Complementary tree nil domination number of
Cartesian Product of Graphs

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Abstract: A set \(D\) of a graph \(G = (V, E)\) is a dominating set, if every vertex in \(V(G) - D\) is adjacent to some vertex in \(D\). The domination number \(\gamma(G)\) of \(G\) is the minimum cardinality of a dominating set. A dominating set \(D\) is called a complementary tree nil dominating set, if the induced subgraph \(< V(G) - D >\) is a tree and also the set \(V(G) - D\) is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of \(G\) and is denoted by \(\gamma_{ctnd}(G)\). In this paper, complementary tree domination numbers of Cartesian product of some standard graphs are found.

Key words: Domination number, Complementary tree nil domination number, Cartesian product.

1. Introduction

Graphs discussed in this paper are finite, undirected and simple connected graphs. For a graph \(G\), let \(V(G)\) and \(E(G)\) denote its vertex set and edge set respectively. A graph \(G\) with \(p\) vertices and \(q\) edges is denoted by \(G(p, q)\). The concept of domination in graphs was introduced by Ore[5]. A set \(D \subseteq V(G)\) is said to be a dominating set of \(G\), if every vertex in \(V(G) - D\) is adjacent to some vertex in \(D\). The cardinality of a minimum dominating set in \(G\) is called the domination number of \(G\) and is denoted by \(\gamma(G)\).

Muthammai, Bhanumathi and Vidhya[5] introduced the concept of complementary tree dominating set. A dominating set \(D \subseteq V(G)\) is said to be a complementary tree dominating set (ctd-set), if the induced subgraph \(< V(G) - D >\) is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of \(G\) and is denoted by \(\gamma_{ctd}(G)\). Any undefined terms in this paper may be found in Harary[2].

The cartesian product of two graphs \(G_1\) and \(G_2\) is the graph, denoted by \(G_1 \times G_2\), with \(V(G_1 \times G_2) = V(G_1) \times V(G_2)\) (where \(x\) denotes the cartesian product of sets) and two vertices \(u = (u_1, u_2)\) and \(v = (v_1, v_2)\) in \(V(G_1 \times G_2)\) are adjacent in \(G_1 \times G_2\) whenever \([u_1 = v_1\) and \((u_2, v_2) \in E(G_2)]\) or \([u_2 = v_2\) and \((u_1, v_1) \in E(G_1)]\). The corona \(G_1 \odot G_2\) of two graphs \(G_1\) and \(G_2\) are defined as the graph \(G\) obtained by taking one copy of \(G_1\) of order \(p_1\) and \(p_1\) copies of \(G_2\) and then joining the \(i^{th}\) vertex of \(G_1\) to every vertex in the \(i^{th}\) copy of \(G_2\). The Corona \(G_1 \odot G_2\) has \(p_1(p_1 + p_2)\) vertices and \(q_1 + p_1q_2 + p_1p_2\) edges. The concept of complementary tree nil dominating set is introduced in [4]. A dominating set \(D \subseteq V(G)\) is said to be a...
complementary tree nil dominating set (ctnd-set), if the induced subgraph $<V(G)\rightarrow D>$ is a tree and the set $V(G) - D$ is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of $G$ and is denoted by $\gamma_{ctnd}(G)$.

In this paper, we find an upper bound for complementary tree nil domination number of Cartesian product of $P_m \times P_n$ and this number found for $K_m \times K_n$, $K_m \times P_n$, $K_m \times C_n$ and $C_m \times P_n$.

2. Main Results

Theorem 2.1:
If $G \cong K_m \times K_n$ ($m, n \geq 3$ and $m \leq n$), then $\gamma_{ctnd}(G) = \begin{cases} \frac{(m-n) + 3}{2}, & \text{if } m = n \\ \frac{(m-n) + 2}{2}, & \text{if } m < n \end{cases}$

Proof:
Let $G \cong K_m \times K_n$.

Let $V(G) = \bigcup_{i=1}^{m} \{v_{i1}, v_{i2}, \ldots, v_{im}\}$ such that $<\{v_{i1}, v_{i2}, \ldots, v_{im}\}> \cong K^i_1$, $i = 1, 2, \ldots, m$ and $<\{v_{ij}, v_{j1}, \ldots, v_{jm}\}> \cong K^j_m$, $j = 1, 2, \ldots, n$, where $K^i_1$ is the $i$th copy of $K_n$ and $K^j_m$ is the $j$th copy of $K_m$ in $K_m \times K_n$. $|V(G)| = mn$.

