THE WHITEHEAD THEOREM FOR CURVED L-INFINITY SPACES

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Abstract. In this paper we develop obstruction theory for $A_\infty/L_\infty$ homomorphisms between curved $A_\infty/L_\infty$ algebras. We also extend the homotopy transfer formula in the case of a curved version of homotopy retraction data. Using these techniques, we prove the Whitehead theorem for curved $L_\infty$ spaces which simultaneously generalizes the inverse function theorem for smooth maps and the Whitehead theorem of $L_\infty$ algebras.

1. Introduction

1.1. Curved $L_\infty$ spaces. The notion of curved $L_\infty$ spaces was introduced by Costello [7], as an alternative approach to derived differential geometry. In loc. cit. an $L_\infty$ space is defined as a pair $(M, \mathcal{G})$ where $M$ is a smooth manifold, and $\mathcal{G}$ is a curved $L_\infty$ algebra over the de Rham algebra $\Omega^*_M$. In this paper, we shall work with a more down-to-earth, but equivalent notion of $L_\infty$ spaces, following analogous constructions in the theory of dg-schemes by Behrend [2][3], and Ciocan-Fontanine-Kapranov [5][6]. Conceptually, the $L_\infty$ approach to derived geometric structures is Koszul dual to the more classical approach using dg (or simplicial) commutative algebras as developed by Toën-Vezzosi [13].

More precisely, throughout the paper, an $L_\infty$ space $\mathbb{M} = (M, \mathfrak{g})$ is given by a pair with $M$ a smooth manifold and $\mathfrak{g}$ a curved $L_\infty$ algebra over the ring of smooth functions $C^\infty(M)$. We also require that $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \cdots \oplus \mathfrak{g}_d,$$

where each $\mathfrak{g}_i$ is a vector bundle in degree $i$. In particular it has minimal degree 2 and maximal degree $d$ for some $d \geq 2$. The curvature term $\mu_0 \in \mathfrak{g}_2$ is a section of the bundle $\mathfrak{g}_2$. Let $p \in \mu_0^{-1}(0)$ be a point in its zero locus. The tangent complex at $p$ is defined as

$$T_p \mathbb{M} := T_p M \xrightarrow{d\mu_0|_p} \mathfrak{g}_2|_p \xrightarrow{\mu_1|_p} \mathfrak{g}_3|_p \xrightarrow{\mu_1|_p} \cdots$$

A morphism between two curved $L_\infty$ spaces $\mathbb{M} = (M, \mathfrak{g})$ and $\mathbb{N} = (N, \mathfrak{h})$ is given by a pair $f = (f, f^2)$ where $f : M \to N$ is a smooth map, and $f^2 = (f_1^2, f_2^2, \ldots) : \mathfrak{g} \to f^* \mathfrak{h}$ an $L_\infty$ algebra map.

Let $f : \mathbb{M} \to \mathbb{N}$ be a morphism and $p \in M$ a point in the zero locus of the curvature of $\mathfrak{g}$. Then it induces a map on the tangent complex $d|_p : T_p \mathbb{M} \to T_{f(p)} \mathbb{N}$, explicitly

\[1\]See [14] for a proof of the equivalence.
The main result of the paper is the following theorem which states that quasi-isomorphisms are “locally invertible”.

**Theorem 1.1.** Assume that the tangent map \( df_p \) is a quasi-isomorphism at \( p \). Then there exist open neighborhoods \( U \) of \( p \) and \( V \) of \( f(p) \) such that the restriction \( f|_U : (U, g|_U) \rightarrow (V, h|_V) \) is a homotopy equivalence \(^2\).

In the case when both \( g \) and \( h \) are trivial, this is simply the inverse function theorem for smooth manifolds. When the \( L_\infty \) bundle \( g \) is concentrated in degree 2, we obtain the notion of \( m \)-Kuranishi neighborhood in the work of Joyce [10], or Kuranishi chart (with trivial isotropy) introduced by Fukaya–Oh–Ohta–Ono [9]. In this special case, our theorem recovers Theorem 4.16 in [10]. In the case when both \( M \) and \( N \) are a point, the above theorem is the so-called Whitehead theorem of (uncurved) \( L_\infty \) algebras which states that quasi-isomorphisms between \( L_\infty \) algebras are homotopy equivalences.

We also formulate a conjecture of the global version of Theorem 1.1. At present, we are able to verify it in the case when both \( g \) and \( h \) are concentrated in degree 2, known as the Kuranishi case, using partition-of-unity arguments from Joyce [10].

**Conjecture 1.2.** Assume that the tangent map \( df_p \) is a quasi-isomorphism for all \( p \in M \), and that the induced map \( f : (\mu^M_0)^{-1}(0) \rightarrow (\mu^N_0)^{-1}(0) \) on the zero loci is a bijection. Then there exist open neighborhoods \( U \) of \( (\mu^M_0)^{-1}(0) \) and \( V \) of \( (\mu^N_0)^{-1}(0) \) such that the restriction

\[ f|_U : (U, g|_U) \rightarrow (V, h|_V) \]

is a homotopy equivalence \(^3\).

A second main result of the paper is the construction of a minimal chart around a point \( p \in \mu^M_0^{-1}(0) \subset M \), generalizing the so-called minimal model construction of \( L_\infty \) algebras. More precisely, we have the following

**Theorem 1.3.** Let \( M = (M, \mathfrak{g}) \) be an \( L_\infty \) space with \( p \in \mu^M_0^{-1}(0) \). Then there exists an \( L_\infty \) space \( \mathbb{W} = (W, \mathfrak{h}) \) with \( W \subset M \) a submanifold containing \( p \), such that

- The tangent complex \( T_p \mathbb{W} \) has zero differential.
- There exists an open neighborhood \( U \subset M \) of \( p \) such that it contains \( W \), and the inclusion map \( i : W \rightarrow U \) extends to a homotopy equivalence

\[ (i, i^! : (W, \mathfrak{h}) \rightarrow (U, g|_U)). \]

Here the map \( i^! : \mathfrak{h} \rightarrow i^* \mathfrak{g} \) is an \( L_\infty \) morphism constructed explicitly using summation over trees. We refer to Subsection 5.2 for details.

\(^2\)The notion of homotopy between \( L_\infty \) spaces is formulated in Definition 4.5.

\(^3\)One can formulate a global version of homotopy equivalence between \( L_\infty \) spaces which is equivalent Definition 4.5 when both \( M \) and \( N \) admit flat and torsion-free connection on the tangent bundle.
1.2. **About the proofs.** The proof of Theorem 1.3 generalizes the homological perturbation formula to the curved setting. More precisely, we construct both the minimal $L_\infty$ structure on $\mathfrak{h}$ and $L_\infty$ homomorphism $\mathfrak{i}^\#$ using summation over trees while under a weaker homotopy retraction data. For example, the homotopy operator $H$ is only assumed to satisfy

\[ \mu_1 H + H \mu_1 = ip - id - H \mu_1^2 H, \]

which reduces to the usual homotopy identity in the un-curved case since $\mu_0 = 0$ implies that $\mu_1^2 = 0$. We refer to Section 3 for more details.

The proof of Theorem 1.1 relies on both Theorem 1.3 and a novel construction of obstruction theory of $L_\infty$ homomorphisms between curved $L_\infty$ algebras. In the un-curved case or filtered case, this is done by [9] (in the $A_\infty$ case). Notably our obstruction theory does not make any additional assumptions on the ground ring or on the curvature term. With the obstruction theory ready, the rest of the proof is analogous to that in [9].

1.3. **Other related works.** A closely related recent preprint [4] of Behrend-Liao-Xu obtained similar results, in the framework of categories of fibrant objects. While the definitions of $L_\infty$ spaces \(^4\) and the tangent complexes are clearly the same as ours, it is not clear at the moment of writing how Behrend-Liao-Xu’s notion of homotopy between morphisms of $L_\infty$ spaces is related ours (see Definition 4.5). Technically speaking, a notable difference is that in Theorem 1.1 we do not assume the underlying map $f$ be a submersion. Also, our approach is more algebraic in the sense that both the obstruction theory and the homological perturbation technique are generalizations of the un-curved situation.

An interesting question is whether Theorem 1.1 and Theorem 1.3 admit generalizations to allow the tangent complex to have components in negative degrees. This is related to the so-called shifted Lie algebroid structure developed by Pym-Safronov [12].

1.4. **Organization of the paper.** In Section 2, we develop the obstruction theory of constructing $A_\infty/L_\infty$ homomorphisms between curved $A_\infty/L_\infty$ algebras. Section 3 generalizes homotopy transfer formula to the curved setting. In Section 4 we recall basic definitions of $L_\infty$ spaces. In particular, we explain explicitly the meaning of a homotopy between two morphisms between $L_\infty$ spaces. In Section 5 we prove Theorem 1.1 and Theorem 1.3.

### 2. Obstruction theory

Let $(A, \mu_0^A, \mu_1^A, \cdots)$ and $(B, \mu_0^B, \mu_1^B, \cdots)$ be two curved $A_\infty/L_\infty$ algebras over a commutative ring $R$. In this section, we study the obstruction theory of $A_\infty/L_\infty$ homomorphisms from $A$ to $B$. Classically, in the non-curved case, this is done by using a pro-nilpotent $L_\infty$ algebra structure on the space $\text{Hom}(T^c A[1], B)$ of cochains on $A$ with values in $B$. In particular, the obstruction class to construct the $(n+1)$-th component of a $A_\infty/L_\infty$ homomorphism $(f_j)_{j=1}^n$ from $A$ to $B$ is a cohomology class

\[ \theta((f_j)_{j=1}^n) \in H^1(\text{Hom}(A[1]^{n+1}, B[1]), d = [\mu_1, -]). \]

\(^4\)In Loc. Cit. the authors use derived manifolds.
determined by the first \( n \) components. However, adding the curvature term spoils the pro-nilpotent structure, and obviously the above obstruction space is not even defined as \( \mu^1_A \) and \( \mu^1_B \) might not square to zero. In this section, we define a variant of the above obstruction space which takes into account the appearance of curvatures, which allows us to extend the obstruction theory of \( A_\infty/L_\infty \) homomorphisms to the curved setting.

2.1. Definition of obstruction spaces. First, using the curvature term \( \mu^0_A \), we may form a complex \( C(A, B) \) of the form

\[
C(A, B) := \cdots \to \text{Hom}(A[1]^\otimes k, B[1]) \xrightarrow{\delta} \text{Hom}(A[1]^\otimes k-1, B[1]) \to \cdots \to B[1] \to 0
\]

\[
\delta(\phi_k)(a_1, \ldots, a_{k-1}) := \sum_{j=0}^{k-1} (-1)^j \phi_k^{(r+|a_1|^r + \cdots + |a_j|^r)}(a_1, \ldots, a_j, \mu^A_0, a_{j+1}, \ldots, a_{k-1})
\]

One verifies that \( \delta^2 = 0 \). We shall denote its cohomology by \( D^k(A, B) \) where \( k \) is the tensor degree. Next, using the operators \( \mu^1_A \) and \( \mu^1_B \), we define another operator \( d \) of the form

\[
d : \text{Hom}(A[1]^\otimes k, B[1]) \to \text{Hom}(A[1]^\otimes k, B[1])
\]

\[
d(\phi_k)(a_1, \ldots, a_k) := \mu^B_0 \phi_k(a_1, \ldots, a_k) - \sum_{j=1}^k (-1)^{j} \phi_k^{(r+|a_1|^r + \cdots + |a_{j-1}|^r)}(a_1, \ldots, a_j, \mu^A_0, a_{j+1}, \ldots, a_k)
\]

where \( * = |a_1|^r + \cdots + |a_{j-1}|^r \). One can verify that we have \( d\delta + \delta d = 0 \). Thus \( d \) induces a map on the \( \delta \)-cohomology, i.e. we obtain maps

\[
d : D^k(A, B) \to D^{k+1}(A, B), \quad \forall k \geq 0.
\]

In general, we don’t have \( d^2 = 0 \). However, we have the following

**Lemma 2.1.** If there exists a \( R \)-linear map \( f : A \to B \) such that \( f(\mu^A_0) = \mu^B_0 \), then we have the composition \( d^2 : D^k(A, B) \to D^{k+1}(A, B) \) equals to zero.

