Axially symmetric multi-baryon solutions and their quantization in the chiral quark soliton model

S. Komori, N. Sawado and N. Shiiki
Department of Physics, Faculty of Science and Technology, Tokyo University of Science, Noda, Chiba 278-8510, Japan
(Dated: December 21, 2018)

In this paper, we study axially symmetric solutions with $B = 2 − 5$ in the chiral quark soliton model. In the background of axially symmetric chiral fields, the quark eigenstates and profile functions of the chiral fields are computed self-consistently. The resultant quark bound spectrum are doubly degenerate due to the symmetry of the chiral field. Upon quantization, various observable spectra of the chiral solitons are obtained. Taking account of the Finkelstein-Rubinstein constraints, we show that our results exactly coincide with the physical observations for $B = 2$ and 4 while $B = 3$ and 5 do not.

PACS numbers: 12.39.Fe, 12.39.Ki, 21.60.-n, 24.85.+p

I. INTRODUCTION

The chiral quark soliton model (CQSM) was developed in 1980’s as a low-energy effective theory of QCD. Since it includes the Dirac sea quark contribution and explicit valence quark degrees of freedom, the model interpolates between the constituent quark model and the Skyrme model [1, 2, 3, 4, 5]. The CQSM is derived from the instanton liquid model of QCD vacuum and incorporates the non-perturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. It has been shown that the $B = 1$ solution provides correct observables as a nucleon including mass, electromagnetic value, spin carried by quarks, parton distributions and octet SU(3) baryon spectra.

For $B = 2$, the stable axially symmetric soliton solution was found in [6]. The solution exhibits doubly degenerate bound spectrum of the quark orbits in the background of the axially symmetric chiral field with winding number two. Upon quantization, various dibaryon spectra were obtained, showing that the quantum number of the ground state exactly coincide with that of physical deuteron [6, 7]. For $B > 2$, the Skyrme model predicts that the solutions have only discrete, crystal-like symmetries [8, 9]. According to the prediction, we studied the CQSM with $B = 3$ tetrahedrally symmetric chiral fields and obtained triply degenerate spectrum of the quark orbits [10, 11]. Its large degeneracy indicates that the tetrahedrally symmetric solution may be the lowest-lying configuration. For $B > 3$, one can also expect that the lowest-lying solutions in the CQSM inherits the discrete symmetries predicted in the Skyrme model. Studying solutions with those symmetries in CQSM is, however, formidable especially for quantization. Thus, before embarking those discrete symmetries, it will be instructive to study axially symmetric solutions which are much simpler. Besides, considering the fact that for some higher baryon numbers, the ground states of the skyrmions do not agree with the experimental observation [12], the possibility that axially symmetric solutions provide correct ground states can not be excluded. Recently, it was found in Ref. [13] that in the BPS monopoles all axially symmetric solutions up to charge five have lower energy than that of discrete symmetries. A cylindrical shape isomer in $^{12}C$ was also found in the Skyrme model framework [14]. Research of axially symmetric solitons is thus in progress.

In Sec. II, we shall obtain axially symmetric classical soliton solutions with $B = 2 − 5$. The solutions exhibit doubly degenerate spectra due to their axial symmetry. Such degeneracy generates large shell gaps and confirms that the solutions are stable local minima. In Sec. III we shall quantize the obtained classical solitons semiclassically. Imposing the Finkelstein-Rubinstein constraints on the states, the ground states of the axially symmetric solitons are constructed and examined if they agree with the experimental observation. In Sec. IV is the detail of the numerical analysis used to obtain the classical and quantum solutions. Conclusions and discussions are in Sec. V.

II. AXIALLY SYMMETRIC CLASSICAL SOLUTIONS

The CQSM is derived from the instanton liquid model of the QCD vacuum and incorporates the nonperturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. The vacuum functional is defined by [1]

$$Z = \int D\pi D\psi D\psi^\dagger \exp \left[ i \int d^4 x \bar{\psi} \left( i \gamma \cdot D - MU^\gamma \right) \psi \right]$$ (1)

where the SU(2) matrix

$$U^\gamma = \frac{1 + \gamma^5}{2} U + \frac{1 - \gamma^5}{2} U^\dagger \quad \text{with} \quad U = \exp \left( i \tau \cdot \pi / f_\pi \right)$$


---

*Electronic address: sawado@ph.noda.tus.ac.jp*

†Electronic address: norikoshiiki@mail.goo.ne.jp
describes chiral fields, $\psi$ is quark fields and $M$ is the constituent quark mass. $f_\pi$ is the pion decay constant and experimentally $f_\pi \sim 93$MeV.

The $B = 1$ soliton solution has been studied in detail at classical and quantum level in [1, 2, 3, 4, 5]. To obtain solutions with $B > 1$, we shall employ the chiral fields with winding number $B$ in the Skyrme Model as the background of quarks, which can be justified as follows.

In Eq. (1), performing the functional integral over $\psi$ and $\psi^\dagger$ fields, one obtains the effective action

$$S_{\text{eff}}(U) = -iN_c \text{Sp} \ln iD = -iN_c \log \det iD,$$  

(2)

where $iD = i\partial - MU_{78}$ is the Dirac operator. The classical solutions can be obtained by the extremum condition of $S_{\text{eff}}$ with respect to $U$. For this purpose, let us consider the derivative expansion of the action [6-9, 11]. Up to quartic terms, we have,

$$S_{\text{eff}} = \int d^4x \left[ \text{Tr}(L_\mu L^\mu) ight. \\
+ \frac{N_c}{32\pi^2} \text{Tr} \left\{ \frac{1}{12} [L_\mu, L_\nu]^2 - \frac{1}{3} (\partial_\mu L^\mu) + \frac{1}{6} (L_\nu L^\nu)^2 \right\},$$  

(3)

where $L_\mu = \partial_\mu U_{78}$. Suitably adjusting the coefficients, one can identify (3) with the Skyrme model action. Therefore, it will be justified to adopt the configurations of the solutions in the Skyrme model to chiral fields in the CQSM.

In the Skyrme model the minimal energy pion configuration with $B = 2$ has an axial symmetry [13] and can be written by

$$U(x) = \cos F(\rho, z) + i\tau \cdot \tilde{n} \sin F(\rho, z),$$  

(4)

where

$$\tilde{n} = (\sin \Theta(\rho, z) \cos m_w \varphi, \sin \Theta(\rho, z) \sin m_w \varphi, \cos \Theta(\rho, z))$$  

(5)

and $m_w$ is the winding number of the pion fields. We shall use this configuration in the backgound to obtain axially symmetric chiral quark solitons.

In the CQSM, the number of valence quark is associated with the baryon number such that the baryon number $B$ soliton consist of $N_c \times B$ valence quarks. If the correlation between quarks is sufficiently strong, their binding energy become large and the valence quarks can not be observed as positive energy particles [23, 24]. Thus, one gets the picture of the topological soliton model in the sense that the baryon number coincide with the winding number of the background chiral field when the valence quarks occupy all the levels diving into negative energy region.

Let us rewrite the effective action in (2) as

$$S_{\text{eff}} = -iN_c \log \det(i\partial - MU_{78}),$$  

$$= -iN_c \log \det(i\partial - H(U_{78}))$$  

(6)

where

$$H(U_{78}) = -i\alpha \cdot \nabla + \beta MU_{78}.$$  

(7)

The classical energy of the soliton can be estimated from the quark determinant in Eq. (6) [25, 26]. We introduce the eigenstates of operators, $i\partial_t - H(U_{78})$ and $H(U_{78})$, such that

$$H(U_{78})\phi_{\mu}(x) = E_\mu \phi_{\mu}(x),$$  

$$(i\partial_t - H(U_{78}))\Psi_{\mu,n} = \lambda_{\mu,n} \Psi_{\mu,n}.$$  

(8)

(9)

where $\Psi_{\mu,n} = e^{-i\omega_n t}\phi_{\mu}$ and $\lambda_{\mu,n} = -E_\mu + \omega_n$. Imposing on $\Psi_{\mu,n}$ the anti-periodicity condition, $\Psi_{\mu,n}(x, T) = -\Psi_{\mu,n}(x, 0)$, reads

$$\omega_n T = (2n + 1)\pi.$$  

(10)

The determinant in Eq. (6) then becomes

$$\det(i\partial_t - H) = \prod_{\mu,n} \lambda_{\mu,n},$$  

$$= \prod_{\mu,n} \left(-E_\mu + \frac{(2n + 1)\pi}{T}\right),$$  

$$= C \prod_{\mu,n \geq 0} \left(1 - \frac{|E_\mu|^2 T^2}{(2n + 1)^2 \pi^2}\right),$$  

$$= C \prod_{\mu} \cos\left(\frac{1}{2} |E_\mu| T\right),$$  

$$= \frac{C}{2} \exp\left(\frac{i}{2} \sum_{\mu} |E_\mu| T\right),$$  

$$\times \prod_{n \geq 0} \left(1 + \exp(-i|E_\mu| T)\right)$$  

(11)

where

$$C = \prod_{n \geq 0} \left(\frac{(2n + 1)^2 \pi^2}{T^2}\right),$$

and the product formula for the cosine function $\cos(z) = \prod_{n \geq 1} (1 - 4z^2/(2n - 1)^2 \pi^2)$ has been used. Inserting (11) into (6), one obtains

$$S_{\text{eff}} = -N_c T \sum_{\mu} n_\mu |E_\mu| + N_c T \frac{1}{2} \sum_{\mu} |E_\mu|$$  

(12)

where $n_\mu$ is the valence quark occupation number which takes values only 0 or 1. Correspondingly, the classical energy is given by

$$E = E_{\text{val}} + E_{\text{field}}$$  

(13)

where

$$E_{\text{val}} = N_c \sum_{\mu} n_\mu |E_\mu|,$$  

$$E_{\text{field}} = -\frac{1}{2} N_c \sum_{\mu} |E_\mu|.$$
representing the valence quark and sea quark contributions to the total energy respectively.

