STABILIZATION OF STOCHASTIC DIFFERENTIAL EQUATIONS
DRIVEN BY G-LÉVY PROCESS WITH DISCRETE-TIME
FEEDBACK CONTROL

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Abstract. The stabilization of stochastic differential equations driven by Brownian motion (G-Brownian motion) with discrete-time feedback controls under Lipschitz conditions has been discussed by several authors. In this paper, we first give the sufficient condition for the mean square exponential instability of stochastic differential equations driven by G-Lévy process with non-Lipschitz coefficients. Second, we design a discrete-time feedback control in the drift part and obtain the mean square exponential stability and quasi-sure exponential stability for the controlled systems. At last, we give an example to verify the obtained theory.

1. Introduction. As non-trivially generalizes the classical Brownian motion, motivated by uncertainty problems, risk measures and the superhedging in finance, Peng [19] set up a framework of a time consistent sublinear expectation associated with G-Brownian motion that draws a lot of attention. G-Brownian motion has a very rich and interesting new structure, the corresponding theory of stochastic analysis has been developed (see, for example, Peng [21, 22], Denis, Hu and Peng [2], Gao [4], Gao and Jiang [5], Soner, Touzi and Zhang [30], Bai and Lin [1], Li and Peng [9] and the references therein).

The natural generalization of G-Brownian motion is to consider a jump process and the uncertainty associated with the drift, the volatility and the jump component. On the other hand, one feels that G-Brownian motion is not sufficient to model the financial world, as both G-Brownian motion and the standard Brownian motion share the same property, which makes them often unsuitable for modelling, namely the continuity of paths. Therefore, Hu and Peng [7] introduced the process with jumps, which they called G-Lévy process and studied the distribution property, i.e., Lévy-Khintchine formula, of a Lévy process under sublinear expectation. Ren [24] considered the representation of a sublinear expectation associated with G-Lévy process. Paczka [16] considered the integration theory for G-Lévy process.

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with finite activity. Paczka [17] studied the properties of the Poisson random measure and the Poisson integral associated with a G-Lévy process. Wang and Yuan [31] obtained the existence of solution for stochastic differential equations driven by G-Lévy process with discontinuous coefficients.

The stability has been one of the most important topics in the study of stochastic differential equations (see Mao [13]). Many authors have discussed the stability for different kinds of stochastic dynamical systems (see, e.g., Hu and Mao [6], Mao [14], Qiu et al. [23], Ren et al. [27], Ren et al. [26], Shao [28], You et al. [33], Li et al. [8]). Recently, Mao [12] concerned with the mean-square exponential stabilization of continuous-time hybrid stochastic differential equations (also known as stochastic differential equations with the Markovian switching) by discrete-time feedback controls. Since then, Song et al. [29] studied the almost sure stabilization of a given unstable hybrid system by nonlinear discrete time stochastic feedback control, Ren et al. [25] considered stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation. Fei et al. [3] tackled the stabilization problem for a class of highly nonlinear hybrid stochastic differential equations. Under some reasonable conditions on the drift and diffusion coefficients, they showed how to design the feedback control. Li and Mao [11] studied the stabilization of highly nonlinear stochastic differential equations (SDEs) driven by G-Lévy Process and proposed the sufficient conditions for the mean exponential stability to the following SDE:

$$dx(t) = f(t, x(t))dt + h(t, x(t))d\langle B \rangle(t) + \sigma(t, x(t))dB(t) + \int_{\mathbb{R}^d} K(t, x(t), z)L(dt, dz), \quad t \geq 0.$$ (1)

The initial data of $x(t)$ is $x_0 \in \mathbb{R}^n$ with $\hat{E}[|x_0|^2] < \infty$, $B(\cdot)$ is $d$-dimensional G-Brownian motion, $\langle B \rangle(\cdot)$ is the quadratic variation process of the G-Brownian motion, $L(\cdot, \cdot)$ is a Poisson random measure associated with the G-Lévy process. The coefficients $f(\cdot, x)$, $h(\cdot, x)$, $\sigma(\cdot, x)$ are in the space $M_G^2([0, T]; \mathbb{R}^n)$, $K(\cdot, x, \cdot, \cdot) \in H_G^2([0, T] \times \mathbb{R}^d; \mathbb{R}^n)$ for any $x \in \mathbb{R}^n$ (the precise definition are given below in Section 2).

In the case when a given SDE (1) is unstable, a natural question is whether we can design a feedback control $u(t, x([t/\tau] \tau))$, where $[t/\tau]$ is the integer part of $t/\tau$, based on the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$ so that the controlled system

$$dx(t) = [f(t, x(t)) + u(t, x([t/\tau] \tau)]dt + h(t, x(t))d\langle B \rangle(t) + \sigma(t, x(t))dB(t) + \int_{\mathbb{R}^d} K(t, x(t), z)L(dt, dz), \quad t \geq 0,$$ (2)

becomes stable? where $\tau > 0$ is a constant which stands for the duration between two consecutive state observations.

The rest of this paper is organized as follows. In Section 2 we introduce preliminary results in the G-framework. In Section 3, we design the discrete time feedback control and obtain the mean square exponentially stability and quasi-sure exponential stability of the solutions for the controlled system driven by G-Lévy process. Finally, in Section 4 we give an example to verify the above content.
2. Preliminaries. In this section, we briefly recall the definition and properties of the G-framework (see, for example, Peng [20], Neufeld and Nutz [15], Paczka [18] and the references therein).

Let Ω be a given set, \( \mathcal{H} \) denotes a linear space of real-valued functions defined on Ω such that if \( X_i \in \mathcal{H}, i = 1, 2, \ldots, d \), then \( \varphi(X_1, \ldots, X_d) \in \mathcal{H} \) for all \( \varphi \in C_{b,lip}(\mathbb{R}_d) \), where \( C_{b,lip}(\mathbb{R}_d) \) is the space of all bounded real-valued Lipschitz continuous functions.

**Definition 2.1.** A sublinear expectation \( \hat{E} \) is a functional \( \hat{E}: \mathcal{H} \to \mathbb{R} \) satisfying the following properties: for all \( X, Y \in \mathcal{H} \), we have

(i) Monotonicity \( \hat{E}(X) \geq \hat{E}(Y) \) if \( X \geq Y \).

(ii) Constant preserving \( \hat{E}(C) = C \) for \( C \in \mathbb{R} \).

(iii) Sub-additivity \( \hat{E}(X + Y) \leq \hat{E}(X) + \hat{E}(Y) \).

(iv) Positive homogeneity \( \hat{E}(\lambda X) = \lambda \hat{E}(X) \) for \( \lambda \geq 0 \).

The tripe \((\Omega, \mathcal{H}, \hat{E})\) is called a sublinear expectation space (compare with a probability space \((\Omega, \mathcal{F}, P)\)). \( \hat{E} \) is called a linear expectation if (iii) and (iv) are replaced by \( \hat{E}[X + \alpha Y] = \hat{E}(X) + \alpha \hat{E}(Y) \) for \( \alpha \in \mathbb{R} \). \( X \in \mathcal{H} \) is called a random variable in \((\Omega, \mathcal{H}, \hat{E})\).

**Definition 2.2.** In a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\), an n-dimensional random vector \( Y = (Y_1, \ldots, Y_n) \), \( Y_i \in \mathcal{H} \) is said to be independent of another m-dimensional random vector \( X = (X_1, \ldots, X_m) \) if for each \( \varphi \in C_{b,lip}(\mathbb{R}^{m+n}) \),

\[
\hat{E}[(\varphi(X, Y)|x = X)] = \hat{E}[(\varphi(x, Y)|x = X)].
\]

It is important to note that under sublinear expectations the condition ‘\( Y \) is independent of \( X \)’ does not implies automatically that ‘\( X \) is independent of \( Y \)’.

**Definition 2.3.** Let \( X_1 \) and \( X_2 \) be two n-dimensional random vectors defined on sublinear expectation spaces \((\Omega_1, \mathcal{H}_1, \hat{E}_1)\) and \((\Omega_2, \mathcal{H}_2, \hat{E}_2)\), respectively. They are called identically distributed if

\[
\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \forall \varphi \in C_{b,lip}(\mathbb{R}^n).
\]

\( Y \) is said to be an independent copy of \( X \), if \( Y \) is identically distributed with \( X \) and independent of \( X \).

