Collapse thresholds for the formation of CDM haloes in the ellipsoidal collapse model

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ABSTRACT
In the excursion set approach to structure formation initially spherical regions of the linear density field collapse if the average density contrast within them exceeds some critical value, $\delta_c$. This model allows one to make predictions for several fundamental aspects of structure formation such as the mass function of dark matter halos, their merger histories and clustering properties. The value of $\delta_c$ is often calculated from the spherical or ellipsoidal collapse model, which provide well-defined predictions given auxiliary properties of the linear tidal field at a given point. We use two cosmological simulations of structure growth in a LCDM cosmology to test a key assumption used in calculating $\delta_c$: that the shapes of the initial Lagrangian patches that eventually collapse (or proto-haloes) are spherical. Our results indicate that the vast majority of dark matter proto-haloes are non-spherical, and have minor-to-major axis ratios that vary from $\langle a_3/a_1 \rangle \sim 0.4$ at the galaxy mass scale to $\sim 0.65$ for rich galaxy clusters. We show that this non-sphericity likely originates from the asymmetry of the linear tidal field which pushes material onto, or away from, local density maxima in the linear density field. We study the implications of these results for the collapse barriers for CDM halo formation inferred from the classic ellipsoidal collapse model. Our results indicate that the “standard” ellipsoidal collapse model commonly adopted in the literature does not provide a full account of the possible collapse thresholds for halo formation, since the model predictions depend sensitively on the assumed shape of the primordial perturbation. We show that an improved model, which accounts for the intrinsic shapes of proto-haloes, provides a much more accurate description of their measured minimum overdensities.

Key words: cosmology: dark matter – methods: numerical

1 INTRODUCTION
In standard models for the growth of large scale structure, such as the Λ-cold dark matter (CDM) model, dark matter haloes form hierarchically from low-amplitude Gaussian density fluctuations seeded during inflation. Galaxy formation is assumed to take place as gas cools and condenses in the centers of these haloes which subsequently “light-up” the underlying large scale matter distribution (e.g., White & Rees, 1978). Although the hydrodynamical processes inherent to the formation of galaxies are difficult to model reliably, they are largely irrelevant to the over-all formation and clustering of the dark matter haloes. A first step towards understanding the statistics of the galaxy distribution is thus a detailed understanding of the spatial distribution and clustering of their haloes.

Since CDM haloes are highly non-linear objects much of our understanding of their properties has come from direct N-body simulations of their formation. Although useful, these simulations are often difficult to interpret and our understanding of the structure formation process can be significantly enhanced by augmenting them with simple physical models for the collapse and virialization of dark matter haloes.

One successful analytic approach to modeling structure formation has become known as the Extended Press-Schechter formalism (see, e.g., Press & Schechter, 1974; Bond et al., 1991). Here one identifies collapsed objects at a given time with regions of the linear density field expected to collapse based on some dynamical model, such as the
spherical (Gunn & Gott, 1972) or ellipsoidal collapse model (e.g., Zel’Dovich, 1970; Ikeda, 1973; White & Silk, 1979; Hoffman, 1986; Lemson, 1993; Bertschinger & Jain, 1994; Bond & Myers, 1996). Although Press & Schechter (1974) originally used this model to calculate the abundance of collapsed objects as a function of mass and redshift, it can be extended to compute a variety of hierarchical clustering statistics, such as halo merger histories (e.g., Lacey & Cole, 1993; Sheth & Lemson, 1999b; van den Bosch, 2002), and spatial correlations (Mo & White, 1996; Mo et al., 1997; Catelan et al., 1998; Sheth & Lemson, 1999a). A related approach, the peaks formalism, relates the abundance of bound objects to the abundance of local density maxima above some threshold in the smoothed linear density field (Peacock & Heavens, 1985; Bardeen et al., 1986; Ludlow & Porciani, 2011).

A fundamental aspect of both the Extended Press-Schechter approach and the peaks formalism is the need to identify which regions of the linear density field are expected to collapse to form bound objects. Here one typically gains insight from dynamical models that can provide physically-motivated thresholds for the linearly extrapolated density that is required for collapse to occur.

One common threshold, based on the spherical collapse model (Gunn & Gott, 1972), assumes that haloes can be identified in the smoothed linear density field by means of a single free parameter: the initial density contrast, $\delta_{sc}$. The basic properties of the halo distribution mentioned above can then be deduced from the trajectories of the linear density field as the spatial scale of the filtering is varied. In this case the threshold for collapse at redshift $z_c$ (in an Einstein-de Sitter universe) is $\delta_{sc} \approx 1.686 \left(1 + z_c\right)$. This result is approximately valid for a large range of cosmologies and is independent of mass scale and initial density contrast. Eke et al. (1996) provide useful analytic formulae for calculating the value of $\delta_{sc}$ for cosmologies with $\Omega_M + \Omega_{\Lambda} = 1$.

Although the simple spherical model works well in a statistical sense its validity is questionable; it is well-known, for example, that perturbations in Gaussian random fields are inherently triaxial (Doroshkevich, 1970; Bardeen et al., 1986). This led to modifications of the spherical model to incorporate the dynamics of ellipsoidal collapse.

Building upon the classic work by Lynden-Bell (1964) and Lin et al. (1965), a number of authors have studied the effect of internal shear on the collapse of a homogeneous ellipsoid in a uniformly expanding background (e.g., Ikeda, 1973; White & Silk, 1979; Peebles, 1980; Watanabe & Inagaki, 1991; Lemson, 1993). Eisenstein & Loeb (1995) and Bond & Myers (1996) generalized this model to follow the collapse of an initially spherical perturbation in the presence of external shear. In this case, the sphere is distorted into an ellipsoid whose principal axes are parallel to those of the external tidal field. One can then ascertain the impact of external tidal shear on the properties of collapsed regions. For example, Monaco (1997) and Sheth et al. (2001) have studied how ellipsoidal collapse modifies the mass function of dark matter haloes in the excursion set formalism (see also Lee & Shandarin, 1998).

In the ellipsoidal collapse model of Bond & Myers (1996) the collapse time depends explicitly on the local ellipticity, $\epsilon$, and prolaticity, $p$, of the tidal field. Less spherical regions of the linear tidal field must overcome additional tidal stretching and thus require higher overdensities to collapse. Sheth et al. (2001) showed that the dependence of the collapse threshold on $\epsilon$ and $p$ could be well-approximated by the solution to

$$\frac{\delta_{sc}}{\delta_{sc}} = 1 + \beta \left[5(e^2 \pm p^2) \left(\frac{\delta_{sc}}{\delta_{sc}}\right)^\gamma\right].$$

Here $\beta$ and $\gamma$ are numerical parameters determined from fits to the model results. Sheth et al. (2001) used the results of Doroshkevich (1970) to compute the most probable values of $\epsilon$ and $p$ as a function of smoothing scale. At any ellipticity, the most probable prolaticity is $p_{mp} = 0$, and for $p = 0$ they find $e_{mp} = \sigma/\delta/\sqrt{5}$. Using these results they showed that

$$\delta_{sc} = \delta_{sc} \left[1 + \beta \left(\frac{\sigma(M)^2}{\delta_{sc}}\right)^\gamma\right],$$

with $\beta = 0.47$ and $\gamma = 0.615$. Using their assumptions, eq. 2 can be expressed as a function of the tidal ellipticity by substituting $\sigma(M) = (\delta/\sqrt{5}) e_{mp}$. Various model-predicted collapse boundaries have been compared directly with N-body simulations by Robertson et al. (2009).)

Adding to this, the mass distribution in the vicinity of the linear density peaks from which haloes form are inherently triaxial (Bardeen et al., 1986). This result also applies to the Lagrangian regions which collapse onto those peaks to form haloes (e.g., Porciani et al., 2002; Ludlow & Porciani, 2011). This implies that, in the absence of any external shear, the internal shear is associated with the initially non-spherical shape of a collapsing region may impact its collapse time, and consequently, the average density required for complete collapse. These subtleties are often overlooked in simple ellipsoidal collapse models, such as those of Bond et al. (1991) and Sheth et al. (2001).

