GRAVITY WATER FLOWS WITH DISCONTINUOUS VORTICITY AND STAGNATION POINTS

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Abstract. We construct small-amplitude steady periodic gravity water waves arising as the free surface of water flows that contain stagnation points and possess a discontinuous distribution of vorticity in the sense that the flows consist of two layers of constant but different vorticities. We also describe the streamline pattern in the moving frame for the constructed flows.

1. Introduction

We present here a study of steady periodic traveling water waves that propagate at the free surface of a two-dimensional inviscid and incompressible fluid of finite depth allowing for stagnation points and for a discontinuous distribution of vorticity. More precisely, we consider water waves interacting with two vertically superposed currents of different constant vorticities.

Confined first to the investigation of waves of small amplitude, which can be satisfactorily approximated by sinusoidal curves within the linear theory, the examination of periodic traveling water waves arising as the free surface of an irrotational flow with a flat bed originates at the beginning of the 19th century. The description of waves that are flatter near the trough and have steeper elevations near the crest necessitates a nonlinear approach, which was in fact conducted in the last decades and led to the first rigorous results concerning the existence of wave trains in irrotational flow, see for instance the case of Stokes waves [34] and the flow beneath them (particle trajectories, behavior of the pressure) cf. [2, 3, 6, 8].

To go beyond irrotational flows and to treat wave current interactions one needs to incorporate vorticity into the problem, cf. [4, 16, 32]. However, the difficulties generated by the presence of the vorticity have prevented a rigorous mathematical development, which appeared only relatively recently in [7], where the existence of small and large amplitude steady periodic gravity water waves with a general (continuous) vorticity distribution was proved.

Of high significance is the investigation of steady periodic rotational waves interacting with currents that possess a rough – that is discontinuous or unbounded – vorticity. Discontinuous vorticities model sudden changes in the underlying current, numerical simulations of such flows being quite recent [18, 19]. Unbounded vorticities on the other hand can describe turbulent flows in channels (see the empirical law in on page 106 of [1]) and are relevant also in the setting of wind generated waves that possess a thin layer of high vorticity adjacent to the wave surface [30, 31]. The discontinuous vorticity distribution was considered in the groundbreaking paper [9] where the existence of steady two-dimensional periodic gravity water waves of small and large amplitude on water flows with an arbitrary bounded (but discontinuous) vorticity was proved. Small amplitude capillary-gravity waves with discontinuous but bounded vorticity were constructed in [25, 29]. Waves with unbounded vorticity

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were first shown to exist in [26] but only when allowing for surface tension as a restoring force. This situation appears in many physical settings one of which being that of wind blowing over a still fluid surface and giving rise to two-dimensional small amplitude wave trains driven by capillarity [17] which grow larger and turn into capillary-gravity waves.

Another striking occurrence in water flows is the presence of stagnation points, that is points where the steady velocity field vanishes, thus making the analysis more intricate, since the usual Dubreil-Jacotin transform which converts the original free boundary problem into a problem in a fixed domain, is no longer available. There is a short list of papers dealing with existence of water flows allowing for stagnation points and for a non-vanishing continuous vorticity, cf. [10, 13, 14, 20, 35] for gravity waves and [22, 23, 24, 28] for waves with capillarity. Under consideration in this paper is a more involved setting where, in addition to permitting stagnation points (whose existence in the fluid is proven), we also allow for a discontinuous distribution of the vorticity. To our best knowledge the incorporation of both stagnation points and of a discontinuous vorticity is a feature that was not rigorously analyzed before.

The governing equations are the Euler equations of motion, together with boundary conditions on the free surface and on the flat bed of the water flow. The discontinuous vorticity that we consider here is of the following type: we assume that the flow has a layer of constant vorticity \( \gamma_2 \) adjacent to the free surface above another layer of constant vorticity \( \gamma_1 \) neighboring the flat bed. Of course, the interesting situation (that we pursue here) is when \( \gamma_1 \neq \gamma_2 \). The unknowns are here the free surface, the interface separating the regions of different vorticities (which can be seen as an internal wave due to the discontinuity in vorticity), the velocity field and the pressure function. In a first step we reduce the number of unknowns by means of the stream function whose utilization converts the problem into a transmission problem along the line of discontinuity of vorticity with fewer unknowns. The second step that we undertake is to consider a flattening transformation which has the advantage that changes the free boundary value problem into a problem in a fixed domain, thus making it more tractable for the analysis. For studying the latter resulted problem we employ the Crandall-Rabinowitz Theorem on bifurcation from simple eigenvalues.

The dispersion relation that we obtain — which is a formula giving the speed at the free surface of the bifurcation inducing laminar flows in terms of the two vorticities \( \gamma_1, \gamma_2 \), the thickness of the two rotational layers and the wavelength — generalizes the one in [10] obtained in the case of a water flow with constant vorticity and allowing for stagnation points. The intricacy of the dispersion relation — a third order algebraic equation — allows us to prove existence of water waves of small wavelength arising as the free surface of water flows with rotational layers of different constant vorticities and containing stagnation points, cf. Theorems 3.4-3.6, 4.2, 4.3. We present also the streamline pattern in the moving frame for the solutions, cf. Figures 1-3. Our results show especially that the ratio of the amplitudes of the surface wave and that of the internal wave — and the fact that the surface wave and the internal wave are in phase or anti phase — is highly influenced by the vorticities of the currents and by the speed at the free surface of the bifurcation inducing laminar flows.

We briefly outline the content of the paper. We present in Section 2 the governing equations together with the analytic setting we work in. Moreover, we also find the dispersion relation whose analysis is undertaken in Section 3 for the case \( \gamma_2 > 0 \), while the more singular case \( \gamma_2 = 0 \) is treated in Section 4. The Appendix contains several technical lemmas.
2. The governing equations

Under consideration is a two-dimensional steady periodic flow, moving under the influence of gravity, such that the surface waves propagate in the positive \( x \)-direction. The water flow occupies the domain \( \Omega \) bounded below by the flat bed \( y = -d \), with \( d > 0 \), and above by the free surface \( y = h(x) \), which is a small perturbation of the flat free surface \( y = 0 \). In a reference frame moving with the wave speed \( c > 0 \), the equations of motion in \( \Omega \) are Euler’s equations

\[
\begin{align*}
(u - c)u_x + vu_y &= -P_x, \\
(u - c)v_x + vv_y &= -P_y - g, \\
u_x + v_y &= 0,
\end{align*}
\]

(2.1a)

where \((u, v)\) denotes the velocity field, \( P \) stands for pressure and \( g \) is the gravity constant. The equations of motion are supplemented by the boundary conditions, which, ignoring surface tension effects, are

\[
\begin{align*}
P &= P_0 & \text{on } y = h(x), \\
v &= (u - c)h' & \text{on } y = h(x), \\
v &= 0 & \text{on } y = -d,
\end{align*}
\]

(2.1b)

with \( P_0 \) being the constant atmospheric pressure.

We are interested in solutions of the problem (2.1) for which the vorticity \( \omega := u_y - v_x \) of the flow presents discontinuities of the following type: we assume that, adjacent to the free surface, the water flow possesses a layer

\[
\Omega(f, h) := \{(x, y) : x \in \mathbb{R}, -d_2 + f(x) < y < h(x)\},
\]

of constant vorticity \( \gamma_2 \), situated above another layer

\[
\Omega(f) := \{(x, y) : x \in \mathbb{R}, -d < y < -d_2 + f(x)\},
\]

which is adjacent to the flat bed and is of constant vorticity \( \gamma_1 \), that is

\[
\omega := \begin{cases} 
\gamma_1, & \text{in } \Omega(f), \\
\gamma_2, & \text{in } \Omega(f, h).
\end{cases}
\]

(2.1c)

We steadily assume that \( \gamma_1 \neq \gamma_2 \) and that \( d_2 > 0 \), \( d - d_2 =: d_1 > 0 \). We note that, additionally to \((u, v, P, h)\), we have a further unknown: the function \( f \) whose graph separates the two currents of different vorticities. By Helmholtz’s law, the vorticity is constant along streamlines of the steady flow, and as a consequence of this \( y = -d_2 + f \) has to be a streamline of the flow. This streamline can be viewed as an internal wave due to the jump in vorticity.

With the help of the stream function \( \psi \), introduced (up to an additive constant) via the relation \( \nabla \psi = (-v, u - c) \) we can reformulate (2.1a)-(2.1c) as the free-boundary problem

\[
\begin{align*}
\Delta \psi_2 &= \gamma_2 & \text{in } \Omega(f, h), \\
\Delta \psi_1 &= \gamma_1 & \text{in } \Omega(f), \\
\psi_2 &= 0 & \text{on } y = h(x), \\
\psi_2 &= \psi_1 & \text{on } y = -d_2 + f(x), \\
\psi_1 &= m & \text{on } y = -d,
\end{align*}
\]

(2.2a)

subjected to the conditions

\[
\begin{align*}
\partial_y \psi_2 &= \partial_y \psi_1 & \text{on } y = -d_2 + f(x), \\
|\nabla \psi_2|^2 + 2g(d + h) &= Q & \text{on } y = h(x),
\end{align*}
\]

(2.2b)
where the constant \(-m\) represents the relative mass flux and \(Q \in \mathbb{R}\) is related with the hydraulic head. Moreover, \(\psi_1 := \psi|_{\Omega(f)}\) and \(\psi_2 := \psi|_{\Omega(f,h)}\), so that from the fourth equation of \((2.2a)\) and the first equation of \((2.2b)\) we see that the \(\nabla \psi\) (hence also the velocity field) is continuous across the interface \(y = -d_2 + f(x)\).

Given \(\alpha \in (0,1)\), it is not difficult to see that any solution

\[
(f, h, \psi_1, \psi_2) \in (C^{3+\alpha}_{\text{per}}(\mathbb{R}))^2 \times C^{3+\alpha}_{\text{per}}(\Omega(f)) \times C^{3+\alpha}_{\text{per}}(\Omega(f,h))
\]

of \((2.2)\) defines a solution

\[
(u, v, P, (f, h)) \in (C^{1}_{\text{per}}(\Omega))^3 \times (C^{3+\alpha}_{\text{per}}(\Omega))^2
\]
\[
\big((u, v)|_{\Omega(f)}, (u, v)|_{\Omega(f,h)}\big) \in (C^{2+\alpha}_{\text{per}}(\Omega(f)))^2 \times (C^{2+\alpha}_{\text{per}}(\Omega(f,h)))^2
\]
\[
(P|_{\Omega(f)}, P|_{\Omega(f,h)}) \in C^{2+\alpha}_{\text{per}}(\Omega(f)) \times C^{2+\alpha}_{\text{per}}(\Omega(f,h))
\]

of \((2.1)\). The subscript \(\text{per}\) stands for functions that are periodic in the horizontal variable, meaning that all the functions considered above are \(L\)-periodic with respect to \(x\), with \(L > 0\) being fixed.

We first determine laminar flow solutions of problem \((2.2)\), that is water flows with a flat free surface and parallel streamlines, meaning that they present no \(x\)-dependence. Of interest are laminar flows that contain stagnation points, more precisely laminar flows that contain streamlines consisting entirely of stagnation points. Then we study when non laminar solutions bifurcate from the laminar flows and describe the qualitative picture of the streamline pattern for the bifurcating solutions.

**Laminar flow solutions.** Because the stream function is constant along the streamline \(y = -d_2 + f(x)\), we use the value of the stream function

\[
\psi_1 = \psi_2 = \lambda \quad \text{on} \quad y = -d_2 + f(x),
\]

(2.3)

to parametrize a family of laminar solutions of \((2.2a)\). Setting \(f \equiv h \equiv 0\) we obtain from \((2.2a)\) that the stream function \(\psi^0 := (\psi_1^0, \psi_2^0)\) satisfies

\[
\psi_1^0(y) = \frac{\gamma_1 y^2}{2} + \frac{\gamma_1 (d + d_2)}{2} + \frac{\lambda - m}{d_1} y + \frac{\lambda d}{d_1} + \frac{\gamma_1 d_2}{2} - \frac{m d_2}{d_1}, \quad y \in [-d, -d_2]
\]

\[
\psi_2^0(y) = \frac{\gamma_2 y^2}{2} + \frac{\gamma_2 d_2}{2} - \frac{\lambda}{d_2} y, \quad y \in [-d_2, 0].
\]

The equations of \((2.2b)\) are equivalent to

\[
m = \frac{\lambda d}{d_2} + d_1 \frac{\gamma_1 d_1 + \gamma_2 d_2}{2}, \quad Q = \left(\frac{\gamma_2 d_2}{2} - \frac{\lambda}{d_2}\right)^2 + 2gd.
\]

(2.4)

In the following we choose the constants \(m\) and \(Q\) in \((2.2a)\) and \((2.2b)\) to be given by \((2.4)\), the constant \(\lambda\) introduced via \((2.3)\) being left as a parameter. Hence, each \(\lambda \in \mathbb{R}\) determines a unique laminar solution \((f, h, \psi_1, \psi_2) := (0, \psi_1^0, \psi_2^0)\) of \((2.2)\) when \(m\) and \(Q\) are defined by \((2.4)\). These are the laminar solutions from which we study bifurcation.

