Partition function of beta-gamma system on orbifolds

Chandrasekhar Bhamidipati*
School of Basic Sciences, Indian Institute of Technology Bhubaneswar
Bhubaneswar 751 007, India.

Koushik Ray†
Department of Theoretical Physics, Indian Association for the Cultivation of Science
Calcutta 700 032, India.

ABSTRACT
Partition function of beta-gamma systems on the orbifolds $\mathbb{C}^2/\mathbb{Z}_N$ and $\mathbb{C}^3/\mathbb{Z}_M \times \mathbb{Z}_N$ are obtained as the invariant part of that on the respective affine spaces, by lifting the geometric action of the orbifold group to the fields. Interpreting the sum over roots of unity as an elementary contour integration, the partition function evaluates to an infinite series counting invariant monomials composed of basic operators of the theory at each mass level.

1 Introduction
A curved beta-gamma ($\beta$-$\gamma$) system is defined as the chiral sector of a certain infinite radius limit of the two-dimensional non-linear sigma model on a complex variety as the target space [1,2]. Since its incipience in the study of sheaves of vertex operators of chiral de Rham complexes [3,4], beta-gamma systems have been studied in the context of topological strings [5,6], geometry of (0, 2) sigma-models [1,7-11], mirror symmetry [12] and pure spinors [13-16]. In the pure spinor formalism, for instance, the non-linear beta-gamma theory appears as a ghost system whose partition function is evaluated as characters of pure spinors [17]. The moments of the partition function of the zero modes is identified with the multiplicities of the ghosts which in turn provide the Virasoro central charge and are related to the current algebra of ghosts.

In this article we consider a beta-gamma system on orbifolds of two and three-dimensional complex affine spaces, namely $\mathbb{C}^2/\mathbb{Z}_N$ and $\mathbb{C}^3/\mathbb{Z}_M \times \mathbb{Z}_N$. The partition function on an affine space is obtained as the generating function of chiral operators, namely the fields $\gamma$, identified as the complex coordinates of the target space, their conjugates and their world-sheet derivatives,
graded by the scaling degree of the fields and mass. Defining a beta-gamma system on a curved
space requires identifying the fields $\gamma$ with the coordinates of the curved space, which can only
be effected on each affine chart. Coordinate transformations made compatible with the operator
products provide the rules for changing patches [2]. If a variety is given as a set of equations in
the coordinate ring of an affine or projective space, defining a beta-gamma system entails imposing
the equations as constraints. This is the case for pure spinors, for example. The orbifolds we consider
may also be defined in this fashion. Equivalently, however, the geometric action of the orbifold
group can be lifted to an action on the fields $\gamma$ through its identification with the coordinates of the
affine space. The scaling symmetry then determines the action on the conjugates $\beta$. We exploit this
for restricting the generating function of chiral operators to the sector invariant under the orbifold
group to obtain the partition function on the orbifolds.

Restricting the generating function to the invariant sector entails a sum over roots of unity. This
is achieved using rudimentary contour integration in a single complex variable and leads to a series
in powers of the modular parameter. The massless or zero-mode part of the partition function can be
easily extracted from the series. It arises from the invariant monomials of the orbifold group in the
coordinate ring of the affine variety. The generating function is then its Molien series [18, 19]. For
simple cases, the Molien series can also be obtained from the syzygies defining the orbifold in the
coordinate ring of the ambient affine space. We show that the zero mode part of the partition function
of the beta-gamma system matches with the Molien series. In principle, evaluating all the moments
of the Molien series enables one to write the partition function of the non-zero modes as well.
However, evaluation of infinite number of moments presents practical difficulties in implementing
this procedure. A series is better suited for explicit evaluation of the partition function.

1.1 Partition function of beta-gamma system on $C^d$

Let us start with a discussion of the partition function of a beta-gamma system on a complex affine
space. A beta-gamma system refers to a two-dimensional conformal field theory with a set of com-
plex fields $\{\gamma^i\}$ and their canonical conjugates, $\{\beta_i\}, i = 1, 2, \cdots, d$. The fields $\gamma$ have vanishing
conformal dimension and $\beta$ are one-forms on the two-dimensional space-time, referred to as the
world-sheet, namely, $\beta_i = \beta_i \xi d\xi + \beta_i \bar{\xi} d\bar{\xi}$, $\xi$ denoting the complex coordinate of the world-sheet
and a bar the complex conjugate. Since the complex affine space $C^d$ can be covered by a single
coordinate chart, the fields are identified with the coordinates as $\gamma^i = x_i$ of the coordinate ring of
$C^d$, viz. $C[x_1, x_2, \cdots, x_d]$ and are assigned the free operator product

$$\gamma^i(\xi)\beta^j(\xi') \sim \delta^i_j \frac{d\xi'}{\xi - \xi'} \quad (1)$$

with their conjugates. The theory is described by an action written in the conformal gauge as

$$S = \frac{1}{2\pi} \int \beta_i \partial \gamma^i \quad (2)$$

where $\partial = \frac{\partial}{\partial \xi}$. The corresponding equations of motion restrict the fields to be holomorphic on
the world-sheet. In considering an orbifold of the affine space, the identification of $\gamma$’s with the
coordinates allows lifting the action of orbifold group to the fields.
The theory possesses two conserved currents, the energy momentum tensor and an $U(1)$ current arising from scaling of the fields as

$$\gamma^i \rightarrow \Lambda_i \gamma^i, \quad \beta_i \rightarrow \Lambda_i^{-1} \beta_i.$$  \hfill (3)

