Remarks on the “non-canonicity puzzle”: Lagrangian symmetries of the Einstein-Hilbert action

N. Kiriushcheva, P.G. Komorowski and S.V. Kuzmin

The Department of Applied Mathematics,
The University of Western Ontario,
London, Ontario, Canada, N6A 5B7

(Dated: July 2, 2013; Received)

Abstract

Given the non-canonical relationship between variables used in the Hamiltonian formulations of the Einstein-Hilbert action (due to Pirani, Schild, Skinner (PSS) and Dirac) and the Arnowitt-Deser-Misner (ADM) action, and the consequent difference in the gauge transformations generated by the first-class constraints of these two formulations, the assumption that the Lagrangians from which they were derived are equivalent leads to an apparent contradiction that has been called “the non-canonicity puzzle”. In this work we shall investigate the group properties of two symmetries derived for the Einstein-Hilbert action: diffeomorphism, which follows from the PSS and Dirac formulations, and the one that arises from the ADM formulation. We demonstrate that unlike the diffeomorphism transformations, the ADM transformations (as well as others, which can be constructed for the Einstein-Hilbert Lagrangian using Noether’s identities) do not form a group. This makes diffeomorphism transformations unique (the term “canonical” symmetry might be suggested). If the two Lagrangians are to be called equivalent, canonical symmetry must be preserved. The interplay between general covariance and the canonicity of the variables used is discussed.

*Electronic address: nkiriush@uwo.ca
†Electronic address: pkomoro@uwo.ca
‡Electronic address: skuzmin@uwo.ca
I. INTRODUCTION

An analysis of the two oldest Hamiltonian formulations of the second-order Einstein-Hilbert (EH) action for metric General Relativity (i.e. Pirani, Schild, and Skinner (PSS) [1]; and Dirac [2]), was completed in [3, 4]. Using the approach of Castellani [5], it was demonstrated that first-class constraints produce a generator for the diffeomorphism invariance\(^1\) - the known gauge symmetry of the Einstein-Hilbert (EH) action. This outcome contradicts the result of the Arnowitt, Deser, and Misner formulation (ADM, or geometrodynamics) [7, 8] where the constraints lead to a different symmetry, one which is known by many names: “spatial diffeomorphism”, “special induced diffeomorphism”, “field-dependent diffeomorphism”, “foliation preserving diffeomorphism”, “one-to-one correspondence”, “one-to-one mapping” (see [4] and references therein). It was shown [4, 9] that the PSS and Dirac Hamiltonians are related by a canonical transformation of the phase-space variables; while the transformation from the Dirac to ADM Hamiltonian is not a canonical change of variables. Canonical transformations must preserve all properties of the Hamiltonian. Because gauge symmetry is an important characteristic of a constrained system, a difference between the symmetries of the PSS (or Dirac) and ADM Hamiltonian formulations indicates that a non-canonical relationship exists between the two formulations. This truism was explicitly confirmed in [4, 9] by the calculation of Poisson brackets (PBs) among the phase-space variables. Waxing poetic, the term “the non-canonicity puzzle” was coined in [10] to describe the results of [4, 9]; but in [11], using more direct language, it is called “the contradiction that again witnesses about the incompleteness of the theoretical foundations”. The source of the “puzzle” or “contradiction” lies in finding how to reconcile the non-equivalence of the two Hamiltonian formulations with their corresponding Lagrangian formulations when “it is supposed” [12] or “it is believed” (as in [11]) “that each of them is equivalent to the Einstein (Lagrangian) formulation” [11, 12].

This belief might lead one to conclude that Dirac’s Hamiltonian formulation of constrained systems is incomplete [11], and proposals ought to follow on how to redefine the primary constraints, on how to use boundary terms, on how to have “two non-canonical transformations that compensate each other” [10], and on how to modify the PBs through

\(^1\) We understand diffeomorphism invariance \((\text{diff})\) as “active” [6] (p. 62) when “coordinates play no role”, i.e. transformations of fields written in the same coordinate system.
the extension of phase space\(^2\) \(^{11, 12}\). But such approaches immediately give rise to a general question: why are such manipulations needed for one formulation (i.e. ADM), but not for the others (i.e. PSS and Dirac)? Hamiltonian solutions of the “puzzle” proposed in \(^{10–12}\) deserve a more detailed discussion; but in this article we shall limit ourselves to the consideration of the Lagrangian formulation and the symmetries of the EH\(^3\) and ADM Lagrangians. We note that the literature on the Hamiltonian formulation of constrained systems contains various treatments with claims that the variables, which appear in the original Lagrangian, have different properties (i.e. they might be dynamical and non-dynamical; or some variables can be treated as Lagrange multipliers; or some variables are canonical, but some not; or some have conjugate momenta, but others do not need them; or that secondary first-class constraints can be “promoted” to primary first-class constraints, et cetera (see \(^4\) and references therein)). This plethora of treatments through which different results may be obtained, depending on the ingenuity of an investigator, leave an impression that the Hamiltonian method for constrained systems is an art, not a defined and unambiguous procedure (or, as suggested in \(^{10–12}\) that it is not yet a procedure).

Instead of relying upon the first-class constraints, as in the Dirac procedure, one may use the Lagrangian formulation of a singular system to derive symmetries from the Noether identities. The differential relationships among the Euler-Lagrange derivatives are linked to the gauge transformations; thus the treatment of the Lagrangian is free from the artistic approaches that have been applied to the Hamiltonian. Lagrangian symmetries describe a transformation in which all fields are treated on the same footing, irrespective of the name assigned to them (e.g. “dynamical” and “non-dynamical”, et cetera). The differential identities (DIs) involve the Euler-Lagrange derivatives with respect to all fields. We must emphasize that the Lagrangian and Hamiltonian methods should have the same mathematical rigor; but the main reason for us to consider Lagrangian symmetries is to aid in developing a criterion for the equivalence of two Lagrangians in light of this “puzzle”.

In \(^4\), an analysis of the Hamiltonian formulations of Dirac and ADM was performed; it

\(^2\) Such an extension is obtained by replacing the classical EH action by “the effective action including gauge and ghost sectors” \(^{11, 12}\).

\(^3\) In PSS \(^{1}\), the gamma-gamma part of the EH action is considered; and Dirac in \(^2\) made some additional manipulations in this Lagrangian. In spite of these modifications, both formulations lead to exactly the same equations of motion as the original EH action; and the metric is an independent field-variable in all of them.
was concluded that if two Hamiltonian formulations are not related by a canonical transformation and if they have different symmetries (i.e. they are not equivalent), then the corresponding Lagrangians are also not equivalent, contrary to the “belief” which forms the basis of the “puzzle”. If two Lagrangians are not equivalent, then the results of [4, 9] are fully consistent, there is no puzzle, and the theoretical foundations are sound. The conclusion drawn that PSS and Dirac are not equivalent to the ADM formulation was based on the belief of the authors of [4] that Dirac’s Hamiltonian method for constrained systems is an unambiguous procedure, applicable to any theory, that leads to a unique symmetry that corresponds exactly to the symmetry present in the Lagrangian.

