Modeling of the molten glass formation zone as a viscoelastic medium

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Abstract. The article studies mathematical models that describe the movement of a viscoelastic medium. The equations of continuity, balance of forces and physical equations describing the glass melt have been created. When studying the behavior of molten glass in the formation zone, it is considered as a viscous liquid, as a viscoelastic liquid (Maxwell's model) and as an elastic-viscous body. A system with boundary conditions is a problem of movement of the viscoelastic medium with a lower movable boundary.

1. Introduction
The theory of viscoelastic flows of an incompressible fluid is of interest for practice and mathematical research [1-2]. The mathematical description of the molten glass formation process is important. A large number of works deal with the process of molten glass formation within the fluid dynamics [3-8]. Glass melt in the temperature softening zone is considered a viscous (Newtonian) liquid. In this case, it is assumed that two processes occur simultaneously in the formation zone: material deformation and hardening. In terms of thermodynamics, a model was built in [3]. The problem is solved by calculating the temperature field in the formation zone. The solution is obtained numerically. Surface tension almost always interferes with obtaining a glass product of the required quality. The softened molten glass takes on a spherical shape. The ratio \( \mu / \sigma \) (\( \mu \) is viscosity, \( \sigma \) is surface tension) serves as an indicator of the material's ability to form. Experiments have shown that the ratio should be 2-6.5. In general, experiments are aimed at studying the zone of molten glass formation. It is believed that the onion-shaped formation zone is an indicator of stability of the molten glass formation process [6].

2. Formation zone models
Mathematical models of zone formation can be constructed based on the general equations of continuum mechanics [9]. The formation zone represents the transition from heated glass melt to glass fiber. We will consider the formation zone as a body of revolution (Figure 1).
Let us take the z-axis as the axis of rotation, and the r-axis perpendicular to z. Two coordinates are sufficient, since all points in the section with z coordinate lie on a circle of radius r and are in the same conditions. The velocities of points of the continuous medium are the sum of velocities of its quasi-rigid and deformational motions.
2.1. Continuity equations

In the section with coordinate \( z \ (0 \leq z \leq L) \), let us single out the elementary volume equal to \( 2 \pi r \cdot \delta r \cdot \delta z \) (Figure 2).

![Figure 1. Fiberglass forming zone](image1.png)

![Figure 2. Cross section of the formation zone](image2.png)

According to the mass conservation condition

\[
\frac{d}{dt}(\delta m) = 0,
\]

where \( \delta m = \rho \ 2 \pi r \cdot \delta r \cdot \delta z \), \( \rho \) is density of the molten glass. We consider \( \rho \) constant.

One of the characteristics of the deformational expansion of the medium is the rate of relative volumetric expansion.

\[
\delta r \delta z \frac{d}{dt} \frac{d r}{d t} + r \frac{d}{dt} (\delta r \delta z) = 0
\]

where \( \delta m = \rho \ 2 \pi r \cdot \delta r \cdot \delta z \), \( \rho \) – fiberglass density.

\[
\frac{d}{dt} (\delta x_1 \delta x_2 \delta x_3) = \text{div } \bar{v} (\delta x_1 \delta x_2 \delta x_3) = \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) (\delta x_1 \delta x_2 \delta x_3),
\]

where \( \delta x_1 \delta x_2 \delta x_3 \) is the elementary volume around a certain point of the medium, \( \delta x_1, \delta x_2, \delta x_3 \) are elementary segments parallel to the coordinate axes, \( x_1, x_2, x_3 \), \( \text{div } \bar{v} \) is the divergence of the velocity of the point.

As a result, we can obtain the equation of continuity:

\[
\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0. \quad (1)
\]

2.2. Equilibrium equations

For each point located on the meridional plane, the area taken in it is the main area of the stress state. The main stress acting on this area is \( p_0 \). In all secant planes, only shear stresses can act. They are parallel to the meridian plane. Let us designate the normal stresses acting in the secant planes through \( p_z \) and \( p_r \), tangents - through \( p_{rz} \) and \( p_{zr} \). Now let us select an infinitely small volume of the medium, for which we can write the static equilibrium equations. These equations should contain the projections of inertial forces on the corresponding axes.

