THE MODULI SPACE OF FANO MANIFOLDS WITH KÄHLER-RICCI SOLITONS

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Abstract. We construct a canonical Hausdorff complex analytic moduli space of Fano manifolds with Kähler-Ricci solitons. This enlarges the moduli space of Fano manifolds with Kähler-Einstein metrics. We discover a moment map picture for Kähler-Ricci solitons, and give complex analytic charts on the topological space consisting of Kähler-Ricci solitons, by studying differential geometric aspects of this moment map. Some stacky words and arguments on Gromov-Hausdorff convergence help to glue them together in the holomorphic manner.

Contents

1. Introduction 1
2. Kähler-Ricci soliton and K-stability 5
  2.1. Kähler-Ricci soliton 5
  2.2. K-stability 7
3. Donaldson-Fujiki set-up 9
  3.1. The moment map 10
  3.2. Local slice 16
  3.3. Completion 20
4. Canonical complex structure 23
  4.1. The moduli stack $\mathcal{K}_{T,\chi}$ 23
  4.2. Main construction 26
  4.3. Consistency 31
  4.4. The promised proof of Proposition 2.17 34
5. Discussions 37
  5.1. On some examples 37
  5.2. Future studies 38
6. Appendix
  A. Can-stack 40
  B. The demonstrations of (5)-(10) in the proof of Proposition 3.1 47
References 49

1. Introduction

After the breakthrough of [CDS, Tian2], the moduli space of Fano manifolds with Kähler-Einstein metrics was constructed in [OSS, Oda1, Oda2, LWX1] and has been explored in more detail by [LWX2] and by [SS, LiuXu] for special cases. The additional stability assumption ‘with Kähler-Einstein metrics’ might be somewhat
EIJI INOUE

stronger but still satisfactory since there are many interesting examples of Fano manifolds with Kähler-Einstein metrics, though many counterexamples are also known.

The existence of a well-behaved moduli space of projective varieties could be considered as a classification of them. Odaka and Okada conjectured in [OO] that every Fano manifold with Picard number one, which is one of the final outcome of the MMP, is K-semistable, so that they are ‘classified’ by the moduli space. But now many counterexamples of this conjecture was constructed by Fujita [Fujit] and Delcroix [Del].

In this paper, we extend the moduli space of Fano manifolds with Kähler-Einstein metrics to the moduli space of Fano manifolds with Kähler-Ricci solitons. Kähler-Ricci soliton, which consists of a Kähler metric and a holomorphic vector field, is a natural generalization of Kähler-Einstein metrics from the viewpoint of Kähler-Ricci flow. There are large amount of known examples of Fano manifolds admitting Kähler-Ricci solitons, including Delcroix’s infinite series of counterexamples of Odaka-Okada conjecture.

To formulate the universality of moduli space, we should consider a category consisting of families of which we intend to construct a moduli space. The category is usually called the moduli stack. In our moduli problem, we do not work with the usual moduli stack consisting of the usual families of Fano manifolds. In order to apply GIT method, we instead consider another new moduli stack $K(n)$ consisting of families of pairs $(X, ξ')$ of Fano manifolds and holomorphic vector fields, which is natural in view of the theory of Kähler-Ricci soliton. The moduli stack $K(n)$ is furthermore divided into clopen (closed and open) substacks $K_{T,χ}$, where the associated holomorphic vector fields are deformed holomorphically. We should review the theory of Kähler-Ricci soliton (in section 2) before explaining this unfamiliar moduli stack, so here we skip the precise description. See section 2 and Definition 4.1. See also Appendix A for generalities on stacks over the category of complex spaces. Example 5.4 might help the readers to better understand the reason why the usual stack does not serve our purpose. The readers will see in Remark 2.7 that the change of our moduli stacks does not affect the sets of what we intend to parametrize $(X$ or $(X, ξ'))$ and these are naturally identified to each other as sets.

Let $KR_{GH}(n)$ be the set of biholomorphism classes of $n$-dimensional Fano manifolds admitting Kähler-Ricci solitons. We can endow $KR_{GH}(n)$ with a natural topology induced by the ‘$J$-enhanced’ Gromov-Hausdorff convergence ([PSS]). In particular, the set $K_{0, GH}(n)$ of biholomorphism classes of $n$-dimensional Fano manifolds admitting Kähler-Einstein metrics forms a clopen subset of $KR_{GH}(n)$. Our main theorem is the following.

**Main Theorem** (Theorem 4.8 + Proposition 4.11). The Hausdorff topological space $KR_{GH}(n)$ admits a natural complex space structure which is uniquely characterized by the following universal property of a natural morphism $K(n) → KR_{GH}(n)$: any morphism $K(n) → X$ to any complex space $X$ holomorphically and uniquely factors through $KR_{GH}(n)$.

The Kähler-Ricci soliton counterpart of the Kähler-Einstein theory developed in [Tian1] and [CDS, Tian2] has been proved by [BW] and [DatSzé]. In contrast to the current known construction ([Oda1, Oda2, LWX1]) of the moduli space of Fano manifolds with Kähler-Einstein metrics, our method of construction actually does not depend on the result in [DatSzé], where they proved the modified K-polystable
Fano manifolds admit Kähler-Ricci solitons. But as some of the readers might prefer algebraic geometric formulation, we formulate things in terms of the modified K-stability, which can be translated into the existence of Kähler-Ricci solitons via [DatSze].

Our main tool for the construction of complex analytic charts on $\mathcal{K}\mathcal{R}_{GH}(n)$ is the following moment map.

**Main Proposition** (Proposition 3.1 + Proposition 3.2). Let $(M,\omega)$ be a $2n$-dimensional $C^\infty$-symplectic manifold underlying a Fano manifold with a Hamiltonian action of a closed real torus $T$. For any $\xi \in \mathfrak{t}$, there is a moment map $S_\xi : \mathcal{J}_T(M,\omega) \to \text{Lie}(\text{Ham}_T(M,\omega))^*$ on the space $\mathcal{J}_T(M,\omega)$ of $T$-invariant almost complex structures with respect to the modified symplectic structure $\Omega_\xi$ (see subsection 3.1) and the action of $\text{Ham}_T(M,\omega)$. Moreover, integrable complex structures in $S_\xi^{-1}(0)$ correspond to Kähler-Ricci solitons.

Actually, we firstly construct charts on the quotient space $(S_\xi^\circ)^{-1}(0)/\text{Ham}_T(M,\omega)$, where $S_\xi^\circ$ denotes the restriction of the moment map $S_\xi$ to the subspace $\mathcal{J}_T^\circ(M,\omega) \subset \mathcal{J}_T(M,\omega)$ consisting of integrable almost complex structures. The quotient space is shown to be identified with a clopen subspace of $\mathcal{K}\mathcal{R}_{GH}(n)$. To compare our constructions with [Oda1, Oda2, LWX1], let us briefly review their methods. They firstly prove the Zariski openness of the set of the K-(semi)stable points in any family of Fano manifolds. It follows that the usual moduli stack is Artin algebraic, so that they can apply the established theory of good moduli spaces of Artin algebraic stacks. Secondly they construct étale local charts on this stack of the form $[V/G]$, where each $V$ is an affine scheme and $G$ is a reductive algebraic group. Each quotient stack $[V/G]$ has the good moduli space $V \G G$. We can glue them together, just applying the gluing theory of good moduli spaces developed in [Alp2]. Technically, the proofs of the Zariski openness and the existence of the étale local charts rely on the argument showing that the set of K-(semi/poly)stable points forms a constructible set of the parameter space in the Zariski topology. The CM-line bundle, whose GIT weight equals to the Donaldson-Futaki invariant ([PT]), is used to prove the constructibleness. (Compare [Don2] for another proof of the Zariski openness.)

However, in the case of Kähler-Ricci soliton, as there is no candidate for the CM-line bundle because of the irrationality of the modified Donaldson-Futaki invariant, we face a problem with the constructibleness. So we will work with the real topology, in other words, with Artin analytic stacks. We can still construct local charts on this Artin analytic stack with good moduli spaces, however, the second nuisance appears when gluing the good moduli spaces together: there is no well-established theory of good moduli spaces for Artin analytic stacks so far. It seems hard to show the uniqueness property of good moduli spaces of Artin analytic stacks, which is obviously a key property for the good gluing theory (cf. [Alp1, Alp2]). This nuisance ultimately comes from the lack of the satisfactory theory of ‘quasi-coherent sheaves’ on complex analytic spaces (cf. [EP]).

Alternatively, we glue our charts by a ‘cooperation of virtual and real’. We construct analytic charts not only on the stack $\mathcal{K}(n)$, but also on the topological spaces $(S_\xi^\circ)^{-1}(0)/\text{Ham}_T(M,\omega)$, which are related in a canonical way. The latter
EIJI INOUE

‘real side’ is studied in section 3 and is used to show that the charts are actually homeomorphisms onto open subsets of \((S^c_\xi)^{-1}(0)/\text{Ham}_T(M,\omega)\). This is not treated in [Oda1, Oda2, LWX1] as they could apply the great Alper’s gluing theory, which works ‘without reality’. The former ‘virtual side’ is studied in section 4 and is used to show the holomorphy of coordinate changes. Finding holomorphic relations between the analytic charts are easier on the stack \(\mathcal{K}(n)\) than on the topological spaces \((S^c_\xi)^{-1}(0)/\text{Ham}_T(M,\omega)\). These ‘virtual’ holomorphic relations descend to the ‘real’ holomorphic relations between the analytic charts on \((S^c_\xi)^{-1}(0)/\text{Ham}_T(M,\omega)\) thanks to the universality of the local moduli spaces and the natural fully faithful embedding (2-Yoneda lemma) of the category \(\text{Can}\) of complex analytic spaces to the 2-category of complex analytic stacks. On the contrary, from this viewpoint, it does not sound realistic to prove the holomorphy ‘within the real side’ as the forgetful functor from \(\text{Can}\) to the category of topological spaces is not full nor faithful.

**Organization.** The remainder of this paper is organized as follows. In section 2, we review some known results on Kähler-Ricci soliton and rearrange K-stability notion modified to the soliton setting so that it fits into our moduli problem. It is explained that the pair \((X,\xi')\) can be converted into the action \(X \ltimes T\), where \(T\) is the torus generated by the holomorphic vector field \(\xi'\). We introduce gentle Fano \(T\)-manifolds as Fano \(T\)-manifolds inseparable from smooth Fano \(T\)-manifolds with Kähler-Ricci solitons, which are expected to be K-semistable. They form an adequate moduli stack in our moduli problem. Finally, we propose Proposition 2.17 which states the uniqueness of the central fiber of gentle degenerations. It will be proved after we complete Proposition 4.7 and play an essential role in the proof of Theorem 4.8 in subsection 4.2.

In section 3, we construct and study an infinite dimensional moment map \(S_\xi\) whose integrable zero points correspond to Kähler-Ricci solitons. We describe that local slices \(\nu : B \rightarrow \mathfrak{k}\) of the moment map actually give charts \(\nu^{-1}(0)/K \approx BK^c//K^c\) on the topological space consisting of Kähler-Ricci solitons. To achieve this, we need to study Banach completions of Fréchet manifolds, where we should pay attention to the completions of \(\text{Ham}_T(M,\omega)\) as they are never Banach Lie groups. We also prove that, in any family of Fano \(T\)-manifolds, the set of gentle Fano \(T\)-manifolds forms an open subset in the parameter space of the family.

In section 4, the main theorem is proved. We introduce the stack \(\mathcal{K}_{T,\chi}\) of gentle Fano \(T\)-manifolds and show that it is an Artin analytic stack. We prove Proposition 2.17 in subsection 4.4, using the results in the former half of subsection 4.2. We use this proposition in the proof of the main theorem. In subsection 4.3, we show that our moduli space is related to the topological space \(\mathcal{K}\mathcal{R}_{GH}(n)\) endowed with the ‘\(J\)-enhanced’ Gromov-Hausdorff topology, which is studied in [PSS].

In section 5, we review some examples of Fano manifolds with Kähler-Ricci solitons and propose future studies. In particular, we find an isotrivial degeneration of a Kähler-Einstein Fano manifold to another Fano manifold with non-Einstein Kähler-Ricci soliton, which implies that the usual moduli stack is not sufficiently separated and hence our new formulation of moduli stacks \(\mathcal{K}(n)\) and \(\mathcal{K}_{T,\chi}\) is essential.

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2. Kähler-Ricci soliton and K-stability

2.1. Kähler-Ricci soliton. A Kähler metric $g$ on a Fano manifold $X$ is called a Kähler-Ricci soliton if it satisfies the following equation:

$$\text{Ric}(g) - L_{\xi'} g = g$$

for some holomorphic vector field $\xi'$. The same term sometimes refers the pair $(g, \xi')$.

A fundamental feature of a Kähler-Ricci soliton $(g, \xi')$ is that it gives an eternal solution of the normalized Kähler-Ricci flow:

$$\partial_t g(t) = -\text{Ric}(g(t)) + g(t).$$

Namely, for the 1-parameter smooth family $\phi_t : X \to X$ generated by $\text{Re}(\xi')$, the following holds:

$$\partial_t (\phi_t^* g) = -\text{Ric}(\phi_t^* g) + \phi_t^* g.$$  

On a Fano manifold admitting Kähler-Ricci soliton, it is shown in [TZ3, TZZZ, DerSzé] that the normalized Kähler-Ricci flow converges to a Kähler-Ricci soliton, starting from any Kähler metric in $2\pi c_1(M)$.

It is shown in [Zhu] that there is a solution $g$ of the equation

$$\text{Ric}(g) - L_{\xi'} g = g_0$$

for any initial Kähler metric $g_0$. Let us consider the following smooth continuity path for Kähler-Ricci soliton:

$$\text{Ric}(g_t) - L_{\xi'} g_t = tg_t + (1-t)g_0.$$  

One can prove that

$$R_{\xi'}(X) := \sup\{t \in [0,1] \mid \text{a solution } g_t \text{ of (1) exists.} \}$$

is independent of the choice of the initial metrics $g_0$ and has the equality

$$R_{\xi'}(X) = \sup\{t \in [0,1] \mid \exists g \text{ s.t. } \text{Ric}(g) - L_{\xi'} g > tg\}.$$  

The proof is in [Szé3] for $\xi' = 0$ case and in [Has] for the general case.

The uniqueness and the existence results analogous to those of the Kähler-Einstein metrics hold also for Kähler-Ricci solitons.

**Theorem 2.1** (uniqueness [TZ1, TZ2] (BW for Q-Fano with $t = 1$)). If $(g_1, \xi'_1)$ and $(g_2, \xi'_2)$ are two Kähler-Ricci solitons on a Fano manifold $X$, then there is an element $\phi \in \text{Aut}^0(X)$ such that

$$g_2 = \phi^* g_1, \quad \xi'_2 = \phi^* \xi'_1,$$

where $\text{Aut}^0(X)$ is the identity component of the group $\text{Aut}(X)$ of biholomorphic automorphisms of $X$. Moreover, a solution $g_t$ of the equation (1) is absolutely unique for any initial metric $g_0$ and $t \in [0,1)$. 


Theorem 2.2 (existence of K-stability). $R_{\xi'}(X) = 1$ for any K-semistable pair $(X, \xi')$. If in addition $(X, \xi')$ is K-polystable, there is a Kähler-Ricci soliton on $X$ with respect to $\xi'$.

We will see the definition of the K-stability of pairs $(X, \xi')$ in the next subsection. The opposite implication is also proved by [BW], ahead of [DatSz], including Q-Fano case.

Theorem 2.3 ([BW]). Let $X$ be a Q-Fano variety. If $X$ has a Kähler-Ricci soliton $(g, \xi')$, then $(X, \xi')$ is K-polystable.

In the case of Kähler-Einstein metrics, i.e. $\xi' = 0$, the twisted K-stability theory developed in [DatSz] shows that $X$ is K-semistable if $R(X) = 1$. So we can summarize as follows.

- $X$ is K-polystable $\iff X$ has a Kähler-Einstein metric.
- $X$ is K-semistable $\iff R(X) = 1$.

The implication from the right to the left handside of the second item is not demonstrated for the soliton setting so far.

There is a version of Futaki invariant suitable for the soliton setting, which is defined in [TZ2]. Define a linear map $\text{Fut}_{\xi'} : H^0(X, \Theta_X) \to \mathbb{C}$ by

$$\text{Fut}_{\xi'}(v') := \int_X v'(h - \theta_{\xi'}) e^{\theta_{\xi'}} \omega^n,$$

where $\omega \in 2\pi c_1(M)$ is a Kähler-form, $h$ is a real valued function satisfying $\sqrt{-1} \partial \bar{\partial} h = \text{Ric}(\omega) - \omega$ and $\theta_{\xi'}$ is a complex-valued function characterized by

$$\begin{cases} L_{\xi'} \omega = \sqrt{-1} \partial \bar{\partial} \theta_{\xi'} \\ \int_X e^{\theta_{\xi'}} \omega^n = \int_X \omega^n. \end{cases}$$

The function $\theta_{\xi'}$ becomes real-valued when $\xi := \text{Im} \xi'$ is a Killing vector.

This linear function is independent of the choice of $\omega$, so it gives an invariant depending only on $X$ and $\xi'$, which is now called the modified Futaki invariant. This invariant obviously vanishes when $X$ admits a Kähler-Ricci soliton with respect to the vector field $\xi'$.

The following is a critical fact in order to formulate our moduli setting.

Proposition 2.4 ([TZ2]). Let $X$ be a Fano manifold, which does not necessarily have a Kähler-Ricci soliton $(g, \xi')$, then $(X, \xi')$ is K-semistable.

Remark 2.5. In general, a reductive algebraic group $K^c$ does not uniquely determine its maximal compact subgroup $K$, but only up to conjugate. When $K^c$ is an algebraic torus, which is isomorphic to $(\mathbb{C}^\times)^k$, its maximal compact subgroup $(U(1))^k$ is uniquely determined. This fact allows us to get away from a formulation relying on structures over the field $\mathbb{R}$ in the next subsection, which may be preferred by algebraic geometers as we can formulate things over any field.

The biholomorphic automorphism group of a Fano manifold $X$ admitting a Kähler-Ricci soliton $(g, \xi')$ is not necessarily reductive. Instead, we have the following.
Theorem 2.6 ([IZ2] ([BW] for \(Q\)-Fano case)). If \(Q\)-Fano variety \(X\) has a Kähler-Ricci soliton \((g, \xi')\), then the subgroup \(\text{Aut}^0(X, \xi') \subset \text{Aut}^0(X)\) consisting of \(\xi'\)-preserving biholomorphic automorphisms is a maximal reductive subgroup of \(\text{Aut}^0(X)\).

Remark 2.7. The reductivity of the automorphism groups of geometric objects of which we intend to find a geometric moduli space, is indispensable if one expects to apply the local or global GIT method to its construction.

The uniqueness of Kähler-Ricci soliton implies that the set consisting of the isomorphism classes of the pairs \((X, \xi')\) with Kähler-Ricci solitons can be naturally identified with the set consisting of the biholomorphism classes of Fano manifolds \(X\) with Kähler-Ricci solitons. So there is no change in the support sets of ‘the moduli spaces’ of the following two moduli stacks: one is the usual moduli stack associated with Fano manifolds \(X\) admitting Kähler-Ricci solitons, and the other is the moduli stack associated with Fano pairs \((X, \xi')\) admitting Kähler-Ricci solitons.

However, there are nice geometric features in the latter stack compared to the former stack, such as a good separatedness property and the reductivity of the stabilizer groups at K-polystable points, which is appropriate for the local GIT construction of the good moduli space.

So we will work with the latter stack, and precisely define it in subsection 4.1, replacing the pairs \((X, \xi')\) with the \(\mathcal{T}_\xi\)-action on \(X\). This may change the topology of the moduli space, but it turns out that the latter stack is correct with regard to the ‘J-enhanced’ Gromov-Hausdorff convergence considered in [PSS].

2.2. K-stability. Here we review the definition of K-stability and formulate it as the stability notion of a Fano manifold with an algebraic torus action. This enables us to introduce an adequate notion of ‘deformations of Fano manifolds with Kähler-Ricci solitons’ and leads us to the proper definition of the stack \(\mathcal{K}(n)\).

Recall that a \(Q\)-Fano variety \(X\) is a reduced irreducible normal complex space \(X\) with the following property: there is a positive integer \(\ell\) such that the sheaf \(i_*((\det \Theta_{X^{reg}})^{\otimes \ell})\), which is denoted by \(\mathcal{O}(-\ell K_X)\), is isomorphic to the sheaf of sections of an ample line bundle on \(X\), and \(X\) has log terminal singularities (see [EGZ]). The minimum \(\ell\) satisfying this property is called the \(\mathcal{Q}\)-Gorenstein index of \(X\). Obviously, \(Q\)-Fano varieties can be embedded into some \(\mathbb{C}P^N\), hence they are also considered as schemes, but we treat them in the category of complex spaces.

A \(Q\)-Fano \(T\)-variety is a \(Q\)-Fano variety \(X\) with a holomorphic action \(\alpha : X \times T \to X\), where we only consider an algebraic torus \(T \cong (\mathbb{C}^*)^k\). When \(X\) is a smooth complex manifold, we call it \(\text{Fano } T\)-manifold. The \(T\)-action canonically lifts to the sheaf \(\mathcal{O}(-m\ell K_X)\) and hence there is an action of \(T\) on the cohomologies of \(\mathcal{O}(-m\ell K_X)\).

Let \(T\) be an algebraic torus. We denote the character lattice of \(T\) by \(M\) and its dual by \(N\). Let \(X\) be a \(Q\)-Fano \(T\)-variety. Set \(h^0_X(m) := \dim H^0(X, \mathcal{O}(-m\ell K_X))\) and \(h^0_{X, u}(m) := \dim H^0_u(X, \mathcal{O}(-m\ell K_X))\) for \(u \in M\), where

\[H^0_u(X, \mathcal{O}(-m\ell K_X)) := \{\sigma \in H^0(X, \mathcal{O}(-m\ell K_X)) \mid t.\sigma = u(t)\sigma \ \forall t \in T\}.

The following definition of Donaldson-Futaki invariant is due to [BW], which is shown to coincide with Fut\(_\xi\) in the previous section, up to a positive factor.

Definition 2.8 (Donaldson-Futaki invariant). For a \(Q\)-Fano \(T\)-variety \(X\) and an element \(\xi \in N_R\), we define the (modified) Donaldson-Futaki invariant \(DF_{X, \xi} : N \to \)
EIJI INOUE

\[ DF_{X,\xi}(\lambda) := - \lim_{m \to \infty} \frac{w_{X,\xi}(m; \lambda)}{m h_0^X(m)}, \]

where

\[ w_{X,\xi}(m; \lambda) := \sum_{u \in M} e^{(u,\xi)/m} h_0^{X,u}(m) u^\ast \langle u, \lambda \rangle. \]

We define the Hilbert character \( \chi : \mathbb{Z} \to \mathbb{Z}[M] \) of a Fano \( T \)-manifold \( X \) by

\[ (3) \quad \chi_m := \sum_{i=0}^{\dim X} (-1)^i \sum_{u \in M} h_i^{X,u}(m) u \in \mathbb{Z}[M]. \]

We call a function \( \chi : \mathbb{Z} \to \mathbb{Z}[M] \) a Fano character if there exists a Fano \( T \)-manifold whose Hilbert character given in (3) is equal to \( \chi \).

**Proposition 2.9 ([TZ2]).** For every Fano \( T \)-manifold \( X \), there exists a unique vector \( \xi \in N_R \) with the vanishing Donaldson-Futaki invariant \( DF_{X,\xi} \equiv 0 \). We call this vector the K-optimal vector of \( (X,T) \).

Obviously from the definition of the Donaldson-Futaki invariant, the K-optimal vector \( \xi \) of a Fano \( T \)-manifold \( X \) depends only on the Hilbert character \( (T,\chi) \). So it also makes sense to say that \( \xi \in N_R \) is the K-optimal vector of a Fano character \( (T,\chi) \).

**Proposition 2.10 ([TZ2]).** If a Fano manifold \( X \) has a Kähler-Ricci soliton \( (g,\xi') \), then the \( \xi' \) is the K-optimal vector with respect to any algebraic torus which contains the algebraic torus generated by \( \xi' \).

**Definition 2.11 (K-optimal action).** We call a Fano character \( (T,\chi) \) K-optimal if there is no proper sub-lattice \( N' \subset N \) with \( \xi \in N'_R \) where \( \xi \in N_R \) is the K-optimal vector of \( (T,\chi) \).

**Remark 2.12.** It is possible that both \( (T_1,\chi_1) \subset (T_2,\chi_2) \) are K-optimal. Actually, there is an example of a Fano manifold \( X \) with Kähler-Einstein metric \( (g,0) \) admitting a degeneration to a Fano \( T \)-manifold \( X_0 \) with Kähler-Ricci soliton \( (g_0,\xi') \) (Example 5.4). This example illustrates that the torus equivariant formulation is essential for the separatedness of the moduli space of Fano manifolds with Kähler-Ricci solitons.

Let \( X \) be a \( \mathbb{Q} \)-Fano \( T \)-variety. A pair \( (\pi : \mathcal{X} \to \mathbb{C}, \theta) \) consisting of the following data is called a special degeneration of \( X \).

(1) \( \mathcal{X} \) is a normal complex space with an action of \( T \times \mathbb{C}^* \) and \( \pi : \mathcal{X} \to \mathbb{C} \) is a \( T \times \mathbb{C}^* \)-equivariant proper flat \( \mathbb{Q} \)-Gorenstein surjective morphism whose central fiber \( \mathcal{X}_0 \) is a \( \mathbb{Q} \)-Fano variety, where \( T \times \mathbb{C}^* \) acts on \( \mathbb{C} \) by \( z \cdot (t,s) = sz \).

(2) \( \theta \) is a \( T \times \mathbb{C}^* \)-equivariant isomorphism \( \theta : \mathcal{X} \times \mathbb{C}^* \cong \pi^{-1}(\mathbb{C}^*) \).