The effective action $S_{\text{eff}}[U]$ is ultraviolet divergent and hence must be regularized. Using the proper-time regularization scheme [27], we can write

$$S_{\text{eff}}[U] = \frac{i}{2} N_c \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp} \left( e^{-D^1 \tau} - e^{-D_0^1 \tau} \right)$$

where $D_0$ and $H_0$ are operators with $U = 1$. The total energy is then given by

$$E_{\text{static}}[U] = E_{\text{val}}[U] + E_{\text{field}}[U] - E_{\text{field}}[U = 1]$$

where

$$E_{\text{val}} = N_c \sum_i E_{\text{val}}^{(i)}$$

$$E_{\text{field}} = N_c \sum_i \left\{ \mathcal{N}(E_\mu)|E_\mu| + \lambda N \exp \left[-\left(\frac{E_\mu}{\lambda}\right)^2\right]\right\}$$

with

$$\mathcal{N}(E_\mu) = -\frac{1}{\sqrt{4\pi}} \Gamma \left(\frac{1}{2},\left(\frac{E_\mu}{\lambda}\right)^2\right)$$

and $E_{\text{val}}^{(i)}$ is the valence energy of the $i$th valence quark. $\lambda$ is a cutoff parameter evaluated by the condition that the derivative expansion of (13) reproduces the pion kinetic term with the correct coefficient, i.e.,

$$f_\pi^2 = \frac{N_c M^2}{4\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} e^{-\tau M^2}.$$  \hspace{1cm} (16)

The extremum conditions for the total energy

$$\frac{\delta}{\delta F(\rho, z)} E_{\text{static}}[U] = 0, \quad \frac{\delta}{\delta \Theta(\rho, z)} E_{\text{static}}[U] = 0$$

yield the following equations of motion for the profile functions,

$$R^T(\rho, z) \cos \Theta(\rho, z) = R^L(\rho, z) \sin \Theta(\rho, z)$$

$$S(\rho, z) \sin F(\rho, z) = P(\rho, z) \cos F(\rho, z)$$

where

$$P(\rho, z) = R^T(\rho, z) \sin \Theta(\rho, z) + R^L(\rho, z) \cos \Theta(\rho, z).$$  \hspace{1cm} (20)

In terms of eigenfunction $\phi$ in Eq. (8), $R^T$, $R^L$, and $S$ are given by

$$R^T(\rho, z) = R_{\text{val}}^T(\rho, z) + R_0^T(\rho, z)$$

$$R^L(\rho, z) = R_{\text{val}}^L(\rho, z) + R_0^L(\rho, z)$$

$$S(\rho, z) = S_{\text{val}}(\rho, z) + S_0(\rho, z)$$

where

$$R_{\text{val}}^T(\rho, z) = \sum_i \int d\phi \chi_i(\rho, \varphi, z) i\gamma_5$$

$$\times (\tau_1 \cos m_\pi \varphi + \tau_2 \sin m_\pi \varphi) \phi_i(\rho, \varphi, z),$$

$$R_0^T(\rho, z) = \sum_{\mu} N(\mu) \text{sgn}(E_\mu) \int d\phi \chi_i(\rho, \varphi, z) i\gamma_5$$

$$\times (\tau_1 \cos m_\pi \varphi + \tau_2 \sin m_\pi \varphi) \phi_i(\rho, \varphi, z),$$

$$R_{\text{val}}^L(\rho, z) = \sum_{\mu} N(\mu) \text{sgn}(E_\mu) \int d\phi \chi_i(\rho, \varphi, z) i\gamma_5$$

$$\times \tau_3 \phi_i(\rho, \varphi, z),$$

$$S_{\text{val}}(\rho, z) = \sum_i \int d\phi \chi_i(\rho, \varphi, z) \phi_i(\rho, \varphi, z),$$

$$S_0(\rho, z) = \sum_{\mu} N(\mu) \text{sgn}(E_\mu) \int d\phi \chi_i(\rho, \varphi, z) \phi_i(\rho, \varphi, z).$$

and subscripts, 0 and val, represent the vacuum and valence quark contributions respectively. The boundary conditions for the profile functions were constructed by Braaten and Carson [22];

$$F(\rho, z) \rightarrow 0 \quad \text{as} \quad \rho^2 + z^2 \rightarrow \infty,$$

$$F(0, 0) = -\pi, \quad \Theta(0, z) = \begin{cases} 0, & z > 0 \\ \pi, & z < 0 \end{cases}.$$  \hspace{1cm} (24)

The procedure to obtain the self-consistent solutions of Eq. (18) and (19) is 1) solve the eigenfunction in Eq. (9) under assumed initial profile functions $F_0(\rho, z), \Theta(0, z)$ which satisfy the boundary conditions eqs. (24), 2) use the resultant eigenfunctions and eigenvalues to calculate $R^T, R^L, S$ and also $P$ in Eq. (20), 3) solve Eq. (13) and (19) to obtain new profile functions, and 4) repeat 1)–3) until the self-consistency is attained.

In Figs. 14 we show the spectrum of the quark orbits in the background of chiral fields with winding number $m_\pi = 2 - 5$, as a function of the size parameter $X$. The profile functions are parameterized by $X$ as

$$F(\rho, z) = -\pi + \pi \sqrt{\rho^2 + z^2}/X \quad \text{for} \quad X \leq \sqrt{\rho^2 + z^2}$$

$$= 0 \quad \text{otherwise},$$

$$\Theta(\rho, z) = \tan^{-1}(\rho/z).$$  \hspace{1cm} (26)

To examine the spectrum in detail, let us consider the Hamiltonian defined in (7). For the axially symmetric chiral field in Eq. (4), this Hamiltonian commutes with the third component of the grand spin operator $K_3$ and the time-reversal operator $T$. These are specifically,

$$K_3 = L_3 + \frac{1}{2} \sigma_3 + \frac{1}{2} m_\pi \tau_3$$

$$T = i\gamma_1 \gamma_5 \cdot i\gamma_1 \gamma_5 C.$$  \hspace{1cm} (28)
where $L_3, \sigma_3, \text{ and } \tau_3$ are respectively the third component of orbital angular momentum, spin, and isospin operator, and $C$ is a charge conjugation operator. The parity operator is defined by $P = \gamma_0$ for odd $B$, and $P = \gamma_0 \gamma_3$ for even $B$. Thus, the eigenvalues of the Hamiltonian can be specified by the magnitude of $K_3$ and the parity $\pi = \pm$. We have $K_3 = 0, \pm 1, \pm 2, \pm 3, \cdots$ for odd $B$, and $K_3 = \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{7}{2}, \cdots$ for even $B$. Since the Hamiltonian is invariant under time reverse, the states of $+K_3$ and $-K_3$ are degenerate in energy.

Fig. 1 shows the quark spectrum with $m_w = 2$. The bound states diving into negative region are doubly degenerate with $K_3 = \pm \frac{3}{2}$. Thus putting $N_c = 3$ valence quarks on each of the bound levels, we have the $B = 2$ soliton solution. For $m_w = 3$, the spectrum of $K_3 = \pm 1^-$ (double degeneracy) and $K_3 = 0^+$ states dive into negative-energy region. Thus, we have the $B = 3$ soliton solution. For $m_w = 4$, the spectrum of $K_3 = \pm \frac{5}{2}^+$ and $K_3 = \pm \frac{7}{2}^-$ (both doubly degenerate) states dive into negative region. Thus, we have the $B = 4$ soliton solution. For $m_w = 5$, the spectrum of $K_3 = \pm 2^+$ (double), $K_3 = \pm 1^-$ (double) and $K_3 = 0^+$ states dive into negative-energy region. Thus we have the $B = 5$ soliton solution. These results confirm that the baryon number of the soliton is identified with the number of diving levels occupied by $N_c$ valence quarks. It can be seen that the degeneracy which occurs due to symmetry of the chiral field reduces the number of states and hence makes large shell gaps. This observation indicates that degeneracy in spectrum contribute to make classical energies of the soliton solutions lower. In fact, our $B = 2$ solution which is considered to be the minimum energy soliton from the study of the $B = 2$ skyrmions provides the maximum degeneracy in spectrum. It will be interesting to study minimum solutions from this point of view.

The baryon number density is defined by the zeroth
FIG. 5: Contour plot of the profile functions with $B = 2$.

FIG. 6: Contour plot of the profile functions with $B = 3$. 
FIG. 7: Contour plot of the profile functions with $B = 4$. 

FIG. 8: Contour plot of the profile functions with $B = 5$. 
The root mean square radius is given by

\[ \langle r^2 \rangle = \frac{1}{m_w} \sum_i \int \rho d\rho d\varphi (\rho^2 + z^2) \phi_i^\dagger (\rho, \varphi, z) \phi_i (\rho, \varphi, z) \]

and

\[ \langle r^2 \rangle_0 = \frac{1}{m_w} \sum_\mu N(E_\mu) \text{sgn}(E_\mu) \times \int \rho d\rho d\varphi (\rho^2 + z^2) \phi_\mu^\dagger (\rho, \varphi, z) \phi_\mu (\rho, \varphi, z) . \]

These values for each baryon number are shown in Table II.

### III. QUANTIZATION

#### A. Rotational Zero Mode Quantization

The solitons that we obtained in the previous section are classical objects and therefore must be quantized to assign definite spin and isospin to them. Quantization of the solitons can be performed semiclassically for their rotational zero modes. For the hedgehog soliton, because of its topological structure, a rotation in isospin space is followed by a simultaneous spatial rotation. For the axially symmetric soliton, there are six rotational zero modes by rotations of iso-degrees of freedom and spatial rotations.