**Proof.** Choose any such \( f \). Given \( [\phi_k] \in D^k(A, B) \), define \( \psi \in \text{Hom}(A[1]^\otimes k+1, B[1]) \) as

\[
\psi := (-1)^{|\phi_k|^r} \sum_{i=0}^{k-1} \phi_k(\text{id}^i \otimes \mu^A_2 \otimes \text{id}^{k-1-i}) + \mu^B_2(\phi_k \otimes f) + \mu^B_2(f \otimes \phi_k).
\]

One can verify that we have \( d^2(\phi_k) = \delta \psi \). Hence \( d^2 = 0 \) in \( \delta \)-cohomology. \( \square \)

**Definition 2.2.** Under the assumption of \( A \) and \( B \) in the above lemma, we set the \( k \)-th obstruction space \( H^k(A, B) \) to be the degree one cohomology of the complex \( (D^k(A, B), d) \), i.e.

\[
H^k(A, B) := H^1(D^k(A, B), d).
\]

2.2. Obstruction classes. Let \( f_j : A[1]^\otimes j \to B[1] \) \( (j = 1, \ldots, n) \) be \( n \) multi-linear maps of cohomological degree zero, such that the following conditions hold:

(i) The \( A_\infty \) homomorphism axiom holds up to \( (n-1) \) inputs, i.e. we have

\[
\sum_{\substack{j \geq 0, i_1 + \cdots + i_j \geq k \atop i_1 \geq 1, \ldots, i_j \geq 1}} \mu^B_j(f_{i_1} \otimes \cdots \otimes f_{i_j}) = \sum_{\substack{r \geq 0, s \geq 0, t \geq k \atop r+s+t = k}} f_{r+t+1}(\text{id}^r \otimes \mu^A_s \otimes \text{id}^t)
\]

for all \( 0 \leq k \leq n - 1 \).
(ii) In the case with \( n \) inputs, we require that
\[
(1) \quad \sum_{j \geq 2, i_1, \ldots, i_j \geq 1} \mu_j^B(f_{i_1} \otimes \cdots \otimes f_{i_j}) - \sum_{r \geq 0, i_1, \ldots, i_r \geq 1} f_{r+1}(\text{id}^r \otimes \mu_s^A \otimes \text{id}^t) + df_{n+1}'
\]
is \( \delta \)-exact, i.e. it lies in the image of \( \delta : C^{n+1}(A, B) \to C^n(A, B) \).

Given such a collection of maps \( f_j : A[1]^{\otimes j} \to B[1] \) \((j = 1, \ldots, n)\), we define its obstruction class as follows. Choose any \( f'_{n+1} \) of cohomological degree zero such that \( \delta f'_{n+1} \) equals the expression in (ii). Then we set
\[
\text{obs}_{n+1} := \sum_{j \geq 2, i_1, \ldots, i_j \geq 1} \mu_j^B(f_{i_1} \otimes \cdots \otimes f_{i_j}) - \sum_{r \geq 0, i_1, \ldots, i_r \geq 1} f_{r+1}(\text{id}^r \otimes \mu_s^A \otimes \text{id}^t) + df_{n+1}'.
\]

**Lemma 2.3.** The above expression \( \text{obs}_{n+1} \) is \( \delta \)-closed. Furthermore, \( d(\text{obs}_{n+1}) \) is \( \delta \)-exact. Thus it represents a well-defined class which we denote by \( \mathfrak{o}(f'_n) \) \( \in H^{n+1}(A, B) \). This class is independent of the choice of \( f'_{n+1} \).

**Proof.** Denote by \( F_n \) the extension of \( \sum_{j=1}^n f_j \) as a coalgebra map \( T(A[1]) \to T(B[1]) \). Similarly, denote by \( \mu_k \) the extension of \( \mu_k \) as coderivations on the tensor coalgebra. Condition (i) implies that we have
\[
\mu^B F_n = F_n \mu^A : A[1]^N \to B[1]^M, \quad \forall -1 \leq N - M \leq n - 2.
\]
Observe also that \( \delta(\phi) = (-1)^{\phi'} \phi \mu_0 \). Then we compute
\[
\delta(\text{obs}_{n+1}) = -\mu \phi F_n \mu F_n + F_n \mu \mu \mu_0
\]
\[
= (-\mu \phi F_n + \mu \mu_0 + \mu \mu_0) + F_n \mu \mu_0 - (\mu \mu F_n + \mu \mu_0 + \mu \mu_0) + (-\mu F_n \mu_1 + F_n \mu_1)
\]
The two terms \(-\mu \phi F_n \) and \(-\mu \mu F_n \) combine to give zero since \( \mu \mu = 0 \). The following three terms give
\[
\mu \phi F_n \mu_1 + \mu \mu_0 + \mu \mu_0 = \mu F_n \mu_0.
\]
Since there are \( n \) inputs, after applying \( \mu \mu_0 \), we are left with at most \( n - 1 \) inputs to apply \( \mu F_n \). In this case, we may use the commutativity \( \mu F_n = F_n \mu \), i.e. we have the above three terms sum up to
\[
\mu F_n \mu_1 = F_n \mu_0.
\]
The last equality follows from \( \mu \mu = 0 \). Putting these back into the calculation of \( \delta(\text{obs}_{n+1}) \), we obtain
\[
\delta(\text{obs}_{n+1}) = F_n \mu \mu_0 + F_n \mu \mu_0 F_n - F_n \mu \mu_0 - F_n \mu \mu_0 = -F_n \mu \mu_0 - F_n \mu \mu_0 = 0
\]
Here we have used the \( A_\infty \) relation that \( \mu \mu_0 = 0 \) and \( \mu_0 \mu_1 + \mu_0 \mu_1 = 0 \). Next, we prove that \( d(\text{obs}_{n+1}) \) is \( \delta \)-exact. We use the notation \( F_n' : T(A[1]) \to T(B[1]) \) to denote the extension of \( f_1, \cdots, f_n, f'_{n+1} \) to the tensor coalgebra. By definition of \( f'_{n+1} \) we have that
\[
\mu F_n' = F_n' \mu : A[1]^N \to B[1]^M, \quad \forall -1 \leq N - M \leq n - 1.
\]
Using the notation \( F_n' \), we may write the obstruction as
\[
\text{obs}_{n+1} = \mu F_n' + F_n' \mu \mu_1 \to A[1]^{n+1} \to B[1].
\]
Apply the operator \( d \) to it yields
\[
d(\text{obs}_{n+1}) = \mu F_n' + F_n' \mu \mu_1 - F_n' \mu \mu_0 = F_n' \mu \mu_0 = 0.
\]
The first and the third terms give
\[ \tilde{\mu}_1 \tilde{F}'_{n+1} + \tilde{\mu} F'_{n+1} \tilde{\mu}_1 = -\tilde{\mu}_2 \tilde{\mu}_1 F'_{n+1} + \tilde{\mu} F'_{n+1} \tilde{\mu}_1 \]
\[ = -\tilde{\mu}_2 F'_{n+1} \tilde{\mu}_1 + \tilde{\mu} F'_{n+1} \tilde{\mu}_1 \]
\[ = -\tilde{\mu}_2 F'_{n+1} \tilde{\mu}_0 - \tilde{\mu}_2 F_n \tilde{\mu}_1 + \tilde{\mu} F_n \tilde{\mu}_1 + \tilde{\mu}_1 f'_{n+1} \tilde{\mu}_1 \]
\[ = -\tilde{\mu}_2 F'_{n+1} \tilde{\mu}_0 - \tilde{\mu}_2 F_n \tilde{\mu}_2 + \tilde{\mu}_1 f'_{n+1} \tilde{\mu}_1 \]

Similarly, we also have
\[ -\tilde{\mu}_1 F'_{n+1} \tilde{\mu}_1 \geq -F'_{n+1} \tilde{\mu}_1 \geq -F'_{n+1} \tilde{\mu}_0 = -\tilde{\mu}_2 F'_{n+1} \tilde{\mu}_1 - F'_{n+1} \tilde{\mu}_0 \]
\[ = -\tilde{\mu}_2 F'_{n+1} \tilde{\mu}_1 + F'_{n+1} \tilde{\mu}_0 + F'_{n+1} \tilde{\mu}_2 \]
\[ = -\tilde{\mu}_2 F'_{n+1} \tilde{\mu}_1 + F'_{n+1} \tilde{\mu}_0 + \tilde{\mu} F_n \tilde{\mu}_2 \]
\[ = F'_{n+1} \tilde{\mu}_0 + \tilde{\mu} F_n \tilde{\mu}_2 - \tilde{\mu}_1 f'_{n+1} \tilde{\mu}_1 \]
\[ = F'_{n+1} \tilde{\mu}_0 + \tilde{\mu}_2 F_n \tilde{\mu}_2 - \tilde{\mu}_1 f'_{n+1} \tilde{\mu}_1 \]

Adding the two equations together yields the desired formula
\[ d(\text{obs}_{n+1}) = -\tilde{\mu}_2 F'_{n+1} \tilde{\mu}_0 + F'_{n+1} \tilde{\mu}_0 = \delta (\tilde{\mu}_2 F'_{n+1} - F'_{n+1} \tilde{\mu}) \]

Finally, to see that the class \( a((f_j)_{j=1}^n) = [\text{obs}_{n+1}] \) is independent of \( f'_{n+1} \), let \( f''_{n+1} \) be another such map. Then we have the two obstructions differ by \( d(f'_{n+1} - f''_{n+1}) \) with \( \delta (f'_{n+1} - f''_{n+1}) = 0 \), this proves the two obstruction classes are equal. \( \square \)

2.3. \( A(n) \) homomorphisms. In the following, we shall refer to a collection of maps \((f_j : A[1] ^{\otimes j} \to B[1])_{j=1}^n \) satisfying conditions (i), (ii) in Subsection 2.2 as an \( A(n) \) homomorphism from \( A \) to \( B \). Obviously, an \( A(n) \) homomorphism is also an \( A(k) \) homomorphism, for any \( k \leq n \). Just as in the case of the usual \( A_\infty \) homomorphisms, one can compose \( A(n) \) homomorphisms, using the formula
\[ (g \circ f)_j := \sum_{i_1 + \cdots + i_l = j} g_l(f_{i_1} \otimes \cdots \otimes f_{i_l}), \]
for \( j = 1, \ldots, n \). If we denote by \( f'_{n+1} \) (and \( g'_{n+1} \)) such that \( \delta f'_{n+1} \) equals the expression in (1) then we can define
\[ (g \circ f)'_{n+1} := g'_{n+1}(f_1 \otimes \cdots \otimes f_1) + g_l(f'_{n+1}) + \sum_{i_1 + \cdots + i_l = n+1} g_l(f_{i_1} \otimes \cdots \otimes f_{i_l}), \]
Please note that this composition is strictly associative.

The crucial observation is that a \( A(n) \) lifts to an \( A(n+1) \) homomorphism if and only if its obstruction class \( a((f_j)_{j=1}^n) \) vanishes.

Next we recall the notion of homotopy between \( A(n) \) homomorphisms. Let \( \Omega^*_{[0,1]} \) be piece-wise polynomial differential forms on the interval \([0,1]\) (as in [9, Definition 4.2.9]). This is an unital dg-algebra, therefore given an \( A_\infty \)-algebra \( B \) we can easily define the tensor product \( B \otimes \Omega^*_{[0,1]} \) (see [1] for details). Moreover there are naive maps of \( A_\infty \)-algebras \( ev_0, ev_1 : B \otimes \Omega^*_{[0,1]} \to B \), given by “evaluating at \( t = 0, 1 \).” Two \( A(n) \) homomorphisms \( f = (f_1, \ldots, f_n), g = (g_1, \ldots, g_n) : A \to B \) are called homotopic, denoted by \( f \cong g \), if there exists an \( A(n) \) homomorphism
\[ F = (F_1, \ldots, F_n) : A \to B \otimes \Omega^*_{[0,1]} \]
such that $ev_0 \circ F = f$ and $ev_1 \circ F = g$. The following properties are standard:

- Homotopy relation "$\cong$" between $A(n)$ morphisms is an equivalence relation.
- If $h : A' \to A$ is another $A(n)$ morphism and $f \cong g$, then $f \circ h \cong g \circ h$.
- If $h : B \to B'$ is another $A(n)$ morphism and $f \cong g$, then $h \circ f \cong h \circ g$.
- Two homotopic $A(n)$ homomorphisms $(f_j)_{j=1}^n$ and $(g_j)_{j=1}^n$ from $A$ to $B$ have the same obstruction class, i.e. we have
  \[ \sigma((f_j)_{j=1}^n) = \sigma((g_j)_{j=1}^n). \]

Indeed, it is clear that we have
\[
\sigma((f_j)_{j=1}^n) = (ev_0) \cdot \sigma((F_j)_{j=1}^n)
\]
\[
\sigma((g_j)_{j=1}^n) = (ev_1) \cdot \sigma((F_j)_{j=1}^n)
\]

Then observe that every cohomology class in $H^{n+1}(A, B \otimes \Omega^*_{[0,1]})$ can be represented by an element of the form $\phi \otimes 1$ with $\phi \in C^{n+1}(A, B)$ and $1$ the constant function in $\Omega^*_{[0,1]}$, which clearly shows that applying the two evaluation maps $ev_0$ and $ev_1$ both yield $[\phi] \in H^{n+1}(A, B)$.

It is useful to spell out the definition of homotopy in the $n = 1$ case.

**Lemma 2.4.** Two $A(1)$-homomorphisms $f_1, g_1 : A \to B$ are $A(1)$-homotopic if and only if there is a map $H : A \to B$ of degree $-1$, such that $H(\mu^A_0) = 0$ and $f_1 - g_1 - \mu^B_1 H - H\mu^A_1$ is $\delta$-exact.