Case 1: $m = n$.

Let $D' = \left( \bigcup_{i=1}^{m-1} \{v_{ii}, v_{i,i+1}\} \right) \cup \{v_{m,m}\}$ and $D = V(G) - D'$. Then $V(G) - D = D'$ and $|D'| = 2(m - 2) + 1 = 2m - 3$. The vertices $v_{ii}, v_{i,i+1}$ in $V(G) - D$ are adjacent to $v_{i1}$ in $D$, $i = 2, 3, \ldots, m$ and the vertex $v_{mn}$ is adjacent to $v_{m1}$ in $D$. Therefore $D$ is a dominating set of $G$. Also $<V(G) - DтроыП_{2(m-2)} + 1 = П_{2m - 3}$. Therefore $D$ is a ctnd-set of $G$ and since $N(v_{11}) \subseteq D$, $D$ is a ctnd-set of $G$. Therefore $\gamma_{ctnd}(G) \leq |D| = |V(G)| - |D'| = mn - (2m - 3) = (m - 2) + 3$.

It is to be noted that, any tree in $G$ is a path and $\delta(G) = m$. Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Then there exists a vertex $u \in D'$ such that $N(u) \subseteq D'$. The longest path that can be obtained from the subgraph of $G$ induced by the vertices of $V(G) - N(u)$ is $P_{2m - 3}$. Therefore $<V(G) - D'> \cong P_{2m - 3}$.

Therefore $D'$ contains at least $mn - (2m - 3) = (m - 2) + 3$ vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq m(n - 2) + 3$.

Hence $\gamma_{ctnd}(G) = m(n - 2) + 3$.

Case 2: $m < n$.

Let $D' = \bigcup_{i=2}^{m} \{v_{ii}, v_{i,i+1}\}$ and $D = V(G) - D'$. Then $V(G) - D = D'$ and $|D'| = 2(m - 1)$. The vertices $v_{ii}, v_{i,i+1}$ ($i = 2, 3, \ldots, m$) are adjacent to $v_{i1}$, $i = 2, 3, \ldots, m$ in $D$. Therefore $D$ is a dominating set of $G$. Also $<V(G) - DтроыП_{2(m-2)} = П_{2m - 2}$. Therefore $D$ is a ctnd-set of $G$ and since $N(v_{11}) \subseteq D$, $D$ is a ctnd-set of $G$.

Therefore $\gamma_{ctnd}(G) \leq |V(G)| - |D'| = mn - (2m - 2) = (m - 2) + 2$. 
As in case 1, any tree in $G$ is a path and $\delta(G) = m$. Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Then there exists a vertex $u \in D'$ such that $N(u) \subseteq D'$. The longest path that can be obtained from the subgraph of $G$ induced by the vertices of $V(G) - N(u)$ is $P_{2m - 2}$.

Therefore $|V(G) - D'| \geq 2m(n - 2)$. Therefore $D'$ contains at least $mn - (2m - 2) = m(n - 2) + 2$ vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq m(n - 2) + 2$.

Hence $\gamma_{ctnd}(G) = \begin{cases} m(n - 2) + 3, & \text{if } m = n \\ m(n - 2) + 2, & \text{if } m < n \end{cases}$.

**Example 2.1:**

For the graph $G$ given in Figure 1.a and Figure 1.b, the set of vertices within the 8 is a minimum $ctnd$-set of $K_n \times K_n$ and $\gamma_{ctnd}(K_4 \times K_4) = 11$ and $\gamma_{ctnd}(K_4 \times K_5) = 14$.

![Figure 1.a](image1.png)  
![Figure 1.b](image2.png)

**Theorem 2.2:**

If $G \cong K_m \times P_n$ ($4 \leq m \leq n$), then $\gamma_{ctnd}(G) = n(m - 2) + 2$.

**Proof:**

Let $G \cong K_m \times P_n$.