We also assume that there is a holomorphic line bundle \( \mathcal{L} \) on \( \mathcal{X} \) with an isomorphism \( \theta^\ast \mathcal{L} \cong p_1^\ast \mathcal{O}(-\ell K_X) \) for some \( \ell \). It is shown in [Ber, Lemma 2.2] that if such \( \mathcal{L} \) exists, then \( -\ell K_X \) becomes \( \mathbb{Q} \)-Cartier and the tensor bundle \( \mathcal{L} \otimes m \mathcal{O}(m) \) is actually isomorphic to \( \mathcal{O}(m\ell K_X/C) \) for some \( m \). So we exclude the datum \( \mathcal{L} \) from the data of special degeneration.
Definition 2.13 (K-stability). Let $\xi \in N_\mathbb{R}$ be the K-optimal vector of a $\mathbb{Q}$-Fano $T$-variety $X$. Set $\xi = (\xi, 0) \in (N \times \mathbb{Z})_\mathbb{R}$, then we call the $\mathbb{Q}$-Fano $T$-variety $X$ K-semistable if for any special degeneration $(\pi : X \to \mathbb{C}, \theta)$ of $X$, the Donaldson-Futaki invariant $DF_{X, \xi}(\pi, \theta) := DF_{X_0, \xi}(\lambda)$ of the central fiber $X_0$ is nonnegative, where $\lambda$ is the one parameter subgroup defined by $\lambda : \mathbb{C}^* \to T \times \mathbb{C}^* : s \mapsto (1, s)$. Furthermore, we call $X$ K-polystable when $X$ is K-semistable and $DF_{X, \xi}(\pi, \theta) = 0$ if and only if there exists a one parameter subgroup $\lambda : \mathbb{C}^* \to \text{Aut}_T(X)$ such that $\theta(t(x\lambda(t)^{-1}, t)$ extends to an isomorphism of the total space $X \times \mathbb{C} \xrightarrow{\sim} \mathbb{X}$, and K-stable when $X$ is K-polystable and $\text{Aut}_T^0(X) = T$.

Remark 2.14. A pair $(X, \xi')$ of a $\mathbb{Q}$-Fano variety $X$ and a holomorphic vector field $\xi'$ is called K-(semi/poly)stable if $\xi := \text{Im} \xi'$ generates a closed real torus $T_\mathbb{R}$ and $(X, T)$ is K-(semi/poly)stable where $T$ denotes the complexification of the closed real torus $T_\mathbb{R}$. In this case, the vector $\xi \in \text{Lie}(T_\mathbb{R}) = N_\mathbb{R}$ is of course K-optimal.

We call X modified K-(semi/poly)stable if there exists a torus action $X \curvearrowright T$ which makes $X$ K-(semi/poly)stable with respect to the action.

Remark 2.15. Note that a K-(semi/poly)stable Fano $T$-manifold is not necessarily a K-(semi/poly)stable Fano manifold (with respect to the trivial torus action), but only a modified K-(semi/poly)stable Fano manifold. However, suppose $X$ is a Fano $T$-manifold, $T' \subset T$ is a subtorus and the K-optimal vector $\xi'$ with respect to the $T'$-action coincides with the K-optimal vector $\xi$ with respect to the $T$-action, then the Fano $T$-manifold $X$ is K-(semi/poly)stable if and only if the Fano $T'$-manifold $X$ is. This is proved in [DatSze] and also proved in [LiXu, LWX3] by an algebraic method for $\xi = 0$.

We introduce a gentle Fano $T$-manifold as a Fano $T$-manifold inseparable from a smooth Fano $T$-manifold admitting Kähler-Ricci soliton.

Definition 2.16 (gentle Fano). A Fano $T$-manifold is called gentle if there is a $T$-equivariant deformation $\mathcal{X} \to \Delta$ with an isomorphism $\mathcal{X}|_{\Delta^*} \cong X \times \Delta^*$ such that its central fiber $\mathcal{X}_0$ is a smooth K-polystable Fano $T$-manifold. We call $\mathcal{X} \to \Delta$ a gentle degeneration.

It is naturally expected that any gentle Fano $T$-manifold is K-semistable. In this paper, we do not pursue this expectation as it is unessential for the construction of our moduli space, while the K-semistability might be philosophically important. Note that we always have $R_\xi(X) = 1$ for a gentle Fano $T$-manifold $X$ with the K-optimal vector $\xi$, thanks to the equality (2).

The following proposition will help the construction of our moduli space.

Proposition 2.17. Let $X$ be a gentle Fano $T$-manifold whose torus action is K-optimal. Then any two gentle degenerations of $X$ have the $T$-equivariant biholomorphic central fibers.

The proposition will be proved at the end of section 4 and will be used in the proof of Theorem 4.8.
\( \omega \)-compatible almost complex structure \( J \) with positive Ricci curvature. Note that \( b^1(M) = 0 \) from familiar Bochner’s theorem.

We denote by \( \text{Symp}(M, \omega) \) the group of symplectic diffeomorphisms and \( \text{Ham}(M, \omega) \) its subgroup generated by Hamiltonian diffeomorphisms. Thanks to Banyaga’s theorem, in the case \( b^1 = 0 \), \( \text{Ham}(M, \omega) \) actually coincides with \( \text{Symp}_0(M, \omega) \), the identity connected component of \( \text{Symp}(M, \omega) \). (Even though it is easy to see that both groups have a natural Fréchet Lie group structures and their Lie algebras coincide, the coincidence at the level of Fréchet Lie group is \textit{non-trivial} because the Fréchet Lie group structures are not locally exponential. See [Neeb] for the generalities on Fréchet Lie groups.) By abuse of notation, we set \( \text{Ham}(M, \omega) := \text{Symp}(M, \omega) \), which is not necessarily connected.

Throughout this section, \( T \) stands for a closed real torus and \( (M, \omega) \) for a symplectic Fano manifold with a Hamiltonian effective action by \( T \).

We consider the space \( \mathcal{J}_T(M, \omega) \) of \( T \)-invariant \( \omega \)-compatible almost complex structures and denote by \( \mathcal{J}^0_T(M, \omega) \) the subspace of integrable complex structures. It is well known that \( \mathcal{J}_T(M, \omega) \) admits a natural Fréchet smooth manifold structure, which is identified with the space of \( T \)-equivariant sections of an associated \( Sp(2n)/U(n) \)-fibre bundle (see [Pal] for instance). The tangent space at \( J \in \mathcal{J}_T(M, \omega) \) can be written as follows.

\[
T_J \mathcal{J}_T(M, \omega) = \{ A \in \Gamma^\infty_T(\text{End}TM) \mid AJ + JA = 0, \omega(A, \cdot) + \omega(\cdot, A) = 0 \}.
\]

Similarly, the group \( \text{Ham}_T(M, \omega) \) of \( T \)-compatible symplectic diffeomorphisms can be endowed with a Fréchet smooth Lie group structure, whose Lie algebra can be identified with \( C^\infty_T(M)/\mathbb{R} \). The left adjoint action is given by

\[
\text{Ham}_T(M, \omega) \times C^\infty_T(M)/\mathbb{R} \to C^\infty_T(M)/\mathbb{R} : (\phi, f) \mapsto f \circ \phi^{-1}.
\]

The following action

\[
\mathcal{J}_T(M, \omega) \times \text{Ham}_T(M, \omega) \to \mathcal{J}_T(M, \omega) : (J, \phi) \mapsto \phi^* J
\]

is also smooth and its derivative is calculated as

\[
C^\infty_T(M)/\mathbb{R} \to T_J \mathcal{J}_T(M, \omega) : f \mapsto L_{X_f} J,
\]

where \( X_f \) is the Hamiltonian vector field of \( f \) \( -df = i(X_f) \omega \).

3.1. **The moment map.** For a given \( \xi \in \mathfrak{t} = \text{Lie}(T) \), we let \( \theta_\xi \) be a real valued function on \( M \) given by

\[
-\theta_\xi = i\xi \omega
\]

with the prescribed normalization

\[
\int_M \theta_\xi e^{-2\theta_\xi} \omega^n = 0.
\]

This function is invariant under the action of \( \text{Ham}_T(M, \omega) \).

Set \( \theta_\xi := -2\theta_\xi \). For each \( J \in \mathcal{J}_T(M, \omega) \),

\[
\xi' := J\xi + \sqrt{-1} \xi \in \mathcal{X}^{1,0}(M, J)
\]

satisfies

\[
\sqrt{-1} \partial \bar{\partial} \theta_\xi = \sqrt{-1} (d(-2\theta_\xi) + \sqrt{-1} J(-2d\theta_\xi))/2 = i\xi' \omega.
\]

We consider a Riemannian metric on \( \mathcal{J}_T(M, \omega) \), twisted by \( \xi \) from usual one, defined as

\[
(A, B)_{\xi} := \int_M g^ij \, g_{kl} A^i_k B_j^l \, e^{-2\theta_\xi} \omega^n
\]
for tangent vectors $A, B \in T_J \mathcal{J}_T(M, \omega)$ and set
\[ \Omega_\xi(A, B) := (JA, B)_\xi. \]

It is easy to see that $\Omega_\xi$ defines a nondegenerate closed 2-form on $\mathcal{J}_T(M, \omega)$. We also consider
\[ (f, g)_\xi := \int_M fg \, e^{-2\theta_\xi} \omega^n \]
for $f, g \in C^\infty_T(M)$, which defines an inner product on the subspace
\[ C^\infty_T(M, \omega) := \{ f \in C^\infty_T(M) \mid \int_M f \, e^{-2\theta_\xi} \omega^n = 0 \} \cong C^\infty_T(M)/\mathbb{R}. \]

Finally, we denote by $s(J)$ the Hermitian scalar curvature of $J$, defined by Donaldson [Don1]. We normalize $s(J)$ by a factor so that it is equal to the Kähler scalar curvature $-g_J^{ij} \partial_{J_i} \partial_{\bar{J}_j} (\log \det g_J)$ for any integrable $J$, which is the half of the Riemannian scalar curvature. Here is the moment map.

**Proposition 3.1.** Fix $\xi, \zeta \in \mathfrak{t}$. For each $J \in \mathcal{J}_T(M, \omega)$, we consider the modified Hermitian scalar curvature
\[ s_{\xi, \zeta}(J) := (s(J) - n) + \Delta_g \theta_\xi' - \xi_j \theta'_\xi - \bar{\theta}_j, \]
where $4$ comes from our convention of the metric $(\cdot, \cdot)_\xi$ (compare [Sze1, Proposition 2.2.1.]). Combined with the following basic identities: (a) $X_{fg} = f X_g + g X_f$,
(b) \( L_{fX}J = fL_XJ - Jdf \otimes X + df \otimes JX \), (c) \( L_\xi J = 0 \), the first term of (4) can be arranged as follows.

\[
\frac{d}{dt} \bigg|_{t=0} (4s(J_t), f e^{-2\theta t}) = (L_{Xf \exp(-2\theta t)} J, JA) = (L_{\exp(-2\theta t)Xf} J, JA) + (-2)(L_{f \exp(-2\theta t)} \xi J, JA) = (e^{-2\theta t}L_{fX}J - Jd(e^{-2\theta t}) \otimes X_f + d(e^{-2\theta t}) \otimes JX_f, JA) + (-2)(Jd\xi J - Jd(e^{-2\theta t}) \otimes \xi + d(e^{-2\theta t}) \otimes J\xi, JA)
\]

Now it suffices to show the following equalities.

\[
\frac{d}{dt} \bigg|_{t=0} ((\Delta_t + (-2)J_t \xi) \theta \xi, f) = (Jd\theta \xi \otimes X_f, JA) = -d(\theta \xi \otimes JX_f, JA) = (Jd\theta \xi \otimes \xi, JA) = -(df \otimes J\xi, JA)
\]

As for (5),

\[
(Jd\theta \xi \otimes X_f, JA) = \int_M g^{ij} g_{kl} (Jd\theta \xi \otimes X_f)_l^j (JA)_k e^{-2\theta n} \omega^n = \int_M (Jd\theta \xi \otimes X_f)_l^j (JA)_k e^{-2\theta n} \omega^n = \int_M -\theta_{i,p} (f_j \omega^{jk}) A^p_k e^{-2\theta n} \omega^n = \int_M -\theta_{i,p} (f_j \omega^{jk}) A^p_k e^{-2\theta n} \omega^n = \frac{d}{dt} \bigg|_{t=0} \int_M g^{ij} (d\theta \xi, df) e^{-2\theta n} \omega^n = \frac{d}{dt} \bigg|_{t=0} ((\Delta_t + (-2)J_t \xi) \theta \xi, f).\]
We can easily obtain (9) as follows.

\[
(f Jd\theta_\xi \otimes \xi, JA)_\xi = \int_M g^{ij}g_{kl}(Jd\theta_\xi \otimes \xi)^k_i (JA)^l_j f e^{-2\theta_\xi} \omega^n = \int_M -\theta_\xi h^k e^k_i (JA)^l_j f e^{-2\theta_\xi} \omega^n = \int_M -\theta_\xi h^k e^k_i (JA)^l_j f e^{-2\theta_\xi} \omega^n = \frac{d}{dt}|_{t=0}(-\theta_\xi(J\theta_\xi, f)\xi) .
\]

The rest of them are demonstrated in the Appendix B. □

Now we observe that our moment map actually corresponds to Kähler-Ricci solitons.

**Proposition 3.2.** For simplicity, we let \(s_\xi, S_\xi\) stand for \(s_{\xi,0}, S_{\xi,0}\), respectively. The following (1)-(3) are equivalent for any integrable \(J \in \mathcal{F}_0^\omega(M, \omega)\).

1. \((g_J, \xi_J')\) is a Kähler-Ricci soliton on \((M, J)\).
2. \(s_\xi(J) = 0\).
3. \(S_\xi(J) = 0\).

**Proof.** Provided that \(g_J\) satisfies the Kähler-Ricci soliton equation \(\text{Ric}(g_J) - L_{\xi_J'} g_J = g_J\). The trace of this formula gives

\[
(11) \quad s(J) + \square \theta_\xi' = n.
\]

Since \(\xi_J'\) is holomorphic, the Lie derivative by \(\xi_J'\) can be arranged as follows.

\[
\sqrt{-1} \partial \bar{\partial} (\theta_\xi' - \xi_J' \theta_\xi) = \sqrt{-1} \partial \bar{\partial} \theta_\xi',
\]

and hence \(\square \theta_\xi' - \xi_J' \theta_\xi - \theta_\xi'\) is constant. Recall that the operator \((\square - \xi_J')\) is a formally self-adjoint elliptic operator with respect to the inner product \(\langle \cdot, \cdot \rangle_{\theta_\xi'}\) (see [Fut]). It follows that the equation \((\square - \xi_J')u = f\) has a solution \(u\) if and only if \(\int_M fe^{\theta_\xi'} \omega^n = 0\). This shows

\[
(12) \quad \square \theta_\xi' - \xi_J' \theta_\xi = \theta_\xi'
\]

under the normalization condition \(\int_M \theta_\xi' e^{\theta_\xi'} \omega^n = 0\). Substituting (12), the equation (11) can be reformulated as

\[
(s(J) - n) + 2\square \theta_\xi' - (\xi_J' \theta_\xi' + \theta_\xi') = 0.
\]

The left hand side of the equation is nothing but \(s_\xi(J)\), and we obtain \(s_\xi(J) = 0\).

Conversely, assume \(S_\xi(J) = 0\). Take a function \(h\) so that \(\sqrt{-1} \partial \bar{\partial} h = \text{Ric}(\omega) - \omega\). Since \(L_{\xi_J'} \omega = \sqrt{-1} \partial \bar{\partial} \theta_\xi'\), it is enough to show that \(h - \theta_\xi'\) is actually constant. Similarly as before, the Lie derivative of \(\sqrt{-1} \partial \bar{\partial} \theta_\xi'\) gives

\[
\sqrt{-1} \partial \bar{\partial} \xi_J' h = \sqrt{-1} \partial \bar{\partial} (\theta_\xi' - \theta_\xi)
\]

and hence

\[
c_1 := \bar{\square} \theta_\xi' - \theta_\xi' - \xi_J' h
\]
is constant. We can rearrange the modified Hermitian scalar curvature as
\[
s_{\xi}(J) = -\Box h + 2\Box \theta_{\xi} - \xi_{f} \theta_{\xi} - \theta_{\xi} = -\Box(h + 2\Box \theta_{\xi} - \xi_{f} \theta_{\xi} - h) + c_{1}.
\]
(13)
Now the assumption $S_{\xi}(J) = 0$ implies that $c_{2} := s_{\xi}(J)$ is a constant. Since 
\[((\Box - \xi_{f})(h - \theta_{\xi}^{s}), 1)_{\theta_{\xi}^{s}} = 0, \text{ the constant } c_{1} - c_{2} = (\Box - \xi_{f})(h - \theta_{\xi}^{s}) \text{ has to be zero and we have shown that } h - \theta_{\xi}^{s} \text{ is constant.}\]

\[\square\]

Remark 3.3. As noted in [Don1] for Kähler-Einstein case, we can interpret or even reproduce the following known results, which were proved in [TZ2], from geometric viewpoint via our moment map.

- The invariance of the modified Futaki invariant.
- The reductivity of $\text{Aut}(X, \xi')$. (cf. [FO])
- The uniqueness of Kähler-Ricci soliton. (cf. [BerBer])

For instance, substituting (13), we obtain

\[
\langle S_{\xi}(J), f \rangle = \int_{M} s_{\xi}(J) f e^{\theta_{\xi}} \omega^{n}
\]
\[
= \int_{M} 4(-\Box - \xi_{f})(h - \theta_{\xi}) + c_{1}) f e^{\theta_{\xi}} \omega^{n}
\]
\[
= -4 \int_{M} (\bar{\partial}(h - \theta_{\xi}) f) e^{\theta_{\xi}} \omega^{n}
\]
\[
= -4 \int_{M} X_{f}^{'s}(h - \theta_{\xi}) e^{\theta_{\xi}} \omega^{n}
\]
\[
= -4 \text{Fut}_{\xi}(X_{f}^{'s})\]

for $X_{f}^{'s} \in \text{Lie(Stab}(J))$, where $c = \int e^{\theta_{\xi}} \omega^{n} / \int \omega^{n}$ is independent of $J$. The invariance can be interpreted as coming from a general fact on moment maps: for any $x \in M$ and $v \in \text{Lie}(K_{\mathfrak{x}})$, $\langle \mu(xg), g^{-1}v \rangle$ is invariant for $g \in K_{\mathfrak{x}}$, where $\mu : M \rightarrow \mathfrak{t}^{*}$ is a moment map.

Proposition 3.1 in particular shows that $\text{Fut}_{\xi_{s}}|_{1}$ is $T$-equivariant complex deformation invariant; it only depends on the $T$-equivariant symplectic structure.

We call a symplectic Fano $T$-manifold $(M, \omega, T)$ $K$-optimal if there is an $\omega$-compatible integrable complex structure $J$ (then $(M, J)$ is a Fano manifold) and whose Hilbert character $(T_{C}, \chi(M, J))$ is $K$-optimal. In this case, using equivariant Hirzebruch-Riemann-Roch formula (McC), its Hilbert character can be written as

\[
\chi_{m}(\exp \eta) = \sum_{k=0}^{\infty} \frac{m^{k}}{k!} \int_{M} (c_{1,1}(\omega, \eta))^{k} Td_{t}(\omega, \eta).
\]

Here the equivariant Chern character and Todd character is given by
\[
c_{1,1}(\omega, \eta) := \text{tr}(F_{\omega}) + 2\pi \sqrt{-1} \langle \mu, \eta \rangle
\]
\[
Td_{t}(\omega, \eta) := \text{det} \left( \frac{\sqrt{-1}F_{t}(\omega, \eta)}{1 - e^{-\sqrt{-1}F_{t}(\omega, \eta)}} \right),
\]
where $\mu$ is a canonical moment map associated to the $T$-action on $(M, L = -K(M, J))$ and
\[
F_{t}(\omega, \eta) := F_{\omega} + 2\pi \sqrt{-1} (L_{\eta} - \nabla \eta).
\]
In summary, though we firstly assume the existence of an integrable complex structure, its Hilbert character is actually calculated by the equivariant Chern classes, which depends only on the equivariant symplectic structure of \((M, \omega, T)\).

We denote by \(S^2_T\) the restriction of the moment map \(S_T : \mathcal{J}_T(M, \omega) \rightarrow C^\infty_T(M, \omega)^*\) to the subspace \(\mathcal{J}_T^2(M, \omega)\), which consists of integrable complex structures.

**Proposition 3.4.** Assume the action of \(T\) on \((M, \omega)\) is K-optimal. Then the following two statements are equivalent for any integrable \(J, J' \in (S^2_T)^{-1}(0)\).

1. There is a \(T\)-equivariant \(C^\infty\)-diffeomorphism \(\phi : M \xrightarrow{\sim} M\) such that \(J = \phi^*J'\).
2. \([J] = [J'] \in (S^2_T)^{-1}(0)/\text{Ham}_T(M, \omega)\).

**Proof.** It follows from the uniqueness of Kähler-Ricci soliton and \(\text{Aut}_T(X) = \text{Aut}(X, \xi)\) for the K-optimal action. \(\square\)

**Remark 3.5.** The above proposition should hold without K-optimal assumption. To achive this, it suffices to prove the following uniqueness result.

If \(g_1, g_2\) are two Kähler-Ricci solitons on a Fano \(T\)-manifold \(X\) which are \(T\)-invariant and have the same soliton vectors \(\xi_1 = \xi_2 \in t\), then there is an element \(\phi \in \text{Aut}_T(X)\) such that \(g_2 = \phi^*g_1\).

In general, verifying the K-optimality of a given torus action on a Fano manifold may be a hard task, especially when the dimension of the center of its maximal reductive subgroup is greater than one. It should be better to consider non K-optimal actions if one expect to get an explicit description of the moduli space of Fano manifolds with Kähler-Ricci solitons in examples.

It follows that the quotient set \((S^2_T)^{-1}(0)/\text{Ham}_T(M, \omega)\) coincides with the set of biholomorphism classes of Fano manifolds admitting Kähler-Ricci solitons with the fixed underlying symplectic structure \((M, \omega)\). Hence this set should be the support of our moduli space. Next we verify the quotient topology on this set is Hausdorff.

**Proposition 3.6.** The action of \(\text{Ham}_T(M, \omega)\) on \(\mathcal{J}_T(M, \omega)\) is proper. In particular, the quotient topological space \((S^2_T)^{-1}(0)/\text{Ham}_T(M, \omega)\) is Hausdorff.

**Proof.** We must show that the map

\[ a : \mathcal{J}_T(M, \omega) \times \text{Ham}_T(M, \omega) \rightarrow \mathcal{J}_T(M, \omega) \times \mathcal{J}_T(M, \omega) : (J, \phi) \mapsto (J, \phi^*J) \]

is proper. Take a sequence \((J_n, \phi_n)\) so that \(J_n, \phi_n^*J_n\) converge to some \(J_\infty, J'_\infty \in \mathcal{J}_T(M, \omega)\) in the given order. It suffices to show that a subsequence of \(\phi_n\) converges to some \(\phi_\infty \in \text{Ham}_T(M, \omega)\) satisfying \(\phi_\infty^*J_\infty = J'_\infty\). Let \(g_\infty, g'_\infty\) denote the Riemannian metrics associated to \(J_\infty, J'_\infty\), respectively.

The first step of the proof is similar to the uniqueness argument in Gromov-Hausdorff convergence. Let us take a dense countable subset \(S\) of \(M\). The diagonal argument shows that we have a subsequence of \(\phi_n\) so that \(\phi_n(x)\) converges for any \(x \in S\). We continue to write \(\phi_n\) for this subsequence. We obtain a distance preserving map \(\phi_{S, \infty} : (S, d_{g_\infty}) \rightarrow (M, d_{g_\infty})\) by putting \(\phi_{S, \infty}(x) := \lim_{n \rightarrow \infty} \phi_n(x)\). Then this map can be uniquely extended to a distance preserving map \(\phi_\infty : (M, d_{g_\infty}) \rightarrow (M, d_{g_\infty})\). Similarly we obtain a distance preserving map \(\psi_\infty : (M, d_{g_\infty}) \rightarrow (M, d_{g_\infty})\) as a limit of \(\phi_n^{-1}\). It follows from [BBI] Theorem 1.6.14 that the distance preserving endomorphism \(\phi_\infty \circ \psi_\infty\) is surjective, and we conclude \(\phi_\infty\) is a continuous bijective map.
Thanks to Myers-Steenrod theorem, we see that $\phi_\infty$ is a $C^\infty$-diffeomorphism with $\phi_\infty^* g_\infty = g_\infty^*$. Moreover, since $\phi_n, \phi_n^* g_n$ respectively converge to $g_\infty, \phi_\infty^* g_\infty$ in $C^\infty$-topology, it can be seen that the coefficients $(d\phi_n)_i^j(d\phi_n)_i^j$ with respect to a $C^\infty$-coordinate converges to $(d\phi_\infty)_i^j(d\phi_\infty)_i^j$ in $C^\infty$-topology. In particular,

\[(\partial_k(d\phi_n)_i^j(d\phi_n)_i^j) \xrightarrow{C^\infty} (\partial_k(d\phi_\infty)_i^j(d\phi_\infty)_i^j)
\]

and $(d\phi_n)_i^j = \sqrt{(d\phi_n)_i^j}^2$ converges to $(d\phi_\infty)_i^j$ in $C^0$-topology. It follows from (14) that the $C^k$-convergence of $d\phi_n$ induces the $C^{k+1}$-convergence of them. This shows that $\phi_n$ converges to $\phi_\infty \in \text{Ham}_T(M, \omega)$ in $C^\infty$-topology and $\phi_\infty^* J_\infty = J_\infty'$.

3.2. Local slice. The materials in this subsection are pararel to Sz2, where the case of constant scalar curvature is treated.

Let $X$ be a Fano $T$-manifold with a Kähler-Ricci soliton $(g, \xi')$, $\phi : (M, J_0) \overset{\sim}{\rightarrow} X$ be a biholomorphism, where $M$ is a $C^\infty$-manifold and $J_0$ is a integrable complex structure on it. Put $\omega := (\phi^g)(J_0, -)$ and $K := \{ h \in \text{Ham}(M, \omega) \mid h^* J_0 = J_0 \}$, which is a compact Lie group, and $\mathfrak{t} := \{ f \in C^\infty(M, \omega) \mid L_\xi, J_0 = 0 \}$, which can be identified with the Lie algebra of $K$. Consider the $L^2_k$-completion of the moment map

$$S_\xi : J_T(M, \omega)_k^2 \rightarrow L^2_{k-2, T}(M, \omega)^*.$$ 

We denote by $\Theta$ the holomorphic tangent sheaf of $X$ and by $H^i_T(X, \Theta)$ the $T$-invariant subspace of the $i$-th cohomology $H^i(X, \Theta)$. Note that $H^1(X, \Theta) = 0$ for $i \geq 2$ since $X$ is Fano.

