Non-holomorphic effective potential in $N = 4$ SU$(n)$ SYM

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Abstract

We compute the one-loop non-holomorphic effective potential for the $N = 4$ SU$(n)$ supersymmetric Yang-Mills theory with the gauge symmetry broken down to the maximal torus $U(1)^{n-1}$. Our approach remains powerful for arbitrary gauge groups and is based on the use of $N = 2$ harmonic superspace formulation for general $N = 2$ Yang-Mills theories along with the superfield background field method.
Extended supersymmetry imposes strong restrictions on the structure of quantum field theories. One of the most prominent examples where extended supersymmetry has played a substantial role is the exact solution for non-perturbative low-energy effective action in the $N = 2$ $SU(2)$ supersymmetric Yang-Mills theory given by Seiberg and Witten \cite{1}. Their construction was generalized to arbitrary gauge groups in \cite{2}. Another intriguing example comes from the $N = 4$ Yang-Mills theory where the powerful symmetry properties allow one to exactly compute some Green’s functions (see \cite{3} and references therein).

In the background field formulation, the effective action of $N = 2$, $D = 4$ super Yang-Mills theories is a manifestly gauge invariant and supersymmetric functional of the covariantly chiral strength $W$ and its conjugate $\bar{W}$ \cite{4}. In the Coulomb branch the effective action is in general reads

$$
\Gamma[W, \bar{W}] = \text{Im} \int d^4x d^4\theta \mathcal{F}(W) + \int d^4xd^8\theta \mathcal{H}(\bar{W}, W) + \ldots
$$

where the first term is integrated over the chiral subspace of $N = 2$ superspace while the second term is integrated over the full $N = 2$ superspace parametrized by $z^M \equiv (x^m, \theta_i^\alpha, \bar{\theta}_i^\dot{\alpha})$. The elipsis denotes all terms involving derivatives of the strengths. The holomorphic potential $\mathcal{F}(W)$ dominates at low energies and presents itself the main object of the Seiberg-Witten theory. The non-holomorphic potential $\mathcal{H}(\bar{W}, W)$ constitutes the next-to-leading finite quantum correction. For finite $N = 2$ Yang-Mills theories with matter, $\mathcal{F}(W)$ coincides with the classical gauge action, and hence $\mathcal{H}(\bar{W}, W)$ is the dominant quantum correction. An important representative of such superconformal models is the $N = 4$ Yang-Mills theory (the first finite quantum field theory found \cite{5}) in which the $N = 2$ matter is realized by a single hypermultiplet in the adjoint representation.

In a recent paper \cite{6} Dine and Seiberg showed that the requirement of scale and chiral invariance severely restricts the possible structure of $\mathcal{H}(\bar{W}, W)$ in the $N = 4$ Yang-Mills theory. For the $N = 4$ $SU(2)$ theory broken down to $U(1)$, they found out the only admissible form for the one-loop non-holomorphic potential:

$$
\mathcal{H}(\bar{W}, W) = c \ln \frac{\bar{W}^2}{\Lambda^2} \ln \frac{W^2}{\Lambda^2}
$$

with some numerical coefficient $c$ and some scale $\Lambda$. The corresponding action given by the second term in (1) turns out to be independent on $\Lambda$. Moreover, Dine and Seiberg argued that $\mathcal{H}(\bar{W}, W)$ gets neither higher-loop perturbative nor instanton corrections, what was confirmed by instantons calculations \cite{7} and two-loop supergraph analysis \cite{8}. The problem of explicit calculation of the coefficient $c$ has been recently solved in Refs.
on the base of different techniques, the final result being \( c = (8\pi)^{-2} \). This value for \( c \) was given in [9] to be the result of calculations based on the use of \( N = 1 \) superspace formulation for the \( N = 4 \) Yang-Mills theory. Gonzalez-Rey and Roček [10] computed, in the framework of \( N = 2 \) projective superspace approach, a special sector of the hypermultiplet low-energy action and then gave some grounds that the non-holomorphic effective potential \( \mathcal{H}(\bar{W}, W) \) should have the same functional form. Finally, in our paper [11] we directly analysed, in the framework of \( N = 2 \) harmonic superspace approach, the effective action corresponding to the \( N = 2 \) gauge multiplet of the full \( N = 4 \) Yang-Mills theory.

