LINES ON CUBIC SURFACES AND WITT INVARIANTS

EVA BAYER-FLUCKIGER AND JEAN-PIERRE SERRE

Introduction

The aim of this note is to give a formula expressing the trace form associated with the 27 lines of a cubic surface. We start with a few properties of the group \text{Weyl}(E_6) and the associated quadratic forms; the application to cubic surfaces is given at the end.

The main tool of the proof is the theory of the Witt invariants and especially their detection via “cubes”, following [Se 03] and [Se 18].

§1. Lattices associated with \text{Weyl}(E_6).

Let $R$ be a root system of type $E_6$, with basis $\{\alpha_1, \ldots, \alpha_6\}$ as in Bourbaki [Bo 68], §4.12, and let $\{\omega_1, \ldots, \omega_6\}$ be the corresponding fundamental weights. Let $Q$ be the root lattice, which is generated by the $\alpha_i$, and let $P$ be the weight lattice, which is generated by the $\omega_i$. We have $Q \subset P$ and $(P : Q) = 3$; let $e : P \to \mathbb{Z}/3\mathbb{Z}$ be the homomorphism with kernel $Q$ such that $e(\omega_1) = 1$. [This choice of $e$ means that we have “oriented” the Dynkin diagram, by choosing one of its extremities.]

The scalar product on $P$ will be denoted by $x.y$; it takes values in $\frac{1}{3}\mathbb{Z}$; for instance $\omega_1.\omega_1 = \frac{4}{3}$. We have $x.y \in \mathbb{Z}$ if $x \in P, y \in Q$; if $\alpha$ is a root, we have $\alpha.\alpha = 2$; we have $\alpha_i.\omega_j = \delta_{ij}$.

Let now $L$ be the sublattice of $\mathbb{Z} \oplus P$ made up of the pairs $(n, p)$ such that $n \equiv e(p) \pmod{3}$. We define a scalar product $q_L$ on $L$ by the formula:

$$q_L(n, p; n', p') = nn'/3 - p.p'.$$

Its values lie in $\mathbb{Z}$, and it is “$\mathbb{Z}$-unimodular”, i.e., it gives an isomorphism of $L$ onto its $\mathbb{Z}$-dual.

[As we shall recall in §7, the lattice $L$ is isomorphic to the Néron-Severi group of a smooth cubic surface, and the scalar product $q_L$ corresponds to the intersection form.]

Note that $Q$ embeds in $L$ by $x \mapsto (0, x)$; this embedding transforms $q_L$ into the opposite of the scalar product of $Q$. In particular, a root $\alpha$ may be viewed as an element of $L$ such that $q_L(\alpha, \alpha) = -2$.

The intersection of $L$ with $\mathbb{Z}$ is generated by the element $h = (3, 0)$. We have $q_L(h, h) = 3$ and $q_L(h, x) = 0$ if $x \in Q$.

Let $G$ be the Weyl group of $R$, i.e., the subgroup of $\text{Aut}(Q \otimes \mathbb{R})$ generated by the reflections $s_\alpha$ associated to the roots $\alpha \in R$. The group $G$ acts on $P$ and $Q$. We extend its action to $\mathbb{Z} \oplus P$, and hence to $L$, by making it act trivially on the factor $\mathbb{Z}$.

\textit{Date:} September 13, 2019.
Proofs of the following theorem can be found in the standard texts on cubic surfaces (cf. [Ma 74], chap.IV, [De 80], [Do 12], chap.9):

**Theorem 1.**

(a) Let \( Y \) be the set of \( y \in L \) such that \( q_L(h, y) = 1 \) and \( q_L(y, y) = -1 \). This set has 27 elements, namely the pairs \((1, \omega)\) where \( \omega \) belongs to the \( G \)-orbit of \( \omega_1 \).

(b) If \( y, y' \) are two distinct elements of \( Y \), then \( q_L(y, y') = 0 \) or 1.

(c) Let \( \Omega \) be the graph with set of vertices \( Y \), two vertices \( y, y' \) being adjacent if \( q_L(y, y') = 1 \). The natural injection \( G \to \text{Aut}(\Omega) \) is bijective.