**Definition 2.4.** (G-Lévy process) A d-dimensional càdlàg process \( X = (X_t)_{t \geq 0} \) defined on a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called a Lévy process if the following properties are satisfied:

(i) \( X_0 = 0 \).

(ii) For each \( s, t \geq 0 \), the increment \( X_{t+s} - X_t \) is independent of \( (X_{t_1}, \ldots, X_{t_n}) \) for every \( n \in \mathbb{N} \) and every partition \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t \).

(iii) The distribution of the increment \( X_{t+s} - X_s, s, t \geq 0 \) does not depend on \( t \).

Moreover, a Lévy process \( X \) is a G-Lévy process if it satisfies the following conditions:

(iv) There exists a 2d-dimensional Lévy process \((X_t^e, X_t^d)_{t \geq 0}\) such that \( X_t = X_t^e + X_t^d \), for each \( t \geq 0 \).

(v) The processes \( X_t^e \) and \( X_t^d \) satisfy:

\[
\lim_{t \downarrow 0} \frac{\hat{E}[|X_t^e|^3]}{t^{-1}} = 0; \quad \hat{E}[|X_t^d|] < C t, \text{ for all } t \geq 0.
\]

Note that the condition (v) implies that \( X_t^e \) is a generalized G-Brownian motion, the jump part \( X_t^d \) is of finite variation (see Hu and Peng [7] for details).
Lemma 2.5. (Lévy-Khintchine representation, [7]) Let $X$ be a $G$-Lévy process in $\mathbb{R}^d$. Defined nonlocal operator $G_X[f(\cdot)] := \lim_{\delta \downarrow 0} E[f(X_\delta)]/\delta^{-1}$, for $f \in C^2_b(\mathbb{R}^d)$ with $f(0) = 0$.

Then, $G_X$ has the following Lévy-Khintchine representation

$$G_X[f(\cdot)] = \sup_{(v,p,Q) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d} f(z)v(dz) + \langle D(f(0), p) + \frac{1}{2} tr[D^2 f(0)QQ^T] \right\},$$

where $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$, $\mathcal{U}$ is a subset $\mathcal{U} \subset V \times \mathbb{R}^d \times Q$, and $V$ is a set of all Borel measures on $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$. $Q$ is a set of all $d$-dimensional positive definite symmetric matrices in $\mathbb{R}^d$ ($S^d$ is the space of all $d \times d$-dimensional symmetric matrices) such that

$$\sup_{(v,p,Q) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d} |v(dz)| + |p| + tr[QQ^T] \right\} < \infty. \quad (3)$$

Lemma 2.6. ([7]) Let $X$ be a $d$-dimensional $G$-Lévy process. For each $\phi \in C_{b, lip}(\mathbb{R}^d)$, we define $u(t,x) := E[\phi(x + X_t)]$. Then, $u$ is the unique viscosity solution of the following integro-partial differential equation (integro-PDE).

$$0 = \partial_t u(t,x) - G_X[u(t,x + \cdot) - u(t,x)]$$

$$= \partial_t u(t,x) - \sup_{v,p,Q} \in \mathcal{U} \left\{ \int_{\mathbb{R}^d} [u(t,x + z) - u(t,x)]v(dz) + \langle Du(t,x), p \rangle + \frac{1}{2} tr[D^2 u(t,x)QQ^T] \right\},$$

with initial condition $u(0,x) = \phi(x)$.

Let $(\Omega, \mathcal{F}, \mathbb{P}_0)$ be a probability space carrying a Brownian motion $W$ and a Lévy process with a Lévy triplet $(0, 0, \mu)$, which is independent of $W$. Let $N(\cdot, \cdot)$ be a Poisson random measure associated with the Lévy process. Define $N_t = \int_{\mathbb{R}^d_0} xN(t,dx)$, which is finite $\mathbb{P}_0$-a.s. as we assume that $\mu$ integrates $|x|$. Moreover, in the finite activity case, i.e. $\lambda = \sup_{v \in V} v(\mathbb{R}^d_0) < \infty$, $V$ is a set of all Borel measures on $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$, we define the Poisson process $M$ with intensity $\lambda$ by considering that $M_t = N(t, \mathbb{R}^d_0)$. We also define the filtration generated by $W$ and $N$:

$${\mathcal{F}}_t := \sigma\{W_s, N_s : 0 \leq s \leq t \} \vee N, \quad {\mathcal{N}} := \{A \in \Omega : P_0(A) = 0\}, \quad {\mathcal{F}} := (F_t)_{t \geq 0}. $$

Lemma 2.7. (Paczyńska [16]) Introduce a set of integrands $\mathcal{A}^{d}_{T,T}$, $0 \leq t \leq T$, associated with $\mathcal{U}$ (satisfying Lévy-Khintchine representation) as a set of all processes $\theta = (\theta^d, \theta^1, \theta^2)$ defined on $(t,T]$ that satisfy the following properties:

(1)$(1,c, \theta^2)$ is $\mathcal{F}$-adapted and $\theta^d$ is $\mathcal{F}$-predictable random field on $(t,T] \times \mathbb{R}^d$.

(2)For $\mathbb{P}_0$-a.a. $\omega \in \Omega$ and a.e.s in $(t,T]$, we have $(\theta^d(s, \cdot)(\omega), \theta^1(\cdot)(\omega), \theta^2(\cdot)(\omega)) \in \mathcal{U}$.

(3) $\theta$ satisfies the following integrability condition

$$E^{\mathbb{P}_0}\left[\int_{t}^{T} \left(|\theta^1_s| + |\theta^2_s| + \int_{\mathbb{R}^d} |\theta^d(s,z)|\mu(dz)\right)ds\right] < \infty.$$

For $\theta \in \mathcal{A}^{d}_{T,T}$, we denote the Lévy-Itô integral as

$$B^{\theta}_{T,t} = \int_{t}^{T} \theta^1_s ds + \int_{t}^{T} \theta^2_s dW_s + \int_{t}^{T} \int_{\mathbb{R}^d} \theta^d(s,z)N(ds, dz).$$
For every \( \xi = \phi(X_{t_1}, X_{t_2-t_1}, \ldots, X_{t_n-t_{n-1}}) \in \text{Lip}(\Omega_T) \),
\[
\hat{E}[\xi] = \sup_{\theta \in \mathcal{A}_{0,\infty}^I} E^{P_\theta}[\phi(B_{t_1}^{0,\theta}, B_{t_2}^{1,\theta}, \ldots, B_{t_n}^{n-1,\theta})].
\]

Let \( \xi \in L^1_G(\Omega) \), we can represent the sublinear expectation as follows:
\[
\hat{E}[\xi] = \sup_{\theta \in \mathcal{A}_{0,\infty}^I} E^{P_\theta}[\xi],
\]
where \( P^\theta := P_0 \circ (B_0^{0,\theta})^{-1}, \theta \in \mathcal{A}_{0,\infty}^I \). We also denote \( \mathfrak{B} := \{ P^\theta : \theta \in \mathcal{A}_{0,\infty}^I \} \).

**Definition 2.8.** For the sublinear expectation \( \hat{E} \), we introduce the capacity \( c \) related to \( \hat{E} \) as
\[
c(A) := \sup_{P \in \mathfrak{B}} P(A), A \in \mathcal{B}(\Omega).
\]

We say that a set \( A \in \mathcal{B}(\Omega) \) is polar if \( c(A) = 0 \) and a property holds quasi-surely (q.s.) if it holds outside a polar set.

**Lemma 2.9.** Let \( X \in L^1_G(\Omega_T) \) and for some \( p > 0 \), \( \hat{E}|X|^p < \infty \). Then, for each \( M > 0 \),
\[
c(|X| > M) \leq \frac{\hat{E}|X|^p}{M^p}.
\]

In this paper, we assume that G-Lévy process \( X \) has finite activity, i.e.,
\[
\lambda := \sup_{v \in \mathcal{V}} v(R^d_0) < \infty.
\]

Without loss of generality we will also assume that \( \lambda = 1 \) and that \( \mu(R^d_0) = 1 \). Let \( X_{u-} \) denotes the left limit of \( X \) at point \( u \), \( \Delta X_u = X_u - X_{u-} \). We define a Poisson random measure \( L(ds,dz) \) associated with the G-Lévy process \( X \) by considering
\[
L((s,t], A) = \sum_{s < u \leq t} 1_A(\Delta X_u), \text{ q.s.},
\]
for any \( 0 < s < t < \infty \) and \( A \in \mathcal{B}(\mathbb{R}^d_0) \). The random measure is well-defined and may be used to define the pathwise integral.

