WEINSTEIN HOMOTOPIES

SAUVIK MUKHERJEE

Abstract. In this paper we discuss a problem mentioned by Eliashberg in his paper [3]. He has asked if two completed Weinstein structures \((\hat{X}, \lambda_0, \phi_0)\) and \((\hat{X}, \lambda_1, \phi_1)\) on the same symplectic manifold \((\hat{X}, \omega)\) can be homotoped through Weinstein structures. We discuss this problem and prove a weak partial result by assuming some additional conditions.

1. Introduction

In this paper we discuss a problem mentioned by Eliashberg in his paper [3]. He has asked if two completed Weinstein structures \((\hat{X}, \lambda_0, \phi_0)\) and \((\hat{X}, \lambda_1, \phi_1)\) on the same symplectic manifold \((\hat{X}, \omega)\) can be homotoped through Weinstein structures. We discuss this problem and prove a weak partial result by assuming some additional conditions.

We begin with the basic definitions. Let \((X, \omega)\) be a 2n-dimensional symplectic domain with boundary with an exact symplectic form \(\omega\) and primitive form \(\lambda\) i.e. \(d\lambda = \omega\).

A Liouville form is a choice of a primitive form \(\lambda\) such that \(\lambda|_{\partial X}\) is a contact form on \(\partial X\) and the orientation on \(\partial X\) by the form \(\lambda \wedge d\lambda|_{\partial X}\) coincides with its orientation as the boundary of \((X, \omega)\). The \(\omega\)-dual vector field \(Z\) of \(\lambda\) is called the Liouville vector field. \(Z\) satisfies \(L_Z \omega = \omega\) and hence its flow is conformally symplectically expanding.

Every Liouville domain \(X\) can be completed in the following way. Set

\[ \hat{X} = X \cup (\partial X \times [0, \infty)) \]

and extend \(\lambda\) on \(\hat{X}\) as \(e^t(\lambda|_{\partial X})\) on the attached end. Given a Liouville domain \(\mathcal{L} = (X, \omega, \lambda)\) consider the compact set

\[ Core(\mathcal{L}) = \cap_{t>0} Z^{-t}(X) \]

It is called Core or the Skeleton of the Liouville domain.

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Let $\lambda_0$ and $\lambda_1$ be two Liouville forms on a fixed symplectic manifold $(X, \omega)$, moreover let $Z_0$ and $Z_1$ be the respective Liouville vector fields. Then obviously $\lambda_1 = \lambda_0 + dh$ for some $h : X \to \mathbb{R}$ and $Z_1 = Z_0 + Z_h$ where $Z_h$ is the hamiltonian vector field for $h$.

A Liouville cobordism $(W, \omega, Z)$ is a cobordism $W$ with an exact symplectic form $\omega$ such that the Liouville vector field $Z$ points inward along $\partial_- W$ and outward along $\partial_+ W$.

**Remark 1.1.** On the infinite end of $\hat{X}$, the Liouville vector field is given by $\partial_s$ irrespective of the choice of the Liouville form $\lambda$ on $X$.

Now we shall define the Weinstein structures. For this we need to recall few notions. A complete vector field is a vector field whose flow exists for all forward and backward time.

Let $\phi$ be a Morse function. A vector field $X$ is called gradient-like for $\phi$ if it satisfies
\[ X . \phi \geq \delta(|X|^2 + |d\phi|^2) \]
for some $\delta > 0$ and $|X|$ is with respect to some Riemannian metric and $|d\phi|$ is with respect to its dual metric.

**Definition 1.2.** ([7]) A Weinstein manifold $(X, \omega, Z, \phi)$ is a symplectic manifold $(X, \omega)$ with a complete Liouville vector field $Z$ which is gradient-like with respect to the exhausting Morse function $\phi$. A Weinstein cobordism $(W, \omega, Z, \phi)$ is a Liouville cobordism $(W, \omega, Z)$ whose Liouville vector field $Z$ is gradient-like with respect to a Morse function $\phi$ which is constant on the boundary. A Weinstein cobordism with $\partial_- W = \emptyset$ (empty) is called a Weinstein domain.

