THE TORAL CONTRACTIONS AND $\Gamma$-DISTINGUISHED $\Gamma$-CONTRACTIONS

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ABSTRACT. Given a bounded domain $\Omega \subset \mathbb{C}^n$, a distinguished variety in $\Omega$ is a nonempty set $W \cap \Omega$, where $W$ is a complex algebraic variety in $\mathbb{C}^n$ such that $W$ exits the domain $\Omega$ through its distinguished boundary $b\Omega$ without intersecting any other part of its topological boundary $\partial \Omega$, i.e. $W \cap \partial \Omega = W \cap b\Omega$. In such case, the complex algebraic variety $W$ is said to define a distinguished variety in $\Omega$. We consider distinguished varieties in the bidisc $\mathbb{D}^2$ and the symmetrized bidisc $G_2$, where

$$G_2 = \{(z_1 + z_2, z_1z_2) : |z_1| < 1, |z_2| < 1\} \subset \mathbb{C}^2.$$ 

A toral polynomial (or a $\Gamma$-distinguished polynomial) is a polynomial $p \in \mathbb{C}[z_1, z_2]$ whose zero set $Z(p)$ defines a distinguished variety in the bidisc $\mathbb{D}^2$ (or in $G_2$). A commuting pair of Hilbert space operators $(S, P)$ for which $\Gamma = G_2$ is a spectral set is called a $\Gamma$-contraction. A commuting pair of contractions $(T_1, T_2)$ is said to be toral if there is a toral polynomial $p \in \mathbb{C}[z_1, z_2]$ such that $p(T_1, T_2) = 0$. Similarly, a $\Gamma$-contraction $(S, P)$ is called $\Gamma$-distinguished if $(S, P)$ is annihilated by a $\Gamma$-distinguished polynomial in $\mathbb{C}[z_1, z_2]$. The main results of this article are the following: we find a necessary and sufficient condition such that a toral pair of contractions dilates to a toral pair of isometries. Also, we characterize all $\Gamma$-distinguished $\Gamma$-contractions that admit $\Gamma$-distinguished $\Gamma$-isometric dilations. We determine the distinguished boundary of a distinguished variety in $\mathbb{D}^2$ and $G_2$. Then we prove that a pair of commuting contractions $(T_1, T_2)$ (or a $\Gamma$-contraction $(S, P)$) dilates to a toral pair of isometries (or a $\Gamma$-distinguished $\Gamma$-isometry) if and only if there is a toral polynomial (or $\Gamma$-distinguished polynomial) $p \in \mathbb{C}[z_1, z_2]$ such that $Z(p) \cap \mathbb{D}^2$ (or $Z(p) \cap G_2$) is a complete spectral set for $(T_1, T_2)$ (or $(S, P)$). Further, we study when a particular minimal $\Gamma$-isometric dilation $(T_0, V_0)$ of a $\Gamma$-distinguished $\Gamma$-contraction $(S, P)$, where $V_0$ is the minimal isometric dilation of $P$, is annihilated by a $\Gamma$-distinguished polynomial. We obtain decomposition results for $\Gamma$-distinguished pure $\Gamma$-isometries under some hypothesis. We provide several examples at places to show the contrasts between the theory of toral contractions and $\Gamma$-distinguished $\Gamma$-contractions.

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1. Introduction

Throughout the paper, all operators are bounded linear operators acting on complex Hilbert spaces. A contraction is an operator with norm not greater than 1. The space of all operators acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. We shall use the following notations: $\mathbb{C}$ denotes the complex plane, $\mathbb{D}$ stands for the open unit disk $\{z : |z| < 1\}$, $\mathbb{T}$ is the unit circle $\{z : |z| = 1\}$ and $\mathbb{E} = \mathbb{C} \setminus \mathbb{D}$, i.e. the complement of $\mathbb{D}$. We define spectral set, complete spectral set, distinguished boundary and rational dilation in Section 2.

A pair of commuting contractions $(T_1, T_2)$ is said to be toral if $p(T_1, T_2) = 0$ for a polynomial $p$ in $\mathbb{C}[z_1, z_2]$ such that the zero set $Z(p)$ of $p$ intersects the bidisc $\mathbb{D}^2$ and exits through the distinguished boundary of $\overline{\mathbb{D}}^2$, the 2-torus $\mathbb{T}^2$ without intersecting any other part of its topological boundary $\partial \mathbb{D}^2$, that is, $Z(p) \cap \mathbb{D}^2 \neq \emptyset$ and $Z(p) \cap \partial \mathbb{D}^2 = Z(p) \cap \mathbb{T}^2$. In the literature [5], such a set $Z(p) \cap \mathbb{D}^2$ is called a distinguished variety in $\mathbb{D}^2$. Also, a polynomial $p$ in $\mathbb{C}[z_1, z_2]$ for which $Z(p) \cap \mathbb{D}^2$ is a distinguished variety in $\mathbb{D}^2$ is called a toral polynomial. Ando’s celebrated theorem [11] tells us that every commuting pair of contractions dilates to a commuting pair of isometries. Thus, it is naturally asked if a toral pair of contractions dilates to a toral pair of isometries. Indeed, it remains an open problem for quite sometime, e.g. see [20]. In this article, we address this problem by finding a necessary and sufficient condition and thus our first main result is the following.

Theorem 1.1. Let $(T_1, T_2)$ be a toral pair of commuting contractions acting on a Hilbert space $\mathcal{H}$. Then $(T_1, T_2)$ dilates to a toral pair of commuting isometries if and only if there is a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, a toral pair of isometries $(D_1, D_2)$ on $\mathcal{H}^\perp = \mathcal{K} \ominus \mathcal{H}$ and $C_1, C_2 \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\perp)$ such that the following hold:

(1) $C_1T_2 + D_1C_2 = C_2T_1 + D_2C_1$
(2) $C_1^*D_1 = C_2^*D_2 = 0$
(3) $C_i^*C_i = D_i^2$ for $i = 1, 2$.

Moreover, if $f$ and $g$ are toral polynomials such that $f(T_1, T_2) = g(D_1, D_2) = 0$, then $Z(fg) \cap \overline{\mathbb{D}}^2$ is a complete spectral set for $(T_1, T_2)$.

Needless to mention that Theorem 1.1 characterizes the class of toral pair of contractions that dilate to toral pair of isometries. We prove this theorem in Section 7. In Section 6 we invoke the theory of analytic variety in a domain (e.g. see [23, 28]) to show that the distinguished boundary of a distinguished variety $Z(p) \cap \mathbb{D}^2$ is equal to $Z(p) \cap \mathbb{T}^2$ and by an application of this we prove in Theorem 7.8 that a pair of commuting contractions $(T_1, T_2)$ dilates to a toral pair of isometries (or...
unitaries) if and only if there is a toral polynomial \( q \) so that \( Z(q) \cap \overline{\mathbb{D}}^2 \) is a complete spectral set for \((T_1, T_2)\). This is the second main result of this paper. As a consequence, we obtain characterizations for toral pair of commuting isometries in Theorem 7.9.

We intend to establish analogous results for the symmetrized bidisc. Indeed, this paper is more of a study of the symmetrized bidisc than the bidisc. The symmetrized bidisc \( G_2 \) is a non-convex but polynomially convex domain in \( \mathbb{C}^2 \) defined by

\[
G_2 := \{(tr(A), det(A)) : A = [a_{ij}]_{2 \times 2}, \|A\| < 1\} \subseteq \mathbb{C}^2.
\]

The symmetrized bidisc has its origin in \( 2 \times 2 \) spectral Nevanlinna-Pick interpolation. The general \( n \times n \) spectral Nevanlinna-Pick interpolation problem states the following: given distinct points \( z_1, \ldots, z_n \) in \( \mathbb{D} \) and \( n \times n \) matrices \( F_1, \ldots, F_n \) in the spectral unit ball \( \Omega_n \) of the space of \( n \times n \) matrices \( \mathcal{M}_n(\mathbb{C}) \), whether or under what conditions it is possible to find an analytic function \( f : \mathbb{D} \rightarrow \Omega_n \) such that \( f(z_i) = F_i, \ i = 1, \ldots, n \). It is obvious that a \( 2 \times 2 \) matrix \( F \) is in \( \Omega_2 \) if and only if its eigenvalues \( \mu_1, \mu_2 \) are in \( \mathbb{D} \) and this happens if and only if \((tr(F), det(F)) = (\mu_1 + \mu_2, \mu_1 \mu_2) \) belongs to \( G_2 \). The symmetrization map \( \pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) is defined by \( \pi(z_1, z_2) = (z_1 + z_2, z_1 z_2) \). The symmetrized bidisc \( G_2 \) and its closure \( \Gamma \), turn out to be the images of the bidisc \( \mathbb{D}^2 \) and its closure \( \overline{\mathbb{D}}^2 \) respectively under \( \pi \), that is,

\[
G_2 = \pi(\mathbb{D}^2) = \{(z_1 + z_2, z_1 z_2) : |z_1| < 1, |z_2| < 1\},
\]

\[
\Gamma = \overline{G}_2 = \pi(\overline{\mathbb{D}}^2) = \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1, |z_2| \leq 1\}.
\]

The main motivation behind studying the symmetrized bidisc is that the \( 2 \times 2 \) Nevanlinna-Pick interpolation problem reduces to a similar interpolation problem of \( G_2 \) in the following way:

**Proposition 1.2** ([33], Proposition 1.1). Let \( \alpha_1, \ldots, \alpha_n \) be distinct points in \( \mathbb{D} \) and let \( F_1 = \lambda_1 I, \ldots, F_k = \lambda_k I \in \Omega_2 \) be scalar matrices. Also let \( F_{k+1}, \ldots, F_n \in \Omega_2 \) be non-scalar matrices. Suppose \( \phi = (\phi_1, \phi_2) : \mathbb{D} \rightarrow G_2 \) is a holomorphic map such that \( \phi(\alpha_j) = \sigma(F_j) = (tr(F_j), det(F_j)) \) for \( j = 1, \ldots, n \). Then there exists a holomorphic map \( \psi : \mathbb{D} \rightarrow \Omega_2 \) satisfying \( \phi = \sigma \circ \psi \) and \( \psi(\alpha_j) = F_j \) for \( j = 1, \ldots, n \) if and only if \( \phi'_i(\alpha_j) = \lambda_j \phi'_i(\alpha_j) \) for \( j = 1, \ldots, k \).

Obviously a bounded domain like \( G_2 \), which has complex-dimension 2, is much easier to deal with a norm-unbounded object like \( \Omega_2 \) which has complex-dimension 4. On the other hand, \( G_2 \) was the first example of a non-convex domain in which the Caratheodory and Kobayashi distances coincide (see [31]). The symmetrized bidisc has a rich literature from complex geometric, function theoretic and operator theoretic points of view, e.g. see [11, 2, 3, 4, 14, 15, 33] and the references therein.

It follows from Ando’s theorem [11] that a pair of commuting operators \( T_1, T_2 \) are contractions if and only if \( \overline{\mathbb{D}}^2 \) is a spectral set for \((T_1, T_2)\). This leads to considering a commuting pair of operators having \( \Gamma \) as a spectral set.

**Definition 1.3.** A commuting pair of operators \((S, P)\) acting on a Hilbert space \( \mathcal{H} \) is said to be 

- (i) a \( \Gamma \)-contraction if \( \Gamma \) is a spectral set for \((S, P)\), that is, the Taylor joint spectrum \( \sigma_T(S, P) \subseteq \Gamma \) and von Neumann’s inequality

\[
\|f(S, P)\| \leq \sup_{(x, p) \in \Gamma} |f(s, p)| = \|f\|_{\infty, \Gamma}
\]

holds for all rational functions \( f = p/q \) with \( p, q \in \mathbb{C}[z_1, z_2] \) and \( q \) having no zeros in \( \Gamma \);

- (ii) a \( \Gamma \)-unitary if \( S, P \) are normal operators and \( \sigma_T(S, P) \subseteq b\Gamma \).
(iii) a $\Gamma$-isometry if there is a Hilbert space $\mathcal{H} \supseteq \mathcal{H}$ and a $\Gamma$-unitary $(\tilde{S}, \tilde{P})$ acting on $\mathcal{H}$ such that $\mathcal{H}$ is a common invariant subspace for $S, P$ and that $S = \tilde{S}|_{\mathcal{H}}, P = \tilde{P}|_{\mathcal{H}}$;

(iv) a pure $\Gamma$-contraction if $(S, P)$ is a $\Gamma$-contraction and $P$ is a pure contraction, i.e. $P^n \to 0$ strongly as $n \to \infty$.

The primary objects of study in this article are distinguished variety, polynomial defining a distinguished variety in $\mathbb{D}^2$ and $\mathbb{G}_2$ and operator pairs annihilated by polynomials. A commuting operator pair $(T_1, T_2)$ that is annihilated by a polynomial $q \in \mathbb{C}[z_1, z_2]$, i.e. satisfying $q(T_1, T_2) = 0$ is called an algebraic pair. Below we define a distinguished variety in a domain in $\mathbb{C}^n$.

**Definition 1.4.** Given a bounded domain $\Omega$ in $\mathbb{C}^n$, a nonempty set $V \subseteq \Omega$ is said to be a distinguished variety in $\Omega$ if there is an algebraic variety $W \subseteq \mathbb{C}^n$ such that $V = W \cap \Omega$ and $W \cap \partial \overline{\Omega} = W \cap b\Omega$, where $b\Omega$ is the distinguished boundary of $\overline{\Omega}$.

The distinguished varieties in domains like bidisc, symmetrized bidisc, tetrablock or even more generally in polydisc and symmetrized polydisc have been extensively studied in past two decades, e.g. see [5] [8] [13] [19] [20] [21] [27] [38] [54] [56] [57]. One of the most pioneering works in operator theory is Andô’s inequality [11], which states that if $(T_1, T_2)$ is a commuting pair of contractions, then for every polynomial $p \in \mathbb{C}[z_1, z_2]$,

$$
\|p(T_1, T_2)\| \leq \sup\{|p(z_1, z_2)| : (z_1, z_2) \in \overline{\mathbb{D}}^2\}.
$$

The main motivation behind studying distinguished varieties is the following improvement of Andô’s inequality by Agler and McCarthy.

**Theorem 1.5** (Agler and McCarthy, [5]). Let $T_1, T_2$ be two commuting contractive matrices, neither of which has an eigenvalue of unit modulus. Then there is a distinguished variety $V$ in $\mathbb{D}^2$ such that

$$
\|p(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in V} |p(z_1, z_2)| \quad \text{for every} \quad p \in \mathbb{C}[z_1, z_2].
$$

A similar result for the symmetrized bidisc was proved by Pal and Shalit in [38]. Also, an explicit description of all distinguished varieties in the bidisc and the symmetrized bidisc were given in [5] and [38] respectively. To go parallel with the bidisc, we shall use the following terminologies for the symmetrized bidisc.

**Definition 1.6.** A polynomial $p \in \mathbb{C}[z_1, z_2]$ is said to be $\Gamma$-distinguished if its zero set $Z(p)$ defines a distinguished variety with respect to $\mathbb{G}_2$, i.e. $Z(p) \cap \mathbb{G}_2 \neq \emptyset$ and $Z(p) \cap \partial \Gamma = Z(p) \cap b\Gamma$. Also, a $\Gamma$-contraction $(S, P)$ is called $\Gamma$-distinguished if it is annihilated by a $\Gamma$-distinguished polynomial, i.e. there is a $\Gamma$-distinguished polynomial $q \in \mathbb{C}[z_1, z_2]$ such that $q(S, P) = 0$.

Every $\Gamma$-contraction dilates to a $\Gamma$-isometry as was shown in [11] and [14]. Below we present a necessary and sufficient condition such that a $\Gamma$-distinguished $\Gamma$-contraction dilates to a $\Gamma$-distinguished $\Gamma$-isometry. This is an analogue of Theorem 1.1 for the symmetrized bidisc and is another main result of this paper.

**Theorem 1.7.** Let $(S, P)$ be a $\Gamma$-distinguished $\Gamma$-contraction acting on a Hilbert space $\mathcal{H}$. Then $(S, P)$ dilates to a $\Gamma$-distinguished $\Gamma$-isometry if and only if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, a $\Gamma$-distinguished $\Gamma$-isometry $(D_1, D_2)$ on $\mathcal{H}^\perp \equiv \mathcal{K} \ominus \mathcal{H}$ and $C_1, C_2 \in \mathcal{B}(\mathcal{H}, \mathcal{K}^\perp)$ such
that the following hold.

\begin{align}
(1) \quad & C_1^*P + D_1C_2 = C_2S + D_2C_1, \\
(2) \quad & S - S^*P = C_1^*C_2, \\
(3) \quad & C_2^*D_2 = 0, \\
(4) \quad & C_2^*C_2 = D_2^2, \\
(5) \quad & C_1 = D_1^*C_2.
\end{align}

Moreover, if \( f \) and \( g \) are \( \Gamma \)-distinguished polynomials that annihilate \((S, P)\) and \((D_1, D_2)\) respectively, then \( Z(fg) \cap \Gamma \) is a complete spectral set for \((S, P)\).

We prove this theorem in Section 8. In Section 6, we determine the distinguished boundary of a distinguished variety \(Z(p) \cap \Gamma\) in the symmetrized bidisc. More precisely, we show that \( b(Z(p) \cap \Gamma) = Z(p) \cap b\Gamma\). Being equipped with this, we prove in Theorem 8.6 that a \( \Gamma \)-contraction \((S, P)\) dilates to a \( \Gamma \)-distinguished \( \Gamma \)-isometry if and only if there is a \( \Gamma \)-distinguished polynomial \( p \) such that \( Z(p) \cap \Gamma \) is a complete spectral set for \((S, P)\). This is another main result of this paper. It is well-known that a \( \Gamma \)-isometry naturally extends to a \( \Gamma \)-unitary. Interestingly, we shall see in Section 8 that every \( \Gamma \)-distinguished \( \Gamma \)-isometry extends to a \( \Gamma \)-distinguished \( \Gamma \)-unitary.

Note that a \( \Gamma \)-contraction \((S, P)\) that admits a dilation to a \( \Gamma \)-distinguished \( \Gamma \)-unitary (or \( \Gamma \)-isometry) \((T, U)\) must be \( \Gamma \)-distinguished. Indeed, if \( p \) is a \( \Gamma \)-distinguished polynomial such that \( p(T, U) = 0 \), then \( p(S, P) = P_{\mathcal{H}} p(T, U) |_{\mathcal{H}} = 0 \). In Examples 4.2 \& 9.8, we construct \( \Gamma \)-contractions that are not \( \Gamma \)-distinguished. Thus, passing through several examples we establish in Section 9 the following:

(i) not every \( \Gamma \)-contraction dilates to a \( \Gamma \)-distinguished \( \Gamma \)-unitary (or \( \Gamma \)-isometry);

(ii) a \( \Gamma \)-contraction can dilate to two different \( \Gamma \)-unitaries (or \( \Gamma \)-isometries) of which one is \( \Gamma \)-distinguished and the other is not.

Also, we show that these two classes of commuting operators, i.e. the \( \Gamma \)-distinguished \( \Gamma \)-contractions and toral pair of contractions can exhibit behaviours that are in stark contrast with each other. For example, Agler, Knese and McCarthy [8] found the following necessary condition for an algebraic pair of commuting pure isometries.

**Theorem 1.8** ([8], Theorem 1.8). For every algebraic pair of commuting pure isometries \((V_1, V_2)\), there is a polynomial \( q \in \mathbb{C}[z_1, z_2] \) such that \( Z(q) \subseteq \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2 \) and \( q(V_1, V_2) = 0 \).

Such a polynomial \( q \) is referred to as an inner toral polynomial in [8]. These polynomials have been further studied in detail by Knese in [27]. Evidently, an inner toral polynomial \( q \) satisfies \( Z(q) \cap \partial \mathbb{D}^2 = Z(q) \cap \mathbb{T}^2 \). Thus multiplication of an inner toral polynomial \( q \) with any toral polynomial \( q_0 \) makes \( q q_0 \) a toral polynomial. It then follows from Theorem 1.8 that every algebraic pair of commuting pure isometries is a toral pair. An analogue of this result fails for the symmetrized bidisc. In Example 4.2 we construct an algebraic pure \( \Gamma \)-isometry that cannot be annihilated by a \( \Gamma \)-distinguished polynomial. The underlying reason is that not every (pure) \( \Gamma \)-isometry can be written as the symmetrization of any commuting pair of (pure) isometries. In Section 4, we characterize all \( \Gamma \)-isometries which arise as the symmetrization of a commuting pair of isometries. Also, in Section 5 we find a necessary and sufficient condition with an additional hypothesis such that an algebraic pure \( \Gamma \)-isometry becomes \( \Gamma \)-distinguished. An explicit construction of a minimal \( \Gamma \)-isometric dilation for a \( \Gamma \)-contraction was shown in Theorem 4.3 in [14]. For that purpose, the notion of the fundamental operator of a \( \Gamma \)-contraction was introduced. In Proposition 5.12
we prove that $\Gamma$-contractions on a finite-dimensional Hilbert space are $\Gamma$-distinguished if the fundamental operator has numerical radius strictly less than 1. We focus on the particular minimal $\Gamma$-isometric dilation constructed in [14] in Section 9. One can ask if this minimal $\Gamma$-isometric dilation of a $\Gamma$-distinguished $\Gamma$-contraction $(S, P)$ is also $\Gamma$-distinguished. Example [9,8] shows that this is not true in general. In Section 9 we provide a necessary and sufficient condition such that this minimal $\Gamma$-isometric dilation of $(S, P)$ is $\Gamma$-distinguished when the fundamental operator of $(S, P)$ is hyponormal. In Section 10 we prove decomposition results for a particular class of $\Gamma$-distinguished pure $\Gamma$-isometry. In Section 2 we accumulate a few useful results from the literature.

2. Preliminaries and a brief literature on $\Gamma$-contractions

In this Section, we recall a few basic concepts from the literature. These facts will be used frequently throughout the paper. We begin with the definition of spectral and complete spectral set.

2.1. Spectral set and complete spectral set. For a compact subset $X$ of $\mathbb{C}^n$, let $\text{Rat}(X)$ be the algebra of rational functions $p/q$, where $p, q \in \mathbb{C}[z_1, \ldots, z_n]$ such that $q$ does not have any zeros in $X$. Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a commuting tuple of operators on a Hilbert space $\mathcal{H}$. The set $X \subseteq \mathbb{C}^n$ is said to be a spectral set for $\mathcal{T}$ if the Taylor joint spectrum $\sigma_{\mathcal{T}}(\mathcal{T})$ of $\mathcal{T}$ is contained in $X$ and

$$\|f(\mathcal{T})\| \leq \|f\|_{\infty, X} = \sup\{|f(\xi)| : \xi \in X\}$$ (2.1)

for every $f \in \text{Rat}(X)$. If (2.1) holds for every matricial rational function $F = [f_{ij}]_{m \times m}$, then $X$ is said to be a complete spectral set for $\mathcal{T}$. Note that for a matricial rational function $F = [f_{ij}]_{m \times m}$, where each $f_{ij} \in \text{Rat}(X)$, we denote by $F(T_1, \ldots, T_n)$ the block matrix of operators $[f_{ij}(T_1, \ldots, T_n)]_{m \times m}$ and in this case the right hand side of the inequality in (2.1) is the following:

$$\|F\|_{\infty, X} = \sup\{|\|f_{ij}(\xi)\|_{m \times m}| : \xi \in X\}.$$

2.2. Distinguished boundary and rational dilation. Let $X \subseteq \mathbb{C}^n$ be a compact set. A boundary for $X$ is a closed subset $C$ of $X$ such that every function in $\text{Rat}(X)$ attains its maximum modulus on $C$. It follows from the theory of uniform algebras that the intersection of all the boundaries of $X$ is also a boundary of $X$ and it is the smallest among all boundaries. This is called the distinguished boundary of $X$ and is denoted by $bX$. For a bounded domain $\Omega \subseteq \mathbb{C}^n$, we denote by $b\Omega$ or $b\Omega$, the distinguished boundary of $\Omega$, and for the sake of simplicity sometimes we call it just the distinguished boundary of $\Omega$.

Let $X$ be a spectral set for a commuting $n$-tuple $\mathcal{T} = (T_1, \ldots, T_n)$. Then $\mathcal{T}$ is said to have a rational dilation or normal $bX$-dilation if there exist a Hilbert space $\mathcal{K}$, an isometry $V : \mathcal{K} \rightarrow \mathcal{K}$ and a commuting $n$-tuple of normal operators $\mathcal{N}$ on $\mathcal{K}$ with $\sigma_{\mathcal{T}}(\mathcal{N}) \subseteq bX$ such that

$$f(\mathcal{T}) = V^*f(\mathcal{N})V$$ (2.2)

for all $f \in \text{Rat}(X)$, or simply $f(\mathcal{T}) = P_{\mathcal{K}}f(\mathcal{N})|_{\mathcal{K}}$ for every $f \in \text{Rat}(X)$ when $\mathcal{H}$ is realized as a closed linear subspace of $\mathcal{K}$ and $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto the space $\mathcal{H}$. The dilation is said to be minimal when

$$\mathcal{K} = \text{span}\{f(\mathcal{N})h : h \in \mathcal{H} \text{ and } f \in \text{Rat}(X)\}.$$

The following two elementary results are well-known and are useful in our context.

**Lemma 2.1.** Let $(T_1, \ldots, T_n)$ be a commuting tuple of normal operators on a Hilbert space $\mathcal{H}$. Then a compact set $K$ is a spectral set for $(T_1, \ldots, T_n)$ if and only if $\sigma_{\mathcal{T}}(T_1, \ldots, T_n) \subseteq K$. 

One can find a proof to this result in the literature, e.g. see [41] or Lemma 5.8 in [36]. Also, the following proposition can be located as Lemma 3.12 in [36].

**Proposition 2.2.** Let $X$ be a compact polynomially convex set in $\mathbb{C}^n$ and let $T = (T_1, \ldots, T_n)$ be a commuting tuple of operators for which $X$ is a spectral set. Then $T$ admits a rational dilation if and only if there exist a Hilbert space $\mathcal{H}$, an isometry $V : \mathcal{H} \to \mathcal{H}$ and a commuting $n$-tuple of normal operators $N$ on $\mathcal{H}$ with $\sigma_T(N) \subseteq bX$ such that

$$p(T) = V^* p(N)V$$

(2.3)

for every polynomial $p$ in $\mathbb{C}[z_1, \ldots, z_n]$.

Naturally, our curiosity extends to ask how these classical concepts (i.e. spectral set, complete spectral set and rational dilation) are related with each other. The following famous theorem due to Arveson combines them beautifully.

**Theorem 2.3** (Arveson, [12]). Let $T = (T_1, \ldots, T_n)$ be a tuple of commuting operators on a Hilbert space $\mathcal{H}$ for which a compact set $X \subseteq \mathbb{C}^n$ is a spectral set. Then $X$ is a complete spectral set for $T$ if and only if $T$ has a normal $bX$-dilation.

### 2.3. Operators associated with the symmetrized bidisc

Recall that a $\Gamma$-contraction is a commuting pair of operators for which the closed symmetrized bidisc $\Gamma$ is a spectral set. Here we outline a few basic facts about the $\Gamma$-contractions from the literature (e.g. [1, 2, 14]). These results will be frequently used in sequel. We begin with the scalar case.

**Theorem 2.4** ([2], Theorem 1.1). Let $(s, p) \in \mathbb{C}^2$. The following are equivalent:

(a) $(s, p) \in \Gamma$;

(b) $|s - \overline{sp}| + |p|^2 \leq 1$ and $|s| \leq 2$;

(c) $2|s - \overline{sp}| + |s^2 - 4p| + |s|^2 \leq 4$;

(d) $|p| \leq 1$ and there exists $\beta \in \mathbb{D}$ such that $s = \beta + \overline{\beta}p$.

The distinguished boundary $b\Gamma$ of $\Gamma$ is the following set:

$$b\Gamma = \{(z_1 + z_2, z_1\overline{z_2}) : |z_1| = |z_2| = 1\}.$$

The following theorem from [2] describes a few characterizations of the distinguished boundary of $\Gamma$.

**Theorem 2.5.** Let $(s, p) \in \mathbb{C}^2$. Then the following are equivalent:

(a) $(s, p) \in b\Gamma$;

(b) $s = \overline{sp}, |s| \leq 2$ and $|p| = 1$;

(c) $|p| = 1$ and there exists $\beta \in \mathbb{T}$ such that $s = \beta + \overline{\beta}p$.

We have several characterizations of a $\Gamma$-contraction from the literature. To understand them, we recall (see Section I.3 of [12]) that $D_P = (I - P^*P)^{\frac{1}{2}}$ and $\mathcal{D}_P = \text{Ran} D_P$ are the defect operator and defect space respectively of a contraction $P$ acting on a Hilbert space $\mathcal{H}$. Given a Hilbert space operator $T$, the numerical radius of $T$ is denoted by $\omega(T)$.

**Theorem 2.6** ([11], Theorem 1.2 & [14], Theorem 4.3). Let $(S, P)$ be a pair of commuting operators on a Hilbert space $\mathcal{H}$. Then the following are equivalent.

(1) $(S, P)$ is a $\Gamma$-contraction;

(2) $\sigma_T(S, P) \subseteq \Gamma$ and $\rho(\alpha S, \alpha^2 P) \geq 0$, for all $\alpha \in \mathbb{D}$, where

$$\rho(S, P) = 2(I - P^*P) - (S - S^*P) - (S^* - P^*S);$$
(3) \((S, P)\) has normal \(b\Gamma\)-dilation;
(4) \(|S| \leq 2, |P| \leq 1\) and the operator equation \(S - S^* P = D_P XD_P\) has a unique solution \(A\) in \(\mathcal{B}(\mathcal{D}_P)\) with \(\omega(A) \leq 1\).

