ELECTROSTATIC SYSTEM WITH DIVERGENCE-FREE BACH TENSOR AND NON-NULL COSMOLOGICAL CONSTANT

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Abstract. We prove that three-dimensional electrostatic manifolds with divergence-free Bach tensor are locally conformally flat, provide that the electric field and the gradient of the lapse function are linearly dependent. Consequently, a three-dimensional electrostatic manifold admits a local warped product structure with a one-dimensional base and a constant curvature surface fiber.

1. Introduction and main results

In this paper, we will consider the following system (cf. [6, 8, 9, 10, 12, 14] and the references therein).

Definition 1.1. Let $(M^3, g)$ be a Riemannian manifold with $E$ a tangent vector field on $M$ and $f \in C^\infty(M)$ satisfying

\begin{align}
\nabla^2 f &= f(Ric - \Lambda g + 2E^\flat \otimes E^\flat - |E|^2 g), \\
\Delta f &= (|E|^2 - \Lambda)f, \quad 0 = \text{div} E \quad \text{and} \quad 0 = \text{curl}(fE).
\end{align}

(1.1)

Here, $\text{Ric}$, $\nabla^2$, $\text{div}$ and $\Delta$ stand for the Ricci tensor, Hessian tensor, divergence and Laplacian operator with respect to the metric $g$, respectively. Moreover, $E^\flat$ is the one-form metrically dual to $E$. We refer to the above equations as electrostatic system with cosmological constant $\Lambda$ for the electrostatic spacetime associated to $(M^3, g, f, E)$.

Remember, the curl stands for an operator that describes the circulation (or rotation) of a vector field. Thus, we have $\text{curl}(fE) = 0$ if and only if

\begin{equation}
\ df \wedge E^\flat + f dE^\flat = 0.
\end{equation}

(1.2)

The smooth function $f$ is called the lapse function, the field $E$ is known as electric field and $M^3$ is the spatial factor for the static electrostatic spacetime. Moreover, $f > 0$ on $M$. If $M$ has boundary $\partial M$, we assume in addition that $f^{-1}(0) = \partial M$ (cf. [8, 9, 10, 12]).

Note that taking the contraction of the first equation and combining it with the Laplacian of $f$ in (1.1), we obtain an useful equation that relates the scalar curvature $R$, the cosmological constant, and the electric field:

\begin{equation}
R = 2(|E|^2 + \Lambda).
\end{equation}

(1.3)

Furthermore, the system (1.1) implies that the electric field and the gradient of the lapse function are linearly dependent on $\partial M = f^{-1}(0)$ (see [10] Lemma 4]).
There are some well-known examples of solutions for the electrostatic system, and we recommend seeing [10] Section 3 for a good overview. For instance, the charged Nariai system is an 3-dimensional space \([0, \frac{\pi}{2}] \times S^2\) with metric tensor 
\[ g = dr^2 + \varphi^2 g_{S^2}, \]
where \(\varphi\) is a constant and \(g_{S^2}\) is the standard metric of the sphere \(S^2\) of radius 1. The electric field and the lapse function are given by

\[ E = \frac{q}{\varphi^2} \partial_r \text{ and } f(r(x)) = \sin(\alpha r(x)), \]

where \(r(x)^2 = x_1^2 + x_2^2 + x_3^2\) such that \((x_1, x_2, x_3)\) are Cartesian coordinates, \(\alpha = \sqrt{\Lambda - \frac{2m}{\varphi^2}}\) and \(\frac{1}{r^2} < \varphi^2 < \frac{1}{\Lambda}\). Moreover, \(0 < m^2 = \frac{1}{16\Lambda} \left[ 1 + 12q^2\Lambda + \sqrt{(1 - 4q^2\Lambda)^3} \right]\) and \(0 < |q| \leq \varphi^2\sqrt{\Lambda}\). It is important to point out that the charged Nariai system is locally conformally flat (see [2] Corollary 1.34). In this work, is also important to remember the cold black hole system and the ultracold black hole system. They also locally conformally flat standard electrostatic models.

The cold black hole is an 3-dimensional space \([0, \infty) \times S^2\) with metric tensor 
\[ g = dr^2 + \varphi^2 g_{S^2}, \]
where \(\varphi\) is a constant and \(g_{S^2}\) is the standard metric of the sphere \(S^2\) of radius 1. The electric field and the lapse function are given by

\[ E = \frac{q}{\varphi^2} \partial_r \text{ and } f(r(x)) = \sinh(\beta r(x)), \]

where \(r(x)^2 = x_1^2 + x_2^2 + x_3^2\), \(\beta = \sqrt{\frac{2m}{\varphi^2} - \Lambda}\) and \(0 < \varphi^2 < \frac{1}{\Lambda}\). Moreover, \(0 < m^2 = \frac{1}{16\Lambda} \left[ 1 + 12q^2\Lambda + \sqrt{(1 - 4q^2\Lambda)^3} \right]\) and \(\varphi^2\sqrt{\Lambda} \leq |q|\).

The ultracold black hole is an 3-dimensional space \([0, \infty) \times S^2\) with metric tensor 
\[ g = dr^2 + \varphi^2 g_{S^2}, \]
where \(\varphi^2 = \frac{1}{4\Lambda} = q^2\). The electric field and the lapse function are given by

\[ E = \sqrt{\Lambda} \partial_r \text{ and } f(r) = r, \]

Moreover, \(m = \frac{1}{3} \sqrt{\frac{2}{\Lambda}}\).

