Robust Covariance Matrix Estimation via Matrix Depth

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Abstract

Covariance matrix estimation is one of the most important problems in statistics. To accommodate the complexity of modern datasets, it is desired to have estimation procedures that not only can incorporate the structural assumptions of covariance matrices, but are also robust to outliers from arbitrary sources. In this paper, we define a new concept called matrix depth and we propose a robust covariance matrix estimator by maximizing the empirical depth function. The proposed estimator is shown to achieve minimax optimal rate under Huber’s \(\epsilon\)-contamination model for estimating covariance/scatter matrices with various structures including bandedness and sparsity.

Keywords. Data depth, Minimax rate, High dimensional statistics, Outliers, Contamination model

1 Introduction

Covariance matrix estimation is one of the most important problems in statistics. The last decade has witnessed the rapid development of statistical theory for covariance matrix estimation under high dimensional settings. Starting from the seminal works of Bickel and Levina [2, 3], covariance matrices with a list of different structures can be estimated with optimal theoretical guarantees. Examples include bandable matrix [6], sparse matrix [28, 5], Toeplitz matrix [8] and spiked matrix [4, 10]. For a recent comprehensive review on this topic, see [9]. However, these works do not take into account the heavy-tailedness of data and the possible presence of outliers. All these methods are based on sample covariance matrix, which is shown to have a \(1/(n+1)\) breakdown point [20]. This means that even if there exists only one arbitrary outlier in the whole dataset, the statistical performance of the estimator can be totally compromised. In this paper, we attempt to tackle the problems of robust covariance matrix estimation under high dimensional settings.

To be more specific, we consider the distribution \((1 - \epsilon)N(0, \Sigma) + \epsilon Q\), where \(Q\) is an arbitrary distribution that models the outliers and \(\epsilon\) is the proportion of contamination.
Given i.i.d. observations \( X_1, \ldots, X_n \) from this distribution, there are approximately \( n\epsilon \) of them distributed according to \( Q \), which can influence the performance of an estimator without robust property. This setting is called \( \epsilon \)-contamination model, first proposed in a path-breaking paper by Huber [26]. In this paper, Huber proposed a robust location estimator and proved its minimax optimality under the \( \epsilon \)-contamination model. His work suggests an estimator that is optimal under the \( \epsilon \)-contamination model must achieve statistical efficiency and resistance to outliers simultaneously. Therefore, we view the \( \epsilon \)-contamination model as a natural framework to develop theories of robust estimation of covariance matrices. The goal of this paper is to propose an estimator of \( \Sigma \) that works optimally for the contaminated data.

To obtain a robust covariance matrix estimator, we propose a new concept called matrix depth. For a \( p \)-variate distribution \( X \sim \mathbb{P} \), the matrix depth of a positive semi-definite \( \Gamma \in \mathbb{R}^{\times p} \) with respect to \( \mathbb{P} \) is defined as

\[
\mathcal{D}(\Gamma, \mathbb{P}) = \min_{||u||=1} \min \left\{ \mathbb{P}\{|u^T X|^2 \leq u^T \Gamma u\}, \mathbb{P}\{|u^T X|^2 \geq u^T \Gamma u\} \right\}.
\]

(1)

We will show that for \( \mathbb{P} = N(0, \Sigma) \), the deepest matrix is \( \beta \Sigma \) for some constant multiplier \( \beta > 0 \). Thus, a natural estimator for \( \Sigma \) is \( \hat{\Gamma}/\beta \) with \( \hat{\Gamma} = \arg \max_{\Gamma \succeq 0} \mathcal{D}(\Gamma, \mathbb{P}_n) \). Here, we use the notation \( \mathbb{P}_n \) to denote the empirical distribution.

Our definition of matrix depth is parallel to Tukey’s depth function [40] for a location parameter. The deepest vector according to Tukey’s depth is a natural extension of median in the multivariate setting, and thus can be used as a robust location estimator. Zuo and Serfling [52] advocated the notion of statistical depth function that satisfies the four properties in [29] and verified that Tukey’s depth indeed satisfies all these properties while many other depth functions [29, 37, 38, 44] do not. The multivariate median defined by Tukey’s depth was shown to have high breakdown point [12, 15, 14]. The original proposal of the depth function in [40] not only provides a way for robust location estimation, but also gives a general way to summarize multivariate data. For example, the depth function can be used to define an index of scatteredness of data [53]. Based on the concept of data depth, a data peeling procedure has been proposed to estimate the covariance matrix. Specifically, one may trim the data points according to their depths and use the remaining ones to estimate the covariance [12, 30]. Though the notion of Tukey’s depth is closely related to covariance matrix estimation, there has not been a concept of depth function directly defined on positive semi-definite matrices, to the best of our knowledge. The matrix depth defined in (1) offers such an option. It provides an alternative way and perhaps a more natural way for robust covariance matrix estimation than the data peeling procedure. Later, we will define several variants of the matrix depth that take into account the high dimensional structures such as bandedness and sparsity. Those matrix depth functions are powerful tools for robust estimation of structured covariance matrices.

