QUADRATIC AND SYMMETRIC BILINEAR FORMS ON MODULES WITH UNIQUE BASE OVER A SEMIRING

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Abstract. We study quadratic forms on free modules with unique base, the situation that arises in tropical algebra, and prove the analog of Witt’s Cancellation Theorem. Also, the tensor product of an indecomposable bilinear module \((U, \gamma)\) with an indecomposable quadratic module \((V, q)\) is indecomposable, with the exception of one case, where two indecomposable components arise.

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Introduction

Recall that a semiring \(R\) is a set \(R\) equipped with addition and multiplication, such that both \((R, +)\) and \((R, \cdot)\) are abelian monoids\(^1\) with elements \(0 = 0_R\) and \(1 = 1_R\) respectively, and multiplication distributes over addition in the usual way. We always assume that \(R\) is a commutative semiring with 1. In other words, \(R\) satisfies all the properties of a commutative ring except the existence of negation under addition. We call a semiring \(R\) a semifield, if every nonzero element of \(R\) is invertible; hence \(R \setminus \{0\}\) is an abelian group.

As in the classical theory, one often wants to consider bilinear forms defined on (semi)modules over a semiring \(R\), often a supertropical semifield, in order to obtain more sophisticated trigonometric information. A module \(V\) over \(R\) is an abelian monoid \((V, +)\) equipped with a scalar multiplication \(R \times V \to V, (a, v) \mapsto av\), such that exactly the same axioms hold as customary for modules if \(R\) is a ring: \(a_1(bv) = (a_1b)v, a_1(v + w) = a_1v + a_1w\),

\(^1\)A monoid means a semigroup that has a neutral element. Here a semiring \(R\) is tacitly assumed to be commutative.

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For any module $V$ that every $x \in V$ is uniquely determined by $V = \sum_{i \in I} x_i$ with scalars $x_i \in R$ and only finitely many $x_i$ nonzero, and we call $(\varepsilon_i \mid i \in I)$ a base of the $R$-module $V$. The obvious example is $V = R^n$, with the classical base. In fact, any free module with a base of $n$ elements is clearly isomorphic to $R^n$, under the map $\sum_{i=1}^n x_i \varepsilon_i \mapsto (x_1, \ldots, x_n)$. The results are decisive when $R$ is a so-called supertropical semiring, cf. Theorem 1.2 below.

As in the classical theory, we are led naturally to our main notion of this paper: For the reader’s convenience, we quote some terminology and results from [6, §1-§4].

**Definition 0.1.** For any module $V$ over a semiring $R$, a **quadratic form** on $V$ is a function $q : V \to R$ with

$$q(ax) = a^2 q(x)$$

for any $a \in R$, $x \in V$, together with a symmetric bilinear form $b : V \times V \to R$ (not necessarily uniquely determined by $q$) such that for any $x, y \in V$

$$q(x + y) = q(x) + q(y) + b(x, y).$$

Every such bilinear form $b$ will be called a **companion** of $q$, and the pair $(q, b)$ will be called a **quadratic pair** on $V$. We also call $V$ a **quadratic module**.

When $R$ is a ring, then $q$ has just one companion, namely, $b(x, y) := q(x + y) - q(x) - q(y)$, but if $R$ is a semiring that cannot be embedded into a ring, this usually is not the case, and it is a major concern of quadratic form theory over semirings to determine all companions of a given quadratic form $q : V \to R$. Much of the paper [6] is devoted to this problem in the case that $V$ is a free $R$-module over a supertropical semiring, but the first four sections of [6] deal with quadratic forms and pairs over an arbitrary semiring, and we draw from these results in the present paper.

A quadratic form $q : V \to R$ is called **quasilinear**, if $q$ has the companion $b = 0$, i.e., $q(x + y) = q(x) + q(y)$ for all $x, y \in V$. Assume that $V$ is free with base $(\varepsilon_i : i \in I)$. Then quasilinearity of $V$ implies that, for any vector $x = \sum_{i \in I} x_i \varepsilon_i$ in $V$,

$$q(x) = \sum_{i=1}^n x_i^2 q(\varepsilon_i),$$

i.e., $q$ has **diagonal form** with respect to the base $(\varepsilon_i : i \in I)$.

Under the assumption that for all $a, b \in R$

$$(a + b)^2 = a^2 + b^2,$$  

we read off from (0.2) that all diagonal forms are quasilinear; so diagonality means the same as quasilinearity. In the present paper we seldom require that $R$ has property (0.4), but “partial quasilinearity,” defined as follows, plays a major role when we consider orthogonal decompositions of quadratic modules.
Definition 0.2. Given subsets $S$ and $T$ of $V$, we say that $q$ is quasilinear on $S \times T$ if
\[ q(x + y) = q(x) + q(y). \]
for all $x \in S$, $y \in T$.

The following fact will be of help below, as special case of [6, Lemma 1.8]. (We write $S + S'$ for $\{s + s' : s \in S, s' \in S'\}$.)

Lemma 0.3. Let $S, S', T$ be subsets of $V$. If $q$ is quasilinear on $S \times T$, $S' \times T$ and $S \times S'$, then $q$ is quasilinear on $(S + S') \times T$.

On the other hand, a quadratic form $q : V \to R$ is called rigid if $q$ has only one companion. For $V$ free with base $(\varepsilon_i : i \in I)$, $q$ is rigid whenever $q(\varepsilon_i) = 0$ for all $i \in I$ [6, Proposition 3.4]. Under the assumption that $R$ has property (0.4) and that, for any $a \in R$,
\[ a + a = 0 \implies a = 0 \quad (0.5) \]
the converse holds too, so by [6, Theorem 3.5] the rigid forms are precisely those with $q(\varepsilon_i) = 0$ for all $i \in I$. {We note in passing that both (0.4) and (0.5) are valid when $R$ is supertropical.}

We say that $V$ is an $R$-module with unique base, if $V$ is a free $R$-module and, given a base $\mathfrak{B} = \{\varepsilon_i : i \in I\}$ of $V$, any other base of $V$ is obtained from $\mathfrak{B}$ by multiplying the $\varepsilon_i$ by units of $R$. In the most important case that $I$ is finite, i.e., $I = \{1, \ldots, n\}$, we have:

Remark 0.4. Any change of base of the free module $R^n$ is attained by multiplication by an invertible $n \times n$ matrix, so having unique base is equivalent to every invertible matrix in $M_n(R)$ being a generalized permutation matrix.

The present paper is devoted to a study of the quadratic and symmetric bilinear forms on $R$-modules with unique base. More specifically we work on orthogonal decompositions of such forms, and on tensor products of two symmetric bilinear forms and of a symmetric bilinear form with a quadratic form.

So our first question is, “What conditions on the semiring $R$ guarantee that $R^n$ has unique base, or equivalently, that every invertible matrix is generalized permutation?” The matrix question was answered by [10, 1]. In their terminology, an “antiring” is a semiring $R$ such that $R \setminus \{0\}$ is closed under addition, and they classify the invertible matrices over antirings. These are just the generalized permutation matrices when $R \setminus \{0\}$ also is closed under multiplication, which they call “entire” (the case in tropical mathematics), and more generally by [1, Theorem 1] (as interpreted in Theorem 1.7) when $R$ is indecomposable, i.e., not isomorphic to a direct product $R_1 \times R_2$ of semirings.

In our proofs of all of our results, we never use the matrix interpretation of the unique base property. {In fact matrices show up only once, in Corollary 4.6.}

Other than the trivial fact that every free $R$-module of rank 1 has unique base, all examples known to us of modules with unique base emanate from Theorem 1.7. But we feel that many arguments and related problems left open in the paper are clearer if, when possible, we assume only that the $R$-modules considered have unique base. (Only in §6 do we deviate from this strategy.)

In §2 we develop the notion of (disjoint) orthogonality of two given disjoint submodules $W_1$ and $W_2$ of a quadratic $R$-module $(V, q)$ (endowed with a fixed quadratic form $q$), which means that $q$ is partially quasilinear on $W_1 \times W_2$. (Note that there is no direct reference to an underlying symmetric bilinear form.) When $V$ has unique base, we look for
orthogonal decompositions $V = W_1 \perp W_2$, more generally $V = \bigoplus_{i \in I} W_i$, where the $W_i$ are **basic submodules** of $V$, i.e., are generated by subsets of a base $\mathfrak{B}$ of $V$.

Theorem 2.6 shows that there is a unique disjoint orthogonal decomposition of $V$ into indecomposable basic submodules.

In §3 we develop the analogous notion of disjoint orthogonality in a **bilinear $R$-module** $(V, b)$ with respect to a fixed symmetric bilinear form $b$ on $V$, and by passing from $q$ to a suitable companion, we obtain the same indecomposable basic submodules (Theorem 3.9). Here it helps to modify $b$ to a related symmetric bilinear form $b_{\text{alt}}$ (Definition 3.5) which facilitates a description of $(V, b)$ as spanned by the connected components of a graph associated with the base.

In §4, these decomposition theories yield an analogue of Witt’s cancellation theorem over fields of characteristic $\neq 2$ [11], given as Theorem 4.9: If $W_1, W_1', W_2, W_2'$ are finitely generated quadratic or bilinear modules with unique base such that $W_1 \cong W_1'$ and $W_1 \perp W_2 \cong W_2 \perp W_2'$, then $W_2 \cong W_2'$ (where $\cong$ means “isometric”). Theorem 4.9 vindicates our somewhat exotic notion of disjoint orthogonality. It actually is given in more general terms, where $W_2$ need not be finitely generated.

The last two sections of the paper are devoted to tensor products of arbitrary $R$-modules over a semiring $R$. While the theory of tensor products of $R$-modules over general semirings can be carried out in analogy to the usual classical construction over rings, it requires the use of congruences, resulting in some technical issues dealt with in [4], for example. But for free modules the basics can be done as easily as over rings, especially when the bases are assumed to be unique (since then one does not need to worry about well-definedness).

In §5 we construct the **tensor product of two free bilinear $R$-modules** over any semiring $R$, analogous to the case where $R$ is a ring, cf. [2, §2], [8, I, §5]. We then take the **tensor product of a free bilinear $R$-module** $U = (U, \gamma)$ with a free quadratic $R$-module $V = (V, q)$. A new phenomenon occurs here, in contrast to the theory over rings. It is necessary first to choose a so called **balanced companion** $b$ of $q$, which always exists, cf. [6, §1], but which is usually not unique. We then define the tensor product $U \otimes_b V$, depending on $b$, by choosing a so called **expansion** $B : V \times V \to R$ of the quadratic pair $(q, b)$ which is a (often non-symmetric) bilinear form $B$ with

$$B(x, x) = q(x), \quad B(x, y) + B(y, x) = b(x, y)$$

for all $x, y \in V$, cf. [6, §1] and then proceeding essentially as in the case of rings, e.g. [8, Definition 1.51], [2, p. 51]^2. The resulting quadratic form $\gamma \otimes_b q$ does not depend on the choice of $B$ but often depends on the choice of $b$. This is apparent already in the case $\gamma = (0 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where the matrix $b$ is stored in the quadratic polynomial $\gamma \otimes_b q$, cf. Example 5.8 below.

In §6 we determine the indecomposable components of tensor products of modules with unique base, first of two indecomposable bilinear (free) $R$-modules, and then of an indecomposable bilinear $R$-module with an indecomposable quadratic $R$-module. For simplicity we assume here that $R \setminus \{0\}$ is an entire antiring, i.e., closed under multiplication and addition, relying on Theorem 1.3 that, in this case, all free $R$-modules have unique base.

Our main result of this section, Theorem 6.16, states that, discarding trivial situations and excluding some pathological semirings, the tensor product of an indecomposable bilinear

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^2For $R$ a ring the “$b$” in the tensor product does not need to be specified since $q$ has only one companion.
module \((U, \gamma)\) with an indecomposable quadratic module \((V, q)\) is again indecomposable, with the exception of one case, where two indecomposable components arise.

The proof of this result, and of the preceding theorems 6.6 and 6.8 about tensor products of indecomposable bilinear modules as well, has a graph theoretic flavor. We work with “paths” and “cycles” in the bases of \(U, V\) and \(U \otimes_R V\), and indeed we could associate graphs to \((U, \gamma), (V, q),\) and \((U, \gamma) \otimes_R (V, q)\) in a way obvious from the arguments, where these paths and cycles get the usual graph-theoretic meaning. We have refrained from appealing to graph theory here, since at this stage no deeper theorems about graphs are needed.

1. \(R\)-modules with unique base and their basic submodules

**Definition 1.1.** An \(R\)-module with unique base is a free \(R\)-module \(V\) in which any two bases \(\mathcal{B}, \mathcal{B}'\) are projectively the same, i.e., we obtain the elements of \(\mathcal{B}'\) from those of \(\mathcal{B}\) by multiplying by units of \(R\).

