Gauge independent approach to chiral symmetry breaking in a strong magnetic field

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Abstract

The gauge independence of the dynamical fermion mass generated through chiral symmetry breaking in QED in a strong, constant external magnetic field is critically examined. We show that the bare vertex approximation, in which the vertex corrections are ignored, is a consistent truncation of the Schwinger-Dyson equations in the lowest Landau level approximation. The dynamical fermion mass, obtained as the solution of the truncated Schwinger-Dyson equations evaluated on the fermion mass shell, is shown to be manifestly gauge independent. By establishing a direct correspondence between the truncated Schwinger-Dyson equations and the 2PI (two-particle-irreducible) effective action truncated at the lowest nontrivial order in the loop expansion as well as in the $1/N_f$ expansion ($N_f$ is the number of fermion flavors), we argue that in a strong magnetic field the dynamical fermion mass can be reliably calculated in the bare vertex approximation.

Key words: Gauge independence, Chiral symmetry, Magnetic field, Schwinger-Dyson equations, $1/N$ expansion
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1 Introduction

The study of chiral symmetry breaking has a long history, beginning with the seminal work of Nambu and Jona-Lasinio [1–3]. Because of its simpler structure, quantum electrodynamics (QED) has widely been used to study...
chiral symmetry breaking in gauge theories [4–12]. Being inherently a nonperturbative phenomenon, the dynamical generation of a fermion mass in gauge theories is usually studied with the help of the Schwinger-Dyson equations truncated in certain schemes. An often used truncation scheme is the rainbow (or quenched ladder) approximation (for a review see Ref. [13]). It is generally difficult to find a consistent truncation scheme such that the resulting dynamical fermion mass is gauge (fixing) independent. A common practice has been to forgo gauge independence in favor of obtaining a nonperturbative solution. Nevertheless, the demonstration of the gauge independence of physical quantities is of fundamental importance in gauge theories [14,15]. In perturbation theory the usual connection between the order of the loop expansion and powers of the gauge coupling guarantees that physical quantities calculated to a given order in the loop expansion are gauge independent. In other words, as is well known, loop expansion is a consistent expansion scheme in perturbation theory. Such a connection however is lost in nonperturbative studies in which contributions of a given order in the gauge coupling arise from every order in the loop expansion. As a result, establishing the gauge independence of physical quantities obtained in a nonperturbative calculation (e.g., the Schwinger-Dyson equations) is a highly nontrivial problem. This is because an infinite subset of diagrams arising from every order in the loop expansion has to be resummed consistently, thus leading to potential gauge dependence of physical quantities whenever not all relevant diagrams are accounted for. Therefore, we emphasize that in gauge theories no truncation schemes of the Schwinger-Dyson equations should be considered consistent unless the gauge independence of physical quantities calculated therein is unequivocally demonstrated.

The problem of chiral symmetry breaking in QED in a strong, constant external magnetic field has received a lot of attention over the past decade [16–29]. It was originally motivated by a proposal [30–33] to explain the correlated $e^+e^-$ peaks observed in heavy ion collision experiments [34,35]. The proposal posits that the $e^+e^-$ peaks are produced from the decay of a bound $e^+e^-$ system formed in a new metastable phase of QED with broken chiral symmetry, which is induced by the strong electromagnetic field present in the neighborhood of the colliding heavy ions. Even though the $e^+e^-$ events were not observed in a later experiment [36], it is still an interesting question to ask whether background fields can induce chiral symmetry breaking. The result of the investigation suggests that it may be relevant for studying the electroweak phase transition in the early universe when a strong magnetic field was present [18,19]. The methodology developed also found interesting applications in the study of color superconductivity in high density QCD [37], superconductivity and magnetic field induced phase transitions in condensed matter systems [38–42], as well as the properties of strongly magnetized astrophysical media [43,44].
In a recent article [45], we have found a consistent truncation scheme for the case of chiral symmetry breaking in a strong, constant external magnetic field. We critically examined the consistency and gauge independence of the bare vertex approximation that has been extensively used in truncating the Schwinger-Dyson equations to calculate the dynamical fermion mass generated through chiral symmetry breaking in weakly coupled QED in a strong magnetic field. In particular, we showed that the bare vertex approximation, in which the vertex corrections are ignored, is a consistent truncation of the Schwinger-Dyson equations in the lowest Landau level approximation. The dynamical fermion mass, obtained as the solution of the truncated Schwinger-Dyson equations evaluated on the fermion mass shell, is manifestly gauge independent. This novel gauge independent approach poses a serious question on the validity of the results and conclusions obtained in earlier studies [18–20,22,24–29]. Most importantly, it leads to a consistent, nonperturbative calculation of the dynamical fermion mass to leading order in the gauge coupling, and allows one to identify the infinite subset of diagrams that contribute to chiral symmetry breaking in a strong magnetic field.

The purpose of this article is twofold. On the one hand, we present the technical details of the results outlined in Ref. [45]. To begin with, we critically review the properties of the so-called $E_p$ functions, first introduced by Ritutus [46,47] in the studies of QED in constant external electromagnetic fields. In doing so, we correct a few mistakes found in the literature [20,22,23,46–48]. We also derive the Ward-Takahashi identity in the bare vertex approximation both within and beyond the lowest Landau level approximation. This allows us to (i) correct a few oversights found in Ref. [23], (ii) verify the conclusion obtained therein on the Ward-Takahashi identity in the bare vertex approximation and within the lowest Landau level approximation, and (iii) show that the Ward-Takahashi identity in the bare vertex approximation can be satisfied only within the lowest Landau level approximation. By utilizing the $E_p$ function formalism and the Ward-Takahashi identity in the bare vertex approximation, we prove that the bare vertex approximation is a consistent truncation of the Schwinger-Dyson equations in the lowest Landau level approximation. In particular, we verify that (i) the truncated vacuum polarization is transverse, (ii) the truncated fermion self-energy evaluated on the fermion mass shell is manifestly gauge independent.

On the other hand, we provide a detailed comparison between our results and those obtained in the literature [18–20,22,24–29]. Based on the gauge independent analysis presented in this article, we argue that to a large extent the results and conclusions obtained in those earlier studies can be attributed to inconsistent truncation schemes, gauge dependent artifacts, or both. Along the way this detailed comparison also establishes several important aspects of the bare vertex approximation: (i) The assumption of a momentum independent fermion self-energy, a consequence of the Ward-Takahashi identity in
the bare vertex approximation, is reliable in the momentum region relevant to chiral symmetry breaking in a strong magnetic field. (ii) While the extension to a momentum dependent fermion self-energy violates the Ward-Takahashi identity in the bare vertex approximation, the resultant gauge dependence of the dynamical fermion mass is of higher order. (iii) In a strong magnetic field the dynamical fermion mass can be reliably calculated in the bare vertex approximation and is gauge independent up to corrections of higher order. (iv) There exists a direct correspondence between the Schwinger-Dyson equations truncated in the bare vertex approximation and the nonperturbative approach based on the 2PI (two-particle-irreducible) effective action truncated at the lowest nontrivial order both in the loop expansion as well as in the $1/N_f$ expansion ($N_f$ is the number of fermion flavors).

The rest of this article is organized as follows. In Sec. 2 we critically review the properties of the Ritus $E_p$ functions. In Sec. 3 the Schwinger-Dyson equations truncated in the bare vertex approximation are discussed, and the Ward-Takahashi identity in the bare vertex approximation is derived both within and beyond the lowest Landau level approximation. In Sec. 4 we present the proof that the bare vertex approximation is a consistent truncation of the Schwinger-Dyson equations in the lowest Landau level approximation. The truncated on-shell Schwinger-Dyson equations are numerically solved for various numbers of fermion flavors, and an analytic fit for the dynamical fermion mass is obtained. In Sec. 5 we compare our results to those found in the literature. We also establish a direct contact with the 2PI effective action truncated at the lowest nontrivial order in the loop expansion as well as in the $1/N_f$ expansion. Finally, we present our conclusions in Sec. 6. In Appendix A we show that the vacuum current vanishes identically in a constant external magnetic field. Appendix B provides a summary of some exact relations derived from the 2PI effective action that are useful to the discussion in the main text.

2 Formalism: Ritus $E_p$ functions

In this section we critically review the properties of the $E_p$ functions, first introduced by Ritus in his seminal work [46,47]. They form a complete set of Dirac matrix-valued orthonormal functions and hence provide a convenient formalism in the studies of QED in the presence of a constant external magnetic field [20,22,23,48]. In doing so, a few mistakes and confusions found in the literature [20,22,23,46–48] are corrected and clarified.

We begin with the free field Dirac equation for a massless fermion in the
presence of a constant external magnetic field\(^1\)

\[ (\gamma \cdot \pi) \psi(x) = 0, \quad (2.1) \]

where \(\pi_\mu = -i \partial_\mu - eA_\mu\) with \(A_\mu\) being the external gauge field and \(\psi\) is a Dirac spinor. Instead of solving Eq. (2.1) directly, we show that there exists a complete set of orthonormal functions, referred to in the literature as the Ritus \(E_p\) functions, such that in the basis spanned by the \(E_p\) functions Eq. (2.1) is rendered formally identical to the Dirac equation for a massless fermion in the absence of external fields.

We will take the constant external magnetic field of strength \(H\) in the \(x_3\)-direction with the corresponding vector potential given by

\[ A_\mu = (0, 0, H x_1, 0), \quad (2.2) \]

where \(\mu = 0, 1, 2, 3\). It is straightforward to verify that the operators \((\gamma \cdot \pi)^2, \Sigma^3 \equiv i\gamma^1\gamma^2\) and \(\gamma^5\) constitute a maximal set of mutually commuting operators.\(^2\) The \(E_p\) functions are constructed in terms of the simultaneous eigenfunctions (eigenvectors) of these mutually commuting operators.

The eigenfunction equation for \((\gamma \cdot \pi)^2\) is given by

\[ -(\gamma \cdot \pi)^2 \phi_p(x) = p^2 \phi_p(x), \quad (2.3) \]

where \(\phi_p(x)\) is the eigenfunction corresponding to the eigenvalue \(p^2\). Note that the subscript \(p\) in \(\phi_p(x)\) denotes symbolically a set of quantum numbers yet to be determined. In the chiral representation in which \(\Sigma^3\) and \(\gamma^5\) are both diagonal with eigenvalues \(\sigma = \pm 1\) and \(\chi = \pm 1\), respectively, the eigenfunctions \(\phi_p(x)\) have the general form

\[ \phi_p(x) = E_{p\sigma}(x) \omega_{\sigma\chi}, \quad (2.4) \]

where \(\omega_{\sigma\chi}\) are bispinors which are the eigenvectors of \(\Sigma^3\) and \(\gamma^5\). The scalar functions \(E_{p\sigma}(x)\) are found to be given by

\[ E_{p\sigma}(x) = N(n) e^{i(p_0 x^0 + p_2 x^2 + p_3 x^3)} D_n(\rho), \quad (2.5) \]

where \(N(n) = (4\pi|eH|)^{1/4}/\sqrt{n!}\) is a normalization factor, and \(D_n(\rho)\) denotes the parabolic cylinder functions [49] with argument \(\rho = \sqrt{2|eH|} (x_1 - p_2/eH)\)

\(^1\) In this article we use the convention in which the metric has the signature \(g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)\), the Dirac matrices obey \(\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\) and \(\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\).

