DIFFERENTIAL GRADED CATEGORIES ARE K-LINEAR
STABLE ∞-CATEGORIES

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Abstract. We describe a comparison between pretriangulated differential
graded categories and certain stable ∞-categories. Specifically, we
use a model category structure on differential graded categories over k
(a field of characteristic 0) where the weak equivalences are the Morita
equivalences, and where the fibrant objects are in particular pretriangul-
ated differential graded categories. We show the underlying ∞-category
of this model category is equivalent to the ∞-category of k-linear stable
∞-categories.

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1. Introduction

In this paper we prove the folklore theorem (e.g. implicitly assumed in works such as [BFN], [G]) that differential graded (dg) categories over \( k \) are \( k \)-linear stable \( \infty \)-categories. The basic setting for our work is the theory of \( \infty \)-categories (and, particularly, stable \( \infty \)-categories), which provide a tractable way to handle “homotopical categories of homotopical categories” as well as homotopically meaningful categories of homotopical functors. Thus, our comparison between dg categories and stable \( \infty \)-categories allows for an interpretation of a “homotopical category of homotopical, algebraic categories.”

Triangulated categories arise naturally in geometry (e.g. as derived categories or as stable homotopy categories). However, triangulated categories lack homotopical properties such as functorial mapping cones. Differential graded categories and stable \( \infty \)-categories are enhancements of triangulated categories that have such homotopical properties. Differential graded categories are advantageous for triangulated enhancements because one can often perform explicit computations in this setting. Stable \( \infty \)-categories are an alternative enhancement of triangulated categories that are more general than dg categories and are amenable to universal properties.

In [To1] and [Tab2], Töen and Tabuada construct a model category structure on the category of differential graded categories, \( \text{Cat}^k_{dg} \), in which the weak equivalences are the Dwyer-Kan (DK)-equivalences. That is, the weak equivalences are dg functors that induce both quasi-isomorphisms on the hom complexes and an ordinary categorical equivalence of homotopy categories. Following [TabM], we also use a “Morita” model structure on dg categories, where the weak equivalences are now the larger class of dg functors that induce DK-equivalences on their categories of modules. In this model category, fibrant dg categories are, in particular, pretriangulated.

One of the impediments to working with these model categories is that while there is a monoidal structure on the category of dg categories given by the tensor product of dg categories, the tensor product does not respect either model category structure. Thus, dg categories do not form a monoidal model
category. There are examples of two cofibrant dg categories whose tensor product is not cofibrant (with respect to both model category structures). This is one reason why our comparison between dg categories and $k$-linear stable $\infty$-categories is not completely trivial. To resolve this issue, we use ideas inspired by [To1], [CT], and [BGT2] to derive the tensor product of dg categories via “flat” dg categories.

Following [Tab1], we use a Quillen equivalence between dg categories over $k$ (with the Morita model structure) and categories enriched in $Hk$-module symmetric spectra (also with the Morita model structure) to reformulate our questions in the language of spectral categories. Here, an $Hk$-module spectrum is a spectrum $A$, with an action map $Hk \wedge A \to A$ that satisfies the usual associativity and identity axioms. This Quillen equivalence may be interpreted as an enriched Dold-Kan correspondence because the mapping complexes in dg categories are chain complexes while the mapping complexes in spectral categories may be modeled using simplicial sets.

Given an $E_\infty$-ring $R$ (e.g. $Hk$), $R$ may be considered as a (coherent homotopy) commutative algebra object in the $\infty$-category of spectra, $Sp$, and also as a commutative algebra object in the category of symmetric spectra $S$. As an object in $S$, we may form its category of modules $R\Mod$, and restrict to the subcategory of cell $R$-modules, $R\Mod^{\text{cell}}$. An $R$-module is cell if it can be iteratively obtained from homotopy cofibers of maps whose sources are wedges of shifts of $R$. Let $\Cat_S$ denote the category of spectral categories. Then a module over $R\Mod^{\text{cell}}$ is a category $\mathcal{C}$ and an action map $R\Mod^{\text{cell}} \wedge \mathcal{C} \to \mathcal{C}$ that preserves colimits in each variable and is equipped with the usual associativity maps. We let $\Mod_{R\Mod^{\text{cell}}}(\Cat_S)$ denote the spectral category of modules over $R\Mod^{\text{cell}}$.

Let $\Cat_S$ denote the category of spectral categories, $\Cat_S^{\text{flat}}$ the category of flat spectral categories, $W$ the collection of Morita equivalences in $\Mod_{R\Mod^{\text{cell}}}(\Cat_S)$, and $W'$ the collection of Morita equivalences in $\Cat_S$. We also let the superscript “$\otimes$” denote a symmetric monoidal $\infty$-category.
Using the notation above, there is a localization of $\infty$-categories (see 4.1)
\[
\theta : N(Cat^\text{flat}_S)^\otimes \rightarrow N(Cat^\text{flat}_S)[W^{-1}]^\otimes.
\]
Since the nerve is monoidal 4.24, the map $\theta$ descends to a well-defined functor
\[
\theta : N(\text{Mod}_{R, \text{Mod}^\text{cell}}(Cat_S)^\text{flat})[W^{-1}] \rightarrow \text{Mod}_{R, \text{Mod}}(N(Cat^\text{flat}_S)[W'^{-1}]^\otimes)
\]
because the nerve of a module category over $R\text{-Mod}^\text{cell}$ in $Cat_S$ is naturally a module category over the $\infty$-category $R\text{-Mod}$. Our main theorem is:

**Theorem 5.1.** The functor
\[
\theta : N(\text{Mod}_{R, \text{Mod}^\text{cell}}(Cat_S)^\text{flat})[W^{-1}] \simeq \text{Mod}_{R, \text{Mod}}(N(Cat^\text{flat}_S)[W'^{-1}]^\otimes)
\]
is an equivalence of $\infty$-categories.

In other words, this theorem asserts an equivalence between module categories over the spectral category $R\text{-Mod}^\text{cell}$ and module categories over the stable presentable $\infty$-category $R\text{-Mod}$ up to Morita equivalence.

We furthermore prove that $\text{Mod}_{R, \text{Mod}^\text{cell}}(Cat_S)$ is equivalent to the category of categories enriched in $R$-module spectra, $Cat_{R, \text{Mod}}$. Using our main theorem above we deduce the following corollary.

**Corollary 5.2.** There is an equivalence of $\infty$-categories
\[
N(Cat_{R, \text{Mod}})[W^{-1}] \simeq \text{Mod}_{R, \text{Mod}}(N(Cat^\text{flat}_S)[W'^{-1}]^\otimes).
\]

In other words, our corollary states that the underlying $\infty$-category of the model category of categories enriched in $R$-module spectra localized at the Morita equivalences, and the $\infty$-category of module categories over the $\infty$-category $R\text{-Mod}$ in $N(Cat^\text{flat}_S)[W'^{-1}]$ are equivalent. Since $N(Cat^\text{flat}_S)[W'^{-1}] \simeq \text{Cat}^\text{perf}_\infty$ by [BGT.2.3.5], this is equivalent to the $\infty$-category of module categories over the $\infty$-category $R\text{-Mod}$ in $\text{Cat}^\text{perf}_\infty$.

Applying this theorem to the Eilenberg-MacLane ring spectrum $Hk$ yields the following corollary.

**Corollary 5.3.** There is an equivalence of $\infty$-categories
\[
N(Cat_{Hk, \text{Mod}})[W^{-1}] \simeq \text{Mod}_{Hk, \text{Mod}}(N(Cat^\text{flat}_S)[W'^{-1}]^\otimes).
\]
We now combine this result with the Quillen equivalence \[ \text{Cat}_{Hk-Mod} \simeq \text{Cat}_{dg}^k. \]

**Corollary 5.4.** There is an equivalence of ∞-categories

\[ N(\text{Cat}_{dg}^k)[W^{-1}] \simeq \text{Mod}_{Hk-Mod}(N(\text{Cat}_{\text{flat}}^S)[W'^{-1}]^\otimes). \]

That is, there is an equivalence between the underlying ∞-category of the model category of dg categories over k localized at the Morita equivalences and the ∞-category of k-linear stable ∞-categories.

This theorem is, first and foremost, a rectification result. That is, given a k-linear stable ∞-category C, there exists a dg category corresponding to C up to Morita equivalence. Thus, the category of dg categories localized at the Morita equivalences can be interpreted as a model for the ∞-category of k-linear stable ∞-categories.

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2. Differential Graded Categories

2.1. Review of Differential Graded Categories.

In this section we review the definition of a differential graded category, give several basic examples, and recall the model category structure on the category of dg categories given by Töen and Tabuada.

We now fix a field k of characteristic 0 for convenience, although we expect similar results hold in characteristic p. Let Ch(k) denote the category of chain complexes of vector spaces over k.

**Definition 2.1.** A differential graded (dg) category \( T \) over k consists of the following data.

- A class of objects \( \text{Ob}(T) \).
- For every pair of objects \( x, y \in T \) a complex \( T(x, y) \in \text{Ch}(k) \).
• For every triple \( x, y, z \in \mathcal{T} \) a composition \( \mathcal{T}(y, z) \otimes \mathcal{T}(x, y) \to \mathcal{T}(x, z) \).

