AUTOMORPHISM GROUPS IN A FAMILY OF K3 SURFACES

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ABSTRACT. A few facts concerning the phrase "the automorphism groups become larger at special points of the moduli of K3 surfaces" are presented. It is also shown that the automorphism groups are of infinite order over a dense subset in any one-dimensional non-trivial family of projective K3 surfaces.

INTRODUCTION

According to the global Torelli Theorem for K3 surfaces due to Piatetski-Shapiro and Shafarevich [PSS] (See also [BPV]), one can describe the automorphism group of a K3 surface as the quotient of the orthogonal group of the Picard lattice by the 2-reflection group, up to finite groups, i.e. up to finite kernel and cokernel, which we denote by \( \equiv \) later. Although an explicit calculation of such a quotient group is rather hard even in a concrete case, beautiful calculations of the automorphism groups of some special but quite remarkable K3 surfaces have been done by Vinberg, Keum, Kondo and others: for instance those of K3 surfaces of \( \rho = 20 \) and of smaller discriminant by [Vi] and [KK], those of generic Kummer surfaces by [Ke] and [Kn3]. In these examples, the automorphism groups are of infinite order. As a counter part, Kondo [Kn1] has also classified the automorphism groups of K3 surfaces \( X \) with \( |\text{Aut}(X)| < \infty \) based on Nukulin’s classification of both the Picard lattices and the dual graphs of smooth rational curves, i.e. generators and relations of the 2-reflection groups of such K3 surfaces ([Ni5] and references therein).

On the other hand, since no polarization on a K3 surface is invariant under \( \text{Aut}(X) \) unless \( |\text{Aut}(X)| < \infty \) and since the behaviour of the Picard lattices is rather unstable under deformation (see for instance [Og]), even when one considers a projective family, one can not relate the full automorphism groups \( \text{Aut}(X) \) well with the automorphism group \( \text{PGL}(n) \) of the ambient space \( \mathbb{P}^n \) nor simultaneously with the orthogonal groups of their Picard lattices, and it does not seem so clear how \( \text{Aut}(X) \) behaves when a K3 surface \( X \) varies in a family.

Aim of this short note is to clarify certain behaviour of the automorphism groups of K3 surfaces under deformation. Saying more explicitly, our interest is to discuss about the phrase, "the automorphism groups of K3 surfaces become larger at special points in their moduli." (See the main Theorem and Corollary below.)

Here and hereafter, we shall work over the complex number field \( \mathbb{C} \). By a K3 surface we mean a simply-connected smooth projective surface \( X \) which admits an everywhere non-vanishing holomorphic 2-from \( \omega_X \).

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In order to state the results in the strongest form, throughout this note, we shall consider a non-trivial smooth family of K3 surfaces over the unit disk:

\[ \varphi : \mathcal{X} \to \Delta := \{ t \in \mathbb{C} \mid |t| < 1 \}. \]

If prefer, one may regard this family either as a 1-dimensional small part of the Kuranishi family of a given K3 surface or as a 1-dimensional small non-trivial part of a global family of K3 surfaces, e.g. a universal family of polarized K3 surfaces of given degree. We regard \( \Delta \) as a topological space by the Euclidean topology unless stated otherwise. We denote the fiber \( \varphi^{-1}(t) \) by \( \mathcal{X}_t \).

For the statement, we choose a marking

\[ \tau : R^2 \varphi_* Z_\mathcal{X} \to \Lambda \times \Delta. \]

Here \( \Lambda \) is the K3 lattice, i.e. the lattice isomorphic to \( U^\oplus 3 \oplus E_8(-1)^\oplus 2 \). Set \( \Lambda_t := \tau_t(\text{NS}(\mathcal{X}_t)) \). By [Og], one can find a primitive sublattice \( \Lambda^0 \subset \Lambda \) and an uncountable dense subset \( \mathcal{G} \subset \Delta \) such that:

1. \( S := \Delta - \mathcal{G} \) is countable and dense;
2. \( \Lambda^0 \subset \Lambda_t \) for all \( t \in \Delta \);
3. \( \Lambda^0 = \Lambda_t \) for all \( t \in \mathcal{G} \);
4. \( \Lambda^0 \neq \Lambda_t \) for all \( t \in S \).

We call a point \( t \in \mathcal{G} \) generic and a point \( s \in S \) special. Special points are nothing but the points \( s \) at which the Picard numbers \( \rho(\mathcal{X}_s) \) jump above.

Under this notation, we can state our main result as follows:

**Main Theorem.** Assume that \( \varphi : \mathcal{X} \to \Delta \) is projective, i.e. there exists a \( \varphi \)-ample invertible sheaf \( \mathcal{L} \). Then:

1. There exist a (possibly empty) finite subset \( \mathcal{F} \subset \mathcal{S} \), a group \( G^0 \), and a positive integer \( N \) depending only on \( \varphi \) such that

   \[ G^0 < \text{Aut}(\mathcal{X}_t) \text{ for all } t \in \Delta - \mathcal{F} \]

   and

   \[ [\text{Aut}(\mathcal{X}_t) : G^0] \leq N \text{ for all } t \in \mathcal{G}. \]

   In particular, the map

   \[ \text{Aut} : \Delta \to \{ \text{groups} \} / \equiv ; t \mapsto [\text{Aut}(\mathcal{X}_t)] \]

   is "upper-semicontinuous" with respect to the co-finite topology of \( \Delta - \mathcal{F} \).

2. There exists a subset \( \mathcal{D} \subset \mathcal{S} \) such that \( \mathcal{D} \) is dense in \( \Delta \) and that

   \[ |\text{Aut}(\mathcal{X}_t)| = \infty \text{ for all } t \in \mathcal{D}. \]

3. There exists a projective family of K3 surfaces \( \varphi : \mathcal{X} \to \Delta \) such that \( \mathcal{F} \neq \emptyset \).