\(^2\) Strictly speaking, this maximal set of mutually commuting operators also contains the obvious operators \(-i\partial_0, -i\partial_2\) and \(-i\partial_3\), which have been omitted here for the sake of presentational simplicity.
and nonnegative integer index \( n = 0, 1, 2, \ldots \) given by

\[
n = l + \frac{\sigma}{2} \text{sgn}(eH) - \frac{1}{2}.
\]  

(2.6)

The nonnegative integer \( l \) in Eq. (2.6) labels the Landau levels with \( l = 0 \) being the lowest Landau level (LLL). Note that in the absence of an external electric field, the scalar functions \( E_{p\sigma}(x) \) do not depend on the chirality \( \chi \) (hence the notation used).

The eigenvalue \( p^2 \) of \(- (\gamma \cdot \pi)^2\) is identified as the momentum squared of a massless fermion in an external magnetic field. This can be understood by writing \((\gamma \cdot \pi)^2 = (\gamma_\parallel \cdot \pi_\parallel)^2 + (\gamma_\perp \cdot \pi_\perp)^2\) and noticing the following properties:

\[
- (\gamma_\parallel \cdot \pi_\parallel)^2 \phi_p(x) \equiv - (\gamma_0^0 \pi_0 + \gamma_3^3 \pi_3)^2 \phi_p(x) \\
= - (\pi_0^0 - \pi_3^3) \phi_p(x) \\
= - (p_0^0 - p_3^3) \phi_p(x) \\
\equiv p_\parallel^2 \phi_p(x),
\]

(2.7)

\[
- (\gamma_\perp \cdot \pi_\perp)^2 \phi_p(x) \equiv - (\gamma_1^1 \pi_1 + \gamma_2^2 \pi_2)^2 \phi_p(x) \\
= (\pi_1^1 + \pi_2^2 - eHS^3) \phi_p(x) \\
= 2|eH|l \phi_p(x) \\
\equiv p_\perp^2 \phi_p(x),
\]

(2.8)

where, here and henceforth, the subscript \( \parallel (\perp) \) refers to the longitudinal: \( \mu = 0, 3 \) (transverse: \( \mu = 1, 2 \)) components. Hence, one finds

\[
p^2 = p_\parallel^2 + p_\perp^2 \\
= - p_0^2 + p_3^2 + 2|eH|l,
\]

(2.9)

with \( l \) being the quantum number of the quantized transverse momentum squared and \( \sqrt{2|eH|} \) the energy gap between adjacent Landau levels (referred to henceforth as the Landau energy).

It is important to note that Eq. (2.6), together with the nonnegativeness of \( n \), imposes a constraint on the allowed value of \( \sigma \) when \( l = 0 \), namely,

\[
\sigma = \begin{cases} 
\text{sgn}(eH) & \text{for } l = 0, \\
\pm \text{sgn}(eH) & \text{for } l > 0.
\end{cases}
\]

(2.10)

This is one of the subtle points that has been overlooked in the previous literature [20,22,23,48] that utilizes the Ritus \( E_p \) functions. Physically, the constraint on \( \sigma \) when \( l = 0 \) means that the spin of the LLL fermions is always aligned parallel to the external magnetic field.

Following Ritus [46,47], we construct the \( E_p \) functions as the matrix of the simultaneous eigenfunctions (eigenvectors) of the maximal set of mutually
commuting operators \{((γ \cdot π)^2, Σ^3, γ^5)\}, viz,

\[ E_p(x) = \sum'_{σ=±1} E_{pσ}(x) \Delta(σ), \tag{2.11} \]

where a prime on the summation symbol means that it is subject to the constraint (2.10), and

\[ \Delta(σ) = \text{diag}(δ_σ1, δ_{σ-1}, δ_{σ1}, δ_{σ-1}) = \frac{1}{2} (1 + σ \Sigma^3). \tag{2.12} \]

Note that the subscript \(p\) in \(E_p(x)\) denotes symbolically the set of quantum numbers \{\(p_0, p_2, p_3, l\)\}.

It is straightforward to verify that (no summation over repeated indices is implied)

\[ \Delta(σ) + Δ(−σ) = 1, \quad Δ(σ) Δ(σ') = δ_{σσ'} Δ(σ), \tag{2.13} \]

hence the matrices \(Δ(σ)\) can be identified as the projection operators on the fermion states with the spin parallel (\(σ = 1\)) and antiparallel (\(σ = −1\)) to the external magnetic field. Furthermore, \(Δ(σ)\) satisfy the following important properties

\[ [Δ(σ), γ_μ^\parallel] = 0, \quad Δ(σ) γ_μ^\parallel = γ_μ^\parallel Δ(−σ), \tag{2.14} \]

which are the key relations that will be frequently used in our calculation.

Using the orthogonal property of the parabolic cylinder functions [49]

\[ \int_{−∞}^{∞} dρ \; D_n'(ρ) \; D_n(ρ) = \sqrt{2\pi} \; n! \; δ_{nn'}, \tag{2.15} \]

we find that the \(E_p\) functions are orthonormal and complete, namely,

\[ \int d^4 x \; \overline{E}_p(x) \; E_{p'}(x) = (2π)^4 \; \tilde{δ}^{(4)}(p − p') \; Π(l), \tag{2.16} \]

\[ \sum_{p} d^4 p \; \overline{E}_p(x) \; E_{p}(y) = (2π)^4 \; δ^{(4)}(x − y), \tag{2.17} \]

where \(\overline{E}_p = γ^0 E^\dagger_p γ^0\). Here we have used the following shorthand notation

\[ Π(l) = \begin{cases} Δ[\text{sgn}(eH)] & \text{for } l = 0, \\ 1 & \text{for } l > 0, \end{cases} \tag{2.18} \]

\[ \sum_{p} d^4 p = \sum_{l=0}^{∞} \int_{−∞}^{∞} dp_0 \int_{−∞}^{∞} dp_2 \int_{−∞}^{∞} dp_3, \tag{2.19} \]

\[ \tilde{δ}^{(4)}(p − p') = δ_{l'w} \; δ(p_0 − p'_0) \; δ(p_2 − p'_2) \; δ(p_3 − p'_3). \tag{2.20} \]
We emphasize that the orthonormal condition of the $E_p$ functions for the LLL fermions (i.e., $l = 0$) in Eq. (2.16) differs from that quoted in the literature [20,22,23,48], which is applicable only for fermions occupying higher Landau levels (i.e., $l > 0$). The presence of the projection operator $\Delta[\text{sgn}(eH)]$ in the orthonormal condition of the $E_p$ functions for the LLL fermions has an important physical consequence that is responsible for an effective dimensional reduction in the dynamics of fermion pairing in a strong external magnetic field [17,19].

Since the $E_p$ functions form a complete set of orthonormal Dirac matrix-valued functions, they can be used as a basis for the Hilbert space of the (perturbative) asymptotic massless fermion states in the presence of a constant external magnetic field. Throughout this article we will refer to the space spanned by the $E_p$ functions simply as the momentum space whenever no confusion may arise.

Explicit calculation shows that the $E_p$ functions satisfy an important property

$$ (\gamma \cdot \pi) E_p(x) = E_p(x) \gamma \cdot p, \quad (2.21) $$

where

$$ p_\mu = (p_0, 0, -\text{sgn}(eH)\sqrt{2|eH|l}, p_3). \quad (2.22) $$

For notational simplicity, in Eq. (2.21) we have used the same notation $p$ to denote both symbolically the set of quantum numbers $\{p_0, p_2, p_3, l\}$ on the left-hand side (LHS) in $E_p(x)$, as well as literally the momentum $p_\mu$ on the right-hand side (RHS) in $\gamma \cdot p$. However, we emphasize that the quantum number $p_2$ should not be confused with the component of the momentum $p$ in the $x_2$-direction, which is determined by the quantum number $l$ as can be seen from Eq. (2.22).

We note that $\gamma \cdot \pi$ does not commute with $\Sigma^3$ and $\gamma^5$, hence it is not diagonal in the basis spanned by the $E_p$ functions, as is evident from Eq. (2.21).

Upon left multiplying Eq. (2.21) by $\overline{E_p}(x)$ and integrating over $x$, one obtains

$$ \int d^4x \overline{E_p}(x) (\gamma \cdot \pi) E_p(x) = (2\pi)^4 \delta^{(4)}(p-p') \Pi(l) \gamma \cdot p. \quad (2.23) $$

Eq. (2.23) reveals clearly the advantage of the $E_p$ functions in the studies of QED in a constant external magnetic field: When expressed in the momentum space spanned by the $E_p$ functions, the (bare) inverse propagator for fermions in a constant external magnetic field is rendered formally identical to that in the absence of external fields (up to the projection operator $\Delta[\text{sgn}(eH)]$ that accounts for the spin alignment of the LLL fermions).

We are now in a position to show that in the momentum space spanned by the $E_p$ functions, Eq. (2.1) is rendered formally identical to the Dirac equation
for a massless fermion in the absence of external fields. This is achieved by considering the Dirac spinor in momentum space $\psi(p)$ defined by

$$\psi(x) = \sum \int \frac{d^4 p}{(2\pi)^4} E_p(x) \psi(p),$$

where the momentum argument $p$ in $\psi(p)$ is given by Eq. (2.22). It follows from Eq. (2.21) that in momentum space Eq. (2.1) reads

$$\langle \gamma \cdot p \rangle \psi(p) = 0,$$

which, as advertised in the beginning of this section, is formally the Dirac equation for a massless fermion in the absence of external fields, and whose solutions are well known. The condition that Eq. (2.25) has nontrivial solutions is given by $p^2 = 0$, which is exactly the mass shell condition for a massless particle. We emphasize that because the $E_p$ functions are not solutions of the free field Dirac equation (2.1), they do not correspond to the (perturbative) asymptotic states of massless fermions in the presence of a constant external magnetic field. Nevertheless, as clearly displayed in Eqs. (2.24) and (2.25), the latter can be expanded in the basis spanned by the $E_p$ functions in terms of the usual Dirac (or Weyl) spinors for massless fermions in the absence of external fields.

So far we have been considering only the noninteracting theory for massless fermions in a constant external magnetic field. Once the interaction with the (quantum) photon field is taken into account, the fermion will receive radiative corrections, which in turn give rise to the self-energy. Consistent with Eq. (2.21), the fermion self-energy in coordinate space $\Sigma(x, x')$ has to satisfy a similar property,

$$\int d^4 x' \Sigma(x, x') E_p(x') = E_p(x) \Sigma(p),$$

where $\Sigma(p)$ is a Dirac matrix-valued function (yet to be determined by explicit calculation) with the momentum argument $p$ given by Eq. (2.22).

