DISTRIBUTION OF THE BAD PART OF THE CLASS GROUP

WEITONG WANG

Abstract. The Cohen-Lenstra-Martinet Heuristics gives a prediction of the distribution of $\text{Cl}_K[p^\infty]$ when $K$ runs over $\Gamma$-fields and $p \nmid |\Gamma|$. In this paper, we prove several results on the distribution of ideal class groups for some $p||\Gamma|$, and show that the behaviour is qualitatively different than what is predicted by the heuristics when $p \nmid |\Gamma|$. We do this by using genus theory and the invariant part of the class group to investigate the algebraic structure of the class group. For general number fields, our result is conditional on a natural conjecture on counting fields. For abelian or $D_4$-fields, our result is unconditional.

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1. Introduction

In this paper, we show some results on the distribution of ideal class groups of number fields that could be compared with the Cohen-Lenstra-Martinet Heuristics (see Cohen and Martinet [CM90, Hypothèse 6.6]). We first try to give an overview of the heuristics in this section. Let $\Gamma$ be a finite group, and let $\Gamma_\infty$ be a fixed subgroup of $\Gamma$. For each prime $p \nmid |\Gamma|$, we can construct a random $\mathbb{Z}(p)[\Gamma]$-module $X$ such that for each finite $\mathbb{Z}(p)[\Gamma]$-module $M$, we have

$$
\mathbb{P}(X \cong M) = \frac{c}{|\text{Aut}_{\Gamma}(M)||\Gamma_\infty|},
$$

where $c > 0$ is a constant (normalizing the measure so that it is a probability space). For two modules $M, N$, let $1_M(N)$ be the indicator of $N \cong M$, i.e., $1_M(N) = 1$ if $N \cong M$, and $1_M(N) = 0$ otherwise. Let $\mathcal{S}$ be the set of $\Gamma$-extensions $K/\mathbb{Q}$ such that its decomposition group at infinity is $\Gamma_\infty$. For each finite $\mathbb{Z}(p)[\Gamma]$-module $M$, we define

$$
\mathbb{P}(\text{Cl}_K[p^\infty] \cong M) := \lim_{x \to \infty} \frac{\sum_{P(K) < x} 1_M(\text{Cl}_K[p^\infty])}{\sum_{P(K) < x} 1}.
$$
where \( K \) runs over all fields in \( \mathcal{S} \). When \( M \) runs over all finite \( \mathbb{Z}_p[\Gamma] \)-module, we obtain the distribution of ideal class group \( \text{Cl}_K[p^\infty] \). The Cohen-Lenstra-Martinet Heuristics predicts that for each finite \( \mathbb{Z}_p[\Gamma] \)-module \( M \), we have

\[
P(\text{Cl}_K[p^\infty] \cong M) = P(X \cong M)
\]

See also [WW21, §3] for details. Since the random module \( X \) is specifically constructed, it is much better understood than ideal class groups. For example, for a finite \( \mathbb{Z}_p[\Gamma] \)-module \( M \), let’s define the \( M \)-moment of ideal class groups as follows:

\[
E(\big| \text{Hom}_\Gamma(\text{Cl}_K, M) \big|) := \frac{\sum_{P(K) < x} |\text{Hom}_\Gamma(\text{Cl}_K, M)|}{\sum_{P(K) < x} 1}
\]

where \( K \) runs over all fields in \( \mathcal{S} \). It is not easy to compute it in general, but according to [WW21, Theorem 1.2], we have the following

\[
E(\big| \text{Sur}_\Gamma(X, M) \big|) = \frac{1}{|M^{\Gamma\infty}|}
\]

where \( \text{Sur} \) means surjective homomorphisms. Note that

\[
|\text{Hom}_\Gamma(\text{Cl}_K, M)| = \sum_{N \subseteq M} |\text{Sur}_\Gamma(X, N)|.
\]

The question whether the heuristics is true or how to prove it could be described as widely open. See the work of Davenport and Heilbronn [DH71] and Datskovsky and Wright [DW88] on the \( \mathbb{Z}/3\mathbb{Z} \)-moment of class groups of quadratic extensions, and the work of Bhargava [Bha05] on \( \mathbb{Z}/2\mathbb{Z} \)-moment of class groups of cubic fields.

The heuristics not only predicts the distribution of ideal class groups in the case of Galois number fields, but also for non-Galois extensions. Let \( \Gamma' \) be a fixed subgroup of \( \Gamma \), and let \( p \) be a prime not dividing \( |\Gamma| \). Let \( \mathcal{S}' \) be the set of fields \( K' \) such that its Galois closure \( K'/\mathbb{Q} \) is a \( \Gamma \)-extension with decomposition group \( \Gamma_\infty \) at infinity and that \( K' = K'^\infty \). We can first construct a maximal \( \mathbb{Z}_p[\Gamma'] \)-order \( \mathfrak{o} \) as a subset of \( \mathbb{Z}_p[\Gamma] \) in the algebra \( \mathbb{Q}[\Gamma'/\Gamma] \). If the heuristics holds for \( \text{Cl}_K[p^\infty] \) when \( K \) runs over all fields in \( \mathcal{S} \), i.e., we can construct some random \( \mathbb{Z}_p[\Gamma] \)-module \( X \) such that for each finite \( \mathbb{Z}_p[\Gamma] \)-module \( M \), we have

\[
P(\text{Cl}_K[p^\infty] \cong M) = P(X \cong M).
\]

Then, we can construct a random \( \mathfrak{o} \)-module \( Y \) out of \( X \), such that for each finite \( \mathfrak{o} \)-module \( N \), we have

\[
P(\text{Cl}_{K'}[p^\infty] \cong N) = P(Y \cong N)
\]

where \( K' \) runs over fields in \( \mathcal{S}' \). See [WW21, §7-8] for details. It is worth mentioning here that when \( \Gamma' \) is a normal subgroup, the random \( \mathfrak{o} \)-module \( Y \) will coincide with a particular random \( \Gamma/\Gamma' \)-module given by the heuristics (see [WW21, §9]). So, the heuristics works uniformly for number fields.

Since we will talk about primes \( q ||\Gamma| \), let’s here emphasize the following fact. If \( A \) is an abelian group, then let \( \text{rk}_p A \) denote the \( p \)-rank of \( A \) where \( p \) is a given prime number. Using

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In this notation, the heuristics implies that for all $r = 0, 1, 2, \ldots$

\[
\mathbb{P}(\text{rk}_p \text{ Cl}_{K'} \leq r) := \lim_{x \to \infty} \frac{\sum_{P(K') < x} 1_{\text{rk}_p \leq r}(\text{Cl}_{K'}[p^\infty])}{\sum_{P(K') < x} 1} > 0,
\]

and

\[
\lim_{r \to \infty} \mathbb{P}(\text{rk}_p \text{ Cl}_{K'} \leq r) = 1,
\]

where $K'$ runs fields in $S'$, and $1_{\text{rk}_p \leq r}(A) = 1$ if $\text{rk}_p A \leq r$ and $1_{\text{rk}_p \leq r}(A) = 0$ otherwise. This can be deduced from (1.1). In addition, the formula of moments (1.2) implies the following. The heuristics implies that for all finite abelian $p$-groups $A$, we have

\[
\mathbb{E}(|\text{Hom}(\text{Cl}_{K'}, A)|) < \infty,
\]

i.e., the $A$-moment is finite for each $A$. Note that here we forget the $\Gamma$-module structure for the convenience of our discussion.

Remark. In the original Cohen-Lenstra-Martinet Heuristics, fields are ordered by discriminant, which was an obvious ordering for number fields. Now we have the question of what kind of ordering one should put on the fields. In some cases, ordering fields by discriminant will contradict what is predicted by the heuristics. See Bartel [BLJ20] for example. The invariants of number fields that have been used for ordering all rely on the combination of ramified primes of different inertia, and we are mainly focused on the product of ramified primes, which is denoted by $P(K)$. See Wood [Woo10] for more discussion on different orderings.

In the above discussions, we restrict the choice of the prime $p$ to the case when $p \nmid |\Gamma|$. Now let’s give an example when this prime divides the order of the Galois group. Recall the genus theory for quadratic number fields, which says that

\[
\omega(P(K)) - 1 \leq \text{rk}_2 \text{ Cl}_K \leq \omega(P(K)),
\]

where $P(K)$ is the product of ramified primes of $K/Q$ and $\omega(n)$ is the number of distinct prime divisors for $n \in \mathbb{Z}$. The distribution of the group $\text{Cl}_K[2^\infty]$ then cannot be described by the approach of the Cohen-Lenstra-Martinet Heuristics. Because, first, the inequality (1.5) basically means that the number of ramified primes determines $\text{rk}_2 \text{ Cl}_K$ (up to 1). This phenomenon could be thought of as “predictable”, hence contradicting the spirit of the Cohen-Lenstra-Martinet Heuristics. Second, we can show that, for all $r \in \mathbb{N}$,

\[
\mathbb{P}(\text{rk}_2 \text{ Cl}_K \leq r) = 0 \quad \text{and} \quad \mathbb{E}(|\text{Hom}(\text{Cl}_K, C_2)|) = +\infty,
\]

which is qualitatively different from what is predicted by the heuristics, i.e., (1.3) and (1.4). According to this example, the statistical behaviour of $\text{Cl}_K[2^\infty]$ should be considered more carefully than other parts. See [FG87, Smi22] for example. On the other hand, we can also try to generalize the genus theory for quadratic number fields to general ones, whose details are given in § 3 following the idea of Ishida [Ish76, Chapter 4]. Here we present the main result in a brief way. Let $K/Q$ be a number field, and let $q$ be a prime. If we have ideal factorization $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}$ such that $\gcd(e_1, \ldots, e_m) \equiv 0 \mod q$, then we call $p$ a ramified prime of type $q$. For any number field $K/Q$, define its genus group $\mathcal{G}$ to be the Galois group of the maximal unramified extension $Kk/K$ obtained by composing with an abelian extension $k/Q$. Then, genus theory implies the following.
Theorem 1.1. Let $K/Q$ be a number field with maximal abelian subextension $K_0/Q$. Fix a rational prime $q$ dividing $n := [K : Q]$. Then, the $q$-rank of the genus group $\mathcal{G}$ admits the following inequality

$$\text{rk}_q \mathcal{G} \geq \# \{ p \text{ is a ramified prime of type } q \text{ and } p \equiv 1 \text{ mod } q \} - \text{rk}_q \text{Gal}(K_0/Q).$$

The genus group of $K$, by definition, is the quotient of the ideal class group. P. Roquette and H. Zassenhaus [RZ69] construct a subgroup of the class group that is associated to ramified primes. The main theorem of their paper says the following.

Theorem 1.2. Let $K$ be a number field of degree $n$ over $Q$ and $q$ be a given prime number, then

$$\text{rk}_q \text{Cl}_K \geq \# \{ p \text{ is a ramified prime of type } q \} - 2(n - 1)$$

The obvious difference compared to genus theory is that we do not require that $p$ be $1 \text{ mod } q$ in the condition. For more details and comparison between these two theories, see § 3.

Since the splitting type of a prime $p$ in a field extension $K/Q$ is determined by its decomposition group $G_p$ and inertia subgroup $I_p$ and so on, it is reduced to a problem of group theory to find out all primes $q$ so that we can apply the above Theorems 1.1 and 1.2 and expect to get nontrivial estimate for $\text{rk}_q \text{Cl}_K$.

Definition 1.3. Let $1 \leq G \leq S_n$ be a finite transitive permutation group. Let $\sigma \in G$ be any permutation. Define $e(\sigma) := \text{gcd}(|\langle \sigma \cdot 1 \rangle|, \ldots, |\langle \sigma \cdot n \rangle|)$, i.e., the greatest common divisor of the size of orbits. We call $q$ a non-random prime for $G$ if $q \mid e(\sigma)$ for some $\sigma \in G$. On the other hand, for a permutation $\sigma \in G$, if $q^l \| e(\sigma)$, then we call $\sigma$ an element of inertia type $q^l$. Define $\Omega(G, q^l)$ to be the subset of $G$ consisting of all elements of inertia type $q^l$. We denote $\bigcup_{l=1}^{\infty} \Omega(G, q^l)$ by $\Omega(G, q^\infty)$.

Example 1.4. First let $G = S_3$. If $K$ is a non-Galois cubic number field, then the permutation action of $G$ on $K \rightarrow \mathbb{C}$ is exactly the conventional action of $S_3$ on $\{1, 2, 3\}$, which induces the isomorphism $G \cong \text{Gal}(\hat{K}/Q)$. By checking the elements of $S_3$, we see that $3$ is a non-random prime for $G$, e.g., $3$ divides the length of $(123)$. Using the language in Theorem 1.2, totally ramified primes satisfy the condition that $e_K(p) \equiv 0 \text{ mod } 3$, in other words totally ramified primes are just ramified primes of type $3$, hence

$$\text{rk}_3 \text{Cl}_K \geq \# \{ p \text{ is totally ramified} \} - 4.$$

This example explains the notion “non-random primes” from the view of Theorem 1.2.

Given a transitive permutation group $1 \leq G \leq S_n$, and a non-random prime $q$ for $G$, we first make the following conjecture on counting fields based on the Malle-Bhargava Heuristics (see [Mal04, Bha07], [ELPS16, p. 291-339] for example) for counting fields with fixed number of ramified primes. For an extension $K/k$ of number fields, we denote its Galois closure by $\hat{K}$.

Definition 1.5. Let $1 \leq G \leq S_n$ be a transitive permutation group, and let $k$ be a number field. Let $\mathcal{S}$ be the set of all number fields $(K/k, \psi)$ such that its Galois closure $(\hat{K}/k, \psi)$ is a $G$-extension (see Definition 2.1), and that $K = \hat{K}^{G_1}$ where $G_1$ is the stabilizer of 1. Suppose
that $\Omega$ is a (nonempty) subset of $G$ that is closed under invertible powering, i.e., if $g^a = h$, $h^b = g$, then $g \in \Omega$ if and only if $h \in \Omega$. Define for the set $\Omega$, and for all $r = 0, 1, 2, \ldots$,

$$1_{(\Omega, r)}(K) := \begin{cases} 1 & \text{if there are exactly } r \text{ primes } p \mid |G| \\ 0 & \text{s.t. } I(p) \cap \Omega \neq \emptyset; \text{ otherwise.} \end{cases}$$

where $I(p)$ here means the inertia subgroup of $p$.

**Conjecture 1.6.** Keep $G, k, S$ as above. Suppose that $id \notin \Omega$ is a (nonempty) subset of $G$ that is closed under invertible powering.

1. For all $r = 0, 1, 2, \ldots$, there exists some $r'$, such that

$$\sum_{K \in S, P(K) < x} 1_{(\Omega, r)}(K) = o \left( \sum_{K \in S, P(K) < x} 1_{(\Omega, r')}(K) \right),$$

In this case we say that the conjecture 1 holds for the pair $(S, \Omega)$.

2. For all $r = 0, 1, 2, \ldots$,

$$\sum_{K \in S, P(K) < x} 1_{(\Omega, r)}(K) = o \left( \sum_{K \in S, P(K) < x} 1 \right)$$

In this case we say that the conjecture 2 holds for the pair $(S, \Omega)$.

Using the conjecture on counting fields, we can present our result on the distribution of class groups. Recall that for a number field $K$, if $L/K$ is an algebraic extension, then let $\text{Cl}(L/K) := \ker(\text{Nm}_{L/K}(\text{Cl}_L \to \text{Cl}_K))$ be the relative class group. We see that $\text{Cl}(K/\mathbb{Q}) = \text{Cl}_K$ is just the usual class group.

**Theorem 1.7.** Let $1 \leq G \leq S_n$ be a transitive permutation group, and let $k$ be a number field. Let $S$ be the set of all number fields $(K/k, \psi)$ such that its Galois closure $(\hat{K}/k, \psi)$ is a $G$-extension, and that $K = \hat{K}^{G_1}$. Let $H \subseteq G$ be a subgroup such that $\hat{K}^H \subseteq K$ for $K \in S$. If $q$ is a non-random prime for $G$ such that $q[[K : \hat{K}^H]]$, and Conjecture 1.6(2) holds for $(S, \Omega)$, where $\Omega := \Omega(G, q^\infty)$, then

$$\mathbb{P}(\text{rk}_q \text{Cl}(K/\hat{K}^H) \leq r) = 0 \quad \text{and} \quad \mathbb{E}(\text{|Hom(}\text{Cl}(K/\hat{K}^H), C_q|)) = +\infty,$$

where $K$ runs over fields in $S$ for the product of ramified primes in $K/\mathbb{Q}$, and $\text{Cl}(K/\hat{K}^H)$ denotes the relative class group.

Roughly speaking, here the set-up is just an extension $\hat{K}^H/K$ of number fields, and the distribution of the relative class groups is very different from the Cohen-Lenstra-Martinet Heuristics if $q$ is non-random. These two phenomena described in the above theorem, zero-probability and infinite moment, justify the notion “non-random prime” from statistics. With the help of Class Field Theory and Tauberian Theorems, we can prove the Conjecture 1.6(1) for abelian extensions. To be precise, we have the following.
Theorem 1.8. Let $\Gamma$ be a finite abelian group with a subgroup $\Lambda$, and let $S$ be the set of all abelian $\Gamma$-extensions $K/\mathbb{Q}$. If $q$ is a prime number such that $q|\Gamma/\Lambda|$, then the Conjecture 1 holds for $(S, \Omega)$, where $\Omega := \Omega(\Gamma, q^\infty)$. In addition, we have

$$\mathbb{P}(\text{rk}_q \text{Cl}(K/K^\Lambda) \leq r) = 0 \quad \text{and} \quad \mathbb{E}(|\text{Hom}(\text{Cl}(K/K^\Lambda), C_q)|) = +\infty,$$

where $K$ runs over fields in $S$ for the product of ramified primes in $K/\mathbb{Q}$.