We apply the proposed robust matrix estimator to the problems of estimating banded covariance matrices, bandable covariance matrices, sparse covariance matrices and sparse principal components. We show that in all these cases, the estimators defined by the matrix depth functions achieve the minimax rates of the corresponding \( \epsilon \)-contamination models un-
nder the operator norm. Therefore, the new estimators enjoy both statistical efficiency and property of resistance to outliers. Interestingly, the minimax rates have a unified expression. That is, $M(\epsilon) \asymp M(0) \lor \omega(\epsilon, \mathcal{F})$, where $M(\epsilon)$ is the minimax rate for the probability class of distributions $(1 - \epsilon)N(0, \Sigma) + \epsilon Q$ ranging over $\Sigma \in \mathcal{F}$ for some covariance matrix class $\mathcal{F}$ and all probability distributions $Q$. The first part $M(0)$ is the classical minimax rate without contamination. The second part is determined by the quantity $\omega(\epsilon, \mathcal{F})$ called modulus of continuity. Its definition goes back to the fundamental works of Donoho and Liu [16] and Donoho [13]. A high level interpretation is that the least favorable contamination distribution $Q$ can be chosen in a way that the parameters within $\omega(\epsilon, \mathcal{F})$ under a given loss cannot be distinguished from each other. We establish this phenomenon rigorously through a general lower bound argument for all $\epsilon$-contamination models.

Besides $\epsilon$-contamination models with Gaussian distributions, we show that our proposed estimators also work for general elliptical distributions. To be specific, the setting $(1 - \epsilon)P_\Gamma + \epsilon Q$ is also considered, where $\Gamma$ is the scatter matrix under the canonical representation of an elliptical distribution. In fact, a characteristic property of the scatter matrix $\Gamma$ of an elliptical distribution is $\mathcal{D}(\Gamma, P_\Gamma) = 1/2$. This property allows the depth function to combine naturally with the elliptical family. The resulting estimators are also shown to achieve the optimal convergence rates. To this end, we can claim that the proposed estimator by matrix depth have two extra robust properties besides its statistical efficiency: resistance to outliers and insensitivity to heavy-tailedness. In fact, there are many works in the literature on scatter matrix estimation for elliptical distributions, including [34, 42] in classical settings and [48, 21, 22, 47, 18, 24, 23, 36, 49] in high dimensional settings. However, the influence of the presence of outliers are not quantified in these works. It still remains open whether these estimators can achieve the minimax rates of the $\epsilon$-contamination models.

Finally, we advocate the use of Huber’s $\epsilon$-contamination model in the analysis of robust estimators by comparing it to the notion of breakdown point. The breakdown point [25, 20, 12] is defined as the smallest proportion of outliers that can totally ruin a given estimator. The concept by itself does not take into account any other statistical property besides robustness. Therefore, robust estimators are proposed to have both high breakdown point and certain statistical properties such as affine invariance [14, 31] and efficiency [50]. In comparison, the $\epsilon$-contamination model is a setting where a successful estimator must achieve statistical efficiency and robustness simultaneously. By considering a population counterpart of the breakdown point which we term as $\delta$-breakdown point, we show that for a given estimator that has convergence rate $\delta$ under the $\epsilon$-contamination model, its $\delta$-breakdown point is at least $\epsilon$. This suggests convergence under Huber’s $\epsilon$-contamination model is a more general notion of robustness than the breakdown point and it provides a unified way to study statistical efficiency and robustness jointly.

The paper is organized as follows. First, we revisit Tukey’s location depth in Section 2 and discuss the statistical property of the associated multivariate median. The matrix depth is introduced in Section 3 and we use it as a tool to solve various robust structured covariance matrix estimation problems. Section 4 presents a general result on minimax lower bound for
the $\epsilon$-contamination model. In Section 5, we discuss the relationship between matrix depth and elliptical distributions. Results of covariance matrix estimation are extended to scatter matrix estimation for elliptical distributions. In Section 6, we discuss some related topics on robust statistics including the connection between breakdown point and the $\epsilon$-contamination model. All proofs of the theoretical results are given in Section 7 and the supplementary material.

We close this section by introducing some notation. Given an integer $d$, we use $[d]$ to denote the set $\{1, 2, \ldots, d\}$. For a vector $u = (u_i)$, $\|u\| = \sqrt{\sum_i u_i^2}$ denotes the $\ell_2$ norm. For a matrix $A = (A_{ij})$, we use $s_k(A)$ to denote its $k$th singular value. The largest and the smallest singular values are denoted as $s_{\max}(A)$ and $s_{\min}(A)$, respectively. The operator norm of $A$ is denoted by $\|A\|_{\text{op}} = s_{\max}(A)$ and the Frobenius norm by $\|A\|_{\text{F}} = \sqrt{\sum_{ij} A_{ij}^2}$. When $A = A^T \in \mathbb{R}^{p \times p}$ is symmetric, $\text{diag}(A)$ means the diagonal matrix whose $(i, i)$th entry is $A_{ii}$. Given a subset $J \subset [p]$, $A_{JJ}$ is an $|J| \times |J|$ submatrix, where $|J|$ means the cardinality of $J$.

The set $S_p^{-1} = \{u \in \mathbb{R}^p : \|u\|_1 = 1\}$ is the unit sphere in $\mathbb{R}^p$. Given two numbers $a, b \in \mathbb{R}$, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. For two positive sequences $\{a_n\}, \{b_n\}$, $a_n \lesssim b_n$ means $a_n \leq C b_n$ for some constant $C > 0$ independent of $n$, and $a_n \asymp b_n$ means $a_n \lesssim b_n$ and $b_n \lesssim a_n$. Given two probability distributions $\mathbb{P}, \mathbb{Q}$, the total variation distance is defined by $\text{sup}_B |\mathbb{P}(B) - \mathbb{Q}(B)|$, and the Kullback-Leibler divergence is defined by $D(\mathbb{P}||\mathbb{Q}) = \int \log \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{P}$.

Throughout the paper, $C, c$ and their variants denote generic constants that do not depend on $n$. Their values may change from line to line.

