Digital Hurewicz Theorem and Digital Homology theory

Samira Sahar Jamil · Danish Ali

Abstract In this paper, we develop homology groups for digital images based on cubical singular homology theory for topological spaces. Using this homology, we present digital Hurewicz theorem for the fundamental group of digital images. We also show that the homology functors developed in this paper satisfy properties that resemble the Eilenberg-Steenrod axioms of homology theory, in particular, the homotopy and the excision axioms. We finally define axioms of digital homology theory.

Keywords digital topology · digital homology theory · digital Hurewicz theorem · cubical singular homology for digital images · digital excision

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1 Introduction

Digital images can be considered as objects in $\mathbb{Z}^n$, where $n$ is 2 for two dimensional images and $n$ is 3 for three-dimensional images. Though these objects are discrete in nature, they model continuous objects of the real world. Researchers are trying to understand whether or not digital images show similar properties as their continuous counterparts. The main motivation behind such studies is to develop a theory for digital images that is similar to the theory of topological spaces in classical topology. Due to discrete nature of digital images, it is difficult to get results that are analogous to those in classical topology. Nevertheless, great effort has been made by researchers to derive a theory that goes in line with general topology. Several notions that are well-studied in general topology and algebraic topology, have been developed for digital images, which include continuity of functions.

S. S. Jamil
Department of Mathematics, University of Karachi, Karachi, Pakistan
Department of Mathematical Sciences, Institute of Business Administration (IBA), Karachi, Pakistan E-mail: ssjamil@iba.edu.pk

D. Ali
Department of Mathematical Sciences, Institute of Business Administration (IBA), Karachi, Pakistan
In algebraic topology, homotopy groups, in particular the fundamental group (which is homotopy group in dimension 1) are important tools used to classify topological spaces. Homotopy groups serve similar purpose and provide a set of topological invariants, which according to Lefschetz, are the most profound and far-reaching creation in all topology. Homotopy groups are functors defined from the category of topological spaces $\text{Top}$ or from a subcategory of $\text{Top}$ to the category $\text{Ab}$ of Abelian groups (see [29] for definitions and details). The construction of homotopy groups is less geometric and less intuitive than that of homotopy groups, but homotopy groups are useful in solving many problems of geometric nature. There are several different but equivalent constructions of homotopy for topological spaces, out of which the well-known ones are simplicial, singular and cellular homology theories. Simplicial homology is the easiest to compute but limited to a class of topological spaces called polyhedra, while singular homology is powerful but complicated in terms of computations. Cubical singular homology discussed in [25], is much similar in construction to singular homology. Hurewicz theorem is one of the central results in algebraic topology, which relates homotopy groups of a topological space to its homology groups via a homomorphism called Hurewicz map. In 1945, Eilenberg and Steenrod [12] introduced an axiomatic approach for homology groups of a topological space. They presented what are now called as Eilenberg-Steenrod axioms, which are considered as a standard set of axioms satisfied by homology functors defined on category of topological pairs $\text{Top}$ [7]. Various homology theories on topological spaces satisfy this set of axioms and this has many interesting consequences. For example, the definition of singular homology is not very useful in computations of homology groups, while excision axiom, which is one of Eilenberg-Steenrod axioms, and its consequences are helpful in computations of homology groups of various spaces.

The idea of fundamental group was first introduced in the field of digital topology by Kong [19]. Boxer [3] adopted a classical approach to define and study fundamental group, which was closer to the methods of algebraic topology. Simplicial homology groups were introduced in the field of digital topology by Arslan et al. [1] and extended by Boxer et al. [6]. Eilenberg-Steenrod axioms for simplicial homology groups of digital images were investigated by Ege and Karaca in [8], where it is shown that all these axioms hold in digital simplicial setting except for homotopy and excision axioms. They demonstrate using an example that Hurewicz theorem does not hold in case of digital simplicial homology groups. Relative homology groups of digital images in simplicial setting were studied in [9]. Karaca and Ege [17] developed the digital cubical homology groups in a similar way as the cubical homology groups of topological spaces in algebraic topology. Unlike the case of algebraic topology, digital cubical homology groups are in general not isomorphic to digital simplicial homology groups studied in [6]. Furthermore, Mayer-Vietoris theorem fails for cubical homology on digital images, which is another contrast to the case of algebraic topology. In [23], singular homology group of digital images were developed.

We do not know of any homology theory on digital images that relates to the fundamental group of digital images developed in [3], as in algebraic topology. This is the main motivation behind work done in this paper. This paper is organized as
follows. We review some of the basic concepts of digital topology in Section 2. We develop homology groups of digital images based on cubical singular homology of topological spaces as given in [25] in Section 3 and give some basic results including the functoriality, additivity and homotopy invariance of cubical singular homology groups. In Section 4, we show that the fundamental group for digital images (given by [3]) is related to our first homology group, and obtain a result that is analogous to Hurewicz theorem of algebraic topology. In Section 5, we prove a result for cubical singular homology on digital images (Theorem 5.4) similar to the excision theorem of algebraic topology except that our result holds only in dimensions less than 3. This result is then generalized and we call this generalization ‘Excision-like property’ for cubical singular homology on digital images (Theorem 5.8). Cubical singular homology groups satisfy properties that are much similar to the Eilenberg-Steenrod axioms of homology theory. We define digital homology theory in Section 6, the axioms of which can be regarded as digital version of Eilenberg-Steenrod axioms in algebraic topology. We also show that cubical singular homology is a digital homology theory. Throughout this paper, we consider finite binary digital images, though most of the results also hold for infinite case.

2 Preliminaries

2.1 Basic concepts of digital topology

Let \( \mathbb{Z}^d \) be the Cartesian product of \( d \) copies of set of integers \( \mathbb{Z} \), for a positive integer \( d \). A digital image is a subset of \( \mathbb{Z}^d \). A relation that is symmetric and irreflexive is called an adjacency relation. In digital images, adjacency relations give a concept of proximity or closeness among its elements, which allows some constructions in digital images that closely resemble those in topology and algebraic topology. The adjacency relations on digital images used in this paper are defined below.

**Definition 2.1** [4] Consider a positive integer \( l \), where \( 1 \leq l \leq d \). The points \( p, q \in \mathbb{Z}^d \) are said to be \( c_l \)-adjacent if they are different and there are at most \( l \) coordinates of \( p \) and \( q \) that differ by one unit.

Usually the notation \( c_1 \) is replaced by number of points \( \kappa \) that are \( c_1 \)-adjacent to a point. For \( \mathbb{Z}^2 \), there are 4 points that are \( c_1 \)-adjacent to a point and there are 8 points that are \( c_2 \)-adjacent to a point, thus \( c_1 = 4 \) and \( c_2 = 8 \). Two points that are \( \kappa \)-adjacent to each other, are said to be \( \kappa \)-neighbors of each other. For \( a, b \in \mathbb{Z}, a < b \), a digital interval denoted as \([a, b]_{\mathbb{Z}}\) is a set of integers from \( a \) to \( b \), including \( a \) and \( b \). The digital image \( X \subseteq \mathbb{Z}^d \) equipped with adjacency relation \( \kappa \) is represented by the ordered pair \((X, \kappa)\).

**Definition 2.2** [19,3] Let \((X, \kappa)\) and \((Y, \lambda)\) be digital images.

(i) The function \( f : X \to Y \) is \((\kappa, \lambda)\)-continuous if for every pair of \( \kappa \)-adjacent points \( x_0 \) and \( x_1 \) in \( X \), either the images \( f(x_0) \) and \( f(x_1) \) are equal or \( \lambda \)-adjacent.

(ii) Digital images \((X, \kappa)\) and \((Y, \lambda)\) are said to be homeomorphic, if there is a \((\kappa, \lambda)\)-continuous bijection \( f : X \to Y \), which has a \((\lambda, \kappa)\)-continuous inverse \( f^{-1} : Y \to X \).
(iii) A $\kappa$-path in $(X, \kappa)$ is a $(2, \kappa)$-continuous function $f : [0,m]_\mathbb{Z} \to X$. We say $f$ is a $\kappa$-path of length $m$ from $f(0)$ to $f(m)$. For a given $\kappa$-path $f$ of length $m$, we define reverse $\kappa$-path $\overline{f} : [0,m]_\mathbb{Z} \to X$ defined by $\overline{f}(t) = f(m - t)$. A $\kappa$-loop is a $\kappa$-path $f : [0,1]_\mathbb{Z} \to X$, with $f(0) = f(m)$.

