E-Eigenvalue Inclusion Theorems for Tensors

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Abstract. Two Z-eigenvalue inclusion theorems for tensors presented by Wang et al. (Discrete Cont. Dyn.-B, 2017, 22(1): 187–198) are first generalized to E-eigenvalue inclusion theorems. And then a tighter E-eigenvalue inclusion theorem for tensors is established. Based on the new set, a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is obtained. Finally, numerical examples are given to verify the theoretical results.

1. Introduction

For a positive integer \(n, n \geq 2\), \(\mathbb{N}\) denotes the set \([1, 2, \cdots, n]\). \(\mathbb{C}(\mathbb{R})\) denotes the set of all complex (real) numbers. We call \(\mathcal{A} = (a_{i_1i_2\cdots i_m})\) a real tensor of order \(m\) dimension \(n\), denoted by \(\mathcal{A} \in \mathbb{R}^{[m,n]}\), if
\[
a_{i_1i_2\cdots i_m} \in \mathbb{R},
\]
where \(i_j \in \mathbb{N}\) for \(j = 1, 2, \cdots, m\). \(\mathcal{A}\) is called nonnegative if \(a_{i_1i_2\cdots i_m} \geq 0\). \(\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}\) is called symmetric \([1]\) if
\[
a_{i_1\cdots i_m} = a_{\pi(i_1)\cdots \pi(i_m)}, \forall \pi \in \Pi_m,
\]
where \(\Pi_m\) is the permutation group of \(m\) indices. \(\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}\) is called weakly symmetric \([2]\) if the associated homogeneous polynomial
\[
\mathcal{A}x^m = \sum_{i_1\cdots i_m \in \mathbb{N}} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}
\]
satisfies \(\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}\), where \(x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n\), and \(\mathcal{A}x^{m-1}\) is an \(n\) dimension vector whose \(i\)th component is
\[
(\mathcal{A}x^{m-1})_i = \sum_{i_2\cdots i_m \in \mathbb{N}} a_{i_2\cdots i_m} x_{i_2} \cdots x_{i_m}.
\]
It is shown in [2] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

Given a tensor $\mathcal{A} = (a_{i_1\cdots i_n}) \in \mathbb{R}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ such that

$$\mathcal{A} x^{m-1} = \lambda x \text{ and } x^T x = 1,$$

then $\lambda$ is called an $E$-eigenvalue of $\mathcal{A}$ and $x$ an $E$-eigenvector of $\mathcal{A}$ associated with $\lambda$. Particularly, if $\lambda$ and $x$ are all real, then $\lambda$ is called a $Z$-eigenvalue of $\mathcal{A}$ and $x$ a $Z$-eigenvector of $\mathcal{A}$ associated with $\lambda$; for details, see [1, 3]. Denote by $\sigma(\mathcal{A})$ (respectively, $E(\mathcal{A})$) the set of all $Z$-eigenvalues (respectively, $E$-eigenvalues) of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq 0$, then the $Z$-spectral radius [2] of $\mathcal{A}$, denoted $\rho(\mathcal{A})$, is defined as

$$\rho(\mathcal{A}) := \max \{ |\lambda| : \lambda \in \sigma(\mathcal{A}) \}.$$

Note here that, Chang et al. in [2] demonstrated by an example that the $Z$-spectral radius $\rho(\mathcal{A})$ of a nonnegative tensor $\mathcal{A}$ may not be itself a positive $Z$-eigenvalue of $\mathcal{A}$, and proved that if $\mathcal{A}$ is a weakly symmetric nonnegative tensor, then $\rho(\mathcal{A})$ is a $Z$-eigenvalue of $\mathcal{A}$; see [2], for details.

The $Z$-eigenvalue problem plays a fundamental role in best rank-one approximation, which has numerous applications in engineering and higher order statistics [1, 4], and neural networks [5]. Recently, much literature has focused on locating all $Z$-eigenvalues of tensors and bounding the $Z$-spectral radius of nonnegative tensors in [6–20]. In 2017, Wang et al. [6] generalized Geršgorin eigenvalue inclusion theorem from matrices to tensors and established the following Geršgorin-type $Z$-eigenvalue inclusion theorem.