The Reissner-Nordström-de Sitter (RNdS) manifold is an important electrostatic system \((M^3, g, f, E)\) where 
\[ M = [r_+, r_c] \times S^2 \]
to some positive constants \(r_+\) and \(r_c\) which are solutions of the lapse function given by

\[ f = \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda r^2}{3} \right)^{\frac{1}{2}}. \]

Also, it is possible to extend \([r_+, r_c]\) to the entire real line. The metric and the electric field are given by

\[ g = f(r)^{-2} dr^2 + r^2 g_{S^2}, \quad E = \frac{q}{r^2} f(r) \partial r. \]

In above, \(q, m\) and \(g_{S^2}\) stand for the charge, mass and the standard metric of unit sphere \(S^2\), respectively. Since the cosmological constant \(\Lambda\) is positive, from (1.3) we can see that \(R > 0\).

The RNdS solution can be rewritten in cosmological coordinates. For instance, the Kastor-Traschen solution represents a \(N\) charge-equal-to-mass, i.e., \(m = |q|\), black holes in a spacetime with a positive cosmological constant \(\Lambda\):

\[ ds^2 = -W^{-2} dt^2 + W^2 (dx_1^2 + dx_2^2 + dx_3^2), \]
where $W = -\sqrt{\frac{\Lambda}{3}} t + \sum_{i=1}^{N} \frac{m_i}{r_i}$. Here, $m_i$ stands for the black hole masses. Moreover, 
$r_i(x) = \sqrt{(x_1 - a_i)^2 + (x_2 - b_i)^2 + (x_3 - c_i)^2}$ is the distance from a fixed point $(a_i, b_i, c_i)$. It is interesting to point out that this solution is time-dependent and correspond to the Majumdar-Papapetrou solution when $\Lambda = 0$.

Keeping the electrostatic solutions in mind, we know that the electric field and the lapse function are related. In fact, from (1.2) the electric field and the gradient of the lapse function must be linearly dependent at the boundary $\partial M$.

There are some well-known classification results of some important geometric structures like static vacuum manifolds and Ricci solitons carrying a metric such that the Bach tensor is free from divergence (cf. [3, 5, 11, 13, 15]). Any three-dimensional a Riemannian manifold is locally conformally flat if, and only if, its Cotton tensor $C$ is identically zero.

In the three dimensional case the Cotton tensor is associated with the Bach tensor, $B$, accordingly to $B = \text{div} C$. The Bach tensor was defined in 1921 by Rudolf Bach and it is connected to general relativity and conformal geometry. This tensor appeared naturally from studies of Huyghens’s principle and has some psychical significance mainly about wave propagation (see for instance [16] and the references therein).

The main goal of this work is to show that an electrostatic system with divergence-free Bach tensor, i.e., $\text{div}^2 B = 0$, must be locally conformally flat. It is important to say that $\text{div}^2 B = 0$ is less restrictive (topologically speaking) than asymptotically flat conditions.

To state our main results we need to define an important function. To that end, we will say that the electric field $E$ and the gradient of the lapse function $\nabla f$ are linearly dependent if there exists a smooth function $\rho$ such that $E = \rho \nabla f$. As an interest consequence of $\text{curl}(fE) = 0$, if $E = \nabla \psi$ for some smooth function $\psi : M \to \mathbb{R}$, then $E$ must be parallel to the gradient of $f$. Also, it is natural to consider the case $fE = \nabla \psi$.

We define the function

$$Q = 2(1 - f^2 \rho^2).$$

**Theorem 1.2.** Let $(M^3, g, f, E)$ be a compact (without boundary) electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and $Q > 0$ (or $Q < 0$). Then, $(M^3, g)$ is locally conformally flat.

The next result proves the noncompact case.

**Theorem 1.3.** Let $(M^3, g, f, E)$ be an electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and $Q > 0$ (or $Q < 0$). If $f$ is a proper function, then $(M^3, g)$ is locally conformally flat.

Now, we are able to provide the geometric structure for an 3-dimensional electrostatic system.

**Theorem 1.4.** Let $(M^3, g, f, E)$ be an electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and $Q > 0$ (or $Q < 0$). If $f$ is a proper function,
around any regular point of \( f \) the manifold is locally a warped product with a one-dimensional base with fiber \((N^2, \mathcal{F})\) of constant curvature, i.e.,

\[
(M^3, g) = (I, dr^2) \times \varphi (N^2, \mathcal{F}),
\]

where \( I \subset \mathbb{R} \) and \( \varphi(r) = c_1 \int \frac{dr}{\sqrt{f(c)}} + c_2 \); \( c_1 \) and \( c_2 \) are constants.

Remark 1.5. It is important to point out that if \( M^3 \) is compact in Theorem 1.4, it is not necessary to ask for \( f \) to be a proper function.

2. Structural lemmas

This section is reserved to some preliminary results to prove the main theorems of this work. We start constructing a covariant \( V \)-tensor similar to the tensor defined in \([\Pi]\).

To that end, first we combine \((1.1)\) with \((1.3)\) to obtain

\[
(2.1) \quad \nabla^2 f = f \left( \text{Ric} + 2E^0 \otimes E^0 - \frac{R}{2} g \right).
\]

On the other hand, it is well known that in any Riemannian manifold we can relate the Riemannian curvature tensor with a smooth function by using the Ricci identity

\[
\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijkl} \nabla^l f.
\]

Since the Hessian operator is symmetric, taking the covariant derivative of \((2.1)\) over \( i \) and \( j \) and then subtract them we get

\[
R_{ijkl} \nabla^l f = \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f
\]

\[
= f(\nabla_i R_{jk} - \nabla_j R_{ik}) - \frac{f}{2}(\nabla_i R_{gjk} - \nabla_j R_{gik}) - \frac{R}{2} (\nabla_i g_{jk} - \nabla_j g_{ik}) + (R_{jk} \nabla_i f - R_{ik} \nabla_j f)
\]

\[
+ 2f(E^0 \nabla_i E^0 k - E^0 \nabla_j E^0 k + \nabla_i E^0 j E^0 k - \nabla_j E^0 i E^0 k)\]

\[
+ 2(\nabla_i f E^0 j - \nabla_j f E^0 i).
\]