(iv) A subset $A \subset X$ is $\kappa$-connected if and only if for all $x, y \in A$, $x \neq y$, there is a $\kappa$-path from $x$ to $y$. A $\kappa$-component of a digital image is the maximal $\kappa$-connected subset of the digital image.

Proposition 2.3 If $f : X \to Y (X, \kappa)$ is a $(\kappa, \lambda)$-continuous function, with $A \subset X$ a $\kappa$-connected subset, then $f(A)$ is $\lambda$-connected in $Y$.

Definition 2.4

(i) Let $f, g : X \to Y$ be $(\kappa, \lambda)$-continuous functions. Suppose there is a positive integer $m$ and a function $H : [0,m]_\mathbb{Z} \times X \to Y$ such that:

- for all $x \in X$, $H(0,x) = f(x)$ and $H(m,x) = g(x)$,
- for all $x \in X$, the function $H_x : [0,m]_\mathbb{Z} \to Y$ defined by $H_x(t) = H(t,x)$ for all $t \in [0,m]_\mathbb{Z}$ is $(2, \lambda)$-continuous,
- for all $t \in [0,m]_\mathbb{Z}$, the function $H_t : X \to Y$ defined by $H_t(x) = H(t,x)$ for all $x \in X$ is $(\kappa, \lambda)$-continuous.

Then $H$ is called $(\kappa, \lambda)$-homotopy from $f$ to $g$ and $f$ and $g$ are said to be $(\kappa, \lambda)$-homotopic, denoted as $f \simeq_{(\kappa, \lambda)} g$. If $g$ is a constant function, $H$ is a null-homotopy and $f$ is null-homotopic.

(ii) Two digital images $(X, \kappa)$ and $(Y, \lambda)$ are homotopically equivalent, if there is a $(\kappa, \lambda)$-continuous function $f : X \to Y$ and $(\lambda, \kappa)$-continuous function $g : Y \to X$ such that $g \circ f \simeq_{(\kappa, \lambda)} 1_X$ and $f \circ g \simeq_{(\lambda, \kappa)} 1_Y$, where $1_X$ and $1_Y$ are identity functions on $X$ and $Y$, respectively.

(iii) Let $H : [0,m]_\mathbb{Z} \times [0,n]_\mathbb{Z} \to X$ be a homotopy between $\kappa$-paths $f, g : [0,n]_\mathbb{Z} \to X$ in $(X, \kappa)$. The homotopy $H$ is said to hold the end-points fixed if $f(0) = H(t,0) = g(0)$ and $f(n) = H(t,n) = g(n)$ for all $t \in [0,m]_\mathbb{Z}$.

2.2 Digital Fundamental group

The concept of fundamental group for digital images was first given by [19], but a more classical approach to define and study digital fundamental group was adopted by Boxer [3]. We briefly explain digital fundamental group as defined in the latter paper.

Definition 2.5

(i) A pointed digital image is a pair $(X, p)$, where $X$ is a digital image and $p \in X$. A pointed digital image $(X, p)$ can be represented as $((X, p), \kappa)$, if one wishes to emphasize the adjacency relation of the digital image $X$.

(ii) Let $f$ and $g$ be $\kappa$-paths of lengths $m_1$ and $m_2$, respectively, in the pointed digital image $(X, p)$, such that $g$ starts where $f$ ends, i.e. $f(m_1) = g(0)$. The ‘product’ $f \ast g$ of two paths is defined as follows:

$$(f \ast g)(t) = \begin{cases} f(t), & \text{if } t \in [0,m_1]_\mathbb{Z} \\ g(t - m_1), & \text{if } t \in [m_1, m_1 + m_2]_\mathbb{Z}. \end{cases}$$
The concept of trivial extension allows stretching the domain of a loop, without changing its homotopy class and thus allows to compare homotopy properties of paths even when the cardinalities of their domain differ.

**Definition 2.6** Let $f$ and $f'$ be $\kappa$-paths in a pointed digital image $(X, p)$. We say that $f'$ is a trivial extension of $f$, if there exist sets of $\kappa$-paths $\{f_1, f_2, \ldots, f_k\}$ and $\{f'_1, f'_2, \ldots, f'_n\}$ in $X$ such that
- $0 < k \leq n$
- $f = f_1 \ast f_2 \ast \cdots \ast f_k$
- $f' = f'_1 \ast f'_2 \ast \cdots \ast f'_n$
- there are indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that:
  - $f'_i = f_j, 1 \leq j \leq k$ and
  - $i \notin \{i_1, i_2, \ldots, i_k\}$ implies $f'_i$ is a constant $\kappa$-path.

(ii) Two $\kappa$-loops $f$ and $g$ with the same basepoint $p \in X$ belong to the same loop class, if there exist trivial extensions of $f$ and $g$, which have homotopy between them that holds the end-points fixed.

**Definition 2.7** Let $\Pi^\kappa_2(X, p)$ be the set of loop classes in $(X, p)$ with basepoint $p$. Let $[f]_n$ denote the loop class of $\kappa$-loop $f$ in $(X, \kappa)$. The product operation $\ast$ defined as:

$$[f]_n \ast [g]_n = [f \ast g]_n$$

is well defined on $\Pi^\kappa_2(X, p)$ as well as associative. The loop class $[c]_n$ of the constant loop is identity in $\Pi^\kappa_2(X, p)$ with respect to taking product. For every loop class $[f]_n$ the loop class $[\overline{f}]_n$, where $\overline{f}$ is the reverse path of $f$, is the inverse of $[f]_n$ with respect to taking product $\ast$. Thus $\Pi^\kappa_2(X, p)$ is a group under $\ast$ and called the digital fundamental group of the pointed digital image $(X, p)$.

### 3 Cubical Singular Homology on Digital images

Consider digital interval $I = [0, 1]^2$. Let $I^n$ be the Cartesian product of $n$ copies of $I$ for $n > 0$. We shall consider $I^n$ as a digital image $(I^n, 2n)$. By definition, $I^0$ is a digital image consisting of single point. For an integer $n \geq 0$, a digitally singular $n$-cube or briefly an $n$-cube in a digital image $(X, \kappa)$ is a $(2n, \kappa)$-continuous map $T : I^n \to X$.

For an integer $n \geq 0$, let $Q^n_0(X)$ denote the free Abelian group generated by the set of all digitally singular $n$-cubes in $(X, \kappa)$. We write $Q_n(X)$ for $Q^n_0(X)$, when the adjacency relation is clear from the context. An element of $Q_0(X)$ is a finite formal linear combination of $n$-cubes. The basis of the group $Q_0(X)$ can be identified with $X$ itself, and one can denote the elements of $Q_0(X)$ as $\sum m_i x_i$, where $x_i \in X$. A digitally singular $n$-cube $T : I^n \to X$ is degenerate if there is an integer $i$, $1 \leq i \leq n$ such that $T(t_1, t_2, \ldots, t_n)$ does not depend on $t_i$. Let $D^n_0(X)$, or simply $D_n(X)$, denote the subgroup of $Q_n(X)$ generated by the set of all degenerate digitally singular $n$-cubes in $(X, \kappa)$. Let $C^n_0(X)$, or simply $C_n(X)$, denote the quotient group $Q_n(X)/D_n(X)$. We say $C_n(X)$ is the group of digitally cubical singular $n$-chains in $(X, \kappa)$ and the elements of $C_n(X)$ are $n$-chains in $(X, \kappa)$. For any digital image $X$, $C_n(X)$ can be shown as free Abelian group generated by non-degenerate $n$-cubes in $X$. 
We define faces of a digitally singular $n$-cube as follows: For an $n$-cube $T : I^n \to X$ and $i = 1, 2, \ldots, n$, we define $(n-1)$-cubes $A_i T, B_i T : I^{n-1} \to X$ as

$$A_i T(t_1, t_2, \ldots, t_{n-1}) = T(t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{n-1}),$$

$$B_i T(t_1, t_2, \ldots, t_{n-1}) = T(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_{n-1}).$$

$A_i T$ and $B_i T$ are called front $i$-face and back $i$-face of $T$, respectively.

