Theoretical properties of the sample generalized codifference function of stable moving average process

I Kharisudin¹* and D Rosadi²
¹Department of Mathematics FMIPA Universitas Negeri Semarang, Indonesia
²Department of Mathematics FMIPA Universitas Gadjah Mada, Indonesia

*Corresponding author: iqbalkharisudin@mail.unnes.ac.id

Abstract. The generalized codifference function as a dependence measure for stationary processes with infinite variance has been proposed as a generalization of the autocorrelation function. In this paper we investigate the theoretical properties of estimator of generalized codifference function of stable moving average process. Some theoretical properties of the sample codifference function of moving average process for small order are discussed.

1. Introduction
Various empirical studies typically show that most of the financial data are leptokurtic (i.e. heavy on the tail and peaked around the center). In other words, empirically derived fact that the chances of occurrence of extreme events and the variations that occur in the data is larger than can be modeled by a normal distribution. Thus, if a practitioner uses a financial model based on the normal distribution, there is a great risk of financial loss due to the emergence of extreme events and the variations in the data that cannot be modeled by this distribution. One of the powerful distributions that can represent the characteristics of such data is the stable distribution (see e.g., Rachev and Mittnik [1] and Rachev et al. [2]). Stable distribution with the normal distribution as a special case of this distribution class, is one of a relatively popular distribution for modeling leptokurtic data [3, 4]. In this context, various empirical studies especially in the economics and finance field showed that the stable non-normal distribution is more suitable for modeling various financial data such as asset returns [3-8].

Most statistical models require the existence of second order moments or distribution with finite variance. Based on the second order moments, the dependencies structure of the model can be assessed. It notice that conditions of the second moments or variances of stable distribution depends on characteristic exponent parameters α. The second moments of stable random variable exist only for α = 2 (Gaussian case) and for 0 < α < 2, one can not use the covariance functions to describe the dependence structure. It is known that the covariance function can completely describe the dependence structure of Gaussian distributed random vectors. Some generalization of the autocovariance function as dependence measure of process with infinite variance have been proposed in the literature, e.g., the
autocovariation (Samorodnitsky and Taqqu [9]) and the codifference function (see e.g., [7], Kokoszka and Taqqu [10], Wylomańska et al. [11], Rosadi [12], and Rosadi and Deistler [6]).

2. Generalized Codifference Function

The generalized codifference function (GCF) of the stationary processes \( \{X_t\} \) at lag \( k \) as proposed in [10] is defined as

\[
\tau(s, -s; k) = -\ln E \exp(is(X_{t+k} - X_t)) + \ln E \exp(isX_{t+k}) + \ln E \exp(-isX_t)
\]

where \( s \in \mathbb{R} \) and \( k \in \mathbb{Z} \). Because the characteristic function always exists, the codifference function requires no moment conditions for the original process \( \{X_t\} \). In the Gaussian case, the codifference function is proportional to the covariance function, i.e., \( \tau(s, -s; k) = -s^2 \gamma(k) \), where \( \gamma(\cdot) \) denotes the covariance function of the stationary process \( \{X_t\} \). Moreover, by defining the normalized codifference function \( I(k) \) as

\[
I(k) = \frac{\tau(s, -s; k)}{\tau(s, -s; 0)}
\]

one directly obtains \( I(k) = \rho(k) \) in the Gaussian case and it does not depend on \( s \), where \( \rho(k) \) denotes the correlation function.

3. Estimation of the GCF

As the generalized codifference function is defined via characteristic function, it can be estimated by empirical characteristic function. Given a sample \( X_1, X_2, ..., X_N \), a consistent estimator for the generalized codifference function at lag \( k \) \( \in \mathbb{Z} \) can be defined for \( s \in \mathbb{R} \), as in [13]

\[
\hat{\tau}(s, -s; k) = \sqrt{\frac{N-k}{N}} \times (-\ln \phi(s, -s; k) + \ln \phi(s, 0; k) + \ln \phi(0, -s; k))
\]

where for \( u, v \in \mathbb{R} \)

\[
\phi(u, v; k) = \begin{cases} (N-k)^{-1} \sum_{t=1}^{N-k} \exp(i(uX_{t+k} + vX_t)) & \text{if } k \geq 0 \\ (N+k)^{-1} \sum_{t=1}^{N+k} \exp(i(uX_{t-k} + vX_t)) & \text{if } k < 0 \\ \end{cases}
\]

Accordingly, \( \hat{I}(k) = \frac{\hat{\tau}(s, -s; k)}{\hat{\tau}(s, -s; 0)} \) can be used as the estimator of the normalized codifference \( I(k) \).

