A STOCHASTIC MODEL OF CONTAGION WITH DIFFERENT INDIVIDUAL TYPES

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ABSTRACT. We develop a stochastic model of contagion with two individual types by extending an archetypal SIS CTMC model. Our results include the analyses of the contagion duration and the number of individual afflictions. Numerical applications with the minority and majority types are provided to consider various contagions.

1. Introduction. Epidemic models have been initially developed to study various contagious diseases in epidemiology. The recent development and extensions of these epidemic models have spawned mathematical epidemiology as a subfield of applied mathematics (see [4]). Mathematical models of contagious diseases can also be applied to study the dynamics of contagion in non-epidemiological contexts. Some potential areas of application include contagious behaviors in finance (see [16] for example), sociopsychology (see [8] for example), and politics (see [14] for example). In this article, we develop and provide a theoretical investigation of an SIS stochastic epidemic model with two individual types to improve our understanding of various contagions.

The SIR and SIS models are two archetypal models of epidemics. The SIR model was developed by Kermack and McKendrick to study the spread of influenza [10]. Three possible states for an individual are known as “susceptible” (S), “infectious” (I), and “removed” (R) (see [6], p. 7.). Susceptible individuals become infected and eventually recover in the SIR model. The SIS model was developed to augment the SIR model by considering the possibility of repetitive infections (see [2], p. 83.). Individuals may veer between the two possible states of

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being susceptible (S) and concurrently being infected and infectious (I). Investigating these models help researchers to develop theoretical hypotheses concerning the duration of a contagion, the number of afflicted individuals, the number of afflictions per individual, the possibility of a cataclysmic outbreak, and more.

The classic mathematical models of epidemiology can alternatively be categorized into deterministic and stochastic models. Deterministic models include [10], [17], and [11]. Stochastic models can be broadly grouped into discrete time Markov chain (DTMC) models [1, 18], continuous time Markov chain (CTMC) models [5, 12, 15], and stochastic differential equation models [13, 9] following Allen’s categorization (see [2], p. 81).

We focus on CTMC models with disparate types of individuals. Existing related models include [15], [3], and [7]. Sani et al. [15] consider a stochastic SIR model with two individual types and approximate the stability of their stochastic SIR model. Ball and Neal [3] develop a stochastic SIR model with two partition levels, which can be construed as two individual types, and obtain an approximation of the final outcome regarding the spread of a contagion. Economou et al. [7] develop a stochastic SIS model with heterogeneous individuals with respect to their propensities to infect and to be infected. Since the number of states in their model exponentially increases as the number of individuals increases, it is difficult to apply their algorithmic procedure to a reasonably large population size.

In this paper, we develop an SIS CTMC model with two individual types to improve our understanding of various contagious processes. Our model attempts to complement the existing CTMC models by providing precise results concerning the contagion duration and the number of afflictions per individual, by providing results that can be quantitatively analyzed, and by allowing an efficient calculation even when the number of majority-type individuals is large. We assume the contact and recovery rates for individuals to be dependent on the individual types. Some numerical applications with the majority and minority types of individuals are provided to illustrate our results and to make tacit suggestions for harnessing contagion.

The remainder of this paper is organized as follows. In Section 2, we describe our stochastic model with two individual types. In Section 3, the Laplace-Stieltjes transform of the contagion duration is obtained. In Section 4, the distributions of the number of individual afflictions for all individual types are obtained. In Section 5, we consider three numerical applications using the results from the previous two sections. We conclude in Section 6.

2. The model. Our SIS CTMC model concerns a population with two types of individuals. The two types are type-1 and type-2. For \( i = 1, 2 \), let \( N_i \) be the number of type-\( i \) individuals in the population. For \( i, j = 1, 2 \), let \( \gamma_i \) be the contact rate of a type-\( i \) individual with type-\( j \) individuals, i.e., \( \gamma_i \) is the mean number of contacts of a type-\( i \) individual with type-\( j \) individuals during a unit of time. Note that the total rate of contacts between type-\( i \) individuals and type-\( j \) individuals is \( N_i \gamma_j \). Thus, the total rate of contacts between type-\( i \) individuals and type-\( 2 \) individuals is \( N_i \gamma_2 \). This should be equal to the total rate of contacts between type-\( 2 \) individuals and type-\( 1 \) individuals, which is \( N_2 \gamma_1 \). That is, it is required that

\[ N_1 \gamma_2 = N_2 \gamma_1. \] (1)

For \( i = 1, 2 \), \( \gamma_i \) denotes the recovery rate of a contagious type-\( i \) individual, i.e., the mean sojourn time of a type-\( i \) individual at the contagious state is \( 1/\gamma_i \).
Mathematical construction. Our model can be mathematically constructed by defining the primitive processes that consist of the contact and the recovery processes. For \( k = 1, 2 \), let us call the individuals of type-\( k \) by individual \((k,1)\), individual \((k,2)\), \ldots, individual \((k,N_k)\). We will define the contact epochs between two different individuals and the recovery times for each individual.

The contact processes. For two different individuals \((k,i)\) and \((l,j)\) and for \( n = 1, 2, \ldots \), let \( t_n^{(k,i),(l,j)} \) denote the \( n \)th contact epoch between the two individuals \((k,i)\) and \((l,j)\). Since the contact rate of a type-\( k \) individual with the other type-\( k \) individuals is \( \beta_{kk} \), the contact rate of a type-\( k \) individual with another particular type-\( k \) individual is \( \frac{\beta_{kk}}{N_k-1} \). For \( k = 1, 2 \) and \( i, j \) with \( 1 \leq i < j \leq N_k \), we define the epochs \( t_n^{(k,i),(k,j)} \), \( n = 1, 2, \ldots \), by a Poisson process with intensity \( \frac{\beta_{kk}}{N_k-1} \). Since the total rate of contacts between type-1 individuals and type-2 individuals is \( N_1 \beta_{12} \) (which equals \( N_2 \beta_{21} \) by requirement (1)), the contact rate of a type-1 individual with a type-2 individual is \( \frac{\beta_{12}}{N_2} \) (which equals \( \frac{\beta_{21}}{N_1} \) by (1)). For \( i \) and \( j \) with \( 1 \leq i \leq N_1 \) and \( 1 \leq j \leq N_2 \), we define the epochs \( t_n^{(1,i),(2,j)} \), \( n = 1, 2, \ldots \), by a Poisson process with intensity \( \frac{\beta_{12}}{N_2} \). The contact epoch processes for the pairs of individuals are assumed to be independent.