Let $V(G) = \bigcup_{i=1}^{n} \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ such that $\{v_{i1}, v_{i2}, \ldots, v_{in}\} \cong K_n^i$, $i = 1, 2, \ldots, m$ and $\{v_{1j}, v_{2j}, \ldots, v_{mj}\} \cong P_m^j$, $j = 1, 2, \ldots, n$, where $K_n^i$ is the $i$th copy of $K_n$ and $P_m^j$ is the $j$th copy of $P_m$ in $K_m \times P_n$.

Let $D' = \begin{cases} \bigcup_{i=1}^{n} \{v_{2i}\} \cup \bigcup_{i=1}^{\frac{n}{2}} \{v_{3,2i}, v_{1,2i+1}\}, & \text{if } n \text{ is odd} \\ \bigcup_{i=1}^{n} \{v_{2i}\} \cup \bigcup_{i=1}^{\frac{n}{2}} \{v_{1,2i-1}, v_{3,2i}\}, & \text{if } n \text{ is even} \end{cases}$.

Then $|D'| = 2(n - 1)$. If $D = V(G) - D'$, then $D$ is a dominating set of $G$ and $N(v_{1i}) \subseteq D$. Also $<V(G) - D> = <D'> \cong P_m^i K_i$. Therefore $D$ is a $ctnd$-set of $G$.

$\gamma_{ctnd}(G) \leq |D| = mn - 2(n - 1) = mn - 2n + 2 = n(m - 2) + 2$.

Hence $\gamma_{ctnd}(G) \leq n(m - 2) + 2$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $D'$ is a $ctnd$-set of $G$, $D'$ contains at least $(m - 2)$ vertices in each of $(n - 1)K_m$’s and since, $V(G) - D'$ is not a dominating set, $D'$ contains
all the vertices of the remaining $K_m$. Hence $D'$ contains at least $(m - 2)(n - 1) + m = mn - m - 2n + 2 + m = n(m - 2) + 2$ vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq n(m - 2) + 3$.

Hence $\gamma_{ctnd}(K_m \times P_n) = n(m - 2) + 2$.

Example 2.2:

For the graph $G$ given in Figure 2, the set of vertices within the $\ominus$ is a minimum $ctnd$-set of $K_m \times K_n$ and $\gamma_{ctnd}(K_4 \times K_9) = 20$.

![Figure 2](image)

Remark 2.1:

In view of Theorem 2.2, $\gamma_{ctnd}(K_m \times C_n) = n(m - 2) + 3$.

Theorem 2.3:

If $G \cong P_m \times P_n$ (m, n $\geq 2$), then $\gamma_{ctnd}(G) \leq \gamma_{ctd}(G) + 2$.

Proof:

Let $G \cong P_m \times P_n$. Then $\delta(G) = 2$.

Let $D$ be a $\gamma_{ctd}$-set of $G$. Let $u \in D$ be a vertex of minimum degree in $G$ and $\deg(u) = \delta(G)$. Then $D' = D \cup N(u)$ is a $ctnd$-set of $G$, since $N(u) \subseteq D'$. Therefore $\gamma_{ctnd}(G) \leq |D'| = |D| + |N(u)| = \gamma_{ctd}(G) + \delta(G) = \gamma_{ctd}(G) + 2$.

Hence $\gamma_{ctnd}(G) \leq \gamma_{ctd}(G) + 2$.

Equality holds, if $G \cong P_2 \times P_n$, n $\geq 3$.

Theorem 2.4:

If $G \cong C_3 \times P_n$, then $\gamma_{ctnd}(G) = n + 2$, n $\geq 3$.

Proof:

Let $G \cong C_3 \times P_n$.

Let $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}\}$ such that $\langle \{v_{1i}, v_{12}, \ldots, v_{1n}\} \rangle \cong P_i$, i = 1, 2, 3 and $\langle \{v_{1j}, v_{2j}, v_{3j}\} \rangle \cong C_3$, j = 1, 2, ..., n, where $P_i$ is the i-th copy of $P_n$ and $C_3$ is the j-th copy of $C_3$ in $C_3 \times P_n$. 
Let $D = \begin{cases} \{v_{11}, v_{21}\} \cup \bigcup_{i=1}^{n-1} \{v_{22i}, v_{32i-1}\}, & \text{if } n \text{ is even} \\ \{v_{11}, v_{21}, v_{31}\} \cup \bigcup_{i=1}^{n-1} \{v_{22i}, v_{32i+1}\}, & \text{if } n \text{ is odd.} \end{cases}$

Then $D$ is a dominating set of $G$ and $N(v_{11}) \subseteq D$. Also $<V(G) - D> \cong P_n \square K_1$.

