Computing sets of graded attribute implications
with witnessed non-redundancy

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Abstract

In this paper we extend our previous results on sets of graded attribute implications with witnessed non-redundancy. We assume finite residuated lattices as structures of truth degrees and use arbitrary idempotent truth-stressing linguistic hedges as parameters which influence the semantics of graded attribute implications. In this setting, we introduce an algorithm which transforms any set of graded attribute implications into an equivalent non-redundant set of graded attribute implications with saturated consequents whose non-redundancy is witnessed by antecedents of the formulas. As a consequence, we solve the open problem regarding the existence of general systems of pseudo-intents which appear in formal concept analysis of object-attribute data with graded attributes and linguistic hedges. Furthermore, we show a polynomial-time procedure for determining bases given by general systems of pseudo-intents from sets of graded attribute implications which are complete in data.

1 Introduction

In this paper, we investigate properties of sets of graded attribute implications and extend the results presented in our recent paper [41]. The graded attribute

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implications, sometimes called fuzzy attribute implications [3], are rules describing if-then dependencies in data with graded attributes. The rules have been proposed and investigated from the point of view of formal concept analysis (shortly, FCA [24]) with linguistic hedges [15]. One of the basic problems in FCA is to extract, given a formal context, a set of attribute implications which is non-redundant and conveys the information about exactly all attribute implications which hold in the given formal context—such sets are called (non-redundant) bases of formal contexts. One of the most profound approaches of determining bases exploits the notion of a pseudo-intent which originated in [28] and has been later utilized, e.g., in [23]. The bases given by pseudo-intents are not only non-redundant but in addition minimal in terms of their cardinality. In our paper, we deal with a general notion of a system of pseudo-intents which appears in the generalization of FCA which includes graded attributes and uses linguistic hedges to reduce the size of concept lattices [8]. By a graded attribute we mean an attribute (property/feature) which may be satisfied (present) to degrees instead of just satisfied/not satisfied (present/not present) as in the ordinary setting. In the past, there have been many approaches to extensions of the traditional concept analysis which accommodate graded attributes [2, 34, 37, 38] and related phenomena. Most of the approaches are focused solely on the structure of concept lattices with little or no attention paid to if-then rules. The exceptions seem the be the early works by Polland [38] and the results made in the framework of FCA with linguistic hedges, see [16] for a survey. In [38], the author proposes generalized pseudo-intents which ensure that the constructed sets of formulas are complete in data, i.e., convey the information about exactly all if-then rules which hold in the data, but are redundant in general. Using a more general setting, [3, 7] show that there is a general notion of a system of pseudo-intents which ensures both the completeness and non-redundancy. Unfortunately, the definition in [3] is not constructive and so far the procedure to find such systems was reduced to finding particular maximal independent sets of vertices in large graphs [6, 10]. In addition, it has been shown that the existence and uniqueness of systems of pseudo-intents is not ensured in the general
setting. Indeed, it follows that the properties of the underlying structures of truth degrees, which together with linguistic hedges determine the semantics of graded attribute implications, substantially affect the properties of such systems. In case of infinite structures of truth degrees, it is known that general systems of pseudo-intents may not exist [16]. The existence in case of finite structures was listed as one of the open problems in [35]. Our paper brings a positive answer to this question and shows that, among other results, that general systems of pseudo-intents can be determined in a polynomial time from any complete set of graded attribute implications. The result is based on some of our recent observations made in [41] where we have put in correspondence bases given by systems of pseudo-intents and non-redundant sets of graded attribute implications with saturated consequents where the non-redundancy of each formula is witnessed by its antecedent.

Detailed description of the problem and the results requires precise introduction of the utilized notions. Therefore, we postpone it to Section 3 after presenting the preliminaries in Section 2. In Section 3 we include the algorithm and comment on its immediate consequences. The soundness of the algorithm is proved in Section 4 which also contains additional remarks and examples. Finally, we present conclusions in Section 5.

2 Preliminaries

In this section, we present the basic notions related to the structures of truth degrees which are used in our paper and recall basic notions of graded attribute implications. We limit ourselves just to the notions which are utilized in this paper. Interested readers can find more details in [16]. Readers familiar with [41] can skip this section and go directly to Section 3.

A residuated lattice [2] [22] is an algebra \( L = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle \) where \( \langle L, \wedge, \vee, 0, 1 \rangle \) is a bounded lattice with 0 and 1 being the least and the greatest elements of \( L \), respectively, \( \langle L, \otimes, 1 \rangle \) is a commutative monoid (i.e., \( \otimes \) is commutative, associative, and 1 is neutral with respect to \( \otimes \)), and \( \otimes \) and \( \rightarrow \) satisfy
the so-called adjointness property: for all \(a, b, c \in L\), we have that \(a \otimes b \leq c\) iff \(a \leq b \to c\). Further in the paper, \(L\) always stands for a residuated lattice of the form \(L = (L, \wedge, \vee, \otimes, \to, 0, 1)\). \(L\) is a complete residuated lattice whenever \((L, \wedge, \vee, 0, 1)\) is a complete lattice, (i.e., infima and suprema exist for arbitrary subsets of \(L\)). If \(L\) is finite, then \(L\) is trivially complete. Examples of (complete) residuated lattices include popular structures defined on the real unit interval using left-continuous triangular norms \([20, 32]\) and their finite substructures. The structures are utilized in mathematical fuzzy logics \([18, 26, 27, 29]\) and their applications \([33]\) as structures of truth degrees with \(\otimes\) and \(\to\) used as truth functions of “fuzzy conjunction” and “fuzzy implication”, respectively.

As usual, a map \(A : Y \to L\) is called an \(L\)-set \(A\) in \(Y\) (or an \(L\)-fuzzy set \([25]\)); \(R : X \times Y \to L\) is called a binary \(L\)-relation between \(X\) and \(Y\), \(R(x, y) \in L\) is interpreted as the degree to which \(x \in X\) and \(y \in Y\) are related by \(R\). The collection of all \(L\)-sets in \(Y\) is denoted by \(L^Y\). Operations with \(L\)-sets are defined componentwise using operations in \(L\). For instance, the union \(A \cup B\) of \(L\)-sets \(A \in L^Y\) and \(B \in L^Y\) is an \(L\)-set in \(Y\) such that \((A \cup B)(y) = A(y) \vee B(y)\); analogously for \(\cap\) and \(\wedge\). If \(a \in L\) and \(A \in L^Y\) then \(a \otimes A\) called the \(a\)-multiple of \(A\), is an \(L\)-set in \(Y\) defined by \((a \otimes A)(y) = a \otimes A(y)\) for all \(y \in Y\). For \(A, B \in L^Y\), we define the degree \(S(A, B)\) to which \(A\) is a subset of \(B\) by

\[
S(A, B) = \bigwedge_{y \in Y} (A(y) \to B(y))
\]

provided that the infimum of \(\{A(y) \to B(y); y \in Y\} \subseteq L\) exists—this condition is satisfied, e.g., if \(L\) is complete or if \(Y\) is finite. Note that \(S\) given by (1) can be understood as a binary \(L\)-relation on \(L\)-sets, i.e., for a fixed \(Y\), it is a map of the form \(S : L^Y \times L^Y \to L\). It is easily seen that \(S(A, B) = 1\) iff \(A(y) \leq B(y)\) for all \(y \in Y\) in which case we write \(A \subseteq B\) and say that \(A\) is a full subset of \(B\).