**Proof.** An $A(1)$-homotopy gives a map $F_1 : A \to B \otimes \Omega^*_{[0,1]}$. We write $F_1(a) = f_1(a) + (-1)^{|a|} h_1(a) dt$, the $A(1)$-homomorphism equation for $F$ is equivalent to:
\[
f_1 = f_1, \quad f_1' = g_1, \quad h_1'(\mu^A_0) = 0 \quad \text{and} \quad -\frac{d}{dt} h_1 + \mu^B_1 h_1 + h_1^2 \mu^A_1 \text{ is } \delta\text{-exact.}
\]
We then get the desired equality by taking $H = \int_0^1 h_1 dt$. \qed

Two $A_\infty$ algebras $A$ and $B$ are called $A(n)$-homotopic, if there exist $A(n)$ homomorphisms $f : A \to B$ and $g : B \to A$ such that both $f \circ g$ and $g \circ f$ are both homotopic to the identity homomorphism.

### 2.4. Homotopy invariance of obstruction theory

The obstruction spaces are natural with respect to $A(1)$ homomorphisms. Namely, fix two curved $A_\infty$ algebras $A$ and $B$. Assume that there exists an $R$-linear map $f : A \to B$ such that $f(\mu^A_0) = \mu^B_0$, so that the obstruction space $H^k(A, B)$ is defined. Let $h : B \to B'$ be an $A(1)$ homomorphism. By definition, we have $hf(\mu^A_0) = h(\mu^B_0) = \mu^B_0$, which shows that the obstruction space $H^k(A, B')$ is also defined. Furthermore, the morphism $h$ induces a push-forward map
\[ h_* : H^k(A, B) \to H^k(A, B'), \]
defined by $[\phi] \mapsto [h \circ \phi]$ where $\phi \in \text{Hom}(A[1] \otimes k, B[1])$ is a representative. Similarly, let $g : A'[1] \to A[1]$ be an $A(1)$ homomorphism. We may define the pull-back map
\[ g^* : H^k(A, B) \to H^k(A', B), \]
by $[\phi] \mapsto [\phi \circ (g \otimes \cdots \otimes g)]$ where we used $k$-copies of $g$ in the tensor product.

**Lemma 2.5.** Assume that $h : B \to B'$ and $g : A' \to A$ are both $A(1)$ homotopy equivalences. Then both $h_*$ and $g^*$ are isomorphisms.
Proof. It follows from Lemma 2.4 that if \( h_0 \) and \( h_1 \) are \( A_{(1)} \)-homotopic, then \( (h_0)_* = (h_1)_* \) and \((h_0)^* = (h_1)^* \). This together with the identities \( (h_0 h_1)_* = (h_0)_*(h_1)_* \) and \((g_0 g_1)^* = (g_0)^*(g_1)^*\) immediately give the result. \(\Box\)

2.5. **Whitehead theorem for curved \( A_\infty \)-algebras.**

**Proposition 2.6.** Let \( f = (f_1, \ldots, f_n) : A \to B \) be an \( A_{(n)} \) homomorphism. Assume that \( o((f_j)_{j=1}^n) = 0 \). Denote by \( \mathcal{L}(f) \) the set of liftings of \( f \) to an \( A_{(n+1)} \) morphism, modulo homotopy equivalence relation. Then \( \mathcal{L}(f) \) carries a natural transitive action by the abelian group \( H^0(D^{n+1}(A, B), d) \).

**Proof.** Let \( f_{n+1} \) be a lift of \( f \) to an \( A_{(n+1)} \) homomorphism. Denote by \([f_{n+1}]\) its equivalence class in \( \mathcal{L}(f) \). Let \( \beta : A[1[^{\otimes n+1}] \to B[1] \) be a map representing an element \([\beta] \in H^0(D^{n+1}(A, B), d) \). We set the action by

\[
[\beta],[f_{n+1}] := [f_{n+1} + \beta]
\]

To see that the action is independent of the choice of \( \beta \), let \( \beta' \) be another representative of the class \([\beta] \). Thus, the difference \( \beta' - \beta = do \) for some \( d \)-closed morphism \( \alpha : A[1[^{\otimes n+1}] \to B[1] \). We may define a homotopy between the two extensions \((f_1, \ldots, f_{n+1} + \beta)\) and \((f_1, \ldots, f_{n+1} + \beta')\) by putting

\[
F : A \to B \otimes \Omega^*_{[0,1]}
\]

\[
F_k = f_k, \quad 1 \leq k \leq n
\]

\[
F_{n+1} = f_{n+1} + t \cdot \beta' + (1 - t) \cdot \beta + \alpha \cdot dt
\]

This shows that the action map (3) is independent of the choice of \( \beta \).

Similarly, assume that \( f_{n+1}' \) is another representative of the lift class \([f_{n+1}]\), i.e. there exists a homotopy \( H : A \to B \otimes \Omega^*_{[0,1]} \) between \((f_1, \ldots, f_n, f_{n+1})\) and \((f_1, \ldots, f_n, f_{n+1}')\). We simply change \( H_{n+1} \) to \( H_{n+1} + \beta \), which gives a homotopy between the two lifts \((f_1, \ldots, f_n, f_{n+1} + \beta)\) and \((f_1, \ldots, f_n, f_{n+1}' + \beta)\). This verifies that the action map is well-defined.

Transitivity of the action map is clear: since any two lifts differ by some \( \beta \) that would represent a class in \( H^0(D^{n+1}(A, B), d) \).

**Lemma 2.7.** Let \( f : A \to B \) be an \( A_{(n)} \) homomorphism. Assume that \( h : B \to B' \) and \( g : A' \to A \) are both \( A_{(n)} \) homomorphisms. Then we have

\[
o\left(\left((hf)_{j}\right)_{j=1}^{n}\right) = (h_1)_* o\left((f_j)_{j=1}^{n}\right)
\]

\[
o\left(\left((fg)_{j}\right)_{j=1}^{n}\right) = (g_1)_* o\left((f_j)_{j=1}^{n}\right)
\]

Moreover, the natural map \( o \circ g : \mathcal{L}(f) \to \mathcal{L}(f \circ g) \) given by composition of \( A_{(n+1)} \) homomorphisms is a homomorphism of \( H^0(D^{n+1}(A, B), d) \)-modules. Here the \( H^0(D^{n+1}(A, B), d) \)-module structure of \( \mathcal{L}(f \circ g) \) is via the group homomorphism

\[
(g_1)^* : H^0(D^{n+1}(A, B), d) \to H^0(D^{n+1}(A', B), d)
\]

**Proof.** The fist statements can be proved as in the uncurved case, see Theorem 4.5.1 in [9]. The second statement follows from the action map (3) and the formula for composition in (2). \(\Box\)

**Theorem 2.8.** An \( A_\infty \) homomorphism \( f = (f_1, f_2, \ldots) : A \to B \) between curved \( A_\infty \) algebras is a homotopy equivalence if and only if the map \( f_1 \) is an \( A_{(1)} \) homotopy equivalence.
Proof. The only if part is trivial, so we prove the if part. Let \( g_1 : B \to A \) be an \( A(1) \) homotopy inverse of \( f_1 : A \to B \). We argue by induction on \( n \) that if we are given an \( A(n) \) homomorphism

\[
g := (g_1, \ldots, g_n) : B \to A
\]

such that \( g \circ f \cong \text{id} \) as \( A(n) \) homomorphisms, then there exists \( g_{n+1} : B[1] \otimes A[n+1] \to A[1] \) that extends \( g \) to an \( A(n+1) \) morphism \( \tilde{g} = (g_1, \ldots, g_n, g_{n+1}) \) such that \( \tilde{g} \circ f \cong \text{id} \) as \( A(n+1) \) homomorphisms.

We first argue that \( o(g) = 0 \). Using the homotopy invariance of obstruction class, we have

\[
f^* o(g) = o(g \circ f) = o(\text{id}) = 0
\]

But \( f \) is an \( A(1) \) homotopy equivalence, thus \( f^* \) is an isomorphism, which shows that \( o(g) = 0 \). Similarly, one can argue that if \( H : A \to A \otimes \Omega^*_0 \) is a \( A(n) \) homotopy between \( \text{id} \) and \( g \circ f \), then we also have \( o(H) = 0 \).

Now consider the following diagram of maps, provided by Proposition 2.6 and Lemma 2.7,

\[
\begin{array}{ccc}
\mathcal{L}(H) & \xrightarrow{(ev_0)^*} & \mathcal{L}(\text{id}_A) \\
\downarrow & & \\
\mathcal{L}(g \circ f) & \xleftarrow{o(f)} & \mathcal{L}(g)
\end{array}
\]

Observe that the upper-right corner admits a canonical lift by \( \text{id}_A \), we claim that there exists a lift \( \tilde{H} \) of \( H \) such that

\[
(ev_0)_* (\tilde{H}) = \text{id}_A
\]

Indeed, let \( \tilde{H}' \) be any lift of \( H \). By the transitivity of the action map, there exists an element \( \beta \in H^0(D^{n+1}(A, A), d) \) such that \( \beta, ((ev_0)_* (\tilde{H}')) = \text{id}_A \).

Since \( ev_0 \) is a homotopy equivalence, there exists \( \gamma \in H^0(D^{n+1}(A, A \otimes \Omega^*_0), d) \) such that \( (ev_0)_* \gamma = \beta \). Using Lemma 2.7 we obtain

\[
(ev_0)_* (\gamma, \tilde{H}') = \beta, ((ev_0)_* (\tilde{H}')) = \text{id}_A
\]

We set \( \tilde{H} := \gamma, \tilde{H}' \). By the same argument, one can show that there exists a lift \( \tilde{g} \) of \( g \) such that

\[
(ev_1)_* (\tilde{H}) = \tilde{g} \circ f
\]

In conclusion, we obtained an \( A(n+1) \) homomorphism \( \tilde{g} : B \to A \) such that \( \tilde{g} \circ f \cong \text{id}_A \).

Finally, we need to prove that \( f \circ \tilde{g} \cong \text{id}_B \). Since \( \tilde{g} \) is also a weak equivalence, the conclusion above implies that there exists an \( A(n+1) \) homomorphism \( f' : A \to B \) extending \( (f_1, \ldots, f_n) \) such that \( f' \circ \tilde{g} \cong \text{id}_B \). Thus we have

\[
f \circ \tilde{g} \cong f' \circ \tilde{g} \circ f \circ \tilde{g} \cong f' \circ \tilde{g} \cong \text{id}_B
\]

which finishes the proof.

\[\square\]

Remark 2.9. Observe that in the un-curved case, according to Lemma 2.4 our notion of \( A(1) \) homotopy between morphisms of chain complexes agrees with the usual one. Furthermore, if we are over a field, quasi-isomorphic chain complexes are in fact homotopy equivalent. Thus, the above theorem easily implies the usual
Whitehead theorem of un-curved $A_\infty$ algebras over a field which states that a quasi-isomorphism between un-curved $A_\infty$ algebras over a field is in fact a homotopy equivalence.

2.6. Curved $L_\infty$ algebras. The previous discussion and results have direct analogous in the $L_\infty$ setting. Let $A$ and $B$ be two $L_\infty$ algebras. In this case, we set the $\delta$-complex $C(A, B)$ as

$$C(A, B) := \cdots \to \text{Hom}(\text{sym}^k A[1], B[1]) \xrightarrow{\delta} \text{Hom}(\text{sym}^{k-1} A[1], B[1]) \to \cdots \to B[1] \to 0$$

$$\delta(\phi_k)((a_1 \cdots a_{k-1})) := (-1)^{|\phi_k|'} \cdot \phi_k(\mu^A_0 \cdot a_1 \cdots a_{k-1})$$

One verifies that $\delta^2 = 0$. As before, we denote its cohomology by $D^k(A, B)$ where $k$ is the tensor degree. Using the operators $\mu^A_1$ and $\mu^B_1$, we define another operator $d$ by

$$d : \text{Hom}(\text{sym}^k A[1], B[1]) \to \text{Hom}(\text{sym}^k A[1], B[1])$$

$$d(\phi_k)(a_1 \cdots a_k) := \mu^B_1 \phi_k(a_1 \cdots a_k) - \sum_{j=1}^k (-1)^r \phi_k(a_1 \cdots \mu^A_1(a_j) \cdots a_k)$$

with $\ast = |\phi_k|' + |a_1|' + \cdots + |a_{j-1}|'$. Here $\text{sym}$ stands for the graded symmetric algebra.

If there exists a $R$-linear map $f : A \to B$ such that $f(\mu^A_0) = \mu^B_0$, we set the $k$-th obstruction space $H^k(A, B)$ to be $H^k(A, B) := H^1(D^k(A, B), d)$.

