HANS WENZL

Abstract. We give a presentation of the centralizer algebras for tensor products of spinor representations of quantum groups via generators and relations. In the even-dimensional case, this can be described in terms of non-standard $q$-deformations of orthogonal Lie algebras; in the odd-dimensional case only a certain subalgebra will appear. In the classical case $q = 1$ the relations boil down to Lie algebra relations.

Classically, representations of Lie groups were studied by decomposing tensor products of a simple generating representation. This worked very well for the vector representations of the general linear group and to a somewhat lesser extent also for the vector representations of orthogonal and symplectic groups. More recently, $q$-versions of these centralizer algebras were studied in connection with Drinfeld-Jimbo quantum groups which have had applications in a number of fields. In this paper, we study centralizer algebras for spinor representations. Some of the possible applications will be given below.

We first give a description of the centralizer algebras for the classical spinor groups. If $N$ is even, this can be comparatively easily deduced from results by Hasegawa [8] for the action of $Pin(N)$. It follows that the commutant of its action on the $l$-th tensor power $S^\otimes l$ of its spinor representation $S$ is given by a representation of $so_l$. Moreover, we give a precise identification of the action of generators which is compatible with embeddings $\text{End}(S^\otimes l) \subset \text{End}(S^\otimes (l+1))$. This is important for studying the corresponding tensor categories and is not immediately obvious from the results in [8]. We also prove an analogous result for the odd-dimensional case, which is more complicated. Here the commutant is generated by a subalgebra of the universal enveloping algebra $Uso_l$.

We then extend these results to the setting of quantum groups. In principle, it should be possible to do this similar to the classical case, using known $q$-deformations of Clifford algebras, see Section 4.1. However, due to their complicated multiplicative structure we determine generating elements via a straightforward approach. They are $q$-deformations of the canonical element $\sum e_i \otimes e_i \subset Cl(V) \otimes Cl(V) \cong \text{End}(S^\otimes 2)$, where $(e_i)_i$ is an orthonormal basis of the vector representation $V$ and $Cl(V)$ is its Clifford algebra. We obtain from this elements which satisfy the relations of generators of another $q$-deformation $U'_q so_l$ of the universal enveloping algebra $Uso_l$. It has appeared before in work of Gavrilik and Klimyk [7], Nouni and Sugitani [20] and Letzter [18]. Unlike the usual $q$-deformation of $Uso_l$, it does not have a Hopf algebra structure. Our main result is that we again have a duality between the actions of $U_q so_N \rtimes \mathbb{Z}/2$
and $U_q^\prime so_l$ acting as each others commutants on $S^\otimes l$ for $N$ is even. Again, the situation is more complicated in the odd-dimensional case where we have to consider a subalgebra of $U_q^\prime so_l$. It is worthwhile mentioning that one of the problems for spinor representations is that already their second tensor power contains an increasing number of irreducible representations. This makes it difficult to characterize the centralizer algebras via braid representations, which worked well for vector representations. A similar problem was encountered by Rowell and Wang in their study of certain braid representations, which they conjecture to be related to spinor representations at certain roots of unity, see [23]. Our results should be useful in studying this question. This and other potential applications are discussed at the end of this paper.

Here is the paper in more detail: We first show how the commutant on the $l$-th tensor product of a spinor representation is related to $so_l$ by fairly elementary methods. While many of the results are not new, it serves as a blueprint for the more difficult quantum group case. We then review basic material from the study of Lie algebras and Drinfeld-Jimbo quantum groups. This is then used to prove the already sketched duality results for quantum groups, where we find generators for $\text{End}_U(S^\otimes l)$ in the third section, and relations in the fourth section.

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1. \textbf{Duality for Spinor Representations}

We assume throughout this paper all the algebras to be defined over the field of complex numbers, with $q$ not being a root of unity. For possible generalizations, see Remark 4.9.

1.1. \textbf{Clifford Algebras and spinor representations.} Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis of the finite dimensional inner product space $V$. Then the Clifford algebra $Cl = Cl(N)$ corresponding to $V$ can be defined via generators, also denoted by $e_i$, and relations

$$e_i e_j + e_j e_i = 2\delta_{ij} 1, \quad \text{for } 1 \leq i, j \leq N.$$ 

It is well-known that $Cl(N)$ has dimension $2^N$. It is isomorphic to $M_{2^{N/2}}$ for $N$ even, and to $M_{2^{(N-1)/2}} \oplus M_{2^{(N-1)/2}}$ for $N$ odd; here $M_d$ denotes the $d \times d$ matrices. The action of an element $g$ in the orthogonal group $O(N)$ on $V$ induces an automorphism $\alpha_g$ on $Cl(N)$, for each $g \in O(N)$. As any automorphism of the $d \times d$ matrices is inner, we obtain a projective representation of $O(N)$ on a $2^{N/2}$ dimensional module $S$ in the \textit{even-dimensional case}. By restriction, the module $S$ becomes a projective $O(N-1)$-module which we will denote by $\hat{S}$. It decomposes into the direct sum of two simple projective $O(N-1)$-modules $\hat{S}_+ \oplus \hat{S}_-$. These two modules are isomorphic as projective $SO(N-1)$-modules and simple.

A more direct way to construct the simple projective $SO(N)$ module for $N$ odd goes as follows: We replace the full Clifford algebra, which is not simple in the odd-dimensional case,
by $Cl_{ev}(N)$, the span of all products of $e_i$'s with an even number of factors. Observe that for $N$ odd, the element $f_N = e_1e_2 \ldots e_N$ is in the center of $Cl(N)$, and $(\gamma f_N)^2 = 1$ for a suitable $\gamma \in \{1, \sqrt{-1}\}$. Hence the map induced by $e_i \mapsto \gamma e_if_N$ defines a homomorphism from $Cl(N)$ into $Cl_{ev}(N)$. If one restricts this homomorphism to the simple subalgebra $M_{2(N-1)/2} \cong Cl(N-1) \subset Cl(N)$, it is obviously not the zero map. Hence it becomes an isomorphism, by dimension count and simplicity of $Cl$ by an element $g$ between $\gamma$ and $\sqrt{-1}$. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups. It will be convenient to consider these modules over $SO$ a projective representation of $O$ covering groups.

The projective representations just mentioned extend to honest representations of suitable covering groups. It will be convenient to consider these modules over $Pin(N)$ for $N$ even, and over $Spin(N)$ for $N$ odd. Here $Pin(N)$ and $Spin(N)$ are the two-fold covering groups of $O(N)$ and $SO(N)$ respectively.

1.2. Hasegawa’s results. Let now $V = \mathbb{C}^Nl = \mathbb{C}^N \otimes \mathbb{C}^l$. Then we have obviously commuting actions of $O(N)$ and $O(l)$ on $V$, acting on the corresponding tensor factors.

**Theorem 1.1.** (Hasegawa [8]) The algebras generated by the actions of $O(N)$ and $O(l)$ on $Cl(Nl)$ are each others commutant.

It is easy to check that $Cl(Nl) \cong Cl(N)^\otimes l \cong Cl(l)^\otimes N$ as vector spaces. This strongly suggests a relationship between the commutant of the action of $Spin(N)$ on the $l$-fold tensor product of its spinor representation, and the group $Spin(l)$. Observations to this extent have been made at the combinatorial level in several papers before, e.g. [1], [2]. However, the precise result we need is a little bit more subtle and does not immediately follow from the results above. In particular, in our context there are nontrivial distinctions between the odd and even-dimensional cases which do not occur in [8].

1.3. Some elementary lemmas. Let $f_m = e_1e_2 \ldots e_m \in Cl(m)$ for $m \in \mathbb{N}$. Moreover, consider the map

$$\Phi : 1 \otimes \ldots \otimes e_i \otimes \ldots \otimes 1 \in Cl(N)^\otimes l \mapsto \begin{cases} f_{(j-1)N}e_{(j-1)N+i} & \text{if } j \text{ is odd,} \\ f_{jN}e_{(j-1)N+i} & \text{if } j \text{ is even,} \end{cases}$$

where $e_i$ is in the $j$-th factor of $Cl(N)^\otimes l$. Then we have the following easy lemma:

**Lemma 1.2.** (a) $f_me_i = (-1)^me_if_m$ for $i > m$, and $f_me_i = -(-1)^me_if_m$ for $i \leq m$.

(b) If $N$ is even, the map $\Phi$ above extends to an algebra and $O(N)$-module isomorphism between $Cl(N)^\otimes l$ and $Cl(Nl)$, and $\Phi(f_N \otimes f_N) = (-1)^{N(N-1)/2}f_{2N}$.

(c) For $N$ odd, the map $\Phi$ defines an embedding of $Cl_{ev}(N)^\otimes l$ into $Cl(Nl)$. It maps the element $e_re_s$, in the $j$-th factor of $Cl_{ev}(N)^\otimes l$, to $e_{(j-1)N+r}e_{(j-1)N+s}$.

**Proof.** Part (a) is straightforward. For part (b), one first checks that the map $\Phi$ indeed defines a nonzero algebra homomorphism. This is straightforward. As both source and target algebras are simple and have the same dimension, $\Phi$ is an isomorphism. As $\Phi$ is an $O(N)$
module morphism on the linear span of the generators, and $O(N)$ acts via algebra automorphisms, $\Phi$ is also an $O(N)$ morphism. The second part of (b) is checked easily using (a) and the definition of $\Phi$, 1.1. Part (c) again is straightforward.

Recall that the Lie algebra $so_l$ is isomorphic to the subset of $l \times l$ matrices spanned by $L_{rs} = E_{rs} - E_{sr}$, $1 \leq r < s \leq l$, where the $E_{rs}$ are matrix units. It can also be defined via generators $L_1, L_2, \ldots, L_{l-1}$ and relations $[L_i, [L_i, L_{i\pm 1}]] = -L_{i\pm 1}$ and $[L_i, L_j] = 0$ for $|i-j| > 1$. Indeed, it is easy to check that these relations are satisfied for $L_i = L_{i,i+1}$. Also observe that one can replace $-L_{i\pm 1}$ by $L_{i\pm 1}$ on the right hand side of the first relation after substituting $L_i$ by $\sqrt{-1}L_i$. Let now $N = 2k$ be even. We define the elements $C_{rs} \in Cl(N)$ by $C_{rs} = \frac{1}{2} \sum_i e_{(r-1)N+i} e_{(s-1)N+i}$ and $C'_{rs} = \frac{1}{2} \sum_i e_{(r-1)N+i} e_{(s-1)N+i}$ for $1 \leq r < s \leq l$. Then we have

**Lemma 1.3.** The elements $C_{rs}$ and $C'_{rs}$ satisfy the commutation relations of the generators of the Lie algebra $so_l$.

