On the Convergence of the ELBO to Entropy Sums

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Abstract
The variational lower bound (a.k.a. ELBO or free energy) is the central objective for many established as well as for many novel algorithms for unsupervised learning. Such algorithms usually increase the bound until parameters have converged to values close to a stationary point of the learning dynamics. Here we show that (for a very large class of generative models) the variational lower bound is at all stationary points of learning equal to a sum of entropies. Concretely, for standard generative models with one set of latents and one set of observed variables, the sum consists of three entropies: (A) the (average) entropy of the variational distributions, (B) the negative entropy of the model’s prior distribution, and (C) the (expected) negative entropy of the observable distribution. The obtained result applies under realistic conditions including: finite numbers of data points, at any stationary point (including saddle points) and for any family of (well behaved) variational distributions. The class of generative models for which we show the equality to entropy sums contains many standard as well as novel generative models including standard (Gaussian) variational autoencoders. The prerequisites we use to show equality to entropy sums are relatively mild. Concretely, the distributions defining a given generative model have to be of the exponential family, and the model has to satisfy a parameterization criterion (which is usually fulfilled). Proving equality of the ELBO to entropy sums at stationary points (under the stated conditions) is the main contribution of this work.

1 Introduction

We consider probabilistic generative models and their optimization based on the variational lower bound. The variational bound is also known as evidence lower bound (ELBO; e.g. [19]) or (variational) free energy [31]. For a given set of data points, a learning algorithm based on generative models usually changes the parameters of the model until no or only negligible parameter changes are observed. Learning is then stopped and parameters will usually lie very close to a stationary point of the learning dynamics, i.e., close to a point where the derivatives of the ELBO objective w.r.t. all optimized parameters are zero.

In this theoretical study, we investigate the ELBO objective at stationary points. As our main result, we show that for many generative models (including very common models), the ELBO becomes equal to a sum of entropies at stationary points. The result is very concise such that it
can be stated here initially. We will consider standard generative models of the form:

\[ \vec{z} \sim p(\vec{z}) \quad \text{and} \quad \vec{x} \sim p(\vec{x} | \vec{z}), \]

where \( \vec{z} \) and \( \vec{x} \) are latent and observed variables, respectively, and where \( \Theta \) denotes the set of model parameters. The distribution \( p(\vec{z}) \) will be denoted the prior distribution, and \( p(\vec{x} | \vec{z}) \) will be referred to as noise model or observable distribution. Given \( N \) data points \( \vec{x}^{(n)} \), the ELBO is given by (compare \[21, 31, 19, 22\]):

\[ F(\Phi, \Theta) = \frac{1}{N} \sum_n \int q^{(n)}(\vec{z}) \log(p(\vec{x}^{(n)} | \vec{z})) \, d\vec{z} - \frac{1}{N} \sum_n D_{KL}[q^{(n)}(\vec{z}) \| p(\vec{z})], \]

where \( q^{(n)}(\vec{z}) \) denote the variational distributions used for optimization.

The main result of this work states that under relatively mild conditions for the generative model in (1), the ELBO decomposes into a sum of entropies at all stationary points of learning. Concretely, we will show that at all stationary points (i.e., at all points of vanishing derivatives of \( F(\Phi, \Theta) \)) applies:

\[ F(\Phi, \Theta) = \frac{1}{N} \sum_n H[q^{(n)}(\vec{z})] - H[p(\vec{z})] - \frac{1}{N} \sum_n \mathbb{E}_{q^{(n)}}[H[p(\vec{x} | \vec{z})]]. \]

Notably, the result applies at any stationary point (including saddle points), for finitely many data points, and for essentially any variational distribution (very weak conditions on \( q^{(n)}(\vec{z}) \)). A reformulation of the ELBO as entropy sums can be useful from theoretical as well as practical perspectives. The entropy decomposition (3) can, for instance, be closed-form also in cases when the original ELBO objective is not (cf. \[11\]). The computation of variational bounds and/or likelihoods can also become easier for other reasons, e.g., for variational autoencoders with linear decoder (an approach closely related to PCA \[10, 24\]) the likelihood can at stationary points be computed from the model parameters alone without knowledge of the data (cf. \[11\]). A number of example applications of Eqn. 3 have been discussed for specific generative models in previous work \[11, 44\], and they range from efficient ELBO estimation and model selection to the analysis of optimization landscapes and posterior collapse in standard variational autoencoders.

Entropy sum reformulations of the ELBO can also be used for learning itself, which has recently been demonstrated for probabilistic sparse coding \[44\]. Learning based on entropy sums such as (3) is made possible because (for many generative models) it is sufficient that only derivatives w.r.t. a subset of the parameters \( \Theta \) vanish. Learning using entropy sums is then advantages for practical reasons, e.g., because (A) an exclusively ELBO-based objective suggests more principled forms of annealing \[44\], or (B) derivatives of entropies can take on especially concise forms. From a theoretical perspective, entropies have themselves deep roots in mathematical statistics and information theory, and their derivatives provide a direct connection to information geometry (e.g. \[13, 2\]).

**Own contributions.** In contrast to previous work on entropy sums (see \[11, 44\]), we will not focus on a specific model or on specific applications of the result of Eqn. 3. Instead, the focus here will be the derivation of Eqn. 3 itself for an as broad class of generative models as possible. More concretely, we aim at formalizing as general as possible conditions that have to apply for a
A generative model in order for its ELBO to converge to an entropy sum. For a given generative model, it will then be sufficient to verify a set of relatively mild and concise conditions that make its ELBO equal to a sum of entropies. One condition that we will require is that the distributions $p_{\Theta}(\tilde{z})$ and $p_{\Theta}(\tilde{x} | \tilde{z})$ are within the exponential family of distributions. Furthermore, we will require conditions for the parameterization of a model in particular for the link between latents and observables.

**Relation to other contributions.** Given an exponential family of distributions, the relation of its maximum likelihood parameter estimation to the (negative) entropy of the distribution family is well-known in mathematical statistics (e.g. [6, 14]). Some of these well-established results have been used for generative models whose joint distributions take the form of an exponential family of distributions (e.g. [15]). The decomposition of variational lower bounds (i.e., ELBOs) into sums of entropies has not been studied to a comparable extent. But at least for idealized conditions it is relatively straightforward to show that at stationary points the ELBO can be written as a sum of entropies (we will discuss such idealized conditions below). At the same time, there is a long-standing interest in stationary points of variational learning that are reached under realistic conditions. For instance, likelihoods, lower bounds, and their stationary points have been studied in detail for (restricted) Boltzmann machines (e.g. [47]), elementary generative models (e.g. [17, 42]) as well as for general graphical models [16, 46, 41, 35]. More recently, stationary points of deep generative models have been of considerable interest. For instance, recent work [24] linked phenomena like mode collapse in variational autoencoders (VAEs) to the stability of stationary points of ELBO-based learning; and other recent work [11] analyzed a series of tractable special cases of VAEs to better understand convergence points of VAE learning. Notably both contributions emphasised the close relation of standard VAEs to principal component analysis (PCA; [38, 42]). Theoretical results on different versions of PCA (e.g. [42, 48, 20], for probabilistic, sparse, and robust PCA) can consequently be of direct relevance for VAEs and vice versa. Furthermore, different theoretical guarantees, e.g. for VAEs, have been derived reconstruction errors [33], and, notably in our context, for convergence rates of VAEs (e.g., [10] and references therein).

Standard VAEs and standard generative models such as probabilistic PCA (p-PCA) or probabilistic sparse coding (e.g. [33]) all assume Gaussian distributed observables (and standard VAEs and p-PCA also assume Gaussian latents). Likewise, another previous contribution on VAEs [11] assumes Gaussians for all distributions. And all previous contributions [25, 11, 44] with results similar to Eqn. 3 derive them for models were at least the observables are Gaussian distributed. The relation of the ELBO to entropy sums in a not as recent contribution [25] was more general regarding the model distributions but required conditions that are unrealistic in practice. Concretely, equality to entropy sums was shown when the following idealized conditions apply: (A) when the limit of infinitely many data points is considered, (B) when the model distribution can exactly match the data distribution, (C) if the variational distributions match the posteriors, and (D) when a global optimum is reached. The authors (of [25]) also discussed relaxations of these unrealistic assumptions (their Sec. 6) but those relaxations relied on properties of specific models with Gaussian observables (p-PCA and sparse coding) when using expectation maximization (EM; [12]). The study by Damm et al. [11] reported a result requiring Gaussian distributions but allowed for very general dependencies of the observables on the latent variables. More con-
cretely, they focused on standard (Gaussian) VAEs, i.e., one specific (but very popular) class of generative models. For standard VAEs, their work shows equality to entropy sums also under realistic conditions (i.e., local optima, imperfect match of model assumptions and data, finite data sets etc.).

In contrast to all such previous work, we in this work do not assume any properties specific to any generative model, i.e., given a generative model, none of its distributions will be considered to have a specific form such as being Gaussian. Instead, we will be interested in how generally the result of Eqn. 3 applies.

We will first specify the required conditions for a generative model in Sec. 2. Based on the conditions, we will then prove two theorems that state the equality of the ELBO and entropy sums at stationary points. The theorems can be used to show, for a given concrete generative model, that its ELBO converges to an entropy sum. In a contribution applying the theorems proven in this work (see [26]), it can thus be shown that the ELBOs of many very well-known generative models converge to entropy sums including: sigmoid belief networks (SBNs, [32, 39]), probabilistic PCA ([38, 42]), factor analysis (e.g. [41]) and Gaussian mixture models (e.g. [30]). For standard (i.e., Gaussian) VAEs, convergence to entropy sums was shown previously ([27] (with the final version of that work [11] already making use of the results presented in this work)).

2 The Class of Considered Generative Models

We first define the class of generative models that we consider, i.e., the models and the properties that we require in order to show convergence to entropy sums. We will also discuss two concrete examples and a counter-example. We keep focusing on generative models of the kind introduced above (Eqn. 1). Such models will be sufficient to communicate the concepts the proof relies on but generalizations will be discussed in Sec. 5. First, we will require a more specific notation for the parameterization of the generative model, however.

**Definition A** (Generative Model). The generative models we consider have prior distribution \( p_{\Psi} \) with parameters \( \Psi \), i.e., all parameters are arranged into one column vector with scalar entries. Analogously, the noise distribution \( p_{\Theta} \) is parameterized by \( \Theta \) such that:

\[
\tilde{z} \sim p_{\Psi}(\tilde{z}),
\]

\[
\bar{x} \sim p_{\Theta}(\bar{x}|\tilde{z}).
\]

Arranging prior parameters and noise model parameters into column vectors will be important for further definitions and for the derivation of the main result. For our purposes, the definition of a generative model in terms of prior and noise model distributions will also be notationally more convenient than using the model’s joint probability.

In general, we will adopt a somewhat less abstract mathematical notation than could have been used. For instance, we will not use the notation of Lebesgue integrals if such a notation is not necessary. The motivation is that then the main derivations can be communicated using conventional integrals (which are more common in the broader Machine Learning literature). Knowledge, e.g., on Lebesgue integrals, Legendre transformations and duality principles are not essential for the main derivations. However, if we later (in Sec. 3) cover the most general form
of our main result, Lebesgue integrals and general measures will be used and will be required for our derivation.

Many standard generative models are of the form as described in Definition A. Examples are those named in the introduction. For our purposes, we will require that prior and noise distribution are of the exponential family of distributions, and we will (at first) require constant base measures of the exponential family distributions. For most standard generative models in Machine Learning of the form stated in Definition A, these conditions are satisfied. But there are exceptions that are not covered. Examples are, e.g., probabilistic sparse coding models with mixture of Gaussians prior [34] or student-t prior [5], i.e., models with a prior not in the exponential family. The subset of generative models defined with the required exponential family distributions will be referred to as exponential family generative models or EF generative models.

Definition B (EF Generative Models). Consider a generative model as given by Definition A. We say the model is an exponential family model (EF model) if there exist exponential family distributions \( p_{\vec{\zeta}}(\vec{z}) \) (for the latents) and \( p_{\vec{\eta}}(\vec{x}) \) (for the observables) such that the generative model can be reparameterized to take the following form:

\[
\vec{z} \sim p_{\vec{\zeta}}(\vec{\Psi}), \tag{6}
\]

\[
\vec{x} \sim p_{\vec{\eta}}(\vec{z}; \vec{\Theta}), \tag{7}
\]

where \( \vec{\zeta}(\vec{\Psi}) \) maps the prior parameters \( \vec{\Psi} \) to the natural parameters of distribution \( p_{\vec{\zeta}}(\vec{z}) \), and where \( \vec{\eta}(\vec{z}; \vec{\Theta}) \) maps \( \vec{z} \) and the noise model parameters \( \vec{\Theta} \) to the natural parameters of distribution \( p_{\vec{\eta}}(\vec{x}) \). Furthermore, we demand that the exponential family distributions \( p_{\vec{\zeta}}(\vec{z}) \) and \( p_{\vec{\eta}}(\vec{x}) \) have a constant base measure.

The natural parameters of the prior can in principle also depend on \( \vec{\Theta} \) because reparameterizations are conceivable that also involve part of the noise model parameters for the reparameterized prior. However, we will use the more intuitive but slightly more restricted definition above.

If a generative model defined by (6) and (7) is an EF generative model, it can be written as follows:

\[
p_{\vec{\zeta}}(\vec{\Psi})(\vec{z}) = h(\vec{z}) \exp\left(\vec{\zeta}^T(\vec{\Psi}) \vec{T}(\vec{z}) - A(\vec{\zeta}(\vec{\Psi}))\right), \tag{8}
\]

\[
p_{\vec{\eta}}(\vec{z}; \vec{\Theta})(\vec{x}) = h(\vec{x}) \exp\left(\vec{\eta}^T(\vec{z}; \vec{\Theta}) \vec{T}(\vec{x}) - A(\vec{\eta}(\vec{z}; \vec{\Theta}))\right), \tag{9}
\]

where we additionally know that \( h(\vec{x}) \) and \( h(\vec{z}) \) are constant. The base measures \( h(\cdot) \) as well as the log-partition functions \( A(\cdot) \) and sufficient statistics \( \vec{T}(\cdot) \) of latent and observable distributions are in general different, of course. To simplify notation, we do not use different symbols for these functions, however, because they will in the following be distinguishable from context (simply by their arguments).

For our derivation of the main result, it will not be sufficient for a generative model to be an EF generative model. An additional property for the mappings to natural parameters (which includes how the latents link to observables) is required. Before we define the property, we introduce further notations that we will require: For an EF generative model as given in Definition B with mappings to natural parameters given by \( \vec{\zeta}(\vec{\Psi}) \) and \( \vec{\eta}(\vec{z}; \vec{\Theta}) \) we define the here used notation for
Jacobian matrices. We denote by $\frac{\partial \vec{\zeta}}{\partial \vec{\Psi}^T}$ the Jacobian matrix of the mapping $\vec{\zeta}(\vec{\Psi})$:

$$\frac{\partial \vec{\zeta}}{\partial \vec{\Psi}^T} := \begin{pmatrix} \frac{\partial \vec{\zeta}}{\partial \Psi_1} ; \ldots ; \frac{\partial \vec{\zeta}}{\partial \Psi_R} \end{pmatrix},$$

(10)

where $R$ denotes the number of prior parameters $\vec{\Psi}$. Furthermore, we denote by $\frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \vec{\theta}^T}$ the Jacobian matrix of the mapping $\vec{\eta}(\vec{z};\vec{\Theta})$:

$$\frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \vec{\theta}^T} := \begin{pmatrix} \frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \theta_1} ; \ldots ; \frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \theta_S} \end{pmatrix},$$

(11)

for which we allow the Jacobian (11) to be constructed using just a subset $\vec{\theta}$ of the parameters $\vec{\Theta}$. The dimension of this subset is referred to as $S$.

We can now define the property we additionally require for EF generative models.

**Definition C (Parameterization Criterion).** Consider an EF generative model as given in Definition B with mappings to natural parameters given by $\vec{\zeta}(\vec{\Psi})$ and $\vec{\eta}(\vec{z};\vec{\Theta})$. We then say that the EF generative model fulfills the parameterization criterion if the following two properties hold:

(A) There exists a vectorial function $\vec{\alpha}(\cdot)$ from the set of parameters $\vec{\Psi}$ to $\mathbb{R}^R$ such that for all $\vec{\Psi}$ holds:

$$\vec{\zeta}(\vec{\Psi}) = \frac{\partial \vec{\zeta}(\vec{\Psi})}{\partial \vec{\Psi}^T} \vec{\alpha}(\vec{\Psi}).$$

(12)

(B) There exists an ($S$-dim) subset $\vec{\theta}$ of $\vec{\Theta}$ and a vectorial function $\vec{\beta}(\cdot)$ from the space of parameters of the observable distribution $\vec{\Theta}$ to $\mathbb{R}^S$ such that for all $\vec{z}$ and $\vec{\Theta}$ holds:

$$\vec{\eta}(\vec{z};\vec{\Theta}) = \frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \vec{\theta}^T} \vec{\beta}(\vec{\Theta}).$$

(13)

The subset $\vec{\theta}$ has to be non-empty ($S \geq 1$) in order for the Jacobian to be defined. Furthermore, notice that $\vec{\beta}(\vec{\Theta})$ only depends on the parameters $\vec{\Theta}$ and not on the latent variable $\vec{z}$.

Part A of the parameterization criterion is usually trivially fulfilled but is required for completeness. For instance, the usual mappings from standard parameters to natural parameters of standard exponential family distributions (Bernoulli, Gaussian, Beta, Gamma etc.) can be shown to fulfill part A of the criterion. Part B can be more intricate because of the additional dependence of the natural parameters $\vec{\eta}$ on the latent variable $\vec{z}$.