Following the same steps that lead to Eq. (2.23), one finds the fermion self-energy in momentum space to be given by

$$\Sigma(p, p') = \int d^4 x d^4 x' \overline{E}_p(x) \Sigma(x, x') E_{p'}(x')$$

$$= (2\pi)^4 \tilde{\delta}^{(4)}(p - p') \Pi(l) \Sigma(p).$$

With Eqs. (2.23) and (2.27), one finds that the full inverse fermion propagator
in momentum space is given by [see Eq. (3.1)]

\[
G^{-1}(p, p') = \int d^4x \, d^4x' \, E_p(x) \, G^{-1}(x, x') \, E_{p'}(x') = (2\pi)^4 \, \delta^{(4)}(p - p') \, \Pi(l) \, [\gamma \cdot p + \Sigma(p)],
\]

which in turn implies that the full fermion propagator in momentum space takes the form

\[
G(p, p') = \int d^4x \, d^4x' \, E_p(x) \, G(x, x') \, E_{p'}(x') = (2\pi)^4 \, \delta^{(4)}(p - p') \, \Pi(l) \, \frac{1}{\gamma \cdot p + \Sigma(p)}.
\]

Eqs. (2.27)-(2.29), together with the orthonormal condition Eq. (2.16) of the \(E_p\) functions, imply that in coordinate space one has

\[
\Sigma(x, x') = \sum \int \frac{d^4p}{(2\pi)^4} \, E_p(x) \, \Sigma(p) \, E_p(x'),
\]

\[
G^{-1}(x, x') = \sum \int \frac{d^4p}{(2\pi)^4} \, E_p(x) \, [\gamma \cdot p + \Sigma(p)] \, E_p(x'),
\]

\[
G(x, x') = \sum \int \frac{d^4p}{(2\pi)^4} \, E_p(x) \, \frac{1}{\gamma \cdot p + \Sigma(p)} \, E_p(x').
\]

The set of equations (2.27)-(2.32) constitutes the main ingredients for the calculation presented in Secs. 3 and 4.

We emphasize that the fermion self-energy \(\Sigma(p)\) in general is not a diagonal matrix, which is contrary to what has been stated in the literature [20,22,23,46–48]. We shall now clarify this confusion. It was argued [20,22,46–48] that in the presence of a constant magnetic field the fermion self-energy operator \(\hat{\Sigma}\) is a Dirac matrix-valued scalar function of the form

\[
\hat{\Sigma} = \hat{\Sigma}[\gamma \cdot \pi, (F_{\mu\nu} \pi^\nu)^2],
\]

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the external field strength. In order to highlight the fact that the self-energy is in general an off-shell quantity, we have used the term self-energy operator instead of the usual term mass operator, which can be misleading.

A comment here is in order. Based on symmetry arguments [46,47], for constant external fields the self-energy operator \(\hat{\Sigma}\) in general depends on two extra operators: \(\gamma^5 \tilde{F}_{\mu\nu} F_{\mu\nu}\) and \(\sigma^{\mu\nu} F_{\mu\nu}\), where \(\tilde{F}_{\mu\nu} = (1/2) \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}\) and \(\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]\). The former vanishes identically in the absence of an external electric field. The latter renders either the fermion self-energy or the position of the fermion pole (i.e., the fermion dispersion relation) explicitly dependent on the sign of the external magnetic field strength, hence is forbidden by a
residual $Z_2$ symmetry of the rotational $O(3)$ symmetry that is broken in the presence of an external magnetic field (see Sec. 3 for a related discussion).

Since $(F_{\mu\nu}\pi^\nu)^2$ commutes with $(\gamma \cdot \pi)^2$, the self-energy operator $\tilde{\Sigma}$ commutes with $(\gamma \cdot \pi)^2$ and hence is diagonal in the basis spanned by the eigenfunctions of $(\gamma \cdot \pi)^2$. While this statement is correct in itself, it does not imply that $\tilde{\Sigma}$ is diagonal in the basis spanned by the $E_p$ functions. This can be understood as follows. We note that even though $\tilde{\Sigma}$ commutes with $(\gamma \cdot \pi)^2$, it does not commute with $\Sigma^3$ and $\gamma^5$. Therefore, $\tilde{\Sigma}$ is not diagonal in the basis spanned by the $E_p$ functions, which are constructed in terms of the simultaneous eigenfunctions (eigenvectors) of the maximal set of mutually commuting operators $\{(\gamma \cdot \pi)^2, \Sigma^3, \gamma^5\}$. This is precisely the same reason why the operator $\gamma \cdot \pi$, in spite of commuting with $(\gamma \cdot \pi)^2$, is not diagonal in the basis spanned by the $E_p$ functions.

Last, but not least, we note that $\tilde{\Sigma}$ does not anticommute with $\gamma^5$. This in turn suggests, as one would already have expected, dynamical chiral symmetry breaking in the full theory of QED in the presence of a constant external magnetic field. While can be confirmed only by explicit nonperturbative calculation, the above symmetry arguments seem to indicate that dynamical chiral symmetry breaking should be a generic phenomenon independent of the number of fermion flavors. Indeed, it is one of the goals of this article to verify in a gauge independent manner that in a constant external magnetic field chiral symmetry is dynamically broken regardless of the number of fermion flavors.

3 Schwinger-Dyson equations and the bare vertex approximation

The Schwinger-Dyson (SD) equations in massless QED in an external magnetic field are well-known in the literature (for reviews see Refs. [50,51]). The equations for the full fermion propagator $G(x,y)$ are given by

$$G^{-1}(x,y) = S^{-1}(x,y) + \Sigma(x,y), \quad \text{(3.1)}$$

$$\Sigma(x,y) = i e^2 \int d^4x'd^4y' \gamma^\mu G(x,x')\Gamma^\nu(x',y,y') D_{\mu\nu}(x,y'), \quad \text{(3.2)}$$

where $S(x,y)$ is the bare fermion propagator in the external gauge field $A_\mu$ given by Eq. (2.2), $\Sigma(x,y)$ is the fermion self-energy and $\Gamma^\nu(x,y,z)$ is the full

$^3$ We note that in writing the fermion self-energy $\Sigma(x,y)$ as in Eq. (3.2), it is implicitly assumed that the vacuum current vanishes. In Appendix A we show that the vacuum current does indeed vanish identically in a constant external magnetic field.
Fig. 1. The SD equations in QED in an external magnetic field. The thin (heavy) line with an arrow denotes the bare (full) fermion propagator in the presence of an external magnetic field. The thin (heavy) wiggly line denotes the free (full) photon propagator. The vertex with(out) a heavy dot denotes the full (bare) fermion-photon interaction vertex.

The full photon propagator $D_{\mu\nu}(x,y)$ satisfies the equations

$$D_{\mu\nu}^{-1}(x,y) = D_{\mu\nu}^{-1}(x,y) + \Pi_{\mu\nu}(x,y),$$

$$\Pi_{\mu\nu}(x,y) = -ie^2 \text{tr} \int d^4x' d^4y' \gamma_{\mu} G(x, x') \Gamma_{\nu}(x', y', y) G(y', x),$$

where $D_{\mu\nu}(x,y)$ is the free photon propagator (defined in covariant gauges) and $\Pi_{\mu\nu}(x,y)$ is the vacuum polarization. The diagrammatical representation of the SD equations (3.1)-(3.4) is depicted in Fig. 1.

Since the dynamics of fermion pairing in a strong magnetic field is dominated by the lowest Landau level (LLL) [18–22,24,25], we will consider the propagation of, as well as radiative corrections originating only from, fermions occupying the LLL. This is referred to as the lowest Landau level approximation (LLLA) in the literature [18–29]. Therefore, except where otherwise explicitly stated, the fermion propagator and the fermion self-energy in the SD equations (3.1)-(3.4) as well as in the rest of this article will be taken to be those for the LLL fermions. For notational simplicity, no separate notation will be introduced.

It is well known that the SD equations (3.1)-(3.4) correspond to an infinite hierarchy of integral equations. This is because the full vertex (i.e., the three-point vertex function) $\Gamma^\mu$ depends on four-point vertex functions which in turn depend on higher-point vertex functions, and so on. In order to reduce the SD equations to a closed system of integral equations that is tractable, a truncation scheme of the SD equations needs to be employed by truncating this infinite hierarchy at some point. Diagrammatically, a truncation scheme of the SD equations corresponds to a resummation of a selected infinite subset of diagrams arising from every order in the loop expansion. The simplest truncation is done at the level of the three-point vertex function by expressing
To this end we will work in the bare vertex approximation (BVA), in which the vertex corrections are completely ignored. This is achieved by replacing the full vertex in the SD equations (3.1)-(3.4) by the bare one, viz,

$$\Gamma^\mu(x,y,z) = \gamma^\mu \delta^{(4)}(x - z) \delta^{(4)}(y - z).$$

(3.5)

The diagrammatical representation of the resultant SD equations truncated in the BVA is depicted in Fig. 2. This truncation is also known as the (improved) rainbow approximation and has been employed extensively in the literature [18–20,22,24–29]. However, we emphasize that unlike what has usually been done in the literature, here we will not confine ourselves to a particular gauge (usually the Feynman gauge) [18–20,22], nor will we make the assumption that the BVA (3.5) is valid only in a certain gauge [24–29]. Instead, it is our aim to prove that the BVA (3.5) is a consistent truncation of the SD equations (3.1)-(3.4) within the LLLA. The dynamical fermion mass, obtained as the solution of the truncated SD equations evaluated on the fermion mass shell, is manifestly gauge independent. In the weak coupling regime that we consider, such a gauge independent approach allows one to resum consistently an infinite subset of diagrams that arise from every order in the loop expansion and whose contributions are of leading order in the gauge coupling, thus leading to a consistent and reliable calculation of the dynamical fermion mass.

The main ingredient in the proof of the gauge independence of physical quantities is the Ward-Takahashi (WT) identity satisfied by the vertex and the inverse fermion propagator. It is known [13,52] that in ordinary QED the BVA is not a consistent truncation of the SD equations because the corresponding WT identity between the full fermion propagator and the bare vertex in general cannot be fulfilled. The situation however changes drastically in the presence of a strong, constant external magnetic field as the motion of low-energy fermions is restricted in directions perpendicular to the magnetic field,
leading to an effective dimensional reduction from \((3 + 1)\) to \((1 + 1)\) in the
dynamics of fermion pairing in a strong magnetic field.

The WT identity in the BVA within the LLLA was first studied in Ref. [23]. It was shown that in order to satisfy the WT identity in the BVA within the LLLA, the LLL fermion self-energy in momentum space has to be a momentum independent constant. However, due to a mistake in Ref. [23] regarding the Dirac matrix structure in the orthonormal condition of the Ritus \(E_p\) functions for the LLL fermions [see Eq. (2.16) and the discussion that follows], the calculation therein requires further investigations. As we will see below, calculation performed by using the correct orthonormal condition shows that the conclusion obtained in Ref. [23] on the WT identity in the BVA remains valid. The reliability of such a momentum independent approximation (in that the fermion self-energy has very weak momentum dependence) and, consequently, of the WT identity in the BVA has been verified within the LLLA in a certain gauge in the momentum region relevant to the dynamics of fermion pairing in a strong magnetic field [24–28] (see further discussions in Sec. 5).