As a direct consequence of the main Theorem, one obtains the following:
Corollary. Let $f : \mathcal{X} \to \Delta$ be a (not necessarily projective) non-trivial family of K3 surfaces. Then, there is a dense subset $\mathcal{D} \subset \Delta$ such that $|\text{Aut}(\mathcal{X}_t)| = \infty$ for all $t \in \mathcal{D}$. In particular, the nef cone $\overline{\text{A}(\mathcal{X}_t)}$ is not finite rational polyhedral if $t \in \mathcal{D}$.

Compare the results with certain boundedness of K3 surfaces with $|\text{Aut}(\mathcal{X})| < \infty$ ([Ni6]). We shall prove the main Theorem and Corollary in Section 1.

The assertion (2) and Corollary are somewhat surprising. For instance, let us take the component $\mathcal{H}$ of the Hilbert scheme consisting of quartic K3 surfaces and consider the universal family $u : \mathcal{U} \to \mathcal{H}$. Then for any sufficiently general $\Delta \to \mathcal{H}$, the induced family $\varphi : \mathcal{X} \to \Delta$ satisfies $\text{Pic}(\mathcal{X}_t) = \mathbb{Z}\mathcal{L}_t$ for $t \in G$, where $\mathcal{L}_t$ is the plane section class of $\mathcal{X}_t \subset \mathbb{P}^3$. So $|\text{Aut}(\mathcal{X}_t)| < \infty$ if $t \in G$. A bit more strongly, there also exists $M$ with $|\text{Aut}(\mathcal{X}_t)| < M$ for all $t \in G$ by the boundedness of the finite subgroups of automorphism groups of K3 surfaces ([Ni1], [Mu] see also [Kn2]). The statement (2) claims, however, that there exists a dense subset $\mathcal{D}$ such that $|\text{Aut}(\mathcal{X}_t)| = \infty$ for all $t \in \mathcal{D}$ even in this family. This also provides an explicit example of a family in which the automorphism groups actually jump above. On the other hand, one has by Kollár([Bo]): If $f : \mathcal{Y} \to \Delta$ is a family of Calabi-Yau manifolds in $|-K_V|$ of a Fano manifold $V$, then the nef cones $\overline{\text{A}(\mathcal{Y}_t)}$ are finite rational polyhedral, whence $|\text{Aut}(\mathcal{Y}_t)| < \infty$, for all $t \in \Delta$ provided that dim $V \geq 4$. Thus, the statements similar to Theorem (2) and Corollary do not hold for Calabi-Yau manifolds of higher dimension.

The assertion (1) mostly justifies the phrase quoted at the beginning, while the assertion (3) denies the phrase in the most strict sense. By the statement (1) and Corollary, one might expect something more geometric behind such as a covariant relation among jumping of automorphism groups, jumping of Picard numbers, those of smooth rational curves and those of elliptic pencils in a family of K3 surfaces. However, there are no such relations in general. Indeed, the second aim of this note is to point out the following:

Proposition. There exist families of projective K3 surfaces $\varphi^i : \mathcal{X}^i \to \Delta$ $(1 \leq i \leq 5)$ such that the central fiber $\mathcal{X}^i_0$ satisfies that $\rho(\mathcal{X}^i_0) = 19$, $|\text{Aut}(\mathcal{X}^i_0)| < \infty$, contains finitely many smooth rational curves and admits finitely many elliptic pencils, but that generic fiber $\mathcal{X}^i_t$ satisfies $\rho(\mathcal{X}^i_t) = 3$ and that:

1. $|\text{Aut}(\mathcal{X}^1_t)| = \infty$ and $\mathcal{X}^1_t$ contains infinitely many smooth rational curves but admits no elliptic pencil.
2. $|\text{Aut}(\mathcal{X}^2_t)| = \infty$ and $\mathcal{X}^2_t$ contains no smooth rational curve but admits infinitely many elliptic pencils.
3. $|\text{Aut}(\mathcal{X}^3_t)| = \infty$ and $\mathcal{X}^3_t$ has infinitely many smooth rational curves and infinitely many elliptic pencils.
4. $|\text{Aut}(\mathcal{X}^4_t)| = \infty$ and $\mathcal{X}^4_t$ has neither smooth rational curve nor elliptic pencil.
5. $|\text{Aut}(\mathcal{X}^5_t)| < \infty$ and $\mathcal{X}^5_t$ has both (necessarily finitely many) smooth rational curves and elliptic pencils.

We shall construct these families in Section 2 by deforming a very remarkable K3 surface found by Nikulin [Ni3] (See also [Kn1]). In our construction, the morphisms $\varphi^i$ are not projective.

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Proof of Theorem (1). Set

\[ l := \tau_t([L_{X_t}]) \in \Lambda_t (\subset \Lambda). \]

Since \( l \) is independent of \( t \), we have \( l \in \Lambda^0 \). We may assume that the class \( l \) is primitive. Let \( A(X_t) \) be the ample cone of \( X_t \). We set:

\[
A_t := \tau_t(A(X_t)) \subset \Lambda_t \otimes \mathbb{R};
\]

\[
A_t^0 := A_t \cap (\Lambda^0 \otimes \mathbb{R});
\]

\[
G_t := \{ g \in O(\Lambda^0) \mid g((\Lambda^0*/\Lambda^0) = \text{id}, g(l) \subset A_t^0 \},
\]

for each \( t \in \Delta \). Here \( \Lambda^0* \) is the dual lattice of \( \Lambda^0 \) and is regarded as an overlattice of \( \Lambda^0 \) via the cup product \((*,**,*)\), which is non-degenerate on \( \Lambda^0 \) by the Hodge index Theorem. Note that \( l \in A_t^0 \) for each \( t \). For simplicity of description, we often identify \( \Lambda_t \) with \( NS(X_t) \), by \( \tau_t \). For instance, we say that an element \( a \in A_t \) is ample on \( X_t \). The next Claim completes the proof of Theorem (1):

**Claim 1.** Choose \( p \in \mathcal{G} \). Then, there exists a (possibly empty) finite subset \( \mathcal{F} \subset \mathcal{S} \) such that

1. \( A_t^0 \subset A_p \) for all \( t \in \Delta \), and \( A_t = A_p \) for all \( t \in \mathcal{G} \).
2. \( G_t \supset G_p \) for all \( t \in \Delta - \mathcal{F} \), and \( G_t = G_p \) for all \( t \in \mathcal{G} \).
3. There is a natural embedding \( \iota_t : G_t \hookrightarrow Aut(X_t) \) for all \( t \in \Delta \). Moreover, there exists a positive integer \( N \) such that \( [Aut(X_t) : \iota_t(G_t)] \leq N \) for all \( t \in \mathcal{G} \).