Our interest in these modules originates from the following two key facts.

**Theorem 1.2.** (cf. [6, Proposition 0.9]) If \(R\) is a supertropical semiring, then every free \(R\)-module has unique base.

**Theorem 1.3.** (cf. [1, §2, Corollary 3], an alternative proof below.) If the set \(R \setminus \{0\}\) is closed under addition and multiplication (i.e., \(a + b = 0 \Rightarrow a = b = 0, a \cdot b = 0 \Rightarrow a = 0\) or \(b = 0\)), then every free \(R\)-module has unique base.

Assume now that \(V\) is a free \(R\)-module and \(\mathcal{B}\) is a fixed base of \(V\).

**Definition 1.4.** We call a submodule \(W\) of \(V\) basic, if \(W\) is spanned by \(\mathcal{B}_W := \mathcal{B} \cap W\), and thus \(W\) is free with base \(\mathcal{B}_W\). Note that then we have a unique direct decomposition \(V = W \oplus U\), where the submodule \(U\) is basic with base \(\mathcal{B} \setminus \mathcal{B}_W\). \(W\) and \(U\) again are \(R\)-modules with unique base. We call \(U\) the complement of \(W\) in \(V\), and write \(U = W^c\).

The theory of basic submodules of \(V\) is of utmost simplicity. All of the following is obvious.

**Scholium 1.5.**

(a) We have a bijection \(W \mapsto \mathcal{B}_W := \mathcal{B} \cap W\) from the set of basic submodules of \(V\) onto the set of subsets of \(\mathcal{B}\).

(b) If \(W_1\) and \(W_2\) are basic submodules of \(V\), then also \(W_1 \cap W_2\) and \(W_1 + W_2\) are basic submodules of \(V\), and

\[
\mathcal{B}_{W_1 \cap W_2} = \mathcal{B}_{W_1} \cap \mathcal{B}_{W_2}, \quad \mathcal{B}_{W_1 + W_2} = \mathcal{B}_{W_1} \cup \mathcal{B}_{W_2}.
\]

(c) If \(W\) is a basic submodule of \(V\), then as stated above,

\[
\mathcal{B}_{W^c} = \mathcal{B} \setminus \mathcal{B}_W.
\]

(d) Finally, if \(W_1 \subseteq W_2\) are basic submodules of \(V\), then \(W_1\) is basic in \(W_2\) and \(W_1^c \cap W_2\) is the complement of \(W_1\) in \(W_2\).

In view of Remark 0.4, Theorem 1.3 follows from Dolzan and Oblak [1, §2, Corollary 3] using matrix arguments within a wider context extending work of Tan [10, Proposition 3.2], which in turn relies on Golan’s book on semirings [3, Lemma 19.4]. We now reprove Theorem 1.3 by a simple matrix-free argument in preparation for a reproof of the more general Theorem 1.7.
Proof of Theorem 1.3. Let $V$ be a free $R$-module and $\mathcal{B}$ a base of $V$. If $x \in V \setminus \{0\}$ is given, we have a presentation

$$x = \sum_{i=1}^{r} \lambda_i x_i$$

with $x_i \in \mathcal{B}$ and $\lambda_i \in R \setminus \{0\}$. We call the set $\{x_1, \ldots, x_r\} \subset \mathcal{B}$ the support of $x$ with respect to $\mathcal{B}$ and denote this set by $\text{supp}_\mathcal{B}(x)$. Note that if $x, y \in V \setminus \{0\}$, then $x + y \neq 0$ and

$$\text{supp}_\mathcal{B}(x + y) = \text{supp}_\mathcal{B}(x) \cup \text{supp}_\mathcal{B}(y) \quad (1.1)$$

due to the assumption that $\lambda + \mu \neq 0$ for any $\lambda, \mu \in R \setminus \{0\}$. Also

$$\text{supp}_\mathcal{B}(\lambda x) = \text{supp}_\mathcal{B}(x) \quad (1.2)$$

for $x \in V \setminus \{0\}, \lambda \in R \setminus \{0\}$, due to the assumption that for $\lambda, \mu \in R \setminus \{0\}$ we have $\lambda \mu \neq 0$.

Now assume that $\mathcal{B}'$ is a second base of $V$. Given $x \in \mathcal{B}$, we have a presentation

$$x = \lambda_1 y_1 + \cdots + \lambda_r y_r$$

with $\lambda_i \in R \setminus \{0\}$ and distinct $y_i \in \mathcal{B}'$. It follows from (1.1) and (1.2) that

$$\{x\} = \text{supp}_\mathcal{B}(x) = \text{supp}_\mathcal{B}(y_1) \cup \cdots \cup \text{supp}_\mathcal{B}(y_r). \quad (1.3)$$

This forces

$$\{x\} = \text{supp}_\mathcal{B}(y_1) = \cdots = \text{supp}_\mathcal{B}(y_r). \quad (1.3)$$

From this, we infer that $r = 1$. Indeed, suppose that $r \geq 2$. Then $y_1 = \mu_1 x, y_2 = \mu_2 x$ with $\mu_1, \mu_2 \in R \setminus \{0\}$. But this implies $\mu_2 y_1 = \mu_1 y_2$, a contradiction since $y_1, y_2$ are different elements of a base of $V$.

Thus $\{x\} = \text{supp}_\mathcal{B}(y)$ for a unique $y \in \mathcal{B}'$, which means $y = \lambda x$ with $\lambda \in R \setminus \{0\}$. By symmetry we have a unique $z \in \mathcal{B}$ and $\mu \in R \setminus \{0\}$ with $x = \mu z$. Then $x = \lambda \mu z$, whence $x = z$ and $\lambda \mu = 1$. Thus $\lambda, \mu \in R^*$ and $x \in R^* y, y \in R^* x$. Of course, $y$ runs through all of $\mathcal{B}'$ if $x$ runs through $\mathcal{B}$, since both $\mathcal{B}$ and $\mathcal{B}'$ span the module $V$. \( \square \)

With further effort, we now obtain a theorem that encompasses both Theorems 1.2 and 1.3.

**Definition 1.6.** We say that the semiring $R$ is indecomposable if $R$ is not isomorphic to a direct product $R_1 \times R_2$ of non-zero semirings $R_1$ and $R_2$; in other words, there do not exist idempotents $\mu_1 \neq 0$ and $\mu_2 \neq 0$ in $R$ with $\mu_1 \mu_2 = 0$ and $\mu_1 + \mu_2 = 1$.

**Theorem 1.7** ([1, Theorem 1]). Assume that $R \setminus \{0\}$ is an indecomposable antiring. Then every free $R$-module has unique base.

**Proof.** Assume that $\mathcal{B}$ and $\mathcal{B}'$ are bases of $V$. Given $x \in V \setminus \{0\}$, we write again

$$x = \lambda_1 y_1 + \cdots + \lambda_r y_r \quad (1.4)$$

with different $y_i \in \mathcal{B}'$, $\lambda_i \in R \setminus \{0\}$. But now, instead of (1.3) we can only conclude that

$$\{x\} = \text{supp}_\mathcal{B}(\lambda_1 y_1) = \cdots = \text{supp}_\mathcal{B}(\lambda_i y_i). \quad (1.5)$$

Thus we have scalars $\mu_i \in R \setminus \{0\}$ such that

$$\lambda_i y_i = \mu_i x \quad \text{for} \quad 1 \leq i \leq r. \quad (1.6)$$

Suppose that $r \geq 2$. Then we have for all $i, j \in \{1, \ldots, r\}$ with $i \neq j$.

$$\mu_j \lambda_i y_i = \mu_j \mu_i x = \mu_i \mu_j x = \mu_i \lambda_j y_j.$$
Since the $y_i$ are elements of a base, this implies $\mu_i \lambda_j = \mu_j \lambda_i = 0$ for $i \neq j$ and then

$$\mu_i \mu_j = 0 \quad \text{for} \quad i \neq j. \quad (1.7)$$

On the other hand, we obtain from (1.4) and (1.6) that

$$x = \mu_1 x + \mu_2 x + \cdots + \mu_r x,$$

and then

$$1 = \mu_1 + \mu_2 + \cdots + \mu_r. \quad (1.8)$$

Multiplying (1.8) with $\mu_i$ and using (1.7), we obtain

$$\mu_i^2 = \mu_i. \quad (1.9)$$

Thus

$$R \cong R\mu_1 \times \cdots \times R\mu_r.$$  

This contradicts our assumption that $R$ is indecomposable.

We have proved that $r = 1$. Thus for every $x \in \mathfrak{B}$ there exist unique $y \in \mathfrak{B}'$ and $\lambda \in R$ with $x = \lambda y$. By the same argument as in the end of proof of Theorem 1.3, we conclude that $\mathfrak{B}$ is projectively unique.

Of course, if $R \setminus \{0\}$ is closed under multiplication, i.e., $R$ has no zero divisors, then $R$ is indecomposable. This also holds when $R$ is supertropical (cf. [7, §3], [6, Definition 0.3]), since then for any two elements $\mu_1, \mu_2$ of $R$ with $\mu_1 + \mu_2 = 1$ either $\mu_1 = 1$ or $\mu_2 = 1$. Thus Theorem 1.7 generalizes both Theorems 1.2 and 1.3.

The following example reveals that Theorem 1.7 is the best we can hope for to guarantee that every free $R$-module has unique base, as long as we stick to the condition that $R$ is an antiring, a natural assumption for the remainder of this paper.

**Example 1.8.** If $R_0$ is an antiring, then $R := R_0 \times R_0$ also is an antiring. Put $\mu_1 = (1, 0), \mu_2 = (0, 1)$. These are idempotents in $R$ with $\mu_1 \mu_2 = 0$ and $\mu_1 + \mu_2 = 1$. Now let $V$ be a free $R$-module with base $\mathfrak{B} = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, $n \geq 2$, choose a permutation $\pi \in S_n$, $\pi \neq 1$, and define

$$\varepsilon'_i := \mu_1 \varepsilon_i + \mu_2 \varepsilon_{\pi(i)} \quad (1 \leq i \leq n).$$

We claim that $\mathfrak{B}' := \{\varepsilon'_1, \ldots, \varepsilon'_n\}$ is another base of $V$.

Indeed, $V$ is a free $R_0$-module with base $(\mu_1 \varepsilon_i | 1 \leq i \leq 2, 1 \leq j \leq n)$. We have

$$\mu_1 \varepsilon'_i = \mu_1 \varepsilon_i, \quad \mu_2 \varepsilon'_i = \mu_2 \varepsilon_{\pi(i)},$$

and thus $(\mu_1 \varepsilon'_i | 1 \leq i \leq 2, 1 \leq j \leq n)$ is a permutation of this base over $R_0$, i.e., regarded as a set, the same base. Thus certainly $\mathfrak{B}'$ spans $V$ as $R$-module.

Given $x \in V$, let $x = \sum a_i \varepsilon'_i$ with $a_i \in R$. We have

$$a_i = a_{i1} \mu_1 + a_{i2} \mu_2 \quad \text{with} \quad a_{i1} \in R_0, \ a_{i2} \in R_0,$$

whence

$$x = \sum_{i=1}^{n} a_{i1}(\mu_1 \varepsilon_i) + \sum_{i=1}^{n} a_{i2}(\mu_2 \varepsilon_{\pi(i)}).$$

This shows that the coefficients $a_{i1}, a_{i2} \in R_0$ are uniquely determined by $x$, whence the coefficients $a_i \in R$ are also uniquely determined by $x$. Our claim is proved.

Since $\text{supp}_{\mathfrak{B}}(\varepsilon'_i)$ has two elements if $\pi(i) \neq i$, $\mathfrak{B}'$ differs projectively from $\mathfrak{B}$. The base of the $R$-module $V$ is not unique.
2. Orthogonal decompositions of quadratic modules with unique base

Assume that $V$ is an $R$-module equipped with a fixed quadratic form $q : V \to R$. We then call $V = (V, q)$ a quadratic $R$-module.

Definition 2.1.

(a) Given two submodules $W_1, W_2$ of the $R$-module $V$, we say that $W_1$ is **disjointly orthogonal** to $W_2$, if $W_1 \cap W_2 = \{0\}$ and $q(x + y) = q(x) + q(y)$ for all $x \in W_1$, $y \in W_2$, i.e., $q$ is quasilinear on $W_1 \times W_2$.

(b) We write $V = W_1 \perp W_2$ if $V = W_1 \oplus W_2$ (as $R$-module) with $W_1$ disjointly orthogonal to $W_2$. We then call $W_1$ an **orthogonal summand** of $W$, and $W_2$ an **orthogonal complement** of $W_1$ in $V$.