**Remark.**

Since \( \omega_1 \) is orthogonal to the \( \alpha_i \) for \( i \geq 2 \), it is fixed by the group \( H \) generated by \( (s_{\alpha_2}, \ldots, s_{\alpha_6}) \), which is a Weyl group of type \( D_5 \), and has index 27 in \( G \). Hence \( Y \simeq G/H \simeq \text{Weyl}(E_6)/\text{Weyl}(D_5) \).

§2. A combinatorial description of the 27-vertices graph \( \Omega \).

Let us recall how one can describe the graph of Theorem 1 in terms of a so-called “double-six”.

Let \( X = \{1, \ldots, 6\} \), and let \( X' \) be a copy of \( X \); if \( x \in X \), the corresponding point of \( X' \) is denoted by \( x' \); let \( S \) be the set of all subsets of \( X \) with 2 elements. The graph \( \Omega \) of Theorem 1 is isomorphic to the graph \( \Omega_X \) whose set of vertices is the disjoint union \( X \cup X' \cup S \), two vertices being adjacent in the following cases (and only in those):

\[
\begin{align*}
& x \in X \text{ adjacent to } y' \in X' \iff x \neq y, \\
& x \in X \text{ adjacent to } s \in S \iff x \in s, \\
& x' \in X' \text{ adjacent to } s \in S \iff x \in s, \\
& s_1 \in S \text{ adjacent to } s_2 \in S \iff s_1 \cap s_2 = \varnothing.
\end{align*}
\]

The group \( \text{Sym}_6 \) of permutations of \( X \) acts on \( \Omega_X \). Let \( \epsilon \) be the automorphism of order 2 of \( \Omega_X \) which fixes the points of \( S \) and exchanges \( x \in X \) with \( x' \in X' \); that automorphism commutes with the action of \( \text{Sym}_6 \), and we thus obtain an embedding of the group \( \{1, \epsilon\} \times \text{Sym}_6 \) into \( \text{Aut}(\Omega_X) \). From the Weyl group point of view, this corresponds to the embedding of \( \text{Weyl}(A_1 \times A_5) \simeq \{1, \epsilon\} \times \text{Sym}_6 \) into \( \text{Weyl}(E_6) \) defined by the inclusion \( A_1 \times A_5 \to E_6 \).

§3. A maximal cube of \( \text{Weyl}(E_6) \).

According to [Se 18], a cube of a Weyl group is an abelian subgroup generated by reflections. In the case of \( G = \text{Weyl}(E_6) \), the maximal cubes of \( G \) have order \( 2^4 \) and are conjugate to each other; there are 135 of them.

In terms of the combinatorial description of §2, we may choose for maximal cube the group \( C \) generated by the following four reflections: the three transpositions \((12),(34),(56)\), and the automorphism \( \epsilon \).

We shall need later:

**Lemma 1.** The action of \( C \) on the set \( Y \) has three fixed points, and six orbits of order 4; these orbits are isomorphic to \( C/C_i \), \( i = 1, \ldots, 6 \), where the \( C_i \) are the six cubes of order 4 contained in \( C \).
Proof. The action of $C$ fixes the points $\{1, 2\}, \{3, 4\}, \{5, 6\}$ of $S$. The four points $\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$ of $S$ make up an orbit isomorphic to $C/C_1$ where $C_1$ is the subgroup of $C$ generated by $\epsilon$ and the transposition $(56)$; there are two other similar orbits in $S$. In $X \cup X'$, the points $1, 1', 2, 2'$ make up an orbit isomorphic to $C/C_4$, where $C_4$ is generated by the transpositions $(34)$ and $(56)$; there are two other similar orbits.

§4. The $G$-quadratic forms $q_6, q_7, q_{27}$ and the $C$-quadratic form $q_4$.

4.1. Recall that, if $G$ is any finite group, a $G$-quadratic form over a commutative ring $A$ is a symmetric bilinear form $q$ over an $A$-module $E$, together with an $A$-linear action of $G$ on $E$ which fixes $q$. In the above case, where $G = \text{Weyl}(E_6)$, we have three such examples, with $A = \mathbb{Z}$, namely:

(i) The bilinear form $\alpha.\beta$ on $Q$.

(ii) The bilinear form $q_L$ on $L$.

(iii) The bilinear form $q_Y$ on $\mathbb{Z}^Y$, given by $q_Y(e_y, e_{y'}) = \delta_{y'}^y$, where $(e_y)_{y \in Y}$ is the natural basis of $Y$.

