Abstract. Using the consistency of some large cardinals we produce a model of Set Theory in which the generalized continuum hypothesis holds and for some torsion-free abelian group $G$ of cardinality $\aleph_{\omega+1}$ and for some torsion group $T$

\[ Bext^2(G,T) \neq 0. \]

Hence G.C.H. is not sufficient for getting the results of [10].

1. Introduction

All groups in this paper are abelian groups. For basic terminology about abelian groups in general we refer the reader to [8]. For terminology concerning Butler groups see [4, 3, 2, 5, 6]. It is commonly agreed that the three major questions concerning the infinite rank Butler groups are:

1. Are $B_1$-groups necessarily $B_2$-groups?
2. Does $Bext^2(G,T) = 0$ hold for all torsion-free groups $G$ and torsion groups $T$?
3. Which pure subgroups of $B_2$-groups are again $B_2$-groups? In particular: is a balanced subgroup of a $B_2$-group a $B_2$-group?

In [2] it is shown that the answer to all these questions is “Yes” for countable groups $G$. In the series of papers [1, 4, 3] it was shown that under the continuum hypothesis the answer is “Yes” to all three questions for groups $G$ of cardinality $\leq \aleph_\omega$.

In [4] it is shown that the answer to question 2 is “No” if the continuum hypothesis fails. In a more recent paper [10] it is shown that in the constructible universe, $L$ the answer is “Yes” to all three questions for arbitrary groups $G$. Actually [10] used only the generalized continuum hypothesis and that the combinatorial principle $\square_{\kappa}$ holds for every singular cardinal $\kappa$ whose cofinality is $\aleph_0$. Is the use made in [10] of the additional combinatorial principle really needed or does the affirmative answer to our three questions follow simply from G.C.H.? Let us mention that a key tool used in [3, 10] was the representation of an arbitrary torsion-free group as the union of a chain of subgroups which are countable unions of balanced subgroups. In [5] it is shown that such a representation is equivalent to a weak version of $\square_{\kappa}$.

In this paper we show that at least for getting an affirmative answer to questions 2 and 3, one needs some extra set theoretic assumptions in addition to G.C.H. We do it by producing a model of Set Theory, satisfying G.C.H., in which for some torsion-free $G$ of cardinality $\aleph_{\omega+1}$ and some torsion $T$, $Bext^2(G,T) \neq 0$. Also in the
same model there will be a balanced subgroup of a completely decomposable group which is not a $B_2$-group. Hence the answer to question 3 in this model is “No”. The construction of the model requires the consistency of some large cardinals, which can not be avoided since getting a model in which $\square_\kappa$ fails for some singular $\kappa$ requires assumptions stronger than the consistency of Set Theory. Let us stress that the status of question 1 is not known and it is possible (though unlikely) that the implication “every $B_1$-group is a $B_2$-group” is a theorem of Set Theory.

Since this paper is aimed at a mixed audience of set theorists and abelian group theorists it is divided into two sections with very different prerequisites. In the next section we describe the construction of the model of Set Theory with certain properties to be listed below. In the following section we shall describe how to use the listed properties to get a group $G$ which will be the counterexample to $\text{Bext}^2(G,T) = 0$. A reader who is not familiar with standard set theoretical techniques, like forcing, can skip the set theoretic section and simply assume the properties of the model listed below. We do assume some basic Set Theory at the level introduced by [6].

We now describe the properties of the model which will be used in the construction of the counterexample to questions 2 and 3. The model will naturally satisfy G.C.H. Hence by standard cardinal arithmetic $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$. Therefore we can enumerate all the $\omega$-sequences from $\aleph_\omega$ in a sequence of order type $\aleph_{\omega+1}$. Let $\langle f_\alpha | \alpha < \aleph_{\omega+1} \rangle$ be this enumeration. Let $F_\alpha$ be the range of $f_\alpha$. The important property of the model is the following:

For some stationary subset $S$ of $\aleph_{\omega+1}$ such that every point of $S$ has cofinality $\aleph_1$, and for some choice of a cofinal set $C_\beta$ in $\beta$ of order type $\omega_1$, for every $\beta \in S$ and for some fixed countable ordinal $\delta$ we have:

1. $\bigcup_{\alpha \in D} F_\alpha$ has order type $\delta$ for every $D \subseteq C_\beta$ which is cofinal subset of $C_\beta$ and for every $\beta \in S$. In particular for $D = C_\beta$

2. $E_\beta = \bigcup_{\alpha \in C_\beta} F_\alpha$ has order type $\delta$.

