Local geodesics for plurisubharmonic functions
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Abstract
We study geodesics for plurisubharmonic functions from the Cegrell class $F_1$ on a bounded hyperconvex domain of $\mathbb{C}^n$ and show that, as in the case of metrics on Kähler compact manifolds, they linearize an energy functional. As a consequence, we get a uniqueness theorem for functions from $F_1$ in terms of total masses of certain mixed Monge-Ampère currents. Geodesics of relative extremal functions are considered and a reverse Brunn-Minkowski inequality is proved for capacities of multiplicative combinations of multi-circled compact sets. We also show that functions with strong singularities generally cannot be connected by (sub)geodesic arcs.

1 Introduction
Starting with pioneer work by Mabuchi [19], a notion of geodesics in the space of Kähler metrics on compact complex manifolds has been playing a prominent role in Kähler geometry and has found a lot of applications. We will not give here any detailed account on this subject; the interested reader can consult, for example, [23], [10], [14], [1], [5], [15], and the bibliography therein. In particular, geodesics in the space of metrics on a compact $n$-dimensional Kähler manifold $(X, \omega)$ have been characterized as solutions to a complex homogeneous equation, which implies linearity of the Mabuchi functional

$$
\mathcal{M}(\psi, \phi_0) = \frac{1}{n+1} \int_X (\psi - \phi_0) \sum_{k=0}^{n} (dd^c \psi)^k \wedge (dd^c \phi_0)^{n-k} \tag{1}
$$

along the geodesics $\psi = \psi_t$ (here $\phi_0$ is a reference metric).

We believe however that a local, flat situation of functions on a bounded pseudoconvex domain $D$ of $\mathbb{C}^n$ deserves independent consideration, at least because of possible applications. The simplest choice here are functions with zero boundary values on $\partial D$ and finite total Monge-Ampère mass. To provide existence of the corresponding boundary problem on $D \times \{1 < |\zeta| < e\}$, we require also finiteness of the Monge-Ampère energy $E(u) = \int_D u(dd^c u)^n$. For such (not necessarily bounded) plurisubharmonic functions we show in Theorem 5.2 that the energy functional $u \mapsto E(u)$ plays role of the Mabuchi functional (1). We use this in proving a uniqueness result (Theorem 3.4 and Corollary 5.3) for functions from the Cegrell class $F_1(D)$ in terms of total masses of $n+1$ mixed Monge-Ampère currents on $D$.

We discuss briefly geodesics connecting relative extremal functions $\omega_{K_j}$ of compact subsets $K_j$ of $D$. In the multi-circled case, a variant of reversed Brunn-Minkowski inequality is proved for the Monge-Ampère capacities of multiplicative combinations of $K_j$. We present a simple example where the geodesic functions $u_t$ are still relative extremal functions, however not of compact sets but of multi-plate condensers.

The case of bounded functions (Theorem 3.3) is close to the classical setting of Kähler metrics, with a modification to handle the boundary effects. The general case requires a
justification for existence of solutions of the corresponding boundary problem like that in [3] and [11]. We show that while this works for $F_1(D)$ (Theorem 5.2), for functions with strong singularities (say, with positive Lelong numbers) such a problem generally has no solution (Theorem 6.2).

2 Energy functional on Cegrell classes

Let $D \subset \mathbb{C}^n$ be a bounded hyperconvex domain. We recall that Cegrell’s class $E_0(D)$ consists of bounded plurisubharmonic functions $u$ in $D$ with zero boundary values on $\partial D$ and finite total Monge-Ampère mass

$$\int_D (dd^c u)^n < \infty;$$

class $E_1(D)$ consists of functions $u$ that are limits of decreasing sequences $u_j \in E_0(D)$ such that

$$\sup_j \int_D |u_j|(dd^c u_j)^n < \infty;$$

if, in addition,

$$\sup_j \int_D (dd^c u_j)^n < \infty,$$

then $u \in F_1(D)$.

If $u \in E_1(D)$, then the current $(dd^c u)^n$ is defined as the limit of $(dd^c u_j)^n$ and is independent of the choice of the approximating sequence $u_j$ [7, Thm. 3.8].