**Conditions for stagnation.** We note that the laminar flows determined above possess stagnation points – that is water particles that travel horizontally with the wave speed – if and only if

\[
\partial_y \psi_2^0(-d_2) \cdot \partial_y \psi_2^0(0) \leq 0 \quad \text{or} \quad \partial_y \psi_1^0(-d_2) \cdot \partial_y \psi_1^0(-d_2) \leq 0.
\]

(2.5)

If \((2.5)\) holds true, then there exists \(y_0 \in [-d, 0]\) such that

\[
\partial_y \psi_i^0(y_0) = 0 \quad \text{for} \quad i = 1 \text{ or } 2.
\]
Diffeomorphisms, we define the linear elliptic operators \( \frac{\partial_y \psi_2}{\partial y} \mid_{y=0} \). For this reason we define \( \lambda \) via (2.8) and use \( \Lambda \) as parameter.

The analytic setting. With \( \Lambda \) as parameter, we are left to seek special values of \( \Lambda \) such that branches of non laminar solutions of (2.2) bifurcate from the curve of laminar flows. For this, we need to recast (2.2) in a suitable analytic setting.

In the following \( \alpha \in (0,1) \) is a fixed Hölder exponent. Because the equations of (2.2a) and (2.2b) are posed on manifolds that depend on the unknown functions \((f, h)\), it is suitable to transform the problem (2.2) on fixed manifolds. For this, we set \( \Omega_1 := \Omega(0) \), \( \Omega_2 := \Omega(0,0) \) and define the mappings

\[
\Phi_f : \Omega_1 \to \Omega(f), \quad \Phi_f(x, y) = \left( x, \frac{d_1 + f(x)}{d_1} y + \frac{d}{d_1} f(x) \right),
\]

\[
\Phi_{(f,h)} : \Omega_2 \to \Omega(f, h), \quad \Phi_{(f,h)}(x,y) = \left( x, \frac{h(x) - f(x) + d_2}{d_2} y + h(x) \right).
\]

It is easy to see that \( \Phi_f \) and \( \Phi_{(f,h)} \) are \( C^{3+\alpha} \)-diffeomorphisms for each \((f, h) \in \mathcal{O})\), whereby the subscript \( e \) referring to the fact that we consider only even function in \( x \). Using these diffeomorphisms, we define the linear elliptic operators

\[
\mathcal{A}(f) : C^{3+\alpha}_{e,\text{per}}(\Omega_1) \to C^{1+\alpha}_{e,\text{per}}(\Omega_1), \quad \mathcal{A}(f)w_1 := \Delta (w_1 \circ \Phi^{-1}_f) \circ \Phi_f,
\]

\[
\mathcal{A}(f, h) : C^{3+\alpha}_{e,\text{per}}(\Omega_2) \to C^{1+\alpha}_{e,\text{per}}(\Omega_2), \quad \mathcal{A}(f, h)w_2 := \Delta (w_2 \circ \Phi^{-1}_{(f,h)}) \circ \Phi_{(f,h)},
\]

and the boundary operators

\[
\mathcal{B}_1 : \mathbb{R} \times \mathcal{O} \times C^{3+\alpha}_{e,\text{per}}(\Omega_2) \to C^{2+\alpha}_{e,\text{per}}(\mathbb{R}),
\]

\[
\mathcal{B}_2 : \mathbb{R} \times \mathcal{O} \times C^{3+\alpha}_{e,\text{per}}(\Omega_1) \times C^{3+\alpha}_{e,\text{per}}(\Omega_2) \to C^{2+\alpha}_{e,\text{per}}(\mathbb{R}),
\]

respectively through

\[
\mathcal{B}_1(\Lambda, (f, h), w_2) := \left( |\nabla (w_2 \circ \Phi^{-1}_{(f,h)})|^2 \circ \Phi_{(f,h)} + 2g(d + h) - Q \right) \mid_{y=0},
\]

\[
\mathcal{B}_2(\Lambda, (f, h))[w_1, w_2] := \left[ (\partial_y (w_2 \circ \Phi^{-1}_{(f,h)})) \circ \Phi_{(f,h)} - (\partial_y (w_1 \circ \Phi^{-1}_f)) \circ \Phi_f \right] \mid_{y=-d_2}.
\]

Observation 2.1. Let \( \Lambda \in \mathbb{R} \), \((f, h, \psi_1, \psi_2) \in \mathcal{O} \times C^{3+\alpha}_{e,\text{per}}(\Omega(f, h)) \times C^{3+\alpha}_{e,\text{per}}(\Omega(f, h)) \), and assume that \( \lambda, m, Q \) are defined by (2.3), (2.4), and (2.8). Then, the tuple \((f, h, \psi_1, \psi_2)\) solves the problem (2.2) if and only if
(i) $w_1 := \psi_1 \circ \Phi_f \in C_{e,\text{per}}^{3+\alpha}(\Omega_1)$ is the unique solution of the Dirichlet problem

$$\begin{cases}
A(f)w_1 = \gamma_1 & \text{in } \Omega_1, \\
w_1 = \lambda & \text{on } y = -d_2, \\
w_1 = m & \text{on } y = -d.
\end{cases} \quad (2.9)$$

(ii) $w_2 := \psi_2 \circ \Phi_{(f,h)} \in C_{e,\text{per}}^{3+\alpha}(\Omega_2)$ is the unique solution of the Dirichlet problem

$$\begin{cases}
A(f,h)w_2 = \gamma_2 & \text{in } \Omega_2, \\
w_2 = 0 & \text{on } y = 0, \\
w_2 = \lambda & \text{on } y = -d_2.
\end{cases} \quad (2.10)$$

(iii) $B_1(\Lambda, (f,h), w_2) = B_2(\Lambda, (f,h))[w_1, w_2] = 0$ in $C_{e,\text{per}}^{3+\alpha}(\mathbb{R})$.

Thanks to Observation 2.1 we can recast the problem (2.2) as a nonlinear and nonlocal equation with $(\Lambda, (f,h))$ as unknown. In order to proceed, we establish first the following result.

**Lemma 2.2.** Given $(\Lambda, (f,h)) \in \mathbb{R} \times \mathcal{O}$, we let $w_1 := w_1(\Lambda, (f,h))$ and $w_2 := w_2(\Lambda, (f,h))$ denote the unique solution of (2.9) and (2.10), respectively, with $\lambda$ given by (2.8). Then, we have $w_i \in C^\omega(\mathbb{R} \times \mathcal{O}, C_{e,\text{per}}^{3+\alpha}(\Omega_i)), i = 1, 2$.

**Proof.** We prove just the real-analyticity of the solution operator $w_1$, the claim for $w_2$ following similarly. By elliptic theory, cf. e.g. [15], we see that $w_1 : \mathbb{R} \times \mathcal{O} \to C_{e,\text{per}}^{3+\alpha}(\Omega_1)$ is well-defined. Moreover, we have that

$$\mathcal{F}(\Lambda, (f,h), w_1(\Lambda, (f,h))) = 0 \quad \text{for all } (\Lambda, (f,h)) \in \mathbb{R} \times \mathcal{O},$$

whereby $\mathcal{F} \in C^\omega(\mathbb{R} \times \mathcal{O} \times C_{e,\text{per}}^{3+\alpha}(\Omega_1), C_{e,\text{per}}^{3+\alpha}(\Omega_1) \times (C_{e,\text{per}}^{3+\alpha}(\mathbb{R}))^2)$ is the operator defined by

$$\mathcal{F}(\Lambda, (f,h), w_1) := (A(f)w_1 - \gamma_1, w_1|_{y=-d_2}, w_1|_{y=-d}).$$

Taking into account that Fréchet derivative

$$\partial_{w_1} \mathcal{F}(\Lambda, (f,h), w_1(\Lambda, (f,h)))[z] := (A(f)z, z|_{y=-d_2}, z|_{y=-d})$$

is an isomorphism, the assertion follows from the implicit function theorem. \hfill \Box

Because $B_i, i = 1, 2$, depend real-analytically on their arguments too, we obtain from Lemma 2.2 and Observation 2.1 that the problem (2.2) is equivalent to the nonlinear and nonlocal equation

$$\Phi(\Lambda, (f,h)) = 0, \quad (2.11)$$

whereby $\Phi := (\Phi_1, \Phi_2) \in C^\omega(\mathbb{R} \times \mathcal{O}, (C_{e,\text{per}}^{3+\alpha}(\mathbb{R}))^2)$ is the operator defined by

$$\Phi(\Lambda, (f,h)) := (B_1(\Lambda, (f,h), w_2(\Lambda, (f,h))), B_2(\Lambda, (f,h))[w_1(\Lambda, (f,h)), w_2(\Lambda, (f,h))]). \quad (2.12)$$

The laminar flow solutions of (2.2) correspond to the trivial solutions $(\Lambda, 0) \in \mathbb{R} \times \mathcal{O}$ of (2.11). In order to find other solutions, we use the theorem on bifurcations from simple eigenvalues due to Crandall and Rabinowitz [11].

**Theorem 2.3** (Crandall and Rabinowitz). Let $X, Y$ be real Banach spaces and let the mapping $\Phi \in C^\omega(\mathbb{R} \times X, Y)$ satisfy:

(a) $\Phi(\Lambda, 0) = 0$ for all $\Lambda \in \mathbb{R};$

(b) There exists $\Lambda_* \in \mathbb{R}$ such that Fréchet derivative $\partial_\Lambda \Phi(\Lambda_*, 0)$ is a Fredholm operator of index zero with a one-dimensional kernel and

$$\text{Ker } \partial_\Lambda \Phi(\Lambda_*, 0) = \text{span } \{x_0\} \quad \text{with } 0 \neq x_0 \in X.$$
Lemma 2.4. Let \( \Phi \) See Appendix.

Proof. Given \( s \) for \( (\Lambda, x) \) : \((-\varepsilon, \varepsilon) \to \mathbb{R} \times \mathbb{K} \) consisting only of solutions of the equation \( \Phi(\Lambda, x) = 0 \). Moreover, as \( s \to 0 \), we have that

\[
\Lambda(s) = \Lambda_0 + O(s) \quad \text{and} \quad x(s) = sx_0 + O(s^2).
\]

Furthermore, there exists an open set \( U \subset \mathbb{R} \times \mathbb{K} \) with \( (\Lambda, x) \in U \) and

\[
\{(\Lambda, x) \in U : \Phi(\Lambda, x) = 0, x \neq 0\} = \{(\Lambda(s), x(s)) : 0 < |s| < \varepsilon\}.
\]

In order to apply this abstract bifurcation result, we need to compute the Fréchet derivative of the operator \( \Phi \). To this end we state the following lemma.

Lemma 2.5. Let \( \Lambda \in \mathbb{R} \) be given. The Fréchet derivative \( \partial_{(f,h)} \Phi(\Lambda, 0) \) is the matrix operator

\[
\partial_{(f,h)} \Phi(\Lambda, 0) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{L}\left(\left(C^{3+\alpha}_{e,\per}(\mathbb{R})\right)^2, \left(C^{2+\alpha}_{e,\per}(\mathbb{R})\right)^2\right).
\]

Given \( 1 \leq i, j \leq 2 \), the operator \( A_{ij} \in \mathcal{L}(C^{3+\alpha}_{e,\per}(\mathbb{R}), C^{2+\alpha}_{e,\per}(\mathbb{R})) \) is the Fourier multiplier with symbol \( m^{ij}(\Lambda) := (m^{ij}_k(\Lambda))_{k \in \mathbb{N}} \) defined by

\[
m^{11}_k(\Lambda) = -2\Lambda(\gamma_2 d_2 - \Lambda) \frac{R_k}{\sinh(R_k d_2)}, \quad m^{12}_k(\Lambda) = 2\left[g + \gamma_2 \Lambda - \Lambda^2 \frac{R_k}{\tanh(R_k d_2)}\right], \quad m^{21}_k(\Lambda) = \gamma_2 - \gamma_1 + (\Lambda - \gamma_2 d_2) \left[\frac{R_k}{\tanh(R_k d_1)} + \frac{R_k}{\tanh(R_k d_2)}\right],
\]

\[
m^{22}_k(\Lambda) = -\Lambda \frac{R_k}{\sinh(R_k d_2)}
\]

for \( k \in \mathbb{N} \), whereby \( R_k := 2k\pi/L \). For \( k = 0 \) the right-hand side of \((2.13)-(2.16)\) should be understood as the limit of the expressions when letting \( R_k \to 0 \).

Proof. With the help of Lemma 2.4 we are now able to determine when the Fréchet derivative \( \partial_{(f,h)} \Phi(\Lambda, 0) \) is a Fredholm operator.