2. If $\beta \neq \gamma$ both in $S$, then $E_\beta \cap E_\gamma$ has order type less than $\delta$.

3. $\delta$ is an indecomposable ordinal, namely $\delta$ can not be represented as a finite sum of smaller ordinals. Or equivalently, $\delta$ is not the finite union of sets of ordinals of order type less than $\delta$.

Denote the conjunction of all the properties above by (*) The main theorem of Section 1 is

**Theorem 1.** Assume the consistency of a supercompact cardinal. Then there is a model of Set Theory in which (*) holds. The model also satisfies the Generalized Continuum Hypothesis.

The construction of the model is very close to the construction in [11]. The main tool that will be used to get in Section 3 an example of a group $G$ satisfying $\text{Bext}^2(G,T) \neq 0$ is the notion of $\aleph_\alpha$-prebalancedness (see [8]). We are rephrasing the original definition in a form which is clearly equivalent to the original definition.
Definition 1. Let $G$ be a pure subgroup of the group $H$. $G$ is said to be $\aleph_0$-prebalanced in $H$ if for every element $h \in H - G$ there are countably many elements $g_0, g_1, \ldots$ of $G$ such that for every element $g$ of $G$ the type (in $H$) of $h - g$ is bounded by the union of finitely many types of the form $lh - g_i$ for some natural number $l$. More explicitly for some $n, l \in \omega$

$$t(h - g) \leq t(lh - g_0) \cup \ldots \cup t(lh - g_n).$$

Also the group $G$ is said to admit an $\aleph_0$-prebalanced chain if $G$ can be represented as a continuous increasing union of pure $\aleph_0$-prebalanced subgroups where at the successor stages the factors are of rank 1.

We shall use the following fundamental result of Fuchs (8):

Theorem 2. A torsion-free group $G$ admits an $\aleph_0$-prebalanced chain if and only if in its balanced projective resolution

$$0 \rightarrow B \rightarrow C \rightarrow G \rightarrow 0$$

(where $C$ is completely decomposable) $B$ is a $B_2$-group. Moreover, if CH holds, then this condition is equivalent to $B \text{ext}^2(G, T) = 0$ for all torsion groups $T$.

The main result of Section 3 will be

Theorem 3. If (*) holds, then there is a torsion-free group $G$ of cardinality $\aleph_{\omega +1}$ which does not admit an $\aleph_0$-prebalanced chain.

Using theorem 2 we get

Corollary 4. If (*) holds, then there is a group $G$ of cardinality $\aleph_{\omega +1}$ such that $B \text{ext}^2(G, T) \neq 0$ for some torsion group $T$.

By using the balanced projective resolution of $G$ we also get

Corollary 5. If (*) holds, then there is a balanced subgroup of a completely decomposable group of cardinality $\aleph_{\omega +1}$ which is not a $B_2$-group.

2. The Consistency of (*)

In this section we shall prove Theorem 3. We assume familiarity with some basic large cardinals notions like supercompact cardinals and some basic forcing techniques. We start from a ground model $V$ having a supercompact cardinal $\kappa$. We can assume without loss of generality that $V$ satisfies G.C.H. We let $\mu = \kappa^{+\omega}$ and $\lambda = \mu^+ = \kappa^{+\omega+1}$. In our final model $\mu$ will be $\aleph_\omega$ and $\lambda$ will be $\aleph_{\omega +1}$. It follows from the results of Menas in [12] that there is a normal ultrafilter $U$ on $P_\kappa(\lambda)$ such that for some set $A \in U$ the map $P \rightarrow \text{sup}(P)$ on $A$ is one-to-one. (Recall that $P_\kappa(\lambda)$ is the set of all subsets of $\lambda$ of cardinality less than $\kappa$). Fix such $U$ and $A$. Also fix an enumeration $\langle g_\alpha \mid \alpha < \lambda \rangle$ of all the $\omega$-sequences in $\mu$. Standard facts about normal ultrafilters on $P_\kappa(\lambda)$ imply that the set of all $P \in P_\kappa(\lambda)$ satisfying the following properties is in $U$:

1. The order type of $P \cap \mu$ is a singular cardinal of cofinality $\omega$ such that the order type of $P$ is its successor.
2. For $\alpha \in \lambda$ the range of $g_\alpha$ is a subset of $P \cap \mu$ if and only if $\alpha \in P$.

Hence we can assume without loss of generality that every $P \in A$ satisfies all the above properties. Again standard arguments show that the set $T = \{\text{sup}(P) \mid P \in A\}$ is a stationary subset of $\lambda$. For $\alpha \in T$, let $P_\alpha$ be the unique $P \in A$ such that
sup(P) = α. Note that for P ∈ A and Q ⊆ P we have that if Q is cofinal in sup(P), then the order type of Q* = ∪{range(gα) | α ∈ Q} is the same as the order type of P ∩ µ. This holds since otherwise Q* has cardinality smaller than δ = the order type of P ∩ µ. Hence, by our G.C.H. assumption, we have less than δ α’s such that the range of gα is in Q*, hence less than the order type of P, which is a regular cardinal. Therefore Q must be bounded in P.

For α ∈ T the map α → the order type of Pα ∩ µ maps T into κ. Hence it is fixed on some subset S which is stationary in λ. Let δ be the fixed value of this map on S. Note that for α ∈ S the order type of Pα is δ+.

**Claim 6.** Let α and β be two different members of S. Then Pα ∩ Pβ ∩ µ has order type less than δ.

**Proof.** Let X = Pα ∩ Pβ ∩ µ. Note that if g is an ω-sequence from X, then g = gρ for some ρ ∈ Pα ∩ Pβ. If X has order type δ, then (using the fact that δ is a singular cardinal of cofinality ω) we have δ+ ω-sequences from X, so that Pα ∩ Pβ must have order type which is at least δ+. Since the order type of both Pα and Pβ is δ+, Pα and Pβ must have the same sup. This is a contradiction.

The model which will witness (*) will be obtained from V by collapsing δ to be countable, followed by the collapsing all the cardinals between δ+ and κ to have cardinality δ++. Denote the resulting model by V1. Note since V satisfies G.C.H. then the resulting model satisfies G.C.H. Also δ is of course countable, δ++ is ℵ1, µ is ℵω and λ is ℵω+1. Since the cardinality of the forcing notion is κ < λ, S is still a stationary subset of λ. Note that now we have for every α ∈ S that the cofinality of α is ℵ1. In order to verify (*) in the resulting model we fix an enumeration ⟨fγ | γ < λ⟩ of all the ω-sequences from ℵω = µ. And as in the previous section let Fγ be the range of fγ. (Note that in V1 there are new ω-sequences so that the enumeration (gγ | γ < λ) we had in V enumerates only a subset of the set of all ω-sequences). For γ < λ let η(γ) be the unique η such that gγ = fη. Without loss of generality (by reducing S to a subset which is still stationary in λ) we can assume that for α ∈ S if γ < α, then η(γ) < α. We can also assume without loss of generality that for α ∈ S, Qα = {η(γ) | γ ∈ Pα} is cofinal in α. This follows since the set {α ∈ S | Qα is bounded in α} is not stationary. So for each α ∈ S pick Cα which is cofinal in Qα and has order type ℵ1 = δ+. We claim that S, δ and ⟨Cα | α ∈ S⟩ are witnesses to the truth of (*) in V1. As in the introduction we put

\[ E_α = \bigcup_{γ ∈ C_α} F_γ. \]

Since we clearly have G.C.H. in V1, since S is stationary and since δ is an indecomposable ordinal (it is a cardinal in V!), we are left with verifying the following claim:

**Claim 7.** In V1

A: For α ≠ β ∈ S Eα ∩ Eβ has order type less than δ.

