MONTEIRO SPACES AND ROUGH SETS DETERMINED
BY QUASIORDER RELATIONS: MODELS FOR NELSON
ALGEBRAS

JOUNI JÄRVINEN AND SÁNDOR RADELECKI

Abstract. The theory of rough sets provides a widely used modern
tool, and in particular, rough sets induced by quasiorders are in the fo-
cus of the current interest, because they are strongly interrelated with
the applications of preference relations and intuitionistic logic. In this
paper, a structural characterisation of rough sets induced by quasiorders
is given. These rough sets form Nelson algebras defined on algebraic
lattices. We prove that any Nelson algebra can be represented as a
subalgebra of an algebra defined on rough sets induced by a suitable
quasiorder. We also show that Monteiro spaces, rough sets induced by
quasiorders and Nelson algebras defined on $T_0$-spaces that are Alex-
drov topologies can be considered as equivalent structures, because they
determine each other up to isomorphism.

1. Rough sets

The theory of rough sets introduced by Pawlak [19] can be viewed as
an extension of the classical set theory. Its fundamental idea is that our
knowledge about the properties of the objects of a given universe of discourse
$U$ may be inadequate or incomplete in a sense that the objects of the universe
$U$ can be observed only within the accuracy of indiscernibility relations.
According to Pawlak’s original definition, an indiscernibility relation $E$ on
$U$ is an equivalence relation (reflexive, symmetric, and transitive binary
relation) interpreted so that two elements of $U$ are $E$-related if they cannot
be distinguished by their properties known by us. Since there is a one-to-one
correspondence between equivalences and partitions, each indiscernibility
relation induces a partition on $U$ such that its blocks consist of objects that
are precisely similar with respect to our information. In this sense, our
ability to distinguish objects can be understood to be blurred – we cannot
distinguish individual objects, only groups of similar objects. But this is
often the case in practice; we may have objects that are indistinguishable
by their properties, but one of them belongs to some set (e.g. people that
have certain disease), while the other one does not.

Each subset $X$ of $U$ can be approximated by two sets: the lower approxi-
mation $X\uparrow$ of $X$ consists of the $E$-equivalence classes that are included in $X$,
and the upper approximation $X\downarrow$ of $X$ contains the $E$-classes intersecting
$X$. The lower approximation $X\uparrow$ can be viewed as the set of elements that are
certainly in $X$ and the upper approximation $X\downarrow$ can be considered as the

Acknowledgements: The research of the second author was carried out as part of the
TAMOP-4.2.1.B-10/2/KONV-2010-0001 project supported by the European Union, co-
financed by the European Social Fund.
set of elements that possibly belong to \( X \). A consequence of this is that the membership functions of sets become three-valued: 1 (the element belongs to the set), 0 (the element is not in the set), \( u \) (unknown borderline case: the element is simultaneously inside and outside the set, at some degree).

Two subsets \( X \) and \( Y \) of \( U \) are defined to be \( \equiv \)-related if both of their approximations are the same, that is, \( X \downarrow = Y \downarrow \) and \( X \uparrow = Y \uparrow \). Clearly, the relation \( \equiv \) is an equivalence, and its equivalence classes are called rough sets. Each element in the same rough set looks the same, when observed through the knowledge given by the indiscernibility relation \( E \). Namely, if \( X \equiv Y \), then exactly the same elements belong certainly and possibly to \( X \) and \( Y \).

Lattice-theoretical study of rough sets was initiated by T. B. Iwiński in [9]. He pointed out that since each rough set is uniquely determined by the lower and the upper approximations of its members, the set of rough sets can be defined as
\[
RS = \{ (X \downarrow, X \uparrow) \mid X \subseteq U \}.
\]
In addition, \( RS \) may be canonically ordered by the coordinatewise order:
\[
(X \downarrow, X \uparrow) \leq (Y \downarrow, Y \uparrow) \iff X \downarrow \subseteq Y \downarrow \text{ and } X \uparrow \subseteq Y \uparrow.
\]

In computer science, rough sets represent a widely used modern tool; they are applied, for instance, to approximative reasoning in feature selection problems, learning theory, and combined with methods of fuzzy sets or of formal concept analysis they are used in data mining also [7].

