A LOCAL TO GLOBAL ARGUMENT ON LOW DIMENSIONAL MANIFOLDS

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Abstract. For an orientable manifold $M$ whose dimension is less than 4, we use the contractibility of certain complexes associated to its submanifolds to cut $M$ into simpler pieces in order to do local to global arguments. One of the deep theorems of Thurston in foliation theory is a homology h-principle theorem that says the natural map

$\text{BHomeo}^\delta(M) \to \text{BHomeo}(M)$,

induces a homology isomorphism where $\text{Homeo}^\delta(M)$ denotes the group of homeomorphisms of $M$ made discrete. In low dimensions, we give a different proof of this theorem without using foliation theory. Secondly, we show that the same method applies to give a different proof of the h-principle theorem in smoothing theory i.e. the map

$\text{BDiff}(M) \to \text{BHomeo}(M)$

is a weak equivalence.

Finally, we give a different proof that the diffeomorphism groups of Haken 3-manifolds with boundary are homotopically discrete.

1. Introduction

Often, in h-principle type theorems (e.g. Smale-Hirsch theory), it is easy to check that the statement holds for the open disks (local data) and then one wishes to glue them together to prove that the statement holds for closed compact manifolds (global statement). But there are cases where one has a local statement for a closed disk relative to the boundary. To use such local data to great effect, instead of covering the manifold by open balls, we use certain “resolutions” associated to submanifolds (see Section 2) to cut the manifold into disks.

The first example of this sort is from smoothing theory. Let $\text{Diff}(D^n, \partial D^n)$ denote the group of compactly supported $C^\infty$-diffeomorphisms of the interior of the disk $D^n$ with $C^\infty$-topology and let $\text{Homeo}(D^n, \partial D^n)$ denote the group of compactly supported homeomorphisms of the interior of the disk $D^n$ with the $C^0$-topology. By the Alexander trick, we know that the group $\text{Homeo}(D^n, \partial D^n)$ is contractible for all $n$. On the other hand, it is a well-known theorem of Smale that $\text{Diff}(D^2, \partial D^2)$ is contractible and by the theorem of Hatcher ([Hat83]) so is $\text{Diff}(D^3, \partial D^3)$. Therefore the natural map between classifying spaces

$\text{BDiff}(D^n, \partial D^n) \to \text{BHomeo}(D^n, \partial D^n),$

is a weak equivalence for $n = 2$ and $n = 3$. The h-principle theorem in smoothing theory for low dimensional manifolds says

**Theorem 1.1** (Earle-Eells, Hamstrom, Cerf). *For a smooth manifold $M$, the map*

(1.2) $\eta : \text{BDiff}(M) \to \text{BHomeo}(M),$

*is a weak equivalence provided $\dim(M) = 2$ (see [Ham74]) or $\dim(M) = 3$ (see [Cer61]). Similar version holds for manifolds with boundary.*
The second example is from foliation theory. Let $\text{Homeo}^\delta(D^n, \partial D^n)$ denote the same group as $\text{Homeo}(D^n, \partial D^n)$ but with the discrete topology. By an infinite repetition trick due to Mather ([Mat71]), it is known that $B\text{Homeo}^\delta(D^n, \partial D^n)$ is acyclic. Therefore, the natural map

$$B\text{Homeo}^\delta(D^n, \partial D^n) \to B\text{Homeo}(D^n, \partial D^n)$$

induced by the identity homomorphism is in particular a homology isomorphism. Thurston generalized Mather’s work on foliation theory in [Thu74a] and as a corollary he obtained the following surprising result.

**Theorem 1.3** (Thurston). For a smooth manifold $M$, the map

$$\iota : B\text{Homeo}^\delta(M) \to B\text{Homeo}(M),$$

induces an isomorphism on homology.

The first proof of this theorem in the literature was given by McDuff following Segal’s program in foliation theory (see [McD80]). Thurston in fact proved a more general homology h-principle theorem for foliations such that Theorem 1.3 is just its consequence for $C^r$-foliations.

In foliation theory, Haefliger defined a topological groupoid $\Gamma^r_q$ whose space of objects are points in $\mathbb{R}^q$ with the usual topology and the space of morphisms between two points is given by germs of $C^r$-diffeomorphisms sending $x$ to $y$ (see [Hae71, Section 1] for more details). The homotopy type of the classifying space of this groupoid, $B\Gamma^r_q$, plays an important role in the classification of $C^r$-foliations (see [Thu74b] and [Thu76]). One of Thurston’s deep theorem in foliation theory relates the homotopy type of $B\Gamma^r_q$ to the group homology of $C^r$-diffeomorphism groups made discrete. For $r = 0$, he first uses Mather’s theorem ([Mat71]) to show that $B\Gamma^0_q$ is weakly equivalent to the classifying space of rank $q$ microbundles, $B\text{Top}(q)$, and as a consequence he deduces that the map $\iota$ in Theorem 1.3 is in fact acyclic.

In the theory of foliations and smoothing theory respectively, people have studied the homotopy fiber of the maps $\iota$ and $\eta$. In fact there are general h-principle theorems in all dimensions that identify the homotopy fibers as holonomic sections of certain section spaces associated to the manifold. Our goal in this paper is to show that in low dimensions one can directly study the maps instead of their homotopy fibers. To do so, we provide the strategy in detail for Theorem 1.3 in low dimensions that does not use any foliation theory and the method is general enough that can unify the proof of Theorem 1.1 for both dimensions 2 and 3.

The reason that we restrict ourselves to low dimensions is that for surfaces and 3-manifolds, there is a procedure to split up the manifold into disks. For the surfaces, this procedure is given by cutting along handles. But for 3-manifolds, it is more subtle. To do so, we use the prime decomposition theorem and Haken’s hierarchy to cut the manifold into disks.

Finally using this technique, we also give a different proof of the contractibility of the identity component of diffeomorphisms in low dimensions.

**Theorem 1.4** (Earle-Schatz, Hatcher). The identity components of diffeomorphism groups of surfaces with boundary (see [ES70]) and Haken manifolds with boundary (see [Hat76]) are contractible.

Instead of working with the diffeomorphisms groups, we work with their classifying spaces. Considering the delooping of these topological groups has the advantage that one can apply homological techniques to the classifying spaces to extract homotopical information about diffeomorphism groups.
1.1. Outline. The paper is organized as follows: in Section 2, we describe semi-simplicial resolutions for the classifying spaces of homeomorphisms using embedded submanifolds. We will treat the case of three manifolds separately because we first have to cut three manifolds into their prime pieces. In Section 3, given the techniques of the previous section, we prove a theorem of Cerf that $\text{Diff}_0(M) \to \text{Homeo}_0(M)$ is a weak homotopy equivalence where $M$ is a three manifold. In Section 4, we give a short proof of the contractibility of the identity component of the diffeomorphism groups for certain low dimensional manifolds.

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2. Resolving classifying spaces by embedded submanifolds

Let us first sketch the idea for Theorem 1.3. Let $M$ be a smooth manifold and let $\text{Homeo}_0(M)$ denote the group of the identity component of the topological group $\text{Homeo}(M)$. Note that the group of connected components $\pi_0(\text{Homeo}(M))$ is a discrete group and sits in a short exact sequence

$$1 \to \text{Homeo}_0(M) \to \text{Homeo}(M) \to \pi_0(\text{Homeo}(M)) \to 1.$$ 

An easy spectral sequence argument reduces Theorem 1.3 to proving that the map

$$\text{BHomeo}_0^0(M) \to \text{BHomeo}_0^0(M),$$

induces a homology isomorphism. To prove this version, we want to inductively reduce Theorem 1.3 to the case of a simpler manifold. Such simpler manifolds are obtained from $M$ by cutting along its submanifolds. Let $\phi$ be an embedding of a manifold into $M$. To cut along this embedding, we construct a semi-simplicial space $A_\bullet(M,\phi)$ on which the topological group $\text{Homeo}_0(M)$ acts (see [ERW17] or [RW16, Section 2] for definitions of (augmented) semi-simplicial objects and their fat realizations). Similarly we construct a semi-simplicial set $A^\delta_\bullet(M,\phi)$ from the underlying semi-simplicial set of the semi-simplicial space $A_\bullet(M,\phi)$ on which the group $\text{Homeo}_0^0(M)$ acts. These semi-simplicial spaces are constructed so that their fat realizations are weakly contractible. Therefore we obtain semi-simplicial resolutions

$$|A^\delta_\bullet(M,\phi)|/\text{Homeo}_0^\delta(M) \xrightarrow{\approx} \text{BHomeo}_0^\delta(M),$$

$$|A_\bullet(M,\phi)|/\text{Homeo}_0^\delta(M) \xrightarrow{\approx} \text{BHomeo}_0(M).$$

We then construct a zig-zag of maps from the space $A^\delta_\bullet(M,\phi)|/\text{Homeo}_0^\delta(M)$ to the space $A_\bullet(M,\phi)|/\text{Homeo}_0(M)$ that induces a homotopy commutative diagram

$$\begin{align*}
H_\ast(|A^\delta_\bullet(M,\phi)|/\text{Homeo}_0^\delta(M);\mathbb{Z}) & \xrightarrow{f_*} H_\ast(|A_\bullet(M,\phi)|/\text{Homeo}_0(M);\mathbb{Z}) \\
\cong & \\
H_\ast(\text{BHomeo}_0^\delta(M);\mathbb{Z}) & \xrightarrow{I_*} H_\ast(\text{BHomeo}_0(M);\mathbb{Z}).
\end{align*}$$

\footnote{For a topological group $G$ acting on a topological space $X$, the homotopy quotient is denoted by $X\rtimes G$ and is given by $X \times G \text{EG}$ where $\text{EG}$ is a contractible space on which $G$ acts freely.}
Therefore, it is enough to prove that $f^*$ is an isomorphism. As we shall see in Section 2.1.4, proving that $f^*$ induces a homology isomorphism is equivalent to the statement of Theorem 1.3 for a manifold that is obtained from $M$ by cutting it along $\phi$. Then, by induction we can reduce Theorem 1.3 to the case of a disk relative to its boundary that

\[
B\text{Homeo}^\delta(D^n, \partial D^n) \to B\text{Homeo}(D^n, \partial D^n),
\]

induces a homology isomorphism. We restricted ourselves to low dimensions, because we still do not know how to make a certain surgery argument in Theorem 2.30 work in dimensions higher than 3.

We want to cut up $M$ into disks in a “contractible space of choices” (e.g. see Proposition 2.13 and Theorem 2.30). As we shall explain at the end of Section 2, the easiest case is when $M$ is homeomorphic to a circle (see also [Jek12, Theorem 4]). For $M$ being a surface, we define certain space of handles to cut the surface along them. Finally if $M$ is a three manifold, we first reduce to the case of irreducible three manifolds and we cut it along incompressible surfaces in a “contractible space of choices”. For this reason, we consider the case of three manifolds separately.

2.1. The case of surfaces. The first step is to reduce the statement of Theorem 1.3 to the case of the surfaces with boundary to be able to remove 1-handles. Hence we first want to remove disks (0-handles) from a closed surface $M$. To consider different choices of 0-handles, we define a semisimplicial spaces. But we give the definitions in all dimensions and restrict to the case of surfaces whenever it is necessary.

Definition 2.1 (0-handle resolutions). We give both topological and discrete versions:

- **Topological versions:** Let $[p]$ denote the set $\{0, 1, ..., p\}$ of $p + 1$ ordered elements. Let

\[
A_p(M) = \text{Emb}(\bigsqcup_{[p]} D^n, M)
\]

denote the subspace of the embedding space (equipped with compact-open topology) consisting of orientation preserving embeddings of $p$ disjoint union of $n$-disks that admit an external collar into the manifold $M$. The collection $A_p(M)$ is a semisimplicial space where the face maps are given by forgetting disks.

We also define an auxiliary semisimplicial space $\overline{A}_p(M)$ whose space of 0-simplices is the same as $A_0(M)$ but its space of $p$-simplices is the subspace of $A_0(M)^{p+1}$ consisting of $(p + 1)$-tuples $(\phi_0, \phi_1, ..., \phi_p)$ where the centers of the embedded disks $\phi_i$ are pairwise disjoint.

Note that the group $\text{Homeo}_0(M)$ acts on $A_p(M)$ transitively. The 0-handle resolution of $B\text{Homeo}_0(M)$ is defined to be the augmented semisimplicial space

\[
X_* = A_* \text{Homeo}_0(M) \to B\text{Homeo}_0(M).
\]

- **Discrete version:** We say two embeddings $g_1$ and $g_2$ in $\text{Emb}(\bigsqcup_{[p]} D^n, M)$ have the same germ if there exists an open neighborhood $U \subset D^n$ around the origin so that $g_1|_{\bigsqcup_{[p]} U} = g_2|_{\bigsqcup_{[p]} U}$. Let

\[
A^\delta_p(M) := \text{Emb}^\delta(\bigsqcup_{[p]} D^n, M),
\]

\[\text{In fact the action is transitive in all dimensions thanks to the annulus theorem and the hypothesis on having external collar for embedded disks is imposed so that the annulus theorem holds.}\]
denote the set of germs of embeddings of disjoint union of $p + 1$ disks compatible with the orientation of $M$. We define an auxiliary semisimplicial set $\overline{\mathcal{T}}_\bullet(M)$ which is given by the underlying set of the semisimplicial space $\overline{\mathcal{A}}_\bullet(M)$.