We define the boundary operator $\partial_n$ on the basis element of $Q_n(X)$ as $\partial_n(T) = \sum_{i=1}^{n} (-1)^i (A_i T - B_i T)$ and extend it by linearity (see [29], for the definition of extension by linearity) to get the homomorphism $\partial_n : Q_n(X) \to Q_{n-1}(X)$, $n \geq 1$. One may write $\partial$ for $\partial_n$ if $n$ is clear from the context. For $n < 0$, let $Q_n(X) = C_n(X) = 0$ and for $n \leq 0$ let $\partial_n = 0$. It can be shown that $\partial_{n-1} \partial_n = 0$, for all integers $n$ (see [25] for details). A cubical singular complex of the digital image $(X, \kappa)$, denoted as $(C^n(X), \partial)$ or $(C_n(X), \partial)$, is the following chain complex:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Let $Z_n(X)$ denote the kernel of $\partial_n$ and $B_n(X)$ denote the image of $\partial_{n+1}$, for all integers $n$. The elements of $Z_n(X)$ and $B_n(X)$ are called $n$-cycles and $n$-boundaries of $(X, \kappa)$, respectively. We define $n^{th}$ cubical singular homology group of the digital image $(X, \kappa)$, as $H_n(X) = H_n(X, \kappa) = Z_n(X)/B_n(X)$, for all non-negative integers $n$. If the adjacency relation $\kappa$ is clear from context, we shall simply write $H_n(X)$ for $H_n(X, \kappa)$.

**$\kappa$-path and 1-cubes:** A 1-cube $T : I \to X$ in a digital image $(X, \kappa)$ can be considered as a $\kappa$-path of length 1. A $\kappa$-path $f$ of length $m$ can be “subdivided” into smaller paths of length 1 or 1-cubes. For a $\kappa$-path $f$ of length $m$, we can associate an element $\sum_{j=1}^{m} f_j$ of $Q_1(X)$ to $f$, where $f_j : I \to X$ as $f_j(t) = f(j + t - 1)$. We say that the element $\sum_{j=1}^{m} f_j$ is subdivision of $f$. Following are some properties of subdivision $\sum_{j=1}^{m} f_j$ of $f$:

1. $f_j$ are degenerate, whenever $f(j + 1) = f(j)$
2. If $f$ is a non-constant path then $\sum_{j=1}^{m} f_j$ is not degenerate, and so $\sum_{j=1}^{m} f_j$ is a nontrivial element in $C_1(X)$, where some $f_j$ might be 0 in $C_1(X)$.
3. $\partial \left( \sum_{j=1}^{m} f_j \right) = f(m) - f(0)$.
4. If $f$ is a $\kappa$-loop then $\sum_{j=1}^{m} f_j$ is a 1-cycle.

**Proposition 3.1** Let $(X, \kappa)$ be a non-empty $\kappa$-connected digital image, then $H_0(X) \approx \mathbb{Z}$.

**Proof.** Consider the map $\varepsilon : C_0(X) \to \mathbb{Z}$ defined as $\sum_i m_i x_i \mapsto \sum_i m_i$. Now for $\sum_i n_i T_i \in C_1(X)$, we have $\varepsilon \circ \partial(\sum_i n_i T_i) = \varepsilon(\sum_i n_i (B_i T_i - A_i T_i)) = \sum_i (n_i - n_i) = 0$. Thus $B_0(X) \subset \ker(\varepsilon)$. The reverse relation also holds for the following reason. Consider $\sum_i m_i x_i \in \ker(\varepsilon)$. We have $\sum_i m_i = 0$. Consider $x \in X$ (X is non-empty) and $\kappa$-paths $f_i$ (X is $\kappa$-connected) from $x$ to $x_i$. These paths can be subdivided to form elements $\sum_i f_{ij} \in C_1(X)$ for each $i$. It can be verified that $\partial(\sum_i f_{ij}) = x_i - x$. Thus $\partial(\sum_i m_i x_i) = \sum_i m_i x_i - (\sum_i m_i) x = \sum_i m_i x_i$, implying $\sum_i m_i x_i \in B_0(X)$. From first isomorphism theorem of groups $H_0(X) = Z_0(X)/B_0(X) = C_0(X)/B_0(X) \approx \mathbb{Z}$. □
Proposition 3.2 Let \( \{X_\alpha | \alpha \in A \} \) be the set of \( \kappa \)-components of the digital image \((X, \kappa)\). Then \( H_n(X) \approx \bigoplus_\alpha H_n(X_\alpha) \).

Proof. The groups \( Q_n(X), D_n(X) \) and \( C_n(X) \) break up to \( \bigoplus_\alpha Q_n(X_\alpha), \bigoplus_\alpha D_n(X_\alpha) \) and \( \bigoplus_\alpha C_n(X_\alpha) \), respectively, because, the image of each \( n \)-cube \( T \) lies entirely in one \( \kappa \)-component of \((X, \kappa)\) (see Section 3.2). We also have \( Z_n(X) = \bigoplus_\alpha Z_n(X_\alpha) \) and \( B_n(X) = \bigoplus_\alpha B_n(X_\alpha) \), and hence \( H_n(X) = \bigoplus_\alpha H_n(X_\alpha) \), because the boundary map \( \partial_\alpha : C_n(X_\alpha) \to C_{n-1}(X_\alpha) \) maps \( C_n(X_\alpha) \) to \( C_{n-1}(X_\alpha) \).

Proposition 3.3 For any digital image \((X, \kappa)\), \( H_0(X) \) is a free Abelian group with rank equal to the number of \( \kappa \)-components of \((X, \kappa)\).

Proof. Follows from Propositions 3.1 and 3.2.

Proposition 3.4 The cubical singular homology group \( H_n(\cdot) \) is a functor from \( \text{Dig} \) to \( \text{Ab} \).

Proof. We define \( H_n(-) \) on morphisms of \( \text{Dig} \) as follows: Consider a \((\kappa, \lambda)\)-continuous function \( f : X \to Y \) from digital image \((X, \kappa)\) to digital image \((Y, \lambda)\). For an \( n \)-cube \( T : I^n \to X \) in \( Q_n(X) \), we have \( f \circ T \in Q_n(Y) \). We define functions \( f_\# : Q_n(X) \to Q_n(Y) \) as \( T \mapsto f \circ T \) and extending by linearity, for integers \( n \geq 0 \). Since \( f_\#(T) \) is degenerate, if \( T \in D_n(X) \), the map \( f_\# \) induces \( f_\# : C_n(X) \to C_n(Y) \), for integers \( n \geq 0 \). It can be shown that \( f_\# \) is a chain map that sends \( n \)-cycles to \( n \)-cycles and \( n \)-boundaries to \( n \)-boundaries, and therefore induces a map \( f_* = H_n(f) : H_n(X) \to H_n(Y) \) defined as \( [T] \mapsto [f_\#(T)] \).

Furthermore, it can be easily shown that for an identity map \( \text{id} : X \to X \) the induced map \( \text{id}_* = H_n(\text{id}) : H_n(X) \to H_n(X) \) is an identity map. Also for functions \( f : X \to Y \) and \( g : Y \to Z \), which are \((\kappa, \lambda)\)- and \((\lambda, \gamma)\)-continuous, we have \( (g \circ f)_* = g_* \circ f_* : H_0(X) \to H_0(Z) \), because \( (g \circ f)_\# = g_\# \circ f_\# : Q_0(X) \to Q_0(Z) \).

The following can be easily proved.

Proposition 3.5 Let \((X, \kappa)\) and \((Y, \lambda)\) be \((\kappa, \lambda)\)-homeomorphic digital images, then \( H_n(X) = H_n(Y) \), for all \( n \).

Proposition 3.6 If \( X = \{x_0\} \) is a one-point digital image, then

\[
H_n(X) = \begin{cases} 
\mathbb{Z}, & \text{if } n = 0 \\
0, & \text{otherwise} 
\end{cases}
\]

Theorem 3.7 Let \( f, g : X \to Y \) be \((\kappa, \lambda)\)-homotopic maps from digital image \((X, \kappa)\) to the digital image \((Y, \lambda)\). Then \( f \) and \( g \) induce the same maps on homology group \( H_n(X) \), i.e. \( f_* = g_* \).