**Proposition 3.7.** There are an open ball $B \subset H^1_T(X, \Theta)$ centered at the origin, a $K$-equivariant holomorphic deformation $\varpi : X \times B \rightarrow X$ of $X$ with a holomorphic morphism $i : X \hookrightarrow X_0$ inducing a biholomorphism to the central fiber, a $K$-equivariant $C^\infty$-smooth map $\mathcal{J} : B \rightarrow J_T(M, \omega)_k^2$ and a $T_k$-equivariant $L^2_k$-regular diffeomorphism $\Phi : B \times M \overset{\sim}{\rightarrow} X$ with the following properties.

1. The holomorphic family $X \xrightarrow{i} X \xrightarrow{\varpi} (B, 0)$ is a semi-universal family of $X$.
2. For each $b \in B$, $\mathcal{J}(b)$ is an $L^2_k$-regular integrable complex structure satisfying $s_\xi(\mathcal{J}(b)) \in \mathfrak{t}$ and $\mathcal{J}(0) = J_0$.
3. The diffeomorphism $\Phi$ satisfies $p_1 \circ \Phi = \varpi$, $\Phi_0 = \phi$. The restricted map $\Phi_b : (M, \mathcal{J}(b)) \rightarrow X_b$ is a biholomorphism for each $b \in B$.

**Proof.** Let $X \rightarrow B$ be the $K$-equivariant Kuranishi family of $T$-equivariant deformation of $X$ (see Kur1, Kur2, Dou1 for its construction). From its construction, we have a holomorphic map $\mu : B \rightarrow J_T(M)^2_k$ whose image $\mu(b)$ is a real analytic complex structure for each $b \in B$ with a biholomorphism $X_b \cong (M, \mu(b))$. Using a $K$-equivariant global basis of the vector bundle $\omega, \mathcal{O}(-\ell K_X/B)$ on $B$, we can uniformly embed these Fano manifolds into some fixed projective space $\mathbb{CP}^N$ (see Lemma 4.6). Pulling back the Fubini-Study metric, we obtain a $K$-equivariant smooth family of Kähler metrics $\{ \omega_b \}_{b \in B}$, where each $\omega_b$ can be identified with a Kähler metric on $(M, \mu(b))$. Taking smaller $B$, we can assume that closed forms $\omega_{b, t} := \omega_0 + t(\omega_b - \omega_0)$ are nondegenerate for each $b \in B$ and $t \in [0, 1]$. Then we can find a $K$-equivariant family of diffeomorphisms $\{ f_b \}_{b \in B}$ so that $f_b^* \omega_b = \omega_0$. Putting $\mathcal{J}'(b) := f_b^* \mu(b)$, we obtain a $K$-equivariant smooth map $\mathcal{J}' : B \rightarrow J_T(M, \omega)_k^2$, whose image $\mathcal{J}'(b)$ is a smooth complex structure for each $b \in B$.

It suffices to show that we can find an equivariant perturbation $\mathcal{J}$ of $\mathcal{J}'$ so that $\mathcal{J}(b) = g_b^* \mathcal{J}'(b)$ for each $b$ and $s_\xi(\mathcal{J}(b)) \in \mathfrak{t}$. Let $U_{k+2}^2 \subset L^2_{k+2}(M, \omega)$ be a small ball
of the origin. For each $\phi \in U_{k+2}$ and an almost complex structure $J \in \mathcal{J}_T(M, \omega)$, we can find an $L_2^2$-regular vector field $X^{\phi,J}_{\omega}$ on $M$ so that $i(X^{\phi,J}_{\omega})(\omega_0 - tdJ\phi) = -J\phi$. This vector field is actually $L_2^2$-regular. In fact, it is sufficient to show that $[X_f, X^{\phi,J}_{\omega}]$ is $L_2^k$-regular for any smooth function $f$. For any smooth vector field $Z$, we have

$$i([X_f, X^{\phi,J}_{\omega}])\omega_t(Z) = -(LX_f, \omega)(X^{\phi,J}_{\omega}, Z) + X_f(\omega(X^{\phi,J}_{\omega}, Z)) - \omega(X^{\phi,J}_{\omega}, [X_f, Z]) = X_f(-Jd\phi(Z)) + Jd\phi([X_f, Z]) \in L_2^k.$$ 

Thus $[X_f, X^{\phi,J}_{\omega}]$ is $L_2^k$, as we expected. The flow $f^{\phi,J}_t$ of this time-dependent vector fields is $L_2^{k+1}$-regular and satisfy $(f^{\phi,J}_t)^* (\omega_0 - tdJ\phi) = \omega_0$. To see the regularity, it is sufficient to show that $(f^{\phi,J}_t)^* Y$ is a $L_2^k$-regular vector field for each smooth vector field $Y$ on $M$. Note that $(d/dt)(f^{\phi,J}_t)^* Y = [X^{\phi,J}_s, Y]$ is $L_2^k$-regular and $(f^{\phi,J}_t)^* Y$ can be written as $\int_0^t [X^{\phi,J}_s, Y] ds$. Then for each $l \leq k$, we obtain the following estimate, so $f^{\phi,J}_t$ is $L_2^{k+1}$-regular.

$$\int_M |\nabla^l (f^{\phi,J}_t)^* Y|^2 \omega^n = \int_M |\int_0^t \nabla^l [X^{\phi,J}_s, Y] ds|^2 \omega^n \leq \int_M t \int_0^t |\nabla^l [X^{\phi,J}_s, Y]|^2 ds \omega^n \leq t \int_0^t ||[X^{\phi,J}_s, Y]||^2_{L_2^k} ds < \infty.$$ 

It follows that $(f^{\phi,J}_t)^* J \in \mathcal{J}_T(M, \omega)_{k+2}^2$. Consider the orthogonal decomposition $L^2 = \mathfrak{t} \oplus \mathfrak{t}_{\perp}$ with respect to $L^2$-norm $(-,-)_\xi$. Put $\mathfrak{t}^2_{k,\perp} := L^2_k \cap \mathfrak{t}_{\perp}$ and let $\Pi_\perp : L^2_{k-2} \to \mathfrak{t}^2_{k-2,\perp}$ be the $L^2$-projection. Note that

$$(D(\phi \mapsto (f^{\phi,J}_t)_0)^* J_0) = \frac{d}{dt} f^{\phi,J}_t J_0 = J_0 P(\psi),$$

where $P$ denotes the linear differential operator $P : \mathfrak{t}^2_{k+2} \to T_J \mathcal{J}_T(M, \omega)_{k+2}^2 : \psi \mapsto L_X, J_0$, and $(D\xi J)(A) = P^* J A$, where $P^*$ is the formal adjoint of $P$ with respect to the norm $(-,-)_\xi$. It follows that

$$G : B \times U \to \mathfrak{t}^2_{k-2,\perp} : (b, \phi) \mapsto \Pi_\perp s_\xi ((f^{\phi,J}_1)^* J'(b))$$

is a $K$-equivariant smooth map with the derivative

$$DG_{(0,0)}(0, \psi) = -P^* P(\psi).$$

Since $P^* P$ is a self-adjoint fourth order elliptic differential operator, it gives the isomorphism $P^* \mathfrak{t}^2_{k+2,\perp} \to \mathfrak{t}^2_{k-2,\perp}$. Applying the implicit function theorem, we can find a new $K$-equivariant smooth map $\mathfrak{J} : B \to \mathcal{J}_T(M, \omega)_{k+2}$ so that $\Pi_\perp s_\xi (\mathfrak{J}(b)) = 0$, hence $s_\xi (\mathfrak{J}(b)) \in \mathfrak{t}$, taking smaller $B$ if necessary.

Pulling back the symplectic structure $\Omega_\xi$ on $\mathcal{J}_T(M, \omega)_{k+2}^2$ by the $K$-equivariant smooth map $\mathfrak{J} : B \to \mathcal{J}_T(M, \omega)_{k+2}^2$, we obtain a $K$-equivariant smooth symplectic structure (by taking smaller $B$ if necessary), which we denote by the same notation. Then $\nu : B \to \mathfrak{t}^* : b \mapsto S_\xi (\mathfrak{J}(b))$ is a moment map with respect to this symplectic structure.
Proposition 3.8 (Please be careful! See PostScript Remark below.). If $B \subset H^1_T(X, \Theta)$ in the Proposition 3.7 is sufficiently small, then the following two statements are equivalent for any $b \in B$.

1. The fiber $X_b$ of the family $\varpi : \mathcal{X} \to B$ has a Kähler-Ricci soliton.
2. The orbit $b \cdot \text{Aut}_T(X) \subset H^1_T(X, \Theta)$ is closed. That is, $b$ is polystable with respect to the $\text{Aut}_T(X)$-action.

PostScript Remark 1. While discussing with R. Dervan and P. Naumann, the author realized that there was a gap in the following proof with regards to the implication “the existence of Kähler-Ricci soliton $\Rightarrow$ GIT-polystability”. To be precise, what we prove here is that the following are equivalent for $b \in B$:

1. $\nu^{-1}(0) \cap B \cap b \cdot G \neq \emptyset$.
2. $\nu^{-1}(0) \cap B \cap b \cdot G \neq \emptyset$.
3. The point $b \in H^1_T(X, \Theta)$ is polystable with respect the $\text{Aut}_T(X)$-action.

Of course, this (1) implies the existence of Kähler-Ricci soliton on $X_b$. On the other hand, however, the existence of Kähler-Ricci soliton on $X_b$ only implies that there is a unique orbit $B \cap b_0 G$ in the closure $B \cap b_0 G$ such that $\nu^{-1}(0) \cap B \cap b_0 G \neq \emptyset$ and $X_{b'}$ is isomorphic to $X_b$ for any $b' \in B \cap b_0 G$ (thanks to $K$-polystability). This in particular implies that $\nu^{-1}(0)/K \approx BK^c / K^c$ (Corollary 3.13 below) can be naturally identified with the isomorphism classes of Fano manifolds admitting Kähler-Ricci solitons who appear in the family $\varpi : \mathcal{X} \to B$ (thanks to Corollary 3.12 below).

The author emphasizes that we do not need the original statement of Proposition 3.8 to prove all the statements in the rest of this paper. So we only need to prove the equivalence stated in this remark. The author leaves the original proof which includes a gap in the last paragraph and leaves it to the readers to check the above equivalence.

Proof of Proposition 3.8. Let $\Omega_0$ be the linearization of $\Omega_\xi$ at $0 \in B$, i.e., $\Omega_0 = (d\varphi_0 - J_0 d\varphi_0 - \xi)$ under the identification $T_b B = \mathbb{H}_1^T = T_0 B$. Consider the map $\nu_0 : \mathbb{H}_1^T \to \mathfrak{k}^*$ defined by

$$\langle f, \nu_0(b) \rangle = \Omega_0(L_{X_b} b, b) = \langle L_{X_b} d\varphi_0 b, J_0 d\varphi_0 b \rangle_\xi.$$ 

Then $\nu_0$ is a moment map with respect to the symplectic structure $\Omega_0$. The Kempf-Ness theorem says that $b \in \mathbb{H}_1^T$ is polystable with respect to $K^c = \text{Aut}_T(X)$ if and only if $bK^c \cap \nu_0^{-1}(0) \neq \emptyset$.

Since

$$\frac{d^2}{dt^2} \big|_{t=0} \langle f, \nu(tb) \rangle = \frac{d^2}{dt^2} \big|_{t=0} \langle f, s_\xi(\varpi(tb)) \rangle_\xi = \frac{d}{dt} \big|_{t=0} \langle L_{X_b} \varpi(tb), \varpi(tb) \varpi(tb) \rangle_\xi = \langle f, \nu_0(b) \rangle,$$

the moment map $\nu : B \to \mathfrak{k}^*$ can be expanded as

$$\nu(tb) = \nu(0) + t d_0 \nu(b) + t^2 \nu_0(b)/2 + O(t^3).$$

Since $0 \in B$ corresponds to Fano manifolds with Kähler-Ricci soliton $(M, J_0, \omega)$, $\nu(0) = S_\xi(J_0) = 0$ from Proposition 3.2. Moreover, since $0$ is a fixed point of the $K$-action, we have $d_0 \nu = 0$. Therefore we get

$$\nu(tb) = t^2 \nu_0(b)/2 + O(t^3).$$
Proof. Suppose the Fano manifold \((\lambda, \sigma_0)\) so that \(\Omega_{b}\) the point \(S\) of complex manifolds, the following subset \(K\) \(b\) is gentle. In particular, for any \(T\) Corollary 3.10.

Now we cite the following general lemma from [Sze2, Proposition 9.] and [Don3, Proposition 17.].

**Lemma 3.9.** Let \((B, \Omega)\) be a symplectic manifolds with a \(K\)-action, \(\nu : B \rightarrow \mathfrak{k}\) be a moment map with respect to the \(K\)-action (\(\mathfrak{k}\) is endowed with a inner product). Suppose \(b \in B\) satisfies \(\nu(b) \in \mathfrak{k}_{b}^\perp\) and \(\lambda, \delta > 0\) with \(\lambda \|\nu(b)\| < \delta\) satisfies 
\[
\|\langle \sigma_{\nu(b)}, \sigma_{v_{\nu(b)}} \rangle \| \leq \lambda \quad \text{for any} \quad v \in \mathfrak{k} \quad \text{with} \quad \|v\| < \delta.
\]
Then there is \(v_{b} \in \mathfrak{k}\) such that \(\nu(v_{b}) = 0\) and \(\|v_{b}\| \leq \lambda \|\nu(b)\|\).

Fix a small \(\delta > 0\) so that there is \(C > 0\) such that for any \(v \in \mathfrak{k}\) with \(\|v\| < \delta\) and any \(f \in \mathfrak{k}^\perp_{v_{\nu(b)}}\)
\[
\|\sigma_{v_{\nu(b)}}(f)\|_{\Omega_{0}}^2 \geq C\|f\|^2
\]
holds. Take smaller \(B\) so that \(\Omega_{\xi} \geq \frac{1}{2}\Omega_{0}\). Since \(\sigma_{t_{\xi}}(f) = t\sigma_{\xi}(f)\) and
\[
\langle \sigma_{t_{\xi}}^{*}, \sigma_{t_{\xi}}(f), f \rangle_{\xi} = \|\sigma_{t_{\xi}}(f)\|_{\Omega}^2 \geq \frac{1}{2}Ct^2\|f\|^2,
\]
we obtain 
\[
\|\langle \sigma_{t_{\xi}}^{*}, \sigma_{t_{\xi}}(f) \rangle \| \leq C\xi^{-2}.
\]
Replacing \(\Omega\) with \(\Omega_{0}\), we obtain the similar estimate for the adjoint of \(\sigma\) with respect to \(\Omega_{0}\).

Suppose \(b \in B\) is polystable. Then there exists a point \(b' \in bK^{c} \cap \nu_{0}^{-1}(0)\). In regards of the linear symplectic form, \(b'\) is given by minimizing the norm of \(b'\) in the \(K^{c}\)-orbit of \(b\), so \(b'\) is also in \(B\). Since the points in the same \(K^{c}\)-orbit give the isomorphic complex structures, we can assume \(\nu_{0}(b) = 0\). It follows that \(\nu(tb) = O(t^{\delta})\). Then we can take \(t\) small so that \(C\xi^{-2}\|\nu(tb)\| < \delta\). Applying the above lemma, we find a point \(tb' \in B\) in the \(K^{c}\)-orbit of \(tb\) satisfying \(\nu(tb') = 0\). It follows that \((M, \mathfrak{J}(tb)) \equiv (M, \mathfrak{J}(tb'))\) admits Kähler-Ricci soliton. Note the polystability of \(b\) and \(tb\) is equivalent as we consider a linear action.

Conversely, suppose \((M, \mathfrak{J}(b))\) admits Kähler-Ricci soliton. Then similarly we can show that there is a point \(b' \in bK^{c}\) satisfying \(\nu_{0}(b'') = 0\). This shows \(b\) is polystable.

The following corollary shows the good geometric feature of the stack \(\mathcal{K}_{T, X}\) we treat in the next section.

**Corollary 3.10.** Any \(T\)-equivariant small deformation of Fano \(T\)-manifold with Kähler-Ricci soliton is gentle. In particular, for any \(T\)-equivariant family \(\mathcal{M} \rightarrow S\) of complex manifolds, the following subset
\[
S^{\circ} := \{s \in S \mid \mathcal{M}_{s} \text{ is a gentle Fano manifold}\}
\]
is an open subset of \(S\) (with respect to the real topology).

**Proof.** Suppose the Fano manifold \((M, \mathfrak{J}(b))\) does not admit Kähler-Ricci soliton for the point \(b \in B\). From the above proposition, \(b \in B\) is not polystable. Then we can find a polystable point \(b_{0} \in B\) in the closure of the orbit \(bK^{c}\) by minimizing the norm \(\Omega_{0}(-, J_{0}-)\). Since \(K^{c}\) is reductive, we can find a regular morphism \(\lambda : \mathbb{C}^{*} \rightarrow \mathbb{H}^{1}_{T}\) so that \(\lambda(t) \rightarrow b_{0}\). We can extend this to a regular morphism \(\tilde{\lambda} : \mathbb{C} \rightarrow \mathbb{H}^{1}_{T}\).
Pulling back the family \( \varpi : \mathcal{X} \to B \), we obtain a \( T \)-equivariant holomorphic family \( \mathcal{M} \to \Delta \) whose central fiber \((M, \mathfrak{J}(b_0))\) has Kähler-Ricci soliton because there is some \( b'_0 \in b_0 K^\circ \) such that \( \nu(b_0) = \mathcal{S}_t(\mathfrak{J}(b'_0)) = 0 \). So \( \mathcal{M} \to \Delta \) gives a gentle degeneration of \( \mathcal{X}_b \), hence \( \mathcal{X}_b \) is gentle. Since the family \( \varpi : \mathcal{X} \to B \) parametrizes all isomorphism classes of complex structures near \( \mathcal{X}_b \) for any \( b \in B \), we have shown the assertion.

3.3. Completion. Take \( \mathfrak{J} : B \to \mathcal{J}_T(M, \omega) \) as in Proposition [3,7]. The topological space \( \text{Ham}_T(M, \omega)^2_{k+1} \) of \( L^2_{k+1} \)-regular symplectic diffeomorphisms admits a natural Banach smooth manifold structure (compare [IKT, KriMic]). The compositions and the inverses of morphisms in \( \text{Ham}_T(M, \omega)^2_{k+1} \) are again in \( \text{Ham}_T(M, \omega)^2_{k+1} \), however, both the maps

\[
\text{Ham}_T(M, \omega)^2_{k+1} \times \text{Ham}_T(M, \omega)^2_{k+1} \to \text{Ham}_T(M, \omega)^2_{k+1} : (\phi, \psi) \mapsto \phi \circ \psi
\]

\[
\text{Ham}_T(M, \omega)^2_{k+1} \to \text{Ham}_T(M, \omega)^2_{k+1} : \phi \mapsto \phi^{-1}
\]

are not differentiable with respect to the Banach smooth manifold structure, but just continuous (see [IKT]). Therefore we can not treat \( \text{Ham}_T(M, \omega)^2_{k+1} \) as a Banach Lie group.

Nevertheless, we can consider the following \( C^1 \)-smooth map

\[
\mathcal{H} : B \times \text{Ham}_T(M, \omega)^2_{k+1} \to \mathcal{J}_T(M, \omega)^2_k : [b, \phi] \mapsto \phi^* \mathfrak{J}(b).
\]

Actually, since the finite dimensional compact Lie group \( K \) acts freely on \( B \times \text{Ham}_T(M, \omega)^2_{k+1} \), the quotient \( B \times \text{Ham}_T(M, \omega)^2_{k+1} := B \times \text{Ham}_T(M, \omega)^2_{k+1}/K \) can be endowed with the unique Banach smooth manifold structure whose quotient map is a submersion. Then the \( C^1 \)-smoothness of \( \mathcal{H} \) follows from the \( C^\infty \)-smoothness of \( \mathfrak{J} : B \to \mathcal{J}_T(M, \omega)^2_k \) and the \( C^1 \)-smoothness of

\[
a_k^2 : \mathcal{J}_T(M, \omega)^2_k \times \text{Ham}_T(M, \omega)^2_{k+1} \to \mathcal{J}_T(M, \omega)^2_k \times \mathcal{J}_T(M, \omega)^2_k : (J, \phi) \mapsto (J, \phi^* J),
\]

which follows from [IKT].

As Proposition [3,6], the map \( a_k^2 \) is proper for any large \( k \) (\( L^2_k \subset C^2 \) is sufficient). To see this, take a sequence \( (J_n, \phi_n) \in \mathcal{J}_T(M, \omega)^2_k \times \text{Ham}_T(M, \omega)^2_{k+1} \) so that \( g_n, g'_{n, g_n} \) converge to \( g', g'' \) in \( L^2_k \)-topology. Construct \( \phi_{\infty} \) as in the proof of Proposition [3,6]. Again, thanks to Myers-Steenrod theorem, \( \phi_{\infty} \) is \( C^2 \)-smooth and satisfies \( \phi_{\infty, g_n} = g'_{n, \phi_n} \). Then \( \phi_{\infty} \) is a harmonic map between \( (M, g_{\infty}) \) and \( (M, g'_{\infty}) \).

Hence it satisfies the elliptic equation

\[
\Delta g'_{\infty} \phi_{\infty} - \Gamma^\alpha_{\beta \gamma} \frac{\partial \phi_{\infty}^\beta}{\partial x^\beta} \frac{\partial \phi_{\infty}^\gamma}{\partial x^\gamma} g''_{ij} = 0,
\]

where the coefficients of the Laplacian \( \Delta g_{\infty} \) and the Levi-Civita connection \( \Gamma^\alpha_{\beta \gamma} \) are \( L^2_{k-1} \)-regular. It follows that \( \phi_{\infty} \) is \( L^2_{k-1} \)-regular.

Let us see that \( \phi_n \) converges to \( \phi_{\infty} \) in \( L^2_{k+1} \)-topology. Since \( g_n \to g_{\infty} \) and \( g'_n := \phi_n^* g_n \to g'_{\infty} \) in \( L^2_k \)-topology, we have \( \Gamma^\alpha_{\beta \gamma, n} \to \Gamma^\alpha_{\beta \gamma} \) and \( \Delta g'_{n, \phi_n} \to \Delta g'_{\infty} \) in \( L^2_{k-1} \)-topology. Now we use the following uniform elliptic estimates for the elliptic operators \( \Delta g_{n} \) (\( n = 1, 2, \ldots, \infty \)) with \( L^2_{k-1} \)-bounded coefficients and \( 0 \leq \ell \leq k - 1 \).

\[
||u||_{L^2_{k-2}(g_n)} \leq C_{k-1} (||\Delta g'_{n} u||_{L^2_{k}(g_n)} + ||u||_{L^2_{k}(g_n)}),
\]

where \( C_{k-1} \) is independent of \( n = 1, 2, \ldots, \infty \) and \( g_0 \) is a fixed reference smooth metric. (Note \( L^2_{k-1} \subset C^1 \). We used this to the above uniform elliptic estimates.
See for instance the proof of the elliptic estimates in the Appendix of [Kod]. Note also Sobolev multiplication works. First, the $C^1$-convergence of $\phi_n \to \phi_\infty$ follows by the same argument as before. Then we know that $\Delta_g^\infty \phi_\infty = \Gamma_{g,\infty}^0 \partial_i \phi_\infty^\alpha \partial_j \phi_\infty^\beta \partial_k g_\infty^{ij}$ converges to $\Delta_{g'}^\infty \phi_\infty^\alpha = \Gamma^\alpha_{\beta,\gamma,n} \partial_i \phi^{\beta \gamma}_n \partial_j \phi_\infty^\alpha \partial_k g_n^{ij}$ in $L^2$-topology (actually in $C^0$-topology). Combined with the $\phi$-convergence to $\Delta_g^\infty \phi_\infty^\alpha \to \Delta_{g'}^\infty \phi_\infty^\alpha$, we obtain $\|\Delta_{g_n}^\infty (\phi_n^\alpha - \phi_\infty^\alpha)\|_{L^2(g_0)} \to 0$. It follows from the above uniform elliptic estimate that

$$\|\phi_n^\alpha - \phi_\infty^\alpha\|_{L^2(g_0)} \leq C_{k-1} (\|\Delta_{g_n}^\infty (\phi_n^\alpha - \phi_\infty^\alpha)\|_{L^2(g_0)} + \|\phi_n^\alpha - \phi_\infty^\alpha\|_{L^2(g_0)}) \to 0,$$

and we obtain $\phi_n \to \phi_\infty$ in $L^2_{k+1}$-topology. We can repeat this process until we conclude the $L^2_{k+1}$-convergence of $\phi_n \to \phi_\infty$.

Now we can prove the following.

**Proposition 3.11.** The $C^1$-smooth map

$$\mathcal{H} : B \times K \text{Ham}_T(M,\omega)^2_{k+1} \to J_T(M,\omega)_k^2$$

is injective for any sufficiently small neighbourhood $B \subset H^1_0(X, \Theta)$ of the origin.

**Proof.** The derivative of $\mathcal{H}$ at $[0, \text{id}]$ is given by

$$\mathbb{H} \times L^2_{T,k+2}(M)_0/\mathbb{R} \to \Omega^{0,1}(T^{1,0})_{k+1} : (\rho, f) \mapsto d\mathcal{J}_0(\rho) + \bar{\partial}X'_f.$$

It is easy to see that this map is injective and has a closed split range. Then the implicit function theorem shows that $\mathcal{H}$ gives an immersion in a neighbourhood of $[0, \text{id}]$. In particular, $\mathcal{H}$ is locally injective at $[0, \text{id}]$.

Suppose $\mathcal{H}$ is not (globally) injective for any sufficiently small $B$. Then we can take sequences $b_n, b'_n \to 0$ in $B$ and $\phi_n, \phi'_n \in \text{Ham}_T(M,\omega)^2_{k+1}$ satisfying

$$[b_n, \phi_n] \neq [b'_n, \phi'_n] \quad \text{and} \quad \mathcal{H}([b_n, \phi_n]) = \mathcal{H}([b'_n, \phi'_n]).$$

In particular, we have $\mathcal{J}(b_n) = (\phi_n' \circ \phi_n^{-1})^* \mathcal{J}(b'_n)$ and both $\mathcal{J}(b_n), \mathcal{J}(b'_n)$ converge to $\mathcal{J}(0)$ in $J_T(M,\omega)_k^2$. From the properness of $\mathcal{J}_k$, we have a subsequence of $\phi'_n \circ \phi_n^{-1}$ which converges to some $\phi_\infty$ in the stabilizer $K$ of $\mathcal{J}(0)$. Hence, after taking a subsequence, both $[b_n, \text{id}]$ and $[b'_n, \phi'_n \circ \phi_n^{-1}]$ converge to the same $[0, \phi_\infty]$ with the same images $\mathcal{H}([b_n, \text{id}]) = \mathcal{H}([b'_n, \phi'_n \circ \phi_n^{-1}])$. Since $\mathcal{H}$ is injective near $[0, \text{id}]$, we conclude $[b_n, \text{id}] = [b'_n, \phi'_n \circ \phi_n^{-1}]$ for sufficiently large $n$. This contradicts to the choice of the sequences $[b_n, \phi_n] \neq [b'_n, \phi'_n]$ and we have shown that $\mathcal{H}$ is injective for some (hence any) sufficiently small $B$.