**Proof.** If for indices $p, q, r, s$ the set $\{r, s\} \cap \{p, q\}$ is empty or has two elements $[L_{rs}, L_{pq}] = 0$. Otherwise, if, say $s = p$, we get $[L_{rs}, L_{sq}] = L_{rq}$. But then we also have $[C_{rs}, C'_{sq}] = \frac{1}{4} \sum_{i \neq j} e_{(r-1)N+i} e_{(s-1)N+i} e_{(q-1)N+j} e_{(1-N+j)N+i} - e_{(s-1)N+j} e_{(q-1)N+j} e_{(r-1)N+i} e_{(1-N+i)N+i} + \frac{1}{4} \sum_i 2e_{(r-1)N+i} e_{(q-1)N+j}.$

One checks that the first sum is equal to $0$, and the second one is equal to $C_{rq}$, which is the required relation. The proof for the $C'_{rs}$ goes the same way.

We shall need the precise preimages of $C_{12}$ and $C_{23}$ under the isomorphism $\Phi$. It follows from Lemma 1.2 that they are given by

$$\Phi^{-1}(C_{12}) = (-1)^{N(N-1)/2} \sum_{i=1}^{N} e_i f_N \otimes e_i f_N \otimes 1 \quad \text{and} \quad \Phi^{-1}(C_{23}) = \sum_{i=1}^{N} 1 \otimes e_i \otimes e_i.$$  

Similarly, the elements $\Phi^{-1}(C'_{12})$ and $\Phi^{-1}(C'_{23})$ are given by the same sums, now only going until $N-1$.

**Corollary 1.4.** The elements $\Phi^{-1}(C_{12})$ and $\Phi^{-1}(C_{23})$ are in $\text{End}_{\text{Pin}(N)}(S^{\otimes 3})$, and the elements $\Phi^{-1}(C'_{12})$ and $\Phi^{-1}(C'_{23})$ are in $\text{End}_{\text{Pin}(N-1)}(\hat{S}^{\otimes 3})$, where $\hat{S}$ is $S$ viewed as a $\text{Pin}(N-1)$-module.

**Proof.** As $g \in O(N)$ fixes $\sum e_i \otimes e_i$, viewed as an element in $V^{\otimes 2}$, one deduces that conjugation of $\Phi^{-1}(C_{23})$ by a lift of $g$ in $\text{Pin}(N)$ leaves it invariant. The other statements follow similarly.
1.4. Eigenvalues of $C$. It remains to determine the structure of the representation of $so_l$ in $\text{Cl}(Nl)$. For this, we define the elements $C$, $C'$ and $\tilde{C}_m$ in $\text{Cl}(N)^{\otimes 2}$ by

\begin{equation}
C = \frac{1}{2} \sum e_i \otimes e_i,
\end{equation}

\begin{equation}
\tilde{C}_m = m! \sum_{i_1 < i_2 < \ldots < i_m} e_{i_1} e_{i_2} \ldots e_{i_m} \otimes e_{i_1} e_{i_2} \ldots e_{i_m},
\end{equation}

and $C'$ is defined as $C$, but with the summation only going until $N - 1$. Observe that $\tilde{C}_1 = 2C$. Moreover, observe that the elements $e_i \otimes e_i$ and $e_j \otimes e_j$ commute also for $i \neq j$. We will also need the polynomials $P_m(N, x)$ defined inductively by $P_0(N, x) = 1$, $P_1(N, x) = x$ and

\begin{equation}
P_{m+1}(N, x) = xP_m(N, x) + m(N + 1 - m)P_{m-1}(N, x).
\end{equation}

Then we have

**Proposition 1.5.** The element $C \in \text{Cl}(N)^{\otimes 2}$ has the eigenvalues $iN/2, i(N/2 - 1), ..., i(1 - N/2), -iN/2$. The same statement holds for the element $C'$, with $N$ replaced by $N - 1$.

**Proof.** Let us first prove the following recursion relation:

\begin{equation}
\tilde{C}_1 \tilde{C}_m = \tilde{C}_{m+1} + m(N + 1 - m)\tilde{C}_{m-1}.
\end{equation}

Observe that if we define $y_j = e_j \otimes e_j$, then $y_j y_i = y_i y_j$ and $y_i^2 = 1$. Moreover, $\tilde{C}_m = m! \sum_{i_1 < i_2 < \ldots < i_m \leq N} y_{i_1} y_{i_2} \ldots y_{i_m}$. Now if one multiplies a monomial $y_{i_1} y_{i_2} ... y_{i_m}$ by $y_i$, we will obtain a monomial with $m + 1$ factors $i \neq \{i_j, 1 \leq j \leq m\}$, and one with $m - 1$ factors otherwise. By symmetry $\tilde{C}_1 \tilde{C}_m$ is a linear combination of $\tilde{C}_{m+1}$ and $\tilde{C}_{m-1}$. It remains to calculate the coefficients of the leading terms $y_1 y_2 ... y_{m+1}$, which is easy.

Comparing 1.5 with 1.4, we see that $\tilde{C}_m = P_m(N, \tilde{C}_1)$ for $m \leq N$. As $\tilde{C}_{N+1} = 0$, it follows that $P_{N+1}(N, \tilde{C}_1) = 0$. As moreover $\tilde{C}_m$ has degree $m$ as a polynomial in the $y_i$s, the elements $\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_N$ are linearly independent. It follows that $P_{N+1}(N, x)$ is the characteristic polynomial of $\tilde{C}$. The statement now follows from Lemma 1.7, which will be proved in the next subsection. The proof for $C'$ goes exactly the same way, with $C'$ defined as in (1.3) with the indices only going until $N - 1$.

1.5. Structure coefficients. This subsection serves to calculate the eigenvalues of the polynomials $P_{N+1}(N, x)$. Moreover, we do some additional calculations which are useful for an explicit description of $\text{End}(S_{\text{pin}}(N)) (S^{\otimes l})$ also in the quantum case. All of this is obtained in a fairly straightforward way from the representation theory of $sl_2$, which is well-known (see e.g. [10]). Presumably, most of the results in this section are known to experts.

Let $H, E, F$ be the usual generators of the Lie algebra $sl_2$, and let $V_N$ be its $(N + 1)$-dimensional simple representation. It can be defined via a basis $\{e_0, e_1, ... e_N\}$ of eigenvectors of $H$ which satisfies

\begin{equation}
E.w_r = (N - r + 1)e_{r-1}, \quad H.e_r = (N - 2r)e_r, \quad F.e_r = (r + 1)e_{r+1}.
\end{equation}
The following lemma is well-known and easy to check (e.g. (a) follows from the fact that $E - F$ is conjugate to $iH$).

**Lemma 1.6.** (a) The element $E - F$ has eigenvalues $(N - 2r)i$, $0 \leq r \leq N$ in the representation $V_N$.

(b) The elements $iH/2$ and $(E \pm F)/2$ satisfy the relations of the generators $L_{rs}$ of the Lie algebra $so_3$ for $1 \leq r < s \leq 3$.

**Lemma 1.7.** (a) The polynomial $P_{N+1}(N, x)$ has the roots $(N - 2r)i$, $0 \leq r \leq N$.

(b) Let $x = (x(\lambda)_r)$ and $y = (y(\lambda)_r)$ be the right and left eigenvectors of $E - F$ for the eigenvalue $\lambda$, with respect to the basis $(e_r)$ and normalized by $x_0 = 1 = y_0$. Then

$$x(\lambda)_r = \frac{(N - r)!P_r(\lambda)}{N!} \quad \text{and} \quad y(\lambda)_r = \frac{P_r(\lambda)}{r!}.$$

**Proof.** Writing $E - F$ as a matrix given by 4.8, we obtain from $(E - F)x(\lambda) = \lambda x(\lambda)$ the recursion relation $x_0 = 1$, $x_1 = \lambda/N$ and

$$x_{r+1} = \frac{1}{N - r}(\lambda x_r + r x_{r-1}).$$

Similarly, one obtains from $y^t(E - F) = \lambda y^t$ the recursion relation $y_0 = 1$, $y_1 = \lambda$ and $y_{r+1} = (-\lambda y + (k + 1 - r)y_{r-1})/(r + 1)$. Comparing this with the recursion relation 1.4, one can easily check claim (b). Moreover, we obtain from the last coordinate in the equation $(E - F)x(\lambda) = \lambda x(\lambda)$ that $\lambda x_N(\lambda) + N x_{N-1}(\lambda) = 0$. Hence

$$0 = (\lambda P_N(N, \lambda) + N \cdot 1P_{N-1}(N, \lambda))/N! = P_{N+1}(N, \lambda)/N!$$

for any eigenvalue $\lambda$ of $E - F$. This together with Lemma 1.6 implies statement (a).

**Proposition 1.8.** (a) Let $N$ be even. Then we obtain a representation of $so_l$ in $\text{End}_{\text{Pin}(N)} S^{\otimes l}$ by mapping the element $L_{rs}$ to the inverse image of $C_{rs}$ under the isomorphism $\Phi$. For $l = 3$, it contains an irreducible $(N + 1)$-dimensional representation of $so_3$.

(b) If $N$ is odd, we obtain a representation of the subalgebra of the universal enveloping algebra $Uso_3$ generated by the elements $L_{rs}^2$, $1 \leq r < s \leq l$, by mapping these generators to the inverse images of the elements $(C_{rs})^2$ in $\text{End}_{\text{Spin}(N)} S^{\otimes l}$. For $l = 3$, it contains an irreducible $(N + 1)/2$-dimensional representation of this subalgebra.

**Proof.** If $N$ is even, we have $\text{End}(S^{\otimes l}) \cong Cl(N)^{\otimes l}$, and $\text{End}_{\text{Pin}(N)} (S^{\otimes l}) \cong Cl(N)^{\otimes l}$, the component of $Cl(N)^{\otimes l} \cong Cl(N)$ on which the multiplicative action of $O(N)$ is trivial. By construction, the elements $C_{rs}$ are fixed by the action of $O(N)$, and they define a representation of $so_l$ by Lemma 1.3. By Prop. 1.5 and Lemma 4.8, the largest eigenvalue of the image of $H$ is $N$, which shows the existence of an irreducible $(N + 1)$-dimensional representation of $so_3 \cong sl_2$ in $Cl(N)^{\otimes 3}$.

