Dynamic Complexity of Group Problems

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Abstract
Dynamic Complexity was introduced by Immerman and Patnaik [14] in the nineties and has seen a resurgence of interest with the positive resolution of their conjecture on directed reachability in DynFO [4]. Since then many natural problems related to reachability and matching have been placed in DynFO and related classes [7, 5, 8]. In this work, we place some dynamic problems from group theory in DynFO.

In particular, suppose we are given an arbitrary multiplication table over \( n \) elements representing an unstructured binary operation (representing a structure called a magma). Suppose the table evolves through a change in one of its \( n^2 \) entries in one step. For a set \( S \) of magma elements which also changes one element at a time, we can maintain enough auxiliary information so that when the magma is a group, we are able to answer the Cayley Group Membership (CGM) problem for \( S \) and a target \( t \) (i.e. “Is \( t \) a product of elements from \( S \)?”) using an FO query at every step. This places the dynamic CGM problem (for groups) when the ambient magma is specified via a table in DynFO.

In contrast, for the table setting, statically CGM was known to be in the class L [1]. Building on the dynamic CGM result, we can maintain the isomorphism of of two magmas, whenever both are Abelian groups, in DynFO.

Our techniques include a way to maintain the powers of the elements of a magma in DynFO using left associative parenthesisation, the notion of cube independence to cube generate a subgroup generated by a set, a way to maintain maximal cube independent sequences in a magma along with some group theoretic machinery available from [13]. The notion of cube independent sequences is new (as far as we know) and may be of independent interest. These techniques are very different from the ones employed in Dynamic Complexity so far.

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1 Introduction

The study of the computational complexity of problems where the input changes dynamically is an important one that has received a lot of interest lately. In the framework of Dynamic Complexity ([14]), one is interested in understanding the complexity of updating an answer.
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Dynamic complexity of various natural problems related to graphs and similar relational structures have been studied recently (see [5, 4, 6, 8]). The goal of this paper is to understand the dynamic complexity of group-theoretic problems in the framework of Immerman and Patnaik. While the complexity of these problems are well-studied in the classical setting, as far as we know this is the first work to study the dynamic complexity of group theoretic problems. As a first step, we look at the fundamental problems of Cayley Group Membership (CGM) and Group Isomorphism (GI) for Abelian groups represented as multiplication tables.

In the CGM problem, we are given a group $G$ as a multiplication table, a set $S \subseteq G$, and an element $t \in G$, and the goal is to test if $t$ is in the group generated by $S$. The Cayley Group Membership problem for general groups was studied by Barrington and McKenzie [1] (see also [10]), who observed that it is in the class $\mathbf{SL}$, and consequently in $\mathbf{L}$. For the special case of Abelian groups, it was subsequently shown in [2] that the CGM problem was in $\mathbf{FOLL}$. The GI problem for arbitrary groups is well-studied, not yet fully settled. Chattopadhyay, Toran and Wagner ([3]) show that for quasi-groups, GI is in $\beta_2 \mathbf{FOLL}$, and Abelian GI is in $\mathbf{TC}^0(\mathbf{FOLL})$. A linear-time algorithm is known for Abelian GI ([12]).

Unlike the static setting of the above results, the dynamic setting causes the additional technical challenge that under updates the multiplication table need not represent a group. In fact, it is possible that the only property satisfied by the binary operation is closure; such a structure is termed as a magma. Thus the CGM and GI problems make sense only when the underlying magma is a group, and this is what we guarantee in the dynamic setting. We extend the group theoretic machinery to magmas by insisting on a left associative parenthesisation of the product so that all the operations become well behaved. This allows us to define the (left associative) logarithmic products and arbitrary (left associative) powers of magma elements. We show that all this allows us to opaquely deal with the group theoretic notion of cube independence which is one of the key primitives that we introduce. Though closely related to cube generating sequences (in groups) [3, 9] it is significantly easier to compute deterministically. As far as we know the only parallel algorithm to compute cube generating sequence uses randomness [3]. However, can compute them via cube independence in $\mathbf{AC}^1$. Further, we show that we can maintain them in $\text{DynFO}(\leq, +, \times)$. CGM is easy to solve with access to cube generating sequences:

**Theorem 1.** Given a dynamic magma $M$ and a dynamic subset $S$ thereof, we can maintain auxiliary data in $\text{DynFO}(\leq, +, \times)$ to be able to answer if a given target element is a product of the elements of $S$, whenever $M$ happens to be a group.

Abelian group isomorphism can now be tackled with results from [13]:

**Theorem 2.** Given two dynamic magmas $M_1$ and $M_2$, we can maintain auxiliary data in $\text{DynFO}(\leq, +, \times)$ to be able to answer if $M_1$ and $M_2$ are isomorphic whenever both the magmas are Abelian groups.

For formal definitions of the problems and the model of computation please see Section 2.

Main technical contribution

Introducing the notion of cube bases, relating them with cube generating sequences (Lemma 9) and showing a deterministic parallel algorithm to compute them (Lemma 13) is the main technical contribution of this work. Note that the construction by [3] of cube generating sequences building on the prior work of [9] is inherently randomised and does not seem to be derandomisable. Also note that while constructing generating sequences is in $\mathbf{L}$, using them to do CGM in $\text{DynFO}(\leq, +, \times)$ seems hopeless.
Organisation of the paper
Section 2 describes the preliminaries related to Dynamic complexity and group theory. In Section 3 we show how to maintain products of slowly changing logarithmically long sequences and maintain powers of magma elements. In Section 4 we introduce the notion of cube independence and relate it with cube generating sequences. We show how to maintain a maximal cube independent sequence and use it to prove Theorem 1. In Section 5 we deal with Abelian group isomorphism and prove Theorem 2. Finally we conclude with Section 6.

2 Preliminaries

Algebraic Preliminaries

In this paper all algebraic structures considered are finite. A magma $M$ is an algebraic structure $(M, \cdot)$ with a binary operation $\cdot : M \times M \rightarrow M$.

Let $M$ be a magma and $S \subseteq M$ be a subset. The Cayley graph associated with the subset $S$ is the graph $X_{M,S}$ such that $V(X_{M,S}) = M$ and $E(X_{M,S}) = \{(g, ga) : g \in M \land a \in S\}$. We abuse notation to write $X_{M,a}$ for $X_{M,\{a\}}$.

Let $G$ be a group, that is an associative magma (i.e. $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$), with an identity (i.e. $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$) and with an inverse for each element (i.e. $\forall a \in G : \exists b \in G : a \cdot b = b \cdot a = e$). $A \subseteq G$ is said to be a subgroup of $G$ if for all $x, y \in A$, $x \cdot y^{-1} \in A$. The order of a $G$, denoted by $|G|$ or $\text{ord}(G)$, is the cardinality of $G$. The order of an element $x \in G$ denoted by $\text{ord}(x)$ or just by $\text{ord}(x)$ is the minimum positive integer $m$ such that $x^m = e$. A group $G$ with $\text{ord}(G) = p^\alpha$ where $p$ is a prime and $\alpha \in \mathbb{N}$ is called a $p$-group. Let $G$ be a group and $S$ be a subset of group $G$ then $\langle S \rangle$ denotes the subgroup generated $S$ i.e. consists of products of (not necessarily distinct) elements of $S$.

A homomorphism from a group $G$ to another group $H$ is a map $\varphi : G \rightarrow H$ such that $\forall g, h \in G, \varphi(gh) = \varphi(g)\varphi(h)$. Two groups $G$ and $H$ are said to be isomorphic if and only if there is a bijective homomorphism from the group $G$ to the group $H$. We have:

Theorem. (The Fundamental Theorem of Finite Abelian Groups) Every finite Abelian group is a product of cyclic $p$-groups. The number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.