For non-abelian extensions, the first obstacle is counting fields. We present here an example, $D_4$-fields. Let $D_4$ be the dihedral group of order 8, and we are going to consider quartic number fields $L/\mathbb{Q}$ whose Galois closure $M/\mathbb{Q}$ are $D_4$-fields. According to the work of S.A.Altug, A.Shankar, I.Varma, K.H.Wilson [ASVW21], the result of counting such fields by the Artin conductor of 2-dimensional irreducible representation of $D_4$ is proven. So, the main result in this case can be summarized as follows.

Theorem 1.9. Let $S$ be the set of quartic number fields $L/\mathbb{Q}$ whose Galois closure are $D_4$-extensions $M/\mathbb{Q}$. We have

$$\mathbb{E}_C(|\text{Hom}(\text{Cl}_L, C_2)|) = +\infty,$$

where the subscript $C$ means that the fields $L \in S$ are ordered by the Artin conductor of 2-dimensional irreducible representation of $D_4$.

From the above genus theory for quadratic number fields, it is not a surprise to obtain some results saying that the $C_2$-moment of the class groups of quartic number fields $L$ in which there is a quadratic subfield $K$ is infinite or large in other senses. So it will be better if we can get more details of $\text{Cl}_L[2^\infty]$. One of the related concepts is the so-called capitulation kernel, which is $\ker(\text{Cl}_K \to \text{Cl}_L)$. See also [CM90, Théorème 7.6] for the discussions on relative class groups. In § 7, we will order quartic fields by product of ramified primes and try to discuss the relation between ramified primes and ideal classes in $\text{Cl}_L[2^\infty]$ under additional hypothesis. We here give the following result. Write $D_4 = \langle \tau, \sigma | \tau^2 = \sigma^4, \tau \sigma \tau^{-1} = \sigma^3 \rangle$.

Theorem 1.10. Let $L/\mathbb{Q}$ be a quartic number field with Galois $D_4$-closure $M/\mathbb{Q}$, let $K$ be the quadratic subfield of $L$, and let $I(p)$ be the inertia subgroup of $p$.

(i) Let $\Omega_1$ be the set $\{\sigma, \sigma^3, \sigma \tau, \sigma^3 \tau\}$. Then we have

$$\text{rk}_2 \text{i}_* \text{Cl}_K \geq |\{p \neq 2 : I(p) \cap \Omega_1 \neq \emptyset\}| - 6.$$

(ii) Let $\Omega_2 := \Omega(D_4, 2^\infty) = \{\sigma, \sigma^3, \sigma^2, \sigma \tau, \sigma^3 \tau\}$. Then we have

$$\text{rk}_2 \text{Cl}(L/K) \geq |\{p \neq 2 : I(p) \cap \Omega_2 \neq \emptyset\}| - 6.$$

To summarize this section, the notion “good prime” in [CM90, Définition 6.1] (see also [WW21, §7]) gives a criterion for us to apply the Cohen-Martinet-Lenstra Heuristics so that we can predict the statistical behaviour of the class groups. In this paper, the notion “non-random prime” predicts the cases when we expect nontrivial subgroup of $\text{Cl}_K$ from ramified primes and qualitatively different statistical results from “good prime” cases. However, Example 4.2 shows that there are primes $p$ that are neither good in the sense of the heuristics nor non-random in this paper. So, such situations may suggest more ideas on the study of class groups.
2. Basic notations

In this section we introduce some of the notations that will be used in the paper. We use some standard notations coming from analytic number theory. For example, write a complex number as $s = \sigma + it$. The notation $q^l || n$ means that $q^l | n$ but $q^{l+1} \nmid n$. Denote the Euler function by $\phi(n)$. Let $\omega(n)$ counts the number of distinct prime divisors of $n$ and so on.

We also follow the notations of inequalities with unspecified constants from Iwaniec and Kowalski [IKS04, Introduction, p.7]. Let’s just write down the ones that are important for us. Let $X$ be some space, and let $f, g$ be two functions. Then $f(x) \ll g(x)$ for $x \in X$ means that $|f(x)| \leq C g(x)$ for some constant $C \geq 0$. Any value of $C$ for which this holds is called an implied constant. We use $f(x) \asymp g(x)$ for $x \in X$ if $f(x) \ll g(x)$ and $g(x) \ll f(x)$ both hold with possibly different implied constants. We say that $f = o(g)$ as $x \to x_0$ if for any $\epsilon > 0$ there exists some (unspecified) neighbourhood $U_\epsilon$ of $x_0$ such that $|f(x)| \leq \epsilon g(x)$ for $x \in U_\epsilon$.

Since there are multiple ways to describe field extensions, we give the following two definitions to make the term like “the set of all non-Galois cubic number fields” precise.

**Definition 2.1.** For a field $k$, by a $\Gamma$-extension of $k$, we mean an isomorphism class of pairs $(K, \psi)$, where $K$ is a Galois extension of $k$, and $\psi : \text{Gal}(K/k) \cong \Gamma$ is an isomorphism. An isomorphism of pairs $(\alpha, m_\alpha) : (K, \psi) \to (K', \psi')$ is an isomorphism $\alpha : K \to K'$ such that the map $m_\alpha : \text{Gal}(K/k) \to \text{Gal}(K'/k)$ sending $\sigma$ to $\alpha \circ \sigma \circ \alpha^{-1}$ satisfies $\psi' \circ m_\alpha = \psi$. We sometimes leave the $\psi$ implicit, but this is always what we mean by a $\Gamma$-extension. We also call $\Gamma$-extensions of $\mathbb{Q}$ $\Gamma$-fields.

**Definition 2.2.** Let $G \subseteq S_n$ whose action on $\{1, 2, \ldots, n\}$ is transitive. Let $k$ be a number field. Let $S(G; k)$ be the set of pairs $(K, \psi)$ such that the Galois closure $(\hat{K}, \hat{\psi})$ is a $G$-extension of $k$ and that $K = \hat{K}^{G_1}$, where $G_1$ is the stabilizer of 1. In other words $\psi$ defines the Galois action of $G$ on $K$ over $k$ and $[K : k] = n$. If the base field $k = \mathbb{Q}$, then we just omit it and write $S(G) := S(G, \mathbb{Q})$.

Then, we give the notation of counting number fields.

**Definition 2.3.** Let $S = S(G; k)$ where $G$ is a transitive permutation group, and let $d : S \to \mathbb{R}^+$ be an invariant of number fields (e.g. discriminant or product of ramified primes). Define

$$N(S, d; x) := \sum_{K \in S, d(K) < x} 1.$$ 

If $f$ is a function defined over $S$, then we define

$$N(S, d; f; x) := \sum_{K \in S, d(K) < x} f(K).$$

In particular, if $f = 1_{(\Omega, r)}$ (see Definition 1.5), then just write

$$N(S, d; (\Omega, r); x) := N(S, d; 1_{(\Omega, r)}; x).$$

Recall that we have defined some indicators on abelian groups. To be precise, if $A, B$ are finite abelian groups, then

$$1_A(B) = \begin{cases} 1 & \text{if } B \cong A; \\ 0 & \text{otherwise.} \end{cases}$$
And if \( q \) is a rational prime, \( r \) is a natural number, then
\[
1_{\text{rk}_q \leq r}(A) = \begin{cases} 
1 & \text{if } \text{rk}_q A \leq r \\
0 & \text{otherwise.} 
\end{cases}
\]

They are both viewed as functions defined on \( \mathcal{S} \), because every \( K \in \mathcal{S} \) admits an ideal class group \( \text{Cl}_K \) (or \( \text{Cl}_K[q^\infty] \)). Using the notations introduced in this section, if \( \mathcal{S} = \mathcal{S}(G) \) is a set of number fields with \( G \) a transitive permutation group, and \( P \) means the product of ramified primes, then we can define for a rational prime \( q \) and an integer \( r \geq 0 \) the following notation, which could be called the probability of the \( p \)-rank of \( \text{Cl}_K \) less than or equal to \( r \) where \( K \) runs over all fields in \( \mathcal{S} \) for the product of ramified primes:
\[
\mathbb{P}(\text{rk}_q \text{Cl}_K \leq r) := \lim_{x \to \infty} \frac{N(\mathcal{S}, P; 1_{\text{rk}_q \leq r}; x)}{N(\mathcal{S}, P; x)},
\]
provided that the limit exists.

3. Non-randomness

The term non-randomness here refers to the case where we can obtain some nontrivial ideal classes in \( \text{Cl}_K \) associated to the ramified primes. There are two main reasons why we are interested in the relation between ideal classes and ramified primes. The first one is that we have the famous example of genus theory for quadratic number fields as in § 1, which implies that the 2-rank is determined by the number of ramified primes (up to 1). We of course want to see if similar phenomena also happen when we look at higher degree fields. The second reason is that product of ramified primes is the main ordering we currently use for number fields. Even in the case people use some other orderings (discriminant, conductor etc.), they are still related to ramified primes. It then makes the results, providing information of the relation between nontrivial ideal classes and ramified primes, more valuable. First recall that we’ve defined the notation ramified prime of type \( q \) for a fixed prime \( q \) in § 1. Here let’s generalize it a little bit as follows.

**Definition 3.1.** Let \( K/\mathbb{Q} \) be a number field, and let \( p, q \) be two rational primes. If \( p \mathcal{O}_K = p_1^{e_1} \cdots p_m^{e_m} \), then let
\[
e_{K}(p) := \gcd(e_1, \ldots, e_m).
\]
If \( e_K(p) \equiv 0 \mod q^l \) for some \( l \geq 1 \), then we call \( p \) a ramified prime of type \( q^l \).

Because of the fundamental identity
\[
[K : \mathbb{Q}] = \sum_{i=1}^{m} e_i f_i
\]
where \( e_i \) is the ramification index and \( f_i \) is the inertia degree for a fixed prime, we only need to discuss ramified primes of type \( q \) for \( q | [K : \mathbb{Q}] \).

**Example 3.2.** If \( K/\mathbb{Q} \) is Galois itself, i.e., \( \text{Gal}(K/\mathbb{Q}) \cong \Gamma \), then for any rational prime \( p \), one has \( p \mathcal{O}_K = \prod_i p_i^{e_i} \). So it is just a question whether \( q \) divides the ramification index \( e \) or not. An example would be a quadratic extension \( K/\mathbb{Q} \) with \( q = 2 \). In this case, \( p \) is a ramified prime of type \( q \) if and only if \( p \) is ramified in \( K/\mathbb{Q} \).

For non-Galois extensions, things become a little more complicated. Let \( K/\mathbb{Q} \) be a non-Galois cubic extension with \( q = 3 \). Then \( p \) is a ramified prime of type 3 if and only if \( p \) is
totally ramified in $K/Q$. Note that there are partially ramified primes for non-Galois cubic extensions, i.e., $pO_K = p^2p_2$, which are not ramified prime of type 3.

3.1. Genus theory. The goal of this section is to prove Theorem 1.1, which requires a brief introduction on genus theory for number fields. The basic question of genus theory is to find out the maximal unramified abelian extension of a number field $K$ obtained by composing with an absolute abelian number field $k$. To be precise, we have the following definition.

**Definition 3.3.** let $K/Q$ be a number field of degree $n$, and let $k/Q$ be the maximal abelian extension such that $Kk/K$ is an unramified extension. We call such a field the genus field $K_k$ over $K$, and call the Galois group $Gal(Kk/K)$ the genus group $G$.

Since $Kk/K$ is an unramified abelian extension, it is a subextension of the Hilbert extension of $K$ whose Galois group is isomorphic to the class group $Cl_K$ of $K$, so the genus group is a quotient group of the class group $Cl_K$. According to Ishida [Ish76, p.33-39], we make a summary of the results on the genus group. Fix a rational prime $q$ such that $q^t || n$ with $t \geq 1$. Let $A \cong Gal(k/Q)$ be the Galois group of the abelian extension $k/Q$, and let $K_0$ be the intersection of $k$ and $K$ which is also the maximal abelian subextension of $K/Q$. See the diagram for summary below.

![Diagram](image)

**Definition 3.4.** Let $K/Q$ be a number field, and let $p$ be a prime number. Let $k(p)$ be the unique subfield of $Q(\zeta_p)$ with degree $gcd(p - 1, e_K(p))$ where $\zeta_p$ is a primitive $p$th root of unity.

Note that $k(p)$ is nontrivial if and only if $gcd(p - 1, e_K(p))$ is nontrivial. Moreover the Galois group of $k(p)/Q$ is cyclic of order $gcd(p - 1, e_K(p))$. The first result is to describe the extension $k/Q$ given the ramified primes of $K/Q$.

**Theorem 3.5.** [Ish76, Chapter IV, Theorem 3] Let $K$ be a number field, and let $k$ be the abelian field such that $Kk$ is the genus field. Let $k_1/Q$ be the composite of $k(p)$ where $p$ runs through all rational prime numbers such that $p || e_K(p)$, and let $k_2/Q$ be the intersection of all inertia subfields of $k$ at $p$ where $p$ runs through all rational prime numbers such that $p || e_K(p)$. Then $k = k_1k_2$ and $k_1 \cap k_2 = Q$. In particular, $A$ admits a subgroup

$$Gal(k/k_2) \cong Gal(k_1/Q) \cong \prod_{p \mid e_K(p)} \mathbb{Z}/gcd(p - 1, e_K(p)).$$

Now that we have a result on the group $A$, the description of $\mathcal{G}$ follows from $A/ Gal(K_0/Q) \cong \mathcal{G}$. In the sense of estimation for $rk_q q^*\mathcal{G}$, we give the following statement.
Theorem 3.6. Let $K/Q$ be a number field with maximal abelian subextension $K_0/Q$. Fix a \emph{rational} prime $q$ dividing $n := [K : Q]$ and some integer $l \geq 1$, the $q$-rank of the group $q^{l-1}G$ admits the following inequality

$$\text{rk}_q q^{l-1}G \geq \# \{ p \mid \text{p is a ramified prime of type } q^l \text{ and } p \equiv 1 \text{ mod } q \} - \text{rk}_q \text{Gal}(K_0/Q).$$

Remark. Theorem 1.1 just follows from this result. Moreover we can see that if $G[q^\infty]$ admits higher torsion part, then $\text{Cl}_K[q^\infty]$ must also have higher torsion part. Also, if there is no prime $p \mid e_K(p)$, then $k_2 = \mathbb{Q}$ and $A \cong \prod_{p \mid e_K(p)} \mathbb{Z}/\text{gcd}(p - 1, e_K(p))$, hence $G \cong \left( \prod_{p \mid e_K(p)} \mathbb{Z}/\text{gcd}(p - 1, e_K(p)) \right)/\text{Gal}(K_0/Q)$.

Example 3.7. Let $K/Q$ be a non-Galois cubic field, and $q$ equal 3. Then the requirement $\text{gcd}(p - 1, e_K(p)) \equiv 0 \text{ mod } 3$ is equivalent to $p$ totally ramified in $K/Q$ and $p \equiv 1 \text{ mod } 3$. In other words, we have

$$\text{rk}_3 \text{Cl}_K \geq \# \{ \text{p is a totally ramified prime and } p \equiv 1 \text{ mod } 3 \}.$$ 

This clearly generalizes genus theory for the quadratic case. See also [Ish76, Chapter 5] for more discussions on the case of odd prime degree.

3.2. The class rank estimate on the invariant part of the class group. In the paper of Roquette and Zassenhaus [RZ69], there is another result on the estimate of the $q$-rank of the class group with respect to ramified primes whose idea is totally different from genus theory. As seen in previous part, what genus theory really cares about is the genus group, which is the quotient of the class group, while the invariant part of the class group we will talk about is a subgroup of the class group.

We will explain briefly the work of Roquette and Zassenhaus by showing the proof of Theorem 3.11. More importantly, we want to show the construction, which is more useful in this paper. Let’s first introduce some notations to present their precise statement. If $K$ is a number field, let $\text{rk}_K$ be its rank of global units. Let $v_q(n)$ denote the normalized exponential valuation associated to the rational prime $q$, i.e., we compute the exponent of $q$ showing up in the factorization of $n$.

Definition 3.8. Let $K/Q$ be a number field whose Galois closure is a $\Gamma$-extension $L/Q$. By viewing the group of fractional ideals $\mathcal{I}_K$ of $K$ as a subgroup of $\mathcal{I}_L$, we define the \emph{invariant part of the class group}, denoted by $C_K^\Gamma$, of $K$ under the action of $\Gamma$ as the image of $\mathcal{I}_L^\Gamma \cap \mathcal{I}_K$ in $\text{Cl}_K$, i.e.,

$$C_K^\Gamma := \text{im}(\mathcal{I}_L^\Gamma \cap \mathcal{I}_K \rightarrow \text{Cl}_K).$$

To give a precise statement on the estimation for the invariant part $C_K^\Gamma$ of the class group, we in addition need the following definition of $q$-radical subfields of $K$.

Definition 3.9 (q-radical subfields). Let $K_q$ be the subfield of $K$ generated by all elements $\xi \in K$ which are $q$-radicals over $\mathbb{Q}$, i.e., $\xi^q \in \mathbb{Q}$. Let $n_q = [K_q : \mathbb{Q}]$ be its degree.

It is clear that $n_q \mid n$, so one has

$$v_q(n_q) \leq v_q(n).$$

But if one wants to ask when $v_q(n_q)$ is strictly less than $v_q(n)$, then the following notation actually gives a condition for it.
Definition 3.10. Define the notation $\delta^{(q)}_K$ to be 1 or 0 according to whether or not the following conditions are simultaneously satisfied:

(i) $q > 2$;
(ii) $K$ contains a primitive $q$-th root of unity $\zeta$;
(iii) There exists some $\eta$ in $K^*$ such that $\zeta - 1 \eta q \in \mathbb{Q}$.

Remark. If all of the above three conditions hold, then one can show that $[K_q(n) : K_q] = q$ so that $n_q \cdot q | n$ and hence $v_q(n_q) + 1 \leq v_q(n)$. See [RZ69, §6].

