Induced Perturbations and Stochastic effects in Collapsing Relativistic Stars

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We present a modified model for relativistic stars which are usually represented by perfect fluids. Fluctuations of the stress tensor act as source in the modified Einstein’s equation, giving it a Langevin equation form. The occurrence of these fluctuations is attributed to the microphysics of the interior of the star and their contribution to statistical properties of the fluid and the induced metric perturbations are argued to be of significance. We also discuss the response of fluctuations of the stress tensor in the interior of the star and possible developments towards fluctuation-dissipation theorem issues in curved spacetime. The aim and further directions of research envisioned are discussed intermittently throughout the manuscript and towards the end.

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I. INTRODUCTION

Perturbations in relativistic stars [1–4] and black holes [5–8] comprise a vast literature, addressing stability issues and oscillations as in the area of asteroseismology and singularities formed as end states of gravitational collapse of massive stars.

These compact configurations of relativistic stars are often modeled by perfect fluid as the matter constituent. In this article we attempt to raise a new aspect, that of taking into account, the fluctuations of the classical stress tensor and its effects in the interior of the star as well as, backreaction on the spacetime geometry in relativistic case. Cowling approximation [4, 9] is often used in asteroseismology (both the non-relativistic and relativistic cases) to ignore the effects of matter perturbations on the spacetime geometry or metric. However this is limited to few specific conditions and certain modes of oscillations and frequencies. We give a full general relativistic treatment of the equations governing perturbations and their backreaction effect on the metric, including stochasticity and do not render the Cowling approximation suitable for a general perturbative analysis as presented here. This is so, because our interest is in exploring the effects of the perturbations and stochasticity in a different context.

In order to include stochasticity of the interior of a relativistic star and look at its backreaction consistently, the idea of a Langevin Equation formalism is adopted from semiclassical stochastic gravity [10, 11]. The semiclassical Einstein-Langevin equation takes account of the fluctuations of the quantum stress tensor and its backreaction, with interesting applications to early universe cosmology [12, 13] and black hole physics [14, 16]. We borrow the idea of setting up a Langevin approach in the classical domain from the semiclassical case. However this has a very different basic formalism regarding mathematical and physical issues to tackle with for us, as well as different domain of applications. While semiclassical Einstein Langevin equation is shown to be of importance to cosmology and quantum aspects of black hole physics, the classical counterpart can be seen to cover the domain of relativistic astrophysics, that of stellar dynamics, asteroseismology and collapse of relativistic stars.

A perfect fluid model, is often the simplest case considered for a relativistic star. The microscopic particles in the perfect fluid collide frequently and their mean free path is short compared to the scale on which density changes. Thus the stress tensor can be defined in elements of the fluid which are small compared to macroscopic length scale but large compared to mean free path. Hence, it is the mean stress tensor defined by $T^{ab}(x)$, with the pressure and density as averaged macroscopic quantities that is usually considered. The stress tensor is given by

$$T^{ab} = u^a u^b (p + \epsilon) + g^{ab} p$$  \hspace{1cm} (1)

Here $p$ denotes pressure and $\epsilon$ the matter energy density.

The average stress tensor thus described leaves scope to define fluctuations given by $T^{ab}(x) - < T^{ab}(x) > = \tau^{ab}(x)$. These may be sourced via various processes in the interior of the star including quantum effects which are partially captured by fluctuations in the density and pressure, the effects being of classical stochastic nature. This model can be considered in terms of small deviation from that of a perfect fluid.

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There can be various additional effects including imperfect and dissipative fluid models, for the study of more realistic relativistic stars. In these models it may be appropriate to associate the fluctuations with dissipative mechanism and thermal effects explicitly. Our formalism here can be extended very easily to all these cases. However in this article, we restrict ourselves to the stress tensor for a perfect fluid as the simplest example in order to set up the basic formalism correctly.

II. THE MODEL USING EINSTEIN LANGEVIN EQUATION

A nonrotating spherically symmetric star in Schwarzschild coordinates is given by

\[ ds^2 = -e^{2\nu(r,t)}dt^2 + e^{2\psi(r,t)}dr^2 + r^2d\Omega^2 \]  

To consistently take into account the fluctuations of the stress tensor for a general relativistic treatment an Einstein Langevin equation can be written as

\[ G^{ab}[g + h](x) = T^{ab}[g + h; \xi](x) + \tau^{ab}[g](x) \]  

where \( \tau^{ab}(x) \) on the background spacetime \( g^{ab}(x) \) is defined stochastically such that \( < \tau^{ab}(x) >= 0 \) and is covariantly conserved, \( \nabla_a \tau^{ab}(x) = 0 \). This follows from the covariance of \( T^{ab}(x) \). Perturbations are defined to be deformations in the fluid elements, by way of shift from the equilibrium configuration and are deterministic in nature. These perturbations in spacetime and fluid variables can be described by \( h_{ab} \) and Lagrangian displacement vector field \( \xi^a \).