In each case, the action of $G$ is the obvious one.

4.2. Let $k$ be a field of characteristic $\neq 2, 3$. The base change $\mathbb{Z} \to k$ transforms the three $G$-quadratic forms above into $G$-quadratic forms over $k$. These forms are non-degenerate (and $q_7, q_{27}$ are also non-degenerate in characteristic $3$). We shall denote them by $q_6, q_7, q_{27}$, but, for convenience we shall divide by 2 the first one. In other words, we put:

$$q_6(\alpha, \beta) = \frac{1}{2} \alpha.\beta,$$

so that $q_6(\alpha, \alpha) = 1$ for every $\alpha \in R$.

With standard notation, we have an isomorphism of $G$-quadratic forms:

$$(4.2.1)\quad q_7 = \langle 3 \rangle \oplus \langle -2 \rangle q_6,$$

with $G$ acting trivially on the rank-1 form $\langle 3 \rangle$.

There is no such simple formula for the $G$-form $q_{27}$, since the underlying linear representation of $G$ is not a linear combination of exterior powers of the standard degree 6 representation.

4.3. The $C$-quadratic form $q_4$.

Let $C$ be the maximal cube introduced in §3. It contains four reflections $s_1, \ldots, s_4$, corresponding to mutually orthogonal roots $\beta_1, \ldots, \beta_4$. Let $V$ be the $k$-vector subspace of $Q \otimes k$ generated by the $\beta_i$, and let $q_4$ denote the restriction of $q_6$ to $V$; we have $q_4(\beta_i, \beta_j) = \delta^i_j$. The group $C$ acts in a natural way on $V$; hence $q_4$ is a $C$-quadratic form of rank 4. The space $V$ splits under the action of $C$ into four 1-dimensional subspaces, orthogonal to each other. This gives a splitting of $q_4$ as:

$$(4.3.1)\quad q_4 = r_1 + \cdots + r_4,$$

where $r_i$ is the $C$-quadratic form of rank 1 generated by $\beta_i$, on which $s_i$ acts by $-1$, and the other $s_j$ act trivially.

4.4. Relations between the $G$-quadratic forms $q_6, q_7, q_{27}$ and the $C$-quadratic form $q_4$. 
Any $G$-quadratic form $q$ defines, by restriction of the action of the group, a $C$-quadratic form, which we shall denote by $q|C$. This applies in particular to $q_6, q_7, q_{27}$. The $C$-quadratic forms so obtained can all be expressed in terms of $q_4$. Namely:

**Theorem 2.** We have the following isomorphisms of $C$-quadratic forms:

\[(4.4.1)\] $q_6|C = q_4 + \langle 2, 6 \rangle$,

\[(4.4.2)\] $q_7|C = \langle -2 \rangle q_4 + \langle -1, -1, 1 \rangle$,

\[(4.4.3)\] $q_{27}|C = \lambda^2 q_4 + 3 q_4 + 9$.

[Here, $\langle 2, 6 \rangle$ means the quadratic form $\langle 2 \rangle \oplus \langle 6 \rangle$ with trivial action of $C$. Similarly, $3q_4$ means $\langle 1, 1, 1 \rangle \otimes q_4 = q_4 \oplus q_4 \oplus q_4$ and 9 means the direct sum of nine copies of $\langle 1 \rangle$. As for $\lambda^2 q_4$, it is the second exterior power of $q_4$, with its natural action of $C$.]

**Proof of (4.4.1).** This is a simple computation in the root lattice $Q$. With Bourbaki’s notation ([Bo 68], §4.12), we may choose for $\beta_i$ the roots $\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_3 + \epsilon_4, \epsilon_3 - \epsilon_4$. They are orthogonal to $\gamma = \epsilon_5$ and $\delta = \epsilon_8 - \epsilon_6 - \epsilon_7$. Moreover, $\gamma$ and $\delta$ are orthogonal to each other, $\frac{1}{2} \gamma, \gamma = \frac{1}{2}$, and $\frac{1}{2} \delta, \delta = \frac{3}{2}$. Hence the orthogonal $V'$ of $V$ in $Q \otimes k$ is quadratically isomorphic to $\langle \frac{1}{2}, \frac{3}{2} \rangle = (2, 6)$, and the action of $C$ on $V'$ is trivial.

