfrak{g}^* \) is a moment map.

Next, let us introduce a \( \mathbb{K}^\times \)-action on \( X \) by \( G \)-equivariant automorphisms. In other words, we need a homomorphism from \( \mathbb{K}^\times \) to the group \( \text{Aut}^G (X) \) of \( G \)-equivariant automorphisms of \( X \). The latter is naturally identified with \( N_G (H) / H \), where \( H \) is the stabilizer of some point \( x \in X \).

Set \( e := \eta (x) \). This is a nilpotent element in \( \mathfrak{g} \). Fix an \( \mathfrak{sl}_2 \)-triple \((e, h, f)\). Let \( \gamma : \mathbb{K}^\times \to G \) denote the composition of the homomorphism \( \text{SL}_2 \to G \) induced by the \( \mathfrak{sl}_2 \)-triple and of the embedding \( \mathbb{K}^\times \hookrightarrow \text{SL}_2, t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \). In particular, \( \gamma (t) \xi = t^i \xi \) provided \([h, \xi] = i \xi \) for \( \xi \in \mathfrak{g}, i \in \mathbb{Z} \). We remark that the image of \( \gamma \) lies in \( N_G (Z_G (e)) \). Moreover, \( \gamma (t) \) normalizes \( H \). Indeed, \( \gamma (t) \) normalizes \( Z_G (e) \) and hence \( Z_G (e)^0 \). This gives rise to a homomorphism \( \mathbb{K}^\times \to \text{Aut} (C(e)) \). But \( \mathbb{K}^\times \) is connected and so the last homomorphism is trivial. It follows that \( \gamma (t) \) normalizes every subgroup between \( Z_G (e)^0 \) and \( Z_G (e) \), in particular, \( H \).

So we can consider the composition \( \gamma : \mathbb{K}^\times \to N_G (H) \to N_G (H) / H \). Define the action of \( \mathbb{K}^\times \) on \( X \) by means of \( \gamma^{-1} \). The moment map \( \mu : X \to \mathfrak{g}^* \) becomes \( \mathbb{K}^\times \)-equivariant, where the action of \( \mathbb{K}^\times \) on \( \mathfrak{g}^* \) is given by \( (t, \alpha) \mapsto t^{-2} \alpha \). We remark that the most obvious \( \mathbb{K}^\times \)-action on \( \mathfrak{g}^* \), \((t, \alpha) \mapsto t^{-1} \alpha \), in general, cannot be lifted to \( X \) : one can take \( \mathfrak{g} = \mathfrak{sl}_2 \) and the universal covering of the principal orbit to get a counter-example.

We finish the subsection recalling the Fedosov construction of quantizations. We are interested in the situation when the symplectic variety \( X \) in interest is affine. In this case a quantization \( \mathcal{D} \) of \( X \) is isomorphic to \( \mathcal{O}_X [[h]] \) as a sheaf of \( \mathbb{K}[[h]] \)-modules. Moreover,
the algebra structure on $\mathcal{D}$ is uniquely recovered from the algebra structure on the space $\Gamma(X, \mathcal{D}) = \mathbb{K}[X][[\hbar]]$ of global sections of $\mathcal{D}$. These claimed are proved analogously to Remarks 1.6,1.7 in [BK]. We note that their setting is a conventional one: they have $\hbar$ while we have $\hbar^2$. Still, the arguments extend easily to our setting. This remark also hold for the results recalled below.

The product on $\mathbb{K}[X][[\hbar]] = \Gamma(X, \mathcal{D})$ is usually called a star-product. Fedosov, [F1],[F2], constructed a star-product in the $C^\infty$-setting starting from a symplectic connection on the tangent bundle and a closed 2-form $\Omega = \sum_{i=0}^{\infty} \omega_i \hbar^i$ with $\omega_0 = \omega$ (the curvature form of a star-product). The same construction works in the algebraic setting provided a symplectic connection exists that is always the case for affine varieties, see, for example, [Lo1]. Also if a reductive group $G$ acts on $X$ by symplectomorphisms, then one can choose a $G$-invariant symplectic connection. The Fedosov construction implies that the star-product produced from a $G$-invariant symplectic connection and a $G$-invariant curvature form is $G$-equivariant.

It turns out that any star-product on $X$ is equivalent to a Fedosov one (meaning that the corresponding quantizations are equivalent), and two Fedosov star-products constructed from curvature forms $\Omega_1, \Omega_2$ are equivalent if and only if $\Omega_1 - \Omega_2$ is exact. The first claim follows, for example, from results of Kaledin and Bezrukavnikov, [BK], the second one was proved by Fedosov in [F2]. In the $G$-equivariant setting (when a star-product is $G$-equivariant/ the curvature forms are $G$-invariant) and equivalence can also be made $G$-equivariant.

In Subsection 3.3 we will need an existence criterium for a quantum comoment map. Such a criterium was obtained by Gutt and Rawnsley in [GR2], Theorem 6.2. Again, their proof transfers to the algebraic setting (and to our definition of a star-product) directly.

**Proposition 3.1.** Let $G$ be a reductive group acting on $X$ by symplectomorphisms. Construct a star-product on $X$ starting from a $G$-invariant symplectic connection and a $G$-invariant curvature form $\Omega$. Then the quantum comoment map for the $G$-action exists if and only if $i_{\xi_x} \Omega$ is exact for all $\xi \in \mathfrak{g}$ ($i_\cdot$ stands for the contraction). Moreover, if the forms are exact, then a $G$-equivariant linear map $\varphi_h : \mathfrak{g} \to \mathbb{K}[X][[\hbar]]$ is a quantum comoment map if and only if $d\varphi_h(\xi) = i_{\xi_x} \Omega$.

**3.2. Quantum jet bundles.** In the proof of Theorem 1.1 an important role is played by the jet bundles of quantizations, compare with [BK]. The material of this subsection should be pretty standard.

We start by recalling the jet bundle $J^\infty \mathcal{O}_X$ of a smooth variety $X$. Let $\pi_1, \pi_2$ denote the projections $X \times X \to X$ to the first and to the second factor. Consider the completion $\mathcal{O}_{X \times X}'$ of $\mathcal{O}_{X \times X} = \pi_2^*(\mathcal{O}_X)$ with respect to the ideal $I_\Delta$ of the diagonal $X \hookrightarrow X \times X$, i.e., $\mathcal{O}_{X \times X}' := \lim_{k \to \infty} \mathcal{O}_{X \times X}/I_\Delta^k$. By definition, $J^\infty \mathcal{O}_X := \pi_1^*(\mathcal{O}_{X \times X}')$. This is a pro-coherent sheaf of $\mathcal{O}_X$-algebras. The jet bundle $J^\infty \mathcal{O}_X$ comes equipped with a flat connection $\nabla$ defined as follows. Pick a vector field $\xi$ on $X$. Define the connection $\nabla$ on $\pi_1^*(\mathcal{O}_{X \times X})$ by $\nabla_\xi(f \otimes g) = (\xi(f) \otimes g) + (\xi(g) \otimes f)$. This connection can be uniquely extended to a continuous (with respect to the $I_\Delta$-adic topology) connection on $J^\infty \mathcal{O}_X$, which is a connection we need. The sheaf of flat sections of $J^\infty \mathcal{O}_X$ is naturally identified with $\mathcal{O}_X$ (or, more precisely, with $\pi_1^*(\pi_2^{-1}(\mathcal{O}_X))$, where $\pi_2^{-1}$ denotes the sheaf-theoretic pull-back). Any fiber of $J^\infty \mathcal{O}_X$ is (non-canonically) identified with the algebra of formal power series in $\dim X$ variables.

Suppose now that $X$ is a symplectic variety. Then $J^\infty \mathcal{O}_X$ comes equipped with a $\mathcal{O}_X$-linear Poisson bracket (extended by continouity from the natural $\pi_1^{-1}(\mathcal{O}_X)$-linear Poisson bracket on $\mathcal{O}_{X \times X}$). The induced bracket on $\mathcal{O}_X$ (considered as the space of flat sections in $J^\infty \mathcal{O}_X$) coincides with the initial one. Any fiber of $J^\infty \mathcal{O}_X$ is isomorphic (as a Poisson algebra) to...
the algebra $A := \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, $n := \frac{1}{2} \dim X$, where a Poisson bracket is defined by $\{x_i, x_j\} = \{y_i, y_j\} = 0$, $\{x_i, y_j\} := \delta_{ij}$.