Below, let us first discuss some examples of generative models. The examples aim at illustrating the parameterization criterion.
Example 1 (Simple SBN). Let us consider a first example, which will be a simple form of a sigmoid belief net (SBN; [33]).

\[ z \sim p_\Psi(z) = \text{Bern}(z; \pi), \quad \text{with } 0 < \pi < 1, \quad (14) \]

\[ \bar{x} \sim p_\Theta(\bar{x} | z) = \text{Bern}(x_1; S(v z)) \text{ Bern}(x_2; S(w z)), \quad \text{where } S(a) = \frac{1}{1 + e^{-a}}, \quad (15) \]

and where \( \text{Bern}(z; \pi) = \pi z (1 - \pi) \) is the standard parameterization of the Bernoulli distribution (\( \pi \) is the probability of \( z \) being one, \( p(z = 1) = \pi \)). The same parameterization is used for the two Bernoulli distributions of the noise model (with the sigmoidal function setting the probability). The prior parameters \( \Psi \) just consist of the parameter \( \pi \in (0, 1) \). The noise model parameters consist of \( v \in \mathbb{R} \) and \( w \in \mathbb{R} \).

The generative model (14) and (15) is an EF generative model because it can be reparameterized as follows:

\[ z \sim p_\zeta(z), \quad \text{where } \zeta(\pi) = \log \left( \frac{\pi}{1 - \pi} \right), \quad (16) \]

\[ \bar{x} \sim p_{\eta}(\bar{x} | \bar{z}), \quad \text{where } \bar{\Theta} = \begin{pmatrix} v \\ w \end{pmatrix} \text{ and } \eta(z; \bar{\Theta}) = \begin{pmatrix} v z \\ w z \end{pmatrix}. \quad (17) \]

\( p_\zeta(z) \) is now the Bernoulli distribution in its exponential family form with natural parameters \( \zeta \). Likewise, \( p_{\eta}(\bar{x}) \) is the Bernoulli distribution (for two observables) with natural parameters \( \eta \). The standard sigmoidal function (15) is the reason for the function to natural parameters, \( \eta(z; \bar{\Theta}) \), to take on a particularly concise form.

For the generative model the parameterization criterion of Definition [34] is fulfilled: From the parameterization as EF model, it follows for part A that the Jacobian \( \frac{\partial \zeta(\pi)}{\partial \Psi} \) of (19) is the scalar \( \frac{1}{\pi (1 - \pi)} > 0 \). Therefore, Eqn. [12] is satisfied by \( \alpha(\pi) = \pi (1 - \pi) \zeta(\pi) \). For part B of the criterion, we use all parameters \( \Theta \) to construct the Jacobian, i.e. \( \bar{\Theta} = \bar{\Theta} = (v, w)^T \). The Jacobian is then a (2 x 2)-matrix given by

\[ \frac{\partial \eta(z; \bar{\Theta})}{\partial \bar{\Theta}^T} = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}. \quad (18) \]

Hence, the vectorial function \( \bar{\beta}(\bar{\Theta}) = \begin{pmatrix} v \\ w \end{pmatrix} \) fulfills Eqn. [13] which shows part B of the criterion.

Example 2 (Simple Factor Analysis). Let us consider as second example a simple form of factor analysis (FA; [15, 7]).

\[ z \sim p_\Psi(z) = N(z; 0, \tilde{\tau}), \quad \text{where } \tilde{\tau} > 0, \quad (19) \]

\[ \bar{x} \sim p_\Theta(\bar{x} | z) = N(\bar{x}; \tilde{w} z, \Sigma), \quad \text{where } \Sigma = \begin{pmatrix} \tilde{\sigma}_1 & 0 \\ 0 & \tilde{\sigma}_2 \end{pmatrix}, \tilde{\sigma}_1, \tilde{\sigma}_2 > 0, \quad (20) \]

and where \( \tilde{w} \in \mathbb{R}^2 \) with \( ||\tilde{w}|| = 1 \).

The parameter vectors are \( \Psi = \Psi = \tilde{\tau} \) and \( \Theta = \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ w_1 \\ w_2 \end{pmatrix} \), for which we abbreviated the variances...
\(\tau^2\) by \(\tilde{\tau}\) and \(\sigma^2_d\) by \(\tilde{\sigma}_d\).

Although we use standard parameters (mean and variance) for the Gaussian distributions (and not natural parameters), the parameterization of the whole model is still somewhat unusual for factor analysis because the prior also has a parameter. However, the vector \(\vec{w}\) is of unit (here Euclidean) norm such that the model parameterizes the same set of distributions as a more conventionally defined FA model. The reasons for the parameterization will become clearer later on.

The model (19) and (20) can be recognized as an EF generative model (Definition A) using the following natural parameters:

\[
\vec{\zeta}(\tilde{\tau}) = \begin{pmatrix} 0 \\ -\frac{1}{2\tilde{\tau}} \end{pmatrix} \text{ and } \vec{\eta}(z; \vec{\Theta}) = \begin{pmatrix} \frac{w_1 z}{\sigma_1} \\ \frac{w_2 z}{\sigma_2} \\ -\frac{1}{2\sigma_1} \\ -\frac{1}{2\sigma_2} \end{pmatrix}.
\] (21)

By computing the Jacobian of the prior natural parameter mapping

\[
\frac{\partial \vec{\zeta}(\vec{\Psi})}{\partial \vec{\Psi}^T} = \frac{\partial \vec{\zeta}(\tilde{\tau})}{\partial \tilde{\tau}} = \begin{pmatrix} 0 \\ \frac{1}{2\tilde{\tau}^2} \end{pmatrix}
\] (22)

and choosing \(\alpha(\tilde{\tau}) = -\tilde{\tau}\), we can see directly that part A of the parameterization criterion is fulfilled. For part B of the criterion, we choose as subset of \(\vec{\Theta}\) the vector \(\vec{\theta} = \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \end{pmatrix}\), i.e. \(\frac{\partial}{\partial \vec{\theta}^T} = (\frac{\partial}{\partial \tilde{\sigma}_1}, \frac{\partial}{\partial \tilde{\sigma}_2})\). The Jacobian is then the \((4 \times 2)\)-matrix

\[
\frac{\partial \vec{\eta}(z; \vec{\Theta})}{\partial \vec{\theta}^T} = \begin{pmatrix} -\frac{w_1 z}{\sigma_1^2} & 0 \\ 0 & -\frac{w_2 z}{\sigma_2^2} \\ \frac{1}{2\sigma_1^2} & 0 \\ 0 & -\frac{w_2 z}{\sigma_2^2} \end{pmatrix}
\] (23)

and \(\vec{\beta}(\vec{\Theta}) = \begin{pmatrix} -\tilde{\sigma}_1 \\ -\tilde{\sigma}_2 \end{pmatrix}\) satisfies Eqn. 13 such that the parameterization criterion is fulfilled.

**Example 3** (Counter-Example: Rigid SBN). As an example of a model for which it can not be shown that the parameterization criterion is fulfilled, we go back to the simple SBN model (Example 1). However, the natural parameters of the noise model now use only one parameter \(v \in \mathbb{R}\) as follows:

\[
z \sim \begin{aligned} p_{\vec{\Psi}}(z) &= \text{Bern}(z; \pi), \\
\vec{x} &\sim \begin{array}{l} p_{\vec{\Theta}}(\vec{x} | z) = \text{Bern}(x_1; S(v z)) \text{ Bern}(x_2; S((v + 1) z)), \end{array}
\] (24, 25)

where \(S(a) = \frac{1}{1 + e^{-a}}\), so \(\vec{\Theta} = v\) (now a scalar) and \(\vec{\eta}(z; \vec{\Theta}) = \begin{pmatrix} v z \\ (v + 1) z \end{pmatrix}\).
Part A of the parameterization criterion remains satisfied. For part B of the criterion, \( \vec{\theta} = \vec{\Theta} = v \) is the only non-empty subset of \( \vec{\Theta} \). The corresponding Jacobian is the \((2 \times 1)\)-matrix

\[
\frac{\partial \vec{\eta}(\vec{z}; \vec{\Theta})}{\partial \vec{\theta}^T} = \left( \begin{array}{c} z \\ z \end{array} \right).
\]  

Consequently, there is no function \( \beta(v) \) satisfying Eqn. 13 in Part B of the parameterization criterion since \( v \neq 1 + v \) for all \( v \in \mathbb{R} \).

The counter-example shows that the parameterization criterion is not necessarily fulfilled for a generative model. Other counter-examples would be simple FA (Example 2) for which one sigma is fixed (e.g. \( \tilde{\sigma}_2 \)). Also generative models with the Poisson distribution would be counter-examples but for another reason: the Poisson distribution does not have a constant base measure (but see Sec. 4 for non-constant base measures). In general, however, one can observe the parameterization criterion to be satisfied for many (maybe most) conventional generative models. A series of concrete models and model classes is systematically addressed elsewhere [26] and using the results we here derive. The contribution also includes the conventional general forms of SBNs and Factor Analysis.

Our general treatment of generative models in this and in the following sections makes assumptions on prior and noise distributions and their interrelation. In this context, let us note that we do not require the model’s joint distribution to be of the exponential family (which would be a further restriction).

### 3 Equality to Entropy Sums at Stationary Points

Based on the preparations in Sec. 2 we will now prove equality to entropy sums for all EF generative models that satisfy the parameterization criterion. Let us, first, consider the optimization of a generative model of Definition A given a set of \( N \) data points \( \vec{x}^{(n)} \). When considering the ELBO objective (2) we, as usual, assume the variational parameters \( \Phi \) to represent a set of parameters different from \( \vec{\Psi} \) and \( \vec{\Theta} \). The variational distributions \( q_{\Phi}^{(n)}(\vec{z}) \) are usually defined to approximate posterior distributions of the generative model well but here we will not require any conditions on the distributions (and we will not use a vector notation for \( \Phi \)). Given distributions \( q_{\Phi}^{(n)}(\vec{z}) \), the ELBO is given by:

\[
\mathcal{F}(\Phi, \vec{\Psi}, \vec{\Theta}) = \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \log(p_{\vec{\Theta}}(\vec{x}^{(n)} | \vec{z}) p_{\vec{\Psi}}(\vec{z})) \, d\vec{z} \\
- \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \log(q_{\Phi}^{(n)}(\vec{z})) \, d\vec{z} \\
= \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \log(p_{\vec{\Theta}}(\vec{x}^{(n)} | \vec{z})) \, d\vec{z} \\
- \frac{1}{N} \sum_n \text{KL}[q_{\Phi}^{(n)}(\vec{z}), p_{\vec{\Psi}}(\vec{z})],
\]  

where the first and second line represent the most common standard forms. The only unusual part of our notation is the use of two separate symbols for prior and noise model parameters (\( \vec{\Psi} \)}
and \( \tilde{\Theta} \) and a notation that denotes them as vectors. For our purposes, and for generative models as defined in Definition A, we first decompose the ELBO into three summands as follows:

\[
F(\Phi, \bar{\Psi}, \tilde{\Theta}) = F_1(\Phi) - F_2(\Phi, \bar{\Psi}) - F_3(\Phi, \tilde{\Theta}),
\]

where

\[
F_1(\Phi) = -\frac{1}{N} \sum_n \int q_\Phi^{(n)}(\bar{z}) \log (q_\Phi^{(n)}(\bar{z})) d\bar{z},
\]

\[
F_2(\Phi, \bar{\Psi}) = -\frac{1}{N} \sum_n \int q_\Phi^{(n)}(\bar{z}) \log (p_\Psi(\bar{z})) d\bar{z},
\]

\[
F_3(\Phi, \tilde{\Theta}) = -\frac{1}{N} \sum_n \int q_\Phi^{(n)}(\bar{z}) \log (p_\Theta(\bar{x}^{(n)} | \bar{z})) d\bar{z}.
\]

To finalize our notational preliminaries, we will use the aggregated posterior (e.g. \([28, 43, 3]\)) as an abbreviation, i.e., we define (noting that the entity is an approximation to the posterior average):

\[
\bar{q}_\Phi(\bar{z}) = \frac{1}{N} \sum_{n=1}^{N} q_\Phi^{(n)}(\bar{z}).
\]

For the proof of our main result, we will require the parameterization criterion given in Definition C. As a preparation, we will show a property that applies for the (transposed) Jacobians \( \partial \zeta^T(\bar{\Psi}) / \partial \bar{\Psi} \) and \( \partial \eta^T(\bar{z}; \tilde{\Theta}) / \partial \bar{\theta} \) if the parameterization criterion in Definition C is fulfilled. The transposed Jacobians are defined in a notation consistent with (10) and (11). Concretely, the transposed Jacobian of \( \zeta(\bar{\Psi}) \) is defined as:

\[
\partial \zeta^T(\bar{\Psi}) := \left( \frac{\partial}{\partial \bar{\Psi}} \zeta_1(\bar{\Psi}), \ldots, \frac{\partial}{\partial \bar{\Psi}} \zeta_K(\bar{\Psi}) \right).
\]

The transposed Jacobian of \( \eta(\bar{z}; \tilde{\Theta}) \) constructed w.r.t. \( \tilde{\Theta} \) (as a subset of \( \tilde{\Theta} \)) is defined by:

\[
\partial \eta^T(\bar{z}; \tilde{\Theta}) := \left( \frac{\partial}{\partial \tilde{\Theta}} \eta_1(\bar{z}; \tilde{\Theta}), \ldots, \frac{\partial}{\partial \tilde{\Theta}} \eta_L(\bar{z}; \tilde{\Theta}) \right).
\]

The following lemma will now derive a property from the parameterization criterion of Definition C that we will need for the proof of our main result.

**Lemma 1.** Consider an EF generative model as given by Definition B and let the dimensionalities of the natural parameter vectors \( \zeta \) and \( \eta \) be \( K \) and \( L \), respectively. Let further \( \partial \zeta^T(\bar{\Psi}) / \partial \bar{\Psi} \) and \( \partial \eta^T(\bar{z}; \tilde{\Theta}) / \partial \bar{\theta} \) denote the transposed Jacobians in (34) and (35), respectively.

If the EF generative model fulfills the parameterization criterion (Definition C), then the following two statements apply:

(A) For any vectorial function \( \bar{f}(\Phi, \bar{\Psi}) \) from the sets of parameters \( \Phi \) and \( \bar{\Psi} \) to the \((K\text{-dim})\) space of natural parameters of the prior it holds:

\[
\frac{\partial \zeta^T(\bar{\Psi})}{\partial \bar{\Psi}} \bar{f}(\Phi, \bar{\Psi}) = \bar{0} \Rightarrow \zeta^T(\bar{\Psi}) \bar{f}(\Phi, \bar{\Psi}) = 0.
\]

(B) For any variational parameter \( \Phi \) there is a non-empty subset \( \tilde{\Theta} \) of \( \tilde{\Theta} \) such that the following applies for all vectorial functions \( \bar{g}_1(\bar{z}; \tilde{\Theta}), \ldots, \bar{g}_N(\bar{z}; \tilde{\Theta}) \) from the parameter set \( \tilde{\Theta} \) and from
the latent space $\Omega_\mathbf{z}$ to the ($L$-dim) space of natural parameters of the noise model:

$$\sum_{n=1}^{N} \int q^{(n)}_\Phi (\mathbf{z}) \frac{\partial \eta^T (\mathbf{z}; \Theta)}{\partial \theta} \tilde{g}_n (\mathbf{z}; \Theta) d\mathbf{z} = 0.$$  \hfill (37)

$$\Rightarrow \sum_{n=1}^{N} \int q^{(n)}_\Phi (\mathbf{z}) \eta^T (\mathbf{z}; \Theta) \tilde{g}_n (\mathbf{z}; \Theta) d\mathbf{z} = 0.$$  \hfill (38)

In the case when $\mathbf{z}$ is a discrete latent variable, the integrals in (37) become a sum over all discrete states of $\mathbf{z}$.

Proof. To verify the first part of the lemma, we choose a function $\bar{\alpha} (\bar{\Psi})$ satisfying Eqn. 12. This is possible since we assume that the parameterization criterion (Definition C) is fulfilled. Moreover, consider an arbitrary vectorial function $\bar{f}(\Phi, \bar{\Psi})$ that satisfies

$$\frac{\partial \bar{\zeta}^T (\bar{\Psi})}{\partial \bar{\Psi}} \bar{f}(\Phi, \bar{\Psi}) = \vec{0}.$$  \hfill (38)

Then we can conclude:

$$\bar{\zeta}^T (\bar{\Psi}) \bar{f}(\Phi, \bar{\Psi}) = \bar{\alpha}^T (\bar{\Psi}) \frac{\partial \bar{\zeta}^T (\bar{\Psi})}{\partial \bar{\Psi}} \bar{f}(\Phi, \bar{\Psi}) = 0.$$  \hfill (39)

Consequently, the first part of the lemma is verified.

For the proof of the second part, let $\Phi$ be any variational parameter. We consider a subset $\bar{\Theta}$ of $\Theta$ and a function $\bar{\beta}(\bar{\Theta})$ satisfying Eqn. 13. Then all vectorial functions $\tilde{g}_1 (\mathbf{z}; \bar{\Theta}), \ldots, \tilde{g}_N (\mathbf{z}; \bar{\Theta})$ fulfilling the equation

$$\sum_{n=1}^{N} \int q^{(n)}_\Phi (\mathbf{z}) \frac{\partial \eta^T (\mathbf{z}; \bar{\Theta})}{\partial \theta} \tilde{g}_n (\mathbf{z}; \bar{\Theta}) d\mathbf{z} = \vec{0}.$$  \hfill (40)

also fulfill the equation

$$\sum_{n=1}^{N} \int q^{(n)}_\Phi (\mathbf{z}) \eta^T (\mathbf{z}; \bar{\Theta}) \tilde{g}_n (\mathbf{z}; \bar{\Theta}) d\mathbf{z} = \bar{\beta}^T (\bar{\Theta}) \sum_{n=1}^{N} \int q^{(n)}_\Phi (\mathbf{z}) \frac{\partial \eta^T (\mathbf{z}; \bar{\Theta})}{\partial \theta} \tilde{g}_n (\mathbf{z}; \bar{\Theta}) d\mathbf{z} = 0,$$  \hfill (41)

which proofs the second part of the lemma. Furthermore, it is easy to see that the proof proceeds analogously in the discrete case, i.e., if the integrals are replace by sums.