The restriction of the map $\mathcal{J} : B \to J_T(M,\omega)_k^2$ gives a continuous map $\nu^{-1}(0) \to (S^2_k)^{-1}(0)^2_k$ and induces another continuous map

$$\nu^{-1}(0)/K \to (S^2_k)^{-1}(0)^2_k/\text{Ham}_T(M,\omega)^2_{k+1}.$$

The following corollaries are essential in the proof of the main theorem.

**Corollary 3.12.** The induced map $\nu^{-1}(0)/K \to (S^2_k)^{-1}(0)^2_k/\text{Ham}_T(M,\omega)^2_{k+1}$ is a homeomorphism onto an open subset.

**Proof.** The injectivity follows from the above Proposition. From the Proposition in section 2 of [Kum], there is a point $b \in B$ such that $(M, \mathcal{J}(b)) \cong (M, J)$ for any integrable $L^2_k$-regular $J$ sufficiently close to $J_0$ in $L^2_k$-topology. Furthermore, since $(M, \mathcal{J}(b')) \cong (M, \mathcal{J}(b))$ for any $b' \in bK^c$ and $bK^c \cap \nu^{-1}(0) \neq \emptyset$ if $J \in (S^2_k)^{-1}(0)_k^2$, we can take such $b$ from $\nu^{-1}(0) \subset B$ for any $J \in (S^2_k)^{-1}(0)_k^2$. Hence the image of $\nu^{-1}(0)/K$ covers the open neighbourhood of $J_0 \in (S^2_k)^{-1}(0)^2_k/\text{Ham}_T(M,\omega)^2_{k+1}$. 
Since \((S_2^b)^{-1}(0)_{k}^2/\text{Ham}_T(M,\omega)_{k+1}^2\) is a Hausdorff space, it follows that the map 
\(\nu^{-1}(0)/K \rightarrow (S_2^b)^{-1}(0)_{k}^2/\text{Ham}_T(M,\omega)_{k+1}^2\) becomes a homeomorphism onto an open subset, by taking smaller \(B\) if necessary.

**Corollary 3.13.** Suppose the torus action on \((M,\omega)\) is K-optimal. The inclusion map \(\nu^{-1}(0) \hookrightarrow BK^c\) induces a homeomorphism \(\nu^{-1}(0)/K \rightarrow BK^c//K^c\).

**Proof.** The analytic GIT quotient \(BK^c//K^c\) is identified as a topological space with the quotient space of \(BK^c\) by the equivalence relation \(b \sim b' \iff \exists \exists b/k^c \neq \emptyset\). Take elements \(b, b' \in \nu^{-1}(0)\) so that \(b \sim b'\). Since both \(b, b'\) are polystable with respect to the \(K^c\)-action, their \(K^c\)-orbits are closed and hence it follows that \(bK^c = b/K^c\). As mentioned before, we obtain \((M,\mathcal{J}(b)) \cong (M,\mathcal{J}(b'))\). Then it follows from Proposition 3.4 that we get \([\mathcal{J}(b)] = [\mathcal{J}(b')] \in (S_2^b)^{-1}(0)_{k}/\text{Ham}_T(M,\omega)_{k+1}^2\). From the above corollary, we obtain \([b] = [b'] \in \nu^{-1}(0)/K\) and we have shown the map \(\nu^{-1}(0)/K \rightarrow BK^c//K^c\) is injective.

We know that \(BK^c//K^c\) consists of closed \(K^c\)-orbits and closed \(K^c\)-orbit has non-empty intersection with \(\nu^{-1}(0)\). This shows the surjectivity of \(\nu^{-1}(0)/K \rightarrow BK^c//K^c\). Since both the spaces are locally compact Hausdorff, by taking smaller \(B\) if necessary, the map becomes a homeomorphism. 

**Corollary 3.14.** For any \(b \in \nu^{-1}(0)\), \(\text{Aut}_T(X_b)\) can be identified with the complexification of the stabilizer group \(K_b\) of the action of \(K\).

**Proof.** From the proof of Theorem 2.6, we know that \(\text{Aut}_T(X_b) \cong \text{Aut}_T(M,\mathcal{J}(t))\) is the complexification of the compact group \(\text{Stab}(\mathcal{J}(b)) \subset \text{Ham}_T(M,\omega)_{k+1}^2\). Since \(\mathcal{J} : B \rightarrow \mathcal{J}_T(M,\omega)_{k+1}^2\) is \(K \subset \text{Ham}_T(M,\omega)_{k+1}^2\)-equivariant, there is an inclusion \(K_b \subset \text{Stab}(\mathcal{J}(b))\). For \(\phi \in \text{Stab}(\mathcal{J}(b))\), we have \(\mathcal{H}([b,\phi]) = \phi*\mathcal{J}(b) = \mathcal{J}(b) = \mathcal{H}([b,\text{id}])\), then the injectivity of \(\mathcal{H}\) shows that \([b,\phi] = [b,\text{id}]\), hence \(\phi \in K_b\).

**PostScript Remark 2.** Corollary 3.14 enables us to prove Corollary 5.12 under the uniqueness statement of (2) in Conjecture 5.11 which we do not follow in this paper. So the last corollary will be never used in any proofs of this paper. Recently, R. Dervan and P. Naumann find an another pure analytic approach to construct the moduli space of cscK manifolds that makes use of this corollary.

In the next section, we will construct complex structures on the following Hausdorff topological spaces.

**Definition 3.15.** We set
\[
K(M,\omega,T)_{k}^2 := (S_2^b)^{-1}(0)_{k}/\text{Ham}_T(M,\omega)_{k+1}^2
\]
and
\[
K_{T,\chi} := \prod_{\chi(M,\omega,T)=\chi} (S_2^b)^{-1}(0)/\text{Ham}_T(M,\omega),
\]
where \((M,\omega,T)\) runs K-optimal symplectic Fano \(T\)-manifolds and \(\chi(M,\omega,T)\) is a Hilbert character of \((M,\omega,T)\) (see the description before Proposition 3.4).

Note that the homeomorphism \(f^* : K(M',\omega',T) \rightarrow K(M,\omega,T)\) induced by a \(T\)-equivariant symplectic diffeomorphism \(f : (M,\omega) \rightarrow (M',\omega')\) is independent of the choice of \(f\), and hence \(K_{T,\chi}\) is free from the choice of the representatives \((M,\omega,T)\) in the symplectic diffeomorphism class \([M,\omega,T]\).
4. Canonical complex structure

4.1. The moduli stack $\mathcal{K}_{T,\chi}$. In this subsection, we introduce the stack $\mathcal{K}_{T,\chi}$ and show its Artin property. The Artin property actually affects our arguments in the next subsection. See Appendix A for generalities on stacks over the category of complex spaces ($\text{Can}$-stacks).

We denote by $\text{Can}$ the category of complex analytic spaces which are not assumed to be reduced nor irreducible. The set of holomorphic morphisms between complex spaces $U$ and $V$ will be denoted by $\text{Holo}(U, V)$. We also denote by $\text{Can}_S$ the category of complex spaces over $S$ and by $\text{Holo}_S(U, V)$ its set of morphisms.

Let $S$ be a complex space. A surjective proper morphism of complex spaces $\pi : \mathcal{M} \to S$ is called a $S$-family of complex manifolds if it is smooth and has connected fibers. Recall that a smooth morphism between singular complex spaces is by definition a morphism of complex spaces $\phi : \mathcal{X} \to \mathcal{Y}$.

There are open coverings $\{V_\alpha \subset Y\}_\alpha$, $\{U_\alpha \subset X\}_\alpha$ of $Y$ and $X$, respectively, an indexed set of smooth complex manifolds $\{W_\alpha\}_\alpha$ and an indexed set of biholomorphisms $\{\phi_\alpha : V_\alpha \times W_\alpha \to U_\alpha\}_\alpha$ satisfying $f \circ \phi_\alpha = p_1$, where the morphism $p_1$ denotes the projection $V_\alpha \times W_\alpha \to V_\alpha$.

Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus. An action of $T$ on an $S$-family $\pi : \mathcal{M} \to S$ is a holomorphic morphism $\alpha : \mathcal{M} \times T \to \mathcal{M}$ which satisfies the following conditions.

1. The morphism $\alpha$ is an $S$-morphism. Namely, $\pi \circ \alpha = \pi \circ p_1 : \mathcal{M} \times T \to S$.
2. $\alpha \circ (\alpha \times \text{id}_T) = \alpha \circ (\text{id}_\mathcal{M} \times \mu) : \mathcal{M} \times T \times T \to \mathcal{M}$, where $\mu : T \times T \to T$ is the multiplication.

An action of $T$ on an $S$-family $\pi : \mathcal{M} \to S$ is called effective if for any $s \in S$ the induced group morphism $T \to \text{Aut}(\mathcal{M}_s)$ is injective. Finally, an $S$-family of complex $T$-manifolds is defined to be an $S$-family of complex manifolds together with an effective $T$-action in the above sense.

Now we introduce the stack $\mathcal{K}_{T,\chi}$. A $\text{Can}$-stack (Definition 6.9) is a category $\mathcal{F}$ together with a functor $\mathcal{F} \to \text{Can}$ satisfying some natural geometric axioms. So we should introduce a category.

**Definition 4.1 (category $\mathcal{K}_{T,\chi}$).** Let $T$ be an algebraic torus and $\chi$ be a Fano character. Object in $\mathcal{K}_{T,\chi}$ is a family of complex $T$-manifolds $\pi : \mathcal{M} \to S$ whose fibers are gentle (see Definition 2.16) Fano $T$-manifolds whose Hilbert characters are $\chi$.

A morphism from $\xi := (\pi : \mathcal{M} \to S, \alpha : \mathcal{M} \times T \to \mathcal{M})$ to $\xi' := (\pi' : \mathcal{M}' \to S', \alpha' : \mathcal{M}' \times T \to \mathcal{M}')$ is a pair $(f, \phi)$ where $f : S \to S'$ is a morphism of complex spaces and $\phi : \mathcal{M} \to \mathcal{M}'$ is a $T$-equivariant morphism which is compatible with $\pi, \pi'$, $f$ and induces a biholomorphism $\tilde{\phi} : \mathcal{M} \to f^* \mathcal{M}'$, where $f^* \mathcal{M}' := S \times_{f, S', \alpha'} \mathcal{M}' \subset S \times \mathcal{M}'$. Here, the morphism $\phi$ is said to be $T$-equivariant if $\alpha' \circ (\phi \times \text{id}_T) = \phi \circ \alpha : \mathcal{M} \times T \to \mathcal{M}'$.

Note that there might be no gentle Fano manifolds whose K-optimal Hilbert characters coincide with a given Fano character $(T, \chi)$, so there might be no object in the category $\mathcal{K}_{T,\chi}$ for a K-optimal Fano character $(T, \chi)$.

We denote by $\mathcal{K}_{T,\chi}^c$ the subcategory of $\mathcal{K}_{T,\chi}$ consisting of families of K-stable Fano $T$-manifolds and by $\mathcal{K}(n)$ the disjoint union of the categories $\mathcal{K}_{T,\chi}$ where $(T, \chi)$ run all the K-optimal Fano characters of $n$-dimensional Fano manifolds.

We denote by $\mathcal{K}_{T,\chi}(S)$ the subcategory consisting of objects $(\pi : \mathcal{M} \to S, \alpha)$ with fixed base $S$ and whose morphisms are pairs $(\text{id}_S, \phi)$. For any two objects
ξ = (M → S, α), η = (N → S, β) ∈ 𝒦_{T,X}(S), we define the contravariant functor 
 Iso_{S}(ξ, η) from Can_{S} to Sets by mapping an object f : U → S to the set 
 Hom_{K_{U}}(f^{*}ξ, f^{*}η) and a morphism h : (U, f) → (V, g) to the map given by the 
 canonical identifications f^{*}ξ ≅ h^{*}g^{*}ξ, f^{*}η ≅ h^{*}g^{*}η.

The categories K_{T,X} and K_{T,X}^{*} naturally form Can-stacks. See Lemma 6.7 and 
6.13 in Appendix A.

Definition 4.2 (Artin Can-stack). A Can-stack F is called an Artin stack if it 
satisfies the following two conditions.

(1) ∆ : F → F × F is representable by complex spaces.

(2) There exists a smooth surjective morphism U → F from a complex space 
U.

Or equivalently,

(1) For every complex space S and any ξ, η ∈ Obj(F), there exists a complex 
space S_{ξ,η} and an isomorphism Holo_{S}(−, S_{ξ,η}) ≅ Iso_{S}(ξ, η) of contravariant 
functors from Can_{S} to Sets.

(2) There exists a morphism U → F of fibred categories from a complex space 
U such that the 2-fibre product U ×_{F} V is isomorphic as stacks to a complex 
space f : V_{U} → V smooth over V with surjective f, for any morphism 
V → F from any complex space V.

Note that the 2-fibre product U ×_{F} V is always isomorphic to some complex 
space W over V from the first condition (cf. [SPA, Tag 045G]).

We frequently use the following representability result in the rest of this paper. 
This is an almost known result and might be proved somewhere in literatures, but 
as the author could not find a reference for the result, we give a proof.

Proposition 4.3 (invariant Hilbert scheme). Suppose f : X → B is a holomorphic 
morphism of complex spaces and α : X × T → X is a holomorphic action with 
f ◦ α = f ◦ p_{1}. Consider the functor Hilb_{T,f} : Can_{B} → Sets given by 

Hilb_{T,f}(S) := \{ Z ⊂ S ×_{B} X \mid Z is a T-invariant closed analytic subspace
and Z → S is a flat family \}.

Then there exists a Hausdorff complex space Hilb_{T,f} representing the functor 
Hilb_{T,f}. Moreover, suppose S = pt and X is projective with an ample line bundle L, then the subfunctor Hilb_{T,X,X} ⊂ Hilb_{T,X} consisting of families Z → S
with a fixed Hilbert polynomial X is representable by a projective complex space 
Hilb_{T,X,X} ⊂ CP^{N}.

Proof. When T is trivial, the existence of the Hausdorff complex space Hilb_{T,f} 
follows from [Dou2] for S = pt case and from [Dou] for general case. The projectivity 
follows from Grothendieck’s existence theorem of the Hilbert scheme, which 
represents an analogical functor defined on the category of schemes Sch_{C}, and 
the coincidence of the functors when they restricted to the subcategory Def_{C} of 
the spectrum of finitely generated Artin algebras over C, which is naturally em-
bbeded into both Can and Sch_{C}. Actually, a morphism f : X → Y between 
complex spaces is an isomorphism if and only if it induces an isomorphism of func-
tors h_{X}|_{Def_{C}} → h_{Y}|_{Def_{C}}. (Note that the local ring O_{X,x} of the structure sheaf O_{X} 
of a complex space X at a point x ∈ X is Noether.)

When T is non-trivial, we can consider the action of T on the set Hilb_{f}(S) for 
each S ∈ Can_{B}, whose fixed point subset is nothing but the subset Hilb_{T,f}(S) ⊂
Hilb_f(S). Then the existence in the category of complex spaces follows from the following two general statements.

(1) Suppose \( H \) is a (not necessarily reduced) complex space with \( T \)-action and \( x \in H \) is a \( T \)-fixed point. Then there is a \( T_R \)-invariant open neighbourhood \( U \subset H \) of \( x \) and a \( T_R \)-equivariant closed embedding \( \varphi : U \to V \) into an open neighbourhood \( V \subset T_x H \) of the origin, where \( T_R \) denotes the maximal compact subgroup of \( T \) and \( T_x H \) denotes the Zariski tangent space (cf. [Akh subsection 2.2]).

(2) Let \( W \subset T_x H \) be the set of \( T \)-invariant points, which forms a \( T \)-invariant linear subspace. Then the complex space \( U_T := U \times_V (W \cap V) \subset U \), considered as a closed subspace of \( U \), enjoys the following universal property: for any holomorphic morphism \( f : S \to H \) invariant under the \( T \)-action on \( H \), the restricted holomorphic morphism \( f|_{f^{-1}(U)} : f^{-1}(U) \to H \) holomorphically and uniquely factors through \( U_T \).

On the other hand, the existence in the category of schemes follows from [Fog]. The rest of the proof is parallel to the first paragraph. \( \square \)

The proof of the next proposition must be a routine work for the readers familiar with Artin stacks.

**Proposition 4.4.** The \( \text{Can} \)-stacks \( \mathcal{K}_{T,\chi} \) is Artin \( \text{Can} \)-stack. If \((T,\chi)\) is K-optimal, then \( \mathcal{K}_{T,\chi} \) is also Artin and is an open substack of \( \mathcal{K}_{T,\chi} \).

**Proof.** By considering the graphs \( M \times_{\phi,N,\text{Id}} N \subset M \times N \) of morphisms \( \phi : M \to N \), the functor \( \text{Isom}_S(\xi,\eta) \) is identified with a subfunctor of \( \text{Hilb}_{T,M \times N/S} \). Then it is easy to see that \( \text{Isom}_S(\xi,\eta) \) is representable by an open subspace of \( \text{Hilb}_{T,M \times N/S} \) (cf. [FGIKNV 5.6.2]).

Next we construct a smooth surjective morphism \( U \to \mathcal{K}_{T,\chi} \). Let us consider a uniform \( T \)-equivariant embedding of Fano manifolds in \( \mathcal{K}_{T,\chi} \). For a sufficiently large \( m \), the direct image sheaf \( \pi_* \langle O(-mK_M) \rangle|_{U_m} \) becomes locally free. Take a covering \( U = \{U_m\}_m \) of \( X \) that trivializes the vector bundle \( \pi_* \langle O(-mK_M) \rangle|_{U_m} \) so that we can consider morphisms \( U_m \to U \) corresponding to trivializations of \( \pi_* \langle O(-mK_M) \rangle|_{U_m} \). There is a unique \( PGL_T \)-equivariant extension \( U_m \times PGL_T \to U \) of these morphisms. Then from the universality of the 2-fibre product \( U \times_{\mathcal{K}_{T,\chi}} U_m \), we get morphisms \( U_m \times PGL_T \to U \times_{\mathcal{K}_{T,\chi}} U_m \). We have the inverse morphisms of these morphisms given as follows. Take an object \((S,\xi : S \to U,\eta : S \to U_m,\phi : \xi^*U \cong \eta^*M|_{U_m})\) of \( U \times_{\mathcal{K}_{T,\chi}} U_m \). Since \( \eta^*M|_{U_m} \) can be considered as being embedded in \( S \times \mathbb{C}P^N \), the isomorphism \( \phi \) corresponds to a morphism \( \tilde{\phi} : S \to PGL_T \). Then we have a morphism \( \eta \times \tilde{\phi} : S \to U_m \times PGL_T \), which gives an object in \( \text{Can}_{U_m \times PGL_T} \). Therefore \( U \times_{\mathcal{K}_{T,\chi}} X \to X \) is locally written as \( U_m \times PGL_T \to U_m \). So it is a smooth morphism.
It is shown in [4] Theorem 3.4 that K-stable Fano manifolds form an open subset in the parameter space of any family of complex manifolds, without introducing the K-stability of Fano T-manifolds. From the exactly same argument as above, we conclude that $K_{T,X}^c$ is Artin and is an open substack of $K_{T,X}^c$.

Note that so far we cannot strengthen the openness of the subset consisting of gentle Fano manifolds in the parameter space of a family to the Zariski openness. This is the reason why we should work in the category of complex spaces rather than the category of algebraic spaces, where the Alper’s theory on good moduli spaces is useful.

4.2. Main construction. In this subsection, we prove our main theorem. First we prepare two general lemmas.

**Lemma 4.5.** Let $K$ be a compact Lie group and $K^c$ be its complexification, $V$ be a representation of $K^c$ and $B \subset V$ be a $K$-invariant Stein open neighbourhood of the origin. Let $s \times t : R \to B \times B$ be a holomorphic groupoid obtained by pulling back the holomorphic action groupoid $a : V \times K^c \to V \times V : (v,k) \mapsto (v,vk)$ along the inclusion $B \times B \subset V \times V$. Then the following holds.

1. The $\text{Can}$-stack $[B/R]$ is naturally isomorphic to $[BK^c/K^c]$ as $\text{Can}$-stacks, where $BK^c$ denotes the $K^c$-invariant open subset $\{bg \in V \mid b \in B, g \in K^c\}$.

2. There is a morphism $[BK^c/K^c] \to BK^c \sslash K^c$ to the analytic GIT quotient $BK^c \sslash K^c$ which enjoys the following universal property: any morphism from the quotient stack $[BK^c/K^c]$ to any complex space $X$ uniquely factors through $BK^c \sslash K^c$.

**Proof.** We identify $[BK^c/K^c]$ with the $\text{Can}$-stack $[BK^c/K^c]$ in Example 6.3. Consider a morphism $\sigma$ from the fibred category $[B/R]_p$ to $[BK^c/K^c]$ sending an object $\xi : S \to X$ in $[B/R]_p$ to the object $(S,S \times K^c, a \circ (\xi \times \text{id}))$ in $[BK^c/K^c]$ (cf. the description right after Example 6.3). Let $S$ be a complex space and $(S,P,\xi')$ be an object in $[BK^c/K^c](S)$. Take a local trivialization $\{P \cong U_\alpha \times K^c\}_\alpha$ of the principal $K^c$-bundle $P$ and consider the associated $K^c$-equivariant morphisms $\xi'_\alpha : U_\alpha \times K^c \to BK^c$. After taking smaller $U_\alpha$, we can find a holomorphic morphism $\xi_\alpha : U_\alpha \to B$ and a holomorphic morphism $g : U_\alpha \to K^c$ so that $\xi_\alpha(x)g(x) = \xi'_\alpha(x,c)$. It follows that the object $(U_\alpha, U_\alpha \times K^c, \xi'_\alpha)$ in $[BK^c/K^c]$ is isomorphic to $\sigma(\xi_\alpha) = (U_\alpha, U_\alpha \times K^c, a \circ (\xi_\alpha g) \times \text{id}_{K^c})$. Moreover, it is easily seen that $\text{Isom}_{[B/R]_p,S}(\xi,\eta) \to \text{Isom}_{[BK^c/K^c],S}(\sigma(\xi),\sigma(\eta))$ is a sheafification of the functor $\text{Isom}_{[B/R]_p,S}(\xi,\eta) : \text{Can}_S \to \text{Sets}$. It follows that $[BK^c/K^c]$ is a stackification of the fibred category $[B/R]_p$. Therefore, it is isomorphic to the stackification $[B/R]$ of $[B/R]_p$.

Since $B$ is a reduced Stein space, $BK^c$ is also a reduced Stein space and there exists a categorical quotient $BK^c \sslash K^c$, which is also a reduced Stein space (see [Hei], [Snow]). Take an object $(S,P,\xi)$ in $[BK^c/K^c]$ and a local trivialization $\{P \cong U_\alpha \times K^c\}_\alpha$ of $P$. Then we have holomorphic morphisms $\xi_\alpha : U_\alpha \to U_\alpha \times K^c \to BK^c \to BK^c \sslash K^c$. Since $\xi_\alpha : U_\alpha \times K^c \to BK^c$ agree on the overlaps $U_\alpha \cap U_\beta$ up to the action of $K^c$, and $BK^c \sslash K^c$ is $K^c$-invariant, holomorphic morphisms $\xi_\alpha$ coincide on the overlaps $U_\alpha \cap U_\beta$ and define a holomorphic morphism $S \to BK^c \sslash K^c$, glued together. This construction gives the morphism $[BK^c/K^c] \to BK^c \sslash K^c$. The universal property follows from the fact that any $K^c$-invariant holomorphic morphism $BK^c \to X$ uniquely factors through $BK^c \sslash K^c$. □
The complex space $BK^c//K^c$ is moreover normal as it is an open subspace of the algebraic GIT quotient $V//K^c$, which is normal whenever $V$ is normal ([MFK]).

**Lemma 4.6.** Let $K$ be a compact Lie group, $B$ be a complex manifold with holomorphic $K$-action and $E \to B$ be a $K$-equivariant holomorphic vector bundle. Suppose $0 \in B$ is a fixed point of $K$-action. Since the fiber $E_0$ can be considered as $K$-representation, we can construct a $K$-equivariant holomorphic vector bundle $E_{0,B,K} := B \times E_0$ whose action is given by $(b, v)k := (bk, vk)$. Then $E$ is $K$-equivariantly isomorphic to $E_{0,B,K}$ on some neighbourhood of $0 \in B$.

**Proof.** Consider the frame bundle $S \to B$, $K$-equivariant holomorphic vector bundle $E$ and fix a point $p_0 \in \pi^{-1}(0) \subset P$. We have a right holomorphic action of $K$ on $P$ defined by

$$ pk : \mathbb{C}^r \xrightarrow{p_0} E_0 \xrightarrow{p_0^{-1}} \mathbb{C}^r \xrightarrow{p} E_b \xrightarrow{k} E_{bk} $$

for $p : \mathbb{C}^r \sim \to E_b \in P$ and $k \in K$. The point $p_0 \in P$ is a fixed point of this action and $\pi : P \to B$ is a $K$-equivariant submersion. So we have a $K$-equivariant holomorphic section $\sigma : B \to P$ with $\sigma(0) = p_0$ by taking smaller $B$ if necessary. Now the map $B \times GL(r) \to P : (b, g) \mapsto \sigma(b)g$ gives a $K$-equivariant isomorphism of principal $GL(r)$-bundles and hence induces a $K$-equivariant isomorphism of the adjoint bundles $\mathcal{C}^r_{B,K} \xrightarrow{p_0} E_{0,B,K}$. \hfill \Box

**Proposition 4.7.** Let $X$ be a Fano $T$-manifold with Kähler-Ricci soliton and the Hilbert character $(T, \chi)$. Then by taking smaller $B$ if necessary, we obtain a natural étale morphism $[B/R] \to K_{T,X}$ with finite fibres, which means that for any morphism $S \to K_{T,X}$ there is an étale morphism $S' \to S$ with finite fibres and an $S$-isomorphism from $S'$ to the 2-base change $S \times_{K_{T,X}} [B/R] \to S$. This morphism is canonically determined by the semi-universal family $\varpi : \mathcal{X} \to B$ in Proposition 3.7.

