DIMENSION OF UNIFORMLY RANDOM SELF-SIMILAR FRACTALS

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Abstract. The purpose of this note is to calculate the almost sure Hausdorff dimension of uniformly random self-similar fractals. These random fractals are generated from a finite family of similarities, where the linear parts of the mappings are independent uniformly distributed random variables at each step of iteration.

1. Introduction

The systematic study of iterated function systems (IFSs) and the corresponding fractal sets was originated by Hutchinson in [13]. He proved that, given a collection of similarities \( \{S_1, \ldots, S_m\} \) of contraction ratios \( \{r_1, \ldots, r_m\} \), satisfying the open set condition, the unique nonempty, compact invariant set \( F \) has dimension equal to the solution \( s \) of \( \sum_{i=1}^{m} r_i^s = 1 \). The open set condition was later proved to be equivalent to the positivity of the \( s \)-dimensional Hausdorff measure of \( F \) by Schief [21]. The corresponding result for self-affine fractals, that is, sets produced from a collection of affine contractions, is due to Falconer [5] and Solomyak [22]. Their result holds for almost all choices of translations. There is another class of self-affine sets, the Bedford-McMullen carpets, whose dimension is calculated in Bedford [3] and McMullen [18]. Notice that in the self-affine case no separation condition guarantees a dimension formula (see for instance [6, Example 9.10]).

There are several ways to randomize the construction of IFS fractals. We mention a few relevant examples, but the list is not meant to be exhaustive. The dimension formula for random self-similar fractals was obtained, independently around the same time by Falconer [4], Graf [10], and Mauldin and Williams [17]. Later on Graf, Mauldin and Williams [11] discovered the gauge function giving positive and finite Hausdorff measure to the random self-similar fractal in its dimension. Dimensional properties of the percolation model were studied by Falconer and Grimmett in [8]. Falconer and Miao [7] calculated the dimensions of random subsets of self-affine sets, Jordan, Pollicott and Simon [15] those of randomly perturbed self-affine sets, and Gatzouras and Lalley [9] those of random Bedford-McMullen carpets. In all of these models the choice for the IFS is done independently and using the same distribution at each step of the construction. We point out that other types probability measures
have also been studied, see Järvenpää et al. [14] and Barnsley, Hutchinson and Stenflo [1, 2], for example.

Dimensional results on random self-similar fractals often require some type of non-overlapping condition, see Barnsley, Hutchinson and Stenflo [2, (2.3)], or Falconer [4, (7.9)], Mauldin and Williams [17, (2), section 1], or Graf [10, Theorem 7.6, condition b)]. In contrast, in the current work we do not assume a step-by-step separation condition. We study a class of random self-similar fractals, which we call uniformly random self-similar sets, meaning that the linear parts of generating similitudes are uniformly distributed at each step of the construction, and independent of each other. The translations are fixed to be different but are otherwise arbitrary.

Another model of random similitude IFSs with fixed translations and uniformly distributed linear parts has been considered by Peres, Simon and Solomyak [20]. They studied general problems related to the absolute continuity and dimension of random projections of Bernoulli type measures to the real line. Their results also imply a dimension result for a class of random self-similar sets on the real line (see [20, Corollary 2.5]). In their model independent, absolutely continuous, multiplicative errors to the IFS are introduced at each level of construction. Their probability structure significantly differs from ours, since in our model the linear parts of the mappings are independent both between levels and inside them.

As the main theorem, Theorem 4.7, we prove that the dimension of a uniformly random self-similar set is almost surely given by a solution of an expectation equation (see Lemma 2.1). The method of proof has been extracted from the dimension theory of self-affine sets: The dimension bounds are obtained from energy estimates, following ideas of Falconer [5], and a key lemma is to prove that a transversality condition, such as in [15, formula (26)], holds (see Lemma 3.2).

The paper is organized as follows: First in Section 2 we introduce the notation used, and prove a preliminary lemma. Section 3 concerns the geometric properties of uniformly random sets. In Section 4 we give the energy estimate and deduce the main theorem, Theorem 4.7, from it. We shortly discuss two related conjectures in Section 5.

2. Preliminaries

We begin by defining $\Omega$, the space of labelled trees. Fix vectors $a_1, \ldots, a_m \in \mathbb{R}^d$, $a_i \neq a_j$ for $i \neq j$, and denote $\min_{i \neq j} |a_i - a_j| = a_-$. Fix numbers $0 < \sigma_- \leq \sigma_+ < 1$. Let $J_k = \{1, \ldots, m\}^k, J_\infty = \{1, \ldots, m\}^\mathbb{N}$, and $J = \cup_{k=1}^\infty J_k$. Denote the orthogonal group of $\mathbb{R}^d$ by $O(d)$. Let $\omega : J \rightarrow ]\sigma_-, \sigma_+[ \times O(d)$ be an $m$-branching tree, edges of which are labelled by $]\sigma_-, \sigma_+[ \times O(d)$. Denote the space of all this kind of labelled trees by $\Omega$.