The parameters $\Lambda_i$ acting on the different fields are not independent. All of them are functions of a single parameter, ergo the symmetry group is not $U(1)^d$ but $U(1)$. The respective charges, namely, $L_0 = \oint \xi \beta \partial \gamma^i$ and $J_0 = \oint \beta_i \gamma^i$, characterize the field theory. Introducing the modular parameter $q$ and another parameter $t$ corresponding to the scaling, then, the partition function of the beta-gamma system is written as

$$Z = \text{Tr} (q^{L_0} t^{J_0}),$$  \hfill (4)

where $\text{Tr}$ signifies a trace with respect to the states of the Hilbert space of the theory.

With the identification alluded to above, the partition function of the beta-gamma system on $C^d$ evaluates to $[9]$

$$Z_{C^d} = (Z_C)^d,$$  \hfill (5)

where

$$Z_C = \frac{1}{1 - t} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n t)(1 - q^n/t)}.$$  \hfill (6)

Let us interpret this combinatorially. First, let us consider the one-dimensional case $C$, with coordinate ring $C[x]$. The partition function is the generating function of products of the fields $\gamma, \beta$ and their derivatives graded by their $q$-degree and $t$-degree, given by the charges $L_0$ and $J_0$, respectively. Each power of $x = \gamma$ furnishes a $t$ and each world-sheet derivative $\partial$ a $q$. The scaling (3) dictates each $\beta$ to contribute $1/t$ while being an one-form it also furnishes a factor of $q$, that is, $q/t$, in total. These contributions are collected in Table 1. A monomial constructed with different powers and combinations of $x$ or $\gamma, \partial$ and $\beta$ furnish an object with a certain $q$- and $t$-degrees. For example, objects of vanishing $q$-degree, namely, the zero modes, arise only from integer powers of $x$; the $t$-degree of a monomial $x^n$ for any positive integer $n$ is $n$. As for objects of higher $q$-degree, each of the combinations $x^5 \partial^3 x, x^4 (\partial^2 x)(\partial x), x^6 (\partial^2 x) \beta, x^9 \beta^3, x^6 (\partial x)(\partial \beta)$ have $q$-degree 3 and $t$-degree 6. The coefficient of $q^r t^s$ in the partition function is the number of combinations with positive integral $q$-degree $r$ and integral $t$-degree $s$.

Accordingly, the generating function of the zero modes given by powers of $x$ is a geometric series in $t$, viz., $1 + t + t^2 + t^3 + \cdots = 1/(1 - t)$, as each monomial appears only once. Then, distributing $n$ derivatives $\partial$ among $k$ number of $x$’s gives combinations with $q$-degree $n$ and $t$-degree

| object | degree |
|--------|--------|
| $x$ or $\gamma$ | $t$ |
| $\partial$ | $q$ |
| $\beta$ | $q/t$ |

Table 1: $q$- and $t$-degrees of fields
generating functions are, respectively, the discrete group on the coordinate ring and the ring of invariants is known as the Molien series \[18, 19\]. Let us consider two examples. Considering the action of a discrete group on the coordinate ring, the Hilbert-Poincaré series of the partition function through the geometric series in \(\omega\), the lone powers of \(\beta\), without derivatives acting on them contribute a factor \(1/(1 - q/t)\) to the partition function through the geometric series in \(\beta\), similar to \(x\). Multiplying all these we obtain the partition function \((5)\). The same consideration is valid for each variable of the coordinate ring of \(\mathbb{C}^d\). Indeed, considering \(\mathbb{C}[x]\) as a graded algebra of monomials, the coordinate ring of \(\mathbb{C}^d\) is \(\mathbb{C}[x_1, x_2, \ldots, x_d] = \mathbb{C}[x_1] \otimes \mathbb{C}[x_2] \otimes \cdots \otimes \mathbb{C}[x_d]\) and the product structure is maintained even when extended with \(\beta\)’s. Thus, the generating function for a beta-gamma system on \(\mathbb{C}^d\) is obtained as the \(d\)-fold product of the generating function on \(\mathbb{C}\), yielding \((5)\). Let us note that it is possible to keep track of the fields by labelling the \(t\)-charges after the variables of the coordinate ring, if necessary, to obtain a refinement of the partition function.

1.2 \(\mathbb{Z}_N\)-invariants and Molien series

The zero mode part of the partition function \(Z_{\mathbb{C}^d}\), viz., \(Z_{\mathbb{C}^d}^{(0)} = (1 - t)^{-d}\) is the Hilbert-Poincaré series of the graded algebra \(\mathbb{C}[x_1, x_2, \ldots, x_d]\), graded by the degree of monomials or the \(t\)-degree. Considering the action of a discrete group on the coordinate ring, the Hilbert-Poincaré series of the ring of invariants is known as the Molien series \([18,19]\). Let us consider two examples.

