A VARIATIONAL REPRESENTATION OF WEAK SOLUTIONS FOR THE PRESSURELESS EULER-POISSON EQUATIONS

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Abstract. We derive an explicit formula for global weak solutions of the one dimensional system of pressure-less Euler-Poisson equations. Our variational formulation is an extension of the well-known formula for entropy solutions of the scalar inviscid Burgers’ equation: since the characteristics of the Euler-Poisson equations are parabolas, the representation of their weak solution takes the form of a “quadratic” version of the celebrated Lax-Oleinik variational formula. Three cases are considered. (i) The variational formula recovers the “sticky particle” solution in the attractive case; (ii) It represents a repulsive solution which is different than the one obtained by the sticky particle construction; and (iii) the result is further extended to the multi-dimensional Euler-Poisson system with radial symmetry.

1. Introduction

We study the system of pressureless Euler-Poisson equations

\begin{align}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x &= \kappa \rho E, \quad E_x = \rho, \quad E(-\infty, t) = 0.
\end{align}

Here \(E\) is the electric field, \(\kappa\) is a given physical constant which signifies the type of the underlying forcing. We distinguish between three different cases depending on the sign on \(\kappa\).

(i) The attractive case, \(\kappa < 0\). Solutions of (1.1) always breakdown at a finite time, \(t = t_c\), where \(u_x(\cdot, t \uparrow t_c) \to -\infty\).

(ii) The repulsive case, \(\kappa > 0\). It was shown in [EnLT01] that in the presence of repulsing forcing, there is a large class of so-called sub-critical initial data, \(u'_0 > -\sqrt{2k\rho_0}\), for which (1.1) admits global smooth solutions governed by

\begin{equation}
\rho_t + uu_x = \kappa E;
\end{equation}

see [LT02, LT03, ChengT08, TW08, ChaeT08, LTW10, We10a, We10b] for the critical threshold phenomena in related Euler and Euler-Poisson systems. For super-critical data, however, the repulsive Euler-Poisson solution breaks-down at a critical time, \(t = t_c < \infty\), after which (1.1b) is no longer equivalent to (1.1).

(iii) Finally, there is the neutral case \(\kappa = 0\), governed by the pressureless Euler equations,

\begin{align}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x &= 0.
\end{align}

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For increasing data, \( u'_0 > 0 \), solutions of \((1.3)\) remains smooth (corresponding to sub-critical data in the limiting case \( \kappa = 0 \)), and are governed by the decoupled inviscid Burgers equation (1.4)

\[
u_t + uu_x = 0.
\]

It is well known that for general non-increasing initial data solutions of \((1.4)\) will lose their initial \( C^1 \) regularity at a finite time \([La57]\). Thereafter, \((1.3)\) and \((1.3b)\) are not equivalent. Solutions of Burgers’ equation \((1.3)\) past the critical time develop shock discontinuities and are given by the celebrated Lax-Oleinik formula \([Ev98]\). Solutions of the pressureless Euler system \((1.3)\) develop \( \delta \)-shocks. Their construction attracted great attention in the 90’s: they were obtained using sticky particles formulation in \([Z70, CPY90, BG98, CKR07]\), using a generalized variational principle in \([ERS96]\), and through a vanishing pressure limiting process \([Bou93, CL03, CL04]\) and the references therein. Uniqueness was proved in \([HW01]\).

Here, we are interested in the global weak solution of the (non-neutral) Euler-Poisson system \((1.1)\), in either the attractive case, \( \kappa < 0 \), or in the repulsive case \( \kappa > 0 \) subject to general initial data, beyond the global regularity in the sub-critical regime studied in \([EnLT01]\). Among the few known results we mention the existence and uniqueness result of global weak solutions for the pressureless Euler-Poisson system \([NTu08]\). In this paper, we construct an explicit formula for weak solutions of the Euler-Poisson \((1.1)\). For the inviscid Burgers’ equation, generic shock develops due to the intersection of straight characteristics, after which entropy solutions are given by the variational Lax-Oleinik formula. For the pressureless Euler-Poisson equations, characteristics become quadratic, and the solution, \( \rho(x,t), pu(x,t) \) will be expressed in terms of the minimizer, \( y(x,t) \) of the weighted quadratic form

\[
y(x,t) = \sup_y \left\{ y \mid y = \arg\inf_y \int_0^y Q_{x,t}(s)\rho_0(s)ds \right\}, \quad Q_{x,t}(s) := s + tu_0(s) + \frac{1}{2}\kappa E_0(s)t^2 - x.
\]

**Theorem 1.1.** \([L^1 \text{ initial density}]\) Consider the pressureless Euler-Poisson system \((1.1)\) subject to initial data \( u_0(x) := u(x,0) \in C^1(\mathbb{R}) \) and \( 0 \leq \rho_0(x) := \rho(x,0) \in L^1(\mathbb{R}) \). Set \( E_0(s) := \int_{-\infty}^s \rho_0(w)dw \) as the corresponding initial electric field. Then, \( (\rho, pu) = (\partial_x R, \partial_x M) \) is a weak solution of \((1.1)\), where \( R(x,t) \equiv R(y(x,t)) \) and \( M(x,t) \equiv M(y(x,t)) \) are given by

\[
R(x,t) = \int_0^{y(x,t)} \rho_0(s)ds, \quad M(x,t) = \int_0^{y(x,t)} \left( u_0(s) + \kappa E_0(s)t \right)\rho_0(s)ds.
\]

**Remark 1.1.** The presence of Poisson potential is responsible for ‘converting’ the straight characteristics familiar from the Burgers’ equation, into parabola in Euler-Poisson equations. This is reflected in the variational formula \((1.6)\) through the additional term \( \frac{1}{2}\kappa E_0(s)t^2 \) in \((1.5)\).

**Remark 1.2.** The representation formula \((1.6)\) applies for all \( \kappa \in \mathbb{R} \). When \( \kappa = 0 \), one recovers the variational formulation of \([HW01]\) for the pressureless Euler equations. In the particular case of \( \kappa = 0 \) and \( \rho_0 \equiv 1 \), \((1.5), (1.6)\) become

\[
y(x,t) := \sup_y \left\{ y \mid y = \arg\inf_y \int_0^y tu_0(s)ds + \frac{(y-x)^2}{2} - \frac{x^2}{2} \right\},
\]

recovering the celebrated Lax-Oleinik formula.

**Remark 1.3.** For \( \kappa \leq 0 \), our representation formula \((1.6)\) gives the “sticky particle” solution of the pressureless Euler-Poisson system. For \( \kappa > 0 \), the weak solution given by \((1.4)\) is different from the one corresponding to the “sticky particle” model. We discuss the details in sections 2 and 3.
We can extend the result of theorem [1.1] to more general initial measure data, \( \rho(x, 0) \in \mathcal{M}_+(\mathbb{R}) \) and \( m(x, 0) = (\rho(x, 0)u(x, 0)) \in \mathcal{M}(\mathbb{R}) \).

**Theorem 1.2.** [Measure initial density] Consider the pressureless Euler-Poisson system [1.1] subject to initial data, \( m(x, 0) = \rho(x, 0)u(x, 0) \in \mathcal{M}(\mathbb{R}) \) and \( \rho_0(x) := \rho(x, 0) \in \mathcal{M}_+(\mathbb{R}) \), such that \( \int_{-\infty}^{\infty} \rho_0(x)dx < \infty \) and \( u_0(x) \) is piecewise continuous. The corresponding initial electric field, \( E_0 \), is given by the average rule

\[
E_0(s) := \frac{1}{2} \left( \int_{-\infty}^{s-} \rho_0(w)dw + \int_{-\infty}^{s+} \rho_0(w)dw \right),
\]

Then, \( (\rho, \rho u) = (\partial_x R, \partial_x M) \) is a weak solution of [1.1], where \( R \equiv R(y(x, t)) \) and \( M \equiv M(y(x, t)) \) are given by

\[
R(x, t) = \begin{cases} 
\int_0^{y(x,t)+} \rho_0(s)ds, & \text{if } Q_{x,t}(y(x,t)) \leq 0, \\
\int_0^{y(x,t)-} \rho_0(s)ds, & \text{if } Q_{x,t}(y(x,t)) > 0,
\end{cases}
\]

and

\[
M(x, t) = \begin{cases} 
\int_0^{y(x,t)+} \left( u_0(s) + \kappa E_0(s)t \right) \rho_0(s)ds, & \text{if } Q_{x,t}(y(x,t)) \leq 0, \\
\int_0^{y(x,t)-} \left( u_0(s) + \kappa E_0(s)t \right) \rho_0(s)ds, & \text{if } Q_{x,t}(y(x,t)) > 0.
\end{cases}
\]

We note in passing that since \( y(\cdot, t) \) is monotonically increasing, consult lemma [2.2] below, the one-sided limits in [1.2], \( y(\cdot, t) \pm \), are well-defined.

These results can be extended to the weighted multi-dimensional Euler/Euler-Poisson systems with symmetry, which is the content of our third main theorem.