Now let us define the notion of a \textit{quantum jet bundle}. By definition, a quantum jet bundle on $X$ is a triple $(\mathfrak{D}, \nabla, \Theta)$ consisting of

- a pro-coherent sheaf $\mathfrak{D}$ of $\mathcal{O}_X[[h]]$-algebras such that $[\mathfrak{D}, \mathfrak{D}] \subset \hbar^2 \mathfrak{D}$,
- a flat $\mathbb{K}[[h]]$-linear connection $\nabla$ on $\mathfrak{D}$ that is compatible with the algebra structure in the sense that $\nabla_{\xi}$ is a derivation of $\mathfrak{D}$ for any vector field $\xi$ on $X$,
- and an isomorphism $\Theta : \mathfrak{D}/h\mathfrak{D} \to J^\infty \mathcal{O}_X$ of flat sheaves of Poisson algebras ("flat" means the $\Theta$ intertwines the connections; the bracket on $\mathfrak{D}/h\mathfrak{D}$ is defined as on $\mathfrak{D}$, see the previous subsection).

By a pro-coherent sheaf we mean an inverse limit of coherent sheaves. Any fiber of $\mathfrak{D}$ is a quantization of the Poisson algebra $A$ isomorphic to the formal Weyl algebra, i.e., the space $D_X$ is a quantization of the Poisson algebra $A$.

An isomorphism of two quantum jet bundles $(\mathfrak{D}_1, \nabla_1, \Theta_1), (\mathfrak{D}_2, \nabla_2, \Theta_2)$ is an isomorphism $\Psi : \mathfrak{D}_1 \to \mathfrak{D}_2$ of $\mathcal{O}_X[[h]]$-algebras intertwining the connections, and such that $\Theta_1 = \Theta_2 \circ \varphi$.

Now let $\mathfrak{D}$ be a quantization of $X$. Then one can define the jet bundle $J^\infty \mathfrak{D}$ of $\mathfrak{D}$ (see [BK], Definition 1.4) similarly to $J^\infty \mathcal{O}_X$. Namely consider the bundle $\mathcal{O}_X \otimes \mathfrak{D}$ on $X \times X$. There is a natural projection (the quotient by $h$) $\mathcal{O}_X \otimes \mathfrak{D} \to \mathcal{O}_X \otimes \mathcal{O}_X$. Let $I_\Delta$ denote the inverse image of $I_\Delta$ under this projection. Set $(\mathcal{O}_X \otimes \mathfrak{D})^\wedge := \lim_{\leftarrow k \to \infty} \mathcal{O}_X \otimes \mathfrak{D}/I_\Delta^k$ and $J^\infty \mathfrak{D} := \pi_{1*}(\mathcal{O}_X \otimes \mathfrak{D})^{\wedge}$. The sheaf $J^\infty \mathfrak{D}$ is equipped with a flat connection $\nabla$ defined completely analogously to the connection $\nabla$ above. There is a natural isomorphism $\Theta : J^\infty \mathfrak{D}/hJ^\infty \mathfrak{D} \to J^\infty \mathcal{O}_X$. It is easy to see that $(J^\infty \mathfrak{D}, \nabla, \Theta)$ is a quantum jet bundle in the sense of the above definition. The sheaf of flat sections of $J^\infty \mathfrak{D}$ is $\pi_{1*}(\pi_{2}^{-1}(\mathfrak{D}))$.

Conversely, let $(\mathfrak{D}, \nabla, \Theta)$ be a quantum jet bundle. Consider the sheaf $\mathfrak{D} := \mathfrak{D}/\nabla$ of flat sheaves of $\mathfrak{D}$. The isomorphism $\Theta : \mathfrak{D}/h\mathfrak{D} \to J^\infty \mathcal{O}_X$ restricts to an embedding $\theta : \mathfrak{D}/h\mathfrak{D} \to (J^\infty \mathcal{O}_X)^\nabla = \mathcal{O}_X$. It is not difficult to show that $\theta$ is also surjective. So $(\mathfrak{D}, \theta)$ is a quantization of $X$.

This discussion can be summarized in the following proposition.

**Proposition 3.2.** The assignments

- $\mathfrak{D} \mapsto J^\infty \mathfrak{D}$
- $\mathfrak{D} \mapsto \mathfrak{D}/\nabla$

define mutually quasi-inverse equivalences between the category of quantizations and the category of quantum jet bundles on $X$.

We need a ramification of this proposition covering graded Hamiltonian $G$-equivariant quantizations. So we need the notion of a graded Hamiltonian $G$-equivariant quantum jet bundle.

Let $G$ be an algebraic group. Suppose $G, \mathbb{K}^\times$ act on $X$ as in the previous subsection. Clearly, a $G$-action induces an action of $G$ on $J^\infty \mathcal{O}_X$ (induced by the diagonal $G$-action on $X \times X$). This action preserves the connection and the algebra structure. We say that a quantum jet bundle $(\mathfrak{D}, \nabla, \Theta)$ is $G$-equivariant if $\mathfrak{D}$ is equipped with a $G$-action such that $h$ and $\nabla$ are $G$-invariant and $\Theta : \mathfrak{D}/h\mathfrak{D} \to J^\infty \mathcal{O}_X$ is $G$-equivariant. The $G$-action gives rise to the homomorphism $\xi \mapsto \xi, g \mapsto \text{Der}(\mathfrak{D})$, of Lie algebras.
For example, let $\mathcal{D}$ be a $G$-equivariant quantization of $\mathcal{O}_X$. The diagonal $G$-action on $\mathcal{O}_X \otimes \mathcal{D}$ extends to $(\mathcal{O}_X \otimes \mathcal{D})^\wedge$ and so gives rise to a $G$-action on $J^\infty \mathcal{D}$ making $J^\infty \mathcal{D}$ a $G$-equivariant quantum jet bundle.

Now let us introduce the notion of a Hamiltonian $G$-equivariant quantum jet bundle. Suppose that there is a moment map $\pi: \tilde{X} \to \mathfrak{g}^*$ for the $G$-action. For $\xi \in \mathfrak{g}$ set $\Phi(\xi) = \pi^*(\mu^*(\xi)) \in \Gamma(X, J^\infty \mathcal{O}_X)^\nabla$. We have the maps $\xi \mapsto \xi_{x \times x}, \xi_{x \times x}, \xi^2_{x \times x}: \mathfrak{g} \to \text{Der}(\mathcal{O}_x \times x)$ associated with the diagonal $G$-action and the actions of $G$ on the first and on the second copy of $X$, respectively. Clearly, $\xi_{x \times x} = \xi^{1}_{x \times x} + \xi^{2}_{x \times x}$. Therefore for the induced derivation $\xi_{x \times x}$ of the jet bundle we have $\xi_{x \times x} = \nabla \xi_x + \{\mu^*(\xi), \cdot\}$. We say that a $G$-equivariant quantum jet bundle $(\mathcal{D}, \nabla, \Theta)$ is Hamiltonian if it is equipped with a map $\Phi_h: \mathfrak{g} \to \Gamma(X, \mathcal{D})^\nabla$ such that

\begin{equation}
\xi_\mathcal{D} = \nabla \xi_x + \frac{1}{\hbar^2} [\Phi_h(\xi), \cdot].
\end{equation}

and $\Theta(\Phi_h(\xi)) = \Phi(\xi)$. Now if $(\mathcal{D}, \theta)$ is a Hamiltonian $G$-equivariant quantization with a quantum comoment map $\varphi_h$, then $J^\infty \mathcal{D}$ is Hamiltonian with $\Phi_h(\xi) = \varphi_h(\xi) \in \Gamma(X, J^\infty \mathcal{D})^\nabla$.