The recovery processes. For an individual \((k,i)\) and for \( n = 1, 2, \ldots \), let \( R_n^{(k,i)} \) denote the recovery duration from the \( n \)th affliction of the individual \((k,i)\). We assume that \( R_n^{(k,i)} \) is exponentially distributed with mean \( \frac{1}{\gamma_k} \). It is assumed that all recovery durations \( R_n^{(k,i)} \) \((k = 1, 2, i = 1, \ldots, N_k, n = 1, 2, \ldots)\) are independent. Furthermore, recovery durations are assumed to be independent of the contact epoch processes.

The Markov process for the epidemic network state. For \( k = 1, 2 \), let \( S_k(t) \) be the number of susceptible type-\( k \) individuals, and let \( I_k(t) \) be the number of contagious type-\( k \) individuals at time \( t \). Then, \((I_1(t), I_2(t))\), \( t \geq 0 \), is a continuous time Markov process with a state space \( E = \{(n,i) : 0 \leq n \leq N_1, 0 \leq i \leq N_2\} \).

The infinitesimal generator \( Q \) of the Markov process \((I_1(t), I_2(t))\) is given by

\[
q(n,i)(n',i') =
\begin{cases}
  0 & \text{if } (n',i') = (n,i),
  n \gamma_1, & \text{if } (n',i') = (n,i-1),
  i \gamma_2, & \text{if } (n',i') = (n+1,i),
  (N_1-n)(\frac{n}{N_1} \beta_{11} + \frac{i}{N_2} \beta_{12}), & \text{if } (n',i') = (n,i+1),
  (N_2-i)(\frac{n}{N_1} \beta_{21} + \frac{i}{N_2} \beta_{22}), & \text{if } (n',i') = (n,i),
  -\left\{n \gamma_1 + i \gamma_2 + (N_1-n)(\frac{n}{N_1} \beta_{11} + \frac{i}{N_2} \beta_{12})
  +(N_2-i)(\frac{n}{N_1} \beta_{21} + \frac{i}{N_2} \beta_{22})\right\}, & \text{otherwise}.
\end{cases}
\]

In the lexicographic order, the infinitesimal generator \( Q \) of the Markov process \((I_1(t), I_2(t))\) is expressed as

\[
Q =
\begin{bmatrix}
  B_0 & A_0 & A_1 \\
  C_1 & B_1 & A_1 \\
  & \cdots & \cdots \\
  & \cdots & \cdots \\
  C_n & B_n & A_n \\
  & \cdots & \cdots \\
  C_{N_1} & B_{N_1} & A_n
\end{bmatrix},
\]
where
\[ C_n = n\gamma_1 I, \]
\[ (A_n)_{ij} = \begin{cases} (N_1 - n)(\frac{n}{N_1}\beta_{11} + \frac{i}{N_2}\beta_{12}), & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \]
\[ (B_n)_{ij} = \begin{cases} i\gamma_2, & \text{if } j = i - 1, \\ (N_2 - i)(\frac{n}{N_1}\beta_{21} + \frac{i}{N_2}\beta_{22}), & \text{if } j = i + 1, \\ -n\gamma_1 + i\gamma_2 + (N_1 - n)(\frac{n}{N_1}\beta_{11} + \frac{i}{N_2}\beta_{12}) + (N_2 - i)(\frac{n}{N_1}\beta_{21} + \frac{i}{N_2}\beta_{22}), & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases} \]

Here, \( I \) is the \((N_2 + 1) \times (N_2 + 1)\) identity matrix.

3. The duration of a contagion. The duration of a contagion is the time until no contagious individuals exist. Let \( D \) be the first hitting time to the state \((0,0)\), i.e., \( D \) is the contagion duration. For \((n,i) \in E\), let \( h_{ni}(s) \) be the conditional Laplace-Stieltjes transform of \( D \), given \((I_1(0), I_2(0)) = (n,i)\), i.e.,
\[ h_{ni}(s) = \mathbb{E}[e^{-sD}|(I_1(0), I_2(0)) = (n,i)]. \]

We will derive the conditional Laplace-Stieltjes transforms of \( h_{ni}(s) \), \((n,i) \in E\).

For \( n = 0, 1, \ldots, N_1 - 1 \), let \( \tau_n \) be the first hitting time to the set of states \( \{(n,i) : 0 \leq i \leq N_2\} \). For \( n = 1, \ldots, N_1 \), let \( G_n(s) \) be the \((N_2 + 1) \times (N_2 + 1)\) matrix whose \((i,j)\)-components \( (G_n(s))_{ij} \), \( 0 \leq i,j \leq N_2 \), are given by
\[ (G_n(s))_{ij} = \mathbb{E}[e^{-s\tau_{n-1}}1_{\{I_1(\tau_{n-1}) = j\}}|I_1(0), I_2(0)) = (n,i)]. \]

It is well known that \( G_n(s) \), \( 1 \leq n \leq N_1 \), can be calculated by the following recursion:
\[ G_{N_1}(s) = (sI - B_{N_1})^{-1}C_{N_1}, \quad (2) \]
\[ G_n(s) = (sI - B_n - A_nG_{n+1}(s))^{-1}C_n, \quad n = N_1 - 1, N_1 - 2, \ldots, 1. \quad (3) \]