The change of variables used by ADM [7, 8] to go from the metric tensor $g_{\mu \nu}$ of the EH action (used in PSS and Dirac) to the ADM variables (lapse $N$, shift $N^i$ and space-space components of the metric tensor $\gamma_{km}$) is

$$N = (-g^{00})^{-1/2}, \quad N^i = -\frac{g^{0i}}{g^{00}}, \quad \gamma_{km} = g_{km},$$

(1)

which is invertible, but not covariant. It is this condition of invariability that some view as sufficient for the EH and ADM Lagrangians to be equivalent. But this conclusion is not an obvious one to make when dealing with singular Lagrangians. In the Hamiltonian formulation of a singular and covariant Lagrangian, gauge symmetries are derived from the first-class constraints; at the Lagrangian level, gauge symmetries are associated with the existence of the Noether identities [13]. The invariability of redefinition (1) allows one, starting from known transformations of one set of variables (e.g. $\delta_{\text{diff}}\{g_{\mu \nu}\}$), to find the transformations for another set (i.e. $\delta_{\text{diff}}\{N, N^i, \gamma_{km}\}$), and vice versa (i.e. from $\delta_{\text{ADM}}\{N, N^i, \gamma_{km}\}$ one finds $\delta_{\text{ADM}}\{g_{\mu \nu}\}$). For example, one can also check whether $\delta_{\text{ADM}}$, which is a symmetry of the ADM Lagrangian, is also a symmetry of the EH Lagrangian (i.e. whether the EH Lagrangian is invariant under transformations given by $\delta_{\text{ADM}}\{g_{\mu \nu}\}$). Direct calculation is difficult; but by the converse of Noether’s theorem [13] (i.e. if an action is invariant under some symmetry, then there exists the corresponding DI) and by assuming that $\delta_{\text{ADM}}\{g_{\mu \nu}\}$ is an invariance, one can find the corresponding DI and directly check it. A shorter ap-

---

4 We employ Greek letters for space-time indices: $\mu = 0, 1, 2, 3$. Latin letters for space indices: $k = 1, 2, 3$, and “0” for the time index.

5 For an English translation of Noether’s paper see [14].

6 Such constructions were described by Schwinger [15], see also [16].
approach is to connect a new DI to a known DI of the EH action; in addition, any DIs that are independent linear combinations of known DIs also describe symmetries. In such a way it is not difficult to demonstrate that \( \delta_{\text{ADM}} \{g_{\mu\nu}\} \) is indeed a symmetry of the EH action. Conversely, one starting from \( \delta_{\text{diff}} \{g_{\mu\nu}\} \) can find \( \delta_{\text{diff}} \{N, N^i, \gamma_{km}\} \) and check that it is a symmetry of the ADM Lagrangian by using its DI (e.g. see p. 17 of [17]). In a similar way, linear combinations of the DIs can be used to construct other transformations that will be symmetries of both the EH and ADM Lagrangians. But do these relationships prove the equivalence of the EH and ADM Lagrangians? One may also ask why it is that when using the Hamiltonian method, one particular symmetry follows from the constraint structure, but in the Lagrangian, we apparently have an infinity of equally good symmetries? Such a non-uniqueness should be a warning sign\(^7\).

The key concept that allows one to distinguish among the numerous symmetries due to the Lagrangian approach may be found in [11], where it is suggested that Dirac’s method is incomplete. According to [11], “the difference in the groups of transformations is the first indication to the inconsistency of the theory” (italic is ours). This key concept can be used to answer the question about how to classify all symmetries that can be constructed for one Lagrangian: which of the symmetries have group properties and thus constitute the “basic”, “true”, or “canonical” symmetries? This also allows one to compare two Lagrangian formulations by matching those symmetries that have group properties for each. A related question is: if only one symmetry has the property to form a group, is it the symmetry that the Hamiltonian formulation produces (or should produce)? At least for the EH Lagrangian, where diffeomorphism is a gauge symmetry with a group property [19], its Hamiltonian formulation [1, 2] leads exactly to this symmetry [3, 4] without any extension of Dirac’s procedure. Now consider going from one Lagrangian to another by performing some invertible change of variables. If the symmetry that had a group property in the original formulation ceases to have a group property in a new formulation, but another symmetry that did not have group properties in the original formulation “develops” group properties

\(^7\) Of course, these questions can be avoided assuming that “Hamiltonian dynamics is not completely equivalent to Lagrangian formulation of the original theory. In Hamiltonian formalism the constraints generate transformations of phase-space variables; however, the group of these transformations does not have to be equivalent to the group of gauge transformations of Lagrangian theory” [18]. Such an assumption eliminates a “non-canonicity puzzle” or, alternatively, it provides the solution: the EH and ADM Lagrangians are equivalent, but the corresponding Hamiltonians just happen to pick different symmetries.
in a new formulation, is this “the first indication to the inconsistency of the theory” \[11\] or is it proof that the two Lagrangian formulations are not equivalent? Perhaps a change of variables that creates such a result should be called “non-canonical”, in analogy with Hamiltonian terminology where a non-canonical change of variables also causes a change of symmetries \[4, 9\].

The investigation of the symmetries of the two Lagrangians (EH and ADM), which are related to each other by the change of variables \[11\], is a less cumbersome calculation to perform compared with the Hamiltonian method; the same is true of the study of whether a symmetry with a group property, of one Lagrangian, is also a symmetry with a group property, of the other Lagrangian. But for non-covariant changes such an investigation is complicated. In particular, to identify a symmetry with a group property, the commutators of two transformations must be considered, and in the case of field-dependent structure functions, higher, nested commutators are needed. For non-covariant variables these calculations must be performed separately for different fields, and the consistency of different commutators must be checked. In this article we discuss the simple parts of the calculation that one may perform in a quasi-covariant form, and consider two symmetries of the EH Lagrangian: transformations of the metric tensor $g_{\mu\nu}$ under $\text{diff}$ $(\delta_{\text{diff}} g_{\mu\nu})$ and under ADM transformations $(\delta_{\text{ADM}} g_{\mu\nu})$. We also compare their group properties. In the next Section we briefly review some results relevant to the $\text{diff}$ invariance of the EH action with an emphasis placed on the role of DIs, their direct connection to the form of the transformations, and the possible construction of additional symmetries by using combinations of DIs (i.e. the results that will be needed for a discussion of $\delta_{\text{ADM}} g_{\mu\nu}$). In Section III we demonstrate the invariance of the EH action under the ADM transformations $\delta_{\text{ADM}} g_{\mu\nu}$, and show that unlike $\text{diff}$, the $\delta_{\text{ADM}} g_{\mu\nu}$ do not constitute a group. More cumbersome calculations of the group properties of the same transformations of the ADM variables, $\delta_{\text{diff}} \{N, N^{i}, \gamma_{km}\}$ and $\delta_{\text{ADM}} \{N, N^{i}, \gamma_{km}\}$, for the ADM Lagrangian are in progress and will be reported elsewhere. In the Conclusion, we summarize our results and discuss the consequences of $\delta_{\text{diff}} \{N, N^{i}, \gamma_{km}\}$ and $\delta_{\text{ADM}} \{N, N^{i}, \gamma_{km}\}$ either having or not having group properties, in all possible combinations. Finally we comment on the role of covariance.
II. SYMMETRIES IN THE LAGRANGIAN APPROACH OF THE EINSTEIN-HILBERT ACTION

