On the Structure of Graphs with Given Odd Girth and Large Minimum Degree

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Abstract: We study minimum degree conditions for which a graph with given odd girth has a simple structure. For example, the classical work of Andrásfai, Erdős, and Sós implies that every $n$-vertex graph with odd girth $2k + 1$ and minimum degree bigger than $\frac{2}{2k+1} n$ must be bipartite. We consider graphs with a weaker condition on the minimum degree. Generalizing results of Häggkvist and of Häggkvist and Jin for the cases $k = 2$ and $3$, we show that every $n$-vertex graph with odd girth $2k + 1$ and minimum degree bigger than $\frac{3}{4k} n$ is homomorphic to the cycle of length $2k + 1$. This is best possible in the sense that there are graphs with minimum degree $\frac{3}{4k} n$ and odd girth $2k + 1$ that are not homomorphic to the cycle of length $2k + 1$. Similar results were obtained by Brandt and Ribe-Baumann. © 2014 Wiley Periodicals, Inc. J. Graph Theory 80: 69–81, 2015

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1. INTRODUCTION

We consider finite and simple graphs without loops and for any notation not defined here we refer to the textbooks [3, 4, 9]. In particular, we denote by $K_r$ the complete graph on $r$ vertices and by $C_r$, a cycle of length $r$. A homomorphism from a graph $G$ into a graph $H$ is a mapping $\phi : V(G) \rightarrow V(H)$ with the property that $\{\phi(u), \phi(w)\} \in E(H)$ whenever $\{u, w\} \in E(G)$. We say that $G$ is homomorphic to $H$ if there exists a homomorphism from $G$ into $H$. Furthermore, a graph $G$ is a blow-up of a graph $H$, if there exists a surjective homomorphism $\phi$ from $G$ into $H$, but for any proper supergraph of $G$ on the same vertex set the mapping $\phi$ is not a homomorphism into $H$ anymore. In particular, a graph $G$ is homomorphic to $H$ if and only if it is a subgraph of a suitable blow-up of $H$. Moreover, we say a blow-up of $H$ is balanced if the homomorphism $\phi$ signifying that $G$ is a blow-up has the additional property that $|\phi^{-1}(u)| = |\phi^{-1}(u')|$ for all vertices $u$ and $u'$ of $H$.

Homomorphisms can be used to capture structural properties of graphs. For example, a graph is $k$-colorable if and only if it is homomorphic to $K_k$. Furthermore many results in extremal graph theory establish relationships between the minimum degree of a graph and the existence of a given subgraph. The following theorem of Andrásfai et al. [2] is a classical result of that type.

**Theorem 1** (Andrásfai, Erdős, and Sós). For every integer $r \geq 3$ and for every $n$-vertex graph $G$ the following holds. If $G$ has minimum degree $\delta(G) > \frac{3r - 7}{3r - 4}n$ and $G$ contains no copy of $K_r$, then $G$ is $(r - 1)$-colorable.

In the special case $r = 3$, Theorem 1 states that every triangle-free $n$-vertex graph with minimum degree greater than $2n/5$ is homomorphic to $K_2$. Several extensions of this result and related questions were studied. For example, motivated by a question of Erdős and Simonovits [10] the chromatic number of triangle-free graphs $G = (V, E)$ with minimum degree $\delta(G) > |V|/3$ was thoroughly investigated in [5, 8, 13, 15, 17] and it was recently shown by Brandt and Thomassé [7] that it is at most four.

Another related line of research (see, e.g., [8, 13, 15, 16]) concerned the question for which minimum degree condition a triangle-free graph $G$ is homomorphic to a graph $H$ of bounded size, which is triangle-free itself. In particular, Häggkvist [13] showed that triangle-free graphs $G = (V, E)$ with $\delta(G) > 3|V|/8$ are homomorphic to $C_5$. In other words, such a graph $G$ is a subgraph of suitable blow-up of $C_5$. This can be viewed as an extension of Theorem 1 for $r = 3$, since balanced blow-ups of $C_5$ show that the degree condition $\delta(G) > 2|V|/5$ is sharp there. Strengthening the assumption of triangle-freeness to graphs of higher odd girth, allows us to consider graphs with a more relaxed minimum degree condition. In this direction Häggkvist and Jin [14] showed that graphs $G = (V, E)$ that contain no odd cycle of length three and five and with minimum degree $\delta(G) > |V|/4$ are homomorphic to $C_7$.

We generalize those results to arbitrary odd girth, where we say that a graph $G$ has odd girth at least $g$, if it contains no odd cycle of length less than $g$.

**Theorem 2.** For every integer $k \geq 2$ and for every $n$-vertex graph $G$ the following holds. If $G$ has minimum degree $\delta(G) > \frac{3k}{4k}$ and $G$ has odd girth at least $2k + 1$, then $G$ is homomorphic to $C_{2k+1}$.

Note that the degree condition given in Theorem 2 is best possible as the following example shows. For an even integer $r \geq 6$ we denote by $M$, the so-called Möbius ladder.
(see, e.g., [12]), i.e., the graph obtained by adding all diagonals to a cycle of length \( r \), where a diagonal connects vertices of distance \( r/2 \) in the cycle. One may check that \( M_{4k} \) has odd girth \( 2k + 1 \), but it is not homomorphmorphic to \( C_{2k+1} \). Moreover, \( M_{4k} \) is 3-regular and, consequently, balanced blow-ups of \( M_{4k} \) show that the degree condition in Theorem 2 is best possible when \( n \) is divisible by \( 4k \).