Here, we are considering \( \{e_i\}_{i=1}^3 \) as a base for the tangent space of \( M \). Moreover, \( E^0 = E^0 (e_i) \). Note that the Cotton tensor over a 3-dimensional Riemannian manifold is defined by

\[
(2.2) \quad C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{4}(\nabla_i R_{gjk} - \nabla_j R_{gik}).
\]

Furthermore, the Riemann curvature tensor is given by

\[
R_{ijkl} = R_{ikgjl} - R_{idgjk} + R_{jilgk} - R_{jikgl} - \frac{R}{2}(g_{ikgjl} - g_{idgjk}).
\]

Therefore, combining these equations we get

\[
(2.3) \quad f C_{ijk} = (R_{jil} \nabla^l f g_{jk} - R_{id} \nabla^l f g_{jk}) + R(\nabla_i f g_{jk} - \nabla_j f g_{ik}) + 2(R_{ik} \nabla_j f - R_{jk} \nabla i f)
\]

\[
- 2f(E^0 \nabla_i E^0 k - E^0 \nabla_j E^0 k + \nabla_i E^0 j E^0 k - \nabla_j E^0 i E^0 k)
\]

\[
- 2E^0 (E^0 \nabla_i f - E^0 \nabla_j f) + \frac{f}{4}(\nabla_i R_{gjk} - \nabla_j R_{gik}).
\]
Now, using $\text{curl}(fE) = 0$ we can infer that
\[
fdE♭(e_i, e_j) = -(df \wedge E♭)(e_i, e_j) = E♭(e_i)df(e_j) - E♭(e_j)df(e_i) = E♭ \nabla_j f - E♭ \nabla_i f.
\]

On the other hand, let $dE♭(e_i, e_j) = E♭ij$, then, by definition we have
\[
(2.4)
E♭ij = \nabla_i E♭j - \nabla_j E♭i.
\]

Further, we can see that
\[
(2.5)
f(\nabla_i E♭j - \nabla_j E♭i) = E♭i \nabla_j f - E♭j \nabla_i f.
\]

We can rewrite (2.3) using $\text{curl}(fE)$. So,
\[
fCijk = (Rjl \nabla_l fgik - Ril \nabla_l fgjk) + 2(Rik \nabla_j f - Rjk \nabla_i f) + R(\nabla_i fgjk - \nabla_j fgik) + \frac{f}{4} (\nabla_i Rgjk - \nabla_j Rgik) - 2f(E♭j \nabla_i E♭k - E♭i \nabla_j E♭k).
\]

Define the covariant 3-tensor $V_{ijk}$ by
\[
V_{ijk} = 2f(E♭i \nabla_j E♭k - E♭j \nabla_i E♭k) + \frac{f}{4} (\nabla_i Rgjk - \nabla_j Rgik) + R(\nabla_i fgjk - \nabla_j fgik) - (Ril \nabla_l fgjk - Rjl \nabla_l fgik) - 2(\nabla_i f Rgjk - \nabla_j f Rgik),
\]

where $E♭i = E♭(e_i)$. The $V$-tensor has the same symmetries as the Cotton tensor $C$ and it is trace-free. Hence, from (2.6) and (2.7) we can conclude our next result.

**Lemma 2.1.** Let $(M^3, g, f, E)$ be an electrostatic system. Then,
\[
(2.8)
fC_{ijk} = V_{ijk}.
\]

Our next results follow the same strategy used by [1], [5] and [13]. We will sketch the proofs here for sake of completeness.

**Lemma 2.2.** Let $(M^3, g, f, E)$ be an electrostatic system. Then,
\[
C_{kji}R^{ik} = \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

**Proof.** In dimension $n = 3$, the Bach tensor is defined by
\[
B_{ij} = \nabla^k C_{kij} = \nabla^k \left( \frac{V_{kij}}{f} \right),
\]

Taking the derivative over $i$, we have
\[
\nabla^i B_{ij} = \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

On other hand,
\[
\nabla^j B_{ij} = -C_{ijk}R^{ik}, \quad C_{ijk} = -C_{jik}, \quad \nabla^k C_{kij} = \nabla^k C_{kji}
\]

and
\[
C_{ijk} + C_{kij} + C_{jki} = 0.
\]

Then, from a straightforward computation, we obtain
\[
\nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right) = \nabla^i B_{ij} = -C_{jik}R^{ik} = -C_{jki}R^{ik} = C_{kji}R^{ik},
\]

which is the expected result. \qed
Lemma 2.3. Let \((M^3, g, f, E)\) be an electrostatic system. Then,
\[
\frac{1}{2}|C|^2 + R^{ik} \nabla^j C_{jki} = -\nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

Proof. Taking the divergence in Lemma 2.2, we get
\[
C_{kji} \nabla^j R^{ik} + R^{ik} \nabla^j C_{kji} = \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

Now, from the symmetries of the \(C\)-tensor and renaming indices, we can infer that
\[
(\nabla^j R^{ik} - \nabla^k R^{ij})C_{jki} = C_{jki} \nabla^j R^{ik} + C_{kji} \nabla^k R^{ij} = 2C_{jki} \nabla^j R^{ik}.
\]

Hence,
\[
\frac{1}{2} C_{kji}(\nabla^j R^{ik} - \nabla^k R^{ij}) + R^{ik} \nabla^j C_{kji} = \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

Now, since the Cotton tensor is trace-free, from (2.2) we obtain
\[
-\frac{1}{2} C_{kji} C^{kji} - R^{ik} \nabla^j C_{jki} = \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

Therefore, the result holds. \(\square\)

From this point, the structure of the electrostatic system plays an important role.