We have the following characterizations of a \(\Gamma\)-unitary from the literature.

**Theorem 2.7** ([2], Theorem 2.2). Let \((S, P)\) be a pair of commuting operators on a Hilbert space \(\mathcal{H}\). Then the following are equivalent.

1. \((S, P)\) is a \(\Gamma\)-unitary;
2. \(S^* P = S\) and \(P\) is unitary and \(|S| \leq 2\);
3. there exists commuting unitaries \(U_1, U_2\) on \(\mathcal{H}\) such that \(S = U_1 + U_2\) and \(P = U_1 U_2\).

Also, the following theorem provides various characterizations of \(\Gamma\)-isometry.

**Theorem 2.8** ([2], Theorem 2.6). Let \((S, P)\) be a pair of commuting operators on a Hilbert space \(\mathcal{H}\). Then the following are equivalent.

1. \((S, P)\) is a \(\Gamma\)-isometry;
2. \(S^* P = S\) and \(P\) is isometry and \(|S| \leq 2\);
3. there exists an orthogonal decomposition \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\) into joint reducing subspaces of \(S\) and \(P\) such that \((S|_{\mathcal{H}_1}, P|_{\mathcal{H}_1})\) is a \(\Gamma\)-unitary and \((S|_{\mathcal{H}_2}, P|_{\mathcal{H}_2})\) is a pure \(\Gamma\)-isometry.

### 3. The algebraic pairs

Recall that an algebraic pair is a commuting pair of operators \((T_1, T_2)\) that is annihilated by a polynomial \(p\) in \(\mathbb{C}[z_1, z_2]\). We begin this Section with various notions of algebraic pairs associated with the bidisc and the symmetrized bidisc. However, the main aim of this Section is to study an algebraic pair associated with the symmetrized bidisc, namely the algebraic pure \(\Gamma\)-isometries. Needless to mention that an algebraic pure \(\Gamma\)-isometry \((S, P)\) is an algebraic pair and a \(\Gamma\)-isometry such that \(P\) is a pure isometry i.e. an isometry with \(P^n \to 0\) strongly as \(n \to \infty\). We recall from the ‘Introduction’ that a polynomial \(p \in \mathbb{C}[z_1, z_2]\) is said to be toral or \(\Gamma\)-distinguished if its zero set defines a distinguished variety in the bidisc or the symmetrized bidisc respectively.

**Note:** We want to emphasize that the term toral polynomial is used in two different contexts in the literature. One definition is attributed to Agler et al. [7], while another one appears in the work [20] by Das et al. To avoid any confusion with the existing literature, we briefly discuss these two definitions here. Let \(p \in \mathbb{C}[z_1, z_2]\). Then \(p\) is called toral (in the sense of Agler et al.) if \(T^2\) is a determining set for the zero set \(Z(p)\) of \(p\), i.e. if \(f\) is a holomorphic function on \(Z(p)\) and \(f|_{Z(p) \cap T^2} = 0\), then \(f = 0\) on \(Z(p)\). On the other hand, if \(Z(p)\) defines a distinguished variety with respect to \(D^2\), then \(p\) is called toral (in the sense of Das et al.). Let us consider, the polynomial \(p(z_1, z_2) = z_1 - z_2\) whose zero set is \(Z(p) = \{(z, z) : z \in \mathbb{C}\}\). Then it is not difficult to see that \(p\) is toral in the sense of Das and Sau. Moreover, if \(f\) is holomorphic on \(Z(p)\), then the map \(\tilde{f} : \mathbb{C} \to \mathbb{C}, \tilde{f}(z) = f(z, z)\) is an entire function. Let \(f|_{Z(p) \cap T^2} = 0\). Then \(\tilde{f} = 0\) on \(T\) and, by identity theorem, \(\tilde{f} = 0\). Consequently, \(f = 0\) on \(Z(p)\). Thus, \(p\) is also toral in the sense of Agler and co-authors. Hence, these two classes of polynomials have a non-empty intersection. However, they are not same, i.e. these two notions of toral polynomials do not coincide. The polynomial \(q(z_1, z_2) = 1 - z_1 z_2\) is not toral in the sense of Das and Sau since \(Z(q) \cap D^2 = \emptyset\). Since \(Z(q)\) is disjoint from \(D^2 \cup (\mathbb{C} \setminus T^2)\), it follows from Theorem 3.5 in [7] that \(p\) is toral in the sense of Agler...
and co-authors. For our results in this article, we adopt the definition of toral polynomial in the sense of Das and Sau as in [20].

The following important class of polynomials appeared in the literature [8, 27].

**Definition 3.1.** A polynomial \( q \in \mathbb{C}[z_1, z_2] \) is said to be inner toral if \( Z(q) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2 \).

Needless to mention, every inner toral polynomial \( q \) satisfies \( Z(q) \cap \partial \mathbb{D}^2 = Z(q) \cap \mathbb{T}^2 \). Thus, an inner toral polynomial is very close to being a toral polynomial. Indeed, the product of an inner toral polynomial with a toral polynomial gives a toral polynomial. However, we mention that not every toral polynomial is inner toral as the following example from [20] clarifies.

**Example 3.2.** Let us consider \( q(z_1, z_2) = z_1 z_2 - 1 \). Then \( Z(q) \cap \partial \mathbb{D}^2 \subset \mathbb{T}^2 \). Therefore, the polynomial \( q_0(z_1, z_2) = (z_1 + z_2) q(z_1, z_2) \) is not inner toral.

In the spirit of inner toral polynomials, we define its analog for the symmetrized bidisc.

**Definition 3.3.** A polynomial \( p \in \mathbb{C}[z_1, z_2] \) is said to be distinguished if \( Z(p) \) is a subset of \( \mathbb{G}_2 \cup b \Gamma \cup \pi(\mathbb{E}^2) \).

One can construct distinguished and \( \Gamma \)-distinguished polynomials from inner toral polynomials and toral polynomial respectively via symmetrization. The subsequent example explains this.

**Example 3.4.** Let \( q(z_1, z_2) \) be a toral polynomial. Evidently, the symmetric polynomial \( q_s(z_1, z_2) = q(z_1, z_2) q(z_2, z_1) \) is also toral. Since \( q_s \) is a symmetric polynomial, there exists \( p \in \mathbb{C}[z_1, z_2] \) such that \( q_s = p \circ \pi \). It can be easily verified that \( p \) is a \( \Gamma \)-distinguished polynomial.

Conversely, if \( p \) is a \( \Gamma \)-distinguished polynomial, then the polynomial \( q = p \circ \pi \) is symmetric and toral as the following example shows.

**Example 3.5.** The polynomial \( q(z_1, z_2) = z_1 - z_2 \) is a toral polynomial whose zero set has empty intersection with \( \partial \mathbb{D}^2 \setminus \mathbb{T}^2 \). Evidently, \( q_s(z_1, z_2) = (z_1 - z_2)(z_2 - z_1) \) is a symmetric toral polynomial with \( Z(q_s) \cap \mathbb{D}^2 \neq \emptyset \). Thus, the polynomial \( \tilde{q} \) satisfying \( q_s = \tilde{q} \circ \pi \) is \( \Gamma \)-distinguished. Some routine calculations yield that \( \tilde{q}(z_1, z_2) = 4z_2 - z_1^2 \) is \( \Gamma \)-distinguished.

In this Section, we prove the existence of a square-free minimal annihilating polynomial for any algebraic pure \( \Gamma \)-isometry. One can ask if the zero set of this minimal polynomial is a distinguished variety in \( \mathbb{G}_2 \). We will see in the next Section that it is not true. We begin with the following notion.

**Definition 3.6.** A \( \Gamma \)-isometry \((S, P)\) is called cyclic if there exists \( u \in \mathcal{H} \) such that the set

\[
\mathbb{C}[S, P] u := \{ q(S, P) u : q \in \mathbb{C}[z_1, z_2] \}
\]

is dense in \( \mathcal{H} \). Such a vector \( u \in \mathcal{H} \) is called a cyclic vector for the cyclic \( \Gamma \)-isometry \((S, P)\).

Given a contraction \( T \) acting on a Hilbert space \( \mathcal{H} \), it is well-known that \( T \mathcal{D}_T = \mathcal{D}_T \cdot T \) (see Section 3 in [32] for details). As a consequence, we have \( \mathcal{T}_T \subset \mathcal{D}_T \) and \( T^* \mathcal{D}_T \subset \mathcal{D}_T \). More precisely, the following relation holds:

\[
\mathcal{D}_T = \overline{T \mathcal{D}_T} + \ker T^*.
\]

Indeed, for any \( x \in \ker T^* \), we have \( x = x - TT^* x = D_T^2 x \). Hence, \( \ker T^* \subset \mathcal{D}_T \). On the other hand, for every \( x \in \mathcal{H} \) and \( y \in \ker T^* \), we have \( \langle T \mathcal{D}_T x, y \rangle = \langle \mathcal{D}_T x, T^* y \rangle = 0 \). This shows that \( \ker T^* \) is orthogonal to \( T \mathcal{D}_T \). Now, for any \( g \in (T \mathcal{D}_T)^\perp \subset \mathcal{D}_T \), we have \( T^* g \perp \mathcal{D}_T \). Again,
Proof. Let \( g \in \mathcal{D}_T^* \) and \( T^*D_{T^*} = D_T T^* \) imply that \( T^*g \in T^*\mathcal{D}_T^* \subseteq \mathcal{D}_T \). Therefore, \( T^*g = 0 \) and hence \( g \in \ker T^* \) which proves that \( (T\mathcal{D}_T) \perp \ker T^* \). So, we have (3.1).

The following lemma shows how an annihilating polynomial gives an estimate of the dimension of the defect space of \( P^* \) for a cyclic \( \Gamma \)-isometry \( (S,P) \). In this connection, we say that a polynomial \( q \in \mathbb{C}[z_1,z_2] \) has degree \((n,m)\) if it has degree \( n \) in \( z_1 \) and \( m \) in \( z_2 \).

**Lemma 3.7.** If \((S,P)\) is a cyclic \( \Gamma \)-isometry on a Hilbert space \( \mathcal{H} \) satisfying \( q(S,P) = 0 \), where \( q \) has degree \((n,m)\), then \( \dim \mathcal{D}_{P^*} \leq n \).

**Proof.** Let \( u \in \mathcal{H} \) be such that \( \mathcal{H} = \text{span}\{p(S,P)u : p \in \mathbb{C}[z_1,z_2]\} \). Since \( P \) is an isometry, we have \( D_P = 0 \). Thus, it follows from (3.1) that \( \mathcal{D}_{P^*} = \ker P^* \). If possible let \( \dim \ker P^* > n \). We can choose some non-zero \( h \in \ker P^* \) that is orthogonal to \( S^i u \) for \( i = 0,1,\ldots,n-1 \). Now, observe that \( 0 = q(S,P)^* h = q(S,0)^* h \). Let \( p \in \mathbb{C}[z_1,z_2] \) and write \( p(z_1,0) = f(z_1)q(z_1,0) + g(z_1) \), where \( g \) has degree less than \( n \). Then,

\[
p(S,P)^* h = p(S,0)^* h = f(S)^* q(S,0)^* h + g(S)^* h = g(S)^* h.
\]

Again, since \( h \) is orthogonal to \( S^i u \) for \( i = 0,1,\ldots,n-1 \) and degree of \( g < n \), we have that

\[
\langle p(S,P)u,h \rangle = \langle g(S)u,h \rangle = 0,
\]

for any \( p \in \mathbb{C}[z_1,z_2] \). Since \( u \) is a cyclic vector, we have that \( h = 0 \) and this leads to a contradiction. Hence, \( \dim \mathcal{D}_{P^*} = \dim (\ker P^*) \leq n \) and the proof is complete.

For a Hilbert space \( \mathcal{H} \), the vectorial Hardy-Hilbert space \( H^2(\mathcal{H}) \) consists of all holomorphic functions from \( \mathbb{D} \) to the Hilbert space \( \mathcal{H} \) with square summable coefficients, that is

\[
H^2(\mathcal{H}) = \left\{ \sum_{n=0}^{\infty} x_n z^n : z \in \mathbb{D}, x_n \in \mathcal{H} \text{ and } \sum_{n=0}^{\infty} \|x_n\|^2 < \infty \right\}.
\]

The **Toeplitz operator** with symbol \( \phi \), denoted by \( T_\phi \), is defined for any bounded analytic function \( \phi : \mathbb{D} \to \mathcal{B}(\mathcal{H}) \) as the multiplication by \( \phi \) on \( H^2(\mathcal{H}) \), i.e. \( T_\phi (f)(z) = \phi(z) (f(z)) \) for every \( f \in H^2(\mathcal{H}) \) and \( z \in \mathbb{D} \). For \( \phi(z) = zI \), we simply write \( T_z \) instead of \( T_{zI} \). The rich operator theory of the symmetrized bidisc is based on one fundamental result from [14]. It states that for every \( \Gamma \)-contraction \( (S,P) \), there is a unique operator \( A \in \mathcal{B}(\mathcal{D}_P) \) with numerical radius \( \omega(A) \) being not greater than 1 such that \( A \) satisfies the following operator equation in \( X \):

\[
S - S^* P = D_P XD_P.
\]

The unique operator \( A \) is called the **fundamental operator** of the \( \Gamma \)-contraction \( (S,P) \). The following theorem gives an explicit model for a pure \( \Gamma \)-isometry in terms of the fundamental operator of its adjoint.

**Theorem 3.8 ([33], Theorem 2.16).** Let \((S,P)\) be a pair of commuting operators on a Hilbert space \( \mathcal{H} \). If \((S,P)\) is a pure \( \Gamma \)-isometry, then there is a unitary operator \( U : \mathcal{H} \to H^2(\mathcal{D}_{P^*}) \) such that

\[
S = U^* T_\phi U \quad \text{and} \quad P = U^* T_{zI} U, \quad \text{where } \phi(z) = F_0^* + F_0 z,
\]

\(F_0 \in \mathcal{B}(\mathcal{D}_{P^*})\) being the fundamental operator of \((S^*,P^*)\).

The following theorem states that every pure \( \Gamma \)-contraction \( (S,P) \) can be modeled as a compression of a pure \( \Gamma \)-isometry which is obtained in terms of the fundamental operator of \((S^*,P^*)\).
**Theorem 3.9** ([15], Theorem 3.1). Let \((S, P)\) be a pure \(\Gamma\)-contraction defined on a Hilbert space \(\mathcal{H}\). Then the operator pair \((T_{F^*}^+ + F_z, T_z)\) on \(H^2(\mathcal{D}_P^*)\) is a minimal isometric dilation of \((S, P)\). Here \(F^*\) is the fundamental operator of \((S^*, P^*)\). Moreover, \(S^* = T_{F^*}^+ + F_z|_{\mathcal{H}}\) and \(P^* = T_z^*|_{\mathcal{H}}\).

The next lemma is a consequence of Theorem 3.8.

**Lemma 3.10.** Let \((S, P)\) on a Hilbert space \(\mathcal{H}\) be a pure \(\Gamma\)-isometry and let \(F^*\) be the fundamental operator of \((S^*, P^*)\). If \(\dim \mathcal{D}_P^* < \infty\), then \((S, P)\) is algebraic. Moreover, if \(\omega(F^*) < 1\), then \((S, P)\) is \(\Gamma\)-distinguished.

**Proof.** By Theorem 3.8 a model for \((S, P)\) is the pair of multiplication operators \((T_{F^*}^+ + F_z, T_z)\) on \(H^2(\mathcal{D}_P^*)\). Since \(\dim \mathcal{D}_P^* < \infty\), we have that \(F^*\) is a matrix. It follows from Theorem 3.5 in [38] that the polynomial \(q(z_1, z_2) = \det(F^* + F^*z_2 - z_1I)\) is \(\Gamma\)-distinguished if \(\omega(F^*) < 1\). It can easily be verified that \(q\) annihilates \((T_{F^*}^+ + F_z, T_z)\).

Thus, we learn that every cyclic pure \(\Gamma\)-isometry is algebraic. We now show that any such pair has a minimal polynomial.

**Lemma 3.11.** Let \((S, P)\) be a cyclic pure \(\Gamma\)-isometry annihilated by an irreducible polynomial \(q \in \mathbb{C}[z_1, z_2]\). Then \(q\) divides any polynomial \(p\) that satisfies \(p(S, P) = 0\).

**Proof.** It follows from Theorem 3.8 that there is a unitary operator operator \(U : \mathcal{H} \to H^2(\mathcal{D}_P^*)\) such that

\[
S = U^*T_\phi U \quad \text{and} \quad P = U^*T_z U,
\]

where \(\phi(z) = F^*_z + F_z\) with \(F_z \in \mathcal{B}(\mathcal{D}_P^*)\) being the fundamental operator of \((S^*, P^*)\). By Lemma 3.7 the space \(\mathcal{D}_P^*\) is finite-dimensional. Thus \(\phi\) is a matrix-valued linear polynomial. It is easy to see that

\[
q(T_\phi, T_z) = Uq(S, P)U^* = 0.
\]

Let the polynomial \(q\) be given by \(q(z_1, z_2) = \sum_{i,j=0}^n q_{ij}z_1^iz_2^j\). Since \(z_1I\) and \(\phi(z_2)\) is a pair of commuting matrices acting on \(\mathcal{D}_P^*\), we have

\[
q(z_1, z_2)I = q(z_1I, z_2I) - q(T_\phi, T_z) = q(z_1I, z_2I) - q(\phi(z_2), z_2I)
\]

\[
= \sum_{i,j=0}^n q_{ij}z_1^iz_2^jI - \sum_{i,j=0}^n q_{ij}\phi(z_2)^iz_2^j
\]

\[
= \sum_{j=0}^n \sum_{i=1}^n q_{ij}(z_1^i - \phi(z_2)^i)z_2^j
\]

\[
= (z_1I - \phi(z_2))Q(z_1, z_2),
\]

for some matricial polynomial \(Q(z_1, z_2)\). The last equality follows from the fact that

\[
A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \ldots + AB^{n-2} + B^{n-1})
\]

for any pair of commuting matrices \((A, B)\) and for any \(n \in \mathbb{N}\). Now, \(Q\) is not identically zero because, otherwise \(q\) would also be equal to 0 then and consequently \(Q\) would have a lower degree in \(z_1\) than \(q\). So, the non-zero entries of the matrix polynomial \(Q\) cannot vanish identically on \(Z(q)\). Since \((z_1I - \phi(z_2))Q(z_1, z_2) = q(z_1, z_2)I = 0\) on \(Z(q)\), we have \(z_1Q(z_1, z_2) = \phi(z_2)Q(z_1, z_2)\) on \(Z(q)\). So, if \(p \in \mathbb{C}[z_1, z_2]\) annihilates \((S, P)\) and hence \((T_\phi, T_z)\), then

\[
p(\phi(z_2), z_2I)Q(z_1, z_2) = p(z_1, z_2)Q(z_1, z_2) = 0 \quad \text{on} \ Z(q).
\]
Since $q$ is irreducible and $Q$ does not vanish identically on $Z(q)$, we have that $q$ divides $p$.

The following theorem holds in general for a cyclic subnormal pair (e.g. see [30, 17, 18]) and thus holds in particular for a cyclic $\Gamma$-isometry. However, we state the theorem here for a cyclic $\Gamma$-isometry for our purpose. Let us mention here that for any compactly supported measure $\mu$ in $C^2$, $P^2(\mu)$ denotes the closure of the polynomials in $L^2(\mu)$.

**Theorem 3.12.** Let $(S, P)$ be a cyclic $\Gamma$-isometry on the Hilbert space $\mathcal{H}$, with cyclic vector $u$. Then there is a positive Borel measure $\mu$ on some compact set $K$ in $C^2$ and a unitary operator $U$ from $\mathcal{H}$ onto $P^2(\mu)$ that maps $u$ to the constant function $1$, and such that $U$ intertwines $V$ with the pair $(M_{z_1}, M_{z_2})$ of multiplication by the coordinate functions.

Theorem 3.12 makes it easy to show that the minimal polynomial of an algebraic pure $\Gamma$-isometry is square-free which we show below.

**Lemma 3.13.** Suppose $(S, P)$ is a pure $\Gamma$-isometry which is annihilated by $p = \prod_{i=1}^{n} p_i^{t_i}$, where the irreducible factors of $p$ are $p_1, \ldots, p_n$ with multiplicities $t_1, \ldots, t_n$ respectively. Let $q = \prod_{i=1}^{n} p_i$. Then $q(S, P) = 0$.

**Proof.** Let $(S, P)$ be a pure $\Gamma$-isometry acting on a Hilbert space $\mathcal{H}$. We need to show that $q(S, P)u = 0$ for every $u \in \mathcal{H}$. For an arbitrary vector $u$ in $\mathcal{H}$, consider the space $\mathcal{H} = \overline{\mathcal{C}[S, P]u}$. Suppose $(S', P') = (S, P)|_{\mathcal{H}}$. By Theorem 3.12, there is a unitary $U : \mathcal{H} \to P^2(\mu)$ such that $(S, P) = (U^*M_{z_1}U, U^*M_{z_2}U)$ for some positive Borel measure $\mu$ on some compact set. Thus

$$0 = p(S, P) = p(S', P') = p(M_{z_1}, M_{z_2})u.$$ 

In particular, $p(M_{z_1}, M_{z_2})1 = p(z_1, z_2) = 0$ on the support of $\mu$. Note that $q$ contains each irreducible factor of $p$. Therefore, $q$ vanishes identically on the support of $\mu$, and so we have

$$\|q(S, P)u\|^2 = \|q(U^*M_{z_1}U, U^*M_{z_2}U)u\|^2 = \|U^*q(M_{z_1}, M_{z_2})Uu\|^2 = \|q(M_{z_1}, M_{z_2})f\|^2_{P^2(\mu)} \quad [f = Uu]$$

$$= \int |q(M_{z_1}, M_{z_2})f|^2 d\mu = \int |q(z_1, z_2).1(f)|^2 d\mu$$

$$\leq \int |q(z_1, z_2)|^2 d\mu \cdot \int |f|^2 d\mu = 0 \quad [\because q = 0 \text{ on the support of } \mu].$$

Hence, $q(S, P)u = 0$. Since $u$ is arbitrary, we have $q(S, P) = 0$.

**Lemma 3.14.** Suppose a pure $\Gamma$-isometry $(S, P)$ is annihilated by a polynomial $q$, where $q$ is a product of distinct irreducible factors and $(S, P)$ is not annihilated by any factor of $q$. Then $q$ divides any polynomial that annihilates $(S, P)$. 
Proof. Let \( q_0 \) be an irreducible factor and write \( q = q_0 q_1 \). Since, \( (S, P) \) is not annihilated by any of these factors, there exists \( u_0 \in \mathcal{H} \) such that \( u := q_1(S, P)u_0 \neq 0 \). Let \( \mathcal{H} \) be the cyclic subspace generated by \( u \), i.e.

\[
\mathcal{H} := \overline{\mathbb{C}[S, P]u} = \text{span}\{g(S, P)u : g \in \mathbb{C}[z_1, z_2]\}.
\]

 Needless to mention, \( \mathcal{H} \) is a joint invariant subspace for \( (S, P) \). Let \( (S', P') = (S|_{\mathcal{H}}, P|_{\mathcal{H}}) \), which is a pure cyclic \( \Gamma \)-isometry annihilated by \( q_0 \). By Lemma 3.11, \( q_0 \) divides every polynomial that annihilates \( (S', P') \). If \( g(S, P) = 0 \) for some \( g \in \mathbb{C}[z_1, z_2] \), then \( g(S', P') = 0 \) and Lemma 3.11 implies that \( q_0 \) divides \( g \). Since \( q_0 \) is arbitrary, every irreducible factor of \( q \) divides \( g \). Hence \( q \) divides \( g \) and the proof is complete.

Combining all these results, we arrive at the following theorem which is the main result of this Section.

**Theorem 3.15.** Let \( (S, P) \) be an algebraic pure \( \Gamma \)-isometry. Then there exists a square-free polynomial \( q \) that annihilates \( (S, P) \). Moreover, if \( p \) is any polynomial that annihilates \( (S, P) \), then \( q \) divides \( p \).

## 4. The Symmetrization Map and Algebraic \( \Gamma \)-Isometries

The literature (e.g. see Theorem 1.8) tells us that every algebraic pair of commuting pure isometries is annihilated by a toral polynomial. A natural question arises, if every algebraic pure \( \Gamma \)-isometry is also annihilated by a \( \Gamma \)-distinguished polynomial. The next example shows that this is not true in general. The main reason behind this is that not every pure \( \Gamma \)-isometry arises as a symmetrization of a pure isometric pair. We justify this by an example after the following lemma.

**Lemma 4.1.** Let \( A \in \mathcal{B}(\mathcal{H}) \). If the pair \( (T_{A+A^*z}, T_z) \) on \( H^2(\mathcal{H}) \) is annihilated by a polynomial \( q \in \mathbb{C}[z_1, z_2] \), then \( q(A, 0) = 0 \).

**Proof.** For any \( h_0 \in \mathcal{H} \), the map \( f_0 : \mathbb{D} \to \mathcal{H} \) given by \( f_0(z) = h_0 \) is in \( H^2(\mathcal{H}) \). By definition, \( T_z(f_0)(0) = 0 \) and \( T_{A+A^*z}(f_0)(0) = A f_0(0) = A h_0 \). Thus, \( q(A, 0) h_0 = q(T_{A+A^*z}, T_z)(f_0(0)) = 0 \) and so \( q(A, 0) = 0 \).

**Example 4.2.** Consider the matrix

\[
A = \begin{pmatrix}
0 & 2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Then the numerical radius \( \omega(A) \) is equal to 1. By Theorem 3.8 the Toeplitz pair \( (T_{A+A^*z}, T_z) \) on \( H^2(\mathbb{C}^3) \) is a pure \( \Gamma \)-isometry. It can be easily verified that the polynomial \( p(z_1, z_2) = \text{det}(A + A^* z_2 - z_1 I) \) annihilates \( (T_{A+A^*z}, T_z) \). If possible let \( (T_{A+A^*z}, T_z) \) be annihilated by a \( \Gamma \)-distinguished polynomial \( q(z_1, z_2) \). Then \( q(A, 0) = 0 \) by Lemma 4.1. It follows from spectral mapping theorem that

\[
\{0\} = \sigma(q(A, 0)) = \{q(\lambda, 0) : \lambda \in \sigma(A)\}.
\]

Since \( \sigma(A) = \{0, 1\} \), we have that \( q(1, 0) = 0 \). Note that the point \( (1, 0) \in \partial \Gamma = \Gamma \setminus \mathbb{G}_2 \) by being the symmetrization of the points 0, 1. The fact that \( q \) is \( \Gamma \)-distinguished implies that

\[
(1, 0) \in Z(q) \cap \partial \Gamma \subseteq b \Gamma.
\]

This is a contradiction as Theorem 2.25 shows that \( (1, 0) \notin b \Gamma \). Hence, \( (T_{A+A^*z}, T_z) \) is an algebraic pure \( \Gamma \)-isometry but no \( \Gamma \)-distinguished polynomial can annihilate it.
As we have mentioned above that the reason behind an algebraic pure $\Gamma$-isometry being not $\Gamma$-distinguished in general is that not every $\Gamma$-isometry arises as the symmetrization of commuting isometries. The pair $(T_{A+\ast z}, T_z)$, in Example 4.2, is a pure $\Gamma$-isometry acting on $H^2(\mathbb{C}^3)$ and it is annihilated by $p(z_1, z_2) = \det(A + A^\ast z_2 - z_1 I)$. We prove that $(T_{A+\ast z}, T_z)$ cannot be the symmetrization of a pair of commuting pure isometries. Let if possible, $(T_{A+\ast z}, T_z) = \pi(V_1, V_2)$ for a pair of commuting pure isometries $(V_1, V_2)$ on $H^2(\mathbb{C}^3)$. Then the polynomial $p \circ \pi$ annihilates $(V_1, V_2)$. We have by Theorem 4.4 that there exists $q \in \mathbb{C}[z_1, z_2]$ such that $q$ annihilates $(V_1, V_2)$ and $Z(q) \subseteq \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$. Thus, $Z(q) \cap \partial \mathbb{D}^2 \subseteq \mathbb{T}^2$. The polynomial $q(z_1, z_2) = q(z_1, z_2)q(z_2, z_1)$ also annihilates $(V_1, V_2)$ with $Z(q_s) \subseteq \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$. Since $q_s$ is symmetric, one can choose $\tilde{q} \in \mathbb{C}[z_1, z_2]$ such that $q_s = \tilde{q} \circ \pi$. By Example 4.4, $\tilde{q}$ is distinguished and so, $Z(\tilde{q}) \cap \partial S_2 \subseteq b\Gamma$. Moreover, $\tilde{q}(T_{A+\ast z}, T_z) = \tilde{q} \circ \pi(V_1, V_2) = q_s(V_1, V_2) = 0$. By Lemma 4.1, $\tilde{q}(A, 0) = 0$. Let $f(z_1, z_2) = z_1 \tilde{q}(z_1, z_2)$. Evidently, $(0, 0) \in Z(f) \cap S_2$. Since $Z(f) = \{0\} \times \mathbb{C} \cup Z(\tilde{q})$, we have for any $(s, p) \in Z(f) \cap \partial \Gamma$ that either $(s, p) \in Z(\tilde{q}) \cap \partial \Gamma \subseteq b\Gamma$ or $s = 0$ and $(0, p) \in \partial \Gamma$. In the latter case, there exists $(z_1, z_2) \in \partial \mathbb{D}^2 = (\mathbb{D} \times \mathbb{T}) \cup (\mathbb{T} \times \mathbb{D})$ such that $z_1 + z_2 = 0$ and $z_1 z_2 = p$. Thus, $|z_1| = |z_2| = 1$ and so, $(0, p) = \pi(z_1, -z_1) \in \pi(\mathbb{T}^2) = b\Gamma$. Consequently, $f$ is a $\Gamma$-distinguished polynomial and that $f(A, 0) = 0$. This gives a contradiction to the fact that $(A, 0)$ cannot be annihilated by any $\Gamma$-distinguished polynomial as shown in Example 4.2. Therefore, $(T_{A+\ast z}, T_z)$ cannot arise as the symmetrization of a commuting pair of pure isometries.