We also remark that Theorem 2 implies that every graph with odd girth at least \( 2k + 1 \) and minimum degree bigger than \( \frac{3n}{4k} \) contains an independent set of size at least \( \frac{kn}{2k+1} \). This answers affirmatively a question of Albertson et al. [1].

Similar results were obtained by Brandt and Ribe-Baumann.

2. FORBIDDEN SUBGRAPHS

In this section, we introduce two lemmas, Lemmas 1 and 2 below, needed for the proof of Theorem 2 given in Section 3. Roughly speaking, in each lemma we show that certain configurations cannot occur in edge-maximal graphs considered in Theorem 2.

We say that a graph \( G \) with odd girth at least \( 2k + 1 \) is edge-maximal if adding any edge to \( G \) (by keeping the same vertex set) yields an odd cycle of length at most \( 2k − 1 \). We denote by \( G_{n,k} \) all edge-maximal \( n \)-vertex graphs satisfying the assumptions of the main theorem, i.e., for integers \( k \geq 2 \) and \( n \) we set

\[
G_{n,k} = \{ G = (V, E) : |V| = n, \delta(G) > \frac{3n}{4k}, \text{ and } G \text{ is edge-maximal with odd girth } 2k + 1 \}.
\]

2.1. Cycles of Length Six with Precisely One Diagonal

For \( k \) fixed, we say an odd cycle is short if its length is at most \( 2k − 1 \). A chord in a cycle of even length \( 2j \) is a diagonal if it joins two vertices at distance \( j \) in the cycle. Given a walk \( W \) we define its length \( l(W) \) as the number of edges, each counted as many times as it appears in the walk. Hence, the lengths of paths and cycles coincide with their number of edges.

**Lemma 1.** Let \( \Phi \) denote the graph obtained from \( C_6 \) by adding exactly one diagonal. For all integers \( k \geq 2 \) and \( n \) and for every \( G \in G_{n,k} \) we have that \( G \) does not contain an induced copy of \( \Phi \).

**Proof.** Suppose, contrary to the assertion, that \( G = (V, E) \) contains \( \Phi \) in an induced way, where \( V(\Phi) = \{ a_i : 0 \leq i \leq 5 \} \subseteq V \) is the vertex set and

\[
E(\Phi) = \{ \{a_i, a_{i+1(\text{mod } 5)}\} : 0 \leq i \leq 5 \} \cup \{a_1, a_4\}.
\]

Note that in fact, the chords of the \( C_6 \) in \( \Phi \) which are not diagonals would create triangles in \( G \) so assuming that \( \Phi \) is induced in \( G \) gives us only information concerning the nonexisting two diagonals. Since \( G \) is edge-maximal, the nonexistence of the diagonal between \( a_0 \) and \( a_3 \) must be forced by the existence of an even path \( P_{03} \) which, together with \( \{a_0, a_3\} \), would yield an odd cycle of length at most \( 2k − 1 \). Consequently, the length of \( P_{03} \) is at most \( 2k − 2 \). Since \( a_0 \) and \( a_3 \) have distance three in \( \Phi \), a shortest path between them in \( \Phi \), together with \( P_{03} \), results in a closed walk with odd length at most \( 2k + 1 \).

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Recall that any odd closed walk is either an odd cycle or it contains a shorter odd cycle, it follows that $P_{03}$ has length exactly $2k - 2$ and its inner vertices are not in $\Phi$. The same reasoning can be applied to the other missing diagonal between $a_2$ and $a_5$ to show that there exists another even path $P_{25}$ of length $2k - 2$ whose inner vertices are disjoint from $V(\Phi)$.

We show that $P_{03}$ and $P_{25}$ are vertex disjoint. Suppose that $V(P_{03}) \cap V(P_{25}) \neq \emptyset$ and let $b$ be the first vertex in $P_{03}$ that is also a vertex of $P_{25}$, i.e., $b$ is the only vertex from $a_0P_{03}b$ that is also contained in $P_{25}$. Consider the walks

$$W_{05} = a_0P_{03}bP_{25}a_5$$

and

$$W_{23} = a_2P_{25}bP_{03}a_3,$$

where we follow the notation from [9], i.e., $W_{05}$ is the walk in $G$ that starts at $a_0$ and follows the path $P_{03}$ up to the vertex $b$ from which the walk continues on the path $P_{25}$ up to the vertex $a_5$. Since $W_{05}$ and $W_{23}$ consist of the same edges (with same multiplicities) as $P_{03}$ and $P_{25}$ their lengths sum up to $4k - 4$. Consequently, one of the walks, say $W_{05}$, has length at most $2k - 2$. If $W_{05}$ is even, then, together with the edge $\{a_0, a_5\}$, it yields an odd closed walk of length at most $2k - 1$ and hence a short odd cycle. Otherwise, if $W_{05}$ and $W_{23}$ are odd, then also the walks

$$W_{02} = a_0P_{03}bP_{25}a_2$$

and

$$W_{35} = a_3P_{03}bP_{25}a_5$$

have an odd length. This implies that one of them, say $W_{02}$, has odd length at most $2k - 3$. Together with the path $a_0a_1a_2$ this results into a closed walk with odd length at most $2k - 1$ that yields the existence of a short odd cycle. Consequently, we derive a contradiction from the assumption that $P_{03}$ and $P_{25}$ are not vertex-disjoint.

