Nominal String Diagrams

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Abstract
We introduce nominal string diagrams as string diagrams internal in the category of nominal sets. This requires us to take nominal sets as a monoidal category, not with the cartesian product, but with the separated product. To this end, we develop the beginnings of a theory of monoidal categories internal in a symmetric monoidal category. As an instance, we obtain a notion of a nominal PROP as a PROP internal in nominal sets. A 2-dimensional calculus of simultaneous substitutions is an application.

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1 Introduction

One reason for the success of string diagrams, see [18] for an overview, can be formulated by the slogan ‘only connectivity matters’ [3, Sec.10.1]. Technically, this is usually achieved by ordering input and output wires and using their ordinal numbers as implicit names. We write \( n = \{1, \ldots, n\} \) to denote the set of \( n \) numbered wires and \( f : n \to m \) for diagrams \( f \) with \( n \) inputs and \( m \) outputs. This approach is particularly convenient for the generalisations of Lawvere theories known as PROPs [13]. In particular, the paper on composing PROPs [11] has been influential [1, 2].

On the other hand, if only connectivity matters, it is natural to consider a formalisation of string diagrams in which wires are not ordered. Thus, instead of ordering wires, we fix a countably infinite set

\[ \mathcal{N} \]

of ‘names’ \( a, b, \ldots, \), on which the only supported operation or relation is equality. Mathematically, this means that we work internally in the category of nominal sets introduced by Gabbay and Pitts [7, 10]. In the remainder of the introduction, we highlight some of the features of this approach.

Partial commutative vs total symmetric tensor. One reason why ordered names are convenient is that the tensor \( \oplus \) is given by the categorical coproduct (addition) in the skeleton \( \mathbb{F} \) of the category of finite sets. Even though \( n \oplus m = m \oplus n \) on objects, the tensor is not commutative but only symmetric, since the canonical arrow \( n \oplus m \to m \oplus n \) is not the identity.
On the other hand, in the category \( n\mathcal{F} \) of finite subsets of \( \mathcal{N} \) (which is equivalent to \( \mathcal{F} \) as an ordinary category), there is a commutative tensor \( A \uplus B \) given by union of disjoint sets. The interesting feature that makes commutativity possible is that \( \uplus \) is partial with \( A \uplus B \) defined if and only if \( A \cap B = \emptyset \).

While it would be interesting to develop a general theory of partially monoidal categories, our approach in this paper is based on the observation that the partial operation \( \uplus : n\mathcal{F} \times n\mathcal{F} \to n\mathcal{F} \) is a total operation \( \uplus : n\mathcal{F} \ast n\mathcal{F} \to n\mathcal{F} \) where \( \ast \) is the separated product of nominal sets [16].

Symmetries disappear in 3 dimensions. From a graphical point of view, the move from ordered wires to named wires corresponds to moving from planar graphs to graphs in 3 dimensions. Instead of having a one dimensional line of inputs or outputs, wires are now sticking out of a plane [10]. As a benefit there are no wire-crossings, or, more technically, there are no symmetries to take care of. This simplifies the rewrite rules of calculi formulated in the named setting. For example, rules such as

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
\]

are not needed anymore. For more on this compare Figs 3 and 4.

Example: Simultaneous Substitutions. Substitutions \([a\mapsto b] \) can be composed sequentially and in parallel as in

\[
[a\mapsto b] ; [b\mapsto c] = [a\mapsto c] \quad [a\mapsto b] \uplus [c\mapsto d] = [a\mapsto b, c\mapsto d].
\]

We call \( \uplus \) the tensor, or the monoidal or vertical or parallel composition. Semantically, the simultaneous substitution on the right-hand side above, will correspond to the function \( f : \{a, c\} \to \{b, d\} \) satisfying \( f(a) = b \) and \( f(c) = d \). Importantly, parallel composition of simultaneous substitutions is partial. For example, \([a\mapsto b] \uplus [a\mapsto c] \) is undefined, since there is no function \( \{a\} \to \{b, c\} \) that maps \( a \) simultaneously to both \( b \) and \( c \).

The advantages of a 2-dimensional calculus for simultaneous substitutions over a 1-dimensional calculus are the following. A calculus of substitutions is an algebraic representation, up to isomorphism, of the category \( n\mathcal{F} \) of finite subsets of \( \mathcal{N} \). In a 1-dimensional calculus, operations \([a\mapsto b] \) have to be indexed by finite sets \( S \)

\[
[a\mapsto b]_S : S \cup \{a\} \to S \cup \{b\}
\]

for sets \( S \) with \( a, b \notin S \). On the other hand, in a 2-dimensional calculus with an explicit operation \( \uplus \) for set-union, indexing with subsets \( S \) is unnecessary. Moreover, while the swapping

\[
\{a, b\} \to \{a, b\}
\]

in the 1-dimensional calculus needs an auxiliary name such as \( c \) in \([a\mapsto c]_{\{b\}} : [b\mapsto a]_{\{c\}} : [c\mapsto a]_{\{b\}} \) it is represented in the 2-dimensional calculus directly by

\[
[a\mapsto b] \uplus [b\mapsto a]
\]

Finally, while it is possible to write down the equations and rewrite rules for the 1-dimensional calculus, it does not appear as particularly natural. In particular, only in the 2-dimensional
calculus, will the swapping have a simple normal form such as \([a\rightarrow b] \uplus [b\rightarrow a]\) (unique up to
commutativity of \(\uplus\)).

**Overview.** In order to account for partial tensors, Section 3 develops the notion of
a monoidal category internal in a symmetric monoidal category. Section 4 is devoted to
eamples, while Section 5 introduces the notion of a nominal prop and Section 6 shows that
the categories of ordinary and of nominal props are equivalent.