Also the 0-handle resolution for $B\text{Homeo}_0^\delta(M)$ is the augmented semisimplicial space

$$X^0_\bullet(M) := \mathcal{A}_\bullet(M) \# \text{Homeo}_0^\delta(M) \to B\text{Homeo}_0^\delta(M).$$

Note that there are natural maps

$$\mathcal{A}_\bullet(M) \xrightarrow{\cong} \overline{\mathcal{T}}_\bullet(M) \leftarrow \overline{\mathcal{A}}_\bullet(M) \to \mathcal{A}_\bullet(M),$$

where the first map is the inclusion (it is easy to see that it induces a weak homotopy equivalence levelwise), the second map is the identity map from the underlying set of a topological space to itself and the last map is induced by taking germs of embeddings of disks at their centers.

2.1.1. The homotopy type of $X_p(M)$ and $X^0_p(M)$. To determine the homotopy type of $X_p(M)$, fix an element $e_p \in A_p(M)$. Let $M \setminus e_p$ denote the manifold obtained from $M$ by removing the interior of the image of $e_p$. The action of $\text{Homeo}_0(M)$ on $e_p$ gives rise to a map

$$(2.2) \quad \text{Homeo}_0(M) \to A_p(M).$$

Let $\text{Homeo}(M, e_p)$ denote those homeomorphisms that are the identity on the image of $e_p$. The fiber over $e_p$ is the topological group $\text{Homeo}_0(M \setminus e_p, \partial(M \setminus e_p))$ whose identity component is $\text{Homeo}_0(M, e_p)$. But given that the embedding $e_p$ has a collar, it is easy to show that the inclusion

$$(2.3) \quad \text{Homeo}_0(M \setminus e_p, \partial(M \setminus e_p)) \to \text{Homeo}_0(M, e_p),$$

is a weak homotopy equivalence.

**Lemma 2.4.** There is a quasi-fibration

$$(2.5) \quad \text{Homeo}_0(M \setminus e_p, \partial(M \setminus e_p)) \to \text{Homeo}_0(M) \to A_p(M).$$

**Proof.** We assume $p = 0$ and the argument for the general case is the same. Let us recall the parametrized isotopy extension theorem in the topological setting ([BL74, page 19]). We consider the simplicial set $\text{Emb}_\bullet^H(D^n, M)$ whose $k$-simplices is given by locally flat embeddings

$$\Delta^k \times D^n \to \Delta^k \times M,$$

that lives over projection to $\Delta^k$. Let $\text{Sing}_\bullet(\text{Homeo}_0(M))$ be the singular set of the homeomorphism group. Fixing an element in $e_0 \in \text{Emb}_\bullet^H(D^n, M)$, we have an evaluation map

$$\text{Sing}_\bullet(\text{Homeo}_0(M)) \to \text{Emb}_\bullet^H(D^n, M),$$

which is a Kan fibration. It is a well known result of Quillen ([Qui68]) that the realization of the Kan fibration is a Serre fibration. Also for codimension zero similar to codimension higher than two ([Las76, Appendix]), the natural map

$$[\text{Emb}_\bullet^H(D^n, M)] \to \text{Emb}(D^n, M) = A_0(M),$$

is a weak homotopy equivalence. And by the theorem of Milnor ([Mil57]), the natural map

$$[\text{Sing}_\bullet(\text{Homeo}_0(M))]) \to \text{Homeo}_0(M),$$

is a weak homotopy equivalence. Hence the evaluation map $\text{Homeo}_0(M) \to A_0(M)$ is a quasi-fibration with the fiber $\text{Homeo}_0(M \setminus e_0, \partial(M \setminus e_0))$. \qed
Recall that for a group $G$ acting on a topological space $X$, we have a natural map $B\text{Stab}(\sigma) \to X \langle \rangle G$ where $\text{Stab}(\sigma)$ is the stabilizer of an element $\sigma \in X$. Note that for $e_p \in A_p(M)$, the group $\text{Homeo}_0(M \langle \rangle e_p, \partial(M \langle \rangle e_p))$ is the stabilizer of $e_p$ for the action of $\text{Homeo}_0(M)$ on $A_p(M)$. Therefore, we have a map

$$h_p : B\text{Homeo}_0(M \langle \rangle e_p, \partial(M \langle \rangle e_p)) \to A_p(M) \langle \rangle \text{Homeo}_0(M) = X_p(M).$$

**Proposition 2.7.** The map $h_p$ is a weak homotopy equivalence.

**Proof.** For brevity, let us denote $\text{Homeo}_0(M \langle \rangle e_p, \partial(M \langle \rangle e_p))$ by $H_p$. Note that we have the following homotopy commutative diagram

$$
\begin{array}{ccc}
H_p \langle \rangle H_p & \to & \text{Homeo}_0(M) \langle \rangle \text{Homeo}_0(M) \\
\downarrow & & \downarrow g \\
B H_p & \xrightarrow{h_p} & A_p(M) \langle \rangle \text{Homeo}_0(M).
\end{array}
$$

The left vertical map is a fibration with $H_p$ as the fiber. If we show that $g$ is a quasi-fibration with $H_p$ as the fiber, then the comparison of the long exact sequence of the (quasi)fibrations induced by vertical maps implies that $h_p$ is a weak homotopy equivalence.

Recall that for a group $G$ acting on a topological space $X$, the two sided bar construction $B_\ast(X, G, \ast) = X \times G^\ast$ is a simplicial space with the usual face maps and degeneracies. If $G$ is a well-pointed topological group, the realization of the two-sided bar construction $B_\ast(X, G, \ast)$ is a model for the homotopy quotient. Since $\text{Homeo}_0(M)$ is a well-pointed group ([EK71]), the map $g$ is induced by taking the geometric realization of the simplicial map

$$g_\ast : B_\ast(\text{Homeo}_0(M), \text{Homeo}_0(M), \ast) \to B_\ast(A_p(M), \text{Homeo}_0(M), \ast).$$

To prove that $g$ is a quasi-fibration whose homotopy fiber is $H_p$, it is enough to prove that the following diagram is homotopy cartesian

$$
\begin{array}{ccc}
\text{Homeo}_0(M) = B_0(\text{Homeo}_0(M), \text{Homeo}_0(M), \ast) & \to & [B_\ast(\text{Homeo}_0(M), \text{Homeo}_0(M), \ast)] \\
\downarrow & & \downarrow g \\
A_p(M) = B_0(A_p(M), \text{Homeo}_0(M), \ast) & \to & [B_\ast(A_p(M), \text{Homeo}_0(M), \ast)].
\end{array}
$$

Using Segal’s gluing lemma ([Seg74, Proposition 1.6]), it is enough to show that the diagram

$$
\begin{array}{ccc}
B_k(\text{Homeo}_0(M), \text{Homeo}_0(M), \ast) & \xrightarrow{d_i} & B_{k-1}(\text{Homeo}_0(M), \text{Homeo}_0(M), \ast) \\
\downarrow g_k & & \downarrow g_{k-1} \\
B_k(A_p(M), \text{Homeo}_0(M), \ast) & \xrightarrow{d_i} & B_{k-1}(A_p(M), \text{Homeo}_0(M), \ast),
\end{array}
$$

is a homotopy cartesian for all face maps $d_i, 0 \leq i \leq k$. Since the multiplication in $\text{Homeo}_0(M)$ has an inverse (see the discussion after [Seg74, Proposition 1.6]), it is enough to show that the diagram

$$
\begin{array}{ccc}
\text{Homeo}_0(M) \times \text{Homeo}_0(M) & \xrightarrow{d_1} & \text{Homeo}_0(M) \\
\downarrow g_1 & & \downarrow g_0 \\
A_p(M) \times \text{Homeo}_0(M) & \xrightarrow{d_1} & A_p(M),
\end{array}
$$

(2.9)
is a homotopy cartesian. For this, note that \( g_0 \) and \( g_1 \) are quasi-fibrations and for example the fiber of \( g_1 \) over \( (e_p, f) \) is \( \text{Homeo}_0(M, e_p) \) (see Equation (2.3)) and the fiber of \( g_0 \) over \( f(e_p) \) is \( \text{Homeo}_0(M, f(e_p)) \). Given that the action of \( \text{Homeo}_0(M) \) on \( A_p(M) \) is transitive, these two groups \( \text{Homeo}_0(M, e_p) \) and \( \text{Homeo}_0(M, f(e_p)) \) are homeomorphic. Therefore, the diagram 2.9 is homotopy cartesian. \( \square \)

**Remark 2.10.** The author does not know if the quasi-fibration 2.5 is a locally trivial bundle similar to the smooth category. If this were true, the space \( A_p(M) \) would become homeomorphic to \( \text{Homeo}_0(M)/H_p \) and therefore the proof of Proposition 2.7 would become much easier.

It is easier to determine the homotopy type of \( X^\delta_p(M) \). To do so, let \( M(e_p) \) denote the manifold \( M \) with \( (p + 1) \) punctures at the centers of the germs of embedding of disks \( e_p \) in \( M \). We may consider \( e_p \) as an element of \( A^\delta_p(M) \), and denote the stabilizer of the element \( e_p \) under the action of \( \text{Homeo}_0^\delta(M) \) on \( A^\delta_p(M) \) by \( \text{Homeo}_{0,c}^\delta(M(e_p)) \). Let \( \text{Homeo}_{0,c}^\delta(M(e_p)) \) be the same group but consider it as a subgroup of \( \text{Homeo}_0(M) \) with the subspace topology. It is useful (in particular in the diagram 2.22) to note that the group of connected components of \( \text{Homeo}_{0,c}^\delta(M(e_p)) \) is the same as the group of components of \( \text{Homeo}_0(M\setminus \partial(M\setminus e_p)) \).

Hence, we have a short exact sequence

\[
\begin{align*}
1 & \to \text{Homeo}_0^\delta(M\setminus e_p) \to \text{Homeo}_{0,c}^\delta(M(e_p)) \to \pi_0(\text{Homeo}_0(M\setminus e_p, \partial(M\setminus e_p))) \to 1.
\end{align*}
\]

Recall that Shapiro’s lemma for discrete groups \( H < G \) says that the natural map \( BH \to (G/H)/fG \) is a weak homotopy equivalence. Therefore, the map

\[
\begin{align*}
\text{BHomeo}_{0,c}^\delta(M(e_p)) & \xrightarrow{\sim} X^\delta_p(M),
\end{align*}
\]

is also a weak homotopy equivalence.

### 2.1.2. A lemma in homotopy theory. Here the goal is to show that \( [\overline{A}^\delta_p(M)] \) and \( [A^\delta_0(M)] \) are weakly contractible. Proving the fat realization of the discrete version \( \overline{A}^\delta_p(M) \) is contractible is easier. Using a lemma in homotopy theory, we show that the contractibility of \( [\overline{A}^\delta_p(M)] \) implies the weak contractibility of \( [A^\delta_0(M)] \). This technique is originally due to Segal ([Seg78, Appendix]) and it is reformulated by Weiss in [Wei05, Lemma 2.2]. In particular, in the setting of semi-simplicial spaces, we use an application of this technique ([GRW17, Proposition 2.8]) due to Galatius and Randal-Williams.

**Proposition 2.13.** The realizations \( [A^\delta_0(M)] \) and \( [\overline{A}^\delta_p(M)] \) are weakly contractible.

**Proof.** We give a proof for weak contractibility of \( [A^\delta_0(M)] \), the case of \( [\overline{A}^\delta_p(M)] \) is similar. Let \( f : S^k \to [A^\delta_0(M)] \) be an element in the \( k \)-th homotopy group of \( [A^\delta_0(M)] \). Since \( [A^\delta_0(M)] \) is a CW-complex and \( S^k \) is compact, the map \( f \) hits finitely many simplices of \( [A^\delta_0(M)] \). Hence, there exists a point \( p \) and an embedded disk \( e(D^n) \) around it such that as an element of \( A^\delta_0(M) \) is not hit by the map \( f \). Thus, we have \( f(S^k) \subset [A^\delta_0(M\setminus e(D^n))] \). Adding the germ of \( e \) at \( p \) to the list of germs of embeddings of disks in \( M\setminus e(D^n) \) gives a semisimplicial null-homotopy for the inclusion \( A^\delta_0(M\setminus e(D^n)) \to A^\delta_0(M) \). Therefore, the element \( f(S^k) \) can be coned off inside \( [A^\delta_0(M)] \). \( \square \)

**Remark 2.14.** Note that because \( [A^\delta_0(M)] \) and \( [\overline{A}^\delta_p(M)] \) have CW structures, they are in fact contractible.
Since the space $A^s_\bullet(M)$ is discrete and $A_\bullet(M)$ is compactly generated weak Hausdorff space, by [RW16, Lemma 2.1], the maps
\begin{align}
|X^s_\bullet(M)| &\rightarrow \text{BHomeo}_0^s(M), \\
|X_\bullet(M)| &\rightarrow \text{BHomeo}_0(M),
\end{align}
are locally trivial fiber bundles with fibers $|A^s_\bullet(M)|$ and $|A_\bullet(M)|$ respectively. Therefore, by Proposition 2.13 the first map $|X^s_\bullet(M)|\rightarrow \text{BHomeo}_0^s(M)$ is a weak homotopy equivalence. To prove that the second map is also a weak homotopy equivalence, we need to show that $|A_\bullet(M)|$ is weakly contractible. To do so, we use the bisimplical technique due to Quillen [Qui73, Proof of Theorem A]. First note that since the map

$$A_\bullet(M) \xrightarrow{\sim} \overline{A}_\bullet(M)$$

is a weak equivalence, it induces a weak homotopy equivalence between the fat realizations. Hence, to show that $|A_\bullet(M)|$ is weakly contractible, it is enough to show that in the zig-zag
\begin{equation}
A_\bullet(M) \xrightarrow{\sim} \overline{A}_\bullet(M) \xrightarrow{\sim} \overline{A}^s_\bullet(M)
\end{equation}
the second map $\beta$ induces a weak homotopy equivalence between fat realizations. Note that $\beta$ is equivariant with respect to the map Homeo_0^s(M) → Homeo_0(M) and the first map is equivariant with respect to the action of Homeo_0(M) on its both sides.