One can then see that the standard form of Einstein’s Equation on averaging of the above equation is obtained.

\[ \delta G^{ab}[\bar{h}](x) = \delta T^{ab}[\bar{h}; \bar{\xi}](x) + \tau^{ab} \]  

The trace reversed metric perturbation defined as

\[ \bar{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab} h; \ h_{ab} = \bar{h}_{ab} - \frac{1}{2} g_{ab} \bar{h} \]

satisfies

\[ \nabla_a \bar{h}^{ab} = 0 \]  

We use this gauge to get a more desirable form of the Einstein Langevin equation which reads

\[ -\frac{1}{2} \Box \bar{h}^{ab}(x) + A^{ab}_{cd}(x)\bar{h}^{cd}(x) = W^{ab}_{cd}(x)\bar{h}^{cd}(x) + E^{ab}_{cd}(x)(\nabla_c \xi^d(x) + \nabla_d \xi^c(x)) + \mathcal{L}_\xi T^{ab}(x) + \tau^{ab}(x) \]  

where

\[ W^{ab}_{cd}(x) = E^{ab}_{cd}(x) - \frac{1}{2} E^{abf}(x)g_{ef}(x)g^{cd}(x) \]

The lhs of equation (6) is written using the Einstein tensor perturbations, while the rhs is that of the stress tensor. The tensors \( A^{ab}_{cd} \) and \( E^{ab}_{cd} \) are given in the appendix, while

\[ \mathcal{L}_\xi T^{ab}(x) = \nabla_c (T^{ab} \xi^c) - T^{ab} \nabla_c \xi^a - T^{ac} \nabla_c \xi^b \]

One can decipher the tensor \( E^{abcd}(x) \) with dissipation in the fluid. This can be shown as arising due to response of the fluctuations causing induced perturbations in the fluid. This is discussed in more detail later. An important point to notice here is that, dissipation seems to be local, unlike many cases where usually a nonlocal dissipation comes out naturally, (including as obtained in semiclassical stochastic gravity). Also it is unlike diffusion constant for ordinary Brownian motion theory in Newtonian case. Thus one may see an example of a local dissipative effect here, which can be interesting for statistical properties of the system.

Now Einstein Langevin equation takes the form,

\[ -\frac{1}{2} \Box \bar{h}^{ab} = [-A^{ab}_{cd} + W^{ab}_{cd}]\bar{h}^{cd} + D^{ab}(x) + \tau^{ab} \]
where we have put,
\[ D^{ab}(x) = E^{abcd}(x)(\nabla_c \xi_d(x) + \nabla_d \xi_c(x)) - \mathcal{L}_\xi T^{ab}(x) \]  
(9)

The aim of such a formalism is to lay a background for doing non equilibrium or near equilibrium statistical physics of isolated relativistic stars during different stages of collapse or including near equilibrium states.

As can be seen, \( D^{ab} \) arises due to \( \xi^a \), we can associate the perturbative effect in the fluid as a response to the stochastic fluctuations.

III. FORM OF SOLUTION OF THE EINSTEIN LANGEVIN EQUATION

Proceeding towards the issues of backreaction of the Langevin noise, equation (5) can be written in the following form
\[ \frac{1}{2} \delta^a_b(x) \delta^c_d(x) \square + A^{ab}_{cd}(x) - W^{ab}_{cd}(x) \tilde{h}^{cd}(x) = D^{ab}(x) + \tau^{ab}(x) \]  
(10)

A general form of solution is thus given by,
\[ \tilde{h}^{cd}(x) = \int G^{abcd}(x, x')[D_{ab}(x') + \tau_{ab}(x')]dx' \]  
(11)

where \( G^{ab}_{cd}(x, x') \) is the Green’s function for the operator \( \left[ \frac{1}{2} \delta^a_b \delta^c_d \square + A^{ab}_{cd} - W^{ab}_{cd} \right] \).