Because we will only use Lemma 1 and not directly the parameterization criterion (Definition C) to show convergence of the ELBO tho entropy sums, the question may arise whether the two statements in Lemma 1 are actually equivalent to the parameterization criterion given in Definition C. For the proof of our main result, we will require Eqns. 39 and 37 which follows (according to Lemma 1) from Definition C. Therefore, instead of verifying for a given generative models that it fulfills the parameterization condition (Definition C), we could also verify Eqns. 39 and 37 directly. This was done in previous work [11] but is much more intricate (see, e.g., their Appendix B), which makes Definition C preferable. A disadvantage of Definition C could be that
it is more restrictive, i.e., that there are generative models that do fulfill Eqs. 36 and 37 but not Definition C. However, given mild assumptions, the conditions of Eqs. 36 and 37 can, indeed, be shown to be equivalent to the conditions of Definition C. We provide the proof in Appendix A. In this context, we do point out that also another criterion can be used as a sufficient condition for (36) and (37). On the other hand, that criterion (as provided by [44], their Appendix A) is not equivalent to Eqs. 36 and 37 (it is more restrictive). But, on the other hand, it has the benefit of being more constructive.

We can now state and prove the main result.

**Theorem 1** (Equality to Entropy Sums). If \( p_\Phi(\tilde{z}) \) and \( p_\Theta(\tilde{x} | \tilde{z}) \) is an EF generative model (Definition B) that fulfills the parameterization criterion of Definition C then at all stationary points of the ELBO (28) it applies that

\[
F(\Phi, \tilde{\Psi}, \tilde{\Theta}) = \frac{1}{N} \sum_{n=1}^{N} H[q_{\Phi}^{(n)}(\tilde{z})] - H[p_\tilde{\Psi}(\tilde{z})] - \mathbb{E}_{q_\Phi} \{ H[p_\Theta(\tilde{x} | \tilde{z})] \}.
\]  

**Proof.** The first summand of the ELBO in Eqn. (29) is already given in the form of an (average) entropy \( (F_1(\Phi) = \frac{1}{N} \sum_{n} H[q_{\Phi}^{(n)}(\tilde{z})]) \). To prove the claim of Theorem 1 we will (at stationary points) show equality to entropies for the second and the third summand of Eqn. (29) and we will show equality to the corresponding entropies separately for the two terms.

\( F_2(\Phi, \tilde{\Psi}) \) at stationary points

We have assumed that the generative model is an EF generative model. Therefore we can rewrite \( F_2(\Phi, \tilde{\Psi}) \) of Eqn. (31) using the reparameterization of \( p_\tilde{\Psi}(\tilde{z}) \) in terms of the exponential family distribution \( p_\zeta(\tilde{z}) \):

\[
F_2(\Phi, \tilde{\Psi}) = -\frac{1}{N} \sum_{n} \int q_{\Phi}^{(n)}(\tilde{z}) \log(p_\tilde{\Psi}(\tilde{z})) \, d\tilde{z}
\]

\[
= -\frac{1}{N} \sum_{n} \int q_{\Phi}^{(n)}(\tilde{z}) \left( -\log(p_\zeta(\tilde{z})) \right) \, d\tilde{z}.
\]  

Using (8), the negative logarithm of \( p_\zeta(\tilde{z}) \) as exponential family distribution can be written as:

\[
-\log(p_\zeta(\tilde{z})) = A(\tilde{\zeta}(\tilde{\Psi})) - \tilde{\zeta}^T(\tilde{\Psi}) \tilde{T}(\tilde{z}) - \log(h(\tilde{z})),
\]  

where \( A \) is the partition function, \( \tilde{T} \) the sufficient statistics and \( h \) is the base measure. We will later require the derivative of the (negative) log probability w.r.t. \( \tilde{\Psi} \). The derivative is given by:

\[
\frac{\partial}{\partial \tilde{\Psi}} \left( -\log(p_\zeta(\tilde{z})) \right) = \frac{\partial \tilde{\zeta}^T(\tilde{\Psi})}{\partial \tilde{\Psi}} \left( \tilde{A}(\tilde{\zeta}(\tilde{\Psi})) - \tilde{T}(\tilde{z}) \right),
\]

\[
\text{where } \tilde{A}(\tilde{\zeta}(\tilde{\Psi})) := \left. \frac{\partial A(\tilde{\zeta})}{\partial \tilde{\zeta}} \right|_{\tilde{\zeta} = \tilde{\zeta}(\tilde{\Psi})}
\]

and where \( \frac{\partial \tilde{\zeta}^T(\tilde{\Psi})}{\partial \tilde{\Psi}} \) is the transposed Jacobian matrix defined in (34).
We will show that the summand $\mathcal{F}_2(\Phi, \bar{\Psi})$ will at stationary points be equal to the prior entropy. Therefore, we first rewrite $-\log(p_{\bar{\Psi}}(\bar{z}))$ using the prior entropy. By using the definition of the entropy

$$\mathcal{H}[p_{\bar{\Psi}}(\bar{z})] = \int p_{\bar{\Psi}}(\bar{z}) \left( - \log(p_{\bar{\Psi}}(\bar{z})) \right) d\bar{z}$$

and by using (44) we obtain:

$$\mathcal{H}[p_{\bar{\Psi}}(\bar{z})] = \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] = A(\tilde{\bar{\Psi}}) - \tilde{\zeta}(\tilde{\bar{\Psi}}) \tilde{A}(\tilde{\bar{\Psi}}) - B(\tilde{\bar{\Psi}})$$

$$\Rightarrow A(\tilde{\bar{\Psi}}) = \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] + \tilde{\zeta}(\tilde{\bar{\Psi}}) \tilde{A}(\tilde{\bar{\Psi}}) + B(\tilde{\bar{\Psi}}),$$

where $B(\tilde{\bar{\Psi}}) := \int p_{\bar{\Psi}}(\bar{z}) \log(h(\bar{z})) d\bar{z}$. 

By combining (44) and (49), the negative logarithm of the prior probability becomes:

$$-\log(p_{\bar{\Psi}}(\bar{z})) = \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] + \tilde{\zeta}(\tilde{\bar{\Psi}}) \tilde{A}(\tilde{\bar{\Psi}}) - T(\bar{z}) + B(\tilde{\bar{\Psi}}) - \log(h(\bar{z}))$$

$$= \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] + \tilde{\zeta}(\tilde{\bar{\Psi}}) \tilde{A}(\tilde{\bar{\Psi}}) - \bar{T}(\bar{z}),$$

where $\bar{T}(\tilde{\bar{\Psi}})$ is defined as in Eqn. 16 and where the last two terms have cancelled because we assumed a constant base measure $h(\bar{z})$. Inserting the negative logarithm (51) back into (43), we therefore obtain for the second summand of the ELBO:

$$\mathcal{F}_2(\Phi, \bar{\Psi}) = \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\bar{z}) \left( \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] + \tilde{\zeta}(\tilde{\bar{\Psi}}) \tilde{A}(\tilde{\bar{\Psi}}) - \bar{T}(\bar{z}) \right) d\bar{z}$$

$$= \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] + \tilde{\zeta}(\tilde{\bar{\Psi}}) \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\bar{z}) (\tilde{A}(\tilde{\bar{\Psi}}) - \bar{T}(\bar{z})) d\bar{z}$$

$$= \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] + \tilde{\zeta}(\tilde{\bar{\Psi}}) \int \bar{T}(\bar{z}) \frac{1}{N} \sum_n q_{\Phi}^{(n)}(\bar{z})$$

$$= \mathcal{H}[p_{\bar{\Psi}}(\bar{z})] + \tilde{\zeta}(\tilde{\bar{\Psi}}) \bar{T}(\bar{z})$$

where $q_{\Phi}(\bar{z}) := \frac{1}{N} \sum_n q_{\Phi}^{(n)}(\bar{z})$ and $\bar{T}(\bar{z}) := A(\tilde{\bar{\Psi}}) - E_{q_{\Phi}} \{ \bar{T}(\bar{z}) \}$. Our aim will now be to show that the second term of (54) vanishes at stationary points of $\mathcal{F}(\Phi, \bar{\Psi}, \bar{\Theta})$. For all stationary points of $\mathcal{F}(\Phi, \bar{\Psi}, \bar{\Theta})$ applies that the derivatives w.r.t. all parameters vanish. We have assumed the parameters $\Phi$, $\bar{\Psi}$ and $\bar{\Theta}$ to be separate sets of parameters. $\mathcal{F}_2(\Phi, \bar{\Psi})$ is, therefore, the only summand of $\mathcal{F}(\Phi, \bar{\Psi}, \bar{\Theta})$ which depends on $\bar{\Psi}$. By using (43) and (45) we obtain at stationary points:

$$\bar{\Theta} = \frac{\partial}{\partial \bar{\Psi}} \mathcal{F}_2(\Phi, \bar{\Psi}) = \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\bar{z}) \left( - \log(p_{\bar{\Psi}}(\bar{z})) \right) d\bar{z}$$

$$= \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\bar{z}) \left( - \log(p_{\bar{\Psi}}(\bar{z})) \right) d\bar{z}$$

$$= \frac{\partial}{\partial \bar{\Psi}} \left( \frac{\partial}{\partial \bar{\Psi}} \mathcal{F}_2(\Phi, \bar{\Psi}) \right)$$

$$= \frac{\partial}{\partial \bar{\Psi}} \left( \tilde{\zeta}(\tilde{\bar{\Psi}}) \bar{T}(\bar{z}) \right)$$

$$= \frac{\partial}{\partial \bar{\Psi}} \left( \tilde{\zeta}(\tilde{\bar{\Psi}}) \bar{T}(\bar{z}) \right)$$
where $\tilde{q}_\Phi$ is defined in (53). Hence, at all stationary points (w.r.t. parameters $\vec{\Psi}$) the following concise condition holds:

$$\frac{\partial \zeta^T(\vec{\Psi})}{\partial \Psi} \vec{f}(\Phi, \vec{\Psi}) = \vec{0},$$

(58)

where $\vec{f}(\Phi, \vec{\Psi})$ is the same function as introduced in (54). We can now apply Lemma 1 as we assumed our generative model to fulfill the criterion given in Definition C. Part A of the lemma is formulated for arbitrary functions $\vec{f}$ with arbitrary parameters $\Phi$ and arbitrary values for $\vec{\Psi}$. So if the parameterization criterion is fulfilled, then (36) of Lemma 1 also applies to the specific function $\vec{f}$ as defined in (54), where $\Phi$ are the variational parameters. Using Lemma 1 we can, therefore, conclude from (58) that at stationary points applies:

$$\zeta^T(\vec{\Psi}) \vec{f}(\Phi, \vec{\Psi}) = 0.$$  

(59)

Using Eqn. (54) we consequently obtain at all stationary points for the summand $\mathcal{F}_2(\Phi, \vec{\Psi})$:

$$\mathcal{F}_2(\Phi, \vec{\Psi}) = \mathcal{H}[p_{\tilde{\zeta}(\vec{\Psi})}(\vec{z})] + \zeta^T(\vec{\Psi}) \vec{f}(\Phi, \vec{\Psi}) = \mathcal{H}[p_{\tilde{\zeta}(\vec{\Psi})}(\vec{z})] = \mathcal{H}[p_\Phi(\vec{z})].$$

(60)

$\mathcal{F}_3(\Phi, \vec{\Theta})$ at stationary points

The proof part for $\mathcal{F}_3(\Phi, \vec{\Theta})$ will be analogous to that of $\mathcal{F}_2(\Phi, \vec{\Psi})$ above but slightly more intricate because the noise distribution is conditional on $\vec{z}$. We have assumed that the generative model is an EF generative model. Therefore we can rewrite $\mathcal{F}_3(\Phi, \vec{\Theta})$ of Eqn.32 using the reparameterization of $p_\vec{\Theta}(\vec{x} | \vec{z})$ in terms of the exponential family distribution $p_{\vec{\eta}(\vec{z}; \vec{\Theta})}(\vec{x})$:

$$\mathcal{F}_3(\Phi, \vec{\Theta}) = -\frac{1}{N} \sum_n \int q_\Phi^{(n)}(\vec{z}) \log(p_\vec{\Theta}(\vec{x}^{(n)} | \vec{z})) \, d\vec{z}$$

$$= -\frac{1}{N} \sum_n \int q_\Phi^{(n)}(\vec{z}) \left( \log(p_{\vec{\eta}(\vec{z}; \vec{\Theta})}(\vec{x}^{(n)})) - \log(h(\vec{x})) \right) \, d\vec{z}.$$  

(61)

As $p_{\vec{\eta}(\vec{x})}$ is an exponential family distribution, its negative logarithm is (see Eqn.9) given by:

$$-\log(p_{\vec{\eta}(\vec{z}; \vec{\Theta})}(\vec{x})) = A(\vec{\eta}(\vec{z}; \vec{\Theta})) - \vec{\eta}^T(\vec{z}; \vec{\Theta}) \vec{T}(\vec{x}) - \log(h(\vec{x})).$$

(62)

We have noted before that we will use the same symbols for sufficient statistics $\vec{T}(\vec{x})$, partition function $A(\vec{\eta})$ and base measure $h(\vec{x})$ as for the prior in Eqn.44.

We will later require the derivative of the (negative) log probability w.r.t. $\vec{\Theta}$, i.e., w.r.t. a subset of the parameters $\vec{\Theta}$. The derivative is given by:

$$\frac{\partial}{\partial \vec{\Theta}} \left( -\log(p_{\vec{\eta}(\vec{z}; \vec{\Theta})}(\vec{x})) \right) = \frac{\partial \eta^T(\vec{z}; \vec{\Theta})}{\partial \vec{\Theta}} (\tilde{A}(\vec{\eta}(\vec{z}; \vec{\Theta})) - \vec{T}(\vec{x})),$$

where $\tilde{A}(\vec{\eta}(\vec{z}; \vec{\Theta})) := \frac{\partial}{\partial \vec{\Theta}} A(\vec{\eta}) \bigg|_{\vec{\eta}=\vec{\eta}(\vec{z}; \vec{\Theta})}$

(63)

and where $\frac{\partial \eta^T(\vec{z}; \vec{\Theta})}{\partial \vec{\Theta}}$ is the transposed Jacobian matrix defined in (35).
We will show that the summand $F_3(\Phi, \Theta)$ will at stationary points be equal to the (expected) noise model entropy. Therefore, we first rewrite $-\log(p_{\eta}(z; \tilde{\Theta}))$ using the noise model entropy. By again using the definition of the entropy

$$H[p_{\eta}(z; \tilde{\Theta})](x) = \int p_{\eta}(z; \tilde{\Theta})(x) \left(-\log p_{\eta}(z; \tilde{\Theta})(x)\right) dx$$  \hspace{1cm} (65)$$

and (62) we obtain:

$$H[p_{\eta}(x | z)] = H[p_{\eta}(z; \tilde{\Theta})] = A(\eta(z; \tilde{\Theta})) - \eta^T_{(z; \tilde{\Theta})} \bar{A}^T(\eta(z; \tilde{\Theta})) - B(\eta(z; \tilde{\Theta}))$$  \hspace{1cm} (66)$$

$$\Rightarrow A(\eta(z; \tilde{\Theta})) = H[p_{\eta}(z; \tilde{\Theta})] + \eta^T_{(z; \tilde{\Theta})} \bar{A}^T(\eta(z; \tilde{\Theta})) + B(\eta(z; \tilde{\Theta})),$$  \hspace{1cm} (67)$$

where we abbreviated $B(\eta) := \int p_{\eta}(x) \log(h(x)) dx.$  \hspace{1cm} (68)$$

By combining (62) and (67), the negative logarithm of the noise distribution becomes:

$$-\log(p_{\eta}(z; \tilde{\Theta}))(x) = H[p_{\eta}(z; \tilde{\Theta})](x) + \eta^T_{(z; \tilde{\Theta})} \bar{A}^T(\eta(z; \tilde{\Theta})) - \bar{T}(x)$$  \hspace{1cm} (69)$$

$$+ B(\eta(z; \tilde{\Theta})) - \log(h(x)) = H[p_{\eta}(z; \tilde{\Theta})](x) + \eta^T_{(z; \tilde{\Theta})} \bar{A}^T(\eta(z; \tilde{\Theta})) - \bar{T}(x),$$  \hspace{1cm} (70)$$

where $\bar{A}^T(\eta(z; \tilde{\Theta}))$ is defined as in Eqn. 64, and where the last two terms cancel again because we assumed a constant base measure $h(x)$ also for the noise model distribution. Inserting the negative logarithm (70) back into (61), we therefore obtain for the third summand of the ELBO:

$$F_3(\Phi, \tilde{\Theta})$$  \hspace{1cm} (71)$$

$$= \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(z) \left(H[p_{\eta}(z; \tilde{\Theta})](x) + \eta^T_{(z; \tilde{\Theta})} \bar{A}^T(\eta(z; \tilde{\Theta})) - \bar{T}(x^{(n)})\right) dz$$  \hspace{1cm} (72)$$

$$= \frac{1}{N} \sum_n \mathbb{E}_{q_{\Phi}^{(n)}} \left\{ H[p_{\eta}(z; \tilde{\Theta})](x) \right\} + \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(z) \eta^T_{(z; \tilde{\Theta})} \bar{A}^T(\eta(z; \tilde{\Theta}))$$  \hspace{1cm} (73)$$

$$- \bar{T}(x^{(n)}) dz$$

$$= \frac{1}{N} \sum_n \mathbb{E}_{q_{\Phi}^{(n)}} \left\{ H[p_{\eta}(z; \tilde{\Theta})](x) \right\} + \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(z) \eta^T_{(z; \tilde{\Theta})} \bar{g}_n(z; \tilde{\Theta}) dz,$$  \hspace{1cm} (74)$$

where $\bar{g}_n(z; \tilde{\Theta}) := A^T(\eta(z; \tilde{\Theta})) - \bar{T}(x^{(n)})$.