## 2 Setting the Scene: String Diagrams and Nominal Sets

We review some of the necessary terminology but need to refer to the literature for details.

### 2.1 String Diagrams

The mathematical theory of string diagrams can be formalised via PROPs as defined by
MacLane [14]. There is also the weaker notion by Lack [11], see Remark 2.9 of Zanasi [20]
for a discussion.

A PROP (products and permutation category) is a symmetric strict monoidal category, with
natural numbers as objects, where the monoidal tensor \(\oplus\) is addition. Moreover, PROPs, along
with strict symmetric monoidal functors, that are identities on objects, form the category
PROP. A PROP contains all bijections between numbers as they can be be generated from
the symmetry (twist) \(1 \oplus 1 \rightarrow 1 \oplus 1\) and from the parallel composition \(\oplus\) and sequential
composition \(;\) (which we write in diagrammatic order).

PROPs can be presented in algebraic form by operations and equations as symmetric monoidal
theories (SMTs) [20].

An SMT \((\Sigma, E)\) has a set \(\Sigma\) of generators, where each generator \(\gamma \in \Sigma\) is given an arity \(m\)
and co-arity \(n\), usually written as \(\gamma: m \rightarrow n\) and a set \(E\) of equations, which are pairs of
\(\Sigma\)-terms. \(\Sigma\)-terms can be obtained by composing generators in \(\Sigma\) with the unit \(id: 1 \rightarrow 1\)
and symmetry \(\sigma : 2 \rightarrow 2\), using either the parallel or sequential composition (see Fig 1).
Equations \(E\) are pairs of \(\Sigma\)-terms with the same arity and co-arity.

**Figure 1** SMT Terms

Given an SMT, we can freely generate a PROP, by taking \(\Sigma\)-terms as arrows, modulo
the equations of Fig 2, together with the smallest congruence (with respect to the two
compositions) of equations in \(E\).
\[ id_m \circ t = t = t \circ id_n \]
\[ (t \circ s) \circ r = t \circ (s \circ r) \]
\[ \sigma_{1,1} \circ \sigma_{1,1} = id_2 \]
\[ (s \circ t) \circ (u \circ v) = (s \circ u) \circ (t \circ v) \]
\[ (t \circ id_z) \circ \sigma_{n,z} = \sigma_{m,z} \circ (id_z \circ t) \]

**Figure 2** Equations of symmetric monoidal categories

**Figure 3** Symmetric monoidal theories

PROPs admit a nice graphical presentation, wherein the sequential composition is modeled by horizontal composition of diagrams, and parallel/tensor composition is vertical stacking of diagrams (see Fig 1). We now present the SMTs of bijections $\mathbb{B}$, injections $\mathbb{I}$, surjections $\mathbb{S}$, functions $\mathbb{F}$, partial functions $\mathbb{P}$, relations $\mathbb{R}$ and monotone maps $\mathbb{M}$. The diagram in Fig 3 shows the generators and the equations that need to be added to the empty SMT, to get a presentation of the given theory. To ease comparison with the corresponding nominal monoidal theories in Fig 4 later we also added on a striped background the equations for wire-crossings that are already implied by the naturality of symmetries, that is, the last equation of Fig 2. These are the equations that are part of the definition of a prop in the sense of MacLane [14] but need to be added explicitly to the props in the sense of Lack [11].

1 The theory of monotone maps $\mathbb{M}$ does not include equations involving the symmetry $\sigma$ and is in fact presented by a so-called PRO rather than a PROP. However, in this paper we will only be dealing with theories presented by PROPs (the reason why this is the case is illustrated in the proof of Proposition 22).
2.2 Nominal Sets

Let $\mathcal{N}$ be a countably infinite set of ‘names’ or ‘atoms’. Let $\mathfrak{S}$ be the group of finite permutations $\mathcal{N} \to \mathcal{N}$. An element $x \in X$ of a group action $\mathfrak{S} \times X \to X$ is supported by $S \subseteq \mathcal{N}$ if $\pi \cdot x = x$ for all $\pi \in \mathfrak{S}$ such that $\pi$ restricted to $S$ is the identity. A group action $\mathfrak{S} \times X \to X$ such that all elements of $X$ have finite support is called a nominal set. We write $\text{supp}(x)$ for the minimal support of $x$ and $\text{Nom}$ for the category of nominal sets, which has as maps the equivariant functions, that is, those functions that respect the permutation action. Our main example is the category of simultaneous substitutions:

► Example 1 (nF). We denote by $\text{nF}$ the category of finite subsets of $\mathcal{N}$ with all functions. While $\text{nF}$ is a category, it also carries additional nominal structure. In particular, both the set of objects and the set of arrows are nominal sets. with $\text{supp}(A) = A$ and $\text{supp}(f) = A \cup B$ for $f : A \to B$. The categories of injections, surjections, bijections, partial functions and relations are further examples along the same lines.

3 Internal monoidal categories

We introduce the notion of an internal monoidal category. Given a symmetric monoidal category $(\mathcal{V}, I, \otimes)$ with finite limits, we are interested in categories $\mathcal{C}$, internal in $\mathcal{V}$, that carry a monoidal structure not of type $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ but of type $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$. This will allow us to account for the partiality of $\uplus$ discussed in the introduction:

► Example 2. The symmetric monoidal (closed) category $(\text{Nom}, 1, \ast)$ of nominal sets with the separated product $\ast$ is defined as follows [10]. $1$ is the terminal object, i.e., a singleton with empty support. The separated product of two nominal sets is defined as $A \ast B = \{(a, b) \in A \times B \mid \text{supp}(a) \cap \text{supp}(b) = \emptyset\}$.

The category $\text{nF}$ (and its relatives) of Example 1 is an internal monoidal category with monoidal operation given by $A \uplus B = A \cup B$ if $A$ and $B$ are disjoint.