As is known that Green’s function in curved spacetime is not easy to get in analytical form and only few very specific examples have been obtained e.g [17] for specialised cases. Since here our intention is to demonstrate the contribution of stochasticity to the perturbations, we do not go forward to develop methods for obtaining the Green’s function in explicit form and leave it as a separate exercise for further developments, which is quite an involved task in itself. We may also later looking at the details of the coupled differential equations that we get here, decide if numerical methods are better suited for a complete solution. This is yet to be investigated and we make no claims here as to the general method of solution for the Einstein Langevin equations, since these many vary from case to case. Rather we give a general form of solution which brings out clearly the two point correlations and stochastic nature of the same.

From equation (11) the two point correlation of the metric perturbations can be written as
\[ \langle \tilde{h}^{ab}(x) \tilde{h}^{cd}(y) \rangle = \int \int G^{ablm}(x, x') G^{cdpq}(y, y') [D_{lm}(x') D_{pq}(y') + N_{lmpq}(x', y')] dx' dy' \]  
(12)

where
\[ N_{lmpq}(x', y') = \langle \tau_{lm}(x') \tau_{pq}(y') \rangle \]  
(13)

is the noise kernel or two point correlation of the stress tensor fluctuations. Towards the end, we discuss more about noise in the system.

From equation (11) one can also see that,
\[ \langle \tilde{h}^{cd}(x) \rangle = \int G^{abcd}(x, x') D_{ab}(x') dx' \]  
(14)

which clearly shows the overall contribution of the fluctuations on metric perturbations seems to be vanishing. While this looks to be quite a consistent result physically, the importance of including the fluctuations in the Einstein’s Equations can be seen clearly in the two point (or higher) correlations (12).

In the coincidence limit equation (12) takes the form
\[ \langle \tilde{h}^{ab}(x) \tilde{h}^{cd}(x) \rangle = \int \int G^{ablm}(x, x') G^{cdpq}(x, y') [D_{lm}(x') D_{pq}(y') + N_{lmpq}(x', y')] dx' dy' \]  
(15)

Hence though the explicit contribution of the stochastic term seems to be vanishing on the mean \( \langle \tilde{h}^{ab}(x) \rangle \), it is not so for the rms of \( \tilde{h}^{ab} \).

Here we attempt to set up the basic theoretical structure, it will be future endeavour to relate these quantities to observations in asterioseismology and oscillations of relativistic stars and have estimates for the magnitude of the contributions.

Now we show more explicitly, the perturbed stochastic equations in comoving coordinates and the form of solutions. Comoving coordinates are useful in modeling collapsing stars, and hence we are interested in getting the expressions in these coordinates.
A. Expressions in Comoving Coordinates

In comoving coordinates a spherically symmetric (non-rotating) relativistic star is given by

$$ds^2 = -e^{2\nu(r,t)}dt^2 + e^{2\psi(r,t)}dr^2 + R^2(t,r)d\Omega$$

(16)

where $R(t,r)$ is the area radius and $t$ and $r$ are the comoving time and radius. The energy momentum tensor for a perfect fluid in these coordinates is of the form:

$$T^{ab}(x) = \left(\begin{array}{ccc}
\rho(x) & 0 & 0 \\
0 & \rho(x) & 0 \\
0 & 0 & \rho(x) - \delta p(x)
\end{array}\right)$$

(17)

We assume $p_r = p_\theta = p_\phi$ for a simple model, and the velocity vector of the comoving observer is given by $u^a = (1,0,0,0)$ which satisfies the condition $u^au_a = -1$.

Here the Lagrangian change for a quantity is denoted by $\Delta z$ and Eulerian change by $\delta z$ such that,

$$\delta z = \Delta z - \mathcal{L} \xi \equiv \Delta z - \xi^a \nabla_a z$$

(18)

