EFFECTIVE EQUIDISTRIBUTION FOR SOME UNIPOTENT FLOWS IN $\text{PSL}(2, \mathbb{R})^k$ MOD COCOMPACT, IRREDUCIBLE LATTICE

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ABSTRACT. Let $k \geq 2$, and let $\Gamma \subset \text{PSL}(2, \mathbb{R})^k$ be an irreducible, cocompact lattice. We prove effective equidistribution for unipotent flows on $\Gamma \backslash \text{PSL}(2, \mathbb{R})^k$ where the generator in $\mathfrak{sl}(2, \mathbb{R})^k$ satisfies the following condition: As a $2k \times 2k$ matrix, the eigenspace of the zero eigenvalue has codimension 1. These are the simplest cases for proving effective equidistribution in $\Gamma \backslash \text{PSL}(2, \mathbb{R})^k$.

1. INTRODUCTION

There has been greater interest recently in making Ratner’s theorems effective. Green-Tao proved all Diophantine nilflows on any nilmanifold become equidistributed at polynomial speed, see [5]. Flaminio-Forni proved rather sharp estimates on the speed of equidistribution for a class of higher step nilmanifolds.

Einsiedler-Margulis-Venkatesh proved effective equidistribution for large closed orbits of semisimple groups on homogeneous spaces in [2]. Strömbergsson and Browning-Vinogradov established a rate of equidistribution for unipotent flows on $\text{SL}(2, \mathbb{Z}) \times \mathbb{Z}^2 \backslash \text{SL}(2, \mathbb{R}) \times \mathbb{R}^2$, see [9] and [1].

We will now describe the setting for our equidistribution result. Let $k \geq 2$ and let $\Gamma \subset \text{PSL}(2, \mathbb{R})^k$ be an irreducible, cocompact lattice, where $\text{PSL}(2, \mathbb{R})^k$ is the direct product of $k$ copies of $\text{PSL}(2, \mathbb{R})$. Let $M = \Gamma \backslash \text{PSL}(2, \mathbb{R})^k$.

The vector fields on $M$ are elements of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})^k$, the direct product of $k$ copies of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. For each $1 \leq i \leq k$, define the vector fields $\{U_i, V_i\}$ in $\mathfrak{sl}(2, \mathbb{R})^k$ by

$$U_i := (0, \ldots, 0, U, 0, \ldots, 0),$$
$$V_i := (0, \ldots, 0, V, 0, \ldots, 0),$$

where $U$ and $V$ are elements of $\mathfrak{sl}(2, \mathbb{R})$ that occur in the $i_{th}$ position of the $k$-tuple, and they are given by

$$U := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

The set $\bigcup_{i=1}^k \{U_i, V_i\}$ is the complete set of generators for unipotent flows on $M$ whose eigenspace for the zero eigenvalue has codimension 1.

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The space of diagonal elements in \( \mathfrak{sl}(2, \mathbb{R})^k \) is \( k \)-dimensional. For each \( 1 \leq i \leq k \), define the diagonal element

\[
X_i := (0, \ldots, 0, X, 0, \ldots, 0) \in \mathfrak{sl}(2, \mathbb{R})^k,
\]

where \( X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \) occurs in the \( i \)th position and generates the geodesic flow on \( \text{PSL}(2, \mathbb{R}) \).

For \( i \in \{1, 2, \ldots, k\} \), we will study the rate of equidistribution of the unipotent flow \( \{\phi^i_t\}_{t \in \mathbb{R}} \) on \( M \) given by

\[
\phi^i_t(x) = xe^{tU_i}.
\]

Our approach is via unitary representations and invariant distributions. The main ingredients for the proof are the work of Flaminio-Forni on the equidistribution of horocycle flows on quotients of \( \text{PSL}(2, \mathbb{R}) \) by a lattice (see [3]) and a result of Kelmer-Sarnak on the spectral gap for irreducible, cocompact lattices \( \Gamma \subset \text{PSL}(2, \mathbb{R})^k \) (see [6]).

Let \( L^2(M) \) be the separable Hilbert space of complex-valued square-integrable functions on \( M \). Let \( C^\infty(M) \) be the space of smooth functions on \( M \) and let \( \mathcal{E}'(M) = (C^\infty(M))' \) be its distributional dual space.

