RATE OF CONVERGENCE FOR A MULTI-SCALE MODEL OF DILUTE EMULSIONS WITH NON-UNIFORM SURFACE TENSION

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To Jeannine Saint Jean Paulin, in Memoriam

Abstract. In this paper we are interested in a problem of dilute emulsions of two immiscible viscous fluids, in which one is distributed in the other in the form of droplets of arbitrary shape, with non-uniform surface tension due to surfactants. The problem includes an essential kinematic condition on the droplets. In the periodic homogenization framework, it can be shown using Mosco-convergence that, as the size of the droplets converges to zero faster than the distance between the droplets, the emulsion behaves in the limit like the continuous phase. Here we determine the rate of convergence of the velocity field for the emulsion to that of the velocity for the one fluid problem and in addition, we determine the corrector in terms of the bulk and surface polarization tensors.

1. Introduction. The better understanding of emulsion behavior is important for many industries. In the pharmaceutical industry, they are used to deliver drugs in a more effective way; in the agricultural industry, emulsions are used as delivery mechanism for pesticides. In general, emulsions allow the delivery of insoluble agents to be uniformly distributed in a more efficient way.

We consider an emulsion of two immiscible fluids, one with viscosity $\mu_0$, forming a connected phase, in which droplets with arbitrary shape of the second fluid of viscosity $\mu_d$ are distributed. We assume the flow to be quasi-static and at low Reynolds numbers. Thus, the system is modeled by a stationary Stokes flow in an open, bounded, Lipschitz domain $\Omega$ in $\mathbb{R}^n$ ($n = 2, 3$). The droplets are periodically distributed with characteristic size $a_\varepsilon$ and with the distance between the droplets’ centers of size $\varepsilon$, and $\varepsilon \gg a_\varepsilon$. Moreover, we assume that the droplets do not intersect the boundary, $\partial \Omega$. Additionally, we consider the presence on the droplets’ surface of a non-uniform surface tension, as in the case when surfactants are present.

The homogenization emulsion problem for the non-dilute case has been studied using two-scale convergence by Lipton and Vernescu [10].

Problems in which the effect of surfactants are important can also involve surfactant transport along the interface. In such models the interface possesses its own rheological properties [13]. In this work we do not consider such a scenario.

2010 Mathematics Subject Classification. Primary: 35J25, 35J20; Secondary: 76D07, 76T20.

Key words and phrases. Emulsions, Stokes flow, surface tension, homogenization.
Our goal is to derive the asymptotic behavior of the velocity of the emulsion, $v^\varepsilon$ as $\varepsilon$ tends to zero. Problems of this kind have been the subject of active research for quite some time starting with the work of Taylor [16]. More recently, work on this topic has been done by Ammari, Garapon, Kang and Lee [2], Bonnetier, Manceau and Triki [4], Nika and Vernescu [14], [15] and many others. Our work is an extension of Nika and Vernescu [15] and Bonnetier, Manceau and Triki [4]. The authors of [4] considered droplets with constant surface tension in an ambient fluid and they established an asymptotic form for the velocity of the emulsion in terms of viscous moment tensor and a curvature moment tensor. However, their equations did not include a kinematic boundary condition on the droplet surface. In [8], as well as in [15], the authors imposed a kinematic condition on the droplet surface in order to ensure that the interface remains a material inter-phase boundary. In [15], the kinematic condition is necessary for recovering G.I. Taylor’s viscosity of an emulsion [16]. In the current work we incorporate a non-uniform surface tension and derive an asymptotic form of the emulsion velocity in terms bulk and surface polarization tensors. We organized the paper the following way: in section 2. we introduce the emulsion problem of a periodic suspension of viscous droplets in a viscous fluid and discuss existence and uniqueness results. Moreover, we provide an overview of the main results of the paper. In section 3. we present the assumptions we made on the surface tension that exists on the droplet surface and provide proofs for the main results.

**Notations.** Throughout the paper we are going to be using the following notations:
- $I$ indicates the $n \times n$ identity matrix, $B(r)$ indicates the ball of radius $r$ and boldface letters indicate vectors in $\mathbb{R}^n$,
- $\nabla_s = \nabla - n(n \cdot \nabla)$ is the surface gradient operator. The symbols $\nabla_{s,x}, \nabla_{s,y}$ sometimes will be used to indicate the variable of differentiation,
- $e(u)$ will indicate the strain rate tensor defined by: $e(u) = \frac{1}{2} (\nabla u + \nabla u^T)$. Often subscripts $e_x(u)$ and $e_y(u)$ will be used throughout the paper to identify the variable of differentiation,
- the inner product between matrices is denoted by $A:B = \text{tr}(A^T B) = \sum_{ij} a_{ij} b_{ij}$,
- the tensor product between two vectors $a, b$ is denoted by $a \otimes b = (a_i b_j)$.