**Proof of Claim 1.** The first assertion is clear if rank \( \Lambda^0 = 1 \). We assume rank \( \Lambda^0 \geq 2 \). Let \( h \in A^0_t \). Then \( (h^2) > 0 \) and \( (h, l) > 0 \). If \( h \) is not ample on \( X_p \), then there is an irreducible curve \( C \) on \( X_p \) such that \( ([C], h) \leq 0 \). This is due to the real version of the Nakai-Moishezon criterion of ampleness by Campana and Peternell [CP]. This \( C \) must be a smooth rational curve by the Hodge index Theorem and the adjunction formula. Moreover, since \( C \) is a curve on \( X_p \), we have \([C] \in \Lambda^0 \). Thus \([C] \in \Lambda_t \) as well. Since \((C^2) = -2\), either \([-C] \text{ or } [C] \) is represented by a (non-effective) effective curve on \( X_t \) by the Riemann-Roch Theorem. Since \(([C], l) > 0 \) by the ampleness of \( l \) plus effectiveness of \( C \) on \( X_p \), and since \( l \) is also ample on \( X_t \), it is \([C] \) that is represented by an effective curve on \( X_t \). (Here the assumption \( \varphi \text{ being projective is essential.} \) However, since \( h \in A_t \), one would then have \(([C], h) > 0\), a contradiction to the previous inequality. Here we used the fact that the values \((*,**,*)\) are independent on which K3 surfaces \( X_t \) we are arguing. Therefore we have \( A^0_t \subset A_p \) for all \( t \in \Delta \). (Note, however, that the other inclusion is false in general as we will see in the proof of (3).) Since \( A^0_t = A_t \) if \( t \in \mathcal{G} \), replacing \( p \) by \( t \), one has \( A_t = A_p \) for all \( t \in \mathcal{G} \). Now, by the definition of \( G_\sigma \), one has \( G_t = G_p \) for all \( t \in \mathcal{G} \).
We shall show the first assertion of (2). From now, we write $A_p$ by $A$. Set

$$\mathcal{F} := \{ t \in S | G_p \not\subset G_t \}.$$  
We need to show that $\mathcal{F}$ is finite. Recall that the group $G_p$ is finitely generated by [BC] (See also [St]). Indeed, $G_p$ is a quotient of an arithmetic subgroup of $O(\Lambda^0)$. Choose a set of generators of $G_p$ and denote it by $\{ g_i | 1 \leq i \leq n \}$. Set for each $i$:

$$\mathcal{F}_i := \{ t \in S | g_i \not\in G_t \}.$$  

It suffices to show that $|\mathcal{F}_i| < \infty$. Choose $i$ and put $g := g_i$. Assume that $t \in \mathcal{F}_i$. Then, by the definition, $g(l) \not\in A_t$. (Note that $g(l) \in \Lambda^0$ by $g \in G_p$.) Since $(g(l)^2) = (l^2) > 0$ and $(g(l), l) > 0$ by $l$, $g(l) \in A$, we see that $g(l)$ is in the positive cone of $X_t$ by the Hodge index Theorem. Then, applying the Riemann-Roch Theorem, one finds an effective divisor $D$ on $X_t$ such that $g(l) = [D]$. Since $g(l)$ is not ample on a K3 surface $X_t$, there exists a smooth rational curve $C_t$ such that $(g(l), C_t) \leq 0$.

Let $\tilde{g}$ be the element of $O(\Lambda)$ such that $\tilde{g}|\Lambda^0 = g$ and $\tilde{g}|\Lambda^{0\perp} = id$, where $\Lambda^{0\perp}$ is the orthogonal complement of $\Lambda^0$ in $\Lambda$. Such an extension exists, because $g|\Lambda^{0\ast}/\Lambda^0 = id$. Since $T_{X_t} \subset \Lambda^{0\perp}$, where $T_{X_t}$ is the transcendental lattice of $X_t$, this $\tilde{g}$ is also a Hodge isometry of $X_t$. Thus, $E := \tilde{g}^{-1}([C_t])$ is an integral $(1, 1)$-class on $X_t$ and satisfies $([E]^2) = -2$. In particular, $[E]$ or $-[E]$ is represented by an effective curve on $X_t$ by the Riemann-Roch Theorem. Therefore, $(g(l), C_t) = (l, E) \neq 0$ by the ampleness of $l$, and we have $(g(l), C_t) < 0$. Then $C_t \subset \text{Supp} \; D$ and one has $(l, C_t) \leq (l, g(l))$. Note that $(l, g(l))$ is independent on $t$. So, if there are infinitely many $t$ such that $t \in \mathcal{F}_i$, then by [Gr], there exists a component $\mathcal{H}$ of the relative Hilbert scheme $\text{Hilb}^1_{X_t/\Delta}$ to which infinitely many of $C_t$ belong. (See also a very nice book [Kl] for the Hilbert schemes.) This $\mathcal{H}$ must dominate $\Delta$ and one has then an effective curve $C$ on $X_p$ such that $([C], g(l)) = ([C_t], g(l)) < 0$ (for $C$ and $C_t$ belong to the same component $\mathcal{H}$). However, this contradicts the ampleness of $g(l)$ on $X_p$. Hence $|\mathcal{F}_i| < \infty$ for each $i$ and we are done for (2).