## 2 Prologue: Robust Location Estimation

We start by the problem of robust location estimation. Consider i.i.d. observations $X_1, \ldots, X_n \sim \mathbb{P}_{(\epsilon, \theta, Q)} = (1 - \epsilon)P_{\theta} + \epsilon Q$, where $P_{\theta} = N(\theta, I_p)$. The goal is to estimate the location parameter $\theta$ from the contaminated data $\{X_i\}_{i=1}^n$. It is known that the sample average will fail because of its sensitivity to outliers. We consider Tukey’s median ([39, 40], see [41] as well) as a robust estimator of the location $\theta$. First, we need to introduce Tukey’s depth function. For any $\eta \in \mathbb{R}^p$ and a distribution $\mathbb{P}$ on $\mathbb{R}^p$, the Tukey’s depth of $\eta$ with respect to $\mathbb{P}$ is defined as

$$D(\eta, \mathbb{P}) = \min_{u \in S_p^{-1}} \mathbb{P}\{u^T X \leq u^T \eta\}, \quad \text{where } X \sim \mathbb{P}.$$ 

Given i.i.d. observations $\{X_i\}_{i=1}^n$, the Tukey’s depth of $\eta$ with respect to the observations $\{X_i\}_{i=1}^n$ is defined as

$$D(\eta, \{X_i\}_{i=1}^n) = D(\eta, \mathbb{P}_n) = \min_{u \in S_p^{-1}} \frac{1}{n} \sum_{i=1}^n \mathbb{I}\{u^T X_i \leq u^T \eta\},$$

where $\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ is the empirical distribution. Then, Tukey’s median is defined to be the deepest point with respect to the observations, i.e.,

$$\hat{\theta} = \arg \max_{\eta \in \mathbb{R}^p} D(\eta, \{X_i\}_{i=1}^n). \quad \text{(2)}$$
When (2) has multiple maxima, \( \hat{\theta} \) is understood as any vector that attains the deepest level. The statistical property of \( \hat{\theta} \) is stated in the following theorem.

**Theorem 2.1.** Consider Tukey’s median \( \hat{\theta} \). Assume \( \epsilon < 1/4 \) and \( p/n < c \) for some sufficiently small constant \( c \). Then, we have

\[
\|\hat{\theta} - \theta\|^2 \leq C \left( \frac{p}{n} \vee \epsilon^2 \right),
\]

with \( P_{(\epsilon, \theta, Q)} \)-probability at least \( 1 - \exp\left(-C'(p + n\epsilon^2)\right) \) uniformly over all \( \theta \) and \( Q \), where \( C, C' > 0 \) are some absolute constants.

**Remark 2.1.** By scrutinizing the proof of Theorem 2.1, the result can hold for any \( \epsilon < 1/3 - c' \) for an arbitrarily small constant \( c' \). The critical threshold \( 1/3 \) has a meaning of the highest breakdown point for Tukey’s median [12, 14]. Further discussion on the connection between the breakdown point and the \( \epsilon \)-contamination model is given in Section 6.

**Remark 2.2.** Theorem 2.1 is valid for identity covariance matrix. For a more general case \( P_\theta = N(\theta, \Sigma) \), as long as \( s_{\text{max}}(\Sigma) \leq M \) with some absolute constant \( M > 0 \), the result remains valid.

Theorem 2.1 says that the convergence rate of Tukey’s median is \( p/n \) when \( \epsilon^2 \lesssim p/n \). Otherwise, the rate is \( \epsilon^2 \). Therefore, as long as the number of outliers from \( Q \) is at the order of \( O(ne) = O(\sqrt{np}) \), the convergence rate of Tukey’s median is identical to the case when \( \epsilon = 0 \). The next theorem shows that Tukey’s median is optimal under the \( \epsilon \)-contamination model in a minimax sense.

**Theorem 2.2.** There are some absolute constants \( C, c > 0 \) such that

\[
\inf_{\hat{\theta}} \sup_{\theta, Q} \mathbb{P}_{(\epsilon, \theta, Q)} \{ \|\hat{\theta} - \theta\|^2 \geq C \left( \frac{p}{n} \vee \epsilon^2 \right) \} \geq c,
\]

for any \( \epsilon \in [0, 1] \).

Theorem 2.2 provides a minimax lower bound for the \( \epsilon \)-contamination model. It implies that as long as \( \epsilon^2 \gtrsim p/n \), the usual minimax rate \( p/n \) for estimating \( \theta \) is no longer achievable. It also justifies the optimality of Tukey’s median from a minimax perspective. To summarize, Theorem 2.1 and Theorem 2.2 jointly provide a framework for robust statistics that characterize both statistical accuracy and resistance to outliers simultaneously.

Another natural robust estimator for location is the coordinate median, defined as \( \hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_p)^T \) with \( \hat{\theta}_j = \text{Median}(\{X_{ij}\}_{i=1}^n) \). We show that the coordinate median has an inferior convergence rate via the following proposition.

**Proposition 2.1.** Consider the coordinate median \( \hat{\theta} \). There are absolute constants \( C, c > 0 \) such that

\[
\sup_{\theta, Q} \mathbb{P}_{(\epsilon, \theta, Q)} \{ \|\hat{\theta} - \theta\|^2 \geq C \left( \frac{1}{n} \vee \epsilon^2 \right) \} \geq c,
\]

for any \( \epsilon \in [0, 1] \).
Obviously, \(p (n^{-1} \lor \epsilon^2)\) is also the upper bound for \(\hat{\theta}\) by applying Theorem 2.1 to each coordinate. Since \(p/n \lor \epsilon^2 \ll p (n^{-1} \lor \epsilon^2)\) when \(\epsilon^2 \geq 1/n\), the coordinate median has a slower convergence rate. It achieves the rate \(p/n\) only when \(n \epsilon = O(\sqrt{n})\). Therefore, to preserve the rate \(p/n\), the coordinate median can tolerate at most \(O(\sqrt{n})\) number of outliers, whereas Tukey’s median can tolerate \(O(\sqrt{pn})\).

