The Quantum Steeplechase

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Abstract

The Quantum Steeplechase is the study of a Luttinger liquid in one dimension in the presence of a finite number of barriers and wells clustered around an origin. The powerful non-chiral bosonization technique (NCBT) is introduced to write down closed formulas for the two-point function and four-point function relevant to the study of Friedel oscillations in this system at zero temperature in the sense of the random phase approximation (RPA). Unlike g-ology based methods that are tied to the translationally invariant, free particle basis, the NCBT explicitly makes use of the translationally non-invariant single particle wavefunctions. The present method that provides the most singular part of the asymptotically exact Green function in a closed form, is in contrast to competing methods that require a combination of renormalization group and/or numerical methods in addition to the bosonization techniques. All the central issues pertaining to this subject viz. the behaviour of the local dynamical density of states, Friedel oscillations of the density correlations, behaviour of the system near the resonant tunnelling condition, computation of the Kubo and tunneling conductance of the wire and so on are revisited using the NCBT formalism and a detailed comparison with existing results are made.

Keywords: Luttinger Liquid, Bosonization, Green’s functions

1. Introduction

In quantum many-body physics, the stated goal is to write down all the “N-point Green’s functions” of a system of many mutually interacting particles in the thermodynamic limit. The N-point particle (hole) Green function is the quantum overlap between two states of a system where each state has N particles added (removed) at various locations and times. An analytical study (as against a numerical one) of mutually interacting quantum particles is beset with formidable technical difficulties and various approximation techniques are used to mitigate these problems. The obvious method that springs to mind is to expand in powers of the interaction potential between the quantum particles. In one dimension, each term in this perturbation series carried out in momentum space, diverges logarithmically at low momenta (known as infra-red divergences). Hence a “non-perturbative” method is called for. For translationally invariant systems, this method, which goes under the name ‘g-ology’ is well established (see e.g. Giamarchi [2]). The g-ology method is tied to the translationally invariant free particle basis and the ‘restricted Hilbert space of states’ in which the Fermi-Bose correspondence is justified makes even the study of free fermions in the presence of barriers/wells (or weak links) quite formidable. By contrast, the present approach which amounts to constructing the ‘restricted Hilbert space of states’ not for free fermions but for free fermions plus these barriers/wells or weak links, is able to study the problem of Luttinger liquids in the presence of these imperfections more easily and is able to provide analytical expressions for the most singular part of the Green’s functions and so on that interpolate between the weak barrier and weak link cases.

The detailed study of transport in Luttinger liquid (LL) in the presence of a weak link was started by Kane and Fisher[8] followed by the study of a LL near a double barrier [9]. Since then a number of papers have appeared that have generalised these ideas using a variety of approaches which include fermionic renormalization[10], path integral approaches[11], functional renormalization[12, 13, 14, 15, 16], flow equations for Hamiltonians[17], functional integral formalism[18], Monte Carlo methods[19] and so on. Other authors [20] have used Bethe ansatz and abelian bosonization in order to extract anomalous exponents.

Luttinger liquid theory [21] which has served as the paradigm for one dimensional systems in presence of interactions is based on linearization of the dispersion relations of the constituent particles. Beyond the low energy
limit, nonlinearity becomes essential and Imambekov et al. used novel methods to tackle these systems which includes ideas from Fermi edge singularity and Fermi liquid theory, perturbation theory, etc. and thus studied 1D quantum fluids beyond the Luttinger paradigm\[22\]. On the other hand, Rozhkov demonstrated that a suitable fine tuning of the interaction between the fermions can stabilize a state in one dimensions, which is neither similar to Fermi liquid nor to Luttinger liquid, which they called quasi Fermi liquid\[23\].

Numerical methods such as the density matrix renormalization group method\[24\] formulated by Steven White \[25\] have also been invented. This method has been used extensively to study different phenomena like impurity in Luttinger liquid\[26\], Friedel oscillations in a ring of fermions with a disorder\[27\], time dependent problems in Luttinger liquids\[28\] and many more\[29, 30\].

Experimental realizations of 1D systems gives a motivational boost to study quantum physics in one dimension. In this regard, Luttinger liquid behavior in carbon nano-tubes\[31, 32\], experimental evidences of resonant tunneling in a Luttinger liquid are worth mentioning\[33\].

The goals of this work are limited to a critique specifically of the standard g-ology based methods as applied to strongly inhomogeneous systems and the introduction of a simple, appealing but powerful analytical alternative that is able provide closed expressions for the most singular parts of the N-point functions of the systems of interest in appropriate limits. This work also makes detailed comparison with the analogous results from existing literature and makes various types of limiting case comparisons in order to validate the NCBT method. Applications to physical phenomena such as the behaviour of the local dynamical density of states, Friedel oscillations of the density correlations, behaviour of the system near the resonant tunnelling condition, computation of the Kubo and tunneling conductance of the wire and so on are also discussed.

2. Problem overview

Consider a Luttinger liquid in one dimension with forward scattering short range mutual interactions \[2\] in the presence of a scalar potential that is localized near an origin. This potential is denoted by a black box indicating that it can be any finite sequence of barriers and wells. In Fig. 2, the scenario is described using playing cards.

The full generic-Hamiltonian of the system(s) under study (before taking the RPA limit) is (are),

\[
H = \int_{-\infty}^{\infty} dx \, \psi^\dagger(x) \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x) + \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, v(x-x') \rho(x)\rho(x')
\]

where \( v(x-x') = \frac{1}{\pi} \sum_q e^{-iq(x-x')} v_q \) (where \( v_q = 0 \) if \( |q| > \Lambda \) for some fixed \( \Lambda \ll k_F \) and \( v_q = v_0 \) is a constant, otherwise) is the forward scattering mutual interaction. Also, \( V(x) \) is the external potential of the forms shown in Fig. 2. Generically they are piecewise constant potentials with compact support: \( V(x) = 0 \) if \( |x| > a > 0 \) and \( V(x) = V_{i+1}; \ a_i < x < a_{i+1}, \ i = -N, -N+1, ..., N-2 \) and \( a_{N-1} = a \) and \( a_{-N} = -a \) where \( \rho(x,t) = \psi^\dagger(x,t)\psi(x,t) - \rho_0 \) (no point splitting is required before taking RPA limit, see Appendix E for a characterization of this model in terms of g-ology couplings). The central goal of this paper is to write down the two point and four point Green functions of these systems at zero and finite temperature in the presence of the potentials described behind each of these black boxes. For an analytical solution to be feasible when mutual interactions are included, it is necessary to confine the study to the so-called RPA limit which means, among other things, working in the limit where the Fermi momentum and the mass of the fermion diverge in such a way that their ratio is finite (i.e. \( k_F, m \to \infty \) but \( k_F/m = v_F < \infty \); units that make \( \hbar = 1 \), so that \( k_F \) is both the Fermi momentum as well as a wavenumber, are used). This amounts to linearizing the energy momentum dispersion near the Fermi surface.

Imagine the black box shown in Fig. 2 has a width \( w \) in addition to smaller subdivisions (depending upon the details present inside) which are in fixed ratios relative to this scale, it is then imperative to define how \( w \) scales
in the RPA limit. The assertion made is that in the RPA limit \( k_F w < \infty \) as \( k_F \to \infty \). Similarly the heights and depths of the various barriers are assumed to be in fixed ratios with the Fermi energy \( E_F = \frac{1}{2} m v_F^2 \) even as \( m \to \infty \) with \( v_F < \infty \).

### 3. Green’s functions of free fermions

Denote the full two-point Green function (also known as single particle Green function) of the system before taking the RPA limit (i.e. with parabolic energy momentum relation) as \( \langle T \psi(x,\sigma,t)\psi^\dagger(x',\sigma',t') \rangle \) where the time ordering decides whether it is particle or hole Green function that is being studied and \( \sigma \) is the spin projection of the individual fermions. In terms of this, the asymptotic or RPA Green function is defined by “smearing out” the positions and times over the scale of the Fermi wavelength and Fermi times as follows,

\[
\langle T \psi(x,\sigma,t)\psi^\dagger(x',\sigma',t') \rangle = \lim_{m\to\infty} \ll \langle T\psi(y,\sigma,\tau)\psi^\dagger(y',\sigma',\tau') \rangle e^{-i k_F \nu (y-y')} e^{i E_F (\tau-\tau')} \gg
\]

where,

\[
\ll f(t) \gg = \frac{1}{2 T_F} \int_{-T_F}^{t+T_F} d\tau \, f(\tau)
\]

\[
\ll g(x) \gg = \frac{1}{2 \lambda_F} \int_{-\lambda_F}^{x+\lambda_F} dy \, g(y)
\]

with \( \lambda_F = 2\pi/k_F \) and \( T_F = 2\pi/E_F, k_F = m v_F \) and \( E_F = (1/2) m v_F^2 \) with \( v_F < \infty \) being held fixed. Also, here \( \nu,\nu' = \pm 1 \) correspond to the right and left Fermi points. When mutual interactions between the fermions are absent it is easy to show that the two-point function has the form (at zero temperature) shown below (here \( \theta(x) \) is Heaviside’s step function),

\[
\langle T \psi(x,\sigma,t)\psi^\dagger(x',\sigma',t') \rangle_0 = \sum_{\gamma,\gamma'=\pm 1} \theta(\gamma x)\theta(\gamma' x') \, g_{\gamma,\gamma'}(\nu,\nu') (\nu x - \nu' x') - v_F (t - t') \delta_{\sigma,\sigma'}
\]

where \( g_{\gamma,\gamma'}(\nu,\nu') \) are given in Table 1.

The values of the transmission\((T)\) and reflection\((R)\) amplitudes are calculated for all the sub-cases shown in Fig. 2 and they are given in Appendix A. Note that for potentials which lack inversion symmetry about any chosen
point, (e.g. asymmetric double deltas) the presence of nontrivial phases in $T$ and $R$ contribute to the expected lack of inversion symmetry in the Green’s functions.

Note that in equation (4), the term $(\nu x - \nu' x') - v_F(t - t')$ appears in the denominator. In general, in a Luttinger liquid with mutual interactions, this term appears with a non-trivial system dependent exponent viz. as $[(\nu x - \nu' x') - v_F(t - t')]^g$. Listing these $g$’s and other similar exponents is one of the main goals of this paper since $g = 1$ is only when mutual interaction between fermions are absent. It is easy to generalise these results to finite temperature since for this a simple replacement viz. $\frac{1}{\beta} \rightarrow \frac{1}{\beta v_F} \cosh[\frac{X}{\beta v_F}]$ is sufficient where e.g. $X \equiv (\nu x - \nu' x') - v_F(t - t')$ and $\beta$ is inverse temperature.

### 3.1. Density density correlation function

The other main goal of this paper to write down the density-density correlation function (DDCF) of the system which is a special case of a 4-point function. The recommended technique for doing this for systems in one spatial dimension is called bosonization which involves inverting the defining formulas for current and number densities viz. $j(x) = Im [\psi(x) \partial_x \psi(x)]$ and $\rho(x) = \psi(x) \psi(x)$ and rewriting $\psi(x)$ in terms of $j$ and $\rho$. Then the continuity equation $\partial_t \rho + \partial_x j = 0$ is invoked to write $\psi(x)$ purely as a (non-local) function of $\rho$ and $\partial \rho$. It follows therefore, that the the N-point function is some combination of the correlations of the density field with itself. Bosonization may be thought of as the “inverse of Wick’s theorem”. While Wick’s theorem - which is valid only for systems with no mutual interactions - seeks to express higher order correlations in terms of lower order ones, bosonization seeks to express the single particle Green function in terms of the higher order density-density correlations. In the RPA sense, the density $\rho(x, t)$ may be “harmonically analysed” as follows.

$$\rho(x, t) = \rho_n(x, t) + e^{2ik_F x} \rho_f(x, t) + e^{-2ik_F x} \rho_f^*(x, t) \quad (5)$$

The DDCF are calculated paying specific attention to what are known as Friedel oscillations which is nothing but a term which oscillates with wavenumber $2k_F$ such as $e^{2ik_F (x-x')} < T \rho_f(x, t) \rho_f^*(x', t') >$. The slowly varying part of the density $\rho_n$ (the average density is subtracted out, so this is really the deviation) has an auto-correlation function which when mutual interactions are absent, may be written down using Wick’s theorem as follows,

$$\langle T \rho_n(x, t) \rho_n(x', t') \rangle_0 = - \sum_{\gamma, \gamma' = \pm 1} \sum_{\nu, \nu' = \pm 1} \frac{|g_{\gamma, \gamma'}(\nu, \nu')|^2 \theta(\gamma x) \theta(\gamma' x')}{[(\nu x - \nu' x') - v_F(t - t')]^2} \quad (6)$$

where $g_{\gamma, \gamma'}(\nu, \nu')$ are given in Table 1.

These three relations viz. equation (4), equation (5) and equation (6) shall be used in the subsequent sections as input to the NCBT scheme in order to enable an explicit evaluation of the two and four point functions.

### 4. Bosonized version of the N-point Green’s functions

Just as the density may be harmonically analysed, the field may also be harmonically analysed so that $\psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x)$. The inversion of the defining relation between current and densities in the standard bosonization scheme that goes by the name g-ology (see the book by Gianarchi [2]) yields the following relation between $\psi_\nu(x, \sigma, t)$ (where $\nu = R(+1)$ or $L(-1)$) and the slowly varying part of the density (this is a mnemonic for generating the N-point functions),

$$\psi_\nu(x, \sigma, t) \sim e^{i\theta_\nu(x, \sigma, t)} \quad (7)$$
with the local phase given by the formula,

$$\theta_{\nu}(x,\sigma,t) = \pi \int_{sgn(x)\infty}^{y} dy \left( \nu \rho_{\nu}(y,\sigma,t) - \int_{sgn(y)\infty}^{y} dy' \partial_{x} \rho_{\nu}(y',\sigma,t) \right)$$

The above prescription in Eq.(7) is valid for nearly translationally invariant systems i.e. with possible external potentials with Fourier components small compared to the Fermi momentum. However if one wishes to study systems of the present paper one has to use the g-ology method and even the undergraduate physics system of free fermions in presence of barriers and wells becomes a formidable sine-Gordon theory. Here it is wished to use an approach where a modification of the correspondence of Eq.(7) is introduced wherein the correlation functions of a system of free fermions plus barriers and wells can be as easily computed in the bosonized language as it is in the original Fermi language and is given explicitly in both approaches.

$$\psi_{\nu}(x_{i},\sigma_{i},t_{i}) \rightarrow \sum_{\gamma_{i}=\pm 1} \sum_{\lambda_{i} \in \{0,1\}} C_{\lambda_{i},\nu_{i},\gamma_{i}}(\sigma_{i}) \theta(\gamma_{i} x_{i}) e^{i\theta_{\nu}(x_{i},\sigma_{i},t_{i}) + 2\pi i \nu_{i} \lambda_{i} \int_{sgn(x_{i})\infty}^{x_{i}} \rho_{\nu}(y_{i},\sigma_{i},t_{i}) dy_{i}}$$

This ‘non-standard harmonic analysis’ is an alternative to the usual one invoked by g-ology community which is valid only for translationally invariant systems whereas the harmonic analysis in Eq.(9) which may be justified easily (see Appendix B) is valid for systems considered in this paper.