Thus $\Delta g_{ab} = h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a$. The perturbation in pressure and density are given by

$$\delta \rho = -\frac{1}{2} \Gamma_1 p q_{cd} \Delta g^{cd} - \xi^a \nabla_a \rho$$

$$\delta p = -\frac{1}{2} \Gamma_1 p q_{cd} \Delta g^{cd} - \xi^a \nabla p$$

(19)

where $q_{ab} = u_a u_b + g_{ab}$ and $\Gamma_1$ is the usual adiabatic index. We consider only radial perturbations here and hence only $\xi^r$ component for the Lagrangian displacement vector being non-zero, while taking only spherically symmetric metric perturbations given by, $h^{00}(x), h^{11}(x), h^{22}(x)$ and $h^{33}(x)$ to be the non-zero for simplicity of solution. The Einstein Langevin equation then gives the following set,

$$\hat{F}_1 \hat{h}^{00} = a_1 \hat{h}^{11} + a_2 \hat{h}^{22} + a_3 \hat{h}^{33} - g^{00} f_1 + \tau^{00}$$

$$\hat{F}_2 \hat{h}^{11} = b_1 \hat{h}^{00} + b_2 \hat{h}^{22} + b_3 \hat{h}^{33} - g^{11} f_2 + \tau^{11}$$

$$\hat{F}_3 \hat{h}^{22} = c_1 \hat{h}^{00} + c_2 \hat{h}^{11} + c_3 \hat{h}^{33} - g^{22} f_2 + \tau^{22}$$

$$\hat{F}_4 \hat{h}^{33} = d_1 \hat{h}^{00} + d_2 \hat{h}^{11} + d_3 \hat{h}^{22} - g^{33} f_2 + \tau^{33}$$

(20)

All the coefficients and operators in the equation are given in the appendix and

$$f_1 = (\epsilon + p) \xi^r + \xi^r \xi^r ; f_2 = p \Gamma_1 \xi^r + \xi^r p'$$

The above set of coupled equations can be solved numerically in general, but here we would be interested in two point correlations of the metric and solution in terms of Langevin approach. In an upcoming article our effort will be to get a complete formal stochastic solution of the above using the Regge Wheeler gauge.

We further assume $\hat{h}^{22} = \hat{h}^{33}$. Substituting the expression for $\hat{h}^{22}$ obtained from the last two equations of (20), in the first two equations of the set, we obtain

$$\hat{D}_1 \hat{h}^{00}(x) = k_1 (x) \hat{h}^{11}(x) + m_1 (x) + k_2 (x) (\tau^{22}(x) - \tau^{33}(x)) + \tau^{00}(x)$$

$$\hat{D}_2 \hat{h}^{11}(x) = p_1 (x) \hat{h}^{00}(x) + m_2 (x) + p_2 (x) (\tau^{22}(x) - \tau^{33}(x)) + \tau^{11}(x)$$

(21)

The operators and coefficients can be verified from the appendix. We show below the expression for $\hat{h}^{00}$, a similar expression for $\hat{h}^{11}$ can be obtained easily. Thus

$$[\hat{D}_2 (\frac{1}{k_1} \hat{D}_1) - p_1] \hat{h}^{00} = Q_1(x) + Q_2(x) (\tau^{22} - \tau^{33}) + Q_3(x) \tau^{00} + \tau^{11}$$

(22)

where

$$Q_1(x) = -\hat{D}_2 (m_2/k_1) + m_2; Q_2(x) = -\hat{D}_2 (k_2/k_1) + p_2; Q_3(x) = -\hat{D}_2 (1/k_1)$$

The final form of $\hat{h}^{00}(x)$ is then obtained as,

$$\hat{h}^{00}(x) = \int G_1(x,x') [Q_1(x') + Q_2(x') (\tau^{22}(x') - \tau^{33}(x')) + Q_3(x') \tau^{00}(x') + \tau^{11}(x')] d^4 x'$$

(23)
where $G_1(x, x')$ is the Green’s function for the operator $\{\tilde{D}_2(\frac{1}{x!}\tilde{D}_1) - p_1\}$. The mean of the metric perturbation is then given by

$$<\tilde{h}^{00}(x)> = \int G_1(x, x')Q_1(x')d^4x'$$  \hspace{1cm} (24)