• A unit for every \( x \in \mathcal{T} \), \( e_x : k \to \mathcal{T}(x, x) \)
These data are required to satisfy the usual associativity and unit conditions.

We say that a dg category is small if its class of objects \( \text{Ob}(\mathcal{T}) \) forms a set. We will denote the category of small dg categories by \( \text{Cat}^k_{\text{dg}} \).

The definition of a dg functor is the usual definition of an enriched functor.

**Definition 2.2.** Let \( \mathcal{T} \) and \( \mathcal{T}' \) be dg categories. A dg functor \( F : \mathcal{T} \to \mathcal{T}' \) consists of:

• A map on objects \( F : \text{Ob}(\mathcal{T}) \to \text{Ob}(\mathcal{T}') \).

• A map of chain complexes \( F_{x,y} : \mathcal{T}(x, y) \to \mathcal{T}'(F(x), F(y)) \) for all \( x, y \in \text{Ob}(\mathcal{T}) \),

which is compatible with the units and compositions in the obvious sense.

Every dg category has an underlying ordinary category given by restricting to the 0th cohomology group of each hom complex.

**Definition 2.3.** Let \( \mathcal{T} \) be a dg category. The homotopy category of \( \mathcal{T} \), denoted by \( [\mathcal{T}] \), is the ordinary category given by:

• The objects of \( [\mathcal{T}] \), \( \text{Ob}([\mathcal{T}]) \), are the objects of \( \mathcal{T} \).

• For every pair of objects \( x, y \in \text{Ob}([\mathcal{T}]) \), the set of morphisms \( [\mathcal{T}](x, y) := H^0(\mathcal{T}(x, y)) \).

The reader is referred to [K] or [To2] for a general introduction to dg categories.

**Example 2.4.** A dg algebra is a dg category with one object.

**Example 2.5.** Chain complexes \( Ch(k) \) form a dg category. Given two chain complexes, one can form a chain complex of morphisms between them.
A fact that we will implicitly use in the sequel is

**Example 2.6.** If $C$ is a dg category, then so is its opposite category $C^{op}$.

**Example 2.7.** The category of flat vector bundles on a differential manifold $X$ forms a dg category. Given two vector bundles $V$ and $W$ with flat connections, we can form a new vector bundle $\text{Hom}(V,W)$ with flat connection. The $n$th component of the complex of morphisms from $V$ to $W$ is given by smooth $n$-forms on $X$ with coefficients in $\text{Hom}(V,W)$.

In [Tab2, 2.2.1], Tabuada shows that the category of dg categories has a model category structure that we recall here.

**Theorem 2.8.** The category of small dg categories, $\text{Cat}_{dg}$, has the structure of a combinatorial model category where

- (W) The weak equivalences are the DK-equivalences. That is, a dg functor $\mathcal{F}: \mathcal{T} \to \mathcal{T}'$ is a weak equivalence if for any two objects $x, y \in \mathcal{T}$ the morphism $\mathcal{T}(x, y) \to \mathcal{T}'(\mathcal{F}(x), \mathcal{F}(y))$ is a quasi-isomorphism of chain complexes and the induced functor $[\mathcal{F}]: [\mathcal{T}] \to [\mathcal{T}']$ is an equivalence of categories.

- (F) The fibrations are the DK-fibrations. That is, a dg functor $\mathcal{F}: \mathcal{T} \to \mathcal{T}'$ is a fibration if for any two objects $x, y \in \mathcal{T}$ the morphism $\mathcal{T}(x, y) \to \mathcal{T}'(\mathcal{F}(x), \mathcal{F}(y))$ is a fibration of chain complexes (i.e. is surjective), and for any isomorphism $u': x' \to y' \in [\mathcal{T}']$ and any $y \in [\mathcal{T}]$ such that $\mathcal{F}(y) = y'$, there is an isomorphism $u: x \to y \in [\mathcal{T}]$ such that $[\mathcal{F}](u) = u'$.

We will refer to this model structure as the Dwyer-Kan (DK)-model structure on $\text{Cat}_{dg}$.

**Remark 2.9.** It is easy to see that $\mathcal{F}: \mathcal{T} \to \mathcal{T}'$ is a DK-equivalence if and only if it induces an equivalence of graded homotopy categories $H^*(\mathcal{F}): H^*(\mathcal{T}) \to H^*(\mathcal{T}')$.

For us, an important aspect of the model category structure is that it does not respect the monoidal structure of dg categories given by the tensor product. For instance, the following is known [To2, Exercise 14].
Example 2.10. Let $\Delta^1_k$ be the dg category with two objects 0 and 1 and with $\Delta^1_k(0,0) = k$, $\Delta^1_k(1,1) = k$, $\Delta^1_k(0,1) = k$, $\Delta^1_k(1,0) = 0$. Then $\Delta^1_k$ is a cofibrant dg category. However, $\Delta^1_k \otimes \Delta^1_k$ is not a cofibrant dg category.

2.2. The Morita Model Category Structure for Differential Graded Categories.

In this section we recall that there is a model category structure on $\text{Cat}^k_{dg}$ where the weak equivalences are the Morita equivalences. Namely, we further localize the model category on $\text{Cat}^k_{dg}$ given in the previous section by the Morita equivalences. We refer to this model category structure as the Morita model category structure on $\text{Cat}^k_{dg}$. The Morita model category structure will be the one we use in our comparison theorem with $k$-linear stable $\infty$-categories. We begin by recalling several definitions in the theory of modules over a dg category.

2.2.1. Differential Graded Categories of Modules.

We give a brief account of dg categories of modules which is largely based on [To2, 2.3].

Definition 2.11. Given a dg category $\mathcal{T}$ the category of $\mathcal{T}$-modules, $\hat{\mathcal{T}}$ is defined to be $\text{Fun}_{dg}(\mathcal{T}^{op}, \text{Ch}(k))$. That is, a $\mathcal{T}$-module is a functor from $\mathcal{T}^{op} \rightarrow \text{Ch}(k)$.

Remark 2.12. The category $\hat{\mathcal{T}}$ can be given the structure of a combinatorial model category in which the weak equivalences are the pointwise equivalences in $\text{Ch}(k)$ and the fibrations are the pointwise fibrations in $\text{Ch}(k)$. The generating cofibrations and generating acyclic cofibrations are the maps $\mathcal{T}(\cdot, x) \otimes f$ for $x \in \mathcal{T}$ and $f$ varying through the generating cofibrations and generating acyclic cofibrations, respectively, of the model structure on chain complexes. The representable $\mathcal{T}$-modules $\mathcal{T}(\cdot, x)$ are both cofibrant and compact.

We will denote by $\hat{\mathcal{T}}^{cf}$ the category of fibrant and cofibrant $\mathcal{T}$-modules, and by $\mathcal{D}(\mathcal{T})$ the derived category of $\mathcal{T}$. As usual, there is an equivalence $[\hat{\mathcal{T}}^{cf}] \simeq \mathcal{D}(\mathcal{T})$. Thus, $\hat{\mathcal{T}}^{cf}$ can be viewed as the dg enhancement of the derived category of $\mathcal{T}$. 


Example 2.13. If $T$ is a dg category with one object, then $T$ has the same data as a dg algebra $A$. The category $T$-modules is equivalent to the category of dg modules over the dg algebra $A$.

2.2.2. Pretriangulated Differential Graded Categories.

We give a brief account of pretriangulated dg categories. Our definition for a pretriangulated dg category is inspired by the definition of a pretriangulated spectral category in [BM, 5.4].

Definition 2.14. A dg category $T$ is called pretriangulated if

1. There is an object $0$ in $T$ such that the right $T$-module $T(\cdot, 0)$ is homotopically trivial (weakly equivalent to the constant functor with value the complex $0$).

2. Whenever a right $T$-module $M$ has the property that $\Sigma M$ is weakly equivalent to a representable $T$-module $T(\cdot, c)$ (for some object $c \in T$), then $M$ is weakly equivalent to a representable $T$-module $T(\cdot, d)$ for some object $d \in T$. (The suspension of a $T$-module $M$ refers to a pointwise shift when evaluated on each object of $T$).

3. Whenever the right $T$-modules $M$ and $N$ are weakly equivalent to representable $T$-modules $T(\cdot, a)$ and $T(\cdot, b)$ respectively, the homotopy cofiber of any map of right $T$-modules $M \to N$ is weakly equivalent to a representable $T$-module.

Example 2.15. If $X$ is a scheme over $k$, then its category of unbounded complexes of sheaves of $\mathcal{O}_X$-modules is a pretriangulated dg category.

Remark 2.16. Note that this definition promises a $0$ object in $\text{Ho}(T)$, as well as shift functors on $\text{Ho}(T)$.

We now relate our notion of pretriangulated, 2.14, to that introduced by Bondal and Kapranov in [BK, Definition 2] and Drinfeld [Drinfeld, 2.4].