An analogy with the anharmonic oscillator problem in undergraduate quantum mechanics may be useful. One could either study this problem in the translationally invariant plane wave basis or more conveniently in a basis next to it which ensure that fermion commutation rules since there is nothing wrong with using the plane wave basis, this really makes the problem quite complicated. Using Eq.(7) to study the problem of fermions in presence of barriers and wells is somewhat like using the plane wave basis to study the anharmonic oscillator. It is much better to use Eq.(9) which is analogous to using the harmonic oscillator basis to study the anharmonic oscillator.

The quantities \( C_{\lambda_{i},\nu_{i},\gamma_{i}}(\sigma_{i}) \) are c-numbers and involve cutoffs and such, which, as in the traditional approach, are not obtainable using these techniques (see section Technical clarifications). The only quantities that have absolute meaning are the anomalous exponents i.e., numbers \( g \) when the term involved appears as \( [\nu x - \nu' x'] - v_{F}(t-t') \)^\( g \). The operators that appear in the exponent in equation (9) are the ones that are really crucial in this approach since they provide the right anomalous exponents. The crucial new ingredient in the modified formula in equation (9) is the term involving \( \rho_{\nu}(y_{i},\sigma_{i},t_{i}) \) that ensures that the effects of backscattering from the external potentials are automatically and naturally taken into account so that the mandated trivial exponents are obtained when equation (9) is used to compute the N-point functions in the sense of the RPA. The addition of these new terms does not spoil fermion commutation rules since there is a prefactor of \( 2\pi i \nu_{i} \) next to it which ensure that fermion commutation relations of the fields are respected. It has also been shown that these new terms also do not spoil the point-splitting constraints for the Fermi bilinears, which is an opaque way of saying that when equation (9) is used to infer the currents and densities - as the latter two are, after all, bilinears of the Fermi fields - the resulting expressions are in accordance with expectations (see Appendix B). In order to extract the anomalous exponents of the system with mutual interactions, two things remain. One is to generalise equation (6) to include mutual interactions.

The other is to derive a prescription for choosing the values of the crucial parameters \( \lambda_{i} = 0, 1 \) which indicates when the traditional form of the field needs modification. It simply involves making sure that the prescription (which is unique) leads to N-point functions of the system (without mutual interactions) identical to what is given by Wick’s theorem. This is done subsequently below. In addition to these \( \lambda_{i}'s \), auto-correlation functions of the slowly varying parts of the density when mutual interactions are present are needed.

Again in the spirit of the RPA the following formula may be obtained \( \langle \rho_{\nu}(x,t) \rangle = \rho_{\nu}(x,\uparrow,t) + \rho_{\nu}(x,\downarrow,t) \) is the “holon” density and \( \rho_{n}(x,t) = \rho_{\nu}(x,\uparrow,t) - \rho_{\nu}(x,\downarrow,t) \) is the “spinon” density and \( a = h \) for holon and \( a = n \) for spinon,

$$\langle T \rho_{\nu}(x_{1},t_{1})\rho_{\nu}(x_{2},t_{2}) \rangle = \frac{v_{F}}{2\pi^{2}a_{\nu}} \sum_{\nu_{i}=\pm 1} \frac{1}{(x_{1}-x_{2} + \nu v_{a}(t_{1}-t_{2}))^{2}} - \frac{v_{F}}{\pi} \frac{sgn(x_{1})sgn(x_{2}) Z_{a}}{(|x_{1}| + |x_{2}| + \nu v_{a}(t_{1}-t_{2}))^{2}}$$

where \( a = n \) or \( h \) and,

$$Z_{a} = \left| \frac{R}{1 - \delta_{a,h} \frac{v_{n} - v_{F}}{v_{h}} |R|^{2}} \right|^{2}$$

Here the spinon velocity is just the Fermi velocity since it is the total density that couples to the short range potential: \( v_{n} = v_{F} \), but the holon velocity is modified by interactions, \( v_{h} = \sqrt{v_{F}^{2} + \frac{2v_{F}^{2}}{4\pi \kappa}} \) where the interaction
between fermions is the two-body short range forward scattering potential which just means the potential between two particles at \( x \) and \( x' \) is \( V(x - x') = \frac{1}{2} \sum_{|q| < \Lambda} e^{-iq(x-x')} v_0 \), where \( \Lambda \) is held fixed as the RPA limit is taken. Finally, \( \langle T \rho_n(x_1,t_1) \rho_n(x_2,t_2) \rangle \equiv 0 \). It can be shown that an expansion of equation (10) in powers of \( v_0 \) matches with the corresponding series obtained by standard perturbation theory so long as one retains only the most singular terms (this notion is defined precisely later).

5. Full two-point Green’s function

The two point (single-particle) Green’s function may be written down using the correspondence in equation (9). These computations are performed in the sense of the RPA. Only the anomalous exponents which refer to the constants \( g \) that appear in terms of the form \( (\psi_1 x_1 - \psi_2 x_2 - \nu_F(t_1 - t_2))^g \) that emerge from this calculation are of interest here. These \( g \)'s are uniquely pinned down once a prescription for deciding which of the \( \lambda \)'s are zero or one and under what circumstances is given. This prescription follows unambiguously by requiring that an evaluation (of the 2M-point function) in the Gaussian (and RPA) sense leads to trivial exponents when mutual interactions between fermions are absent. Of lesser importance are the coefficients \( C \)'s which depend on the details of the potentials and cutoffs and other such non-universal features, as is also the case in the conventional approach. The prescription for obtaining the \( \lambda \)'s is simple. Consider a general 2M-point function. Imagine mentally pairing up one annihilation operator with one creation operator and create \( M \) such pairs. This is simply a mental activity since this pairing (Wick’s theorem) is not valid when mutual interactions are present. Consider one such pair and let the two \( \lambda \)'s of this pair be \( (\lambda_m, \lambda_k) \) where \( k > m \). The constraints are as follows:

\[
\lambda_m = \begin{cases} 
\lambda_k & \text{if } (\nu_m, \nu_k) = (\gamma_m, \gamma_k) \text{ or } (\nu_m, \nu_k) = (-\gamma_m, -\gamma_k) \\
1 - \lambda_k & \text{if } (\nu_m, \nu_k) = (-\gamma_m, \gamma_k) \text{ or } (\nu_m, \nu_k) = (\gamma_m, -\gamma_k)
\end{cases}
\]

This (unique) prescription guarantees the right trivial exponents in the right places when mutual interactions are turned off. The full Green’s function in presence of interactions are as follows (Notation: \( X_i \equiv (x_i, \sigma_i, t_i) \), also, in order to remove ambiguities associated with cutoff dependent quantities in translationally non-invariant systems, the notion of weak equality denoted by \( A[X_1, X_2] \sim B[X_1, X_2] \) is introduced which really means \( \partial_t \text{Log}[A[X_1, X_2]] = \partial_t \text{Log}[B[X_1, X_2]] \) assuming of course, A and B do not vanish identically. Furthermore the finite temperature versions of the formulas below are obtained by replacing \( \text{Log}[Z] \) by \( \text{Log}[	ext{Sinh} \frac{E}{T}] \) where \( Z \sim (\nu x_1 - \nu' x_2 - v_a(t_1 - t_2)) \) and singular cutoffs ubiquitous in this subject are suppressed in this notation for brevity - they have to be understood to be present. The notion of weak equality is unable to pin down possible prefactors in the Green functions that may even be spatially inhomogeneous (but time independent) in addition to being singular. A more careful study is able to reveal this information which is relegated to Appendix C. The inhomogeneous prefactors are nothing but terms such as \( e^{\frac{1}{2} (A^2)} \) and \( e^{\frac{1}{2} (B^2)} \) that come about when evaluating \( \langle e^{A e^{B}} \rangle = e^{\frac{1}{2} (A^2)} e^{\frac{1}{2} (B^2)} e^{\langle A B \rangle} \) when \( \langle A B \rangle \propto \text{Log}[(\nu x_1 - \nu' x_2) - v_a(t_1 - t_2)] \). It must be stressed that these inhomogenous prefactors are important for extracting the exponents associated with tunneling conductance and the local dynamical density of states.):

**Case I :** \( x_1 \) and \( x_2 \) on the same side of the origin

\[
\langle T \psi_{\nu_1}(X_1)\psi_{\nu_2}^\dagger(X_2) \rangle \sim \text{Exp}[- \sum_{\nu, \nu' = \pm 1, a=\hbar,n} Q_{\nu_1,\nu_2}(\nu, \nu'; a) \log((\nu x_1 - \nu' x_2) - v_a(t_1 - t_2))]
\]

(13)

**Case II :** \( x_1 \) and \( x_2 \) on opposite sides of the origin

\[
\langle T \psi_{\nu}(X_1)\psi_{\nu}^\dagger(X_2) \rangle \sim \text{Exp}[- \sum_{\nu, \nu' = \pm 1, a=\hbar,n} S_{1,1}(\nu, \nu'; a; 1) \log((\nu x_1 - \nu' x_2) - v_a(t_1 - t_2))]
\]

(14)
the table are shown in section Anomalous Exponents.

Table 2: Anomalous exponents \( Q_{x_1,x_2}(\nu, \nu'; a) \) for \( x_1 \) and \( x_2 \) on the same side of the origin. The analytical formulas for the entries in the table are shown in section Anomalous Exponents.

| \( Q_{x_1,x_2}(\nu, \nu'; a) \) | \( \nu_1=1, \nu_2=1; a=h \) | \( \nu_1=-1, \nu_2=-1; a=h \) | \( \nu_1=1, \nu_2=-1; a=h \) | \( \nu_1=-1, \nu_2=1; a=h \) |
|---|---|---|---|
| \( \nu_1=1, \nu_2=1; a=n \) | P | Q | X | X |
| \( \nu_1=-1, \nu_2=-1; a=n \) | Q | P | X | X |
| \( \nu_1=-1, \nu_2=-1; a=n \) | 0 | 0.5 | 0 | 0 |
| \( \nu_1=1, \nu_2=-1; a=n \) | S | S | Y | Z |
| \( \nu_1=-1, \nu_2=1; a=n \) | 0 | 0 | 0.5 | 0 |
| \( \nu_1=-1, \nu_2=1; a=n \) | 0 | 0 | 0 | 0.5 |

The anomalous exponents \( Q_{x_1,x_2}(\nu, \nu'; a) \) and \( S_{x_1,x_2}(\nu, \nu'; a) \) are listed (in shortened form) in Table 2 and Table 3, respectively. The highlight of this work are the formulas described in Case II above. It is easy to see that even after setting \( |R| = 0 \), these Green functions do not correspond to the translationally invariant Luttinger liquid. This implies that even a small reflection coefficient changes the properties of the system drastically when the two points are on opposite sides of the origin. In Appendix C it is shown how these formulas are consistent with trivial checks such as turning off the external potential or turning off the mutual interactions. The Green function at resonant tunneling condition and for the half line case are explicitly written down and a comparison is made with...
Table 3: Anomalous exponents $S_{\nu_1,\nu_2}(\nu,\nu';\alpha;j)$ for $x_1$ and $x_2$ on opposite sides of the origin. Explicit expressions for the entries in the table are given in section Anomalous Exponents.

| $S_{\nu_1,\nu_2}(\nu,\nu';\alpha;j)$ | $\nu' = 1$; $\nu = -1$; | $\nu' = -1$; $\nu = 1$ | $\nu' = 1$; $\nu = -1$ | $\nu' = -1$; $\nu = 1$ |
|--------------------------------------|------------------------|------------------------|------------------------|------------------------|
| $\nu_1 = 1$, $\nu_2 = 1$, $\alpha = h$, $j = 1$ | A | B | C | D |
| $\nu_1 = 1$, $\nu_2 = 1$, $\alpha = n$, $j = 1$ | 0.5 | 0 | 0 | $-0.5$ |
| $\nu_1 = 1$, $\nu_2 = 1$, $\alpha = h$, $j = 2$ | A | B | D | C |
| $\nu_1 = 1$, $\nu_2 = 1$, $\alpha = n$, $j = 2$ | 0.5 | 0 | $-0.5$ | 0 |
| $\nu_1 = -1$, $\nu_2 = -1$, $\alpha = h$, $j = 1$ | B | A | D | C |
| $\nu_1 = -1$, $\nu_2 = -1$, $\alpha = n$, $j = 1$ | 0 | 0.5 | $-0.5$ | 0 |
| $\nu_1 = -1$, $\nu_2 = -1$, $\alpha = h$, $j = 2$ | B | A | C | D |
| $\nu_1 = -1$, $\nu_2 = -1$, $\alpha = n$, $j = 2$ | 0 | 0.5 | 0 | $-0.5$ |

existing literature [2, 39, 38] when such data is available and a comparison is feasible and helpful. Also, the Green functions far away from the cluster of external potentials are shown to reduce to expected forms.