For $N$ odd, we obtain a representation of $so_l$ in $\text{End}_{\text{Pin}(N)} (S^{\otimes l})$ by Lemma 1.3 and its corollary. Up to a common sign, the elements $\Phi^{-1}((C_{rs})^2)$ coincide with the elements $\Phi^{-1}(C_{rs}^2) \in Cl_{ev}(N)^{\otimes l} \cong \text{End}(S^{\otimes l})$. Moreover, it is easy to see that $(E - F)^2$ and $H^2$ leave the spans of the even and of the odd basis vectors invariant, and that these are irreducible submodules.
Definition 1.9. (a) We define the algebra $U(l, k)$ via generators $D_1, D_2, \ldots, D_{l-1}$ and relations 
\[ [D_i, [D_i, D_{i \pm 1}]] = D_{i \pm 1}, [D_i, D_j] = 0 \text{ if } |i - j| > 1 \text{ and by } \prod_{j=1}^{k} (D_i - j) = 0. \]
(b) We define the algebra $Uo(l, k)$ as a subalgebra, generated by $D_1^2, D_2^2, \ldots, D_{l-1}^2$, of an algebra with generators $D_1, D_2, \ldots, D_{l-1}$, where the $D_i$ satisfy the relations $[D_i, [D_i, D_{i \pm 1}]] = D_{i \pm 1}, [D_i, D_j] = 0$ if $|i - j| > 1$ and by $\prod_{j=-k+1}^{k} (D_i - j - 1/2) = 0$.

Observe that the algebra $U(l, k)$ is a quotient of the universal enveloping algebra $Uso_l$ of the Lie algebra $so_l$, while $Uo(l, k)$ is a quotient of a subalgebra of $Uso_l$.

Corollary 1.10. (a) If $N = 2k$ is even, the image of $Uso_l$ in $End_{Pin(2k)}(S^{\otimes l})$ is a quotient of $U(l, k)$.
(b) If $N = 2k - 1$, the image of $Uso_l$ in $End_{Spin(2k-1)}(S^{\otimes l})$ is a quotient of $Uo(l, k)$.

2. Lie algebras and quantum groups

2.1. Quantum groups. We list some basic information about quantum groups (see e.g. [12], [19]). Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra given by a Coxeter graph $X$ with $k$ vertices, with generators $e_i$ and $f_i$, $1 \leq i \leq k$; eventually, we will only be interested in orthogonal Lie algebras. We denote the simple roots by $\alpha_i$, $i = 1, \ldots, k$. Fix an invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$, and define $\tilde{\alpha}_i = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$. If all the roots have the same length, we assume $\langle \cdot, \cdot \rangle$ to be normalized such that $\tilde{\alpha}_i = \alpha_i$. If $\langle \cdot, \cdot \rangle$ is nondegenerate, we define the fundamental weights $\Lambda_j$ by $\langle \tilde{\alpha}_i, \Lambda_j \rangle = \delta_{ij}$. If $\alpha \in \mathfrak{h}^*$, the reflection $s_\alpha$ on $\mathfrak{h}^*$ is defined by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha$. We denote by $U = U_q \mathfrak{g}$ the Drinfeld-Jimbo quantum group corresponding to the semisimple Lie algebra $\mathfrak{g}$. It is well-known that for $q$ not a root of unity, the representation theory of $U$ is essentially the same as the one of $\mathfrak{g}$, i.e. same labeling set of simple representations, character formulas etc. So we will sometimes state results only for $\mathfrak{g}$ when its generalization to $U$ is obvious.

2.2. Gradation via Lie subalgebra. Let $\mathfrak{g}_0$ be a Lie subalgebra of $\mathfrak{g}$ corresponding to the graph obtained from $X$ by removing the vertex labeled by 1. If $\lambda = \sum_{i=1}^{N} a_i \Lambda_i$ is a weight of $\mathfrak{g}$, we denote by $\hat{\lambda} = \sum_{i=2}^{N} a_i \Lambda_i$ the corresponding weight of $\mathfrak{g}_0$ (after obvious identifications of the fundamental weights of $\mathfrak{g}_0$ with a subset of fundamental weights of $\mathfrak{g}$). Let $V$ be a finite dimensional module of $\mathfrak{g}$ such that $\langle \Lambda, \Lambda \rangle = c$, a constant for all highest weights $\Lambda$ of $V$; here $\Lambda_1$ is the fundamental weight corresponding to 1. We denote by $V[0]$ the $\mathfrak{g}_0$-module generated by the highest weight vectors of $V$. More generally, we define the level $i$ subspaces $V^{\otimes n}[i]$ for tensor powers of $V$ and for $i = 0, 1, 2, \ldots$ by 
\[ V^{\otimes n}[i] = \text{span}\{V^{\otimes n}[\mu], \langle n\Lambda - \mu, \Lambda_1 \rangle = i\}. \]
It is easy to see that $V^{\otimes n}[0] = (V[0])^{\otimes n}$ for all $n \in \mathbb{N}$. Conversely, we say that a weight $\mu$ has level $i$ in $V^{\otimes n}$ if $V^{\otimes n}[\mu] \subset V^{\otimes n}[i]$; in this case we denote the level of $\mu$ by $lev_n(\mu)$ or just $lev(\mu)$ if no confusion arises.

Lemma 2.1. Let $V$ be a $\mathfrak{g}$-module as just described. Then
(a) $V^{\otimes n}[0] \cong V^{\otimes n}_\Lambda$ as a $\mathfrak{g}_0$-module.
Moreover, the multiplication in $B_\mathfrak{g}$ extends easily to our slightly more general setting. Part (b) is an easy consequence of part (a).

(b) Let $W$ be the $\mathfrak{g}$-module generated by $V^\otimes n[0]$. Then $\text{End}_U(W) \cong \text{End}_{\mathfrak{g}_0}(V^\otimes n[0])$. In particular, $\text{mult}_{\mathfrak{g}_\mu}(V^\otimes n) = \text{mult}_{\mathfrak{g}_\mu}(V^\otimes n[0])$ for any weight $\mu$ with $\text{lev}_{\mathfrak{g}_\mu}(\mu) = 0$.

Proof. Part (a) was already shown in [30], Lemma 1.1 if $V$ is irreducible. The proof carries over easily to our slightly more general setting. Part (b) is an easy consequence of part (a).

2.3. Traces and contractions. The following material can be found in e.g. [15], [27] and [21], Section 1.4. Let $W$ be a $U$-module, and let $a \in \text{End}_U(W)$. Then the categorical trace or $q$-trace $\text{Tr}_q(a)$ is given by $\text{Tr}_q(a) = Tr(q^a)$; here $Tr$ is the ordinary trace on $\text{End}(W)$, and $q^a$ acts on the weight vector $w \in W$ with weight $\mu$ by the scalar $q^{(\mu, \mu)}$. Let $W = V_\lambda$ be an irreducible module with highest weight $\lambda$. Using the notation $[n] = (q^n - q^{-n})/(q - q^{-1})$, we can explicitly write the $q$-dimension as

$$\dim_q V_\lambda = \prod_{\alpha > 0} \frac{[\langle \lambda + \rho, \alpha \rangle]}{[\langle \rho, \alpha \rangle]}.$$  

In particular, if $e$ is a minimal idempotent in $\text{End}_U(W)$ projecting onto an irreducible submodule $\cong V_\lambda$ of $W$, we have $\text{Tr}_q(e) = \dim_q V_\lambda$. The normalized trace $\text{tr}_q$ is defined by $\text{tr}_q = (1/\dim_q W)\text{Tr}_q$.

Let $A \subset B$ be finite-dimensional semisimple algebras with a nondegenerate normalized trace $\text{tr}$ on $B$ such that also its restriction to $A$ is nondegenerate; nondegenerate here means that the bilinear form $\langle b_1, b_2 \rangle = \text{tr}(b_1b_2)$ is nondegenerate. Then the orthogonal projection from $B$ onto $A$ with respect to this bilinear form is usually called the trace preserving conditional expectation $\varepsilon_A$. Its values are uniquely determined by $\text{tr}(ae(b)) = \text{tr}(ab)$ for all $a \in A$ and all $b \in B$.

In the setting above, one can define an algebra extension $B_1$ of $B$ with respect to the inclusion $A \subset B$, Jones’ basic construction, as follows: It is generated by $B$, acting on itself via left multiplication and the projection $\varepsilon_A$ coming from $\varepsilon_A$, viewed as a linear operator on $B$. It is well-known that $B_1$ is isomorphic as a vector space to $Be_A B$ (here we identify $B$ with $\Lambda_B$, the algebra of linear operators on $B$ coming from left-multiplication by elements of $B$). Moreover, the multiplication in $Be_A B$ is defined by

$$\text{(b)} \ e_A b_2\text{(b)}e_A b_1 = b_1 \varepsilon_A (b_2b_3) e_A b_1.$$  

Assume now that the trivial representation $1$ appears in the second tensor power of the representation $V$ with multiplicity $1$, and it only appears in even tensor powers of $V$. Decomposing $V^\otimes n$ as a direct sum of simple $U = U_q\mathfrak{g}$-modules, we define $V^\otimes n_{\text{old}}$ to be the direct sum of those simple modules which already appeared in $V^\otimes n-2$. By semisimplicity and definition of $V^\otimes n_{\text{old}}$, we have a unique decomposition $V^\otimes n = V^\otimes n_{\text{old}} \oplus V^\otimes n_{\text{new}}$ in these cases. The following result has already more or less appeared before in various publications; in the form below, see e.g. [30], Prop. 4.10.

**Proposition 2.2.** Let $C_n = \text{End}_U(V^\otimes n)$. Then the algebra $\text{End}_U(V^\otimes n_{\text{old}})$ is isomorphic to Jones’ basic construction for $C_{n-1} \subset C_n$. In particular, it is isomorphic as a vector space to $(C_n \otimes 1)p_n(C_n \otimes 1)$ where $p$ is the projection onto $1 \subset V^\otimes 2$, and $p_n = 1_{n-1} \otimes p \in \text{End}_U(V^\otimes n_{\text{old}})$.
We also need the following well-known property of categorical traces.

**Lemma 2.3.** If $V,W$ are $U$-modules with $V$ being irreducible, $a \in \text{End}_U(V^\otimes 2)$ and $b \in \text{End}_U(W \otimes V)$, then $\text{tr}((b \otimes 1_V)(1_W \otimes a)) = \text{tr}(b)\text{tr}(a)$.

**Proof.** This follows from the categorical definition of $\text{tr}$ and is well-known. E.g. the proof of [21], Prop. 1.4(c) can easily be modified to prove the claim.