**Proof of (4.4.2).** We have

\[q_7|C = \langle 3 \rangle + \langle -2 \rangle q_6|C\] by (4.2.1),

\[= \langle 3 \rangle + \langle -2 \rangle (q_4 + \langle 2, 6 \rangle)\] by (4.4.1),

\[= \langle -2 \rangle q_4 + \langle 3, -1, -1 \rangle,\]

\[= \langle -2 \rangle q_4 + \langle -1, -1, 1 \rangle \text{ since } \langle 3, -3 \rangle = \langle 1, -1 \rangle.

**Proof of (4.4.3).** Lemma 1 gives a decomposition of $q_{27}|C$ as the orthogonal sum of $\langle 1, 1, 1 \rangle$ with trivial action, and six $C$-quadratic forms $q_i'$ associated with the permutation sets $C/C_i$, where the $C_i$ are the six cubes of order 4 contained in $C$. Consider for instance the case where $C_i$ is generated by $s_1, s_2$, as in §4.3. In that case, one finds that $q_i' = 1 + r_3 + r_4 + r_3r_4$, where the $r_i$ are the $C$-quadratic forms of rank 1 occurring in (4.3.1). The other $q_i'$ correspond similarly to the pairs (13), (14), (23), (24), (34). Adding up gives:

\[q_{27}|C = 3 + 6 + 3 \sum_{n=1}^{4} r_n + \sum_{1 \leq m < n \leq 4} r_mr_n,\]

By (4.3.1), we have $\sum_{n=1}^{4} r_n = q_4$ and $\sum_{1 \leq m < n \leq 4} r_mr_n = \lambda^2 q_4$. We thus obtain (4.4.3).

**§5. Twists.**

We keep the assumption that the characteristic of $k$ is $\neq 2, 3$. Let $k_s$ be a separable closure of $k$. Let $\Gamma_k = \text{Gal}(k_s/k)$ be the “absolute Galois group” of $k$. If $\varphi : \Gamma_k \rightarrow G$ is a continuous homomorphism, we can use $\varphi$ to twist (cf. [Se 94], chap.III, §1) any $G$-quadratic form $q$ over $k$ and we thus find a quadratic form $q_\varphi$ over $k$. This applies in particular to the $G$-forms $q_6, q_7, q_{27}$ above. Relation (4.2.1) implies:

\[(5.1)\] $q_7, \varphi = \langle 3 \rangle + \langle -2 \rangle q_6, \varphi,$
where the + sign means addition in the Witt-Grothendieck group of \( k \).

What is less obvious is that there is also a formula for \( q_{27,\varphi} \) in terms of \( q_{6,\varphi} \):

**Theorem 3.** We have:

\[
q_{27,\varphi} = \lambda^2(q_{6,\varphi}) + \langle 3 \rangle q_{6,\varphi} + 6.
\]

The proof will be given in §6.

**Remarks.**

1. By using (4.2.1), we may express \( q_{27,\varphi} \) in terms of \( q_{7,\varphi} \). The result is:

\[
q_{27,\varphi} = \lambda^2(q_{7,\varphi}) + \langle -1 \rangle - \langle 2 \rangle q_{7,\varphi} + 7 - \langle -2 \rangle.
\]

This less appetizing formula has the advantage of making sense (and being true) also in characteristic 3, in which case it reduces to \( q_{27,\varphi} = \lambda^2(q_{7,\varphi}) + 6 \).

2. The quadratic form \( q_{27,\varphi} \) may also be viewed as a trace form. Indeed, the group \( \Gamma_k \) acts on \( Y \) via \( \varphi \), and this defines an étale algebra of rank 27 over \( k \) whose trace form is \( q_{27,\varphi} \), cf. e.g., [BS 94], §1.

3. We have here chosen the case of \( Y = G/H \), where \( H = \text{Weyl}(D_5) \). A similar method can be applied to any \( G \)-set; it gives a formula expressing the corresponding trace form as a linear combination of the \( \lambda^n(q_{7,\varphi}) \), \( n = 0, 1, ..., 4 \), the coefficients being \( \mathbb{Z} \)-linear combinations of \( 1 \), \( \langle -1 \rangle \), \( \langle 2 \rangle \), and \( \langle -2 \rangle \). What is remarkable in the case \( H = \text{Weyl}(D_5) \) is that the higher exterior powers \( \lambda^3 \) and \( \lambda^4 \) do not occur in the formula.