The purpose of the present paper is to investigate how simple modifications to the classic ellipsoidal collapse model affect the predicted shape of collapse barriers. Our analysis focuses on two cosmological simulations of structure formation, the results of which we compare directly with predictions from the model. The rest of the paper is organized as follows. In Section 2 we introduce our simulations and briefly describe our main analysis techniques. Section 3 describes some of the basic characteristics of proto-haloes in our simulations, focusing on their inertial and tidal tensors. We use these results in Section 4 to motivate a closer look at the ellipsoidal collapse model, and in particular, how the predicted collapse barriers depend on the assumed shape of the collapsing region. The predictions are compared directly with the results of two cosmological simulations in Section 5. Finally, we end with a brief discussion in Section 6 before summarizing our main results in Section 7.
2.1 The simulations

We focus our analysis on two high-resolution cosmological simulations of structure formation in the standard LCDM cosmology. Our runs adopted cosmological parameters consistent with the WMAP five-year data release (Komatsu et al., 2009). These parameters are as follows: \( \Omega_M = 0.279 \), \( \Omega_L = 1 - \Omega_M = 0.721 \), \( \sigma_8 = 0.817 \), \( n_s = 0.96 \), and a Hubble constant \( H_0 \equiv H(z = 0) = 73 \) km s\(^{-1}\) Mpc\(^{-1}\).

Both runs followed the evolution of the dark matter component in comoving coordinates using a lean version of the simulation code GADGET (Springel, 2005). The linear density field was sampled using 1024\(^3\) equal mass particles in a periodic box with side-lengths equal to \( l_{box} = 150 \) h\(^{-1}\) Mpc and 1200 h\(^{-1}\) Mpc. Initial conditions were generated using the Zel’dovich approximation, adopting a starting redshift \( z_s = 70 \) for the 150 h\(^{-1}\) Mpc box run and \( z_s = 50 \) for the 1200 h\(^{-1}\) Mpc box (a thorough discussion of our initial conditions can be found in Pillepich et al. (2010)). For these choices the particles masses are 2.43\( \times 10^{9} \) h\(^{-1}\) M\(_{\odot}\) and 1.24\( \times 10^{11} \) h\(^{-1}\) M\(_{\odot}\) in the 150 h\(^{-1}\) Mpc and 1200 h\(^{-1}\) Mpc boxes respectively. The corresponding softening lengths are 3 h\(^{-1}\) kpc and 20 h\(^{-1}\) kpc, which are kept fixed in comoving coordinates during the simulation.

2.2 The analysis

We study some of the basic properties of the Lagrangian regions that eventually collapse to form haloes. We refer to these as proto-haloes for convenience. Dark matter haloes were identified at \( z = 0 \) using a friends-of-friends (FOF) halo finder with linking-length \( \ell = 0.2 \) times the mean inter-particle separation; the “proto-haloes” are identified in the simulation initial conditions by tracing backwards in time the subset of particles that belong to these FOF haloes.

The proto-halo mass is defined as the total mass associated with a given FOF halo. We have verified that our results are insensitive to our adopted halo definition by repeating the analysis with dark matter haloes identified with a spherical-overdensity halo finder. These haloes were identified at a density contrast level of 200 \( \times \rho \).

We quantify proto-halo shapes using the following description of their mass distribution (Cole & Lacey, 1996; Bailin & Steinmetz, 2005; Hopkins et al., 2005; Shaw et al., 2006; Bett et al., 2007):

\[
I_{ij} = \sum_k \frac{r_{i,k}r_{j,k}}{r_k^3},
\]

where \( r_{i,k} \) is the distance from the center of mass to particle \( i \), and the factor of \( 1/r_k^2 \) is present to prevent particles at large distances from dominating the shape estimates. The center of mass is defined, at the initial redshift of the simulation, as the geometric center of all FOF particles associated with the \( z = 0 \) halo.

Eq. 3 can be diagonalized and the axis lengths, \( a_1 \geq a_2 \geq a_3 \), determined from the square roots of the eigenvalues; the eigenvectors coincide with the principal axis frame of the system. Once the axis lengths \( a_i \) have been found the shapes of the proto-haloes can be characterized in terms of their ratios: \( a_2/a_1 \) measures the intermediate-to-major axis ratio, and \( a_3/a_1 \) the minor-to-major.

In addition to shapes, we also measure properties of the linear tidal field at the geometric center of each proto-halo. The linear tidal field is calculated from the deformation tensor:

\[
D_{ij} = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}.
\]

Here \( \Phi(x) \) is the peculiar gravitational potential at position \( x \), which is related to the density contrast through Poisson’s equation: \( \nabla^2 \Phi = \delta \). The ordered eigenvalues of \( D \), \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \), determine the geometry of the linear density field, and their signs indicate whether the flow around a given point is of compression (+) or dilution (−).

The eigenvalues of \( D \) can also be expressed in terms of the tidal ellipticity, \( e \), and prolateness, \( p \), defined

\[
e = \frac{\lambda_1 - \lambda_3}{2\delta},
\]

and

\[
p = \frac{\lambda_1 - 2\lambda_2 + \lambda_3}{2\delta},
\]

where \( \delta = \lambda_1 + \lambda_2 + \lambda_3 \). For \( \delta > 0 \), we have \( e \geq 0 \) and \( -e \leq p \leq e \).

In what follows we will measure the eigenvalues of the linear deformation tensor at proto-halo barycenters after averaging the relevant quantities over a spherical Lagrangian volume enclosing the halo mass.

3 PROTO-HALOES IN COSMOLOGICAL SIMULATIONS

In Fig 1 we plot the mass dependence of the axis ratios, \( a_2/a_1 \) and \( a_3/a_1 \), for all proto-haloes in both of our simulations. Colored points show the median trends; boxes and whiskers highlight the 10\(^{th}\), 25\(^{th}\), 75\(^{th}\) and 90\(^{th}\) percentiles of the distribution. Results for haloes in our 150 h\(^{-1}\) Mpc box are plotted in blue, and red points are used those in our 1200 h\(^{-1}\) Mpc box.

There is a clear mass dependence to the mean proto-halo shape. Judging their sphericity by the value of \( a_3/a_1 \), we find that, on average, \( (a_3/a_1) \) varies from \( \sim 0.4 \) for haloes with \( M_{FOF} \approx 10^{11} h^{-1} M_\odot \) to \( \sim 0.65 \) for rich galaxy clusters, albeit with considerable scatter an any given mass scale. It is important to note that spherical proto-haloes are quite rare in our sample. In our 150 h\(^{-1}\) Mpc box only \( \sim 0.4\% \) of proto-haloes containing more than 500 particles have \( a_3/a_1 \geq 0.8 \). For our larger box, which probes higher mass scales, \( \sim 2.9\% \) of \( (N_{FOF} \geq 500) \) haloes have \( a_3/a_1 \geq 0.8 \). Thus, the Lagrangian regions subject to the collapse constraint are highly non-spherical.

For haloes containing more than 500 particles (indicated by vertical arrows in Figure 1) the median trends for each run agree quite well and can be approximated by:

\[
a_{3/1} = Q \times 10^{\log_{10}(M_{FOF}/[10^{11} h^{-1} M_\odot])} + S_0,
\]

as shown by the dotted lines in Fig 1*. The best-fit parameters, \( Q \), and \( S_0 \) are listed in table 3. Note also that

* Note that the best-fit parameters in eq 7 vary only slightly when halos are identified as spherical-overdensities with density contrast 200 \( \times \rho \). Best-fit parameters for both FOF and SO halos are given in Table 3.
proto-haloes are nearly triaxial, as indicated by the thick black lines in the lower panel of Fig 1. This line shows the ratio \(a_3/a_1\) expected for “perfectly triaxial” ellipsoids (i.e. those with \(a_2/a_1 = a_3/a_2\)) given the median values of \(a_2/a_1\) in the upper panel. In what follows we shall only consider haloes with \(N_{\text{FOF}} \geq 500\), unless explicitly stated otherwise.

What determines the shape of a proto-halo? Several studies have assumed that proto-halo boundaries can be identified with a fixed-level iso-density contour drawn around local density maxima in the linear density field (e.g., Bardeen et al., 1986; Hoffman, 1986, 1988; Heavens & Peacock, 1988; Catelan & Theuns, 1996). However, as discussed by Porciani et al. (2002), this scenario is not true for proto-haloes, in general; the boundaries of proto-haloes are not fully determined by the self-gravity of the local mass distribution, but rather by the tidal field generated by the external mass distribution.

In Figure 2 we show an example of this. Here we plot the linearly extrapolated density field, \(1 + \delta\) centered on a \(2.99 \times 10^{13} h^{-1} M_\odot\) proto-halo (red dots) identified in the initial conditions of our 150 \(h^{-1}\) Mpc box. Isodensity contours corresponding to fixed density contrasts of \(\delta = 1.0\) and \(\delta = 1.686\) are also shown. The iso-density contours enclosing the peak around which the halo collapses provides an intuitive sense of the (projected) spatial distribution of dark matter surrounding the peak: the peak is extended along a high density ridge connecting two massive structures and separating two low density voids. The long axis of the proto-halo, however, lies transverse to that of the density ridge in which it forms. It is easy to understand why. Material flowing out of the voids leads to maximal compression along an axis roughly perpendicular to the density ridge, whereas tides from the surrounding large scale structure cause dilation along the ridge. The net result is a push-and-pull of material that sets the shape of the proto-halo such that its long axis coincides with the direction of maximum compression, or infall, and its shortest axis to direction of maximum dilation. The grey arrows in Figure 2 show the linear velocity field in the vicinity of the proto-halo, and help to clarify the impact of large-scale tides on gravitational collapse around density peaks.