There are statements in the literature such as: “...one of the advantages of the Hamiltonian formulation is that one does not have to specify the gauge symmetries \textit{a priori}. Instead, the structure of the Hamiltonian constraints provides an essentially algorithmic way in which the correct gauge symmetry structure is determined automatically” [20]. We note that this is not a special or exclusive property of the Hamiltonian method\textsuperscript{8}. The Lagrangian approach also provides an algorithm, which is due to Noether’s second theorem for finding gauge symmetries [13, 14], that connects these symmetries with the DIs - combinations of Euler-Lagrange derivatives that are identically equal to zero (off-shell). The Hamiltonian method provides an algorithm for finding and classifying constraints (all first-class constraints are needed to find a symmetry); in the Lagrangian approach, DIs can be built using an iterative procedure. For the Einstein-Cartan (EC) action, which has richer symmetry properties than EH, such a construction was performed in [16]. In the same way, DIs can be built for the EH action [4]. The relative simplicity of such calculations is due to the covariance of the theories considered. It would be a much more complicated procedure to try to find DIs in non-covariant theories, or non-covariant DIs for covariant theories. Of course, for any theory for which there is no \textit{a priori} knowledge of the existence of gauge symmetries, it is unproductive to search for identities without preliminary analysis. The first step is to determine if a Lagrangian is singular, by evaluating its Hessian:

\[
H^{\alpha\beta} = \frac{\delta^2 L}{\delta \dot{Q}_\alpha \delta \dot{Q}_\beta},
\]

where \(\dot{Q}_\alpha\) are the time derivatives of \(Q_\alpha\) - the independent fields of the Lagrangian. If the determinant of the Hessian is zero, then the Lagrangian is singular; the rank of the Hessian is related to the number of independent DIs that can be found. It should be noted that singularity of the Lagrangian is a necessary condition to have a gauge symmetry, but not sufficient (the simplest example is the massive vector (Proca) field where the Lagrangian is singular, but has no gauge symmetry). The rank of the Hessian provides only an upper

\textsuperscript{8} It has to be admitted that applications of Hamiltonian methods can lead to very long and cumbersome calculations, which in some cases, are not straightforward.
bound on the maximum number of independent gauge symmetries. The Hessian is often written for velocities, even for the Lagrangian of covariant theories; but time is not special for covariant theories and singling it out is unnecessary.

In the Hamiltonian approach, knowledge of the first-class constraints is sufficient to restore gauge invariance: for example, by using the Castellani procedure \cite{5}. Although there are some modifications of the Castellani procedure, they must be used with care (see \cite{21}). Similarly for Lagrangians, if the DIs are known, then transformations can be found using the explicit connections of the DIs and the transformations \cite{13, 15}. The approach described for finding \textit{a priori} unknown Lagrangian symmetries is general; but for the EH action, a well-known covariant DI had already appeared, along with the EH action\textsuperscript{9} itself, in Hilbert’s paper \cite{23, 24}.

We shall briefly illustrate the application of this general procedure to the EH action. These results will be needed, used, and compared in the next Section, where the ADM symmetry is discussed. The Einstein-Hilbert action is \cite{25, 26}

\[ S_{EH} = \int L \, d^4x = \int \sqrt{-g} R \, d^4x, \tag{3} \]

where \( g = \det g_{\mu\nu}, \) \( L \) is the scalar density (Lagrangian density) and the Ricci scalar \( R, \) Ricci tensor \( R_{\mu\nu}, \) and Christoffel symbol \( \Gamma^\alpha_{\mu\nu} \) are:

\[ R = g^{\mu\nu} R_{\mu\nu}, \quad R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\alpha\nu}, \tag{4} \]

\[ \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left( g_{\beta\nu,\mu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta} \right). \tag{5} \]

The variational, Euler-Lagrange derivative (ELD) of the EH action is

\[ E^{\alpha\beta} = \frac{\delta L_{EH}}{\delta g_{\alpha\beta}} = \sqrt{-g} \left( \frac{1}{2} g^{\alpha\beta} R - R^{\alpha\beta} \right) = -\sqrt{-g} G^{\alpha\beta}, \tag{6} \]

where \( G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \) is the Einstein tensor.

It is not difficult to find the DI by using a general construction similar to that performed for the Einstein-Cartan action \cite{16}. Under the reasonable assumption that a covariant theory

\textsuperscript{9} So, it is not easy to recognize this due to Hilbert’s presentation and also complications with coupling to Mie’s electrodynamics. In addition, this identity was known before any connection was made to Euler-Lagrange derivatives of the EH action - this is simply the contracted Bianchi identity \cite{22}.

\textsuperscript{10} For an English translation see \cite{24}.
should also have covariant identities, and given the rank of the Hessian for the EH action, the DI follows almost immediately. The rank of the Hessian is six; and because the second-rank metric tensor has ten independent components, there should be four independent DIs. The four covariant identities that one can build from the ELDs, which are covariant symmetric second-rank tensor densities, consist of either four scalars or one vector. It is impossible to construct four scalars from the ELDs; but to find a true vector, one may take a covariant derivative of a second-rank tensor to yield:

\[ I^\mu = E_{\nu}^{\mu} = E_{\nu}^{\mu} + \Gamma_{\alpha\beta}^{\mu} E^{\alpha\beta} \equiv 0. \]  
(7)

By direct substitution, one can easily confirm that this combination is identically zero.

Schwinger’s paper [15] contains a description of how to construct the DIs from known gauge transformations; and in correspondence to Noether’s theorem, this process also applies in inverse order, through the converse relationship between DIs and transformations. One forms a scalar from a vector DI (7) by using the gauge parameters of appropriate tensorial dimension followed by equating the scalar to variations of the action. One writes

\[ \delta S_{EH} = \int \delta g_{\mu\nu} E^{\mu\nu} d^4x = \int \xi_{\mu} I^\mu d^4x , \]  
(8)

and performing integration by parts yields

\[ \int \xi_{\mu} I^\mu d^4x = \int \xi_{\mu} (E_{\nu}^{\mu} + \Gamma_{\alpha\beta}^{\mu} E^{\alpha\beta}) d^4x = \int \left( -\frac{1}{2} \xi_{\mu,\nu} - \frac{1}{2} \xi_{\nu,\mu} + \Gamma_{\mu\nu}^{\alpha} \xi_{\alpha} \right) E^{\mu\nu} d^4x , \]  
(9)

then one obtains

\[ \delta_{\text{diff}} g_{\mu\nu} = -\frac{1}{2} \xi_{\mu,\nu} - \frac{1}{2} \xi_{\nu,\mu} + \Gamma_{\mu\nu}^{\alpha} \xi_{\alpha} . \]  
(10)

Note that the coefficient \( \frac{1}{2} \) also appears when symmetries are restored in the Hamiltonian approach [3, 4] but this result is usually presented in a different form. The constant \( \frac{1}{2} \) can be incorporated into a gauge parameter without any effect on the results; and we will use the shorter form

\[ \delta_{\text{diff}} g_{\mu\nu} = -\xi_{\mu,\nu} - \xi_{\nu,\mu} + 2 \Gamma_{\mu\nu}^{\alpha} \xi_{\alpha} = -\xi_{\mu,\nu} - \xi_{\nu,\mu} , \]  
(11)

which is a manifestly covariant expression, a consequence of using a covariant DI. Knowledge of this transformation allows one to also find transformations for any combination built from the metric, for example,
\[ \delta_{\text{diff}} \Gamma^\alpha_{\mu \nu} = -\xi^\beta \Gamma^\alpha_{\mu \nu, \beta} + \Gamma^\beta_{\mu \nu} \xi^\alpha_{,\beta} - \Gamma^\alpha_{\mu \beta} \xi^\beta_{,\nu} - \Gamma^\alpha_{\nu \beta} \xi^\beta_{,\mu} - \xi^\alpha_{,\mu \nu} , \]  
\[ \delta_{\text{diff}} R_{\mu \nu} = -\xi^\rho R_{\mu \nu, \rho} - \xi^\rho_{,\mu} R_{\nu \rho} - \xi^\rho_{,\nu} R_{\mu \rho} , \quad \delta R = -\xi^\rho R_{,\rho} , \]  
and

\[ \delta_{\text{diff}} G_{\mu \nu} = -\xi^\rho G_{\mu \nu, \rho} - \xi^\rho_{,\mu} G_{\nu \rho} - \xi^\rho_{,\nu} G_{\mu \rho} . \]  

We note that gauge parameters in general, and the \( \xi_\mu \) of the EH action in particular, are field-independent, as was explicitly stated by Hilbert [23], Noether [13], and others, and by Rosenfeld, in the first discussion on the Hamiltonian formulation for a singular Lagrangian [27]. Further, the methods used to restore gauge symmetries, such as the Castellani approach [5], are also based on the condition that the gauge parameters be field-independent.