Proof. Let \( F : [0, m] \times X \to Y \) be the homotopy from \( f \) to \( g \). The homotopy \( F \) can be subdivided into functions \( F_j : I \times X \to Y \) defined as \( F_j(t, x) = F(j + t - 1, x) \) for \( j \in [1, m] \). Observe that \( F_1(0, x) = f(x) \) and \( F_m(1, x) = g(x) \). In order to show that \( f_* = g_* \), we follow the standard method of algebraic topology, which is, to construct a map \( \Phi_n : Q_n(X) \to Q_{n+1}(Y) \) that contains similar information as the Homotopy \( F \), and satisfies:

\[
g_\# - f_\# = \partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n
\]
Define $\Phi_n : Q_n(X) \rightarrow Q_{n+1}(Y)$ as $T \mapsto \sum_{j=1}^{m} F_j(id \times T)$ and extending by linearity, where $id : [0,1]_Z \rightarrow [0,1]_Z$ is identity function. We need to compute the boundary $\partial \Phi$ to verify eq. \ref{eq:boundary}. One can observe the following:

\begin{align}
A_i \Phi_n(T) &= f_\# (T) + \sum_{j=2}^{m} F_j(0, T) \quad \text{and} \quad B_i \Phi_n(T) = \sum_{j=1}^{m-1} F_j(1, T) + g_\#(T) \quad \text{(2)}
\end{align}

\begin{align}
A_i \Phi_n(T) &= \Phi_{n-1} A_{i-1} T, \quad \text{and} \quad B_i \Phi_n(T) = \Phi_{n-1} B_{i-1} T, \quad i \in [2, n+1]_Z \quad \text{(3)}
\end{align}

\begin{align}
F_j(1, T) &= F_{j+1}(0, T), \quad j \in [1, m-1]_Z \quad \text{(4)}
\end{align}

Using these equations we can calculate the boundary of $\Phi$:

\begin{align}
\partial \Phi_n(T) &= \sum_{i=1}^{n+1} (-1)^i (A_i \Phi_n(T) - B_i \Phi_n(T)) \\
&= g_\#(T) - f_\#(T) + \sum_{i=2}^{n+1} (-1)^i (A_i \Phi_n(T) - B_i \Phi_n(T)) \\
&\quad \text{using eqs. (2) and (4) for } i = 1, \text{ and using eqs. (3)} \\
&\quad \text{and substituting } j = i - 1 \text{ for } i > 1 \\
&= g_\#(T) - f_\#(T) - \Phi_{n-1} \partial T \quad \text{by definition of } \partial(T)
\end{align}

It can be shown that $\Phi$ maps degenerate $n$-cubes in $(X, \kappa)$ to degenerate $n+1$-cubes in $(Y, \lambda)$, inducing a homomorphism $\phi_n : C_n(X) \rightarrow C_{n+1}(Y)$. If we choose $T$ to be a non-degenerate $n$-cycle, i.e. $T \in Z_n(X)$, then we get $g_\#(T) - f_\#(T) \in B_n(Y)$.

Therefore in $H_n(Y)$ we have,

\begin{align}
[g_\#(T) - f_\#(T)] = g_\#([T]) - f_\#([T]) = 0 \Rightarrow g_* = f_*
\end{align}

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\text{\hfill \small \textcircled{$\square$}}
\end{flushright}

**Corollary 3.8** If $(X, \kappa)$ and $(Y, \lambda)$ be homotopically equivalent digital images, then $H_n(X) \approx H_n(Y)$.

**Proof.** Follows from Proposition \ref{prop:main} and functoriality of $H_n$.

\begin{flushright}
\text{\hfill \small \textcircled{$\square$}}
\end{flushright}

**Example 3.9** A digital image is said to be $\kappa$-contractible \cite{3}, if its identity map is $(\kappa, \kappa)$-homotopic to a constant function $c_p$ for some $p \in X$. For a $\kappa$-contractible digital image $(X, \kappa)$, one can compute the homology groups using Propositions \ref{prop:homology} and \ref{prop:boundary} as $H_n(X) = \begin{cases} Z, \quad \text{if } n = 0 \\ 0, \quad \text{otherwise,} \end{cases}$ because a $\kappa$-contractible digital image is homotopy equivalent to a point \cite{3}.
4 Digital Hurewicz theorem

**Lemma 4.1** Let \((X, p, \kappa)\) be a digital image with basepoint \(p\) and \(\kappa\)-adjacency relation and \(\Pi^\kappa_1(X, p)\) be the fundamental group. Then there is a homomorphism \(\phi : \Pi^\kappa_1(X, p) \to H_1(X)\) given by \([f]_n \mapsto \left[\sum_{j=1}^{m} f_j\right] \), where \(\sum_{j=1}^{m} f_j\) is the subdivision of \(\kappa\)-loop \(f\).

**Proof.** Well-defined: We need to show that \(\phi\) is a well-defined. Consider \(\kappa\)-loops \(f\) and \(g\) of lengths \(m_1\) and \(m_2\), respectively, both based at point \(p \in X\) such that \([f]_n = [g]_n \in \Pi^\kappa_1(X, p)\). Now \(f\) and \(g\) are in the same loop class implies that there are trivial extensions \(f'\) and \(g'\) of \(f\) and \(g\), respectively such that there exists a homotopy \(H : [0, m]_\mathbb{Z} \times [0, M]_\mathbb{Z} \to X\) with: \(H(0, s) = f'(s), H(1, s) = g'(s)\), for all \(s \in [0, M]_\mathbb{Z}\), the function \(H_t(s) = H(t, s)\) for all \(t \in [0, m]_\mathbb{Z}\) is \((2, \kappa)\)-continuous for all \(t \in [0, m]_\mathbb{Z}\), the function \(H_1(s) = H(t, s)\) for all \(s \in [0, M]_\mathbb{Z}\) is \((2, \kappa)\)-continuous, where \(M\) is the length of the trivial extensions \(f'\) and \(g'\). Subdivide \(H_1\) into 2-cubes \(H_{j,k} : I^2 \to X\) defined as \((s, t) \mapsto H(j + s - 1, k + t - 1)\), for \(j \in [1, m]_\mathbb{Z}\) and \(k \in [1, M]_\mathbb{Z}\) (see Figure 1). We shall show that the boundary \(\partial \left(\sum_{j,k} H_{j,k}\right)\) is equal to the difference of \(\sum_{j=1}^{m_1} f_j\) and \(\sum_{j=1}^{m_2} g_j\), which implies that the classes of these subdivisions are equal in the homology group \(H_1(X)\). Before computing \(\partial \left(\sum_{j,k} H_{j,k}\right)\), note that the following equations hold:

\[
\begin{align*}
M \sum_{k=1}^{M} A_1 H_{1,k} &= \sum_{j=1}^{M} f'_j = \sum_{j=1}^{m_1} f_j \quad \text{and} \quad M \sum_{k=1}^{M} B_1 H_{1,k} = \sum_{j=1}^{M} g'_j = \sum_{j=1}^{m_2} g_j \quad (5)
\end{align*}
\]

The only difference between \(f\) and its trivial extension \(f'\) is that \(f'\) pauses more frequently for rest than \(f\) and whenever a path pauses for rest, its subdivision is trivial at that point in \(C_1(X)\) (being degenerate in \(Q_1(X)\)).

\[
\begin{align*}
A_1 H_{j,k} &= B_1 H_{j-1,k}, \quad j \in [2, m]_\mathbb{Z}, k \in [1, M]_\mathbb{Z} \\
A_2 H_{j,k} &= B_2 H_{j-1,k}, \quad j \in [1, m]_\mathbb{Z}, k \in [2, M]_\mathbb{Z} \quad (6)
\end{align*}
\]

\[
\begin{align*}
A_2 H_{j,1} &= B_2 H_{j,M} = c_p, \quad j \in [1, m]_\mathbb{Z}, \quad (7)
\end{align*}
\]

where \(c_p\) is the constant path of length 1 at basepoint \(p \in X\). Using eqs. [5] to [7] it can be shown that \(\partial \left(\sum_{j,k} H_{j,k}\right) = \sum_{j=1}^{m_2} g_j - \sum_{j=1}^{m_1} f_j \in C_1(X)\). This proves that \(\phi\) is well-defined.

**Homomorphism:** Consider \(\kappa\)-loops \(f\) and \(g\) of lengths \(m_1\) and \(m_2\), respectively, both based at point \(p \in X\). Then

\[
\phi([f]_n \ast [g]_n) = \phi([f \ast g]_n) = \left[\sum_{j=1}^{m_1+m_2} (f \ast g)_j\right] = \left[\sum_{j=1}^{m_1} (f \ast g)_j + \sum_{j=m_1+1}^{m_1+m_2} (f \ast g)_j\right]
\]

\[
= \left[\sum_{j=1}^{m_1} f_j + \sum_{j=1}^{m_2} g_j\right] = \left[\sum_{j=1}^{m_1} f_j\right] + \left[\sum_{j=1}^{m_2} g_j\right] = \phi([f]_n) + \phi([g]_n)
\]

\[
\square
\]

We say that the map \(\phi\) defined in Lemma 4.1 is Digital Hurewicz map.
Fig. 1 Domain of $H$ (proof of Lemma 4.1)
(a) Subdivision of $H$ into 2-cubes $H_{jk}$ (b) 1-cubes involved in $\partial(H_{jk})$

(a) Domain of $H$ with images labeled on each pixel (b) Schematic representation of $T$ (c) Domain of $H$ with images labeled on each pixel

Lemma 4.2 Let $(X, \kappa)$ be a digital image.