Anomalous exponents

The explicit expressions of the anomalous exponents mentioned in Table 2. and Table 3 are listed below.

$$Q = \frac{(v_h - v_f)^2}{8v_h v_f} ; \quad X = \frac{|R|^2(v_h - v_f)(v_h + v_f)}{8v_h(v_h - |R|^2(v_h - v_f))} ; \quad C = \frac{v_h - v_f}{4v_h}$$  \hspace{1cm} (16)

The other exponents can be expressed in terms of the above expressions.

$$P = \frac{1}{2} + Q ; \quad S = \frac{Q}{C} \left( \frac{1}{2} - C \right) ; \quad Y = \frac{1}{2} + X - C ;$$

$$Z = X - C ; \quad A = \frac{1}{2} + Q - X ; \quad B = Q - X ;$$

$$D = -\frac{1}{2} + C ; \quad \gamma_1 = X ; \quad \gamma_2 = -1 + X + 2C ;$$

Plots of anomalous exponents

All the anomalous exponents are plotted versus the interaction parameter $v_0$ and the barrier strength $V_0$ for a single delta function potential in Fig. 3.

The Fig. 3(a-c) displays the anomalous exponents (for holons) of the two-point functions as a function of the interaction parameter $v_0$ of a system with a single delta potential, the plots (a) and (b) showing those for both the points $x_1$ and $x_2$ on the same side of the origin while the plot (c) showing the same for the points on opposite sides (Table 2. and Table 3.). On the other hand, the anomalous exponents for spinons are either 0 or ±0.5. For the two points on the same side of origin, only one combination out of the four $[(\nu_1 x_1 - \nu_2 x_2) - \nu(t_1 - t_2)]$ survives when $v_0 = 0$ which is expected. But for points on opposite sides of the origin, the rule for choosing $\lambda_1$ in equation (9) causes two such independent terms to be present in the final form of the two-point function in an additive fashion, the denominator being the same in both. When mutual interactions are absent, the numerators add up in such a way as to become time independent which can then be modified by adjusting the c-numbers coefficients (which are both cutoff dependent as well as spatially inhomogenous) to obtain the proper form of the Green function known from elementary considerations (see also Technical clarifications). When mutual interactions are absent, this Green function becomes the translationally invariant one when $|R| = 0$. But when mutual interactions are present, this Green function does not reduce to the translationally invariant one when $|R|$ is made smaller and smaller implying that when both $v_0 \neq 0$ and $|R| \neq 0$, the most singular part of the asymptotic Green function is discontinuous in $|R|$ near $|R| = 0$. This anomaly is the present work’s version of the metaphor introduced by Kane and Fisher[8] who suggested that a $v_0 > 0$ will “cut the wire” when $|R| > 0$. They arrive at their conclusions in a rather convoluted manner by invoking renormalization group methods and so on, but the closed formulas of the present work are more compelling.

Description of figures: The Fig. 3(d-f) shows the variation of the anomalous exponents for a single impurity as a function of the strength of the impurity $V_0$, the interaction parameter $v_0$ being held constant. When $V_0$ is made
zero, the exponents take standard values of translationally invariant systems that are obtained using conventional bosonization for the two points on the same side of the origin (see Giamarchi [2]). The plots of some anomalous exponents for the asymmetric double barrier and symmetric triple barrier are shown in Fig. 4 and Fig. 5 respectively.

Figure 4: Anomalous exponents for double barrier: The anomalous exponents $X$ and $A$ as functions of impurity strength $V_1$ and $V_2$ for an asymmetric double delta potential. Near resonance (cross lines), the system has the same colour it has when both $V_1$ and $V_2$ are zero.

Figure 5: Anomalous exponents for triple barrier: The anomalous exponents $X$ and $B$ as functions of impurity strength $V_0$ (at the center) and $V_1$ (each at $\pm a$) for a symmetric triple delta potential. Resonance can be seen if $V_0$ is zero such that for a specific value of $V_1$ the anomalous exponent takes the same value it takes when $V_0 = V_1 = 0$ as shown by the black line on the surface.

5.1. Four-point functions (Friedel oscillations)

The prescription of equation (12) for pinning down the form of $\lambda_i$ in equation (9) leads to the correct general four-point functions when mutual interactions are absent. However, the general expressions for four-point functions with mutual interactions are quite formidable. Fortunately, there is a special case viz. the density-density correlation function which is much simpler and also more important physically. Friedel oscillations are the rapid spatial variation ($\sim e^{\frac{2\pi k_F x}{a}}$) of the otherwise homogeneous density profile in a Luttinger liquid in response to a spatially localised impurity. In the Kubo formalism it is given as the density-density correlation function[34],[35]. Egger and Grabert have studied Friedel oscillations in a Luttinger liquid with arbitrary interactions and arbitrary strengths of impurities[36]. Consider the expression in equation (4). Define $\tilde{\rho}_f \equiv \rho_f - \langle \rho_f \rangle$. The prescription for choosing $\lambda_i$ in equation (9) leads to the unambiguous conclusion that,

$$\langle T \tilde{\rho}_f(X_1)\tilde{\rho}_f(X_2) \rangle \sim \langle \exp \left[ \sum_{\nu, \nu' = \pm 1}^{a = h, n} \Gamma(\nu, \nu'; a) \log[(\nu x_1 - \nu' x_2) - \nu a(t_1 - t_2)] - 1 \right] \rangle$$

$$\langle T \tilde{\rho}_f(X_1)\tilde{\rho}^*_f(X_2) \rangle \sim \langle \exp[- \sum_{\nu, \nu' = \pm 1}^{a = h, n} \Gamma(\nu, \nu'; a) \log[(\nu x_1 - \nu' x_2) - \nu a(t_1 - t_2)] - 1] \rangle$$

(17)

One should remember that this really means the time derivative of the logarithms of both sides are equal to each other. The values of the anomalous scaling exponents $\Gamma(\nu, \nu'; a)$ can be obtained from the expression below.

$$\Gamma(\nu, \nu'; a) = \left( \frac{v_F}{2v_h} \delta_{\nu, h} + \frac{1}{2} \delta_{\nu, n} \right) (\delta_{\nu, \nu'} - \delta_{\nu, -\nu'})$$

(18)
5.2. Resonant tunneling across a double barrier

Resonant tunneling is well-known in elementary quantum mechanics. Typically, this phenomenon is studied in a double-barrier system. When the Fermi wavenumber bears a special relation with the inter-barrier separation and height, the reflection coefficient becomes zero and the Green functions of the system behave as if they are those of a translationally invariant system. Consider a symmetric double delta-function with strength $V$ and height, the reflection coefficient becomes zero and the Green functions of the system behave as if they are those of a translationally invariant system. Define, $\xi_0 = k_F d$. The resonance condition in this case is well-known to be,

$$V_0 \sin [\xi_0] + v_F \cos [\xi_0] = 0 \quad (19)$$

The anomalous exponents $Q_{1,1}(\nu, \nu'; a)$ (Fig. 6 (a,b)) and $S_{1,1}(\nu, \nu'; a)$ (Fig. 6(c)) are plotted in the vicinity of resonance to see the signatures of resonance tunneling on the Luttinger liquid Green function.

From Fig. 6, it may be seen that when the system is at resonance (depicted by the vertical line), all the anomalous exponents take exactly the same value that they take when there is no barrier at all (also depicted in Fig. 4 and Fig. 5).

5.3. Dynamical density of states

In this section the final results for the dynamical density of states (DDOS) $D_x(\omega)$ at location $x$ is presented. Physically, $D_x(\omega) d\omega$ is the number of quasiparticle states per unit length with energy between $h\omega$ and $h(\omega + d\omega)$ relative to the Fermi energy. It may be seen using the explicit formulas for the Green functions of this work that a closed formula for $D_x(\omega)$ may be derived especially in interesting limits viz. when $x$ is far away from the cluster of barriers and wells and also when $x$ is near or at the location of barriers and wells. The final results are as follows. The detailed plots that help in visualizing these results are relegated to the Figure below. First a dimensionless parameter proportional to the location $x$ is defined viz. $\xi = \frac{2\omega|x|}{v_F \hbar}$. The results for DDOS at zero temperature may be written in general as (the static - i.e. at the Fermi level - low density of states at finite temperature is obtained by simply replacing $\omega$ below by $k_BT$ viz. the temperature),

$$D_\xi(\omega) \sim \omega^{\alpha(\xi)} \quad (20)$$

where $0 < \xi \ll 1 \approx 2Q + 2X$ (Q and X are given in equation (16)) and $\alpha(\xi \gg 1) \approx \frac{(v_h - v_F)^2}{4v_F v_h} = 2Q = \frac{1}{4}(K_\rho + \frac{4}{K_\rho} - 2)$ which is precisely the exponent found in the textbooks for fermions with spin (Giamarchi, Eq.(7.27)).

In the case of the half line, Kane and Fisher [8] have remarked that for spinless fermions the density of states is $\rho_{end}(\epsilon) \sim e^{\frac{\epsilon}{\hbar\omega}}$ where $g = \frac{v_h}{v_F}$. For fermions with spin the exponent may be inferred as half of this as done earlier viz. $D_{\text{half-line}}(\omega) \sim \omega^{\frac{1}{2}\left(\frac{7}{3}-1\right)}$. Setting $|R| = 1$ for half-line, $Z_h = \frac{v_h}{v_F}$ hence $\alpha(\xi \equiv 0) = \frac{(v_h - v_F)^2}{2v_F} = \frac{1}{2}(\frac{1}{K_\rho} - 1)$ since $K_\rho = \frac{2\omega}{v_h}$. For repulsive interactions, $v_h > v_F$ the exponent $\alpha(0 < \xi \ll 1) > 0$. But for attractive interactions $v_h < v_F$ and for $|R|^2 > |R_{c1}|^2 = \frac{v_F - v_h}{2v_F v_h}$ the exponent $\alpha(0 < \xi \ll 1) > 0$. Lastly for attractive interactions with $|R| > |R_{c1}|$ the exponent $\alpha(0 < \xi \ll 1) < 0$.

The main advancement of the present work is being able to provide simple analytical expressions for exponents such as these that interpolate between the no-barrier and strong barrier cases which competing methods are unable to do with resorting to difficult RG methods. The novel technical framework that abandons the g-ology framework in favor of a non-chiral bosonization method with non-standard harmonic analysis of the field operator enables an exact treatment of free fermions plus impurity problem.
The exponent near the impurity becomes negative in some regions (see Appendix D) signifying that the density of states diverges at low energies near the impurity i.e. the impurity together with attractive interactions effectively brings back low-energy quasiparticles which ought not to be there in a Luttinger liquid. This phenomenon may also been seen in the tunneling conductance derived in Appendix E (also see below). Note that contributions that are analogous to Friedel oscillations in the DDOS have been ignored.

5.4. Conductance

Finally, the general formula for the conductance of a quantum wire (obtained from Kubo’s formula that relates it to current-current correlations) without leads but with electrons experiencing forward scattering short-range mutual interactions and in the presence of a finite number of barriers and wells clustered around an origin is (in proper units),

\[ G = \frac{e^2}{h} \frac{v_F}{v_h} \left( 1 - \frac{v_F}{v_h} Z_h \right) \]

(21)

where \( Z_h \) is given by Eq.(11). See Appendix E for more details such as plots and so on that pictorially depict this result.

Alternatively, conductance may also be thought of the outcome of a tunneling experiment [8]. In this case the results depend on the length of the wire \( L \) and a cutoff \( L_\omega = \frac{k_B T}{v_F} \) that may be regarded either as inverse temperature or inverse frequency (in case of a.c. conductance). The result derived in Appendix E is

\[ G \sim \left( \frac{L}{L_\omega} \right)^{-4Q} \left( \frac{L}{L_\omega} \right)^{8X} \]

(22)

It is important to stress that the present work has carefully defined tunneling conductance and it is not simply the square of the dynamical density of states of either the bulk or the half line (Appendix E). Of particular interest is the weak link limit where \(|R| \to 1\). The limiting case of the weak link are two semi-infinite wires. In this case,

\[ G_{\text{weak-link}} \sim \left( \frac{L}{L_\omega} \right)^{(v_h+v_F)^2-4v_h^2} \]

(23)

Hence the d.c. conductance scales as \( G_{\text{weak-link}} \sim (k_B T) \frac{(v_h+v_F)^2-4v_h^2}{2v_h v_F} \). This formula is consistent with the assertions of Kane and Fisher [8] that show that at low temperatures \( k_B T \to 0 \) for a fixed \( L \), the conductance vanishes as a power law in the temperature if the interaction between the fermions is repulsive (\( v_h > v_F > 0 \)) and diverges as a power law if the interactions between the fermions is attractive (\( v_F > v_h > 0 \)). Their result is applicable to spinless fermions without leads \( G_{\text{weak-link-nospin}} \sim (k_B T)^{\frac{2}{3}} \) to compare with the result of the present work.
this exponent has to be halved \( G_{\text{weak-link-with-spin}} \sim (k_B T)^{\frac{1}{2}} \). But this is arrived at by assuming that the conductance is proportional to the square of the dynamical density of states for a half line near the weak link. The present work shows that this is not really correct. However the qualitative conclusions they arrive at are consistent with the present work. In general, the claim of the present work is that the temperature dependence of the tunneling d.c. conductance of a wire with no leads in the presence of barriers and wells and mutual interaction between particles is,

\[
G \sim (k_B T)^{\eta}; \quad \eta = 8X - 4Q
\]

When \( \eta > 0 \) the conductance vanishes at low temperatures as a power law - characteristic of a weak link. However when \( \eta < 0 \) the conductance diverges at low temperature as a power law - characteristic of a clean quantum wire. Of special interest is the situation \( \eta = 0 \) where the conductance is independent of temperature. This crossover from a conductance that vanishes as a power law at low temperatures to one that diverges as a power law occurs at the critical exponent \( \eta_c \). For attractive interactions, \( \eta < 0 \) which means the conductance always diverges as a power law at low temperatures. This means attractive interactions heal the chain for all reflection coefficients including in the extreme weak link case. On the other hand for repulsive interactions, for \( |R| > |R_c2|, \eta > 0 \) the chain is broken (conductance vanishes) at low temperatures. For \( |R| < |R_c2|, \eta < 0 \) and even though the interactions are repulsive the chain is healed (conductance diverges).

It is hard to overstate the importance of these results. They show that it is possible to analytically interpolate between the weak barrier and weak link limits without involving RG techniques.