Our goal is to compare the diff transformations with the ADM transformations of the same EH action; therefore, transformations of the same variables (i.e. the metric tensor) under ADM are needed. For this purpose we find them from the transformations of ADM variables by using their connection to the metric (1). The lapse and shift are expressed in terms of contravariant components, so it is easier to find the transformations of contravariant components of the metric from the ADM transformations. Of course, transformations of covariant and contravariant components of the metric have a simple relationship due to \( g_\mu \nu g^{\nu \alpha} = \delta^\alpha_\mu \); but for our discussion, it is preferable to know the DIs that lead directly to \( \delta_{\text{diff}} g^{\mu \nu} \). There are a few ways to find a DI that is expressed in terms of a covariant ELD; we might use a DI that is already known, and consider

\[ I_\alpha = g_{\alpha \mu} I^\mu , \]  

which is also an identity. After performing some simple rearrangement, we obtain

\[ I_\alpha = g_{\alpha \mu} I^\mu = -2 (g^{\mu \nu} E_{\mu \alpha})_{,\nu} - g^{\mu \nu} E_{\mu \alpha} . \]  

Repeating steps (8) - (11) for a covariant DI, \( I_\alpha \), and using

\[ \delta S_{\text{EH}} = \int \delta g^{\mu \nu} E_{\mu \nu} d^4x = \int \xi_\alpha I_\alpha d^4x , \]  

11 For an English translation see [28].
we obtain

$$
\delta_{\text{diff}} g^{\mu \nu} = \xi^\nu g^{\mu \alpha} + \xi^\mu g^{\nu \alpha} - g^{\mu \nu} \xi^\alpha
$$

(18)
(here we also incorporated the constant $\frac{1}{2}$ into the gauge parameter).

But do these transformations form a group? To answer this question, the commutator of two transformations is needed (i.e. $[\delta_2, \delta_1]$) for which it should be possible to present the result in the form of a single transformation, but with a new parameter $\delta_{[1,2]}$:

$$
[\delta_2, \delta_1] g^{\mu \nu} = (\delta_2 \delta_1 - \delta_1 \delta_2) g^{\mu \nu} = \delta_{[1,2]} g^{\mu \nu} .
$$

(19)

This result was found by Bergmann and Komar [19] in the following form:

$$
\xi^\alpha_{[1,2]} = \xi^\beta_{1} \xi^\alpha_{1,\beta} - \xi^\beta_{2} \xi^\alpha_{2,\beta}.
$$

(20)

To shorten the notation, we henceforth eliminate the subscript $\text{diff}$. Because of the anti-symmetry of this expression, this combination is equivalent to one with covariant derivatives

$$
\xi^\alpha_{[1,2]} = \xi^\beta_{2} \xi^\alpha_{1,\beta} - \xi^\beta_{1} \xi^\alpha_{2,\beta} ;
$$

(21)

this form explicitly shows that the new parameter, $\xi^\alpha_{[1,2]}$, preserves its vector form.

Because there are no fields in parameter redefinition (20), it remains unaltered when we consider a double commutator, i.e.

$$
\xi^\alpha_{[[1,2],3]} = \xi^\beta_{3} \xi^\alpha_{1,3,\beta} - \xi^\beta_{2} \xi^\alpha_{2,3,\beta} .
$$

(22)

In general, the field-independence of a new parameter is a sufficient condition to have a group, but not a necessary condition. Should fields appear, additional calculations of the double commutators must be performed to find a definite answer. From (21) one may conclude that the $\text{diff}$ transformations form a group. Therefore the Jacobi identity follows for any transformation with group properties; that is

$$
([[\delta_2, \delta_1] , \delta_3] + [[\delta_3, \delta_2] , \delta_1] + [[\delta_1, \delta_3] , \delta_2]) g^{\mu \nu} \equiv 0 ,
$$

(23)

which is equivalent to a simple relationship for the gauge parameters of the double commutators,
\[ \xi^\alpha_{[[1,2],[3]]} + \xi^\alpha_{[[3,1],[2]]} + \xi^\alpha_{[[2,3],[1]]} \equiv 0. \quad (24) \]

Note that all of the above expressions (ELDs, DIs, transformations, and even the redefinition of parameters that support group properties) are written in manifestly covariant form; the expectation that the results must be covariant in a covariant theory can be used to obtain some solutions to avoid direct calculation (e.g. construction of a covariant DI). We can use these properties to find symmetries by using the Lagrangian approach; but a different method can be found in the literature that has the name “the Lagrangian approach” (see [29] and references therein). It proceeds by singling out one coordinate, time, followed by a long sequence of calculations to find symmetries. For the covariant theories discussed in [29] this approach has unnecessary over-complications; the Noether DI is used as an input in this method anyway. If the Noether DI is known, then to find a transformation requires a one-line calculation (as (8) - (11)), not the pages of calculation as outlined in [29]. This particular “Lagrangian approach” resembles the Hamiltonian one, and it may be of use to those who want to trace down explicit connections between the Lagrangian and Hamiltonian methods at different stages in the calculation.

We note that Lagrangian methods are general and unrestricted by the covariance of an action; therefore, all of the results above can be obtained without making reference to or taking guidance from covariance, although considerable difficulties may arise. But for any Lagrangian with an \textit{a priori} unknown gauge symmetry, one should be able to find a DI by using only ELDs of a given action.

According to Noether’s second theorem [13], there is a maximum number of independent DIs, but apparently there are no additional restrictions. Are the DIs (7) and (16) for the EH action and \( \delta_{\text{diff}} \) transformation (18) unique? Keeping covariance, there is little freedom to construct a new DI (see (18)) and its corresponding transformation; and for the EH action the covariant DI (7) is unique. But for the EC action there is greater freedom to construct covariant DIs (see [16]). If the restriction of covariance is lifted, then the number of new combinations of DIs and new gauge transformations become unlimited; they can be found by using a very simple manipulation, without the need for further calculation (of course this is true if only the transformations are of interest; but it could be a complicated task to find, for example, a commutator like (19), or to calculate the Jacobi identity). If the DIs are known (e.g. (16)), one can start to build combinations of them that are obviously also DIs. And
by using the approach of [15] one may obtain the corresponding transformations. Despite considerations of simplicity and the manifest covariance of the DIs and transformations, are all such transformations equally good? According to Noether’s theorem, which is a general result, the existence of a maximum number of independent DIs is an important characteristic of a singular Lagrangian. From the rank of the Hessian of the EH action, we know that the maximum number of independent DIs is four. So one can obtain four new DIs by using,

\[ \tilde{I}(\nu) = F^\mu_{(\nu)} (g_{\alpha\beta}) I_\mu, \quad (25) \]

where \( I_\mu \) is a known DI, \((\nu)\) is not a covariant index, just a numbering of the DI, and \( F^\mu_{(\nu)} (g_{\alpha\beta}) \) are some functionals of the metric that also need not be covariant. The only restriction on (25) is that the combinations be linearly independent, that is,

\[ \det \left| \frac{\partial \tilde{I}(\nu)}{\partial I_\mu} \right| \neq 0. \quad (26) \]