The asymptotic properties of the sample generalized codifference function for a class of linear time series models was investigated in [13]. It can be shown that for a stationary linear process \( X_t = \sum_{j=0}^\infty c_j \varepsilon_{t-j} \) where the coefficients \( c_j \)’s are real and majored by geometrically bounded weights and \( \{\varepsilon_t\} \) is i.i.d \( \mathcal{S} \alpha \mathcal{S} \), by applying Theorem 1 in [13], we obtain

\[
\hat{\tau}(s, -s; k) \rightarrow \tau(s, -s; k)
\]

where the consistency is in the probability sense.

We summarize the following results from [13] regarding the asymptotic behavior of the sample generalized codifference function for a class of linear processes. Let \( X_t = \sum_{j=0}^\infty c_j \varepsilon_{t-j} \) be a stationary symmetric \( \alpha \)-stable linear processes. Then for \( h \in \mathbb{N}, s \in \mathbb{R} \),

\[
\left[ \frac{\text{Re} \hat{I}(s, 1)}{\text{Im} \hat{I}(s, 1)}, \frac{\text{Re} \hat{I}(s, 2)}{\text{Im} \hat{I}(s, 2)}, ..., \frac{\text{Re} \hat{I}(s, h)}{\text{Im} \hat{I}(s, h)} \right]^T
\]

is

\[
A N \left[ \frac{l_r \hat{I}(1)}{l_r 0}, \frac{l_r \hat{I}(2)}{l_r 0}, ..., \frac{l_r \hat{I}(h)}{l_r 0} \right]^T \frac{1}{N^2} W
\]
where Re and Im stand for the real and imaginary parts, \( l_r = [1,1,\ldots,1]^T \in \mathbb{R}^r \), and the \((i, j)\)th element of the matrix \( W \) is
\[
W_{ij} = \begin{bmatrix}
D_{11}^{11} m_{00}^{R} D_{11}^{11} + D_{l(i+1)}^{11} m_{00}^{R} D_{j(j+1)}^{11} + D_{l(i+1)}^{11} m_{0}^{R} D_{j(j+1)}^{11} + D_{l(i+1)}^{11} m_{0}^{R} D_{j(j+1)}^{11}
& 0_r \\
0_r & D_{l(i+1)}^{11} m_{0}^{R} D_{j(j+1)}^{11}
\end{bmatrix}
\]
and for \( l = 1,\ldots,h \),
\[
D_{l1}^{11} = \text{diag} \left[ \frac{-I(l)}{\tau(s_1,-s_1;0)}, \frac{-I(l)}{\tau(s_2,-s_2;0)}, \ldots, \frac{-I(l)}{\tau(s_r,-s_r;0)} \right]
\]
\[
D_{l(i+1)}^{11} = \text{diag} \left[ \frac{1}{\tau(s_1,-s_1;0)}, \frac{1}{\tau(s_2,-s_2;0)}, \ldots, \frac{1}{\tau(s_r,-s_r;0)} \right]
\]
Here \( \text{diag} [\cdot] \) stands for a diagonal matrix. \( m_{pq}, p, q \geq 0 \) is the \((p, q)\)th block element of covariance matrix
\[
M = \left[ \lambda L_2^{P} V_{pq} L_2^{q} A^T \right]_{p, q=0,\ldots,h'}
\]
where
\[
\lambda = \left( L_1 \otimes \lambda_1 \quad 0 \right)
\]
with \( L_r \) denotes the matrix identity of size \( r \) and \( \lambda_1 = (1\ 1 - 1) \). Inside the covariance matrix (8),
\[
V_{pq} = \begin{pmatrix}
V_{pq}^{RR} & V_{pq}^{RI} \\
V_{pq}^{IR} & V_{pq}^{II}
\end{pmatrix},
\]
where
\[
V_{pq}^{RR} = \text{cov} \left( \begin{pmatrix}
\text{Re} \left( \phi_1(s_i, p) \right) \\
\text{Re} \left( \phi_2(s_i, p) \right) \\
\text{Re} \left( \phi_3(s_i, p) \right)
\end{pmatrix}, \begin{pmatrix}
\text{Re} \left( \phi_1(s_i, q) \right) \\
\text{Re} \left( \phi_2(s_i, q) \right) \\
\text{Re} \left( \phi_3(s_i, q) \right)
\end{pmatrix} \right).
\]
Here \( V_{pq}^{RR} \) and \( V_{pq}^{II} \) and \( V_{pq}^{IR} \) denote the partitions of \( V_{pq} \) which correspond to the real and the imaginary components, respectively. The components \( V_{pq}^{II} \) and \( V_{pq}^{IR} \) are defined similarly as \( V_{pq}^{RR} \). In (10), \( \phi_1, \phi_2, \phi_3 \), are
\[
\phi_1(s_i, p) = \frac{1}{N} \sum_{t=1}^{N} \exp \left( -is_i X_{t} \right)
\]
\[
\phi_2(s_i, p) = \frac{1}{N} \sum_{t=1}^{N} \exp \left( is_i X_{t+p} \right)
\]
\[
\phi_3(s_i, p) = \frac{1}{N} \sum_{t=1}^{N} \exp \left( is_i (X_{t+p} - X_t) \right)
\]
\( L_2^k \) defines as
where $d^k$ is diagonal matrix with elements are also diagonal matrix $d_i^k, i = 1, \ldots, r$ and the elements of $d_i^k$ are