**Theorem 1.3.** Consider the \( n \)-dimensional weighted Euler-Poisson equations

\[
\rho_t + \nabla \cdot (\rho u) = 0, \quad u(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{R}^n,
\]

\[
(\rho u)_t + \nabla \cdot (\rho u \otimes u) = \kappa \rho |x|^{n-1} \nabla V, \quad \Delta V = \rho,
\]

subject to spherically symmetric initial data, \( \rho_0(x)u_0(x) := \rho_0(|x|)u_0(|x|) \frac{x}{|x|} \), such that \( 0 \leq \rho(x, 0) = \rho_0(|x|) \in L^1(\mathbb{R}^n) \), and \( u_0(|x|) \in C^1(\mathbb{R}_+) \). Then the system [1.9] admits a weak radial solution, \( (\rho, \rho u) \), of the form

\[
\rho(x, t) = \frac{1}{r^{n-1}} \frac{\partial R(r, t)}{\partial r}, \quad \rho(x, t)u(x, t) = \frac{x}{r} \frac{1}{r^{n-1}} \frac{\partial M(r, t)}{\partial r}, \quad r := |x|.
\]

Here, \( R(r, t) \equiv R(y(r, t)) \) and \( M(r, t) \equiv M(y(r, t)) \) are given by,

\[
R(r, t) = \int_0^{y(r,t)} \rho_0(s)s^{n-1}ds, \quad M(r, t) = \int_0^{y(r,t)} \rho_0(s) \left( w_0(s) + \kappa E_0(s)t \right) s^{n-1}ds,
\]

where \( E_0 \) is the initial electric field,

\[
E_0(r) := n^{-1}V_r(|x|, 0) = \int_0^r s^{n-1} \rho_0(s)ds, \quad r = |x|,
\]
and \( y = y(r, t) \) is determined in terms of the quadratic form \( Q_{r,t}(s) = s + tu_0(s) + \frac{1}{2}\kappa E_0(s)t^2 - r \),

(1.10d) \[ y(r, t) := \sup_y \left\{ y \mid y = \arg\inf_y \int_0^y Q_{r,t}(s)s^{n-1}\rho_0(s)ds \right\} \]

**Remark 1.4.** If \( \kappa = 0 \), theorem (1.1) yields the formula for a global weak solution of the multi-dimensional pressureless Euler system with symmetry. If \( n = 1 \), it recovers the formula for the 1D pressureless Euler-Poisson system, that is, theorem (1.1) with a anti-symmetric \( E_0 \).

The paper is organized as follows. In section 2, we develop the formula for the attractive pressureless Euler-Poisson system. In section 3, we extend the formula to the repulsive system, and explain different physical meanings of the weak solutions for \( \kappa < 0 \) and \( \kappa > 0 \). In section 4, we extend the formula to the multidimensional weighted Euler-Poisson system.

2. Weak solutions of the attractive pressureless Euler-Poisson system

2.1. From continuum to particles. We begin with the transport of the center of mass in the smooth case.

**Lemma 2.1.** Assume that \( (\rho(\cdot, t), \rho u(\cdot, t)) \) is a smooth solution of the pressureless Euler-Poisson system (2.1) for \( t \in [0, T] \). Let \( x(\alpha, t) \) denote the particle path emanating from \( \alpha \in \mathbb{R} \). Then the center of mass of particles emanating from the interval \([a, b]\) is given by the parabola

\[
X(t) = X_0 + U_0t + \frac{1}{2}\kappa \tilde{E}_0 t^2, \quad t \in [0, T]
\]

where

\[
X_0 = \frac{\int_a^b x\rho_0(x)dx}{\int_a^b \rho_0(x)dx}, \quad U_0 = \frac{\int_a^b \rho_0(x)u_0(x)dx}{\int_a^b \rho_0(x)dx}, \quad \tilde{E}_0 = \frac{1}{2} \left( \int_{-\infty}^a \rho_0(x)dx + \int_b^{\infty} \rho_0(x)dx \right).
\]

**Proof.** We use the method of characteristics to obtain an explicit solution of (2.1), (EnLT01).

Along the particle trajectory, the equations of \( x \) and \( u \) are

(2.3a) \[ \frac{dx(\alpha, t)}{dt} = u(x(\alpha, t), t), \quad x(\alpha, 0) = \alpha, \]

and

(2.3b) \[ \frac{du(x(\alpha, t), t)}{dt} = \kappa V_x(x(\alpha, t), t) = \kappa E(x(\alpha, t), t), \]

where

(2.4) \[ E(x(\alpha, t), t) = \int_{-\infty}^{x(\alpha, t)} \rho(\xi)d\xi. \]

Since

\[
\frac{d}{dt}E\left(x(\alpha, t), t\right) = \frac{d}{dt}\left(x(\alpha, t)\right) \cdot \rho\left(x(\alpha, t), t\right) + \int_{-\infty}^{x(\alpha, t)} \rho_t(\xi, t)d\xi
\]

\[ = u(\alpha, t)\rho\left(x(\alpha, t), t\right) - \int_{-\infty}^{x(\alpha, t)} \left(\rho(\xi, t)u(\xi, t)\right)_{\xi} d\xi = 0, \]

the electric field remains constant along \( x(\alpha, t) \). Let

\[ E_0(\alpha) = \int_{-\infty}^{x(\alpha, 0)} \rho_0(s)ds, \quad u_0 = u(x(\alpha, 0), 0), \quad \rho_0 = \rho(x(\alpha, 0), 0). \]
Then (2.3b) can be simplified to
\[ \frac{du(x(\alpha, t), t)}{dt} = \kappa E_0, \]
which yields
\[ (2.5) \quad u(x(\alpha, t), t) = u_0 + \kappa E_0 t. \]
This together with (2.3a) yield the equation of the particle path
\[ (2.6) \quad x(\alpha, t) = \alpha + u_0 t + \frac{\kappa E_0 t^2}{2}. \]
Let
\[ \Gamma(\alpha, t) := \frac{\partial x}{\partial \alpha} = 1 + u'_0 t + \frac{\kappa \rho_0 t^2}{2}, \quad u'_0 := \frac{\partial u_0(\alpha)}{\partial \alpha}. \]
Taking the \( x \) derivative of \( u(x(\alpha, t), t) \) yields
\[ (2.7) \quad u_x(x(\alpha, t), t) = \frac{\partial u}{\partial \alpha} \frac{\partial x}{\partial \alpha} = \frac{u'_0 + \kappa \rho_0 t}{1 + u'_0 t + \frac{\kappa \rho_0 t^2}{2}} = \frac{\Gamma_t(\alpha, t)}{\Gamma(\alpha, t)}. \]
Plugging (2.7) into the mass equation (1.1a) yields
\[ \frac{d}{dt} \rho(x(\alpha, t), t) = -u_x \rho = -\frac{\Gamma_t(\alpha, t)}{\Gamma(\alpha, t)} \rho(x(\alpha, t), t). \]
Solving this equation, we obtain
\[ (2.8) \quad \rho(x(\alpha, t), t) = \frac{\rho_0}{\Gamma(\alpha, t)}. \]
Equipped with (2.5), (2.6) and (2.8), we can find the position \( X(t) \) of the center of gravity of the mass on \([a, b]\) at time \( t \)
\[
X(t) = \frac{\int_{x(a, t)}^{x(b, t)} \xi \rho(\xi, t) d\xi}{\int_{x(a, t)}^{x(b, t)} \rho(\xi, t) d\xi} = \frac{\int_{x(a, t)}^{x(b, t)} x(\alpha, t) \rho(x(\alpha, t), t) \Gamma(\alpha, t) d\alpha}{E_0(b) - E_0(a)}
\]
\[
= \frac{\int_{a}^{b} (\alpha + u_0(\alpha)t + \frac{\kappa E_0(\alpha)t^2}{2}) \rho_0(\alpha) d\alpha}{E_0(b) - E_0(a)} = X_0 + U_0 t + \frac{\kappa t^2 \int_{a}^{b} E_0(\alpha) \rho_0(\alpha) d\alpha}{2(E_0(b) - E_0(a))}
\]
\[
= X_0 + U_0 t + \frac{\kappa t^2 \int_{a}^{b} E_0(\alpha) dE_0(\alpha)}{2(E_0(b) - E_0(a))} = X_0 + U_0 t + \frac{\kappa t^2 (E_0(b)^2 - E_0(a)^2)}{4(E_0(b) - E_0(a))}
\]
\[
= X_0 + U_0 t + \frac{1}{2} \kappa \tilde{E}_0 t^2. \]

Lemma 2.1 tells us that if we replace the initial mass along \([a, b]\) by a Dirac mass, \( \int_{a}^{b} \rho_0(x) dx \), situated at location \( X_0 \), give it velocity \( U_0 \), and apply to this particle an electrical field \( \tilde{E}_0 \), then the trajectory of this particle is the same as the trajectory of the center of mass of the whole initial interval \([a, b]\). Motivated by this lemma, we continue to deal with measure densities. In
particular, if the density $\rho(\cdot, t)$ at location $x$ has a Dirac mass with strength $\overline{m}(x)$, then we set the electric field at that point to be

$$E(x, t) := \int_{-\infty}^{x^-} \rho(s, t)ds + \frac{\overline{m}(x)}{2};$$

otherwise, $E(x, t) := \int_{-\infty}^{x} \rho(s, t)ds$. Combining both cases, we arrive at the following definition of the electric field, which is in agreement with (1.7).

**Definition 2.1.** [The electric field] We define the electric field, $E(x, t)$, as

$$E(x, t) := \frac{1}{2} \left( \int_{-\infty}^{x^-} \rho(s, t)ds + \int_{-\infty}^{x^+} \rho(s, t)ds \right),$$

Equipped with definition 2.1, we will show later that the conclusion of Lemma 2.1 remains valid even after collision takes place. Therefore, setting $E$ as in (2.9) guarantees that the trajectory of the center of mass is independent of whether there is a collision or not. This is a key point which explains the validity behind our approach.

