On a nonlinear elasticity problem with friction and Sobolev spaces with variable exponents

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Abstract

We consider a nonlinear elasticity problem in a bounded domain, its boundary is decomposed in three parts: lower, upper, and lateral. The displacement of the substance, which is the unknown of the problem, is assumed to satisfy the homogeneous Dirichlet boundary conditions on the upper part, and not homogeneous one on the lateral part, while on the lower part, friction conditions are considered. In addition, the problem is governed by a particular constitutive law of elasticity system with a strongly nonlinear strain tensor. The functional framework leads to using Sobolev spaces with variable exponents. The formulation of the problem leads to a variational inequality, for which we prove the existence and uniqueness of the solution of the associated variational problem.

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1 Introduction

The study of partial differential equation problems with variable exponents comes from the theory of nonlinear elasticity, elastic mechanics, fluid dynamics, electrorheological fluids, image processing, etc. (see [2, 15, 19]).

First, we introduce the notations needed in this article. Let Ω be a connected open bounded domain of \( \mathbb{R}^N \) (\( N = 3 \)) with Lipschitz boundary \( \Gamma \). To a given field of displacement \( u \), we associate a nonlinear deformation tensor \( E \) defined by

\[
E(\nabla u(x)) = \frac{1}{2} (\nabla u + \nabla u^T),
\]

whose components are

\[
E_{ij}(\nabla u(x)) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3} \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right), \quad 1 \leq i, j \leq 3.
\]
The corresponding nonlinear constraints tensor \( \sigma(u) = (\sigma_i^j(u(x)))_{1 \leq i,j \leq 3} \) is then given by

\[
\sigma^j_i(u(x)) = \sum_{k,h=1}^{3} a^i_{jkh}(x) E_{kh}(\nabla u(x)), \quad 1 \leq i,j \leq 3,
\]

(1.2)

which describes a nonlinear relation between the stress tensor \((\sigma_i^j)_{1 \leq i,j \leq 3}\) and the deformation tensor \((E_{ij})_{1 \leq i,j \leq 3}\). The coefficients of elasticity \(a^i_{jkh}\) (see [3]) satisfy the following symmetry properties:

\[
a^i_{jkh} = a^i_{jkh} = a^j_{ikh} \text{ for all } 1 \leq i,j,k,h \leq 3.
\]

(1.3)

The aim of this paper is to prove the existence and uniqueness of weak solution for the following nonlinear problem, encountered in the theory of nonlinear elasticity [3]: Let \(w\) be a bounded domain in \(\mathbb{R}^2\) situated in the plane of equation \(x_3 = 0\). We suppose that \(w\) represents the lower surface of the domain occupied by the substance. The upper surface \(\Gamma_1\) is defined by

\[
\{(x', x_3) \in \mathbb{R}^3, x_3 = h(x') \text{ and } x' \in w\},
\]

where \(h\) is a function defined and bounded on \(w\), that is to say, there exist \(h_\ast\) and \(h^\ast\) in \(\mathbb{R}\) such that

\[
0 < h_\ast \leq h(x') \leq h^\ast, \quad \forall \ (x',0) = (x_1, x_2, 0) \in w.
\]

We study the displacement of a substance in

\[
\Omega = \{(x', x_3) \in \mathbb{R}^3 : (x',0) \in w \text{ and } 0 < x_3 < h(x')\}
\]

the boundary \(\partial \Omega = \Gamma = \partial \Omega_w \cup \Gamma_1 \cup \Gamma_L\), where \(\Gamma_L\) is the lateral surface of \(\Omega\).

The outer normal vector unitary on \(\Gamma\) is denoted by \(n = (n_1, n_2, n_3)\). The outer normal vector unitary on \(w\) is the vector \((0, 0, -1)\).

Einstein’s convention, which consists of making the sum on the repeated indices, will be used unless otherwise stated.

We define the normal and the tangential components \(u_n\) and \(u_t = (u_{t1}, u_{t2}, u_{t3})\), of the displacement variable \(u\) by

\[
u_n = u.n = u_i n_i, \quad u_t = u_i - u_n n_i, \quad i \text{ and } j = 1, 2, 3.
\]

(1.4)

For normal and tangential components \(\sigma_n\) and \(\sigma_t = (\sigma_{t1}, \sigma_{t2}, \sigma_{t3})\) of the strain tensor, the definition is as follows:

\[
\sigma_n = (\sigma.n)n = \sigma_i n_i n_j, \quad \sigma_{ti} = \sigma_i n_j - \sigma_n n_i \quad i \text{ and } j = 1, 2, 3.
\]

(1.5)

In this section, we are interested in the following equation:

\[
-\frac{\partial}{\partial x_j} \sigma_{ij}(u(x)) = f_i(x), \quad i \text{ and } j = 1, 2, 3,
\]

(1.6)

where \(f = (f_1, f_2, f_3)\) represents a mass density of the external forces.
For boundary conditions, it is assumed that

\begin{align*}
  u &= 0 \quad \text{on } \Gamma_1, \quad (1.7) \\
  u &= g \quad \text{on } \Gamma_L, \quad (1.8) \\
  u.n &= 0 \quad \text{on } w. \quad (1.9)
\end{align*}

Condition (1.9) means that there is a tangential force exerted by the surface \( w \) on the substance. This tangential effort cannot exceed a certain threshold. The Tresca law assumes that this threshold is fixed and known:

\[ |\sigma_t| \leq K \quad \text{on } w, \quad (1.10) \]

where \( K \) is a given positive function called coefficient of friction and \( |\sigma_t| \) is the modulus of the tangential constraint defined on \( w \) by (1.5).

As long as the tangential constraint \( \sigma_t \) has not reached the threshold \( K \), the substance moves with a given displacement \( s \), which is the displacement of the lower surface \( w \) (adhesion). When the threshold is reached, the substance and the surface move tangentially relative to each other and there is proportional sliding. What can be summarized as follows [8]:

\[
\begin{align*}
  |\sigma_t| < K & \Rightarrow \quad u_t = s, \\
  |\sigma_t| = K & \Rightarrow \exists \lambda \geq 0 \quad \text{such that } u_t = s - \lambda \sigma_t,
\end{align*}
\]

where the positive real \( \lambda \) is unknown.