In analogy to the derivation for $F_2(\Phi, \tilde{\Psi})$, our aim will now be to show that the second term of (74) vanishes at stationary points of $F(\Phi, \tilde{\Psi}, \tilde{\Theta})$. For all stationary points of $F(\Phi, \tilde{\Psi}, \tilde{\Theta})$ applies that the derivatives w.r.t. all parameters vanish. We have assumed the parameters $\Phi, \tilde{\Psi}$ and $\tilde{\Theta}$ to be separate sets of parameters. $F_3(\Phi, \tilde{\Theta})$ is therefore the only summand of $F(\Phi, \tilde{\Psi}, \tilde{\Theta})$ which depends on $\tilde{\Theta}$. At stationary points we can consequently conclude for any subset $\tilde{\theta}$ of $\tilde{\Theta}$ that
\[ \frac{\partial}{\partial \theta} F_3(\Phi, \tilde{\Theta}) = 0. \] By using (61) and (63) we obtain at any stationary point and for any subset \( \tilde{\theta} \):

\[ 0 = \frac{\partial}{\partial \theta} F_3(\Phi, \tilde{\Theta}) = \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \frac{\partial}{\partial \theta} \left( -\log(p_{\tilde{\theta}}(\vec{z}; \vec{\phi}(n))) \right) d\vec{z} \]

\[ = \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \frac{\partial \tilde{\eta}_T(\vec{z}; \tilde{\Theta})}{\partial \theta} \left( \tilde{A}'(\vec{y}(\vec{z}; \tilde{\Theta})) - \tilde{T}(\vec{x}^{(n)}) \right) d\vec{z}. \quad (76) \]

In expression (76) we recognize the functions \( \tilde{g}_n(\vec{z}; \tilde{\Theta}) \) as defined in Eqn. (75). Therefore, expression (76) means that at all stationary points and for all subsets \( \tilde{\theta} \) of \( \tilde{\Theta} \) the following holds:

\[ \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \frac{\partial \tilde{\eta}_T(\vec{z}; \tilde{\Theta})}{\partial \theta} \tilde{g}_n(\vec{z}; \tilde{\Theta}) d\vec{z} = 0. \quad (77) \]

As (77) applies for all subsets \( \tilde{\theta} \), it also applies for the specific subset \( \tilde{\theta} \) of \( \tilde{\Theta} \) that exists according to the parameterization criterion (Definition C, part B) or Lemma 1 (part B). Using Part B of Lemma 1, we can consequently conclude from (77) that at all stationary points applies:

\[ \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \tilde{\eta}_T(\vec{z}; \tilde{\Theta}) \tilde{g}_n(\vec{z}; \tilde{\Theta}) d\vec{z} = 0. \quad (78) \]

By going back to (74) and by inserting (78), we consequently obtain at all stationary points for the summand \( F_3(\Phi, \tilde{\Theta}) \):

\[ F_3(\Phi, \tilde{\Theta}) = \frac{1}{N} \sum_n \mathbb{E}_{q_{\Phi}^{(n)}} \left\{ \mathcal{H}[p_{\tilde{\theta}}(\vec{z}; \vec{\phi})] \right\} + \frac{1}{N} \sum_n \int q_{\Phi}^{(n)}(\vec{z}) \tilde{\eta}_T(\vec{z}; \tilde{\Theta}) \tilde{g}_n(\vec{z}; \tilde{\Theta}) d\vec{z} \]

\[ = \frac{1}{N} \sum_n \mathbb{E}_{q_{\Phi}^{(n)}} \left\{ \mathcal{H}[p_{\tilde{\theta}}(\vec{z}; \vec{\phi})] \right\} = \frac{1}{N} \sum_n \mathbb{E}_{q_{\Phi}^{(n)}} \left\{ \mathcal{H}[p_{\tilde{\theta}}(\vec{x} | \vec{z})] \right\}. \quad (79) \]

**Conclusion**

We rewrote the ELBO to consist of three terms, \( F(\Phi, \tilde{\Psi}, \tilde{\Theta}) = F_1(\Phi) - F_2(\Phi, \tilde{\Psi}) - F_3(\Phi, \tilde{\Theta}) \), with \( F_1(\Phi), F_2(\Phi, \tilde{\Psi}) \) and \( F_3(\Phi, \tilde{\Theta}) \) as defined by Eqns. (30), (31) and (32), respectively. The first term is by definition an average entropy, \( F_1(\Phi) = \frac{1}{N} \sum_{n=1}^{N} \mathcal{H}[q_{\Phi}^{(n)}(\vec{z})] \). We have shown (see Eqn. (61)) that \( F_2(\Phi, \tilde{\Psi}) \) becomes equal to the entropy of the prior distribution at stationary points. Furthermore, we have shown (see Eqn. (74)) that \( F_3(\Phi, \tilde{\Theta}) \) becomes equal to an expected entropy at stationary points. By using the results (60) and (79), we have shown that at all stationary points the following applies for the ELBO (28):

\[ F(\Phi, \tilde{\Psi}, \tilde{\Theta}) = \frac{1}{N} \sum_{n=1}^{N} \mathcal{H}[q_{\Phi}^{(n)}(\vec{z})] - \mathcal{H}[p_{\tilde{\theta}}(\vec{z})] - \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_{q_{\Phi}^{(n)}} \left\{ \mathcal{H}[p_{\tilde{\theta}}(\vec{x} | \vec{z})] \right\}, \quad (80) \]

from which (42) follows by using (33) for the last summand. \( \square \)

Theorem 1 can be regarded as the main result. Most generative models of the form given by Definition A are usually defined using exponential family distributions, and usually those distributions have a constant base measure (Bernoulli, Categorical, Exponential, Gaussian, Gamma etc.). Still, it may be a bit unsatisfactory that not all exponential family distributions can be treated, which is why we consider a generalization of Theorem 1 in the following.
4 Convergence for General Exponential Families

While Theorem 1 applies for a broad range of generative models, the condition on exponential family distributions to have constant base measures (Definition B) excludes practically relevant distributions such as the Poisson distributions. Also from a theoretical perspective, it may not feel satisfactory to only consider a subset of exponential family distributions. Therefore, in this section, we will treat general exponential family distributions. The treatment will, in principle, be analogous to the previous section. However, we will in this section explicitly use Lebesgue integrals and general (non-negative) measures, which we have avoided for notational simplicity, so far.

As distributions $p_{\tilde{\Psi}}(\tilde{z})$ and $p_{\tilde{\Theta}}(\tilde{x} \mid \tilde{z})$ we now demand exponential family distributions but without restrictions to constant base measure.

**Definition D (General EF Models).** Given a generative model as given by Definition A, we say the generative model is a general EF model if there exist exponential family distributions $p_{\tilde{\zeta}}(\tilde{z})$ (for the latents) and $p_{\tilde{\eta}}(\tilde{x} \mid \tilde{z})$ (for the observables) such that the generative model can be reparameterized to take the following form:

\begin{align}
\tilde{z} & \sim p_{\tilde{\zeta}}(\tilde{\Psi}), \\
\tilde{x} & \sim p_{\tilde{\eta}(\tilde{z}; \tilde{\Theta})}(\tilde{x}),
\end{align}

where $\tilde{\zeta}(\tilde{\Psi})$ maps the prior parameters $\tilde{\Psi}$ to the natural parameters of distribution $p_{\tilde{\zeta}}(\tilde{z})$, and where $\tilde{\eta}(\tilde{z}; \tilde{\Theta})$ maps $\tilde{z}$ and the noise model parameters $\tilde{\Theta}$ to the natural parameters of distribution $p_{\tilde{\eta}}(\tilde{x})$.

The main idea for the generalization of Theorem 1 is using the base measures of the distributions in Definition D to define new measures for Lebesgue integration. The derivations for the general case can then be done in analogy to the proof of Theorem 1 except of using other Lebesgue integrals. Before, however, we require some preparations. Concretely, we introduce the new measures and define entropies and log-likelihood in the spaces with the new measures.

**Definition E (New Measures and Densities).** We consider a probability space with measure $\mu(\tilde{z})$. In this space let $p_{\tilde{\zeta}}(\tilde{z})$ be a probability density of an exponential family with base measure $h(\tilde{z})$. The density $p_{\tilde{\zeta}}(\tilde{z})$ is normalized w.r.t. the measure $\mu(\tilde{z})$ (e.g., but not necessarily, the standard Lebesgue measure of $\mathbb{R}^H$). We now define a new measure and the corresponding probability density as follows:

\begin{align}
\forall \tilde{z}: \quad & \tilde{\mu}(\tilde{z}) = h(\tilde{z}) \mu(\tilde{z}) \quad \text{and} \quad \forall \tilde{z}: \quad \tilde{p}_{\tilde{\zeta}}(\tilde{z}) = \frac{1}{h(\tilde{z})} p_{\tilde{\zeta}}(\tilde{z}).
\end{align}

Analogously, consider a probability space with measure $\mu(\tilde{x})$. In this space let $p_{\tilde{\eta}}(\tilde{x})$ be a probability density of an exponential family with base measure $h(\tilde{x})$. The density $p_{\tilde{\eta}}(\tilde{x})$ is normalized w.r.t. the measure $\mu(\tilde{x})$ (this can, for instance, be the standard Lebesgue measure of $\mathbb{R}^D$). We now define:

\begin{align}
\forall \tilde{x}: \quad & \tilde{\mu}(\tilde{x}) = h(\tilde{x}) \mu(\tilde{x}) \quad \text{and} \quad \forall \tilde{x}: \quad \tilde{p}_{\tilde{\eta}}(\tilde{x}) = \frac{1}{h(\tilde{x})} p_{\tilde{\eta}}(\tilde{x}).
\end{align}
It immediately follows that if $p_\zeta(z)$ is given by (83) and if $p_\eta(x)$ is given by (84) then we obtain:

$$
\tilde{p}_\zeta(z) = \exp\left(\zeta^T \tilde{T}(z) - A(\zeta)\right) \quad \text{and} \quad \tilde{p}_\eta(x) = \exp\left(\eta^T \tilde{T}(x) - A(\eta)\right),
$$

(85)
i.e., $\tilde{p}_\zeta(z)$ and $\tilde{p}_\eta(x)$ have unit base measures in the new probability spaces. So, $\tilde{p}_\zeta(z)$ and $\tilde{p}_\eta(x)$ simply have the form of the original distributions but with base measures equal one (in that sense Eqns. (85) are formally more rigorous than (83) and (84) because the latter use divisions by $h(z)$ and $h(x)$, respectively, and the base measures can be zero at some points). It is also immediately obvious that $\tilde{p}_\zeta(z)$ and $\tilde{p}_\eta(x)$ are normalized to one in the new spaces. For completeness, we also define the probabilities w.r.t. the new measures but using the original parameters $\vec{\Psi}$ and $\vec{\Theta}$:

$$
\tilde{p}_\zeta(z) = \tilde{p}_{\zeta(\vec{\Psi})}(z) \quad \text{and} \quad \tilde{p}_{\eta(\vec{\Theta})}(x | z) = \tilde{p}_{\eta(z; \vec{\Theta})}(x),
$$

(86)
where the functions $\vec{\zeta}(\vec{\Psi})$ and $\vec{\eta}(\vec{z}; \vec{\Theta})$ remain unchanged, i.e., the change of measures has no effect on the mappings to the natural parameters.

As a further preparation, we will require entropies and likelihood defined in the new probability spaces. Let us start with entropy definitions. For an exponential family distribution $p_\vec{\phi}(\vec{z}) = p_{\zeta(\vec{\Psi})}(\vec{z})$ as given in Definition D we define the pseudo entropy of $p_\vec{\phi}(\vec{z})$ in the same way as the conventional entropy but using the new measure:

$$
\tilde{H}[p_\vec{\phi}](\vec{z}) = -\int \tilde{p}_\vec{\phi}(\vec{z}) \log \tilde{p}_\vec{\phi}(\vec{z}) \, d\tilde{\mu}(\vec{z}) = -\tilde{\zeta}(\vec{\Psi}) \tilde{A}'(\tilde{\zeta}(\vec{\Psi})) + A(\tilde{\zeta}(\vec{\Psi})),
$$

(87)
where $\tilde{A}'(\tilde{\zeta}(\vec{\Psi})) := \frac{\partial}{\partial \vec{\zeta}} A(\tilde{\zeta}) \bigg|_{\vec{\zeta} = \vec{\zeta}(\vec{\Psi})}$ is the gradient of the log-partition function with respect to the natural parameters. The pseudo entropy can be expressed concisely using natural parameters and log-partition function (see right-hand-side of Eqn. (87) which is derived in Appendix C.1). Analogously, we define the pseudo entropy of the observable distribution $p_{\vec{\Theta}}(x | \vec{z}) = p_{\eta(z; \vec{\Theta})}(x)$ as follows:

$$
\tilde{H}[p_{\vec{\Theta}}](x | \vec{z}) = -\int \tilde{p}_{\vec{\Theta}}(x | \vec{z}) \log \tilde{p}_{\vec{\Theta}}(x | \vec{z}) \, d\tilde{\nu}(\vec{z}) = -\tilde{\eta}(\vec{z}; \vec{\Theta}) \tilde{A}'(\tilde{\eta}(\vec{z}; \vec{\Theta})) + A(\tilde{\eta}(\vec{z}; \vec{\Theta})),
$$

(88)
where $\tilde{A}'(\tilde{\eta}(\vec{z}; \vec{\Theta})) := \frac{\partial}{\partial \vec{\eta}} A(\tilde{\eta}) \bigg|_{\vec{\eta} = \tilde{\eta}(\vec{z}; \vec{\Theta})}$ is, similarly, the gradient of the log-partition function with respect to the natural parameters (see again Appendix C.1 for the derivation of the right-hand-side of Eqn. (88)). Although we consider pseudo entropies here only for exponential families, they can be defined for any (well-behaved) distributions (see Appendix C.1 for details). Specifically, the pseudo entropy of the variational distribution $q_\phi^{(n)}(\vec{z})$ is defined as

$$
\tilde{H}[q_\phi^{(n)}](\vec{z}) = -\int q_\phi^{(n)}(\vec{z}) \log q_\phi^{(n)}(\vec{z}) \, d\tilde{\mu}(\vec{z}) = -\int q_\phi^{(n)}(\vec{z}) \log \tilde{q}_\phi^{(n)}(\vec{z}) \, d\mu(\vec{z}),
$$

(89)
where $q_\phi^{(n)}(\vec{z})$ and $\tilde{q}_\phi^{(n)}(\vec{z})$ are related by the equation $q_\phi^{(n)}(\vec{z}) = h(\vec{z}) \tilde{q}_\phi^{(n)}(\vec{z})$, with $h(\vec{z})$ representing the base measure of the prior distribution.
We now define the log-likelihood w.r.t. the new measure $\tilde{\mu}(\vec{z})$. In analogy to the standard log-likelihood $L(\vec{\Psi}, \vec{\Theta})$, we define the pseudo log-likelihood function $\tilde{L}(\vec{\Psi}, \vec{\Theta})$ to be given by:

$$\tilde{L}(\vec{\Psi}, \vec{\Theta}) = \frac{1}{N} \sum_n \log \left( \int \tilde{p}_{\vec{\Theta}}(\vec{x}^{(n)} | \vec{z}) \tilde{p}_{\vec{\Psi}}(\vec{z}) \, d\tilde{\mu}(\vec{z}) \right)$$  \hspace{1cm} (90)

$$= \frac{1}{N} \sum_n \log \left( \int \tilde{p}_{\tilde{\eta}(\vec{z}, \vec{\Theta})}(\vec{x}^{(n)}) \tilde{p}_{\tilde{\zeta}(\vec{\Psi})}(\vec{z}) \, d\tilde{\mu}(\vec{z}) \right).$$  \hspace{1cm} (91)

It is straightforward to relate the pseudo log-likelihood $\tilde{L}(\vec{\Psi}, \vec{\Theta})$ to the standard log-likelihood. They differ only by an additive constant, which does not depend on any parameter (see Appendix C.1 for details). Therefore, parameter optimization using the pseudo log-likelihood is equivalent to parameter optimization using the standard log-likelihood.

We can treat the pseudo log-likelihood analogously to our treatment of the conventional log-likelihood. The main technical difference will be the use of probability measures $\tilde{\mu}(\vec{z})$ and $\tilde{\mu}(\vec{x})$.