### 3. Spinors for quantum groups

#### 3.1. Roots and weights for orthogonal Lie groups.

For information about roots and weights, see e.g. [10],[14], and about spinor representations, see e.g. [31]. Let $\{\epsilon_i, 1 \leq i \leq k\}$ be the usual standard basis of $\mathbb{R}^k$. We represent the simple roots $(\alpha_i), i = 1, 2, \ldots, k$ of Lie types $B_k$ and $D_k$ as usual by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < k$, and as $\alpha_k = \epsilon_k$ for Lie type $B_k$ and as $\alpha_k = \epsilon_{k-1} + \epsilon_k$ for Lie type $D_k$. With these notations, the weight lattice is given by $\mathbb{Z}^k \cup (\mathbb{Z}^k + \epsilon)$, where $\epsilon = (1/2, \ldots, 1/2)$. The irreducible representations of the corresponding Lie algebras are labeled by the dominant weights $\lambda = (\lambda_i)$ which can be explicitly described as the set of all weights $\lambda$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ for type $B_k$ resp. $\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_k|$ for type $D_k$.

#### 3.2. Pin groups.

The unique compact connected and simply connected Lie group corresponding to the root systems $B_k$ and $D_k$ is the spin group $Spin(N)$ with $N = 2k + 1$ for type $B_k$ and with $N = 2k$ for type $D_k$. It is a 2-fold covering of the orthogonal group $SO(N)$. As usual, we denote the corresponding covering group of $O(N)$ by $Pin(N)$. We embed $g \in O(2k) \to (g,\epsilon) \in O(2k) \times \mathbb{Z}/2 \subset SO(2k + 1)$, where the sign is chosen so that we obtain determinant one. This embedding carries over to an embedding of $Pin(2k)$ into $Spin(2k+1)$, i.e. we can consider $Pin(2k)$ as a subgroup of $Spin(2k+1)$. As already indicated in the previous section, it will be convenient to consider $Pin(2k)$ instead of $Spin(2k)$.

Algebraically, $Pin(2k)$ is a semidirect product of $Spin(2k)$ with $\mathbb{Z}/2$. On the Lie algebra level, the $\mathbb{Z}/2$-action is given by interchanging the generators labeled by $k-1$ and $k$, i.e. the generators belonging to the endpoints of the $D_k$ graph next to its triple vertex. This $\mathbb{Z}/2$-action induces a linear map $\lambda \mapsto \bar{\lambda}$ on the weight space determined by permuting the roots $\alpha_{k-1}$ and $\alpha_k$, and leaving the other simple roots fixed. It is easy to check that if $\lambda = (\lambda_1, \ldots, \lambda_k)$, then $\bar{\lambda} = (\lambda_1, \ldots, \lambda_{k-1}, -\lambda_k)$. The connection between irreducible $Spin(2k)$ and irreducible $Pin(2k)$-modules is described easily as follows:

- If $\lambda \neq \bar{\lambda}$ (i.e. $\lambda_k \neq 0$), then there exists a unique irreducible $Pin(2k)$-module whose restriction to $Spin(2k)$ decomposes as a direct sum of highest weight modules labeled by $\lambda$ and $\bar{\lambda}$. We shall denote this $Pin(2k)$-module by $V_\lambda$ with $\lambda$ the dominant weight satisfying $\lambda_k > 0$.

- If $\lambda = \bar{\lambda}$, there exist exactly two irreducible nonisomorphic $Pin(2k)$-modules, denoted by $V_\lambda$ and $V_\lambda^\dagger$ whose restriction to $Spin(2k)$ is isomorphic to the highest weight module labeled by $\lambda$. Observe that in this case $\lambda$ can be identified with a Young diagram and one takes for $\lambda^\dagger$ the Young diagram with the same columns as $\lambda$ except that the first one now has $2k - \lambda_1^\dagger$
3.3. Spinors. Let $S$ be the spinor module as constructed via the Clifford algebra in Section 1. In the odd-dimensional case, Lie type $B_k$ it is the irreducible representation with highest weight $\Lambda_k = \varepsilon$, the fundamental weight dual to $\alpha_k$. In the even-dimensional case, Lie type $D_k$, the module $S$ remains irreducible as a $Pin(2k)$-module, but decomposes into the direct sum of two irreducible $Spin(2k)$-modules whose highest weights are the fundamental weights $\Lambda_{k-1} = \varepsilon - \epsilon_k$ and $\Lambda_k = \varepsilon$.

The module $S$ has the following properties: Its weights are given by \{
\omega, \omega = \tfrac{1}{2} \sum_{i=1}^k \pm \epsilon_i \},
which holds for $S$ being viewed as a $Spin(2k+1)$ module as well as a $Pin(2k)$-module. The following tensor product rules for spinor groups are well-known and follow easily from general theory. More specialized treatments can also be found in e.g. [1], [2] to name but a few. If $V_\lambda$ is an irreducible module with highest weight $\lambda$ for Lie type $B_k$, then

$$V_\lambda \otimes S \cong \bigoplus_{\mu} V_\mu,$$

where the summation goes over all dominant weights $\mu = \lambda + \omega$ with $\omega$ a weight of $V$. For Lie type $D_k$, we have the following modification:

$$V_\lambda \otimes S \cong V_{\lambda^\dagger} \otimes V \cong \bigoplus_{\mu} V_\mu \oplus \bigoplus_{\mu=\mu^\dagger} V_{\mu^\dagger},$$

where the summation goes over all dominant weights $\mu = \lambda + \omega$ with $\omega$ a weight of $S$ and with $\mu_k \geq 0$.

In the following we use Young diagram notation for labeling the irreducible representations of the orthogonal groups, with the convention for $\lambda^\dagger$ as described above. In particular we will write $[1^r]$ for the Young diagram with $r$ boxes in one column. We will need the following straightforward examples, which are elevated to the rank of a lemma.

**Lemma 3.1.** We have the following decompositions:

(a) $S^{\otimes 2} \cong \bigoplus_{s=0}^k V_{[1^k-s]}$ for Lie type $B_k$,

(b) $S^{\otimes 2} \cong \bigoplus_{s=-k}^k V_{[1^k-s]}$ for Lie type $D_k$,

(c) $S^{\otimes 3} \cong \bigoplus_{r=0}^k m_r V_{[r+1]}$, where the multiplicity $m_r$ is equal to $r+1$ for Lie type $B_k$, and it is equal to $2r+1$ for Lie type $D_k$,

(d) For $Pin(2) \cong O(2)$ (see discussion in Section 3.6) and $Spin(3) \cong SU(2)$, $S^{\otimes n}_{\text{new}}$ consists of one irreducible representation, except for $n = 2$ in the $Pin(2)$ case, where it is the direct sum of two irreducible representations.

3.4. $q$-Dimensions. Recall that the $q$-dimension of a representation is given by Eq. 2.1. We need more explicit formulas for certain representations. We use the notation $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. Let $U$ be equal to $U_q so_N$ (for $N$ odd) or the semidirect product of
$U_q so_N$ with \( \mathbb{Z}/2 \) as in Section 3.6 for \( N \) even. It is well-known that for the \( U \) module \( V_{[r]} \) with highest weight \((1, \ldots, 1, 0, \ldots, 0)\) (with \( r \) 1s), we have

\[
\dim_q V_{[r]} = d(r, N) = \binom{N-1}{r}_q + \binom{N-1}{r-1}_q = \left(\frac{q^{N-r} + q^{-r-k}}{q^k + q^{-k}}\right),
\]

where \( \binom{n}{m}_q = \frac{[n]_q!/[m]_q![n-m]_q!}{q^{n-m}} \), and where \([n]_q! = \prod_{i=1}^n [i]_q\). This can be derived from the character formulas (see Eq. 2.1) or it can be read of as a special case from the formulas in Section 5 of [28] for \( \lambda = [1^r] \). Also observe that the dimension \( \dim_q S = [2^k]_{qq} \) of the spinor module \( S \) is given by

\[
[2^k]_{qq} = \begin{cases} (q^{1/2} + q^{-1/2})(q^{3/2} + q^{-3/2}) \cdots (q^{k-1/2} + q^{1-k/2}) & \text{if } N = 2k + 1 \text{ is odd}, \\ 2(q + q^{-1})(q^2 + q^{-2}) \cdots (q^{k-1} + q^{1-k}) & \text{if } N = 2k \text{ is even}, \end{cases}
\]

3.5. Symmetric representations of \( C_3 \). The calculation of the structure coefficients will be significantly simplified by the existence of a certain involutive antihomomorphism \( T \) on \( C_3 \). In fact, it can be defined for all \( C_n, n \in \mathbb{N} \). It satisfies the involutive property \( (cT)^T = c \) for all \( c \in C_{n}, n \in \mathbb{N} \) and the functorial property

\[
(c_1 \otimes c_2)^T = c_1^T \otimes c_2^T \quad c_1 \in C_{n_1}, c_2 \in C_{n_2}.
\]

The existence of this antihomomorphism is a consequence of Kashiwara’s inner product on modules of quantum groups (see e.g. [30], Section 1.4 for details).

Lemma 3.2. For any simple module of \( \text{End}_U(S_{\text{old}}^{\otimes 3}) \subset C_3 \), we can find a basis \( (v_r) \) of simultaneous eigenvectors of \( C_2 \otimes 1 \) for which each element in \( 1 \otimes C_2 \) is given by a symmetric matrix. In particular, if \( p^{(k)} \) is the projection onto \( 1 \subset S^{\otimes 2} \), we can choose the basis such that \( p_2^{(k)} = 1 \otimes p^{(k)} \) is given by the matrix

\[
(\sqrt{\dim_q V_{[1]}}, \sqrt{\dim_q V_{[1]}/\dim_q S})_{ij}.
\]

Proof. The first statement is a special case of e.g. [30], Lemma 1.9. In our special case of \( C_3 \) path basis vectors would just be simultaneous eigenvectors for the projections \( p^{(s)}_1 \), with \( 0 \leq |s| \leq k \), and the argument of that Lemma works for any element \( A_2 \) in \( 1 \otimes C_2 \). The matrix of \( p_2^{(k)} \) with respect to this basis can be calculated by observing that it coincides with the projection of Jones’ basic construction for \( C_1 \subset C_2 \) under the isomorphism stated in Prop. 2.2. Indeed, matrices for such projections have been calculated in [24] or [22] in terms of the weight vectors of the trace; for the latter, see Sections 2.3 and 3.4. The second statement follows from this.