Definition 2.17. To a dg category $A$, Bondal and Kapranov associate another dg category, $A^{pre-tr}$, where the objects are formal expressions

$$\{(\oplus_{i=1}^n C_i[r_i], q), C_i \in A, r_i \in \mathbb{Z}, q = (q_{ij})\},$$
where each \( q_{ij} \in \text{Hom}(C_i, C_j)[r_j-r_i] \) is homogenous of degree 1 and \( dq + q^2 = 0 \). If \( C, C' \in \mathcal{A}^{\text{pre-tr}} \), \( C = (\oplus_{j=1}^n C_j[r_j], q) \) and \( C' = (\oplus_{i=1}^n C'_i[r'_i], q') \), then the complex \( \text{Hom}(C, C') \) is the space of matrices \( f = (f_{ij}), f_{ij} \in \text{Hom}(C_j, C'_i)[r'_i - r_j] \) and composition is given by matrix multiplication.

The differential \( d : \text{Hom}(C, C') \to \text{Hom}(C, C') \) is given by

\[
df := d'f + q'f - (-1)^l fq, \quad \text{if } \deg(f_{ij}) = l,
\]

where \( d' := (df_{ij}) \).

Given a morphism \( f : A \to B \) in \( \mathcal{A} \), the cone on \( f \), \( \text{cone}(f) \), is defined to be the object \( (A[1] \oplus B, q) \in \mathcal{A}^{\text{pre-tr}} \), where \( q_{12} = f \) and \( q_{11} = q_{21} = q_{22} = 0 \).

The dg category \( \mathcal{A} \) is said to be pretriangulated if 0 \( \in \mathcal{A}^{\text{pre-tr}} \) is homotopic to an object in \( \mathcal{A} \), if for every \( A \in \mathcal{A} \), the object \( A[n] \in \mathcal{A}^{\text{pre-tr}} \), is homotopic to an object in \( \mathcal{A} \), and for every closed morphism \( f \) of degree 0 in \( \mathcal{A} \), \( \text{cone}(f) \in \mathcal{A}^{\text{pre-tr}} \), is homotopic to an object in \( \mathcal{A} \).

The concept behind definitions 2.14 and 2.17 of a pretriangulated dg category is for our dg categories to be pointed, closed under shifts, and closed under the formation of mapping cones. Thus

**Proposition 2.18.** The two notions of pretriangulated dg categories 2.14 and 2.17 agree.

**Proof.** There is a functor

\[
\alpha : \mathcal{A}^{\text{pre-tr}} \to \text{Fun}_{dg}(\mathcal{A}^{\text{op}}, \text{Ch}(k))
\]

by sending \( K = (\oplus_{i=1}^n C_i[r_i], q) \) to the functor \( \alpha(K) : \mathcal{A}^{\text{op}} \to \text{Ch}(k) \), where \( \alpha(K)(B) = \oplus \text{Hom}_A(B, C_i)[r_i] \) with differential \( d + q \), where \( d \) is the differential on \( \oplus \text{Hom}_A(B, C_i)[r_i] \).

The functor \( \alpha \) is an embedding of \( \mathcal{A}^{\text{pre-tr}} \) as a full dg subcategory of \( \hat{\mathcal{A}} = \text{Fun}_{dg}(\mathcal{A}^{\text{op}}, \text{Ch}(k)) \) and it sends the element \( \text{cone}(f) \in \mathcal{A}^{\text{pre-tr}} \) to the element \( \text{cone}(\alpha(f)) \in \hat{\mathcal{A}} \). If \( K \) is homotopic to an object in \( K' \in \mathcal{A} \), then \( \alpha(K) \) is certainly weakly representable by \( \mathcal{A}(-, K') \). Thus, 2.17 implies 2.14. Conversely, if the functor \( \alpha(K) \) is weakly representable by an object \( K' \in \mathcal{A} \), then \( K \) is homotopic to \( K' \) in \( \mathcal{A}^{\text{pre-tr}} \). This follows since \( \mathcal{A}(-, K') \) is a cofibrant-fibrant object of \( \hat{\mathcal{A}} \) and \( \mathcal{A}^{\text{pre-tr}} \) is a full subcategory.
Thus, a weak homotopy equivalence \( \mathcal{A}(-, K') \to \alpha(K) \) is a homotopy equivalence in \( \mathcal{A}^{pre-tr} \). We conclude that 2.14 implies 2.17. \( \square \)

**Corollary 2.19.** Since the homotopy category of a pretriangulated dg category as in 2.17 is triangulated by [BK, Proposition 2], so is the homotopy category of a pretriangulated dg category as in 2.14.

We now wish to say that any small dg category embeds in a small pretriangulated category. However, we must first define an embedding in the dg category setting.

**Definition 2.20.** A dg functor \( \mathcal{F} : \mathcal{T} \to \mathcal{T}' \) is a Dwyer-Kan embedding or DK-embedding if for any objects \( x, y \in Ob(\mathcal{T}) \), the map \( \mathcal{T}(x, y) \to \mathcal{T}'(\mathcal{F}(x), \mathcal{F}(y)) \) is a quasi-isomorphism.

**Remark 2.21.** This is equivalent to the definition of quasi-fully faithful in [To2, 2.3].

**Proposition 2.22.** Any small dg category DK-embeds in a small pretriangulated dg category.

**Proof.** We use an argument similar to [BM, 5.5]. Given any dg category \( \mathcal{T} \), its category of modules, \( \hat{\mathcal{T}} \), is pretriangulated. We restrict to a small full subcategory of \( \hat{\mathcal{T}} \) as follows: For any set \( U \), write \( U\hat{\mathcal{T}} \) for the full subcategory of \( \hat{\mathcal{T}} \) consisting of functors taking values in \( Ch(k) \) whose underlying sets are in \( U \). Then \( U\hat{\mathcal{T}} \) is a small dg category, and if we choose \( U \) to be the power set of a sufficiently large cardinal, then \( U\hat{\mathcal{T}} \) will be closed under the usual constructions of homotopy theory in \( \hat{\mathcal{T}} \), including the small objects argument constructing factorizations. In particular, \( U\hat{\mathcal{T}} \) is a model category with cofibrations, fibrations, and weak equivalences the maps that are such in \( \hat{\mathcal{T}} \). We also have a closed model category \( Fun_{dg}(U\hat{\mathcal{T}}, Ch(k)) \) of modules over \( U\hat{\mathcal{T}} \). Let \( \tilde{\mathcal{T}} \) be the full subcategory of \( U\hat{\mathcal{T}} \) consisting of the cofibrant-fibrant objects. Then we also have a closed model category \( Fun_{dg}(\tilde{\mathcal{T}}, Ch(k)) \) of modules over \( \tilde{\mathcal{T}} \). Properties (1) and (2) for \( \tilde{\mathcal{T}} \) in the definition of a pretriangulated dg category are clear. For property (3), consider a map of \( \tilde{\mathcal{T}} \)-modules \( \mathcal{M} \to \mathcal{N} \). Since the model structure on \( Fun_{dg}(\tilde{\mathcal{T}}, Ch(k)) \) is left
proper, after replacing $\mathcal{M}$ and $\mathcal{N}$ with fibrant replacements, we obtain an equivalent homotopy cofiber, and so we can assume without loss of generality that $\mathcal{M}$ and $\mathcal{N}$ are fibrant. We assume that $\mathcal{M}$ is weakly equivalent to $\tilde{T}(\cdot, a)$ and $\mathcal{N}$ is weakly equivalent to $\tilde{T}(\cdot, b)$ for objects $a, b \in \tilde{T}$. Since $\tilde{T}(\cdot, a)$ and $\tilde{T}(\cdot, b)$ are cofibrant and $\mathcal{M}$ and $\mathcal{N}$ are fibrant, we can choose weak equivalences $\tilde{T}(\cdot, a) \to \mathcal{M}$ and $\tilde{T}(\cdot, b) \to \mathcal{N}$. Furthermore, as $\tilde{T}(a, b)$ and $\mathcal{N}(a)$ are both fibrant, we can lift the composite map $\tilde{T}(\cdot, a) \to \mathcal{N}$ to a homotopic map $\tilde{T}(\cdot, a) \to \tilde{T}(\cdot, b)$. We obtain a weak equivalence on the homotopy cofibers. The map $\tilde{T}(\cdot, a) \to \tilde{T}(\cdot, b)$ is determined by the map $a \to b$ by the Yoneda lemma. A fibrant replacement of the homotopy cofiber in $U\hat{T}$ is in $\tilde{T}$ and represents the homotopy cofiber of $\mathcal{M} \to \mathcal{N}$ in $\tilde{T}$.

Moreover, we have that

**Proposition 2.23.** A dg functor between pretriangulated dg categories induces a triangulated functor on its homotopy categories. Moreover, a dg functor between pretriangulated dg categories is a DK-equivalence if and only if it induces an equivalence of homotopy categories.

Now that we have developed the notion of pretriangulated dg categories, we wish to construct a model category structure on dg categories where every fibrant dg category is in particular pretriangulated. The following definitions lie in the model for $\hat{T}$ constructed in the proof above.