To create the equation of the balance of forces, we use the theorem on the change in the main vector of the system momentum [10]:

\[
\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0. \quad (1)
\]
\[
\frac{d\vec{K}}{dt} = \vec{F}_{\text{vol}} + \vec{F}_{\text{surf}},
\]

where \(\vec{F}_{\text{vol}}\) and \(\vec{F}_{\text{surf}}\) are the main vector of external volumetric and surface forces, respectively. Here \(\vec{K} = \rho \vec{\Phi} \delta \tau\), \(\vec{F}_{\text{vol}} = \rho \vec{F} \delta \tau\), \(\vec{F}_{\text{surf}} = \vec{p}_n \delta S\), where \(\delta \tau\) is the elementary volume of the medium containing a point with velocity \(\vec{\Phi}\), \(\delta S\) is the element of the surface bounding the volume \(\delta \tau\). \(\vec{F}\) is density of distribution of volumetric forces, \(\vec{p}_n\) is the main stress vector applied to the area.

According to the Cauchy equality

\[
\vec{p}_n = n_1 \vec{p}_1 + n_2 \vec{p}_2 + n_3 \vec{p}_3,
\]

where \(\vec{p}_1\), \(\vec{p}_2\), \(\vec{p}_3\) are stress vectors on the main sites. They are perpendicular to the corresponding coordinate axes \((x, y, z)\); \(n_1, n_2, n_3\) - direction cosines.

According to the Gauss-Ostrogradsky theorem

\[
\int_{S} \vec{p}_n \delta S = \int_{S} \left( n_1 \vec{p}_1 + n_2 \vec{p}_2 + n_3 \vec{p}_3 \right) \delta S = \int_{\tau} \text{div} P \delta \tau,
\]

where \(\text{div} P\) is stress tensor divergence equal to \(\text{div} P = \frac{\partial \vec{p}_1}{\partial x} + \frac{\partial \vec{p}_2}{\partial y} + \frac{\partial \vec{p}_3}{\partial z}\), \(\text{div} P\) is the ratio of the main vector of surface forces applied to \(\delta S\), limiting volume \(\delta \tau\), to this volume.

Let us integrate equation (1) over the entire volume \(\tau\)

\[
\frac{d}{dt} \int_{\tau} \rho \vec{\Phi} \delta \tau = \int_{\tau} \rho \vec{F} \delta \tau + \int_{\tau} \left( \frac{\partial \vec{p}_1}{\partial x} + \frac{\partial \vec{p}_2}{\partial y} + \frac{\partial \vec{p}_3}{\partial z} \right) \delta \tau,
\]

It can be written as

\[
\frac{d}{dt} (\rho \vec{\Phi} \delta \tau) = \rho \vec{F} \delta \tau + \left( \frac{\partial \vec{p}_1}{\partial x} + \frac{\partial \vec{p}_2}{\partial y} + \frac{\partial \vec{p}_3}{\partial z} \right) \delta \tau.
\]

The equilibrium equation in stresses is

\[
\rho \frac{d\vec{\Phi}}{dt} = \rho \vec{F} + \text{div} P
\]

(2)

Use the Lagrangian derivative

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \left( \vec{\Phi} \cdot \vec{\nabla} \right),
\]

where \(\vec{\nabla}\) is the nabla vector operator.

Equation (2) in projections onto the coordinate axes can be written as

\[
\rho \frac{d\vec{v}_r}{dt} = \rho \left( \frac{\partial \vec{v}_r}{\partial t} + \vec{v}_r \frac{\partial \vec{v}_r}{\partial r} + \vec{v}_\theta \frac{\partial \vec{v}_r}{\partial \theta} + \vec{v}_\phi \frac{\partial \vec{v}_r}{\partial \phi} \right) + \text{div} P,
\]

\[
\rho \frac{d\vec{v}_\theta}{dt} = \rho \left( \frac{\partial \vec{v}_\theta}{\partial t} + \vec{v}_r \frac{\partial \vec{v}_\theta}{\partial r} + \vec{v}_\theta \frac{\partial \vec{v}_\theta}{\partial \theta} + \vec{v}_\phi \frac{\partial \vec{v}_\theta}{\partial \phi} \right) + \text{div} P,
\]

\[
\rho \frac{d\vec{v}_\phi}{dt} = \rho \left( \frac{\partial \vec{v}_\phi}{\partial t} + \vec{v}_r \frac{\partial \vec{v}_\phi}{\partial r} + \vec{v}_\theta \frac{\partial \vec{v}_\phi}{\partial \theta} + \vec{v}_\phi \frac{\partial \vec{v}_\phi}{\partial \phi} \right) + \text{div} P.
\]

2.3. Physical equations

Physical equations depend on the medium. The formation zone can be represented as a Newtonian viscous fluid. In some cases, the model of an elastic-viscous body can be more accurate. Maxwell's model is based on the ability of the medium to return to the equilibrium state. If we consider a one-dimensional flow only along coordinate \(z\), we can write
\[ \lambda \dot{p}_z + p_z = \mu \frac{\partial \nu_z}{\partial z}, \]

where \( \lambda = \mu/G \) is relaxation time; \( \dot{p}_z = \frac{d p_z}{d t} \); \( G \) - shear modulus.