2. **Problem statement and outline of main results.** For the homogenization setting of the emulsion problem we define $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$), to be a bounded open set with Lipschitz boundary $\partial \Omega$, and let $Y = \left( \begin{array}{c} 0 \ 0 \end{array} \right)$ be the unit cube in $\mathbb{R}^n$. For every $\varepsilon > 0$, let $N^\varepsilon$ be the set of all points $\ell \in \mathbb{Z}^n$ such that $\varepsilon(\ell + Y)$ is strictly included in $\Omega$ and denote by $|N^\varepsilon|$ their total number. Let $T$ be the closure of an open connected set with Lipschitz boundary, compactly included in $Y$. For every $\varepsilon > 0$ and $\ell \in N^\varepsilon$ we consider the set $T^\varepsilon_\ell \subset \varepsilon(\ell + Y)$, where $T^\varepsilon_\ell = \varepsilon \ell + a_\varepsilon T$. The set $T^\varepsilon_\ell$ represents one of the droplets suspended in the fluid, and $\Gamma^\varepsilon_\ell = \partial T^\varepsilon_\ell$ denotes its surface. We now define the following subsets of $\Omega$:

$$\Omega_{1\varepsilon} = \bigcup_{\ell \in N^\varepsilon} T^\varepsilon_\ell, \quad \Omega_{2\varepsilon} = \Omega \setminus \Omega_{1\varepsilon},$$

where $\Omega_{1\varepsilon}$ is the domain occupied by the droplets of viscosity $\mu_\varepsilon$, and $\Omega_{2\varepsilon}$ is the domain occupied by the surrounding fluid, of viscosity $\mu_0$. Let $n$ be the unit normal on the boundary of $\Omega_{2\varepsilon}$ that points outside the domain. In this setting the emulsion
problem is described by

\[-\text{div} \left( 2 \mu^\varepsilon \varepsilon (\varepsilon) - p^\varepsilon I \right) = f + (\lambda^\varepsilon \kappa^\varepsilon n - \nabla_s \lambda^\varepsilon) \chi_{\varepsilon} \]  

\text{in } \Omega, \quad (2.1a)

\[\text{div } \varepsilon = 0 \]  

\text{in } \Omega, (2.1b)

\[\left[ \varepsilon \right] = 0 \]  

\text{on } S^\varepsilon, \quad (2.1c)

\[\varepsilon = V_{\ell}^\varepsilon + \omega_{\ell}^\varepsilon \times (x - x_{c}) \]  

\text{on } S^\varepsilon, \quad (2.1d)

\[\varepsilon = 0 \]  

\text{on } \partial \Omega, \quad (2.1e)

where \( \varepsilon \) represents the velocity field, \( p^\varepsilon \) the pressure, \( \varepsilon(\varepsilon) \) the strain rate, \( f \) the body forces, \( \lambda^\varepsilon \) the surface tension, \( \kappa^\varepsilon \) the mean curvature, \( n \) the exterior normal to the droplets, \( \mu^\varepsilon(x) = \mu_0 \) if \( x \in \Omega_1^\varepsilon \) and \( \mu^\varepsilon(x) = \mu_0 \) if \( x \in \Omega_2^\varepsilon \). Moreover, for simplicity, we will assume that \( f \) is \( C^\infty(\Omega) \).

We note here that the second term on the right hand side of (2.1a) is due to the presence of surface tension as the jump of the normal stress vector on \( S^\varepsilon \) is of the form

\[\left[ \sigma^\varepsilon n \right] = \lambda^\varepsilon \kappa^\varepsilon n - \nabla_s \lambda^\varepsilon,\]

where the stress is \( \sigma^\varepsilon = -p^\varepsilon I + 2 \mu^\varepsilon(x) \varepsilon(\varepsilon) \). The stress jump above satisfies the balance of forces and torques on each droplet, as explained in subsection 2.1. Condition (2.1d) represents the kinematic condition with \( V_{\ell}^\varepsilon \) and \( \omega_{\ell}^\varepsilon \) representing respectively the unknown droplet surface translational and angular velocities, for droplet \( \ell \) with \( x_{c} \) the position vector of its center of mass.

2.1. Balance of forces and torques. In system (2.1) we have imposed the jump of the stress vector on the surface of the droplets. Let us verify that under this assumption the droplets are in equilibrium. To that end we recall a classical result from calculus on surfaces (see Gilbarg and Trudinger [7]).

**Proposition 1.** Assume that \( T \) has a \( W^{2,\infty} \) boundary and that \( g \in W^{2,1}(T) \) and \( \varepsilon \in (C^{1}(\mathbb{R}^{n}))^{n} \) uniformly bounded then we have the following integration by parts on the surface \( S = \partial T \):

\[\int_{S} (\varepsilon \cdot \nabla g + g \text{div}_{s} \varepsilon) ds = \int_{S} \left( \frac{\partial g}{\partial n} + \kappa g \right) \varepsilon \cdot n ds, \quad (2.2)\]

where \( \text{div}_{s} \varepsilon = \text{div } \varepsilon - \nabla \varepsilon \cdot n \).

In particular, a manipulation of the expression in (2.2) yields

\[\int_{S} (g \kappa n - \nabla_{s} g) \cdot \varepsilon ds = \int_{S} g \text{div}_{s} \varepsilon ds, \quad (2.3)\]

Let us now apply (2.3) with \( g = \lambda \) and \( \varepsilon = e_i \) we obtain that

\[\int_{S} \left[ \sigma \varepsilon n \right] ds = \int_{S} (\lambda \kappa n - \nabla_{s} \lambda) ds = 0, \]

which represents the balance of forces on each droplet.