**Theorem 1.1.** [6, Theorem 3.1] Let $\mathcal{A} = (a_{i_1\cdots i_n}) \in \mathbb{R}^{[m,n]}$. Then

$$\sigma(\mathcal{A}) \subseteq K(\mathcal{A}) = \bigcup_{i \in N} K_i(\mathcal{A}),$$

where

$$K_i(\mathcal{A}) = \{ z \in \mathbb{C} : |z| \leq R_i(\mathcal{A}) \} \text{ and } R_i(\mathcal{A}) = \sum_{j_1,\cdots,j_n \in N} |a_{i_1j_1\cdots i_n}|.$$

Based on the set $K(\mathcal{A})$, the following upper bound for $\rho(\mathcal{A})$ presented in [7] is obtained easily.

**Theorem 1.2.** [7, Corollary 4.5] Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be nonnegative. Then

$$\rho(\mathcal{A}) \leq \max_{i \in N} R_i(\mathcal{A}).$$

To get a tighter $Z$-eigenvalue inclusion set than $K(\mathcal{A})$, Wang et al. [6] obtained the following Brauer-type $Z$-eigenvalue inclusion theorem for tensors.

**Theorem 1.3.** [6, Theorem 3.3] Let $\mathcal{A} = (a_{i_1\cdots i_n}) \in \mathbb{R}^{[m,n]}$. Then

$$\sigma(\mathcal{A}) \subseteq M(\mathcal{A}) = \bigcup_{i,j \in N \neq i} \left( M_{i,j}(\mathcal{A}) \cup H_{i,j}(\mathcal{A}) \right),$$

where

$$M_{i,j}(\mathcal{A}) = \{ z \in \mathbb{C} : (|z| - R_i(\mathcal{A}) - |a_{ij-}|)(|z| - P_{i}(\mathcal{A})) \leq |a_{ij-}|(R_i(\mathcal{A}) - P_{i}(\mathcal{A})) \},$$

$$H_{i,j}(\mathcal{A}) = \{ z \in \mathbb{C} : R_i(\mathcal{A}) - |a_{ij-}| < |z| < P_{i}(\mathcal{A}) \},$$

and

$$P_{i}(\mathcal{A}) = \sum_{j_1,\cdots,j_n \in N \neq i} |a_{ji_1\cdots i_n}|.$$

Based on the set $M(\mathcal{A})$, Wang et al. [6] obtained a better upper bound than that in Theorem 1.2.
Theorem 1.4. [6, Theorem 4.6] Let $A = (a_{i_1, \ldots, i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then
\[
\varphi(A) \leq \Psi(A) = \max_{i,j \in [N], i \neq j} \left\{ \frac{1}{2} \left( R_i(A) - a_{i,j} + P_{ij}(A) + \Lambda_{ij}^2(A) \right) , R_i(A) - a_{i,j} - P_{ij}(A) \right\},
\]
where
\[
\Lambda_{ij}(A) = (R_i(A) - a_{i,j} - P_{ij}(A))^2 + 4a_{i,j}(R_i(A) - P_{ij}(A)).
\]

Due to various new and important applications of $E$-eigenvalue problem in numerical multilinear algebra [21], image processing [22], higher order Markov chains [23], spectral hypergraph theory, the study of quantum entanglement, and so on, some properties of $E$-eigenvalues have been studied systematically; see [8] for details. However, characterizations of inclusion set for $E$-eigenvalue are still underdeveloped. This stimulates us to establish some inclusion theorems to identify the distribution of $E$-eigenvalues.

In the sequel, we research on the $E$-eigenvalue localization problems for tensors and their applications. First, Theorems 1.1 and 1.3 are extended to $E$-eigenvalue inclusion theorems. Second, a new $E$-eigenvalue inclusion set for tensors is presented and proved to be tighter than those in Theorems 1.1 and 1.3. Finally, as an application of the new set, a new upper bound for the $E$-spectral radius of weakly symmetric nonnegative tensors is given and proved to be sharper than those in Theorems 1.2 and 1.4.