We now derive the WT identity in the BVA both within and beyond the LLLA. This allows us to (i) correct a few oversights found in Ref. [23], (ii) verify the conclusion obtained therein on the WT identity in the BVA within the LLLA, and (iii) show that the WT identity in the BVA can be satisfied only within the LLLA. It is noted that in the rest of this section the fermion propagator and the fermion self-energy will in general not be restricted to those for the LLL fermions.

Following the standard procedure for deriving the WT in QED [53,54,23], one finds the WT identity in the BVA in the presence of an external gauge field to be given by

\[-\partial_{\mu}[^{\delta^{(4)}(x - z) \delta^{(4)}(y - z) \gamma^\mu]} = i[^{\delta^{(4)}(y - z) - \delta^{(4)}(x - z)} G^{-1}(x, y)], \quad (3.6)\]

where it is noted that \(G(x, y)\) is the full fermion propagator in an external gauge field. Since the inverse fermion propagator in the above equation does not depend on \(z\), performing the Fourier transform on the variable \(z\) leads to

\[e^{-iq \cdot x} \delta^{(4)}(x - y) \gamma \cdot q = (e^{-iq \cdot x} - e^{-iq \cdot y}) G^{-1}(x, y) , \quad (3.7)\]

where \(q^\mu\) is the momentum carried by the photon. Upon left and right multiplying Eq. (3.7) by \(\bar{E}_p(x)\) and \(E_{p'}(y)\), respectively, and integrating over \(x\) and \(y\), we obtain the following identity

\[\int d^4x e^{-iq \cdot x} \bar{E}_p(x) \gamma \cdot q E_{p'}(x) = \int d^4x d^4y (e^{-iq \cdot x} - e^{-iq \cdot y}) \bar{E}_p(x) G^{-1}(x, y) E_{p'}(y) , \quad (3.8)\]

where the subscripts \(p\) and \(p'\) denote the sets \(\{p_0, p_2, p_3, l\}\) and \(\{p'_0, p'_2, p'_3, l'\}\), respectively.
Let us first derive the WT identity in the BVA within the LLLA, namely, \( l = l' = 0 \) in Eq. (3.8). Using the properties of the \( E_p \) functions [see Eqs. (2.11), (2.12) and (2.5)], one finds that the LHS of the WT identity (3.8) is simplified to

\[
\int d^4x e^{-i q \cdot x} \overline{E}_p(x) \gamma \cdot q E_{p'}(x) = (2\pi)^4 \delta^{(3)}(p + q - p') e^{-q^2/4|eH|} e^{-i q_1(p_2 + p'_2)/2eH} \times \Delta[\text{sgn}(eH)] \gamma \parallel \cdot q_\parallel \Delta[\text{sgn}(eH)],
\]

where used has been made of Eq. (2.31), and the momenta \( p \) to (2.12) and (2.5), we find that the LHS of the WT identity (3.8) is simplified to

\[
\int d^4x d^4y (e^{-i q \cdot x} - e^{-i q \cdot y}) \overline{E}_p(x) G^{-1}(x,y) E_{p'}(y)
\]

\[
= (2\pi)^4 \delta^{(3)}(p + q - p') e^{-q^2/4|eH|} e^{-i q_1(p_2 + p'_2)/2eH} \times \Delta[\text{sgn}(eH)] [\gamma \parallel \cdot (p' - p)_\parallel + \Sigma(p'_\parallel) - \Sigma(p_\parallel)] \Delta[\text{sgn}(eH)],
\]

where used has been made of Eq. (2.31), and the momenta \( p \) and \( p' \) on the RHS in the square brackets are given by \( p_\mu = (p_0, 0, 0, p_3) \) and \( p'_\mu = (p'_0, 0, 0, p'_3) \), respectively [see the remarks after Eq. (2.22)]. Upon collecting the above results and using the delta functions to express \( p'_\parallel \) in terms of \( (p + q)_\parallel \), we can rewrite the WT identity (3.8) in the following compact form

\[
\Delta[\text{sgn}(eH)] \left\{ \gamma \parallel \cdot q_\parallel - \left[ \gamma \parallel \cdot (p + q)_\parallel + \Sigma(p_\parallel + q_\parallel) \right] + \left[ \gamma \parallel \cdot p_\parallel + \Sigma(p_\parallel) \right] \right\} \Delta[\text{sgn}(eH)] = 0.
\]

In the above expression, we have purposely kept the tree-level terms, which cancel among themselves, to highlight the most general form of the WT identity in the BVA within the LLLA.

To proceed further, one needs to know the Dirac matrix structure of the fermion self-energy \( \Sigma(p) \). Based on symmetry arguments [23], for a constant external magnetic field \( \Sigma(p) \) takes the following form

\[
\Sigma(p) = A_\parallel(p) \gamma \parallel \cdot p_\parallel + A_\perp(p) \gamma _\perp \cdot p_\perp + B(p),
\]

where \( A_\parallel, A_\perp(p) \) and \( B(p) \) are functions of the longitudinal and transverse momentum squared as well as the magnitude of the external magnetic field, yet to be determined by explicit calculation.

A comment here is in order. It was shown in Ref. [23] that for a constant external magnetic field along the \( x_3 \)-direction, the most general structure of \( \Sigma(p) \) contains two extra terms that are proportional to \( H \Sigma^3 \), where \( H \) is the strength of the constant external magnetic field. Detailed analysis indicates
that such terms, if present, lead to either the fermion self-energy or the position of the fermion pole being explicitly dependent on the sign of the external magnetic field strength $H$, hence are forbidden by a residual $Z_2$ symmetry (i.e., $H \rightarrow -H$) of the rotational $O(3)$ symmetry that is broken in the presence of an external magnetic field (see Sec. 2 for a related discussion).

Upon substituting Eq. (3.13) into Eq. (3.12), one finds that the transverse components proportional to $\gamma_\perp \cdot p_\perp$ in Eq. (3.13) decouple from the WT identity. Furthermore, the requirement that the WT identity is fulfilled for arbitrary momenta $p$ and $q$ regardless of the sign of $H$, leads to the conclusion that $\Sigma(p_\parallel)$ has to be a momentum independent constant.

Before ending this section, let us derive the WT identity in the BVA but beyond the LLLA, where one of the fermions occupies the lowest Landau level while the other fermion occupies a higher Landau level. In other words, we consider Eq. (3.8) with $l = 0$ and $l' > 0$. We shall show that the WT identity in the BVA can be satisfied only within the LLLA. This is an important conclusion that allows for an unequivocal refutation of the results and conclusions obtained in Ref. [29], which can be attributed to an inconsistent truncation of the SD equations beyond the LLLA (see Sec. 5 for details).

Following similar steps as above, one can rewrite the WT identity (3.8) in the BVA but beyond the LLLA in the following compact form ($l' > 0$)

$$\Delta[\text{sgn}(eH)] \left\{ K_{l'}(q_\perp^2, \varphi) \left[ \gamma_\parallel \cdot q_\parallel - \gamma_\perp \cdot p_\perp + \gamma_\parallel \cdot p_\parallel - \Sigma(p') + \Sigma(p_\parallel) \right] + K_{l'-1}(q_\perp^2, \varphi) \gamma_\perp \cdot q_\perp \right\} = 0,$$

(3.14)

where use has been made of the properties of $\Delta(\sigma)$ in Eq. (2.14) as well as the Dirac matrix structure of the fermion self-energy in Eq. (3.13), and the momenta $p$, $p'$ and $q$ are respectively given by [see the remarks after Eq. (2.22)]

$$p_\mu = (p_0, 0, 0, p_3),$$

$$p'_\mu = (p'_0, 0, -\text{sgn}(eH) \sqrt{2|eH| l'}, p'_3),$$

$$q_\mu = (p'_0 - p_0, q_1, p'_2 - p_2, p'_3 - p_3).$$

In the above expression, we have use the shorthand notation

$$K_{l'}(q_\perp^2, \varphi) = \frac{1}{\sqrt{l'}} \exp[-i \text{sgn}(eH) l' \varphi \left( i \text{sgn}(eH) \left( \frac{q_\perp^2}{2|eH|} \right)^{1/2} \right)^{l'}],$$

(3.16)

where $\varphi = \arctan(q_2/q_1)$. It is an easy exercise to verify that for either a momentum independent or a momentum dependent fermion self-energy, Eq. (3.14) cannot be satisfied for arbitrary momenta $p$ and $p'$ regardless of the sign of $H$. This result is not unexpected because, as is evident from Eq. (3.14), there is no longer an effective dimensional reduction in the dynamics of fermion
pairing beyond the LLLA. The situation is then similar to that in ordinary QED, in which it is known that the BVA is not a consistent truncation of the SD equations [13,52]. Consequently, we conclude that (i) the WT identity in the BVA can be satisfied only within the LLLA with a momentum independent fermion self-energy; (ii) in order to go beyond the LLLA one has to use truncation schemes of the SD equations that consistently account for vertex corrections. To the best of our knowledge, such a consistent truncation of the SD equations beyond the LLLA has not appeared in the literature.

4 On-shell gauge independence in the bare vertex approximation

As per the WT identity in the BVA and within the LLLA (3.12), we can write the self-energy for the LLL fermion as

\[ \Sigma(p_\parallel) = m(\xi), \]  

(4.1)

where \( m(\xi) \) is a momentum independent but gauge dependent constant, with \( \xi \) being the gauge fixing parameter in covariant gauges. It is important to note that \( m(\xi) \) depends implicitly on \( \xi \) through the full photon propagator \( D_{\mu\nu} \) in Eq. (3.2). We emphasize that because of its implicit \( \xi \)-dependence, \( m(\xi) \) should not be taken for granted to be the dynamical fermion mass, which is a gauge independent physical quantity.

We now begin the proof that the BVA is a consistent truncation of the SD equations (3.1)-(3.4), in which \( m(\xi) \) is \( \xi \)-independent and hence can be identified unambiguously as the dynamical fermion mass, if and only if the truncated SD equation for the fermion self-energy is evaluated on the fermion mass shell.

We first recall that, as proved in Ref. [15], in gauge theories the singularity structures (i.e., the positions of poles and branch singularities) of gauge boson and fermion propagators are gauge independent when all contributions of a given order of a systematic expansion scheme are accounted for. Consequently, this means the dynamical fermion mass has to be determined by the pole of the full fermion propagator obtained in a consistent truncation scheme.

