MAURER–CARTAN ELEMENTS AND HOMOTOPICAL PERTURBATION
THEORY

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Abstract. Let $L$ be a (pro-nilpotent) curved $L_\infty$-algebra, and let $h : L \to L[-1]$ be a homotopy between $L$ and a subcomplex $M$. Using homotopical perturbation theory, Fukaya constructed from this data a curved $L_\infty$-structure on $M$. We prove that projection from $L$ to $M$ induces a bijection between the set of Maurer–Cartan elements $x$ of $L$ such that $hx = 0$ and the set of Maurer–Cartan elements of $M$.

Homological perturbation theory is concerned with the following situation, which we call a context: a pair of complete filtered cochain complexes $(V^*, \delta)$ and $(W^*, d)$, filtered morphisms of complexes $f : W \to V$ and $g : V \to W$ such that $gf = 1_W$, and a map $h : V \to V[-1]$, compatible with the filtration, such that

$$1_V = fg + \delta h + h\delta.$$ 

In addition, we assume the side conditions

$$h^2 = hf = gh = 0.$$ 

For a review of this subject and its history, see Gugenheim and Lambe [6].

If $\mu : V \to V[1]$ is a deformation of the differential on $V$, in the sense that $\delta \mu^2 = 0$, where $\delta \mu = \delta + \mu$, and if $\mu$ has strictly positive filtration degree, then $\mu$ induces a deformation of the above context, with

$$h_\mu = \sum_{n=0}^{\infty} (-h\mu)^n h \quad d_\mu = d + \sum_{n=0}^{\infty} g(-\mu h)^n \mu f$$

$$f_\mu = \sum_{n=0}^{\infty} (-h\mu)^n f \quad g_\mu = \sum_{n=0}^{\infty} g(-\mu h)^n.$$ 

These expressions are convergent, by the hypothesis that $\mu$ has strictly positive filtration degree and that the filtrations on $V$ and $W$ are complete.

Homotopical perturbation theory considers a more general perturbation, in which $\mu$ is not only a perturbation of the differential on $V$, but deforms $V$ to a non-trivial algebraic structure, such as an associative algebra, commutative algebra, or Lie algebra. The outcome is a homotopy algebraic structure of the same type on $W$, and a morphism $F_\mu$ of homotopy algebras from $W$, with the transported structure, to $V$. For associative, respectively commutative and Lie algebras, a homotopy algebra is known as an $A_\infty$-algebra, $C_\infty$-algebra and $L_\infty$-algebra respectively. This theory was pioneered by Kadeishvili [7], in the special case of associative algebras, and under the assumption that the differential $d$ on $V$ and the deformation of the differential $\delta$ on $W$ vanish. The case of Lie algebras was taken

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up by Fukaya [4]: he generalizes the construction considerably, permitting the deformation on $V$ to be a \textbf{curved $L_\infty$-algebra}. For an overview of homotopical perturbation theory in this context and others, see also Bandiera [1], Berglund [2] and Loday–Vallette [9].

In this paper, we work in Fukaya’s setting. A complete filtered complex is a complex $(L, \delta)$ with a decreasing filtration of finite length

$$L = F_0 L \supset F_2 L \supset F_3 L \supset \ldots$$

which is a complete metric space with respect to the metric

$$d_c(x, y) = \inf \{ c^{-k} | x - y \in F_k L \}.$$ 

Here, $c$ may be any real number greater than 1.

A curved $L_\infty$-algebra $G^\bullet$ is a graded vector space with a complete filtration, together with a sequence of graded antisymmetric brackets

$$[x_1, \ldots, x_n]_n : (G^i)^{i_1} \times \cdots \times (G^i)^{i_n} \to (G^i)^{i_1 + \cdots + i_n + 2 - n}$$

of degree $2 - n$, $n \geq 0$, satisfying certain relations which we will recall below. Here,

$$[x_1, x_2] = [x_1, x_2]_2 : (G^i)^i \times (G^j)^j \to (G^i)^{i+j},$$

generalizes the bracket of graded Lie algebras,

$$\delta x_1 = [x_1]_1 : (G^i)^i \to (G^i)^{i+1}$$

is an operator analogous to a differential, and $R = [\ ]_0 \in G^2$ is the curvature of $\delta$, in the sense that

$$\delta^2 x = [R, x].$$

Following Fukaya, we assume that $R \in F_1 G^2$ is an element of strictly positive filtration degree.