Let us introduce the “dynamically rotated” chiral fields

\[ \tilde{U}(x, t) = A(t) U(x') A(t) \]

where

\[ \tilde{Z}_j^i [B(t)] = \frac{1}{2} Tr [\sigma^j B(t) \sigma_i B(t)^\dagger] , \]

and \( A(t) \) and \( B(t) \) are time-dependent SU(2) matrices generating an iso-rotation and a spatial rotation respectively. By transforming the rotating frame of reference, the Dirac operator with Eq. (32) can be written as

\[ i \vec{D} = i \partial_t - MU^\gamma (x, t) = A(t) S(t)^\dagger \gamma^0 [i\partial_t + i\gamma^0 \gamma^k \partial^k - U^\gamma (x')] + i A^\dagger \hat{A} + i S^\dagger \hat{S}^j [S(t) A(t) \dagger] \]

where

\[ \gamma^\mu = \Lambda^\mu_\nu S \gamma^\nu S^\dagger = \left( \begin{array}{c} \gamma^0 \\ \gamma^k + (r' \times \theta) \gamma^k \end{array} \right) \]

and \( S(t) \) is the rotation operator for the Dirac field and \( \theta \) is an angle of the spatial rotation. Note that the gamma matrices \( \gamma^\mu \) explicitly depend on the coordinates and do not transform as a contravariant vector. Substituting Eq. (35) into Eq. (34), one obtains

\[ i \vec{D} = A(t) S(t)^\dagger \gamma^0 [i\partial_t - H(U^\gamma) + \Omega_A + \Omega_B] S(t) A(t)^\dagger \]

where

\[ \Omega_A = \int \rho d\rho d\varphi (\rho^2 + z^2) \phi_i^\dagger (\rho, \varphi, z) \phi_i (\rho, \varphi, z) \]

### TABLE I: The classical mass for \( B = 1 \sim 5 \) ([MeV]).

| \( B \) | Valence | Vacuum | Total |
|---|---|---|---|
| 1 | 173 | 674 | 1192 |
| 2 | 173 | 1166 | 2204 |
| 3 | 173 | 298 | 3493 |
| 4 | 106 | 232 | 232 | 2727 | 4753 |
| 5 | 145 | 319 | 409 | 2537 | 6543 |

### TABLE II: The mean radius and mean root square radius in each baryon number for \( B = 1 \sim 5 \). (The result of \( B = 1 \) is quoted by Ref. 2).

| \( B \) | \( \langle \rho \rangle [fm] \) | \( \sqrt{\langle r^2 \rangle} [fm] \) |
|---|---|---|
| 1 | 0.785 | |
| 2 | 0.821 | |
| 3 | 0.854 | |
| 4 | 1.140 | |
| 5 | 1.225 | |
where

$$\Omega_A = iA^\dagger \dot{A} = \frac{1}{2} \Omega_3^a \tau_a$$

$$\Omega_B = iS^\dagger \dot{S} + (r \times p) \cdot \theta = \Omega_3^B J_b$$

with $J_a = 1/2\epsilon_{abc} \gamma^b \gamma^c - i(r \times \nabla)_a$. $\Omega_A$ and $\Omega_B$ are the angular velocity operators for an isorotation and for a spatial rotation respectively. Assuming that the rotation of the soliton is adiabatic, we shall expand the effective action $S_{eff}$ around the classical solution $U_S$ with respect to the angular momentum velocity $\Omega_A$ and $\Omega_B$ up to second order

$$S_{eff}(U) = S_{eff}(U_S) - iN_c \text{Sp} \left[ \log(i\partial_t - H(U_S^\gamma)) + \Omega_A + \Omega_B \right]$$

$$- \text{Sp} \left[ \log(i\partial_t - H(U_S^\gamma)) \right]$$

(39)

With the proper-time regularization, we have

$$S_{reg}^{eff}(U) = S_{reg}^{eff}(U_S)$$

$$+ \frac{1}{2} \int dt \left[ I^{AA}_{0,ab} \Omega_a^a(t) \Omega_A^b(t) + I^{AB}_{0,ab} \Omega_a^a(t) \Omega_B^b(t) + I^{BA}_{0,ab} \Omega_a^a(t) \Omega_A^b(t) + I^{BB}_{0,ab} \Omega_a^a(t) \Omega_B^b(t) \right]$$

(40)

where $I_{0S}$ are the vacuum sea contributions to the moments of inertia defined by

$$I^{AA}_{0,ab} = \frac{1}{8} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n|\tau_a|m\rangle \langle m|\tau_b|n\rangle$$

$$I^{AB}_{0,ab} = \frac{1}{4} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n|\tau_a|m\rangle \langle m|J_b|n\rangle$$

$$I^{BA}_{0,ab} = \frac{1}{4} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n|J_a|m\rangle \langle m|\tau_b|n\rangle$$

$$I^{BB}_{0,ab} = \frac{1}{2} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n|J_a|m\rangle \langle m|J_b|n\rangle$$

FIG. 9: baryon number density [fm$^{-3}$].
with the cutoff function \( f(E_m, E_n, \Lambda) \)

\[
f(E_m, E_n, \Lambda) = \frac{2\Lambda}{\sqrt{\pi}} \frac{e^{-E_m^2/\Lambda^2} - e^{-E_n^2/\Lambda^2}}{E_m^2 - E_n^2} + \frac{\text{sgn}(E_m)\text{erfc}(|E_m|/\Lambda) - \text{sgn}(E_n)\text{erfc}(|E_n|/\Lambda)}{E_m - E_n}.
\]

(41)

Similarly, for the valence quark contribution to the moments of inertia, we have

\[
I_{\text{val},ab}^{AA} = \frac{1}{2}N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | \tau_a | m \rangle \langle m | \tau_b \rangle \text{val}}{E_m - E_{\text{val}}},
\]

\[
I_{\text{val},ab}^{ab} = N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | \tau_a | m \rangle \langle m | \tau_b \rangle \text{val}}{E_m - E_{\text{val}}},
\]

\[
I_{\text{val},ab}^{BA} = N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | J_a | m \rangle \langle m | J_b \rangle \text{val}}{E_m - E_{\text{val}}},
\]

\[
I_{\text{val},ab}^{BB} = 2N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | J_a | m \rangle \langle m | J_b \rangle \text{val}}{E_m - E_{\text{val}}}.
\]

(42)

The total moments of inertia are then given by the sum of the vacuum and valence as

\[
I_{\text{vac},ab}^{AA} = I_{\text{val},ab}^{AA} + I_{0,ab}^{AA},
\]

\[
I_{\text{vac},ab}^{ab} = I_{\text{val},ab}^{ab} + I_{0,ab}^{ab},
\]

\[
I_{\text{vac},ab}^{BA} = I_{\text{val},ab}^{BA} + I_{0,ab}^{BA},
\]

\[
I_{\text{vac},ab}^{BB} = I_{\text{val},ab}^{BB} + I_{0,ab}^{BB}.
\]

From axial symmetry of the system, following relations are derived

\[
I_{ij} = 0, \quad i \neq j,
\]

\[
I_{11}^{AA} = I_{22}^{AA}, \quad I_{11}^{BB} = I_{22}^{BB},
\]

\[
I_{12}^{AA} = I_{21}^{AA}, \quad I_{12}^{BB} = I_{21}^{BB},
\]

\[
I_{13}^{AA} = m_1^2 I_{33}^{AA}, \quad I_{13}^{BB} = I_{33}^{BB} = -m_1 I_{33}^{AA}.
\]

(43)

Theoretically, these moments of inertia can be computed using the eigenstates of Eq. (43). However, due to the difference of the boundary conditions between the initial and final states of the matrix element, the moments of inertia acquire nonzero values with vanishing pion fields. To overcome this problem, we make the following replacement

\[
\langle n | J_a | m \rangle \rightarrow \langle n | [H(U_s^\dagger), J_a] | m \rangle \langle E_n - E_m \rangle = \langle n | [MU_s, J_a] | m \rangle \langle E_n - E_m \rangle
\]

(44)

where \( l_a = -i(r \times \nabla)_a \). Unless the Hamiltonian explicitly depend on the coordinates, the numerator vanishes with vanishing pion fields. The spurious contributions to the moment of inertia can be removed in this way.

| B | Valence | Sea | Total |
|---|---|---|---|
| 2 | \( I_{11}^{AA} \) | 0.00773 | 0.00363 | 0.01136 |
| 3 | \( I_{11}^{BB} \) | 0.01141 | 0.00464 | 0.01605 |
| 4 | \( I_{11}^{AA} \) | 0.00429 | 0.00125 | 0.00554 |
| 5 | \( I_{11}^{BB} \) | 0.01231 | 0.00280 | 0.01511 |
| 6 | \( I_{11}^{BB} \) | 0.02174 | 0.00384 | 0.02558 |
| 7 | \( I_{11}^{AA} \) | 0.00594 | 0.00027 | 0.00622 |
| 8 | \( I_{11}^{BB} \) | 0.04108 | 0.00959 | 0.05066 |
| 9 | \( I_{11}^{AB} \) | 0.04272 | 0.01245 | 0.05517 |
| 10 | \( I_{11}^{AA} \) | 0.01172 | 0.00074 | 0.01246 |
| 11 | \( I_{11}^{BB} \) | 0.02786 | 0.00716 | 0.03502 |
| 12 | \( I_{11}^{AB} \) | 0.12124 | 0.01112 | 0.13236 |
| 13 | \( I_{11}^{AA} \) | 0.01368 | 0.00007 | 0.01375 |