In the present paper we extend the results of our work [11] to the case of \( N = 4 \) \( SU(n) \) Yang-Mills theory with the gauge group broken down to \( U(1)^{n-1} \). Our method of computing \( \mathcal{H}(\bar{W}, W) \) is equally powerful for arbitrary semi-simple gauge groups and naturally leads to a nice algebraic structure encoded in \( \mathcal{H}(\bar{W}, W) \).

It is interesting to note that \( \mathcal{H}(\bar{W}, W) \) is in general unambiguously defined when \( W \) lies along the flat directions of the \( N = 2 \) Yang-Mills potential

\[
[W, W] = 0. \tag{3}
\]

Otherwise, the following identity [4]

\[
\{\mathcal{D}_i^\alpha, \mathcal{D}_j^\beta\}W = 2i \varepsilon_{\alpha\beta} \varepsilon^{ij} [\bar{W}, W] \tag{4}
\]

implies that some higher derivative terms, which are denoted by the dots in (1), can also contribute to \( \mathcal{H}(\bar{W}, W) \). Such problems do not appear when eq. (3) takes place.

As is well known, the most powerful approach to investigate quantum supersymmetric field theories is to make use of an unconstrained superfield formulation. Unfortunately, such a manifestly supersymmetric formulation for the \( N = 4 \) Yang-Mills theory is not known. For our present purpose, however, it is sufficient to realize the \( N = 4 \) Yang-Mills theory as a theory of \( N = 2 \) unconstrained superfields. The \( N = 2 \) harmonic superspace [12] is the only manifestly supersymmetric formalism developed to describe general \( N = 2 \) Yang-Mills theories in terms of unconstrained (analytic) superfields. This approach has been successfully applied for investigating effective action in various \( N = 2 \) supersymmetric models in recent papers [13, 14, 15, 8, 11].

From the point of view of \( N = 2 \) supersymmetry, the \( N = 4 \) Yang-Mills theory describes coupling of the \( N = 2 \) vector multiplet to the hypermultiplet in the adjoint representation. In the harmonic superspace approach, the vector multiplet is realized by
an unconstrained analytic gauge superfield $V^{++}$. As concerns the hypermultiplet, it can be described either by a real unconstrained analytic superfield $\omega$ ($\omega$-hypermultiplet) or by a complex unconstrained analytic superfield $q^+$ and its conjugate $\bar{q}^+$ ($q$-hypermultiplet).

In the $\omega$-hypermultiplet realization, the classical action of $N = 4$ Yang-Mills theory reads

$$S[\omega^{++}, \omega] = \frac{1}{2g^2} \text{tr} \int d^4xd^4\theta W^2 - \frac{1}{2g^2} \text{tr} \int d\zeta(-4) \nabla^{++} \omega \nabla^{++} \omega$$  \hspace{1cm} (5)$$

where the second term describes the $\omega$-hypermultiplet action and is integrated over the analytic subspace of harmonic superspace (see Refs. [12, 15, 11] for more details and notation). The first term in (5) is the pure $N = 2$ Yang-Mills action. The explicit expression for the strength $W$ via the prepotential $V^{++}$ is given in [16]. The theory with action (5) is manifestly $N = 2$ supersymmetric. However, the action (5) turns out to be invariant under two hidden supersymmetric transformations [12]

$$\delta V^{++} = u_i^+ \left( \epsilon^{\alpha i} \theta_{\dot{\alpha}}^{+} + \epsilon_{\alpha}^{\dot{\alpha}} \bar{\theta}^{+\dot{\alpha}} \right) \omega$$
$$\delta \omega = -\frac{1}{8} u_i^- \left\{ (D^+)^2(\epsilon^i \theta - W_\lambda) + (\bar{D}^+)^2(\bar{\epsilon}^i \bar{\theta} - \bar{W}_{\dot{\lambda}}) \right\}.$$  \hspace{1cm} (6)$$