**Definition 2.24.** Given a dg category $\mathcal{T}$, a $\mathcal{T}$-module $X : \mathcal{T}^{\text{op}} \to Ch(k)$ is called a finite cell object if it can be obtained from the initial $\mathcal{T}$-module by a finite sequence of pushouts along generating cofibrations in $\tilde{T}$.

**Definition 2.25.** Given a dg category $\mathcal{T}$, its triangulated closure, $\hat{T}_{\text{tri}}$, is defined to be the dg subcategory in $\hat{T}^{\text{cf}}$ of objects that have the homotopy type of finite cell objects. Given a dg category $\mathcal{T}$, the category of perfect modules, $\hat{T}_{\text{perf}}$, is the thick closure of $\hat{T}_{\text{tri}}$. That is, it is the smallest full dg subcategory of $\hat{T}$ containing objects that have the homotopy type of retracts of finite cell objects.

**Remark 2.26.** The category $\hat{T}_{\text{perf}}$ is an idempotent-complete pretriangulated dg category.
Definition 2.27. A dg functor $F : \mathcal{T} \to \mathcal{T}'$ is called a Morita equivalence if $F$ induces a DK-equivalence of dg categories $\hat{T}_{\text{perf}} \to \hat{T}'_{\text{perf}}$.

Using the definitions above, the following is proved in [TabM, 5.1].

Proposition 2.28. The category $\text{Cat}^k_{dg}$ admits the structure of a combinatorial model category whose weak equivalences are the Morita equivalences and whose cofibrations are the same as those in the DK-model structure on $\text{Cat}^k_{dg}$.

Definition 2.29. The model category structure defined above will be referred to as the Morita model category structure on $\text{Cat}^k_{dg}$.

3. Spectral Categories and the Enriched Dold-Kan Correspondence

The theory of spectral categories has a similar flavor to the theory of dg categories. The reader is encouraged to read [BGT1], [BGT2], and [BM] for further background in this direction.

3.1. Review of Spectral Categories.

We let $\mathcal{S}$ denote the symmetric monoidal simplicial model category of symmetric spectra [HSM]. When relevant, we will be using the stable model structure on $\mathcal{S}$.

Definition 3.1. A spectral category $\mathcal{A}$ is given by

- A class of objects $\text{Ob}(\mathcal{A})$.
- For each pair of objects $x, y \in \text{Ob}(\mathcal{A})$, a symmetric spectrum $\mathcal{A}(x, y)$.
- For each triple $x, y, z \in \text{Ob}(\mathcal{A})$ a composition in $\mathcal{S}$, $\mathcal{A}(y, z) \wedge \mathcal{A}(x, y) \to \mathcal{A}(x, z)$.
- For any $x \in \text{Ob}(\mathcal{A})$, a map $e_x : \mathcal{S} \to \mathcal{A}(x, x)$ in $\mathcal{S}$.

satisfying the usual associativity and unit conditions.
A spectral category is said to be small if its class of objects forms a set. We write $\text{Cat}_S$ for the category of small spectral categories and spectral (enriched) functors. As with dg categories, there is a DK-model structure on $\text{Cat}_S$, where the weak equivalences are the DK-equivalences [TabS, 5.10].

**Definition 3.2.** Let $\mathcal{A}$ be a spectral category. The homotopy category of $\mathcal{A}$, denoted by $[\mathcal{A}]$, is the ordinary category given by:

- The objects of $[\mathcal{A}]$, $\text{Ob}(\mathcal{A})$, are the objects of $\mathcal{A}$.
- For every pair of objects $x, y \in \text{Ob}(\mathcal{A})$, the set of morphisms $[\mathcal{A}](x, y) := \pi_0(T(x, y))$.

**Definition 3.3.** A spectral functor $\mathcal{F} : \mathcal{A} \to \mathcal{A}'$ is a DK-equivalence if

- For every pair of objects $x, y \in \text{Ob}(\mathcal{A})$, the morphism in $\mathcal{S}$, $\mathcal{F}(x, y) : \mathcal{A}(x, y) \to \mathcal{B}(x, y)$ is a stable equivalence of symmetric spectra.
- The induced functor $[\mathcal{F}] : [\mathcal{A}] \to [\mathcal{B}]$ is an equivalence of ordinary categories.

Moreover, we have [BGT2, 2.2.4].

**Proposition 3.4.** The category $\text{Cat}_S$ with the DK-model structure is a combinatorial model category. Moreover, $\text{Cat}_S$ can be replaced by a Quillen equivalent simplicial model category.

**Remark 3.5.** As with dg categories, there are similar notions of a pretriangulated spectral category and a Morita equivalence of spectral categories. We can use the machinery of Bousfield localization in this setting to obtain a combinatorial model category structure on $\text{Cat}_S$ where the weak equivalences are the Morita equivalences.

### 3.2. Categories of $R$-Module Spectra.

In this section we set up the relevant properties of the categories of $R$-module spectra used in our main theorem. Let $R$ be an $E_\infty$-ring spectrum and $R\text{-Mod}$ the category of $R$-module spectra. There is a cofibrantly generated model category structure on $R\text{-Mod}$ by [HSM].
Remark 3.6. More explicitly, start with the stable model category on symmetric spectra [HSM, 6.3]. Applying the machinery of [SS1], one then constructs a model category structure on modules over $R$. Since the former is cofibrantly generated, so is the latter. The generating cofibrations and generating acyclic cofibrations for symmetric spectra are given by the sets $FI_\partial$ and $K \cup FI_\Lambda$ respectively. Here, $FI_\partial$ is the set

$$F_n \partial \Delta[m]^+ \to F_n \Delta[m]^+,$$

$K$ is the set of pushouts products

$$\Delta[m]^+ \wedge F_{n+1} S^1 \cup_{\partial \Delta[m]^+ \wedge F_{n+1} S^1} \partial \Delta[m]^+ \wedge Z(\lambda_n) \to \Delta[m]^+ \wedge Z(\lambda_n),$$

and $FI_\Lambda$ is the set of horn inclusions

$$F_n \Lambda^k[m]^+ \to F_n \Delta[m]^+.$$

The functor $F_n$ is the left adjoint of the evaluation functor $Ev_n$ and $Z(\lambda_n)$ is the mapping cylinder of the natural map $F_{n+1} S^1 \to F_n S^0$. The generating cofibrations for $R$-$Mod$ are given by the set $R \wedge FI_\partial$ and the generating acyclic cofibrations are given by $R \wedge (K \cup FI_\Lambda)$. For a more detailed description, please see [HSM, 3.4].

We now let $Cat_{R-Mod}$ denote the combinatorial model category of categories whose morphism spaces are enriched in $R$-$Mod$. In particular, each category in $Cat_{R-Mod}$ is a category enriched in spectra. There is a monoidal structure on $Cat_{R-Mod}$ given by the smash product spectral categories.

**Definition 3.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be two spectral categories. The smash product $\mathcal{C} \wedge \mathcal{D}$ is the spectral category specified by

- The objects of $\mathcal{C} \wedge \mathcal{D}$, $Ob(\mathcal{C} \wedge \mathcal{D})$, are given by pairs of objects $(c, d)$, where $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
- For every pair of objects $(c, d), (c', d') \in \mathcal{C} \wedge \mathcal{D}$, the morphism spectrum $\mathcal{C} \wedge \mathcal{D}((c, d), (c', d')) := \mathcal{C}(c, c') \wedge \mathcal{D}(d, d')$.

We will need the notion of cell $R$-modules.

**Definition 3.8.** An $R$-module $M$ is called cell if $M$ is the union of an expanding sequence of sub $R$-modules $M_i$ such that $M_0 = \ast$ and $M_{j+1}$ is
the homotopy cofiber of a map $\phi_j : F_j \to M_j$, where $F_j$ is a wedge of shifts of $R$ as a module over itself. We denote the category of cell $R$-modules by $R$-$\text{Mod}^{\text{cell}}$.

**Remark 3.9.** The definition of cell $R$-modules given above agrees with the usual definition of cell objects using the model structure in the stable model category of symmetric spectra mentioned in the previous remark. Let $\mathcal{I} = \{R\wedge F_n S^0 \to R\wedge F_n D^1\}$ be the set of inclusions of sphere $R$-modules into disk $R$-modules. These maps are stable cofibrations of symmetric spectra by [HSM, 3.4.2]. It suffices to show that a map $f : X \to Y$ of $R$-modules has the right lifting property with respect to $\mathcal{I}$ if and only if the map $f$ is a trivial stable fibration of symmetric spectra. However, by [HSM, 3.4.5] a map is a stable trivial fibration if and only if it is a level trivial fibration, that is, if each $f_n : X_n \to Y_n$ is a trivial fibration of simplicial sets. This implies that $\mathcal{I}$ satisfies the properties of a set of generating cofibrations for the model category structure on $R$-$\text{Mod}$ [Hovey, 2.1.19]. Thus, we obtain wedges of shifts of $R$ as claimed for the construction of cell objects.