In the literature can be found numerous studies on rough sets that are determined by different types of relations reflecting distinguishability or indistinguishability of the elements of the universe of discourse \( U \) (see e.g. [5]). If \( R \subseteq U \times U \) is an arbitrary binary relation, then the lower and upper approximations of a set \( X \subseteq U \) are defined as follows. For any \( x \in U \), we denote \( R(x) = \{ y \in U \mid x R y \} \). The lower approximation of \( X \) is
\[
X \downarrow = \{ x \in U \mid R(x) \subseteq X \},
\]
and \( X \)'s upper approximation is
\[
X \uparrow = \{ x \in U \mid R(x) \cap X \neq \emptyset \}.
\]

If \( R \) is reflexive, then \( X \downarrow \subseteq X \subseteq X \uparrow \). In the case \( R \) is a quasiorder, that is, \( R \) is a reflexive and transitive binary relation on \( U \), we have \( x R y \iff R(y) \subseteq R(x) \), and the map \( X \mapsto X \uparrow \) is a topological closure operator and \( X \mapsto X \downarrow \) is a topological interior operator on the set \( U \) (see [10]).

Rough sets induced by quasiorders are in the focus of current interest; see [11–13, 16], for example. Let us denote by \( \wp(U) \) the power set of \( U \), that is, the set of all subsets of \( U \). It was shown by J. Järvinen, S. Radeleczki, and L. Veres [13] that \( RS \) is a complete sublattice of \( \wp(U) \times \wp(U) \) ordered by the coordinatewise set-inclusion relation, which means that \( RS \) is an algebraic completely distributive lattice such that
\[
\bigwedge \{ (X \downarrow, X \uparrow) \mid X \in \mathcal{H} \} = \left( \bigcap_{X \in \mathcal{H}} X \downarrow, \bigcap_{X \in \mathcal{H}} X \uparrow \right)
\]
and
\[ \bigvee \{ (X^\updownarrow, X^\downarrow) \mid X \in \mathcal{H} \} = \left( \bigcup_{X \in \mathcal{H}} X^\updownarrow, \bigcup_{X \in \mathcal{H}} X^\downarrow \right) \]
for all \( \mathcal{H} \subseteq \wp(U) \). Since \( RS \) is a completely distributive complete lattice, it is a Heyting algebra, that is, a lattice with 0 such that for each \( a, b \), there is a greatest element \( x \) with \( a \land x \leq b \). This element is the relative pseudocomplement of \( a \) with respect to \( b \), and is denoted \( a \Rightarrow b \).

Constructive logic with strong negation was introduced by Nelson [17] and independently by Markov [14]. It is often called simply as Nelson logic. It is an extension of the intuitionistic propositional logic by strong negation \( \sim \). The intuitive reading of \( \sim A \) is “a counterexample of \( A \)”. As described in [23], one sentence \( A \) may have many counterexamples and each of them needs to contradict \( A \). For instance, a counterexample of the sentence “This apple is red” is “This apple is green” or “This apple is yellow”. The axioms of Nelson logic can be interpreted as “algorithms” of constructing counterexamples of compound sentences by means of given counterexamples of their components, and the name strong negation comes from the fact that the formula \( \sim A \rightarrow \sim A \) is a theorem of the logic. Nelson logic is axiomatized by extending intuitionistic logic with the formulas:

(NL1) \( \sim A \rightarrow (A \rightarrow B) \)
(a counterexample of \( A \) contradicts \( A \), that is, \( A \land \sim A \) implies everything)

(NL2) \( \sim(A \rightarrow B) \leftrightarrow A \land \sim B \)
(a counterexample of \( A \rightarrow B \) can be constructed by the conjunction of \( A \) with a counterexample of \( B \))

(NL3) \( \sim(A \land B) \leftrightarrow \sim A \lor \sim B \)
(a counterexample of a conjunction can be constructed as a disjunction of counterexamples of its components)

(NL4) \( \sim(A \lor B) \leftrightarrow \sim A \land \sim B \)
(a counterexample of a disjunction can be can be constructed as a conjunction of counterexamples of its components)

(NL5) \( \sim \sim A \leftrightarrow A \)
(\( A \) is a counterexample of \( \sim A \))

(NL6) \( \sim \sim A \leftrightarrow A \)
(\( A \) is a counterexample of a counterexample of \( A \))