Here $W_\lambda$ denotes the strength in the $\lambda$-frame [12, 15]. In the $q$-hypermultiplet realization, the $N = 4$ Yang-Mills theory is given by the action

$$S[V^{++}, q^+, \bar{q}] = \frac{1}{2g^2} \text{tr} \int d^4xd^4\theta W^2 - \frac{1}{2g^2} \text{tr} \int d\zeta(-4) q^{++} \nabla^{++} q_i^+$$  \hspace{1cm} (7)$$

where

$$q_i^+ = (q_i^+, \bar{q}_i^+) , \quad q^{+i} = \epsilon^{ij} q_j^+ = (\bar{q}^+,-q^+) .$$  \hspace{1cm} (8)$$

This model is manifestly $N = 2$ supersymmetric. It also possesses two hidden supersymmetries

$$\delta V^{++} = \left( \epsilon^{\alpha i} \theta_{\dot{\alpha}}^{+} + \epsilon_{\alpha}^{\dot{\alpha}} \bar{\theta}^{+\dot{\alpha}} \right) q_i^+$$
$$\delta q^{+i} = -\frac{1}{4} \left\{ (D^+)^2(\epsilon^i \theta - W_\lambda) + (\bar{D}^+)^2(\bar{\epsilon}^i \bar{\theta} - \bar{W}_{\dot{\lambda}}) \right\} .$$  \hspace{1cm} (9)$$

To provide manifest gauge invariance and supersymmetry at the quantum level, we study the effective action for the classically equivalent theories (3) and (7) within the $N = 2$ superfield background field method [13, 8]. In accordance with [15, 14, 13], the one-loop effective action in both realizations is given by

$$\Gamma^{(1)}[V^{++}] = \frac{i}{2} \text{Tr}(2,2) \ln \square - \frac{i}{2} \text{Tr}(4,0) \ln \square$$  \hspace{1cm} (10)$$
where $\Box$ is the analytic d’Alambertian introduced in \[15\]
\[
\Box = D^m D_m + \frac{i}{2}(D^+ W)D^-_\alpha + \frac{i}{2}(D^-_\alpha W)D^+ - \frac{i}{4}(D^+ D^-_\alpha W)D^- - \frac{1}{8}[D^+ , D^-_\alpha]W + \frac{1}{2}\{W, W\}.
\] (11)

The formal definitions of the Tr\(_{(2, 2)}\)ln $\Box$ and Tr\(_{(4, 0)}\)ln $\Box$ are given in Ref. \[11\].

For computing $\mathcal{H}(\bar{W}, W)$ it is sufficient in fact to consider a special background
\[
D^\alpha (D^-)^{\alpha} W = 0.
\] (12)

Then, one can get the following path integral representation for $\Gamma^{(1)}$ \[11\]
\[
\exp\left(\frac{i}{2}\operatorname{tr} \int d\zeta \zeta \Box F^{++}\right) \frac{\exp\left\{-i\frac{1}{2}\operatorname{tr} \int d\zeta \zeta F^{++}\right\}}{\int \exp\left\{-i\frac{1}{2}\operatorname{tr} \int d\zeta \zeta F^{++}\right\}}.
\] (13)

The superfield $F^{++}(z, u)$ belonging to the adjoint representation looks like $F^{++}(z, u) = F^{ij}(z) u^+_i u^+_j$, with $F^{ij} = F^{ji}$ satisfying the constraints
\[
D^{(\alpha}_\alpha F^{jk)} = D^{(\alpha}_\alpha F^{jk)} = 0, \quad \overline{F}^{ij} = F_{ij}.
\] (14)