Theorem 2.4. Let \((M^3, g, f, E)\) be an electrostatic system. For every \(C^2\)-function \(\phi : \mathbb{R} \to \mathbb{R}\), with \(\phi(f)\) having compact support \(K \subseteq M\) such that \(K \cap \partial M = \emptyset\) we have
\[
\frac{1}{4} \int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki} + \int_M \phi(f) E^b \nabla^k E^a C_{jki}.
\]

Proof. From Lemma 2.2 we obtain
\[
\frac{1}{2} |C|^2 \phi(f) + \phi(f) R^{ik} \nabla^j C_{jki} = -\phi(f) \nabla^j \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

Integrating this expression, we get
\[
\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} = \int_M \phi(f) \nabla^j f \nabla^i \nabla^k \left( \frac{V_{kij}}{f} \right).
\]

Thus, from Lemma 2.2 we have
\[
\frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \phi(f) R^{ik} \nabla^j C_{jki} = -\int_M \phi(f) R^{ik} \nabla^j f C_{jki}.
\]

We will perform integration in some parts of the above equation, separately, using (2.1) and the fact that \(C_{ijk}\) is trace-free and skew-symmetric. First,
\[
\int_M \phi(f) R^{ik} \nabla^j C_{jki} = \int_M \frac{\phi(f)}{f} \nabla^i \nabla^k f \nabla^j C_{jki} - 2 \int_M \phi(f) E^b \nabla^i \nabla^j C_{jki}
\]
\[
= \int_M \frac{\phi(f)}{f} \nabla^i \nabla^k f \nabla^j C_{jki} + 2 \int_M \phi(f) f \nabla^i \nabla^j C_{jki} + 2 \int_M \phi(f) \nabla^j (E^b \nabla^k) C_{jki}.
\]
On the other hand,
\[ \int_M \phi(f) R^{ik} \nabla^j f C_{jki} = \int_M \frac{\phi(f)}{f} \nabla^j f \nabla^i \nabla^k f C_{jki} - 2 \int_M \phi(f) \nabla^j f \theta^{ki} E^h C_{jki}. \]

Note that, since the Hessian tensor is symmetric
\[ 2 \nabla^j \nabla^k f C_{jki} = \nabla^k \nabla^j f C_{jki} + \nabla^i \nabla^k f C_{kji} = \nabla^k \nabla^j f (C_{jki} + C_{kji}) = 0. \]

Hence,
\[ \frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f} \nabla^i \nabla^k f \nabla^j f C_{jki} + 2 \int_M \phi(f) \nabla^j (E^h E^k) C_{jki} = - \int_M \frac{\phi(f)}{f} \nabla^j f \nabla^k f C_{jki} = - \int_M \frac{\phi(f)}{f} \nabla^j f \nabla^i g_{jki} = - \int_M \frac{\phi(f)}{f} \nabla^j f \nabla^k f C_{jki} + \int_M \frac{\phi(f)}{f} \nabla^i \nabla^k f \nabla^j f C_{jki} + \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i C_{jki}. \]

Therefore, we get
\[ \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j f C_{jki} = \frac{1}{2} \int_M |C|^2 \phi(f) + \int_M \frac{\phi(f)}{f^2} \nabla^i \nabla^k f \nabla^j f C_{jki} + 2 \int_M \phi(f) \nabla^j (E^h E^k) C_{jki}. \]  
(2.9)

Then, since the Cotton tensor is trace-free and skew-symmetric, another integration by parts gives us
\[ \int_M \frac{\phi(f)}{f^2} \nabla^i \nabla^k f \nabla^j f C_{jki} = \int_M \frac{\phi(f)}{f^2} \nabla^j \nabla^i f \nabla^k f C_{kji} = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^j f C_{jki} + 2 \int_M \phi(f) \nabla^j (E^h E^k) C_{jki}. \]

We used (2.1) in the last equality. Thus, (2.9) can be rewrite in the following form:
\[ \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j f C_{jki} = \frac{1}{2} \int_M |C|^2 \phi(f) + 2 \int_M \phi(f) \nabla^j (E^h E^k) C_{jki} + 2 \int_M \phi(f) \nabla^k f \nabla^j f C_{jki} + \int_M \phi(f) \nabla^j (E^h E^k) C_{jki}. \]

Now, from (2.7) and (2.8), we have
\[ R^{ij} \nabla^k f C_{kji} = \frac{1}{2} C_{kji} (\nabla^k f R^{ij} - \nabla^j f R^{ki}) = - \frac{1}{2} f C_{kji} \left[ \frac{1}{2} C^{kji} + (E^h \nabla^k E^i - E^h \nabla^j E^i) \right] = - \frac{1}{4} f |C|^2 - \frac{1}{2} f \left( E^h \nabla^k E^i - E^h \nabla^j E^i \right) C_{kji}. \]
Thus,

\[
\frac{1}{4} \int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki} + \frac{1}{4} \int_M \Phi(f) \left( \nabla^i E^k f - f \nabla^i (E^k E^s) + \frac{1}{4} \left( E^k \nabla^j E^s - E^s \nabla^k E^j \right) \right) C_{jki}.
\]

Furthermore, from (2.5) we have

\[
E^k \nabla^i f - E^k \nabla^i = f (\nabla^i E^k - \nabla^k E^i).
\]

Combining the last two equations and the fact that Cotton tensor is skew-symmetric, yields to

\[
\frac{1}{4} \int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki} + \frac{1}{4} \int_M \Phi(f) \left( E^k \nabla^i E^k - E^k \nabla^i E^s \right) C_{jki}.
\]

Note that

\[
E^k \nabla^i E^k C_{jki} = -E^k \nabla^i E^k C_{jki}, \quad E^k \nabla^j E^k C_{jki} = -E^k \nabla^j E^k C_{jki};
\]

\[
E^k \nabla^i E^k C_{jki} = -E^k \nabla^i E^k C_{jki} \quad \text{and} \quad E^k \nabla^j E^k C_{jki} = -E^k \nabla^j E^k C_{jki}.
\]