Static and Dynamic Complexity

We assume familiarity with relational structures, first-order logic $\mathbf{FO}$ and other notions from finite model theory and descriptive complexity [11]. Our treatment closely follows that of [12] in this section.

We write $\mathbf{FO}(\leq, +, \times)$ to denote that formulas have access to built-in relations $\leq$, $+$, $\times$ which are interpreted as linear order, addition and multiplication on the domain of the underlying structure and is known to equal $\mathbf{DLOGTIME}$-uniform $\mathcal{AC}^0$ (see [15, Theorem 4.69] for details). In particular, the following operations are in $\mathbf{FO}(\leq, +, \times)$: (a) Product and sum of $(\log n)^{O(1)}$ numbers bounded by $n^{O(1)}$ (b) Prime factorisation of numbers bounded by $n^{O(1)}$. (c) Sorting $\log^O(n)$ many $O(\log n)$-bit integers.

We also need to refer to uniform $\mathbf{AC}^1$ – it contains all queries that can be computed by (families of $\mathbf{DLOGTIME}$-uniform) circuits of depth $O(\log n)$, consisting of polynomially many $\land, \lor, \neg$ gates where $\land$ and $\lor$ gates may have unbounded fan-in. Equivalently, $\mathbf{AC}^1$ is characterised as the problems that can be expressed by applying a first-order formula $O(\log n)$
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In other words: DLOGTIME-uniform $AC^1 = IND[\log n]$ problems that can be expressed by applying a $\text{FO}(\leq, +, \times)$ formula for $O(\log n)$ times.

The goal of a dynamic program is to answer a given query on an input structure subjected to insertion or deletion of tuples. The program may use an auxiliary data structure over the same domain. The domain is fixed during each run of the program.

A dynamic program has a set of update rules that specify how auxiliary relations are updated after a change of the input database. An update rule for updating an auxiliary relation $T$ is basically a formula $\phi$. As an example, if $\phi(x, y)$ is the update rule for auxiliary relation $T$ under insertions into input relation $R$, then the new version of $T$ after insertion of a tuple $\bar{a}$ to $R$ is $T := \{\bar{b} | (I, Aux) \models \phi(\bar{a}, \bar{b})\}$ where $I$ and $Aux$ are the current input and auxiliary data structures. For a state $S = (I, Aux)$ of the dynamic program $P$ with input structure $I$ and auxiliary data structure $Aux$ we denote the state of the program after applying a sequence $\alpha$ of changes by $P_\alpha(S)$. The dynamic program $P$ maintains a $k$-ary query $q$ if, for each non-empty sequence $\alpha$ of changes and the initial input structure $I_{\text{init}}$, a designated auxiliary relation $Q$ in $P_\alpha(S_{\text{init}})$ and $q(\alpha(I_{\text{init}}))$ coincide. Here, $S_{\text{init}} = (I_{\text{init}}, Aux_{\text{init}})$ where $Aux_{\text{init}}$ denotes the initial auxiliary data structure over the domain of $I_{\text{init}}$, and $\alpha(I_{\text{init}})$ is the input structure after applying $\alpha$.

Unlike what is usual in Dynamic Complexity (where initially both input and auxiliary structures are empty), in our case, the input structure is initially populated with some default values and the auxiliary data structure with values that are consistent with these.

There are several ways to adjust the dynamic setting to restricted classes $C$ of structures. Sometimes it is possible that a dynamic program itself detects that a change operation would yield a structure outside the class $C$ or it may be possible to simply disallow change sequences that construct structures outside $C$. Instead we allow structures outside the target class $C$ and say that a program maintains $q$ for a class $C$ of structures, if $Q$ contains its result whenever a change sequence $\alpha$ is such that the application of $\alpha$ to $I_{\text{init}}$ yields a structure from $C$.

The class of queries that can be maintained by a dynamic program with $\text{FO}(\leq, +, \times)$ update formulas is called $\text{DynFO}(\leq, +, \times)$. We say that a query $q$ is in $\text{DynFO}(\leq, +, \times)$ for a class $C$ of structures, if there is such a dynamic program that maintains $q$ for $C$.

To encode magmas relationally, the input structure has a ternary relation $\cdot$ that is a formula for $\text{DynFO}(\leq, +, \times)$ over the domain $U \times U \to U$ and expressed as: $a \cdot b = c$ where $a, b, c \in U$. The size of the domain $U$ is fixed and assumed to be $n$. Our structures may contain one or two monadic relations i.e. $S \subseteq U$ depending on the problem.

In the single change regime, both the magma operation $\cdot$ and the generator set $S$ (in case of CGM, see below) are dynamic. In other words, for some $a, b \in U$, $a \cdot b$ changes from $c$ to $c'$. and the set $S$ changes by the insertion of deletion of a single element.

In the dynamic setting the function $\cdot$ is assumed to be initialised to the constant function such that for every $a, b \in U$ it is the case that $a \cdot b = c_0$ for an element $c_0 \in U$. $S$ is initialised to the empty set. We deal with several kinds of queries:

1. Dynamic Powering in groups: Given a dynamic magma $M$, arbitrary elements $a, b \in M$ and a positive integer $i < n = \text{ord}(M)$, is $a^i = b$ whenever $M$ is a group?
2. Dynamic CGM: Given a dynamic magma $M$, a dynamic subset $S$ thereof, and an arbitrary element $t \in M$, if $M$ is a group then is $t \in \langle S \rangle$?
3. Dynamic Abelian Subgroup Isomorphism: Given a dynamic magma and dynamic sets $S_1, S_2 \subseteq G$, is $\langle S_1 \rangle \cong \langle S_2 \rangle$ when the magma is an Abelian group?
4. Dynamic Abelian Group Isomorphism: Given two dynamic magmas $M_1, M_2$ on $n$ elements each, whenever both are Abelian groups, are they isomorphic?
For a complexity class \( C \) and a function \( f : \mathbb{N} \to \mathbb{N} \), a query \( q \) is called \((C, f)\)-maintainable if there is a dynamic program \( P \) (with first-order definable updates) and a \( C \)-algorithm \( A \) such that for each input database \( I \) over a domain of size \( n \), and each linear order \( \leq \) on the domain, and each change sequence \( \alpha \) of length \( |\alpha| \leq f(n) \), the relation \( Q \) in \( P_n(S) \) and \( q(\alpha(I)) \) coincide where \( S = (I, A(I, \leq)) \).

A version of the muddling lemma suitable for our needs is the following. A sketch of its proof is contained in Appendix A.

> **Lemma 3.** Every \((AC^1, \log n)\) maintainable query is in \( \text{DynFO}(\leq, +, \times) \) if the auxiliary data structure is initialisable in \( \text{FO}(\leq, +, \times) \) and the input structure contains no isolated elements.

### 3 Maintaining small products and powers of magma elements

We show that we can maintain products of logarithmically many elements and powers of each element in \( \text{DynFO}(\leq, +, \times) \). Notice that both these operations are in \( \text{FOLL} \) \[2\] but not in \( \text{FO} \). These primitives are useful in maintaining \( \text{CGM} \) as well as Abelian Group Isomorphism.

#### 3.1 Maintaining products of logarithmically many elements

Let \( a = a_1, \ldots, a_\ell \) be a sequence of elements from a magma.

For a set \( S \subseteq [\ell] \), a prefix \( S' \) of \( S \) is a subset of \( S \) such that if \( s' \in S' \) then for every \( s \in S \) such that \( s \leq s' \), it must be that \( s \in S' \). We will consider products of the elements of a subset \( S \subseteq [\ell] \) under a left associative parenthesis with the left most element in the product being a variable (that we replace with an arbitrary element of the magma). In other words, parenthesizations of the form \( \Pi_S(x) := (\ldots (x \cdot a_{s_1}) \cdot \ldots a_{s_\ell}) \) where \( s_1, s_\ell \) are the smallest and largest indices in \( S \) and \( x \in M \). For convenience we allow \( x \) to take the value \( \varepsilon \) with the intended interpretation that \( \varepsilon \cdot y = y \) for every \( y \in M \).