For any function $u \in E_1(D)$, consider its energy functional

$$E(u) = (n + 1) \mathcal{M}(u, 0) = \int_D u(dd^c u)^n. \quad (2)$$

For any sequence $u_j$ from the definition of $E_1(D)$, we have $E(u_j) \to E(u)$ [7 Thm. 3.8].

Similarity with the Mabuchi functional [11] for metrics on compact manifolds becomes visible from the following important identity.

**Proposition 2.1** For any $u, v \in E_1(D)$,

$$E(u) - E(v) = \int_D (u - v) \sum_{k=0}^{n} (dd^c u)^k \wedge (dd^c v)^{n-k}. \quad (3)$$

**Proof.** This easily follows from the integration by parts formula

$$\int_D u dd^c v \wedge T = \int_D v dd^c u \wedge T \quad (4)$$

valid for $u, v \in E_1$ and positive closed currents $T$ [3 Cor. 3.4].

**Corollary 2.2** If $u, v \in E_1(D)$ satisfy $u \leq v$, then $E(u) \leq E(v)$. If, in addition, $u \in F_1(D)$ and $E(u) = E(v)$, then $u = v$ on $D$.  

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Proof. The inequality is well known (see, for example, [7 Thm. 3.8]) and follows, in particular, directly from Proposition 2.1.

The condition $E(u) = E(v)$ gives us, by (3), $(dd^cu)^n = 0$ on the set $A = \{z: u(z) < v(z)\}$. We claim that this implies $u = v$ everywhere in $D$. In [6], this was proved for locally bounded $u$ and $v$; we adapt the proof to our case. Let $P(z) = |z|^2 - C$ $C$ $v(z)\}$ has positive Lebesgue measure for some $\eta > 0$. By [7 Lemma 4.4],

$$\eta^n \int_{A_\eta} (dd^cP)^n \leq \int_{A_\eta} (dd^c(\eta P + v)^n) \leq \int_{A_\eta} (dd^cu)^n \leq \int_{\{\eta < v\}} (dd^cu)^n = 0,$$

which contradicts the positivity of the Lebesgue measure of $A_\eta$. 

Remark. The second statement of Corollary 2.2 remains true if the condition $u \in F_1(D)$ is replaced by $u \in F_1(D)$ and $u(z) = 0$ as $z \to \partial D$. In this case (increasing, if needed, the constant $C$ in the definition of the function $P$), the set $A_\eta$ is compactly supported in $D$ and thus both $u$ and $v$ have finite Monge-Ampère mass on a neighborhood of $\overline{A_\eta}$, so [7 Lemma 4.4] still can be applied.

3 Geodesics for the class $E_0$

Let $S$ be the annulus $\{\zeta \in \mathbb{C}: 1 < |\zeta| < e\}$ bounded by the circles $S_0 = \{|\zeta| = 1\}$ and $S_1 = \{|\zeta| = e\}$. Given two functions $u_0, u_1 \in E_0(D)$, consider the class $W(u_1, u_2)$ of all functions $u \in \text{PSH}^{-}(D \times S)$ such that $\limsup u(z, \zeta) \leq u_j(z)$ as $\zeta \to S_j$. The class is not empty because, for example, is contains $u_0 + u_1$.

Denote

$$\hat{u}(z, \zeta) = \sup\{u(z, \zeta): u \in W(u_1, u_2)\}.$$ 

Since its u.s.c. regularization $\hat{u}^*$ belongs to $W(u_1, u_2)$, we have $\hat{u} = \hat{u}^*$. Moreover, being a maximal plurisubharmonic function, it satisfies the homogeneous Monge-Ampère equation

$$(dd^c\hat{u})^{n+1} = 0 \text{ on } D \times S. \quad (5)$$

Evidently, $\hat{u}(z, \zeta) = \hat{u}(z, |\zeta|)$ on $D \times S$, so the function $u_t(z) := \hat{u}(z, et)$ is convex in $t \in (0, 1)$; we will call it the geodesic of $u_0$ and $u_1$. Similar to [5], we get

**Proposition 3.1** The geodesic $u_t$ of $u_0, u_1 \in E_0(D)$ has the following properties:

(i) $u_t(z) \to 0$ as $z \to \partial D$;

(ii) $u_t \to u_j$ as $t \to j$, uniformly on $D$ ($j = 0, 1$);

(iii) $u_t \leq U_t := (1 - t)u_0 + tu_1$;

(iv) $u_t \geq s_t := \max\{u_0 - M_1 t, u_1 - M_0 (1 - t)\}$, where $M_j = ||u_j||_\infty$. 