Having established that $V(P_{03}) \cap V(P_{25}) = \emptyset$, we deduce that $G$ contains the following graph $\Phi'$ consisting of a cycle of length $4k$

$$a_0a_1a_2P_{25}a_5a_4a_3P_{03}a_0$$

with three diagonals $\{a_0, a_5\}$, $\{a_1, a_4\}$, and $\{a_2, a_3\}$.

We remark that it follows from [14, Lemma 2] that such a graph $\Phi'$ cannot occur as a subgraph in any $G \in G_{n,k}$. However, for a self-contained presentation we include a proof below.

We show that no vertex in $G$ can be joined to four vertices in $\Phi'$. Suppose, for a contradiction, that there exists a vertex $x$ in $G$ such that $|N_G(x) \cap V(\Phi')| \geq 4$. Recall that $x$ can be joined to at most two vertices of a cycle of length $2k + 1$ and, if so, then these vertices must have distance two in that cycle. Since each of the three diagonals splits the cycle of length $4k$ of $\Phi'$ into two cycles of length $2k + 1$, we have that $x$ cannot have more than four neighbors in $\Phi'$. Moreover, the only way to pick four neighbors is to choose two vertices from each of these cycles and none from their intersection, i.e., the ends of the diagonal. By applying this argument to each of the three diagonals, we infer that no vertex from $V(\Phi)$ can be a neighbor of $x$, therefore two neighbors $b_1$ and $b_2$ are
some inner vertices of $P_{03}$ and the two other neighbors $c_1$ and $c_2$ are inner vertices of $P_{25}$. Consider the vertex disjoint paths

$$P_1 = b_1P_{03}a_0a_1a_2P_{25}c_1$$

and

$$P_2 = b_2P_{03}a_3a_4a_5P_{25}c_2.$$  

Since $b_1$ and $b_2$ as well as $c_1$ and $c_2$ have distance two on the cycle of length $4k$ in $\Phi'$, both path lengths have the same parity and their lengths sum up to $4k - 4$. If both lengths are odd, one must have length at most $2k - 3$ and, together with $x$, this yields a short odd cycle. If, on the other hand, both lengths are even, then the paths

$$P_1' = b_1P_{03}a_0a_5P_{25}c_2$$

and

$$P_2' = b_2P_{03}a_3a_2P_{25}c_1$$

have odd length. Since their lengths sum up to $4k - 6$, together with $x$, this yields the existence of a short odd cycle. Therefore, every vertex of $G$ is joined to at most three vertices of $\Phi'$, which leads to the following contradiction

$$3n = 4k \frac{3n}{4k} < \sum_{u \in V(\Phi')} |N_G(u)| = \sum_{x \in V} |N_G(x) \cap V(\Phi')| \leq 3|V| = 3n.$$ 

This concludes the proof of Lemma 1.

### 2.2. Tetrahedra with Odd Faces

In the next lemma, we will show that graphs $G \in \mathcal{G}_{n,k}$ contain no graph from the following family, which can be viewed as tetrahedra with three faces formed by cycles of length $2k + 1$, i.e., a particular odd subdivision of $K_4$ (see, e.g., [11]).

**Definition 1** ($(2k + 1)$-tetrahedra). Given $k \geq 2$ we denote by $T_k$ the set of graphs $T$ consisting of

(i) one cycle $C_T$ with three branch vertices $a_T$, $b_T$, and $c_T \in V(C_T)$,  
(ii) a center vertex $z_T$, and  
(iii) internally vertex disjoint paths (called spokes) $P_{az}$, $P_{bz}$, $P_{cz}$ connecting the branch vertices with the center.

Furthermore, we require that each cycle in $T$ containing $z$ and exactly two of the branch vertices must have length $2k + 1$ and two of the spokes have length at least two.

It follows from the definition that for $T \in T_k$ we have that the cycle $C_T$ has odd length and if $T \subseteq G$ for some $G \in \mathcal{G}_{n,k}$, then $T$ consists of at least $4k$ vertices. In fact, the length of $C_T$ equals the sum of the lengths of the three cycles containing $z$ minus twice the sum of the lengths of the spokes. Since all three cycles containing $z$ have an odd length, the length of $C_T$ must be odd as well. In particular, if $T \subseteq G$ for some $G \in \mathcal{G}_{n,k}$, then the length of $C_T$ must be at least $2k + 1$. Summing up the lengths of all four cycles, counts
every vertex twice, except the branch vertices and the center vertex, which are counted
three times. Consequently,

\[ |V(T)| \geq \frac{1}{2} (4 \cdot (2k + 1) - 4) = 4k \] (1)

for every \( T \in T_k \) with \( T \subseteq G \) for some \( G \in \mathcal{G}_{n,k} \).

We will also use the following further notation. For a cycle containing distinct vertices
\( u, v, \) and \( w \) we denote by \( P_{uvw} \) the unique path on the cycle with endvertices \( u \) and \( w \) that
contains \( v \) and, similarly, we denote by \( P_{u\gamma} \) the path from \( u \) to \( w \) that does not contain \( v \).