The operator $\Box$ acts on $F^{ij}$ as follows
\[
\Box F^{ij} = (D^m D_m + \frac{1}{2}\{\bar{W}, W\}) F^{ij} + \frac{i}{3} D^{\alpha(i} W D_{\alpha[k]} F^{jk)} + \frac{i}{3} \bar{D}^{(\alpha}_\alpha \bar{W} \bar{D}^{\alpha)}_{\alpha[k]} F^{jk)}.
\] (15)

Representation (13) involves path integrals over constrained $N = 2$ superfields. Our aim now is to transform these path integrals to those over unconstrained $N = 1$ superfields. We introduce $N = 1$ Grassmann coordinates $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ by the rule $\theta^\alpha = \theta^1_\alpha$, $\bar{\theta}^{\dot{\alpha}} = \bar{\theta}^1_{\dot{\alpha}}$, the corresponding gauge covariant derivatives $D_\alpha = D^1_\alpha$, $\bar{D}^{\dot{\alpha}} = \bar{D}^1_{\dot{\alpha}}$ and then define the $N = 1$ projection of an arbitrary $N = 2$ superfield $f(z^M)$ by the standard rule $f| = f(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})|_{\theta^2 = \bar{\theta}^2 = 0}$. As is well known, from the $N = 2$ Yang-Mills strength $W$ one obtains two $N = 1$ covariantly chiral superfields $\Phi = W|$ and $2iW_\alpha = D^2_{\alpha} W|$. The $N = 1$ projections of $F^{ij}$ read
\[
\Psi = F^{22}|, \quad \bar{\Psi} = F^{11}|, \quad F = F = -2iF^{12}|
\] (16)

and satisfy the constraints
\[
D_\alpha \Psi = 0, \quad -\frac{1}{4} D^2 F + [\Phi, \Psi] = 0
\] (17)
Therefore, $\Psi$ is a covariantly chiral $N = 1$ superfield while the real superfield $F$ is subject to a modified linear constraint.

Until this point, the $N = 2$ Yang-Mills strength was constrained only by eq. (12). Now, we specify $W$ to belong to the Cartan subalgebra and, hence, to satisfy eq. (3). Moreover, we require the $N = 1$ components of $W$ to be (covariantly) constant,

$$D_\alpha \Phi = 0, \quad D_\alpha W_\beta = 0.$$  \hspace{1cm} (18)

Such a background is still sufficient for calculating $\mathcal{H}(\bar{W}, W)$, since the identity

$$\int d^4x d^8\theta \mathcal{H}(\bar{W}, W) = \int d^8z W^\alpha \bar{W}_\dot{\alpha} \bar{W}_\dot{\beta} \frac{\partial^4 \mathcal{H}(\Phi, \bar{\Phi})}{\partial \Phi^2 \partial \bar{\Phi}^2} + \text{derivatives}$$  \hspace{1cm} (19)

along with the requirement of scale and chiral invariance allow us to uniquely restore $\mathcal{H}(\bar{W}, W)$. Here $d^8z$ denotes the full $N = 1$ superspace measure. For the background chosen the operator $\bar{\square}$ does not mix the superfields $\Psi$, $\bar{\Psi}$ and $F$

$$\Delta(F^{ij}) = (\bar{\square} F^{ij})|.$$  \hspace{1cm} (20)

where

$$\Delta = D^m D_m - W^\alpha D_\alpha + \bar{W}_\dot{\alpha} \bar{D}^{\dot{\alpha}} + \frac{1}{2} \{\Phi, \bar{\Phi}\}.$$  \hspace{1cm} (21)