The symmetrization of a pair of commuting pure isometries is always a pure $\Gamma$-isometry but the converse does not hold as discussed above. Also, we present below a simple example showing that not every pure $\Gamma$-isometry arises as the symmetrization of commuting isometries.

**Example 4.3.** Consider the pair $(0, T_z)$ on $H^2(\mathbb{D})$ which is annihilated by the polynomial $q(z_1, z_2) = z_1$. By part-(2) of Theorem 2.8, $(0, T_z)$ is a pure $\Gamma$-isometry. If this pair is symmetrization of a pair of commuting isometries, say $(V_1, V_2)$ on $H^2(\mathbb{D})$, then $V_1 + V_2 = 0$ and $V_1 V_2 = T_z$ and consequently we have $-V_2^2 = T_z$. This is a contradiction, because the shift operator $T_z$ cannot be written as a square of an operator.

The above examples lead to the question, if we can characterize all $\Gamma$-isometries which are symmetrization of commuting isometries. We borrow techniques as in [35] by the first named author of this article and obtain a necessary and sufficient condition for the same.

**Theorem 4.4.** Let $(S, P)$ be a $\Gamma$-isometry acting on a Hilbert space $\mathcal{H}$. Then $S = V_1 + V_2$ and $P = V_1 V_2$ for a pair of commuting isometries $V_1, V_2$ on $\mathcal{H}$ if and only if $S^2 - 4P$ has a square root $\Delta$ such that

1. $\Delta$ commutes with $S$ and $P$;
2. $\Delta = S^2 \pm \Delta$ are isometries.

**Proof.** Let $(V_1, V_2)$ be two commuting isometries such that $S = V_1 + V_2$ and $P = V_1 V_2$. Then $S^2 - 4P = (V_1 - V_2)^2$ and hence, $S^2 - 4P$ has a square root, say, $\Delta = V_1 - V_2$ which commutes with $S$ and $P$. Clearly,

$$
\left( \frac{1}{2}(S + \Delta), \frac{1}{2}(S - \Delta) \right) = (V_1, V_2)
$$

which is a commuting pair of isometries. Conversely, suppose $S^2 - 4P$ has a square root $\Delta$ that commutes with $S, P$ and $\Delta = S^2 \pm \Delta$ are isometries. Set

$$
V_1 = \frac{1}{2}(S + \Delta) \quad \text{and} \quad V_2 = \frac{1}{2}(S - \Delta).
$$
Then \((V_1, V_2)\) is a commuting pair of isometries such that \(S = V_1 + V_2\) and \(P = V_1 V_2\).

The following corollary is an easy consequence of the above theorem.

**Corollary 4.5.** Let \((S, P)\) be a pure \(\Gamma\)-isometry acting on a Hilbert space \(\mathcal{H}\). Then \(S = V_1 + V_2\) and \(P = V_1 V_2\) for a pair of commuting pure isometries \(V_1, V_2\) on \(\mathcal{H}\) if and only if \(S^2 - 4P\) has a square root \(\Delta\) such which commutes with \(S\) and \(P\) and \(\frac{1}{2}(S \pm \Delta)\) are pure isometries.

The conditions that \(\frac{1}{2}(S \pm \Delta)\) are pure in Corollary 4.5 cannot be ignored and the following example explains this.

**Example 4.6.** Let us consider the following matrices:

\[
A = \begin{pmatrix}
0 & 2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
E = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

As mentioned in Example 4.2 and discussion thereafter, \(\omega(A) = 1\) and the pure \(\Gamma\)-isometry \((S, P) = (T_{A+Az}, T_z)\) on \(H^2(\mathbb{C}^3)\) cannot arise as the symmetrization of a commuting pair of pure isometries. Here, we show that \(S^2 - 4P\) has a square root \(\Delta\) which commutes with \(S, P\) and \(\frac{1}{2}(S \pm \Delta)\) are commuting isometries. It is evident that \(A^2 = A^* A = E\) and \(A^* A + AA^* + 2E = 4I\). Define \(\Delta := T_{E - Ez} = V^2(\mathbb{C}^3)\) whose replica in \(l^2(\mathbb{C}^3)\) is given by the following block matrix:

\[
\Delta = \begin{pmatrix}
E & 0 & 0 & 0 & \ldots \\
-E & E & 0 & 0 & \ldots \\
0 & -E & E & 0 & \ldots \\
0 & 0 & -E & E & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}.
\]

A routine calculation (which is provided in the Appendix) yields the following:

\[
S^2 - 4P = \Delta^2.
\]

Thus \(S^2 - 4P\) has a square root \(\Delta\). We now have that (see a proof in the Appendix) \(S\Delta = \Delta S\).

Similarly, \(\Delta P = P\Delta\). Again by some routine computations, we have

\[
S + \Delta = \begin{pmatrix}
A + E & 0 & 0 & 0 & \ldots \\
A^* - E & A + E & 0 & 0 & \ldots \\
0 & A^* - E & A + E & 0 & \ldots \\
0 & 0 & A^* - E & A + E & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}, \quad S - \Delta = \begin{pmatrix}
A - E & 0 & 0 & 0 & \ldots \\
A^* + E & A - E & 0 & 0 & \ldots \\
0 & A^* + E & A - E & 0 & \ldots \\
0 & 0 & A^* + E & A - E & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}.
\]

Again, by a few steps of calculations (see the Appendix) we have that \((S + \Delta)^*(S + \Delta) = 4I\).

Similarly, we have \((S - \Delta)^*(S - \Delta) = 4I\).

5. **The \(\Gamma\)-Distinguished Pure \(\Gamma\)-Contractions**

In this Section, we study the \(\Gamma\)-distinguished pure \(\Gamma\)-contractions. Recall that a \(\Gamma\)-contraction \((S, P)\) is said to be \(\Gamma\)-distinguished if \((S, P)\) is annihilated by a \(\Gamma\)-distinguished polynomial in \(\mathbb{C}[z_1, z_2]\). Also, we consider the following class of \(\Gamma\)-contractions.
Definition 5.1. A \( \Gamma \)-contraction \((S, P)\) is said to be distinguished if there is some distinguished polynomial that annihilates \((S, P)\).

We have an immediate lemma that clarifies the fact that the class of \( \Gamma \)-distinguished \( \Gamma \)-contractions is bigger than that of distinguished \( \Gamma \)-contractions.

Lemma 5.2. Every distinguished \( \Gamma \)-contraction is \( \Gamma \)-distinguished.

Proof. Let \((S, P)\) be a \( \Gamma \)-contraction acting on a Hilbert space \( \mathcal{H} \) and let \( q \in \mathbb{C}[z_1, z_2] \) be a distinguished polynomial that annihilates \((S, P)\). Let us define \( p(z_1, z_2) = z_1 q(z_1, z_2) \). Evidently, \((0, 0) \in Z(p) \cap \mathbb{G}_2 \). Let \((z_1, z_2) \in Z(p) \cap \partial \Gamma \). Either \( z_1 = 0 \) or \((z_1, z_2) \in Z(q) \). In either case, \((z_1, z_2) \in \mathbb{G}_2 \cup b \Gamma \cup \pi(\mathbb{E}^2) \) and thus \((z_1, z_2) \in Z(p) \cap b \Gamma \). The proof is complete.

Now we present a few simple but important observations in the form of lemmas below.

Lemma 5.3. Let \((S, P)\) be a commuting pair of operators acting on a Hilbert space \( \mathcal{H} \). Then \((S, P)\) is distinguished or \( \Gamma \)-distinguished if and only if \((S^*, P^*)\) is distinguished or \( \Gamma \)-distinguished respectively.

Proof. Let \( q(z_1, z_2) = \sum a_{ij} z_1^i z_2^j \). Consider the polynomial \( \tilde{q}(z_1, z_2) = \sum b_{ij} z_1^i z_2^j \) for which we have that \( Z(\tilde{q}) = \{ (z_1, z_2) : (z_1, z_2) \in Z(q) \} \). Hence, \( \tilde{q} \) is distinguished or \( \Gamma \)-distinguished if and only if \( q \) is distinguished or \( \Gamma \)-distinguished respectively. Also, \( q(S, P) = 0 \) and only if \( \tilde{q}(S^*, P^*) = 0 \).

Lemma 5.4. Let \((S, P)\) be a distinguished pure \( \Gamma \)-isometry. Then there is a square-free distinguished polynomial \( q \) that annihilates \((S, P)\). Moreover, if \( p \) is any polynomial that annihilates \((S, P)\) then \( q \) divides \( p \).

Proof. It follows from Theorem 3.15 that there exists a square-free minimal polynomial \( q \) that annihilates \((S, P)\). Let \( f \) be a distinguished polynomial such that \( f(S, P) = 0 \). Then \( q \) divides \( f \) and consequently, \( Z(q) \subseteq Z(f) \subseteq \mathbb{G}_2 \cup b \Gamma \cup \pi(\mathbb{E}^2) \). It completes the proof.

Lemma 5.5. If \((S_1, P_1)\) and \((S_2, P_2)\) are two unitarily equivalent \( \Gamma \)-contractions on the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively i.e. there is a unitary operator \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) such that \( S_1 = U^* S_2 U \) and \( P_1 = U^* P_2 U \). Then \((S_1, P_1)\) is distinguished or \( \Gamma \)-distinguished if and only if \((S_2, P_2)\) is distinguished or \( \Gamma \)-distinguished respectively.

Proof. The conclusion follows from the fact that if \( p \in \mathbb{C}[z_1, z_2] \) is a polynomial annihilating \((S_2, P_2)\) then \( p(S_1, P_1) = U^* p(S_2, P_2) U \).

A major part of operator theory on the distinguished varieties in the symmetrized bidisc is based on the \( \Gamma \)-contractions \((S, P)\) for which \( \mathcal{D}_p \) or \( \mathcal{D}_{p^*} \) is finite dimensional, e.g. see [38]. So, we now explore relations of these classes of \( \Gamma \)-contractions and their associated fundamental operators with \( \Gamma \)-distinguished polynomials. More precisely, we provide below a set of necessary and sufficient conditions for such pure \( \Gamma \)-isometry to be \( \Gamma \)-distinguished. Before that we must recall the following result from the literature.

Theorem 5.6 ([19], Theorem 2.5). For a matrix \( F \) with numerical radius atmost 1, the set
\[
\mathcal{V}_F = \{ (z_1, z_2) \in \mathbb{C}^2 : \det(F^* + z_2 F - z_1 I) = 0 \}
\]
is an algebraic variety that intersects \( \Gamma \). Moreover, the following are equivalent:

(i) \( \mathcal{V}_F \) is a distinguished variety in \( \mathbb{G}_2 \).
Let \( F^\ast \) for every polynomial \( f \).

(iii) \( \mathcal{W}_F \subseteq \pi(\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2) \).

**Theorem 5.7.** Let \((S, P)\) be a pure \( \Gamma \)-isometry on a Hilbert space \( \mathcal{H} \) such that \( \dim \mathcal{D}_{P^\ast} < \infty \) and let \( F_* \in \mathcal{B}(\mathcal{D}_{P^\ast}) \) be the fundamental operator of \((S^\ast, P^\ast)\). Then the following are equivalent:

1. \((F_0, 0)\) is \( \Gamma \)-distinguished;
2. \((S, P)\) is \( \Gamma \)-distinguished;
3. \( r(F_0) < 1 \).

**Proof.** The model theory of pure \( \Gamma \)-isometry (see Theorem 3.8) tells us that \((S, P)\) is unitarily equivalent to the pure \( \Gamma \)-isometry \((T_{F_0}^\ast + F_* z_2 - z_1 I, T_z)\) on \( H^2(\mathcal{D}_{P^\ast}) \). Since \( \dim \mathcal{D}_{P^\ast} < \infty \), we have that \( p(z_1, z_2) = \text{det}(F_0^\ast + F_* z_2 - z_1 I) \) defines a polynomial. Moreover, \( p(T_{F_0}^\ast + F_* z_2, T_z^\ast) = 0 \). It follows from Lemma 5.5 that \( p(S, P) = 0 \). We prove: (1) \( \implies \) (3) \( \implies \) (2) \( \implies \) (1).

(1) \( \implies \) (3). Assume that \( f(F_0, 0) = 0 \) for some \( \Gamma \)-distinguished polynomial \( f \). Since \( F_* \) is the fundamental operator of \((S^\ast, P^\ast)\), we have \( r(F_0) \leq \omega(F_0) \leq 1 \). Let if possible \( r(F_0) = 1 \). Since \( F_* \) is a matrix, there exists some \( \lambda \in \sigma(F_* \cap \mathbb{T} \). Spectral mapping theorem implies that

\[
\{0\} = \mathcal{O}_F(f(F_0, 0)) = f(\mathcal{O}_F(F_0), 0) = f(\mathcal{O}_F(F_0) \times \{0\})
\]

Since \( \lambda \in \sigma(F_* \cap \mathbb{T} \), we have \( f(\lambda, 0) = 0 \) and \( f \) being \( \Gamma \)-distinguished implies that

\[
\pi(\lambda, 0) = (\lambda, 0) \in Z(f) \cap \partial G_2 \subseteq b \Gamma \implies \pi(\lambda, 0) \in b \Gamma,
\]

which is a contradiction. Hence, \( r(F_0) < 1 \).

(3) \( \implies \) (2). Let \( r(F_0) < 1 \). Since \( \sigma(F_0) \subseteq \mathbb{D} \) and \( \omega(F_0) \leq 1 \), it follows from Theorem 5.6 that \( p(z_1, z_2) \) is a distinguished polynomial. Thus \( p_0(z_1, z_2) = z_1 p(z_1, z_2) \) is a \( \Gamma \)-distinguished polynomial so that \( p_0(z_1, z_2) \) annihilates \((T_{F_0}^\ast + F_* z_2, T_z)\).

(2) \( \implies \) (1). Let \((S, P)\) be \( \Gamma \)-distinguished. By Lemma 5.5 \((T_{F_0}^\ast + F_* z_2, T_z)\) is also \( \Gamma \)-distinguished. Let \( q \) be a \( \Gamma \)-distinguished polynomial such that \( q(T_{F_0}^\ast + F_* z_2, T_z) = 0 \). We have by Lemma 4.1 that \( q(F_0, 0) = 0 \) and so \((F_0, 0)\) is \( \Gamma \)-distinguished. The proof is now complete.

**Theorem 5.8.** Let \((S, P)\) be a pure \( \Gamma \)-contraction on a Hilbert space \( \mathcal{H} \) such that \( \dim \mathcal{D}_{P^\ast} < \infty \). Let \( F_* \in \mathcal{B}(\mathcal{D}_{P^\ast}) \) be the fundamental operator of \((S^\ast, P^\ast)\). Then \((S, P)\) is \( \Gamma \)-distinguished if \((F_0, 0)\) is \( \Gamma \)-distinguished or \( r(F_0) < 1 \).

**Proof.** From Theorem 4.6 in [14], we have that the \( \Gamma \)-isometry \((T, V) = (T_{F_0}^\ast + F_* z_2, T_z)\) on \( H^2(\mathcal{D}_{P^\ast}) \) dilates \((S, P)\) which means that

\[
(f(S, P) = P_{\mathcal{H}} f(T_{F_0}^\ast + F_* z_2, T_z))_{|_{\mathcal{H}}}
\]

for every polynomial \( f \in \mathbb{C}[z_1, z_2] \). The pair \((T, V)\) is a pure \( \Gamma \)-isometry with \( F_* \) as the fundamental operator of \((S^\ast, P^\ast)\). Since \( \dim \mathcal{D}_{P^\ast} < \infty \), Theorem 5.7 yields that \((T_{F_0}^\ast + F_* z_2, T_z)\) is \( \Gamma \)-distinguished if and only if \((F_0, 0)\) is \( \Gamma \)-distinguished or \( r(F_0) < 1 \). In either case, we have a \( \Gamma \)-distinguished polynomial annihilating \((T, V)\) and hence \((S, P)\).

In Example 9.8 we shall see that even if \((S, P)\) is a \( \Gamma \)-distinguished pure \( \Gamma \)-contraction, then the pair \((F_0, 0)\) need not be \( \Gamma \)-distinguished.

**Theorem 5.9.** Let \((S, P)\) be a \( \Gamma \)-contraction defined on a Hilbert space \( \mathcal{H} \) such that \((S^\ast, P^\ast)\) is pure and \( \dim \mathcal{D}_{P^\ast} < \infty \). Let \( A \) be the fundamental operator of \((S, P)\) with \( \omega(A) < 1 \). Then \((S, P)\) is a \( \Gamma \)-distinguished \( \Gamma \)-contraction.
Proof. By applying Theorem 3.9 to \((S^\ast, P^\ast)\), we have that \(\mathcal{H} \subseteq H^2(\mathcal{D}_P)\) and that \(S = T_{A^\ast+Az}^\ast|_{\mathcal{H}}\) and \(P = T_{_\ast}^\ast|_{\mathcal{H}}\). Following the proof of Theorem 3.5 in [38], we have that \(f(z_1, z_2) = \text{det}(A^\ast + Az_2 - z_1I)\) is \(\Gamma\)-distinguished since \(\omega(A) < 1\). Note that \(f(T_{A^\ast+Az}, T_\ast) = 0\) and so \((T_{A^\ast+Az}, T_\ast)\) is a \(\Gamma\)-distinguished \(\Gamma\)-isometry. By Lemma 5.3, \((T_{A^\ast+Az}, T_\ast)\) is a \(\Gamma\)-distinguished \(\Gamma\)-contraction. The desired conclusion now follows.

5.1. \(\Gamma\)-distinguished \(\Gamma\)-contractions on finite-dimensional spaces. Recall from Section 2 that for a \(\Gamma\)-contraction \((S, P)\), we have

\[
\rho(S, P) = 2(I - P^\ast P) - (S - S^\ast P) - (S^\ast - P^\ast S).
\]

A commuting pair \((S, P)\) is said to be strict \(\Gamma\)-contraction if there is a constant \(c > 0\) such that \(\rho(\alpha S, \alpha^2 P) \geq cI\) for all \(\alpha \in \overline{D}\). Below we mention some useful properties of a strict \(\Gamma\)-contraction from [38].

(a) If \((S, P)\) is a strict \(\Gamma\)-contraction, then \(P\) is a strict contraction.
(b) If \(A\) is the fundamental operator of a strict \(\Gamma\)-contraction, then \(\omega(A) < 1\).

Proposition 5.10. Strict \(\Gamma\)-contractions on a finite-dimensional Hilbert space are \(\Gamma\)-distinguished.

Proof. Let \((S, P)\) be a strict \(\Gamma\)-contraction on a finite-dimensional Hilbert space \(\mathcal{H}\) and let \(A\) be the fundamental operator of \((S, P)\). Then \((S^\ast, P^\ast)\) is a pure \(\Gamma\)-contraction and \(\omega(A) < 1\). It follows from Theorem 5.9 that \((S, P)\) is \(\Gamma\)-distinguished.

The first named author of this article and Shalit gave an explicit description of a distinguished variety in the symmetrized bidisc in [38]. We recall that result for our purpose.

Theorem 5.11 ([38], Theorem 3.5). Let \(A\) be a square matrix with \(\omega(A) < 1\), and let \(W\) be the subset of \(\mathbb{G}_2\) defined by

\[
W = \{(z_1, z_2) \in \mathbb{G}_2 : \text{det}(A + z_2A^\ast - z_1I) = 0\}.
\]

Then \(W\) is a distinguished variety. Conversely, every distinguished variety in \(\mathbb{G}_2\) has the form \(\{ (z_1, z_2) \in \mathbb{G}_2 : \text{det}(A + z_2A^\ast - z_1I) = 0 \}\), for some matrix \(A\) with \(\omega(A) \leq 1\).

Proposition 5.12. Let \((S, P)\) be a \(\Gamma\)-contraction on a finite-dimensional Hilbert space \(\mathcal{H}\) and let \(A\) be the fundamental operator of \((S, P)\) with \(\omega(A) < 1\). Then \((S, P)\) is \(\Gamma\)-distinguished.

Proof. The idea of the proof is similar to that of Theorem 4.7 in [38]. Let \(\{r_n\}_{n \in \mathbb{N}}\) be a sequence of positive numbers converging to 1 from below. Then \(\Sigma_n = (S_n, P_n) = (r_nS, r_n^2P)\) gives a sequence of pure \(\Gamma\)-contractions with \(\text{dim} \mathcal{D}_{P_n} < \infty\). Let \(A_n\) denote the fundamental operator of \(\Sigma_n\), and let

\[
A\Sigma_n = \{(z_1, z_2) \in \Gamma : \text{det}(A_n^\ast + z_2A_n - z_1I) = 0\}.
\]

We consider all the operators \(A_n\) on the finite-dimensional Hilbert space \(\mathcal{H}\). By passing to a subsequence, we may assume that the sequence \(A_n\) converges to some operator \(F\) in the operator norm topology. We next show that \(F\) is the fundamental operator of \((S, P)\). Since \(A_n\) is the fundamental operator of \(\Sigma_n\), we have that

\[
S_n - S_n^\ast P_n = (I - P_n^\ast P_n)^{\frac{1}{2}} A_n(n - P_n^\ast P_n)^{\frac{1}{2}}.
\]

As \(n \to \infty\), we get

\[
S - S^\ast P = (I - P^\ast P)^{\frac{1}{2}} F(I - P^\ast P)^{\frac{1}{2}}
\]
in the operator norm-topology. Thus, by the uniqueness of the fundamental operator, $F = A$. By Theorem 5.11, $f(z_1, z_2) = det(A^* + z_2 A - z_1 I)$ is $\Gamma$-distinguished since $\omega(A) < 1$. Moreover, we have by Theorem 4.7 in [38] that $\|f(S, P)\| \leq \sup\{|f(z_1, z_2)| : det(A^* + z_2 A - z_1 I) = 0\} = 0$. Therefore, the $\Gamma$-distinguished polynomial $f$ annihilates $(S, P)$. The proof is complete.

We cannot drop the hypothesis $\omega(A) < 1$ in the above proposition and the reason is explained in Example 4.2.

6. THE DISTINGUISHED BOUNDARY OF A DISTINGUISHED VARIETY IN $\mathbb{D}^2$ AND $\mathbb{G}_2$

Recall that the distinguished boundary $bX$ of a compact set $X \subset \mathbb{C}^n$ is the smallest closed subset of $X$ on which every member in $Rat(X)$ attains its maximum modulus. Also, $bX$ is the Shilov boundary of the algebra $Rat(X)$. It follows from the maximum principle that $bX$ is always contained in the topological boundary $\partial X$. However, $bX$ can be thinner than $\partial X$ in higher dimensions. For example, the closed disc $\overline{D}$ has distinguished boundary $\mathbb{T}$ which is also its topological boundary but for the closed bidisc $\overline{D^2}$ the topological boundary is $(\overline{D} \times \mathbb{T}) \cup (\mathbb{T} \times \overline{D})$ which is much bigger than the distinguished boundary $\mathbb{T}^2$. Clearly, if there is a function $f \in Rat(X)$ and a point $x \in X$ such that $f(x) = 1$ and $|f(y)| < 1$ for all $y \in X \setminus \{x\}$, then $x \in bX$. Such a point $x$ is said to be a peak point of $X$ and the function $f$ is called a peaking function for $x$.

The distinguished boundary plays a significant role in both complex-function theory and operator theory associated with a domain. A seminal work due to Arveson (e.g. see Corollary to Theorem 1.2.2 in [12] or Theorem 2.3 of this paper) states that a commuting operator tuple $T = (T_1, \ldots, T_n)$ has $X$ as a complete spectral set if and only if $T$ possesses a normal dilation $\mathcal{N} = (N_1, \ldots, N_n)$ such that the Taylor joint spectrum $\sigma_T(\mathcal{N}) \subseteq bX$. We intend to study Arveson’s theorem when a pair of toral contractions or a $\Gamma$-distinguished $\Gamma$-contraction admits a toral unitary dilation or a $\Gamma$-distinguished $\Gamma$-unitary dilation respectively. Note that such a study, even if succeeds, does not guarantee the success or failure of rational dilation on a distinguished variety in $\mathbb{D}^2$ or $\mathbb{G}_2$. However, before getting into complete spectral set versus normal $bX$-dilation in our setting it is necessary to determine the distinguished boundary of a distinguished variety in $\mathbb{D}^2$ and $\mathbb{G}_2$.

There are several techniques in the literature (e.g. see [10]) to determine distinguished boundaries of bounded domains in $\mathbb{C}^n$. However, these techniques vary from domain to domain and till date there is no fixed algorithm to find the distinguished boundary of any compact set in $\mathbb{C}^n$. Unfortunately, none of these techniques seem to work for a distinguished variety in $\mathbb{D}^2$ or $\mathbb{G}_2$. A probable underlying reason is that the compact sets of type $Z(p) \cap \overline{D^2}$ or $Z(p) \cap \Gamma$ are too thin. Here we shall apply the theory of analytic variety in a domain [23] for our purpose. Indeed, by an application of this theory we shall show in this Section that $b(Z(p) \cap \overline{D^2}) = Z(p) \cap \mathbb{T}^2$ and $b(Z(p) \cap \Gamma) = Z(p) \cap b\Gamma$, when $p$ is a toral or $\Gamma$-distinguished polynomial respectively. We begin with a brief theory of analytic variety in a domain.

6.1. Analytic variety and maximum principle. Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A subset $V$ is said to be an analytic variety or an analytic subset or a subvariety of $\Omega$ if for every point $z \in \Omega$, there is a neighborhood $U_z$ of $z$ and functions $f_1, \ldots, f_k$ holomorphic in $U_z$ such that $U_z \cap V = \{w \in U_z : f_1(w) = 0, \ldots, f_k(w) = 0\}$.
Let $M$ be a complex manifold. By a globally analytic variety or a globally analytic subset of the manifold $M$, we mean any set of the form

$$V = \{ z \in M : f_1(z) = 0, \ldots, f_k(z) = 0 \},$$

where $f_1, \ldots, f_k$ are holomorphic on $M$. A subset $Z$ of a manifold $M$ is called an analytic subset of $M$ if every point of $M$ has an open neighborhood $U$ such that the set $Z \cap U$ is a global analytic subset of $U$. Since every domain $\Omega$ in $\mathbb{C}^n$ is a complex manifold of dimension $n$, one can define an analytic subset of $\Omega$. It is easy to see that, in this case, the concept of an analytic subset of $\Omega$ coincides with that of a subvariety of $\Omega$. An interested reader is referred to Chapter II, Section 3 in [28] and Chapter II, Section E in [23] for a detailed study. We mention a few important definitions and results which will be used in sequel.

**Lemma 6.1** ([28], Chapter IV, Section 1). Connected components of an analytic subset of a complex manifold $M$ are analytic subsets of $M$.

An immediate consequence of Lemma 6.1 is that if $V$ is a subvariety of a domain $\Omega$, then each connected component of $V$ is again a subvariety of $\Omega$.

**Definition 6.2.** Let $f$ be a real-valued function defined on a domain $\Omega$ in $\mathbb{C}^n$. The function $f$ is said to be plurisubharmonic if $f$ is upper semi-continuous and has the following property: if $\phi : D \to \Omega$ is holomorphic, then $f \circ \phi$ is subharmonic.

It is well-known that if $f$ is a holomorphic function on a domain $\Omega \subseteq \mathbb{C}^n$, then $|f|$ is a plurisubharmonic function on $\Omega$. We now mention that plurisubharmonic functions obey the following maximum principle.

**Theorem 6.3** ([23], Chapter IX, Section C, Proposition 3). Let $f$ be a continuous plurisubharmonic function defined in a domain $\Omega$, and let $V$ be a closed connected subvariety of $\Omega$. If $f|_V$ attains its maximum at some point of $V$, then $f|_V$ is constant.

The following theorem, a consequence of Cartan’s prominence work on analytic sheaves, was originally proved in [16]. Comprehensive proofs are provided in the classical texts [23] and [25] (Theorem 7.4.8). By a holomorphic function on an analytic variety $V$, we mean a function that locally agrees with the restriction of a holomorphic function on an open set containing $V$.

**Theorem 6.4** (H. Cartan). If $V$ is an analytic variety in a domain of holomorphy $\Omega$ and if $f$ is a holomorphic function on $V$, there is a holomorphic function $F$ on $\Omega$ such that $F = f$ on $V$.

Begin armed with the theory of analytic variety in a domain, we now proceed to finding the distinguished boundary of a distinguished variety in the bidisc and the symmetrized bidisc. We start with a simple lemma.

**Lemma 6.5.** Let $K$ be a polynomially convex set in $\mathbb{C}^n$ and let $p \in \mathbb{C}[z_1, \ldots, z_n]$ be such that $Z(p) \cap K \neq \emptyset$. Then $Z(p) \cap K$ is a polynomially convex set in $\mathbb{C}^n$.

**Proof.** Let $X = Z(p) \cap K$. To prove $X$ is polynomially convex, it suffices to show that for any $x \notin X$, there exists a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\|f\|_{\infty, X} < |f(x)|$. Take any $x \notin X$. We discuss the two cases here depending on whether $x$ lies in $K$ or not. Let $x \notin K$. Then there exists $f \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\|f\|_{\infty, K} < |f(x)|$ since $K$ is polynomially convex. Thus $\|f\|_{\infty, X} \leq \|f\|_{\infty, K} < |f(x)|$. For $x \in K$, we have $x \notin Z(p)$ and so, $|p(x)| > 0 = \|p\|_{\infty, X}$. The proof is complete.

