A variational principle for time of arrival of null geodesics

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Normally the issue or question of the time of arrival of light rays at an observer coming from a
given source is associated with Fermat’s Principle of Least Time which yields paths of extremal time.
We here investigate a related but different problem. We consider an observer receiving light from
an extended source that has propagated in an arbitrary gravitational field. It is assumed from the
start that the propagation is along null geodesics. Each point of the extended source is sending out
a light-cones worth of null rays and the question arises which null rays from the source arrive first
at the observer. Stated in a different fashion, a pulse of light comes from the source with a wave-
front as the leading edge, which rays are associated with that leading edge. In vacuum flat-space we
have, from Huygen’s principle, that the rays normal to the source constitute the leading edge and
hence arrive first at an observer. We here investigate this issue in the presence of a gravitational
field. Though it is not obvious, since the rays bend and are focused by the gravitational field and
could even cross, in fact it is the normal rays that arrive earliest. We give two proofs both involving
the extremization of the time of arrival, one based on an idea of Schrodinger for the derivation of
gravitational frequency shifts and the other based on V.I. Arnold’s theory of generating families.

Dedicated to Jayant Narlikar.

I. INTRODUCTION

If one considers the problem of the “time of arrival” of a light signal coming from an extended source (a two-
surface) and arriving at an observer who is moving along an arbitrary time-like world-line, one normally invokes
Fermat’s Principle of Least Time to determine the path of the light-ray from each point on the source to the observer.
We will discuss a different but related issue - at each source point there is a light-cones worth of rays that are emitted
and the question is what are the directions of the rays for them to have the earliest “time of arrival” at the observer.
Stated in a different manner, if the source suddenly lights-up, then from each point there will be a sphere’s worth of rays
leaving it resulting in a pulse arriving at the observer from different source points with a leading edge followed by
a tail. The question then is which rays arrive earliest? From the start we assume that the paths of the rays are given
by null geodesics, so that there is no need to invoke Fermat’s Principle. In flat vacuum space-time it is clear that it
is the normal rays that arrive earliest, i.e., they travel the shortest distance. However in a Lorentzian manifold with
a time varying metric the issue is not as obvious. The problems are that there is no well defined distance between
source and observer, the rays can bend and be focused by the field and could even cross or even more than one ray
could arrive at the observer from the same source point. Nevertheless we will show that it is the rays that are emitted
normal to the source that are the ones that arrive earliest, i.e., the time of arrival function is extremized by the normal
rays.

We will give two alternative proofs of this. The first proof, given in Sec.II, involves a modification of the beautiful
derivation by Schrodinger of the gravitational frequency shift while the second proof, in Sec.III, uses the theory of
generating families of V.I. Arnold.

Probably the most powerful technique for the study of wavefronts and their related characteristic (or null) surfaces
in Lorentzian space-times and the associated difficulties in the analytic description of the development of caustics
and crossover regions is Arnold’s theory of Lagrangian and Legendre submanifolds and the associated Lagrange and
Legendre maps\cite{1}. One of the main ingredients in this theory is the construction of what has been referred to as
generating families. They are, in general, two-point functions, $F(x,s)$ (chosen from, perhaps, different spaces, $x$ and $s$,
with perhaps different dimensions), that are to be constructed from physical arguments and which are stationary with
respect to variations in one of the two different spaces. In our case we give an example of this construction where the
two-point function is the time-of-arrival function of light rays which begin from points on a two-surface, embedded in
a four dimensional space-time, (thought of as a source of radiation), and which end at points on a one-dimensional
manifold, a curve (thought of as the world-line of an observer of that radiation) also embedded in the same four-space.

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Though we will not amplify on it here, this example appears to be, in principle if not in practice, of generic use in the theory of gravitational lensing in any Lorentzian space-time.

II. THE TIME OF ARRIVAL FUNCTION

We are concerned with the travel time of light signals from an extended source to a localized observer. For our purposes, the source lights up instantaneously, in its own rest frame, emitting photons in all directions from every point on its (closed) surface. There is one photon that arrives first at the observer’s location, in the observer’s proper time. If the metric is stationary, then this photon is, intuitively, the one that takes the shortest spatial path, perpendicularly to the surface of the source. In the following, we make these notions more precise, extending them to the case of arbitrary metrics.