### 2.2. The dynamics of two Dirac masses.

Given the density $\rho(\cdot, t)$ as a non-negative measure and the velocity $u(\cdot, t)$ which is uniformly bounded, we introduce the corresponding mass and momentum of the system, which play an important role throughout the paper,

$$R(x, t) := \int_{-\infty}^{x^+} \rho(s, t)ds, \quad M(x, t) := \int_{-\infty}^{x^+} \rho(s, t)u(s, t)ds$$

To illustrate the construction of a weak solution solely from the physical principles, we start with the simplest example of two particles governed by an attractive force, $\kappa \leq 0$. We consider two particles with masses $m_1, m_2$ at initial positions $y_1(0) < y_2(0)$, and initial velocity $u_1(0), u_2(0)$, respectively. Thus, the initial density and momentum consist of two Dirac masses

$$\rho_0(x) = m_1 \delta(x - y_1(0)) + m_2 \delta(x - y_2(0)), \quad \rho_0(x)u_0(x) = m_1 u_1(0) \delta(x - y_1(0)) + m_2 u_2(0) \delta(x - y_2(0)).$$

If there is no collision, the electrical fields which are associated with the first and second particles are, respectively, $E_1 := \frac{m_1}{2}$ and $E_2 := m_1 + \frac{m_2}{2}$. Hence the velocity and location of the first particle are

$$u_1(t) = u_1(0) + \kappa \frac{m_1}{2}t, \quad y_1(t) = y_1(0) + u_1(0)t + \frac{1}{2} \kappa \frac{m_1}{2}t^2.$$

The velocity and location of the second particle are

$$u_2(t) = u_2(0) + \kappa \left( m_1 + \frac{m_2}{2} \right)t, \quad y_2(t) = y_2(0) + u_2(0)t + \frac{1}{2} \kappa \left( m_1 + \frac{m_2}{2} \right)t^2,$$

Therefore the velocity and location of the center of gravity of the system are

$$u(t) = \frac{m_1 u_1(t) + m_2 u_2(t)}{m_1 + m_2} = \frac{m_1 u_1(0) + m_2 u_2(0)}{m_1 + m_2} + \frac{1}{2} \kappa \frac{m_1^2 + 2m_1m_2 + m_2^2}{m_1 + m_2} t$$

$$u(0) = \frac{m_1 u_1(0) + m_2 u_2(0)}{m_1 + m_2},$$

$$u(0) \Delta := \frac{m_1 u_1(0) + m_2 u_2(0)}{m_1 + m_2}.$$
and
\[ y(t) = \frac{m_1 y_1(t) + m_2 y_2(t)}{m_1 + m_2} \]
\[ = \frac{m_1 y_1(0) + m_2 y_2(0)}{m_1 + m_2} + m_1 u_1(0) + m_2 u_2(0) t + \frac{1}{4} \kappa \left( \frac{m_1}{m_1 + m_2} + \frac{m_2}{m_1 + m_2} \right) t^2 \]
\[ = y(0) + u(0) t + \frac{1}{2} \kappa \left( \frac{m_1 + m_2}{2} \right) t^2, \quad y(0) := \frac{m_1 y_1(0) + m_2 y_2(0)}{m_1 + m_2}. \]

If collision takes place at time \( \bar{t} \), then a new particle will be formed at \( y(\bar{t}) \) with velocity \( u(\bar{t}) \): the motion of this new particle is determined by
\[(2.13a) \quad u_{\text{new}}(t) = u(\bar{t}) + \kappa \frac{m_1 + m_2}{2} (t - \bar{t}) = u(t), \]
and
\[(2.13b) \quad y_{\text{new}}(t) = y(\bar{t}) + u(\bar{t}) (t - \bar{t}) + \frac{1}{2} \kappa \left( \frac{m_1 + m_2}{2} \right) t^2 = y(t). \]

Hence, when and where the collision occurs, it will not change the trajectory of the center of mass of this system.

The standard way to describe the state of this two-particle system at any time \( t \), would be to provide the following information:

(i) if the two particles have not collided before time \( t \): provide the position and velocity of each particle at time \( t \);

(ii) if the two particles collided at some time before \( t \): then they are “stuck” as one particle thereafter, and we provide the position and velocity of this new particle at time \( t \).

Alternatively, we can provide a complete description of the two-particle system in terms of the mass \( R(x, t) \) and momentum \( M(x, t) \). To this end, take the difference of (2.12) and (2.11), to find
\[ y_2(t) - y_1(t) = \frac{1}{2} \kappa \left( \frac{m_1}{2} + \frac{m_2}{2} \right) t^2 + (u_2(0) - u_1(0)) t + (y_2(0) - y_1(0)). \]

Since the two particle system is attractive, \( \kappa \leq 0 \), there is a positive \( t_c > 0 \), such that,
\[
\begin{cases} 
  y_2(t) > y_1(t) & 0 \leq t < t_c, \\
  y_2(t) = y_1(t) & t = t_c, \\
  y_2(t) < y_1(t) & t > t_c.
\end{cases}
\]

Then, there are four possibilities to consider, depending on the relative positions of \( q_j(t) := y_j + u_j t + \frac{1}{2} \kappa E_j t^2, \ j = 1, 2 \):

(i) if both \( q_1(t) \) and \( q_2(t) \) are to the left of \( x \): \( q_1(t), q_2(t) \leq x \), then \( R(x, t) = m_1 + m_2 \) and \( M(x, t) = m_1 (u_1 + \kappa E_1 t) + m_2 (u_2 + \kappa E_2 t) \);

(ii) if both \( q_1(t) \) and \( q_2(t) \) are to the right \( x \): \( q_1(t), q_2(t) > x \), then \( R(x, t) = 0 \), and \( M(x, t) = 0 \);

(iii) if \( q_1(t) \leq x \) and \( q_2(t) > x \), then \( R(x, t) = m_1 \), and \( M(x, t) = m_1 (u_1 + \kappa E_1 t) \);

(iv) finally, if \( q_1(t) > x \) and \( q_2(t) < x \), this means that collision occurred earlier, and there is a new particle with mass \( m = m_1 + m_2 \). The velocity and location of this new particle at time \( t \) are given by (2.13a), (2.13b),
\[ u(t) = \frac{m_1 (u_1 + \kappa E_1 t) + m_2 (u_2 + \kappa E_2 t)}{m_1 + m_2}, \quad y(t) = \frac{m_1 q_1(t) + m_2 q_2(t)}{m_1 + m_2}. \]
Therefore, in this case $R(x, t)$ and $M(x, t)$ are determined by the sign of the expression
\[
\frac{m_1 q_1(t) + m_2 q_2(t)}{m_1 + m_2} - x = \frac{m_1 (q_1(t) - x) + m_2 (q_2(t) - x)}{m_1 + m_2},
\]
namely
\[
\begin{cases}
  R(x, t) = m_1 + m_2 \\
  M(x, t) = m_1 (u_1 + \kappa E_1 t) + m_2 (u_2 + \kappa E_2 t) \\
  R(x, t) = M(x, t) \equiv 0,
\end{cases}
\]
if $m_1 (q_1(t) - x) + m_2 (q_2(t) - x) \leq 0$
otherwise.

Summarizing the four cases above we observe:
(i) if $q_1(t) \leq x$, then no matter whether collision happened or not, $m_1$ will contribute to the mass $R(x, t)$;
(ii) if $q_1(t) > x$ then the contribution of $m_1$ to the mass $R(x, t)$ depends on whether the second particle is slow enough, that is, whether $m_1 (q_1(t) - x) + m_2 (q_2(t) - x) \leq 0$.

2.3. **The dynamics of general mass distribution.** In this subsection, we extend the definition of $R$ and $M$ to the general attractive case. Suppose that there exists $Y_1$ such that

\[
q_s(t) := s + u_0(s)t + \frac{1}{2} \kappa E(s, 0)t^2 \leq x, \quad \forall s \leq Y_1.
\]

Then, independently whether collision occurred or not, the particles emanating from $y \leq Y_1$ will end up to the left side of $x$ at time $t$, and therefore, the part of the mass $\int_{-\infty}^{Y_1} \rho_0(s)ds$ will be on the left side of $x$ at time $t$. If, on the other hand,

\[
q_s(t) = s + u_0(s)t + \frac{1}{2} \kappa E(s, 0)t^2 > x, \quad \forall s \in (Y_1, Y_2],
\]

then the position of the mass of this part, $\int_{Y_1}^{Y_2} \rho_0(s)ds$, relative to $x$, depends on whether there is enough slow material which will collide with this part. That is, if there exists $Y_3 > Y_2$, such that

\[
\int_{Y_1}^{Y_2} q_s(t) \rho_0(s)ds + \int_{Y_2}^{Y_3} q_s(t) \rho_0(s)ds \leq \int_{Y_1}^{Y_3} q_s(t) \rho_0(s)ds,
\]

then the mass on $(Y_1, Y_2]$ will be end up to be on the left side of $x$. We can rewrite (2.14) in the equivalent form

\[
\int_{Y_1}^{Y_3} Q_{x,t}(s) \rho_0(s)ds \leq 0, \quad Q_{x,t}(s) = q_s(t) - x.
\]

This is similar to the case (iv) of the two particle system, where $\int_{Y_1}^{Y_2} \rho_0(s)ds$ corresponds to the first particle, and $\int_{Y_2}^{Y_3} \rho_0(s)ds$ corresponds to the second particle. Continuing this process, we find that the exact amount of mass which ends on the left side of $x$, is given by

\[
R(x, t) := \int_{-\infty}^{y(x,t)} \rho_0(s)ds,
\]
where $y(x,t)$ is determined by (1.5),

$$y(x,t) = \sup_y \left\{ y \mid y = \arg\inf_y \int_0^y Q_{x,t}(s)\rho_0(s)ds \right\}, \quad Q_{x,t}(s) = q_s(t) - x.$$  

The momentum then follows

$$M(x,t) := \int_{-\infty}^{y(x,t)} \rho_0(s)\left(u_0(s) + \kappa E_0(s)t\right)ds. \tag{2.17}$$

2.4. **Proof of theorem** \[\text{Lemma 2.3} \] **for the attractive case.** As a preparation for the proof of theorem \[\text{Lemma 2.3} \] with $\kappa \leq 0$, we first characterize the entropy solution of (1.1) in terms of a one-sided Lipschitz condition. This is the content of the following two lemmas.