For a tetrahedron \( T \in T_k \) we denote by \( C_{ab} \) the cycle containing \( z \) and the two branch
vertices \( a \) and \( b \). Similarly, we define \( C_{ac} \) and \( C_{bc} \). Note that the union of two cycles, for
instance \( C_{ab} \) and \( C_{ac} \), contains an even cycle

\[ C_{ab} \oplus C_{ac} = C_{ab} \cup C_{ac} - P_{ac} = aP_{abc}zP_{zca}a, \]

where \( P_{abc} \) is a path on the cycle \( C_{ab} \) and \( P_{zca} \) a path on the cycle \( C_{ac} \). Clearly, the length
of \( C_{ab} \oplus C_{ac} \) equals

\[ l(C_{ab} \oplus C_{ac}) = l(C_{ab}) + l(C_{ac}) - 2l(P_{ac}) = 4k + 2 - 2l(P_{ac}). \] (2)

**Lemma 2.** For all integers \( k \geq 2 \) and \( n \) and for every \( G \in \mathcal{G}_{n,k} \) we have that \( G \) does
not contain any \( T \in T_k \) as a (not necessarily induced) subgraph.

**Proof.** Suppose, contrary to the assertion, that \( G = (V, E) \) contains a graph from \( T_k \). Fix that graph
\( T \in T_k \) contained in \( G \) having the shortest length of \( C_T \). We shall prove that no vertex in \( G \) can be joined to four vertices in \( T \) and we will obtain a contradiction to the minimum degree assumption on \( G \).

Suppose that there exists a vertex \( x \in V \) such that \( |N_G(x) \cap V(T)| \geq 4 \) and fix four
of those neighbors. Since \( T \) consists of the union of three cycles of length \( 2k + 1 \) one
of those cycles must contain exactly two of those neighbors. This implies that we can
either pick two of those cycles that contain the four neighbors (see Claim 1 below), or
we have at least two ways to pick two such cycles that contain exactly three neighbors
(see Claim 2 below).

Recall that the vertices on the spokes belong to two cycles and the center \( z \) belongs to
all three cycles \( C_{ab}, C_{ac}, \) and \( C_{bc} \). If \( z \) is a neighbor of \( x \), then one more neighbor \( z' \) must
be on a spoke, because it must have distance two from \( z \) and \( T \) has at least two spokes
of length at least two. This means that two cycles already have two neighbors \( z \) and \( z' \), and
the third cycle already has one neighbor, namely \( z \). Therefore there cannot be two more
neighbors of \( x \) in \( T \). A similar argument shows that at most two neighbors of \( x \) can lie on
all the spokes of \( T \) all together.

Before we proceed to analyze the two cases, note that \( x \) can also be a vertex in \( T \). It is
easy to check that \( x \) cannot be \( z \), since it would have three neighbors on the three spokes,
which we just excluded. Furthermore, \( x \) cannot be one of the branch vertices. Indeed,
suppose \( x = a \). Then three neighbors \( y_1, y_2, y_3 \) of \( a \) are placed at distance 1 from \( a \) on
\( P_{azb}, P_{aza}, \) and \( P_{azc} \) respectively, and a neighbor \( y_4 \) can only be on \( P'_{azc} \), the interior of \( P_{azc} \).

Consider the paths

\[ P_{24} = y_2P_{az}zP_{zy_4}y_4 \]

and

\[ P'_{24} = y_2P_{az}zP_{zy_4}y_4, \]

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Since the subpaths \( zP_{db}y_4 \) and \( zP_{cy}y_4 \) cover the cycle \( C_{bc} \), which has length \( 2k + 1 \), the lengths of the paths \( P_{24} \) and \( P_{24}' \) have different parity. Suppose that \( P_{24} \) has odd length. Let \( P_{34} \) be the path \( y_3P_{ac}y_4 \) in \( C_{ac} \oplus C_{bc} \). Then both \( P_{24} \) and \( P_{34} \) have length \( 2k - 1 \), because

\[
l(P_{24}) + l(P_{34}) = l(C_{ac} \oplus C_{bc}) - 2 = 4k - 2l(P_{cz}) \leq 4k - 2
\]

and together with \( x \) each of the paths \( P_{24} \) and \( P_{34} \) create an odd cycle. The graph obtained from \( T \) by replacing the cycle \( C_{ab} \) with the cycle \( ay_2P_{24}y_4a \) of length \( 2k + 1 \) results in a graph \( T' \in T_k \), with branch vertices \( a, y_4, a \) and center \( z \). Since the spoke \( P_{zb} \) of \( T \) is replaced by the larger spoke \( P_{zy} = zP_{db}y_4 \) in \( T' \), we have that the cycle \( C_{T'} \) has shorter length than \( C_T \). This contradicts the choice of \( T \subseteq G \).

Summarizing the above, from now on we can assume that \( x \in V \setminus \{z, a, b, c\} \). Moreover, if \( x \in V(T) \), then \( x \) lies in one of the cycles \( C_{ab}, C_{ac}, \) or \( C_{bc} \) and two of the four neighbors of \( x \) in \( T \) must be direct neighbors on this cycle.

We now consider the aforementioned cases in Claim 1 and Claim 2 below.

**Claim 1.** The four neighbors of \( x \) in \( T \) cannot be contained in only two of the cycles \( C_{ab}, C_{ac}, \) and \( C_{bc} \).