6. Technical clarifications

In this section, some technical points related to the implementation of the scheme outlined in the main text are clarified. In particular, care must be taken while using the modified prescription in equation (9) to write down the general N-point function. While doing so one encounters c-numbers such as \( \langle C_{\lambda_1,\nu_1,\sigma_1} C_{\lambda_2,\nu_2,\sigma_2} C_{\lambda_3,\nu_3,\sigma_3} \rangle \) and so on. Rather than thinking of these as products of complex numbers \( C \) and its complex conjugate \( C^* \), one is required to think of them as one single object each viz. \( F_2(\lambda_1,\nu_1,\sigma_1;\lambda_2,\nu_2,\sigma_2) \), \( F_4(\lambda_1,\nu_1,\sigma_1;\lambda_2,\nu_2,\sigma_2;\lambda_3,\nu_3,\sigma_3) \) and so on. This may seem unique to the present technique but it is not. The standard g-ology method also employs this contraction - indeed it is crucial there otherwise none of those formulas make sense. To understand why, recall that in g-ology, the field operator has a bosonic description such as [2]

\[
\psi_R(x,t) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\epsilon}} e^{i\theta_R(x,t;\epsilon)}
\]

The equation (24) is merely a mnemonic for writing down the N-point functions of a translationally invariant system as this argument shows. Taken literally, equation (24) means,

\[
\langle \psi_R(x,t)\psi_R^\dagger(x',t') \rangle = \lim_{\epsilon,\epsilon' \to 0} \frac{1}{\sqrt{4\epsilon\epsilon'}} \langle e^{i\theta_R(x,t;\epsilon)} e^{-i\theta_R(x',t';\epsilon')} \rangle
\]

and

\[
\{\psi_R(x,t),\psi_R^\dagger(x',t')\} = \lim_{\epsilon,\epsilon' \to 0} \frac{1}{\sqrt{4\epsilon\epsilon'}} \{e^{i\theta_R(x,t;\epsilon)} e^{-i\theta_R(x',t';\epsilon')} \}
\]

It is easy to see that equation (25) and equation (26) make no sense at all unless one “contracts” the two \( \epsilon \)'s and set them equal to each other before taking the limit. Hence the cutoff-dependent prefactor \( (4\epsilon\epsilon')^{-\frac{1}{2}} \) should be regarded as one single entity viz. \( (4\epsilon^2)^{-\frac{1}{2}} \) rather than the product of two independent singular quantities.

The second technical point is that the presence of additional operators in the exponent of equation (9) such as \( 2\pi i \int \rho(-y,t)dy \) appears to violate the point splitting constraint which is important in order to recover the expressions for the current and density operators from the field operator. The point splitting condition is as follows.

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \langle \psi_R^\dagger(x',t')\psi_R(x,a,\sigma,t) \rangle = \langle \psi_R^\dagger(x,a,\sigma,t+\epsilon)\psi_R(x,a,\sigma,t) \rangle - \langle \psi_R^\dagger(x,a,\sigma,t)\psi_R(x,a,\sigma,t) \rangle
\]

\[
= \frac{1}{2\rho} \langle \rho_e(x,a,\sigma,t) + \int_{\epsilon \to 0} dy' \partial_{y'} \rho_e(y',a,\sigma,t) \rangle
\]

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \langle \psi_R^\dagger(x',t')\psi_R(x,a,\sigma,t) \rangle = \langle \psi_R^\dagger(x,a,\sigma,t+\epsilon)\psi_R(x,a,\sigma,t) \rangle - \langle \psi_R^\dagger(x,a,\sigma,t)\psi_R(x,a,\sigma,t) \rangle
\]

\[
= \frac{1}{2\rho} \langle \rho_e(x,a,\sigma,t) + \int_{\epsilon \to 0} dy' \partial_{y'} \rho_e(y',a,\sigma,t) \rangle
\]
Define,
\[ \sum_{\gamma_1, \gamma_2 = \pm 1} \theta(\gamma_1 x_1) \theta(\gamma_2 x_2) F_2(\lambda_1, \nu_1, \gamma_1, x_1, \sigma_1; \lambda_2, \nu_2, \gamma_2, x_2, \sigma_2) \equiv C_2(\lambda_2, \nu_2, x_2, \sigma_2; \lambda_1, \nu_1, x_1, \sigma_1) \]

The point-splitting condition in equation (27) is obeyed provided the c-numbers, \( C_2 \) satisfy the following constraints \((a \to 0, a > 0)\),
\[ C_2(1, \nu, x, \sigma; 0, \nu, x + a, \sigma) = C_2(0, \nu, x, \sigma; 1, \nu, x + a, \sigma) \]
\[ C_2(1, \nu, x, \sigma; 1, \nu, x + a, \sigma) = 0 \]
\[ C_2(0, \nu, x, \sigma; 0, \nu, x + a, \sigma) = \frac{1}{2\pi a} \]

One practical consequence of this is that while evaluating the two-point function, a term where both the field operators have the additional \( \rho(−y, t) \) operator in the exponent is never necessary.

7. Discussion

The results of the present work are compelling for the following reasons.
(i) The correspondence between the non-local Fermi field and the boson-like variables in the formalism reproduce the proper Fermi (anti) commutation rules even though this correspondence differs from the one used for studying homogeneous (or weakly inhomogeneous) systems.
(ii) Current algebra, point splitting constraints c.e.t.c. continue to be obeyed by this new Fermi-Bose correspondence.
(iii) The slow part of the density correlations are rigorously seen to be the resummation to all orders (in the mutual coupling) of the most singular parts of the RPA contributions.
(iv) Unlike in the g-ology based approaches, the source of the strong inhomogeneities are treated exactly in the NCBT. The asymptotically exact continuous one particle Green function and the concomitant trivial exponents for a system without mutual interactions are naturally obtained. This is in contrast to the usual approach where even this model is a complicated sine-Gordon theory.
(v) It is shown that in addition, retaining the most singular part of the RPA Green function in the perturbative expansion using Fermi algebra yields terms that match exactly with the terms obtained by expanding the closed analytical expressions obtained using NCBT.
(vi) The formalism described can also be successfully used to obtain the correlation functions of a fermionic one step ladder system [37].

8. Conclusions

In this work, the formalism of the non-chiral bosonization technique (NCBT) has been laid down which is superior to the conventional g-ology based methods especially for systems that are strongly inhomogeneous as the former is capable of providing the most singular parts of the asymptotically exact Green functions of such systems as closed analytical expressions. These formulas interpolate between the weak barrier and weak link extreme cases that are studied in the literature. Unlike the competing methods that can only study these extreme limits reliably, the present approach is able to connect the two regimes using analytical means. The results thus obtained match fully with various obvious as well as non-trivial limiting cases of the corresponding Green functions found in the literature. The four point function needed for the study of Friedel oscillations have also been computed. For double barrier systems, the phenomenon of resonant tunneling has been highlighted and the behavior of the anomalous exponents in its vicinity elucidated. The non-uniform nature of the local density of states in these systems and the associated exponents is also highlighted. Finally, closed analytical formulas for both the Kubo-conductance and exponents associated with tunneling conductance of a wire with impurities are provided.

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The idea, formalism and calculations of the present work belong to GSS. Co-author JPD verified all the claims made in this work independently and found alternative routes to obtaining the same results. Co-author CC was involved in a similar verification for part of the work in its early stages.

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APPENDICES

APPENDIX A: Transmission and Reflection Amplitudes

The amplitude reflection and transmission amplitudes for various situations are listed below. These are in turn used to write explicit expressions for the Green function of the translationally non-invariant system when mutual interactions between fermions are absent. These latter results are then used as inputs in the NCBT formalism. The transmission and reflection amplitudes of the six cases shown in the “playing card” Fig.2 are shown below:

(a) Single delta-function

\[ T = \frac{1}{1 + V_0 \frac{v_F}{v_F}} \quad ; \quad R = - \frac{iV_0}{v_F} \left( 1 + V_0 \frac{v_F}{v_F} \right) \]  

(A.1)

(b) Symmetric double delta-function

\[ T = \frac{1}{1 + V_0 \frac{v_F}{v_F}} - \frac{\sqrt{V_0 \frac{v_F}{v_F}} \sin[\xi_0] + iV_0}{v_F} \left( 1 + V_0 \frac{v_F}{v_F} \right) \quad ; \quad R = \frac{2iV_0}{v_F} \cos[\xi_0] \]  

(A.2)

(c) Asymmetric double delta-function

\[ T = \frac{1}{1 + iV_1 + V_2 + iV_0 \frac{v_F}{v_F}} + \frac{iV_0 \frac{v_F}{v_F}}{v_F} \cos[\xi_0] \quad ; \quad R = \frac{2iV_1 V_2}{v_F} \sin[\xi_0] + \frac{2iV_0}{v_F} \left( \frac{V_1 e^{i\xi_0} + V_2 e^{-i\xi_0}}{v_F} \right) \]  

(A.3)

(d) Symmetric triple delta-function

\[ T = \frac{1}{1 - iV_0 \frac{v_F}{v_F} - 2V_0 \frac{v_F}{v_F} + iV_0 \frac{v_F}{v_F} + 2iV_0 \frac{v_F}{v_F}} + \frac{iV_0 \frac{v_F}{v_F}}{v_F} \cos[\xi_0] \quad ; \quad R = \frac{2iV_0}{v_F} \left( \frac{V_0 \frac{v_F}{v_F} - i\cos[\xi_0] \frac{V_0 \frac{v_F}{v_F}}{v_F} + 2V_0 V_1 + e^{i\xi_0} V_2 \right) \]  

(A.4)

Note: \( \lambda = \frac{v_F}{v_F} \) is fixed while taking the RPA limit and \( V \) is the well depth or barrier height and \( E_F = \frac{1}{2} m v_F^2 \) is the Fermi energy. \( \xi_0 = 2k_F a \) where \( k_F \) is the Fermi momentum and the barrier(or well) goes from ‘-a’ to ‘a’.

(e) Finite barrier tunneling

\[ T = \frac{4e^{-i\xi_0} \sqrt{X - 1}}{4\sqrt{X - 1} \cosh[\xi_0 \sqrt{X - 1}] + 2(2 - \lambda) \sinh[\xi_0 \sqrt{X - 1}]} \quad ; \quad R = \frac{e^{-i\xi_0} 2\lambda \sin[\xi_0 \sqrt{X - 1}]}{4\sqrt{X - 1} \cosh[\xi_0 \sqrt{X - 1}] + 2(2 - \lambda) \sinh[\xi_0 \sqrt{X - 1}]} \]  

(A.5)

(f) Finite well

\[ T = \frac{4e^{-i\xi_0} \sqrt{X + 1}}{4\sqrt{X + 1} \cos[\xi_0 \sqrt{X + 1}] - 2(2 + \lambda) \sin[\xi_0 \sqrt{X + 1}]} \quad ; \quad R = \frac{e^{-i\xi_0} 2\lambda \sin[\xi_0 \sqrt{X + 1}]}{4\sqrt{X + 1} \cos[\xi_0 \sqrt{X + 1}] - 2(2 + \lambda) \sin[\xi_0 \sqrt{X + 1}]} \]  

(A.6)

It can be shown that on taking proper limits, it is possible to obtain one from the another, for example, from finite barrier to single delta, from asymmetric double delta to symmetric double delta and so on.
APPENDIX B: Universal field, non-standard harmonic analysis

In this section, the technique used in the main text is explained. It consists of two steps. i) Prove all the statements below with the aim of justifying the central correspondence in Eq.(B.11) which is valid universally and not just in the RPA limit. ii) Extracting the slow parts of the field in Eq.(B.11) involves invoking a model-dependent harmonic analysis of the density. For piecewise constant potentials with compact support discussed in this paper, the prescription is as follows (chosen to ensure trivial exponents when mutual interactions are absent).

**Harmonic analysis of density fluctuation:** The non-standard harmonic analysis of the density that is central to the sort of systems considered in this work may be written as follows ($\rho(x) \to \psi^+(x)\psi(x)$ with the spin index absorbed into the symbol “$x$”),

\[
\rho(x) = \rho_0 + \rho_s(x) + F_1 e^{2ik_Fx} e^{2\pi i sgn(x)\pi} dy \rho_s(y) + F_2 e^{2ik_Fx} e^{2\pi i sgn(x)\pi} dy \rho_s(y) + \rho_s(-y) + c.c.
\]  

(B.1)

Here $\rho_0$ is the average density which diverges in the RPA limit, $\rho_s(x) \sim \psi^+_L(x)\psi_R(x) + \psi^+_R(x)\psi_L(x)$ is the slow part of the density. The remaining terms in Eq.(B.1) form the harmonic analysis part. The term next to $F_1$ is the usual form seen in Haldane’s harmonic analysis whereas the new ingredient in this work is the “non-standard harmonic analysis” viz. the term $F_2$. This Eq.(B.1) is then inserted into Eq.(B.11) and the slow part of the field is extracted. The quantities $F_1$ and $F_2$ are cutoff dependent objects similar to $\frac{1}{\sqrt{2\pi}}$ found in Eq.(24). Just as two such quantities $\frac{1}{\sqrt{2\pi}}$ and $\frac{1}{\sqrt{2\pi'}}$ when multiplied should not be thought of as the product of two numbers viz. $\frac{1}{\sqrt{4\pi'}}$ but one single number obtained through a “contraction” viz. $\frac{1}{\sqrt{2\pi'}}$, here too the contraction of the objects $F_1$ and $F_2$ that appear in the evaluation of density-density correlations such as $\langle F_1 F_1 \rangle$, $\langle F_1 F_2 \rangle$ et c. are to be thought of as operate quantities to be retained or discarded based upon whether such terms do or do not lead to spurious non-trivial exponents for the systems of free fermions plus the cluster of barriers and wells. More convincingly, the rapidly varying parts of the density-density correlations may be evaluated as a special case of the general four-point function using the slow part of the field shown in Eq.(9). This new ingredient with the operator $\rho_s(-y) + \rho_s(y)$ in the exponent is crucial in order to recover the mandatory trivial exponents for a system of free fermions plus impurities. To see this, imagine that Haldane’s harmonic analysis is applicable even for systems with barriers and wells. In this case, the fast part of the density density correlations have the form (i.e. the part that oscillates as $e^{2ik_F(x-x')}$, $\rho_f = \psi^+_L\psi_R$ and $\tilde{\rho}_f = \rho_f - \langle \rho_f \rangle$),

\[
<\rho_f(x,t)\rho_f^*(x',t')>_{\text{chiral-method}} \sim \langle e^{2\pi i sgn(x)\pi} dy \rho_s(y,t) e^{-2\pi i sgn(x)\pi} dy \tilde{\rho}_s(y,y') \rangle
\]  

(B.2)

$\tilde{\rho}_f(x,t)\rho_f^*(x',t')$ may be evaluated for the system of free fermions interacting with the cluster of barriers and wells by evaluating Eq.(B.2) using the cumulant expansion. The terms that appear in the exponent are nothing but the slow part of the density-density correlations given by Eq.(6) (for concreteness consider a single delta function $\delta(x)$). The non-standard harmonic analysis of the density. For piecewise constant potentials with compact support discussed in this paper, the prescription is as follows (chosen to ensure trivial exponents when mutual interactions are absent).