Let $f_j : \text{sym}^j A[1] \to B[1]$ ($j = 1, \ldots, n$) be $n$ multi-linear maps of cohomological degree zero. We call the sequence $(f_1, \ldots, f_n)$ an $L_{(n)}$ morphism if the following conditions hold:

(i) The $L_\infty$ homomorphism axiom holds up to $(n-1)$ inputs, i.e. for all $0 \leq m \leq n-1$ we have

$$\sum_{k \geq m} \frac{1}{k!} \sum_{\sigma} \epsilon_\sigma \cdot \mu^B_k(f_{i_1}(a_{\sigma(1)}) \cdots f_{i_k}(a_{\sigma(m)}))$$

$$= \sum_{r \geq 0} \sum_{\tau} \epsilon_\tau \cdot f_{m-r+1}(\mu^A_r(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(m)})$$

where $\sigma$ is a $(i_1, \ldots, i_k)$ type shuffle, and $\tau$ is a $(r, n-r)$ type shuffle, and $\epsilon_\sigma$ and $\epsilon_\tau$ are Koszul signs associated with these permutations.

(ii) In the case with $n$ inputs, we require that

$$\sum_{k \geq 2} \frac{1}{k!} \mu^B_k(f_{i_1} \otimes \cdots \otimes f_{i_k}) \text{Sh}_\sigma - \sum_{r \geq 1, \tau} f_{n-r+1}(\mu^A_r \otimes \text{id}^{\otimes n-r}) \text{Sh}_\tau$$

is $\delta$-exact, i.e. it lies in the image of $\delta : C^{n+1}(A, B) \to C^n(A, B)$.

Given an $L_{(n)}$ morphism $(f_1, \ldots, f_n) : A \to B$, we define its obstruction class as follows. Choose any $f'_{n+1}$ of cohomological degree zero such that $\delta f'_{n+1}$ equals the expression in (ii). Then we set

$$\text{obs}_{n+1} :=$$

$$\sum_{k \geq 2, \sigma} \frac{1}{k!} \mu^B_k(f_{i_1} \otimes \cdots \otimes f_{i_k}) \text{Sh}_\sigma - \sum_{r \geq 2, \tau} f_{n-r+2}(\mu^A_r \otimes \text{id}^{\otimes n+1-r}) \text{Sh}_\tau + d f'_{n+1}$$
and define the obstruction class by
\[ \sigma((f_j)_{j=1}^n) = [\text{obs}_{n+1}] \in H^{n+1}(A, B) \]
Again, one can verify (similar to Lemma 2.3) that this class is well-defined and independent of the choice of \( f'_n \).

The formal properties of the obstruction theory still holds in the \( L_\infty \) case, with which we deduce the following result for curved \( L_\infty \) algebras. Again, in the uncurved case and over a field, this result immediately implies the classical Whitehead theorem of \( L_\infty \) algebras that quasi-isomorphisms are also homotopy equivalences.

**Theorem 2.10.** An \( L_\infty \) homomorphism \( f = (f_1, f_2, \ldots) : A \to B \) between curved \( L_\infty \) algebras is a homotopy equivalence if and only if \( f_1 \) is an \( L_{(1)} \) homotopy equivalence.

### 3. Homotopy transfer of curved algebras

In this section, we prove a curved version of the homological perturbation lemma. This works for both \( A_\infty \) and \( L_\infty \) algebras in the presence of a new version of homotopy retraction data in the curved case. This is used to construct minimal charts of \( L_\infty \) spaces in Subsection 5.2.

#### 3.1. Curved homotopy retraction data

Let us consider the following situation: we are given a curved \( A_\infty \)-algebra \((A, m_k)\), a graded vector space \( V \), \( R \)-linear maps \( i : V \to A \) and \( p : A \to V \) and a \( R \)-linear map \( H : A \to A \) of degree \(-1\). Assume there is \( C \in V \) of degree 2 such \( i(C) = m_0 \) and moreover:
\begin{align*}
\tag{5} m_1 H + H m_1 &= ip - \text{id}_A - H m_2^2 H, \\
\tag{6} pm_1 H &= 0, \quad H m_1 i = 0.
\end{align*}

We have the following

**Theorem 3.1.** In the situation described, \( V \) has the structure of curved \( A_\infty \)-algebra with \( \mu_0 = C \) and \( \mu_1 = pm_1 i \). Moreover there is an \( A_\infty \)-homomorphism \( \varphi : (V, \mu_k) \to (A, m_k) \) with \( \varphi_1 = i \).

A common application of this theorem is to construct “minimal” algebras. In that case, we have the side conditions \( Hi = pH = 0 \) and \( pi = id \) which, together with (5) imply (6). Before we go into the proof we describe the maps \( \mu_k \), for \( k \geq 2 \):
\[ \mu_k = \sum_{T \in \Gamma_k} \mu_T, \]
where \( \Gamma_k \) is the set of rooted directed stable planar trees with \( k \)-leaves

Now we use \( T \) as a flow chart to define a map
\[ \mu_T : V^\otimes k \to V. \]
We assign to each \( v \in V(T) \) the map \( m_{val(v)} \); to the internal edges we assign \( H \); and finally we assign \( p \) to the root and \( i \) to the leaves. For examples, the tree in the pictures gives the map
\[ \mu_T(u_1, u_2, u_3, u_4, u_5, u_6) = p \circ m_3(H \circ m_3(i(u_1), i(u_2), i(u_3)), i(u_4), H \circ m_2(i(u_5), i(u_6))). \]

To prove these maps define an \( A_\infty \)-algebra we will need the following auxiliary maps. Let \( T \in \Gamma_k \), denote by \( E(T) \) the set of edges of \( T \) and by \( e(T) \) the set of
internal edges of $T$. For each $T \in \Gamma_k$, we define $\overline{T}$ as the tree $T$ with one additional vertex in each internal edge of $T$. Given $e \in E(\overline{T})$ we define $\hat{\mu}_{\overline{T},e}$ in the same way as $\mu_T$ with the extra assignment of $m_1$ to the edge $e$. Given $e \in e(T)$ we define $\hat{\mu}_{\Pi,}\hat{\mu}_{id},\hat{\mu}_{\gamma}$ in the same way as $\mu_T$, but with $\Pi = id$ (respectively $id$ and $\gamma := Hm_1^2H$) assigned to $e$ instead of $H$.

Proof of Theorem 3.1. One can easily check the first two $A_\infty$ equations for $\mu_k$ using equations (5, 6). For two or more inputs we define

$$\hat{\mu}_{k,\beta}(u_1, \ldots, u_k) = \sum_{T \in \Gamma_k(\beta)} (-1)^{|T_e|} \hat{\mu}_{T,e}(u_1, \ldots, u_k)$$

where $|T_e| = \sum_{i=1}^{m_e} |u_i|$ with $m_e$ defined as the smallest $1 \leq j \leq k$ such that the path from the $i^{th}$ leaf to the root does not include $e$, for all $i < j$. Then given $e \in e(T)$, denote by $E_-$ and $E_+$ the edges of $\overline{T}$ contained in $e$. The equation (5) implies

$$\hat{\mu}_{T,E_-} + \hat{\mu}_{T,E_+} = \mu_{\Pi} - \mu_{id} - \mu_{\gamma}.$$ 

Therefore

$$\hat{\mu}_k = \sum_{T \in \Gamma_k} \sum_{e \in E(\overline{T}) \setminus e(T)} (-1)^{|T_e|} \hat{\mu}_{T,e} + \sum_{T \in \Gamma_k} \sum_{e \in e(\overline{T})} (-1)^{|T_e|} (\mu_{\Pi} - \mu_{id} - \mu_{\gamma}).$$

(7)  

On the other hand,

$$\hat{\mu}_k = \sum_{T \in \Gamma_k} \sum_{e \in V(T)} \sum_{v \in \partial e} (-1)^{|T_e|} \hat{\mu}_{T,e},$$

and the $A_\infty$-equation implies that

$$\hat{\mu}_k = \sum_{T \in \Gamma_k} - \sum_{e \in V(T)} \sum_{S \in \Gamma_k} (-1)^{|S_e|} \hat{\mu}_{S,e} + \sum_{S \in \Gamma_{k+1}} (-1)^{|S_e|} \mu_S(...u_{i-1}, \mu_0, u_i...).$$

(8)
Here \( S/e \) is the tree obtained from \( S \) by collapsing the edge \( e \) and \( S/i \) is the tree obtained by deleting the \( i \)-th leaf of \( S \). Putting (7) and (8) together we conclude

\[
\sum_{T \in \Gamma_k} \left( \sum_{e \in e(T)} (-1)^{|T_e|} (\mu_{T,e} - \mu_{T,e}^\gamma) + \mu_{T,r} + \sum_{i=1}^k (-1)^{|T_e|} \hat{\mu}_{T,i} \right)
\]

(9)

\[
+ \sum_{T \in \Gamma_k} \sum_{S \in \Gamma_{k+1}} (-1)^{|S|} \mu_S(...u_{i-1}, \mu_0, u_i,...) = 0,
\]

where \( r \) is the edge adjacent to the root and the \( l_i \) are the edges adjacent to the leaves of \( T \). It follows from the definition of \( \mu_T \) that

\[
\sum_{T \in \Gamma_k} \sum_{e \in e(T)} (-1)^{|T_e|} \mu_{T,e} = \sum_{k_1 \neq 0, 1} \mu_{k_2} (u_1, \ldots, u_k).
\]

Equations (5) and (6) imply \( pm_1 = \mu_1 p - pm_1^2 \). This combined with the \( A_\infty \) equation gives

\[
\hat{\mu}_{T,r} = \mu_1 \mu_T + \mu_{C_2 \circ_T} (\mu_0, \ldots) + (-1)^* \mu_{C_2 \circ_T} (\ldots, \mu_0),
\]

where \( C_2 \) is the unique tree with two leaves and \( C_2 \circ_T \) is the tree obtained by grafting the root of \( T \) to the \( i \)-th leaf of \( C_2 \).

Analogously, the identity \( m_1 i = i \mu_1 - Hm_1^2 i \) implies

\[
\sum_{i=1}^k (-1)^{|T_i|} \hat{\mu}_{T,i} = \sum_{i} \mu_T(..., \mu_1 (u_i), \ldots) + \sum_{i} (-1)^* \mu_{T_0, C_2} (... , \mu_0, u_i, ...)
\]

Finally, using the fact that \( -\gamma(u) = Hm_2 (m_0, H(u)) + (-1)^{|u|} m_2 (H(u), m_0) \) we have

\[
(-1)^{|T_1|+1} \mu_{T,e} = (-1)^* \mu_{T_0, C_2 \circ_{T_1} (\ldots, \mu_0, u_i, ...)} + (-1)^* \mu_{T_0, C_2 \circ_{T_2} (\ldots, \mu_0, u_i, ...)}
\]

where \( T_1 \) and \( T_2 \) are the trees obtained from cutting \( T \) along the edge \( e \) and \( j \) is the number of leaves in \( T_2 \). These last four identities prove that Equation (9) is equivalent to the \( A_\infty \)-algebra equation for the \( \mu_k \).

The construction of map \( \varphi : V \rightarrow A \) is similar. We put \( \varphi_1 = i \) and \( \varphi_k = \sum_{T \in \Gamma_k} \varphi_T \) where map \( \varphi_T \) is defined in the same way as \( \mu_T \), the only difference is that we assign \( H \) to the root vertex (instead of \( p \) as in the case of \( \mu_T \)). Similarly we define the auxiliary maps

\[
\hat{\varphi}_k = \sum_{T \in \Gamma_k} (-1)^{|T|} \varphi_{T,e},
\]

and for each \( e \in e(T) \) we define \( \varphi_{T,e}^{id}, \varphi_{T,e}^{\varphi} \) and \( \varphi_{T,e}^\gamma \).

The same argument we used above applies to show

\[
\sum_{T \in \Gamma_k} \left( \sum_{e \in e(T)} (-1)^{|T_e|} (\mu_{T,e} - \mu_{T,e}^\gamma) + \varphi_{T,r} + \sum_{i=1}^k (-1)^{|T_e|} \hat{\varphi}_{T,i} \right)
\]

(10)

\[
+ \sum_{T \in \Gamma_k} \sum_{S \in \Gamma_{k+1}} (-1)^{|S|} \varphi_S(...u_{i-1}, \mu_0, u_i,...) = 0,
\]

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Now using (5) again we see that
\[ \hat{\varphi}_{T,r} = -m_1 \circ \varphi_T + i \circ \mu_T - \varphi_{T,r}^{id} - \varphi_{T,r}^\gamma, \]
and
\[ \varphi_{T,r}^{id} = m_j(\varphi_{T_1}, \ldots, \varphi_{T_j}), \]
where \( j \) is the valency of the vertex of \( T \) closest to the root and \( T_i \) are the trees obtained from cutting \( T \) at the incoming edges at that vertex. One can now see by the same argument that Equation (10) is equivalent to the \( A_\infty \) homomorphism equation for \( \varphi \):
\[
\sum_{j, i_1+\ldots+i_j=k} m_j(\varphi_{i_1}(u_1, \ldots, u_{i_1}), \ldots, \varphi_{i_j}(\ldots, u_k)) - \sum_{0 \leq j \leq k} \sum_{0 \leq i \leq k-j} (-1)^i \varphi_{k-j+1}(u_1, \ldots, \mu_j(u_{i+1}, \ldots, u_{i+j}), \ldots, u_k) = 0.
\]

Remark 3.2. In the case of uncurved \( A_\infty \)-algebras, there are also explicit formulas for a homomorphism \( \psi : (A, m_k) \rightarrow (V', \mu_k) \) with \( \psi_1 = p \) and a homotopy \( H : (A, m_k) \rightarrow (A, m_k) \) between \( \varphi \circ \psi \) and \( \text{id}_A \). See [11] for this construction.