$(\text{nF}, \emptyset, \uplus)$ as defined in the previous example is not a monoidal category, since $\uplus$, being partial, is not an operation of type $\text{nF} \times \text{nF} \to \text{nF}$. The purpose of this section is to show that $(\text{nF}, \emptyset, \uplus)$ is an internal monoidal category in $(\text{Nom}, 1, \ast)$ with $\uplus$ of type

$$\uplus : \text{nF} \times \text{nF} \to \text{nF}.$$ 

To this end we need to extend $\ast : \text{Nom} \times \text{Nom} \to \text{Nom}$ to

$$\ast : \text{Cat(Nom)} \times \text{Cat(Nom)} \to \text{Cat(Nom)}$$

where we denote by $\text{Cat(Nom)}$, the category of (small) internal categories in $\text{Nom}$.

The necessary (and standard) notation from internal categories is reviewed in Appendix A.

► Remark 3. Let $\mathcal{C}$ be an internal category in a symmetric monoidal category $(\mathcal{V}, I, \otimes)$ with finite limits. Since $\otimes$ need not preserve finite limits, we cannot expect that defining $(\mathcal{C} \otimes \mathcal{C})_0 = \mathcal{C}_0 \otimes \mathcal{C}_0$ and $(\mathcal{C} \otimes \mathcal{C})_1 = \mathcal{C}_1 \otimes \mathcal{C}_1$ results in $\mathcal{C} \otimes \mathcal{C}$ being an internal category. Consequently, putting $(\mathcal{C} \otimes \mathcal{C})_1 = \mathcal{C}_1 \otimes \mathcal{C}_1$ does not extend $\otimes$ to an operation $\text{Cat(V)} \times \text{Cat(V)} \to \text{Cat(V)}$. To show what goes wrong in a concrete instance is the purpose of the next example.

2 A permutation is called finite if it is generated by finitely many transpositions.
Example 4. Define a binary operation \( nF \ast nF \) as \((nF \ast nF)_0 = nF_0 \ast nF_0 \) and \((nF \ast nF)_1 = nF_1 \ast nF_1 \). Then \( nF \ast nF \) cannot be equipped with the structure of an internal category. Indeed, assume for a contradiction that there was an appropriate pullback \((nF \ast nF)_2\) and arrow \(\text{comp} \) such that the two diagrams commute:

\[
\begin{array}{ccc}
(nF \ast nF)_2 & \xrightarrow{\text{comp}} & nF_1 \ast nF_1 \\
\pi_1 & & \pi_2 \downarrow \downarrow \downarrow \downarrow \\
nF_1 \ast nF_1 & \xrightarrow{\text{dom}} & nF_0 \ast nF_0
\end{array}
\]

Let \( \delta_{xy} : \{x\} \rightarrow \{y\} \) be the unique function in \( nF \) of type \( \{x\} \rightarrow \{y\} \), which can be depicted as

\[
\begin{array}{ccc}
\{a\} & \xrightarrow{\delta_{ac}} & \{c\} \\
\{b\} & \xrightarrow{\delta_{bd}} & \{d\} \xrightarrow{\delta_{da}} \{a\}
\end{array}
\]

is in the pullback \((nF \ast nF)_2\), but there is no \(\text{comp} \) such that the two squares above commute, since \(\text{comp}(\delta_{ac}, \delta_{bd}), (\delta_{cb}, \delta_{da})) \) would have to be \((\delta_{ab}, \delta_{ba})\), which do not have disjoint support and therefore are not in \( nF_1 \ast nF_1 \).

The solution to the problem consists in assuming that the given symmetric monoidal category with finite limits \((V, 1, \otimes)\) is semi-cartesian (aka affine), that is, the unit \(1\) is the terminal object. In such a category there are canonical

\[
j : A \otimes B \rightarrow A \times B
\]

and we can use them to define arrows \(j_1 : (C \otimes C)_1 \rightarrow C_1 \times C_1\) that give us the right notion of tensor on arrows. From our example \(nF\) above, we know that we want arrows \((f, g)\) to be in \((C \otimes C)_1\) if \(\text{dom}(f) \cap \text{dom}(g) = \emptyset\) and \(\text{cod}(f) \cap \text{cod}(g) = \emptyset\). We now turn this observation into a category theoretic definition.

Let \(C\) and \(D\) be internal categories in \(V\). Our first task is to define \((C \otimes D)_1\). This is accomplished by stipulating that \((C \otimes D)_1\) is the limit in the diagram below

\[
\begin{array}{ccc}
(C \otimes D)_1 & \xrightarrow{j_1} & C_1 \times D_1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
C_0 \otimes D_0 & \xrightarrow{j} & C_0 \times D_0
\end{array}
\]

In the following we abbreviate the diagram above to

\[
\begin{array}{ccc}
(C \otimes D)_1 & \xrightarrow{j_1} & C_1 \times D_1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
(C \otimes D)_0 & \xrightarrow{j} & C_0 \times D_0
\end{array}
\]
We are now in the position to extend the monoidal operation \( \otimes : V \times V \to V \) to a monoidal operation \( \otimes : \text{Cat}(V) \times \text{Cat}(V) \to \text{Cat}(V) \).

\[ \text{Definition 5.} \]

Let \((V, 1, \otimes)\) be a monoidal category where the unit is the terminal object. The operation \(\otimes : \text{Cat}(V) \times \text{Cat}(V) \to \text{Cat}(V)\) is defined as follows.