Similar results were discussed previously by Porciani et al. (2002), who found a very strong alignment between the eigenvectors of proto-halo’s inertia tensors and those of the tidal shear at the proto-halo’s center. Their results, based on one cosmological N-body simulation, indicate that the the geometry of the linear tidal field has a substantial impact on the geometry of Lagrangian regions which are destined

### Table 1. Best-fit parameters describing the average axis ratios, \(a_2/a_1\) and \(a_3/a_1\), for dark matter proto-haloes as a function of mass. Results for both FOF and SO halos are provided.

| axis ratio | Halo Def. | \(Q\) | \(S_0\) |
|-----------|-----------|------|------|
| \(a_2/a_1\) | FOF       | 0.0286 | 0.666 |
| \(a_3/a_1\) | SO        | 0.0618 | 0.345 |
| \(a_2/a_1\) | FOF       | 0.0277 | 0.671 |
| \(a_3/a_1\) | SO        | 0.0629 | 0.336 |

Figure 1. Mass dependence of the axis ratios, \(a_2/a_1\) and \(a_3/a_1\), computed for proto-haloes in our simulation initial conditions. Points show the median trends; boxes and whiskers the 10\(^{\text{th}}\), 25\(^{\text{th}}\), 75\(^{\text{th}}\), and 90\(^{\text{th}}\) percentiles of the distribution in each mass bin. Blue symbols correspond to our 150 \(h^{-1}\) Mpc box simulation; red to our 1200 \(h^{-1}\) Mpc box. Vertical arrows indicate the lower 500-particle limit imposed in our halo sample. The dotted lines show the best-fit (eq. 7) to the combined median trends for mass bins corresponding to haloes with more than 500 particles. The thick black lines in the lower panel show the minor-to-major axis ratios expected for “perfectly triaxial ellipsoids”: \(a_2/a_1 = a_3/a_2\).
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Figure 2. Linear density and velocity fields in the vicinity of a proto-halo identified in our 150 $h^{-1}$ Mpc box simulation. The proto-halo is shown using red dots; thick contours highlight fixed thresholds, $\delta = 1.0$ and $\delta = 1.686$, of the linear density field, which has been smoothed on the halo mass scale and linearly extrapolated to $z = 0$. The halo forms in the vicinity of a density peak of the same characteristic mass, yet the mass distribution of the proto-halo and its associated density peak are very different; the geometry of the peak is determined only by the density threshold, whereas the proto-halo shape and orientation are determined by the external tidal field generated by the surrounding large scale structure.

to form haloes. Whether or not the inertia tensors of density peaks (above a given density contrast) correlate with their local tidal field remains a pending issue which holds particular relevance for theories of galaxy formation.

From a dynamical perspective the in and outflow of material around proto-halo centers can be characterized by the signs of the eigenvalues of the deformation tensor measured at their centers of mass. These indicate whether the initial flows along the principle axes are of compression (+) or dilation (−).

If proto-halo boundaries are set by the tidal field then both the relative magnitudes and the signs of the $\lambda$s determine their shape. In Figure 3 we plot the ratios $a_i/a_j$ versus $\lambda_i/\lambda_j$ for proto-haloes in our simulations, showing separately two different mass bins. The $\lambda$s are measured at each proto-halo’s geometric center after smoothing the linear tidal field on the halo mass scale with a spherical tophat filter. In our 150 $h^{-1}$ Mpc box we select haloes in the range $10^{12} \leq M_{\text{FOF}}/(h^{-1}M_\odot) \leq 5 \times 10^{12}$; the contours show the 50 and 75 percentiles of their distribution. In our 1200 $h^{-1}$ Mpc box 2686 haloes are selected from a higher mass bin spanning $5 \times 10^{14} h^{-1} M_\odot$ to $2 \times 10^{15} h^{-1} M_\odot$. The medians of the joint distributions for these mass bins are shown as blue circles and red triangles, respectively. The Spearman rank correlation coefficients for each of these samples are provided in the panels. In the high mass bin all proto-haloes have $\lambda_1 > \lambda_2 > \lambda_3 > 0$; in the lower mass bin $\lambda_1 > \lambda_2 > 0$ for all haloes, but $\lambda_3 > 0$ for only 41.9%†.

The trends in Figure 3 can be interpreted in terms of the $D - I$ correlation discussed at length by Porciani et al. (2002) and Lee & Pen (2000). Consider each of the columns, in turn. In the left-hand panels we plot the various axis ratios $a_i/a_j$ versus $\lambda_2/\lambda_1$. For $\lambda_2 \sim \lambda_1$ roughly equal compression

† In general, we find that essentially all proto-haloes with $N_{\text{FOF}} \geq 100$ have $\lambda_1 > \lambda_2 > 0$, but that there is a strong mass dependence to the fraction with $\lambda_3 > 0$. Below $10^{12} h^{-1} M_\odot$, for example, $\gtrsim 87\%$ of proto-haloes have $\lambda_3 < 0$; above $10^{14} h^{-1} M_\odot$ $\gtrsim 97\%$ have $\lambda_3 > 0$; $\lambda_3 > 0$ for 50% of haloes at $M_{\text{FOF}} \approx M_\star$, the characteristic mass for collapse at $z = 0$. 

occurs along the major and intermediate axes of the tidal field. In this case we might expect the proto-halo to be approximately spherical (in the case that $\lambda_3 \lesssim \lambda_2$), or oblate (if $\lambda_3 \lesssim 0$). The Massive haloes (red curves) with $\lambda_3 > 0$ are, on average, more spherical than the lower mass ones: the average value of $(a_2/a_1)$ is $\sim 0.80$, and for $(a_3/a_1) \sim 0.65$, with very little mass-dependence to the mean. In the lower mass bin, however, haloes have $-0.5 \lesssim \lambda_2/\lambda_1 \lesssim 0.5$ which leads to weak trends between the axis ratios and the ratio $\lambda_2/\lambda_1$. These trends can be interpreted as follows. For $\lambda_2 \sim \lambda_1$ haloes are slightly oblate; $a_3/a_2$ decreases as $\lambda_2 \rightarrow \lambda_1$, while $a_2/a_1$ increases. As $\lambda_2 \rightarrow 0$, however, the systems then become more prolate, since $\lambda_1 > \lambda_2 \approx 0$, which guarantees $\lambda_3 \lesssim 0$. In this case, $a_2/a_1$ decreases while $a_3/a_2$ increases (more prolate configurations).

Next we consider the far right panels, which plot the ratios $a_i/a_j$ against $\lambda_i/\lambda_j$. Again certain correlations are seen amongst these parameters. For $\lambda_3 \sim \lambda_2$ proto-haloes are close to spherical (if $\lambda_1 \approx \lambda_2 \approx \lambda_3$) or prolate (if $\lambda_1 > \lambda_2 \approx \lambda_3$). For prolate systems $a_3/a_2$ increases as $\lambda_3 \rightarrow \lambda_2$; for spherical systems both $a_2/a_1$ and $a_3/a_2$ increase. For small or negative values of $\lambda_3$ we see a sharp decrease in the ratios $a_3/a_1$ and $a_3/a_2$ as the systems become less spherical, and less prolate, respectively.

Finally we consider the middle panels, which plot $a_i/a_j$ versus $\lambda_i/\lambda_1$; these show the strongest and least mass-dependent correlations of all. The trends can be understood as follows. In cases where $\lambda_3/\lambda_1$ is large and positive the compression exerted by external tidal forces is most isotropic and $c \rightarrow b \rightarrow a$ (recall that in the spherical limit $\lambda_{1,2,3} = \delta/3$). As $\lambda_3$ becomes large and negative, however, there is substantial dilation along the minor axis of the tidal field. Since $\lambda_2 \geq 0$ for all of our haloes we expect to see a sharp decrease in the axis ratios $a_3/a_1$ and $a_3/a_2$ due to the tidal dilation along the minor axis of the proto-halo.

It should be emphasized that these correlations are subject to the fact that the proto-haloes are destined to collapse to form a single object by $z = 0$. The Lagrangian geometries of other present-day structures, such as filament, sheets or voids may differ substantially from these.