Consider, in an arbitrary Lorentzian four-dimensional manifold, a given closed space-like two-surface, \( S \), described by

\[
x^a = x_0^a(s^1, s^2)
\]

where \( x^a \) are space-time coordinates in the neighborhood of the source, and \( s^J = (s^1, s^2) \) parametrize the surface \( S \). In addition, consider a timelike worldline, \( L \). In the neighborhood of the worldline, with no loss of generality, let the local coordinates be such that \( L \) is given by \( (\tau, X^i) \) where the \( X^i \) are three constants, the spatial location of the observer, and \( \tau \) is the proper time along the worldline.

From each point \( s^J \) of \( S \), construct its future lightcone, \( C_{s^J} \). In general, in the absence of horizons, the line \( L \) intersects each \( C_{s^J} \) at least once. The intersection takes place at a particular value of the proper time \( \tau \) for each point \( s^J \) on the surface. This means that there is a two-point function

\[
\tau = T(X^i, s^J).
\]

that represents the proper time of arrival at \( L \) of light signals from \( S \). One explicit way of constructing such a function is as follows. The lightcone \( C_{s^J} \) is foliated by light-rays from the point \( x_0^a(s^J) \), which are solutions

\[
x^a = \gamma^a(r; s^J, \theta, \phi)
\]

of the geodesic equation with initial data labeled by the initial point \( s^J \) and the initial direction \( (\theta, \phi) \) of the ray. Here \( r \) can be thought of as an affine parameter along the null geodesics. The intersection of the lightcone with the worldline \( L \) takes place at points where

\[
\gamma^i(r; s^J, \theta, \phi) = X^i
\]

and the time \( (x^0 = \tau) \) at which the light ray reaches the observer is

\[
\tau = \gamma^0(r; s^J, \theta, \phi)
\]

where the values of \( (r; s^J, \theta, \phi) \) in the right-hand side are restricted by \( (3) \). In cases where \( (3) \) is invertible for every value of \( s^J \), it provides \( (r, \theta, \phi) \) as functions of \( (X^i, s^J) \), which can be inserted into \( (5) \) to yield \( (2) \).
The proper time of arrival, $T$, at $L$, is extremized by those rays that leave $S$ perpendicularly to it. In other words, the points $s^J$ such that

$$\frac{\partial T}{\partial s^J}(X^i, s^J) = 0$$

are connected to $L$ by light-rays that are normal to $S$.

**Proof:** The proof is based on the standard variational principle for null geodesics, and is an extension of a similar result in [4]. Consider the action

$$I(p, q) = \int_0^1 g_{ab} \dot{x}^a \dot{x}^b dr,$$

where $\dot{x}^a \equiv dx^a/dr$ is the tangent vector to an affinely parametrized null geodesic between the points $p = x^a(0)$ and $q = x^a(1)$, and $s$ is the affine parameter. Consider the variation of $I$ constructed by taking the difference between two neighboring null geodesics with different initial points, $p_1$ and $p_2$, and different final points, $q_1$ and $q_2$. Since $I$ evaluates identically to zero in both instances, its variation is zero as well,

$$0 = \Delta I = I(p_2, q_2) - I(p_1, q_1).$$

The variation is

$$\Delta I = \int_0^1 g_{ab,c} \dot{x}^a \dot{x}^b \delta x^c + 2g_{ab} \dot{x}^a \delta \dot{x}^b \, dr$$

$$= \int_0^1 \left( (g_{ab,c} - 2g_{cb,a}) \dot{x}^a \dot{x}^b - 2g_{cb} \ddot{x}^b \right) \delta x^c + 2 \frac{d}{dr} (g_{ab} \dot{x}^a \delta \dot{x}^b) \, dr.$$

Since the curves are null geodesics, the term proportional to $\delta x^c$ in the integrand vanishes, and we are left with
\[
\Delta I = 2 \left( g_{ab} \dot{x}^a \delta x^b \right)_{r=0}.
\]  
(11)

By (8) and (11), we have
\[
g_{ab} \dot{x}^a \delta x^b \bigg|_{r=1} = g_{ab} \dot{x}^a \delta x^b \bigg|_{r=0}
\]  
(12)