**Proof.** The family $\varpi : \mathcal{X} \to B$ in Proposition 3.7 defines a morphism $B \to K_{T,X}$. Now we will show that this morphism factors through the quotient morphism $B \to [B/R]$. It is equivalent to the existence of a natural $T$-equivariant $R$-biholomorphism $s^*\mathcal{X} \sim \to t^*\mathcal{X}$. We prove this by relating our analytic family to an algebraic family as groupoids. As a consequence, the induced morphism $[B/R] \to K_{T,X}$ is shown to be étale with finite fibres.

Since $\varpi : \mathcal{X} \to B$ is a $K$-equivariant family and $O(-K_{\mathcal{X}}/B)$ is relatively ample, we can find a large $\ell \in \mathbb{N}$ so that the direct image sheaf $\varpi_*O(-\ell K_{\mathcal{X}}/B)$ is $K$-equivariantly isomorphic to the sheaf of sections of a $K$-equivariant holomorphic vector bundle $E$. Lemma 4.6 shows that there is a $K$-equivariant isomorphism $H^0(X, \mathcal{O}(\ell K_{\mathcal{X}}))_{B,K} \cong E$, so we have a $K$-equivariant $B$-embedding $\mathcal{X} \hookrightarrow B \times \mathbb{P}^N$, where we identify $\mathbb{P}^N$ with $\mathbb{P}(H^0(X, \mathcal{O}(\ell K_{\mathcal{X}})))$. From the universality of $\text{Hilb}_{T,\mathbb{P}^N}$, we obtain a $K$-equivariant holomorphic morphism $h : B \to \text{Hilb}_{T,\mathbb{P}^N}$ together with an isomorphism $h^*\mathcal{U} \cong \mathcal{X}$.

From the Euler sequence

$$ 0 \to O_{\mathbb{P}^N} \to O(1)^{\otimes N+1} \to \Theta_{\mathbb{P}^N} \to 0, $$

...
we obtain $H^1_T(X, i^*\Theta_{\mathbb{P}^N}) = 0$ and $H^0_T(X, i^*\Theta_{\mathbb{P}^N}) \cong H^0_T(\mathbb{P}^N, \Theta_{\mathbb{P}^N})$. Combining this with the following exact sequence
\[
0 \to H^0_T(X, \Theta_X) \to H^0_T(X, i^*\Theta_{\mathbb{P}^N}) \to H^0_T(X, N_{X/\mathbb{P}^N}) \\
\to H^1_T(X, \Theta_X) \to H^1_T(X, i^*\Theta_{\mathbb{P}^N}) \to H^1_T(X, N_{X/\mathbb{P}^N}) \to 0
\]
shows that the sequence
\[
0 \to H^0_T(X, \Theta_X) \to H^0_T(\mathbb{P}^N, \Theta_{\mathbb{P}^N}) \to H^0_T(X, N_{X/\mathbb{P}^N}) \to H^1_T(X, \Theta_X) \to 0
\]
is exact and $H^1_T(X, N_{X/\mathbb{P}^N})$ vanishes. So we conclude that $\text{Hilb}_{T, \mathbb{P}^N}$ is smooth at $[X] = h(0) \in \text{Hilb}_{T, \mathbb{P}^N}$, whose tangent space is given by $H^0_T(X, N_{X/\mathbb{P}^N})$ (cf. [FGIKNV, subsection 6.4]).

Now we work in the category of algebraic spaces in the blink of an eye. Since $\text{Aut}_T(X)$ is reductive, we can apply the étale slice theorem [AHR, Theorem 2.1], which generalizes the Luna’s étale slice theorem to non-affine cases, and then obtain the following: a smooth affine $\text{Aut}_T(X)$-variety $(W, w)$, an $\text{Aut}_T(X)$-equivariant morphism $\phi : (W, w) \to (\text{Hilb}_{T, \mathbb{P}^N}, h(0))$ which induces a $\text{PGL}_T(N + 1)$-equivariant étale morphism $W \times^{\text{Aut}_T(X)} \text{PGL}_T \to \text{Hilb}_T$, and a $\text{Aut}_T(X)$-equivariant étale morphism $(W, w) \to (H^1_T(X, \Theta), 0)$.

Note that the quotient morphism $W \times \text{PGL}_T \to W \times^{K^c} \text{PGL}_T$ is a $K^c$-equivariant submersion, under the right action of $K^c = \text{Aut}_T(X)$ on $W \times \text{PGL}_T$ defined by $(x, g_0)g_1 = (xg_0, g_1^{-1}g_0g_1)$ and on $W \times^{K^c} \text{PGL}_T$ defined by $[x, g_0]g_1 = [x, g_0g_1]$. Since the point $(w, e) \in W \times \text{PGL}_T$ is fixed by this $K^c$-action, we obtain a $K^c$-equivariant holomorphic section $\sigma$ from a neighbourhood of $[w, e] \in W \times^{K^c} \text{PGL}_T$ with $\sigma([w, e]) = (w, e)$. Therefore, taking smaller $B$ if necessary, we can assume that $h : B \to \text{Hilb}_T$ factors through $W \times \text{PGL}_T \to \text{Hilb}_T$. We can moreover assume that the composed morphism $(B, 0) \to (W, w)$ of a lifting $B \to W \times \text{PGL}_T$ passing through $(w, e)$ with the projection to the first factor is $K$-equivariant holomorphic open embedding. Note that we do not know whether this morphism $(B, 0) \to (W, w)$ is a section of the étale morphism $(W, w) \to (H^1_T(X, \Theta_X), 0)$.

Set $\text{Hilb}^\circ_T := h(B) \cdot \text{PGL}_T \subset \text{Hilb}_T$. Since $\varpi : X \to B$ is a complete family at any point $b \in B$, $\text{Hilb}^\circ_T$ is an open subset. The restricted $\text{PGL}_T$-equivariant universal family $U^\circ \to \text{Hilb}^\circ_T$ parametrizes only gentle Fano $T$-manifolds and hence induces an open embedding $[\text{Hilb}^\circ_T / \text{PGL}_T] \to \mathcal{K}_{T,X}$. We fix this subset $\text{Hilb}^\circ$ while we later take smaller $B$.

It follows from [Snow, Proposition 5.1] that we have a $K^c$-invariant open neighbourhood $W^\circ \subset \phi^{-1}(\text{Hilb}^\circ_T)$ of $w$ so that the restriction $W^\circ \to H^1_T(X, \Theta)$ is a $K^c$-invariant open embedding. Taking smaller $B$, we have the restricted morphism $B \to W^\circ \times \text{PGL}_T$. Let $g : B \to \text{PGL}_T$ be the composition of this morphism with the projection to the second factor. Denote by $h_\circ : B \to \text{Hilb}^\circ$ the composition $\alpha_{\text{Hilb}^\circ} \circ (h \times g^{-1}) : B \to \text{Hilb}^\circ_T \times \text{PGL}_T \to \text{Hilb}^\circ_T$. Then the holomorphic morphism $h_\circ$ is $K$-equivariant and factors through the $K^c$-equivariant holomorphic morphism $W^\circ \to \text{Hilb}^\circ_T$. Moreover, we have an induced isomorphism $h_\circ^* U \cong X$. 

Since the differential of the induced morphism $B \to W^c$ at $0 \in B$ is a $K$-equivariant isomorphism, we can assume that $B \to W^c$ is a $K$-equivariant open embedding. Let us denote by $\beta : B \to H^1_B(X, \Theta)$ the composition of this morphism $B \to W^c$ with $W^c \to H^1_B(X, \Theta)$. Then $\beta$ is also a $K$-equivariant open embedding.

Note that both $BK^c \subset H^1_B(X, \Theta)$ and $\beta(B)K^c \subset H^1_B(X, \Theta)$ are the complexification, in the sense of [Hei], of $B$ with respect to the action of $K$. From the uniqueness of the complexification, there is a $K^c$-equivariant biholomorphism $\gamma : BK^c \to \beta(B)K^c$ which is compatible with the $K$-equivariant morphisms $\subset B \to BK^c$ and $\beta : B \to \beta(B)K^c$.

Now we have the following cartesian diagrams

\[
\begin{array}{ccc}
R & \xrightarrow{\times} & BK^c \times K^c \\
\downarrow s \times t & & \downarrow \alpha_{BK^c} \\
B \times B & \xrightarrow{\alpha \circ \gamma} & H^1_T \times H^1_T
\end{array}
\]

\[
\begin{array}{ccc}
R_W & \xrightarrow{\beta(B)K^c \times K^c} & (\gamma \circ \id)^{-1}(H^1_T \times H^1_T) \\
\downarrow s_W \times t_W & & \downarrow \alpha_{\beta(B)K^c} \\
B \times B & \xrightarrow{\beta(B)K^c \times \beta(B)K^c} & H^1_T \times H^1_T
\end{array}
\]

Since $\gamma : BK^c \to \beta(B)K^c$ is $K^c$-equivariant, it satisfies $(\gamma \circ \alpha_{BK^c}) = \alpha_{\beta(B)K^c} \circ (\gamma \circ \id)^{-1}$ and hence gives an isomorphism of the groupoids $(\rho_1 \circ \alpha_{BK^c}, \rho_2 \circ \alpha_{BK^c}) : BK^c \times K^c \to BK^c$ and $(\alpha_1 \circ \alpha_{\beta(B)K^c}, \alpha_2 \circ \alpha_{\beta(B)K^c}) : \beta(B)K^c \times K^c \to \beta(B)K^c$. It follows that there is an isomorphism $(\rho, \id_B) : (R, B) \xrightarrow{\sim} (R_W, B)$ of the groupoids $s \times t : R \to B \times B$ and $s_W \times t_W : R_W \to B \times B$. Hence there is an isomorphism $[B/R] \cong [B/R_W]$ of the quotient $\can$-stacks.

On the other hand, since $\beta : B \to H^1_B$ factors through the $K^c$-equivariant open embedding $W^c \to H^1_T$, the groupoids $s_W \times t_W : R_W \to B \times B$ also appears in the following cartesian diagram.

\[
\begin{array}{ccc}
R_W & \xrightarrow{W^c \times K^c} & W^c \times W^c \\
\downarrow s_W \times t_W & & \downarrow \alpha_{W^c} \\
B \times B & \xrightarrow{\beta \circ \id} & W^c \times W^c
\end{array}
\]

Therefore we obtain an open embedding of the quotient $\can$-stacks $[B/R_W] \hookrightarrow [W^c/K^c]$.

Moreover, the étale finite morphism $W^c \times K^c PGL_T \to \hilb$ induces an étale finite morphism of the quotient $\can$-stacks $[W^c/K^c] \cong [W^c \times K^c PGL_T/PGL_T]$.

Now combining all, we obtain an étale morphism $[B/R] \to \mathcal{K}_{T, \chi}$ with finite fibers, which obviously commutes with $B \to \mathcal{K}_{T, \chi}$ and $B \to [B/R]$ from its construction.

Here is our main theorem.

**Theorem 4.8.** There exists a Hausdorff complex analytic space $\mathcal{K}_{T, \chi}$, which we call the moduli space of Fano manifolds with Kähler-Ricci solitons, and a morphism $\mathcal{K}_{T, \chi} \to \mathcal{K}_{T, \chi}$ from the Artin $\can$-stack $\mathcal{K}_{T, \chi}$ such that any morphism from
\( \mathcal{K}_{T,X} \) to any complex space \( X \) holomorphically and uniquely factors through \( \mathcal{K}_{T,X} \). Moreover, this moduli space enjoys the following property.

1. The complex space \( \mathcal{K}_{T,X} \) is normal and homeomorphic to the space \( K_{T,X} \) in Definition 3.15 (see also Proposition 4.11).

2. The morphism \( \mathcal{K}_{T,X} \to \mathcal{K}_{T,X} \) induces a bijection \( |\mathcal{K}_{T,X}|/\sim \to |\mathcal{K}_{T,X}| \) where \( |\mathcal{K}_{T,X}| \) denotes the set of points of the stack \( \mathcal{K}_{T,X} \), which is canonically identified with the set of the isomorphism classes of gentle Fano manifolds, and \( [X] \sim [X'] \) if the central fibers of the gentle degenerations of gentle Fano manifolds \( X \) and \( X' \) coincide.

The following proof relies on Proposition 2.17 which we will prove in subsection 4.4, making use of the above proposition.

**Proof.** The image of the étale morphism \( [B/R] \to \mathcal{K}_{T,X} \) defines an open substack \( \text{Im}[B/R] \subset \mathcal{K}_{T,X} \). Object in \( \text{Im}[B/R] \) is an object \( (\pi : \mathcal{M} \to S, \alpha) \) in \( \mathcal{K}_{T,X} \) whose fibers are gentle Fano \( T \)-manifolds appearing in the Kuranishi family \( \mathcal{X} : T \to B \).

First we prove that the morphism \( [B/R] \to BK^c//K^c \) in Lemma 4.5 factors through \( \text{Im}[B/R] \). We construct a morphism \( \text{Im}[B/R] \to BK^c//K^c \). Take an object \( (\pi : \mathcal{M} \to S, \alpha) \) in \( \text{Im}[B/R] \) and consider the following cartesian diagram.

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\phi} & [B/R] \\
\downarrow & & \downarrow \\
S & \xrightarrow{\alpha} & \text{Im}[B/R]
\end{array}
\]

Since \( [B/R] \to \text{Im}[B/R] \) is étale, \( \tilde{S} \to S \) is also étale. Then we can take local slices \( s_{\alpha} : U_{\alpha} \to \tilde{S} \) of \( \tilde{S} \to S \) so that \( \{U_{\alpha}\}_{\alpha} \) covers \( S \) and obtain morphisms \( U_{\alpha} \to [B/R] \), hence also obtain holomorphic morphisms \( \phi_{\alpha} : U_{\alpha} \to BK^c//K^c \). From its construction, we know that \( x \in U_{\alpha} \) maps to the point \( \phi_{\alpha}(x) \in BK^c//K^c = \nu^{-1}(0)/K \) representing the central fiber of some gentle degeneration of \( \mathcal{M}_x \), which is unique due to Proposition 2.17. So if \( S \) is reduced, these morphisms \( \phi_{\alpha} : U_{\alpha} \to BK^c//K^c \) coincide on the overlaps, hence they give a holomorphic morphism \( \phi : S \to BK^c//K^c \), glued together. When \( S \) is not reduced, since any Fano \( T \)-manifold has reduced semi-universal family, we can locally extend the morphism \( S \to \text{Im}[B/R] \) to some \( T \to \text{Im}[B/R] \) with reduced \( T \). Take a point \( x \in U_{\alpha} \cap U_{\beta} \) and a small neighbourhood \( U \) of \( x \) so that \( U \to \text{Im}[B/R] \) extends to \( T \to \text{Im}[B/R] \) with reduced \( T \). Taking smaller \( T \), sections \( s_{\alpha}|_U, s_{\beta}|_U \) extend to some sections \( t_{\alpha}, t_{\beta} : T \to \tilde{T} \). Therefore the morphisms \( \phi_{\alpha}|_U, \phi_{\beta}|_U : U \to BK^c//K^c \) extend to some morphisms \( \psi_{\alpha}, \psi_{\beta} : T \to BK^c//K^c \). As we have already observed that \( \psi_{\alpha} \) and \( \psi_{\beta} \) coincide, \( \phi_{\alpha}|_U, \phi_{\beta}|_U : U \to BK^c//K^c \) also coincide as holomorphic morphisms. Therefore we obtain a morphism \( \phi : S \to BK^c//K^c \) by gluing the morphisms \( \phi_{\alpha} : U_{\alpha} \to BK^c//K^c \). It is easy to see that this construction is functorial, so we obtain the expected morphism \( \text{Im}[B/R] \to BK^c//K^c \), which inherits the universal property from the morphism \( [B/R] \to BK^c//K^c \).

Now consider two morphisms \( [B/R] \to \mathcal{K}_{T,X} \) and \( [B'/R'] \to \mathcal{K}_{T,X} \) with different domains. We also consider two maps \( i : BK^c//K^c \to K(M, \omega, T)^c_k \) and \( i' : BK''^c//K''^c \to K''(M, \omega, T)^c_k \). For any point \( x \in K(M, \omega, T)^c_k \) in the overlaps \( \text{Im}i \cap \text{Im}i' \), we can find another étale morphism \( [B''/R''] \to \mathcal{K}_{T,X} \) and a map \( i'' : BK''^c//K''^c \to K''(M, \omega, T)^c_k \) with \( i''([0]) = x \) so that \( \text{Im}[B''/R''] \subset \text{Im}[B/R] \cap \text{Im}[B'/R'] \subset \mathcal{K}_{T,X} \) and \( \text{Im}i'' \subset \text{Im}i \cap \text{Im}i' \). Especially we have a natural inclusion morphism
The moduli space of Fano manifolds with Kähler-Ricci solitons

Im[B''/R''] \to \operatorname{Im}[B'/R'] and hence obtain a morphism \( \operatorname{Im}[B''/R''] \to BK^c / \! / K^c \).

From the universality of the morphism \( \operatorname{Im}[B''/R''] \to B''K'' / K''^c \), we obtain a holomorphic morphism \( B''K'' / K''^c \to BK^c / K^c \). This holomorphic morphism is clearly compatible with \( i'' \) and \( i \) as maps, so especially it is a homeomorphism onto its open image, after taking smaller \( B'' \) if necessary. Since the analytic GIT quotient spaces \( BK^c / K^c \) are normal, this holomorphic homeomorphism is actually a biholomorphism. This argument shows that the coordinate change \( i''^{-1} \circ i \) is biholomorphic. Thus we obtain a complex space \( K(M, \omega, T) \) by giving a complex structure on the topological space \( K(M, \omega, T) \mathbb{D}^2 \) defined from the above holomorphic charts. Set \( K_{T, \chi} := \bigsqcup_{\chi(M, \omega, T) = \chi} K(M, \omega, T) \). Clearly from its construction, there is a morphism \( K_{T, \chi} \to K_{T, \chi} \) enjoying the universal property. It follows from section 3 and Proposition 2.17 that this morphism induces a bijection \( |K_{T, \chi}| \sim \to K_{T, \chi} \).

We prove in the next subsection that the space \( K(M, \omega, T) \), which is homeomorphic to \( K(M, \omega, T) \mathbb{D}^2 \), is actually homeomorphic to \( K(M, \omega, T) \).

\[ \square \]

**Corollary 4.9.** The \textbf{Can}-stack \( K_{T, \chi}^c \) admits a tame moduli space \( K_{T, \chi}^c \to K_{T, \chi}^s \) (see [Alp1, Definition 7.1]), with the same universal property as the moduli space \( K_{T, \chi} \to K_{T, \chi} \). Moreover, the complex space \( K_{T, \chi}^s \) is a Hausdorff complex orbifold.

This corollary follows from the construction in the proof of the main theorem, the openness property of the K-stable Fano \( T \)-manifolds and the fact that \( [B'/R'] \to K_{T, \chi}^c \) is an open embedding in this case, which is an easy consequence of the injectivity of the map \( \|B'/R'\| \to BK^c / K^c \approx \nu^{-1}(0)/K \to K_{T, \chi} \) and the bijection \( |K_{T, \chi}| \to K_{T, \chi} \). The orbifold coordinates are given by open neighbourhoods of the origin in the spaces \( H^1_2(X, \Theta) \! / \! / (\operatorname{Aut}_T(X))/T \). We can also consider a separated smooth Deligne-Mumford \textbf{Can}-stack \( K_{T, \chi}^c \) associated to the \textbf{Can}-stack \( K_{T, \chi}^c \).

**4.3. Consistency.** In the previous section, we constructed a complex analytic space structure on the spaces \( K(M, \omega, T) \mathbb{D}^2_k = (S^c_k)^{-1}(0)_{\mathbb{D}^2_k}/\operatorname{Ham}_T(M, \omega)^2_{k+1} \) and proved that it has a certain universality independent of \( k \), which is described in terms of the stack \( K_{T, \chi} \). Since the universality determines a complex space uniquely up to biholomorphisms, the complex spaces \( (S^c_k)^{-1}(0)_{\mathbb{D}^2_k}/\operatorname{Ham}_T(M, \omega)^2_{k+1} \) are all canonically biholomorphic to each other. In particular, we deduce that they are all homeomorphic through the following natural maps

\[ \mathcal{I}_{l,k} : (S^c_k)^{-1}(0)_{\mathbb{D}^2_k}/\operatorname{Ham}_T(M, \omega)^2_{k+1} \to (S^c_l)^{-1}(0)_{\mathbb{D}^2_l}/\operatorname{Ham}_T(M, \omega)^2_{l+1} : [J] \mapsto [J]. \]

Now we show the continuous map

\[ \mathcal{I}_k : (S^c_k)^{-1}(0)/\operatorname{Ham}_T(M, \omega) \to (S^c_k)^{-1}(0)_{\mathbb{D}^2_k}/\operatorname{Ham}_T(M, \omega)^2_{k+1} : [J] \mapsto [J] \]

is also homeomorphic, using that \( \mathcal{I}_{l,k} \) is homeomorphic.

**Proposition 4.10.** The continuous map \( \mathcal{I}_k \) is a homeomorphism.

**Proof.** Take two elements \( J, J' \in (S^c_k)^{-1}(0) \) and suppose there is a \( L^2_{k+1} \)-regular map \( \phi \in \operatorname{Ham}_T(M, \omega)^2_{k+1} \) such that \( \phi^* J = J' \). Then \( \phi \) is \( C^\infty \)-smooth by Myers-Steenrod theorem. This shows that \( \mathcal{I}_k \) is injective.

Next we show the surjectivity. It is sufficient to show that for any \( J \in (S^c_k)^{-1}(0)_{\mathbb{D}^2_k} \), there is a \( L^2_{k+1} \)-regular map \( \phi \in \operatorname{Ham}_T(M, \omega)^2_{k+1} \) such that \( \phi^* J \in (S^c_k)^{-1}(0) \). Take a large integer \( m \geq 2 \) and \( l \) so that \( L^2_l \subset C^{m-1,1} \subset C^{m-1} \subset L^2_{k+1} \). Since \( \mathcal{I}_{l,k} \) is a homeomorphism, we can find a \( L^2_{k+1} \)-regular map \( \phi_0 \in \operatorname{Ham}_T(M, \omega)^2_{k+1} \)
so that $\phi_0^* J \in (S_2^2)^{-1}(0)$.

Then it follows from [NW] that there is a $C^{m,n}$-smooth

diffeomorphism $\phi_0 : M' \to M$ such that $\phi_0^* \phi_1^* J$ is a $C^\infty$-smooth integrable

complex structure, where on the other hand $\phi_0^* \phi_2^* \omega$ and $\phi_0^* g_J$ is only $C^{m-1,n}$-regular, in particular, $L^2_{k+1}$-regular.

We can choose a $C^\infty$-smooth diffeomorphism $\phi_2 : M \to M'$, which we can additionally suppose that it is sufficiently close to $\phi_1^{-1}$ in $C^m$-topology. Note that $\phi_0 \circ \phi_1 \circ \phi_2$ is sufficiently close to $\phi_0$ in $L^2_{k+1}$-topology.

The pull-back metric $\phi_2^* (\phi_0 \circ \phi_1)^* g_J$ is a $L^2_{k+1}$-regular metric which is a Kähler-Ricci soliton with respect to $C^\infty$-smooth integrable complex structure $\phi_2^* (\phi_0 \circ \phi_1)^* J$.

The elliptic regularity argument shows that $(\phi_0 \circ \phi_1 \circ \phi_2)^* g_J$ is in fact $C^\infty$-smooth. Hence $(\phi_0 \circ \phi_1 \circ \phi_2)^* \omega$ is also $C^\infty$-smooth. Since we further assume that $(\phi_0 \circ \phi_1 \circ \phi_2)^* \omega$ is close to $\phi_0^* \omega = \omega$ in $L^2_{k+1}$-topology, both $C^\infty$-smooth symplectic form $\omega$, $(\phi_0 \circ \phi_1 \circ \phi_2)^* \omega$ have the same cohomology classes and $\omega_t := t \omega + (1-t)(\phi_0 \circ \phi_1 \circ \phi_2)^* \omega$ is nondegenerate for any $t \in [0,1]$.

From Moser’s theorem, we obtain a $C^\infty$-smooth diffeomorphism $\phi_3$ satisfying $\phi_3^* (\phi_0 \circ \phi_1 \circ \phi_2)^* \omega = \omega$, which is close to $\text{id}_M$ in $L^2_{k+1}$-topology as in the proof of Proposition 3.7. Now we have obtained the expected $L^2_{k+1}$-regular map $\phi := \phi_0 \circ \phi_1 \circ \phi_2 \circ \phi_3$.

From the construction, we know $\phi$ can be taken sufficiently close to $\phi_0$ in $L^2_{k+1}$-topology.

Finally we prove that $\mathcal{I}_k$ is actually a homeomorphism. Take a convergent sequence $J_n \to J_\infty \in (S_2^2)^{-1}(0)$.

It suffices to show that there are elements $\phi_n, \phi_\infty \in \text{Ham}_T(M, \omega)^2_{k+1}$ such that $\phi_n^* J_n, \phi_\infty^* J_\infty$ belong to $(S_2^2)^{-1}(0)$ and the sequence $\phi_n^* J_n$ converges to $\phi_\infty^* J_\infty$ in the $C^\infty$-topology, by taking a subsequence if necessary (injectivity of $\mathcal{I}_k$). Since $\mathcal{I}_k$ is surjective, we can find an element $\phi_\infty \in \text{Ham}_T(M, \omega)^2_{k+1}$ so that $\phi_\infty^* J_\infty$ is $C^\infty$-smooth and hence there is no loss of generality in supposing $C^\infty$-smoothness of $J_\infty$ from the beginning. Since $\mathcal{I}_{k,l}$ is a homeomorphism, we can find a sequence $\phi_{n,t} \in \text{Ham}_T(M, \omega)^2_{k+1}$ so that $\phi_{n,t}^* J_n \in (S_2^2)^{-1}(0)$ and $\phi_{n,t}^* J_n$ converges to $J_\infty$ in the $L^2_{k+1}$-topology.

We define a set

$$\Sigma_l(J) := \{ \phi \in \text{Ham}_T(M, \omega)^2_{k+1} \mid \phi^* J \in (S_2^2)^{-1}(0) \}$$

for $J \in (S_2^2)^{-1}(0)$. Since $\mathcal{I}_k$ is surjective, $\Sigma_l(J)$ is a non-empty set. Moreover, the density of $\text{Ham}_T(M, \omega) \subset \text{Ham}_T(M, \omega)^2_{k+1}$, which we can deduce from Weinstein’s tubular neighbourhood theorem, shows that $\Sigma_l(J) \subset \text{Ham}_T(M, \omega)^2_{k+1}$ is also dense.