**Lemma 6.6.** Let $q$ be a toral polynomial. Then every point of $Z(q) \cap \mathbb{T}^2$ is a peak point of $Z(q) \cap \mathbb{D}^2$. 


that there is a holomorphic map
\[ f(z_1, z_2) = \left( \frac{e^{ix}}{2e^{ix} - z_1} \right) \left( \frac{e^{iy}}{2e^{iy} - z_2} \right) \]
is holomorphic on \( \mathbb{D}^2 \) and \( g(e^{ix}, e^{iy}) = 1 \). For every \( z \in \partial \mathbb{D} \), we have that \( 2e^{ix} - z \geq 2 - |z| \geq 1 \) and so,
\[ |g(z_1, z_2)| = \left| \frac{e^{ix}}{2e^{ix} - z_1} \right| \cdot \left| \frac{e^{iy}}{2e^{iy} - z_2} \right| \leq 1 \quad \text{for } z_1, z_2 \in \partial \mathbb{D}.
\]Assume that \( |g(z_1, z_2)| = 1 \) for some \( (z_1, z_2) \in \mathbb{D}^2 \). Then \( 2e^{ix} - z_1 = 1 = 2e^{iy} - z_2 \) which is possible if and only if \( (z_1, z_2) = (e^{ix}, e^{iy}) \). Hence, \( g \) is a peaking function for \( (e^{ix}, e^{iy}) \). The restriction of \( g \) to \( Z(q) \cap \partial \mathbb{D}^2 \) is a map in \( \text{Rat}(Z(q) \cap \partial \mathbb{D}^2) \) that peaks at \( (e^{ix}, e^{iy}) \) and the proof is complete.

The following theorem determines the distinguished boundary of a distinguished variety in \( \mathbb{D}^2 \) or even more. This is a main result of this Section.

**Theorem 6.7.** Let \( q \in \mathbb{C}[z_1, z_2] \) be a toral polynomial. Then the following are equivalent:

1. \( (z_1, z_2) \in Z(q) \cap \mathbb{T}^2; \)
2. \( (z_1, z_2) \) is a peak point of \( Z(q) \cap \partial \mathbb{D}^2; \)
3. \( (z_1, z_2) \in b(Z(q) \cap \partial \mathbb{D}^2). \)

Moreover, \( b(Z(q) \cap \partial \mathbb{D}^2) = Z(q) \cap \mathbb{T}^2. \)

**Proof.** We prove \((1) \implies (2) \implies (3) \implies (1)\). We shall use the following notations: \( X = Z(q) \cap \mathbb{D}^2 \) and \( V = Z(q) \cap \partial \mathbb{D}^2 \).

(1) \implies (2). Let \((z_1, z_2) \in Z(q) \cap \mathbb{T}^2. \) It follows from Lemma [6.6] that there exists \( f \in \text{Rat}(X) \) that peaks at \((z_1, z_2)\).

(2) \implies (3). It follows from the definition of peak point and the distinguished boundary that every peak point of \( Z(q) \cap \partial \mathbb{D}^2 \) must belong to \( b(Z(q) \cap \partial \mathbb{D}^2) \).

(3) \implies (1). First we prove that \( Z(q) \cap \mathbb{T}^2 \) is a closed boundary of \( X \), i.e. every \( g \in \text{Rat}(X) \) attains its maximum modulus over \( Z(q) \cap \partial \mathbb{D}^2 \) at some point in \( Z(q) \cap \mathbb{T}^2 \). Take any \( g \in \text{Rat}(X) \) and some \( x \in X \) such that \( \|g\|_{\infty, X} = |g(x)| \). We show that there is \( y \in Z(q) \cap \mathbb{T}^2 \) such that \( \|g\|_{\infty, X} = |g(y)| \). If \( x \in Z(q) \cap \mathbb{T}^2 \), then we are done. We assume that \( x \notin Z(q) \cap \mathbb{T}^2 \). Note that
\[
X = Z(q) \cap (\mathbb{D}^2 \cup \partial \mathbb{D}^2) = V \cup (Z(q) \cap \partial \mathbb{D}^2) = V \cup (Z(q) \cap \mathbb{T}^2),
\]where the last equality follows from the fact that \( q \) is toral. Thus \( x \in V \). We can re-write \( V = \bigsqcup_{i \in I} V_i \), where each \( V_i \) is a connected component of \( V \) such that \( V_i \cap V_j = \emptyset \) for all \( i \neq j \). Indeed, each \( V_i \) is a connected component of \( V \) that exits \( \mathbb{D}^2 \) through its distinguished boundary, the 2-torus \( \mathbb{T}^2 \). Therefore, the closure of each \( V_i \) in \( \mathbb{C}^2 \), denoted by \( \overline{V}_i \), must intersect \( \mathbb{T}^2 \). Since \( V \) is an analytic subset or subvariety of \( \mathbb{D}^2 \), it follows from Lemma [6.1] that each \( V_i \) is a subvariety of \( \mathbb{D}^2 \). Also, it follows that each \( V_i \) is a closed connected subvariety of \( \mathbb{D}^2 \). Since \( x \in V \), we must have \( x \in V_i \) for some \( i \in I \). The function \( g|_{V_i} \) is holomorphic on \( V_i \) and hence, it follows from Theorem [6.4] that there is a holomorphic map \( h \) on \( \mathbb{D}^2 \) such that \( h|_{V_i} = g|_{V_i} \). The function \( f : \mathbb{D}^2 \to \mathbb{R} \) defined by \( f(z) = |h(z)| \) is a continuous plurisubharmonic function on \( \mathbb{D}^2 \) such that \( f|_{V_i} \) (which is same as \( |g||_{V_i} \) attains its maximum at \( x \in V_i \). It follows from Theorem [6.3] that \( f|_{V_i} \) is constant and hence,
Theorem 6.9. \( \frac{|g(z)|}{|g(x)|} \) for all \( z \in \mathcal{V}_i \). Using continuity argument, we have that \( |g(z)| = |g(x)| \) for all \( z \in \overline{\mathcal{V}_i} \).

Since \( \mathcal{V}_i \cap \mathbb{T}^2 \neq \emptyset \), we have that \( |g(x)| = |g(y)| \) for some \( y \in \mathcal{V}_i \cap \mathbb{T}^2 \). Hence, we have

\[
\|g\|_{\infty, \mathcal{X}} = |g(x)| = |g(y)|
\]

for some \( y \in Z(q) \cap \mathbb{T}^2 \). Since \( g \in \text{Rat}(X) \) is arbitrary, every function in \( \text{Rat}(X) \) attains its maximum modulus at some point in \( Z(q) \cap \mathbb{T}^2 \). Therefore, \( Z(q) \cap \mathbb{T}^2 \) is a closed boundary for \( X \). Since \( b(Z(q) \cap \mathbb{D}^2) \) is the smallest closed boundary of \( X \), we have \( b(Z(q) \cap \mathbb{D}^2) \subseteq Z(q) \cap \mathbb{T}^2 \) and the proof is complete.

Needless to mention that the above theorem may fail if the concerned polynomial is not a toral polynomial. Below we show it by an example.

Example 6.8. The polynomial \( p(z_1, z_2) = z_2 \) is not toral. It is easy to see that

\[
Z(p) \cap \mathbb{D}^2 = \mathbb{D} \times \{0\} \quad \text{and} \quad Z(p) \cap \mathbb{T}^2 = \emptyset.
\]

Take any \( (e^{i\theta}, 0) \in \mathbb{T} \times \{0\} \). Following the proof of Lemma [6.6], the rational function given by

\[
f : Z(p) \cap \mathbb{D}^2 \rightarrow \mathbb{C}, \quad f(z_1, z_2) = \frac{e^{i\theta}}{2e^{i\theta} - z_1}
\]

peaks at \( (e^{i\theta}, 0) \). Thus \( (e^{i\theta}, 0) \in b(Z(p) \cap \mathbb{D}^2) \) and so, \( Z(p) \cap \mathbb{T}^2 = b(Z(p) \cap \mathbb{D}^2) \).

Evidently, the distinguished boundary of the symmetrized bidisc \( b\Gamma \) is the symmetrization of the 2-torus \( \mathbb{T}^2 \) (see [2] for details). Theorem [2.5] characterizes the points in \( b\Gamma \) in several ways. Also, by Proposition 8.1 in [9], if \( x \in b\Gamma \) then there is a function \( f \) that is holomorphic on \( \mathbb{G}_2 \) and continuous on \( \Gamma \) such that \( f \) peaks at \( x \). For our purpose, we need a peaking function \( f \) in \( \text{Rat}(\Gamma) \) and the next theorem provides such a function. We give a proof to this theorem here, because we could not locate it in the literature. However, our proof is based on the techniques that are used in the proof of Theorem 8.4 in [9].

Theorem 6.9. Let \( (s_0, p_0) \in b\Gamma \). Then there exists \( f \in \text{Rat}(\Gamma) \) such that \( f \) peaks at \( (s_0, p_0) \).

Proof. Consider \( (s_0, p_0) = (z_1 + z_2, z_1 z_2) \) for some \( z_1, z_2 \in \mathbb{T} \). If \( z_1 = z_2 \), then \( (s_0, p_0) = (2z_1, z_1^2) \) and the function

\[
f(s, p) = \frac{1}{2} \left( 1 + \frac{s}{s_0} \right)
\]

peaks at \( (s_0, p_0) \). To see this, note that \( f(s_0, p_0) = 1 \) and \( |f(s, p)| \leq \frac{1}{2} \left( 1 + |s|/2 \right) \leq 1 \) for every \( (s, p) \in \Gamma \). If \( |f(s, p)| = 1 \) for \( (s, p) = (\alpha_1 + \alpha_2, \alpha_1 \alpha_2) \in \Gamma \), then \( |1 + s/s_0| = 2 \). Since \( s/s_0 \in \overline{\mathbb{D}} \), we must have \( s_0 = s \). Then \( 2z_1 = \alpha_1 + \alpha_2 \) and so, \( |\alpha_1| + |\alpha_2| = 2 \) which gives \( |\alpha_1| = |\alpha_2| = 1 \), as \( \alpha_1, \alpha_2 \in \overline{\mathbb{D}} \). Again, \( \alpha_1 + \alpha_2 = 2z_1 \) implies that \( |1 + \alpha_1 \alpha_2| = 2 \), which is possible only when \( \alpha_1 \alpha_2 = 1 \). Thus, \( \alpha_1 = \alpha_2 = z_1 \) and hence, \( (s, p) = (s_0, p_0) \). Let \( z_1 \neq z_2 \). Choose an automorphism \( v \) of \( \mathbb{D} \) such that \( v(z_1) = 1 \) and \( v(z_2) = -1 \). The map

\[
T_v : \mathbb{G}_2 \rightarrow \mathbb{G}_2, \quad T_v(w_1 + w_2, w_1 w_2) = (v(w_1) + v(w_2), v(w_1)v(w_2))
\]

defines an automorphism of \( \mathbb{G}_2 \) which extends continuously from \( \Gamma \) onto \( \Gamma \) in a bijective manner such that \( T_v(b\Gamma) = b\Gamma \). Note that \( T_v(s_0, p_0) = (0, -1) \). The map

\[
g : \Gamma \rightarrow \mathbb{C}, \quad g(s, p) = \frac{1}{2} \left( 1 + \frac{s_0^2 - 4p}{4} \right)
\]
is a polynomial with \(g(0, -1) = 1\). It follows from Theorem 6.4 that every \((s, p) \in \Gamma\) satisfies the following inequality:

\[
\frac{|s - \bar{z}p|}{2} + \frac{|s^2 - 4p|}{4} + |s|^2 \leq 1.
\]

(6.1)

Using (6.1), we have

\[
|g(s, p)| \leq \frac{1}{2} \left(1 + \frac{|s^2 - 4p|}{4}\right) \leq 1
\]

for every \((s, p) \in \Gamma\). Let \(|g(s, p)| = 1\) for some \((s, p) \in \Gamma\). Then

\[
2 = \left|1 + \frac{s^2 - 4p}{4}\right| \quad \text{and so,} \quad \frac{s^2 - 4p}{4} = 1
\]

(6.2)

which gives \(s = 0\) by the virtue of (6.1). It follows from (6.2) that \((s, p) = (0, -1)\). Therefore, \(g\) peaks at \((0, -1)\) and consequently, \(f = g \circ T_p\) peaks at \((s_0, p_0)\). Evidently, \(f\) is a rational function that has no singularity in \(\Gamma\). The proof is now complete.

We now determine the distinguished boundary of a distinguished variety in \(\mathbb{G}_2\), i.e. an analogue of Theorem 6.7. Obviously, one can imitate the proof of Theorem 6.7 too here. However, we present an alternative proof leveraging the symmetrization map.

**Theorem 6.10.** Let \(p \in \mathbb{C}[z_1, z_2]\) be a \(\Gamma\)-distinguished polynomial. Then the following are equivalent:

1. \((s_0, p_0) \in Z(p) \cap b\Gamma\);
2. \((s_0, p_0)\) is a peak point of \(Z(p) \cap \Gamma\);
3. \((s_0, p_0) \in b(Z(p) \cap \Gamma)\).

Moreover, \(b(Z(p) \cap \Gamma) = Z(p) \cap b\Gamma\).

**Proof.** (1) \(\implies\) (2) follows from Theorem 6.9 and (2) \(\implies\) (3) follows from the definition of peak points. We show that \(b(Z(p) \cap \Gamma) \subseteq Z(p) \cap b\Gamma\) proving (3) \(\implies\) (1). Indeed, we prove that \(Z(p) \cap b\Gamma\) is a closed boundary of \(X\). Let us define

\[
q(z_1, z_2) = p(z_1 + z_2, z_1z_2) = p \circ \pi(z_1, z_2).
\]

Following Example 3.4 we conclude that \(q\) is a toral polynomial. Take \(g \in \text{Rat}(Z(p) \cap \Gamma)\) and define \(f := g \circ \pi\). It is evident that \(f \in \text{Rat}(Z(q) \cap \mathbb{T}^2)\). It follows from Theorem 6.7 that there exists \(x \in Z(q) \cap \mathbb{T}^2\) such that

\[
|f(x)| = \|f\|_{\infty, Z(q) \cap \mathbb{T}^2}.
\]

(6.3)

It is not difficult to see that \(\pi(Z(q) \cap \mathbb{T}^2) = Z(p) \cap \Gamma\). Thus

\[
|g \circ \pi(x)| = |f(x)| = \sup\{|g \circ \pi(z_1, z_2)| : (z_1, z_2) \in Z(q) \cap \mathbb{T}^2\} \quad \text{[By (6.3)]}
\]

\[
= \sup\{|g \circ \pi(z_1, z_2)| : \pi(z_1, z_2) \in Z(p) \cap \Gamma\}
\]

\[
= \sup\{|g(\vec{s}, \vec{p})| : (\vec{s}, \vec{p}) \in Z(p) \cap \Gamma\}
\]

\[
= \|g\|_{\infty, Z(p) \cap \Gamma}.
\]

Note that \(\pi(x) \in Z(p) \cap b\Gamma\), as \(x \in Z(q) \cap \mathbb{T}^2\). Therefore, \(g\) attains its maximum modulus at a point in \(Z(p) \cap b\Gamma\). Since \(g \in \text{Rat}(Z(p) \cap \Gamma)\) is arbitrary, we have that \(Z(p) \cap b\Gamma\) is a closed boundary for \(Z(p) \cap \Gamma\). Consequently, \(b(Z(p) \cap \Gamma) \subseteq Z(p) \cap b\Gamma\) and the proof is complete.

\[\blacksquare\]
We cannot remove the hypothesis that \( q \) is \( \Gamma \)-distinguished in the above theorem as the following example explains.

**Example 6.11.** The polynomial \( q(z_1, z_2) = z_2 \) is not \( \Gamma \)-distinguished. Let \( (s_0, 0) \in Z(q) \cap \Gamma \). By Theorem 2.4, \( |s_0 - \overline{\lambda} p| \leq 1 - |p_0|^2 \). Thus \( |s_0| \leq 1 \) and so, \( Z(q) \cap \Gamma \subseteq \mathbb{D} \times \{0\} \). For any \( \lambda \in \mathbb{D} \times \{0\} \), we have \( (\lambda, 0) = \pi(\lambda, 0) \in \Gamma \). Therefore, \( Z(q) \cap \Gamma = \mathbb{D} \times \{0\} \). Moreover, \( Z(q) \cap b\Gamma = \emptyset \) since \( |p_0| = 1 \) for \( (s_0, p_0) \in b\Gamma \). Using the same computations as seen in Example 6.8, we have that \( Z(q) \cap b\Gamma \neq b(Z(q) \cap \Gamma) \).

7. Dilation of Toral Contractions

The aim of this Section is two-fold. First we prove Theorem 7.1 which provides a necessary and sufficient condition in order to answer the question raised in [20]: if a toral pair of contractions dilates to a toral pair of isometries. Next, we establish a vice-versa relation of the phenomenon of dilation of a pair of commuting contractions to a toral pair of unitaries or isometries with Arveson’s celebrated theorem (see Theorem 2.3) in the setting of a distinguished variety in the bidisc. Note that it was proved in [20] that every toral pair of isometries extends to a toral pair of unitaries. Thus, dilating a pair of commuting contractions to a toral pair of isometries is same as dilating it to a toral pair of unitaries. Arveson’s theorem profoundly found a connection between the two significant classical notions: normal \( bX \)-dilation of a commuting operator tuple associated with a compact set \( X \in \mathbb{C}^n \) and \( X \) being a complete spectral set. Indeed, it tells us that they are equivalent to each other. Here our aim is not to investigate if rational dilation succeeds on a distinguished variety in \( \mathbb{D}^2 \). Rather, we would like to investigate an answer to the following question.

**Question.** Let \( (T_1, T_2) \) be a toral pair of contractions that dilates to a toral pair of unitaries \( (U_1, U_2) \). Let \( p \in \mathbb{C}[z_1, z_2] \) be that toral polynomial for which \( p(U_1, U_2) = 0 \). Is \( Z(p) \cap \mathbb{D}^2 \) a complete spectral set for \( (T_1, T_2) \), i.e. \( \sigma_T(T_1, T_2) \subseteq Z(p) \cap \overline{\mathbb{D}}^2 \) and

\[
\|f(T_1, T_2)\| \leq \sup \left\{ \|f(z_1, z_2)\| : (z_1, z_2) \in Z(p) \cap \overline{\mathbb{D}}^2 \right\}
\]

for every matricial rational function \( f = [f_{i,j}] \) with singularities off \( Z(p) \cap \overline{\mathbb{D}}^2 \)?

We answer this question completely in this Section. In this connection let us mention that a tuple \( (S_1, \ldots, S_n) \) of commuting operators on a Hilbert space \( \mathcal{H} \) is said to have a **commuting normal extension** or simply c.n.e. if there is a tuple \( (N_1, \ldots, N_n) \) of commuting normal operators on some larger Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \) such that \( \mathcal{H} \) is invariant under \( N_1, \ldots, N_n \) and \( N_i|\mathcal{H} = S_i \) for \( i = 1, \ldots, n \). If we take

\[
\mathcal{K} = \text{span} \left\{ N_1^{m_1} \cdots N_n^{m_n} h : h \in \mathcal{H}, m_1, \ldots, m_n \in \mathbb{N} \cup \{0\} \right\},
\]

the minimal common reducing subspace of \( N_1, \ldots, N_n \) containing \( \mathcal{H} \), then \( (N_1, \ldots, N_n) \) is unique up to unitary equivalence and is called the **minimal commuting normal extension** or simply minimal c.n.e. of \( (S_1, \ldots, S_n) \).

**Lemma 7.1** ([29], Corollary 2). Let \( S = (S_1, \ldots, S_n) \) be a commuting tuple of subnormal operators on a Hilbert space \( \mathcal{H} \) and let \( \mathcal{N} = (N_1, \ldots, N_n) \) be the minimal c.n.e. of \( S \). Then \( p(\mathcal{N}) \) is unitarily equivalent to the minimal normal extension of \( p(S) \) for any polynomial \( p \) in \( \mathbb{C}[z_1, \ldots, z_n] \).

We start our campaign with Proposition 4.4 in [20] which tells us that every toral pair of isometries extends to a toral pair of unitaries. We provide an alternative proof to this result here.
**Proposition 7.2** ([20], Proposition 4.4). Every toral pair of isometries extends to a toral pair of unitaries.

**Proof.** Let \((V_1, V_2)\) be a pair of commuting isometries on a Hilbert space \(\mathcal{H}\) and let \(q\) be a toral polynomial such that \(q(V_1, V_2) = 0\). By Ando’s theorem, there exists a pair of commuting unitaries \((\tilde{U}_1, \tilde{U}_2)\) acting on a Hilbert space \(\tilde{\mathcal{H}} \supseteq \mathcal{H}\) such that \((V_1, V_2) = (\tilde{U}_1|_{\mathcal{H}}, \tilde{U}_2|_{\mathcal{H}})\). Let

\[
\mathcal{K} = \overline{\text{span}}\{\tilde{U}_1^m\tilde{U}_2^n h : h \in \mathcal{H} & m, n \in \mathbb{N} \cup \{0\}\}.
\]

Evidently, \(\mathcal{K}\) is a closed subspace of \(\tilde{\mathcal{H}}\) that reduces \(\tilde{U}_1, \tilde{U}_2\) and contains \(\mathcal{H}\). Let us define

\[
(U_1, U_2) = (\tilde{U}_1|_{\mathcal{H}}, \tilde{U}_2|_{\mathcal{H}})
\]

which is a commuting pair of unitaries. Moreover, \((U_1, U_2)\) is the minimal c.n.e. of \((V_1, V_2)\). It follows from Lemma 7.1 that \(q(U_1, U_2)\) is unitarily equivalent to the minimal normal extension of \(q(V_1, V_2)\). Thus \(q(U_1, U_2) = 0\) and the desired conclusion follows.

As discussed earlier, for a commuting pair of Hilbert space contractions \((T_1, T_2)\) annihilated by a toral polynomial \(q\), spectral mapping theorem gives that \(\sigma_T(T_1, T_2) \subseteq Z(q) \cap \mathbb{D}^2\). A natural question arises here if \(Z(q) \cap \mathbb{D}^2\) is a spectral set for \((T_1, T_2)\). We show by an example below that it is too much to expect for general contractions. However, we have an affirmative answer when \(T_1, T_2\) are commuting isometries. To begin with, we have the following result for commuting unitaries.

**Proposition 7.3** ([20], Proposition 4.5). A pair of commuting unitaries \((U_1, U_2)\) is toral if and only if its joint spectrum is contained in a distinguished variety in \(\mathbb{D}^2\).

By Proposition 7.3 a pair of commuting unitaries \((U_1, U_2)\) is annihilated by \(q \in \mathbb{C}[z_1, z_2]\) if and only if \(Z(q) \cap \mathbb{D}^2\) is a spectral set for \((U_1, U_2)\). The underlying reason is that \(\sigma_T(N)\) is a spectral set for a commuting tuple \(N\) of normal operators.

**Proposition 7.4.** Let \((V_1, V_2)\) be a toral pair of isometries and let \(q \in \mathbb{C}[z_1, z_2]\) be a toral polynomial. Then \(q(V_1, V_2) = 0\) if and only if \(Z(q) \cap \mathbb{D}^2\) is a spectral set for \((V_1, V_2)\).

**Proof.** It is evident that \(q(V_1, V_2) = 0\) if \(Z(q) \cap \mathbb{D}^2\) is a spectral set for \((V_1, V_2)\). Conversely, let \(q(V_1, V_2) = 0\). By spectral mapping theorem, \(\sigma_T(V_1, V_2) \subseteq Z(q) \cap \mathbb{D}^2\). Following the proof of Proposition 7.2, we have that there is a pair of commuting unitaries \((U_1, U_2)\) acting on a Hilbert space \(\mathcal{K} \supseteq \mathcal{H}\) such that \((V_1, V_2) = (U_1|_{\mathcal{H}}, U_2|_{\mathcal{H}})\) and \(q(U_1, U_2) = 0\). By Proposition 7.3 \(Z(q) \cap \mathbb{T}^2\) is a spectral set for \((U_1, U_2)\). For any \(g \in \text{Rat}(Z(p) \cap \overline{\mathbb{D}}^2)\), we have

\[
\|g(V_1, V_2)\| = \|g(U_1, U_2)\| \leq \|g(U_1, U_2)\| \leq \|g\|_{\infty, Z(q) \cap \mathbb{T}^2} \leq \|g\|_{\infty, Z(q) \cap \overline{\mathbb{D}}^2}.
\]

Therefore, \(Z(p) \cap \overline{\mathbb{D}}^2\) is a spectral set for \((V_1, V_2)\).

Next, we show that for a pair \((T_1, T_2)\) of commuting contractions annihilated by a toral polynomial \(q(z_1, z_2)\), the set \(Z(q) \cap \overline{\mathbb{D}}^2\) need not be a spectral set for \((T_1, T_2)\).

**Example 7.5.** Let

\[
T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]
It is not difficult to verify that \((T_1, T_2)\) is a commuting pair of contractions acting on \(\mathbb{C}^3\). Moreover, 
\[ p(z_1, z_2) = (z_1 - z_2)^3 \]
is a toral polynomial such that
\[ p(T_1, T_2) = (T_1 - T_2)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Let \(g(z_1, z_2) = z_1 - z_2\). Since \(T_1 - T_2 \neq 0\) and \(Z(p) = Z(g)\), we have
\[ \|g(T_1, T_2)\| > 0 = \|g\|_{\infty, Z(p) \cap \mathbb{D}^2}. \]

Therefore, \(Z(p) \cap \mathbb{D}^2\) is not a spectral set for \((T_1, T_2)\) though \((T_1, T_2)\) is a toral pair of commuting contractions annihilated by \(p(z_1, z_2)\).

The set
\[ W = \{ (z_1, z_2) \in \mathbb{D}^2 : z_1^2 = z_2^3 \} \]
is an example of a distinguished variety in \(\mathbb{D}^2\) which is known as Neil parabola. Dritschel, Jury and McCullough proved (see Theorem 1.1 and Corollary 3.2 in [21]) that rational dilation fails on the Neil parabola by assuring the existence of a contractive representation of \(\text{Rat}(W)\) which is not completely contractive. Capitalizing their ideas, we produce our next example to show that there is a commuting pair of contractions \((X, Y)\) annihilated by an irreducible toral polynomial \(q\) but \(Z(q) \cap \mathbb{D}^2\) is not a spectral set for \((X, Y)\).

**Example 7.6.** Let \(A(\mathbb{D})\) be the algebra of functions that are continuous on \(\overline{\mathbb{D}}\) and holomorphic in \(\mathbb{D}\). Let \(\mathcal{A}_0\) be the subalgebra of \(A(\mathbb{D})\) generated by the polynomials \(z^2\) and \(z^3\). Then every element in \(\mathcal{A}_0\) is of the form
\[ g(z) = g_1(z)z^2 + g_2(z)z^3 \]
for some \(g_1, g_2 \in A(\mathbb{D})\). Let \(\mathcal{A}\) be the closure of \(\mathcal{A}_0\) in \(A(\mathbb{D})\). A unital representation \(\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})\) is said to be contractive if
\[ \|\Phi(f)\| \leq \|f\|_{\infty, \overline{\mathbb{D}}} \quad \text{for all } f \in \mathcal{A} \]
where \(\|f\|_{\infty, \overline{\mathbb{D}}} = \sup\{|f(z)| : z \in \overline{\mathbb{D}}\}\). It follows from Corollary 3.2 in [21] that there exists a pair of commuting contractions \(X\) and \(Y\) acting on \(\mathbb{C}^k\) (for some \(k \in \mathbb{N}\)) with \(X^3 = Y^2\) such the representation of \(\mathcal{A}\) given by
\[ \Phi : \mathcal{A} \to \mathcal{B}(\mathbb{C}^k), \quad \Phi(z^2) = X \quad \text{and} \quad \Phi(z^3) = Y \]
is bounded but not contractive. Then there is some \(f \in \mathcal{A}\) such that \(\|\Phi(f)\| > \|f\|\). Since \(\mathcal{A} = \overline{\mathcal{A}_0}\), there is a sequence \(\{f_n\} \subset \mathcal{A}_0\) such that \(\|f_n - f\|_{\infty, \overline{\mathbb{D}}} \to 0\) as \(n \to \infty\). Using continuity of \(\Phi\), it follows that there exists some \(N \in \mathbb{N}\) such that \(\|\Phi(f_N)\| > \|f_N\|_{\infty, \overline{\mathbb{D}}}\). By definition of \(\mathcal{A}_0\), there exist \(f_{N,1}, f_{N,2} \in A(\mathbb{D})\) such that
\[ f_N(z) = f_{N,1}(z)z^2 + f_{N,2}(z)z^3. \]
Since polynomials are dense in \(A(\mathbb{D})\), one can obtain sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\) of polynomials such that
\[ \lim_{n \to \infty} \|f_{N,1} - \alpha_n\|_{\infty, \overline{\mathbb{D}}} = 0 = \lim_{n \to \infty} \|f_{N,2} - \beta_n\|_{\infty, \overline{\mathbb{D}}}. \]
Consequently, we have that the sequence of polynomials \(\{\gamma_n\}\) given by \(\gamma_n(z) = \alpha_n(z)z^2 + \beta_n(z)z^3\) converges to \(f_N\) uniformly over \(\overline{\mathbb{D}}\). Again by continuity of \(\Phi\), it follows that there is some \(N_0\) such
that \(|\Phi(\gamma_0)| > |\gamma_0|\). Let us define \(p(z) := \gamma_0(z)\). Then there exist polynomials \(p_1(z), p_2(z)\) such that
\[
p(z) = p_1(z)z^2 + p_2(z)z^3.
\]
So, combining all facts together we have that
\[
|\Phi(p)| > |p|_{\infty,D}. \tag{7.1}
\]
It follows from the definition of \(p(z)\) that we can rewrite \(p(z) = \tilde{p}(z^2, z^3)\) for some polynomial \(\tilde{p}\) (not necessarily unique) in two variables. Then it follows from the definition of the representation \(\Phi\) that
\[
\Phi(p) = \tilde{p}(X,Y). \tag{7.2}
\]
We claim that
\[
||p||_{\infty,D} = ||\tilde{p}||_{\infty,Z(q)\cap D^2} \tag{7.3}
\]
where \(q(z_1z_2) = z_1^3 - z_2^2\). To see this, note that there is some \(\alpha \in \mathbb{D}\) such that \(|p|_{\infty,D} = |p(\alpha)|\) and so,
\[
||p||_{\infty,D} = |p(\alpha)| = |\tilde{p}(\alpha^2, \alpha^3)| \leq \sup\{|\tilde{p}(z_1, z_2)| : z_1^3 = z_2^2 & (z_1, z_2) \in \mathbb{D}^2\} = ||\tilde{p}||_{\infty,Z(q)\cap D^2}.
\]
Let \((\alpha, \beta) \in Z(q) \cap D^2\) such that \(||\tilde{p}||_{\infty,Z(q)\cap D^2} = |\tilde{p}(\alpha, \beta)|\). To prove the reverse of inequality in (7.3), we need to show that there is some \(t_0 \in \mathbb{D}\) such that \(|\tilde{p}(\alpha, \beta)| = |p(t_0)|\). Let \(\alpha = re^{i\theta}\) for some \(r \in [0,1]\) and \(-\pi < \theta \leq \pi\). Define \(t = \sqrt{r}e^{i\theta/2}\) which is in \(\mathbb{D}\) and \(t^2 = \alpha\). For this choice of \(t\), we have \(\beta^2 = \alpha^3 = r^3e^{3i\theta}\). Thus either \(\beta = r\sqrt{r}e^{3i\theta/2} = t^3\) or \(\beta = -r\sqrt{r}e^{3i\theta/2} = -t^3\) and consequently, either \((\alpha, \beta) = (t^2, t^3)\) or \((\alpha, \beta) = ((-t)^2, (-t)^3)\). In either case, we have that there is some \(t_0 \in \mathbb{D}\) such that \((\alpha, \beta) = (t_0^2, t_0^3)\). Consequently,
\[
||\tilde{p}||_{\infty,Z(q)\cap D^2} = |\tilde{p}(\alpha, \beta)| = |\tilde{p}(t_0^2, t_0^3)| = |p(t_0)| \leq ||p||_{\infty,D}
\]
and thus (7.3) holds. Therefore, it follows from (7.1)-(7.3) that
\[
||\tilde{p}(X,Y)|| = ||\Phi(p)|| > ||p||_{\infty,D} = ||\tilde{p}||_{\infty,Z(q)\cap D^2}.
\]
Therefore, \(Z(q) \cap D^2\) is not a spectral set for the commuting pair \((X,Y)\) of contractions but \(q(z_1, z_2)\) is an irreducible toral polynomial such that \(q(X,Y) = X^3 - Y^2 = 0\).