Caution. If $V = W_1 \perp W_2$, we may choose a companion $b$ of $q$ such that $b(W_1, W_2) = 0$, but note that it could well happen that the set of all $x \in V$ with $b(x, W_1) = 0$ is bigger than $W_2$, even if $R$ is a semifield and $q|W_1$ is anisotropic (e.g., if $q$ itself is quasilinear). Our notion of orthogonality does not refer to any bilinear form.

We now also define infinite orthogonal sums. This seems to be natural, even if we are originally interested only in finite orthogonal sums. Indeed, even if $R$ is a semifield, a free $R$-module with finite base often has many submodules which are not finitely generated.

Definition 2.2. Let $(V_i \mid i \in I)$ be a family of submodules of the quadratic module $V$. We say that $V$ is the **orthogonal sum of the family** $(V_i)$, and then write

$$V = \bigoplus_{i \in I} V_i,$$

if for any two different indices $i, j$ the submodule $V_i$ is disjointly orthogonal to $V_j$, and moreover $V = \bigoplus_{i \in I} V_i$.

N.B. Of course, then for any subset $J \subset I$, the module $V_J = \sum_{i \in J} V_i$ is the orthogonal sum of the subfamily $(V_i \mid i \in J)$; in short,

$$V_J = \bigoplus_{i \in J} V_i.$$

We state a fact which, perhaps contrary to first glance, is not completely trivial.

Proposition 2.3. Assume that we are given an orthogonal decomposition $V = \bigoplus_{i \in I} V_i$. Let $J$ and $K$ be two disjoint subsets of $I$. Then the submodule $V_J = \bigoplus_{i \in J} V_i$ of $V$ is disjointly orthogonal to $V_K = \bigoplus_{i \in K} V_i$, and thus

$$V_{J \cup K} = V_J \perp V_K.$$

Proof. It follows from Lemma 0.3 above that for any three different indices $i, j, k$ the form $q$ is quasilinear on $V_i \times (V_j + V_k)$, and thus $V_i$ is orthogonal to $V_j \perp V_k$. By iteration, we see that the claim holds if $J$ and $K$ are finite. In the general case, let $x \in V_J$ and $y \in V_K$. There exist finite subsets $J', K'$ of $J$ and $K$ with $x \in V_{J'}$, $y \in V_{K'}$, and thus $q(x + y) = q(x) + q(y)$. This proves that $V_J$ is orthogonal to $V_K$. \qed

\footnote{Later we say “orthogonal” for short, instead of “disjointly orthogonal”, when it is clear a priori that $W_1 \cap W_2 = \{0\}$.}
In the rest of this section, we assume that \( V \) has unique base. Then a basic orthogonal summand \( W \) of \( V \) has only one basic orthogonal complement, namely, \( W^\perp \), equipped with the form \( q|W^\perp \).

**Definition 2.4.** If the quadratic module \( V \) has a basic orthogonal summand \( W \neq V \), we call \( V \) decomposable. Otherwise we call \( V \) indecomposable. More generally, we call a basic submodule \( X \) of \( V \) decomposable if \( X \) is decomposable with respect to \( q|X \), and otherwise we call \( X \) indecomposable.

Our next goal is to decompose the given quadratic module \( V \) orthogonally into indecomposable basic submodules. Therefore, we choose a base \( \mathcal{B} \) of \( V \) (unique up to multiplication by scalar units). We then choose a companion \( b \) of \( q \) such that \( b(\varepsilon, \eta) = 0 \) for any two different \( \varepsilon, \eta \in \mathcal{B} \) such that \( q \) is quasilinear on \( R\varepsilon \times R\eta \), cf. [6, Theorem 6.3]. We call such a companion a quasiminimal companion of \( q \).

**Comment.** In important cases, e.g., if \( R \) is supertropical or more generally “upper bound” (cf. [6, Definition 5.8]), the set of companions of \( q \) can be partially ordered in a natural way. The prefix “quasi” here is a reminder that we do not mean minimality with respect to such an ordering.

**Lemma 2.5.** Let \( W \) and \( W' \) be basic submodules of \( V \) with \( W \cap W' = \{0\} \). If \( b \) is any quasiminimal companion of \( q \), then \( W \) is (disjointly) orthogonal to \( W' \) iff \( b(W, W') = 0 \).

**Proof.** If \( b(W, W') = 0 \), then \( q(x+y) = q(x) + q(y) \) for any \( x \in W \) and \( y \in W' \), which means by definition that \( W \) is orthogonal to \( W' \). (This holds for any companion \( b \) of \( q \).)

Conversely, if \( W \) is orthogonal to \( W' \), then for base vectors \( \varepsilon \in \mathcal{B}_W, \eta \in \mathcal{B}_{W'} \) the form \( q \) is quasilinear on \( R\varepsilon \times R\eta \) and thus \( b(\varepsilon, \eta) = 0 \). This implies that \( b(W, W') = 0 \). \( \Box \)

We now introduce the following equivalence relation on the set \( \mathcal{B} \). We choose a quasiminimal companion \( b \) of \( q \). Given \( \varepsilon, \eta \in \mathcal{B} \), we put \( \varepsilon \sim \eta \), iff either \( \varepsilon = \eta \), or there exists a sequence \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r \) in \( \mathcal{B} \), \( r \geq 1 \), such that \( \varepsilon = \varepsilon_0, \eta = \varepsilon_r \), and \( \varepsilon_i \neq \varepsilon_{i+1} \), \( b(\varepsilon_i, \varepsilon_{i+1}) \neq 0 \) for \( i = 0, \ldots, r-1 \).

**Theorem 2.6.** Let \( \{ \mathcal{B}_k \mid k \in K \} \) denote the set of equivalence classes in \( \mathcal{B} \) and, for every \( k \in K \), let \( W_k \) denote the submodule of \( V \) having base \( \mathcal{B}_k \).

(a) Then every \( W_k \) is an indecomposable basic submodule of \( V \) and

\[
V = \bigoplus_{k \in K} W_k.
\]

(b) Every indecomposable basic submodule \( U \) of \( V \) is contained in \( W_k \), for some \( k \in K \) uniquely determined by \( U \).

(c) The modules \( W_k, k \in K \), are precisely all the indecomposable basic orthogonal summands of \( V \).

**Proof.** (a): Suppose that \( W_k \) has an orthogonal decomposition \( W_k = X \perp Y \) with basic submodules \( X \neq 0, Y \neq 0 \). Then \( \mathcal{B}_k \) is the disjoint union of the non-empty sets \( \mathcal{B}_X \) and \( \mathcal{B}_Y \). Choosing \( \varepsilon \in \mathcal{B}_X \) and \( \eta \in \mathcal{B}_Y \), there exists a sequence \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r \) in \( \mathcal{B}_k \) with \( \varepsilon = \varepsilon_0, \eta = \varepsilon_r \) and \( b(\varepsilon_{i-1}, \varepsilon_i) \neq 0, \varepsilon_{i-1} \neq \varepsilon_i \), for \( 1 \leq i \leq r \). Let \( s \) denote the last index in \( \{1, \ldots, r\} \) with \( \varepsilon_s \in \mathcal{B}_X \). Then \( s < r \) and \( \varepsilon_{s+1} \in \mathcal{B}_Y \). But \( b(X, Y) = 0 \) by Lemma 2.5 and thus \( b(\varepsilon_s, \varepsilon_{s+1}) = 0 \), a contradiction. This proves that \( W_k \) is indecomposable. Since \( \mathcal{B} \) is the
disjoint union of the sets $\mathfrak{B}_k$, we have

$$V = \bigoplus_{k \in K} W_k.$$  

Finally, if $k \neq \ell$, then $b(W_k, W_\ell) = 0$ by the nature of our equivalence relation. Thus

$$V = \bigoplus_{k \in K} W_k.$$  

(b): Given an indecomposable basic submodule $U$ of $V$, we choose $k \in K$ with $\mathfrak{B}_U \cap \mathfrak{B}_k \neq \emptyset$. Then $U \cap W_k \neq 0$. From $V = W_k \oplus W_k^c$, we conclude that $U = (U \cap W_k) \oplus (U \cap W_k^c)$, and then have $U = (U \cap W_k) \perp (U \cap W_k^c)$ because $W_k$ is orthogonal to $W_k^c$. Since $U$ is indecomposable and $U \cap W_k \neq 0$, it follows that $U = U \cap W_k$, i.e., $U \subset W_k$. Since $W_k \cap W_\ell = 0$ for $k \neq \ell$, it is clear that $k$ is uniquely determined by $U$.

(c): If $U$ is an indecomposable basic orthogonal summand of $V$, then $V = U \perp U^c$. We have $U \subset W_k$ for some $k \in K$, and obtain $W_k = U \perp (U^c \cap W_k)$, whence $W_k = U$. \hfill \Box

**Definition 2.7.** We call the submodules $W_k$ of $V$ occurring in Theorem 2.6 the **indecomposable components** of the quadratic module $V$.

The following facts are easy consequences of the theorem.

**Remark 2.8.**

(i) If $U$ is a basic orthogonal summand of $V$, then the indecomposable components of the quadratic module $U = (U, q|U)$ are the indecomposable components of $V$ contained in $U$.

(ii) If $U$ is any basic submodule of $V$, then

$$U = \bigoplus_{k \in K} (U \cap W_k),$$

and every submodule $U \cap W_k \neq \{0\}$ is an orthogonal sum of indecomposable components of $U$.

3. **Orthogonal decomposition of bilinear modules with unique base**

We now outline a theory of symmetric bilinear forms analogous to the theory for quadratic forms given in §2. The bilinear theory is easier than the quadratic theory due the fact that, in contrast to quadratic forms, on a free module we do not need to distinguish between “functional” and “formal” bilinear forms cf. [6, §1]. As before, $R$ is a semiring.

Assume in the following that $V$ is an $R$-module equipped with a fixed symmetric bilinear form $b : V \times V \rightarrow R$. We then call $V = (V, b)$ a **bilinear $R$-module**. If $X$ is a submodule of $V$, we denote the restriction of $b$ to $X \times X$ by $b|X$.

**Definition 3.1.**

(a) Given two submodules $W_1, W_2$ of the $R$-module $V$, we say that $W_1$ is **disjointly orthogonal** to $W_2$, if $W_1 \cap W_2 = \{0\}$ and $b(W_1, W_2) = 0$, i.e., $b(x, y) = 0$ for all $x \in W_1$, $y \in W_2$.

(b) We write $V = W_1 \perp W_2$ if $W_1$ is disjointly orthogonal to $W_2$ and moreover $V = W_1 \oplus W_2$ (as $R$-module). We then call $W_1$ an **orthogonal summand** of $V$ and $W_2$ an **orthogonal complement** of $W_1$ in $V$. 
Definition 3.2. Let \((V_i \mid i \in I)\) be a family of submodules of the bilinear module \(V\). We say that \(V\) is the orthogonal sum of the family \((V_i)\), and then write
\[
V = \bigoplus_{i \in I} V_i,
\]
if for any two different indices \(i, j\) the submodule \(V_i\) is disjointly orthogonal to \(V_j\), and moreover \(V = \bigoplus_{i \in I} V_i\).

In contrast to the quadratic case, the exact analogue of Proposition 2.3 is now a triviality.

Proposition 3.3. Assume that \(V = \bigoplus_{i \in I} V_i\). Let \(J\) and \(K\) be disjoint subsets of \(I\). Then \(V_J = \bigoplus_{i \in J} V_i\) is disjointly orthogonal to \(V_K = \bigoplus_{i \in K} V_i\), and
\[
V_J \perp V_K = V_J \cap V_K.
\]

In the following, we assume again that \(V\) has unique base. Then again a basic orthogonal summand \(W\) of \(V\) has only one basic orthogonal complement in \(V\), namely, \(W^c\) equipped with the bilinear form \(b|W^c\).

For \(X\) a basic submodule of \(V\), we define the properties "decomposable" and "indecomposable" in exactly the same way as indicated by Definition 2.4 in the quadratic case.

We start with a definition and description of the "indecomposable components" of \(V = (V, b)\) in a similar fashion as was done in §2 for quadratic modules. We choose a base \(\mathcal{B}\) of \(V\) and again introduce the appropriate equivalence relation on the set \(\mathcal{B}\), but now we adopt a more elaborate terminology than in §2. This will turn out to be useful later on.

Definition 3.4. We call the symmetric bilinear form \(b\) alternate if \(b(\varepsilon, \varepsilon) = 0\) for every \(\varepsilon \in \mathcal{B}\).