Therefore we can perturb $\phi_n,t$ as small as $n$ goes to the infinity, so that $\phi_{n,t}^* J_n$ are $C^\infty$-smooth and preserve the $L^2_{k+1}$-convergence $\phi_{n,t}^* J_n \to J_\infty \in (S_2^2)^{-1}(0)$. Now we can proceed to the diagonal argument with respect to $(n,l)$ and conclude that a subsequence $\phi_{n,t}^* J_n$ converges to $J_\infty$ in $C^\infty$-topology.

There is another topological space consisting of biholomorphism classes of Fano manifolds with Kähler-Ricci solitons, which is considered in [PSS].

$$K_{\text{Ric}}(n, F) := \{ (M, J, \xi) \mid (M, J, g, \xi') \text{ is a Fano manifold } (M, J) \text{ with }$$

$$\text{a Kähler-Ricci soliton } (g, \xi') \text{ and } \int_M |\xi'|^2 \omega^n \leq F \}$$

In [PSS], a topological compactification $\overline{K_{\text{Ric}}}(n, F)$ of this space is considered in regard to the ‘$J$-enhanced’ Gromov-Hausdorff convergence. It was open whether the space $K_{\text{Ric}}(n, F)$ is stable for large $F$, which was expected in [PSS]. This is equivalent to say that the invariant $\text{Fut}(\xi') = 2 \int_M |\xi'|^2 \omega^n$ is uniformly bounded from above for $n$-dimensional Fano manifolds with Kähler-Ricci solitons.
We show that this invariant is actually bounded, moreover, for all \( n \)-dimensional Fano \( T \)-manifolds with the maximal \( K \)-optimal action. Furthermore, we compare \( \mathcal{K}R_{GH}(n, F) \) with our \( K(M, \omega, T) \), especially \( K_{T, \chi} \).

**Proposition 4.11.** Set

\[
\mathcal{K}R_{GH}(n) := \left\{ [M, J, g, \xi] \mid (M, J, g, \xi) \text{ is a Fano manifold } (M, J) \text{ with a Kähler-Ricci soliton } (g, \xi) \right\}.
\]

Then \( \mathcal{K}R_{GH}(n) = \mathcal{K}R_{GH}(n, F) \) for large \( F \) and the map

\[
K(M, \omega, T) \to \mathcal{K}R_{GH}(n) : [J] \mapsto [M, J, g, \xi]
\]

gives a homeomorphism onto a clopen (closed and open) subset of \( \mathcal{K}R_{GH}(n) \), for any \( 2n \)-dimensional symplectic Fano manifold \( (M, \omega) \) with \( K \)-optimal \( T \)-action.

**Proof.** Since Fano manifolds are bounded [KMM], we have a sufficiently large Hilbert scheme \( \text{Hilb of } \mathbb{C}P^N \) with bounded Hilbert polynomial so that for any Fano manifold \( X \) we can find a point \( [X] \in \text{Hilb} \) representing an anticanonically embedded \( X \subset \mathbb{C}P^N \). We denote by \( \text{Hilb}_{Fano} \) the Zariski open locus parametrizing the anticanonically embedded Fano manifolds. The action of the group \( \text{Aut}(\mathbb{P}^N) = PGL(N + 1) \) preserves \( \text{Hilb}_{Fano} \).

Fix a maximal algebraic torus \( T \) of \( PGL(N + 1) \) and consider its action on \( \text{Hilb}_{Fano} \). Since maximal tori are unique up to conjugate, we can find a point \( [X] \in \text{Hilb}_{Fano} \) so that \( \text{Stab}([X]) \cap T \subset \text{Stab}([X]) \cong \text{Aut}(X) \) is a maximal torus.

Next, consider the normalization \( \text{Hilb} \to \text{Hilb} \), where \( \text{Hilb} \) is a normal projective variety and the morphism is a finite surjective morphism. Then we have a \( T \)-equivariant embedding of \( \text{Hilb} \) into some \( \mathbb{P}(V) \), where \( V \) is a \( T \)-representation ([MFK Corollary 1.6]). Since \( V \) decomposes into 1-dimensional representations as \( V \cong \mathbb{C}_{\chi_1} \oplus \cdots \oplus \mathbb{C}_{\chi_{\dim V}} \), the stabilizer \( T_x \subset T \) of any point \( x \in \mathbb{P}(V) \) can be written as \( \chi_{i_1}^{-1}(1) \cap \cdots \cap \chi_{i_k}^{-1}(1) \), hence the possibilities are finite. It also follows that every fiber \( S_{T}^{-1}(T') \) of the following map

\[
S_T : \mathbb{P}(V) \to \{ \text{sub torus of } T \} : x \mapsto T_x
\]

is a (possibly non irreducible) subvariety in \( \mathbb{P}(V) \). Therefore, we obtain a finite stratification \( \{ S_{T}^{-1}(T) \subset \mathbb{P}(V) \} \) and \( \{ H_i \subset \text{Hilb}_{Fano} \} \) by its restriction. We refine this stratification by taking connected components of each \( H_i \) and continue to write \( \{ H_i \subset \text{Hilb}_{Fano} \} \). Since the pull-back family \( U|_{H_i} \to H_i \) gives a family of Fano \( T_i \)-manifolds, we can consider the \( K \)-optimal vector \( \xi_i \in (N_i)_\mathbb{R} \) with respect to the \( T_i \)-action on the Fano manifolds \( X_s (s \in H_i) \), which is independent of \( s \in H_i \). Let \( T'_i \subset T_i \) be the subtorus generated by \( \xi_i \).

Now from the construction, every Fano manifold \( X \) with a maximal \( K \)-optimal \( T \)-action finds some \( H_i \) satisfying \( [X] \in H_i \) and \( T' = T'_i \). Since the Futaki invariant of \( \xi'_i \) on \( X_s \) is independent of the choice of \( s \in H_i \), we conclude that there are only finitely many possibilities of the values of \( \text{Fut}_X(\xi'_i) \) for the pairs \( (X, \xi'_i) \) of Fano manifolds with vanishing modified Futaki invariant \( \text{Fut}_X(\xi'_i) \). In particular, \( \text{Fut}_X(\xi'_i) \) is bounded for \( (X, g, \xi'_i) \in \mathcal{K}R_{GH}(n) \) and hence \( \mathcal{K}R_{GH}(n, F) = \mathcal{K}R(n) \) for large \( F \).

Next we show that the given map \( K(M, \omega, T) \to \mathcal{K}R_{GH}(n) \) is a homeomorphism. The continuity of the map is obvious. For every \( [M, J, g] \in \mathcal{K}R_{GH}(n) \) and any two representatives \( (M_1, J_1, g_1), (M_2, J_2, g_2) \in [M, J, g] \), we have a diffeomorphism \( \phi : M_1 \to M_2 \) satisfying \( \phi^* J_2 = J_1, \phi^* g_2 = g_1 \) and \( (\phi^{-1})^* \xi'_2 = \xi'_1 \), where \( \xi'_i \) is the unique holomorphic vector field satisfying \( \text{Ric}(g_i) = -\xi'_i g_i = g_i \). It follows
that the map \( K(M, \omega, T) \to \mathcal{KR}_{GH}(n) \) is injective for \( K \)-optimal \((M, \omega, T)\), and the images of \( K(M_1, \omega_1, T_1), K(M_2, \omega_2, T_2) \to \mathcal{KR}_{GH}(n) \) intersect if there is an isomorphism \( \theta : T_1 \cong T_2 \) and a \((T_1, T_2)\)-equivariant symplectic diffeomorphism \((M_1, \omega_1) \cong (M_2, \omega_2)\).

Since the images of the maps for distinct pairs \((M_1, \omega_1, T_1), (M_2, \omega_2, T_2)\) are disjoint, it suffices to prove that the maps are closed. Actually, if the maps are closed, then the maps are homeomorphisms onto their images and the images are open from the above finiteness of the possibilities of the \( K \)-optimal pairs \((M, \omega, T)\). To see the closedness, take a sequence \([J_n] \in K(M, \omega, T)\) which has the convergent images \([M_n, J_n, g_{J_n}] \to [M_\infty, J_\infty, g_{\infty}]\) in \( \mathcal{KR}_{GH}(n) \). As remarked before Proposition 6.1 in [PSS], we have a sequence \([M_n, J_n, g_{J_n}] \in \text{Hilb}_T^o\) which converges to \([M_\infty, J_\infty, g_{\infty}] \in \text{Hilb}_T^o\), where \( \text{Hilb}_T^o \) denotes the open subset of \( \text{Hilb}_T \) parametrizing gentle Fano \( T \)-manifolds with bounded Hilbert polynomial. Now we have a canonical continuous (holomorphic) map \( \text{Hilb}_T^o \to K(M, \omega, T) \subset \mathcal{KR}_{T,X} \) induced by the universality of \( K_{T,X} \). The image of the sequence \([M_n, J_n, g_{J_n}]\) is nothing but the original sequence \([J_n]\), so we obtain the convergence of \([J_n]\) to the image of \([M_\infty, J_\infty, g_{\infty}]\) in \( K(M, \omega, T) \).

\[\square\]

Remark 4.12. The compactification \( \overline{\mathcal{KR}_{GH}(n)} \) of \( \mathcal{KR}_{GH}(n) \) constructed in [PSS] is a compact Hausdorff space with a countable basis (cf. [DonSim2]) and the boundary \( \overline{\mathcal{KR}_{GH}(n)} \setminus \mathcal{KR}_{GH}(n) \) is closed. The closedness of the boundary is easily confirmed as follows. Suppose \([X_n, g_n]\) is a sequence in \( \overline{\mathcal{KR}_{GH}(n)} \setminus \mathcal{KR}_{GH}(n) \) converging to \([X_\infty, g_\infty]\) in \( \mathcal{KR}_{GH}(n) \). Take a sequence \([X_{n,i}, g_{n,i}] \in \mathcal{KR}_{GH}(n)\) for each \( n \) converging to \([X_n, g_n]\) in \( \overline{\mathcal{KR}_{GH}(n)} \). We can suppose that \( \text{Hilb}(X_{n,i}, g_{n,i}) \to \text{Hilb}(X_n, g_n) \) in \( \text{Hilb}_T \). Then we can find a subsequence of \([X_n, g_n]\) so that \( \text{Hilb}(X_{n,i}, g_{n,i}) \to \text{Hilb}(X_\infty, g_\infty) \) in \( \text{Hilb}_T \) by the diagonal argument. Since the subset of \( \text{Hilb}_T \) parametrizing singular subspaces of \( \mathbb{C}P^n \) forms a closed subset, \([X_\infty, g_\infty]\) should also be singular, hence \([X_\infty, g_\infty] \in \overline{\mathcal{KR}_{GH}(n)} \setminus \mathcal{KR}_{GH}(n) \).

4.4. The promised proof of Proposition 2.17. If \( X \) is a gentle Fano \( T \)-manifold, then \( R_{\xi^t}(X) = 1 \) for the \( K \)-optimal vector \( \xi^t \). So there exists a unique solution \( \omega_t = \omega_t(\alpha) \) of the following equation

\[\text{Ric}(\omega_t) - L_\xi \omega_t = t \omega_t + (1 - t) \alpha\]

for every \( t \in [0, 1) \) and any initial metric \( \alpha \).

Lemma 4.13. Let \( X \to \Delta \) be a family of Fano \( T \)-manifolds with \( R_{\xi^t}(X_\sigma) = 1 \) for the \( K \)-optimal vector \( \xi^t \) over a compact disc \( \Delta \) and \( \alpha \) be a smooth family of \( T_\mathbb{R} \)-invariant Kähler metrics \( \omega_\sigma \) on \( X_\sigma \). Then there is a sufficiently divisible \( k \in \mathbb{N} \) and a positive constant \( c > 0 \) which depend only on the pair \((X, \alpha)\) such that for any \( \sigma \in [0, 1] \) and \( t \in [0, 1) \) the following uniform partial \( C^0 \)-estimate holds.

\[\rho_{X_\sigma, \omega_t(\alpha_\sigma), k} \geq c_t\]

where \( \rho_{X_\sigma, \omega_t(\alpha_\sigma), k} \) denotes the function on \( X_\sigma \) defined by

\[\rho_{X_\sigma, \omega_t(\alpha_\sigma), k}(x) := \max \{ s(x) | h_{X_\sigma, \omega_t(\alpha_\sigma), k} \},\]

where \( s \) runs over \( s \in H^0(X_\sigma, \mathcal{O}(-kK_X)) \) with \( \int_{X_\sigma} |s|^2 h_{X_\sigma, \omega_t(\alpha_\sigma), k} = 1 \) and \( h_{X_\sigma, \omega_t(\alpha_\sigma), k} \) denotes a metric on \( -kK_{X_\sigma} \) whose curvature is \( k\omega_t(\alpha_\sigma) \).

Proof. This follows from estimates in the proof of Lemma 5.6, Lemma 5.7 and Lemma 5.8 in [F. Wang, X. Zhu]. Note that we can uniformly take constants \( C \in \mathbb{R} \)
Lemma 5.6, B in Lemma 5.7 and c₁, C in Lemma 5.8 independent of αₗ, since the constants of Theorem A in [Mab] and of Corollary 5.3 in [Zhu] can be uniformly taken. Then it follows that any sequence \((X_\alpha, \omega_\alpha)\) \((t_\alpha \to 1)\) is a sequence of almost Kähler-Ricci solitons in the sense of [F. Wang, X. Zhu] Definition 5.1. Now we can deduce our estimate from [JWZ, Cororally 1.4], [DonSun1, Lemma 3.4] and the argument after the lemma. □

Now we can apply the arguments in [DonSun1] to the metric family
\[
\{(X, \omega_t(\alpha))\}_{t,s \in [0,1] \times [0,1]},
\]
under the above partial \(C^0\)-estimate. Thus we have a sufficiently divisible number \(k \in \mathbb{N}\) with the following properties.

1. The pair \((X_\alpha, \omega_t(\alpha))\) defines a point \(\text{Hilb}(X_\alpha, \omega_t(\alpha))\) in the compact Hausdorff topological space \(\text{Hilb}_{T/U_T}\) by embedding \(X_\alpha\) into \(\mathbb{C}P^N\) using a unitary basis of \(H^0(X_\alpha, \mathcal{O}(-kK_{X_\alpha}))\) with respect to the metric \(\omega_t(\alpha)\).
2. For any sequence \((\sigma_t, t_i) \in \Delta \times [0,1]\), we have a subsequence such that \((X_\sigma, \omega_t(\alpha))\) converges in the ‘\(J\)-enhanced’ Gromov-Hausdorff topology to some \(\mathbb{Q}\)-Fano variety \(X_\infty\) with a Kähler-Ricci soliton \((\omega_\infty, \xi_\infty)\).
3. After taking a further subsequence, the sequence \(\text{Hilb}(X_\sigma, \omega_t(\alpha))\) \(\in\) \(\text{Hilb}_{T/U_T}\) converges in \(\text{Hilb}_{T/U_T}\) to the point \(\text{Hilb}(X_\infty, \omega_\infty)\) which is similarly defined using a unitary embedding \(X_\infty \hookrightarrow \mathbb{C}P^N\).

**Proof of Proposition 2.17** Let \(\{(X, \omega_t(\alpha))\}_{t \in [0,1]}\) be the family of solutions of the continuity method with an initial metric \(\alpha\). Suppose there is a smooth Fano \(T\)-manifold with Kähler-Ricci soliton \((X_\infty, \omega_\infty)\) which is the limit of a Gromov-Hausdorff convergent subsequence \((X, \omega_t(\alpha))\). First we show that the limit \((X_\infty, \omega_\infty)\) is uniquely determined independent of the choice of the initial metrics \(\alpha\) and the subsequences \((X, \omega_t(\alpha))\). Suppose \(\alpha'\) is another Kähler metric on \(X\) and \((X, \omega_t(\alpha')) \to (X'_\infty, \omega'_\infty)\) be a convergent subsequence to a \(\mathbb{Q}\)-Fano \(T\)-variety with Kähler-Ricci soliton. Set \(\alpha_s := s\alpha' + (1-t)\alpha\). As we noted right before this proof, we can find a sufficiently divisible number \(k_\alpha \in \mathbb{N}\) so that all \((X, \omega_t(\alpha))\) can be uniformly embedded using the unitary basis of \(H^0(X, \mathcal{O}(-k_\alpha K_X))\) with respect to \(\omega_t(\alpha)\), which defines a point \(\text{Hilb}(X, \omega_t(\alpha))\) \(\in\) \(\text{Hilb}_{T/U_T}\). Moreover, we can assume \((X_\infty, \omega_\infty)\) and \((X'_\infty, \omega'_\infty)\) also define points \(\text{Hilb}(X_\infty, \omega_\infty) \in \text{Hilb}_{T/U_T}, \text{Hilb}(X'_\infty, \omega'_\infty) \in \text{Hilb}_{T/U_T}\) respectively, and \(\text{Hilb}(X, \omega_t(\alpha)) \to \text{Hilb}(X_\infty, \omega_\infty) \in \text{Hilb}_{T/U_T}, \text{Hilb}(X, \omega_t(\alpha')) \to \text{Hilb}(X'_\infty, \omega'_\infty) \in \text{Hilb}_{T/U_T}\). These embeddings clearly define a continuous map \([0,1] \times [0,1] \to \text{Hilb}_{T/U_T} : (t, s) \mapsto \text{Hilb}(X, \omega_t(\alpha))\).

Suppose \(X_\infty \not\cong X'_\infty\). If \(\text{Hilb}(X_\infty, \omega_\infty)^{\text{PGL}_T} \cap \text{Hilb}(X'_\infty, \omega'_\infty)^{\text{PGL}_T} \neq \emptyset\), then we obtain a test configuration of \(X_\infty\) with the central fiber \(X'_\infty\) from the reductivity of the stabilizer \(\text{Aut}_T(X'_\infty)\), which allows to apply the étale slice theorem [AHR, Theorem 2.1] and the Hilbert-Mumford theorem. Since the central fiber admits a Kähler-Ricci soliton, the modified Donaldson-Futaki invariant of this test configuration is zero. However, as \(X_\infty\) has Kähler-Ricci soliton and hence K-polystable, \(X'_\infty\) must be isomorphic to \(X_\infty\). This contradicts to our assumption.

So we have \(\text{Hilb}(X_\infty, \omega_\infty)^{\text{PGL}_T} \cap \text{Hilb}(X'_\infty, \omega'_\infty)^{\text{PGL}_T} = \emptyset\). Then in particular we can take open neighbourhoods \(B_{\epsilon}(\text{Hilb}(X_\infty, \omega_\infty)^{\text{PGL}_T}), B_{\epsilon}(\text{Hilb}(X'_\infty, \omega'_\infty)^{\text{PGL}_T})\) separating the two closed subsets \(\text{Hilb}(X_\infty, \omega_\infty)^{\text{PGL}_T}\) and \(\text{Hilb}(X'_\infty, \omega'_\infty)^{\text{PGL}_T}\). Here we use a \(U_T\)-invariant distance on \(\text{Hilb}_{T/U_T}\) to consider \(B_{\epsilon}\) and fix this distance. Take \(U_T\)-invariant open neighbourhoods \(V \subset V' \subset \text{Hilb}_{T/U_T}\) of \(\text{Hilb}(X_\infty, \omega_\infty)^{\text{PGL}_T}\) so that
$U|_{V'} \to V'$ parametrizes Fano $T$-manifolds appearing in the family $\varpi: \mathcal{X} \to B$ with central fiber $X_0 = X_\infty$. We can assume $\text{Hilb}(X, \omega_\epsilon(\alpha)) \in V/U_T$. From the finiteness of the fibers of the morphism $[B/R] \to \mathcal{K}_{T,X}$ in Proposition 1.27, there are only finitely many isomorphism classes of Fano $T$-manifolds with Kähler-Ricci solitons in this family that can be the central fiber of some gentle degeneration of $X$. Putting $\omega_\epsilon(\sigma) := \omega_{\epsilon_1'(1-\epsilon)}(t_\epsilon(\alpha))$, we have a continuous curve $\text{Hilb}(X, \omega_\epsilon(-)) : [0, 1] \to \text{Hilb}_T/U_T$. Furthermore, putting

$$\sigma_i := \sup\{\sigma \in [0, 1] \mid \text{Hilb}(X, \omega_\epsilon(-))|_{\{0, \sigma\}} \subset B_{\epsilon}(\text{Hilb}(X_\infty, \omega_\infty)PGL_T/U_T)\},$$

we obtain a sequence of almost Kähler-Ricci solitons in the sense of [F. Wang, X. Zhu].

So after taking a subsequence, we have a sequence $(X, \omega_\epsilon(\sigma))$ converging to some Q-Fano $T$-variety admitting Kähler-Ricci soliton $(X''_\infty, \omega''_\infty)$ with the convergent corresponding sequence $\text{Hilb}(X, \omega_\epsilon(\sigma)) \to \text{Hilb}(X''_\infty, \omega''_\infty)$ in $\text{Hilb}_T/U_T$. Replacing $\epsilon$ with $\epsilon/2^k$, we can construct $\sigma_{i,k}$ and $X''_{\epsilon/k, k}$ by the same process.

Suppose there is infinitely many $i$ for each $k$ such that $\text{Hilb}(X, \omega_i(\sigma_{i,k})) \notin VPGL_T/U_T$. After taking subsequence, we know that

$$\text{Hilb}(X, \omega_i(\sigma_{i,k})) \in \partial(VPGL_T/U_T) \cap B_{\epsilon/2^{k-1}}(\text{Hilb}(X_\infty, \omega_\infty)PGL_T/U_T)$$

for

$$\sigma_{i,k} := \sup\{\sigma \in [0, \sigma_{i,k}] \mid \text{Hilb}(X, \omega_\epsilon(-))|_{\{0, \sigma\}} \subset VPGL_T/U_T\}.$$

Since $(X, \omega(\sigma'_{i,k}))$ is a sequence of almost Kähler-Ricci solitons for each $k$, we can assume $(X, \omega(\sigma'_{i,k})) \to (X''_{\epsilon/k, k}, \omega''_{\epsilon/k, k})$ for some Q-Fano $T$-variety with Kähler-Ricci soliton $(X''_{\epsilon/k, k}, \omega''_{\epsilon/k, k})$. The diagonal argument shows that there is a subsequence $\{(X, \omega_{i,k}(\sigma'_{i,k}))\}_{k=1}^\infty$ of $(X, \omega(\sigma'_{i,k}))_{i,k}$ and a Q-Fano $T$-variety $X''_{\epsilon/k, \infty}$ such that $(X, \omega_{i,k}(\sigma'_{i,k})) \to (X''_{\epsilon/k, \infty}, \omega''_{\epsilon/k, \infty})$ and $\text{Hilb}(X, \omega_{i,k}(\sigma'_{i,k})) \to \text{Hilb}(X''_{\epsilon/k, \infty}, \omega''_{\epsilon/k, \infty})$.

Now from the property (15), we conclude $\text{Hilb}(X''_{\epsilon/k, \infty}, \omega''_{\epsilon/k, \infty}) \in \text{Hilb}(X_\infty, \omega_\infty)PGL_T \setminus \text{Hilb}(X_\infty, \omega_\infty)PGL_T$. But this is absurd in the same way as we have seen before.

Therefore we can assume that for any large $k$, $\text{Hilb}(X, \omega_i(\sigma_{i,k}))$ is in the neighbourhood $VPGL_T/U_T$ for all but only finitely many $i$. In this case, the convergent sequence $(X, \omega_i(\sigma_{i,k})) \to (X''_{\epsilon/k, k}, \omega''_{\epsilon/k, k})$ defines a convergent sequence $\text{Hilb}(X, \omega_i(\sigma_{i,k})) \to \text{Hilb}(X''_{\epsilon/k, k}, \omega''_{\epsilon/k, k})$ in $\text{Hilb}_T/U_T$ that is uniformly away from $\text{Hilb}(X_\infty, \omega_\infty)PGL_T$ because $\text{Hilb}(X, \omega_i(\sigma_{i,k})) \in \partial B_{\epsilon/2^k}(\text{Hilb}(X_\infty, \omega_\infty)PGL_T/U_T)$. It follows that $X''_{\epsilon/k, k} \neq X_\infty$. Since $\text{Hilb}(X''_{\epsilon/k, k}, \omega''_{\epsilon/k, k}) \in VPGL_T/U_T$ and each there is a gentle degeneration of $X$ with its central fiber $X''_{\epsilon/k, k}$, there is only finitely many isomorphism classes in $\{X''_{\epsilon/k, k}\}_{k=1}^\infty$. So we can assume $X''_{\epsilon/k, k}$ is all isomorphic after taking subsequence. From the uniqueness of Kähler-Ricci soliton, the sequence $(X''_{\epsilon/k, k}, \omega''_{\epsilon/k, k})$ is constant and hence converges to the limit $(X''_{\epsilon/\infty, \infty}, \omega''_{\epsilon/\infty, \infty}) \cong (X''_{0, k}, \omega''_{0, k})$. It follows that $\text{Hilb}(X''_{\epsilon/\infty, \infty}, \omega''_{\epsilon/\infty, \infty}) \in \text{Hilb}(X_\infty, \omega_\infty)PGL_T$ from the fact

$$\text{Hilb}(X''_{\epsilon/\infty, \infty}, \omega''_{\epsilon/\infty, \infty}) \in B_{\epsilon/2^k}(\text{Hilb}(X_\infty, \omega_\infty)PGL_T).$$

This is the last contradiction in this argument, which is now familiar to us. Finally, we conclude $X''_{0, k} \cong X_\infty$, so $X''_{0, k}$ is independent of the choice of the initial metrics $\alpha$ and the subsequences $t_i$.