A Maurer–Cartan element of a curved $L_\infty$-algebra is an element $x$ of $F_1 G^1$ satisfying the equation

$$\sum_{n=0}^\infty \frac{1}{n!} [x, \ldots, x]_n = 0.$$ 

The sum makes sense because the filtration degree of the terms $[x, \ldots, x]_n$ converges to infinity with $n$, owing to the hypothesis that the filtration degree of $x$ is strictly positive. Denote by $MC(G)$ the set of Maurer-Cartan elements of $G$.

Now suppose that $G$ is endowed in addition with a homological perturbation theory context

$$f : (\mathfrak{G}, d) \rightleftarrows (G, \delta) : g,$$

with homotopy $h : G \to G[-1]$. Fukaya associates to these data a curved $L_\infty$-structure on $\mathfrak{G}$. Consider the sets $MC(\mathfrak{G})$ and

$$MC(G, h) = \{ x \in MC(G) | hx = 0 \}.$$ 

We call $MC(G, h)$ the Kuranishi set of $(G, h)$. The goal of this paper is to give a self-contained proof of the following result.

\textbf{Theorem 1.} \textit{The morphism $g$ induces a bijection from $MC(G, h)$ to $MC(\mathfrak{G})$.}
The $L_\infty$-morphism from $\mathcal{S}$ to $\mathcal{G}$ constructed by Fukaya induces a map from $MC(\mathcal{S})$ to $MC(\mathcal{G})$, whose image is actually seen by inspection to lie in $MC(\mathcal{G}, h)$. Thus, our task will be to show that $g$, on restriction to $MC(\mathcal{G}, h)$, gives an inverse to this map.

In [8], Kuranishi considers the following analogue of the above situation. The differential graded Lie algebra $\mathcal{G}_\bullet$ is the Dolbeault resolution $\mathcal{A}^0_i(X, T)$ of the sheaf of holomorphic vector fields on a compact complex manifold $X$, the complex $\mathcal{S}_\bullet$ is the subspace of harmonic forms $\mathcal{H}^0(X, T) \subset \mathcal{A}^0_i(X, T)$ with respect to a choice of Hermitian metric on $X$, and has vanishing differential, $f$ is the inclusion, $g$ is the orthogonal projection with respect to the $L^2$-inner product, and $h$ is the operator $h = (\bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*)^{-1} \bar{\partial}^*$. We have neglected the Banach completions which are needed in order to make sense of the infinite sums in the formulas, and are considerably more complicated to describe than the non-Archimedean Banach spaces, or complete filtered vector spaces, considered in this paper. For example, one may take the completion of $\mathcal{A}^0_i(X, T)$ consisting of differential forms whose coefficients are in the Sobolev space of functions with square-integrable derivatives up to order $\text{dim}_\mathbb{C}(X) - i + 1$.

In this setting, $MC(\mathcal{G})$ is the space of solutions of the Kodaira–Spencer equation on $\mathcal{A}^{0,1}(X, T)$, $\bar{\partial} x + \frac{1}{2} [x, x] = 0$, and $MC(\mathcal{G}, h)$ is the space of solutions of the Kodaira–Spencer equation that also satisfy the Kuranishi gauge condition $\bar{\partial}^* s = 0$.

Kuranishi shows that this subset is a slice for the foliation of $MC(\mathcal{G})$ induced by the action of $\Gamma(X, T) = \mathcal{A}^{0,0}(X, T)$ on $\mathcal{A}^{0,1}(X, T)$: the tangent space to the leaf through $x \in MC(\mathcal{G})$ is the subspace \[ \{ \bar{\partial} z + [x, z] \mid z \in \Gamma(X, T) \}. \]

In the algebraic setting, this was proved (and considerably generalized) in [5], in the case where the filtration on $L$ is finite, and extended to the case of complete filtrations by Dolgushev and Rogers [3], with essentially the same proof.