Lemma 2.5. Let \( \Lambda \in \mathbb{R} \) be given. We have:

(i) If \( \Lambda \in \{0, \gamma_2 d_2\} \), then \( \partial_{(f,h)} \Phi(\Lambda, 0) \) is not a Fredholm operator.

(ii) If \( \Lambda \notin \{0, \gamma_2 d_2\} \), then \( \partial_{(f,h)} \Phi(\Lambda, 0) \) is a Fredholm operator of index zero.

Proof. In order to prove (i), we infer from \((2.13)\) and \((2.14)\) that for \( \Lambda = 0 \) we have

\[
\partial_{(f,h)} \Phi_1(\Lambda, 0) = 2gh \quad \text{for all } (f, h) \in \left(C^{3+\alpha}_{e,\per}(\mathbb{R})\right)^2,
\]

meaning that \( \text{Im} \partial_{(f,h)} \Phi_1(\Lambda, 0) = C^{3+\alpha}_{e,\per}(\mathbb{R}) \). Since \( C^{3+\alpha}_{e,\per}(\mathbb{R}) \) is not a closed subspace of \( C^{2+\alpha}_{e,\per}(\mathbb{R}) \), the assertion is evident. Furthermore, if \( \Lambda = \gamma_2 d_2 \neq 0 \), then

\[
\partial_{(f,h)} \Phi_2(\Lambda, 0) = (\gamma_2 - \gamma_1) f + Ch \quad \text{for all } (f, h) \in \left(C^{3+\alpha}_{e,\per}(\mathbb{R})\right)^2,
\]

whereby

\[
K \sum_{k \in \mathbb{N}} b_k \cos(R_k x) = -\Lambda \sum_{k \in \mathbb{N}} \frac{R_k}{\sinh(R_k d_2)} b_k \cos(R_k x).
\]
It is not difficult to see that $K\subset C^\infty_{e,\text{per}}(\mathbb{R})$, so that $\text{Im} \partial_{(f,h)} \Phi_2(\Lambda, 0) = C^3_{e,\text{per}}(\mathbb{R})$. Hence, $\text{Im} \partial_{(f,h)} \Phi(\Lambda, 0)$ is not a closed subspace of $(C^3_{e,\text{per}}(\mathbb{R}))^2$. This proves (i).

To prove (ii), we choose $\Lambda \notin \{0, \gamma_2 d_2\}$ and set

$$D(k, \Lambda) := m_k^{11}(\Lambda) m_k^{22}(\Lambda) - m_k^{12}(\Lambda) m_k^{21}(\Lambda), \quad k \in \mathbb{N}.$$  

From (2.13)-(2.16) it is clear that there exists $k_0 \in \mathbb{N}$ such that $D(k, \Lambda) \neq 0$ for all $k \geq k_0$.

Defining the symbols $\tilde{m}_{ij}(\Lambda)$ by $\tilde{m}_{ij}(\Lambda) = m_{ij}(\Lambda)$ for $k \geq k_0$ and $1 \leq i, j \leq 2$ and

$$\tilde{m}_k^{11}(\Lambda) = \tilde{m}_k^{22}(\Lambda) = 1, \quad \tilde{m}_k^{12}(\Lambda) = \tilde{m}_k^{21}(\Lambda) = 0 \quad \text{for } 0 \leq k \leq k_0 - 1,$$

we see that $\partial_{(f,h)} \Phi(\Lambda, 0)$ is a compact perturbation of the operator

$$T := \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \in \mathcal{L}((C^3_{e,\text{per}}(\mathbb{R}))^2, (C^3_{e,\text{per}}(\mathbb{R}))^2),$$

where $\tilde{A}_{ij} \in \mathcal{L}(C^3_{e,\text{per}}(\mathbb{R}), C^3_{e,\text{per}}(\mathbb{R}))$ is the Fourier multiplier defined by $\tilde{m}_{ij}(\Lambda), 1 \leq i, j \leq 2$.

Because $\tilde{D}(k, \Lambda) := \tilde{m}_k^{11}(\Lambda) \tilde{m}_k^{22}(\Lambda) - \tilde{m}_k^{12}(\Lambda) \tilde{m}_k^{21}(\Lambda) \neq 0$ for all $k \in \mathbb{N}$, we can define the formal inverse of $T$ by

$$S := \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{pmatrix}. $$

Here, $\tilde{B}_{ij}$ is the Fourier multiplier corresponding to the symbol $b_{ij}$, $1 \leq i, j \leq 2$, whereby

$$b_k^{11} := \frac{\tilde{m}_k^{22}(\Lambda)}{\tilde{D}(k, \Lambda)}, \quad b_k^{12} := -\frac{\tilde{m}_k^{12}(\Lambda)}{\tilde{D}(k, \Lambda)}, \quad b_k^{21} := -\frac{\tilde{m}_k^{21}(\Lambda)}{\tilde{D}(k, \Lambda)}, \quad b_k^{22} := \frac{\tilde{m}_k^{11}(\Lambda)}{\tilde{D}(k, \Lambda)} \quad \text{for } k \in \mathbb{N}. $$

Using now [21, Theorem 2.1], we see that a Fourier multiplier

$$\sum_{k \in \mathbb{N}} \alpha_k \cos(R_k x) \rightarrow \sum_{k \in \mathbb{N}} \lambda_k \alpha_k \cos(R_k x)$$

belongs to $\mathcal{L}(C^3_{e,\text{per}}(\mathbb{R}), C^3_{e,\text{per}}(\mathbb{R}))$ if

$$\sup_{k \in \mathbb{N}} |k \lambda_k| < \infty \quad \text{and} \quad \sup_{k \in \mathbb{N}} k^2 |\lambda_{k+1} - \lambda_k| < \infty.$$  

Because of this, it is a matter of direct computation to see that $\tilde{B}_{ij} \in \mathcal{L}(C^3_{e,\text{per}}(\mathbb{R}), C^3_{e,\text{per}}(\mathbb{R}))$ for all $1 \leq i, j \leq 2$. Hence, $T$ is an isomorphism, and therefore $\partial_{(f,h)} \Phi(\Lambda, 0)$ is a Fredholm operator of index zero.

Because of Lemma 2.5 (i) it is clear that we cannot apply the Crandall-Rabinowitz bifurcation theorem at $(\Lambda, 0)$ with $\Lambda \in \{0, \gamma_2 d_2\}$. As a consequence of this, the laminar flows from which we show that non laminar waves bifurcate will not possess stagnation points on the wave surface or on the interface separating the two layers of constant vorticity, cf. (2.6)-(2.7), but only inside the layers. This is different than in the case of internal waves propagating between two layers of constant but different density, where in the presence of capillarity stagnation points may be located also on the internal wave, cf. [27].

It is now evident that potential bifurcation values for $\Lambda \notin \{0, \gamma_2 d_2\}$ are to be looked for among the solutions of

$$D(k, \Lambda) = 0 \quad (2.17)$$

for some integer $k \geq 1$. Since in Theorem 2.3 the Fréchet derivative $\partial_{(f,h)} \Phi(\Lambda, 0)$ needs to be a Fredholm operator of index zero with a one-dimensional kernel, we need to find $\Lambda$ such
that \((2.17)\) has exactly one root \(1 \leq k \in \mathbb{N}\). Plugging the expressions \((2.13)-(2.16)\) in \((2.17)\), we rediscover the dispersion relation
\[
\Lambda^3 - \frac{1}{R_k} \left[ \gamma_2 \left( R_k d_2 + \frac{\sinh(R_k d_2) \cosh(R_k d_1)}{\cosh(R_k d)} \right) + \gamma_1 \frac{\sinh(R_k d_1) \cosh(R_k d_2)}{\cosh(R_k d)} \right] \Lambda^2 \\
+ \tanh(R_k d) \left[ \frac{\gamma_2^2 d_2 - g}{R_k} \gamma_2 (\gamma_1 - \gamma_2) \frac{\sinh(R_k d_1) \sinh(R_k d_2)}{R_k^2 \sinh(R_k d)} \right] \Lambda \\
+ \frac{g \tanh(R_k d)}{R_k^2} \left[ (\gamma_1 - \gamma_2) \frac{\sinh(R_k d_1) \sinh(R_k d_2)}{\sinh(R_k d)} + \gamma_2 d_2 R_k \right] = 0,
\] (2.18)

found also in [25, Equation (5.11)] (with \(\sigma = 0\)). This relation has been analyzed in the setting of flows without stagnation points in [5] for \(\gamma_1 \neq 0 = \gamma_2\) and in [9] for \(\gamma_1 = 0 \neq \gamma_2\). Herein, we assume only that \(\gamma_1 \neq \gamma_2\) and restrict the analysis to the complementary case when stagnation points are included.

In studying the dispersion relation \((2.18)\) we will make use of the following remark, which allows us to restrict our attention to a few of relevant cases, the remaining ones being analogous.

**Remark 2.6.** Note that \((2.18)\) possesses the following symmetry property: \(k\) is a solution of \((2.18)\) for some \(\Lambda \notin \{0, \gamma_2 d_2\}\) and \((\gamma_1, \gamma_2) \in \mathbb{R}^2\) if and only if \(k\) is a solution of \((2.18)\) for \(-\Lambda \notin \{0, -\gamma_2 d_2\}\) and \((-\gamma_1, -\gamma_2) \in \mathbb{R}^2\). Because additionally the inequalities \((2.6)\) and \((2.7)\) are invariant under the transformation \((\Lambda, (\gamma_1, \gamma_2)) \mapsto (-\Lambda, (-\gamma_1, -\gamma_2))\), we are left only with the two cases:

(i) \(\gamma_2 > 0\) and \(\gamma_1 \neq \gamma_2\);

(ii) \(\gamma_2 = 0\) and \(\gamma_1 < 0\).

3. **Analysis of the dispersion relation: the case \(\gamma_2 > 0\) and \(\gamma_1 \neq \gamma_2\)**

Because the dispersion relation is highly nonlinear in \(k\), the study of the roots of \((2.18)\) when keeping \(\Lambda\) fixed seems to be very difficult. Therefore, we consider the inverse problem of determining the zeros \(\Lambda_1, \Lambda_2, \Lambda_3\) of this cubic equation when keeping \(k\) fixed, and then to study the properties of the mappings \(k \mapsto \Lambda_i(k), 1 \leq i \leq 3\). We will do this for small wavelength \(L\), because then we can use asymptotic expansions and Cardano’s formula in order to determine the roots \(\Lambda_i\) of \((2.18)\). This small wavelength regime corresponds to the setting \(t \to \infty\), where
\[
t := R_k = \frac{2\pi k}{L} \in \mathbb{R}.
\]

Plugging \(t\) for \(R_k\), the equation \((2.18)\) can be written in the more concise form
\[
\Lambda^3 + A(t)\Lambda^2 + B(t)\Lambda + C(t) = 0.
\] (3.1)

We will show in the sequel that equation \((3.1)\) has three real roots when \(t\) is sufficiently large. To this end, we note first that the coefficient functions \(A = A(t), B = B(t)\), and \(C = C(t)\) and their first derivatives have the following asymptotic expansions for \(t \to \infty\):
\[
A = -\gamma_2 d_2 - \frac{\gamma_1 + \gamma_2}{2t} + o\left(\frac{1}{t^2}\right), \quad A' = \frac{\gamma_1 + \gamma_2}{2} \cdot \frac{1}{t^2} + o\left(\frac{1}{t^3}\right),
\]
\[
B = \frac{\gamma_2^2 d_2 - g}{t} + \frac{\gamma_2 (\gamma_1 - \gamma_2)}{2t^2} + o\left(\frac{1}{t^3}\right), \quad B' = -\frac{\gamma_2^2 d_2 - g}{t^2} - \frac{\gamma_2 (\gamma_1 - \gamma_2)}{t^3} + o\left(\frac{1}{t^4}\right),
\]
\[
C = \frac{g\gamma_2^2 d_2}{t} + \frac{g(\gamma_1 - \gamma_2)}{2t^2} + o\left(\frac{1}{t^3}\right), \quad C' = -\frac{g\gamma_2^2 d_2}{t^2} - \frac{g(\gamma_1 - \gamma_2)}{t^3} + o\left(\frac{1}{t^4}\right).
\] (3.2)
Letting $z := \Lambda + A/3$, we find that $z$ solves the depressed cubic equation

$$z^3 + pz + q = 0,$$

with

$$\frac{p}{3} = \frac{B}{3} - \frac{A^2}{9} = -\frac{(\gamma_2d_2)^2}{9} + \frac{(2\gamma_2 - \gamma_1)}{36t^2} + o\left(\frac{1}{t^3}\right)$$

and

$$\frac{q}{2} = \frac{AB}{6} + C - \frac{A^3}{27} = -\frac{(\gamma_2d_2)^3}{6} + \frac{6g + 2\gamma_2(2\gamma_2 - \gamma_1)}{18} - \left[\frac{\gamma_2d_2\gamma_1^2 - \gamma_1\gamma_2 + \gamma_2^2}{36} + g(2\gamma_2 - \gamma_1)\right] \frac{1}{t^2}$$

$$+ \frac{9\gamma_2(\gamma_1^2 - \gamma_2^2)}{216} - (\gamma_1 + \gamma_2)^2 \frac{1}{t^3} + o\left(\frac{1}{t^4}\right).$$