Using the notations introduced above, we can first state an estimation for $C_K$ due to Roquette and Zassenhaus. Though in the paper [RZ69], the main goal is to prove the result for the $q$-rank of the class group. We here instead describe the structure of the subgroup $C^\Gamma_K[q^\infty] \subseteq \text{Cl}_K[q^\infty]$, which is also due to Roquette and Zassenhaus.

Theorem 3.11. Let $K/Q$ be a (not necessarily Galois) extension of degree $n$ whose Galois group is $\Gamma$, and let $q$ be a given prime number, and let $l \geq 1$ be an integer, then

$$
\#\{p \text{ is a ramified prime of type } q^l\} - \left(\text{rk}_K + v_q(n_q) + \delta^{(q)}_K\right)
\leq \text{rk}_q q^l C^\Gamma_K \leq \#\{p \text{ is a ramified prime of type } q^l\}.
$$

We prove the theorem by two lemmas. The first lemma gives a detailed description of the elements in $C^\Gamma_K$ in the sense of fractional ideals.

Lemma 3.12. [RZ69, Equation (8)] Let $K/Q$ be a number field whose Galois closure a $\Gamma$-extension $L/Q$. For each prime $p$, define $a(p)$ to be the ideal such that $p\mathcal{O}_K = a(p)^{e_K(p)}$. Then $\mathcal{I}_K \cap \mathcal{I}^\Gamma_L$ is a free abelian group generated by $\{a(p)\}$. Let $\mathcal{P}_k$ be the group of principal ideals of a number field $k$. Then the group $\tilde{C}^\Gamma_K := \mathcal{I}_K \cap \mathcal{I}^\Gamma_L / \mathcal{P}_Q$ is given by

$$
\tilde{C}^\Gamma_K \cong \prod_p \mathbb{Z} / e_K(p).
$$

The second lemma is to estimate the difference between principal ideals of $K$ and $Q$.

Lemma 3.13. [RZ69, Equation(11)] Let $K/Q$ be a number field whose Galois closure a $\Gamma$-extension $L/Q$, and let $\mathcal{P}_k$ be the group of principal ideals of a number field $k$. Then

$$
\text{rk}_q \mathcal{P}^\Gamma_K / \mathcal{P}_Q \leq \text{rk}_K + v_q(n_q) + \delta^{(q)}_K,
$$

where $\mathcal{P}^\Gamma_K := \mathcal{P}_K \cap \mathcal{I}^\Gamma_L$.

Now let’s prove the theorem.

Proof of Theorem 3.11. It is clear that Lemma 3.12 gives the upper bound of $\text{rk}_q q^l C^\Gamma_K$, because $\mathcal{P}_Q \subseteq \mathcal{P}_K$. According to the short exact sequence

$$
0 \to \mathcal{P}^\Gamma_K / \mathcal{P}_Q \to \mathcal{I}^\Gamma_K / \mathcal{P}_Q \to C^\Gamma_K \to 0,
$$

the inequality (3.1) tells us the lower bound directly. \qed

We can apply the theorem to the class group $\text{Cl}_K$, for $C^\Gamma_K \subseteq \text{Cl}_K$. 

Corollary 3.14. Let $K$ be a number field of degree $n$ over $\mathbb{Q}$, and let $q$ be a prime number, and let $l \geq 1$ be an integer, then
\[ \text{rk}_q q^{l-1} \text{Cl}_K \geq \# \{ p \text{ is a ramified prime of type } q^l \} - \left( \text{rk}_K + v_q(n_q) + \delta^{(q)}_K \right). \]

Remark. One can show that the number $\text{rk}_K + v_q(n_q) + \delta^{(q)}_K$ is always smaller than $2(n-1)$, i.e., the above inequality has a weaker but shorter expression
\[ \text{rk}_q q^{l-1} \text{Cl}_K \geq \# \{ \text{ramified primes of type } q^l \} - 2(n-1), \]
which proves the statement of Theorem 1.2.

One of the advantages of this theory, as mentioned above, is that $C^*_K$ is a subgroup of $\text{Cl}_K$. So, we can even try to discuss the relative class group here, i.e., for a subfield $K' \subseteq K$, we want to give a description for $C^*_K \cap \text{Cl}(K/K')$.

Theorem 3.15. Let $K/\mathbb{Q}$ be a number field of degree $n$ whose Galois group is $\Gamma$. If $K' \subseteq K$ such that $q^l \mid [K : K']$ where $q$ is a rational prime, and $l \geq 1$, then
\[ \text{rk}_q q^{l-1} \text{Cl}(K/K') \geq \# \{ p \text{ is a ramified prime of type } q^l \} - 2(n-1). \]

Proof. For each prime $p$, recall that $a(p)$ is the ideal of $K$ such that $p\mathcal{O}_K = a(p)^{e_K(p)}$. We’ve shown in Lemma 3.12 that $a(p)$ is fixed by the action of $\Gamma$, viewed as an element of $\mathcal{J}_L$. So, we have the following computation
\[ \text{Nm}_{K/K'}(a(p)) = a(p)^{[K:K']}, \]
where $a(p)^{[K:K']}$ is treated as an ideal of $K'$. If $q^l \mid e_K(p)$, and $b(p) := (a(p))^{e_K(p)/q^l}$, then $\text{Nm}_{K/K'}(b(p))$ becomes a power of $p\mathcal{O}_{K'}$, hence a principal ideal of $K'$. So, $b(p)$ represents an ideal class of $\text{Cl}(K/K')$. By Lemma 3.12, the group $C^*_K$ is generated by ideals of the form $a(p)$, we therefore know that the subgroup $B^*_K := \langle b(p) \mid p \text{ is a ramified prime of type } q^l \rangle/\mathcal{P}_K \subseteq C^*_K[q]$ is contained in $\text{Cl}(K/K')$. In particular, $\langle b(p) \rangle/\mathcal{P}_Q \cong \prod_{i=1}^{s} \mathbb{Z}/q^i$, where $s = \# \{ p \text{ is a ramified prime of type } q^i \}$. Then by Lemma 3.13, we know that $\text{rk}_q \mathcal{P}_K/\mathcal{P}_Q$ is bounded above by a constant $\text{rk}_K + v_q(n_q) + \delta^{(q)}_K \leq 2(n-1)$, hence the result. □

4. Non-random primes

In Theorem 1.1 and Theorem 1.2, we see that the $q$-rank $\text{rk}_q \text{Cl}_K$ of class group is related to ramified primes of type $q$. If we fix the Galois group, a question we can ask is for which primes $q$ we can get ramified primes of type $q$? In a $G$-extension (see Definition 2.1), let’s first answer this question by the following lemma.

Lemma 4.1. Let $G$ be a finite transitive permutation group, and let $q$ be a fixed prime. If $(K/\mathbb{Q}, \psi)$ is an extension of number fields such that its Galois closure $(\hat{K}, \hat{\psi})$ is a $G$-field and that $K = \hat{K}^{G_1}$, then a prime $p \mid |G|$ is a ramified prime of type $q$ if and only if $I(p) \cap \Omega(G, q^\infty) \neq \emptyset$ where $I(p)$ is the inertia subgroup of $p$.

Proof. If $\mathfrak{P}$ is a prime of $\hat{K}$ lying above $p$, then $\mathfrak{P} \cap K$ is of course a prime of $K$ above $p$. Since $G$ acts on all primes of $\hat{K}$ above $p$, we know that $\sigma \mathfrak{P} \cap K$ will also run over all primes of $K$ above $p$ when $\sigma$ runs over all elements in $G$. But if $\sigma \in G_1$, then $\sigma \mathfrak{P} \cap K = \mathfrak{P} \cap K$. So, we obtain a $G$-set $G_1 \backslash G$, i.e., $\sigma_1 \sim \sigma_2$ if
\[ \sigma_1 \mathfrak{P} \cap K = \sigma_2 \mathfrak{P} \cap K, \]
which means that \( G_1 \sigma_1 = G_1 \sigma_2 \). This can also be induced by the permutation action of \( G \) on the embeddings \( K \to \mathbb{C} \).

One can check that if \( G(\mathfrak{P}) \) is the decomposition group of \( \mathfrak{P} \), then the orbits of \( G(\mathfrak{P}) \) on \( G/G_1 \) correspond to the primes \( p \) of \( K \) above \( p \), and the size of orbits of \( I(\mathfrak{P}) \) corresponds to the ramification index respectively.

If \( p \) is a tamely ramified prime, then we can just write \( I(p) \) instead of \( I(\mathfrak{P}) \), because \( I(p) \) is cyclic, generated by a single element \( \sigma \). So, \( p \) is a ramified prime of type \( q \) if and only if the size of orbits of the inertia generator has common divisor \( q \), i.e., \( q | e(\sigma) \) using the notations in Definition 1.3, which means that \( I(p) \cap \Omega(G, q^\infty) \neq \emptyset \). \( \square \)

Recall that \( \mathcal{S}(G) \) is the set of \( G \)-fields (see Definition 2.2).

**Example 4.2** ("Counter-example"). We here show an example where the prime \( q \) divides the order of the extension \( K/\mathbb{Q} \) but it is *not* a non-random prime. Consider the permutation group \( G \) defined as the image of the following morphism \( A_4 \to S_6 \), \((12)(34) \mapsto (1)(2)(34)(56) \), and \((123) \mapsto (134)(265) \). In the sense of number theory, \( \mathcal{S}(G) \) contains number fields \( K/\mathbb{Q} \) of degree 6, whose Galois closure \( \hat{K}/\mathbb{Q} \) are \( A_4 \)-extensions, so that the action of \( \text{Gal}(\hat{K}/\mathbb{Q}) \) on \( K/\mathbb{Q} \) induces a map \( A_4 \to S_6 \). The stabilizer of 1 is the image of \( \{1,(12)(34)\} \) in this case. One can check that the group \( G \) has no element required in Definition 1.3 to turn the prime 2 into a non-random prime, despite the fact that 2 divides the order of \( [K : \mathbb{Q}] \). Note that according to Cohen and Martinet [CM90], the prime 2 is not "good", i.e., 2 is not good but not non-random. See also [WW21, §7] for details on the concept "good primes".

Our most important goal in this section is to prove Theorem 1.7, which can justify the notion non-random prime from the view of statistics. Recall that if \( f : \mathcal{S}(G) \to \mathbb{R} \) is a function, then

\[
N(\mathcal{S}(G), d; f; x) = \sum_{K \in \mathcal{S}, \frac{d(K)}{d} < x} f(K)
\]

is the notation for the sum of \( f \) over \( G \)-fields with respect to the invariant \( d \) (see Definition 2.3). We reach the theorem in several steps. Recall that we have the Conjectures 1.6 in § 1. Step zero is to prove the relation between these two conjectures, i.e., Conjecture 1.6(1) implies Conjecture 1.6(2).

**Lemma 4.3.** Let \( G \) be a transitive permutation group and \( k \) be a fixed number field, and let \( \mathcal{S} := \mathcal{S}(G, k) \). Let \( d \) be an invariant of number fields in \( \mathcal{S} \). Suppose that \( \Omega \) is a (nonempty) subset of \( G \) that is closed under invertible powering. If for all \( r = 0, 1, 2, \ldots \), there exists some \( r' \) such that

\[
N(\mathcal{S}, d; (\Omega, r); x) = o(N(\mathcal{S}, d; (\Omega, r'); x))
\]

then for all \( r = 0, 1, 2, \ldots \), we have

\[
N(\mathcal{S}, d; (\Omega, r); x) = o(N(\mathcal{S}, d; x)).
\]

**Proof.** Since all fields counted by \( N(\mathcal{S}, d; (\Omega, r); x) \) and \( N(\mathcal{S}, d; (\Omega, r'); x) \) are also counted by \( N(\mathcal{S}, d; x) \), we have \( N(\mathcal{S}, d; (\Omega, r'); x) \leq N(\mathcal{S}, d; x) \). The lemma then follows. \( \square \)

The first step is to show that Conjecture 1.6(2), the weaker one, implies zero-probability.
Theorem 4.4. Let $1 \leq G \leq S_n$ be a transitive permutation group, and let $S := S(G)$. Let $G_1 \subseteq H \subseteq G$, where $G_1$ is the stabilizer of 1 that fixes $K \in S$. Let $q$ be a non-random prime for $G$ such that $q^2 \mid [H : G_1]$, where $l \geq 1$, and let $\Omega := \bigcup_{j=1}^{\infty} \Omega(G, q^j)$. Let $d$ be an invariant of the number fields in $S$. If for all $r = 0, 1, 2, \ldots$, we have

$$N(S, d, (\Omega, r); x) = o(N(S, d; x)),$$

then for all $r = 0, 1, 2, \ldots$, we have

$$\mathbb{P}(\text{rk}_q q^{l-1} \text{Cl}(K/\hat{K}^H) \leq r) := \lim_{x \to \infty} \frac{N(S, d; \text{rk}_q \leq r(\text{Cl}(K/\hat{K}^H)); x)}{N(S, d; x)} = 0,$$

where $K$ runs over fields in $S$ for the invariant $d$, and $\hat{K}$ is the Galois closure of $K$.

Proof. First of all, recall that we have for all finite abelian group $A$ and for all $r = 0, 1, 2, \ldots$,

$$1_{\text{rk}_q \leq r}(A) = \begin{cases} 1 & \text{if } \text{rk}_q A \leq r \\ 0 & \text{otherwise}. \end{cases}$$

According to Theorem 3.15, if there are at least $r + 2n$ ramified primes $p$ for $K/\mathbb{Q}$ contained in the set $\{p \mid |G| : I(p) \cap \Omega \neq \emptyset\}$, then

$$\text{rk}_q q^{l-1} \text{Cl}(K/\hat{K}^H) > r.$$ 

If $N(S, d, (\Omega, r); x) = o(N(S, d; x))$ for all $r$, then this implies that

$$\mathbb{P}(\text{rk}_q q^{l-1} \text{Cl}(K/\hat{K}^H) \leq r) = \lim_{x \to \infty} \frac{N(S, d; \text{rk}_q \leq r(\text{Cl}(K/\hat{K}^H)); x)}{N(S, d; x)} \leq \lim_{x \to \infty} \frac{N(S, d; \sum_{i=0}^{r+2n} 1_{(\Omega, i)}; x)}{N(S, d; x)} = 0.$$

Then let’s prove that bounded probability, hence also zero-probability, implies infinite moment.

Theorem 4.5. Let $1 \leq G \leq S_n$ be a transitive permutation group, and let $S := S(G)$. Let $G_1 \subseteq H \subseteq G$, where $G_1$ is the stabilizer of 1 that fixes $K \in S$. Let $d$ be an invariant of the number fields in $S$. Let $q$ be a rational prime and $l \geq 1$ be an integer. If there exists some constant $0 \leq c < 1$ such that for all $r = 0, 1, 2, \ldots$, we have

$$\mathbb{P}(\text{rk}_q q^{l-1} \text{Cl}(K/\hat{K}^H) \leq r) \leq c,$$

then,

$$\mathbb{E}(\text{Hom}(q^{l-1} \text{Cl}(K/\hat{K}^H), C_q)) := \lim_{x \to \infty} \frac{N(S, d; \text{Hom}(q^{l-1} \text{Cl}(K/\hat{K}^H), C_q); x)}{N(S, d; x)} = +\infty,$$

where $K$ runs over fields in $S$ for the invariant $d$, and $\hat{K}$ is the Galois closure of $K$, and $C_q$ is the cyclic group of order $q$. 

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Proof. According to similar idea, \(1_{\text{rk}_q=r}(A)\) to be the indicator that tells us if \(\text{rk}_q A = r\). By definition of the \(C_q\)-moment and the probability of the \(q\)-rank, for all \(r \geq 0\), we have

\[
\mathbb{E}(|\text{Hom}(q^{l-1}\text{Cl}(K/\hat{K}^H), C_q)|) = \lim_{x \to \infty} \frac{N(S, d; |\text{Hom}(q^{l-1}\text{Cl}(K/\hat{K}^H), C_q); x)}{N(S, d; x)}
\]

\[
= \lim_{x \to \infty} \sum_{n=0}^{\infty} q^n \frac{N(S, d; 1_{\text{rk}_q=n}(q^{l-1}\text{Cl}(K/\hat{K}^H))); x)}{N(S, d; x)}
\]

\[
\geq q^r \lim_{x \to \infty} \left(1 - \frac{N(S, d; 1_{\text{rk}_q \leq r-1}(q^{l-1}\text{Cl}(K/\hat{K}^H))); x)}{N(S, d; x)}\right)
\]

\[
= q^r \cdot (1 - \mathbb{P}(\text{rk}_q q^{l-1}\text{Cl}(K/\hat{K}^H) < r)) \geq q^r(1 - c).
\]

For any number \(N > 0\), by taking a large enough \(r > 0\), we have

\[
\mathbb{E}(|\text{Hom}(q^{l-1}\text{Cl}(K/\hat{K}^H), C_q)|) > N,
\]

hence \(\mathbb{E}(|\text{Hom}(q^{l-1}\text{Cl}(K/\hat{K}^H), C_q)|) = +\infty\). \(\square\)

In short, we can just say that whenever the Conjecture 1.6 holds, then we have zero-probability and infinite moment, i.e., Theorem 1.7 is true. We will show in later sections that for abelian extensions, the Conjecture 1.6(1) holds (see Proposition 6.7). Here we discuss an example where the fields are not ordered by product of ramified primes.

**Example 4.6** (Ordering by discriminant). Let’s consider the set \(S := S(S_3, \{1, (23)\})\) of non-Galois cubic extensions. Let \(\Omega := \Omega(S_3, 3) = \{(123), (132)\}\). In this case we want to show that the analogous statement of Conjecture 1.6(1) is *false* for \((S; \Omega)\) when the fields ordered by *discriminant*, despite the fact that 3 is non-random for \(S_3\).

