THE ISOMETRY GROUP OF OUTER SPACE

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ABSTRACT. We prove analogues of Royden’s Theorem for the Lipschitz metrics of Outer Space, namely that $\text{Isom}(CV_n) = \text{Out}(F_n)$.

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1. INTRODUCTION

For $n \geq 2$ let $F_n$ be the free group of rank $n$, and $\text{Out}(F_n)$ be the group of outer automorphisms of $F_n$. The Culler-Vogtmann Outer Space, $CV_n$, is the analogue of Teichmüller space for $\text{Out}(F_n)$ and is a space of metric graphs with fundamental group of rank $n$.

As for Teichmüller space, one can define the Lipschitz metric of $CV_n$ with a resulting metric which is not symmetric. This non-symmetric metric is geodesic and seems natural in terms of capturing the dynamics of free group automorphisms; for instance the axes of iwip automorphisms ([1]). However the non-symmetric version also lacks some properties one might want; it fails to be complete, for instance, while the symmetrised version turns $CV_n$ into a proper metric space (see [11, 1, 2], and also [12] for a different approach.)

The group $\text{Out}(F_n)$ naturally acts on $CV_n$ and the action is by isometries. It is also easy to see that this action is faithful for $n \geq 3$ but
not faithful for \( n = 2 \). The reason for this is that \( \text{Out}(F_2) \cong GL(2, \mathbb{Z}) \)
has a central element of order 2, namely \(-I_2\), which is in the kernel of the action. If one picks a basis, \( x_1, x_2 \) for \( F_2 \) the automorphism which sends each \( x_i \) to \( x_i^{-1} \) is a pre-image in \( \text{Aut}(F_2) \) of \(-I_2\).

In this paper, we prove an analogue of Royden’s Theorem for both metrics, and any rank, so that \( \text{Isom}(CV_n) = \text{Out}(F_n) \) (see below for exact statements).

There are many of this kind of results in literature, for instance

- The Fundamental Theorem of projective geometry (If a field \( F \) has no non-trivial automorphisms, the group of incidence-preserving bijections of the projective space of dimension \( n \) over \( F \) is precisely \( PGL(n, F) \)).
- Tits Theorem: Under suitable hypotheses, the full group of simplicial automorphisms of the spherical building associated to an algebraic group is equal to the algebraic group ([22]).
- Ivanov’s Theorem: The group of simplicial automorphisms of the curve-complex of a surface \( S \) of genus at least two is the mapping class group of \( S \) ([15]).
- Royden’s Theorem: The isometry group of the Teichmuller space of \( S \) is the mapping class group of \( S \) ([20]).
- Bridson and Vogtmann’s Theorem: For \( n \geq 3 \) the group of simplicial automorphisms of the spine of \( CV_n \) is \( \text{Out}(F_n) \) ([6]).
- Aramayona and Souto’s Theorem: For \( n \geq 3 \), the group of simplicial automorphisms of the free splitting graph is \( \text{Out}(F_n) \); ([3]).

Our main results are:

**Theorem 1.1.** With respect to the symmetric Lipschitz distance,

\[
\text{Isom}(CV_n) = \text{Out}(F_n) \text{ for } n \geq 3.
\]

For \( n = 2 \),

\[
\text{Isom}(CV_2) = PGL(2, \mathbb{Z}).
\]

We note that replacing the symmetric distance by its non-symmetrised version one gets the same result.

**Theorem 1.2.** For both non-symmetric Lipschitz distances \( d_R \) and \( d_L \),

\( \text{Isom}(CV_n) \) is \( \text{Out}(F_n) \) for \( n \geq 3 \) and \( PGL(2, \mathbb{Z}) \) for \( n = 2 \).

This kind of result has immediate corollaries of fixed-point type (see for example [5, 6]).

**Corollary 1.3.** Let \( G \) be a semisimple Lie group with finite centre and no compact factors and suppose the real rank of \( G \) is at least two. Let \( \Gamma \) be a non-uniform, irreducible lattice in \( G \). Then every isometric action of \( \Gamma \) on \( CV_n \) has a global fixed point.
As we note above, there already exists a result of this kind for the spine of $CV_n$; [6], which states that the simplicial automorphism group of the spine of $CV_n$ is equal to $Out(F_n)$ for $n \geq 3$. At a first glance, Theorem 1.1 could appear to be a direct consequence of [6] after some easy remarks (using, for instance, Lemma 4.1) and in fact that was exactly the thought of the authors when this work started.

However, the main difficulty in the paper is precisely moving from a statement that an isometry preserves the simplicial structure of $CV_n$ to the statement that it is the identity. For instance, once one knows that an isometry leaves some simplex invariant, it is not clear, a priori, that the centre of the simplex is fixed (in fact it is not true in general if one simply looks at isometries of a simplex rather than the restriction of a global isometry). And even when one has that a given isometry leaves every simplex invariant, it is not clear how to deduce that the isometry is in fact the identity - obviously, this is in sharp contrast to the piecewise Euclidean metric.

Let us emphasise this contrast. Suppose that one wants to prove Theorem 1.1 for the piecewise Euclidean metric. First, note that simplices corresponding to graphs with disconnecting edges are an obvious obstruction. However, one always may to restrict to a “reduced” Outer Space by removing such simplices. Now, looking at the incidence structure of ideal vertices, one can prove that any isometry (w.r.t. the piecewise Euclidean metric) maps ideal vertices to ideal vertices and thus simplices to simplices because isometries are local PL-maps. (Lemma 4.1 is no longer true, as stated, for this reduced Outer Space as one can easily see in the rank-2 case.) Then, invoking the Bridson-Vogtmann result, one gets that up to composing with automorphisms, simplices are not permuted, and the PL-structure now completes the job.

Now let us return to the Lipschitz metric. There are four key facts in Theorem 1.1. First, the study of local isometries. The main point is that in general, the isometry group of a fixed simplex of $CV_n$ is in fact much bigger than its stabiliser in $Out(F_n)$.

The second fact is that $CV_n$ is highly non-homogeneous. This allows one to find particular points in simplices of $CV_n$ that are invariant under isometries, so that one can characterise those isometries that are restrictions of global ones.

Third, there is the fact that asymptotic behavior of distances from a particular set of points determines the distance between points of $CV_n$. This is a non-trivial issue, that we like to paraphrase saying that Busemann functions of ideal vertices are coordinates for $CV_n$. The main consequence of this fact is that one can deduce that an isometry that does not permute simplices is in fact the identity.

Lastly, there is a permutation issue, similar to the one faced in [6], that we solve metrically using our “Busemann functions”.

We also remark that Theorem 1.1 holds for any rank and includes the study of simplices with disconnecting edges. The complete schema of the proof of Theorem 1.1 is described in Section 3.

Acknowledgments. The first named author wishes to thank the UFF of Niterói (RJ, Brazil), the CRM of Barcelona (Spain) and the SOTON (Southampton, UK) for their kind hospitality and the great work environment they provide.

This work was inspired by the beautiful articles [6, 5], and most of the material of this introduction was picked from there.

2. Preliminaries

In this section we fix terminology, give basic definitions, and recall some known facts (and prove some easy ones) that we shall need for the rest of the paper. Experienced readers may skip directly to next section and refer to present one just for notation.

2.1. Outer Space. First of all, we recall what Culler-Vogtmann space or “Outer Space” is. We refer to the pioneer work [9] and beautiful surveys [23, 24] for more details.

For any $n \geq 2$ let $F_n$ be the free group of rank $n$ which we identify with the fundamental group of $R_n = S^1 \wedge \cdots \wedge S^1$ (the product taken $n$ times).

Consider finite graphs $X$ whose vertices have valence at least three, this means that each vertex has at least three germs of incident edges. We require that $X$ has rank $n$, that is to say, $\pi_1(X) \simeq F_n$ and that $X$ comes equipped with a metric. Giving a metric on $X$ is equivalent to giving positive lengths for the edges of $X$.

We also require $X$ to be a marked graph, which is to say that it comes with a fixed marking. A marking on $X$ is a continuous map $\tau : R_n \to X$ which induces an isomorphism $\tau_* : F_n \simeq \pi_1(R_n) \to \pi_1(X)$. Two marked metric graphs $(A, \tau_A)$ and $(B, \tau_B)$ are considered equivalent if there exists a homothety, $h : A \to B$, such that the following diagram commutes up to free homotopy,

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{\tau_A} & & \downarrow{\tau_B} \\
R_n & \xrightarrow{\tau_A} & X
\end{array}
\]

Culler Vogtmann Space of $F_n$ or Outer Space of rank $n$ is the set $CV_n$ of equivalence classes of marked metric graphs of rank $n$.

It is common to consider standard representative of a given class by taking volume one graphs (here volume means total edge length.)
However, we usually do not normalise metric graphs, and when we will do it we will use different normalisations depending on the calculations we are making.

We note that since the equivalence allows homothety, given a point \([X]\) in CV\(_n\), we only have the metric on \(X\) up to scaling constants. If one instead only considers the equivalence up to isometry, then one obtains unprojectivised CV\(_n\) and the metric on the graph corresponding to a point there is determined by the point.

**Remark 2.1.** In the following, if there is no ambiguity, we will not distinguish between a metric graph \(X\) and its class \([X]\). If we need to choose a particular representative of \([X]\) we will explicitly declare that.

### 2.2. The Topology of CV\(_n\).

Outer space is endowed with topology induced by edge-lengths of graphs.

Given any marked graph \(A\), we can look at the universal cover \(T_A\) which is an \(\mathbb{R}\)-tree on which \(\pi_1(R_n)\) acts by isometries, via the marking \(\tau_A\). Conversely, given any minimal free action of \(F_n\) by isometries on a simplicial \(\mathbb{R}\)-tree, we can look at the quotient object, which will be a graph, \(A\), and produce a homotopy equivalence \(\tau_A : R_n \to A\) via the action. Equivalence of graphs in CV\(_n\) corresponds to actions which are equivalent up to equivariant homothety.

Thus, points in CV\(_n\) can be thought of as equivalence classes of minimal free isometric actions on simplicial \(\mathbb{R}\)-trees. Given an element \(w\) of \(F_n\) and a point \(A\) of the unprojectivised CV\(_n\), with universal cover \(T_A\) whose metric we denote by \(d_A\), we may consider,

\[
L_A(w) := \inf_{p \in T_A} d_A(p, wp).
\]

It is well known that this infimum is always obtained and that, for a free action, it is non-zero for the non-identity elements of the group. In this context, \(L_A(w)\) is called the translation length of the element \(w\) in the corresponding tree and clearly depends only on the conjugacy class of \(w\) in \(F_n\). If we look at graph \(A\), then \(L_A(w)\) is the length of the geodesic representative of \(w\) in \(A\), that is to say, the length of shortest closed loop representing free homotopy class of \(\tau_A \ast (w)\) as an element of \(\pi_1(A)\). Thus for any point, \(A\), in CV\(_n\) we can associate the sequence \((L_A(w))_{w \in F_n}\) and it is clear that equivalent marked metric graphs will produce two sequences, one of which is a multiple of the other by a positive real number (the homothety constant). Moreover, it is also the case that inequivalent points in CV\(_n\) will produce sequences which are not multiples of each other \([8]\). Thus, we have an embedding of CV\(_n\) into \(\mathbb{R}^{F_n}/\sim\), where \(\sim\) is the equivalence relation of homothety. The space CV\(_n\) is given the subspace topology induced by this embedding.

Finally it is clear we can realise any automorphism, \(\phi\), of \(F_n\) as a homotopy equivalence, also called \(\phi\), of \(R_n\). Thus the automorphism
group of $F_n$ acts on $CV_n$ by changing the marking. That is, given a point $(A, \tau_A)$ of $CV_n$ the image of this point under $\phi$ is $(A, \tau_A\phi)$.

Since two automorphisms which differ by an inner automorphism always send equivalent points in $CV_n$ to equivalent points, we actually have an action of $Out(F_n)$ on $CV_n$, and this space is called Outer Space for this reason.

2.3. Simplicial Subdivision of $CV_n$. Given a rank-$n$, marked, metric graph $X$ whose edges are labelled $e_1, \ldots, e_k$, we can consider all marked metric graphs homeomorphic to $X$ and with same marking. Such subset of $CV_n$ can be embedded in $\mathbb{R}^k$ by

$$X \mapsto (L_X(e_1), \ldots, L_X(e_k)).$$

If we consider standard normalisation with volume one, we obtain standard open $(k-1)$-simplex of $\mathbb{R}^k$, i.e. the set $\{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_i > 0, \sum x_i = 1\}$.

This gives us a natural subdivision of $CV_n$ into open simplices.

**Definition 2.2.** Let $\Delta$ be an open simplex of $CV_n$. The (marked) graph underlying of $\Delta$ is the (marked) topological type of graphs corresponding to points of $\Delta$.

Simplices of $CV_n$ will have some ideal faces and some true faces. More precisely, in an abstract way, if $\Delta$ is a simplex with underlying graph $X$, a face $\delta$ of $\Delta$ is obtained by setting to zero the lengths of some of the edges of $X$. This topologically corresponds to collapsing such edges. If the resulting graph has still rank $n$, then $\delta$ exists as a simplex of $CV_n$, and in this sense it is a true face. On the other hand, if the rank decreases, then $\delta$ is not in $CV_n$ (and in fact belongs to the boundary at infinity of $CV_n$) and in this case we say that $\delta$ is an ideal face of $\Delta$.

In what follows we always deal with true faces.

**Definition 2.3.** Let $\Delta$ be a simplex of $CV_n$ with underlying marked graph $X$. A face of $\Delta$ is a simplex of $CV_n$ whose underlying marked graph is obtained from $X$ by collapsing some edges. The codimension of the face of $\Delta$ is the number of collapsed edges.