Therefore $D$ is a ctnd-set of $G$.

$$\gamma_{ctnd}(G) \leq |D| = \begin{cases} \left(\frac{n}{2}\right) + 2 = n + 2, & \text{if } n \text{ is even} \\ \left(\frac{n-1}{2}\right) + 3 = n + 2, & \text{if } n \text{ is odd}. \end{cases}$$

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $C_1 \times P_n$ contains at least one vertex from each cycle, $D'$ contains at least $n$ vertices. Also, since $V(G) - D'$ is not a dominating set, the remaining vertices of first cycle $C_3$ in $C_3 \times P_n$ must be included in $D'$.

Therefore $D'$ contains at least $n + 2$ vertices and $\gamma_{ctnd}(G) = |D'| \geq n + 2$.

Hence $\gamma_{ctnd}(C_1 \times P_n) = n + 2, n \geq 3$.

**Theorem 2.5:**

If $G \cong C_4 \times P_n$, then $\gamma_{ctnd}(G) = \left\lfloor \frac{3n+4}{2} \right\rfloor, n \geq 2$.

**Proof:**

Let $G \cong C_4 \times P_n$ and $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}\}$ such that $\langle v_{1i}, v_{2i}, ..., v_{ni}\rangle \cong P_n$, $i = 1, 2, 3, 4$ and $\langle v_{11}, v_{21}, v_{31}, v_{41}\rangle \cong C_4, j = 1, 2, ..., n$, where $P_n$ is the $i$th copy of $P_n$ and $C_4$ is the $j$th copy of $C_4$ in $C_4 \times P_n$ and $|V(G)| = 4n$.

**Case 1:** $n$ is even.

Let $D' = \{v_{31}, v_{3n}\} \cup \bigcup_{i=1}^{n-2} \{v_{12i+1}, v_{2i}, v_{32i+1}\} \cup \bigcup_{j=1}^{n} \{v_{22j}, v_{2j+1}\}$ and $D = V(G) - D'$. Then $|D'| = 2 + 3 \left(\frac{n-2}{2}\right) + n - 1 = 5n - 4$. Then $D$ is a dominating set of $G$ and $N(v_{11}) \subseteq D$. Also $<V(G) - D> \cong D'$ is a tree obtained from a path $P_n = \langle v_{21}, i = 2, 3, ..., n \rangle \geq 2$ by attaching $P_3$ at each of the vertices $v_{22}, v_{23}, v_{25}, ..., v_{2,n-1}$ and attaching a pendant edge at each of the vertices $v_{24}, v_{26}, ..., v_{2,n}$. Therefore $D$ is a ctnd-set of $G$.

$$\gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 4n - \left(\frac{5n-4}{2}\right) = \frac{3n+4}{2}.$$

Hence $\gamma_{ctnd}(G) \leq \frac{3n+4}{2}$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $<V(G) - D'>$ is not a dominating set, $D'$ contains a vertex $u$ such that $N(u) \subseteq D$. $u$ is taken to be a vertex of minimum degree $\delta(G) = 3$ in $G$. The blocks $A, B, C$ are constructed as given below.
G is obtained by concatenating the blocks $A, B^{\frac{n-2}{2}}$, and $C$. That is, $G \cong A \bowtie B^{\frac{n-2}{2}} \bowtie C$. The vertices with the symbol $\bigcirc$ in each of the blocks represent the vertices that are to be included in $D'$.

Therefore $D'$ contains 3 vertices from block $A$ and at least 3 vertices from each block $B$ of $B^{\frac{n-2}{2}}$ and 2 vertices from block $C$. Therefore $\gamma_{ctnd}(G) = |D'| \geq 3 + 3 \left(\frac{n-2}{2}\right) + 2 = \frac{3n+4}{2}$.

and hence $\gamma_{ctnd}(G) = \frac{3n+4}{2}$.