To start, observe that it is straightforward to derive a lower bound (ELBO) of the pseudo log-likelihood. By applying the probabilistic version of Jensen’s inequality (e.g. [4]) we obtain (see Appendix C.1 for details):

$$\tilde{L}(\vec{\Psi}, \vec{\Theta}) = \frac{1}{N} \sum_n \log \left( \int \tilde{p}_{\vec{\Theta}}(\vec{x}^{(n)} | \vec{z}) \tilde{p}_{\vec{\Psi}}(\vec{z}) \, d\tilde{\mu}(\vec{z}) \right)$$

$$= \frac{1}{N} \sum_n \int \tilde{q}_{\vec{\Phi}}^{(n)}(\vec{z}) \log \left( \tilde{p}_{\vec{\Theta}}(\vec{x}^{(n)} | \vec{z}) \tilde{p}_{\vec{\Psi}}(\vec{z}) \right) \, d\tilde{\mu}(\vec{z}) - \frac{1}{N} \sum_n \int \tilde{q}_{\vec{\Phi}}^{(n)}(\vec{z}) \log \tilde{q}_{\vec{\Phi}}^{(n)}(\vec{z}) \, d\tilde{\mu}(\vec{z})$$

$$= \tilde{F}(\Phi, \tilde{\Psi}, \vec{\Theta}),$$

where the variational distributions $\tilde{q}_{\vec{\Phi}}^{(n)}(\vec{z})$ are normalized w.r.t. the new measure $\tilde{\mu}$. Similar to the log-likelihood and pseudo log-likelihood, the lower bound $\tilde{F}(\Phi, \tilde{\Psi}, \vec{\Theta})$ differs from the conventional ELBO only by an additive constant that does not depend on any model parameters (see Appendix C.2 for details).

In a preparation of the main result, we prove a similar statement as in Lemma 1. Since the parameterization criterion given in Definition C does not refer to the base measures $h(\vec{z})$ and $h(\vec{x})$, it does not change for general EF models, and hence we assume the same criterion as in Sec. 3.

Lemma 2. Consider a general EF model as given in Definition D and let the dimensionalities of the natural parameter vectors $\tilde{\zeta}$ and $\tilde{\eta}$ be $K$ and $L$, respectively. Let further $\frac{\partial \tilde{\zeta}^T(\vec{\Psi})}{\partial \vec{\Psi}}$ and $\frac{\partial \tilde{\eta}^T(\vec{\Psi}, \vec{\Theta})}{\partial \vec{\Theta}}$ denote the transposed Jacobians in (34) and (35), respectively.

If the general EF model fulfills the parameterization criterion (Definition C), then the following two statements apply:

(A) For any vectorial function $\tilde{f}(\Phi, \tilde{\Psi})$ from the sets of parameters $\Phi$ and $\tilde{\Psi}$ to the ($K$-dim) space of natural parameters of the prior it holds:

$$\frac{\partial \tilde{\zeta}^T(\vec{\Psi})}{\partial \tilde{\Psi}} \tilde{f}(\Phi, \tilde{\Psi}) = \vec{0} \Rightarrow \tilde{\zeta}^T(\vec{\Psi}) \tilde{f}(\Phi, \tilde{\Psi}) = 0.$$  \hspace{1cm} (93)
(B) For any variational parameter $\Phi$ there is a non-empty subset $\tilde{\Theta}$ of $\Theta$ such that the following applies for all vectorial functions $\vec{g}_1(\vec{z}; \tilde{\Theta}), \ldots, \vec{g}_N(\vec{z}; \tilde{\Theta})$ from the parameter set $\tilde{\Theta}$ and from the latent space $\Omega_\vec{z}$ to the ($L$-dim) space of natural parameters of the noise model:

$$\sum_{n=1}^{N} \int \tilde{q}_\Phi(n)(\vec{z}) \frac{\partial \vec{\eta}^T(z; \tilde{\Theta})}{\partial \vec{\theta}} \vec{g}_n(\vec{z}; \tilde{\Theta}) \, d\tilde{\mu}(\vec{z}) = 0 \quad (94)$$

$$\Rightarrow \sum_{n=1}^{N} \int \tilde{q}_\Phi(n)(\vec{z}) \vec{\eta}^T(z; \tilde{\Theta}) \vec{g}_n(\vec{z}; \tilde{\Theta}) \, d\tilde{\mu}(\vec{z}) = 0. \quad (95)$$

**Proof.** The proof is essentially the same as that of Lemma 1. In fact, nothing changes for part A. For part B, we only need the linearity of the integral, which also applies in the case of a general measure. So we can infer from Eqn. (94) that

$$\sum_{n=1}^{N} \int \tilde{q}_\Phi(n)(\vec{z}) \vec{\eta}^T(z; \tilde{\Theta}) \vec{g}_n(\vec{z}; \tilde{\Theta}) \, d\tilde{\mu}(\vec{z}) = 0.$$

Just like in Sec. 3 we will use Lemma 2 instead of the parameterization criterion (Definition C) to proof our main result. In Appendix A we will again show that under mild assumptions the criterions given in Lemma 2 are actually equivalent to the parameterization criterion.

We are now in the position to prove the generalization of Theorem 1.

**Theorem 2** (Equality to Sums of Pseudo Entropies). Let $p_\vec{\Psi}(\vec{z})$ and $p_\vec{\Theta}(\vec{x}|\vec{z})$ be a general EF model as given by Definition D and let the model fulfill the parameterization criterion of Definition C. Then at all stationary points of the ELBO (92) it applies that

$$\tilde{\mathcal{F}}(\Phi, \vec{\Psi}, \vec{\Theta}) = \frac{1}{N} \sum_{n=1}^{N} \tilde{\mathcal{H}}[q_\Phi^{(n)}(\vec{z})] - \tilde{\mathcal{H}}[p_\vec{\Psi}(\vec{z})] - \mathbb{E}_{\vec{q}_\Phi} \{ \tilde{\mathcal{H}}[p_\vec{\Theta}(\vec{x}|\vec{z})] \}, \quad (96)$$

where $\tilde{\mathcal{H}}[\cdot]$ are pseudo entropies as defined in (87), (88) and Appendix C.1, and where the expectation can be computed w.r.t. the original measure, i.e., $\mathbb{E}_{\vec{q}_\Phi} \{ f(\vec{z}) \} = \int \vec{q}_\Phi(\vec{z}) f(\vec{z}) \, d\mu(\vec{z})$.

The proof of the theorem is provided in Appendix B.

It can be verified that Theorem 2 is a genuine generalization of Theorem 1 (see Appendix C.3). As Theorem 2 applies for all general EF models (i.e., no restrictions apply for the exponential family distributions), the class of generative models for which Theorem 2 applies is a proper super-set of the models for which Theorem 1 applies. For instance, Theorem 2 can not be applied to mixtures of Poisson distributions, while Theorem 2 is applicable as the Poisson distribution is an exponential family distribution.
5 Discussion

We have provided the first general proof for the ELBO to be equal to a sum of entropies at stationary points. The resulting theorems (Theorem 1 and 2) apply for a very broad class of models with only mild conditions that have to be satisfied. As detailed in the introduction, our investigation of entropy convergence has its roots in two earlier papers ([25, 11]). Both of those contributions focused on models with Gaussian distributions while the here presented results show convergence to entropy sums much more generally: our results show that convergence to entropy sums is a very common feature of generative models, which does not require properties specific to the Gaussian distribution. Instead, for generative models with one set of latents and one set of observable variables, it suffices if the variables are distributed according to an exponential family distribution. Otherwise only mild assumptions have to be satisfied and, as an integral part of our contribution, we have defined the conditions that imply the ELBO of a generative model to converge to an entropy sum (Definitions C and D).

Simplified SBN (Example 1) and simplified factor analysis (Example 2) are examples for how the criterion can be verified for concrete models. A series of standard generative models can now be analyzed using the here derived theorems. Convergence to entropy sums can thus be shown (see [26]) for standard SBNs, for probabilistic PCA, factor analysis, Gaussian Mixture Models (GMMs), Poisson Mixture Models and, in general, for mixtures of exponential family distributions. Also the ELBO of standard probabilistic sparse coding [33] was recently shown to converge to an entropy sum by applying Theorem 1 (see [44]). Furthermore, the technical proof for Gaussian VAEs [11] can be much simplified using Theorem 1 and the theorem provides the deeper theoretical reason why ELBOs of standard VAEs converge to entropy sums. Furthermore, VAEs with non-Gaussian observable distributions or non-Gaussian prior distributions can now be addressed relatively easily.

A still other line of research could investigate the use of entropy sums as learning objectives. Considering entropy sums as objectives may not be intuitive because the entropy sums as they appear in Theorems 1 and 2 are by themselves no learning objectives. However, considering the Theorems and their proofs, it can be observed that only a subset of parameters has to be at stationary points. All remaining parameters can then be learned using an entropy sum objective. For probabilistic sparse coding, it was, for instance, shown that analytic solutions can be derived for those parameters of the ELBO that need to be at stationary points for equality to entropy sums. Using the analytic solution and Theorem 1 then allows for using an objective solely based on entropies. Future work will further investigate those purely entropy-based objectives. Optimization landscapes can, for instance, be investigated on such grounds: by using Eqns. 87 and 88 further analysis could exploit the relation of derivatives of entropies to Fisher-Rao metrics in information geometry ([36, 2]). Entropy objectives would also represent a potentially appealing basis for generalizations of ELBO objectives. For instance, [23] have suggested generalized ELBO objectives using Rényi divergences as generalizations of the Kullback-Leibler divergence. Based on an entropy sum objective, generalizations would be possible by directly using Rényi entropies.

Future work may also be able to exploit the relation between studies of convergence guarantees for ELBO objectives and Theorems 1 and 2. One recent such contribution provides convergence

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1 Indeed, [37] first introduced Rényi entropies before introducing Rényi divergences.
rates for Gaussian VAEs \cite{40}. The theoretically quantified convergence rates imply the same rates for the convergence of the ELBO to entropy sums (as explicitly noted in \cite{40}). Vice versa, information about the optimization landscape in the vicinity of stationary points could be provided through entropy sum expressions, which may be useful for the study of convergence guarantees.

Finally, while the class of models that can be addressed using Theorems 1 and 2 includes many very standard generative models \cite{26} and with VAEs also prominent deep models, future work will aim at still further extending the class of considered models. For instance, similar results can be aimed at for generative models defined based on more intricate directed graphical models, which would include deep generative models with stochastic layers (deep SBNs, stacked VAEs, diffusion models etc). Also generative models defined based on undirected graphical models could be addressed, which would include prominent models such as deep Boltzmann machines \cite{1,18} and other deep generative models \cite{8}.

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Appendix

A Parameterization Criterion

Here we show that the two statements of Lemma 1 (or Lemma 2 in the case of a general EF model) are actually equivalent to the parameterization criterion given in Definition C if we assume few mild additional conditions. Consequently, we do not introduce a restriction when using the parameterization criterion instead of Lemma 1 (or Lemma 2) directly. Since we treat general measures in Sec. 4 we adopt the same approach here. However, the following statements and proofs also apply when we use the notation from Sec. 3. We just need to replace each $d\mu(\vec{z})$ by $d\vec{z}$. First, we claim that part A of Lemma 1 (or Lemma 2) is equivalent to part A of the parameterization criterion. For this we do not need any additional conditions as we can see in the following lemma:

Lemma 3 (Equivalence for the Prior). Consider an EF generative model as given in Definition B or a general EF generative model as given in Definition D with a mapping to the natural parameters of the prior given by $\vec{z}(\vec{\Psi})$. The dimensionalities of the vectors $\vec{z}$ and $\vec{\Psi}$ are $K$ and $R$, respectively. Then the following two statements are equivalent:

(A) There exists a vectorial function $\vec{\alpha}(\cdot)$ from the set of parameters $\vec{\Psi}$ to $\mathbb{R}^R$ such that for all $\vec{\Psi}$ holds:

$$\vec{z}(\vec{\Psi}) = \frac{\partial \vec{z}(\vec{\Psi})}{\partial \vec{\Psi}} \cdot \vec{\alpha}(\vec{\Psi}).$$

(97)
(B) For any vectorial function $\vec{f}(\Phi, \vec{\Psi})$ from the sets of parameters $\Phi$ and $\vec{\Psi}$ to the $(K\text{-dim})$ space of natural parameters of the prior it holds:

$$\frac{\partial \vec{\zeta}^T(\vec{\Psi})}{\partial \vec{\Psi}} \vec{f}(\Phi, \vec{\Psi}) = \vec{0} \Rightarrow \vec{\zeta}^T(\vec{\Psi}) \vec{f}(\Phi, \vec{\Psi}) = 0. \tag{98}$$

**Proof.** We may rewrite statement B by using linear algebra. In fact, for fixed $\Phi$ and $\vec{\Psi}$, the first equation in (98) is equivalent to the statement that the vector $\vec{f}(\Phi, \vec{\Psi})$ is an element of the kernel of the transposed Jacobian matrix $\frac{\partial \vec{\zeta}^T(\vec{\Psi})}{\partial \vec{\Psi}}$. Thus, because $\vec{\zeta}^T(\vec{\Psi}) \vec{f}(\Phi, \vec{\Psi})$ is the standard Euclidean inner product, statement B is equivalent to

$$\vec{\zeta}(\vec{\Psi}) \perp \ker \frac{\partial \vec{\zeta}^T(\vec{\Psi})}{\partial \vec{\Psi}}. \tag{99}$$

In other words, $\vec{\zeta}(\vec{\Psi})$ is in the orthogonal complement of the transposed Jacobian, i.e.

$$\vec{\zeta}(\vec{\Psi}) \in \left( \ker \frac{\partial \vec{\zeta}^T(\vec{\Psi})}{\partial \vec{\Psi}} \right)^\perp.$$

Denoting the image or range of a matrix $A$ by im $A$ and using the well known identities $\ker A = (\text{im } A^T)^\perp$ and $((\text{im } A)^\perp)^\perp = \text{im } A$ for an arbitrary matrix $A$, we can rewrite statement B of this lemma in the form

$$\vec{\zeta}(\vec{\Psi}) \in \text{im } \frac{\partial \vec{\zeta}(\vec{\Psi})}{\partial \vec{\Psi}^T}.$$

By definition of im $\frac{\partial \vec{\zeta}(\vec{\Psi})}{\partial \vec{\Psi}^T}$, this is equivalent to statement A and the lemma is proved. \hfill \square

The equivalence of part B of Lemma 1 (or Lemma 2) and part B of the parameterization criterion (Definition C) is somewhat more intricate. For the proof we interpret the left integral in (37) (or (94)) as a bounded linear operator between two suitable Hilbert spaces and compute its adjoint. More precisely, if we denote the latent space by $\Omega_{\vec{z}}$ and the dimensionality of the natural parameter vector $\vec{\eta}$ by $L$, we can define for any fixed variational parameter $\Phi$ and any $n \in \{1, \ldots, N\}$ the Hilbert space $^2$

$$L^2_{n,\Phi}(\Omega_{\vec{z}}, \mathbb{R}^L) := \{ \vec{g}_n : \Omega_{\vec{z}} \rightarrow \mathbb{R}^L \mid \int q^{(n)}_\Phi(\vec{z}) |\vec{g}_n(\vec{z})|^2 d\mu(\vec{z}) < \infty \}$$

of square integrable functions w.r.t. the weight $q^{(n)}_\Phi(\vec{z})$ and equip it with the inner product

$$\langle \vec{f}_n, \vec{g}_n \rangle_{n,\Phi} := \int q^{(n)}_\Phi(\vec{z}) \vec{f}_n(\vec{z})^T \vec{g}_n(\vec{z}) d\mu(\vec{z}), \quad \vec{f}_n, \vec{g}_n \in L^2_{n,\Phi}(\Omega_{\vec{z}}, \mathbb{R}^L).$$

As mentioned above, we here use a general measure $\mu(\vec{z})$ on the latent space $\Omega_{\vec{z}}$. But if we treat the standard Lebesgue measure in the continuous case we can simply replace the $d\mu(\vec{z})$ by $d\vec{z}$.

\footnote{Although we should actually consider equivalence classes instead of functions, we will not do this for reasons of convenience. But we keep in mind that the functions may only be defined almost everywhere.}
Analogously, we can replace the integrals by sums, when the latent variable \( \tilde{z} \) is discrete (in this case \( \mu(\tilde{z}) \) is the counting measure). Furthermore, in Sec. 3 the measure \( \mu(\tilde{z}) \) would play the role of the new measure \( \tilde{\mu}(\tilde{z}) \) as given in Definition 11 (not of the \( \mu(\tilde{z}) \) used there) and \( q_{\Phi}^{(n)}(\tilde{z}) \) of the new variational distributions \( \tilde{q}_{\Phi}^{(n)}(\tilde{z}) \) normalized w.r.t. the measure \( \tilde{\mu}(\tilde{z}) \). However, for convenience and to be consistent with Sec. 3 we omit the tilde here. Either way, the Cartesian product

\[
L_{\Phi}^{2}(\Omega_{\tilde{z}}, \mathbb{R}^{L}) := L_{1,\Phi}^{2}(\Omega_{\tilde{z}}, \mathbb{R}^{L}) \times \cdots \times L_{N,\Phi}^{2}(\Omega_{\tilde{z}}, \mathbb{R}^{L})
\]

of the individual Hilbert spaces together with the inner product

\[
\langle (\tilde{f}_{1}, \ldots, \tilde{f}_{N}), (\tilde{g}_{1}, \ldots, \tilde{g}_{N}) \rangle_{\Phi} := \sum_{n=1}^{N} \langle \tilde{f}_{n}, \tilde{g}_{n} \rangle_{n,\Phi}
\]

is also a Hilbert space. Since we have to treat the transposed Jacobian matrix \( \frac{\partial \tilde{g}^{T}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}} \), we also introduce for any fixed variational parameter \( \Phi \) and any \( n \in \{1, \ldots, N\} \) the Banach space

\[
L_{n,\Phi}^{2}(\Omega_{\tilde{z}}, \mathbb{R}^{S \times L}) := \{ J : \Omega_{\tilde{z}} \to \mathbb{R}^{S \times L} \mid \int q_{\Phi}^{(n)}(\tilde{z}) \| J(\tilde{z}) \|^2 d\mu(\tilde{z}) < \infty \},
\]

where \( S \) is the dimension of the subset \( \tilde{\Theta} \) of the noise parameters \( \tilde{\Theta} \) we use to construct the Jacobian, and \( \| \cdot \| \) an arbitrary matrix norm on \( \mathbb{R}^{S \times L} \).