This problem models the behavior of a heterogeneous material with the above Tresca friction free boundary condition. The consideration of this general material is in no way restrictive. Indeed, we can apply this study to the most particular elastic materials, but this particular case makes it easy to describe the different stages of this work. The tensor of the constraints considered here is nonlinear and grouped, as special cases, some models used in Ciarlet [3], Lions [12], and Dautray and Lions [4]. Let us cite by way of example (see [3, 12]).

The complete problem \((P_0)\) is therefore to find the displacement field \( u \), satisfying the following equation and boundary conditions:

\[
\begin{align*}
  -\frac{1}{2} \sigma_{ij}(u(x)) &= f_i(x) \quad \text{in } \Omega, 1 \leq i \leq 3, \\
  \sigma_{ij}(u(x)) &= \sum_{k,h=1}^{3} a_{ijkl}(x) E_{kh}(\nabla u(x)) \quad \text{in } \Omega, 1 \leq i,j \leq 3, \\
  E_{ij}(\nabla u(x)) &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{m=1}^{3} \frac{\partial u_i}{\partial x_m} \frac{\partial u_j}{\partial x_m} \right) \quad \text{in } \Omega, 1 \leq i,j \leq 3, \\
  u &= 0 \quad \text{on } \Gamma_1, \\
  u &= g \quad \text{on } \Gamma_L, \\
  u.n &= 0 \\
  |\sigma_t| < K & \Rightarrow \quad u_t = s, \\
  |\sigma_t| = K & \Rightarrow \exists \lambda \geq 0 \quad \text{such that } u_t = s - \lambda \sigma_t.
\end{align*}
\]

\((P_0)\)

We consider the functional framework of the considered problem \((P_0)\) using Lebesgue and Sobolev spaces with variable exponents, see for example [6]. However, it is not necessary to use this notion of Lebesgue and Sobolev spaces with variable exponents to study
this problem. But we see it as a good generalization to the same study with Lebesgue and Sobolev spaces with fixed exponents.

Several authors studied the system of elasticity with laws of particular behavior and using various techniques in Sobolev spaces with constant exponents. For example, in [3] Ciarlet used the implicit function theorem to show the existence and uniqueness of a solution; in [4] Dautray and Lions studied the linear problem in a regular boundary domain; in [21] Zoubai and Merouani studied the existence and uniqueness of the solutions of the nonlinear elasticity system by topological degree; and in [13, 20] Zoubai and Merouani studied the existence and uniqueness of the solution of Dirichlet’s and Neumann’s problems in Sobolev spaces with variable exponents.

In Sect. 2, we recall some definitions and properties of Lebesgue and Sobolev spaces with variable exponents (see for example [5–7, 10, 11] for the proofs and more details). This notion of Sobolev spaces with variable exponents is also used in many works (see for example [1, 9, 14]).

The need to work with the concept of Sobolev spaces with variable exponents is motivated by the appearance of these spaces when modeling electrorheological and thermorheological fluids (see [16]) and in image restoration (see [2]).

In Sect. 3, using this notion of Sobolev spaces with variable exponents, we give the convenient functional framework for the considered problem \((P_0)\) to lead to variational problem 3.1. Then we prove in Theorem 3.2 the existence part by checking all hypotheses of Theorem 8.1 page 251 in [12]. And finally, in Sect. 4, we obtain also the uniqueness of the solution to variational problem \((3.1)\).

2 Generalized Lebesgue and Sobolev spaces

Let \(\Omega \subset \mathbb{R}^N\), let \(p(\cdot) : \Omega \longrightarrow [1, +\infty]\) be a measurable function, called the variable exponent. In the following, we adopt the following notations:

\[
C_+ = \{ p(\cdot) \in C(\overline{\Omega}), p(x) \geq 1 \text{ for all } x \in \overline{\Omega} \},
\]

and

\[
p^- = \text{ess inf}_{x \in \Omega} p(x), \quad p^+ = \text{ess sup}_{x \in \Omega} p(x).
\]

We define the generalized Lebesgue space \(L^{p(\cdot)}(\Omega)\), also called Lebesgue space with variable exponent, as the set of measurable functions \(u : \Omega \rightarrow \mathbb{R}\) for which the convex modulus

\[
\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx
\]

is finished.

For \(x \in \Omega\), \(p(x) > 1\), the function of \(\mathbb{R}_+ \rightarrow \mathbb{R}\) given by \(Y \mapsto Y^{p(x)}\) is convex, so also the function \(u \mapsto \rho_{p(x)}(u)\).

Moreover, for \(1 < p^+ < +\infty\), we put the function

\[
u \mapsto \| u \|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{p(x)} \left( \frac{u}{\lambda} \right) = \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}.
\]
Note that \( \|u\|_{L^p(\Omega)} = 0 \) implies that \( \lambda = 0 \), then we must have \( u = 0 \), so that this inf bound is finite. For every \( \alpha \in \mathbb{R} \) and \( u \in L^p(\Omega) \), using the convexity of \( \rho_p(\cdot) \), we have

\[
\|\alpha u\|_{L^p(\Omega)} = \inf \left\{ \lambda > 0 : \rho_p\left( \frac{\alpha u + (1 - \alpha)0}{\lambda} \right) \leq 1 \right\} = \inf \left\{ \lambda > 0 : |\alpha| \rho_p\left( \frac{u}{\lambda} \right) \leq 1 \right\} = |\alpha| \inf \left\{ \lambda > 0 : \rho_p\left( \frac{u}{\lambda} \right) \leq 1 \right\} = |\alpha| \|u\|_{L^p(\Omega)}.
\]

Also let \( u \) and \( v \) be in \( L^p(\Omega) \) such that

\[
\rho_p\left( \frac{u}{\|u\|_{L^p(\Omega)}} \right) \leq 1 \quad \text{and} \quad \rho_p\left( \frac{v}{\|v\|_{L^p(\Omega)}} \right) \leq 1,
\]

so

\[
\rho_p\left( \frac{u + v}{\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}} \right) \leq \frac{\|u\|_{L^p(\Omega)}}{\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}} \rho_p\left( \frac{u}{\|u\|_{L^p(\Omega)}} \right) + \frac{\|v\|_{L^p(\Omega)}}{\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}} \rho_p\left( \frac{v}{\|v\|_{L^p(\Omega)}} \right) \leq 1.
\]

So with \( \lambda = \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \) we obtain

\[
\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}
\]

therefore the given function (2.2) defines a norm of \( L^p(\Omega) \), called the norm of Luxembourg [5].