4 NON-LINEAR DYNAMICS: ELLIPSOIDAL COLLAPSE OF A TRIAXIAL OVERDENSITY

Based on these results, we consider here modifications to the classic ellipsoidal collapse model of Bond & Myers (1996) in order to follow the collapse of an initially triaxial (rather than spherical) perturbation in the presence of an external tidal field. In principle, this situation is much more complicated since the external shear need not be aligned with the principal axes of the initial ellipsoid. However, in their systematic study of proto-halo shapes Porciani et al. (2002) found a nearly perfect alignment between the principal axes of the total shear and the inertia tensor of proto-haloes in numerical simulations (see also Lee & Pen, 2001; Lee et al., 2009). This suggests that either the internal shear dominates over the external, or that the shapes of proto-haloes are dictated by the external shear.

In our model we consider a uniform density perturbation on top of a flat Friedmann-Robertson-Walker background whose energy content is dominated by the matter density, $\rho_m$, and a cosmological constant, $\Lambda$. We model the perturbation as an homogeneous ellipsoid with semi-axes of physical length $a_i$ ($i = 1, 2, 3$) and density contrast $\delta$, and adopt Cartesian coordinates, $x_i$, measured along the principal axes of the ellipsoid. In this coordinate system, the internal gravitational potential of the ellipsoid can be written

$$\Phi_{\text{int}}(x) = \pi G (1 + \delta) \rho_m \sum_{i=1}^{3} b_i x_i^2,$$

where the coefficients $b_i$ are given by Carlson’s elliptic integrals (Kellogg, 1929; Chandrasekhar, 1969):

$$b_i = a_1 a_2 a_3 \int_0^\infty \frac{\, \mathrm{d} \tau}{\tau + a_i^2 \prod_{j=1}^{3} (\tau + a_j^2)^{1/2}}.$$

For a sphere $b_1 = b_2 = b_3 = 2/3$ and, in general, $\sum b_i = 2$ as a consequence of Poisson’s equation. The potential generated by the uniform distribution of matter outside the perturbation can be written

$$\Phi(x) = -\pi G \rho_m \sum_{i=1}^{3} \left[ b_i (1 + \delta) + \beta_i + \frac{\Delta c^2}{3} \right] x_i^2.$$

![Figure 3.]( attachment)
The early evolution of the internal gravitational potential of the ellipsoid can be determined by combining eq.s \ref{eq:linear} with \ref{eq:initial_conditions}. The principal axes of the quadrupole, the tidal gravitational potential is then $\Phi_{\text{tid}}(y) = \sum_i \lambda_i^{(\text{ext})} \eta_i^1$, with $\sum_i \lambda_i^{(\text{ext})} = 0$.

Assuming that the tidal shear is parallel to the principal axes of the density perturbation (which naturally arises when one starts with a sheared spherical perturbation), we can write the equation of motion for the semi-axis lengths of the ellipsoid as

$$\ddot{a}_i = \left[ \frac{\Lambda c^2}{3} - 4\pi G \rho_a \left( \frac{1 + \delta}{3} + \frac{\beta_i}{2} \delta + \lambda_i^{(\text{ext})} \right) \right] a_i ,$$

where dots denote time derivatives.

Since the potential is quadratic, and the acceleration is linear in the coordinates, all ellipsoids with the same axes aspect ratios evolve self-similarly, and the density perturbation remains homogeneous. Given the comoving coordinates of the axes extrema, $q_i$, the evolution thus preserves $a_1 a_2 a_3 (1 + \delta) = q_1 q_2 q_3 a^3 = \text{constant}$, and so

$$\delta = \frac{q_1 q_2 q_3}{a_1 a_2 a_3} a^3 - 1 .$$

Initial conditions can be set at some early time, $t_0$, using the Zel’dovich approximation:

$$a_i(t_0) = q_i \left[ 1 - \lambda_i(t_0) \right] a(t_0) ,$$

$$\dot{a}_i(t_0) = H(t_0) a_i(t_0) - q_i H_D(t_0) \lambda_i(t_0) a(t_0) .$$

Here $H = \dot{a}/a$ is the Hubble parameter; $D$ the linear growth factor; $H_D \equiv \dot{D}/D$, and $\lambda_i(t) \propto D(t)$ are the eigenvalues of the linear deformation tensor. In order to match the last expressions with eq. \ref{eq:linear_approx}, we also derive the eigenvalues of the deformation tensor at $t_0$ from the total gravitational potential:

$$\lambda_i^{(\text{tot})}(t_0) = \delta(t_0) \left( \frac{1}{3} + \frac{\beta_i(t_0)}{2} \right) + \lambda_i^{(\text{ext})}(t_0) .$$

Only the components of this expression that scale as $D(t)$ will contribute to the $\lambda_i$ terms appearing in the Zel’dovich approximation.

The early evolution of the internal gravitational potential of the ellipsoid can be determined by combining eq.s \ref{eq:linear} and \ref{eq:initial_conditions} and then performing a Taylor expansion in powers of $D(t)$:

$$\beta_i(t) = \beta_i^{(0)} + \Delta \beta_i D(t) + O[D^2(t)] .$$

Here $\beta_i^{(0)}$ is evaluated using eq. \ref{eq:initial_conditions} with $a_i = q_i$. The second term in the expansion is given by

$$\Delta \beta_i(t) = \sum_{j=1}^3 \alpha_{ij} \lambda_j ,$$

where

$$\alpha_{ij} = q_1 q_2 q_3 \int_0^\infty \frac{(2q_i^2 - \tau) d\tau}{(q_i^2 + \tau)^2 \prod_{i=1}^3 (q_i^2 + \tau)^{1/2}}$$

and, for $j \neq k$,

$$\alpha_{ik} = -q_1 q_2 q_3 \int_0^\infty \frac{\tau d\tau}{(q_i^2 + \tau)(q_k^2 + \tau)^{1/2} \prod_{i=1}^3 (q_i^2 + \tau)^{1/2}} .$$

When all the $q_i$’s are equal we have $\alpha_{ij} = 8/15$ when $i = j$.

**Figure 4.** Evolution of the axis lengths for collapsing ellipsoids. Left panels assume the initial overdensity is a uniform sphere; right-hand panels adopt an initially triaxial overdensity with axis ratios $a_2/a_1 = a_3/a_2 = 0.8$. The evolution of the axis lengths is determined by the total tidal field, $(e, p, \delta)$, as shown in each panel. Left panels, the initially triaxial perturbation we modify the external tidal field to account for the initially non-zero internal component, such that the total tidal field is the same in the left and right-hand panels.

Here $\beta_i = b_i - 2/3$ and $c$ is the speed of light.

In a realistic cosmological setting the perturbation will be immersed in a randomly fluctuating density field and will be subject to external tidal forces. The net effect of these tides can be approximated by expanding the external potential in multipoles and focusing on the quadrupole term. The dipole term generates a translation of the ellipsoid’s center of mass but does not alter its shape; contributions from higher-order moments are generally subdominant (Quinn & Binney, 1992)). In a coordinate system which is parallel to
Figure 5. Comparison of the collapse boundaries predicted by the ellipsoidal collapse model for various initial perturbation shapes. The different panels show separately the linearly extrapolated over-density required for collapse along the primary axis (top-left), the intermediate axis (top-right), and the minor axis (bottom-left) of the initial perturbation. The bottom-right panel shows the barrier height required for all three axes to collapse. The shaded and hatched regions show the effect of varying the prolaticity of the tidal field over the range $-\epsilon/2 \leq p \leq \epsilon/2$. Orange, blue and white regions show results for different initial perturbation shapes: $a_2/a_1 = a_3/a_2 = 1$, $a_2/a_1 = a_3/a_2 = 0.8$, and $a_2/a_1 = a_3/a_2 = 0.6$, respectively. In all cases, the major axis of the tidal field is assumed to be aligned with the major axis of the initial perturbation; the corresponding intermediate and minor axes are also aligned.

and $-4/15$ when $i \neq j$. Note that this reduces to the result of Bond & Myers (1996), $\dot{\beta}_i \simeq \Delta \dot{\beta}_i D(t) = (4/5) t_i$ for $q_1 = q_2 = q_3$. Thus, the term $\beta \delta/2$ in eq. 15 is of second order in $D(t)$ and does not contribute to the linear shear tensor, which is fully generated by the external tides. However, contrary to the case discussed by Bond & Myers (1996), for initially triaxial perturbations the linear velocity shear is composed of two terms

$$\lambda_i = \frac{\delta}{3} + \left[ \frac{\beta_i^{(0)}}{2} + \lambda_i^{(\text{ext})} \right].$$

(20)

For the typical axes ratios of proto-haloes in our simulations ($q_2/q_1 \simeq q_3/q_2 \simeq 0.8$) we find

$$\beta_i^{(0)} \simeq (-0.170, -0.011, 0.182),$$

(21)

showing that internal shear slows down collapse along the long axis and speeds it up along the short axis.