We shall show the assertion (3). As before, each element $g_t \in G_t$ extends a Hodge isometry $\tilde{g}_t$ of $X_t$ such that $\tilde{g}_t|\Lambda^0 = g_t$ and $\tilde{g}_t|\Lambda^{0\perp} = id$. This $\tilde{g}_t$ is also effective, because

$$\tilde{g}_t(l) = g_t(l) \in \Lambda^0_t \subset A_t.$$  

There then exists a unique $f_t \in \text{Aut}(X_t)$ such that $f_t^*|\Lambda^0 = g_t$ and $f_t^*|\Lambda^{0\perp} = id$ by the global Torelli Theorem for K3 surfaces. The map $\iota_t$ is given by $g_t \mapsto f_t$. Now the assertion follows from the next facts:

(1) $\tau_t(\text{NS}(X_t)) = \Lambda^0$ for all $t \in G$;
(2) $\Lambda^{0\ast}/\Lambda^0$ is finite (and is independant of $t$);
(3) The subgroup $\text{Ker}(\text{Aut}(X_t) \to O(\text{NS}(X_t)^\perp))$ of $\text{Aut}(X_t)$ is of finite index less than or equal to $66 \times |\text{Aut}(\Lambda^{0\ast}/\Lambda^0)|$ if $t \in G$ ([Ni1], see also [St]).  

Now we are done for Theorem (1).  

Proof of Theorem (2). We may consider the case $|\text{Aut}(X_t)| < \infty$ for all $t \in G$. (If otherwise, one has $|\text{Aut}(X_t)| = \infty$ for all $t \in \Delta - \mathcal{F}$ by Theorem (1) and may put $\mathcal{D} = \Delta - \mathcal{F}$.) Since $G$ is dense, the statement will follow from the next:
Claim 2. For any given $p \in \mathcal{G}$ and for any open neighbourhood $p \in U$, there exists $t \in S \cap U$ such that $|\text{Aut}(X_t)| = \infty$.

Proof of Claim 2. By abuse of notation, we shall write again $U = \Delta$ and $\varphi|U = \varphi$. Assuming $|\text{Aut}(X_t)| < \infty$ for all $t \in \Delta$, we shall derive a contradiction. Set $r_0 := \text{rank } \Lambda^0$. By shrinking, we may assume that $\mathcal{F} = \emptyset$. We shall argue dividing into the following two cases (1) and (2) (by shrinking again, we see that these two cover all the possible cases):

1. There exists a sequence $\{t_n\}_{n \geq 1} (\subset \mathcal{S})$ such that $\text{rank } \Lambda_{t_n} \geq 3$ for all $n$ and that $\lim_{n \to \infty} t_n = p$;
2. $r_0 = 1$ and $\text{rank } \Lambda_t = 2$ for all $t \in \mathcal{S}$.

Case (1). Since $\text{rank } \Lambda_{t_n} \geq 3$ and $|\text{Aut}(X_{t_n})| < \infty$, by [PSS], $X_{t_n}$ contains only finitely many smooth rational curves, say,

$$C_{n_1}, C_{n_2}, \ldots, C_{nk(n)}.$$  

By Kovacs [Kv], the classes $[C_{ni}]$ generate the Kleiman-Mori cone $N_{n}(:= \overline{NE}(X_{t_n}))$ of $X_{t_n}$. Recall also that the isomorphism classes of both the Picard lattices and the dual graphs of smooth rational curves, or in other words, the fundamental domains of the 2-reflection groups, of K3 surfaces with $|\text{Aut}(X)| < \infty$ and $\rho(X) \geq 3$ are finite. This deep result is due to Nikulin [Ni 3, 4, 5] and it is this finiteness that we need the case assumption essentially. (Note that such finiteness is clearly false if $\rho \leq 2$.) Therefore we may assume that $\rho := \text{rank } \Lambda_{t_n}$ and $k := k(t_n)$ are constant and the intersection matrices $(C_{ni}, C_{nj})$ are also independent of $n$. (However, we do not know a priori whether the classes $[C_{ni}]$ form constant subsystems of $\Lambda \times \Delta$.)

Consider first the case of $r_0 \geq 3$. Then $X_p$ has finitely many smooth rational curves as well by $|\text{Aut}(X_p)| < \infty$. Denote all of them by $D_j$ ($1 \leq j \leq k'$). Then one has $l = \sum_{j=1}^{k'} a_j D_j$, where $a_j$ are non-negative rational numbers. Since $[D_j] \in \Lambda^0 \subset \Lambda_t$, each $D_j$ extends $\mathbb{P}^1$-bundles $\mathcal{D}_j \to \Delta$ near $p$. (See for instance [Og].) Thus, for large $n$, one has $\mathcal{D}_{jt_n} \in \{C_{ni}\}_{i=1}^k$ and $l = \sum_{j=1}^{k'} a_j \mathcal{D}_{jt_n}$. Then the set $\{(C_{ni}, l)\}_{n \geq 1}$ is bounded for all $i$ by the finiteness of dual graphs. Thus, using the existence of relative Hilbertscheme and arguing as before, one finds effective divisors $C_{0i}$ ($i = 1, 2, \ldots, k$) on $X_p$ whose dual graph is same as $C_{ni}$. However, one would then have rank $\Lambda^0 \geq \text{rank } \Lambda_t > r_0$, a contradiction.

Next consider the case of $r_0 = 2$. By $|\text{Aut}(X_p)| < \infty$, the Kleiman-Mori cone $N_p$ of $X_p$ is generated by either (1) two classes of smooth rational curves, (2) two classes of elliptic pencils, (3) the class of a smooth rational curve and the class of elliptic pencil, or (4) the class of a smooth rational curve and an irrational ray $\mathbb{R}_{\geq 0}e$ with $(e^2) = 0$ (See for instance [Kv]). In the first three cases, the same argument as above immediately gives a contradiction if we notice the following facts:

1. Each elliptic pencil on $X_p$ extends a family of elliptic pencils near $p$. This is because $\Lambda^0 \subset \Lambda_t$. (See for instance [Og]).
2. Each elliptic pencil on $X_{t_n}$ has at least one fiber consisting of smooth rational curves. This follows from Shioda’s formula of the Mordell-Weil rank of the Jacobian (see for instance [Sh]) and the assumption rank $\Lambda_{t_n} \geq 3$. Indeed, the Mordell-Weil rank must be 0 by $|\text{Aut}(X_{t_n})| < \infty$. 