The space \((L^p(\Omega), \|\cdot\|_{L^p(\Omega)})\) is a Banach space and \( D(\Omega) \) is dense in \( L^p(\Omega) \). Moreover, if \( p > 1 \), \( L^p(\Omega) \) is [6] uniformly convex and therefore reflexive, and its dual is isomorphic to \( L^{p'}(\Omega) \) where \( \frac{1}{p} + \frac{1}{p'} = 1 \) for \( x \in \Omega \).

We also have the following inequality called Hölder type inequality:

\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p} + \frac{1}{p'} \right) \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)} \leq 2 \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}
\]

for all \( u \in L^p(\Omega) \) and all \( v \in L^{p'}(\Omega) \).

Now we define the generalized Sobolev space also called Sobolev space with variable exponent

\[
W^{1,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega); \nabla u \in (L^{p(\cdot)}(\Omega))^N \}
\]

which endowed with the norm

\[
u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}
\]

is a Banach space.

The space \( W^{1,p(\cdot)}_0(\Omega) \) denotes the completion of \( C_0^{\infty}(\Omega) \) in \( W^{1,p(\cdot)}(\Omega) \).
Let $p(\cdot) \in C_{s}(\overline{\Omega})$, $p^{-} \geq 1$, and $u \in W^{1,p(\cdot)}_{0}(\Omega)$, we have the inequality of Poincaré

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

where $C$ depends on $p(\cdot)$ and therefore on $\Omega$.

In particular, see for example in [6] Theorem 8.1.6 page 249, see also [5, 7, 10, 11] that if $p^{-} > 1$, the space $W^{1,p(\cdot)}_{0}(\Omega)$ is a separable, reflexive Banach space, and uniformly convex. Its dual space is denoted by $W^{-1,p(\cdot)}(\Omega)$.

In the writing of variational formulations, the convex modulus $\rho_{p(\cdot)}$ appears, which leads us to state the following results.

**Proposition 2.1** ([11]) If $u_{n}, u \in L^{p(\cdot)}(\Omega)$ and $p^{+} < +\infty$, the following relationships are true:

1. $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ (resp. $= 1$, $> 1$) $\iff \rho_{p(\cdot)}(u) < 1$ (resp. $= 1$, $> 1$),
2. $\|u\|_{L^{p(\cdot)}(\Omega)} \geq 1 \implies \|u\|_{L^{p^{+}(\cdot)}(\Omega)} \leq \rho_{p^{+}(\cdot)}(u) \leq \|u\|_{L^{p^{+}(\cdot)}(\Omega)},$
3. $\|u\|_{L^{p^{-}(\cdot)}(\Omega)} < 1 \implies \|u\|_{L^{p^{+}(\cdot)}(\Omega)} \leq \rho_{p^{+}(\cdot)}(u) \leq \|u\|_{L^{p^{+}(\cdot)}(\Omega)},$
4. $\lim_{n \to \infty} \rho_{p^{+}(\cdot)}(u_{n}) = 0 \iff \lim_{n \to \infty} \|u_{n}\|_{L^{p^{+}(\cdot)}(\Omega)} = 0.$

**Proposition 2.2** ([11]) If $q \in C_{s}(\overline{\Omega})$ and if for all $x \in \overline{\Omega}$, $q(x) < p^{+}(x)$, then the injection of $W^{1,p^{+}(\cdot)}(\Omega)$ into $L^{q^{\cdot}(\cdot)}(\Omega)$ is continuous and compact, where

$$p^{\ast}(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

In particular, the injection of $W^{1,p^{\ast}(\cdot)}_{0}(\Omega)$ into $L^{q^{\cdot}(\cdot)}(\Omega)$ is continuous and compact.

**Proposition 2.3** ([18]) We note

$$p^{\ast}(\cdot) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Let $q \in C_{s}(\partial\Omega)$. If for all $x \in \partial\Omega$, $q(x) < p^{\ast}(x)$, then the following injections of $W^{1,p^{\ast}(\cdot)}(\Omega) \hookrightarrow L^{p^{\ast}(\cdot)}(\partial\Omega) \hookrightarrow L^{q^{\cdot}(\cdot)}(\partial\Omega)$ are continuous and compact.

**Definition 2.1** The continuous function $p : \overline{\Omega} \to [1, +\infty)$ satisfies Hölder’s continuity condition if there is a constant $C$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|} \quad \forall x, y \in \overline{\Omega} \text{ with } |x - y| < \frac{1}{2}.$$

**Remark 2.1** Although this condition of regularity is not necessary to define Lebesgue and Sobolev spaces with variable exponents, it proves to be very useful for these spaces to introduce some properties, such that $C^{\infty}(\overline{\Omega})$ is dense in $W^{1,p^{\ast}(\cdot)}(\Omega)$ and $W^{1,p^{\ast}(\cdot)}_{0}(\Omega) = W^{1,1}(\Omega)$.

Moreover, if $1 < p^{-} \leq p^{+} < N$, then the Sobolev injection of $W^{1,p^{\ast}(\cdot)}(\Omega)$ into $L^{q^{\cdot}(\cdot)}(\Omega)$ remains true for $q(\cdot) = p^{\ast}(\cdot)$ (for more details, see [5]).
Remark 2.2 According to the Poincaré inequality, it is obvious that the norms
\[ u \mapsto \|\nabla u\|_{L^p(\Omega)} \quad \text{and} \quad u \mapsto \|u\|_{W_0^{1,p}(\Omega)} \]
are equivalent on \( W_0^{1,p}(\Omega) \).