3.2. The \( L_\infty \) case. The discussion in the \( L_\infty \) case is very much the same as the \( A_\infty \) case, except that instead of using planar stable rooted trees in the formulas, one uses isomorphism classes of stable rooted trees. We refer to the article [8, Section 4] for a detailed treatment.

4. The category of curved \( L_\infty \) spaces

In this section, we recall basic definitions of curved \( L_\infty \) spaces, morphisms between these spaces and describe the notion of homotopy between morphisms.

4.1. Curved \( L_\infty \) spaces. A curved \( L_\infty \) space is given by a pair \((M, g)\) where \( M \) is a smooth manifold, and \( g \) is a \( \mathbb{Z} \)-graded vector bundle over \( M \) of the form
\[ g = g_2 \oplus g_3 \oplus \cdots \oplus g_d \]
for some \( d \geq 2 \), together with bundle maps \( \mu_k : \text{sym}^k(g[1]) \rightarrow g[1] \) of degree one such that the \( L_\infty \) equation holds:
\[
\sum_{k=0}^n \sum_{\sigma \in \text{Sh}(k, n-k)} \epsilon_{\sigma} \cdot \mu_{n-k+1}(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}) = 0
\]
where \( \text{Sh}(k, n-k) \) consists of \((k, n-k)\)-type shuffles, and \( \epsilon_{\sigma} \) is the Koszul sign associated with the permutation \( a_1 \otimes \cdots \otimes a_n \mapsto a_{\sigma(1)}, \ldots, a_{\sigma(n)} \) with the \( a \)'s considered as elements of \( g[1] \).

In order to formulate a good notion of homotopy between morphisms of \( L_\infty \) spaces we need to have “special” connections. Therefore we make the following assumption: \( T_M \) has a torsion-free, flat connection and \( g \) has a flat connection.\(^5\)

The main results in this paper are local, meaning \( M \) is an open ball in \( \mathbb{R}^n \), therefore such connections always exist.

\(^5\)In fact, it would be enough to have these connections on an open neighborhood of \( \mu_0^{-1}(0) \).
A morphism from \((M, g)\) to \((N, h)\) is a pair \(\mathcal{f} = (f, f^\sharp)\) where \(f : M \to N\) is a smooth map, and \(f^\sharp : g \to f^* h\) is a morphism of \(L_\infty\) algebras. This means a sequence of (degree zero) bundle maps \(f_k^\#: \Sigma^k(g[1]) \to f^* h[1]\) satisfying
\[
\sum_k \frac{1}{k!} \sum_\sigma \epsilon_\sigma \cdot \mu_k(f_k^\#(a_{\sigma(1)} \cdots a_{\sigma(n)})) = \sum_r \sum_\tau \epsilon_\tau \cdot f_{n-r+1}^\#(\mu_r(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(n)}),
\]
where \(\sigma\) is a \((i_1, \ldots, i_k)\) type shuffle, and \(\tau\) is a \((r, n-r)\) type shuffle. On the left-hand side there is an abuse of notation: \(\mu_k\) stands for \(f^* \mu_k\).

Morphisms of \(L_\infty\) spaces can be composed similarly to the algebra case. Given \(L_\infty\) morphisms \(\epsilon : (M', g') \to (M, g)\) and \(f : (M, g) \to (N, h)\) we define \(\mathcal{f} \circ \epsilon := (f \circ e, f^\sharp \circ e^\sharp)\) where
\[
(f^\sharp \circ e^\sharp)(a_1 \cdots a_k) = \frac{1}{k!} \sum_\sigma \epsilon_\sigma \cdot e^\#(f_k^\#(a_{\sigma(1)} \cdots a_{\sigma(n)}))
\]
As in the algebra case we define also \(L_{(n)}\) morphisms between curved \(L_\infty\) spaces.

4.2. Extensions of \(L_\infty\) structures. Let \((M, g)\) be an \(L_\infty\) space. By our assumptions, we can choose a torsion-free, flat connection on \(T_M\) and also a flat connection on the bundle \(g\). We set
\[
\tilde{g} := T_M \oplus g,
\]
with \(T_M\) at cohomological degree one. The \(L_\infty\) structure on \(g\) naturally extends to \(\tilde{g}\) by inductively applying the following formula
\[
\mu_{k+1}(X \cdot a_1 \cdots a_k) := \nabla_X \mu_k(a_1 \cdots a_k) - \sum_{j=1}^k \mu_k(a_1 \cdots a_j \nabla_X a_{j+1} \cdots a_k)
\]
Using the torsion freeness and the flatness, one can verify that when pulling out tangent vectors using the above formula, the choice of order does not matter, i.e. we have that
\[
\mu_{k+2}(X \cdot Y \cdot a_1 \cdots a_k) = \mu_{k+2}(Y \cdot X \cdot a_1 \cdots a_k)
\]
for any two tangent vectors \(X, Y \in T_M\).

Lemma 4.1. Equation (12) defines an \(L_\infty\) algebra structure on \(\tilde{g}\).

Proof. We prove the \(L_\infty\) identity by induction on the total number of tangent vectors. Indeed, when there is no tangent vector, the \(L_\infty\) identity holds since \(g\) forms an \(L_\infty\) algebra to begin with. We want to verify the \(L_\infty\) identity:
\[
\sum_{k=1}^n \sum_{\sigma \in \operatorname{Sh}(k, n-k)} \epsilon_\sigma \cdot \mu_{n-k+1}(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}) = 0
\]
We may also assume that all the inputs \(a\)'s are flat with respect to the chosen connection \(\nabla\). To this end, let us pick a tangent vector, say \(a_1\) among the inputs, and apply Equation (12) to pull it out of the inputs. If \(a_1\) falls into \(a_{\sigma(1)}, \ldots, a_{\sigma(k)}\), we obtain terms of the form
\[
\sum \sum \epsilon_\sigma \mu_{n-k+1}(\nabla a_1 \mu_{k-1}(\cdots), \cdots)
\]
When \(a_1\) falls into \(a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}\), we get
\[
\nabla_{a_1} \left( \sum \sum \epsilon_\sigma \mu_{n-k}(\mu_k(\cdot), \cdot) \right) - \sum \sum \epsilon_\sigma \mu_{n-k+1}(\nabla_{a_1} \mu_{k-1}(\cdot), \cdot)
\]
Thus, their sum yields \(\nabla_{a_1} \left( \sum \sum \epsilon_\sigma \mu_{n-k}(\mu_k(\cdot), \cdot) \right)\) which vanishes by induction.

\(L_\infty\) morphisms between \(L_\infty\) spaces can also be extended to the tangent bundles. More precisely, let \((f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})\) be a morphism of \(L_\infty\) spaces and, as above choose torsion free and flat connections on both spaces, we can extend the \(L_\infty\) morphism \(f^\sharp\) which we still denote by
\[
f^\sharp : \mathfrak{g} \to f^* \mathfrak{h}
\]
The formula of extension is the same as in Equation (12), i.e. we inductively define
\[
f^\sharp_{k+1}(X \cdot a_1 \cdots a_k) := \nabla_X f^\sharp_k(a_1, \cdots, a_k) - \sum_{j=1}^k f^\sharp_k(a_1, \cdots, \nabla_X a_j, \cdots, a_k)
\]
A minor difference is that here we need \(k \geq 1\). Thus, we need to define by hand the map \(f^\sharp_k : T_M \to f^* T_N\). For this, we set it to be the tangent map \(d\!
abla\).

**Lemma 4.2.** The maps defined in Equation (13) form an \(L_\infty\) morphism \(f^\sharp : \mathfrak{g} \to f^* \mathfrak{h}\).

**Proof.** We need to verify that
\[
\sum_k \frac{1}{k!} \sum_\sigma \epsilon_\sigma \cdot \mu_k \left( f^\sharp_{i_1}(a_{\sigma(1)}), \cdots, f^\sharp_{i_k}(\cdot, \cdots, a_{\sigma(n)}) \right) = \sum_r \sum_\tau \epsilon_r \cdot f^\sharp_n(a_{\tau(1)} \cdots a_{\tau(r)} \cdots a_{\sigma(n)})
\]
Let us pick up a tangent vector, say \(a_1\) among the inputs. Also we assume that all the input vectors are flat. If \(a_1\) is inside \(f_{i_j}\), and \(i_j = 1\), we get
\[
\sum_k \frac{1}{(k-1)!} \sum_\sigma \epsilon_\sigma \cdot \nabla_{a_1, \mu_{k-1}} \left( f^\sharp_{i_1}(a_{\sigma(1)}), \cdots, f^\sharp_{i_1}(a_1), \cdots, f^\sharp_{i_k}(\cdot, a_{\sigma(n)}) \right)
\]
\[
- \sum_k \frac{1}{(k-1)!} \sum_\sigma \epsilon_\sigma \cdot \mu_{k-1} \left( f^\sharp_{i_1}(a_{\sigma(1)}), \cdots, f^\sharp_{i_1}(a_1), \cdots, \nabla_{a_1, f^\sharp_{i_1}}(\cdot, \cdots, f^\sharp_{i_k}(\cdot, a_{\sigma(n)}) \right)
\]
The coefficient becomes \(\frac{1}{(k-1)!}\) since there are \(k\) possible choice of \(j\). The second term cancels precisely the terms with \(a_1\) inside \(f_{i_j}\) with \(i_j \geq 2\). Thus the left hand side is equal to (by induction on the total number of tangent vectors)
\[
\nabla_{a_1} \left( \sum_k \frac{1}{(k-1)!} \sum_\sigma \epsilon_\sigma \cdot \mu_{k-1} \left( f^\sharp_{i_1}(a_{\sigma(1)}), \cdots, f^\sharp_{i_1}(a_1), \cdots, f^\sharp_{i_k}(\cdot, a_{\sigma(n)}) \right) \right)
\]
\[
= \nabla_{a_1} \left( \sum_r \sum_\tau \epsilon_r \cdot f^\sharp_n(a_1, \cdots, a_{\tau(r)} \cdots a_{\sigma(n)}) \right)
\]
which is exactly the right hand side. \(\square\)

These extensions of \(L_\infty\) spaces induced by the choice of connections are in fact independent of these choices up to isomorphism.
Lemma 4.3. Let $\nabla$ and $\nabla'$ be torsion free and flat connections. Let $\tilde{g}$ and $\tilde{g}'$ be the associated extended $L_\infty$ algebras. Then there is an isomorphism

$$\Phi^\varnothing : \tilde{g} \to \tilde{g}'$$

defined by $\Phi^\varnothing_1 = \text{id}, \Phi^\varnothing_2(X,\alpha) = (\nabla'_X - \nabla_X)(\alpha)$, and for $k \geq 3$ by the recursive formula

$$\Phi^\varnothing_k(X \cdot \alpha_1 \cdots \alpha_{k-1}) := \nabla'_X \Phi^\varnothing_{k-1}(\alpha_1 \cdots \alpha_{k-1}) - \sum_{j=1}^{k-1} \Phi^\varnothing_k(\alpha_1 \cdots \nabla_X \alpha_j \cdots \alpha_k).$$

Moreover this isomorphism is natural: given a $L_\infty$ morphism $(f, f') : (M, g) \to (N, h)$ and different choices of connections, the induced extended $L_\infty$ morphisms $f^\varnothing : \tilde{g} \to f'^\# \tilde{g}$ and $(f')' : \tilde{g}' \to f'^\# \tilde{g}'$ satisfy $\Phi^\varnothing \circ f^\varnothing = (f')' \circ \Phi^\varnothing$.