Then,

\[
\frac{1}{4} \int_M \phi(f)|C|^2 = \int_M \phi(f) \left[ 2E^k \nabla^i E^k - 2E^k \nabla^j E^k + E^k \nabla^j E^s \right] C_{jki} + \frac{1}{4} \int_M \Phi(f) \left( \nabla^k f \nabla^i \nabla^j C_{jki} \right).
\]

Since from (2.4) we have

\[
2E^k \nabla^j E^s C_{jki} = E^k E^j E^s C_{jki},
\]
we can infer that

\[
\frac{1}{4} \int_M \phi(f)|C|^2 = \int_M \phi(f) \left[ -2E^a_\nu \nabla^\nu E^a_j - E^a_i E^a_{ij} + E^a_i \nabla^j E^a_j \right] C_{jki} \\
+ \int_M \phi(f) \frac{\partial M}{\partial f} \nabla^k f \nabla^i \nabla^j C_{jki}
\]

\[
= \int_M \phi(f) \left[ -E^a_i \nabla^i E^a_k + E^a_i E^a_{ik} + E^a_k E^a_{ij} \right] C_{jki} \\
+ \int_M \phi(f) \frac{\partial M}{\partial f} \nabla^k f \nabla^i \nabla^j C_{jki}
\]

\[
= \int_M \phi(f) \left[ E^a_i E^a_{ij} + E^a_j E^a_{ik} + E^a_k E^a_{ij} \right] C_{jki} \\
+ \int_M \phi(f) \frac{\partial M}{\partial f} \nabla^k f \nabla^i \nabla^j C_{jki}.
\]

On the other hand, since \( E^a \wedge dE^a = 0 \), from (2.5) we get

\[
f(E^a_i E^a_{ij} + E^a_j E^a_{ik} + E^a_k E^a_{ij}) = E^a_i E^a_{ij} \nabla^i f - E^a_i E^a_{ik} \nabla^j f + E^a_j E^a_{ik} \nabla^i f - E^a_j E^a_{ij} \nabla^k f = 0.
\]

Finally,

\[
\frac{1}{4} \int_M \phi(f)|C|^2 = \int_M \phi(f) \frac{\partial M}{\partial f} \nabla^k f \nabla^i \nabla^j C_{jki} + \int_M \phi(f) E^a_i \nabla^k E^a_j C_{jki}.
\]

\[
\square
\]

3. Proof of the main results

It is well-known (see [10, Lemma 4]) that in \( \partial M = f^{-1}(0) \) the electric field and the gradient of the lapse function are linearly dependent (LD). Motivated by the charged Nariai solution and the cold black hole system, we assume that both fields are linearly dependent on \( M \), that is, there exists a smooth function \( \rho \) such that \( E = \rho \nabla f \). Thus we can rewrite the \( V \)-tensor (2.7) as follow.

**Lemma 3.1.** Let \((M^3, g, f, E)\) be an electrostatic system in which \( E = \rho \nabla f \).

Then the \( V \)-tensor is given by

\[
V_{ijk} = f \rho \nabla f \nabla^2 (\nabla_i \rho g_{jk} - \nabla_j \rho g_{ik}) + \left( R - \frac{1}{2} R f^2 \rho^2 - f^2 \rho^2 \Delta \right) (\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\
+ 2(f^2 \rho^2 - 1)(\nabla_i f R_{jk} - \nabla_j f R_{ik}) + (f^2 \rho^2 - 1)(R_{il} \nabla^l f g_{jk} - R_{jl} \nabla^l f g_{ik}).
\]
Proof. Since the electric field and the lapse function are linearly dependent (LD), there exists a smooth function \( \rho \) such that \( E = \rho \nabla f \). Using (2.1) we get
\[
\nabla_i E^j = \nabla_i (\rho \nabla_j f) = \nabla_i \rho \nabla_j f + \rho \nabla_i \nabla_j f = \nabla_i \rho \nabla_j f + 2 f \rho^i \nabla_j f \nabla_i f + f \rho R_{ij} - \frac{f}{2} \rho R_{ij}.
\]

On other hand, from (1.3) we get
\[
\nabla_i R = 2\nabla_i \|E\|^2 = 4 \rho \|\nabla f\|^2 \nabla_i \rho + 2 \rho^2 \nabla_i \|\nabla f\|^2.
\]

From (2.1) we know that
\[
\nabla_i \|\nabla f\|^2 = 2f \left( R_{d} \nabla^i f + 2 \rho^2 \|\nabla f\|^2 \nabla_i f - \frac{R}{2} \nabla_i f \right).
\]

Combining the last two equations and using (1.3), we have
\[
\nabla_i R = 4 \rho \|\nabla f\|^2 \nabla_i \rho + 4 \rho^2 \left( R_{d} \nabla^i f + 2 \rho^2 \|\nabla f\|^2 \nabla_i f - \frac{R}{2} \nabla_i f \right).
\]

Then, from (2.7) it follows that
\[
V_{ijk} = 2f \rho (\nabla_i f \nabla_j \rho \nabla_k f - \nabla_j f \nabla_i \rho \nabla_k f) + f \rho \|\nabla f\|^2 (\nabla_i \rho g_{jk} - \nabla_j \rho g_{ik}) + 2(f^2 \rho^2 - 1)(\nabla_i f R_{jk} - \nabla_j f R_{ik}) + (f^2 \rho^2 - 1)(R_{il} \nabla^l f g_{jk} - R_{jl} \nabla^l f g_{ik})
\]
\[
+ \left(1 - \frac{3}{2} f^2 \rho^2 \right) R + 2 f^2 \rho^4 \|\nabla f\|^2 (\nabla_i f g_{jk} - \nabla_j f g_{ik}).
\]

Since \( \text{curl}(fE) = 0 \) implies that \( E^k = 0 \) (the fields are LD, i.e., \( df \wedge E^k = 0 \)), we have \( \nabla_i E^j = \nabla_j E^i \). So,
\[
\nabla_i \rho \nabla_j f = \nabla_j \rho \nabla_i f.
\]

Finally, combining (1.3) with the last two identities the result follows. \( \square \)

Moreover, from Theorem 2.4 we obtain the following corollary.