### 3 Robust Covariance Matrix Estimation

In this section, we consider the problem of estimating covariance matrices under the \(\epsilon\)-contamination model. The model is represented as \(P(\epsilon, \Sigma, Q) = (1 - \epsilon)P\Sigma + \epsilon Q\), where \(P\Sigma = N(0, \Sigma)\) and \(Q\) is any distribution. Motivated by Tukey’s depth function for location parameters, we introduce a new concept called matrix depth. The robust matrix estimator is defined as the deepest covariance matrix with respect to the observations. This estimator achieves minimax optimal rates under the \(\epsilon\)-contamination model.

#### 3.1 Matrix Depth

The main idea of Tukey’s median is to project multivariate data onto all one-dimensional subspaces and obtain the deepest point by evaluating depths in those one-dimensional subspaces. Such an idea can be used to estimate covariance matrices. Formally speaking, for \(X \sim N(0, \Sigma)\), the population median of \(|u^T X|^2\) is \(\beta u^T \Sigma u\) for every \(u \in S^{p-1}\) with some absolute constant \(\beta\) defined later. Thus, an estimator of \(\Sigma\) can be obtained by estimating variance on every direction.

Inspired by the above idea, we define the matrix depth of a positive semi-definite \(\Gamma \in \mathbb{R}^{p \times p}\) with respect to a distribution \(\mathbb{P}\) as

\[
D(\Gamma, \mathbb{P}) = \min_{u \in S^{p-1}} \min \left\{ \mathbb{P}\{|u^T X|^2 \leq u^T \Gamma u\}, \mathbb{P}\{|u^T X|^2 \geq u^T \Gamma u\} \right\}, \quad \text{where } X \sim \mathbb{P}.
\]

For computational reasons, it is also helpful to define matrix depth by a subset of the directions \(S^{p-1}\). Given a subset \(\mathcal{U} \subset S^{p-1}\), the matrix depth of \(\Gamma\) with respect to a distribution \(\mathbb{P}\) relative to \(\mathcal{U}\) is defined as

\[
D_{\mathcal{U}}(\Gamma, \mathbb{P}) = \min_{u \in \mathcal{U}} \min \left\{ \mathbb{P}\{|u^T X|^2 \leq u^T \Gamma u\}, \mathbb{P}\{|u^T X|^2 \geq u^T \Gamma u\} \right\}, \quad \text{where } X \sim \mathbb{P}.
\]

We adopt the notation \(D_{S^{p-1}}(\Gamma, \mathbb{P}) = D(\Gamma, \mathbb{P})\). At the population level, the following proposition shows that the true covariance matrix, multiplied by a scalar, is the deepest positive semi-definite matrix.

**Proposition 3.1.** Define \(\beta\) through the equation

\[
\Phi(\sqrt{\beta}) = \frac{3}{4}, \quad (3)
\]

where \(\Phi\) is the cumulative distribution function of \(N(0, 1)\). Then, for any \(\mathcal{U} \subset S^{p-1}\), we have \(D_{\mathcal{U}}(\beta \Sigma, P\Sigma) = \frac{1}{2}\).
Given i.i.d. observations \( \{X_i\}_{i=1}^n \) from \( \mathbb{P} \), the matrix depth of \( \Gamma \) with respect to \( \{X_i\}_{i=1}^n \) is defined as
\[
D_\mathcal{U}(\Gamma, \{X_i\}_{i=1}^n) = \min_{u \in \mathcal{U}} \min \left\{ \frac{1}{n} \sum_{i=1}^n I\{|u^T X_i|^2 \leq u^T \Gamma u\}, \frac{1}{n} \sum_{i=1}^n I\{|u^T X_i|^2 \geq u^T \Gamma u\} \right\}. \tag{4}
\]
A general estimator for \( \beta \Sigma \) is given by
\[
\hat{\Gamma} = \arg \max_{\Gamma \succeq 0} D_\mathcal{U}(\Gamma, \{X_i\}_{i=1}^n), \tag{5}
\]
where \( \mathcal{F} \) is some matrix class to be specified later. The estimator of \( \Sigma \) is
\[
\hat{\Sigma} = \hat{\Gamma} / \beta, \tag{6}
\]
where \( \beta \) is defined through (3).

### 3.2 General Covariance Matrix

Consider the following covariance matrix class with bounded spectra
\[
\mathcal{F}(M) = \left\{ \Sigma = \Sigma^T \in \mathbb{R}^{p \times p} : \Sigma \succeq 0, s_{\max}(\Sigma) \leq M \right\},
\]
where \( \Sigma \succeq 0 \) means \( \Sigma \) is positive semi-definite and \( M > 0 \) is some absolute constant that does not scale with \( p \) or \( n \).

To define an estimator, we need to specify a subset \( \mathcal{U} \subset S^{p-1} \) in the depth function. Let \( \mathcal{U}_p \) be a \((1/4)\)-net of the unit sphere \( S^{p-1} \) in the Euclidean space in \( \mathbb{R}^p \). This means for any \( u \in S^{p-1} \), there exists a \( u' \in \mathcal{U}_p \) such that \( \|u - u'\| \leq 1/4 \). According to [45], such \( \mathcal{U}_p \) can be picked with cardinality bounded by \( 9^p \). Define
\[
\hat{\Gamma} = \arg \max_{\Gamma \succeq 0} D_{\mathcal{U}_p}(\Gamma, \{X_i\}_{i=1}^n). \tag{7}
\]
When (7) has multiple maxima, \( \hat{\Gamma} \) is understood as any positive semi-definite matrix that attains the deepest level. A final estimator of \( \Sigma \) is defined by \( \hat{\Sigma} = \hat{\Gamma} / \beta \) as in (6). The statistical property of \( \hat{\Sigma} \) is stated in the following theorem.