The proof follows the method used in [Se 03] for the symmetric groups, and generalized in [Se 18] to all Weyl groups: checking first the case where \( \varphi : \Gamma_k \to G \) takes values in the maximal cube \( C \), and then showing that this special case implies the general one.

The quadratic form \( q_{5,\varphi} \).

**Lemma 2.** The quadratic form \( q_{6,\varphi} \) represents 6.

(Hence, there is a uniquely defined \( q_{5,\varphi} \) such that \( q_{5,\varphi} = q_{5,\varphi} + \langle 6 \rangle \).

**Proof.** By a standard theorem on quadratic forms, it is enough to prove this after replacing \( k \) by any odd-degree finite extension. Since \( H \) has odd degree in \( G \), this means that we may assume that \( \varphi \) maps \( \Gamma_k \) into \( H \). The group \( H = \text{Weyl}(D_5) \) has a natural representation of degree 5 (namely on its root lattice), which is isomorphic to the standard one (on \( k^5 \), by permutations and odd number of sign changes on the natural basis \( e_1, ..., e_5 \) of \( k^5 \)); let \( q_5 \) be the unit quadratic form \( \langle 1, 1, 1, 1, 1 \rangle \) on \( k^5 \); it is invariant by \( H \), hence its twist \( q_{5,\varphi} \) by \( \varphi \) makes sense. We have the following isomorphism of \( H \)-quadratic forms:

\[
(*) \quad q_6|H = q_5 + \langle 6 \rangle,
\]

where \( q_6|H \) means \( q_6 \), viewed as an \( H \)-form.

**§6. Proof of Theorem 3.**

6.1. **Proof of Theorem 3 when \( \varphi \) maps \( \Gamma_k \) into \( C \).**

By (4.4.1) and (4.4.3), we have:

\[
q_{6,\varphi} = q_{4,\varphi} + \langle 2, 6 \rangle,
\]
(6.1.2) \( q_{27,\varphi} = \lambda^2 q_{4,\varphi} + 3q_{4,\varphi} + 9. \)

The first formula implies:

(6.1.3) \( \lambda^2 q_{6,\varphi} = \lambda^2 q_{4,\varphi} + \langle 2, 6 \rangle q_{4,\varphi} + \langle 3 \rangle, \)

hence:

(6.1.4) \( \lambda^2 q_{6,\varphi} + \langle 3 \rangle q_{6,\varphi} + 6 = \lambda^2 q_{4,\varphi} + \langle 2, 3, 6 \rangle q_{4,\varphi} + \langle 2, 3, 6 \rangle + 6; \)

since \( \langle 3, 6 \rangle \) represents 1 (because \( 3 + 6 = 3^2 \)), we have \( \langle 3, 6 \rangle = \langle 2, 1 \rangle \), hence \( \langle 2, 3, 6 \rangle = \langle 2, 2, 1 \rangle = (1, 1, 1) = 3 \), and we may rewrite (6.1.4) as:

(6.1.5) \( \lambda^2 q_{6,\varphi} + \langle 3 \rangle q_{6,\varphi} + 6 = \lambda^2 q_{4,\varphi} + 3q_{4,\varphi} + 9. \)

By comparing (6.1.2) and (6.1.5) we obtain (5.2).

### 6.2. The Witt-Grothendieck invariants defined by \( q_6, q_7, q_{27} \).

(In what follows, we use freely the definitions and the elementary properties of the “invariant” of an algebraic group given in the first sections of [Se 03].)

The cohomology set \( H^1(k, G) \) of all \( G \)-torsors over \( k \) can be canonically identified with the conjugation classes of continuous homomorphisms \( \varphi : \Gamma_k \to G \), cf. e.g. [BS 94], §1. Since the Galois twists defined by conjugate homomorphisms are the same, we may interpret the maps \( \varphi \mapsto q_{6,\varphi}, q_{7,\varphi}, q_{27,\varphi} \) as maps of \( H^1(k, G) \) into the Witt-Grothendieck ring \( WGr(k) \) of \( k \); let us denote them by \( a_{6,k}, a_{7,k}, a_{27,k} \). This construction applies to all the field extensions \( K \) of \( k \), and we thus obtain three Witt-Grothendieck invariants \( a_6, a_7, a_{27} \) of \( G \), i.e. three elements of the group \( \text{Inv}(G, WGr) \), cf. [Se 03], VIII. For every subgroup \( H \) of \( G \), there is a natural restriction map \( \text{Inv}(G, WGr) \to \text{Inv}(H, WGr) \), cf. [Se 03], §13.