Remark 2.3 Let \( a \geq 0, b \geq 0 \) and let \( 1 \leq p^- \leq p^+ < +\infty \), then for all \( x \in \Omega \)
\[ (a + b)^{p(x)} \leq 2^{p^+-1} (a^{p(x)} + b^{p(x)}). \]

3 Variational problem and existence result

We introduce now the following functional space:
\[ V(\Omega) = \{ \phi \in (W^{1,p} \Gamma_1 \cup \Gamma_2(\Omega))^3 \cap (W^{2,p}(\Omega))^3 : \phi \cdot n = 0 \text{ on } \Gamma_1, \Gamma_2 \} \]
with
\[ \| \cdot \|_{V(\Omega)} = \| \cdot \|_{(W^{1,p} \Gamma_1 \cup \Gamma_2(\Omega))^3}. \]

\[ G \in (W^{2,p}(\Omega))^3 \quad \text{such that} \quad G|_{\Gamma_1 \cup \Gamma_2} = g \quad \text{and} \quad G \cdot n = 0 \text{ on } w. \quad (3.1) \]

The variational formulation of problem \((P_0)\), see for example [8], leads to the following variational problem.

**Problem 3.1** Let \( f \in (L^p(\Omega))^3 \) and \( G \) satisfying \((3.1)\) be given.

Find \( u \) such that \( u - G \in V(\Omega) \) and satisfying the following variational inequality hold
\[
\sum_{i,j,k,h=1}^3 \int_{\Omega} a_{ijkh}(x) E_{ijk}(\nabla u) \frac{\partial}{\partial x_j} (\psi_i - (u_i - G_i)) \, dx \, dx_3 + J(\psi) - J(u - G) \\
\geq \int_{\Omega} f(\psi - (u - G)) \, dx' \, dx_3, \quad \forall \psi \in V(\Omega), \quad (3.2)
\]
where
\[ J(\psi) = \int_w K|\psi - s| \, dx'. \]

To prove the existence of a solution to Problem 3.1, let us assume the following assumptions:

\[ (H_1) \quad 3 < p(x) < +\infty \quad \text{for } x \in \Omega, \]
\[ (H_2) \quad \exists \alpha_0 > 0; \quad \exists \beta > 0 \quad \text{such that} \quad \alpha_0 \leq a_{ijkh}(x) \leq \beta \quad \text{a.e. in } \Omega, \quad \forall i,j,k,h = 1 \text{ to } 3 \]
\[ (H_3) \quad f = (f_1, f_2, f_3) \in (L^p(\Omega))^3, \quad p'(x) = \frac{p(x)}{p(x) - 1}. \]
Taking
\[ W_{1,1}^{p(\cdot)}(\Omega) = \left( W_{1,1}^{p(\cdot)}(\Omega) \right)^3 \cap \left( W^{2,p(\cdot)}(\Omega) \right)^3, \]
we need the three properties of the operator \( E_{kh} \) in the following theorem.

**Theorem 3.1** For \( u \) such that \( u \in G \in W_{1,1}^{p(\cdot)}(\Omega) \), with for \( x \in \Omega \) \( 3 < p(x) < +\infty \), the components \( E_{kh} \) of the deformation tensor of \( S_1 \) Vanent E satisfy the following properties:

1. (Continuity) \( E_{kh} \) is a continuous function, \( k, h = 1 \) to 3;
2. For all \( v \in W_{1,1}^{p(\cdot)}(\Omega) \), we have \( E_{kh}(\nabla u)^{\sum_{i,j} v_{ij}} \in L^1(\Omega), \forall i, j, k, h = 1 \) to 3;
3. (Coercivity) \( \exists \alpha > 0; \) such as \( E_{kh}(\xi) \xi \geq \alpha |\xi|^{p(\cdot)}, \forall k, h = 1 \) to 3.

**Proof** First 1. For \( x \in \Omega, p(x) > 3 \), and \( u, v \in W^{1,p(\cdot)}(\Omega) \), we have \( u v \in W^{1,p(\cdot)}(\Omega) \).

So, for \( v \in W_{1,1}^{p(\cdot)}(\Omega) \), we have
\[
\frac{\partial v_h}{\partial x_k} \frac{\partial v_k}{\partial x_h} \text{ and } \sum_{m=1}^3 \frac{\partial v_m}{\partial x_k} \frac{\partial v_m}{\partial x_h} \in W^{1,p(\cdot)}(\Omega),
\]
thus \( E_{kh}(\nabla v) \in W^{1,p(\cdot)}(\Omega) \). Moreover, for \( p(x) > 3 \), we have the continuous injection of \( W^{1,p(\cdot)}(\Omega) \) in \( C(\Omega) \), thus 1. holds.

Second 2. For \( x \in \Omega \),
\[
|E_{kh}(\nabla u)|^{p(x)} = \left( \frac{1}{2} \right)^{p(x)} \left| \frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h} + \sum_{m=1}^3 \frac{\partial u_m}{\partial x_k} \frac{\partial u_m}{\partial x_h} \right|^{p(x)},
\]
using Remark 2.3, we have
\[
|E_{kh}(\nabla u)|^{p(x)} \leq \left( \frac{1}{2} \right)^{p(x)} \times 2^{p^{(\cdot)}-1} \left( \left| \frac{\partial u_h}{\partial x_k} \right|^{p(x)} + \left| \frac{\partial u_k}{\partial x_h} \right|^{p(x)} + \sum_{m=1}^3 \left| \frac{\partial u_m}{\partial x_k} \frac{\partial u_m}{\partial x_h} \right|^{p(x)} \right).
\]
Using again Remark 2.3, we obtain
\[
\left| \frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h} \right|^{p(x)} \leq 2^{p^{(\cdot)}-1} \left( \left| \frac{\partial u_h}{\partial x_k} \right|^{p(x)} + \left| \frac{\partial u_k}{\partial x_h} \right|^{p(x)} \right),
\]
thus \( E_{kh}(\nabla u) \in L^{p(\cdot)}(\Omega) \) for \( h, k = 1 \) to 3. As \( p(x) > p'(x) \), as soon as \( p(x) > 3 \) and \( \Omega \) is bounded, we get \( E_{kh}(\nabla u) \in L^{p(\cdot)}(\Omega) \) for \( h, k = 1 \) to 3.