Observe that the discriminant for (3.3) is

$$D := \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = -\frac{9g(\gamma_2d_2)^2}{243t} + O\left(\frac{1}{t^2}\right) < 0$$

for $t \to \infty$, property which implies, cf. [33], that (3.3), and hence also (3.1), has three real roots. They are given by the relation $z = r \cos(\beta)$, whereby

$$r = \sqrt{-4p/3} = \frac{2\gamma_2d_2}{3} - \frac{\gamma_2d_2(2\gamma_2 - \gamma_1)}{3\gamma_2d_2t} + O\left(\frac{1}{t^2}\right)$$

and $\beta$ is one of the solution of

$$\cos(3\beta) = -\frac{q}{2} \sqrt{-\frac{27}{p^3}} = 1 - \frac{3^3g}{2(\gamma_2d_2)^2t} + O\left(\frac{1}{t^2}\right).$$

Thus, choosing $\beta := 3^{-1} \arccos\left(-\frac{q/2}{\sqrt{-27/p^3}}\right)$ we see that $\beta(t) \underset{t \to \infty}{\to} 0$ and the roots of (3.1) are

$$\Lambda_1 = r \cos(\beta) - \frac{A}{3},$$

$$\Lambda_2 = r \cos\left(\beta - \frac{2\pi}{3}\right) - \frac{A}{3} = -r \cos\left(\beta + \frac{\pi}{3}\right) - \frac{A}{3},$$

$$\Lambda_3 = r \cos\left(\beta + \frac{2\pi}{3}\right) - \frac{A}{3} = -r \cos\left(\beta - \frac{\pi}{3}\right) - \frac{A}{3}.$$  

(3.5)

Together with (3.2) and (3.4), it follows at once that for $t \to \infty$ we have

$$\Lambda_1(t) \to \gamma_2d_2, \quad \Lambda_2(t) \to 0, \quad \Lambda_3(t) \to 0.$$

(3.6)

Let us also observe that since $C(t) > 0$ for $t \to \infty$, it must hold that $\Lambda_2(t)\Lambda_3(t) < 0$ for $t \to \infty$. Moreover, it is clear from (3.5) that $\Lambda_2(t) > \Lambda_3(t)$, hence

$$\Lambda_2(t) > 0 > \Lambda_3(t) \quad \text{for} \ t \to \infty.$$ 

3.1. **Existence of water flows bifurcating from $\Lambda_1$.** In order to consider the bifurcation problem for (2.11), we need to study first the properties of the mapping $[t \mapsto \Lambda_1(t)]$.

**Lemma 3.1.** There is a constant $t_0 \geq 0$ such that the function

$$[t_0, \infty) \ni t \mapsto \Lambda_1(t) \in (0, \infty)$$

is strictly monotone.
Proof. Let $\phi(t, \Lambda) := \Lambda^3 + A(t)\Lambda^2 + B(t)\Lambda + C(t)$ for $\Lambda \in \mathbb{R}$ and $t \geq 0$. Since for $t \to \infty$ we have

$$
\phi_\Lambda(t, \Lambda_1(t)) = (\Lambda_1(t) - \Lambda_2(t))(\Lambda_1(t) - \Lambda_3(t)) > 0,
$$

we conclude that $\Lambda_1$ is differentiable with respect to $t$. On the other hand

$$
t^2\Phi_\Lambda(t, \Lambda_1(t)) = t^2(\Lambda_1^2(t)A'(t) + \Lambda_1(t)B'(t) + C'(t)) \to_{t \to \infty} \frac{(\gamma_2d_2)^2(\gamma_1 - \gamma_2)}{2}.
$$

Since $\Lambda_1'(t) = -\phi_\Lambda(t, \Lambda_1(t))/\phi_\Lambda(t, \Lambda_1(t))$, we conclude that $\Lambda_1'$ has the same sign as $\gamma_2 - \gamma_1$. The constant $t_0$ is defined as $t_0 := \inf\{t > 0 : |\Lambda_1'| > 0 \text{ on } (t, \infty)\}$. □

From Lemma 3.1 it follows at once that

• if $\gamma_1 < \gamma_2$, then $\Lambda_1(t)$ satisfies (2.6) for $t \geq t_0$;
• if $\gamma_1 > \gamma_2$, then $\Lambda_1(t)$ satisfies (2.7) for $t \geq t_0$.

We look now for bifurcation solutions when choosing $\Lambda_1$ as the bifurcation point. Therefore, we choose $t_0 > 0$ in Lemma 3.1 large enough to guarantee additionally that

$$
\inf_{[t_0, \infty)} \Lambda_1^2 > \sup_{(t_0, \infty)} \left(\Lambda_2^2 + \Lambda_3^2\right),
$$

$$
D(0, \Lambda_1(t)) \neq 0 \quad \text{for all } t \geq t_0.
$$

Let

$$
L_0 := 2\pi/t_0,
$$

fix $L \leq L_0$, and set $\Lambda_1 := \Lambda_1(2\pi/L)$. Then, since $\phi(2\pi/L, \Lambda_1) = 0$, we get $D(1, \Lambda_1) = 0$. Due to the choice of $t_0$, the equation $D(\cdot, \Lambda_1)$ has no solutions $k \in \mathbb{N}$ other than $k = 1$. Consequently, since $\Lambda_1 \notin \{0, \gamma_2d_2\}$, the derivative $\partial_{(f, h)}\Phi_1(\Lambda_1, 0)$ is a Fredholm operator with a one-dimensional kernel

$$
\text{Ker} \partial_{(f, h)}\Phi_1(\Lambda_1, 0) = \text{span} \left\{(m^{22}_1(\Lambda_1), -m^{21}_1(\Lambda_1)) \cos(2\pi x/L)\right\}.
$$

In order to apply Theorem 2.3 to this particular setting, it remains to study whether the transversality condition is satisfied. To this end, we obtain the following characterization of $\text{Im} \partial_{(f, h)}\Phi_1(\Lambda_1, 0)$.

Lemma 3.2. Let $L_0$ be given by (3.9), $L \leq L_0$, and set $\Lambda_1 := \Lambda_1(2\pi/L)$. Then, we have

$$
\text{Im} \partial_{(f, h)}\Phi(\Lambda_1, 0) = \left\{ (\xi, \eta) = \left(\sum_{k \in \mathbb{N}} \xi_k \cos(R_k x), \sum_{k \in \mathbb{N}} \eta_k \cos(R_k x) \right) : \gamma_1 = \frac{m^{11}_1(\Lambda_1)}{m^{21}_1(\Lambda_1)} \eta_1 \right\}.
$$

Proof. To prove the claim, let $(f, h) = \left(\sum_{k \in \mathbb{N}} a_k \cos(R_k x), \sum_{k \in \mathbb{N}} b_k \cos(R_k x)\right)$ be such that

$$
\partial_{(f, h)}\Phi(\Lambda_1, 0)(f, h) = (\xi, \eta).
$$

Then, obviously

$$
\begin{cases}
m^{11}_1(\Lambda_1)a_1 + m^{12}_1(\Lambda_1)b_1 = \gamma_1, \\
m^{21}_1(\Lambda_1)a_1 + m^{22}_1(\Lambda_1)b_1 = \eta_1.
\end{cases}
$$

Because $D(1, \Lambda_1) = 0$, we find from (2.13)-(2.16) that

$$
\frac{m^{11}_1(\Lambda_1)}{m^{21}_1(\Lambda_1)} = \frac{m^{12}_1(\Lambda_1)}{m^{22}_1(\Lambda_1)} =: \mu \neq 0.
$$

Hence, $(\xi, \eta)$ is an element of the set defined by the right-hand side of (3.11). Because the latter set is a closed subspace of $(C^{2+\alpha}_{\text{per}}(\mathbb{R}))^2$ of codimension one that contains $\text{Im} \partial_{(f, h)}\Phi(\Lambda_1, 0)$, the conclusion follows from Lemma 2.5. □

We are now at the point of showing the transversality condition (c) from Theorem 2.3.
Lemma 3.3. We have that
\[ \partial_{\lambda(f,h)}\Phi(\Lambda_1,0)[(m_1^{22}(\Lambda_1), -m_1^{21}(\Lambda_1)) \cos(2\pi x/L)] \neq \Im \partial_{(f,h)}\Phi(\Lambda_1,0). \]

Proof. Since for \(a, b \in \mathbb{R}\), it holds that
\[ \partial_{\lambda(f,h)}\Phi(\Lambda_1,0)[(a, b) \cos(2\pi x/L)] = (am_1^{11}(\Lambda_1) + bm_1^{12}(\Lambda_1), am_1^{21}(\Lambda_1) + bm_1^{22}(\Lambda_1)) \cos(2\pi x/L), \]
we are left to show that
\[ m_1^{22}(\Lambda_1)m_1^{11}(\Lambda_1) - m_1^{21}(\Lambda_1)m_1^{12}(\Lambda_1) \neq \frac{m_1^{11}(\Lambda_1)}{m_1^{21}(\Lambda_1)}(m_1^{22}(\Lambda_1)m_1^{21}(\Lambda_1) - m_1^{21}(\Lambda_1)m_1^{22}(\Lambda_1)), \]
or equivalently that
\[ m_1^{22}(\Lambda_1)m_1^{11}(\Lambda_1) - m_1^{21}(\Lambda_1)m_1^{12}(\Lambda_1) \neq m_1^{21}(\Lambda_1)m_1^{22}(\Lambda_1) - m_1^{11}(\Lambda_1)m_1^{22}(\Lambda_1). \]

Hence, we need to show that \( D_\Lambda(1, \Lambda_1) \neq 0 \). Recalling the definition of the mapping \( \phi \) from the proof of Lemma 3.1, we have that \( D(1, \Lambda) = \phi(2\pi/L, \Lambda) \), and therefore
\[ D_\Lambda(1, \Lambda_1) = \Phi_\Lambda(2\pi/L, \Lambda_1(2\pi/L)) > 0, \]
which is the desired property. \( \square \)

Theorem 3.4 (Bifurcation from \( \Lambda_1 \)). Let \( \gamma_2 > 0 \), \( \gamma_1 \neq \gamma_2 \) and let \( \alpha \in (0, 1) \) be given. Furthermore, let \( L_0 \) be the constant defined by (3.9) and \( L \leq L_0 \). Then, there exists a real-analytic curve \( (\Lambda(s), (f, h)(s)) : (-\varepsilon, \varepsilon) \rightarrow (0, \infty) \times \mathcal{O} \) consisting only of solutions of the problem (2.11). This curve contains exactly one trivial solution of (2.11), and for \( s \rightarrow 0 \) we have that
\[ \Lambda(s) = \Lambda_1 + O(s), \quad (f, h)(s) = s(m_1^{22}(\Lambda_1), -m_1^{21}(\Lambda_1)) \cos(2\pi x/L) + O(s^2), \]
whereby \( \Lambda_1 := \Lambda_1(2\pi/L) \). The flow determined by \( (\Lambda(s), (f, h)(s)), s \in (-\varepsilon, \varepsilon) \), contains a critical layer consisting of closed streamlines very close to the internal wave

(i) in the layer adjacent to the wave surface if \( \gamma_1 < \gamma_2 \), or
(ii) in the bottom layer if \( \gamma_1 > \gamma_2 \).

Moreover, the amplitude of the internal wave is much larger than that of the surface wave, cf. Figure 1.

Proof. It remains only to show that the amplitude of the internal wave is much larger than that of the surface wave. To this end, we note that due to \( D(1, \Lambda_1) = 0 \), we have
\[ \frac{-m_1^{22}(\Lambda_1)}{m_1^{21}(\Lambda_1)} \rightarrow_{L \rightarrow 0} \text{sign} (\gamma_1 - \gamma_2) \infty, \]
since
\[ \lim_{L \rightarrow 0} \frac{m_1^{12}(\Lambda_1)}{m_1^{11}(\Lambda_1)} = \lim_{t \rightarrow \infty} \frac{g + \gamma_2\Lambda_1(t) - \Lambda_2(t)}{\text{tanh} (td_2)} \frac{t}{\text{sign} (\gamma_2 \Lambda_1(t) - \Lambda_1(t))} = \text{sign} (\gamma_2 - \gamma_1) \infty. \]
\( \square \)
Figure 1. This figure illustrates the streamlines in the moving frame for the solutions found in Theorem 3.4 for $\gamma_1 > \gamma_2 > 0$ (left) and $\gamma_1 < \gamma_2, \gamma_2 > 0$ (right), cf. Lemmas A.1-A.2. The thick streamlines represent the wave surface, the internal wave, and the flat bad, respectively. The blue streamlines are separatrices which bound the critical layer and the dashed line consists of points where the $y$–derivative of the stream function vanishes. This line contains in both cases exactly three stagnation points: two located at $x = 0$ and $x = L$, and a third one inside the critical layer at $x = L/2$.