So, we may consider the last case (4). Since the smooth rational curve \( C \) on \( X_p \) extends sideways (by \([C] \in \Lambda^0\)), we may assume that \([C] = [C_{n1}]\) for large \( n \). This \([C]\) generates one of the two boundary rays of \( N_p \). Since \( A_{f_n} \) is finite rational polyhedral, so is \( A_{f_n}^0 := A_{f_n} \cap (\Lambda^0 \otimes \mathbb{R}) \). Therefore, by the case assumption, one has \( A_{f_n}^0 \neq A_p \) (while one has now \( A_{f_n}^0 \subset A_p \) by Claim 1 (1).) There then exist a nef and big divisor \( H_n \) on \( X_{f_n} \) and a smooth rational curve \( C_{n2} \) on \( X_{f_n} \), such that \((H_n, C_{n2}) = 0\) and that \([H_n] \in A_p\). (Note that \(((H_n), [C_{n1}]) = ([H_n], [C]) > 0\) because \([H_n] \in A_p\).) Choose a euclidean norm \( \|\cdot\| \) on \( \Lambda \otimes \mathbb{R} \). If the set \( \{\|C_{n2}\|\}_{n \geq 1} \) is bounded, then, since \( \Lambda \) is finite rational polyhedral, so is \( \Lambda \) as well. Then, by the Hodge index Theorem, \( X_p \) contains an effective divisor \( D_{n2} \) such that \([D_{n2}] = [C_{n2}]\). However, the class \([D_{n2}]\) gives then another rational boundary of \( N_p \), a contradiction to the case assumption. Therefore, \( \{\|C_{n2}\|\}_{n \geq 1} \) is unbounded. Then, passing to a subsequence, we have \( \lim_{n \to \infty} \|C_{n2}\| = \infty \). Set \( x_{n2} := C_{n2}/\|C_{n2}\| \). Since \( \|x_{n2}\| = 1 \), by passing to some subsequence, we find the limit \( x_{02} := \lim_{n \to \infty} x_{n2} \) in \( \Lambda \otimes \mathbb{R} \). We calculate that \( \|x_{02}\| = 1 \) and that
\[
(x_{02})^2 = \lim_{n \to \infty} -2/\|C_n\| = 0.
\]
Moreover, we calculate that
\[
(x_{02}, [\omega X_p]) = \lim_{n \to \infty} (x_{n2}, [\omega X_{f_n}]) = 0.
\]
Here \( \omega \) is the 2-form on \( X_p \) given by the base change of a local section \( \omega \) of \( \varphi_* \Omega^2_{X/\Delta} \) around \( p \). Thus, \( x_{02} \) is also a real \((1,1)\)-class on \( X_p \). Set \( h_n := H_n/\|H_n\| \). Since \( h_n \in A_p \), passing to a subsequence, we have the limit
\[
h_0 := \lim_{n \to \infty} h_n \in \overline{A_p}.
\]
Since \((h_n, x_{n2}) = 0\), one has \((h_0, x_{02}) = 0\) as well. Then, by the Hodge index Theorem, one has that \( x_{02} = \alpha h_0 \) and \( x_{02} \in \overline{A_p} \). In particular, \( R_{>0} x_{02} \) gives another boundary of \( N_p \). On the other hand, since \((C_{n1}, C_{n2})\) is constant, one calculates
\[
([C], x_{02}) = \lim_{n \to \infty} ([C_{n1}], [C_{n2}])/\|C_{n2}\| = 0.
\]
Since \( \Lambda^0 \) is of rank 2, this implies \( R x_{02} = [C]^{-1} \). However, \( R_{>0} x_{02} \) would be then a rational boundary, a contradiction to the case assumption.

Finally consider the case where \( r_0 = 1 \). If \( \{\|C_{ni}\|\}_{n \geq 1} \) is bounded for some \( i \), then, as before, \( \{(C_{ni}, l)\}_{n \geq 1} \) is bounded and we have an effective curve \( D \) such that \([D] = [C_{ni}]\) for large \( n \). This class \([D]\) satisfies \([D] \in \text{NS}(X_p)\) and \((D^2) = -2\). However, this is impossible, because \( X_p \) is projective and \( \text{rank NS}(X_p) = r_0 = 1 \). Therefore the sets \( \{\|C_{ni}\|\}_{n \geq 1} \) are unbounded for each \( i \). We set \( x_{ni} := C_{ni}/\|C_{ni}\| \). Passing to a subsequence, we have \( \lim_{n \to \infty} \|C_{ni}\| = \infty \) and get the limit points \( x_{0i} := \lim_{n \to \infty} x_{ni} \). Since \((C_{ni}, C_{nj})\) are all bounded, we have that \((x_{0i}^2) = (x_{0i}, x_{0j}) = 0\) for all \( i \) and \( j \). Moreover, by the same argument as in the previous case, we see that \( x_{0i} \) are all real \((1,1)\)-classes on \( X_p \). Then, by the Hodge index Theorem, there exists \( \epsilon \) with \((\epsilon^2) = 0\) such that \( x_{0i} \in \mathbb{R} \epsilon \) for all \( i \). Then, the Kleiman-Mori cones \( N_{\epsilon} \), which are finite polyhedral cones generated by \( x_{0i} \).
(1 \leq i \leq k), accumulate to the ray \( R e \) if \( n \to \infty \). However, since \( l \in N_n \), we would then have \( l \in R e \), a contradiction to \((l^2) > 0\). Now we are done in Case 1. \( \square \)