A basic fact about invariants is:

**Theorem 4.** Assume that the characteristic of \( k \) is \( \neq 2 \). Let \( G \) be a Weyl group, and let \( a \in \text{Inv}(G, WGr) \). Assume that the restriction of \( a \) to every cube of \( G \) is 0. Then \( a = 0 \).

This is proved in [Se 03], §29, when \( G \) is a symmetric group (for Witt invariants - the case of the Witt-Grothendieck invariants follows). The proof for an arbitrary Weyl group is similar; it only requires some extra arguments for the small characteristics, such as 3 for \( G_2, E_6, E_7, E_8 \) and 5 for \( E_8 \). Details will hopefully be given in [Se ??].

### 6.3. End of the proof of Theorem 3.

We apply Theorem 4 with \( G = G \), and with \( a \in \text{Inv}(G, WGr) \) defined by:

(6.3.1) \( a = a_{27} - \lambda^2 a_6 - \langle 3 \rangle a_6 - 6. \)

By §6.1, the restriction of \( a \) to \( C \) is 0. Since every cube \( C' \) of \( G \) is conjugate to a subgroup of \( C \), the restriction of \( a \) to \( C' \) is 0. By Theorem 4, this implies \( a = 0 \); hence (5.2).

**Remark.** The same method can be used to give the structure of \( \text{Inv}(G, WGr) \). The result is simpler to state for the Witt invariant ring \( \text{Inv}(G, W) \): this ring is a free \( W(k) \)-module with basis the five elements \( \lambda^i a_6, i = 0, \ldots, 4 \).

§7. The cubic surfaces and their 27 lines.
7.1. Here, we drop our assumptions on the ground field $k$, i.e., we accept \( \text{char}(k) = 2 \) or 3.

Let $V \subset \mathbb{P}_3$ be a smooth cubic surface over $k$. It is well known that, over a suitable field extension of $k$, it contains 27 lines, cf. [Do 12], chap.9 and [Ma 74], chap.IV. These lines are rational over $k$, cf. [Co 88], Theorem 1 and [KW 17], Corollary 52. Let $L_V$ be the Néron-Severi group of $V$ over $k_s$, equipped with the symmetric $\mathbb{Z}$-bilinear form “intersection product”. It is a lattice of rank 7, and it contains the following elements:

(a) The class $h_V$ of the hyperplane sections of $V$; we have $h_V.h_V = 3$.

(b) The set $Y_V$ of the classes of the 27 lines; if $y \in Y_V$, we have $y.y = -1$ and $h_V.y = 1$; if $y' \in Y_V$ is distinct from $y$, we have $y.y' = 0$ if the corresponding lines are disjoint, and $y.y' = 1$ if they meet.

It is well known that the triple $T_V = (L_V, h_V, \text{intersection product})$ is isomorphic to the triple $T = (L, h, q_L)$ of §1; see the above references. More precisely, let $\Theta_V$ denote the set of isomorphisms $\theta : T_V \to T$. We have a left action of $G = \text{Weyl}(\mathbb{E}_6) = \text{Aut}(T)$ on $\Theta_V$, by $g\theta = g \circ \theta$; that action is free, and transitive. On the other hand, we have a right action of $\Gamma_k$ on $\Theta_V$, by $\theta \gamma = \theta \circ \gamma$, for $\gamma \in \Gamma_k$. These two actions commute. We may view such a situation in the following equivalent ways (cf. [BS 94], §1.3):

\begin{enumerate}
\item[(7.1.1)] The action of $\Gamma_k$ on $\Theta_V$ defines an étale algebra $E_V$, on which $G$ acts, and one hence gets a $G$-Galois algebra over $k$.
\item[(7.1.2)] The $k$-finite étale scheme Spec $E_V$ is a $G$-torsor over $k$, whose set of $k_s$-points is $\Theta_V$.
\item[(7.1.3)] If we choose a point $\theta$ of $\Theta_V$, for every $\gamma \in \Gamma_k$ there is a unique element $\varphi(\gamma)$ of $G$ such that $\varphi(\gamma)\theta = \theta\gamma$, and the map $\gamma \mapsto \varphi(\gamma)$ is a continuous homomorphism $\varphi : \Gamma_k \to G$. Changing the choice of $\theta$ replaces $\varphi$ by a conjugate.
\end{enumerate}