Let \( H^S_G([0,T] \times \mathbb{R}^d_0) \) be a space of all the elementary random fields on \([0,T] \times \mathbb{R}^d_0\) of the form
\[
K(r,z)(w) = \sum_{k=1}^n \sum_{l=1}^m F_{k,l}(w) [I_{(t_k,t_{k+1})}(r) \psi_l(z)], n, m \in \mathbb{N},
\]
where \( 0 \leq t_1 < \cdots < t_n \leq T \) is a partition of \([0,T]\), \( \{ \psi_l \}_{l=1}^m \subset C_{b,\text{lip}}(\mathbb{R}^d) \) are functions with disjoint supports such that \( \psi_l(0) = 0 \) and \( F_{k,l} = \phi_{k,l}(X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}) \), \( \phi_{k,l} \in C_{b,\text{lip}}(\mathbb{R}^{d \times k}) \). We introduce the norm on this space
\[
||K||^p_{H^S_G([0,T] \times \mathbb{R}^d_0)} := \hat{E} \left[ \int_0^T \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d_0} |K(r,z)|^p v(dz) dr \right], p = 1, 2.
\]

**Definition 2.10.** Let \( 0 \leq s < t \leq T \). The Itô integral of \( K \in H^S_G([0,T] \times \mathbb{R}^d_0) \) with respect to the jump measure \( L \) is defined as
\[
\int_s^t \int_{\mathbb{R}^d_0} K(r,z)L(dr,dz) := \sum_{s < r \leq t} K(r, \Delta X_r), \text{ q.s.}.
\]
It is worth noting that for every $K \in H_G^2([0, T] \times \mathbb{R}_0^d)$, $\int_0^T \int_{\mathbb{R}_0^d} K(r, z)L(dr, dz)$ is an element of $L_B^2(\Omega_T)$ and $L_C^2(\Omega_T)$.

Let $H_G^p([0, T] \times \mathbb{R}_0^d)$ denotes the topological completion of $H_G^2([0, T] \times \mathbb{R}_0^d)$ under the norm $\| \cdot \|_{H_G^p([0, T] \times \mathbb{R}_0^d)}$, $p = 1, 2$. Then Itô integral can be continuously extended to the whole space $H_G^p([0, T] \times \mathbb{R}_0^d)$, $p = 1, 2$. Moreover, the extended integral takes values in $L_B^p(\Omega_T), p = 1, 2$. The formula from Definition 2.10 still holds for all $K \in H_G^2([0, T] \times \mathbb{R}_0^d)$. Now, we introduce the BDG-type inequality for the integral with respect to jump measure $L$.

**Lemma 2.11.** (Wang and Gao [32]) Let $Y(t) := \int_0^t \int_{\mathbb{R}_0^d} K(r, z)L(dr, dz)$, $K(r, z) \in H_G^2([0, T] \times \mathbb{R}_0^d)$. Then there exists a càdlàg modification $\tilde{Y}(t)$ of $Y(t)$ for all $t \in [0, T]$ such that

$$\hat{E} \left[ \sup_{0 \leq t \leq T} |\tilde{Y}(t)|^2 \right] \leq C_T \hat{E} \left[ \int_0^T \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_0^d} K^2(r, z)v(z)dzdr \right],$$

where $C_T > 0$ is a constant depending on $T$.

For $K(r, z) \in H_G^2([0, T] \times \mathbb{R}_0^d)$, we know that

$$M(t) := \int_0^T \int_{\mathbb{R}_0^d} K(r, z)L(dr, dz) - \int_0^T \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_0^d} K(r, z)v(z)dzdr,$$

is a G-martingale, hence we have

$$\hat{E} \left[ \sup_{0 \leq t \leq T} |M(t)|^2 \right] \leq C \hat{E} \left[ \int_0^T \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_0^d} K^2(r, z)v(z)dzdr \right],$$

where $C$ is a positive constant.

Next, we consider the following type of simple process: for a given partition $\pi_T = t_0, t_1, \cdots, t_N$ of $[0, T]$, let

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{(t_k, t_{k+1})}(t),$$

where $\xi_k \in L_G^p(\Omega_T), k = 0, 1, \cdots, N - 1$ are given. The collection of these processes is denoted by $M_G^{p, 0}(0, T)$. Let $M_G^{p}(0, T)$ denotes the completion of $M_G^{p, 0}(0, T)$ under the norm

$$\| \eta \|_{M_G^{p}(0, T)} = \left( \int_0^T \hat{E}[|\eta(t)|^p]dt \right)^{\frac{1}{p}}.$$

For a process $\eta \in M_G^{p}(0, T)(p \geq 2)$, the stochastic integral can be defined with respect to the G-Brownian motion $B(t)$, which is $\int_0^t \eta(s)dB(s)$. Similarly for $\eta \in M_G^{p}(0, T)(p \geq 1)$ one can define integrals $\int_0^t \eta(s)ds$ and $\int_0^t \eta(s)d(B(s)$, respectively, where $\langle B \rangle(t)$ is the quadratic variation process of G-Brownian motion $B(t)$. Moreover, all these integrals belong to $L_B^p(\Omega)$ for $p \geq 1$ (see Peng [19]).

The following two lemmas are the BDG-type inequalities for the G-stochastic calculus with respect to $B(t)$ and $\langle B \rangle(t)$, respectively.

**Lemma 2.12.** (Gao [4]) For $p \geq 2, \eta \in M_G^{p}(0, T)$. Then

$$\hat{E} \left[ \sup_{0 \leq u \leq T} \left| \int_0^u \eta(r)dB(r) \right|^p \right] \leq C_p T^{\frac{p-2}{2}} \int_0^T \hat{E}[|\eta(r)|^p]dr,$$

where $C_p > 0$ is a constant that is only dependent on $p$. 
Lemma 2.13. (Gao [4]) For $p \geq 1, \eta \in M^p_G([0,T])$. Then there exists a constant $C_p > 0$, such that

$$
\mathbb{E} \left[ \sup_{0 \leq u \leq T} \left| \int_0^u \eta(r)d(B)(r)^p \right| \right] \leq C_p T^{p-1} \int_0^T \mathbb{E} |\eta(r)|^p dr.
$$

Let $M^p_G([0,T]; \mathbb{R}^n)$ and $H^p_G([0,T] \times \mathbb{R}^d; \mathbb{R}^n)$ be the spaces of $n$-dimensional stochastic process with each element belonging to $M^p_G([0,T])$ and $H^p_G([0,T] \times \mathbb{R}^d)$, respectively.

3. The mean square exponential stability and quasi-sure exponential stability. In this section, we firstly give sufficient conditions for the mean square exponential instability of the solution for the SDE (1) (in short, GSDEs).

Definition 3.1. The solution $x(t)$ of GSDEs is mean square exponentially stable if for any initial $x_0$, the solution $x(t)$ satisfies

$$
\mathbb{E}[|x(t)|^2] \leq C \mathbb{E}[|x_0|^2]e^{-\gamma(t-t_0)},
$$

where $\gamma$ and $C$ are positive constants that are independent of $t_0$.

Definition 3.2. The solution $x(t)$ of GSDEs is quasi-sure exponentially stable if for any initial $x_0$, the solution $x(t)$ satisfies

$$
\limsup_{t \to \infty} \frac{1}{t} \ln |x(t)| \leq -\gamma, \text{ q.s.,}
$$

where $\gamma$ is a positive constant.

Definition 3.3. The function $V(t,x)$ is said to belong to the class $\mathcal{V}_0$, if $V(t,x) \in C^{1,2}([t_0, +\infty) \times \mathbb{R}^n; \mathbb{R}^n)$, that is, $V_t, V_x, V_{xx}$ are continuous on $[t_0, +\infty) \times \mathbb{R}^n$ and $V_{xx}$ satisfies the local Lipschitz condition, where

$$
V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \quad V_x(t,x) = \left( \frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \ldots, \frac{\partial V(t,x)}{\partial x_n} \right),
$$

$$
V_{xx}(t,x) = \left( \frac{\partial^2 V(t,x)}{\partial x_i \partial x_j} \right)_{n \times n}.
$$

We assume that the functions $f, h, \sigma$ and $K$ satisfy all the necessary conditions in Wang and Gao [32] for the existence and uniqueness of a solution for the GSDEs (1), $t \geq t_0$.