Proof. As before we prove the $L_\infty$ homomorphism equation by induction on the number of inputs that are tangent vectors. When there is no tangent vector, the operations $\mu_k$ and $\mu'_k$ agree and $\Phi$ is just the identity. Let us now pick up a tangent vector, say $a_1$ among the inputs. For simplicity we assume that all the inputs are flat with respect to $\nabla$. Let’s consider the left-hand side of the $L_\infty$ equation, when $a_1$ is inside $\Phi_{i_j}$ with $i_j = 1$, we get

$$\sum_k \frac{1}{(k-1)!} \sum_{\sigma} \epsilon_\sigma \cdot \nabla'_{a_1} \mu'_{k-1}(\Phi_{i_1}(a_{\sigma(1)}) \cdots \Phi_{i_1}(a_1) \cdots \Phi_{i_k}(\cdots a_{\sigma(n)}))$$

$$- \sum_k \frac{1}{(k-1)!} \sum_{\sigma} \epsilon_\sigma \cdot \mu'_{k-1}(\Phi_{i_1}(a_{\sigma(1)}) \cdots \Phi_{i_1}(a_1) \cdots \nabla'_{a_1} \Phi_{i_1}(\cdots) \cdots \Phi_{i_k}(\cdots a_{\sigma(n)})),$$

using the definition of $\mu'_k$ and the fact $\Phi_1 = \text{id}$. The second sum above exactly cancels with the other terms in the left-hand side of the $L_\infty$-homomorphism equation with $a_1$ inside $f_{i_j}$ with $i_j \geq 2$. This is because the $a_1$ are $\nabla$ flat and $\Phi_2(a_1, a_j) = \nabla'_{a_1} a_j$. Therefore, by induction hypothesis, the left-hand side equals

$$\nabla'_{a_1} \left( \sum_r \sum_{\tau} \epsilon_\tau \cdot \Phi_{n-r} \left( \mu_{r}(a_{\tau(2)} \cdots a_{\tau(n)}) \right) \right) = \nabla'_{a_1} \left( \mu_{n-1}(a_2 \cdots a_n) \right) +$$

$$+ \sum_{r \leq n-2} \sum_{\tau} \epsilon_\tau \cdot \Phi_{n+1-r} \left( \mu_{r}(a_{\tau(2)} \cdots a_{\tau(n)}) \right)$$

(14)

$$+ \sum_{r \leq n-2} \sum_{\tau} \epsilon_\tau \cdot \Phi_{n-r} \left( \nabla'_{a_1} \left( \mu_{r}(a_{\tau(2)} \cdots a_{\tau(n)}) \right) \right).$$

Here the first term equals $\Phi_2(a_1, \mu_{n-1}(a_2 \cdots a_n))$ and in the third term we have $\nabla'_{a_1} (\mu_{r}(a_{\tau(2)} \cdots)) = \mu_{r+1}(a_1 \cdot a_{\tau(2)} \cdots)$. Hence (14) equals

$$\sum_r \sum_{\tau} \epsilon_\tau \cdot \Phi_{n-r+1} \left( \mu_{r}(a_{\tau(1)} \cdots a_{\tau(r)}) \cdots a_{\tau(n)}) \right)$$

More precisely, the first two terms in (14) correspond above to the terms where $a_1$ is outside the $\mu_r$.

The proof of the naturality statement is entirely analogous and we omit it. □
4.3. Homotopy of $L_\infty$ morphisms. In order to define a good notion of homotopy we need to consider the space version of tensoring with $\Omega^*_\Gamma[0,1]$ as in Subsection 2.3. Let $(M, g)$ and $(N, h)$ be two $L_\infty$ spaces and let $F : M \times [0,1] \to N$ be a smooth map. Consider the graded bundle $F^*\pi_\Gamma[0,1] := F^*\pi \otimes \pi_2^*\Omega^*_\Gamma[0,1]$, where $\pi_2 : M \times [0,1] \to [0,1]$ is the projection and denote $\mu^\natural_k = (f^*)^\natural \mu_k$, $f^t = F(-, t)$. On $F^*\pi_\Gamma[0,1]$ we define the operations

$$
\mu^\natural_0 := \mu_0 - (dF/dt)dt
$$

$$
(15) \quad \mu^\natural_i(x(t) + y(t)dt) := \mu^\natural_i(x(t)) + \mu^\natural_i(y(t)) dt + (-1)^{|x(t)|} \nabla_{dF/dt(x(t))} dt
$$

$$
\mu^\natural_k(..., x_i(t) + y_i(t)dt, ...) := \mu^\natural_k(..., x_i(t), ...) + \sum_i (-1)^{|\pi_i|} \mu^\natural_k(x_1(t), ..., y_i(t), ..., x_k(t)) dt,
$$

for $k \geq 2$, where $|\pi_i| = \sum_{a=\pm 1} |x_a|'$.

Lemma 4.4. The operations $\mu^\natural_k$ define a curved $L_\infty$ algebra structure on $F^*\pi_\Gamma[0,1]$.

Proof. The proof is standard, it follows from the fact that the tensor product of a $L_\infty$-algebra and a commutative dg-algebra is again a $L_\infty$-algebra together with the relation

$$
\nabla_{a\mu^\natural_k(a_1, \ldots, a_k)} = \mu^\natural_{k+1}(\frac{\partial F}{\partial t} dt, a_1, \ldots, a_k).
$$

We are now ready to define homotopy.

Definition 4.5. Two maps $(f^0, f^{0,\natural})$ and $(f^1, f^{1,\natural})$ from $(M, g)$ to $(N, h)$ are homotopic if there exists a map $F : M \times [0,1] \to N$, together with an $L_\infty$ homomorphism

$$
F^\natural : \pi_1^*g \to F^*\pi_\Gamma[0,1],
$$

where $\pi_1 : M \times [0,1] \to M$ is the projection map. We also require the following:

- It’s compatible with the connection, i.e.

$$
F^\natural_{k+1}(X \cdot \alpha_1 \cdot \ldots \cdot \alpha_k) := \nabla_X F^\natural_k(\alpha_1, \ldots \cdot \alpha_k) - \sum_{j=1}^k F^\natural_k(\alpha_1 \cdot \ldots \cdot \nabla_X \alpha_j \cdot \ldots \cdot \alpha_k)
$$

and $F^\natural_1(X) = dF(X)$, $\forall X \in T_M$.

- The following boundary conditions hold:

$$
(F, F^\natural)|_{t=0} = (f^0, f^{0,\natural})
$$

$$
(F, F^\natural)|_{t=1} = (f^1, f^{1,\natural})
$$

Note that by Lemma 4.3, this definition is independent of the choice of $\nabla$.

It will be helpful to unwind this definition. The compatibility condition implies the morphism $F^\natural$ is determined by its values on the elements of $\pi_1^*g$. We write

$$
F^\natural_k(a_1, \ldots, a_k) = f^\natural_k(a_1, \ldots, a_k) + (-1)^{\sum_i |a_i|'} \pi_\Gamma[h_k(a_1, \ldots, a_k)] dt.
$$

Then the $L_\infty$ morphism equation for $F^\natural$ is equivalent to

(1) The maps $(f^\natural_k)_{k \geq 1}$ define an $L_\infty$ morphism $g \to (f^\natural)^* h$;
The maps $h_i$ satisfy the equations $h_i^t(\mu_0) = \frac{\partial \rho}{\partial t}$ and for $n \geq 1$
\[ \sum_k \frac{1}{(k - 1)!} \sum_{\sigma} \epsilon_\sigma \mu_k^t(h_i(a_{\sigma(1)} \cdots) f_{i_1}^t \cdots f_{i_k}^t (\cdots a_{\sigma(n)})) + \]
(16) \[ \sum_{j \geq 0} \sum_{\tau} \epsilon_\tau h_{n-j+1}(\mu_j(a_{\tau(1)} \cdots a_{\tau(r)} \cdots a_{\tau(n)}) = \nabla \frac{d}{dt} f_n^t(a_1, \ldots, a_n) \]
where $\sigma$ is a $(i_1, \cdots, i_k)$ type shuffle, $\tau$ is a $(r, n-r)$ type shuffle and the $a_i$ are flat.

**Proposition 4.6.**

(a) Homotopy of $L_\infty$ morphisms is an equivalence relation.
(b) If $(f^0, f^{0,2})$ and $(f^1, f^{1,2})$ homotopic, then $(f^0, f^{0,2}) \circ (d, d^2)$, $(f^1, f^{1,2}) \circ (d, d^2)$ are homotopic and $(e, c^2) \circ (f^0, f^{0,2})$, $(e, c^2) \circ (f^1, f^{1,2})$ are homotopic, for any composable $L_\infty$ morphisms $(d, d^2), (e, c^2)$.

**Proof.** For (a) first note that a diffeomorphism $\rho : [0, 1] \to [0, 1]$ induces, by pull-back, a $L_\infty$ homomorphism $\rho^* : F^* h_{[0,1]} \to (F^\rho)^* h_{[0,1]}$, where $F^\rho := F \circ (\text{id} \times \rho)$. Now given a homotopy $(F, F^2)$ from $(f^0, f^{0,2})$ to $(f^1, f^{1,2})$, take $\rho(t) = 1-t$ and consider the pair $(F^\rho, F^2)$ which is a homotopy $(f^1, f^{1,2})$ to $(f^0, f^{0,2})$, which shows symmetry of the homotopy relation. For transitivity let $\rho$ be a non-decreasing diffeomorphism which is constant in neighborhoods of 0 and 1 in $[0, 1]$. For this choice of $\rho$, $\mathfrak{H}_\rho := (F^\rho, F^2)$ is a new homotopy from $(f^0, f^{0,2})$ to $(f^1, f^{1,2})$. Give a homotopy $(G, G^2)$ from $(f^1, f^{1,2})$ to $(f^2, f^{2,2})$ we consider $\mathfrak{H}_\rho \circ \mathfrak{H}_\rho$ by
\[ F^\rho \bullet G^\rho(x,t) = \begin{cases} F^\rho(x, 2t), & t \leq 1/2 \\ G^\rho(x, 2t - 1), & t \geq 1/2 \end{cases} \]
and analogously $F^2 \bullet G^2$. By our choice of $\rho$ these are smooth maps and can be easily seen to determine a homotopy from $(f^0, f^{0,2})$ to $(f^2, f^{2,2})$.

For (b) we prove only the second statement as they are analogous. Let $(F, F^2)$ be a homotopy from $(f^0, f^{0,2})$ to $(f^1, f^{1,2})$ and $c^2 : h \to c^2 h'$ be a $L_\infty$ homomorphism. It is easy to check that there is an induced $L_\infty$ homomorphism $c^2 : F^* h_{[0,1]} \to (e \circ F)^* h_{[0,1]}$. Now the pair $(e \circ F, c^2 \circ F^2)$ defines the required homotopy from $(e, c^2) \circ (f^0, f^{0,2})$ to $(e, c^2) \circ (f^1, f^{1,2})$.

**5. Whitehead theorem for $L_\infty$ spaces**

In this section, we first adapt the obstruction theory of Section 2 to the case of $L_\infty$ spaces. Then we prove Theorem 1.1 and Theorem 1.3.

5.1. **Obstruction theory for morphisms between $L_\infty$ spaces.** Much of the discussion on the obstruction theory for $A_\infty$ homomorphisms in Section 2 translates without significant changes to the $L_\infty$ space setting. When we are given two $L_\infty$ spaces $(M, g)$, $(N, h)$ and a smooth map $f : M \to N$, we can define a differential $\delta$ on $\sum_k \text{Hom}(\text{Sym}^k(g[1]), h[1])$, as in Subsection 2.1,
\[ \delta(\phi_k)(a_1, \ldots, a_{k-1}) := (-1)^{|\phi_k|} \phi_k(\mu_0, a_1, \ldots, a_{k-1}). \]
We denote by $D^k(g, f^*h)$ the $\delta$ cohomology and, assuming there is a map $f^1_i$ satisfying $d\phi_i^k(\mu_0) = \mu_0$, we define the differential
d$\phi(a_1, \ldots, a_k) := \mu_1 \phi(a_1, \ldots, a_k) - (-1)^i |\phi'|^i |a_i| \sum_{i=1}^k |a_i| \phi(\mu_1(a_i), a_1, \ldots, \hat{a}_i, \ldots, a_k).

As in Definition 2.2, we define the obstruction space

$$H^k(g, f^*h) := H^1(D^k(g, f^*h), d).$$

We also define a sequence of maps $(f^2_j, \ldots, f^n_k) : g \to f^*h$ (together with $f$) to be an $L_{(n)}$-morphism, if it satisfies the $L_\infty$ homomorphism equation for $0 \leq k \leq n-1$ and for $n$ inputs up to a $\delta$ exact term. We define, in the same way as in Section 2, an obstruction class

$$\phi(f^1_j)_{j=1} \in H^{n+1}(g, f^*h).$$

This obstruction class vanishes if and only if the map can be lifted to a $L_{(n+1)}$ morphism.

**Lemma 5.1.** The obstruction class $\phi(f^1_j)_{j=1} \in H^{n+1}(g, f^*h)$ vanishes if and only if the corresponding class (in the extended $L_\infty$ algebras) $\phi(f^1_j)_{j=1} \in H^{n+1}(\tilde{g}, f^*\tilde{h})$ vanishes.