Let \(\mathcal{R} = \mathbb{C}[x_1, x_2]\) be the coordinate ring of \(\mathbb{C}^2\). Let us consider the action of the discrete group \(\mathbb{Z}_N\) on \(\mathcal{R}\) as

\[
(x_1, x_2) \mapsto (\omega x_1, \omega^{-1} x_2),
\]

where \(\omega = e^{2\pi i/N}\) denotes an \(N\)-th root of unity. Then the ring of invariants is \(\mathcal{R}^{\mathbb{Z}_N} = \mathbb{C}[y_1, y_2, y_3]\), with \(y_1 = x_1^N\), \(y_2 = x_2^N\), \(y_3 = x_1 x_2\), and a syzygy \(y_1 y_2 = y_3^N\). Since the coordinates \(y_1\) and \(y_2\) are of degree \(N\), \(y_3\) is of degree 2 and the syzygy has degree of homogeneity 2\(N\), the Molien series is given by

\[
\mathcal{P}_{\mathbb{C}^2/\mathbb{Z}_N} = \frac{(1 - t^{2N})}{(1 - t^N)^2(1 - t^2)} = \frac{(1 + t^N)}{(1 - t^N)(1 - t^2)}
\]

Similarly, let us consider the three-dimensional orbifold \(\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}'_N\), given by the action of the discrete group on the coordinate ring \(\mathcal{R} = \mathbb{C}[x_1, x_2, x_3]\) of \(\mathbb{C}^3\) as

\[
\mathbb{Z}_N : (x_1, x_2, x_3) \mapsto (\omega x_1, \omega^{-1} x_2, x_3),
\]

\[
\mathbb{Z}'_N : (x_1, x_2, x_3) \mapsto (x_1, \omega' x_2, \omega'^{-1} x_3),
\]

\[\text{(11)}\]
where \( \omega \) and \( \omega' \) are \( N \)-th roots of unity. The ring of invariants is \( \mathcal{R}^{\mathbb{Z}_N \times \mathbb{Z}'_N} = C[y_1, y_2, y_3, y_4] \), with \( y_1 = x_1^N, y_2 = x_2^N, y_3 = x_3^N, y_4 = x_1x_2x_3 \), with syzygy \( y_1y_2y_3 = y_4^N \). Since \( y_1, y_2 \) and \( y_3 \) have degree \( N \), \( y_4 \) has degree 3 and the syzygy is of degree \( 3N \), the Molien series is given by \[ P_{C^3/\mathbb{Z}_N \times \mathbb{Z}'_N} = \frac{(1 - t^{3N})}{(1 - t^3)^3(1 - t^3)}. \] (12)

We shall consider the partition function of beta-gamma system on the same orbifolds. However, imposing the syzygies as constraints is difficult to implement for the higher mass modes. The strategy of finding the partition function for all modes is to restrict it by the orbifold action directly to the invariant combinations of the fields. Towards this goal let us note that the Molien series of the above two instances can be obtained as

\[
P_{C^2/\mathbb{Z}_N} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{(1 - \omega^k t)(1 - \omega^{-k} t)},
\]

(13)

and

\[
P_{C^3/\mathbb{Z}_N \times \mathbb{Z}'_N} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{k'=0}^{N-1} \frac{1}{(1 - \omega^k t)(1 - \omega^{-k} \omega^{k'} t)(1 - \omega^{-k'} t)}.
\]

(14)

by effecting the orbifold actions (9) and (11), respectively, on the zero mode part of the partition function (5) and keeping track of contribution from each coordinate. We obtain the partition function on the orbifolds by projecting with the orbifold actions on the complete partition function (5). We shall obtain formulas for more general orbifolds, by evaluating the sums.

### 1.3 Sum over roots of unity as a contour integral

Writing the partition functions (13), (14) as well as their generalization to higher mass levels calls for evaluating sums over roots of unity. For a fixed \( N \) we need to sum over all the \( N \)-th roots of unity. We describe a general procedure for evaluating such sums using elementary contour integration. Let \( g(z) \) be a function on the complex \( z \)-plane possessing a countable number of poles inside the unit circle. For a positive integer \( N \) let \( \omega \) denote an \( N \)-th root of unity. Then the sum

\[
S = \frac{1}{N} \sum_{k=0}^{N-1} g(\omega^k)
\]

(15)

can be written as the integral

\[
\mathcal{I} = \frac{1}{N} \oint_{\bigcup_{k=0}^{N-1} C_k} \frac{g(z)}{z - \omega^k}
\]

(16)

where \( C_k \) are circles centered at \( \omega^k, k = 0, 1, 2, \ldots, N - 1 \), with radii sufficiently small so as not to include any other pole of the integrand, as shown in the first diagram of Figure [1]. By deforming the contour to \( \tilde{C} \) with two disjoint components, as shown in the second diagram of Figure [1], we rewrite the integral as

\[
\mathcal{I} = \oint_{\tilde{C}} \frac{z^{N-1}}{z^N - 1} g(z)
\]