A Nelson algebra is a structure \( \mathcal{A} = (A, \lor, \land, \rightarrow, \sim, 0, 1) \) such that \((A, \lor, \land, 0, 1)\) is a bounded distributive lattice and for all \( a, b, c \in A \):

(N1) \( \sim \sim a = a \),

(N2) \( a \leq b \) if and only if \( \sim b \leq \sim a \),

(N3) \( a \land \sim a \leq b \lor \sim b \),

(N4) \( a \land c \leq \sim a \lor b \) if and only if \( c \leq a \rightarrow b \),

(N5) \( (a \land b) \rightarrow c = a \rightarrow (b \rightarrow c) \).

Nelson algebras provide models for constructive logic with strong negation, as shown by H. Rasiowa [21].

In each Nelson algebra, an operation \( \neg \) can be defined as \( \neg a = a \rightarrow 0 \). The operation \( \neg \) is called weak negation. A Nelson algebra \( \mathcal{A} \) is semi-simple if \( a \lor \neg a = 1 \) for all \( a \in A \). It is well known that semi-simple Nelson
We denote this Nelson algebra by $RS_L$ of some compact elements of a complete lattice with the property that any element of it is equal to the join by a quasiorder, the bounded distributive lattice $RS$ forms a Nelson algebra.

We denote this Nelson algebra by $RS$, and the operations are defined by:

$$
(X^v, X^a) \lor (Y^v, Y^a) = (X^v \cup Y^v, X^a \cup Y^a),
$$
$$
(X^v, X^a) \land (Y^v, Y^a) = (X^v \cap Y^v, X^a \cap Y^a),
$$
$$
\neg(X^v, X^a) = (-X^v, -X^a),
$$
$$
(X^v, X^a) \rightarrow (Y^v, Y^a) = ((-X^v \cup Y^v)^v, -X^v \cup Y^a),
$$

where $-X$ denotes the set-theoretical complement $U \setminus X$ of the subset $X \subseteq U$. The 0-element is $(\emptyset, \emptyset)$ and $(U, U)$ is the 1-element (see also [11]).

We showed in [12] that if $A$ is a Nelson algebra defined on an algebraic lattice, then there exists a set $U$ and a quasiorder $R$ on $U$ such that $A$ and the Nelson algebra $RS$ are isomorphic. Note that an algebraic lattice $L$ is a complete lattice with the property that any element of it is equal to the join of some compact elements of $L$. In [11], we proved an algebraic completeness theorem for Nelson logic in terms of finite rough set-based Nelson algebras determined by quasiorders.

2. Monteiro spaces

An Alexandrov topology $T$ on $X$ is a topology in which an arbitrary intersection of open sets is open, or equivalently, every point $x \in X$ has the least neighbourhood $N(x) \subseteq T$. For an Alexandrov topology $T$, the least neighbourhood of a point $x$ is $N(x) = \bigcap\{B \in T \mid x \in B\}$. We denote by $C$ and $I$ the closure and the interior operators of $T$, respectively. Then, $T = \{I(B) \mid B \subseteq X\}$. Additionally, $B_T = \{N(x) \mid x \in X\}$ forms a smallest base of $T$, implying that for all $B \in T$, $B = \bigcup_{x \in B} N(x)$. Note that a complete ring of sets means exactly the same thing as Alexandrov topology $[12]$. For an Alexandrov topology $T$ on $X$, we may define a quasiorder $R_T$ on $X$ by $x \overset{T}{\rightarrow} R y$ if and only if $y \in N(x)$.

On the other hand, let $R$ be a quasiorder on $X$. The set of all $R$-closed subsets of $X$ forms an Alexandrov topology $T_R$, meaning that $B \in T_R$ if and only if $x \in B$ and $x \overset{R}{\rightarrow} R y$ imply $y \in B$. Since the set $R(x)$ of $R$-successors is $R$-closed, we have $N(x) = R(x)$ in $T_R$. In addition, $I(B) = \{x \in X \mid R(x) \subseteq B\} = B^v$ and $C(B) = \{x \in X \mid R(x) \cap B \neq \emptyset\} = B^*$ for any $B \subseteq X$.