Solving eq. 11 requires knowledge of the time evolution of the external tidal field, $\lambda_i^{(\text{ext})}$. Bond & Myers (1996) proposed two limiting approximations. In one case, the external structure is assumed to grow independently of the ellipsoid and the shear scales as the linear growth factor,

$$\lambda_i^{(\text{ext})}(t) = \frac{D(t)}{D(t_0)} \lambda_i^{(\text{ext})}(t_0).$$

(22)

This approximation is accurate when the shear is dominated by the contribution from large scales. Alternatively, one can assume that the external shear is dominated by the contribution from the regions immediately surrounding the ellipsoid, and is completely driven by the shape of the ellipsoid itself. In this case, $\lambda_i^{(\text{ext})}(t) = (5/4) \beta_i$ (note that this reduces to
eq. 22 at early times). The relation above only holds for an initially spherical perturbation but can be generalized to triaxial ones using

$$\lambda_i^{(\text{ext})}(t) = \sum_j \alpha^{-1}_{ij} \left( \beta_j - \beta_j^{(0)} \right),$$

where the $\alpha^{-1}_{ij}$ are the elements of the inverse matrix of $\alpha_{ij}$. A hybrid model which interpolates between these asymptotic regimes has been recently proposed by Angrick & Bartelmann (2010). In this case one follows the non-linear growth proposed by Bond & Myers (1996) until the corresponding axis turns around while the linear growth is used afterwards. For simplicity, we consider only the linear evolution of the external tides in this work.

The ellipsoidal model for halo collapse is not fully self consistent. For example, the presence of the ellipsoidal perturbation will produce inhomogeneities in the background (the potential is not quadratic outside the perturbation) which will, in turn, perturb the density distribution within it. This effect is usually neglected by assuming that both the perturbation and the background remain homogeneous at all times (e.g., Icke, 1973; White & Silk, 1979; Eisenstein & Loeb, 1995). Furthermore, external tides cannot be generated by a smooth background, which should be interpreted as an effective mean field. The presence of substructure (which are not considered in the ellipsoidal model) can also modify (and in particular slow down) the collapse of a perturbation (e.g., Davis & Peebles, 1977). Finally, the equations of motion do not account for the physics that results in the virialization of the collapsing structure. These relaxation processes are crudely mimicked by “freezing” the collapse along each axis once a critical radius $a_{\alpha_i j} = f a q_i$ is reached. Bond & Myers (1996) suggested using the radial freeze-out factor $f = 0.177$, which reproduces the virial contrast of $\delta \approx 178$ characterizing the collapse of a spherical perturbation in an Einstein-de Sitter universe. (Introducing this radial freeze-out factor also modifies slightly the characteristic density required for collapse; in the spherical limit $\delta_s \approx 1.686$ for collapse to an infinite density point, $\delta_s \approx 1.609$ for $f = 0.177$.) More recently, Angrick & Bartelmann (2010) have used the tensor virial theorem to predict more realistic values of $f$. However, the last phase of collapse is generally rapid, and altering the value of $f$ does not strongly affect the estimated epoch of virialization. For ease of comparison with previous work, all applications of this model presented below have assumed an Einstein-de Sitter background cosmology, and a freeze-out factor $f = 0.177$.

Figure 4 plots the trajectories of the three axes lengths, in physical units, for the ellipsoidal collapse model of Bond & Myers (1996) (left panels) and for the modified model described above (right). We start our numerical calculations at $t_0 = 0.250$ Myr, at which point $\delta(t_0) = 1.686$ sets the overdensity based on the spherical collapse criterion. The top two panels show the evolution of an initial spheroidal (left) and ellipsoidal perturbation with axis lengths $a_2/a_1 = a_3/a_2 = 0.8$ (right) when no external tides are present. The lower panels show, for the same initial set-up, the effect of varying the external tidal field on the evolution of the axis lengths. In all cases the external tides are imposed at $z_t$ so that the total tidal field acting on the overdensity is the same in both models.

Figure 4 highlights a number of things worth noting. First, for a given $e$, the Bond & Myers (1996) model predicts the formation of pancake-like structures when $p > 0$; for $p < 0$ filamentary structures form. Thus, for a given $(e, p)$ the ellipsoidal collapse model predicts the axis ratios of the object at all times, and in particular, at the final time. These results have been used recently by Rossi et al. (2011) to model the shape distribution of collapsed structures within the excursion set theory. Notice, however, that relaxing the assumption of spherical symmetry at $z_t$ substantially changes the evolution of the axis lengths, even when the total tidal field is chosen to be identical. In particular, the ordering of the axes lengths can change; what is initially the primary axis, for example, can swap between intermediate and minor, and then back to primary again. In addition, the turnaround and freeze-out times also differ in the initially triaxial model, a result which, as we show below, has important implications for modeling collapse boundaries for halo formation.

In Figure 5 we illustrate the role played by the initial shape of the ellipsoidal perturbation in setting the collapse thresholds. Orange, blue and white regions show the results for $a_2/a_1 = a_3/a_2 = 1.0$ and 0.6, respectively; the shading shows the effect of varying the tidal prolaticity over the range $-c/2 \leq p \leq e/2$. Each panel plots separately the linearly extrapolated density contrast required for collapse along each of the three principle axes of the initial perturbation (the first axis corresponds to the major axis of inertia; the second to the minor axis). In the lower-right panel we plot the “collapse boundary”, defined as the linear density contrast required for all three axes of the initial ellipsoid to collapse.

In the absence of external tidal forces an initially non-spherical perturbation requires higher density contrasts for collapse than the spherical case. This is because the internal shear acting along the long axis of the ellipsoid always slows collapse, and external tides are required in order to balance the internal forces. Because $p = 0$ corresponds to $\lambda_2 = 0$, collapse along the intermediate axis is not strongly affected by tides; only the (sub-dominant) internal shear induced by the aspherical nature of the initial perturbation matters for this. Along the minor axis of the initial ellipsoid collapse barriers are always lower than the spherical case. This is because the internal shear acting along this axis always aids in collapse, and very large values of tidal ellipticity (corresponding to very negative values of $\lambda_3$) are required to suppress this.

The collapse barriers inferred from the ellipsoidal collapse model depend strongly on the assumed initial shape of the collapsing ellipsoid. In the presence of the same tidal field the time required for a given axis to collapse is sensitive to its initial displacement from the perturbation center. Consequently, the linearly extrapolated overdensity required for collapse can vary significantly for different initial shapes.

$^1$ Recall that the primary axis of the perturbation is assumed to align perfectly with the major axis of the tidal field, so large ellipticities correspond to greater compression along this axis.
Figure 6. Smoothed linear overdensities, extrapolated to $z = 0$, plotted as a function of the ellipticity of the tidal shear for proto-haloes containing more than 500 particles. Both quantities are measured at the proto-halo center of mass and are averaged over a spherical Lagrangian region containing a mass $M_{\text{FOF}}$. Colored circles show the predicted collapse thresholds derived from the ellipsoidal collapse model for several different initial perturbation shapes. In each case, the initial perturbation is assumed to be "perfectly triaxial" (i.e. $a_2/a_1 = a_3/a_2$) and we have set $p = 0$ for simplicity. The solid lines (of the same color) show the best-fit collapse boundary given by eq. 24; best-fit parameters are given in Table 2; residuals from the best-fits are shown in the bottom panel. For comparison, we also show the Sheth et al. (2001) barrier using a dashed line.

Figure 7. Same as Figure 6 but for haloes in four narrow mass bins. Contours enclose the 50 and 75 percentiles of the measured collapse boundaries for haloes in each mass bin. In each panel the dot-dashed curve shows the ellipsoidal collapse boundary for an initially spherical perturbation acted on by external tidal forces with $p = 0$ and ellipticity $e$. The solid curves show the barrier heights for an initially triaxial perturbation with axis ratios consistent with the average axis ratio of haloes in each mass bin. The total tidal field (internal plus external) is the same in both cases. For comparison, we also show the fitting formula of Sheth et al. (2001) as a dashed line, converted to the most probable halo ellipticities via $e_{\text{mp}} = \sigma/\delta/\sqrt{5}$.

| $a_2/a_1$ | $A$  | $B_1$  | $B_2$  |
|-----------|------|--------|--------|
| 1.0       | 0.4079 | 1.4913 | 1.7588 |
| 0.9       | 0.4212 | 2.0327 | 1.5374 |
| 0.8       | 0.4670 | 2.5969 | 1.2453 |
| 0.7       | 0.5489 | 3.0730 | 0.9278 |
| 0.6       | 0.6861 | 3.4729 | 0.6133 |
| 0.5       | 0.9332 | 3.8446 | 0.2978 |
| 0.4       | 1.4157 | 4.7276 | 0.0418 |

Table 2. Best-fit parameters, $A$, $B_1$, and $B_2$, for the ellipsoidal collapse barriers (eq. 24) shown in Figure 6. The model results are based on the collapse of "perfectly triaxial" ellipsoids (i.e. $a_2/a_1 = a_3/a_2$) embedded in a tidal field for which $p = 0$.