\[
d_i^k(1,1) = (\text{Re}\Phi(0,-s_i;k))^{-1}
\]

\[
d_i^k(2,2) = (\text{Re}\Phi(s_i,0;k))^{-1}
\]

\[
d_i^k(3,3) = (\text{Re}\Phi(s_i,-s_i;k))^{-1},
\]

and equal 0 for otherwise. Notice that $\Phi(u,v;k) = E\left\{\exp\left(i(uX_{t+k} - vX_t)\right)\right\}$ for any $u, v \in \mathbb{R}$.

4. Asymptotic Property of the Sample GCF

Let us consider the univariate discrete time process $\{X_t, t \in \mathbb{Z}\}$ which is a moving average process of order 1 (MA(1)) with symmetric $\alpha$-stable innovation, i.e.,

\[
X_t = \varepsilon_t + c\varepsilon_{t-1}
\]  

(12)

Here $\varepsilon_t$ is i.i.d. symmetric $\alpha$-stable (S$\alpha$S) distributed, i.e., it has a characteristic function of the form

\[
E \exp(i\varepsilon) = \exp(-\sigma^\alpha |s|^\alpha)
\]

where $\alpha$ denotes the index of stability ($0 < \alpha \leq 2$) and $\sigma \geq 0$ denotes the scale parameter. In the following theorem, we derive the limiting distribution property of the sample normalized codifference function for moving average (MA) process of order 1 with symmetric $\alpha$ stable (S$\alpha$S) innovation [see 13].

**Theorem 1.** Let $\{X_t, t \in \mathbb{Z}\}$ is MA(1) process (12). The limiting distribution of the sample normalized codifference functions for $k > 1$ is

\[
\text{Re}\{s,k\} \sim AN(0, N^{-1}W_1),
\]

\[
\text{Im}\{s,k\} \sim AN(0, N^{-1}W_2),
\]

(13)

(14)

where the $(i,j)$th elements of the matrix $W_1$ and $W_2$ are

\[
W_1(i,j) = \frac{f_{ij}}{g_{ij}} \quad \text{and} \quad W_2(i,j) = \frac{h_{ij}}{g_{ij}}, i,j = 1, \ldots, r
\]  