In (12), \(\delta x^b\) represents an arbitrary (up to the condition that \(p\) and \(q\) can be connected by a null geodesic) displacement at \(r = 0\) and \(r = 1\) between the null geodesic with tangent \(\dot{x}^b\) and a neighboring one. We now particularize (12) to our case of interest, in which, initially, neighboring null geodesics are connected by displacements on the surface \(S\); i.e.,
\[
\delta x^b \bigg|_{r=0} = \left( \frac{\partial x^b}{\partial s^J} \right) ds^J.
\]  
(13)

where \(\frac{\partial x^b}{\partial s^J}\) are the two coordinate tangent vectors to \(S\) and \(ds^J\) is arbitrary. The final displacement must be tangent to \(L\), i.e.,
\[
\delta x^b \bigg|_{r=1} = v^b d\tau
\]  
(14)

with \(v^b\) the tangent vector to the curve \(L\). However, because the two null geodesics arriving at \(r = 1\) and separated by \(\delta x^b \big|_{s=1}\) must be the same pair of null geodesics leaving \(r = 0\) separated by \(\delta x^b \big|_{r=0}\) then \(d\tau\) is not arbitrary, but
\[
d\tau = dT \bigg|_{X^i} = \frac{\partial T}{\partial s^J} ds^J.
\]  
(15)

With (13), (14) and (15), Eq. (12) reads
\[
\left( g_{ab} \dot{x}^a \frac{\partial x^b}{\partial s^J} - g_{ab} \dot{x}^a v^b \frac{\partial T}{\partial s^J} \right) ds^J = 0.
\]  
(16)

Since \(ds^J\) is arbitrary, and since \(g_{ab} \dot{x}^a v^b\) can not vanish as long as \(\dot{x}^a\) and \(v^b\) are tangent to a null and a timelike curve, respectively, then (16) is equivalent to
\[
\frac{\partial T}{\partial s^J} = \left( \frac{1}{g_{cd} \dot{x}^c v^d} \right) g_{ab} \dot{x}^a \frac{\partial x^b}{\partial s^J}.
\]  
(17)

This implies that, at each point of \(S\), \(\partial T/\partial s^J\) vanishes if and only if the null ray \(\dot{x}^a\) is normal to the surface at that point. This proves the theorem.□

Note that since no property of the line \(L\) was used, the theorem can be restated as follows. Given a time foliation of a Lorentzian manifold with local coordinates chosen as \((\tau, X^i)\), and a source, a closed two-surface that “lights up”, the time \(\tau = T(x^i, s^J)\) of arrival at any spatial point \(X^i\), of light signals from a surface point \(s^J\), is extremized by the light-rays leaving the surface perpendicularly.

### III. EIKONALS AND THE TIME OF ARRIVAL

In the following, we provide an alternative method for obtaining the time of arrival function which is based entirely on the use of the eikonal equation - with Arnold’s generating families - and specifically on knowledge of a two-parameter family of solutions of the eikonal equation,
\[
g^{ab}(x^c) \partial_a Z \partial_b Z = 0;
\]  
(18)
i.e., it is assumed that a solution, with the two parameters \(\alpha^A = (\alpha^1, \alpha^2)\)
\[
u = Z(x^a, \alpha^A)
\]  
(19)
to Eq. (18) is known. Then for each value of \(\alpha^A\) the level surfaces of \(Z\) are null (i.e., \(\partial_a Z\) is a null covector). Furthermore it is assumed that at each point \(x^c\), \(\partial_a Z\) sweeps out the entire null cone at \(x^a\) as \(\alpha^A\) goes through its range.
Remark. We point out and emphasize that the level surfaces of the solutions to \( \text{Eq.}(18) \) though referred to as “null or characteristic surfaces” are not strictly speaking surfaces; they can have self-intersections and in general are only piece-wise smooth. Though Arnold refers to them as “big-wave-fronts” we will continue to call them null surfaces. The intersection of a big wave front with a generic three surface yields a two-dimensional (small) wave front.

The first thing that we want to show is that the light-cone, \( \mathcal{C}_{x_0} \), from an arbitrary space-time point \( x_0 \) can be constructed from knowledge of the function \( Z \) of \( \text{Eq.}(19) \).

One sees immediately, from \( \text{Eqs.}(18) \) and \( (19) \), that the function

\[
S^*(a, x_0, \alpha^A) = Z(a, \alpha^A) - Z(x_0, \alpha^A) = 0
\]

defines a two-parameter set of surfaces which all pass thru the point \( x_0 \) and which, furthermore, are all null surfaces. The envelope of this family is constructed by demanding that

\[
\partial_{\alpha^A} S^*(a, x_0, \alpha^A) = 0
\]

where \( \partial_{\alpha^A} \) denote the derivatives with respect to the \( \alpha^A \). Assuming for the moment that \( (21) \) could be solved for the \( \alpha^A = \alpha^A(a^a) \), then when they are substituted into \( (21) \) one obtains the function

\[
S(a, x_0) = Z(a, \alpha^A(a^a)) - Z(x_0, \alpha^A(a^a)) = 0.
\]  