**Theorem 3.1.** Assume that \( \epsilon < 1/4 \) and \( p/n < c \) for some sufficiently small constant \( c \). Then, we have
\[
\|\hat{\Sigma} - \Sigma\|_{op}^2 \leq C \left( \frac{p}{n} \vee \epsilon^2 \right),
\]
with \( \mathbb{P}_{(\epsilon, \Sigma, Q)} \)-probability at least \( 1 - \exp \left(-C'(p + n \epsilon^2)\right) \) uniformly over all \( Q \) and \( \Sigma \in \mathcal{F}(M) \), where \( C, C' > 0 \) are some absolute constants.

**Remark 3.1.** Due to the computational consideration, we chose a \((1/4)\)-net \( \mathcal{U}_p \) of \( S^{p-1} \) and defined our estimator via the matrix depth relative to \( \mathcal{U}_p \). In fact, it can be shown that the result in Theorem 3.1 also holds if we define \( \hat{\Gamma} = \arg \max_{\Gamma \succeq 0} D(\Gamma, \{X_i\}_{i=1}^n) \) relative to \( S^{p-1} \).
The convergence rate for the deepest covariance is \( p/n \lor \epsilon^2 \) under the squared operator norm. This rate is minimax optimal over the matrix class \( \mathcal{F}(M) \) under the \( \epsilon \)-contamination model.

**Theorem 3.2.** There are some constants \( C, c > 0 \) such that

\[
\inf_{\hat{\Sigma}} \sup_{\Sigma \in \mathcal{F}(M)} \sup_Q \mathbb{P}_{(\epsilon, \Sigma, Q)} \left\{ ||\hat{\Sigma} - \Sigma||_{op}^2 \geq C \left( \frac{p}{n} \lor \epsilon^2 \right) \right\} \geq c,
\]

for any \( \epsilon \in [0, 1] \).

### 3.3 Bandable Covariance Matrix

In many high-dimensional applications such as time series data in finance, the covariates of data are collected in an ordered fashion. This leads to a natural banded estimator of the covariance matrix \([3, 6]\). Define the class of covariance matrices with a banded structure by

\[
\mathcal{F}_k = \{ \Sigma = (\sigma_{ij}) \succeq 0 : \sigma_{ij} = 0 \text{ if } |i - j| > k \}.
\]

Next, we propose a notion of matrix depth function relative to some subset \( \mathcal{U}_k \subset S^{p-1} \) defined particularly for the class \( \mathcal{F}_k \). For any \( l_1, l_2 \in [p] \), define \( \mathcal{V}_{[l_1, l_2]} = \{ u = (u_i) \in S^{p-1} : u_i = 0 \text{ if } i \notin [l_1, l_2] \} \). Then \( \mathcal{V}_{[l_1, l_2]} \) is equivalent to \( S^{l_2-l_1} \) on the coordinates \( \{l_1, ..., l_2\} \). There exists a \((1/4)\)-net of \( \mathcal{V}_{[l_1, l_2]} \), denoted by \( \tilde{\mathcal{V}}_{[l_1, l_2]} \), whose cardinality can be bounded by \( 9^{l_2-l_1+1} \). The depth function is defined relatively to the following subset

\[
\mathcal{U}_k = \bigcup_{l=1}^{p+1-2k} \tilde{\mathcal{V}}_{[l,l+2k-1]} \text{ if } 2k \leq p, \quad \text{and} \quad \mathcal{U}_k = \tilde{\mathcal{V}}_{[1,p]} \text{ if } 2k > p.
\]

Then, a robust covariance matrix estimator with banded structure is defined as

\[
\hat{\Gamma} = \arg \max_{\Gamma \in \mathcal{F}_k} D_{\mathcal{U}_k}(\Gamma, \{X_i\}_{i=1}^n).
\]

An estimator for \( \Sigma \) is \( \hat{\Sigma} = \hat{\Gamma}/\beta \) as in (6).

**Remark 3.2.** The cardinality of \( \mathcal{U}_k \) is bounded by \( p \times 9^{2k} \). When \( k \ll p \), this is significantly smaller than that of the set used in (7), because we have taken advantage of the banded structure.

To study the statistical property of \( \hat{\Sigma} \), we consider the class \( \mathcal{F}_k(M) = \mathcal{F}_k \cap \mathcal{F}(M) \). The convergence rate of \( \hat{\Sigma} \) under the \( \epsilon \)-contamination model is stated in the following theorem.

**Theorem 3.3.** Assume that \( \epsilon < 1/4 \) and \( (k+\log p)/n < c \) for some sufficiently small constant \( c \). Then, we have

\[
||\hat{\Sigma} - \Sigma||_{op}^2 \leq C \left( \frac{k + \log p}{n} \lor \epsilon^2 \right),
\]

with \( \mathbb{P}_{(\epsilon, \Sigma, Q)} \)-probability at least \( 1 - \exp \left( C' (k + \log p + ne^2) \right) \) uniformly over all \( Q \) and \( \Sigma \in \mathcal{F}_k(M) \), where \( C, C' > 0 \) are some absolute constants.
Theorem 3.3 states that the convergence rate for \( \hat{\Sigma} \) under the class \( F_k(M) \) is \( \frac{k + \log p}{n} \vee \epsilon^2 \). When \( \epsilon^2 \lesssim \frac{k + \log p}{n} \), this is exactly the minimax rate in [6]. Therefore, Theorem 3.3 extends the result of [6] to a robust setting. If the rate \( \frac{k + \log p}{n} \) is pursued, then the maximum number of outliers that \( \hat{\Sigma} \) can tolerate is \( O(\sqrt{n(k + \log p)}) \).