Putting everything together, we have the following result.

**Lemma 7.7.** Let \((T_1, T_2)\) be a commuting pair of Hilbert space contractions and let \(q(z_1, z_2)\) be a toral polynomial. If \(Z(q) \cap D^2\) is a spectral set for \((T_1, T_2)\), then \(q(T_1, T_2) = 0\) but the converse does not hold even if \(q\) is an irreducible polynomial.

Now we present a main result of this Section.

**Theorem 7.8.** Let \((T_1, T_2)\) be a pair of commuting contractions acting on a Hilbert space \(\mathcal{H}\). Then \((T_1, T_2)\) dilates to a toral pair of unitaries if and only if there is a toral polynomial \(q(z_1, z_2)\) such that \(Z(q) \cap D^2\) is a complete spectral set for \((T_1, T_2)\).

**Proof.** Assume that \((T_1, T_2)\) admits dilation to a toral pair of unitaries \((U_1, U_2)\) on \(\mathcal{H} \supsetneq \mathcal{H}\). Let \(q\) be a toral polynomial such that \(q(U_1, U_2) = 0\). It follows from Proposition 7.3 and Theorem 6.7.
that $Z(q) \cap T^2 = b(Z(p) \cap \mathbb{T}^2)$ is a spectral set for $(U_1, U_2)$. Furthermore, for every $f \in \text{Rat}(\mathbb{T}^2)$, we have

$$f(T_1, T_2) = P_{\mathcal{K}} f(U_1, U_2)|_{\mathcal{K}}. \quad (7.4)$$

In particular, $(7.4)$ holds for every polynomial $f$ in two variables. By Proposition 2.2 and Lemma 6.5, $(7.4)$ holds for every function $f$ in $\text{Rat}(Z(q) \cap \mathbb{T}^2)$. Consequently, there is a commuting pair of normal operators $(U_1, U_2)$ on $\mathcal{K} \supseteq \mathcal{H}$ such that $\sigma_T(U_1, U_2) \subseteq b(Z(q) \cap \mathbb{T}^2)$ and

$$f(T_1, T_2) = P_{\mathcal{K}} f(U_1, U_2)|_{\mathcal{K}}$$

for every $f \in \text{Rat}(Z(q) \cap \mathbb{T}^2)$. It follows from Theorem 2.3 (Arveson’s theorem) that $Z(q) \cap \mathbb{T}^2$ is a complete spectral set for $(T_1, T_2)$.

Conversely, assume that $Z(q) \cap \mathbb{T}^2$ is a complete spectral set for $(T_1, T_2)$ for some toral polynomial $q(z_1, z_2)$. Again by Arveson’s theorem, there is a commuting pair of normal operators $(U_1, U_2)$ on $\mathcal{H} \supseteq \mathcal{K}$ such that $\sigma_T(U_1, U_2) \subseteq b(Z(q) \cap \mathbb{T}^2)$ and $f(T_1, T_2) = P_{\mathcal{K}} f(U_1, U_2)|_{\mathcal{K}}$ for every holomorphic polynomial $f$ in two variables. It follows from Theorem 6.7 that $b(Z(q) \cap \mathbb{T}^2) = Z(q) \cap \mathbb{T}^2$ and so, $\sigma_T(U_1, U_2) \subseteq Z(q) \cap T^2$. By Proposition 7.3, $Z(q) \cap \mathbb{T}^2$ is a spectral set for $(U_1, U_2)$ and $q(U_1, U_2) = 0$. Thus $(U_1, U_2)$ is a toral pair of unitaries that dilates $(T_1, T_2)$. The proof is now complete.

The following theorem is an obvious corollary of Proposition 7.2 and Theorem 7.8.

**Theorem 7.9.** Let $(V_1, V_2)$ be a pair of commuting isometries acting on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

1. $(V_1, V_2)$ is a toral pair;
2. $(V_1, V_2)$ has an extension to a toral pair of commuting unitaries;
3. There is a toral polynomial $q$ such that $Z(q) \cap \mathbb{T}^2$ is a spectral set for $(V_1, V_2)$;
4. There is a toral polynomial $q$ such that $Z(q) \cap \mathbb{T}^2$ is a complete spectral set for $(V_1, V_2)$.

We now focus on proving Theorem 1.1, one of the main results of this article. It demands a few preparatory results and they are provided below.

**Proposition 7.10.** If a pair of (toral) commuting contractions $(T_1, T_2)$ on a Hilbert space $\mathcal{H}$ admits a dilation to a (toral) pair of commuting isometries, then it has a minimal dilation to a (toral) pair of commuting isometries.

**Proof.** Let $(\tilde{V}_1, \tilde{V}_2)$ on $\mathcal{K} \supseteq \mathcal{H}$ be an isometric dilation of $(T_1, T_2)$. We can always have such a dilation by Andô’s dilation theorem [11]. Also, let $q$ be a toral polynomial such that $q(\tilde{V}_1, \tilde{V}_2) = 0$. Define

$$\mathcal{K} = \text{span}\{\tilde{V}_i^j V_2^j h : h \in \mathcal{K} \text{ and } i, j \in \mathbb{N} \cup \{0\}\}.$$

It is easy to see that $\mathcal{K}$ is invariant under $\tilde{V}_1^j$ and $\tilde{V}_2^i$ for any non-negative integers $i, j$. We denote by $(V_1, V_2)$ the following pair: $(V_1, V_2) := (V_1|_{\mathcal{K}}, \tilde{V}_2|_{\mathcal{K}})$. Then $(V_1, V_2)$ is also a commuting pair of isometries and $q(V_1, V_2) = 0$. Moreover,

$$\mathcal{K} = \text{span}\{V_1^i V_2^j h : h \in \mathcal{H} \text{ and } i, j \in \mathbb{N} \cup \{0\}\}.$$

Thus, $P_{\mathcal{K}}(V_1^i V_2^j h) = T_1^i T_2^j h$ for any $h \in \mathcal{H}$ and $i, j \in \mathbb{N} \cup \{0\}$. Therefore, $(V_1, V_2)$ on $\mathcal{K}$ is a minimal isometric dilation of $(T_1, T_2)$ with $q(V_1, V_2) = 0$. The proof is now complete.
Proposition 7.11. Let \((T_1, T_2)\) be a commuting pair of contractions acting on a Hilbert space \(\mathcal{H}\) and let \((V_1, V_2)\) on \(\mathcal{K} \supseteq \mathcal{H}\) be a minimal isometric dilation of \((T_1, T_2)\). Then \((V_1^*, V_2^*)\) is a co-isometric extension of \((T_1^*, T_2^*)\). Conversely, if \((V_1^*, V_2^*)\) is a co-isometric extension of \((T_1^*, T_2^*)\), then \((V_1, V_2)\) is an isometric dilation of \((T_1, T_2)\).

Proof. We first prove that \(T_1 P_{\mathcal{H}} = P_{\mathcal{H}} V_1\) and \(T_2 P_{\mathcal{H}} = P_{\mathcal{H}} V_2\). By definition, we have

\[
\mathcal{K} = \operatorname{span}\{V_i^j V_j^k h : h \in \mathcal{H} \text{ and } i, j \in \mathbb{N} \cup \{0\}\}.
\]

Now for \(h \in \mathcal{K}\), we have

\[
T_1 P_{\mathcal{H}}(V_1^i V_2^j h) = T_1(T_1^i T_2^j h) = T_1^i T_2^j h = P_{\mathcal{H}}(V_1^i V_2^j h) = P_{\mathcal{H}} V_1(V_1^i V_2^j h).
\]

By continuity argument we have \(T_1 P_{\mathcal{H}} = P_{\mathcal{H}} V_1\) and similarly \(T_2 P_{\mathcal{H}} = P_{\mathcal{H}} V_2\). Also, for \(h \in \mathcal{H}\) and \(k \in \mathcal{K}\), we have

\[
\langle T_1^i h, k \rangle = \langle P_{\mathcal{H}} T_1^i h, k \rangle = \langle T_1^i h, P_{\mathcal{H}} k \rangle = \langle h, T_1 P_{\mathcal{H}} k \rangle = \langle h, P_{\mathcal{H}} V_1 k \rangle = \langle V_1^i h, k \rangle.
\]

Hence, \(T_1^i = V_1^i |_{\mathcal{H}}\) and similarly \(T_2^i = V_2^i |_{\mathcal{H}}\). The converse part is obvious.

Lemma 7.12. Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be Hilbert spaces. Let \(V_1 = \begin{bmatrix} T_1 & 0 \\ C_1 & D_1 \end{bmatrix}\) and \(V_2 = \begin{bmatrix} T_2 & 0 \\ C_2 & D_2 \end{bmatrix}\) be commuting contractions acting on \(\mathcal{H}_1 \oplus \mathcal{H}_2\). If \(f(T_1, T_2) = 0 = g(D_1, D_2)\) for \(f, g \in \mathbb{C}[z_1, z_2]\), then \(f(V_1, V_2)g(V_1, V_2) = 0\). Moreover, \((V_1, V_2)\) is a toral pair if and only if both \((T_1, T_2)\) and \((D_1, D_2)\) are toral pairs.

Proof. For any \(p \in \mathbb{C}[z_1, z_2]\), a routine calculation gives

\[
p(V_1, V_2) = \begin{bmatrix} p(T_1, T_2) & 0 \\ * & p(D_1, D_2) \end{bmatrix}.
\] (7.5)

Therefore, we have

\[
f(V_1, V_2)g(V_1, V_2) = g(V_1, V_2)f(V_1, V_2)
\]

\[
= \begin{bmatrix} g(T_1, T_2) & 0 \\ * & g(D_1, D_2) \end{bmatrix} \begin{bmatrix} f(T_1, T_2) & 0 \\ * & f(D_1, D_2) \end{bmatrix}
\]

\[
= \begin{bmatrix} g(T_1, T_2) & 0 \\ * & 0 \end{bmatrix} \begin{bmatrix} f(T_1, T_2) & 0 \\ * & f(D_1, D_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

\[
\text{Let } q \text{ be a toral polynomial such that } q(V_1, V_2) = 0. \text{ By (7.5), } q(T_1, T_2) = 0 \text{ and } q(D_1, D_2) = 0. \text{ Conversely, let } f \text{ and } g \text{ be toral polynomials such that } f(T_1, T_2) = 0 = g(D_1, D_2). \text{ Then, } f(V_1, V_2)g(V_1, V_2) = 0. \text{ Since } Z(fg) = Z(f) \cup Z(g), \text{ it follows that } fg \text{ is a toral polynomial. Combining everything together, we have that } (V_1, V_2) \text{ is a toral pair annihilated by } fg \text{ and the proof is complete.}
\]

Now we are in a position to present the proof of the main result of this Section.

Proof of Theorem 1.1. Suppose \((T_1, T_2)\) dilates to a toral pair of commuting isometries \((\tilde{V}_1, \tilde{V}_2)\) on \(\tilde{\mathcal{K}} \supseteq \mathcal{H}\). Let \(q\) be a toral polynomial that annihilates \((\tilde{V}_1, \tilde{V}_2)\). Let us define \((V_1, V_2) := (\tilde{V}_1 |_{\mathcal{K}}, \tilde{V}_2 |_{\mathcal{K}})\), where \(\mathcal{K} = \operatorname{span}\{V_i^j V_j^k h : h \in \mathcal{H} \text{ and } i, j \in \mathbb{N} \cup \{0\}\}\), the minimal isometric
dilation space of \((T_1, T_2)\) with respect to the dilation \((\tilde{V}_1, \tilde{V}_2)\). It follows from Proposition 7.10 that \((V_1, V_2)\) on \(\mathcal{K}\) is an isometric dilation of \((T_1, T_2)\) and \(q(V_1, V_2) = 0\). By Proposition 7.11 \((V_1^*, V_2^*)\) is a co-isometric extension of \((T_1^*, T_2^*)\). Let \(\mathcal{K}^\perp = \mathcal{K} \ominus \mathcal{K}\). Then there exist \(C_1, C_2 \in \mathcal{B}(\mathcal{K}, \mathcal{K}^\perp)\) and \(D_1, D_2 \in \mathcal{B}(\mathcal{K}^\perp)\) such that

\[
V_1 = \begin{bmatrix} T_1 & 0 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} T_2 & 0 \\ C_2 & D_2 \end{bmatrix}
\]

with respect to the orthogonal decomposition \(\mathcal{K} = \mathcal{K} \ominus \mathcal{K}^\perp\). The fact that \((V_1, V_2)\) is a pair of commuting isometries gives the following:

- \(V_1 V_2 = V_2 V_1\)
- \(V_1^* V_1 = I = V_2^* V_2\).

Straightforward computations show that

\[
V_1 V_2 = V_2 V_1 \iff \begin{bmatrix} T_1 & 0 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} T_2 & 0 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} T_2 & 0 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ C_1 & D_1 \end{bmatrix}
\]

\(\iff \begin{bmatrix} T_1 T_2 \\ C_1 T_2 + D_1 C_2 \\ D_1 D_2 \end{bmatrix} = \begin{bmatrix} T_2 T_1 & 0 \\ C_2 T_1 + D_2 C_1 & D_2 D_1 \end{bmatrix}\) \quad (7.6)

and

\[
V_i^* V_i = I \iff \begin{bmatrix} T_i^* & C_i^* \\ 0 & D_i^* \end{bmatrix} \begin{bmatrix} T_i & 0 \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} T_i^* T_i + C_i^* C_i & C_i^* D_i \\ D_i^* C_i & D_i^* D_i \end{bmatrix} \iff \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (i = 1, 2).\) \quad (7.7)

It follows from (7.6) and (7.7) that \((D_1, D_2)\) is a commuting pair of isometries on \(\mathcal{K}^\perp\). Moreover, \(q(D_1, D_2) = q(V_1, V_2)\) on \(\mathcal{K}\) and so, \((D_1, D_2)\) is a toral pair. Again by (7.6) and (7.7), the operator equations in the necessary part of this theorem follow immediately.

We now prove the converse. Suppose there exist a Hilbert space \(\mathcal{K} \supsetneq \mathcal{H}\), a toral pair of isometries \((D_1, D_2)\) on \(\mathcal{K} = \mathcal{K} \ominus \mathcal{K}\) and \(C_1, C_2 \in \mathcal{B}(\mathcal{K}, \mathcal{K}^\perp)\) such that the operator equations given in the statement of the theorem hold. Set

\[
V_1 = \begin{bmatrix} T_1 & 0 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} T_2 & 0 \\ C_2 & D_2 \end{bmatrix}
\]

with respect to orthogonal decomposition \(\mathcal{K} = \mathcal{K} \ominus \mathcal{K}^\perp\). It follows from (7.6) and (7.7) that \((V_1, V_2)\) is a pair of commuting isometries. Evidently, \(T_1^* = V_1^*|_\mathcal{K}, T_2^* = V_2^*|_\mathcal{K}\) and hence \((V_1, V_2)\) dilates \((T_1, T_2)\). We now prove that \((V_1, V_2)\) is a toral pair. Since \((T_1, T_2)\) and \((D_1, D_2)\) are toral pairs, there are toral polynomials \(f\) and \(g\) such that \(f(T_1, T_2) = 0 = g(D_1, D_2)\). It follows from Lemma 7.12 that the toral polynomial \(fg\) annihilates \((V_1, V_2)\) and so, \((V_1, V_2)\) is a toral pair. By Proposition 7.2 \((V_1, V_2)\) extends a pair of commuting unitaries \((U_1, U_2)\) such that \(f(U_1, U_2)g(U_1, U_2) = 0\) and so, \((T_1, T_2)\) dilates \((U_1, U_2)\). It follows from Theorem 7.8 that \(Z(fg) \cap \mathbb{D}^2\) is a complete spectral set for \((T_1, T_2)\). The proof is now complete.

\[\blacksquare\]

8. DILATION OF \(\Gamma\)-DISTINGUISHED \(\Gamma\)-CONTRACTIONS

In this Section, we first find analogues of the results of Section 7 in the symmetrized bidisc setting. Then we investigate more about dilation of a \(\Gamma\)-distinguished \(\Gamma\)-contraction \((S, P)\) when the defect spaces \(\mathcal{D}_P, \mathcal{D}_P^*\) are finite dimensional. We begin with the most expected and natural lemma that guarantees that every \(\Gamma\)-distinguished \(\Gamma\)-isometry extends to a \(\Gamma\)-distinguished \(\Gamma\)-unitary.
Proposition 8.1. Every \( \Gamma \)-distinguished \( \Gamma \)-isometry admits a \( \Gamma \)-distinguished \( \Gamma \)-unitary extension.

Proof. Let \((T, V)\) be a \( \Gamma \)-isometry acting on a Hilbert space \( \mathcal{H} \) and \( f(T, V) = 0 \) for some \( \Gamma \)-distinguished polynomial \( f(z_1, z_2) \). Let \((N_1, N_2)\) be a \( \Gamma \)-unitary extension of \((T, V)\) and also let,

\[
\mathcal{H} = \overline{\text{span}} \left\{ N_1^n N_2^j h : h \in \mathcal{H}, i, j \in \mathbb{N} \cup \{0\} \right\}.
\]

We now prove that the minimal c.n.e. \((U_1, U_2) = (N_1|_\mathcal{H}, N_2|_\mathcal{H})\) of \((T, V)\) is a \( \Gamma \)-distinguished \( \Gamma \)-unitary. Since \((N_1, N_2)\) is a \( \Gamma \)-unitary and \( \mathcal{H} \) is a common reducing subspace of \( N_1, N_2 \), thus, \((U_1, U_2)\) is a \( \Gamma \)-unitary on \( \mathcal{H} \). It follows from Lemma 7.1 that \( f(U_1, U_2) \) is unitarily equivalent to the minimal normal extension of \( f(T, V) \) and thus, \( f(U_1, U_2) = 0 \). Therefore, \((U_1, U_2)\) is a \( \Gamma \)-distinguished \( \Gamma \)-unitary on \( \mathcal{H} \) so that \( U_1|_\mathcal{H} = T \) and \( U_2|_\mathcal{H} = V \).

\[\blacksquare\]

After the above theorem, it is evident that dilation of a \( \Gamma \)-contraction to a \( \Gamma \)-distinguished \( \Gamma \)-isometry implies and is implied by dilation to a \( \Gamma \)-distinguished \( \Gamma \)-unitary. The next corollary is an easy consequence of the above proposition.

Corollary 8.2. A \( \Gamma \)-isometry is \( \Gamma \)-distinguished if and only if it extends to a \( \Gamma \)-distinguished \( \Gamma \)-unitary.

We look a little deeper into the nature of a \( \Gamma \)-distinguished \( \Gamma \)-unitary and have the following.

Proposition 8.3. Let \((U_1, U_2)\) be a \( \Gamma \)-unitary and let \( p \in \mathbb{C}[z_1, z_2] \). Then \( p(U_1, U_2) = 0 \) if and only if \( Z(p) \cap b\Gamma \) is a spectral set for \((U_1, U_2)\).

Proof. Suppose that a polynomial \( p \) annihilates a \( \Gamma \)-unitary \((U_1, U_2)\). Then the spectral mapping theorem implies that

\[
p(\sigma_T(U_1, U_2)) = \sigma(p(U_1, U_2)) = \{0\}.
\]

Thus, \( \sigma_T(U_1, U_2) \subseteq Z(p) \cap b\Gamma \). Conversely, let us assume that \( p(\sigma_T(U_1, U_2)) = \{0\} \). Since \( U_1 \) and \( U_2 \) are commuting normal operators, \( p(U_1, U_2) \) is a normal operator too and we have

\[
\|p(U_1, U_2)\| = \sup\{|\lambda| : \lambda \in \sigma(p(U_1, U_2))\} = \sup\{|\lambda| : \lambda \in p(\sigma_T(U_1, U_2))\} = 0.
\]

Therefore, \( p \) annihilates \((U_1, U_2)\).

\[\blacksquare\]

It follows from the above proposition that a \( \Gamma \)-unitary \((U_1, U_2)\) is annihilated by a \( \Gamma \)-distinguished polynomial \( p \) if and only if \( \sigma_T(U_1, U_2) \subseteq Z(p) \cap b\Gamma \). The latter is possible if and only if \( Z(p) \cap b\Gamma \) is a spectral set for \((U_1, U_2)\). We prove an analogue of the above result for the \( \Gamma \)-isometries. If \((V_1, V_2)\) is a \( \Gamma \)-isometry on a Hilbert space \( \mathcal{H} \) then it has a \( \Gamma \)-unitary extension \((U_1, U_2)\) on some space \( \mathcal{H}' \supseteq \mathcal{H} \). For any polynomial \( p(z_1, z_2) \), it follows that \( p(V_1, V_2) = p(U_1, U_2)|_\mathcal{H} \). This shows that \( p(V_1, V_2) \) is a subnormal operator and hence a hyponormal operator. It is a well-known result [40] that for a hyponormal operator \( T \), \( \|T\| \) is the spectral radius of \( T \). Using this, we have that \( \|p(V_1, V_2)\| \) is same as the spectral radius of \( p(V_1, V_2) \). Being armed with this observation, we now prove the following result.

Proposition 8.4. A \( \Gamma \)-isometry \((V_1, V_2)\) is \( \Gamma \)-distinguished if and only if there is a \( \Gamma \)-distinguished polynomial \( p \) such that \( Z(p) \cap \Gamma \) is a spectral set for \((V_1, V_2)\).

Proof. Let \((V_1, V_2)\) be annihilated by a \( \Gamma \)-distinguished polynomial \( p \). Then \( \sigma_T(V_1, V_2) \subseteq Z(p) \cap \Gamma \). Since \( \sigma_T(V_1, V_2) \) is a spectral set for \((V_1, V_2)\), we have that \( Z(p) \cap \Gamma \) is a spectral set for \((V_1, V_2)\). Conversely, if there is a \( \Gamma \)-distinguished polynomial \( p \) such that \( Z(p) \cap \Gamma \) is a spectral set for
(V_1, V_2), then \( \sigma(p(V_1, V_2)) = \{0\} \). Since \( V_1 \) and \( V_2 \) have commuting normal extensions, it follows that \( p(V_1, V_2) \) is a subnormal operator. Hence, \( \|p(V_1, V_2)\| = \sup\{|z| : z \in \sigma(p(V_1, V_2))\} = 0 \). This gives that \( p(V_1, V_2) = 0 \).

Next, we provide a counter-example to show that the above results are not true in general for a \( \Gamma \)-contraction. Indeed, we show that there is a \( \Gamma \)-contraction \((S, P)\) that is annihilated by a \( \Gamma \)-distinguished polynomial \( q \) but \( Z(q) \cap \Gamma \) is not a spectral set for \((S, P)\).

**Example 8.5.** Consider the commuting operators acting on \( \mathbb{C}^4 \) given by

\[
A_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

We define \( T_1 = rA_1 \) and \( T_2 = rA_2 \), where \( r \neq 0 \) is chosen in such a way that \( \|T_1\|, \|T_2\| \leq 1 \). The commuting pair of operators \((S, P) = (T_1 + T_2, T_1 T_2)\) is a \( \Gamma \)-contraction and is given by

\[
S = r \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
1 & -1 & 1 & 0
\end{pmatrix}, \quad P = r^2 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

We have

\[
4P - S^2 = 4r^2 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} - r^2 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0
\end{pmatrix} = r^2 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0
\end{pmatrix}.
\]

This shows that the \( \Gamma \)-distinguished polynomial \( p(z_1, z_2) = (4z_2 - z_1^2)^2 \) annihilates \((S, P)\). Let if possible, \( Z(p) \cap \Gamma \) be a spectral set for \((S, P)\) then we must have that

\[
\|f(S, P)\| \leq \|f\|_{\infty, Z(p) \cap \Gamma} = \sup\{|f(z_1, z_2)| : (z_1, z_2) \in Z(p) \cap \Gamma\}
\]

for every \( f \in \mathbb{C}[z_1, z_2] \). In particular, the von Neumann’s inequality must hold for \( f(z_1, z_2) = 4z_2 - z_1^2 \). Since \( Z(f) = Z(p) \), we have \( \|f\|_{\infty, Z(p) \cap \Gamma} = 0 \) and \( \|f(S, P)\| = \|4P - S^2\| > 0 \). Hence, \( Z(p) \cap \Gamma \) is a not a spectral set for \((S, P)\).

The following is an analogue of Theorem 7.8 and is a main result of this Section.

**Theorem 8.6.** Let \((S, P)\) be a \( \Gamma \)-contraction on a Hilbert space \( \mathcal{H} \). Then \((S, P)\) admits a \( \Gamma \)-distinguished \( \Gamma \)-unitary dilation if and only if there is a \( \Gamma \)-distinguished polynomial \( p(z_1, z_2) \) such that \( Z(p) \cap \Gamma \) is a complete spectral set for \((S, P)\).

**Proof.** Assume that \((S, P)\) admits a \( \Gamma \)-unitary dilation \((U_1, U_2)\) on \( \mathcal{H} \supseteq \mathcal{H} \) such that \( p(U_1, U_2) = 0 \) for some \( \Gamma \)-distinguished polynomial \( p \). It follows from Proposition 8.3 and Theorem 6.10 that \( b(Z(p) \cap \Gamma) = Z(p) \cap b\Gamma \) and that \( b(Z(p) \cap \Gamma) \) is a spectral set for \((U_1, U_2)\). Furthermore, for every \( f \in \text{Rat}(\Gamma) \), we have

\[
f(S, P) = P_{\mathcal{H}}f(U_1, U_2)|_{\mathcal{H}}.
\]

In particular, \( (8.1) \) holds for every polynomial \( f \) in two variables. By Proposition 2.2 and Lemma 6.5, \( (8.1) \) holds for every function \( f \) in \( \text{Rat}(Z(p) \cap \Gamma) \). Consequently, there is a commuting pair of normal operators \((U_1, U_2)\) on \( \mathcal{H} \supseteq \mathcal{H} \) with \( \sigma_f(U_1, U_2) \subseteq b(Z(p) \cap \Gamma) \) such that

\[
f(S, P) = P_{\mathcal{H}}f(U_1, U_2)|_{\mathcal{H}}
\]
for every \( f \in \text{Rat}(Z(p) \cap \Gamma) \). By Theorem 2.3 (Arveson’s theorem), \( Z(p) \cap \Gamma \) is a complete spectral set for \((S, P)\).

Conversely, assume that \( Z(p) \cap \Gamma \) is a complete spectral set for \((S, P)\) for some \( \Gamma \)-distinguished polynomial \( p \). Again by Arveson’s theorem, there is a commuting pair of normal operators \((U_1, U_2)\) on \( \mathcal{H} \supsetneq \mathcal{H} \) such that \( \sigma_f(U_1, U_2) \subseteq b(Z(p) \cap \Gamma) \) and \( f(S, P) = P|f(U_1, U_2)|_{\mathcal{H}} \) for every holomorphic polynomial \( f \) in two variables. It follows from Theorem 6.10 that \( b(Z(p) \cap \Gamma) = Z(p) \cap b\Gamma \) and so, \( \sigma_f(U_1, U_2) \subseteq Z(p) \cap b\Gamma \). Consequently, by Proposition 8.3 we have that \((U_1, U_2)\) is a \( \Gamma \)-unitary and \( p(U_1, U_2) = 0 \). Hence, \((U_1, U_2)\) is a \( \Gamma \)-distinguished \( \Gamma \)-unitary that dilates \((S, P)\). The proof is now complete.