**Definition 2.17.** Let $A_{\bullet,q}(M)$ be the bisemisimplicial space such that $A_{p,q}(M)$ is the subspace of $\overline{A}_p^s(M) \times \overline{A}_q(M)$ consisting of those $(p + q + 2)$-tuples

$$(a_0,\ldots,a_p,c_0,\ldots,c_q),$$

where the centers of the disks $a_i$ and the disks $c_j$ are pairwise disjoint.

The bisemisimplicial space $A_{\bullet,q}(M)$ is augmented in two different directions

$$\epsilon_p : A_{p,\bullet}(M) \rightarrow \overline{A}_p^s(M),$$
$$\delta_q : A_{\bullet,q}(M) \rightarrow \overline{A}_q(M).$$

Similar to [GRW17, Lemma 5.8], one can show that the following diagram is homotopy commutative
\begin{equation}
\begin{tikzcd}
|A^s_\bullet(M)| \ar[r, \epsilon] \ar[dr, \delta] & |\overline{A}_\bullet(M)| \\
|A_{\bullet,q}(M)| \ar[r, \sim] & |\overline{A}_\bullet(q)|
\end{tikzcd}
\end{equation}

**Proposition 2.19.** The fat realization $|A_\bullet(M)|$ is weakly contractible.

*Proof.* Since $|A_{\bullet,q}(M)| \xrightarrow{\sim} |\overline{A}_\bullet(M)|$, we instead show that $|\overline{A}_\bullet(M)|$ is weakly contractible. Because the diagram 2.18 is homotopy commutative and $|\overline{A}_\bullet(M)|$ is weakly contractible, if we show that the map $\delta$ is a weak homotopy equivalence, we then deduce that $|\overline{A}_\bullet(M)|$ is also weakly contractible.

Let $\mathcal{Q}$ be in $\overline{A}_q(M)$. By the definition of the bisemisimplicial space $A_{\bullet,q}(M)$, the fiber of the map $\delta_q$ over $\mathcal{Q}$ is

$$\delta_q^{-1}(\mathcal{Q}) = \overline{A}_p(M)\text{centers of }\mathcal{Q}).$$
Note that by Proposition 2.13, we know that $|\delta_q^{-1}(Q)|$ is contractible. Using [GRW17, Proposition 2.8], we deduce that the map

$$\delta_q : |A_{\bullet q}(M)| \to \mathcal{T}_q(M),$$

is a microfibration with a contractible fiber, hence it is a fibration (see [Wei05, Lemma 2.2] or [GRW17, Proposition 2.6]). Therefore $|\delta_q|$ induces a weak equivalence. By realizing in $q$-direction of both sides of the map $|\delta_q|$ in (2.20), we deduce that $\delta$ is also a weak homotopy equivalence.

\[\square\]

2.1.3. Reducing Theorem 1.3 to the case of manifolds with boundary. Recall that the maps in (2.15) are weak homotopy equivalence so the semi-simplicial spaces $X_{\bullet}(M)$ and $X^\delta_{\bullet}(M)$ are resolutions for $\text{BHomeo}_0(M)$ and $\text{BHomeo}^0_0(M)$ respectively. Therefore, to compare $\text{BHomeo}_0(M)$ and $\text{BHomeo}^0_0(M)$, we need to compare their resolutions. But there is no direct map between them. We, however, show that there is a map on the level of homology induced by the zig-zag of maps

$$X^\delta_{\bullet}(M) \leftarrow \mathcal{T}_{\bullet q}(M)/\text{BHomeo}^0_0(M) \to X_{\bullet}(M),$$

which in turn is induced by the zig-zag $\mathcal{A}_{\bullet q}(M) \leftarrow \mathcal{T}_{\bullet q}(M) \to \mathcal{A}_{\bullet}(M)$.

Given any $p$-simplex $\sigma$ in $\mathcal{T}_{\bullet q}(M)$, we have a map

$$\text{BStab}(\sigma) \to \mathcal{T}_{\bullet q}(M)/\text{BHomeo}^0_0(M),$$

where $\text{Stab}(\sigma)$ is the stabilizer of $\sigma$ under the action $\text{Homeo}^0_0(M)$. Recall that we fixed an element $e_p \in A_p(M)$ in the quasi-fibration 2.5, we can consider the same element $e_p \in \mathcal{T}_{\bullet q}(M)$ and therefore the stabilizer of $e_p$ is the group $\text{Stab}_p(M)$. Thus we have a map

$$\text{BHomeo}^0_0(M\setminus e_p, \partial(M\setminus e_p)) \to \mathcal{T}_{\bullet q}(M)/\text{BHomeo}^0_0(M).$$

Given the weak equivalences 2.5, 2.12 and the zig-zag 2.21, we have a homotopy commutative diagram

$$X^\delta_{\bullet}(M) \leftarrow \mathcal{T}_{\bullet q}(M)/\text{BHomeo}^0_0(M) \to X_{\bullet}(M) \uparrow \uparrow \uparrow \uparrow$$

$$\text{BHomeo}^0_0(M\setminus e_p, \partial(M\setminus e_p)) \leftarrow \text{BHomeo}^0_0(M\setminus e_p, \partial(M\setminus e_p)) \to \text{BHomeo}^0_0(M\setminus e_p, \partial(M\setminus e_p)).$$

The short exact sequence 2.11 and [Nar17, Corollary 2.3] \(^3\) implies that the bottom left horizontal map induces a homology isomorphism. Therefore, we have a zig-zag

$$X^\delta_{\bullet}(M) \xrightarrow{H_{\bullet-iso}} \text{BHomeo}^0_0(M\setminus e_p, \partial(M\setminus e_p)) \to X_{\bullet}(M),$$

which induces a map $\alpha_* : H_\bullet(X^\delta_{\bullet}(M)) \to H_\bullet(X_{\bullet}(M))$. Since the map $\alpha_*$ commutes with the face maps, we have an induced map between the spectral sequences given by the skeletal filtration of the realizations.

\(^3\)This corollary that says certain pushing collar maps between diffeomorphism groups induce homology isomorphisms also works for homeomorphisms
\[
\begin{array}{cccc}
H_q(X_p^q(M)) & \xrightarrow{\alpha_*} & H_q(X_p(M)) \\
\downarrow & & \downarrow \\
H_{p+q}(|X_p^q(M)|) & \xrightarrow{\iota_*} & H_{p+q}(|X_p(M)|) \\
\downarrow & & \downarrow \\
H_{p+q}(\wedge_\text{Homeo}(M)) & \xrightarrow{\iota_*} & H_{p+q}(\wedge_\text{Homeo}(M)).
\end{array}
\]

To reduce Thurston’s theorem to the case of manifolds with boundary, we need the following lemma.

**Proposition 2.25.** Given Thurston’s theorem 1.3 for manifolds with boundary, the map \(\alpha_*\) is an isomorphism.

**Proof.** Given the diagram 2.22, proving \(\alpha_*\) is an isomorphism is equivalent to proving the map

\[
\wedge_\text{Homeo}(\delta(M\setminus e_p, \partial(M\setminus e_p))) \to \wedge_\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p))
\]

induces a homology isomorphism. On the other hand, by the hypothesis, we know that the map

\[
\wedge_\text{Homeo}(\delta(M\setminus e_p, \partial(M\setminus e_p))) \to \wedge_\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p)),
\]

induces a homology isomorphism. Recall that the identity component of the topological group \(\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p))\) is weakly homotopy equivalent to the group \(\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p))\). Now using the comparison of Serre spectral sequences for the fibrations

\[
\begin{array}{cccc}
\wedge_\text{Homeo}(\delta(M\setminus e_p, \partial(M\setminus e_p))) & \xrightarrow{\iota^*} & \wedge_\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p)) \\
\downarrow & & \downarrow \\
\wedge_\text{Homeo}(\delta(M\setminus e_p, \partial(M\setminus e_p))) & \xrightarrow{\iota^*} & \wedge_\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p)) \\
\downarrow & & \downarrow \\
\wedge_\pi_0(\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p))) & \xrightarrow{\iota^*} & \wedge_\pi_0(\text{Homeo}(M\setminus e_p, \partial(M\setminus e_p))),
\end{array}
\]

we readily conclude that the middle horizontal map induces a homology isomorphism. \(\square\)

Hence, if we prove Thurston’s theorem for manifolds without 0-handles or more generally for manifolds with boundary, the comparison map on the \(E^1\)-page

\[
E^1_{p,q}(X^q_p(M)) \xrightarrow{\alpha_*} E^1_{p,q}(X_p(M)),
\]

is an isomorphism, so is on the \(E^\infty\)-page. Hence, we deduce that \(\iota\) in the diagram 2.24 induces a homology isomorphism which implies Thurston’s theorem for the closed manifold \(M\).

### 2.1.4. Higher dimensional handles.

Now we want to cut along the core of the higher dimensional handles. To do so, we use the same notation for the handlebody decomposition as in \([\text{CLM}]\). To recall their notation, let \(W\) be a manifold with boundary. To attach a handle of index \(q\), let \(\phi^q : S^{n-1} \times D^{n-q} \to \partial W\) be an embedding that admits an external collar similar to the 0-handle case. Let \(W + (\phi^q)\) denote the manifold \(W \cup_{\phi^q} D^n \times D^{n-q}\). And let \(\phi^q\) denote the embedding \(D^n \times D^{n-q} \to W + (\phi^q)\).
Definition 2.26. We say two handles $\phi_1, \phi_2 : D^q \times D^{n-q} \to M$ have the same germ around the core if there exists $\epsilon > 0$ such that

$$\phi_1|_{D^q \times D^{n-q}} = \phi_2|_{D^q \times D^{n-q}}$$

where $D^{n-q}$ consists of all $x \in D^{n-q}$ with the norm $|x| \leq \epsilon$. We denote the class of the germ of $\phi_i$ by $[\phi_i]$.

Given what we proved in the previous section, we can assume that $M$ is a manifold with boundary whose boundary components are in fact homeomorphic to spheres.

To reduce the problem to a manifold with fewer number of handles, we use the same idea as 0-handle resolutions. We shall define semisimplicial spaces encoding the space of choices of removing a handle.

Definition 2.27 (q-handle resolutions). There are versions with different topologies as Definition 2.1.

- **Discrete versions:** Let $\phi^q : D^q \times D^{n-q} \to M$ be a q-handle with an external collar such that $\phi^q(D^q \times D^{n-q}) \cap \partial M = \phi^q(S^{q-1} \times D^{n-q})$. We first define a semisimplicial set $H^q_\bullet(M, \phi^q)$ associated to $\phi^q$ as follows:
  - Let $e_{q+1}$ be the $(q+1)$-st standard basis element. The set of 0-simplices $H^0_\bullet(M, \phi^q)$, consists of pairs $(t, [\phi])$ where $[\phi]$ is a germ of a q-handle $D^q \times D^{n-q} \to M$ so that for a small $\epsilon$ we have
    \begin{equation}
    \phi|_{S^{q-1} \times D^{n-q}} = \phi^q|_{S^{q-1} \times (D^{n-q} + t.e_{q+1})},
    \end{equation}
    and $\phi(D^q \times \{0\})$ is isotopic to $\phi^q(D^q \times \{t.e_{q+1}\})$ relative to the boundary.
  - The set of p-simplices $H^p_\bullet(M, \phi^q)$, consists of $(p+1)$-tuples
    $$(t_0, [\phi_0]), (t_1, [\phi_1]), \ldots, (t_p, [\phi_p]),$$
    in $H^0_\bullet(M, \phi^q)^{p+1}$ so that $t_0 < t_1 < \cdots < t_p$ and the embedded cores $\phi_i(D^q \times \{0\})$ are disjoint.
  - Let $H^*_\bullet(M, \phi^q)$ be the semisimplicial set whose 0-simplices consist of pairs $(t, \phi)$ where $(t, [\phi]) \in H^0_\bullet(M, \phi^q)$. Note that the difference here is we consider actual embeddings not just their germs around the core. And let p-simplices be the subset of $H^p_\bullet(M, \phi^q)^{p+1}$ consisting of those $(p+1)$-tuples
    $$(t_0, \phi_0), (t_1, \phi_1), \ldots, (t_p, \phi_p),$$
    where the cores of $\phi_i$’s are pairwise disjoint.

