NUCLEAR MATTER ASPECTS OF SKYRMIONS

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ABSTRACT

As an alternative approach to the infinite-array description of dense matter in the Skyrme model, we report about the properties of a single skyrmion on a compact 3-sphere of finite radius. The density of this matter can be increased by decreasing the hypersphere radius. As in the array calculations one encounters a transition to a distinct high density phase characterized by a delocalization in energy and baryon charge and by increased symmetries. We will argue that the high density phase has to be interpreted as chirally restored one. The arguments are based on the formation of complete chiral multiplets and the vanishing of the pionic massless Goldstone modes in the high-density fluctuation spectrum. We show that the restoration of chiral symmetry is common to any chirally invariant extension of the usual Skyrme model - whether via higher-order contact terms or via the introduction of stabilizing vector mesons which act over a finite range.

1. Introduction

In this lecture we will present some studies about the high-density behavior of baryonic configurations in the Skyrme model and its variants. These models belong to a class of effective models which treat the baryon stabilization (and hence the baryon structure) and the interaction between baryons on the same footing. There is no principal difference between the stabilization and the interaction mechanism, the models just act in different topological sectors: in the baryon (=winding) number $B = 1$ sector for the structure physics and in the $B > 1$ sector for the interaction physics. For $B \rightarrow \infty$ one has a new parameter (the density) at ones disposal in order to tune the interplay between the stabilization and the interaction mechanisms. Because of these features the Skyrme model and its variants have the inherent possibility to allow for radical changes when baryonic matter is compressed to high densities: the baryons which at low densities are well-separated and clearly defined objects might completely loose their identity and the baryon matter can become uniform. There are at least two distinct ways for investigating the high-density behavior of Skyrme-type models.

The first approach which was pioneered by Igor Klebanov uses infinite periodic arrays of skyrmions with baryon number (winding number) of one per unit cell in real space-time. The details of the lattice structure and the periodic boundary conditions to which the mesonic fields are subject can be chosen as to avoid nearest neighbor frustration at low densities. The aim is to find the classical static field configura-

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tion which minimizes the energy per cell volume for a fixed baryon charge per array cell. The equations of motion (which are non-linear partial differential equations) are solved numerically on a lattice grid subject just to two constraints: the choice of the crystal and the form of the twisted periodic boundary conditions which guarantee the periodic structure and minimize the frustrations in the field gradients (and thus the energy) between neighboring cells in the asymptotic low density region. The boundary conditions are then extrapolated without alteration to high densities as well. At low densities the skyrmions in the periodic arrays are well-separated, ensembled in a phase of weakly interacting baryons. With increased density, however, they grow in size as measured by their r.m.s. radius until at a critical density - as first observed by Wüst, Brown and Jackson 7 - they “melt” to a distinct high-density phase where the skyrmions completely lose their identity. The new phase is characterized by the approximate uniformity in the baryon as well as the energy density, by the fact that the averages of the $\sigma$- and $\pi$-fields vanish over the cell volume and that there appears an additional symmetry at the critical density: the “half-skyrmion symmetry” of Goldhaber and Manton.8

The short-comings of these calculations are: they are by fiat of numerical nature, the rotational and translational symmetries are explicitly broken down to discrete ones by the artificial crystalline grid structure. Finally the calculations depend on the form of the crystal and the twisted periodic boundary conditions which are extracted in the asymptotic low density regime, but nevertheless applied at high densities. For instance, the order of the phase transition can depend on these choices: For the same simple cubic lattice it is first order for Klebanov’s choice of boundary terms, while it is second order for the modified rectangular boundary terms of Jackson and Verbaarschot 9 where a preferred direction in space is singled out. But the fact that there are phase transitions are common for all these calculations - and even the critical densities are approximately the same. Klebanov’s crystal is not of minimal energy, a face-centered cubic arrangement of skyrmions (at low densities) is preferable. Again, as the density increases there is a second-order phase transition and the half-skyrmion symmetry emerges.10 The crystal of minimal energy is in this higher density phase and has energy per baryon only of 3.8% above the topological lower bound. Further generalizations are the inclusion of temperature and the derivation of the equation of state for the Skyrme matter in the case of the simple cubic lattice 11 as well as the fcc structure 12 by T.S. Walhout. He has also studied periodic arrays using an $\omega$-stabilized variant of the Skyrme model.13

Nevertheless, the question might arise whether these phase transitions are lattice artifacts or not. Fortunately, there is an alternative approach to study the same phenomena by replacing the periodic arrays in flat space $\mathbb{R}^3$ by few-skyrmion-systems (or even a single skyrmion) on the compact manifold $S^3(L)$ pioneered by Nick Manton.14 15 The finite baryon number on $S^3(L)$ corresponds to a finite baryon density in $\mathbb{R}^3$(with infinite baryon number), so that one obtains a model for skyrmion

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This symmetry is characterized by families of planes on which the scalar field, $\sigma$ is zero. The pionic fields have a reflection symmetry about these planes while the scalar field is reflection antisymmetric.
matter which is appreciably simpler to study than any lattice model. The density of this matter can be increased by decreasing the hypersphere radius $L$, the radius of the 3-sphere in four dimensions. In the limit $L \to \infty$ a skyrmion localized on the hypersphere has basically the same properties as an isolated skyrmion in flat space or in a periodic array for infinite separation. For a large hypersphere it is still well localized and dominated by its stabilization mechanism, but because of the curvature and finite size effects of the hypersphere it acts as if the tail of another skyrmion (here of course nothing else than itself seen via the opposite pole of the 3-sphere) is present. By shrinking the hypersphere radius this interaction mechanism will become stronger and stronger. Finally it can become so strong that the stabilizing mechanism cannot localize the skyrmion any longer: the skyrmion will be smeared (=delocalized) over the whole 3-sphere. The connection between both approaches can be made by identifying the averaged baryon densities: in the periodic array calculation the baryon density is averaged over the cell volume, on the hypersphere it is given by the ratio $B/(2\pi^2 L_{\text{dim}}^3)$ of the baryon charge $B$ and the (surface-) volume of the 3-sphere in terms of the dimensioned radius $L_{\text{dim}}$. The $S^3(L)$ approach has the advantage that it is technically far simpler than the period array one and that it allows for mathematically rigorous results for the properties of the “high-density” (or “small $L$”) phase of a single skyrmion on $S^3(L)$. Furthermore neither the (continuous) rotational nor the translational symmetry are broken. The disadvantage is obvious: The physical space is not a hypersphere - at least not a small one. Both approaches supplement each other in a complementary way: the first is realized in normal space-time and for infinite systems (a precondition for a phase transition), but is technically complicated and plagued by its crystalline nature, by the ambiguities in choosing the array itself and the boundary terms; the other is simple and exact, but is set up in an unphysical world and limited to finite systems which therefore should not be identified with an infinite periodic array, but just with one array cell.

The talk is organized as follows. In sect. 2 we will give Manton’s general description of the Skyrme model on the 3-sphere and review some of the $B = 1$ properties. In sect. 3 we present arguments why the phase transition has to be interpreted as chiral restoration. In sect. 4 we report on the generalization to variants of the Skyrme-model which involve higher-order contact terms or which are stabilized by vector mesons. Sect. 5 contains a discussion about the order of the phase transition. We end the talk with a short discussion section.