The quantization conditions for the collective coordinates, \( A(t) \) and \( B(t) \), define a body-fixed isospin operator \( K \) and a body-fixed angular momentum operator \( L \) as

\[
I_{ab}^{AA} \Omega_A + I_{ab}^{BB} \Omega_B \rightarrow 0 \quad \text{tr} \left( A \frac{\partial}{\partial A} - \frac{\partial}{\partial B} \right) = -K_a \]

(45)

\[
I_{ab}^{AA} \Omega_A + I_{ab}^{BB} \Omega_B \rightarrow \text{tr} \left( \frac{\sigma}{2} B \frac{\partial}{\partial B} \right) = -L_a.
\]

(46)

These are related to the usual coordinate-fixed isospin operator \( I_a \) and coordinate-fixed angular momentum \( J_a \) operator by transformations,

\[
I_a = -i \chi_a [A(t)] K_b, \quad J_a = -i \chi_a [B(t)] K_b.
\]

(47)

To estimate the quantum energy corrections, let us introduce the basis functions of the spin and isospin operators which were inspired from the cranking method for nuclei

\[
\langle A, B | i \delta k_3, j \delta k_3 \rangle = \frac{\sqrt{(2i + 1)(2j + 1)}}{8\pi^2} \frac{D_i^{(j)}(A)D_j^{(j)}(B)}{s_{33}^{(j)}},
\]

\[
\text{where } D \text{ is the Wigner rotation matrix. Then, we find the quantized energies of the soliton as}
\]

\[
E = E_{\text{static}} + \frac{1}{2I_{11}^{AB}} i(i + 1) + \frac{1}{2I_{11}^{AA}} j(j + 1)
\]

\[
+ \frac{1}{2} \left( \frac{1}{I_{11}^{AB}} - \frac{1}{I_{11}^{AA}} - \frac{m_1^2}{I_{11}^{BB}} \right) k_3^2
\]

(48)

where \( i(i + 1), j(j + 1) \) and \( k_3 \) are eigenvalues of the Casimir operators \( I^2 \) and \( J^2 \), and the operator \( K_3 \), respectively.

On Table III are the results of our calculation of moments of inertia, \( I_{11}^{AA}, I_{11}^{BB} \) and \( I_{33}^{AA} \), with \( B = 2 \). It is instructive to compare our results with the Skyrme model \( 23 \) where \( U_{11} = 0.104, V_{11} = 0.163 \) and \( U_{33} = 0.0709 \) which are correspondingly our \( I_{11}^{AA}, I_{11}^{BB} \) and \( I_{33}^{AA} \). They are qualitatively in good agreement.
B. Finkelstein-Rubinstein constraints

If a multi-skyrmion describes atomic nuclei upon quantization, it has to be quantized as a boson or as a fermion whether $B$ is even or odd. This requirement is implemented in the form of Finkelstein-Rubinstein (FR) constraints. The FR constraints for the rational map ansatz was constructed in \cite{33} and \cite{13} and applied to predict the ground states of skyrmions up to $B = 22$. In this section, we shall apply the FR constraints for the rational map ansatz directly to our axially symmetric multi-skyrmions and obtain their ground states.

Following the notation in \cite{33}, let $g$ be a rotation by $\alpha$ around $\mathbf{n}$ followed by an isorotation by $\beta$ around $\mathbf{N}$. Then the FR constraints can be defined as

$$\exp(-i\alpha \mathbf{n} \cdot \mathbf{J}) \exp(-i\beta \mathbf{N} \cdot \mathbf{I}) \psi = \chi_{FR}(g) \psi \quad (49)$$

where

$$\chi_{FR}(g) = \begin{cases} 1 & \text{if contractible} \cr -1 & \text{otherwise.} \end{cases}$$

and, $\mathbf{J}$ and $\mathbf{I}$ are space-fixed spin and isospin operators respectively. $\psi$ is the wave function which transforms under a tensor product of rotations and isorotations. In particular, a closed loop is noncontractible for odd $B$ and contractible for even $B$, which is consistent with spin statistics. Consequently, quantum numbers $I$ and $J$ are half-integers for odd $B$ and integers for even $B$.

In order to construct the ground states for a given baryon number $B$, let us define $N(L(\alpha, \beta))$ as a homotopy invariant for a loop $L$ generated by rotations by $\alpha$ and isorotations by $\beta$. Then, for the axially symmetric rational map of degree $B$, it is given by \cite{33}

$$N(L(\alpha, \beta)) = \frac{B}{2\pi}(B\alpha - \beta). \quad (50)$$

It can be shown that $N \mod 2$ determines if the loop is contractible or not in the same sense as $B \mod 2$. Therefore, $N \mod 2$ gives the FR constraints for each generator of the symmetry group of the rational map.

The axially symmetric rational map with degree $B$ is given by

$$R(z) = \frac{1}{z^B}. \quad (51)$$

There are two symmetric generators for this rational map. One is a rotation by $\alpha$ followed by an isorotation by $\beta = B\alpha$. Substituting it into (50), one obtains $N(L(\alpha, B\alpha)) = 0$. The FR constraints for this loop is thus given by

$$e^{-i\pi(L_3 - BK_3)} \psi = \psi. \quad (52)$$

where we introduced the body-fixed spin ($L$) and isospin ($K$) operators related to the space-fixed operators by orthogonal transformations. The other symmetry is $C_2$ with transformation

$$z \to \frac{1}{z}, \quad R(z) \to \frac{1}{R(z)}. \quad (53)$$

This corresponds to $\alpha = \beta = \pi$ and hence $N(L(\pi, \pi)) = B(B - 1)/2$. The FR constraints for this loop is

$$e^{-i\pi(L_3 + K_3)} \psi = (-1)^{B(B-1)/2} \psi. \quad (54)$$

In the following we construct the ground states consistent with the derived FR constraints (52) and (54) for $B = 2 - 5$ with axial symmetry.

- **$B = 2$**
  We find the FR constraints

$$e^{-i\pi(L_3 - 2K_3)} \psi = \psi$$

$$e^{-i\pi(L_3 + K_3)} \psi = -\psi. \quad (56)$$

This gives the ground state as $|J, L_3 \rangle |I, K_3 \rangle = |1, 0 \rangle |0, 0 \rangle$.

- **$B = 3$**
  We find the FR constraints

$$e^{-i\pi(L_3 - 3K_3)} \psi = \psi$$

$$e^{-i\pi(L_3 + K_3)} \psi = -\psi. \quad (58)$$

This gives the ground state as $|J, L_3 \rangle |I, K_3 \rangle = |\frac{5}{2}, \frac{3}{2} \rangle |\frac{1}{2}, \frac{1}{2} \rangle$.

- **$B = 4$**
  We find the FR constraints

$$e^{-i\pi(L_3 - 4K_3)} \psi = \psi$$

$$e^{-i\pi(L_3 + K_3)} \psi = \psi. \quad (60)$$

This gives the ground state as $|0, 0 \rangle |0, 0 \rangle$.

- **$B = 5$**
  We find the FR constraints

$$e^{-i\pi(L_3 - 5K_3)} \psi = \psi$$

$$e^{-i\pi(L_3 + K_3)} \psi = -\psi. \quad (62)$$

This gives the ground state as $|J, L_3 \rangle |I \rangle = |\frac{7}{2}, \frac{3}{2} \rangle |\frac{1}{2}, \frac{1}{2} \rangle$.

Thus, for even $B$, the axially symmetric solitons are possible candidates of the ground states of $B$ atomic nuclei as is the case of the deuteron and $^4\text{He}$ while for odd $B$ they emerge only as excited states.
IV. NUMERICAL ANALYSIS

A. Eigen Equations

In this subsection, we show the numerical analysis of the eigen equations in detail. To solve the eigenvalue equation of the form,

$$[-i \alpha \cdot \nabla + \beta M (\cos F(p, z) + i \gamma \tau \cdot \hat{n} \sin F(p, z))] \phi_{\mu}(x) = E_{\mu} \phi_{\mu}(x), \quad (63)$$

we introduce the deformed harmonic oscillator spinor basis which was originally constructed by Gambhir et al., in the relativistic mean field theory for deformed nuclei \[34\]. The upper and lower components of the Dirac spinors are expanded separately by the basis as

$$\phi_{\mu}(x) = \left( \begin{array}{c} f_{\mu}(x) \\ i g_{\mu}(x) \end{array} \right) = \left( \sum_{\alpha} f_{\alpha} \Phi_{\alpha}(x, s) \right) \chi_{m_{s}}, \quad (64)$$

where $\Phi_{\alpha}(x, s, \tau)$ are the eigenvectors of a deformed harmonic oscillator potential

$$V_{osc}(p, z) = \frac{1}{2} M \omega_{p}^{2} p^{2} + \frac{1}{2} M \omega_{z}^{2} z^{2}, \quad (65)$$

and defined by

$$\Phi_{\alpha}(x, m_{s}) = \frac{1}{\sqrt{2\pi}} \phi^{[\omega]}(\rho) \phi_{n_{z}}(z) e^{i u \phi} \chi_{m_{s}}, \quad (66)$$

with

$$\phi^{[\omega]}(\rho) = \frac{N_{\mu}^{\omega}}{\sqrt{\omega \rho}} e^{-\frac{1}{2} \alpha_{\rho} \rho^{2}} L_{n_{\rho}}^{\alpha}(\rho)$$

$$n_{\rho} = 0, 1, 2, \ldots, N_{imax}$$

$$\phi_{n_{z}}(z) = N_{n_{z}} e^{-\frac{1}{2} \alpha_{n_{z}} z^{2}} H_{n_{z}}(\sqrt{\alpha_{n_{z}} z})$$

$$n_{z} = 1, 3, \ldots, 2N_{zmax} + 1 \text{ or } 0, 2, \ldots, 2N_{zmax},$$

and

$$\chi_{+} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \chi_{-} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \quad (67)$$

depending on if the eigenvalues of the third components of the spin $m_{s}$ (isospin $m_{\tau}$) takes $+1$ or $-1$. The functions, $L_{n_{\rho}}^{\alpha}$ and $H_{n_{z}}$, are the associated Laguerre polynomials and the Hermite polynomials with the normalization constants

$$N_{n_{\rho}^{[\omega]}} = \sqrt{\frac{2 \alpha_{\rho} n_{\rho}!}{(n_{\rho} + |\omega|)!}}, \quad N_{n_{z}} = \frac{1}{\sqrt{2^{n_{z}} n_{z}! (n_{z}) !}}. \quad (68)$$