B: If D ⊆ Cα is cofinal in α, then ∪{Fγ | γ ∈ D} has order type δ.

**Proof.** Clause A follows immediately from the fact that for α ∈ S, Eα ⊆ Pα ∩ µ, hence Eα ∩ Eβ ⊆ Pα ∩ Pβ ∩ µ and the last set has order type less than δ if α ≠ β.

For proving B note that if D ⊆ Cα is cofinal in α, then the set F = {γ | η(γ) ∈ D} is a subset of Pα of cardinality ℵ1 = δ+. Our forcing is an iteration of two forcing
notions where the first is of cardinality (in V) \( \delta \) and the second is \( \delta^{++} \) closed, hence it introduces no new sets of ordinals of order type \( \delta^{+} \). So \( F \) contains a subset \( Q \in V \) of cardinality \( \delta^{+} \), \( Q \) must be cofinal in \( P_{\alpha} \) since \( P_{\alpha} \) has order type \( \delta^{+} \), so by a previous remark \( \cup \{ \text{range}(y_{\gamma}) \mid \gamma \in Q \} \) has order type \( \delta \). But this last set is clearly a subset of \( \cup \{ F_{\rho} \mid \rho \in D \} \), so this set clearly has order type at least \( \delta \). It can not have order type greater than \( \delta \) since it is a subset of \( P_{\alpha} \cap \mu \).

\[ \Box \]

3. A GROUP WHICH DOES NOT ADMIT AN \( \aleph_{0} \)-PREBALANCED CHAIN

In this section we prove Theorem 3. So we assume (*). Fix the enumeration \( \langle f_{\alpha} \mid \alpha < \aleph_{\omega+1} \rangle \) of the \( \omega \)-sequences from \( \aleph_{\omega} \). Let \( F_{\alpha} \) be the range of \( f_{\alpha} \). Also fix the stationary subset \( S \) of \( \aleph_{\omega+1} \), the countable ordinal \( \delta \) and for \( \beta \in S \) a set \( C_{\beta} \) cofinal in \( \beta \), which witness the truth of (*). As in the statement of (*) (for \( \beta \in S \)) let

\[ E_{\beta} = \bigcup_{\alpha \in C_{\beta}} F_{\alpha}. \]

We know that the order type of \( E_{\beta} \) is \( \delta \). Since \( \delta \times \omega \) is countable we can assign to every pair \( \mu < \delta, n < \omega \) a unique prime number \( p_{\mu}^{n} \).

We are ready to define the group \( G \) that will not admit a chain of \( \aleph_{0} \)-prebalanced subgroups. For each \( \alpha < \aleph_{\omega+1} \) and \( \beta \in S \) fix distinct symbols \( x_{\alpha} \) and \( y_{\beta} \). The group \( G \) is a subgroup of

\[ \sum_{\alpha < \aleph_{\omega+1}} \oplus Qx_{\alpha} \oplus \sum_{\beta \in S} \oplus Qy_{\beta}. \]

\( G \) is generated by \( x_{\alpha} \) for \( \alpha < \aleph_{\omega+1} \), by \( y_{\beta} \) for \( \beta \in S \) and by \( \frac{1}{p_{\mu}^{n}}(y_{\beta} - x_{\alpha}) \) provided \( \alpha \) is in \( C_{\beta} \) and the \( f_{\alpha}(n) \) is the \( \mu \)-th member of \( E_{\beta} \). For \( \delta < \aleph_{\omega+1} \) let \( G_{\delta} \) be the subgroup of \( G \) generated by \( x_{\alpha}, y_{\gamma} \) and \( \frac{1}{p_{\mu}^{n}}(y_{\gamma} - x_{\alpha}) \) where \( \alpha \) and \( \gamma \) are less than \( \delta \).