**Corollary 3.2.** Let \((M^3, g, f, E)\) be an electrostatic system where the electric field and gradient of the lapse function are linearly dependent. For every \( \phi : \mathbb{R} \rightarrow \mathbb{R} \), \( C^2 \) function with \( \phi(f) \) having compact support \( K \subseteq M \) we have
\[
\frac{1}{2} \int_M \frac{1}{Q} |C|^2 \phi(f) = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki},
\]
where \( Q = 2(1 - f^2 \rho^2) \neq 0 \).

**Proof.** Taking in accounting that \( E = \rho \nabla f \) in Theorem 2.4 since the Cotton tensor is skew-symmetric and trace-free we obtain
\[
\frac{1}{4} \int_M \phi(f) |C|^2 = \int_M \frac{\phi(f)}{f} \nabla^k f \nabla^i \nabla^j C_{jki} + \int_M \phi(f) \rho^2 \nabla^j f R_{ki} C_{jki},
\]
where we used that \( \text{curl}(fE) = 0 \), i.e.,
\[
\nabla^k \rho \nabla^i f = \nabla^i \rho \nabla^k f.
\]

On other hand, from Lemma 3.1 we have
Now, we are able to demonstrate our main results.

**Theorem 3.3** (Theorem 1.2). Let \((M^3, g, f, E)\) be a compact electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and \(Q\) has defined sign everywhere. Then \((M^3, g)\) is locally conformally flat.

**Proof.** Considering \(M\) is compact without boundary and \(\phi(f) = f^4\), from Corollary 3.2 we obtain

\[
\frac{1}{2} \int_M \frac{1}{Q} |C|^2 f^4 = \int_M f^3 \nabla^k f \nabla^i \nabla^j C_{jki} = \frac{1}{4} \int_M \nabla^i f^4 \nabla^k \nabla^j C_{jki} = -\frac{1}{4} \int_M f^4 \nabla^i \nabla^k \nabla^j C_{jki}.
\]

Since \(\text{div}^2 B = 0\), by definition, \(\text{div}^3 C = 0\), then the right-hand side is identically zero, i.e.,

\[
\int_M \frac{1}{Q} |C|^2 f^4 = 0,
\]

Since \(f > 0\) over \(M\) and \(Q\) has defined sign everywhere, \(f^2 \rho^2 \neq 1\), that is, the integral has always the same sign, therefore \(C\) must be identically zero, thus the result holds.

\(\square\)

**Theorem 3.4** (Theorem 1.3). Let \((M^3, g, f, E)\) be an electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and \(Q\) has defined sign everywhere. If \(f\) is a proper function, then \((M^3, g)\) is locally conformally flat.

**Proof.** Let \(s > 0\) be a real number fixed, and so we take \(\chi \in C^3 \) a real non-negative function defined by \(\chi = 1\) in \([0, s]\), \(\chi' \leq 0\) in \([s, 2s]\) and \(\chi = 0\) in \([2s, +\infty]\). Since \(f\) is a proper function, we have that \(\phi(f) = f^4 \chi(f)\) has compact support in \(M\) for
From Corollary 3.2, we get
\[-\frac{1}{2} \int_M \frac{1}{Q} |C|^2 f^4 \chi(f) = \int_M f^3 \chi(f) \nabla^k f \nabla^l \nabla^j C_{jkl}\]
\[= \frac{1}{4} \int_M \chi(f) f^4 \nabla^k \nabla^l \nabla^j C_{jkl}\]
\[= \frac{1}{4} \int_M \chi(f) f^4 \nabla^l \nabla^k \nabla^j C_{jkl} + \frac{1}{4} \int_M \dot{\chi}(f) f^4 \nabla^k \nabla^l \nabla^j C_{jkl}.
\]

In the last equality we used integration by parts. Now, since \(\text{div} B = 0\) and taking \(\phi(f) = f^3 \dot{\chi}(f)\) in Corollary 3.2 one more time, we obtain
\[-\frac{1}{2} \int_M \frac{1}{Q} |C|^2 f^4 \chi(f) = \frac{1}{8} \int_M \frac{1}{Q} |C|^2 f^5 \dot{\chi}(f),\]
i.e.,
\[\int_M \frac{1}{Q} f^4 |C|^2 \chi(f) + \frac{1}{4} f \dot{\chi}(f) = 0.
\]

Let be \(M_s = \{x \in M; f(x) \leq s\}\). Thus, by definition, \(\chi(f) + \frac{1}{4} f \dot{\chi}(f) = 1\) on the compact set \(M_s\). Thus, on \(M_s\),
\[\int_{M_s} \frac{1}{Q} f^4 |C|^2 = 0.
\]
Therefore, since \(Q\) has defined sign everywhere and \(f\) is positive, \(C = 0\) in \(M_s\).
Taking \(s \to +\infty\), we obtain that \(C = 0\) on \(M\). □

4. The Warped Product Structure

In this section we will prove the warped product structure of a locally conformally flat electrostatic system following the ideas of [3, 4]. Consequently, the proof of Theorem 1.4 is given.

