AN IDENTITY OF THE SYMMETRY FOR THE
FROBENIUS-EULER POLYNOMIALS ASSOCIATED WITH
THE FERMIONIC \( p \)-ADIC INVARIANT \( q \)-INTEGRALS ON \( \mathbb{Z}_p \)

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Abstract. The main purpose of this paper is to prove an identity of symmetry for the
Frobenius-Euler polynomials. It turns out that the recurrence relation and multiplication
theorem for the Frobenius-Euler polynomials which discussed in [ K. Shiratani, S.
Yamamoto, On a \( p \)-adic interpolation function for the Euler numbers and its deri-
vatives, Memo. Fac. Sci. Kyushu University Ser.A, 39(1985), 113-125]. Finally we investigate
several further interesting properties of symmetry for the fermionic \( p \)-adic invariant
\( q \)-integral on \( \mathbb{Z}_p \) associated with the Frobenius-Euler polynomials and numbers.

§1. Introduction

The \( n \)-th Frobenius-Euler numbers \( H_n(q) \) and the \( n \)-th Frobenius-Euler polynomi-
als \( H_n(q, x) \) attached to an algebraic number \( q \neq 1 \) may be defined by the exponential
generating functions

\[
\sum_{n=1}^{\infty} H_n(q) \frac{t^n}{n!} = \frac{1 - q}{e^t - q}, \quad \text{see [6,7]},
\]

\[
\sum_{n=0}^{\infty} H_n(q, x) \frac{t^n}{n!} = \frac{1 - q}{e^t - q} e^x t.
\]

It is easy to show that \( H_n(q, x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} H_l(q) \). Let \( p \) be a fixed prime.
Throughout this paper \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}, \) and \( \mathbb{C}_p \) will, respectively, denote the ring of \( p \)-adic rational integers, the field of \( p \)-adic rational numbers, the complex number field,
and the completion of algebraic closure of \( \mathbb{Q}_p \). When one talks of \( q \)-extension, \( q \)
is variously considered as an indeterminate, a complex \( q \in \mathbb{C} \), or a \( p \)-adic number
\( q \in \mathbb{C}_p \), see [9-22]. If \( q \in \mathbb{C} \), then we assume \( |q| < 1 \). If \( q \in \mathbb{C}_p \), then we assume

\text{Key words and phrases.} fermionic \( p \)-adic \( q \)-integral, Frobenius-Euler number.

2000 AMS Subject Classification: 11B68, 11S80
This paper is supported by Jangjeon Mathematical Society(JJMS-10R-2008)
$|1 - q|_p < 1$. For $x \in \mathbb{Q}_p$, we use the notation $[x]_q = \frac{1 - q^x}{1 - q}$, and $[x]_{-q} = \frac{1 - (-q)^x}{1 + q}$, see [5-6]. The normalized valuation in $\mathbb{C}_p$ is denoted by $|\cdot|_p$ with $|p|_p = \frac{1}{p}$. We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$ have a limit $l = f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{0 \leq x < dp^N} f(x) q^x,$$

Thus, we note that

$$qI_q(f_1) = I_q(f) + (q - 1)f(0) + \frac{q - 1}{\log q} f'(0),$$

where $f_1(x) = f(x + 1)$. 

The fermionic $p$-adic invariant $q$-integral on $\mathbb{Z}_p$ is defined as

$$(2) \quad I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x,$$

In [8], H.J.H. Tuenter provided a generalization of the Bernoulli number recurrence

$$B_m = \frac{1}{a(1 - a^m)} \sum_{j=0}^{m-1} a^j \binom{m}{j} B_j \sum_{i=0}^{a-1} i^{m-j},$$

where $a, m \in \mathbb{Z}$ with $a > 1$ and $m \geq 1$, attributed to E.Y. Deeba and D.M. Rodriguez[2] and to I. Gessel[3]. Define $S_m(k) = 0^m + 1^m + \cdots + k^m$, where $a, m \in \mathbb{Z}$, with $a \geq 0$ and $m \geq 0$. H.J.H. Tuenter proved that the quantity

$$\sum_{j=0}^{m} \binom{m}{j} a^{j-1} B_j b^{m-j} S_{m-j}(a - 1),$$

is symmetric in $a$ and $b$, provided $a, b, m \in \mathbb{Z}$, with $a > 0, b > 0$ and $m \geq 0$. In this paper we prove an identity of symmetry for the Frobenius-Euler polynomials. It
turns out that the recurrence relation and multiplication theorem for the Frobenius-Euler polynomials which discussed in [7]. Finally we investigate the several further interesting properties of the symmetry for the fermionic $p$-adic invariant $q$-integral on $\mathbb{Z}_p$ associated with the Frobenius-Euler polynomials and numbers.