In momentum space the full propagator for the LLL fermion is given by [see Eq. (2.29)]

\[ G(p_\parallel) = \frac{1}{\gamma_\parallel \cdot p_\parallel + \Sigma(p_\parallel)} \Delta[\text{sgn}(eH)], \]  

(4.2)

where, as noted in Sec. 2, \( \Delta[\text{sgn}(eH)] \) is the projection operator on the fermion states with the spin parallel to the external magnetic field. Assume for the moment that the BVA is a consistent truncation of the SD equations in the
LLLA, such that the position of the pole of the LLL fermion propagator $G(p_{\parallel})$ is gauge independent. It follows that in accordance with the WT identity in the BVA (3.12) or, equivalently, the momentum independence of the LLL fermion self-energy, we have

$$\Sigma(p_{\parallel}) = \Sigma(p_{\parallel}^2 = -m^2) = m, \quad (4.3)$$

where $m$ is the gauge independent, physical dynamical fermion mass, yet to be determined by solving the truncated SD equations self-consistently. With the LLL fermion self-energy given by Eq. (4.3), the WT identity in the BVA (3.12) reduces to a tree-like form

$$\gamma_{\parallel} \cdot q_{\parallel} = (\gamma_{\parallel} \cdot p_{\parallel} + m) - [\gamma_{\parallel} \cdot (p - q)_{\parallel} + m]. \quad (4.4)$$

What remains to be verified in our proof that the BVA is a consistent truncation of the SD equations in the LLLA is the following statements: (i) the truncated vacuum polarization is transverse, (ii) the truncated fermion self-energy when evaluated on the fermion mass shell ($p_{\parallel}^2 = -m^2$) is manifestly gauge independent. We highlight that the fermion mass shell condition is one of the most important points that has gone unnoticed in the literature, where the truncated fermion self-energy used to be evaluated off the fermion mass shell at, say, $p_{\parallel}^2 = 0$ [18–20,22,24–29].

Let us first consider the vacuum polarization, which in the BVA reads

$$\Pi^{\mu\nu}(x,y) = -ie^2 \text{tr} \gamma^\mu G(x,y) \gamma^\nu G(y,x), \quad (4.5)$$

where the LLL fermion propagator $G(x,y)$ is given by Eq. (2.32) with the summation over $l$ restricted to the $l = 0$ term and, as per the WT identity in the BVA (3.12), a momentum independent self-energy given by Eq. (4.3). In other words, one has

$$G(x,y) = \sum' \int \frac{d^4p}{(2\pi)^4} E_p(x) \frac{1}{\gamma_{\parallel} \cdot p_{\parallel} + m} E_p(y), \quad (4.6)$$

where a prime on the summation symbol means the summation over $l$ is restricted to the $l = 0$ term. Taking the Fourier transform of Eq. (4.5) and using the properties in Eq. (2.14), we find the vacuum polarization in momentum space to be given by

$$\Pi^{\mu\nu}(q) = -ie^2 \frac{1}{2\pi} |eH| \exp\left(-\frac{q_\perp^2}{2|eH|}\right) \text{tr} \int \frac{d^2p_{\parallel}}{(2\pi)^2} \frac{1}{\gamma_{\parallel} \cdot p_{\parallel} + m} \gamma^\mu \gamma^\nu \frac{1}{\gamma_{\parallel} \cdot (p - q)_{\parallel} + m} \Delta[\text{sgn}(eH)]. \quad (4.7)$$

The presence of $\Delta[\text{sgn}(eH)]$ in Eq. (4.7) is a consequence of the LLLA, which, as explicitly displayed in Eq. (4.7), leads to an effective dimensional reduction.
from $(3+1)$ to $(1+1)$ as the LLL fermions couple only to the longitudinal components of the photon field.

The WT identity in the BVA (4.4) guarantees that the vacuum polarization $\Pi_{\mu\nu}(q)$ is transverse, viz,

$$q^\mu \Pi_{\mu\nu}(q) = 0. \quad (4.8)$$

Explicit calculation in dimensional regularization shows that the $1/\epsilon$ pole corresponding to an ultraviolet logarithmic divergence cancels, leading to

$$\Pi_{\mu\nu}(q) = \Pi(q_\parallel^2, q_\perp^2) \left( g_{\parallel}^{\mu} q_{\parallel}^{\nu} - \frac{q_{\parallel}^{\mu} q_{\parallel}^{\nu}}{q_\parallel^2} \right). \quad (4.9)$$

This in turn implies that the full photon propagator takes the following form in covariant gauges ($\xi = 1$ is the Feynman gauge):

$$D_{\mu\nu}(q) = \frac{1}{q^2 + \Pi(q_\parallel^2, q_\perp^2)} \left( g_{\parallel}^{\mu} q_{\parallel}^{\nu} - \frac{q_{\parallel}^{\mu} q_{\parallel}^{\nu}}{q_\parallel^2} \right) + g_{\perp}^{\mu} q_{\perp}^{\nu} + \frac{2}{q_2 q_\perp^2} + (\xi - 1) \frac{1}{q^2} \frac{q_\parallel q_\parallel}{q^2}. \quad (4.10)$$

The polarization function $\Pi(q_\parallel^2, q_\perp^2)$ is given by

$$\Pi(q_\parallel^2, q_\perp^2) = \frac{2\alpha}{\pi} |eH| \exp \left( -\frac{q_\perp^2}{2|eH|} \right) F \left( \frac{q_\parallel^2}{4m^2} \right), \quad (4.11)$$

where $\alpha = e^2/4\pi$ is the fine-structure constant and

$$F(u) = 1 - \frac{1}{2u\sqrt{1+1/u}} \log \frac{\sqrt{1+1/u}+1}{\sqrt{1+1/u}-1}. \quad (4.12)$$

Our result for $\Pi(q_\parallel^2, q_\perp^2)$ agrees with those obtained in the literature [19,21,55]. It is worth noticing that the result in Ref. [21] is valid only in the limit $m \to 0$.

A detailed analysis of the analytic structure of the function $F(u)$ shows that $F(u)$ has an imaginary part for $u < -1$, viz,

$$\text{Im} F(u) = -\frac{\pi}{2u\sqrt{1+1/u}} \theta(-u-1), \quad (4.13)$$

where $\theta(x)$ is the Heaviside step function. Hence the polarization function $\Pi(q_\parallel^2, q_\perp^2)$ is complex for $q_\parallel^2 < -4m^2$, with an imaginary part given by

$$\text{Im} \Pi(q_\parallel^2, q_\perp^2) = -\frac{2\alpha}{\pi} |eH| \text{sgn}(q_0) \exp \left( -\frac{q_\perp^2}{2|eH|} \right) \text{Im} F \left( \frac{q_\parallel^2}{4m^2} \right). \quad (4.14)$$

We note that the extra factor of $-\text{sgn}(q_0)$ is due to the analytic continuation to the complex $q^0$-plane, namely, $q^0 \to q^0 + i\epsilon$ with $\epsilon \to 0^+$. As will be
seen momentarily, the property that \( \text{Im}\Pi(q^2_\parallel, q^2_\perp) \) is an odd function of \( q_0 \) has an important consequence on the dynamical fermion mass. The real and imaginary parts of \( F(u) \) are depicted in Fig. 3. The real part of \( F(u) \) has the following asymptotic behavior:

\[
\text{Re} F(u) = \begin{cases} 
\frac{2u}{3} + \mathcal{O}(u^2) & \text{for } |u| \ll 1, \\
1 - \frac{1}{2u}\log(4|u|) + \mathcal{O}\left(\frac{1}{u^2}\right) & \text{for } |u| \gg 1.
\end{cases}
\]

(4.15)

The polarization effects modify drastically the propagation of virtual photons in a constant external magnetic field. While photons of momenta \( |q^2_\parallel| \ll m^2 \) remain unscreened, photons of momenta \( m^2 \ll |q^2_\parallel| \ll |eH| \) and \( q^2_\perp \ll |eH| \) are screened with a characteristic length \( L = (2\alpha|eH|/\pi)^{-1/2} \). It is noted that the upper limit \( |eH| \) in the range of \( |q^2_\parallel| \) is due to the LLLA. As will be seen in Sec. 5, this screening effect renders the rainbow approximation \([18–20,22]\), in which the bare vertex as well as the free photon propagator are used, completely unreliable in this problem.

We next consider the fermion self-energy, which in the BVA reads

\[
\Sigma(x, y) = ie^2 \gamma^\mu G(x, y) \gamma^\nu D_{\mu\nu}(x, y),
\]

(4.16)

where \( G(x, y) \) is the LLL fermion propagator given by Eq. (4.6) and \( D_{\mu\nu}(x, y) \) is the full photon propagator

\[
D_{\mu\nu}(x, y) = \int \frac{d^4q}{(2\pi)^4} e^{iq(x-y)} D_{\mu\nu}(q),
\]

(4.17)

with \( D_{\mu\nu}(q) \) given by Eq. (4.10).
Following similar steps in Sec. 3 in the derivation of the WT identity by left and right multiplying Eq. (4.16), respectively, by $E_p(x)$ and $E_p'(y)$ corresponding to the LLL fermion (i.e., $l = l' = 0$), and integrating over $x$ and $y$, we find that in momentum space the fermion self-energy evaluated on the fermion mass shell is given by

$$m \Delta[\text{sgn}(eH)] = i \epsilon^2 \int \frac{d^4q}{(2\pi)^4} \exp \left( -\frac{q_\perp^2}{2|eH|} \right) \frac{1}{\gamma_\parallel \cdot (p - q)_\parallel + m} \gamma_\mu^\nu \times D_{\mu\nu}(q) \Delta[\text{sgn}(eH)] \bigg|_{q_\perp^2 = -m^2},$$

(4.18)

where use has been made of Eq. (2.14). The presence of $\Delta[\text{sgn}(eH)]$ in (4.18) is again a consequence of the LLLA.

We note that the RHS of Eq. (4.18) appears to contain a gauge dependent part that arises from the gauge dependent term in $D_{\mu\nu}(q)$ [see Eq. (4.10)]. The WT identity in the BVA (4.4) guarantees that this would-be gauge dependent contribution to the fermion self-energy (denoted symbolically as $\Sigma_\xi$) is proportional to $(\gamma_\parallel \cdot p_\parallel + m)$. An explicit calculation shows that $\Sigma_\xi$ is given by

$$\Sigma_\xi = \alpha (\xi - 1) (\gamma_\parallel \cdot p_\parallel + m) \int_0^1 dx \int \frac{d^2q_\perp}{(2\pi)^2} \exp \left( -\frac{q_\perp^2}{2|eH|} \right) \left[ \frac{(1 - x)q_\perp^2 - x m (\gamma_\parallel \cdot p_\parallel - m)}{[(1 - x)q_\perp^2 + x(1 - x)p_\parallel^2 + xm^2]^2} \Delta[\text{sgn}(eH)] \right].$$

(4.19)

Since $\Sigma_\xi$ is proportional to $(\gamma_\parallel \cdot p_\parallel + m)$, it vanishes identically on the fermion mass shell $p_\parallel^2 = -m^2$ or, equivalently, $\gamma_\parallel \cdot p_\parallel + m = 0$. This, together with the transversality of the vacuum polarization, completes our proof that the BVA is a consistent truncation of the SD equations. Consequently, the dynamical fermion mass, obtained as the solution of the truncated SD equations evaluated on the fermion mass shell, is manifestly gauge independent.