(15)

with

$$
f_{ij} = e^{\sigma^\alpha(\sqrt{|s_i|}\alpha + \sqrt{|s_j|}\alpha - |s_i-s_j|^\alpha)(1+|\varepsilon|^\alpha)} \left\{ \frac{1}{2} e^{\sigma^\alpha(\sqrt{|s_i|}\alpha + \sqrt{|s_j|}\alpha - |s_i-s_j|^\alpha)(1+|\varepsilon|^\alpha)} - 1 \right\} + 1, \quad h_{ij} = e^{\sigma^\alpha(\sqrt{|s_i|}\alpha + \sqrt{|s_j|}\alpha - |s_i-s_j|^\alpha)(1+|\varepsilon|^\alpha)} \left\{ \frac{1}{2} e^{\sigma^\alpha(\sqrt{|s_i|}\alpha + \sqrt{|s_j|}\alpha - |s_i-s_j|^\alpha)(1+|\varepsilon|^\alpha)} + 1 \right\} - e^{\sigma^\alpha(\sqrt{|s_i|}\alpha + \sqrt{|s_j|}\alpha - |s_i-s_j|^\alpha)(1+|\varepsilon|^\alpha)} \left\{ \frac{1}{2} e^{\sigma^\alpha(\sqrt{|s_i|}\alpha + \sqrt{|s_j|}\alpha - |s_i-s_j|^\alpha)(1+|\varepsilon|^\alpha)} + 1 \right\} 
$$

and

\[
g_{ij} = 4\sigma^2\alpha|s_i|\alpha|s_j| (1 + |\varepsilon|^\alpha)^2.
\]

**Proof.** According to (5), we obtain that $w_{kk}, k > q$ is reduced to

\[
w_{kk} = D_{k(k+1)}m_{kk}D_{k(k+1)}
\]

(16)

where matrix $D_{k(k+1)}$ as given in (7), with
\[ D_{k(k+1)}^{11} = \text{diag} \left[ \frac{1}{-2\sigma|s_1|^2(1+|c|^2)}, \ldots, \frac{1}{-2\sigma|s_1|^2(1+|c|^2)} \right] \] (17)

and matrix \( m_{kk} \) as given in (8), i.e.,

\[ m_{kk} = \lambda L_2^k V_{kk} L_2^k \lambda^T \] (18)

where \( \lambda \) as given in (9), with elements of the matrix \( L_2^k \) and covariance matrix \( V_{kk} \) given below.

The elements of \( L_2^k \) as given in (11), where elements \( d_i^k, i = 1, \ldots, r \) are

\[ d_i^k(1,1) = \rho^\alpha_{|s_i|^2(1+|c|^2)}, \quad d_i^k(2,2) = \rho^\alpha_{|s_i|^2(1+|c|^2)}, \quad \text{and} \quad d_i^k(3,3) = e2\sigma|s_i|^2(1+|c|^2). \]

In order to find the component of \( V_{kk} \), we write

\[ V_{kk}^{RR}(i,j) = \text{cov} \left( \text{Re}\phi_m(s_i,k), \text{Re}\phi_n(s_j,k) \right) \]

And

\[ V_{kk}^{II}(i,j) = \text{cov} \left( \text{Im}\phi_m(s_i,k), \text{Im}\phi_n(s_j,k) \right) \]

denote as \((i,j)\)th block elements of \( V_{kk}^{RR} \) and \( V_{kk}^{II} \) respectively, with their component as follows

\[ V_{pq}^{RR}(1,1) = \frac{1}{2} \left( e^{-\sigma|s_i-s_j|^2} - e^{-\sigma|s_i-s_j|^2} - e^{-\sigma|s_i+s_j|^2} + e^{-\sigma|s_i+s_j|^2} \right). \]

\[ V_{pq}^{RR}(1,3) = e^{-\sigma|s_i|^{2}} + e^{-\sigma|s_i|^{2}} - e^{-\sigma|s_i|^{2}} + e^{-\sigma|s_i|^{2}} - e^{-\sigma|s_i+s_j|^2} + e^{-\sigma|s_i+s_j|^2}. \]

\[ V_{pq}^{RR}(3,1) = e^{-\sigma|s_i|^{2}} - e^{-\sigma|s_i|^{2}} - e^{-\sigma|s_i|^{2}} - e^{-\sigma|s_i|^{2}} + e^{-\sigma|s_i+s_j|^2} + e^{-\sigma|s_i+s_j|^2}. \]