(17)
The contour \( \tilde{C} \) is then deformed as in the third diagram of the figure to two concentric circles \( C_1 \) and \( C_2 \) with radii \( 1 - \epsilon \) and \( 1 + \epsilon \), respectively, with \( \epsilon \ll 1 \) such that all the poles of \( g(z) \) are inside the smaller circle \( C_1 \) and all the poles of \( g(1/z) \) are outside the bigger circle \( C_2 \). The integral then becomes

\[
I = \oint_{C_2} \frac{z^{N-1}}{z^N - 1} g(z) - \oint_{C_1} \frac{z^{N-1}}{z^N - 1} g(z).
\]  

(18)

We then transform \( z \mapsto 1/z \) in the first integral, which is along the outer contour \( C_2 \), thereby exchanging the poles outside and inside the circle as well as changing the radius of the circle to less than unity. The new circle may be further deformed, if necessary, to another circle with radius \( 1 - \epsilon' \) with \( \epsilon' \ll 1 \), so as to contain all the poles inside. Finally, the integral becomes

\[
I = \oint_{C} \frac{dz}{z} \frac{z^N g(z) + g(1/z)}{1 - z^N},
\]  

(19)

where \( C \) is a circle with radius \( 1 - \min(\epsilon, \epsilon') \), and the poles to be taken into account are the ones inside \( C \) for both \( g(z) \) and \( g(1/z) \), in addition to the pole at \( z = 0 \).

Now since in (15) we sum over \textit{all} the \( N \)-th roots of unity, we can as well write the sum as over the inverse of the roots of unity, that is,

\[
S = \frac{1}{N} \sum_{k=0}^{N-1} g(\omega^{-k}).
\]  

(20)

This sum can be evaluated exactly as above by considering the function \( g(1/z) \), resulting in the expression

\[
I = \oint_{C} \frac{dz}{z} \frac{z^N g(1/z) + g(z)}{1 - z^N},
\]  

(21)

From the two expressions (19) and (21) the integral can be written in a more symmetric form as

\[
I = \frac{1}{2} \oint_{C} \frac{dz}{z} \frac{1 + z^N}{1 - z^N} (g(z) + g(1/z)).
\]  

(22)

We shall use this formula to evaluate the full partition function on the orbifolds, including all the massive modes.
Partition function of beta-gamma system on $\mathbb{C}^2/\mathbb{Z}_N$

In this section we evaluate the partition function of a beta-gamma system on the two-dimensional orbifold $\mathbb{C}^2/\mathbb{Z}_N$ by restricting the partition function on $\mathbb{C}^2$ to the invariants of the orbifold group $\mathbb{Z}_N$. As discussed above, the partition function of a beta-gamma system on $\mathbb{C}^2 = \mathbb{C}[x_1, x_2]$ is

$$Z_{\mathbb{C}^2} = \left(\frac{1}{1-t} \prod_{n=1}^{\infty} \frac{1}{(1-q^n t)(1-q^n/t)}\right)^2,$$

where each factor corresponds to a coordinate in the coordinate ring. The action of the discrete group $\mathbb{Z}_N$ on the coordinates is given as

$$(x_1, x_2) \mapsto (\omega x_1, \omega^{-a} x_2) \quad (\omega = e^{2\pi i/N} \text{ denoting an } N\text{-th root of unity and } 1 \leq a < N \text{ an integer}).$$

This action is lifted to the fields by embedding the $\mathbb{Z}_N$ into the $U(1)$ scaling symmetry group through identification of $\gamma$’s as $x$’s, leading to the orbifold action

$$(\gamma_1, \gamma_2) \mapsto (\omega \gamma_1, \omega^{-a} \gamma_2), \quad (\beta_1, \beta_2) \mapsto (\omega^{-1} \beta_1, \omega^a \beta_2).$$

on the fields of the beta-gamma system. We shall often denote the group $\mathbb{Z}_N$ as $\frac{1}{N}(1, a)$ and label the partition function with this alone, not mentioning $\mathbb{C}^2$. The partition function on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$ is obtained as the invariant part of $Z_{\mathbb{C}^2}$ as

$$Z_{\frac{1}{N}(1, a)} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{(1-\omega^k t)(1-\omega^{-ak} t)} \prod_{n=1}^{\infty} \frac{1}{(1-\omega^k q^n t)(1-\omega^{-ak} q^n t)(1-\omega^{-k} q^n t)(1-\omega^{ak} q^n t)}.$$

This can be thought of as a generalized Molien series extending over to the non-zero modes. The sum is evaluated by applying (22) with

$$g(z) = \frac{1}{(1-z t)(1-t/z^a)} \prod_{n=1}^{\infty} \frac{1}{(1-z q^n t)(1-z^{-a} q^n t)(1-z^{-1} q^n t)(1-z^a q^n t)}.$$

Let us first consider the special case of unit $a$ before treating the general case.