These polynomials can be calculated by following recursion relations

$$x \frac{d}{dx} L_{n}^{\alpha}(x) = n L_{n-1}^{\alpha}(x) \quad (69)$$

and

$$H_{n+1}(x) - 2x H_{n}(x) + 2 \omega_{z} H_{n-1}(x) = 0 \quad (70)$$

where constants $\alpha_{p}$ and $\alpha_{n}$ can be expressed by the oscillator as

$$\alpha_{p} = \frac{M \omega_{p}}{\hbar}, \quad \alpha_{n} = \frac{M \omega_{z}}{\hbar} \quad (73)$$

which are free parameters chosen optimally. The $N_{imax}$ and $N_{zmax}$ are increased until convergence is attained. The parity transformation rule of $\Phi_{\alpha}$ is given by

$$\Phi_{\alpha}(\rho, \varphi + \pi, -z; s, t) = (-1)^{w+n_{z}} \Phi_{\alpha}(\rho, \varphi, z; s, t) \quad (74)$$

where

$$H_{n_{z}}(-\sqrt{\alpha_{n_{z}} z}) = (-1)^{n_{z}} H_{n_{z}}(\sqrt{\alpha_{n_{z}} z}) \quad (75)$$

has been used. The parity is $+ \text{ for } \omega + n_{z} = \text{ odd}$, and $- \text{ for } \omega + n_{z} = \text{ even}$.

There are two sets of the complete basis for each parity. One is the natural basis with $K_{s}^{p} = 0^{+}, 1^{-}, 2^{+}, \ldots$, for odd $B$ and $K_{s}^{p} = \frac{1}{2}^{+}, \frac{3}{2}^{+}, \frac{5}{2}^{+}, \ldots$ for even $B$. Another is the unnatural basis with $K_{s}^{p} = 0^{-}, 1^{+}, 2^{-}, \ldots$, for odd $B$ and $K_{s}^{p} = \frac{1}{2}^{-}, \frac{3}{2}^{-}, \frac{5}{2}^{-}, \ldots$ for even $B$. The natural basis is given by

$$\phi_{(n)}^{[\omega]}(x) = \left( \begin{array}{c} \sum_{\alpha(0), \beta} f_{\alpha(0), \beta} \Phi_{\alpha(0)}(x, \uparrow S) + \sum_{\alpha(1), \beta} f_{\alpha(1), \beta} \Phi_{\alpha(1)}(x, \downarrow S) \\ \sum_{\beta(0)} i g_{\beta(0), \alpha} \Phi_{\beta(0)}(x, \uparrow S) + \sum_{\beta(1)} i g_{\beta(1), \alpha} \Phi_{\beta(1)}(x, \downarrow S) \end{array} \right) \chi^{I}_{u} + \left( \begin{array}{c} \sum_{\alpha(2), \beta} f_{\alpha(2), \beta} \Phi_{\alpha(2)}(x, \uparrow S) + \sum_{\alpha(3), \beta} f_{\alpha(3), \beta} \Phi_{\alpha(3)}(x, \downarrow S) \\ \sum_{\beta(2)} i g_{\beta(2), \alpha} \Phi_{\beta(2)}(x, \uparrow S) + \sum_{\beta(3)} i g_{\beta(3), \alpha} \Phi_{\beta(3)}(x, \downarrow S) \end{array} \right) \chi^{I}_{d} \quad (76)$$

where

$$\alpha(0) = \{ n_{r}, n_{z} : \text{odd}, \omega_{0} = K_{3} - 1/2 - m_{w}/2 \}$$

$$\alpha(1) = \{ n_{r}, n_{z} : \text{even}, \omega_{1} = K_{3} + 1/2 - m_{w}/2 \}$$

$$\alpha(2) = \{ n_{r}, n_{z} : \text{odd}, \omega_{2} = K_{3} - 1/2 + m_{w}/2 \}$$

$$\alpha(3) = \{ n_{r}, n_{z} : \text{odd}, \omega_{3} = K_{3} + 1/2 + m_{w}/2 \}$$

and

$$\beta(0) = \{ n_{r}, n_{z} : \text{even}, \omega_{0} = K_{3} - 1/2 - m_{w}/2 \}$$

$$\beta(1) = \{ n_{r}, n_{z} : \text{odd}, \omega_{1} = K_{3} + 1/2 - m_{w}/2 \}$$

$$\beta(2) = \{ n_{r}, n_{z} : \text{odd}, \omega_{2} = K_{3} - 1/2 + m_{w}/2 \}$$

$$\beta(3) = \{ n_{r}, n_{z} : \text{even}, \omega_{3} = K_{3} + 1/2 + m_{w}/2 \}. \quad (76)$$
The unnatural basis \( \phi_{\mu}^{(u)} \) is given by replacing, \( \alpha \leftrightarrow \beta \) in (76).

B. Matrix elements of the eigenequation

By using the natural and unnatural basis, the eigenvalue problem in Eq. (43) can be reduced to a symmetric matrix diagonalization problem.

Let us calculate the matrix elements of the Hamiltonian below. For the kinetic term

\[
\alpha \cdot p = \begin{pmatrix}
0 & \sigma \cdot p \\
\sigma \cdot p & 0
\end{pmatrix},
\]

we have

\[
\langle \Phi_{\alpha(0)} | \sigma \cdot p | \Phi_{\beta(0)} \rangle = \frac{1}{2\pi} \int d^3 x \phi_{\alpha(0)}^{\dag} (\rho) \phi_{\alpha(0)} (z) e^{-i \omega_0 \phi} \\
\times (\frac{\partial}{\partial \phi}) \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z) e^{i \omega_0 \phi}
\]

\[
= \delta_{\alpha, \alpha'} (N_{n_{1}} N_{n_{1}'} \sqrt{\alpha_{n_{1}} n_{1}'} - \frac{1}{N_{n_{1}'}^2} \delta_{n_{1} n_{1}'} - 1) \\
- \frac{1}{2} N_{n_{1}} N_{n_{1}'} \sqrt{\alpha_{n_{1}} n_{1}'} \delta_{n_{1} n_{1}'} + 1)
\]

and

\[
\langle \Phi_{\alpha(0)} | \sigma \cdot p | \Phi_{\beta(1)} \rangle
\]

\[
= \frac{1}{2\pi} \int d^3 x \phi_{\alpha(0)}^{\dag} (\rho) \phi_{\alpha(0)} (z) e^{-i \omega_0 \phi} \\
\times e^{-i \phi} \left( \frac{\partial}{\partial \rho} - i \frac{\partial}{\partial \phi} \right) \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z) e^{i \omega_0 \phi}
\]

\[
= \left\{ \begin{array}{l}
\delta_{n_{1} n_{1}'} \sqrt{\alpha_{n_{1}'} n_{1}'} \left( \sqrt{n_{1} + \omega_{0} + \frac{1}{2} \delta_{n_{1} n_{1}'} + \sqrt{n_{1} - 1} \delta_{n_{1} n_{1}'} - 1} \right) \\
- \delta_{n_{1} n_{1}'} \sqrt{\alpha_{n_{1}'} n_{1}'} \left( \sqrt{n_{1} - \omega_{0} \delta_{n_{1} n_{1}'} + \sqrt{n_{1} + 1} \delta_{n_{1} n_{1}'} - 1} \right). \\
\end{array} \right.
\]

In the natural basis, quantum numbers \((n_{1}, n_{1}')\) takes values \((1, 2), (3, 4), \ldots\) for the upper part and \((1, 0), (3, 2), \ldots\) for the lower part. In the unnatural basis, \((n_{1}, n_{1}') = (0, 1), (2, 3), \ldots\) for the upper part and \((n_{1}, n_{1}') = (2, 1), (3, 2), \ldots\) for the lower part.

For the potential term

\[
\beta M \cos F(\rho, z) + i \gamma \sigma \cdot \hat{n} \sin F(\rho, z)
\]

\[
= M \left( \begin{array}{c}
\cos F(\rho, z) \\
-i \sigma \cdot \hat{n} \sin F(\rho, z)
\end{array} \right),
\]

we have

\[
\langle \Phi_{\alpha(0)} \chi_{\mu}^{f} | M \cos F(\rho, z) | \Phi_{\alpha(0)} \chi_{\mu}^{f} \rangle = \int \rho d\rho dz M \cos F(\rho, z)
\]

\[
\times \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z) \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z)
\]

\[
\langle \Phi_{\alpha(0)} \chi_{\mu}^{f} | M \sigma \cdot \hat{n} \sin F(\rho, z) | i \Phi_{\beta(0)} \chi_{\mu}^{f} \rangle
\]

\[
= - \int \rho d\rho dz M \sin \Theta(\rho, z) \sin F(\rho, z)
\]

\[
\times \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z) \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z)
\]

and

\[
\langle \Phi_{\alpha(0)} \chi_{\mu}^{f} | M \sigma \cdot \hat{n} \sin F(\rho, z) | i \Phi_{\beta(2)} \chi_{\mu}^{f} \rangle
\]

\[
= - \int \rho d\rho dz M \sin \Theta(\rho, z) \sin F(\rho, z)
\]

\[
\times \phi_{\beta(2)}^{\dag} (\rho) \phi_{\beta(2)} (z) \phi_{\beta(2)}^{\dag} (\rho) \phi_{\beta(2)} (z).
\]

Other elements can be calculated in the same manner. In Appendix A we shall present these calculations of the matrix elements in more detail.