5 COLLAPSE THRESHOLDS FROM ELLIPSOIDAL COLLAPSE

In this Section we wish to give a more quantitative assessment of the effects previously discussed. In what follows, we will consider the collapse barriers inferred from the ellipsoidal collapse model and express them as a function of the ellipticity of the tidal field, $e$. We will compare these directly with the actual overdensities and tidal ellipticities measured at proto-halo centers of mass, with both quantities averaged over the appropriate Lagrangian volume. Unless explicitly stated otherwise, we shall follow Sheth et al. (2001) and assume $p = 0$ when calculating barrier heights from the ellipsoidal collapse model.

5.1 Collapse barriers from the ellipsoidal collapse model

Our adaptation of the ellipsoidal collapse model of Bond & Myers (1996) yields a barrier height that is both a function of the statistics of the tidal field at any given point, $(e, p)$, and also of the sphericity of the collapsing region. To reduce the number of degrees of freedom in this model we make the following simplifying assumptions: a) CDM proto-haloes can be modelled as "perfectly triaxial" ellipsoids (i.e. $a_2/a_1 = a_3/a_2$), and b) the prolaticity of the tidal field takes on the mean value at any mass scale, i.e. $p = 0$.

Figure 6 plots the various collapse boundaries, $\delta(e)$, inferred from our ellipsoidal collapse model for several initial perturbation shapes. Different colored points correspond to initial overdensities with different axis ratios, which are indicated on the plot. The data are well reproduced by the simple function...
\[ \delta = A \left[ \exp(-B_1 \epsilon) + \exp(B_2 \epsilon) \right]^2, \]  

as can be seen by the solid curves in Figure 6, whose colors are chosen to match the model predictions. These curves differ markedly from the Sheth et al. (2001) barrier, which is shown with a dashed black line. The best-fit parameters vary systematically with the initial shape of the ellipsoidal, and are listed for each of the plotted models in Table 2.

In the lower panel of Figure 6 we plot the residuals from the best-fit to eq. 24. The residuals are typically small, \( \lesssim 5\% \) over a wide range of ellipticities. Note, however, that the model predictions have a much sharper upturn than the best-fit profile; beyond some characteristic ellipticity the fits diverge from the model predictions. However, as we have already seen, the average minor-to-major axis ratio varies from \( \langle a_3/a_1 \rangle \sim 0.4 \) at halo masses \( \sim 10^{13} h^{-1} M_{\odot} \) to \( \sim 0.65 \) on the scale of rich galaxy clusters. The mass dependence of the average value of \( \langle a_2/a_1 \rangle \) is weaker, ranging from \( \sim 0.7 \) to \( \sim 0.8 \) over the same mass range. On average, the most appropriate barriers to compare to the simulation results are, therefore, the yellow and green curves in Figure 6. For these, the barrier heights predicted by the ellipsoidal model are accurate to within \( \sim 5\% \) for ellipticities between roughly 0.65 and 0.8; in our 150 h\(^{-1}\) Mpc-box simulation only \( \sim 6.8\% \) of haloes have \( e > 0.65 \); 2.7\% have \( e > 0.8 \). The typical mass of these objects is low, with an average of \( \sim 32 \times 10^{10} h^{-1} M_{\odot} \). For the higher mass scales probed by our 1200 h\(^{-1}\) Mpc run, there are no haloes with \( e > 0.65 \). Our fits are therefore accurate over the range of ellipticities and halo masses probed by our simulations. We discuss this issue further in the following subsections.

The shaded area and contours show the distribution of \( (\delta_b, \epsilon) \) for all proto-haloes with \( N_{\text{FOF}} \geq 500 \) in both of our simulations. Clearly the very large scatter in the average linear overdensity at dark matter proto-halo centers is not captured by the standard version of the ellipsoidal collapse model. Considering the large variation in barrier heights consistent with the ellipsoidal collapse model it is interesting to investigate the extent to which the variation is induced by initial shapes of proto-haloes. We turn to this next.

### 5.2 Mass dependence of barrier heights

Figure 7 plots the dependence of the Barrier height on tidal ellipticity for haloes selected to lie in several narrow mass bins spanning the range \( M_{\text{FOF}} \sim 10^{13} h^{-1} M_{\odot} \) to \( \sim 6 \times 10^{14} h^{-1} M_{\odot} \). Red contours enclose the 50 and 75 percentiles of the distributions for our simulated halo data; the curves show various theoretical estimates of the barrier height. The dashed curves show the fit given by Sheth et al. (2001) assuming \( \epsilon = \epsilon_{\text{mp}} = \sigma/\delta/\sqrt{5}; \) the dot-dashed curves show the results of the “standard” ellipsoidal collapse model (with \( p = 0 \) and \( q_1 = q_2 = q_3 \)); the solid curves show the barrier height computed using the average \( (q_2/q_1) \) and \( (q_3/q_1) \) for haloes in each bin. The percentage of haloes whose measured overdensities exceed these barriers are provided next to each curve.

Clearly the minimum barrier height for haloes of any given mass is more accurately described by the ellipsoidal collapse model once the initial perturbation shape is chosen to match the mean shape in each mass bin. At the lowest masses, where proto-haloes can have the most flattened geometries, only \( \sim 49.6\% \) of proto-haloes lie above the standard ellipsoidal collapse barrier, whereas \( \sim 93.8\% \) lie above the collapse barrier inferred for proto-haloes with \( a_2/a_1 = 0.70 \) and \( a_3/a_1 = 0.45 \); the mean values for this mass. Towards higher masses, where proto-haloes are intrinsically more spherical, we find \( \sim 73.2\% \) lie above the dot-dashed curve (56.0% above the Sheth et al. (2001) barrier) and \( \sim 95.5\% \) above the barrier when modelled to account for the average initial shape of the Lagrangian proto-halos.

Note, however, the large scatter in \( (\epsilon_b, \delta_b) \) at any given mass scale. Within the standard ellipsoidal collapse model described by Bond et al. (1991), the scatter in barrier heights at any mass and ellipticity is completely determined by the prolaticity of the tidal field in the vicinity of the proto-halo. The observed scatter in \( \delta_b \) at fixed \( \epsilon_b \) shown in Figures 6 and 7 can not be accounted for by variation in prolaticity alone. By comparing the scatter amongst the red points in Figure 7 to the grey shaded regions in the lower-right panel of Figure 5, it is clear that changes in prolaticity alone cannot account for the variety of measured barrier heights at fixed halo mass. In the next section we explore whether the variation in proto-halo shape can account for some of this scatter.

### 5.3 Shape dependence of barrier heights

Some insight into the origin of scatter in barrier heights can be gained by considering how the measured proto-halo boundaries vary as a function of proto-halo shape at fixed mass. In Figure 8 we plot the joint distribution of \( (\delta_b, \epsilon_b) \) for proto-haloes in our 150 h\(^{-1}\) Mpc box simulation that lie in the mass range \( 24.3 < M_{\text{FoF}}/(10^{13} h^{-1} M_{\odot}) < 48.7 \) (the same proto-halo sample plotted in the upper-left panel of Figure 7). These proto-haloes have been split into four separate subsamples based on the value of their axis ratios. In order to make a more meaningful comparison with the predicted collapse barrier, which we have computed for a “perfectly triaxial” ellipsoid, we also select proto-haloes to have \( a_2/a_1 \approx a_3/a_2 \) spanning a narrow range. Proto-haloes in the upper-left panel are the most spherical in this mass bin, and \( a_3/a_1 > 0.8 \); in the upper-right panel all proto-haloes have \( 0.65 < a_2/a_1, a_3/a_2 < 0.75 \). Those shown in the lower left panel have \( 0.55 < a_2/a_1, a_3/a_2 < 0.65 \), and those in the lower-right panel are highly flattened systems, with \( 0.45 < a_2/a_1, a_3/a_2 < 0.55 \). For comparison, we also plot in each panel the collapse barriers associated with the Bond et al. (1991) ellipsoidal collapse model (dot-dashed line), the Sheth et al. (2001) fit (dashed line), and the “true” ellipsoidal collapse barrier consistent with the quoted proto-halo shape (solid curve). As in Figure 7, we mark each curve with the percentage of haloes that lie above it.