we see a vanishing contribution of the fluctuation terms here as expected. There is an overall smearing effect which is physically desirable for such a stochasticity. Though the stochastic term itself vanishes, but we see that it induces an effect (in equation (24)) given by $Q_1(x')$ in the fluid variables comprising of the Lagrangian displacement $\xi$ and the metric perturbations in the gravity sector. Here we emphasise that there is no external agency giving rise to these perturbative effects in the relativistic star. In general theory of perturbations for stars, there is either an internal rotational effect of the fluid or and external agency through which the perturbations and thus oscillations are induced. Here the physical picture is drastically different, in that, the stochastic effects inside the star give rise to similar perturbative effects as can be seen to arise due to the former mentioned sources. This is consistent with the linear response theory in statistical physics. Hence we propose that even in isolated relativistic stars, either undergoing dynamical collapse or near equilibrium configurations of compact configurations, one can see these perturbative effects, which can give rise to oscillations inside the star. We do not at present claim any estimate of the frequency of oscillations or magnitude of pertubations, but assume these to be strong enough to have backreaction effects. The significance of even the slightest disturbances or low magnitude of these perturbations, both in the fluid and the spacetime geometry can have consequences near critical phases of collapse,where they may affect important, decisive physical results of the collapsing cloud. Further, the correlations of the metric perturbations thus obtained , near critical points are expected to enable one to do statistical analysis of spacetime geometry near these phases. This is one of our the prime aims, (along with study of oscillations set by the stochastic effects) in establishing such a formalism as in this article.

One can see that, the two point correlation takes the form

$$<\tilde{h}^{00}(x)\tilde{h}^{00}(y)> = \int \int G_1(x, x')G_1(y, y')Q_1(x')Q_1(y') + S(x', y')d^4x'd^4y'$$  \hspace{1cm} (25)

here $S(x', y')$ is the stochastic part and is given in the appendix. While in the coincident limit we would get the mean square of the metric perturbations and can be easily followed from the above expression by taking $x \to y$.

**IV. NOISE, FLUCTUATION AND DISSIPATION**

Below we discuss a few aspects of noise.

The two point correlation of the fluctuations of the stress tensor can be given by

$$N^{abcd}(x, y) = <\tau^{ab}(x)\tau^{cd}(y)> = N^{cdab}(y, x)$$  \hspace{1cm} (26)

from the definition of $\tau^{ab}(x) >$ it follows that,

$$N^{abcd}(x, y) = <T^{ab}(x)T^{cd}(y)> - <\tau^{ab}(x)><T^{cd}(y)>$$  \hspace{1cm} (27)

Since $\nabla_a\tau^{ab} = 0$, as mentioned earlier, the noise satisfies the following property,

$$\nabla_aN^{abcd}(x, y) = \nabla_bN^{abcd}(x, y) = \nabla_cN^{abcd}(x, y) = \nabla_dN^{abcd}(x, y) = 0$$  \hspace{1cm} (28)

For the spherically symmetric spacetime and stress tensor in comoving coordinates given by equation (17), the non-zero
The nature of dissipation being local is certainly interesting, and we would attempt to find out consequences of the same in the statistical properties of a collapsing relativistic star in future work.

A fluctuation dissipation theorem can be written as

$$N^{abcd}(x, x') = \int K(x, x'; y) E^{abcd}(y) dy$$

(31)

here $K(x, x'; y)$ is the fluctuation-dissipation kernel. The locality of the dissipation kernel here, may thus lay some restriction on the model of noise that is physically allowed for such systems. This needs further investigations.
V. CONCLUSIONS

As concluding remarks, we highlight few additional issues that can be raised over the stochasticity introduced here in the model for relativistic stars.

A collapsing star goes through different phases, where classical and quantum effects become important to study the interior regions of the star and overall dynamics. The formulation presented in this manuscript may also be useful for investigations in the following directions.

One can try to explore if there is mesoscopic regime between classical and quantum effects in the intermediate phases of gravitational collapse. These classical fluctuations, then can take over before quantum effects set in as the singularity is reached, and can lead to short term where mesoscopic physics becomes important during the collapse of the star. In addition to this, stochasticity in the strong field regime due to microscopic or partially captured quantum effects can be probably linked to gravitational decoherence [18] which is an upcoming area. This would be relevant towards the end states of collapse which result either in singularities or more exotic compact objects rather than stable neutron stars.

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APPENDIX

\[ A^{abcd} = \frac{1}{2} \left[ g^{ab} R^{cd} + g^{cd} R^{ab} - g^{ac} R^{bd} - g^{ad} R^{bc} + \left( g^{ac} g^{bd} - \frac{1}{2} g^{ab} g^{cd} \right) \right] - 2 R^{acbd} \]

\[ E^{abcd} = \frac{1}{2} (\epsilon + p) u^a u^b u^c u^d + \frac{1}{2} p (4 g^{ab} g^{cd} - g^{ac} g^{bd} - g^{ad} g^{bc}) - \frac{1}{2} \nabla \cdot p q^a q^c \]