Comment. Beware that this does not imply that \(b(x, x) = 0\) for every \(x \in V\). The classical notion of an alternating bilinear form is of no use here since in the semirings under consideration here (cf. §1) \(\alpha + \beta = 0\) implies \(\alpha = \beta = 0\), whence \(b(x + y, x + y) = 0\) implies \(b(x, y) = 0\). An alternating bilinear form in the classical sense would be identically zero.

Definition 3.5. We associate to the given symmetric bilinear form \(b\) an alternate bilinear form \(b_{alt}\) by the rule
\[
b_{alt}(\varepsilon, \eta) = \begin{cases} 
b(\varepsilon, \eta) & \text{if } \varepsilon \neq \eta \\
0 & \text{if } \varepsilon = \eta \end{cases}
\]
for any \(\varepsilon, \eta \in \mathcal{B}\).

Lemma 3.6. Let \(W\) and \(W'\) be basic submodules of \(V\) with \(W \cap W' = \{0\}\). Then \(W\) is (disjointly) orthogonal to \(W'\) iff \(b_{alt}(W, W') = 0\).

Proof. This can be seen exactly as with the parallel Lemma 2.5. Just replace in its proof the quasiminimal companion of \(q\) by \(b_{alt}\).

Definition 3.7.
(a) A path \(\Gamma\) in \(V = (V, b)\) of length \(r \geq 1\) in \(\mathcal{B}\) is a sequence \(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r\) of elements of \(\mathcal{B}\) with
\[
b_{alt}(\varepsilon_i, \varepsilon_{i+1}) \neq 0 \quad (0 \leq i \leq r - 1).
\]
In essence this condition does not depend on the choice of the base $B$, since $B$ is unique up to multiplication by units, and so we also say that $\Gamma$ is a path in $V$. We say that the path runs from $\varepsilon := \varepsilon_0$ to $\eta := \varepsilon_r$, or that the path connects $\varepsilon$ to $\eta$. A path of length 1 is called an edge. This is just a pair $(\varepsilon, \eta)$ in $B$ with $\varepsilon \neq \eta$ and $b(\varepsilon, \eta) \neq 0$.

(b) We define an equivalence relation on $B$ as follows. Given $\varepsilon, \eta \in B$, we declare that $\varepsilon \sim \eta$ if either $\varepsilon = \eta$ or there runs a path from $\varepsilon$ to $\eta$.

It is now obvious how to mimic the theory of indecomposable components from the end of §2 in the bilinear setting.

**Scholium 3.8.** Theorem 2.6 and its proof remain valid for the present equivalence relation on $B$. We only have to replace the quasiminimal companion $b$ of $q$ there by $b_{\text{alt}}$ and to use Lemma 3.6 instead of Lemma 2.5. Again we denote the set of equivalence classes of $B$ by $\{B_k \mid k \in K\}$ and the submodule of $V$ with base $B_k$ by $V_k$, and again we call the $V_k$ the indecomposable components of $V$. Also the analog to Remark 2.8 remains valid.

We state a consequence of the parallel between the two decomposition theories.

**Theorem 3.9.** Assume that $(V, q)$ is a quadratic module with unique base and $b$ is a quasi-minimal companion of $q$. The indecomposable components of $(V, q)$ coincide with the indecomposable components of $(V, b)$.

**Proof.** The equivalence relation used in Theorem 2.6 is the same as the equivalence relation in Definition 3.7. \(\square\)

We add an easy observation on bilinear modules.

**Proposition 3.10.** Assume that $(V, b)$ is a bilinear $R$-module with unique base. A basic submodule $W$ of $V$ is indecomposable with respect to $b$, iff $W$ is indecomposable with respect to $b_{\text{alt}}$.

**Proof.** The equivalence relation on $B$ just defined (Definition 3.7) does not change if we replace $b$ by $b_{\text{alt}}$. \(\square\)

### 4. ISOMETRIES, ISOTYPICAL COMPONENTS, AND A CANCELLATION THEOREM

Let $R$ be any semiring.

**Definition 4.1.**

(a) For quadratic $R$-modules $V = (V, q)$ and $V' = (V', q')$, an isometry $\sigma : V \to V'$ is a bijective $R$-linear map with $q'(\sigma x) = q(x)$ for all $x \in V$. Likewise, if $V = (V, b)$ and $(V', b')$ are bilinear $R$-modules, an isometry is a bijective $R$-linear map $\sigma : V \to V'$ with $b'(\sigma x, \sigma y) = b(x, y)$ for all $x, y \in V$.

(b) If there exists an isometry $\sigma : V \to V'$, we call $V$ and $V'$ isometric and write $V \cong V'$. We then also say that $V$ and $V'$ are in the same isometry class.

In the following we study quadratic and bilinear $R$-modules with unique base on an equal footing.

It would not hurt if we supposed that the semiring $R$ satisfies the conditions in Theorem 1.7, so that every free $R$-module has unique base, but the simplicity of all of the arguments in the present section becomes more apparent if we do not rely on Theorem 1.7.
Notation/Definition 4.2.
(a) Let \((V^\lambda_0 | \lambda \in \Lambda)\) be a set of representatives of all isometry classes of indecomposable quadratic (resp. bilinear) \(R\)-modules with unique base \(^4\).
(b) If \(W\) is such an \(R\)-module, where \(W \cong V^\lambda_0\) for a unique \(\lambda \in \Lambda\), we say that \(W\) has **type** \(\lambda\) (or: \(W\) is **indecomposable of type** \(\lambda\)).
(c) We say that a quadratic (resp. bilinear) module \(W \neq 0\) with unique base is **isotypical** of type \(\lambda\), if every indecomposable component of \(V\) has type \(\lambda\).
(d) Finally, given a quadratic (resp. bilinear) \(R\)-module with unique base, we denote the sum of all indecomposable components of \(V\) of type \(\lambda\) by \(V^\lambda\) and call the \(V^\lambda \neq 0\), the **isotypical components of** \(V\).

The following is now obvious from §2 and §3 (cf. Theorem 2.6 and Scholium 3.8).

**Proposition 4.3.** If \(V\) is a quadratic or bilinear \(R\)-module with unique base, then

\[ V = \bigoplus_{\lambda \in \Lambda'} V^\lambda \]

with \(\Lambda' = \{ \lambda \in \Lambda | V^\lambda \neq 0 \}\).

Since our notion of orthogonality for basic submodules of \(V\) is encoded in the linear and quadratic, resp. bilinear, structure of \(V\), the following fact also is obvious, but in view of its importance will be dubbed a “theorem”.

**Theorem 4.4.** Assume that \(V\) and \(V'\) are quadratic (resp. bilinear) \(R\)-modules with unique bases and \(\sigma : V \to V'\) is an isometry. Let \(\{V_k | k \in K\}\) denote the set of indecomposable components of \(V\):

(a) \(\{\sigma(V_k) | k \in K\}\) is the set of indecomposable components of \(V'\).
(b) If \(V_k\) has type \(\lambda\), then \(\sigma(V_k)\) has type \(\lambda\), and so \(\sigma(V^\lambda) = V^\lambda_{\sigma}\) for every \(\lambda \in \Lambda\).

Also in the remainder of the section, we assume that the quadratic or bilinear modules have unique base.

**Definition 4.5.** Let \(O(V)\) denote the group of all isometries \(\sigma : V \to V\) (i.e., automorphisms) of \((V, q)\), resp. \((V, b)\). As usual, we call \(O(V)\) the **orthogonal group** of \(V\).

Theorem 4.4 has the following immediate consequence.

**Corollary 4.6.** Every \(\sigma \in O(V)\) permutes the indecomposable components of \(V\) of fixed type \(\lambda\), and so \(\sigma(V^\lambda) = V^\lambda_{\sigma}\) for every \(\lambda \in \Lambda\).

We have a natural isomorphism

\[ O(V) \xrightarrow{1:1} \prod_{\lambda \in \Lambda'} O(V^\lambda), \]

sending \(\sigma \in O(V)\) to the family of its restrictions \(\sigma|V^\lambda \in O(V^\lambda)\).

**Definition 4.7.**
(a) Let \(\lambda \in \Lambda\). We denote the cardinality of the set of indecomposable components of \(V^\lambda\) by \(m_\lambda(V)\), and we call \(m_\lambda(V)\) the **multiplicity** of \(V^\lambda\). \{N.B. \(m_\lambda(V)\) can be infinite or zero.\}

\(^4\)of rank bounded by the cardinality of \(V\), in order to avoid set-theoretical complications
(b) If \( m_\lambda \in \mathbb{N}_0 \) for every \( \lambda \in \Lambda \), we say that \( V \) is isotypically finite.

**Theorem 4.8.** If \( V \) and \( V' \) are quadratic or bilinear \( R \)-modules with unique bases, then \( V \cong V' \) iff \( m_\lambda(V) = m_\lambda(V') \) for every \( \lambda \in \Lambda \).

**Proof.** This follows from Proposition 4.3 and Theorem 4.4. \( \square \)

We are ready for a main result of the paper.

**Theorem 4.9.** Assume that \( W_1, W_2, W'_1, W'_2 \) are quadratic or bilinear modules with unique base and that \( W_1 \) is isotypically finite. Assume furthermore that \( W_1 \cong W'_1 \) and that \( W_1 \perp W_2 \cong W'_1 \perp W'_2 \). Then \( W_2 \cong W'_2 \).

**Proof.** For every \( \lambda \in \Lambda \), clearly \( m_\lambda(V) = m_\lambda(W_1) + m_\lambda(W_2) \) and \( m_\lambda(V') = m_\lambda(W'_1) + m_\lambda(W'_2) \). Since \( V \cong V' \), the multiplicities \( m_\lambda(V) \) and \( m_\lambda(V') \) are equal, and since \( W_1 \cong W'_1 \), the same holds for the multiplicities \( m_\lambda(W'_1) \). Since \( m_\lambda(W_1) = m_\lambda(W'_1) \) is finite, it follows that \( m_\lambda(W_2) = m_\lambda(W'_2) \). By Theorem 4.8 this implies that \( W_2 \cong W'_2 \). \( \square \)

**Remark 4.10.** If the free \( R \)-module \( W_1 \) has finite rank, then certainly \( W_1 \) is isotypically finite. Thus Theorem 4.9 may be viewed as the analogue of Witt’s cancellation theorem from 1937 [11] proved for quadratic forms over fields.

The assumption of isotypical finiteness in Theorem 4.9 cannot be relaxed. Indeed if \( m_\lambda(W_1) \) is infinite for at least one \( \lambda \in \Lambda \), then the cancellation law becomes false. This is evident by Theorem 4.8 and the following example.

**Example 4.11.** Assume that \( V \) is the orthogonal sum of infinitely many copies \( V_1, V_2, \ldots \) of an indecomposable quadratic or bilinear module \( V_0 \) with unique base. Consider the following submodules of \( V \):

\[
W_1 := V_2 \perp V_3 \perp \cdots, \quad W_2 := V_1, \\
W'_1 := V_3 \perp V_4 \perp \cdots, \quad W'_2 := V_1 \perp V_2.
\]

Then \( W_1 \perp W_2 = V = W'_1 \perp W'_2 \), and \( W_1 \cong W'_1 \). But \( W_2 \) is not isometric to \( W'_2 \).

5. EXPANSIONS AND TENSOR PRODUCTS

Let \( q : V \rightarrow R \) be a quadratic form on an \( R \)-module \( V \). We recall from [6, §1] that, when \( V \) is free with base \( (\varepsilon_i : i \in I) \), then \( q \) admits a (not necessarily unique) balanced companion, i.e., a companion \( b : V \times V \rightarrow R \) such that \( b(x, x) = 2q(x) \) for all \( x \in V \), and that it suffices to know for this that \( b(\varepsilon_i, \varepsilon_i) = 2q(\varepsilon_i) \) for all \( i \in I \) [6, Proposition 1.7]. Balanced companions are a crucial ingredient in our definition below of a tensor product of a free bilinear module and a free quadratic module. They arise from “expansions” of \( q \), defined as follows, cf. [6, Definition 1.9].

**Definition 5.1.** A bilinear form \( B : V \times V \rightarrow R \) (not necessarily symmetric) is an expansion of a balanced pair \((q, b)\) if \( B + B' = b \), i.e.,

\[
B(x, y) + B(y, x) = b(x, y) \tag{5.1}
\]

for all \( x, y \in V \), and

\[
q(x) = B(x, x) \tag{5.2}
\]

for all \( x \in V \). If only the form \( q \) is given and (5.2) holds, we say that \( B \) is an expansion of \( q \).
As stated in the [6, §1], every bilinear form \( B : V \times V \rightarrow R \) gives us a balanced pair \((q, b)\) via (5.1) and (5.2), and, if the \( R \)-module \( V \) is free, we obtain all such pairs \((q, b)\) in this way. But we will need a description of all expansions of \((q, b)\) in the free case.