Let \( S \) be a subset such that the product \( \Pi_{S'}(x) \) and \( \Pi_{S''}(x) \) are distinct whenever \( S', S'' \) are distinct prefixes of \( S \). \( P_S(x, a) \) is defined to be defined as \( \Pi_S(x) \) (and is undefined if for some distinct prefixes \( S', S'' \) of \( S \) the two products are identical. We will have occasion to omit the second argument \( a \) in \( P_S(x, a) \) as it remains fixed during the entire proof.

We say that \( a \) is good with respect to \( S, x \) if the value \( P_S(x) \) is defined. We show:

> **Lemma 4.** For any sequence \( a \) of length \( \ell = O(\log n) \) drawn from a dynamic magma of \( n \) elements we can maintain the pairs \( S, x \) with respect to which \( a \) is good, in \( \text{DynFO}(\leq, +, \times) \).

**Proof.** Suppose for a sequence, \( a \), we know the values \( P_S(x) \) wherever defined. We have to show how to update this database when one entry of the table changes. Suppose, the entry \( g_1 \cdot g_2 = g \) changes to \( g_1 \cdot g_2 = g' \). We first find all \( S_1, S_2 \subseteq [\ell] \) and \( x \in M \cup \{\varepsilon\} \) such that (a) \( S_1 \) is a prefix of \( S_2 \) (b) \( P_{S_1}(x) = g_1 (\varepsilon) a_{s_2} = g_2 \), where \( s_2 \) is the minimum index in \( S_2 \setminus S_1 \).

For all such \( S_1, S_2, x \), purge the entry for \( P_{S_2}(x) \) from the database. In the modified database, we find all \( S_1, S_2 \subseteq [\ell] \) and an element \( x \) of the magma such that: (a) \( S_1 \) is a prefix of \( S_2 \) (b) \( P_{S_1}(x) = g_1 (\varepsilon) a_{s_2} = g_2 \), where \( s_2 \) is the minimum index in \( S_2 \setminus S_1 \). (d) \( P_{S_2 \setminus \{S_1 \cup \{s_2\}\}}(g') \) is defined (e) for every prefix \( S' \) of \( S_1 \), and every prefix \( S'' \) of \( S_2 \setminus \{S_1 \cup \{s_2\}\} \), it is the case that \( P_{S'}(x) \) is distinct from \( P_{S''}(g') \) [4].

If so, augment the database with the following \( P_{S_2}(x) := P_{S_2 \setminus \{S_1 \cup \{s_2\}\}}(g') \).

\[1\] Notice that we do not separately have to specify the definability of \( P_S(x) \) and \( P_{S'}(g') \) because \( P_S(x) \) and \( P_{S_2 \setminus \{S_1 \cup \{s_2\}\}}(g') \) are already assumed to be defined.
The purge and the augment operations are both in \( \text{FO}(\leq, +, \times) \) since the involved sets have cardinality \( O(\log n) \) and are represented as bit vectors and the corresponding \( P_S(x) \) entries are stored in a table with columns \( S, x \).

The correctness of the two operations is easy to see since every purge removes an entry that is no longer correct while every augmentation correctly sets an entry. Next we need to show that the operations are complete in the sense that after performing them every \( P_S(x) \) that is distinct from each \( P_{S'}(x) \) for \( S' \) prefix of \( S \), is actually defined.

Suppose, \( P_{S_2}(x) \) is not defined on changing \( g_1 \cdot g_2 \) to \( g' \) from \( g \). If it were defined before the purge and augment operations then it must have been (correctly) purged but not added back in the augmentation. Thus there must be a prefix \( S_1 \) of \( S_2 \) such that \( P_{S_1}(x) = g_1 \) and \( a_s = g_2 \) (where \( s \) is the smallest index in \( S_2 \setminus S_1 \)). Thus we would redefine \( P_{S_2}(x) \) to be \( P_{S_2 \setminus \{S_1 \cup \{s\}\}}(g') \) unless there exists some prefixes \( S', S'' \) of respectively, \( S_1, S_2 \setminus (S_1 \cup \{s\}) \) such that \( \Pi_{S'}(x) \) and \( \Pi_{S''}(g') \) are both equal, thus \( \Pi_{S'}(x) = \Pi_{S_1 \cup \{s\} \cup S''}(x) \) after the change. Hence again, \( P_{S_2}(x) \) must not be defined, since \( S' \) and \( S'' \) both prefixes of \( S_2 \).

Hence not adding back \( P_{S_2}(x) \) is the correct decision in the augmentation phase ensuring that the operations are complete. ▶

Next we consider the case when \( a \) is gradually increasing in length but remains of length \( O(\log n) \). See Appendix [D] for a proof.

\[\text{Lemma 5.} \text{ For any sequence } a \text{ of length } \ell = O(\log n) \text{ drawn from a dynamic magma of } n \text{ elements which changes in length by one via insertion/deletion of one magma element in a step for } O(\log n) \text{ steps, we can maintain the pairs } S, x \text{ with respect to which } a \text{ is good, in } \text{DynFO}(\leq, +, \times).\]

### 3.2 Maintaining Powers in a Magma

Consider the graph \( X_{M,(g)} \) (that we will abbreviate to \( X_g \)). This has a vertex set \( V(X_g) = M \) and edges \( E = \{x, x \cdot g\} \) for every \( x \in M \). This is an outdegree one graph and the vertices reachable by a given vertex contain a unique directed cycle – thus it forms a so-called pseudo-forest with each weakly connected component containing exactly one cycle (which is also a directed cycle in the original graph) along with directed trees incident on some vertices of the cycle. It is easy to see that there is at most one directed path from a vertex \( s \) to a vertex \( t \) in the graph. We aim to maintain this path in the graph \( X_g \).

In other words, we maintain a predicate \( Q \) of arity 4: such that for every \( g_0, g, g_1 \in M \) and \( i \in [[M]] \) it is the case that \( Q(g_0, g, g_1, i) \) holds iff there is a simple path of length \( i \) from \( g_0 \) to \( g_1 \) in \( X_g \). Equivalently, \( g_0 \cdot g^i = g_1 \) where we assume a parenthesisisation in the product that is left associative.

Suppose the magma changes by one table entry, say \( a \cdot b \) changes from \( c \) to \( c' \). Then if \( b \neq g \) there is no change in \( X_g \). On the other hand if \( b = g \), then we can define the new values \( Q(g_0, g, g_1, j) \) (for every \( g_0, g_1 \in M, j \in [[M]] \)) iff

\[
\begin{align*}
(Q(g_0, g, g_1, j) \land \neg (\exists i < j Q(g_0, g, a, i))) \lor \\
\exists i_1, i_2 \in [[M]] (Q(g_0, g, a, i_1) \land ((j = i_1 + 1 \land g_1 = c') \lor (Q(c', g, g_1, i_2) \land j = i_1 + i_2 + 1 \land \neg (\exists i' < i_2, i'' \in [[M]] (Q(g_0, g, g', i') \land Q(c', g, g', i''))))))
\end{align*}
\]

In simple words, we just concatenate the list containing \( g_0 \cdot g^a \) up to the point that \( a \) is seen, with the list \( c' \cdot g^i \) till just before a value is repeated.
Generalising the notion of ord(·) in a group to a magma we say that ord(g) is the length of the unique cycle reachable from g in Xg. We also define the notion of index ind(·) to be the number of edges in the path from g to the unique cycle reachable from g in Xg. Thus we have the following:

- **Lemma 6.** Maintaining left parenthesised powering and ord(·), ind(·) in a magma are in DynFO(≤, +, ×).