Using \( (21) \) it is easy to see that \( \partial_a S = \partial_a S^* \) so that again \( S(a, x_0) \) is a null surface thru the point \( x_0 \); its gradient at \( x_0 \), namely \( \partial_a S = Z(x_0, \alpha^A) \) spans the light-cone at \( x_0 \) [at \( x_0 \), \( \text{Eq.}(21) \) can not be solved for the \( \alpha^A = \alpha^A(a^a) \); all values of \( \alpha^A \) are allowed.] We thus see that \( \text{Eq.}(22) \) represents the light-cone \( \mathcal{C}_{x_0} \). The assumption that \( \text{Eq.}(21) \) could be solved for \( \alpha^A = \alpha^A(a^a) \) depended on the non-vanishing of the determinant \( J_{ij} \equiv \partial_i \partial_j S^*(a^a, x_0, \alpha^A) \).

\( J \) does vanish at the singularities of the “surface” \( S(a, x_0) \), e.g., at the apex \( x^a = x_0^a \). In general, however even when \( J = 0 \), \( \text{Eqs.}(21) \) and \( (24) \) can be solved for other variables, namely some set of three (say \( x^a \);which might be different in different regions) of the four \( a^a \), in terms of the fourth one (say \( x^* \)) and the \( \alpha^A \), i.e.,

\[
x^a = x^a(x_0^a, x^*, \alpha^A).
\]  

Note the important point that if the coordinates \( x^a \) are such that three of them are space-like and one of them is a time coordinate, \( x^0 \), then \( \text{Eq.}(23) \) has a stronger version, namely

\[
x^j = x^j(x_0^a, x^*, \alpha^A),
\]

\[
x^0 = x^0(x_0^a, x^*, \alpha^A)
\]

where the two \( x^j \) and the \( x^* \) are the three space-like coordinates. That one can solve for the \( x^0 = x^0(x_0^a, x^*, \alpha^A) \) follows from the fact that \( \text{Eq.}(24) \) can always be solved, from the implicit function theorem, for \( x^0 \) since \( S^* \) satisfies the eikonal equation and hence \( \partial S^*/\partial x^0 \neq 0 \).

\( \text{Eqs.}(24) \) and \( (25) \) are a parametric representation of \( \mathcal{C}_{x_0} \) via the null geodesics that rule it. For the different given values of the \( \alpha^A \), they are the null geodesics thru \( x_0^a \).

We thus have the result that the \( \mathcal{C}_{x_0} \) can be given either via the surface \( (22) \) or by its geodesics \( (24) \) and \( (25) \). We will return to \( \text{Eq.}(23) \) later.
If we now allow the \( x_0^a \) to lie on a space-like two-surface \( S \) described by \( x_0^a = x_0^a(s^J) \), parametrized by the two parameters \( s^J \), then the previous construction of light-cones yields the family of light-cones of all the points of \( S \) via \( x_0^a = x_0^a(s^J) \). The intersection of the set of all the light-cones with a constant-time slice \( x^0 = \text{constant} \), is a family of individual (small, two-dimensional) wavefronts emanating from each point on the surface; they are denoted as “Huygen’s wavelets”. By Huygen’s principle, the envelope of all the wavelets, at \( x^0 = \text{constant} \), is the two-dimensional wavefront from the source \( S \). (see Fig. 2). The evolution, as \( x^0 \) changes, of these wavefronts yields a new characteristic surface (big wave-front). It is equivalent to the envelope of the family of light-cones of all the points of \( S \); the envelope corresponding to the stationary variation of the family of light-cones with respect to variations in the \( s^J \).

More precisely, the envelope is the three-surface defined, first by Eqs. (20) and (21), [the conditions for the light-cones of \( x \), \( x \)]

\[
S^*(x^a, x_0^a(s^J), \alpha^A) = Z(x^a, \alpha^A) - Z(x_0^a(s^J), \alpha^A) = 0, \tag{26}
\]

augmented by the \( s^J \) variations, i.e., by

\[
\partial_A S^*(x^a, x_0^a(s^J), \alpha^A) = 0 \tag{27}
\]

These are five conditions on the eight variables \((x^a, s^J, \alpha^A)\) thus forming a three-surface in the eight dimensional space; this when projected down to the space-time results in the aforementioned envelope. It is easily seen from Eqs. (27) and (28) that this surface, which we will denote by

\[
N(x^a) = 0 \tag{29}
\]

is a characteristic surface and hence satisfies the eikonal equation, \([18]\). Though almost everywhere it can be given in the form of the vanishing of a function of \( x^a \), i.e., by Eq. (29), there will be lower dimensional regions where it must be given parametrically. See e.g., Eqs. (23) or Eqs. (24) and (25).