**Case 2:** $n$ is odd.

Let $D' = \{v_{21}\} \cup \left(\bigcup_{i=1}^{\frac{n-2}{2}} \{v_{3i-1+1}, v_{4,i-1+1}, v_{5,3i}\}\right) \cup \left(\bigcup_{i=2}^{n} \{v_{2i}\}\right)$.

Then $|D'| = 1 + 3 \left(\frac{n-1}{2}\right) + n - 1 = \frac{5n-3}{2}$ and $D = V(G) - D'$. Then $D$ is a dominating set of $G$ and $N(v_{i1}) \subseteq D$. Also $<V(G) - D> = <D'>$ is a tree obtained from a path $P_{n-1} = <v_{2,1}, i = 2, 3, ..., n>$, $(n \geq 2)$ by attaching $P_3$ at each of the vertices $v_{22}, v_{23}, v_{25}, ..., v_{2,n}$ and attaching a pendant edge at each of the vertices $v_{24}, v_{26}, ..., v_{2,n-1}$. Therefore $D$ is a $ctnd$-set of $G$.

$\gamma_{ctnd}(G) \leq |D'| = |V(G) - D'| = 4n - \left(\frac{5n-3}{2}\right) = \frac{3n+3}{2}$.

Hence $\gamma_{ctnd}(G) \leq \left\lfloor \frac{3n+3}{2} \right\rfloor = \left\lfloor \frac{3n+4}{2} \right\rfloor$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $<V(G) - D'>$ is not a dominating set, $D'$ contains a vertex $u$ such that $N(u) \subseteq D$. $u$ is taken to be a vertex of minimum degree $\delta(G) = 3$ in $G$. The blocks $A, B$ are constructed as in case 1.
Complementary tree nil domination number of Cartesian Product of Graphs

G is obtained by concatenating the blocks A and $B^{\frac{n-1}{2}}$ as in case 1. That is, 
$G \cong AB^{\frac{n-1}{2}}$. The vertices with the symbol $\bullet$ in each of the blocks represent the vertices 
that are to be included in $D'$. 

Therefore $D'$ contains 3 vertices from block A and atleast 3 vertices from each 
block B of $B^{\frac{n-1}{2}}$.

Therefore $Y_{ctn}(G) = |D'| \geq 3 + 3 \left(\frac{n-1}{2}\right) = \frac{3n+3}{2} = \frac{3n+4}{2}$.

Hence $Y_{ctn}(C_4 \times P_n) = \left[\frac{3n+4}{2}\right], n \geq 2$.

**Theorem 2.6:**

If $G \cong C_5 \times P_n$, then $Y_{ctn}(G) = 2n + 1, n \geq 3$.

**Proof:**

Let $G \cong C_5 \times P_n$ and $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}\}$ such that 
$<\{v_{1i}, v_{2i}, ..., v_{ni}\} \cong P_n, i = 1, 2, 3, 4, 5$ and $<\{v_{4j}, v_{5j}, v_{4j}, v_{5j}\} > \cong C_5$, 
j = 1, 2, ..., n, where $P_n$ is the $i$th copy of $P_n$ and $C_5$ is the $j$th copy of $C_5$ in $C_5 \times P_n$. 
$|V(G)| = 5n$.

**Case 1:** n is odd

Let $D = \{ v_{21}, v_{32} \} \cup \bigcup_{i=1}^{n+1} \{v_{1,2i-1}, v_{5,2i-1}\} \cup \bigcup_{i=2}^{n-1} \{v_{3,2i}, v_{4,2i}\}$. 
Then $|D| = 3 + 2 \left(\frac{n+1}{2}\right) + 2 \left(\frac{n-3}{2}\right) = 2n + 1$.

Consider the blocks
Then $G \cong AB^{n-2}C$. Let $D$ be the set of vertices with the symbol $\bigcirc$ in each of the blocks $A$, $B^{n-2}$ and $C$. $D$ contains 5 vertices from block $A$, and 4 vertices from each block $B$ of $B^{n-2}$ and 2 vertices from block $C$. Then $D$ is a dominating set of $G$ and the vertex $v_{11}$ is such that $N(v_{11}) \subseteq D$ and $<V(G) - D> \cong T$, where $T$ is a tree constructed as below.