**Proof.** Given g we can compute the largest i such that Q(g, g, g1) is defined for some g1. Next we find j = ind(g) such that Q(g, g, g1 · g, j) is defined. Then the length of the unique cycle ord(g) is k = i − j + 1. To find gα we need to consider two cases – one where α < ind(g) + ord(g) then Q(g, g, g1 · g, α) is defined for some g1 which is gα. On the other hand if α ≥ ind(g) + ord(g) we consider β = ind(g) + ((α − ind(g)) mod ord(g)) and gα is the unique g1 such that Q(g, g, g1 · g, β) is defined.

We conclude this section with a technical lemma to be used later whose proof can be found in Appendix D.1. Let c be a dynamic sequence (that evolves via single insertions or deletions in a step) of distinct elements from a dynamic magma of variable length k = O(log n) that can be maintained in DynFO(≤, +, ×). Let α be a dynamic sequence from \{0, \ldots, n\} of length k in which insertions and deletions take place at the same point as in c (i.e. if an element is added in c at the i-th place a similar insertion or deletion of a small integer occurs in α). Additionally, the value of at most one αi can change in a time step.

- **Lemma 7.** The sequence d = \{c^αi\}i∈[k] (the left associative powers of each element in c) is also a dynamic sequence which evolves via single changes in each step and can be maintained in DynFO(≤, +, ×).

## 4 Cube independence and cube bases

Let c = c1, \ldots, ck be a sequence in a magma M. Let S be a set in M and let \langle S \rangle be the set generated from S (by taking products of elements from S). For a vector \epsilon \in \{0, 1\}^k we write \epsilon^c as short hand for \prod_{i∈[k]} c_i^{\epsilon_i}. The cubic span \langle\langle c\rangle\rangle is defined as \{\epsilon^c : \epsilon \in \{0, 1\}^k\}.

For a sequence c = c1, \ldots, ck from a group G we let c^{-1} denote the sequence \epsilon^{-1}c^{-1}, \ldots, \epsilon^{-1}c^{-1}. Further for two sequences c, c′ from a magma, we let c △ c′ denote the concatenation of the two sequences. We have the following important definitions. c is said to

- be cube-generate \langle S \rangle if \langle S \rangle = \langle\langle c\rangle\rangle the cubic span of c,
- be cube-independent, if for all \epsilon \in \{0, 1\}^k, it is the case that all c^\epsilon are distinct elements of M △ \{\epsilon\},
- be a cube-base of a subset S if it is a cube independent contained in \langle S \rangle such that for any m ∈ \langle S \rangle it must be the case that d = c △ \{m\} is not cube-independent.

Notice that, c is a cube-base if it is (a) contained in \langle S \rangle, (b) cube independent, and, (c) maximal with respect to the previous two properties. Also note that we do not claim that all cube-bases of S have the same cardinality but have the following upper bound which is true because there are 2^k distinct \epsilon’s if \epsilon \in \{0, 1\}^k.

- **Proposition 8.** A cube base of a set S ⊆ M has at most \lceil log ord(M) \rceil elements.

In a group, a cube-base of a set S is close to being a cube-generating sequence for the set \langle S \rangle in a sense that we make precise in the following lemma.

- **Lemma 9.** Let c be a cube-base for a set S, then \tilde{c} = c^{-1} △ c is a cube-generating sequence for \langle S \rangle.
Proof. For any element \( g \in \langle S \rangle \) we know that there exist \( c', c'' \), subsequences of \( c \) such that \( \Pi_{c'} g = \Pi_{c''} \). Thus, \( g = \Pi_{c'-1} \cdot \Pi_{c''} \) and hence \( g \) can be written as \( c' \) for some \( \epsilon \in \{0, 1\}^{2k} \) completing the proof.

Let \( a \in M \). Let \( \langle a \rangle = \{ a^i : i \in \mathbb{N} \} \), be the set of powers of \( a \) under the left associative magma product. Then, we show that a cube base for \( \langle a \rangle \) can be computed in \( \text{FO}(\leq, +, \times) \) given access to auxiliary relations.

Let \( \langle a \rangle \) be of index \( m \) and order \( r \), i.e \( a^m = a^{m+r} \). Let \( b_n = a, a^2, \ldots, a^{2^r} \), where \( s = m + r - 1 \).

\[ \text{Lemma 10.} \quad b_n \text{ is a cube base for the left associative powers } \langle a \rangle. \]

Proof. We can see that for any \( i < 2^{i+1} \), \( a^i \) can be represented as a product of the subsequence of \( b_n \), viz., \( a^i = \prod_{j=1}^{i} c_j a^{2^j} \) where \( c_j \in \{0, 1\} \). Every subsequence of \( b_n \) corresponds to a unique binary number such that \( j \)th bit is one if \( a^{2^j} \) is in the subsequence. Hence, \( b_n \) is cube independent.

Clearly, we can’t include any \( a^i \) in \( b_n \) such that \( i < 2^{i+1} \) without destroying the cube independence of \( b_n \). We show that in fact we can’t include any \( a^i \) for \( 2^{i+1} \leq i < m + r \) in \( b_n \) without affecting its cube independence. We know that \( k = m + r - i < 2^{i+1} \). So, \( a^k \) can be represented by product of elements in a subsequence of \( b_n \). In the case that \( 2^{i+1} \leq i > m \) and \( i \leq m \), if we include \( a^i \) in the \( b_n \), then \( b_n \) loses its cube independence, since \( a^i a^{m-r} = a^{i+m-r} \) and the fact that \( m - i, m + r - i < 2^{i+1} \). If \( i > m \) then also we can’t include \( a^i \) because \( a^i a^i \) since \( a^i \) is in the cycle part of \( \langle a \rangle \) with order \( r \). In the case that \( m < 2^{i+1} \leq i \) we can’t include \( a^i \) in \( b_n \) without affecting its cube independence because \( a^i a^{m-r} = a^m \) and the fact that \( m + r - i, m < 2^{i+1} \). So, \( b_n \) is maximal and hence is a cube base.

\[ \text{Lemma 11.} \quad b_n \text{ can be maintained in } \text{DynFO}(\leq, +, \times). \]

Proof. Notice that we can obtain the order and the index of \( \langle a \rangle \) just by looking up the powers of \( a \) relation. That is check amongst all the \( i, j \) such that \( a^j = a^{i+j} \) and choose the smallest such \( i, j \) and declare \( i \) to be the index and \( j \) to be the order of \( \langle a \rangle \). Now, for the cube base, we need to compute \( s \) which is equal to \( |\log(m+r-1)| \). This we can do since arithmetic on small numbers is in \( \text{FO}(\leq, +, \times) \). The \( i \)th element \( (i \in \{0\} \cup \{s\}) \) in the cube base \( b_n \) is \( a^{2^i} \) which can be accessed from the powers of \( a \) relation that we maintain.

We will merge cube bases \( b_n \) for single elements \( a_i \) to form cube bases for a set \( \{a_1, a_2, \ldots\} \) of elements. Thus each element of the merged cube base will be of the form \( a^{2^i} \) as described in the definition of \( b_n \), just before Lemma 11. We say an element \( a_i^{2^j} \) arises from \( b_n \).

\[ \text{Lemma 12.} \quad \text{Given two cube bases } c_1, c_2 \text{ a cube base } c \text{ for the set } c_1 \circ c_2 \text{ can be computed in } \text{FO}(\leq, +, \times). \text{ Moreover, we can compute a function } \rho_c \text{ defined on elements of } c \text{ such that } \rho_c(x) = a \text{ if the element } x \text{ arises from } b_n. \]

Proof. Note that \( c_2^{-1} \circ c_1^{-1} \circ c_1 \circ c_2 \) is a cube generating sequence for \( \langle c_1 \circ c_2 \rangle \). So, \( c_1 \circ c_2 \) contains a cube base.