Using the approach of [15], one must consider combinations of these four new DIs, \( \tilde{I}(\nu) \), with four gauge functions, \( \varepsilon^{(\nu)} \); after performing an integration by parts, as in (8) - (11), one can easily read-off the new transformations with four gauge parameters. One may equally well perform the following rearrangements:

\[ \varepsilon^{(\nu)} \tilde{I}(\nu) = \varepsilon^{(\nu)} F^\mu_{(\nu)} (g_{\alpha\beta}) I_\mu \equiv \tilde{\xi}^\mu I_\mu, \quad (27) \]

after which, transformations of the metric would have the same form as before, but with a different, field-dependent, gauge parameter:

\[ \tilde{\xi}^\mu = \varepsilon^{(\nu)} F^\mu_{(\nu)} (g_{\alpha\beta}). \quad (28) \]

Therefore (28) is a different transformation from (18); for example, even its form cannot be preserved in calculations of the commutators (19) of two such transformations. The independence of gauge parameters, stated in [13, 23, 27], and [5], is not contradicted by (28), because it is merely a short form of presentation of the results. In a full expression, the field-independent parameter would appear \( (\varepsilon^{(\nu)}) \). But the “quasi-covariant” form (28) could be useful in performing calculations. This idea will be explained in the next Section, where one particular case of (25)-(28) is discussed: the ADM transformations.
We note that Noether’s theorem and the explicit connection between DIs and gauge transformations (as in \[15\]) considerably simplifies the analysis of singular Lagrangians. For example, the construction of new transformations, as outlined above, can also be used to check the validity of some proposed or “guessed” transformations. Assuming that a transformation is correct, one would follow \[15\] to construct a corresponding DI candidate that can be checked by direct substitution of the ELDs. If the candidate is a true DI, then by the converse of Noether’s theorem, it is a symmetry. By checking an identity one may manage expressions of any complexity because terms of different types can be considered separately; this property is very important for dealing with non-covariant expressions (i.e. all terms with a particular derivative of a particular field should be zero independently of the rest of an expression). This method is simpler if some DIs are already known; in such a case, one may express the new DIs as combinations of known identities, as in (27) (e.g. see \[17\]), which is sufficient confirmation of the correctness of the proposed transformations.

III. ADM SYMMETRY OF THE EH ACTION

The transformations of the ADM variables, \(\delta_{ADM} \{N, N^i, \gamma_{km}\}\), that follow from the constraints of the ADM Hamiltonian are well-known; and using (1) allows one to find the transformations of the metric tensor, \(\delta_{ADM} \{g^\mu\nu\}\). They can be presented in the following form:

\[
\delta_{ADM} g^{\mu\nu} = \tilde{\xi}^\nu_{\alpha} g^{\mu\alpha} + \tilde{\xi}^\mu_{\alpha} g^{\nu\alpha} - g^{\mu\nu}_{\alpha} \tilde{\xi}^\alpha
\]

with \(\tilde{\xi}^\alpha\) given by

\[
\tilde{\xi}^\nu = \delta^\nu_{\nu} (-g^{00})^{-\frac{1}{2}} \varepsilon^\perp + \delta^\nu_k \left[ \varepsilon^k + g^{0k}_{00} (-g^{00})^{-\frac{1}{2}} \varepsilon^\perp \right]
\]

(e.g. see appendix of \[5\], for more detailed calculations in \[4\] and also \[19\]).

This representation helps one to see the origin of some of the names of the ADM transformations: “specific metric-dependent diffeomorphisms” \[30\], or “a one-to-one correspondence between the diffeomorphisms and the gauge variations” \[31\], or “diffeomorphism-induced gauge symmetry” \[32\]. From the discussion at the end of the previous Section, one may see that \[30\] is one of many possible “field-dependent diffeomorphisms” and “one-to-one correspondence”. Using the transformations \[29\], the corresponding DIs can be restored.
Because they are combinations of the known covariant DIs and they are also the identities, the transformations (29) represent the gauge symmetry of the EH Lagrangian. So whatever the field dependence of the transformations of the form shown in (30) might be, these transformations are guaranteed to be a symmetry of the EH Lagrangian. We can also explicitly find separate identities that correspond to each parameter \((\varepsilon^{\perp}, \varepsilon^{k})\) of the ADM transformations:

\[
\tilde{\xi}^\nu I_\nu = \varepsilon^{\perp} \left[ \frac{g^{0k}}{g^{00}} (g^{00})^{\frac{1}{2}} I_k + (-g^{00})^{\frac{1}{2}} I_0 \right] + \varepsilon^k I_k , \tag{31}
\]

which in turn, give two DIs to describe the ADM transformations:

\[
\tilde{\xi}^\nu I_\nu = \varepsilon^{\perp} \tilde{I}_\perp + \varepsilon^k \tilde{I}_k ,
\]

with

\[
\tilde{I}_\perp = \frac{g^{0k}}{g^{00}} (g^{00})^{\frac{1}{2}} I_k + (-g^{00})^{\frac{1}{2}} I_0 , \tag{32}
\]

and

\[
\tilde{I}_k = I_k . \tag{33}
\]

These are obviously DIs since they are linear combinations of the components of the covariant DI.

The names “field-dependent diffeomorphism” and “one-to-one correspondence” are misleading. This transformation is different from diffeomorphism and even its resemblance to \(\text{diff}\) in “form” disappears if one were to calculate the commutator of two such transformations. The previous relation for \(\text{diff}\) (20) changes and a simple substitution of (31) into (20) is not equivalent to the direct calculation of the commutator

\[
\tilde{\xi}^\alpha_{[1,2]} \neq \tilde{\xi}^\beta_2 \tilde{\xi}_1^\alpha - \tilde{\xi}^\beta_2 \tilde{\xi}_1^{\alpha,\beta}
\]

in which extra contributions appear. This was noticed in [32]: “It is impossible to get for \(\xi_3\) [our \(\tilde{\xi}^\alpha_{[1,2]}\)] the standard diffeomorphism rule” and so the transformation with parameters (30) is not a field-dependent diffeomorphism, but a different symmetry. Even the formal resemblance of \(\text{diff}\) transformations does not survive in the commutator.