3.2. Existence of water flows bifurcating from $\Lambda_2$. For $t \to \infty$ we have that
- $\Lambda_2(t)$ satisfies $(2.6)$;
- if $\gamma_1 d_1 + \gamma_2 d_2 \leq 0$, then $\Lambda_2(t)$ satisfies also $(2.7)$.

Letting $\phi = \phi(t, \Lambda)$ be the function defined in the proof of Lemma 3.1, we note that for large $t$ we have

$$\phi_\Lambda(t, \Lambda_2(t)) = (\Lambda_2(t) - \Lambda_1(t))(\Lambda_2(t) - \Lambda_3(t)) < 0.$$ 

Hence, $\Lambda_2$ is differentiable with respect to $t$. Moreover, it follows from (3.2) that

$$t^2 \phi_t(t, \Lambda_2(t)) \to_{t \to \infty} -g\gamma_2 d_2,$$

and therefore $\Lambda'_2(t) = -\phi_t(t, \Lambda_2(t))/\phi_\Lambda(t, \Lambda_2(t)) < 0$ when $t$ is large. Defining

$$t_0 := \inf \{ t > 0 : \Lambda'_2 < 0 \text{ on } (t, \infty) \},$$

we see that $[t_0, \infty) \ni t \mapsto \Lambda_2(t) \in (0, \infty]$ is decreasing. In view of this property, we can choose $t_0 > 0$ large enough to guarantee that

$$\sup_{[t_0, \infty)} \Lambda_2 < \inf_{[t_0, \infty)} \Lambda_1,$$

$$D(0, \Lambda_2(t)) \neq 0 \quad \text{for all } t \geq t_0. \quad (3.12)$$

Then, we set

$$L_0 := 2\pi/t_0, \quad (3.13)$$

we fix $L \leq L_0$, and define $\Lambda_2 := \Lambda_2(2\pi/L)$. Recalling that $\phi(2\pi/L, \Lambda_2) = 0$, we obtain that $D(1, \Lambda_2) = 0$. In fact, the equation $D(\cdot, \Lambda_2) = 0$ has $k = 1$ as the only integer solution, cf. (3.12). Because of $\Lambda_2 \in (0, \gamma_2 d_2)$, $\partial_{(f,h)} \Phi_1(\Lambda_2, 0)$ is a Fredholm operator with a one-dimensional kernel

$$\text{Ker} \partial_{(f,h)} \Phi_1(\Lambda_2, 0) = \text{span} \{ (m_1^{22}(\Lambda_2), -m_1^{21}(\Lambda_2)) \cos(2\pi x/L) \}. $$
Using the same arguments as in the proof of Lemma 3.2, we see that
\[
\text{Im} \partial_{(f,h)} \Phi(\Lambda_2,0) = \left\{ (\xi,\eta) = \left( \sum_{k \in \mathbb{N}} \xi_k \cos(R_k x), \sum_{k \in \mathbb{N}} \eta_k \cos(R_k x) \right) : \gamma_1 = \frac{m_1^{11}(\Lambda_2)}{m_2^{21}(\Lambda_2) \eta_1} \right\}.
\]
Moreover, the transversality condition
\[
\partial_{\Lambda(f,h)} \Phi(\Lambda_2,0) \left[ \left( m_2^{22}(\Lambda_2), -m_1^{21}(\Lambda_2) \right) \cos(2\pi x/L) \right] \notin \text{Im} \partial_{(f,h)} \Phi(\Lambda_2,0)
\]
reduces to showing that
\[
D_\Lambda(1,\Lambda_2) = \phi_\Lambda(2\pi/L,\Lambda_2(2\pi/L)) \neq 0,
\]
relation which holds true. We conclude with the following result.

\[\text{Figure 2.}\] This figure illustrates the streamlines in the moving frame for the solutions found in Theorem 3.5 for \(\gamma_1 d_1 + \gamma_2 d_2 > 0\) (left) and \(\gamma_1 d_1 + \gamma_2 d_2 \leq 0\) (right), cf. Lemmas A.3-A.4.

**Theorem 3.5** (Bifurcation from \(\Lambda_2\)). Let \(\gamma_2 > 0\), \(\gamma_1 \neq \gamma_2\) and let \(\alpha \in (0,1)\) be given. Furthermore, let \(L_0\) be the constant defined by (3.13) and \(L \leq L_0\). Then, there exists a real-analytic curve \((\Lambda,(f,h)) : (-\varepsilon,\varepsilon) \to (0,\infty) \times \mathcal{O}\) consisting only of solutions of the problem (2.11). This curve contains exactly one trivial solution of (2.11), and for \(s \to 0\) we have that

\[
\Lambda(s) = \Lambda_2 + O(s), \quad (f,h)(s) = s \left( m_1^{22}(\Lambda_2), -m_1^{21}(\Lambda_2) \right) \cos(2\pi x/L) + O(s^2),
\]
whereby \(\Lambda_2 := \Lambda_2(2\pi/L)\). The flow determined by \((\Lambda(s),(f,h)(s)), s \in (-\varepsilon,\varepsilon)\), contains a critical layer consisting of closed streamlines

(i) in the layer adjacent to the wave surface if \(\gamma_1 d_1 + \gamma_2 d_2 > 0\);
(ii) in each of the layers if \(\gamma_1 d_1 + \gamma_2 d_2 \leq 0\).

The vortex in the top layer is located right beneath the wave surface. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 2.

**Proof.** It remains only to show that the amplitude of the surface wave is much larger than that of the internal wave. This follows from (2.15)-(2.16), as we have

\[
-\frac{m_1^{21}(\Lambda_2)}{m_1^{22}(\Lambda_2)} \to_{L \to 0} -\infty.
\]

□
3.3. Existence of water flows bifurcating from $\Lambda_3$. Since $0 > \Lambda_3(t) \to t \to \infty 0$ we see that $\Lambda_3(t)$ satisfies (2.7) provided that $\gamma_1 d_1 + \gamma_2 d_2 < 0$. Because for large $t$
\[
\phi_\Lambda(t, \Lambda_3(t)) = (\Lambda_3(t) - \Lambda_1(t))(\Lambda_3(t) - \Lambda_2(t)) > 0,
\]
the function $\Lambda_3$ is differentiable with respect to $t$. Since $t^2 \phi_\Lambda(t, \Lambda_3(t)) \to t \to \infty -g\gamma_2 d_2$, we conclude that $\Lambda_3'(t) > 0$ when $t$ is large. Defining $t_0 := \inf\{t > 0 : \Lambda_3'(t) > 0 \text{ on } (t, \infty)\}$, we see that $[[t_0, \infty) \ni t \mapsto \Lambda_3(t) \in (-\infty, 0)]$ is increasing. In view of this property, we can choose $t_0 > 0$ large enough to guarantee that
\[
D(0, \Lambda_3(t)) \neq 0 \quad \text{for all } t \geq t_0.
\] (3.14)
Let
\[
L_0 := 2\pi/t_0,
\]
choose $L \leq L_0$, and define $\Lambda_3 := \Lambda_3(2\pi/L)$. Since $\phi(2\pi/L, \Lambda_3) = 0$, we get that $k \in \mathbb{N}$ solves $D(k, \Lambda_3) = 0$ if and only if $k = 1$. Moreover, since $\Lambda_3 < 0$, $\partial_{(f,h)}\Phi_1(\Lambda_3, 0)$ is a Fredholm operator with a one-dimensional kernel
\[
\text{Ker } \partial_{(f,h)}\Phi_1(\Lambda_3, 0) = \text{span } \{(m_1^{22}(\Lambda_3), -m_1^{21}(\Lambda_3)) \cos(2\pi x/L)\}.
\]
As in Lemma 3.2, we find that
\[
\text{Im } \partial_{(f,h)}\Phi(\Lambda_3, 0) = \left\{(\xi, \eta) = \left(\sum_{k \in \mathbb{N}} \xi_k \cos(R_k x), \sum_{k \in \mathbb{N}} \eta_k \cos(R_k x)\right) : \gamma_1 = \frac{m_1^{22}(\Lambda_3)}{m_1^{21}(\Lambda_3)} \eta_1\right\},
\]
the transversality condition
\[
\partial_{(f,h)}\Phi(\Lambda_3, 0)\left[(m_1^{22}(\Lambda_3), -m_1^{21}(\Lambda_3)) \cos(2\pi x/L)\right] \notin \text{Im } \partial_{(f,h)}\Phi(\Lambda_3, 0)
\]
being equivalent to $D_\Lambda(1, \Lambda_3) = \phi_\Lambda(2\pi/L, \Lambda_3(2\pi/L)) \neq 0$. This shows that all the assumptions of Theorem 2.3 are satisfied. Consequently, we have the following result.

Theorem 3.6 (Bifurcation from $\Lambda_3$). Let $\gamma_2 > 0$, $\alpha \in (0, 1)$, and assume $\gamma_1 d_1 + \gamma_2 d_2 < 0$. Furthermore, let $L_0$ be the constant defined by (3.15) and $L \leq L_0$. Then, there exists a real-analytic curve $(\Lambda, (f, h)) : (-\varepsilon, \varepsilon) \to (0, \infty) \times \mathcal{O}$ consisting only of solutions of the problem (2.11). This curve contains exactly one trivial solution of (2.11), and for $s \to 0$ we have that
\[
\Lambda(s) = \Lambda_3 + O(s), \quad (f, h)(s) = s(m_1^{22}(\Lambda_3), -m_1^{21}(\Lambda_3)) \cos(2\pi x/L) + O(s^2),
\]
whereby $\Lambda_3 := \Lambda_3(2\pi/L)$. The flow determined by $(\Lambda(s), (f, h)(s))$, $s \in (-\varepsilon, \varepsilon)$, contains a critical layer consisting of closed streamlines in the layer adjacent to the bed. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 3.

Proof. The claim concerning the amplitude of the surface and internal waves follows from (2.15)-(2.16), as we have
\[
-\frac{m_1^{21}(\Lambda_2)}{m_1^{22}(\Lambda_2)} \to L \to 0 \infty.
\]
\[\square\]
4. **Analysis of the dispersion relation: the case \( \gamma_2 = 0 \) and \( \gamma_1 < 0 \)**

Because of \( \gamma_2 = 0 \), the inequality (2.6) reduces to \( \Lambda = 0 \), situation when \( \partial_{(f,h)} \Phi(\Lambda, 0) \) is not even a Fredholm operator, cf. Lemma 2.5. For this reason we consider the bifurcation problem for (2.11) just for values of \( \Lambda \) which satisfy (2.7). Hence, the flows that we construct will have stagnation points only in the bottom layer.

With the notation from Section 3, we determine for the depressed cubic equation (3.3) that

\[
D = \left( \frac{p}{3} \right)^3 + \left( \frac{q}{2} \right)^2 = -\frac{g^3}{27} t^{-3} + O(t^{-4}) < 0 \quad \text{for } t \to \infty,
\]

hence (3.1) has again three positive roots. They are given by the relation \( z = r \cos(\beta) \), whereby

\[
r = \sqrt{-\frac{4p}{3}} = 2\sqrt{\frac{g}{3}} t^{-1/2} + \frac{\gamma_1^2}{36} \sqrt{\frac{3}{g}} t^{-3/2} + O(t^{-5/2})
\]

and \( \beta \) is one of the solutions of

\[
\cos(3\beta) = \frac{-q}{2} \sqrt{\frac{27}{p^3}} = -\gamma_1 \sqrt{\frac{3}{g}} t^{-1/2} + \frac{10\gamma_1^3}{72g} \sqrt{\frac{3}{g}} t^{-3/2} + O(t^{-5/2}).
\]

Setting \( \beta := 3^{-1} \arccos \left( -\frac{q}{2} \sqrt{-27/p^3} \right) \), we see that \( \beta(t) \to_{t \to \infty} \pi/6 \) and the roots of (3.1) are

\[
\Lambda_1 = r \cos(\beta) - \frac{A}{3}, \\
\Lambda_2 = r \cos \left( \beta - \frac{2\pi}{3} \right) - \frac{A}{3} = -r \cos \left( \beta + \frac{\pi}{3} \right) - \frac{A}{3}, \\
\Lambda_3 = r \cos \left( \beta + \frac{2\pi}{3} \right) - \frac{A}{3} = -r \cos \left( \beta - \frac{\pi}{3} \right) - \frac{A}{3}.
\]
It follows now easily from (3.2), (4.1), and (4.2) that \( \Lambda_i \to t \to \infty 0 \) for \( i = 1, 2, 3 \) and that \( \Lambda_3 < \Lambda_2 < 0 < \Lambda_1 \) for \( t \to \infty \). Thus, we can find \( t_0 > 0 \) such that
\[
\gamma_1 d_1 \Lambda_3 < \Lambda_2 < 0 < \Lambda_1 \quad \text{on } [t_0, \infty),
\]
\[
D(0, \Lambda_i(t)) \neq 0 \quad \text{for all } t \geq t_0, \ i = 2, 3.
\]
(4.3)

In view of \( \gamma_1 < 0 \), the relation (2.7) is equivalent to \( \Lambda \in (\gamma_1 d_1, 0) \) and therefore just flows bifurcating from negative \( \Lambda \) may contain stagnation points. For this reason, we only investigate in the following the functions \( \Lambda_2 \) and \( \Lambda_3 \).