Let \( c = c_1 \circ c_2' \), where \( c_2' \) is a maximal length subsequence of \( c_2 \) such that \( c_1 \circ c_2' \) is cube independent. This, by definition qualifies as a cube base. Since, the length of any cube base is logarithmically bounded we can go over all polynomially many subsequences of \( c_2 \) and check their cube independence. Selecting the maximal candidate comes down to comparing \( \log n \) bit numbers which can be done \( \text{FO}(\leq, +, \times) \) with bit predicates.

Finding \( \rho_c(x) \) is easy since we know \( \rho_{c_1}(x), \rho_{c_2}(x) \) for each \( x \in c_1, c_2 \) respectively.
Lemma 13. A cube base $c$ for a subset $S$ of the magma can be computed in $AC^1$. The function $\rho_c$ can also be computed in $AC^1$.

Proof. We know that the set $S$ is ordered; let $S = \{s_1, s_2, ..., s_k\}$ where $s_1 < s_2 < \ldots < s_k$. We initialise $c = b_{s_1}$. After some number of steps suppose have the cube base $c$ for $S_i = \{s_1, \ldots, s_i\}$. For each $s_j \in S \setminus S_i$ we check if $b_{s_j}$ is in $(\langle c \rangle)$. We pick the first $s_j$ for which $b_{s_j}$ is not contained in $(\langle c \rangle)$ and merge $c$ and $b_{s_j}$ using Lemma 12. We set the new $i = j$ and repeat. We have to show that each step of the procedure is in $FO(\leq, +, \times)$ and there are $O(\log n)$ stages.

Throughout the procedure we maintain $c^\epsilon$ for all possible $\epsilon$’s. Updating it in $FO(\leq, +, \times)$ in one step is easy because we need to take the product of $c^\epsilon$ with $b_{s_j}^{\epsilon'}$ for all possible $\epsilon, \epsilon'$. The first value is available and the second is a small power of $s_j$ and each power is available by repeated squaring in $AC^1$ (even in FOLL by [2]) at the beginning of the procedure. Secondly we add exactly one element at every stage till we run out of elements to add. Since a cube base is of length at most $O(\log n)$ the number of stages is at most $\log n$.

Further we can compute the function $\rho_c$ as we can carry around the $\rho$’s while merging cube bases using Lemma 12.

At this point, for a cube base $c$ of a set $S$, let us define the root base, $\hat{c}$ of the set $S$ as the set $\{\rho_c(x) : x \in c\}$. It is easy to see that $\langle \hat{c} \rangle = \langle \langle c \rangle \rangle$. Also notice that if $c$ is a cube sequence for a set $S$ then even though $c$ might not be contained in $S$ (since it can contain elements from $b_a \setminus S$ for some $a \in S$) it must be the case that the constructed root base $\hat{c} \subseteq S$.

4.1 Dynamic Maintenance of a cube base

Suppose we have a dynamic subset $S$ of a magma $M$ for which we want to maintain a cube base. We will refer to the instantaneous set $S$ at time instance $t$ as $S_t$. Notice that due to the use of muddling and the static computation of the cube base $c_0$ in $AC^1$, we need to consider $t < t := \log n$ only as explained in the proof of the (muddling) Lemma 3. We handle insertions and deletions separately. For an inserted element $a$ Lemma 16 merges (using Lemma 12) its cube base (found via Lemma 10) to the existing cube base of previously inserted elements. For deletions we “precompute” using Lemma 14 the polynomially many deletion possibilities and the outcomes. During the dynamic phase of muddling (in Lemma 15) we just monitor the deletions to see which of these possibilities occur and use the outcome of the appropriate pre-computed one. Lemma 17 then merges the effects of insertions and deletions with some finer adjustments.

Fix a deletion sequence. Let $\Delta_0 \subseteq \hat{c}_0$ be the elements of the root base $\hat{c}_0$ that are deleted from the set $S$ in the given deletion sequence. Let $c_1$ be the cube base for $S \setminus \rho_{c_0}(\Delta_0)$. Similarly, let $\Delta_1 \subseteq \hat{c}_1$ be the elements in the modified root base $\hat{c}_1$ that are deleted. Notice that $\Delta_0 \cap \Delta_1 = \emptyset$ because we include all the elements that are ever deleted from $\hat{c}_0$ in $\Delta_0$ thus, $\Delta_1 \subseteq \hat{c}_1 \setminus \hat{c}_0$. In general, we let $\Delta_j \subseteq \hat{c}_j$ be the elements deleted from the $j$-th root base $\hat{c}_j$ and thus $\Delta_j \subseteq \hat{c}_j \setminus \hat{c}_{j-1}$. Hence, again $\Delta_j \cap \Delta_{j'} = \emptyset$ for $j \neq j'$.

Let us count the number of possible sequences of root bases $\hat{c}_1, \hat{c}_2, \ldots$ assuming that $\ell_j, \tilde{\ell}_j$ are the number of elements in of $c_j, \hat{c}_j$ respectively. We also abbreviate $|\Delta_j|$ to $i_j$ in the following. There are $\binom{\ell_0}{i_0}$ ways to choose $i_0 := |\Delta_0|$ elements to delete from the original root base $\hat{c}_0$. Next we will complete the cube base $c_1$ from $S_0 \setminus \Delta_0$, and the root base $\hat{c}_1$.

The number of ways to delete $i_1$ elements from $\hat{c}_1$: $\binom{\ell_0 + (\ell_1 - i_0)}{i_1}$ because only the elements not in $\hat{c}_0 \setminus \Delta_0$ can be deleted next and the number of new elements included in $\hat{c}_0 \setminus \Delta_0$ to yield $\hat{c}_1$ is precisely $i_0 + (\ell_1 - \ell_0)$ since $i_0$ elements are deleted from $\ell_0, \ell_1 - (\ell_0 - i_0)$
is the number of new elements in $\hat{c}_1$, assuming that $\hat{t}_1 > \hat{t}_0$. Continuing this way, the number of new elements included in $\hat{c}_{j-1} \setminus A_{j-1}$ to yield $\hat{c}_j$ is precisely $i_{j-1} + (\hat{t}_j - \hat{t}_{j-1})$. The sequence of root base sets $\hat{c}_0, \hat{c}_1, \ldots$ can be described by a bit vector of length at most $\sum_{j=0}^{k} (i_{j-1} + (\hat{t}_j - \hat{t}_{j-1})) \leq 2\ell$ that has a bit $b_{j\ell}$ set to 1 iff the $\ell$-th element (where we assume the standard ordering on the elements of the magma) in the root base $\hat{c}_j$ is deleted. However, we might need superlinear in $\ell$ many bits to describe $c_0, c_1, \ldots$. Nevertheless, the proof of Lemma 14 below tells us that these latter sequences are available to us in the beginning of the muddling phase.

Lemma 14. The number of possible root base sequences $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_k$ is bounded by $n^2$. We can compute all possible such sequences as well as the cube bases $c_1, c_2, \ldots, c_k$ in $AC^1$, along with $c_i^{\epsilon_i}$ for all possible $\{0,1\}$-vectors $\epsilon_i$ of length equal to $c_i$.

Proof. The bound of $n^2$ follows from the fact that the possible root bases can be described a bit sequence of length $2\ell$. To prove the $AC^1$ bound just notice that the (parallel) time taken to compute $c_j$ from $c_{j-1}$ needs at most $i_{j-1} + (\hat{t}_j - \hat{t}_{j-1})$ applications of Lemma 11 and Lemma 12 each of which is in $FO(\leq,+,\times)$. Thus, the total parallel time taken is $\sum_{j=0}^{k} (i_{j-1} + (\hat{t}_j - \hat{t}_{j-1})) \leq 2\ell$. Since $\ell = \log n$, this completes the proof of computation of the sequence $c_1, c_2, \ldots, c_k$. For this to be possible we have access to all $c_i^{\epsilon_i}$ throughout. Converting $c_j$ to $\hat{c}_j$ is, of course, in $FO(\leq,+,\times)$ and that completes the proof.