Remark 1. Let us note that the notion of a graded subsethood \([1]\) defined using the residuated implication has been proposed by Goguen \([25, 26]\) and plays an important role in the interpretation of the if-then rules we consider in this paper. This corresponds with the fact that the usual inclusion of sets is used to define
the interpretation of the classic attribute implications. Indeed, if \( Y \) is a non-empty set of attributes (symbolic names), any formula of the form \( A \Rightarrow B \) where \( A, B \subseteq Y \) is called an attribute implication [24]. Moreover, it is considered true given \( M \subseteq Y \), written \( M \models A \Rightarrow B \), whenever \( A \subseteq M \) implies \( B \subseteq M \). If we depart from the classic setting to the graded setting and replace the classic sets by \( L \)-sets \( A, B, M \in L^Y \), there are several possible ways how to define the notion of “\( A \Rightarrow B \) being true in \( M \)” which all collapse to the ordinary notion when \( L \) is the two-element Boolean algebra. As it is described in detail in [16], two borderline (and both interesting) cases can be based on the graded and the full inclusion of \( L \)-sets.

The framework we use in our paper enables us to reason with several different interpretations of inclusion of \( L \)-sets, and thus several different ways of understanding the interpretation of data dependencies, using a single formalism which is based on additional parameterization of the semantics of the rules. Namely, we use the approach based on linguistic hedges [16]. In a more detail, given a non-empty and finite set \( Y \) of attributes and a complete residuated lattice \( L \), a graded attribute implication in \( Y \) is an expression \( A \Rightarrow B \) where \( A, B \in L^Y \); \( A \) is called the antecedent of \( A \Rightarrow B \), \( B \) is called the consequent of \( A \Rightarrow B \). Furthermore, let \( * \) be a map \( * : L \rightarrow L \). For \( A, B, M \in L^Y \), the degree \( ||A \Rightarrow B||_M^* \) to which \( A \Rightarrow B \) is true in \( M \) (under \( L \) and \( * \)) is defined by

\[
||A \Rightarrow B||_M^* = S(A, M)^* \rightarrow S(B, M).
\]

In particular, \( * \) used in our considerations is always an idempotent truth-stressing linguistic hedge [16], i.e., it satisfies the following conditions:

\[
1^* = 1, \quad \text{(3)}
\]

\[
a^* \leq a, \quad \text{(4)}
\]

\[
(a \rightarrow b)^* \leq a^* \rightarrow b^*, \quad \text{(5)}
\]

\[
a^{**} = a^*, \quad \text{(6)}
\]

for all \( a, b \in L \). Now, two important cases of (2) result by setting \( * \) to the identity (i.e., \( * \) is a map such that \( a^* = a \) for all \( a \in L \)) and the so-called
globalization in which case \( a^* = 1 \) for \( a = 1 \) and \( a^* = 0 \) for \( a < 1 \). We refrain from detailed description of the role of hedges because it is presented elsewhere, see [42, 46, 47] for the initial development of hedges in fuzzy logic in the wide sense, [30, 21] for the treatment in the context of mathematical fuzzy logics, and [16] for the explanation of the role of hedges as parameters of the interpretation of if-then rules. Also note that the approach to parameterization of the interpretation of if-then rules by hedges can be seen as a particular case of a more general approach which utilizes systems of isotone Galois connections and is described in [42].

In this paper, we utilize the notion of the semantic entailment of graded attribute implications. Each set of graded attribute implications (in a fixed \( Y \)) is called a theory and is denoted by \( \Sigma, \Gamma, \Delta, \ldots \). An \( L \)-set \( M \in L^Y \) is called a model of \( \Sigma \) whenever \( ||A \Rightarrow B||_M^* = 1 \) for all \( A \Rightarrow B \in \Sigma \). The set of all models of \( \Sigma \) is denoted by \( \text{Mod}^*(\Sigma) \). The degree \( ||A \Rightarrow B||_\Sigma^* \) to which \( A \Rightarrow B \) is semantically entailed by \( \Sigma \) is defined by

\[
||A \Rightarrow B||_\Sigma^* = \bigwedge_{M \in \text{Mod}^*(\Sigma)} ||A \Rightarrow B||_M^*. 
\]

We put \( \Sigma \sqsubseteq \Gamma \) and say that \( \Gamma \) is stronger than \( \Sigma \) whenever

\[
||A \Rightarrow B||_\Sigma \preceq ||A \Rightarrow B||_\Gamma^*
\]

for all \( A, B \in L^Y \). Furthermore, \( \Sigma \) and \( \Gamma \) are equivalent, written \( \Sigma \equiv \Gamma \), whenever \( \Sigma \sqsubseteq \Gamma \) and \( \Gamma \sqsubseteq \Sigma \). It is a well-known fact that \( \Gamma \equiv \Sigma \iff \text{Mod}^*(\Sigma) = \text{Mod}^*(\Gamma) \), see [9, 10]. Theory \( \Sigma \) is called redundant whenever there is \( \Gamma \subset \Sigma \) such that \( \Sigma \equiv \Gamma \); otherwise \( \Sigma \) is called non-redundant. Theory \( \Sigma \) is called minimal whenever for each \( \Gamma \) such that \( \Sigma \equiv \Gamma \), we have \( |\Sigma| \leq |\Gamma| \). One of the basic properties of non-redundant sets of graded attribute implications is the following: \( \Sigma \) is non-redundant iff \( ||A \Rightarrow B||_{\Sigma \setminus \{A \Rightarrow B\}}^* < 1 \) for all \( A \Rightarrow B \in \Sigma \), see [12]. Furthermore, if \( ||A \Rightarrow B||_{\Sigma \setminus \{A \Rightarrow B\}}^* = 1 \) for \( A \Rightarrow B \in \Sigma \), we say that \( A \Rightarrow B \) is redundant in \( \Sigma \).

Recall from [41] that if \( \Sigma \) is non-redundant, we say that the non-redundancy of \( \Sigma \) is witnessed (by the antecedents of the formulas in \( \Sigma \)) whenever for every
$A \Rightarrow B \in \Sigma$, we have that $A \in \text{Mod}^*(\Sigma \setminus \{A \Rightarrow B\})$. This is one of the key notions used in the present paper since we seek a procedure to transform a given set of graded attribute implications into an equivalent one which is non-redundant and its non-redundancy is witnessed.