- \((C \otimes D)_0\) and \((C \otimes D)_1\) and \(\text{cod}, \text{dom} : (C \otimes D)_1 \to (C \otimes D)_0\) as in the diagram above.
- \(i : (C \otimes D)_0 \to (C \otimes D)_1\) is the arrow into the limit \((C \otimes D)_1\) given by

\[
\begin{array}{c}
(C \otimes D)_0 \\
\downarrow \id \quad \downarrow \id \\
(C \otimes D)_1 \\
\downarrow \text{dom} \quad \downarrow \text{cod} \\
(C \otimes D)_0 \\
\end{array}
\]

from which one reads off

\[\text{dom} \circ i = \id_{(C \otimes D)_0} = \text{cod} \circ i\]

- \((C \otimes D)_2\) is the pullback

\[
\begin{array}{c}
(C \otimes D)_2 \\
\downarrow \pi_1 \quad \downarrow \pi_2 \\
(C \otimes D)_1 \\
\downarrow \text{dom} \quad \downarrow \text{cod} \\
(C \otimes D)_0 \\
\end{array}
\]

Recalling the definition of \(j_1\) from (1), there is also a corresponding \(j_2 : (C \otimes D)_2 \to C_2 \times D_2\) due to the fact that the product of pullbacks is a pullback of products

\[
\begin{array}{c}
(C \otimes D)_2 \\
\downarrow \pi_1 \quad \downarrow \pi_2 \\
(C \otimes D)_1 \\
\downarrow \text{dom} \quad \downarrow \text{cod} \\
(C \otimes D)_0 \\
\end{array}
\]

Recall the definition of the limit \((C \otimes D)_1\) from (1). Then \(\text{comp} : (C \otimes D)_2 \to (C \otimes D)_1\) is the arrow into \((C \otimes D)_1\)

\[
\begin{array}{c}
(C \otimes D)_2 \\
\downarrow \text{comp} \quad \downarrow (\text{comp} \times \text{comp}) \circ j_2 \\
(C \otimes D)_1 \\
\downarrow \text{dom} \quad \downarrow \text{cod} \circ \pi_1 \quad \downarrow \text{cod} \circ \pi_2 \\
(C \otimes D)_0 \\
\end{array}
\]
from which one reads off

\[
\text{dom} \circ \text{comp} = \text{dom} \circ \pi_1 \quad \text{cod} \circ \text{comp} = \text{cod} \circ \pi_2
\]

- The equations \(\text{comp} \circ (i \circ \text{dom}, id_{(C \otimes D)_1}) = id_{(C \otimes D)_1} = \text{comp} \circ (id_{(C \otimes D)_1}, i \circ \text{cod})\) are proved in Proposition 7.
- The equation \(\text{comp} \circ \text{compl} = \text{comp} \circ \text{compr}\) will be shown in Proposition 8.

This ends the definition of \(C \otimes D\) and the next few pages are devoted to showing that it is indeed an internal category. To prove the next propositions, we will need the following lemma, which can be skipped for now. It is a consequence of the general fact that the isomorphism \([I, C](K_A, D) \cong C(A, \text{lim } D)\) defining limits is natural in \(A\) and \(D\).

**Lemma 6.** If in the diagram

![Diagram](https://via.placeholder.com/150)

\(f\) and \(f'\) are cones commuting with \(j_1\) and \(k\), that is, if

\[
\text{cod} \circ f_1 = \text{dom} \circ f_2 \\
\text{(cod} \times \text{cod}) \circ f'_1 = (\text{dom} \times \text{dom}) \circ f'_2 \\
j_1 \circ f_i = f'_i \circ k
\]

and \(h, h'\) are the respective unique arrows into the pullbacks, then also

\[
h' \circ k = j_2 \circ h
\]

holds.

Using the lemma, the next two propositions have reasonably straightforward proofs.

**Proposition 7.** \(\text{comp} \circ (i \circ \text{dom}, id_{(C \otimes D)_1}) = id_{(C \otimes D)_1} = \text{comp} \circ (id_{(C \otimes D)_1}, i \circ \text{cod})\).

**Proposition 8.** \(\text{comp} \circ \text{compl} = \text{comp} \circ \text{compr}\)

This finishes the verification that \(C \otimes D\) is an internal category. We next show that \(1\) carries the structure of an internal monoidal category.

**Proposition 9.** Let \((\mathcal{V}, 1, \otimes)\) be a monoidal category where the unit is the terminal object. \(1\) carries the structure of an internal monoidal category \(1\) which is the neutral element wrt to the internal tensor \(\otimes: \text{Cat}(\mathcal{V}) \times \text{Cat}(\mathcal{V}) \to \text{Cat}(\mathcal{V})\) of Definition 5.

The next step is to show that the \(\otimes: \text{Cat}(\mathcal{V}) \times \text{Cat}(\mathcal{V}) \to \text{Cat}(\mathcal{V})\) of Definition 5 can be extended to a functor.
Proposition 10. Let $(V, 1, \otimes)$ be a monoidal category with finite limits where the unit is the terminal object. The internal tensor $\otimes : \text{Cat}(V) \times \text{Cat}(V) \to \text{Cat}(V)$ of Definition 5 is functorial.

The main result of the section is

Theorem 11. Let $(V, 1, \otimes)$ be a (symmetric) monoidal category with finite limits where the unit is the terminal object and $\otimes : \text{Cat}(V) \times \text{Cat}(V) \to \text{Cat}(V)$ the internal tensor of Definition 5. Then $(\text{Cat}(V), 1, \otimes)$ is a (symmetric) monoidal category.

Finally, internal strict monoidal categories organise themselves in a (2-)category.

Definition 12. We denote by $\text{Mon}(\text{Cat}(V), 1, \otimes))$, or briefly, $\text{Mon}(\text{Cat}(V))$, the category of monoids in $(\text{Cat}(V), 1, \otimes))$.

Theorem 13. $\text{Mon}(\text{Cat}(V), 1, \otimes))$ is a 2-category.