The quantum comoment map $\Phi_h$ extends to a certain sheaf homomorphism which will be important in the sequel. Namely, define the graded universal enveloping algebra $U_h$ of $\mathfrak{g}$ as the quotient of $T(\mathfrak{g})[h]$ by the relations $\xi \otimes \eta - \eta \otimes \xi - h^2 [\xi, \eta], \xi, \eta \in \mathfrak{g}$. Then $\Phi_h$ extends to an algebra homomorphism $U_h \to \Gamma(X, \mathcal{D}^\nabla)$. Extend $\Phi_h$ to a homomorphism $\mathcal{O}_X \otimes U_h \to \mathcal{D}$ by $\mathcal{O}_X$-linearity. Further, we have a homomorphism

\[ \mathcal{O}_X \otimes U_h/h\mathcal{O}_X \otimes U_h \to \mathcal{O}_X \otimes \mathbb{K}[X] \]

given by $f \otimes x \mapsto f \otimes \mu^*(x)$, where $f$ is a local section of $\mathcal{O}_X$ and $x \in S(\mathfrak{g}) = U_h/hU_h$. Let $I_{\mu, \Delta}$ denote the inverse image of $I_{\Delta}$ in $\mathcal{O}_X \otimes U_h/h\mathcal{O}_X \otimes U_h$ and $\tilde{I}_{\mu, \Delta}$ be the inverse image of $I_{\mu, \Delta}$ in $\mathcal{O}_X \otimes U_h$. Consider the completion

\[ J^\infty U_h := \lim_{k \to \infty} \mathcal{O}_X \otimes U_h/(\mathcal{O}_X \otimes U_h)\tilde{I}_{\mu, \Delta} \]

The homomorphism $\Phi_h: \mathcal{O}_X \otimes U_h \to \mathcal{D}$ is continuous in the $\tilde{I}_{\mu, \Delta}$-adic topology and so extends to a continuous homomorphism $J^\infty U_h \to \mathcal{D}$ (also denoted by $\Phi_h$) in a unique way. We remark that $J^\infty U_h$ comes equipped with a natural connection and the homomorphism $\Phi_h$ intertwines the connections.

Finally, let us define graded Hamiltonian $G$-equivariant quantum jet bundles. We have a natural $\mathbb{K}^\times$-action on $J^\infty \mathcal{O}_X$. We say that a Hamiltonian $G$-equivariant quantum jet bundle $(\mathcal{D}, \nabla, \Theta)$ is graded, if $\mathcal{D}$ is equipped with a $\mathbb{K}^\times$-action by algebra automorphisms such that

- the action commutes with $G$,
- $t.\hbar = \hbar t, t.\Phi_h(\xi) = t^2 \Phi_h(\xi)$,
- $\Theta$ is $\mathbb{K}^\times$-equivariant and $\nabla$ is $\mathbb{K}^\times$-invariant.

If $\mathcal{D}$ is graded, then $J^\infty \mathcal{D}$ has a natural $\mathbb{K}^\times$-action and is graded with respect to this action.

The previous discussion implies the following corollary of Proposition 3.2.

**Corollary 3.3.** The equivalences of Proposition 3.2 define mutually inverse equivalences between

- the category of graded Hamiltonian $G$-equivariant quantizations and
- the category of graded Hamiltonian $G$-equivariant quantum jet bundles.
3.3. Existence of quantum comoment map. In this subsection we show that under some conditions (satisfied for all coverings of $\mathcal{O}$) any $G$-equivariant quantization possesses a unique quantum comoment map.

The main result is the following proposition.

**Proposition 3.4.** Let $G$ be a semisimple algebraic group and $X$ be a symplectic variety. Let $G$ act on $X$ by symplectomorphisms. Suppose $H^1_{DR}(X) = \{0\}$. Let $\mathcal{D}$ be a $G$-equivariant quantization of $X$. Then there exists a unique quantum comoment map $\varphi_h : \mathfrak{g} \to \Gamma(X, \mathcal{D})$.

**Proof.** We remark that this is a standard fact that a moment map for the action of $G$ on $X$ exists and is unique.

Now let us check that there is a linear map $\varphi'_h : \mathfrak{g} \to \Gamma(X, \mathcal{D})$ such that $\frac{1}{\sqrt{\hbar}}[\varphi'_h(\xi), \cdot] = \xi_D$. Then for $\varphi_h$ we can take the $G$-invariant component of $\varphi'_h$.

It is enough to show that an element $f \in \Gamma(X, \mathcal{D})$ with $\frac{1}{\sqrt{\hbar}}[f, \cdot] = \xi_D$ exists for any element $\xi \in \mathfrak{g}$ lying in the Lie algebra of a one-dimensional torus, say $T$, of $G$.

There is a $T$-stable open affine covering $X = \bigcup_i X_i$. The restriction of $\mathcal{D}$ to $X_i$ is, of course, a $T$-equivariant quantization of $X_i$. Therefore the restriction is isomorphic to the Fedosov quantization of $X_i$ with a $T$-invariant curvature form, say, $\Omega_i$. Recall that for $f_i \in \mathbb{K}[X_i][[\hbar]]$ the condition $\frac{1}{\sqrt{\hbar}}[f_i, \cdot] = \xi_{X_i}$ is equivalent to $df_i = \iota_{\xi_{X_i}}\Omega_i$, see Proposition 3.1.

The restrictions of $\Omega_i, \Omega_j$ to $X_i \cap X_j$ have the same cohomology class. So one can find $T$-invariant 1-forms $\alpha_{ij}$ on $X_i \cap X_j$ such that $d\alpha_{ij} = \Omega_i - \Omega_j$ and $\alpha_{ij} = -\alpha_{ji}$. Set $\beta_i := \iota_{\xi_{X_i}}\Omega_i, g_{ij} := \iota_{\xi_{X_i}}\alpha_{ij}$. Let us check that there are $h_{ij} \in \mathbb{K}$ such that $\beta_i, g_{ij} + h_{ij}$ form a De Rham 1-cocycle.

The equalities $d\beta_i = 0, dg_{ij} = \beta_i - \beta_j$ follow from the condition that $\Omega_i, \alpha_{ij}$ are $T$-invariant and so vanish under the Lie derivative of $\xi$. It remains to check that $h_{ijk} := g_{ij} + g_{jk} + g_{ki} = 0$. Remark that $dh_{ijk} = 0$. So $h_{ijk}$ is a 2-cocycle on $X$ (in the Zariski topology) with coefficients in $\mathbb{K}$. Such a cocycle is a coboundary (from the irreducibility of $X$): there are constants $h_{ij}$ with $h_{ijk} = h_{ij} + h_{jk} + h_{ki}$. These are constants we need.

Since $H^1_{DR}(X) = \{0\}$, we see that $\beta_i, g_{ij} + h_{ij}$ is a coboundary and the existence of $f_i$ follows.

Abusing the notation we denote the element of $\Gamma(X_i, \mathcal{D})$ corresponding to $f_i$ again by $f_i$. We remark that $f_i - f_j$ lies in the center of $\Gamma(X_i \cap X_j, \mathcal{D})$. The latter coincides with $\mathbb{K}[[\hbar]]$. So $f_i - f_j$ form a 1-cocycle with coefficients in $\mathbb{K}[[\hbar]]$. So we can add elements of $\mathbb{K}[[\hbar]]$ to the $f_i$’s to glue $f_i$ into a global section $f$ of $\mathcal{D}$.

We have just proved the existence of a quantum comoment map. As in the classical setting the uniqueness follows from the fact that $\mathfrak{g}$ has no nontrivial central extensions. □

**Corollary 3.5.** We preserve the conventions of Proposition 3.4 Suppose $\mathbb{K}^\times$ acts on $X$ as in Subsection 3.1 and $\mathcal{D}$ is graded. Then the quantum comoment map $\varphi_h : \mathfrak{g} \to \Gamma(X, \mathcal{D})$ satisfies $t^2\varphi_h(\xi) = t^2\varphi(\hbar)$.

**Proof.** We remark that $\xi \mapsto t^{-2}(t\varphi_h(\xi))$ is again a quantum comoment map. Now the claim follows from the uniqueness of the quantum comoment map. □

Now let $X$ be a covering of a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$. Since $X$ is a homogeneous space of a simply connected semisimple algebraic group, we have $H^1_{DR}(X) = \{0\}$. So any $G$-equivariant quantization of $X$ has a unique quantum comoment map.
4. W-algebras

In this section we recall some results about W-algebras. In Subsection 4.1 we define a W-algebra \( \mathcal{W} \) following [GG] (the definition given there is very close to the original definition of Premet, [Pr1]) and recall a category equivalence proved by Skryabin in the appendix to [Pr1].

Subsection 4.2 is devoted to a basic result on W-algebras from [Lo1], the so called decomposition theorem. It says that a certain completion \( \mathcal{U}_h^\wedge \) of the algebra \( \mathcal{U}_h \) decomposes into the completed tensor product of the completed W-algebra and of a formal Weyl algebra. In Subsection 4.3 we use this result to establish a correspondence between the sets of ideals in \( \mathcal{W}, \mathcal{U}, \mathcal{U}_h^\wedge \) also originally obtained in [Lo1].