Similarly by denoting \( y = x - x_{c} \) from (2.3) for \( g = \lambda \) and \( \varepsilon = e_i \times y \) we have

\[\int_{S} (x - x_{c}) \times \left[ \sigma \varepsilon n \right] \cdot e_i ds = \int_{S} (\lambda \kappa n - \nabla_{s} \lambda) \cdot (e_i \times y) ds = \int_{S} \lambda \text{div}_{s}(e_i \times y) ds = 0, \]

which verifies the balance of torques on each droplet.
2.2. Limit problem. The weak formulation as well as existence and uniqueness results for the periodic problem are given in Nika and Vernescu [14]. As \( \varepsilon \) tends to zero and \( a_\varepsilon = O(\varepsilon^{-n/2}) \) \( (n = 3) \), the sequence \( v^\varepsilon \) was shown to converge strongly in \( (H^1_0(\Omega))^3 \) to \( v \) solution to
\[
\begin{align*}
\text{div} (2 \mu_0 e(v) - p I) &= f \quad \text{in } \Omega, \\
\text{div } v &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

The approach in [14] involved recognizing the solution of (2.1), \( v^\varepsilon \), as the unique minimizer of an energy functional associated to (2.1) that Mosco converges (see Attouch [3]) to a limit functional associated to (2.4) with \( v \) as its unique minimizer.

2.3. Outline of main results. In this paper we determine the first two terms in the asymptotics of the velocity in the case of dilute emulsions of arbitrary shape. The rate of convergence is shown to be of order \( a_\varepsilon^{n/2} \):
\[
\| v^\varepsilon - v \|_{(H^1_0(\Omega))^n} \leq C a_\varepsilon^{n/2}.
\]

Furthermore, we improve the \( L^2 \) estimate by identifying the next term in the velocity field expansion which is of order \( a_\varepsilon^n \) and estimate the error:
\[
v^\varepsilon(z) = v_t(z) + a_\varepsilon^n (e_\varepsilon(G_i(x_c,z)) \cdot P e_\varepsilon(v)(x_c) - e_\varepsilon(G_i(x_c,z)) \cdot S) + O(a_\varepsilon^{n+\frac{1}{2}}),
\]

where \( G \) is the Green’s tensor defined in (3.2) associated with the homogenized flow in (2.4), \( P \) and \( S \) are the bulk and surface polarization tensors corresponding to a droplet.

The asymptotics of the velocity of a similar problem but for constant surface tension \( \lambda = \text{const} \), has been considered in Nika and Vernescu [15]; in this case
\[
\| \sigma^\varepsilon n \| = \lambda_\varepsilon \kappa_\varepsilon n,
\]
and while the asymptotic form is similar, the techniques and the local problems are different to allow for a full stress jump. The techniques used are inspired from the paper of Bonnetier, Manceau and Triki [4] where the authors consider a constant surface tension problem, but do not include the kinematic condition on the droplet surface, thus the expansion in their case has an extra term. In Nika and Vernescu [15], it was shown that the kinematic condition is essential for recovering G. I. Taylor’s celebrated formula for the viscosity of a suspension of spherical fluid droplets in a viscous fluid. We also use stress estimates derived in Maris and Vernescu [12].

3. Proofs of the main results.

3.1. Scaling of surface tension and mean curvature. On the droplet surface there is a stress jump \( \llbracket \sigma n \rrbracket \neq 0 \). The stress jump can be obtained from the principle that the forces on an element of inter-facial area of arbitrary shape and size must be in equilibrium, because the interface is assumed to have zero thickness and thus zero mass. One can thus obtain (see Leal [9])
\[
\llbracket \sigma^\varepsilon n \rrbracket = \lambda_\varepsilon \kappa_\varepsilon n - \nabla_s \lambda_\varepsilon,
\]
where \( \lambda \) is the surface tension, and \( \kappa \) is the mean curvature. For a droplet of size \( a_\varepsilon \) we propose the following scaling of the surface tension, \( \lambda_\varepsilon \), so that the surface
energy remain bounded:
\[
\lambda_\varepsilon(x) = \lambda_\varepsilon(a_\varepsilon y + x_\varepsilon) = a_\varepsilon \lambda(y) \quad \text{for all } y \in S,
\]
where \( \lambda(y) \) is the surface tension on \( S \). This is in agreement with Tolman’s scaling when the surface tension is uniform [18]. Moreover, the mean curvature on \( S^\varepsilon \), \( \kappa_\varepsilon \), scales the following way:
\[
\kappa_\varepsilon(x) = \kappa_\varepsilon(a_\varepsilon y + x_\varepsilon) = \frac{1}{a_\varepsilon} \kappa(y) \quad \text{for all } y \in S,
\]
where \( \kappa(y) \) is the mean curvature on \( S \). Thus, for a droplet of size \( a_\varepsilon \) the jump of the stress is the following:
\[
\lambda_\varepsilon(x) \kappa_\varepsilon(x) n_x - \nabla_{n,\varepsilon} \lambda_\varepsilon(x) = \lambda(y) \kappa(y) n_y - \nabla_{n,y} \lambda(y).
\]

3.2. \( H^1 \) rate of convergence for the velocity field. We are interested in deriving the order of convergence of the emulsion velocity \( v^\varepsilon \) to the velocity field \( v \) of the unperturbed flow. Since we are studying the dilute case it is sufficient to consider the problem for a single droplet. In doing so we will drop the subscripts \( \ell \) from \( T_\varepsilon^\ell \) and \( S_\varepsilon^\ell \) and simply write \( T^\varepsilon \) and \( S^\varepsilon \) for their unscaled counterparts.