It can be observed that
\[ h_{ni}(s) = G_n(s)G_{n-1}(s)\cdots G_1(s)h_0(s), \quad n = 1, \ldots, N_1, \quad (4) \]
where \( h_n(s) \) is the \((N_2 + 1)\)-dimensional column vector whose components are \( h_{ni}(s) \), \( 0 \leq i \leq N_2 \). Therefore, it remains to obtain \( h_0(s) \) for the calculation of the conditional Laplace-Stieltjes transforms \( h_{ni}(s) \), \((n,i) \in E\), of the duration \( D \) of the contagion. Clearly, the first component \( h_{00}(s) \) of the vector \( h_0(s) \) equals 1. Let us split the vector \( h_0(s) \) as follows:
\[ h_0(s) = \begin{bmatrix} 1 \\ \tilde{h}_0(s) \end{bmatrix}, \quad (5) \]
where \( \tilde{h}_0(s) \) is the \(N_2\)-dimensional column vector whose components are \( h_{0i}(s) \), \( 1 \leq i \leq N_2 \). To derive an expression for \( \tilde{h}_0(s) \), we introduce the following matrices and column vectors:
- \( A_0 \) is the \(N_2 \times N_2\) matrix whose components are \((A_0)_{ij}, 1 \leq i, j \leq N_2\);
- \( B_0 \) is the \(N_2 \times N_2\) matrix whose components are \((B_0)_{ij}, 1 \leq i, j \leq N_2\);
- \( b_0 \) is the \(N_2\)-dimensional column vector whose components are \((B_0)_{0i}, 1 \leq i \leq N_2\);
- \( G_1(s) \) is the \(N_2 \times N_2\) matrix whose components are \((G_1(s))_{ij}, 1 \leq i, j \leq N_2\);
\( \tilde{g}_1(s) \) is the \( N_2 \)-dimensional column vector whose components are \((G_1(s))_{i0}, 1 \leq i \leq N_2 \).

Then, we have
\[
\tilde{h}_0(s) = \left( s \tilde{I} - \tilde{B}_0 - \tilde{A}_0 \tilde{G}_1(s) \right)^{-1} (\tilde{b}_0 + \tilde{A}_0 \tilde{g}_1(s)),
\]
where \( \tilde{I} \) is the \( N_2 \times N_2 \) identity matrix. A detailed derivation of (6) is given in Appendix A.

In summary, the conditional Laplace-Stieltjes transforms \( h_{ni}(s), (n, i) \in E, \) of the duration \( D \), given \((I_1(0), I_2(0)) = (n, i)\), can be computed by (4), (5), and (6). Here, the matrices \( G_n(s), 1 \leq n \leq N_1 \), can be computed by the recursion (2) and (3). The numerical values of the conditional probability density functions or the conditional cumulative distributed functions for the duration \( D \), given \((I_1(0), I_2(0)) = (n, i)\), can be computed by numerically inverting the conditional Laplace-Stieltjes transforms, \( h_{ni}(s), (n, i) \in E \).

4. The distribution of the number of individual afflictions. Choose an arbitrary individual. We shall call the chosen individual a tagged individual. When the tagged individual is of type-\( k \), \( k = 1, 2 \), let \( N_k(t) \) be the number of recoveries until \( t \) for the tagged individual. Note that \( N_k(D) \) is the number of afflictions for the type-\( k \) tagged individual during the duration of the contagion. Let \( \xi_k(t) \) be the disease state of the type-\( k \) tagged individual; \( \xi_k(t) = S \) if the tagged individual is susceptible at time \( t \), and \( \xi_k(t) = I \) if the tagged individual is contagious at time \( t \).

4.1. The number of afflictions for a type-1 individual. For \((n, i) \in E \) with \( n < N_1 \), let \( \phi_{n,i}^{1S}(z) \) be the conditional probability generating function of \( N_1(D) \), given that \((I_1(0), I_2(0), \xi_1(0)) = (n, i, S)\), i.e.,
\[
\phi_{n,i}^{1S}(z) = E \left[ z^{N_1(D)} \mid (I_1(0), I_2(0), \xi_1(0)) = (n, i, S) \right] .
\]
For \((n, i) \in E \) with \( n \geq 1 \), let \( \phi_{n,i}^{1I}(z) \) be the conditional probability generating function of \( N_1(D) \), given that \((I_1(0), I_2(0), \xi_1(0)) = (n, i, I)\), i.e.,
\[
\phi_{n,i}^{1I}(z) = E \left[ z^{N_1(D)} \mid (I_1(0), I_2(0), \xi_1(0)) = (n, i, I) \right] .
\]

For the derivation of the conditional probability generating functions \( \phi_{n,i}^{1S}(z) \) and \( \phi_{n,i}^{1I}(z) \), we need to introduce the following notations. For \( n = 1, \ldots, N_1 \), let \( \Psi_{n}^{1SS}(z) \) and \( \Psi_{n}^{1SI}(z) \) be the \((N_2 + 1) \times (N_2 + 1)\) matrices whose \((i, j)\)-components are respectively given by
\[
(\Psi_{n}^{1SS}(z))_{ij} = E \left[ z^{N_1(\tau_{n-1})} \mathbbm{1}_{(I_2(\tau_{n-1}) = j, \xi_1(\tau_{n-1}) = S)} \mid (I_1(0), I_2(0), \xi_1(0)) = (n, i, I) \right] ,
\]
\[
(\Psi_{n}^{1SI}(z))_{ij} = E \left[ z^{N_1(\tau_{n-1})} \mathbbm{1}_{(I_2(\tau_{n-1}) = j, \xi_1(\tau_{n-1}) = I)} \mid (I_1(0), I_2(0), \xi_1(0)) = (n, i, I) \right] .
\]
Similarly, for \( n = 1, \ldots, N_1 - 1 \), let \( \Psi_{n}^{1SS}(z) \) and \( \Psi_{n}^{1SI}(z) \) be the \((N_2 + 1) \times (N_2 + 1)\) matrices whose \((i, j)\)-components are respectively given by
\[
(\Psi_{n}^{1SS}(z))_{ij} = E \left[ z^{N_1(\tau_{n-1})} \mathbbm{1}_{(I_2(\tau_{n-1}) = j, \xi_1(\tau_{n-1}) = S)} \mid (I_1(0), I_2(0), \xi_1(0)) = (n, i, S) \right] ,
\]
\[
(\Psi_{n}^{1SI}(z))_{ij} = E \left[ z^{N_1(\tau_{n-1})} \mathbbm{1}_{(I_2(\tau_{n-1}) = j, \xi_1(\tau_{n-1}) = I)} \mid (I_1(0), I_2(0), \xi_1(0)) = (n, i, S) \right] .
\]
For simplicity of notation, we define \( \Psi_{N_1}^{1SS}(z) \) and \( \Psi_{N_1}^{1SI}(z) \) as the \((N_2 + 1) \times (N_2 + 1)\) zero matrix.
We can derive the following equations:

\[
\Phi_{1,1}^{\text{IS}}(z) = z \gamma_1 (-B_{N_1})^{-1}, \tag{7}
\]

\[
\Phi_{1,1}^{\text{II}}(z) = \gamma_1 (N_1 - 1) (-B_{N_1})^{-1}. \tag{8}
\]