The following theorem follows as an immediate corollary of Proposition 8.1 and Theorem 8.6.

**Theorem 8.7.** Let \((V_1, V_2)\) be a \( \Gamma \)-isometry on a space \( \mathcal{H} \). Then the following are equivalent:

1. \((V_1, V_2)\) is a \( \Gamma \)-distinguished;
2. \((V_1, V_2)\) has a \( \Gamma \)-distinguished \( \Gamma \)-unitary extension;
3. There is a \( \Gamma \)-distinguished polynomial \( p \) such that \( Z(p) \cap \Gamma \) is a spectral set for \((V_1, V_2)\);
4. There is a \( \Gamma \)-distinguished polynomial \( p \) such that \( Z(p) \cap \Gamma \) is a complete spectral set for \((V_1, V_2)\).

We now present a few examples of \( \Gamma \)-distinguished \( \Gamma \)-contractions that admit a \( \Gamma \)-distinguished \( \Gamma \)-isometric dilation. Recall from the literature (e.g., see [4]) that the royal variety in the symmetrized bidisc is defined to be the set \( R := \{ (2z, z^2) : z \in \mathbb{D} \} \), which is a distinguished variety in the symmetrized bidisc.

**Example 8.8.** Let \( f(z_1, z_2) = z_1^2 - 4z_2 \) which is a \( \Gamma \)-distinguished polynomial. Then

\[ Z(f) \cap \Gamma = \{ (z_1, z_2) \in \Gamma : z_1^2 = 4z_2 \} = \{ (2z, z^2) : z \in \mathbb{D} \} = \overline{R}. \]

Let \((S, P)\) be a \( \Gamma \)-contraction such that \( f(S, P) = 0 \) and so, \( P = S^2/4 \). Take any matricial polynomial \([f_{ij}]_{1 \leq i, j \leq n}\) and define \( g_{ij}(z) = f_{ij}(2z, z^2) \) for every \( i, j \). Then

\[
\| [f_{ij}(S, P)]_{i, j} \| = \| [f_{ij}(S, S^2/4)]_{i, j} \| = \| [g_{ij}(S/2)]_{i, j} \|
\leq \max_{z \in \overline{D}} \| [g_{ij}(z)]_{i, j} \|
\leq \max \{ \| [f_{ij}(2z, z^2)]_{i, j} \| : z \in \mathbb{D} \}
\leq \max \{ \| [f_{ij}(z_1, z_2)]_{i, j} \| : (z_1, z_2) \in Z(f) \cap \Gamma \}.
\]

Thus \( Z(f) \cap \Gamma \) is a complete spectral set for \((S, P)\). It follows from Theorem 8.6 that \((S, P)\) dilates to a \( \Gamma \)-distinguished \( \Gamma \)-isometry.

**Example 8.9.** For \( a \in \mathbb{D} \), define \( f(z_1, z_2) := z_1 - \overline{a}z_2 - a \). It follows from Theorem 5.11 that \( f \) is a \( \Gamma \)-distinguished polynomial. Let \((S, P)\) be a \( \Gamma \)-contraction such that \( f(S, P) = 0 \) and so, \( S = \overline{a}P + aI \). Observe that if \( z \in \overline{D} \), then \((s, p) = (\overline{a}z + a, z) \in \Gamma \) since \( |s| \leq 2 \) and

\[
|s - \overline{p}| = |\overline{a}z + a - (\overline{a}z + a)| = |a(1 - |z|^2)| < 1 - |z|^2 = 1 - |p|^2.
\]

Take any matricial polynomial \([f_{ij}]_{1 \leq i, j \leq n}\) and define \( h_{ij}(z) = f_{ij}(\overline{a}z + a, z) \) for every \( i, j \). Then
Proof. It follows from the definition of minimality that
\[
\| [f_{ij}(S,P)]_{i,j} \| = \| [f_{ij}(\mathbf{a}P + aI,P)]_{i,j} \| = \| [h_{ij}(P)]_{i,j} \|
\]
\[
\leq \max_{z \in \overline{D}} \| [h_{ij}(z)]_{i,j} \|
\]
\[
\leq \max \{ \| [f_{ij}(z,z)]_{i,j} \| : z \in \overline{D} \}
\]
\[
\leq \max \{ \| [f_{ij}(z_{1},z_{2})]_{i,j} \| : (z_{1},z_{2}) \in Z(f) \cap \Gamma \}
\]
Therefore, \( Z(f) \cap \Gamma \) is a complete spectral set for \((S,P)\). By Theorem 8.6, \((S,P)\) dilates to a \(\Gamma\)-distinguished \(\Gamma\)-isometry.

We now move to prepare for giving a proof to Theorem 1.7, the main result of this Section and an analogue of Theorem 1.4. Needless to mention, we shall develop similar preparatory results as in the previous section. So, let us begin with the following proposition.

**Proposition 8.10.** Let \((S,P)\) be a \(\Gamma\)-distinguished \(\Gamma\)-contraction acting on a Hilbert space \(\mathcal{K}\). If \((S,P)\) has a \(\Gamma\)-distinguished \(\Gamma\)-isometric dilation, then it has a minimal \(\Gamma\)-distinguished \(\Gamma\)-isometric dilation.

**Proof.** Let \((\tilde{T},\tilde{V})\) be a \(\Gamma\)-distinguished \(\Gamma\)-isometry acting on a Hilbert space \(\tilde{\mathcal{K}} \supseteq \mathcal{K}\) such that \((\tilde{T},\tilde{V})\) dilates \((S,P)\). Let \(p\) be a \(\Gamma\)-distinguished polynomial that annihilates \((\tilde{T},\tilde{V})\). Let us consider
\[
\mathcal{K} = \text{span}\{ \tilde{T}^{i} \tilde{V}^{j} h : h \in \mathcal{K} \text{ and } i, j \in \mathbb{N} \cup \{0\} \}.
\]
Evidently, \(\mathcal{K}\) is invariant under \(\tilde{V}_{1}^{i}\) and \(\tilde{V}_{2}^{j}\) for any non-negative integers \(i, j\). We denote by
\[
(T,V) = (\tilde{T}|_{\mathcal{K}},\tilde{V}|_{\mathcal{K}})
\]
and so, \((T,V)\) is a \(\Gamma\)-isometry and \(q(T,V) = 0\). Moreover,
\[
\mathcal{K} = \text{span}\{ T^{i} V^{j} h : h \in \mathcal{K} \text{ and } i, j \in \mathbb{N} \cup \{0\} \}.
\]
Thus \(P_{\mathcal{K}}(T^{i} V^{j})h = S^{i} P^{j} h\) for any \(h \in \mathcal{K}\) and \(i, j \in \mathbb{N} \cup \{0\}\). Therefore, \((T,V)\) on \(\mathcal{K}\) is a minimal isometric dilation of \((S,P)\) with \(q(T,V) = 0\) and the proof is complete.

**Proposition 8.11.** Let \((S,P)\) be a \(\Gamma\)-contraction acting on a Hilbert space \(\mathcal{K}\) and let \((T,V)\) be its minimal \(\Gamma\)-isometric dilation acting on a Hilbert space \(\mathcal{K}\). Then \((T^{*},V^{*})\) is a \(\Gamma\)-co-isometric extension of \((S^{*},P^{*})\). Conversely, if \((T^{*},V^{*})\) is a \(\Gamma\)-co-isometric extension of \((S^{*},P^{*})\) then \((T,V)\) is a \(\Gamma\)-isometric dilation of \((S,P)\).

**Proof.** It follows from the definition of minimality that
\[
\mathcal{K} = \text{span}\{ T^{i} V^{j} h : h \in \mathcal{K} \text{ and } i, j \in \mathbb{N} \cup \{0\} \}.
\]
We show that \(S P_{\mathcal{K}} = P_{\mathcal{K}} T\) and \(P P_{\mathcal{K}} = P_{\mathcal{K}} V\). For any \(h \in \mathcal{K}\), we have
\[
S P_{\mathcal{K}}(T^{i} V^{j} h) = S(S^{i} P^{j} h) = S^{i+1} P^{j} h = P_{\mathcal{K}}(T^{i+1} V^{j} h) = P_{\mathcal{K}} T(T^{i} V^{j} h).
\]
Using continuity argument, it follows that \(S P_{\mathcal{K}} = P_{\mathcal{K}} T\) and similarly, \(P P_{\mathcal{K}} = P_{\mathcal{K}} V\). Also for \(h \in \mathcal{K}\) and \(k \in \mathcal{K}\), we have
\[
\langle S^{*} h, k \rangle = \langle P_{\mathcal{K}} S^{*} h, k \rangle = \langle S^{*} h, P_{\mathcal{K}} k \rangle = \langle h, S P_{\mathcal{K}} k \rangle = \langle h, P_{\mathcal{K}} T k \rangle = \langle T^{*} h, k \rangle.
\]
Hence, \(S^{*} = T^{*}|_{\mathcal{K}}\) and similarly \(P^{*} = V^{*}|_{\mathcal{K}}\). The converse part is obvious.
Lemma 8.12. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces. Let \( V_1 = \begin{bmatrix} T_1 & 0 \\ C_1 & D_1 \end{bmatrix} \) and \( V_2 = \begin{bmatrix} T_2 & 0 \\ C_2 & D_2 \end{bmatrix} \) be commuting operators acting on \( \mathcal{H}_1 \oplus \mathcal{H}_2 \). If \( f(T_1, T_2) = 0 = g(D_1, D_2) \) for \( f, g \in \mathbb{C}[z_1, z_2] \), then \( f(V_1, V_2)g(V_1, V_2) = 0 \). Moreover, \( (V_1, V_2) \) is a \( \Gamma \)-distinguished \( \Gamma \)-contraction if and only if both \( (T_1, T_2) \) and \( (D_1, D_2) \) are \( \Gamma \)-distinguished \( \Gamma \)-contractions.

Proof. The proof is similar to that of Lemma 7.12 and thus we skip it.

Being armed with all the preparatory result we now give a proof to the main result of this Section. Needless to mention that the proof here will follow arguments analogous to that given in the proof of Theorem 1.1 in the previous section.

Proof of Theorem 1.7 Suppose \((S, P)\) dilates to a \( \Gamma \)-distinguished \( \Gamma \)-isometry \((\tilde{T}, \tilde{V})\) acting on a space \( \tilde{K} \supseteq \mathcal{H} \). Then there is a \( \Gamma \)-distinguished polynomial \( p \in \mathbb{C}[z_1, z_2] \) such that \( p(\tilde{T}, \tilde{V}) = 0 \). Let

\[
(T, V) = (\tilde{T}|_{\mathcal{H}}, \tilde{V}|_{\mathcal{H}})
\]

where \( \mathcal{H} = \overline{\text{span}}\{\tilde{T}^j\tilde{V}^ih : h \in \mathcal{H} \text{ and } i, j \in \mathbb{N} \cup \{0\}\} \), the minimal isometric dilation space of \((S, P)\) with respect to the dilation \((\tilde{T}, \tilde{V})\). It follows from Proposition 8.10 that \((T, V)\) on \( \mathcal{H} \) is a \( \Gamma \)-isometric dilation of \((S, P)\) and \( p(T, V) = 0 \). By Proposition 8.11, \((T^*, V^*)\) is a \( \Gamma \)-co-isometric extension of \((S^*, P^*)\). Let \( \mathcal{H}^\perp = \mathcal{H} \cap \mathcal{H}^\perp \). Then there exist \( C_1, C_2 \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\perp) \) and \( D_1, D_2 \in \mathcal{B}(\mathcal{H}^\perp) \) such that

\[
T = \begin{bmatrix} S & 0 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} P & 0 \\ C_2 & D_2 \end{bmatrix}
\]

with respect to the orthogonal decomposition \( \mathcal{H} = \mathcal{H} \oplus \mathcal{H}^\perp \). A straightforward computation shows that \( TV = VT \) if and only if

\[
\begin{bmatrix} S & 0 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} P & 0 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} S & 0 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} P & 0 \\ C_2 & D_2 \end{bmatrix} \iff \begin{bmatrix} SP & 0 \\ C_1P + D_1C_2 & D_1D_2 \end{bmatrix} = \begin{bmatrix} PS & 0 \\ C_2S + D_2C_1 & D_2D_1 \end{bmatrix}.
\]

Similar computation implies that \( T = T^*V \) if and only if

\[
\begin{bmatrix} S & 0 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} S^* & C_1^* \\ 0 & D_1^* \end{bmatrix} \begin{bmatrix} P & 0 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} S^*P + C_1^*C_2 & C_1^*D_2 \\ D_1^*C_2 & D_1^*D_2 \end{bmatrix}.
\]

Again by routine calculations, we have that \( V^*V = I \) if and only if

\[
\begin{bmatrix} P^* & C_2^* \\ 0 & D_2^* \end{bmatrix} \begin{bmatrix} P & 0 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} P^*P + C_2^*C_2 & C_2^*D_2 \\ D_2^*C_2 & D_2^*D_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]

Combining the equality of the block matrices in all three cases above and applying Theorem 2.8 we have that \((T, V)\) is a \( \Gamma \)-isometry if and only if \( ||T|| \leq 2 \) and the following hold:

1. \( C_1P + D_1C_2 = C_2S + D_2C_1 \),
2. \( D_1D_2 = D_2D_1 \),
3. \( S - S^*P = C_1^*C_2 \),
4. \( C_1^*D_2 = 0 \),
5. \( C_1 = D_1^*C_2 \),
6. \( D_1 = D_1^*D_2 \),
7. \( C_2^*C_2 = D_2^*D_2 \),
8. \( D_2^*D_2 = I \).

Since \((T, V)\) is a \( \Gamma \)-isometry and \( (D_1, D_2) = (T|_{\mathcal{H}^\perp}, V|_{\mathcal{H}^\perp}) \), it follows that \( (D_1, D_2) \) is a \( \Gamma \)-isometry too. The operator equations in (11.1) follow from (8.2). The necessary part of this theorem is now established.
Conversely, let us assume that there is a Hilbert space $\mathcal{H}$ containing $\mathcal{H}$, a $\Gamma$-distinguished $\Gamma$-isometry $(D_1, D_2)$ on $\mathcal{H}^\perp = \mathcal{H} \oplus \mathcal{H}'$ and $C_1, C_2 \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\perp)$ such that (1.1) holds. Set

$$T = \begin{bmatrix} S & 0 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} P & 0 \\ C_2 & D_2 \end{bmatrix}$$

on $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}^\perp$. We have by Lemma 1 in \cite{14} that $\sigma(T) \subseteq \sigma(S) \cup \sigma(D_1)$ and thus $r(T) \leq \max\{r(S), r(D_1)\}$. Since $(S, P)$ and $(D_1, D_2)$ are $\Gamma$-contractions, we must have $r(S) \leq 2$ and $r(D_1) \leq 2$ by Theorem 4.4 in \cite{14}. Thus $r(T) \leq 2$. Theorem 2.14 in \cite{14} states that $(T, V)$ is a $\Gamma$-isometry if and only if $r(T) \leq 2$ and the following hold: $TV = VT$, $T = T^*V$ and $V^*V = I$. With the given hypothesis that $(D_1, D_2)$ is a $\Gamma$-isometry and (1.1) holds, we have that (8.2) holds if and only if $C_1^*D_2 = 0$. In other words, $(T, V)$ is a $\Gamma$-isometry if and only if $C_1^*D_2 = 0$. By part-(5) of (1.1) and part-(3) of (1.1), it follows that

$$C_1^*D_2 = C_2^*D_1D_2 = C_2^*D_2D_1 = 0.$$ 

Consequently, $(T, V)$ is a $\Gamma$-isometry. It is evident that $S^* = T^*|_{\mathcal{H}'}$, $P^* = V^*|_{\mathcal{H}'}$ and so, $(T, V)$ is a $\Gamma$-isometric dilation of $(S, P)$. By Lemma 8.12, $(T, V)$ is $\Gamma$-distinguished. Moreover, if $f$ and $g$ are $\Gamma$-distinguished polynomials that annihilate $(S, P)$ and $(D_1, D_2)$ respectively, then $fg$ annihilates $(T, V)$. Then it follows from Theorem 8.6 that $Z(fg) \cap \Gamma$ is a complete spectral set for $(S, P)$. The proof is now complete.

Now we investigate a few special cases when a $\Gamma$-distinguished $\Gamma$-contraction $(S, P)$ dilates to a $\Gamma$-distinguished $\Gamma$-unitary, especially when $\mathcal{D}_p$ or $\mathcal{D}_p^*$ is finite dimensional. We start with the following result due to Pal and Shalit \cite{33}.

**Theorem 8.13** \cite{33}, Theorem 4.5). Let $(S, P)$ be a $\Gamma$-contraction on a Hilbert space $\mathcal{H}$ such that $(S^*, P^*)$ is pure, and suppose that $\dim \mathcal{D}_p < \infty$. Let $F$ be the fundamental operator of $(S, P)$ and let

$$\Lambda = \{(z_1, z_2) \in \Gamma : \det(F^* + z_2F - z_1I) = 0\}.$$ 

Then for every matrix-valued polynomial $f$,

$$\|f(S, P)\| \leq \max\{\|f(z_1, z_2)\| : (z_1, z_2) \in \Lambda \cap b\Gamma\}.$$ 

Moreover, if $\omega(F) < 1$, then $\Lambda \cap G_2$ is a distinguished variety in $G_2$.

**Lemma 8.14.** Let $(S, P)$ be a $\Gamma$-contraction on a Hilbert space $\mathcal{H}$. Then $(S, P)$ dilates to $\Gamma$-distinguished $\Gamma$-unitary if and only if $(S^*, P^*)$ dilates to a $\Gamma$-distinguished $\Gamma$-unitary.

**Proof.** It suffices to prove the necessary part. Assume that $(S, P)$ has a dilation to a $\Gamma$-distinguished $\Gamma$-unitary $(U_1, U_2)$ on $\mathcal{H}$ containing $\mathcal{H}$. Then the $\Gamma$-unitary $(U_1^*, U_2^*)$ dilates $(S^*, P^*)$ and it follows from Lemma 8.3 that $(U_1^*, U_2^*)$ is $\Gamma$-distinguished. The proof is complete.

**Theorem 8.15.** Let $(S, P)$ be a $\Gamma$-contraction on a Hilbert space $\mathcal{H}$ such that $(S^*, P^*)$ is pure, and suppose that $\dim \mathcal{D}_p < \infty$. Let $F$ be the fundamental operator of $(S, P)$ such that $\omega(F) < 1$. Then $(S, P)$ dilates to a $\Gamma$-distinguished $\Gamma$-unitary.

**Proof.** Applying Theorem 3.9 to $(S^*, P^*)$, we have that $\mathcal{H} \subseteq H^2(\mathcal{D}_p)$ and that $S = T^*_\phi|_{\mathcal{H}'}$ and $P = T^*_z|_{\mathcal{H}'}$, where $\phi(z) = F^* + Fz$ and $F$ is the fundamental operator of $(S, P)$. Moreover, $F$ is a matrix since $\mathcal{D}_p$ is finite-dimensional. Note that the pair $(T, z)$ acting on $H^2(\mathcal{D}_p)$ is a $\Gamma$-isometric dilation of $(S^*, P^*)$. We show that $(T, z)$ is $\Gamma$-distinguished. Define

$$\Lambda = \{(z_1, z_2) \in \Gamma : \det(F^* + z_2F - z_1I) = 0\}.$$
Let $f$ be any matrix-valued polynomial. Then it follows from the proof of Theorem 4.5 in [38] that
\[ \|f(T^*_\phi, T^*_z)\| \leq \max \{|f(z_1, z_2)| : (z_1, z_2) \in \Lambda \cap b\Gamma \}. \tag{8.3} \]
Since $\omega(F) < 1$, it follows from Theorem 8.13 that $q(z_1, z_2) = \det(F^* + z_2F - z_1I)$ is a $\Gamma$-distinguished polynomial. By (8.3), we have
\[ \|q(T^*_\phi, T^*_z)\| \leq \max \{|q(z_1, z_2)| : (z_1, z_2) \in \Lambda \cap b\Gamma \} = 0. \]
Hence, $(T^*_\phi, T^*_z)$ is $\Gamma$-distinguished. By Lemma 5.3, $(T^*_\phi, T^*_z)$ is $\Gamma$-distinguished too. Therefore, $(S^*, P^*)$ dilates to the $\Gamma$-distinguished $\Gamma$-isometry $(T^*_\phi, T^*_z)$. By Proposition 8.1, we have that $(T^*_\phi, T^*_z)$ has an extension to a $\Gamma$-distinguished $\Gamma$-unitary. Putting everything together, we have that $(S^*, P^*)$ has a $\Gamma$-distinguished $\Gamma$-unitary dilation. The desired conclusion follows from Lemma 8.14.

The following result is a direct consequence of Lemma 8.14 and Theorem 8.15.

**Corollary 8.16.** Let $(S, P)$ be a $\Gamma$-contraction on a Hilbert space $\mathcal{H}$ such that $(S, P)$ is pure, and suppose that $\dim \mathcal{D}_P < \infty$. Let $F_*$ be the fundamental operator of $(S^*, P^*)$ such that $\omega(F_*) < 1$. Then $(S, P)$ dilates to a $\Gamma$-distinguished $\Gamma$-unitary.

It follows from Proposition 5.10 that every strict $\Gamma$-contraction on a finite-dimensional Hilbert space is $\Gamma$-distinguished. We prove a stronger version of this result.

**Corollary 8.17.** Every strict $\Gamma$-contraction on a finite-dimensional Hilbert space admits a $\Gamma$-distinguished $\Gamma$-unitary dilation.

**Proof.** Let $(S, P)$ be a strict $\Gamma$-contraction on a finite-dimensional space $\mathcal{H}$ and let $F$ be the fundamental operator of $(S, P)$. By Lemma 4.2 in [38] and Proposition 4.3 [38], we have that $(S^*, P^*)$ is a pure $\Gamma$-contraction and $\omega(F) < 1$. It follows from Theorem 8.15 that $(S, P)$ dilates to a $\Gamma$-distinguished $\Gamma$-unitary.

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**9. Minimal dilation of a $\Gamma$-distinguished $\Gamma$-contraction**

In this Section, we study in detail when a minimal $\Gamma$-isometric dilation of a $\Gamma$-contraction $(S, P)$ is annihilated by a $\Gamma$-distinguished polynomial, especially when dimension of $\mathcal{D}_P$ or $\mathcal{D}_{P^*}$ is finite. It was shown by Agler and Young in [11] that every $\Gamma$-contraction admits a $\Gamma$-isometric dilation. Also, a minimal $\Gamma$-isometric dilation of a $\Gamma$-contraction was explicitly constructed in [14] with the help of the fundamental operator (see (3.2)). Also, that dilation was minimal and acted on the minimal isometric dilation space of $P$. We recall that dilation theorem from [14] described a new way of characterizing $\Gamma$-contractions.

**Theorem 9.1** ([14], Theorem 4.3). Let $(S, P)$ be a $\Gamma$-contraction on a Hilbert space $\mathcal{H}$. Let $A$ be the unique fundamental operator of the fundamental equation $S - S^*P = D_PAD_P$. Consider the operators $T_A, V_0$ defined on $\mathcal{H} \oplus l^2(\mathcal{D}_P)$ by
\[
T_A(x_0, x_1, x_2, \ldots) = (Sx_0, A^*D_Px_0 + Ax_1, A^*x_1 + Ax_2, A^*x_2 + Ax_3, \ldots),
\]
\[
V_0(x_0, x_1, x_2, \ldots) = (Px_0, D_Px_0, x_1, x_2, \ldots).
\]
Then up to unitary $(T_A, V_0)$ is the unique $\Gamma$-isometric dilation of $(S, P)$ on $\mathcal{H} \oplus l^2(\mathcal{D}_P)$. 
Note that the operators $T_A, V_0$ have the following matrix form with respect to the decomposition $\mathcal{H} \oplus \mathcal{D}_p \oplus \mathcal{D}_p \oplus \ldots$ of the space $\mathcal{H} \oplus l^2(\mathcal{D}_p)$:

$$T_A = \begin{bmatrix} S & 0 & 0 & 0 & \ldots \\ A^* D_p & A & 0 & 0 & \ldots \\ 0 & A^* & A & 0 & \ldots \\ 0 & 0 & A^* & A & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad V_0 = \begin{bmatrix} P & 0 & 0 & 0 & \ldots \\ D_p & 0 & 0 & 0 & \ldots \\ 0 & I & 0 & 0 & \ldots \\ 0 & 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$  

The pair $(T_A, V_0)$ can also have the following $2 \times 2$ block matrix representation.

$$T_A = \begin{bmatrix} S & 0 \\ C & D \end{bmatrix} \quad \text{and} \quad V_0 = \begin{bmatrix} P & 0 \\ B & E \end{bmatrix}, \quad (9.1)$$

where $(D, E)$ is a commuting pair of operators on $l^2(\mathcal{D}_p)$ defined by

$$D := \begin{bmatrix} A & 0 & 0 & \ldots \\ A^* & A & 0 & \ldots \\ 0 & A^* & A & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad E := \begin{bmatrix} 0 & 0 & 0 & \ldots \\ I & 0 & 0 & \ldots \\ 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \quad (9.2)$$

Note that $(D, E)$ on $l^2(\mathcal{D}_p)$ is unitarily equivalent to $(T_{A^*+A^*}, T_z)$ on $H^2(\mathcal{D}_p)$, where, $\omega(A) \leq 1$. Indeed, this is a standard model for a pure $\Gamma$-isometry, viz. Theorem 3.8. In this Section, we mainly investigate the following questions.

Q1. Is the pair $(T_A, V_0)$ always $\Gamma$-distinguished?

Q2. Given a $\Gamma$-distinguished $\Gamma$-contraction $(S, P)$, is the $\Gamma$-isometry $(T_A, V_0)$ always $\Gamma$-distinguished?

Q3. If $(S, P)$ and $(T_A, V_0)$ are both $\Gamma$-distinguished, then does every $\Gamma$-distinguished polynomial annihilating $(S, P)$ also annihilate $(T_A, V_0)$?

About Q1, we would like to say that if this is to happen then $(S, P)$ is also $\Gamma$-distinguished, because, for every $f \in \mathbb{C}[z_1, z_2]$ we have $f(S, P) = P_{\mathcal{H}} f(T_A, V_0)|_{\mathcal{H}}$. Since every $\Gamma$-contraction $(S, P)$ dilates to such a $\Gamma$-isometry $(T_A, V_0)$ as in (9.2), it is evident that the answer to this question is negative in general. There are $\Gamma$-contractions which are not $\Gamma$-distinguished as is shown in Example 4.2.

Remark 9.2. With the notations as in (9.2), it follows from Lemma 8.12 that $(T_A, V_0)$ is $\Gamma$-distinguished if and only if $(S, P)$ and $(D, E)$ are $\Gamma$-distinguished.

If $(S, P)$ is a $\Gamma$-distinguished $\Gamma$-contraction, then the above remark implies that its $\Gamma$-isometric dilation $(T_A, V_0)$ is $\Gamma$-distinguished if and only if the pure $\Gamma$-isometry $(D, E)$ is $\Gamma$-distinguished. Thus, the question if $(T_A, V_0)$ is $\Gamma$-distinguished boils down to whether or not the pure $\Gamma$-isometry $(D, E)$ is annihilated by a $\Gamma$-distinguished polynomial provided that $(S, P)$ is $\Gamma$-distinguished. We already have some results in this direction in Section 5. For example, Theorem 5.7 provides a necessary and sufficient condition so that a pure $\Gamma$-isometry becomes $\Gamma$-distinguished under some additional hypothesis. Using the theory developed in Section 5, we have the following theorem.

Theorem 9.3. Let $(S, P)$ be a $\Gamma$-distinguished $\Gamma$-contraction and $\dim \mathcal{D}_p < \infty$. Then TFAE:

1. The minimal dilation $(T_A, V_0)$ is $\Gamma$-distinguished;
2. $(A, 0)$ is $\Gamma$-distinguished;
3. $r(A) < 1$. 

Proof. By Remark 9.2, it suffices to prove that the pure $\Gamma$-isometry $(D, E)$ is $\Gamma$-distinguished. We also have that $(D, E)$ on $l^2(\mathcal{D}_p)$ is unitarily equivalent to the pure $\Gamma$-isometry $(T_A + A^*z, T_z)$ on $H^2(\mathcal{D}_p)$. As $A^*$ is the fundamental operator of $(T_A^*, T_z^*)$ and $\dim \mathcal{D}_z = \dim \mathcal{D}_p$ which is finite, Theorem 5.7 yields that the following statements are equivalent:

(a) $(T_A + A^*z, T_z)$ is $\Gamma$-distinguished;
(b) $(A^*, 0)$ is $\Gamma$-distinguished;
(c) $r(A^*) < 1$.

The desired conclusion follows from Lemma 5.3. □

Let $(S, P)$ be a $\Gamma$-distinguished $\Gamma$-contraction on a Hilbert space $\mathcal{H}$. We show by an example that even if $\dim \mathcal{D}_p$ is not finite, its minimal dilation $(T_A, V_0)$ (as specified in Theorem 9.1) can still be $\Gamma$-distinguished.