Having proved the gauge independence of the dynamical fermion mass in the BVA, we are now ready to find $m$ by solving Eq. (4.18) self-consistently. Note that the transverse components in $D_{\mu\nu}(q)$ decouple in the LLLA. Following the same argument given above in the proof of the on-shell gauge independence of the fermion self-energy, one can verify fairly easily that contributions from the longitudinal components in $D_{\mu\nu}(q)$ proportional to $g_{\parallel\parallel}^\mu q_{\parallel\perp}^\nu / q_{\parallel\perp}^2$ are also proportional to $(\gamma_\parallel \cdot p_\parallel + m)$, hence they vanish identically on the fermion mass shell. Therefore, only the first term in $D_{\mu\nu}(q)$ proportional to $g_{\parallel\parallel}^\mu$ contributes to the on-shell SD equation (4.18). Consequently, the Dirac matrix structures...
on both sides of Eq. (4.18) are consistent. With these remarks we obtain

\[
m = -\frac{2e}{2\pi^2} \int \frac{d^4q}{(p-q)^2} \sum_{\mu=1}^N \frac{m}{q_\mu^2 + m^2} \exp \left( -\frac{q_0^2}{2|eH|} \right) \frac{q^2 + \Re \Pi(q_\parallel^2, q_\perp^2) - i\Im \Pi(q_\parallel^2, q_\perp^2)}{[q^2 + \Re \Pi(q_\parallel^2, q_\perp^2)]^2 + [\Im \Pi(q_\parallel^2, q_\perp^2)]^2} \bigg|_{p_\parallel = -m^2}.
\]  

(4.20)

Since \( \Im \Pi(q_\parallel^2, q_\perp^2) \) is an odd function of \( q_0 \) [see Eq. (4.14)], the imaginary part of the above integral vanishes identically. As a consequence, the dynamical fermion mass determined by the on-shell SD equation (4.20) is a real-valued quantity as it should be. Anticipating that the dynamical fermion mass is much less than the Landau energy, i.e., \( m^2 \ll |eH| \), we find that the real part of the above integral is dominated by the region of momentum \( m^2 \ll |q_0^2| \ll |eH| \) and \( q_\perp^2 \ll |eH| \). Therefore, the imaginary part of the polarization function \( \Im \Pi(q_\parallel^2, q_\perp^2) \) in the denominator in Eq. (4.20) can be neglected. Using the mass shell condition \( p_\mu = (m, 0) \) that corresponds to a LLL fermion at rest and performing a Wick rotation to Euclidean space, we find that Eq. (4.20) in Euclidean space reads

\[
m = \frac{\alpha}{2\pi^2} \int d^2q_\parallel \sum_{\mu=1}^N \frac{m}{q_\parallel^2 + (q_1 - m)^2 + m^2} \int_0^\infty dq_\perp^2 \frac{\exp(-q_\perp^2/2|eH|)}{q_\parallel^2 + q_\perp^2 + \Re \Pi(q_\parallel^2, q_\perp^2)}.
\]  

(4.21)

where \( q_\parallel^2 = q_3^2 + q_4^2 \).

Before proceeding further we note that, like what has usually been done in the literature, Eq. (4.21) can be obtained directly from Eq. (4.18) by making a Wick rotation to Euclidean space. While, as remarked in Ref. [53], this procedure is easy to justify in perturbation theory by neglecting the possibility of dynamically generated singularities in the first and third quadrants of the complex energy plane, it is highly nontrivial in nonperturbative studies and must be performed with care. Our detailed analysis of the photon polarization in the BVA shows clearly that no such singularities will be generated dynamically in the first and third quadrants of the complex \( q_0 \)-plane, thus leading to a justification for the Wick rotation to Euclidean space in this problem.

The generalization of our result to the case of QED with \( N_f \) fermion flavors can be done straightforwardly by the replacement \( \Pi(q_\parallel^2, q_\perp^2) \rightarrow N_f \Pi(q_\parallel^2, q_\perp^2) \) in Eqs. (4.10), (4.20) and (4.21). We have numerically solved Eq. (4.21) to obtain \( m \) as a function of \( \alpha \) for several values of \( N_f \). The results are displayed in Fig. 4. Numerical analysis shows that the solution of Eq. (4.21) can be fit by the following analytic expression:

\[
m = a \sqrt{2|eH|} \beta(\alpha) \exp \left( -\frac{\pi}{\alpha \log(b/N_f \alpha)} \right),
\]  

(4.22)

where \( a \) is a constant of order one, \( b \simeq 2.3 \), and \( \beta(\alpha) \simeq N_f \alpha \).
It is instructive to consider the case of QED with a large number of fermion flavors, i.e., the $1/N_f$ expansion in large-$N_f$ QED [21,28]. The advantage of the $1/N_f$ expansion is that it allows for a calculation of the dynamical fermion mass without a restriction to small $\alpha$. Specifically, we take the limit $N_f \rightarrow \infty$ with $N_f \alpha$ finite and fixed. This can be achieved by a rescaling of the coupling constant $e \rightarrow e/\sqrt{N_f}$, which together with Eq. (4.22) leads to the following result for the dynamical fermion mass in the large-$N_f$ limit:

$$m \simeq a \frac{\alpha}{\sqrt{2|eH|}} \exp \left[ - \frac{N_f \pi}{\alpha \log(b/\alpha)} \right], \quad (4.23)$$

where $\alpha$ is the rescaled fine-structure constant, which remains finite and fixed in the limit $N_f \rightarrow \infty$ as noted above. We will argue in the next section that Eq. (4.23) can also be obtained from the 2PI (two-particle-irreducible) effective action truncated at next-to-leading order in the $1/N_f$ expansion, which resums exactly the same infinite subset of diagrams as the SD equations truncated in the BVA.

We end this section by showing that the on-shell gauge independence of the fermion self-energy in the BVA can also be understood in terms of the wave function renormalization, i.e., the residue at the fermion pole. Since the gauge dependent contribution as well as contributions from the terms proportional to $q_{\parallel}^\mu q_{\parallel}^\nu / q_{\parallel}^2$ in the full photon propagator are proportional to $(\gamma_\parallel \cdot p_\parallel + m)$, they contribute only to the wave function renormalization in the full fermion propagator if the fermion self-energy were evaluated off the fermion mass shell. Hence they do not affect the position of the fermion pole, which is at $p_{\parallel}^2 = -m^2$ and gauge independent. This is precisely the reason that the physical dynamical fermion mass has to be determined by the pole of the full fermion propagator in a consistent truncation. The wave function renormalization is
gauge dependent, but this is perfectly fine since it is not a physical observable. Furthermore, the wave function renormalization can be absorbed into a redefinition of the fermion field. This in turn implies that up to a field redefinition the physical fermion pole at $p_\parallel^2 = -m^2$ has unit residue.

5 Comparison to previous works and discussions

Our results for the dynamical fermion mass differ from those obtained in the BVA in Refs. [18–20,22,24–29], in which the truncated fermion self-energy were evaluated exclusively off the fermion mass shell at $p_\parallel^2 = 0$. Consequently, the results for the dynamical fermion mass obtained in these earlier studies are inevitably gauge dependent and cannot be identified unambiguously as the physical dynamical fermion mass. We now argue that to a large extent those earlier results can be attributed to inconsistent truncation schemes, gauge dependent artifacts, or both.

In Refs. [18–20,22] chiral symmetry breaking in a strong magnetic field was first studied in the so-called rainbow approximation, in which the bare vertex and the free photon propagator were used. Apart from the gauge dependence, these earlier results are found to have a very different functional dependence on the gauge coupling. This is a consequence of an inconsistent truncation of the SD equations in that, as displayed in Fig. 2, the photon propagator that enters the SD equations in the BVA is the full one, hence could be replaced by the bare one if and only if the vacuum polarization effects are shown to be negligible. However, this is not the case for the problem under consideration. As discussed in Sec. 4, there is a strong screening effect in the longitudinal components of the photon propagator for photons with momenta $m^2 \ll |q_\parallel^2| \ll |eH|$ and $q_\perp^2 \ll |eH|$. This is precisely the reason that the numerical values of the dynamical fermion mass found in Refs. [18–20,22] tend to be overestimated by several orders of magnitude when compared to what we have obtained in Sec. 4.

More recently, chiral symmetry breaking in a strong magnetic field has been studied in the so-called improved rainbow approximation [24,25,28,29], in which the bare vertex and the full photon propagator were used. We note that the improved rainbow approximation used in Refs. [24,25,28,29] is exactly the same as the bare vertex approximation used in this article. It can be verified fairly easily that the truncated SD equations in both approximations resum identically the same infinite subset of diagrams.

The authors of Refs. [24,25,28] claimed that (i) in covariant gauges there are one-loop vertex corrections arising from the term $q_\mu q_\nu / q^2 q_\parallel^2$ in the full photon propagator that are not suppressed by powers of $\alpha$ (up to logarithms) and
hence need to be accounted for; (ii) there exists a noncovariant and nonlocal
gauge in which, and only in which, the BVA is a reliable truncation of the
SD equations that consistently resums these one-loop vertex corrections. The
gauge independent analysis in the BVA, as presented in this article, shows
clearly that such contributions vanish identically on the fermion mass shell.
Furthermore, it is evident that if one expands diagrammatically the SD equa-
tions in the BVA, one finds by induction that to all orders in the loop expansion
there is not a single diagram with vertex corrections being resummed by the
SD equations (see Fig. 2). This statement is true in any gauge (be it covariant
or otherwise), because gauge fixing affects only the explicit form of the (free
or full) photon propagator but not the topology of the Feynman diagrams.
Therefore, both the above quoted conclusions are incorrect.

We emphasize that the WT identity is a necessary condition for establishing
the gauge independence of the dynamical fermion mass, but it is far from suf-
ficient. While the WT identity guarantees the truncated vacuum polarization
is transverse, it guarantees only the truncated on-shell fermion self-energy is
gauge independent. Hence, the dynamical fermion mass is gauge independent
only when determined by the position of the fermion pole obtained in a con-
sistent truncation. This is tantamount to evaluating the truncated fermion
self-energy on the fermion mass shell. Even though the WT identity in the
BVA is verified in Refs. [24,25] in a particular noncovariant and nonlocal
gauge, this does not guarantee that the dynamical fermion mass obtained
therein from the truncated fermion self-energy evaluated off the fermion mass
shell will be gauge independent. In fact, the authors of Refs. [24,25,28] simply
chose a particular noncovariant and nonlocal gauge such that the gauge de-
pendent contribution happens to cancel contributions from terms proportional
to $q^\mu q^\nu / q^2$ in the full photon propagator [56]. Our gauge independent analysis
in the BVA reveals clearly that such a gauge fixing not only is ad hoc and
unnecessary, but also leaves the issue of gauge independence unaddressed.