According to the work of Bhargava, Shankar, and Tsimerman [BST13, Theorem 8], there exists some positive constants \(c_1, c_2\) such that

\[
N(S, \text{Disc}; (\Omega, 0); x) \sim c_1 x \quad \text{and} \quad N(S, \text{Disc}; x) \sim c_2 x,
\]

i.e., counting nowhere totally ramified cubic fields ordered by discriminant admits a main term \(c_1 x\) and counting all non-Galois cubic fields gives a main term \(c_2 x\). This already contradicts the analogous statement of Conjecture 1.6(2) when ordering fields by discriminant, the weaker one. On the other hand, Proposition 6.11 shows that, under some hypothesis, Conjecture 1.6(2) holds for \((S; \Omega)\). This shows that the conjecture, and possibly the statistical behaviors of non-random primes are dependent on the ordering of number fields.

### 5. Dirichlet series and Tauberian Theorem

In this section, we discuss the analytic properties of some Dirichlet series, which could be treated as useful tools in later sections. Let’s first present a Tauberian Theorem that is used in the paper repeatedly. Recall that for the complex variable \(s \in \mathbb{C}\), we write \(s = \sigma + it\).

**Theorem 5.1** (Delange-Ikehara). [Nar14, Appendix II Theorem I] Assume that the coefficients of a Dirichlet series are real and non-negative, and that it converges in the half-plane...
σ > 1, defining a regular function \( f(s) \). Assume, moreover, that in the same half-plane one can write

\[
f(s) = \sum_{j=0}^{q} g_j(s) \log^{b_j} \left( \frac{1}{s-1} \right) (s-1)^{-\alpha_j} + g(s),
\]

where functions \( g, g_0, \ldots, g_q \) are regular in the closed half plane \( \sigma \geq 1 \), the \( b_j \)-s are non-negative rational integers, \( \alpha_0 \) is a positive real number, \( \alpha_1, \ldots, \alpha_q \) are complex numbers with \( \Re \alpha_j < \alpha_0 \), and \( g_0(1) \neq 0 \).

Then for the summatory function \( S(x) = \sum_{n<x} a_n \) we have, for \( x \) tending infinity,

\[
S(x) = \left( \frac{g_0(1)}{\Gamma(\alpha_0)} + o(1) \right) x (\log x)^{\alpha_0-1} (\log \log x)^{b_0}.
\]

If \( f \) satisfies the same assumptions, except that \( \alpha_0 = 0 \) and \( b_0 \geq 1 \), then

\[
S(x) = (b_0 g_0(1) + o(1)) x (\log \log x)^{b_0-1} \log x.
\]

Let’s introduce a notation, which is inspired by problems of counting fields.

**Definition 5.2.** For a pair \((m,n)\) of relatively prime numbers, let

\[\mathcal{P}(m,n) := \{ p \text{ is a prime natural number} | p \equiv n \mod m \}.\]

Moreover, let \( \mathcal{P} \) be the set of all rational primes. Then define

\[
\zeta(m,n; s) = \prod_{p \in \mathcal{P}(m,n)} (1 - p^{-s})^{-1}
\]

when \( \Re(s) > 1 \).

Let’s show that a suitable power of the function \( \zeta(m,n; s) \) admits a meromorphic extension to the closed half plane \( \sigma \geq 1 \). Recall that \( \phi \) is the Euler function, i.e.,

\[
\phi(n) = |(\mathbb{Z}/n)^*|.
\]

**Lemma 5.3.** Let \((m,n)\) be a pair of relatively prime numbers. The function \( \zeta(m,n; s)\phi(m) \) admits a meromorphic continuation to the half plane \( \sigma > 1/2 \) with a simple pole at \( s = 1 \). To be precise, \( \zeta(m,n; s) \) defines a regular function in the open half-plane \( \sigma > 1 \), and there exists some regular function \( f_0(s) \) in the closed half plane \( \sigma \geq 1 \) such that

\[
\zeta(m,n; s)\phi(m) = f_0(s) \frac{1}{s-1}.
\]

In addition, \( f_0(s) \) and \( L(\chi, s) \) has no zero along the line \( \sigma = 1 \) for all Dirichlet characters \( \chi \).

**Proof.** Let \( \langle \mathcal{P}(m,n) \rangle \) denote the semi-subgroup of \( \langle \mathbb{Z}_{>0}, \cdot \rangle \) generated by \( \mathcal{P}(m,n) \). Since

\[
\sum_{\substack{n \leq x \\text{prime} \in \langle \mathcal{P}(m,n) \rangle}} \frac{1}{|n^s|} \leq \sum_{n \leq x} \frac{1}{|n^s|},
\]
we know that $\zeta(m, n; s)$ converges absolutely and uniformly in $\sigma > 1 + \delta$ for any $\delta > 0$. Then for any $\sigma > 1$, we have

$$\log \zeta(m, n; s) = \sum_{p \in \mathcal{P}(m, n)} p^{-s} + \frac{1}{2} \sum_{p \in \mathcal{P}(m, n)} p^{-2s} + \cdots.$$  

(5.1)

Similarly for any Dirichlet character $\chi \mod m$, we have

$$\log L(s, \chi) = \sum_{p \in \mathcal{P}} \frac{\chi(p)}{p^s} + \frac{1}{2} \sum_{p \in \mathcal{P}} \frac{\chi(p^2)}{p^{2s}} + \cdots.$$  

Therefore

$$\log \zeta(m, n; s) = \frac{1}{\phi(m)} \sum_{\chi} \chi(n) \log L(s, \chi) + g(s)$$  

for all $\sigma > 1$, where $\chi$ runs over all Dirichlet characters modulo $m$, and $g(s)$ is given by

$$g(s) = \sum_{n=2}^{\infty} \sum_{p \in \mathcal{P}(m, n)} \left( p^{-ns} - \frac{1}{\phi(m)} \sum_{\chi} \chi(n) \chi(p^n) p^{-ns} \right) = \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p^n \not\equiv n \mod m} p^{-ns}$$

Note that $g(s)$ is absolutely convergent in $\sigma > 1/2$. By taking the exponent, we have

$$\zeta(m, n; s)^{\phi(m)} = \prod_{\chi} L(s, \chi)^{\chi(n)} : h(s)$$

where $h(s)$ is a non-vanishing function in $\sigma > 1/2$. Therefore the pole behaviour of $\zeta(m, n; s)$ is the same as $\prod_{\chi} L(s, \chi)^{\chi(n)}$, hence the same as $L(s, \chi_0)$ where $\chi_0$ is the principal Dirichlet character modulo $m$ (see for example [MV06, 4.8,4.9]), which concludes the proof for the formula.

According to [IKS04, Theorem 5.26], the Dirichlet $L$-functions have no zero along the line $\sigma = 1$. 

It is a classical result saying that

$$\sum_{p \in \mathcal{P}(m, n)} p^{-s} = \frac{1}{\phi(m)} \log \left( \frac{1}{s-1} \right) + O(1),$$

where $\gcd(m, n) = 1$ and $\sigma > 1$, see [IKS04, Equation (2.25)] for example. We’ll turn it into a statement of analytic continuation so that it satisfies the condition of the Tauberian Theorem 5.1.

**Lemma 5.4.** For a given pair of coprime numbers $(m, n)$, the series $\sum_{p \in \mathcal{P}(m, n)} p^{-s}$ defines a regular function in the open half-plane $\Re(s) > 1$, and there is some regular function $g(s)$ in the closed half plane $\sigma \geq 1$, such that

$$\sum_{p \in \mathcal{P}(m, n)} p^{-s} = \frac{1}{\phi(m)} \log \left( \frac{1}{s-1} \right) + g(s).$$
Proof. Compared to Riemann zeta function, we know that the series \( \sum_{p \in \mathcal{P}(m,n)} p^{-s} \) converges in the open half-plane \( \sigma > 1 \), hence defining a regular function. Let’s consider the formula coming from (5.1):

\[
(5.3) \quad \sum_{p \in \mathcal{P}(m,n)} p^{-s} = \log \zeta(m,n; s) - \frac{1}{2} \sum_{p \in \mathcal{P}(m,n)} p^{-2s} - \cdots .
\]

Since Lemma 5.3, or more precisely [IKS04, Theorem 5.26], says that Dirichlet L-functions have no zero along the line \( \sigma = 1 \), we know that the formula (5.2) says that \( \log \zeta(m,n; s) \) admits an analytic continuation to the closed half-plane \( \sigma \geq 1 \). The other terms on the right-hand side of (5.3) defines a regular function in the closed half-plane \( \sigma \geq 1 \). So, there exists some regular function \( g(s) \) in the closed half-plane \( \sigma \geq 1 \) such that

\[
\sum_{p \in \mathcal{P}(m,n)} p^{-s} = \frac{1}{\phi(m)} \log \left( \frac{1}{s-1} \right) + g(s).
\]

\( \square \)

Then let’s show the main result of this section.

**Proposition 5.5.** Let \( m \) be a fixed natural number, and let \( a : \mathbb{N} \to \mathbb{R}_{\geq 0} \) be a multiplicative function such that \( a(p) = a(q) \) for all rational primes \( p \equiv q \mod m \), and that \( a(p) > 0 \) for at least one prime \( p \). Then for any natural number \( r = 1, 2, \ldots \), the following Dirichlet series

\[
\sum_n a_{n,r} n^{-s} := \sum_{n \text{ squarefree}} a(n) n^{-s}
\]

defines a regular function in the open half-plane \( \sigma > 1 \), and there exists regular functions \( f_0, \ldots, f_r \) in the closed half-plane \( \sigma \geq 1 \) such that

\[
\sum_n a_{n,r} n^{-s} = \sum_{i=0}^r f_i(s) \log^{r-i} \left( \frac{1}{s-1} \right) .
\]

**Proof.** By comparing with a large enough power of Riemann zeta function, we can see that \( \sum_n a_{n,r} n^{-s} \) defines a regular function in the open half-plane \( \sigma > 1 \), i.e., there exists some \( N > 0 \) such that for each \( \sigma > 0 \) we have \( \sum_n a_{n,r} n^{-\sigma} \leq \zeta(\sigma)^N \). So, we are focused on showing its analytic continuation.

If \( r = 1 \), then we can write

\[
\sum_n a_{n,1} n^{-s} = \sum_{n \mod m} \sum_{p \equiv n \mod m} a(p) p^{-s} .
\]

Lemma 5.4 implies first that there exists some regular function \( g_n(s) \) in the closed half-plane \( \sigma \geq 1 \) such that \( \sum_{p \equiv n \mod m} a(p) p^{-s} = g_n(s) \log(1/(s-1)) \), then we have

\[
\sum_n a_{n,1} n^{-s} = f_0(s) \log \left( \frac{1}{s-1} \right) .
\]
by taking the sum over $n$ modulo $m$. In other words, the statement holds when $r = 1$. Assume that the proposition is true for $1, \ldots, r$. Let’s compute the following Dirichlet series,

$$
\sum_{p \in \mathcal{P}(m,n)} a(p)p^{-s} \sum_{r_1 < \cdots < r_{r+1}} \prod_{i=1}^{r} a(p_i)p_i^{-s}
$$

= \sum_{r_1 < \cdots < r_{r+1}} \prod_{i=1}^{r+1} \left( (r+1)a(p_i)p_i^{-s} + \sum_{i=1}^{r} \sum_{r_1 < \cdots < r_{r+1}} a(p_i)^{r+1} p_i^{-(r+1)s} \prod_{j=1, j \neq i}^{r} a(p_j)p_j^{-s} \right).

On the other hand, if $r \geq 2$, then for each $2 \leq l \leq r$, we have

$$
\sum_{p \in \mathcal{P}(m,n)} a(p)^l p^{-ls} \sum_{r_1 < \cdots < r_{r+1-l}} \prod_{i=1}^{r+1-l} a(p_i)p_i^{-s}
$$

= \sum_{r_1 < \cdots < r_{r+2-l}} a(p_i)^{r+2-l} p_i^{-ls} \prod_{i=1}^{r+2-l} a(p_j)p_j^{-s} + \sum_{r_1 < \cdots < r_{r+1-l}} a(p_i)^{r+1-l} p_i^{-(r+1)s} \prod_{j=1, j \neq i}^{r+1-l} a(p_j)p_j^{-s}.

So by comparing the coefficients, for $r \geq 2$, we can get the following formula,

$$
\sum_{n} a_{n,r+1}n^{-s} = \sum_{r_1 < \cdots < r_{r+1}} \prod_{i=1}^{r+1} a(p_i)p_i^{-s}
$$

= \frac{1}{r+1} \sum_{p \in \mathcal{P}(m,n)} a(p)p^{-s} \sum_{r_1 < \cdots < r_{r+1}} \prod_{i=1}^{r} a(p_i)p_i^{-s}

+ \frac{r}{r+1} \sum_{l=2}^{r} \frac{(-1)^{l-1}}{a(p_i)^{l-s}} \sum_{r_1 < \cdots < r_{r+1-l}} \prod_{i=1}^{r+1-l} a(p_i)p_i^{-s}

+ \frac{(-1)^{r}}{r+1} \sum_{p \in \mathcal{P}(m,n)} a(p)^{r+1} p^{-(r+1)s}.

If $r = 1$, we have

$$
\sum_{n} a_{n,2}n^{-s} = \frac{1}{2} \left( \sum_{p \in \mathcal{P}(m,n)} a(p)p^{-s} \right)^2 - \frac{1}{2} \sum_{p \in \mathcal{P}(m,n)} a(p)^2 p^{-2s}.
$$

Note that when $l \geq 2$, the Dirichlet series $\sum_{p \in \mathcal{P}(m,n)} a(p)^{l-s} p^{-ls}$ defines a regular function in the closed half-plane $\sigma \geq 1$ by comparing with $\sum_{p} p^{-ls}$. Then by induction on $r$, we know that there exists some regular functions $f_0, \ldots, f_{r+1}$ in the closed half plane $\sigma \geq 1$ such that

$$
\sum_{n} a_{n,r+1}n^{-s} = \sum_{i=0}^{r+1} f_i(s) \log^{r-i} \left( \frac{1}{s-1} \right)
$$
in the open-half plane $\sigma > 1$. And the proof is done by induction on $r$. \qed
Corollary 5.6. Let $m$ be a fixed natural number, and let $a : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a multiplicative function such that $a(p) = a(q)$ for all rational primes $p \equiv q \mod m$, and that $a(p) > 0$ for at least one prime $p$. For each prime $p$, assign a real coefficient $b_p$ such that

$$0 \leq b_p \leq c,$$

where $c$ is a constant. Let

$$g(s) := \sum_{\substack{p \in \mathcal{P} \ni p_1 \cdots < p_r}} \prod_{i=1}^{r} \frac{a(p_i)}{p_i^s + b_{p_i}}.$$

Then $g(s)$ defines a regular function in the open half-plane $\sigma > 1$, and there exists regular functions $f_0, \ldots, f_r$ in the closed half plane $\sigma \geq 1$ such that

$$g(s) = \sum_{j=0}^{r} f_j(s) \log^j \left( \frac{1}{s-1} \right).$$

Proof. Note that

$$\frac{1}{p^s + b_p} \leq \frac{1}{p^{-\sigma}}.$$

By comparing with a large enough power of Riemann zeta function, we know that $g(s)$ defines a regular function in the open half-plane $\sigma > 1$.

We prove the statement of analytic continuation by induction on $r$. In the case of $r = 1$, for all prime $p$, we have

$$\frac{1}{p^s} - \frac{1}{p^s + b_p} = \frac{b_p}{p^s(p^s + b_p)}.$$

The expression $\sum_{p \in \mathcal{P}} \frac{b_p a(p)}{p^s(p^s + b_p)}$ defines a regular function in the closed half plane $\sigma \geq 1$, because it is absolutely convergent by comparing with $\sum_{p} \frac{b_p a(p)}{p^{2s}}$. We therefore see that the corollary holds in the case when $r = 1$ by Proposition 5.5.

Assume that the corollary holds for $1, 2, \ldots, r$ with $r \geq 1$. Then let’s do the following computation:

$$\sum_{\substack{p \in \mathcal{P}^{r+1} \ni p_1 \cdots < p_{r+1}}} \left( \sum_{i=1}^{r+1} \frac{a(p)}{p_i^s + a_{p_i}} \cdots \frac{a(p)}{p_r^s + b_{p_r}} \right) - (r + 1) \cdot \frac{a(p)}{p_1^s + b_p} \cdots \frac{a(p)}{p_r^s + b_{p_r}}$$

$$= \sum_{\substack{p \in \mathcal{P}^{r+1} \ni p_1 \cdots < p_{r+1}}} \sum_{i=1}^{r+1} \frac{b_p a(p)}{p_i^s \prod_{j=1}^{r+1} (p_j^s + b_{p_j})}$$

$$= \sum_{l=1}^{r+1} (-1)^{l-1} \sum_{p \in \mathcal{P}^{r+1-l}} \frac{b_p a(p)^l}{p^s(p^s + b_p)^l} \sum_{\substack{p \in \mathcal{P}^{r+1-l} \ni p_1 \cdots < p_{r}}} \frac{a(p)}{p_i^s + b_{p_i}}$$

$$= \sum_{i=0}^{r} g_i(s) \log^i \left( \frac{1}{s-1} \right),$$
where the last step is obtained by induction assumption. Since
\[
\sum_{p \in \mathcal{P} \atop p_1 < \cdots < p_r+1} \prod_{i=1}^{r+1} \frac{a(p)}{p_i^s + a_{p_i}} = \sum_{p \in \mathcal{P} \atop p_1 < \cdots < p_r} \prod_{i=1}^{r} \frac{a(p)}{p_i^s + a_{p_i}} \prod_{i=1}^{r+1} \frac{a(p)}{p_i^s + a_{p_i}},
\]
according to Proposition 5.5 and the induction assumption, there exists regular functions \( h_i, i = 1, \ldots, r + 1 \), in the closed half-plane \( \sigma \geq 1 \), such that
\[
\sum_{p \in \mathcal{P} \atop p_1 < \cdots < p_r+1} \prod_{i=1}^{r+1} \frac{a(p)}{p_i^s + a_{p_i}} = \sum_{i=0}^{r+1} h_i(s) \log^i \left( \frac{1}{s-1} \right).
\]
Combining these two parts, we see that the statement is true for \( r + 1 \). So the proof is done by induction. \( \Box \)

Finally we give a result that is inspired by Hardy-Littlewood Tauberian Theorem (see for example [MV06, Theorem 5.11]), relating \( \sum_{n < x} a_n \) and \( \sum_{n < x} a_n/n \). Using summation by parts (see in particular [Ten15, Theorem 0.3]), it is straightforward to prove the following statement.