**Proof.** If there exists $f^2_{n+1} : \text{sym}^{n+1}(g[1]) \to f^*h[1]$, we may extend it using Equation (13) to obtain an $L_{(n+1)}$ map on the extended $L_\infty$ algebras, which implies that $\phi(f^1_j)_{j=1} \in H^{n+1}(\tilde{g}, f^*\tilde{h})$ vanish. Conversely, if the latter obstruction class vanishes, we simply restrict the map $f^1_j$ to $\text{sym}^{n+1}(g[1]) \to f^*h[1]$. □

Extra work is needed to formulate the homotopy invariance of obstruction spaces and classes. Let $(F, F^0)$ be a homotopy between two $L_{(1)}$ morphisms $(f^0, f^{0,2})$ and $(f^1, f^{1,2})$. Denote by $\iota_a : M \to M \times [0,1]$ the inclusion map $\iota_a(x) = (x, a)$. For an element $\phi \in \text{Hom}(\text{sym}^{n+1}(\tilde{g}[1], F^*\tilde{h}[1][1]))$, we have $\iota_a^*(\phi) \in \text{Hom}(\text{sym}^{n+1}(\tilde{g}[1], (f^a)^*\tilde{h}))$, for $a = 0, 1$. It is easy to see this assignment induces a map on obstruction spaces $ev_a : H^{n+1}(\pi_1\tilde{g}, F^*\tilde{h}[1,1]) \to H^{n+1}(\tilde{g}, (f^a)^*\tilde{h})$, which we call the evaluation map.

**Proposition 5.2.** The evaluation map

$$ev_a : H^{n+1}(\pi_1\tilde{g}, F^*\tilde{h}[1,1]) \to H^{n+1}(\tilde{g}, (f^a)^*\tilde{h})$$

is an isomorphism.

**Proof.** Both cases are identical, we will prove the statement for $a = 0$, by constructing $i$ a homotopy inverse to $ev_0$. Given $\phi \in \text{Hom}(\text{sym}^{n+1}(\tilde{g}[1], (f^0)^*\tilde{h}))$, we define $i\phi(a_1, \ldots, a_k)$ by taking the (unique) flat extension of $\phi(a_1, \ldots, a_k)$ in the $t$-direction and so obtain an element of $F^*\tilde{h}$. It is clear $i$ commutes with the $\delta$ differential and hence it induces a map from $D^k(\tilde{g}, (f^0)^*\tilde{h})$ to $D^k(\pi_1\tilde{g}, F^*\tilde{h}[1,1])$.

We pick a flat frame of the bundle $F^*\tilde{h}$ and compute, for a $\delta$-closed $\phi$,

$$d(i(\phi)) - i(d(\phi)) = (\mu_1 - \mu_0) \circ i\phi = \int_0^t \nabla_{d^i/ds} \mu_1 \circ i\phi \, ds = \int_0^t \mu_2^0 \frac{\partial h_1}{\partial s} \cdot i\phi \, ds = \int_0^t \mu_2^0(h_1^1(\mu_0), i\phi) \, ds = \delta(\int_0^t \mu_2^0(h_1^1 \cdot i\phi) \, ds),$$

for $\mu_0$ the identity function.
Here $h^1$ is the map coming from the definition of homotopy in (16). Hence $i$ induces a map between the corresponding obstruction spaces. Clearly, we have $ev_0 \circ i = id$. Let

$$K : D^k(\pi_1^* \tilde{F}, F^* \tilde{b}_{[0,1]}) \to D^k(\pi_1^* \tilde{F}, F^* \tilde{b}_{[0,1]})$$

be the map induced by the integration map

$$K(\varphi^t + \psi^t dt)(a_1, \ldots, a_k) = (-1)^{|\psi|^+} \int_0^t \psi^s(a_1, \ldots, a_k) ds,$$

where $\star := |a_1|' + \ldots + |a_k|'$. We claim that $i \circ ev_0 - id = dK + Kd$.

**Proof.** The morphism $(f_0, f_0^\sharp)$ is defined in the same manner as in the algebra case. We have

$$\text{Corollary 5.3.}$$

Let $(f_0, f_0^\sharp)$ and $(f^1, f^1, f^2)$ be two homotopic $L(n)$ morphisms. Then $\text{Corollary 5.3}$ lifts to a $L(n+1)$ morphism if and only if $(f^1, f^1, f^2)$ does.

**Proof.** The morphism $(f_0, f_0^\sharp)$ lifts to a $L(n+1)$ morphism if and only if $o((f_j^a)^n_{j=1})$ vanishes. Let $(F, F^\sharp)$ be the homotopy between the two $L(n)$ morphisms. We can easily see

$$o((f_j^a)^n_{j=1}) = ev_n(o(F_j^a)^n_{j=1}).$$

By the previous proposition, $ev_n$ is an isomorphism therefore $o((f_j^a)^n_{j=1})$ vanishes if and only if $o(F_j^a)^n_{j=1}$ does.

The push-forward $(\varepsilon)^*$ and pull-back $(\delta)^*$ maps on the obstruction space, under a $L(1)$ morphism, are defined in the same manner as in the algebra case. We have the analogue to Lemma 2.5.

**Lemma 5.4.** Let $\varepsilon = (e, e^f_1) : (N, h) \to (N', h')$ and $\delta = (d, d^f_1) : (M', g') \to (M, g)$ be $L(1)$ homotopy equivalences. Then both maps

$$(\varepsilon)^* : H^n(g, f^* h) \to H^n(g, (e \circ f)^* h'),$$

$$(\delta)^* : H^n(g, f^* h) \to H^n(g', (f \circ d)^* h)$$

are isomorphisms.
Proof. Let $\mathcal{E}$ be a $L_{(1)}$ homotopy between two $L_{(1)}$ morphisms $e^0$ and $e^1$. Observe that $ev_a \circ (e^0)_* = (e^0)_*$ for $a = 0, 1$. Since $ev_a$ is an isomorphism, by Proposition 5.2, we conclude $(e^0)_*$ is an isomorphism if and only if $(e^1)_*$ is an isomorphism. Now let $\tilde{e}$ be a $L_{(1)}$ homotopy inverse for $e$, then by the previous argument $(e \circ \tilde{e})_*$ (and $\tilde{e} \circ e_*$) is an isomorphism. Hence we conclude $(e)_*$ is an isomorphism from the equality $(e \circ \tilde{e})_* = (e)_* \circ (\tilde{e})_*$.

The same argument proves the statement for $(d)_*$.

With these preparations, we may deduce the following result which is the analogue of Theorem 2.8 in the case of $L_\infty$ spaces. Its proof is essentially the same: using the previous results we prove analogues of Proposition 2.6 and Lemma 2.7 which lead to the following theorem. Since this involves only minor modifications we omit its proof.

**Theorem 5.5.** An $L_\infty$ space homomorphism $f = (f, f^2) : (M, g) \to (N, h)$ is a homotopy equivalence if and only if $(f, f^2)$ is an $L_{(1)}$ homotopy equivalence.

**Remark 5.6.** Unlike in the algebraic case (see Remark 2.9) the Whitehead theorem of $L_\infty$ spaces (Theorem 1.1) does not immediately follow from the above result. The proof that quasi-isomorphisms are $L_{(1)}$ homotopies in the curved situation is considerably harder. In the remaining part of the section, we shall first prove Theorem 1.3 on the existence of minimal charts. Then we make use of the minimal charts to prove the desired Whitehead theorem.

**5.2. Minimal charts.** Let $\mathcal{M} = (M, g)$ be a $L_\infty$ space and let $p \in M$ be a point in the zero-set of $\mu_0$. As in the Introduction we define the tangent complex of $(M, g)$ at $p$ to be

$$T_p \mathcal{M} := T_p M \xrightarrow{\nabla^{\mu_0}} g_2 \xrightarrow{\mu_1} g_3 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_n} g_N |_p.$$  \hspace{1cm} (17)

The fact that this is indeed a complex follows from the $L_\infty$ algebra equation together with the condition $\mu_0 |_p = 0$. Also note that the first map is independent of the connection.

Let $f = (f, f^2) : (M, g) \to (N, h)$ be a $L_\infty$ morphism and $p \in \mu_0^{-1}(0)$. It easily follows from the definition of morphism that $df$ and $f^2$ induce a chain map $T_p f : T_p \mathcal{M} \to T_{f(p)} \mathcal{N}$.

**Definition 5.7.** Let $(M, g)$ be a $L_\infty$ space and let $p \in \mu_0^{-1}(0)$. We say $(M, g)$ is minimal at $p$ if all the maps in the complex $T_p \mathcal{M}$ are zero.

A morphism $f : (M, g) \to (N, h)$ is called a quasi-isomorphism at $p$ if the chain map $T_p f$ induces an isomorphism in cohomology.

We have the following easy lemma.

**Lemma 5.8.** A $L_\infty$ morphism which is a $L_{(1)}$ homotopy equivalence is a quasi-isomorphism at any point.

For any open set $W \subseteq M$ we can restrict the $L_\infty$ structure to $W$ and so obtain a new $L_\infty$ space $(W, g|_W)$. We define a chart at $p$ to be a $L_\infty$ space $(N, h)$, with $n_p \in \mu_0^{-1}(0)$, together with a $L_\infty$ homotopy equivalence $i = (i, i^2) : (N, h) \to (W, g|_W)$ for some neighborhood $W$ of $p$ in $M$, such that $i(n_p) = p$. We say the chart is minimal if $(N, h)$ is minimal at $n_p$.

The main step in the proof of the inverse function theorem for $L_\infty$ spaces is the construction of minimal charts. We will do it in two steps.


Proposition 5.9. Let \((M, g)\) be a \(L_\infty\) space and \(q \in \mu_0^{-1}(0)\). There is a chart at \(q\), \((N, h)\) with the property \(\nabla \mu_0|_q = 0\).

Proof. In a neighborhood \(U\) of \(q\) with coordinates \((x_1, \ldots, x_n)\), trivialize the bundle \(g_2\) and write \(\mu_0 = s = (s_1, \ldots, s_m) : U \rightarrow \mathbb{R}^m\). If \(\nabla \mu_0|_q \neq 0\), there is \(i,j\) such that \(\frac{\partial s_i}{\partial x_j}(q) \neq 0\). Hence \(N = s_i^{-1}(0) \cap W\), for some small open set \(W \subset U\), is a smooth submanifold. It follows from the inverse function theorem (for smooth manifolds) that we can find coordinates on \(W\), \((x_1, \ldots, x_{n-1}, y)\) such that \(N = \{(x_1, \ldots, x_{n-1}, 0)\}\). Moreover, we can decompose the bundle \(g_2|_N = E_2 \oplus C_2\) such that \(s(x, y) = (\nu(x, y), y)\). We define \(E_k = g_k|_N\) for \(k \geq 3\). We define the map \(\iota : N \rightarrow M\) as \(\iota(x) = (x, 0)\). Additionally we denote by \(i\) the inclusion \(E \rightarrow g\) and by \(p\) the projection \(g \rightarrow E\). We claim the operations \(\lambda_0 := \nu|_N\) and \(\lambda_k = \mu_k|_E\), \(k \geq 1\) define a \(L_\infty\) space. Indeed this is a degenerate case of Theorem 3.1 where we take \(H = 0\). Please note that even though Equation (21) does not hold on \(g_2\), the theorem still holds since it is enough to have Equation (21) hold on \(g_{\geq 3}\), since this is the only situation where it is applied.