1. Consider a singular 1-cube $T \in C_1(X)$ and let $\overline{T}$ denote the ‘reverse’ of $T$, i.e. $\overline{T} \in C_1(X)$, $\overline{T}(t) = T(1 - t)$. Then class of $T + \overline{T}$ is trivial in $H_1(X)$.

2. Consider singular 2-cube $T \in C_2(X)$ in $X$ be a singular 2-cube in $X$. Define $\kappa$-paths $T_0, T_1, T_2$ and $T_3$ to be $A_1T, A_2T, B_1T$ and $B_2T$, respectively. Then $T_0$ is homotopic to $T_1 * T_2 * T_3$.

Proof.

1. Let $S : I^2 \rightarrow X$ be a basis element of $C_2(X)$ defined as $S(t, 0) = T(t)$ and $S(t, 1) = T(0)$, for $t = 0, 1$ (see Figure 2(a)). Note that the back 1-face $B_1S = T$ (see Figure 2(a)) and thus the boundary $\partial S = T + \overline{T}$ in $C_1(X)$ making the class of $T + \overline{T}$ trivial in $H_1(X)$.

2. Consider the homotopy $H$ defined as $H : [0, 3]Z \times I \rightarrow X$ as $H(0, 0) = T_0(0)$, $H(1, 0) = T_2(0)$, $H(2, 0) = T_3(1)$, $H(3, 0) = T_3(0)$, $H(0, 1) = H(1, 1) = T_0(0)$, $H(2, 1) = H(3, 1) = T_0(1)$ (see Figure 2(b) and (c)).

Clearly, $H(t, 0) = T_1 * T_2 + T_3(t)$ and $H(t, 1)$ is a trivial extension of $T_0$. 

Fig. 2 (a) 2-cube $S$, and (b) 2-cube $T$, (c) Homotopy $H$ (proof of Lemma 4.2)

(a) Domain of $S$ with images labeled on each pixel (b) Schematic representation of $T$ (c) Domain of $H$ with images labeled on each pixel
The following Lemma (quoted from [29] with some minor changes) is required in the proof of digital Hurewicz theorem (Theorem 4.4).

**Lemma 4.3** Substitution Principle
Let $F$ be a free Abelian group with basis $B$, let $x_0, x_1, \ldots, x_N$ be a list of elements in $B$, possibly with repetitions and assume that $\sum_{i=0}^{k} m_i x_i = \sum_{i=k+1}^{N} m_i x_i$, where $m_i \in \mathbb{Z}$ and $0 \leq k < N$. If $G$ is any Abelian group and $y_0, y_1, \ldots, y_N$ is a list of elements in $G$ such that $x_i = x_j \Rightarrow y_i = y_j$, then $\sum_{i=0}^{k} m_i y_i = \sum_{i=k+1}^{N} m_i y_i$ in $G$.

**Proof.** Define a function $\eta : B \to G$ with $\eta(x_i) = y_i$ for all $i = 1, 2, \ldots, N$ and $\eta(x) = 0$, otherwise ($\eta$ is well-defined because of the given hypothesis). Extend the map $\eta$ by linearity to $\sigma : F \to G$. Thus $0 = \eta \left( \sum_{i=0}^{k} m_i x_i - \sum_{i=k+1}^{N} m_i x_i \right) = \sum_{i=0}^{k} m_i y_i - \sum_{i=k+1}^{N} m_i y_i$. \qed

**Theorem 4.4** Digital Hurewicz Theorem
If $(X, \kappa)$ is a $\kappa$-connected digital image with $p \in X$ then the digital Hurewicz map (defined in Lemma 4.1) is surjective with ker$\phi$ as commutator subgroup of the fundamental group $\Pi_1(X, p)$. Hence, Abelianized Fundamental group is isomorphic to $H_1(X)$.

**Proof.** Surjectivity: Consider $[z] \in H_1(X)$, with $z = \sum_{i=0}^{m} n_i T_i$, where $T_i : I \to X$ is a non-degenerate 1-cube, for all $i$. Though $n_i \in \mathbb{Z}$, we can assume, without loss of generality, that $n_i = 1, \forall i$, for the following reason: If $n_i = 0$, no contribution is made to $z$ by $n_i T_i$ and if $n_i < 0$ then we can replace $n_i T_i$ by $-n_i T_i$ without changing the class $[z]$, using Lemma 4.2(1). Thus we can assume $n_i > 0, \forall i$, but then each $n_i T_i$ can be written as $T_i + T_i + \cdots + T_i$ ($n_i$ terms). Therefore, $z = \sum_{i=0}^{m} T_i$. Since $z$ is a cycle, we have

$$\partial z = \partial \left( \sum_{i=0}^{m} T_i \right) = 0 \quad \Rightarrow \quad \sum_{i=0}^{m} (B_1 T_i - A_1 T_i) = 0.$$  \hspace{1cm} (8)

For every $i \in [0, m]$, there exists $j \in [0, m]$ and $B_1 T_i = A_1 T_j$, so that the sum in eq. 8 is 0, but $i \neq j$, because in case $i = j$, $T_i$ would be degenerate. Let $\rho$ be the permutation on elements of $[0, m]$, satisfying the condition that $A_1 T_{\rho(i+1)} = B_1 T_{\rho(i)}$ for all $i \in [0, M]$, where arguments of $\rho$ are read mod $(M + 1)$. We can take product of $\kappa$-paths $T_{\rho(i)}$ to get a $\kappa$-loop $\prod_{i=0}^{m} T_{\rho(i)}$ based at point $T_{\rho(0)}(0) \in X$. Since the digital image $(X, \kappa)$ is $\kappa$-connected, we can take $\kappa$-path $\sigma$ from $p$ to $T_{\rho(0)}(0)$. We get:

$$\phi \left( \left[ \sigma \prod_{i=0}^{m} T_{\rho(i)} \right] \sigma \right) = \left[ \sum_{l=1}^{M} \sigma_l + \sum_{i=0}^{m} T_{\rho(i)} + \sum_{l=1}^{M} \sigma_l \right] = \left[ \sum_{l=1}^{M} \sigma_l + \sum_{i=0}^{m} T_{\rho(i)} - \sum_{l=1}^{M} \sigma_l \right], \text{ using Lemma 4.2(1)}$$

$$= \sum_{i=0}^{m} T_i = [z].$$

Kernel of $\phi$: Let $\Pi'$ denote the commutator subgroup of $\Pi_1^f(X,p)$ and $\Pi$ denote the Abelianized fundamental group, i.e. $\Pi$ is the quotient group $\Pi_1^f(X,p)$ modulo the commutator subgroup $\Pi'$. Since $H_1(X)$ is an Abelian group, $\Pi' \subset \ker \phi$. We claim that the reverse inequality also holds. Consider a $\kappa$-loop $f$ of length $m$ such that $[f]_n \in \ker \phi$. It suffices to show that $[f]$ is identity in $\Pi$, where $[f] \in \Pi$. Since $\phi([f]_n) = 0$, the cycle $\sum_{j=1}^m f_j$ lies in the boundary group $B_1(X)$, i.e. there is $N \in \mathbb{Z}$ and $T_i : j^2 \to X$ are 2-cubes. We assume without loss of generality that $n_i = 1, \forall i$. Let's denote $A_1T_i, A_2T_i, B_1T_i$ and $B_2T_i$ as $T_i, T_{i1}, T_{i2}$ and $T_{i3}$, respectively, for $i \in [1, N]$. We get

$$\sum_{j=1}^m f_j = \sum_{i=1}^M (-T_{i0} + T_{i2} + T_{i1} - T_{i3})$$  \hspace{1cm} (9)