Now we proceed to prove the uniqueness of the central fibers of gentle degenerations of $X$. Let $\mathcal{X} \to \Delta$ be a gentle degeneration. We have a smooth family of Kähler metrics $\alpha_\epsilon$ on $\mathcal{X}$ which extends the Kähler-Ricci soliton $\alpha_0$ on the central fiber $X_0$, thanks to the stability argument of the Kähler condition in any sufficiently
small deformation (see the last chapter of [KodMor]). The uniqueness of the continuity path, proved in [TZ1], shows that $\omega_t(\alpha_0) = \alpha_0$, so we can find a sequence $t_i \to 1$ and $s_i \to 0 \in \Delta$ so that $(X, \omega_{t_i}(\alpha_{s_i}))$ converges to $(X_0, \alpha_0)$. We can show that the sequence $(X, \omega_{t'}(\alpha_{s'_i}))$ also converges to $(X_0, \alpha_0)$ for any sequence $t'_i \to 1$ by a similar argument as above (compare [LWX1, Lemma 6.9. (1)]). Consider some convergent sequence $(X, \omega_{t_m}(\alpha_{s_i})) \xrightarrow{m \to 1} (X_{\infty, i}, \omega_{\infty, i})$ and a sequence $t'_i \to 1$ so that $d_{GH}((X_{\infty, i}, \omega_{\infty, i}), (X, \omega_{t'_i}(\alpha_{s'_i}))) < 1/i$. The diagonal argument shows that $(X_{\infty, i}, \omega_{\infty, i}) \to X_0$. Since $X_0$ is a smooth Fano $T$-manifold, $X_{\infty, i}$ is also smooth for large $i$. From what we have shown in the above argument, it follows that for any fixed Kähler metric $\alpha$ on $X$, we obtain $(X, \omega_t(\alpha)) \xrightarrow{t \to 1} (X_{\infty, i}, \omega_{\infty, i})$ for each $i$, so especially $(X_{\infty, i}, \omega_{\infty, i})$ are all isomorphic to each other. Now we conclude $(X, \omega_t(\alpha)) \xrightarrow{t \to 1} (X_0, \alpha_0)$ where the limit is independent of the choice of the initial metrics $\alpha$ and the sequence is also independent of the choice of the central fibers $(X_0, \alpha_0)$ of the gentle degenerations. So for another central fiber $(X'_0, \alpha'_0)$ of another gentle degeneration $X'' \to \Delta$ of $X$, we also have $(X, \omega_t(\alpha)) \xrightarrow{t \to 1} (X'_0, \alpha'_0)$. It follows that $(X'_0, \alpha'_0)$ is isomorphic to $(X_0, \alpha_0)$ from the uniqueness of the limit. This is what we expected.

5. Discussions

5.1. On some examples. Here we observe step by step some known examples of Fano manifolds admitting Kähler-Ricci solitons. As far as the author knows, no explicit descriptions of the associated holomorphic vector fields $\xi'$ are given in any case.

Example 5.1. The blowing-up of $\mathbb{C}P^2$ at one point is a typical example of Fano manifolds admitting non-Einstein Kähler-Ricci solitons. This might be the first example of a compact complex manifold proved to admit Kähler-Ricci solitons, which was found by Koiso [Koi] and Cao [Cao], independently.

Example 5.2 (toric Fano manifolds). It is shown in [X-J. Wang, X. Zhu] and re-proved by [DatSze] from the K-stability viewpoint that every toric Fano manifold admit Kähler-Ricci soliton and it is Kähler-Einstein if and only if the barycenter of the canonical polytope coincides with the origin. Note that the maximal torus action on a toric Fano manifold is not necessarily K-optimal.

Every toric Fano manifold is rigid, i.e. $H^1(X, \Theta_X) = 0$, where $\Theta_X$ denotes the tangent sheaf ([BieBru, Proposition 4.2.1.]). It follows that toric Fano manifolds give discrete points in the moduli space $K\mathcal{R}_{GH}(n)$.

Example 5.3 (Fano homogeneous toric bundles). It is shown in [PS] that Fano homogeneous toric bundles have Kähler-Ricci solitons. This is a generalization of the main result in [X-J. Wang, X. Zhu]. It is again proved in [BieBru, Proposition 4.2.] that Fano homogeneous toric bundles are rigid (see also [BieBru, Proposition 2.2.1.], [Del, Example 3.10.]).

Example 5.4 (horospherical Fano manifolds). It is shown in [Del] from the K-stability viewpoint and reproved by [Frai] that every horospherical Fano manifold admits Kähler-Ricci solitons. This is a generalization of one of the main result in [PS]. Horospherical Fano manifolds with Picard number one ($b^1 = 1$) are classified in [Pas]. There is one example in this class, which is related to the complex $G_2$.
group, that has a non-trivial small deformation. It is shown in [PP] that the Kuranishi family of this Fano manifold $X_0$ is an isotrivial degeneration $\mathcal{X} \to \mathbb{C}$ of the orthogonal Grassmanian $Gr_q(2, 7)$. Since $Gr_q(2, 7)$ is homogeneous, it admits Kähler-Einstein metrics ([Mar]). Then from the separatedness property of Kähler-Einstein Fano manifolds ([SSY], [LWX1]), we conclude that $X_0$ cannot admit Kähler-Einstein metrics. It follows that $X_0$ must admit non-Einstein Kähler-Ricci solitons.

This example shows that the family $\mathcal{X} \to \mathbb{C}$ is not in the category $K(n)$, though any fibers in the family, which are isomorphic to either $Gr_q(2, 7)$ or $X_0$, admit Kähler-Ricci solitons. We should separate them into two pieces $\mathcal{X}^* \to \mathbb{C}^*$ and $X_0 \to \{0\}$ as the associated holomorphic vector fields jump at the origin.

It seems interesting to study whether any horospherical Fano manifolds are $K$-rigid, which means $H^1_T(X, \Theta_X) = 0$ for a $K$-optimal action $X \acts T$.

**Example 5.5** (Fano manifolds with complexity one). It is shown in [IS] and [CabSüss] that complexity one Fano threefolds of type 2.30, 2.31, 3.8*, 3.18, 3.21, 3.22, 3.23, 3.24, 4.5* and 4.8 from Mori and Mukai's classification [MM] admit non-Einstein Kähler-Ricci soliton.

Especially 3.8 and 4.5 admit deformations, so $H^1_T(X, \Theta) \not\parallel \text{Aut}_T(X)$ might be not mere a point.

The product $X \times Y$ of two Fano manifolds $X, Y$ with Kähler-Ricci solitons admits Kähler-Ricci solitons. So for instance, suppose $X$ is a Del Pezzo surface of degree $1 \leq d \leq 4$ and $Y$ is the blowing-up of $\mathbb{C}P^2$ at one point, then $X \times Y$ admits non-Einstein Kähler-Ricci solitons. By deforming $X$ while fixing $Y$, we get a $T$-equivariant deformation of $X \times Y$ where $X \times Y \acts T$ is induced from the K-optimal action $Y \acts T$. So $X \times Y$ provides a non discrete point in the moduli space $KR_{GH}(n)$ outside of the subset $K_{0, GH}(n)$ consisting of Kähler-Einstein Fano manifolds.

Dancer-Wang’s examples [DW] should also provide non discrete points in the moduli space.

**5.2. Future studies.**

**5.2.1. Questions on the structure of the moduli space.**

**Question 5.6.** Find an example of $(T, \chi)$ or $(M, \omega, T)$ with non-trivial $T$ so that the moduli space $K_{T, \chi}$ or $K(M, \omega, T)$ is positive dimensional and we can concretely describe its structure.

The author does not have any concrete description of positive dimensional moduli spaces $K(M, \omega, T)$ so far. Related studies in the Kähler-Einstein case ($T = 0$ case) are explored by [OSS], [SS], [LiuXu].

**Question 5.7.** Is the complex analytic space $K_{T, \chi}$ actually quasi-projective?

This question is related to the result in [LWX2] where the quasi-projectivity of the moduli space of Fano manifolds with Kähler-Einstein metrics is proved.

When $T$ is non-trivial, even the finiteness of the number of the connected components of $K_{T, \chi}$ is still hard to prove, though it has a natural topological compactification as a moduli space.

**Question 5.8.** Is there a canonical complex analytic structure on the compact topological space $KR_{GH}(n)$? How about on the space $KR_{GH}(n) \setminus KR_{GH}(n)$? Can we identify them with algebraic spaces, or moreover with projective schemes?
This is related to the work of [Oda2, LWX1]. The techniques in this paper do not work in the singular setting.

**Question 5.9.** Is there a canonical complex analytic (or algebraic) moduli space of $\mathbb{Q}$-Fano varieties with Kähler-Ricci solitons?

In all questions, we should investigate modified K-stability from more algebro-geometric perspectives, possibly with the help of differential geometry.

### 5.2.2. Questions related to the extent of the moduli space.

**Question 5.10.** Are there any non-gentle (or modified K-unstable) examples of a Fano manifolds with Picard number one? How about birationally rigid Fano manifolds with Picard number one?

This is a refined question related to the Odaka-Okada conjecture [OO]. Two modified K-unstable examples are given in [Del], but both have the Picard number greater than one.

**Conjecture 5.11.** Let $X$ be a $\mathbb{Q}$-Fano variety.

1. If $X$ is not modified K-semistable, there is a (non-equivariant) $\mathbb{R}$-degeneration (cf. [DerSze, CSW]) of $X$ whose central fiber is a modified K-semistable $\mathbb{Q}$-Fano variety whose Donaldson-Futaki invariant attains the infimum of the Donaldson-Futaki invariants over all $\mathbb{R}$-degenerations. Moreover, these degenerations are unique up to isomorphisms.

2. If $X$ is modified K-semistable with respect to a torus action $T$, then there is a $T$-equivariant degeneration $X' \to \Delta$ of $X$ whose central fiber $X'_0$ is a K-polystable $\mathbb{Q}$-Fano $T$-variety (modified K-polystable with respect to the $T$-action). Moreover, any two such $T$-equivariant degenerations $X'_1 \to \Delta, X'_2 \to \Delta$ are equivalent in the sense of the $T$-equivariant version of [BHJ, Definition 6.1].

This conjecture is related to [CSW, Conjecture 3.7] and is an analogy of the Harder-Narasimhan filtration for torsion-free coherent sheaves and the Jordan-Hölder filtration for semistable coherent sheaves (see [IL]) as already observed in [DerSze, Remark 3.6]. We include the singular case for the future application to the Question 5.9.

For the first item, [CSW] shows that every smooth Fano manifold $X$ has an $\mathbb{R}$-degeneration with the $\mathbb{Q}$-Fano central fiber $X_0$ and there is another degeneration $X'' \to \Delta$ of $X_0$ with the modified K-polystable $\mathbb{Q}$-Fano central fiber $X''_0$ with the K-optimal vector $\xi'$, which can be extended to $X_0$ with the vanishing modified Futaki invariant (see also [DerSze]). So as for the existence, it suffices to prove the modified K-semistability of $X_0$. Since $(X'_0, \xi')$ is K-polystable, the problem is reduced to the ‘stability of K-semistability in small deformations’, which is related to the Artin property of the $\text{Can}$-stack consisting of K-semistable $\mathbb{Q}$-Fano $T$-varieties, as in Proposition 4.4. It is remarkable that if $X$ is K-unstable (with respect to the trivial torus action), then $X'_0$ should be endowed with non-Einstein Kähler-Ricci solitons ([CSW], p. 17).

The existence part of the second item is confirmed in [DatSze] for smooth modified K-semistable Fano $T$-manifolds. The uniqueness of the central fiber in this case could be demonstrated by the same methods in [LWX1], which is a role model.
of our proof of Proposition 2.17. (We worked with the smooth central fiber because
the author thought it should make arguments clear.)

The uniqueness assertion in the second item is stronger than the uniqueness of
the central fiber. This stronger uniqueness (for every smooth gentle Fano $T$-manifold $X$) has the following application.

**Corollary 5.12** (of Conjecture 5.11). The moduli space $K_{T,\chi} \to K_{T,\chi}$ we con-
structed in Theorem 4.8 is good in the sense of Alper [Alp1]. (In our case, the
cohomological affineness should be defined as the exactness of the pushforward func-
tor $\text{Coh}(K_{T,\chi}) \to \text{Mod}(K_{T,\chi}).$ )

Actually, using the uniqueness of the degeneration in the sense of [BHJ], we
 can show that the étale morphism $B/R \to K_{T,\chi}$ is an open embedding. Then the
corollary follows from the fact that $[BK^c/K^c] \to BK^c/K^c$ is a good moduli space.
Recall that we already have shown the central fiber of the degeneration is unique,
which we used to prove that the morphism $[B/R] \to BK^c/K^c$ factors through $\text{Im}[B/R] \subset K_{T,\chi}$. There might be other ways to show this naturally expected
corollary.

6. Appendix

**A. Can-stack.** In this Appendix A, we briefly review on definitions and examples
related to stacks, which we needed in section 4. As we work only over the category
(or more precisely, the site) $\text{Can}$ of complex spaces, we do not introduce stacks in
full generality, which actually work over any site such as the étale sites of schemes
or algebraic spaces, the site of $C^\infty$-manifolds and so on. The interested readers
should also refer to [SPA, FGIKNV] for stacks in full generality.

**A-1. Fibred category.** Recall that we denote by $\text{Can}$ the category of complex
spaces, which are not assumed to be reduced nor irreducible. The set of holomorphic
morphisms between complex spaces $U$ and $V$ is denoted by $\text{Holo}(U,V)$.

**Definition 6.1** (fibred category). Let $\mathcal{F}$ be a category and $p : \mathcal{F} \to \text{Can}$ be a
functor to the category of complex spaces. The functor $p : \mathcal{F} \to \text{Can}$ is called a
fibred category over $\text{Can}$ if it satisfies the following properties. For any holomorphic
morphism $f : X \to Y$ between complex spaces and any object $\eta \in \text{Obj}(\mathcal{F})$, there
exists an object $\xi \in \text{Obj}(\mathcal{F})$ with $p(\xi) = X$ and a strongly cartesian morphism $\phi : \xi \to \eta \in \text{Mor}(\mathcal{F})$
over $f$.

Here the morphism $\phi : \xi \to \eta$ is called **strongly cartesian** if it enjoys the following
universal property: for any complex space $X'$, any holomorphic morphism $g : X' \to X$, any object $\xi' \in \text{Obj}(\mathcal{F})$ with $p(\xi') = X'$ and any morphism $\phi' : \xi' \to \eta \in \text{Mor}(\mathcal{F})$ with $p(\phi') = f \circ g$, there exists a unique morphism $\chi : \xi' \to \xi$ such that
$\phi' = \phi \circ \chi$ and $p(\chi) = g$.

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \eta \\
\downarrow^p & \downarrow & \downarrow \\
\text{Can} & \xrightarrow{f \circ g} & Y \\
\end{array}
\]

Let $X$ be a complex space and $p : \mathcal{F} \to \text{Can}$ be a fibred category. We denote by
$\mathcal{F}(X)$ the subcategory of $\mathcal{F}$ consisting of objects $\xi \in \text{Obj}(\mathcal{F})$ with $p(\xi) = X$ and
morphisms \( \phi \) with \( p(\phi) = \id_X \). We call \( \mathcal{F} \) (or more precisely \( \mathcal{F} \to \text{Can} \)) a \textit{category fibred in groupoids} if morphisms in \( \mathcal{F}(X) \) are all invertible for any complex space \( X \).

A functor \( f : \mathcal{F} \to \mathcal{G} \) between two fibred categories is called a \textit{morphism} of fibred categories if \( p_{\mathcal{F}} = p_{\mathcal{G}} \circ f \) (strictly) and \( f \) maps strongly cartesian morphisms in \( \mathcal{F} \) to strongly cartesian morphisms in \( \mathcal{G} \). We can also consider \textit{2-morphisms} between two (1-)morphisms \( f, g : \mathcal{F} \to \mathcal{G} \) which are just natural transformations \( t : f \to g \) satisfying \( p_{\mathcal{G}}(t_\xi : f(\xi) \to g(\xi)) = \id_{p_{\mathcal{F}}(\xi)} \) for all \( \xi \in \text{Obj}(\mathcal{F}). \)

The functor \( \text{Can}_X \to \text{Can} : (\xi : S \to X) \mapsto S \), where \( \text{Can}_X \) denotes the category of holomorphic morphisms \( \xi : S \to X \), is a typical example of category fibred in groupoids (actually in sets). A holomorphic morphism of complex spaces \( f : X \to Y \) gives the morphism \( \text{Can}_X \to \text{Can}_Y \) which maps an object \( \xi : S \to X \) to the object \( f \circ \xi : S \to Y \). On the other hand, a morphism \( f : \text{Can}_X \to \text{Can}_Y \) as fibred categories gives a holomorphic morphism \( f(\id_X) : X \to Y \). Therefore, we have a canonical fully faithful embedding of \( \text{Can} \) to the (2-)category of fibred categories. So we often abbreviate \( \text{Can}_X \) as \( X \).

Example 6.2. Let \( a : X \times G \to X \) be a holomorphic action of a complex Lie group \( G \) to a complex space \( X \). We denote by \([X/G]_p\)\(^1\) the fibred category (in groupoids) defined as follows.

(1) Its objects are holomorphic morphisms \( \xi : S \to X \) from some complex spaces \( S \).

(2) Its morphisms \( \xi_S \to \eta_T \) are the pairs \( (f, \phi) \) of holomorphic morphisms \( f : S \to T \) and \( \phi : S \to X \times G \) satisfying \( p_1 \circ \phi = \xi \) and \( a \circ \phi = \eta \circ f \).

(3) Its functor \([X/G]_p \to \text{Can} \) maps objects \( \xi_S \) to \( S \) and morphisms \( (f, \phi) : \xi_S \to \eta_T \) to \( f : S \to T \).

Objects in the fibred category \([X/G]_p\) coincide with \( X = \text{Can}_X \), but morphisms are different. For instance, two objects \( x, y : pt \to X \) in \([X/G]_p\) are isomorphic if and only if there exists an element \( g \in G \) with \( xg = y \). We have the morphism \( X \to [X/G]_p \) of fibred categories defined by \( \xi_S \mapsto \xi_S \).

There is another related fibred category \([X/G] \) with a good geometric feature.

Example 6.3. We denote by \([X/G] \) the fibred category (in groupoids) defined as follows.

(1) An object consists of a triple \((S, P, \xi)\) where \( S \) is a complex space, \( P \) is a principal \( G \)-holomorphic bundle over \( S \) and \( \xi : P \to X \) is a \( G \)-equivariant holomorphic morphism.

(2) A morphism \((S, P, \xi) \to (T, Q, \eta)\) is a pair \((f, \phi)\) where \( f : S \to T \) is a holomorphic morphism and \( \phi : P \to Q \) is a \( G \)-equivariant holomorphic morphism over \( f \) which induces a biholomorphism \( P \cong S \times_T Q \), and satisfies \( \xi = \xi' \circ \phi \).

(3) Its functor \([X/G] \to \text{Can} \) maps objects \((S, P, \xi)\) to \( S \) and morphisms \((f, \phi) : (S, P, \xi) \to (T, Q, \eta)\) to \( f : S \to T \).

We have the morphism \([X/G]_p \to [X/G] \) of fibred categories which maps an object \( \xi : S \to X \) to the object \((S, S \times G, a \circ (\xi \times \id_G))\) and a morphism \((f, \phi) : \xi_S \to \eta_T \) to the morphism \((f, f \times \phi)\). This is a typical example of the ‘stackification’ we treat in the next subsection. The fibred category \([X/G] \) is called a \textit{quotient stack}.

\(^1\)The symbol \( p \) means that this fibred category is not a stack in general; it is just a prestack.
When the action is proper free, then there exists a complex space $X/G$, a holomorphic morphism $X \to X/G$ and an isomorphism $X/G \cong \mathbb{M}[X/G]$ of fibred categories, which is compatible with $X \to X/G$ and $X \to \mathbb{M}[X/G]$.

Let us consider one more example. A holomorphic groupoid consists of the following data $(X, R, s, t, c)$:

1. $X$ and $R$ are complex spaces.
2. $s$ and $T$ are holomorphic morphisms from $R$ to $X$.
3. $c : R \times_{s,X,t} R \to R$ is a holomorphic morphism.

These data are to satisfy the following rules for any complex space $S$:

1. For every holomorphic morphism $\xi \in \text{Holo}(S, X)$, there exists a holomorphic morphism $e_\xi \in \text{Holo}(S, R)$ such that $c \circ (e_\xi \times \phi) = \phi$ and $c \circ (\psi \times e_\xi) = \psi$ for any pairs $(e_\xi, \phi), (\psi, e_\xi)$ with $s \circ e_\xi = t \circ \phi$ and $s \circ \psi = t \circ e_\xi$.
2. The equality $c \circ ((c \circ (\phi \times \psi)) \times \chi) = c \circ (\phi \times (c \circ (\psi \times \chi)))$ holds for any $\phi, \psi, \chi \in \text{Holo}(S, R)$ with $s \circ \phi = t \circ \psi$ and $s \circ \psi = t \circ \chi$.
3. For any $\phi \in \text{Holo}(S, R)$, there exists a $\psi \in \text{Holo}(S, R)$ such that $s \circ \phi = t \circ \psi = \xi, s \circ \psi = t \circ \phi = \eta$ and $c \circ (\phi \times \psi) = e_\eta, c \circ (\psi \times \phi) = e_\xi$.

This condition is equivalent to say that $\text{Holo}(S, X)$ forms an abstract groupoid whose morphisms $\xi \to \eta$ are $\phi \in \text{Holo}(S, R)$ with $s \circ \phi = \xi$ and $t \circ \phi = \eta$, and composition is given by $c$.

A holomorphic group action $a : X \times G \to X$ gives an example of holomorphic groupoid with $R = X \times G$, $s = p_1$, $t = a$ and $c = \text{id} \times \mu : X \times G \times G \to X \times G$. If $u : U \to X$ is a holomorphic morphism, then we can consider the pull-back holomorphic groupoid $(U, (U \times U) \times_{u \times_X u, u \times_X s \times X, s \times t} R, s', t', c')$.

**Example 6.4.** We denote by $[X/R]_p$ the fibred category (in groupoids) defined as follows.

1. Its objects are holomorphic morphisms $\xi : S \to X$ from some complex spaces $S$.
2. Its morphisms $\xi_S \to \eta_T$ are the pairs $(f, \phi)$ of holomorphic morphisms $f : S \to T$ and $\phi : S \to R$ satisfying $s \circ \phi = \xi$ and $t \circ \phi = \eta \circ f$.
3. Its functor $[X/R]_p \to \text{Can}$ maps objects $\xi_S$ to $S$ and morphisms $(f, \phi) : \xi_S \to \eta_T$ to $f : S \to T$.

Here is our interested fibred category from Definition 4.1.

**Lemma 6.5.** The category $\mathcal{K}_{T,\chi}$ and $\mathcal{K}_{T,\chi}^p$ forms a fibred category by the functor $\mathcal{K}_{T,\chi}^{(s)} \to \text{Can} : (\pi : \mathcal{M} \to S, \alpha) \mapsto S$.

This is just because the following cartesian diagram gives a cartesian morphism for any holomorphic morphism $f : X \to Y$ between complex spaces and any object $(\pi : \mathcal{M} \to Y, \alpha) \in \mathcal{K}_{T,\chi}^{(s)}$.

\[
\begin{array}{ccc}
X \times_Y \mathcal{M} & \longrightarrow & \mathcal{M} \\
\downarrow & \downarrow \pi \\
X & \longrightarrow & Y \end{array}
\]

The correspondence $(\pi : \mathcal{M} \to Y, \alpha) \mapsto (f^* \pi : X \times_Y \mathcal{M} \to X, f^* \alpha)$ gives a functor $\mathcal{K}_{T,\chi}(Y) \to \mathcal{K}_{T,\chi}(X)$. It looks like that this provides a functor $X \mapsto \mathcal{K}_{T,\chi}(X)$ from the category $\text{Can}$ to the “category” of groupoids, but actually does not. This nuisance comes from the set theoretical fact that $X \times_{f,Y} (Y \times_{g,Z} \mathcal{M}) \neq$
\(X \times_{\mathcal{M}} \mathcal{M}\); they are not exactly the same objects but just naturally isomorphic. This is the reason why we should formulate things in terms of fibred category.

**A-2. Descent data.** We introduce descent data of a fibred category over \(\text{Can}\).

**Definition 6.6** (descent data). Let \(p : \mathcal{F} \to \text{Can}\) be a fibred category, \(X\) be a complex space, \(\mathcal{U} := \{i_\alpha : U_\alpha \hookrightarrow X\}_{\alpha \in A}\) be an open cover of \(X\) (in the real topology). We denote by \(u_{\alpha,\beta} : U_\alpha \cap U_\beta \hookrightarrow U_\alpha\) the inclusion morphism to the first factor and by \(u_{\alpha,\beta,\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \hookrightarrow U_\alpha \cap U_\beta\) the inclusion morphism to the intersection of the first and second factor \((u_{\alpha,\beta,\gamma} = u_{\beta,\alpha,\gamma})\). Put \(A_2 := A^2/S_2\), \(A_3 := A^3/S_3\).

A descent datum of \(\mathcal{F}\) over \((X, \mathcal{U})\) consists of the following data \(\mathcal{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)\):

\[\Xi_1 := \{\xi_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha \in A}\]
\[\Xi_2 := \{\xi_{\alpha\beta} \in \mathcal{F}(U_\alpha \cap U_\beta)\}_{(\alpha, \beta) \in A_2}\]
\[\Xi_3 := \{\xi_{\alpha\beta\gamma} \in \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma)\}_{(\alpha, \beta, \gamma) \in A_3}\]

are sets of objects in \(\mathcal{F}\) and

\[\Theta_2 := \{\theta_{\alpha,\beta} : \xi_{\alpha\beta} \to \xi_\alpha \mid \theta_{\alpha,\beta}\ \text{is cartesian over } u_{\alpha,\beta}\}_{(\alpha, \beta) \in A_2}\]
\[\Theta_3 := \{\theta_{\alpha\beta,\gamma} : \xi_{\alpha\beta\gamma} \to \xi_{\beta}\ \mid \theta_{\alpha\beta,\gamma}\ \text{is cartesian over } u_{\alpha\beta,\gamma}\}_{(\alpha, \beta, \gamma) \in A_2 \times A}\]

are sets of cartesian morphisms in \(\mathcal{F}\). These data must satisfy

\[\theta_{\alpha,\beta} \circ \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\gamma} \circ \theta_{\gamma\alpha,\beta}\]

for any \(\alpha, \beta, \gamma \in A\).

\[\begin{array}{ccccccc}
\mathcal{F} & \xrightarrow{p} & \text{Can} \\
\xi_{\alpha\beta\gamma} & \xrightarrow{\theta_{\alpha\beta\gamma}} & \xi_{\beta\gamma} & \xrightarrow{\theta_{\beta\gamma}} & U_{\alpha\beta\gamma} & \xrightarrow{u_{\alpha\beta\gamma}} & U_{\beta\gamma} \\
\downarrow \phi_{\alpha\beta,\gamma} & & \downarrow \phi_{\beta,\gamma} & & \downarrow \phi_{\alpha,\beta} & & \downarrow \phi_{\alpha,\gamma} \\
\xi_{\alpha\beta} & \xrightarrow{\theta_{\alpha\beta}} & \xi_{\beta} & \xrightarrow{\theta_{\beta}} & U_{\alpha\beta} & \xrightarrow{u_{\alpha\beta}} & U_{\beta} \\
\downarrow \theta_{\alpha,\beta} & & \downarrow \theta_{\alpha} & & \downarrow \theta_{\alpha,\gamma} & & \downarrow \theta_{\alpha} \\
\xi_{\alpha} & \xrightarrow{\theta_{\alpha}} & \xi_{\alpha} & \xrightarrow{\theta_{\alpha}} & U_{\alpha} & \xrightarrow{u_{\alpha}} & X
\end{array}\]

A descent datum \(\mathcal{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)\) is called effective if there exists an object \(\xi \in \mathcal{F}(X)\) and a set of morphisms

\[\Theta_1 := \{\theta_\alpha : \xi_\alpha \to \xi \mid \theta_\alpha\ \text{is cartesian over } i_\alpha\}_{\alpha \in A}\]

satisfying

\[\theta_\alpha \circ \theta_{\alpha,\beta} = \theta_{\beta} \circ \theta_{\beta,\alpha}\]

for any \(\alpha, \beta \in A\). We define an effective descent datum of \(\mathcal{F}\) over \((X, \mathcal{U})\) to be an object consisting of data \(\mathcal{D}_+ = (\mathcal{D}, \xi, \Theta_1) = (\xi, \Xi_1, \Xi_2, \Xi_3, \Theta_1, \Theta_2, \Theta_3)\) as above.