The correspondences $T \mapsto R_T$ and $R \mapsto T_R$ are mutually inverse bijections between the class of all Alexandrov topologies and the class of the quasiorders on the set $X$. In addition, it is known that a quasiorder $R$ is a partial order if and only if $T_R$ satisfies the $T_0$-separation axiom, that is, for any two different points $x$ and $y$, there is an open set which contains one of these points and not the other. Topologies satisfying the $T_0$-separation axiom are called the $T_0$-spaces. Therefore, there is a one-to-one correspondence between partial orders and Alexandrov topologies that are $T_0$-spaces.
For each topology $T$ on $X$, the lattice $(T, \subseteq)$ forms a Heyting algebra such that the relative pseudocomplement of $B, C \in T$ is $B \Rightarrow C = T(\neg B \cup C)$. In particular, for a quasiorder $R$, the relative pseudocomplement in $T_R$ can be expressed as
\[ B \Rightarrow C = \{ x \in X \mid x R y \text{ and } y \in B \text{ imply } y \in C \}. \]

Let $(X, \leq, g)$ be a structure such that $(X, \leq)$ is a partially ordered set and $g$ is a map on $X$ satisfying the following conditions for all $x, y$:

\begin{align*}
\text{(J1)} & \text{ if } x \leq y, \text{ then } g(y) \leq g(x), \\
\text{(J2)} & \text{ } g(g(x)) = g(x), \\
\text{(J3)} & \text{ if } x \leq g(x), \text{ or } g(x) \leq x, \\
\text{(J4)} & \text{ if } x, y \leq g(x), g(y), \text{ then there is } z \in X \text{ such that } x, y \leq z \leq g(x), g(y).
\end{align*}

According to D. Vakarelov \cite{22}, these systems are called Monteiro spaces, because A. Monteiro was the first who introduced them in \cite{15}.

Let $\leq$ be a partial order on $X$. It is typical that $\leq$-closed sets are called upsets. We denote by $U(X)$ the set of all upsets of $X$. By the above, $U(X)$ forms a $T_0$-space.

As proved by D. Vakarelov \cite{22}, each Monteiro space $M = (X, \leq, g)$ defines a Nelson algebra
\[ N_M = (U(X), \lor, \land, \rightarrow, \sim, 0, 1), \]
where the operations are defined by:
\begin{align*}
0 &= \emptyset, & 1 &= X, \\
A \lor B &= A \cup B, & A \land B &= A \cap B, \\
\sim A &= \{ x \in X \mid g(x) \notin A \}, & A \rightarrow B &= A \Rightarrow (\sim A \lor B); \\
\end{align*}

note that the operation $\Rightarrow$ is defined in $U(X)$ by:
\[ B \Rightarrow C = \{ x \in X \mid x \leq y \text{ and } y \in B \text{ imply } y \in C \}. \]

Let $A = (A, \lor, \land, \rightarrow, \sim, 0, 1)$ be a Nelson algebra. We denote by $F_P$ the set of prime filters of $A$. We define for any $P \in F_P$ the set of elements
\[ g(P) = \{ x \in A \mid \sim x \notin P \}. \]

The set $g(P)$ is known to be a prime filter of $A$, and the mapping $g$ on $F_P$ satisfies the conditions (J1)–(J4) with respect to the set-inclusion order (see also \cite{3}). Thus, the structure $M = (F_P, \subseteq, g)$ is a Monteiro space and it determines a Nelson algebra
\[ N_M = (U(F_P), \cup, \cap, \rightarrow, \sim, 0, F_P). \]

For any $x \in A$, we define a set of prime filters as
\[ h(x) = \{ P \in F_P \mid x \in P \}. \]
If $P \in h(x)$ and $P \subseteq Q$, then $x \in P \subseteq Q$, that is, $Q \in h(x)$. Therefore, $h(x) \in U(F_P)$. Because $(A, \subseteq)$ is a distributive lattice, for all $x \neq y$, there exists a prime filter $P$ such that $x \in P$ and $y \notin P$, or $x \notin P$ and $y \in P$ by the well-known “prime filter theorem” of distributive lattices. This means that $h(x) \neq h(y)$, and hence $h$ is an injection from $A$ to $U(F_P)$.

Next we will show that $h$ is a Nelson-algebra homomorphism:
• $h(0) = \emptyset$, because prime filters must be proper filters. Therefore, 0 does not belong to any prime filter.