2. The Skyrme Model on the Hypersphere

As mentioned there is considerable interest in studying the Skyrme model

\[ L_{2,4} = \frac{f_\pi^2}{4} \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right) + \frac{\epsilon_4^2}{4} \text{Tr} \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2, \]

(1)

on a 3-dimensional hypersphere of radius $L$, $S^3(L)$.\footnote{In fact most of the calculations can be done analytically or involve at most the numerical task of solving ordinary non-linear differential equations.} We will report here espe-
cially on the findings of refs. In order to get parameterization-independent results it is useful to relate the quaternion representation $U$ of the Skyrme model to a cartesian representation $\{\Phi^a\} = (\Phi^0, \Phi^x, \Phi^x, \Phi^y)$ with the imposed constraint $\Phi^a\Phi^a = 1$ as $U = \Phi^0 + i\vec{r} \cdot \vec{\Phi}$ and to introduce the strain tensor \[ K_{ij} \equiv \partial_i \Phi^a \partial_j \Phi^a \] (with $\{\Phi^a\} = (\sigma, \pi^z, \pi^x, \pi^y)$) \[ = -\frac{1}{2} \text{Tr} \{ U^\dagger \partial_i U, U^\dagger \partial_j U \} \] where $i$ and $j$ label the space coordinates. This is a symmetric $3 \times 3$ matrix with three positive eigenvalues which we will denote as $\lambda_a^2, \lambda_b^2, \lambda_c^2$. Manton has shown that the $\lambda_i$ have a simple geometrical interpretation. They correspond to the length changes of the images of any orthonormal system in a given space manifold (here $\mathbb{R}^3$ or $\mathbb{S}^3(L)$) under the conformal map, $U$, onto the group manifold, here $SU(2) \cong \mathbb{S}_3$. Therefore the name strain tensor which refers to such a general “rubber-sheet” geometry. Among the invariants of $K_{ij}$, three are fundamental and have a simple geometric meaning:

\[
\begin{align*}
\text{Tr}(K) &= \lambda_a^2 + \lambda_b^2 + \lambda_c^2 = \sum \text{length}^2 \\
\frac{1}{2} \left\{ \left( \text{Tr}(K) \right)^2 - \text{Tr}(K^2) \right\} &= \lambda_a^2 \lambda_b^2 + \lambda_b^2 \lambda_c^2 + \lambda_c^2 \lambda_a^2 = \sum \text{area}^2 \\
\text{det}(K) &= \lambda_a^2 \lambda_b^2 \lambda_c^2 = \text{volume}^2.
\end{align*}
\]

The first one measures the sum of the squared length changes of the mapped orthonormal frame, the second one the sum of the squared area changes and the third the squared volume change. With the help of Eq.(2) it is easy to see that these invariants are (modulo normalization factors) the static energy densities of the second-order non-linear $\sigma$ model, $\frac{f_0^2}{2} \text{Tr}(\partial_i U \partial_i U^\dagger)$, the fourth-order Skyrme term, $-(\epsilon_2^2/4) \text{Tr}[U^\dagger \partial_i U, U^\dagger \partial_j U]$, and the sixth-order term proportional to the square of the baryon density (see Eq.(24) in section 4). The static energy of a general skyrmion configuration on the hypersphere has in this language the form \[ E = \int_{\mathbb{S}^3(L)} dV \left( \lambda_a^2 + \lambda_b^2 + \lambda_c^2 + \lambda_b^2 \lambda_c^2 + \lambda_c^2 \lambda_a^2 + \lambda_a^2 \lambda_b^2 \right) \]

\[ = \int_{\mathbb{S}^3(L)} dV \left( \lambda_a - \lambda_b \lambda_c \right)^2 + (\text{cycl. perm.}) + 6 \int_{\mathbb{S}^3(L)} dV \lambda_a \lambda_b \lambda_c. \]

The last term is just $12\pi^2 B$ with the baryon (winding) number $B$, since $\lambda_a \lambda_b \lambda_c/(12\pi^2)$ is the Jacobian of the map $\mathbb{S}^3(L) \rightarrow \mathbb{S}^3(1) \cong SU(2)$ - in other words the baryon (winding) number density. Because of the positive definiteness of the terms in the last expression of (4) it is obvious that any skyrmion configuration has to respect the topological bound

\[ E \geq 12\pi^2 B. \]

\[ \text{All other invariants can be constructed from them.} \]

\[ \text{We have adopted a dimensionless form of the Skyrme lagrangian. To obtain the dimensioned quantities, one divides the } \lambda_i \text{'s and multiplies } L \text{ by } 2\sqrt{f_4}/f_\pi \text{ and multiplies } E \text{ by } \sqrt{f_4} f_\pi. \]
Furthermore, one can immediately see that there is exactly one possibility to satisfy the topological bound: The insertion of the identity map, \( \lambda_a = \lambda_b = \lambda_c \), with \( \lambda_i = 1 \) which corresponds to an isometric mapping from the spatial manifold into the target manifold and which has baryon (winding) number \( B = 1 \). Specifying to the case at hand where the target manifold is \( SU(2) \cong S^3(1) \), the unit 3-sphere, we see that the only way for saturating the topological bound (the absolute minimum of any skyrmion configuration with a non-zero winding number) is the isometric mapping of the spatial \( S^3(L) \) with \( L = 1 \) onto the target iso-spin sphere \( S^3(1) \). We can therefore conclude that the topological bound can never be saturated by any \( B > 1 \) system or by any skyrmion configuration in flat space, \( \mathbb{R}^3 \). After minimizing \( E[\lambda_a, \lambda_b, \lambda_c] \) under the constraint

\[
\int_{S^3(L)} dV \lambda_a \lambda_b \lambda_c = 2\pi^2 B \quad (6)
\]

Manton \cite{Manton} and Loss \cite{Loss} found that for \( L < 1 \) there is even a stronger bound on any \( B = 1 \) skyrmion configuration,

\[
E_{\text{identity}} = 6\pi^2 \left( L + \frac{1}{L} \right), \quad (7)
\]

which is still satisfied by the identity map \( \lambda_a = \lambda_b = \lambda_c = 1/L \) independently of the parameterization of the \( B = 1 \) configuration. In other words for \( L \leq 1 \) the identity map is the absolute minimum of any \( B = 1 \) configuration.

Finally, it is now simple to show in full generality \cite{Manton} \cite{Loss} that for \( L < \sqrt{2} \) the identity map is stable for all allowed \( (\delta B = 0) \) small-amplitude perturbations \( \lambda_i \rightarrow (1/L) + \delta_i(x) \), whereas it becomes a saddle for \( L > \sqrt{2} \). Under the small perturbation given above the energy is given to second-order as

\[
E[\lambda_a, \lambda_b, \lambda_c] = E_{\text{identity}} + \left( \frac{2}{L} + \frac{4}{L^3} \right) \int I_1 + \left( 1 + \frac{2}{L^2} \right) \int I_2 + \frac{4}{L^2} \int I_3 + O(\delta^3) \quad (8)
\]

with \( I_1 = \delta_a + \delta_b + \delta_c \), \( I_2 = \delta_a^2 + \delta_b^2 + \delta_c^2 \) and \( I_3 = \delta_a \delta_b + \delta_b \delta_c + \delta_c \delta_a \). Using the \( B \)-constraint \cite{Manton} we have up to third order corrections the relation

\[
\frac{1}{L^2} \int_{S^3(L)} dV I_1 = -\frac{1}{L} \int_{S^3(L)} dV I_3 + O(\delta^3).
\]

After inserting this relation in Eq. (8) and using the fact that the \( \delta_i \)'s are local variations, one can finally find that the identity map is stable against small-amplitude fluctuations, i.e. \( E[\lambda_a, \lambda_b, \lambda_c] > E_{\text{identity}} \), as long as the following local inequality holds

\[
\left( 1 + \frac{2}{L^2} \right) (\delta_a^2 + \delta_b^2 + \delta_c^2) - 2(\delta_a \delta_b + \delta_b \delta_c + \delta_c \delta_a) > 0, \quad (9)
\]

Of course, we cannot exclude the possibility that such a configuration might come infinitesimally close to the topological bound.