Expressing the $N = 2$ integration variables in (13) via their $N = 1$ projections, we obtain the following representation for $\Gamma^{(1)}$ in terms of path integrals over (still constrained) $N = 1$ superfields \footnote{It is worth pointing out that we deduce eq. (22) from the representation (13) for $\Gamma^{(1)}$ which is manifestly $N = 2$ supersymmetric and invariant with respect to the automorphism $SU(2)_R$ symmetry. That is why it is in our power to make use of any useful technique in order to compute special contributions to $\Gamma^{(1)}$, in particular, to reduce $\Gamma^{(1)}$ to $N = 1$ superfields. This is completely different to the case when the $N = 2$ or $N = 4$ theories are formulated from the very beginning in $N = 1$ superfields, when only $N = 1$ supersymmetry is realized off-shell; in such a case the effective action possesses $N = 1$ supersymmetry only. By construction, our approach is manifestly $N = 2$ supersymmetric, in spite of the comments given in [13].}

$$\exp\left(i\Gamma^{(1)}\right) = \frac{\int D\bar{\Psi}D\Psi DFD \exp\left\{i \operatorname{tr} \int d^8z (-\bar{\Psi} \Delta \bar{\Psi} + \frac{i}{2} F \Delta F)\right\}}{\int D\bar{\Psi}D\Psi DFD \exp\left\{i \operatorname{tr} \int d^8z (-\bar{\Psi} \bar{\Psi} + \frac{i}{2} F^2)\right\}}.$$  \hspace{1cm} (22)

Our next step is to evaluate the right hand side of eq. (22).

Until now the gauge group was completely arbitrary. Let us specialize our consideration to the case of $SU(n)$. To start with we make a quick tour through the corresponding...
Lie algebra $su(n)$ consisting of hermitian traceless matrices. We introduce the Weyl basis $\{e_{kl}\}$ of $su(n)$

$$(e_{kl})_{pq} = \delta_{kp}\delta_{lq}, \quad k, l, p, q = 1, 2, \ldots, n \tag{23}$$

Then an arbitrary element $a \in su(n)$ looks like

$$a = \sum_{k=1}^{n} a^k e_{kk} + \sum_{k \neq l} a^{kl} e_{kl}, \quad a^{kl} = \overline{a^{lk}}, \quad \sum_{k=1}^{n} a^k = 0 \tag{24}$$

with $a^i$ being real. The elements $r$ of the Cartan subalgebra are

$$r = \sum_{k=1}^{n} r^k e_{kk} = \text{diag}(r^1, r^2, \ldots, r^n), \quad \sum_{i=1}^{n} r^i = 0 . \tag{25}$$

For any elements of the Weyl basis we have

$$\text{tr}(e_{pq} e_{kl}) = 2n \text{tr}_F(e_{pq} e_{kl}) = 2n \delta_{pl} \delta_{qk} . \tag{26}$$

Here ‘$\text{tr}_F$’ denotes the trace in the fundamental representation. From here one gets important consequences

$$\text{tr}(e_{kl} e_{lk}) = 2n ; \quad \text{tr}(e_{pq} e_{kl}) = 0 , \quad p \neq l, q \neq k . \tag{27}$$

Given an element $r$ of the Cartan subalgebra, one finds

$$[r, e_{kl}] = (r^k - r^l)e_{kl} \tag{28}$$

with the eigenvalues $(r^k - r^l)$ defining the roots of $su(n)$.

For the gauge group chosen, the strengths $W$ and $\bar{W}$ lie in the Cartan subalgebra of $su(n)$

$$W = \text{diag}(W^1, W^2, \ldots, W^n) , \quad \sum_{k=1}^{n} W^k = 0 . \tag{29}$$

Since we are interested in the situation when the gauge group $SU(n)$ is broken down to the maximal torus $U(1)^{n-1}$, we should have $W^k - W^l \neq 0$ for $k \neq l$. In the opposite case, when several eigenvalues $W^k$ coincide, some nonabelian group $H \in SU(n)$ remains unbroken. Introducing the $N = 1$ projections $\Phi = W|$ and $W_\alpha = -\frac{i}{2}D^2_\alpha W|$ associated with $W$, we obtain the $N = 1$ superfield roots $\Phi^k - \Phi^l$ and $W^k_\alpha - W^l_\alpha$. The above restrictions on $W^k$ are equivalent to $\Phi^k - \Phi^l \neq 0$ for $k \neq l$.