Finally, we have to recall the notion of least models which is useful in the characterization of the semantic entailment. Recall that each $\text{Mod}^*(\Sigma)$ is an $L^*$-closure system [5, 12]. As a consequence, we may introduce the least model $[M]_{\Sigma}^*$ of $\Sigma$ which contains $M$:

$$[M]_{\Sigma}^* = \bigcap\{N \in \text{Mod}^*(\Sigma); M \subseteq N\}. \quad (9)$$

According to [16, Theorem 3.11], $||A \Rightarrow B||_{\Sigma}^* = S(B, [A]_{\Sigma}^*)$ for any $\Sigma$ and all $A, B \in L^Y$. Furthermore, $[\cdot \cdot]_{\Sigma}^*$ is an $L^*$-closure operator [5], i.e., it satisfies the following conditions

$$A \subseteq [A]_{\Sigma}^*; \quad (10)$$
$$S(A, B)^* \leq S([A]_{\Sigma}^*, [B]_{\Sigma}^*), \quad (11)$$
$$[[A]_{\Sigma}^*]_{\Sigma}^* = [A]_{\Sigma}^*, \quad (12)$$

for all $A, B \in L^Y$. Note that (10) and (12) are the usual extensivity and idempotency; (11) is a stronger form of monotony and, in particular, (11) implies that $[A]_{\Sigma}^* \subseteq [B]_{\Sigma}^*$ provided that $A \subseteq B$. Using least models, we can introduce graded attribute implications whose consequents are largest with respect to a given theory as follows: we say that $A \Rightarrow B \in \Sigma$ has a saturated consequent (with respect to $\Sigma$) whenever $B = [A]_{\Sigma}^*$. A theory where each formula has a saturated consequent is called a theory with saturated consequents. According to [41, Lemma 4], for each $\Gamma$ and

$$\Sigma = \{A \Rightarrow [A]_{\Gamma}^*; A \Rightarrow B \in \Gamma\}, \quad (13)$$

we have $\Gamma \equiv \Sigma$. Therefore, for each theory there is an equivalent theory consisting of graded attribute implications with saturated consequents.
3 Problem setting and results

In our previous paper [41], we have shown that considering the globalization as the hedge, each theory can be transformed into a theory which is equivalent, non-redundant, and its non-redundancy is witnessed by antecedents of the formulas in the theory. Namely, if $\bullet$ denotes globalization on $L$, for any theory $\Delta$, we can consider

$$\Gamma \subseteq \{ A \Rightarrow [A]_{\Delta}; A \Rightarrow B \in \Delta \},$$

(14)

such that $\Gamma$ is non-redundant and $\Gamma \equiv \Delta$, see [41, Lemma 4]. Furthermore, [41, Theorem 9] yields that

$$\Sigma = \{ [A]_{\Pi \setminus \{ A \Rightarrow [A]_{\Gamma} \}} \Rightarrow [A]_{\Gamma}; A \Rightarrow [A]_{\Gamma} \in \Gamma \}.$$

(15)

is non-redundant, its non-redundancy is witnessed, and $\Sigma \equiv \Gamma \equiv \Delta$, i.e., $\Sigma$ is the desired theory. Since $\Delta \equiv \Sigma$, $\Sigma$ can be interpreted as a theory which conveys the same information as $\Delta$. As a consequence of having $\bullet$ as the globalization, $\Sigma$ is minimal in terms of the number of formulas it contains and may be significantly smaller than $\Delta$. In addition, $\Sigma$ may be considered more informative than $\Delta$ because (i) its consequents are saturated, i.e., they are the largest possible $L$-sets of attributes, and (ii) its non-redundancy is witnessed—the non-redundancy of each formula in $\Sigma$ is directly observed from its antecedent because it is a model of the remaining formulas in $\Sigma$. Therefore, computing $\Sigma$ from $\Delta$ is desirable from several points of view.

Remark 2. Note for readers familiar with non-redundant bases generated from data: [41] shows that for all equivalent theories with saturated consequents, $\Sigma$ given by (15) is uniquely given (considering $\bullet$ as the globalization) and the set of its antecedents forms a system of pseudo-intents of any formal $L$-context whose intents are exactly the models of $\Sigma$. As a consequence, if one finds a set of graded attribute implications which is complete in a given formal $L$-context considering globalization as the linguistic hedge, then the system of pseudo-intents of the formal $L$-context can be computed from the complete set of graded attribute
implications in a polynomial time. Indeed, as we have outlined here and as it is presented in [41], the procedure involves computations of least models for each formula in the complete set which may be done in polynomial time and efficient algorithms for computation of closures exist [41]. This is in contrast with computing the systems of pseudo-intents directly from a formal $L$-context which is known to be hard even in the classic case [19].

The goal of this paper is to show that equivalent non-redundant theories with saturated consequents and witnessed non-redundancy can be computed for any theory considering an arbitrary idempotent truth-stressing hedge and finite $L$, thus extending the previous results in [41] which was presented only for globalization. In such a setting, the semantics of graded attribute implications is very specific and all truth degrees that appear in antecedents and consequents of formulas can be seen as hard thresholds. Indeed, from (2) and the fact that $\bullet$ is globalization, it follows that

$$||A \Rightarrow B||_M = \begin{cases} 1, & \text{if } S(A, M) < 1 \text{ (i.e., } A \nsubseteq M), \\ S(B, M), & \text{otherwise.} \end{cases}$$

(16)

In particular, $||A \Rightarrow B||_M = 1$ means $A \subseteq M$ implies $B \subseteq M$. In words, if each $y \in Y$ is present in $M$ at least to the degree to which it is present in $A$, then each $y \in Y$ is present in $M$ at least to the degree to which it is present in $B$. Therefore, the degrees $A(y)$ and $B(y)$ can be seen as “hard thresholds” for the presence of $y \in Y$. In contrast, when $*$ is a hedge other than the globalization, the thresholds are not so strict and may be seen as “soft thresholds”. This is best seen in the case of $*$ being identity where $||A \Rightarrow B||_M^* = 1$ means that the degree to which $B$ is included in $M$ is at least as high as the degree to which $A$ is included in $M$. In general, we have $||A \Rightarrow B||_M^* \leq ||A \Rightarrow B||_M^\bullet$ but not vice versa. For some structures of truth degrees, it seems that non-redundant bases computed from data using general $*$ tend to be smaller than bases computed using $\bullet$ as it follows from the preliminary experimental evaluation we include in Section 4. Thus, in addition to solving the problem of existence of general systems of pseudo-intents, this may be seen as a practical motivation because
users usually want to infer bases as small as possible.