Example 9.4. Consider the Hilbert space $\mathcal{H} = l^2(\mathbb{N})$, i.e.

$$\mathcal{H} = \{(x_0, x_1, x_2, \ldots) : x_j \in \mathbb{C} \text{ and } \sum_{j=0}^{\infty} |x_j|^2 < \infty\}$$

and take any $r \in (0, 1)$. Consider the pair of commuting scalar contractions $(rI, rI)$ on $\mathcal{H}$, where $I$ is the identity operator on $\mathcal{H}$. We take the symmetrization of this pair and define $(S, P) := \pi(rI, rI) = (2rI, r^2I)$ on $\mathcal{H}$. Evidently, this is a $\Gamma$-contraction. Moreover, $(S, P)$ is annihilated by the polynomial $4z_2 - z_1^2$ which is a $\Gamma$-distinguished polynomial as seen in Example 3.5. Hence, $(S, P)$ is a $\Gamma$-distinguished $\Gamma$-contraction. Let $A$ be the fundamental operator of this $\Gamma$-contraction. Since $D_p = \sqrt{1 - r^4}I$, we have $\mathcal{D}_p = \mathcal{H}$ and so $\mathcal{D}_p$ is not finite-dimensional. Also, $S - S^*P = 2r(1 - r^2)$ and $D_pAD_p = (1 - r^4)A$. From the uniqueness of the fundamental operator, it follows that $A = \frac{2r}{1 + r^2}I$ and $\omega(A) = \frac{2r}{1 + r^2} < 1$ as $0 < r < 1$. Now we show that $(T_A, V_0)$ is $\Gamma$-distinguished. By Lemma 9.2 it suffices to show that the pure $\Gamma$-isometry $(D, E)$ is $\Gamma$-distinguished. Putting $a = \frac{2r}{1 + r^2}$, we have

$$D = \begin{bmatrix} A & 0 & 0 & \ldots \\ A^* & A & 0 & \ldots \\ 0 & A^* & A & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} al & 0 & 0 & \ldots \\ al & al & 0 & \ldots \\ 0 & al & al & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$ 

The polynomial $f(z_1, z_2) = z_1 - az_2 - a$ annihilates the pair $(D, E)$. We prove that $f$ is a $\Gamma$-distinguished polynomial. Since $a < 1$, we have $\pi(a, 0) = (a, 0) \in Z(f) \cap \mathbb{G}_2$. For any $(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) \in Z(f) \cap \partial \mathbb{G}_2$, we must have that either $\lambda_1$ or $\lambda_2$ is in $\mathbb{T}$. Without loss of generality, let $\lambda_1 \in \mathbb{T}$. Note that

$$f(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) = \lambda_1 + \lambda_2 - a\lambda_1 \lambda_2 - a = 0 \quad \text{i.e.} \quad \lambda_2 = \frac{-\lambda_1 - a}{1 - a\lambda_1}.$$ 

Since $z \mapsto \frac{z - a}{1 - az}$ with $a < 1$ defines an automorphism of the unit disc which sends points of $\mathbb{T}$ onto $\mathbb{T}$, we have that $\lambda_2 \in \mathbb{T}$. Hence, $(\lambda_1, \lambda_2) \in \mathbb{T}^2$ which shows that $(\lambda_1 + \lambda_2, \lambda_1 \lambda_2) \in b\Gamma$. Thus, $Z(f) \cap \partial \mathbb{G}_2 \subseteq b\Gamma$. So, $(T_A, V_0)$ is a $\Gamma$-distinguished $\Gamma$-isometry and the dimension of $\mathcal{D}_p$ is not finite. □

Remark 9.5. Every $\Gamma$-distinguished polynomial that annihilates $(S, P)$ need not annihilate $(T_A, V_0)$ even if $(T_A, V_0)$ itself is $\Gamma$-distinguished. In the example above, the $\Gamma$-distinguished polynomial
$4z_2 - z_1^2$ annihilates $(S, P)$ but it does not annihilate $(T_A, V_0)$. Let if possible, $(D, E)$ be also annihilated by the same polynomial $4z_2 - z_1^2$. Then $D^2 = 4E$ and a routine calculation shows that

$$D^2 = \begin{bmatrix} a^2I & 0 & 0 & \ldots \\ \ast & a^2I & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots \\ \end{bmatrix}, \quad 4E = \begin{bmatrix} 0 & 0 & 0 & \ldots \\ 4I & 0 & 0 & \ldots \\ 0 & 4I & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$  

The equation $D^2 = 4E$ leads to $a^2 = 0$, which contradicts the fact that $0 < a < 1$.

We now focus on the following question: Is the $\Gamma$-isometry dilation $(T_A, V_0)$ of a $\Gamma$-distinguished $\Gamma$-contraction $(S, P)$ itself $\Gamma$-distinguished? Addressing this question requires a crucial observation which is essential for constructing a potential counter-example later.

**Proposition 9.6.** If $(T_A, V_0)$ is $\Gamma$-distinguished, then the following hold:

1. $(A, 0)$ and $(S, P)$ are annihilated by the same polynomial that annihilates $(T_A, V_0)$;
2. the fundamental operator $A$ is algebraic;
3. the spectrum of $A$ is finite that lies strictly inside $\mathbb{D}$.

**Proof.** Assume that $f(T_A, V_0) = 0$ for some $\Gamma$-distinguished polynomial $f(z_1, z_2)$. It follows from Lemma 8.12 that $f(S, P) = 0$ and $f(D, E) = 0$. Using the block matrix form (9.2), we rewrite the pair $(D, E)$ in the $2 \times 2$ block matrix form as

$$D = \begin{bmatrix} A & 0 \\ G & H \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 \\ K & L \end{bmatrix}$$

with respect to $\mathcal{D}_p \oplus l^2(\mathcal{D}_p)$. Again by Lemma 8.12, $f(A, 0) = 0$. Since $f$ is $\Gamma$-distinguished, $f(z_1, z_2)$ cannot be of the form $z_2g(z_1, z_2)$ for any $g \in \mathbb{C}[z_1, z_2]$. Otherwise, $f(1, 0) = 0$ which implies that $(1, 0) = \pi(1, 0) \in Z(f) \cap \partial \mathbb{G}_2 \subseteq b\Gamma$. So, $(1, 0) \in b\Gamma$ which is a contradiction. Thus, $f(z_1, z_2)$ has the following form:

$$f(z_1, z_2) = a_0 + a_1 z_1 + \cdots + a_n z_1^n + z_2 g(z_1, z_2)$$

for some $g \in \mathbb{C}[z_1, z_2]$, $n \in \mathbb{N} \cup \{0\}$ and $a_n \neq 0$. Hence, the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is a non-constant polynomial and $p(A) = f(A, 0) = 0$. By the spectral mapping theorem (e.g. see [31]), we have

$$p(\sigma(A)) = \sigma(p(A)) = \sigma(\{0\}) = 0.$$  

Thus, $\sigma(A) \subseteq Z(p)$ which is a finite set. Since $A$ is the fundamental operator of $(S, P)$, we have that $r(A) \leq \omega(A) \leq 1$. If $r(A) = 1$, there exists some $\lambda \in \mathbb{T} \cap \sigma(A)$. Again, it follows from spectral mapping theorem that

$$0 = \sigma_T(f(A, 0)) = f(\sigma_T(A, 0)) = f(\sigma(A) \times \{0\}).$$

Hence, $f(\lambda, 0) = 0$. Since $(\lambda, 0) \in \partial \mathbb{D}^2$, we have $\pi(\lambda, 0) \in \partial \mathbb{G}_2$. Now, $f$ being $\Gamma$-distinguished implies that

$$\pi(\lambda, 0) = (\lambda, 0) \in Z(f) \cap \partial \mathbb{G}_2 \subseteq b\Gamma,$$

which is a contradiction since $|p| = 1$ for every $(s, p) \in b\Gamma$. Consequently, $r(A) < 1$ and so, $\sigma(A)$ is a finite subset of $\mathbb{D}$. \hfill $\blacksquare$
Remark 9.7. Following the proof of the above proposition, one can easily see that if \((T,0)\) is a \(\Gamma\)-distinguished \(\Gamma\)-contraction, then \(\sigma(T)\) is a finite subset of \(\mathbb{D}\). In particular, for the fundamental operator \(A\), if the \(\Gamma\)-contraction \((A,0)\) is \(\Gamma\)-distinguished, then \(\sigma(A)\) is a finite subset of \(\mathbb{D}\).

Being armed with the above proposition, we are now in a position to show that the minimal \(\Gamma\)-isometric dilation \(T_A, V_0\) of a \(\Gamma\)-distinguished \(\Gamma\)-contraction need not be \(\Gamma\)-distinguished.

Example 9.8. Consider the right-shift operator \(W\) on \(\mathcal{H} = \ell^2(\mathbb{N})\) given by
\[
W(x_0, x_1, x_2, x_3, \ldots) = (0, x_0, x_1, x_2, \ldots)
\]
and take any \(r \in (0, 1)\). Consider the pair of commuting contractions \((rW, rW)\) on \(\mathcal{H}\) and we take the symmetrization of this pair and define \((S, P) := \pi(rW, rW) = (2rW, r^2W^2)\). Evidently, this is a \(\Gamma\)-contraction annihilated by the \(\Gamma\)-distinguished polynomial \(4z^2 - z^2\) as shown in Example 3.5. Hence, \((S, P)\) is a \(\Gamma\)-distinguished \(\Gamma\)-contraction which is also a pure \(\Gamma\)-contraction. Since \(D^2_P = I - r^4W^*W^2W = (1 - r^4)I\), we have \(D_P = \sqrt{1 - r^4}I\) and \(\mathcal{D}_P = \mathcal{H}\). We find the fundamental operator \(A\) of \((S, P)\). Note that
\[
S - S^*P = 2rW - 2r^3W^*W^2 = 2r(1 - r^2)W \quad \text{and} \quad D_P AD_P = (1 - r^4)A.
\]
The uniqueness of fundamental operator gives that \(A = \frac{2r}{1 + r^2}W\) and \(\omega(A) = \frac{2r}{1 + r^2} < 1\) as \(0 < r < 1\). Suppose, if possible \((T_A, V_0)\) is \(\Gamma\)-distinguished. Proposition 9.6 implies that \(A = \omega(A)W\) is an algebraic operator with a finite spectrum. As the numerical radius \(\omega(A)\) is a non-zero scalar, the spectral mapping theorem yields that \(W\) also possesses a finite spectrum. This contradicts the fact that \(\sigma(W) = \mathbb{D}\). Hence, \((T_A, V_0)\) cannot be annihilated by a \(\Gamma\)-distinguished polynomial even though it is a \(\Gamma\)-isometric dilation of a \(\Gamma\)-distinguished \(\Gamma\)-contraction. However, it follows from Example 8.8 that \((S, P)\) admits a \(\Gamma\)-distinguished \(\Gamma\)-isometric dilation since \(S^2 - 4P = 0\).

We have seen in Proposition 9.6 that a necessary condition for \((T_A, V_0)\) to be annihilated by a \(\Gamma\)-distinguished polynomial is the existence of a \(\Gamma\)-distinguished polynomial that annihilates both \((S, P)\) and \((A,0)\). We shall prove a partial converse to that. Indeed, we show the existence of a sequence of \(\Gamma\)-distinguished \(\Gamma\)-contractions that converges to \((T_A, V_0)\) in the strong operator topology when both \((S, P)\) and \((A,0)\) are \(\Gamma\)-distinguished.

Lemma 9.9. Let \((S, P)\) be a \(\Gamma\)-contraction on a Hilbert space \(\mathcal{H}\) with fundamental operator \(A\). Suppose both \((S, P)\) and \((A,0)\) are \(\Gamma\)-distinguished \(\Gamma\)-contractions. Then there is a sequence of \(\Gamma\)-distinguished \(\Gamma\)-contractions on \(\mathcal{H} \oplus \ell^2(\mathcal{D}_P)\) which converges to \((T_A, V_0)\) in the strong operator topology.

Proof. Consider the sequence of commuting pair of operators \((T_n, V_n)\) on \(\mathcal{H} \oplus \ell^2(\mathcal{D}_P)\) defined by
\[
T_n(x_0, x_1, x_2, \ldots) = (Sx_0, A^*D_p x_0 + Ax_1, A^*x_1 + Ax_2, \ldots, A^*x_{n-1} + Ax_n, 0, 0, \ldots);
\]
\[
V_n(x_0, x_1, x_2, \ldots) = (Px_0, D_p x_1, \ldots, x_{n-1}, 0, 0, \ldots).
\]
With respect to the decomposition \(\mathcal{H} \oplus \ell^2(\mathcal{D}_P)\), the 2 \times 2 block matrix form of \((T_n, V_n)\) is given by
\[
T_n = \begin{bmatrix} S & 0 \\ C & D_n \end{bmatrix} \quad \text{and} \quad V_n = \begin{bmatrix} P & 0 \\ B & E_n \end{bmatrix}
\]
(9.3)
where the operators \(C\) and \(B\) are the same as defined in (2.1). The operator pair \((D_n, E_n)\) on \(\ell^2(\mathcal{D}_P)\) is defined as
\[
D_n := \begin{bmatrix} \hat{A}_n & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_n := \begin{bmatrix} \hat{I}_n & 0 \\ 0 & 0 \end{bmatrix}
\]
(9.4)
where the operator pair \((\hat{A}_n, \hat{I}_n)\) is given by
\[
\hat{A}_n = \begin{bmatrix} A & 0 & 0 & \ldots & 0 \\ A^* & A & 0 & \ldots & 0 \\ 0 & A^* & A & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A^* & A \end{bmatrix}_{n \times n}, \quad \hat{I}_n = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ I & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{bmatrix}_{n \times n}.
\] (9.5)

It follows from Lemma 8.12 in the Appendix that each \((T_n, V_n)\) is a \(\Gamma\)-contraction on \(\mathcal{H} \oplus \ell^2(\mathcal{D}_p)\). We show that each \((T_n, V_n)\) is \(\Gamma\)-distinguished. It is clear that each \((D_n, E_n)\) is \(\Gamma\)-distinguished if and only if each \((\hat{A}_n, \hat{I}_n)\) is \(\Gamma\)-distinguished. Since \((A, 0)\) is \(\Gamma\)-distinguished, a repeated application of Lemma 8.12 yields that each \((\hat{A}_n, \hat{I}_n)\) is \(\Gamma\)-distinguished. With the help of similar computations involved in the proof of Lemma 8.12, we can show that each \((T_n, V_n)\) is \(\Gamma\)-distinguished as \((S, P)\) and \((D_n, E_n)\) are \(\Gamma\)-distinguished for every \(n \in \mathbb{N}\). Next, we show that the sequences \(\{T_n\}_{n \in \mathbb{N}}\) and \(\{V_n\}_{n \in \mathbb{N}}\) converge to \(T_A\) and \(V_0\) respectively in the strong operator topology. Given \(x = (x_0, x_1, x_2, \ldots) \in \mathcal{H} \oplus \ell^2(\mathcal{D}_p)\), we have that
\[
\|T_A x - T_n x\|^2 = \sum_{j=n}^{\infty} \|A^* x_j + Ax_{j+1}\|^2 \to 0 \text{ as } n \to \infty
\]
because the above sum is a tail of a convergent series with limit \(\|T_A x\|^2\) and similarly, we have that
\[
\|V_0 x - V_n x\|^2 = \sum_{j=n}^{\infty} \|x_j\|^2 \to 0 \text{ as } n \to \infty
\]
as it is a tail of a convergent series with limit \(\|x\|^2\). The proof is now complete.

The \(\Gamma\)-isometric dilation \((T_A, V_0)\) of a \(\Gamma\)-distinguished \(\Gamma\)-contraction \((S, P)\) may or may not be \(\Gamma\)-distinguished. In Example 9.8 we have already encountered one such case. It is worth finding out a necessary and sufficient conditions such that the \(\Gamma\)-isometry \((T_A, V_0)\) becomes \(\Gamma\)-distinguished. The following result is a first step in this direction.

**Theorem 9.10.** Let \((S, P)\) be a \(\Gamma\)-contraction acting on a Hilbert space \(\mathcal{H}\). Then the following are equivalent.

1. \((T_A, V_0)\) is a \(\Gamma\)-distinguished \(\Gamma\)-isometry.
2. There is a \(\Gamma\)-distinguished polynomial that annihilates every \(\Gamma\)-contraction in the sequence \(\{(T_n, V_n)\}_{n \in \mathbb{N}}\) as defined in (9.3).
3. \((S, P)\) is \(\Gamma\)-distinguished and there is a \(\Gamma\)-distinguished polynomial annihilating each \(\Gamma\)-contraction in the sequence \(\{(\hat{A}_n, \hat{I}_n)\}_{n \in \mathbb{N}}\) given in (9.5).

**Proof.** We will prove that (1) \(\implies\) (2) \(\implies\) (3) \(\implies\) (2) \(\implies\) (1).

(1) \(\implies\) (2). Let \((T_A, V_0)\) be a \(\Gamma\)-distinguished \(\Gamma\)-isometry. From the 2 \(\times\) 2 block matrix form of \((T_n, V_n)\) as given in (9.3), it follows that \((T_n, V_n)\) is \(\Gamma\)-distinguished if and only if \((S, P)\) and \((D_n, E_n)\) are \(\Gamma\)-distinguished. Since \((T_A, V_0)\) is \(\Gamma\)-distinguished, as a consequence of Lemma 8.12 and Proposition 9.6, it follows that the \(\Gamma\)-distinguished polynomial annihilating \((T_A, V_0)\) also annihilates \((S, P)\) and \((\hat{A}_n, \hat{I}_n)\). From the definition of \((D_n, E_n)\), it is clear that the same is true for \((D_n, E_n)\).
(2) $\implies$ (3). Let $f$ be a $\Gamma$-distinguished polynomial such that $f(T_n, V_n) = 0$ for every $n \in \mathbb{N}$. Given the $2 \times 2$ block matrix form of $(T_n, V_n)$ in (9.3), it follows from Lemma 8.12 that each $f(D_n, E_n) = 0$ and $f(S, P) = 0$. Applying Lemma 8.12 on (9.4), we have $f(\hat{A}_n, \hat{I}_n) = 0$ for every $n \in \mathbb{N}$.

(3) $\implies$ (2). Let $g$ and $h$ be $\Gamma$-distinguished polynomials such that $g(S, P) = 0$ and $h(\hat{A}_n, \hat{I}_n) = 0$ for every $n \in \mathbb{N}$. By Lemma 8.12 and (9.4), $h(D_n, E_n) = 0$ for all $n \in \mathbb{N}$. It follows from Lemma 8.12 that the $\Gamma$-distinguished polynomial $f(z_1, z_2) = g(z_1, z_2)h(z_1, z_2)$ annihilates each $(T_n, V_n)$.

(2) $\implies$ (1). Let $f$ be a $\Gamma$-distinguished polynomial such that $f(T_n, V_n) = 0$ for every $n \in \mathbb{N}$. It follows from Lemma 8.9 that $\{T_n\}_{n \in \mathbb{N}}$ and $\{V_n\}_{n \in \mathbb{N}}$ converge to $T_A$ and $V_0$ respectively in the strong operator topology. Moreover, $\|T_n\| \leq 2$ and $\|V_n\| \leq 1$ for each $n \in \mathbb{N}$. Consequently, $\{f(T_n, V_n)\}_{n \in \mathbb{N}}$ converges to $f(T_A, V_0)$ strongly and so, $f(T_A, V_0) = 0$. The proof is complete.

The above result can be improved further if we assume an additional hypothesis on the fundamental operator of a $\Gamma$-contraction. In Proposition 9.6 we proved that if $(T_A, V_0)$ is $\Gamma$-distinguished, then $(A, 0)$ is $\Gamma$-distinguished as well. The subsequent results show that the converse is also true if one assumes that $A$ is hyponormal.

**Theorem 9.11.** Assume that $(S, P)$ is a $\Gamma$-distinguished $\Gamma$-contraction and the fundamental operator $A$ of $(S, P)$ is normal. Then the $\Gamma$-isometry $(T_A, V_0)$ is $\Gamma$-distinguished if and only if $(A, 0)$ is $\Gamma$-distinguished.

**Proof.** Assume that $(A, 0)$ is $\Gamma$-distinguished. Following the proof of Proposition 9.6, one can deduce that there is a non-constant minimal polynomial $p(z)$ which annihilates $A$. Thus, we have

$$p(A) = (\overline{\alpha_1} - A)(\overline{\alpha_2} - A) \ldots (\overline{\alpha_m} - A) = 0,$$

where $\alpha_j$’s need not be distinct. Define $X_j = (\overline{\alpha_j} - A)$ and $Y_j = X_j^* = (\alpha_j - A^*)$ for $1 \leq j \leq m$. For $Q_j = X_j$ or $Y_j (1 \leq j \leq m)$, we show that $Q_1Q_2 \ldots Q_m = 0$. To prove this, we consider the operators on $D_P$ given by

$$R_j = \begin{cases} X_j & \text{if } Q_j = Y_j \\ Y_j & \text{if } Q_j = X_j \end{cases} \quad (1 \leq j \leq m).$$

Note that $X_j$ and $Y_i$ commute since $A$ is normal. This gives that

$$0 = p(A)p(A)^* = (X_1X_2 \ldots X_m)(Y_mY_{m-1} \ldots Y_1) = (Q_1Q_2 \ldots Q_m)(R_mR_{m-1} \ldots R_1) = (Q_1Q_2 \ldots Q_m)(Q_1Q_2 \ldots Q_m)^*,$$

and hence, $Q_1Q_2 \ldots Q_m = 0$. From Remark 9.7, it follows that $r(A) < 1$ and so $|\alpha_j| < 1$ for each $j$. Hence, the polynomial $\overline{\alpha_j} + \alpha_jz_2 - z_1$ is $\Gamma$-distinguished for each $j$ as shown in Example 9.4. This means that the polynomial defined by

$$f(z_1, z_2) = \prod_{j=1}^{m} (\overline{\alpha_j} + \alpha_jz_2 - z_1)$$

is normal. This gives that $f(z_1, z_2)$ is $\Gamma$-distinguished for each $j$ as shown in Example 9.4. This means that the polynomial defined by
Remark 9.2 that

\[ \text{The proof is now complete.} \]

and since

\[ Q \]

Every

\[ \text{operators, we have that} \]

\[ A \]

Assume that

\[ \text{Theorem 9.12.} \]

the fundamental operator

\[ A \]

\[ f(A + A^* z, z) = \prod_{j=1}^{m} \left( \overline{\alpha}_j + \alpha_j z - (A + A^* z) \right) \]

\[ = \prod_{j=1}^{m} \left( \overline{\alpha}_j - A + (\alpha_j - A^*) z \right) \]

\[ = \prod_{j=1}^{m} (X_j + Y_j z) \]

\[ = \prod_{j=1}^{m} X_j + z \prod_{j=1}^{m} Q_j^{(1)} + z^2 \prod_{j=1}^{m} Q_j^{(2)} + \cdots + z^m \prod_{j=1}^{m} Y_j, \]

where \( Q_j^{(k)} \) is either \( X_j \) or \( Y_j \). Hence, \( \prod_{j=1}^{m} Q_j^{(k)} = 0 \) for each \( k \). Consequently, \( f \) annihilates \( (T_{A + A^* z}, T_z) \) and since \( (D, E) \) is unitarily equivalent to \( (T_{A + A^* z}, T_z) \), we have \( f(D, E) = 0 \). Now, it follows from Remark 9.2 that \( (T_A, V_0) \) is \( \Gamma \)-distinguished. The converse is a direct consequence of Remark 9.2. The proof is now complete.

Recall that an operator \( T \) defined on a Hilbert space \( \mathcal{H} \) is said to be hyponormal if \( T^* T - TT^* \geq 0 \), or equivalently \( \|T^* x\| \leq \|Tx\| \) for every \( x \) in \( \mathcal{H} \). Next, we prove that the above result holds when the fundamental operator of a \( \Gamma \)-distinguished \( \Gamma \)-contraction is assumed to be hyponormal.

**Theorem 9.12.** Assume that \( (S, P) \) is a \( \Gamma \)-distinguished \( \Gamma \)-contraction and the fundamental operator \( A \) of \( (S, P) \) is hyponormal. Then \( (T_A, V_0) \) is \( \Gamma \)-distinguished if and only if \( (A, 0) \) is \( \Gamma \)-distinguished.

**Proof.** A hyponormal operator annihilated by a polynomial is normal. This fact is an easy consequence of Corollary 2 in [40]. The desired conclusion follows immediately from Theorem 9.11.

We conclude this section with the following corollary.

**Corollary 9.13.** Assume that \( (S, P) \) is a normal strict \( \Gamma \)-distinguished \( \Gamma \)-contraction with fundamental operator \( A \). Then \( (T_A, V_0) \) is \( \Gamma \)-distinguished if and only if \( (A, 0) \) is \( \Gamma \)-distinguished.

**Proof.** For a strict \( \Gamma \)-contraction \( (S, P) \), the defect operator \( D_P \) is invertible since \( \|P\| < 1 \). Hence, the fundamental operator \( A \) of \( (S, P) \) is given by \( A = D_P^{-1} (S - S^* P) D_P^{-1} \). Since \( S \) and \( P \) are normal operators, we have that \( A \) is normal. The conclusion now follows from Theorem 9.12.

### 10. Decomposition of pure \( \Gamma \)-isometries annihilated by distinguished polynomials

Every \( \Gamma \)-isometry admits a Wold-type decomposition, e.g. see [2]. Indeed, if \( (T, V) \) is a \( \Gamma \)-isometry acting on a Hilbert space \( \mathcal{H} \), then there is an orthogonal decomposition of \( \mathcal{H} \) into closed joint reducing subspaces \( \mathcal{H}_u \) and \( \mathcal{H}_p \) such that \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_p \) and

(a) \( (T|_{\mathcal{H}_u}, V|_{\mathcal{H}_u}) \) is a \( \Gamma \)-unitary;

(b) \( (T|_{\mathcal{H}_p}, V|_{\mathcal{H}_p}) \) is a pure \( \Gamma \)-isometry.
It follows that if \((T, V)\) is \(\Gamma\)-distinguished, then so are \((T|_{\mathcal{H}_u}, V|_{\mathcal{H}_u})\) and \((T|_{\mathcal{H}_p}, V|_{\mathcal{H}_p})\). Naturally, the polynomial that annihilates \((T, V)\) also annihilates these two pairs. In other words, we have the following.

**Lemma 10.1.** Let \((T, V)\) be a \(\Gamma\)-distinguished \(\Gamma\)-isometry on \(\mathcal{H}\). Then there is an orthogonal decomposition of \(\mathcal{H}\) into closed joint reducing subspaces \(\mathcal{H}_u\) and \(\mathcal{H}_p\) such that \(\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_p\) and

\[
\begin{align*}
(a) & \quad (T|_{\mathcal{H}_u}, V|_{\mathcal{H}_u}) \text{ is a } \Gamma\text{-unitary}; \\
(b) & \quad (T|_{\mathcal{H}_p}, V|_{\mathcal{H}_p}) \text{ is a } \Gamma\text{-distinguished pure } \Gamma\text{-isometry}.
\end{align*}
\]

In this Section, we present some decomposition results for the class of pure \(\Gamma\)-isometries that are annihilated by distinguished polynomials. It follows from Lemma 5.2 that any such pure \(\Gamma\)-isometry is a \(\Gamma\)-distinguished one. To begin with, we prove an important result that will be used frequently in this Section.

**Lemma 10.2.** Let \((T, U)\) be a \(\Gamma\)-unitary acting on a Hilbert space \(\mathcal{H}\) annihilated by a distinguished polynomial \(q\). Let \(q_1\) and \(q_2\) be distinct factors of \(q\) such that \(q = q_1q_2\). Then

\[
q_1(T, U)\mathcal{H} \perp q_2(T, U)\mathcal{H}.
\]

**Proof.** Since \((T, U)\) is a \(\Gamma\)-unitary on \(\mathcal{H}\), there is a commuting pair \(\mathcal{U} = (U_1, U_2)\) of unitaries on \(\mathcal{H}\) such that \(\pi(U_1, U_2) = (T, U)\). By Example 3.4, the polynomial \(p = q \circ \pi\) is inner toral and \(p(\mathcal{U}) = 0\). Let \(p_i = q_i \circ \pi\) for \(i = 1, 2\) so that \(p = p_1 p_2\). Since \(p_1\) is also inner toral, we have that

\[
z_1^m z_2^n p_1(1/z_1, 1/z_2) = \alpha p_1(z_1, z_2),
\]

where \(\alpha \in \mathbb{T}\) and \((n, m)\) is the degree of \(p_1\). In fact, we may assume \(\alpha = 1\) by replacing \(p_1\) with an appropriate constant multiple (see [27] for further details). Consequently,

\[
p_1(z_1, z_2) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} z_1^i \bar{z}_2^j
\]

and \(a_{ij} = a_{(n-i)(m-j)}\). This can be used to show that \(p_1(\mathcal{U})^* U_1^n U_2^m = p_1(\mathcal{U})\), as \(U_1, U_2\) are commuting unitaries. Consequently,

\[
p_1(\mathcal{U})^* p_2(\mathcal{U}) = U_1^n U_2^m p_1(\mathcal{U}) p_2(\mathcal{U}) = U_1^n U_2^m p(\mathcal{U}) = 0.
\]

It then follows that for any \(h_1, h_2 \in \mathcal{H}\),

\[
\langle q_1(T, U) h_1, q_2(T, U) h_2 \rangle = \langle p_1(\mathcal{U}) h_1, p_2(\mathcal{U}) h_2 \rangle = \langle h_1, p_1(\mathcal{U})^* p_2(\mathcal{U}) h_2 \rangle = 0.
\]

Thus, we have proved that \(q_1(T, U)\mathcal{H} \perp q_2(T, U)\mathcal{H}\). \(\blacksquare\)

Now, we present the first decomposition result for a \(\Gamma\)-unitary \((T, U)\) annihilated by a distinguished polynomial \(q\). The decomposition is determined by the factors of the polynomial \(q = q_1 q_2 \ldots q_N\). To explain the idea of the proof, we first prove the decomposition result for \(q = q_1 q_2\). We fix the following notations in this section, to explain the proofs easily.