Suppose \( C_{ab} \) and \( C_{ac} \) contain four neighbors of \( x \). Then the spoke \( P_{ac} \) shared by both cycles does not contain any neighbor of \( x \). Let \( y_1, y_2 \in N_G(x) \cap P_{ab} \) and \( y_3, y_4 \in N_G(x) \cap P_{ac} \), where \( y_1 \) and \( y_3 \) are the neighbors of \( x \) coming first on the respective paths \( P_{ab} \) and \( P_{ac} \) starting at \( a \). Consider the paths

\[
P_{13} = y_1P_{ba}aP_{acz}y_3
\]

and

\[
P_{24} = y_2P_{abc}zP_{ca}y_4.
\]

Since the neighbors in the same \( (2k + 1) \)-cycle have distance two and \( l(C_{ab} \oplus C_{ac}) \) is even, we infer that \( P_{13} \) and \( P_{24} \) have the same parity and

\[
l(P_{13}) + l(P_{24}) = 2(2k + 1) - 2l(P_{ac}) - 4 \leq 4k - 4.
\]

If \( P_{13} \) and \( P_{24} \) have odd length, then one of them must have length at most \( 2k - 3 \), thus, together with \( x \), it yields the existence of a short odd cycle. This implies that \( P_{13} \) and \( P_{24} \) have even length. Consequently, the paths

\[
P_{14} = y_1P_{zba}aP_{acz}y_3P_{za}y_4
\]

and

\[
P_{23} = y_2P_{abc}zP_{ca}y_3P_{za}
\]

have odd length and we have that

\[
l(P_{14}) + l(P_{23}) = 2(2k + 1) - 4 = 4k - 2.
\]

Therefore, because of the odd girth of \( G \), they must have both length \( 2k - 1 \).

Suppose that one path, say \( P_{14} \), has no endpoints inside the spokes \( P_{bc} \) and \( P_{cz} \) (here the branch vertices \( b \) and \( c \) are allowed to be neighbors of \( x \)) and \( x \) itself is not a vertex of \( P_{bc} \) and \( P_{cz} \).
In this case consider the \((2k + 1)\)-cycle \(C_{y_c}\) given by \(xy_1P_{14}y_4x\). As a result the graph obtained from \(T\) by replacing \(C_{ac}\) with \(C_{y_c}\) is a graph \(T' \in T_k\) with \(l(C_{T'}) < l(C_T)\), since the spoke \(P_{za}\) is replaced by the longer spoke \(P_{zy_1} = zP_{xy_1}\). This contradicts the choice of \(T\). Furthermore, if \(x\) would be on one of the spokes \(P_{bc}\) or \(P_{cz}\), then it must lie on \(P_{bc}\), since otherwise \(x\) would lie between \(y_3\) and \(y_4\) and then \(y_4\) would be contained in the interior of \(P_{cz}\), which we excluded here. Consequently, we arrive at the situation that \(y_1 = b\) and both \(y_2\) and \(x\) are inside \(P_{bc}\). Hence, the four neighbors of \(x\) are also contained in the cycle \(C_{ac} \oplus C_{bc}\), which also contains \(P_{23}\). Next we consider the path

\[ P'_{14} = y_1P_{y_1ca}y_4 \]

in \(C_{ac} \oplus C_{bc}\). Since \(l(C_{ac} \oplus C_{bc})\) is even and \(l(P_{23})\) is odd we have \(l(P'_{14}) = l(C_{ac} \oplus C_{bc}) - l(P_{23}) - 4\) is also odd. Recalling, that \(l(P_{23}) = 2k - 1\) we obtain

\[ l(P'_{14}) = 2(2k + 1) - 2l(P_{cz}) - l(P_{23}) - 4 = 2k - 1 - 2l(P_{cz}) \leq 2k - 3. \]

Hence, we arrive at the contradiction that \(P'_{14}\) together with \(x\) yields a short odd cycle in \(G\).

Thus both of the paths \(P_{13}\) and \(P_{24}\) must have an end vertex on one of the spokes \(P_{bc}\) and \(P_{cz}\). If both paths have an end vertex on the same spoke, say \(P_{bc}\), then we can repeat the last argument (considering \(P'_{14}\)).

Therefore, it must be that both \(P_{bc}\) and \(P_{cz}\) contain one neighbor of \(x\) each, namely \(y_2\) and \(y_4\). Since \(y_2\) and \(y_4\) are in the same \((2k + 1)\)-cycle \(C_{bc}\), they also have distance two in \(T\). This means that \(T\) contains a path \(y_1by_2z\) which, together with \(x\), results in cycle \(x_1by_2z\) of length six. Note that the diagonal \(\{y_2, x\}\) is present. Owing to Lemma 1 at least one of the other diagonals \(\{y_1, z\}\) and \(\{b, y_4\}\) must be an edge of \(G\). But both these edges are chords in cycles \(C_{ab}\) and \(C_{bc}\) of length \(2k + 1\), which contradicts the odd girth assumption on \(G\). This concludes the proof of Claim 1.

**Claim 2.** Three neighbors of \(x\) in \(T\) cannot be contained in only two of the cycles \(C_{ab}\), \(C_{ac}\), and \(C_{bc}\).

Let \(T \subseteq G\) chosen in the beginning of the proof violate the claim. First, we will show that we may assume that \(T\) also has the following properties:

(a) all four neighbors of \(x\) are contained in \(C_T\),  
(b) the two cycles can be chosen in such a way, that the spoke shared by them contains no neighbor of \(x\) and has length at least two, and  
(c) the cycle containing one neighbor of \(x\) has the property that this neighbor is not one of the two branch vertices contained in that cycle.