4 Examples

Before we give a formal definition of nominal PROPs and nominal monoidal theories (NMTs) in the next section, we present as examples those NMTs that correspond to the SMTs of Fig 3. The nominal monoidal theories of Fig 4 should be immediately recognizable, indeed the significant differences are that wires now carry labels and there is a new generator which allows us to change the label of a wire.

Theorem 14. The calculi of Fig 4 are complete.
The proof of the theorem shows that the categories presented by Fig 4 are isomorphic to the categories of finite sets with the respective maps. These proofs seem easier for NMTs than the corresponding proofs for SMTs (see eg Lafont [12]) because NMTs have no wire crossings. For example, in the case of bijections, it is immediate that every nominal diagram rewrites to a normal form, which is a parallel composition of diagrams of the form \[ \begin{array}{c} a \\ \downarrow \\ b \end{array} \]. Completeness then follows, as usual, from the possibility to rewrite every diagram into normal form. The other cases are only slightly more complicated.

## 5 Nominal monoidal theories and nominal PROPs

In this section, we introduce nominal PROPs as internal monoidal categories in nominal sets. We first spell out the details of what that means in elementary terms and then discuss the notion of diagrammatic alpha-equivalence.

### 5.1 Nominal monoidal theories

A nominal monoidal theory \((\Sigma, E)\) is given by a nominal set \(\Sigma\) of generators and a nominal set \(E\) of equations. A generator \(\gamma : A \rightarrow B\) has finite sets \(A, B\) of names as types and \(\Sigma\) is closed under permutations \(\pi \cdot \gamma : \pi \cdot A \rightarrow \pi \cdot B\). The set of terms is given by closing under the operations of Fig 5, which should be compared with Fig 1.

Every NMT freely generates a monoidal category internal in nominal sets by quotienting the generated terms by the equations \(E\) as well as by equations describing that terms form a monoidal category and a nominal set. The equations of an internal monoidal category are given in Fig 6. The main difference with the equations in Fig 2 is that the interchange law for \(\uplus\) is required to hold only if both sides are defined and that the two laws involving symmetries are replaced by the commutativity of \(\uplus\).

```
\[
\begin{align*}
\overline{\gamma} : A \rightarrow B & \in \Sigma \\
\text{id}_a : \{a\} & \rightarrow \{a\} \\
\delta_{ab} : \{a\} & \rightarrow \{b\} \\
t : A & \rightarrow B \\
t' : A' & \rightarrow B' \\
t \uplus t' : A \uplus A' & \rightarrow B \uplus B' \\
t : B & \rightarrow C \\
t ; s : A & \rightarrow C \\
(a \ b) t : (a \ b) \cdot A & \rightarrow (a \ b) \cdot B
\end{align*}
\]
```

**Figure 5** NMT Terms

For terms to form a nominal set, we need the usual equations between permutations (not listed here) to hold, as well as the equations of Fig 7 that specify how permutations act on

```
\[
\begin{align*}
id_A : t = t = t ; id_B \\
(t ; s) ; r = t ; (s ; r) \\
t \uplus s = s \uplus t \\
(t \uplus s) \uplus r = t \uplus (s \uplus r) \\
(s ; t) \uplus (u ; v) = (s \uplus u) ; (t \uplus v)
\end{align*}
\]
```

**Figure 6** NMT Equations of internal monoidal categories
The equations of Fig 7 introduce a notion of diagrammatic alpha-equivalence, which allows us to rename ‘internal’ names and to contract renamings.

**Definition 15.** Two terms of a nominal monoidal theory are alpha-equivalent if their equality follows from the equations in Fig 7.

**Notation:** Every permutation $\pi$ of names gives rise to bijective functions $\pi_A : A \to \pi[A] = \{\pi(a) \mid a \in A\} = \pi \cdot A$. Any such $\pi_A$, as well as the inverse $\pi_A^{-1}$, are parallel compositions of $\delta_{ab}$ for suitable $a, b \in N$. In fact, we have $\pi_A = \bigcup_{a \in A} \delta_{\pi(a)}$. We may therefore use the $\pi_A$ as abbreviations in terms.

**Proposition 16.** Let $t : A \to B$ be a term of a nominal monoidal theory. The equations in Fig 7 entail that $\pi \cdot t = (\pi_A)^{-1} \cdot t \cdot \pi_B$.

**Corollary 17.** Let $t : A \sqcup \{c\} \to B \sqcup \{c\}$ be a term of a nominal monoidal theory and $d \# t$. Then $t = (\delta_{cd} \sqcup \text{id}_A) \cdot (c \cdot d) \cdot t \cdot (\delta_{dc} \sqcup \text{id}_B)$.

**Corollary 18.** Let $t : A \to B$ be a term of a nominal monoidal theory. Modulo the equations of Fig 7, the support of $t$ is $A \cup B$.

The last corollary shows that internal names are bound by sequential composition. Indeed, in a composition $A \xrightarrow{t} C \xrightarrow{s} B$, the names in $C \setminus (A \cup B)$ do not appear in the support of $t \cdot s$.

### 5.3 Nominal PROPs

From the point of view of Section 5.2, a nominal PROP is an internal strict monoidal category in $(\text{Nom}, 1, \ast)$ that has finite sets of names as objects and at least all bijections as arrows. We spell this out in detail.
A nominal prop has a nominal set of objects and a nominal set of arrows. To define translations between ordinary and nominal monoidal theories we introduce some auxiliary notation. We denote lists that contain each letter at most once by bold letters. If \( \mathbf{a} = [a_1, \ldots, a_n] \) is a list, then \( \mathbf{a} = \{a_1, \ldots, a_n\} \). Given lists \( \mathbf{a} \) and \( \mathbf{a}' \) with \( \mathbf{a} = \mathbf{a}' \), we abbreviate bijections in PROP (also called symmetries) mapping \( i \mapsto a_i = a'_j \mapsto j \) as \( \langle \mathbf{a}\rangle \langle\mathbf{a}'\rangle \). Given lists \( \mathbf{a} \) and \( \mathbf{b} \) of the same length we write \( [\mathbf{a}][\mathbf{b}] = \big\| \delta_{a_i,b_j} \big\| \) for the bijection \( a_i \mapsto b_i \) in an nPROP.