Let us also recall that in [41, Example 4], we have presented observations showing that the procedure of obtaining \( \Sigma \) as in (15) is not directly applicable in case of general theories and general hedges because \( \Sigma \) may not be equivalent to \( \Gamma \) and the initial theory \( \Delta \). In this paper, we introduce an alternative approach which is conceptually close to that in [41] but always ensures that the results are equivalent to the initial theories and have all the desired properties. The approach is based on a procedure which sequentially modifies the initial theory and in finitely many steps produces the result. An elementary step of the transformation of theories is formalized by a binary relation \( \supset^* \) on the set of all theories and it is defined as follows: For theories \( \Delta \) and \( \Gamma \), we put

\[
\Delta \supset^* \Gamma
\]

whenever \( \Delta \neq \Gamma \) and there are two distinct formulas \( A \Rightarrow B \in \Delta \) and \( C \Rightarrow D \in \Delta \) such that

\[
\Gamma = (\Delta \setminus \{A \Rightarrow B\}) \cup \{A \cup (S(C, A)^* \otimes D) \Rightarrow B\}. \tag{18}
\]

Furthermore, \( \Delta \) is called irreducible (under \( \supset^* \)) whenever there is no \( \Gamma \) such that \( \Delta \supset^* \Gamma \). By definition, \( \supset^* \) is a relation on the set of all theories. The fact \( \Delta \supset^* \Gamma \) can be read “\( \Delta \) reduces to \( \Gamma \) (in a single step)” and it represents an elementary step in a procedure which converts given theory to a theory with the desired properties. For convenience, we introduce the reflexive and transitive closure of \( \supset^* \) in the usual way: We put \( \Delta \supset^*_0 \Delta \) for any \( \Delta \) and, for any natural \( n \), we put \( \Delta \supset^*_n \Gamma \) whenever there is some \( \Xi \) such that \( \Delta \supset^*_{n-1} \Xi \) and \( \Xi \supset^* \Gamma \). Finally, \( \supset^*_\infty \) is the union of all \( \supset^*_n \) for all non-negative \( n \).

**Theorem 1.** Let \( L \) be finite, \( \ast \) be any idempotent truth-stressing hedge, \( \Gamma \) be a non-redundant/minimal set of graded attribute implications in \( Y \) with saturated consequents. Then, for each irreducible \( \Sigma \) such that \( \Gamma \supset^*_\infty \Sigma \), the following conditions are satisfied:

(i) \( \Sigma \equiv \Gamma \),
(ii) $\Sigma$ is non-redundant/minimal,

(iii) the non-redundancy of $\Sigma$ is witnessed.

Before we elaborate the proof in Section 4 let us comment on some immediate consequences of Theorem 1. The theorem implies that we can always start with any $\Gamma$ and convert it, in finitely many steps, into $\Sigma$ with properties (i)–(iii). This is a consequence of two facts: First, using (13), for any $\Gamma$ there is an equivalent and non-redundant set $\Gamma'$ with saturated consequents which can be computed in finitely many steps just by a repeated computation of least models in order to saturate the consequents of formulas in $\Gamma$ and to remove redundant formulas. Clearly, such a procedure is polynomial. Second, we can apply Theorem 1 with $\Gamma'$ to get the desired $\Sigma$—it remains to prove that an irreducible $\Sigma$ such that $\Gamma' \models^\infty \Sigma$ and (i)–(iii) hold can be found in finitely many steps but this fact comes almost immediately. Indeed, observe that $\Delta \models^* \Xi$ means that $\Xi$ differs from $\Delta$ in a single formula, namely in its antecedent, which is larger (in sense of the inclusion $\subseteq$ of $L$-sets). Since both $L$ and $Y$ are assumed to be finite, reduction of any theory always terminates by an irreducible element and the number of elementary steps is, again, polynomial in the size of the initial theory. Therefore, it remains to prove that the outcome of the reduction is indeed a theory with the desired properties—this is shown in Section 4.

An important consequence of Theorem 1 is for the existence of general systems of pseudo-intents of object-attribute data with graded attributes. Let us recall from [41] that for non-empty finite sets $X$ (set of objects) and $Y$ (set of attributes), and a binary $L$-relation $I: X \times Y \rightarrow L$, the triplet $I = \langle X, Y, I \rangle$ is called a formal $L$-context [2]. The degree $|A \Rightarrow B|_I$ to which $A \Rightarrow B$ ($A, B \in L^Y$) is true in $I$ (under $*$), see [2], is defined as

$$|A \Rightarrow B|_I = \bigwedge_{x \in X} |A \Rightarrow B|_{I_x}$$

(19)

where $I_x \in L^Y$ such that $I_x(y) = I(x, y)$ for all $x \in X$ and $y \in Y$. Thus, if $I$ is considered in the usual way as a table consisting of rows corresponding to objects and columns corresponding to attributes, $I_x$ represents the $L$-set of
attributes of object $x \in X$, i.e., it represents a single row of the table. Thus, $||A \Rightarrow B||_I^*$ is indeed a degree to which the following condition is true: “For each object $x \in X$, if (it is very true that) the object has all the attributes from $A$, then it has all the attributes from $B$”. If $\Sigma$ satisfies

$$||A \Rightarrow B||_I^* = ||A \Rightarrow B||_I$$

(20)

for all $A, B \in L_Y$, then $\Sigma$ is called complete in $I$. In addition, if $\Sigma$ is non-redundant then it is called a (non-redundant) base of $I$; if $\Sigma$ is minimal then it is called a minimal base of $I$. In the description of bases, we consider the well-known concept-forming operators with a linguistic hedge [15]. Namely, we use a couple of operators $\uparrow: L^X \rightarrow L^Y$ and $\downarrow: L^Y \rightarrow L^X$ defined by

$$A^\uparrow(y) = \bigwedge_{x \in X}(A(x)^* \rightarrow I(x, y)), \quad (21)$$

$$B^\downarrow(x) = \bigwedge_{y \in Y}(B(y) \rightarrow I(x, y)), \quad (22)$$

for all $A \in L^X$, $B \in L^Y$, $x \in X$, and $y \in Y$. Importantly, the composition $\downarrow \uparrow$ of $\downarrow$ and $\uparrow$ is an $L^*$-closure operator [5] and it follows that $\Sigma$ is complete in $I$ iff Mod$(\Sigma)$ is the set of all fixed points of $\downarrow \uparrow$, see [9, 16]. Therefore, using this fact, we immediately get the following consequence of Theorem 1:

**Corollary 2.** Let $I = \langle X, Y, I \rangle$ be a formal $L$-context and let $\Gamma$ be a non-redundant/minimal base of $I$ which consists of formulas with saturated consequents. If $L$ is finite then any irreducible $\Sigma$ such that $\Gamma \supseteq^*_\infty \Sigma$ is a non-redundant/minimal base of $I$ whose non-redundancy is witnessed.