**Lemma 2.2.** Let $y(x,t)$ be the minimizer in (1.5). Then $y(\cdot, t)$ is non-decreasing

$$y(x_1, t) \leq y(x_2, t), \quad \forall x_1 < x_2.$$  

In particular, $y(x+, t) := \lim_{z \to x^+} y(z, t)$ is well-defined and $y(x, t) = y(x+, t), \quad \forall x.$

**Proof.** Assume there exists $x_1 < x_2$ such that $y(x_1, t) > y(x_2, t)$. Since $y(x_1, t)$ minimizes $\int_0^y Q_{x_1,t}(s)\rho_0(s)ds$, we have $\int_{y(x_1, t)}^{y(x_2, t)} Q_{x_1,t}(s)\rho_0(s)ds \leq 0$. It follows that

$$\begin{align*}
\int_{y(x_1, t)}^{y(x_2, t)} Q_{x_2,t}(s)\rho_0(s)ds &= \int_{y(x_1, t)}^{y(x_2, t)} Q_{x_1,t}(s)\rho_0(s)ds + \int_{y(x_2, t)}^{y(x_1, t)} (x_1 - x_2)\rho_0(s)ds \\
&\leq 0.
\end{align*}$$

That is,

$$\int_0^{y(x_1, t)} Q_{x_2,t}(s)\rho_0(s)ds \leq \int_0^{y(x_2, t)} Q_{x_2,t}(s)\rho_0(s)ds, \quad y(x_1, t) > y(x_2, t),$$

which is a contradiction to the definition of $y(x_2, t)$.

Next, assume there exists $x$ such that $y(x+, t) > y(x, t)$. Then $\int_{y(x, t)}^{y(x+, t)} Q_{x,t}(s)\rho_0(s)ds > 0$.

Therefore, for $\epsilon > 0$ small enough, we have

$$\begin{align*}
\int_{y(x, t)}^{y(x+, t)} Q_{x+,t}(s)\rho_0(s)ds &= \int_{y(x, t)}^{y(x+, t)} Q_{x,t}(s)\rho_0(s)ds + \int_{y(x+, t)}^{y(x+, t)} Q_{x+,t}(s)\rho_0(s)ds \\
&= \int_{y(x, t)}^{y(x+, t)} Q_{x,t}(s)\rho_0(s)ds + \epsilon(s)\rho_0(s)ds + \int_{y(x+, t)}^{y(x+, t)} Q_{x+,t}(s)\rho_0(s)ds > 0.
\end{align*}$$

That is

$$\int_0^{y(x, t)} Q_{x+,t}(s)\rho_0(s)ds < \int_0^{y(x+, t)} Q_{x+,t}(s)\rho_0(s)ds,$$

which is a contradiction to the definition of $y(x + \epsilon, t)$. \hfill \Box

**Lemma 2.3.** ([One sided Lipschitz condition]) Consider the attractive Euler-Poisson system (1.1) with $\kappa \leq 0$. If $y(x, t_0) > y(x-, t_0)$, then the values in the open interval $\gamma \in (y(x-, t_0), y(x, t_0))$, cannot be reached by evolving (1.1) along particle path, namely, $\gamma \neq$ a minimizer $y(x, t)$.

Thus, if $y(x, t_0) > y(x-, t_0)$, then according to (1.6), $\rho(x, t_0)$ will be a Dirac mass and the lemma 2.3 tells us that once a Dirac mass is formed, it will never split.
Proof. By the definition \([1,5]\), to prove the Lemma, it is enough to show \(\forall \gamma \in (y(x-,t_0),y(x,t_0))\), \(\forall z \in \mathbb{R}, \forall t > t_0\), there exists \(w\) such that \(\int_0^w Q_{z,t}(s)ds < \int_0^\gamma Q_{z,t}(s)ds\), that is, \(\gamma\) does not minimize \(\int_0^y Q_{z,t}(s)\rho_0(s)ds\). For every \(\gamma \in (y(x-,t_0),y(x,t_0))\), we have

\[
\int_\gamma^{y(x,t_0)} Q_{x,t_0}(s)\rho_0(s)ds \leq 0 \leq \int_\gamma^{y(x-,t_0)} Q_{x,t_0}(s)\rho_0(s)ds,
\]

These inequalities can be rewritten in terms of the quadratics \(f_1(t)\) and \(f_2(t)\),

\[
f_2(t_0) \leq x \leq f_1(t_0), \quad f_j(t) := a_jt^2 + b_jt + c_j,
\]

where the coefficients of \(f_j(t)\)'s are given by in terms of \(m_1 := \int_\gamma^{y(x-,t_0)} \rho_0(s)ds\) and \(m_2 := \int_\gamma^{y(x,t_0)} \rho_0(s)ds\):

\[
a_1 = \frac{1}{m_1} \int_\gamma^{y(x-,t_0)} \frac{1}{2}\kappa E_0(s)\rho_0(s)ds, \quad b_1 = \frac{1}{m_1} \int_\gamma^{y(x-,t_0)} u_0(s)\rho_0(s)ds, \quad c_1 = \frac{1}{m_1} \int_\gamma^{y(x-,t_0)} sp_0(s)ds,
\]

and

\[
a_2 = \frac{1}{m_2} \int_\gamma^{y(x,t_0)} \frac{1}{2}\kappa E_0(s)\rho_0(s)ds, \quad b_2 = \frac{1}{m_2} \int_\gamma^{y(x,t_0)} u_0(s)\rho_0(s)ds, \quad c_2 = \frac{1}{m_2} \int_\gamma^{y(x,t_0)} sp_0(s)ds.
\]

Notice that \(a_2 < \kappa E_0(\gamma) < a_1 < 0\), and \(c_2 > \gamma > c_1\). Setting \(f_1(t) = f_2(t)\), we find two solutions, one positive and one negative. Denote the positive one by \(t_c\), then

\[
\left\{ \begin{array}{l}
f_1(t) < f_2(t), \quad 0 < t < t_c \\
f_1(t) > f_2(t), \quad t > t_c.
\end{array} \right.
\]

It follows that \(t_0 \geq t_c\). For every \(t > t_0\) and \(z \in \mathbb{R}\) we have

\[
\int_\gamma^{y(x-,t_0)} Q_{z,t}(s)\rho_0(s)ds = m_1(a_1t^2 + b_1t + c_1 - z) = m_1(f_1(t) - z),
\]

\[
\int_\gamma^{y(x,t_0)} Q_{z,t}(s)\rho_0(s)ds = m_2(a_2t^2 + b_2t + c_2 - z) = m_2(f_2(t) - z).
\]

If \(f_1(t) > z\), then

\[
\int_0^\gamma Q_{z,t}(s)\rho_0(s)ds > \int_0^{y(x-,t_0)} Q_{z,t}(s)\rho_0(s)ds;
\]

if \(f_1(t) \leq z\), then \(f_2(t) < f_1(t) \leq z\), which implies

\[
\int_0^\gamma Q_{z,t}(s)\rho_0(s)ds > \int_0^{y(x,t_0)} Q_{z,t}(s)\rho_0(s)ds.
\]

Therefore \(\gamma\) does not minimize \(\int_0^y Q_{z,t}(s)\rho_0(s)ds\).

Remark 2.1. It is straightforward to verify that for smooth initial data, \(u_0(x) \in C(\mathbb{R})\), we have \(Q_{x,t}(y(x,t)) = 0\), i.e.,

\[
y(x,t) + tu_0(y(x,t)) + \frac{1}{2}\kappa E_0(y(x,t))t^2 = x, \quad \forall x.
\]
Remark 2.2. Consider two adjacent discontinuous points \( a \) and \( b \) of \( y(x,t) \), i.e., \( y(a-,t) < y(a,t) \), \( y(b-,t) < y(b,t) \) and \( y(x-,t) = y(x,t) \) for every \( x \in (a,b) \). Then \( R \) and \( M \) are continuous on \( (a,b) \) at time \( t \). Moreover, combining lemma 2.2 and remark 2.1, we find that the characteristics emanate from the interval \( (y(a,t),y(b,t)) \) at \( t = 0 \),

\[
x(t) = x_0 + u_0(x_0)t + \frac{1}{2} \kappa E_0(x_0)t^2, \quad \forall x_0 \in (y(a,t),y(b,t)),
\]

will not intersect before \( t \).

As a final preparation for the proof of theorem 1.1 we calculate the distributional derivative of jump discontinuities across curves over surfaces, which will be useful when dealing the singular part of \( R_t \) and \( M_x \). We summarize this calculation in the following lemma.