There are a couple of things in Figure 8 worth highlighting. First, note that the distribution in \( (\epsilon_b, \delta_b) \) at fixed halo mass depend sensitively on the the intrinsic shape of the proto-halo. For example, of objects with \( a_2/a_1 \) and \( a_3/a_1 > 0.8 \) all have \( e \lesssim 0.42 \). For the strongly triaxial proto-haloes this distribution is shifted to the right; for example, we find that only 21.8% of all proto-haloes with \( 0.45 < a_2/a_1, a_3/a_2 < 0.55 \) have \( e < 0.42 \). For all haloes in this mass bin only about half have \( e < 0.42 \). The reason for this is clear in the context of ellipsoidal collapse dynamics. Perturbations that are initially very elongated require
external tidal compression along their major axis to balance the suppressive internal shear. Increasing the ellipticity of the tidal field in our model results in larger external compression along the long axis of the ellipsoid and ultimately reduces the density contrast required for collapse.

Another important point to note is that, for any given proto-halo shape, the minimum barrier heights are described surprisingly well by our simple modification to the classic ellipsoidal collapse model of Bond et al. (1991). The nearly spherical proto-haloes plotted in the upper left panel of Figure 8 are essentially always above the barrier inferred for the collapse of a spherical region in the presence of an external tidal field; indeed, the maximum ellipticity allowed for spherical proto-haloes is \( \sim 0.53 \) (for \( p = 0 \)), for \( e \) greater than this value collapse along the minor axis of the tidal field is impossible, and so “full collapse” never occurs. Relaxing the assumption that the proto-halo should be spherical has a substantial impact on the inferred collapse boundary. For example, of the proto-haloes in this mass range with \( 0.65 < a_2/a_1, a_3/a_2 < 0.75 \) nearly 44% have average internal density contrasts below what is required by the Bond et al. (1991) model for their measured ellipticities. Indeed, for the proto-haloes with the most extreme axis ratios (lower-right panel) we find that only about one in five come from regions of the linear density field that meet the collapse requirement based on the “standard” ellipsoidal collapse model, starting from initially spherical Lagrangian patches.

These results will have important implications for analytic predictions of the halo mass function and clustering properties based on the excursion set method, and may help improve our understanding of structure formation in general. However, in order to be useful one must have a way to anticipate the typical shape of a proto-halo that is expected to form at a given position, smoothing scale, etc. We discuss this next.

6 DISCUSSION

In the ellipsoidal model the collapse threshold depends explicitly on the shape of the tidal field at a given point \( x \); the inferred collapse threshold can then be compared with the true overdensity at that location to decide whether or not collapse should take place. The approach of Sheth et al.
Collapse thresholds for the formation of CDM haloes in the ellipsoidal collapse model

Figure 9. The fraction of haloes of a given mass whose smoothed linearly extrapolated overdensities exceed the threshold for collapse. Different colored curves show results for two simulations; red lines for our 1200 h^{-1} Mpc box simulation; blue curves to our 150 h^{-1} Mpc box run. Dashed lines (with triangles) adopt the mass-dependence of the standard ellipsoidal collapse barrier of Sheth & Tormen; solid curves (with circles) adopt a collapse barrier computed independently for each mass bin assuming an initial density perturbation with axis ratios consistent with the average shape of proto-haloes in each bin. The vertical lines mark the 500 particle limit, below which shape estimates may be biased by low particle numbers.

(2001) was to adopt, for every spatial location, the most probabilistic values of $e$ and $p$ in Gaussian random fields, resulting in a barrier height that depends explicitly on filtering scale: $e_{mp} = \sigma(M)/\delta/\sqrt{\pi}$ (since $p_{mp} = 0$ for any $\sigma(M)$). The resulting barrier height depends on only one variable: the characteristic mass scale under consideration.

We have shown that relaxing the assumption that the assumption of sphericity for the collapsing Lagrangian regions leads to very different predictions for the collapse barriers at any $e$, $p$, or mass scale. But what is the characteristic shape of a proto-halo and how is it related to the external tides, $(e, p)$, the characteristic mass scale, the filtering choice, etc.? In this section we describe two simple methods for estimating in advance the potential shapes of collapsing proto-haloes at any position $x$.

One possibility is to estimate the characteristic shape of collapsing proto-haloes directly from the mass dependence of the characteristic collapse barrier at every characteristic mass scale, since the axis ratios are themselves smooth functions of mass. In order to make the calculation simpler we have assumed that $p = 0$ for all $x$ (so that the tidal field is fully characterized by its ellipticity, $e$) and that dark matter proto-haloes can be approximated as “perfectly triaxial ellipsoids”, i.e. $a_2/a_1 = a_3/a_2$, so that a single axis ratio is sufficient for describing their shape.

In Figure 9 we plot the fraction of proto-haloes whose linear density contrast exceeds the threshold for collapse based on the method outlined above. Here we estimate, at any given halo mass scale, the typical value of $a_2/a_1$ from eq. 7, set $a_3/a_1 = (a_2/a_1)^2$, and compute the corresponding barrier as a function of tidal ellipticity in the usual fashion, by assuming $p = 0$. At any given mass scale, the fraction of points that lie above this threshold are shown as solid circles connected by lines. Triangles connected by dashed lines show the fraction of haloes that lie above the barrier advocated by Sheth et al. (2001). We use blue and red lines to distinguish results obtained from our 150 h^{-1} Mpc and 1200 h^{-1} Mpc box runs, respectively.

Taking into account the average mass-dependence of proto-halo shapes we find that $\gtrsim 90\%$ now lie above the ellipsoidal collapse boundary. Adopting the Sheth et al. (2001) barrier instead yields very different results: at $\sim 10^{13} h^{-1} M_\odot$, for example, only about 70% of proto-haloes should collapse given their averaged ellipticities and density contrasts. At the highest mass scales, where haloes are initially more spherical and have low ellipticities, roughly one half of proto-haloes should not collapse in the standard approach to ellipsoidal collapse. This is a strong indication that carefully accounting for proto-halo shapes (and ultimately their boundaries) is key to realistic excursion set modeling.

Another possibility is to attach potential proto-halo shapes to points in the smoothed linear density field by exploiting the correlations between the ratios $a_i/a_j$ and $\lambda_i/\lambda_j$ of the eigenvalues of the inertia and tidal tensors (see Figure 3). After some experimentation we found a relatively well-defined and mass-independent trend between the ratio $a_3/a_1$ and $\lambda_3/\lambda_1$; we show the median trends and their associated one-sigma scatter for several different mass bins in Figure 10. Note that the haloes selected for this plot span more than four orders of magnitude in mass, yet at any given mass scale the average minor-to-major axis ratios, $a_3/a_1$, are highly correlated with the relative compression factors along the minor and major axes of the tidal field. The main trend can be reasonably well described by the simple function

$$a_3/a_1 = C \times \tanh \left( \frac{\lambda_3}{\lambda_1} \right) + \psi, \tag{25}$$

where $C = 0.465$ and $\psi = 0.488$ are the best-fit parameters. Eq. 25 is shown as a dotted-dashed line in Fig. 10.

Given eq. 25, one can estimate the axis ratios $a_3/a_1$ of potential proto-haloes completely in terms of the tidal field measured at a given point, without making any explicit reference to mass scale. Given an estimate of $a_3/a_1$ we can set $a_2/a_1 = \sqrt{a_3/a_1}$ (i.e. assume perfectly triaxial proto-haloes) and compute the shape-dependent collapse boundaries on a point by point basis with knowledge of only the eigenvalues of the linear tidal field: $\lambda_1 \geq \lambda_2 \geq \lambda_3$. The collapse barrier can then be compared to the true overdensity at that point in the same sense as before.
in this way. Haloes are binned according to the ratio $\lambda$, average overdensities exceed the collapse threshold inferred by the model of Sheth et al. (2001). Different colored curves show results for our two simulations; red lines for our 1200 $h^{-1}$ Mpc box simulation; blue for our 150 $h^{-1}$ Mpc box run.

In Figure 11 we plot the fraction of proto-haloes whose measured density contrasts exceed the collapse threshold inferred from eq. 25. The curves shown in Figure 11 adopt the same colors and line-styles as those in Figure 9. Solid curves with circles correspond to the fraction of proto-haloes with measured density contrasts exceeding the barrier height inferred from our model; dashed curves plot the fraction of haloes exceeding the model of Sheth et al. (2001). Different colored curves show results for our two simulations; red lines for our 1200 $h^{-1}$ Mpc box simulation; blue for our 150 $h^{-1}$ Mpc box run.