**Lemma 4.1.** There exists \( t_0 > 0 \) such that (4.3) holds and \( \Lambda_i : [t_0, \infty) \to (-\infty, 0), i = 2, 3 \), are both increasing functions.

**Proof.** Note first that \( \phi_\Lambda(t, \Lambda_2(t)) < 0 \) and \( \phi_\Lambda(t, \Lambda_3(t)) > 0 \) for \( t \geq t_0 \). Hence, \( \Lambda_i, i = 2, 3 \), are differentiable on \( [t_0, \infty). \) Moreover, it is easy to see from (4.1) and (3.2) that
\[
\lim_{t \to \infty} t^{5/2} \phi_\Lambda(t, \Lambda_3(t)) = \lim_{t \to \infty} t^{5/2} \Lambda_3(t) B'(t) = -g^{3/2}.
\]
Hence, we may chose \( t_0 \) large to ensure the assertion for the mapping \( \Lambda_3 \).

This argument does no longer work for \( \Lambda_2 \) as \( \cos(\beta + \pi/3) \to t, \to \infty 0 \). Hence, we have to determine an expansion for \( \cos(\beta + \pi/3) \). Let
\[
z_0 := \frac{\sqrt{3}}{2} - \frac{\gamma_1}{6} \sqrt{\frac{3}{g}} t^{-1/2},
\]
and observe that
\[
|\cos(\beta) - z_0| \cos(\beta + z_0 \cos(\beta) + z_0^2) = |4 \cos^{3}(\beta) - 3 \cos(\beta) - 4z_0^3 + 3z_0| = \left| \cos(3\beta) + \gamma_1 \sqrt{\frac{3}{g}} t^{-1/2} \right| + O(t^{-1}) = O(t^{-1}).
\]
Therewith \( \cos(\beta) = z_0 + O(t^{-1}) \) for \( t \to \infty \). It is now easy to see that for \( t \to \infty \) we have
\[
\sin(\beta) = \frac{1}{2} + \frac{\gamma_1}{2\sqrt{g}} t^{-1/2} + O(t^{-1}),
\]
from which it follows easily that
\[
\Lambda_2(t) := \frac{5\gamma_1}{6} t^{-1} + O(t^{-3/2}) \quad \text{and} \quad \Lambda_3(t) := -\sqrt{g} t^{-1/2} - \frac{\gamma_1}{6} t^{-1} + O(t^{-3/2}).
\]
(4.4)
The expansion (4.4) combined with (3.2) shows that
\[
\lim_{t \to \infty} t^3 \phi_\Lambda(t, \Lambda_2(t)) = \lim_{t \to \infty} t^3 (\Lambda_2(t) B'(t) + C'(t)) = -g \gamma_1 / 6 > 0,
\]
relation which proves the claim. \( \square \)

**Theorem 4.2** (Bifurcation from \( \Lambda_3 \)). Let \( \gamma_2 = 0, \gamma_1 < 0, \) and \( \alpha \in (0, 1) \). Furthermore, let \( L_0 := 2\pi / t_0 \) and \( L \leq L_0 \). Then, there exists a real-analytic curve \( (\Lambda, (f, h)) : (\varepsilon, \varepsilon) \to (0, \infty) \times \mathcal{O} \) consisting only of solutions of the problem (2.11). This curve contains exactly one trivial solution of (2.11), and for \( s \to 0 \) we have that
\[
\Lambda(s) = \Lambda_3 + O(s), \quad (f, h)(s) = s(m^{22}_1(\Lambda_3), -m^{21}_1(\Lambda_3)) \cos(2\pi x / L) + O(s^2),
\]
whereby \( \Lambda_3 := \Lambda_3(2\pi / L) \). The flow determined by \( (\Lambda(s), (f, h)(s)), s \in (-\varepsilon, \varepsilon), \) contains a critical layer consisting of closed streamlines in the layer adjacent to the bed just below the internal wave. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 3.
Proof. Because $t := 2\pi/L \geq t_0$, we know from (4.3) and Lemma 2.5 that $\partial_{(f,h)} \Phi_1(\Lambda_3,0)$ is a Fredholm operator. To determine its kernel we need to solve $D(k, \Lambda_3) = \phi(kt, \Lambda_3) = 0$. As $\Lambda_3 = \Lambda_3(t)$, we see that $D(1, \Lambda_3) = 0$, while (4.3) ensures that $D(0, \Lambda_3) = 0$. Recalling Lemma 4.1, we see that $D(k, \Lambda_3) \neq 0$ for all $k \geq 2$. Indeed, for $k \geq 2$, $\Lambda_3(kt) > \Lambda_3$, and if $\Lambda_3 = \Lambda_2(kt)$, then $\Lambda_3 = \Lambda_2(kt) > \Lambda_2(t) > \Lambda_3$, a contradiction. Hence, $\partial_{(f,h)} \Phi_1(\Lambda_3,0)$ is a Fredholm operator with a one-dimensional kernel

$$\text{Ker} \partial_{(f,h)} \Phi_1(\Lambda_3,0) = \text{span}\{(m_1^{22}(\Lambda_3), -m_1^{21}(\Lambda_3)) \cos(2\pi x/L)\}.$$ 

Similarly as before we have

$$\text{Im} \partial_{(f,h)} \Phi(\Lambda_3,0) = \{(\xi, \eta) = \left(\sum_{k \in \mathbb{N}} \xi_k \cos(R_kx), \sum_{k \in \mathbb{N}} \eta_k \cos(R_kx)\right) : \gamma_1 = \frac{m_1^{11}(\Lambda_3)}{m_1^{21}(\Lambda_3)} \eta_k\},$$

and one can verify that the transversality condition

$$\partial_{(f,h)} \Phi(\Lambda_3,0) \left[(m_1^{22}(\Lambda_3), -m_1^{21}(\Lambda_3)) \cos(2\pi x/L)\right] \notin \text{Im} \partial_{(f,h)} \Phi(\Lambda_3,0)$$

is also satisfied. We are thus in the position of applying Theorem 2.3. To finish the proof, we infer from (2.15)-(2.16) and (4.4), that

$$- \frac{m_1^{21}(\Lambda_3)}{m_1^{22}(\Lambda_3)} \to L \to 0 \infty.$$

□

When considering bifurcation from $\Lambda_2$ the situation is more complicated because the derivative $\partial_{(f,h)} \Phi_1(\Lambda_2,0)$ may possess a two-dimensional kernel if $\Lambda_3(2\pi k/L) = \Lambda_2(2\pi/L)$ for some $L \geq L_0$ and some integer $k \geq 2$. When this happens, the integer $k$ is unique, cf. Lemma 4.1, so that we can conclude the existence of a curve of bifurcating solutions from Theorem 4.2. When $\Lambda_3(2\pi k/L) \neq \Lambda_2(2\pi/L)$ for all $k \geq 2$, we can apply again Theorem 2.3.

**Theorem 4.3 (Bifurcation from $\Lambda_2$).** Let $\gamma_2 = 0$, $\gamma_1 < 0$, and let $\alpha \in (0,1)$. Furthermore, let $L_0 := 2\pi/t_0$ and $L \leq L_0$.

(i) Assume that $\Lambda_3(2\pi k/L) \neq \Lambda_2(2\pi/L)$ for all integers $k \geq 2$. Then, there exists a real-analytic curve $(\Lambda, (f, h)) : (-\varepsilon, \varepsilon) \to (0, \infty) \times \mathcal{O}$ consisting only of solutions of the problem (2.11). This curve contains exactly one trivial solution of (2.11), and for $s \to 0$ we have that

$$\Lambda(s) = \Lambda_2 + O(s), \quad (f, h)(s) = s(m_1^{22}(\Lambda_2), -m_1^{21}(\Lambda_2)) \cos(2\pi x/L) + O(s^2),$$

whereby $\Lambda_2 := \Lambda_2(2\pi/L)$.

(ii) Assume that $\Lambda_3(2\pi k/L) = \Lambda_2$ for some integer $k \geq 2$. Then the assertion of Theorem 4.2 holds true, but with $L$ replaced by $L/k$.

The flow determined by $(\Lambda(s), (f, h)(s))$, $s \in (-\varepsilon, \varepsilon)$, contains a critical layer consisting of closed streamlines in the layer adjacent to the bed just beneath the internal wave. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 3.

**Proof.** Setting $t := 2\pi/L \geq t_0$, we know from (4.3) and Lemma 2.5 that $\partial_{(f,h)} \Phi_1(\Lambda_2,0)$ is a Fredholm operator. To determine its kernel we need to solve $D(k, \Lambda_2) = \Phi(kt, \Lambda_2) = 0$. A solution of this equation is $k = 1$ as $\Lambda_2 = \Lambda_2(t)$. Equation (4.3) ensures additionally that $D(0, \Lambda_2) = 0$. Because $\Lambda_3$ is increasing to zero, there may exist a (unique) integer $k \geq 2$ such that $\Lambda_3(2\pi k/L) = \Lambda_2(2\pi/L)$, hence $D(k, \Lambda_2) = 0$. In this case we are in the situation (ii) and the proof is obvious. If $\Lambda_3(2\pi k/L) \neq \Lambda_2(2\pi/L)$ for all integers $k \geq 2$, then we are in the case (i) and the proof is similar to that of Theorem 4.2. □
Remark 4.4. Since the properties of the functions $\Lambda_i(t)$, for $i = 1, 2, 3$ were essential in finding the branches of solutions to the water wave problem, we summarize them in the Table 1 below.

| $\gamma_2$ | $\gamma_1 > 0$ | $\gamma_1 = 0$ | $\gamma_1 < 0$ |
|----------|----------------|----------------|----------------|
| $\gamma_2 > 0$ | $\Lambda_3 < 0 < \Lambda_2 < \Lambda_1$ | $\Lambda_1 \to \gamma_2 d_2$, $\Lambda_1 \to 0$, $i \in \{2, 3\}$ | |
| $\gamma_2 = 0$ | $\Lambda_3 < 0 < \Lambda_2 < \Lambda_1 < \gamma_1 d_1$ | $\gamma_1 d_1 < \Lambda_3 < \Lambda_2 < 0 < \Lambda_1$ | $\Lambda_i \to 0$, $i \in \{1, 2, 3\}$ |
| $\gamma_2 < 0$ | $\Lambda_3 < \Lambda_2 < 0 < \Lambda_1$ | $\Lambda_3 \to \gamma_2 d_2$, $\Lambda_1 \to 0$, $i \in \{1, 2\}$ | |

Table 1. Properties of the roots $\Lambda_i, i \in \{1, 2, 3\}$, of the dispersion relation (2.18) in dependence of the vorticity constants $\gamma_i, i \in \{1, 2\}$ for $\gamma_1 \neq \gamma_2$ and for large $R_k = 2\pi k/L$. Our analysis is dedicated to the cases: (i) $\gamma_2 > 0$ and $\gamma_1 \neq \gamma_2$; and (ii) $\gamma_2 = 0$ and $\gamma_1 < 0$. The analysis in the other two cases: (iii) $\gamma_2 < 0$ and $\gamma_1 \neq \gamma_2$; and (iv) $\gamma_2 = 0$ and $\gamma_1 > 0$ is similar to that for (i) and (ii), respectively (see Remark 2.6).