For \( n = 1, 2, \ldots, N_1 - 1 \),

\[
B_n \Psi_{n}^{1, \text{SS}}(z) + C_n + \frac{N_1 - n - 1}{N_1 - n} A_n \left( \Psi_{n+1}^{1, \text{SS}}(z) \Psi_{n}^{1, \text{SS}}(z) + \Psi_{n+1}^{1, \text{SI}}(z) \Psi_{n}^{1, \text{IS}}(z) \right)
\]

\[
+ \frac{1}{N_1 - n} A_n \left( \Psi_{n+1}^{1, \text{IS}}(z) \Psi_{n}^{1, \text{SS}}(z) + \Psi_{n+1}^{1, \text{II}}(z) \Psi_{n}^{1, \text{IS}}(z) \right) = O, \tag{9}
\]

\[
B_n \Psi_{n}^{1, \text{SI}}(z) + \frac{N_1 - n - 1}{N_1 - n} A_n \left( \Psi_{n+1}^{1, \text{SS}}(z) \Psi_{n}^{1, \text{SI}}(z) + \Psi_{n+1}^{1, \text{SI}}(z) \Psi_{n}^{1, \text{II}}(z) \right)
\]

\[
+ \frac{1}{N_1 - n} A_n \left( \Psi_{n+1}^{1, \text{IS}}(z) \Psi_{n}^{1, \text{SI}}(z) + \Psi_{n+1}^{1, \text{II}}(z) \Psi_{n}^{1, \text{II}}(z) \right) = O, \tag{10}
\]

\[
B_n \Psi_{n}^{1, \text{IS}}(z) + \frac{N_1 - n - 1}{n} C_n + A_n \left( \Psi_{n+1}^{1, \text{IS}}(z) \Psi_{n}^{1, \text{SS}}(z) + \Psi_{n+1}^{1, \text{II}}(z) \Psi_{n}^{1, \text{IS}}(z) \right) = O, \tag{11}
\]

\[
B_n \Psi_{n}^{1, \text{II}}(z) + \frac{n - 1}{n} C_n + A_n \left( \Psi_{n+1}^{1, \text{IS}}(z) \Psi_{n}^{1, \text{SS}}(z) + \Psi_{n+1}^{1, \text{II}}(z) \Psi_{n}^{1, \text{II}}(z) \right) = O. \tag{12}
\]

Equations (9)-(12) can be written as, for \( 1 \leq n \leq N_1 - 1 \),

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} \otimes B_n \Psi_{n}(z) + \begin{bmatrix}
1 \\
\frac{1}{n} - 1
\end{bmatrix} \otimes C_n 
+ \begin{bmatrix}
\frac{N_1 - n - 1}{N_1 - n} \\
0
\end{bmatrix} \otimes A_n \Psi_{n+1}(z) \Psi_{n}(z) = O, \tag{13}
\]

where

\[
\Psi_{n}(z) = \begin{bmatrix}
\Psi_{n}^{1, \text{IS}}(z) & \Psi_{n}^{1, \text{II}}(z)
\end{bmatrix}.
\]

We can rewrite (13) as follows:

\[
\Psi_{n}(z) = - \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \otimes B_n + \begin{bmatrix}
\frac{N_1 - n - 1}{N_1 - n} & 0 \\
0 & 1
\end{bmatrix} \otimes A_n \Psi_{n+1}(z) \Psi_{n}(z) \right) \right)^{-1}
\]

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} \otimes C_n. \tag{14}
\]

From (7), (8), and (14), the transform matrices \( \Psi_{n}(z) \), \( n = 1, 2, \ldots, N_1 \), can be obtained by backward induction on \( n \).

Let \( \phi_{n}^{1,\text{S}}(z) \) and \( \phi_{n}^{1,\text{I}}(z) \) be the \( (N_2 + 1) \)-dimensional column vectors whose components are respectively \( \phi_{n}^{1,\text{S}}(z) \) and \( \phi_{n}^{1,\text{I}}(z) \). It can be observed that

\[
\phi_{n}^{1}(z) = \Psi_{n}(z) \phi_{n+1}^{1,\text{S}}(z) \cdots \Psi_{1}(z) \left( \begin{bmatrix}
1 & 0
\end{bmatrix}^{\top} \otimes \phi_{0}^{1,\text{S}}(z) \right), \tag{15}
\]

where

\[
\phi_{n}(z) = \begin{bmatrix}
\phi_{n}^{1,\text{S}}(z) \\
\phi_{n}^{1,\text{I}}(z)
\end{bmatrix}, \quad 1 \leq n \leq N_1.
\]
A detailed derivation of (17) is given in Appendix B.

\[ \phi_1(z) = \left[ \begin{array}{c} z \\ \phi_0(z) \end{array} \right], \]

(16)

where \( \phi_0(z) \) is the \( N_2 \)-dimensional column vector whose components are \( \phi_{0i} \), \( 1 \leq i \leq N_2 \). To derive an expression for \( \phi_0(z) \), we introduce the following matrices and column vectors:

\( \tilde{\Psi}_1(z) \) is the \( N_2 \times N_2 \) matrix whose components are \( (\tilde{\Psi}_1(z))_{ij} \), \( 1 \leq i, j \leq N_2 \); \( \tilde{\Psi}_1(z) \) is the \( N_2 \)-dimensional column vector whose components are \( (\tilde{\Psi}_1(z))_{(0)} \), \( 1 \leq i \leq N_2 \); \( \tilde{\Psi}_1(z) \) is the \( N_2 \times N_2 \) matrix whose components are \( (\tilde{\Psi}_1(z))_{ij} \), \( 1 \leq i, j \leq N_2 \); \( \tilde{\Psi}_1(z) \) is the \( N_2 \)-dimensional column vector whose components are \( (\tilde{\Psi}_1(z))_{(0)} \), \( 1 \leq i \leq N_2 \).