We now turn to the proof of Theorem 1. For technical reasons, we shift the degrees of our curved $L_\infty$-algebras down by one: this has the effect of giving all of the brackets $[x_1, \ldots, x_n]$ degree one, and considerably simplifies the signs arising in the formulas. Introduce the shifted brackets on $L = \mathcal{G}[1]$: if $x_j \in L^j$ and $s x_j \in \mathcal{G}^{j+1}$ correspond to each other under the identification of $\mathcal{G}$ as the suspension of $L$, then \[ \lambda_n(x_1, \ldots, x_n) = (-1)^{\sum_{i=1}^n (n-i) |x_i|} [s x_1, \ldots, s x_n]. \]

The shifted brackets on $L$ are graded symmetric, and in terms of them, the Maurer–Cartan equation on $F_1 L^0$ becomes \[ \sum_{n=0}^\infty \frac{1}{n!} \lambda_n(x, \ldots, x) = 0. \]
Given a pair \(L\) and \(M\) of complete filtered complexes, we consider the complete filtered complex \(S^{n,i}(L, M)\), where

\[ S^{n,i}(L, M) = \{ \text{filtered graded symmetric } n\text{-linear maps from } L \text{ to } M \text{ of degree } i \}. \]

Here, an \(n\)-linear map \(a_n\) is graded symmetric if for all \(1 \leq j < n\),

\[ a_n(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n) = (-1)^{|x_j||x_{j+1}|} a_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n), \]

filtered if

\[ a_n(F_{k_1} L, \ldots, F_{k_n} L) \subset F_{k_1 + \cdots + k_n} M, \]

and has degree \(i\) if

\[ |a_n(x_1, \ldots, x_n)| = |x_1| + \cdots + |x_n| + i. \]

Let \(S^*(L, M)\) be the complex of inhomogeneous multilinear maps

\[ S^i(L, M) = F_1 M \times \prod_{n=1}^{\infty} S^{n,i}(L, M) \subset \prod_{n=1}^{\infty} S^{n,i}(L, M), \]

with filtration

\[ F_k S^i(L, M) = \{(a_0, a_1, \ldots) \in S^i(L, M) | a_n(F_{k_1} L, \ldots, F_{k_n} L) \subset F_{k_1 + \cdots + k_n + k} M\}. \]

There is a binary operation from \(S^i(L, M) \times S^j(L, L)\) to \(S^{i+j}(L, M)\), defined by the formula

\[ (a \circ b)_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} a_{n-k+1}(b_k(x_{\sigma_1}, \ldots), \ldots, x_{\sigma_n}). \]

For example,

\[ (a \circ b)_0 = a_1(b_0) \]

and

\[ (a \circ b)_1(x) = a_2(b_0, x) + a_1(b_1(x)). \]

This product satisfies Gerstenhaber’s pre-Lie algebra axiom:

\[(a \circ b) \circ c - (-1)^{|b||c|} (a \circ b) \circ c = a \circ (b \circ c) - (-1)^{|b||c|} a \circ (b \circ c).\]

If \(\lambda \in S^1(L, L)\) is an element of degree 1, consider the operation

\[ \delta \lambda f = \delta f + \lambda \circ f - (-1)^{|f|} f \circ \lambda. \]

By (1), we have

\[ \delta^2 f = (\delta \lambda + \lambda \circ \lambda) \circ f - f \circ (d\lambda + \lambda \circ \lambda). \]

**Definition 1.** A curved \(L_\infty\)-algebra is a filtered complex \(L\) together with an element \(\lambda \in S^1(L, L)\) such that \(\delta \lambda + \lambda \circ \lambda = 0\). It is an \(L_\infty\)-algebra if its curvature \(\lambda_0\) vanishes. It is pro-nilpotent if \(\lambda \in F_1 S^1(L, L)\).