Finally, in Subsection 4.4 we recall some known results on 1-dimensional \( \mathcal{W} \)-modules.

4.1. Generalities. In this subsection we will sketch a definition of a W-algebra associated to \((\mathfrak{g}, \mathbb{O})\) (due to Premet, [Pr1]).

Let \( e \in \mathbb{O} \). Recall an \( \mathfrak{sl}_2 \)-triple \((e, h, f)\), the subgroup \( Q := Z_G(e, h, f) \) and a homomorphism \( \gamma : \mathbb{K}^\times \to G \) considered in Subsection 3.1.

A W-algebra \( \mathcal{W} \) can be defined as a quantum Hamiltonian reduction

\[
(\mathcal{U}/\mathcal{U}m_\chi)^{\text{ad}m} := \{ a + \mathcal{U}m_\chi | [\xi, a] \in \mathcal{U}m_\chi, \forall \xi \in m \},
\]

where \( m \subset \mathfrak{g} \) is a subalgebra, \( \chi : m \to \mathbb{K} \) is a character, both to be specified below, \( m_\chi := \{ \xi - \langle \chi, \xi \rangle, \xi \in m \} \).

The subalgebra \( m \subset \mathfrak{g} \) and the character \( \chi \) are constructed as follows. Consider the grading \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \), where \( \mathfrak{g}(i) := \{ [h, \xi] = i\xi \} \). Set \( \chi := (e, \cdot) \in \mathfrak{g}^* \) and consider the skew-symmetric form \( \omega_\chi (\xi, \eta) = \langle \chi, [\xi, \eta] \rangle \) on \( \mathfrak{g} \). The restriction of \( \omega_\chi \) to \( \mathfrak{g}(-1) \) is non-degenerate. Pick a lagrangian subspace \( l \subset \mathfrak{g}(-1) \) and set \( m := l + \bigoplus_{i \leq -2} \mathfrak{g}(i) \). The restriction of \( \chi \) to \( m \) is a character of \( m \).

The algebra \( \mathcal{W} \) has a nice filtration called the Kazhdan filtration. It is inherited from a Kazhdan filtration on \( \mathcal{U} \) defined as follows. Let \( F_k^s \mathcal{U} \) denote the standard PBW filtration on \( \mathcal{U} \) and \( \mathcal{U}(j) := \{ u \in \mathcal{U} | [h, u] = ju \} \). Then the Kazhdan filtration \( F_i \mathcal{U} \) on \( \mathcal{U} \) is defined by \( F_i \mathcal{U} := \sum_{j+k \leq i} F_k^s \mathcal{U} \cap \mathcal{U}(j) \). We have the induced filtrations \( F_i(\mathcal{U}/\mathcal{U}m_\chi), F_i \mathcal{W} \) on \( \mathcal{U}/\mathcal{U}m_\chi, \mathcal{W} \), respectively. We remark that \( F_0(\mathcal{U}/\mathcal{U}m_\chi) \) is spanned by the image of \( 1 \in \mathcal{U} \). It follows that \( F_0 \mathcal{W} \) is spanned by the unit of \( \mathcal{W} \).

The associated graded algebra of \( \mathcal{W} \) has the following nice description. Define a Slodowy slice \( S \) by \( S := e + \mathfrak{g}(f) \). This is an affine subspace in \( \mathfrak{g} \) but it will be more convenient for us to consider it as a subspace as in \( \mathfrak{g}^* \). Consider a Kazhdan \( \mathbb{K}^\times \)-action on \( \mathfrak{g}^* \) defined by \( t.\alpha = t^{-2}\gamma(t)\alpha \), where \( \gamma : \mathbb{K}^\times \to G \) was defined in Subsection 3.1. We remark that \( \chi \) is \( \mathbb{K}^\times \)-invariant and that \( \mathbb{K}^\times \) preserves \( S \). Moreover, the \( \mathbb{K}^\times \)-action on \( S \) is contracting: \( \lim_{t \to \infty} t.s = \chi \) for any \( s \in S \). So \( \mathbb{K}[S] \) comes equipped with a positive grading \( \mathbb{K}[S] = \bigoplus_{i \geq 0} \mathbb{K}[S]_i \) with \( \mathbb{K}[S]_0 = \mathbb{K} \). As Premet proved in [Pr1], \( \mathfrak{g} \mathcal{W} \cong \mathbb{K}[S] \).

The \( S(\mathfrak{g}) \)-module \( \text{gr}\mathcal{U}/\mathcal{U}m_\chi \) also has a nice description, see [GG]. Namely, a natural homomorphism \( S(\mathfrak{g})/S(\mathfrak{g})m_\chi \to \text{gr}\mathcal{U}/\mathcal{U}m_\chi \) is a bijection. Also, as was shown by Gan and Ginzburg, there is a natural identification \( S(\mathfrak{g})/S(\mathfrak{g})m_\chi \cong \mathbb{K}[M] \otimes \mathbb{K}[S] \), where \( M \) is the unipotent subgroup of \( G \) with Lie algebra \( m \). This identification preserves the gradings, where the grading on \( \mathbb{K}[M] \) is induced by the \( \mathbb{K}^\times \)-action \( (t, m) \mapsto \gamma(t)m\gamma(t)^{-1}, t \in \mathbb{K}^\times, m \in M \).

To finish the subsection let us recall the Skryabin equivalence, see the appendix to [Pr1]. This is an equivalence between the category \( \mathcal{W}\text{-Mod} \) of left \( \mathcal{W} \)-modules and the category \( \text{Wh} \) of Whittaker \( \mathcal{U} \)-modules. By definition, a left \( \mathcal{U} \)-module \( M \) is called Whittaker if \( m_\chi \)
acts on $M$ by locally nilpotent endomorphisms. According to Skryabin, the functor $N \mapsto (U/Um) \otimes W N$ is an equivalence between the categories $W$-Mod and Wh. A quasiinverse equivalence sends $M \in Wh$ to the space $M^{m_x}$ of $m_x$-invariants.

Now let $N$ be a finitely generated $W$-module. Equip $N$ with a filtration that is compatible with the Kazhdan filtration on $W$. Assume in addition that $gr N$ is a finitely generated $K[S]$-module. Then one can equip $U/Um \otimes W N$ with the product filtration. The following result was obtained in the proof of Theorem 6.1 in [GG].

**Lemma 4.1.** The natural homomorphism
\[ K[M] \otimes gr N = (gr U/Um) \otimes K[S] gr N \to gr ((U/Um) \otimes W N) \]

is an isomorphism.

4.2. **Decomposition theorem.** The most crucial property of $W$-algebras we need is the decomposition theorem, see [Lo1], Subsection 3.3, and [Lo2], Subsection 2.3. This theorem asserts that, up to a suitably understood completion, the universal enveloping algebra $U$ of $\mathfrak{g}$ is decomposed into the tensor product of the $W$-algebra and of certain Weyl algebra. It will be convenient for us to work with “homogeneous” versions of our algebras.

Equip $U$ with the ”doubled” standard filtration $F_i U$, where $F_i U$ is spanned by all monomials $\xi_1 \ldots \xi_k$ with $2k \leq i$. Then form the Rees algebra $R_\ell(U) = \bigoplus_{i \geq 0} \ell^i F_i U \subset U[\ell]$. This Rees algebra is naturally isomorphic to the algebra $U_\ell$ introduced in Subsection 3.2. We have a natural identification $U_\ell / h U_\ell = S g = K[g^*]$. Let $I_\chi$ denote the maximal ideal of $\chi$ in $S g$ and $\tilde{I}_\chi$ be its preimage under the natural projection $U_\ell \to S g$.

A completion of $U_\ell$ we need is $U_\ell^\wedge := \varprojlim \}_{k \to \infty} U/I_\ell^k$. As we have seen in [Lo2], Subsection 2.4, $U_\ell^\wedge$ can be considered as the space $K[g^*/I_\ell^k] \otimes_{K[g]} K[\ell]$ equipped with a new (deformed) product, here $K[g^*/I_\ell^k] \otimes_{K[g]} K[\ell]$ is the algebra of formal power series in the neighborhood of $\chi$.