Any element of the Lie algebra \( \mathfrak{sl}(2, \mathbb{R})^k \) acts on \( \mathcal{E}'(M) \) via the right regular representation. The Laplacian operator \( \triangle \) is a second-order, elliptic element in the enveloping algebra of \( \mathfrak{sl}(2, \mathbb{R})^k \). It is an essentially self-adjoint differential operator on \( L^2(M) \) and is given by

\[
\triangle := \triangle_i + \triangle_0,
\]

where

\[
\triangle_i := -X_i^2 - 1/2(U_i^2 + V_i^2) \quad \text{and} \quad \triangle_0 := -\sum_{i \neq j} X_j^2 + 1/2(U_j^2 + V_j^2).
\]

The Sobolev space of order \( s \in \mathbb{R}^+ \) is the maximal domain \( W^s(M) \) of the inner product

\[
\langle f, g \rangle_s := \langle (I + \triangle)^s f, g \rangle,
\]

where \( I \) is the identity element of \( \text{PSL}(2, \mathbb{R})^k \). The space of \( s \)-order distributions on \( W^s(M) \) is \( W^{-s}(M) = (W^s(M))' \). Notice that \( C^\infty(M) = \bigcap_{s>0} W^s(M) \) and \( \mathcal{E}'(M) = \bigcup_{s>0} W^{-s}(M) \).

The center of the enveloping algebra of \( \mathfrak{sl}(2, \mathbb{R})^k \) contains the second-order differential operator

\[
\Box_i := [-X_i^2 - 1/2(U_iV_i + V_iU_i)].
\]

A result by Kelmer-Sarnak shows \( \Box_i \) has a spectral gap on \( L^2(M) \) in the sense that the positive eigenvalues of \( \Box_i \) are bounded below by some \( \mu_0 > 0 \) (see Theorem 2 and Section 1.3 of [6]).
Our estimates are given in terms of Sobolev norms involving a finite number of derivatives. In what follows, we let
\[ s > \frac{3k}{2} + 1, \quad \text{and fix} \]
\[ i \in \{1, 2, \ldots, k\}. \]
Let \( W_s(M) \) be the maximal domain of the operator \((I + \triangle_0)\) on \( L^2(M) \) with inner product
\[ (f, g)_{W_s(M)} := \langle (I + \triangle_0)^s f, g \rangle_{L^2(M)}. \]

Let \( \pi : \text{PSL}(2, \mathbb{R}) \to \mathcal{U}(W_s(M)) \) be a unitary representation of \( \text{PSL}(2, \mathbb{R}) \) defined by
\[ (1) \quad \pi(g)f (\Gamma(a, h, b)) = f (\Gamma(a, hg, b)), \]
for any \((a, h, b) \in \text{PSL}(2, \mathbb{R})^{i-1} \times \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})^{k-i}\). Let \( d\pi \) be the derived representation of \( \pi \). The representation \( d\pi \) is related to the derived representation of the right regular representation by the following simple lemma.

**Lemma 1.1.** Let \( Q \in \mathfrak{sl}(2, \mathbb{R}) \), and let \( Q_i := (0, \ldots, 0, Q, 0, \ldots, 0) \in \mathfrak{sl}(2, \mathbb{R})^k \), where \( Q \) is in the \( i \)-th position. Then
\[ d\pi(Q) = Q_i. \]

**Proof.** Let \( x \in M \) and \( f \in C^\infty(M) \). For any \( t \in \mathbb{R} \),
\[ f(x(I^{i-1}, \exp(tQ), t^{k-i})) = f(x \exp(tQ_i)). \]
Differentiating at \( t = 0 \) gives the lemma. \( \square \)

Hence,
\[ (2) \quad U_i = d\pi(U) \quad \text{and} \quad (I + \triangle_i)f = d\pi(I - X^2 - 1/2(U^2 + V^2)). \]

Then with respect to a positive Stieltjes measure the unitary representation \( \pi \) has the following direct integral decomposition
\[ W_s(M) = \int_{\oplus \mu \in \text{spec}(\square_i)} \mathcal{H}_{\mu, s}dm(\mu), \]
where there Casimir element \( \square_i \) acts as the constant \( \mu \) on each unitary representation space \( \mathcal{H}_{\mu, s} \), and \( d\beta(\mu) \) is a positive Stieltjes measure. Each representation space \( \mathcal{H}_{\mu, s} \) is a direct sum of an at most countable number of irreducible unitary representation spaces.

By irreducibility and (2), the vector fields \( U_i, X_i \) and \( V_i \) are decomposable into the irreducible representations of \( \pi \) in the sense that
\[ (3) \quad W^s(M) = \int_{\oplus \mu \in \text{spec}(\square_i)} \mathcal{H}_{\mu}^s dm(\mu), \]
where \( \mathcal{H}_{\mu}^s < \mathcal{H}_{\mu, s} \) inherits the inner product from \( W^s(M) \).