Owing to Claim 1 we know that any pair of two out of the three cycles \(C_{ab}, C_{ac}\), and \(C_{bc}\) contains at most three of the four neighbors of \(x\) in \(T\). Consequently, the spokes \(P_{ac}\), \(P_{bc}\), and \(P_{cz}\) all together can contain at most one neighbor of \(x\). Suppose \(v\) is a neighbor of \(x\) on the spoke \(P_{ac}\). Since we already showed that \(z\) cannot be a neighbor of \(x\), property (a) follows, by showing that \(v\) is not contained in \(P_{ac}'\) the interior of \(P_{ac}\). If \(v \neq a\), then the two neighbors \(y_1\) and \(y_2\) of \(x\) contained in \(C_{ab}\) and \(C_{ac}\) would have distance two from \(v\). Consequently, \(v\) would have to be a neighbor of \(a\) in \(P_{ac}\) and \(y_1\) and \(y_2\) would also have to be neighbors of \(a\) in \(T\). Hence, replacing \(a\) by \(x\) would give a rise to a subgraph \(T' \in T_k\) of \(G\), where \(x\) is a branch vertex. This yields a contradiction as shown before in Claim 1 and, hence, property (a) must hold.
Furthermore, if none of the neighbors is a branch vertex, then one cycle would contain two neighbors and the other two would contain one neighbor. Since at least two spokes have length at least two, we can select two cycles containing three neighbors in such a way that properties (b) and (c) hold.

If one neighbor is a branch vertex, say $b$, then the two cycles $C_{ab}$ and $C_{bc}$ contain two neighbors and $C_{ac}$ contains one neighbor of $x$. In particular the spokes $P_{ac}$ and $P_{cz}$ contain no neighbor and one of them has length at least two. This implies that we can select one of the cycles $C_{ab}$ or $C_{bc}$ together with $C_{ac}$ such that properties (b) and (c) also hold in this case.

Without loss of generality, we may, therefore, assume that the cycle $C_{ab}$ contains two neighbors $y_1$ and $y_2 \in P_{a \rightarrow b} \setminus \{a\}$ (where $y_1$ is closer to $a$ and $y_2$ is closer to $b$), that the cycle $C_{ac}$ contains one neighbor $y_3 \in P_{a \rightarrow c}$, and that the spoke $P_{az}$ has length at least two. In $C_{ab} \oplus C_{ac}$ we consider the paths

$$P_{13} = y_1P_{bac}y_3$$

and

$$P_{23} = y_2P_{abc}zP_{zca}y_3.$$ 

Since $P_{ac}$ has length at least two, we have that

$$l(P_{13}) + l(P_{23}) = 2(2k + 1) - 2l(P_{ac}) - 2 \leq 4k - 4.$$ 

Therefore, if $P_{13}$ and $P_{23}$ have odd length, then one has length at most $2k - 3$ and, together with $x$, it yields the existence of a short odd cycle. This implies that $P_{13}$ and $P_{23}$ have even length. Consequently, the paths

$$P'_{13} = y_1P_{baz}zP_{zca}y_3$$

and

$$P'_{23} = y_2P_{abc}zP_{zca}y_3$$

have odd length, and we have that

$$l(P'_{13}) + l(P'_{23}) = 2(2k + 1) - 2 = 4k.$$ 

Therefore, one of these paths, say $P'_{23}$ has length $2k - 1$. Set $C_{23} = xy_2P'_{23}y_3x$. The graph $T'$ obtained from $T$ by replacing $C_{ab}$ with $C_{23}$ is a again member of $T_k$. Since the spoke $P_{az}$ is replaced by the longer spoke $P_{yz} = y_3P_{ax}z$, we have $l(C_{T'}) < l(C_T)$ This contradicts the minimal choice of $T$, which concludes the proof of Claim 2.

Claim 2 yields that every vertex $x$ in $G$ is joined to at most three vertices of $T$. Recall that every $T \in T_k$ with $T \subseteq G$ consists of at least $4k$ vertices (see (1)). Similarly, as in the proof of Lemma 1, we obtain the following contradiction

$$3n = 4k \frac{3n}{4k} < \sum_{u \in V(T)} |N_G(u)| = \sum_{x \in V} |N_G(x) \cap V(T)| \leq 3|V| = 3n.$$ 

3. PROOF OF THE MAIN RESULT

In this section, we deduce Theorem 2 from Lemmas 1 and 2.
Proof of Theorem 2. Let \( G = (V, E) \) be a graph from \( \mathcal{G}_{n,k} \). We may assume that \( G \) is not a bipartite graph and we will show that it is a blow-up of a \((2k + 1)\)-cycle.

First we observe that \( G \) contains a cycle of length \( 2k + 1 \). Indeed, suppose for a contradiction that for some \( l > k \) a cycle \( C = a_0 \ldots a_{2l} \) is a smallest odd cycle in \( G \). Since \( G \) is edge-maximal, the nonexistence of the chord \( \{a_0, a_{2k}\} \) is due to the fact that it creates an odd cycle of length at most \( 2k - 1 \). Therefore \( a_0 \) and \( a_{2k} \) are linked by an even path \( P \) of length at most \( 2k - 2 \) which, together with the path \( P' = a_{2k}a_{2k+1} \ldots a_2a_0 \) yields the existence of an odd closed walk and, hence, of an odd cycle, of length at most \( 2l - 1 \), which contradicts the minimal choice of \( C \).

Let \( B \) be a vertex-maximal blow-up of a \((2k + 1)\)-cycle contained in \( G \). Let \( A_0, \ldots, A_{2k} \) be its vertex classes, labeled in such a way that every edge of \( B \) is contained in \( E_G(A_i, A_{i+1}) \) for some \( i \in \{0, \ldots, 2k\} \). Here and below addition in the indices of \( A \) is taken modulo \( 2k + 1 \). Clearly, the sets \( A_0, \ldots, A_{2k} \) are independent sets in \( G \). We will show \( B = G \). Suppose, for a contradiction, that there exists a vertex \( x \in V \setminus V(B) \).