In [3] Eliashberg has asked the following question.

**Problem:** Let $(\hat{X}, \lambda_0, \phi_0)$ and $(\hat{X}, \lambda_1, \phi_1)$ be two completed Weinstein structures on the same symplectic manifold $(\hat{X}, \omega)$. Are they homotopic as Weinstein structures?

Obviously if $Z_0$ and $Z_1$ are the respective Liouville vector fields then $Z_1 = Z_0 + Z_h$ for $h$ satisfying $\lambda_1 = \lambda_0 + dh$ and hence
\[ Z_t = Z_0 + Z_{th} = Z_0 + tZ_h, \quad t \in [0, 1] \]
gives a homotopy of Liouville vector fields. However \((X, \omega, Z_t)\) may not be a Liouville homotopy. We refer the reader [7] for a precise definition of Liouville homotopy.

On Weinstein cobordisms a similar result has been proved in [7] although the Weinstein structures need to flexible. We refer the reader to [7] for a precise definition of flexible Weinstein structures.

**Theorem 1.3.** ([7]) Let \((W, \omega_0, \lambda_0, \phi_0)\) and \((W, \omega_1, \lambda_1, \phi_1)\) be two flexible Weinstein structures on the same cobordism \(W\) with dimension \(2n > 4\) which coincide on \(\partial W\). Let \(\eta_t\) be a homotopy rel \(\partial W\) of non-degenerate two forms on \(W\) connecting \(\omega_0\) and \(\omega_1\). Then there exists a homotopy of flexible Weinstein structures connecting the given ones.

Let us now return to the question asked by Eliashberg. We assume that all the zeros of \(Z_t\) are non-degenerate for all \(t \in [0, 1]\). So the zeros of \(Z_t\) executes curves \(\gamma_i(t), i = 1, ..., k\) (say). We consider \(\tilde{X} = \tilde{X} \times [0, 1]\) and define vector field \(Z(x, t) = Z_t(x)\) on \(\tilde{X}\). The curves \(\gamma_i\)'s define curves \(\Gamma_i\)'s on \(\tilde{X}\) as follows
\[
\Gamma_i(t) = (\gamma_i(t), t)
\]
Consider two tubular neighborhoods of \(\Gamma_i\) as \(\Gamma_i \subset N_i' \subset N_i''\). Let \(\Psi_i : \tilde{X} \rightarrow \mathbb{R}\) be cutoff functions such that \(\Psi_i = 1\) on \(N_i'\) and \(\Psi_i = 0\) outside \(N_i''\). Define \(\tilde{Z}\) on \(\tilde{X}\) by canonically removing the zeros of \(Z\) as follows. Define \(\tilde{Z}\) close to \(\Gamma_i\) as
\[
\tilde{Z}(x, t) = \Psi_i(x, t)\partial_t + (1 - \Psi_i(x, t))Z(x, t)
\]
Let \(\tilde{F}\) be the foliation defined by \(\tilde{Z}\). Then \(\tilde{F}\) is a regular foliation.

**Definition 1.4.** We call the homotopy of the Liouville vector field \(Z_t\) uniformly open if it satisfies

1. All the zeros of \(Z_t\) are non-degenerate for all \(t \in [0, 1]\)

2. The foliation \(\tilde{F} \times \tilde{F}\) on \(\tilde{X} \times \tilde{X}\) is uniformly open

Please see 3.1 below for the definition of Uniformly open foliation. Now we state the main theorem of this paper.

**Theorem 1.5.** Let \((\hat{X}, \lambda_0, \phi_0)\) and \((\hat{X}, \lambda_1, \phi_1)\) be two completed Weinstein structures on the same symplectic manifold \((\hat{X}, \omega)\) and let the homotopy of the Liouville vector field \(Z_t\) is
uniformly open \((\mathcal{I}_A)\). Then \((\hat{X}, \lambda_0, \phi_0)\) and \((\hat{X}, \lambda_1, \phi_1)\) can joined by a homotopy of Weinstein structures for which the underlying symplectic structure \(\omega\) remains fixed.