Besides matrices with exact banded structure, we also consider the following class of bandable matrices. That is, \( F_\alpha(M, M_0) = \{ \Sigma = (\sigma_{ij}) \in F(M) : \max_j \sum_{\{i : |i-j| > k\}} |\sigma_{ij}| \leq M_0 k^{-\alpha} \} \).

Theorem 3.4. Consider the robust banded estimator \( \hat{\Sigma} \) with \( k = \lceil n^{\frac{1}{2\alpha + 1}} \rceil \). Assume that \( \epsilon < \frac{1}{4} \) and \( n^{-\frac{2\alpha}{2\alpha + 1}} + \frac{\log p}{n} < c \) for some sufficiently small constant \( c \). Then, we have

\[
\| \hat{\Sigma} - \Sigma \|_{op}^2 \leq C \left( \min \left\{ n^{-\frac{2\alpha}{2\alpha + 1}} + \frac{\log p}{n}, p \right\} \vee \epsilon^2 \right),
\]

with \( \mathbb{P}_{(\epsilon, \Sigma, Q)} \)-probability at least \( 1 - \exp \left( -C'(\min \left\{ n^{-\frac{2\alpha}{2\alpha + 1}} + \log p, p \right\} + n\epsilon^2) \right) \) uniformly over all \( Q \) and \( \Sigma \in F_\alpha(M, M_0) \), where \( C, C' > 0 \) are some absolute constants.

To close this section, we show in the following theorem that both rates in Theorem 3.3 and Theorem 3.4 are minimax optimal under the \( \epsilon \)-contamination model.

Theorem 3.5. Assume \( p \leq \exp(\gamma n) \) for some \( \gamma > 0 \). There are some constants \( C, c > 0 \) such that

\[
\inf_{\Sigma} \sup_{\Sigma \in F_k(M)} \sup_Q \mathbb{P}_{(\epsilon, \Sigma, Q)} \left\{ \| \hat{\Sigma} - \Sigma \|_{op}^2 \geq C \left( \frac{k + \log p}{n} \vee \epsilon^2 \right) \right\} \geq c,
\]

and

\[
\inf_{\Sigma} \sup_{\Sigma \in F_\alpha(M, M_0)} \sup_Q \mathbb{P}_{(\epsilon, \Sigma, Q)} \left\{ \| \hat{\Sigma} - \Sigma \|_{op}^2 \geq C \left( \min \left\{ n^{-\frac{2\alpha}{2\alpha + 1}} + \frac{\log p}{n}, p \right\} \vee \epsilon^2 \right) \right\} \geq c,
\]

for any \( \epsilon \in [0, 1] \).

### 3.4 Sparse Covariance Matrix

We consider sparse covariance matrices in this section. For a subset of coordinates \( S \subset [p] \), define \( G(S) = \{ G = (g_{ij}) \in \mathbb{R}^{p \times p} : g_{ij} = 0 \text{ if } i \notin S \text{ or } j \notin S \} \). Define \( G(s) = \cup_{S \subset [p] : |S| \leq s} G(S) \).

Then, the sparse covariance class is

\[
F_s = \{ \Sigma \succeq 0 : \Sigma - \text{diag}(\Sigma) \in G(s) \}.
\]

In other words, there are \( s \) covariates in a block that are correlated with each other. The remaining covariates are independent from this block and from each other. Such sparsity structure has been extensively studied in the problem of sparse principal component analysis.
[27, 32, 46, 7], and is different from the notion of degree sparsity studied in [2, 5]. Estimating the whole covariance matrix under such sparsity was considered by [10].

To take advantage of the sparsity structure, we define a subset $U_s \subset S_{p-1}$ for the matrix depth function. For any $S \subset [p]$, define $V_S = \{u = (u_i) \in S_{p-1} : u_i = 0 \text{ if } i \notin S\}$. Then there exists a $(1/4)$-net of $V_S$, denoted by $\tilde{V}_S$, whose cardinality can be bounded by $9|S|$. The depth function is defined relatively to the following subset

$$U_s = \bigcup_{S \subset [p] : |S| = 2s} \tilde{V}_S.$$  

A robust sparse covariance matrix estimator is defined by

$$\hat{\Gamma} = \arg \max_{\Gamma \in F_s} D_{U_s}(\Gamma, \{X_i\}_{i=1}^n).$$  

(9)

An estimator for $\Sigma$ is $\hat{\Sigma} = \hat{\Gamma} / \beta$ as in (6).

The statistical property of $\hat{\Sigma}$ is studied in the class $F_s(M) = F_s \cap F(M)$ under the $\epsilon$-contamination model.

**Theorem 3.6.** Assume that $\epsilon < 1/4$ and $s \log(ep/s)/n < c$ for some sufficiently small constant $c$. Then, we have

$$\|\hat{\Sigma} - \Sigma\|_{op}^2 \leq C \left( \frac{s \log \frac{ep}{s}}{n} \vee \epsilon^2 \right),$$

with $P_{(\epsilon, \Sigma, Q)}$-probability at least $1 - \exp\left(-C'(s \log(ep/s) + ne^2)\right)$ uniformly over all $Q$ and $\Sigma \in F_s(M)$, where $C, C' > 0$ are some absolute constants.

The next theorem shows that the upper bound in Theorem 3.6 is optimal under the $\epsilon$-contamination model.

**Theorem 3.7.** Assume $p/n \leq C_1$ for some constant $C_1 > 0$. There are some constants $C, c > 0$ such that

$$\inf_\Sigma \sup_{\Sigma \in F_s(M)} \sup_Q P_{(\epsilon, \Sigma, Q)} \left\{ \|\hat{\Sigma} - \Sigma\|_{op}^2 \geq C \left( \frac{s \log \frac{ep}{s}}{n} \vee \epsilon^2 \right) \right\} \geq c,$$

for any $\epsilon \in [0, 1]$.