**Construction 5.2.** Assume that \( V \) is a free \( R \)-module and \((\varepsilon_i \mid i \in I)\) is a base of \( V \). When \((q, b)\) is a balanced pair on \( V \), we obtain all expansions \( B : V \times V \rightarrow R \) of \((q, b)\) as follows.

Let \( \alpha_i := q(\varepsilon_i) \), \( \beta_{ij} := b(\varepsilon_i, \varepsilon_j) \) for \( i, j \in I \). We have \( \beta_{ij} = \beta_{ji} \). We choose a total ordering on \( I \) and for every \( i < j \) two elements \( \chi_{ij}, \chi_{ji} \in R \) with
\[
\beta_{ij} = \chi_{ij} + \chi_{ji}, \quad (i < j).
\]

We furthermore put
\[
\chi_{ii} := \alpha_i,
\]
and define \( B \) by the rule
\[
B(\varepsilon_i, \varepsilon_j) = \chi_{ij}
\]
for all \((i, j)\) \( \in I \times I \).

In practice one usually chooses \( \chi_{ij} = \beta_{ij} \), \( \chi_{ji} = 0 \) for \( i < j \), i.e., takes the unique “triangular” expansion \( B \) of \((q, b)\), cf. [6, §1], but now we do not want to depend on the choice of a total ordering of the base \((\varepsilon_i \mid i \in I)\). We used such an ordering above only to ease notation.

**Tensor products** over semirings in general require the use of congruences [4], but for free modules the basics can be done precisely as over rings, and we leave the formal details to the interested reader. We only state here that, given two free \( R \)-modules \( V_1 \) and \( V_2 \), with bases \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), the \( R \)-module \( V_1 \otimes_R V_2 \) “is” the free \( R \)-module with base \( \mathcal{B}_1 \otimes \mathcal{B}_2 \), which is a renaming of \( \mathcal{B}_1 \times \mathcal{B}_2 \), writing \( \varepsilon \otimes \eta \) for \((\varepsilon, \eta)\) with \( \varepsilon \in \mathcal{B}_1, \eta \in \mathcal{B}_2 \). If
\[
\mathcal{B}_1 = \{\varepsilon_i \mid i \in I\}, \quad \mathcal{B}_2 = \{\eta_j \mid j \in J\}
\]
and \( x = \sum_{i \in I} \varepsilon_i \in V_1 \) and \( y = \sum_{j \in J} \eta_j \in V_2 \), we define, as common over rings,
\[
x \otimes y := \sum_{(i, j) \in I \times J} x_i y_j (\varepsilon_i \otimes \eta_j), \quad (5.3)
\]
and this vector is independent of the choice of the bases \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \). If \( B_1 \) and \( B_2 \) are bilinear forms on \( V_1 \) and \( V_2 \) respectively, we have a well defined bilinear form on \( V_1 \otimes_R V_2 \), denoted by \( B_1 \otimes B_2 \), such that for any \( x_i \in V_1, y_j \in V_2 (i, j \in \{1, 2\}) \)
\[
(B_1 \otimes B_2)(x_1 \otimes x_2, y_1 \otimes y_2) = B_1(x_1, y_1) B_2(x_2, y_2). \quad (5.4)
\]
If \( b_1 \) and \( b_2 \) are symmetric bilinear forms on \( V_1 \) and \( V_2 \) respectively, then \( b_1 \otimes b_2 \) is symmetric. Then we call the bilinear module \((V_1 \otimes_R V_2, b_1 \otimes b_2)\) the **tensor product of the bilinear modules** \((V_1, b_1)\) and \((V_2, b_2)\).

We next define the tensor product of a free bilinear and a free quadratic module. The key fact which allows us to do this in a reasonable way is as follows.

**Proposition 5.3.** Let \( \gamma : U \times U \rightarrow R \) be a symmetric bilinear form and \((q, b)\) a balanced quadratic pair on \( V \). Assume that \( B \) and \( B' \) are two expansions of \((q, b)\). Then the bilinear forms \( \gamma \otimes B \) and \( \gamma \otimes B' \) on \( U \otimes V \) yield the same balanced pair \((\tilde{q}, \tilde{b})\) on \( U \otimes V \). We have \( \tilde{b} = \gamma \otimes b \), whence for \( u_1, u_2 \in U, v_1, v_2 \in V \),
\[
\tilde{b}(u_1 \otimes v_1, u_2 \otimes v_2) = \gamma(u_1, u_2) b(v_1, v_2). \quad (5.5)
\]
Furthermore, for \( u \in U \) and \( v \in V \),
\[
\tilde{q}(u \otimes v) = \gamma(u, u)q(v). \tag{5.6}
\]

**Proof.** \( \gamma \otimes B + (\gamma \otimes B)^t = \gamma \otimes B + \gamma^t \otimes B = \gamma \otimes B + \gamma \otimes B^t = \gamma \otimes (B + B^t) = \gamma \otimes b \). Also \( \gamma \otimes B' + (\gamma \otimes B')^t = \gamma \otimes b \). Furthermore,
\[
(\gamma \otimes B)(u \otimes v, u \otimes v) = \gamma(u, u)B(v, v) = \gamma(u, u)q(v) = (\gamma \otimes B')(u \otimes v, u \otimes v)
\]
for any \( u \in U, v \in V \). Together these equations imply \((\gamma \otimes B)(z, z) = (\gamma \otimes B')(z, z)\) for any \( z \in U \otimes V \).

**Definition 5.4.** We call \( \tilde{q} \) the tensor product of the bilinear form \( \gamma \) and the quadratic form \( q \) with respect to the balanced companion \( b \) of \( q \), and write
\[
\tilde{q} = \gamma \otimes_b q,
\]
and we also write \( \tilde{V} = U \otimes_b V \) for the quadratic \( R \)-module \( \tilde{V} = (U \otimes V, \tilde{q}) \).

**Remark 5.5.** If \( q \) has only one balanced companion, we may suppress the “\( b \)” here, writing \( \tilde{q} = \gamma \otimes q \). Cases in which this happens are: \( q \) is rigid, \( V \) has rank one, \( R \) is embeddable in a ring.

**Proposition 5.6.** If \( U = (U, \gamma) \) has an orthogonal decomposition \( U = \bigoplus_{i \in I} U_i \), then
\[
U \otimes_b V = \bigoplus_{i \in I} U_i \otimes_b V.
\]

**Proof.** It is immediate that \((\gamma \otimes b)(U_i \otimes V, U_j \otimes V) = 0\) for \( i \neq j \).

We proceed to explicit examples. For this we need notation from [6, §1] which we recall for the convenience of the reader.

Assume that \( V \) is free of finite rank \( n \) and \( \mathcal{B} \) is a base of \( V \) for which we now choose a total ordering, \( \mathcal{B} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \). Then we identify a bilinear form \( B \) on \( V \) with the \((n \times n)\)-matrix
\[
B = \begin{pmatrix}
\beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\
\beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{n1} & \cdots & \beta_{nn}
\end{pmatrix}, \tag{5.7}
\]
where \( \beta_{ij} = B(\varepsilon_i, \varepsilon_j) \). In particular, a bilinear \( R \)-module \((V, \beta)\) is denoted by a symmetric \((n \times n)\)-matrix, namely its Gram matrix \( b = (\beta_{ij})_{1 \leq i, j \leq n} \), where \( \beta_{ij} = \beta_{ji} = b(\varepsilon_i, \varepsilon_j) \).

Given a quadratic module \((V, q)\), we choose a triangular expansion
\[
B = \begin{pmatrix}
\alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\
0 & \alpha_2 & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha_n
\end{pmatrix}, \tag{5.8}
\]
of \( q \) and denote \( q \) by the triangular scheme
\[
q = \begin{pmatrix}
\alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_2 & \cdots & \alpha_{2n} \\
\vdots & \ddots & \vdots \\
\alpha_n
\end{pmatrix}. \tag{5.9}
\]
so that \( q \) is given by the polynomial

\[
q(x) = \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{i<j}^{n} \alpha_{ij} x_i x_j.
\]

(Such triangular schemes have already been used in the literature when \( R \) is a ring, e.g. \([9, \text{I} \S 2]\).)

In the case that \( q \) is diagonal, i.e., all \( \alpha_{ij} \) with \( i < j \) are zero, we usually write instead of (5.8) the single row

\[
q = [\alpha_1, \alpha_2, \ldots, \alpha_n].
\]

(5.10)

Analogously we use for a diagonal symmetric bilinear form \( b \) (i.e., \( b(\varepsilon_i, \varepsilon_j) = 0 \) for \( i \neq j \)) the notation

\[
b = \langle \beta_{11}, \beta_{22}, \ldots, \beta_{nn} \rangle.
\]

(5.11)

We note that the quadratic form (5.9) has the balanced companion

\[
b = \begin{pmatrix}
\alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{12} & \alpha_2 & & \\
& \vdots & \ddots & \\
\alpha_{1n} & & & \alpha_n
\end{pmatrix}
\]

(5.12)

and (5.10), being diagonal, has the unique (!) balanced companion

\[
b = \langle 2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n \rangle.
\]

(5.13)

Example 5.7. If \( a_1, \ldots, a_n, c \in R \), then

\[
\langle a_1, \ldots, a_n \rangle \otimes [c] = [a_1 c, \ldots, a_n c].
\]

(5.14)

This is evident from Proposition 5.6 and the rule \( \langle a \rangle \otimes [c] = [ac] \) for one-dimensional forms which holds by (5.6). In particular

\[
[a_1, \ldots, a_n] = \langle a_1, \ldots, a_n \rangle \otimes [1].
\]

(5.15)

Example 5.8. (As before, \( R \) is any semiring.) Assume that \( V = (V, q) \) has dimension \( n \), and take a base \( \eta_1, \ldots, \eta_n \) of \( V \). Let

\[
(U, \gamma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

with base \( \varepsilon_1, \varepsilon_2 \). We choose a balanced companion \( b \) of \( V \), written as a symmetric \((n \times n)\)-matrix \( \langle b(\eta_i, \eta_j) \rangle \). We see by the use of the rules (5.5) and (5.6) that

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes_b q = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}
\]

(5.16)

written with respect to the base

\[
\varepsilon_1 \otimes \eta_1, \ldots, \varepsilon_1 \otimes \eta_n, \varepsilon_2 \otimes \eta_1, \ldots, \varepsilon_2 \otimes \eta_n.
\]

This example illustrates dramatically that in general the tensor product of \( \gamma \) and \( q \) depends on the chosen balanced companion \( b \) of \( q \): tensoring \( q \) by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) produces the symmetric matrix of \( b \).

Remark 5.9. If \( \gamma_1 \) and \( \gamma_2 \) are bilinear forms on the same free \( R \)-module \( U \), then the rules (5.5) and (5.6) imply for any \( \lambda_1, \lambda_2 \in R \) that

\[
(\lambda_1 \gamma_1 + \lambda_2 \gamma_2) \otimes_b q = \lambda_1 (\gamma_1 \otimes_b q) + \lambda_2 (\gamma_2 \otimes_b q).
\]

(5.17)
Example 5.10. Using (5.17) with
\[ \gamma_1 = \langle a_1, a_2 \rangle, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = \lambda, \]
we obtain from Proposition 5.6 and Example 5.7 that
\[
\left( \begin{array}{c} a_1 \\ \lambda \\ a_2 \end{array} \right) \otimes_b q = \begin{pmatrix} a_1q & \lambda b \\ a_2q \end{pmatrix}
\]
(5.18)

Example 5.11. Let
\[ q = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & a_{n-1,n} \end{bmatrix} \]
with \( a_{ij} \in R \) (\( i < j \)). Then \( q \) is rigid (cf. [6, Proposition 3.4]; no assumption on \( R \) is needed here). Furthermore, let
\[ \gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{pmatrix} \]
with \( \gamma_{ij} = \gamma_{ji} \in R \). Then we obtain by the rules (5.5) and (5.6) that
\[
\gamma \otimes q = \begin{bmatrix} 0 & a_{12} \gamma & \cdots & a_{1n} \gamma \\ 0 & 0 & \cdots & a_{2n} \gamma \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1,n} \gamma & \cdots & 0 \end{bmatrix}
\]
(5.19)

More precisely, if the presentations of \( q \) and \( \gamma \) above refer to ordered bases \( (\eta_1, \ldots, \eta_n) \) and \( (\varepsilon_1, \ldots, \varepsilon_m) \), respectively, then (5.19) refers to the ordered base
\[ (\varepsilon_1 \otimes \eta_1, \ldots, \varepsilon_m \otimes \eta_1, \varepsilon_1 \otimes \eta_2, \ldots, \varepsilon_m \otimes \eta_n). \]

We now consider the tensor product \( \gamma \otimes [a] = \gamma \otimes_b [a] \), cf. Equation (5.10), where \( b \) is the unique balanced companion of \([a]\), (5.13). Our starting point is a definition which makes sense for any semiring \( R \) and any \( R \)-module \( U \).