A differential graded Lie algebra \(\mathfrak{g}^*\) gives rise to an \(L_\infty\)-algebra by setting \(L = \mathfrak{g}[1]\), with the discrete filtration \(F_0 L = L\) and \(F_1 L = 0\), and with

\[ \lambda_n(x_1, \ldots, x_n) = \begin{cases} (-1)^{|x_1|} [x_1, x_2], & n = 2, \\ 0, & n \neq 2. \end{cases} \]

For example, if \(E\) is a complex of vector bundles on a manifold \(M\) with differential \(\delta\), and \(D\) is a flat connection on \(E\), preserving degree, such that the covariant derivative of \(\delta\) vanishes, then the total complex of the bicomplex \(\Omega^*(M, \text{End}(E))\) of differential forms on
$M$ with values in the bundle of graded algebras $\text{End}(E)$ and differentials $\text{ad}(D)$ and $\text{ad}(\delta)$ is a differential graded Lie algebra, with bracket equal to the graded commutator.

On the other hand, if the connection $D$ is not flat, but has curvature

$$R \in \Omega^2(M, \text{End}(E)),$$

then $\Omega^*(M, \text{End}(E))$ is a curved differential graded Lie algebra, with curvature $E$.

Curved $L_\infty$-algebras are a common generalization of $L_\infty$-algebras and curved differential graded Lie algebras.

Given $a \in S^i(L, M)$ and $b \in S^0(K, L)$, define the composition $a \bullet b \in S^i(K, M)$ by the formula

$$(a \bullet b)_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^\varepsilon \sum_{k=0}^n \frac{1}{k!} \sum_{n_1 + \cdots + n_k = n} \frac{1}{n_1! \cdots n_k!} a_k(b_{n_1}(x_{\sigma_1}), \ldots, b_{n_k}(x_{\sigma_n})).$$

It is in order for this operation be well-defined that we have imposed the restriction that $b_0 \in F_1 L$.

The operation $a \bullet b$ is associative: if $a \in S^i(L, M)$, $b \in S^0(K, L)$ and $c \in S^0(J, K)$, then

$$(a \bullet b) \bullet c = a \bullet (b \bullet c) \in S^i(J, M).$$

It has $1_L \in S^0(L, L)$ as a right inverse

$$a \bullet 1_L = a,$$

and $1_M$ as a left inverse

$$1_M \bullet a = a.$$

We generalize the product $a \circ b$ as follows. Extend the ground ring to the exterior algebra with a generator $\varepsilon$ of degree $-1$. If $a \in S^i(L, M)$, $b \in S^0(K, L)$ and $\beta \in S^1(K, L)$, define $a \circ_\beta \beta$ by the formula

$$a \circ_\beta (b + \beta \varepsilon) = a \circ (b \circ_\beta \beta) \varepsilon.$$ 

In particular, if $K = L$, we have

$$a \circ (1_L + \beta \varepsilon) = a + (a \circ \beta) \varepsilon.$$ 

If $c \in S^0(J, K)$, then by the associativity of $\circ$, we have

$$(a \circ_\beta \beta) \circ c = a \circ_{\beta \circ c} (\beta \circ c).$$

Also, we have

$$\delta(a \bullet b) = (\delta a) \bullet b + (-1)^{|a|} a \circ_b (\delta b).$$

**Definition 2.** Let $(L, \lambda)$ and $(M, \mu)$ be curved $L_\infty$-algebras. An $L_\infty$-**morphism** $\phi : L \to M$ (sometimes called a shmap) is an element $\phi \in S^0(L, M)$ satisfying the equation

$$\delta \phi + \mu \bullet \phi = \phi \circ \lambda.$$ 

**Proposition 1.** The composition $\psi \bullet \phi \in S^0(K, M)$ of two $L_\infty$-morphisms $\phi \in S^0(K, L)$ and $\psi \in S^0(L, M)$ is an $L_\infty$-**morphism**.
Proof. We argue as follows:
\[
\delta(\psi \bullet \phi) + \mu \bullet (\psi \bullet \phi) = (\delta \psi + \mu \bullet \psi) \bullet \phi + \psi \circ \phi \delta \phi
\]
\[
= (\psi \circ \lambda) \bullet \phi + \psi \circ \phi \delta \phi
\]
\[
= \psi \circ \phi (\delta \phi + \lambda \bullet \phi)
\]
\[
= \psi \circ \phi (\phi \circ \kappa) = (\psi \bullet \phi) \circ \kappa.
\]
□