In this coordinate system, the physical equations for Maxwell’s model are

\[ \begin{align*}
(p_z - p_{az}) + \lambda (\dot{p}_z - \dot{p}_{az}) &= 2\mu \frac{\partial \nu_z}{\partial z}; \\
(p_r - p_{ar}) + \lambda (\dot{p}_r - \dot{p}_{ar}) &= 2\mu \frac{\partial \nu_r}{\partial r}; \\
(p_0 - p_{av}) + \lambda (\dot{p}_0 - \dot{p}_{av}) &= 2\mu \frac{\nu_r}{r}; \\
p_{z\tau} + \lambda \dot{p}_{z\tau} &= \mu \left( \frac{\partial \nu_z}{\partial r} + \frac{\partial \nu_r}{\partial z} \right).
\end{align*} \tag{4} \]

3. The one-dimensional flow model

Assuming that velocity, viscosity and stresses in the formation zone are functions of time and axial coordinates, we can write the continuity equation (1) as

\[ \frac{d}{d t} (S \cdot \delta z) = 0, \]

where \( S \) is the area of the current section of the formation zone (flow), \( \delta z \) is an elementary segment parallel to the \( z \) axis.

Taking into account the rate of linear expansion, the equation for the rate of a one-dimensional model can be written as

\[ \frac{d}{d t} (\delta z) = \frac{\partial \nu_z}{\partial z} \delta z, \]

where \( \nu_z = \nu_z \) is the velocity of glass melt in the section with the \( z \) coordinate.

Using the Lagrangian derivative, we reduce the continuity equation to the form

\[ \frac{\partial S}{\partial t} + \frac{\partial (S \nu)}{\partial z} = 0. \]

Using the theorem on the change in the momentum \( K \) of volume \( S \cdot \delta z \), the equation of balance of forces can be written as

\[ \frac{d K}{d t} = \rho \frac{d (\nu \cdot S \delta z)}{d t} = \rho \left( \frac{d (\nu S)}{d t} \delta z + \nu S \frac{d}{d t} (\delta z) \right). \]

The bulk forces are forces of gravity \( \rho S g \delta z \). The surface forces are equal to \( \frac{\partial (p S)}{\partial z} \delta z \), where \( p = p_z \) is pressure on \( S \) with the coordinate \( z \).

The balance of forces equation can be written as

\[ \rho \left( \frac{d (\nu S)}{d t} + \frac{\partial (\nu^2 S)}{\partial z} \right) = \rho S g + \frac{\partial (p S)}{\partial z}. \tag{5} \]
Using the Lagrangian derivative, the physical equations are reduced to one equation

\[ p + \lambda \left( \frac{\partial p}{\partial t} + \upsilon \frac{\partial p}{\partial z} \right) = 3 \mu \frac{\partial \upsilon}{\partial z}. \]  

For the one-dimensional model (Figure 3), we have a system of three equations

\[ \begin{cases} \frac{\partial S}{\partial t} + \frac{\partial (\upsilon S)}{\partial z} = 0; \\ \rho \left[ \frac{\partial (\upsilon S)}{\partial t} + \frac{\partial (\upsilon^2 S)}{\partial z} \right] = \rho S g + \frac{\partial (p S)}{\partial z}; \\ p + \lambda \left( \frac{\partial p}{\partial t} + \upsilon \frac{\partial p}{\partial z} \right) = 3 \mu \frac{\partial \upsilon}{\partial z}. \end{cases} \]  

The boundary conditions are determined by feed rate \( \upsilon_f \) of the workpiece and pulling velocity \( \upsilon_p \)

\[ z = 0: \ \upsilon = \upsilon_f; \quad z = L: \ \upsilon = \upsilon_p. \]  

System (7) with boundary conditions (8) is the problem of motion of the viscoelastic medium with moving boundaries [11,12].