**Lemma 5.7.** Let \( \{a_n\} \) be a non-negative sequence.

(i) If there exists a constant \( \alpha > 0 \) and an integer \( \beta \geq 0 \) such that
\[
\sum_{n < x} a_n = (\alpha + o(1)) x \log^\beta x,
\]
Then we have
\[
\sum_{n < x} \frac{a_n}{n} = \left( \frac{\alpha}{\Gamma(\beta + 2) + o(1)} \right) (\log x)^{\beta + 1}.
\]

(ii) If there exists some integers \( \beta_1 \geq -1, \beta_2 \geq 0 \) such that for \( x > N \), where \( N > e \) is a constant,
\[
\sum_{n < x} a_n \ll_N x(\log x)^{\beta_1}(\log \log x)^{\beta_2},
\]
then for \( x > N \), we have
\[
\sum_{n < x} \frac{a_n}{n} \ll_N (\log x)^{\beta_1+1}(\log \log x)^{\beta_2'},
\]
where \( \beta_2' = \beta_2 \) if \( \beta_1 > -1 \), and \( \beta_2' = \beta_2 + 1 \) if \( \beta_1 = -1 \).

The proof itself is left to the reader.

### 6. Dihedral extensions

In this section, we first try to estimate dihedral extensions with local specifications, and then use examples to illustrate their applications towards distribution of class groups. We use the following notation to explain what we mean by dihedral extensions in this section.

**Definition 6.1.** Given a finite group \( G \), we call \( G \) a dihedral group, if \( G = H \rtimes F \), where \( H, F \) are both finite abelian groups such that \( fhf^{-1} = h^{-1} \) for all \( h \in H \) and \( \text{id} \neq f \in F \).
For example, $G$ is just abelian group itself, i.e., $G = H$ and $F = \{id\}$. In other words, we treat abelian extensions as a special case of dihedral ones. For another example, $G = D_q$ where $q$ is an odd prime.

6.1. **Algebraic theory.** Let $L/\mathbb{Q}$ be a Galois $G$-field (see Definition 2.1) with $G = H \rtimes F$ a dihedral group, and let $K$ be the fixed field of $H$. Since $L/K$ is an abelian extension, we can try to apply Class Field Theory here. Let $\mathcal{J}_K$ be the idèle class group of $K$, and let $\mathcal{O}_K^*$ be the group of global units. For any $F$-module $A$, $B$, let $\text{Hom}_F(A, B)$ be the group of $F$-morphisms from $A$ to $B$. The following short exact sequences

$$1 \to \mathcal{O}_K^* \to \prod_v \mathcal{O}_v^* \to \mathcal{O}_K^*/\mathcal{O}_K^* \to 1$$

$$1 \to \left( \prod_v \mathcal{O}_v^* \right)/\mathcal{O}_K^* \xrightarrow{i} \mathcal{J}_K \to \text{Cl}_K \to 1$$

where we denote the embedding $(\prod_v \mathcal{O}_v^*)/\mathcal{O}_K^* \to \mathcal{J}_K$ by $i$, induce long exact sequences respectively, i.e., for any $F$-module $A$, we have

$$0 \to \text{Hom}_F\left(\prod_v \mathcal{O}_v^*/\mathcal{O}_K^*, A\right) \to \text{Hom}_F\left(\prod_v \mathcal{O}_v^*, A\right) \to \text{Hom}_F(\mathcal{O}_K^*, A)$$

$$\to \text{Ext}^1_F\left(\prod_v \mathcal{O}_v^*/\mathcal{O}_K^*, A\right) \to \cdots$$

(6.1)

$$0 \to \text{Hom}_F(\text{Cl}_K, A) \to \text{Hom}_F(\mathcal{J}_K, A) \xrightarrow{i^*} \text{Hom}_F\left(\prod_v \mathcal{O}_v^*/\mathcal{O}_K^*, A\right)$$

$$\to \text{Ext}^1_F(\text{Cl}_K, A) \to \cdots$$

**Remark.** The exact sequences show that for some $\rho \in \text{Hom}_F(\prod_v \mathcal{O}_v^*, A)$, if there is $\tilde{\rho} \in \text{Hom}_F(\mathcal{J}_K, A)$ such that $\tilde{\rho}$ coincides with $\rho_v$ on $\mathcal{O}_v^*$ for each $v$, then the number of such $\tilde{\rho} \in \text{Hom}_F(\mathcal{J}_K, A)$ is exactly $|\text{Hom}_F(\text{Cl}_K, A)|$.

We give a notation recording the order of an element in a group.

**Definition 6.2.** If $g \in G$, then define $\gamma_g$ to be its order.

With the help of this notation, the following result describes the application of Galois theory in this specific case, especially to the ramified primes.

**Lemma 6.3.** Let $L/\mathbb{Q}$ be a Galois $G$-extension with $G = H \rtimes F$ a dihedral group. Let $p \nmid |G|$ be a rational prime, and let $p$ be a prime of $L$ lying above $p$ with decomposition group $G_p$. Let $y_p$ be a generator of the inertia subgroup $I_p$, and let $x_p \in G_p$ be the Frobenius element. If $y_p \in H$, and $x_p = hf$ with $h \in H$ and $f \in F$ such that $fy_p f^{-1} = y_p^a$, then

$$p \equiv a \mod \gamma_{y_p}.$$  

**Proof.** Note that $x_p y_p x_p^{-1} = y_p^b$. Recall that $fhf^{-1} = h^b$ with $b = \pm 1$ by definition of $G$. If $h^bn = h$ for some integer $n$, then we have $hf = fh^n$. Since $H$ is abelian, we have

$$x_p y_p x_p^{-1} = fh^n y_p h^{-n} f^{-1} = y_p^a.$$

Therefore $a \equiv p \mod \gamma_{y_p}$ as required.  

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Inspired by the Galois action we make the following notation.

**Definition 6.4.** Let $p$ be a rational prime. Let $S$ be a subset of $H$. Then define

$$H_p(S) := \{ h \in S | p \equiv \pm 1 \mod \gamma_h \}.$$ 

Let $h_p(S) := |H_p(S)|$ denote its order.

6.2. **Counting fields.** We first make a notation inspired by Wood [Woo10, Theorem 3.1]. Recall that for $g \in G$, we denote its order by $\gamma_g$.

**Definition 6.5.**

(i) For any abelian group $\Gamma$, let $\Omega$ be a subset of $\Gamma$ closed under invertible powering. Define

$$\beta(\Omega) := \sum_{id \neq h \in \Omega} [\mathbb{Q}(\zeta_{\gamma_h}) : \mathbb{Q}]^{-1}$$

(ii) Let $G = H \rtimes F$ be a dihedral group, and let $\Omega$ be a subset of $H$ closed under invertible powering and conjugation. Define

$$\beta(F, \Omega) := \sum_{id \neq h \in \Omega} c(h)[\mathbb{Q}(\zeta_{\gamma_h}) : \mathbb{Q}]^{-1}$$

where $c(g)$ is the number of elements conjugate to $g$.

One can check that the notations $\beta(\Omega), \beta(F, \Omega)$ are always integers. Note that if $F = \{id\}$ is trivial, then $\beta(F, \Omega)$ coincides with $\beta(\Omega)$. Recall the notation $S(G)$ for a transitive permutation group $G \subseteq S_n$ from Definition 2.2, and the notation $N(S(G), P; f; x)$ for a function $f : S(G) \to \mathbb{R}$ from Definition 2.3. In this section, we show the following two theorems on counting fields.

**Proposition 6.6.** Let $G = H \rtimes F$ be a dihedral group. We view $G, H, F$ as transitive permutation groups via its action on itself, i.e., $G \to S_{|G|}, H \to S_{|H|},$ and $F \to S_{|F|}$. Let $\Omega \subseteq H \setminus \{id\}$ be a subset that is closed under invertible powering and conjugation. Let $M > e$ be a constant. If there exists some integer $\beta_1$ such that for $x > M$, we have

$$N(S(F), P; |\text{Hom} (\text{Cl}(K), H)|^2; x) \ll x(\log x)^{\beta_1},$$

then for all $r = 0, 1, 2, \ldots$, and $x > M$, we have

$$N(S(G), P; (\Omega, r); x) \ll \begin{cases} x(\log x)^{\beta(F,H;\Omega)+\frac{1}{2}(\beta(F)+\beta_1)(\log \log x)^\frac{1}{2}r+1} & \text{if } \Omega \neq \emptyset \\ x(\log x)^{\beta(F,H;\Omega)+\frac{1}{2}(\beta(F)+\beta_1)} & \text{Otherwise.} \end{cases}$$

The implied constant is dependent on $G, \Omega, r, M$.

Recall from § 2 that for two functions $a, b : S \to \mathbb{R}$, where $S$ is a space, we say that $a(x) \asymp b(x)$ for $x \in S$, if $a(x) \ll b(x)$ and $b(x) \ll a(x)$ are both true.

**Proposition 6.7.** Let $G$ be an abelian group, viewed as a transitive permutation group by the action on itself. Let $M > e$ be a constant. If $id \notin \Omega \subseteq G$ is a nonempty subset closed under invertible powering, then for each $r = 0, 1, \ldots$, and $x > M$, we have

$$N(S(G), P; (\Omega, r); x) \ll x(\log x)^{\beta(G;\Omega)-1}(\log \log x)^r.$$ 

In addition, if $r > |G|$, then for $x > M$, we have

$$N(S(G), P; (\Omega, r); x) \asymp x(\log x)^{\beta(G;\Omega)-1}(\log \log x)^{r-\delta(\beta(G;\Omega))},$$
where $\delta$ is the indicator of $-1$, i.e., $\delta(-1) = 1$, and $\delta(n) = 0$ otherwise. The implied constants are dependent on $G, \Omega, r, M$.

We need many notations. Let’s first use the following definitions to explain the computations of product of ramified primes and $1_{(\Omega, r)}$ (see Definition 1.5) when it comes to $G$-extensions using Class Field Theory.

**Definition 6.8.** Let $G = H \rtimes F$ be a dihedral group, and let $K$ be an $F$-field. Let $\rho \in \text{Hom}_F(\prod_v \mathcal{O}_v^*, H)$ where $v$ runs over all places of $K$. For each $p \nmid P(K)$, let $\rho_p \in \text{Hom}_F(\prod_{v|p} \mathcal{O}_v^*, H)$ be the corresponding local morphism. We define

$$P(\rho_p) = \begin{cases} p & \text{if } \rho_p \text{ is nontrivial;} \\ 1 & \text{otherwise.} \end{cases}$$

Then define

$$P(\rho) := P(K) \prod_{p|P(K)} P(\rho_p).$$

Let $L/K$ be an $H$-extension such that $L$ is a $G$-field. There exists an Artin reciprocity map $\tilde{\rho} \in \text{Hom}_F(\mathcal{J}_K, H)$ corresponding to $L$, i.e., $L$ is the class field of $\tilde{\rho}$. Then define $P(\tilde{\rho}) := P(L)$. If $\tilde{\rho}$ agrees with $\rho$ on $\prod_v \mathcal{O}_v^*$ for each place $v$ of $K$, then we have

$$P(\tilde{\rho}) = P(\rho) = P(L).$$

**Definition 6.9.** Let $id \notin \Omega$ be a subset of $H$ closed under invertible powering and conjugation, and let $r$ be a natural number. Let $\rho_p \in \text{Hom}_F(\prod_{v|p} \mathcal{O}_v^*, H)$ be a local morphism where $p|G|$ is a rational prime. Define

$$1^{(p)}_{\Omega}(\rho_p) = \begin{cases} 1 & \text{if the image of } \rho_p \text{ has nontrivial intersection with } \Omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then for $\rho \in \text{Hom}_F(\prod_v \mathcal{O}_v^*, H)$, define $1_{(\Omega, r)}(\rho) = 1$ if there are exactly $r$ rational primes $p|G|$ such that $1^{(p)}_{\Omega}(\rho_p) = 1$, and 0 otherwise. On the other hand, if $\tilde{\rho} \in \text{Hom}_F(\mathcal{J}_K, H)$ corresponding to the class field $L$, then just define $1_{(\Omega, r)}(\tilde{\rho}) := 1_{(\Omega, r)}(L)$. One can check that if $\tilde{\rho}_v = \rho_v$ for all $v$ of $K$, then $1_{(\Omega, r)}(\tilde{\rho}) = 1_{(\Omega, r)}(\rho)$. By abuse of notation, we just call them $1_{(\Omega, r)}$.

Recall that, in Definition 5.2, for a pair of relative prime integers $(m, n)$, we have defined the notation

$$\mathcal{P}(m, n) = \{p \text{ rational prime} | p \equiv n \mod m\} \quad \text{and} \quad \mathcal{P} := \{p | p \text{ is a rational prime}\}.$$ 

Then let’s present a lemma, which is the key of the above theorems.

**Lemma 6.10.** Let $G = H \rtimes F$ be a dihedral group, viewed as a transitive permutation group by the action on itself, and let $K$ be an $F$-field. Let $id \notin \Omega$ be a subset of $H$ closed under invertible powering and conjugation.

(i) Let

$$h(p) := \max_{[K_v \geq \mathbb{Q}_{\rho}]} |\text{Hom}(\mathcal{O}_{v}^*, G)|,$$
where \( \mathcal{O}_v \) is the valuation ring of \( K_v \), and let
\[
\sum_n b_n(\Omega, r) n^{-s} := \prod_{p|G} (1 + h(p)p^{-s}) \prod_{p \not| G} (1 + h_p(H|\Omega)p^{-s})
\]
\[
\cdot \left(1 + \sum_{p \not| G} \prod_{i=1}^r h_p(\Omega)p_i^{-s}\right).
\]

Then \( \sum_n b_n(\Omega, r) n^{-s} \) defines a regular function in the open half-plane \( \Re(s) > 1 \). If \( \Omega \neq \emptyset \), then there exists some regular functions \( \lambda_0(s), \ldots, \lambda_r(s) \) in the closed half plane \( \Re(s) \geq 1 \) such that
\[
\sum_n b_n(\Omega, r) n^{-s} = \sum_{i=0}^r \lambda_i(s)(s-1)^{\beta(F,H|\Omega)} \log^{r-i} \left(\frac{1}{s-1}\right).
\]

Else if \( \Omega = \emptyset \), then there exists some regular function \( \lambda(s) \) in the closed half plane \( \Re(s) \geq 1 \) such that
\[
\sum_n b_n(\Omega, r) n^{-s} = \lambda(s)(s-1)^{\beta(F,H)}.
\]

(ii) Let
\[
\sum_n b_n(K, (\Omega, r)) n^{-s} := |\text{Hom}(\text{Cl}_K, H)|P(K)^{-s} \sum_n b_n(\Omega, r) n^{-s}.
\]

Then, we have
\[
\sum_{L \in S(G)} \sum_{K \subseteq L, P(L) = n} 1_{(\Omega, r)}(L) n^{-s} \leq \sum_n b_n(K, (\Omega, r)) n^{-s}.
\]

Proof. We start by proving (ii). By Definition 6.8 and Definition 6.9, the groups \( \text{Hom}_F(J_K, H) \), \( \text{Hom}_F(\prod_v \mathcal{O}_v^*, H) \) are both sets of morphisms with an invariant \( P \) and functions \( 1_{(\Omega, r)} \) for each \( r = 0, 1, 2, \ldots \). According to Class Field Theory (6.1), for an integer \( n \), we have
\[
\sum_{L \in S(G)} \sum_{K \subseteq L, P(L) = n} 1_{(\Omega, r)}(L) \leq \sum_{\tilde{\rho} \in \text{Hom}_F(J_K, H)} 1_{(\Omega, r)}(\tilde{\rho})
\]
\[
\leq |\text{Hom}(\text{Cl}_K, H)| \cdot \sum_{\rho \in \text{Hom}_F(\prod_v \mathcal{O}_v^*, H)} 1_{(\Omega, r)}(\rho).
\]

First of all, if \( p|P(K) \) but \( p|G \), and \( w \) is a prime dividing \( p \) of a \( G \)-field \( L \) such that \( K \subset L \), then the inertia subgroup \( I_w \) of \( w \) in \( G = \text{Gal}(L/\mathbb{Q}) \) is cyclic and let \( y_w \) be the generator. We can write \( y_w = hf \) where \( h \in H \) and \( f \in F \). In this set-up, \( f \) must be non-trivial, because \( p \) is ramified in the \( F \)-field \( K \). However, \( G \) is a dihedral group with \( fh = h^{-1}f \), hence \( (hf)^2 = f^2 \), which implies that \( y_w \) has the same order as \( f \) (the order of \( f \) must be even), say \( y_w^e = f^e = \text{id} \). This implies that \( p \) has ramification index \( e \) in the \( F \)-extension
implies that for any $\rho \in \Hom_F(\prod_v \mathcal{O}_v^*, H)$, if $p|P(K)$ then $P(\rho_p) = 1$. Let

$$S_n(\Omega, r) := \{\rho \in \Hom_F(\prod_v \mathcal{O}_v^*, H) : \prod_{p \text{ rational}} P(\rho_p) = n \text{ and } 1_{(\Omega, r)}(\rho) = 1\}.$$ 

Because of the above discussion on primes $p|P(K)$, the formula $\prod_{p \text{ rational}} P(\rho_p) = P(\rho)/P(K)$ holds. To conclude (ii), it suffices to prove that for each square-free $n$, the set $S_n(\Omega, r)$ is nonempty, then we can write $n = m_1m_2p_1 \cdots p_r$, such that

- if $p|m_1$ then $p|G$ and $1_{\Omega}(\rho_p) = 0$;
- if $p|m_2$ then $p||G|$;
- and $p_i|G$ is a rational prime such that $1_{\Omega}(\rho_p) = 1$ for all $i = 1, \ldots, r$.