The above prediction of standard chiral bosonization is wrong since this quantity cannot have nontrivial system dependent exponents. This follows from the observation that this quantity may be trivially evaluated using Wick’s theorem as follows,

\[
<\tilde{\rho}_f(x,t)\tilde{\rho}_f^*(x',t')>_{\text{correda}} = - T \psi_L(x',t')\psi^+_L(x,t) + T \psi_L(x,t)\psi^+_L(x',t')
\]  

Substituting the formulas for the Green function found in Eq.(4) the correct expression has only trivial exponents.

\[
<\tilde{\rho}_f(x,t)\tilde{\rho}_f^*(x',t')>_{\text{correda}} = \frac{1}{(2\pi)^2((x-x')^2 - V^2_F(t-t')^2)}
\]  

(B.3)

The way to modify Eq.(B.2) so that it correctly gives Eq.(B.3) when barriers and wells are present is to think of it as

\[
<\tilde{\rho}_f(x,t)\tilde{\rho}_f^*(x',t')>_{\text{non-chiral-method}} = <\tilde{U}_{nc}(x,t)\tilde{U}^*_c(x',t')> + <\tilde{U}_c(x,t)\tilde{U}^*_c(x',t')>
\]  

(B.4)

where $\tilde{U}_{nc}(x,t) = U_{nc}(x,t) - <U_{nc}(x,t)>$ and $\tilde{U}_c(x,t) = U_c(x,t) - <U_c(x,t)>$ and,

$U_{nc}(x,t) = e^{2\pi i sgn(x)\pi} dy (\tilde{\rho}_s(y,t) + \tilde{\rho}_s(-y,t))$
Theorem FBD: Eq.(B.10) together with Eq.(B.7) implies Eq.(B.9).

Proof: Straightforward provided

Fermi-Bose Correspondence: Conjecture: The Eq.(B.11) inserted into Eq.(B.6) leads to an identity together with Eq.(B.10) provided the following identification is made,

e
where

Proof: Well-known, left to the reader.

Theorem F2: Field-Current/Density commutators:

Forward relation:

Proof: Well-known, left to the reader.

2.1. Fermi Language

Fermi fields: Let there be \( N \) species of fermions \( \psi_j(x) \) where \( j = 1, 2, \ldots, N \).

\[
\{\psi_j(x), \psi_k(x')\} = 0 ; \quad \{\psi_j(x), \psi_k^\dagger(x')\} = \delta_{j,k} \delta(x-x')
\]  

(B.5)

Forward relation:

\[
j_k(x) = Im[\psi_k^\dagger(x) \partial_x \psi_k(x)] ; \quad \rho_k(x) = \psi_k^\dagger(x) \psi_k(x)
\]  

(B.6)

Theorem F1: Current Algebra:

\[
[j_k(x), \rho_l(x')] = i \delta_{k,l} \rho_l(x') \partial_x \delta(x-x')
\]

(B.7)

Proof: Well-known, left to the reader.

2.2. Bose Language

Boson Fields: Define self-adjoint \( \pi_j(x) \) and \( \rho_j(x) \), \( j = 1, 2, 3, \ldots, N \) obeying canonical commutation rules.

\[
[j_k(x), \rho_l(x')] = 0 ; \quad [\pi_j(x), \pi_k(x')] = 0 ; \quad [\pi_j(x), \rho_k(x')] = i\delta_{j,k} \delta(x-x')
\]  

(B.9)

Forward relation:

\[
j_k(x) = -\rho_k(x) \partial_x \pi_k(x)
\]  

(B.10)

Theorem FBD: Eq.(B.10) together with Eq.(B.7) implies Eq.(B.9).

Proof: Straightforward provided \( \rho_k(x) \) vanishes nowhere since division by this quantity is needed.

Conjecture: Fermi-Bose Correspondence:

\[
\psi_k(x) = e^{i\frac{\pi}{N} \sum_{l<k} \int_{-\infty}^{\infty} dp \rho_l(p) \frac{1}{\sqrt{N^0}} \sum_p n_F(p) e^{i\xi(p)} - i\pi sgn(p) \int_{-\infty}^{\infty} dy \rho_k(y) e^{-i\pi k(x)} \sqrt{\rho_k(x)}}
\]

(B.11)

where \( n_F(p) = \theta(k_F - |p|) \) and \( N^0 = \sum_p n_F(p) \).

Theorem FB0: The Eq.(B.11) inserted into Eq.(B.6) leads to an identity together with Eq.(B.10) provided the following identification is made, \( e^{i\xi(p)} e^{-i\xi(p')} = \delta_{p,p'} \) (imagine e.g. \( \xi(p) \neq \xi(p') \)) when \( p \neq p' \) to be a real quantity.
that tends to infinity for all \( p \).

**Proof:** Straightforward.

**Theorem FB1:** The conjecture in Eq.(B.11) obeys fermion commutation rules Eq.(B.5) in conjunction with Eq.(B.9) and \( e^{i\xi(p)} e^{-i\xi(p')} = \delta_{pp'} \).

**Proof:** Some effort needed but straightforward.

**Theorem B1: Field-Current/Density commutators:**
Lastly, the identities in Eq.(B.12) below are obeyed regardless of whether these commutators are evaluated in the usual Fermi language or using the conjecture Eq.(B.11) and the canonical commutators Eq.(B.9).

\[
\begin{align*}
[\psi_k(x), \rho_l(x')] &= \delta_{k,l} \delta(x - x') \psi_k(x); \\
[\psi_k(x), j_l(x')] &= \frac{1}{2t} (\delta_{k,l} \delta(x - x') (\partial_{x'} \psi_l(x')) - \delta_{k,l} (\partial_{x'} \delta(x - x')) \psi_l(x'))
\end{align*}
\]

(B.12)

**Proof:** Straightforward again provide it is permissible to divide by \( \rho_k(x) \).

**APPENDIX C: Limiting case checks**

Here the focus is to convince the reader that the formulas for the two-point functions derived using the NCBT approach (demonstration for four point functions are similar and will be omitted) viz. Eq.(13), Eq.(14) and Eq.(15) reduce to well -understood limits under the following situations.

a) Standard Luttinger liquid result when there are no barriers or wells (equivalently at resonant tunneling) \( |R| = 0 \) when the two points are on the same side of the origin.

b) Two-point functions of free electrons in the presence of barriers and wells when mutual interactions are absent.

c) Two-point functions of a Luttinger liquid in a half-line when reflection coefficient is unity \( R = -1 \) (imagine the single delta function case and set \( V_0 = \infty \)).

d) Two-point function of a standard Luttinger liquid when the two points are on the same side of the cluster of barriers and wells and far away from this cluster.

The following are the closed expressions for two point functions obtained using NCBT technique for a cluster of impurities along with the forward scattering.

**Case I: \( x \) and \( x' \) on the same side of the origin**
\[
\langle T \psi_R(x, \sigma, t) \psi_R^\dagger(x', \sigma', t') \rangle = \left( g_{1.1}(1, 1) \theta(x) \theta(x') + g_{-1, -1}(1, 1) \theta(-x) \theta(-x') \right) e^{\gamma_1 \log \left[ 4x' \right]} e^{-\frac{1}{2} \log \left( \left( x-x' \right) - v_F (t-t') \right)} e^{-P \log \left[ \left( x-x' \right) - v_F (t-t') \right]} e^{-Q \log \left[ \left( x-x' \right) + v_F (t-t') \right]} e^{-X \log \left[ \left( x+x' \right) - v_h (t-t') \right]} e^{-X \log \left[ \left( x+x' \right) + v_h (t-t') \right]}
\]
\[
\langle T \psi_L(x, \sigma, t) \psi_L^\dagger(x', \sigma', t') \rangle = \left( g_{1.1}(1, 1) \theta(x) \theta(x') + g_{-1, -1}(1, 1) \theta(-x) \theta(-x') \right)
\]
\[
\frac{1}{2} \left( 1 + \gamma_2 \right) e^{\gamma_1 \log \left[ 2x \right]} e^{\gamma_2 \log \left[ 2x' \right]} e^{-\frac{1}{2} \log \left[ \left( x-x' \right) - v_F (t-t') \right]} e^{-P \log \left[ \left( x-x' \right) - v_F (t-t') \right]} e^{-Q \log \left[ \left( x-x' \right) + v_F (t-t') \right]} e^{-X \log \left[ \left( x+x' \right) - v_h (t-t') \right]} e^{-X \log \left[ \left( x+x' \right) + v_h (t-t') \right]}
\]

Case II: \( x \) and \( x' \) on opposite sides of the origin

\[
\langle T \psi_R(x, \sigma, t) \psi_R^\dagger(x', \sigma', t') \rangle = \frac{1}{2} \left( x + x' \right) e^{\gamma_1 \log \left[ 2x \right]} e^{\gamma_2 \log \left[ 2x' \right]}
\]

\[
\left( g_{1.1}(1, 1) \theta(x) \theta(x') + g_{-1, 1}(1, 1) \theta(-x) \theta(x') \right) e^{-\frac{1}{2} \log \left( \left( x-x' \right) - v_F (t-t') \right)} e^{-P \log \left( \left( x-x' \right) - v_F (t-t') \right)} e^{-X \log \left( \left( x+x' \right) - v_h (t-t') \right)} e^{-X \log \left( \left( x+x' \right) + v_h (t-t') \right)}
\]

\[
\left( g_{1.1}(1, 1) \theta(x) \theta(x') + g_{-1, 1}(1, 1) \theta(-x) \theta(x') \right) e^{-\frac{1}{2} \log \left( \left( x-x' \right) - v_F (t-t') \right)} e^{-P \log \left( \left( x-x' \right) - v_F (t-t') \right)} e^{-X \log \left( \left( x+x' \right) - v_h (t-t') \right)} e^{-X \log \left( \left( x+x' \right) + v_h (t-t') \right)}
\]

The values of the all anomalous exponents \( P, Q, A, B \) etc are given in the section “Anomalous exponents” of the main paper. The values of the \( \gamma_1, \gamma_2 (\nu, \nu') \) can be obtained from the table 1 of the main text and the \( T \)'s and \( R \)'s for the different subcases are given in Appendix A.

The cases (a),(b),(c) and (d) are proved one by one.

(a) When \( |R| = 0 \) then \( g_{1.1}(1, -1) = g_{-1,-1}(1, -1) = g_{1.1}(-1, 1) = g_{-1,-1}(-1, 1) = 0 \). This means even in NCBT formalism \( \langle T \psi_R(x_1, t_1) \psi_R^\dagger(x_2, t_2) \rangle = \langle T \psi_L(x_1, t_1) \psi_L^\dagger(x_2, t_2) \rangle = 0 \). When \( |R| = 0 \),

\[
P = \frac{(v_h + v_F)^2}{8v_h v_F} ; Q = \frac{(v_h - v_F)^2}{8v_h v_F} ; X = 0 ; \gamma_1 = 0
\]

and \( g_{1.1}(1, 1) = g_{1.1}(-1, -1) = g_{-1,-1}(1, 1) = g_{-1,-1}(-1, -1) = \frac{i}{2\pi} \). Hence the only non-vanishing parts of the
NCBT two-point function for points on the same side the origin are,
\[
\langle T\psi_R(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') \rangle = \left(g_{11}(1,1)\theta(x)\theta(x') + g_{-1,-1}(1,1)\theta(-x)\theta(-x')\right) e^{-\frac{1}{2}\log[|x-x'|-v_F(t-t')]}
\]
\[
e^{-\frac{(v_h-v_F)^2}{8v_hv_F}\log[|x-x'|-v_h(t-t')]} e^{-\frac{(v_h+v_F)^2}{8v_hv_F}\log[|x-x'|+v_h(t-t')]} \tag{C.3}
\]

These are precisely the standard Luttinger liquid two-point functions of a translationally invariant system (for example the expected answer is given in of Giamarchi’s book, “Quantum Physics in One Dimension” Oxford (2003) after inserting his Eq.(7.22) and similar version for spin into Eq.(7.21)). It is reassuring to set \(v_h = v_F\) in the above formulas and verify that expected results are obtained for the elementary case of free fermions including the correct prefactors since \(g_{11}(1,1) = g_{11}(-1,-1) = g_{-1,-1}(1,1) = g_{-1,-1}(-1,-1) = \frac{1}{2\pi}\).