Remark 2.29. Note that by definition, for every pair $(t, [\phi]) \in H^0_\bullet(M, \phi^q)$, the real number $t$ is uniquely determined by $\phi$. We denote this t-coordinate by $t_\phi$.

The group $Homeo_0^\delta(M, \partial M)$ acts on $H^q_\bullet(M, \phi^q)$ and $H^*_\bullet(M, \phi^q)$. The q-handle resolution associated to $\phi^q$ in this case is

$$X^q_\bullet(M, \phi^q) := H^q_\bullet(M, \phi^q) \times_{Homeo_0^\delta(M, \partial M)} BHomeo_0^\delta(M, \partial M).$$

- **Topological versions:** Let $H_\bullet(M, \phi^q)$ be the semisimplicial space whose 0-simplices as a set consists of pairs $(t, \phi)$ where $(t, [\phi]) \in H^0_\bullet(M, \phi^q)$. We topologize $H_\bullet(M, \phi^q)$ as the subspace of real numbers times the space of embeddings of a q-handle into $M$ equipped with the compact-open topology. The space of p-simplices $H_p(M, \phi^q)$ is a subspace of $H_0(M, \phi^q)^{p+1}$ consisting of $(p+1)$-tuples
    $$(t_0, \phi_0), (t_1, \phi_1), \ldots, (t_p, \phi_p),$$
    so that $t_0 < t_1 < \cdots < t_p$ and the embedded handles $\phi_i(D^q \times D^{n-q})$ are disjoint. It is topologized with the subspace topology.
Note that \( \text{Homeo}_0(M, \partial M) \) acts on \( \mathcal{H}_q(M, \phi^\ell) \) and we define the \( q \)-handle resolution associated to \( \phi^\ell \) in this case to be the augmented semisimplicial space:

\[
X_q(M, \phi^\ell) := \mathcal{H}_q(M, \phi^\ell) \{ \text{Homeo}_0(M, \partial M) \} \xrightarrow{f_{\phi^\ell}} B\text{Homeo}_0(M, \partial M).
\]

We want to prove that \( f_{\phi^\ell} \) and \( g_{\phi^\ell} \) induce weak homotopy equivalences. Similar to semi-simplicial resolutions 2.15 and Proposition 2.19, it is enough to show that \( \mathcal{H}_q^\ell(M, \phi^1) \) is contractible.

**Theorem 2.30.** Let \( M \) be a surface with boundary and let \( \phi^1 \) be a 1-handle, then the fat realizations \( \mathcal{H}_q^\ell(M, \phi^1) \) and \( \mathcal{H}_q^\ell(M, \phi^1) \) are weakly contractible.

**Remark 2.31.** In fact for a manifold \( M \) whose dimension is larger than 4, one can show that \( \mathcal{H}_q^\ell(M, \phi^1) \) is contractible if the handle index \( q \leq \dim(M)/2 \). If one shows that \( \mathcal{H}_q^\ell(M, \phi^1) \) is contractible for handle indices larger than the middle dimension, one could deduce Theorem 1.3 in all dimensions without resorting to foliation theory.

**Proof.** We give a proof that \( \mathcal{H}_q^\ell(M, \phi^1) \) is contractible and the proof for contractibility of \( \mathcal{H}_q^\ell(M, \phi^1) \) is the same. To show that a continuous map \( f : S^k \rightarrow \mathcal{H}_q^\ell(M, \phi^1) \) is nullhomotopic, we fix a triangulation \( K \) of \( S^k \) and without loss of generality, we assume that \( f \) is a PL-map from \( K \) to \( \mathcal{H}_q^\ell(M, \phi^1) \).

Let \( f : K \rightarrow \mathcal{H}_q^\ell(M, \phi^1) \) be a PL-map from a triangulation \( K \) of the sphere \( S^k \). To show that \( f \) is nullhomotopic, we show that there exists \( [\phi] \in \mathcal{H}_q^\ell(M, \phi^1) \) so that one can homotope the image \( f(K) \) into \( \text{Star}([\phi]) \). Note that for every \( v \in K \), the core of the germ of the embedded 1-handle \( f(v) \in \mathcal{H}_q^\ell(M, \phi^1) \) has a normal (micro)bundle since every core comes equipped with the germ of its cocore.

First we show that we can homotope \( f \) so that the cores of \( f(v) \) for all \( v \in K \) are pairwise transverse to each other. This part of the argument works for higher dimensional manifolds. But in dimension 2, the transversality argument is easier and in higher dimensions, one has to use the transversality in the sense of [KS77, Essay 3, section 1].

To do the first step, we need to consider the parallel copies of the handles. To explain what we mean by parallel copies, let \( f(v) = \phi_0 \in f(K) \) be a vertex and \( \{\phi_1, \phi_2, \ldots, \phi_n\} \) be all the vertices in \( f(K) \) that are connected to \( \phi_0 \). Recall that by definition of \( \mathcal{H}_q^\ell(M, \phi^1) \), there exists a small positive \( \epsilon \) such that

\[
\phi_0|_{S^2 \times D^1} = \phi^1|_{S^2 \times (D^1 + \epsilon \cdot e_2)}.
\]

A nearby parallel copy \( \phi'_0 : D^3 \times D^1 \rightarrow M \) can be described by \( \phi_0 \) restricted to \( D^3 \times (D^1 + \epsilon \cdot e_2) \). Note that \( [\phi'_0] \) is a vertex in \( \mathcal{H}_q^\ell(M, \phi^1) \) and since the cores of \( \phi'_0 \) and \( \phi_0 \) are disjoint, the vertices \( [\phi'_0] \) and \( [\phi_0] \) are connected in \( \mathcal{H}_q^\ell(M, \phi^1) \).

Let us enumerate the vertices of \( f(K) \) by \( [\psi_1], [\psi_2], \ldots, [\psi_m] \). First we choose a parallel copy of \( [\psi_2] \) and perturb it by a small isotopy to obtain \( [\psi'_2] \) so that its core becomes transverse to the core of \( [\psi_1] \). If the isotopy is small enough the core of \( [\psi'_2] \) is disjoint from the core of \( \psi_2 \) and the core of all vertices in \( f(K) \) that \( [\psi_2] \) was disjoint from. Therefore, there is a homotopy replacing \( [\psi'_2] \) with \( [\psi_2] \) and fixing the image of other vertices. Thus we may assume that \( [\psi_1] \) and \( [\psi_2] \) have transverse cores. Hence the intersection of their cores is a set of points.

Now we move on to \( [\psi_3] \). Similarly by choosing a nearby copy of \( [\psi_3] \) and a small perturbation \( [\psi'_3] \) of this nearby copy, we obtain a handle whose core is disjoint from the points in the intersection of the previous two handles \( [\psi_1] \) and \( [\psi_2] \). Hence we can choose a small neighborhood \( U \) of the intersection of the cores \( [\psi_1] \) and \( [\psi_2] \) such that the core of \( [\psi'_3] \) is also disjoint from \( U \). Now by Quinn’s transversality, we can find a small isotopy whose support is away from \( U \) and we obtain a handle
the vertices of \( f \) continuing this process we can change \( \phi \) core of all vertices of \( f \) points in the intersection of the cores \( \{\psi_j\} \)

Remark 2.33. Note that in general if \( q < \dim(M)/2 \), the realization of \( H^k_\ast(M, \phi^1) \) is contractible by the same argument as Proposition 2.13. Because transversality in this codimension implies disjointness.

Lemma 2.34. Let \( M \) be a surface with boundary and let \( \phi^1 \) be a 1-handle, the fat realization \( [H_\ast(M, \phi^1)] \) is weakly contractible.

\[ \text{Proof.} \quad \text{Similar to Proposition 2.19.} \]

Now since the fibers of the maps \([f_{\phi^1}] : X^\ast(M, \phi^1) \to \text{BHomeo}_0(M, \partial M)\) and \([g_{\phi^1}] : X^\ast(M, \phi^1) \to \text{BHomeo}_0^0(M, \partial M)\) are weakly contractible, by [RW16, Lemma 2.1] they induce weak equivalences.

2.1.5. The homotopy type of \( X^\ast(M, \phi^q) \) and \( X^\ast(M, \phi^q) \). Given that the boundary condition 2.28 is fixed by the action of \( \text{Homeo}_0(M, \partial M) \), unlike the case of the disk resolutions 3.5, the action of \( \text{Homeo}_0(M, \partial M) \) on \( H_0(M, \phi^q) \) and the action of \( \text{Homeo}_0^0(M, \partial M) \) on \( H^k_0(M, \phi^q) \) are not transitive. But there is a bijection between the set of the orbits of these actions.

- **Action of** \( \text{Homeo}_0(M, \partial M) \) on \( H_\ast(M, \phi^q) \): To determine the weak homotopy type of \( X_\ast(M, \phi^q) \), we shall first describe the orbits of the action \( \text{Homeo}_0(M, \partial M) \) on \( H_\ast(M, \phi^q) \). To do so, let us first introduce few notations. Let \( \sigma = (\phi_0, \phi_1, \ldots, \phi_p) \) be a \( p \)-simplex in \( H_p(M, \phi^q) \) and let \( M \backslash \sigma \) be the manifold obtained from \( M \) by removing the handles \( \phi_i(D^q \times \text{int}(D^{n-q})) \). Let also

\[ \text{Homeo}_0(M \backslash \sigma, \partial(M \backslash \sigma)), \]
denote the identity component of the compactly supported homeomorphisms of \( \text{int}(M) \setminus \cup_i \phi_i(D^q \times \text{int}(D^{n-q})) \). Note that the submanifold \( M \setminus \sigma \to M \) might have different connected components. Similar to Lemma 2.4, the sequence
\[
\text{Stab}(\sigma) \to \text{Homeo}_0(M, \partial M) \to \text{orbit}(\sigma),
\]
is a quasi-fibration where the topological group \( \text{Stab}(\sigma) \) is naturally identified with the group \( \text{Homeo}_0(M, \sigma, \partial(M \setminus \sigma)) \) whose identity component has the same homotopy type as \( \text{Homeo}_0(M \setminus \sigma, \partial(M \setminus \sigma)) \) (similar to the inclusion 2.3). Therefore, choosing once and for all, a \( p \)-simplex in each orbit, we obtain the map
\[
\bigsqcup_{\sigma} (\text{BHHomeo}_0(M, \sigma, \partial(M \setminus \sigma))) \xrightarrow{\sim} X_p(M, \phi^0),
\]
which is a weak homotopy equivalence (see Proposition 2.7).

\begin{itemize}
  \item **Action of Homeo_0^\delta(M, \partial M) on \( H_\delta^p(M, \phi^0) \):** To determine the weak homotopy type of \( X_\delta^p(M, \phi^0) \), we shall first describe the orbits of the action \( \text{Homeo}_0^\delta(M, \partial M) \) on \( H_\delta^p(M, \phi^0) \). Let \( [\sigma] = ([\phi_0], [\phi_1], \ldots, [\phi_p]) \) be a \( p \)-simplex in \( H_\delta^p(M, \phi^0) \) and let us denote \( \text{int}(M) \setminus \cup_i \phi_i(D^q \times \{0\}) \) by \( M([\sigma]) \). Let \( \text{Homeo}_0^\delta(M([\sigma])) \) denote the stabilizer of \( \sigma \) as an element of \( H_\delta^p(M, \phi^0) \) acted on by \( \text{Homeo}_0^\delta(M, \partial M) \). Similar to the short exact sequence 2.11, we have
\[
1 \to \text{Homeo}_0^\delta(M([\sigma])) \to \text{Homeo}_0^\delta(M([\sigma])) \to \pi_0(\text{Homeo}_0^\delta(M, \partial(M \setminus \sigma))) \to 1.
\]

Given that there is a bijection between the set of the orbits of these two actions, we shall consider the germs of the representatives of the first action, as a representative set of the orbits of the action of \( \text{Homeo}_0^\delta(M, \partial M) \) on \( H_\delta^p(M, \phi^0) \). Hence, by Shapiro’s lemma, we obtain a map
\[
\bigsqcup_{[\sigma]} (\text{BHHomeo}_0^\delta(M([\sigma]))) \xrightarrow{\sim} X_\delta^p(M, \phi^0),
\]
which is a weak equivalence.

\end{itemize}

**Remark 2.35.** Recall that the inclusion \( M \setminus \sigma \to M([\sigma]) \) induces a natural map
\[
\text{BHHomeo}_0^\delta(M, \sigma, \partial(M \setminus \sigma)) \xrightarrow{\sim} \text{BHHomeo}_0^\delta(M([\sigma])),
\]
which is a weak equivalence (similar to the map 2.3). The same map between discrete homeomorphisms
\[
\text{BHHomeo}_0^\delta(M, \sigma, \partial(M \setminus \sigma)) \xrightarrow{H_\delta^{\text{iso}}} \text{BHHomeo}_0^\delta(M([\sigma])),
\]
induces a homology isomorphism by the same argument as [Nar17, Corollary 2.3].