Then, we have

\[ \tilde{\phi}_0(z) = \left( \tilde{A}_0 \tilde{\Psi}_1(z) \right)^{-1} \left( \tilde{b}_0 + \frac{N_1 - 1}{N_1} \tilde{A}_0 \tilde{\Psi}_1(z) \tilde{A}_0 \tilde{\Psi}_1(z) \right). \]

(17)

A detailed derivation of (17) is given in Appendix B.

In summary, the conditional probability generating functions \( \phi_{n1}(z) \), \( 0 \leq n \leq N_1 \), \( 0 \leq i \leq N_2 \), given \( (I_1(0), I_2(0), \xi_1(0)) = (n, i, S) \) and the conditional probability generating functions \( \phi_{n1}^{1h}(z) \), \( 0 < n \leq N_1 \), \( 0 \leq i \leq N_2 \), given \( (I_1(0), I_2(0), \xi_1(0)) = (n, i, I) \), can be computed by (15), (16), and (17). Here, the matrices \( \Psi_{n1}(z) \), \( 1 \leq n \leq N_1 \), can be computed by the recursion (7), (8), and (14). The numerical values of the conditional probability mass functions of the number of afflictions \( N_1(D) \) for a tagged type-1 individual can be computed by numerically inverting the conditional probability generating functions, \( \phi_{n1}^{1S}(z) \) and \( \phi_{n1}^{1I}(z) \).

4.2. The number of afflictions for a type-2 individual. For \( (n, i) \in E \) with \( i < N_2 \), let \( \phi_{ni}^{2S}(z) \) be the conditional probability generating function of \( N_2(D) \), given that \( (I_1(0), I_2(0), \xi_2(0)) = (n, i, S) \), i.e.,

\[ \phi_{ni}^{2S}(z) = E \left[ z^{N_2(D)} | (I_1(0), I_2(0), \xi_2(0)) = (n, i, S) \right]. \]

For \( (n, i) \in E \) with \( i \geq 1 \), let \( \phi_{ni}^{2I}(z) \) be the conditional probability generating function of \( N_2(D) \), given that \( (I_1(0), I_2(0), \xi_2(0)) = (n, i, I) \), i.e.,

\[ \phi_{ni}^{2I}(z) = E \left[ z^{N_2(D)} | (I_1(0), I_2(0), \xi_2(0)) = (n, i, I) \right]. \]

We will derive the conditional probability generating functions \( \phi_{ni}^{2S}(z) \) and \( \phi_{ni}^{2I}(z) \).

For \( n = 1, \ldots, N_1 \), let \( \Psi_{n}^{2SS}(z) \), \( \Psi_{n}^{2SI}(z) \), \( \Psi_{n}^{2IS}(z) \), and \( \Psi_{n}^{2II}(z) \) be the \( N_2 \times N_2 \) matrices whose \((i, j)\)-components are respectively given by

\[ (\Psi_{n}^{2SS}(z))_{ij} = E \left[ z^{N_2(\tau_{n-1})} 1_{(I_2(\tau_{n-1}) = i, \xi_2(\tau_{n-1}) = S)} | (I_1(0), I_2(0), \xi_2(0)) = (n, i, S) \right], \]

\[ 0 \leq i \leq N_2 - 1, 0 \leq j \leq N_2 - 1; \]
To obtain the matrix $\Psi^2(z)$, $\Psi^2_{SS}(z)$, $\Psi^2_{SI}(z)$, and $\Psi^2_{HI}(z)$, we consider the Markov process $(I_1(t), I_2(t), \xi_2(t))$, $t \geq 0$. The state space of the Markov process $(I_1(t), I_2(t), \xi_2(t))$ is

$$\{(n, i, S) : 0 \leq n \leq N_1, 0 \leq i \leq N_2 - 1 \} \cup \{(n, i, I) : 0 \leq n \leq N_1, 1 \leq i \leq N_2 \}.$$ 

For the state space, we adopt the order defined by the following rule.

- $(n, i, \xi)$ precedes $(n', i', \xi')$ if and only if one of the following holds:
  - $n < n'$;
  - $n = n'$, $\xi = S$, and $\xi' = I$;
  - $n = n'$, $\xi = \xi'$, and $i < i'$.

Using this order, the infinitesimal generator of the Markov process $(I_1(t), I_2(t), \xi_2(t))$ is given by

$$Q_2 = \begin{bmatrix}
\hat{B}_0 & D_0 & \hat{A}_0 \\
\gamma_2 \hat{B}_0 & \hat{A}_0 \\
\hat{C}_1 & \hat{B}_1 & D_1 & \hat{A}_1 \\
\hat{C}_1 & \gamma_2 \hat{B}_1 & \hat{A}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hat{C}_n & \hat{B}_n & D_n & \hat{A}_n \\
\hat{C}_n & \gamma_2 \hat{B}_n & \hat{A}_n \\
\hat{C}_N & \hat{B}_N & D_N & \hat{A}_N \\
\hat{C}_N & \gamma_2 \hat{B}_N & \hat{A}_N
\end{bmatrix}$$

Here,

- $\hat{A}_n$ is the $N_2 \times N_2$ matrix whose components are $(A_n)_{ij}$, $0 \leq i, j \leq N_2 - 1$;
- $\hat{A}_n$ is the $N_2 \times N_2$ matrix whose components are $(A_n)_{ij}$, $1 \leq i, j \leq N_2$;
- $\hat{C}_n$ is the $N_2 \times N_2$ matrix whose components are $(C_n)_{ij}$, $0 \leq i, j \leq N_2 - 1$;
- $\hat{C}_n$ is the $N_2 \times N_2$ matrix whose components are $(C_n)_{ij}$, $1 \leq i, j \leq N_2$;
- $\hat{B}_n$ is the $N_2 \times N_2$ matrix whose components are $(\hat{B}_n)_{ij}$, $0 \leq i, j \leq N_2 - 1$,

where

$$(\hat{B}_n)_{ij} = \begin{cases} 
(N_2 - i - 1)\left(\frac{n_i}{N_1} \beta_{21} + \frac{n_{i+1}}{N_1} \beta_{22}\right), & \text{if } j = i + 1, \\
(\hat{B}_n)_{ij}, & \text{otherwise};
\end{cases}$$
If we write we can derive following equations:

From (18) and (20), the transform matrices $\Psi$ and, for Equation (19) is rewritten as

where

Therefore, it remains to obtain $\hat{\phi}_n^2(z)$.