• $h(1) = \mathcal{F}_p$, because 1 must belong to all prime filters.

• $P \in h(x \lor y) \iff x \lor y \in P \iff x \in P \text{ or } y \in P \iff P \in h(x)$ or $P \in h(y) \iff P \in h(x) \cup h(y)$.

• $P \in h(x \land y) \iff x \land y \in P \iff x \in P \text{ and } y \in P \iff P \in h(x) \text{ and } P \in h(y) \iff P \in h(x) \cap h(y)$.

• $P \in h(\neg x) \iff \neg x \in P \iff x \notin g(P) \iff g(P) \notin h(x) \iff P \in \neg h(x)$.

D. Vakarelov [22] has proved that for any $P \in \mathcal{F}_p$, $a \to b \in P$ if and only if for all $Q \in \mathcal{F}_p$, $P \subseteq Q$, and $a \in g(Q)$ imply $b \in Q$. Therefore,

• $P \in h(x \to y) \iff x \to y \in P \iff$ for all $Q \in \mathcal{F}_p$, $P \subseteq Q$, $x \in Q$, and $x \in g(Q)$ imply $y \in Q \iff$ for all $Q \in \mathcal{F}_p$, $P \subseteq Q$, $Q \in h(x)$, and $Q \notin \neg h(x)$ imply $Q \in h(y) \iff$ for all $Q \in \mathcal{F}_p$, $P \subseteq Q$ and $Q \in h(x)$ imply $Q \in \neg h(x) \cup h(y) \iff P \in h(x) \Rightarrow (\neg h(x) \cup h(y)) = h(x) \to h(y)$.

We have now proved that $h$ is an injective homomorphism $A \to \mathcal{U}(\mathcal{F}_p)$. Thus, $h$ is an Nelson-algebra embedding and we can write the following proposition that appears already in [22].

**Proposition 2.1.** Let $\mathcal{A}$ be a Nelson algebra. Then, $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{N}_M$, where $M$ is the Monteiro space $(\mathcal{F}_p, \subseteq, g)$.

For a Nelson algebra $\mathcal{A}$, the family of sets $\mathcal{U}(\mathcal{F}_p)$ is an Alexandrov topology. Therefore, $(\mathcal{U}(\mathcal{F}_p), \subseteq)$ forms an algebraic lattice. As we already noted, we showed in [12] that if $\mathcal{A}$ is a Nelson algebra such that its underlying lattice is algebraic, then there exists a universe $U$ and a quasiorder $R$ on $U$ such that $\mathcal{A} \cong \mathcal{R}S$. This means that $\mathcal{N}_M$, where $M$ is the Monteiro space $(\mathcal{F}_p, \subseteq, g)$, is isomorphic to some rough set Nelson algebra $\mathcal{R}S$, and let us denote by $\varphi$ this isomorphism in question.

It is now obvious that the mapping $\varphi \circ h$ is an embedding from $\mathcal{A}$ to $\mathcal{R}S$, and we can write the following theorem.

**Theorem 2.2.** Let $\mathcal{A}$ be a Nelson algebra. Then, there exists a set $U$ and a quasiorder $R$ on $U$ such that $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{R}S$.

3. **ALEXANDROV SPACES AND ROUGH SETS**

Let $\mathcal{A} = (A, \lor, \land, \neg, \to, 0, 1)$ be a Nelson algebra such that the lattice $(A, \leq)$ is algebraic. In the lattice $(A, \leq)$, each element of $A$ can be represented as the join of completely join-irreducible elements $J$ below it (see [12]). Let us define an order $\triangleleft$ on $J$ by setting

\[ x \triangleleft y \iff y \leq x \text{ in } A. \]

Let $\mathcal{U}(J)$ be the set of upsets with respect to $\triangleleft$. Then, $\mathcal{U}(J)$ is an Alexandrov topology and a $T_0$-space (because $\triangleleft$ is a partial order on $J$). It is now clear that for all $x, y \in J$,

\[ x \leq y \iff N(x) \subseteq N(y); \]

note that $N(x) = \{y \in J \mid x \triangleleft y\}$. 
It is known that the set of completely join-irreducible elements of $\mathcal{U}(\mathcal{J})$ is $\mathcal{B} = \{ N(x) \mid x \in \mathcal{J} \}$. We define a map 
\[ \varphi: \mathcal{J} \rightarrow \mathcal{B}, x \mapsto N(x). \]