It is readily checked by an Euler characteristic count that simplices of maximal dimension of $CV_n$ correspond to trivalent graphs, and that such graphs have $3n - 3$ edges and $2n - 2$ vertices. Therefore their dimension is $3n - 4$ as $CV_n$ is the projectivised outer space. Looking at the topology of graphs we see that in general,
Lemma 2.4. $k$-dimensional simplices of (projectivised) $CV_n$ correspond to graphs with $k + 1$ edges and $k - n + 2$ vertices.

Next we consider the $i$-skeleton of $CV_n$.

Definition 2.5. For $i \leq 3n - n$, the $i$-skeleton $CV^i_n$ of $CV_n$ is the set of simplices of $CV_n$ of dimension at most $i$.

An easy but important fact is that $i$-simplices correspond to smooth points of the $i$-skeleton.

Definition 2.6. A point $x \in CV^i_n$ is smooth if it has a neighbourhood in $CV^i_n$ homeomorphic to $\mathbb{R}^i$.

Lemma 2.7. Open $i$-simplices of $CV_n$ are exactly the connected components of the set of smooth points of $CV^i_n$. That is to say

$$\{x \in CV^i_n : x \text{ is smooth} \} = \bigcup_{\Delta \text{ open } i\text{-simplex}} \Delta$$

Proof. It is enough to show that any $i - 1$ simplex is the face of at least three different $i$-simplices. Let $X$ be a point of an $(i - 1)$-simplex. Then $X$ is obtained by collapsing to zero an edge $e$ of a point $\bar{X}$ of an $i$-simplex. Let $v_-$ and $v_+$ be the endpoints of $e$. Clearly $v_- \neq v_+$ because otherwise the collapse would decrease the rank. By definition, both $v_-$ and $v_+$ have valence at least three, and they are identified in $X$ to the same vertex $v$ which therefore has valence at least four.

For any subdivision of the set of germs of edges at $v$ in two subsets of at least two germs, we can form a different $i$-simplex, having $X$ in one of its faces, by separating such subsets and inserting a new edge between them. Clearly, different subdivisions give different $i$-simplices, and we have at least three such subdivisions because the valence of $v$ is at least four. \hfill $\square$

2.4. Roses and Multi-thetas. Our result will be based on a detailed study of isometries of two particular classes of marked graphs. Namely roses and multi-theta graphs.

Definition 2.8. A rose simplex is a simplex $\Delta$ of $CV_n$ whose underlying graph is a rose, i.e. a bouquet of $n$ copies of $S^1$. Edges of such a graph are also called petals. The centre of $\Delta$ is the symmetric graph, that is to say one whose petals all have the same length.

One should note that in the definition above, the centre is only defined by specifying that the edges have the same length without saying what that length is. We shall usually take a representative whose petals all have length 1 but the reader should be aware that as long as all the petals have the same length, the point in $CV_n$ will be the same.

By Lemma 2.4, rose simplices are those simplices of lowest dimension of $CV_n$. 
**Definition 2.9.** A multi-theta simplex is a simplex $\Delta$ of $CV_n$ whose underlying graph has only two vertices and $n + 1$ edges joining them (such graph is called a multi-theta.) The centre of $\Delta$ is the symmetric graph, that is to say, the one whose edges have all same length.

![Figure 1. A multi-theta graph in $CV_4$](image)

**Definition 2.10.** A rose-face of a simplex $\Delta$ of $CV_n$ is a rose simplex which is a face of $\Delta$.

Formally speaking, simplices are open, so the rose-face of a simplex is not subset of it. Nonetheless, it is readily checked that any isometry of a simplex extends to its faces and rose-faces, though it may permute them. As we are interested in studying isometries, by abuse of notation, we will consider the rose-faces of a simplex as subsets of it.

**Remark 2.11.** Let $\Delta$ be a simplex of $CV_n$ with underlying graph $X$. Then any rose-face of $\Delta$ is obtained by collapsing a maximal tree $T$ of $X$, and different trees give rise to different faces. Therefore rose-faces of $\Delta$ are in correspondence with maximal trees of $X$ (for instance, in case of multi-theta simplices, rose-faces are in correspondence with edges).

**2.5. Distances and stretching factors.** We recall here the definitions of — both the symmetric and non-symmetric — Lipschitz distances on $CV_n$. These are defined via stretching factors of maps between points of outer space. Stretching factors, outer space and related topics are widely studied by many authors, and literature on the matter is huge (see for instance [4, 13, 16, 21, 9, 11, 14, 17, 10, 18, 19,].)

**Definition 2.12.** For any two points $X$ and $Y$ in $CV_n$, normalised to have volume one, we define the right stretching factor as

$$\Lambda_R(X, Y) = \sup_{\gamma} \frac{L_X(\gamma)}{L_Y(\gamma)}$$

where the supremum is taken over all loops (or, equivalently over all conjugacy classes in $F_n$.) Similarly, the left stretching factor is

$$\Lambda_L(X, Y) = \Lambda_R(Y, X) = \sup_{\gamma} \frac{L_Y(\gamma)}{L_X(\gamma)}.$$
**Definition 2.13.** For any two points \(X, Y \in CV_n\), normalised to have volume one, the right and left distances are defined by
\[
d_R(X, Y) = \log(\Lambda_R(X, Y)) \quad d_L(X, Y) = \log(\Lambda_L(X, Y)).
\]

**Definition 2.14.** For any two points \(X, Y \in CV_n\), not necessarily normalised, the symmetric bi-Lipschitz metric, is defined by
\[
d(X, Y) = d_R(X, Y) + d_L(X, Y) = \log\sup_{\gamma} \frac{L_X(\gamma)}{L_Y(\gamma)} \sup_{\gamma} \frac{L_Y(\gamma)}{L_X(\gamma)}.
\]

We refer to [11] for a detailed discussion on such metrics. We recall some basic facts. Firstly, the suprema in definitions are actually maxima. Also, we recall that \(Out(F_n)\) acts faithfully by isometries on \(CV_n\) (for \(n \geq 3\), in rank two the kernel of the action is \(\mathbb{Z}_2\)) endowed with any of above metrics. Finally we note that the symmetric metric is scale invariant, while the non-symmetric ones require normalisation.

The main tool for studying such distances is the so-called sausages lemma, which allows us to quickly compute stretching factors, and which we will use extensively throughout the paper (see [11] for the proof).

**Definition 2.15** (Almost simple closed curves). Let \(X\) be a point of \(CV_n\). A simple closed curve (s.c.c. for short) is an embedding of \(S^1\) to \(X\). A figure-eight curve is an embedding to \(X\) of the bouquet \(S^1 \wedge S^1\) of two circles. A barbell curve is roughly speaking an embedding to \(X\) of the space: \(O—O\). More precisely, let \(Q = \{(x, y) \in \mathbb{R}^2 : \sup(|x|, |y|) = 1\}\), then a barbell curve is an immersion \(c : Q \rightarrow X\) such that \(c(x, y) = c(x', y')\) if and only if \(x = x'\) and \(|y| = |y'| = 1\).

An almost simple closed curve (a.s.c.c. for short) is a curve which is either an s.c.c., or a figure-eight or a barbell curve.

**Lemma 2.16** (Sausages Lemma). For any two marked metric graphs \(X, Y\)
\[
\sup_{\gamma} \frac{L_Y(\gamma)}{L_X(\gamma)}
\]
is realised by an a.s.c.c. of \(X\). Moreover, If both \(X\) and \(Y\) are roses, then the supremum is realised by petals.

We notice that the Sausages Lemma not only allows to actually compute distances, but is also important from a theoretical view-point. Indeed, the fact that lengths of a.s.c.c. determine distances, and therefore points of outer space, is a key-point in the proof of Theorem 1.1 (see in particular Theorems 6.7 and 6.2).

Another simple but somewhat surprising result that we will need in the sequel is the following (whose proof can be found in [11])

**Lemma 2.17.** Suppose \(\sigma\) is a \(d\)-geodesic between two points \(X, Y\) of \(CV_n\). Let \(Z\) be a point in \(\sigma\). A loop \(\gamma_0\) is maximally (resp. minimally)
stretched from $X$ to $Y$ — that is to say, it realises $\sup_\gamma L_Y(\gamma)/L_X(\gamma)$ — if and only if the same is true from $X$ to $Z$ and from $Z$ to $Y$.

3. Schema of proof of Theorem 1.1

We briefly describe here the strategy for proving our main result. We recall that we aim to show that any isometry $\Phi$ of $C\nu_n$ is induced by some element of $\text{Out}(F_n)$.

(1) For topological reasons, $\Phi$ maps simplices to simplices. Moreover, it maps rose simplices to rose simplices and multi-theta simplices to multi-theta simplices.

(2) Computation of isometry group of rose simplices (it will be $\mathbb{R}^n \rtimes$ a finite group.)

(3) For a point $X$ in a simplex $\Delta$ of $C\nu_n$, the asymptotic behaviour of distances from $X$ to points in rose-faces of $\Delta$ determine lengths of simple closed curves of $X$. This being true not only for points of $\Delta$ but also for points in any other simplex having the same rose-faces as $\Delta$.

(4) For a point $X$ in a simplex $\Delta$ (or in other simplices sharing rose-faces with $\Delta$) the lengths of simple closed curves and the asymptotic behaviour of distances from $X$ to points in rose-faces of $\Delta$, determine lengths of almost simple closed curves of $X$ (whence asymptotic distances determine lengths of a.s.c.c.)

(5) Study of isometries of multi-theta simplices. We show that any isometry of a multi-theta simplex fixes its centre. How:

(a) Study of those pairs of points joined by a unique geodesic, showing that for any point $X$ in the interior of $\Delta$, there is a standard set of “rigid” geodesics emanating from $X$.

(b) Show that for any point other than the centre, there is at least one more “rigid” geodesic, while for the centre, the standard set is all we have. This characterises the centre of $\Delta$ from a metric point of view.

(c) Finally, for the centre of any rose-face of $\Delta$ there is a unique “rigid” geodesic joining it to the centre.

(d) In particular, any isometry fixes the centre, and if it does not permute rose-faces, it fixes also such “rigid” geodesics.

(6) Combining this with the knowledge of isometries of roses, we get that if an isometry of a multi-theta simplex, $\Delta$, does not permute its rose-faces, then it point-wise fixes them and hence point-wise fixes $\Delta$ by $\text{Out}(F_n)$ above (and we always can reduce to the situation where $\Phi$ leaves some multi-theta $\Delta$ and all its rose faces invariant, by composing with an appropriate element of $\text{Out}(F_n)$).

(7) Show that any isometry that fixes a multi-theta simplex, also fixes all rose simplices of $C\nu_n$ (not only its faces.)
(8) Show that simplices that are possibly permuted by \( \Phi \) share their rose-faces, and that simplices that share rose-faces “have the same set of simple closed curves and the same set of almost simple closed curves”. As asymptotic distances from rose-faces determine points in such simplices, it follows a posteriori that they cannot be permuted.

4. Topological constraints for homeomorphisms

In this section we prove first step of our strategy, that isometries of \( CV_n \) respect its simplicial and incidence structure. That result does not require any metric structure, just the fact that isometries are homeomorphisms.

**Lemma 4.1.** Any homeomorphism of \( CV_n \) maps \( k \)-dimensional simplices to \( k \)-dimensional simplices.

**Proof.** The proof goes by induction on the codimension. Open top dimensional simplices coincide with smooth points (Lemma 2.7.) Clearly, to be a smooth point is invariant under homeomorphisms. Again, by Lemma 2.7 open top-dimensional simplices are exactly connected components of set of smooth points. Therefore homeomorphisms map open top-dimensional simplices to open top-dimensional simplices.

Suppose the claim true for dimensions greater than \( i \). By induction, any homeomorphism \( \Phi \) of \( CV_n \) induces a homeomorphism of \( i \)-skeleton \( CV^i_n \). Open codimension-(\( n - i \)) simplices are now connected components of smooth part of \( CV^i_n \), and therefore \( \Phi \) maps \( i \)-simplices to \( i \)-simplices. \( \Box \)

**Lemma 4.2.** Any homeomorphism of \( CV_n \) maps rose-simplices to rose-simplices, and multi-theta simplices to multi-theta simplices.

**Proof.** This is just a dimensional argument. Clearly, homeomorphisms preserve dimension. By Lemma 2.4, \( n - 1 \)-dimensional simplices are exactly rose-simplices, and the first claim follows. If we look at \( n \) dimensional simplices, we see that multi-theta simplices are characterised by having exactly \( n + 1 \) rose-faces. So the homeomorphic image of a multi-theta simplex still is a multi-theta simplex. \( \Box \)

5. Isometries of roses

In this section, we compute the isometry groups of rose simplices. In rank two, it is immediate to see that a rose simplex is isometric to \( \mathbb{R} \), so its isometries are known. For the general case we prove,

**Theorem 5.1.** The isometry group of a rose-simplex \( R \) of \( CV_{n+1} \) is \( \mathbb{R}^n \rtimes \mathfrak{F} \), where \( \mathfrak{F} \) is finite and stabilises the centre, and \( \mathbb{R}^n \) acts transitively. Moreover, for \( n \geq 2 \), the group \( \mathfrak{F} \) is \( S_{n+1} \times \mathbb{Z}_2 \), where \( S_{n+1} \) is
the symmetric group on \( n + 1 \) letters and is induced by permutations of petals. For \( n = 1 \) (i.e. in the rank-two case) \( \mathcal{F} = \mathbb{Z}_2 = S_2 \).

**Proof.** Any point of \( R \) is determined by the lengths of its petals, that we label \( e_0, \ldots, e_n \). We identify the unprojectivised \( R \) with \( \mathbb{R}^{n+1} \) as follows. To any \((x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\) is associated the graph \( X \) such that

\[
L_X(e_i) = e^{x_i}
\]

Note that origin of \( \mathbb{R}^{n+1} \) corresponds to centre of \( R \). Moreover, scaling-equivalence on \( CV_{n+1} \) descends to relation

\[
x \sim y \quad \text{if and only if} \quad x - y = \lambda(1, \ldots, 1).
\]

The pull back of the (pseudo) metric \( d \) to \( \mathbb{R}^{n+1} \) is then

\[
d((x_0, \ldots, x_n), (y_0, \ldots, y_n)) = \sup_i (x_i - y_i) + \sup_i (y_i - x_i).
\]

This immediately implies that translations of \( \mathbb{R}^{n+1} \) are isometries, and that translations along vector \((1, \ldots, 1)\) are in fact the only ones inducing the identity of the projectivised \( R \). So we have that

\[
\mathbb{R}^n = \mathbb{R}^{n+1}/ < (1, \ldots, 1) >
\]

acts freely and transitively on \( R \).