Theorem 3.4. Assume that there exists a $V \in \mathcal{V}_0$, positive constants $C_1, C_2$ and $\lambda$, such that

(a) $C_1|x|^2 \leq V(t,x) \leq C_2|x|^2$, for any $t \geq t_0, x \in \mathbb{R}^n$,

(b) $LV(t,x) \geq \lambda V(t,x)$.

Then, the solution $x(t)$ of GSDEs (1) is mean square exponentially unstable.

Proof. For $t \in [t_0, T]$, applying the G-Itô formula (see Paczka [16]) to $e^{-\lambda t}V(t,x(t))$, we have

$$
d(e^{-\lambda t}V(t,x(t))) = e^{-\lambda t}[-\lambda V(t,x(t)) + V_t(t,x(t)) + \langle V_x(t,x(t)), f(t,x(t)) \rangle]dt
$$

$$
+ e^{-\lambda t} \langle V_x(t,x(t)), \sigma(t,x(t)) \rangle dB(t) + e^{-\lambda t} \langle V_x(t,x(t)), h(t,x(t)) \rangle dB(t)
$$

$$
+ \frac{1}{2} e^{-\lambda t} \langle V_{xx}(t,x(t)) \sigma(t,x(t)), \sigma(t,x(t)) \rangle dB(t)
$$
Hence,
\[
e^{-\lambda t} V(t,x(t)) = e^{-\lambda t_0} V(t_0,x_0) - \int_{t_0}^{t} -e^{-\lambda r} [-\lambda V(r,x(r)) + LV(r,x(r))] dr
\]
\[
- \int_{t_0}^{t} -e^{-\lambda r} (V(r,x(r)),\sigma(r,x(r))) dB(r) - (M^t_{t_0} - (P^t_{t_0})
\]
where
\[
M^t_s := \int_s^t -e^{-\lambda r} (V_x(r,x(r)),h(r,x(r))) + \frac{1}{2}(V_{xx}(r,x(r)))\sigma(r,x(r)) dr
\]
\[
\sigma(r,x(r))) d(\mathcal{B})(r) - \int_s^t -e^{-\lambda r} \sup_{Q \in \mathcal{Q}} tr(V_x(r,x(r)),h(r,x(r)))
\]
\[
+ \frac{1}{2}(V_{xx}(r,x(r)))\sigma(r,x(r)),\sigma(r,x(r)))QQ^T) dr,
\]
\[
P^t_s := \int_s^t \int_{\mathbb{R}^d} -e^{-\lambda r} [V(r,x(r^-) + K(r,x(r),z) - V(r(x^-))] L(dr, dz)
\]
\[
- \int_s^t \sup_{V \in \mathcal{V}} \int_{\mathbb{R}^d} -e^{-\lambda r} [V(r,x(r^-) + K(r,x(r),z) - V(r(x^-))] v(dz) dr.
\]

It comes from Peng [19] and Paczka [18] that \(\{M^t_s\}, \{P^t_s\}\) are G-martingale, so we take the expectation on both sides and obtain
\[
\hat{E}[e^{-\lambda t} V(t,x(t))] \geq \hat{E}[e^{-\lambda t_0} V(t_0,x_0)] - \hat{E}[\int_{t_0}^{t} -e^{-\lambda r} [-\lambda V(r,x(r)) + LV(r,x(r))] dr]
\]

From condition (b), we have
\[
\hat{E}[e^{-\lambda t} V(t,x(t))] \geq \hat{E}[e^{-\lambda t_0} V(t_0,x_0)].
\]

As \(V(t,x) \geq C_1|\sigma|^2\), it holds that \(\hat{E}[V(t_0,x_0)] \geq C_1 \hat{E}[|\sigma_0|^2]\) and
\[
\hat{E}[e^{-\lambda t} V(t,x(t))] \geq C_1 \hat{E}[|\sigma_0|^2]e^{-\lambda t_0}.
\]

So,
\[
\hat{E}[|x(t)|^2] \geq \frac{\hat{E}[V(t,x(t))]}{C_2} \geq \frac{C_1}{C_2} \hat{E}[|\sigma_0|^2]e^{\lambda(t-t_0)}.
\]

It implies that the solution \(x(t)\) of GSDEs (1) is mean square exponentially unstable. This completes the proof.

The following Theorem finds a feedback control \(u(x([t/\tau]\tau))\), based on the discrete time observations of the state \(x(t)\) at times \(0,\tau,2\tau,\cdots\) so that the controlled system
\[
dx(t) = [f(t,x(t)) + u(t,x([t/\tau]\tau))] dt + h(t,x(t)) dB(t)
+ \sigma(t,x(t)) dB(t) + \int_{\mathbb{R}^d} K(t,x(t),z) L(dt,dz), t \geq 0 \tag{5}
\]
becomes stable, where \(\tau > 0\) is a constant which stands for the duration between two consecutive state observations, and \([t/\tau]\) is the integer part of \(t/\tau\).
Note that the system (5) is a stochastic differential equation driven by G-Lévy process with a bounded variable delay. In fact, we can define $\xi : [0, \infty) \to [0, \tau)$ by
$$
\xi(t) = t - k\tau, \ k\tau \leq t < (k + 1)\tau, \ k = 0, 1, 2, \ldots .
$$
Thus the system (5) can be rewritten as
$$
dx(t) = [f(t, x(t)) + u(t, x(t - \xi(t)))]dt + h(t, x(t))dB(t) \\
+ \sigma(t, x(t))dB(t) + \int_{\mathbb{R}^d} K(t, x(t), z) L(dt, dz), \ t \geq 0.
$$
In order to consider the stability, we suppose that
$$
f(t, 0) = h(t, 0) = u(t, 0) = \sigma(t, 0) = K(t, 0, z) = 0, \ t \geq 0,
$$
which implies that $x(t) \equiv 0$ is the trivial solution of the system (5). To study the stability of the system (5), the generating operator $L$ can be defined as follows:
$$
LV := V_t(t, x) + \langle V_x(t, x), f(t, x) + u(t, x) \rangle + \sup_{Q \in \mathbf{Q}} \text{tr}[\langle V_x(t, x), h(t, x) \rangle] \\
+ \frac{1}{2} \langle V_{xx}(t, x)\sigma(t, x), \sigma(t, x) \rangle QQT] + \sup_{v \in \mathbf{V}} \int_{\mathbb{R}^d} (V(t, x + K(t, x, z)) - V(t, x)) v(dz).
$$

**Assumption 1.** For every fixed $t \in [t_0, T]$ and $x_1, x_2 \in \mathbb{R}^d$, the functions $f(t, x), u(t, x), h(t, x), \sigma(t, x), K(t, x, z)$ satisfy
$$
|f(t, x_1) - f(t, x_2)|^2 + |u(t, x_1) - u(t, x_2)|^2 + |h(t, x_1) - h(t, x_2)|^2 \\
+ |\sigma(t, x_1) - \sigma(t, x_2)|^2 + \sup_{v \in \mathbf{V}} \int_{\mathbb{R}^d} |K(t, x_1, z) - K(t, x_2, z)|^2 v(dz)
$$
$$
\leq \kappa(|x_1 - x_2|^2),
$$
where $\kappa(\cdot)$ is a continuous, concave, nondecreasing function. Moreover, $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$, $\int_0^\infty \frac{du}{\kappa(u)} = \infty$.

**Example 1.** We give a few concrete examples of the function $\kappa(\cdot)$. Let $K > 0$, and let $\delta \in (0, 1)$ be small enough. We define
$$
\kappa_1(u) = Ku, \ u \geq 0, \\
\kappa_2(u) = \begin{cases}
  u \log(u^{-1}), & \text{if } 0 \leq u \leq \delta, \\
  \delta \log(\delta^{-1}) + \kappa'_2(\delta -)(u - \delta), & \text{if } u > \delta,
\end{cases} \\
\kappa_3(u) = \begin{cases}
  u \log(u^{-1}) \log \log(u^{-1}), & \text{if } 0 \leq u \leq \delta, \\
  \delta \log(\delta^{-1}) \log \log(\delta^{-1}) + \kappa'_3(\delta -)(u - \delta), & \text{if } u > \delta,
\end{cases}
$$
where $\kappa'$ denotes the derivative of the function $\kappa$. They are all concave nondecreasing functions satisfying $\int_0^\infty \frac{du}{\kappa(u)} = \infty$. Furthermore, we observe that the Lipschitz condition is a special case of our proposed conditions.