This equation has basis elements of the free Abelian group $C_1(X)$ on both sides. We shall apply substitution principle (Lemma 4.3) to obtain an analogous equation in $\Pi$. We need for each term in eq. 9 an element in $\Pi$, satisfying the hypothesis of substitution principle. For each $x \in X$, choose a $\kappa$-path from $p$ to $x$, denoted by $\beta_x$, such that for the base point $p$, $\beta_p = c_p$ is a constant $\kappa$-path at $p$. For each $j \in [0, m] \mathbb{Z}$, define $\kappa$-loops, $L_j = \beta_{f(j-1)} * f_j * \overline{\beta_f(j)}$ based at $p$ corresponding to each $f_j$ (see Figure 3(a)). Similarly, define $\kappa$-loops $L_{iq} = \beta_{T_{iq}(0)} * T_{iq} * \overline{\beta_{T_{iq}(1)}}$ based at $p$, corresponding to each $T_{iq}$ (see Figure 3(b)). We get the following in $\Pi_1^f(X,p)$:

$$[\Pi_0 \ast L_1 \ast L_{i2} \ast L_{i3}]_n$$

$$= [\beta_{T_{iq}(0)} * T_{i0} * \overline{T_{iq}(0)} * \beta_{T_{i1}(0)} * T_{i1} * \beta_{T_{i2}(0)} * T_{i2} * \beta_{T_{i3}(0)} * T_{i3} * \overline{T_{iq}(0)}]_n$$

$$= [\beta_{T_{i1}(0)} * T_{i0} * T_{i1} * T_{i2} * T_{i3} * \overline{T_{iq}(0)}]_n, \text{ using Lemma 4.2}$$

$$= [\beta_{T_{i1}(0)} * \overline{T_{iq}(0)}]_n = [c_p]_n$$  \hspace{1cm} (10)

Second equality above follows because $T_{i0}(0) = T_{i3}(0) \Rightarrow \beta_{T_{i0}(0)} = \beta_{T_{i3}(0)}$. $T_{i1}(1) = T_{i2}(0) \Rightarrow \beta_{T_{i1}(1)} = \beta_{T_{i2}(0)}$, $T_{i2}(1) = T_{i3}(1) \Rightarrow \beta_{T_{i2}(1)} = \beta_{T_{i3}(1)}$ and $T_{i3}(0) = T_{i0}(1) \Rightarrow \beta_{T_{i3}(0)} = \beta_{T_{i0}(1)}$. Similarly, $[\Pi_0 \ast \beta_{f(j-1)} * f_j * \overline{f_f(j)}]_n = [\Pi_0 \ast f_j]_n = [f]_n$ in $\Pi_1^f(X,p)$, because $\beta_{f(0)} = \overline{\beta_f(m)}$ is the constant path $c_p$ at $p$. Therefore, we get the following in $\Pi$,

$$[f] = \left[ \prod_{j=1}^m f_j \right] = \left[ \prod_{j=1}^m \beta_{f(j-1)} * f_j * \overline{f_f(j)} \right]$$

$$= \left[ \prod_{i=1}^M L_{i0} \ast L_{i1} \ast L_{i2} \ast L_{i3} \right]$$

by applying substitution principle (Lemma 4.3) to eq. 9 for the free Abelian group $C_1(X)$ and the multiplicative Abelian group $\Pi$. Using eq. 10 $[f]$ is trivial in $\Pi$ and
Fig. 3 Schematic representation of paths $\beta_x$ (proof of Theorem 4.4)

Paths $\beta_x$ are shown in blue color (a) from $p$ to $T_{i0}(1)$, $T_{i1}(0)$, $T_{i2}(0)$ and $T_{i3}(1)$, and (b) from $p$ to $f(j), j \in [0, m - 1]_\mathbb{Z}$.

$$[f]_n \in \Pi'$$ Therefore, the kernel of the digital Hurewicz map is the commutator of $\Pi'_1(X, p)$, and $\Pi \approx H_1(X)$, using first isomorphism theorem of groups. □

5 Relative Homology and Excision

For a digital image $(X, \kappa)$ and $A \subset X$, $(A, \kappa)$ is a digital image in its own right. Let $((X, A), \kappa)$ or briefly, $(X, A)$ denote digital image pair with $\kappa$-adjacency. A map of pairs $f : (X, A) \rightarrow (Y, B)$ between digital image pairs $((X, A), \kappa)$ and $((Y, B), \lambda)$ is a map $f : X \rightarrow Y$, with $f(A) \subset B$. We say that $f : (X, A) \rightarrow (Y, B)$ is $(\kappa, \lambda)$-continuous if $f : X \rightarrow Y$ is $(\kappa, \lambda)$-continuous. It can be verified that $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ maps $C_n(A)$ to $C_{n-1}(A)$. If $C_n(X, A)$ denotes the quotient group $C_n(X)/C_n(A)$, then $\partial_n$ induces homomorphism $\partial_n : C_n(X, A) \rightarrow C_{n-1}(X, A)$ satisfying $\partial_{n-1} \circ \partial_n = 0$, and making up a chain complex $(C_\bullet(X, A), \partial)$, given as:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \cdots$$

Lets denote the homology of this chain complex as $H_n(X, A)$, i.e.

$$H_n(X, A) = \frac{\ker(\partial_n : C_n(X, A) \rightarrow C_{n-1}(X, A))}{\operatorname{Im}(\partial_{n+1} : C_{n+1}(X, A) \rightarrow C_n(X, A))}$$

We say that $H_n(X, A)$ is relative cubical singular homology group of the digital image pair $(X, A)$. Clearly, $H_n(X) = H_n(X, \emptyset)$. 
5.1 Excision-like property for digital images

**Definition 5.1** Let \((X, \kappa)\) be a digital image. We define operators \(\text{Int}_\kappa : \mathcal{P}(X) \to \mathcal{P}(X)\) and \(\text{Cl}_\kappa : \mathcal{P}(X) \to \mathcal{P}(X)\) as follows:

\[
\text{Int}_\kappa(A) = \{ x \in A \mid N_\kappa(x, X) \subset A \}, \\
\text{Cl}_\kappa(A) = \{ x \in X \mid N_\kappa(x, X) \cap A \neq \emptyset \},
\]

where \(N_\kappa(x, X) = \{ y \in X \mid x \text{ is } \kappa\text{-adjacent or equal to } y \}\).

We say that \(\text{Int}_\kappa(A)\) is \(\kappa\)-interior of \(A\) in \((X, \kappa)\) and \(\text{Cl}_\kappa(A)\) is \(\kappa\)-closure of \(A\) in \((X, \kappa)\) and the set \(N_\kappa(x, X)\) is neighborhood of \(x\) in \((X, \kappa)\).

Notions similar to above appear in [20] and [13] and also, the \(\kappa\)-interior and \(\kappa\)-closure operators defined above are very closely related to dilation and erosion operators, respectively, used in [13]. The following proposition shows that these operators satisfy many relations that are similar to those satisfied by their counterparts in topology.

**Proposition 5.2** Let \((X, \kappa)\) be a digital image, \(A, B \subset X\) and \(x, y \in X\). Then:

(i) \(A \subset \text{Cl}_\kappa(A), \text{Int}_\kappa(A) \subset A\)

(ii) \(\text{Int}_\kappa(X - A) = X - \text{Cl}_\kappa(A), X - \text{Int}_\kappa(A) = \text{Cl}_\kappa(X - A)\)

(iii) \(A \subset B \Rightarrow \text{Cl}_\kappa(A) \subset \text{Cl}_\kappa(B)\) and \(\text{Int}_\kappa(A) \subset \text{Int}_\kappa(B)\)

(iv) \(X = \text{Int}_\kappa(A) \cup \text{Int}_\kappa(B) \Leftrightarrow \text{Cl}_\kappa(X - B) \subset \text{Int}_\kappa(A)\)

**Proof.** The proofs are simple and follow easily from Definitions 5.1.

The \(\kappa\)-interior and \(\kappa\)-closure operators for digital images are not idempotent, i.e. \(\text{Int}_\kappa \circ \text{Int}_\kappa \neq \text{Int}_\kappa\) and \(\text{Cl}_\kappa \circ \text{Cl}_\kappa \neq \text{Cl}_\kappa\), unlike interior and closure operators in topology, as shown in the following example.

**Example 5.3** Consider the digital image \((X, 4)\) and \(A \subset X\) shown in Figure 4(a). The interiors \(\text{Int}_4(A)\) and \(\text{Int}_4^2(A) = \text{Int}_4(\text{Int}_4(A))\) are shown in Fig 4(b) and (c), respectively, and the closures \(\text{Cl}_4(A)\) and \(\text{Cl}_4^2(A) = \text{Cl}_4(\text{Cl}_4(A))\) in \(X\) in Figure 5(a) and (b), respectively. Clearly, \(\text{Int}_4 \circ \text{Int}_4(A) \neq \text{Int}_4(A)\) and \(\text{Cl}_4 \circ \text{Cl}_4(A) \neq \text{Cl}_4(A)\).