4. Stationary configurations of the formation zone

Let is write the continuity equation as

\[ S \upsilon = Q = \text{const}, \]  

from the equation of equilibrium of forces we have (without taking into account the mass term)

\[ p S - \rho \upsilon^2 S = C, \]  

where \( C = F_s \) is the constant of integration.

Having \( p \) determined from the third equation of system (7) and substituted into (10), we obtain

\[ F_s = 3 \mu S \frac{d\upsilon}{dz} - \lambda \upsilon^2 S \rho \frac{d\upsilon}{dz} - \rho \upsilon^2 S, \]  

where \( S(z) \) is the cross-sectional area of the formation zone with the coordinate \( z \); \( \rho \) is density of the molten glass.

The first term on the right-hand side of expression (11) depends on the viscous resistance of the molten glass, the second - on its elastic properties, and the third determines the inertial properties.
When solving the problem, the following ranges were identified: speed $v_p = 1 \text{ m/min} \ldots 10 \text{ m/s}$; glass viscosity $\mu = 10^6 \ldots 10^9 \text{ Pa}$; the radius of glass fiber is $r_g = 0.5 \ldots 2 \text{ mm}$.

For these parameters, the dominant influence is exerted by the first term in expression (11). For example, the value of the inertial component at a speed of $1 \text{ m/s}$ reaches 0.1% of the viscous term.

Let us introduce the dimensionless parameters,

$$\xi = z/L; \quad a = A^{-1} = S_0/S = r_0^2/r^2 = v/v_0,$$

where $L$ is length of the formation zone.

Assuming the viscosity constant in the formation zone, equation (11) takes the form

$$\frac{d a}{d \xi} - Na = Ra^2,$$  \hspace{1cm} (12)

where $N = \frac{C L}{3 \mu_0 r_0^2 v_0}$; $R = \frac{\rho Q L}{3 \mu_0 r_0^2}$.

We accept $R = 0$ and, using the boundary condition $a(0) = 1$, we obtain a solution of the zero approximation in the form

$$\ln a(0) = N \xi.$$

At $\xi = 1 \quad a(0) = \frac{S_0}{S} = k$, where $k$ is the overstretching coefficient, $\ln k = N$.

The zero-order solution takes the form

$$a(0) = \exp(\xi \ln k).$$ \hspace{1cm} (13)

Let us complicate the model by introducing both the inertial term and the surface tension forces.

Taking into account the surface tension (Figure 4), equation (10) can be written as

$$3\mu r^2 \frac{d v}{d z} - \rho v^2 r^2 - \sigma r = C,$$ \hspace{1cm} (14)

$$\frac{d a}{d \xi} - Na = Ra^2 - Wa^{0.5},$$ \hspace{1cm} (15)

where $W = \frac{\sigma L}{3 \mu_0 r_0^2 v_0}$, $\sigma$ is surface tension [N/m].

Let us introduce solution (13) into equation (15) and obtain the solution in the first approximation

$$a'(0) - Na(0) = Ra(0)^2 - Wa(0)^{0.5},$$ \hspace{1cm} (16)

where $a'(0) = \frac{d a(0)}{d \xi}$.

After transformations, we have

$$a'(0) = a(0) \left[ 1 + \frac{R}{\ln k} (a(0) - 1) + \frac{2W}{\ln k} (a(0)^{0.5} - 1) \right].$$ \hspace{1cm} (17)

The number of approximations is estimated using numerical calculations based on the convergence of results obtained.

For the selected ranges of parameters $R = 10^{-2} \ldots 10^{-3}$, $W = 0.1 \ldots 0.5$.

Figure 5 shows graphs $a^{-1}(\xi)$ by expression (17) for $W = 0$; $R = 0$; $R = 0.01$; $R = 0.02$; and Figure 6 shows graphs for $R = 0$; $W = 0$; $W = 0.1$; $W = 0.5$; $k = 100$. 

5. Conclusion
Based on the molten glass formation zone modelling, the effect of surface tension and flow inertia on the configuration of the zone has been identified [13, 14]. The graphical dependence of the formation zone on velocity and viscosity has been established taking into account the surface tension. The dependence of the formation zone on glass melt viscosity, which is a function of time and an axial coordinate, has been shown. The solution has been reduced to three partial differential equations (the one-dimensional flow model). The formation process has been described by a system of first-order quasilinear partial differential equations. The use of piecewise linear approximations made it possible to obtain solutions to the stationary problem for the configuration of the formation zone in a steady state [15-17].

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