Lemma 15. We can compute a cube base $c$ for a set $S$ in a deletion only regime (where the table remains unchanged) in $DynFO(\leq,+,\times)$ per step, at the end of $\log n$ steps. We can also simultaneously compute all $c^{\epsilon}$'s.

Proof. In the muddling phase (after the completion of the static computation) we keep track of which elements are actually deleted so that at the end of the muddling phase we know all the deleted elements and then we can just pick the pattern that matches the sequence of root bases $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_k$ and output the corresponding cube $c = c_k$ and all $c^{\epsilon_i}$'s which are available from Lemma 14.

Lemma 16. We can compute a cube base $a$ for a set $S$ in an insertion only regime (where the table remains unchanged) and initially the set $S$ is empty in $DynFO(\leq,+,\times)$ per step for $\log n$ steps. We can also maintain all $a^{\epsilon}$'s.

Proof. We initialise $a_1$ to $b_{a_1}$ where $a_1$ is the first element added. Subsequently after the $i$-th step when the element $a_{i+1}$ is added we take the maintained $b_{a_{i+1}}$ and merge it with the current cube base $a_i$ using Lemma 12 in $FO(\leq,+,\times)$ to yield the cube base $a_{i+1}$ of $S = \{a_1, \ldots, a_{i+1}\}$. This completes the proof.

Lemma 17. We can maintain a cube base $c$ for a set $S$ in the fully dynamic regime (where the table and the set $S$ both evolve via single changes) in $DynFO(\leq,+,\times)$ for $\log n$ steps. Further we can maintain $c^{\epsilon}$ for all $\{0,1\}$-vectors $\epsilon$.

Proof. We first make the simplifying assumptions that:

1. there is a gap of more than $\log n$ steps between inserting an element $s$ into $S$ and deleting it from $S$, and,
2. Symmetrically there is a gap of more than $\log n$ steps between deleting an element from $S$ and adding it back to $S$, and,
3. the table does not change.
Thus we can separately use Lemma 15 and Lemma 16 to find cube bases of the sets $S_0 \setminus \Delta^-$ and $\Delta^+$, where the sets $\Delta^-$ (respectively $\Delta^+$) correspond to deletions (respectively, insertions). Let $c, a$ be the cube bases for the two cases. Since there is no intersection between $\Delta^+, \Delta^-$ we can just use Lemma 12 to merge $c, a$ yielding the final cube base.

Next we relax the first assumption. When the element is added we create two “branches” of execution – one in which the element is added as usual and in the other there is no insertion. In all, this would create $2^{\log n} = O(n)$ many branches and at the end we would have the results for all the possible subsequences of insertions. Thus if some inserted elements are deleted within the $\log n$ steps there would be a branch corresponding precisely to the correct final insertions and we would use that one. The second assumption may be relaxed in a very similar way.

Finally, to handle table changes, notice that if $a \cdot b = c$ changes to $c'$ then this might affect powers of $b$ as well as the cubic span of cube bases that have $b$ in them. To deal with we can delete $b$ from the root base (and $S$) and reinsert it back in $S$ after altering the table. Thus, only the members of $b_t$ that are also members of the cube base of $S$ are affected. Since we have dealt with both deletion and insertion earlier it can use calls to the insertion and deletion routines.

Note that we not only obtained the cube base $c$ after the muddling step but also $c^t$ for all $t \in \{0, 1\}^t$ where $t$ is the final length of $c$. These were necessary to verify that $c$ was indeed a cube base. This completes the proof of the lemma.

Now, we can complete the proof of Theorem 1.

Proof. (of Theorem 1) Let $c$ be the maintained cube base for the set $S$ from Lemma 17. From Lemma 9 this reduces to checking that the target $t$ is cube generated by an element of $c^{-1}$ and $c$. This, in turn, involves checking if the target equals the product of some two elements $c^{-c}$ and $c^{c}$ which are available from Lemma 17. This is in $\text{FO}(\leq, +, \times)$ and completes the proof.

5 Dynamic Abelian group isomorphism

Before proving Theorem 2 we state the following lemma for the subgroup isomorphism problem to which we reduce the isomorphism problem (by a truth table reduction preserving the number of changes). Then we show how the theorem about the isomorphism problem for Abelian groups follows from it.

Lemma 18. Given a dynamic magma $M$, and sets $S_1, S_2 \subseteq M$, the isomorphism between $\langle S_1 \rangle$ and $\langle S_2 \rangle$ can be checked in $\text{DynFO}(\leq, +, \times)$ when the magma $M$ is an Abelian group.

Proof. (of Theorem 2) Given two dynamic magmas $M_1$ and $M_2$, we maintain the set of magmas $(\{a_1\} \times M_2)_{a_1 \in M_1}$ and $(M_1 \times \{a_2\})_{a_2 \in M_2}$. Notice that when the magmas are groups, then $M_2 \cong M_2$ iff the subgroups $\{e_1\} \times M_2$ and $M_1 \times \{e_2\}$ of the group $M_1 \times M_2$ are isomorphic. Here $e_1$ and $e_2$ are the identities of $M_1$ and $M_2$ respectively. Thus, we can use Lemma 18 to check this in $\text{DynFO}(\leq, +, \times)$.

We need the following two lemmas from [13], followed by a three lemmas (see Appendix E for proofs), in preparation for the proof of Lemma 18. The idea is to invoke the decomposition of Abelian groups into a direct product of $p$-groups (see Section 2) and use the fact that two Abelian groups are isomorphic if each of the corresponding $p$-groups are isomorphic. The $p$-group isomorphism is handled by Lemma 20 to reduce it to finding the orders of some
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$p$-groups. Lemma 19 uses Lemma 18 to reduce the order finding to certain CGM questions along with Powering. The reductions are composed in Lemma 23.

Lemma 19. (Proposition 6.6) Let $G = \langle g_1, \ldots, g_r \rangle$ be a finite Abelian group. Then, $\text{ord}(G) = t_1 t_2 \cdots t_r$ where $t_j$ is the least positive integer such that $g_j^{t_j} \in \langle g_{j+1}, \ldots, g_r \rangle$ for $1 \leq j \leq r$. (such a $t_j$ must exist for each $j$).

Lemma 20. (Proposition 6.4) Let $G = \langle g_1, \ldots, g_r \rangle$ and $H = \langle h_1, \ldots, h_s \rangle$ be Abelian $p$-groups, and let $p^k = \max_{i \in [r], j \in [s]} \text{ord}(g_i), \text{ord}(h_j)$, then $G \cong H$ iff for every $i \in \{0, \ldots, k-1\}$: $\text{ord}(\langle g_i^r, \ldots, g_i^{r^k} \rangle) = \text{ord}(\langle h_i^r, \ldots, h_i^{r^k} \rangle)$.

Lemma 21. Let $g_1, \ldots, g_r$ be a cube-generating sequence for an Abelian group $G$. For any prime $p$, the Sylow $p$-subgroup is cube generated by $g_1^{q_1 \alpha_1}, \ldots, g_r^{q_r \alpha_r}$ where $\alpha_j$ is the largest factor of $\text{ord}(g_j)$ not divisible by $p$ and $\alpha_j'$ is its multiplicative inverse modulo the largest power of $p$ dividing $\text{ord}(G)$.

Lemma 22. Let $T$ be a dynamic subset that changes by at most one element at a time and of size $|T| = O(\log n)$. Then there is a $\text{FO}(\leq, +, \times)$ truth table reduction from the order of $\text{ord}(T)$ to the CGM and Powering problems. Further, if there is a single change in the generating set then in the reduction there are $O(1)$ changes in the instances of CGM and Powering.