The sequence \( \langle G_{\delta} \mid \delta < \aleph_{\omega+1} \rangle \) is a filtration of \( G \) into a continuous chain of smaller cardinality. If \( G \) allows an \( \aleph_{0} \)-prebalanced chain, then by standard arguments, the set of \( \delta < \aleph_{\omega+1} \) such that \( G_{\delta} \) appears in the \( \aleph_{0} \)-prebalanced chain contains a closed unbounded subset of \( \aleph_{\omega+1} \). This will imply, since \( S \) is stationary in \( \aleph_{\omega+1} \), that for some \( \beta \in S \), \( G_{\beta} \) is \( \aleph_{0} \) prebalanced in \( G \). The fact that we get a contradiction and that \( G \) does not allow an \( \aleph_{0} \)-prebalanced chain follows from:

Claim 8. For \( \beta \in S \), \( G_{\beta} \) is not an \( \aleph_{0} \)-prebalanced subgroup of \( G \).

Proof. Assume that for some fixed \( \beta \in S \), \( G_{\beta} \) is \( \aleph_{0} \)-prebalanced in \( G \). We apply the definition of \( \aleph_{0} \)-prebalancedness for \( y_{\beta} \) and get a sequence of elements \( z_{\alpha} \in G_{\beta} \) such that for every element \( z \) of \( G_{\beta} \) there are \( e \) and \( l \) such that

\[ t(y_{\beta} - z) \leq t(l y_{\beta} - z_{0}) \cup \ldots \cup t(l y_{\beta} - z_{e}). \]

\( C_{\beta} \) has order type \( \aleph_{1} \) and hence for some fixed \( e \) and \( l \) we get that the set

\[ D = \{ \alpha \in C_{\beta} \mid t(y_{\beta} - x_{\alpha}) \leq t(l y_{\beta} - z_{0}) \cup \ldots \cup t(l y_{\beta} - z_{e}) \} \]

is unbounded in \( C_{\beta} \). It means that for \( \alpha \in D \) there is a natural number \( d_{\alpha} \) such that if \( p \) is a prime number greater than \( d_{\alpha} \) and \( p \) divides \( y_{\beta} - x_{\alpha} \), then \( p \) divides \( l y_{\beta} - z_{i} \) for some \( 0 \leq i \leq e \). Without loss of generality we can assume that for \( \alpha \in D, d_{\alpha} \) is some fixed natural number \( d \). Let \( D^{*} = \cup_{\gamma \in D} F_{\gamma} \). We know that \( D^{*} \subseteq E_{\beta} \) and that the order type of \( D^{*} \) is \( \delta \). We need the following lemma.
Lemma 9. Let $z$ be a member of $G_\beta$ with
\[ z = \sum_{i=1}^{k} r_i x_{\alpha_i} + \sum_{j=1}^{g} s_j y_{\beta_j}, \]
where $r_i, s_j \in Q$ and $\alpha_i, \beta_j < \beta$ for $1 \leq i \leq k, 1 \leq j \leq g$. Assume also that $ly_\beta - z$ is divisible (in $G$) by $p_\mu^n$ where $p_\mu^n > l$. Then either for some $1 \leq j \leq g$, the $\mu$-th member of $E_\beta$ is the same as the $\mu$-th member of $E_{\beta_j}$ or for some $1 \leq i \leq k$, the $\mu$-th member of $E_\beta$ is in $F_{\alpha_i}$.

Proof. By assumption $ly_\beta - z$ is divisible by $p = p_\mu^n$ in $G$. Hence
\[ ly_\beta - z = p(\sum_{m=1}^{f} r_m x_{\gamma_m} + \sum_{i=1}^{u} s_i y_{\eta_i} + \sum_{q=1}^{v} w_q (y_{\nu_q} - x_{\xi_q})). \tag{2} \]
where the $r_m$'s, the $s_i$'s and the $w_q$'s are integers.