C. Matrix Elements for the Moments of Inertia

To compute the matrix elements of the moments of inertia, we shall evaluate \( \langle n | r_1 | m \rangle, \langle n | r_2 | m \rangle \) and \( \langle n | J_1 | m \rangle \). For \( \langle n | r_1 | m \rangle \), only following elements survive

\[
\langle \Phi_{\alpha(0)} \chi_{\mu}^{f} | \Phi_{\alpha(2)} \chi_{\mu}^{f} \rangle = \langle \Phi_{\alpha(0)} \chi_{\mu}^{f} | \Phi_{\beta(2)} \chi_{\mu}^{f} \rangle
\]

\[
= \frac{1}{2\pi} \int d^3 x \phi_{\alpha(0)}^{\dag} (\rho) \phi_{\alpha(0)} (z) e^{-i \omega_0 \phi} | \phi_{\beta(2)}^{\dag} (\rho) \phi_{\beta(2)} (z) e^{i \omega_0 \phi}
\]

\[
= \delta_{\alpha, \alpha'} \delta_{n_{1} n_{1}'} \delta_{K_{1}} - \frac{i}{\kappa_{1}} \kappa_{1}^{*} + \frac{m}{\kappa_{1}}
\]

\[
\langle \Phi_{\alpha(1)} \chi_{L}^{f} | \Phi_{\beta(2)} \chi_{L}^{f} \rangle = \langle \Phi_{\alpha(1)} \chi_{L}^{f} | \Phi_{\beta(1)} \chi_{L}^{f} \rangle = \langle \Phi_{\beta(1)} \chi_{L}^{f} | \Phi_{\beta(2)} \chi_{L}^{f} \rangle
\]

\[
= \frac{1}{2\pi} \int d^3 x \phi_{\alpha(1)}^{\dag} (\rho) \phi_{\alpha(1)} (z) e^{-i \omega_0 \phi} | \phi_{\beta(1)}^{\dag} (\rho) \phi_{\beta(1)} (z) e^{i \omega_0 \phi}
\]

\[
= \delta_{\alpha, \alpha'} \delta_{n_{1} n_{1}'} \delta_{K_{1}} - \frac{i}{\kappa_{1}} \kappa_{1}^{*} + \frac{m}{\kappa_{1}}
\]

\[
\langle \Phi_{\alpha(3)} \chi_{D}^{f} | \Phi_{\alpha(0)} \chi_{L}^{f} \rangle = \langle \Phi_{\alpha(3)} \chi_{D}^{f} | \Phi_{\beta(0)} \chi_{L}^{f} \rangle
\]

\[
= \frac{1}{2\pi} \int d^3 x \phi_{\alpha(3)}^{\dag} (\rho) \phi_{\alpha(3)} (z) e^{-i \omega_2 \phi} | \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z) e^{i \omega_2 \phi}
\]

\[
= \delta_{\alpha, \alpha'} \delta_{n_{1} n_{1}'} \delta_{K_{3}} - \frac{i}{\kappa_{3}} \kappa_{3}^{*} + \frac{m}{\kappa_{3}}
\]

\[
\langle \Phi_{\alpha(1)} \chi_{D}^{f} | \Phi_{\beta(0)} \chi_{L}^{f} \rangle = \langle \Phi_{\beta(1)} \chi_{D}^{f} | \Phi_{\beta(0)} \chi_{L}^{f} \rangle
\]

\[
= \frac{1}{2\pi} \int d^3 x \phi_{\alpha(1)}^{\dag} (\rho) \phi_{\alpha(1)} (z) e^{-i \omega_1 \phi} | \phi_{\beta(0)}^{\dag} (\rho) \phi_{\beta(0)} (z) e^{i \omega_1 \phi}
\]

\[
= \delta_{\alpha, \alpha'} \delta_{n_{1} n_{1}'} \delta_{K_{3}} - \frac{i}{\kappa_{3}} \kappa_{3}^{*} + \frac{m}{\kappa_{3}}.
\]
For $\langle n|\gamma_3|m \rangle$, we shall evaluate $\langle n|[H(U_0^{\gamma_3}),J_1]|m \rangle$ instead with the replacement of $\Phi$.

$$\langle n|[H(U_0^{\gamma_3}),J_1]|m \rangle = \langle n|\beta M (\cos F(\rho,z)+i\gamma_5 \tau \cdot \hat{n} \sin F(\rho,z)),l_1|l_1 \rangle_{|m \rangle}$$

$$= -\langle n|\beta M (\cos F(\rho,z)+i\gamma_5 \tau \cdot \hat{n} \sin F(\rho,z))|m \rangle_{|n \rangle}$$

For $\langle n|\gamma_3|m \rangle$, the components of the angular momentum operator $l$ in cylindrical coordinates. Using eqs. (84), one obtains

$$l_1 = -\frac{1}{2} e^{i\phi} \left( \rho \frac{\partial}{\partial z} - z \frac{\partial}{\partial \rho} + i \frac{z}{\rho} \frac{\partial}{\partial \phi} \right) - e^{-i\phi} \left( \rho \frac{\partial}{\partial z} - z \frac{\partial}{\partial \rho} + i \frac{z}{\rho} \frac{\partial}{\partial \phi} \right)$$

$$l_2 = -\frac{1}{2} e^{i\phi} \left( \rho \frac{\partial}{\partial z} - z \frac{\partial}{\partial \rho} - i \frac{z}{\rho} \frac{\partial}{\partial \phi} \right) + e^{-i\phi} \left( \rho \frac{\partial}{\partial z} - z \frac{\partial}{\partial \rho} - i \frac{z}{\rho} \frac{\partial}{\partial \phi} \right)$$

$$l_3 = -\frac{\partial}{\partial \phi}$$

are the components of the angular momentum operator $l$ in cylindrical coordinates. Using eqs. (84), one obtains

$$\langle \Phi^{(n)}_{\alpha(0)} e^{\pm i\phi} \left( \rho \frac{\partial}{\partial z} - z \frac{\partial}{\partial \rho} \pm i \frac{z}{\rho} \frac{\partial}{\partial \phi} \right) \cos F(\rho,z) \left| \Phi^{(n)}_{\alpha'(0)} \right\rangle = -\int d\rho d\phi \rho |\omega_0| N_{n_r'} N_{n_z'} (\sqrt{\alpha_0}) e^{-\alpha_0 \rho^2} N_{n_z} N_{n_t'} e^{-\alpha_2 z^2}$$

$$\times \left\{ \rho |\omega_0| + |\omega_0| + 1 L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2) L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2) \right\}$$

$$\times \sqrt{\alpha_z} \left[ n_z H_{n_z-1} (\sqrt{\alpha_2 z}) - \frac{1}{2} H_{n_z+1} (\sqrt{\alpha_2 z}) \right] H_{n_z'} (\sqrt{\alpha_2 z})$$

$$+ H_{n_z} (\sqrt{\alpha_2 z}) \left[ n_z' H_{n_z-1} (\sqrt{\alpha_2 z}) - \frac{1}{2} H_{n_z+1} (\sqrt{\alpha_2 z}) \right]$$

$$- \rho |\omega_0| + |\omega_0|^{-1} \left[ (2n_r + |\omega_0| - \alpha_2 \rho^2) L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2) - 2(n_r + |\omega_0|) L_{n_r'-1}^{(\omega_0)} (\alpha_0 \rho^2) \right] L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2)$$

$$+ L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2) \left[ (2n_r' + |\omega_0| - \alpha_2 \rho^2) L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2) - 2(n_r' + |\omega_0|) L_{n_r'-1}^{(\omega_0)} (\alpha_0 \rho^2) \right] \pm L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2) L_{n_r'}^{(\omega_0)} (\alpha_0 \rho^2)$$

$$\times H_{n_z} (\sqrt{\alpha_2 z}) H_{n_z'} (\sqrt{\alpha_2 z}) \cos F(\rho,z) \delta_{K_3 \pm \frac{1}{2} K_4 \pm \frac{3}{2}}$$