§2. An identity of symmetry for the Frobenius-Euler polynomials

From (2) we can derive

$$(3) \quad q I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1(x) = f(x+1).$$

By continuing this process, we see that

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \text{ where } f_n(x) = f(x+n).$$

When $n$ is an odd positive integer, we obtain

$$(4) \quad q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l f(l) q^l.$$

If $n \in \mathbb{N}$ with $n \equiv 0 \pmod{2}$, then we have

$$(5) \quad q^n I_{-q}(f_n) - I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} f(l) q^l.$$

From (1) and (3) we derive

$$(6) \quad \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{1 - (-q)^{-1}}{e^t - (-q)^{-1}} = \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}.$$

Thus, we note that

$$\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = H_n(-q^{-1}), \text{ and } \int_{\mathbb{Z}_p} (y + x)^n d\mu_{-q}(x) = H_n(-q^{-1}, x).$$

Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$. Then we obtain

$$[2]_q \sum_{l=0}^{n-1} (-1)^l q^l t^m = q^n H_m(-q^{-1}, n) + H_m(-q^{-1}).$$
For \( n \in \mathbb{N} \) with \( n \equiv 0 \pmod{2} \), we have

\[
q^n H_m(-q^{-1}, n) - H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^{l+1} q^l l^m.
\]

By substituting \( f(x) = e^{xt} \) into (4), we can easily see that

\[
(7) \quad \int_{\mathbb{Z}_p} q^n e^{(x+n)t} d\mu_q(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) = [2]_q \frac{q^n e^{nt} + 1}{qe^t + 1} = [2]_q \sum_{l=0}^{n-1} (-1)^{l+1} q^l e^{lt}.
\]

Let \( S_{k,q}(n) = \sum_{l=0}^{n} (-1)^l l^k q^k \). Then \( S_{k,q}(n) \) is called by the alternating sums of powers of consecutive \( q \)-integers. From the definition of the fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \), we can derive

\[
(8) \quad \int_{\mathbb{Z}_p} q^n e^{(x+n)t} d\mu_q(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) = \frac{[2]_q \int_{\mathbb{Z}_p} e^{xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} e^{nxt} q^{(n-1)x} d\mu_q(x)}.
\]

By (8), we easily see that

\[
\int_{\mathbb{Z}_p} q^{(n-1)x} e^{nxt} d\mu_q(x) = \frac{1 + q}{q^n e^{nt} + 1}.
\]

Let \( w_1, w_2 (\in \mathbb{N}) \) be odd. By using double fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \), we obtain

\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} e^{(w_1 x_1 + w_2 x_2)t} d\mu_q(x_1) d\mu_q(x_2) = \frac{[2]_q (q^{w_1 w_2 e^{w_1 w_2 t}} + 1)}{(qe^{w_1 t} + 1)(qe^{w_2 t} + 1)}.
\]

Now we also consider the following fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) associated with Frobenius-Euler polynomials.

\[
I = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x_2)t} d\mu_q(x_1) d\mu_q(x_2)\]

\[
= \frac{[2]_q e^{w_1 w_2 x_1 t} (q^{w_1 w_2 e^{w_1 w_2 t}} + 1)}{(qe^{w_1 t} + 1)(qe^{w_2 t} + 1)}.
\]

From (9) and (8), we can derive
Theorem 1. Let \( w_1, w_2 (\in \mathbb{N}) \) be odd and let \( n(\geq 0) \) with \( n \equiv 1 (\mod 2) \). Then we have

\[
\sum_{i=0}^{n} \binom{n}{i} H_i(-q^{-1}, w_2 x) S_{n-i, q w_2} (w_1 - 1) w_1^i w_2^{n-i}
\]

where \( H_n(q, x) \) are the \( n \)-th Frobenius-Euler polynomials.