We denote the space of distributions in \( W^{-s}(M) \) that are invariant under \( U_i \) by
\[ \mathcal{I}^s(M) := \{ D \in W^{-s}(M) : U_i D = 0 \}. \]
Let $I(M) := \bigcup_{s > 0} J^s(M)$. By (2), the classification of $U_i$-invariant distributions in $I(M)$ into irreducible, unitary representation spaces is given by the corresponding classification of $d\pi(U)$-invariant distributions from Theorem 1.1 of [3].

Let $\sigma_{pp}$ be the eigenvalues of $\triangle_i$ on $L^2(M)$. Theorem 1.1 (Flaminio-Forni). The space $I(M)$ has infinite countable dimension. There is a decomposition

$$I(M) = \bigoplus_{\mu \in \sigma_{pp}} J_{\mu} \oplus \bigoplus_{n \in \mathbb{Z}^+} J_n \oplus \bigoplus_{c \in \mathcal{C}} J_c,$$

where

- for $\mu = 0$, the space $J_0$ is spanned by the $\text{PSL}(2, \mathbb{R})$-invariant volume;
- for $0 < \mu < 1/4$, there is a splitting $J_{\mu} = J^+_{\mu} \oplus J^-_{\mu}$, where $J^\pm_{\mu} \subset W^{-s}(M)$ if and only if $s > \frac{1 + \sqrt{1 - 4\mu}}{2}$, and each subspace has dimension equal to the multiplicity of $\mu \in \sigma_{pp}$;
- for $\mu \geq 1/4$, the space $J_{\mu} \subset W^{-s}(M)$ if and only if $s > 1/2$, and it has dimension equal to twice the multiplicity of $\mu \in \sigma_{pp}$;
- for $n \in \mathbb{Z}^+ \geq 2$, the space $J_n \subset W^{-s}(M)$ if and only if $s > n/2$ and it has dimension equal to twice the rank of the space of holomorphic sections of the $n_{th}$ power of the canonical line bundle over $M$.
- for $c \in \mathcal{C}$, the space $J_c \subset W^{-s}(M)$ if and only if $s > 1/2$, and it has infinite countable dimension.

For $s > 1/2$, Theorem 1.4 of [3] shows $I(M)$ has a basis $\mathcal{B}^s(M)$ of unit normed (in $W^{-s}(M)$), generalized eigenvectors for the geodesic flow $\{e^{tX_i}\}_{t \in \mathbb{R}}$. This basis is countable, because $W^{-s}(M)$ is separable.

Let

$$\mathcal{B}^s_+ := \bigcup_{\mu \in \sigma_{pp}/\{\frac{1}{4}\}} \mathcal{B}^s(M) \cap J^s_{\mu},$$

be a basis of $U_i$-invariant distributions for $(\bigoplus_{\mu \in \sigma_{pp}} J^s_{\mu})/J^s_{1/4}$. Let

$$\mathcal{B}^s_- := \left(\bigcup_{\mu \geq \frac{1}{4}} \mathcal{B}^s(M) \cap J^s_{\mu}\right)/\mathcal{B}^s_+.$$

be a basis of invariant distributions for the rest of principal series. It will also be convenient to define

$$\mathcal{B}^s_{1/4} := \mathcal{B}^s(M) \cap J^s_{1/4}.$$

For $D \in W^{-s}(M)$, let

$$s_D := \begin{cases} 
1 + \sqrt{1 - 4\mu} & \text{if } D \in J^s_{\mu}, \mu > 0; \\
\frac{n}{2} & \text{if } D \in J_n, n \in \mathbb{Z} \geq 2; \\
\frac{1}{2} & \text{if } D \in J_c.
\end{cases}$$

For $T \geq 1$, let $\log^+ T := \max\{1, \log T\}$. 
Theorem 1.2. Let $s > 3k/2 + 1$ and $k \geq 2$. Let $\Gamma \subset \text{PSL}(2, \mathbb{R})^k$ be a cocompact, irreducible lattice.

Then there is a constant $C_s := C_s(\Gamma) > 0$ such that for all $(x, T) \in M \times \mathbb{R}_{\geq 1}$, there are real-valued functions \{\(c_D(x, T)\)\}_{D \in B_+^s \cup B_-^s} on $\(M, \mathbb{R}^+\)$ and distributions $D^s_{x,T}, R^s_{x,T} \in W^{-s}(M)$ such that the following estimate holds.