We will also need a crucial property of cell $R$-modules.

**Proposition 3.10.** Every $R$-module is functorially weakly equivalent to a cell $R$-module.

*Proof.* Retracts of cell objects are the cofibrant objects in any cofibrantly-generated model category. Moreover, the standard use of the small object argument can be used to make this choice functorial. \hfill \Box

Now, let $\text{Mod}_{R$-$\text{Mod}^{\text{cell}}(\text{Cat}_S)}$ be the category of module categories over the category $R$-$\text{Mod}^{\text{cell}}$. There is a monoidal structure on $\text{Mod}_{R$-$\text{Mod}^{\text{cell}}(\text{Cat}_S)}$ given by the tensor product of module categories.

We have the following:

**Lemma 3.11.** The two categories $\text{Mod}_{R$-$\text{Mod}^{\text{cell}}(\text{Cat}_S)}$ and $\text{Cat}_{R$-$\text{Mod}}$ are equivalent.
Proof. Let \( C \) be a category in \( \text{Mod}_{R \text{-Mod}^{\text{cell}}(\text{Cat}_S)} \), then \( C \) is equipped with a bifunctor \( \alpha : R \text{-Mod}^{\text{cell}} \land C \to C \), that preserves colimits in each variable and has the property that \( R \), the monoidal unit in \( R \text{-Mod} \), acts as the unit on objects in \( C \). That is, \( \alpha(R, x) = x \) for all objects \( x \in C \). We see that \( C \) is thus equipped with maps \( \text{Hom}_R(R, R) \land C(x, y) \to C(x, y) \), which is equivalent to \( R \land C(x, y) \to C(x, y) \), so each mapping space in \( C \) is indeed an \( R \)-module.

Similarly, starting with an action of \( R \) on each mapping space \( C(x, y) \), we can construct a bifunctor \( \alpha : R \text{-Mod}^{\text{cell}} \land C \to C \), that preserves colimits in each variable and has the property that \( \alpha(R, x) = x \) for all objects \( x \in C \).

Moreover, if we start with \( \beta : R \text{-Mod}^{\text{cell}} \land C \to C \), restrict to the action on each hom spectra \( \text{act}_\beta : R \land C(x, y) \to C(x, y) \), then the natural transformation \( \text{Lan}_i(\text{act}_\beta) \to \beta \) is additionally a natural isomorphism. The natural transformation on a cell \( R \)-module \( M \) is given by \( \text{colim}_{f : R \to M} \text{act}_\beta \to \beta(M) \). Since \( M \in R \text{-Mod}^{\text{cell}} \), we have that \( M \simeq \text{colim}_{f : R \to M} R \). Thus, \( \text{Lan}_i(\text{act}_\beta) \) and \( \beta \) give the data of isomorphic module categories in \( \text{Mod}_{R \text{-Mod}^{\text{cell}}(\text{Cat}_S)} \).

Corollary 3.12. The category \( \text{Mod}_{R \text{-Mod}^{\text{cell}}(\text{Cat}_S)} \) is a combinatorial model category.

Proof. The equivalence in the lemma allows us to use that \( \text{Cat}_{R \text{-Mod}} \) is a combinatorial model category.

3.3. The Enriched Dold-Kan Correspondence.

In [SS2], Schwede-Shipley generalize the Dold-Kan correspondence and prove
that the category of modules over the Eilenberg-MacLane symmetric spectrum $Hk$, $Hk$-$Mod$, is Quillen equivalent to chain complexes of $k$-modules, $Ch(k)$-$Mod$. In [Tab1], Tabuada further generalizes this result and establishes a Quillen equivalence between dg categories over $k$, $Cat_k$, and categories enriched in $Hk$-module symmetric spectra, $Cat_{Hk}$-$Mod$, where both categories have the DK-model structure. We exploit this equivalence between dg categories and categories enriched in $Hk$-module spectra in our comparison theorem by using machinery that was developed for spectral categories in [BGT1] and [BGT2].

Moreover, one can conclude that:

**Proposition 3.13.** A dg category $T$ is pretriangulated if and only if its associated spectral category is pretriangulated. Similarly, two dg categories are Morita equivalent if and only if their associated spectral categories are Morita equivalent.

### 4. Review of $\infty$-Categories

We now work with the theory of quasicategories, a well-developed model of $\infty$-categories. These first appeared in the work of Boardman and Vogt, where they were referred to as weak Kan complexes. The theory was subsequently developed by Joyal and then extensively studied by Lurie. In this section we give a brief review of the relevant background regarding the theory of $\infty$-categories. Our basic references for this material are Jacob Lurie’s books [HA], [T]. An extremely brief introduction to the definition of an $\infty$-category is given in [L3].

**4.1. From Model Categories to $\infty$-Categories.**

There are a number of options for producing the “underlying” $\infty$-category of a category equipped with a notion of “weak equivalence.” The most structured setting is that of a simplicial model category $C$, where the $\infty$-category can be obtained by restricting to the full simplicial subcategory $C^{cf}$ of cofibrant-fibrant objects and then applying the simplicial nerve functor $N$. More generally, if $C$ is a category equipped with a subcategory of
weak equivalences \( wC \), the Dwyer-Kan simplicial localization \( LC \) provides a corresponding simplicial category, and then \( N((LC)^f) \), where \( (-)^f \) denotes fibrant replacement in simplicial categories, yields an associated \( \infty \)-category. Lurie has given a version of this approach in [HA, 1.3.3]: we associate to a (not necessarily simplicial) category \( C \) with weak equivalences \( W \) an \( \infty \)-category \( N(C)[W^{-1}] \). Here, the notation \( N(C)[W^{-1}] \) refers to the universal \( \infty \)-category equipped with a map \( N(C) \to N(C)[W^{-1}] \) such that for another \( \infty \)-category \( D \), the functor induced by precomposition

\[
\text{Fun}(N(C)[W^{-1}], D) \to \text{Fun}(N(C), D)
\]

is a fully faithful embedding whose essential image is the collection of functors from \( C \) to \( D \) that map the image of morphisms in \( W \) to equivalences in \( D \) [HA, 1.3.4.1]. If \( C \) is a model category, it is usually convenient to restrict to the cofibrant objects \( C^c \) and consider \( N(C^c)[W^{-1}] \).

All of these constructions produce equivalent \( \infty \)-categories if \( C \) is a model category [HA, 1.3.7]. We will refer to this construction as the underlying \( \infty \)-category of a model category.

Furthermore, all of these constructions are functorial. In the sequel, we will need that given a Quillen adjunction \((F, G)\), there is an induced adjunction of functors on the level of the associated \( \infty \)-categories [T, 5.2.4.6]. Moreover, a Quillen equivalence will induce an equivalence of \( \infty \)-categories.

We will often be interested in \( \infty \)-categories for which we have set theoretic control. Lurie provides a thorough treatment of the theory of presentable and accessible \( \infty \)-categories in [T, 5.4] and [T, 5.5]. Briefly, an \( \infty \)-category \( \mathcal{A} \) is accessible if it is locally small and has a good supply of filtered colimits and compact objects. A great source of examples of accessible \( \infty \)-categories are Ind-categories of small \( \infty \)-categories. An \( \infty \)-category \( \mathcal{A} \) is presentable if it is accessible and furthermore admits all small colimits.

We will also need [HA, 1.3.4.22]:

**Proposition 4.1.** Let \( C \) be a combinatorial model category, then the underlying \( \infty \)-category of \( C \) is a presentable \( \infty \)-category.
One of the most important properties of the category of spectra, or the category of chain complexes, $Ch(k)$, is that it is stable, for the functors that shift complexes up and down are certainly equivalences. Thus, our image of dg categories in the category of small $\infty$-categories, $Cat_\infty$, must also be stable.

**Definition 4.2.** An $\infty$-category is stable [HA, 1.1.1.9] if it has finite limits and colimits and pushout and pullback squares coincide [HA, 1.1.3.4]. Let $Cat^\text{ex}_\infty$ denote the $\infty$-category of small stable $\infty$-categories and exact functors (i.e., functors which preserve finite limits and colimits) [HA, 1.1.4]. The $\infty$-category of exact functors between $A$ and $B$ is denoted by $\text{Fun}^\text{ex}(A, B)$.

**Remark 4.3.** For a small stable $\infty$-category $C$, the homotopy category $\text{Ho}(C)$ is triangulated, with the exact triangles determined by the cofiber sequences in $C$ [HA, 1.1.2.13]. This is why we use the Morita model structure on dg categories (and on $Cat_{Hk-Mod}$).

Recall that an $\infty$-category $C$ is idempotent-complete if the image of $C$ under the Yoneda embedding $C \to \text{Fun}(C, N(\text{Set}^f_\Delta))$ is closed under retracts, where $N(\text{Set}^f_\Delta)$ is the $\infty$-category of spaces. Let $Cat^\text{perf}_\infty$ denote the $\infty$-category of small idempotent-complete stable $\infty$-categories. There is an idempotent completion functor given as the left adjoint to the inclusion $Cat^\text{perf}_\infty \to Cat^\text{ex}_\infty$ [HA, 5.1.4.2], which we denote by $\text{Idem}(\cdot)$.