That means that the identity map is at least a local minimum for \( 1 < L < \sqrt{2} \).
in other words as long as $L < \sqrt{2}$. This concludes Manton’s general proof - independent of any parameterization - that the identity map is at least a local minimum up to $L_c = \sqrt{2}$ and becomes a saddle for $L > \sqrt{2}$.

So far we have not specified any parameterization. In the following we will specialize to the familiar hedgehog parameterization. It has the following properties: (a) according to a general theorem by Palais \cite{Palais} about reduced variational equations for symmetric fields a variational solution of hedgehog form is a solution of the full set of Euler-Lagrange equations of the Skyrme model, (b) the hedgehog solution is the energetically lowest solution known so far in the $B = 1$ sector in $\mathbb{R}^3$ and (c) it is numerically proven that the $B = 1$ hedgehog solution is at least a local minimum. Hedgehog configurations on the hypersphere are conveniently described in terms of the conventional ‘polar’ coordinates $\mu$, $\theta$, $\phi$ on $S^3(L)$ with $0 \leq \mu, \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, so that a typical point on $S^3(L)$ has Cartesian coordinates $(L \cos \mu, L \sin \mu \cos \theta, L \sin \mu \sin \theta \cos \phi, L \sin \mu \sin \theta \sin \phi)$.

In these coordinates the metric is $ds^2 = L^2(d\mu^2 + \sin^2 \mu d\theta^2 + \sin^2 \mu \sin^2 \theta d\phi^2)$ and the volume element is $dV = L^3 \sin^2 \mu d\mu \sin \theta d\theta d\phi$.

The field components of a hedgehog of baryon number $B$ on $S^3(L)$ have then the form

$$
\begin{align*}
\Phi^0 &= \cos f(\mu) \\
\Phi^z &= \sin f(\mu) \cos \theta \\
\Phi^x &= \sin f(\mu) \sin \theta \cos \phi \\
\Phi^y &= \sin f(\mu) \sin \theta \sin \phi,
\end{align*}
$$

where the “radial” profile function $f(\mu)$ is subject to the boundary condition $f(0) = 0$ and $f(\pi) = B\pi$, $B$ integer. The topological winding-number $B$ is of course the baryon charge of the corresponding field configuration. The field components, $f_\pi \Phi^\alpha$ ($f_\pi$ is the pion-decay constant) should be identified with the familiar $\sigma, \pi^z, \pi^x, \pi^y$ fields.

In the normal quaternion representation the hedgehog ansatz reads

$$U \equiv \Phi^0 + i \hat{r} \cdot \Phi = \exp(i \hat{r} \cdot \hat{r}(\theta, \phi) f(\mu))$$

where $\hat{r}(\theta, \phi)$ is the usual radial $S^2$ unit vector. Note that for the hedgehog configuration the eigenvalues of the strain tensor are simply $\lambda^2_a = f^2/L^2$, $\lambda^2_b = \lambda^2_c = \sin^2 f/(L^2 \sin^2 \mu)$ where $f'$ stands for $df/d\mu$.

The energy of the hedgehog field configuration \((13)\) on $S^3(L)$ is \(E\)

$$
E = 4\pi L \int_0^\pi \sin^2 \mu d\mu \left( f'^2 + 2 \frac{\sin^2 f}{\sin^2 \mu} \right) \\
+ 4\pi \frac{1}{L} \int_0^\pi \sin^2 \mu d\mu \left( \frac{\sin^2 f}{\sin^2 \mu} \left[ 2 f'^2 + \frac{\sin^2 f}{\sin^2 \mu} \right] \right).
$$

\(\text{Note in order to obtain the dimensioned quantities, one multiplies } L \text{ in } (14) \text{ by } 2\sqrt{2}e_4/f_\pi \text{ and } E \text{ by } \sqrt{2}e_4f_\pi.\)
Variation of (14) with respect to the radial profile function \( f(\mu) \) leads us to the Euler equation

\[
\frac{d^2 f}{d\mu^2} + \frac{\sin 2\mu}{\sin^2 \mu} \frac{df}{d\mu} - \frac{\sin 2f}{\sin^2 \mu} + \frac{2}{L^2} \frac{\sin^2 f}{\sin^2 \mu} \left( \left( \frac{df}{d\mu} \right)^2 - \frac{\sin^2 f}{\sin^2 \mu} \right) = 0.
\]

(15)

Again because of Palais’ general theorem \(^{18}\), solutions of (15) yield solutions of the full set of Euler-Lagrange equations of the Skyrme model on \( S^3(L) \) when substituted into (12) or (13). It is very simple to check that one solution of Eq. (15) (in the case \( B = 1 \)) is the map \( f(\mu) = \mu \) which corresponds to the uniform mapping of \( S^3(L) \) onto the isospin manifold \( S^3(1) \approx SU(2) \). One can easily see that for the uniform hedgehog map, \( f(\mu) = \mu \), the eigenvalues of the strain tensor (2) are simply \( \lambda^2_a = \lambda^2_b = \lambda^2_c = 1/L^2 \). In other words it is a special parameterization of the identity map. The uniform map, \( f(\mu) = \mu \), is a solution for all \( L \), but has no finite \( L \to \infty \) limit. The energy associated with the uniform map is of course given by (7). As mentioned for \( 1 < L < \sqrt{2} \), the uniform map is still at least a local minimum. For \( L > \sqrt{2} \) there exist a lower energy configuration which represents a skyrmion localized about one point, \( i.e. f(\mu) \neq \mu \) in the hedgehog parameterization \(^{12} \) \( ^{13} \) \( ^{17} \). In fact in this parameterization the identity map bifurcates into two solutions of the same energy concentrated around the north or south pole of the hypersphere which are related by \(^{17} \)

\[
f_N(\mu) = \pi - f_S(\pi - \mu).
\]

(16)

In the limit \( L \to \infty \), one recovers the usual flat space skyrmion.\(^{17} \) The identity map has O(4) symmetry \((\text{see Eq. (13)})\) and is therefore completely uniform in energy and baryon density. For \( L > \sqrt{2} \) the symmetry of the energetically most favorable solution is broken to O(3) indicating that the solution is localized in energy and baryon distribution. The bifurcation at \( L = \sqrt{2} \) is of the standard “pitchfork” type corresponding to a second order phase transition. In Landau’s scheme the bifurcation is characterized by a symmetric quartic polynomial whose quadratic term changes sign.