We will need two group actions on $U_\ell^\wedge$. Let $Q := Z_G(e,h,f)$ be the centralizer of $(e,h,f)$ in $G$. There is a natural $Q$-action on $U_\ell$ by graded algebra automorphisms. This action stabilizes $I_\ell$ and so uniquely extends to a $Q$-action on $U_\ell^\wedge$ by topological algebra automorphisms. Also there is a Kazhdan action of $K^\times$ on $U_\ell^\wedge$ defined as follows: an element $t \in K^\times$ acts on $h^i F_i U$ via $(t,u) \mapsto t^i \gamma(t) u$. This action fixes $\tilde{I}_\chi$ and again lifts to $U_\ell^\wedge$.

Next, consider the homogeneous version $W_\ell = R_\ell(W)$ of the $W$-algebra. Similarly to the previous paragraph, define the completion $W_\ell^\wedge$ of $W_\ell$ with respect to the maximal ideal of $\chi$ in $S$. There is a $K^\times$-action on $W_\ell$ given by $t h^i w = t^i h^i w$ that can again be naturally extended to $W_\ell^\wedge$. An important difference of this action from that on $U_\ell^\wedge$ is that $W_\ell^\wedge$ is isomorphic to the subalgebra of $W_\ell^\wedge$ consisting of all $K^\times$-finite (locally finite in the terminology of [Lo2]) elements.

The group $Q$ also acts on $W_\ell^\wedge$ as follows. There is a $Q$-action on $W$ by filtered algebra automorphisms, see [Pr2], 1.2 or [Lo1], Subsections 3.1, 3.3. This action gives rise to a $Q$-action on $W_\ell$ and the latter extends to $W_\ell^\wedge$.

Finally, we need a completed version of the Weyl algebra of an appropriate symplectic vector space. A vector space we need is $V := \im(\ad(f))$. The restriction of $\omega_x$ to $V$ is non-degenerate. We equip $V$ with the $Q$-action restricted from the coadjoint $G$-action and with the $K^\times$-action given by $t v = \gamma(t)^{-1} v$. In particular, $t \in K^\times$ multiplies $\omega_x$ by $t^2$.

By definition, the homogeneous Weyl algebra $A_\ell$ of $V$ is the quotient of $T(V)[h]$ by $u \otimes v - v \otimes u = h^2 \omega_x(u,v), u, v \in V$. Let $A_\ell^\wedge$ denote the completion of $A_\ell$ at 0.

The algebras $U_\ell^\wedge, W_\ell^\wedge, A_\ell^\wedge$ have natural topologies: the topologies of inverse limits.
Consider the completed tensor product $A_h^\wedge(W_h^\wedge) := A_h^\wedge \otimes_{K[[\hbar]]} W_h^\wedge$ (i.e., we first take the usual tensor product of the topological algebras and then complete it with respect to the induced topology). The decomposition theorem is the following statement.

**Proposition 4.2.** There is a $Q \times K^\times$-equivariant isomorphism $\Psi_h : U_h^\wedge \to A_h^\wedge(W_h^\wedge)$ of topological $K[[\hbar]]$-algebras.

For the proof see [Lo1], Theorem 3.3.1 and the discussion after the theorem.

In the sequel we identify $U_h^\wedge$ with $A_h^\wedge(W_h^\wedge)$ by means of $\Psi_h$.

### 4.3. Correspondence between ideals

We need to relate two-sided ideals in $U_h^\wedge$ and $W$. More precisely, we consider two sets: the set $\mathfrak{I}(W)$ of two-sided ideals in $W$ and the set $\mathfrak{I}_h(U_h^\wedge)$ consisting of all $K^\times$-stable $\hbar$-saturated two-sided ideals in $U_h^\wedge$ (an ideal $\mathfrak{I}_h \subset U_h^\wedge$ is said to be $\hbar$-saturated if $\hbar a \in \mathfrak{I}_h$ implies $a \in \mathfrak{I}_h$, equivalently, if the quotient $U_h^\wedge/\mathfrak{I}_h$ is a flat $K[[\hbar]]$-module). Construct a map $\mathfrak{I} \mapsto \mathfrak{I}^\#: \mathfrak{I}(W) \to \mathfrak{I}_h(U_h^\wedge)$ as follows. Pick $\mathfrak{I} \in \mathfrak{I}(W)$. Form the Rees ideal $\mathfrak{I}_h := \bigoplus_{i \geq 0} (F_i W \cap \mathfrak{I})h^i$. Then take the closure $\mathfrak{I}_h^\wedge \subset W_h^\wedge$ of $\mathfrak{I}_h$. We set $\mathfrak{I}^\# := A_h^\wedge \otimes_{K[[\hbar]]} \mathfrak{I}_h^\wedge \subset A_h^\wedge(W_h^\wedge) = U_h^\wedge$.

**Lemma 4.3.** The map $\mathfrak{I} \mapsto \mathfrak{I}^\# : \mathfrak{I}(W) \to \mathfrak{I}_h(U_h^\wedge)$ is a bijection. The inverse map sends $\mathcal{J}_h \in \mathfrak{I}_h(U_h^\wedge)$ to the image, denoted by $\mathcal{J}_h^\#$, of $\mathcal{J}_h \cap W_h$ under the natural epimorphism $W_h \twoheadrightarrow W$.

This is proved in [Lo2], Proposition 3.3.1.

**Remark 4.4.** It follows from the definition that the map $\mathfrak{I} \mapsto \mathfrak{I}^\#$ is $Q$-equivariant. We remark that any element of $\mathfrak{I}_h(U_h^\wedge)$ is $Z_G(e)^o$-stable because the differential of the $Z_G(e)$-action on $U_h^\wedge$ is given by $\xi \mapsto \frac{1}{\hbar} [\xi, \cdot]$. In particular, it follows that any element of $\mathfrak{I}(W)$ is $Q^o$-stable. So we see that the component group $C(e) := Q/Q^o = Z_G(e)/Z_G(e)^o$ acts on both $\mathfrak{I}(W)$ and $\mathfrak{I}_h(U_h^\wedge)$.

The following lemma follows directly from the construction of the map $\mathfrak{I} \mapsto \mathfrak{I}^\#$.

**Lemma 4.5.** $\dim W/\mathfrak{I} = \chi$ if and only if $(U_h^\wedge/\mathfrak{I}^\#)h(U_h^\wedge/\mathfrak{I}^\#)$ coincides with the completion $K[O]_\chi$ of $K[O]$ at $\chi$.

Following [Lo1], define a map $\bullet^\dagger : \mathfrak{I}(W) \to \mathfrak{I}(U)$ by setting $\mathfrak{I}^\dagger$ to be the image of $\mathfrak{I}^\# \cap U_h$ in $U$. The map $\mathfrak{I} \mapsto \mathfrak{I}^\dagger$ has the following properties (see [Lo1], Theorem 1.2.1 and [Lo2], Conjecture 1.2.1).

**Proposition 4.6.**

1. If $\mathfrak{I}$ is primitive, then so is $\mathfrak{I}^\dagger$.
2. If $\dim W/\mathfrak{I} < \infty$, then the associated variety $V(U/\mathfrak{I}^\dagger)$ coincides with $O$.
3. Let $\mathcal{J}$ be a primitive ideal in $U$ with $V(U/\mathcal{J}) = \overline{\mathcal{J}}$. Then the set of $\mathfrak{I} \in \mathfrak{I}(W)$ with $\dim W/\mathfrak{I} < \infty$ and $\mathfrak{I}^\dagger = \mathcal{J}$ is a single $C(e)$-orbit.

### 4.4. 1-dimensional representations

In this subsection we will explain known results about 1-dimensional representations of $W$-algebras. Let $\mathfrak{I}^1(W)$ denote the set of two-sided ideals of codimension 1 in $W$. We start with the existence theorem.

**Theorem 4.7.**

1. If $g$ is classical, then $\mathfrak{I}^1(W)^{C(e)} \neq \emptyset$.
2. If $g$ is $G_2, F_4, E_6, E_7$ and $e$ is arbitrary, or $g$ is $E_8$ and $e$ is not rigid, then $\mathfrak{I}^1(W) \neq \emptyset$. 

The first assertion was proved in [Lo1], Theorem 1.2.3. However, most ingredients there were not essentially new. The construction of an ideal \( J \subset U \) such that \( \emptyset \) is open in \( V(U/J) \) and the multiplicity of \( U/J \) on \( \emptyset \) is 1 given there was first obtained by Brilynski in [Br]. That such an ideal has the form \( I^\dagger \) for \( \dim W/I = 1 \) was essentially observed by Moeglin in [Mo]. Finally, the claim that \( I \) must be \( C(e) \)-invariant follows from [Lo1], Theorem 1.2.2.