3.6. Centralizers. In the following we denote by \( U \) either the Drinfeld-Jimbo \( q \)-deformation \( U_q so_{2k+1} \) of the universal enveloping algebra of \( so_{2k+1} \) or the semidirect product of \( U_q so_{2k} \) with \( \mathbb{Z}/2 \) as described in Section 3.2; the action described there carries over to the quantum
group level in the obvious way. So, in particular, the action of \( t \) on a weight vector \( v_\omega \) of the spinor representation \( S \) or the vector representation \( V \) in the usual normalization is given by

\[
(3.3) \quad t : v_\omega \mapsto \bar{v}_\omega.
\]

We define the algebras \( C^{(0)}_n = \mathbb{C} \) and \( C^{(N)}_n = \text{End}_U(S^{\otimes n}) \), where \( S \) is the spinor representation of type \( B_k \) (for \( N = 2k + 1 \) odd) or \( D_N \) (for \( N = 2k \) even). We will give an inductive procedure how to determine the structure of these algebras. To get the induction started, let us review the cases for \( Pin(2) \) and \( Spin(3) \), which are well-known:

**Pin(2):** It is easy to check that \( Pin(2) \) is isomorphic to the orthogonal group \( O(2) \): The two-fold covering of a circle is again a circle, which defines an isomorphism between \( Spin(2) \) and \( SO(2) \). This isomorphism extends to one between \( Pin(2) \) and \( O(2) \). Moreover, the spinor representation can be identified with the usual two-dimensional representation of \( O(2) \) under this isomorphism. It is well-known that in the group case the centralizer algebra \( C^{(2)}_n \) is isomorphic to a quotient of Brauer’s centralizer algebra \( D_n(2) \), whose simple components are labeled by Young diagrams whose first two columns contain at most two boxes. It is also well-known that we do not have a quantum deformation in this case, which was shown on the categorical level in e.g. [26], Lemma 7.5. Hence we have in general that

\[
C^{(2)}_2 \cong \mathbb{C}^3 \quad \text{and} \quad C^{(2)}_{n+1} \quad \text{is isomorphic to a direct sum of Jones’ basic construction for} \quad C^{(2)}_n \subset C^{(2)}_{n-1} \quad \text{and a one-dimensional direct summand labeled by the Young diagram} \quad \nu_{n+1} \quad \text{(see [5], [29])}.
\]

**Spin(3) \cong SU(2):** Here the spinor representation \( S \) corresponds to the two-dimensional representation of \( SU(2) \). The centralizer algebras are well-understood in this case in the classical as well as in the quantum case. They are given as quotients of Hecke algebras of type \( A \), which are also known as Temperley-Lieb algebras. Again, also in the type \( B \) case, \( C^{(3)}_{n+1} \) can be determined inductively as a direct sum of Jones’ basic construction for \( C^{(3)}_n \subset C^{(3)}_{n-1} \) and a one-dimensional direct summand (see [13]).

We will need the following notations for the general induction: The element in \( C^{(N)}_n \) which projects onto the submodule \( V_{[k-s]} \subset V^{\otimes 2} \) will be denoted by \( p^{(s)} \). Observe that \( p^{(k)} \) is the projection onto the trivial representation, and that \( s \) can be negative for type \( D \) (see the end of Section 3.3). Moreover, if \( a \in C^{(N)}_2 \), we define

\[
a_i = 1_{i-1} \otimes a \otimes 1_{n-i-1} \in \text{End}_U(S^{\otimes n}) = C^{(N)}_n.
\]

**Theorem 3.3.** The structure of \( C^{(N)}_n \), as defined above, is determined for \( n = 2 \) by Lemma 3.1, and it is determined inductively for \( n > 2 \) and \( N > 0 \) by

\[
C^{(N)}_{n+1} \cong C^{(N-2)}_{n+1} \oplus C^{(N)}_n p^{(N)}_n C^{(N)}_n,
\]

as a direct sum of algebras, with the multiplicative structure as in Eq 2.2. In particular, \( C^{(N)}_n \) is generated by the elements \( a_i, \ i = 1, 2, \ldots n-1 \), with \( a \in \text{End}_U(S^{\otimes 2}) \).

**Proof.** We shall prove this theorem by induction on both \( N \) and \( n \). Observe that the theorem follows from the discussion above for \( N = 2 \) and \( N = 3 \). Also observe that the trivial
representation appears in the second tensor power of the spinor representation $S$ in general for arbitrary $N$. Hence, by Prop. 2.2, $\text{End}_{U}(S_{\text{old}}^\otimes n+1) \cong C_n^{(N)} p_n^{(N)} C_n^{(N)}$. So it suffices to show that $\text{End}_{U}(S_{\text{new}}^\otimes n+1) \cong C_{n+1}^{(N-2)}$. The statement about the generators follows by induction on $n$ and $N$.

By Lemma 2.1(a), the commutant of the action of the quantum group $U'$ of type $B_{k-1}$ resp. type $D_k$ on $S_{\text{old}}^\otimes n[0]$ is isomorphic to $C_n^{(N-2)}$. Hence it suffices to show that the $U'$-module generated by $V_{\text{new}}^\otimes n[0]$ is equal to $V_{\text{new}}^\otimes n+1$, where now $U$ is the quantum group of type $B_k$ resp. type $D_k$; indeed as every irreducible $U'$ submodule in $S_{\text{old}}^\otimes n[0]$ generates an irreducible $U$ module, we have $C_{n+1}^{(N-2)} \cong \text{End}_{U'}(S_{\text{old}}^\otimes n[0]) \cong \text{End}_{U}(S_{\text{new}}^\otimes n+1)$.

As $|\omega| \leq 1/2$ for any weight $\omega$ of $S$, it follows from the tensor product rules for $S$ (see Section 3.3) by induction that also $|\lambda_1| \leq n/2$ for any highest weight $\lambda$ in $S^\otimes n$. On the other hand, it is easy to check by induction, that any module $V_\lambda$ resp $V_{\lambda'}$ labeled by a dominant weight $\lambda$ with $|\lambda_1| \leq n/2$ does indeed occur in $S^\otimes n$: the claim is obviously true for $n = 1$, and given a dominant weight $\lambda$ with $\lambda_1 \geq 1$, we can always find a weight $\omega$ of $S$ such that $\lambda' = \lambda - \omega$ is dominant. As $V_{\lambda'} \subset V^\otimes (n-1)$ by induction assumption, we obtain $V_{\lambda'} \subset V_{\lambda'} \otimes S \subset S_{\text{new}}^\otimes n$. Hence $S_{\text{new}}^\otimes n$ is a direct sum of highest weight modules $V_\lambda$ such that $\lambda_1 = n/2$, and any irreducible submodule of $S_{\text{new}}^\otimes n$ with such a highest weight is contained in $S_{\text{new}}^\otimes n$.

As $n/2 = n(\lambda_1, \lambda)$, it follows that all these highest weight vectors are contained in $S_{\text{new}}^\otimes n[0]$. Hence $S_{\text{new}}^\otimes n$ is contained in the $U$-module generated by $S_{\text{new}}^\otimes n[0]$. The other inclusion follows from the fact that any highest weight vector in $V^\otimes n[0]$ has a weight $\lambda$ satisfying $\langle \lambda, A_1 \rangle = n/2$ (see the discussion before Lemma 2.1). Hence it is contained in $S_{\text{new}}^\otimes n$. This finishes the proof.

**Corollary 3.4.** If $N = 2k$ is even, the algebra $\text{End}_{\text{Pin}(2k)}(S_{\otimes l})$ is a quotient of $U(k, l)$, as defined in Def. 1.9. If $N = 2k + 1$, the algebra $\text{End}_{\text{Spin}(2k+1)}(S_{\otimes l})$ is a quotient of $Uo(k, l)$.

**Proof.** We have seen that the element $C$ has $N + 1$ distinct eigenvalues, and, for $N$ odd, the element $C^2$ has $(N + 1)/2$ distinct eigenvalues. Hence $C$ resp $C^2$ generates $\text{End}_{\text{Pin}(N)}(S_{\otimes 2})$ for $N$ even, resp $\text{End}_{\text{Spin}(N)}(S_{\otimes 2})$ for $N$ odd. The rest follows from Theorem 3.3.

### 4. Structure Coefficients

**4.1. Invariant elements, Clifford approach.** We have seen in Section 1 that the element $2C = \sum e_i \otimes e_i \in Cl_{\otimes 2} \cong \text{End}(S_{\otimes 2})$ generates the commutant of the action of $\text{Pin}(2k)$, where $(e_i)_i$ is an orthonormal basis. For the odd-dimensional case, it is convenient to consider the restriction of the action on the module in the last sentence to $\text{Pin}(N - 1)$. We now denote this $\text{Pin}(N - 1)$ module by $\tilde{S}$. It decomposes into a direct sum of irreducible $\text{Pin}(N - 1)$-modules $S_+ \oplus S_-$, which are isomorphic as $\text{Spin}(N - 1)$-modules. Here we take as invariant element the canonical element $C$ for the inner product of an $N - 1$-dimensional subspace of $V$. If we take the usual weight vectors as before, we can express $C$ in the form

\begin{equation}
C(v_\lambda \otimes v_\mu) = \frac{1}{2} \alpha^{(N/2)}_{\lambda, \mu} v_\lambda \otimes v_\mu + \sum_{1 \leq j < N/2, \; \lambda_j \neq \mu_j} \alpha^{(j)}_{\lambda, \mu} v_{\lambda_j} \otimes v_{\mu_j}.
\end{equation}
here $\bar{\gamma}^j$ is defined to coincide with $\gamma$ except for a sign change in the $j$-th coordinate for any weight $\gamma$ of $S$ resp. $\tilde{S}$ (here we use the notation for weights for $Pin(N)$). In the even-dimensional case, the expression for $C$ is as above, except that the sum goes until $N/2$ without the special case for $j = N/2$. It is not hard to calculate the coefficients $\alpha_{\lambda,\mu}^{(j)}$ which are equal to $\pm 1$.

In principle, this approach can be extended to the setting of quantum groups, using a $q$-Clifford algebra, which has already been studied (see [9], [6]). However, in this context, one would also have to deform the multiplication of the second tensor power of the $q$-Clifford algebra in a nontrivial way. This makes calculations cumbersome. Instead, we shall produce the $q$-analog of the invariant element $C$ by a straightforward calculation of the coefficients $a_{\lambda,\mu}^{(j)}$ in the quantum case.