Definition 5.12. Let \( \gamma : U \times U \to R \) be a symmetric bilinear form. The norm form of \( \gamma \) is the quadratic form \( n(\gamma) : U \to R \) with
\[ n(\gamma)(x) := \gamma(x, x) \]
for any \( x \in U \).

Remark 5.13. The norm form \( n(\gamma) \) has the expansion \( \gamma : U \times U \to R \) and the associated balanced companion \( \gamma + \gamma^t = 2\gamma \). The norm forms are precisely all the quadratic forms which admit a symmetric expansion. If \( U \) has a finite base \( \varepsilon_1, \ldots, \varepsilon_n \), then with respect to this base
\[
n(\gamma) = \begin{bmatrix} \gamma_{11} & 2\gamma_{12} & \cdots & 2\gamma_{1m} \\ \gamma_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ \gamma_{mm} \end{bmatrix},
\]
(5.20)
where \( \gamma_{ij} := \gamma(\varepsilon_i, \varepsilon_j) \).
Proposition 5.14. Assume that $U = (U, \gamma)$ is a free bilinear $R$-module and $a \in R$. Then
\[ U \otimes [a] \cong (U, a n(\gamma)). \] (5.21)

Proof. We realize the form $[a]$ as a quadratic module $(V, q)$ with $V = R\eta$ free of rank 1 and $q(\eta) = a$. (\{q has the unique balanced companion $b : V \times V \to R$, with $b(\eta, \eta) = 2a.\}) The form $\tilde{q} := \gamma \otimes q = \gamma \otimes_b q$ is given by
\[ \tilde{q}(x \otimes \eta) = \gamma(x, x)a = (an(\gamma))(x). \]

The claim is obvious. \hfill \Box

Example 5.15. Assume that $U$ has base $\varepsilon_1, \ldots, \varepsilon_m$. Let $\gamma_{ij} := \gamma(\varepsilon_i, \varepsilon_j)$. Then
\[ \gamma \otimes [a] \cong (a \gamma) \otimes [1], \]
and
\[ \gamma \otimes [1] = \begin{bmatrix} \gamma_{11} & 2\gamma_{12} & \cdots & 2\gamma_{1n} \\ & \gamma_{22} & \ddots & \vdots \\ & & \ddots & \gamma_{mn} \\ & & & \gamma_{mm} \end{bmatrix}, \] (5.22)

where the right hand side refers to the base $\varepsilon_1 \otimes \eta, \varepsilon_2 \otimes \eta, \ldots, \varepsilon_m \otimes \eta$.

At a crucial point in §6, we will need an explicit description of the tensor products $\gamma \otimes_b q$ with $q$ indecomposable of rank 2. We start with a general fact.

Proposition 5.16. Assume that $\gamma$ is a symmetric bilinear form on a free $R$-module $U$ and $q_1, q_2$ are quadratic forms on a free $R$-module $V$. Let $b_1, b_2$ be balanced companions of $q_1$ and $q_2$, respectively. Let $q := \lambda_1 q_1 + \lambda_2 q_2$ with $\lambda_1, \lambda_2 \in R$. Then $b := \lambda_1 b_1 + \lambda_2 b_2$ is a balanced companion of $q$, and
\[ \gamma \otimes_b q = \lambda_1 (\gamma \otimes_b q_1) + \lambda_2 (\gamma \otimes_b q_2). \] (5.23)

This form has the balanced companion $\gamma \otimes b$ (as we know) and
\[ \gamma \otimes b = \lambda_1 (\gamma \otimes b_1) + \lambda_2 (\gamma \otimes b_2). \] (5.24)

Proof. An easy check by use of (5.5) and (5.6). \hfill \Box

Example 5.17. We take a free module $V$ with base $\eta_1, \eta_2$, and choose with respect to this base
\[ q_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} = [a_1, a_2], \quad q_2 = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \]

with $a_1, a_2, c \in R, c \neq 0$, and the balanced companions
\[ b_1 = \begin{pmatrix} 2a_1 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ c \end{pmatrix}. \]

Then
\[ q := q_1 + q_2 = \begin{bmatrix} a_1 & c \\ c & a_2 \end{bmatrix} \]

has the balanced companion
\[ b := b_1 + b_2 = \begin{pmatrix} 2a_1 & c \\ c & 2a_2 \end{pmatrix}. \]
For
\[ \gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{pmatrix} \]
on a free module $U$ with to the base $\varepsilon_1, \ldots, \varepsilon_m$, we get
\[ \gamma \otimes b_1 q_1 = \begin{bmatrix} a_1 n(\gamma) \\ 0 \\ a_2 n(\gamma) \end{bmatrix}, \quad \gamma \otimes b_2 \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \gamma \\ 0 & 0 \end{bmatrix}, \quad \text{cf. (5.19)}, \]
and finally
\[ \gamma \otimes b \begin{bmatrix} a_1 & c \\ a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_1 n(\gamma) & c \gamma \\ a_2 n(\gamma) & 0 \end{bmatrix}, \quad (5.25) \]
with respect to the base
\[ \varepsilon_1 \otimes \eta_1, \ldots, \varepsilon_m \otimes \eta_1, \varepsilon_1 \otimes \eta_2, \ldots, \varepsilon_m \otimes \eta_2. \]

**Remark 5.18.** From (5.25) and (5.18), we obtain the useful formula
\[ \gamma \otimes b \begin{bmatrix} a_1 & c \\ a_2 & a_3 \end{bmatrix} = \left( \begin{bmatrix} a_1 & c \\ a_2 & a_3 \end{bmatrix} \otimes_{2\gamma} n(\gamma) \right), \quad (5.26) \]
by use of Example 5.10 for the quadratic pair $(n(\gamma), 2\gamma)$.

From now on, we assume that $V$ has unique base. {We do not need that $U$ has unique base.}

**Definition 5.19.** We call a companion $b$ of $q$ **faithful** if $b$ is balanced and quasiminimal.

**Proposition 5.20.** Assume that $b$ is a faithful companion of $q$, and that $V = W_1 \perp W_2$ is an orthogonal decomposition of $V$. Then, writing $U \otimes b W_i$ instead of $U \otimes (b|_{W_i}) W_i$, we have
\[ U \otimes b V = U \otimes b W_1 \perp U \otimes b W_2 \]
for any bilinear $R$-module $U$.

**Proof.** $b(W_1, W_2) = 0$, since $b$ is quasiminimal. It follows that
\[ (\gamma \otimes b)(U \otimes W_1, U \otimes W_2) = 0. \]
Thus, $\tilde{q} = \gamma \otimes b q$ is quasilinear on $(U \otimes W_1) \times (U \otimes W_2)$. \qed

**Example 5.21.** Our assumption, that $b$ is faithful, is necessary here. If $V = W_1 \perp W_2$, and $b$ is balanced, but $b(W_1, W_2) \neq 0$, then
\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes b V = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \]
is not the orthogonal sum of
\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes b W_1 \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes b W_2. \]

**Example 5.22.** Let $q = [a_1, a_2, \ldots, a_n]$ be a diagonal quadratic form. The diagonal symmetric bilinear form
\[ b := \langle 2q_1, \ldots, 2a_n \rangle \]
is the unique faithful companion of $q$. For any bilinear $R$-module $(U, \gamma)$, we have
\[ \gamma \otimes b q = \gamma \otimes [a_1] \perp \cdots \perp \gamma \otimes [a_n]. \quad (5.27) \]
Concerning the forms $\gamma \otimes [a_i]$, recall Proposition 5.14 and Example 5.15.
6. Indecomposability in tensor products

In this section, we assume for simplicity that \( R \setminus \{0\} \) is an entire antiring. So every free \( R \)-module has unique base (cf. Theorem 1.3), and \( R \) has no zero divisors. We discuss decomposability first in tensor products of (free) bilinear modules, later in tensor products of bilinear modules with quadratic modules.

Let \( V_1 = (V_1, b_1) \) and \( V_2 = (V_2, b_2) \) be indecomposable free (symmetric) bilinear modules over \( R \), and let \( V := V_1 \otimes V_2 = (V_1 \otimes V_2, b) \) with \( b := b_1 \otimes b_2 \). We take bases \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \) of the \( R \)-modules \( V_1, V_2 \) respectively and then have the base

\[
\mathfrak{B} = \mathfrak{B}_1 \otimes \mathfrak{B}_2 := \{ \varepsilon \otimes \eta \mid \varepsilon \in \mathfrak{B}_1, \eta \in \mathfrak{B}_2 \}
\]

of \( V \). Our task is to determine the indecomposable components of \( V \). First we discuss the “trivial” cases.

**Remark 6.1.** Assume that \( V_1 \) has dimension (= rank) one, so \( V_1 \cong \langle a \rangle \) with \( a \in R \). If \( a \neq 0 \), then \( V \) is clearly indecomposable. If \( a = 0 \), then \( b_1 \otimes b_2 = 0 \), whence \( V \) is indecomposable only if also \( \dim V_2 = 1 \). Then \( V = \langle 0 \rangle \).

In all the following, we assume that \( V_1 \neq \langle 0 \rangle \), \( V_2 \neq \langle 0 \rangle \).

We resort to §3 to describe bases of the indecomposable components of \( V = (V, b) \) as the classes in

\[
\mathfrak{B} = \{ \varepsilon \otimes \eta \mid \varepsilon \in \mathfrak{B}_1, \eta \in \mathfrak{B}_2 \}
\]

of an equivalence relation given by “paths”, cf. Definition 3.7. So a path of length \( r \geq 1 \) in \( V \), i.e., in \( \mathfrak{B} \), is a sequence

\[
\Gamma = (\varepsilon_0 \otimes \eta_0, \varepsilon_1 \otimes \eta_1, \ldots, \varepsilon_r \otimes \eta_r)
\]

with

\[
b_1(\varepsilon_i, \varepsilon_{i+1})b_2(\eta_i, \eta_{i+1}) \neq 0
\]

and

\[
\varepsilon_i \neq \varepsilon_{i+1} \quad \text{or} \quad \eta_i \neq \eta_{i+1}
\]

for \( 0 \leq i \leq r - 1 \).

Let us first assume that both \( b_1 \) and \( b_2 \) are alternate, whence also \( b = b_1 \otimes b_2 \) is alternate. Now condition (6.3) is a consequence of (6.2) and thus can be ignored. We read off from (6.2) that

\[
\Gamma_1 = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r), \quad \Gamma_2 = (\eta_0, \eta_1, \ldots, \eta_r)
\]

are paths in \( V_1 \) and \( V_2 \) respectively of same length \( r \). Conversely, given such paths \( \Gamma_1 \) and \( \Gamma_2 \), they combine to a path \( \Gamma \) of length \( r \) in \( V \), as written in (6.1). {Here we use the assumption that \( R \) has no zero divisors.} We write

\[
\Gamma = \Gamma_1 \otimes \Gamma_2.
\]

We will speak of “cycles” in \( \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B} \), in the following obvious way:

**Definition 6.2.** Let \( \mathfrak{C} \) be a base of a free bilinear \( R \)-module \( W \).

(a) We denote the length of a path \( \Gamma \) in \( \mathfrak{C} \) by \( \ell(\Gamma) \).

(b) A cycle \( \Delta \) in \( W \) with base point \( \zeta \in \mathfrak{C} \) is a path \( (\zeta_0, \zeta_1, \ldots, \zeta_r) \) in \( \mathfrak{C} \) with \( \zeta_0 = \zeta_r = \zeta \).

We say that the cycle \( \Delta \) is **even** (resp. **odd**) if \( \ell(\Delta) \) is **even** (resp. **odd**). We say that \( \Delta \) is a 2-cycle if \( \ell(\Delta) = 2 \), whence \( \Delta = (\zeta, \zeta', \zeta) \) with \( (\zeta, \zeta') \) an edge.