Next we will give a probability measure $\mathbb{P}$ on $\Omega$. Let $\theta$ be the unique uniformly distributed probability on $O(d)$ (that is, $\theta$ is the Haar measure, see [12, Chapter XI]). Notice that $\theta$ has the property that for $A \subset S^{d-1}, x \in S^{d-1},$

$$\tag{2.1} \theta\{g \in O(d) \mid g(x) \in A\} = \sigma_{d-1}(A),$$

where $\sigma_{d-1}$ is the normalized surface measure on $S^{d-1}$ (see [16, Theorem 3.7]). Let $\lambda$ be the normalized Lebesgue measure on $]\sigma_-, \sigma_+[$. Taking the product $(\lambda \times \theta)^J$
over the tree defines a probability measure $P$ on $\Omega$. Denote the mapping $\omega(i) \in ]\sigma_-, \sigma_+ \times \mathcal{G}(d)$ by $T_i^\omega = r_i^\omega Q_i^\omega$. Notice that for $i \neq j \in J$ the labels $T_i^\omega$ and $T_j^\omega$ are independent with respect to $P$. Denote the expectation by $E$.

Now we are able to make precise the notion of uniformly random self-similar sets. For $i \in J$ or $J_\infty$, denote by $i_k$ the initial word of $i$ of length $k$ and by $i_k$ the $k$-th symbol in $i$. Put $f_j^\omega(x) = T_j^\omega(x) + a_{j|i}$ for all $j \in J$ and $x \in \mathbb{R}^d$, and let $\pi : J_\infty \times \Omega \to \mathbb{R}^d$ be the mapping

$$
\pi(i, \omega) = \lim_{k \to \infty} \left( a_{i_1} + T_{i_1}^\omega(a_{i_2}) + \cdots + T_{i_k}^\omega \cdots T_{i_2}^\omega(0) \right),
$$

and define the uniformly random self-similar fractal $F(\omega)$ as

$$
F(\omega) = \bigcup_{i \in J_\infty} \pi(i, \omega).
$$

Next we introduce some more notation related to the sequence space $J$. Denote by $|i|$ the length, or the number of indices of $i$. If $i$ and $j$ are finite or infinite words such that $i_{|i|} = j_{|j|}$, then write $i \leq j$. If neither $i \leq j$ nor $j \leq i$, we say that $i$ and $j$ are incomparable and write $i \perp j$. Let $i \wedge j$ be the word of maximal length such that $i \wedge j \leq i$ and $i \wedge j \leq j$. Notice that $i \wedge j$ can be empty. For $i \in J$, let

$$[i] = \{ j \in J_\infty \mid j_{|j|} = i \}. $$

For the sake of brevity, for every $i \in J$, let

$$ r_i^\omega = r_{i_1}^\omega r_{i_2}^\omega \cdots r_{i_{|i|}}^\omega, Q_i^\omega = Q_{i_1}^\omega \circ Q_{i_2}^\omega \circ \cdots \circ Q_{i_{|i|}}^\omega $$

and

$$ T_i^\omega = T_{i_1}^\omega \circ T_{i_2}^\omega \circ \cdots \circ T_{i_{|i|}}^\omega. $$

When there is no threat of misunderstanding, we may suppress the relation to $\omega$.

We end the section with the following simple lemma. The proof is standard, see [4] for instance, but we give a short proof for the sake of completeness.

**Lemma 2.1.** There exists a unique number $s$ satisfying

$$
E(\sum_{i=1}^m r_i^s) = 1.
$$

Furthermore, $E(\sum_{i=1}^m r_i^t) < 1$ for all $t > s$.

**Proof.** The function $E(\sum_{i=1}^m r_i^t)$ is continuous and strictly decreasing in $s$, by dominated convergence and the fact $0 < \sigma_- < r_i < \sigma_+ < 1$. For $s = 0$ it attains the value $m$ and decreases to 0 when $s \to \infty$. Thus a unique value satisfying the equation (2.3) exists and the latter claim becomes apparent. \qed

3. **Geometric lemmas**

In this section we prove that a transversality condition holds for our model. We will need a bit of more notation and begin with an observation.
Observation 3.1. Fix \( i, j \in J_\infty \) with \(|i \wedge j| = p\). Notice that

\[
|\pi(i, \omega) - \pi(j, \omega)| = r_{i,j}^\omega \mathcal{Q}_{i|j}^\omega (a_{i,p+1} - a_{j,p+1} + T_{i,p+1}^\omega (x(i, \omega, p)) - T_{j,p+1}^\omega (x(j, \omega, p)))
\]

or

\[
= r_{i,j}^\omega |a_{i,p+1} - a_{j,p+1} + T_{i,p+1}^\omega (x(i, \omega, p)) - T_{j,p+1}^\omega (x(j, \omega, p))|,
\]

where the random variables \( x(i, \omega, p) \in \mathbb{R}^d \) and \( x(j, \omega, p) \in \mathbb{R}^d \) are independent of each other, since \(|i \wedge j| = p\) and \( x(a, \omega, p) \) only depends on \( T_{a,k}^\omega \) for \( k > p + 1\).