In Ref. [29] its authors claimed that (i) in QED with $N_f$ fermion flavors a
critical number $N_{cr}$ exists for any value of $\alpha$, such that chiral symmetry re-
mains unbroken for $N_f > N_{cr}$; (ii) the dynamical fermion mass is generated
with a double splitting for $N_f < N_{cr}$. As can be gleaned clearly from Fig. 4
or from the large-$N_f$ result, Eq. (4.23), both the conclusions quoted above are
incorrect. They are gauge dependent artifacts of an inconsistent truncation:
On the one hand, the SD equation for the momentum independent fermion
self-energy was obtained (in an unspecified “appropriate” gauge) in the BVA
within the LLLA. On the other hand, the vacuum polarization was calculated
in the BVA but beyond the LLLA. This however is not a consistent truncation
of the SD equations since, as shown in the end of Sec. 3, the WT identity in
the BVA can be satisfied only within the LLLA.

The results of Ref. [29] suggest that in the inconsistent truncation as well
as in the unspecified “appropriate” gauge used there, the unphysical, gauge dependent contributions from higher Landau levels are so large that they become dominant over the physical, gauge independent contribution from the LLL and lead to the authors’ incorrect conclusions. Hence, we emphasize that the LLL dominance in a strong magnetic field should be understood in the context of consistent truncation schemes as follows. Contributions to the dynamical fermion mass from higher Landau levels that are obtained in a (yet to be determined) consistent truncation of the SD equations are subleading when compared to that from the LLL obtained in the consistent BVA truncation. Our analysis of the WT identity in the BVA but beyond the LLLA indicates that in order to go beyond the LLLA, one has to use truncation schemes that consistently account for vertex corrections. To the best of our knowledge, consistent truncations of the SD equations that include vertex corrections either within or beyond the LLLA have not appeared in the literature. In view of this, a detailed study of the general structure as well as the analytic properties of the full vertex akin to that in Ref. [57] but in the presence of a constant external magnetic field is of crucial importance.

Extension to a momentum dependent fermion self-energy in the BVA has been studied in a certain gauge by neglecting corrections to the vacuum polarization [24–28]. It was found that the fermion self-energy has very weak momentum dependence in the region relevant to fermion pairing in a strong magnetic field. Since a momentum dependent fermion self-energy cannot fulfill the WT identity in the BVA, the results obtained in these studies are inevitably gauge dependent. It is, however, important to note that when compared to the corresponding results obtained using a momentum independent fermion self-energy, these results lend support to the reliability of a momentum independent fermion self-energy and, consequently, of the WT identity in the BVA in the momentum region $|p_\parallel^2| \ll |eH|$ that is relevant to the dynamics of fermion pairing in a strong magnetic field. Similar study has also been carried out in a certain gauge in the case of QED with an extra charge neutral, self-interacting scalar field that couples to the fermion through the Yukawa interaction [58,59].

Despite obtained from gauge dependent calculations in a certain gauge, as will be argued momentarily, the conclusion from these studies [24–28] that a momentum independent fermion self-energy or, equivalently, the WT identity in the BVA is reliable in the momentum region relevant to chiral symmetry breaking in a strong magnetic field remains valid. In fact, this is precisely the underlying physical reason that the BVA is a consistent truncation of the SD equations in the LLLA.

In the BVA the presence of momentum dependent terms in the fermion self-energy leads to violation of the WT identity, hence introduces gauge dependent contributions to the SD equations even though the latter are evaluated on the
fermion mass shell. We now argue that such gauge dependent contributions are of higher orders when compared to the dynamical fermion mass obtained from a momentum independent fermion self-energy. Therefore, within the LLLA the dynamical fermion mass can be reliably calculated in the BVA by using a momentum independent fermion self-energy.

To begin with, we first note that the infinite subset of diagrams being re-summed in Refs. [24–28] are identical to, as well as in the same kinematic region as, those in the present article. The only difference is that in those references the calculation is carried out in a certain gauge and the fermion self-energy is evaluated off the fermion mass shell at \( p_\parallel^2 = 0 \). Since the dynamical fermion mass is much less than the Landau energy, i.e., \( m^2 \ll |eH| \), the LLL fermion self-energy \( \Sigma(p_\parallel) \) with momentum \( |p_\parallel^2| \ll |eH| \) can be expanded in powers of \( p_\parallel^2/|eH| \) around the fermion mass shell in the BVA (i.e., \( p_\parallel^2 = -m^2 \)) as [see Eq. (3.13)]

\[
A_\parallel(p_\parallel) = a(\xi) x + O(x^2), \tag{5.1}
\]
\[
B(p_\parallel) = m [1 + b(\xi) x + O(x^2)], \tag{5.2}
\]

where \( x = (p_\parallel^2 + m^2)/|eH| \), and \( a(\xi) \) and \( b(\xi) \) are \( \xi \)-dependent coefficients presumably of order one. This, together with the gauge dependent results obtained in the LLLA [24–28] regarding the reliability of a momentum independent fermion self-energy in the BVA in the momentum region \( |p_\parallel^2| \ll |eH| \), indicates clearly that the gauge dependent contributions to the dynamical fermion mass arising from the momentum dependent terms in \( A_\parallel(p_\parallel) \) and \( B(p_\parallel) \) are of order \( |p_\parallel^2|/|eH| \). Hence physical quantities sensitive only to the kinematic region of momenta \( |p_\parallel^2| \ll |eH| \) can be reliably calculated in the BVA, and are gauge independent up to corrections of order \( |p_\parallel^2|/|eH| \ll 1 \). This in turn means that, consistent with the underlying assumption of the LLLA, the kinematic regime of low-energy photon exchange \( (|q_\parallel^2|, q_\perp^2 \ll |eH|) \) is precisely that which determines the dynamics of fermion pairing in a strong magnetic field.

We are now in a position to establish direct contact between the SD equations truncated in the BVA and the 2PI (two-particle-irreducible) effective action truncated at the lowest nontrivial order in the loop expansion as well as in the \( 1/N_f \) expansion. Since the 2PI effective action is constructed in terms of the full propagators and bare vertices, the corresponding WT identities usually cannot be satisfied. In general, any truncation of the 2PI effective action is inevitably gauge dependent and hence, strictly speaking, an inconsistent one. Nevertheless, it has been shown recently in Refs. [60–62] that the truncated on-shell 2PI effective action has a controlled gauge dependence with the explicit gauge dependent terms always appearing at higher order. This is reminiscent of the situation that the dynamical fermion mass obtained from the on-shell SD equations truncated in the BVA with a momentum dependent fermion
Fig. 5. (a) Contribution to the 2PI effective action at two-loop order in the loop expansion or at next-to-leading order in the $1/N_f$ expansion. (b) The corresponding fermion self-energy and vacuum polarization at the same order. All internal lines denote the full propagators and all vertices are the bare ones.

self-energy that violates the corresponding WT identity, is gauge independent up to corrections of higher order

In Appendix B we briefly summarize some exact relations derived from the 2PI effective action that are useful to our discussion here. The interested reader is referred to the literature [63] for further details. For QED in a constant external magnetic field, the 2PI effective action is given by

$$\Gamma[A, G, D] = S_0[A] - i \text{Tr} \log G^{-1} - i \text{Tr} S^{-1}(G - S)$$

$$+ \frac{i}{2} \text{Tr} \log D^{-1} + \frac{i}{2} \text{Tr} D^{-1}(D - D) + \Gamma_2[G, D], \quad (5.3)$$

where $S_0[A]$ is the classical Maxwell action with $A_\mu$ being the external gauge field, and the propagators are the same as those in Sec. 3. Specifically, we consider the LLLA hence the fermion propagators are those for the LLL fermions. In actual applications, the 2PI effective action is truncated at some order in a chosen expansion parameter. For the purpose of our discussion, it suffices to consider only the lowest nontrivial order truncation, namely, at two-loop order in the loop expansion in QED or at next-to-leading order [i.e., $\mathcal{O}(N_f^0)$] in the $1/N_f$ expansion in large-$N_f$ QED.

At the lowest nontrivial order in the loop expansion or in the $1/N_f$ expansion $\Gamma_2[G, D]$ is given by

$$\Gamma_2[G, D] = -\frac{ie^2}{2} \int d^4x \, d^4y \, \text{tr} \gamma^\mu G(x, y) \gamma^\nu G(y, x) D_{\mu\nu}(x, y), \quad (5.4)$$
where in the $1/N_f$ expansion $e$ is understood as the rescaled coupling that remains finite and fixed in the limit $N_f \to \infty$. Functional differentiations of $\Gamma[A,G,D]$ with respect to the full propagators in the absence of sources leads to the SD equations for the respective propagators. The corresponding fermion self-energy and vacuum polarization are obtained by varying $\Gamma_2[G,D]$ with respect to the corresponding full propagators. The resultant SD equations are found to be exactly the same as those in the BVA with the corresponding fermion self-energy and vacuum polarization given by Eqs. (4.16) and (4.5), respectively. A diagrammatical representation of the 2PI effective action is depicted in Fig. 5, in which it is noted that the vertices that appear in $\Gamma_2[G,D]$ as well as in the corresponding fermion self-energy and vacuum polarization are the bare ones. With the above observation one can conclude fairly easily that the 2PI effective action truncated at the lowest nontrivial order in the loop expansion or in the $1/N_f$ expansion resums identically the same infinite subset of diagrams as the SD equations truncated in the BVA.

The direct correspondence with the 2PI effective action truncated at the lowest nontrivial order has the following important consequences. On the one hand, the WT identity in the BVA (4.4) guarantees that the truncation at the lowest nontrivial order in the loop expansion or in the $1/N_f$ expansion with a momentum independent fermion self-energy is a consistent truncation of the 2PI effective action in the momentum region relevant to chiral symmetry breaking in a strong magnetic field. On the other hand, the facts that vertex corrections in the 2PI effective action start to appear at the next-to-lowest-nontrivial order in the loop expansion or in the $1/N_f$ expansion and that the truncated on-shell 2PI effective action (in general with a momentum dependent fermion self-energy) has a controlled gauge dependence with the explicit gauge dependent terms always appearing at higher order, provide an unequivocal justification of our argument that in a strong magnetic field the dynamical fermion mass to leading order in the gauge coupling or in the $1/N_f$ expansion can be reliably calculated in the BVA, and is gauge independent up to corrections of higher order. Whereas a direct verification by explicit calculation is indispensable, to the best of our knowledge consistent truncations of the SD equations that include vertex corrections either within or beyond the LLLA have not appeared in the literature. Clearly such a task is beyond the scope of the current article, and will be the subject of further investigations.