2. $E$-eigenvalue inclusion sets for tensors

In this section, we first generalized those sets in Theorems 1.1 and 1.3 to $E$-eigenvalue inclusion sets. And then we present a new $E$-eigenvalue inclusion set for tensors and establish the comparison among these three sets. Firstly, similar to the proof of Theorems 3.1 and 3.3 of [6], the following theorem is obtained easily.

Theorem 2.1. Let $A = (a_{i_1, \ldots, i_m}) \in \mathbb{R}^{[m,n]}$. Then
\[
E(A) \subseteq K(A), \text{ and } E(A) \subseteq M(A).
\]

Next, a new $E$-eigenvalue inclusion theorem for tensors is presented.

Theorem 2.2. Let $A = (a_{i_1, \ldots, i_m}) \in \mathbb{R}^{[m,n]}$. Then
\[
E(A) \subseteq \Omega(A) = \bigcup_{i,j \in [N], i \neq j} \left( \tilde{\Omega}_{ij}(A) \cup \left( \tilde{\Omega}_{ij}(A) \cap K(A) \right) \right),
\]
where
\[
\tilde{\Omega}_{ij}(A) = \left\{ z \in \mathbb{C} : |z| < P_{ij}(A) \text{ and } |z| < P_{ij}(A) \right\}
\]
and
\[
\tilde{\Omega}_{ij}(A) = \left\{ z \in \mathbb{C} : \left( |z| - P_{ij}(A) \right) \left( |z| - P_{ij}(A) \right) \leq \left( R_i(A) - P_{ij}(A) \right) \left( R_i(A) - P_{ij}(A) \right) \right\}.
\]

Proof. Let $\lambda$ be an $E$-eigenvalue of $A$ with corresponding $E$-eigenvector $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$, i.e.,
\[
Ax^{m-1} = \lambda x, \text{ and } ||x||_2 = 1.
\]
Let $|x_i| \geq |x_s| \geq \max_{i \in [N], i \neq s} |x_s|$. Obviously, $0 < |x_i|^{m-1} \leq |x_s|^{m-1} \leq |x_i| \leq 1$. From (1), we have
\[
\lambda x_i = \sum_{j \neq i, j \in [N]} a_{ij, i_m} x_j \cdots x_{i_m} + \sum_{j \neq i, j \in [N]} a_{ij, i_m} x_j \cdots x_{i_m}.
\]
Taking modulus in the above equation and using the triangle inequality give

\[
|\lambda| |x_i| \leq \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_1}| \cdots |x_{i_m}|
\]

\[
\leq \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_2}|^{m-2} + \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_1}|^{m-1}
\]

\[
\leq \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_2}| + \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_1}|
\]

\[
= (R_e(A) - P^e(A))|x_i| + P^e(A)|x_i|
\]

i.e.,

\[
(|\lambda| - P^e(A))|x_i| \leq (R_e(A) - P^e(A))|x_i|.
\]

By (2), it is not difficult to see $|\lambda| \leq R_e(A)$, that is, $\lambda \in \mathcal{K}_e(A)$. If $|x_i| = 0$, then $|\lambda| - P^e(A) \leq 0$ as $|x_i| > 0$. When $|\lambda| - P^e(A) = 0$, obviously, $\lambda \in (\Omega_e(A) \cap \mathcal{K}_e(A)) \subseteq \Omega(A)$. And when $|\lambda| - P^e(A) < 0$, if $|\lambda| \geq P^e(A)$, then we have

\[
(|\lambda| - P^e(A))(|\lambda| - P^e(A)) \leq 0 \leq (R_e(A) - P^e(A))(R_e(A) - P^e(A)),
\]

which implies $\lambda \in (\Omega_e(A) \cap \mathcal{K}_e(A)) \subseteq \Omega(A)$; if $|\lambda| < P^e(A)$, then we have $\lambda \in \Omega_e(A) \subseteq \Omega(A)$.