We consider an orthonormal frame \(\{e_1, e_2, e_3\}\) diagonalizing the Ricci tensor \(\text{Ric}\) at a regular point \(p \in \Sigma = f^{-1}(c)\), with associated eigenvalues \(R_{kk}\), \(k = 1, 2, 3\), respectively. That is, \(R_{ij}(p) = R_{ii}\delta_{ij}(p)\). Now, from Theorem 1.2 and Theorem 1.3 we can infer that \(V_{ijk} = 0\) (since \((M, g)\) is locally conformally flat). Then, from Lemma 3.1 for all \(i \neq j\) we get
\[0 = V_{ijj} = f \rho |\nabla f|^2 \nabla_i \rho + (f^2 \rho^2 - 1)(2R_{jj} + R_{ii}) \nabla_i f + \left(R - \frac{1}{2} R f^2 \rho^2 - f^2 \rho^2 \Lambda \right) \nabla_i f.
\]

Without loss of generalization, consider \(\nabla_i f \neq 0\) and \(\nabla_i f = 0\) for all \(i \neq j\). Observe that \(\text{Ric}(\nabla f) = R_{ii} \nabla f\), i.e., \(\nabla f\) is an eigenvector for \(\text{Ric}\). From (4.1), we obtain that \(R_{ii}\) and \(R_{jj}\), \(j \neq i\), have multiplicity 1 and 2, respectively. In fact,
\[-f \rho |\nabla f|^2 \nabla_i \rho - \left(R - \frac{1}{2} R f^2 \rho^2 - f^2 \rho^2 \Lambda \right) \nabla_i f = (f^2 \rho^2 - 1) R_{ii} = 2(f^2 \rho^2 - 1) R_{jj},
\]
for \(j = 2, 3\).
Moreover, suppose that $\nabla_i f \neq 0$ for at least two distinct directions. Assume $\nabla_1 f \neq 0$, $\nabla_2 f \neq 0$ and $\nabla_3 f = 0$. So, for instance, we have

$$-f \rho |\nabla f|^2 \nabla^2 f \rho - \left(R - \frac{1}{2} R f^2 \rho^2 - f^2 \rho^2 \Lambda \right) - (f^2 \rho^2 - 1) R_{11} = 2(f^2 \rho^2 - 1) R_{33}$$

and

$$-f \rho |\nabla f|^2 \nabla^2 f \rho - \left(R - \frac{1}{2} R f^2 \rho^2 - f^2 \rho^2 \Lambda \right) - (f^2 \rho^2 - 1) R_{22} = 2(f^2 \rho^2 - 1) R_{33}.$$  

Then, using that $\text{curl}(fE) = 0$, i.e.,

$$\nabla^k \rho \nabla^i f = \nabla^i \rho \nabla^k f.$$  

We can conclude that

$$\frac{\nabla^1 \rho}{\nabla^1 f} = \frac{\nabla^2 \rho}{\nabla^2 f}.$$  

Thus, $R_{11} = R_{22}$. Analogously, if $\nabla_i f \neq 0$ for all $i \in \{1, 2, 3\}$. Then, $R_{11} = R_{22} = R_{33}$. So, we can conclude that $Ric$ has at most two distinct eigenvalues $\lambda$ and $\mu$ with one of them having multiplicity 2.

Therefore, in any case we have that $\nabla f$ is an eigenvector for $Ric$. From the above discussion we can take $\{e_1 = \frac{\nabla f}{|\nabla f|}, e_2, e_3\}$ as an orthonormal frame for $\Sigma$ diagonalizing the Ricci tensor $Ric$ for the metric $g$.

Now, from (2.1) we have

$$\nabla_a |\nabla f|^2 = 2f \left(R_{ab} \nabla^i f + 2\rho^2 |\nabla f|^2 \nabla_a f - \frac{R}{2} \nabla_a f \right); \quad a \in \{2, 3\}.$$  

Hence, $|\nabla f|$ is a constant in $\Sigma$. Thus, we can express locally the metric $g$ in the form

$$g_{ij} = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta_a d\theta_b,$$

where $g_{ab}(f, \theta) d\theta_a d\theta_b$ is the induced metric and $(\theta_2, \theta_3)$ is any local coordinate system on $\Sigma$. We can find a good overview of the level set structure in $[3, 13]$.

Observe that there is no open subset $\Omega$ of $M^n$ where $\{\nabla f = 0\}$ is dense. In fact, if $f$ is constant in $\Omega$ and $M^n$ is complete, we have that $f$ is analytic, which implies $f$ is constant everywhere. Thus, we consider $\Sigma$ a connected component of the level surface $f^{-1}(c)$ (possibly disconnected) where $c$ is any regular value of the function $f$. Suppose that $I$ is an open interval containing $c$ such that $f$ has no critical points in the open neighborhood $U_I = f^{-1}(I)$ of $\Sigma$. For sake of simplicity, let $U_I \subset M \setminus \{f = 0\}$ be a connected component of $f^{-1}(I)$. Then, we can make a change of variables

$$r(x) = \int \frac{df}{|\nabla f|}$$

such that the metric $g$ in $U_I$ can be expressed by

$$g_{ij} = dr^2 + g_{ab}(r, \theta) d\theta_a d\theta_b.$$  

Let $\nabla r = \frac{\partial}{\partial r}$, then $|\nabla r| = 1$ and $\nabla f = f'(r) \frac{\partial}{\partial r}$ on $U_I$. Note that $f'(r)$ does not change sign on $U_I$. Thus, we may assume $I = (-\varepsilon, \varepsilon)$ with $f'(r) > 0$ for $r \in I$. Moreover, we have $\nabla \partial_r \partial r = 0$. 