(b) In this case \(v_h = v_F\),
\[
P = \frac{1}{2} ; \quad Q = 0 ; \quad X = 0 ; \quad S = 0 ;
\]
\[
Y = \frac{1}{2} ; \quad Z = 0 ; \quad A = \frac{1}{2} ; \quad B = 0 ;
\]
\[
C = 0 ; \quad D = -\frac{1}{2} ; \quad \gamma_1 = 0 ; \quad \gamma_2 = -1
\]

Using these the NCBT two-point functions may be written as,

**Case I: \(x\) and \(x'\) on the same side of the origin**
\[
\langle T\psi_R(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') \rangle = e^{-\log[|x-x'|-v_F(t-t')]\left(g_{11}(1,1)\theta(x)\theta(x') + g_{-1,-1}(1,1)\theta(-x)\theta(-x')\right)}
\]
\[
\langle T\psi_R(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') \rangle = e^{-\frac{1}{2}\log[|x-x'|-v_F(t-t')]} \tag{C.4}
\]

**Case II: \(x\) and \(x'\) on opposite sides of the origin**
\[
< T\psi_R(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') > = \frac{1}{2(x+x')} \left(g_{11}(-1,1)\theta(x)\theta(-x') + g_{-1,1}(1,1)\theta(-x)\theta(x')\right) e^{-\frac{1}{2}\log[|x-x'|-v_F(t-t')]} e^{-\log[|x-x'|-v_F(t-t')]} e^{-\frac{1}{2}\log[|x-x'|+v_F(t-t')]} e^{-\frac{1}{2}\log[|x-x'|+v_F(t-t')]} 
\]
\[
< T\psi_L(x,\sigma,t)\psi_R^\dagger(x',\sigma,t') > = 0
\]
\[
< T\psi_R(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') > = 0 \tag{C.5}
\]

which is also precisely what is expected from elementary physics. It is hard to overstate the importance of this observation. Only the present approach correctly solves the case of free fermions in the presence of a finite number of barriers and wells in the bosonization language without resorting to RG flow equations. This problem has a trivial solution in undergraduate physics and it is crucial that at the very least, any other technique that is recommended is able to solve this system with equal ease and get all the aspects fully right.

c) Two-point functions of a Luttinger liquid in a half-line.
In this case $R = -1$. In this case there is no need to consider the two points on opposite sides of the origin.

$$P = \frac{(v_h + v_F)^2}{8v_h v_F}; Q = \frac{(v_h - v_F)^2}{8v_h v_F}; X = \frac{(v_h - v_F)(v_h + v_F)}{8v_h v_F}$$

$$S = \frac{(v_h - v_F)(v_h + v_F)}{8v_h v_F}; Y = \frac{(v_h + v_F)^2}{8v_h v_F}; Z = \frac{(v_h - v_F)^2}{8v_h v_F}$$

$$A = \frac{v_h + v_F}{4v_h}; B = -\frac{(v_h - v_F)}{4v_h}$$

$$C = \frac{v_h - v_F}{4v_h}; D = -\frac{v_h + v_F}{4v_h}; \gamma_1 = \frac{(v_h - v_F)(v_h + v_F)}{8v_h v_F}$$

$$\gamma_2 = \frac{[(v_h - v_F) - 4v_F](v_h + v_F)}{8v_h v_F}$$

**Case I: $x$ and $x'$ on the same side of the origin**

$$\langle T \psi_R(x, \sigma, t) \psi_R^\dagger(x', \sigma, t') \rangle$$

$$= \left( g_{1,1}(1,1) \theta(x) \theta(x') + g_{-1,-1}(1,1) \theta(-x) \theta(-x') \right) e^{\gamma_1 \log [4x x']} e^{-\frac{1}{2} \log \left[ (x-x') - v_F(t-t') \right]}$$

$$e^{-P \log [(x-x') - v_h(t-t')] e^Q \log \left[ (x-x') - v_h(t-t') \right]} e^{-X \log \left[ (x+x') - v_h(t-t') \right]} e^{-X \log \left[ -(x+x') - v_h(t-t') \right]}$$

$$\langle T \psi_L(x, \sigma, t) \psi_L^\dagger(x', \sigma, t') \rangle$$

$$= \left( g_{1,1}(-1,-1) \theta(x) \theta(x') + g_{-1,-1}(-1,-1) \theta(-x) \theta(-x') \right) e^{\gamma_1 \log [4x x']} e^{-\frac{1}{2} \log \left[ (x-x') - v_F(t-t') \right]}$$

$$e^{-Q \log \left[ (x-x') - v_h(t-t') \right]} e^{-P \log \left[ (x-x') - v_h(t-t') \right]} e^{-X \log \left[ (x+x') - v_h(t-t') \right]} e^{-X \log \left[ -(x+x') - v_h(t-t') \right]}$$

$$\langle T \psi_R(x, \sigma, t) \psi_L^\dagger(x', \sigma, t') \rangle$$

$$= \left( g_{1,1}(-1,1) \theta(x) \theta(x') + g_{-1,-1}(-1,1) \theta(-x) \theta(-x') \right) e^{\gamma_1 \log [4x x']} e^{-\frac{1}{2} \log \left[ (x-x') - v_F(t-t') \right]}$$

$$+ e^{\gamma_1 \log [2x] e^{(1+\gamma_2) \log [2x]} e^{-\frac{1}{2} \log \left[ (x-x') - v_F(t-t') \right]}$$

$$e^{-Q \log \left[ (x-x') - v_h(t-t') \right]} e^{-Q \log \left[ (x+x') - v_h(t-t') \right]} e^{-P \log \left[ -(x+x') - v_h(t-t') \right]}$$

One of the important checks that has to be done is to see if the two-point function vanishes (called open boundary conditions) when one of the points is at the location of the impurity when the reflection amplitude $R = -1$ as is mandatory (when mutual interactions are not attractive). More precisely since $R = -1$ implies $\psi_R(x = 0, t) = \psi_L(x = 0, t), < T \psi_R(x = 0, t)(...) > = - < T \psi_L(x = 0, t)(...) >$. It may be seen that the equations Eq.(C.6) obey this constraint as,

$$g_{1,1}(1,1) = g_{-1,-1}(1,1) = g_{1,1}(-1,1)$$

$$g_{1,1}(-1,1) = g_{-1,-1}(-1,1) = -g_{1,1}(1,1) = -g_{-1,-1}(1,1)$$

$$= -g_{-1,-1}(1,1) = -g_{-1,-1}(-1,1) = \frac{i}{2\pi}$$

When the mutual interactions are not attractive $v_h \geq v_F$ so that $\gamma_1 \geq 0$ and $1 + \gamma_2 \geq 0$. This means if mutual interactions exist and are repulsive NCBT predicts $< T \psi_R(x = 0, t)(...) > = - < T \psi_L(x = 0, t)(...) > = 0$. If mutual interactions are not present then NCBT predicts $< T \psi_R(x = 0, t)(...) > = - < T \psi_L(x = 0, t)(...) > \neq 0$. The other check one can do is study the form of the two-point functions far away from the impurity. Instead of doing this crucial check only for $R = -1$ (perfect reflection) it is important to do it in general which is done next. Before this is done it is worthwhile to see if the above NCBT two-point functions for the half-line case are in conformity with other results found in the literature. The main works with which the present work is compared are a) S. Eggert, H. Johannesson and A. Mattsson, Phys. Rev. Lett. 76 1505 (1996) [38] and b) A. Mattsson, S.
In the notation of this work, the $g$-ology coupling constants of Ref.b) are in an external potential obtained from Fermi algebra. This exercise yields spurious system dependent fractional external potential all the exponents are mandated to be trivial (i.e. lead to first order simple poles). Hence Eq.(11)

\[< T \psi_L^1(x,t)\psi_R(y,0) >= G(x,y,t) ; \]
\[< T \psi_R^1(x,t)\psi_L(y,0) >= G(-x,-y,t) ; \]
\[< T \psi_L^1(x,t)\psi_R(y,0) >= -G(x,-y,t) ; \]
\[< T \psi_R^1(x,t)\psi_L(y,0) >= -G(-x,y,t) \]

These expectations are largely consistent with the NCBT results Eq.(C.6). Note that the central Eq.(11) of Ref.b) upon which Eq.(C.7) is based has been criticised in the present work. In particular, evaluating the propagator from Eq.(11) of Ref.b) involves computing the correlations between the chiral currents in the exponents. If the system is the undergraduate system of free fermions plus external impurity potential, these current-current correlations may be evaluated simply using Wick’s theorem and the well known two-point functions of free fermions in an external potential obtained from Fermi algebra. This exercise yields spurious system dependent fractional exponents when evaluating $< T \psi_R(x,t)\psi_L(y,0) >$ using Eq.(11) of Ref.b) whereas for a system of free fermions plus external potential all the exponents are mandated to be trivial (i.e. lead to first order simple poles). Hence Eq.(11) is inapplicable to translationally non-invariant systems (it is only applicable for calculating $< T \psi_R^1(x,t)\psi_R(y,0) >$ and $< T \psi_L^1(x,t)\psi_L(y,0) >$, that too only when $x$ and $y$ are on the same side of the impurity. In all other situation a modification of Eq.(11) has to be invoked using non-standard harmonic analysis i.e. Eq.(B.1). The alternative, viz. the present approach (NCBT), does in fact give the correct trivial exponents for a system of free fermions plus impurity unlike chiral bosonization where an RG approach is invoked even to study this trivial system. Bottom line is, the restriction $\psi_R(x,t) = -\psi_L(-x,t)$ is too strong for a half line. For a half line it is only necessary that $\psi_R(0,t) = -\psi_L(0,t)$ which is also true in the NCBT approach (typically, in these subjects, $A = B$ really stands for $< A(...) >= < B(...) >$ so that such weak equalities are not very sacred and inequivalent approaches are possible). The correct alternative to Eq.(11) of Ref.b) for translationally non-invariant systems is obtained by performing a non-standard harmonic analysis i.e. Eq.(B.1) on the universal field i.e. Eq.(B.11).

In Ref.(a,b) the propagator $G(x,y,t) = < \psi_L^1(x,t)\psi_L(y,0) >$ is given as follows,

\[G(x,y,t) \propto \prod_{\nu = e,s} \exp \left( \frac{i}{2}(K_0^2 - K_s^2) \log(4xy) \right) \exp \left( -\frac{i}{2}(K_0^2 + K_s^2) \log(4x^2v + y^2) \right) \exp \left( -\frac{i}{2}(K_0^2 - K_s^2) \log(4x^2v - y^2) \right) \exp \left( -\frac{i}{2}(K_0^2 - K_s^2) \log(-4x^2v + y^2) \right) \]

(C.8)

In the notation of this work, the g-ology coupling constants of Ref.b) are $g_{4\parallel} = \frac{1}{2}v_0$, $g_{4\perp} = \frac{1}{2}v_0$, $g_{1\parallel} = 0$, $g_{2\parallel} = v_0$ and $g_{2\perp} = v_0$. Hence $g_{c} = g_{2\parallel} + g_{2\perp} = 2v_0$ and $g_{s} = g_{2\parallel} - g_{2\perp} = 0$ and $g_{s}^* = -\sqrt{g_{c}^2 - g_{2\perp}} = 0$. This means in Ref.b) the following substitutions have to be made in order to compare with the present results viz. to first order in $v_0$, $K_s = \sqrt{1 - \frac{g_{2\perp}^2}{2\pi v_0}} = 1$ and $K_c = \sqrt{1 - \frac{g_{c}^2}{2\pi v_0}}$.

\[G(x,y,t) \propto \exp \left( -\frac{i}{2} \log(4x^2v - y^2v) \right) \exp \left( \frac{i}{2\pi v_0} \log(4xy) \right) \exp \left( -\frac{i}{2} \log(4x^2v + y^2) \right) \exp \left( -\frac{i}{2} \log(-4x^2v + y^2) \right) \]

(C.9)

To leading order in $v_0$, $v_h \approx v_F + \frac{v_0}{\pi}$. The exponents of the present work may be simplified to leading order as follows,

\[P = \frac{1}{2} ; \quad Q = 0 ; \quad X = \frac{v_0}{4\pi v_F} ; \quad S = \frac{v_0}{4\pi v_F} ; \quad Y = \frac{1}{2} ; \quad Z = 0 ; \quad A = \frac{1}{2} - \frac{v_0}{4\pi v_F} ; \quad B = -\frac{v_0}{4\pi v_F} ; \quad C = \frac{v_0}{4\pi v_F} ; \quad D = -\frac{1}{2} + \frac{v_0}{4\pi v_F} ; \quad \gamma_1 = \frac{v_0}{4\pi v_F} ; \quad \gamma_2 = -1 + \frac{3v_0}{4\pi v_F} \]
Inserting into Eq.(C.6),
\[ \langle T\psi_L(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') \rangle \sim e^{\frac{\nu_0}{v_F} \log [4\pi x'y]} e^{-\frac{1}{2} \log [(x-x')-v_F(t-t')] e^{-\frac{1}{2} \log [-(x-x')-v_h(t-t')]} \]
\[ e^{-\frac{\nu_0}{v_F} \log [(x+x')-v_h(t-t')] e^{-\frac{1}{2} \log [-(x+x')-v_h(t-t')]} } \]  
(C.10)

Since \( G(x,y,t) \sim \langle T \psi_L(y,\sigma,0)\psi_L^\dagger(x,\sigma,t) \rangle \), the result of the present work viz. Eq.(C.10) and the result of Ref b) viz. Eq.(C.9) are identical.

(d) Consider the situation when \( x > 0, x' > 0 \). Set \( X_{cm} = \frac{x+x'}{2} \) and \( y = x - x' \). This means \( x = X_{cm} + \frac{y}{2} \) and \( x' = X_{cm} - \frac{y}{2} \). Imagine that \( y \) is held fixed as \( X_{cm} \to \infty \). This is the region far away from the impurity. In such a situation it is expected that Green functions such as \( < T \psi_R(x,t)\psi_R^\dagger(x',t') > \) and \( < T \psi_L(x,t)\psi_L^\dagger(x',t') > \) to be immune to the presence or absence of the cluster of impurities. However the parts \( < T \psi_R(x,t)\psi_R(x',t') > \) and \( < T \psi_R(x,t)\psi_L^\dagger(x',t') > \) that are non-zero only because of the impurity have no such restriction. In passing it is noted that while the opposite choice viz. holding \( X_{cm} \) fixed while making \( y \to \infty \) also makes the two points far from the impurity, since the region where the impurity is present has to be traversed, this Green function certainly will not be immune to the presence of the impurity. Inserting this notation in Eq.(C.1) yields,

**Case I: \( x \) and \( x' \) on the same side of the origin**

\[ \langle T\psi_R(x,\sigma,t)\psi_R^\dagger(x',\sigma,t') \rangle \sim e^{\gamma_1 \log [4X_{cm}^2-y^2]} e^{-\frac{1}{2} \log [y-v_F(t-t')] } \]
\[ e^{-P \log [y-v_h(t-t')] -Q \log [y-v_h(t-t')] -X \log [2X_{cm}-v_h(t-t')] -X \log [-2X_{cm}-v_h(t-t')] } \]
\[ \langle T\psi_L(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') \rangle \sim e^{\gamma_1 \log [4X_{cm}^2-y^2]} e^{-\frac{1}{2} \log [-y-v_F(t-t')] } \]
\[ e^{-P \log [y-v_h(t-t')] -Q \log [y-v_h(t-t')] -X \log [2X_{cm}-v_h(t-t')] -X \log [-2X_{cm}-v_h(t-t')] } \]  
(C.11)

For a general cluster of impurities (not necessarily half line) the following identities hold, \( P - Q = \frac{1}{2} \) and \( \gamma_1 = X \). Performing said limit \( X_{cm} \to \infty \) and holding everything else fixed reduces Eq.(C.11) to,

\[ \langle T\psi_R(x,\sigma,t)\psi_R^\dagger(x',\sigma,t') \rangle \sim e^{-\frac{1}{4} \log [y-v_F(t-t')] -\frac{1}{2} \log [y-v_h(t-t')] -Q \log [-y-v_h(t-t')] } \]
\[ \langle T\psi_L(x,\sigma,t)\psi_L^\dagger(x',\sigma,t') \rangle \sim e^{-\frac{1}{4} \log [-y-v_F(t-t')] -\frac{1}{2} \log [-y-v_h(t-t')] -Q \log [-y-v_h(t-t')] } \]  
(C.12)

Since \( Q = \frac{(v_h-v_F)^2}{8v_h v_F} \), Eq.(C.12) is precisely the Green function of a translationally invariant Luttinger liquid. In other words these Green functions are immune to the presence of the impurity.