2.1 The case $a = 1$

In the special case when the discrete group $\mathbb{Z}_N$ acts on the fields with $a = 1$ in (25), the function $g(z)$ in (27) equals its inverse, $g(1/z) = g(z)$. Using (22) this results in a simplified expression for the partition function, namely,

$$Z_{\frac{1}{N}(1, 1)} = \oint_{\mathcal{C}} \frac{dz}{z} \frac{1}{1 + z^N} \prod_{n=1}^{\infty} \frac{1}{(1-z q^n t)(1-z^{-1} q^n t)(1-z^{-1} q^n t)(1-z q^n t)}.$$

(28)
The poles to be considered in evaluating the integral are the ones inside the closed contour $\mathcal{C}$. The residue of the integrand at the simple pole at $z = 0$ vanishes. Whether the other poles fall inside the contour depends on the magnitude of $q$ and $t$. Let us note at this point that the expression (6) converges if $|t| < 1$ and $|q| \ll 1$, so that $|q/t| < 1$. In this domain of the parameters the poles of $g(z)$ inside the contour are all simple and located at $z = q^n t$ and $z = q^n/t$. Evaluating residues at the poles leads to the infinite series for the partition function

$$
\mathcal{Z}_{\chi^{(1,1)}} = \chi(t, q) \left( \sum_{m=0}^{\infty} \frac{1 + (q^m t)^N}{1 - (q^m t)^N} q^{m^2} t^{2m} - \sum_{m=1}^{\infty} \frac{1 + (q^m/t)^N}{1 - (q^m/t)^N} q^{m^2/t^{2m}} \right),
$$

(29)

where we defined

$$
\chi(t, q) = \frac{1}{1 - t^2} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^2 (1 - q^n t^2)(1 - q^n/t^2)}.
$$

(30)

Rearranging the second term (29) can be written as

$$
\mathcal{Z}_{\chi^{(1,1)}} = \chi(t, q) \left( \sum_{m=0}^{\infty} \frac{1 + (q^m t)^N}{1 - (q^m t)^N} q^{m^2} t^{2m} + \sum_{m=1}^{\infty} \frac{1 + (t/q^m)^N}{1 - (t/q^m)^N} q^{m^2/t^{2m}} \right).
$$

(31)

Writing the second term of (31) as a sum over the negative integers the partition function assumes a compact form

$$
\mathcal{Z}_{\chi^{(1,1)}} = \chi(t, q) \sum_{m=-\infty}^{\infty} \frac{1 + (q^m t)^N}{1 - (q^m t)^N} q^{m^2} t^{2m}.
$$

(32)

Let us note that the change of sign of $m$ does not affect the convergence of the series due to the presence of $q^{m^2}$ in the numerator. The zero mode part, obtained as the term with $m = 0$ and ignoring the terms depending on $q$ in $\chi(q, t)$, reproduces the Molien series (10).

In terms of theta functions with rational characteristics

$$
\vartheta_{[0]}^{[a]}(\nu, \tau) = \sum_{m=-\infty}^{\infty} e^{i\pi\tau(m+a)^2 + 2i\pi(m+a)(\nu+b)}
$$

(33)

the partition function (32) can be expressed as

$$
\mathcal{Z}_{\chi^{(1,1)}} = \frac{e^{i\pi(-\nu+\tau/6)} \vartheta_{[0]}^{[0]}(\nu, \tau) \vartheta_{[0]}^{[0]}(0, -\tau N^2/4)}{i\eta(\tau/2) \vartheta_{[1/2]}^{[1/2]}(\nu, \tau/2)},
$$

(34)

if $N$ is even and as

$$
\mathcal{Z}_{\chi^{(1,1)}} = \frac{e^{i\pi(-\nu+\tau/6)} \left( \vartheta_{[0]}^{[0]}(\nu, \tau) \vartheta_{[0]}^{[0]}(0, -\tau N^2) + \vartheta_{[1/2]}^{[1/2]}(\nu, \tau/2) \right)}{i\eta(\tau/2) \vartheta_{[1/2]}^{[1/2]}(\nu, \tau/2)},
$$

(35)

if $N$ is odd, where we defined $q = e^{i\pi\tau}$ and $t = e^{i\pi\nu}$.
2.2 The case $1 \leq a < N$

As mentioned above, the general partition function in the case of $a$ being a positive integer less than $N$ is given by the integral (22) with the function $g(z)$ as given in (27). In this case, poles arising from both $g(z)$ and $g(1/z)$ have to be taken into account, separately. Defining

$$\gamma^a = t, \quad \alpha_n^a = q^a t, \quad \beta_n^a = q^n t$$

first let us write $g(z)$ as

$$g(z) = \frac{1}{(1-z\gamma^a)(1-z\alpha_n^a z^a)} \prod_{n=1}^\infty \frac{1}{(1-z\alpha_n^a)(1-\alpha_n^a z^a)(1-\beta_n^a/z)(1-z\beta_n^a)^{\gamma}}.$$  \hfill (37)