In order to discriminate the delocalized (high-density) phase from the localized (low-density) phase in a way which can be generalized to few-skyrmion systems on the hypersphere or to flat space array-calculations, several candidates for an order parameter were studied in ref. \(^{17} \). One candidate is the integrated squared deviation of the local energy/baryon density from the averaged density normalized by the squared averaged density times the hypersphere volume. It converges to the value 1 for the localized solution in the limit \( L \to \infty \), for smaller \( L \) it decreases and shows a square-root bifurcation at \( L = \sqrt{2} \) to the value zero belonging to the identity map. The interpretation would be that this order parameter signalled deconfinement in the energy and baryon distribution. This is, however, a special property of the \( B = 1 \) system on \( S^3(L) \). For few-skyrmion systems \(^{17} \) \(^{19} \) and flat space arrays \(^{4} \) \(^{10} \) this
parameter never becomes zero in the high-density phase, although there is still a sizable decrease in its value from the low-density to the high-density phase. The second suggestion of ref. [17] on the other hand works also in these cases. It is the squared chiral expectation value

$$\langle \sigma/f_π \rangle^2 + \langle \vec{\pi}/f_π \rangle^2,$$  \hspace{1cm} (17)

where the $\sigma$ and pion fields are averaged over the hypersphere volume (or the cell volume for periodic arrays). Because of the residual $O(3)$ symmetry for the hedgehog (or discrete symmetries in the array calculations) which cause $\langle \vec{\pi} \rangle$ to be zero it has the same content as the parameter $\langle \sigma \rangle$ alone. For the identity map it is obvious (see Eq. (12)) that the parameter (17) vanishes. In the case of the localized solution it approaches the value 1 with increasing $L$, since in most of the space the $U$ field of the localized skyrmion is approximately close to the vacuum value $U_0 = 1$ and only markedly deviates from this in the small region where the skyrmion is located. The interpretation is that a zero in this order parameter signals chiral symmetry restoration or at least “chiral democracy”, since there is no escape from the “chiral-circle” constraint $\sigma^2 + \vec{\pi}^2 = f_π^2$. But nevertheless as shown for few-skyrmion systems on the hypersphere [17] [19] and for flat space arrays [8] [10] the $\sigma$ and pion profiles of the high-density phase are so equally distributed about the chiral circle that their averages vanish.

Let me summarize the $B = 1$ results on $S^3(L)$: There is a second-order transition at a baryon density $1/(2\pi^2L_c^3)$ for $L_c = \sqrt{2}$ from a localized and chirally broken low-density solution to a “chirally restored” high-density solution. Restoring dimensionful parameters into the Skyrme lagrangian (by using the empirical value of the pion decay constant $f_π = 93$ MeV and an $\epsilon_4 = 0.0743$ guaranteeing a reasonable value for $g_A$) one finds

$$\rho_c = \left( \frac{f_π}{2\sqrt{2}\epsilon_4} \right)^3 \frac{1}{2^2\epsilon_4} \approx 0.20\text{fm}^{-3},$$  \hspace{1cm} (18)

which is far too low (remember that the nuclear matter density is $\rho_{nm} = 0.16\text{fm}^{-3}$). The important result, however, is that the $S^3(L)$ value for $\rho_c$ is more or less the same than the one found in the periodic array calculations [7] [9] under the same input parameters, approximately $\rho_c = 0.17\text{fm}^{-3}$. Furthermore, the corresponding transition densities $\rho_c$ in the few-skyrmion calculation [17] [19] interpolate between these two numbers. Thus the identification of the hypersphere formalism with the periodic array calculation seems to be justified even quantitatively.

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\*This might improve if the stabilization term is replaced by more realistic ones. Furthermore all contributions from kinetic terms are neglected. Finally, there is the problem of the missing central attraction\(^\ddagger\) which is probably linked to loop corrections\(^\ddagger\) or missing $1/N_c$ corrections\(^\ddagger\). Any additional attractive term tends to localize the skyrmion and will therefore move the transition point up to higher densities.

\(^\ddagger\)
3. Indications for a Chiral Symmetry Restoration

In this section I will present some arguments (see refs. 22, 23 for more details) why the transition reviewed in the preceding section should be interpreted as chiral symmetry restoration. As mentioned the complete uniformity of the high-density $B = 1$ solution on $S^3(L)$ is linked to the strong condition $f(\mu) = \mu$ and does not generalize to few-skyrmion systems 17, 19 on $S^3(L)$ or periodic arrays 7, 8, 9, 10 in flat space. However, the “chiral democracy” as discussed at the end of the preceding section can even be achieved via a weaker condition

$$f(\mu) = \pi - f(\pi - \mu)$$

(19)

which indicates that in the high-density phase there is a symmetry about the hypersphere equator between the “northern” and the “southern” hemisphere. In fact this is just the half-skyrmion symmetry 8 which is a common signal of the high-density phase for few-skyrmion systems on the hypersphere 17, 19 as well as for periodic arrays in flat space 8, 24. So the chiral symmetry restoration seems to be prior to the delocalization. In fact this can be tested by adding a (pion mass) term which explicitly breaks chiral symmetry:

$$L_{SB} = \frac{m_\pi^2}{4} \frac{f^2}{f_\pi^2} \mathrm{Tr}(U + U^\dagger - 2).$$

(20)

When this term is added, the phase transition is not a sharp one any longer, but it is smeared out. The chiral order parameter (17) still approaches zero for higher and higher densities, but never actually becomes exactly zero. Of course, this is just a consequence of the explicit breaking of the chiral symmetry. A second effect of adding the term (20) is a small shift of the now approximate phase transition to higher densities, since the term induces an attractive force.

In ref. 22 it is furthermore argued that the vanishing of the order parameter $\langle \sigma \rangle$ (for the Skyrme model without explicit symmetry breaking of course) implies the vanishing of the matrix element for pion decay $\langle 0 | A^a_\mu(x) | \pi^b \rangle$ in case the axial current $A^a_\mu$ is calculated in the mean-field approximation in the frame-work of a $\sigma$-model, and therefore the vanishing of the “effective” pion decay constant and the quark condensate.

The most convincing arguments why the phase transitions in the Skyrme model ought to be identified with a chiral symmetry restoration are the following two: In the $B = 1$ case on $S^3(L)$ it can be shown that the transition from the low-density phase to the high-density phase is accompanied

(i) by the formation of complete chiral multiplets in the fluctuation spectrum

(ii) and by the vanishing of the three pionic Goldstone modes from the spectrum.

These two points are treated at length in ref. 23 where the local stability of the uniform and the localized skyrmion solution on $S^3(L)$ and thus their small-amplitude normal modes are investigated. If all normal modes have positive energy, the solution
is locally stable. If some normal modes have negative energy, the solution is a saddle point. The study of the small-amplitude fluctuations about the $B = 1$ solutions on the hypersphere revealed the following scenario:

(i) In general, the modes about a hedgehog solution have just the usual $O(3)$ degeneracies, i.e. a degeneracy $2N + 1$ in terms of the principal quantum number $N \geq 1$ of the modes.