Clearly, this map is an order-isomorphism between $(\mathcal{J}, \leq)$ and $(\mathcal{B}, \subseteq)$. This means that $\varphi$ can be canonically extended to a lattice-isomorphism $\Phi: A \rightarrow \mathcal{U}(\mathcal{J})$ by 
\[ \Phi(x) = \bigcup \{ \varphi(j) \mid j \in \mathcal{J} \text{ and } j \leq x \} \]
\[ = \bigcup \{ N(j) \mid j \in \mathcal{J} \text{ and } j \leq x \}. \]

Obviously, $\Phi(0) = \emptyset$, $\Phi(1) = \mathcal{J}$, and since $A$ and $\mathcal{U}(\mathcal{J})$ are Heyting algebras, the relative pseudocomplement satisfies $\Phi(x \Rightarrow y) = \Phi(x) \Rightarrow \Phi(y)$. This is because the relative pseudocomplement is unique in the sense that it depends only on the order of the Heyting algebra in question, and now the ordered sets $(A, \leq)$ and $(\mathcal{U}(\mathcal{J}), \subseteq)$ are isomorphic.

Note that for all $x \in A$ and $j \in \mathcal{J}$, 
\[ j \in \Phi(x) \iff j \leq x. \]

Namely, if $j \in \Phi(x)$, then $j \in N(k)$ for some $k \in \mathcal{J}$ such that $k \leq x$. Thus, $k \triangleleft j$ and $j \leq k$, which give $j \leq x$. On the other hand, if $j \leq x$, then $j \in N(j)$ gives $j \in \Phi(x)$.

We may now define a map $g: \mathcal{J} \rightarrow \mathcal{J}$ by setting 
\[ g(j) = \bigwedge \{ x \in A \mid x \not\triangleleft j \}. \]

By our work [12], $(\mathcal{J}, \triangleleft, g)$ forms a Monteiro space. Thus, the structure 
\[ (\mathcal{U}(\mathcal{J}), \cup, \cap, \rightarrow, \sim, \emptyset, \mathcal{J}) \]
is a Nelson algebra. Because the operation $\rightarrow$ is defined in terms of $\Rightarrow$ and $\sim$, to show that this is isomorphic to $\mathcal{A}$, it suffices to show that 
\[ \Phi(\sim x) = \sim \Phi(x) \]
for all $x \in A$. Now, 
\[ \Phi(\sim x) = \{ j \in \mathcal{J} \mid j \leq \sim x \}. \]

On the other hand, by the definition of $\sim$ in $\mathcal{U}(\mathcal{J})$, we have: 
\[ \sim \Phi(x) = \{ j \in \mathcal{J} \mid g(j) \not\in \Phi(x) \} \]
\[ = \{ j \in \mathcal{J} \mid g(j) \not\in x \}. \]

We have noted in [12] that for all $x \in A$ and $j \in \mathcal{J}$, $g(j) \not\in x$ iff $j \leq \sim x$. Therefore, we have proved the following theorem.

**Theorem 3.1.** Let $\mathcal{A}$ be a Nelson algebra such that the lattice $(A, \leq)$ is algebraic. Then, $\mathcal{A}$ and $(\mathcal{U}(\mathcal{J}), \cup, \cap, \rightarrow, \sim, \emptyset, \mathcal{J})$ are isomorphic.

We end this work by presenting the following theorem that shows how certain structures studied in this work can be considered as equivalent structures.

**Theorem 3.2.** The following structures determine each other “up-to-isomorphism” (and hence they can be considered equivalent):

(i) Rough sets induced by quasiorders;
Nelson algebras defined on algebraic lattices;
(iii) Nelson algebras defined on $T_0$-spaces that are Alexandrov topologies;
(iv) Monteiro spaces.