**Proposition 21.** For any PROP \( S \), there is an nPROP

\[
NOM(S)
\]

that has for all arrows \( f : \mathbf{a} \to \mathbf{b} \) of \( S \), and for all lists \( \mathbf{a} = [a_1, \ldots, a_n] \) and \( \mathbf{b} = [b_1, \ldots, b_m] \) arrows \( [\mathbf{a}]f([\mathbf{b}]) \). These arrows are subject to equations

\[
\begin{align*}
[a]f: g(c) &= [a]f([b]): [b]g(c) \\
[a + c]f \oplus g(b + d) &= [a]f([b]) \oplus [c]g([d]) \\
[a]id([b]) &= [a][b] \\
[a] \langle b \rangle \langle b' \rangle; f(c) &= [a][b]; [b']f(c) \\
[a]f([b \langle b' \rangle]) ([c]) &= [a]f([b]); [b']f([c])
\end{align*}
\]

**Proof.** To show that \( NOM(S) \) is well-defined, we need to check that the equations of \( S \) are respected. We only have space here for the most interesting case which is the naturality of
symmetries given by the last equation in Fig. 2. We write \( a^m \) for a list of \( a \)'s of length \( m \).

\[
[a^m + a^\omega] \ (t \oplus id_z) : \sigma_{n,z} \langle b^z + b^n \rangle = ([a^m] \ t \langle x^n \rangle \cup [a^\omega] \ i_d \langle x^z \rangle) : [x^n + x^z] \ \sigma_{n,z} \langle b^z + b^n \rangle
\]

\[
= ([a^\omega] \ i_d \langle x^z \rangle \cup [a^m] \ t \langle x^n \rangle) : [x^n + x^z] \ \sigma_{n,z} \langle b^z + b^n \rangle
\]

\[
= [a^\omega + a^m] \ i_d \langle x^z \rangle \oplus t \langle x^n + x^z \rangle : [x^n + x^z] \ \sigma_{n,z} \langle b^z + b^n \rangle
\]

\[
= [a^\omega + a^m] \ i_d \langle x^z \rangle \oplus t \langle x^n + x^z \rangle : [x^n + x^z] \ \langle x^n + x^z \rangle \ \langle b^z + b^n \rangle
\]

\[
= [a^\omega + a^m] \ i_d \langle x^z \rangle \oplus t \langle x^n + x^z \rangle : [x^n + x^z] \ \langle b^z + b^n \rangle
\]

\[
= [a^\omega + a^m] \ i_d \langle x^z \rangle \oplus t \langle b^z + b^n \rangle
\]

\[
= [a^m + a^\omega] \ [a^m + a^\omega] \ i_d \langle x^z \rangle \oplus t \langle b^z + b^n \rangle
\]

\[
= [a^m + a^\omega] \ [a^m + a^\omega] \ i_d \langle x^z \rangle \oplus t \langle b^z + b^n \rangle
\]

\[
= [a^m + a^\omega] \ [a^m + a^\omega] \ i_d \langle x^z \rangle \oplus t \langle b^z + b^n \rangle
\]

Note how commutativity of \( \oplus \) is used to show that naturality of symmetries is respected. ▶

**Proposition 22.** For any nPROP \( T \) there is a PROP

\( ORD(T) \)

that has for all arrows \( f : A \to B \) of \( T \), and for all lists \( a = [a_1, \ldots, a_n] \) and \( b = [b_1, \ldots, b_m] \) arrows \( \langle a \rangle f \langle b \rangle \). These arrows are subject to equations

\[
\langle a \rangle f : g \langle c \rangle = (a) f [b] ; (b) g [c]
\]

\[
\langle a \rangle + \langle a \rangle f \ \cup \ g \ [b] + [b] = (a) f \ [b] \ \oplus \ (a) g \ [b]
\]

\[
\langle a \rangle \ i_d \langle a \rangle = id
\]

\[
\langle a \rangle \ [a'] \ [b] : f \ [c] = a) \ [a'] \ [b] \ [f] \ [c]
\]

\[
\langle a \rangle f \ [b] \ [c'] = (a) f \ [b] \ [c']
\]

**Proof.** To show that \( ORD \) is well defined we need to show that the equations of an NMT are respected. The most interesting case here is the commutativity of \( \cup \) since the \( \oplus \) of SMTs is not commutative.

\[
\langle a_i + a_s \rangle t \cup s \ [b_i + b_s] = (a_i) t \ [b_i] \ \oplus \ (a_s) s \ [b_s]
\]

\[
= (\langle a_i \rangle t \ [b_i] ; \ i_d [b_i] \ \oplus \ (a_s) s \ [b_s])
\]

\[
= (\langle a_i \rangle t \ [b_i] \ \cup \ i_d [b_i] \ \oplus \ (a_s) s \ [b_s])
\]

\[
= (\langle a_i \rangle t \ [b_i] \ \cup \ i_d [b_i] \ \cup \ (a_s) s \ [b_s])
\]

\[
= (\langle a_i \rangle t \ [b_i] \ \cup \ i_d [b_i] \ \cup \ (a_s) s \ [b_s])
\]

\[
= (\langle a_i \rangle t \ [b_i] \ \cup \ i_d [b_i] \ \cup \ (a_s) s \ [b_s])
\]

\[
= (\langle a_i \rangle t \ [b_i] \ \cup \ i_d [b_i] \ \cup \ (a_s) s \ [b_s])
\]

\[
= (\langle a_i \rangle t \ [b_i] \ \cup \ i_d [b_i] \ \cup \ (a_s) s \ [b_s])
\]

\[
= (\langle a_i \rangle t \ [b_i] \ \cup \ i_d [b_i] \ \cup \ (a_s) s \ [b_s])
\]