\footnote{From now on, small Latin letters are used for $SU(n)$ indices.}
Let us return to eq. (22). Since the strengths $\Phi$ and $W_\alpha$ belong to the Cartan subalgebra, the components of the quantum superfields $\bar{\Psi}$, $\Psi$, $F$ which lie in the Cartan subalgebra do not interact with the background field and therefore they completely decouple. On the other hand, the components of $\Psi$ and $\bar{\Psi}$ out of the Cartan subalgebra are expressed via $F$ and $\bar{F}$ with the aid of constraints (17)

$$
\Psi_{kl} = \frac{\mathcal{D}^2 F^{kl}}{4(\Phi^k - \Phi^l)}, \quad \bar{\Psi}_{kl} = \frac{\mathcal{D}^2 F^{kl}}{4(\Phi^k - \Phi^l)},
$$

(30)

and these expressions are nonsingular in the case under consideration. As a result, we can transform the right hand side of eq. (22) to path integrals over unconstrained superfields

$$
V^{kl} \equiv F^{kl}, \quad \bar{V}^{kl} \equiv F^{lk}, \quad k < l.
$$

(31)

Taking into account eqs. (27) and (30), we can transform the integral in the denominator of eq. (22) as follows

$$
\text{tr} \int d^8z (-\bar{\Psi} \Psi + \frac{1}{2} F^2) = 2n \int d^8z \sum_{k<l} \bar{V}^{kl} B_{kl} V^{kl}
$$

(32)

where

$$
B_{kl} = \frac{1}{16} \left\{ \mathcal{D}^2, \mathcal{D}^2 \right\} |\Phi^k - \Phi^l|^2 + 1.
$$

(33)

It is worth pointing out that the sum in (32) is taken over half the roots and we can choose the positive roots to contribute to (32). As a result

$$
\int \mathcal{D} \Psi \mathcal{D} \bar{\Psi} \mathcal{D} F \exp \left\{ i \text{tr} \int d^8z (-\bar{\Psi} \Psi + \frac{1}{2} F^2) \right\}
$$

$$
= \int \mathcal{D} \bar{V}^{kl} \mathcal{D} V^{kl} \exp \left\{ 2n i \int d^8z \sum_{k<l} \bar{V}^{kl} B_{kl} V^{kl} \right\} = \prod_{k<l} \text{Det}^{-1}(B_{kl}).
$$

(34)

Next we turn to the nominator in (22). First of all we find the action of $\Delta$ (21) on the superfields $F^{kl}$. The result reads

$$
\Delta(F^{kl} \dot{e}_{kl}) = (\Delta_{kl} F^{kl}) \dot{e}_{kl} \quad \text{(no sum)}
$$

(35)

where

$$
\Delta_{kl} = \mathcal{D}^m \mathcal{D}_m - (W^{\alpha \bar{\alpha}} - W^{l \bar{\alpha}}) \mathcal{D}_\alpha + (\bar{W}^{\bar{\alpha} \bar{\alpha}} - \bar{W}^{l \bar{\alpha}} \bar{\mathcal{D}}^{\bar{\alpha}} + |\Phi^k - \Phi^l|^2.
$$

(36)

Using (30) and fulfilling straightforward calculations we get

$$
\text{tr} \int d^8z \left( \frac{1}{2} F \Delta F - \bar{\Psi} \Delta \Psi \right) = 2n \int d^8z \sum_{k<l} \bar{V}^{kl} B_{kl} \Delta_{kl} V^{kl}
$$