**Definition 4.4.** Let $A$ and $B$ be small stable $\infty$-categories. Then we will say that $A$ and $B$ are Morita equivalent if $\text{Idem}(A)$ and $\text{Idem}(B)$ are equivalent.

We now use the combinatorial model structures on $Cat_S$ and $Cat_{R-Mod}$ to pass from spectral categories to $\infty$-categories using these constructions. Recall that the $\infty$-category $Pr^L_{st}$ of presentable stable $\infty$-categories admits a symmetric monoidal structure with unit the $\infty$-category $Sp$ of spectra, and the $\infty$-category $Cat^\text{perf}_\infty$ of idempotent-complete small stable $\infty$-categories inherits a symmetric monoidal structure from this structure [HA, 6.3.1].

It is shown in [BGT1, 4.23] that:
Theorem 4.5. The ∞-category $\text{Cat}^\text{perf}_\infty$ is the ∞-category associated to the category $\text{Cat}_S$ endowed with the Morita model category structure. That is, the notion of Morita equivalence for spectral categories is compatible with the notion of Morita equivalence for stable ∞-categories.

Thus, pretriangulated spectral categories can be interpreted as a model of idempotent-complete small stable ∞-categories.

4.2. Review of ∞-Operads.

In this section we briefly recall the basic definitions and properties of ∞-operads, the ∞-categorical notion of a colored operad following [HA, Chapter 2]. Note that ordinary operads are colored operads with only one color. Thus, there is a slight abuse of terminology. We use the language of ∞-operads to define symmetric monoidal ∞-categories and also algebra and module objects within a symmetric monoidal ∞-category. We then use the language of ∞-operads to show the nerve functor is monoidal.

We will need a few technical definitions before we can define an ∞-operad. Let $\Gamma$ denote the category with objects the pointed sets $\langle n \rangle = \{ *, 1, 2, \ldots, n \}$ and morphisms those functions which preserve the base point $\ast$.

Definition 4.6. Let $\langle n \rangle^\circ = \{ 1, 2, \ldots, n \}$, we will say a morphism $f : \langle m \rangle \to \langle n \rangle$ is inert if for $i \in \langle n \rangle^\circ$, the inverse image $f^{-1}(i)$ has exactly one element.

Thus, $f : \langle m \rangle \to \langle n \rangle$ is inert if $\langle n \rangle$ is obtained from $\langle m \rangle$ by identifying some subset of $\langle m \rangle^\circ$ with the base point $\ast$.

An example we will use is

Example 4.7. For $1 \leq i \leq n$, let $\rho^i : \langle n \rangle \to \langle 1 \rangle$ denote the inert morphism

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ \ast & \text{otherwise.} \end{cases}$$

Definition 4.8. Let $p : X \to S$ be an inner fibration of simplicial sets (i.e. the fiber over any simplex of $S$ is an ∞-category), and let $f : x \to y$ be an edge of $X$, then $f$ is p-coCartesian if the natural map $X_{x/} \to X_{y/} \times_{S_{p(x)/}} S_{p(f)/}$ is a trivial Kan fibration (or if $f$ satisfies the properties of [T, 2.4.1.8]).
**Remark 4.9.** Informally, if $\bar{f} : s \to s'$ is an edge in $S$ and $f : x \to x'$ lifts $\bar{f}$, then if $f$ is $p$-coCartesian it is determined up to equivalence by $\bar{f}$ and its source $x$.

**Definition 4.10.** Let $p : X \to S$ be an inner fibration of simplicial sets, $p$ is a coCartesian fibration of simplicial sets if for every edge $\bar{f} : s \to s'$ in $S$, and every vertex $x \in X$ with $p(x) = s$, there exists a $p$-coCartesian edge $f : x \to x'$ with $p(f) = \bar{f}$.

**Definition 4.11.** An $\infty$-operad is an $\infty$-category $O^\otimes$ and a functor 

$$p : O^\otimes \to \mathcal{N}(\Gamma)$$

satisfying the following conditions [HA, 2.1.1.10]:

- The map $p$ is a coCartesian fibration. Thus, for every inert morphism $f : \langle m \rangle \to \langle n \rangle$ in $\Gamma$ and every object $C \in O^\otimes_{\langle m \rangle}$, there is a $p$-coCartesian morphism $\tilde{f} : C \to C'$ in $O^\otimes$ lifting $f$.

- Let $C \in O^\otimes_{\langle m \rangle}$ and $C' \in O^\otimes_{\langle n \rangle}$ be objects, let $f : \langle m \rangle \to \langle n \rangle$ be a morphism in $\Gamma$, and let $\text{map}^f_{O^\otimes}(C, C')$ denote the union of the components of $\text{map}_{O^\otimes}(C, C')$ which lie over $f \in \text{Hom}_{\Gamma}(\langle m \rangle, \langle n \rangle)$. Choose $p$-coCartesian morphisms $C' \to C'_i$ lying over the morphism $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for each $1 \leq i \leq n$. The induced map

$$\text{map}^f_{O^\otimes}(C, C') \to \prod_i \text{map}^{f \circ \rho^i}_{O^\otimes}(C, C'_i)$$

is a homotopy equivalence.

- For every finite collection $C_1, C_2, \ldots, C_n$ of $O^\otimes_{\langle 1 \rangle}$, there exists an object $C$ of $O^\otimes_{\langle n \rangle}$ and $p$-coCartesian morphisms $C \to C_i$ covering each $\rho^i$.

**Example 4.12.** The identity map $N(\Gamma) \to N(\Gamma)$ is an $\infty$-operad. It will be denoted by $\text{Comm}^\otimes$, and it is the analogue of the $E_\infty$ operad. More generally, for each $1 \leq n \leq \infty$, there is a topological category $\tilde{E}[n]$ [HA, 5.1.0.2] with a natural functor $N(\tilde{E}[n]) \to N(\Gamma)$, which results in analogues of the $E_n$ operads.
Example 4.13. Let $O$ be a colored operad, then [HA, 2.1.1.7] constructs a category $O^\otimes$ whose objects are finite sequences of colors in $O$ and a map $O^\otimes \to \Gamma$. The properties of $O^\otimes$ imply that the induced map $N(O^\otimes) \to N(\Gamma)$ is an $\infty$-operad.

Given an $\infty$-operad $q : O^\otimes \to N(\Gamma)$ and a coCartesian fibration $p : C^\otimes \to O^\otimes$, we will say that $p : C^\otimes \to O^\otimes$ is an $O$-monoidal $\infty$-category if the composite $q \circ p : C^\otimes \to N(\Gamma)$ is an $\infty$-operad. Such a map $p$ is called a coCartesian fibration of $\infty$-operads.

Definition 4.14. A symmetric monoidal $\infty$-category [HA, 2.1.2.18] is an $\infty$-category $C$ and a coCartesian fibration of $\infty$-operads $p : C^\otimes \to N(\Gamma)$. The underlying $\infty$-category $C$ is obtained by $C = p^{-1}(\langle 1 \rangle)$. In abuse of terminology, we say that $C$ is a symmetric monoidal $\infty$-category.

Thus, a symmetric monoidal $\infty$-category is a coCartesian fibration $p : C^\otimes \to N(\Gamma)$ which induces equivalences of $\infty$-categories $C^\otimes_{(n)} \simeq C^n$, where $C$ is the $\infty$-category $C^\otimes_{(1)}$. Moreover the morphisms $\alpha : \langle 0 \rangle \to \langle 1 \rangle$ and $\beta : \langle 2 \rangle \to \langle 1 \rangle$ determine functors

$$\Delta^0 \to C \text{ and } C \times C \to C$$

which are well-defined up to a contractible space of choice. These maps induce a unit object $1 \in C$ and a monoidal structure $\otimes$ which satisfy all of the properties of a symmetric monoidal category up to homotopy.

Example 4.15. If $C$ is a symmetric monoidal category, then $N(C)$ is a symmetric monoidal $\infty$-category.

We now develop the definition of an algebra over an $\infty$-operad. We start with a technical definition.

Definition 4.16. A morphism $f$ in an $\infty$-operad $p : O^\otimes \to N(\Gamma)$ is called inert if $p(f)$ is inert and $f$ is p-coCartesian.

Definition 4.17. A morphism of $\infty$-operads is a map of simplicial sets $f : O^\otimes \to O'^\otimes$ over $N(\Gamma)$ such that $f$ takes inert morphisms in $O^\otimes$ to inert morphisms in $O'^\otimes$. The $\infty$-category of $\infty$-operad maps is denoted $Alg_O(O')$ and is considered as a full subcategory of $Fun_{N(\Gamma)}(O^\otimes, O'^\otimes)$.
More generally, we have

**Definition 4.18.** If \( p : C^\otimes \rightarrow O^\otimes \) is a fibration of \( \infty \)-operads (i.e. a categorical fibration [T, 2.2.5.1]) and \( f : O^\otimes \rightarrow O^\otimes \) is a map of \( \infty \)-operads, let \( Alg\mathcal{O}/O(C) \) be the full subcategory of \( Fun_\mathcal{O}(O^\otimes, C^\otimes) \) spanned by the maps of \( \infty \)-operads. When \( O = O' \) and \( f \) is the identity, we will denote this \( \infty \)-category by \( Alg\mathcal{O}(C) \), the \( \infty \)-category of \( O \)-algebra objects in \( C \).