**Proof of Theorem 1.3 for \( \text{dim}(M) \leq 2 \):** In Proposition 2.25, we reduced the theorem to manifolds with boundary. Let us fix a handle decomposition of \( M \)
\[
M = \partial_0 M \times [0, 1] + (\phi_1^0) + (\phi_2^0) + \cdots + (\phi^0).
\]
We want to reduce the statement of the theorem for \( M \) to the case of a disk (Mather’s theorem [Mat71]) by cutting handles from \( M \). First suppose \( \text{dim}(M) = 1 \). Since we already reduced the theorem to the case of manifolds with boundary and in this dimension, a manifold with boundary is homeomorphic to the union of disks for which the theorem holds by Mather’s theorem ([Mat71]). Now we assume \( M \) is a surface with boundary.

**Claim 2.36.** If Thurston’s theorem holds for \( M \setminus \sigma \) for all \( \sigma \in \mathcal{H}_p(M, \phi^1) \) and for all \( p \), it also holds for \( M \).
Proof of the claim: Similar to Proposition 2.25, we have a zig-zag of maps from $X_p^\delta(M,\phi^1)$ to $X_p(M,\phi^1)$ which corresponds to the zig-zag

$$\text{BHomeo}_\delta^\delta(M([\sigma])) \xrightarrow{H_\ast \text{-iso}} \text{BHomeo}_\delta^\delta(M\setminus \sigma,\partial(M\setminus \sigma)) \rightarrow \text{BHomeo}_\delta^\delta(M\setminus \sigma,\partial(M\setminus \sigma)).$$

Given the hypothesis of the lemma, the above zig-zag induces a homology isomorphism between $\text{BHomeo}_\delta^\delta(M([\sigma]))$ and $\text{BHomeo}_\delta^\delta(M\setminus \sigma,\partial(M\setminus \sigma))$. Hence the induced map between $H_\ast(X_p^\delta(M,\phi^1))$ and $H_\ast(X_p(M,\phi^1))$ is an isomorphism. Therefore, by the comparison of the spectral sequences

$$H_\ast(X_p^\delta(M,\phi^1)) \xrightarrow{\cong} H_\ast(X_p(M,\phi^1))$$

we conclude that $\iota_\ast$ is an isomorphism. \(\blacksquare\)

For a 1-handle $[\sigma^1]$ and a $p$-simplex $\sigma \in H_p(M,\phi^1)$, the manifold $M\setminus \sigma$ is homeomorphic to the union of $p$ disks and $M\setminus \phi^1(D^3 \times \text{int}(D^3))$. Thus using the claim, the theorem holds for $M$ if it holds for $M\setminus \phi^1(D^3 \times \text{int}(D^3))$. Therefore, by removing $1$-handles inductively from the surface $M$, we can reduce the theorem for $M$ to the case of a disk, which is given by Mather’s theorem ([Mat71]). \(\Box\)

Remark 2.37. In fact using Remark 2.31 and a similar argument as above, one can show that Thurston’s theorem for a manifold $M$ whose dimension is larger than 4 is equivalent to Thurston’s theorem for a trivial bordism $N \times D^3$ where $N$ is a manifold whose dimension is $\dim(M) - 1$.

2.2. The case of three manifolds. To do exactly similar argument as the case of surfaces, we need to find contractible semi-simplicial spaces that cut the manifold into union of disks. Doing an inductive process to cut a three manifold into disks, however, is harder than the case of surfaces.

For certain types of three-manifolds, namely for Haken 3-manifolds, this process of cutting into disks is well known. Recall that $M$ is Haken if it is irreducible and contains a properly embedded two sided incompressible surface. Being an irreducible 3-manifold means that every embedded 2-sphere bounds a ball. The existence of this ball allows us to do a similar surgery argument as we did for isotopic arcs in a surface. Recall that a compact connected surface $S$, not $S^2$, in $M$ is an incompressible surface, if it is properly embedded $S \cap \partial M = \partial S$, and the normal bundle of $S$ is trivial and the inclusion $S \hookrightarrow M$ is $\pi_1$ injective. Given the Haken manifold theory, there is a finite sequence of incompressible surfaces that as we cut a Haken manifold $M$ along those surfaces, we obtain disjoint union of balls.

The idea is to induct on the number of prime factors in a prime decomposition of $M$ to reduce Thurston’s theorem to the case of Haken manifolds and then use the hierarchy of Haken manifolds to reduce it to the case of disks.

Let $M \cong kP\# N$ be the connected sum of $k$ copies of a prime manifold $P$ and a manifold $N$ where $N$ has no prime factor homeomorphic to $P$. We will define semi-simplicial spaces with contractible realizations that encode different ways of cutting $M$ into the union of $k$ copies of $P\setminus \text{int}(D^3)$ and $N\setminus \cup_{j=1}^k \text{int}(D^3)$ and copies of $S^2 \times [0,1]$. By the same argument as the previous section, a spectral sequence argument shows that Thurston’s theorem holds for $M$ if it does for $P\setminus \text{int}(D^3)$ and $N\setminus \cup_{j=1}^k \text{int}(D^3)$ and $S^2 \times [0,1]$. We then show that Thurston’s theorem for Haken manifolds implies that the theorem holds for $S^2 \times [0,1]$ and for prime manifolds with
any number of disks removed. Therefore, we deduce that it holds for $P \setminus \text{int}(D^3)$ and by induction for $N \setminus \cup_{i=1}^k \text{int}(D^3)$.

2.2.1. **Cutting along separating spheres.** Let $\phi : S \to M$ be an embedding of a surface $S$ in $M$ with a trivial normal bundle. To cut along this embedding, similar to Definition 2.27, we define different semisimplicial spaces.

**Definition 2.38.** Fix an embedding of the two-sided collar $\psi : S \times [-1, 1] \to M$. The germ of embeddings of $S$ into $M$ is defined similar to Definition 2.26 and we define the core of $\psi$ to be its restriction to $S \times \{0\}$. We consider the following semi-simplicial spaces associated to $\psi$.

- **Discrete version:** Let $K^d_\psi(M, \psi)$ be a semisimplicial set defined as follows:
  - If $S$ has no boundary, the set of 0-simplices $K^d_0(M, \psi)$, consists of germs of embeddings $[\phi]$ so that the core of $\phi$ is isotopic to a parallel copy of the core of $\psi$. And the set of $p$-simplices $K^d_p(M, \psi)$, consists of $(p + 1)$-tuples $([\phi_0], [\phi_1], \ldots, [\phi_p])$ of germs of embeddings so that their cores are disjoint.
  - If $S$ has a boundary, the set of 0-simplices $K^d_0(M, \psi)$, consists of pairs $(t, [\phi])$ where $[\phi]$ is a germ of an embedding $S \times [-1, 1] \to M$ so that for a small $\epsilon$ we have
    $$\phi(\partial S \times (-\epsilon, \epsilon)) = \psi(\partial S \times (-\epsilon, t + \epsilon)),$$
    and $\phi(S \times \{0\})$ is isotopic to $\psi(S \times \{t\})$ relative to the boundary.
  - The set of $p$-simplices $K^d_p(M, \psi)$, consists of $(p + 1)$-tuples
    $$((t_0, [\phi_0]), (t_1, [\phi_1]), \ldots, (t_p, [\phi_p])),\)$$
    in $K^d_0(M, \psi)^{p+1}$ so that $t_0 < t_1 < \cdots < t_p$ and the embedded surfaces $\phi_i(S \times \{0\})$ are disjoint. The face maps are given by forgetting the embeddings.
    Note that for every $(t, \phi) \in K^d_0(M, \psi)$, the coordinate $t$ is uniquely determined by $\phi$. We might just write $\phi$ for a vertex and refer to its $t$-coordinate by $t_\phi$.
  - The face maps are given by omitting the coordinates.

- **Topological versions:** For a surface $S$ with boundary, let $K_\ast(M, \psi)$ be the semi-simplicial space whose 0-simplices as a set consists of pairs $(t, \phi)$ where $(t, [\phi]) \in K^d_0(M, \psi)$. We topologize $K_0(M, \psi)$ as the subspace of real numbers times the space of embeddings the collared surface $S \times [-1, 1]$ into $M$ equipped with the compact-open topology. The space of $p$-simplices $K_p(M, \psi)$ is a subspace of $K_0(M, \psi)^{p+1}$ consisting of $(p + 1)$-tuples
    $$((t_0, \phi_0), (t_1, \phi_1), \ldots, (t_p, \phi_p)),\)$$
    so that $t_0 < t_1 < \cdots < t_p$ and the embedded collared surfaces $\phi_i(S \times [-1, 1])$ are disjoint. It is topologized with the subspace topology. The case of the closed surface $S$ is defined similarly without the $t$-coordinate. The face maps are given by omitting the coordinates.

**Definition 2.39.** Let $\psi : S \times [-1, 1] \to M$ for $1 \leq i \leq k$ be a fixed set of disjoint proper embeddings. We define the semi-simplicial set $\mathcal{K}^d_i(M; \psi_1, \psi_2, \ldots, \psi_k)$ whose $p$-simplices

$$K^d_p(M; \psi_1, \psi_2, \ldots, \psi_k) \subset K^d_p(M, \psi_1) \times K^d_p(M, \psi_2) \times \cdots \times K^d_p(M, \psi_k)$$

consist of those $k$-tuples whose cores are pairwise disjoint. We define the topological version $\mathcal{K}_i(M; \psi_1, \psi_2, \ldots, \psi_k)$ similarly.

We assume that $M$ is orientable, the non-orientable case is similar. Now for $1 \leq i \leq k$, let

$$\phi_i : S^2 \times [-1, 1] \to M$$

(2.40)
be embeddings whose cores cut \( M \) into \( k+1 \) connected components that are homeomorphic to the disjoint union of \( k \) copies of \( P \backslash \text{int}(D^3) \) and \( N \cup \bigcup_{i=1}^{k} \text{int}(D^3) \). Since in the prime decomposition of \( N \) there is no factor homeomorphic to \( P \), for a \( p \)-simplex \( \sigma_p \in \mathcal{K}_p(M; \phi_1, \phi_2, \ldots, \phi_k) \), the manifold \( M \backslash \sigma_p \) is homeomorphic to the disjoint union of \( k \) copies of \( P \backslash \text{int}(D^3) \), \( N \cup \bigcup_{i=1}^{k} \text{int}(D^3) \) and \( pk \) copies of \( S^2 \times [0,1] \).

To show that the realization the semi-simplicial set \( \mathcal{K}_\bullet(M; \phi_1, \phi_2, \ldots, \phi_k) \) is contractible, we need to assume that all prime manifolds in the prime decomposition of \( M \) are irreducible. So let us first reduce to the case that this assumption holds. To secure this assumption, we need to cut out solid tori from \( M \) by defining certain semi-simplicial spaces.

**Definition 2.41.** Let \( \phi : S^1 \times D^2 \to M \) be a \( \pi_1 \)-injective embedding. Let \( \mathcal{T}_0(M; \phi) \) be the space of embeddings of a solid torus whose core is isotopic to the core of \( \phi \) and \( \mathcal{T}_p(M; \phi) \subset \mathcal{T}_0(M; \phi)^{p+1} \) is a subspace consisting of \( p+1 \) tuples of disjoint embeddings. We define the discrete version \( \mathcal{T}_\bullet^D(M; \phi) \) similar to Definition 2.38, by taking germs of embeddings with the discrete topology.

**Lemma 2.42.** The fat realizations \( |\mathcal{T}_\bullet^D(M; \phi)| \) and \( |\mathcal{T}_\bullet^D(M; \phi)| \) are weakly contractible.

**Proof.** It is enough to show that \( |\mathcal{T}_\bullet^D(M; \phi)| \) is contractible (see Proposition 2.19). Similar to Theorem 2.30, to show that a continuous map \( f : S^k \to |\mathcal{T}_\bullet^D(M; \phi)| \) is nullhomotopic, we fix a triangulation \( K \) of \( S^k \) and without loss of generality, we assume that \( f \) is a PL-map from \( K \) to \( |\mathcal{T}_\bullet^D(M; \phi)| \). Again by the similar argument as Theorem 2.30, we can assume that the core of vertices in the image of \( f \) are pairwise transverse. But note that in this case the codimension of the core of a solid torus is 2 so transversality in this codimension implies disjointness. Therefore, by applying transversality we can find a vertex \( v \) in \( \mathcal{T}_0^D(M; \phi) \) whose core is disjoint from the core of vertices in the image of the map \( f \) which implies that \( f(K) \subset \text{Star}(v) \). Hence, the map \( f \) is null-homotopic. \( \square \)

**Proposition 2.43.** If Theorem 1.3 holds for those three manifolds that are homeomorphic to a connected sum of irreducible manifolds, then it also holds for any three manifold.

**Proof.** It is well-known (see [Hat, Proposition 1.4]) that the only orientable prime 3-manifold that is not irreducible is \( S^1 \times S^2 \). For \( 1 \leq i \leq n \), let \( \theta_i : S^1 \times D^2 \to S^1 \times S^2 \) be \( \pi_1 \)-injective embeddings of solid tori. If the embeddings \( \theta_i \)'s are disjoint, it is easy to see that \( S^1 \times S^2 \cup \bigcup_{i=1}^{n} \theta_i(S^1 \times \text{int}(D^2)) \) is irreducible.