(85)
\( \langle \Phi_{\alpha(0)}^{(n)} | e^{i \pm i \delta} \left[ \left( \frac{\partial}{\partial z} - z \frac{\partial}{\partial \rho} + i z \frac{\partial}{\partial \varphi} \right) \sin F(\rho, z) \cos \Theta(\rho, z) \right] | \Phi_{\beta(0)}^{(n)} \rangle \)

\[
= \int pdpdz N_{n_r}^{|\omega|} N_{n_{\rho'}}^{|\omega'|} e^{-\alpha \rho^2} N_{n_r} N_{n_{\rho'}} e^{-\alpha z^2} \times \left\{ \rho^{|\omega|+|\omega'|+1} L_{n_r}^{(|\omega|)} (\alpha \rho^2) L_{n_{\rho'}}^{(|\omega'|)} (\alpha \rho^2) \right. \\
\left. \times \sqrt{\alpha_z} \left[ \left( n_z H_{n_z-1}(\sqrt{\alpha_z}) - \frac{1}{2} H_{n_z+1}(\sqrt{\alpha_z}) \right) H_{n_{\rho'}}(\sqrt{\alpha_z}) \\
+ H_{n_z}(\sqrt{\alpha_z}) \left( n_{\rho'} H_{n_{\rho'}-1}(\sqrt{\alpha_z}) - \frac{1}{2} H_{n_{\rho'}+1}(\sqrt{\alpha_z}) \right) \right] \\
\right.
\]

\[
- \rho^{|\omega|+|\omega'|-1} \left[ \left(2 n_r + |\omega| - \alpha \rho^2\right) L_{n_r}^{(|\omega|)} (\alpha \rho^2) - 2(n_r + |\omega|) L_{n_{\rho'}-1}^{(|\omega'|)} (\alpha \rho^2) \right] L_{n_{\rho'}}^{(|\omega'|)} (\alpha \rho^2) \\
\left. + L_{n_r}^{(|\omega|)} (\alpha \rho^2) \left( 2 n_r' + |\omega| - \alpha \rho^2 \right) L_{n_{\rho'}-1}^{(|\omega'|)} (\alpha \rho^2) - 2(n_r' + |\omega|) L_{n_{\rho'}-1}^{(|\omega'|)} (\alpha \rho^2) \right] \pm L_{n_r}^{(|\omega|)} (\alpha \rho^2) L_{n_{\rho'}}^{(|\omega'|)} (\alpha \rho^2) \\
\times H_{n_z}(\sqrt{\alpha_z}) z H_{n_{\rho'}}(\sqrt{\alpha_z}) \right) \sin F(\rho, z) \cos \Theta(\rho, z) \delta K_3 \pm \frac{1}{2} K_3 \pm \frac{1}{2}.
\]

(86)

Other elements can be obtained in the same manner. In Appendix B we shall present these calculations in more detail.

D. Numerical Convergence

In this subsection, we show the convergence of the soliton energy \( E_{\text{static}} \) with respect to \( T) K_3 \) with the basis number fixed, and \( U \) the number of the basis (discretized momentum number) with \( K_3 \) fixed.

Fig 11 shows the case of \( T) \) with \( B = 3 \). As can be seen, the energy is almost convergent at the \( (K_3)_{\text{max}} = 10 \). In the case of \( U \), the energy is not perfectly convergent up to \( N_{r_{\text{max}}} = N_{z_{\text{max}}} = 22 \) (see Fig 11). Therefore, all our results have \( 1 - 2 \% \) uncertainty.

V. CONCLUSIONS

In this paper, we studied axially symmetric soliton solutions with \( B = 2 - 5 \) in the chiral quark soliton model. The one-quark spectral flow analysis indicates that the number of diverging states from positive continuum to negative coincide with the baryon number. As is shown in Table II the valence quark spectra contain double degeneracy, realising lower energy than non-degenerate states. Therefore, our solitons are stable although they are not necessarily minimal energy one, which confirms that they are good saddle point solutions.

Upon quantization, we computed zero-mode rotational corrections to the classical energy. The study of the
Finkelstein-Rubinstein constraints indicate that the axially symmetric solution with even $B$ has the same quantum number as the physically observed nuclei. These results are shown in Table IV and V. Some of the states may be observed in experiments. For odd $B$, the constraint of $C_2$ in Eq. (52) seems to assure the validity of the ansatz. Indeed, it provides the ground state as $i = j = 1/2$ for $B = 3$ and as $i = 1/2, j = 3/2$ for $B = 5$, which exactly coincide with physical observations. This seems to make sense since in the minimal energy configurations with discrete symmetries, the solutions tend to have $i = j = 1/2$ due to their shell-like structure. However, unfortunately the constraint in Eq. (52) forbids such states. Consequently, the axially symmetric solitons with odd $B$ emerge only as excited states. The resultant lowest state is $E = 3657$ MeV with $i = 1/2, j = 5/2$ for $B = 3$, and is $E = 6591$ MeV with $i = 1/2, j = 7/2$ for $B = 5$.

Recently, we also studied classical multi-baryonic systems with discrete symmetries in the CQSM and found larger degeneracy of the quark orbits than of the axially symmetric states. For example, triply degenerate bound spectrum is obtained in the $B = 3$ tetrahedral soliton background. The interesting point is that the corresponding energy of the soliton is $E \sim 210$ MeV which is higher than the axially symmetric, $E = 173$ MeV (see Table I). Likewise, for the $B = 4$ minimal energy soliton with cubic symmetry, the valence quark spectrum shows four-fold degeneracy with $E \sim 170$ MeV while for the axially symmetric, $E = 106, 232$ MeV. Thus, although the degeneracy of the spectrum indicates the stability of the solutions, other factors should be also taken into account in regard to minimization of their classical energies. More detailed discussions on this subject will be made elsewhere.

\[ E_{max} \begin{bmatrix} 3350 \\ 3360 \\ 3370 \\ 3380 \\ 3390 \\ 3400 \end{bmatrix} \]

\[ (K_3)_{max} \begin{bmatrix} 3450 \\ 3460 \\ 3470 \\ 3480 \\ 3490 \end{bmatrix} \]

\[ (i,j) \begin{bmatrix} 3500 \\ 3510 \\ 3520 \\ 3530 \end{bmatrix} \]

\[ (K_3)_{max} \begin{bmatrix} 3540 \\ 3550 \\ 3560 \\ 3570 \end{bmatrix} \]

\[ (i,j,k) \begin{bmatrix} 3650 \\ 3660 \\ 3670 \end{bmatrix} \]
We thank S.Oryu for useful discussions. We are also grateful to N.S.Manton to inform the paper of S.Krusch (ref.[33]). One of us (Sawado) also thanks M.Kawabata and K.Saito for their help of numerical computations.

APPENDIX A: EVALUATION OF THE MATRIX ELEMENTS

In this appendix, we shall present detailed calculations of the matrix elements, $\int d^3x \Phi_{\eta(0)}|\sigma \cdot p| \Phi_{\rho(0)}$.

For the kinetic term of the elements, one gets

$$\langle \Phi_{\sigma(0)}| \sigma \cdot p | \Phi_{\rho(0)} \rangle = \frac{1}{2\pi} \int d^3x \phi_{\nu_{\alpha}}(\rho) \phi_{\nu_{\alpha}}(z) e^{-i\omega z}$$

or

$$\int \rho dp dz |\phi_{\nu_{\alpha}}|^{2} \phi_{\nu_{\alpha}}^{\dagger} N_{\eta_{3}} e^{-\alpha z^{2}} H_{\eta_{3}}(\sqrt{\alpha z})$$

All the other 3 functions are equal to:

$$\delta_{\nu_{13}} \delta_{\omega \omega_{\nu_{1}} \nu_{1}} N_{\eta_{3}} N_{\eta_{2}} \frac{1}{N_{\eta_{3}}}$$

or

$$\delta_{\nu_{13}} \delta_{\omega \omega_{\nu_{1}} \nu_{1}} N_{\eta_{3}} N_{\eta_{2}} \frac{1}{N_{\eta_{3}}}$$

For $(n_{z}, n_{z}') = (1,2), (3,4), \cdots$ or $(n_{z}, n_{z}') = (0,1), (2,3), \cdots$, the right-hand side of Eq. (A1) becomes

$$\delta_{n_{z}n_{z}'-1} \cdot \frac{N_{n_{z}n_{z}'}}{N_{n_{z}}} \sqrt{\alpha z_{n_{z}'}} \delta_{n_{z}n_{z}'+1} \cdot (A2)$$

For $(n_{z}, n_{z}') = (1,0), (3,2), \cdots$ or $(n_{z}, n_{z}') = (2,1), (4,3), \cdots$, the right-hand side of Eq. (A1) becomes

$$\delta_{n_{z}n_{z}'+1} \cdot \frac{N_{n_{z}n_{z}'}}{N_{n_{z}}} \sqrt{\alpha z_{n_{z}'}} \delta_{n_{z}n_{z}'} \cdot (A3)$$

[1] D. I. Diakonov, V. Yu. Petrov, and P. V. Pobylitsa, Nucl. Phys. B306, 809 (1988).
[2] H. Reinhardt and R. Wünsch, Phys. Lett. B 215, 577 (1988).
[3] Th. Meissner, F. Grümmer, and K. Goeke, Phys. Lett. B 227, 296 (1989).
[4] For detailed reviews of the model see: Th. Meissner, F. Grümmer, and K. Goeke, Phys. Lett. B 256, 235 (1987).
\[
\langle \Phi_{\alpha(0)} | \sigma \cdot p | i \Phi_{\beta'(1)} \rangle = \frac{1}{\pi} \int d^3 x \phi_{n_r}^*(\rho) \phi_{n'_r}(z) e^{-i\omega_0 p} e^{-i\varphi} \\
\times \left( \frac{\partial}{\partial \rho} - i \frac{\partial}{\partial \varphi} \right) \phi_{n'_{r'}i}^*(z) e^{i\omega_1 \varphi} \\
= \delta_{\omega_0, \omega'_1} \delta_{n_r, n'_{r'}} \int \rho d\rho \phi_{n_r}^*(\rho) \frac{\partial}{\partial \rho} \phi_{n'_{r'}}(\rho) \\
= \delta_{\omega_0, \omega'_1} \delta_{n_r, n'_{r'}} \int \rho d\rho (\sqrt{\alpha p} | \omega_0 + | \omega'_1 + \omega_1 - \alpha \rho^2 \rangle \\
\times N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1} L_{n_r}^{\omega_0} (\alpha \rho^2) \left( 2n'_r + | \omega'_1 + \omega_1 - \alpha \rho^2 \right) \\
\times L_{n'_{r'}}^{\omega'_1} (-\alpha \rho^2) - 2(n'_r + | \omega'_1) N_{n'_{r'}-1}(\alpha \rho^2) \right). \quad (A4)
\]