If $p \nmid |G|$ and $v$ is a place of $K$ above $p$, then the local morphism $\rho_v : \mathcal{O}_v^* \to H$ always factors through the group of roots of unity $\langle \mu \rangle$ in $\mathcal{O}_v^*$, hence totally determined by the image of the generator $\mu$. Let $T \subseteq G$ be a subset, according to Lemma 6.3, if $\rho_v(\mu) = y$, for some $y \in H \cap T$, then $y \in H_p(T)$. For each $\rho_v \in \Hom(\mathcal{O}_v^*, H)$, there is at most one $\rho_p \in \Hom(\prod_v \mathcal{O}_v^*, H)$ such that $\rho_p|_v = \rho_v$. So we see that for any $p|m_1$, resp. $p_i$ where $i = 1, \ldots, r$, when $\rho$ runs over all morphisms in $S_n(\Omega, r)$, there are at most $h_p(H\setminus \Omega)$, resp. $h_{p_i}(\Omega)$, many different local morphisms $\rho_p|_v$, resp. $\rho_{p_i}|_v$.

Let $p|m_2$ be a prime. Since for each $\rho_v \in \Hom(\mathcal{O}_v^*, H)$, there is at most one $\rho_p \in \Hom(\prod_v \mathcal{O}_v^*, H)$ such that $\rho_p|_v = \rho_v$, we know that

$$|\Hom_F(\prod_{v|p} \mathcal{O}_v^*, H)| \leq |\Hom(\mathcal{O}_v^*, H)|.$$

Therefore, if $\rho$ runs over all morphisms in $S_n(\Omega, r)$, then there are at most $h(p)$ many different local morphisms $\rho|_p$. Combining all situations above, we finally have

$$|S_n(\Omega, r)| \leq \prod_{p|m_1} h_p(H\setminus \Omega) \prod_{p|m_2} h(p) \prod_{i=1}^r h_{p_i}(\Omega) \leq b_n(\Omega, r).$$

This proves that

$$\sum_n \sum_{L \in \mathcal{S}(G) \atop k \leq L, P(L) = n} 1_{(\Omega, r)}(L)n^{-s} \leq \sum_n b_n(K, (\Omega, r))n^{-s},$$

and we are done for the proof of (ii).

Then, let’s prove (i), the analytic properties of the Dirichlet series $\sum_n b_n(\Omega, r)n^{-s}$. We first consider the following Euler product

$$\sum_n c_n(\Omega, r)n^{-s} := \begin{cases} \prod_{h \in H_{\Omega}} \zeta(\gamma_h, 1; s) & \text{if } F \neq \emptyset \\ \prod_{h \in H_{\Omega}} \zeta(\gamma_h, 1; s) & \text{otherwise.} \end{cases}$$
where $\zeta(m, n; s) = \prod_{p \in \mathcal{P}(m, n)} (1 - p^{-s})^{-s}$ (see also Definition 5.2). We can compute the order of pole at $s = 1$ by the following expression:

$$\sum_{id \neq h \in H \setminus \Omega} c(h) [\mathbb{Q}(\zeta_h) : \mathbb{Q}]^{-1} = \beta(F, H \setminus \Omega),$$

where $c(h)$ denote the number of elements conjugate to $h$. The analytic continuation is given by Lemma 5.3, i.e., there exists a regular function $\varphi(s)$ in the closed half-plane $\Re(s) \geq 1$ such that

$$\sum_n c_n(\Omega, r) n^{-s} = \varphi(s)(s - 1)^{\beta(F, H \setminus \Omega)}.$$

On the other hand, for a fixed rational prime $p$, we have

$$h_p(H \setminus \Omega) = |\{h \in H \setminus \Omega : p \equiv \pm 1 \mod \gamma_h\}|.$$

This shows that

$$\sum_n c_n(\Omega, r) n^{-s} = \prod_p (1 - p^{-s})^{-h_p(H \setminus \Omega)},$$

hence

$$(6.2) \quad \sum_n c_n(\Omega, r) n^{-s} = \tilde{\varphi}(s) \prod_{p \mid |G|} (1 + h_p(H \setminus \Omega)p^{-s})$$

for some regular function $\tilde{\varphi}$, where $\tilde{\varphi}(1) \neq 0$, in the closed half-plane $\Re(s) \geq 1$. This actually proves (i) when $\Omega = \emptyset$, because $h_p(\Omega) = 0$ for all rational prime $p$ in this case.

Then, let’s show the analytic properties of the series

$$(6.3) \quad \sum_{p \in \mathcal{P} \setminus \mathcal{P}_r} \prod_{i=1}^{r} h_{p_i}(\Omega)p_i^{-s}. $$

Assume without loss of generality that $\Omega \neq \emptyset$. Otherwise $h_p(\Omega) = 0$ for all $p$. According to the definition of $h_p(\Omega)$ (see Definition 6.4), its value is totally determined by $p$ modulo $|H|$. Therefore, we first obtain a function $h : \mathcal{P} \to \mathbb{R}_{\geq 0}$ such that $h(p) = h(q)$ if $p \equiv q \mod |H|$. The function $h$ can be extended to $\mathbb{N}$ as a multiplicative function. So, by Proposition 5.5, we know that the above series $(6.3)$ defines a regular function in the open half-plane $\Re(s) > 1$ and there exists some regular functions $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_r$ in the closed half-plane $\Re(s) \geq 1$ such that

$$(6.3) = \sum_{i=0}^{r} \tilde{\lambda}_i(s) \log^i \left( \frac{1}{s - 1} \right).$$

Combining with $(6.2)$, when $\Omega \neq \emptyset$, we finally obtain regular functions $\lambda_0, \ldots, \lambda_r$ in the closed half-plane $\Re(s) \geq 1$ such that

$$\sum_n b_n(\Omega, r) n^{-s} = \sum_{i=0}^{r} \lambda_i(s)(s - 1)^{\beta(F, H \setminus \Omega)} \log^{r-i} \left( \frac{1}{s - 1} \right).$$

And we are done for the proof. \qed

Then let’s give the proof of the propositions.
Proof of Proposition 6.6. Let

\[ \sum_{n} A_n(\Omega, r)n^{-s} := \sum_{n} \sum_{L \in S_{P(L)=n}} 1_{(\Omega, r)}(L)n^{-s} \]

where \( \Omega \subseteq H\setminus\{\text{id}\} \) is closed under invertible powering and conjugation.

Let \( K \) be an \( F \)-field, then define

\[ \sum_{n} a_n(K; (\Omega, r)) := \sum_{n} \sum_{L \in S(G)} K \subseteq L, P(L) = n 1_{(\Omega, r)}(L)n^{-s} \].

According to Lemma 6.10, we know that

\[ \sum_{n} a_n(K; (\Omega, r))n^{-s} \leq \sum_{n} b_n(K; (\Omega, r))n^{-s}, \]

i.e., \( a_n(K; (\Omega, r)) \leq b_n(K; (\Omega, r)) \) for each \( n \). Let \( K \) runs over all \( F \)-fields, we have

\[ \sum_{n} A_n(\Omega, r)n^{-s} = \sum_{K \in S(F)} \sum_{n} a_n(K; (\Omega, r))n^{-s} \]

\[ \leq \sum_{K \in S(F)} \sum_{n} b_n(K; (\Omega, r))n^{-s} =: \sum_{n} B_n(\Omega, r)n^{-s} \]

Then, by Cauchy-Schwartz Inequality, for \( x > M \), we have

\[ \sum_{n<x} A_n(\Omega, r) \leq \sum_{n<x} B_n(\Omega, r) \]

\[ = \sum_{P(K)<n} |\text{Hom}(\text{Cl}_K, H)| \sum_{n<x/P(K)} b_n(\Omega, r) \]

\[ \leq \left( \sum_{P(K)<n} |\text{Hom}(\text{Cl}_K, H)|^2 \frac{x}{P(K)} \right)^{1/2} \left( \sum_{n<x} (b_n(\Omega, r))^2 N(S(F), P; x/n) \right)^{1/2} \]

\[ \ll \left( \sum_{P(K)<n} |\text{Hom}(\text{Cl}_K, H)|^2 \frac{x}{P(K)} \right)^{1/2} \left( \sum_{n<x} (b_n(\Omega, r))^2 \frac{x}{n} (\log x)^{\beta(F)-1} \right)^{1/2} \]

By our assumption, for \( x > M \), we have

\[ \sum_{P(K)<n} |\text{Hom}(\text{Cl}_K, H)|^2 \ll x(\log x)^{\beta_1}. \]

Then, by Lemma 5.7, we know that

\[ x \sum_{P(K)<n} |\text{Hom}(\text{Cl}_K, H)|^2 P(K)^{-1} \ll x(\log x)^{\beta_1+1}. \]

Since \( b_n(\Omega, r)^2 \) is the coefficient of the Dirichlet series

\[ \prod_p (1 + h_p(H\setminus\Omega)^2 p^{-s}) \sum_{d=p_1}^{p_r} \prod_{i=1}^r h_{p_i}(\Omega)^2 p_i^{-s}, \]

where \( \prod_{p} (1 + h_p(H\setminus\Omega)^2 p^{-s}) \sum_{d=p_1\cdots p_r} \prod_{i=1}^r h_{p_i}(\Omega)^2 p_i^{-s}, \)]
Lemma 6.10 and Theorem 5.1 implies that for \( x > M \), we have
\[
\sum_{n<x} (b_n(\Omega, r))^2 \ll \begin{cases} 
  x (\log x)^{2\beta(F,H) - 1} (\log \log x)^{r+1} & \text{if } \Omega \neq \emptyset \\
  x (\log x)^{2\beta(F,H) - 1} & \text{otherwise.}
\end{cases}
\]

By Lemma 5.7 again, for \( x > M \), we have
\[
x (\log x)^{\beta(F) - 1} \sum_{n<x} (b_n(\Omega, r))^2 n^{-1} \ll \begin{cases} 
  x (\log x)^{2\beta(F,H) + \beta(F) - 1} (\log \log x)^{r+2} & \text{if } \Omega \neq \emptyset \\
  x (\log x)^{2\beta(F,H) + \beta(F) - 1} & \text{otherwise.}
\end{cases}
\]

Combining all components of the inequality, we obtain the desired result, i.e.,
\[
N(S(G), P; (\Omega, r); x) \ll \begin{cases} 
  x (\log x)^{\beta(F,H) + \frac{1}{2}(\beta(F) + \beta_1)} (\log \log x)^{\frac{1}{2}r + 1} & \text{if } \Omega \neq \emptyset \\
  x (\log x)^{\beta(F,H) + \frac{1}{2}(\beta(F) + \beta_1)} & \text{otherwise.}
\end{cases}
\]

Finally let’s prove Proposition 6.7, the Conjecture 1.6(1) for abelian extensions.

**Proof of Proposition 6.7.** Let \( \beta := \beta(G\setminus \Omega) \), and \( \delta \) be the indicator of \(-1\), i.e., \( \delta(-1) = 1 \) and \( \delta(n) = 0 \) otherwise. Let
\[
\sum_n a_n(\Omega, r) n^{-s} := \sum_{n} \sum_{K \in S(G)} 1_{(\Omega, r)}(K) n^{-s}.
\]

Since \( F = \{\text{id}\} \) is trivial, the long exact sequence of Class Field Theory (6.1) implies that for every surjective morphism \( \rho \in \text{Hom}(\prod_p \mathbb{Z}_p^*, G) \), where \( p \) runs over all rational primes, there exists a unique morphism \( \tilde{\rho} \in \text{Hom}(\mathcal{J}_G, G) \) such that \( \tilde{\rho}_p = \rho_p \). This shows that, using the notation in Definition 6.8 and Definition 6.9, we have
\[
(6.4) \quad \sum_{K \in S(G)} 1_{(\Omega, r)}(K) = \sum_{\tilde{\rho} \in \text{Sur}(\mathcal{J}_G, G)} 1_{(\Omega, r)}(\tilde{\rho}) = \sum_{\rho \in \text{Sur}(\prod_p \mathbb{Z}_p^*, G)} 1_{(\Omega, r)}(\rho),
\]

where \( \text{Sur} \) means surjective morphisms. Inspired by Lemma 6.10, we first consider the following Dirichlet series. If \( |G\setminus \Omega| = r_0 \), then let \( q_1 < \cdots < q_{r_0} \) be the smallest \( r_0 \) primes in \( \mathcal{P}(|G|, 1) \), i.e., any other prime \( p \) that is \( 1 \) mod \( |G| \) satisfies \( p > q_{r_0} \). Denote \( q_1 \cdots q_{r_0} \) by \( D \).

Define
\[
\sum_n b_n(\Omega, r) n^{-s} := \prod_{p||G|} (1 + h(p)p^{-s}) \prod_{p||G|} (1 + h_p(G\setminus \Omega)p^{-s}) \sum_{\rho \in \mathcal{P}^r} \prod_{i=1}^r h_{\rho_i}(\Omega)p_i^{-s}.
\]

\[
\sum_n c_n(\Omega, r) n^{-s} := D^{-s} \sum_{\rho \in \mathcal{P}(|G|, 1)^r} \sum_{q_{r_0} < p_1 < \cdots < p_r} (p_1 \cdots p_r)^{-s} \prod_{p||G|D_{p_1 \cdots p_r}} (1 + h_p(G\setminus \Omega)p^{-s})
\]

where \( h(p) = |\text{Hom}(\mathbb{Z}_p^*, G)| \), and \( \mathcal{P}(m, n) = \{ p \in \mathbb{N} : p \equiv n \mod m \} \). Note that \( c_n(\Omega, r) = 0 \) if \( n \) is not square-free. Our goal is to show that \( a_n(\Omega, r) \leq b_n(\Omega, r) \) for each \( n \), and in addition
if \( r > |G| \), then \( c_n(\Omega, r) \leq a_n(\Omega, r) \) for each \( n \). Let
\[
S_n(\Omega, r) := \{ \rho \in \text{Hom}(\bigtimes_p \mathbb{Z}_p^*, G) | 1_{(\Omega, r)}(\rho) = 1 \text{ and } P(\rho) = n \}.
\]
We see from (6.4) that \( a_n(\Omega, r) = |S_n(\Omega, r)| \). Let’s use the similar strategy to prove the statement. If \( n \) is a square-free number such that \( S_n(\Omega, r) \) is nonempty, then we can write \( n = m_1m_2p_1 \cdots p_r \) such that

- if \( p|m_1 \) then \( p || G \);
- if \( p|m_2 \) then \( p || G \), and \( 1_{\tilde{\Omega}}^{(p)}(\rho_p) = 0 \);
- and \( p_i || G \) with \( 1_{\tilde{\Omega}}^{(p_i)}(\rho_{p_i}) = 1 \) for all \( i = 1, \ldots, r \).

We discuss them separately. Let \( p || G \) be a rational prime. Since \( F = \{ \text{id} \} \) in this case, for any \( S \subseteq G \), the computation of \( h_p(S) \) gives
\[
h_p(S) = |\{ g \in S : g \neq \text{id} \text{ and } p \equiv 1 \mod \gamma_y \}|.
\]
In addition, if \( \mu \) the generator of the group of roots of unity of \( \mathbb{Z}_p^* \) and \( \rho_p(\mu) = y \) for some \( y \in S \), then this says that \( p \equiv 1 \mod \gamma_y \). This implies that if \( S \) is closed under invertible powering, then
\[
h_p(S) = |\{ \rho_p \in \text{Hom}(\mathbb{Z}_p^*, G) : \text{im} \rho \cap S \neq \emptyset \}|.
\]
So, we can conclude that when \( \rho \) runs over all morphisms in \( S_n(\Omega, r) \), there are exactly \( h_p(G\setminus \Omega) \), resp. \( h_p(\Omega) \), many different local morphisms \( \rho_p \) for \( p|m_2 \), resp. \( \rho_{p_i} \) for each \( i = 1, \ldots, r \). On the other hand, if \( p || G \), the notation \( h(p) \) is exactly the number of different local morphisms \( \rho_p \in \text{Hom}(\mathbb{Z}_p^*, G) \). In other words, when \( \rho \) runs over all morphisms in \( S_n(\Omega, r) \), there are at most \( h(p) \) different local morphisms \( \rho_p \). So, we first have
\[
\prod_{p|m_2} h_p(G\setminus \Omega) \leq \sum_{\rho \in S_n(\Omega, r) \atop 1_{\tilde{\Omega}}^{(p_i)}(\rho) = 1, i = 1, \ldots, r} \leq \prod_{p|m_1} h_p(G\setminus \Omega) \prod_{i=1}^r h_p(\Omega).
\]
Now, let \( m_1, m_2, p_1, \ldots, p_r \) run over all possible different situations, we see that
(6.5) \[
c_n(\Omega, r) \leq a_n(\Omega, r) \leq b_n(\Omega, r),
\]
when \( a_n(\Omega, r) > 0 \).