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Proof. Since \( u_t \geq u_0 + u_1 \), we have (i). Relation (iii) follows because \( U_0 = u_0, U_1 = u_1 \) and \( U_t \) is harmonic in \( t \) (while \( u_t \) is convex in \( t \)). The lower bound (iv) is evident because \( s(z, \zeta) := s_{\log |\zeta|}(z) \) belongs to \( W(u_0, u_1) \). Finally, (iii) and (iv) imply (ii). \( \square \)

A family of functions \( v_t \in \mathcal{E}_0(D) \), \( 0 < t < 1 \), will be called a subgeodesic for \( u_0 \) and \( u_1 \) if \( \tilde{v}(z, \zeta) := v_{\log |\zeta|}(z) \in W(u_0, u_1) \).

Let us study values of the energy functional \( E \) on curves in \( \mathcal{E}_0(D) \). Here again we get its properties as in the case of compact manifolds.

**Proposition 3.2** The functional \( v \mapsto E(v) \) is concave on \( \mathcal{E}_0(D) \).

Proof. Let \( U_t = (1-t)u_0 + tu_1, 0 < t < 1 \). By Proposition 2.1,
\[
\frac{d}{dt} E(U_t) = (n + 1) \int_D (u_1 - u_0)(dd^c U_t)^n,
\]
so
\[
\frac{1}{n+1} \frac{d^2}{dt^2} E(U_t) = n \int_D (u_1 - u_0) \wedge dd^c (u_1 - u_0) \wedge (dd^c U_t)^{n-1} = -n \int_D d(u_1 - u_0) \wedge d^c (u_1 - u_0) \wedge (dd^c U_t)^{n-1} \leq 0,
\]
which proves the claim. \( \square \)

It also turns out that, on the other hand, the function \( E(v_t) \) is convex along subgeodesics.

**Theorem 3.3** Let \( v_t \) be a subgeodesic for \( u_0, u_1 \in \mathcal{E}_0(D) \). Then the function \( t \mapsto E(v_t) \) is convex, and it is linear if and only if the subgeodesic \( v_t \) is a geodesic.

Proof. The idea of the proof is similar to that for Proposition 3.2, however it needs more technicalities.

Convexity of \( E(v_t) \) is equivalent to subharmonicity of the function
\[
\tilde{E} = E(\tilde{v}) = \int_D \tilde{v}(dz \wedge d^c z)^n,
\]
and the linearity of \( E \) corresponds to the harmonicity of \( \tilde{E} \). The corresponding result for the Mabuchi functional (1) on a compact manifold \( X \) follows from the formula
\[
d_{\zeta} d_{\zeta}^c \tilde{E} = \int_X (d^c \tilde{v})^{n+1}
\]
(see, for example, [1]), and one gets then the claims from the plurisubharmonicity of the subgeodesics and equation (5).

In the case of functions from \( \mathcal{E}_0(D), D \subset \mathbb{C}^n \), one can argue as follows. By [9 Thm. 1.2], \( \tilde{v} \) is the limit of a decreasing sequence of smooth functions \( \tilde{v}^{(j)} \) from \( \mathcal{E}_0(D \times S) \); clearly, they can be assumed to be independent of the argument of \( \zeta \). Furthermore, since \( v_t^{(j)} \in \mathcal{E}_0(D) \),
decrease to \( v_t \in \mathcal{E}_0(D) \), we have \( \mathbf{E}(v_t^{(j)}) \to \mathbf{E}(v_t) \) by [7] Thm. 3.8. So, we can assume \( \widehat{v} \in \mathcal{E}_0(D \times S) \cap C^\infty(D \times S) \).

Note that the aforementioned approximation theorem rests on the following result from [18], see also [9] Lem. 2.2: If \( \varphi, \psi \in \text{PSH}(\Omega) \) and \( b : \mathbb{R} \to \mathbb{R}_+ \) is a smooth convex function with \( b(x) = |x| \text{ for all } |x| > \epsilon > 0 \), then \( \max(\varphi, \psi) := \varphi + \psi + b(\varphi - \psi) \in \text{PSH}(\Omega) \).