From now on, for any fixed \( \tilde{\Theta} \), we will assume the mapping \( \tilde{g}(\tilde{z}; \tilde{\Theta}) \) to be an element of all Hilbert spaces \( L_{n,\Phi}^{2}(\Omega_{\tilde{z}}, \mathbb{R}^{L}) \), and the transposed Jacobian \( \frac{\partial \tilde{g}^{T}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}} \) to be an element of all Banach spaces \( L_{n,\Phi}^{2}(\Omega_{\tilde{z}}, \mathbb{R}^{S \times L}) \). This assumptions are quite mild, because often only some expectation values w.r.t. the variational distributions \( q_{\Phi}^{(n)}(\tilde{z}) \) need to exist, which is usually the case in machine learning. Especially, if the latent space \( \Omega_{\tilde{z}} \) is finite, since in this case the integrals are replaced by finite sums, which always exist.

After these preparations, we can define the linear bounded operator we want to use for rewriting statement B of Lemma 1 (or Lemma 2). As the functions \( \tilde{g}_{n}(\tilde{z}; \tilde{\Theta}) \) arising there are arbitrary, we can omit their parameter \( \tilde{\Theta} \) to get the same statement (allowing for the treatment of functions \( \tilde{g}_{n}(\tilde{z}) \)). Hence, for all parameters \( \Phi \) and \( \tilde{\Theta} \) we define the linear operator

\[
A_{\Phi, \tilde{\Theta}} : L_{\Phi}^{2}(\Omega_{\tilde{z}}, \mathbb{R}^{L}) \to \mathbb{R}^{S}, \quad \tilde{g} = (\tilde{g}_{1}, \ldots, \tilde{g}_{N}) \mapsto \sum_{n=1}^{N} \int q_{\Phi}^{(n)}(\tilde{z}) \frac{\partial \tilde{g}^{T}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}} \tilde{g}_{n}(\tilde{z}) d\mu(\tilde{z}).
\]

Remember that the transposed Jacobian \( \frac{\partial \tilde{g}^{T}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}} \) is a \( (S \times L) \)-matrix, where \( L \) is the dimension of the natural parameter vector \( \tilde{\eta} \) and \( S \) the number of parameters \( \tilde{\theta} \) we use to construct the transposed Jacobian \( \frac{\partial \tilde{g}^{T}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}} \). Using the triangle inequality and Hölder’s inequality it is easy to see, that \( A_{\Phi, \tilde{\Theta}} \) is well-defined and bounded. The boundedness is important for the adjoint to be defined on the hole Hilbert space \( \mathbb{R}^{S} \).
Lemma 4 (The Adjoint of $A_{\Phi,\vec{\Theta}}$). The adjoint operator of $A_{\Phi,\vec{\Theta}}$ is given by

$$A^\dagger_{\Phi,\vec{\Theta}} : \mathbb{R}^S \rightarrow L^2_\Phi(\Omega_\vec{z}, \mathbb{R}^L), \quad \vec{\beta} \mapsto A^\dagger_{\Phi,\vec{\Theta}} \vec{\beta},$$

in which $(A^\dagger_{\Phi,\vec{\Theta}} \vec{\beta})(\vec{z}) = \left( \frac{\partial \vec{\eta}(\vec{z},\vec{\Theta})}{\partial \theta^T} \vec{\beta}, \ldots, \frac{\partial \vec{\eta}(\vec{z},\vec{\Theta})}{\partial \theta^T} \vec{\beta} \right)$.

Proof. For any $\vec{g} = (\vec{g}_1, \ldots, \vec{g}_N) \in L^2_\Phi(\Omega_\vec{z}, \mathbb{R}^L)$ and any $\vec{\beta} \in \mathbb{R}^S$ we have to show that the equation

$$\langle A_{\Phi,\vec{\Theta}} \vec{g}, \vec{\beta} \rangle_{\mathbb{R}^S} = \langle \vec{g}, A^\dagger_{\Phi,\vec{\Theta}} \vec{\beta} \rangle_{\Phi}$$

is fulfilled. We can do this by a direct computation:

$$\langle A_{\Phi,\vec{\Theta}} \vec{g}, \vec{\beta} \rangle_{\mathbb{R}^S} = \sum_{n=1}^N \int q_n^{(n)}(\vec{z}) \frac{\partial \vec{\eta}^T(\vec{z};\vec{\Theta})}{\partial \theta} \vec{g}_n(\vec{z}) \, d\mu(\vec{z}), \vec{\beta}_{\mathbb{R}^S}$$

$$= \sum_{n=1}^N \int q_n^{(n)}(\vec{z}) \left( \frac{\partial \vec{\eta}^T(\vec{z};\vec{\Theta})}{\partial \theta} \vec{g}_n(\vec{z}), \vec{\beta} \right)_{\mathbb{R}^S} \, d\mu(\vec{z})$$

$$= \sum_{n=1}^N \int q_n^{(n)}(\vec{z}) \left( \vec{g}_n(\vec{z}), \frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \theta^T} \vec{\beta} \right)_{\mathbb{R}^L} \, d\mu(\vec{z})$$

$$= \langle \vec{g}, A^\dagger_{\Phi,\vec{\Theta}} \vec{\beta} \rangle_{\Phi}. \quad \Box$$

After these preparations we are now able to show the remaining equivalence of part B of Lemma 1 (or Lemma 2) and part B of the parameterization criterion as given in Definition C.

Lemma 5 (Equivalence for the Noise Model). For an EF generative model as given in Definition B or a general EF generative model as given in Definition D, let the natural parameter vector $\vec{\eta}(\vec{z};\vec{\Theta})$ be in the Hilbert space $L^2_{\mathbb{R}^S \times L}(\Omega_\vec{z}; \mathbb{R}^L)$ and its transposed Jacobian $\frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \theta^T}$ in the Banach space $L^2_{\mathbb{R}^S \times L}(\Omega_\vec{z}; \mathbb{R}^{S \times L})$ for all parameters $\Phi$ and $\vec{\Theta}$ and all $n \in \{1, \ldots, N\}$. Then the following two statements are equivalent:

(A) There are a non-empty subset $\vec{\theta}$ of $\vec{\Theta}$, whose dimension we denote by $S$, and a vectorial function $\vec{\beta}(\vec{\Theta})$ from the set of hole parameters $\vec{\Theta}$ to the $(S\text{-dim})$ space of $\vec{\theta}$ that fulfills for almost every $\vec{z}$ the equation

$$\vec{\eta}(\vec{z},\vec{\Theta}) = \frac{\partial \vec{\eta}(\vec{z};\vec{\Theta})}{\partial \theta^T} \vec{\beta}(\vec{\Theta}). \quad (100)$$

Here “for almost every $\vec{z}$” means that the set of all $\vec{z}$ which do not fulfill the equation has measure zero w.r.t. the weighted measures $\mu_n(\vec{z}) := q_n^{(n)}(\vec{z}) \mu(\vec{z})$ for all $n$. 

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(B) For any variational parameter $\Phi$ there is a non-empty subset $\tilde{\theta}$ of $\tilde{\Theta}$ such that the following applies for all vectorial functions $\vec{g}_1(\tilde{z}; \tilde{\Theta}), \ldots, \vec{g}_N(\tilde{z}; \tilde{\Theta})$ from the parameter set $\tilde{\Theta}$ and from the latent space $\Omega_{\tilde{z}}$ to the ($L$-dim) space of natural parameters of the noise model that are square-integrable $g_n(z; \tilde{\Theta}) \in L^2_{n,\Phi}(\Omega_{\tilde{z}}, \mathbb{R}^L)$ w.r.t. the latent variable $\tilde{z}$:

$$\sum_{n=1}^{N} \int q^{(n)}_\Phi(\tilde{z}) \frac{\partial \vec{\eta}^T(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}} g_n(\tilde{z}; \tilde{\Theta}) \, d\mu(\tilde{z}) = 0 \quad (101)$$

$$\Rightarrow \sum_{n=1}^{N} \int q^{(n)}_\Phi(\tilde{z}) \vec{\eta}^T(\tilde{z}; \tilde{\Theta}) g_n(\tilde{z}; \tilde{\Theta}) \, d\mu(\tilde{z}) = 0.$$

Here $\mu(\tilde{z})$ is a general measure on the latent space $\Omega_{\tilde{z}}$. But in the case of the standard Lebesgue measure, $d\mu(\tilde{z})$ can be replaced by $d\tilde{z}$ and in the case when $\tilde{z}$ is a discrete latent variable, the lemma is true as well if the integral is replaced by a sum over all discrete states of $\tilde{z}$. Furthermore, $\mu(\tilde{z})$ and $q^{(n)}_\Phi(\tilde{z})$ can be replaced in the context of Sec. 4 by $\tilde{\mu}(\tilde{z})$ and $q^{(n)}_\Phi(\tilde{z})$, respectively.

**Proof.** Since the functions $\vec{g}_n(z; \tilde{\Theta})$ are arbitrary anyway, it is straightforward to realise that we can omit the dependence of the parameters $\tilde{\Theta}$ in (B) to get the same statement. Hence, we can treat them w.l.o.g. as functions $\vec{g}_n(z) \in L^2_{n,\Phi}(\Omega_{\tilde{z}}, \mathbb{R}^L)$ not depending on $\tilde{\Theta}$. By using the operator $A_{\Phi,\tilde{\Theta}}$ and the shorter notation $\vec{g} = (\vec{g}_1, \ldots, \vec{g}_N)$, the first equation in (101) can be rewritten as $\vec{g} \in \text{ker } A_{\Phi,\tilde{\Theta}}$. Hence, because the second equation in (101) is an inner product in the Hilbert space

$$L^2_{\Phi}(\Omega_{\tilde{z}}, \mathbb{R}^L) = L^2_{1,\Phi}(\Omega_{\tilde{z}}, \mathbb{R}^L) \times \cdots \times L^2_{N,\Phi}(\Omega_{\tilde{z}}, \mathbb{R}^L),$$

statement B is equivalent to

$$(\vec{\eta}(\tilde{z}; \tilde{\Theta}), \ldots, \vec{\eta}(\tilde{z}; \tilde{\Theta})) \in \text{ker } A_{\Phi,\tilde{\Theta}}.$$  \hspace{1cm} (102)

For a bounded linear operator $A$ between two Hilbert spaces with a finite-dimensional image, the two identities $\text{ker } A = (\text{im } A^\dagger)^\perp$ and $(\text{im } A^\dagger)^\perp = \text{im } A^\dagger$ hold. As before, we denote by $\text{im } A^\dagger$ the image or range of the adjoint operator $A^\dagger$. Therefore, (102) is equivalent to

$$(\vec{\eta}(\tilde{z}; \tilde{\Theta}), \ldots, \vec{\eta}(\tilde{z}; \tilde{\Theta})) \in \text{im } A^\dagger_{\Phi,\tilde{\Theta}}.$$ \hspace{1cm} (103)

By definition and by Lemma 4 there is a vector $\vec{\beta} \in \mathbb{R}^S$ such that

$$(\vec{\eta}(\tilde{z}; \tilde{\Theta}), \ldots, \vec{\eta}(\tilde{z}; \tilde{\Theta})) = \begin{pmatrix} \frac{\partial \vec{\eta}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}^T} \vec{\beta}, & \ldots, & \frac{\partial \vec{\eta}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}^T} \vec{\beta} \end{pmatrix}.$$ \hspace{1cm} (104)

Because this is an equality in $L^2_{\Phi}(\Omega_{\tilde{z}}, \mathbb{R}^L)$, Eqn. (103) is equivalent to

$$\vec{\eta}(\tilde{z}; \tilde{\Theta}) = \frac{\partial \vec{\eta}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}^T} \vec{\beta},$$ \hspace{1cm} (104)

for almost every $\tilde{z}$. Since the operator $A_{\Phi,\tilde{\Theta}}$ and its adjoint $A^\dagger_{\Phi,\tilde{\Theta}}$ depend on the parameters $\Phi$ and $\tilde{\Theta}$, the vector $\vec{\beta}$ may also depend on $\Phi$ and $\tilde{\Theta}$. However, the dependency on $\Phi$ does not matter, as in (104), the natural parameters $\vec{\eta}(\tilde{z}; \tilde{\Theta})$ and their Jacobian $\frac{\partial \vec{\eta}(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\theta}^T}$ do not depend on $\Phi$. \hfill \Box
B Proof of Theorem 2

We here provide the proof of Theorem 2. In a preparation and in analogy to Sec. 3 (see Eqns 29 to 32), we first decompose the ELBO (102) into three summands:

\[ \hat{F}(\Phi, \Psi, \Theta) = \hat{F}_1(\Phi) - \hat{F}_2(\Phi, \Psi) - \hat{F}_3(\Phi, \Theta), \quad \text{where} \]
\[ \hat{F}_1(\Phi) = -\frac{1}{N} \sum_n \int \hat{q}_n(\hat{z}) \log(\hat{q}_n(\hat{z})) \, d\hat{\mu}(\hat{z}), \]
\[ \hat{F}_2(\Phi, \Psi) = -\frac{1}{N} \sum_n \int \hat{q}_n(\hat{z}) \log(\hat{\mu}_\Psi(\hat{z})) \, d\hat{\mu}(\hat{z}), \]
\[ \hat{F}_3(\Phi, \Theta) = -\frac{1}{N} \sum_n \int \hat{q}_n(\hat{z}) \log(\hat{\mu}_\Theta(\hat{x}(n) | \hat{z})) \, d\hat{\mu}(\hat{z}). \]

Proof of Theorem 2. The first summand of the ELBO (105) we observe to take the form of (an average of) a pseudo entropy \( \hat{F}_1(\Phi) = \frac{1}{N} \sum_n \hat{H}[q_{\Phi}^{(n)}(\hat{z})] \).

It remains to generalize the proof of Theorem 1. For this, observe that all terms of the ELBO (see Eqn. 105) are defined analogously as in (29) only the base measure has changed. Second, observe that all derivation steps of the proof of Theorem 1 used the standard properties of integration that apply to any measure for integration. As \( \hat{\mu}_\Psi(\hat{z}) \) and \( \hat{\mu}_\Theta(\hat{x}) \) are probability distributions for the measures \( \hat{\mu}(\hat{z}) \) and \( \hat{\mu}(\hat{x}) \), respectively, they represent probability distributions with constant (and unit) base measure in the probability spaces defined by these measures, respectively. The proof can consequently proceed along the same lines as for Theorem 1.

\( \hat{F}_2(\Phi, \Psi) \) at stationary points

As in the proof of Theorem 1 we want to rewrite the second summand \( \hat{F}_2(\Phi, \Psi) \) of the ELBO (see Eqn. 107) at all stationary points. To do so, we again compute the negative logarithm of \( \hat{\mu}_\Psi(\hat{z}) \) that is now given by

\[ -\log(\hat{\mu}_\Psi(\hat{z})) = -\log(\hat{\mu}_\Psi(\hat{z})) = A(\hat{\zeta}(\Psi)) - \hat{\zeta}_T(\hat{\Psi}), \]

since \( \hat{\mu}_\Psi(\hat{z}) \) has unit base measure. Furthermore, the derivative of the negative logarithm w.r.t. \( \Psi \) is given by

\[ \frac{\partial}{\partial \Psi} \left( -\log(\hat{\mu}_\Psi(\hat{z})) \right) = \frac{\partial \hat{z}_T(\Psi)}{\partial \Psi} \left( \hat{A}(\hat{\zeta}(\Psi)) - \hat{T}(\hat{z}) \right), \]

where \( \hat{A}(\hat{\zeta}(\Psi)) := \frac{\partial}{\partial \zeta} A(\zeta) \bigg|_{\zeta = \hat{\zeta}(\Psi)}. \]

Here \( A \) is again the log-partition function, \( \hat{T} \) the sufficient statistics and \( \frac{\partial \hat{z}_T(\Psi)}{\partial \Psi} \) the transposed Jacobian matrix defined in (34). By using Eqn. 87 to compute the pseudo entropy of the prior

\[ \hat{H}[p_{\Phi}(\hat{z})] = \hat{H}[p_{\zeta}(\hat{\Psi})(\hat{z})] = -\hat{\zeta}_T(\hat{\Psi}) \hat{A}(\hat{\zeta}(\hat{\Psi})) + A(\hat{\zeta}(\hat{\Psi})) \]

and by using (109) we obtain:

\[ -\log(\hat{\mu}_\Psi(\hat{z})) = \hat{H}[p_{\zeta}(\hat{\Psi})(\hat{z})] + \hat{\zeta}_T(\hat{\Psi}) (\hat{A}(\hat{\zeta}(\hat{\Psi})) - \hat{T}(\hat{z})). \]
Inserting the negative logarithm of the prior probability density (Eqn. 113) into $\tilde{F}_2(\Phi, \tilde{\Psi})$ (see Eqn. 107), we therefore obtain for the second summand of the ELBO:

$$\tilde{F}_2(\Phi, \tilde{\Psi}) = -\frac{1}{N} \sum_n \int q_{\Phi}(\tilde{z}) \log (\tilde{p}_{\tilde{\zeta}(\tilde{\Psi})}(\tilde{z})) \, d\tilde{\mu}(\tilde{z}) = H[p_{\tilde{\zeta}(\tilde{\Psi})}(\tilde{z})] + \zeta_2^T \tilde{f}(\Phi, \tilde{\Psi}), \quad (114)$$

where $\tilde{f}(\Phi, \tilde{\Psi}) := \tilde{A}(\zeta(\tilde{\Psi})) - \frac{1}{N} \sum_n E_{q_{\Phi}(n)} \tilde{T}(\tilde{z})$. \quad (115)

So, it remains to show that the last term of (114) disappears at all stationary points of $\tilde{F}(\Phi, \tilde{\Psi}, \tilde{\Theta})$. Precisely in the same way as in the proof of Theorem 1, the only summand of $\tilde{f}_2(\Phi, \tilde{\Psi})$ depends on $\vec{F}$.