Therefore we have an \(L_\infty\) space

\[
(N, \mathfrak{E} := \oplus_{k \geq 2} E_k, \lambda_k).
\]

Moreover there is a \(L_\infty\) homomorphism \(\iota^* : \mathfrak{E} \rightarrow \iota^* g\), with \(\iota^*_i = i|_N\). We now construct a \(L(1)\) homotopy inverse to \((\iota, \iota^*)\). For this purpose, we define the maps

\[
\Pi : M \times [0, 1] \rightarrow M, \Pi(x, y, t) = (x, ty)
\]

and \(\pi_1^{t,\sharp} : g \rightarrow (\Pi^t)^* g\) by the formula

\[
\left[ \text{id} - \int_0^1 \frac{\partial}{\partial y}(x, sy) ds \right] : g_2 \rightarrow (\Pi^t)^* g_2,
\]

and \(\pi_1^{t,\sharp} = \text{id}\) on \(g_{\geq 3}\). We claim the pair \((P, P_1^{t,\sharp})\), where \(P(x, y) = x\) and \(P_1^{t,\sharp} = p\pi_1^{t,\sharp}\), is a \(L(1)\) morphism from \((M, g)\) to \((N, \mathfrak{E})\), and moreover it is a \(L(1)\) homotopy inverse to \((\iota, \iota^*)\). We first show that \(\pi_1^{t,\sharp}\) is a \(L(1)\) homomorphism. An easy computation gives

\[
\pi_1^{t,\sharp}(\mu_0) = (\nu(x, ty), ty) = (\Pi^t)^*(\mu_0).
\]

In the decomposition \(g_2|_N = E_2 \oplus C_2\), we write \(\mu_1 = (\varphi, \alpha)\) and compute

\[
(\pi_1^{t,\sharp} \mu_1 - \mu_1^{t,\sharp} \pi_1^{t,\sharp})|_{g_2} = (\varphi - \varphi^t, \alpha - t\alpha^t + \varphi^t \cdot \int_t^1 \frac{\partial \varphi}{\partial y}(sy) ds).
\]

We claim this is \(\delta\)-exact. We define

\[
P_2 = \int_t^1 \mu_2(K \otimes \text{id}) - \mu_2(\text{id} \otimes K) ds,
\]

where \(K : g_2 \rightarrow T_M\) is the map defined as \(K(e, c) = c \frac{\partial}{\partial y}\), and compute

\[
\delta(P_2)(e, c) = \int_t^1 \mu_2(y \frac{\partial}{\partial y}, (e, c)) - \mu_2(\mu_0, c \frac{\partial}{\partial y}) ds
\]

\[
= \int_t^1 \nabla \frac{\partial}{\partial y} \mu_1((e, c)) - c \frac{\partial \mu_1}{\partial y}(x, sy)(\mu_0) ds
\]

\[
= (\varphi - \varphi^t, \alpha - \alpha^t - \int_t^1 \frac{\partial \varphi}{\partial y}(sy) \cdot \nu(x, y) ds - \int_t^1 \frac{\partial \alpha}{\partial y}(sy)y ds)(e, c).
\]
Therefore $\delta(P_2) = (\pi_1^t \mu_1 - \mu_1^t \pi_1^t)|_{\g_2}$ is equivalent to the following identity

$$\int_t^1 \varphi(x, ty) \frac{\partial \nu}{\partial y}(sy) \, ds + \int_t^1 \frac{\partial \nu}{\partial y}(sy) \cdot \nu(x, y) \, ds = t \alpha^t - \alpha,$$

which in turns follows from the fact that the left-hand side equals

$$\int_t^1 \frac{\partial (\varphi \cdot \nu)}{\partial y}(x, sy) \, ds.$$

The $L_{\infty}$ relation $\mu_1(\mu_0) = 0$ implies $\varphi \cdot \nu = -\alpha \cdot y$, hence

$$\int_t^1 \frac{\partial (\varphi \cdot \nu)}{\partial y}(x, sy) \, ds = - \int_t^1 \frac{\partial \alpha}{\partial y}(x, sy) sy + \alpha(x, sy) \, ds = t \alpha(x, ty) - \alpha(x, y).$$

Here the last equality is given by integration by parts. Similarly one shows that $(\pi_1^t \mu_1 - \mu_1^t \pi_1^t)|_{\g_2 \infty} = \delta(\int_t^1 \mu_2^t(K \odot \text{id}))$. Hence we conclude that $\pi_1^t$ is a $L_{(1)}$ homomorphism.

We observe that $P^t \circ \iota^t = \text{id}$, $\pi_1^t = \iota^t \circ P^t$ and $\pi_1^0 = \text{id}$. Therefore, in order to conclude that $(P, P_2)$ is a $L_{(1)}$ homotopy inverse to $(\iota, \iota^t)$ it is enough, by (16), to define

$$h_1^t = \begin{cases} K, & g_2 \\ 0, & g_{2 \infty} \end{cases}$$

and check the following identities

$$h_1^t(\mu_0) = \frac{\partial \Pi}{\partial t}, \quad \frac{\partial \pi_1^t}{\partial t} = \mu_1^t h_1^t + h_1^t \mu.$$

We have thus proved that the $L_{\infty}$ morphism $(\iota, \iota^t)$ is a $L_{(1)}$ homotopy equivalence. It now follows from Theorem 5.5 that $(\iota, \iota^t)$ is a $L_{\infty}$ homotopy equivalence and therefore $(N, \xi)$ is a chart at $q$. By construction, the rank of $\nabla \mu_0|_q$ in $n$ is strictly smaller than in the original space $M$, hence by applying the previous construction finitely many times we can find a chart at $q$ such that $\nabla \mu_0|_q$ has rank zero as claimed in the statement.

**Theorem 5.10.** Let $(M, g)$ be a $L_{\infty}$ space and $q \in \mu_0^{-1}(0)$. There is a minimal chart at $q$.

**Proof.** Proposition 5.9 implies that we can assume $\nabla \mu_0|_q = 0$. We pick a sub-bundle $E_3 \subseteq g_3$ such that $E_3|_q \oplus \text{im} (\mu_1|_q) = g_3|_q$. Then $\tilde{\mu}_1 : g_2 \to g_3/E_3$ is surjective at $q$, since $\text{im} \mu_1|_q = \text{im} \tilde{\mu}_1|_q$. Hence $\tilde{\mu}_1$ is surjective on some neighborhood of $q$, which we denote by $U$. This implies that $A_2 := \ker \tilde{\mu}_1|_U$ is a sub-bundle of $g_2$. We pick a complement $A_2 \oplus B_2 = g_2$ and define $C_3 = \mu_1(B_2) \subseteq g_3|_U$. By construction, $\mu_1|_{B_2}$ is injective and therefore $C_3$ is a bundle. Moreover $g_{3|_U} = E_3 \oplus C_3$.

Next we pick $E_4 \subseteq g_4$ such that $E_4|_q \oplus \text{im} (\mu_1|_q) = g_4|_q$ and define $A_3 := \ker \tilde{\mu}_1 : E_3 \to g_4/E_4$. We pick a complement $A_3 \oplus B_3 = E_3$ and define $C_4 := \mu_1(B_3)$. We repeat this argument, for all $k$ and obtain a decomposition $g_k = A_k \oplus B_k \oplus C_k$ in a neighborhood of $q$, here $C_2 = 0$. In this decomposition, we the map $\mu_1$ takes the form

$$\mu_1 = \begin{bmatrix} \varphi & 0 & \alpha \\ \psi & 0 & \beta \\ 0 & \epsilon & \gamma \end{bmatrix}$$
Note that $\epsilon$ is an isomorphism, so we can define the degree $-1$ map

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon^{-1} \\ 0 & 0 & 0 \end{bmatrix} : \mathfrak{g} \to \mathfrak{g}.$$ 

On $\mathfrak{g}_2$, $H$ is defined to be zero. We denote by $i$ the inclusion $A \to \mathfrak{g}$ and by $p$ the projection onto $A$. We have the following identities on $\mathfrak{g}$, which are easy to check,

(20) \hfill $H \circ i = 0, \ p \circ H = 0$

(21) \hfill $H \mu_1 + \mu_1 H = i \circ p - \text{id}_{\mathfrak{g}} - H \mu_1^2 H$.

The first $L_\infty$ equation $\mu_1(\mu_0) = 0$ implies that $\mu_0 = (\nu, 0)$, since $\epsilon$ is an isomorphism. Hence $i(\nu)|_N = \nu^* \mu_0$ and we have all the data and conditions in Theorem 3.1. Therefore Theorem 3.1 constructs an $L_\infty$ space

$$(M, \mathfrak{A} := \oplus_{k \geq 2} A_k, \lambda_k),$$

with $\lambda_0 = \nu$ and $\lambda_1 = \rho \mu_1 i$. Moreover there is a $L_\infty$ morphism $(\iota, \iota^\sharp) : (U, \mathfrak{A}) \to (U, \mathfrak{g}^\nu|_U)$, with $\iota = \text{id}$ and $\iota^\sharp = i$. Also observe that, by definition of $A_k$, $\mu_1|_q(A_k) = 0$ and thus $(U, \mathfrak{A})$ is minimal at $q$.

The last step is to construct a $L_{(1)}$ homotopy inverse to $(\iota, \iota^\sharp)$ and then appeal to Theorem 5.5 to conclude that $(\iota, \iota^\sharp)$ is a $L_\infty$ homotopy equivalence. For this purpose, we define the maps $\pi_{\iota^\sharp}^{\mu_1} : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\begin{bmatrix} \text{id} & 0 & 0 \\ 0 & t \cdot \text{id} & 0 \\ 0 & 0 & t \cdot \text{id} \end{bmatrix} : \mathfrak{g} \to \mathfrak{g}.$$ 

We first show that $\pi_{\iota^\sharp}^{\mu_1}$ is a $L_{(1)}$ homomorphism. Since $\mu_0 = (\nu, 0)$ we have $\pi_{\iota}^{\mu_1}(\mu_0) = \mu_0$. Next we define the map

$$P_2 = (1 - t) \left( H \mu_2(p \otimes p) + p \mu_2(H \otimes \text{id}) - p \mu_2(\text{id} \otimes H) \right).$$

A simple computation using (21) gives

$$\mu_1 \pi_{\iota^\sharp}^{\mu_1} - \pi_{\iota^\sharp}^{\mu_1} \mu_1 = \delta(P_2).$$

In particular, we have shown that $P_{\iota^\sharp}^{\mu_1} := \Pi \pi_{\iota^\sharp}^{\mu_1}$ (where $\Pi : \mathfrak{g} \to A$ is the projection) is a $L_{(1)}$ morphism from $\mathfrak{g}$ to $\mathfrak{A}$.

It is obvious that $P_1^{\iota^\sharp} \circ \iota^\sharp = \text{id}_{\mathfrak{A}}$. Finally, we need to show that $\iota^\sharp \circ P_1^{\iota^\sharp}$ is $L_{(1)}$ homotopic to the identity. First note $\pi_{\iota}^{\iota^\sharp} = \text{id}_\mathfrak{g}$ and $\pi_{\iota^\sharp}^{\iota} = \iota^\sharp \circ P_1^{\iota^\sharp}$. We define $h_1^t = -H$ and easily check

$$h_1^t(\mu_0) = 0, \ \frac{\partial \pi_{\iota}^{\iota^\sharp}}{\partial t} = h_1^t \mu_1 + \mu_1 h_1^t - \delta(Q_2),$$

where $Q_2 = H \mu_2(H \otimes \text{id}) + H \mu_2(\text{id} \otimes H)$. This completes the proof that $(\iota, \iota^\sharp)$ is a $L_{(1)}$ and therefore a $L_\infty$ homotopy equivalence. \hfill $\square$

**Theorem 5.11.** Let $(M, \mathfrak{g})$ be a $L_\infty$ space and $\mathfrak{f} = (f, f^\sharp) : (M, \mathfrak{g}) \to (N, \mathfrak{h})$ be a $L_\infty$ morphism. Assume that $\mathfrak{f}$ is a quasi-isomorphism at $q \in \mu_0^{-1}(0)$. Then there are neighborhoods $q \in U \subseteq M$ and $f(q) \in V \subseteq N$ such that $f(U) \subseteq V$ and

$$f|_U : (U, \mathfrak{g}|_U) \to (V, \mathfrak{h}|_V)$$

is a $L_\infty$ homotopy equivalence. \hfill 25
Proof. Theorem 5.10 provides minimal charts at \( p \) and \( f(p) \) and hence we have the following diagram

\[
\begin{array}{ccc}
(L, a) & \xrightarrow{i} & (U, g|_U) \\
\tilde{J}^{-1} & \downarrow \tilde{b} & \downarrow f_U \\
(L, \tilde{a}) & \xrightarrow{\tilde{i}} & (V, h|_V)
\end{array}
\]

Here the pairs \( i, \tilde{p} \) and \( \tilde{i}, \tilde{\tilde{p}} \) are homotopy inverses and \( \tilde{J} := \tilde{p} \circ f_U \circ i \). It follows from Lemma 5.8 that \( \tilde{J} \) is a quasi-isomorphism at \( n_q := p(q) \), but since \((L, a)\) and \((\tilde{L}, \tilde{a})\) are minimal at \( n_q \) and \( J(n_q) \), we conclude that \( J \) is a local diffeomorphism and \( J^1_{|n_q} \) is an isomorphism. After restricting to small neighborhoods \( U' \) and \( V' \) of \( n_q \) and \( J(n_q) \) we have that \( J \) is a diffeomorphism and \( J^1 \) is an isomorphism of bundles. Therefore we can solve Equation (11) inductively on \( n \) to find a strict \( L_\infty \) inverse to \( J^1 \), which we denote by \( J^{-1} \).

We make the neighborhoods \( U \) and \( V \) smaller, if necessary, to ensure the restrictions of \( i, \tilde{p} \) (and \( \tilde{i}, \tilde{\tilde{p}} \)) are homotopy inverses on \((U', a|_{U'})\) (and \((V', \tilde{a}|_{V'})\)). We define \( \mathcal{R} := i \circ \tilde{J}^{-1} \circ \tilde{p} \). By construction \( \tilde{J}^{-1} \circ \tilde{p} \circ f_U \cong p \), therefore

\[
\mathcal{R} \circ f_U \cong i \circ \tilde{J}^{-1} \circ \tilde{p} \circ f_U \cong i \circ p \cong \text{id}.
\]

Similarly, \( f_U \circ i \circ \tilde{J}^{-1} \cong \tilde{i} \), hence \( f_U \circ \mathcal{R} \cong \tilde{i} \circ \tilde{p} \cong \text{id} \). Thus \( \mathcal{R} \) is a homotopy inverse to \( f_U \).

\[\square\]

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