(ii) The modes about the identity map on the other hand can be classified according to the following representations of the group $O(4)$: the symmetric tensor representation $(N, 0)$ which has $(N + 1)^2$ degenerate states and as static eigenvalues

$$\lambda_{N,0} = L \{ N(N + 2) - 4 \} + 2 \{ N(N + 2) - 2 \} / L,$$

and the $(N, 1)$ representation with a $2((N + 1)^2 - 1)$ degeneracy and the static eigenvalues

$$\lambda_{N,1} = \{ N(N + 2) - 3 \} (L + 1/L).$$

Thus their degeneracy exceeds by far the one of the modes about localized hedgehog solutions. Furthermore for each mode about the identity map with a given parity, there exists at least one degenerate mode of opposite parity. Using the covering group $SU(2)_L \times SU(2)_R$ of $SO(4)$ it can be shown (see ref. 23) that the $(N, 0)$ modes belong to a trajectory based on the four-fold degenerate $(1/2, 1/2)$ representations of $SU(2)_L \times SU(2)_R$, while the $(N, 1)$ modes belong to a trajectory based on the six-fold degenerate $(1, 0) + (0, 1)$ representation. Thus the modes form complete multiplets of chiral symmetry either by the additional $(\sigma, \vec{\pi})$ degeneracy of the $(N, 0)$ modes (21) or by the parity doubling of the $(N, 1)$ modes (22). Note that in ref. 23 the $(N, 0)$ modes were identified as purely “electric” grand-spin modes (but with a degeneracy between e.g. grand spin $K = 0^+$ and $K = 1^-$ modes), while the $(N, 1)$ modes were identified as degenerate “magnetic” and “electric” modes of the same grand spin $K$, but opposite parity. The $(N = 1, 1)$ modes are the sixth zero modes about the identity map.

(iii) At high densities ($L < \sqrt{2}$) all normal modes about the identity map have positive energies, the identity map is stable. There are four degenerate low-lying modes in the spectrum, the four $(N = 1, 0)$ modes, one with $\sigma$ quantum numbers and three with pion ones indicating that the $\sigma$ and all three $\pi$-fields are degenerate and treated on the same footing. They have the static eigenvalue

$$\lambda_{N=1,0} = -L + 2/L.$$  

(iv) The four $(N = 1, 0)$ modes cross zero at $L = \sqrt{2}$ and become negative for $L > \sqrt{2}$ signalling the instability of the identity map.

---

1 We use the phrase “static eigenvalue” as a short-hand notation for the change of the static energy under the small perturbation of the fluctuation-mode.
(v) At $L = \sqrt{2}$ there is the bifurcation into the normal localized hedgehog solution with the reduced $O(3)$ symmetry. All modes about the localized solution are stable for $L \geq \sqrt{2}$. The modes with $N \geq 1$ have (with the exception of some accidental degeneracies for special values of $L$) only the usual $O(3)$ degeneracy $2N + 1$. However, there exist 9 zero modes. The three additional zero modes (compared with the number of zero modes for the identity map) correspond to the three unstable pionic $(N = 1, 0)$ modes of the identity map. Their fourth partner, the $\sigma$-like mode, corresponds to a positive stable excitation about the localized hedgehog (to an infinitesimally small conformal variation in the profile function $f(\mu) \rightarrow f(\mu) + \delta(\mu)$). The breakdown of the continuous symmetry $O(4)$ of the identity map at $L = \sqrt{2}$ to the $O(3)$ symmetry of the localized for $L > \sqrt{2}$ energetically favorable hedgehog solution is therefore closely linked to the appearance of three new zero modes.

(vi) These three zero modes should be in fact interpreted as the three pionic Goldstone modes linked to the spontaneous breakdown of chiral symmetry. The arguments for this identification are the following: Whereas six of the nine zero modes stay normalizable even in the limit $L \rightarrow \infty$, three have no finite $L \rightarrow \infty$ limit. This is consistent with the fact that the flat space hedgehog has only six zero modes. Furthermore when the chiral symmetry breaking term (20) is added to the Skyrme lagrangian, the above mentioned normalizable zero modes stay zero modes, whereas the three additional zero modes are shifted in energy by the pion mass. These are the three pionic Goldstone modes which become massive when an explicitly chiral symmetry breaking term is added and which show up as non-normalizable plain wave excitations of the vacuum in the flat space limit.

Since the three Goldstone modes disappear for the high-density delocalized phase and the high-density modes form complete chiral multiplets, the chiral symmetry restoration is established.

4. Generalizations

4.1. Contact Terms

In the following we will show that the form of the Skyrme model ([1] and especially the stabilization by the fourth-order Skyrme term is not a necessary precondition for the very existence of the chiral phase transition. Let us for instance replace the fourth-order stabilizing term in ([1] by a stabilizing term of sixth-order (see e.g. ref. 25)

$$\mathcal{L}_{2,6} = \frac{f_s^2}{4} \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right) - c_6 B_\mu B^\mu$$

Note that a hedgehog in flat space has only 6 zero modes: 3 rotational (which cannot be separated from iso-rotational ones because of the hedgehog symmetry) and 3 translation ones.
(note \(c_6 > 0\)) where \(B_\mu\) is the topological baryon current
\[
B_\mu = \frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2} \text{Tr} \left( (U^\dagger_\nu U) (U^\dagger_\alpha U) (U^\dagger_\beta U) \right).
\]  

(25)

As reported in section 2 the fourth-order Skyrme term as well as the sixth-order term and the second-order non-linear sigma model term have a geometrical meaning when expressed in the strain tensor language. One can apply therefore Manton’s general machinery to show that the sixth-order term allows for the identity map as a solution and to construct the critical value \(L_c\) where the identity map of the model (24) becomes unstable and bifurcates into a localized solution. In fact all qualitative phenomena show up as before. Naturally the value of \(L_{\text{min}}\) for the minimum of the static energy is in general not the same for both models and more importantly the critical hypersphere radius \(L_c\) where the chiral restoration occurs is
\[
L_c = (3)^{\frac{1}{4}} L_{\text{min}}
\]

(26)

for the model (24) instead of \(L_c = \sqrt{2} L_{\text{min}}\) as it was the case for the usual Skyrme model (1).

As shown in ref. 26 any lagrangian which (a) is a polynomial of the fundamental invariants (3) of the strain tensor, (b) which has the usual vacuum properties and (c) which allows for (locally) stable \(B = 1\) Skyrmion solutions, leads to a chirally restored phase at high densities. In addition the position of the phase transition can be uniquely derived just from the lagrangian without solving any equation of motion. Let us discuss for this purpose a lagrangian of the analytic form
\[
\mathcal{L}_G \equiv -G(-\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6) = - \sum_{j_1, j_2, j_3=0}^{\infty} A_{j_1 j_2 j_3} (-\mathcal{L}_2)^{j_1} (-\mathcal{L}_4)^{j_2} (-\mathcal{L}_6)^{j_3}
\]

(27)

where \(j_1, j_2, j_3\) are integer indices and \(A_{j_1 j_2 j_3}\) Taylor coefficients. The arguments of the analytic function \(G\) and the signs are chosen in such a way, that the corresponding static energy density has the simple form
\[
\mathcal{E} \equiv \mathcal{G}(\mathcal{E}_2, \mathcal{E}_4, \mathcal{E}_6) = \sum_{j_1, j_2, j_3=0}^{\infty} A_{j_1 j_2 j_3} \mathcal{E}_2^{j_1} \mathcal{E}_4^{j_2} \mathcal{E}_6^{j_3}.
\]

(28)

The function \(G\) should be of course positive definite to ensure a positive energy density and \(G(0, 0, 0) \equiv 0\) (e.g., \(A_{000} = 0\)) to ensure triviality in the \(B = 0\) sector. By inserting \(f = 0\) or \(f' = 0\) one furthermore learns that \(G(x, 0, 0) \geq 0\) and \(G(x, x^2/4, 0) \geq 0\) for any real \(x \geq 0\). The first inequality signals that a possible negative term expressed solely by the second-order term can neither be compensated by adding fourth- or sixth-order terms in any combination to the static energy density. Therefore Skyrme-type models which at fourth-order are not positive definite cannot be saved by the addition of a sixth-order term proportional to \(B_\mu B^\mu\) or any power of this term.