Thus, for \( v \in W^{p(\cdot)}(\Omega) \), we have \( \frac{\partial v_h}{\partial x_k} \in L^{p(\cdot)}(\Omega) \) for \( i, j = 1 \) to 3. Hence, by the Hölder inequality, we obtain that 2. holds. The third 3. property comes from \([17]\).

**Theorem 3.2** Let given \( f \in (L^{p(\cdot)}(\Omega))^3 \), \( G \) satisfying (3.1), and the hypotheses (\( H_1 \)) to (\( H_3 \)) hold. Then there exists a solution \( u \) to Problem 3.1.

**Proof** For the existence part, we apply Theorem 8.5 page 251 in \([12]\) and the first three properties of \( E_{kh} \) cited in Theorem 3.1. First, we rewrite variational inequality (3.2) in the form of this Theorem 8.5 page 251 in \([12]\).
As we see for example in [6] Theorem 8.1.6 page 249, \((W^{1,p}_1(\Omega)) \cap (W^{2,p}_1(\Omega))^3\) is a separable and reflexive Banach space, then its closed subspace \(V(\Omega)\) is also a separable and reflexive Banach space.

The application \(V(\Omega) \to \mathbb{R}\) defined by
\[
\phi \mapsto \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) E_{kh}(\nabla u) \frac{\partial \phi_i}{\partial x_j} \, dx' \, dx_3
\]
is linear and continuous, so it is an element of \(V'(\Omega)\). We note by \(T(u-G)\) this application, so we have
\[
\langle T(u-G), \phi \rangle_{V'(\Omega), V(\Omega)} = \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) E_{kh}(\nabla u) \frac{\partial \phi_i}{\partial x_j} \, dx' \, dx_3.
\]

Similarly, we have the application \(V(\Omega) \to \mathbb{R}\), which associates
\[
\phi \mapsto \int_{\Omega} f \, dx' \, dx_3
\]
is linear and continuous, so is an element \(V'(\Omega)\). We note by \(f\) this application, so we have
\[
\langle f, \phi \rangle_{V'(\Omega), V(\Omega)} = \int_{\Omega} f \, dx' \, dx_3,
\]
therefore, problem (3.2) becomes
\[
\langle T(u-G) - f, \phi - (u-G) \rangle_{V'(\Omega), V(\Omega)} + f(\phi) - f(u-G) \geq 0, \quad \forall \phi \in V(\Omega),
\]
with
\[
f(\phi) = \int_{w} K |\phi - s| \, dx'.
\]

We check now that the operator \(T\) is pseudo-monotonic.

a) Let \(u\) be bounded in \(V(\Omega)\), we have
\[
\| T(u) \|_{V'(\Omega)} = \sup_{\| \phi \|_{V(\Omega)} = 1} \| (T(u), \phi) \|
\leq \sup_{\| \phi \|_{V(\Omega)} = 1} \left| \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) E_{kh}(\nabla u) \frac{\partial \phi_i}{\partial x_j} \, dx' \, dx_3 \right|
\leq \sup_{\| \phi \|_{V(\Omega)} = 1} \left| \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) E_{kh}(\nabla u) \frac{\partial \phi_i}{\partial x_j} \, dx' \, dx_3 \right|.
\]

Let \(u \in V(\Omega)\), by Remark 2.3 we get \(E_{kh}(\nabla u) \in L^{p_k}(\Omega)\), and as \(p(x) > p'(x)\), as soon as \(p(x) > 3\) and \(\Omega\) bounded, we have
\[
E_{kh}(\nabla u) \in L^{p_k}(\Omega) \quad \forall 1 \leq k, h \leq 3.
\]
Using now hypothesis (H₂) and the H"older inequality, with (3.3) we obtain

\[ \| T(u) \|_{V'(\Omega)} \leq 2\beta \sup_{\| \psi \|_{V(\Omega)} = 1} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \| E_{kh}(\nabla u) \|_{L^{p'}(\Omega)} \| \frac{\partial \psi_j}{\partial x_j} \|_{L^{p}(\Omega)} \]

\[ \leq 6\beta \sup_{\| \psi \|_{V(\Omega)} = 1} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \| E_{kh}(\nabla u) \|_{L^{p'}(\Omega)} \| \psi_i \|_{W^{1,p}(\Omega)} \]

\[ \leq 18\beta \sup_{\| \psi \|_{V(\Omega)} = 1} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \| E_{kh}(\nabla u) \|_{L^{p'}(\Omega)} \| \psi \|_{W^{1,p}(\Omega)} \]

\[ \leq 18\beta \sum_{i,j=1}^{3} \| E_{kh}(\nabla u) \|_{L^{p'}(\Omega)} \]

From (3.3) and (2.1) we get

\[ \int_{\Omega} |E_{kh}(\nabla u)|^{\rho(x)} \, dx = \rho \int_{\Omega} |E_{kh}(\nabla u)|^{\rho(x)} \, dx < \infty, \quad \forall 1 \leq h, k \leq 3, \]

and from (ii)–(iii) of Proposition 2.1 we have

\[ \min \{ \| E_{kh}(\nabla u) \|_{L^{p'}(\Omega)}^{\rho \rho(x)}, \| E_{kh}(\nabla u) \|_{L^{p'}(\Omega)}^{\rho \rho(\omega)} \} \leq \rho \| E_{kh}(\nabla u) \|_{L^{p'}(\Omega)} < \infty. \]

For all \( h \) and all \( k \in \{1, 2, 3\} \), we get that \( \| E_{kh} \|_{L^{p'}(\Omega)} \) is bounded for all \( h \) and all \( k \in \{1, 2, 3\} \), consequently \( \| T(u) \|_{V'(\Omega)} \) is bounded.