Each of these points of view show that \textit{we have associated to $V$ a $G$-torsor over $k$, i.e. an element $e_V$ of $H^1(k, G)$}.

7.2. Assume now that $\text{char}(k) \neq 2$. As in §5, we may twist the quadratic forms $q_7$ and $q_{27}$ by the torsor $e_V$ defined above. If we define $\varphi : \Gamma_k \to G$ as in (7.1.3), we obtain the quadratic forms $q_{7, \varphi}$ and $q_{27, \varphi}$; if, moreover $\text{char}(k) \neq 3$, we obtain similarly a quadratic form $q_{6, \varphi}$. Since these forms depend only of the cubic surface $V$, we may denote them by $q_{7,V}$, $q_{27,V}$ and $q_{6,V}$. By (5.2) and (5.3), we have:

**Theorem 5.**

\begin{enumerate}
\item[(7.2.1)] $q_{27,V} = \lambda^2(q_{6,V}) + \langle 3 \rangle q_{6,V} + 6$, if $\text{char}(k) \neq 3$.
\item[(7.2.2)] $q_{27,V} = \lambda^2(q_{7,V}) + \langle -1 - 2 \rangle q_{7,V} + 7 - \langle -2 \rangle$.
\end{enumerate}

7.3. **Interpretations of $q_{7,V}$ and $q_{27,V}$**.

(a) The case of $q_{7,V}$.

Let us denote by $V_s$ the $k_s$-variety deduced from $V$ by the base change $k \to k_s$. The Néron-Severi group $L_V$ is equal to the divisor class group $H^1(V_s, \mathcal{O}^*_{V_s})$. The map $f \mapsto df/f$ gives a homomorphism $\mathcal{O}^*_{V_s} \to \Omega^1_{V_s}$; we thus get an homomorphism $L_V \to H^1(V_s, \Omega^1_{V_s})$, hence also $k_s \otimes \mathbb{Z} L_V \to H^1(V_s, \Omega^1_{V_s})$. 


Since $V_s$ is a smooth rational surface, this map is an isomorphism. Moreover, it transforms the intersection form on $k_s \otimes \mathbb{Z} L_V$ into the cup-product $H^1(V_s, \Omega^1_{V_s}) \otimes H^1(V_s, \Omega^1_{V_s}) \to H^2(V_s, \Omega^2_{V_s}) \simeq k_s$. By descent, this gives an interpretation of $q_{7,V}$ as the quadratic space $H^1(V_s, \Omega^1_{V_s})$ endowed with its natural cup-product form.

(b) The case of $q_{27,V}$.

That quadratic form is the trace form of the étale algebra $A_{27}$ defined by the $\Gamma$-set $Y_V$ of the 27 lines. One may also view $A_{27}$ as the subalgebra of the $G$-Galois algebra $E_V$ of (7.1.1) which is fixed by the subgroup $H \simeq \text{Weyl}(D_5)$ defined at the end of §1.

7.4. A question.

It would be interesting to be able to compute the quadratic form $q_{7,V}$ (and hence also $q_{27,V}$) directly from the knowledge of the cubic equation $F$ defining $V$; this would certainly involve using the invariants of $F$ defined by Sylvester. A first step in that direction is the computation of the first Stiefel-Whitney class $w_1(q_{7,V})$ by T. Saito ([Sa 12], §5.4)): it is the element of $H^1(k, \mu_2) = k^*/k^{*2}$ defined by $-\text{disc}_d(F)$, where $\text{disc}_d$ is the “divided discriminant”; this amounts to computing the homomorphism $\Gamma_k \to G \to \{1, -1\}$ associated with $V$. The next step would be the determination of $w_2(q_{7,V})$.

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