Note that the energy density (28) is the most general symmetric function of the eigenvalues, \(\lambda_i^2\), of the strain tensor \(K_{ij}\) (3) and includes all forms which can come
form lagrangians which involve arbitrary combinations of powers of first derivatives of the fields to even order.

In the chirally restored (uniform) regime on the hypersphere the energy densities of the second-, fourth- and sixth-order term, \( \mathcal{E}_2 \), \( \mathcal{E}_4 \) and \( \mathcal{E}_6 \), have the simple form (modulo a prefactor which can be incorporated into the Taylor coefficients \( A_{(j_1j_2j_3)} \))

\[
\mathcal{E}_{2i} = \frac{3}{L^{2i}}, \quad i = 1, 2, 3.
\] (29)

In ref. \(^{26}\) the eigenvalues of all possible small amplitude (static) perturbations around the identity map were obtained as the following analytical result:

\[
\lambda_{N,a}^G = \left\{ \sum_{i=1,2,3} \lambda_{i,N,a} \frac{\partial}{\partial \mathcal{E}_{2i}} + \lambda_{3,N,a} \frac{2L^2}{3} \left( \frac{\partial}{\partial \mathcal{E}_2} + \frac{2}{L^2} \frac{\partial}{\partial \mathcal{E}_4} + \frac{3}{L^4} \frac{\partial}{\partial \mathcal{E}_6} \right) \right\} \mathcal{G}(\mathcal{E}_2, \mathcal{E}_4, \mathcal{E}_6)
\] (30)

with

\[
\lambda_{i,N,a} = i \left\{ N(N+2) + a^2 - 4 - \frac{i-1}{2} \left( \{a(N+1)\}^2 - 4 \right) \right\} L^{3-2i}.
\] (31)

where the indices \( N \geq 1 \) and \( a = 0, 1 \) characterize the allowed \( O(4) \) representations: \((N,a = 0)\), the symmetric tensor representations which have \((N+1)^2\) degenerate states and the \((N,a = 1)\) representations which have a \(2\{(N+1)^2 - 1\}\) degeneracy (see refs. \(^{23,25}\) for further details). Note that in the case \( a = 1 \) the expression for \( \lambda_{N,a=1}^G \) simplifies to

\[
\lambda_{N,a=1}^G = L\{N(N+2) - 3\} \left( \frac{\partial}{\partial \mathcal{E}_2} + \frac{1}{L^2} \frac{\partial}{\partial \mathcal{E}_4} \right) \mathcal{G}(\mathcal{E}_2, \mathcal{E}_4, \mathcal{E}_6).
\] (32)

A negative value for any eigenvalue \( \lambda_{N,1} \) in (32) would mean that infinitely many of the \((N,1)\) modes would pass through zero energy simultaneously. This is probably indicative of a non-perturbative configuration of lower dimension, e.g., a string-like configuration. Fortunately, one can exclude this pathological instability, since it violates the constraint that the model \((27)\) should have the same vacuum properties as the nonlinear \( \sigma \) model (see ref. \(^{25}\) for a discussion about this point). So the \((N,1)\) fluctuations cannot lead to an instability. In case the identity map is (locally) stable in the high-density region (i.e. all \( \lambda_{N,a}^G \geq 0 \)), the monotonic increase of the eigenvalues \( \lambda_{N,a}^G \) of the small-amplitude normal modes with increasing \( N \) is guaranteed. Then, the critical \( L_c \) where the identity map becomes unstable against small-amplitude perturbations is given by the value of \( L \) where the first \( \lambda_{N,0}^G \) in Eq. \((31)\) becomes negative.

Finally, let

\[
\mathcal{G}(\mathcal{E}_2, \mathcal{E}_4, \mathcal{E}_6) \to \mathcal{E}_2 \quad \text{for} \quad \mathcal{E}_{2i} \to 0
\] (33)

and

\[
\mathcal{G}(\mathcal{E}_2, \mathcal{E}_4, \mathcal{E}_6) \to C_1 \mathcal{E}_2^2 + C_2 \mathcal{E}_4 \quad \text{for} \quad \mathcal{E}_{2i} \to \infty
\] (34)

\(^k\)See section 3 for the corresponding results for the usual Skyrme model \([1]\).
where $i$ runs from 1 to 3 and $C_1 \geq 0$ and $C_2 > 0$ are fixed positive coefficients. The condition (33) enforces that asymptotically for low densities (large $L$) the lagrangian (27) is dominated by the non-linear sigma model lagrangian, whereas for high densities (small $L$) it scales as a free Fermi gas (i.e. $\mathcal{E}_G \propto 1/L^4$). The low density behavior is of course indisputable, the high density constraint on the other hand requires the additional input that such effective models can be applied even for densities where the underlying theory, QCD, becomes to leading order a free Fermi gas.

Back to the stability analysis: Note that the signs of the static eigenvalues $\lambda_{G,N,0}$, therefore the stability of the uniform regime follow from the signs of the derivatives acting on $\dot{G}$ and from the signs of the coefficients $\lambda_{i,N,a}$ (31). From all possible $\lambda_{i,N,a}$ with $i = 1, 2, 3$, $N \geq 1$ and $a = 0, 1$ only the term $\lambda_{1,1,0}$ is negative, all the other coefficients are guaranteed to be larger than zero or at most equal zero. (To the latter category belong the six $O(4)$ zero modes with $N = 1$ and $a = 1$. Furthermore all the terms $\lambda_{i=3,N,1}$ are zero indicating that the sixth-order term itself has infinitely many zero modes.) Now taking the high-density (34) and the low-density behavior (33) into account, we can conclude that at high densities the uniform $B = 1$ skyrmion solution on the hypersphere is bound to be stable whereas at sufficiently low densities, where the non-linear $\sigma$ model term becomes the dominant one, the uniform solution is unstable since $\lambda_{N=1,0}^G$ becomes negative eventually. Furthermore we have to take into account that in the stable regime - even infinitesimally close to the instability - all $\lambda_{N,0}^G$ have to increase monotonically with $N$, that the $\lambda_{N,1}^G$ modes have to be stable by fiat and that the eigenvalues of the modes have to be smooth functions of $L$. Then the existence of a critical value of $L$ is guaranteed where $\lambda_{1,0}^G$ becomes negative such that the uniform high density phase becomes unstable. There is a bifurcation from the uniform solution which is chirally restored to the usual, localized and chirally broken $B = 1$ hedgehog solution. So even for the most general geometric lagrangian (27) built form first order field derivatives the existence of the chiral phase transition (as discussed at length in the preceding chapters for the normal Skyrme model) is guaranteed under the assumption of a few reasonable constraints (the vacuum structure, Eqs. (33) and (34)) on the form of the lagrangian.