Assertion (ii) follows from [GRU] and [Pr3]. According to [GRU], a one-dimensional \( W \)-module exists for all rigid elements in \( G_2, F_4, E_6, E_7 \) and some (relatively small) rigid elements in \( E_8 \). Here the term “rigid” refers to the Lusztig-Spaltenstein induction, see [LS]. The main result of [Pr3] is that the existence of a one-dimensional module for a \( W \)-algebra is preserved by the induction. Hence (ii). The relation between 1-dimensional modules and the Lusztig-Spaltenstein induction is also discussed in [Lo3].

In fact, for a rigid element \( e \) one can describe the set of ideals \( \mathcal{I}^\dagger \) with \( \dim W/I = 1 \) in terms of highest weights. Fix a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) and a system \( \Pi \) of simple roots. Recall that according to Duflo, every primitive ideal in \( U \) has the form \( J(\lambda) := \text{Ann}_U L(\lambda) \), where \( L(\lambda) \) denotes the irreducible highest weight module with highest weight \( \lambda \) (we do not use the \( \rho \)-shift here, so \( L(0) \) stands the trivial one-dimensional module). An important remark is that \( \lambda \) is not recovered from \( J(\lambda) \) uniquely.

In [Lo3] we have found some conditions on \( \lambda \) such that the ideals \( J(\lambda) \) exhaust the set of ideals \( \mathcal{I}^\dagger \) with \( \dim W/I = 1 \) under the condition that the algebra \( \mathfrak{q} = \mathfrak{z}_0(e, h, f) \) is semisimple that is always the case for rigid elements. These conditions are (in a sense) combinatorial provided \( e \) is of principal Levi type.

In more detail, let \( \mathfrak{t} \) be a Cartan subalgebra in \( \mathfrak{q} \). Set \( I := \mathfrak{z}_0(\mathfrak{t}) \) and let \( L \) be the Levi subgroup of \( G \) with Lie algebra \( I \). Conjugating the triple \( (e, h, f) \) one may assume that \( \mathfrak{h} \subset I \) and that \( \mathfrak{t} \) contains a dominant element. Let \( W^0 \) denote the \( W \)-algebra constructed for the pair \( (I, e) \). For this \( W \)-algebra we have a map \( \bullet^{I_0} \) from the set of primitive ideals of finite codimension in \( W^0 \) to the set of primitive ideals \( J_0 \subset \mathcal{U}^0 := \mathcal{U}(I) \) with \( V(\mathcal{U}^0/J_0) = \mathcal{L}e \).

We need a certain element \( \delta \in \mathfrak{h}^* \). Let \( \Delta^{<0} \) denote the set of all negative roots in \( \Delta \) that are not roots of \( I \). Set

\[
\delta := \sum_{\alpha \in \Delta^{<0}, \langle \alpha, h \rangle = 1} \frac{1}{2} \alpha + \sum_{\alpha \in \Delta^{<0}, \langle \alpha, h \rangle \geq 2} \alpha.
\]

In [Lo3], Subsection 5.3, we have proved the following result.

**Corollary 4.8.** Suppose \( \mathfrak{q} \) is semisimple.

1. Let \( \lambda \in \mathfrak{h}^* \) satisfy the following four conditions:
   (A) \( V(\mathcal{U}^0/J_0(\lambda)) = \mathcal{L}e \).
   (B) \( \dim V(\mathcal{U}/J(\lambda)) \leq \dim \emptyset \).
   (C) \( \lambda - \delta \) vanishes on the center \( z(I) \) of \( I \).
   (D) \( J_0(\lambda) = \mathcal{I}_0^{I_0} \) for some ideal \( \mathcal{I}_0 \) of codimension 1 in \( W^0 \).
   Then \( J(\lambda) = \mathcal{I}^\dagger \) for some ideal \( \mathcal{I} \subset W \) of codimension 1.

2. For any \( \mathcal{I} \subset W \) of codimension 1 there is \( \lambda \in \mathfrak{h}^* \) satisfying (A)-(D) and such that \( J(\lambda) = \mathcal{I}^\dagger \).

When \( e \) is principal in \( I \) the condition (A) means that \( \lambda \) is antidominant for \( I \), while the condition (D) becomes vacuous. The condition (B) is still very difficult to check.
5. Quantizations of nilpotent orbits

This section is the main part of the paper. In Subsection 5.1 we prove Theorem 1.1. In Subsection 5.2 we give a description of the algebra of global sections of a quantization \( D \) of \( X \). Finally, in Subsection 5.3 we compare our results with Moeglin’s, [Mo].

5.1. Proof of Theorem 1.1. Recall that \( G \) is a simply connected semisimple algebraic group, and \( \mathfrak{g} \) is its Lie algebra.

In this subsection we consider a \( G \)-equivariant covering \( X \) of a nilpotent orbit \( O \subset \mathfrak{g} \cong \mathfrak{g}^\bullet \). Let \( \mathcal{W} \) denote the \( \mathcal{W} \)-algebra associated to \( O \). Pick a point \( \chi \in O \) and a point \( x \in X \) lying over \( \chi \). Set \( H := G_x, \Gamma := H/H^\circ \subset C(e) \). We will construct mutually inverse bijections between the set of isomorphism classes of homogeneous Hamiltonian \( G \)-equivariant quantum jet bundles and the set \( \mathfrak{J}^{\Gamma}(\mathcal{W})^\Gamma \). Thanks to Corollary 3.3 and the results of Subsection 3.3, this will imply Theorem 1.1.

Recall the flat sheaf \( J^\infty \mathcal{U}_h \) on \( X \). We start with a standard lemma describing various properties of \( J^\infty \mathcal{U}_h \).

**Lemma 5.1.**

1. The fiber of \( J^\infty \mathcal{U}_h \) at \( x \) is naturally identified with \( \mathcal{U}_h^x \).
2. Let \( \mathfrak{J}(J^\infty \mathcal{U}_h) \) denote the set of homogeneous \( h \)-saturated ideals in \( J^\infty \mathcal{U}_h \). Taking the fiber of an ideal at \( x \) defines a bijection between \( \mathfrak{J}(J^\infty \mathcal{U}_h) \) and \( \mathfrak{J}(\mathcal{U}_h^x)^{\Gamma} \). The inverse map is given by \( J^\infty \mathcal{U}_h \ni I \mapsto I \cap \mathcal{U}_h^x \). This will follow if we prove (2).
3. Any element of \( \mathfrak{J}(J^\infty \mathcal{U}_h) \) is stable with respect to the connection \( \nabla \).

**Proof.** Assertion (1) follows from the observation that the completion functor is right exact.

Let us proceed to the proof of (3). Recall the equality \( \xi_{J^\infty \mathcal{U}_h} = \nabla_{\xi_x} + 1/\hbar [\xi, \cdot], \xi \in \mathfrak{g} \). Let \( I \in \mathfrak{J}(J^\infty \mathcal{U}_h) \). Being an \( h \)-saturated two-sided ideal, \( I \) is stable with respect to \( 1/\hbar [\xi, \cdot] \). Being \( G \)-stable, \( I \) is stable with respect to \( \xi_{J^\infty \mathcal{U}_h} \). So \( I \) is \( \nabla_{\xi_x} \)-stable. But the vector fields \( \xi_x \) generated the tangent sheaf of \( X \). So \( I \) is \( \nabla \)-stable.

Let us prove (2). First of all, we recall that \( J^\infty \mathcal{U}_h \) is a \( G \)-equivariant pro-coherent sheaf of \( \mathcal{O}_X \)-algebras. Consider the category of all such algebras. Then the functor of taking the fiber at \( x \) defines an equivalence between this category and the category of \( H \)-equivariant pro-finite dimensional algebras. A quasi-inverse equivalence is \( \mathcal{A} \rightarrow \pi_*(\mathcal{O}_G \otimes \mathcal{A})^H \).

It remains to prove that \( J^\infty \mathcal{U}_h/I \) is pro-coherent for any \( I \in \mathfrak{J}(\mathcal{U}_h^x) \). This will follow if we check that \( I \) is closed in \( J^\infty \mathcal{U}_h \). But any left ideal in \( J^\infty \mathcal{U}_h \) is closed, compare with Lemma 2.4.4 in [Lo2], this lemma generalizes to the sheaf setting directly.