Suppose in the prime decomposition of \( M \) there are \( k \) copies of \( S^1 \times S^2 \). We inductively reduce to the case with fewer copies of \( S^1 \times S^2 \)'s. To do so, we want to cut out solid tori from these summands. Let \( \phi : S^1 \times D^2 \to M \) be a \( \pi_1 \)-injective embedding whose image is in one of the copies of \( S^1 \times S^2 \). Note that for all \( \sigma \in \mathcal{T}_\bullet(M; \phi) \), the manifold \( M \backslash \sigma \) obtained from \( M \) by removing the interior of the solid tori in \( \sigma \), has fewer non-irreducible summand in its prime decomposition. Therefore, the argument in Claim 2.36 implies if we have Thurston's theorem for those 3-manifolds with irreducible summands, we have the theorem for all 3-manifolds. \( \square \)

Now that we can assume the prime factors in \( M \) are all irreducible, we prove the contractibility of the semi-simplicial spaces of separating spheres \( \phi_i \) in 2.40.

**Theorem 2.44.** If \( M \) is a connected sum of irreducible 3-manifolds, the fat realizations \( |\mathcal{K}_\bullet^D(M; \phi_1, \phi_2, \ldots, \phi_k)| \) and \( |\mathcal{K}_\bullet(M; \phi_1, \phi_2, \ldots, \phi_k)| \) are weakly contractible.

**Proof.** We give the proof for the case where \( k = 1 \) and for the general \( k \), the argument is the same. Recall from Proposition 2.13 that the contractibility of \( |\mathcal{K}_\bullet(M; \phi_1)| \) implies the weak contractibility of \( |\mathcal{K}_\bullet(M; \phi_1)| \). So it is enough to prove the former.
We can represent an element of the $k$-th homotopy group of $[K^d_\bullet(M;\phi_1)]$ by a PL map $f: K \rightarrow [K^d_\bullet(M;\phi_1)]$ where $K$ is a triangulation of $S^k$. By the similar argument as Theorem 2.30, we can assume that the core of vertices in the image of $f$ are pairwise transverse. Let $v_1 \in K^d_\bullet(M;\phi_1)$ be a vertex whose core is transverse to the core of vertices in $f(K)$. To show that $f$ is null-homotopic, we homotope $f$ to a map $g$ so that $g(K) \subset \text{Star}(v_1)$. To do this we need to consider all separating spheres in the prime decomposition at once. Let $\{w_1, w_2, \ldots, w_m\}$ be a set of separating spheres in a prime decomposition of $M$ where $w_1$ is the core of $v_1$. We also assume that all $w_i$'s are transverse to the cores of the vertices in $f(K)$.

**Claim 2.45.** Let $w$ be an embedded sphere in $M$ that is isotopic to $w_1$ and is transverse to all $w_i$'s. Let $C$ be an innermost circle in the intersection of $w$ and $w_i$'s, i.e. it bounds a disk $D$ in $w$ so that the interior of $D$ does not intersect $w_i$ for any $i$. If $C$ is in the intersection of $w_1$ and $w$, it bounds a disk $D'$ in $w_1$ so that the embedded sphere $D \cup D'$ bounds a ball in $M$.

We call the ball whose boundary is $D \cup D'$, the Whitney ball. Because we can push $D$ along the ball to remove the intersection $C$.

**Proof of the claim:** Since int$(D)$ does not intersect any of the spheres $w_i$'s, it lies entirely in one of the irreducible components, say $P_j \setminus \text{int}(D^3)$ whose boundary is the sphere $\partial w_j$. Because $P_j$ is irreducible either of two disks in $w_j$ that bounds $C$ union $D$ is an embedded sphere in $P_j$, hence bounds a ball but one of these balls lies entirely in $P_j \setminus \text{int}(D^3)$. Therefore, the circle $C$ bounds a disk $D'$ in $w_j$ so that the sphere $D' \cup D$ bounds a ball in $M$. ■

By doing surgery similar to Theorem 2.30, we want to homotope the map $f$ to reduce the number of circles in the intersection of the core of vertices of $f(K)$ with the spheres $\{w_1, w_2, \ldots, w_m\}$. By the claim for the core of any vertex in $f(K)$ that intersect the union of $w_i$'s, there exists a Whitney ball. Let $\theta_0 = f(s_0) \in f(K)$ be a vertex whose Whitney ball is innermost, i.e. if $\{\theta_1, \theta_2, \ldots, \theta_n\}$ are the vertices that are connected to $\theta_0$ in $f(K)$, they do not intersect the Whitney ball of $\theta_0$. Therefore, by pushing the core of $\theta_0$ along the the Whitney ball, we could obtain a vertex $\theta'_0$ whose core is still disjoint from the core of vertices $\{\theta_1, \theta_2, \ldots, \theta_n\}$. By considering a near parallel copy of $\theta'_0$, we can assume that the core of $\theta'_0$ is also disjoint from the core of $\theta_0$. Therefore, we can homotope the map $f$ to a map $g$ so that it takes the same value on all vertices in $K$ but $s_0$ and $g(s_0) = \theta'_0$. By repeating this process, we reduce the number of circles in the intersection of the cores of $f(K)$ with the spheres $\{w_1, w_2, \ldots, w_m\}$ until we homotope $f$ into the star of the vertex $v_1$.

The contractibility of these semi-simplicial spaces, similar to the case of surfaces, reduces Thurston’s theorem to the case of $P \setminus \text{int}(D^3)$ and $N \setminus \bigcup_{i=1}^n \text{int}(D^3)$ and copies of $S^2 \times [0,1]$.

2.2.2. **Reducing Thurston’s theorem to the case of Haken manifolds.** Let us first consider the case $S^2 \times [0,1]$.

**Proposition 2.46.** Theorem 1.3 holds for $M = S^2 \times [0,1]$ if it holds for Haken 3-manifolds.

**Proof.** Choose a 1-handle $\phi: D^1 \times D^2 \rightarrow S^2 \times [0,1]$ so that $\phi([0] \times D^2) \subset S^2 \times \{0\}$ and $\phi([1] \times D^2) \subset S^2 \times \{1\}$. Note that the codimension of this 1-handle is less than the half of the dimension of the ambient manifold. Therefore, similar to Lemma 2.42, transversality implies that $[H^d_\phi(S^2 \times [0,1], \phi)]$ is contractible. Hence, as in Claim 2.36, Thurston’s theorem holds for $S^2 \times [0,1]$ if it holds for $S^2 \times [0,1] \setminus \sigma$ for all $p$-simplexes $\sigma \in H^d_\phi(S^2 \times [0,1], \phi)$ and all $p$. But for a $p$-simplex $\sigma$, the manifold $S^2 \times [0,1] \setminus \sigma$ is a handle-body, so it is Haken. ■
As the general strategy is to cut along submanifolds, we always get manifolds with boundary. Furthermore, an irreducible 3-manifold with boundary is Haken. But note that the sphere boundaries in \( P(\text{int}(D^3)) \) and \( N \setminus \bigcup_{i=1}^k \text{int}(D^3) \) destroys the irreducibility. So to reduce Thurston’s theorem for \( P(\text{int}(D^3)) \) and \( N \setminus \bigcup_{i=1}^k \text{int}(D^3) \) to the case for Haken manifolds, we first cut along certain 1-handles to reduce the number of sphere boundaries. To do so, we need to show that \( P(\text{int}(D^3)) \) and \( N \setminus \bigcup_{i=1}^k \text{int}(D^3) \) are not simply connected.

**Lemma 2.47.** If a 3-manifold \( M \) with boundary is simply connected, it is obtained from \( S^3 \) by removing the interior of a union of disjoint balls in \( S^3 \).

**Proof.** It is enough to show that the boundary \( \partial M \) is homeomorphic to union of \( S^2 \)'s. Because if we fill in the sphere boundaries by balls, we obtain a simply connected closed 3-manifold which has to be homeomorphic to \( S^3 \) by Perelman’s theorem ([Per02, Per03]). Since \( M \) is simply connected, we have \( H_1(M) = 0 \), so by the Poincaré-Lefschetz duality, we also have \( H_2(M, \partial M) \cong H^1(M) = 0 \). The homology long exact sequence for the pair \((M, \partial M)\) implies that \( H_2(M, \partial M) \to H_1(\partial M) \to H_1(M) \) is exact. Therefore, \( H_1(\partial M) = 0 \) which implies that \( \partial M \) is homeomorphic to a union of \( S^2 \)'s.

Let \( Q \) be the manifold obtained from \( P \) by removing the interior of \( m \) disjoint balls in \( P \). To prove Thurston’s theorem for \( Q \), we want to cut 1-handles from \( Q \) to make it irreducible. Not that since \( P \) is not simply connected and is not homeomorphic to sphere, so by Lemma 2.47, the manifold \( Q \) is not simply connected either. Let \( \partial_i Q \) be the \( i \)-th boundary component. We choose an arc \( \gamma_i \) with the two ends on \( \partial_i Q \) so that the arc \( \gamma_i \) with a path between its two ends on the boundary is non-trivial in the fundamental group of \( P \).

Let \( \phi_i : D^1 \times D^2 \to Q \) be a 1-handle whose core is \( \gamma_i \). Let us denote the manifold obtained from \( Q \) by removing the interior of the handle \( \phi_i \) by \( Q \setminus \bigcup_{i=1}^m \phi_i \). Given that \( P \) is irreducible, it is easy to see that \( Q \setminus \bigcup_{i=1}^m \phi_i \) is also irreducible. Because every embedded sphere in \( Q \setminus \bigcup_{i=1}^m \phi_i \) bounds a ball in \( P \). If this ball contains any of the boundary components with the 1-handle attached to it, then the core union the path between the two ends of the core on the boundary would be trivial in the fundamental group of \( P \), which is a contradiction.

**Proposition 2.48.** Thurston’s theorem 1.3 holds for \( Q \), if it does for Haken manifolds.

**Proof.** By the above discussion, if we remove at least one handle in \( H^0_{\partial_i}(Q, \phi_i) \) from \( Q \) for each \( i \), we obtain a Haken manifold. Because it is an irreducible manifold whose boundary components have positive genus. We inductively reduce the number of sphere boundary components by cutting along 1-handles whose cores are isotopic to a parallel copy of the core of \( \phi_i \)’s.

Similar to Claim 2.36, we want to show that Theorem 1.3 holds for \( Q \) if it holds for \( Q \sigma \) for all \( \sigma \in H^0_{\partial_i}(Q, \phi_i) \) and all \( p \). To do this, it is enough to show the semi-simplicial sets \( H^0_{\partial_i}(Q, \phi_i) \) has contractible realization. Since the codimension of the cores is larger than half of the dimension of \( Q \), transversality implies that \( |H^0_{\partial_i}(Q, \phi_i)| \) is contractible (see Lemma 2.42).

Similarly, we can reduce Theorem 1.3 for \( N \setminus \bigcup_{i=1}^k \text{int}(D^3) \) the case where all boundary components have positive genus. Now we can apply prime decomposition for 3-manifolds with boundary ([Hem04, Section 3]). Note that all the prime factors are irreducible again, so we can apply Theorem 2.44 to inductively reduce to the case with fewer prime factors. Hence, Theorem 1.3 for \( N \setminus \bigcup_{i=1}^k \text{int}(D^3) \) is also deduced from Proposition 2.48.
2.2.3. Theorem 1.3 for Haken 3-manifolds. By the theory of Haken manifolds ([Hak62]), we know that they have a hierarchy, where they can be split up into 3-balls along incompressible surfaces. Let \( \psi : S \times [-1, 1] \to M \) be a proper embedding of an incompressible surface with its trivial normal bundle. Given the case of Haken manifolds which are lower compared to \( M \) in the Haken hierarchy, we inductively prove Theorem 1.3 for \( M \) by considering the semi-simplicial set \( K_0^2(M, \psi) \) (see Definition 2.38 to recall its definition).

Note that for any \( \sigma \in K_0^2(M, \psi) \), the manifold \( M \setminus \sigma \) is homeomorphic to the disjoint union of \( M \setminus \psi(S \times \{0\}) \) with \( p \) copies of \( S \times [-1, 1] \). By induction on the Haken hierarchy, we can assume that Theorem 1.3 holds for \( M \setminus \psi(S \times \{0\}) \). To apply Claim 2.36, we need also to know Theorem 1.3 for \( S \times [-1, 1] \). But \( S \times [-1, 1] \) is a handlebody (a 3-ball with 1-handles attached) so there are finitely many properly embedded 2-disks (that are in fact incompressible surface in \( S \times [-1, 1] \)) such that if we cut along those disks, we obtain a 3-ball. Hence, it is a special case of Haken manifolds. Therefore, to finish the proof of Theorem 1.3 for three manifolds, it is left to prove the following proposition.

**Proposition 2.49.** Let \( M \) be a Haken manifold with boundary. The fat realization \( |K_0^2(M, \psi)| \) is contractible.

**Proof.** Let us represent an element of the homotopy group \( f : S^k \to |K_0^2(M, \psi)| \) by a PL map with respect to some triangulation \( K \) on \( S^k \). Similar to Theorem 2.30 we can homotope \( f \) so that the core of the vertices of \( f(K) \) are pairwise transverse. Also by the same argument, we can choose \( \phi \in K_0^2(M, \psi) \) so that the collared embedding \( \phi(S) \) is transverse to the core of vertices of \( f(K) \) and its \( t \)-coordinate \( t_\phi \) is different from that of vertices of \( f(K) \). We want to homotope \( f \) to a PL map \( g : K \to |K_0^2(M, \psi)| \) so that \( g(K) \subset \text{Star}(\phi) \), hence \( f \) becomes nullhomotopic.