**Assumption 2.** Suppose that there exist two positive numbers $\lambda_1, \lambda$ such that $LV(t, x) + \lambda_1 V(t, x)^2 \leq -\lambda \kappa(|x|^2)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

**Proposition 1.** Let us suppose Assumption 1 and 2 hold. If $\tau$ is small enough for
$$
\rho = \frac{2a}{\lambda_1(1 - a)}, \lambda_2 > \frac{2a\tau}{\lambda_1(1 - a)}(6\tau + C_p'), \ C_p + C_T, \tau^2 \leq \frac{1 - a}{32a},
$$
Then the solution of controlled system (5) satisfies
$$
\int_0^\infty \hat{E}(\kappa(|x(s)|^2))ds < +\infty,
$$
for any initial \( x_0 \in \mathbb{R}^n \).

**Proof.** We can construct a G-Lyapunov functional on the segment \( x_t := \{ x(t + r) : -\tau \leq r \leq 0 \} \) for \( t \geq 0 \). For the well defined \( x_t \), \( 0 \leq t \leq \tau \), we set \( x(r) = x_0 \) for \( -\tau \leq r \leq 0 \). Let

\[
\dot{V}(t, x_t) = V(t, x(t)) + \rho \int_{t-\tau}^t \int_{x} [f(w, x(w)) + u(w, x(\delta_w))]^2 + C_p^2|\tau h(w, x(w))|^2 + C_p|\sigma(w, x(w))|^2 + C_T \sup_{v \in \mathbb{V}} \int_{\mathbb{R}^d} K^2(w, x(w), z) v(dz)] dw dr,
\]

where \( \rho \) is a positive constant to be determined later, \( \delta_w = [w/\tau] \tau \).

Then applying the G-Itô formula to \( V(t, x(t)) \) and \( \dot{V}(t, x_t) \) respectively, we have

\[
dV(t, x(t)) := V_t(t, x) dt + V_x(t, x)[f(t, x(t)) + u(t, x(t))] dt + V_x(t, x(t)) \sigma(t, x(t)) dB(t)
\]

\[
+ \sup_{Q \in \mathbb{Q}} \text{tr}[(V_x(t, x), h(t, x)) + \frac{1}{2}(V_{xx}(t, x) \sigma(t, x), \sigma(t, x))]QQ^T] dt
\]

\[
+ \sup_{v \in \mathbb{V}} \int_{\mathbb{R}^d} (V(t, x + K(t, x, z)) - V(t, x)) v(dz) dt - dM^0_t - dP^0_t,
\]

and

\[
d\dot{V}(t, x_t) = L\dot{V}(t, x_t) dt + \dot{V}_x(t, x_t) \sigma(t, x(t)) dB_t - dM^0_t - dP^0_t,
\]

where \( M^0_t, P^0_t \) are G-martingale.

\[
d(\rho \int_{t-\tau}^t \int_{x} [f(w, x(w)) + u(w, x(\delta_w))]^2 + C_p^2|\tau h(w, x(w))|^2 + C_p|\sigma(w, x(w))|^2
\]

\[
+ C_T \sup_{v \in \mathbb{V}} \int_{\mathbb{R}^d} K^2(w, x(w), z) v(dz)] dw dr)
\]

\[
= (\rho\tau[f(t, x(t)) + u(t, x(\delta_t))]^2 + C_p^2|\tau h(t, x(t))]^2 + C_p|\sigma(t, x(t))]^2
\]

\[
+ C_T \sup_{v \in \mathbb{V}} \int_{\mathbb{R}^d} K^2(t, x, z) v(dz)] - \rho \int_{t-\tau}^t [f(r, x(r)) + u(r, x(\delta_r))]^2
\]

\[
+ C_p^2|\tau h(r, x(r))]^2 + C_p|\sigma(r, x(r))]^2 + C_T \sup_{v \in \mathbb{V}} \int_{\mathbb{R}^d} K^2(r, x, z) v(dz)] dr) dt.
\]

Moreover,

\[
LV(t, x(t)) := V_t(t, x) + (V_x(t, x), f(t, x(t)) + u(t, x(t)))
\]

\[
+ \sup_{Q \in \mathbb{Q}} \text{tr}[(V_x(t, x), h(t, x)) + \frac{1}{2}(V_{xx}(t, x) \sigma(t, x), \sigma(t, x))]QQ^T]
\]

\[
+ \sup_{v \in \mathbb{V}} \int_{\mathbb{R}^d} (V(t, x + K(t, x, z)) - V(t, x)) v(dz),
\]
and

\[
L\tilde{V}(t, x_t) := V_t(t, x) + \langle V_x(t, x), f(t, x(t)) + u(t, x(\delta_t)) \rangle
+ \sup_{Q \in \mathcal{Q}} \text{tr}[(V_x(t, x), h(t, x)) + \frac{1}{2}(V_{xx}(t, x)\sigma(t, x), \sigma(t, x))]QQ^T
+ \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_d^d} (V(t, x + \mathcal{K}(t, x, z)) - V(t, x))v(dz)
+ \rho\tau[f(t, x(t)) + u(t, x(\delta_t))]^2 + C_p^2\tau[h(t, x(t))]^2 + C_p\sigma(t, x(t))]^2
+ C_T \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_d^d} K^2(v, x(v), z)v(dz)]
- \rho \int_{t-\tau}^t \tau[f(r, x(r)) + u(r, x(\delta_r))]^2 + C_p^2\tau[h(r, x(r))]^2 + C_p\sigma(r, x(r))]^2
+ C_T^2 \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_d^d} K^2(r, x(r), z)v(dz)]dr.
(12)
\]

Hence, we have

\[
L\tilde{V}(t, x_t) = LV(t, x(t)) - V_x(t, x)[u(t, x) - u(t, x(\delta_t))]
+ \rho\tau[f(t, x(t)) + u(t, x(\delta_t))]^2 + C_p^2\tau[h(t, x(t))]^2 + C_p\sigma(t, x(t))]^2
+ C_T \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_d^d} K^2(v, x(v), z)v(dz)] - \rho \int_{t-\tau}^t \tau[f(r, x(r)) + u(r, x(\delta_r))]^2
+ C_p^2\tau[h(r, x(r))]^2 + C_p\sigma(r, x(r))]^2 + C_T^2 \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_d^d} K^2(r, x(r), z)v(dz)]dr.
(13)
\]

By (7), we have

\[
-V_x(t, x(t))[u(t, x) - u(t, x(\delta_t))]] = -2\sqrt{\lambda_1}V_x(t, x(t)) \cdot \frac{1}{2\sqrt{\lambda_1}}[u(t, x) - u(t, x(\delta_t))]
\leq (\sqrt{\lambda_1}V_x(t, x(t))]^2 + \frac{1}{2\sqrt{\lambda_1}}[u(t, x) - u(t, x(\delta_t))]^2
\leq \lambda_1[V_x(t, x(t))]^2 + \frac{1}{4\lambda_1}\kappa([x(t) - x(\delta_t)]^2).
(14)
\]
From (14) and (7), we further derive that

\[ L \dot{V}(t, x_t) \]

\[ \leq LV(t, x(t)) + \lambda_1 |V_x(t, x(t))|^2 + \frac{1}{4\lambda_1} \kappa(|x(t) - x(\delta_t)|^2) + \rho \tau |\tau (2\kappa(|x(t)|^2) + 4\kappa(|x(t)|^2) + 4\kappa(|x(t) - x(\delta_t)|^2)) + C_p' \tau |h(t, x(t))|^2 + C_p |\sigma(t, x(t))|^2 \]

\[ + C_T \sup_{v \in V} \int_{\mathbb{R}^d} K^2(v, x(v), z)v(dz) - \rho \int_{t-\tau}^t [\tau f(r, x(r)) + u(r, x(\delta_t))|^2 \]

\[ + C_p' \tau |h(r, x(r))|^2 + C_p |\sigma(r, x(r))|^2 + C_T \sup_{v \in V} \int_{\mathbb{R}^d} K^2(r, x(r), z)v(dz)]dr \]  

(15)

By (18), we have

\[ 0 \leq \dot{V}(t, x_t) \leq \int_0^t \dot{E} \hat{L} \dot{V}(s, x_s)ds \leq C_1 - \lambda \int_0^t \dot{E}(\kappa(|x(s)|^2))ds, \]