Here we again use the same symbols for sufficient statistics $\tilde{T}(\tilde{x})$ and partition function $A(\tilde{\eta})$ as for the prior in Eqn. 109. Accordingly, for the derivative of the negative logarithm applies:

$$\frac{\partial}{\partial \tilde{\Psi}} \left( \frac{1}{N} \sum_n \int q_{\Phi}(\tilde{z}) \frac{1}{\log (p_{\tilde{\zeta}(\tilde{\Psi})}(\tilde{z}))} \, d\tilde{\mu}(\tilde{z}) \right) = \frac{1}{N} \sum_n \int q_{\Phi}(\tilde{z}) \frac{1}{\log (p_{\tilde{\zeta}(\tilde{\Psi})}(\tilde{z}))} \left( \tilde{A}(\zeta(\tilde{\Psi})) - \tilde{T}(\tilde{z}) \right) \, d\tilde{\mu}(\tilde{z})$$

Using Eqn. (114) we consequently obtain at all stationary points for the summand $\tilde{F}_2(\Phi, \tilde{\Psi})$:

$$\tilde{F}_2(\Phi, \tilde{\Psi}) = H[p_{\tilde{\zeta}(\tilde{\Psi})}(\tilde{z})] + \zeta_2^T \tilde{f}(\Phi, \tilde{\Psi}) = H[p_{\tilde{\zeta}(\tilde{\Psi})}(\tilde{z})] = H[p_{\tilde{\zeta}(\tilde{\Psi})}(\tilde{z})]. \quad (118)$$

$\tilde{F}_2(\Phi, \tilde{\Theta})$ at stationary points

To show convergence for the third summand of the ELBO (105), nothing significant changes compared to the second summand $\tilde{F}_2(\Phi, \tilde{\Theta})$ or to the proof of Theorem 1. In fact, the negative logarithm of $\tilde{p}_{\tilde{\Theta}}(\tilde{x} | \tilde{z})$ takes the form

$$-\log (\tilde{p}_{\tilde{\Theta}}(\tilde{x} | \tilde{z})) = -\log (\tilde{p}_{\tilde{\eta}(\tilde{z}; \tilde{\Theta})}(\tilde{x})) = A(\tilde{\eta}(\tilde{z}; \tilde{\Theta})) - \tilde{\eta}^T(\tilde{x}). \quad (119)$$

Here we again use the same symbols for sufficient statistics $\tilde{T}(\tilde{x})$ and partition function $A(\tilde{\eta})$ as for the prior in Eqn. 109. Accordingly, for the derivative of the negative logarithm applies:

$$\frac{\partial}{\partial \tilde{\Theta}} \left( -\log (\tilde{p}_{\tilde{\eta}(\tilde{z}; \tilde{\Theta})}(\tilde{x})) \right) = \frac{\partial \tilde{\eta}^T(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\Theta}} \left( A(\tilde{\eta}(\tilde{z}; \tilde{\Theta})) - \tilde{T}(\tilde{x}) \right), \quad (120)$$

where $\tilde{\eta}(\tilde{z}; \tilde{\Theta}) := \frac{\partial}{\partial \tilde{\Theta}} A(\tilde{\eta}) \bigg|_{\tilde{\eta}=\tilde{\eta}(\tilde{z}; \tilde{\Theta})} \quad (121)$

and where $\frac{\partial \tilde{\eta}^T(\tilde{z}; \tilde{\Theta})}{\partial \tilde{\Theta}}$ is the transposed Jacobian matrix defined in (35). Furthermore, by using Eqn. 88 to compute the pseudo entropy of the noise model

$$\tilde{H}[p_{\tilde{\Theta}}(\tilde{x} | \tilde{z})] = \tilde{H}[p_{\tilde{\eta}(\tilde{z}; \tilde{\Theta})}(\tilde{x})] = A(\tilde{\eta}(\tilde{z}; \tilde{\Theta})) - \tilde{\eta}^T(\tilde{z}; \tilde{\Theta}) \tilde{A}(\tilde{\eta}(\tilde{z}; \tilde{\Theta})), \quad (122)$$
the negative logarithm of the probability density then takes the form
\[
- \log\left(p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x})\right) = \tilde{\mathcal{H}}[p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x})] + \tilde{\eta}_T^T(\tilde{z}; \Theta) \left(\tilde{A}(\tilde{\eta}(\tilde{z}; \Theta)) - \tilde{T}(\tilde{x})\right),
\]
(123)
where \(\tilde{A}(\tilde{\eta}(\tilde{z}; \Theta))\) is defined as in Eqn. 121. Inserting the negative logarithm into \(\tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta})\) (see Eqn. 108), we therefore obtain for the third summand of the ELBO:
\[
\tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta}) = \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \left( - \log\left(p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x}^{(n)})\right) \right) d\tilde{\mu}(\tilde{z})
= \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \left( \tilde{\mathcal{H}}[p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x})] + \tilde{\eta}_T^T(\tilde{z}; \Theta) \left(\tilde{A}(\tilde{\eta}(\tilde{z}; \Theta)) - \tilde{T}(\tilde{x}^{(n)})\right) \right) d\tilde{\mu}(\tilde{z})
= \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \tilde{\mathcal{H}}[p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x})] d\mu(\tilde{z})
+ \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \tilde{\eta}_T^T(\tilde{z}; \Theta) \tilde{g}_n(\tilde{z}; \Theta) d\mu(\tilde{z})
= \frac{1}{N} \sum_n \mathbb{E}_{q_\Phi^{(n)}} \{ \tilde{\mathcal{H}}[p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x})] \} + \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \tilde{\eta}_T^T(\tilde{z}; \Theta) \tilde{g}_n(\tilde{z}; \Theta) d\mu(\tilde{z}),
\]
(124)
where \(\tilde{g}_n(\tilde{z}; \Theta) := \tilde{A}(\tilde{\eta}(\tilde{z}; \Theta)) - \tilde{T}(\tilde{x}^{(n)})\).
(125)
In analogy to the derivation for \(\tilde{\mathcal{F}}_2(\Phi, \tilde{\Psi})\) and to the proof of Theorem 11, the only summand of the ELBO that depends on the noise model parameters \(\tilde{\Theta}\) is \(\tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta})\). Therefore, at stationary points we can conclude for any subset \(\tilde{\Theta}\) of \(\Theta\) that \(\frac{\partial}{\partial \tilde{\Theta}} \tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta}) = 0\). By using the definition of \(\tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta})\) (see Eqn. 108 and Eqn. 120), we obtain at any stationary point and for any subset \(\tilde{\Theta}\):
\[
\tilde{\theta} = \frac{\partial}{\partial \tilde{\Theta}} \tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta}) = \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \frac{\partial \left( - \log\left(p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x}^{(n)})\right) \right)}{\partial \tilde{\theta}} d\tilde{\mu}(\tilde{z})
= \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \frac{\partial \tilde{\eta}_T^T(\tilde{z}; \Theta)}{\partial \tilde{\theta}} \left(\tilde{A}(\tilde{\eta}(\tilde{z}; \Theta)) - \tilde{T}(\tilde{x}^{(n)})\right) d\tilde{\mu}(\tilde{z})
= \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \frac{\partial \tilde{\eta}_T^T(\tilde{z}; \Theta)}{\partial \tilde{\theta}} \tilde{g}_n(\tilde{z}; \Theta) d\tilde{\mu}(\tilde{z}),
\]
(126)
where \(\tilde{g}_n(\tilde{z}; \Theta)\) are the same functions as introduced in 125. Again, we can use Lemma 2 and conclude from 126 that at all stationary points applies:
\[
\frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \tilde{\eta}_T^T(\tilde{z}; \Theta) \tilde{g}_n(\tilde{z}; \Theta) d\tilde{\mu}(\tilde{z}) = 0.
\]
(127)
By going back to 124 and by inserting 127, we consequently obtain at all stationary points for the summand \(\tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta})\):
\[
\tilde{\mathcal{F}}_3(\Phi, \tilde{\Theta}) = \frac{1}{N} \sum_n \mathbb{E}_{q_\Phi^{(n)}} \{ \tilde{\mathcal{H}}[p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x})] \} + \frac{1}{N} \sum_n \int \tilde{q}_\Phi^{(n)}(\tilde{z}) \tilde{\eta}_T^T(\tilde{z}; \Theta) \tilde{g}_n(\tilde{z}; \Theta) d\tilde{\mu}(\tilde{z})
= \frac{1}{N} \sum_n \mathbb{E}_{q_\Phi^{(n)}} \{ \tilde{\mathcal{H}}[p_{\tilde{\eta}(\tilde{z}; \Theta)}(\tilde{x})] \} + \frac{1}{N} \sum_n \mathbb{E}_{q_\Phi^{(n)}} \{ \tilde{\mathcal{H}}[p_{\tilde{\Theta}}(\tilde{x} | \tilde{z})] \}.
\]
(128)
Conclusion

We conclude as in the proof of Theorem 1, noting that we started by rewriting the ELBO consisting of three terms. In this case: 
\[ \tilde{F}(\Phi, \bar{\Psi}, \bar{\Theta}) = \tilde{F}_1(\Phi) - \tilde{F}_2(\Phi, \bar{\Psi}) - \tilde{F}_3(\Phi, \bar{\Theta}), \]
with \( \tilde{F}_1(\Phi), \tilde{F}_2(\Phi, \bar{\Psi}) \)
and \( \tilde{F}_3(\Phi, \bar{\Theta}) \) as defined by (106), (107) and (108), respectively. The first term is by definition an average of pseudo entropies, 
\[ \tilde{F}_1(\Phi) = \frac{1}{N} \sum_{n=1}^{N} \tilde{H}[q_{\Phi}^{(n)}(\bar{z})]. \]
We have shown (see Eqn. 118) that \( \tilde{F}_2(\Phi, \bar{\Psi}) \) becomes equal to the pseudo entropy of the prior distribution at stationary points. Furthermore, we have shown (see Eqn. 128) that \( \tilde{F}_3(\Phi, \bar{\Theta}) \) becomes equal to an expected pseudo entropy at stationary points. By inserting the results (118) and (128), we have shown that at all stationary points the following applies for the ELBO (92):
\[ \tilde{F}(\Phi, \bar{\Psi}, \bar{\Theta}) = \frac{1}{N} \sum_{n=1}^{N} \tilde{H}[q_{\Phi}^{(n)}(\bar{z})] - \tilde{H}[p_{\bar{\Psi}}(\bar{z})] - \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}_{q_{\Phi}^{(n)}}\{ \tilde{H}[p_{\bar{\Theta}}(\bar{x} | \bar{z})]\}, \]
from which (96) follows by using the aggregated posterior (see Eqn. 33) for the last summand.

C Entropies, Log-Likelihood, and Free Energy in the New Probability Spaces

In this section, we again provide the definitions of entropies, log-likelihood and free energy with respect to the new measures introduced in Section 4. However, the definition of the entropy here is somewhat more general than the one from Section 4, as we do not restrict ourselves to exponential family distributions. This generalization is necessary, since we do not demand that the variational distribution \( q_{\Phi}^{(n)}(\bar{z}) \) belong to an exponential family. Moreover, we discuss the connections to the conventional quantities.

C.1 Definitions and Basic Properties

We start with the definition of entropies in the new probability space that we define in analogy to standard entropies.

**Definition F (Pseudo Entropies).** Consider a general EF model from Definition D with probability densities \( p_{\zeta}(\bar{z}) \) and \( p_{\tilde{\eta}}(\bar{x}) \) defined w.r.t. the measures \( \mu(\bar{z}) \) and \( \mu(\bar{x}) \), respectively. Now consider the new measures \( \tilde{\mu}(\bar{z}) \) and \( \tilde{\mu}(\bar{x}) \) introduced by Definition E.

If \( p(\bar{z}) \) is a probability density defined w.r.t. the original measure \( \mu(\bar{z}) \), then \( \tilde{p}(\bar{z}) = \frac{1}{\tilde{\mu}(\bar{z})} p(\bar{z}) \) is a probability density w.r.t. the new measure \( \tilde{\mu}(\bar{z}) \). We can then define the pseudo entropy of \( p(\bar{z}) \) as follows:
\[ \tilde{\mathcal{H}}[p(\bar{z})] = - \int \tilde{p}(\bar{z}) \log \tilde{p}(\bar{z}) \, d\tilde{\mu}(\bar{z}). \]  
(130)

Analogously, if \( p(\bar{x}) \) is a probability density defined w.r.t. the original measure \( \mu(\bar{x}) \), then \( \tilde{p}(\bar{x}) = \frac{1}{\tilde{\mu}(\bar{x})} p(\bar{x}) \) is a probability density w.r.t. the new measure \( \tilde{\mu}(\bar{x}) \). We can then define the pseudo entropy of \( p(\bar{x}) \) as follows:
\[ \tilde{\mathcal{H}}[p(\bar{x})] = - \int \tilde{p}(\bar{x}) \log \tilde{p}(\bar{x}) \, d\tilde{\mu}(\bar{x}). \]  
(131)
Pseudo entropies can, hence, be defined for any (well-behaved) distributions. For the exponential family distributions \( p_\zeta (\vec{z}) \) and \( p_\eta (\vec{x}) \), the pseudo entropies take on a particularly concise and convenient form, however.

**Lemma 6.** For a general EF model as given in Definition D consider the new measures \( \tilde{\mu} (\vec{z}) \) and \( \tilde{\mu} (\vec{x}) \) introduced in Definition E. Then it applies for the pseudo entropies of Definition F that

\[
\tilde{H} [ p_\tilde{\mu} (\vec{z}) ] = \tilde{H} [ p_\tilde{\mu} (\vec{x} | \vec{z}) ] = - \vec{\zeta}^T (\vec{\Psi}) \tilde{A} (\vec{\zeta} (\vec{\Psi})) + A (\vec{\zeta} (\vec{\Psi})) \quad \text{and} \\
\tilde{H} [ p_\tilde{\eta} (\vec{x} | \vec{z}) ] = \tilde{H} [ p_\tilde{\eta} (\vec{x} | \vec{z}; \vec{\Theta}) ] = - \vec{\eta}^T (\vec{\Theta}) \tilde{A} (\vec{\eta} (\vec{x} | \vec{\Theta})) + A (\vec{\eta} (\vec{x} | \vec{\Theta})),
\]

where \( \tilde{A} (\vec{\zeta} (\vec{\Psi})) := \frac{\partial}{\partial \vec{\zeta}} A (\vec{\zeta}) \bigg|_{\vec{\zeta} = \vec{\zeta} (\vec{\Psi})} \) and \( \tilde{A} (\vec{\eta} (\vec{x} | \vec{\Theta})) := \frac{\partial}{\partial \vec{\eta}} A (\vec{\eta}) \bigg|_{\vec{\eta} = \vec{\eta} (\vec{x} | \vec{\Theta})} \) are the gradients of the log-partition functions with respect to the natural parameters.

**Proof.** We only show Eqn. 133 since Eqn. 132 follows along the same lines. Abbreviating \( \vec{\eta} = \vec{\eta} (\vec{x} | \vec{\Theta}) \) we obtain:

\[
\tilde{H} [ p_\tilde{\eta} (\vec{x}) ] = - \int p_\tilde{\eta} (\vec{x}) \log p_\tilde{\eta} (\vec{x}) \, d\tilde{\mu} (\vec{x}) = - \int p_\tilde{\eta} (\vec{x}) \log \tilde{p}_\eta (\vec{x}) \, d\mu (\vec{x}) \\
= - \int p_\tilde{\eta} (\vec{x}) \left( \vec{\eta}^T \tilde{A} (\vec{x}) + A (\vec{\eta}) \right) \, d\mu (\vec{x}) = - \vec{\eta}^T \tilde{A} (\vec{\eta}) + A (\vec{\eta}),
\]

where we applied the standard result \( \tilde{A} (\vec{\zeta}) = \frac{\partial}{\partial \vec{\eta}} A (\vec{\eta}) = \int p_\eta (\vec{x}) \tilde{T} (\vec{x}) \, d\mu (\vec{x}) \) (note that \( \mu (\vec{x}) \) is the original/old measure again).

Considering Lemma 6, the pseudo entropies of distributions \( p_\tilde{\mu} (\vec{z}) \) and \( p_\tilde{\mu} (\vec{x} | \vec{z}) \) can simply be computed from the mappings to natural parameters and their corresponding log-partition functions.

We now define the log-likelihood w.r.t. the new measure \( \tilde{\mu} (\vec{z}) \).

**Definition G (Pseudo Log-Likelihood).** Consider a general EF model as given by Definition D Let the measures \( \tilde{\mu} (\vec{z}) \) and \( \tilde{\mu} (\vec{x}) \) as well as the densities \( \tilde{p}_\zeta (\vec{z}) \) and \( \tilde{p}_\eta (\vec{x}) \) be defined as in Definition E. In analogy to the standard log-likelihood \( \mathcal{L} (\vec{\Psi}, \vec{\Theta}) \), we define the **pseudo log-likelihood** function \( \tilde{\mathcal{L}} (\vec{\Psi}, \vec{\Theta}) \) to be given by:

\[
\tilde{\mathcal{L}} (\vec{\Psi}, \vec{\Theta}) = \frac{1}{N} \sum_n \log \left( \int \tilde{p}_\tilde{\Theta} (\vec{x}^{(n)} | \vec{z}) \, \tilde{p}_\zeta (\vec{z}) \, d\tilde{\mu} (\vec{z}) \right) \\
= \frac{1}{N} \sum_n \log \left( \int \tilde{p}_\tilde{\Theta} (\vec{x}^{(n)} | \vec{z}) \, \tilde{p}_\zeta (\vec{\Psi}) (\vec{z}) \, d\tilde{\mu} (\vec{z}) \right),
\]

It is straightforward to relate the pseudo log-likelihood \( \tilde{\mathcal{L}} (\vec{\Psi}, \vec{\Theta}) \) to the standard log-likelihood.