Lemma 23. We can reduce Abelian subgroup isomorphism with for $p$-groups (for, $p|n$) where the size of the generating sets remains $O(\log n)$, to CGM and Powering using $\text{FO}(\leq, +, \times)$ truth table reductions. Further, if there is a single change in the generating sets for the subgroups then in the reduction there are $O(1)$ changes in the instances of CGM and Powering.

Proof. (of Lemma 18) From Lemma 17 (which allows us to maintain a cube-base sequence in $\text{DynFO}(\leq, +, \times)$) and Lemma 9 we see that a cube generating sequence for $(S)$ can be maintained in $\text{DynFO}(\leq, +, \times)$. Lemma 21 (which allows us to extract a cube generating sequence for the Sylow $p$-subgroup for every prime $p$) we can get a sequence of $p$-elements say $c(p)$ that cube generate the $p$-Sylow subgroup $G_p$ of $M$ whenever $M$ is an Abelian group, for every prime $p$ dividing $n$.

This list is the result of powering the elements $c_i$ of a cube generating sequence $c$ of $G$ to $\alpha_i \alpha_i'$. This latter integer sequence can be maintained in $\text{DynFO}(\leq, +, \times)$ because of Lemma 6 and the fact that arithmetic on small numbers is in $\text{FO}(\leq, +, \times)$. At this point we use Lemma 33 to assert that $c(p)$ evolves via single changes as well 2. Thus, the $P_k(\varepsilon)$ of all good pairs of the form $S, \varepsilon$ of $c(p)$ can be maintained in $\text{DynFO}(\leq, +, \times)$ via Lemma 5.

This enables us to use Lemma 23 (which allows us to do isomorphism for the Sylow $p$-subgroups for various primes $p$). Notice that we require $\text{AC}^1$ initialisation to populate the auxiliary database with all the powers and also for the products of $O(\log n)$ elements needed for initialising the cube generating sequences.

6 Conclusion and Open Ends

We have shown that we can maintain a $\text{DynFO}$ bound for two problems – (1) the dynamic CGM problem when the magma specified as a table is a group and (2) the dynamic isomorphism problem when the magma specified as a table is an Abelian group.

2 The elements of $c$ might not be pairwise distinct since $c$ consists of the union of distinct elements and their inverses and there could be overlap between the two lists. Therefore we just maintain the products of the appropriate powers of the cube base and separately of their inverses.
The next natural question to ask is if it is possible to extend the dynamic bounds to other group theoretic problems such as the ones parallelised in [13]. Another direction to pursue would be to extend the bound on dynamic Abelian group isomorphism to more general classes of groups. Unfortunately, our methods seem intricately tied to the group being Abelian. The notion of cube independence is new, and it would be interesting to find other applications of this to group theoretic problems.

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A Proof of the Muddling Lemma

A version of the Muddling Lemma [6] states that

Lemma 24. ([6, Theorem 4.1]) Every \((\AC^1, \log n)\)-maintainable, almost domain independent query is in DynFO.

Here almost independent query refers to one whose answer does not depend on the number of isolated elements (i.e. those elements of the domain that do not appear in the input database) beyond a constant. Since we are dealing with the case that the input structure contains a total function \((\cdot : U \times U \to U)\) at any point of time, the number of isolated elements is zero. The modified version of the lemma suitable for our needs is:

Lemma 25. (Originally Lemma 3) Every \((\AC^1, \log n)\) maintainable query is in DynFO(\((\leq, +, \times)\)) if the auxiliary data structure is initialisable in FO(\((\leq, +, \times)\)) and the input structure contains no isolated elements.

Proof. (Sketch) In this extended abstract we only sketch the important points in the full proof which is a variant of the proof of [6, Theorem 4.2].

Assume we have an \(\AC^1\) algorithm \(A\) and a dynamic program \(P\) which witness that a query \(q\) is \((\AC^1, \log n)\)-maintainable. We need to exhibit a dynamic program \(P'\) that witnesses \(q \in \text{DynFO}(\((\leq, +, \times)\))\). We restrict ourselves to a structure encoding a magma as well as constantly many sets of domain elements, for simplicity. We present the intuitive version of \(P'\) and leave the details for the full paper.

We consider each application of one change as a time step and refer to the structure after time step \(t\) as \(S_t\). After each time step \(t\), the program \(P'\) starts a thread that in charge of answering the query at time point \(t + \log n\). Each thread works in two phases, each lasting \(\log n\) time steps. Roughly speaking, the first phase is in charge of simulating \(A\) and in the second phase \(P\) is used to apply all changes that occur from time step \(t + 1\) to time step \(t + \log n\). Using \(\log n\) many threads, \(P'\) is able to answer the query from time point \(\log n\) onwards.

For the first phase let \(\psi\) be the FO(\((\leq, +, \times)\)) formula that applied \(d \log n\) times is equivalent to the algorithm \(A\) - this exists because DLOGTIME-uniform \(\AC^1\) is the same as IND[\(\log n\)]. The program \(P'\) applies \(\psi\) to \(S_t\), \(2d\) times in each time step and thus the result of \(A\) on \(S_t\) is obtained after \(\frac{\log n}{2}\) steps. The change operations that occur during these steps are not applied to \(S_t\) directly but rather stored in some additional relation. If some tuple is changed multiple times, the stored change for the tuple is adjusted accordingly.

During the second phase the \(f = \frac{\log n}{2}\) stored change operations and the \(f\) change operations that happen during the next \(f\) steps are applied to the state after phase 1. To this end, it suffices for \(P'\) to apply two changes during each time step by simulating two update steps of \(P'\). Observe that \(P'\) processes the changes in a different order than they actually occur. However, both change sequences result in the same input structures. ◆
Reach is the problem of checking whether there is a directed path from a vertex $s$ to a vertex $t$ in a directed graph $G$. The following result is known:

**Theorem 26.** ([4, 7]) Reach is in DynFO($\leq$, $+$, $\times$) under batch changes of $O(\frac{\log n}{\log \log n})$ edges.

This immediately implies the following:

**Corollary 27.** Strong connectivity in a directed graph can be maintained in DynFO($\leq$, $+$, $\times$) under batch changes of $O(\frac{\log n}{\log \log n})$ edges.

### B.1 Known Results

The following set of results/observations can be found in the paper by Barrington et al. [2] with proofs.

**Lemma 28.** ([2]) Let $M$ be a magma specified via a multiplication table. Then the following properties can be checked in FO:

1. $M$ is a group.
2. $M$ is Abelian.

### B.2 Cayley Group Membership problem with a fixed set of generators

We start with a result that is an easy consequence of the Reachability in DynFO result from [7] (see also [4]). In the bulk change regime, there is only one type of operation permitted viz. Bulk dynamic table: $\ell = O(\frac{\log n}{\log \log n})$ many values associated with $\cdot$ change i.e. there are $\ell$ distinct pairs of elements $(a_i, b_i)$ for $a_i, b_i \in U$ and $i \in [\ell]$, such that $a_i \cdot b_i$ changes from $c_i$ to $c'_i$. The relation $S$ does not change.

**Theorem 29.** The CGM(Abelian) problem for a fixed set $S$ of generators in DynFO whenever the underlying magma is a group and the group $\langle S \rangle$ is Abelian under bulk changes of size $O(\frac{\log n}{\log \log n})$ at every step.

**Proof.** Consider the Cayley graph $X_{M,S}$ of the magma under the given set of generators $S$. This (multi) graph has the elements of $M$ as vertices and every vertex $m \in M$ has $|S|$ edges i.e. $(m, m \cdot s)$ for each $s \in S$ emanating from it. Notice that $\ell$ changes in the table for the magma causes at most $\ell$ edges to change in $X_{M,S}$ since the set $S$ is fixed. Thus reachability in this graph can be maintained in DynFO for $\ell = O(\frac{\log n}{\log \log n})$.