**Lemma 6.3.** Let \( \varepsilon, \varepsilon' \in \mathfrak{B}_1 \) and \( \eta, \eta' \in \mathfrak{B}_2 \). Let \( \Gamma_1 \) be a path from \( \varepsilon \) to \( \varepsilon' \) of length \( r \) and \( \Gamma_2 \) a path from \( \eta \) to \( \eta' \) of length \( s \), and assume that \( r \equiv s \pmod{2} \). Then \( \varepsilon \otimes \eta \sim \varepsilon' \otimes \eta' \).
Proof. Assume, without loss of generality, that \( s \geq r \), whence \( s = r + 2t \) with \( t \geq 0 \). If \( t = 0 \), then \( \Gamma_{1} \otimes \Gamma_{2} \) is a path from \( \varepsilon \otimes \eta \) to \( \varepsilon' \otimes \eta' \) in \( V \). If \( t > 0 \), we replace \( \Gamma_{1} = (\varepsilon_{0}, \varepsilon_{1}, \ldots, \varepsilon_{r}) \) by
\[
\tilde{\Gamma}_{1} = (\varepsilon_{0}, \varepsilon_{1}, \ldots, \varepsilon_{r}, \varepsilon_{r-1}, \varepsilon_{r}, \ldots)
\]
adjointing \( t \) copies of the 2-cycle \((\varepsilon_{r}, \varepsilon_{r-1}, \varepsilon_{r})\) to \( \Gamma_{1} \). Now \( \tilde{\Gamma}_{1} \otimes \Gamma_{2} \) runs from \( \varepsilon \otimes \eta \) to \( \varepsilon' \otimes \eta' \).

**Theorem 6.4.** Assume that both \( b_{1} \) and \( b_{2} \) are alternate (and \( V_{1} \neq \langle 0 \rangle \), \( V_{2} \neq \langle 0 \rangle \), as always).

a) If \( V_{1} \) or \( V_{2} \) contains an odd cycle, then \( V_{1} \otimes V_{2} \) is indecomposable.

b) Otherwise \( V_{1} \otimes V_{2} \) is the orthogonal sum of two indecomposable components.

**Proof.** a): We assume that \( V_{1} \) contains an odd cycle \( \Delta \) with base point \( \delta \). Let \( \varepsilon \otimes \eta \) and \( \varepsilon' \otimes \eta' \) be different elements of \( \mathcal{B} \). We want to verify that \( \varepsilon \otimes \eta \sim \varepsilon' \otimes \eta' \). We choose a path \( \Gamma_{1} \) from \( \varepsilon \) to \( \varepsilon' \) in \( V_{1} \) and a path \( \Gamma_{2} \) from \( \eta \) to \( \eta' \) in \( V_{2} \). If \( \ell(\Gamma_{1}) = \ell(\Gamma_{2}) \) (mod 2), then we know by Lemma 6.3 that \( \varepsilon \otimes \eta \sim \varepsilon' \otimes \eta' \). Now assume that \( \ell(\Gamma_{1}) \) and \( \ell(\Gamma_{2}) \) have different parity. We choose a new path \( \tilde{\Gamma}_{1} \) from \( \varepsilon \) to \( \varepsilon' \) as follows: We first take a path \( H \) from \( \varepsilon \) to the base point \( \delta \) of \( \Delta \), then we run through \( \Delta \), then we take the path inverse to \( H \) (in the obvious sense) from \( \delta \) to \( \varepsilon \), and finally we run through \( \Gamma_{1} \). The length \( \ell(\tilde{\Gamma}_{1}) \) has different parity than \( \ell(\Gamma_{1}) \) and thus the same parity as \( \ell(\Gamma_{2}) \). We conclude again that \( \varepsilon \otimes \eta \sim \varepsilon' \otimes \eta' \).

b): Now assume that both \( V_{1} \) and \( V_{2} \) contain only even cycles. This means that both in \( V_{1} \) and \( V_{2} \) all paths from a fixed start to a fixed end have length of the same parity. Given \( \varepsilon \otimes \eta \) and \( \varepsilon' \otimes \eta' \) in \( \mathcal{B} \), every path \( \Gamma \) from \( \varepsilon \otimes \eta \) to \( \varepsilon' \otimes \eta' \) has the shape \( \Gamma_{1} \otimes \Gamma_{2} \) with \( \Gamma_{1} \) running from \( \varepsilon \) to \( \varepsilon' \), \( \Gamma_{2} \) running from \( \eta \) to \( \eta' \), and \( \ell(\Gamma_{1}) = \ell(\Gamma_{2}) \). Thus, if the paths from \( \varepsilon \) to \( \varepsilon' \) have length of different parity than those from \( \eta \) to \( \eta' \), then \( \varepsilon \otimes \eta \) cannot be connected to \( \varepsilon' \otimes \eta' \) by a path. But \( \varepsilon \otimes \eta \) can be connected to \( \varepsilon' \otimes \eta'' \), where \( \eta'' \) arises from \( \eta' \) by adjoining an edge at the endpoint of \( \eta' \). We fix some \( \varepsilon_{0} \in \mathcal{B}_{1} \), and \( \eta_{0}, \eta_{1} \in \mathcal{B}_{2} \) with \( b_{2}(\eta_{0}, \eta_{1}) = 1 \). Then every element of \( \mathcal{B} \) can be connected by a path to \( \varepsilon_{0} \otimes \eta_{0} \) or to \( \varepsilon_{0} \otimes \eta_{1} \), but not to both. \( V \) has exactly two indecomposable components.

**Remark 6.5.** Assume again that \( b_{1} \) and \( b_{2} \) are alternate and \( \mathcal{B}_{1} \) and \( \mathcal{B}_{2} \) both contain only even cycles. Let \( \varepsilon, \varepsilon' \in \mathcal{B}_{1} \) and \( \eta, \eta' \in \mathcal{B}_{2} \), and choose paths \( \Gamma_{1} \) from \( \varepsilon \) to \( \varepsilon' \) and \( \Gamma_{2} \) from \( \eta \) to \( \eta' \). As the proof of Theorem 6.4.b has shown, \( \varepsilon \otimes \eta \) and \( \varepsilon' \otimes \eta' \) lie in the same indecomposable component of \( V_{1} \otimes V_{2} \) iff \( \ell(\Gamma_{1}) \) and \( \ell(\Gamma_{2}) \) have the same parity.

There remains the case that \( b_{1} \) or \( b_{2} \) is not alternate.

**Theorem 6.6.** Assume that \( b_{1} \) is not alternate and – as before – that \( V_{1} = (V_{1}, b_{1}) \) and \( V_{2} = (V_{2}, b_{2}) \) are indecomposable. Then \( (V_{1} \otimes V_{2}, b_{1} \otimes b_{2}) \) is indecomposable.

**Proof.** Every path in \( V := V_{1} \otimes V_{2} \) with respect to \((b_{1})_{alt} \otimes (b_{2})_{alt}\) is also a path with respect to \( b_{1} \otimes b_{2} \), as is easily checked, and the paths in \( V_{i} \) with respect to \( b_{i} \) are the same as those with respect to \((b_{i})_{alt}\) (\( i = 1, 2 \)). Thus we are done by Theorem 6.4, except in the case that all cycles in \( V_{1} \) and \( V_{2} \) are even. Then \( V \) has two indecomposable components \( W', W'' \) with respect to \((b_{1})_{alt} \otimes (b_{2})_{alt}\). The base
\[
\mathcal{B} = \mathcal{B}_{1} \otimes \mathcal{B}_{2} := (\varepsilon \otimes \eta \mid \varepsilon \in \mathcal{B}_{1}, \eta \in \mathcal{B}_{2})
\]
of \( V_{1} \otimes V_{2} \) is the disjoint union of sets \( \mathcal{B}', \mathcal{B}'' \) which are bases of \( W' \) and \( W'' \). Any two elements of \( \mathcal{B}' \) are connected by a path with respect to \((b_{1})_{alt} \otimes (b_{2})_{alt}\), hence by a path with respect to \( b_{1} \otimes b_{2} \), and the same holds for the set \( \mathcal{B}'' \).

We choose some \( \rho \in \mathcal{B}_{1} \) with \( b_{1}(\rho, \rho) \neq 0 \) and an edge \((\eta_{0}, \eta_{1})\) in \( \mathcal{B}'' \). Since \( R \) has no zero divisors, it follows that \((\rho \otimes \eta_{0}, \rho \otimes \eta_{1})\) is an edge in \( \mathcal{B} \) with respect to \( b_{1} \otimes b_{2} \). Perhaps
interchanging $W'$ and $W''$, we assume that $\rho \otimes \eta_0 \in \mathfrak{B}'$. Suppose that also $\rho \otimes \eta_1 \in \mathfrak{B}'$. Then there exists a path $\Gamma$ in $\mathfrak{B}'$ with respect to $(b_1)_{alt} \otimes (b_2)_{alt}$ running from $\rho \otimes \eta_0$ to $\rho \otimes \eta_1$. $\Gamma$ has the form $\Gamma_1 \otimes \Gamma_2$, with $\Gamma_1$ a cycle in $V_1$ with base point $\rho$, and $\Gamma_2$ a path in $V_2$ running from $\eta_0$ to $\eta_1$. We have $\ell(\Gamma_1) = \ell(\Gamma_2)$ and $\ell(\Gamma_2)$ is even. But there exists the path $(\eta_0, \eta_1)$ from $\eta_0$ to $\eta_1$ of length 1. Since all paths in $V_2$ from $\eta_0$ to $\eta_1$ have the same parity, we infer that $\ell(\Gamma_2)$ is odd, a contradiction.

We conclude that $\rho \otimes \eta_1 \in \mathfrak{B}'$. The elements $\rho \otimes \eta_0 \in \mathfrak{B}'$ and $\rho \otimes \eta_1 \in \mathfrak{B}'$ are connected by a path with respect to $b_1 \otimes b_2$, and thus all elements of $\mathfrak{B}$ are connected by paths with respect to $b_1 \otimes b_2$. \hfill $\Box$

Turning to a study of indecomposable components of tensor products of bilinear and quadratic modules, we need some more terminology. Let $V = (V, q)$ be a free quadratic $R$-module and $\mathfrak{B}$ a base of $V$. We focus on balanced companions of $q$.

**Definition 6.7.**

(a) We call a companion $b$ of $q$ **faithful** if $b$ is balanced and quasiminimal (cf. §2 above), whence $b(\varepsilon, \varepsilon) = 2q(\varepsilon)$ for all $\varepsilon \in \mathfrak{B}$ and $b(\varepsilon, \eta) = 0$ for $\varepsilon \neq \eta$ in $\mathfrak{B}$ such that $q$ is quasilinear on $R\varepsilon \times R\eta$.

(b) Given a balanced companion $b$ of $q$, we define a new bilinear form $b_f$ on $V$ by the rule that, for $\varepsilon, \eta \in \mathfrak{B}$,

$$b_f(\varepsilon, \eta) = \begin{cases} 0 & \text{if } \varepsilon \neq \eta \text{ and } q \text{ is quasilinear on } R\varepsilon \times R\eta, \\ b(\varepsilon, \eta) & \text{else.} \end{cases}$$

It is clear from [6, Theorem 6.3] that again $b_f$ is a companion of $q$. By definition, this companion is quasiminimal. $b_f$ is also balanced, since $b_f(\varepsilon, \varepsilon) = b(\varepsilon, \varepsilon) = 2q(\varepsilon)$ for all $\varepsilon \in \mathfrak{B}$, cf. [6, Proposition 1.7], and so $b_f$ is faithful. We call $b_f$ the **faithful companion of $q$ associated to $b$.**

**Theorem 6.8.** Assume that $b$ is a balanced companion of $q$, and that $W$ is a basic submodule of $V$. Then $W$ is indecomposable with respect to $q$ iff $W$ is indecomposable with respect to $b_f$.

**Proof.** This is a special case of Theorem 3.9, since $b_f|W = (b|W)_f$ is a quasiminimal companion of $q|W$. \hfill $\Box$

**Definition 6.9.**

(a) We say that $q$ is **diagonally zero** if $q(\varepsilon) = 0$ for every $\varepsilon \in \mathfrak{B}$.

(b) We say that $q$ is **anisotropic** if $q(\varepsilon) \neq 0$ for every $\varepsilon \in \mathfrak{B}$.

**Remarks 6.10.**

(i) If $q$ is diagonally zero, then $q$ is rigid, cf. [6, Proposition 3.4]. Conversely, if $q$ is rigid and the quadratic form [1] is quasilinear, i.e., $(\alpha + \beta)^2 = \alpha^2 + \beta^2$ for any $\alpha, \beta \in R$, then $q$ is diagonally zero, as proved in [6, Theorem 3.5].

(ii) If $q$ is anisotropic, then $q(x) \neq 0$ for every $x \in V \setminus \{0\}$. So our definition of anisotropy here coincides with the usual meaning of anisotropy for quadratic forms (which makes sense, say, for $R$ a semiring without zero divisors and $V$ any $R$-module).

**Definition 6.11.** In a similar vein, we call a symmetric bilinear form $b$ on $V$ **anisotropic** if $b(\varepsilon, \varepsilon) \neq 0$ for every $\varepsilon \in \mathfrak{B}$, and then have $b(x, x) \neq 0$ for every $x \in V \setminus \{0\}$. 
Note that, if \( b \) is a balanced companion of \( q \), then \( b \) is anisotropic iff \( q \) is anisotropic.