(19)
Then the solution $x(t)$ of system (5) satisfies that

$$C_1 = V(0, x_0) + \rho \int_{-\tau}^{0} \int_{-\tau}^{0} |r| f(v, x(v)) + u(v, x(\delta_v))|^2 + C'_p |h(v, x(v))|^2 + C_p |\sigma(v, x(v))|^2 + C_T \sup_{v \in \mathbb{V}} \int_{\mathbb{R}_d} K^2(v, x_v, z) v(dz) dv dr.$$  

Thus we have

$$\int_{0}^{t} \dot{E}(\kappa(|x(s)|^2)) ds \leq \frac{C_1}{\lambda}.$$ 

Hence,

$$\int_{0}^{\infty} \dot{E}(\kappa(|x(s)|^2)) ds < +\infty.$$ 

This completes the proof. \(\square\)

**Lemma 3.5.** Let (7) hold, $L(\tau) = \alpha r[10\tau + 10C'_p + 10C_p + 10C_T + 5\tau]e^{10\alpha\tau[r + C'_p + C_p + C_T]}$, $M(\tau) = \alpha e^{10\alpha\tau[r + C'_p + C_p + C_T]}$ for $\tau > 0$. If $\tau$ is small enough such that $2L(\tau) < 1$. Then the solution $x(t)$ of system (5) satisfies that

$$\dot{E}(\kappa(|x(t) - x(\delta_t)|^2)) \leq \frac{2L(\tau)}{1 - 2L(\tau)} \dot{E}(\kappa(|x(\delta_t)|^2)) + \frac{M(\tau)}{1 - 2L(\tau)}.$$  

**Proof.** For $lT \leq t < (l + 1)\tau$, $l \geq 0$ is any integer, we have $\delta_t = [t/\tau] \tau = l\tau$. By (5), it is easy to obtain that

$$|x(t) - x(\delta_t)|^2 = |x(t) - x(l\tau)|^2 = \int_{l\tau}^{t} (f(r, x(r)) + u(r, x(\delta_r))) dr + h(r, x(r))dB_r + \int_{\mathbb{R}_d} K(r, x(r), z)L(dr, dz)^2$$

$$\leq 5\int_{l\tau}^{t} f(r, x(r))dr \leq |2 + 5| \int_{l\tau}^{t} u(r, x(\delta_r))dr |2 + 5| \int_{l\tau}^{t} h(r, x(r))dB_r dr$$

$$+ 5\int_{l\tau}^{t} \sigma(r, x(r))dB_r dr + 5\int_{l\tau}^{t} K(r, x(r), z)L(dr, dz)^2.$$ 

It follows from that Hölder inequality, Lemma 2.11—2.13, we have

$$\dot{E}|x(t) - x(\delta_t)|^2$$

$$\leq 5\tau \int_{l\tau}^{t} \dot{E}(\kappa(|x|^2)) dr + 5\tau \int_{l\tau}^{t} \dot{E}(\kappa(|x(\delta_r)|^2)) dr + 5\tau C'_p \int_{l\tau}^{t} \dot{E}(\kappa(|x|^2)) dr + 5C_p \int_{l\tau}^{t} \dot{E}(\kappa(|x|^2)) dr + 5C_T \int_{l\tau}^{t} \dot{E}(\kappa(|x|^2)) dr$$

$$\leq 10|\tau + C'_p + C_p + C_T| \int_{l\tau}^{t} \dot{E}(\kappa(|x(t) - x(l\tau)|^2)) dr$$

$$+ \tau[10\tau + 10C'_p + 10C_p + 10C_T + 5\tau] \dot{E}(\kappa(|x(\tau)|^2)) dr.$$ 

So,

$$\dot{E}(\kappa(|x(t) - x(\delta_t)|^2))$$

$$= \dot{E}(a + a|x(t) - x(\delta_t)|^2) \leq a + a\dot{E}(|x(t) - x(\delta_t)|^2)$$
Let Assumption 1 and 2 hold. If there exist two positive constants $C_3$ and $C_4$ such that

$$C_3|x|^2 \leq V(t, x) \leq C_4|x|^2, \quad (t, x) \in R^+ \times R^n,$$

then the solution of controlled system (5) is mean square exponentially stable, which satisfies that

$$\dot{\hat{E}}(\langle x(t) \rangle^2) \leq Ce^{-\gamma(t-t_0)},$$

for any initial data $x_0 \in R^n$, where $C > 0$. Here $\tau$, $\lambda$ and $\rho$ are the same as in Proposition 1 and $\gamma$ is the unique positive solution shown in the following

$$\gamma = K\gamma + \frac{\gamma C_4}{a},$$

with

$$K = \rho \sigma + 1.$$

Proof. By G-Itô formula, we have

$$d(e^{\gamma t}\hat{V}(t, x_t)) = e^{\gamma t}[\gamma \hat{V}(t, x_t) + \hat{V}_t(t, x_t) + <\hat{V}_x(t, x_t), f(t, x(t)) + u(t, x(\delta_t))>dt$$

$$+ e^{\gamma t}(\hat{V}_x(t, x_t), \sigma(t, x(t)))dB(t) + e^{\gamma t}(\hat{V}_z(t, x_t), h(t, x(t)))d(B)(t)$$

$$+ \frac{1}{2} e^{\gamma t}(\hat{V}_{xx}(t, x_t)\sigma(t, x(t)), \sigma(t, x(t)))d(B)'(t)$$

$$+ \int_{\mathbb{R}^d} e^{\gamma t}[\hat{V}(t, x_t - J(t, x(t), z)) - \hat{V}(t, x_t)]L(dt, dz).$$
Hence,
\[ e^{\gamma t} \tilde{V}(t, x_t) = \tilde{V}(0, x_0) + \int_0^t e^{\gamma r} [\gamma \tilde{V}(r, x_r) + LV(r, x_r)]dr + \int_0^t e^{\gamma r} (\tilde{V}(r, x_r), \sigma(r, x(r)))dB(r) + N_t^0 + D_t^0, \]
where
\[ N_t^s := \int_s^t e^{\gamma r} (\tilde{V}_x(r, x_r), h(r, x_r)) + \frac{1}{2} (\tilde{V}_{xx}(r, x_r) \sigma(r, x_r), \sigma(r, x_r))d(B)(r) \]
\[ - \int_s^t e^{\gamma r} \sup_{Q \in \mathcal{Q}} tr[(\tilde{V}_z(r, x_r), h(r, x_r)) + \frac{1}{2} (\tilde{V}_{zz}(r, x_r) \sigma(r, x_r), \sigma(r, x_r))QQ^T]dr, \]
\[ D_t^s := \int_s^t \int_{\mathbb{R}^d} e^{\gamma r} (\tilde{V}(r, x_r - K(r, x_r, z)) - \tilde{V}(r, x_r))L(dr, dz) \]
\[ - \int_s^t \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} e^{\gamma r} (\tilde{V}(r, x_r - K(r, x_r, z)) - \tilde{V}(r, x_r))v(dz)dr. \]
So,
\[ \dot{E}[e^{\gamma t} \tilde{V}(t, x_t)] = \dot{\tilde{V}}(0, x_0) + \dot{E} \int_0^t e^{\gamma r} [\gamma \tilde{V}(r, x_r) + LV(r, x_r)]dr. \]
Combining (18) with (20), we obtain
\[ \dot{E}[e^{\gamma t} \tilde{V}(t, x_t)] \leq C_1 + \int_0^t e^{\gamma r} [\gamma \dot{E} \tilde{V}(r, x_r) - \lambda \dot{E}(\kappa(|x(r)|^2))]dr. \]
By (8) and (22), we get
\[ \dot{E} \tilde{V}(r, x_r) \leq C_4 \dot{E}(|x(r)|^2) + \dot{E}(U(r, x_r)), \]
where
\[ U(r, x_r) = \rho \int_{r-\tau}^r \int_{r-\tau}^r [\tau |f(w, x(w)) + u(w, x(\delta_w))|^2 \]
\[ + C_p' \tau |h(w, x(w))|^2 + C_p |\sigma(w, x(w))|^2 \]
\[ + C_T \sup_{v \in \mathcal{V}} \int_{\mathbb{R}^d} K^2(w, x(w), z)v(dz)] dw dr. \]
By Assumption 1, it is easy to get
\[ \dot{E} U(t, x_t) \leq \rho \tau \int_{t-\tau}^t [(6\tau + C_p' \tau + C_p + C_T) \dot{E}(\kappa(|x(r)|^2)) + 4\tau \dot{E}(\kappa(|x(r) - x(\delta_r)|^2))]dr. \]
For \( t \geq \tau \), by Lemma 3.5, we have
\[ \dot{E} U(t, x_t) \]
\[ \leq \rho \tau \int_{t-\tau}^t [(6\tau + C_p' \tau + C_p + C_T) \dot{E}(\kappa(|x(r)|^2)) + \frac{8\tau L(\tau)}{1 - 2L(\tau)} \dot{E}(\kappa(|x(r)|^2))] \]
\[ + \frac{8\tau M(\tau)}{1 - 2L(\tau)}]dr \]
\[ \leq K \int_{t-\tau}^t \dot{E}(\kappa(|x(r)|^2))dr + H, t \geq \tau, \]
where \( H = \frac{8r^3 \rho_M(r)}{1 - 2L(r)} \). Combining (22), (28), (29) and (31), we obtain