**b)** Let \( u, v, w \in V(\Omega) \) and \( \lambda \in \mathbb{R} \), we check that the application of \( \mathbb{R} \) in \( \mathbb{R} \):

\[ \lambda \mapsto \langle T(u + \lambda v), w \rangle \]

is continuous. For this, let us consider \( \{\lambda_n\} \) to be a sequence of \( \mathbb{R} \) that converges to \( \lambda \). Let us denote

\[ \mathcal{F}_n(x) = \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u + \lambda_n \nabla v) \frac{\partial w_j}{\partial x_j} \]

and

\[ \mathcal{F}(x) = \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u + \lambda \nabla v) \frac{\partial w_j}{\partial x_j} \]

The \( E_{kh} \) being continuous, we therefore have, for all \( h \) and all \( k \in \{1, 2, 3\} \),

\[ \mathcal{F}_n(x) = \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u + \lambda_n \nabla v) \frac{\partial w_j}{\partial x_j} \]

converges to

\[ \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u + \lambda \nabla v) \frac{\partial w_j}{\partial x_j} \text{ a.e. in } \Omega, \]
and we have also with (H$_2$) and the definition of $E_{kh}$:

$$\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) E_{kh}(\nabla u + \lambda_n \nabla v) \frac{\partial w_i}{\partial x_j}$$

$$\leq \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \beta |E_{kh}(\nabla u + \lambda_n \nabla v)| \frac{\partial w_i}{\partial x_j}$$

$$\leq \frac{1}{2} \beta \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \left( |\frac{\partial u_h}{\partial x_k} + \lambda_n \frac{\partial v_h}{\partial x_k}| + |\frac{\partial u_k}{\partial x_h} + \lambda_n \frac{\partial v_k}{\partial x_h}| \right)$$

$$+ \sum_{m=1}^{3} \left( |\frac{\partial u_m}{\partial x_k} + \lambda_n \frac{\partial v_m}{\partial x_k}| \left( \frac{\partial u_m}{\partial x_h} + \lambda_n \frac{\partial v_m}{\partial x_h} \right) \right) \frac{\partial w_i}{\partial x_j}.$$ 

As the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}$, then $\exists m \in \mathbb{R} : |\lambda_n| \leq m$, so

$$|F_n(x)| \leq \frac{1}{2} \beta \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \left( |\frac{\partial u_h}{\partial x_k} + m \frac{\partial v_h}{\partial x_k}| + |\frac{\partial u_k}{\partial x_h} + m \frac{\partial v_k}{\partial x_h}| \right)$$

$$+ \sum_{m=1}^{3} \left( |\frac{\partial u_m}{\partial x_k} + m \frac{\partial v_m}{\partial x_k}| \left( \frac{\partial u_m}{\partial x_h} + m \frac{\partial v_m}{\partial x_h} \right) \right) \frac{\partial w_i}{\partial x_j}.$$ 

We define now the function $L$ by

$$L(x) = \frac{1}{2} \beta \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \left( |\frac{\partial u_h}{\partial x_k} + m \frac{\partial v_h}{\partial x_k}| + |\frac{\partial u_k}{\partial x_h} + m \frac{\partial v_k}{\partial x_h}| \right)$$

$$+ \sum_{m=1}^{3} \left( |\frac{\partial u_m}{\partial x_k} + m \frac{\partial v_m}{\partial x_k}| \left( \frac{\partial u_m}{\partial x_h} + m \frac{\partial v_m}{\partial x_h} \right) \right) \frac{\partial w_i}{\partial x_j}.$$ 

We obtain that $L \in L^P(\Omega)$. 

Thus from Lebesgue's dominated convergence theorem, we deduce that

$$\langle T(u + \lambda_n v), w \rangle \longrightarrow \langle T(u + \lambda v), w \rangle,$$

which shows that $T$ is hemi-continuous.

c) By hypothesis (H$_2$) and the monotony of $E_{kh}$, we have

$$\langle T(u) - T(v), u - v \rangle = \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) (E_{kh}(\nabla u)$$

$$- E_{kh}(\nabla v)) \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j} \right) dx^i \geq 0,$$

so $T$ is monotonous. As it is also bounded and hemi-continuous, then $T$ is pseudo-monotonic.

- The functional $f$ is proper, convex, and lower semi-continuous on $V(\Omega)$. Indeed, let $u$ and $v$ be two elements of $V(\Omega)$, and $\lambda \in [0, 1]$, we have
d) \( J \) is convex, indeed

\[
J(\lambda u + (1 - \lambda)v) = \int_\Omega K(\lambda u + (1 - \lambda)v - s) \, dx' = \int_\Omega K(\lambda(u - s) + (1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \int_\Omega K(\lambda(u - s)) \, dx' + \int_\Omega K((1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \lambda \int_\Omega K(u - s) \, dx' + (1 - \lambda) \int_\Omega |v - s| \, dx' = \lambda J(u) + (1 - \lambda)J(v).
\]

\[\square\]

\[
J(\lambda u + (1 - \lambda)v) = \int_\Omega K(\lambda u + (1 - \lambda)v - s) \, dx' = \int_\Omega K(\lambda(u - s) + (1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \int_\Omega K(\lambda(u - s)) \, dx' + \int_\Omega K((1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \lambda \int_\Omega K(u - s) \, dx' + (1 - \lambda) \int_\Omega |v - s| \, dx' = \lambda J(u) + (1 - \lambda)J(v).
\]

\[\square\]

\[
J(\lambda u + (1 - \lambda)v) = \int_\Omega K(\lambda u + (1 - \lambda)v - s) \, dx' = \int_\Omega K(\lambda(u - s) + (1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \int_\Omega K(\lambda(u - s)) \, dx' + \int_\Omega K((1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \lambda \int_\Omega K(u - s) \, dx' + (1 - \lambda) \int_\Omega |v - s| \, dx' = \lambda J(u) + (1 - \lambda)J(v).
\]

\[\square\]

\[
J(\lambda u + (1 - \lambda)v) = \int_\Omega K(\lambda u + (1 - \lambda)v - s) \, dx' = \int_\Omega K(\lambda(u - s) + (1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \int_\Omega K(\lambda(u - s)) \, dx' + \int_\Omega K((1 - \lambda)(v - s)) \, dx'
\]