First, we will construct a test function \( \tilde{w}^\varepsilon \) and then obtain some estimates for the difference of \( v^\varepsilon - \tilde{w}^\varepsilon \) as well as \( v - w^\varepsilon \). For this we need the following results (see [5], [19]). In what follows, then proofs will be done for \( n = 3 \).

Lemma 3.1. Let \( r > 0 \) and \( u \in (L^2(B(r)))^3 \), \( \text{div } u = 0 \) in \( B(r) \). Then there exists \( \bar{u} \in (H^1(B(r)))^3 \) such that
\[
\text{div } \bar{u} = 0, \quad \text{curl } (\bar{u}) = u \text{ in } B(r), \quad \bar{u} \cdot n = 0 \text{ on } \partial B(r),
\]
\[
\|\bar{u}\|_{(L^2(B(r)))^3} \leq C r \|u\|_{(L^2(B(r)))^3}, \quad \|\nabla \bar{u}\|_{(L^2(B(r)))^{3 \times 3}} \leq C \|u\|_{(L^2(B(r)))^3}.
\]

Lemma 3.2. Let \( \bar{v} \) be the function associated with \( v(x) - v(x_\varepsilon) \) then
\[
\text{curl } (\bar{v} \varphi_\varepsilon) \to 0 \text{ strongly in } (H^1(\Omega))^3,
\]
where \( \varphi_\varepsilon \) is a cut-off function with \( \text{supp } \varphi_\varepsilon \subset B(2a_\varepsilon) \) and \( \varphi_\varepsilon(x) = 1 \) on \( T^\varepsilon \).

Using the above lemmas we define the vector field \( w^\varepsilon \),
\[
w^\varepsilon(x) = \begin{cases} 
v(x) & \text{in } \Omega - B(2a_\varepsilon), \\
v(x) - \text{curl } (\varphi \varphi_\varepsilon) & \text{in } B(2a_\varepsilon) - T^\varepsilon, \\
v(x_\varepsilon) & \text{in } T^\varepsilon. \end{cases}
\]

Remark 1. The construction of \( w^\varepsilon \) makes it an appropriate test function for problem (2.1). Given the general nature of the stress for problem (2.1), \( w^\varepsilon \) makes the variational formulation of (2.1) more accessible for constructing the estimates mentioned in the above paragraph.

Lemma 3.3. There exists a constant \( C > 0 \), independent of \( \varepsilon \), such that
\[
\|v^\varepsilon - w^\varepsilon\|_{(H^1_0(\Omega))^n} \leq C a_\varepsilon^{n/2}.
\]

Proof. Using \( v^\varepsilon - w^\varepsilon \) as a test function in (2.1) we have
\[
\int_\Omega 2\mu^\varepsilon e(v^\varepsilon) : e(v^\varepsilon - w^\varepsilon) \, dx = \int_\Omega f \cdot (v^\varepsilon - w^\varepsilon) \, dx,
\]
where the surface term is zero due to the fact that \( \mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon \) is a rigid body motion on \( S^\varepsilon \) and by using the balance of forces and torques condition. Moreover, we compute the following integral

\[
\int_\Omega 2 \mu^\varepsilon e(\mathbf{w}^\varepsilon) : e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) \, d\mathbf{x}
\]

\[
= \int_{\Omega - B(2a_\varepsilon)} 2 \mu^\varepsilon e(\mathbf{v}) : e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) \, d\mathbf{x} + \int_{B(2a_\varepsilon) - T^\varepsilon} 2 \mu^\varepsilon e(\mathbf{v} - \text{curl } (\mathbf{v} \phi_\varepsilon)) : e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) \, d\mathbf{x}
\]

\[
= \int_\Omega f : (\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) \, d\mathbf{x} - \int_{B(2a_\varepsilon) - T^\varepsilon} 2 \mu_0 e(\text{curl } (\mathbf{v} \phi_\varepsilon)) : e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) \, d\mathbf{x}
\]

\[
- \int_{T^\varepsilon} 2 \mu_0 e(\mathbf{v}) : e(\mathbf{w}^\varepsilon) \, d\mathbf{x}
\]

Hence, as the \( e(\text{curl } (\mathbf{v} \phi_\varepsilon)) = e(\mathbf{v}) \) on \( T^\varepsilon \), we obtain

\[
\int_\Omega 2 \mu^\varepsilon e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) : e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) \, d\mathbf{x} = \int_{B(2a_\varepsilon)} 2 \mu_0 e(\text{curl } (\mathbf{v} \phi_\varepsilon)) : e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon) \, d\mathbf{x}
\]

\[
\leq C \|e(\text{curl } (\mathbf{v} \phi_\varepsilon))\|_{(L^2(B(2a_\varepsilon)))^n} \|e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon)\|_{(L^2(B(2a_\varepsilon)))^n}.
\]

Using results from elliptic regularity theory we have that the curl \( (\mathbf{v} \phi_\varepsilon) \) is bounded in \( \Omega \), therefore we obtain:

\[
\|e(\mathbf{v}^\varepsilon - \mathbf{w}^\varepsilon)\|_{(L^2(\Omega))^{n \times n}} \leq C a_\varepsilon^{n/2},
\]

and hence, using Korn’s inequality we obtain the desired estimate. \( \square \)