Then $g(z)$ has simple poles at $z = \lambda^k \gamma, \lambda^k \alpha_n, \beta_n$, while the poles of $g(1/z)$, all simple, are located at $z = \gamma^a, \alpha_n^a, \lambda^k \beta_n$ for every $n = 1, 2, \cdots, \infty$ and $k = 0, 1, \cdots, a - 1$. Here $\lambda$ is taken to denote an $a$-th root of unity, $\lambda = e^{2\pi i/a}$. Evaluating the integral (22) using residues at these poles leads to the partition function

$$Z_{(1,a)}^{(1)} = \frac{1}{2}(Z^{(1)} + Z^{(2)}),$$ \hfill (38)

where the poles of $g(z)$ contribute to the first term

$$Z^{(1)} = \frac{1}{a} \sum_{k=0}^{a-1} \sum_{m=-\infty}^\infty \frac{1 + (\lambda^k \alpha_m)^N}{1 - (\lambda^k \alpha_m)^N} \frac{1}{1 - \lambda^k \alpha_m \gamma^a} \frac{(-1)^m q^{m(a+1)/2}}{\prod_{n=1}^\infty (1 - \lambda^k \alpha_m \alpha_n^a)(1 - \lambda^{-k} \beta_n^a / \alpha_m)(1 - q^n)^2},$$ \hfill (39)

while the second term

$$Z^{(2)} = \frac{1}{1 - t^{a+1}} \sum_{m=-\infty}^\infty \frac{1 + (q^m t)^N}{1 - (q^m t)^N} \frac{(-1)^m q^{m(a+1)} q^{m(a+1)/2 + ma(ma-1)/2} t^{ma(a+1)}}{\prod_{n=1}^\infty (1 - q^n t^{a+1})(1 - q^n t^{a+1})}$$ \hfill (40)

contains the share of the poles of $g(1/z)$. It appears that $Z^{(1)}$ contains fractional powers of $q$. This is not the case due to the sum over all the $a$-th roots of unity, which can be verified explicitly using the contour integral (22) once again. However, we refrain from displaying the expressions as they are rather cumbersome without being particularly illuminating otherwise.

The partition function (38) reduces to (29) if we set $a = 1$. It also matches with the direct counting by construction of $Z_N$-invariant monomials to arbitrary given orders in $t$ and $q$. In particular, the Molien series of some orbifolds obtained from the zero modes of (39) and (40) are

$$Z_{(1,2)}^{(0)} = \frac{1 - t + t^3}{1 - t - t^3 + t^6},$$ \hfill (41)

$$Z_{(1,2)}^{(0)} = \frac{1 - t + t^3 - t^4 + t^5}{1 - t - t^3 + t^8},$$ \hfill (42)

$$Z_{(1,3)}^{(0)} = \frac{1 - t + t^4}{1 - t - t^2 + t^5}.$$ \hfill (43)
3 Partition function of beta-gamma system on $C^3/Z_M \times Z_N$

In this section we obtain the partition function of a beta-gamma system on the threefold $C_{MN} = C^3/Z_M \times Z_N$ of $C^3$. According to (5) the partition function of a beta-gamma system on the affine space $C^3 = C[x_1, x_2, x_3]$ is,

$$Z_{C^2} = \left( \frac{1}{(1-t)} \prod_{n=1}^{\infty} \frac{1}{(1-q^n t)(1-q^n/t)} \right)^3.$$  \hspace{1cm} (44)

Let us consider the action of the orbifold group on the coordinate ring as

$$Z_M : (x_1, x_2, x_3) \mapsto (\lambda x_1, \lambda^{-1} x_2, x_3), \hspace{1cm} (45)$$

$$Z_N : (x_1, x_2, x_3) \mapsto (x_1, \omega x_1, \omega^{-1} x_3), \hspace{1cm} (46)$$

where $\omega = e^{2\pi i/N}$ is an $N$-th root of unity and $\lambda = e^{2\pi i/M}$ is an $M$-th root of unity. As in the previous case, by embedding the discrete group $Z_M \times Z_N$ into the group of scaling transformations, we have the action of the orbifold group on the fields, namely,

$$Z_M : (\gamma^1, \gamma^2, \gamma^3) \mapsto (\lambda \gamma^1, \lambda^{-1} \gamma^2, \gamma^3), \hspace{1cm} (\beta_1, \beta_2, \beta_3) \mapsto (\lambda^{-1} \beta_1, \lambda \beta_2, \beta_3), \hspace{1cm} (47)$$

$$Z_N : (\gamma^1, \gamma^2, \gamma^3) \mapsto (\gamma^1, \omega \gamma^2, \omega^{-1} \gamma^3), \hspace{1cm} (\beta_1, \beta_2, \beta_3) \mapsto (\beta_1, \omega^{-1} \beta_2, \omega \beta_3). \hspace{1cm} (48)$$

Restriction of the partition function (44) by the orbifold group to the invariant part is

$$Z_{C_{MN}} = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{r=0}^{N-1} \frac{1}{(1-\lambda^k t)(1-\lambda^{-k} \omega^r t)(1-\omega^{-r} t)} \prod_{n=1}^{\infty} \frac{1}{(1-\lambda^k q^n t)(1-\lambda^{-k} \omega^r q^n t)(1-\omega^{-r} q^n t)} \prod_{n=1}^{\infty} \frac{1}{(1-\lambda^k q^n t)(1-\lambda^{-k} \omega^{-r} q^n t)(1-\omega^{-r} q^n t)}.$$  \hspace{1cm} (49)