If we take here \( \Omega = D \times S, \varphi = \widehat{v} - 2\epsilon, \) and \( \psi = \rho/\epsilon \) for a smooth exhaustion function \( \rho \) of \( D \) (which exists by [9] Cor. 1.3), then \( \max(\varphi, \psi) \in \mathcal{E}_0(D \times S) \cap C^\infty(D \times S) \). Moreover, it coincides with \( \rho/\epsilon \) near \( \partial D \times S \), so it is independent of \( \zeta \) there. Since \( \max_u(\varphi, \psi) \to \widehat{v} \) uniformly as \( \epsilon \to 0 \), we can thus also assume \( d_z^c \widehat{v} = 0 \) near \( \partial D \).

By Proposition 2.1

\[
d_z^c \widehat{E} = (n + 1) \int_D d_z^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n,
\]

so

\[
\frac{1}{n + 1} d_z d_z^c \widehat{E} = \int_D d_z d_z^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n + n \int_D d_z^c \widehat{v} \wedge d_z (d_z d_z^c \widehat{v}) \wedge (d_z d_z^c \widehat{v})^{n-1} = \int_D d_z d_z^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^n - n \int_D d_z d_z^c \widehat{v} \wedge d_z d_z^c \widehat{v} \wedge (d_z d_z^c \widehat{v})^{n-1} = \frac{1}{n + 1} \int_D (d_z d_z^c \widehat{v})^n+1,
\]

where the second equality follows from Stokes’ theorem because \( d_z d_z^c \widehat{v} = 0 \) near \( \partial D \), and the last one by direct calculation with \( d = d_z + d_z^c, \ d^c = d_z^c + d_z^c \).

Finally, let \( v_j = \lim v_t \) as \( t \to j \) for \( j = 0, 1 \), and let \( w_t \) be the geodesic of \( v_0, v_1 \). If \( \mathbf{E}(v_t) \) is linear, then \( \mathbf{E}(v_t) = \mathbf{E}(w_t) \), so \( v_t = w_t \) for all \( t \) by Corollary 2.2.

Now we can prove the following uniqueness result.

**Theorem 3.4** Let \( u_0, u_1 \in \mathcal{E}_0(D) \) satisfy

\[
\int_D u_0 (d d^c u_0)^k \wedge (d d^c u_1)^{n-k} = \mathbf{E}(u_1), \quad k = 0, \ldots, n. \tag{7}
\]

Then \( u_0 = u_1 \) in \( D \).

**Proof.** By (4), condition (7) implies

\[
\int_D u_1 (dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = \mathbf{E}(u_1), \quad k = 0, \ldots, n,
\]

as well, so

\[
\int_D (u_1 - u_0)(dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = 0, \quad k = 0, \ldots, n. \tag{8}
\]

Denote \( U_t = (1 - t)u_0 + tu_1 \). By (8) and a computation in the proof of Proposition 3.2 the function \( \mathbf{E}(U_t) \) is linear on \([0, 1]\), so \( \mathbf{E}(U_t) = \mathbf{E}(u_0) \).

On the other hand, by Proposition 3.1 the geodesic \( u_t \) of \( u_0 \) and \( u_1 \) satisfies \( u_t \leq U_t \) and, by Theorem 3.3 \( \mathbf{E}(u_t) = \mathbf{E}(u_0) \) as well. By Corollary 2.2 we get \( u_t = U_t \) for any \( t \).
Therefore, the function \( \widehat{U}(z, \zeta) = (1 - \log |\zeta|) u_0(z) + \log |\zeta| u_1(z) \) is plurisubharmonic in \( D \times S \). Then
\[
\frac{\partial}{\partial s_k}(u_1 - u_0) = 0
\]
for all \( k \), so \( u_1 - u_0 \) is analytic in \( D \), equal to 0 on \( \partial D \), and thus is identical 0. \( \Box \)

**Remark.** If \( u \in \mathcal{E}_0(D) \) and \( u_j = \max\{u, -\alpha_j\} \), then we have
\[
\int_D (dd^c u_0)^k \wedge (dd^c u_1)^{n-k} = \int_D (dd^c u_1)^n, \quad k = 0, \ldots, n,
\]
for any \( \alpha_0, \alpha_1 > 0 \). Therefore, using the mixed energy functionals in Theorem 3.4 is essential.