**Example 4.19.** If \( C \) is a symmetric monoidal category, the \( \infty \)-category \( Alg\mathcal{N}(\Gamma)(C^\otimes) = Alg\mathcal{Comm}(C^\otimes) \) is the \( \infty \)-category of commutative algebra objects in \( C \).

We now use the language of \( \infty \)-operads to show the nerve functor is monoidal. This will be used to construct the functor in our main theorem in the sequel.

**Definition 4.20.** We define a colored operad \( LM \) [HA, 4.2.1.4] as follows:

- The set of objects of \( LM \) has two objects \( a \) and \( m \).
- Let \( \{X_i\}_{i \in I} \) be a finite collection of objects in \( LM \), and let \( Y \) be another object. If \( Y = a \), then \( Mul_{LM}(\{X_i\}, Y) \) is the collection of all orderings on \( I \) provided that each \( X_i = a \) and is empty otherwise. If \( Y = m \), then \( Mul_{LM}(\{X_i\}, Y) \) is the collection of all orderings \( \{i_1 < \ldots < i_n\} \) on the set \( I \) such that \( X_{i_n} = m \) and \( X_{i_j} = a \) for \( j < n \).
- The composition law on \( LM \) is determined by composing linear orderings in the natural way.

**Remark 4.21.** Restricting to the object \( a \) we get a subcolored operad isomorphic to the associative operad \( Ass \) of [HA, 4.1.1.1]. Thus, if \( C \) is a symmetric monoidal category, \( F : LM \rightarrow C \) a map of colored operads, \( F(a) = A \in C \), \( F(m) = M \in C \), then the unique \( \phi \in Mul_{LM}(\{a, m\}, m) \) determines a map \( \phi : A \otimes M \rightarrow M \), which exhibits \( M \) as a left \( A \)-module, where \( A \) is an associative algebra object in \( C \).

**Definition 4.22.** Applying [HA, 2.1.1.7] to the colored operad \( LM \), we obtain a category \( LM^\otimes \) and a map \( LM^\otimes \rightarrow \Gamma \). Thus, the nerve \( LM^\otimes := \)
\(N(\text{LM}^\otimes) \to N(\Gamma)\) is the \(\infty\)-operad which controls left module objects over an associative algebra. There is also an \(\infty\)-operad \(\text{Ass}^\otimes\) obtained by applying the same construction to \(\text{Ass}\).

**Definition 4.23.** Let \(C^\otimes \to \text{Ass}^\otimes\) be a fibration of \(\infty\)-operads, and \(M\) and \(\infty\)-category. A weak enrichment of \(M\) over \(C^\otimes\) is a fibration of \(\infty\)-operads \(q : O^\otimes \to \text{LM}^\otimes\) such that \(O^a_\otimes \simeq C^\otimes\) and \(O^m_\otimes \simeq M\). We let \(LMod(M) := \text{Alg}_{\text{LM}}(O)\) be the \(\infty\)-category of left module objects in the \(\infty\)-category \(M\) [HA, 4.2.1.13].

We will need the following result we will need in our comparison theorem.

**Proposition 4.24.** Let \(C\) be a monoidal category, and \(s : \text{LM} \to C\) a left module object of \(C\), then \(N(s) : \text{LM}^\otimes \to N(C)^\otimes\) defines a left module object of \(N(C)\).

**Proof.** Let \(N(C)^\otimes \to \text{Ass}^\otimes\) be a monoidal \(\infty\)-category. Then

\[O^\otimes = N(C)^\otimes \times_{\text{Ass}^\otimes} \text{LM}^\otimes\]

exhibits the \(\infty\)-category \(N(C)\) as weakly enriched over \(N(C)^\otimes\). The map \(s : \text{LM} \to C\) of colored operads induces a map \(s : \text{LM}^\otimes \to C^\otimes\) of categories, which in turn induces a map \(N(s) : \text{LM}^\otimes \to N(C)^\otimes\) of \(\infty\)-operads. This gives an element of \(N(s) \in \text{Fun}_{\text{LM}}(\text{LM}, O)\), and furthermore, \(N(s)\) sends inert morphisms to inert morphism, and hence, \(N(s) \in \text{Alg}_{\text{LM}}(O)\) determines a left module object in \(N(C)\). \( \square \)

4.3. **From Cofibrant Spectral Categories to Flat Spectral Categories.** In this section we will denote a symmetric monoidal \(\infty\)-category \(\mathcal{A}^\otimes\) by additionally having the superscript “\(\otimes\)”. When \(C\) is a symmetric monoidal model category such that the weak equivalences are preserved by the product, the underlying \(\infty\)-category of \(C\) produces a symmetric monoidal \(\infty\)-category \(N(C^c)[W^{-1}]^\otimes\) by [HA, 4.1.3.4], [HA, 4.1.3.6]. For instance, when \(C\) is a symmetric monoidal model category, the cofibrant objects \(C^c\) form a symmetric monoidal category with
weak equivalences preserved by the product, and this symmetric monoidal category induces a symmetric monoidal structure on $N(C^c)[W^{-1}]$, which we will denote by $N(C^c)[W^{-1}]^\otimes$.

The category $\text{Cat}_S$ has a closed symmetric monoidal structure given by the smash product of hom spectra. However, the smash product of cofibrant spectral categories is not necessarily cofibrant as is also the case for dg categories. Consequently, the model structure on $\text{Cat}_S$ is not monoidal.

To resolve this problem, we will use the notion of flat objects and functors following [BGT2, Chapter 3]. A functor between model categories is flat if it preserves weak equivalences and colimits. An object $X$ of a model category (whose underlying category is monoidal) is then said to be flat if the functor $X \otimes (\cdot)$ is a flat functor. For instance, cofibrant spectra are in particular flat. The following definition given in [BGT2, Chapter 3] satisfies this notion of flatness.

**Definition 4.25.** A spectral category $C$ is pointwise-cofibrant if each morphism spectrum $C(x, y)$ is a cofibrant spectrum.

The following proposition summarizes the properties of pointwise-cofibrant spectral categories that we will need [BGT2, 3.2].

**Proposition 4.26.** The following are properties of pointwise-cofibrant spectral categories:

- Every spectral category is functorially Morita equivalent to a pointwise-cofibrant spectral category with the same objects.
- The subcategory of pointwise-cofibrant spectral categories is closed under the smash product.
- A pointwise-cofibrant spectral category is flat with respect to the smash product of spectral categories.
- If $C$ and $D$ are pointwise-cofibrant spectral categories, the smash product $C \wedge D$ computes the derived smash product $C \wedge^L D$. 

Therefore, we use the subcategory $\text{Cat}_{S}^{\text{flat}}$ of pointwise-cofibrant spectral categories to produce a suitable symmetric monoidal model of the $\infty$-category of idempotent-complete small stable $\infty$-categories. For instance, we have that [BGT2, 3.4]:

**Proposition 4.27.** The functor induced by cofibrant replacement $\text{Cat}_{S}^{\text{flat}} \to \text{Cat}_{S}^{\text{c}}$ induces a categorical equivalence $N(\text{Cat}_{S}^{\text{flat}})[W^{-1}] \simeq N(\text{Cat}_{S}^{\text{c}})[W^{-1}]$, where $W$ is the class of Morita equivalences.

**Corollary 4.28.** We have an equivalence $N(\text{Cat}_{S}^{\text{flat}})[W^{-1}] \simeq \text{Cat}_{\infty}^{\text{perf}}$.

Moreover, combined with [HA, 4.1.3.4] and [HA, 4.1.3.6], we see that:

**Proposition 4.29.** The $\infty$-category $N(\text{Cat}_{S}^{\text{flat}})[W^{-1}]$ can be promoted to a symmetric monoidal $\infty$-category $N(\text{Cat}_{S}^{\text{flat}})[W^{-1}] \otimes$.

Then, [BGT2, 3.5] also states that:

**Proposition 4.30.** There is an equivalence of symmetric monoidal $\infty$-categories $N(\text{Cat}_{S}^{\text{flat}})[W^{-1}] \otimes \simeq (\text{Cat}_{\infty}^{\text{perf}}) \otimes$.

Thus, combined with:

**Proposition 4.31.** Let $R$ be an $E_{\infty}$-ring spectrum, then the category $R\text{-Mod}$ is a commutative algebra object in $(\text{Cat}_{\infty}^{\text{perf}}) \otimes$.

We see that the notion of module categories over $R\text{-Mod}$ is well-defined. In particular, by [HA, 4.2.3.7]:

**Corollary 4.32.** The category $\text{Mod}_{R\text{-Mod}}(N(\text{Cat}_{S}^{\text{flat}})[W^{-1}])$ is an idempotent-complete presentable stable $\infty$-category.