The sums over the two sets of roots of unity can be performed in two steps. Let us define $g$ as a function of two complex variables $\zeta$ and $z$, namely,

$$g(\zeta, z) = \frac{1}{(1-t\zeta)(1-tz/\zeta)(1-t/z)} \prod_{n=1}^{\infty} \frac{1}{(1-q^n t\zeta)(1-q^n tz/\zeta)(1-q^n t/z)} \prod_{n=1}^{\infty} \frac{1}{(1-q^n t\zeta)(1-q^n t\zeta/tz)(1-zq^n t)}.$$  \hspace{1cm} (50)

First we evaluate the sum over the $N$-th roots of unity, using (22) with $g(\zeta, z)$ treated as a function of $z$. This is done by working out the contour integral (22) over $z$ by evaluating the residues of

\[\text{the notation here is different from the previous section where } \lambda \text{ denoted an } a\text{-th root of unity}\]
in (55) to expand the invariant part can now be performed by keeping only the terms in which powers of \(\zeta\) to the \(N\)-th roots of unity in (55) is beleaguered by the existence of poles of the integrand at \(z = \lambda^k q^m t\), which brings back a sum over the \(M\)-th roots of unity. In order to obtain a series not involving the roots of unity we use the identity \([21]\)

\[
\chi_{\zeta}(t, q) = \chi_0 \left( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{mn} \zeta^{n-m} t^{n+2m} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q^{mn} \zeta^{n-m} t^{-(m+2n)} \right)
\]

in (55) to expand \(\chi_{\zeta}(t, q)\) in powers of \(z\). The sum over the \(N\)-th roots of unity in (55) to restrict to the invariant part can now be performed by keeping only the terms in which powers of \(z\) vanish.

\[
\chi_{\zeta}(t, q) = \chi_0 \left( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{mn} \zeta^{n-m} t^{n+2m} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q^{mn} \zeta^{n-m} t^{-(m+2n)} \right)
\]

in (55) to expand \(\chi_{\zeta}(t, q)\) in powers of \(z\). The sum over the \(N\)-th roots of unity in (55) to restrict to the invariant part can now be performed by keeping only the terms in which powers of \(z\) vanish.
modulo $N$. This leads to a series expansion of $Z_2$ as

$$Z_2 = \chi_0 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{s=-\infty}^{\infty} q^{m^2+n^2+mn+Mp(m+n)-Nms} t^{3(m+n)+2Mp-Ns}$$

$$+ \chi_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \sum_{s=-\infty}^{\infty} q^{m^2+n^2+mn+Mp(m-n)-Nms} t^{-(3m+n)+2Mp-Ns}$$

$$- \chi_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \sum_{s=-\infty}^{\infty} q^{m^2+n^2+mn+Mp(m+n)-Nms} t^{-(3m+M-2Ns)}$$

$$- \chi_0 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \sum_{s=-\infty}^{\infty} q^{m^2+n^2+mn+Mp(m+n)-Nms} t^{3m+M-2Ns}$$

$$- \frac{1}{2} \chi_0 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} q^{m^2+n^2+mn-Nms} t^{3(m+n)-Ns}$$

$$- \frac{1}{2} \chi_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} q^{m^2+n^2+mn-Nms} t^{-(3m+n)-Ns}$$

$$+ \frac{1}{2} \chi_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} q^{m^2+n^2+mn-Nms} t^{-(3m-2Ns)}$$

$$+ \frac{1}{2} \chi_0 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} q^{m^2+n^2+mn-Nms} t^{3m-2Ns},$$

where $[r]$ denotes the largest integer less than a rational number $r$. Using this expression in (56) along with (53) gives a series for the partition function in terms of $q$ and $t$.

### 3.1 Molien series from zero modes

Molien series of various orbifolds can be evaluated from (56) by collecting the zero mode parts from (53) and (58). Here is a list of examples

$$Z_{C_{2,2}}^{(0)} = \frac{1 - t + t^2}{(1 - t)^3(1 + t)^2}, \quad Z_{C_{2,2}}^{(0)} = \frac{1 - t + 2t^3 - t^5 + t^6}{(1 - t)^3(1 + t)^2(1 + t^2 + t^4)}, \quad Z_{C_{3,3}}^{(0)} = \frac{1 - t^9}{(1 - t)^3},$$

$$Z_{C_{5,5}}^{(0)} = \frac{t^8 - t^7 + t^5 - t^4 + t^3 - t + 1}{(1 - t)^3(t^4 + t^3 + t^2 + t + 1)^2},$$

$$Z_{C_{3,5}}^{(0)} = -\frac{(t^4 - t^2 + 1)((t - 1)t(t^3 + t + 1) + 1)(t(t^4 - t^3 + t - 1) + 1)}{t(t((t - 1)^2t^3 - 1) + 2) - 1}.$$

These expressions match with the direct computation of invariant monomials.
4 Conclusion