If we write

we can derive following equations:

and, for $n=1, 2, \ldots, N_1-1$,

Equation (19) is rewritten as

From (18) and (20), the transform matrices $\Psi_n^2(z)$, $n=1, 2, \ldots, N_1$, can be obtained by the backward induction on $n$.

Let $\phi_n^{2,8}(z)$ and $\phi_n^{2,1}(z)$ be the $N_2$-dimensional column vectors whose components are respectively $\phi_{ni}^{2,8}(z)$, $0 \leq i \leq N_2-1$, and $\phi_{ni}^{2,1}(z)$, $1 \leq i \leq N_2$. It can be observed that

where

Therefore, it remains to obtain $\phi_0^{2,8}(z)$ for the calculation of the conditional probability generating function $\phi_{ni}^{2,8}(z)$, $0 \leq n \leq N_1$, $0 \leq i \leq N_2-1$ and $\phi_{ni}^{2,1}(z)$, $0 \leq n \leq N_1$, $1 \leq i \leq N_2$. Clearly, $\phi_{0i}^{2,8}(z)$ equals 1. Let us split the vector $\phi_0^{2,8}(z)$ as follows:

where $\phi_0^{2,8}(z)$ is the $N_2$-dimensional column vector whose components are $\phi_{ni}^{2,8}(z)$, $1 \leq i \leq N_2-1$.

To obtain $\phi_0^{2,8}(z)$ and $\phi_0^{2,1}(z)$, we split the matrices $\hat{A}_0$, $\hat{B}_0$, and $D_0$ as follows:

where $\hat{A}_0$ and $\hat{B}_0$ are $(N_2-1) \times N_2$, $\hat{b}_0$ is $(N_2-1) \times 1$, $\hat{B}_0$ is $(N_2-1) \times (N_2-1)$, and $\mathbf{0}$s are the zero column vectors with appropriate sizes. Further, let $\Psi_1^2(z)$ be
different initial numbers of contagious individuals. From the figure, we observe

the matrix obtained by deleting the first column from $\Psi^2_1(z)$, and let $\tilde{\psi}^2_1(z)$ be the first column of $\Psi^2_1(z)$. Then, we have

$$
\begin{bmatrix}
\phi^2_{0,1}(z)
\end{bmatrix}^{-1} = -\begin{bmatrix}
B_0 & D_0
\end{bmatrix} \begin{bmatrix}
\hat{A}_0 & O
\end{bmatrix}
\begin{bmatrix}
\tilde{\psi}^2_1(z)
\end{bmatrix},
$$

(23)

where $e_1$ is the $N_2$-dimensional column vector whose first component is one and others are zero, and $\mathbf{1}$ is the $N_2 \times (N_2 - 1)$ matrix obtained by deleting the first column from $\mathbf{1}$.

In summary, the conditional probability generating functions $\phi^{2S}_{ni}(z)$, $0 \leq n \leq N_1$, $0 \leq i < N_2$, given $(I_1(0), I_2(0), \xi_2(0)) = (n, i, S)$, and the conditional probability generating functions $\phi^{2I}_{ni}(z)$, $0 \leq n \leq N_1$, $0 < i \leq N_2$, given $(I_1(0), I_2(0), \xi_2(0)) = (n, i, I)$, can be computed by (21), (22), and (23). Here, the matrices $\Psi^2_{ni}(z)$, $1 \leq n \leq N_2$, can be computed by the recursion (18) and (20). The numerical values of the conditional probability mass functions of the number of afflictions $N_2(D)$ for a tagged type-2 individual can be computed by numerically inverting the conditional probability generating functions, $\phi^{2S}_{ni}(z)$ and $\phi^{2I}_{ni}(z)$.

5. Numerical applications. In this section, we investigate three numerical applications of interest. The subsequent applications will vary along one of the following dimensions: (1) the contact rate, (2) the recovery rate, and (3) the cross-contact propensity. In all three applications, there will be 100 type-1 individuals ($N_1 = 100$) and 5 type-2 individuals ($N_2 = 5$). This majority-minority divide is, in part, motivated by the observation that many contagious processes of interest have minority individuals with special characteristics. These special minorities can be thought of as elites, opinion leaders, or special interests. The majority-minority divide is also suitable for this paper’s model and utilizes the model’s advantage in its ability to deal with a large number of majority individuals.

Application 1. A population with active minority individuals. In this application, type-2 individuals constitute active minorities. We consider the following two cases: (i) $\beta_{11} = 0.2$, $\beta_{12} = 0.1313$, $\beta_{21} = 2.625$, and $\beta_{22} = 1.25$ and (ii) $\beta_{11} = 0.15$, $\beta_{12} = 0.1313$, $\beta_{21} = 2.625$, and $\beta_{22} = 2.25$. In case (i), type-2 individuals have a higher intragroup contact rate than type-1 individuals. In case (ii), type-2 individuals have a much higher intragroup contact rate than type-1 individuals vis-à-vis case (i). Note that in all cases, the total rate of contacts $c_1$ is constant, where

$$
c_1 = \frac{N_1(\beta_{11} + \beta_{12}) + N_2(\beta_{21} + \beta_{22})}{2}.
$$

The recovery rate for each contagious individual is assumed to be 1 regardless of their type, i.e., $\gamma_1 = \gamma_2 = 1$. The parameter values are summarized in Table 1.