Let us define a (bipartite) graph $P$, whose nodes are all the symbols ($x$'s and $y$'s) appearing in equation 2, where $y_p$ is connected by an edge to $x_\xi$ iff for some $1 \leq q \leq v$, $p = \nu_q$, $\xi = \xi_q$ and $p_q = p$. Let $W$ be the connected component of $y_\beta$ in $P$ and let $a \in Q$ be the sum of all the coefficients in the right side of equation 2 of symbols in $W$. $a$ is easily seen to be a member of $pQ_p$, where $Q_p$ is the ring of rationals whose denominators are prime to $p$. This is true because the only summands on the right side of 2 that can possibly add to $a$ a rational number which is not in $pQ_p$ is of the form $y_{\nu_q} - x_\xi$ where $p_q = p$. But in this case $y_{\nu_q}$ and $x_\xi$ are connected by an edge of $P$, so they are both in $W$ or both outside of $W$. In both cases the contribution of this summand to $a$ is 0.

We use the fact that the sum of the coefficients of symbols in $W$ must be the same for the left side and the right side of 2. Of course $y_\beta \in W$ and its coefficient in equation 2 is $l$ which is not in $pQ_p$, so there must be a symbol in $W$ appearing in the representation of $z$, so that either $x_\alpha \in W$ for some $1 \leq i \leq k$, or $y_{\beta_j} \in W$ for some $1 \leq j \leq g$. Our lemma will be verified if we prove

Claim 10. 1. If $y_\eta \in W$, then the $\mu$-th member of $E_\eta$ is the same as the $\mu$-th member of $E_\beta$.

2. If $x_\gamma \in W$, then $f_\gamma (n)$ is the $\mu$-th member of $E_\beta$.

Proof. The proof is by induction on the length of the path in $P$ leading from $y_\beta$ to the symbol $y_\eta$ and $x_\gamma$, respectively. If this length is 0, we are in the case where the symbol is $y_\eta = y_\beta$, and the claim is obvious. For the induction step, in the first case we are given $y_\eta$. Let $x_\gamma$ be the element preceding $y_\eta$ in the path leading from $y_\beta$ to $y_\eta$. By the induction assumption $f_\gamma (n)$ is the $\mu$-th member of $E_\beta$. $x_\gamma$ and $y_\beta$ are connected by an edge of $P$, so that $\frac{1}{p_\mu^n}(y_\eta - x_\gamma)$ is one of the generators of $G$.

Hence $\gamma \in C_\eta$ and $f_\gamma (n)$ is the $\mu$-th member of $E_\eta$, and the claim is verified in this case. The other case (the $x_\gamma$ case) is argued similarly where $y_\eta$ is now the element in the path preceding $x_\gamma$.

For $z \in G_\beta$ let $S(z)$ be the set of all elements $\gamma$ of $E_\beta$ such that for some $\mu < \delta$ and $n \in \omega, \gamma$ is the $\mu$-th member of $E_\beta$ and $ly_\beta - z$ is divisible in $G$ by $p_\mu^n$ where $p_\mu^n > l$. It follows from Lemma 9 that for $z \in G_\beta$, $S(z)$ is included in a finite union of singletons and of sets of the form $E_\eta \cap E_\beta$ for $\eta < \beta$. So $S(z)$ is a finite union
of sets of order type less than $\delta$. $\delta$ is an indecomposable ordinal, so for $z \in G_\beta$ the order type of $S(z)$ is less than $\delta$. By definition of $D$, every element of $D^*$, except possibly finitely many, is in $\bigcup_{0 \leq i \leq e} S(z_i)$. This is because there are only finitely many members of $E_\beta$ such that if $\gamma$ is the $\mu$-th member of $E_\beta$, then $p^n_\mu \leq \max(d, l)$ for some $n$. So if $\gamma \in D^*$ is not one of these finitely many elements, say $\gamma$ is the $\mu$-th member of $E_\beta$, then $p^n_\mu > \max(d, l)$. Now $\gamma = f_\alpha(n)$ for some $\alpha \in D$ and a natural number $n$, and hence $p^n_\mu$ divides $y_\beta - x_\alpha$, which implies by equation 1 and the definition of $d$ that $p^n_\mu$ divides $ly_\beta - z_i$ for some $1 \leq i \leq e$. We got that $D^*$ is a finite union of sets of order type less than $\delta$, and hence $D^*$ has order type less than $\delta$. We got a contradiction.

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