**Notations.** A \(q\)-\(\Gamma\)-contraction means a \(\Gamma\)-contraction annihilated by a polynomial \(q(z_1, z_2)\). For a commuting pair \(\Sigma = (T_1, T_2)\) acting on a Hilbert space \(\mathcal{H}\) and a polynomial \(q\), let \(q(\Sigma)\) and \(q(\Sigma)^*\) denote the operators \(q(T_1, T_2)\) and \((q(T_1, T_2))^*\) respectively. For a closed joint invariant subspace \(\mathcal{L} \subseteq \mathcal{H}\) of \(T_1, T_2\), we denote \((T_1|_{\mathcal{L}}, T_2|_{\mathcal{L}})\) by \(\Sigma|_{\mathcal{L}}\).
Proposition 10.3. Let $\Sigma = (T, U)$ be a $\Gamma$-unitary on $\mathcal{H}$ annihilated by a distinguished polynomial $q$ and let $q_1, q_2$ be distinct factors of $q$ such that $q = q_1q_2$. Then there exist closed joint reducing subspaces $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{H}$ such that

1. $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and
2. each $\Sigma|_{\mathcal{H}_j}$ is a $\Gamma$-unitary annihilated by $q_j$ for $j = 1, 2$.

Proof. It follows from Lemma 10.2 that $q_1(\Sigma)\mathcal{H} \perp q_2(\Sigma)\mathcal{H}$. The space $\mathcal{L}_1 = \overline{q_2(\Sigma)\mathcal{H}}$ is a closed joint reducing subspace of $\mathcal{H}$ and $\Sigma|_{\mathcal{L}_1}$ is a $q_1$-$\Gamma$-unitary. The space $\mathcal{H}_2 = q_2(\Sigma)\mathcal{H}$ is orthogonal to $\mathcal{L}_1$ and is a joint reducing subspace too. Hence, $\Sigma|_{\mathcal{H}_2}$ is a $q_2$-$\Gamma$-unitary. Thus, we have the following orthogonal decomposition of $\mathcal{H}$ into the closed joint reducing subspaces:

$$\mathcal{H} = \mathcal{L}_1 \oplus \mathcal{H}_2 \oplus \mathcal{L'},$$

where, $\mathcal{L'} = (\text{Ran } q_1(\Sigma) \oplus \text{Ran } q_2(\Sigma))^\perp = \text{Ker } q_1(\Sigma)^* \cap \text{Ker } q_2(\Sigma)^*$. Since $q_1 \circ \pi$ and $q_2 \circ \pi$ are inner toral, we can prove the following (similar to the proof of Lemma 10.2).

1. $q_1 \circ \pi(U_1, U_2)^* = U_1^{*j}U_2^{*k}q_1 \circ \pi(U_1, U_2)$
2. $q_2 \circ \pi(U_1, U_2)^* = U_1^{*m}U_2^{*m}q_2 \circ \pi(U_1, U_2),$  

where, $U_1, U_2$ are commuting unitaries on $\mathcal{H}$ such that $\pi(U_1, U_2) = (T, U)$ and $j, k, n, m \geq 0$. Consequently, we have that

$$q_1(\Sigma)^* = U_1^{*j}U_2^{*k}q_1(\Sigma) \quad \text{and} \quad q_2(\Sigma)^* = U_1^{*m}U_2^{*m}q_2(\Sigma).$$

This shows that for any $x \in \mathcal{L'}$, we have $U_1^{*j}U_2^{*k}q_1(\Sigma)x = q_1(\Sigma)^*x = 0$. Hence, $q_1(\Sigma)|_{\mathcal{L'}} = 0$. The space $\mathcal{H}' = \mathcal{L}_1 \oplus \mathcal{L'}$ is a closed joint reducing subspace and $\Sigma|_{\mathcal{H}'}$ is a $\Sigma$-$\Gamma$-unitary. The desired conclusion follows.

The following generalized version of our previous result follows from mathematical induction and Proposition 10.3.

Theorem 10.4. Let $\Sigma = (T, U)$ be a $\Gamma$-unitary acting on a Hilbert space $\mathcal{H}$ annihilated by a distinguished polynomial $q$ and let $q_1, \ldots, q_N$ be the distinct irreducible factors of $q$. Then there is an orthogonal decomposition of $\mathcal{H}$ into closed joint reducing subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_N$ such that each $\Sigma|_{\mathcal{H}_j}$ is a $\Gamma$-unitary annihilated by $q_j$.

As we have seen, a $\Gamma$-isometry is an orthogonal sum of a $\Gamma$-unitary and a pure $\Gamma$-isometry. We already have a decomposition result for a $\Gamma$-unitary. Now, we obtain some decomposition results for a pure $\Gamma$-isometry. Recall that every pure $\Gamma$-isometry $(T, V)$ is unitarily equivalent to the restriction of the $\Gamma$-unitary $(M_\kappa, M_\zeta)$ acting on $L^2(\mathcal{D}_p)$ to the joint invariant subspace $H^2(\mathcal{D}_p)$ and if $(T, V)$ is annihilated by a distinguished polynomial, Lemma 5.4 ensures that there is a square-free minimal distinguished polynomial $q$ that annihilates $(T, V)$. We follow this development to find a decomposition result for a distinguished pure $\Gamma$-isometry.

Theorem 10.5. Let $\Sigma = (T, V)$ be a pure $\Gamma$-isometry on $\mathcal{H}$. Let $q$ be a distinguished polynomial that annihilates $(T, V)$ and let $q_1, \ldots, q_N$ be the distinct irreducible factors of $q$. Then there exist $(N + 1)$ closed orthogonal disjoint subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_N, \mathcal{H}'$ of $\mathcal{H}$ that are invariant under both $T$ and $V$ such that

1. $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_N \oplus \mathcal{H}'$,
2. each $\Sigma|_{\mathcal{H}_j}$ is a pure $\Gamma$-isometry annihilated by $q_j$.

Moreover, the spaces $\mathcal{H}_j$ are given by $r_j(\Sigma)\mathcal{H}$ for the polynomial $r_j = q/q_j$ for $j = 1, \ldots, N$. 
Proof. For $q = q_1q_2 \ldots q_N$, let $p = q_2q_3 \ldots q_N$. We will show that $\mathcal{H}$ has a closed joint invariant subspace restricted to which $\Sigma$ can be written as the direct sum of a $q_1$-$\Gamma$-isometry and a $p$-$\Gamma$-isometry. The desired conclusion then follows from mathematical induction. By Theorem 3.8, $(T,V)$ is unitarily equivalent to the restriction of the $\Gamma$-unitary $U = (M_\phi,M_\zeta)$ acting on $\mathcal{H} = L^2(D_p)$ to the joint invariant subspace $H^2(D_p)$. Without loss of generality, we assume that $\mathcal{H} = H^2(D_p)$ and $(T,V) = (M_\phi|_{\mathcal{H}},M_\zeta|_{\mathcal{H}})$. It follows from the proof of Proposition 8.1 that $q(U) = 0$ since $(M_\phi,M_\zeta)$ is (upto unitary equivalence) the minimal normal extension of $(T,V)$. By Proposition 10.3, there is an orthogonal decomposition of $\mathcal{H}$ into disjoint closed subspaces $\mathcal{H}_1, \mathcal{H}_2$ reducing both $M_\phi,M_\zeta$ such that $U|_{\mathcal{H}_1}$ is a $q_1$-$\Gamma$-unitary and $U|_{\mathcal{H}_2}$ is a $p$-$\Gamma$-unitary. The subspaces are given by

$$\mathcal{H}_1 = p(U)\mathcal{H} \oplus \left[ \text{Ker } q_1(U)^* \cap \text{Ker } p(U)^* \right] \quad \text{and} \quad \mathcal{H}_2 = q_1(U)\mathcal{H}.$$ 

The subspaces of $\mathcal{H}$ defined by $\mathcal{H}_1 = p(\Sigma)\mathcal{H}$ and $\mathcal{H}_2 = q_1(\Sigma)\mathcal{H}$ are invariant under $T$ and $V$. Since $p(\Sigma)\mathcal{H} = p(U)\mathcal{H} \subseteq \mathcal{H}_1$ and $q_1(\Sigma)\mathcal{H} = q_1(U)\mathcal{H} \subseteq \mathcal{H}_2$, we have that $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Hence, $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint orthogonal subspaces such that $\Sigma|_{\mathcal{H}_1}$ and $\Sigma|_{\mathcal{H}_2}$ are pure $\Gamma$-isometries annihilated by $q_1$ and $p$ respectively. The pair $\Sigma_2 = (T_2,V_2) = (T|_{\mathcal{H}_2},V|_{\mathcal{H}_2})$ is a pure $\Gamma$-isometry and $p(T_2,V_2) = 0$. Again, applying Theorem 3.8 the model theorem for a pure $\Gamma$-isometry, we have that $(T_2,V_2)$ is unitarily equivalent to $(T_{\phi_2},T_{\zeta_2})$ on $H^2(D_{V_2})$ and $p(T_{\phi_2},T_{\zeta_2}) = 0$. This yields that the $\Gamma$-unitary $U_2 = (M_{\phi_2},M_{\zeta_2})$ on $\mathcal{H}_2 = L^2(D_{V_2})$ is annihilated by $p = q_2 \ldots q_N$. Let $p_2 = q_3 \ldots q_N$. Proposition 10.3 shows that $\mathcal{H}_2$ can be written as an orthogonal decomposition of reducing subspaces $\mathcal{H}_{21}$ and $\mathcal{H}_{22}$ such that $\mathcal{H}_2|_{\mathcal{H}_{21}}$ is a $q_2$-$\Gamma$-unitary and $\mathcal{H}_2|_{\mathcal{H}_{22}}$ is a $p_2$-$\Gamma$-unitary. Moreover, the structures of these two subspaces are as follows.

$$\mathcal{H}_{21} = p_2(U_2)\mathcal{H}_2 \oplus \left[ \text{Ker } q_2(U_2)^* \cap \text{Ker } p_2(U_2)^* \right] \quad \text{and} \quad \mathcal{H}_{22} = q_2(U_2)\mathcal{H}_2.$$ 

One can refer to the proof of Proposition 10.3 to obtain the above two subspaces. Now, the spaces $\mathcal{H}_{21} = p_2(\Sigma)\mathcal{H}_2$ and $\mathcal{H}_{22} = q_2(\Sigma)\mathcal{H}_2$ are (unitarily equivalent to) closed joint subspaces of $\mathcal{H}_{21}$ and $\mathcal{H}_{22}$ respectively which are both invariant under $T$ and $V$. Thus, $\mathcal{H}_2$ and $\mathcal{H}_2'$ are orthogonal and it is not difficult to see that $\Sigma_2|_{\mathcal{H}_{21}}$ and $\Sigma_2|_{\mathcal{H}_{22}}$ are pure $\Gamma$-isometries annihilated by $q_2$ and $p_2$ respectively. Needless to mention that $\mathcal{H}_2$ is the same as $\overline{q_1(\Sigma)p_2(\Sigma)}\mathcal{H}$. Thus, we have proved that there are closed disjoint orthogonal subspaces $\mathcal{H}_{11}, \mathcal{H}_{12}$ and $\mathcal{H}_{21}$ that are invariant under $T$ and $V$. The space $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21}$ and $\Sigma|_{\mathcal{H}_{21}}$ is a pure $\Gamma$-isometry that $q_j$ annihilates for $j = 1,2$. Continuing this process for finitely many steps, we have the desired conclusion.

11. Appendix

In this Section, we prove some results that we used in the Section 9 but did not provide proofs there due to long computations. The notations are the same as Section 9.
For $n \in \mathbb{N}$, we consider the pair of operators $(\hat{A}_n, \hat{I}_n)$ on $\bigoplus_{i=1}^{n} \mathcal{D}_p = \mathcal{D}_p \oplus \mathcal{D}_p \oplus \cdots \oplus \mathcal{D}_p$ defined by

$$
\hat{A}_n := \begin{bmatrix}
A & 0 & 0 & \cdots & 0 \\
A^* & A & 0 & \cdots & 0 \\
0 & A^* & A & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & A^* & A
\end{bmatrix}_{n \times n}, \quad \hat{I}_n := \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
I & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & I & 0
\end{bmatrix}_{n \times n}.
$$

**Lemma 11.1.** The pair $(\hat{A}_n, \hat{I}_n)$ on $\bigoplus_{i=1}^{n} \mathcal{D}_p$ is a $\Gamma$-contraction for every $n \in \mathbb{N}$.

**Proof.** We shall prove that the pair $(\hat{A}_n, \hat{I}_n)$ commutes, $\|\hat{A}_n\| \leq 2$, $\|\hat{I}_n\| \leq 1$ and there is a unique solution to the fundamental equation

$$
\hat{A}_n - \hat{A}_n^* \hat{I}_n = D_{I_n} X_n D_{\hat{I}_n}
$$

for some $X_n \in B(\mathcal{D}_n)$ with $\omega(X_n) \leq 1$. Then Theorem 4.4 in [14] gives that $(\hat{A}_n, \hat{I}_n)$ is a $\Gamma$-contraction. Some routine computations give that

$$
\hat{A}_n \hat{I}_n = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
A & 0 & 0 & \cdots & 0 \\
A^* & A & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & A^* & A
\end{bmatrix}_{n \times n} = \hat{I}_n \hat{A}_n.
$$

Following the proof of Theorem 4.3 in [14], we have $\|\hat{A}_n\| \leq 2$. It is easy to see that $\|\hat{I}_n\| = 1$. The LHS $\hat{A}_n - \hat{A}_n^* \hat{I}_n$ in (11.1) equals

$$
\hat{A}_n - \begin{bmatrix}
A^* & A & 0 & \cdots & 0 \\
0 & A^* & A & \cdots & 0 \\
0 & 0 & A^* & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A^*
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
I & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & I & 0
\end{bmatrix}
= \begin{bmatrix}
A & 0 & 0 & \cdots & 0 \\
A^* & A & 0 & \cdots & 0 \\
0 & A^* & A & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & A^* & A
\end{bmatrix}
= \begin{bmatrix}
o_{n-1} & 0 \\
0 & A
\end{bmatrix},
$$

where, $O_{n-1}$ denotes the zero block matrix of order $(n-1) \times (n-1)$. Next, we compute the defect operator and the defect space corresponding to the operator $\hat{I}_n$. The operator $I - \hat{I}_n^* \hat{I}_n$ equals

$$
I - \begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
I & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & I & 0
\end{bmatrix}
= \begin{bmatrix}
I & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I
\end{bmatrix}
= \begin{bmatrix}
o_{n-1} & 0 \\
0 & I
\end{bmatrix}.
$$

Hence, $D_{I_n} = (I - \hat{I}_n^* \hat{I}_n)^{1/2} = \begin{bmatrix}
o_{n-1} & 0 \\
0 & I
\end{bmatrix}$ and $\mathcal{D}_n = \bigoplus_{i=1}^{n-1} \mathcal{D}_p \oplus \cdots \oplus \mathcal{D}_p$. The operator $X_n : \mathcal{D}_n \rightarrow \mathcal{D}_n$ given by

$$
X_n(0,0,\ldots,0,x) = (0,0,\ldots,0,Ax)
$$

and
has numerical radius, $\omega(X_n) = \omega(A) \leq 1$. The matrix form of $X_n$ with respect to the decomposition
\[
\mathcal{D}_n = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \mathcal{D}_p \quad n\text{-times}
\]
is $X_n = \begin{bmatrix} O_{n-1} & 0 \\ 0 & A \end{bmatrix}$. It follows from (11.2) that
\[
D_{\mathcal{I}_n} X_n D_{\mathcal{I}_n} = \begin{bmatrix} O_{n-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} O_{n-1} & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} O_{n-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} O_{n-1} & 0 \\ 0 & A \end{bmatrix} = \hat{A}_n - \hat{A}_n^* \mathcal{I}_n.
\]
Therefore, $(\hat{A}_n, \mathcal{I}_n)$ is a $\Gamma$-contraction on $\bigoplus \mathcal{D}_p$ for every $n \in \mathbb{N}$.

In Section 9 we construct a sequence of $\Gamma$-contractions on the bigger space $\mathcal{H} \oplus \mathbb{C}^2(\mathcal{D}_p)$ which will converge in strong operator topology. We recall that construction here. For every $n \in \mathbb{N}$, we define the operators on $\mathbb{C}^2(\mathcal{D}_p)$ given by
\[
D_n = \begin{bmatrix} \hat{A}_n & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_n = \begin{bmatrix} \mathcal{I}_n & 0 \\ 0 & 0 \end{bmatrix}
\]
and $(D_n, E_n)$ on $\mathbb{C}^2(D_p)$ is a sequence of $\Gamma$-contractions which follows from Lemma 11.1. In the matrix form, the $\Gamma$-isometry $(T_A, V_0)$ can be written as
\[
T_A = \begin{bmatrix} S & 0 \\ C & D \end{bmatrix} \quad \text{and} \quad V_0 = \begin{bmatrix} P & 0 \\ B & E \end{bmatrix}
\]
with respect to $\mathcal{H} \oplus \mathbb{C}^2(\mathcal{D}_p)$ and we define the following sequence of bounded linear operators on $\mathcal{H} \oplus \mathbb{C}^2(\mathcal{D}_p)$ as follows.
\[
T_n = \begin{bmatrix} S & 0 \\ C & D_n \end{bmatrix} \quad \text{and} \quad V_n = \begin{bmatrix} P & 0 \\ B & E_n \end{bmatrix}
\]
In other words, for every $(x_0, x_1, \ldots) \in \mathcal{H} \oplus \mathbb{C}^2(\mathcal{D}_p)$, we define
(a) $T_n(x_0, x_1, \ldots) = (Sx_0, A^* D_p x_0 + Ax_1, A^* x_1 + Ax_2, \ldots, A^* x_{n-1} + Ax_n, 0, 0, \ldots)$;
(b) $V_n(x_0, x_1, \ldots) = (P x_0, D_p x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots)$.

**Lemma 11.2.** The pair $(T_n, V_n)$ on $\mathcal{H} \oplus \mathbb{C}^2(\mathcal{D}_p)$ is a $\Gamma$-contraction for every $n \in \mathbb{N}$.

**Proof.** We shall again use Theorem 4.4 in [14] to show that $(T_n, V_n)$ is indeed a $\Gamma$-contraction. We prove that $(T_n, V_n)$ is a commuting pair with spectral radius of $T_n$ atmost 2 and the fundamental equation
\[
T_n - T_n^* V_n = D V_n T_n D V_n
\]
has a solution $Y_n$ with $\omega(Y_n) \leq 1$. Note that
\[
T_n V_n(x_0, x_1, \ldots) = T_n(P x_0, D_p x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots)
\]
\[
= (S P x_0, A^* D_p x_0 + A x_1, A^* x_1 + A x_2, \ldots, A^* x_{n-2} + A x_{n-1}, 0, 0, \ldots)
\]
and
\[
V_n T_n(x_0, x_1, \ldots) = V_n(S x_0, A^* D_p x_0 + A x_1, A^* x_1 + A x_2, \ldots, A^* x_{n-1} + A x_n, 0, 0, \ldots)
\]
\[
= (P S x_0, D_p x_0, A^* D_p x_0 + A x_1, A^* x_1 + A x_2, \ldots, A^* x_{n-2} + A x_{n-1}, 0, 0, \ldots).
\]
Since $(S, P)$ is a $\Gamma$-contraction and $A$ is the fundamental operator of $(S, P)$, one can show that $D_p(A^* D_p + A D_p) = D_p^2 S$. Thus, it follows that $A^* D_p P + AD_p = D_p S$ and hence $T_n V_n = V_n T_n$. 


For any \( x = (x_0, x_1, x_2, \ldots) \in \mathcal{H} \oplus \ell^2(\mathcal{D}_p) \), it follows that

\[
\|V_n x\|^2 = \|P x_0\|^2 + \|D_p x_0\|^2 + \|x_1\|^2 + \cdots + \|x_n\|^2 \leq \|x\|^2
\]

and

\[
\|T_n x\|^2 = \|S x_0\|^2 + \|A^* D_p x_0 + x_1\|^2 + \|A^* x_1 + x_2\|^2 + \cdots + \|A^* x_{n-1} + A x_n\|^2
\]

\[
\leq \|S x_0\|^2 + \|A^* D_p x_0 + x_1\|^2 + \sum_{j=1}^{\infty} \|A^* x_j + x_{j+1}\|^2
\]

\[
= \|T_A(x)\|^2 \leq 4\|x\|^2.
\]

This gives that \( \|V_n\| \leq 1 \) and \( \|T_n\| \leq 2 \) for each \( n \). The LHS in (11.3) is given by

\[
T_n - T_n^* V_n = \begin{bmatrix} S & 0 \\ C & D_n \end{bmatrix} - \begin{bmatrix} S^* & C^* \\ 0 & D_n^* \end{bmatrix} \begin{bmatrix} P & 0 \\ B & E_n \end{bmatrix} = \begin{bmatrix} S - S^* P - C^* B & -C^* E_n \\ C - D_n^* B & D_n - D_n^* E_n \end{bmatrix}.
\]

(11.4)

We further compute the operators appearing in the above 2 \times 2 block matrix representation of \( T_n - T_n^* V_n \). Since \( C = [D_p A 0 0 \ldots]^* \) and \( B = [D_p 0 0 \ldots]^* \), we have

\[
C^* E_n = [D_p A 0 0 \ldots] : \cdots : 0 0 0 0 \cdots : I \cdots : 0 0 0 0 \cdots = 0,
\]

and

\[
D_n^* B = \begin{bmatrix} A^* & A & 0 & \cdots & 0 & 0 & \cdots \\ 0 & A^* & A & \cdots & 0 & 0 & \cdots \\ 0 & 0 & A^* & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A^* & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & A^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} D_p \\ [A^* D_p] \\ \vdots \end{bmatrix} = \begin{bmatrix} C \end{bmatrix}.
\]

Lastly, we compute \( D_n - D_n^* E_n \) which has the following 2 \times 2 block representation with respect to the same decomposition as in the matrix form of \( (T_n, V_n) \) given in (11.4).

\[
D_n - D_n^* E_n = \begin{bmatrix} \hat{A}_n & 0 \\ 0 & \hat{A}_n^* \end{bmatrix} - \begin{bmatrix} \hat{A}_n^* & 0 \\ 0 & \hat{A}_n \end{bmatrix} \begin{bmatrix} \hat{I}_n \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{A}_n - \hat{A}_n^* \hat{I}_n \\ 0 \end{bmatrix}.
\]

It follows from (11.2) and (11.4) that

\[
T_n - T_n^* V_n = \begin{bmatrix} 0 & 0 \\ 0 & D_n - D_n^* E_n \end{bmatrix} = [a_{ij}]_{i,j=0}^\infty
\]

where \( a_{n,n} = A \) and other blocks and entries are zero. For any \( x = (x_0, x_1, \ldots) \in \mathcal{H} \oplus \ell^2(\mathcal{D}_p) \), we have that

\[
(T_n - T_n^* V_n)(x_0, x_1, x_2, \ldots) = (0, 0, \ldots, 0, A x_n, 0, 0, \ldots)
\]
where, $Ax_n$ is at the $(n+1)$-th position (counting from zero) and before that the $n$ entries are zero. We compute the defect operator and the defect space for $V_n$.

$$D^{2}_{V_n} = I - V^{*}_n V_n = I - \begin{bmatrix} P^* & B^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & E_n \end{bmatrix} = I - \begin{bmatrix} P^* P + B^* B & B^* E_n \\ E_n^* B & E_n^* E_n \end{bmatrix}$$

We compute each block appearing in the above block matrix representation of $D^{2}_{V_n}$. Since $B^* B = D^{2}_{P}$, we must have that $P^* P + B^* B = I$. Note that

$$B^* E_n = \begin{bmatrix} D_P & 0 & 0 & \ldots \end{bmatrix}$$

We shall use the $2 \times 2$ block matrix representation of $E_n$.

$$E_n^* E_n = \begin{bmatrix} \hat{I}_n^* & 0 \\ 0 & \hat{I}_n \end{bmatrix} \begin{bmatrix} \hat{I}_n^* & 0 \\ 0 & \hat{I}_n \end{bmatrix} = \begin{bmatrix} \hat{I}_n^* \hat{I}_n & 0 \\ 0 & \hat{I}_n \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we show that $E_n^* E_n = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$. We shall use the $2 \times 2$ block matrix representation of $E_n$.

Therefore, we get that

$$D^{2}_{V_n} = I - \begin{bmatrix} I & 0 \\ 0 & E_n^* E_n \end{bmatrix} = I - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} O_n & 0 \\ 0 & I \end{bmatrix}.$$ 

Hence, $D_{V_n} = \begin{bmatrix} O_n & 0 \\ 0 & I \end{bmatrix}$ and the defect space $\mathcal{D}_{V_n} = \{0 \oplus \cdots \oplus 0\} \oplus \mathcal{D}_P \oplus \mathcal{D}_P \oplus \cdots$ i.e.

$$\mathcal{D}_{V_n} = \{(x_0, x_1, \ldots) \in \mathcal{H} \oplus \ell^2(\mathcal{D}_P) : x_0 = x_1 = \cdots = x_{n-1} = 0\}.$$ 

Now, we finally define our candidate for the fundamental operator of $(T_n, V_n)$. Consider the operator $Y_n : \mathcal{D}_{V_n} \to \mathcal{D}_{V_n}$ given by

$$Y_n(0, 0, \ldots, 0, x_n, x_{n+1}, x_{n+2}, \ldots) = (0, 0, \ldots, 0, Ax_n, 0, 0, \ldots),$$

where $Ax_n$ is at the $(n+1)$-th position (counting from zero) and the first $n$ entries are zero. Since $\omega(Y_n) = \omega(A)$, we have $\omega(Y_n) \leq 1$. The RHS in (11.3) becomes

$$D_{V_n} Y_n D_{V_n}(x_0, x_1, \ldots) = D_{V_n} Y_n(0, 0, \ldots, 0, x_n, x_{n+1}, x_{n+2}, \ldots)$$

$$= D_{V_n}(0, 0, \ldots, 0, Ax_n, 0, 0, \ldots)$$

$$= (0, 0, \ldots, 0, Ax_n, 0, 0, \ldots)$$

$$= (T_n - T_n^* V_n)(x_0, x_1, \ldots),$$

for any $x = (x_0, x_1, \ldots) \in \mathcal{H} \oplus \ell^2(\mathcal{D}_P)$. This shows that

$$D_{V_n} Y_n D_{V_n} = T_n - T_n^* V_n$$

with $\omega(Y_n) \leq 1$ and $Y_n \in B(\mathcal{D}_{V_n})$.

Therefore, $(T_n, V_n)$ on $\mathcal{H} \oplus \ell^2(\mathcal{D}_P)$ is a sequence of $\Gamma$-contractions. The proof is complete.
Proof of Equation-(4.1). We have that

\[
S^2 - 4P = \begin{bmatrix} A & 0 & 0 & 0 & \ldots \\ A^* & A & 0 & 0 & \ldots \\ 0 & A^* & A & 0 & \ldots \\ 0 & 0 & A^* & A & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A & 0 & 0 & 0 & \ldots \\ A^* & A & 0 & 0 & \ldots \\ 0 & A^* & A & 0 & \ldots \\ 0 & 0 & A^* & A & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} - 4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ I & 0 & 0 & 0 & \ldots \\ 0 & I & 0 & 0 & \ldots \\ 0 & 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

\[
= \begin{bmatrix} A^2 & 0 & 0 & 0 & \ldots \\ A^*A + AA^* - 4I & A^2 & 0 & 0 & \ldots \\ A^2 & A^*A + AA^* - 4I & A^2 & 0 & \ldots \\ 0 & A^* & A^*A + AA^* - 4I & A^2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

We have that

\[
E = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ -2E & 0 & 0 & 0 & \ldots \\ 0 & -2E & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

\[
= \Delta^2.
\]

Proof of Equation-(4.2). Note that \(E\) commutes with \(A\) and \(A^*\). With the help of this we show that \(\Delta\) commutes with \(S\) and \(P\). We have

\[
S\Delta - \Delta S
\]

\[
= \begin{bmatrix} A & 0 & 0 & 0 & \ldots \\ A^* & A & 0 & 0 & \ldots \\ 0 & A^* & A & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} E & 0 & 0 & 0 & \ldots \\ -E & E & 0 & 0 & \ldots \\ 0 & -E & E & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} E & 0 & 0 & 0 & \ldots \\ -E & E & 0 & 0 & \ldots \\ 0 & -E & E & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A & 0 & 0 & 0 & \ldots \\ A^* & A & 0 & 0 & \ldots \\ 0 & A^* & A & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

\[
= \begin{bmatrix} AE & 0 & 0 & 0 & \ldots \\ A^*E - AE & AE & 0 & 0 & \ldots \\ 0 & -A^*E & A^*E - AE & AE & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} EA & 0 & 0 & 0 & \ldots \\ EA^* - EA & EA & 0 & 0 & \ldots \\ 0 & -EA^* & EA^* - EA & EA & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

\[
= 0.
\]

Proof of Equation-(4.3). We have that

\[
(S + \Delta)^*(S + \Delta)
\]

\[
= \begin{bmatrix} A^* + E & A - E & 0 & 0 & \ldots \\ 0 & A^* + E & A - E & 0 & \ldots \\ 0 & 0 & A^* + E & A - E & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A + E & 0 & 0 & 0 & \ldots \\ A^* - E & A + E & 0 & 0 & \ldots \\ 0 & A^* - E & A + E & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

\[
= \begin{bmatrix} A^* + AA^* + 2E & A^2 - E & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ A^2 - E & A^*A + AA^* + 2E & A^2 - E & 0 & \ldots \\ 0 & A^2 - E & A^*A + AA^* + 2E & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

\[
= 4I.
\]
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