**4 Example: geodesics of relative extremal functions**

Here we consider a particular case of the construction above. Recall that the relative extremal function of a set \( K \subset D \) is
\[
\omega_K(z) = \limsup_{x \to z} \sup\{u(x) : u \in \text{PSH}^-(D), u|_K \leq -1\} \in \mathcal{E}_0(D).
\]

We will be interested in the following: Given two relatively compact subsets \( K_0, K_1 \) of \( D \), let \( u_j = \omega_{K_j} \) for \( j = 1, 2 \), what can be said about their geodesic \( u_t \)? In particular, is \( u_t \) for any fixed \( t \) a relative extremal function on \( D \) and if not, how far is it from being such?

Note that
\[
E(\omega_K) = \int_D \omega_K (dd^c \omega_K)^n = - \int_D (dd^c \omega_K)^n = -\text{Cap}(K), \quad (9)
\]
the Monge-Ampère capacity of \( K \) with respect to \( D \). We have, by Theorem 3.3, the following

**Proposition 4.1** If \( u_t \) is the geodesic for a pair of relative extremal functions \( \omega_{K_j} \), then
\[
E(u_t) = (t - 1) \text{Cap}(K_0) - t \text{Cap}(K_1).
\]

Denote \( L_t = \{z \in D : u_t(z) = -1\} \), then we have \( u_t \leq \omega_{L_t} \). By Corollary 2.2 and Proposition 4.1, this implies

**Proposition 4.2** In the conditions of Proposition 4.1,
\[
\text{Cap}(L_t) \leq (1 - t) \text{Cap}(K_0) + t \text{Cap}(K_1),
\]
and the inequality becomes equality if and only if \( \omega_{L_t} \) is the geodesic.

Now let us assume \( D \) to be a bounded complete logarithmically convex Reinhardt domain of \( \mathbb{C}^n \), that is, \( y \in D \) provided \( z \in D \) and \( |y_l| \leq |z_l| \) for all \( l \), and such that the set \( \log D = \{s \in \mathbb{R}^n : \text{Exp} s \in D_j\} \) is a convex subset of \( \mathbb{R}^n \); here \( \text{Exp} s = (e^{s_1}, \ldots, e^{s_n}) \). In
addition, let $K_j$, $j = 0, 1$, be compact Reinhardt subsets of $D$. In this setting, the functions $\omega_{K_j}$ are toric (multi-circled) and so, the function

$$\tilde{u}(s,t) := u_t(\text{Exp } s)$$

is convex in $(s,t) \in \mathbb{R}^n \times (0, 1)$. Denote

$$K_t = K_0^{-t}K_1^t = \{ z \in \mathbb{D}^n : |z_l| = |\eta|^{1-t} |\xi|^t, \ 1 \leq l \leq n, \ \eta \in K_0, \ \xi \in K_1 \}, \quad 0 < t < 1; \quad (10)$$

in other words, $\log K_t = (1-t) \log K_0 + t \log K_1$. Note that $K_t \subset D$ because $\log D$ is convex.

Recall that volumes $|\cdot|$ of convex combinations $(1-t)P_0 + tP_1$ of two bodies $P_j \subset \mathbb{R}^n$ satisfy

$$|(1-t)P_0 + tP_1| \geq |P_0|^{1-t}|P_1|^t,$$

the Brunn-Minkowski inequality (in multiplicative form). In our case, the sets $\log K_j$ typically are of infinite volume. Instead of the volumes, we have a reversed Brunn-Minkowski inequality for the capacities of $K_t$ (multiplicative combinations of $K_j$), in additive form.

**Theorem 4.3** In the Reinhardt situation, the capacities of the sets $K_t$ defined by (10) satisfy

$$\text{Cap}(K_t) \leq (1-t) \text{Cap}(K_0) + t \text{Cap}(K_1).$$

**Proof.** By the convexity of $\tilde{u}$, we have $\tilde{u}(s,t) \leq -1$ when $s \in (1-t) \log K_0 + t \log K_1 = \log K_t$. Therefore, $K_t \subset L_t$, and the result follows from Proposition 1.2. \qed

Evidently, $\omega_{K_t}$ is the geodesic if and only if $\tilde{\omega}_{K_t}(s)$ is convex in $(s,t)$. It turns out that the latter need not be true.