For the CGM(Abelian) problem we first verify that $M$ is a group and $\langle S \rangle$ is an Abelian group, both of which predicates are in FO. In the process we also discover the identity $e$ of the group $M$. Checking if the target $t$ is reachable from $e$ in the graph $X_{M,S}$ allows us to test for the CGM(Abelian) problem.

We will next aim to make $S$ dynamic as well as relax the Abelian restriction on groups. The next couple of sections are devoted to collecting tools to achieve that.

### C An application of powering in a magma

We now have the following direct application of our Lemma [5] on powering in magmas.

**Theorem 30.** Dynamic nilpotent group testing is in DynFO($\leq$, $+$, $\times$).
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Proof. We can check in FO if a magma $M$ is a group from Lemma 28. In particular we can find the (unique) identity $e$ of the group if it is indeed a group. Whenever it is a group, using Lemma 6, we have access to $\text{ord}(a)$ for every $a \in M$ in DynFO$(\leq, +, \times)$.

Thus we can identify the $p$-elements of $G$ (i.e. $g$ such that $o(g)$ is a power of $p$) for every $p|n$. We can now check in FO if the $p$-elements form a group just as in Lemma 28. Thus we are done using the following lemma from [2].

- Lemma 31. [2, Lemma 3.1] A finite group $G$ is nilpotent iff, for each prime $p$ dividing $\text{ord}(G)$, the set of $p$-elements in $G$ is a group.

D Maintaining products of logarithmically many elements

- Lemma 32. (Originally Lemma 3) For any sequence $a$ of length $\ell = O(\log n)$ drawn from a dynamic magma of $n$ elements which changes in length by one via insertion/deletion of one magma element in a step for $O(\log n)$ steps, we can maintain the pairs $S, x$ with respect to which $a$ is good, in DynFO$(\leq, +, \times)$.

Proof. The proof extends that of Lemma 4. Suppose that before some step $a$ consists of $\ell$ magma elements and a new element is inserted in the list $a$ at index $i$ to yield $a'$. We write $S + k = \{s + k : s \in S\}$ as a notational convenience. We have to consider $S' \subseteq [\ell + 1]$. Define, $S_\ell = S' \cap \{1, \ldots, i-1\}$ and $S_{\ell+1} = S \cap \{i+1, \ldots, \ell + 1\} - 1$. Finally, let $S = S_\ell \cup S_{\ell+1}$. Now, if $i \notin S'$ then $\Pi_{S'}(x)$ is just $P_S(x)$. Else, let $y = \Pi_{S_\ell}(x), z = y \cdot a'_i$ and $w = \Pi_{S_{\ell+1}}(z)$. Then $\Pi_{S'}(x)$ equals $w$. Now its easy to check in FO$(\leq, +, \times)$ if $S', x$ is good for $a'$ by checking if $P_{S'}(x)$ is defined or not which in turn is easy in FO$(\leq, +, \times)$ given the $\Pi_{S''}(x)$ for all $S'' \subseteq [\ell + 1]$.

D.1 Maintaining powers in a magma

- Lemma 33. The sequence $d = \{e_i a_i\}_{i \in [k]}$ (the left associative powers of each element in $c$) is also a dynamic sequence which evolves via single changes in each step and can be maintained in DynFO$(\leq, +, \times)$.

Proof. Suppose $a \circ b$ changes from $c$ to $c'$. Then the only change in the sequence $c$ is for that $i$ for which $a_i = c$. This is because $x^a$ (under a left associative product) remains unchanged unless $x = c$. Also a single change in $a$, say to $a_i$, causes only $e_i a_i$ to change and this last is available from Lemma 6, completing the proof.

E Omitted parts of Section 5

We start with a simple result:

- Lemma 34. (18, Lemma 3.8) Let $G = \langle g_1, \ldots, g_r \rangle$ be Abelian. For any prime $p$, the Sylow $p$-subgroup equals $\langle g_1^{\alpha_1}, \ldots, g_r^{\alpha_r} \rangle$ where $\alpha_j$ is the largest factor of $\text{ord}(g_j)$ not divisible by $p$.

We adapt the above lemma for our purposes:

- Lemma 35. (Originally Lemma 21) Let $g_1, \ldots, g_r$ be a cube-generating sequence for an Abelian group $G$. For any prime $p$, the Sylow $p$-subgroup is cube generated by $g_1^{\alpha_1 a_i}, \ldots, g_r^{\alpha_r a_i}$, where $\alpha_j$ is the largest factor of $\text{ord}(g_j)$ not divisible by $p$ and $\alpha_j$ is its multiplicative inverse modulo the largest power of $p$ dividing $\text{ord}(G)$.
Proof. For any element \( g \in G \), write \( g = g^{(p)} \cdot g^{(\bar{p})} \) where \( g^{(p)} \) is a \( p \)-element (i.e. has order divisible by a power of \( p \)) and \( g^{(\bar{p})} \) is a \( \bar{p} \)-element (i.e. has order relatively prime to \( p \)). It is easy to see that this factorisation is unique (since if \( g^{(p)} \cdot g^{(p)} = a \cdot b \) where \( a \) is a \( p \)-element and \( \text{ord}(b) \) is relatively prime to \( p \) then \( a^{-1} \cdot g^{(p)} = b \cdot (g^{(\bar{p})})^{-1} \). But the left hand side is a \( p \)-element and the right hand side has order relatively prime to \( p \) implying both are identity).

Thus, \( g_i^{\alpha_i \alpha'_i} = (g^{(p)}_i)^{\alpha_i \alpha'_i} \cdot ((g^{(p)}_i)^{\alpha_i})^{\alpha'_i} \). Now \( \alpha_i = \text{ord}(g^{(p)}_i) \) and \( \alpha_i \alpha'_i = qp^\alpha + 1 \) where \( p^\alpha \) is the highest power of \( p \) dividing \( n \) and \( q \) is some integer. Thus, \( (g^{(p)}_i)^{\alpha_i \alpha'_i} = g^{(p)}_i \) while, \( g^{(p)}_i \) is the identity. Hence, \( g_i^{\alpha_i \alpha'_i} = g^{(p)}_i \).

Notice that for any \( g \in G \) there exists an \( \epsilon \in \{0,1\}^r \) such that \( g = \prod_{i \in [r]} g_i^{\epsilon_i} \). Thus \( g^{(p)} = \prod_{i \in [r]} (g^{(p)}_i)^{\epsilon_i} \) from the unique factorisation into a \( p \)-element and a \( \bar{p} \)-element. This completes the proof since every element of the Sylow \( p \)-subgroup is \( g^{(p)} \) for some \( g \in G \).

▼ Lemma 36. (Originally Lemma 22) Let \( T \) be a dynamic subset that changes by at most one element at a time and of size \( |T| = O(\log n) \). Then there is a \( \text{FO}(\leq, +, \times) \) truth table reduction from the order \( \text{ord}(\langle T \rangle) \) to the CGM and Powering problems. Further, if there is a single change in the generating set then in the reduction there are \( O(1) \) changes in the instances of CGM and Powering.

Proof. Follows from Lemma 19 because we can check for each \( j \in [r], t \in [\text{ord}(g_j)] \) if \( g_j^t \in \langle g_{j+1}, \ldots, g_r \rangle \). This can be done via CGM whenever magma is a group. Finding the smallest \( t \) is in \( \text{FO}(\leq, +, \times) \). Note that \( g_j^t \) and \( \text{ord}(g_j) \) are from Powering.

▼ Lemma 37. (Originally Lemma 23) We can reduce Abelian subgroup isomorphism with for \( p \)-groups (for, \( p \mid n \)) where the size of the generating sets remains \( O(\log n) \), to CGM and Powering using \( \text{FO}(\leq, +, \times) \) truth table reductions. Further, if there is a single change in the generating sets for the subgroups then in the reduction there are \( O(1) \) changes in the instances of CGM and Powering.

Proof. Immediate from composing Lemma 20 and Lemma 22

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