**Remark 1.6.** In \((\mathcal{I}_A)\) the underlying symplectic structure is not fixed.

2. \(h\)-Principle

This section does not have any new result, we just recall some facts from the theory of \(h\)-principle which we shall need in our proof.

Let \(X \to M\) be any fiber bundle and let \(X^{(r)}\) be the space of \(r\)-jets of jerms of sections of \(X \to M\) and \(j^r f : M \to X^{(r)}\) be the \(r\)-jet extension map of the section \(f : M \to X\). If \(X = M \times N\) then \(X^{(r)}\) is denoted as \(J^r (M, N)\). A section \(F : M \to X^{(r)}\) is called holonomic if there exists a section \(f : M \to X\) such that \(F = j^r f\). In the following we use the notation \(Op(A)\) to denote a small open neighborhood of \(A \subset M\) which is unspecified.

Let \(R\) be a subset of \(X^{(r)}\). Then \(R\) is called a differential relation of order \(r\). \(R\) is said to satisfy \(h\)-principle if any section \(F : M \to R \subset X^{(r)}\) can be homotopped to a holonomic section \(\tilde{F} : M \to R \subset X^{(r)}\) through sections whose images are contained in \(R\). Put differently, if the space of sections of \(X^{(r)}\) landing into \(R\) is denoted by \(Sec R\) and the space of holonomic sections of \(X^{(r)}\) landing into \(R\) is denoted by \(Hol R\) then \(R\) satisfies \(h\)-principle if the inclusion map \(Hol R \to Sec R\) induces a epimorphism at 0-th homotopy group \(\pi_0\). \(R\) satisfies parametric \(h\)-principle if \(\pi_k (Sec R, Hol R) = 0\) for all \(k \geq 0\).

Let \(p : X \to M\) be a fiber bundle and by \(Diff_M X\) we denote the fiber preserving diffeomorphisms \(h_X : X \to X\), i.e, \(h_X \in Diff_M X\) if and only if there exists diffeomorphism \(h_M : M \to M\) such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{h_X} & X \\
p & & p \\
\downarrow \quad & & \downarrow \\
M & \xrightarrow{h_M} & M
\end{array}
\]

Let \(\pi : Diff_M X \to Diff M\) be the projection \(h_X \mapsto h_M\). We call a fiber bundle \(p : X \to M\) natural if there exists a homomorphism \(j : Diff M \to Diff_M X\) such that \(\pi \circ j = id\). For a natural fiber bundle \(p : X \to M\) the associated jet bundle \(X^{(r)} \to M\) is also natural. The
lift is given by
\[ j^r : \text{Diff}M \to \text{Diff}M X^{(r)}, \ h \mapsto h_* \]
where \( h_*(s) = J^r_{j(h)(s)}(h(m)), \ s \in X^{(r)}, \ m = p'(s) \in M \) and \( \tilde{s} \) is a local section near \( m \) which represents the \( r \)-jet \( s \). Observe \( (h^{-1})_* = (h_*)^{-1} \) and hence define \( h^s = h_*^{-1}. \)

For a natural fiber bundle \( X \to M \), a differential relation \( R \subset X^{(r)} \) is called \( \text{Diff}M \)-invariant if the action \( s \mapsto h_* s, \ h \in \text{Diff}M, \) leaves \( R \) invariant.

**Theorem 2.1.** ([4]) *If a relation \( R \) is open and \( \text{Diff}M \)-invariant on an open manifold \( M \) then it satisfies parametric \( h \)-principle.*

3. **Bertelson’s Uniformly Open Foliations**

In this section we recall some result from [1] and [2].

**Definition 3.1.** ([1]) *A foliated manifold \((M, \mathcal{F})\) is called uniformly open if there exists a function \( f : M \to [0, \infty) \) such that*

1. \( f \) is proper,
2. \( f \) has no leafwise local maxima,
3. \( f \) is \( \mathcal{F} \)-generic.