Thus, it remains to determine the stabiliser of the origin.

Clearly, permutations of coordinates are isometries that fix origin. Finally, we have the reflection

\[
\sigma : (x_0, \ldots, x_n) \mapsto (-x_0, \ldots, -x_n).
\]

In the rank-two case, that is to say when \( n = 1 \), we are studying isometries of \( R \) that fix origin. Therefore in rank-two, the stabiliser of the origin consists of the reflection about the origin and the identity: note that this reflection (the map \( \sigma \), above, in other words) is induced by the map which interchanges the two petals of our rank 2 rose.

For \( n > 1 \), our claim is that the stabiliser of origin is

\[
\mathcal{F} = S_{n+1} \times < \sigma > .
\]

For that, we need some work. First of all, note that the (pseudo) metric \( d \) on \( \mathbb{R}^{n+1} \) is induced by the (pseudo) norm

\[
\|x\| = d(0, x).
\]

In order to make \( \| \cdot \| \) a norm and \( d \) a metric, for any point \( x \in \mathbb{R}^{n+1} \) we choose the \( \sim \)-representative of \( x + \mathbb{R}(1, \ldots, 1) \) that has 0 as the first coordinate. We can do that because \((x_0, \ldots, x_n) \sim (x_0, \ldots, x_n) - x_0(1, \ldots, 1)\). This gives an isometry between \( R \) and \( \mathbb{R}^n \) with the following metric (still denoted by \( d \))

\[
d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := d((0, x_1, \ldots, x_n), (0, y_1, \ldots, y_n)).
\]

We give now a more explicit description of that metric.
**Lemma 5.2.** For any set $I \subseteq \{1, \ldots, n\}$ let $R^I$ be the sector of $\mathbb{R}^n$ such that either $x_i \geq 0$ for all $i \in I$ and $x_i \leq 0$ for all $i \notin I$, or vice versa. Then, for $x \in R^I$

$$||x|| = ||x||_{\infty,I} + ||x||_{\infty,I^c}$$

where $||x||_{\infty,I} = \sup_{i \in I} |x_i|$ and $I^c$ is the complement of $I$ in $\{1, \ldots, n\}$.

**Proof.** This is a straightforward calculation. Indeed, by definition

$$||x|| = \sup_{i=1,...,n} \left( \sup_{i \in I} |x_i| + \sup_{i \notin I} |x_i| \right)$$

and, when $x \in R^I$, that equals $||x||_{\infty,I} + ||x||_{\infty,I^c}$. \hfill \Box

Our next step is an idea that we will return to throughout the paper, and it is that the “unique” geodesics are rather rare and allow one to determine the possible isometries.

**Remark 5.3.** Note that $l^1$-norms naturally present phenomena of non-uniqueness of geodesics. Namely, consider two geodesic spaces $(X_1, d_1)$ and $(X_2, d_2)$, and their cartesian product equipped with the sum metric $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$. Then any geodesic $\gamma : [0, 1] \to X_1 \times X_2$ is of the form $\gamma = (\gamma_1, \gamma_2)$, and, up to reparametrisation,

$$t \mapsto \begin{cases} (\gamma_1(t), \gamma_2(0)) & t \in [0, 1] \\ (\gamma_1(1), \gamma_2(t - 1)) & t \in [1, 2] \end{cases}$$

are two different geodesics whenever neither $\gamma_1$ nor $\gamma_2$ is the constant map. This situation is exactly the one arising in each sector $R^I$ as above, where, by Lemma 5.2, we have the sum of two $l^\infty$-norms.

**Proposition 5.4.** Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, equipped with the metric $d$ above. Then there exists a unique geodesic joining the origin to $x$ if and only if there exists a real number $\lambda$ such that for all $i$, $x_i = \lambda$ or $x_i = 0$. This geodesic is given (up to reparametrisation) by the path $\gamma_x$ whose $i^{th}$ coordinate at time $t$ is $tx_i$.

Equivalently, a point $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ represents a point in $\mathbb{R}^n$ joined to the origin by a unique geodesic if and only if there exist $\lambda, \mu$ such that each $x_i$ is equal to either $\lambda$ or $\mu$.

**Proof.** The last statement follows trivially from the first, on taking the representative with $x_0 = 0$, obtained by subtracting one of $\lambda$ or $\mu$ from each coordinate.

Next, let $O$ denote the origin of $\mathbb{R}^n$. For any $x, y$ let $\overline{xy}$ denote the path whose $i^{th}$ coordinate at time $t$ is $x_i + t(y_i - x_i)$, $t \in [0, 1]$. By Remark 5.3 if there is a set of indices $I$ such that $||x||_{\infty,I}||y||_{\infty,I^c} \neq 0$ then $x$ is joined to $O$ by at least two different geodesics. Thus, up to rearranging coordinates and possibly applying the isometry $\sigma$ above, we can suppose $0 \leq x_1 \leq \cdots \leq x_n$. Clearly, $d(\gamma_x(s), \gamma_x(t)) = x_n|t - s|$, so that $\gamma_x = \overline{Ox}$ is a geodesic. Suppose there is $i$ such that $0 < x_i <
Then, consider the point \( x_\varepsilon = (x_1/2, \ldots, x_i/2 + \varepsilon, \ldots, x_n/2) \). For small enough \( \varepsilon \) the path \( \gamma_\varepsilon \) resulting on the union of \( Ox_\varepsilon \) and \( x_\varepsilon O \) is a geodesic from \( x \) to \( O \) as \( d(\gamma_\varepsilon(s), \gamma_\varepsilon(t)) = x_n|t - s|/2 \). Also, \( \gamma_\varepsilon \) is not a reparametrisation of \( \gamma_\varepsilon \) because they differ in their middle points.

Conversely, suppose that there is \( i \) so that \( x_j = 0 \) for \( j < i \) and \( x_j = x_n \) for \( j \geq i \). Let \( \gamma \) be a geodesic between \( O \) and \( x \). If there is a time \( t \) such that the \( j \)th coordinate of \( \gamma(t) \) is different from 0 for some \( j < i \), then a direct calculation shows that \( d(O, \gamma(t)) + d(\gamma(t), x) \) is strictly bigger than \( x_n \) (while \( d(O, x) = x_n \)). Thus, the \( j \)th coordinates of \( \gamma(t) \) all vanish for \( j < i \). The very same argument shows that for \( j \geq t \) the \( j \)th coordinate of \( \gamma(t) \) equals the \( n \)th one, this showing that \( \gamma_\varepsilon \) is the unique geodesic from 0 to \( x \).

Proposition 5.4 is a translation of the fact that two roses in the same simplex are joined by a unique geodesic if and only if there are only two possible stretching factors for petals.

We note that Proposition 5.4 gives us a collection of geodesics which are permuted by any isometry fixing the origin. Using this fact, we now proceed to calculate the stabiliser of the origin. Since we already have that these geodesics must be permuted by any isometry fixing the origin, we shall proceed by studying points on these geodesics at fixed distance 1 from the origin. These are also permuted and will give us the information we need about the stabiliser.

For any \( I \subseteq \{1, \ldots, n\} \) we define points \( p^+_I \) and \( p^-_I \) in \( \mathbb{R}^n \) by

\[
p^+_I = (x_1, \ldots, x_n) : x_i = \begin{cases} 
  \pm 1 & i \in I \\
  0 & i \in I^c
\end{cases}
\]

such points are equivalents to points \( P_I \) of \( \mathbb{R}^{n+1} \)

\[
P_I = (x_0, \ldots, x_n) : x_i = \begin{cases} 
  1 & i \in I \\
  0 & i \in I^c
\end{cases}
\]

where \( p^+_I \) is equivalent to \( P_{0 \cup I} \), and \( p^-_I \) is equivalent to \( P_{I^c} \) (the complement here is made in \( \{0, \ldots, n\} \).

Lemma 5.5. For any distinct \( I, J \subseteq \{0, \ldots, n\} \) we have

\[
d(P_I, P_J) = \begin{cases} 
  1 & \text{if } I \subseteq J \text{ or } J \subseteq I \\
  2 & \text{otherwise}
\end{cases}
\]

Moreover, if \( d(P_I, P_J) = 2 \), then the points of \( \mathbb{R}^n \) corresponding to \( P_I \) and \( P_J \) are joined by a unique geodesic if and only if \( I = J^c \).

Proof. The first part is a simple calculation. For the second part, we use the fact that translations are isometries. Translate the point \( P_I \) to the origin and look at the image of \( P_J \), which we call denote \( x = (x_0, \ldots, x_n) \). Then there will be a unique geodesic between \( P_I \) and \( P_J \) if and only if there is a unique geodesic between \( x \) and the origin.
However, is clear what each \( x_i \) will be. Namely,

\[
x_i = \begin{cases} 
0 & \text{if } i \in I \cap J \\
0 & \text{if } i \in I^c \cap J^c \\
1 & \text{if } i \in I^c \cap J \\
-1 & \text{if } i \in I \cap J^c 
\end{cases}
\]

As \( d(P_I, P_J) = 2 \), we cannot have either \( I \subseteq J \) or \( J \subseteq I \) and hence both 1 and \(-1\) must be taken by some of the \( x_i \). So by Proposition 5.4, \( P_I \) and \( P_J \) will be joined by a unique geodesic if and only if no \( x_i \) is equal to zero, which is the same as saying \( I \cap J = \emptyset = I^c \cap J^c \). Equivalently, \( I = J^c \).

As stated, by Proposition 5.4 any isometry that fixes the origin must permute the \( P_i \)'s. For such an isometry \( F \) and \( I \subseteq \{0, \ldots, n\} \), we denote by \( F(I) \) the set corresponding to point \( F(P_I) \).

From Lemma 5.5 we get

\begin{align*}
(1) \quad (I \subseteq J \text{ or } J \subseteq I) & \iff (F(I) \subseteq F(J) \text{ or } F(J) \subseteq F(I)) \\
(2) \quad F(I^c) & = F(I)^c
\end{align*}

Remark 5.6. The isometry \( \sigma \) corresponds to \( I \mapsto I^c \).

Lemma 5.7. For any isometry \( F \), the cardinality \( |F(I)| \) is either \( |I| \) or \( n + 1 - |I| \).

Proof. By (1) sets \( I \) and \( F(I) \) must have the same numbers of subsets and supersets. For \( I \) such number is \( 2^{|I|} + 2^{n+1-|I|} - 1 \), whence

\[
2^{|I|} + 2^{n+1-|I|} = 2^{|F(I)|} + 2^{n+1-|F(I)|}.
\]

Set \( x = \min\{|I|, n + 1 - |I|\} \) and \( y = \min\{|F(I)|, n + 1 - |F(I)|\} \). We have

\[
2^x(1 + 2^k) = 2^y(1 + 2^h)
\]

for some non-negative numbers \( k, h \). Whence \( x = y \) and the claim follows. \( \square \)

Remark 5.8. Up to possibly composing with \( \sigma \) we may suppose, as we do, that there is \( i_0 \) such that \(|F(\{i_0\})| = 1 \).

Lemma 5.9. If there is \( i_0 \) such that \(|F(\{i_0\})| = 1 \), then for all \( i \) we have that \(|F(\{i\})| = 1 \).

Proof. Note that by (2), \(|F(\{i_0\}^c)| = |F(\{i_0\})^c| = n \). Now consider some \( i \neq i_0 \), whence \( \{i\} \subseteq \{i_0\}^c \). If \( \{i\} = \{i_0\}^c \) then \( n = 1 \) and the lemma is proved. So we can suppose \( \{i\} \neq \{i_0\}^c \), so \( F(\{i\}) \neq F(\{i_0\}) \). Thus, by (1) and Lemma 5.7 we have that \( F(\{i\}) \) is strictly contained in \( F(\{i_0\})^c \). We therefore have \(|F(\{i\})| \leq n - 1 \), which implies \(|F(\{i\})| = 1 \) because of Lemma 5.7. \( \square \)
Remark 5.10. When \(|F(\{i\})| = 1\) for all \(i\), we can define an element \(f\) of \(S_{n+1}\) by
\[ F(\{i\}) = \{f(i)\}. \]

We show now that the permutation \(F\) is actually induced by \(f\).

Lemma 5.11. Suppose \(|F(\{i\})| = 1\) for all \(i\). For all \(I \subseteq \{0, \ldots, n\}\) we have
\[ F(I) = \{f(i) : i \in I\}. \]

Proof. For any \(i \in I\) we have that \(\{f(i)\}\) is either contained in or contains \(F(I)\), so we must have \(f(i) \in F(I)\). The same holds for \(I^c\). \(\square\)

An immediate consequence of all these facts is the following fact.

Proposition 5.12. Up to possibly composing with \(\sigma\) and an element of \(S_{n+1}\), any isometry of \(R\) that fixes origin also fixes all points \(p_I^\pm\).

Proof. Let \(F\) be an isometry of \(R\) fixing the origin. We shall also use \(F\) to denote the induced permutation of \(\{0, \ldots, n\}\), so that \(F(P_I) = P_{F(I)}\).

By remark 5.8 and Lemma 5.9 we may suppose that for all \(i\) we have \(|F(\{i\})| = 1\). Hence by Lemma 5.11, \(F\) is induced by some permutation, \(f\). We can think of this permutation as an isometry of \(R\) which permutes the petals of the rose. By composing \(F\) with the inverse of this isometry, we get that \(F(I) = I\) for all subsets \(I\) of \(\{0, \ldots, n\}\). Thus \(F\) fixes all the points \(P_I\) and thus all the \(p_I^\pm\). \(\square\)

Next lemma is a simple case of a general asymptotic argument (see Section 6 and compare in particular with Proposition 6.4).