---

**Fig. 4** (a) Digital image \((X, 4)\), its subset \(A\) and (b) \(\text{Int}_4(A)\), (c) \(\text{Int}_4^2(A)\)

Digital image \(X\), \(A\) and interiors are shown in blue, dark blue and grey color, respectively.
The following theorem, which is like Excision axiom of homology theory holds only to dimensions upto 2.

**Theorem 5.4** [[Excision in dimension $n \leq 2$]]

Let $(X, \kappa)$ be a digital image.

- For subsets $A, W \subset X$ such that $\text{Cl}_\kappa(W) \subset \text{Int}_\kappa(A)$, the inclusion $(X - W, A - W) \to (X, A)$ induces isomorphisms $H_n(X - W, A - W) \to H_n(X, A)$, for integers $n \leq 2$.

Equivalently,

- For subsets $A, B \subset X$ such that $X = \text{Int}_\kappa(A) \cup \text{Int}_\kappa(B)$, the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$, for integers $n \leq 2$.

We need some notations and a result (Lemma 5.5) to prove Theorem 5.4. For any subsets $A, B \subset X$, let $Q_n^{A,B}(X)$ be the subgroup of $Q_n(X)$ generated by singular $n$-cubes $T : I^n \to X$ such that the image $\text{Im}(T) \subset A$ or $\text{Im}(T) \subset B$. Let $D_n^{A,B}(X) = D_n(X) \cap Q_n^{A,B}(X)$, and $C_n^{A,B}(X) = Q_n^{A,B}(X)/D_n^{A,B}(X)$. It can be shown that if $T \in Q_n(X)$ and $\text{Im}(T) \subset A$ then $\partial T \in Q_{n-1}(A)$. Therefore, the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ takes $C_n^{A,B}(X)$ to $C_{n-1}^{A,B}(X)$, making up a chain complex $(C_\bullet^{A,B}(X), \partial)$, given as:

$$
\cdots \xrightarrow{\partial_{n+1}} C_n^{A,B}(X) \xrightarrow{\partial_n} C_{n-1}^{A,B}(X) \xrightarrow{\partial_{n-1}} \cdots
$$

**Lemma 5.5** Let $(X, \kappa)$ be a digital image, with subsets $A$ and $B$ such that $X = \text{Int}_\kappa(A) \cup \text{Int}_\kappa(B)$. Then $C_n^{A,B}(X) = C_n(X)$, for integers $n \leq 2$.

**Proof.** For integers $n < 0$, $C_n^{A,B}(X) = C_n(X) = 0$. It suffices to show that $C_n(X) \subset C_n^{A,B}(X)$, $n \in \{0, 1, 2\}$, because the reverse inclusion holds by definition of $C_n^{A,B}(X)$. Consider a singular $n$-cube $T \in C_n(X)$ and the following cases for $n$:

Case: $n = 0$ In this case $\text{Im}(T)$ consists of single element, say $x_0$, of $X$. Thus $x_0 \in \text{Int}_\kappa(A)$ or $x_0 \in \text{Int}_\kappa(B)$, implying $\text{Im}(T) \subset A$ or $\text{Im}(T) \subset B$. In each case
Let $T \in C^{A,B}_0(X)$. Case: $n = 1$ In this case, the set $\text{Im}(T) \subset X$ comprises two elements, namely, $T(0)$ and $T(1)$. We can assume without loss of generality that the element $T(0) \in \text{Int}_\kappa(A)$. From definition of $\text{Int}_\kappa$ operator, $T(1) \in A$, for being $\kappa$-neighbor of $A$. Since $\text{Int}_\kappa(A) \subset A$ (Proposition 5.2(i)), we get $\text{Im}(T) \subset A$ implying $T \in C^{A,B}_1(X)$.

Case: $n = 2$ In this case, the set $\text{Im}(T) \subset X$ comprises at most four distinct elements, namely, $T(0,0)$, $T(0,1)$, $T(1,0)$ and $T(1,1)$. We can assume without loss of generality that the element $T(0,0) \in \text{Int}_\kappa(A)$. From definition of $\text{Int}_\kappa$ operator, $T(0,1), T(1,0) \in A$, for being $\kappa$-neighbors of $T(0,0)$. Now $T(1,1)$ may or may not lie in $A$. If $T(1,1) \in A$, then $\text{Im}(T) \subset A$ implying $T \in C^{A,B}_2(X)$. If $T(1,1) \notin A$, then we claim that $\text{Im}(T) \subset B$ also implying $T \in C^{A,B}_2(X)$. Our claim follows from the following argument: From the definition of $\text{Cl}_\kappa$ operator, $T(1,1) \in X – A$ implies that $T(0,1)$ and $T(1,0)$ (for being $\kappa$-neighbors of $T(1,1)$) both lie in $\text{Cl}_\kappa(X – A)$, which is a subset of $\text{Int}_\kappa(B)$ by Proposition 5.4(iv). Therefore, $T(0,1), T(1,0) \in \text{Int}_\kappa(B) \Rightarrow T(0,0) \in B \Rightarrow \text{Im}(T) \subset B$. 

Proof (Theorem 5.4).
The equivalence of the two statements follows from Proposition 5.4(iv) by taking $B = X – W$, which implies $W = X – B$ and $A – W = A \cap B$.

One can verify that for all $n$, $C_n^{A,B}(X) = C_n(A) + C_n(B)$ and $C_n(A) \cap C_n(B) = C_n(A \cap B)$. Furthermore, the map $\frac{C_n(A) + C_n(B)}{C_n(A)} \rightarrow \frac{C_n(A \cap B)}{C_n(A)}$ induced by inclusion is an isomorphism by second isomorphism theorem of groups. Therefore we get:

$$C_n(B, A \cap B) = \frac{C_n(B)}{C_n(A \cap B)} = \frac{C_n(B)}{C_n(A) \cap C_n(B)} \approx \frac{C_n(A) + C_n(B)}{C_n(A)} = \frac{C_n(X)}{C_n(A)} = C_n(X, A),$$

where only the second last equality is restricted to $n \leq 2$ by Lemma 5.5. It follows that $H_n(X, A) \cong H_n(B, A \cap B)$, for integers $n \leq 2$. 

Theorem 5.4 is restricted to dimensions upto 2. There is a need to extend this idea to higher dimensions. The first of the two versions of Theorem 5.4 states that there is no change in the relative homology groups of the digital image pair $(X, A)$ upto dimension 2, if we excise out a subset $W$, which is contained well-inside $A$. In order to extend this idea to higher dimensions, we need the subset $W$ to be contained deeper inside $A$. This can be done by iteratively applying interior and closure operators. This gives rise to the following definitions and results similar to those in Proposition 5.2.

**Definition 5.6** Let $(X, \kappa)$ be a digital image and $A \subset X$. We define the operators $\text{Int}_\kappa^i : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ and $\text{Cl}_\kappa^i : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, for non-negative integers $i$, recursively, as follows:

- $\text{Int}_\kappa^0(A) = A$, $\text{Int}_\kappa^i(A) = \text{Int}_\kappa(\text{Int}_\kappa^{i-1}(A))$, for positive integer $i$,
- $\text{Cl}_\kappa^0(A) = A$, $\text{Cl}_\kappa^i(A) = \text{Cl}_\kappa(\text{Cl}_\kappa^{i-1}(A))$, for positive integer $i$. 

Proposition 5.7 Let \((X, \kappa)\) be a digital image, \(A, B \subset X\) and \(x, y \in X\). Then:
\(\begin{align*}
(i) \quad Cl^i_k(A) &\subset Cl^{i+1}_k(A), \quad Int^{i+1}_k(A) \subset Int^i_k(A) \\
(ii) \quad Int^i_k(X - A) &\equiv X - Cl^i_k(A), \quad X - Int^i_k(A) = Cl^i_k(X - A) \\
(iii) \quad X = Int^i_k(A) \cup Int^i_k(B), \quad \Leftrightarrow Cl^i_k(X - B) \subset Int^i_k(A)
\end{align*}\)

Proof. The proofs are simple and follow easily from Definitions 5.1 and 5.6, and Proposition 5.2.

Theorem 5.8 [Excision-like property]
Let \((X, \kappa)\) be a digital image.

- For subsets \(A, W \subset X\) such that there is a positive integer \(i\), with \(Cl^i_k(W) \subset Int^i_k(A)\), the inclusion \((X-W, A-W) \rightarrow (X, A)\) induces isomorphisms \(H_n(X-W, A-W) \rightarrow H_n(X, A)\), for integers \(n \leq i + 1\).