Since the intersections of \( \phi(S) \) with the core of vertices of \( f(K) \) are transverse and also they do not intersect on the boundary \( \partial M \), all intersections are circles. We want to do surgery on the image of \( f \) to remove these circles. We first do surgery on the circles that are nullhomotopic in \( M \).

**Case 1:** Since \( \phi(S) \) is incompressible, any nullhomotopic circle in the intersection of \( \phi(S) \) and the core of the vertices of \( f(K) \) is in fact nullhomotopic in \( \phi(S) \). Therefore such circles bound a disk \( D \) in \( \phi(S) \). Choose a metric on the surface \( \phi(S) \) and among the nullhomotopic circles in the intersection, let \( C \) be the one whose interior has the minimal area. Suppose \( C \) is in the intersection of \( \phi(S) \) and \( \phi_0(S) \) where \( \phi_0 = f(v) \in f(K) \) is a vertex in the image of \( f \) and \( \{\phi_1, \phi_2, \ldots, \phi_n\} \) is the set of all the vertices in \( f(K) \) that are connected to \( \phi_0 \).

Again by the incompressibility the circle \( C \) bounds a disk \( D_0 \) in \( \phi_0(S) \). Since \( M \) is irreducible, the sphere \( D \cup D_0 \) bounds a ball \( B \) in \( M \). Note that by the choice of the circle \( C \), the ball \( B \) does not intersect \( \phi_i(S) \) for \( 1 \leq i \leq n \). By pushing \( D \) across \( B \) to \( D_0 \) and considering a nearby parallel copy, we obtain \( \phi'_0 \in K_0^2(M, \psi) \) whose core is disjoint from \( \phi_0(S) \) and the core of all vertices that are connected to \( \phi_0 \) in \( f(K) \). Therefore, we obtain a homotopy \( F : K \times [0, 1] \to |K_0^2(M, \psi)| \) where \( F(-, 0) = f \), \( F(v, 1) = \phi' \) and \( F(-, 1) \) is the same as \( f \) on vertices other than \( v \). But also note that \( \phi'_0(S) \) has fewer circle component in its intersection with \( \phi(S) \).

Hence, by repeating this process, we can eliminate all nullhomotopic intersections.

**Case 2:** Now suppose none of the circles in the intersection of the cores of the vertices of \( f(K) \) and \( \phi(S) \) is nullhomotopic. We use Hatcher’s idea ([Hat76, Page 342]) to deal with this case. Let \( p : M \to \tilde{M} \) be the covering corresponding to the subgroup \( \pi_1(\phi(S), a) \subset \pi_1(M, a) \) where \( a \) is a base point in \( \phi(S) \). Let \( S_0 \) be the homeomorphic lift of \( \phi(S) \) passing through the base point \( \tilde{a} \) of \( \tilde{M} \). Let \( \{S_i\} \) be the components of \( p^{-1}(\phi(S)) \). Each \( S_i \) separates \( \tilde{M} \) into two components. Let
\(M_i\) be the closure of the component that does not contain the boundary of \(S_0\). Let \(M_{0i}\) be minimal with respect to inclusion among \(M_i\)'s that intersect the lifts of the vertices of \(f(K)\).

Suppose that for a vertex \(v \in K\), we denote the lift of the surface \(f(v)(S)\) that intersects \(M_{0i}\) by \(\hat{f}(v)(S)\), if it exists. Let \(C_v\) be a component of \(f(v)(S) \cap M_{0i}\).

Laudenbach (see [Lau74, Corollary II.4.2] and also [Hat99, page 8]) showed that there is a unique trivial h-cobordism \(W_v\) in \(M_{0i}\) whose one end is \(C_v\) and the other end lies on a lift of \(\phi(S)\). Furthermore, \(p\) is a homeomorphism restricted to \(W_v\).

Among the trivial cobordisms, for a vertex \(v \in K\), let \(W_v\) be minimal with respect to inclusion. Let \(\{v_1, v_2, \ldots, v_n\}\) be the vertices connected to \(v\) in \(K\). Since \(f(v)(S)\) and \(f(v_i)(S)\) are disjoint for all \(i\) and \(W_v\) is minimal, one can see that \(W_v\) does not intersect \(W_{v_i}\). Now the trivial cobordism \(p(W_v)\) plays the role of the ball \(B\) in the previous case.

By pushing \(p(C_v)\) across the trivial bordism to its other boundary and considering a nearby parallel copy, we obtain \(\phi'_0 \in \mathcal{K}_0^\phi(M, \psi)\) whose core is disjoint from \(\phi_0(S)\) and the core of all vertices that are connected to \(\phi_0\) in \(f(K)\). Therefore, we get a homotopy \(F : K \times [0, 1] \to [\mathcal{K}_0^\phi(M, \psi)]\) where \(F(-, 0) = f, F(v, 1) = \phi'_0\) and \(F(-, 1)\) is the same as \(f\) on vertices other than \(v\). Now the number of circles in the intersections of \(\phi(S)\) and the vertices of \(F(-, 1)\) has been reduced by one. By repeating this process, we can eliminate all the remaining intersections. \(\square\)

3. Smoothing Theory in Low Dimensions

Let \(M\) be a smooth manifold of dimension 2 or 3. To use a similar technique to prove Theorem 1.1 which says

\[
\text{BDiff}(M) \to \text{BHomeo}(M),
\]

is a weak equivalence, first one needs to show that

\[
\pi_0(\text{Diff}(M)) = \pi_0(\text{Homeo}(M)).
\]

This is not hard for surfaces ([Bol99]) but for 3-manifolds, it follows from a theorem of Cerf ([Cer61]). Assuming that \(\text{Diff}(M)\) and \(\text{Homeo}(M)\) have the same group of connected components, the statement is reduced to showing that the map

\[
\eta : \text{BDiff}_0(M) \to \text{BHomeo}_0(M),
\]

induces a weak homotopy equivalence. But note that both spaces \(\text{BDiff}_0(M)\) and \(\text{BHomeo}_0(M)\) are simply connected, therefore it is enough to show that \(\eta\) induces a homology isomorphism.

Similar to the previous section, by using certain semi-simplicial spaces, we want to cut the manifold into pieces until we get to the disks. And the case for disks is a corollary of Smale’s theorem ([Sma59]) for 2-disks and Hatcher’s theorem ([Hat83]) for 3-disks. But the only difference to the previous section is instead of having discrete and topologized versions of semisimplicial spaces, we would have the smooth version and the topological version with a slight modification. For those semi-simplicial spaces that involve the boundary of the manifold, we have to control the behavior near the boundary.

**Condition.** Let \(M\) be a smooth manifold with boundary and let \(c : \partial M \times [0, 1] \to M\) be a collar neighborhood. For all semi-simplicial spaces (topological versions) that involved the boundary, namely \(\mathcal{H}_\ast(M, \phi^\partial)\), we impose an extra condition of being smooth near the boundary: for example a vertex \([\phi] \in \mathcal{H}_0(M, \phi^\partial)\) which is given by an embedding \(\phi : D^n \times D^{n-}\partial \to M\) that restricts to a smooth embedding in a neighborhood of the boundary \(c(\partial M \times [0, \epsilon])\) for some \(\epsilon\). Note that still \(\text{Homeo}_0(M, \partial M)\) acts on \(\mathcal{H}_\ast(M, \phi^\partial)\) since \(\text{Homeo}_0(M, \partial M)\) consists of homeomorphisms whose supports are away from the boundary.
**Definition 3.3.** We can define the smooth version of the topological version of all semi-simplicial spaces considered in the previous section and we denote them by superscript sm. For example, let $\mathcal{H}_{\ast}^{sm}(M, \phi^{q}) \in \mathcal{H}_{\ast}(M, \phi^{q})$ be a sub-semisimplicial space consisting of smooth handles with an induced $C^{0}$-topology rather than $C^{\infty}$-topology.

**Lemma 3.4.** The maps from the smooth version of the semi-simplicial spaces to the corresponding topological versions are equivariant with respect to the map $\text{Diff}_{0}(M) \rightarrow \text{Homeo}_{0}(M)$ and induce bijections between the set of orbits of the corresponding actions.

*Proof.* We give the proof for the map

$$\mathcal{H}_{\ast}^{sm}(M, \phi^{q}) \rightarrow \mathcal{H}_{\ast}(M, \phi^{q}),$$

and the other cases are similar. Note that in every orbit of the action of $\text{Homeo}_{0}(M)$ on $\mathcal{H}_{\ast}(M, \phi^{q})$, there is a smooth handle. Hence, the induced map between orbits is surjective. To show that it is also injective, we want to show that for two smooth handles $\phi, \phi' \in \text{Emb}_{\phi}^{sm}(D^{q} \times D^{n-q}, M) \subset \text{Emb}_{\phi}(D^{q} \times D^{n-q}, M)$, if $\phi$ and $\phi'$ are in the same orbit of the action of $\text{Homeo}_{0}(M, \partial M)$ on $\text{Emb}_{\phi}(D^{q} \times D^{n-q}, M)$, then they are in the same orbit of the action of $\text{Diff}_{0}(M, \partial M)$ on $\text{Emb}_{\phi}^{sm}(D^{q} \times D^{n-q}, M)$.

Now by the isotopy extension theorems for homeomorphisms and diffeomorphisms, we have maps between quasi-fibrations

$$\text{Diff}(M \setminus \partial M, \partial M) \rightarrow \text{Emb}_{\phi}^{sm}(D^{q} \times D^{n-q}, M)$$

By our assumption 3.2, since the first two vertical maps induce bijection on $\pi_{0}$ so does the third vertical map. So, $\phi$ and $\phi'$ are in the same path component of $\text{Emb}_{\phi}^{sm}(D^{q} \times D^{n-q}, M)$. Therefore, by the isotopy extension theorem again there exists an element in $\text{Diff}_{0}(M, \partial M)$ that sends $\phi$ to $\phi'$.

Similar to Theorem 2.30, one can prove that the smooth version of semi-simplicial spaces are weakly contractible. Hence, a similar argument as the previous section reduces Theorem 1.1 to the fact that the map

(3.5) $\text{BDiff}(D^{n}, \partial D^{n}) \rightarrow \text{BHomeo}(D^{n}, \partial D^{n}),$

is a weak equivalence in these dimensions.

**Remark 3.6.** In dimension 3, Hatcher ([Hat83]) proved that the map

$$\text{SO}(4) \xrightarrow{\sim} \text{Diff}(S^{3}),$$

is a weak equivalence. It is standard to see that this version of Hatcher’s theorem is equivalent to the weak equivalence 3.5 for $n = 3$. Cerf in ([Cer61]) also used a different method to prove that the weak equivalence

$$\text{Diff}(M) \xrightarrow{\sim} \text{Homeo}(M),$$

can be reduced to Hatcher’s theorem.

4. **Contractibility of the Identity Component of the Diffeomorphism Group for Certain Low Dimensional Manifolds**

Note that for a manifold $M$, the (weak) contractibility of $\text{Diff}_{0}(M)$ is equivalent to the acyclicity of the classifying space $\text{BDiff}_{0}(M)$. Similar to previous sections, we obtain a semisimplicial resolution for $\text{BDiff}_{0}(M)$ by cutting the manifold into simpler pieces. To show that $\text{BDiff}_{0}(M)$ is acyclic, we then study the spectral sequence associated to the semisimplicial resolutions.
4.1. **Contractibility of \( \text{Diff}_0(\Sigma, \partial \Sigma) \) for a surface \( \Sigma \) with boundary.** We sketch a new proof of the contractibility of the identity component of the diffeomorphisms of a surface with boundary by first showing that \( \text{BDiff}_0(\Sigma, \partial \Sigma) \) is acyclic. Therefore, by the Whitehead theorem it should be contractible. By the argument of the previous section, we deduce that \( \text{Homeo}_0(\Sigma, \partial \Sigma) \) is also weakly contractible.

For a closed surface \( \Sigma \) with a negative Euler number, the contractibility of \( \text{Diff}_0(\Sigma) \) was first proved by Earle and Eells ([EE69]) using Teichmüller theory and it was later extended to the surfaces with boundary by Earle and Schatz ([ES70]). Therefore, by the techniques of the previous sections, the contractibility of \( \text{Homeo}_0(\Sigma) \) which is a theorem of Hamstrom [Ham74] can be reduced to the contractibility of \( \text{Diff}_0(M) \) which has a more concrete proof.

Gramain ([Gra73]) gave a topological proof of contractibility of \( \text{Diff}_0(\Sigma) \) for \( \Sigma \) with a negative Euler number and hence found a new proof of the contractibility of the Teichmüller space. As was explained in Hatcher’s exposition ([Hat 11, Appendix B]), the case of the closed surface can be easily reduced to the case of a surface with boundary.

Gramain’s proof reduces to the case of a disk by proving that certain space of embeddings of arcs into a surface is contractible. But the advantage of working with semi-simplicial sets is that proving the contractibility of their realizations is often easier and more combinatorial. Having the contractibility of such semi-simplicial sets, it was a homotopy theory lemma (Proposition 2.19) that implies that the realization of the corresponding semi-simplicial spaces is weakly contractible.