**Example 4.4** Let $n = 1$, $D = \mathbb{D}$, $K_0 = \{ z : |z| \leq e^{-1} \}$ and $K_1 = \{ z : |z| \leq e^{-2} \}$. Then $K_t = \{ z : |z| \leq e^{-1-t} \}$ and the function

$$\tilde{\omega}_{K_t}(s) = \max \left\{ \frac{s}{1+t}, -1 \right\}$$

is not convex in $(s,t)$, so $\omega_{K_t}$ is not geodesic. It is easy to check that

$$\tilde{u}(s,t) = \max \left\{ s, \frac{s + t - 1}{2}, -1 \right\},$$

so $K_t = L_t$ and $u_t$ is not a relative extremal function at all.

Note also that $E(\omega_{K_t}) = -\text{Cap}(K_t) = -(1+t)^{-1}$ is far from being linear. Finally, $E(u_t) = t/2 - 1$, as expected.

In this example, the geodesics $u_t$ still pertain some features of relative extremal functions. Namely, recall that a pluriregular condenser $(K_1, ..., K_m, \sigma_1, ..., \sigma_m)$ is a system of pluriregular compact sets $K_m \subset K_{m-1} \subset \ldots \subset K_1 \subset D \subset \overline{D} = K_0$ and numbers $\sigma_m < \sigma_{m-1} < \ldots < \sigma_1 < 0$ such that there is a continuous plurisubharmonic function $\omega$ on $D$ with zero boundary values, $K_i = \{ z \in D : \omega \leq \sigma_i \}$ and $\omega$ is maximal on the complement of $K_i$ in the interior of $K_{i-1}$, see [20]. In our case, $u_t$ is the extremal function for the condenser $(K_{1,t}, K_{2,t}, t-1, -1)$, where $K_{1,t} = \{ z : |z| \leq e^{1-t} \}$ and $K_{2,t} = \{ z : |z| \leq e^{-1-t} \}$, and $E(u_t)$ is the energy of the condenser.

It would be nice to know if anything similar holds in the general case of geodesics of relative extremal functions.
5 Geodesics on $\mathcal{F}_1$

One cannot apply the above construction to functions from $\mathcal{F}_1(D)$ directly, because they need not be bounded from below and thus existence of the ‘good’ envelope $\hat{\nu}$ is not guaranteed (in the next section, we will show that generally there are no geodesics for plurisubharmonic functions with nonzero Lelong numbers).

Let $u_j \in \mathcal{F}_1(D)$, $j = 0, 1$ and let $u_{j,N} \in \mathcal{E}_0(D)$ decrease to $u_j$ as $N \to \infty$. Then their geodesics $u_{t,N} \in \mathcal{E}_0(D)$ linearize the functional $E$:

$$ E(u_{t,N}) = (1 - t) E(u_{0,N}) + t E(u_{1,N}). $$

Since $u_{t,N} \geq u_1 + u_2 \in \mathcal{F}_1(D)$ for any $N$, the functions $u_{t,N}$ decrease to functions $v_t \in \mathcal{F}_1(D)$ and $E(u_{t,N})$ decrease to $E(v_t)$ for $0 < t < 1$ while $E(u_{j,N})$ decrease to $E(u_j)$ for $j = 0, 1$ by [7, Thm. 3.8]. Therefore,

$$ E(v_t) = (1 - t) E(u_0) + t E(u_1). \quad (11) $$

Nor also that since $\hat{\nu}_N(z, \zeta) = u_{\log |\zeta|,N}(z)$ satisfy $(dd^c\hat{\nu}_n)^n + 1 = 0$ on $D \times S$ and decrease to $\hat{\nu}(z, \zeta)$, we have $(dd^c\hat{\nu})^n + 1 = 0$ as well.

To have a complete analogy with the bounded case, we need to establish the relations $v_t \to u_j$ as $t \to j$ for $j = 0$ and 1. Since $v_t$ are convex in $t$ and $v_t \geq u_1 + u_2$, the functions $v_j = \limsup_{t \to j} v_t$ are weak limits of $v_t$ and belong to $\mathcal{F}_1(D)$. By construction, $v_j \leq u_j$.

Denote $V_t = (1 - t) v_0 + t v_1$. Then a direct computation shows $E(V_t) \to E(v_j)$ as $t \to j$.