Denote by \( \mathbb{P}_i \) the probability on node \( i \), and let \( \mathbb{P}_i = (\lambda \times \theta)^{J \setminus \{i\}} \). The statement of the following lemma was inspired by [15, (26)], and the proof influenced by [15, Lemma 5.1].

Lemma 3.2. The following transversality condition holds: Fix \( i, j \in J_\infty \) with \(|i \wedge j| = p\) and \( \rho > 0\). Assume \(|x(i, \omega, p)| \geq |x(j, \omega, p)|\). Then

\[
\mathbb{P}_i^{p+1} (|\pi(i, \omega) - \pi(j, \omega)| < \rho) \leq C' \frac{\rho^d}{(r_{i,j}^\omega)^d},
\]

where \( C' = C'(\alpha, d, \sigma_-) \).

Proof. Denote \( a = a_{i,p+1} - a_{j,p+1} \), \( x = x(i, \omega, p) \) and \( y = x(j, \omega, p) \). Then \(|a| \geq a_- > 0\). Notice that throughout the proof the notions \( a, T_{j,p+1}, x \) and \( y \) are fixed, since they don’t depend on the label at node \( i_{p+1} \). Recalling Observation 3.1 the probability we want to estimate is the probability of the event

\[
A = \{ T_{i,p+1} \in |\sigma_-, \sigma_+| \times \mathcal{O}(d) \mid |a + T_{i,p+1}(x) - T_{j,p+1}(y)| < \gamma := \frac{\rho}{r_{i,j}^\omega} \},
\]

Firstly, if \( r_{i,p+1} / \in \{ |x|^{-1}(|a - T_{j,p+1}(y)| - \gamma), |x|^{-1}(|a - T_{j,p+1}(y)| + \gamma) \} =: I \), then

\[
|a + T_{i,p+1}(x) - T_{j,p+1}(y)| \geq ||T_{i,p+1}(x)| - |a - T_{j,p+1}(y)||
\]

\[
= |r_{i,p+1}| |x| - |a - T_{j,p+1}(y)||
\]

\[
> \gamma.
\]

Here \( \lambda(I) = 2\gamma |x|^{-1} \). Notice that, since \(|x| \geq |y|\), \( A = \emptyset \) whenever \(|x| < \frac{1}{2}(a_- - \gamma)\), and we may assume that the opposite inequality holds. Similarly \(|a - T_{j,p+1}^\omega(y)| \geq \sigma_- |x| - \gamma \). We now have \( \lambda(I) \leq 4\gamma(a_- - \gamma)^{-1} \).

Denote the open ball of radius \( \delta \) and centre \( z \) by \( B(z, \delta) \) and the cone of direction \( v \) and opening angle \( \alpha \) by \( \mathcal{V}(v, \alpha) \). Further, let \( \beta = \gamma/a - T_{j,p+1}^\omega(y)|^{-1} \). Then, for all \( r_{i,p+1} \), for \( \beta < 1 \), that is, for all \( \gamma \) satisfying \( \gamma |a - T_{j,p+1}^\omega(y)|^{-1} < 1 \), we have

\[
\{ Q_{i,p+1} \mid |a + Q_{i,p+1}(r_{i,p+1})x) - T_{j,p+1}^\omega(y) < \gamma \}
\]

\[
= \{ Q_{i,p+1} \mid Q_{i,p+1}(r_{i,p+1} x) \in B(a - T_{j,p+1}^\omega(y), \gamma) \}
\]

\[
\subset \{ Q_{i,p+1} \mid Q_{i,p+1}(r_{i,p+1} x) \in \mathcal{V}(a - T_{j,p+1}^\omega(y), \arcsin \beta) \}
\]

\[
= \{ Q_{i,p+1} \mid Q_{i,p+1}(\frac{x}{|x|}) \in \mathcal{V}(a - T_{j,p+1}^\omega(y), \arcsin \beta) \cap S^{d-1} \}
\]