(37)
where $B_{kl}$ is given by eq. (33). From here we obtain
\[
\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}F \exp \left\{ i \text{tr} \int d^8z \left( -\bar{\Psi}\Delta\Psi + \frac{1}{2} F\Delta F \right) \right\} = \int \mathcal{D}\bar{V}^{kl}V^{kl} \exp \left\{ 2n i \int d^8z \sum_{k<l} \bar{V}^{kl}B_{kl}\Delta_{kl}V^{kl} \right\} = \prod_{k<l} \text{Det}^{-1}(B_{kl})\text{Det}^{-1}(\Delta_{kl}) .
\] (38)

We result with
\[
e^{i\Gamma^{(1)}} = \prod_{k<l} \text{Det}^{-1}(\Delta_{kl}) .
\] (39)

It is seen that the one-loop correction $\Gamma^{(1)}$ to effective action is determined by the functional determinant of the operator (36) on the space of unconstrained $N = 1$ superfields under the Feynman boundary conditions. Eq. (39) can be rewritten as follows
\[
\Gamma^{(1)} = \sum_{k<l} \Gamma_{kl} , \quad \Gamma_{kl} = i \text{Tr} \ln \Delta_{kl} .
\] (40)

The $SU(n)$-operator $\Delta_{kl}$ (36) has the same structure as the $SU(2)$-operator $\Delta$ introduced in \[11\]. Therefore we can apply the technique developed in \[11\] and obtain
\[
\Gamma_{kl} = \frac{1}{(4\pi)^2} \int d^8z \frac{W^{\alpha kl}W^{\bar{k}l\bar{\alpha}}W^{k\bar{\alpha}}W^{\bar{l}\alpha}}{(\Phi^{kl})^2(\bar{\Phi}^{kl})^2}
\] (41)

where
\[
\Phi^{kl} = \Phi^k - \Phi^l , \quad W^{\alpha kl} = W^{k\alpha} - W^{l\alpha} .
\] (42)

Eqs. (40–42) define the non-holomorphic effective potential $\mathcal{H}(\bar{W}, W)$ of the $N = 4$ Yang-Mills theory in terms of the $N = 1$ projections of $W$ and $\bar{W}$.

From eqs. (19) and (40–42) one can easily restore $\mathcal{H}(\bar{W}, W)$:
\[
\Gamma^{(1)} = \int \text{d}^4x \text{d}^8\theta \mathcal{H}(\bar{W}, W)
\]
\[
\mathcal{H}(\bar{W}, W) = \frac{1}{(8\pi)^2} \sum_{k<l} \ln \left( \frac{\bar{W}^k - \bar{W}^l}{\Lambda} \right)^2 \ln \left( \frac{W^k - W^l}{\Lambda} \right)^2
\] (43)

where the strengths $W^k$ are chosen as in (29), with $W^k - W^l \neq 0$ for $k \neq l$. Eq. (43) is our final result. Similar to the holomorphic effective potential $\mathcal{F}(W)$ \[4\], the non-holomorphic effective potential is constructed in terms of the roots of $SU(n)$ and obviously invariant under the Weyl group. Some bosonic contributions to $\Gamma^{(1)}$ were discussed in \[18\].

It is necessary to point out that our method to compute the non-holomorphic effective potential $\mathcal{H}(\bar{W}, W)$ is general and perfectly works for arbitrary semi-simple gauge groups, for instance, $SO(n)$. The starting point is representation (13). Then, one has to specify
the Cartan subalgebra and Weyl basis for the gauge group in field and, finally, it remains to repeat the technical steps described. Given a semi-simple rank-$r$ gauge group $G$, we introduce its Weyl basis $\{h_i, e^{+\alpha}, e^{-\alpha}\}$, where the elements $h_i$ span the Cartan subalgebra, $i = 1, \ldots, r$, and $\pm\alpha$ are the positive (negative) roots. When the gauge group is broken down to its maximal torus $U(1)^r$, the $N = 2$ strength looks like $W = \sum W_i h_i$, $[W, e^{+\alpha}] = W^{+\alpha} e^{+\alpha}$, with all $W^{+\alpha}$ being non-vanishing. The non-holomorphic effective potential reads