Therefore, from the existence of non-redundant bases with witnessed non-redundancy, we can directly derive, using [41, Theorem 10], the existence of general systems of pseudo-intents. Let us recall [6, 10, 16] that denoting

$$U = \{P \in L_Y; P \neq P^\uparrow\},$$

(23)

we call $P \subseteq U$ a system of pseudo-intents (of $I$) whenever for each $P \in U$, we have

$$P \in \mathcal{P} \text{ iff } ||Q \Rightarrow Q^\uparrow||_P^* = 1 \text{ for any } Q \in \mathcal{P} \text{ such that } Q \neq P.$$  

(24)
If $\mathcal{P}$ exists, then

$$\Sigma = \{ P \Rightarrow P^{i\uparrow}; P \in \mathcal{P} \}$$

is a non-redundant base of $I$, see [7, Theorem 10], and its non-redundancy is witnessed [41] and, obviously, each formula in $\Sigma$ has a saturated consequent. So far, the existence of such systems of pseudo-intents was proved only for finite $L$ with globalization, see [3] but [41, Theorem 10] shows that $\Sigma$ is a non-redundant base of $I$ consisting of graded attribute implications with saturated consequents such that its non-redundancy is witnessed iff

$$\mathcal{P} = \{ A \in L^Y; A \Rightarrow A^{i\uparrow} \in \Sigma \}$$

is a system of pseudo-intents of $I$. Therefore, we conclude:

**Corollary 3.** Let $L$, $X$, and $Y$ be finite. For arbitrary $*$ satisfying (3) – (6), any formal $L$-context $I = \langle X, Y, I \rangle$ has at least one system of pseudo-intents which determines a minimal base.

In addition, it follows from our previous remarks that a system of pseudo-intents of $I$ can be computed, in a polynomial time, from any complete set in $I$. Thus, the procedure is tractable. As we have said before, this is in contrast with approaches to finding systems of pseudo-intents directly from $I$ which are known, even in the classic case, to be hard, see [19] for the results on complexity related to enumerating pseudo-intents.

**Remark 3.** We have presented the solution of the problem of existence of general systems of pseudo-intents using formal $L$-contexts as structures in which we evaluate formulas. In a similar setting, we could do the same with other semantic structures which yield the same notion of semantic entailment. For instance, in relational similarity-based databases [11], we can think of bases of similarity-based functional dependencies which are true in data tables over domains with similarities or their ranked extensions [14]. In all such cases, the important point is that systems of models of such rules can be identified with systems of fixed points of $L^*$-closure operators and, therefore, the procedure introduced in
our paper, can also be applied in these settings where the notion of a system of pseudo-intents is present, see, e.g., [11].

4 Proofs and notes

In this section, we present the proof of Theorem 1 and present further notes, including an illustrative example, and some experimental observations which illustrate that in the case of other hedges than the globalization, non-redundant bases given by systems of pseudo-intents may be smaller. In order to prove the main assertion of the paper, we investigate properties of $\lll^*$. In all the subsequent lemmas, we assume that $\mathcal{L}$ is a finite residuated lattice and $*$ satisfies (3)–(6).

Lemma 4. If $\Delta \lll^* \Gamma$, then $\Delta \equiv \Gamma$.

Proof. Suppose that $\Delta \lll^* \Gamma$, i.e., there are distinct $A \Rightarrow B \in \Delta$ and $C \Rightarrow D \in \Delta$ such that $\Gamma$ is of the form (18). Thus, $\Gamma$ results from $\Delta$ by removing $A \Rightarrow B$ and adding the formula $A \cup (S(C, A)^* \otimes D) \Rightarrow B$. Trivially, for its antecedent $A \cup (S(C, A)^* \otimes D)$, we have $A \subseteq A \cup (S(C, A)^* \otimes D)$, i.e., $\Gamma \subseteq \Delta$. Indeed, for any $M \in L^Y$, using the antitony of $\Rightarrow$ and $S$ in their first arguments together with the isotony of $*$, we get that

$$||A \Rightarrow B||_M^* = S(A, M)^* \Rightarrow S(B, M)$$

$$\leq S(A \cup (S(C, A)^* \otimes D), M)^* \Rightarrow S(B, M)$$

$$= ||A \cup (S(C, A)^* \otimes D) \Rightarrow B||_M^*.$$

Therefore, $\text{Mod}^*(\Delta) \subseteq \text{Mod}^*(\Gamma)$ and thus $\Gamma \subseteq \Delta$, i.e., $\Delta$ is stronger than $\Gamma$. It remains to show that $\Delta \subseteq \Gamma$ holds as well. Take any $M \in \text{Mod}^*(\Gamma)$. Obviously, $M \in \text{Mod}^*(\Delta \setminus \{A \Rightarrow B\})$. By the adjointness and (1), we get $S(C, A)^* \otimes C \subseteq A$. Therefore, using properties of residuated lattices and hedges together with the
fact that \(||C \Rightarrow D||_M^* = 1\), we get

\[
S(A, M)^* \leq S(S(C, A)^* \otimes C, M)^* = (S(C, A)^* \rightarrow S(C, M))^*
\]

\[
\leq S(C, A)^{**} \rightarrow S(C, M)^* = S(C, A)^* \rightarrow S(C, M)^*
\]

\[
\leq S(C, A)^* \rightarrow S(D, M) = S(S(C, A)^* \otimes D, M).
\]

Therefore, we may write

\[
S(A, M)^* = S(A, M) \wedge S(A, M)^* \leq S(A, M) \wedge S(S(C, A)^* \otimes D, M)
\]

\[
= S(A \cup (S(C, A)^* \otimes D), M)
\]

From the last inequality, applying \(||A \cup (S(C, A)^* \otimes D) \Rightarrow B||_M^* = 1\), we get

\[
S(A, M)^* \leq S(A \cup (S(C, A)^* \otimes D), M)^* \leq S(B, M),
\]

which proves that \(||A \Rightarrow B||_M^* = 1\), i.e., \(M \in \text{Mod}^*(\Delta)\). Altogether, \(\Delta \equiv \Gamma\). 

**Remark 4.** Readers familiar with Armstrong-style axiomatic systems [1] of the logic of graded attribute implications parameterized by linguistic hedges, see [4, 13, 17, 40], may easily see that Lemma 4 can be proved by arguments about provability and utilizing the completeness theorems. Indeed, \(\Gamma \subseteq \Delta\) is a consequence of the fact that the rule of weakening (also known as augmentation [1, 36]; from \(A \Rightarrow B\) we infer \(A \cup E \Rightarrow B\) for \(E = S(C, A)^* \otimes D\)) is a derivable deduction rule in the logic. In addition, \(\Delta \subseteq \Gamma\) follows by the rules of multiplication (from \(C \Rightarrow D\) we infer \(a^* \otimes C \Rightarrow a^* \otimes D\) for \(a = S(C, A)\)) and cut (also known as pseudo-transitivity [31, 36]: from \(S(C, A)^* \otimes C \Rightarrow S(C, A)^* \otimes D\) and \(A \cup (S(C, A)^* \otimes D) \Rightarrow B\) we infer \(A \cup (S(C, A)^* \otimes C) \Rightarrow B\) which equals to \(A \Rightarrow B\).)