Since $v_t \leq V_t$, we get

$$ E(u_j) = \lim_{t \to j} E(v_t) \leq \lim_{t \to j} E(V_t) = E(v_j), $$

which implies $u_j = v_j$ by Corollary [2].

So, from now on we rename the functions $v_t$ to $u_t$ since they have $u_j$ as there endpoints, for the moment as upper limits when $t \to j$. We claim that actually $u_t \to u_j$ in capacity, that is, for any $\epsilon > 0$, we have $\text{Cap} \ A_{\epsilon,t} \to 0$, where $A_{\epsilon,t} = A_{\epsilon,t,j} = \{ z \in D : |u_t(z) - u_j(z)| > \epsilon \}.$

We will prove the claim for $j = 0$, the case $j = 1$ being completely similar. By subadditivity of the capacity, it suffices to show that $\text{Cap} \ A_{\epsilon,t}^+ \to 0$, where $A_{\epsilon,t}^+ = \{ z : u_t(z) - u_0(z) > \epsilon \}$ and $A_{\epsilon,t}^- = \{ z : u_t(z) - u_0(z) < -\epsilon \}$. Moreover, since $u_t \geq u_0 + u_1$ and $\text{Cap} \{ z : u_0(z) + u_1(z) < -N \} = o(N^{-n})$ as $N \to \infty$ [2, Lemma 2.1], we can assume $u_0 + u_1 \geq -N$ on $A_{\epsilon,t}$. Since $u_t \leq V_t$, we have $u_t - u_0 \geq \epsilon / t$ on $A_{\epsilon,t}^+$, so for any $\psi \in \text{PSH}(D), -1 \leq \psi \leq 0$,

$$ \int_{A_{\epsilon,t}^+} (dd^c\psi)^n \leq t^{-1} \int_{A_{\epsilon,t}^+} (u_1 - u_0)(dd^c\psi)^n \leq t N \epsilon^{-1} \int_{A_{\epsilon,1/2}^+} (dd^c\psi)^n \leq t N \epsilon^{-1} \text{Cap} A_{\epsilon,1/2}^+ $$

for any $t < 1/2$, which implies $\text{Cap} A_{\epsilon,t}^+ \to 0$ as $t \to 0$.

To work with the set $A_{\epsilon,t}^-$ is more tricky because, in the unbounded case, there are no straightforward subgeodesics with good behavior at the endpoints. We will use here an envelope technique introduced (in the Kähler setting) in [22] and developed in [11] (especially in Theorems 4.3 and 5.2 of the latter paper).

Given $u, v \in \text{PSH}(D)$, denote the largest plurisubharmonic minorant of $\min\{u, v\}$ in $D$ by $P(u, v)$. If $u_0, u_1 \in \mathcal{F}_1(D)$ and $C \geq 0$, then

$$ u_0 + u_1 \leq p_C := P(u_0, u_1 + C) \leq u_0, $$

8
Corollary 5.3 The uniqueness result of Theorem \[3.4\] remains true for \(u_0, u_1 \in \mathcal{F}_1(D)\).
6 Case of strong singularities

The Monge-Ampère current \((dd^c u)^n\) of functions from the class \(\mathcal{F}_1\) cannot charge pluripolar sets. If functions \(u_j \in \text{PSH}^- (D)\) are allowed to have stronger singularities, the process of constructing geodesics generally fails. The breaking point is that the presumed ‘geodesic’ \(u_t\) can have \(\lim_{t \to j} u_t < u_j\).

We start with a simple observation. Let \(a \in D\) and let \(G_a\) be the pluricomplex Green function of \(D\) with pole at \(a\).

**Lemma 6.1** If \(\Phi \in \text{PSH}^- (D \times S)\) is such that \(\limsup_{|\zeta| \to \infty} \Phi (z, \zeta) \leq G_a (z)\) for all \(z \in D\) as \(|\zeta| \to \infty\), then \(\Phi (z, \zeta) \leq G_a (z)\) for all \(z \in D\) and all \(\zeta \in S\).

*Proof.* The functions \(\psi_N (z, \zeta) = \max \{ G_a (z), -N \log |\zeta| \} \in \text{PSH}^- (D \times S)\) are equal to 0 on \(\partial D \times S\). We also have \(\psi_N (z, \zeta) \to u_{N,0} (z) = 0\) when \(|\zeta| \to 1\), and \(\psi_N (z, \zeta) \to u_{N,1} (z) = \max \{ G_a (z), -N \} \) when \(|\zeta| \to \infty\).