Lemma 2.4. Consider an open region, \( V \subset \mathbb{R}^2 \), and a curve, \( C(t,x(t)) \) in the \( t-x \) plane which divides \( V \) into two parts, \( V^l \) and \( V^r \), and assume that a function \( S(t,x) \) is smooth on either side of this curve \( C \), with values \( S^l \) on \( V^l \) and \( S^r \) on \( V^r \). The weak derivative of \( S \) is given by,

\[
S_x = S^l_x + S^r_x + (S^r - S^l) \nu_2 \eta_C, \quad S_t = S^l_t + S^r_t + (S^r - S^l) \nu_1 \eta_C.
\]

Here, \( \nu \) is the outward unit normal vector of \( V^l \) on boundary \( C \),

\[
\nu = (\nu_1, \nu_2) := \left( \frac{\dot{x}(t)}{\sqrt{\dot{x}^2(t) + 1}}, \frac{-1}{\sqrt{\dot{x}^2(t) + 1}} \right),
\]

and \( \eta_C(t,x) \) is a surface measure supported on the curve \( C \), satisfying

\[
\int_{\mathbb{R}^2} \phi(t,x) \eta_C(t,x) dt dx = \int_a^b \phi(t,x(t)) \sqrt{\dot{x}^2(t) + 1} dt, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).
\]

Proof. For every \( \phi(x,t) \in C_0^\infty(V) \),

\[
\int_V \phi_x(x,t) S(x,t) dt dx = \int_{V^l} \phi_x(x,t) S^l(x,t) dt dx + \int_{V^r} \phi_x(x,t) S^r(x,t) dt dx
= - \int_{V^l} \phi(x,t) S^l_x(x,t) dt dx + \int_C \phi(t,x(t)) S^l(t,x(t)) \nu_2 \eta_C - \int_{V^r} \phi(x,t) S^r_x(x,t) dt dx
\]

\[
- \int_C \phi(t,x(t)) S^r(t,x(t)) \nu_2 \eta_C \quad \text{(where} \quad (dC = \sqrt{(dx/dt)^2 + 1})
= - \int_{V^l} \phi(x,t) S^l_x(x,t) dt dx - \int_{V} \phi(x,t) S^l_x(x,t) dt dx + \int_C \phi(t,x(t)) \left( S^r(t,x(t)) - S^l(t,x(t)) \right) \nu_2 \eta_C dt dx
= - \int_{V^l} \phi(x,t) S^l_x(x,t) dt dx - \int_{V} \phi(x,t) S^l_x(x,t) dt dx
\]

Hence \( S_x = S^l_x + S^r_x + (S^r - S^l) \nu_2 \eta_C \). Similarly,

\[
\int_V \phi_t(x,t) S(x,t) dt dx = \int_{V^l} \phi_t(x,t) S^l(x,t) dt dx + \int_{V^r} \phi_t(x,t) S^r(x,t) dt dx
= - \int_{V^l} \phi(x,t) S^l_t(x,t) dt dx + \int_C \phi(t,x(t)) S^l(t,x(t)) \nu_1 \eta_C - \int_{V^r} \phi(x,t) S^r_t(x,t) dt dx
\]

\[
- \int_C \phi(t,x(t)) S^r(t,x(t)) \nu_1 \eta_C \quad \text{(where} \quad (dC = \sqrt{(dx/dt)^2 + 1})
= - \int_{V^l} \phi(x,t) S^l_t(x,t) dt dx - \int_{V} \phi(x,t) S^l_t(x,t) dt dx + \int_C \phi(t,x(t)) \left( S^r(t,x(t)) - S^l(t,x(t)) \right) \nu_1 \eta_C dt dx.
\]
Hence \( S_t = S_t^i + S_t^r + (S^r - S^i)\nu_t\eta_C. \)

Equipped with lemmas 2.3 and 2.4 we are now ready to complete the proof of theorem 1.1 in the attractive case, \( \kappa \leq 0. \)

**Proof of theorem 1.1 with \( \kappa < 0. \) Step \#1 [the mass equation].** First, we show \( \rho \) and \( \rho u \) satisfy the mass equation (1.1a) in the weak sense. We need to verify that

\[
\int_0^\infty \int_{-\infty}^{\infty} \rho_t \phi + (\rho u)_x \phi \, dx \, dt = \int_0^\infty \int_{-\infty}^{\infty} R_x \phi + M_{xx} \phi \, dx \, dt = \int_0^\infty \int_{-\infty}^{\infty} R\phi_{xt} + M\phi_{xxt} \, dx \, dt
\]

vanishes for all test functions \( \phi \in C_c^\infty([0, \infty) \times (-\infty, \infty)) \). To this end, we claim that

\[
\frac{\partial R}{\partial t} = -\frac{\partial M}{\partial x}.
\]

It follows (2.20) that there exists a function \( G \) such that

\[
R(x, t) = -\frac{\partial G(x, t)}{\partial x}, \quad M(x, t) = \frac{\partial G(x, t)}{\partial t}.
\]

By plugging (2.21) into (2.19) we obtain

\[
\int_0^\infty \int_{-\infty}^{\infty} \rho_t \phi + (\rho u)_x \phi \, dx \, dt = \int_0^\infty \int_{-\infty}^{\infty} G\phi_{xt} - G\phi_{xxt} \, dx \, dt = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+).
\]

We decompose \( R_t \) and \( M_x \) into the sum of an absolutely continuous measure and a singular measure. We denote

\[
R_t = R_t^a + R_t^s, \quad M_x = M_x^a + M_x^s,
\]

where \( R_t^a \) and \( M_x^a \) are absolutely continuous measures, \( R_t^s \) and \( M_x^s \) are singular measures. More precisely, if \( (x, t) \) is a jumping point (that is, \( y(x-), t < y(x, t) \)), then \( R_t \) and \( M_x \) are singular at \( (x, t) \); if \( y(x-, t) = y(x, t) \), then \( R \) and \( M \) are continuous, hence \( R_t \) and \( M_x \) are not singular.

The detailed proof of (2.20) is carried out by verifying that \( R_t^a = -M_x^a \) and \( R_t^s = -M_x^s \) in steps \#1(a) and \#1(b) below.

**Step \#1(a) [the regular part of the mass equation].** We show that \( R_t^a = -M_x^a \). If \( (x, t) \) is a jumping point, then \( R_t^a = -M_x^a = 0 \). Otherwise, \( y(x, t) \) is continuous at \( (x, t) \), and by the definition of \( R \) and \( M \), we have

\[
R_t^a = \rho_0(y) \frac{\partial y}{\partial t}, \quad M_x^a = \rho_0(y) \left(u_0(y) + \kappa E_0(y) t\right) \frac{\partial y}{\partial x}.
\]

By remark 2.2 the equality \( Q_{x,t}(y(x, t)) = 0 \) is valid in the neighborhood of \( (x, t) : Q_{x,t}(y(x, t)) = tu_0(y) + y + \frac{1}{2}\kappa E_0(y) t^2 - x \equiv 0 \). Taking partial derivatives with respect to \( t \) and \( x \) yields

\[
u_0(y) + \kappa E_0(y) t + \frac{\partial Q_{x,t}(y)}{\partial y} \frac{\partial y}{\partial t} = 0 \quad \text{and} \quad -1 + \frac{\partial Q_{x,t}(y)}{\partial y} \frac{\partial y}{\partial x} = 0.
\]

Hence

\[
\frac{\partial y}{\partial t} = -\frac{\nu_0(y) + \kappa E_0(y) t}{\partial Q_{x,t}(y)}, \quad \frac{\partial y}{\partial x} = \frac{1}{\partial Q_{x,t}(y)},
\]

and therefore

\[
\frac{\partial y}{\partial t} = -\left(u_0(y) + \kappa E_0(y) t\right) \frac{\partial y}{\partial x}.
\]

Combining (2.21) and (2.23) we obtain \( R_t^a(x, t) = -M_x^a(x, t) \).
Step #1(b) [the singular part of the mass equation]. We show that $R^s_t = -M^s_x$. If $y(z-, t) = y(z, t)$, then $R^s_t = -M^s_x = 0$ at $(z, t)$. Otherwise, $(z, t)$ is a jumping point where we have

$$
\int_{y(z-, t)}^{y(z, t)} \left( s + tu_0(s) + \frac{1}{2} \kappa E_0(s)t^2 - z \right) \rho_0(s) ds = 0,
$$

(2.25)

$$
y(z, t) + tu_0(y(z, t)) + \frac{1}{2} \kappa E_0(y(z, t))t^2 - z = 0,
$$

(2.26)

$$
y(z-, t) + tu_0(y(z-, t)) + \frac{1}{2} \kappa E_0(y(z-, t))t^2 - z = 0.
$$

(2.27)

Denote the trajectory of this jumping point by $C(t, z(t))$. According to Lemma 2.4, we have

$$
R^s_t(z, t) = B(z, t)\eta_C \frac{\dot{z}(t)}{\sqrt{1 + \dot{z}^2(t)}}
$$

(2.28)

and

$$
M^s_x(z, t) = -B(z, t)u(z, t)\eta_C \frac{1}{\sqrt{1 + \dot{z}^2(t)}};
$$

(2.29)

where

$$
B(z, t) = \int_{y(z-, t)}^{y(z, t)} \rho_0(s) ds, \quad u(z, t) = \frac{\int_{y(z-, t)}^{y(z, t)} \rho_0(s) \left( u_0(s) + \kappa E_0(s)t \right) ds}{B(z, t)}.
$$

Thus, to prove $R^s_t = -M^s_x$ at the jumping point $(z, t)$, it remains to show that the propagation speed of this jumping point, $\dot{z}(t) =: \bar{u}(z, t)$ is actually given by $u(z, t)$. We provide the details of $\bar{u}(z, t) = u(z, t)$ below.