**Remark 6.7.** Note that

- Every descent datum of \(\text{Can}_X\) is effective.
- There are descent data of \([X/G]_p\) which are not effective, in general.
• Every descent datum of \([X/G]\) is effective.

As for the second item, consider the fibred category \([([C^2 \setminus \{0\})/C^*]_p)\) for example. More explicitly, let \(U\) be an open cover of \(CP^1\) defined by two open subsets \(U_\alpha := \{(z_1 : z_2) \mid z_2 \neq 0\}, U_\beta := \{(z_1 : z_2) \mid z_1 \neq 0\}\) and let \(\xi_\alpha : U_\alpha \to C^2 \setminus \{0\}, \xi_\beta : U_\beta \to C^2 \setminus \{0\}\) be morphisms defined by \(\xi_\alpha(z_1 : z_2) := (z_1/z_2, 1), \xi_\beta(z_1 : z_2) := (1, z_2/z_1)\), respectively. Consider a descent datum over \((CP^1, U)\) with \(\Xi_1 := \{\xi_\alpha, \xi_\beta\}\) given by an obvious way. In order to be effective, this descent datum should define a non-constant morphism \(CP^1 \to C^2 \setminus \{0\}\) which is isomorphic (not equal) to \(\xi_\alpha, \xi_\beta\) when restricted to each open set, but this is impossible as every holomorphic map \(CP^1 \to C^2 \setminus \{0\}\) is constant. So this descent datum is not effective in this fibred category.

On the other hand, the corresponding descent datum in the fibred category \([([C^2 \setminus \{0\})/C^*])\) becomes effective, completed by the object \(CP^1 \xrightarrow{\id} ([C^2 \setminus \{0\})/C^*])\) in \([([C^2 \setminus \{0\})/C^*])\). Actually, \([([C^2 \setminus \{0\})/C^*])\) is isomorphic to \(CP^1\) as fibred categories.

**Definition 6.8** (descent data). Let \(\mathcal{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)\), \(\mathcal{D}' = (\Xi'_1, \Xi'_2, \Xi'_3, \Theta'_2, \Theta'_3)\) be two descent data of \(\mathcal{F}\) over \((X, U)\). A morphism from \(\mathcal{D}\) to \(\mathcal{D}'\) is a triple \(\Phi = (\Phi_1, \Phi_2, \Phi_3)\) of sets of morphisms

\[
\Phi_1 := \{\phi_\alpha : \xi_\alpha \to \xi'_\alpha \in \mathcal{F}(U_\alpha) \mid \xi_\alpha \in \Xi_1, \xi'_\alpha \in \Xi'_1\}_{\alpha \in A}
\]

\[
\Phi_2 := \{\phi_{\alpha\beta} : \xi_{\alpha\beta} \to \xi'_{\alpha\beta} \in \mathcal{F}(U_\alpha \cap U_\beta) \mid \xi_{\alpha\beta} \in \Xi_2, \xi'_{\alpha\beta} \in \Xi'_2\}_{(\alpha, \beta) \in A_2}
\]

\[
\Phi_3 := \{\phi_{\alpha\beta\gamma} : \xi_{\alpha\beta\gamma} \to \xi'_{\alpha\beta\gamma} \in \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma) \mid \xi_{\alpha\beta\gamma} \in \Xi_3, \xi'_{\alpha\beta\gamma} \in \Xi'_3\}_{(\alpha, \beta, \gamma) \in A_3}
\]

in \(\mathcal{F}\) satisfying

\[
\phi_\alpha \circ \theta_{\alpha, \beta} = \theta_{\alpha', \beta} \circ \phi_{\alpha\beta} \quad \text{and} \quad \phi_{\alpha\beta} \circ \theta_{\alpha, \beta, \gamma} = \theta_{\alpha', \beta, \gamma} \circ \phi_{\alpha\beta\gamma}
\]

for all \(\alpha, \beta, \gamma \in A\). Descent data naturally form a category with these morphisms. We denote by \(\mathcal{F}_{\text{des}}(X, U)\) the category of descent data.

Let \(\mathcal{D}_+ = (\mathcal{D}, \xi, \Theta_1)\), \(\mathcal{D}'_+ = (\mathcal{D}', \xi', \Theta'_1)\) be two effective descent data of \(\mathcal{F}\) over \((X, U)\). A *morphism* from \(\mathcal{D}_+\) to \(\mathcal{D}'_+\) is a quadruple \(\Phi_+ = (\phi, \Phi_1, \Phi_2, \Phi_3)\) where \((\Phi_1, \Phi_2, \Phi_3)\) gives a morphism of corresponding descent data and \(\phi : \xi \to \xi'\) is a morphism in \(\mathcal{F}(X)\) satisfying

\[
\phi \circ \theta_\alpha = \theta_{\alpha, \beta} \circ \phi_\alpha
\]

for any \(\alpha \in A\). We denote by \(\mathcal{F}_{\text{eff}}(X, U)\) the category of effective descent data.

**A-3. Can-stack.** We can consider the forgetful functors \(\mathcal{F}_{\text{eff}}(X, U) \to \mathcal{F}_{\text{des}}(X, U)\) defined by \(\mathcal{D}_+ = (\mathcal{D}, \xi, \Theta_1) \mapsto \mathcal{D}\) and \(\mathcal{F}_{\text{eff}}(X, U) \to \mathcal{F}(X)\) defined by \(\mathcal{D}_+ = (\mathcal{D}, \xi, \Theta_1) \mapsto \xi\). The latter functor \(\mathcal{F}_{\text{eff}}(X, U) \to \mathcal{F}(X)\) is fully faithful and essentially surjective. Therefore there is an inverse functor \(\mathcal{F}(X) \to \mathcal{F}_{\text{eff}}(X, U)\) (assuming the axiom of global choice). As for our fibred category \(\mathcal{K}_{T, X}\), there is a canonical choice\(^2\) of the inverse functor defined by

\[
(\pi : M \to S) \mapsto (M \to S, \{\pi^{-1}(U_\alpha) \to U_\alpha\}_\alpha, \{\pi^{-1}(U_\alpha \cap U_\beta) \to U_\alpha \cap U_\beta\}_{\alpha, \beta, \ldots})
\]

\(^2\)This is well-defined because \(U_\alpha \cap U_\beta = U_\beta \cap U_\alpha\) as complex spaces, in particular as sets. On the other hand, \(U_\alpha \times_X U_\beta \neq U_\beta \times_X U_\alpha\) as sets, though they are canonically isomorphic, because of the set theoretical fact \((a, b) = \{(a), (a, b)\} \neq \{(b), (b, a)\} = (b, a)\).
However, in general there is no canonical choice of this inverse functor; there needs an additional choice of \((\mathcal{O}, \Theta_1)\) compatible to \(\xi\), which is not unique as object but unique only up to isomorphisms.

**Definition 6.9 (Can-stack).** A fibred category \(p : \mathcal{F} \to \text{Can} \) is called a **Can-stack** if it satisfies the following two conditions for any complex space \(X\) and any open cover \(U\) of \(X\).

1. The functor \(\mathcal{F}_{\text{eff}}(X,U) \to \mathcal{F}_{\text{des}}(X,U)\) is fully faithful.
2. The functor \(\mathcal{F}_{\text{eff}}(X,U) \to \mathcal{F}_{\text{des}}(X,U)\) is essentially surjective.

**Remark 6.10.** If we have a choice of pull back \(f \mapsto f^*\xi\) with a morphism \(f^*\xi \to \xi\) so that it is cartesian over \(f : S \to X\), we can consider a contravariant functor defined by

\[
\text{Mor}_X(\xi, \eta) : \text{Can}^\text{op}_X \to \text{Sets} : (f : S \to X) \mapsto \text{Hom}_{\mathcal{F}(S)}(f^*\xi, f^*\eta),
\]

where \(\text{Can}^\text{op}_X\) stands for the opposite category of \(\text{Can}_X\). Then the first condition of the above definition is equivalent to say that the functor \(\text{Mor}_X(\xi, \eta)\) is a sheaf on the site \(\text{Can}_X\).

It is customary to denote by \(\text{Isom}_X(\xi, \eta)\) the functor \(\text{Mor}_X(\xi, \eta)\) when \(\mathcal{F}\) is a category fibred in groupoids, as every morphism in \(\mathcal{F}(X)\) is an isomorphism.

We can consider a related fibred category \(\text{Mor}_X(\xi, \eta)\) (in sets) without a choice of pull back. The category consists of objects \((f_S, \phi_\xi, \phi_\eta)\) where \(f : S \to X\) is a holomorphic morphism of complex spaces and \(\phi_\xi : \xi_S \to \xi, \phi_\eta : \eta_S \to \eta \in \text{Mor}(\mathcal{F})\) are cartesian arrows over \(f\). Its morphisms \((f_S, \phi_\xi, \phi_\eta) \to (f_{S'}, \phi_\xi', \phi_\eta')\) are triples \((g, \psi_\xi, \psi_\eta)\) where \(g : S \to S'\) is a holomorphic morphism of complex spaces and \(\psi : \xi_S \to \xi_{S'}, \psi_\eta : \eta_S \to \eta_{S'}\) are cartesian arrows over \(g\) satisfying \(f_S = f_{S'} \circ g\), \(\phi_\xi = \phi_\xi' \circ \psi_\xi\) and \(\phi_\eta = \phi_\eta' \circ \psi_\eta\).

For any fibred category, we can always associate a stack in a canonical way. Here is the fact from [SPA] TAG 02ZN, 0435.

**Proposition 6.11.** Let \(p : \mathcal{F} \to \text{Can}\) be a fibred category. Suppose we have a choice of pull back \((f, \xi) \mapsto f^*\xi\) (just for simplicity). Then there exists a Can-stack \(\mathcal{F}'\) (with a choice of pull back) and a morphism \(s : \mathcal{F} \to \mathcal{F}'\) of fibred categories with the following properties.

1. For every complex space \(X\) and any \(\xi, \eta \in \text{Obj}(\mathcal{F}(X))\), the morphism of presheaf \(\text{Mor}_X(\xi, \eta) \to \text{Mor}_X(s(\xi), s(\eta))\) is a sheafification of \(\text{Mor}_X(\xi, \eta)\).
2. For every complex space \(X\) and any \(\xi' \in \text{Obj}(\mathcal{F}'(X))\), there exists an open cover \(U = \{i_\alpha : U_\alpha \to X\}_\alpha\) of \(X\) such that \(i_\alpha^*\xi'\) is isomorphic to \(s(\xi'_\alpha)\) for some \(\xi'_\alpha \in \text{Obj}(\mathcal{F})\) for every \(\alpha\).
3. Given a Can-stack \(\mathcal{G}\) and a morphism \(g : \mathcal{F} \to \mathcal{G}\) of fibred categories, there exists a morphism \(g' : \mathcal{F}' \to \mathcal{G}\) of fibred categories such that there exists a 2-isomorphism between \(g\) and \(g' \circ s\).

The last property actually follows from the first two properties. A stack \(\mathcal{F}'\) with the last property is called a **stackification** of \(\mathcal{F}\) and the stack \(\mathcal{F}'\) constructed in the proof of this proposition as the **stackification** of \(\mathcal{F}\) (a fixed construction is in mind). We denote by \([X/G]\), \([X/R]\) the stackification of the fibred category \([X/G]_p\), \([X/R]_p\) respectively and call them the **quotient stack**. The stack \([X/G]\) is a stackification of the fibred category \([X/G]_p\), so it is (canonically) isomorphic to the stackification \([X/G]\).
A 2-fibre product \([\text{SPA}_003Q]\) of fibred categories can be calculated as follows. We refer to this construction as the 2-fibre product of fibred categories.

**Proposition 6.12.** Let \(\mathcal{F}, \mathcal{G}, \mathcal{H}\) be fibred categories over \(\text{Can}\) and \(f : \mathcal{F} \rightarrow \mathcal{H}, g : \mathcal{G} \rightarrow \mathcal{H}\). The fibred category \(\mathcal{E}\) defined as follows enjoys the universal property of 2-fibre product.

1. An object of \(\mathcal{E}\) is a quadruple \((X, \xi, \eta, \phi)\) where \(X\) is a complex space, \(\xi\) is an object in \(\mathcal{F}(X)\), \(\eta\) is an object in \(\mathcal{G}(X)\) and \(\phi : f(\xi) \rightarrow g(\eta)\) is an isomorphism in \(\mathcal{H}(X)\).

2. A morphism \((X, \xi, \eta, \phi) \rightarrow (Y, \xi', \eta', \phi')\) is a pair \((\sigma, \tau)\) where \(\sigma : X \rightarrow Y\) and \(\tau : \eta \rightarrow \eta'\) is a morphism in \(\mathcal{G}\) satisfying \(p(\sigma) = p(\tau) : X \rightarrow Y\) and \(g(\tau) \circ \phi = \phi' \circ f(\sigma)\).

When \(\mathcal{F}, \mathcal{G}, \mathcal{H}\) are all \(\text{Can}\)-stacks, the stackification of the fibred category \(\mathcal{E}\) is denoted by \(\mathcal{F} \times_f \mathcal{G}\). The \(\text{Can}\)-stack \(\mathcal{F} \times_f \mathcal{G}\) satisfies the universal property of 2-fibre product in the 2-category of \(\text{Can}\)-stacks.

The following verifying process might help the readers’ better understanding of the notion of descent. See Definition 4.1 for the definition of \(\mathcal{K}_{T,\chi}\).

**Lemma 6.13.** The fibred categories \(\mathcal{K}_{T,\chi}, \mathcal{K}_{T,\chi}^r\) are \(\text{Can}\)-stacks.

**Proof.** For abbreviation, we let \(\mathcal{M}\) stand for \((\pi : \mathcal{M} \rightarrow S, \alpha) \in \mathcal{K}_{T,\chi}\). Let \(S\) be a complex space, \(\mathcal{U} = \{U\}_\alpha\) be an open cover of \(S\) and \(\mathcal{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)\) be a descent datum of \(\mathcal{K}_{T,\chi}\) over \((X, \mathcal{U})\). Since \(\theta_{\alpha, \beta} \in \Theta_2\) is cartesian, it induces an isomorphism

\[\tilde{\theta}_{\alpha, \beta} : \mathcal{M}_{\alpha \beta} \xrightarrow{\sim} \mathcal{M}_{\alpha \beta | U_{\alpha \cap U_{\beta}}}\]

So we obtain an isomorphism

\[\theta'_{\beta \alpha} := \tilde{\theta}_{\beta, \alpha} \circ \tilde{\theta}_{\alpha, \beta}^{-1} : \mathcal{M}_{\alpha \beta | U_{\alpha \cap U_{\beta}}} \xrightarrow{\sim} \mathcal{M}_{\beta \alpha | U_{\alpha \cap U_{\beta}}}\]

Similarly, we obtain an isomorphism

\[(\tilde{\theta}_{\beta, \alpha} | U_{\alpha \beta} \circ \tilde{\theta}_{\alpha, \beta} | U_{\alpha \beta} \circ \tilde{\theta}_{\alpha, \beta, \gamma})^{-1} : \mathcal{M}_{\alpha \beta | U_{\alpha \cap U_{\beta}}} \xrightarrow{\sim} \mathcal{M}_{\beta \alpha | U_{\alpha \cap U_{\beta}}}\]

which we denote by \(\theta'_{\beta \alpha, \gamma}\), from the cartesian arrow \(\theta_{\alpha, \beta, \gamma} \in \Theta_3\).

From the condition \(\theta_{\alpha, \beta, \gamma} \circ \theta'_{\beta \alpha, \gamma} = \theta_{\gamma, \alpha, \beta} \circ \theta'_{\beta \alpha, \gamma}\), we obtain \(\theta'_{\beta \alpha, \gamma} \circ \theta'_{\beta \alpha, \gamma} = \theta'_{\gamma, \alpha, \beta}\) and \(\theta'_{\beta \alpha, \gamma} | U_{\alpha \beta} \circ \theta'_{\beta \alpha, \gamma} = \theta'_{\beta \alpha, \gamma}\). So we can glue \(\mathcal{M}_{\alpha}\) together by gluing maps \(\theta'_{\beta \alpha}\) and obtain a complex space \(\mathcal{M}\) with a natural set of morphisms \(\Theta_1 := \{\theta_{\alpha} : \mathcal{M}_{\alpha} \rightarrow \mathcal{M}\}_{\alpha \in A}\) such that \((\mathcal{M}, \Xi_1, \Xi_2, \Xi_3, \Theta_1, \Theta_2, \Theta_3)\) is an effective descent datum. Therefore the forgetful functor \(\mathcal{F}_{\text{eff}}(S, \mathcal{U}) \rightarrow \mathcal{F}_{\text{des}}(S, \mathcal{U})\) is essentially surjective.

It is easy to see that \(\text{Mor}_S(\mathcal{M}, \mathcal{M}')\) is a sheaf on the site \(\text{Can}_X\). \(\square\)

**Example 6.14.** This example is cited from \([\text{Alp1}_003Q]\) Example 8.2] and must help the readers to understand the nontriviality of gluing good moduli spaces. Consider the \(\mathbb{C}^*\)-action on \(\mathbb{C}^2\) by the scalar multiplication. The quotient stack \([(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*]\) is naturally an open substack of the quotient stack \([\mathbb{C}^2/\mathbb{C}^*]\). Both stacks admit good moduli spaces \([(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*] \rightarrow \mathbb{C}P^1, [\mathbb{C}^2/\mathbb{C}^*] \rightarrow \mathbb{C}^2 / \mathbb{C}^* = \text{pt} \) respectively. Inspite of the openness of the morphism \([(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*] \rightarrow [\mathbb{C}^2/\mathbb{C}^*], the induced morphism \(\mathbb{C}P^1 \rightarrow \text{pt}\) of good moduli spaces is not open.
B. The demonstrations of (5)-(10) in the proof of Proposition 3.1. Let $\omega$ be a symplectic form, $J$ be a $\omega$-compatible almost complex structure, $g = \omega(-, J-)$ be the Riemannian metric associated to $(\omega, J)$ on a manifold $M$. Let $A$ be an endomorphism of the tangent bundle $TM$ satisfying $JA + AJ = 0$ and $\omega(AX, Y) = \omega(AY, X)$.

On a local coordinate, we denote by $\omega^{kj}$ the tensor satisfying $\omega^{kj}\omega_{ij} = \delta_i^k$ and by $g^{kj}$ the tensor satisfying $g^{kj}g_{ij} = \delta_i^k$. The following rules are basic for our calculations.

A. (a) $\omega_{ij} = -\omega_{ji}$, (b) $J^i_j J^j_k = -\delta^i_k$, (c) $g_{ij} = g_{ji}$.
B. (a) $\omega_{ij} = g_{pq} J^p_i J^q_j = -g_{ij}$, (b) $g_{ij} = \omega_{pq} J_p^i J_q^j = -\omega_{ij} J_i^j$.
C. (a) $\omega^{kj} = -g_{pq} J_p^k J^q_j = -\omega^{kj}$, (b) $g^{kj} = \omega_{pq} J^k_p J^k_q = -\omega^{kj} J_p^j = g^{jk}$.
D. (a) $\omega^{kj}\omega_{ij} = \omega^{kj} \delta_i^j = \delta_i^k$, (b) $g^{kj} g_{ij} = g^{kj} g_{ji} = \delta_i^k$.
E. $f_j = -X^i_j g_{ij} = -X^i_j g_{pi} J^p_i$, $f_j g_{ij} = -J^j_i g_{ij}$.
F. $X^i_j = -f_j \omega^{kj}$, $J^k_j = -f_j J^k_i$.
G. (a) $(JA)^i_k = J^i_k A^j_p = -J^i_p A^j_k$, (b) $\omega_{kj} A^k_i = \omega_{kj} A^j_k$.
H. $g^{ij} g_{ik}(JA)^l_j = (JA)^l_j$.

Now we verify (5)-(10) in detail. As for (5),

\[
(Jd\theta_\xi \otimes X_f, JA)_\xi = \int_M g^{ij} g_{ik}(Jd\theta_\xi \otimes X_f)^k_j (JA)^l_j e^{-2\theta_\xi} \omega^n = \int_M (Jd\theta_\xi \otimes X_f)^k_j (JA)^l_j e^{-2\theta_\xi} \omega^n = \int_M -\theta_\xi, f (f \omega^{jk} A^j_k) e^{-2\theta_\xi} \omega^n = \int_M -\theta_\xi, f \omega^{jk} A^j_k e^{-2\theta_\xi} \omega^n = \frac{d}{dt} \bigg|_{t=0} \int_M g^{\ast} (d\theta_\xi, df) e^{-2\theta_\xi} \omega^n = \frac{d}{dt} \bigg|_{t=0} ((\Delta_t + (-2) J_t \theta_\xi, f_\xi),
\]

where the last equality comes from the following calculation

\[
(d\theta_\xi, df)_\xi = (e^{-2\theta_\xi} d\theta_\xi, df) = (d^* (e^{-2\theta_\xi} d\theta_\xi), f) = (-d (e^{-2\theta_\xi} \ast (d\theta_\xi)), f) = (-2) e^{-2\theta_\xi} \ast (d\theta_\xi \wedge \ast (d\theta_\xi), f) + (d^* d\theta_\xi, f)_\xi = \frac{d}{dt} (g^{ij} \theta_\xi, \ast (d\theta_\xi)_j, f_\xi + (\Delta \theta_\xi, f)_\xi = \frac{d}{dt} ((-2) (g^{ij} \theta_\xi, \ast (d\theta_\xi)_j, f_\xi + (\Delta \theta_\xi, f)_\xi = \frac{d}{dt} ((-2) (-\xi^k J^{j_k}_i \theta_\xi, j_\xi + (\Delta \theta_\xi, f)_\xi = ((-2) J_\xi \theta_\xi, f_\xi) + (\Delta \theta_\xi, f)_\xi.\]
Similarly, we can arrange (6) as follows:

\[-(d\theta \otimes JX_f, JA)_\xi = - \int M (d\theta \otimes JX_f)_k^i (JA)_k^j e^{-2\theta \omega^n} \]
\[= - \int M \theta_{\xi, i} J^k_m X^m_f J^j_p A^p_k e^{-2\theta \omega^n} \]
\[= \int M J^p_p \theta_{\xi, i} X^m_f J^j_p A^p_m e^{-2\theta \omega^n} \]
\[= \int M (Jd\theta \otimes JX_f)_p^m (JA)_m^e e^{-2\theta \omega^n} \]
\[= (Jd\theta \otimes X_f, JA)_\xi. \]

The rest of them are calculated as follows:

(7)

\[(Jdf \otimes \xi, JA)_\xi = \int M g^{ij} g_{kl} (Jdf \otimes \xi)_k^i (JA)_l^j e^{-2\theta \omega^n} \]
\[= \int M g^{ij} g_{kl} (X^m_p \theta_{\xi, r} g^{sr} J^k_s) (Jdf)_l^j e^{-2\theta \omega^n} \]
\[= \int M g_{kl} X^m_f \theta_{\xi, r} g^{ks} J^r_s (JA)_l^j e^{-2\theta \omega^n} \]
\[= \int M \theta_{\xi, r} J^p_p X^m_f J^r_r^j A^p_m e^{-2\theta \omega^n} \]
\[= \int M (d\theta \otimes JX_f)_p^m (JA)_m^e e^{-2\theta \omega^n} \]
\[= -(d\theta \otimes X_f, JA)_\xi. \]

(8)

\[-(df \otimes J\xi, JA)_\xi = - \int M (df \otimes JX_f)_k^i (JA)_k^j e^{-2\theta \omega^n} \]
\[= - \int M f_{ij} J^k_m X^m_f J^j_p A^p_k e^{-2\theta \omega^n} \]
\[= \int M J^j_p J^p_r X^m_f J^r_s A^p_m e^{-2\theta \omega^n} \]
\[= \int M (Jdf \otimes JX_f)_p^m (JA)_m^e e^{-2\theta \omega^n} \]
\[= (Jdf \otimes X_f, JA)_\xi. \]
\[(f J d\theta \otimes \xi, JA)_\xi = \int_M g^{ij} g_{kl} (J d\theta \otimes \xi)^k_i (JA)^l_j f e^{-2\theta} \omega^n
\]
\[= \int_M (J d\theta \otimes \xi)^k_i (JA)^l_j f e^{-2\theta} \omega^n
\]
\[= \int_M -\theta \xi^k A^k_p f e^{-2\theta} \omega^n
\]
\[= \frac{d}{dt} \bigg|_{t=0} \int_M (-J I \xi) \theta \xi f e^{-2\theta} \omega^n
\]
\[= \frac{d}{dt} \bigg|_{t=0} \int_M (f J d\theta \otimes \xi, JA)_\xi.
\]

\[-(f d\theta \otimes J_\xi, JA)_\xi = -\int_M (d\theta \otimes J_\xi)^k_i (JA)^l_j f e^{-2\theta} \omega^n
\]
\[= -\int_M \theta \xi^k J^m_j \xi^m J^i_p A^p_k f e^{-2\theta} \omega^n
\]
\[= \int_M J^i_p \theta \xi^k \xi^m J^p_k A^m_i f e^{-2\theta} \omega^n
\]
\[= \int_M (J d\theta \otimes \xi)^m_p (JA)^p_m f e^{-2\theta} \omega^n
\]
\[= (f J d\theta \otimes \xi, JA)_\xi.
\]

We have done it.

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