**Lemma 5.** If \(\Delta\) has saturated consequents and \(\Delta \models^* \Gamma\), then so has \(\Gamma\).

**Proof.** Assume that \(A \Rightarrow B \in \Delta\) and \(C \Rightarrow D \in \Delta\) are the formulas from (18). Since \(\Delta\) consists of formulas with saturated consequents, then \(A \Rightarrow B\) and \(C \Rightarrow D\) are in fact in the form \(A \Rightarrow [A]_\Delta^*\) and \(C \Rightarrow [C]_\Delta^*\), respectively. Furthermore, using Lemma 4 \(\Delta \equiv \Gamma\), i.e., \([E]_\Delta^* = [E]_\Gamma^*\) for any \(E \in L^\gamma\). Thus, each formula

\[
\]
in $\Gamma \setminus \{A \cup (S(C,A)^* \otimes [C]_\Gamma^*) \Rightarrow [A]_\Gamma^* \}$ has a saturated consequent. Therefore, it remains to check this fact for the formula $A \cup (S(C,A)^* \otimes [C]_\Gamma^*) \Rightarrow [A]_\Gamma^*$. By (10), we get $A \subseteq [A]_\Gamma^*$. Applying (11), we get $S(C,A)^* \leq S([C]_\Gamma^* \setminus [A]_\Gamma^*)$ and so the adjointness property gives $S(C,A)^* \otimes [C]_\Gamma^* \subseteq [A]_\Gamma^*$. Therefore, using (11) and (12), the previous inclusions yield

$$[A \cup (S(C,A)^* \otimes [C]_\Gamma^*)]_\Gamma^* \subseteq [A]_\Gamma^*. \quad (27)$$

The converse inclusion to that in (27) follows directly from (11). As a consequence, $A \cup (S(C,A)^* \otimes [C]_\Gamma^*) \Rightarrow [A]_\Gamma^*$ has a saturated consequent.

**Lemma 6.** If $\Delta$ is non-redundant and $\Delta \supseteq \Gamma$, then so is $\Gamma$.

**Proof.** Again, we assume that $\Gamma$ is a theory of the form (18) and $\Delta$ is non-redundant. First, note that $A \cup (S(C,A)^* \otimes D) \Rightarrow B \notin \Delta$ because otherwise $A \cup (S(C,A)^* \otimes D) \Rightarrow B$ would be redundant in $\Delta$ because $A \Rightarrow B \in \Delta$—a contradiction. Now, take arbitrary $E \Rightarrow F \in \Gamma \setminus \{A \cup (S(C,A)^* \otimes D) \Rightarrow B\}$. The fact that $E \Rightarrow F$ is non-redundant in $\Delta$ means that $||E \Rightarrow F||_{\Delta \setminus \{E \Rightarrow F\}} < 1$. By similar arguments as in the proof of Lemma 4, it follows that $\Gamma \setminus \{E \Rightarrow F\} \subseteq \Delta \setminus \{E \Rightarrow F\}$. Therefore, we have

$$||E \Rightarrow F||_{\Gamma \setminus \{E \Rightarrow F\}}^* \leq ||E \Rightarrow F||_{\Delta \setminus \{E \Rightarrow F\}}^* < 1$$

which proves that $E \Rightarrow F$ is non-redundant in $\Gamma$. Thus, it remains to prove that $A \cup (S(C,A)^* \otimes D) \Rightarrow B$ is non-redundant in $\Gamma$. By contradiction, let

$$||A \cup (S(C,A)^* \otimes D) \Rightarrow B||_{\Gamma \setminus \{A \cup (S(C,A)^* \otimes D) \Rightarrow B\}}^* = 1$$

Since $\Gamma$ and $\Delta$ differ only in the formula $A \Rightarrow B \in \Delta$ (and $A \cup (S(C,A)^* \otimes D) \Rightarrow B \in \Gamma$), we immediately get

$$||A \cup (S(C,A)^* \otimes D) \Rightarrow B||_{\Delta \setminus \{A \Rightarrow B\}}^* = 1.$$

Therefore, for any $M \in \text{Mod}^*(\Delta \setminus \{A \Rightarrow B\})$, the previous fact yields $M \in \text{Mod}^*(\Gamma)$. Since $\Gamma \equiv \Delta$, we get that $M \in \text{Mod}^*(\Delta)$, i.e., $\text{Mod}^*(\Delta \setminus \{A \Rightarrow B\}) \subseteq \text{Mod}^*(\Delta)$ which contradicts the fact that $\Delta$ is non-redundant. \[\square\]
Lemma 7. Let $\Sigma$ be a non-redundant irreducible theory. Then the non-redundancy of $\Sigma$ is witnessed.

Proof. Take any $A \Rightarrow B \in \Sigma$. It suffices to show that $A \in \text{Mod}^*(\Sigma \setminus \{A \Rightarrow B\})$. Take any $C \Rightarrow D \in \Sigma \setminus \{A \Rightarrow B\}$, i.e., $A \Rightarrow B$ and $C \Rightarrow D$ are distinct formulas. Now, observe that $A \cup (S(C, A)^* \otimes D) \Rightarrow B \not\in \Sigma$ because otherwise the non-redundancy of $\Sigma$ would be violated owing to $A \Rightarrow B \in \Sigma$. Since $\Sigma$ is irreducible, we must have $S(C, A)^* \otimes D \subseteq A$ which by adjointness yields $S(C, A)^* \leq S(D, A)$, i.e., $A \in \text{Mod}^*(\{C \Rightarrow D\})$. As a consequence, $A \in \text{Mod}^*(\Sigma \setminus \{A \Rightarrow B\})$, which finishes the proof.

Proof of Theorem 1. The proof results by putting together the observations in the previous lemmas. Because of the finiteness of $L$ and $Y$, the existence of irreducible $\Sigma$ such that $\Gamma \vDash^* \Sigma$ is ensured. Then, by induction, Lemma 4 yields $\Sigma \equiv \Gamma$, Lemma 6 yields that $\Sigma$ is non-redundant, Lemma 5 yields that $\Sigma$ has saturated consequents. Finally, Lemma 7 shows that the non-redundancy of $\Sigma$ is witnessed. This proves Theorem 1 for $\Gamma$ being non-redundant. If $\Gamma$ is in addition minimal, it is easily seen that $\Sigma$ is minimal as well—this follows directly by the form of (18).

We conclude this section by illustrative examples and some experimental observations on sizes of non-redundant bases determined from data using different hedges. In the examples we use the usual notation for writing $L$-sets on finite universe sets, namely, $\{a_i/y_1, \ldots, a_n/y_n\}$ denotes an $L$-set $A$ in $Y = \{y_1, \ldots, y_n\}$ such that $A(y_i) = a_i$ for all $i = 1, \ldots, n$. Optionally, we omit $a_i/y_i$ whenever $a_i = 0$ and write just $y_i$ instead of $a_i/y_i$ whenever $a_i = 1$.