Otherwise, $|x_i| > 0$. By (1), we can get

\[
|\lambda| |x_i| \leq \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_1}| \cdots |x_{i_m}|
\]

\[
\leq \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_2}|^{m-1} + \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_1}|^{m-1},
\]

\[
\leq \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_2}| + \sum_{i_1, \ldots, i_m \in \{1, \ldots, n\}} |a_{i_1-2, \ldots, i_m}| |x_{i_1}|
\]

i.e.,

\[
(|\lambda| - P^e(A))|x_i| \leq (R_e(A) - P^e(A))|x_i|.
\]

When $|\lambda| \geq P^e(A)$ or $|\lambda| \geq P^e(A)$ holds, multiplying (2) with (3) and noting that $|x_i|x_i| > 0$, we have

\[
(|\lambda| - P^e(A))(|\lambda| - P^e(A)) \leq (R_e(A) - P^e(A))(R_e(A) - P^e(A)),
\]

which implies $\lambda \in (\Omega_e(A) \cap \mathcal{K}_e(A)) \subseteq \Omega(A)$. And when $|\lambda| < P^e(A)$ and $|\lambda| < P^e(A)$ hold, we have $\lambda \in \Omega_e(A) \subseteq \Omega(A)$. Hence, the conclusion $\sigma(A) \subseteq \Omega(A)$ follows immediately from what we have proved.

Next, a comparison theorem is given for Theorems 2.1 and 2.2.

**Theorem 2.3.** Let $A = (a_{i_1-2, \ldots, i_m}) \in \mathbb{R}^{m,n}$. Then

\[
\Omega(A) \subseteq M(A) \subseteq \mathcal{K}(A).
\]
Proof. By Corollary 3.2 in [6], $M(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$ holds. Hence, we only prove $\Omega(\mathcal{A}) \subseteq M(\mathcal{A})$. Let $z \in \Omega(\mathcal{A})$. Then there are $t, s \in N$ and $t \neq s$ such that $z \in \hat{\Omega}_{t,s}(\mathcal{A})$ or $z \in (\hat{\Omega}_{t,s}(\mathcal{A}) \cap K_{t}(\mathcal{A}))$. We divide the proof into two parts.

Case I: If $z \in \hat{\Omega}_{t,s}(\mathcal{A})$, that is, $|z| < P_{t}^{s}(\mathcal{A})$ and $|z| < P_{t}^{s}(\mathcal{A})$. Then, it is easily to see that

$$|z| < P_{t}^{s}(\mathcal{A}) \leq R_{t}(\mathcal{A}) - |a_{ts-s}|$$

which implies that $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq M(\mathcal{A})$, consequently, $\Omega(\mathcal{A}) \subseteq M(\mathcal{A})$.

Case II: If $z \notin \hat{\Omega}_{t,s}(\mathcal{A})$, that is,

$$|z| \geq P_{t}^{s}(\mathcal{A})$$

or

$$|z| \geq P_{t}^{s}(\mathcal{A})$$

then $z \in (\hat{\Omega}_{t,s}(\mathcal{A}) \cap K_{t}(\mathcal{A}))$, i.e.,

$$|z| \leq R_{t}(\mathcal{A})$$

and

$$(|z| - P_{t}^{s}(\mathcal{A}))(|z| - P_{t}^{s}(\mathcal{A})) \leq (R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A}))(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})).$$

(i) Assume $(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A}))(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})) = 0$. When (4) holds, by (7), we have

$$\left(|z| - (R_{t}(\mathcal{A}) - |a_{ts-s}|)\right)(|z| - P_{t}^{s}(\mathcal{A})) \leq \left(|z| - P_{t}^{s}(\mathcal{A})\right)(|z| - P_{t}^{s}(\mathcal{A})) \leq (R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A}))(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})) = 0 \leq |a_{ts-s}||R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})|,$$

which implies that $z \in M_{t,s}(\mathcal{A}) \subseteq M(\mathcal{A})$. On the other hand, when (5) holds, we only prove $z \in M(\mathcal{A})$ under the case that $|z| < P_{t}^{s}(\mathcal{A})$. When