As the input to our proof, we also use the contractibility of \( \text{Diff}(D^2, \partial D^2) \) and for a non-separating arc \( \sigma \) between two points on the boundary, we use a \( \pi_0 \)-statement that the the map between the mapping class groups

\[
\pi_0(\text{Diff}(\Sigma|\sigma, \partial \Sigma|\sigma)) \to \pi_0(\text{Diff}(\Sigma, \partial \Sigma)),
\]

is injective where \( \Sigma|\sigma \) is a surface obtained from \( \Sigma \) by cutting along \( \sigma \).

**Theorem 4.2.** Let \( \Sigma \) be a surface with a boundary, \( \text{BDiff}_0(\Sigma, \partial \Sigma) \) is acyclic.

**Proof.** Similar to section 2.1.5, we use induction on handles to reduce to the case of the disk. For a 1-handle \( \phi \) and a \( p \)-simplex \( \sigma_p \in \mathcal{H}_p(\Sigma, \phi) \), by the induction hypothesis, \( \text{BDiff}(\Sigma|\sigma_p, \partial (\Sigma|\sigma_p)) \) is acyclic. Note that the surface \( \Sigma|\sigma_p \) is a union of \( p \) disjoint disks and a surface that is diffeomorphic to \( \Sigma|\phi \).

By the isotopy extension theorem, we have a fibration

\[
\text{Diff}(\Sigma|\sigma_p, \partial (\Sigma|\sigma_p)) \to \text{Diff}(\Sigma, \partial \Sigma) \to \text{Emb}_0(\sigma_p, \Sigma).
\]

Given the injectivity of the map 4.1, we deduce that

\[
\text{Diff}_0(\Sigma|\sigma_p, \partial (\Sigma|\sigma_p)) \to \text{Diff}_0(\Sigma, \partial \Sigma) \to \text{orb}(\sigma_p),
\]

is also a fibration. Hence, there is a weak equivalence

\[
\bigcup_{\sigma_p} \text{BDiff}_0(\Sigma|\sigma_p, \partial (\Sigma|\sigma_p)) \simeq X_p(\Sigma, \phi),
\]

where the disjoint union is over a representative set of orbits. Given the induction hypothesis that \( \text{BDiff}_0(\Sigma|\sigma_p, \partial (\Sigma|\sigma_p)) \) is acyclic, the spectral sequence

\[
E^1_{p,q} = H_q(X_p(\Sigma, \phi)) \Rightarrow H_{p+q}(\text{BDiff}_0(\Sigma, \partial \Sigma); \mathbb{Z}),
\]

is concentrated in the first row \( q = 0 \). We have

\[
H_0(X_p(\Sigma, \phi)) = \mathbb{Z}[\text{the set of the orbits of the } p\text{-simplices}].
\]

Note that the set of orbits of the action of \( \text{Diff}_0(\Sigma, \partial \Sigma) \) on \( \mathcal{H}_p(\Sigma, \phi) \) is in bijection with \( (p+1)\)-tuples \( (t_0, t_1, \ldots, t_p) \) in \( \phi(\{0\} \times \text{int}(D^1)) \subset \partial \Sigma \). Therefore, we can denote the set of the orbits by the semi-simplicial set \( \text{Conf}(\bullet) \) where \( \text{Conf}(p) \) is the...
set of $p + 1$ points in $\mathbb{R}$. Let us denote the first differential of the spectral sequence by $\delta$ which is given by the alternating sum $\Sigma(-1)^i d_i$. of the maps induced by the face maps, $a_i$, of the semi-simplicial set $\text{Conf}(\bullet)$. Hence, it is enough to prove the following claim:

**Claim 4.3.** The chain complex $(\mathbb{Z}[\text{Conf}(\bullet)], d)$ is acyclic.

**Proof of the claim:** To prove the claim, let us recall a theorem attributed to Moore in [MP73, Theorem 4.1]. For a topological space $X$, let $S_p(X)$ denote the the group of singular $p$-chains of $X$ with coefficients in $\mathbb{Z}$. For a semisimplicial space $X_\bullet$, let $\delta : S_\bullet(X_\bullet) \to S_{\bullet-1}(X_\bullet)$ denote the map given by the alternating sum of maps induced by the face maps. Let $d : S_\bullet(X_\bullet) \to S_{\bullet-1}(X_\bullet)$ be the singular boundary maps. Recall the total differential is $D = d + (-1)^p \delta$ for elements in $S_p(X_\bullet)$. Hence, we obtain a total chain complex $(S_\bullet(X_\bullet), D)$. Moore ([MP73, Theorem 4.1]) proved that there is a natural chain equivalence

$$f : (S_\bullet(X_\bullet), D) \to (S_\bullet([X_\bullet]), d).$$

Let us apply this theorem for $X_\bullet = \text{Conf}(\bullet)$. To compute the homology of the double complex $(S_\bullet(\text{Conf}(\bullet)), D)$, we first filter it in the simplicial direction. The first page of the associated spectral sequence is $E_{1,q}^p = H_q(\text{Conf}(p) ; \mathbb{Z})$. But $\text{Conf}(\bullet)$ is a discrete space, therefore it is concentrated in the first row $q = 0$. Note that the chain complex $E_{1,0}^1$ is the same as $(\mathbb{Z}[\text{Conf}(\bullet)], \delta)$. Since this spectral sequence collapses, by the Moore theorem, we have

$$E_{p,0}^2 = E_{p,0}^\infty = H_p([\text{Conf}(\bullet)] ; \mathbb{Z}).$$

But similar to Proposition 2.13, one can show that $[\text{Conf}(\bullet)]$ is weakly contractible, therefore the chain complex $E_{1,0}^1 = (\mathbb{Z}[\text{Conf}(\bullet)], \delta)$ must be acyclic. $\square$

### 4.2. Contractibility of $\text{BDiff}_0(M, \partial M)$ for a Haken manifold $M$ with boundary

Hatcher computed the homotopy type of the space of PL homeomorphisms of Haken manifolds in [Hat76]. Given his proof of Smale’s conjecture, his computation of PL homeomorphisms carries over to diffeomorphisms of the Haken manifolds ([Hat99]). Here, we simplify his proof of the contractibility of $\text{BDiff}_0(M, \partial M)$ for a Haken manifold $M$ with boundary using the same idea as the proof of Theorem 4.2. Hatcher improved Laudenbach’s surgery techniques ([Lau74, Chapter 2.5]) to a parametrized surgery on the space of incompressible surfaces. In a way, we simplify Hatcher’s proof by avoiding his parametrized surgery argument.

Let $\psi : S \times [-1, 1] \to M$ be an embedding of a two-sided collar of an incompressible surface. Recall in Definition 2.38, we defined $K_\bullet(M, \psi)$ whose realization is weakly contractible as the corollary of Proposition 2.49. Hence, we can define an augmented semi-simplicial space $X_\bullet(M, \psi) \to \text{BDiff}_0(M, \partial M)$ as follows

$$X_\bullet(M, \psi) := K_\bullet(M, \psi) \cup \text{BDiff}_0(M, \partial M).$$

Given that $|K_\bullet(M, \psi)|$ is weakly contractible, $X_\bullet(M, \psi)$ is a semisimplicial resolution for $\text{BDiff}_0(M, \partial M)$ i.e. the induced map

$$|X_\bullet(M, \psi)| \to \text{BDiff}_0(M, \partial M),$$

is a weak equivalence.

**Theorem 4.4.** If $M$ is a Haken manifold with boundary, the classifying space $\text{BDiff}_0(M, \partial M)$ is acyclic.

**Proof.** Note that for every $p$-simplex $\sigma_p \in K_p(M, \psi)$, the space $M \setminus \sigma_p$ is diffeomorphic to the disjoint union of $M \setminus \psi(S)$ with $p$ copies of $S \times [-1, 1]$. By the induction on the Haken hierarchy, we can assume that $\text{BDiff}_0(M \setminus \psi(S), \partial(M \setminus \psi(S)))$ is acyclic. Recall that $S \times [-1, 1]$ is a handlebody so there are finitely many embedded 2-disks
such that if we cut along those disks, we obtain a 3-ball. Thus contractibility of \( \text{Diff}_0(S \times [-1,1], \partial(S \times [-1,1])) \) is in fact a special case of Theorem 4.4. Therefore, we can assume that for all \( \sigma_p \), the space \( \text{BDiff}_0(M \setminus \partial(M \setminus \sigma_p)) \) is acyclic.

To identify the weak homotopy type of \( X_p(M, \psi) \), we need to determine the homotopy type of \( \text{Stab}(\sigma_p) \) for each \( \sigma_p \in \mathcal{K}_p(M, \psi) \). Recall by the isotopy extension theorem, we have a fibration

\[
\text{Diff}(M \setminus \sigma_p, \partial(M \setminus \sigma_p)) \to \text{Diff}(M, \partial M) \to \text{Emb}_0(\sigma_p, M).
\]

By [Lau74, Chapter 2, Section 7.2], the fundamental group of \( \text{Emb}_0(\sigma_p, M) \) is trivial, therefore we have an injection

\[
\pi_0(\text{Diff}(M \setminus \sigma_p, \partial(M \setminus \sigma_p))) \to \pi_0(\text{Diff}(M, \partial M)).
\]

Thus we have a fibration

\[
\text{Diff}_0(M \setminus \sigma_p, \partial(M \setminus \sigma_p)) \to \text{Diff}_0(M, \partial M) \to \text{orb}(\sigma_p),
\]

which implies that there is a weak equivalence

\[
\coprod_{\sigma_p} \text{BDiff}_0(M \setminus \sigma_p, \partial(M \setminus \sigma_p)) \cong X_p(M, \psi),
\]

where the disjoint union is over a representative set of orbits. Therefore, similar to the proof of Theorem 4.2, the spectral sequence

\[
E^{1}_{p,q} = H_q(X_p(M, \psi)) \Rightarrow H_{p+q}(\text{BDiff}_0(M, \partial M); \mathbb{Z}),
\]

implies that \( \text{BDiff}_0(M, \partial M) \) is acyclic.

**Remark 4.5.** We end with a question about hyperbolic three manifolds. Let \( M \) be closed hyperbolic 3-manifold. Gabai in [Gab01] used his high powered “insulator” machinery (see [Gab97]) and minimal surface theory to prove that \( \text{Diff}_0(M) \) is contractible by reducing to the case of Haken manifolds with boundary. We wondered if the same techniques could prove Gabai’s theorem without using high powered tools in geometry. To find a semisimplicial resolution for \( \text{BDiff}_0(M) \) let \( \gamma \) be a closed geodesic in \( M \). Fix a parametrized tubular neighborhood of \( \gamma \) by embedding \( \phi : D^2 \times S^1 \to M \) so that \( \phi(\{(0,0) \times S^1\}) = \gamma \).

**Definition 4.6.** Let \( B_\bullet(M) \) be a semisimplicial space whose space of 0 simplices is given by the space of oriented closed curves that are isotopic to \( \gamma \). We define \( B_p(M) \) as a subspace of \( B_0(M)^{p+1} \) to be the space of \( (p+1) \)-tuples \( \sigma_p = (\gamma_0, \gamma_1, \ldots, \gamma_p) \) so that there exists a diffeomorphism \( f_{\sigma_p} \in \text{Diff}_0(M) \) where \( f_{\sigma_p}(\gamma_i) = \phi((\{t_i,0\}) \times S^1) \) for a \( t_i \) such that \( t_0 < t_1 < \cdots < t_p \). The \( i \)-th face maps is given by forgetting the \( i \)-th curve.

**Question 4.7.** Is \( |B_\bullet(M)| \) weakly contractible?

Note that similar to Proposition 2.19, it is enough to show that realization of the semi-simplicial set \( B_\bullet(M)^{ \delta } \) is contractible. If the answer to this question is affirmative, one could give a simpler proof of Gabai’s theorem as follows: Consider the semisimplicial resolution

\[
B_\bullet(M) \# \text{Diff}_0(M) \to \text{BDiff}_0(M).
\]

Since the action of \( \text{Diff}_0(M) \) on \( B_\bullet(M) \) is transitive, for a \( p \)-simplex \( \sigma_p \) in \( B_p(M) \), we have \( B_p(M) \# \text{Diff}_0(M) \cong \text{BStab}(\sigma_p) \). Given that the complement of \( \sigma_p \) in \( M \) is a Haken manifold, the identity component of \( \text{Stab}(\sigma_p) \) is contractable, therefore \( \text{BStab}(\sigma_p) \cong \pi_0(\text{Stab}(\sigma_p)) \). On the other hand, using JSJ decomposition and some hyperbolic geometry, it is not hard to show that \( \pi_0(\text{Stab}(\sigma_p)) \) is isomorphic to the pure braid group \( \text{PBBr}_{p+1} \). Hence, one might have a spectral sequence

\[
E^{1}_{p,q} = H_q(\text{PBBr}_{p+1}) \Rightarrow H_{p+q}(\text{BDiff}_0(M); \mathbb{Z}),
\]
but recall that a model for $BPBr_{p+1}$ is an ordered configuration space $Emb([p], D^2)$. Thus the above spectral sequence converges to the realization of the semi-simplicial space $Emb([\bullet], D^2)$. Now from Proposition 2.19 we know that the realization of $Emb([\bullet], D^2)$ is weakly contractible, therefore the above spectral sequence converges to zero in positive degrees.

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