Figure 1 shows the probability density functions and the complementary cumulative distribution functions of the contagion duration $D$ for the two cases with different initial numbers of contagious individuals. From the figure, we observe

| $\beta_{11}$ | $\beta_{12}$ | $\beta_{21}$ | $\beta_{22}$ | $\gamma_1$ | $\gamma_2$
|---------------|--------------|--------------|--------------|------------|------------|
| (i)           | 0.2          | 0.1313       | 2.625        | 1.25       | 1          |
| (ii)          | 0.15         | 0.1313       | 2.625        | 2.25       | 1          |

Table 1. Parameter values for Application 1.
that the contagion duration stochastically increases as $\beta_{22} - \beta_{11}$ increases. Figure 2 shows the probability mass functions of the number of afflictions $N_k(D)$ for $k = 1, 2$ of an individual for the two cases with the initial condition $(I_1(0), I_2(0), \xi_k(0))$. We observe that the number of individual afflictions stochastically increases as $\beta_{22} - \beta_{11}$ increases. We observe that the range for the type-1’s number of individual affliction is smaller than the range for the type-2’s. It is possibly interesting to see that even when the contact rate for the majority individuals decreases from 0.2 in (i) to 0.15 in (ii), the increase in the contact rate for the active minority individuals from 1.25 in (i) to 2.25 in (ii) is impactful enough to generate the stochastic increase in the contagion duration and the number of afflictions per individual as can be seen from Figures 1 and 2.

Application 2. A population with recovery-resistant minority individuals. We consider the following two cases: (i) $\gamma_1 = \gamma_2 = 1$ and (ii) $\gamma_1 = 1.02$ and $\gamma_2 = 0.6$. In case (i), both types have an identical recovery rate. In case (ii), type-2 individuals have a lower recovery rate than type-1 individuals. Type-2 individuals in case (ii) are recovery-resistant minorities. While type-2 individuals have lower recovery rate in case (ii) than in case (i), it is noteworthy that type-1 individuals have higher recovery rate in case (ii) than in case (i). In all cases, the total rate of recoveries $c_2$ is constant, where

$$c_2 = N_1\gamma_1 + N_2\gamma_2.$$ 

All individuals in each case have fixed contact rates where $\beta_{11} = \beta_{22} = 0.25$, $\beta_{12} = 0.1313$, and $\beta_{21} = 2.625$. The parameter values are summarized in Table 2.

Figure 3 shows the probability density functions and the complementary cumulative distribution functions of the contagion duration $D$ for the two cases with different initial numbers of contagious individuals. From the figure, we observe that the contagion duration in case (i) is stochastically smaller than that in case (ii) which has recovery-resistant minorities and recovery-prone majorities. Figure 4 shows the probability mass functions of the number of individual afflictions $N_k(D)$, $k = 1, 2$, of an individual for the three cases with the initial condition $(I_1(0), I_2(0), \xi_k(0))$. We observe that the number of individual afflictions $N_k(D)$, $k = 1, 2$, of case (i) is stochastically smaller than that of case (ii). We observe that the range for the type-1’s number of individual afflictions is smaller than the range for the type-2’s.

Application 3. The impact of different cross-contact rates. In this application, we consider the following three cases: (i) $\beta_{11} = \beta_{22} = 0.5$, $\beta_{12} = \beta_{21} = 0$; (ii) $\beta_{11} = \beta_{22} = 0.25$, $\beta_{12} = 0.1313$, and $\beta_{21} = 2.625$; (iii) $\beta_{11} = 0$, $\beta_{22} = 5.25$, $\beta_{12} = 0.2625$, and $\beta_{21} = 0$.

### Table 2. Parameter values of Application 2.

|        | $\beta_{11}$ | $\beta_{12}$ | $\beta_{21}$ | $\beta_{22}$ | $\gamma_1$ | $\gamma_2$ |
|--------|--------------|--------------|--------------|--------------|------------|------------|
| (i)    | 0.25         | 0.1313       | 2.625        | 0.25         | 1          | 1          |
| (ii)   | 0.25         | 0.1313       | 2.625        | 0.25         | 1.02       | 0.6        |

### Table 3. Parameter values of Application 3.

|        | $\beta_{11}$ | $\beta_{12}$ | $\beta_{21}$ | $\beta_{22}$ | $\gamma_1$ | $\gamma_2$ |
|--------|--------------|--------------|--------------|--------------|------------|------------|
| (i)    | 0.5          | 0            | 0            | 0.5          | 1          | 1          |
| (ii)   | 0.25         | 0.1313       | 2.625        | 0.25         | 1          | 1          |
| (iii)  | 0            | 0.2625       | 5.25         | 0            | 1          | 1          |
\( \beta_{22} = 0.25, \beta_{12} = 0.1313, \text{ and } \beta_{21} = 2.625; \) and (iii) \( \beta_{11} = \beta_{22} = 0, \beta_{12} = 0.2625, \text{ and } \beta_{21} = 5.25. \) In case (i), each type of individuals only contacts its own type. In case (ii), each type of individuals contacts all types. In case (iii), each type of individuals only contacts its opposite type. Note that in all three cases, the total rate of contacts \( c_3 \) is constant as was in Application 1. The recovery rate for each contagious individual is assumed to be 1 regardless of its type, i.e., \( \gamma_1 = \gamma_2 = 1. \) The parameter values are summarized in Table 3.

Figure 5 shows the probability density functions and the complementary cumulative distribution functions of the contagion duration \( D \) for the three cases with different initial numbers of contagious individuals. From the figure, we observe that group separation in which each individual only interacts with other individuals of identical type can reduce the contagion duration. Figure 6 shows the probability mass functions of the number of individual afflictions \( N_k(D) \), \( k = 1, 2 \) of an individual for the three cases with the initial condition, \((I_1(0), I_2(0), \xi_k(0))\). We observe that the number of individual afflictions can be reduced when the two types are separated.

6. Conclusion. In this paper, we have developed an SIS CTMC model with two individual types to consider contagions. We have investigated three numerical applications with the majority and minority individual types given the substantive and methodological relevances of such individual types. In addition to investigating the contagion duration and the number of individual afflictions, we can additionally consider the total count of afflicted individuals, the likelihood of a devastating outbreak, and other relevant considerations in our future research. This paper’s model can and is likely to be fine-tuned for a theoretical analysis of a particular contagious process in the future.