$$P_{t}^{s}(\mathcal{A}) \leq |z| < R_{t}(\mathcal{A}) - |a_{ts-s}|,$$

we have $z \in \mathcal{H}_{t,s}(\mathcal{A}) \subseteq M(\mathcal{A})$. And when

$$R_{t}(\mathcal{A}) - |a_{ts-s}| \leq |z| \leq R_{t}(\mathcal{A}),$$

from

$$\left(|z| - (R_{t}(\mathcal{A}) - |a_{ts-s}|)\right)(|z| - P_{t}^{s}(\mathcal{A})) \leq 0 \leq |a_{ts-s}||R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})|,$$

we have $z \in M_{t,s}(\mathcal{A}) \subseteq M(\mathcal{A})$.

(ii) Assume $(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A}))(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})) > 0$. Then dividing both sides by $(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A}))(R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A}))$ in (7), we have

$$\frac{|z| - P_{t}^{s}(\mathcal{A})}{R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})} \leq \frac{|z| - P_{t}^{s}(\mathcal{A})}{R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})} \leq 1.$$

If $|a_{ts-s}| > 0$, let $a = |z|, b = P_{t}^{s}(\mathcal{A}), c = R_{t}(\mathcal{A}) - |a_{ts-s}| - P_{t}^{s}(\mathcal{A})$ and $d = |a_{ts-s}|$, by (6) and Lemma 2.2 in [24], we have

$$\frac{|z| - (R_{t}(\mathcal{A}) - |a_{ts-s}|)}{|a_{ts-s}|} = \frac{a - (b + c)}{d} \leq \frac{a - b}{c + d} = \frac{|z| - P_{t}^{s}(\mathcal{A})}{R_{t}(\mathcal{A}) - P_{t}^{s}(\mathcal{A})}.$$
When (4) holds, by (11) and (12), we have
\[
\frac{|z| - (R_s(A) - |a_{ts-s}|)}{|a_{ts-s}|} \leq \frac{|z| - P_s'(A)}{R_s(A) - P_s'(A)} \leq \frac{|z| - P_s'(A)}{R_s(A) - P_s'(A)} \leq 1,
\]
equivalently,
\[
(z - (R_s(A) - |a_{ts-s}|))(z - P_s'(A)) \leq |a_{ts-s}|(R_s(A) - P_s'(A)),
\]
which implies that \(z \in M_{ts,s}(A) \subseteq M(A)\). On the other hand, when (5) holds, we only prove \(z \in M(A)\) under the case that \(|z| < P_s'(A)\). If (8) holds, then \(z \in H_{ts,s}(A) \subseteq M(A)\). And if (9) holds, by (10), we have \(z \in M_{ts,s}(A) \subseteq M(A)\).

If \(|a_{ts-s}| = 0\), by \(|z| \leq R_s(A)\), we have \(|z| - (R_s(A) - |a_{ts-s}|) \leq 0 = |a_{ts-s}|\). When (4) holds, we can obtain
\[
(z - (R_s(A) - |a_{ts-s}|))(z - P_s'(A)) \leq 0 = |a_{ts-s}|(R_s(A) - P_s'(A)),
\]
which implies that \(z \in M_{ts,s}(A) \subseteq M(A)\). On the other hand, when (5) holds, we only prove \(z \in M(A)\) under the case that \(|z| < P_s'(A)\). If (8) holds, then \(z \in H_{ts,s}(A) \subseteq M(A)\). And if (9) holds, by (13), we have \(z \in M_{ts,s}(A) \subseteq M(A)\). The conclusion follows from Case I and Case II. \(\square\)

**Remark 2.4.** Theorem 2.3 shows that the set \(\Omega(A)\) in Theorem 2.2 is tighter than \(K(A)\) and \(M(A)\) in Theorem 2.1, that is, \(\Omega(A)\) can capture all E-eigenvalues of \(A\) more precisely than \(K(A)\) and \(M(A)\).