Lemma 5.13. For any \(i = 1, \ldots, n\) and \(t \in \mathbb{R}\), let \(x_i(t)\) be the point of rose \(R\), identified with \(\mathbb{R}^n\), whose coordinates are zero except for \(i^{th}\) which is \(t\):
\[ x_i(t) = (0, \ldots, t, \ldots, 0), \quad t \text{ at the } i^{th} \text{ place} \]

and let \(x_0(t) = (t, \ldots, t)\). Then, points of \(R\) are determined by distances from points \(x_i(t)\)'s.

Proof. Let \(y = (y_1, \ldots, y_n)\) be a point of \(R\). Clearly, for large enough \(t\), we have
\[ d(y, x_i(t)) = t - y_i + \max\{0, \sup_{j \neq i} y_j\}. \]

Therefore, by knowing such distances, we know for each \(i\)
\[ -y_i + \max\{0, \sup_{j \neq i} y_j\}. \]

Note that the \(y_i\)'s are all negative numbers if and only if such quantities are all positive, and in that case they give exactly \(-y_i\). On the other hand, if some non-positive quantity appears, then indices \(i\) for which \(y_i\) is maximum are characterised by the fact that \(i^{th}\) quantity is not
positive. Thus, varying $i > 0$ we know all differences $y_i - y_j$ for any $i, j$.

Finally, consider distances from $x_0(t)$ as $t \to -\infty$

$$d(y, x_0(t)) = \max\{0, \sup_i (t - y_i)\} + \max\{0, \sup_i (y_i - t)\} = \max y_i - t.$$ 

This gives knowledge of $\max_i y_i$, and since we know those indices for which $y_i$ is maximum, and all differences $y_i - y_j$, we get all the $y_i$’s. $\square$

Note that such a result can be re-paraphrased by saying that Busemann functions of ideal vertices determines points.

We are now able to finish proof of Theorem 5.1. Let $\phi$ be an isometry of $R$. Up to composing with a translation of $\mathbb{R}^n$, we can suppose that $\phi$ fixes the origin. By Proposition 5.12 after possibly composing with elements of $S_{n+1} \times \langle \sigma \rangle$, we can suppose that $\phi$ fixes all the points $p^\pm_1$. Therefore, by Proposition 5.3, $\phi$ must fix all the points $x_i(t)$ and $x_0(t)$. Lemma 5.13 now implies that $\phi$ is the identity. $\square$

We conclude this part anticipating results of subsequent sections. We have seen what the isometry group of a rose simplex is, and we have seen in particular that there are isometries which are not induced by elements of $Out(F_{n+1})$. This seems, a priori, to count as evidence against our final result. However, no such isometry arises as the restriction of a global isometry of $CV_{n+1}$. Indeed, we will show that translations of $\mathbb{R}^n$ and reflection $\sigma$ are not restrictions of global isometries. On the other hand, any isometry in $S_{n+1}$ is induced by a permutation of generators and hence by an element of $Out(F_{n+1})$ (see Sections 6 and 7, in particular Remark 7.8 and Lemma 7.9).

Note that this is not enough to show that $\text{Isom}(CV_{n+1})$ is $Out(F_{n+1})$. Indeed, it could be possible that an isometry permutes simplices, and second, that restrictions of an isometry to different simplices are restrictions of different elements of $Out(F_{n+1})$. We will see that this is not the case (Section 7).

6. Asymptotic distances from roses and global isometries

In this section we generalise calculations made in Section 5 about the asymptotic behaviour of distances. The underlying philosophy is that Busemann functions of ideal vertices are enough to distinguish points of outer space.

What we have in mind is to prove the following fact, that if an isometry fixes all rose-simplices of $CV_n$ then it must be the identity. This, together with results of next section, opens the way towards Theorem 1.1.

The first point in proving that result is that a priori, an isometry that fixes all rose-faces, could permute other simplices.
For that, we have to understand simplices that are possibly not invariant under the action of such an isometry. Lemma 6.1 below will tell us that any two putatively permuted simplices must have the same rose-faces.

Then, our aim will be to show that a point \( X \) is determined by asymptotic distances from points in the rose-faces of the simplex containing \( X \). More precisely, we show how such distances determine the lengths, in \( X \), of every almost simple closed curve.

We emphasise that the results we are proving here (out of necessity, due to Lemma 6.1) depend only on the set of rose-faces, and not on the simplex containing \( X \).

That is to say, suppose \( X \) and \( Y \) are points of simplices \( \Delta_1 \) and \( \Delta_2 \) who share their rose-faces. If for any \( p \) in any rose-face we have \( d(X, p) = d(Y, p) \), then we show that for any two a.s.c.c. \( \gamma_1 \) and \( \gamma_2 \), \( L_X(\gamma_1)/L_X(\gamma_2) = L_Y(\gamma_1)/L_Y(\gamma_2) \) (so lengths of a.s.c.c. are equal up to scaling.) Of course, we need also to show that whenever \( \Delta_1 \) and \( \Delta_2 \) share rose-faces, then a loop \( \gamma \) is a.s.c.c. in \( \Delta_1 \) if and only if the same happens in \( \Delta_2 \).

Since the distance from \( X \) to \( Y \) is computed using only a.s.c.c. (because of Lemma 2.16) we deduce that this implies \( X = Y \).

We start by studying simplices possibly permuted by isometries that fix roses.

**Lemma 6.1.** Let \( \Phi \) be an isometry of \( CV_n \) that fixes all rose-simplices. If \( \Delta \) is any simplex of \( CV_n \), then \( \Delta \) and \( \Phi(\Delta) \) have the same rose-faces.

**Proof.** Let \( R \) be a rose-face of \( \Delta \), then \( d(R, \Delta) = 0 \), so

\[
0 = d(R, \Delta) = d(\Phi(R), \Phi(\Delta)) = d(R, \Phi(\Delta)).
\]

Thus, \( R \) is a rose-face also of \( \Phi(\Delta) \). Using \( \Phi^{-1} \) we get the converse. \( \square \)

Now, we show that two simplices that share rose-faces have the same a.s.c.c.

**Theorem 6.2.** Let \( \Delta_1 \) and \( \Delta_2 \) be two simplices of \( CV_n \) that share their rose-faces. Then they have the same set of almost simple closed curves. More precisely, if \( \gamma \) is a conjugacy-class in \( F_n \), then its geodesic representative in \( \Delta_1 \) is simple if and only if it is simple in \( \Delta_2 \), and it is a figure-eight or bar-bell curve in \( \Delta_1 \) if and only if the same is true in \( \Delta_2 \) (possibly bar-bells become figure-eight curves and vice versa.)

**Proof.** Let \( G_1 \) and \( G_2 \) be marked graphs corresponding to simplices \( \Delta_1 \) and \( \Delta_2 \). Any rose-face of \( \Delta_i \) is obtained by collapsing a maximal tree in \( G_i \).

We first prove that a loop is simple in \( G_1 \) if and only if it is simple in \( G_2 \). Let \( \gamma \) be a simple loop in \( G_1 \), and let \( e_1 \) be an edge of \( \gamma \). As \( e_1 \) is part of a simple loop, it does not disconnect \( G_1 \). Extend \( \gamma \setminus e_1 \)
to a maximal tree $T_1$ in $G_1$. Let $R$ be the rose obtained by collapsing $T_1$. The class of $\gamma$ in $R$ is represented by a petal $p$ (the image of $e_1$.) As $\Delta_1$ and $\Delta_2$ share rose-faces, $R$ is obtained by collapsing a maximal tree $T_2$ in $G_2$. So the class of $\gamma$ in $G_2$ is represented by an edge $e_2$ corresponding to the petal $p$ plus a path in $T_2$. As $T_2$ is a tree, such path is unique and its union with $e_2$ is simple. Thus, $\gamma$ is represented by a simple loop also in $G_2$.

Now, we deal with figure-eight and barbell curves. Let $\gamma$ be such a curve in $G_1$. Let $\alpha$ and $\beta$ be the two simple loops of $\gamma$.

**Lemma 6.3.** Let $R$ be any rose-face of $G_1$, then $\alpha \cup \beta$ is represented in $R$ by a union of petals, each petal appearing at most once. In particular the representatives of $\alpha$ and $\beta$ in $R$ have no common petal.

**Proof.** Let $T$ be the maximal tree of $G_1$ collapsed in order to obtain $R$. Since $T$ is a tree, and it is maximal, it cannot contain the whole $\alpha$, nor the whole $\beta$. As $\alpha$ and $\beta$ have no common edge, their images in $R$ share no petal. Moreover, since $\alpha$ and $\beta$ are simple, no petal can occur twice. $\square$

Note that Lemma 6.3 would fail if $\alpha \cup \beta$ were a theta curve. We can now conclude proof of Theorem 6.2. Let $e_\alpha$ be an edge of $\alpha$ and $e_\beta$ be an edge of $\beta$. As $e_\alpha$ and $e_\beta$ are part of simple loops with no common edges, we have that $e_\alpha \cup e_\beta$ does not disconnect $G_1$. Extend $\gamma \setminus (e_\alpha \cup e_\beta)$ to a maximal tree $T_1$, and let $R$ be the rose obtained by collapsing $T_1$. Let $T_2$ be the tree of $G_2$ whose collapsing gives $R$. The loop $\alpha$ is represented in $G_2$ by an edge corresponding to $e_\alpha$, which we still denote by $e_\alpha$, and a path $\sigma_\alpha$ in $T_2$ joining the end-points of $e_\alpha$. The same (with the same notation) for $\beta$. The paths $\sigma_\alpha$ and $\sigma_\beta$ have connected intersection because $T_2$ is a tree. It follows that the representative of $\gamma$ in $G_2$ is either a figure-eight or a barbell, or a theta-curve. We show now that the case of theta-curve cannot arise.

Indeed, suppose representative of $\gamma$ in $G_2$ is a theta-curve. This is equivalent to saying that $\sigma_\alpha \cap \sigma_\beta$ contains at least one edge $e_0$. Clearly, $e_0$ does not disconnect $G_2$. We can therefore find a maximal tree $T_0$ not containing $e_0$. Collapsing $T_0$ we get a rose $R$ with a petal $p_0$ corresponding to $e_0$. In $R$, loops representing $\alpha$ and $\beta$ share petals $p_0$. By Lemma 6.3 $R$ cannot be obtained from $G_1$, in contradiction with hypothesis that $\Delta_1$ and $\Delta_2$ have same rose-faces. $\square$

Our next goal is to show that asymptotic distances from rose-faces determine points. First, we show how to determine lengths of simple closed curves using distances from rose-faces. After, we will deal with a.s.c.c.

**Proposition 6.4** (Distances from roses determine simple loops). Let $G_1$ and $G_2$ be the underlying graphs of two simplices $\Delta_1$ and $\Delta_2$ having
the same rose-faces. Let $X_1, X_2 \in \Delta_1 \cup \Delta_2$ such that $d(X_1, Y) = d(X_2, Y)$ for any point $Y$ of any rose face of $\Delta_1$ (or $\Delta_2$).

Now fix a conjugacy class $\gamma_0$ which is a simple loop in $G_1$ (and hence $G_2$) and suppose that $X_1, X_2$ are the representatives for which $L_{X_1}(\gamma_0) = L_{X_2}(\gamma_0) = 1$.

Then, for any conjugacy class $\gamma$ in $F_n$ which is represented by a simple loop in $G_1$,

$$L_{X_1}(\gamma) = L_{X_2}(\gamma).$$

Recall that $\gamma$ is simple in $G_1$ if and only if it is simple in $G_2$ because of Theorem 6.2. Proposition 6.4 will follow from next lemma.

**Lemma 6.5.** Let $R$ be a rose simplex in $CV_n$ and let $e, e_0$ be petals in the underlying graph of $R$. Set $Y_t$ to be the ray in $R$, consisting of roses in $R$ all of whose edges except $e, e_0$ have length 1, and such that at time $t$

$$L_{Y_t}(e) = t \quad L_{Y_t}(e_0) = \frac{1}{t}.$$

Now consider an $X \in \Delta$, where $\Delta$ is a simplex of $CV_n$ whose underlying graph is $G$ and such that $R$ is a rose-face of $\Delta$.

Let $\gamma$ be the simple closed curve in $X$ corresponding to $e_0$. Also let $\gamma_e$ be an a.s.c.c. in $X$ which minimises $\frac{L_{Y_t}(\gamma_e)}{n_e(\gamma_e)}$, where $n_e(\gamma_e)$ is the number of times $\gamma_e$ crosses $e$ (when projected to $R$.) Then,

$$L_{X}(\gamma) = C(e, X) \lim_{t \to \infty} \frac{e^{d(X, Y_t)}}{t^2}$$

where $C(e, X) = \frac{L_{X}(\gamma_e)}{n_e(\gamma_e)}$ as above.

**NOTE:** The ray $Y_t$ depends only on the edges $e, e_0$, and the rose simplex $R$.

**Proof.** Let $T$ be the maximal tree in $G$ corresponding to the projection of $\Delta$ to $R$. We can find lifts of the edges $e, e_0$ in $G$. We continue to call these edges $e$ and $e_0$.

Now consider the ray $Y_t$. We let $t \to \infty$ and study the asymptotic behaviour of $d(X, Y_t)$.

We claim that for sufficiently large $t$, the loop $\gamma$ is maximally shrunk from $X$ to $Y_t$. Indeed, if $\sigma = e_1 \ldots e_k$ is a loop, then

$$L_{Y_t}(\sigma) = \sum_{i: e_i = e_0} \frac{1}{t} + \sum_{i: e_i = e} \frac{t}{\sum_{i: e_i \notin (T \cup e_0)}} + \sum_{i: e_i \notin (T \cup e_0)} 1.$$

Since $T$ is a tree, it cannot contain loops. Thus, if in $\sigma$ there is some $e_0 \neq e_i \notin T$ the above stretching factor is bounded below uniformly on $t$.