For all $f \in W^s(M)$,
\[
\frac{1}{T} \int_0^T f \circ \phi_t(x) dt - \text{vol}(f) = \sum_{D \in B_+^s} c_D(x, T)D(f)T^{-s_D} + \sum_{D \in B_-^s} c_D(x, T)D(f)T^{-s_D} \log T + \text{const}
\]
where for all $(x, T) \in M \times \mathbb{R}_{\geq 1}$,
\[
\sum_{D \in B_+^s \cup B_-^s} |c_D(x, T)|^2 + \|D^s_{x,T}\|^2_{-s} + \|R^s_{x,T}\|^2_s \leq C_s.
\]

Additionally, we have the following lower bound. For every $D \in B_+$, there is a constant $C_s(D) > 0$ such that for all sufficiently large $T \geq 1$,
\[
\|c^s_D(\cdot, T)\|_{L^2(M)} \geq C_s(D).
\]

Remark 1.1. For $T \geq 1$ sufficiently large, the uniform estimate in Theorem 1.2 is sharp up to possibly a multiplication by $\log(T)$.

2. Cohomological equation

For any $s > 0$, let
\[
\text{Ann}^s(M) := \{ f \in W^s(M) : D(f) = 0 \text{ for all } D \in \mathcal{D}^s(M) \}.
\]

As a consequence of Theorem 1.2 of [3], we derive

Theorem 2.1. Let $0 \leq r < s - 1$. Then for all $f \in \text{Ann}^s(M)$, there exists a unique $g \in W^r(M)$ (up to additive constants) such that
\[
U_ig = f.
\]

Moreover, there is a constant $C_{r,s} := C_{r,s}(\Gamma) > 0$ such that
\[
\|g\|_r \leq C_{r,s}\|f\|_s.
\]

Proof. First say $r$ is an integer. By Theorem 1.2 of [3], for any $f \in \text{Ann}^s(M)$ there is a $g \in W^r(M)$ such that
\[
U_ig = f
\]
and
\[
\|(I + \Delta)g\|_{L^2(M)} \leq C_{r,s}\|(I + \Delta^s)f\|_{L^2(M)}.
\]
Now fix \( f \) and \( g \) as in the theorem. For \( \{c_n\}_{n=0}^{r} \subset \mathbb{Z}^+ \), write

\[
(I + \triangle)^r = \sum_{n=0}^{r} c_n \Delta^n.
\]

and observe that

\[
\Delta^n = \sum_{m=0}^{n} \Delta^m_0 \Delta^{n-m}_0.
\]

Because \( U_i \) commutes with \( \Delta_0 \), we have

\[
U_i \Delta_0 g = \Delta_0 f.
\]

Then for any \( \epsilon > 0 \), (4) gives

\[
\| (I - \Delta_i)^m \Delta_0^{n-m} g \|_{L^2(M)} \leq C\epsilon \| (I - \Delta_i)^{m+1+\epsilon} \Delta_0^{n-m} f \|_{L^2(M)}.
\]

By Lemma 6.3 of [8], for all \( \alpha, \beta \in \mathbb{Z}_{\geq 0} \), there is a constant \( C_{\alpha+\beta} > 0 \) such that

\[
(I - \Delta_i)^{\alpha} \Delta_0^{\beta} \leq C_{\alpha+\beta} (I + \triangle)^{\alpha+\beta}.
\]

By interpolation, the same holds for all \( \alpha, \beta \in \mathbb{R}_{\geq 0} \), see [7]. Hence,

\[
(I - \Delta_i)^{2(m+1+\epsilon)} \Delta_0^{2(n-m)} \leq C\epsilon (I + \triangle)^{2n+1+\epsilon}.
\]

Letting \( \epsilon < s - 2n - 1 \), and combining (5), (6) and (7), we have

\[
\langle (1 + \triangle)^{2r} g, g \rangle \leq C \langle (1 + \triangle)^{2s} f, f \rangle.
\]

This finishes the proof of Theorem 2.1 in the case when \( r \in \mathbb{Z}_{\geq 0} \). The general case for \( r \geq 0 \) follows by interpolation.

\[ \square \]

3. PROOF OF THEOREM 1.2

For all \( (x, T) \in M \times \mathbb{R}^+ \), write \( \gamma_{x,T} \) as

\[
\gamma_{x,T}(f) := \frac{1}{T} \int_0^T f \circ \phi^t(x) dt.
\]

We may project \( \gamma_{x,T} \) onto a basis of invariant distributions described in Theorem 1.1