Owing to the odd girth assumption on \( G \), the vertex \( x \) can have neighbors in at most two of the vertex classes of \( B \) and if there are two such classes, then they must be of the form \( A_{i-1} \) and \( A_{i+1} \) for some \( i = 0, \ldots, 2k \). The following claim, which follows from Lemma 1 shows that \( x \) can have neighbors in at most one of the vertex classes of \( B \).

Claim 3. If the neighbors of \( x \) in \( G \) belong to exactly two vertex classes \( A_{i-1} \) and \( A_{i+1} \), then \( x \in A_i \).

Moreover, we will apply Lemma 2 to show that \( x \) cannot have neighbors in only one class of \( B \).

Claim 4. The neighbors of \( x \) in \( G \) cannot belong to exactly one vertex class \( A_i \).

As a consequence every \( x \in V \setminus V(B) \) has no neighbor in \( B \). Therefore, \( V \setminus V(B) \) would be disconnected from \( B \), which violates the edge-maximality of \( G \). Consequently, \( V \setminus V(B) = \emptyset \) and \( G = B \), which (up to the verification of Claims 3 and 4) concludes the proof of Theorem 2.

Proof of Claim 3. Let \( x \in V \) have neighbors \( a_{i-1} \in A_{i-1} \) and \( a_{i+1} \in A_{i+1} \). In order to show that \( x \in A_i \), we shall prove that \( x \) is joined to all the vertices from \( A_{i-1} \) and to all the vertices from \( A_{i+1} \). Suppose that this is not the case and there is some vertex \( b_{i-1} \in A_{i-1} \), which is not a neighbor of \( x \). The argument for the other case, when there is such a vertex in \( A_{i+1} \) is identical.

Fix vertices \( a_{i-2} \in A_{i-2} \) and \( a_i \in A_i \) arbitrarily. This way we fixed a cycle

\[
C = x a_{i+1} a_{i-1} a_{i-2} a_{i-1} x
\]

of length six in \( G \). Owing to the choice of \( b_{i-1} \) the diagonal \( \{x, b_{i-1}\} \) is missing in \( C \). Moreover, the diagonal \( \{a_{i+1}, a_{i-2}\} \) is also not present, since together with a path from \( a_{i-2} \) to \( a_{i+1} \) through the vertex classes \( A_{i-3}, \ldots, A_1, A_0, A_{2k-1}, \ldots, A_{i+2} \) it would create an odd cycle of length \( 2k - 1 \). On the other hand, since \( B \) is a blow-up, the edge \( \{a_i, a_{i-1}\} \) is contained in \( B \subseteq G \), which is a diagonal in \( C \). Consequently, precisely one diagonal of \( C \) is present, which contradicts Lemma 1. Therefore, such a vertex \( b_{i-1} \) cannot exist, which yields the claim.

We will appeal to Lemma 2 to verify Claim 4.

Proof of Claim 4. Let \( \emptyset \neq N_G(x) \cap V(B) \subseteq A_i \) and fix some neighbor \( a_i \) of \( x \) in \( A_i \). Moreover, for every \( j \neq i \) fix a vertex \( a_j \in A_j \) arbitrarily. Since \( B \) is a blow-up of \( C_{2k+1} \),
those vertices span a cycle $C = a_0a_1 \ldots a_2a_0$ of length $2k + 1$. Moreover, since $x$ has no neighbors in $A_{i-2} \cup A_{i+2}$, the vertex $x$ is neither joined to $a_{i-2}$ nor to $a_{i+2}$.

The edge-maximality of $G \in \mathcal{G}_{n,k}$ implies the existence of paths $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$ in $G$ with an even length of at most $2k - 2$. Under all choices of such paths we pick two that minimize the number of edges together with $C$, i.e., we pick paths $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$ of even length at most $2k - 2$ such that

$$E(C) \cup E(P_{a_{i-2}x}) \cup E(P_{xa_{i+2}})$$

has minimum cardinality and we set

$$T = C \cup P_{a_{i-2}x} \cup P_{xa_{i+2}} \subseteq G.$$  

We shall show that $T$ is a tetrahedron from $T_k$ with center vertex $a_i$. Hence, Lemma 2 gives rise to a contradiction and no such vertex $x$ can exist.

Owing to the path $xa_i a_{i-1} a_{i-2} x$ of length three the path $P_{a_{i-2}x}$ must have length $2k - 2$. Similarly, $a_{i+2} a_{i+1} a_i x$ yields that $P_{xa_{i+2}}$ has length $2k - 2$. Moreover, $P_{a_{i-2}x}$ and $P_{a_{i+2}x}$ are disjoint from $\{a_{i-1}, a_i, a_{i+1}\}$. We set

$$C' = a_{i-2} P_{a_{i-2}x} xa_i a_{i-1} a_{i-2} \quad \text{and} \quad C'' = a_{i+2} a_{i+1} a_i x P_{xa_{i+2}} a_{i+2}.$$  

We just showed that $C'$ and $C''$ both have length $2k + 1$. In order to show that $T$ is a tetrahedron we have to show that the cycles $C$, $C'$, and $C''$ intersect pairwise in spokes with center $a_i$.

Consider the intersection $P$ of the cycles $C'$ and $C''$. We will show that $P$ is a path with one end vertex being $a_i$. Indeed every vertex in $a \in V(P) \setminus \{a_i\}$ is a vertex in the paths $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$. Owing to the minimal choice of $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$ it suffices to show that $a$ has the same distance to $x$ in both paths.