\[
\leq \lambda \int_\Omega K(u - s) \, dx' + (1 - \lambda) \int_\Omega |v - s| \, dx' = \lambda J(u) + (1 - \lambda)J(v).
\]

\[\square\]
and consequently, $\forall i,j = 1 \text{ to } 3$,
\[
\{E_{ij}(u) - E_{ij}(v), u - v\} \geq \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), u - v
\]
\[
- \frac{1}{4} \left( \sum_{m=1}^{3} \left( \frac{\partial u_m}{\partial x_i} \right)^2 + \sum_{m=1}^{3} \left( \frac{\partial v_m}{\partial x_i} \right)^2 \right)
\]
\[
- \left( \sum_{m=1}^{3} \left( \frac{\partial u_m}{\partial x_i} \right)^2 + \sum_{m=1}^{3} \left( \frac{\partial v_m}{\partial x_i} \right)^2 \right), u - v \right) \geq \frac{1}{2} \left( \frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j} \right). (4.1)
\]
To conclude, we must prove that the second member of (4.1) is positive. For that, we separate the second member of (4.1) in linear and nonlinear parts.

Let the linear function $\Omega \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by
\[
(A_{ij} \circ f_x)(x) = A_{ij} \left( \frac{\partial u_i}{\partial x_j}(x), \frac{\partial u_j}{\partial x_i}(x) \right) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right), i,j = 1 \text{ to } 3,
\]
and the nonlinear function $\Omega \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by
\[
(B_{ij} \circ f_x)(x) = B_{ij} \left( \frac{\partial u_i}{\partial x_j}(x), \frac{\partial u_j}{\partial x_i}(x) \right) = -\frac{1}{4} \left( \sum_{i=1}^{3} \left( \frac{\partial u_i}{\partial x_i}(x) \right)^2 + \sum_{i=1}^{3} \left( \frac{\partial u_i}{\partial x_i}(x) \right)^2 \right).
\]

The functions $A_{ij}$ and $B_{ij}$ are continuous for $p(\cdot) > 3$. It remains to show that, $\forall i,j = 1 \text{ to } 3$, the $A_{ij}$ are increasing on $\mathbb{R}$, the $B_{ij}$ are increasing on $\mathbb{R}^-$, and the $A_{ij} + B_{ij}$ are increasing on $]-\infty, \frac{1}{3} [$.  

1. **Let us show that the $A_{ij}$ are increasing:** let the function
\[
\Omega \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R} \text{ be defined by } \left( \frac{\partial u_i}{\partial x_j} \circ f_x \right)(x) = \frac{\partial u_i}{\partial x_j}(x), i,j = 1 \text{ to } 3.
\]

We note
\[
\frac{\partial u_i}{\partial x_j}(x) = t_{ij} \text{ and } \frac{\partial u_j}{\partial x_i}(x) = \tau_{ij}
\]
and
\[
\frac{\partial u_i}{\partial x_j}(x) = t_{ij} \text{ and } \frac{\partial u_j}{\partial x_i}(x) = \tau_{ij}.
\]

The function $t \mapsto \frac{1}{t}$ of $\mathbb{R} \mapsto \mathbb{R}$, being increasing on $\mathbb{R}$, we have
\[
\frac{1}{2} \left( \frac{\partial u_j}{\partial x_j} - \frac{\partial v_j}{\partial x_j} \right)^2 + \frac{1}{2} \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j} \right) \left( \frac{\partial u_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i} \right) dx
\]
\[
= \frac{1}{2} \left( \frac{\partial u_j}{\partial x_j} - \frac{\partial v_j}{\partial x_j} \right)^2 \geq 0.
\]

Therefore, the $A_{ij}$ are increasing.
2. Let us show that the $B_{ij}$ are increasing: let the function

$$
\Omega \xrightarrow{f} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{B_{ij}} \mathbb{R}
$$

be defined by

$$(B_{ij} \circ f)(x) = B_{ij}(t_j, \tau_i) = -\frac{1}{4} \left( \sum_{j=1}^{3} \left( \frac{\partial u_j}{\partial x_j}(x) \right)^2 + \sum_{i=1}^{3} \left( \frac{\partial u_i}{\partial x_i}(x) \right)^2 \right), \quad i, j = 1 \text{ to } 3.$$

As in point 1., we note

$$t_j = \frac{\partial u_j}{\partial x_j}(x), \quad \tau_i = \frac{\partial u_i}{\partial x_i}(x), \quad \forall i, j = 1 \text{ to } 3,$$

so

$$B_{ij}(t_j, \tau_i) = -\frac{1}{4} \left( \sum_{j=1}^{3} t_j^2 + \sum_{i=1}^{3} \tau_i^2 \right) \geq -\frac{1}{4} \left( 6 \times \max_{1 \leq i,j \leq 3} (t_{ij}^2, \tau_{ji}^2) \right) = -\frac{3}{2} x^2.$$

For the function $f(x) = -\frac{3}{2} x^2$ being continuous and increasing on $\mathbb{R}^-$, we deduce that the $B_{ij}$ are increasing on $\mathbb{R}^-$.  

3. We show that the $A_{ij} + B_{ij}$ are increasing: the proofs of points 1. and 2. imply that the sum $A_{ij} + B_{ij}$, $\forall i, j = 1 \text{ to } 3$, corresponds to the sum of the two functions $f(x) + g(x) = x - \frac{3}{2} x^2$, $\mathbb{R} \to \mathbb{R}$, obviously continuous and increasing on $]-\infty, \frac{1}{3}]$, as the derivative of the convex function $h(x) = \frac{1}{2} x^2 - \frac{1}{3} x^3$ on $]-\infty, \frac{1}{3}]$. So, (4.1) is verified, and consequently

$$\langle E_{ij}(u) - E_{ij}(v), u - v \rangle \geq \langle (A_{ij} + B_{ij})(u) - (A_{ij} + B_{ij})(v), u - v \rangle \geq 0, \quad \forall i, j = 1 \text{ to } 3.$$

In other words, the $E_{ij}(u), i, j = 1 \text{ to } 3$, are monotonic $W^{1,1}(\Omega)$ in $(W^{1,1}(\Omega))'$, $i, j = 1$ to 3.