The location of this jumping point at time $t + \Delta t$ is $(t + \Delta t, z(t + \Delta t))$. Hence

$$
\int_{y(z(\Delta t), t + \Delta t)}^{y(z(t + \Delta t), t + \Delta t)} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z(t + \Delta t) \right) \rho_0(s) ds = 0.
$$

(2.30)

Combining,

$$
y(z(t + \Delta t), t + \Delta t) = y(z(t), t + \Delta t) = y(z(t), t) + O(\Delta t),
$$

(2.31)

$$
y(z(t + \Delta t), t + \Delta t) = y(z(t), t) + O(\Delta t),
$$

$$
z(t + \Delta t) = z + \bar{u}(z, t)\Delta t + O(\Delta t^2).
$$

with (2.30) and (2.25), we obtain

$$
0 = \int_{y(z(\Delta t), t + \Delta t)}^{y(z(t + \Delta t), t + \Delta t)} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z(t + \Delta t) \right) \rho_0(s) ds
$$

$$
= \int_{y(z-, t)}^{y(z, t)} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z - \bar{u}(z, t)\Delta t + O(\Delta t^2) \right) \rho_0(s) ds
$$

$$
+ \int_{y(z(\Delta t), t - \Delta t)}^{y(z(t + \Delta t), t - \Delta t)} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z - \bar{u}(z, t)\Delta t \right) \rho_0(s) ds
$$

$$
+ \int_{y(z(t + \Delta t), t + \Delta t)}^{y(z(t + \Delta t), t + \Delta t)} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z - \bar{u}(z, t)\Delta t \right) \rho_0(s) ds.
$$
For every $s \in \left( y(z(t + \Delta t)) - t + \Delta t, y(z(t + \Delta t)) \right)$, using Taylor expansion and (2.26), we obtain

$$s + tu_0(s) + \frac{1}{2} \kappa E_0(s)t^2 - z = y(z(t) + tu_0(y(z(t))) + \frac{1}{2} \kappa E_0(y(z(t)))t^2 - z + O(\Delta t) = O(\Delta t).$$

Hence

$$\int_{y(z(t + \Delta t)) - t + \Delta t}^{y(z(t + \Delta t)) + \Delta t} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z - \bar{u}(z(t)\Delta t) \right) \rho_0(s)ds = o(\Delta t).$$

Similarly

$$\int_{y(z(t + \Delta t)) - t + \Delta t}^{y(z(t + \Delta t)) + \Delta t} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z - \bar{u}(z(t)\Delta t) \right) \rho_0(s)ds = o(\Delta t).$$

Therefore

$$0 = \int_{y(z(t + \Delta t)) - t + \Delta t}^{y(z(t + \Delta t)) + \Delta t} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z(t + \Delta t) \right) \rho_0(s)ds$$

$$= \int_{y(z(t)) - t + \Delta t}^{y(z(t)) + \Delta t} \left( s + (t + \Delta t)u_0(s) + \frac{1}{2} \kappa E_0(s)(t + \Delta t)^2 - z - \bar{u}(z(t)\Delta t) \right) \rho_0(s)ds + o(\Delta t)$$

$$= \int_{y(z(t)) - t + \Delta t}^{y(z(t)) + \Delta t} \left( s + tu_0(s) + \frac{1}{2} \kappa E_0(s)t^2 - z \right) \rho_0(s)ds + o(\Delta t)$$

$$+ \int_{y(z(t)) - t + \Delta t}^{y(z(t)) + \Delta t} \left( u_0(s)\Delta t + \frac{1}{2} \kappa E_0(s)(2t\Delta t + \Delta t^2) - \bar{u}(z(t)\Delta t) \right) \rho_0(s)ds + o(\Delta t)$$

$$= \Delta t \int_{y(z(t)) - t + \Delta t}^{y(z(t)) + \Delta t} \left( u_0(s) + \kappa E_0(s)t - \bar{u}(z(t)) \right) \rho_0(s)ds + o(\Delta t).$$

This concludes our argument that $\bar{u}(z(t)) = u(z(t))$,

$$\bar{u}(z(t)) = \frac{\int_{y(z(t))}^{y(z(t)) + \Delta t} \rho_0(s) \left( u_0(s) + \kappa E_0(s)t \right) ds}{\int_{y(z(t))}^{y(z(t)) + \Delta t} \rho_0(s) ds} = u(z(t)),$$

and $R^t_x = -M_x^2$ follows. Thus, (2.20) holds, and the mass equation (1.1a) is satisfied in the weak sense (2.22).

**Step #2** [the momentum equation]. Next, we verify the momentum equation (1.1b) in a similar way. We apply test functions $\phi \in C_0^\infty([0, \infty) \times (-\infty, \infty))$ to (1.1b), then

$$(2.32) \int_0^\infty \int_{-\infty}^\infty (pu)_t \phi + (pu^2)_x \phi - \kappa pE \phi dx dt$$

$$= \int_0^\infty \int_{-\infty}^\infty M_{xt} \phi + W_{xx} \phi - Z_{xx} \phi dx dt = \int_0^\infty \int_{-\infty}^\infty M_{xt} + (W - Z) \phi_{xx} dx dt,$$

where

$$W(x, t) := \int_0^x \frac{M^2(s, t)}{R_x(s, t)} ds, \quad Z_{xx}(x, t) = \kappa pE.$$
We will show below, in steps #2(a)-2(b), that
\begin{equation}
\frac{\partial M}{\partial t} = -\frac{\partial (W - Z)}{\partial x}.
\end{equation}
This yields the existence of \( \Psi \) such that
\begin{equation}
M = \frac{\partial \Psi}{\partial x}, \quad W - Z = -\frac{\partial \Psi}{\partial t},
\end{equation}
which in turn, implies that the momentum equation, (1.1b), holds in its weak formulation (2.32).

**Step #2(a).** The main claim here is that \( Z_x \) is given by \( \int_{y(x,t)}^{y(x,t)} \kappa \rho_0(s) E_0(s) ds \), that is, if we let \( F \) denote \( Z_x \) then
\begin{equation}
F_x(x,t) = \kappa \rho(x,t) E(x,t), \quad F(x,t) := \int_{0}^{y(x,t)} \kappa \rho_0(s) E_0(s) ds.
\end{equation}
As before, we distinguish between two cases. In the case \( x \) is a continuity point, \( y(x-,t) = y(x,t) \), then
\begin{equation}
\frac{\partial F}{\partial x} = \kappa \rho_0(y) E_0(y) \frac{\partial y}{\partial x}.
\end{equation}
Combining \( \rho(x,t) = \frac{\partial}{\partial x} R(x,t) \) and (1.1b), we obtain
\begin{equation}
\rho_0(y) \frac{\partial y}{\partial x} = \frac{\partial}{\partial x} \int_{0}^{y(x,t)} \rho_0(s) ds = \frac{\partial R(x,t)}{\partial x} = \rho(x,t),
\end{equation}
and
\begin{equation}
E(x,t) = \int_{-\infty}^{x} \rho(w,t) dw = \int_{-\infty}^{x} \frac{\partial}{\partial w} \left( \int_{0}^{y(w,t)} \rho_0(s) ds \right) dw
= \int_{0}^{y(x,t)} \rho_0(s) ds - \int_{-\infty}^{0} \rho_0(s) ds = \int_{0}^{y(x,t)} \rho_0(s) ds = E_0(y).
\end{equation}
Plugging (2.36b) and (2.36c) into (2.36a), we obtain (2.35), \( F_x = \kappa \rho(x,t) E(x,t) \).

Next, we consider the case of a jump discontinuity at \( x = z \), where \( y(z,t) > y(z-,t) \). Then
\begin{equation}
\frac{\partial F}{\partial x} \bigg|_{x=z} = \frac{\partial}{\partial x} \int_{-\infty}^{y(z,t)} \kappa \rho_0(s) E_0(s) ds \bigg|_{x=z} = \delta(x-z) \int_{y(z-,t)}^{y(z,t)} \kappa \rho_0(s) E_0(s) ds
= \delta(x-z) \int_{y(z-,t)}^{y(z,t)} \kappa E_0(s) ds = \delta(x-z) \frac{\kappa}{2} \left( E_0(y(z,t))^2 - E_0(y(z-,t))^2 \right).
\end{equation}
On the other hand
\begin{equation}
k \rho(z,t) E(z,t) = \kappa \int_{0}^{y(z,t)} \rho_0(s) ds \bigg|_{x=z} \frac{1}{\kappa} \left( \int_{-\infty}^{z} \rho(s,t) ds + \int_{-\infty}^{\infty} \rho(s,t) ds \right)
= \frac{\kappa}{2} \delta(x-z) \int_{y(z-,t)}^{y(z,t)} \rho_0(s) ds \left( \int_{-\infty}^{z} \frac{\partial R(s,t)}{\partial s} ds + \int_{-\infty}^{\infty} \frac{\partial R(s,t)}{\partial s} ds \right)
= \frac{\kappa}{2} \delta(x-z) \int_{y(z-,t)}^{y(z,t)} \rho_0(s) ds \left( \int_{-\infty}^{y(z-,t)} \rho_0(s) ds + \int_{y(z-,t)}^{y(z,t)} \rho_0(s) ds \right)
= \frac{\kappa}{2} \delta(x-z) \left( E_0(y(z,t)) - E_0(y(z-,t)) \right) \left( E_0(y(z-,t)) + E_0(y(z,t)) \right)
= \delta(x-z) \frac{\kappa}{2} \left( E_0(y(z,t))^2 - E_0(y(z-,t))^2 \right).
\end{equation}
Thus, recover (2.35) \( F_x = \kappa \rho(x, t)E(x, t) \) also at any jumping point \( z \),

\[
Z_x = F = \int_0^{y(x,t)} \kappa \rho_0(s)E_0(s)ds.
\]

**Step #2(b).** To prove (2.33), we decompose \( M_t \) and \((W - Z)_x\) into the sum of an absolutely continuous measure and a singular measure, denoting

\[
M_t = M^a_t + M^s_t; \quad (W - Z)_x = (W - Z)^a_x + (W - Z)^s_x,
\]

where \( \{ \}^a \) denotes an absolutely continuous measure and \( \{ \}^s \) denotes a singular measure. The absolutely continuous measure of \((W - Z)_x\) is given by \((\tilde{W}_x - F)\), where

\[
\tilde{W} := \int_0^{y(x,t)} \rho_0(s)\left( u_0(s) + E_0(s) t \right)^2 ds.
\]

Since \( y(x, t) \) is continuous almost everywhere, we have \( W_x = \tilde{W}_x \) almost everywhere.