Suppose the distances have different parity. This implies that the closed walks

$$a P_{a_{i-2}x} P_{xa_{i+2}} a \quad \text{and} \quad a_i a_{i-1} a_{i-2} P_{a_{i-2}x} a P_{xa_{i+2}} a_{i+2} a_{i+1} a_i$$

have odd length. Since those walks cover the edges (with multiplicity) of $C'$ and $C''$ with the only exception of $xa_i$, the sum of their lengths is $l(C') + l(C'') - 2$. Hence, one of the closed walks would have an odd length of at most $2k - 1$, which yields a contradiction. If the distances between $a$ and $x$ are different, but have the same parity, then replacing the longer path by the shorter one in the corresponding cycle yields an odd cycle of length at most $2k - 1$. This again contradicts the assumptions on $G$ and, hence, $P = C' \cap C''$ is indeed a path with end vertex $a_i$.

In the same way one shows that $C \cap C'$ and $C \cap C''$ are paths with end vertex $a_i$. Since those two paths contain $a_i a_{i-1} a_{i-2}$ and $a_{i+2} a_{i+1} a_i$, respectively, their length is at least two. Therefore, $T$ is a tetrahedron from $T_k$ with center $a_i$ and spokes $C' \cap C''$, $C \cap C'$, and $C \cap C''$.

4. CONCLUDING REMARKS

We close with a few remarks.

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Extremal case in Theorem 2

A more careful analysis yields that the unique \( n \)-vertex graph with odd girth at least \( 2k + 1 \) and minimum degree exactly \( \frac{3n}{4k} \), which is not homomorphic to \( C_{2k+1} \), is the balanced blow-up of the Möbius ladder \( M_{4k} \). In fact, the proofs of Lemmas 1 and 2 can be adjusted in such a way that for maximal graphs \( G \) with \( \delta(G) \geq \frac{3n}{4k} \) they either exclude the existence of \( \Phi \) resp. \( T \) in \( G \) or they yield a copy of \( M_{4k} \) in \( G \). In the former case, one can repeat the proof of Theorem 2 based on those lemmas and obtains that \( G \) is homomorphic to \( C_{2k+1} \). In the latter case, one uses the degree assumption to deduce that \( G \) is isomorphic to a balanced blow-up of \( M_{4k} \).

Open questions

It would be interesting to study the situation, when we further relax the degree condition in Theorem 2. It seems plausible that if \( G \) has odd girth at least \( 2k + 1 \) and \( \delta(G) \geq \left( \frac{1}{2k-1} - \varepsilon \right)n \) for sufficiently small \( \varepsilon > 0 \), then the graph \( G \) is homomorphic to \( M_{4k} \). In fact, this seems to be true until \( \delta(G) > \frac{4n}{4k-1} \). At this point blow-ups of the \((6k-1)\)-cycle with all chords connecting two vertices of distance \( 2k \) in the cycle added, would show that this is best possible. For \( k = 2 \) such a result was proved by Chen et al. [8] and for \( k = 3 \) it was obtained by Brandt and Ribe-Baumann [6].

More generally, for \( l \geq 2 \) and \( k \geq 3 \) let \( F_{l,k} \) be the graph obtained from a cycle of length \( (2k-1)(l-1)+2 \) by adding all chords that connect vertices with distance of the form \( j(2k-1)+1 \) in the cycle for some \( j = 1, \ldots, [(l-1)/2] \). Note that \( F_{2,k} = C_{2k+1} \) and \( F_{3,k} = M_{4k} \). For every \( l \geq 2 \) the graph \( F_{l,k} \) is \( l \)-regular, has odd girth \( 2k + 1 \), and it has chromatic number three. Moreover, \( F_{l+1,k} \) is not homomorphic to \( F_{l,k} \), but contains it as a subgraph.

A possible generalization of the known results would be the following: if an \( n \)-vertex graph \( G \) has odd girth at least \( 2k + 1 \) and minimum degree bigger than \( \frac{lnn}{(2k-1)(l-1)+2} \), then it is homomorphic to \( F_{l-1,k} \). However, this is known to be false for \( k = 2 \) and \( l > 10 \), since such a graph \( G \) may contain a copy of the Grötzsch graph that (due to having chromatic number four) is not homomorphically embeddable into any \( F_{l,k} \). However, in some sense this is the only exception for that statement. In fact, with the additional condition \( \chi(G) \leq 3 \) the statement is known to be true for \( k = 2 \) (see, e.g., [8]). To our knowledge it is not known if a similar phenomenon happens for \( k > 2 \) and it would be interesting to study this further.

The discussion above motivates the following question, which asks for an extension of the result of Łuczak for triangle-free graphs from [16]. Note that for fixed \( k \) the degree of \( F_{l,k} \) divided by its number of vertices tends to \( \frac{1}{2k-1} \) as \( l \to \infty \). Is it true that every \( n \)-vertex graph with odd girth at least \( 2k + 1 \) and minimum degree at least \( \left( \frac{1}{2k-1} + \varepsilon \right)n \) can be mapped homomorphically into a graph \( H \) that also has odd girth at least \( 2k + 1 \) and \( V(H) \) is bounded by a constant \( C = C(\varepsilon) \) independent of \( n \)? Łuczak proved this for \( k = 2 \) and we are not aware of a counterexample for larger \( k \).

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