Case (2). Observe that if rank \( \Lambda_t = 2 \), then \( |\text{Aut}(\mathcal{X}_t)| = \infty \) if and only if the quadratic form of \( \Lambda_t \) represents neither 0 nor \(-2\). Let us consider the period map

\[
\pi : \Delta \to \mathcal{P} := \{[\omega] \in \mathcal{P}(\Lambda \otimes \mathbb{C})|(\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}.
\]

For each \( a \in \Lambda \), we set

\[
H_a := \{[\omega] \in \mathcal{P}|(a, \omega) = 0\} \subset \mathcal{P}.
\]

Then \( \pi(\Delta) \subset H_l \). Moreover, by the case assumption, we have that \( t \in S \) if and only if there exists \( a \in \Lambda - Zl \) such that \( \pi(t) \in H_a \) and that \( \Lambda_t = Z \langle l, a \rangle \). (Recall that \( l \) is assumed to be primitive.) For this \( t \), one has the following:

**Claim 3.** There exist an element \( h \in \Lambda \) and arbitrarily large positive integers \( n \) such that \( Z \langle l, na + h \rangle \) is primitive in \( \Lambda \) and represents neither 0 nor \(-2\).

**Proof of Claim 3.** Set \( (l, l) = 2A, (l, a) = B \) and \( (a, a) = 2C \). Since the primitive embedding \( \Lambda_t \hookrightarrow \Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \) is unique up to isomorphism by the primitive embedding Theorem \([\text{Ni}2, \text{Theorem 1.14.4}] \), we may assume that

\[
l = e_{11} + Ae_{12} , \quad a = Be_{12} + e_{21} + Ce_{22} ,
\]

where \( e_{11} \) and \( e_{i2} \) \((1 \leq i \leq 3)\) are the standard basis of \( U^{\oplus 3} \). Set \( h := e_{31} - me_{32} \). Then \( (h^2) = -2m \) and \( (h, l) = (a, l) = 0 \). Put:

\[
L_{m, n} := Z \langle l, na + h \rangle.
\]

Then this lattice \( L_{m, n} \) is primitive in \( \Lambda \) and its quadratic form is

\[
Q(x, y) := 2\{Ax^2 + nBxy + (n^2C - m)y^2\}.
\]

If \( A \geq 2 \), we take \( n = AN \) and \( m = AM \). Then \( Q(x, y) \) is divisible by \( 2A \). Thus \( L_{m, n} \) does not represent \(-2\). Moreover, the discriminant of \( Q \) is

\[
4A^2\{N^2(B^2 - 4AC) + 4M\}.
\]

This is not square if \( M \) and \( N \) are general. Thus \( L_{m, n} \) does not represent 0, either. If \( A = 1 \), we set \( n = 4N \) and \( m = 8M \). Then \( Q(x, y) \) and its discriminant are

\[
Q(x, y) = 2x^2 + 8Nxy + 8(4N^2C - M)y^2 ;
\]

\[
D := 64\{N^2(B^2 - 4C) + M\}.
\]

Then, \( Q(x, y) \equiv 2 \) or 0 modulo 8. Thus \( L_{m, n} \) does not represent \(-2\). Moreover, \( D \) is not square if \( M \) and \( N \) are general. Thus \( L_{m, n} \) does not represent 0, either. \( \square \)

Let us return back to the original situation. If the lattice \( Z \langle l, a \rangle \) (for \( t \)) represents neither 0 nor \(-2\) then we are done. If otherwise, using Claim 3, we find an element \( h \in \Lambda \) and an element \( n \in Z \) such that \( Z \langle l, na + h \rangle \) is primitive in \( \Lambda \) and represents neither 0 nor \(-2\). Since \( H_{na + h} \in H_{na} \), the hyperplane \( H_{na} \) meets \( \pi(\Delta) \) at
\[ \pi(t'), \] where \( t' \) is a point very close to \( t \), if \( n \) is sufficiently large. For this \( t' \), one has \( \Lambda_{t'} = \mathbb{Z}(l, na + h) \) and \( |\text{Aut}(\mathcal{X}_{t'})| = \infty \). □

Now we are done for Theorem (2). □

**Proof of Theorem (3).** Let \( T \) be a K3 surface such that \( \text{NS}(T) = U \oplus A_1(-1) \) and \( |\text{Aut}(T)| < \infty \). Such a K3 surface exists by [Ni3]. Let us consider the Kuranishi family \( u : \mathcal{U} \to \mathcal{K} \) of \( T \). Choosing a marking

\[ \tau : R^2u_*\mathbb{Z}_{\mathcal{U}} \to \Lambda \times \mathcal{K}, \]

one can define the period map

\[ \pi : \mathcal{K} \to \mathcal{P}. \]

Since \( \pi \) is a local isomorphism by the local Torelli Theorem for K3 surfaces, we may identify \( \mathcal{K} \) with a small open set of the period domain \( \mathcal{P} \) (denoted again by \( \mathcal{P} \)) through \( \pi \). Let \( \Pi \) be a general primitive sublattice of rank 2 of \( U \oplus A_1(-1) \). Then \( \Pi \) represents neither 0 nor \(-2\). Let us take the subset \( \mathcal{K}_0 \subset \mathcal{K} = \mathcal{P} \) defined by the equations

\[ (a, \omega) = (b, \omega) = 0, \]

where \( a \) and \( b \) form an integral basis of \( \Pi \). Consider the induced family \( u^0 : \mathcal{U}^0 \to \mathcal{K}_0 \). Here \( \mathcal{K}_0 \) is of dimension 18. By construction, we have \( 0 \in \mathcal{K}_0 \) and \( \text{NS}(\mathcal{U}_t^0) \simeq \Pi \), whence \( |\text{Aut}(\mathcal{U}_t^0)| = \infty \) for generic \( t \). Then, the induced family \( \varphi : \mathcal{X} \to \Delta \) for a generic linear section \( 0 \in \Delta \subset \mathcal{K}_0 \) satisfies \( 0 \in \mathcal{F} \). In this example, we have \( A_0^0 \neq A_t \) for \( t \) being generic. Indeed, \( \partial A_0^0 = \partial (A_0 \cap \Pi) \) is rational, while \( \partial A_t \) is irrational. □