On the other side, if $\sigma = \gamma$, the stretching factor goes to zero as $t \to \infty$. Finally, if in $\sigma$ there is no edge $e_i \notin T \cup e_0$, then $\sigma$ is a multiple
of $\gamma$ because $T$ is a tree, and so $\gamma$ is the only way to obtain a simple loop from $e_0$ by adding edges of $T$.

Now, we look for maximally stretched loops. As above, we compute $L_Y(\sigma)/L_X(\sigma)$ for a generic a.s.c.c. $\sigma$ using (4). If $\sigma$ does not contain $e$, then there is an upper bound to the stretching factor and, as $t \to \infty$, it is readily checked that if (4) is maximised, then for big enough $t$ the ratio of $L_X(\sigma)$ over the number of occurrences of $e$ in $\sigma$ is minimised, hence

$$\frac{L_X(\sigma)}{n_e(\sigma)} = \frac{L_X(\gamma_e)}{n_e(\gamma_e)} := C(e, X).$$

It follows, that for sufficiently large $t$, if $\sigma$ is maximally stretched we have

$$d(X, Y_t) = \log \frac{L_Y(\sigma) L_X(\gamma)}{L_X(\sigma) L_Y(\gamma)} = \log \frac{L_X(\gamma)}{L_X(\sigma)} t(n_e(\sigma)t + b + n_{e_0}(\sigma) \frac{1}{t})$$

where $n_e(\sigma)$, $n_{e_0}(\sigma)$ are either 1 or 2 as $\sigma$ is a.s.c.c., $b$ is the number of edges of $\sigma$ not belonging to $T \cup e_0 \cup e$. Whence,

$$\frac{L_X(\gamma)}{L_X(\sigma)} = \lim_{t \to \infty} \frac{e^{d(X, Y_t)}}{n_e(\sigma)t^2}$$

so

$$L_X(\gamma) = \frac{L_X(\sigma)}{n_e(\sigma)} \lim_{t \to \infty} \frac{e^{d(X, Y_t)}}{t^2} = \frac{L_X(\gamma_e)}{n_e(\gamma_e)} \lim_{t \to \infty} \frac{e^{d(X, Y_t)}}{t^2} = C(e, X) \lim_{t \to \infty} \frac{e^{d(X, Y_t)}}{t^2}.$$

and the lemma is proved. $\square$

**Proof of Proposition 6.4.** Let $X$ be a point of either $\Delta_1$ or $\Delta_2$. Let $\gamma$ and $\eta$ be two simple closed curves in $G_1$. Choose an edge $e_0$ in $\gamma$ (but not in $\eta$) and an edge $f_0$ in $\eta$ (but not in $\gamma$). We can then find a maximal tree $T$ in $G$ which extends $\gamma \cup \eta - (e_0 \cup f_0)$. Let $R$ be the corresponding rose face of $\Delta$. Note that in any rose within this simplex $\gamma$ and $\eta$ each project to a single petal, which we will call $e_0$ and $f_0$ (these petals are also the projections of those edges).

Now assume that $n \geq 3$ so that we can find yet another petal, $e$, distinct from $e_0, f_0$.

By Lemma 6.5 there is a ray $Y_t$ such that,

$$L_X(\gamma) = C(e, X) \lim_{t \to \infty} \frac{e^{d(X, Y_t)}}{t^2}.$$

Similarly, there is a ray $Z_t$ such that,

$$L_X(\eta) = C(e, X) \lim_{t \to \infty} \frac{e^{d(X, Z_t)}}{t^2}.$$
Hence,

\[ \frac{L_X(\gamma)}{L_X(\eta)} = \lim_{t \to \infty} \frac{e^{d(X,Y_t)}}{e^{d(X,Z_t)}}. \]  

Moreover, by Lemma 6.5 this last equation must hold for any \( X \) which has \( R \) as a rose face (where we simply interpret \( \gamma, \eta \) as conjugacy classes of \( F_n \)) and thus certainly for any \( X \in \Delta_1 \cup \Delta_2 \). Thus, for the \( X_1, X_2 \) in the statement of the Proposition,

\[ \frac{L_{X_1}(\gamma)}{L_{X_1}(\eta)} = \frac{L_{X_2}(\gamma)}{L_{X_2}(\eta)} \]

for any two loops \( \gamma, \eta \) which are simple in \( G_1 \) (and hence \( G_2 \)). Putting \( \eta = \gamma_0 \) proves Proposition 6.4 when \( n \geq 3 \).

Now consider the case \( n = 2 \). Note that here, distinct simplices have different collections of rose faces (so we need not worry about \( \Delta_2 \)). When the underlying graph of \( X \) is a rose, there are exactly two simple loops in \( X \), \( \gamma \) and \( \eta \) and Lemma 6.5 produces exactly two different rays \( Y_t \) with limits, as in 6, \( L_{X_1}(\gamma) \) and \( L_{X_2}(\eta) \) and the order of these is independent of \( X \). So Proposition 6.4 is true in this case.

Similarly, if the underlying graph of \( X \) is a barbell, then there is exactly one rose face and exactly two simple loops, \( \gamma, \eta \). Again, the limits from 6 will give \( L_{X_1}(\gamma) \) and \( L_{X_2}(\eta) \) and the lemma is again true in this case.

Finally, if the underlying graph of \( X \) is a theta curve, then \( X \) has exactly 3 edges, \( x, y, z \), 3 rose faces and 3 simple closed curves, \( x\overarrow{y}, x\overarrow{z}, y\overarrow{z} \). There are then 6 possible rays as in Lemma 6.5. However, each limit,

\[ \lim_{t \to \infty} \frac{e^{d(X,Y_t)}}{t^2}, \]

is equal to one of \( \frac{L_X(x\overarrow{y})}{C(x,X)}, \frac{L_X(x\overarrow{z})}{C(y,X)}, \frac{L_X(y\overarrow{z})}{C(x,X)} \). Also note that \( C(x, X) \) is simply the length of the shortest simple loop in \( X \) which crosses \( x \), since an a.s.c.c. in \( X \) is actually a simple loop. Hence, \( C(x, X) \) is equal to either \( L_X(x\overarrow{y}) \) or \( L_X(x\overarrow{z}) \).

Thus, if \( x\overarrow{y} \) is the shortest simple loop in \( X \), then \( \frac{L_X(x\overarrow{y})}{C(x,X)} \) will be the smallest of the three limits, \( C(x, X) = C(y, X) = L_X(x\overarrow{y}) \) and conversely. From these observations, the Proposition follows easily. Take \( X_1, X_2 \) with the same distances to rose faces. Then the limits above, for \( X_1, X_2 \) respectively, produce the same ordered results (however, the \( C \) terms need to be evaluated in different \( X_i \)'s).

Nevertheless, if without loss of generality, \( x\overarrow{y} \) is the shortest loop in \( X_1 \), then the limit \( L_{X_1}(x\overarrow{y})/C(z, X_1) \) will be least, and thus so will \( L_{X_2}(x\overarrow{y})/C(z, X_2) \) and hence \( x\overarrow{y} \) must also be the shortest loop in \( X_2 \).

The Proposition now readily follows. \( \square \)
Remark 6.6. We note that the constant $C$ depends on $X$ and on $e$, but not on $\gamma$ or $e_0$. Such a dependence is thus cancelled when we consider the ratio $L_X(\gamma)/L_X(\eta)$, which therefore actually depends only on asymptotic distances from $X$ to rose-faces.

We now have sufficient tools for proving that asymptotic distances from $X$ to rose-faces determine the lengths of all a.s.s.c., whence determine $X$.

**Theorem 6.7.** Let $\Delta_1$ and $\Delta_2$ be simplices of $CV_n$ with the same set of rose-faces. Let $G_1$ and $G_2$ be the underlying graphs of $\Delta_1$ and $\Delta_2$ respectively. Let $\gamma_0$ be a simple loop in $G_1$ (whence its representative in $G_2$ is a simple loop as well.) For any class $[X]$ of metric graphs in $\Delta_1 \cup \Delta_2$ consider the representative $X$ so that $L_X(\gamma_0) = 1$. Now consider two such representatives, $X_1, X_2 \in \Delta_1 \cup \Delta_2$ such that $d(X_1, Y) = d(X_2, Y)$ for any $Y$ in any rose face of $\Delta_1$. Then, for any a.s.c.c. $\gamma$ in $G_1$ (and hence $G_2$),

$$L_{X_1}(\gamma) = L_{X_2}(\gamma).$$

**Proof.** The proof is in the same spirit as Proposition 6.4, but the situation now it is a little more complicated.

Let $X$ be a point in either $\Delta_1$ or $\Delta_2$. It will be sufficient to show that we can calculate the length of any a.s.c.c. in $X$ by only using distances to rose faces.

By Proposition 6.4, we know that lengths in $X$ of simple loops are determined via asymptotic distances to particular sequences of points, not depending on $X$. Thus, we can suppose that we already know the lengths of all simple loops in $X$, because we have normalised so that $L_X(\gamma_0) = 1$. Thus what remains is to deal with figure-eight and barbell curves. Clearly, the length of a figure-eight is determined via Proposition 6.4. On the other hand, Theorem 6.2 tells that a figure-eight curve in $\Delta_1$ may become a barbell in $\Delta_2$. For this reason we treat figure-eight and barbell curves at the same time, considering a figure-eight as a barbell whose central segment is reduced to a point.

In order to do this, we proceed as in Proposition 6.4: for any given barbell curve, we build an appropriate sequence of points $Y_t$ in some rose-face, such that the asymptotic distances from $Y_t$ determine the length of the barbell.

**Remark 6.8.** At this point, the reader should be aware of the subtle difference in the argument from that in Proposition 6.4. Indeed, the points $Y_t$ we constructed in Lemma 6.5 do not depend on $X$, but just on $e_0$ (hence on $\gamma$) and $e$. Here, the ray $Y_t$ we shall define will actually depend on $X$, or at least seem to, and thus present a logical obstacle to our argument.

More precisely, the ray $Y_t$ here will depend on lengths of simple loops in $X$. Intuitively speaking, the ray $Y_t$ escapes to infinity in a rose face...
and the “slope” of the this ray is determined by the lengths of simple loops in $X$. However, this is sound because of Proposition 6.4. So for any barbell curve, the ray we chose for computing its length is the same for both $X_1$ and $X_2$, thus barbells have same lengths in $X_1$ and $X_2$, and Theorem 6.7 will be proved.

The rank-two case is easy and left to the reader (just use the following argument without the need to introduce the edge $e$ and the loop $\gamma_e$.) Suppose $n \geq 3$.

Let $\gamma$ be a barbell curve, possibly degenerate to a figure-eight curve, say in $G_1$. Let $\gamma_1$ and $\gamma_2$ be the two simple loops of $\gamma$, and let $e_1 \in \gamma_1$ and $e_2 \in \gamma_2$ be two edges. Clearly, $G_1 \setminus (e_1 \cup e_2)$ is connected. Extend $\gamma \setminus (e_1 \cup e_2)$ to a maximal tree $T$, and consider the rose $R_T$ obtained by collapsing $T$. Since $n > 2$ there is an edge $e \notin (T \cup \gamma)$. Also, there is a simple loop not containing $e$ (for instance, $\gamma_1$.)

In $R_T$ we still denote by $e, e_1, e_2$ the petals corresponding to $e, e_1, e_2$ respectively.

Now, look at simplex $\Delta_2$. Since $R_T$ is a rose-face also of $G_2$, it is obtained by collapsing a maximal tree $T'$ in $G_2$. Therefore, petals $e, e_1, e_2$ correspond to edges of $G_2 \setminus T'$, and the representative of $\gamma$ in $G_2$ is disjoint from $e$.

Now let $Y_t$ be the point of $R_T$ whose petals have length 1 except $e, e_1, e_2$ for which we set

$$L_{Y_t}(e) = t \quad L_{Y_t}(e_1) = \frac{L_X(\gamma_1)}{t} \quad L_{Y_t}(e_2) = \frac{L_X(\gamma_2)}{t}.$$

We now let $t \to \infty$. If $\sigma = l_1 \ldots l_k$ is a loop, then (replace $T$ with $T'$ if $X \in \Delta_2$)

$$L_{Y_t}(\sigma) = \frac{\sum_{i:d_i=e_1} L_X(\gamma_1) + \sum_{i:d_i=e_2} L_X(\gamma_2) + \sum_{i:d_i=e} t + \sum_{i:d_i(\neq (T \cup e_1 \cup e_2)} 1}{\sum_{i} L_X(e_i)}.$$

Note that by Lemma 2.16 the loop $\sigma$ minimising the equation above is realised by an a.s.c.c. in $Y_t$ (note this statement is independent of $t$) and inspection of the equation 7 shows that the only possible candidates are $e_1, e_2$ and $e_1 e_2$. It is then easy to see that $e_1 e_2$ is a loop which realises the minimum, and this is exactly the realisation of $\gamma$ in $Y_t$. (Note that when the barbell is actually a figure-eight, all three loops give the same answer, but our statement remains true.)

As in Lemma 6.4 one also checks that, for large enough $t$, any maximally stretched loop $\sigma$ from $X$ to $Y_t$ (i.e. one that maximises 7) must minimise the ratio of $L_X(\sigma)$ over the number of occurrences of $e$ in $\sigma$, among all loops. Such a ratio is exactly the constant $C(e, X)$ introduced in Lemma 6.4.

Distances may then be computed, and we obtain an expression of the form,
where $a$ is the number of occurrences of $e$ in $\sigma$. Thus

$$L_X(\gamma) = \lim_{t \to \infty} \frac{e^{d(X,Z_t)}}{t^2} = C(e, X) \lim_{t \to \infty} \frac{e^{d(X,Y_t)}}{t^2}$$

Since $L_X(\gamma_1)$ and $L_X(\gamma_2)$, are known, we just need to determine $C(e, X)$, which is given by Lemma 6.5 in terms of asymptotic distances. Namely, if $(Z_i)$ is the sequence of points given by Lemma 6.5 for computing the length of $\gamma_1$ we get $L_X(\gamma_1) = C(e, X) \lim_{t \to \infty} e^{d(X,Z_i)} / t^2$.