Now let \( \mathcal{O} \) be a homogeneous Hamiltonian \( G \)-equivariant quantum jet bundle on \( X \).

**Lemma 5.2.** The morphism \( \Phi_h : J^\infty \mathcal{U}_h \rightarrow \mathcal{O} \) is surjective.

**Proof.** Both \( J^\infty \mathcal{U}_h \) and \( \mathcal{O} \) are \( G \)-equivariant and \( \mathcal{O}_X \)-pro-coherent. Therefore it is enough to show that the induced homomorphism of fibers at \( x \) is surjective. Both fibers are complete and separated in the \( \hbar \)-adic topology. Therefore it remains to prove the surjectivity modulo \( \hbar \). Here we have the homomorphism \( \mathbb{K}[\mathfrak{g}^*]_{\chi}^\wedge \rightarrow \mathbb{K}[X]_{x}^\wedge \) of the completions induced by the comoment map \( \mu^* : \mathbb{K}[\mathfrak{g}^*] \rightarrow \mathbb{K}[X] \). But \( \mu \) is just the composition of the covering \( X \rightarrow O \) and the inclusion \( O \hookrightarrow \mathfrak{g}^* \). So \( \mu \) is unramified. Hence the surjectivity claim.

Let \( \mathfrak{J} \) denote the kernel of \( \Phi_h \). This is a homogeneous \( G \)-stable \( h \)-saturated (and automatically closed) ideal in \( J^\infty \mathcal{U}_h \). Let \( \mathcal{I}_h^\infty \) be the fiber of \( \mathfrak{J} \) at \( x \). Then \( \mathcal{I}_h^\infty \) is a homogeneous \( H \)-stable \( h \)-saturated ideal in \( \mathcal{U}_h^\infty \). Then, by the construction, \( (\mathcal{U}_h^\infty /\mathcal{I}_h^\infty) / h(\mathcal{U}_h^\infty /\mathcal{I}_h^\infty) = \mathbb{K}[O]_{\chi}^\wedge \).
Set $I_D := I_{h\ell}$. By Lemma 4.5, dim $W/I_D = 1$. The inclusion $I_D \in J(W)^{\Gamma}$ follows from Remark 4.4. So we have got a map in one direction.

Let us describe a map in the opposite direction. Pick $I \in J(W)^{\Gamma}$. Remark 4.4 implies $I^\#$ is $H$-stable. Also there is a natural isomorphism $\theta : (U_h^\# / I^\#) / h(U_h^\# / I^\#) \cong K[O]^\wedge_X = \mathbb{K}[X]_x^\wedge$. Now let $J$ be the ideal in $J^\infty U_h$ corresponding to $I^\#$. Set $D_I := J^\infty U_h / J$. In other words, $D_I = \pi_*((O_G \otimes K_X[I])^H)$. By assertion (3) of Lemma 5.1, $J$ is $\nabla$-stable. It follows the sheaf $D_I$ comes equipped with a flat connection $\nabla$ induced from the connection on $J^\infty U_h$. Since $J^\infty O_X = \pi_*((O_G \otimes K_X[I])^H)$, we see that $\theta$ gives rise to an isomorphism $\Theta : D_I / hD_I \cong J^\infty O_X$. It is straightforward to verify that $(D_I, \nabla, \Theta)$ is a homogeneous Hamiltonian $G$-equivariant quantum jet bundle.

Also it is clear that the maps $D \mapsto D_I$, $I \mapsto D_I$ are mutually inverse. This completes the proof of Theorem 1.1.

5.2. Global sections. Let $D$ be a homogeneous Hamiltonian $G$-equivariant quantization of $X$. The goal of this subsection is to describe the algebra $\Gamma(X, D)$ of global sections.

Let $I'_D \in J(W)(U_h)$. We say that an element $a \in U_h^\# / I'_D$ is finite if it lies in a finite dimensional $\mathfrak{g}$- and $\mathbb{K}^x$-stable subspace. It is clear that all finite elements form a subalgebra in $U_h^\# / I'_D$. We denote this subalgebra by $(U_h^\# / I'_D)_{\text{fin}}$.

Let $x \in X$, $\Gamma \subset C(e)$ be as above. Let us introduce a $\Gamma$-action on $(U_h^\# / I'_D)_{\text{fin}}$. The last algebra has two $H$-actions: the action $\rho$ induced from the $H$-action on $U_h^\# / I'_D$ and the action $\rho'$ restricted from the $G$-action on $(U_h^\# / I'_D)_{\text{fin}}$. Similarly to [Lo2], Subsection 3.2, $\rho \circ \rho'^{-1}$ descends to a $\Gamma = H / H^e$-action on $(U_h^\# / I'_D)_{\text{fin}}$ commuting with $G \times \mathbb{K}^x$.

The main result is as follows.

Proposition 5.3. Let $I'_D := I_{h\ell}^\#$ be the ideal in $U_h^\#$ corresponding to $D$. The algebra $\Gamma(X, D)$ is naturally identified with the $h$-adic completion of $(U_h^\# / I'_D)_{\text{fin}}$.

Proof. First of all let us produce an algebra homomorphism $\Gamma(X, D) \to U_h^\# / I'_D$. The algebra $\Gamma(X, D)$ coincides the algebra $\Gamma(X, D)^{\overline{\nabla}}$ of the global flat section of $D := J^\infty D$. By the construction of the previous subsection, $U_h^\# / I'_D$ is the fiber $D_x$ of $D$ at $x$. A homomorphism we need is $\Gamma(X, D) \to \Gamma(X, D) \
abla \to D_x = U_h^\# / I'_D$.

Let us check that this homomorphism is injective. Let $K$ stand for the kernel. It follows from (3.1) that on $\Gamma(X, D)$ the derivation $\xi \delta$ coincides with $\frac{1}{r^2}[\Phi_h(\xi), \cdot]$. Being an $h$-saturated two-sided ideal in $\Gamma(X, D)$, the kernel $K$ is $\xi \delta$-stable for any $\xi \in \mathfrak{g}$. This means that $K$ is $G$-stable. Therefore any element of $K \subset \Gamma(X, D)$ vanishes in every point of $X$. So $K = \{0\}$.

The subalgebra $\Gamma(X, D)_{\text{fin}}$ of finite elements of $\Gamma(X, D)$ is dense in the $h$-adic topology. So it remains to check that the embedding $\Gamma(X, D) \hookrightarrow U_h^\# / I'_D$ maps $\Gamma(X, D)_{\text{fin}}$ onto $(U_h^\# / I'_D)_{\text{fin}}$.

Pick an irreducible $G \times \mathbb{K}^x$-module $L$. It is enough to show a natural map

$$\text{Hom}_{G \times \mathbb{K}^x}((L, \Gamma(X, D))) \to \text{Hom}_{G \times \mathbb{K}^x}((L, (U_h^\# / I'_D)_{\text{fin}}))$$

is an isomorphism onto $\text{Hom}_{G \times \mathbb{K}^x}((L, (U_h^\# / I'_D)_{\text{fin}}))$.

Consider the bundle $D_L := D \otimes L^*$ on $X$. This is a $D$-bimodule (the direct sum of several copies of $D$). We have a connection $\nabla_L := \nabla \otimes \text{id}$. Of course, $D_L \nabla_L = D \otimes L^*$ and so $\Gamma(X, D_L)^{\nabla_L} = \Gamma(X, D \otimes L^*) = \text{Hom}(L, \Gamma(X, D))$. The group $G \times \mathbb{K}^x$ acts naturally on $D_L$. This action gives rise to a map $\xi \mapsto \xi_{D_L} : \mathfrak{g} \to \text{End}(D_L)$. Then

$$\xi_{D_L} = \nabla_{L, \xi} + \frac{1}{h^2}[\Phi_h(\xi), \cdot] + \xi_L^*.$$
Consider the restriction map $\Gamma(X, \mathcal{D}_L) \to \mathcal{D}_{L,x}$. Since $\mathcal{D}_L$ is a pro-coherent $G$-equivariant $\mathcal{O}_X$-module, the restriction map descends to an isomorphism
\[
(5.3) \quad \Gamma(X, \mathcal{D}_L)^G \cong \mathcal{D}_{L,x}^H = (\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge \otimes L^*)^H.
\]
Thanks to (5.2), on $\Gamma(X, \mathcal{D}_L)^G$ the connection $\nabla_L$ coincides with $-\frac{1}{\hbar} [\Phi_h(\xi), \cdot] - \xi L^*$. Under the isomorphism (5.2) the last operator corresponds to $-\frac{1}{\hbar} [\xi, \cdot] - \xi L^*$. So, restricting (5.3) to $\Gamma(X, \mathcal{D}_L)^G$, we get an isomorphism
\[
(5.4) \quad \Gamma(X, \mathcal{D}_L)^G \cong (\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge \otimes L^*)^g \cap (\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge \otimes L^*)^H,
\]
where $\mathfrak{g}$ acts on $\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge \otimes L^*$ by $\xi \mapsto \frac{1}{\hbar} [\xi, \cdot] + \xi L^*$.