**Lemma 7.** Consider a general EF model as given by Definition D and the corresponding pseudo log-likelihood as given in Definition G Then the difference to the standard log-likelihood is a constant not depending on any parameters:

\[
\tilde{\mathcal{L}} (\vec{\Psi}, \vec{\Theta}) = \mathcal{L} (\vec{\Psi}, \vec{\Theta}) - \frac{1}{N} \sum_n \log \left( h (x^{(n)}) \right) .
\]
\textbf{Proof.} Starting from the standard log-likelihood, we obtain:

\[
\mathcal{L}(\tilde{\Psi}, \tilde{\Theta}) = \frac{1}{N} \sum_n \log \left( \int p_{\tilde{\eta}(\tilde{z}; \tilde{\Theta})}(x^{(n)}) p_{\tilde{\psi}(\tilde{\psi})} (\tilde{z}) \, d\mu(\tilde{z}) \right) \\
= \frac{1}{N} \sum_n \log \left( h(x^{(n)}) \int \tilde{p}_{\tilde{\eta}(\tilde{z}; \tilde{\Theta})}(x^{(n)}) \tilde{p}_{\tilde{\psi}(\tilde{\psi})} (\tilde{z}) h(\tilde{z}) \, d\mu(\tilde{z}) \right) \\
= \frac{1}{N} \sum_n \log h(x^{(n)}) + \frac{1}{N} \sum_n \log \int \tilde{p}_{\tilde{\eta}(\tilde{z}; \tilde{\Theta})}(x^{(n)}) \tilde{p}_{\tilde{\psi}(\tilde{\psi})} (\tilde{z}) \, d\tilde{\mu}(\tilde{z}).
\]

If we subtract the term \(\frac{1}{N} \sum_n \log h(x^{(n)})\) on both sides, together with the Definition of the pseudo log-likelihood (Definition \[E\]), we obtain the claim. \(\square\)

By virtue of Lemma \[D\] parameter optimization using the pseudo log-likelihood is equivalent to parameter optimization using the standard log-likelihood. The main technical difference will be the use of probability measures \(\tilde{\mu}(\tilde{z})\) and \(\tilde{\mu}(\tilde{x})\). Furthermore, we can also derive a lower bound for the pseudo log-likelihood, just as we do for the standard log-likelihood. By applying the probabilistic version of Jensen’s inequality (e.g. \[D\]) we obtain:

\[
\tilde{\mathcal{L}}(\tilde{\Psi}, \tilde{\Theta}) \\
= \frac{1}{N} \sum_n \log \left( \int \tilde{p}_{\tilde{\Theta}}(\tilde{x}^{(n)} | \tilde{z}) \tilde{p}_{\tilde{\psi}} (\tilde{z}) \, d\tilde{\mu}(\tilde{z}) \right) \\
= \frac{1}{N} \sum_n \log \left( \mathbb{E}_{\tilde{q}_{\phi}} \left\{ \frac{\tilde{p}_{\tilde{\Theta}}(\tilde{x}^{(n)} | \tilde{z}) \tilde{p}_{\tilde{\psi}} (\tilde{z})}{\tilde{q}_{\phi}^{(n)} (\tilde{z})} \right\} \right) \geq \frac{1}{N} \sum_n \mathbb{E}_{\tilde{q}_{\phi}} \left\{ \log \left( \frac{\tilde{p}_{\tilde{\Theta}}(\tilde{x}^{(n)} | \tilde{z}) \tilde{p}_{\tilde{\psi}} (\tilde{z})}{\tilde{q}_{\phi}^{(n)} (\tilde{z})} \right) \right\} \\
= \frac{1}{N} \sum_n \int \tilde{q}_{\phi}^{(n)} (\tilde{z}) \log \left( \frac{\tilde{p}_{\tilde{\Theta}}(\tilde{x}^{(n)} | \tilde{z}) \tilde{p}_{\tilde{\psi}} (\tilde{z})}{\tilde{q}_{\phi}^{(n)} (\tilde{z})} \right) \, d\tilde{\mu}(\tilde{z}) \\
= \frac{1}{N} \sum_n \int \tilde{q}_{\phi}^{(n)} (\tilde{z}) \log (\tilde{p}_{\tilde{\Theta}}(\tilde{x}^{(n)} | \tilde{z}) \tilde{p}_{\tilde{\psi}} (\tilde{z})) \, d\tilde{\mu}(\tilde{z}) - \frac{1}{N} \sum_n \int \tilde{q}_{\phi}^{(n)} (\tilde{z}) \log \tilde{q}_{\phi}^{(n)} (\tilde{z}) \, d\tilde{\mu}(\tilde{z}) \\
=: \tilde{\mathcal{F}}(\Phi, \tilde{\Psi}, \tilde{\Theta}),
\]

where the variational distributions \(\tilde{q}_{\phi}^{(n)} (\tilde{z})\) are normalized w.r.t. the new measure \(\tilde{\mu}(\tilde{z})\).

\textbf{C.2 Relation to Entropy and Free Energy}

In the following two lemmas we provide, first, the relation between the pseudo entropies of a general EF generative model from Sec. \[D\] and the conventional entropies, and, second, the relation between the ELBOs \(\mathcal{F}(\Phi, \tilde{\Psi}, \tilde{\Theta})\) and \(\tilde{\mathcal{F}}(\Phi, \tilde{\Psi}, \tilde{\Theta})\).

\textbf{Lemma 8 (Relation between Entropies and Pseudo Entropies).} Consider a general EF model as given in Definition \[D\] with densities \(p_{\psi}(z)\) and \(p_{\theta}(\tilde{x})\) w.r.t. measures \(\mu(z)\) and \(\mu(\tilde{x})\). Let \(\tilde{\mu}(\tilde{z})\) and \(\tilde{\mu}(\tilde{x})\) be the new measures as given in Definition \[E\]. If the variational distributions \(q_{\phi}^{(n)} (\tilde{z})\) and \(\tilde{q}_{\phi}^{(n)} (\tilde{z})\) w.r.t. the measures \(\mu(\tilde{z})\) and \(\tilde{\mu}(\tilde{z})\), respectively, are related by the equation \(q_{\phi}^{(n)} (\tilde{z}) = h(\tilde{z}) \tilde{q}_{\phi}^{(n)} (\tilde{z})\), then the following applies:
Let \( \tilde{\mathcal{H}}[p_{\zeta}(\tilde{z})] = \mathcal{H}[p_{\zeta}(\tilde{z})] + \int p_{\zeta}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z}) \).

(B) \( \tilde{\mathcal{H}}[p_{\eta}(\tilde{x})] = \mathcal{H}[p_{\eta}(\tilde{x})] + \int p_{\eta}(\tilde{x}) \log (h(\tilde{x})) \, d\mu(\tilde{x}) \),

(C) \( \tilde{\mathcal{H}}[q_{\Phi}^{(n)}(\tilde{z})] = \mathcal{H}[q_{\Phi}^{(n)}(\tilde{z})] + \int q_{\Phi}^{(n)}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z}) \).

**Proof.** In order to proof the first equation we derive:

\[
\mathcal{H}[p_{\zeta}(\tilde{z})] = - \int p_{\zeta}(\tilde{z}) \log (p_{\zeta}(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= - \int p_{\zeta}(\tilde{z}) \log (\tilde{p}_{\zeta}(\tilde{z})) \, d\tilde{\mu}(\tilde{z}) - \int p_{\zeta}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= - \int \tilde{p}_{\zeta}(\tilde{z}) \log (\tilde{p}_{\zeta}(\tilde{z})) \, d\tilde{\mu}(\tilde{z}) - \int p_{\zeta}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= - \int \tilde{p}_{\zeta}(\tilde{z}) \log (\tilde{p}_{\zeta}(\tilde{z})) \, d\tilde{\mu}(\tilde{z}) - \int p_{\zeta}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= \tilde{\mathcal{H}}[p_{\zeta}(\tilde{z})] - \int p_{\zeta}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z}).
\]

By adding the second term of the last equation on both sides we get statement A. The other two statements can be shown analogously. We only have to replace the density \( p_{\zeta}(\tilde{z}) \) by the densities \( p_{\eta}(\tilde{x}) \) and \( q_{\Phi}^{(n)}(\tilde{z}) \). In the case of statement B, we must also replace the measures \( \mu(\tilde{z}) \) and \( \tilde{\mu}(\tilde{z}) \) by \( \mu(\tilde{x}) \) and \( \tilde{\mu}(\tilde{x}) \), respectively.

\[\square\]

**Lemma 9 (Relation between ELBOs w.r.t. different measures).** Consider a general EF generative model as given in Definition 13 with densities \( p_{\zeta}(\tilde{z}) \) and \( p_{\eta}(\tilde{x}) \) w.r.t. measures \( \mu(\tilde{z}) \) and \( \mu(\tilde{x}) \). Let \( \tilde{\mu}(\tilde{z}) \) and \( \tilde{\mu}(\tilde{x}) \) be new measures as given in Definition 14. If the variational distributions \( q_{\Phi}^{(n)}(\tilde{z}) \) and \( \tilde{q}_{\Phi}^{(n)}(\tilde{z}) \) w.r.t. the measures \( \mu(\tilde{z}) \) and \( \tilde{\mu}(\tilde{z}) \), respectively, are related by the equation

\[
q_{\Phi}^{(n)}(\tilde{z}) = h(\tilde{z})\tilde{q}_{\Phi}^{(n)}(\tilde{z}),
\]

then the following applies:

\[
\mathcal{F}(\Phi, \tilde{\Psi}, \tilde{\Theta}) = \mathcal{F}(\Phi, \tilde{\Psi}, \tilde{\Theta}) - \frac{1}{N} \sum_{n} \log (h(\tilde{x}^{(n)})).
\]

**Proof.** In Eqns. 29 and 105, we have decomposed the ELBOs into three summands, respectively. We now show a relationship between each summand of \( \mathcal{F}(\Phi, \tilde{\Psi}, \tilde{\Theta}) \) and \( \tilde{\mathcal{F}}(\Phi, \tilde{\Psi}, \tilde{\Theta}) \). For the first one, we can easily apply Lemma 8 since this is the average (pseudo) entropy of the variational distribution:

\[
\mathcal{F}_1(\Phi) = \frac{1}{N} \sum_{n} \tilde{\mathcal{H}}[q_{\Phi}^{(n)}(\tilde{z})] = \frac{1}{N} \sum_{n} \tilde{\mathcal{H}}[q_{\Phi}^{(n)}(\tilde{z})] + \frac{1}{N} \sum_{n} \int q_{\Phi}^{(n)}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= \mathcal{F}_1(\Phi) + \frac{1}{N} \sum_{n} \int q_{\Phi}^{(n)}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z}).
\]

For the second summand, we explicitly calculate:

\[
\mathcal{F}_2(\Phi, \tilde{\Psi}) = - \frac{1}{N} \sum_{n} \int q_{\Phi}^{(n)}(\tilde{z}) \log (p_{\zeta}(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= - \frac{1}{N} \sum_{n} \int q_{\Phi}^{(n)}(\tilde{z}) \log (\tilde{p}_{\zeta}(\tilde{z})) \, d\tilde{\mu}(\tilde{z}) - \frac{1}{N} \sum_{n} \int q_{\Phi}^{(n)}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= - \frac{1}{N} \sum_{n} \int \tilde{q}_{\Phi}^{(n)}(\tilde{z}) \log (\tilde{p}_{\zeta}(\tilde{z})) \, d\tilde{\mu}(\tilde{z}) - \frac{1}{N} \sum_{n} \int \tilde{q}_{\Phi}^{(n)}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z})
\]
\[
= \tilde{\mathcal{F}}_2(\Phi, \tilde{\Psi}) - \frac{1}{N} \sum_{n} \int \tilde{q}_{\Phi}^{(n)}(\tilde{z}) \log (h(\tilde{z})) \, d\mu(\tilde{z}).
\]

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Consequently, the two expressions $F_1(\Phi) - F_2(\Phi, \Psi)$ and $\tilde{F}_1(\Phi) - \tilde{F}_2(\Phi, \Psi)$ coincide. So, the difference originates from the third summand of the ELBOs:

$$F_3(\Phi, \Theta) = -\frac{1}{N} \sum_n \int q^{(n)}(\tilde{z}) \log (p_{\tilde{\eta}}(\tilde{x}^{(n)})) \, d\mu(\tilde{z})$$

$$= -\frac{1}{N} \sum_n \int q^{(n)}(\tilde{z}) \log (\tilde{p}_{\tilde{\eta}}(\tilde{x}^{(n)})) \, d\mu(\tilde{z})$$

$$- \frac{1}{N} \sum_n \int q^{(n)}(\tilde{z}) \log (h(\tilde{x}^{(n)})) \, d\mu(\tilde{z})$$

$$= -\frac{1}{N} \sum_n \int q^{(n)}(\tilde{z}) \log (\tilde{p}_{\tilde{\eta}}(\tilde{x}^{(n)})) \, d\mu(\tilde{z}) - \frac{1}{N} \sum_n \log (h(\tilde{x}^{(n)}))$$

$$= \tilde{F}_3(\Phi, \Theta) - \frac{1}{N} \sum_n \log (h(\tilde{x}^{(n)})) .$$

If we insert everything into the ELBOs, we finally get:

$$\tilde{\mathcal{F}}(\Phi, \Psi, \Theta) = \tilde{F}_1(\Phi) - \tilde{F}_2(\Phi, \Psi) - \tilde{F}_3(\Phi, \Theta)$$

$$= F_1(\Phi) - F_2(\Phi, \Psi) - F_3(\Phi, \Theta) - \frac{1}{N} \sum_n \log (h(\tilde{x}^{(n)}))$$

$$= F(\Phi, \Psi, \Theta) - \frac{1}{N} \sum_n \log (h(\tilde{x}^{(n)})) .$$

\[ \square \]

### C.3 Theorem 2 as Generalization of Theorem 1

We have not verified that Theorem 2 is a genuine generalization of Theorem 1 yet. In order to do so, let us apply Theorem 2 to a model that also fulfills the conditions for Theorem 1. Concretely, we consider an EF model (Def. B) that fulfills the parameterization condition (Def. C). An EF model has exponential family distributions $p_{\Psi}(\tilde{z})$ and $p_{\Theta}(\tilde{x} | \tilde{z})$ with constant base measures. For such a model, the lower bound used by Theorem 2, i.e. $\tilde{\mathcal{F}}(\Phi, \Psi, \Theta)$, and the lower bound used by Theorem 1, i.e. $\mathcal{F}(\Phi, \Psi, \Theta)$, are related as follows (see Lemma 9):

$$\tilde{\mathcal{F}}(\Phi, \Psi, \Theta) = \mathcal{F}(\Phi, \Psi, \Theta) - \log (h_0) .$$

Analogously, the pseudo entropies are equal to the corresponding standard entropies except of offsets given by the corresponding base measures:

$$H[q^{(n)}(\tilde{z})] = H[q^{(n)}(\tilde{z})] + \log (h_0^{\text{prior}}) ,$$

$$H[p_{\tilde{\eta}}(\tilde{z})] = H[p_{\tilde{\eta}}(\tilde{z})] + \log (h_0^{\text{prior}}) ,$$

$$H[p_{\Theta}(\tilde{x} | \tilde{z})] = H[p_{\Theta}(\tilde{x} | \tilde{z})] + \log (h_0) ,$$

where we have used $h_0^{\text{prior}}$ to distinguish the constant base measure of the prior distribution, $h(\tilde{z}) = h_0^{\text{prior}}$, from the constant base measure of the noise distribution, $h(\tilde{x}) = h_0$.

Now for the considered EF model Theorem 2 applies. If we insert the relations (138) to (141).
into (96) of the theorem, we obtain:

\[ \tilde{F}(\Phi, \vec{\Psi}, \vec{\Theta}) = \frac{1}{N} \sum_{n=1}^{N} \tilde{\mathcal{H}}[q_{\phi}^{(n)}(\vec{z})] - \tilde{\mathcal{H}}[p_{\vec{\Phi}}(\vec{z})] - \mathbb{E}_{\eta_{\Phi}} \{ \tilde{\mathcal{H}}[p_{\vec{\Phi}}(\vec{x}|\vec{z})] \}, \]

\[ = \frac{1}{N} \sum_{n=1}^{N} \mathcal{H}[q_{\phi}^{(n)}(\vec{z})] - \mathcal{H}[p_{\vec{\Phi}}(\vec{z})] - \mathbb{E}_{\eta_{\Phi}} \{ \mathcal{H}[p_{\vec{\Phi}}(\vec{x}|\vec{z})] \} - \log(h_0) \]

\[ \Rightarrow \mathcal{F}(\Phi, \vec{\Psi}, \vec{\Theta}) = \frac{1}{N} \sum_{n=1}^{N} \mathcal{H}[q_{\phi}^{(n)}(\vec{z})] - \mathcal{H}[p_{\vec{\Phi}}(\vec{z})] - \mathbb{E}_{\eta_{\Phi}} \{ \mathcal{H}[p_{\vec{\Phi}}(\vec{x}|\vec{z})] \}, \]  

(142)

i.e., applying Theorem 2 recovers the result of Theorem 1 for any model fulfilling the prerequisites of Theorem 1.

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