Note how naturality of symmetries is used to show that the definition of \( ORD \) respects commutativity of \( \cup \). ▶
Remark 23. The following equations can be obtained from the ones above:

\[
\begin{align*}
[a] f; ⟨b|b⟩ & ; g(c) = [a] f ⟨b⟩; [b] g(c) \\
[a] ⟨b|b⟩ & ; c = [a]⟨b|b⟩ & ; c \\
⟨a⟩ f; ⟨b|c⟩; g[d] & = (⟨a⟩ f; ⟨c⟩ g[d]) \\
⟨a⟩ f; ⟨b|c⟩ & ; g & = (⟨a⟩ f; ⟨c⟩ g[d]) \\
⟨a⟩ f; ⟨b|c⟩ & ; g & = (⟨a⟩ f; ⟨c⟩ g[d]) \\
\end{align*}
\]

Proposition 24. \( NOM : \text{PROP} \to \text{nPROP} \) is a functor mapping an arrow of PROPs \( F : S \to S \) to an arrow of nPROPs \( NOM(F) : NOM(S) \to NOM(S) \) defined by

\[
NOM(F)([a] g ⟨b⟩) = [a] Fg ⟨b⟩.
\]

Proposition 25. \( ORD \) is a functor mapping an arrow of nPROPs \( F : T \to T \) to an arrow of PROPs \( ORD(F) : ORD(T) \to ORD(T) \) defined by

\[
ORD(F)(⟨a⟩ f ⟨b⟩) = ⟨a⟩ Ff ⟨b⟩.
\]

Proposition 26. For each PROP \( S \), there is an isomorphism of PROPs, natural in \( S \),

\[
S \to ORD(NOM(S))
\]

mapping \( f \in S \) to \( ⟨a⟩ [a] f ⟨b|b⟩ \) for some choice of \( a, b \).

Proposition 27. For each nPROP \( T \), there is an isomorphism of nPROPs, natural in \( T \),

\[
NOM(ORD(T)) \to T
\]

mapping the \( [c] ⟨a⟩ f ⟨b|d⟩ \) generated by an \( f : a \to b \) in \( T \) to \( [c] [a] f ; [b|d] \).

Since the last two propositions provide an isomorphic unit and counit of an adjunction, we obtain

Theorem 28. The categories \( \text{PROP} \) and \( \text{nPROP} \) are equivalent.

Remark 29. If we generalise the notion of prop from MacLane [14] to Lack [11], in other words, if we drop the last equation of Fig 2 expressing the naturality of symmetries, we still obtain an adjunction, in which \( NOM \) is left-adjoint to \( ORD \). Nominal props then are a full reflective subcategory of ordinary props. In other words, the (generalised) props \( S \) that satisfy naturality of symmetries are exactly those for which \( S \cong ORD(NOM(S)) \).

7 Conclusion

The equivalence of nominal and ordinary props (Theorem 28) has a satisfactory graphical interpretation. Indeed, comparing Figs 3 and 4 we see that both share, modulo different labellings of wires mediated by the functors \( ORD \) and \( NOM \), the same core of generators and equations while the difference lies only in the equations expressing, on the one hand, that \( \oplus \) has natural symmetries and, on the other hand, that generators are a nominal set.

There are several directions for future research. First, the notion of an internal monoidal category has been developed because it is easier to prove the basic results in general rather
than only in the special case of nominal sets. Nevertheless, it would be interesting to explore whether there are other interesting instances of internal monoidal categories.

Second, internal monoidal categories are a principled way to build monoidal categories with a partial tensor. For example, by working internally in the category of nominal sets with the separated product we can capture in a natural way constraints such as the tensor $f \oplus g$ for two partial maps $f, g : N \to V$ being defined only if the domains of $f$ and $g$ are disjoint. This reminds us of the work initiated by O’Hearn and Pym on categorical and algebraic models for separation logic and other resource logics, see eg [15, 8, 6]. It seems promising to investigate how to build categorical models for resource logics based on internal monoidal theories. In one direction, one could extend the work of Curien and Mimram [4] to partial monoidal categories.

Third, there has been substantial progress in exploiting Lack’s work on composing PROPs [11] in order to develop novel string diagrammatic calculi for a wide range of applications, see eg [1, 2]. It will be interesting to explore how much of this technology can be transferred from props to nominal props.

Fourth, various applications of nominal string diagrams could be of interest. The original motivation for our work was to obtain a convenient calculus for simultaneous substitutions that can be integrated with multi-type display calculi [6] and, in particular, with the multi-type display calculus for first-order logic of Tzimoulis [19]. Another direction for applications comes from the work of Ghica and Lopez [9] on a nominal syntax for string diagrams. In particular, it would of interest to add various binding operations to nominal props.

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Some internal category theory

See eg Borceux, Handbook of Categorical Algebra, Volume 1, Chapter 8 and the nlab.