Let $H$ be the graph obtained by subdividing each of the pendant edges of $P_{2n+2}$ exactly once and $T$ be the tree obtained from $H$ by attaching a pendant edge at one pendant vertex say $v$ of $P_{n+2}$ and then joining a vertex of degree 2 of $P_i$ by an edge to a pendant vertex at a distance 2 from $v$.

Therefore $D$ is a ctd-set of $G$.

$\gamma_{ctnd}(G) \leq |D'| = 2n + 1$.

Let $D'$ be a $\gamma_{ctnd}$ set of $G$. Since $\gamma(C_5) = 2$, $D'$ contains 2 vertices from each of $n$ cycles and $D'$ contains one more vertex from a cycle $C_5$ and hence $D'$ contains at least $2n+1$ vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 2n + 1$.

Hence $\gamma_{ctnd}(G) = 2n + 1, n \geq 2$.

**Case 2:** $n$ is even

Let $D = \{v_{11}, v_{12}, v_{21}, v_{32}, v_{51}\} \cup \{U_{i=2}^{n} \{v_{1,2i-1}, v_{3,2i}, v_{4,2i}, v_{5,2i-1}\}\}$. Then $|D| = 5 + 4 \left(\frac{n-2}{2}\right) = 2n + 1$.

$G$ is obtained by concatenating the blocks $A$, $B^{n-2}$. That is $G \cong AB^{n-2}$. Let $D$ be the set of vertices with the symbol $\bigcirc$ in each of the blocks $A$ and $B^{n-2}$. $D$ contains 5 vertices from block $A$, and 4 vertices from each block $B$ of $B^{n-2}$. Then $D$ is a dominating set of $G$ and the vertex $v_{11}$ is such that $N(v_{11}) \subseteq D$ and $<V(G) - D> \cong T$, where $T$ is a tree constructed as in case 1.

Therefore $D$ is a ctd-set of $G$ and $\gamma_{ctnd}(G) \leq |D| = 2n + 1$.

Let $D'$ be a $\gamma_{ctnd}$ set of $G$. Since $\gamma(C_5) = 2$, $D'$ contains 2 vertices from each of $n$ cycles and since $V(G) - D$ is not a dominating set of $G$, $D'$ contains one more vertex from a cycle $C_5$ and hence $D'$ contains at least $2n+1$ vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 2n + 1$.

Hence $\gamma_{ctnd}(G) = 2n + 1, n \geq 2$.

**Theorem 2.7:**

If $G \cong C_5 \times P_n$ then $\gamma_{ctnd}(G) = 5$. 
Proof:

Let $G \boxtimes C_6 \times P_n$ and $V(G) = \bigcup_{i=1}^n \{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}\}$ such that $\langle \{v_{1i}, v_{2i}\} \rangle \geq P_n^i$, $i = 1, 2, 3, 4, 5$ and $\langle \{v_{4i}, v_{5i}\} \rangle \geq C_5^j$, $j = 1, 2$, where $P_n^i$ is the $i$th copy of $P_n$ and $C_5^j$ is the $j$th copy of $C_5$ in $C_6 \times P_n$. Let $D = \{v_{11}, v_{21}, v_{31}, v_{41}, v_{12}\}$. Then $N(v_{11}) \subseteq D$ and $D$ is a dominating set of $G$. Also $V(G) - D = \{v_{31}, v_{22}, v_{33}, v_{44}, v_{55}\}$ and $\langle V(G) - D \rangle$ is a graph obtained from $P_3$ by attaching 2 pendant edges at a pendant vertex of $P_3$. Therefore $D$ is a ctnd-set of $G$.

$\gamma_{ctnd}(G) \leq |D| = 5$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. $D'$ contains 4 vertices from $C_5^1$ and at least one vertex from $C_5^2$.

Therefore $D'$ contains at least 5 vertices. $\gamma_{ctnd}(G) = |D'| \geq 5$.

Hence $\gamma_{ctnd}(G) = 5$.

**Theorem 2.8:**

If $G \boxtimes C_6 \times P_n$, then $\gamma_{ctnd}(G) = \left\lceil \frac{5n+1}{2} \right\rceil$, $n \geq 3$.