4.2. Invariant elements, direct approach. As motivated in the previous subsection, we now determine a special element $C \in \text{End}(S^\otimes 2)$ by finding suitable coefficients for the expression in Eq. 4.1.

Proposition 4.1. Let $U = U_q so_{2k} \rtimes \mathbb{Z}/2$. The element $C \in \text{End}(S^\otimes 2)$, defined by

$$C(v_\mu \otimes v_\nu) = \sum_j \delta_{\mu_j, -\nu_j} (-q)^{(\nu-\mu)_1} v_{\mu j} \otimes v_{\nu j},$$

commutes with the action of $U$. Here $\{\gamma\}_{j=1}^N = \sum_{i=1}^{j-1} \gamma_i$ for any $\gamma \in \mathbb{R}^k$. Moreover $\bar{\gamma}^j$ is defined to coincide with $\gamma$ except for a sign change in the $j$-th coordinate.

Proof. It is easy to check that $C$ leaves invariant the weight spaces. So it does commute with the generators $K_i$ of $U$. Also, as the action of $C$ on $v_\mu \otimes v_\nu$ only depends on the last coordinates of $\mu$ and $\nu$ as far as whether they are equal or not, $C$ also commutes with the generator $t$ of $\mathbb{Z}/2$ in $U$ (see Eq 3.3). It remains to check the equation

$$C\Delta(X_j)(v_\mu \otimes v_\nu) = \Delta(X_j)C(v_\mu \otimes v_\nu),$$

for $X_j = E_j, F_j$ and $1 \leq j \leq k$. Recall that the coproduct is defined by

$$\Delta(X_j) = K_j^{1/2} \otimes X_j + X_j \otimes K_j^{-1/2}.$$ 

It will be convenient to use the following notations for the matrix coefficients of $C$:

$$a_{\mu,\nu}^{(j)} = C_{\mu,\nu}^{\mu,\nu} = \delta_{\mu_j, -\nu_j} (-q)^{(\nu-\mu)_1}.$$

Comparing the coefficients of the vector $v_{\mu+\epsilon_j} \otimes v_{\nu-\epsilon_{j+1}}$ in 4.10 for $X_j = E_j$, we obtain

$$q^{-\langle \nu, \alpha_j \rangle/2} a_{\mu+\alpha_j, \nu}^{(j+1)} + q^{\langle \mu, \alpha_j \rangle/2} a_{\mu, \nu+\alpha_j}^{(j)} = q^{-\langle \nu-\epsilon_{j+1}, \alpha_j \rangle} a_{\mu+\epsilon_{j+1}, \nu}^{(j+1)} + q^{\langle \mu+\epsilon_j, \alpha_j \rangle/2} a_{\mu, \nu}^{(j)}.$$

A similar equation follows if we consider the coefficients of the vector $v_{\mu-\epsilon_{j+1}, \nu+\epsilon_j}$.

Let us first consider the case $U = U_q so_{2k}$. We can write the weights and basis vectors of $S \otimes S$ as written as a vector
with two such pairs, such as e.g. 

\[ (+++,++) \]. Then we deduce the following identities from Eq 4.10:

\[
\begin{align*}
    a_{(++,+-)}^{(j+1)} &= -q^{-1} a_{(++,+-)}^{(j)}, & a_{(++,+-)}^{(j+1)} &= -q^{-1} a_{(++,+-)}^{(j)}; \\
    a_{(++,++)}^{(j+1)} &= a_{(++,++)}^{(j)}, & a_{(++,++)}^{(j+1)} &= a_{(++,++)}^{(j)}.
\end{align*}
\]

(4.5)

Indeed, e.g. the equations in the first line follow from Eq. 4.10 for the vector 

\[-(+-,++)\] by comparing the coefficients of the vectors 

\[-(+-,++)\] and 

\[-(++,++)\], see also Eq 4.4. The other equations can be derived similarly, using generators 

\[ E_{1}, E_{2}, F_{1}, F_{2} \]. Essentially, these are calculations within 

\[ U_{q}sl_{2} \], applied to tensor products of vectors of weights \( \pm 1 \) or 0.

The general case with \( U = U_{q}so_{2k} \), with \( k > 2 \) is not much more complicated. If we check the claim in Eq. 4.10 with \( \langle \epsilon_{j}, \alpha_{r} \rangle \neq 0 \), i.e. \( j \in \{ j, j+1 \} \) for \( j < k \), only the \( j \)-th and \( (j+1) \)-st coordinates of \( \mu \) and \( \nu \) are relevant for checking Eq. 4.10, up to a common multiple for both sides. We again get equations as in 4.5 from which we can determine the coefficients \( a_{(j)}^{(1)} \) by induction on \( j \), starting with \( a_{(j)}^{(1)} = \delta_{(j)} \).

**Lemma 4.2.** (a) If \( N = 2k \) even, the eigenvalues of the map \( C \) are \( [j] = (q^{j} - q^{-j})/(q - q^{-1}) \) for \( -k \leq j \leq k \). The corresponding eigenspaces are \( V_{[2j-k]} \subset S_{k}^{\otimes 2} \), see Lemma 3.1.

(b) If \( N = 2k + 1 \) is odd, \( C \) has the eigenvalues \( [j + 1/2] \) for \( -k + 1 \leq j \leq k \).

**Proof.** Let \( \mathbf{v} = \sum_{\lambda}(q^{-\lambda})^{(k)}v_{\lambda} \otimes v_{-\lambda} \), where \( \rho = (k - i)_{i} \), and \( \varepsilon \) is the highest weight vector of \( S \). Then the \( v_{\lambda} \otimes v_{-\lambda} \) coordinate of \( C \mathbf{v} \) is given by

\[
\sum_{j} a_{(j)}^{(1)} v_{(j)}^{(k)}(\rho) = (q^{-\lambda})^{(k)} \sum_{j} q^{(j)}(\rho) = \delta_{\mu,\nu} \sum_{j} q^{(j)}(\rho) = \delta_{\mu,\nu} \sum_{j} q^{(j)}(\rho),
\]

where we used the fact that \( \langle \lambda, \rho \rangle + j - 1 \) is even. Hence \( \mathbf{v} \) is an eigenvector of \( C \) if we can show that the set of exponents of \( q \), namely \( \{ 2\lambda_{j}(k - j) - \{2\lambda\}_{j-1}, 1 \leq j \leq k \} \) coincides with the set of numbers \( k + 1 - 2r, 1 \leq r \leq k \). This is easily shown by induction on \( k \), by using the induction assumption for the weight \( \mu = (\lambda_{2}, \lambda_{2}, ..., \lambda_{k}) \) and observing that for \( j = 1 \) we get \( \pm(k - 1) \) depending on the sign of \( \lambda_{1} \). This shows that \( [k] \) is an eigenvalue. Changing the sign of the coefficient of \( v_{\lambda} \otimes v_{-\lambda} \) for which \( \lambda \) is a weight in \( S_{-} \), one also sees that \( -[k] \) is an eigenvalue.

To prove the claim for the other values, observe that \( C \) leaves invariant the span \( S_{r}^{\otimes 2} \) spanned by vectors \( v_{\mu} \otimes v_{\mu} \) for which \( \mu_{j} = \nu_{j} = + \) if \( j > r \). Moreover, the action onto this subspace coincides with the one of the element \( C \) for \( so_{2r} \). Hence we also have the eigenvalues \( \pm[r] \) for any \( 0 \leq r < k \). As \( C \in End_{U}(S_{r}^{\otimes 2}) \) can have at most \( 2k + 1 \) distinct eigenvalues, the claim follows.

Part (b) can be shown similarly.

**4.3. Odd-dimensional case.** The same method also works in the odd-dimensional case. We shall do the case \( O(3) \) in detail. We shall consider a faithful representation of \( Cl(3) \) on a simple
module of $Cl(4)$. We again use notation $(++)$, $(-+)$ ... for the basis vectors. Then it is easy to see that the maps

$$E: (--) \mapsto (++) \quad \text{and} \quad (--) \mapsto (+-),$$

and $F$ being the transposed of $E$ with respect to this basis define a representation of $U_q\mathfrak{sl}_2$. It is the direct sum of two simple two-dimensional representations with highest weight vectors $(++)$ and $(+-)$. Using the coproduct as in the proof of Prop. 4.1, one can determine a commuting operator $C$ as in Eq. 4.1. If we set $\alpha_{\lambda,\mu}^{(1)} = 1$ for any $\lambda, \mu$ with $\lambda_1 = -\mu_1$, we can determine $\alpha_{\lambda,\mu}^{(2)} = 1/[2]$ if both $v_\lambda$ and $v_\mu$ are highest weight vectors, and

$$\alpha_{(+-,+-)}^{(2)} = -q^{-1}/[2], \quad \alpha_{(-+,++)}^{(2)} = -q/2,$$

where $[2] = q^{1/2} + q^{-1/2}$. Moreover, the coefficients for any tensor product of two basis vectors are one of the above, where it only depends whether the tensor factors are a highest or a lowest weight vector.

**Proposition 4.3.** If $N = 2k + 1$ is odd, we can determine coefficients $\alpha_{\lambda,\mu}^{(j)}$, $1 \leq j \leq k$ for $C$ as in Eq. 4.1 such that $C$ commutes with $U = U_q\mathfrak{so}_N$ on $\tilde{S}^{\otimes 2}$, and that $C$ has the eigenvalues $[j + 1/2] = (q^{j+1/2} - q^{-j-1/2})/(q - q^{-1})$ for $-k - 1 \leq j \leq k$.

**Proof.** If $j \leq k$, we take for $\alpha_{\lambda,\mu}^{(j)}$ the value as in Prop. 4.1. For $\alpha_{\lambda,\mu}^{(k+1)}$, we define $\varepsilon, \kappa$ to be the ‘vectors’ consisting of the $k$-th and $(k + 1)$-st components of $\lambda$ and $\mu$ respectively. If $\alpha_{\lambda,\mu}^{(k)} \neq 0$, we multiply it by $\alpha_{\varepsilon,\kappa}^{(2)} = -q^{\pm 1}/[2]$ as in the $O(3)$-case to get $\alpha_{\lambda,\mu}^{(k+1)}$. If $\alpha_{\lambda,\mu}^{(k)} = 0$, we set $\alpha_{\lambda,\mu}^{(k+1)} = \alpha_{\lambda,\mu}^{(k)}/[2]$, where $\tilde{\lambda}^k$ coincides with $\lambda$ except for the $k$-th coordinate. The claim about the coefficients of $C$ now follows from Prop. 4.1 and the calculations for the $O(3)$ case.