In the following, an example is given to verify Remark 2.4.

**Example 2.5.** Let \(A = (a_{i,j}) \in \mathbb{R}^{[3,3]}\) with entries defined as follows:
\[
\begin{align*}
A(:,1) = \begin{pmatrix} 0 & 3 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}, & A(:,2) = \begin{pmatrix} 2 & 0.5 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & A(:,3) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

We now locate all E-eigenvalues of \(A\). By Theorem 2.1, we have
\[
K(A) = \{z \in \mathbb{C} : |z| \leq 14.5000\} \quad \text{and} \quad M(A) = \{z \in \mathbb{C} : |z| \leq 14.2228\}.
\]

By Theorem 2.2, we have
\[
\Omega(A) = \{z \in \mathbb{C} : |z| \leq 11.5000\}.
\]

The E-eigenvalue inclusion sets \(K(A)\), \(M(A)\), \(\Omega(A)\) and all E-eigenvalues \(-6.3796, -3.2536, -1.8154, -0.8351, -0.7011 - 0.8430i, -0.7011 + 0.8430i, -0.4608, 0.4608, 0.7011 - 0.8430i, 0.7011 + 0.8430i, 0.8351, 1.8154, 3.2536, 6.3796\) are drawn in Figure 1, where \(K(A)\), \(M(A)\), \(\Omega(A)\) and the exact E-eigenvalues are represented by black solid boundary, blue dashed boundary, red solid boundary and black “+”, respectively. It is easy to see that \(\sigma(A) \subseteq \Omega(A) \subseteq M(A) \subset K(A)\), that is, \(\Omega(A)\) can capture all E-eigenvalues of \(A\) more precisely than \(M(A)\) and \(K(A)\).

3. A sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors

As an application of the set \(\Omega(A)\) in Theorem 2.2, a new upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is given.

**Theorem 3.1.** Let \(A = (a_{i,j}) \in \mathbb{R}^{[m,n]}\) be a weakly symmetric nonnegative tensor. Then
\[
\phi(A) \leq \Omega_{\max} = \max \left\{ \Omega_{\max}, \Omega_{\max} \right\},
\]
Figure 1: Comparisons of $K(A), M(A)$ and $\Omega(A)$.

where

$$\Omega_{\max} = \max_{i,j \in N, i \neq j} \min\{P_j^i(A), P_i^j(A)\},$$

$$\tilde{\Omega}_{\max} = \max_{i,j \in N, i \neq j} \min\{R_i^j(A), \Delta_{ij}(A)\},$$

and

$$\Delta_{ij}(A) = \frac{1}{2} \left( P_j^i(A) + P_i^j(A) + \sqrt{(P_j^i(A) - P_i^j(A))^2 + 4(R_i^j(A) - P_j^i(A))(R_j^i(A) - P_i^j(A))} \right).$$

Proof. As stated in Section 1, if $A$ is weakly symmetric and nonnegative, then $\rho(A)$ is the largest $Z$-eigenvalue of $A$. Hence, by Theorem 2.2, we have

$$\rho(A) \in \bigcup_{i,j \in N, i \neq j} \left( \tilde{\Omega}_{ij}(A) \cup \tilde{\Omega}_{ij}(A) \cap K_i(A) \right),$$

that is, there are $t, s \in N, t \neq s$ such that $\rho(A) \in \tilde{\Omega}_{ts}(A)$ or $\rho(A) \in \left( \tilde{\Omega}_{ts}(A) \cap K_t(A) \right)$. If $\rho(A) \in \tilde{\Omega}_{ts}(A)$, i.e., $\rho(A) < P_s^t(A)$ and $\rho(A) < P_t^s(A)$, we have $\rho(A) < \min\{P_t^s(A), P_s^t(A)\}$. Furthermore, we have

$$\rho(A) \leq \max_{i,j \in N, i \neq j} \min\{P_j^i(A), P_i^j(A)\}. \quad (14)$$