Likewise, for \( \omega_0 < 0 \) (\( \omega'_1 = \omega_0 + 1 \geq 0 \)), the righthand side of Eq. (A3) becomes

\[
\delta_{n_r, n'_{r'}} \int \rho d\rho (\sqrt{\alpha p})^{2\omega_0 + 1} \rho^{2\omega_0} e^{-\alpha \rho^2} N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1+1} \\
\times L_{n_r}^{\omega_0} (\alpha \rho^2) \left( 2n'_r + 2\omega_0 + 2 \right) L_{n'_{r'}}^{\omega'_1+1}(\alpha \rho^2) \\
\delta_{n_r, n'_{r'}} \int \rho d\rho (\sqrt{\alpha p})^{2\omega_0 + 1} \rho^{2\omega_0 + 2} e^{-\alpha \rho^2} N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1+1} \\
\times \left( L_{n_r}^{\omega_0+1}(\alpha \rho^2) - L_{n'_{r'}-1}^{\omega'_1}(\alpha \rho^2) \right) L_{n'_{r'}}^{\omega'_1+1}(\alpha \rho^2) \\
= \delta_{n_r, n'_{r'}} \sqrt{\alpha p} N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1+1} \left( 2n'_r + 2\omega_0 + 2 \right) \frac{1}{N_{n_r}^{\omega_0+2}} \delta_{n_r, n'_{r'}} \\
- \delta_{n_r, n'_{r'}} \sqrt{\alpha p} N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1+1} \\
\times \left( \frac{1}{N_{n_r}^{\omega_0+2}} \delta_{n_r, n'_{r'}} - \frac{1}{N_{n_r}^{\omega_0+2}} \delta_{n_r, -1n'_{r'}} \right) \\
= \delta_{n_r, n'_{r'}} \sqrt{\alpha p} \left( 2\sqrt{n_r + \omega_0} + 1 \delta_{n_r, n'_{r'}} \right) \\
- \sqrt{n_r + \omega_0 + 1 \delta_{n_r, n'_{r'}}} \\
= \delta_{n_r, n'_{r'}} \sqrt{\alpha p} \left( \sqrt{n_r + \omega_0 + 1 \delta_{n_r, n'_{r'}}} + \sqrt{n_r - 1 \delta_{n_r, n'_{r'}}} \right) \quad (A5)
\]

where

\[
N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1+1} \delta_{n_r, n'_{r'}} = \frac{\delta_{n_r, n'_{r'}}}{\sqrt{n_r + \omega_0 + 1}} \quad (A6)
\]

\[
N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1+1} \delta_{n_r, n'_{r'}} = \sqrt{n_r + \omega_0 + 1} \delta_{n_r, -1n'_{r'}} \quad (A7)
\]

\[
N_{n_r}^{\omega_0} N_{n'_{r'}}^{\omega'_1+1} \delta_{n_r, -1n'_{r'}} = \sqrt{n_r} \delta_{n_r, -1n'_{r'}} \quad (A8)
\]

have been used.

For the potential term in the Hamiltonian given by

\[
M \left( \cos F(\rho, z) \quad i \tau \cdot \hat{n} \sin F(\rho, z) \quad - \cos F(\rho, z) \right)
\]
only following matrix elements survive:

\[ \langle \Phi_{\alpha(0)} | M \cos F(\rho, z) | \Phi_{\alpha'(0)} \rangle \]

\[ = \int \rho d\rho dz M \cos F(\rho, z) \]

\[ \times \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \]  

(A15)

\[ \langle \Phi_{\alpha(0)} | iM \cdot \hat{n} \sin F(\rho, z) | \Phi_{\alpha'(0)} \rangle \]

\[ = - \int \rho d\rho dz M \cos \Theta(\rho, z) \sin F(\rho, z) \]

\[ \times \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \]  

(A16)

\[ \langle \Phi_{\alpha(0)} | iM \tau \cdot \hat{n} \sin F(\rho, z) | \Phi_{\alpha'(0)} \rangle \]

\[ = - \int \rho d\rho dz M \sin \Theta(\rho, z) \sin F(\rho, z) \]

\[ \times \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \]  

(A17)

**APPENDIX B: EVALUATION OF THE MOMENTS OF INERTIA**

In this appendix, we shall give the detailed derivation to obtain matrix elements in Eq. (86).

For the first term in Eq. (86), one obtains

\[ \langle \Phi_{\alpha(0)}^{(n)} | e^{\pm i\varphi} \left( \rho \frac{\partial}{\partial z} \cos F(\rho, z) \right) | \Phi_{\alpha'(0)}^{(n')} \rangle \]

\[ = \frac{1}{2\pi} \int d^3 \phi \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \]

\[ \times \left( \rho \frac{\partial}{\partial z} \cos F(\rho, z) \right) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha'}}^{(\omega_{\alpha})}(\rho) \delta_{K_{3}+\pm K_{3}'} \delta_{K_{2}+\pm K_{2}'} \].  

(B1)

Performing partial integration with respect to \( z \) in Eq. (B1),

\[ = - \int \rho d\rho dz \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \rho \frac{\partial}{\partial z} \left( \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \phi_{n_{\alpha}}^{(\omega_{\alpha})}(\rho) \right) \cos F(\rho, z) \delta_{K_{3}+\pm K_{3}' + \pm} \]

\[ = - \int \rho d\rho dz N_{n_{\alpha}}^{(\omega_{\alpha})} N_{n_{\alpha'}}^{(\omega_{\alpha})} \left( \sqrt{\alpha_{\rho}} \right)^{|\omega_{\alpha}|+|\omega_{\alpha}'|+1} e^{-\alpha_{\rho}z^2} \frac{L_{|\omega_{\alpha}|}(\alpha_{\rho} r^2) L_{|\omega_{\alpha}'|}(\alpha_{\rho} r'^2)}{H_{n_{\alpha}}(\sqrt{\alpha_{\rho} z}) H_{n_{\alpha'}}(\sqrt{\alpha_{\rho} z})} \]

\[ \times n_{\alpha} N_{n_{\alpha'}} \sqrt{\alpha_{\rho}} e^{-\alpha_{\rho}z^2} \left( n_{\alpha} H_{n_{\alpha}-1}(\sqrt{\alpha_{\rho} z}) - \frac{1}{2} H_{n_{\alpha}+1}(\sqrt{\alpha_{\rho} z}) \right) \frac{1}{2} H_{n_{\alpha}+1}(\sqrt{\alpha_{\rho} z}) \]  

\[ \times H_{n_{\alpha}'}(\sqrt{\alpha_{\rho} z}) \left( n_{\alpha}' H_{n_{\alpha}'-1}(\sqrt{\alpha_{\rho} z}) - \frac{1}{2} H_{n_{\alpha}'+1}(\sqrt{\alpha_{\rho} z}) \right) \cos F(\rho, z) \delta_{K_{3}+\pm K_{3}' + \pm} \].  

(B2)
Similarly, performing partial integration with respect to $\rho$, one obtains
\[
\langle \Phi^{(n)}_{\alpha(0)} \rangle [e^{\pm i\varphi} \left( z \frac{\partial}{\partial \rho} \cos F(\rho, z) \right) | \Phi^{(u)}_{\alpha'(0)} \rangle \\
= \frac{1}{2\pi} \int d^3x \phi_{\nu}^{(\omega)}(\rho) \phi_{n_{\nu}}(z) e^{-i\omega_0} e^{\pm i\varphi} \left( z \frac{\partial}{\partial \rho} \cos F(\rho, z) \right) \phi_{n_{\nu}'(r)}(\rho) \phi_{n_{\nu}'}(z) e^{i\omega_0} e^{\pm i\varphi} \\
= \int \rho d\rho d\phi_{\nu}^{(\omega)}(\rho) \phi_{n_{\nu}}(z) \left( z \frac{\partial}{\partial \rho} \cos F(\rho, z) \right) \phi_{n_{\nu}'}(\rho) \phi_{n_{\nu}'}(z) \delta_{K_3} \mp \frac{e}{2} \left( K_3 \mp \frac{e}{2} \right) \\
= -\int d\rho d\phi_{\nu}^{(\omega)}(\rho) \phi_{n_{\nu}}(z) \phi_{n_{\nu}'}(\rho) \phi_{n_{\nu}'}(z) \cos F(\rho, z) \delta_{K_3} \mp \frac{e}{2} \left( K_3 \mp \frac{e}{2} \right) \\
= -\int \rho d\rho d\phi_{\nu}^{(\omega)}(\rho) \phi_{n_{\nu}}(z) \phi_{n_{\nu}'}(\rho) \phi_{n_{\nu}'}(z) \cos F(\rho, z) \delta_{K_3} \mp \frac{e}{2} \left( K_3 \mp \frac{e}{2} \right) \times N_{n_{\nu}} N_{n_{\nu}'} e^{-\alpha \rho^2} H_n(z) \cos F(\rho, z) \delta_{K_3} \mp \frac{e}{2} \left( K_3 \mp \frac{e}{2} \right) .
\]

Finally, we reach the final answer
\[
\langle \Phi^{(n)}_{\alpha(0)} \rangle [e^{\pm i\varphi} \left( \rho \frac{\partial}{\partial \rho} - z \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \varphi} \right) \cos F(\rho, z) | \Phi^{(u)}_{\alpha'(0)} \rangle \\
= -\int \rho d\rho d\phi_{\nu}^{(\omega)}(\rho) \phi_{n_{\nu}}(z) \phi_{n_{\nu}'}(\rho) \phi_{n_{\nu}'}(z) \cos F(\rho, z) \delta_{K_3} \mp \frac{e}{2} \left( K_3 \mp \frac{e}{2} \right) \times N_{n_{\nu}} N_{n_{\nu}'} e^{-\alpha \rho^2} H_n(z) \cos F(\rho, z) \delta_{K_3} \mp \frac{e}{2} \left( K_3 \mp \frac{e}{2} \right) .
\]