The claim about the eigenvalues is shown as in Lemma 4.2, where we now pick as eigenvector $v = \sum (-q)^{(\varepsilon - \lambda, \rho)} v_\lambda \otimes v_{-\lambda}$, where $\rho = (k + 1/2 - i)$ and $\varepsilon$ is the highest weight vector of $S$.

**4.4. Action in third tensor power.** The main result of this subsection is listed in Lemma 4.4. It is elementary. We will first deal with the slightly easier case $N$ even. Recall that the $i$-th antisymmetrization $\Lambda^i V$ of the vector representation $V$ of $O(N)$ appears with multiplicity 1 in $S^{\otimes 2}$ and that $\Lambda^i V \otimes S$ contains a unique summand which is isomorphic to $S_i$, which we denote by $S_i$, i.e. we have a direct summand $S_i \subset S^{\otimes 3}$ defined by

$$S \cong S_i \subset \bigwedge^i V \otimes S \subset S^{\otimes 3}, \quad 0 \leq i \leq N.$$ 

Let now $(v_i)$ be an orthonormal basis of highest weight vectors $v_i \in S_i$ of weight $\varepsilon$. Their span is a module of the commutant of the $U$ action. Let

$$(4.6) \quad v_i = \sum \alpha_{i,\mu_2,\mu_3}^{(i)} v_{\mu_1} \otimes v_{\mu_2} \otimes v_{\mu_3},$$

with $v_{\mu_j}$ a weight vector of $S$ for all indices $j$. We extend the partial order of weights to tensor products of weight vectors in alphabetic order, i.e. the order structure is determined by the first factor for which the weights are not the same. Let $\Lambda_i$ be the highest weight of $\bigwedge^i V$ for
i \neq N/2 (in this case it is also irreducible as a $U_q so_N$ module). Then $v_i \otimes v_{\lambda_i-\epsilon} \otimes v_{\epsilon-\Lambda_i}$ and $v_i \otimes v_{\lambda_i-\epsilon} \otimes v_{\epsilon-\Lambda_i}$ are two maximal vectors with nonzero coefficients in the linear combination of $v_i$ and $v_{N-i}$ for $i < N/2$. Hence $(1 \otimes C)v_i$ and $(1 \otimes C)v_{N-i}$ are linear combinations of the vectors $v_j$ with $j \leq i + 1$ or $j \geq N - i - 1$.

Moreover recall that $S = S_+ \oplus S_-$ as a $U_q so_N$ module. Then $\bigwedge^i V$ is contained in $S^{\otimes 2} \oplus S_-^{\otimes 2}$ if $N/2 - i$ is even, and in $S_+ \otimes S_- \otimes S_- \otimes S_+$ if $N/2 - i$ is odd. Hence we also have that $(1 \otimes C)v_i$ is a linear combination of vectors $v_j$ such that $j - i$ is odd. We have set up everything for $N$ even for the following Lemma.

**Lemma 4.4.** Let $C$ be the linear map in $\text{End}_U(S^{\otimes 2})$ (for $N$ even) resp. in $\text{End}_U(\tilde{S}^{\otimes 2})$ as defined in the previous sections. The vector $(1 \otimes C)v_i$ is a linear combination of $v_{i-1}$ and $v_{i+1}$.

**Proof.** It follows from the definitions that $t$ has to map a highest weight vector $v_\lambda$ of an $so_N$ module to a highest weight vector. As $\bigwedge^i V$ remains irreducible as an $so_N$ module for $i \neq N/2$ and as $t^2 = 1$, we have $tv_\lambda = \pm v_\lambda$ in this case. One can now check directly for the highest weight vectors of $\bigwedge^i V$ that it is $+1$ for $i < N/2$ and $-1$ for $i > N/2$. Hence we can normalize the vectors $v_i$ and $v_{N-i}$ such that in their basis expansions the vector $v_i \otimes v_{\lambda_i-\epsilon} \otimes v_{\epsilon-\Lambda_i}$ has the same coefficient, and the coefficients of the vector $v_i \otimes v_{\lambda_i-\epsilon} \otimes v_{\epsilon-\Lambda_i}$ differ by a sign. It follows from the definition of $C$ that $(1 \otimes C)v_i$ is a linear combination of the vectors $v_j$ with $j \leq i + 1$ and with $j - i$ odd, for $i < N/2$. A similar result also holds for $(1 \otimes C)v_{N-i}$.

Finally, we use Kashiwara’s inner product (see e.g. [19], Section 3.5 or also [30], Section 1.4) for representations of quantum groups. We actually only need the fact that it is multiplicative for tensor products, and that its adjoint maps $\text{End}_{U_q g}(W)$ into itself for any $U_q g$ module $W$.

As the weight spaces are mutually orthogonal and $\text{End}_U(S^{\otimes 2})$ is abelian, one deduces that $C^T = C$, and hence also $(1 \otimes C)$ is self-adjoint. Hence, after suitably normalizing the mutually orthogonal vectors $v_i$, we can assume $C$ to be symmetric. The claim for $N$ even follows from this and the statements at the end of the last paragraph. The proof for $N$ odd goes exactly the same way, after the notation is set up the right way. This will be done in the remainder of this subsection.

If $N$ is odd, we consider the module $\tilde{S} \cong \tilde{S}_+ \oplus \tilde{S}_-$, where $\tilde{S}_\pm \cong S$ with highest weight vectors $\varepsilon_+ = \varepsilon$ and $\varepsilon_- = \bar{\varepsilon}$; this is exactly the same decomposition of the corresponding $Pin(N + 1)$-module into a direct sum of irreducible $Spin(N + 1)$-modules. Then we have as before that as $O(N)$-modules, $S^{\otimes 2}_+ \cong \bigoplus_{i=0}^k \bigwedge^{2i} V$ and $S_+ \otimes S_- \cong \bigoplus_{i=0}^k \bigwedge^{N-2i} V$. If $(N + 1)$ is divisible by 4, also $v_{-\varepsilon} \in \tilde{S}_-$. In this case, we define $v_{2i}$ to be a highest weight vector of the unique module $S \cong S_{2i} \subset \bigwedge^{2i} V \otimes \tilde{S}_+ \subset \tilde{S}_+^{\otimes 2}$, and we define $v_{N-2i}$ to be a highest weight vector of the unique module $S \cong S_{N-2i} \subset \bigwedge^{N-2i} V \otimes \tilde{S}_+ \subset \tilde{S}_+ \otimes \tilde{S}_+^{\otimes 2}$. If $(N + 1)$ is not divisible by 4, we define the vectors $v_{2i} \in \tilde{S}_+^{\otimes 2} \otimes \tilde{S}_-$ and $v_{N-2i} \in \tilde{S}_+ \otimes \tilde{S}_- \otimes \tilde{S}_+$. Using the fact that $C$ maps $\tilde{S}_+^{\otimes 2}$ to $\tilde{S}^{\otimes 2}$ and $\tilde{S}_+ \otimes \tilde{S}_-$ to $\tilde{S}_- \otimes \tilde{S}_+$, it should now be no problem for the reader to adapt the proof of Lemma 4.4 for the case $N$ odd.
4.5. **Technical lemma.** Let \( \{i\} = q^i + q^{-i} \) and define \( b_i \) up to a sign by
\[
b_i^2 = \binom{N}{i} q^{\{N/2 - i\}/q^{\{N/2\}}}.
\]
Then we have the following lemma.

**Lemma 4.5.** Let \( A \) be a symmetric \((N + 1) \times (N + 1)\) matrix with \( a_{ij} = 0 \) unless \( |i - j| = 1 \). Then the entries of \( A \) are completely determined by one eigenvalue \( \lambda \) and its corresponding eigenvector \( b = (b_i) \), where \( b_i \) is as defined above, then
\[
a^2_{i,i+1} = a^2_{i+1,i} = \frac{[i + 1][N - i]}{\{N/2 - i\}\{N/2 - i - 1\}}.
\]

**Proof.** It is straightforward to show by induction on \( i \), using the equation \( Ab = \lambda b \), that
\[
a_{i,i+1} = \frac{\lambda}{b_i b_{i+1}} (b_i^2 - b_{i-1}^2 + b_{i-2}^2 \ldots).
\]
Now let \( \lambda = [N/2] \) and let its eigenvector \( b \) be given as above. Then we also have
\[
b_i^2 = \binom{N - 1}{i}_q \binom{N - 1}{i - 1}_q,
\]
from which we get
\[
a^2_{i,i+1} = \frac{[N/2]^2 \binom{N-1}{i}^2}{b_i^2 b_{i+1}^2}.
\]
Using the definitions, it now is straightforward to show the claim.

Let \( H, E, K^{\pm 1} \) be the usual generators of the Drinfeld-Jimbo quantum group \( U_q \text{sl}_2 \), and let \( V_N \) be its \((N + 1)\)-dimensional simple representation. It can be defined via a basis \( \{e_o, e_1, \ldots, e_N\} \) of eigenvectors of \( K \) which satisfies
\[
E.e_r = ([N - r + 1][r])^{1/2} e_{r-1}, \quad K.e_r = q^{N-2r} e_r, \quad F.e_r = ([N - r][r + 1])^{1/2} e_{r+1}.
\]

**Corollary 4.6.** The element \( 1 \otimes C \in \text{End}_U(S^{\otimes 3}) \) is given with respect to the basis \( (v_r) \) in Lemma 3.2 by
\[
A = (K^{1/2} + K^{-1/2})^{-1/2} (E + F)(K^{1/2} + K^{-1/2})^{-1/2},
\]
where the elements on the right hand side stand for the matrices representing these quantum group elements in \( V_N \).

**Proof.** Let \( N \) be even. It follows from Lemmas 3.2, 4.2 and 4.4 that the matrix \( A \) representing \( 1 \otimes C \) with respect to the basis \( (v_r) \) satisfies the conditions of the lemma; for the statement about \( b \) observe that \( p^{(k)}_2 \) is an eigenprojection of \( C_2 \) in the notation of Lemma 3.2. For \( N \) odd, we get the appropriate eigenvectors for \( C^2 \), restricted to the basis vectors in \( S^{\otimes 3}_+ \), as well as for the eigenvectors in \( S_+ \otimes S^{\otimes 2}_- \), by Lemma 3.2. The claim can now be shown also for \( C \), using its block structure with respect to our basis.
4.6. Relations for centralizer algebras. We will need the following nonstandard $q$-deformation of the universal enveloping algebra of $so_N$. It was defined by Gavriliuk and Klimyk (see e.g. [7]) and by Noumi and Sugitani [20]. It is also a special case of the co-ideal subalgebras of $U_qsl_N$ defined by Letzter for $\Theta = id$, see [18], Remark 2.4. It is not isomorphic to the usual Drinfeld-Jimbo quantum group, see [18], Remark 2.3.