Proof. Cases (i), (ii) and (iii) are all “equivalent”, as we have seen: each Nelson algebra defined on an algebraic lattice can be represented up to isomorphism as (i) and (iii). Each Monteiro space induces a Nelson algebra defined on an algebraic lattice, and each Nelson algebra defined on an algebraic lattice induces a Monteiro space that determines an Alexandrov topology Nelson algebra isomorphic to the original algebra. Thus, (ii) and (iv) are equivalent. □

In our next example, we illustrate the isomorphisms between different structures.

Example 3.3. As we already noted, for a Nelson algebra $A$ defined on an algebraic lattice, its each element can be represented as the join of completely join-irreducible elements below it. Therefore, concerning the structure of $A$, the essential thing is how its completely join-irreducible elements are related to each other. In addition, isomorphisms between Nelson algebras defined on algebraic lattices are completely defined by maps on completely join-irreducible elements.

The map $g: J \rightarrow J$, defined by

\[(\star) \quad g(j) = \bigwedge \{x \in A \mid x \not\sim j\},\]

satisfies conditions (J1)–(J4) as noted in page $\ref{page}$. Particularly, we have by (J3) that $j \leq g(j)$ or $g(j) \leq j$ for any $j \in J$. We define for every $j \in J$ a “representative” $\rho(j)$ by

\[\rho(j) = \begin{cases} j & \text{if } j \leq g(j) \\ g(j) & \text{otherwise.} \end{cases}\]

In terms of $\rho$, we define a quasiorder $R$ on $U = J$ by setting $x \sim y \iff \rho(x) \leq \rho(y)$.

In $\cite{ref}$, we showed that for this quasiorder $R$ on $J$, $\mathbb{R}S$ and $A$ are isomorphic Nelson algebras. If $J(RS)$ denotes the set of completely join-irreducible elements of $RS$, then the isomorphism $\varphi: J \rightarrow J(RS)$ is defined by

\[\varphi(j) = \begin{cases} (\emptyset, \{j\}^\mathbb{A}) & \text{if } j \leq g(j) \\ (R(j), R(j)^\mathbb{A}) & \text{otherwise.} \end{cases}\]

Consider the Nelson algebra $\mathbb{A}$ of Figure $\ref{fig}(a)$. Because it is finite, it is trivially defined on an algebraic lattice. Suppose that the operation $\sim$ is defined by $\sim 0 = 1$, $\sim a = f$, $\sim b = e$, and $\sim c = d$. The completely join-irreducible elements $\mathcal{J}$ are marked by filled circles, and we have $g(a) = e$, $g(b) = f$, and $g(d) = d$. The induced quasiorder on $U = J = \{a, b, d, e, f\}$ is given in Figure $\ref{fig}(b)$ and the corresponding rough set structure $RS$ is depicted in Figure $\ref{fig}(c)$. Recall that the operation $\sim$ is defined in $\mathbb{R}S$ by $\sim(X^\mathbb{A}, X^\mathbb{A}) = (X^\mathbb{A}, -X^\mathbb{A})$.

On the other hand, the partially ordered set $(\mathcal{J}, \preceq)$ induced by $\mathbb{A}$ is given in Figure $\ref{fig}(d)$. The corresponding structure of its upsets $U(\mathcal{J})$ can be seen in Figure $\ref{fig}(e)$. If $\mathcal{B}$ denotes the set of completely join-irreducible elements...
Figure 1.

of \(U(\mathcal{J})\), then the mapping \(\psi: \mathcal{J} \to \mathcal{B}\) defined by \(j \mapsto N(j)\) is an order-isomorphisms, where \(N(j)\) is the principal filter of \(j\) with respect to \(\triangleleft\). The map \(g\) in the Monteiro space \((\mathcal{J}, \triangleleft, g)\) is defined by \(\star\), and the operation \(\sim\) in the Nelson algebra \((\mathcal{U}(\mathcal{J}), \cup, \cap, \to, \sim, \emptyset, \mathcal{J})\) is given by \(\sim X = \{j \in \mathcal{J} \mid g(j) \notin X\}\).

4. SOME CONCLUDING REMARKS

We end this paper by noting that there is a perfect analogy between the algebraic counterparts of classical logic and constructive logic with strong negation: The basic algebraic structures of classical logic are Boolean algebras, and by the well-known representation theorem of M. H. Stone, any Boolean algebra is isomorphic with a field of sets, that is, with a subalgebra of the Boolean algebra defined on a power set. Analogously, the algebraic counterparts of constructive logic with strong negation are Nelson algebras, and any Nelson algebra is isomorphic to a subalgebra of the Nelson algebra defined on a rough set lattice, according to Theorem 2.2.