Remark 30 (internal category). In a category with finite limits an internal category is a diagram

\[
\begin{array}{c}
A_2 \xrightarrow{\text{comp}} A_1 \\
\downarrow^{\text{dom}} \quad \downarrow^i \quad \downarrow_{\text{cod}} \\
A_0
\end{array}
\]

where

1. \(A_2\) is a pullback

2. \(\text{dom} \circ \text{comp} = \text{dom} \circ \pi_1\) and \(\text{cod} \circ \text{comp} = \text{cod} \circ \pi_2\),

3. \(\text{dom} \circ i = \text{id}_{A_0} = \text{cod} \circ i\),

4. \(\text{comp} \circ (i \circ \text{dom}, \text{id}_{A_1}) = \text{id}_{A_1} = \text{comp} \circ (\text{id}_{A_1}, i \circ \text{cod})\)

5. \(\text{comp} \circ \text{compl} = \text{comp} \circ \text{compr}\)

where

- \((i \circ \text{dom}, \text{id}_{A_1}) : A_1 \to A_2\) and \((\text{id}_{A_1}, i \circ \text{cod}) : A_1 \to A_2\) are the arrows into the pullback

- \(A_2\) pairing \(i \circ \text{dom}, \text{id}_{A_1} : A_1 \to A_1\) and \(\text{id}_{A_1}, i \circ \text{cod} : A_1 \to A_1\), respectively.

- the “triple of arrows”-object \(A_3\) is the pullback

\[
\begin{array}{c}
A_3 \\
\downarrow^{\text{right}} \quad \downarrow^\pi_1 \\
A_2 \xrightarrow{\pi_2} A_1
\end{array}
\]

where left “projects out the left two arrows” and right “projects out the right two arrows”

- \(\text{compl}\) is the arrow composing the “left two arrows”

\[
\begin{array}{c}
A_3 \\
\downarrow^\pi_2 \circ \text{right} \\
A_2 \\
\downarrow^{\text{left}} \\
A_1 \xrightarrow{\pi_1} A_0
\end{array}
\]

- \(\text{compr}\) is the arrow composing the “right two arrows”

\[
\begin{array}{c}
A_1 \\
\downarrow^{\pi_1 \circ \text{left}} \\
A_0
\end{array}
\]

1. and 2. define \(A_2\) as the ‘object of composable pairs of arrows’ while 3. and 4. express that the ‘object of arrows’ \(A_1\) has identities and 5. formalises associativity of composition.
Remark 31. A morphism $f : A \to B$ between internal categories, an \textit{internal functor}, is a triple $(f_0, f_1, f_2)$ of arrows such that the two diagrams (one for $\text{dom}$ and one for $\text{cod}$)

$$
\begin{array}{ccc}
A_2 & \xrightarrow{\text{comp}} & A_1 \\
\downarrow & & \downarrow \\
B_2 & \xrightarrow{\text{comp}} & B_1 \\
\end{array}
\quad
\begin{array}{ccc}
A_0 & \xrightarrow{\text{dom}} & A_1 \\
\downarrow & & \downarrow \\
B_0 & \xrightarrow{\text{cod}} & B_1 \\
\end{array}
$$

(7)

commute and identities are preserved. Since $A_2$ is a pullback, $f_2$ is uniquely determined by $f_1$ (in other words, the existence of $f_2$ is a property rather than a structure). In more detail, if $\Gamma \to A_2$ is any arrow then, because $A_2$ is a pullback, it can be written as a pair $\langle l, r \rangle$ of arrows $l, r : \Gamma \to A_1$ and $f_2$ is determined by $f_1$ via

$$
f_2 \circ \langle l, r \rangle = \langle f_1 \circ l, f_1 \circ r \rangle
$$

(8)

Remark 32. A natural transformation $\alpha : f \to g$ between internal functors $f, g : A \to B$, an \textit{internal natural transformation}, is an arrow $\alpha : A_0 \to B_1$ such that

$$
\begin{align*}
\text{dom} \circ \alpha &= f_0 \\
\text{cod} \circ \alpha &= g_0 \\
\text{comp} \circ (f_1, \alpha \circ \text{cod}) &= \text{comp}(\alpha \circ \text{dom}, g_1)
\end{align*}
$$

Remark 33. Internal categories with functors and natural transformations form a 2-category. We denote by $\text{Cat}(\mathcal{V})$ the category or 2-category of categories internal in $\mathcal{V}$. The forgetful functor $\text{Cat}(\mathcal{V}) \to \mathcal{C}$ mapping an internal category $A$ to its object of objects $A_0$ has both left and right adjoints and, therefore, preserves limits and colimits. Moreover, a limit of internal categories is computed componentwise as $(\lim D)_j = \lim(D_j)$ for $j = 0, 1, 2$.

Remark 34. A monoidal category can be thought of both as a monoid in the category of categories and as a category internal in the category of monoids. To understand this in more detail, note that both cases give rise to the diagram

$$
\begin{array}{ccc}
A_2 \times A_2 & \xrightarrow{\text{comp} \times \text{comp}} & A_1 \times A_1 \\
\downarrow & & \downarrow \\
A_2 & \xrightarrow{\text{comp}} & A_1 \\
\end{array}
\quad
\begin{array}{ccc}
A_0 \times A_0 & \xrightarrow{\text{dom} \times \text{dom}} & A_1 \times A_1 \\
\downarrow & & \downarrow \\
A_0 & \xrightarrow{\text{cod} \times \text{cod}} & A_1 \\
\end{array}
$$

where

- in the case of a monoid $A$ in the category of internal categories, $m = (m_0, m_1, m_2)$ is an internal functor $A \times A \to A$ and, using that products of internal categories are computed componentwise, we have $\text{comp} \circ m_2 = m_1 \circ (\text{comp} \times \text{comp})$, which gives us the interchange law

$$
(f; g) \cdot (f'; g') = (f \cdot f') ; (g \cdot g')
$$

by using (8) with $m$ for $f$ and writing $;$ for $\text{comp}$ and $\cdot$ for $m_1$;

- in the case of a category internal in monoids we have monoids $A_0, A_1, A_2$ and monoid homomorphisms $i, \text{dom}, \text{cod}, \text{comp}$ which, if spelled out, leads to the same commuting diagrams as the previous item.