7 SUMMARY
We have used two cosmological simulations of structure formation to test a key assumption commonly used in excursion set modeling of structure formation, namely, that the regions of the linear density field that collapse to form bound objects are spherical. We used our results to motivate a deeper look into the collapse boundaries of dark matter halos predicted by the ellipsoidal collapse model. Our main results can be summarized as follows.

- The shapes of the Lagrangian patches that eventually collapse to form CDM halos are highly non-spherical. Over the mass scale $\sim 10^{11} h^{-1} M_\odot$ to $\sim 10^{15} h^{-1} M_\odot$, the average intermediate-to-major axis ratio varies from $\langle a_2/a_1 \rangle \sim 0.7$ to $\sim 0.8$, and $\langle a_3/a_1 \rangle$ from roughly $\sim 0.4$ to $\sim 0.65$. Above the characteristic mass scale for collapse (roughly $10^{15} h^{-1} M_\odot$) proto-haloes are, on average, "perfectly triaxial", with axis ratios $a_2/a_1 \approx a_3/a_2$.

- The shapes of dark matter proto-haloes are primarily determined by the relative compression factors associated with the linear tidal field acting upon them (see Fig 3). For example, haloes that form in regions of the linear density field where $\lambda_1$ is large and positive, but $\lambda_3$ is large and negative, tend to accrete material from very flattened and elongated regions. This is because material is compressed along one axis of the tidal field, and dilated along an orthogonal direction. The net effect of this push-and-pull of material is a strong alignment between the principle axes of the inertia tensors of proto-haloes and those of the tidal field measured at their geometric centers (Porciani et al., 2002; Lee & Pen, 2000). The origin of this correlation is rooted in the tidal field, which sets the boundaries of regions of the linear density field that are able to collapse by a given time.

- The alignment between the eigenvectors of the linear tidal field and the inertia tensor of proto-haloes can be exploited to simplify the application of ellipsoidal collapse to triaxial density perturbations. In Section 4 we modified the...
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standard ellipsoidal collapse model of Bond et al. (1991) in order to account for the non-spherical nature of the Lagrangian patches that collapse to form halos. Relaxing the assumption that these regions are initially spherical yields very different predictions for the thresholds for non-linear structure formation.

- In the absence of external tidal forces any non-spherical density perturbation requires a higher density contrast for collapse than the corresponding spherical perturbation. This is because the internal tidal forces suppress collapse along the major axis of the perturbation, and correspondingly higher densities are needed to counter this. When external tidal forces are introduced as described in Section 4, there is a well-defined minimum barrier height for any non-spherical perturbation; for a given tidal prolateness, \( p \), this minimum occurs when the density contrast for collapse along the major and minor axis of the density perturbation are equal (See Fig 5).

- We provided two possible methods for estimating the typical shape of a proto-halo that may form at any given location in the smoothed linear density field. The first involves measuring the average shapes of proto-haloes as a function of their mass, and using these shapes to compute explicitly the average barrier height for gravitational collapse as a function of halo mass. Another possibility is to estimate it directly the proto-halo shape by exploiting the correlation between the axis ratio \( a_3/a_1 \) and the relative tidal compression factors \( \lambda_3/\lambda_1 \) (see Fig. 10). Since proto-haloes are roughly triaxial (\( a_3/a_2 = a_2/a_1 \)) one can compute the typical shape of a proto-halo that will form at a given location based solely on the ratio \( \lambda_3/\lambda_1 \) measured at that location. As many as \( \gtrsim 90\% \) of (\( N_{\text{OF}} \gtrsim 500 \)) dark matter proto-haloes have measured linear density contrasts that exceed the collapse thresholds inferred in these ways.

All in all, our results indicate that the standard ellipsoidal collapse model of Bond et al. (1991) provides a rather incomplete census of the possible collapse thresholds for the formation of CDM halos from Gaussian random fields. In addition to the specifics of the linear tidal field at a given location, the collapse threshold depends on the assumed shape of the Lagrangian region that eventually collapses to form a bound object. This is because the linear overdensity required for collapse in a given tidal field is sensitive to the initial displacement of a given mass element from the perturbation center, thus changing how the collapse barriers depend on auxiliary properties of the tidal field. These results will have important implications for excursion set modeling of the halo mass function, spatial clustering and mass accretion histories, and may provide new insight into the complex process of dark matter halo formation.

Bond J. R., Cole S., Efstathiou G., Kaiser N., 1991, ApJ, 379, 440
Bond J. R., Myers S. T., 1996, ApJS, 103, 1
Catelan P., Matarrese S., Porciani C., 1998, ApJL, 502, L1+
Catelan P., Theuns T., 1996, MNRAS, 282, 455
Chandrasekhar S., 1969, Ellipsoidal figures of equilibrium
Cole S., Lacey C., 1996, MNRAS, 281, 716
Davis M., Peebles P. J. E., 1977, ApJS, 34, 425
Doroshkevich A. G., 1970, Astrofizika, 6, 581
Eisenstein D. J., Loeb A., 1995, ApJ, 439, 520
Eke V. R., Cole S., Frenk C. S., 1996, MNRAS, 282, 263
Gunn J. E., Gott III J. R., 1972, ApJ, 176, 1
Heavens A., Peacock J., 1988, MNRAS, 232, 339
Hoffman Y., 1986, ApJ, 308, 493
Hoffman Y., 1988, ApJ, 329, 8
Hopkins P. F., Bahcall N. A., Bode P., 2005, ApJ, 618, 1
Icke V., 1973, A&A, 27, 1
Kellogg O. D., 1929, Foundations of Potential Theory
Komatsu E., Dunkley J., Nolta M. R., Bennett C. L., Gold B., Hinshaw G., Jarosik N., Larson M., Page L., Spergel D. N., Halpern M., Hill R. S., Kogut A., Meyer S. S., Tucker G. S., Weiland J. L., Wollack E., Wright E. L., 2009, ApJS, 180, 530
Lacey C., Cole S., 1993, MNRAS, 262, 627
Lee J., Hahn O., Porciani C., 2009, ApJ, 707, 761
Lee J., Pen U., 2000, ApJL, 532, L5
Lee J., Pen U., 2001, ApJ, 555, 106
Lee J., Shandarin S. F., 1998, ApJ, 500, 14
Lemson G., 1993, MNRAS, 263, 913
Lin C. C., Mestel L., Shu F. H., 1965, ApJ, 142, 1431
Ludlow A. D., Porciani C., 2011, MNRAS, 413, 1961
Lynden-Bell D., 1964, ApJ, 139, 1195
Mo H. J., Jing Y. P., White S. D. M., 1997, MNRAS, 284, 189
Mo H. J., White S. D. M., 1996, MNRAS, 282, 347
Monaco P., 1997, MNRAS, 290, 439
Peacock J. A., Heavens A. F., 1985, MNRAS, 217, 805
Peebles P. J. E., 1980, The large-scale structure of the universe
Pillepich A., Porciani C., Hahn O., 2010, MNRAS, 402, 191
Porciani C., Dekel A., Hoffman Y., 2002, MNRAS, 332, 339
Press W. H., Schechter P., 1974, ApJ, 187, 425
Quinn T., Binney J., 1992, MNRAS, 255, 729
Robertson B. E., Kravtsov A. V., Tinker J., Zentner A. R., 2009, ApJS, 696, 636
Rossi G., Sheth R. K., Tormen G., 2011, MNRAS, pp 1032–
Shaw L. D., Weller J., Ostriker J. P., Bode P., 2006, ApJ, 646, 815
Sheth R. K., Lemson G., 1999a, MNRAS, 304, 767
Sheth R. K., Lemson G., 1999b, MNRAS, 305, 946
Sheth R. K., Mo H. J., Tormen G., 2001, MNRAS, 323, 1
Springel V., 2005, MNRAS, 364, 1105
van den Bosch F. C., 2002, MNRAS, 331, 98
Watanabe T., Inagaki S., 1991, PASJ, 43, 413
White S. D. M., Rees M. J., 1978, MNRAS, 183, 341
White S. D. M., Silk J., 1979, ApJ, 231, 1
Zel’Dovich Y. B., 1970, A&A, 5, 84

REFERENCES

Angrick C., Bartelmann M., 2010, A&A, 518, A38+
Bailin J., Steinmetz M., 2005, ApJ, 627, 647
Bardeen J. M., Bond J. R., Kaiser N., Szalay A. S., 1986, ApJ, 304, 15
Bertschinger E., Jain B., 1994, ApJ, 431, 486
Bett P., Eke V., Frenk C. S., Jenkins A., Helly J., Navarro J., 2007, MNRAS, 376, 215