Example 1. Suppose that $L$ is a residuated lattice with $L = \{0, 0.5, 1\}$ and $\otimes = \land$ (i.e., $L$ is a the three-element Gödel chain $[2, 27, 32]$) and let $*$ be the identity. Furthermore, we take theory $\Gamma$ from Example 4:

$$\Gamma = \{\{0.5/p\} \Rightarrow \{0.5/p, 0.5/q, r\}, \{p\} \Rightarrow \{p, q, r\}\}.$$
For this theory on \( Y = \{p, q, r\} \) and the considered \( L \) and *, [II, Example 4] shows that \( \Sigma \) given by

\[
\Sigma = \{ [A]_T^*, \\{ A \Rightarrow [A]_T^* \} \in \Gamma \}.
\]

which equals to

\[
\Sigma = \{ \{0.5/p, 0.5/q, 0.5/r\} \Rightarrow \{0.5/p, 0.5/q, r\}, \{p, 0.5/q, r\} \Rightarrow \{p, q, r\} \}
\]

is not equivalent to \( \Gamma \). This is a particular example of a theory which cannot be converted into an equivalent one with witnessed non-redundancy using the method described in [II]. Using the method shown in the present paper, for \( \Gamma \) we may find an irreducible theory \( \Sigma' \) such that \( \Gamma \equiv_{\infty}^* \Sigma' \). Actually, this can be done in a single step: For \( \{0.5/p\} \Rightarrow \{0.5/p, 0.5/q, r\} \in \Gamma \) and \( \{p\} \Rightarrow \{p, q, r\} \in \Gamma \), we can easily see that

\[
S(\{p\}, \{0.5/p\})^* \otimes \{p, q, r\} = 0.5 \otimes \{p, q, r\} = \{0.5/p, 0.5/q, 0.5/r\},
\]

i.e., \( \Gamma \) can be reduced to

\[
\Gamma' = \{ \{0.5/p, 0.5/q, 0.5/r\} \Rightarrow \{0.5/p, 0.5/q, r\}, \{p\} \Rightarrow \{p, q, r\} \}
\]

and \( \Gamma' \) is already irreducible. Thus, Theorem II yields that \( \Gamma' \) is equivalent to \( \Gamma \), it is non-redundant, has saturated consequents, and its non-redundancy is witnessed. This example demonstrates that the outcome of the algorithm studied in the previous paper [II] is unnecessary weak in case of hedges other than the globalization.

Example 2. In this example, we show a transformation of a non-redundant theory with saturated consequents into an equivalent theory with witnessed non-redundancy which takes more elementary steps. We assume that \( L \) is an equidistant five-element Lukasiewicz chain, i.e., \( L = \{0, 0.25, 0.5, 0.75, 1\} \) and the adjacent operations \( \otimes \) and \( \rightarrow \) are defined as \( a \otimes b = \max(0, a + b - 1) \) and
Thus, $\Gamma \Rightarrow A$ are not irreducible. Indeed, there is $\Gamma$ such that $\Gamma \Rightarrow 0$ is non-redundant and has saturated consequents. Also, it is evident that the non-redundancy of $\Gamma_0$ is not witnessed because it is not irreducible. Indeed, there is $\Gamma_1$ such that $\Gamma_0 \supseteq \Gamma_1$. Namely, we can take the formulas $A \Rightarrow B$ and $C \Rightarrow D$ which appear in $\text{KS}$ as follows:

$A \Rightarrow B$ is $\{0.25/r, 0.25/s \} \Rightarrow \{0.75/p, 0.75/q, 0.75/r, 0.25/s \}$,

$C \Rightarrow D$ is $\{0.5/p, 0.25/q, 0.5/s \} \Rightarrow \{0.75/p, 0.75/q, 0.5/r, 0.5/s \}$,

$S(C, A)^* \otimes D = 0.5 \otimes \{0.75/p, 0.75/q, 0.5/r, 0.25/s \} = \{0.25/p, 0.25/q, 0.25/s \}$,

replace $A \Rightarrow B$ in $\Gamma_0$ by $\{0.25/p, 0.25/q, 0.5/r, 0.25/s \} \Rightarrow \{0.75/p, 0.75/q, 0.75/r, 0.25/s \}$.

Thus, $\Gamma_1 = (\Gamma_0 \setminus \{A \Rightarrow B\}) \cup \{\{0.25/p, 0.25/q, 0.5/r, 0.25/s \} \Rightarrow B\}$. For the new formula, we can repeat the procedure with the same $C \Rightarrow D$. That is,

$A \Rightarrow B$ is $\{0.25/p, 0.25/q, 0.5/r, 0.25/s \} \Rightarrow \{0.75/p, 0.75/q, 0.75/r, 0.25/s \}$,

$C \Rightarrow D$ is $\{0.5/p, 0.25/q, 0.5/s \} \Rightarrow \{0.75/p, 0.75/q, 0.5/r, 0.5/s \}$,

$S(C, A)^* \otimes D = 0.75 \otimes \{0.75/p, 0.75/q, 0.5/r, 0.25/s \} = \{0.5/p, 0.5/q, 0.75/r, 0.25/s \}$,

replace $A \Rightarrow B$ in $\Gamma_1$ by $\{0.5/p, 0.5/q, 0.75/r, 0.25/s \} \Rightarrow \{0.75/p, 0.75/q, 0.75/r, 0.25/s \}$,

i.e., $\Gamma_2 = (\Gamma_1 \setminus \{A \Rightarrow B\}) \cup \{\{0.5/p, 0.5/q, 0.75/r, 0.25/s \} \Rightarrow B\}$. In the next step, we update the antecedent of the formula which so far played the role of $C \Rightarrow D$:

$A \Rightarrow B$ is $\{0.5/p, 0.25/q, 0.5/s \} \Rightarrow \{0.75/p, 0.75/q, 0.5/r, 0.25/s \}$,

$C \Rightarrow D$ is $\{0.25/p, s \} \Rightarrow \{0.75/p, q, r, s \}$,

$S(C, A)^* \otimes D = 0.5 \otimes \{0.75/p, q, r, s \} = \{0.25/p, 0.5/q, 0.5/r, 0.5/s \}$,

replace $A \Rightarrow B$ in $\Gamma_2$ by $\{0.5/p, 0.5/q, 0.5/r, 0.5/s \} \Rightarrow \{0.75/p, 0.75/q, r, 0.5/s \}$.