Furthermore, they satisfy \((dd^c \psi_N)^{n+1} = 0\) everywhere in \(D \times S\). Therefore, \(\psi_{N,t}\) is the geodesic for \(u_{N,0}\) and \(u_{N,1}\). Since \(\Phi \leq \psi_N\) for any \(N\), the proof is complete. \(\square\)

A bit more generally, let \(u \in \text{PSH}^- (D)\) be such that \(A = \{ z : u (z) = -\infty \}\) is a closed subset of \(D\) and \(u \in L^\infty_{\text{loc}} (D \setminus A)\). Then the function

\[
g_u (z) = \limsup_{x \to z} \sup \{ v (x) : v \in \text{PSH}^- (D), v \leq u + O (1) \}
\]

is plurisubharmonic in \(D\), locally bounded outside \(A\) and satisfying \((dd^c g_u)^n = 0\) there. When \(A\) is a single point, then \(g_u \equiv 0\) if and only if \((dd^c u)^n = 0\) there.

As is easy to see, \(g_u \not\equiv 0\) if \(u\) has nonzero Lelong number at some point of \(A\); we do not know if the converse is true.

By repeating the arguments of the proof of Lemma 6.1 we get

**Theorem 6.2** If \(\Phi \in \text{PSH}^- (D \times S)\) is such that

\[
\limsup_{|\zeta| \to j} \Phi (z, \zeta) \leq u_j (z) \quad \forall z \in D, \ j = 0, 1,
\]

then \(\Phi (z, \zeta) \leq P (z)\) for all \(\zeta \in S\), where \(P = P (g_{u_0}, g_{u_1})\) is the largest plurisubharmonic minorant of the function \(\min_j g_{u_j}\). In particular, if each \(u_j = g_{u_j}\), then the largest \(\Phi\) satisfying (12) coincides with \(P\) (and thus is independent of \(\zeta\)).

**Example 6.3** Let \(A\) be a finite subset of \(D\) and let \(u_j\) equal the multi-pole Green function of \(A\) with weights \(m_{j,k} \geq 0\) at \(a_k \in A\). Then the best function \(\Phi\) satisfying (12) is the multi-pole Green function of \(A\) with weights \(M_k = \max_j m_{j,k}\) at \(a_k \in A\).

**Remark.** The situation changes if one replaces the segment \(0 < t < 1\) with the ray \(-\infty < t < 0\). For example, let \(\varphi_j = u_j + w_j\) such that \(u_j \in \mathcal{E}_1 (D)\) and \(w_0 = w_1 + w\), where \(w \in \text{PSH}^- (D)\) has zero boundary values. If \(u_t, 0 < t < 1\), is the geodesic arc for \(u_0\) and \(u_1\), then

\[
\varphi_t = u_t + w_1 + \max \{ w, t \}, \quad -\infty < t < 0,
\]

is a subgeodesic ray with \(\varphi_t \to \varphi_j\) as \(t \to \log j, \ j = 0, 1\).
7 Relations to the Kähler case

Let $(X, \omega)$ be a compact Kähler manifold. An upper semicontinuous function $\varphi$ on $X$ is called $\omega$-plurisubharmonic if $\omega + dd^c \varphi \geq 0$. Cegrell’s classes were generalized to such functions in [17]. A corresponding class $\mathcal{E}_1(X, \omega)$ was introduced, and it has turned to be a natural frame for studying the Mabuchi functional [3]; see also a nice presentation in [10], where, in addition, toric geodesics on toric manifolds are considered.

Some of problems studied in recent papers by T. Darvas with co-authors (e.g., [4], [11], [12], [13]) in the Kähler setting are close to those treated here. In particular, Proposition 4.2 from [4] is a complete analog of our Corollary 2.2. Theorem 5.2 from [11] characterizes $\omega$-plurisubharmonic functions that can be joined by a weak geodesic in terms of a technique from [22], which is closely related to our Theorem 6.2. Finally, we have borrowed the idea of using the envelope technique for proving convergence in capacity in Theorem 5.2 from Theorems 4.3 and 4.3 of [11].

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