Let us try to find the commutator of two ADM transformations. From now on we shall eliminate the subscript ADM, and use \(\delta_{\text{ADM}g^{\mu\nu}} = \tilde{\delta}g^{\mu\nu}\) to abbreviate the notation. The
Lagrangian is analyzed. The components of a contravariant tensor at once; this is impossible to do when the ADM results from the previous Section, and then to consider the transformations of all of the quasi-covariant form of (30) allows one to simplify the calculations by using some of the \( \tilde{\xi}^\alpha \) (which is an abbreviated form (30), not a field-independent parameter). Consider

\[
\left( \tilde{\delta}_2 \tilde{\delta}_1 - \tilde{\delta}_1 \tilde{\delta}_2 \right) g^{\mu\nu} = \tilde{\xi}_{1,\alpha}^{\nu} \delta_2 g^{\mu\alpha} + \tilde{\xi}_{1,\alpha}^{\nu} \delta_2 g^{\mu\alpha} - \left( \delta_2 g^{\mu\nu} \right)_{,\alpha} \tilde{\xi}_{1,1}^{\alpha} - \tilde{\xi}_{2,\alpha}^{\nu} \delta_1 g^{\mu\alpha} - \tilde{\xi}_{2,\alpha}^{\nu} \delta_1 g^{\mu\alpha} + \left( \delta_1 g^{\mu\nu} \right)_{,\alpha} \tilde{\xi}_{2,2}^{\alpha}
\]

In performing the calculation of the commutator of the ADM transformation, that is

\[
\left( \delta_2 \tilde{\xi}_1^{\nu} - \delta_1 \tilde{\xi}_2^{\nu} \right) g^{\mu\alpha} + \left( \delta_2 \tilde{\xi}_1^{\mu} - \delta_1 \tilde{\xi}_2^{\mu} \right) g^{\nu\alpha} - \delta_\alpha g^{\mu\nu} \delta_2 \tilde{\xi}_2^{\alpha} - \delta_\alpha g^{\mu\nu} \delta_2 \tilde{\xi}_2^{\alpha} - \left( \delta_2 \tilde{\xi}_1^{\mu} - \delta_1 \tilde{\xi}_2^{\mu} \right) g^{\nu\alpha} - \left( \delta_2 \tilde{\xi}_1^{\nu} - \delta_1 \tilde{\xi}_2^{\nu} \right) g^{\mu\alpha}
\]

the terms in the first line (no contributions with \( \tilde{\xi}^\alpha \)) give the same result as that for the known diff (with \( \tilde{\xi}^\alpha \)); the second line produces additional contributions that can be combined into the following form:

\[
\left( \delta_2 \tilde{\xi}_1^{\nu} - \delta_1 \tilde{\xi}_2^{\nu} \right) g^{\mu\alpha} + \left( \delta_2 \tilde{\xi}_1^{\mu} - \delta_1 \tilde{\xi}_2^{\mu} \right) g^{\nu\alpha} - \delta_\alpha g^{\mu\nu} \left( \delta_2 \tilde{\xi}_1^{\alpha} - \delta_1 \tilde{\xi}_2^{\alpha} \right)
\]

After making some simple rearrangement, we obtain a general expression:

\[
\tilde{\xi}_{[1,2]}^\alpha = \tilde{\xi}_2^{\beta} \tilde{\xi}_1^{\alpha,\beta} - \tilde{\xi}_2^{\beta} \tilde{\xi}_1^{\alpha,\beta} + \tilde{\xi}_2^{\alpha} - \tilde{\xi}_1^{\alpha}
\]

with additional contributions that must be explicitly calculated for the particular field dependence of the parameters.

In the first two terms of (34), we merely substitute (30); and in the last two terms (which are zero if parameters are “field-independent”), we have

\[
\delta_2 \tilde{\xi}_1^{\alpha} - \delta_1 \tilde{\xi}_2^{\alpha} = \delta_0^{\alpha} \tilde{\xi}_2^{\beta} - \tilde{\xi}_1^{\alpha,\beta} + \delta_2^{\alpha} \tilde{\xi}_1^{\beta} - \tilde{\xi}_2^{\alpha,\beta}
\]

(35)
(Note that in (35) $\varepsilon^k$ is absent, since it enters (30) without field-dependent coefficients.) The final result for (34) can be presented in the same form as (29),

$$\tilde{\xi}^\alpha_{[1,2]} = \delta_0^\alpha \left( -g^{00} \right)^{\frac{1}{2}} \varepsilon^\alpha_{[1,2]} + \delta_k^\alpha \left[ \varepsilon_k^{1,2} + \frac{g^{0k}_{00}}{g_{00}^{00}} \left( -g^{00} \right)^{\frac{1}{2}} \varepsilon^\alpha_{[1,2]} \right]$$

where

$$\varepsilon^\alpha_{[1,2]} = \varepsilon^{\alpha}_{2,1,k} - \varepsilon^{\alpha}_{1,2,k}$$

and

$$\varepsilon^{k}_{[1,2]} = \varepsilon^{m,k}_{2,1,m} - \varepsilon^{m,k}_{1,2,m} + \left( \varepsilon^{\perp}_{1,m} - \varepsilon^{\perp}_{2,m} \right) e^{mk}.$$  

(36)

(37)

Here the combination, $e^{mk}$, which found in Dirac’s Hamiltonian analysis of the EH action, is formed

$$e^{mk} = g^{mk} - \frac{g^{0m}_0 g^{0k}}{g_{00}^{00}}$$

Due to the presence of fields in (37), one might conclude that this “soft algebra” structure signifies that the symmetry transformations no longer form a group. This is a possible outcome when fields appear in the structure constant, but not always. The field independence of the parameters in a commutator of two transformations is a sufficient condition to have an algebra, but not a necessary one.

With the appearance of fields, such as in (37), the double commutator must be checked by direct calculation. Again, we can use the general form of the results and consider the double commutator. We return to (34), which is a general expression whatever the field dependence of the gauge parameters might be, and by making the changes $1 \to [1,2]$ and $2 \to 3$, we obtain

$$\tilde{\xi}^\alpha_{[[1,2],3]} = \tilde{\xi}^\beta_{[1,2],3} \xi^\alpha_{[1,2],\beta} - \tilde{\xi}^\beta_{[1,2],3} \xi^\alpha_{[1,2],\beta} + \delta_3^\alpha \xi^\alpha_{[1,2]} - \delta_{[1,2]}^\alpha \xi^\alpha_{3}.$$  

(38)

The evaluation of the first two terms is straightforward; but the second pair, because of the presence of $\xi^\alpha_{[1,2]}$ (with fields, see (37)), produces an additional contribution as compared to the simple change of indices ($1 \to [1,2]$ and $2 \to 3$) in (35):

$$\delta_3^\alpha \xi^\alpha_{[1,2]} - \delta_{[1,2]}^\alpha \xi^\alpha_{3} = \delta^\alpha_0 \varepsilon^\alpha_{[1,2]} \delta^\alpha_3 \left( -g^{00} \right)^{\frac{1}{2}} + \delta^k_\alpha \varepsilon^\alpha_{[1,2]} \delta^k_\alpha \left[ \frac{g^{0k}_{00}}{g_{00}^{00}} \left( -g^{00} \right)^{\frac{1}{2}} \right] + \delta^\alpha_0 \delta^k_3 \varepsilon^k_{[1,2]}$$

17
\[- \delta_0^{\alpha} \varepsilon_{3}^{\alpha} \delta_{[1,2],3} (-g^{00})^{\frac{1}{2}} - \delta_k^{\alpha} \varepsilon_{3}^{\alpha} \delta_{[1,2]} \left[ \frac{g^{0k}}{g^{00}} (-g^{00})^{\frac{1}{2}} \right] \] 

(39)

The last contribution in the first line was absent from (35) because \( \varepsilon^k \) is all field-independent. This additional contribution is

\[ \varepsilon_{[1,2],3} = \delta_3^{\alpha} \varepsilon_{[1,2],3} = (\varepsilon_1^{\frac{1}{2}} - \varepsilon_2^{\frac{1}{2}} + \varepsilon_3^{\frac{1}{2}}) \delta_3 e^{mn}. \]