4.2. Vector Meson Stabilization

We have discussed so far lagrangians which ensured the stabilization of skyrmions by higher-order (contact) terms in the field derivatives. Naturally the question arises whether the existence of the chiral phase transition is linked to the stabilization by contact terms or whether vector mesons can be added which act over a finite range, as e.g. $\omega$ (and $\rho$) mesons. Let us take as an example a Skyrme-type variant with $\omega$ meson stabilization. (See ref for a periodic array study of this model.) The corresponding lagrangian has the structure (27)

$$
\mathcal{L}_{2,\omega} = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{m_\omega^2}{2} \omega_\mu \omega^\mu + g_\omega \omega_\mu B^\mu
$$

(35)

where the first part is the usual non-linear $\sigma$ model term, the second one the $\omega$ kinetic term expressed through the $\omega$ field-strength tensor $\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$, the third part
is the $\omega$ mass term and the last one is the coupling of the $\omega$ field to the topological baryon current (24). As shown by Adami\textsuperscript{28} the lagrangian (24) has stable $B = 1$ solitons.

By putting an $\omega$-stabilized $B = 1$ hedgehog on the hypersphere one can easily find that one solution of the equations of motion is always the uniform solution $f(\mu) = \mu$ for the hedgehog profile and

$$\omega_0 = -\frac{g_\omega}{m_\omega} B_0 = -\frac{g_\omega}{m_\omega^2} \frac{1}{2\pi^2 L^3}$$

for the $\omega_0$-component. The $\omega_0$-component is in this case spatially constant and depends only on the hypersphere radius $L$. (As in flat space the spatial components $\omega_i$ are identically zero because of the static hedgehog form of the soliton profile.) When the uniform solutions $f(\mu) = \mu$ and (36) are reinserted into the corresponding energy density of the lagrangian (35), one recovers the same structure as for a model stabilized by a sixth-order term with $\mathcal{L}_6 = -(g_\omega^2/2m_\omega^2)B_\mu B^\mu$. This should not come as a surprise, since in the limit $m_\omega, g_\omega \to \infty$, under a constant ratio $g_\omega/m_\omega$, the lagrangian (35) converges to the lagrangian (24) with $c_6 = (g_\omega^2/2m_\omega^2)$. Both lagrangians have the same high-density phase in the static sector. The physical reason for the identical behavior of the static solutions of both lagrangians is that at high densities it is energetically favorable that the $\omega_0$-component becomes spatially constant, otherwise the “costs” in energy from a finite spatial gradient term in this component becomes higher and higher with increasing density. This together with the equation of motion for the $\omega_0$-component necessarily lead to Eq.(36) which in turn guarantees the identical high-density behavior of both lagrangians. However, the $\omega$-stabilized model bifurcates at a smaller $L_c$ and the low-density behavior of both lagrangians is different. Under increasing parameters $g_\omega$ and $m_\omega$ with the ratio $g_\omega/m_\omega$ kept constant the critical density and the low density phase of the $\omega$ stabilized model will approach the corresponding quantities of the model (24). So the inclusion of finite range vector mesons leads to an increase in the transition density in comparison to the corresponding contact-term model, see also ref.\textsuperscript{13}. In summary, also the $\omega$-stabilized Skyrme model has a critical hypersphere radius $L_c$ where the uniform (chirally restored) phase on the hypersphere becomes unstable and bifurcates into a localized (chirally broken) phase which has the usual $B = 1$ hedgehog as flat space limit.

A similar behavior follows when the $\mathcal{L}_{2,\omega}$ model is extended by the introduction of $\rho$ mesons. This can be done in different ways. For our purposes the only essential precondition is that the total lagrangian is still explicitly chiral symmetric. For simplicity let us consider in the following the hidden $\rho$ meson coupling à la Bando et al.\textsuperscript{29} in a minimal way\textsuperscript{30}, i.e.

$$\mathcal{L}_{2,\omega,\rho} = \frac{f^2}{4} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) - a \frac{f^2}{4} \text{Tr}(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger - 2ig_\rho p_\mu)^2 - \frac{1}{4} p_\mu p^\mu$$

$$- \frac{1}{4} \omega_\mu \omega^\mu + \frac{m_\omega^2}{2} \omega_\mu \omega^\mu + g_\omega \omega_\mu B^\mu + (\omega \rho \pi \text{ coupling terms})$$

(37)
with $\xi \equiv \sqrt{U}$, and $\rho_{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu - ig[\rho_\mu, \rho_\nu]$, the non-abelian field-strength tensor. The second term in (37) is responsible for the generation of the $\rho$ mass and the $\rho \pi \pi$ coupling with a coupling constant $g$. (The standard choice for the parameter $a$ is $a = 2$, see ref. 29.) Note that under the static hedgehog ansatz only the spatial $\rho_i$ components are excited. Again it is easy to show that the equations of motion of this model are satisfied by the uniform profiles

$$
\begin{align*}
    f(\mu) &= \mu \\
    \omega_0 &= -\frac{g_\omega}{m_\omega^2} B_0 \\
    \rho_i &= \frac{1}{2ig} (\xi^\dagger \partial_i \xi - \xi \partial_i \xi^\dagger) \quad \text{with} \quad \xi = \exp(i\tau \cdot \hat{r}_\mu/2).
\end{align*}
$$

The last equation guarantees that the second term in (37) vanishes and that the $\rho$ kinetic term behaves as fourth-order Skyrme term with a coefficient $\epsilon_4^2 = 1/(8g^2)$. The lagrangian (37) (without extra $\omega \rho \pi$ couplings) has therefore the same high density behavior as the simple contact lagrangian

$$
\mathcal{L}_{2,4,6} = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32g^2} \text{Tr}[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 - \frac{1}{2} \left( \frac{g_\omega}{m_\omega} \right)^2 B_\mu B^\mu.
$$

This behavior presented here for the simplest $\omega \rho \pi$-model is generic – just the coefficient of the sixth-order term in (11) may change when $\omega \rho \pi$-coupling terms are introduced. For all models with vector-meson stabilization which have been studied so far 2 31 32 the following is true: It can be numerically shown that

(i) the only existing $B = 1$ solution at high densities is the uniform (delocalized) hedgehog one, the static solutions are identical to those of contact-term lagrangians with suitable coefficients and the static vector meson fields are equal to their driving pionic currents (see e.g. (36) or (40)),

(ii) there exists a critical model-dependent hypersphere-radius $L_c$ where the uniform solution becomes unstable and bifurcates into the usual localized configuration and

(iii) at low densities the solutions on the hypersphere $S^3(L)$ converge asymptotically to the corresponding solutions in the flat space.

The essential conditions on these models are two-fold: they should be explicitly chiral symmetric and they should allow for locally stable $B = 1$ hedgehog solitons. In case these two conditions are met, there will be necessarily a chiral phase transition at high densities with the same features as already discussed. Still, under realistic values for the mesonic input parameters (e.g. $f_\pi$, $\epsilon_4$, $c_6$, see ref. 23, or $g_\omega$ and $g$, see refs. 24 24 3)) the transition densities $\rho_c$ come out systematically too low, $\rho_c \approx \rho_{nm}$ (0.16 fm$^{-3}$). Apparently these models lack terms which generate additional attraction. Note that the hedgehog mass under such realistic parameters is systematically too high ($\sim 1.6$ GeV) whereas there is not sufficient central attraction in the $B = 2$
If one would tune the stabilizing parameters – while keeping $f_\pi = 93\text{MeV}$ fixed – such that the classical hedgehog mass would be 0.87 GeV (consistent with a nucleon mass of 0.94 GeV), then the transition density $\rho_c$ would be approximately three times the nuclear matter density.