**Remark 3.2.** *Observe that if \( \dim \mathcal{F} = 1 \) then \((M, \mathcal{F})\) can not be uniformly open as on a one dimensional manifold, a critical point will be either a local maximum or minimum.*

So let us explain the notion \( \mathcal{F} \)-generic. In order to do so we need to define the singularity set \( \Sigma^{(i_1, i_2, \ldots, i_k)}(f) \) for a map \( f : M \to W \). \( \Sigma^{(i_1)}(f) \) is the set

\[ \{ p \in M : \dim(\ker(df)_p) = i_1 \} \]

It was proved by Thom [6] that for most maps \( \Sigma^{(i_1)}(f) \) is a submanifold of \( M \). So we can restrict \( f \) to \( \Sigma^{(i_1)}(f) \) and construct \( \Sigma^{(i_1, i_2)}(f) \) and so on. In [6] it has been proved that there exists \( \Sigma^{(i_1, \ldots, i_k)} \subset J^k(M, W) \) such that \( (j^k f)^{-1}\Sigma^{(i_1, \ldots, i_k)} = \Sigma^{(i_1, \ldots, i_k)}(f) \).
Let us set \( W = \mathbb{R} \) as this is the only situation we need. Let \((M, \mathcal{F})\) be a foliated manifold with a leaf \( F \). Define the restriction map

\[
    r_F : J^k(M, \mathbb{R}) \to J^k(F, \mathbb{R}) : j^k f(x) \mapsto j^k(f|_F)(x)
\]

Define foliated analogue of the singularity set as

\[
    \Sigma_{\mathcal{F}}^{(i_1, i_2, \ldots, i_k)} := \{ F \text{ leaf of } \mathcal{F} \} r_F^{-1} \Sigma_{\mathcal{F}}^{(i_1, i_2, \ldots, i_k)}
\]

**Definition 3.3.** \((\text{[1]}\)) A smooth real valued function \( f : M \to \mathbb{R} \) is called \( \mathcal{F} \)-generic if the first jet \( j^1 f \ni \Sigma_{\mathcal{F}}^{(n)} \) and the second jet \( j^2 f \ni \Sigma_{\mathcal{F}}^{(i_1, i_2)} \) for all \((i_1, i_2)\).

**Definition 3.4.** \((\text{[1]}\)) An isotopy of the manifold \( M \) is a family \( \psi_t, t \in [0, 1] \) of diffeomorphisms of \( M \) such that the map \( \psi : M \times [0, 1] \to M : (x, t) \mapsto \psi_t(x) \) is smooth and \( \psi_0 = \text{id}_M \).

Consider a foliation \( \mathcal{F} \) on \( M \). A foliated isotopy of \((M, \mathcal{F})\) is an isotopy \( \psi_t \) of \( M \) that preserves the foliation \( \mathcal{F} \), that is, \((\psi_t)_* (T \mathcal{F}) = T \mathcal{F} \) for all \( t \in [0, 1] \). A relation \( R \) is called foliated invariant on \((M, \mathcal{F})\) if the action by foliated isotopies leaves \( R \) invariant.

**Theorem 3.5.** \((\text{[1]}\)) On an uniformly open foliated manifold, any open, foliated invariant differential relation satisfies the parametric \( h \)-principle.

In \([2]\) Bertelson has contructed counter examples that without the uniformly open condition \(\text{5.5} \) fails.

### 4. Main Theorem

In this section we prove \(\text{1.5} \). Let us first set some notations. First of all we have the Liouville vector fields \( Z_0 \) and \( Z_1 = Z_0 + Z_h \) and let \( Z_t = Z_0 + t Z_1 \) be the homotopy of uniformly open Liouville vector field. So we have

1. \( Z_0.\phi_0 \geq \delta (|Z_0|^2 + |d\phi_0|^2) \)

2. \( Z_0.\phi_1 + Z_h.\phi_1 \geq \delta' (|Z_0 + Z_h|^2 + |d\phi_1|^2) \)

With equality occurs in the above inequalities at the zeros of \( Z_0 \) and \( Z_0 + Z_h \). Define

\[
    \phi_t = (1 - t)\phi_0 + t\phi_1
\]