\[
=: V.
\]
Recall that \(|x| \geq \frac{1}{2}(a_- - \gamma)|\), and \(|a - T_{j_{p+1}}(y)| \geq \sigma \frac{1}{2}(a_- - \gamma) - \gamma|\), so that \(\beta < 1\) for all \(\gamma < \frac{1}{5}a_- \sigma_- \leq \frac{1}{5}a_-\). By elementary geometry, recalling \((2.1)\),

\[
\theta(V) = \sigma_{d-1}|V(a - T_{j_{p+1}}(y), \arcsin \beta) \cap S^{d-1}| \leq C''(d) \beta^{d-1}
\]

\[
\leq C''(d)\gamma^{d-1}(\frac{1}{5}\sigma_- a_-)^{-d+1}
\]

for all \(\gamma < \frac{1}{5}a_- \sigma_- \leq \frac{1}{5}a_-\).

By the above considerations, \(A \subset I \times V\), and

\[
\mathbb{P}^{b_{p+1}}(A) \leq \lambda(I) \theta(V) \leq C''(d) \gamma^{d} \sigma^{-a_-} a_-^{-d+1}
\]

whenever \(\gamma < \frac{1}{5}a_- \sigma_- \leq \frac{1}{5}a_-\). This proves the claim with the constant \(C' = 5C''(d) \sigma^{-a_-} a_-^{-d+1}\) for all \(\rho < \frac{1}{5}\sigma_- a_- \). Larger \(\rho\)'s can be dealt with by further increasing \(C'\) to satisfy \(C' \geq (\frac{1}{5} \sigma_- a_-)^{-d}\), since \(\mathbb{P}\) is a probability measure. \(\square\)

The following lemma is a simplification of \([15, \text{Lemma } 4.5]\).

**Lemma 3.3.** Fix \(i \neq j \in J_{\infty}\) with \(|i \wedge j| = p\), and \(t < d\). Assume \(|x(i, \omega, p)| \geq |x(j, \omega, p)|\). Then

\[
\int_{\Omega} \frac{d \mathbb{P}^{b_{p+1}}(\omega)}{|\pi(i, \omega) - \pi(j, \omega)|^{t}} \leq Cr_{i \wedge j}^{-t},
\]

for some \(C = C(t, d, a_-, \sigma_-)\).

**Proof.** Using first \([16, \text{Theorem } 1.15]\), and then Lemma \(3.2\) for \(\rho \leq r_{i \wedge j}\) and the trivial estimate for \(\rho \geq r_{i \wedge j}\)

\[
\int_{\Omega} \frac{d \mathbb{P}^{b_{p+1}}(\omega)}{|\pi(i, \omega) - \pi(j, \omega)|^{t}} = t \int_{0}^{\infty} \mathbb{P}^{b_{p+1}}(\omega) \mid |\pi(i, \omega) - \pi(j, \omega)| < \rho) \rho^{-t-1} d \rho
\]

\[
\leq C't \int_{0}^{r_{i \wedge j}} \rho^{d} \rho^{-t-1} d \rho + t \int_{r_{i \wedge j}}^{\infty} \rho^{-t-1} d \rho
\]

\[
\leq \left(\frac{C'}{d - t} + 1\right) r_{i \wedge j}^{-t},
\]

where \(C' = C''(a_-, d, \sigma_-)\) from Lemma \(3.2\). \(\square\)

4. **Proof of the Main Theorem**

Recall number \(s\) from Lemma \(2.1\). In this section we prove as the main theorem, Theorem \(4.7\), that for \(\mathbb{P}\)-almost all \(\omega \in \Omega\), \(\dim_{\mathcal{H}} F(\omega) = \min\{s, d\}\). We first prove the upper bound as Proposition \(4.2\). We then define random measures on \(J_{\infty}\), almost surely projecting onto \(F\) as measures of finite energy. Lower bound for the dimension is then an easy consequence of the energy estimate.

Fix \(0 < t < s\) for the time being. For all \(k\), denote by \(\mathcal{F}_k\) the sigma-algebra generated by the random variables \(T_a\) for all \(|a| \leq k\).

The proofs of Lemma \(4.1\), Proposition \(4.2\) and Lemma \(4.3\) are essentially from the proof of \([6, \text{Theorem } 15.1]\) (also see \([17\) and \([10]\)). For the convenience of the reader, and since the exposition in \([6]\) is not overly detailed, we repeat the necessary arguments here.
Lemma 4.1. For all $u > 0$ and $k \in \mathbb{N}$,

$$
\mathbb{E} \left( \sum_{|i|=k+1} r_i^u \right) = \mathbb{E} \left( \left( \sum_{i=1}^m r_i^u \right)^{k+1} \right).
$$