Proof:

Let $G \boxtimes C_6 \times P_n$ and $V(G) = \bigcup_{i=1}^n \{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i}\}$ such that $\langle \{v_{1i}, v_{2i}, \ldots, v_{ni}\} \rangle \geq P_n^i$, $i = 1, 2, 3, 4, 5, 6$ and $\langle \{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}, v_{6j}\} \rangle \geq C_6^j$, $j = 1, 2, \ldots, n$, where $P_n^i$ is the $i$th copy of $P_n$ and $C_6^j$ is the $j$th copy of $C_6$ in $C_6 \times P_n$ and $|V(G)| = 6n$.

Case 1: $n$ is odd.

Let $D' = \{v_{31}, v_{51}, v_{32}, v_{52}, v_{62}\} \cup \bigcup_{i=1}^{n-1} \{v_{1i+1, 2i+1, v_{5i+1, 2i+1, v_{6i+1}\}}\} \cup \bigcup_{i=2}^n \{v_{2i}, v_{4i}, v_{2i}, v_{4i}\}$.

Then $|D'| = 5 + 3 \left( \frac{n-1}{2} \right) + n - 1 + 2 \left( \frac{n-3}{2} \right) = \frac{7n-1}{2}$ and $D = \langle V(G) - D' \rangle$. Then $D$ is a dominating set of $G$ and $N(v_{2i}) \subseteq D$. Also $\langle V(G) - D \rangle$ is a tree obtained from a path $P_{n-1} = \langle v_{2i, 1} \rangle$, $i = 1, 2, \ldots, n$, $n \geq 2$ by attaching $P_i$ at each of the vertices $v_{2i}, v_{2i+1}, v_{2i+2}, \ldots, v_{2i+n-1}$. Therefore $D$ is a ctnd-set of $G$.

$\gamma_{ctnd}(G) \leq |D| = \langle V(G) - D' \rangle = 6n - \left( \frac{7n-1}{2} \right) = \frac{5n+1}{2}$.

Hence $\gamma_{ctnd}(G) \leq \frac{5n+1}{2}$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. Since $\langle V(G) - D' \rangle$ is not a dominating set. Therefore $D'$ contains a vertex of $u$ such that $N(u) \subseteq D$. $u$ is taken to be a vertex of minimum degree $\delta(G) = 3$ in $G$. The blocks $A$, $B$, $C$ are constructed as given below.
G is obtained by concatenating the blocks A, B \(\frac{n-3}{2}\) and C. That is, \(G \cong A \ B \frac{n-3}{2} \ C\). The vertices with the symbol \(\bigcirc\) in each of the blocks represent the vertices that are to be included in \(D\). Therefore \(D\) contains 6 vertices from block A and at least 5 vertices from each block B of \(B \frac{n-3}{2}\) and 2 vertices from block C. Therefore \(\gamma_{ctnd}(G) = |D| \geq 6 + 5\left(\frac{n-3}{2}\right) + 2 = \frac{5n+1}{2}\) and hence \(\gamma_{ctnd}(G) = \frac{5n+1}{2}\).

Case 2: \(n\) is even.

Let \(D = \{v_{31}, v_{41}, v_{51}, v_{32}, v_{62}\} \cup \bigcup_{1 \leq i \leq n-2} \{v_{1,2i+1}, v_{5,2i+1}, v_{6,2i+1}\}\).

Then \(|D| = 5 + 3\left(\frac{n-2}{2}\right) + n - 1 + 2\left(\frac{n-2}{2}\right) = \frac{7n-2}{2}\) and \(D = V(G) - D\). Then D is a dominating set of G and \(N(v_{i}) \subseteq D\). Also \(<V(G) - D > = \langle D\rangle\) is a tree obtained from a path \(P_{n-1} = \langle v_{2,1}, i = 2, 3, ..., n \rangle\). \((n \geq 2)\) by attaching \(P_{1}\) at each of the vertices \(v_{23}, v_{25}, v_{27}, ..., v_{2,n-1}\) and attaching \(P_{3}\) at each of the vertices \(v_{24}, v_{26}, ..., v_{2,n}\). Therefore D is a ctnd-set of G.

\(\gamma_{ctnd}(G) \leq |D| = |V(G) - D| = 6n - \left(\frac{7n-2}{2}\right) = \frac{5n+2}{2}\).

Hence \(\gamma_{ctnd}(G) \leq \frac{5n+2}{2}\).