Setting \( x = 0 \) in (13), we obtain the following corollary.
Corollary 2. Let \( w_1, w_2 \in \mathbb{N} \) be odd and let \( n \in \mathbb{Z}_+ \) be an odd. Then we have

\[
\sum_{i=0}^{n} \binom{n}{i} H_i(-q^{-1}) S_{n-i, q} (w_1 - 1) w_1^{n-i} w_2^{n-i} = \sum_{i=0}^{n} \binom{n}{i} H_i(-q^{-1}) S_{n-i, q} (w_2 - 1) w_2^{n-i} w_1^{n-i},
\]

where \( H_i(-q^{-1}) \) are the \( n \)-th Frobenius-Euler numbers.

If we take \( w_2 = 1 \) in (13), then we have

\[
H_n(-q^{-1}, w_1 x) = \sum_{i=0}^{n} \binom{n}{i} H_i(-q^{-1}, x) S_{n-i, q} (w_1 - 1) w_1^i.
\]

Setting \( x = 0 \) in (14), we obtain the following corollary.

Corollary 3. Let \( w_1 (> 1) \) be an odd integer and let \( n \in \mathbb{Z}_+ \) with \( n \equiv 1 \pmod{2} \). Then we have

\[
H_n(-q^{-1}) = \frac{1}{1 - w_1^n} \sum_{i=0}^{n-1} \binom{n}{i} H_i(-q^{-1}) S_{n-i, q} (w_1 - 1) w_1^i.
\]

From (7) and (8), we derive

\[
I = \left( \frac{e^{w_1 w_2 x t}}{[2]q} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1) \right) \left( \frac{[2]q \int_{\mathbb{Z}_p} e^{w_2 x_2 t} d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q(w_1 w_2 - 1)x d\mu_{-q}(x)} \right) \\
= \left( \frac{e^{w_1 w_2 x t}}{[2]q} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} d\mu_{-q}(x_1) \right) \left( [2]q \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} e^{w_2 l t} \right) \\
= \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} \int_{\mathbb{Z}_p} e^{(x_1 + w_2 x + \frac{w_2}{w_1} l) t w_1} d\mu_{-q}(x_1) \\
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{w_1-1} (-1)^l q^{w_2 l} H_n(-q^{-1}, w_2 x + \frac{w_2}{w_1} l) \right) \frac{t^n}{n!}.
\]

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On the other hand,

\[
I = \left( \frac{e^{w_1w_2xt}}{[2]_q} \int_{\mathbb{Z}_p} e^{w_2x^d}d\mu_q(x_2) \right) ^ \left( \frac{[2]_q \int_{\mathbb{Z}_p} e^{w_1x_1l}d\mu_q(x_1)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt}q(w_1w_2^{-1})x_2d\mu_q(x)} \right) \\
= \left( \frac{1}{[2]_q} \int_{\mathbb{Z}_p} e^{w_2x^d}d\mu_q(x_2) \right) ^ \left( \frac{\sum_{l=0}^{w_2-1} (l)q^{w_1l}e^{(w_1l+w_1w_2)t}}{[2]_q} \right) \\
= \sum_{l=0}^{w_2-1} (-1)^lq^{w_1l} \int_{\mathbb{Z}_p} e^{(x_2+w_1x+w_2l)t}d\mu_q(x_2) \\
= \sum_{n=0}^{\infty} \left( \frac{w_2^n}{n!} \sum_{l=0}^{w_2-1} (-1)^lq^{w_1l}H_n(-q^{-1}, x+w_1x+w_2l) \right) t^n.
\]

(16)

By comparing the coefficients on the both sides of 915) and (160, we obtain the following theorem.

**Theorem 4.** Let \(w_1, w_2(\in \mathbb{N})\) be odd and let \(n \in \mathbb{Z}_+\) with \(n \equiv 1\pmod{2}\). Then we have

\[
w_1^n \sum_{l=0}^{w_2-1} (-1)^lq^{w_2l}H_n(-q^{-1}, w_2x+w_2l) = w_2^n \sum_{l=0}^{w_2-1} (-1)^lq^{w_1l}H_n(-q^{-1}, w_1x+w_1l).
\]

Setting \(w_2 = 1\) in Theorem 4, we get the multiplication theorem for the Frobenius-Euler polynomials as follows:

\[
H_n(-q^{-1}, w_1x) = w_1^n \sum_{l=0}^{w_2-1} (-1)^lq^{l}H_n(-q^{-1}, x+l)w_1.
\]

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