6 Summary and conclusions

In this article we presented a critical and detailed study of chiral symmetry breaking in QED in a constant external magnetic field. The main goal is to determine in a gauge independent manner the dynamical fermion mass generated through chiral symmetry breaking.
We focused on the strong field limit as well as on the weak coupling regime. The former leads to a wide separation of energy scales such that the dynamics of fermion pairing is dominated by the lowest Landau level. The latter allows for a controlled, nonperturbative calculation of the dynamical fermion mass, in the sense that there exists a systematic expansion in powers of the gauge coupling (up to logarithms) such that contributions of leading order in the gauge coupling that arise from every order in the loop expansion can be consistently accounted for, while subleading contributions are suppressed by powers of the gauge coupling (up to logarithms). Consequently, in the weak coupling regime a gauge independent, consistent truncation of the SD equations is tantamount to a good and reliable approximation.

The WT identity in the BVA is at the heart of our proof that the BVA is a consistent truncation of the SD equations within the LLLA. We first verified that, with the corrected orthonormal condition of the $E_p$ functions for the LLL fermions, the conclusion of Ref. [23] that in order to satisfy the WT identity in the BVA within the LLLA, the fermion self-energy has to be a momentum independent constant remains valid. Furthermore, we showed that the WT identity in the BVA can be satisfied only within the LLLA. The proof then proceeds with the assumption that the BVA is a consistent truncation such that the position of the fermion pole obtained therein is gauge independent. With this assumption, the WT identity in the BVA implies that the pole of the LLL fermion propagator is located at $p_\parallel^2 = -m^2$, with $m$ being the momentum independent fermion self-energy as well as the physical, gauge independent dynamical fermion mass. The proof is completed by verifying that such an assumption does not lead to any inconsistency. This is achieved by showing that (i) the truncated vacuum polarization is transverse, (ii) the truncated fermion self-energy evaluated on the fermion mass shell is manifestly gauge independent. In particular, we showed that the would-be gauge dependent contribution to the truncated fermion self-energy that arises from the gauge dependent term in the full photon propagator vanishes identically on the fermion mass shell. This detailed study leads to a gauge independent, nonperturbative calculation of the dynamical fermion mass to leading order in the gauge coupling. Furthermore, it allows one to identify unambiguously the infinite subset of diagrams that contribute to chiral symmetry breaking in a strong magnetic field.

We made a detailed comparison between our results and those obtained in previous works [18–20,22,24–29]. The gauge independent analysis we presented in this article shows clearly that the results as well as the conclusions of those earlier studies are inevitably gauge dependent and can be attributed to inconsistent truncation schemes, gauge dependent artifacts, or both. Nevertheless, based on the gauge dependent results [24–28] obtained within the LLLA regarding the reliability of a momentum independent fermion self-energy in the BVA in the momentum region $|p_\parallel^2| \ll |eH|$, we argued that in a strong mag-
netic field the dynamical fermion mass can be reliably calculated in the BVA and is gauge independent up to corrections of higher order. This is consistent with the underlying assumption of the LLLA that the kinematic region of low-energy photon exchange ($|q^2_\parallel|, q^2_\perp \ll |eH|$) is precisely that which determines the dynamics of fermion pairing in a strong magnetic field.

Motivated by the fact that the gauge dependence of the dynamical fermion mass calculated in the BVA but with a momentum dependent fermion self-energy is subleading, we established a direct contact between the SD equations truncated in the BVA and the 2PI effective action truncated at the lowest nontrivial order in the loop expansion as well as in the $1/N_f$ expansion. This direct correspondence, together with the facts that vertex corrections in the 2PI effective action start to appear at the next-to-lowest-nontrivial order in the loop expansion or in the $1/N_f$ expansion and that the truncated on-shell 2PI effective action has a controlled gauge dependence with the explicit gauge dependent terms always appearing at higher order, provides the justification of our argument that in a strong magnetic field the dynamical fermion mass can be reliably calculated in the BVA to leading order in the gauge coupling or in the $1/N_f$ expansion.

In conclusion, the presence of a strong, constant external magnetic field provides a consistent truncation of the Schwinger-Dyson equations in terms of the bare vertex approximation within the lowest Landau level approximation. This allows us to study in a gauge independent manner the physics of chiral symmetry breaking in QED in an external magnetic field and to obtain a solution for the physical dynamical fermion mass. The gauge independent approach to the Schwinger-Dyson equations in gauge theories that we presented in this article is quite general in nature, and hence not specific to the problem of chiral symmetry breaking in a strong magnetic field discussed here. We believe this approach will be useful in other areas of physics that also require a nonperturbative understanding of gauge theories, such as the studies of hadronic structure and the connection between quark confinement and chiral symmetry breaking in nuclear and particle physics [50,64], the physics of superconductivity and superfluidity in condensed matter systems [65,66], as well as the equilibrium and nonequilibrium properties of hot and dense matter [67,68].

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A Vacuum current in a constant magnetic field

Using the Ritus $E_p$ functions and the general structure of the fermion self-energy, we shall show in a nonperturbative manner that the vacuum current vanishes identically in a constant external magnetic field. The vacuum current is defined to be

$$J^\mu = ie \text{tr} \gamma^\mu G(x, x)$$

$$= ie \text{tr} \sum \frac{d^4 p}{(2\pi)^4} E_p(x) \gamma^\mu E_p(x) \frac{1}{\gamma \cdot p + \Sigma(p)},$$

where in the second equality we have used the momentum representation of the full fermion propagator given by Eq. (2.32). It is noted that in this appendix the fermion propagator and the fermion self-energy will not be restricted to those for the LLL fermions.

With the properties (2.14) and (2.15), the integration over $p_2$, one of the spin sums, as well as the trace can be done easily, leading to

$$J^\mu = \frac{ie}{\pi} |eH| \sum_{l=0}^{\infty} \sum_{\sigma = \pm 1} \int \frac{d^2 p_\parallel}{(2\pi)^2} \frac{[1 + A_\parallel(p)] p_\parallel^\mu}{(1 + A_\parallel(p))^2 p_\parallel^2 + [1 + A_\perp(p)]^2 p_\perp^2 + B^2(p)},$$

where $p_\parallel^2 = -p_0^2 + p_3^2$, $p_\perp^2 = 2|eH|l$, and use has been made of the Dirac matrix structure of the fermion self-energy in Eq. (3.13). Since $A_{\parallel,\perp}(p)$ and $B(p)$ are functions of the longitudinal and transverse momentum squared, one finds that $J^\mu$ vanishes identically. The same conclusion also applies to the vacuum current in the LLLA, in which the sum over $l$ is restricted to the $l = 0$ term and that over $\sigma$ is restricted to $\sigma = \text{sgn}(eH)$.

We note that the vanishing of the vacuum current in a constant external magnetic field is a consequence of the Lorentz boost invariance along, as well as the rotational invariance about, the direction of the constant external magnetic field, namely, the $x_3$-direction.
B 2PI effective action

In this appendix we briefly summarize some exact relations derived from the 2PI (two-particle-irreducible) effective action [63] that are useful to our discussion. For presentational simplicity we consider here only bosonic fields, the generalization to fermionic fields is straightforward. The generating functional for connected correlation functions is defined as

\[ Z[J, K] = e^{i W[J, K]} = \int D\varphi e^{i(S[\varphi] + J_i \varphi^i + \frac{1}{2} \varphi^i K_{ij} \varphi^j)}, \quad (B.1) \]

where \( S[\varphi] \) is the classical action, \( \varphi^i \) denotes bosonic fields and \( J \) \((K)\) is the auxiliary local (bilocal) source. We use a shorthand notation in which Latin indices stand for space-time variables as well as internal indices, and summation and integration over repeated indices are understood, e.g.,

\[ J_i \varphi^i = \int d^4x \ J(x) \ \varphi(x). \quad (B.2) \]

The mean field \( \phi^i = \langle \varphi^i \rangle \) and the connected two-point function \( G^{ij} = \langle T \varphi^i \varphi^j \rangle - \langle \varphi^i \rangle \langle \varphi^j \rangle \) are obtained by functional differentiations of \( W[J, K] \) with respect to the local source \( J \) as

\[ i \frac{\delta W}{\delta (i J_i)} = \phi^i, \quad i \frac{\delta^2 W}{\delta (i J_i) \delta (i J_j)} = G^{ij}. \quad (B.3) \]

Functional differentiations of \( W[J, K] \) with respect to the bilocal source \( K \) may generate also disconnected correlation functions. For example, differentiating once with respect to \( K \) leads to

\[ i \frac{\delta W}{\delta (i K_{ij})} = \frac{1}{2}(\phi^i \phi^j + G^{ij}). \quad (B.4) \]

A functional Legendre transform in the mean field \( \phi \) and the connected two-point function \( G \) leads to the 2PI effective action

\[ \Gamma[\phi, G] = W[J, K] - J_i \phi^i - \frac{1}{2} K_{ij} (\phi^i \phi^j + G^{ij}). \quad (B.5) \]

From Eqs. (B.3) and (B.4) one can derive the relations

\[ \frac{\delta \Gamma[\phi, G]}{\delta \phi^i} = -J_i - K_{ij} \phi^j, \quad \frac{\delta \Gamma[\phi, G]}{\delta G^{ij}} = -\frac{1}{2} K_{ij}. \quad (B.6) \]

The 2PI effective action can be cast into a very convenient form which has a simple diagrammatical interpretation in terms of 2PI diagrams (for fermionic...
fields the factors of $1/2$ before the trace terms are replaced by $-1$)

$$
\Gamma[\phi, G] = S_0[\phi] + \frac{i}{2} \text{Tr} \log G^{-1} + \frac{i}{2} \text{Tr} G_0^{-1} (G - G_0) + \Gamma_2[\phi, G],
$$
\text{(B.7)}

where $S_0$ is the free part of the classical action and $G_0$ is the bare two-point function

$$
G_{0ij} = \left( -i \frac{\delta^2 S_0[\phi]}{\delta \phi^i \delta \phi^j} \right)^{-1}.
$$
\text{(B.8)}

The functional $\Gamma_2[\phi, G]$ is the sum of all 2PI skeleton vacuum diagrams with bare vertices and full propagators. Here, skeleton diagrams are those without self-energy insertions.

The equations of motion for the mean field $\phi$ and the connected two-point function $G$ are determined by the stationary condition, i.e., from the implicit functional equation (B.6) for vanishing sources $J$ and $K$, as

$$
\frac{\delta \Gamma[\phi, G]}{\delta \phi^i} = 0, \quad \frac{\delta \Gamma[\phi, G]}{\delta G^{ij}} = 0.
$$
\text{(B.9)}

With Eq. (B.7) the equation of motion for the connected two-point function in Eq. (B.9) becomes

$$
G_{ij}^{-1} = G_{0ij}^{-1} + \Sigma_{ij},
$$
\text{(B.10)}

where $\Sigma$ is the one-particle-irreducible self-energy (for fermionic fields the factor of $-2$ is replaced by 1)

$$
\Sigma_{ij} = -2i \frac{\delta \Gamma_2[\phi, G]}{\delta G^{ij}}.
$$
\text{(B.11)}

It is noted that Eq. (B.10) is the Schwinger-Dyson equation for the full propagator $G$.

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