Therefore all the limiting cases both trivial (only forward scattering or only impurity) and nontrivial (i.e. half line with forward scattering) are properly recovered. In the conventional gology approach, only the half line case and translationally invariant Luttinger liquid are recovered correctly. The case of free electrons in the presence of impurities continues to be complicated in the gology approach but is trivial in the NCBT approach.

**APPENDIX D: Dynamical density of states**

In elementary physics, the dynamical density of states (DDOS) at zero temperature relative to the Fermi energy is defined through the correspondence,

\[ D(\omega) = \frac{1}{V} \sum_k \delta(\omega - \epsilon_k + E_F) \]  
(D.1)

\( D(\omega)d\omega \) is the number of states between \( \omega \) and \( \omega + d\omega \) per unit volume. In one dimension for example, this yields the following result for \( \omega \) small compared to Fermi energy,

\[ D(\omega) = \frac{1}{\pi v_F} \]  
(D.2)
This is the standard result that the density of states near the Fermi energy is constant for a Fermi liquid. This is also the case in higher dimensions. But for a Luttinger liquid, the DDOS $D(\omega) \sim \omega^\alpha$ where $\alpha$ is the density of states exponent. $\alpha$ depends on the forward scattering interaction strength and vanishes when this is zero. To prove this, it is important to generalize the idea of density of states to interacting many-body systems. The generalization is given below.

$$D(x, \omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i(t(x) + t_F)} \langle \{\psi(x, t), \psi^\dagger(x, 0)\} \rangle$$

Thus the density of states for translationally non-invariant systems becomes the local density of states (for fermions and, this is the standard result that the density of states near the Fermi energy is constant for a Fermi liquid. This

$$\langle T \psi_R(x_1) \psi^\dagger_R(x_2) \rangle = \frac{1}{2\pi} e^{-\sum_{n,h} \frac{v_F}{h} \sum_{\nu=\pm 1} (\frac{\nu}{v_F} - \frac{\nu}{v_a})^2 \log(x_1 - x_2 + \nu v_a(t_1 - t_2))}$$

and,

$$\langle T \psi_L(x_1) \psi^\dagger_L(x_2) \rangle$$

$$\langle T \psi_L(x_1) \psi^\dagger_L(x_2) \rangle \rangle = \langle \psi_R(x_1) \psi^\dagger_R(x_2) \rangle + \langle \psi_L(x_1) \psi^\dagger_L(x_2) \rangle$$

These are sufficient to extract the local DDOS including terms that corresponding to Friedel oscillations since in the present case,
same density of states exponent one gets when external potentials are absent which is the standard result predicted by Luttinger liquid theory. Hence an important consistency check has been verified.

Now the important next step is to see how \( D_x(\omega) \) depends on \( x \). The parameter that determines this is the dimensionless quantity \( \xi = \frac{2|v|}{v_h} \). The limit that has already been studied is \( D_{\xi \gg 1}(\omega) \approx D_\infty(\omega) \). The other limit of interest is \( D_{0 < \xi < 1}(\omega) \) i.e. close to the cluster of barriers and wells. Set \( \xi = \frac{2|v|}{v_h} , 2|x| = \frac{\xi v_h}{\omega} \). From the above propagators it is straightforward to deduce,

\[
D_{0 < \xi < 1}(\omega) \sim \left( \frac{\Omega_h}{\omega} \right)^{\frac{1}{2}} Z_h(1+\frac{v_F^2}{\pi^2})-\frac{v_F^2+\omega^2}{\pi^2 v_h^2} + \frac{1}{2} \sim \omega^{\alpha}
\]

where \( \alpha(0 < \xi \ll 1) \approx \frac{(v_h-v_F)(v_F Z_h(v'_F+v_h)+v_h(v_h-v_F))}{4v_F v_h^2} \).

**APPENDIX E: Conductance of a quantum wire**

In this section, the conductance of a quantum wire with no leads is discussed first using Kubo’s formula and next using the idea that it is the outcome of a tunneling experiment.

### 5.3. Landauer’s formula

The electric field is \( E(x,t) = \frac{V_g}{L} \) between \(-\frac{L}{2} < x < \frac{L}{2}\) and \( E(x,t) = 0 \) for \( |x| > \frac{L}{2} \). Here \( V_g \) is the Voltage between two extreme points. Thus a d.c. situation is being considered right from the start. This corresponds to a vector potential,

\[
A(x,t) = \begin{cases} 
\frac{V_g}{L}(ct), & \quad -\frac{L}{2} < x < \frac{L}{2} \\
0, & \text{otherwise.}
\end{cases}
\]

This means (since \( j \approx j_s \),

\[
< j(x,\sigma,t) >= \frac{ie}{\hbar} \sum_{\sigma'} \int_{-L/2}^{L/2} dx' \int_{-\infty}^{t} dt' \frac{V_g}{L}(ct') < [j(x,\sigma,t),j(x',\sigma',t')] >_{LL}
\]

### 5.4. Clean wire: |\( R \)| = 0 but \( v_0 \neq 0 \)

\[
< [j_s(x,\sigma,t),j_s(x',\sigma',t')] > = -\frac{v_F^2}{8\pi^2} \sum_{\nu=\pm 1} (2\pi i) \partial_{\nu F' \nu} (\delta(x-x'+\nu v_h(t-t')) + \sigma \sigma' \delta(x-x'+\nu v_F(t-t')))
\]

Hence,

\[
< j(x,\sigma,t) >= \frac{ie}{\hbar} \sum_{\sigma'} \int_{-L/2}^{L/2} dx' \int_{-\infty}^{t} dt' \frac{V_g}{L}(ct')
\]

\[
\left( -\frac{v_F^2}{8\pi^2} \sum_{\nu=\pm 1} (2\pi i) \partial_{\nu F' \nu} (\delta(x-x'+\nu v_h(t-t')) + \sigma \sigma' \delta(x-x'+\nu v_F(t-t'))) \right)
\]

Or

\[
< j(x,\sigma,t) > = -\frac{V_g e}{2\pi} \frac{v_F}{v_h}
\]

or,

\[
I = (-e) < j(x,\sigma,t) > = \frac{V_g e^2}{2\pi} \frac{v_F}{v_h}
\]
This gives the formula for the conductance (per spin) for a clean quantum wire with interactions,

\[ G = \frac{e^2 v_F}{2\pi v_h} \]

or in proper units,

\[ G = \frac{e^2 v_F}{2\pi \hbar} = \frac{e^2 v_F}{h} \]

A comparison with standard g-ology with the present chosen model gives the following identifications (Eq.(2.105) of Giamarchi).

\[ g_{1,\perp} = g_{1,\parallel} = 0 \]
\[ g_{2,\perp} = g_{2,\parallel} = g_{4,\perp} = g_{4,\parallel} = v_0 \]
\[ g_\rho = g_{1,\parallel} - g_{2,\parallel} - g_{2,\perp} = 0 - v_0 - v_0 = -2v_0 \]
\[ g_\sigma = g_{1,\parallel} - g_{2,\parallel} + g_{2,\perp} = 0 - v_0 + v_0 = 0 \]
\[ g_{4,\rho} = g_{4,\parallel} + g_{4,\perp} = 2v_0 \]
\[ g_{4,\sigma} = g_{4,\parallel} - g_{4,\perp} = 0 \]
\[ y_\rho = g_\rho/(\pi v_F) = -\frac{2v_0}{\pi v_F} \]
\[ y_\sigma = g_\sigma/(\pi v_F) = 0 \]
\[ y_{4,\rho} = g_{4,\rho}/(\pi v_F) = g_{4,\rho}/(\pi v_F) = 2v_0/(\pi v_F) \]
\[ y_{4,\sigma} = g_{4,\sigma}/(\pi v_F) = 0 \]

\[ u_\rho = v_F \sqrt{(1 + y_{4,\rho}/2)^2 - (y_\rho/2)^2} = v_F \sqrt{1 + 2v_0/(\pi v_F)} = v_h \]

\[ K_\rho = \sqrt{\frac{1 + y_{4,\rho}/2 + y_\rho/2}{1 + y_{4,\rho}/2 - y_\rho/2}} = \frac{v_F}{v_h} \]

\[ u_\sigma = v_F \sqrt{(1 + y_{4,\sigma}/2)^2 - (y_\sigma/2)^2} = v_F \]

\[ K_\sigma = \sqrt{\frac{1 + y_{4,\sigma}/2 + y_\sigma/2}{1 + y_{4,\sigma}/2 - y_\sigma/2}} = 1 \]

This gives,

\[ G = \frac{e^2 v_F}{h} = \frac{e^2}{h} K_\rho \]

which is the standard result for a clean quantum wire.

**5.5. The general case:** $|R| > 0$ and $v_0 \neq 0$

Thus,

\[ < [j_s(x, \sigma, t), j_s(x', \sigma', t')] > \]
\[ = -(2\pi i) \frac{v_F v_F}{8\pi^2 v_h} \partial_{\nu_{x',t'}} \sum_{\nu = \pm 1} \left( \delta(\nu(x - x') + v_h(t - t')) - \frac{v_F}{v_h} Z_h \delta(\nu(|x| + |x'|) + v_h(t - t')) \right) \]
\[ - (2\pi i) \frac{\sigma' v_F}{8\pi^2 v_h} \partial_{\nu_{x',t'}} \sum_{\nu = \pm 1} \left( \delta(\nu(x - x') + v_F(t - t')) - |R|^2 \delta(\nu(|x| + |x'|) + v_F(t - t')) \right) \]

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which means,

\[
< j(x, \sigma, t) > = i e \sum_{\sigma'} \int_{-L/2}^{L/2} dx' \int_{-\infty}^{t} dt' \partial_{v_h t'} \frac{V_p}{L} (2\pi i) \frac{v_F}{8\pi^2} \sum_{\nu=\pm 1} \left( \theta(-\nu(x-x') - v_h(t-t')) - \frac{v_F}{v_h} Z_h \theta(-\nu(|x|+|x'|) - v_h(t-t')) \right)
\]

therefore,

\[
< j(x, \sigma, t) > = 2 i e \sum_{\sigma'} \int_{-\infty}^{t} dt' \partial_{v_h t'} \frac{V_p}{L} (2\pi i) \frac{v_F}{8\pi^2} \left( 1 - \frac{v_F}{v_h} Z_h \right)
\]

The conductance of a quantum wire without leads but in the presence of barriers and wells is,

\[
G = \frac{e^2}{(2\pi) v_h} \left( 1 - \frac{v_F}{v_h} Z_h \right)
\]

where,

\[
Z_h = \frac{|R|^2}{1 - \frac{(v_h-v_F)^2}{v_h^2} |R|^2}
\]

Hence the general formula for the conductance of a quantum wire without leads but with electrons experiencing forward scattering short-range mutual interactions and in the presence of a finite number of barriers and wells clustered around an origin is (in proper units),

\[
G = \frac{e^2 v_F}{h v_h} \left( 1 - \frac{v_F}{v_h} Z_h \right)
\]

(E.1)

The above general formula agrees with the three well known limiting cases.

(i) when \( v_h = v_F \), Landauer’s formula \( G = \frac{e^2}{h} |T|^2 \) is recovered.

(ii) when \( |R| = 0 \), the formula \( G = \frac{e^2}{h} K_\rho \) is also recovered.

(iii) when \( |R| = 1 \), \( G = 0 \) regardless of what \( v_h \) is.

From the Fig. E.1 it can be seen that when the reflection coefficient becomes unity (\( |R| = 1 \)), then the conductance vanishes irrespective of the interaction parameter. On the other hand, for any fixed value of \( |R| \), the conductance increases as the mutual interaction becomes more and more attractive (negative \( v_0 \)) and decreases as the interaction becomes more and more repulsive (positive \( v_0 \)). On the other hand for a fixed value of interaction parameter, the conductance decreases with increase in the reflection parameter.

5.6. Conductance from a tunneling experiment

If the conduction process is envisaged as a tunneling phenomenon as against the usual Kubo formula based approach which involves relating conductance to current-current correlation, a qualitatively different formula for the conductance is obtained.