Then observe that

\[
    Z_0.d\phi_t = (1 - t)Z_0.d\phi_0 + tZ_0.d\phi_1
\]
Now consider
\[ Z_0.d\phi_t + Z_{th}.d\phi_1 = (1 - t)Z_0.d\phi_0 + t[Z_0.d\phi_1 + Z_h.d\phi_1] \]

\[ \geq (1 - t)\delta(|Z_0|^2 + |d\phi_0|^2) + t\delta'(|Z_0 + Z_h|^2 + |d\phi_1|^2) \]

\[ \geq \min(\delta, \delta')(1 - t)|Z_0|^2 + t|Z_0 + Z_h|^2 + (1 - t)|d\phi_0|^2 + t|d\phi_1|^2 \]

So we get
\[ \frac{(Z_0.d\phi_t + Z_{th}.d\phi_1)}{|Z_0 + tZ_h|^2 + |d\phi_t + d\phi_1|^2} \geq \min(\delta, \delta')\left[\frac{(1 - t)|Z_0|^2 + t|Z_0 + Z_h|^2}{|Z_0 + tZ_h|^2 + |d\phi_t + d\phi_1|^2} + \frac{2(1 - t)|Z_0||Z_0 + Z_h| + |d\phi_t + d\phi_1|^2}{|Z_0 + tZ_h|^2 + |d\phi_t + d\phi_1|^2}\right] \]

Recall that (according to 1.1) on the infinite end of \( \tilde{X} \) the Liouville vector fields \( Z_0 \) and \( Z_0 + Z_h \) are equal to \( \partial_t \). So the right hand side of the above inequality is bounded and hence the right hand side is equal to \( \tilde{d} \) (say). So we get
\[ (Z_0.d\phi_t + Z_{th}.d\phi_1) \geq \tilde{d}|Z_0 + tZ_h|^2 + |d\phi_t + d\phi_1|^2 \]

Without loss of generality we assume that \( Z_0 \cap Z_h \) otherwise we can use a relative version of \( h \)-principle.

We replace \( t \) by a new parameter \( t' = f(t) \) where \( f : [0, 1] \rightarrow [0, 1] \) is such that \( f = 0 \) on \([0, \epsilon] \) and \( f = 1 \) on \([1 - \epsilon, 1] \). We can replace the parameter in the above inequality.

Define one forms \( \alpha_{t'} \) and \( \eta_{t'} \) as follows. First \( \alpha_{t'}(Z_0) = d\phi_{t'}(Z_0), \ Z_h \in ker(\alpha_{t'}), \ for \ t \in [\epsilon, 1 - \epsilon], \ t' = f(t) \) and \( \alpha_0 = d\phi_0, \ \alpha_1 = d\phi_1 \). Similarly \( \eta_{t'}(Z_{t'h}) = \eta_{t'}(t'Z_h) = d\phi_1(t'Z_h), \ Z_0 \in ker\eta_{t'} \ for \ t \in [\epsilon, 1 - \epsilon], \ t' = f(t) \) and \( \eta_0 = d\phi_1 = \eta_1 \). So we have
\[ (\alpha_{t'} + \eta_{t'})(Z_0 + t'Z_h) = (\alpha_{t'}(Z_0) + \eta_{t'}(Z_{t'h})) \geq \tilde{d}|Z_0 + t'Z_h|^2 + |\alpha_{t'} + \eta_{t'}|^2 \]

Now extending on \( \tilde{X} \) and regularizing \( Z_{t'} = Z_0 + t'Z_h \) we get \( \tilde{Z} \) as in 1. We extend \( \alpha_{t'} \) and \( \eta_{t'} \) to \( \tilde{X} \) as \( \alpha'(x, t') = \alpha_{t'}(x) \) and \( \eta'(x, t') = \eta_{t'}(x) \). Adjusting \( \alpha' \) and \( \eta' \) near \( \Gamma_i \)’s (1) to \( \tilde{\alpha} \) and \( \tilde{\eta} \) so that
\[ (\tilde{\alpha} + \tilde{\eta})(\tilde{Z}) > \tilde{d}|\tilde{Z}|^2 + |\tilde{\alpha} + \tilde{\eta}|^2 \]

Now we come to the \( h \)-principle part. Consider \( M = \tilde{X} \times \tilde{X} \) and the trivial bundle \( P \times P : M \times \mathbb{R}^2 = \tilde{X} \times \mathbb{R} \times \tilde{X} \times \mathbb{R} \rightarrow M \) where \( P : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X} \) is the projection on the first factor. Observe that
\[ (M \times \mathbb{R}^2)^{(1)} = (\tilde{X} \times \mathbb{R})^{(1)} \times (\tilde{X} \times \mathbb{R})^{(1)} \]
Note that this does not happen in case of higher order jet extensions as there will be mixed
derivatives.