**Appendix A.**

We present herein the proof of Lemma 2.4 and additionally we rigorously prove that the streamline pattern for the solutions that we found is as shown in Figures 1-3, respectively. To this end, we determine first explicit expressions for the elliptic and boundary operators introduced right before Observation 2.1. Given $(f, h) \in \mathcal{O}$, it is easy to see that

\[
A(f) = \partial_{xx} - 2 \frac{d + y}{d_1 + f} f' \partial_{xy} + \frac{(d + y) f'' + d_1^2}{(d_1 + f)^2} \partial_{yy} - (d + y) \frac{(d_1 + f) f'' - 2f'^2}{(d_1 + f)^2} \partial_v, \quad (A.1)
\]

\[
A(f, h) = A(f) - 2 \frac{d_2 h' + (h - f)' y}{h - f + d_2} \partial_{xy} + \frac{d_2 h' + (h - f)' y^2 + d_2^2}{(h - f + d_2)^2} \partial_{yy} - \frac{d_2 h'' + (h - f)' y + (h' - f')^2 y}{(h - f + d_2)^2} \partial_y, \quad (A.2)
\]

respectively, given $(w_1, w_2) \in C^{3+\alpha}_{\text{e,per}}(\Omega_1) \times C^{3+\alpha}_{\text{e,per}}(\Omega_2)$ and $\Lambda \in \mathbb{R}$, we have that

\[
B_1(\Lambda, (f, h), w_2) = \left. \left[ \partial_x w_2^2 - 2 \frac{d_2 h'}{h - f + d_2} \partial_x w_2 \partial_y w_2 + \frac{d_2^2 (1 + h^2)}{(h - f + d_2)^2} \partial_y w_2^2 \right] \right|_{y=0} = 2g(d + h) - Q(\Lambda), \quad (A.3)
\]

\[
B_2(\Lambda, (f, h))[w_1, w_2] = \left. \frac{d_2}{h - f + d_2} \partial_y w_2 \right|_{y=-d_2} - \left. \frac{d_1}{d_1 + f} \partial_y w_1 \right|_{y=-d_2}. \quad (A.4)
\]

**Proof of Lemma 2.4.** Since

\[
\partial_{(f,h)} \Phi(\Lambda, 0)((f, h)) = \begin{pmatrix} \partial_f \Phi_1(\Lambda, 0)[f] & \partial_h \Phi_1(\Lambda, 0)[h] \\ \partial_f \Phi_2(\Lambda, 0)[f] & \partial_h \Phi_2(\Lambda, 0)[h] \end{pmatrix}
\]
we only need to determine the entries in the matrix \( \partial_{f,h} \Phi(\Lambda,0) \).

The derivative \( \partial_f \Phi_1(\Lambda,0) \) Using the definition of \( \Phi_1 \), we see that

\[
\partial_f \Phi_1(\Lambda,0)[f] = 2 \left[ \frac{f}{d_2} |\partial_y \psi_2^0|^2 + \partial_y \psi_2^0 \partial_y (\partial_f w_2(\Lambda,0)[f]) \right] \bigg|_{y=0},
\]

whereby \( \partial_y \psi_2^0 \big|_{y=0} = \Lambda \) and \( z := \partial_f w_2(\Lambda,0)[f] \) is, in view of (2.10), the solution of the Dirichlet problem

\[
\begin{align*}
\Delta z &= -\partial_f A(0,0)[f] \psi_2^0 \quad \text{in } \Omega_2, \\
\partial_n z &= 0 \quad \text{on } \partial \Omega_2.
\end{align*}
\]  

(A.6)

A routine calculation shows now that

\[
\partial_f A(0,0)[f] \psi_2^0 = \frac{2 \gamma_2 f}{d_2} + \left( \frac{\gamma_2}{d_2} y^2 + \frac{\Lambda}{d_2} y \right) f''.
\]

Expanding \( f \) and \( z(y), y \in [-d_2,0] \), by their Fourier series, we have

\[
f = \sum_{k \in \mathbb{N}} a_k \cos(R_k x) \quad \text{and} \quad z(y) = \sum_{k \in \mathbb{N}} a_k A_k(y) \cos(R_k x).
\]

The coefficients \( A_k \) solve, in view of (A.6), the following boundary value problem

\[
\begin{align*}
A''_k - R_k^2 A_k &= -\frac{2 \gamma_2}{d_2} + R_k^2 \left( \frac{\gamma_2}{d_2} y^2 + \frac{\Lambda}{d_2} y \right) \quad \text{in } (-d_2,0), \\
A_k(-d_2) &= A_k(0) = 0,
\end{align*}
\]

and therefore

\[
A_k(y) = (\Lambda - \gamma_2 d_2) \frac{\sinh(R_k y)}{\sinh(R_k d_2)} - \left( \frac{\gamma_2}{d_2} y^2 + \frac{\Lambda}{d_2} y \right).
\]

Using the relation (A.5) we obtain now that

\[
\partial_f \Phi_1(\Lambda,0)[f] = \sum_{k \in \mathbb{N}} m_{k1}^1 a_k \cos(R_k x),
\]

whereby \( \{m_{k1}^1\}_{k \in \mathbb{N}} \) is defined by (2.13).

The derivative \( \partial_h \Phi_1(\Lambda,0) \) We have that

\[
\partial_h \Phi_1(\Lambda,0)[h] = 2 \left[ -\frac{h}{d_2} |\partial_y \psi_2^0|^2 + \partial_y \psi_2^0 \partial_y (\partial_h w_2(\Lambda,0)[h]) \right] \bigg|_{y=0} + 2 gh,
\]

with \( z := \partial_h w_2(\Lambda,0)[h] \) solving the Dirichlet problem

\[
\begin{align*}
\Delta z &= -\partial_h A(0,0)[h] \psi_2^0 \quad \text{in } \Omega_2, \\
\partial_n z &= 0 \quad \text{on } \partial \Omega_2,
\end{align*}
\]

cf. (2.10). Recalling (A.2), we compute that

\[
\partial_h A(0,0)[h] \psi_2^0 = -\frac{2 \gamma_2}{d_2} h - \left( \frac{\gamma_2}{d_2} y^2 + \frac{2 \gamma_2 d_2 + \Lambda}{d_2} y + \Lambda \right) h''.
\]

Using Fourier expansions as before, that is

\[
h = \sum_{k \in \mathbb{N}} b_k \cos(R_k x) \quad \text{and} \quad z(y) = \sum_{k \in \mathbb{N}} b_k B_k(y) \cos(R_k x) \quad \text{for } y \in [-d_2,0],
\]

we obtain that the coefficients \( B_k \) satisfy

\[
\begin{align*}
B''_k - R_k^2 B_k &= -\frac{2 \gamma_2}{d_2} - R_k^2 \left( \frac{\gamma_2}{d_2} y^2 + \frac{2 \gamma_2 d_2 + \Lambda}{d_2} y + \Lambda \right) \quad \text{in } (-d_2,0), \\
B_k(-d_2) &= B_k(0) = 0.
\end{align*}
\]
The solution of this boundary value problem is

\[ B_k(y) = -\Lambda \left( \frac{\sinh (R_k y)}{\tanh (R_k d_2)} + \cosh (R_k y) \right) + \frac{\gamma_2}{d_2} y^2 + \frac{\gamma_2 d_2 + \Lambda}{d_2} y + \Lambda, \]

and the desired representation for the derivative \( \partial \Phi_1(\Lambda, 0) \) follows at once.

The derivative \( \partial \Phi_2(\Lambda, 0) \) From the definition of \( \Phi_2 \) we obtain that

\[ \partial \Phi_2(\Lambda, 0)[f] = \left[ \frac{f}{d_2} \partial_y \psi_2^0 + \frac{f}{d_1} \partial_y \psi_1^0 + \partial_y (\partial_f w_2(\Lambda, 0)[f] - \partial_f w_1(\Lambda, 0)[f]) \right] \big|_{y=-d_2}, \]

whereby, in the equality above, \( z := \partial_f w_1(\Lambda, 0)[f] \) solves the Dirichlet problem

\[ \begin{aligned} \Delta z &= -\partial_f A(0)[f] \psi_1^0 \quad \text{in } \Omega_1, \\ z &= 0 \quad \text{on } \partial \Omega_1. \end{aligned} \]

In view of (A.1), we compute that

\[ \partial_f A(0)[f] \psi_1^0 = -\frac{2\gamma_1 f}{d_1} - \left[ \frac{\gamma_1}{d_1} y^2 + \left( \frac{d + d_2}{d_1} \right) \gamma_1 + \left( \frac{d - \gamma_2 d_2}{d_1} \right) + \frac{d \Lambda}{d_1} \right] f''. \]

Expanding \( f \) and \( z(y), y \in [-d, -d_2] \), by their Fourier series

\[ f = \sum_{k \in \mathbb{N}} a_k \cos (R_k x) \quad \text{and} \quad z(y) = \sum_{k \in \mathbb{N}} a_k C_k(y) \cos (R_k x), \]

we find that the coefficient \( C_k \) is the solution of

\[ \begin{cases} \frac{C''_k - R_k^2 C_k}{d_1} = \frac{2\gamma_1}{d_1} y^2 + \left( \frac{d + d_2}{d_1} \right) \gamma_1 + \left( \frac{d - \gamma_2 d_2}{d_1} \right) + \frac{d \Lambda}{d_1} + \frac{d d_2 (\gamma_1 - \gamma_2)}{d_1} \quad \text{in } (-d, -d_2), \\ C_k(-d) = C_k(-d_2) = 0, \end{cases} \]

whence

\[ C_k(y) = \frac{\Lambda \sinh ((d + y) R_k)}{\sinh (R_k d_1)} + \frac{\gamma_1}{d_1} y^2 + \left( \frac{d + d_2}{d_1} \right) \gamma_1 + \left( \frac{d - \gamma_2 d_2}{d_1} \right) + \frac{d \Lambda}{d_1} + \frac{d d_2 (\gamma_1 - \gamma_2)}{d_1}. \]

The representation of \( \partial \Phi_2(\Lambda, 0) \) as a Fourier multiplier follows now easily.

The derivative \( \partial \Phi_2(\Lambda, 0) \) Observing that

\[ \partial \Phi_2(\Lambda, 0)[h] = \frac{\gamma_2 d_2 - \Lambda}{d_2} h + \partial_y (\partial_f w_2(\Lambda, 0)[h]) \big|_{y=-d_2}, \]

the desired representation for \( \partial \Phi_2(\Lambda, 0) \) follows by using the expression for \( \partial_f w_2(\Lambda, 0)[h] \) determined in the second part of this proof.

In the remaining part we establish the Lemmas A.1-A.6 that provide the justification for the streamlines pattern, as seen from a reference frame moving with the wave, as shown in Figures 1-3. Because the proofs of Lemmas A.1-A.6 use similar arguments, we present herein only the proof for Lemma A.1. For this, it is important to note that because there is no time dependence in problem (2.1) (or (2.2)), the particle trajectories and the streamlines corresponding to the solutions found in Theorems 3.4-3.6, 4.2, 4.3 coincide with the level curves of the corresponding stream function. They are parametrized by solutions of the system of ordinary differential equations

\[ \begin{cases} x' = u - c = \psi_y, \\ y' = v = -\psi_x, \end{cases} \quad (A.7) \]

stagnation points of the flows corresponding to equilibria of (A.7). Hence, our task is to determine the level curves of the stream function. The direction of motion of the particles along the level curves is determined by the sign of \( u - c \) or \( v \).
Lemma A.1. Assume that $\gamma_1 > \gamma_2 > 0$ and let

$$(f,h),\psi_1,\psi_2) \in \left(C_{\text{per}}^{\alpha}(\mathbb{R})\right)^2 \times C_{\text{per}}^{\alpha}\left(\Omega(f)\right) \times C_{\text{per}}^{\alpha}\left(\Omega(f,h)\right)$$

be a solution of (2.2) that is determined by a point $(\Lambda(s), (f,h)(s))$ on the bifurcation curve found in Theorem 3.4. Provided that $s$ is small enough, the following assertions are true:

(i) $f' > 0$ and $h' > 0$ on $(0,L/2)$;
(ii) $\partial_x \psi_2 < 0$ in $\{x, y\} \in \Omega(f,h) : x \in (0,L/2)$ and $\partial_y \psi_2 > 0$ in $\Omega(f,h)$;
(iii) $\partial_x \psi_1 < 0$ in $\{x, y\} \in \Omega(f) : x \in (0,L/2)$;
(iv) There is a smooth curve $\{(x, \xi(x)) : x \in [0,L/2]\}$ with $-d < \xi(x) < -d + f(x)$ for all $x \in [0,L/2]$ and satisfying additionally:

(a) Given $x \in [0,L/2]$, it holds that: $\partial_y \psi_1(x,\xi(x)) = 0$, $\partial_y \psi_1(x, y) < 0$ for all $y \in [-d, \xi(x)]$, and $\partial_y \psi_1(x, y) > 0$ for all $y \in (\xi(x), -d + f(x))$;
(b) $\xi$ is strictly decreasing on $(0,L/2)$;
(c) The function $[x \mapsto \psi_1(x,\xi(x))]$ is strictly decreasing on $(0,L/2)$.

Proof. Since $\Lambda(0) = \Lambda_1 \in (\gamma_2d_2, \gamma_1d_1 + \gamma_2d_2)$, for small $s$ it holds $\Lambda(s) \in (\gamma_2d_2, \gamma_1d_1 + \gamma_2d_2)$. Recalling that

$$f(s) = s m_{12}^2(\Lambda_1) \left(\frac{2\pi}{L} x\right) + O(s^2), \quad h(s) = -s m_{11}^2(\Lambda_1) \left(\frac{2\pi}{L} x\right) + O(s^2), \quad (A.8)$$

with $m_{12}^2(\Lambda_1) < 0$ and $m_{11}^2(\Lambda_1) > 0$, the claim (i) follows by using the same arguments as in the proof of [35, Lemma 4.2].