$$\mathcal{H}(\bar{W}, W) = \frac{1}{(8\pi)^2} \sum_{\text{pos. roots}} \ln \left( \frac{\bar{W}^{+\alpha}}{\Lambda} \right)^2 \ln \left( \frac{W^{+\alpha}}{\Lambda} \right)^2$$

and this is similar to the structure of perturbative holomorphic effective potential \[2\].

When this work was completed, there appeared recent papers \[19, 20\] where similar results were obtained by different methods.

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**References**

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; B430 (1994) 485.

[2] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B344 (1995) 169;
   P.C. Argyres and A.E. Farragi, Phys. Rev. Lett. 74 (1995) 3931; A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. A11 (1996) 1929.

[3] P.S. Howe, E. Sokatchev and P.C. West, 3-Point Functions in $N = 4$ Yang-Mills, hep-th/9808162.

[4] R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B133 (1978) 275.

[5] M. Sohnius and P. West, Phys. Lett. B100 (1981) 45; M. Grisaru and W. Siegel, Nucl.
   Phys. B201 (1982) 292; S. Mandelstam, Nucl. Phys. B213 (1983) 149; P.S. Howe, K.S.
   Stelle and P.K. Townsend, Nucl. Phys. B214 (1983) 519; Nucl. Phys. B236 (1984)
   125; L. Brink, O. Lindgren and B. Nilsson, Nucl. Phys. B212 (1983) 401.
[6] M. Dine and N. Seiberg, Phys. Lett. B409 (1997) 239.

[7] N. Dorey, V.V. Khoze, M.P. Mattis, J. Slater and W.A. Weir, Phys. Lett. B408 (1997) 213; D. Bellisai, F. Fucito, M. Matone and G. Travaglini, Phys. Rev. D56 (1997) 5218.

[8] I.L. Buchbinder, S.M. Kuzenko and B.A. Ovrut, Phys. Lett. B433 (1998) 335.

[9] V. Periwal and R. von Unge, Phys. Lett. B430 (1998) 71.

[10] F. Gonzalez-Rey and M. Rocek, Phys. Lett. B434 (1998) 303.

[11] I.L. Buchbinder and S.M. Kuzenko, Mod. Phys. Lett. A13 (1998) 1629.

[12] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1 (1984) 469; A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 2 (1985) 601; 617.

[13] I.L. Buchbinder, E.I. Buchbinder, E.A. Ivanov, S.M. Kuzenko and B.A. Ovrut, Phys. Lett. B412 (1997) 309; E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov and S.M. Kuzenko, Mod. Phys. Lett. A13 (1998) 1071.

[14] S.V. Ketov, Phys. Lett. B399 (1997) 83; E. Ivanov, S. Ketov and B. Zupnik, Nucl. Phys. B509 (1997) 53; S. Ketov, Phys. Rev. D57 (1998) 1277.

[15] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko and B.A. Ovrut, Phys. Lett. B417 (1998) 61.

[16] B. Zupnik, Phys. Lett. B183 (1987) 175.

[17] A.O. Barut, R. Raczka, Theory of Group Representations and Applications, PWN - Polish Scientific Publishers, Warszawa, 1977.

[18] I. Chepelev and A.A. Tseytlin, Nucl. Phys. B511 (1998) 629.

[19] F. Gonzalez-Rey, B. Kulik, I.Y. Park and M. Roček, Self-Dual Effective Action for N = 4 Super-Yang Mills, [hep-th/9810152] v2.

[20] D.A. Lowe and R. von Unge, Constraints on Higher Derivative Operators in Maximal Supersymmetric Gauge Theories, [hep-th/9811017].