Assume now that \( U := (U, \gamma) \) is a free bilinear module, \( V := (V, q) \) is a free quadratic module, and \( b \) is a balanced companion of \( q \). Let

\[
\tilde{V} := (\tilde{V}, \tilde{q}) := (U \otimes V, \gamma \otimes_b q).
\]

We want to determine the indecomposable components of \( \tilde{V} \). Discarding trivial cases, we assume that \( U \neq \langle 0 \rangle \), \( V \neq [0] \).

We choose bases \( B_1 \) and \( B_2 \) of the \( R \)-modules \( U \) and \( V \), respectively, and introduce the subsets

\[
\begin{align*}
B_1^+ & := \{ \varepsilon \in B_1 \mid \gamma(\varepsilon, \varepsilon) \neq 0 \}, \\
B_1^0 & := \{ \varepsilon \in B_1 \mid \gamma(\varepsilon, \varepsilon) = 0 \}, \\
B_2^+ & := \{ \eta \in B_1 \mid q(\eta) \neq 0 \}, \\
B_2^0 & := \{ \eta \in B_1 \mid q(\eta) = 0 \},
\end{align*}
\]

of \( B_1 \) and \( B_2 \), respectively, and furthermore the basic submodules \( U^+, U^0, V^+, V^0 \) respectively spanned by these sets.

**Lemma 6.12.**

a) If \( \varepsilon \in B_1^+ \), then the indecomposable components of the basic submodule \( \varepsilon \otimes V := (R\varepsilon) \otimes V \) of \( U \otimes V \) with respect to \( \tilde{q} \) are the submodules \( \varepsilon \otimes W \) with \( W \) running through the indecomposable components of \( V \) with respect to \( q \).

b) If \( \eta \in B_1^+ \), then the indecomposable components of \( U \otimes \eta := U \otimes (R\eta) \) with respect to \( \tilde{q} \) are the modules \( U \otimes \eta \) with \( U' \) running through the indecomposable components of \( U \) with respect to the norm form \( \eta(\gamma) \) of \( \gamma \) (cf. Definition 5.12).

**Proof.** This follows from the formulas \( \tilde{q}(\varepsilon \otimes y) = \gamma(\varepsilon, \varepsilon)q(y) \) for \( y \in V \) and \( \tilde{q}(x \otimes \eta) = \gamma(x, x)q(\eta) \) for \( x \in U \) (cf. (5.6)), since \( \gamma(\varepsilon, \varepsilon) \neq 0 \), \( q(\eta) \neq 0 \). \( \square \)

**Lemma 6.13.** Assume that \( (V, q) \) is indecomposable. Let \( a, c \in R \setminus \{0\} \). Then

\[
\begin{pmatrix} a & c \\ c & 0 \end{pmatrix} \otimes_b V = \begin{bmatrix} aq & cb \\ cb & 0 \end{bmatrix}
\]

(cf. (5.19)) is indecomposable.

**Proof.** Let

\[
(U, \gamma) = \begin{pmatrix} a & c \\ c & 0 \end{pmatrix}
\]

with respect to a base \( \varepsilon_1, \varepsilon_2 \) and assume for notational convenience that \( V \) has a finite base \( \eta_1, \ldots, \eta_n \). By Lemma 6.12.a, we have

\[
\varepsilon_1 \otimes \eta_1 \sim \varepsilon_1 \otimes \eta_2 \sim \cdots \sim \varepsilon_1 \otimes \eta_n.
\]

For given \( \varepsilon_1 \otimes \eta_i, \varepsilon_2 \otimes \eta_j \) with \( i \neq j \), \( \gamma \otimes_b q \) has the value table

\[
\begin{bmatrix} aq(\eta_i) & cb(\eta_i, \eta_j) \\ 0 & \end{bmatrix}.
\]

Starting with \( \varepsilon_2 \otimes \eta_j \), we find some \( \eta_i, i \neq j \), with \( b(\eta_i, \eta_j) \neq 0 \), because \( (V, q) \) is indecomposable. Since \( R \) has NQL, it follows that \( R(\varepsilon_i \otimes \eta_i) + R(\varepsilon_j \otimes \eta_j) \) is indecomposable with respect to \( \tilde{q} \), whence \( \varepsilon_1 \otimes \eta_i \sim \varepsilon_2 \otimes \eta_j \). Thus all \( \varepsilon_k \otimes \eta_\ell \) are equivalent. \( \square \)
In order to avoid certain pathologies concerning indecomposability in tensor products $U \otimes_b V$, we henceforth will assume that our semiring has the following property:

For any $a$ and $c$ in $R \setminus \{0\}$ there exists some $\mu \in R$ with $a + \mu c \neq a$. \hfill (NQL)

Clearly, this property means that every free quadratic module $[x, \delta]$ with $c \neq 0$ is not quasilinear on $(R \eta_1) \times (R \eta_2)$, where $(\eta_1, \eta_2)$ is the associated base, whence the label “NQL”.

Examples 6.14.

(a) In the important case that $R$ is supertropical the condition (NQL) holds iff all principal ideals in $eR$ are unbounded with respect to the total ordering of $eR$. In particular, the “multiplicatively unbounded supertropical semirings” appearing in [6, §4] have NQL.

(b) If $R$ is any entire antiring, then the polynomial ring $R[t]$ in one variable (and so in any set of variables) has NQL.

(c) The polynomial function semirings over supersemirings appearing in [7, §4] have NQL.

Lemma 6.15. Assume that $(U, n(\gamma))$ is indecomposable. Let $a, c \in R \setminus \{0\}$. Then the tensor product $U \otimes_b \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix}$, taken with respect to $b = \begin{pmatrix} 2a & c \\ c & 0 \end{pmatrix}$, is indecomposable.

Proof. By formula (5.26)

$$\gamma \otimes_b \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} a & c \\ c & 0 \end{pmatrix} \otimes_{2, \gamma} n(\gamma).$$

Now Lemma 6.13 with $(V, q) := (U, n(\gamma))$ gives the claim. \hfill \Box

We are ready for the main result of this section. Recall that $U := (U, \gamma)$.

Theorem 6.16. Assume that $R$ has NQL. Assume furthermore that both $(U, n(\gamma))$ and the quadratic free module $V = (V, q)$ are indecomposable, and $U \neq \{0\}$, $V \neq \{0\}$. Assume moreover that $V$ is indecomposable. Let $b$ be a balanced companion of $q$. Then the quadratic module $U \otimes_b V := (U \otimes V, \gamma \otimes_b q)$ is indecomposable, except in the case that $\gamma$ is alternate, $q$ is diagonally zero, $U$ and $V$ contain only even cycles with respect to $\gamma$ and $b$. Then $U \otimes_b V$ has exactly two indecomposable components, and these coincide with the indecomposable components of $U \otimes V$ with respect to $\gamma \otimes b$, and also with respect to $\gamma \otimes b_f$.

Proof. Of course, indecomposability of $(U, n(\gamma))$ implies indecomposability of $(U, \gamma)$. As before, let $\tilde{q} := \gamma \otimes_b q$. We distinguish three cases.

1) Assume that $V^+ \neq \{0\}$, i.e., there exist anisotropic base vectors in $V$. Our claim is that all elements of $\mathcal{B}_1 \otimes \mathcal{B}_2$ are equivalent, whence $U \otimes_b V$ is indecomposable.

We choose $\eta_0 \in \mathcal{B}_1^+$. By Lemma 6.12.b, the module $(U \otimes \eta_0, \tilde{q}) := (U \otimes \eta_0, \tilde{q} | U \otimes \eta_0)$ is indecomposable, and thus all elements of $\mathcal{B}_1 \otimes \eta_0$ are equivalent.

Let $\varepsilon \otimes \eta \in \mathcal{B}_1 \otimes \mathcal{B}_2$. We verify the equivalence of $\varepsilon \otimes \eta$ with some element of $\mathcal{B}_1 \otimes \eta_0$, and then will be done. If $\gamma(\varepsilon, \varepsilon) \neq 0$, then by Lemma 6.12.a, all elements of $\varepsilon \otimes \mathcal{B}_2$ are equivalent, whence $\varepsilon \otimes \eta \sim \varepsilon \otimes \eta_0$. Assume now that $\gamma(\varepsilon, \varepsilon) = 0$. Since $(U, \gamma)$ is indecomposable, there exists some $\varepsilon' \in \mathcal{B}_1$ with $c := \gamma(\varepsilon', \varepsilon) \neq 0$. Let $a := \gamma(\varepsilon', \varepsilon')$. We choose a base $\eta_1, \ldots, \eta_n$ of $V$, assuming for notational convenience that $V$ has finite rank. By Example 5.10,

$$(Re' + R\varepsilon) \otimes_b V = \begin{bmatrix} aq & cq \\ 0 & 0 \end{bmatrix}$$
with respect to the base $\varepsilon' \otimes \eta_1, \ldots, \varepsilon' \otimes \eta_n, \varepsilon \otimes \eta_1, \ldots, \varepsilon \otimes \eta_2$. Now Lemma 6.13 tells us that $(R\varepsilon' + R\varepsilon) \otimes_b V$ is indecomposable, whence all elements $\varepsilon \otimes \eta, \varepsilon' \otimes \eta'$ with $\eta, \eta' \in \mathcal{B}_2$ are equivalent. In particular, $\varepsilon \otimes \eta \sim \varepsilon' \otimes \eta_0$.

2) Assume that $U^+ \neq \{0\}$, i.e., there exist an anisotropic base vector in $U$ with respect to $n(\gamma)$. Our claim again is that all elements of $\mathcal{B}_1 \otimes \mathcal{B}_2$ are equivalent, whence $U \otimes_b V$ is indecomposable. We choose $\varepsilon_0 \in \mathcal{B}_1^+$, and then know by Lemma 6.12.a that all elements of $\varepsilon_0 \otimes \mathcal{B}_2$ are equivalent.

Let $\varepsilon \otimes \eta \in \mathcal{B}_1 \otimes \mathcal{B}_2$ be given. We verify equivalence of $\varepsilon \otimes \eta$ with some element of $\varepsilon_0 \otimes \mathcal{B}_2$, and then will be done. If $q(\eta) \neq 0$, then by Lemma 6.12.a all elements of $\mathcal{B}_1 \otimes \eta$ are equivalent, and thus $\varepsilon \otimes \eta \sim \varepsilon_0 \otimes \eta$.

Hence, we may assume that $q(\eta) = 0$. Since $(V, q)$ is indecomposable, there exists some $\eta' \in \mathcal{B}_2$ with $c := b(\eta, \eta') \neq 0$. Let $a := q(\eta')$. Then

$$(R\eta' + R\eta, q) = \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix}.$$ 

Let $b' := b(R\eta' + R\eta) = \begin{bmatrix} a & c \\ c & 0 \end{bmatrix}$. Then we see from (5.25) that

$$\gamma \otimes b' \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} an(\gamma) & c\gamma \\ 0 & 0 \end{bmatrix}. $$

By Lemma 6.15, this quadratic module is indecomposable, whence all elements $\varepsilon \otimes \eta, \varepsilon' \otimes \eta'$ with $\varepsilon, \varepsilon' \in \mathcal{B}_1$ are equivalent. In particular, $\varepsilon \otimes \eta \sim \varepsilon_0 \otimes \eta'$.

3) The remaining case: $U = U^0$, and $V = V^0$, i.e., $\gamma$ is alternate and $q$ is diagonally zero. Now $(U \otimes V, \tilde{q})$ is rigid. By Theorem 6.8, the indecomposable components of $(U \otimes V, \tilde{q})$ coincide with those of $(U \otimes V, (\gamma \otimes b)_f)$. But $\tilde{q}$ has only one companion, whence $(\gamma \otimes b)_f = \gamma \otimes b = \gamma \otimes b_f$. Invoking Theorem 6.4, we see that the assertion of the theorem also holds in the case under consideration, where $\gamma$ is alternate and $b$ is diagonally zero. \hfill \Box

In general, let $\{U_i \mid i \in I\}$ denote the set of indecomposable components of $(U, n(\gamma))$. Then

$$U \otimes_b V = \bigoplus_{i \in I} U_i \otimes_b V$$

by Proposition 5.6, whence, applying Theorem 6.16 to each summand $U_i \otimes_b V$, we obtain a complete list of all indecomposable components of $U \otimes_b V$. In particular, if $q$ is not diagonally zero, or if $(V, b)$ contains an odd cycle, then the $U_i \otimes_b V$ themselves are the indecomposable components of $U \otimes_b V$. 

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