Now we are done. Q.E.D. of the main Theorem. □

**Proof of Corollary.** Since \( \Lambda^0 \subset \Lambda_t \), an ample divisor on \( \mathcal{X}_p \) extends sideways (as a line bundle) if \( p \in \mathcal{G} \). In particular, \( f \) is locally projective around \( p \). The result now follows from the Theorem together with the density of \( \mathcal{G} \). □

**2. Proof of Proposition**

We shall construct explicit examples with required properties by deforming a remarkable K3 surface found by Nikulin and Kondo ([Ni3], [Ko1]). This is a K3 surface \( S \) which satisfies the following properties:

1. \( \rho(S) = 19 \) and \( \text{NS}(S) = U \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1) \) as a lattice,
2. \( S \) contains exactly 24 smooth rational curves and finitely many \((>0)\) elliptic pencils;
3. \( \text{Aut}(S) \) is a finite group (and is indeed isomorphic to \( S_3 \times \mu_2 \)).

As before, let us take the Kuranishi family \( u : \mathcal{U} \to \mathcal{K} \) of \( S \) and choose a marking

\[ \tau : R^2u_*\mathbb{Z}_{\mathcal{U}} \to \Lambda \times \mathcal{K}. \]

Then \( S = \mathcal{U}_0 \) and one has the period map

\[ \pi : \mathcal{K} \to \mathcal{P}. \]

Again as before, we identify \( \mathcal{K} \) with a small open set of the period domain \( \mathcal{P} \) (denoted again by \( \mathcal{P} \)) through \( \pi \). Let

\[ g \circ f : \mathbb{C}^8 (\text{all } 1 \leq i \leq 8), \]

be the elements of $\Lambda$ corresponding to the standard integral basis of $\text{Pic}(S) = U \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1)$.

**Proof of (1).** Let us take the subset $0 \in K^1 \subset K = \mathcal{P}$ defined by the equations

$$(e + 3nf, \omega) = (v_{11}, \omega) = (v_{21}, \omega) = 0,$$

where $n$ is an integer non-divisible by 3. Consider the induced family $u^1 : U^1 \to K^1$. Here $K^1$ is of dimension 17. By construction, $\text{NS}(U^1_t) \cong \langle e + 3nf, v_{11}, v_{21} \rangle$ and the intersection matrix is:

$$
\begin{pmatrix}
6n & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{pmatrix},
$$

if $t \in K^1$ is generic. The discriminant of this matrix is $24n$. Since the quadratic form

$$Q_1(x, y) := 6nx^2 - 2y^2 - 2z^2$$

does not represent 0, $U^1_t$ has no elliptic pencil. Since there are only finitely many isomorphism classes of lattices of rank 3 which are isomorphic to the Picard lattices of K3 surfaces $X$ with $|\text{Aut}(X)| < \infty$ [Ni4], one has $|\text{Aut}(U^1_t)| = \infty$ for $n$ being sufficiently large. Moreover, $U^1_t$ admits at least one smooth rational curve. This follows from the fact that $Q_1(x, y)$ represents $-2$ plus the Riemann-Roch Theorem. Then $U^1_t$ contains infinitely many smooth rational curves by $|\text{Aut}(U^1_t)| = \infty$ and by [PSS]. (See also [St] and [Kv].) Now the induced family over a generic linear section $0 \in \Delta \subset K^1$ satisfies the required property of (1). □

**Proof of (2).** Let us take the subset $0 \in K^2 \subset K = \mathcal{P}$ defined by the equations

$$(e + 2f, \omega) = (v_{11} + v_{13}, \omega) = (v_{21} + v_{23}, \omega) = 0.$$

Consider the induced family $u^2 : U^2 \to K^2$. Here $K^2$ is of dimension 17. By construction, $\text{NS}(U^2_t) \cong \langle e + 2f, v_{11} + v_{13}, v_{21} + v_{23} \rangle$ and the intersection matrix is:

$$
\begin{pmatrix}
4 & 0 & 0 \\
0 & -4 & 0 \\
0 & 0 & -4
\end{pmatrix},
$$

if $t \in K^2$ is generic. Since the quadratic form

$$Q_2(x, y) := 4x^2 - 4y^2 - 4z^2$$

does not represent $-2$, $U^2_t$ contains no smooth rational curve. In particular, the ample cone coincides with the positive cone. This already implies $|\text{Aut}(U^2_t)| = \infty$, because the automorphism group is then isomorphic to the orthogonal group of the Picard lattice up to finite groups [PSS]. Moreover, since $Q_2(x, y)$ admits infinitely many primitive integral zero’s, $U^2_t$ admits infinitely many elliptic pencils. Now the induced family over a generic linear section $0 \in \Delta \subset K^2$ satisfies the required property of (2). □

**Proof of (3).** Let us take the subset $K^3 \subset K = \mathcal{P}$ defined by the equations

$$(e + 4f, \omega) = (v_{11} + v_{13} + v_{15} + v_{17}, \omega) = 0.$$

Consider the induced family $u^3 : U^3 \to K^3$. Here $K^3$ is of dimension 17. By construction, one has $\text{NS}(U^3) \simeq U \oplus \langle -8 \rangle$ if $t \in K^3$ is generic. Then $X$ admits a Jacobian pencil $f : X \to \mathbf{P}^1$, i.e. an elliptic pencil with section, given by $U$. Here we set $X := U^3_t$. Since $f$ has no reducible fibers, the Mordell-Weil lattice $M(f)$ of $f$ is isomorphic to the lattice $\langle 8 \rangle$. (See the excellent survey of Shioda [Sh] for basic properties of the Mordell-Weil lattice.) Let $g$ be the generator of $M(f)$ corresponding to $0$. Then as before, the ample cone coincides with the positive cone and one has $|\{ f_n \}_{n \geq 1} | = \infty$. Combining all these together, one finds that $U^3_t$ has $\rho(U^3_t) = 3$, infinitely many smooth rational curves, infinitely many elliptic pencils, and satisfies $|\text{Aut}(U^3_t)| = \infty$ if $t \in K^3$ is generic. Now the induced family over a generic linear section $0 \in \Delta \subset K^3$ satisfies the required property of (3).