Before looking at the time of arrival function, we first look at Eq. (28) more closely. Substituting Eq. (26) into Eq. (28) and taking the required derivatives we have

\[
\partial_J Z(x_0^a(s^J), \alpha^A) \frac{\partial x_0^a}{\partial s^J} = 0 \tag{30}
\]

which is the statement that for the null ray leaving \( S \) at the point \( x_0^a(s^J) \), \( \partial_J Z(x_0^a(s^J), \alpha^A) \) must be normal to the tangent vectors \( \frac{\partial x_0^a}{\partial s^J} \) at \( S \) and thus normal to \( S \). Eq. (30), hence, chooses among all the rays forming the light-cone at \( x_0^a(s^J) \), i.e., the rays parametrized by \( \alpha^A \), just the appropriate \( \alpha^A \) so that the ray is the (unique) normal to \( S \). We thus have the result that (30) can be solved by

\[
\alpha^A = \alpha^A(s^J). \tag{31}
\]

Using Eq. (31), we have that Eqs. (26) and (27) become

\[
S^*(x^a, x_0^a(s^J), \alpha^A(s^J)) = Z(x^a, \alpha^A(s^J)) - Z(x_0^a(s^J), \alpha^A(s^J)) = 0, \tag{32}
\]

\[
\partial_A S^*(x^a, x_0^a(s^J), \alpha^A)|_{\alpha^A = \alpha^A(s^J)} = 0 \tag{33}
\]

Using the same argument that led to Eq. (25), namely the implicit function theorem and \( \partial S^*/\partial x^0 \neq 0 \), we see that Eq. (32) is equivalent to

\[
x^0 = T(x^a, x_0^a(s^J), \alpha^A(s^J)) \tag{34}
\]

If we take the three \( x^a = X^a \) as the “constant spatial position” of the world-line of Sec. II, we have the time of arrival function. Since, interpreting Eq. (32) as defining Eq. (34) implicitly, we have that

\[
\frac{\partial S^*}{\partial x^0} \partial_J T + \partial_J S^* = 0, \tag{35}
\]
which, since $\partial S^*/\partial x^0 \neq 0$, implies that

$$\partial_J S^* = 0 \Rightarrow \partial_J T = 0.$$  \hfill (36)

Thus the extremization of $S^*$ implies the extremization of $T(x^\alpha, x^0_0(s^j), \alpha^A(s^j))$ as was to be proved. This proof is not affected by the difficulties in Sec. II of the possible multivaluedness of the earlier $T$.

In the terminology of Arnold, these results follow from his theory of Legendre submanifolds and maps, where

$$Z(x^\alpha, \alpha^A) - Z(x^0_0(s^j), \alpha^A) = 0$$  \hfill (37)

from Eq. (26), defines a generating family $F(x^\alpha, \alpha^A, s^j)$ and Eqs. (27) and (28) define the Legendre map.

IV. DISCUSSION

We have given two derivations of a variational principle for the time of arrival of null geodesics at an observer. Superficially, it appears as if it were a version of Fermat’s principle; in actuality it is quite different. Fermat’s principle leads to local evolutionary laws for the rays while here we have from the start assumed that the rays are given by null geodesics. Our variational principle gives the initial direction of the ray. Often Fermat’s principle is invoked to derive the equations of gravitational lensing [5,6]. A paper is now in preparation, using the techniques discussed here, in which a universal lensing equation valid in all situations is obtained.

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[1] V. I. Arnol’d, Mathematical Methods of Classical Mechanics, 2nd ed. (Springer-Verlag, New York, 1978).
[2] P. Schneider, J. Ehlers, and E. E. Falco, Gravitational Lenses (Springer-Verlag, New York, 1992).
[3] V. I. Arnol’d, S. M. Gusein-Zade, and V. A. N, Singularities of Differentiable Maps (Birkhäuser, Boston, 1985), Vol. I.
[4] E. Schrödinger, Expanding Universes (Cambridge University Press, Cambridge, 1956), pp. 41–45.
[5] V. Faraoni, Astrophysical Journal 398, 425 (1992).
[6] V. Perlick, Class. Quantum Grav. 7, 1849 (1990).