Proof. Notice that, recalling the definitions of $r_i$ and $r_i$ from Section 2

$$
\mathbb{E} \left( \sum_{|i|=k+1} r_i^u \mid \mathcal{F}_k \right) = \mathbb{E} \left( \sum_{|i|=k} \sum_{i=1}^m r_i^u r_i^u \mid \mathcal{F}_k \right)
$$

Iterating this the claim follows for $\mathbb{E}(\sum_{|i|=k+1} r_i^u) = \mathbb{E}(\mathbb{E}(\sum_{|i|=k+1} r_i^u \mid \mathcal{F}_k))$. \hfill \Box

Proposition 4.2. For $\mathbb{P}$-almost all $\omega \in \Omega$

$$
\dim_\mathcal{H} F(\omega) \leq \min\{s, d\}.
$$

Proof. Certainly $\dim_\mathcal{H} F \leq d$, and we only need to check that $\dim_\mathcal{H} F \leq s$.

Define a sequence of random variables $X_k = \sum_{|i|=k} r_i^s$. We have, for all $\omega \in \Omega$, for the Hausdorff measure of $F(\omega)$

$$
\mathcal{H}^s(F(\omega)) \leq R \liminf_{k \to \infty} X_k(\omega),
$$

since $F(\omega) \subset \cup_{i \in I_k} \pi([i], \omega)$, where the diameter of $\pi([i], \omega)$ is bounded from above by $R r_i$ for a constant $R > 0$ independent of $\omega$. We prove that $X_k$ is an $L^2$-bounded martingale with respect to the sequence of sigma-algebras $\mathcal{F}_k$. Firstly, by the choice of $s$, for any $i \in J$,

$$
\mathbb{E} \left[ \sum_{i=1}^m r_i^s \right] = 1
$$

so that by calculation [4.1] above $\mathbb{E}(X_{k+1} \mid \mathcal{F}_k) = X_k$ immediately. Thus $X_k$ is a martingale. Furthermore,

$$
\mathbb{E}(X_k^2 \mid \mathcal{F}_{k-1}) = \mathbb{E} \left[ (\sum_{|i|=k} r_i^s)^2 \mid \mathcal{F}_{k-1} \right]
$$

$$
= \mathbb{E} \left[ \sum_{|i|=k-1} r_i^{2s} \sum_{i=1}^m \sum_{j=1}^m r_i^s r_j^s \mid \mathcal{F}_{k-1} \right]
$$

$$
+ \mathbb{E} \left[ \sum_{|i|=k-1} \sum_{|a|=k-1, a \neq i} r_i^s r_a^s \sum_{i=1}^m \sum_{j=1}^m r_i^s r_j^s \mid \mathcal{F}_{k-1} \right]
$$

$$
= \sum_{|i|=k-1} r_i^{2s} \mathbb{E} \left[ \sum_{i=1}^m \sum_{j=1}^m r_i^s r_j^s \right] + \sum_{|i|=k-1} \sum_{|a|=k-1, a \neq i} r_i^s r_a^s \mathbb{E} \left[ \sum_{i=1}^m \sum_{j=1}^m r_i^s r_j^s \right].
$$
For \( a \neq i, \, |a| = |i| \), the random variables \( r_{ii} \) and \( r_{aj} \) are independent for all \( i, j = 1, \ldots, m \). Thus, by choice of \( s \),

\[
\mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j=1}^{m} r_{ii}^s r_{aj}^s \right] = \mathbb{E} \left[ \sum_{i=1}^{m} r_{ii}^s \right] \mathbb{E} \left[ \sum_{j=1}^{m} r_{aj}^s \right] = 1.
\]

Notice that \( \mathbb{E}(\sum_{i=1}^{m} r_{ii}^s) \) does not depend on \( i \in J \). Denote this quantity by \( \lambda \), and notice that \( \lambda < 1 \) by Lemma 2.1. Then, again by the definition of \( s \),

\[
\mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j=1}^{m} r_{ii}^s r_{ij}^s \right] = \mathbb{E} \left[ \sum_{i=1}^{m} r_{ii}^{2s} \right] + \mathbb{E} \left[ \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} r_{ii}^s r_{ij}^s \right] \\
\leq \lambda + 1.
\]

From the above calculations

\[
\mathbb{E}(X_k^2 \mid \mathcal{F}_{k-1}) \leq \sum_{|i|=k-1} r_{i}^{2s} \lambda + X_{k-1}^2.
\]

By Lemma 4.1,

\[
\mathbb{E}( \sum_{|i|=k-1} r_{i}^{2s} ) = \lambda^{k-1},
\]

and hence,

\[
\mathbb{E}(X_k^2) \leq \lambda^k + \mathbb{E}(X_{k-1}^2) \leq \sum_{k=1}^{\infty} \lambda^k + 1 < \infty.
\]