If one likes exact formulae, one would have to introduce sequences $(Z^1_i)$ and $(Q^1_i)$, given by Lemma 6.5 for the ratio $L_X(\gamma_1)/L_X(\gamma_0)$; then look at sequences $(Z^2_i)$ and $(Q^2_i)$, given by Proposition 6.4 for the ratio $L_X(\gamma_2)/L_X(\gamma_0)$, and get (remembering the normalisation $L_X(\gamma_0) = 1$, and noting that the edge $e$ may occur in $\gamma_0$ so that all the sequences below may be different)

$$L_X(\gamma) = \lim_{t \to \infty} \frac{e^{d(X,Y_i)}}{e^{d(X,Z_i)}} \frac{e^{d(X,Z^1_i)}}{e^{d(X,Q^1_i)}} \frac{e^{d(X,Z^2_i)}}{e^{d(X,Q^2_i)}}$$

Finally, we are able to deal with global isometries of $CV_n$, proving that isometries are determined by their restrictions to rose-simplices.

**Theorem 6.9.** The only isometry of $CV_n$ that fixes all rose-simplices is the identity.

**Proof.** Let $\Phi$ be such an isometry. Let $X$ be a point of a simplex $\Delta_1$ of $CV_n$, and let $\Delta_2 = \Phi(\Delta_1)$. By Lemma 6.1, $\Delta_1$ and $\Delta_2$ share their rose-faces. By Theorem 6.2 simple loops in $\Delta_1$ are also simple in $\Delta_2$. In particular we can choose a simple loop $\gamma_0$ and consider representatives of metric graphs of $\Delta_1$ and $\Delta_2$ by imposing that the length of $\gamma_0$ is 1.

By Theorem 6.2 $\Delta_1$ and $\Delta_2$ have the same almost simple closed curves. Since $\Phi$ fixes points in rose simplices, for any $Y$ in a rose-face of $\Delta_1$, we have $d(X,Y) = d(\Phi(X), \Phi(Y)) = d(\Phi(X), Y)$. Then, Theorem 6.4 says that the lengths of almost simple closed curves are the same in $X$ and $\Phi(X)$. Therefore, the Sausages Lemma 2.16 implies $X = \Phi(X)$ (whence $\Delta_1 = \Delta_2$).

7. **ISOMETRIES OF MULTI-THETA SIMPLICES AND THEIR EXTENSIONS**

Recall that our main result is that isometries of Outer Space are all induced by automorphisms of the free group. By Theorem 6.9 it is enough to show that up to composing with automorphisms, we can
reduce to the case of isometries that point-wise fix every rose-simplex, and we do that by studying isometries of multi-theta simplices.

Our first main result of this section is that isometries of multi-theta simplices are induced by permutations of edges. Thus we have no translations or inversions as in rose-simplices. In particular this also shows that translations and inversions of rose-simplices cannot arise as restrictions of global isometries of $CV_n$.

Then, we will prove that situation is in fact even more rigid. Indeed, we show that if two isometries coincide on a multi-theta simplex, then they coincide on all rose-simplices of $CV_n$ (not only on faces of that simplex). This will basically conclude Theorem 1.1.

We start by proving following theorem.

**Theorem 7.1.** Let $\Delta$ be a multi-theta simplex, and let $\Phi$ be an isometry of $\Delta$. Then $\Phi$ fixes the centre of $\Delta$ (recall definition 2.9). Moreover, if $\Phi$ leaves invariant all the rose-faces of $\Delta$, then it actually fixes them point-wise, and in that case $\Phi$ is the identity map on $\Delta$.

Before proving Theorem 7.1 we need to establish some preliminary technical lemmas. We follow the strategy sketched in schema of Section 3, focusing on the study of those pairs of points that are joined by a unique geodesic. We recall that Outer Space is not a geodesic space; nevertheless, in any simplex, segments (for the linear structure of the simplex) are geodesic. More precisely, if we are in $CV_n$, then the points within a multi-theta simplex $\Delta$ are specified by $n+1$ positive reals (giving an open $n$-simplex, since one further needs to projectivise), corresponding to the lengths of the $n+1$ edges. Then, given $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$ we can consider the segment $xy := (1-t)x + ty$ in $\Delta$. This turns out to be a geodesic with respect to the symmetric Lipschitz metric. See [11] for details and proofs.

However geodesics, even within a given simplex, are in general not unique. Our strategy is broadly to determine sufficiently many “unique” geodesics.

**Definition 7.2.** A geodesic segment $\sigma$ of $CV_n$ is rigid if for any two points on it, the restriction of $\sigma$ is the unique (unparameterised) geodesic joining them.

We fix now a multi-theta simplex $\Delta$, and we denote by $e_0, \ldots, e_n$ the (oriented) edges of underlying graph of $\Delta$. Any point $x$ in $\Delta$ is thus determined by lengths $L_x(e_i)$ of $e_i$ in $x$. As usual, we denote by $\bar{e}_i$ the edge $e_i$ with the inverse orientation.

We begin by describing a set of standard rigid geodesics of $\Delta$. 
**Lemma 7.3** (Standard rigid geodesics). Let $x \neq y$ be metric graphs in $\Delta$. For any $i$ let $\lambda_i$ be the stretching factor of $e_i$ from $x$ to $y$:

$$\lambda_i = \frac{L_y(e_i)}{L_x(e_i)}$$

If the set of such stretching factors contains exactly two elements, none of them with multiplicity 2, then the segment between $x$ and $y$ is rigid.

**Proof.** This is a consequence of Lemma 2.16 and Lemma 2.17.
Indeed, up to rearranging edges, we can suppose $\lambda_0 = \cdots = \lambda_k = \mu$ and $\lambda_{k+1} = \cdots = \lambda_n = \lambda$ for two numbers $\mu < \lambda$. By scaling the graph $y$ by $\mu$, we may reduce to the case where $\lambda_0 = \cdots = \lambda_k = 1 < \lambda_{k+1} = \cdots = \lambda_n = \lambda$. In particular, we have scaled $y$ so that the edges $e_0, \ldots, e_k$ have the same length in both $x$ and $y$.

Let $z$ be a point in a geodesic joining $x$ and $y$. We claim that, up possibly to scaling, the edges $e_0, \ldots, e_k$ are not stretched from $x$ to $z$, while the edges $e_{k+1}, \ldots, e_n$ are stretched all by the same amount between 1 and $\lambda$. That is to say, we scale $z$ so that the length of $e_0$ in $z$ is equal to the length of $e_0$ in both $x$ and $y$. Now we claim that if $z$ belongs to a geodesic joining $x$ and $y$, then it belongs to the segment between $x$ and $y$, which therefore is rigid.

Let us examine our claim. First, suppose $k > 1$. Then, the loops $e_i \tilde{e}_j$ with $i, j \leq k$ are minimally stretched from $x$ to $y$. Thus, by Lemma 2.17 the same must be true from $x$ to $z$. In particular all such loops are stretched the same from $x$ to $z$. As we have at least three such loops (because $k > 1$) this implies that the edge-stretching factors $L_z(e_0)/L_x(e_0), \ldots, L_z(e_k)/L_x(e_k)$ all coincide.

This fact is also trivially true if $k = 0$, while the case $k = 1$ is impossible because the multiplicity of $\mu$ was supposed different from 2. So, up possibly to scaling, the edges $e_0, \ldots, e_k$ are not stretched from $x$ to $z$ (they have the same length in each metric graph).

The same argument, now with maximally stretched loops, shows that edges $e_{k+1}, \ldots, e_n$ are all stretched the same amount (as above, $k \neq n - 2$ because the multiplicity of $\lambda$ is not 2) and by an amount which is between 1 and $\lambda$.

Note that rigid segments of type just described, always emanate from any point $x$ of $\Delta$. Indeed it suffices to consider a set $I$ of edges and consider a point $y$ whose edge-lengths equal those of $x$ for edges in $I$ and, say, double those of $x$ for remaining edges. As above, this will be a rigid geodesic which obviously extends to a rigid geodesic ray. This is why we call such geodesic “standard”.

One can think these geodesics as being a standard set in the tangent space at $x$. Our objective now is to see that points of $\Delta$ can have more rigid geodesics emanating from them, and that such a set of “rigid” directions is minimal when $x$ is the centre of $\Delta$. We notice that this
is a substantial difference with respect to case of rose-simplices, which are homogeneous as there is transitive action of translations.

**Lemma 7.4** (Rigid geodesics from the centre, in the case of rank at least 3). Suppose $x$ is the centre of $\Delta$. If $n \geq 3$, then any rigid geodesic through $x$ is of the type described in Lemma 7.3.

*Proof.* We scale $x$ so that its edges have length one. We have to show that for any point $y$, if the segment $xy$ is rigid, then the set of edge-stretching factors contains exactly two elements, none of them with multiplicity two.

Suppose first that we have two edge-stretching factors, one of them with multiplicity two. Up to scaling $y$ and rearranging edges, we can suppose that the stretching factors of edges $e_i$ are $1$ for $i = 0, \ldots, n - 2$ and $\lambda$ for $i = n - 1, n$. We show that in that case the segment from $x$ to $y$ is not rigid.

Without loss of generality we can suppose $\lambda > 1$. Let $z$ be the middle point of such segments, that is to say

$$1 = L_z(e_0) = \cdots = L_z(e_{n-2}) \quad L_z(e_{n-1}) = L_z(e_n) = \frac{1 + \lambda}{2}.$$

Since $\lambda > 1$, the loops $e_i \overline{e}_j$ with $i, j < n - 1$ (whose existence is guaranteed because $n \geq 3$) are minimally stretched, and $e_{n-1} \overline{e}_n$ is maximally stretched, both from $x$ to $y$, form $x$ to $z$ and from $z$ to $y$.

Moreover, since the inequalities in play are all strict, the same remains true if we slightly perturb the length of $e_n$ (note that maximally and minimally stretched loops have no common edges). That is to say, if $z_\varepsilon$ denote the graph whose edge-lengths equal those of $z$ except for $e_n$, for which we set $L_{z_\varepsilon}(e_n) = L_z(e_n) + \varepsilon$, for small enough $\varepsilon$, it is still true that loops $e_i \overline{e}_j$ with $i, j < n - 1$ are minimally stretched, and $e_{n-1} \overline{e}_n$ is maximally stretched, both from $x$ to $y$, form $x$ to $z_\varepsilon$ and from $z_\varepsilon$ to $y$. This implies

$$d(x, z_\varepsilon) + d(z_\varepsilon, y) = d(x, y).$$

Thus, as segments are geodesics, the union $\sigma_\varepsilon$ of segments $xz_\varepsilon$ and $z_\varepsilon y$ is a geodesic between $x$ and $y$. On the other hand it is clear that $z_\varepsilon$ does not belong to segment $xy$, so $\sigma_\varepsilon$ is different from $xy$ which is therefore not rigid.

It now remains to show that if we have at least three different stretching factors, then we can find a geodesic between $x$ and $y$ which is not a segment. As above, we can scale $y$, and rearrange edges so that $1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$.

Since $L_{\lambda_i}(e_i) = 1$ for all $i$, the minimally stretched loops from $x$ to $y$ are all the $e_i \overline{e}_j$ for which $\lambda_i = \lambda_0 = 1$ and $\lambda_j = \lambda_1$, and maximally stretched ones are those $e_i \overline{e}_j$ for which $\lambda_i = \lambda_{n-1}$ and $\lambda_j = \lambda_n$.

Let $z$ be the middle point of the segment from $x$ to $y$. Let $\lambda \in \{\lambda_i\}$ be an edge-stretching factor such that $1 \neq \lambda \neq \lambda_n$. Let $z_\varepsilon$ be a
metric graph whose edge-lengths equal those of \( z \), except that for edges stretched by \( \lambda \), for which differ by \( \varepsilon \)

\[
L_{z\varepsilon}(e_i) = \begin{cases} 
L_z(e_i) & \lambda_i \neq \lambda \\
L_z(e_i) + \varepsilon & \lambda_i = \lambda 
\end{cases}
\]

and let \( \sigma_{\varepsilon} \) be the union of segments \( \overline{xz\varepsilon} \) and \( \overline{z\varepsilon y} \).

It is clear — because we have at least three stretching factors — that \( z\varepsilon \) does not belong to the segment \( \overline{xy} \), whence \( \sigma_{\varepsilon} \neq \overline{xy} \). If we show that \( \sigma_{\varepsilon} \) is a geodesic we are done. As above, it is enough to show that

\[
d(x, z\varepsilon) + d(z\varepsilon, y) = d(x, y).
\]

For that, we have to prove that there are loops \( \gamma_0 \) and \( \gamma_1 \) that are respectively minimally and maximally stretched from \( x \) to \( z\varepsilon \) and from \( z\varepsilon \) to \( y \). This easily follows, for small enough \( \varepsilon \), by the choice of \( \lambda \). Indeed, it suffices (since the other cases are easier) to look at the situation when the stretching factors are \( 1, \lambda, \ldots, \lambda, \lambda_n \). Here, min. and max. loops-stretching factors form \( x \) to \( y \) are \((1 + \lambda)/2 \) and \((\lambda + \lambda_n)/2 \), realised by \( e_0\overline{e_i} \) and \( e_i\overline{e_n} \) for \( i = 1, \ldots, n \). Such loops are therefore min and max stretched both from \( x \) to \( z \) and from \( z \) to \( y \), and perturbing \( \lambda \) a little such loops remain min. and max. stretched.

Now, we show how Lemma 7.4 provides (in rank bigger than two) a metric characterisation of the centre of \( \Delta \) as the point having the minimum number of rigid geodesics passing through it.

**Lemma 7.5.** For any point \( x \) other than centre of \( \Delta \), there is at least one rigid geodesic emanating from \( x \) which is not of the type described in Lemma 7.3.