Denote the category of curved \(L_\infty\)-algebras and \(L_\infty\)-morphisms by \(L_\infty\). The curved \(L_\infty\)-algebra 0 is a terminal object of this category, and a point \(x \in L_\infty(0, L)\) of a curved \(L_\infty\)-algebra is called a Maurer–Cartan element: in other words, \(x\) is an element of \(F_1 L\) such that

\[
\lambda \bullet x = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_n(x, \ldots, x) = 0.
\]

The Maurer–Cartan set \(MC(L)\) is the set of all Maurer–Cartan elements, in other words, the set of points \(L_\infty(0, L)\) of \(L\). We see that \(MC\) is a functor from the category of curved \(L_\infty\)-algebras to the category of sets.

When \(F_1 L^0\) is finite-dimensional, \(MC(L)\) is an algebraic subvariety of the affine space \(F_1 L^0\). When \(L^0\) and \(M^0\) are finite-dimensional and \(\phi : L \to M\) is an \(L_\infty\)-morphism, \(MC(\phi) : MC(L) \to MC(M)\) is an algebraic morphism of affine varieties.

Now suppose that \(L\) is endowed with a homological perturbation theory context

\[
f : (M, d) \leftrightarrow (L, \delta) : g,
\]

with homotopy \(h : L \to L[-1]\). Given a curved \(L_\infty\)-structure \(\lambda\) on \(L\), Fukaya defines a curved \(L_\infty\)-structure \(\mu\) on \(M\), and an \(L_\infty\)-morphism \(F : L \to M\) such that \(gF = 1_M\). The formulas for \(\mu\) and \(F\) are determined by the solution of a fixed-point problem: in order for this solution to exist, we must assume that \(\lambda\) is pro-nilpotent.

**Theorem 2** (Fukaya). There is a unique solution in \(S^0(M, L)\) of the fixed-point equation

\[
F = f - h\lambda \bullet F.
\]

Furthermore, \(\mu = g\lambda \bullet F \in S^1(M, M)\) is a curved \(L_\infty\)-structure on \(M\), and \(F\) is an \(L_\infty\)-map from \((M, d, \mu)\) to \((L, \delta, \lambda)\).

Proof. The existence and uniqueness of \(F\) follows from the hypothesis of pro-nilpotence of \(\lambda\), since the map \(F \mapsto f - h\lambda F\) is a contraction mapping.

It remains to show the vanishing of the quantities

\[
\alpha = \delta \bullet F + \lambda \bullet F - F \circ \delta - F \circ \mu \in S^1(M, L)
\]

and

\[
\beta = \delta \mu + \mu \circ \mu \in S^1(M, M).
\]

This follows from the equations

\[
\begin{cases}
\alpha = -h(\lambda \circ F \alpha) \\
\beta = -g(\lambda \circ F \alpha)
\end{cases}
\]

which may be proved by explicit calculation. Applying the contraction mapping theorem once more, we see that \(\alpha = 0\) is the unique solution of the first equation, from which we conclude from the second equation that \(\beta\) vanishes as well.
It is now easy to prove Theorem 1. Applying $g$ to both sides of (2), we see that $gF = 1_M$. It is also clear that if $x$ is a Maurer–Cartan element of $M$, that the Maurer–Cartan element $F(x) \in \MC(L)$ satisfies the gauge condition $hF(x) = 0$:

$$hF(x) = hf(x) - h^2 \lambda \bullet F(x)$$

vanishes by the side conditions $hf = 0$ and $h^2 = 0$. It remains to verify that if $x \in \MC(L, h)$, then $F(gx) = x$. In fact, an explicit calculation shows that if $x$ is an element of $F_1L^0$ satisfying $hx = 0$, then

$$x - F(gx) = h\lambda(F(gx)) - h\lambda(x).$$

The vanishing of $x - F(gx)$ follows by the contraction mapping theorem, since the map $y \mapsto -h\lambda(y)$ is a contraction on $L^0$, by the mean value theorem. To see that this map is a contraction, it suffices to observe that its partial derivatives raise filtration degree at every point.

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