### 3.5 Sparse Principal Component Analysis

As an application of Theorem 3.6, we consider sparse principal component analysis. We adopt the spiked covariance model [27, 4]. That is,

$$\Sigma = V \Lambda V^T + I_p,$$

where $V \in \mathbb{R}^{p \times r}$ is an orthonormal matrix and $\Lambda$ is a diagonal matrix with elements $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0$. When $V$ has $s$ nonzero rows [7, 10], $\Sigma$ is in the class $F_s$. The goal is to estimate the subspace projection matrix $VV^T$. We propose a robust estimator by applying
singular value decomposition to \( \hat{\Gamma} \) in (9). That is, \( \hat{\Gamma} = \hat{V} \hat{D} \hat{V}^T \). Then, \( \hat{V} \hat{V}^T \) is a robust estimator of \( V V^T \).

To study the statistical property of \( \hat{V} \), define the covariance matrix class as

\[
F_{s,\lambda}(M,r) = \{ \Sigma = V \Lambda V^T + I_p : \lambda \leq \lambda_r \leq \ldots \lambda_1 \leq M, V \in O(p,r), |\text{supp}(V)| \leq s \},
\]

where \( O(p,r) \) is the class of \( p \times r \) orthonormal matrices and \( \text{supp}(V) \) is the set of nonzero rows of \( V \). The rank \( r \) is assumed to be bounded by a constant.

**Theorem 3.8.** Assume that \( \frac{s \log \frac{ep}{n \lambda^2}}{\lambda^2} \sqrt{\frac{\epsilon^2}{\lambda^2}} \leq c \) for some sufficiently small constant \( c \) and \( r \leq C_1 \) for some constant \( C_1 > 0 \). Then, we have

\[
\| \hat{V} \hat{V} - V V^T \|_F^2 \leq C \left( \frac{s \log \frac{ep}{n \lambda^2}}{n \lambda^2} + \frac{\epsilon^2}{\lambda^2} \right),
\]

with \( \mathbb{P}(e,\Sigma,Q) \)-probability at least \( 1 - \exp(-C'(s \log(ep/s) + \epsilon^2)) \) uniformly over all \( Q \) and \( \Sigma \in F_{s,\lambda}(M,r) \), where \( C,C' > 0 \) are some absolute constants.

According to Theorem 3.8, the convergence rate for principal subspace estimation is \( \frac{s \log \frac{ep}{n \lambda^2}}{\lambda^2} \sqrt{\frac{\epsilon^2}{\lambda^2}} \). We have the rate \( \epsilon^2/\lambda^2 \) instead of the usual \( \epsilon^2 \) to account for the outliers in the previous cases. As shown in the next theorem, the rate \( \epsilon^2/\lambda^2 \) is in fact optimal for sparse principal component analysis.

**Theorem 3.9.** There are some constants \( C,c,c' > 0 \) such that

\[
\inf_{\Sigma} \sup_{\Sigma \in F_{s,\lambda}(M,r)} \sup_{Q} \mathbb{P}(e,\Sigma,Q) \left\{ \| \hat{V} \hat{V} - V V^T \|_F^2 \geq C \left( \frac{s \log \frac{ep}{n \lambda^2}}{n \lambda^2} + \frac{\epsilon^2}{\lambda^2} \right) \wedge c' \right\} \geq c,
\]

for any \( \epsilon \in [0,1] \).

## 4 A General Minimax Lower Bound

In this section, we provide a general minimax theory for \( \epsilon \)-contamination model. Given a general statistical experiment \( \{ P_{\theta} : \theta \in \Theta \} \), recall the notation \( \mathbb{P}(e,\theta,Q) = (1 - \epsilon)P_{\theta} + \epsilon Q \).

If we denote the minimax rate for the class \( \{ \mathbb{P}(\theta,Q) : \theta \in \Theta, Q \} \) under some loss function \( L(\theta_1,\theta_2) \) by \( \mathcal{M}(\epsilon) \), then most rates we obtained in Section 2 and Section 3 can be written as \( \mathcal{M}(\epsilon) \approx \mathcal{M}(0) \vee \epsilon^2 \). The only exception is \( \mathcal{M}(\epsilon) \approx \mathcal{M}(0) \vee (\epsilon^2/\lambda^2) \) for sparse principal component analysis. Therefore, a natural question is whether we can have a general theory for the \( \epsilon \)-contamination model that governs those minimax rates. The answer for this question lies in a key quantity called modulus of continuity, whose definition goes back to the seminal works of Dohono and Liu [16] and Donoho [13].

The modulus of continuity for the \( \epsilon \)-contamination model is defined as

\[
\omega(\epsilon, \Theta) = \sup \{ L(\theta_1,\theta_2) : \text{TV}(P_{\theta_1}, P_{\theta_2}) \leq \epsilon/(1 - \epsilon); \theta_1, \theta_2 \in \Theta \}.
\]

The quantity \( \omega(\epsilon, \Theta) \) measures the ability of the loss \( L(\theta_1,\theta_2) \) to distinguish two distributions \( P_{\theta_1} \) and \( P_{\theta_2} \) that are close in total variation at the order of \( \epsilon \). A high level interpretation is
that two distributions \( P_{\theta_1} \) and \( P_{\theta_2} \) as close as \( O(\epsilon) \) under total variation distance cannot be distinguished at the presence of arbitrary outlier distribution \( Q \). A general minimax lower bound depending on the modulus of continuity is stated in the following theorem.