If \( S_n(\Omega, r) \) is empty and \( r > |G| \), then this means that there is no extension \( K/\mathbb{Q} \) with product of ramified primes \( P(K) = n \) such that \( 1_{(\Omega, r)}(K) = 1 \). Since \( b_n(\Omega, r) \geq 0 \) for all \( n \), we can assume for contradiction that \( c_n(\Omega, r) > 0 \). Then there exists \( p_1 < \cdots < p_r \) such that \( p_i | n \) for \( i = 1, \ldots, r \) and that for each \( p | n/(p_1 \cdots p_r) \), we have \( h_p(G\setminus \Omega) > 0 \). More importantly, \( q_j | n \), for all \( j = 1, \ldots, r_0 \), and \( q_j \equiv 1 \mod |G| \). Since \( r > |G| \) and \( r_0 > |G\setminus \Omega| \), we know that each \( i = 1, \ldots, r \) and \( j = 1, \ldots, r_0 \), we can construct local morphisms \( \rho_{q_j} \in \text{Hom}(\mathbb{Z}_q^*, G) \) and \( \rho_{p_i} \in \text{Hom}(\mathbb{Z}_{p_i}^*, G) \) such that the image of \( \rho_{p_i} \) has nontrivial intersection with \( \Omega \) for each \( i \), and that the images of \( \rho_{q_j} \)-s and \( \rho_{p_i} \)-s generate the whole group \( G \). For example, for each nontrivial element of \( \Omega \), we associate it with a morphism \( \rho_{p_i} \), and for each nontrivial element of \( G\setminus \Omega \), associate it with some \( \rho_{q_j} \). So, we first obtain a surjective morphism \( \rho \in \text{Hom}(\prod_p \mathbb{Z}_p^*, G) \) by combining local morphisms. Then, by Class Field Theory 6.1, we know that there exists a class field with Galois group \( G \) and product of ramified primes \( n \), which contradicts the assumption that \( S_n(\Omega, r) = \emptyset \). So, we prove that
(6.6) \[
0 = c_n(\Omega, r) = a_n(\Omega, r) \leq b_n(\Omega, r),
\]
when \( a_n(\Omega, r) = 0 \).

Note that by definition, \( \beta = \beta(F, H \setminus \Omega) \), because \( F \) is trivial and \( G = H \). So, by Lemma 6.10, we know that the series \( \sum_n b_n(\Omega, r)n^{-s} \) defines a regular function in the open half-plane \( \Re(s) > 1 \) and there exists regular functions \( \lambda_0, \ldots, \lambda_r \) in the closed half-plane \( \Re(s) \geq 1 \) such that
\[
\sum_n b_n(\Omega, r)n^{-s} = \sum_{j=0}^{r} \lambda_j(s)(s-1)^{\beta}(\log \frac{1}{s-1})^{r-j}.
\]

By Tauberian Theorem 5.1 and (6.5) and (6.6), we know that for all \( r = 0, 1, 2, \ldots \), we have
\[
N(S(G), P; (\Omega, r); x) \ll x(\log x)^{\beta-1}(\log \log x)^{r+\delta(\beta)}.
\]

Let’s rewrite the second Dirichlet series as follows:
\[
\sum_n c_n(\Omega, r)n^{-s} = (q_1 \cdots q_{r_0})^{-s} \prod_{p \in P(G|1)} (1 + h_p(G \setminus \Omega)p^{-s}) \prod_{p \in P(G|1)} \frac{1}{p^s + h_p(G \setminus \Omega)}.
\]

Then Lemma 6.10 and Corollary 5.6 implies that \( \sum_n c_n(\Omega, r)n^{-s} \) defines a regular function in the open half-plane \( \Re(s) > 1 \), and there exists some regular functions \( \varphi_0, \ldots, \varphi_r \) in the closed half-plane \( \Re(s) \geq 1 \) such that
\[
\sum_n c_n(\Omega, r)n^{-s} = \sum_{j=0}^{r} \varphi_j(s)(s-1)^{\beta}(\log \frac{1}{s-1})^{r-j}.
\]

By Tauberian Theorem 5.1 and (6.5) and (6.6) again, we know that for all \( r = 0, 1, 2, \ldots \), we have
\[
N(S(G), P; (\Omega, r); x) \gg x(\log x)^{\beta-1}(\log \log x)^{r+\delta(\beta)}.
\]

And we are done for the proof. \( \square \)

6.3. Applications. Let’s first use the following result to explain why the above discussions are useful for dihedral extensions. Recall that given \( G = H \times F \) and \( S(G) \), we have, from Theorem 4.4 and Theorem 4.5, the notations
\[
\mathbb{P}(\text{rk}_q q^{l-1} \text{Cl}(K/K^H) \leq r) \quad \text{and} \quad \mathbb{E}(|\text{Hom}(q^{l-1} \text{Cl}(K/K^H), C_q)|),
\]
where \( K \) runs over all fields in \( S(G) \) for the product of ramified primes in \( K/\mathbb{Q} \).

**Proposition 6.11.** Let \( q \) be an odd prime, and let \( \Omega := \Omega(D_q, q) \), where we view the dihedral group \( D_q \) as a subgroup of \( S_{2q} \). If the Cohen-Lenstra Heuristics hold for quadratic number fields, or to be precise, for \( x > 1 \), we have
\[
N(S(C_2), P; |\text{Hom(Cl}_{K_2}, C_q)|^2; x) \ll x,
\]
and the Malle-Bhargava Heuristics holds for cubic fields, or to be precise, for \( x > 1 \), we have
\[
N(S(D_q), P; x) \gg x \log x,
\]
then, we have
\[
N(S(D_q), P; (\Omega, r); x) = o(N(S(D_q), P; x))
\]
for all \( r = 1, 2, \ldots \). Also, for all \( r = 0, 1, 2, \ldots \), we have
\[
\mathbb{P}(\text{rk}_q \text{Cl}_K \leq r) = 0 \quad \text{and} \quad \mathbb{E}(|\text{Hom(Cl}_K, C_q)|) = +\infty,
\]

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where \( K \) runs over all fields in \( S \) for the product of ramified primes in \( K/\mathbb{Q} \).

**Proof.** If we write \( D_q = C_q \times C_2 \), then let \( \Omega = C_q \backslash \{\text{id}\} \). Therefore, \( \beta(C_2, C_q \backslash \Omega) = 0 \). Also the condition says that \( \beta_1 = 0 \). And \( \beta(C_2) = 1 \) by counting quadratic number fields via product of ramified primes. So, Proposition 6.6 says that, for each \( r \), we have

\[
N(S(D_q), P; (\Omega, r); x) \ll x (\log x)^{\frac{1}{2}} (\log \log x)^{\frac{1}{2}r + 1},
\]

So by the Malle-Bhargava Heuristics, we have

\[
N(S(D_q), P; (\Omega, r); x) = o(N(S(D_q), P; x))
\]

Then the rest just follows from Theorem 4.4 and Theorem 4.5. \( \square \)

**Remark.** The statement itself is exactly Conjecture 1.6(2) for \( D_q \)-extensions. But we need two hypothesis: the Cohen-Lenstra Heuristics for quadratic number fields and an estimate for counting quadratic number fields with its \( D_q \)-Galois closure, this implies that the same statement holds when \( D_q \) is viewed as a subgroup of \( S_q \).

Then let’s show Theorem 1.8, the result of relative class groups for abelian extensions, as promised in § 1.

**Theorem 6.12.** Let \( G \) be a finite abelian group with a subgroup \( H \), and let \( S := S(G) \). If \( q \) is a prime number such that \( q || G/H \), then \( q \) is a non-random prime for \( G \). In addition, for all \( r = 0, 1, \ldots \), we have

\[
\mathbb{P}(\text{rk}_q \Cl(K/K^H) \leq r) = 0 \quad \text{and} \quad \mathbb{E}(|\Hom(\Cl(K/K^H), C_q)|) = +\infty,
\]

where \( K \) runs over all fields in \( S \) for the product of ramified primes in \( K/\mathbb{Q} \).

**Proof.** Let \( \beta := \beta(G \backslash \Omega) \). We view the abelian group \( G \) as a transitive permutation group via its action on itself. Since \( q || G \), the set \( \Omega := \Omega(G, q^\infty) \) is nontrivial. This shows that \( q \) is a non-random prime for \( G \). The set \( \Omega \) is closed under invertible powering (and conjugation). So, Proposition 6.7 first implies that for all \( r = 0, 1, \ldots \), we have

\[
N(S(G), P; (\Omega, r); x) \ll x (\log x)^{\beta-1} (\log \log x)^r
\]

and that for \( r > |G| \), we have

\[
N(S(G), P; (\Omega, r); x) \asymp x (\log x)^{\beta-1} (\log \log x)^{r-\delta}\beta
\]

where \( \delta \) is the indicator of \(-1 \), i.e., \( \delta(-1) = 1 \) and \( \delta(n) = 0 \) otherwise. This implies that for \( r > |G| \), we have

\[
N(S(G), P; (\Omega, r); x) = o(N(S(G), P; (\Omega, r + 1); x)),
\]

and that for \( r \leq |G| \), we have

\[
N(S(G), P; (\Omega, r); x) = o(N(S(G), P; (\Omega, |G| + 1); x)).
\]

This means that, Conjecture 1.6(1) holds for \((S, \Omega)\). Then, Lemma 4.3, Theorem 4.4 and Theorem 4.5 show that the statements of zero-probability and infinite moment are true. \( \square \)

**Remark.** The proof actually shows that if \( G \) is abelian and the prime \( q || G \), then Conjecture 1.6(1) will hold for \((S(G), \Omega(G, q^\infty))\).
7. $D_4$ EXTENSIONS

In this section, let $D_4 = \langle \sigma, \tau | \sigma^4 = 1 = \tau^2, \tau^{-1}\sigma\tau = \sigma^3 \rangle$ be the dihedral group of order 8. Recall that $S(G)$ is the set of $G$-fields where $G$ is a transitive permutation group (see Definition 2.1 and 2.2). Define $S := S(D_4)$ with $D_4$ being a subgroup of $S_4$, i.e., we are focused on the quartic extensions whose Galois closure are $D_4$-fields. According to the Definition 1.3 of non-random primes, we know by checking the cycles in $D_4 \to S_4$ that the prime 2 is the only non-random prime for the permutation group $D_4$.

7.1. The distribution of $\text{Cl}_L[2^\infty]$ when ordered by conductor. We first introduce the definition of the conductor for a quadratic extension of a quadratic number field, which will be used here as the invariant of the number fields.

**Definition 7.1 (Conductor).** If $K$ is a quadratic number field and $L$ is a quadratic extension of $K$, define the conductor of the pair $(L, K)$ as

$$C(L, K) := \frac{\text{Disc}(L)}{\text{Disc}(K)}.$$  

If $L$ is a $D_4$-field and $K$ denotes its (unique) quadratic subfield, then $C(L, K) = C(L)$ (the conductor of $L$).

Note that the notation given by the above definition agrees with the Artin conductor for the irreducible 2-dimensional representation of $D_4$, if the quartic field has $D_4$-Galois closure. See also [ASVW21, §2.3] for details. We here follow this definition for the convenience of both computation and generalization to other quartic fields. Recall that $S = S(D_4, \langle \tau \rangle) = \{(L, \psi)\}$. Note that $L$ admits a unique quadratic subfield $K$. Let $q = 2$, which is the only non-random prime for $(D_4, \langle \tau \rangle)$. We want to study the statistical behaviour of $\text{Cl}_L[2^\infty]$ where $L \in S$ for the conductor. Let’s first prove a lemma whose statement is similar to [ASVW21, Lemma 5.1].

**Lemma 7.2.** For any $0 < \epsilon < \frac{1}{2}$, we have

$$\sum_{0 < D < X \text{ squarefree}} \frac{2^{\omega(D)}}{D} \cdot \left( \sum_{m=1}^{\infty} \sum_{n=1}^{D^{\frac{1}{2} + \epsilon}} \frac{\mu(m)}{m^2 n} \left( \frac{D}{mn} \right) \right) = o(X),$$

where $(\cdot)$ means Legendre symbol here.

**Proof.** The proof itself is straightforward. First of all, the Riemann zeta function admits an analytic continuation to the complex plane except a pole of order 1 at $s = 1$. Similarly, the Dirichlet series

$$\prod_{p \in \mathcal{P}} (1 + 2p^{-s})$$

admits an analytic continuation to the closed half-plane $\Re(s) \geq 1$ with a pole of order 2 at $s = 1$. Then, according to [MV06, Theorem 5.11], we have

$$\sum_{n < X} \frac{1}{n} \sim c_1 \log X \quad \text{and} \quad \sum_{0 < D < X \text{ squarefree}} \frac{2^{\omega(D)}}{D} \sim c_2 \log^2 X$$
for some constant \( c_1, c_2 > 0 \). So, we know that

\[
\left| \sum_{0 < D < X \atop D \text{ squarefree}} \frac{2^\omega(D)}{D} \cdot \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(m)}{m^2 n} \left( \frac{D}{mn} \right) \right) \right| 
\leq \sum_{0 < D < X \atop D \text{ squarefree}} \frac{2^\omega(D)}{D} \cdot \left( \sum_{m=1}^{\infty} \sum_{n=1}^{X} \frac{1}{m^2 n} \right) 
\ll \log^3(X) = o(X), \text{ as } x \to \infty.
\]

\[
\square
\]

Using the above lemma, we can prove the following proposition, which is similar to [ASVW21, Proposition 5.2]

**Proposition 7.3.** We have:

\[
\sum_{[K:Q]=2 \atop 0 < \text{Disc}(K) < x} \frac{L(1, K/Q)}{L(2, K/Q)} \cdot \frac{2^\omega(\text{Disc}(K))}{\text{Disc}(K)} = \sum_{[K:Q]=2 \atop 0 < \text{Disc}(K) < x} \frac{2^\omega(\text{Disc}(K))}{\text{Disc}(K)} = \sum_{0 < a, b < \infty \atop (\text{Disc}(K), ab)=1} \frac{\mu(a)}{a^3 b^2} + o(x)
\]

(7.1)

\[
\sum_{[K:Q]=2 \atop -x < \text{Disc}(K) < 0} \frac{L(1, K/Q)}{L(2, K/Q)} \cdot \frac{2^\omega(\text{Disc}(K))}{\text{Disc}(K)} = \sum_{[K:Q]=2 \atop 0 < \text{Disc}(K) < x} \frac{2^\omega(\text{Disc}(K))}{\text{Disc}(K)} = \sum_{0 < a, b < \infty \atop (\text{Disc}(K), ab)=1} \frac{\mu(a)}{a^3 b^2} + o(x)
\]

where \( L(s, K/Q) = \sum_n \frac{\chi_K(n)}{n^s} \) and \( \chi_K \) is the quadratic character associated to \( K \).

**Proof.** The proof is similar to that of [ASVW21, Proposition 5.2]. According to [ASVW21, (17)], we have

\[
\frac{L(1, K/Q)}{L(2, K/Q)} = \frac{1}{L(2, K/Q)} \cdot \sum_{n=1}^{\text{Disc}(K)} \frac{\chi_{K}(n)}{n} + O_\epsilon \left( \frac{\log(|\text{Disc}(K)|)}{|\text{Disc}(K)|^\epsilon} \right).
\]

Let’s rewrite the left-hand sides of 7.1 as

\[
\sum_{[K:Q]=2 \atop -x < \text{Disc}(K) < 0} \frac{2^\omega(\text{Disc}(K))}{\text{Disc}(K)} \cdot \frac{1}{L(2, K/Q)} \cdot \sum_{n=1}^{\text{Disc}(K)} \frac{\chi_{K}(n)}{n} + O_\epsilon \left( \frac{\log(|\text{Disc}(K)|)}{|\text{Disc}(K)|^\epsilon} \right);
\]

(7.2)

\[
\sum_{[K:Q]=2 \atop -x < \text{Disc}(K) < 0} \frac{2^\omega(\text{Disc}(K))}{\text{Disc}(K)} \cdot \frac{1}{L(2, K/Q)} \cdot \sum_{n=1}^{\text{Disc}(K)} \frac{\chi_{K}(n)}{n} + O_\epsilon \left( \frac{\log(|\text{Disc}(K)|)}{|\text{Disc}(K)|^\epsilon} \right)
\]

Note that

\[
\sum_{[K:Q]=2 \atop 0 < |\text{Disc}(K)| < x} \frac{2^\omega(\text{Disc}(K))}{\text{Disc}(K)} \log(|\text{Disc}(K)|) = O_\epsilon(1),
\]

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we focus on the remaining terms. Then by [ASVW21, (20)], we know that
\[(7.3)\]
\[
\frac{1}{L(2, K/Q)} \sum_{n=1}^\infty \frac{\chi_K(n)}{n} = \sum_{0 < a, b < \infty \atop \text{Disc}(K, ab) = 1} \frac{\mu(a)}{a^3b^2} + \sum_{n=1}^\infty \frac{\chi_K(n)}{n} \sum_{m=1 \atop mn \neq \Box}^\infty \frac{\mu(m)\chi_K(m)}{m^2}.
\]

By substituting (7.3) back to (7.2), we have
\[
\sum_{[K:Q]=2 \atop 0 < \text{Disc}(K) < x} \frac{2^{\omega(\text{Disc}(K))}}{\text{Disc}(K)} \left( \sum_{0 < a, b < \infty \atop \text{Disc}(K, ab) = 1} \frac{\mu(a)}{a^3b^2} + \sum_{n=1}^\infty \frac{\chi_K(n)}{n} \sum_{m=1 \atop mn \neq \Box}^\infty \frac{\mu(m)\chi_K(m)}{m^2} \right)
\]
\[
\sum_{[K:Q]=2 \atop -x < \text{Disc}(K) < 0} \frac{2^{\omega(\text{Disc}(K))}}{\text{Disc}(K)} \left( \sum_{0 < a, b < \infty \atop \text{Disc}(K, ab) = 1} \frac{\mu(a)}{a^3b^2} + \sum_{n=1}^\infty \frac{\chi_K(n)}{n} \sum_{m=1 \atop mn \neq \Box}^\infty \frac{\mu(m)\chi_K(m)}{m^2} \right)
\]
\[
\sum_{[K:Q]=2 \atop 0 < \text{Disc}(K) < x} \frac{2^{\omega(\text{Disc}(K))}}{\text{Disc}(K)} \sum_{n=1}^\infty \frac{\chi_K(n)}{n} \sum_{m=1 \atop mn \neq \Box}^\infty \frac{\mu(m)\chi_K(m)}{m^2} = o(x).
\]

Similarly for the case when \(\text{Disc}(K) > 0\). And we are done for the proof. \(\square\)

Recall that \(S = S(D_4, \{1, \tau\})\). For any function \(f : S \to \mathbb{R}\), we have defined the notation
\[
N(S, C; f; x) := \sum_{L \in S, C(L) < x} f(L),
\]

where \(C\) means the conductor of \(L \in S\) (see also Definition 2.3). In addition, here we can even generalize the usual counting fields notation as follows.