**Proof.** We denote by \( x_i \) the lengths \( L_x(e_i) \). Up to scaling \( x \) and rearranging edges, we can suppose that

\[
1 = x_0 \geq x_1 \geq \cdots \geq x_n.
\]

We want to find stretching factors

\[
1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n
\]

at least three of them being different, such that segment between \( x \) and point \( y \) corresponding to graph whose edges have length \( \lambda_ix_i \), is rigid. As three of the \( \lambda_i \) are different, this will prove the lemma.

Let us start by making the simplifying assumption that \( x_n \neq x_1 \).

Stretching factors, from \( x \) to \( y \), of loops \( e_i\overline{e_j} \) are \( \frac{\lambda_i x_i + \lambda_j x_j}{x_i + x_j} \), and if

\[
1 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n,
\]

an immediate calculation shows that whenever \( j \geq i \) we have

\[
\frac{1 + \lambda_i x_i}{1 + x_i} \leq \frac{\lambda_i x_i + \lambda_j x_j}{x_i + x_j} \leq \frac{\lambda_j x_j + \lambda_n x_n}{x_j + x_n}.
\]
This implies that if we are searching for minimally (respectively maxi-
mally) stretched loops, we can restrict to loops of the form $e_0 \bar{e}_i$ (re-
spectively $e_i \bar{e}_n$.)

The idea is now to force such loops to have the same stretching factors. We impose conditions

$$\lambda_1 = \frac{1 + x_1}{x_1}$$

and, for $i > 0$

$$\frac{1 + \lambda_i x_i}{1 + x_i} = \frac{1 + \lambda_1 x_1}{1 + x_1} = \frac{2 + x_1}{1 + x_1}$$ (8)

We remark that the assumption on $\lambda_1$ is for simplifying calculations, we only need $\lambda_1 > 1$.

We can solve these equations getting

$$\lambda_i = \frac{(2 + x_1)(1 + x_i)}{(1 + x_1)x_i} - \frac{1}{x_i} = 1 + \frac{1 + x_i}{(1 + x_1)x_i}$$

thus $\lambda_i \geq \lambda_1$, with equality if and only if $x_i = x_1$, and $\lambda_i \leq \lambda_j$ for $j \geq i$, with equality if and only if $x_i = x_j$. In particular, under our simplifying assumption, we have $\lambda_0 = 1 < \lambda_1 < \lambda_n$, so at least three of the $\lambda_i$’s are different.

So we get numbers $\lambda_i$’s with the requested properties. Now, let $y$ be the point of $\Delta$ given by

$$L_y(e_i) = \lambda_i x_i$$

and let $z$ be any point in a geodesic between $x$ and $y$, scaled so that

$$L_z(e_0) = 1.$$ We define $\mu_i$ by

$$L_z(e_i) = \mu_i x_i.$$

Loops $e_0 \bar{e}_i$ are minimally stretched from $x$ to $y$. Thus, we must have that such loops are minimally stretched from $x$ to $z$ and from $z$ to $y$. This forces the edge-stretching factors $\mu_i$ to satisfy condition (8), which allows us to obtain $\mu_i$ as a function of $\mu_1$ exactly as $\lambda_i$ is obtained from $\lambda_1$. This implies that, if $z'$ is the point in the geodesic line between $x$ and $y$ with first edge-stretching factor equal to $\mu_1$, we have that $z = z'$.

So that $z$ belongs the segment $\overline{xy}$ which is hence rigid, and not of the type described in Lemma 7.3.

We are now left with the case in which $x_n = x_1$ so $x_i = x_j$ for any $i, j \neq 0$. As we are supposing that $x$ is not the centre of $\Delta$, we must have $x_0 \neq x_1$. Up to scaling $x$ and rearranging edges, this case is equivalent to

$$(x_0, \ldots, x_n) = (1, \ldots, 1, c)$$

with $c > 1$.

We choose $y$ of the form

$$y = (1, \lambda, \ldots, \lambda, \mu c)$$
Stretching factors of simple loops are
\[
\frac{1 + \lambda}{2}, \quad \frac{1 + \mu c}{1 + c}, \quad \lambda, \quad \frac{\lambda + \mu c}{1 + c}.
\]
Now, we impose conditions
\[
\mu c = 1, \quad \frac{1 + \lambda}{2} = \frac{1 + \mu c}{1 + c}
\]
which imply that \(\lambda \neq \mu\) because \(c \neq 1\), and \(\lambda < 1\). Whence
\[
\frac{1 + \lambda}{2} = \frac{1 + \mu c}{1 + c} > \max(\lambda, \frac{\lambda + \mu c}{1 + c}).
\]

So all the loops \(e_0 \vec{e}_i\) are maximally stretched from \(x\) to \(y\) (and in particular, stretched by the same amount). Now we argue as before: the same must be true for any point \(z\) on any geodesic from \(x\) to \(y\), and this forces \(z\) to be of the form (once scaled so that \(L_z(e_0) = 1\))
\[
z = (1, \bar{\lambda}, \ldots, \bar{\lambda}, \bar{\mu} c)
\]
with
\[
\frac{1 + \bar{\lambda}}{2} = \frac{1 + \bar{\mu} c}{1 + c}.
\]

As above, this implies that \(z\) belongs to the segment \(xy\), which is then rigid and it is not of the type described in Lemma 7.3 because \(1 \neq \lambda \neq \mu \neq 1\).

**Lemma 7.6 (Rigid geodesics in rank two).** Let \(x \neq y\) be two marked metric graphs in \(\Delta\). Suppose \(n = 2\), so that \(\Delta\) has exactly three different (unoriented) simple loops. Then the segment \(xy\) is rigid if and only if two of the three simple loops are stretched the same from \(x\) to \(y\).

**Proof.** The proof uses same arguments of higher rank case, but takes into account the peculiarities of rank two.

If the three simple loops are stretched by three different factors, then for any point \(w\) close enough to the middle point \(z\) of \(xy\), the maximally and minimally stretched loops do not change from \(x\) to \(w\) from \(w\) to \(y\) and from \(z\) to \(y\). So that \(xy\) is not rigid.

On the other hand, if two simple loops are stretched by the same factor, we may rearrange the edges so that \(e_0\) is the edge shared by such loops, and scale graphs so that \(L_x(e_0) = L_y(e_0) = 1\). Moreover, as we have only three simple loops, \(e_0 \vec{e}_1\) and \(e_0 \vec{e}_2\) are either maximally or minimally stretched from \(x\) to \(y\). So the same must be true from \(x\) to \(z\) and from \(z\) to \(y\) for any point \(z\) in a geodesic between \(x\) and \(y\). If \(x = (1, a, b)\) and \(y = (1, \lambda a, \mu b)\), we have
\[
\frac{1 + \lambda a}{1 + a} = \frac{1 + \mu b}{1 + b}
\]
and the same relation holds for the edge stretching factors of point \(z\) which therefore belongs to the segment \(xy\). \(\square\)
Lemma 7.7. For any rose-face of \(\Delta\) there is a unique rigid geodesic from the centre of \(\Delta\) to that face.

Proof. By Lemma 7.4 and 7.6, a rigid geodesic emanating from the centre is of the type described in Lemma 7.3 (and 7.6 in the rank-2 case). A rose-face corresponds to collapsing an edge, say \(e_0\). So in a rigid geodesic from the centre to that face we have \(\lambda_0 = t\) and \(\lambda_i = 1\) for \(i > 0\), with \(t \in [1, 0]\). Therefore such geodesic is unique. \(\square\)

Now, we continue with proof of Theorem 7.1. We begin by examining the first claim in the rank-two case. Since permutations of edges of \(\Delta\) are isometries that fix its centre and permute its rose-faces, up to composing \(\Phi\) with such a permutation we can suppose that \(\Phi\) does not permute rose-faces of \(\Delta\). If the restriction of \(\Phi\) to a rose-face has a translational part, then for any point \(x\) in that face we see that the distance of \(\Phi^n(x)\) from at least one of the remaining two rose faces of \(\Delta\) goes to infinity, this being impossible because \(\Phi\) is an isometry. It follows that \(\Phi\) fixes the centres of rose-faces of \(\Delta\). Explicit calculations (using Lemma 7.6, see the Appendix) show that the centre of \(\Delta\) is the unique point which is joined to the centres of the three rose-faces by rigid geodesics. Thus \(\Phi\) fixes the centre of \(\Delta\), and the first claim of Theorem 7.1 is proved for \(n = 2\).

If \(n \geq 3\), Lemma 7.4 and Lemma 7.5 imply that any isometry \(\Phi\) of \(\Delta\) must fix its centre, so first claim of theorem is proved. Moreover, if \(\Phi\) does not permute rose-faces, then by Lemma 7.7 it must fix point-wise rigid geodesics emanating from \(x\) and going to rose-faces. In particular, \(\Phi\) fixes centres of rose-faces.

Remark 7.8. Note that we have proved that if \(\Phi\) does not permute rose-faces of \(\Delta\), then its restriction to any rose-face has no translational parts, which is to say that it fixes the centre of rose-face.

Therefore, by Theorem 5.1 restriction of \(\Phi\) to rose-faces of \(\Delta\) is an element of \(S_n \times \langle \sigma \rangle\). In the next lemma we show that such an element must be the identity. We first introduce some terminology.

Let \(R_i\) denote the rose-face of \(\Delta\) obtained by collapsing edge \(e_i\), and let \(C_i\) denote its centre. Also, for \(i \neq j\) we let \(\Gamma_j^i(\epsilon)\) denote the point of \(R_j\) all of whose petals have length 1 except for \(e_i\) which has length \(\epsilon\). For \(i \neq j\), straightforward calculations show

\[
(9) \quad d(C_i, \Gamma_j^k(2)) = \begin{cases} 
\log 6 & i = k \quad \text{in any rank} \\
\log 6 & i \neq k \quad \text{in rank bigger that 2} \\
\log 3 & i \neq k \quad \text{in rank 2}
\end{cases}
\]
\[
(10) \quad d(C_i, \Gamma_j^k(0.5)) = \begin{cases} 
\log 3 & i = k \text{ in any rank} \\
\log 8 & i \neq k \text{ in rank bigger than 2} \\
\log 6 & i \neq k \text{ in rank 2} 
\end{cases}
\]

Note that in the rank-2 case, for \( i \neq j \neq k \) we have \( \Gamma_j^k(2) = \Gamma_j(0.5) \), up to scaling.

**Lemma 7.9.** Let \( \Phi \) be an isometry of a multi-theta simplex \( \Delta \) which fixes the centres of its rose-faces. Then \( \Phi \) is the identity on each rose face.

**Proof.** By Theorem 5.1 the restriction of \( \Phi \) to any rose face is an element of \( S_n \times \langle \sigma \rangle \). Hence the image of the point \( \Gamma_j^i(2) \) is either \( \Gamma_j^k(2) \) or \( \Gamma_j^i(0.5) \) for some \( k \). However, by (9) and (10), since each \( C_i \) is fixed by \( \Phi \) the distances to \( \Gamma_j^i(2) \) are preserved and we must have that \( \Gamma_j^i(2) \) is actually fixed by \( \Phi \). Since this is true for every \( i \neq j \), and since the only element of \( S_n \times \langle \sigma \rangle \) which fixes all these is the identity, we get that \( \Phi \) restricts to the identity on any \( R_j \). \( \square \)

We can now finish proof of Theorem 7.1. We proved that any isometry of \( \Delta \) fixes it centre, and that if it does not permute rose-faces \( R_i \), then it fixes their centres \( C_i \). By Lemma 7.9 this implies that \( \Phi \) point-wise fixes rose-faces of \( \Delta \). Now let \( x \in \Delta \). For any \( y \) in some \( R_i \), we have \( d(x, y) = d(\Phi(x), y) \) (because \( R_i \) is fixed.) Therefore, Theorem 6.7 tells us that lengths of a.s.c.c. in \( x \) and \( \Phi(x) \) coincide. Thus, by Lemma 2.16 we have \( d(x, \Phi(x)) = 0 \). It follows that \( \Phi \) is the identity of \( \Delta \), and the proof of Theorem 7.1 is concluded. \( \square \)

We come now the other main result of this section, that is that for an isometry of \( CV_n \), what happens on a single multi-theta simplex determines the isometry on the whole \( CV_n \). The first step is to show that if an isometry of a multi-theta simplex is the identity on a rose-face, then it is the identity of the multi-theta simplex. Our claim will follow then by an argument of connection.

**Lemma 7.10.** Let \( \Phi \) be an isometry of a multi-theta simplex \( \Delta \) which restricts to the identity on one of the rose-face of \( \Delta \). Then \( \Phi \) restricts to the identity on each rose-face of \( \Delta \).

**Proof.** Let \( R_0 \) be the rose-face fixed by hypothesis. By Lemma 7.9 it is sufficient to show that \( \Phi \) fixes each centre \( C_i \). By first claim of Theorem 7.1 we know that the centre of \( \Delta \) is fixed. By Lemma 7.7 there is a unique rigid geodesic from the centre to each rose face, ending in \( C_i \). Hence, the \( C_i \) are permuted by \( \Phi \).

However, the stabiliser in \( Out(F_n) \) of \( \Delta \) contains a subgroup isomorphic to \( S_{n+1} \), by simply permuting the edges of the underlying graph of \( \Delta \), and this subgroup will induce every permutation of the \( n + 1 \)
rose-faces of $\Delta$. Hence, by Lemma 7.9, $\Phi$ is equal to the restriction of some element of $S_{n+1}$ (in fact, some such element which fixes the edge corresponding to the fixed rose-face). But the only element of this sort which restricts to the identity in a rose face is the identity. (This also follows from (9) and (10)). □

Theorem 7.11. Let $\Phi$ be an isometry of $CV_n$ that point-wise fixes a multi-theta simplex. Then it point-wise fixes all rose and multi-theta simplices of $CV_n$.