**Proof of proposition.** It is shown in [HA, 6.3.2] that the presentable stable-$\infty$-category of spectra, $\text{Sp}$, admits a symmetric monoidal product $\otimes : \text{Sp} \times \text{Sp} \to \text{Sp}$. If $R$ is an $E_{\infty}$-ring, then $R\text{-Mod}$ is stable and presentable by [HA, 8.1.2.1]. That $R\text{-Mod}$ is, furthermore, a commutative algebra object in $\text{Cat}_{\infty}^{\text{perf}}$ follows from [HA, 8.1.2.6].  

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4.4. The $\infty$-categorical Barr-Beck Theorem.

Given an adjunction of ordinary categories $F : C \xrightleftharpoons{} D : G$, the classic Barr-Beck theorem gives an equivalence between $T$-algebras and $D$ where $T$ is the monad $T = G \circ F$, provided that $G$ is conservative and preserves certain colimits. The Barr-Beck theorem is a useful tool for identifying categories as categories of modules.

In [HA, 6.6.2.5], Lurie provides an $\infty$-categorical version of the Barr-Beck theorem. Namely, if $F : C \xrightleftharpoons{} D : G$ is an adjunction of $\infty$-categories, then there is an equivalence between $T$-algebras and $D$ as long as $G$ is conservative and preserves certain colimits (i.e. so-called $G$-split simplicial objects).

A corollary, [HA, 6.6.2.14], of the $\infty$-categorical Barr-Beck theorem will provide the machinery for our main theorem, we record it here for use in the next section.

**Theorem 4.33.** Suppose we are given a commutative diagram of $\infty$-categories

\[
\begin{array}{ccc}
C & \xrightarrow{U} & C' \\
\downarrow{G} & & \downarrow{G'} \\
D & \xleftarrow{G'} & C'
\end{array}
\]

Assume that:

1. The functors $G$ and $G'$ admit left adjoints $F$ and $F'$.

2. $C$ admits geometric realizations of simplicial objects, which are preserved by $G$.

3. $C'$ admits geometric realizations of simplicial objects, which are preserved by $G'$.

4. The functors $G$ and $G'$ are conservative.

5. For each object $d \in D$, the unit map $d \to GF(d) \simeq G'(UF(d))$ induces an equivalence $G'F'(d) \to GF(d)$ in $D$.

Then $U$ is an equivalence of $\infty$-categories.
5. The Main Theorem

We now prove the main theorem of this paper. We show that if $R$ is an $E_{\infty}$-ring spectrum, then the $\infty$-category of module categories over $R$-$\text{Mod}$ in $(\text{Cat}_{\infty}^{\text{perf}})\otimes$ corresponds to the underlying $\infty$-category of categories enriched in $R$-module spectra. Note that in particular, $Hk$ is an $E_{\infty}$-ring spectrum, and this will provide the link between dg categories over $k$ and $k$-linear stable $\infty$-categories. We will discuss this more in the next section. We first state and prove our theorem.

Let $W'$ be the class of Morita equivalences in $\text{Cat}_S$ and let $W$ be the class of Morita equivalences in $\text{Mod}_{R,\text{Mod}^{\text{cell}}}(\text{Cat}_S)$. Note that these Morita equivalences are determined by the forgetful functor to $\text{Cat}_S$. The natural monoidal functor

$$\theta : N(Cat_{S}^{\text{flat}})^{\otimes} \to N(Cat_{S}^{\text{flat}})[W^{-1}]^{\otimes}$$

defined by the localization of symmetric monoidal $\infty$-categories (which exists by [HA, 4.1.3.4]) determines a natural map

$$\theta : N(\text{Mod}_{\text{R,Mod}^{\text{cell}}}(\text{Cat}_S)^{\text{flat}})[W^{-1}] \to \text{Mod}_R \text{-Mod}(N(Cat_{S}^{\text{flat}})[W^{-1}]^{\otimes}).$$

This map $\theta$ is well-defined because the nerve of a module category over $R$-$\text{Mod}^{\text{cell}}$ in $\text{Cat}_S$ is naturally a module category over the $\infty$-category $R$-$\text{Mod}$. This follows by 4.24 and by the observations that the $\infty$-category associated to $R$-$\text{Mod}^{\text{cell}}$ is $R$-$\text{Mod} \in \text{Cat}_{\infty}^{\text{perf}}$ by 3.8.

**Theorem 5.1.** The functor

$$\theta : N(\text{Mod}_{R,\text{Mod}^{\text{cell}}}(\text{Cat}_S)^{\text{flat}})[W^{-1}] \to \text{Mod}_R \text{-Mod}(N(Cat_{S}^{\text{flat}})[W^{-1}]^{\otimes})$$

is an equivalence of presentable stable $\infty$-categories.

**Proof.** As discussed in the previous section, the proof of this theorem is an application of the $\infty$-categorical Barr-Beck theorem. Namely, we show that this diagram satisfies the hypotheses of [HA, 6.2.2.14]:

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Consider the diagram

\[
\begin{array}{c}
N(\text{Mod}_{R-\text{Mod}}(\text{Cat}_S)^{\text{flat}})[W^{-1}] \\
\downarrow G \\
N(\text{Cat}_S^{\text{flat}})[W'^{-1}] \\
\end{array}
\xrightarrow{\theta}
\begin{array}{c}
\text{Mod}_{R-\text{Mod}}(N(\text{Cat}_S^{\text{flat}})[W'^{-1}]^\otimes) \\
\downarrow G' \\
\end{array}
\]

(1) The two \(\infty\)-categories given by the source and target of \(\theta\) admit geometric realizations of simplicial objects. In fact, both of these categories are presentable \(\infty\)-categories. The former is a presentable \(\infty\)-category since \(\text{Mod}_{R-\text{Mod}}(\text{Cat}_S)\) is a combinatorial model category. Since the underlying \(\infty\)-category of a combinatorial model category is a presentable \(\infty\)-category, \(N(\text{Mod}_{R-\text{Mod}}(\text{Cat}_S)^{\text{flat}})[W^{-1}]\) is a presentable \(\infty\)-category. The latter is a presentable \(\infty\)-category by [HA, 4.2.3.7], that is, categories of modules over a presentable \(\infty\)-category is a presentable \(\infty\)-category.

(2) The functors \(G\) and \(G'\) admit left adjoints \(F\) and \(F'\). The existence of a left adjoint to \(G\) follows from the fact that \(G\) is determined by a right Quillen functor. The existence of a left adjoint to \(G'\) follows from [HA, 4.2.4.8].

(3) The functor \(G'\) is conservative and preserves geometric realizations. The former is true by [HA, 4.2.3.2]. The latter is true by [HA, 4.2.3.5].

(4) The functor \(G\) is conservative and preserves geometric realizations. The first assertion is immediate from the definition of the weak equivalences in \(\text{Mod}_{R-\text{Mod}}(\text{Cat}_S)\), and the second follows from the fact that \(G\) is also a left Quillen functor and by [T, 5.5.8.17].

(5) The morphism \(G' \circ F' \to G \circ F\) is an equivalence. This follows because the natural map \(\mathcal{C} \to R-\text{Mod} \otimes \mathcal{C}\) induces an equivalence \(F'(\mathcal{C}) \cong R-\text{Mod} \otimes \mathcal{C}\) by [HA, 4.2.4.8].

Therefore, the functor is an equivalence. \(\square\)
Recall that $\text{Mod}_{R,\text{Mod}^{\text{cat}}}(\text{Cat}_S)$ is equivalent to the category of categories enriched in $R$-module spectra, $\text{Cat}_{R,\text{Mod}}$. Using our main theorem above we deduce the following corollary.

**Corollary 5.2.** There is an equivalence of $\infty$-categories

$$N(\text{Cat}_{R,\text{Mod}})[W^{-1}] \simeq \text{Mod}_{R,\text{Mod}}(N(\text{Cat}^{\text{flat}}_S)[W'^{-1}]^\otimes).$$

Applying this theorem to the Eilenberg-MacLane ring spectrum $Hk$ yields the following corollary.

**Corollary 5.3.** There is an equivalence of $\infty$-categories

$$N(\text{Cat}_{Hk,\text{Mod}})[W^{-1}] \simeq \text{Mod}_{Hk,\text{Mod}}(N(\text{Cat}^{\text{flat}}_S)[W'^{-1}]^\otimes).$$

We now combine this result with the Quillen equivalence [Tab1] $\text{Cat}_{Hk,\text{Mod}} \simeq \text{Cat}^k_{\text{dg}}$.

**Corollary 5.4.** There is an equivalence of $\infty$-categories

$$N(\text{Cat}^k_{\text{dg}})[W^{-1}] \simeq \text{Mod}_{Hk,\text{Mod}}(N(\text{Cat}^{\text{flat}}_S)[W'^{-1}]^\otimes).$$

In other words, the underlying $\infty$-category of the Morita model category structure on $\text{Cat}^k_{\text{dg}}$ is equivalent to the $\infty$-category of $k$-linear stable $\infty$-categories.

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