By martingale convergence theorem, see [19, Theorems 12.24 and 12.28], the \( L^2 \)-boundedness of the martingale \( (X_k) \) implies that the sequence of random variables converges (almost surely and in \( L^2 \)) to a random variable \( X \) and also that

\[
\mathbb{E}(X \mid \mathcal{F}_k) = X_k,
\]

most importantly giving \( \mathbb{E}(X) = 1 \). This means that \( X(\omega) < \infty \) for almost every \( \omega \), and thus the upper bound for the dimension follows. \( \square \)

**Lemma 4.3.** There exists a random measure \( \mu^\omega \) on \( J_\infty \) having the properties

1. almost surely \( 0 < \mu^\omega(J_\infty) < \infty \),
2. \( \mathbb{E}(\mu^\omega[i] \mid \mathcal{F}_k) = r_i^s \) for all \( i \in J_k \), and
3. for all \( k \), \( \mathbb{E}(\sum_{|i|=k} \mu^\omega[i]) = 1 \).

**Proof.** For \( i \in J \), define a sequence of random variables

\[
\mu_k[i] = \sum_{j \in |i| \supseteq J_k} r_j^i.
\]

Exactly the same proof as above for \( X_k \) shows that also \( \mu_k[i] \) is an \( L^2 \)-bounded martingale, and hence converges to a \( \tilde{\mu}[i] \) with \( 0 \leq \tilde{\mu}[i] < \infty \) almost everywhere, and

\[
\mathbb{E}(\tilde{\mu}[i] \mid \mathcal{F}_i) = r_i^s.
\]

Furthermore, since \( \tilde{\mu}[i] = \sum_{i=1}^{m} \tilde{\mu}[i] \) for all \( i \in J \), almost surely the cylinder function \( \tilde{\mu} \) extends naturally to a Borel measure \( \mu^\omega \) on \( J_\infty \) with \( \mu^\omega[i] = \tilde{\mu}[i] \) for all \( i \in J \).
Now, $\tilde{\mu}[i] = 0$ with probability $q < 1$ and, on the other hand, $\tilde{\mu}[i] = 0$ if and only if $\tilde{\mu}[i] = 0$ for all $i = 1, \ldots, m$. Notice that by self-repeating nature of the probability, $\tilde{\mu}[i]$ and $r^{-s}_{k} \tilde{\mu}[i]$ have the same distribution. By independence of $\tilde{\mu}[i] = 0$ and $\tilde{\mu}[ij] = 0$ for $i \neq j$, this leads to $q^m = q$ and hence $\tilde{\mu}[i] > 0$ almost surely. Then $0 < \mu^\omega(J_{\infty}) < \infty$.

Lemma 4.1 and the definition of $s$ give the last claim, since for all $i \in J$ we have $E(\mu^\omega[i]) = E(E(\mu^\omega[i] | F[i])) = E(r^i_s)$. □

The following easy lemma will be the key to proving an energy estimate for the measure $\mu^\omega$.

**Lemma 4.4.** Let $\omega, \omega' \in \Omega$ and $i \in J$. If $T^\omega_a = T^\omega'_a$ for all $a \neq i$, then

$$\mu^\omega|_i = \left(\frac{r^\omega_i}{r^\omega'_i}\right)^s \mu^\omega'|_i$$

and for all $j \perp i$, in fact $\mu^\omega|_j = \mu^\omega'|_j$.

**Proof.** For all $a$ with $i \leq a$ we have

$$r^\omega_a = \frac{r^\omega'_a}{r^\omega'_i} r^\omega_i,$$

and for all $a$ with $i \perp j \leq a$, we have $r^\omega_a = r^\omega'_a$. By definition of the measures $\mu^\omega$ and $\mu^\omega'$ the claim follows. □

Denote by $I_t(\nu)$ the t-energy of a measure $\nu$ with support $E$, that is, let

$$I_t(\nu) = \int \int_{E \times E} |x - y|^{-t} d\nu(x) d\nu(y).$$

We verify that the expectation of $I_t(\pi_* \mu^\omega)$ is finite for all $t < s$ as Theorem 4.6. Here the image of the measure $\mu^\omega$ under $\pi(\cdot, \omega)$ is denoted by $\pi_* \mu^\omega$. For properties of energies of measures, including their connection to the dimension of the supporting set, see [16, Chapter 8].