We have obtained explicit expressions for the partition function of curved beta-gamma system on orbifolds of affine spaces in two and three dimensions, namely, $\mathbb{C}^2/\mathbb{Z}_N$ and $\mathbb{C}^3/\mathbb{Z}_M \times \mathbb{Z}_N$. Since an affine space can be covered with a single chart, the fields $\gamma$ of a free beta-gamma system can be identified with the indeterminates of the coordinate ring of the affine space. This in turn allows lifting the geometric action of the orbifold group to the fields. Its action on the conjugates is then determined by requiring the invariance of the action under scaling. The partition function on the orbifold is then obtained by restricting the partition function on the affine spaces to the part invariant under the orbifold group. The partition function so obtained is a generalized Molien series in the sense that its zero mode part furnishes the Molien series of the orbifolds. While the Molien series may be evaluated by other means, e.g., using the syzygies describing the orbifolds as algebraic varieties in the coordinate rings of the affine spaces, imposing the constraint on the full partition function turns out to be computationally cumbersome at the least. Restriction to the invariant sector on the other hand, entails summing over roots of unity. For simple cases the summation may be performed order by order in $q$. This however becomes extremely involved even for $N$ bigger than 2 for $\mathbb{C}^2/\mathbb{Z}_N$, for instance, rendering the task of obtaining formulas at higher orders in $q$ difficult. We evaluated the sum by expressing it as a contour integral in a single complex variable. While this elementary treatment suffices for the surfaces, the partition function on the threefolds involves sums over two sets of roots of unity. In that case a combination of a contour integration and an infinite series identity is employed to obtain a series for the partition function. We note that the contour integration employed above can straightforwardly be generalized to higher dimensions. A series for the partition function obtained after taking care of summation over the roots of unity is effective for its explicit evaluation. We hope the formulas presented to be of use in obtaining physical quantities for beta-gamma systems. In particular, evaluation of various moments of the partition function, which yield different physical quantities, will be facilitated.

References

[1] E. Witten, “Two-dimensional models with (0,2) supersymmetry: Perturbative aspects,” Adv. Theor. Math. Phys. 11 (2007) [hep-th/0504078].

[2] N. A. Nekrasov, “Lectures on curved beta-gamma systems, pure spinors, and anomalies,” [hep-th/0511008].

[3] F. Malikov, V. Schechtman and A. Vaintrob, “Chiral de Rham complex,” Commun. Math. Phys. 204 (1999) 439 [math.ag/9803041].

[4] F. Malikov, V. Schechtman and A. Vaintrob, “Chiral de Rham complex II,” Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2 194 (1999) 149 – 188, [math.ag/9901065].
[5] A. Kapustin, “Chiral de Rham complex and the half-twisted sigma-model,” [hep-th/0504074].

[6] N. Berkovits, “Pure spinor formalism as an N=2 topological string,” JHEP 0510, 089 (2005) [hep-th/0509120].

[7] M. -C. Tan, “Two-dimensional twisted sigma models and the theory of chiral differential operators,” Adv. Theor. Math. Phys. 10, 759 (2006) [hep-th/0604179].

[8] M. -C. Tan, “The Half-Twisted Orbifold Sigma Model and the Chiral de Rham Complex,” Adv. Theor. Math. Phys. 12, 547 (2008) [hep-th/0607199].

[9] P. A. Grassi and G. Policastro, “Curved beta-gamma systems and quantum Koszul resolution,” [hep-th/0602153].

[10] P. A. Grassi, G. Policastro and E. Scheidegger, “Partition Functions, Localization, and the Chiral de Rham complex,” [hep-th/0702044].

[11] Y. Aisaka and E. A. Arroyo, “Hilbert space of curved beta gamma systems on quadric cones,” JHEP 0808, 052 (2008) [arXiv:0806.0586 [hep-th]].

[12] E. Frenkel, A. Losev, “Mirror symmetry in two steps: A-I-B,” Commun. Math. Phys. 269, 39 (2006), [hep-th/0505131].

[13] N. Berkovits, “Super Poincare covariant quantization of the superstring,” JHEP 0004, 018 (2000), [hep-th/0001035].

[14] P. A. Grassi and J. F. Morales Morera, “Partition functions of pure spinors,” Nucl. Phys. B 751, 53 (2006) [hep-th/0510215].

[15] E. Aldo Arroyo, “Pure Spinor Partition Function Using Pade Approximants,” JHEP 0807, 081 (2008) [arXiv:0806.0643 [hep-th]].

[16] Y. Aisaka, E. A. Arroyo, N. Berkovits and N. Nekrasov, “Pure Spinor Partition Function and the Massive Superstring Spectrum,” JHEP 0808, 050 (2008) [arXiv:0806.0584 [hep-th]].

[17] N. Berkovits and N. Nekrasov, “The Character of pure spinors,” Lett. Math. Phys. 74, 75 (2005) [hep-th/0503075].

[18] R. P. Stanley, “ Invariants of finite groups and their applications to combinatorics”, Bull. Amer. Math. Soc. 1 (1979), 475.

[19] L. Smith, “Polynomial invariants of finite groups a survey of recent developments,” Bull. Amer. Math. Soc. (New Series) 34 (1997) 211.

[20] B. Feng, A. Hanany and Y. -H. He, “Counting gauge invariants: The Plethystic program,” JHEP 0703, 090 (2007) [hep-th/0701063].

[21] V. G. Kac and P. Cheung “Quantum Calculus,” Springer, 2002; p. 54.