\[
C_3 e^{\gamma t} \hat{E}(|x(t)|^2) \leq C_1 + \gamma C_4 \int_0^t e^{\gamma r} \hat{E}(|x(r)|^2) dr + \int_0^t e^{\gamma r} H dr
\]

\[
+ \int_0^t e^{\gamma r} K \int_r^\tau \hat{E}(\kappa(|x(v)|^2)) dv dr - \int_0^t e^{\gamma r} \lambda \hat{E}(\kappa(|x(r)|^2)) dr
\]

\[
\leq C_1 + \gamma C_4 \int_0^t e^{\gamma r} \hat{E}(|x(r)|^2) dr + \int_0^t e^{\gamma r} H dr
\]

\[
+ K \tau \gamma \int_0^t e^{\gamma r} \hat{E}(a + a|x(r)|^2) dr - \int_0^t e^{\gamma r} \lambda \hat{E}(a + a|x(r)|^2) dr
\]

\[
\leq (C_1 + \int_0^t e^{\gamma r} H dr + aK \tau \gamma e^{r \gamma} - a\lambda e^{r \gamma})
\]

\[
+ (\gamma C_4 + aK \tau \gamma - a\lambda) \int_0^t e^{\gamma r} \hat{E}(|x(r)|^2) dr.
\]

(32)

Under the condition (24), we get

\[
C_3 e^{\gamma t} \hat{E}(|x(t)|^2) \leq C,
\]

\[
C = C_1 + \int_0^t e^{\gamma r} H dr + aK \tau \gamma e^{r \gamma} - a\lambda e^{r \gamma}, \forall t \geq \tau.
\]

This completes the proof. \( \square \)

**Theorem 3.7.** Let Assumption 1, Assumption 2 and the conditions of Theorem 3.6 hold, and

\[
|f(t,x)|^2 + |h(t,x)|^2 + |\sigma(t,x)|^2 + \sup_{v \in v} \int_\mathbb{R}^d |K(t,x,z)|^2 v(dz) \leq \kappa(|x|^2).
\]

(33)

Then the solution of controlled system (5) is quasi-sure exponentially stable.

**Proof.** By Theorem 3.6, the solution is mean square exponentially stable. Therefore, there exists a positive constant \( C_5 \) such that

\[
\hat{E}(|x(t)|^2) \leq C_5 e^{-\gamma t}.
\]

(34)

Moreover,

\[
x(t+s) = x(t) + \int_t^{t+s} [f(r,x(r)) + u(r,x(\delta r))] dr + \int_t^{t+s} h(r,x(r)) d\langle B \rangle (r)
\]

\[
+ \int_t^{t+s} \sigma(r,x(r)) dB(r) + \int_t^{t+s} \int_\mathbb{R}^d K(r,x(r),z) L(dr,dz).
\]

Hence,

\[
|x(t+s)|^2 \leq 6||x(t)||^2 + \left| \int_t^{t+s} f(r,x(r)) dr \right|^2
\]

\[
+ \left| \int_t^{t+s} u(r,x(\delta r)) dr \right|^2 + \left| \int_t^{t+s} h(r,x(r)) d\langle B \rangle (r) \right|^2
\]

\[
+ \left| \int_t^{t+s} \sigma(r,x(r)) dB(r) \right|^2 + \left| \int_t^{t+s} \int_\mathbb{R}^d K(r,x(r),z) L(dr,dz) \right|^2.
\]
Thus we have
\[
\hat{E}\left[ \sup_{0 \leq s \leq h} |x(t + s)|^2 \right] \\
\leq 6(\hat{E}[x(t)]^2 + \hat{E}\left[ \int_t^{t+h} |f(r, x(r))|dr \right]^2 + \hat{E}\left[ \int_t^{t+h} |u(r, x(\delta_r))|dr \right]^2 \\
+ \hat{E}\left[ \sup_{0 \leq s \leq h} |\int_t^{t+s} h(r, x(r))d(B)(r)^2 \right] + \hat{E}\left[ \sup_{0 \leq s \leq h} \int_t^{t+s} |\sigma(r, x(r))dB(r)|^2 \right] \\
+ \hat{E}\left[ \sup_{0 \leq s \leq h} \int_t^{t+s} \int_{R^2} K(r, x(r), z)L(dr, dz)|^2 \right],
\]
(35)

where \( h \) is a positive constant. By Hölder inequality, Lemma 2.11-2.13 and (34), we have
\[
\hat{E}\left[ \int_t^{t+h} |f(r, x(r))|dr \right]^2 \leq h \int_t^{t+h} \hat{E}[f(r, x(r))]^2 dr \leq h \int_t^{t+h} \hat{E}[\kappa(|x(r)|^2)] dr \\
\leq h \int_t^{t+h} \hat{E}[a + a|x(r)|^2] dr \leq h^2 a + ah \int_t^{t+h} C_5 e^{-\gamma t} dr \\
\leq h^2 a + \frac{C_5 ha}{\gamma} e^{-\gamma t}.
\]
(36)

Using the same way as (36), we have
\[
\hat{E}\left[ \sup_{0 \leq s \leq h} |u(r, x(\delta_r))|dr \right]^2 \leq h^2 a + \frac{C_5 ha}{\gamma} e^{-\gamma t},
\]
(37)
\[
\hat{E}\left[ \sup_{0 \leq s \leq h} |\int_t^{t+s} h(r, x(r))d(B)(r)^2 \right] \leq C_p'h^2 a + \frac{C_p'C_5 ha}{\gamma} e^{-\gamma t},
\]
(38)
\[
\hat{E}\left[ \sup_{0 \leq s \leq h} \int_t^{t+s} |\sigma(r, x(r))dB(r)|^2 \right] \leq C_p ha + \frac{C_p C_5 a}{\gamma} e^{-\gamma t},
\]
(39)
\[
\hat{E}\left[ \sup_{0 \leq s \leq h} \int_t^{t+s} \int_{R^2} K(r, x(r), z)L(dr, dz)|^2 \right] \leq C_T ha + \frac{C_T C_5 a}{\gamma} e^{-\gamma t}.
\]
(40)

Substituting (36)–(40) into (35), we have
\[
\hat{E}\left[ \sup_{0 \leq s \leq h} |x(t + s)|^2 \right] \leq C_6 e^{-\gamma t} + C_T,
\]
where \( C_6 \) is a positive constant, \( C_T = 6ha(2h+C_p'h+C_p+C_T) \) is a positive constant.