If $\rho(A) \in \left( \tilde{\Omega}_{ts}(A) \cap K_t(A) \right)$, i.e., $\rho(A) \leq R_t(A)$ and

$$\left( \rho(A) - P_t^s(A) \right) \left( \rho(A) - P_s^t(A) \right) \leq \left( R_t(A) - P_t^s(A) \right) \left( R_s(A) - P_s^t(A) \right), \quad (15)$$
then solving \( \varrho(\mathcal{A}) \) in (15) gives

\[
\varrho(\mathcal{A}) \leq \frac{1}{2} \left( P_s(\mathcal{A}) + P_t(\mathcal{A}) + \sqrt{(P_s(\mathcal{A}) - P_t(\mathcal{A}))^2 + 4[R_s(\mathcal{A}) - P_s(\mathcal{A})(R_s(\mathcal{A}) - P_t(\mathcal{A})]} \right) = \Delta_{ts}(\mathcal{A}),
\]

and furthermore

\[
\varrho(\mathcal{A}) \leq \min \{ R_s(\mathcal{A}), \Delta_{ts}(\mathcal{A}) \} \leq \max \min \{ R_i(\mathcal{A}), \Delta_{ij}(\mathcal{A}) \}. \tag{16}
\]

The conclusion follows from (14) and (16). \( \square \)

By Theorem 2.3 and Corollary 4.2 in [6], the following comparison theorem can be derived easily.

**Theorem 3.2.** Let \( \mathcal{A} = (a_{i_1\ldots i_n}) \in \mathbb{R}^{[m,n]} \) be a weakly symmetric nonnegative tensor. Then the upper bound in Theorem 3.1 is sharper than those in Theorems 1.2 and 1.4, that is,

\[
\varrho(\mathcal{A}) \leq \Omega_{\max} \leq \Psi \leq \max_{i \in N} R_i(\mathcal{A}).
\]

Finally, we show that in some cases the upper bound in Theorem 3.1 is sharper than those in [6, 7, 9–15] by an example.

**Example 3.3.** Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]} \) be a symmetric tensor defined by

\[
a_{1111} = \frac{1}{2}, \quad a_{2222} = 3, \quad a_{ijkl} = \frac{1}{3} \text{ elsewhere}.
\]

By computation, we obtain \((\rho(\mathcal{A}), x) = (3.1092, (0.1632, 0.9866))\). By Corollary 4.5 of [7], we have

\[
\varrho(\mathcal{A}) \leq 5.3333.
\]

By Theorem 2.7 of [15], we have

\[
\varrho(\mathcal{A}) \leq 5.2846.
\]

By Theorem 3.3 of [11], we have

\[
\varrho(\mathcal{A}) \leq 5.1935.
\]

By Theorem 4.5, Theorem 4.6 and Theorem 4.7 of [6], we all have

\[
\varrho(\mathcal{A}) \leq 5.1822.
\]

By Theorem 3.5 of [12] and Theorem 6 of [13], we both have

\[
\varrho(\mathcal{A}) \leq 5.1667.
\]

By Theorem 7 of [9], we have

\[
\varrho(\mathcal{A}) \leq 5.0437.
\]

By Theorem 2.9 of [14], we have

\[
\varrho(\mathcal{A}) \leq 4.5147.
\]

By Theorem 5 of [10], we have

\[
\varrho(\mathcal{A}) \leq 4.4768.
\]

By Theorem 3.1, we obtain

\[
\varrho(\mathcal{A}) \leq 4.3971,
\]

which shows that this upper bound is better.
4. Conclusion

In this paper, we first generalize two Z-eigenvalue inclusion sets $\mathcal{K}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ presented by Wang et al. in [6] to E-eigenvalue localization sets. And then we establish a new E-eigenvalue localization set $\Omega(\mathcal{A})$ and prove that it is tighter than $\mathcal{K}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$. Based on the set $\Omega(\mathcal{A})$, we obtain a new upper bound $\Omega_{\text{max}}$ for the Z-spectral radius of weakly symmetric nonnegative tensors and show that it is better than those in [6, 7, 9–15] in some cases by a numerical example.

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