**Remark 4.5.** By Lemma 4.3, for all $i \in J$ the function $\omega \mapsto \mu^\omega[i]$ is a measurable function. Since all open sets of $J_{\infty}$ are disjoint finite unions of cylinder sets, also $\omega \mapsto \mu^\omega(A)$ is measurable for all open and closed sets. Since all continuous functions $f$ on $J_{\infty}$ are limits of sequences of simple functions of the form $\sum_{i=1}^n c_i \chi_{A_i}$ for characteristic functions of open and closed sets $A_i$, also $\omega \mapsto \int f d\mu^\omega$ is measurable. Presenting $|\pi(i, \omega) - \pi(j, \omega)|^{-t}$ as a limit $\lim_{k \to \infty} \min\{|\pi(i, \omega) - \pi(j, \omega)|^{-t}, k\}$ of continuous functions, we see that $\omega \mapsto I_t(\mu^\omega)$ is measurable.

**Theorem 4.6.** Let $0 < t < \min\{s, d\}$. Then the expectation of the t-energy of measure $\pi_* \mu^\omega$ is finite.

**Proof.** Denote by $v(s)$ the number $(\frac{r}{s})^s$. By Lemma 4.3 the measure $\mu^\omega$ is well-defined for almost all $\omega \in \Omega$. Below we only consider $\omega$’s which are typical in this
sense. Since $\mu^\omega$'s don’t have atoms, we have for the expectation of the energy,

$$
\mathbb{E}(I_t(\pi_\ast \mu^\omega)) = \int \int \int \frac{d\mu^\omega(i)}{|\pi(i,\omega) - \pi(j,\omega)|^t} \frac{d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t} = \int \int \int \frac{d\mu^\omega(i)}{|\pi(i,\omega) - \pi(j,\omega)|^t} \frac{d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t}
$$

(4.3)

$$
\leq \sum_{k=0}^{\infty} \sum_{|q|=k} \sum_{i \neq j} \int \int |q_i| \int d\mu^\omega(i) \frac{d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t}.
$$

For a while, fix $k$, $|q| = k$ and $i \neq j \in \{1, \ldots, m\}$, and furthermore, fix $T_a^\omega$ for all $a \neq q_i$ and all $a \neq q_j$. Notice that then, given $i \in [q_i]$ and $j \in [q_j]$, the vectors $s(i,\omega,k)$ and $s(j,\omega,k)$ from Observation 3.1 are fixed. Let $T_0 = T_0(\omega) = \{i,j} \in [q_i] \times [q_j] \mid |s(i,\omega,k)| \geq |s(j,\omega,k)|\}$. Let $T_0 \cap [\sigma_+ \times \sigma_+ \times \sigma_+ \times \sigma_+] = T_0$. Furthermore, by Lemma 3.4 and Lemma 3.3

$$
\int \int \int_X \frac{d\mu^\omega(i)}{|\pi(i,\omega) - \pi(j,\omega)|^t} \frac{d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t} = \int \int \int_X \frac{d\mu^\omega(i)}{|\pi(i,\omega) - \pi(j,\omega)|^t} \frac{d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t}
$$

Since $\mu^\omega$ and $\mu^\omega|_{[q]}$ do not depend on $T_a^\omega$, by Lemma 3.4.

$$
\int v(s) C r_q^{-t} \mu^{\omega_q[q]}[q_i] \mu^{\omega_q[q]}[q_i] \mathbb{P}(q) = \int v(s) C r_q^{-t} \mu^{\omega_q[q]}[q_i] \mu^{\omega_q[q]}[q_i] \mathbb{P}(q)
$$

(4.4)

$$
\leq \int v(s)^2 C r_q^{-2t} \mu^{\omega_q[q]}[q_i] \mu^{\omega_q[q]}[q_i] \mathbb{P}(q).
$$

Furthermore, using the fact that $\mu^\omega[q_i]$ and $\mu^\omega[q_j]$ are independent when conditioned on $\mathcal{F}_k$, by Lemma 3.3

$$
v(s)^2 C \mathbb{E}(\mathbb{E}(r_q^{-t} \mu^{\omega_t}[q_i] \mu^{\omega_t}[q_j] \mid \mathcal{F}_k)) = v(s)^2 C \mathbb{E}(r_q^{-t} \mathbb{E}(\mu^{\omega_t}[q_i] \mid \mathcal{F}_k) \mathbb{E}(\mu^{\omega_t}[q_j] \mid \mathcal{F}_k))
$$

$$
\leq v(s)^2 C \mathbb{E}(r_q^{-t} r_q^{+} \mathbb{E}(\mu^{\omega_t}[q_i] \mid \mathcal{F}_k))
$$

$$
\leq v(s)^2 C \sigma_+^{k(s-t)} \mathbb{E}(\mu^{\omega_t}[q_j]).
$$

Combining the above calculations gives

(4.4)