\[ \tilde{F}_1 = \{ -\frac{1}{2} \tilde{e} + \frac{1}{4} (\epsilon + 3p) + \epsilon^0_{\ 00} \} \]
\[ \tilde{F}_2 = \{ -\frac{1}{2} \tilde{e} - \frac{1}{2} (1 - \frac{1}{2} \Gamma_1) p + A^{11}_{\ 11} \} \]
\[ \tilde{F}_3 = \{ -\frac{1}{2} \tilde{e} + \frac{1}{2} \frac{1}{2} (1 - \frac{1}{2} \Gamma_1) p + A^{22}_{\ 22} \} \]
\[ \tilde{F}_4 = \{ -\frac{1}{2} \tilde{e} + \frac{1}{2} (1 - \frac{1}{2} \Gamma_1) p + A^{33}_{\ 33} \} \]

\[ a_1 = -A^{00}_{\ 11} - g^{00} g^{11} \left( \frac{1}{4} (3 \epsilon + p) \right), \]
\[ a_2 = -A^{00}_{\ 22} - g^{00} g^{22} \left( \frac{1}{4} (3 \epsilon + p) \right), \]
\[ b_1 = -A^{11}_{\ 00} - \frac{1}{2} g^{00} g^{11} \frac{p \Gamma_1}{1 + \frac{3}{2} \Gamma_1} (p + \frac{1}{2} \Gamma_1) \]
\[ b_2 = -A^{11}_{\ 22} \frac{g^{11}}{2} (1 + \frac{1}{2} \Gamma_1) p g^{11}, \]
\[ b_3 = -A^{11}_{\ 33} + \frac{1}{2} g^{11} g^{33} (1 + \frac{1}{2} \Gamma_1) \]
\[ c_1 = -A^{22}_{\ 00} + \frac{g^{22}}{2} (1 + \frac{1}{2} \Gamma_1) p g^{22}, \]
\[ c_2 = -A^{22}_{\ 11} + \frac{g^{22}}{2} (1 + \frac{1}{2} \Gamma_1) p g^{22}, \]
\[ c_3 = -A^{22}_{\ 33} + \frac{g^{22}}{2} (1 + \frac{1}{2} \Gamma_1) p g^{22}, \]
\[ d_1 = -A^{33}_{\ 00} + g^{33} \frac{1}{2} (1 + \frac{3}{2} \Gamma_1) p g^{33}, \]
\[ d_2 = -A^{33}_{\ 22} + g^{33} \frac{1}{2} (1 + \frac{1}{2} \Gamma_1) p g^{33}, \]
\[ d_3 = -A^{33}_{\ 33} + g^{33} \frac{1}{2} (1 + \frac{1}{2} \Gamma_1) p g^{33}, \]
\[ s_1 = A^{22}_{\ 22} + A^{22}_{\ 33} + \frac{1}{2} (1 - \frac{1}{2} \Gamma_1) \]
\[ s_2 = A^{22}_{\ 00} + \frac{g^{22}}{2} g^{00} (1 + \frac{3}{2} \Gamma_1) \]
\[ s_3 = A^{22}_{\ 11} + \frac{g^{22}}{2} g^{11} (1 + \frac{1}{2} \Gamma_1) \]
\[ s_4 = A^{33}_{\ 33} + \frac{1}{2} g^{33} p g^{22}, \]
\[ l_1 = \frac{1}{2} g^{33} p g^{22}, \]
\[ l_2 = -A^{33}_{\ 00} - \frac{1}{2} g^{33} p e^{22}, \]
\[ k_1 = a_1 + (a_2 + a_3) \frac{(s_2 - l_2)}{(s_1 - l_1)}, \]
\[ k_2 = \frac{(a_2 + a_3)}{(s_1 - l_1)} \]
\[ p_1 = b_1 + \frac{b_2 + b_3}{(s_1 - l_1)(s_1 - l_1)} \]
\[ p_2 = \frac{(s_2 - l_2)}{(s_1 - l_1)} \]
\[ m_1 = -g^{00} f_{11} - k_2 f_2, \]
\[ m_2 = -(g^{11} + p_{12}) f_2 \]

\[ S(x, y) = Q_2(x) Q_3(y) < \tau^{22}(x) \tau^{22}(y) > + < \tau^{33}(x) \tau^{33}(y) > - < \tau^{22}(x) \tau^{33}(y) > - < \tau^{33}(x) \tau^{22}(y) > \]

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