First observe that the quantity \( |T|^2 \) and \( K_\rho \) both serve as a “transmission coefficient” - the former when mutual interactions are absent but barriers and wells are present and the latter vice versa. Both these may be related to spectral function of the field operator (single particle spectral function) as follows.

\[
v_F \int_{-\infty}^{\infty} dt < \{ \psi_\nu(x, \sigma, t), \psi_\nu^\dagger(x', \sigma, 0) \} >= - (2\pi i) \sum_{\gamma, \gamma' = \pm 1} \theta(\gamma x) \theta(\gamma' x') g_{\gamma, \gamma'}(\nu, \nu)
\]

\[
v_F \int_{-\infty}^{\infty} dt < \{ \psi_\nu(\nu L/2, \sigma, t), \psi_\nu^\dagger(-\nu L/2, \sigma, 0) \} >= - (2\pi i) g_{\nu, -\nu}(\nu, \nu)
\]

\[
v_F \int_{-\infty}^{\infty} dt < \{ \psi_\nu(-\nu L/2, \sigma, t), \psi_\nu^\dagger(\nu L/2, \sigma, 0) \} >= T
\]
Figure E.1: Conductance as a function of the absolute value of the reflection amplitude as well the interaction parameter ($v_F = 1$)

$$|T|^2 \rightarrow |v_F \int_{-\infty}^{\infty} dt < \{ \psi_R(\frac{L}{2}, \sigma, t), \psi_R^\dagger(-\frac{L}{2}, \sigma, 0) \} >^2$$

From this point of view, the conductance is,

$$G = \frac{e^2}{h} |v_F \int_{-\infty}^{\infty} dt < \{ \psi_R(\frac{L}{2}, \sigma, t), \psi_R^\dagger(-\frac{L}{2}, \sigma, 0) \} >^2$$  \hspace{1cm} (E.2)

Note that the above formula is not related to the square of the dynamical density of states unlike what was surmised by Kane and Fisher (C. L. Kane and Matthew P. A. Fisher Phys. Rev. Lett. 68, 1220 (1992)). The dynamical density of states is equal-space and unequal time Green function. For tunneling, an electron is injected at $x = -L/2$ and collected at $x = +L/2$ as is the case here which is unequal-space unequal-time Green function i.e. the Green function for the electron traversing the impurity. Technically speaking, the g-ology community are able to handle only the no barrier case and the half line case properly hence for a weak link they are forced to surmise that conductance has something to do with dynamical density of states for a half line near the weak link. The present approach is not only different but physically more sensible and compelling.

Since Eq.(E.2) is conceptually new and different from what is discussed in the literature the exponents are not going to match fully. However their qualitative behavior is going to be identical to the one found by Kane and Fisher as may be seen below.

In the case of a homogeneous system, the relative sign of $x$ and $x'$ is irrelevant since a single formula is applicable in all cases viz. (this matches with Eq.(7.23) of Giamarchi’s textbook)

$$\langle T \psi_R(\frac{L}{2}, \sigma, t) \psi_R^\dagger(-\frac{L}{2}, \sigma, 0) \rangle = \frac{i}{2\pi} e^{-\frac{1}{2} \log |L-v_F t|} e^{-\frac{1}{2} \log |L-v_h t|} e^{-\frac{(v_h-v_F)^2}{4(v_h+v_F)} \log \frac{L^2-(v_h)^2}{L^2}}$$

Hence,

$$\langle \{ \psi_R(\frac{L}{2}, \sigma, t), \psi_R^\dagger(-\frac{L}{2}, \sigma, 0) \} \rangle$$

$$= \frac{i}{2\pi} e^{-\frac{1}{2} \log |L-v_F(t-i\epsilon)|} e^{-\frac{1}{2} \log |L-v_h(t-i\epsilon)|} e^{-\frac{(v_h-v_F)^2}{4(v_h+v_F)} \log \frac{L^2-(v_h)^2}{L^2}}$$

$$- \frac{i}{2\pi} e^{-\frac{1}{2} \log |L-v_F(t+i\epsilon)|} e^{-\frac{1}{2} \log |L-v_h(t+i\epsilon)|} e^{-\frac{(v_h-v_F)^2}{4(v_h+v_F)} \log \frac{L^2-(v_h)^2}{L^2}}$$

while integrating over $t$ the only regions that contribute are $L - v_F t \approx 0$ and $L - v_h t \approx 0$. When $v_h \neq v_F$ these two are different regions. Set $L - v_F t = y$ then $L - v_h t = L - \frac{v_h}{v_F}(L-y)$ and $L + v_h t = L + \frac{v_h}{v_F}(L-y)$. The

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implication is, integration over $t$ is now integration over $y$ and this is important only when $y$ is close to zero. Next set $L - y f t = y'$ then $L + y f t = 2L - y'$ and $L - y ft = L - \frac{y}{y} (L - y')$ and the integrals are important only when $y'$ is close to zero. This means,

$$v_F \int_{-\infty}^{\infty} dt \left\{ \psi_R(L/2, \sigma, t), \psi_R^\dagger(-L/2, \sigma, 0) \right\} =$$

$$\int_{-\infty}^{\infty} dy \frac{i}{2\pi} \left( e^{-\frac{1}{2} \log [y + y f t]} - e^{-\frac{1}{2} \log [y - y f t]} \right) e^{-\frac{1}{2} \log [L(1 - y f t)]} e^{-\frac{1}{2} \log [y f t] e^{-\frac{1}{2} \log [L - y']}}$$

$$+ \frac{v_F}{v_h} \int_{-\infty}^{\infty} dy \frac{i}{2\pi} \left( e^{-\frac{1}{2} \log [y' + y f t]} - e^{-\frac{1}{2} \log [y' - y f t]} \right) e^{-\frac{1}{2} \log [L(1 - y f t)]} e^{-\frac{1}{2} \log [y' f t] e^{-\frac{1}{2} \log [L - y']}}$$

Only the dependence on $L$ is of interest. Write $y = L$ and $y' = L s'$. Hence,

$$v_F \int_{-\infty}^{\infty} dt \left\{ \psi_R(L/2, \sigma, t), \psi_R^\dagger(-L/2, \sigma, 0) \right\} \sim e^{-\frac{(y_f - y_p)^2}{2v_h y_f v_F} \log \left[ \frac{L^2}{2} \right]}$$

This means the tunneling conductance of a clean (no barrier) quantum wire scales as,

$$G_{\text{clean}} \sim \frac{e^2}{h} e^{-\frac{(y_f - y_p)^2}{2v_h y_f v_F} \log \left[ \frac{L^2}{2} \right]} \sim \left( \frac{L}{L_\text{w}} \right)^{\frac{1}{2}(K_F + \frac{1}{2} - 2)}$$

where $L_\omega = \frac{v_F}{K_F T}$ is the length scale associated with temperature (or frequency since $k_B T$ is interchangeable with $\omega$). It says that at low temperatures, the tunneling d.c. conductance of a clean quantum wire with no leads but with interactions ($v_h \neq v_F$) diverges as a power law with exponent $\frac{1}{2}(K_F + \frac{1}{2} - 2) > 0$. Fortuitously, the magnitude of this exponent matches with the exponent of the square of the dynamical density of states of a clean wire (no impurity). However when impurities (or a weak link) is present, there is no guarantee that this coincidence will persist. For a clean wire there is nothing for a electron to tunnel across so this exercise is pointless. What should be studied is tunneling across a weak link. However the general case involves including a finite number finite barriers and wells clustered around the origin. This case is solved elegantly here where a closed formula for the conductance exponents may be obtained unlike in competing approaches found in the literature where a combination of RG and other approaches are needed that fail well short of providing a closed expression for the exponents. More importantly, the present approach is able to provide an analytical interpolation from the weak barrier limit (see above) to the weak link limit to be discussed below - something the competing approaches are incapable of doing without solving complicated RG flow equations, often numerically.

In the general case with the barriers and wells, the Green function for points on opposite sides of the origin has a form that is qualitatively different from the form when the points are on the same side of the origin. This is the really striking prediction of this work.

5.7. With the impurities

Consider the general Green function derived earlier for $xx' < 0$ (Eq.(C.2)). From that it is possible to conclude ($W = g_{1,-1}(1, 1)\theta(x)\theta(-x') + g_{-1,1}(1, 1)\theta(-x)\theta(x')$),

$$< T \psi_R(L/2, \sigma, t), \psi_R(\sigma, 0) > = \frac{v_F + v_h}{2v_F v_h} g_{1,-1}(1, 1) e^{(2X + 2C) \log [L]} e^{-\frac{1}{2} \log [L - v_F t]} e^{-\frac{1}{2} \log [L - v_h t]} e^{(Q - X) \log [L^2 - (v_h t)^2]}$$

$$e^{-\frac{1}{2} \log [L - v_h t]} e^{-(Q - X) \log [L^2 - (v_h t)^2]} e^{-C \log [L^2 - (v_h t)^2]}$$

Since $G \sim |v_F \int_{-\infty}^{\infty} dt < \psi_R(L/2, \sigma, t), \psi_R(\sigma, 0) | >^2$ it is possible to read off the conductance exponent as follows,

$$G \sim \left( \frac{L_\omega}{L} \right)^{-4Q} \left( \frac{L}{L_\omega} \right)^{8X}$$

(E.4)

where $Q = \frac{v_h v_F}{8v_h v_F}$ and $X = \frac{R^2 (v_h v_F)(v_F v_h)}{8v_h v_F |R|^2 (v_h^2 + v_F^2)}$.

It is easy to see that for a vanishing barrier $|R| \to 0$, the earlier result of the conductance of a clean quantum wire is recovered. The other interesting limit is the weak link limit where $|R| \to 1$. The limiting case of the weak link are two semi-infinite wires. In this case,

$$G_{\text{weak-link}} \sim \left( \frac{L}{L_\omega} \right)^{\frac{(v_h + v_F)^2 - 4v_h^2}{2v_h v_F}}$$

(E.5)
Hence the d.c. conductance scales as \( G_{\text{weak-link}} \sim (k_B T)^{\frac{v_h+v_F}{2v_h^2v_F}} \). This formula is consistent with the assertions of Kane and Fisher (C. L. Kane and Matthew P. A. Fisher Phys. Rev. Lett. 68, 1220 (1992)) that show that at low temperatures \( k_B T \to 0 \) for a fixed \( L \), the conductance vanishes as a power law in the temperature if the interaction between the fermions is repulsive \((v_h > v_F > 0)\) and diverges as a power law if the interactions between the fermions is attractive \((v_F > v_h > 0)\). Their result is applicable to spinless fermions without leads \( G_{\text{weak-link-no spin}} \sim (k_B T)^{\frac{1}{2}} \). But this is arrived at by assuming that the conductance is proportional to the square of the dynamical density of states for a half line near the weak link. The present work shows that this is not really correct. However the qualitative conclusions they arrive at are consistent with the present work. In general, the claim of the present work is that the temperature dependence of the tunneling d.c. conductance of a wire with no leads in the presence of barriers and wells and mutual interaction between particles is,

\[ G \sim (k_B T)^{\eta}; \quad \eta = 8X - 4Q \]

When \( \eta > 0 \) the conductance vanishes at low temperatures as a power law - characteristic of a weak link. However when \( \eta < 0 \) the conductance diverges at low temperature as a power law - characteristic of a clean quantum wire. Of special interest is the situation \( \eta = 0 \) where the conductance is independent of temperature. This crossover from a conductance that vanishes as a power law at low temperatures to one that diverges as a power law occurs at reflection coefficient \( |R| \). This is valid only for repulsive interactions \( v_h > v_F \). For attractive interactions, \( \eta < 0 \) for any \( |R| \) which means the conductance always diverges as a power law at low temperatures. This means attractive interactions heal the chain for all reflection coefficients including in the extreme weak link case. On the other hand for repulsive interactions, for \( |R| > |R_c|, \eta > 0 \) and the chain is broken (conductance vanishes) at low temperatures. For \( |R| < |R_c|, \eta < 0 \) and even though the interactions are repulsive the chain is healed (conductance diverges).

**APPENDIX F: Perturbative comparison of Green’s functions and density density correlation functions**

The Green functions obtained from NCBT technique are claimed to be the most singular part of the full Green function. To verify this claim the obtained expressions are expanded in powers of the interaction parameter \( v_0 \). Also, the single particle Green function is perturbatively expanded in powers of the interaction parameter with the source of inhomogeneities being treated exactly. Both these formulas match exactly at the zeroth order (this is already an improvement over standard g-ology based methods). The first order term of both these approaches, viz, perturbative approach and NCBT technique for various cases are calculated and the most singular parts are retained and are found to be identical to one another.

\[
G_{RR} : x, x' > 0
\]

\[
G_{RR}(x, x', t - t') = \frac{i}{2\pi} \frac{1}{(x - x') - v_P(t - t')} + \frac{i(t - t')}{4\pi^2((x - x') - v_P(t - t'))^2} v_0
\]
The first term is regarded as more singular than the second (if

\[ A \frac{\tau}{(\tau - a)^2} + \frac{B}{(\tau - a_1)(\tau - a_2)} \]

The first term is regarded as more singular than the second (if \( a_1 \neq a_2 \)) since the former is a second order pole whereas the latter when partial fraction expanded are a sum of two first order poles. In the perturbative expansion of the slow part of the density density correlations, pretending that Wick’s theorem applies at the level of the density fluctuations is tantamount to retaining second order poles and discarding poles of a lower order. The same rule applies when deciding what to retain and what to discard in the perturbation expansion of the single-particle Green function. The density density correlation functions in presence of interactions obtained using the NCBT is nothing but the most singular part of the asymptotically exact density density correlation functions under RPA. The density density correlation functions as mentioned in equation (10) is perturbatively expanded in powers of the interaction parameter \( v_0 \). The zeroth order term has an exact match with that of the non interacting density density correlation function as mentioned in equation (6) of the main text. The most singular part of the first order term of the density density correlation functions obtained from conventional perturbation theory is as follows:

**Case I :** \( x_1 \) and \( x_2 \) are on the same side of the origin

\[ \delta <\rho_h(x_1,t_1)\rho_h(x_2,t_2)> = \frac{1}{2\pi^2 v_F} \left( \frac{2(t_1 - t_2)v_F}{(x_1 - x_2 + v_F(t_1 - t_2))^3} - \frac{2(t_1 - t_2)v_F}{(x_1 - x_2 - v_F(t_1 - t_2))^3} + \frac{2|R|^2(t_1 - t_2)v_F}{(x_1 + x_2 + v_F(t_1 - t_2))^3} - \frac{2|R|^2(t_1 - t_2)v_F}{(x_1 + x_2 - v_F(t_1 - t_2))^3} \right) \]

**Case II :** \( x_1 \) and \( x_2 \) are on opposite sides of the origin

\[ \delta <\rho_h(x_1,t_1)\rho_h(x_2,t_2)> = 1 - \frac{|R|^2}{2\pi^2 v_F} \left( \frac{2(t_1 - t_2)v_F}{(x_1 - x_2 + v_F(t_1 - t_2))^3} - \frac{2(t_1 - t_2)v_F}{(x_1 - x_2 - v_F(t_1 - t_2))^3} \right) \]

The above expressions are precisely the same as those obtained by expanding equation (10) in powers of the interaction strength \( v_0 \) for both the cases, i.e. ‘\( x_1 \) and \( x_2 \) on the same side’ and ‘\( x_1 \) and \( x_2 \) on opposite sides’ and retaining the most singular part of the first order term.

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