Thus, $\Gamma_3 = (\Gamma_2 \setminus \{A \Rightarrow B\}) \cup \{\{0.5/p, 0.5/q, 0.5/r, 0.5/s \} \Rightarrow B\}$. In the next step, we flip the roles of $A \Rightarrow B$ and $C \Rightarrow D$ from the previous step and make an
update of the antecedent of \(\{0.25/p, s\} \Rightarrow \{0.75/p, q, r, s\}\):

\[A \Rightarrow B\] is \(\{0.25/p, s\} \Rightarrow \{0.75/p, q, r, s\}\),
\[C \Rightarrow D\] is \(\{0.5/p, 0.5/q, 0.5/r, 0.5/s\} \Rightarrow \{0.75/p, 0.75/q, r, 0.5/s\}\),
\[S(C, A)^{\ast} \otimes D = 0.5 \otimes \{0.75/p, 0.75/q, r, 0.5/s\} = \{0.25/p, 0.25/q, 0.5/r\}\),

replace \(A \Rightarrow B\) in \(\Gamma_3\) by \(\{0.25/p, 0.25/q, 0.5/r, s\} \Rightarrow \{0.75/p, q, r, s\}\).

i.e., \(\Gamma_4 = (\Gamma_3 \setminus \{A \Rightarrow B\}) \cup \{\{0.25/p, 0.25/q, 0.5/r, s\} \Rightarrow B\}\). In the last two steps, we use the same formulas for update, i.e.,
\[A \Rightarrow B\] is \(\{0.25/p, 0.25/q, 0.5/r, s\} \Rightarrow \{0.75/p, q, r, s\}\),
\[C \Rightarrow D\] is \(\{0.5/p, 0.5/q, 0.5/r, 0.5/s\} \Rightarrow \{0.75/p, 0.75/q, r, 0.5/s\}\),
\[S(C, A)^{\ast} \otimes D = 0.75 \otimes \{0.75/p, 0.75/q, r, 0.5/s\} = \{0.5/p, 0.5/q, 0.75/r, 0.25/s\}\),

replace \(A \Rightarrow B\) in \(\Gamma_4\) by \(\{0.5/p, 0.5/q, 0.75/r, s\} \Rightarrow \{0.75/p, q, r, s\}\).

Thus, \(\Gamma_5 = (\Gamma_4 \setminus \{A \Rightarrow B\}) \cup \{\{0.5/p, 0.5/q, 0.75/r, s\} \Rightarrow B\}\). Finally, we can see that \(C\) is now fully included in \(A\), i.e.,
\[A \Rightarrow B\] is \(\{0.5/p, 0.5/q, 0.75/r, s\} \Rightarrow \{0.75/p, q, r, s\}\),
\[C \Rightarrow D\] is \(\{0.5/p, 0.5/q, 0.5/r, 0.5/s\} \Rightarrow \{0.75/p, 0.75/q, r, 0.5/s\}\),
\[S(C, A)^{\ast} \otimes D = 1.0 \otimes \{0.75/p, 0.75/q, r, 0.5/s\} = \{0.75/p, 0.75/q, r, 0.5/s\}\),

replace \(A \Rightarrow B\) in \(\Gamma_5\) by \(\{0.75/p, 0.75/q, r, s\} \Rightarrow \{0.75/p, q, r, s\}\).

Now, the last computed theory \(\Gamma_5\):
\[
\Gamma_5 = \{\{0.75/p, 0.75/q, r, s\} \Rightarrow \{0.75/p, q, r, s\}\},
\{0.5/p, 0.5/q, 0.5/r, 0.5/s\} \Rightarrow \{0.75/p, 0.75/q, r, 0.5/s\}\),
\{0.5/p, 0.5/q, 0.75/r, 0.25/s\} \Rightarrow \{0.75/p, 0.75/q, 0.75/r, 0.25/s\}\}

is irreducible, i.e., its non-redundancy is witnessed.

*Example 3.* In the last example, we present results of some experimental observations on sizes of non-redundant bases with witnessed non-redundancy computed for randomly generated formal L-contexts. Fig. shows the mean sizes
Figure 1: Mean sizes of non-redundant bases of formal L-contexts with 5 objects and 10 attributes computed for structures of degrees with $|L| = 11$ using globalization as the hedge, and using Gödel and Łukasiewicz operations with identity as the hedge.

Figure 2: Mean sizes of non-redundant bases of sparse formal L-contexts with 20 objects and 50 attributes for $|L| = 5$. 
of bases computed from $L$-contexts with 5 object and 10 attributes. The set of truth degrees used in the experiment is an 11-element equidistant subchain of $[0, 1]$. The $x$-axis in Fig. 1 indicates the density of input data which if for $I = \langle X, Y, I \rangle$ expressed in percents by

$$\frac{\sum_{x \in X} \sum_{y \in Y} I(x, y)}{|X| \cdot |Y|} \cdot 100.$$  

The $y$-axis indicates the mean number of formulas in a non-redundant base with witnessed non-redundancy. For each randomly generated $L$-contexts, we have computed a base using the globalization (which completely supersedes the role of $\otimes$ and $\to$) and identity as the hedge. In the latter case, we have used the standard Lukasiewicz and Gödel operations on $[0, 1]$. The total number of generated formal $L$-contexts for the experiment is 58,300. As it turns out, the Lukasiewicz operations with identity as the hedge yield smaller bases which is most evident if the density of the input data is about 50%. Interestingly, in most cases, the Gödel operations with identity yield greater bases than in the case of globalization with the exception of sparse data sets but this may be just a deviation from a typical behavior which is due to the small size of the data. In Fig. 2, we have included the results of a similar experiment on randomly generated data with 20 objects, 50 attributes, and using 5 truth degrees. Here, even if the data is sparse, the sizes of bases for the Gödel operations with identity are bigger than that for globalization. Even in this case, the Lukasiewicz operations with identity seem to produce the smallest bases. Let us note that in order to produce Fig. 2 we have used 300 randomly generated $L$-contexts.

5 Conclusion

Our paper deals with transformations of sets of if-then rules describing dependencies between graded attributes which generalize the ordinary attribute implications in a setting where presence of attributes is expressed by degrees. The transformations we deal with allow us to transform a set of such dependencies into an equivalent one which is non-redundant and satisfies a stronger
condition of witnessed non-redundancy. The witnessed non-redundancy may be seen as a desirable property because the non-redundancy of every rule in the set can be directly observed from its antecedent. Compared to our previous results, the proposed algorithm works for arbitrary idempotent truth-stressing linguistic hedge which may serve as a parameter of interpretation of the rules. As one of the consequences of the theoretical result, we obtained that every finite object-attribute data table with graded attributes (formal L-context) admits at least one system of pseudo-intents provided that the utilized structure of degrees is finite and, in addition, it admits a system of pseudo-intents which determines a minimal base of the data table. By this, we have closed one of the open problems listed in [35]. We have also presented some initial experimental evidence that bases of if-then rules computed using different hedges than the globalization can be smaller and thus more interesting for users. Clearly, in the graded setting, the topics related to non-redundancy and minimality of bases are considerably more involved than in the classic setting and further investigation focused on theory, algorithms, and experiments is needed.

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