**Lemma 3.4.** There exists a constant \( C > 0 \), independent of \( \varepsilon \), such that

\[
\|\mathbf{v} - \mathbf{w}^\varepsilon\|_{(H_0^1(\Omega))^{n}} \leq C a_\varepsilon^{n/2}.
\]

**Proof.** We start by evaluating the following integral:

\[
\int_\Omega e(\mathbf{v} - \mathbf{w}^\varepsilon) : e(\mathbf{v} - \mathbf{w}^\varepsilon) \, d\mathbf{x}
\]

\[
= \int_{B(2a_\varepsilon) - T^\varepsilon} e(\text{curl } (\mathbf{v} \phi_\varepsilon)) : e(\text{curl } (\mathbf{v} \phi_\varepsilon)) \, d\mathbf{x} + \int_{T^\varepsilon} e(\mathbf{v}) : e(\mathbf{v}) \, d\mathbf{x}
\]

\[
= \int_{B(2a_\varepsilon)} e(\text{curl } (\mathbf{v} \phi_\varepsilon)) : e(\text{curl } (\mathbf{v} \phi_\varepsilon)) \, d\mathbf{x} = \|e(\text{curl } (\mathbf{v} \phi_\varepsilon))\|_{(L^2(B(2a_\varepsilon)))^n}^2 \leq C a_\varepsilon^n,
\]

where the last inequality was obtained following the same arguments as in the previous lemma. Using Korn’s inequality we obtain the desired estimate. \( \square \)

**Theorem 3.5.** There exists a constant \( C > 0 \), independent of \( \varepsilon \), such that

\[
\|\mathbf{v}^\varepsilon - \mathbf{v}\|_{(H_0^1(\Omega))^{n}} \leq C a_\varepsilon^{n/2}.
\]

**Proof.** Follows from Lemma 3.3, Lemma 3.4, and an application of the triangle inequality. \( \square \)

### 3.3. Asymptotics of the velocity in the \( L^2 \) norm.

**Theorem 3.5.** shows that the convergence of \( \mathbf{v}^\varepsilon \) to \( \mathbf{v} \) in the \( (H_0^1(\Omega))^{n} \) norm is of order \( O(a_\varepsilon^{n/2}) \). As a next step we are interested in computing the next term in the asymptotic expansion of the velocity vector field.
The approach we follow here is similar to the one in Bonnetier, Manceau and Triki [4]. We denote by \( (G, F) \) the Green’s tensors associated with the homogenized flow in (2.4)

\[
-\text{div} \left( 2 \mu_0 e_x (G_i)(x, z) - F_i (x, z) I \right) = e_i \delta (x - z) \quad \text{in } \Omega, \\
\text{div}_z (G_i)(x, z) = 0 \quad \text{in } \Omega, \\
G_i (x, z) = 0 \quad \text{on } \partial \Omega.
\]

Moreover, we introduce the following problems centered at the origin,

\[
-\text{div} \left( 2 \mu e(\phi^{kl}) - s^{kl} I \right) = (\lambda \kappa n_y - \nabla_{s,y} \lambda) \chi_S \quad \text{in } \mathbb{R}^n, \\
\text{div} \phi^{kl} = 0 \quad \text{in } \mathbb{R}^n, \\
[\phi^{kl}] = 0 \quad \text{on } S, \\
\phi^{kl} = \phi^{c,kl} + \omega^{kl} \times y \quad \text{on } S, \\
\phi^{kl} \to B^{kl} \quad \text{at } \infty,
\]

\[
-\text{div}_y \left( 2 \mu e_y (V^\varepsilon) - q^\varepsilon I \right) = (\lambda \kappa n_y - \nabla_{s,y} \lambda) \chi_S \quad \text{in } \Omega_\varepsilon, \\
\text{div}_y V^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\
[V^\varepsilon] = 0 \quad \text{on } S, \\
V^\varepsilon = V^{c,\varepsilon} + \omega \times y \quad \text{on } S, \\
V^\varepsilon = e(v)(x_\varepsilon) y \quad \text{on } \partial \Omega_\varepsilon,
\]

\[
-\text{div}_y \left( 2 \mu e_y (V) - q I \right) = (\lambda \kappa n_y - \nabla_{s,y} \lambda) \chi_S \quad \text{in } \mathbb{R}^n, \\
\text{div}_y V = 0 \quad \text{in } \mathbb{R}^n, \\
[V] = 0 \quad \text{on } S, \\
V = V^\varepsilon + \omega \times y \quad \text{on } S, \\
V \to e(v)(x_\varepsilon) y \quad \text{at } \infty,
\]

where \( B^{kl} = (e_{1y_k} + e_{2y_l} - \frac{2}{n} y^{kl}) \omega^{kl} \), \( \omega^{kl} \) are unknown constant vectors in \( \mathbb{R}^n \), \( \phi^{c,kl} \), \( V^{c,\varepsilon} \) and \( V^\varepsilon \) are defined in similar spirit as in Lipton and Vernescu [10], and \( \Omega_\varepsilon \) is the scaled domain defined the following way

\[
\Omega_\varepsilon := \left\{ y = \frac{x - x_\varepsilon}{\alpha_\varepsilon} \mid x \in \Omega \right\}.
\]

For simplicity we have taken \( \Omega_\varepsilon \) to be a ball in \( \mathbb{R}^n \). We further remark that the balance of forces and torques for each problem is automatically satisfied by virtue of formula (2.3), whilst existence and uniqueness of the above problems follows from the theory of pseudo-monotone operators in Hilbert spaces in Lions [11].