$$
\int \int \int_X \frac{d\mu^\omega(i)}{|\pi(i,\omega) - \pi(j,\omega)|^t} \frac{d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t} \mathbb{P}(q) \leq v(s)^2 C \sigma_+^{k(s-t)} \mathbb{E}(\mu^{\omega_t}[q_j]).
$$
Using (4.3), (4.4), the counterpart of (4.4) for \( Y = Y(\omega) = \{(i,j) \in [q]^i \times [q]^j \mid x(i,\omega,k) < x(j,\omega,k)\} \), and Lemma 4.3 (3), we obtain
\[
E(I_t(\pi_*\mu^\omega)) = \sum_{k=0}^\infty \sum_{|q|=k} \sum_{i \neq j} \int \int \int_X \frac{d\mu^\omega(i) d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t} + \int \int \frac{d\mu^\omega(i) d\mu^\omega(j)}{|\pi(i,\omega) - \pi(j,\omega)|^t} d\mathbb{P}(\omega)
\]
\[
\leq \sum_{k=0}^\infty \sum_{|q|=k} 2mv(s)^2C\sigma^k(s-t)E(\mu^\omega[q])
\]
\[
= \sum_{k=0}^\infty 2mv(s)^2C\sigma^k(s-t) < \infty,
\]
where the sum converges since \( \sigma_+ < 1 \). □

We can now prove the main theorem.

**Theorem 4.7.** For the probability \( \mathbb{P} \) from Section 2 we have \( \text{dim}_F F(\omega) = \min\{s, d\} \) almost surely, where \( F(\omega) \) is the uniformly random self-similar set of (2.2), and \( s \) is from Lemma 2.1.

**Proof.** By Lemma 4.2, \( \text{dim}_F F(\omega) \leq \min\{s, d\} \) almost surely.

Fix \( 0 < t < \min\{s, d\} \). By Lemma 4.3 and Theorem 4.6 there exists a finite Borel measure \( \mu^\omega \) on \( J_\infty \) with
\[
(4.5) \quad E(\int \int_{F(\omega) \times F(\omega)} \frac{d(\pi_*\mu^\omega(x)) d(\pi_*\mu^\omega(y))}{|x - y|^t} ) < \infty.
\]
Thus, almost surely, \( \text{dim}_F F(\omega) \geq t \). (See [5, Lemma 5.2].) Approaching \( s \) along a sequence will result in \( \text{dim}_F F(\omega) \geq s \), almost surely. □

5. An example and further problems

We begin this section by giving an example that shows sharpness of Theorem 4.7 in a sense. This is an example of a fractal set in \( \mathbb{R}^d \) such that for the generating similitudes contraction ratios are uniformly distributed, but rotations deterministic. The dimension of such a set can be strictly less than the number \( \min\{s_0, d\} \) of Theorem 4.7.

**Example 5.1.** Fix \( m \) so large that \( \frac{1}{m} < \sigma_- \). Consider a system of \( m \) similarities in \( \mathbb{R} \) having the random structure described in Section 2, that is, each \( r_i \) is uniformly distributed in \( ]\sigma_-, \sigma_+[ \). Then for the limiting set \( F \)
\[
\text{dim}_F F \leq 1 = \min\{1, s_0\},
\]
where \( s_0 > 1 \) satisfies \( m\sigma^{s_0} = 1 \). Now, embed this set in \( \mathbb{R}^d \) for \( d \geq 2 \). Still
\[
\text{dim}_F F \leq 1 < \min\{d, s_0\},
\]
and the claim of Theorem 4.7 does not hold for the set \( F \).

We then present ways of generalizing the result. Instead of similarities one could also consider affine mappings, and conjecture
Conjecture 5.2. The Hausdorff dimension of a uniformly random self-affine set is a constant number almost surely.

Here the uniform distribution on the space of contractive bijective linear mappings is the normalized Lebesgue measure $\theta$. The probability is defined to be the product over the tree $J$, as in Section 2. The number giving the dimension can be defined using singular value functions instead of $r_i$. (For definition of singular value function, see [5], for example.) The main problem in proving Conjecture 5.2 follows from this; unlike in the similitude case, in the affine case the singular value function is not multiplicative, and multiplicativity is needed in multiple places in the proof of Theorem 4.7.

Let us next consider a somewhat more general, related problem. Let $T = \{T_1, \ldots, T_m\}$ be a collection of independent, $\theta$-distributed linear mappings, and fix $a = \{a_1, \ldots, a_m\}$, $a_i \in \mathbb{R}^d$ with $a_1 \neq \cdots \neq a_m$. Denote by $F(T, a)$ the limiting set corresponding to the IFS $f_1, \ldots, f_m$, $f_i(x) = T_i(x) + a_i$. Then one can ask the question

Question 5.3. Is it true that $\dim F(T,a) = \dim F(T)$, for almost all $T$?

This question is, in a sense, a more natural continuation of Falconer’s [5] than Conjecture 5.2. However, it seems to be somewhat difficult to verify the answer one way or the other.

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