After performing a transformation \( \delta_3 e^{mn} \), it leads to

\[ \varepsilon_{[1,2],3}^{\frac{1}{2}} = (\varepsilon_1^{\frac{1}{2}}, \varepsilon_2^{\frac{1}{2}} - \varepsilon_3^{\frac{1}{2}}, \varepsilon_3^{\frac{1}{2}}) \{ \varepsilon_{3,n} e^{kn} + \varepsilon_{3,n} e^{mn} - \varepsilon_{3,n} e^{kn} \} \]

(40)

\[ + (-g^{00})^{\frac{1}{2}} \left[ \frac{g^{0k}}{g^{00}}, e^{mn} + \frac{g^{0m}}{g^{00}}, e^{kn} - \varepsilon_{3,m}^{\alpha} e^{kn} \right] \varepsilon_{3}^{\alpha} \}

The remaining contributions (see (38) and (39)) are the same as those found in the previous calculations, so we can use (37) with \((1 \rightarrow [1,2] \text{ and } 2 \rightarrow 3)\), as before, to obtain:

\[ \varepsilon_{[1,2],3} = -\varepsilon_{[1,2],3} e_{3,i}^{k} + \varepsilon_{[1,2],m} e_{3,i}^{k} + \varepsilon_{[1,2],m} e_{3,j}^{k} e^{mk} - \varepsilon_{3,m} e_{[1,2]}^{k} e^{mk}. \]

(41)

After the substitution of \( \varepsilon_{[1,2],3}^{i} \), we have:

\[ \varepsilon_{[1,2],3}^{i} = - (\varepsilon_{1}^{m}, \varepsilon_{2}^{m} - \varepsilon_{3}^{m}, \varepsilon_{1}^{m} e^{mi} - \varepsilon_{3,m} e_{[1,2]} e^{mk} \}

\[ + \varepsilon_{3}^{i} (\varepsilon_{1}^{m}, \varepsilon_{2}^{m} e^{mk} - \varepsilon_{3,m} e_{[1,2]} e^{mk} \}

\[ + (\varepsilon_{1}^{n}, \varepsilon_{2}^{n} e^{mk} - \varepsilon_{3,m} e_{[1,2]} e^{mk} \}

(42)

Combining (40) and (42) leads to some simplification; but the condition, which must be satisfied for the Jacobi identities to be correct, does not hold:

\[ \varepsilon_{[1,2],3}^{i} + \varepsilon_{[2,3],1}^{i} + \varepsilon_{[3,1],2}^{i} \neq 0 \]

(one contribution that prevents cancellation is the term in (40) proportional to \( e_{n}^{km} e_{3}^{mn} \)).

The EH action is invariant under the ADM transformations, but unlike \( \text{diff, } \delta_{ADM} \{ g^{\mu \nu} \} \)

do not form a group. This result illustrates that all possible symmetries, which can be
constructed easily from various combinations of DIs, are not equally good. There is one transformation (in general, some restricted class of transformations) that forms a group; and such transformations constitute the “basic” or “true” gauge symmetry of the Lagrangian. In analogy with the Hamiltonian formulation, one might call a symmetry that can form a group a “canonical” symmetry of the Lagrangian.

IV. CONCLUSION

The application of Dirac’s method to derive the Hamiltonian formulations of the EH Lagrangian, \( L_{EH} (g^{\mu\nu}) \), and the ADM Lagrangian, \( L_{ADM} (N, N^i; \gamma_{km}) \), leads to two different gauge symmetries; because of this difference in symmetries, it is no surprise that their Hamiltonian formulations are not related by a canonical transformation \([4, 9]\). If the Hamiltonian method is considered to be an algorithm that allows one to restore a gauge symmetry, and if the Lagrangian and Hamiltonian methods are equivalent, then one might conclude that the two Lagrangians are not equivalent \([4]\). The expression ”non-canonicity puzzle” was coined to describe this result \([10]\). But if equivalence of two Lagrangians is assumed, then one might alternatively conclude that the Hamiltonian method is not an algorithm (at least in its currently known form or for this particular case); thus Dirac’s method must be modified \([11, 12]\).

In this paper we offer a preliminary answer to the question of how to compare the symmetries of two Lagrangians which differ by invertible change of variables. Before such an undertaking is made, it is essential to understand how to distinguish two symmetries for the same Lagrangian. Based on Noether’s theorem, we demonstrate that both symmetries (\( \text{diff} \) and \( \text{ADM} \)) are symmetries of the EH Lagrangian, when written for the same variables; we also demonstrate that more symmetries can be constructed using the Lagrangian method. But a study of their group properties reveals that only one symmetry, \( \text{diff} \), has group properties; and neither \( \text{ADM} \) nor any other symmetries, constructed by using a so-called field-dependent redefinition of gauge parameters, have such a property. Therefore, for the EH Lagrangian, only one distinct symmetry with a group property exists (canonical symmetry).

To call two Lagrangians equivalent, any and all canonical symmetries should be presented in both formulations. The ADM symmetry, which follows from the Hamiltonian formulation
of the ADM action, is not a canonical symmetry of the EH action. Of course, the question whether the ADM formulation possesses canonical symmetry needs to be answered. Such calculations are straightforward, but extremely cumbersome (the penalty for working with non-covariant variables); and the relatively simple calculations presented in this article, which use a quasi-covariant form to allow one to consider transformations for all components of metric at once, are impossible in the case of the ADM Lagrangian. The calculations must be performed separately for all fields, and the redefinition of the gauge parameters must be the same for all fields. The DIs are also much more complicated, especially for the transformation of the ADM variables under diffeomorphism; and such transformations must also be checked to determine if they correspond to symmetries with a group property for the ADM Lagrangian.

From the analysis of the invariance of the EH Lagrangian performed in this paper, it follows that $\delta_{\text{diff}}$ has a group property; but $\delta_{\text{ADM}}$ does not. We are currently undertaking an investigation of the properties of these two symmetries for the ADM Lagrangian. There are four possible cases, all of which lead to contradictions and further questions. For the ADM Lagrangian, these cases are:

(a) both transformations form groups;
(b) neither transformation forms a group;
(c) $\delta_{\text{ADM}}$ forms a group, but not $\delta_{\text{diff}}$;
(d) $\delta_{\text{diff}}$ forms a group, but not $\delta_{\text{ADM}}$.

The first three of these cases lead to the non-equivalence of the Lagrangians. Cases (a) and (b) raise a question about the uniqueness of Dirac’s procedure. The two transformations both form groups (case (a)), or neither of them forms a group (case (b)); but only one symmetry is chosen by the Hamiltonian procedure. Case (c) is consistent with the uniqueness of the Hamiltonian method, as for the EH action, it selects a symmetry with a group property; but the Lagrangians (ADM and EH) cannot be equivalent.