Then

\[
\gamma_{x,T} = \left( \sum_{D \in B^+_{1/4} \cup B^+_{1/4}} D_{\gamma_{x,T}} \right) \oplus C_{\gamma_{x,T}} \oplus R_{\gamma_{x,T}}.
\]

It follows from Lemma 5.2 of [3] that for some constant \( C_s := C_s(\Gamma) > 0 \), the quantity

\[
\sum_{D \in B^+_{1/4} \cup B^+_{1/4}} \| D_{\gamma_{x,T}} \|_{-s}^2 + \| C_{\gamma_{x,T}} \|_{-s}^2 + \| R_{\gamma_{x,T}} \|_{-s}^2.
\]

satisfies

\[
C_s^{-2} \| \gamma_{x,T} \|_{-s}^2 \leq [9] \leq C_s^2 \| \gamma_{x,T} \|_{-s}^2.
\]
Now fix \((x, T)\) as in Theorem 1.2 and set
\[ \gamma := \gamma_{x,T} \]
We prove Theorem 1.2 by controlling each of the terms in (9).

We begin by estimating \(\| R_\gamma \|_{-s} \).

**Lemma 3.1.** Let \(s > \frac{3k}{2} + 1\). There is a constant \(C_s := C_s(\Gamma) > 0\) such that for all \(T \geq 1\),
\[ \| R_\gamma \|_{-s} \leq C_s \frac{T}{T} . \]

**Proof.** Let \(f \in \text{Ann}^s(M)\). Then by Theorem 2.1, for any \(\frac{3k}{2} < r < s - 1\), there is a constant \(C_{r,s} := C_{r,s}(\Gamma) > 0\) and a function \(g \in W^s(M)\) satisfying \(U_i g = f\) and
\[ \| g \|_r \leq C_{r,s} \| f \|_s . \]

Then as in Lemma 5.5 of [3],
\[ |R_\gamma(f)| = \frac{1}{T} \int_0^T f \circ \phi_t(x) dt | \]
\[ = \frac{1}{T} \int_0^T U_i g \circ \phi_t(x) dt | \]
\[ = \frac{|g \circ \phi_t(x) - g(x)|}{T} \]
\[ \leq \frac{C_r}{T} \| g \|_r \leq \frac{C_{r,s}}{T} \| f \|_s . \]

The dependence of \(C_{r,s}\) on \(r\) can be removed by taking \(r = s/2 + (3k - 2)/4\).

**Lemma 3.2.** Let \(s > \frac{3k}{2} + 1\), and let \(C(\gamma)\) be as in [3]. Then
\[ \| C(\gamma) \|_{-s} \leq C_s T^{-1/2} \log(T) . \]

**Proof.** The unipotent arc \(\{ \phi_t \}_{t=0}^T \) contracts under right multiplication by the map \(e^\alpha X_i\), for \(\alpha > 0\). Setting \(\alpha = \log T\), we get
\[ e^{-\log T X_i} \gamma = \gamma_{x e^\log T X_i,1} . \]

By (10) and the Sobolev embedding theorem, the projection of \(\gamma_{x e^\log T X_i,1}\) onto the component with continuous spectrum satisfies
\[ \| C(\gamma_{x e^\log T X_i,1}) \|_{-s} \leq C_s, \Gamma . \]

Lemma 5.1 of [3] gives a constant \(C_s > 0\) such that
\[ \| \exp(t X_i) \exp(C(\gamma_{x e^\log T X_i,1})) \|_{-s} \leq C_s (1 + |t|) e^{-t/2} . \]

Then the lemma would now be immediate except that the orthogonal splitting \(\mathcal{J}^s(M) \oplus \mathcal{J}^s(M) \uparrow\) is not preserved under the geodesic flow. However, by iteratively applying the geodesic map \(e^{X_i}\) to the arc \(\gamma_{x e^\log T X_i,1}\) and using Lemma 3.1 to control any additional contribution to the remainder distribution, Lemma 3.2 follows. See Section 5.3 of [3] for details.
Proof of Theorem 1.2. Upper bounds for the other invariant distributions in the splitting (8) follow in the same way as Lemma 3.2 by using Theorem 1.4 of [3] (instead of Theorem 5.1). Combining this with Lemma 3.1 and Lemma 3.2 gives the upper bound in the proof Theorem 1.2.

Lower bounds can be obtained by an argument in [3] involving the $L^2$ version of the Gottschalk-Hedlund Lemma. See Lemma 5.7, Lemma 5.8, Lemma 5.9 and Lemma 5.13 of this paper for details.

4. ACKNOWLEDGEMENTS

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