Observe that the section space \( \Gamma(\tilde{X} \times \mathbb{R}) = C^\infty(\tilde{X}, \mathbb{R}) \). There is a natural affine fibration
\( L : (\tilde{X} \times \mathbb{R})^{(1)} \to T^*(\tilde{X} \times \mathbb{R}) \) given by \( L(j^1 f(x)) = df_x \) where \( f \in C^\infty(\tilde{X}, \mathbb{R}) \). Define the
relation \( R \subset (M \times \mathbb{R}^2)^{(1)} \) as
\[
R = \{(j^1 f_0, j^1 f_1) \in (M \times \mathbb{R}^2)^{(1)} : L(j^1 f_i)(\tilde{Z}) > \tilde{\delta} |\tilde{Z}|^2 + |L(j^1 f_i)|^2 \} \quad \text{for} \quad i = 0, 1, \quad \text{for some} \quad \tilde{\delta} > 0
\]
Obviously \( (\tilde{\alpha} + \tilde{\eta}, \tilde{\alpha} + \tilde{\eta}) \in SecR \). Next we shall show that \( R \) is open and invariant under
\( \tilde{F} \times \tilde{F} \)-foliated isotopy. This will conclude the proof of \([3.5]\) in view of \([3.5]\). Only thing one
needs to do is the following. Let \((f_0, f_1)\) is a resulting solution. Choose either \( f_0 \) or \( f_1 \) say \( f_0 \). Then define \( f_t \) as
\[
f_t(x) = f_0(x,t)
\]
Now we have to re-introduce the singularities. Let \( g_t \) be a family of Morse functions defined
near \( \Gamma_i \) with index same as the index of \( Z \) along \( \Gamma_i \). Let \( \beta \) be a cutoff function such that
\( \beta = 1 \) on a tubular neighborhood \( \mathcal{N}_i \supset \mathcal{N}_{i'} \) and \( \beta = 0 \) outside \( \mathcal{N}_i \supset \mathcal{N}_{i'} \). Let \( \beta_i(x) = \beta(x,t) \).
Observe
\[
Z_t(\beta_t g_t + (1 - \beta_t)f_t) = [\beta_t Z_t(g_t) + (1 - \beta_t)Z_t(f_t)] + g_t Z_t(\beta_t) - f_t Z_t(\beta_t)
\]
Observe that \( Z_t(\beta_t) \) has compact support and \( g_t \) is of the form
\[
a + x_1^2 + ... + x_k^2 - x_{k+1}^2 + ... + x_{2n}^2
\]
So if we take \( a \) large enough then \((g_t Z_t(\beta_t) - f_t Z_t(\beta_t)) > 0 \) and obviously compactly sup-
ported. So we get the desired result.

**Lemma 4.1.** The relation \( R \) is open and invariant under the action of \( \tilde{F} \times \tilde{F} \)-foliated isoto-
pies.

**Proof.** Openness of \( R \) follows directly from the definition of \( R \).

For second part we see \( \psi_*(df)(\tilde{Z}) = df(d\psi_*(\tilde{Z})) \geq cdf(\tilde{Z}) \), where \( c \) is a positive real
number. Positive as \( \psi_0 = id \) and \( M \times \mathbb{R}^2 \) is connected. So
\[
\psi_*(df)(\tilde{Z}) \geq cdf(\tilde{Z}) > c\tilde{\delta} |\tilde{Z}|^2 + |df|^2
\]
\( \square \)
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Presidency University, Kolkata, India., e-mail: mukherjeesauvik@gmail.com,