We re-define problems (3.3), (3.4), (3.5) by introducing the vector fields \( \hat{\phi}^{kl} := \phi^{kl} - B^{kl} \), \( W^\varepsilon := V^\varepsilon - e(v)(x_\varepsilon) y \), and \( W := V - e(v)(x_\varepsilon) y \) that decay to zero at infinity. Hence, we obtain the following corresponding problems for (3.3), (3.4), and (3.5),

\[
-\text{div} \left( 2 \mu e(\hat{\phi}^{kl}) - s^{kl} I \right) = (\lambda \kappa n_y - \nabla_{s,y} \lambda) \chi_S \\
- (\mu d - \mu_0) (e_l n_k + e_k n_l - \frac{2}{n} n_y \delta_{kl}) \chi_S \quad \text{in } \mathbb{R}^n, \\
\text{div} \hat{\phi}^{kl} = 0 \quad \text{in } \mathbb{R}^n,
\]
There exists a constant $C$, independent of $\varepsilon$, such that
\[ \| e_y (v^\varepsilon (a_x y + x_c) - v (a_x y + x_c) - a_x W^\varepsilon (y)) \|_{\ell^2(\Omega^\varepsilon))} \leq C a_x^2. \]

**Lemma 3.6.** There exists a constant $C$, independent of $\varepsilon$, such that
\[ \| e_y (v^\varepsilon (a_x y + x_c) - v (a_x y + x_c) - a_x W^\varepsilon (y)) \|_{\ell^2(\Omega^\varepsilon))} \leq C a_x^2. \]

**Proof.** Define $\Xi^\varepsilon (y) = v^\varepsilon (a_x y + x_c) - v (a_x y + x_c) - a_x W^\varepsilon (y)$ and consider
\[ \int_{\Omega^\varepsilon} 2 \mu | e_y (\Xi^\varepsilon (y))|^2 \, dy = \int_{\Omega^\varepsilon} 2 \mu e_y (v^\varepsilon (a_x y + x_c)) : e_y (\Xi^\varepsilon (y)) \, dy \]
\[ - \int_{\Omega^\varepsilon} 2 \mu e_y (v (a_x y + x_c)) : e_y (\Xi^\varepsilon (y)) \, dy \]
\[ - a_x \int_{\Omega^\varepsilon} 2 \mu e_y (W^\varepsilon (y)) : e_y (\Xi^\varepsilon (y)) \, dy. \]

If $x \in \Omega$, set $\xi^\varepsilon (x) := \Xi^\varepsilon \left( \frac{\varepsilon - x}{a_x} \right) = \Xi^\varepsilon (y)$. We further note that $\Xi^\varepsilon$ is divergence free and $\xi^\varepsilon = 0$ on $\partial \Omega$. Thus, considering each integral separately we have,
\[ \int_{\Omega^\varepsilon} 2 \mu e_y (v^\varepsilon (a_x y + x_c)) : e_y (\Xi^\varepsilon (y)) \, dy \]
\[ = a_x^{-n} \int_{\Omega} 2 \mu e_x (v^\varepsilon) : e_x (\xi^\varepsilon) \, dx \]
\[ = a_x^{-n} \int_{\Omega} f : \xi^\varepsilon \, dx + a_x^{-n} \int_{S^x} (\lambda_x \kappa_x n_x - \nabla_{s,x} \lambda_x) \cdot \xi^\varepsilon (x) \, ds_x \]
\[ = a_x^{-n} \int_{\Omega} f : \xi^\varepsilon \, dx + a_x \int_{S} (\lambda \kappa y - \nabla_{s,y} \lambda) \cdot \Xi^\varepsilon (y) \, ds_y, \]
\[
\int_{\Omega_\varepsilon} 2 \mu_\varepsilon e_y(v(a_\varepsilon y + x_c)) : e_y(\Xi^\varepsilon(y)) \, dy
\]
\[
= a_\varepsilon^{2-n} \int_\Omega 2 \mu^\varepsilon e_x(v) : e_x(\xi^\varepsilon) \, dx
\]
\[
= a_\varepsilon^{2-n} \int_\Omega 2 \mu \varepsilon e_x(v) : e_x(\xi^\varepsilon) \, dx + a_\varepsilon^{2-n} \int_{T^\varepsilon} 2 (\mu_d - \mu_0) e_x(v) : e_x(\xi^\varepsilon) \, dx
\]
\[
= a_\varepsilon^{2-n} \int_\Omega \mathbf{f} : \xi^\varepsilon \, dx + a_\varepsilon \int_{T^\varepsilon} 2 (\mu_d - \mu_0) e_x(v)(a_\varepsilon y + x_c) : e_y(\Xi^\varepsilon(y)) \, dy.
\]
Therefore,
\[
\int_{\Omega_\varepsilon} 2 \mu |e_y(\Xi^\varepsilon)|^2 \, dy = a_\varepsilon \int_{T^\varepsilon} 2 (\mu_0 - \mu_d) (e_x(v)(a_\varepsilon y + x_c) - e_x(v)(x_c)) : e_y(\Xi^\varepsilon(y)) \, dy.
\]
Using elliptic regularity results we can conclude that since \( v \) is smooth inside \( \Omega \),
\[
\|e_y(\Xi^\varepsilon)\|_{(L^2(\Omega_\varepsilon))^{n \times n}} \leq C a_\varepsilon^2.
\]

\[\square\]