Consider the absolutely continuous parts: if \( y(x-, t) < y(x, t) \), then \( M^a_t(x, t) = -(W - Z)^a_x(x, t) = 0 \); otherwise, if \( y(x-, t) = y(x, t) \), then

\[
M^a_t(x, t) = \rho_0(y(x, t)) \left( u_0(y(x, t)) + \kappa E_0(y(x, t)) \right) \frac{\partial y}{\partial t} + \int_0^{y(x,t)} \kappa \rho_0(s)E_0(s)ds
\]

\[
= \rho_0(y(x, t)) \left( u_0(y(x, t)) + \kappa E_0(y(x, t)) \right) \frac{\partial y}{\partial t} + F,
\]

and

\[
(W - Z)^a_x(x, t) = \tilde{W}^a_x - F = \rho_0(y(x, t)) \left( u_0(y(x, t)) + \kappa E_0(y(x, t)) \right) \frac{\partial y}{\partial x} - F.
\]

Combining (2.39a), (2.39b) and (2.24), we obtain \( M^a_t = -(W - Z)^a_x \).

For the singular parts: if \( y(x-, t) = y(x, t) \), then \( M^s_t(x, t) = -(W - Z)^s_x(x, t) = 0 \); otherwise, at any jumping point \((z, t)\) we have, (by the definition of \( R, M \) and \( Z \)),

\[
M^s_t(z, t) = B(z, t)u(z, t)\eta C \frac{\hat{z}(t)}{\sqrt{1 + \hat{z}(t)^2}}.
\]

\[
(W - Z)_x^s(z, t) = W^s_x(z, t) = -B(z, t)u^2(x, t)\eta C \frac{1}{\sqrt{1 + \hat{z}(t)^2}}.
\]

When deriving \((W - Z)_x^s\), we used the fact that (i) \( Z_x^s = 0 \) (since \( Z_x = F \) is bounded everywhere), and (ii) we have already shown, \( \hat{z} = u(z, t) \). Hence, we also have for the singular part \( M^s_t = -(W - Z)^s_x \). Thus, the momentum equation (1.16) holds in the weak sense. \(\square\)

The proof of theorem 1.1 covers of course the case of smooth solutions and in particular, the variational formula (1.6) describes the globally-in-time smooth solutions in the sub-critical case [EnLT01]. In this sense, (1.6) could be viewed as an extension that covers both sub-critical and super-critical initial configurations. We close this section by reproducing the proof of theorem 1.1 for the simpler case of smooth solutions.

**Lemma 2.5.** Consider (1.1) with a smooth solution on \([0, T]\). Then \((\rho, pu)\) given in (1.1) is that smooth solution.

**Proof.** We denote the smooth solution by \((\tilde{\rho}, \tilde{u})\), the solution given by (1.6) by \((\rho, u)\), and we show \((\tilde{\rho}(\cdot, t), \tilde{u}(\cdot, t)) = (\rho(\cdot, t), u(\cdot, t))\), \( t \in [0, T] \). We can solve the equation by the method of characteristics, and we have shown the details in the proof of Lemma 2.1. Since the solution is smooth, no characteristics will intersect. For every given \( x \in \mathbb{R} \), \( t \in [0, T] \), there is an unique \( \tilde{y}(x, t) \), such that the characteristic which emanates from \( \tilde{y}(x, t) \) arrives \( x \) at time \( t \). We claim \( \tilde{y}(x, t) \) is the unique minimizer of \( \int_0^\infty Q_{x,t}(s)\rho_0(s)ds \), that is,
\[ \tilde{y}(x,t) = \arg \inf_y \int_0^y Q_{x,t}(s) \rho_0(s) ds. \]

Otherwise there exists \( w \neq \tilde{y} \) such that
\[
\int_0^{\tilde{y}(x,t)} Q_{x,t}(s) \rho_0(s) ds \geq \int_0^w Q_{x,t}(s) \rho_0(s) ds,
\]
i.e.,
\[
\int_w^{\tilde{y}(x,t)} Q_{x,t}(s) \rho_0(s) ds \geq 0, \quad \text{if } w < \tilde{y}(x,t),
\]
\[
\int_w^{\tilde{y}(x,t)} Q_{x,t}(s) \rho_0(s) ds \leq 0, \quad \text{if } w > \tilde{y}(x,t).
\]
Combining the above inequalities with \( Q_{x,t}(\tilde{y}) = 0 \), we obtain: if \( w < \tilde{y} \), then \( \exists z \in (w, \tilde{y}) \), such that \( Q_{x,t}(z) \geq 0 \), which implies the characteristics emanate form \( z \) and \( \tilde{y} \) must intersect no latter than \( t \); otherwise \( w > \tilde{y} \), then \( \exists z \in (\tilde{y}, w) \), such that \( Q_{x,t}(z) \leq 0 \), which implies the characteristics emanate form \( z \) and \( \tilde{y} \) must intersect no later than \( t \). This is a contradiction to smooth solution exists up to \( T > t \). Therefore,
\[ R(x,t) := \int_0^{\tilde{y}(x,t)} \rho_0(s) ds, \quad M(x,t) := \int_0^{\tilde{y}(x,t)} \left( u_0(s) + \kappa E_0(s)t \right) \rho_0(s) ds, \]
and
\[ \rho(x,t) = \frac{\partial \tilde{y}(x,t)}{\partial x} \rho_0(\tilde{y}), \quad \rho(x,t) u(x,t) = \frac{\partial \tilde{y}(x,t)}{\partial x} \rho_0(\tilde{y}) \left( u_0(\tilde{y}) + \kappa E_0(\tilde{y})t \right). \]
Thus
\[ u(x,t) = u_0(\tilde{y}) + \kappa E_0(\tilde{y})t = \tilde{u}(x,t). \]
To verify \( \rho(x,t) = \tilde{\rho}(x,t) \), it is enough show that
\[ \tag{2.40} \int_{-\infty}^{x} \tilde{\rho}(s,t) ds = \int_{-\infty}^{\tilde{y}(x,t)} \rho_0(s) ds = R(x,t) = \int_{-\infty}^{x} \rho(s,t) ds, \quad \forall x, \quad \forall t < T. \]
We denote the inverse function of \( \tilde{y} \) by \( \tilde{x}(y,t) \), which means the characteristic starts from \( y \) arrives \( \tilde{x} \) at time \( t \). Then (2.40) is equivalent to
\[ \tag{2.41} \int_{-\infty}^{\tilde{x}(y,t)} \tilde{\rho}(s,t) ds = \int_{-\infty}^{y} \rho_0(s) ds, \quad \forall y, \quad \forall t < T. \]
Notice that \( y = \tilde{x}(y,0) \), so (2.41) equivalents to: for every fixed \( y \), \( E(y,t) := \int_{-\infty}^{\tilde{x}(y,t)} \tilde{\rho}(s,t) ds \) is a constant. Physically, this is clear: since no charge can across the particle path, by conservation of charge, \( E \) remains constant along \( \tilde{x}(y,t) \). To show \( E \) is a constant along \( \tilde{x}(y,t) \), we take the time derivative of \( E \), then
\[
\frac{\partial}{\partial t} E(y,t) = \frac{d \tilde{x}(y,t)}{dt} \tilde{\rho}(\tilde{x},t) + \int_{-\infty}^{\tilde{x}(y,t)} \frac{\partial \tilde{\rho}(s,t)}{\partial t} ds
\]
\[ = \tilde{u}(\tilde{x},t) \tilde{\rho}(\tilde{x},t) - \int_{-\infty}^{\tilde{x}(y,t)} \left( \tilde{\rho}(s,t) \tilde{u}(\tilde{x},t) \right)_x ds = \tilde{u} \tilde{\rho} - \tilde{\rho} \tilde{u} = 0. \]
3. Weak solutions of the repulsive pressureless Euler-Poisson system

For \( \kappa > 0 \), (1.6) still yields a weak solution of the Euler-Poisson system, although it may be different from the one corresponding to the “sticky particle model”. The following example demonstrates this point.

**Example 3.1.** Let \( \kappa = 1 \). We consider a system of two initial Dirac masses: both have mass 1, their initial positions are 0 and 1, their initial velocities are 2 and 0, respectively. Hence the initial density and momentum are \( \rho_0(x) = \delta(x) + \delta(x-1) \), and \( \rho_0(x)u_0(x) = 2\delta(x) \). The characteristics of them are

\[
y_1(t) = 2t + \frac{1}{4}t^2, \quad y_2(t) = 1 + \frac{3}{4}t^2.
\]

If we set \( y_1(t) = y_2(t) \), then it yields two solutions: \( t_1 = 2 - \sqrt{2} \) and \( t_2 = 2 + \sqrt{2} \). The dynamic of the weak solution given by our formula is: the two particles collides at time \( t_1 \), then they stick as one particle, when this new particle arrives the location \( y_1(t_2) = y_2(t_2) \) at time \( t_2 \), it splits into two particles again.

Therefore, in the repulsive case \( \kappa > 0 \), the weak solution given by (1.6) subject to smooth initial data — even if the data is super-critical, will eventually become smooth again; consult remark 3.2 below.