**Definition 4.7.** (a) The algebra $U'_qso_l$ is defined via generators $B_1, B_2, \ldots, B_{l-1}$ and relations $B_iB_j = B_jB_i$ for $|i - j| > 1$ and

$$B_i^2B_{i\pm 1} - (q + q^{-1})B_iB_{i\pm 1}B_i + B_{i\pm 1}B_i^2 = B_{i\pm 1}.\quad (4.6)$$

(b) The algebra $U_q(l, k)$ is the quotient of $U'_qso_l$ defined via the additional relation $\prod_{j=k}^l(B_i - [j]) = 0$.

(c) The algebra $U_{oq}(l, k)$ is the quotient of the subalgebra of $U'_qso_l$ generated by $B_i^2$, $1 \leq i < l$ defined via the additional relation $\prod_{j=1}^k(B_i^2 - [j - 1/2]^2) = 0$.

It is clear that for $q = 1$ we obtain the relations for the universal enveloping algebra of the Lie algebra $so_l$, see e.g. the remarks before Lemma 1.3.

**Theorem 4.8.** (a) If $N = 2k$ is even, we have representations of $U = U_qso_N \rtimes \mathbb{Z}/2$ and $U'_qso_l$ on $S^{\otimes k}$ which are each others commutant. Moreover, the image of $U'_qso_l$ factors through $U_q(l, k)$.

(b) If $N = 2k + 1$ is odd, we have actions of $U = U_qso_N$ and $U_{oq}(l, k)$ on $S^{\otimes l}$ which are each others commutant.

**Proof.** In case (a), it follows from Lemma 4.2 that $C$ has $N + 1$ eigenvalues, and hence generates $\text{End}_U(S^{\otimes 2})$. It follows from Theorem 3.3 that $\text{End}_U(S^{\otimes l})$ is generated by the elements $C_i$, $1 \leq i < l$.

It can easily be seen that the claim will follow if the commutation relations between $C_1$ and $C_2$ are checked, using Lemma 4.2. Now observe that by Lemma 4.2 and by Cor. 4.6 the elements $C_1$ and $C_2$ act on the span of the vectors $(v_i) \subset S^{\otimes 3}$ via the matrices representing the elements $D_1 = (k^{1/2} - k^{-1/2})/(q - q^{-1})$ and $A$ in $U_qsl_2$. To check the commutation relations between $C_1$ and $C_2$, observe that

$$[k, E] = (q^2 - 1)Ek = (1 - q^{-2})kE, \quad [k^{-1}, E] = (1 - q^2)k^{-1}E = (q^{-2} - 1)Ek^{-1}.\quad (4.8)$$

The relations involving $F$ are obtained from above by substituting $E$ by $F$ and $q$ by $q^{-1}$. Setting $D_1 = (k^{1/2} - k^{-1/2})/(q - q^{-1})$, we obtain

$$[D_1, E] = q^{1/2}Ek^{1/2} + q^{-1/2}Ek^{-1/2} = E(q^{1/2}k^{1/2} + q^{-1/2}k^{-1/2}),$$

and a similar expression with $E$ replaced by $F$. Hence

$$[D_1, [D_1, E + F]] = E(q^{1/2}k^{1/2} + q^{-1/2}k^{-1/2})^2 + F(q^{-1/2}k^{1/2} + q^{1/2}k^{-1/2})^2$$

$$= (q^{1/2} + q^{-1/2})^2(E + F) + (k^{1/2} - k^{-1/2})(E + F)(k^{1/2} - k^{-1/2}).$$
This can be rewritten as

$$D_1^2(E + F) - (q + q^{-1})D_1(E + F)D_1 + (E + F)D_1^2 = (E + F).$$

As $(k^{1/2} + k^{-1/2})^{-1/2}$ commutes with $D_1$, and obviously also $C_i$ commutes with $C_j$ provided $|i - j| > 1$, we have proved the relation in Def. 4.7 for $i$ and $i + 1$ for $i = 1$. Proving the relation with $B_1$ and $B_2$ interchanged can again be done via a calculation in $U_q$-$sl_2$, or one checks that one gets the same matrices in our set-up if we interchange $C_1$ and $C_2$. This shows that the relation holds for the summand $C_2^{(N)} p_2^{(N)} c_2^{(N)}$ of $C_3^{(N)}$ in Theorem 3.3. The general claim follows from that theorem and induction by $N$, observing that the projection $p_n^{(N)}$ is an eigenprojection of $C_n$.

Again, the proof for the case $N$ odd goes along the same lines.

**Remark 4.9.** We have stated our results here only for $q$ generic over the ground field of the complex numbers. It is not hard to generalize them to more general fields; e.g. even though square roots appear in our proofs, we do not expect them to be essential. Moreover the definition of the algebra $U_q's$ only involves elements in the ring $\mathbb{Z}[q, q^{-1}]$. So a similar result should also hold over that ring; indeed, by general results of Lusztig’s also centralizer algebras of suitably defined quantum groups have canonical bases defined over $\mathbb{Z}[q, q^{-1}]$, see [19], Section 27.3.

**Remark 4.10.** In previous work [16], [25] tensor categories were classified whose Grothendieck semiring was the one of a unitary, orthogonal or symplectic group. There an intrinsic description of centralizer algebras via braid group representations played a crucial role. For spinor representations, it is more difficult to describe these braid representations as the standard generators have too many eigenvalues. This made it necessary to consider a new description which generalizes the braid relations. Indeed, in many ways, the algebra $U_q's$ can be considered an algebraic object of type $A_l$. They should be useful in proving similar classification results for spinor groups.

**Remark 4.11.** In the recent publication [23] Rowell and Wang study certain representations of braid groups which they call Gaussian representations. They are of particular interest in their studies motivated by quantum computing. They conjecture that these representations are related to the centralizer algebras of quantum groups for spinor representations for certain roots of unity. Again, as already noted in Remark 4.10, it is difficult to determine these braid representations because of the increasing number of eigenvalues. The detailed analysis of representations of $C_3$ in this paper could be useful in solving this problem, see e.g. the proof of Theorem 4.8.

**Remark 4.12.** It is not hard to deduce from Theorem 3.3 that for $N$ odd the $R$-matrices generate $\text{End}_q(S^{\otimes l})$ for all $l$. Indeed, using Drinfeld’s quantum Casimir, one shows that the $R$ matrix has $(N + 1)/2$ distinct eigenvalues, and hence generates $\text{End}(S^{\otimes 2})$. Similar results have been shown by Lehrer and Zhang for vector representations of classical type and certain representations in type $A_1$ and $G_2$, see [17].
References

[1] F. L. Bauer, Zur Theorie der Spingruppen. Math. Ann. 128, (1954). 228–256.
[2] G. Benkart and J. Stroomer, Tableaux and insertion schemes for spinor representations of the orthogonal Lie algebra so(2r + 1, C). J. Combin. Theory Ser. A 57 (1991), no. 2, 211–237.
[3] J. Birman; H. Wenzl, Braids, link polynomials and a new algebra, Trans. AMS 313 (1989) 249-273.
[4] N. Bourbaki, Groupes et algèbres de Lie, ch. 3,4,5, Masson
[5] R. Brauer, On algebras which are connected with the semisimple continuous groups, Ann. of Math. 63 (1937), 854-872.
[6] Ding, Jin Tai; Frenkel, Igor B. Spinor and oscillator representations of quantum groups. Lie theory and geometry, 127165, Progr. Math., 123, Birkhuser Boston, Boston, MA, 1994.
[7] Gavrilik, A. M.; Klimyk, A. U. q-deformed orthogonal and pseudo-orthogonal algebras and their representations. Lett. Math. Phys. 21 (1991), no. 3, 215220.
[8] K. Hasegawa, Spin module versions of Weyl’s reciprocity theorem for classical Kac-Moody Lie algebras— an application to branching rule duality. Publ. Res. Inst. Math. Sci. 25 (1989), no. 5, 741–828.
[9] T. Hayashi, q-analogues of Clifford and Weyl algebras - spinor and oscillator representations of quantum enveloping algebras, Comm. Math. Phys. 127 (1990) 129-144.
[10] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer Verlag, 1978.
[11] J.C. Jantzen, Lectures on quantum groups. Graduate Studies in Mathematics, 6. American Mathematical Society, 1996.
[12] V.F.R. Jones, Index for subfactors, Invent. Math 72 (1983) 1–25.
[13] V. Kac, Infinite dimensional Lie algebras, 3rd edition, Cambridge University Press.
[14] Ch. Kassel, Quantum groups, Springer 1995.
[15] D. Kazhdan; H. Wenzl, Reconstructing monoidal categories, Adv. in Soviet Math., 16 (1993) 111-136.
[16] Lehrer, G. I.; Zhang, Hechun; Zhang, R. B. Strongly multiplicity free modules for Lie algebras and quantum groups, J. Algebra, 306 (2006) 138-174
[17] Letzter, Gail, Subalgebras which appear in quantum Iwasawa decompositions, Can. J. Math. Vol. 49 (6), 1997 pp. 1206-1223.
[18] M. Noumi and T. Sagita, Quantum symmetric spaces and related q-orthogonal polynomials. Group Theoretical Methods in Physics (ICGTMP) (Toyonaka, Japan, 1994), World Sci. Publishing, River Edge, NJ, 1995, 2840.
[19] R. Orellana, H. Wenzl, 0-centralizer algebras for spin groups. J. Algebra 253 (2002), no. 2, 237–275.
[20] A. Ram and H. Wenzl Matrix units for centralizer algebras J. Algebra. 102 (1992), 378-395.
[21] E. Rowell and Zh. Wang, Localization of unitary braid group representations, arXiv:1009.0241
[22] Sunder, V. S. A model for AF algebras and a representation of the Jones projections. J. Operator Theory 18 (1987), no. 2, 289301.
[23] H. Wenzl, Quantum groups and subfactors of Lie type B, C and D, Comm. Math. Phys. 133 (1990) 383-433.
[24] H. Wenzl, Braids and invariants of 3-manifolds, Invent. math. 114 (1993) 235-275.
[25] H. Wenzl, On tensor categories of Lie type $E_N$, $N \neq 9$, preprint
[26] H. Weyl, The classical groups, Princeton University Press.
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