Take $\mathbb{K}^\times$-invariants in (5.4). The left hand side becomes $\text{Hom}_{G \times \mathbb{K}^\times} (L, \Gamma(X, \mathcal{D}))$. By the definition of the $\Gamma$-action on $(\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge)_{\text{fin}}$, the space $\mathbb{K}^\times$-invariants in the right hand side of is nothing else but $\text{Hom}_{G \times \mathbb{K}^\times} (L, (\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge)_{\text{fin}})$. The corresponding map
\[
\text{Hom}_{G \times \mathbb{K}^\times} (L, \Gamma(X, \mathcal{D})) \to \text{Hom}_{G \times \mathbb{K}^\times} (L, (\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge)_{\text{fin}}) \hookrightarrow \text{Hom}_{G \times \mathbb{K}^\times} (L, (\mathcal{U}_h^\wedge / \mathcal{T}_h^\wedge)_{\text{fin}})
\]
coincides with (5.1). $\square$

5.3. Comparison with Moeglin’s results. An alternative language to speak about quantizations of coverings of nilpotent orbits is that of Dixmier algebras. Recall that a Dixmier algebra $\mathcal{A}$ over $\mathcal{U} = U(\mathfrak{g})$ is an associative algebra equipped with a (rational) $G$-action and a $G$-equivariant homomorphism $\mathcal{U} \to \mathcal{A}$ such that $\mathcal{A}$ is a finitely generated left $\mathcal{U}$-module. Then automatically the differential of the $G$-action coincides with the adjoint action of $\mathfrak{g}$.

Let us define a notion of a filtered Dixmier algebra quantizing a covering of $\mathcal{O}$. By definition, this is a pair $(\mathcal{A}, F^i, A)$, where $\mathcal{A}$ is a Dixmier algebra and $F^i, A, i \geq 0$, is a $G$-stable increasing exhaustive algebra filtration on $\mathcal{A}$ satisfying the following conditions:

- The induced filtration on the image of $\mathcal{U}$ is compatible with the filtration induced from the PBW filtration of $\mathcal{U}$.
- The associated graded algebra $\text{gr} \mathcal{A}$ is a finitely generated commutative domain.
- There is a $G$-equivariant embedding $\text{gr} \mathcal{A} \hookrightarrow \mathbb{K}[\mathcal{O}]$, where $\mathcal{O}$ is the universal covering of $\mathcal{O}$, intertwining the natural homomorphisms $S(\mathfrak{g}) \to \text{gr} \mathcal{A}, S(\mathfrak{g}) \to \mathbb{K}[\mathcal{O}]$ (the homomorphism $S(\mathfrak{g}) \to \text{gr} \mathcal{A}$ is induced from a linear map $\mathfrak{g} \to \text{gr} \mathcal{A}$ that comes from the Lie algebra homomorphism $\mathfrak{g} \to \mathcal{A}$).

One can introduce a partial order on the set of isomorphism classes of filtered Dixmier algebras: $(\mathcal{A}, F^i, A) \preceq (\mathcal{A}', F^i, A')$ if there is a $G$-equivariant embedding $\iota : \mathcal{A} \hookrightarrow \mathcal{A}'$ that is strictly compatible with the filtrations: $\iota^{-1}(F^i, A') = F^i, A$ for all $i$.

Moeglin, [Mo], related maximal filtered Dixmier algebras quantizing a covering of $\mathcal{O}$ to primitive ideals in $\mathcal{U}$ “admitting a Whittaker model”. Let us explain her result.

Recall the subgroup $M \subset G$ and the $\mathcal{U}$-module $\mathcal{U}/\mathcal{U}m_\chi$ equipped with a Kazhdan filtration, see Subsection 4.1. We remark that each $\mathcal{U}$-submodule in $\mathcal{U}/\mathcal{U}m_\chi$ is automatically $M$-stable. Let $\mathcal{J}$ be a primitive ideal in $\mathcal{U}$. Following Moeglin, we say that an irreducible quotient $N$ of $\mathcal{U}/\mathcal{U}m_\chi$ is a Whittaker model for $\mathcal{J}$ if $\mathcal{J}$ annihilates $N$ and $\text{gr} N$ is isomorphic to $\mathbb{K}[M]$ as a graded $M$-module (the grading on $\mathbb{K}[M]$ was introduced in Subsection 4.1). Under the Skryabin equivalence a quotient $N$ of $\mathcal{U}/\mathcal{U}m_\chi$ with $\text{gr} N = \mathbb{K}[M]$ corresponds to a one-dimensional $\mathcal{W}$-module. This follows from Lemma 4.1.

Let $N$ be a Whittaker model for $\mathcal{J}$. Let $L(N, N)$ denote the space of $\mathfrak{g}$-finite maps $N \to N$. This is an algebra equipped with a homomorphism $\mathcal{U} \to L(N, N)$. Consider the filtration
on \( L(N,N) \) induced by the filtration on \( N \): \( \varphi(L(N,N)) \) consists of all maps \( \varphi \) such that \( \varphi(F_j N) \subset F_{i+j} N \) for all \( j \). Then, according to Moeglin, the pair \( (L(N,N), \varphi) \) is a maximal filtered Dixmier algebra quantizing a covering of \( \mathcal{O} \) (see [Mo], Theorem 15).

Let us briefly explain the relation between our construction and Moeglin’s.

First of all, let \( (\mathcal{A}, \mathcal{F}, \mathcal{A}) \) be a filtered Dixmier algebra. Form the Rees algebra \( R_\mathbb{h}(\mathcal{A}) \) and complete it with respect to the \( \mathbb{h} \)-adic topology. Then we can localize this completion on \( \text{Spec} \text{-} \mathcal{A} \mathbb{h} \mathcal{A} \) to get a sheaf of algebras. The restriction of this sheaf to the open \( G \)-orbit \( \mathcal{O} \) is a homogeneous Hamiltonian \( G \)-equivariant quantization \( \mathcal{D} \) of \( \mathcal{O} \). Now let \( X \to \mathcal{O} \) be a \( G \)-equivariant covering of \( \mathcal{O} \). It is possible to show that there is a unique quantization of \( X \) that lifts \( \mathcal{D} \) (in an appropriate). Also it is not difficult to see that if \( (\mathcal{A}, \mathcal{F}, \mathcal{A}) \cong (\mathcal{A}', \mathcal{F}', \mathcal{A}') \), then the quantizations of the open \( G \)-orbit \( \text{Spec}(\mathcal{A}') \) given by \( \mathcal{A} \) and by \( \mathcal{A}' \) are isomorphic.

On the other hand, let \( \mathcal{D} \) be a quantization of the universal cover \( X \to \mathcal{O} \). Consider the algebra \( \Gamma(X, \mathcal{D}) \) and its finite part \( \Gamma(X, \mathcal{D})\text{fin} \). Set \( \mathcal{A}_{\mathcal{D}} := \Gamma(X, \mathcal{D})\text{fin}/(\mathbb{h}-1)\Gamma(X, \mathcal{D})\text{fin} \). By Proposition 5.3, \( \mathcal{A}_{\mathcal{D}} = (\mathcal{W}/\mathcal{I})^\dagger \) in the notation of [Lo2], Remark 3.5.1. Using the techniques of [Lo2] it is easy to show that \( \mathcal{A}_{\mathcal{D}} \) (with its natural filtration induced by the \( \mathbb{K}^\times \)-action on \( \Gamma(X, \mathcal{D})\text{fin} \)) is a maximal filtered Dixmier algebra quantizing a covering of \( \mathcal{O} \). The construction of the previous paragraph shows that, conversely, any maximal filtered Dixmier algebra has the form \( \mathcal{A}_{\mathcal{D}} \).

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