Let \(D\) be a \(\gamma_{ctnd}\)-set of G. Since \(<V(G) - D > = \langle D\rangle\) is not a dominating set, \(D\) contains a vertex of \(u\) such that \(N(u) \subseteq D\). \(u\) is taken to be a vertex of minimum degree \(\delta(G) = 3\) in G. The blocks A, B are constructed as in case 1.

**Figure 5**
G is obtained by concatenating the blocks $A$ and $B^{\frac{n-2}{2}}$. That is, $G \cong AB^{\frac{n-2}{2}}$. The vertices with the symbol $\Theta$ in each of the blocks represent the vertices that are to be included in $D'$.

Therefore $D'$ contains 6 vertices from block $A$ and at least 5 vertices from each block $B$ of $B^{\frac{n-2}{2}}$. Therefore $\gamma_{ctnd}(G) = |D'| \geq 6 + 5\left(\frac{n-2}{2}\right) = \frac{5n+2}{2}$ and hence

$$\gamma_{ctnd}(G) = \frac{5n+2}{2}.$$  

Hence $\gamma_{ctnd}(C_6 \times P_n) = \frac{5n+1}{2}$, $n \geq 2$.

Theorem 2.9:
If $G \cong C_6 \times P_2$, then $\gamma_{ctnd}(G) = 5$.

Proof:

$G \cong C_6 \times P_n$ and $V(G) = \bigcup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i}\}$ such that $<v_{1i}, v_{2i}, \ldots, v_{16}> \cong P_n$, $i=1, 2, 3, 4, 5, 6$ and $<v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}, v_{6j}> \cong C_6^j$, $j=1, 2$, where $P_n$ is the $i$th copy of $P_n$ and $C_6^j$ is the $j$th copy of $C_6$ in $C_6 \times P_2$.

Let $D = \{v_{11}, v_{21}, v_{61}, v_{12}, v_{42}\}$. Then $N(v_{11}) \supseteq D$ and $D$ is a dominating set of $G$. Also $V(G) - D = \{v_{31}, v_{41}, v_{51}, v_{22}, v_{32}, v_{44}, v_{52}, v_{62}\}$ and $<V(G) - D> \cong P_7$.

Therefore $D$ is a ctnd-set of $G$, $\gamma_{ctnd}(G) = |D| = 5$.

Let $D'$ be a $\gamma_{ctnd}$-set of $G$. $D'$ contains 3 vertices from $C_6^1$ and at least 2 vertices from $C_6^2$.

Therefore $D'$ contains at least 5 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 5$.

Hence $\gamma_{ctnd}(G) = 5$.

Remark 2.2:
In view of Theorem 2.4, Theorem 2.5, Theorem 2.6, and Theorem 2.8,
1. $\gamma_{ctnd}(C_3 \times C_n) = n+3$, $n \geq 3$.
2. $\gamma_{ctnd}(C_4 \times C_n) = \left\lceil \frac{3n+6}{2} \right\rceil$, $n \geq 3$.
3. $\gamma_{ctnd}(C_5 \times C_n) = 2n+3$, $n \geq 3$.
4. $\gamma_{ctnd}(C_6 \times C_n) = 3n$, $n \geq 3$.

Remark 2.3:
1. If $G_1 \cong K_n$ and $G_2 \cong K_n$ then $\gamma_{ctnd}(G_1 + G_2) = m + n$.
2. If $G_1$ and $G_2$ are any two non-complete connected graphs of order $m$ and $n$ respectively, with minimum degree at least two, then $\gamma_{ctnd}(G_1 + G_2) \leq m + n - 1$.

Equality holds, if $G_1 \cong K_m - e$, $G_2 \cong K_n - e$. 

3. For any two connected graphs \( G_1 \) and \( G_2 \) of order \( m \) and \( n \) respectively,
\[
\gamma_{ctnd}(G_1 \Box G_2) \leq m + n - 1.
\]
Equality holds, if \( G_1 \cong P_2 \) and \( G_2 \cong nK_1 \).

4. For any two nontrivial connected graphs \( G_1 \) and \( G_2 \) with the of order \( m \) and \( n \)
respectively,
\[
\gamma_{ctnd}(G_1 \Box G_2) \leq m + n - 2.
\]
Equality holds, if \( G_1 \cong P_2 \) and \( G_2 \cong C_3 \).

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