It follows that
\[
\hat{E}\left[ \sup_{nh \leq t \leq (n+1)h} |x(t)|^2 \right] \leq C_6 e^{-\gamma nh} + C_T, n = 1, 2, \cdots
\]
Hence, for an arbitrary \( \varepsilon \in (0, \gamma) \), from Lemma 2.9, we have
\[
c(\omega) \sup_{nh \leq t \leq (n+1)h} |x(t)|^2 \leq e^{-(\gamma-\varepsilon)nh} \leq e^{-\varepsilon nh}(C_6 + C_T e^{\gamma nh}).
\]
It follows from the Borel-Cantelli lemma, there exists a \( N(\omega) \) such that for almost all \( \omega \in \Omega, n > N(\omega) \), we have
\[
\sup_{nh \leq t \leq (n+1)h} |x(t)|^2 \leq e^{-(\gamma-\varepsilon)nh}, q.s.
\]
This gives

\[ \limsup_{t \to \infty} \frac{\ln |x(t)|}{t} \leq -\frac{(\gamma - \varepsilon)}{2}, \quad q.s.. \]

If \( \varepsilon \to 0 \), from Definition 3.2, we get the desired results, this completes the proof. \( \Box \)

4. Example.

**Example 2.** Let us consider the following stochastic differential equation driven by the G-Lévy process

\[ dx(t) = f(t, x(t))dt + h(t, x(t))dB(t) + \sigma(t, x(t))dB(t) \]

+ \( \int_{\mathbb{R}_0^d} K(t, x(t), z)L(dt, dz) \), \( t \geq 0 \), \hspace{1cm} (41)

where

\[ f(t, x(t)) = x(t)\sin t, \quad h(t, x(t)) = x(t), \quad \sigma(t, x(t)) = \sqrt{2}x(t), \quad K(t, x(t), z) = 2x(t)R(z). \]

We assume that the function \( R(z) \) satisfies

\[ 1 - \sup_{Q \in \mathcal{Q}} \sup_{Q \in \mathcal{Q}} \int_{\mathbb{R}_0^d} |R(z)|^2 + R(z)v(dz) < 2 - \sup_{Q \in \mathcal{Q}} \sup_{Q \in \mathcal{Q}} \int_{\mathbb{R}_0^d} |IQQT|^2, \]

where \( I \) is a \( d \times d \) dimensional matrix whose elements are all one.

Let us consider \( V(t, x(t)) = |x|^2 \), we have

\[ LV = V_t(t, x) + \langle V_x(t, x), f(t, x) \rangle + \sup_{Q \in \mathcal{Q}} \sup_{Q \in \mathcal{Q}} \int_{\mathbb{R}_0^d} (2|x|^2 + 2|x|^2)QQ^T \]

+ \( \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_0^d} (|x(t)|^2 + 2|x(t)|^2)|x(t)|^2 \]

\[ \geq 2|x(t)|^2. \]

According to Theorem 3.4, the system (41) is mean square exponentially unstable.

Let \( u(t, x(t)) = kx(t), k = -8 \). It follows that

\[ LV = V_t(t, x) + \langle V_x(t, x), f(t, x) + u(t, x) \rangle + \sup_{Q \in \mathcal{Q}} \sup_{Q \in \mathcal{Q}} \int_{\mathbb{R}_0^d} (2|x|^2 + 2|x|^2)QQ^T \]

+ \( \sup_{v \in \mathcal{V}} \int_{\mathbb{R}_0^d} (|x(t) + 2x(t)R(z)|^2 - |x(t)|^2)v(dz) \)

\[ = (2\sin t + 2k + 4) \sup_{Q \in \mathcal{Q}} \sup_{Q \in \mathcal{Q}} \int_{\mathbb{R}_0^d} (|R(z)|^2 + R(z)v(dz))|x(t)|^2 \]
\[ \leq -6|x(t)|^2. \]

Thus, the conditions in Theorem 3.6 are satisfied. Hence the system (41) is mean square exponentially stable.

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REFERENCES

[1] X.-P. Bai and Y.-Q. Lin, On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with integral-Lipschitz coefficients, Acta Math. Appl. Sin. Engl. Ser., 30 (2014), 589–610.

[2] L. Denis, M. S. Hu and S. G. Peng, Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths, Potential Analysis, 34 (2011), 139–161.

[3] C. Fei, W. Y. Fei, X. R. Mao, D. F. Xia and L. T. Yan, Stabilisation of highly nonlinear hybrid systems by feedback control based on discrete-time state observations, Institute of Electrical and Electronics Engineers. Transactions on Automatic Control, (2019), http://dx.doi.org/10.1109/TAC.2019.2933604.

[4] F. Q. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and Their Applications, 119 (2009), 3356–3382.

[5] F. Q. Gao and H. Jiang, Large deviations for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and Their Applications, 120 (2010), 2212–2240.

[6] L. J. Hu and X. R. Mao, Almost sure exponential stabilisation of stochastic systems by state-feedback control, Automatica J. IFAC, 44 (2008), 465–471.

[7] M. S. Hu and S. G. Peng, G-Lévy processes under Sublinear Expectations, arXiv:0911.3533v1.

[8] M. L. Li, F. Q. Deng and X. R. Mao, Basic theory and stability analysis for neutral stochastic functional differential equations with pure jumps, Science China Information Sciences, 62 (2019), 012204, 15 pp.

[9] X. P. Li and S. G. Peng, Stopping times and related Itô’s calculus with G-Brownian motion, Stochastic Processes and Their Applications, 121 (2011), 1492–1508.

[10] X. Y. Li and X. R. Mao, Lyapunov-type conditions and stochastic differential equations driven by G-Brownian motion, Journal of Mathematical Analysis and Applications, 439 (2016), 235–255.

[11] X. Y. Li and X. R. Mao, Stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control, Automatica J. IFAC, 112 (2020), 108657, 11 pp.

[12] X. R. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, Automatica J. IFAC, 49 (2013), 3677–3681.

[13] X. R. Mao, Stochastic differential equations and their applications (2nd ed.), Elsevier, (2007).

[14] X. R. Mao, Almost sure exponential stabilization by discrete-time stochastic feedback control, IEEE Trans. Automat. Control, 61 (2016), 1619–1624.

[15] A. Neufeld and M. Nutz, Nonlinear Lévy processes and their characteristics, Transactions of the American Mathematical Society, 369 (2017), 69–95.

[16] K. Paczka, Itô calculus and jump diffusions for G-Lévy processes, arXiv:1211.2973v3.

[17] K. Paczka, On the properites of Poisson random measures associated with a G-Lévy process, arXiv:1411.4660v1.

[18] K. Paczka, G-martingale representation in the G-Lévy setting, arXiv:1404.2121v1.

[19] S. G. Peng, Nonlinear Expectations and Stochastic Calculus under Uncertainty: With Robust CLT and G-Brownian Motion, Probability Theory and Stochastic Modelling, 95. Springer, Berlin, 2019, arXiv:1002.4546v1.

[20] S. G. Peng, Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, Science in China. Series A, 52 (2009), 1391–1411.

[21] S. G. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Itô type, Stochastic Analysis and Applications, Abel Symp., Springer, Berlin, 2 (2007), 541–567.
[22] S. G. Peng, Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, *Stochastic Processes and Their Applications*, 118 (2008), 2223–2253.

[23] Q. Qiu, W. Liu, L. Hu and J. Lu, Stabilisation of hybrid stochastic systems under discrete observation and sample delay, *Control Theory and Applications*, 33 (2016), 1024–1030.

[24] L. Y. Ren, On representation theorem of sublinear expectation related to G-Lévy process and paths of G-Lévy process, *Statistics and Probability Letters*, 83 (2013), 1301–1310.

[25] Y. Ren, W. S. Yin and R. Sakthivel, Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation, *Automatica J. IFAC*, 95 (2018), 146–151.

[26] Y. Ren, W. S. Yin and D. J. Zhu, Stabilisation of SDEs and applications to synchronisation of stochastic neural network driven by G-Brownian motion with state-feedback control, *International Journal of Systems Science*, 50 (2019), 273–282.

[27] Y. Ren, X. J. Jia and R. Sakthivel, The pth moment stability of solutions to impulsive stochastic differential equations driven by G-Brownian motion, *Applicable Analysis*, 96 (2017), 988–1003.

[28] J. H. Shao, Stabilization of regime-switching processes by feedback control based on discrete time state observations, *SIAM Journal on Control and Optimization*, 55 (2017), 724–740.

[29] G. F. Song, Z. Y. Lu, B.-C. Zheng and X. R. Mao, Almost sure stabilization of hybrid systems by feedback control based on discrete-time observations of mode and state, *Science China Information Sciences*, 61 (2018), 070213, 16 pp.

[30] H. M. Soner, N. Touzi and J. F. Zhang, Martingale representation theorem for the G-expectation, *Stochastic Processes and Their Applications*, 121 (2011), 265–287.

[31] B. J. Wang and M. X. Yuan, Existence of solution for stochastic differential equations driven by G-Lévy processes with discontinuous coefficients, *Advances in Difference Equations*, 2017 (2017), 13 pp.

[32] B. J. Wang and H. J. Gao, Exponential stability of solutions to stochastic differential equations driven by G-Lévy Process, *Applied Mathematics and Optimization*, (2019).

[33] S. R. You, W. Liu, J. Q. Liu, X. R. Mao and Q. W. Qiu, Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM Journal on Control and Optimization*, 53 (2015), 905–925.

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