**Lemma 3.7.** There exists a constant \( C \), independent of \( \varepsilon \), such that
\[
\|e_y(\mathbf{w}^\varepsilon(a_\varepsilon y + x_c) - \mathbf{v}(a_\varepsilon y + x_c) - a_\varepsilon \mathbf{W}(y))\|_{(L^2(\Omega_\varepsilon))^{n \times n}} \leq C a_\varepsilon^{3/2}.
\]

**Proof.** In view of **Lemma 3.6** it is enough to show that
\[
\|e_y(\mathbf{W} - \mathbf{W}^\varepsilon)\|_{(L^2(\Omega_\varepsilon))^{n \times n}} \leq C a_\varepsilon^{1/2}.
\]
This result follows as a special case from **Lemma 5.7** in Maris and Vernescu [12]. The authors of [12] obtain a bound for the stress for a problem similar to (3.7). The concept they use is to transform (3.7) into a system in free space. Then by making use of the regularity lemma in Temam [17] and the fundamental solution of the transformed Stokes system, which they obtain by way of the Fourier transform, are able to get an estimate for the stress in terms of the droplet radius. We refer the interested reader to Maris and Vernescu [12] for a detailed proof. \[\square\]

**Theorem 3.8.** For any \( z \in \Omega \) at a distance \( d > 0 \) away from \( T^\varepsilon \) we have
\[
v_1^\varepsilon(z) = v_1(z) + a_\varepsilon^n (e_x(G_i)(x_c, z) : \mathbb{P} e_x(v)(x_c) - e_x(G_i)(x_c, z) : \mathbb{S}) + \mathcal{O}\left(a_\varepsilon^{n+\frac{1}{2}}\right),
\]
where \( \mathbb{P} \) and \( \mathbb{S} \) are the bulk and surface polarization tensors defined by:
\[
\mathbb{P}_{ijkl} = 2(\mu_0 - \mu_d) \int_T e_{yij}(\phi^{kl}(y)) \, dy, \quad \mathbb{S} = \int_S \lambda(y) n_y \otimes n_y \, ds_y.
\]

**Proof.** Using (3.2) and (2.4) we have
\[
v_1(z) = \int_{\Omega} 2 \mu e_x(G_i) : e_x(v) \, dx = \int_{\Omega} \mathbf{f} \cdot G_i \, dx.
\]
Similarly, using (3.2) and (2.1a) – (2.1e) we have
\[
v_i'(z) = \int_\Omega 2 \mu_0 e_x(G_i)(x, z) : e_x(v^r) \, dx
\]
\[
= \int_\Omega 2 \mu^r e_x(v^r) : e_x(G_i)(x, z) \, dx
\]
\[
+ \int_{T^r} 2(\mu_0 - \mu_d) e_x(v^r) : e_x(G_i)(x, z) \, dx
\]
\[
= \int_\Omega f \cdot G_i \, dx + \int_{S^r} (\lambda \kappa n - \nabla s \lambda) \cdot G_i(x, z) \, ds_x
\]
\[
+ \int_{T^r} 2(\mu_0 - \mu_d) e_x(v^r) : e_x(G_i)(x, z) \, dx.
\]

Thus we get,
\[
(v_i' - v_i)(z) = \int_{S^r} (\lambda \kappa n - \nabla s \lambda) \cdot G_i(x, z) \, dx
\]
\[
+ \int_{T^r} 2(\mu_0 - \mu_d) e_x(v^r) : e_x(G_i)(x, z) \, dx. \tag{3.10}
\]

We compute the integral over the surface first and we get,
\[
\int_{S^r} (\lambda \kappa n - \nabla s \lambda) \cdot G_i(x, z) \, ds_x
\]
\[
= a_z^{n-1} \int_S (\lambda \kappa n_y - \nabla s y \lambda) \cdot G_i(a_z y + x_c, z) \, ds_y
\]
\[
= a_z^{n-1} \int_S (\lambda \kappa n_y - \nabla s y \lambda) \cdot G_i(x_c, z) \, ds_y
\]
\[
+ a_z^n \int_S (\lambda \kappa n_y - \nabla s y \lambda) \cdot \nabla x G_i(x_c, z) y \, ds_y + O(a_z^{n+1}) \tag{3.11}
\]
\[
= a_z^n \int_S (\lambda \kappa n_y - \nabla s y \lambda) \cdot \nabla x G_i(x_c, z) y \, ds_y + O(a_z^{n+1})
\]
\[
= -a_z^n \int_S \lambda \nabla x G_i(x_c, z) n_y \cdot n_y \, ds_y + O(a_z^{n+1}),
\]

where we used an expansion on \(G_i\) and formula (2.3). To compute the term over the droplet \(T^r\) we define the following,
\[
R_r(y) = v^r(a_z y + x_c) - v(a_z y + x_c) - a_z W(y), \quad r_r(x) = R_r \left( \frac{x - x_c}{a_z} \right) = R_r(y).
\]