5. Second- Versus First-Order

Whereas in the periodic array calculations second as well as first order phase transitions were found depending on the choice of the lattice and the twisted boundary conditions, we have so far encountered only second order transitions between the delocalized and the localized phase on the 3-sphere. This does not need to be the case in general: All the Skyrme-type lagrangians presented are explicitly not scale invariant and do not “know” (yet) about the trace anomaly in QCD. Let us therefore try to incorporate the same scaling behavior as in QCD into these lagrangians (see refs. [33, 34, 35]), e.g. consider the lagrangian

$$L = \chi^2 \sum f_\pi^2 \frac{\partial^4}{\partial \phi^4} + \frac{\chi^2}{\lambda^2} \frac{\partial^4}{\partial \phi^4} + \frac{\chi^2}{\lambda^2} \frac{\partial^4}{\partial \phi^4} + \frac{B_B}{2} \left(1 + \frac{\chi}{\lambda_0} \right)^4 \log \left(\frac{\chi^4}{\lambda_0^4}\right)$$

(42)

as simplest extension of the usual Skyrme lagrangian. The scalar field $\chi$ with the vacuum expectation value $\chi_0 = \langle 0 | \chi | 0 \rangle$ is introduced with the purpose of making the first term in (42) scale invariant (the Skyrme term and the $\chi$-kinetic term are already scale invariant), whereas the last term in (42), the $\chi$-potential term, is adjusted to fit the trace anomaly of QCD. $B_B$ is the “bag constant” which can be expressed in terms of the gluon condensate as $B_B = (9/32) \langle 0 | (\alpha_s/\pi) G^2 | 0 \rangle$. For values of the parameters $\chi_0$ and $B_B$ see ref. [34]. (Note that the fluctuations of the $\chi$-field correspond to glueball-excitations.) In ref. [36] it was numerically shown that the lagrangian (42) again possesses a phase transition to a uniform delocalized solution at high densities with $f(\mu) = \mu$ and a constant $\chi$ profile which – depending on the choice of parameters – can be even $\chi = 0$. Because of the insufficient accuracy in the numerics, the authors of ref. [36] missed the fact that the phase transition is of first order. The latter is more or less obvious from the form of the $\chi$-potential term which is adjusted to be minimal for $\chi = \chi_0$. Thus it is impossible to find a smooth transition between the constant $\chi < \chi_0$ (high-density) profile and the localized (low density) $\chi$ profile (which at the south pole on the 3-sphere is exactly $\chi_0$) without violating the $\chi$ equation of motion:

$$\chi'' + 2 \frac{\cos \mu}{\sin \mu} \chi' - \frac{f_\pi^2}{\lambda_0^2} \frac{\partial^4}{\partial \phi^4} \left(f'^2 + 2 \frac{\sin^2 f}{\sin^2 \mu} \right) - 4B_B \frac{\chi^3}{\lambda_0^4} \log \left(\frac{\chi^4}{\lambda_0^4}\right) = 0.$$

(43)

For a non-vanishing nonlinear sigma term (as in the present case since $f(\mu) \neq 0$) the $\chi$ field cannot both be constant and equal to $\chi_0$. So the $\chi$ field can only do the transition from the low- to the high-density phase and vice versa by a jump. Thus by
incorporating a scalar field à la Schechter in the Skyrme model (or its extensions) the second order phase transition can be easily changed to a first order one. See also ref. for a different mechanism to achieve the same thing.

It is rather easy to extend the lagrangian to the most general with the QCD trace anomaly consistent geometrical form involving only first derivatives to even order:

\[
\mathcal{L}_{G,\chi} = -\left(\frac{\chi}{\chi_0}\right)^4 \mathcal{G} \left( -(\frac{\chi_0}{\chi})^2 \mathcal{L}_2, -(\frac{\chi_0}{\chi})^4 \mathcal{L}_4, -(\frac{\chi_0}{\chi})^6 \mathcal{L}_6 \right) + \frac{1}{2} (\partial_\mu \chi)^2 - B_B \left( 1 + \left(\frac{\chi}{\chi_0}\right)^4 \log \left(\frac{\chi^4}{\chi_0^4}\right) \right).
\]

Note that the restoration of scale invariance, \(\chi = 0\), is only consistent with the lagrangian if \(G\) has the asymptotics, \(i.e.\)

\[
\left(\frac{\chi}{\chi_0}\right)^4 \mathcal{G} \left( -(\frac{\chi_0}{\chi})^2 \mathcal{L}_2, -(\frac{\chi_0}{\chi})^4 \mathcal{L}_4, -(\frac{\chi_0}{\chi})^6 \mathcal{L}_6 \right) \to C_1(-\mathcal{L}_2)^2 + C_2(-\mathcal{L}_4) \quad \text{if} \quad \chi \to 0.
\]

We know from what was said before that one can expect in this case a first-order phase transition to a chirally restored phase. However, the \(\chi\)-profile in the high-density phase does not need to become zero, it can equally well be just a constant \(\chi < \chi_0\). Scale invariance implies chiral restoration, but not vice versa. Nevertheless, just the possibility to have a scale invariant limit at high densities leads to strong constraints on the form of the effective model. If a lagrangian is considered which includes any term higher than fourth-order in the derivatives of the pion fields, terms of all orders have to be included as well both to satisfy the high-density asymptotic behavior and to ensure the stability of the chiral symmetric high-density phase. The necessity for infinitely many terms is consistent with the large \(N_c\) philosophy. Because of asymptotic freedom, the effective model which follows from QCD to leading order in the \(1/N_c\) expansion must involve infinitely many mesons. The hope is – as shown for the \(\omega\) and \(\rho\) meson case – that the high density phase of this infinite tower of meson resonances can be approximated by infinitely many suitable contact terms. This might lead to “dreams” of Regge trajectories.

6. Discussion

Even effective models of the Skyrme class (which do not possess any quark degrees of freedom) can have a phase transitions to a high-density phase with restored chiral symmetry. There are essentially just two conditions on such a model: It must have the relevant symmetries (especially the lagrangian must be explicitly chirally invariant) and it must treat the baryon structure and interaction on the same footing. Normally, the phase transitions are of second order, but one can easily change them to first-order ones by incorporating the QCD trace anomaly for instance. This allows also depending on the input parameters - for a scale invariant high density limit. Scale invariance implies chiral symmetry restoration, but not vice versa.
Under the same parameter input (from the meson sector) the periodic array calculations as well as the 3-sphere calculations give approximately the same values for the transition densities. The predictions, however, are for both systematically too low, $\rho_c \approx \rho_{nm}$. This problem may be connected with two other deficiencies which plague the Skyrme-like models: the missing central attraction in the $B = 2$ channel and the predicted high value for the hedgehog mass. Quantum (higher loop or higher $1/N_c$) corrections may have a chance to cure all three problems at the same time by providing extra non-local terms which generate attraction.

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