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Appendix.

Appendix A. Derivation of (6). Let \( \sigma \) be the first transition epoch of the Markov process \((I_1(t), I_2(t))\), i.e.,

\[ \sigma = \inf\{t > 0 : (I_1(t), I_2(t)) \neq (I_1(t-), I_2(t-))\}. \]

Since the Markov process \((I_1(t), I_2(t))\) has the strong Markov property and the stopping time \( \sigma \) is independent of \((I_1(\sigma), I_2(\sigma))\), for \( i = 1, \ldots, N_2, \)

\[
h_{0i}(s) = E[e^{-s\sigma}e^{-s(D-\sigma)}|(I_1(0), I_2(0)) = (0, i)]
= E[e^{-s\sigma}(|I_1(0), I_2(0)) = (0, i)]E[e^{-s(D-\sigma)}|(I_1(0), I_2(0)) = (0, i)]
= \frac{-(B_0)_{ii}}{s - (B_0)_{ii}}E[e^{-s(D-\sigma)}|(I_1(0), I_2(0)) = (0, i)]. \quad (24)
\]

Conditioning on \((I_1(\sigma), I_2(\sigma))\), the expectation on the right-hand side of (24) is given by

\[ E[e^{-s(D-\sigma)}|(I_1(0), I_2(0)) = (0, i)] \]
Figure 1. The probability density functions and the complementary cumulative distribution functions of the contagion duration in Application 1.

\[ = \mathbb{P}(I_1(\sigma), I_2(\sigma)) = (0, 0) | (I_1(0), I_2(0)) = (0, i) \]

\[ + \sum_{j=1}^{N_2} \mathbb{P}(I_1(\sigma), I_2(\sigma)) = (0, j) | (I_1(0), I_2(0)) = (0, i)) \mathbb{E}[e^{-\alpha D} | (I_1(0), I_2(0)) = (0, j)] \]
Figure 2. The probability mass functions of the number of individual afflictions in Application 1.

\[ P(I_1(\sigma), I_2(\sigma)) = (1, i) | (I_1(0), I_2(0)) = (0, i) \] \[ \mathbb{E}[e^{-sD} | (I_1(0), I_2(0)) = (1, i)] \]

Substituting (25) into (24) leads to

\[ sh_0(s) = (B_0)_{i0} + \sum_{j=1}^{N_2} (B_0)_{ij} h_0(s) + (A_0)_{ii} h_1(s), \quad i = 1, \ldots, N_2. \] (26)

The above system of equations can be written in a matrix form as follows:

\[ \tilde{sh}_0(s) = \tilde{b}_0 + \tilde{B}_0 \tilde{h}_0(s) + \tilde{A}_0 \tilde{h}_1(s), \] (26)

where \( \tilde{h}_1(s) \) is the \( N_2 \)-dimensional column vector whose components are \( (h_1(s))_i, \quad i = 1, \ldots, N_2 \). By (4),

\[ \tilde{h}_1(s) = \tilde{g}_1(s) + \tilde{G}_1(s) \tilde{h}_0(s). \] (27)

Substituting (27) into (26) yields (6).

**Appendix B. Derivation of (17).** Let \( \tilde{\sigma} \) be the first transition epoch of the Markov process \( (I_1(t), I_2(t), \xi_1(t)) \), i.e.,

\[ \tilde{\sigma} = \inf \{ t > 0 : (I_1(t), I_2(t), \xi_1(t)) \neq (I_1(t-), I_2(t-), \xi_1(t-)) \}. \]
Figure 3. The probability density functions and the complementary cumulative distribution functions of the contagion duration in Application 2.

Since the Markov process \((I_1(t), I_2(t), \xi_1(t))\) has the strong Markov property, we have, for \(i = 1, \ldots, N_2\),

\[
\phi_{0i}^{1, S}(z) = \mathbb{E}[z^{N_1(D)} - N_1(\tilde{\sigma}) | (I_1(0), I_2(0), \xi_1(0)) = (0, i, S)] \\
= \mathbb{P}(\{I_1(\tilde{\sigma}), I_2(\tilde{\sigma}), \xi_1(\tilde{\sigma}) = (0, 0, S) | (I_1(0), I_2(0), \xi_1(0)) = (0, i, S))
\]
This is written as

\[(B_0)_{i0} + \sum_{j=1, j \neq i}^{N_2} (B_0)_{ij} \phi_{0j}^{1S}(z) + \frac{N_1 - 1}{N_1} (A_0)_{ii} \phi_{1i}^{1S}(z) + \frac{1}{N_1} (A_0)_{ii} \phi_{1i}^{1I}(z) = 0, \quad i = 1, \ldots, N_2.\]
The above system of equations can be written in a matrix form as follows:

$$
\tilde{b}_0 + \tilde{B}_0 \phi_0^{S} \phi_1^S(z) + \frac{N_1 - 1}{N_1} \tilde{A}_0 \phi_1^{S} \phi_2^S(z) + \frac{1}{N_1} \tilde{A}_0 \phi_1^{S} \phi_1^I(z) = 0,
$$

(29)
Figure 6. The probability mass functions of the number of individual afflictions in Application 3.

where \( \tilde{\phi}_1^{1,S}(z) \) and \( \tilde{\phi}_1^{1,I}(z) \) are the \( N_2 \)-dimensional column vectors where components are \( \phi_1^{1,S}(z) \), \( 1 \leq i \leq N_2 \) and \( \phi_1^{1,I}(z) \), \( 1 \leq i \leq N_2 \), respectively. By (15),

\[
\tilde{\phi}_1^{1,S}(z) = \psi_1^{1,SS}(z) + \psi_1^{1,SS}(z) \tilde{\phi}_0^{1,S}(z),
\]
\[
(30)
\]
\[
\tilde{\phi}_1^{1,I}(z) = \psi_1^{1,IS}(z) + \psi_1^{1,IS}(z) \tilde{\phi}_0^{1,S}(z).
\]
\[
(31)
\]

Substituting (30) and (31) into (29) yields (17).

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