Thus,
\[
\int_{T^r} 2(\mu_0 - \mu_d) e_x(G_i)(x, z) : e_x(v^r) \, dx
\]
\[
= \int_{T^r} 2(\mu_0 - \mu_d) e_x(G_i)(x, z) : e_x(v^r(x)) \, dx
\]
\[
+ \int_{T^r} 2(\mu_0 - \mu_d) e_x(G_i)(x, z) : e_x \left( v(x) + a_z W \left( \frac{x - x_c}{a_z} \right) \right) \, dx.
\]

The first integral above, in view of Lemma 3.7, becomes
\[
\left| \int_{T^r} 2(\mu_0 - \mu_d) e_x(G_i)(x, z) : e_x(v^r(x)) \, dx \right|
\]
For the second integral we have,

\begin{equation}
\int_T 2 (\mu_0 - \mu_d) e_x(G_i)(a_x \mathbf{y} + \mathbf{x_c} \cdot \mathbf{z}) : e_y(R^*(\mathbf{y})) \, dy
\end{equation}

\begin{equation}
\leq C \varepsilon^{-1/2} = \varepsilon^{-3/2} = C \varepsilon^{-n+1/2}.
\end{equation}

For the second integral we have,

\begin{equation}
2 (\mu_0 - \mu_d) \varepsilon \int_T e_x(G_i)(a_x \mathbf{y} + \mathbf{x_c} \cdot \mathbf{z}) : e_x(\mathbf{v} + a_x W \left( \frac{\mathbf{x} - \mathbf{x_c}}{a_x} \right) ) \, dx
\end{equation}

\begin{equation}
2 (\mu_0 - \mu_d) \varepsilon T e_x(G_i)(a_x \mathbf{y} + \mathbf{x_c} \cdot \mathbf{z}) : (e_x(\mathbf{v})(a_x \mathbf{y} + \mathbf{x_c}) + e_y(W)(\mathbf{y})) \, dy
\end{equation}

\begin{equation}
2 (\mu_0 - \mu_d) \varepsilon T e_x(G_i)(\mathbf{x_c} \cdot \mathbf{z}) : \left\{ e_x(\mathbf{v})(\mathbf{x_c}) + e_y(W)(\mathbf{y}) \right\} \, dy + O(a_x^{n+1}).
\end{equation}

To complete the proof, we write \( W(\mathbf{y}) \) as a linear combination of \( \hat{\phi}^{kl} \), solution to (3.6), the following way,

\begin{equation}
W(\mathbf{y}) = \sum_{k,l=1}^n e_{xkl}(\mathbf{v})(\mathbf{x_c}) \hat{\phi}^{kl}(\mathbf{y}).
\end{equation}

Replacing \( W(\mathbf{y}) \) in (3.13) we get,

\begin{equation}
2 (\mu_0 - \mu_d) \varepsilon T e_x(G_i)(\mathbf{x_c} \cdot \mathbf{z}) : e_x(\mathbf{v} + a_x W \left( \frac{\mathbf{x} - \mathbf{x_c}}{a_x} \right) ) \, dx
\end{equation}

\begin{equation}
2 (\mu_0 - \mu_d) \varepsilon T e_x(G_i)(\mathbf{x_c} \cdot \mathbf{z}) : \left\{ e_x(\mathbf{v})(\mathbf{x_c}) + \sum_{k,l=1}^n e_{xkl}(\mathbf{v})(\mathbf{x_c}) e_y(\hat{\phi}^{kl})(\mathbf{y}) \right\} \, dy
\end{equation}

\begin{equation}
+ O(a_x^{n+1}).
\end{equation}

Combining the above result, (3.10), (3.11), (3.12), (3.13), and substituting \( \hat{\phi}^{kl}(\mathbf{y}) = \phi^{kl}(\mathbf{y}) - B^{kl} \) we get

\begin{equation}
(v^c_i - v_i)(\mathbf{z}) = 2 (\mu_0 - \mu_d) \varepsilon \sum_{k,l=1}^n e_x(G_i)(\mathbf{x_c} \cdot \mathbf{z}) : \left\{ \sum_{k,l=1}^n e_{xkl}(\mathbf{v})(\mathbf{x_c}) e_y(\phi^{kl})(\mathbf{y}) \right\} \, dy
\end{equation}

\begin{equation}
- a_x^2 e_x(G_i)(\mathbf{x_c} \cdot \mathbf{z}) : \int_{S} \lambda(\mathbf{y}) \mathbf{n}_y \otimes \mathbf{n}_y \, ds_y + O\left( a_x^{n+1/2} \right).
\end{equation}

\[\square\]

4. Conclusions. We consider a problem of dilute emulsions of two immiscible viscous fluids, in which one is distributed in the other in the form of droplets of arbitrary shape, with non-uniform surface tension due to surfactants. The problem includes an essential kinematic condition on the droplets. In the periodic homogenization framework, it can be shown using Mosco convergence that, as the size of the droplets converges to zero faster than the distance between the droplets, the emulsion behaves in the limit like the continuous phase. In Theorem 3.5 we demonstrate the order of convergence, in the \( H^1 \) norm, of the velocity vector field \( \mathbf{v}^c \) to \( \mathbf{v} \) to be \( O(a_x^{n+1/2}) \). Moreover, in Theorem 3.8 we determine the first correction term in the velocity expansion which is described in terms of the bulk and surface polarization tensors.
Acknowledgments. The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work was supported by the NSF grant DMS-1109356.

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Received June 2015; 1st revision October 2015; 2nd revision November 2015.
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