Proof. We start by doing a simple calculation. Let $\Delta$ be a multi-theta simplex of $CV_n$, with edges oriented and labelled $e_0, e_1, \ldots, e_n$. For an edge $e$, we denote by $\overline{e}$ the edge $e$ with inverse orientation. Let $R_0$ be a rose face of $\Delta$ obtained by collapsing $e_i$. We will label the edges of $R_i$, $e_0^i, e_1^i, \ldots, e_{i-1}^i, e_{i+1}^i, \ldots, e_n^i$.

Now let us explicitly write down the homotopy equivalences between $R_0$ and $R_i$ in terms of these edges. The map from $R_0$ to $R_i$ is given by the following,

\begin{align*}
  e_0^0 &\mapsto e_0^i, j \neq i \\
  e_i^0 &\mapsto e_i^0.
\end{align*}

Similarly, the map from $R_i$ to $R_0$ is given by,

\begin{align*}
  e_j^i &\mapsto e_j^0, j \neq 0 \\
  e_0^i &\mapsto e_0^0.
\end{align*}

This in particular implies that the sub-complex of $CV_n$ consisting of multi-theta and rose-simplices is connected, as we realised Nielsen automorphisms passing from a rose-face to another in a multi-theta simplices.

Now, it would seem that we are done simply by starting from our initial fixed multi-theta simplex and extending our results, via Lemma 7.10, over the whole of $CV_n$. The only problem is that we do not know, a priori, that $\Phi$ does not induce some non-trivial permutation of the multi-theta simplices. Therefore, we need to rule out this possibility.

Remark 7.12. The next Lemma is an “elementary” proof of the fact that permutations of multi-theta simplices do not occur. The calculations it involves are somewhat tedious and the reader may prefer to invoke the result of Bridson and Vogtmann [6] asserting that simplicial actions on the spine of $CV_n$ (see [6] for definitions and details) come from automorphisms (for $n \geq 3$). Then, she could show that isometries naturally induce such actions on the spine, and since the spine encodes the combinatoric of roses and multi-theta incidences, get the desired result. We present here the proof of Lemma 7.13 as follows because it is self-contained and more in the spirit of the techniques of the present work.
Lemma 7.13. Let $\Delta$ be a multi-theta simplex, $R$ a rose face of it. Suppose that $\Delta_1, \ldots, \Delta_k$ are all the other multi-theta simplices in $CV_n$ which are incident to $R$. Let $\Phi$ be an isometry of $CV_n$ which point-wise fixes $\Delta$ (and therefore $R$). Then $\Phi$ leaves each $\Delta_i$ invariant.

Proof. Consider our multi-theta simplex $\Delta$ which is given by a graph with 2 vertices and $n + 1$ edges, ordered and labelled $e_0, \ldots, e_{n+1}$. As usual, for an edge $e$ we denote by $\overline{e}$ the one with inverse orientation. Moreover, we chose orientations so that the $e_i$'s share the same initial vertex (so they also share the terminal vertex). We will let $R$ denote the rose simplex obtained by collapsing the edge $e_0$.

It is now an easy exercise to see that there are $2^{n-1}$ multi-theta simplices incident to $R$. Therefore, the result is trivial in $CV_2$ and we shall restrict our attention to $CV_n$ for $n \geq 3$.

We shall describe the set of multi-thetas incident to $R$ by listing the homotopy equivalences from $\Delta$. Specifically, choose some $I \subseteq \{1, \ldots, n\}$ and consider the homotopy equivalence on $\Delta$ given by,

\[
\begin{align*}
e_0 & \mapsto e_0 \\
e_i & \mapsto e_i, \ i \notin I \\
e_i & \mapsto e_0 \overline{e_i}, \ i \in I
\end{align*}
\]

It is then clear that the set of all multi-thetas incident to $R$ will be given by these maps. However, we note that replacing $I$ by its complement gives the same simplex, so we have counted each twice. From now we will make a choice between $I$ and $I^c$ so that $|I| \geq |I^c|$ (or $I = \emptyset$) — if $|I| = |I^c|$ the choice will be arbitrary. Hence if $I$ is not empty it will have at least two elements, and its complement will be non-empty. Let $\Delta_I$ denote the multi-theta simplex obtained via the map above. This gives us our $2^{n-1}$ multi-thetas, with $\Delta = \Delta_\emptyset$.

Now, we will show that the distances from $\Delta$ will determine the $\Delta_I$. Note that since we are dealing with multi-theta graphs, by the Sausages Lemma (2.10) the maximally and minimally stretched loops can be taken to be simple closed curves, which are straightforward to enumerate. Below, we present a list of curves. On the left side, we have curves in $\Delta$ and on the right side their image in $\Delta_I$ so that each simple closed curve in either $\Delta$ or $\Delta_I$ appears somewhere on the list (up to orientation). Throughout, we have that $i, j \neq 0$.

\[
\begin{align*}
\Delta & \quad \Delta_I \\
e_i e_0 & \mapsto e_i e_0, \ i \notin I \\
e_i e_0 & \mapsto e_0 \overline{e_i}, \ i \in I \\
e_i \overline{e_j} & \mapsto e_i \overline{e_j}, \ i, j \notin I \\
& \quad \mapsto e_0 \overline{e_i e_j e_0}, \ i, j \in I \\
& \quad \mapsto e_0 \overline{e_i e_0 e_j}, \ i \in I, j \notin I \\
\overline{e_i} e_0 \overline{e_j} e_0 & \mapsto e_i \overline{e_j}, \ i \in I, j \notin I
\end{align*}
\]
Now let us assign edge lengths and calculate distances. For each $\Delta_I \neq \Delta$, we will let all edge lengths equal 1, since we know that isometries preserve the centres. Next choose some $J \subseteq \{1, \ldots, n\}$ and let $\Delta(J, 1/3)$ be the graph $\Delta$ where each edge has length 1 except for the $e_j$ which has length $1/3$ for all $j \in J$. Moreover, let us stipulate that $J \neq \emptyset, \{1, \ldots, n\}$. It is then an easy exercise to check the stretching factors for each of the simple loops in $\Delta(J, 1/3)$ and $\Delta_I$. Clearly, this depends on the relationship between $J$ and $I$. We list below, the possible stretching factors between $\Delta(J, 1/3)$ and $\Delta_I$, with the condition which allows it. Some stretching factors can occur in more than one way, in which case we have removed the redundancy (an empty condition means the stretching factor is always realisable).

So distance is computed by taking the log of the ratio of the maximum over the minimum of the allowed factors. Recall that by the choice we made for $I$, we always have $|I| \geq 2$ and $|I^c| \geq 1$.

| Stretching factor | Condition |
|-------------------|-----------|
| 1                 |           |
| $3/2$             |           |
| 3                 | $|I \cap J| \geq 2$ or $|I^c \cap J| \geq 2$ |
| 2                 | $|I \cap J| \geq 1$ and $|I^c \cap J^c| \geq 1$ |
| 3                 | $|I \cap J| \geq 1$ and $|I^c \cap J^c| \geq 1$ |
| 3                 | $|I \cap J^c| \geq 1$ and $|I^c \cap J| \geq 1$ |
| 6                 | $|I \cap J| \geq 1$ and $|I^c \cap J^c| \geq 1$ |
| $1/2$             | $|I \cap J^c| \geq 1$ and $|I^c \cap J| \geq 1$ |
| $3/5$             | $|I \cap J| \geq 1$ and $|I^c \cap J^c| \geq 1$ |
| $3/5$             | $|I \cap J^c| \geq 1$ and $|I^c \cap J| \geq 1$ |
| $3/4$             | $|I \cap J| \geq 1$ and $|I^c \cap J^c| \geq 1$ |

We now apply these conditions to calculate the distances from $\Delta(J, 1/3)$ to $\Delta_I$ when $J$ has exactly 2 elements. Specifically,

- If $|J| = 2$, $|J \cap I| = |J \cap I^c| = 1$, and $|I^c| \geq 2$, then maximal and minimal stretching factors are 6 and $1/2$, so the distance is log 12.
- If $|J| = 2$, $|J \cap I| = |J \cap I^c| = 1$, and $|I^c| = 1$, then the max and min stretching factors are 6 and $3/5$, whence the distance is log 10.
- If $|J| = 2$ and $J \subseteq I$ or $J \subseteq I^c$, then the maximal stretching factor is always 3, and the distance is log 5 or log 6, depending on the sizes of $I$ and $I^c$.

Hence, we may determine $I$ and $I^c$. More precisely, the set $\{1\} \cup \{i \neq 1 : d(\Delta(\{1, i\}, 1/3), \Delta_I) = \log 5\} \cup \{i \neq 1 : d(\Delta(\{1, i\}, 1/3), \Delta_I) = \log 6\}$ is equal to either $I$ or $I^c$.

Note that this doesn’t let us distinguish which one we picked, but since $\Delta_I$ only depended on the pair $I, I^c$, this is sufficient to distinguish the simplex and proves Lemma 7.13.

\[\square\]
Now Theorem 7.11 follows.

8. Proof of Theorem 1.1 and main results

We prove here results stated in Section 1.

Proof of Theorem 1.1. Our claim is that the isometry-group of outer space of rank-\(n\) free group, is just \(\text{Out}(F_n)\) (for \(n \geq 3\)). Clearly, \(\text{Out}(F_n)\) acts faithfully on \(CV_n\) for \(n \geq 3\) and this action is by isometries (see for instance [11]). Thus, we have an inclusion of \(\text{Out}(F_n)\) into the group of isometries of \(CV_n\). For \(n = 2\), we still have an isometric action, but this is no longer faithful. However, up to this small kernel (a group of order 2 consisting of the identity and the automorphism which inverts each basis element), we still have a map from \(\text{Out}(F_2)\) to the isometry group of \(CV_2\).

Our goal is to show that this exhausts the isometry group of \(CV_n\) (in either case).

Let \(\Phi\) be an isometry of \(CV_n\). We shall compose \(\Phi\) with elements of \(\text{Out}(F_n)\) until we obtain the identity.

By Lemma 4.2, \(\Phi\) maps multi-theta simplices to multi-theta simplices. Therefore, since the action of \(\text{Out}(F_n)\) on multi-theta simplices is transitive, we may suppose that \(\Phi\) leaves invariant a multi-theta simplex \(\Delta\). In fact, the stabiliser in \(\text{Out}(F_n)\) of \(\Delta\) will induce any permutation of the \(n + 1\) rose faces of \(\Delta\) and so we may also assume that \(\Phi\) leaves both \(\Delta\) and every rose-face of \(\Delta\) invariant.

Theorem 7.1 then implies that \(\Phi\) is the identity of \(\Delta\). Then, by Theorem 7.11 \(\Phi\) point-wise fixes all rose-simplices. And Theorem 6.9 implies that \(\Phi\) is the identity.

Proof of Theorem 1.2. Let us denote by \(\text{Isom}_R(CV_n)\) the group of isometries of \(CV_n\) for the non-symmetric metric \(d_R\), and by \(\text{Isom}(CV_n)\) the group of isometries of \(CV_n\) for the symmetric metric \(d\).

Let \(\Phi\) be an isometry of \(CV_n\) for \(d_R\). Then

\[
\forall x, y \quad d_R(x, y) = d_R(\Phi x, \Phi y).
\]

Since \(d_L(x, y) = d_R(y, x)\), we have that \(\Phi\) is also an isometry for \(d_L\), whence \(\Phi\) is an isometry for the symmetric Lipschitz metric \(d\). Thus, \(\text{Isom}_R(CV_n) \subseteq \text{Isom}(CV_n)\).

As for the symmetric case, one has that \(\text{Out}(F_n) \subseteq \text{Isom}_R(CV_n)\) (with a small adjustment for rank 2). By Theorem 1.1 we have that \(\text{Isom}(CV_n) = \text{Out}(F_n)\). Thus

\[
\text{Out}(F_n) \subseteq \text{Isom}_R(CV_n) \subseteq \text{Isom}(CV_n) = \text{Out}(F_n).
\]

The same for \(d_L\). (The argument for rank 2 is the same.)
Proof of Corollary 1.3. Every homomorphism from $\Gamma$ to $Out(F_n)$ has finite image by [5], and every finite subgroup of $Out(F_n)$ has a fixed point in its action on $CV_n$ by [7]. □

Appendix A. Rigid geodesics in rank two

Here we explicitly calculate rigid geodesics emanating from centres of rose-simplices and pointing into theta-simplices, for the rank-two case. Showing that for any theta-simplex, its centre is the unique point simultaneously joined to centres of all rose-faces by rigid geodesics.

We fix a theta-simplex $\Delta$ and we parametrise its points by (projective classes of) triples of positive numbers $(x, y, z)$. Such simplex is a triangle with vertices removed, as can be seen by taking representatives unitary volume.

Let $(1, 0, 1)$ be the centre of a rose face of $\Delta$ and let $(1, z, y)$ be a point joined to it by a rigid segment, scaled so that $x = 1$. Stretching factors are

$$1 + z, z + y, \frac{1 + y}{2}$$

(the loop with stretching factor $z + (1 + y)/2$ is not relevant)

By Lemma 7.6 we must have only two stretching factors from $(1, 0, 1)$ to $(1, z, y)$. Possible cases are $1 + z = z + y, \frac{1 + y}{2}, z + y = \frac{1 + y}{2}$.

If

$$1 + z = z + y$$

then $y = 1$ and this is the rigid geodesic going to the centre of $\Delta$. If

$$1 + z = \frac{1 + y}{2}$$

then $y = 1 + 2z$, then $(1, y, z) = (1, z, 1 + 2z)$. We want to know where such geodesic hits other rose-faces. Letting $z \to \infty$ and scaling by $z$ we get $(1/z, 1, 2 + 1/z)$ which ends up to the point $(0, 1, 2)$. Finally,

$$z + y = \frac{1 + y}{2}$$

gives $z = (1 - y)/2$, so that $(1, y, z) = (1, (1 - y)/2, y)$. Letting $y \to 0$ we get $(1, 0.5, 0)$. The picture of rigid geodesics through the centres is therefore as follows
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