Case (d) would imply an equivalence of the canonical symmetries of the ADM and EH Lagrangians, and that $\text{diff}$ is a symmetry with group properties for the ADM Lagrangian; but such a conclusion contradicts the widely quoted statement of Isham and Kuchar [33]: “the full group of spacetime diffeomorphisms has somehow got lost in making the transition

\[12\] In addition, such DIs are not covariant, and because of this, cannot be true in all coordinate systems.
from the Hilbert action to the Dirac-ADM action” (italic is ours)\textsuperscript{13}. Such statements were based on the results of the Hamiltonian formulation of the ADM Lagrangian with ADM gauge transformations. And one can often find claims that only spatial \textit{diff} is a symmetry of the ADM formulation. (Such statements are not compatible at all with equivalence of the EH and ADM actions.) So, case (d) would inexorably lead one to conclude that Dirac’s Hamiltonian method does not work for ADM variables (for the metric formulation of the EH action, it picks the symmetry with group properties, but for the ADM action it fails to do so). This outcome would force one to reconsider the “theoretical foundations”; to be more precise, to reconsider Dirac’s method, as suggested in \cite{11,12}, and to doubt its validity as an algorithm (at least in its current form). An algorithm should work without an \textit{a priori} knowledge of the gauge symmetry, and not demand modification of the method to adjust its outcome to the results that are known, \textit{a priori}, for a particular Lagrangian (e.g. ADM). Note also that such a formulation should be expected to be connected by a canonical transformation to the Hamiltonians of PSS and Dirac. We plan to continue this discussion after completing the analysis of the group properties of two transformations for the ADM Lagrangian.

There is another solution to the “puzzle”, but it would probably not be well accepted or considered seriously in view of the movement to devalue the importance of general covariance. This historical change of views on covariance is expressed perfectly by Norton \cite{34}: “When Einstein formulated his General Theory of Relativity, he presented it as the culmination of his search for a generally covariant theory. That this was the signal achievement of the theory rapidly became the orthodox conception. A dissident view, however, tracing back at least to objections raised by Eric Kretschmann in 1917, holds that there is no physical content in Einstein’s demand for general covariance. That dissident view has grown into the mainstream. Many accounts of general relativity no longer even mention a principle or requirement of general covariance.”

Considering the EH action and its original variables (the metric tensor), the Hamiltonian method (innately non-covariant) or combinations of DIs (which can be chosen to be

\textsuperscript{13} The name of Dirac is used incorrectly in this statement because Dirac’s Hamiltonian is not canonically related to the ADM Hamiltonian and, in addition, Dirac’s modification of the EH action is performed in the way to preserve Einstein’s equations. Moreover, if case (d) is correct, then neither for the Dirac nor for the ADM action the diffeomorphism “got lost”.

21
unrestricted by covariance) both single out the one unique, covariant symmetry. Covariance is neither demanded nor encoded in either of these methods; but when they are applied to covariant actions only covariant results are “somehow” produced. Many statements can be found in the literature that are similar to the recent one in [35]: “one of the beauties of general relativity is that it is difficult to deform it without running into inconsistencies”. Maybe, the solution to the “puzzle” is simple: do not destroy covariance - “one of the beauties” of Einstein’s theory; and do not deform it by using non-covariant variables. Heeding these caveats will prevent one from “running into inconsistencies”, finding contradictions, and facing such “puzzles”. Further, instead of being on the horns of a dilemma, to choose “canonical or covariant” [10], one might simply conclude: only covariant results are canonical for General Relativity.

V. ACKNOWLEDGMENT

We would like to thank A. Frolov, L.A. Komorowski, D.G.C. McKeon, and A.V. Zvelindovsky for discussions.

[1] Pirani, F.A.E., Schild, A., Skinner, R.: Phys. Rev. 87, 452 (1952)
[2] Dirac, P.A.M.: Proc. R. Soc. A 246, 333 (1958)
[3] Kiriushcheva, N., Kuzmin, S.V., Racknor, C., Valluri, S.R.: Phys. Lett. A 372, 5101 (2008)
[4] Kiriushcheva, N., Kuzmin, S.V.: Central Eur. J. Phys. 9, 576 (2011)
[5] Castellani, L.: Ann. Phys. 143, 357 (1982)
[6] Rovelli, C.: Quantum Gravity. Cambridge University Press, Cambridge (2004)
[7] Arnowitt, R., Deser, S., Misner, C.W.: In: Witten, L. (Ed.), Gravitation: An Introduction to Current Research. Wiley, New York. 227 (1962); arXiv:gr-qc/0405109
[8] Arnowitt, R., Deser, S., Misner, C.W.: Gen. Relativ. Gravit. 40, 1997 (2008)
[9] Frolov, A.M., Kiriushcheva, N., Kuzmin, S.V.: arXiv:0809.1198 [gr-qc]
[10] Cianfrani, F., Lulli, M., Montani, G.: arXiv:1104.0140 [gr-qc]
[11] Shestakova, T.P.: Class. Quantum Grav. 28, 055009 (2011)
[12] Shestakova, T.P.: Grav. and Cosm. 17, 67 (2011)
[13] Noether, E.: Nachrichten von der König. Gesellsch. der Wiss. zu Göttingen, Math.-Phys. Klasse, 2, 235 (1918)

[14] Noether, E. (M.A. Tavel’s English translation): arXiv:physics/0503066

[15] Schwinger, J.: Phys. Rev. 130, 1253 (1963)

[16] Kiriushcheva, N., Kuzmin, S.V.: Gen. Rel. Grav. 42, 2613 (2010)

[17] Banerjee, R., Gangopadhyay, S., Mukherjee, P., Roy, D.: JHEP 1002, 075 (2010)

[18] Shestakova, T.P.: Proceedings of Russian summer school - seminar on Gravitation and Cosmology GRACOS - 2007. Kazan, p. 179 (2007); arXiv:0801.4854 [gr-qc]

[19] Bergmann, P.G., Komar, A.: Int. J. Theor. Phys. 1, 15(1972)

[20] Hofava, P., Melby-Thompson, C.M.: Phys. Rev. D 82, 064027 (2010)

[21] Kiriushcheva, N., Kuzmin, S.V.: Eur. Phys. J. C 70, 389 (2010)

[22] Bianchi, L.: Lezioni di geometria differenziale, 2nd ed., v. 1. Spoerri, Pisa (1902)

[23] Hilbert, D.: Nachrichten von der König. Gesellsch. der Wiss. zu Göttingen, Math.-phys. Klasse, 8, 395 (1916)

[24] Hilbert, D.: In: Renn, J. (Ed.): The Genesis of General Relativity, 4, 1003 (2007)

[25] Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields. Fourth ed., Pergamon Press, Oxford (1975)

[26] Carmeli, M.: Classical Fields, General Relativity and Gauge Theory. World Scientific, New Jersey (2001)

[27] Rosenfeld, L.: Annalen der Physik, 397, 113 (1930)

[28] Salisbury, D.: Preprint 381, Max Planck Institute for the History of Science (2009)

[29] Samanta, S.: Int. J. Theor. Phys. 48, 1436 (2009)

[30] Pons, J.M.: Class. Quantum Grav. 20, 3279 (2003)

[31] Mukherjee, P., Saha, A.: Int. J. Mod. Phys. A 24, 4305 (2009)

[32] Pons, J.M., Salisbury, D.C., Shepley, L.C.: Phys. Rev. D 55, 658 (1997)

[33] Isham, C.J., Kuchar, K.V.: Ann. Phys. 164, 316 (1985)

[34] Norton, J.D.: In: Brading, K., Castellani, E. (Eds.): Symmetries in Physics: Philosophical Reflections. Cambridge University Press, Cambridge, p. 110 (2003)

[35] Henneaux, M., Kleinschmidt, A., Gómez, G.L.: arXiv:1004.3769 [hep-th]; to appear in the proceedings of the conference “Gauge Fields: Yesterday, Today, Tomorrow”, dedicated to the 70th anniversary of Professor A.A. Slavnov