**Theorem 4.2** Variational inequality (3.2) has a unique solution satisfying Theorem 3.2.

**Proof** Suppose now that variational inequality (3.2) has two solutions $u_1, u_2$ satisfying Theorem 3.2, and $(E_{ib}(\xi) - E_{ib}(\eta))(\xi - \eta) = 0$ if and only if $\xi = \eta$. So we have

$$\sum_{i,j=1}^{3} \int_{\Omega} a_{ijb}(x) E_{ijb}(\nabla u_1) \frac{\partial}{\partial x_j} \left( \varphi_i - (u_{1i} - G_i) \right) \, dx' \, dx_3$$

$$\quad + \int_{\omega} K(\varphi - s - |(u_1 - G_1) - s|) \, dx'$$

$$\geq \int_{\Omega} f(\varphi - (u_1 - G)) \, dx' \, dx_3, \quad \forall \varphi \in V(\Omega) \quad (4.2)$$

2. Let us show that the $B_{ij}$ are increasing: let the function

$$\Omega \xrightarrow{f} \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{B_{ij}} \mathbb{R}$$

be defined by

$$(B_{ij} \circ f)(x) = B_{ij}(t_j, \tau_i) = -\frac{1}{4} \left( \sum_{j=1}^{3} \left( \frac{\partial u_j}{\partial x_j}(x) \right)^2 + \sum_{i=1}^{3} \left( \frac{\partial u_i}{\partial x_i}(x) \right)^2 \right), \quad i, j = 1 \text{ to } 3.$$
and
\[
\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) E_{kh}(\nabla u_2) \frac{\partial}{\partial x_j} \left( \phi_i - (u_{2i} - G_i) \right) \, dx_3 \, dx_j \\
+ \int_{\Omega} K \left( |\phi - s| - |(u_2 - G) - s| \right) \, dx' \\
\geq \int_{\Omega} f \left( \phi - (u_2 - G) \right) \, dx_3, \quad \forall \phi \in V(\Omega). \tag{4.3}
\]

We take \( \phi = u_2 - G \) in (4.2) and \( \phi = u_1 - G \) in (4.3), so
\[
\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) \left( E_{kh}(\nabla u_1) - E_{kh}(\nabla u_2) \right) \frac{\partial}{\partial x_j} \left( (u_{1i} - G_i) - (u_{2i} - G_i) \right) \leq 0,
\]
but according to Theorem 4.1 on the monotony of \( E_{kh} \), we have
\[
\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) \left( E_{kh}(\nabla u_1) - E_{kh}(\nabla u_2) \right) \frac{\partial}{\partial x_j} (u_{1i} - u_{2i}) \geq 0,
\]
so
\[
\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \int_{\Omega} a_{ijkh}(x) \left( E_{kh}(\nabla u_1) - E_{kh}(\nabla u_2) \right) \frac{\partial}{\partial x_j} (u_{1i} - u_{2i}) = 0.
\]
Thus \( \nabla u_1 = \nabla u_2 \) in \( (L^p(\Omega))^3 \), and by the inequality of Poincaré, we get \( u_1 = u_2 \) in \( (L^p(\Omega))^3 \), and so \( u_1 = u_2 \) dans \( (W^{1,p}(\Omega))^3 \). \( \square \)

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No abbreviations used.

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Declarations

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The authors declare that they have no competing interests.

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References

1. Azroul, E., Benboubker, M.B., Rhoudaf, M.: Entropy solution for some $p(x)$-quasilinear problems with right-hand side measure. Afr. Diaspora J. Math. 13(2), 23–44 (2012)
2. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. 66, 1383–1406 (2006)
3. Ciarlet, P.G.: Mathematical Elasticity, Vol. I: Three-Dimensional Elasticity. North-Holland, Amsterdam (1988)
4. Dautray, R., Lions, J.L.: Analyse mathématique et calcul numérique pour les sciences et les techniques, vol. 1. Masson, Paris (1984)
5. Diening, L.: Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{k,p(x)}$. Math. Nachr. 268, 31–43 (2004)
6. Diening, L., Harjulehto, P., Hästö, P., Ruzicka, M.: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2017. Springer, Heidelberg (2011)
7. Diening, L., Hästö, P., Nekvinda, A.: Open problems in variable exponent Lebesgue and Sobolev spaces. In: FSDONA04 Proceedings, Milovy, Czech Republic, pp. 38–58 (2004)
8. Duvaut, G., Lions, J.L.: Les inéquations en mécanique et physique. Dunod, Gauthiers-Villars, Paris (1972)
9. El Hachimi, A., Maatouk, S.: Existence of periodic solutions for some quasilinear parabolic problems with variable exponents. Arab. J. Math. 6, 263–280 (2017). https://doi.org/10.1007/s40065-017-0178-0
10. Fan, X., Shen, J., Zhao, D.: Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. 262, 424–446 (2001)
11. Lions, J.L.: Quelques méthode de résolution des problèmes aux limites non linéaires. Dunod, Paris (1969)
12. Mihăilescu, M., Rădulescu, V.: On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. Proc. Am. Math. Soc. 135(9), 2929–2937 (2007)
13. Rădulescu, M.: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Math., vol. 1784. Springer, Berlin (2000)
14. Ruzicka, M.: Electrorheological Fluids, Modelling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2002)
15. Tinsley Oden, J.: Existence theorems for a class of problems in nonlinear elasticity. J. Math. Anal. Appl. 69, 51–83 (1979)
16. Yao, J.: Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators. Nonlinear Anal. 68, 1271–1283 (2008)
17. Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR, Izv. 29, 33–36 (1987)
18. Zoubai, F., Merouani, B.: On a pure traction problem for the nonlinear elasticity system in Sobolev spaces with variable exponents. Stud. Univ. Babeş-Bolyai Math. (2019) (accepted)
19. Zoubai, F., Merouani, B.: Study of a mixed problem for a nonlinear elasticity system by topological degree. Stud. Univ. Babeş-Bolyai, Math. 66(3), 537–551 (2021). https://doi.org/10.24193/subbmath.2021.3.10