Then the second fundamental formula on $\Sigma$ is given by

$$h_{ab} = -\langle e_1, \nabla_a e_b \rangle = \frac{\nabla_a \nabla_b f}{|\nabla f|}$$

(4.2)

$$= \frac{1}{|\nabla f|} \left( f R_{ab} - \frac{Rf}{2} g_{ab} \right) = \frac{f}{|\nabla f|} \left( \mu - \frac{R}{2} \right) g_{ab} = \frac{H}{2} g_{ab},$$

where $H = H(r)$, since $H$ is constant in $\Sigma$. In fact, contracting the Codazzi equation

$$R_{1cab} = \nabla_a h_{bc} - \nabla_b h_{ac}$$

over $c$ and $b$, it gives

$$R_{1a} = \nabla_a (H) - \frac{1}{2} \nabla_a (H) = \frac{1}{2} \nabla_a (H).$$

On the other hand, since $R_{1a} = 0$, we conclude that $H$ is constant in $\Sigma$.

For what follows, we fix a local coordinate system

$$(x_1, x_2, x_3) = (r, \theta_2, \theta_3)$$

in $U_1$, where $(\theta_2, \theta_3)$ is any local coordinate system on the level surface $\Sigma_c$. Considering that $a, b, c, \cdots \in \{2, 3\}$, we have

$$h_{ab} = -g(e_1, \nabla_a \partial_b) = -g(e_1, \Gamma^1_{ab} \partial_r) = \frac{-1}{|\nabla f|} \Gamma^1_{ab}.$$ 

Now, by definition

$$\Gamma^1_{ab} = \frac{1}{2} g^{11} \left( -\frac{\partial}{\partial r} g_{ab} \right) = \frac{1}{2} |\nabla f| \frac{\partial}{\partial r} g_{ab}.$$ 

Then,

$$\frac{\partial}{\partial r} g_{ab} = -H(r) g_{ab}$$

implies that

$$g_{ab}(r, \theta) = \varphi(r)^2 g_{ab}(r_0, \theta),$$

where $\varphi(r) = e^{-\int_{r_0}^r H(s) ds}$ and the level set $\{ r = r_0 \}$ corresponds to the connected component $\Sigma$ of $f^{-1}(c)$.

Now, we can apply the warped product structure (see [2]). Hence, considering

$$(M^3, g) = (I, dr^2) \times_{\varphi} (N^2, \mathcal{G}),$$

where $g = dr^2 + \varphi^2 \mathcal{G}$. The Ricci tensor of $(M^3, g)$ is

(4.3)

$$R_{11} = -2 \frac{\varphi''}{\varphi}, \quad R_{1a} = 0$$

and

$$R_{ab} = \mathcal{R}_{ab} - \left[ (\varphi')^2 + \varphi \varphi'' \right] g_{ab} \quad (a, b \in \{2, 3\}).$$

Since $\mathcal{R}_{ab} = \frac{\mathcal{R}}{2} g_{ab}$,

$$R_{ab} = \left[ \frac{\mathcal{R}}{2} - (\varphi')^2 - \varphi \varphi'' \right] g_{ab}.$$
On the other hand, since
\[ R = \varphi^{-2} \mathring{R} - 2 \left( \frac{\varphi'}{\varphi} \right)^2 - 4 \frac{\varphi''}{\varphi}, \]
we get
\[ \mathring{R} = \varphi^2 R + 2(\varphi')^2 + 4\varphi\varphi''. \]

Since \( R = 2(\rho^2 |\nabla f|^2 + \Lambda) \) we get
\[ (4.4) \quad \mathring{R} = 2\varphi^2 \rho^2 (f')^2 + 2(\varphi')^2 + 4\varphi\varphi'' + 2\varphi^2 \Lambda. \]

Moreover, from (2.1) we know that
\[ \frac{1}{|\nabla f|^2} \langle \nabla |\nabla f|^2, \nabla f \rangle = 2f \left( R_{11} + 2\rho^2 |\nabla f|^2 - \frac{R}{2} \right). \]

That is, from (1.3) and (4.3) we get
\[ \langle \nabla |\nabla f|^2, \nabla f \rangle = 2f (f')^2 \left[ \rho^2 (f')^2 - 2\frac{\varphi''}{\varphi} - \Lambda \right]. \]

Hence, using that \( \nabla f = f' \partial_r \), we obtain
\[ 2(f')^2 f'' = 2f (f')^2 \left[ \rho^2 (f')^2 - 2\frac{\varphi''}{\varphi} - \Lambda \right]. \]

So,
\[ \rho^2 = \frac{1}{(f')^2} \left[ \frac{f''}{f} + 2\frac{\varphi''}{\varphi} + \Lambda \right]. \]

Combining the above identity with (4.4) we can conclude that \( \mathring{R} \) does not depend on \( \theta \). Therefore, \( \mathring{R} \) is a constant.

Furthermore, from (4.2) we have
\[ \frac{1}{2} |\nabla f| H g_{ab} = \nabla_a \nabla_b f = f \left( R_{ab} + 2\rho^2 \nabla_a f \nabla_b f - \frac{R}{2} g_{ab} \right). \]

Thus,
\[ \left( \frac{1}{2} |\nabla f| H + \frac{R f}{2} \right) \varphi^2 \mathring{g}_{ab} = f R_{ab}. \]

On the other hand,
\[ f R_{ab} = f \left[ \frac{\mathring{R}}{2} - (\varphi')^2 - \varphi\varphi'' \right] \mathring{g}_{ab}. \]

Then,
\[ f \left[ \frac{1}{2} \varphi^2 R + \varphi\varphi'' \right] = \left( \frac{1}{2} f' H + \frac{R f}{2} \right) \varphi^2, \]
i.e.,
\[ H = 2 \frac{f}{f'} \frac{\varphi''}{\varphi}. \]

Since \( \varphi = e^{-\int_{r_0}^r H(s) ds} \), we conclude
\[ \varphi' + 2 \frac{f}{f'} \varphi'' = 0 \quad \Rightarrow \quad \varphi(r) = c_1 f(r)^{-1/2}, \]
where \( c_1 \in \mathbb{R} \).
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