Equivalently,

- For subsets \(A, B \subset X\) such that there is a positive integer \(i\), with \(X = Int^i_k(A) \cup Int^i_k(B)\), the inclusion \((B, A \cap B) \rightarrow (X, A)\) induces isomorphisms \(H_n(B, A \cap B) \rightarrow H_n(X, A)\), for integers \(n \leq i + 1\).

We need the following result (Lemma 5.9) to prove the above theorem.

Lemma 5.9 Let \((X, \kappa)\) be a digital image, with subsets \(A\) and \(B\) such that there is a positive integer \(i\) with \(X = Int^i_k(A) \cup Int^i_k(B)\). Then \(C^{A,B}_n(X) = C_n(X)\), for integers \(n \leq i + 1\).

Proof. For \(n < 0\), clearly \(C^{A,B}_n(X) = C_n(X) = 0\). It suffices to show that \(C_n(X) \subset C^{A,B}_n(X)\) because the reverse inclusion holds by definition of \(C^{A,B}_n(X)\). Consider a singular \(n\)-cube \(T \subset C_n(X)\), \(n \in \{0, 1, \ldots, i + 1\}\). The set \(Im(T) \subset X\) can be partitioned into sets \(S_j\) for \(j = 0, 1, \ldots, n\) defined as follows:

\[S_j = \{\tau(x) \mid x = (x_1, x_2, \ldots, x_n) \in I^n \text{ with } \sum_{i=1}^n x_i = j\}\]

Note that for all \(j\), elements of \(S_j\) are \(\kappa\)-neighbors of elements \(S_{j+1}\) and \(S_{j-1}\) and that \(S_0\) and \(S_n\) are singletons.

Case: \(n = 0\) In this case, the partition of \(Im(T)\) consists of single set \(S_0 \subset X\). Thus \(S_0 \subset Int^0_k(A)\) or \(S_0 \subset Int^0_k(B)\), implying \(Im(T) \subset A\) or \(Im(T) \subset B\) in each case \(T \in C^{A,B}_n(X)\).

Case: \(0 < n < i + 1\) We can assume without loss of generality that the singleton \(S_0 \subset Int^0_k(A)\). Then by definition of \(Int^0_k\) operator, \(S_j \subset Int^{n_j}_k\), for \(j = 1, 2, \ldots, n\). Thus for all \(j, S_j \subset A\), since \(Int^0_k(A) \subset A\) from Proposition 5.7(i). Therefore, \(Im(T) \subset A\) and \(T \in C^{A,B}_n(X)\).

Case: \(n = i + 1\) Again, we can assume without loss of generality that the set \(S_0 \subset Int^0_k(A)\). From the definition of \(Int^i_k\), for \(j = 1, 2, \ldots, n - 1\), \(S_j \subset Int^{n_j-1}_k\). Now \(S_n\) may or may not lie in \(A\). If \(S_n \subset A\), then \(Im(T) \subset A\) implying \(T \in C^{A,B}_n(X)\). If \(S_n \subset X - A\), then we claim that \(Im(T) \subset B\) also implying \(T \in C^{A,B}_n(X)\). Our claim follows from the following argument: From the definition of \(Cl_k\) operator, \(S_n \subset X - A\) implies \(S_{n-1}\) is contained in \(Cl_k(X - A)\). Using Proposition 5.7(ii) and (iii) we get the following:

\[Cl_k(X - A) \subset Cl^i_k(X - A) \subset Int^i_k(B)\]
We prove the axioms of digital homology theory one-by-one:

**Proof.**

Theorem 6.3

The relative cubical singular homology groups $H_n^{\text{Dig}}$ are defined for each digital image pair $(X, A)$. The homotopy axiom ensures that if $f, g : (X, A) \rightarrow (Y, B)$ are homotopically equivalent, then $f_*, g_* : H_n(X, A) \rightarrow H_n(Y, B)$ are equal maps. The exactness axiom guarantees that for each digital image pair $(X, A)$ and subset $W \subset A$ such there is a positive integer $i$ with $\text{Cl}_i(W) \subset \text{Int}_i(A)$, the inclusion $(X - W, A - W) \rightarrow (X, A)$ induces isomorphism $H_n(X - W, A - W) \rightarrow H_n(X, A)$ for $0 \leq n \leq i + 1$.

**Definition 6.2**

Digital homology theory consists of functors $H_n(-,-)$ from the category of digital image pairs $\text{Dig}^2$ to the category of Abelian groups $\text{Ab}$ along with natural transformations $\partial_* : H_n(X, A) \rightarrow H_{n-1}(A)$, where $H_{n-1}(A, \emptyset)$ is denoted as $H_n(-, -)$ satisfying following axioms:

- **Homotopy axiom.** If $f, g : (X, A) \rightarrow (Y, B)$ are homotopically equivalent, then $f_*, g_* : H_n(X, A) \rightarrow H_n(Y, B)$ are equal maps.

- **Exactness axiom.** For each digital image pair $(X, A)$, and inclusion maps $i : A \rightarrow X$ and $j : (X, A) \rightarrow (X, A)$, there is a long-exact sequence:

\[
\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} \cdots
\]

- **Excision axiom.** For a digital image pair $(X, A)$ and a subset $W \subset A$ such there is a positive integer $i$ with $\text{Cl}_i(W) \subset \text{Int}_i(A)$, the inclusion $(X - W, A - W) \rightarrow (X, A)$ induces isomorphism $H_n(X - W, A - W) \rightarrow H_n(X, A)$ for $0 \leq n \leq i + 1$.

- **Dimension axiom.** If $X = \{x_0\}$ is a one-point digital image, $H_n(X) = 0$, for all $n > 0$.

**Additivity axiom.** Let $\{(X_\alpha, \kappa) \mid \alpha \in A\}$ be a collection of disjoint digital images and $(X, \kappa)$ be the digital image $X = \bigcup_\alpha X_\alpha$, then $H_n(X) \approx \bigoplus_\alpha H_n(X_\alpha)$.

**Theorem 6.3** The relative cubical singular homology groups $H_n(-,-)$ form a digital homology theory.

**Proof.** We prove the axioms of digital homology theory one-by-one:

- **Homotopy axiom** It can be shown, using Theorem 3.7 that if $f, g : (X, A) \rightarrow (Y, B)$ are homotopically equivalent, then $f$ and $g$ induce the same map $f_* = g_*$ from $H_n(X, A)$ to $H_n(Y, B)$. 
[Exactness axiom] For a digital image pair \((X, A)\), we have chain complexes \((C_n(A), \partial)\), \((C_n(X), \partial)\) and \((C_n(X, A), \partial)\). We also have chain maps \(i_* : C_n(A) \to C_n(X)\) and \(j_* : C_n(X) \to C_n(X, A)\), induced by inclusions \(i : A \hookrightarrow X\) and \(j : (X, \emptyset) \hookrightarrow (X, A)\). This gives the following short exact sequence of chain-complexes:

\[
0 \rightarrow C_n(A) \xrightarrow{i_*} C_n(X) \xrightarrow{j_*} C_n(X, A) \rightarrow 0
\]

The above short-exact sequence induces the following long-exact sequence of homology groups:

\[
\cdots \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} \cdots
\]

by zig-zag lemma ([15], Lemma 24.1). The zig-zag lemma also asserts the existence and uniqueness of the homomorphism \(\partial_* : C_n(X, A) \to C_{n-1}(A)\).

[Excision axiom] See Theorem 5.8
[Dimension axiom] See Proposition 3.6
[Additivity axiom] See Proposition 3.2

7 Conclusion

We have developed cubical singular homology for digital images as functors from the category of digital images \(\text{Dig}\) to the category of Abelian groups. We showed that the Abelianized fundamental group of digital images developed by Boxer [3] is isomorphic to our first homology group. Furthermore, we show that the sequence of functors satisfy axioms that can be regarded as digital analogue to Eilenberg-Steenrod axioms. We also define digital version of homology theory.

Singular homology for topological spaces is in general difficult to compute and the same is true for our case of cubical singular homology for digital images. More theoretical study is required to make computations possible to some extent. This work can be extended in various directions. Based on our work, cohomology theory for digital images can be developed. Our work is restricted to black-and-white digital images, one might extend this work to develop homology theory for grey-scale and colored digital images. Currently, we do not know to what extent this work is applicable to the unbounded digital images or to homology theory for graphs.

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