Proof of (4). Let us take the subset $K^4 \subset K = \mathcal{P}$ defined by the equations

$$ (e + 6f, \omega) = (v_{11} + v_{13}, \omega) = (v_{21} + v_{23}, \omega) = 0. $$

Consider the induced family $u^4 : U^4 \to K^4$. Here $K^4$ is of dimension 17. By construction, $\text{NS}(U^4_t) \simeq \langle e + 6f, v_{11} + v_{13}, v_{21} + v_{23} \rangle$ and the intersection matrix is:

$$
\begin{pmatrix}
12 & 0 & 0 \\
0 & -4 & 0 \\
0 & 0 & -4
\end{pmatrix},
$$

if $t \in K^4$ is generic. Since the quadratic form

$$ Q_4(x, y) := 12x^2 - 4y^2 - 4z^2 $$

represents neither 0 nor $-2$, $U^4_t$ has neither smooth rational curve nor elliptic pencil. Then as before, the ample cone coincides with the positive cone and one has $|\text{Aut}(U^4_t)| = \infty$. Now the induced family over a generic linear section $0 \in \Delta \subset K^4$ satisfies the required property of (4).

Proof of (5). Let us take the subset $K^5 \subset K = \mathcal{P}$ defined by the equations

$$ (e, \omega) = (f, \omega) = (v, \omega) = 0. $$

Consider the induced family $u^5 : U^5 \to K^5$. Here $K^5$ is of dimension 17 and one has $\text{NS}(U^5_t) = U \oplus A_1(-1)$ for generic $t \in K^5$ and $U^5_t$ enjoys all the properties required in (5) by [Ni3]. Now the induced family over a generic linear section $0 \in \Delta \subset K^5$ satisfies the required property of (5). Note that no element $h \in U \oplus A_1(-1)$ is ample on $S = U^5_t$, because such $h$ is perpendicular to $E_8(-1) \oplus E_8(-1)$. Therefore, the family is not projective. 

Now we are done. Q.E.D. for Proposition.
References

[Bo] C. Borcea, Homogeneous vector bundles and families of Calabi-Yau threefolds, II, Proc. Sym. Pure Math. 52 (1991), 83–91.

[BC] A. Borel and H. Chandra, Arithmetic subgroups of algebraic groups, Ann. Math 75 (1962), 485–535.

[BPV] W. Barth, C. Peters, A. Van de Ven, Compact complex surfaces, Springer-Verlag (1984).

[CP] F. Campana and T. Peternell, Algebraicity of the ample cone of projective varieties, J. reine angew. Math. 407 (1990), 160–166.

[Gr] A. Grothendieck, Fondements de la Géométrie Algébrique, Sec. Math. Paris (1962).

[Ke] J.H. Keum, Automorphisms of Jacobian Kummer surfaces, Compositio Math. 107 (1997), 269–288.

[KK] J.H. Keum and S. Kondo, The automorphism groups of Kummer surfaces associated with the product of two elliptic curves, Trans. Amer. Math. Soc. 353 (2001), 1469–1487.

[Kl] J. Kollár, Rational curves on algebraic varieties. A series of Modern Surveys in Mathematics, Springer-Verlag 32 (1996).

[Kn1] S. Kondo, Algebraic K3 surfaces with finite automorphism groups, Nagoya Math. J. 116 (1989), 1–15.

[Kn2] S. Kondo, Niemeier Lattices, Mathieu groups, and finite groups of symplectic automorphisms of K3 surfaces, Duke Math. J. 92 (1998), 593–598.

[Kn3] S. Kondo, The automorphism groups of a generic Kummer surface, J. Alg. Geom. 7 (1998), 589 – 609.

[Kv] S. Kovacs, The cone of curves of a K3 surfaces, Math. Ann. 300 (1994), 681–691.

[Mu] S. Mukai, Finite groups of automorphisms of K3 surfaces and the Mathieu group, Invent. Math. 94 (1988), 183–221.

[Ni1] V. V. Nikulin, Finite automorphism groups of Kähler K3 surfaces, Trans. Moscow Math. Soc. 38 (1980), 71–135.

[Ni2] V. V. Nikulin, Integral symmetric bilinear forms and some of their geometric applications, Math. USSR Izv. 14 (1980), 103–167.

[Ni3] V. V. Nikulin, On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by the 2-reflections, J. Soviet Math. 22 (1983), 1401–1476.

[Ni4] V. V. Nikulin, Surfaces of type K3 with finite automorphism groups and a Picard group of rank three, Proc. Steklov Institute Math. 3 (1985), 131–155.

[Ni5] V. V. Nikulin, Discrete reflection groups in Lobachevsky spaces and algebraic surfaces: in Proceedings of the International Congress of Mathematicians, Berkley 1986, Amer. Math. Soc. (1987), 654–671.

[Ni6] V. V. Nikulin, A remark on algebraic surfaces with polyhedral Mori cone, Nagoya Math. J. 157 (2000), 73–92.

[Og] K. Oguiso, Picard numbers in a family of hyperkähler manifolds - a supplement to the article of R. Borchers, L. Katzarkov, T. Pantev, N. I. Shepherd-Barron, preprint (2000; AG/0011259).

[PSS] I. Piatetski-Shapiro and I. R. Shafarevich, A Torelli Theorem for algebraic surfaces of type K3, Math. USSR Izv. 5 (1971), 547–587.

[Sh] T. Shioda, Theory of Mordell-Weil lattices: in Proceedings of the International Congress of Mathematicians (Kyoto 1990), Math. Soc. Japan (1991), 473 – 489.

[St] H. Sterk, Finiteness results for algebraic K3 surfaces, Math. Z. 189 (1985), 507-513.

[Vi] E. B. Vinberg, The two most algebraic K3 surfaces, Math. Ann. 265 (1983), 1–21.