For (ii), we see first that $\partial_y \psi_2^0 = \gamma_2y + \Lambda_1 > \gamma_2y + \gamma_2d_2 > 0$ in $\Omega(f,h)$. Therefore, $\partial_y \psi_2 > 0$ in $\Omega(f,h)$ provided that $s$ is small. Using now (i) and the fact that $\psi$ is constant on $\partial\Omega(f,h)$ and even with respect to $x$, it is easy to see that $\partial_x \psi_2 \leq 0$ on the boundary of the set $\{(x, y) \in \Omega(f,h) : x \in (0,L/2)\}$. Observing that $\partial_x \psi_2(x, h(x)) < 0$ for all $x \in (0,L/2)$ and $\Delta \psi_x = 0$ in $\Omega(f,h)$, elliptic maximum principles ensure that $\partial_x \psi_2 < 0$ in $\{(x, y) \in \Omega(f,h) : x \in (0,L/2)\}$. The claim (iii) is obtained in a similar manner.

For (iv), we remark that

$$\partial_y \psi_1^0 |_{y = -d} < 0, \quad \partial_y \psi_1^0 |_{y = -d + f(x)} > 0, \quad \partial_y \psi_1^0 > 0 \text{ in } \Omega_1.$$  

Therewith, for small $s$ the function $\psi_1$ satisfies the similar inequalities

$$\partial_y \psi_1 |_{y = -d} < 0, \quad \partial_y \psi_1 |_{y = -d + f(x)} > 0, \quad \partial_y \psi_1 > 0 \text{ in } \Omega(f). \quad (A.9)$$

Hence, for each $x \in [0,L/2]$, there exists a unique $\xi(x) \in (-d, -d + f(x))$ such that $\partial_y \psi_1(x, \xi(x)) = 0$. Due to the third inequality in (A.9) we conclude from the implicit function theorem that $\xi$ is smooth and

$$\partial_x \psi_1(x, \xi(x)) + \xi'(x) \partial_{yy} \psi_1(x, \xi(x)) = 0 \quad \text{for all } x \in [0,L/2]. \quad (A.10)$$

We are going to determine now the sign of $\partial_x \psi_1$. To this end note that $\psi_1 = w_1 \circ \Phi_f^{-1}$ where $w_1 \in C_{\text{per}}^{\alpha}(\Omega_1)$ is the unique solution of the problem (2.9), that is $w_1 := w_1(\Lambda(s), (f,h)(s))$. By the chain rule we get

$$\partial_x \psi_1 = -\frac{d_1 f'}{(d_1 + f)^2} \partial_y w_1 \circ \Phi_f^{-1} + \frac{d_1}{d_1 + f} \partial_{xy} w_1 \circ \Phi_f^{-1} - \frac{d_2 f' y}{(d_1 + f)^3} \partial_{yy} w_1 \circ \Phi_f^{-1}$$

$$+ \frac{dd_1 f'}{(d_1 + f)^3} \partial_{yy} w_1 \circ \Phi_f^{-1} - \frac{dd_1 f'}{(d_1 + f)^2} \partial_{yy} w_1 \circ \Phi_f^{-1}.$$  

On the other hand we have the following expansion

$$w_1(\Lambda(s), (f,h)(s)) = w_1(\Lambda_1, 0) + \partial_1 w_1(\Lambda_1, 0)[\Lambda(s) - \Lambda_1] + \partial_2 w_1(\Lambda_1, 0)[f(s)] + O(s^2)$$
in $C^{3+\alpha}_{e,\text{per}}(\Omega_1)$. Observing that
\[
\begin{align*}
\partial_y w_1 \circ \Phi_f^{-1} &= \partial_y \psi_1^0 + O(s), \\
\partial_{xy} w_1 \circ \Phi_f^{-1} &= \partial_{xy} (\partial_f w_1(1,0)[f]) + O(s^2), \\
\partial_{yy} w_1 \circ \Phi_f^{-1} &= \gamma_1 + O(s),
\end{align*}
\]
in $C^{2+\alpha}_{e,\text{per}}(\Omega(f))$, a lengthy calculation leads us to
\[
\partial_{xy} \psi_1 = -sm_2 f \left( \psi_1 \right) \frac{\cosh(L_1(d + y))}{\sinh(L_1d_1)} \sin (L_1x) + O(s^2) \quad \text{in } C^{1+\alpha}_{e,\text{per}}(\Omega(f)).
\]
A similar argument to the one used in (i) shows that $\partial_{xy} \psi_1 > 0$ in $\Omega(f)$ if $s > 0$ is sufficiently small. The latter property together with (A.9) and (A.10) implies that $\xi^t < 0$ in $x \in (0, L/2)$. This proves the claim in (b). Since (c) is an obvious consequence of (iii) we have completed the proof.

$$\square$$

It follows now readily from Theorem 3.4 and Lemma A.1 that the streamline pattern in the moving frame for the non laminar solutions found in Theorem 3.4 for $\gamma_1 > \gamma_2$ is as in Figure 1 (left picture). The next lemma justifies the right picture of Figure 1.

**Lemma A.2.** Assume that $\gamma_1 < \gamma_2, \gamma_2 > 0$ and let

\[
((f,h),\psi_1,\psi_2) \in (C^{3+\alpha}_{\text{per}}(\mathbb{R}))^2 \times C^{3+\alpha}_{\text{per}}(\Omega(f)) \times C^{3+\alpha}_{\text{per}}(\Omega(f,h))
\]

be a solution of (2.2) that is determined by a point $(\Lambda(s),(f,h)(s))$ on the bifurcation curve found in Theorem 3.4. Provided that $s$ is small enough, the following assertions are true:

(i) $f' > 0$ and $h' < 0$ on $(0, L/2)$;
(ii) $\partial_x \psi_1 > 0$ in $\{(x,y) \in \Omega(f) : x \in (0, L/2)\}$ and $\partial_y \psi_1 < 0$ in $\Omega(f)$;
(iii) $\partial_x \psi_2 > 0$ in $\{(x,y) \in \Omega(f,h) : x \in (0, L/2)\}$;
(iv) There is a smooth curve $\{(x,\xi(x)) : x \in [0, L/2]\}$ with $-d_2 + f(x) < \xi(x) < h(x)$ for all $x \in [0, L/2]$ and satisfying additionally:

(a) Given $x \in [0, L/2]$, it holds that: $\partial_y \psi_2(x,\xi(x)) = 0$, $\partial_y \psi_2(x,y) < 0$ for all $y \in [-d_2 + f(x),\xi(x))$, and $\partial_y \psi_2(x,y) > 0$ for all $y \in (\xi(x),h(x))$;
(b) $\xi$ is strictly decreasing on $[0, L/2]$;
(c) The function $[x \mapsto \psi_2(x,\xi(x))]$ is strictly increasing on $[0, L/2]$.

The next lemma provides a justification for the left picture of Figure 2.

**Lemma A.3.** Assume that $\gamma_2 > 0, \gamma_1 d_1 + \gamma_2 d_2 > 0$ and let

\[
((f,h),\psi_1,\psi_2) \in (C^{3+\alpha}_{\text{per}}(\mathbb{R}))^2 \times C^{3+\alpha}_{\text{per}}(\Omega(f)) \times C^{3+\alpha}_{\text{per}}(\Omega(f,h))
\]

be a solution of (2.2) that is determined by a point $(\Lambda(s),(f,h)(s))$ on the bifurcation curve found in Theorem 3.5. Provided that $s$ is small enough, the following assertions are true:

(i) $f' > 0$ and $h' < 0$ on $(0, L/2)$;
(ii) $\partial_x \psi_1 > 0$ in $\{(x,y) \in \Omega(f) : x \in (0, L/2)\}$ and $\partial_y \psi_1 < 0$ in $\Omega(f)$;
(iii) $\partial_x \psi_2 > 0$ in $\{(x,y) \in \Omega(f,h) : x \in (0, L/2)\}$;
(iv) There is a smooth curve $\{(x,\xi(x)) : x \in [0, L/2]\}$ with $-d_2 + f(x) < \xi(x) < h(x)$ for all $x \in [0, L/2]$ and satisfying additionally:

(a) Given $x \in [0, L/2]$, it holds that: $\partial_y \psi_2(x,\xi(x)) = 0$, $\partial_y \psi_2(x,y) < 0$ for all $y \in [-d_2 + f(x),\xi(x))$, and $\partial_y \psi_2(x,y) > 0$ for all $y \in (\xi(x),h(x))$;
(b) $\xi$ is strictly decreasing on $[0, L/2]$;
(c) The function $[x \mapsto \psi_2(x,\xi(x))]$ is strictly increasing on $[0, L/2]$.

We provide now a justification for the right picture of Figure 2.
Lemma A.4. Assume that $\gamma_2 > 0, \gamma_1 d_1 + \gamma_2 d_2 \leq 0$ and let

$$(f, h, \psi_1, \psi_2) \in \left( C_{\text{per}}^{3+\alpha}(\mathbb{R}) \right)^2 \times C_{\text{per}}^{3+\alpha}(\Omega(f)) \times C_{\text{per}}^{3+\alpha}(\Omega(f, h))$$

be a solution of (2.2) that is determined by a point $(\Lambda(s), (f, h)(s))$ on the bifurcation curve found in Theorem 3.5. Provided that $s$ is small enough, the following assertions are true:

(i) $f' > 0$ and $h' < 0$ on $(0, L/2)$;
(ii) $\partial_x \psi_1 > 0$ in $(x, y) \in \Omega(f) : x \in (0, L/2)$;
(iii) $\partial_x \psi_2 > 0$ in $(x, y) \in \Omega(f, h) : x \in (0, L/2)$;
(iv) There is a smooth curve $\{(x, \xi(x)) : x \in [0, L/2]\}$ with $-d < \xi(x) < -d_2 + f(x)$ for all $x \in [0, L/2]$ and satisfying additionally:

(a) Given $x \in [0, L/2]$, it holds that: $\partial_y \psi_1(x, \xi(x)) = 0$, $\partial_y \psi_1(x, y) > 0$ for all $y \in [\xi(x), -d_2 + f(x)]$; $\partial_y \psi_1(x, y) < 0$ for all $y \in (\xi(x), -d_2 + f(x)]$;
(b) $\xi(x)$ is strictly increasing on $[0, L/2]$;
(c) The function $[x \mapsto \psi_1(x, \xi(x))]$ is strictly increasing on $[0, L/2]$.

We consider now the non laminar flows corresponding to the bifurcation solutions found in Theorem 3.6 and prove the following result which justifies the left picture of Figure 3.

Lemma A.5. Assume that $\gamma_2 > 0, \gamma_1 d_1 + \gamma_2 d_2 < 0$ and let

$$(f, h, \psi_1, \psi_2) \in \left( C_{\text{per}}^{3+\alpha}(\mathbb{R}) \right)^2 \times C_{\text{per}}^{3+\alpha}(\Omega(f)) \times C_{\text{per}}^{3+\alpha}(\Omega(f, h))$$

be a solution of (2.2) that is determined by a point $(\Lambda(s), (f, h)(s))$ on the bifurcation curve found in Theorem 3.6. Provided that $s$ is small enough, the following assertions are true:

(i) $f' < 0$ and $h' < 0$ on $(0, L/2)$;
(ii) $\partial_x \psi_2 < 0$ in $(x, y) \in \Omega(f, h) : x \in (0, L/2)$ and $\partial_y \psi_2 < 0$ in $\Omega(f, h)$;
(iii) $\partial_x \psi_1 > 0$ in $(x, y) \in \Omega(f) : x \in (0, L/2)$;
(iv) There is a smooth curve $\{(x, \xi(x)) : x \in [0, L/2]\}$ with $-d < \xi(x) < -d_2 + f(x)$ for all $x \in [0, L/2]$ and satisfying additionally:

(a) Given $x \in [0, L/2]$, it holds that: $\partial_y \psi_1(x, \xi(x)) = 0$, $\partial_y \psi_1(x, y) > 0$ for all $y \in [\xi(x), -d_2 + f(x)]$; $\partial_y \psi_1(x, y) < 0$ for all $y \in (\xi(x), -d_2 + f(x)]$;
(b) $\xi(x)$ is strictly increasing on $[0, L/2]$;
(c) The function $[x \mapsto \psi_1(x, \xi(x))]$ is strictly decreasing on $[0, L/2]$.

Finally, we have the following result which justifies the right picture of Figure 3.

Lemma A.6. Assume that $\gamma_2 = 0, \gamma_1 < 0$, and let

$$(f, h, \psi_1, \psi_2) \in \left( C_{\text{per}}^{3+\alpha}(\mathbb{R}) \right)^2 \times C_{\text{per}}^{3+\alpha}(\Omega(f)) \times C_{\text{per}}^{3+\alpha}(\Omega(f, h))$$

be a solution of (2.2) that is determined by a point $(\Lambda(s), (f, h)(s))$ on one of the bifurcation curves found in Theorems 4.2-4.3. Then, the assertions from Lemma A.5 hold verbatim.

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