Theorem 3.2 means that a rough set system determined by a quasiorder can be treated as a family of upsets of a partially ordered set, which is a well-studied structure in the literature. For any partially ordered set, its upsets form a complete lattice, but in our case, upsets form an Alexandrov
T₀-space provided with operations of a Nelson algebra. This also opens new approaches for further research concerning the representation of particular objects in the category of Nelson algebras.

REFERENCES

[1] P. Alexandroff, Diskrete räume, Matematičeskij Sbornik 2 (1937), 501–518.
[2] G. Birkhoff, Rings of sets, Duke Mathematical Journal 3 (1937), 443–454.
[3] R. Cignoli, The class of Kleene algebras satisfying an interpolation property and Nelson algebras, Algebra Universalis 23 (1986), 262–292.
[4] S. D. Comer, On connections between information systems, rough sets, and algebraic logic, Algebraic methods in logic and computer science, 1993, pp. 117–124.
[5] S. P. Demri and E. S. Orłowska, Incomplete information: Structure, inference, complexity, Springer, 2002.
[6] M. Gehrke and E. Walker, On the structure of rough sets, Bulletin of Polish Academy of Sciences. Mathematics 40 (1992), 235–245.
[7] A. E. Hassanien, Z. Suraj, D. Slezak, and P. Lingras (eds.), Rough computing: Theories, technologies and applications, IGI Global, 2007.
[8] L. Iurriroz, Rough sets and three-valued structures, Logic at work. essays dedicated to the memory of helena rasiowa, 1999, pp. 596–603.
[9] T. B. Iwiński, Algebraic approach to rough sets, Bulletin of Polish Academy of Sciences. Mathematics 35 (1987), 673–683.
[10] J. Järvinen, Lattice theory for rough sets, Transactions on Rough Sets VI (2007), 400–498.
[11] J. Järvinen, P. Pagliani, and S. Radeleczki, Information completeness in Nelson algebras of rough sets induced by quasiorders, Studia Logica 101 (2013), 1073–1092.
[12] J. Järvinen and S. Radeleczki, Representation of Nelson algebras by rough sets determined by quasiorders, Algebra Universalis 66 (2011), 163–179.
[13] J. Järvinen, S. Radeleczki, and L. Veres, Rough sets determined by quasiorders, Order 26 (2009), 337–355.
[14] A. A. Markov, Constructive logic (in russian), Uspekhi Matematicheskih Nauk 5 (1950), 187–188.
[15] A. Monteiro, Construction des algèbres de Nelson finies, Bulletin de l’Académie Polonaise des Sciences 11 (1963), 359–362.
[16] E. Nagarajan and D. Umadevi, A method of representing rough sets system determined by quasi orders, Order 30 (2013), 313–337.
[17] D. Nelson, Constructible falsity, Journal of Symbolic Logic 14 (1949), 16–26.
[18] P. Pagliani and M. Chakraborty, A geometry of approximation. Rough set theory: Logic, algebra and topology of conceptual patterns, Springer, 2008.
[19] Z. Pawlak, Rough sets, International Journal of Computer and Information Sciences 11 (1982), 341–356.
[20] J. Pomykała and J. A. Pomykała, The Stone algebra of rough sets, Bulletin of Polish Academy of Sciences. Mathematics 36 (1988), 495–512.
[21] H. Rasiowa, An algebraic approach to non-classical logics, North-Holland, Amsterdam, 1974.
[22] D. Vakarelov, Notes on N-lattices and constructive logic with strong negation, Studia Logica 36 (1977), 109–125.
[23] ______, Nelsons negation on the base of weaker versions of intuitionistic negation, Studia Logica 80 (2005), 393–430.

J. JÄRVINEN, Sirkankuja 1, 20810 Turku, Finland
E-mail address: Jouni.Kalervo.Jarvinen@gmail.com
URL: http://sites.google.com/site/jounikalervojarvinen/

S. RADELECZKI, Institute of Mathematics, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary
E-mail address: matradi@uni-miskolc.hu
URL: http://www.uni-miskolc.hu/~matradi/