For \( p = q \), we find

\[ V_{pq}^{RR}(3,3) = \frac{1}{2} e^{-2\sigma|s_i-s_j|^2} + \frac{1}{2} e^{-2\sigma|s_i-s_j|^2} + e^{-\sigma|s_i|^{2}} - 3e^{-\sigma|s_i|^{2}}. \]

The components of \( V_{pq}(m,n) \) are:

\[ V_{pq}^{II}(1,1) = \frac{1}{2} \left( e^{-\sigma|s_i-s_j|^2} - e^{-\sigma|s_i-s_j|^2} \right). \]

\[ V_{pq}^{II}(1,3) = V_{pq}^{II}(3,1) = \frac{1}{2} e^{-2\sigma|s_i-s_j|^2} + e^{-\sigma|s_i+s_j|^2} + e^{-\sigma|s_i+s_j|^2} + e^{-\sigma|s_i+s_j|^2}. \]

As from (18), we find matrix \( m_{kk}^{RR} \) as:

\[ m_{kk}^{RR}(i,j) = \lambda_1 d_i^k V_{kk}^{RR}(i,j) d_j^k \lambda_1^T \]

\[ = e^{\alpha\rho^\alpha_{|s_i+|s_j|^2}} \left\{ \frac{1}{2} e^{\alpha\rho^\alpha_{|s_i+|s_j|^2}} - e^{\alpha\rho^\alpha_{|s_i+|s_j|^2}} \right\} + 1. \]
and matrix $m_{kk}^{ij}(i,j) = e^{\sigma_a^{\alpha}(|s_i + s_j|^{\alpha} - |s_i - s_j|^{\alpha})(1 + |c|^{\alpha})} \left( \frac{1}{2} e^{\sigma_a^{\alpha}(|s_i + s_j|^{\alpha} - |s_i - s_j|^{\alpha})(1 + |c|^{\alpha})} + 1 \right) - e^{\sigma_a^{\alpha}(|s_i|^{\alpha} + |s_j|^{\alpha})(1 + |c|^{\alpha})} \left( \frac{1}{2} e^{\sigma_a^{\alpha}(|s_i|^{\alpha} + |s_j|^{\alpha})(1 + |c|^{\alpha})} + 1 \right)$.

By using (16) we have completes the proof.

References
[1] Rachev S T and Mittnik S 2000 Stable Paretian Models in Finance (Wiley)
[2] Rachev S T Menn C and Fabozzi F J 2005 Fat-Tailed and Skewed Asset Return Distributions (Wiley)
[3] Laha A K and Raja A P 2019 Modeling Commodity Market Returns: The Challenge of Leptokurtic Distributions. In Advances in Analytics and Applications (Singapore: Springer)
[4] Mwaniki I J 2019 J. Appl. finance bank. 9 1
[5] Sousa J and Sousa R M 2017 J. Int. Financial Mark. Inst. Money 50 204
[6] Oden J, Hurt K and Gentry S 2017 Res. Bus. Manag. 4 13
[7] Broda S A, Krause J and Paolella M S 2018 Econom. stat. 8 184
[8] Liu R, Shao Z, Wei G and Wang W 2017 J. Account. Bus. Finance Res. 1 71
[9] Samorodnisky G and Taqqu M S 1994 Stable Non Gaussian Processes: Stochastic Models with Infinite Variance (Chapman & Hall)
[10] Kokoszka P S and Taqqu M S 1995 Stoch. Process. their Appl. 60 19
[11] Wyłomańska A, Chechkin A and Gajda J 2015 Physica A 421 412
[12] Rosadi D 2009 Comput. Stat. Data Anal. 53 4516
[13] Rosadi D and Deistler M 2011 Metrika 73 395
[14] Kharisudin I, Rosadi D, Abdurakhman and Suhartono 2016 Far East J. Math. Sci. 99 1297