**Definition 7.4.** For \(S = S(D_4, \{1, \tau\})\), and any function \(f : S \to \mathbb{R}\), define
\[
N(S, C; f; x, y) := \sum_{n < x} \sum_{L \in S \atop C(L) = n, \text{Disc}(K) < y} f(L).
\]

In other words we put some restrictions on the discriminant of \(K/Q\) by this notation. See [ASVW21, Theorem 4.3] for its usage in counting \(D_4\)-fields. Here, we mainly focus on its application in the problems related to the distribution of class groups.

**Theorem 7.5.** For each \(L \in S\), let \(K\) be its unique quadratic subfield. We have
\[
\mathbb{E}_C(\text{Hom}(\text{Cl}_{L}, C_2)) = \lim_{x \to \infty} \frac{N(S, C; |\text{Hom}(\text{Cl}_{L}, C_2)|; x)}{N(S, C; x)} = +\infty
\]

where \(L\) runs over all fields in \(S\) for the conductor \(C\).
Proof. The basic idea of the proof is similar to that of [ASVW21, Theorem 2]. First of all, by Theorem 3.11, it suffices to prove that

$$E_C(f) = \lim_{x \to \infty} \frac{N(S, C; f; x)}{N(S, C; x)} = +\infty$$

where \( f(L) = 2^{\omega(\text{Disc}(K))} \) for each \( L \in S \), because \( |\Hom(\text{Cl}_L, C_2)| \geq 2^{-6} \cdot f(L) \). Second, let \( \chi_{[0,1]} \) is the characteristic function of \([0,1]\). Then we can write

$$N(S, C; f; x, x^\beta) = \sum_{[K:Q]=2} \sum_{\substack{[L:Q]=2 \atop |\text{Disc}(K)| < x^\beta \atop L \in S}} 2^{\omega(\text{Disc}(K))} \chi_{[0,1]} \left( \frac{\text{Disc}(K) \text{Nm}_{K/Q}(\text{Disc}(L/K))}{x} \right),$$

where \( 0 < \beta < 1 \). For \( \epsilon > 0 \), there exists some compactly supported smooth functions \( \varphi^\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that \( \varphi^\pm - \chi_{[0,1]} \) takes value in \( \mathbb{R}^\pm \) and that \( \text{Vol}(\varphi^\pm) = 1 \pm \varphi \). Then [ASVW21, Lemma 4.5 and (13)] implies that

$$\sum_{[K:Q]=2} \sum_{\substack{[L:Q]=2 \atop |\text{Disc}(K)| < x^\beta \atop L \in S}} 2^{\omega(\text{Disc}(K))} \varphi^\pm \left( \frac{\text{Disc}(K) \text{Nm}_{K/Q}(\text{Disc}(L/K))}{x} \right)$$

$$= \sum_{[K:Q]=2} \frac{1 \pm \epsilon}{\zeta(2)} \frac{L(1, K/Q)}{L(2, K/Q)} \cdot \frac{2^{-r_2(K)}}{|\text{Disc}(K)|} x + O_\epsilon \left( \sum_{[K:Q]=2} |\text{Disc}(K)|^{-\frac{1}{4}+\epsilon} x^{\frac{1}{2}+\epsilon} \right),$$

where \( r_2(K) \) means the number of complex embeddings of \( K \). Here \( \varphi^\pm \) gives upper bound, resp. lower bound, of \( N(S, C; f; x, x^\beta) \). Note that the error term is bounded by \( O_\epsilon (x^{1/2+3\beta/4+\epsilon}) \).

So, when \( \beta < 2/3 \), the error term is bounded by \( o(X) \). By letting \( \epsilon \) goes to 0, we are reduced to proving that

$$\sum_{[K:Q]=2} \frac{L(1, K/Q)}{L(2, K/Q)} \frac{2^{\omega(\text{Disc}(K))}}{|\text{Disc}(K)|} \sim c(\log y)^2.$$

Now Proposition 7.3 says that it suffices to prove that (7.5)

$$\sum_{[K:Q]=2} \frac{L(1, K/Q)}{L(2, K/Q)} \frac{2^{\omega(\text{Disc}(K))}}{|\text{Disc}(K)|} = \sum_{[K:Q]=2} \frac{2^{\omega(\text{Disc}(K))}}{|\text{Disc}(K)|} \sum_{0<a,b<\infty} \frac{\mu(a)}{a^3b^2} \sim c_1(\log y)^2 \quad (\text{Disc}(K), ab)=1$$

$$\sum_{[K:Q]=2} \frac{L(1, K/Q)}{L(2, K/Q)} \frac{2^{\omega(\text{Disc}(K))}}{|\text{Disc}(K)|} = \sum_{[K:Q]=2} \frac{2^{\omega(\text{Disc}(K))}}{|\text{Disc}(K)|} \sum_{0<a,b<\infty} \frac{\mu(a)}{a^3b^2} \sim c_2(\log y)^2 \quad (\text{Disc}(K), ab)=1$$

Using [ASVW21, (21)],

$$\sum_{0<a,b<\infty} \frac{\mu(a)}{a^3b^2} = \frac{\zeta(2)}{\zeta(3)} \prod_{p} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}},$$

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we get
\[
\lim_{y \to \infty} \frac{\zeta(2)}{\zeta(3) \log(y)^2} \sum_{1 < D < y \atop D \text{ squarefree}} \frac{2\omega(D)}{D} \prod_{p \mid D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} = \zeta(2) \prod_{p \in \mathcal{P}} \left( 1 - \frac{3}{p^2} - \frac{1}{p^3} + \frac{6}{p^5} - \frac{3}{p^7} \right).
\]

We can do the similar computation when the squarefree \( D \) takes \( D \equiv \pm 1 \mod 4 \) and so on, i.e., discriminant of quadratic number fields belongs to different conjugacy classes (see [ASVW21, (23)] for example). Similar result holds when \( \text{Disc}(K) < 0 \). This proves (7.5), hence the theorem.

7.2. Further discussion with Malle-Bhargava Heuristics. The main topic in this section is slightly different from the previous one. For a quartic number field \( L \in \mathcal{S} \), it admits a unique quadratic number field \( K \). We first try to describe the relation between \( \text{Cl}_L[2^\infty] \) and ramified primes in \( L/\mathbb{Q} \), and then prove the related statistical results when ordering the fields by product of ramified primes. Let’s consider the group homomorphism
\[
i : \mathcal{I}_K \to \mathcal{I}_L
\]

between fractional ideals. It induces a group homomorphism on the class groups \( \text{Cl}_K \to \text{Cl}_L \), which we denote by \( i_* \). We already know by genus theory for quadratic number fields that \( \text{rk}_2 \text{Cl}_K \) admits a good estimate using the number of ramified primes in \( K/\mathbb{Q} \). So, it would not be surprising that \( \text{rk}_2 \text{Cl}_L \) has a similar algebraic expression. So, let’s try to give a result estimating the difference between \( \text{Cl}_K[2^\infty] \) and \( \text{Cl}_L[2^\infty] \).

Recall that if \( M \) is the Galois closure, then we view the group \( \mathcal{I}_L \) of fractional ideals of \( L \) as a subgroup of \( \mathcal{I}_M \), and define
\[
C^{D_4}_L := (\mathcal{I}_L \cap \mathcal{I}^{D_4}_M) \cdot \mathcal{P}_L/\mathcal{P}_L,
\]

where \( \mathcal{P}_L \) is the group of principal ideals. Similar definition works for \( C^{D_4}_K \). Let’s present the result on their difference based on the method of Theorem 3.15.

**Theorem 7.6.** Let \( L/\mathbb{Q} \) be a quartic number field with Galois \( D_4 \)-closure \( M/\mathbb{Q} \), let \( K \) be the quadratic subfield of \( L \), and let \( I(p) \) be the inertia subgroup of \( p \).

(i) Let \( \Omega_1 \) be the set \( \{\sigma, \sigma^3, \sigma \tau, \sigma^3 \tau\} \). Then we have
\[
|\{p \neq 2 : I(p) \cap \Omega_1 \neq \emptyset\}| \geq \text{rk}_2 i_* C^{D_4}_K \geq |\{p \neq 2 : I(p) \cap \Omega_1 \neq \emptyset\}| - 6.
\]

(ii) Let \( \Omega_2 := \Omega(D_4, 2^\infty) = \{\sigma, \sigma^3, \sigma^2, \sigma \tau, \sigma^3 \tau\} \). Then we have
\[
\text{rk}_2 \text{Cl}(L/K) \geq |\{p \neq 2 : I(p) \cap \Omega_2 \neq \emptyset\}| - 6.
\]

(iii) Let \( \Omega_3 \) be the set \( \{\sigma^2\} \). Then we have
\[
\text{rk}_2 C^{D_4}_L/i_*(C^{D_4}_K) \geq |\{p \neq 2 : I(p) \cap \Omega_3 \neq \emptyset\}| - 6.
\]

(iv) Let \( \Omega_4 \) be the set \( \{\sigma, \sigma^3\} \). Then we have
\[
|\{p \neq 2 : I(p) \cap \Omega_4 \neq \emptyset\}| \geq \text{rk}_2 2C^{D_4}_L \geq |\{p \neq 2 : I(p) \cap \Omega_4 \neq \emptyset\}| - 6.
\]

**Proof.** We view the group of fractional ideals \( \mathcal{I}_K, \mathcal{I}_L \) as subgroups of \( \mathcal{I}_M \).

(i): First of all, an odd prime \( p \) is ramified in the quadratic extension \( K/\mathbb{Q} \) if and only if \( I(p) \cap \Omega_1 \neq \emptyset \). In other words,
\[
C^{D_4}_K = \langle p | p \neq 2, I(p) \cap \Omega_1 \neq \emptyset \rangle \mathcal{P}_K/\mathcal{P}_K.
\]
where $p^2 = p\mathcal{O}_K$ and $\mathcal{P}_K$ is the group of principal ideal of $K$. This already gives the upper bound of the 2-rank. We have $\Omega_1 \subseteq \Omega(D_4, 2^\infty)$, and

$$i_*C_{K_i}^{D_4} = \langle p | p \neq 2, I(p) \cap \Omega_1 \neq \emptyset \rangle \mathcal{P}_L / \mathcal{P}_L$$

is a subgroup of $C_{L_i}^{D_4}$. So, according to Lemma 3.13, the lower bound of $\text{rk}_2 i_*C_{K_i}^{D_4}$ comes from

$$\text{rk}_2 \mathcal{P}_L / \mathcal{P}_K \leq \text{rk}_2 \mathcal{P}_L / \mathcal{P}_Q \leq 6.$$  

(ii): This is just the application of Theorem 3.15 with $\Gamma = D_4$ and $[L : K] = 2 = q$.  

(iii): By comparing with the choice of $\Omega_1$, it is not hard to see that if an odd prime $p$ has inertia in $\Omega_3$, then this means that $p$ is unramified in $K/\mathbb{Q}$, and then prime(s) lying above $p$ are ramified in $L/K$. Whatever the specific splitting type of $p$ is, we see that $e_L(p) \equiv 0 \text{ mod } 2$, and $p$ is a ramified prime of type 2. This shows that $\Omega_3 \subseteq \Omega(D_4, 2^\infty)$, i.e.,

$$\langle a(p) | p \neq 2 \text{ and } I(p) \cap \Omega_3 \neq \emptyset \rangle / \mathcal{P}_L^{D_4},$$

where $a(p) = (p\mathcal{O}_L)^{e_L(p)}$, is a subgroup of $C_{L_i}^{D_4}$. Also, since $p$ is unramified in $K/\mathbb{Q}$, it is not contained in $i_*C_{K_i}^{D_4}$. So, by Lemma 3.13 again, we obtain the result directly.

(iv): An odd prime $p$ has inertia in $\Omega_4$ means that it is totally ramified in $L/\mathbb{Q}$. The conclusion itself follows from Theorem 3.11 directly. \hfill $\square$

This lemma actually shows the relation between $\Omega_i \subseteq D_4$, defined in the statement, and the different subgroups of $C_{L_i}^{D_4} \subseteq \text{Cl}_L$. Now let's see what happens for counting fields. When ordered by product of ramified primes, the Malle-Bhargava Heuristics (see [Mal04, Bha07] and [ELPS16, p.291-339]) predict that

$$(7.6) \quad N(S, P; x) \gg x \log^3 x.$$  

**Theorem 7.7.** Recall the definition of $\Omega_i$ in the above Theorem 7.6, where $i = 1, 2$. We show that

$$N(S, P; (\Omega_i, r); x) \ll x(\log x)^2(\log \log x)^{r+1}$$

for $i = 1, 2$.

**Proof.** The proof for $\Omega_1$ and $\Omega_2$ are similar to each other. The choice of $\Omega_1$ means that if $p$ has inertia in $\Omega_1$ then $p$ is ramified in the quadratic subextension $K/\mathbb{Q}$ of $L/\mathbb{Q}$. First we define the Dirichlet series

$$\sum_n a_n n^{-s} := \sum_n \left( \sum_{L \in S, P(L) = n} 1_{\text{rk}_2 \mathcal{P}_L / \mathcal{P}_K \leq 6} \right) n^{-s}.$$  

Given a quadratic number field $K/\mathbb{Q}$ of product of ramified primes $P(K)$, let

$$\sum_n b_n(K) n^{-s} := \sum_n \sum_{L \in S(D_4)} 1(K) n^{-s}.$$  

Recall the definition of $P(\rho), P(\rho)$ for $\rho \in \text{Hom}(J_K, C_2)$ and $\rho \in \text{Hom}(\Pi_v \mathcal{O}_v^*, C_2)$ from Definition 6.8. Let

$$S_n(\Omega, r) := \{ \rho \in \text{Hom}(J_K, C_2) | P(\rho) = n \}. 38$$
Consider the following Dirichlet series
\[
\sum_n \tilde{b}_n(K)n^{-s} := |\text{Hom}(\text{Cl}_K, C_2)|2^{\omega(K)}P(K)^{-s}(1 + 128 \cdot 2^{-s}) \prod_{p \mid 2P(K)} (1 + 3p^{-s}).
\]
we want to show that \(b_n(K) \leq \tilde{b}_n(K)\) for all \(n\). Let \(n\) be a square-free number such that \(b_n(K)\) is non-zero. If \(p \nmid 2P(K)\) and \(p \mid n\), then there are at most 3 local morphisms \(p|_{\rho}\) in the set \(\text{Hom}(\prod_{\rho \mid P(K)} \mathcal{O}_{\mathcal{V}}^*, C_2)\), as \(\rho\) runs over all morphisms \(\rho \in S_n(\Omega, r)\). Let \(p\) be an odd prime such that \(p \mid P(K)\) and \(p \mid n\). Let \(v\) be the place of \(K\) such that \(v \mid p\). If \(L/K\) is a quadratic extension such that \(w|v\), then \(L_{w}/K_{v}\) is either totally ramified or unramified. Finally if \(2|n\), then just consider \(\max_{|K:Q|=2}|\text{Hom}(\prod_{\rho \mid 2} \mathcal{O}_{\mathcal{V}}^*, C_2)|\). We then can see that
\[
b_n(K) \leq |\text{Hom}(\text{Cl}_K, C_2)|128 \cdot 2^{\omega(K)} \prod_{p \text{ odd}} \prod_{p \mid P(K)} 3 = \tilde{b}_n(K).
\]
As mentioned above, we require that \(K\) admits exactly \(r\) tamely ramified primes, so let’s consider the following Dirichlet series
\[
\sum_n c_n n^{-s} := \sum_{p_0 < p_1 < \cdots < p_r} \sum_{P(K)=p_0 p_1 \cdots p_r} \sum_n b_n(K)n^{-s}.
\]
Since \(|\text{Hom}(\text{Cl}_K, C_2)| \leq 2^{\omega(K)}\), we have
\[
\sum_{n < x} c_n n^{-s} \leq \sum_{d=p_1 \cdots p_r} 2^{2r+2}d^{-s}(1 + 128 \cdot 2^{-s}) \prod_{p} (1 + 3p^{-s})
\]
\[
= g(s) + g_0(s) \log^{2r} \left( \frac{1}{s-1} \right) (s-1)^{-3} + \cdots
\]
where \(g(s), g_0(s), \ldots\) are regular functions in the closed half plane \(\Re(s) \geq 1\), whose existence follows from Proposition 5.5. So by Theorem 5.1, we have
\[
\sum_{n < x} c_n \ll x \log^2 (x \log \log x)^{2r}.
\]
In particular, since \(\sum_n a_n n^{-s} \leq \sum_n c_n n^{-s}\), we’ve shown that
\[
N(S, P; (\Omega_1, r); x) \ll x \log^2 (x \log \log x)^{2r},
\]
which is \(o(x \log^3 x)\).

This statement on the other hand says that we find out that the relative class group and the class group of the subfield embedded in \(\text{Cl}_L\) are both “large” in the sense of statistics. Consequently we have the following statistical results, whose proof are just a direct application of Theorem 4.4 and Theorem 4.5.

**Corollary 7.8.** Assume that the Malle-Bhargava Heuristics holds for \(N(S, P; x)\), or (7.6) holds, then for all \(r = 1, 2, 3, \ldots\), we have
\[
\mathbb{P}(\text{rk}_2 i_*(\text{Cl}_K) \leq r) = 0 \quad \text{and} \quad \mathbb{E}(|\text{Hom}(i_*(\text{Cl}_K), C_2)|) = +\infty
\]
\[
\mathbb{P}(\text{rk}_2 \text{Cl}(L/K) \leq r = 0) \quad \text{and} \quad \mathbb{E}(|\text{Hom}(\text{Cl}(L/K), C_2)|) = +\infty
\]
where \(L \in S\) for the product of ramified primes in \(L/\mathbb{Q}\) and \(K\) is the quadratic subfield of \(L\).
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