To further clarify the different behavior of solutions to the attractive and repulsive pressureless Euler-Poisson, consider a two particles Euler-Poisson system with mass \( m_1, m_2 \), initial velocity \( u_1, u_2 \), initial position \( y_1 < y_2 \). If \( \kappa < 0 \), then

\[
(3.1) \quad y_1(t) = y_1 + u_1 t + \frac{1}{2} \kappa m_1 t^2 = y_2(t) = y_2 + u_2 t + \frac{1}{2} \kappa (m_1 + m_2) t^2
\]

always yields one and only one positive solution \( t_c \), such that

\[
y_1(t) < y_2(t), \quad 0 \leq t < t_c
\]

\[
y_1(t) > y_2(t), \quad t > t_c.
\]

If \( \kappa > 0 \), however, then (3.1) can either have no positive solution or two positive solutions, and for this is the reason, our representation formula gives a “non-sticky particle” solution for the repulsive model.

We now turn to the proof of theorem 1.1 in the repulsive case, \( \kappa > 0 \).

**Proof.** To verify the mass equation (1.1.a), we decompose \( R_t \) and \( M_x \) into two parts: an absolutely continuous measure and a singular measure. We distinguish between two cases.

(1). If \( y(x-, t) = y(x, t) \), one can trace \( x \) backward along the characteristic at least for a short time, and then applies the argument of Step #1(a) in the proof of the attractive case, \( \kappa \leq 0 \) case.

(2) If \( y(x-, t) < y(x, t) \), then there is a \( \delta \)-shock. The dynamics of the \( \delta \)-shock for \( \kappa > 0 \) is a little different from the one for \( \kappa \leq 0 \) in that the shock may split or disappear. But before it splits or disappears, the argument of Step #1(b) in the proof of \( \kappa \leq 0 \) case is still valid. If the \( \delta \)-shock splits, we apply the argument to each sub-shock, the conclusion still holds.

In a similar way, we can verify the moment equation for \( \kappa > 0 \).

\( \square \)

**Remark 3.1.** Using the same technique, we can easily extend Theorem 1.1 to measure density initial data, that is, Theorem 1.2.

**Remark 3.2.** There is a global smooth solution if and only if \( x(\alpha, t) = \alpha + u_0(\alpha) t + \frac{1}{2} \kappa E_0(\alpha) t^2 \) remains a monotonically increasing function with respect to \( \alpha \), that is, the solution remains...
smooth as long as
\[
(3.2) \quad \frac{\partial x(\alpha, t)}{\partial \alpha} = 1 + u_0'(\alpha)t + \frac{1}{2}\kappa \rho_0(\alpha)t^2 > 0, \quad \forall \alpha \in \mathbb{R}, t > 0.
\]
Solving (3.2), we obtain the following critical condition: the repulsive Euler-Poisson system admits a global smooth solution if and only if \( u_0'(\alpha)t + \frac{1}{2}\kappa \rho_0(\alpha)t^2 = 0 \), \( \forall \alpha \in \mathbb{R} \).

Remark 3.3. The \( \delta \)-shock splits for \( \kappa > 0 \) because the particles have “memory” back to time \( t = 0 \). More precisely, for \( \kappa > 0 \), we have the following two constructions which may yield different solutions at time \( t > 0 \):

1. Use (1.6) with initial data \((\rho_0, u_0)\) to compute \( \rho(\cdot, t) \) and \( \rho u(\cdot, t) \) directly.
2. Fix \( 0 < t_1 < t \). First use (1.6) with initial data \((\rho_0, u_0)\) to construct \( \rho(\cdot, t_1) \) and \( \rho u(\cdot, t_1) \). Then, solve the Euler-Poisson system subject to initial data \( \rho(\cdot, t_1), \rho u(\cdot, t_1) \) to obtain \( \rho(\cdot, t) \) and \( \rho u(\cdot, t) \). That is, apply (1.6) with initial data \((\rho, u)(\cdot, t_1)\) to obtain \( \rho(\cdot, t) \) and \( \rho u(\cdot, t) \). In this second construction, particles are losing their “history” before \( t_1 \). That is, if particles collided before \( t_1 \), they will “forget” they used to be separate and therefore stick together forever.

To enforce the particles to lose their “history” at every moment (hence once collision happens, the particles will forget they were separate, and stick as one particle thereafter), one can impose the one sided Lipschitz condition of lemma 2.3 as an entropy condition: to this end, (1.6) should be changed into
\[
(3.3) \quad y(x, t) := \sup y \left\{ y = \arg \inf \int_0^y Q_{x,t}(s)\rho_0(s)ds \right\},
\]
where \( D(t) = \{ z | \exists x \in \mathbb{R}, \exists \bar{t} < t, \text{ s.t. } y(x, \bar{t}) < z < y(x, \bar{t}) \} \).

4. Weak solution of multi-dimensional system with symmetry

In this section, we extend the result to the multi-dimensional system with symmetry with explicit formulation of global weak solutions for the weighted Euler-Poisson system outlined in theorem 1.3.

Proof. of theorem 1.3. Since the initial data of (1.9) are spherically symmetric, the solution of (1.9) will remain spherically symmetric, \( \rho(x, t) = \rho(r, t) \), \( u(x, t) = w(r, t)\frac{x}{r} \). Plugging these into (1.9a), we obtain
\[
\rho_t + \frac{\partial(\rho w)}{\partial r} \left( \frac{x}{r}, \frac{x}{r} \right) + \rho w(-\frac{1}{r^2} \frac{x}{r} \cdot x) + \rho w \frac{n}{r} = 0,
\]
that is
\[
(4.1) \quad \rho_t + \frac{\partial(\rho w)}{\partial r} + \frac{n-1}{r} \rho w = 0.
\]
Multiplying (4.1) by \( r^{n-1} \), we have
\[
(4.2) \quad (r^{n-1}\rho)_t + \frac{\partial(r^{n-1}\rho w)}{\partial r} = 0.
\]
Since \( \rho \) is spherically symmetric, so does \( V \), i.e., \( V(x,t) = V(r,t) \). Therefore, \( \nabla V = \frac{\partial V}{\partial r} \frac{x}{r} \), and \( \Delta V = V_{rr} + \frac{1}{r} V_r = \rho \). Multiplying \( V_{rr} + \frac{1}{r} V_r \) by \( \rho \) we obtain \( \rho V_{rr} = \rho \). Plugging \( \rho(r,t), u(r,t) \) and \( V(r,t) \) into (4.3), we have

\[
\left( \rho w \frac{x}{r} \right)_t + \nabla \cdot \left( \rho w^2 \frac{x}{r^2} \otimes x \right) = \kappa \rho r n^{-1} \nabla V,
\]

which is

\[
\frac{1}{r} \rho w (n + 1) \mathbf{x} + \left( \rho w^2 \right) \frac{1}{r^2} \mathbf{x} = \kappa \rho r n^{-1} V \frac{\mathbf{x}}{r}.
\]

Further simplification yields

\[
\rho w t + \frac{\rho w^2}{r} (n + 1) + \left( \rho w^2 \right) \frac{r^2}{r} = \rho w t + \frac{\rho w^2}{r} (n - 1) + \left( \rho w^2 \right) \frac{r}{r} = \kappa \rho r n^{-1} V_r.
\]

Multiplying the above equation by \( r^{-1} \), we obtain

(4.3)
\[
(r^{-1} \rho w) t + (r^{-1} \rho w^2) = \kappa r^{-1} \rho r n^{-1} V_r.
\]

Therefore, let \( \varsigma(|x|, t) = |x|^{-1} \rho(|x|, t) \), then (4.1) and (4.3) can be rewritten as

(4.4)
\[
\varsigma_t + (\varsigma w)_r = 0,
\]

\[
(\varsigma w)_t + (\varsigma w^2)_r = \kappa \varsigma r n^{-1} V_r = \varsigma E, \quad \frac{\partial E}{\partial r} = \frac{\partial (\rho r^{-1} V_r)}{\partial r} = \varsigma.
\]

Consider (4.4) with symmetric initial density \( \varsigma < -r, 0 > = \varsigma(r, 0) = r^{-1} \rho(r, 0) \) and anti-symmetric initial velocity \( w(-r, 0) = -w(r, 0) \). Let

(4.5)
\[
E(r, t) = \frac{1}{2} \left( \int_0^{-r} \varsigma(s,t) ds + \int_0^{r} \varsigma(s,t) ds \right).
\]

The difference between the electric field (4.5) and the previous one (which is defined by \( E(s,t) = \frac{1}{2} \left( \int_0^{-r} \varsigma(s,t) ds + \int_0^{r} \varsigma(s,t) ds \right) \)) is a constant \( \int_{-\infty}^0 \varsigma(s,0) ds \). Physically, it corresponds to a Galilean transformation. For symmetric initial data, choosing \( E \) as (4.5) is natural and convenience, since in such setting the particle located at the origin will not move and \( (\varsigma, w) \) will stay symmetric. Applying theorem (1.2) with the electric field (4.5), we obtain a weak solution for \( (\varsigma, w) \). Then we recover \( \left( \rho(x,t), (u(x,t)) \right) \) from \( (\varsigma(r,t), w(r,t)) \), \( r \geq 0 \)

\[
\square
\]

5. Concluding remarks

We have constructed a global weak solution for the 1D pressureless Euler-Poisson system. For the weighted multi-dimensional pressureless Euler-Poisson system with symmetry, which is essentially a 1D system, we have constructed a global weak solution in the same manner. The open question is: is it possible to at least extend the method to the real multi-dimensional Euler system and obtain a weak solution from the physical laws directly?

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