**Theorem 4.1.** Suppose there is some \( M(0) \) such that

\[
\inf_{\hat{\theta}} \sup_{\theta, Q} \sup_{(\epsilon, \theta, Q)} \{ L(\hat{\theta}, \theta) \geq M(\epsilon) \} \geq c
\]

holds for \( \epsilon = 0 \). Then for any \( \epsilon \in [0,1] \), (11) holds for \( M(\epsilon) \sim M(0) \vee \omega(\epsilon, \Theta) \).

Theorem 4.1 shows that the quantity \( \omega(\epsilon, \Theta) \) is the price of robustness one has to pay in the minimax rate. To illustrate this result, let us consider the location model in Section 2 where \( P_{\theta} = N(\theta, I_p) \). Since \( \|\theta_1 - \theta_2\|^2 = 2D(P_{\theta_1}, P_{\theta_2}) \geq 4TV(P_{\theta_1}, P_{\theta_2})^2 \), we have \( \omega(\epsilon, \Theta) \geq \epsilon^2 \). Besides, it is well known that \( M(0) \sim p/n \) for the location model, and thus we obtain the rate \( p/n \vee \epsilon^2 \) as the lower bound, which implies Theorem 2.2. Similar calculation can also be done for the covariance model. In particular, for sparse principal component analysis, we get \( \omega(\epsilon, \Theta) \sim (\epsilon/\lambda)^2 \). The details of derivation are given in Section 7.3.

### 5 Extension to Elliptical Distributions

In Section 3, we considered estimating the covariance matrix under the Gaussian distribution \( P_{\Sigma} = N(0, \Sigma) \). Though we show that our covariance estimator via matrix depth function is robust to arbitrary outliers, it is not clear whether such property also holds under more general distributions. In real applications, the data may not follow a Gaussian distribution and can have very heavy tails. In this section, we extend the Gaussian setting in Section 3 to general elliptical distributions. We show that at the population level, the scatter matrix of an elliptical distribution achieves the maximum of the matrix depth function. This fact motivates us to use the matrix depth estimator (5) in the elliptical distribution setting. Indeed, all statistical properties we prove under the Gaussian distribution continue to hold under the elliptical distributions. Therefore, the proposed estimator is also adaptive to the shape of the distribution. We start by introducing the definition of an elliptical distribution.

**Definition 5.1** ([19]). A random vector \( X \in \mathbb{R}^p \) follows an elliptical distribution if and only if it has the representation \( X = \mu + \xi A U \), where \( \mu \in \mathbb{R}^p \) and \( A \in \mathbb{R}^{p \times r} \) are model parameters. The random variable \( U \) is distributed uniformly on the unit sphere \( S^{p-1} \) and \( \xi \geq 0 \) is a random variable in \( \mathbb{R} \) independent of \( U \). Letting \( \Sigma = AA^T \) and we denote \( X \sim EC(\mu, \Sigma, \xi) \).

For simplicity, we consider the model with \( \mu = 0 \). We want to remark two points on this definition. First, the representation \( EC(0, \Sigma, \xi) \) is not unique. This is because \( EC(0, \Sigma, \xi) = EC(0, a^{-2} \Sigma, a \xi) \) for any \( a > 0 \). Secondly, for an elliptical random variable \( X \sim EC(0, \Sigma, \xi) \) with \( s_{\min}(\Sigma) > 0 \), given any unit vector \( u \in S^{p-1} \), the distribution of \( u^T X / \sqrt{u^T \Sigma u} \) is independent of \( u \). In other words, \( \Sigma^{-1/2} X \) is spherically symmetric. Motivated by these two points, we define the canonical representation of an elliptical distribution as follows.
Definition 5.2. For a non-degenerate elliptical distribution $EC(0, \Sigma, \xi)$ in the sense that $s_{\text{min}}(\Sigma) > 0$, $EC(0, \Gamma, \eta)$ is its canonical representation if and only if $\Gamma = a^{-2} \Sigma$ and $\eta = a \xi$ for some $a > 0$, and $P_\Gamma \left( \frac{|u^T X|^2}{u^T \Gamma u} \leq 1 \right) = \frac{1}{2}$, where $P_\Gamma = EC(0, \Gamma, \eta)$. From now on, whenever we use $P_\Gamma = EC(0, \Gamma, \eta)$, it always denotes the canonical representation.

To guarantee the existence and uniqueness of the canonical representation, we need the following assumption on the marginal distribution. Define the distribution function $G(t) = P_\Gamma \left( \frac{|u^T X|^2}{u^T \Gamma u} \leq t \right)$. Note that $G(t)$ does not depend on the specific direction $u \in S^{p-1}$ used in the definition. We assume that $G(t)$ is continuous at $t = 1$ and there exist some $\tau \in (0, 1/2)$ and $\alpha, \kappa > 0$ such that

$$ \inf_{|t| \geq \alpha} |G(1) - G(1 + t)| \geq \tau \quad \text{and} \quad \inf_{|t| < \alpha} \frac{|G(1) - G(1 + t)|}{|t|} \geq \kappa^{-1/2}. $$

Proposition 5.1. For an elliptical distribution $EC(0, \Gamma, \eta)$ that satisfies (13), its canonical representation exists and is unique.

The existence and uniqueness of the canonical representation of $EC(0, \Gamma, \eta)$ imply that the matrix $\Gamma$ is a well-defined object. We call $\Gamma$ the scatter matrix. The following proposition shows that the scatter matrix $\Gamma$ is actually the deepest one with respect to the matrix depth function.

Proposition 5.2. For any subset $U \subset S^{p-1}$, we have $D_U(\Gamma, P_\Gamma) = \frac{1}{2}$.

When $X \sim EC(0, \Gamma, \eta)$ has a density function, it must have the form $p(x) = f(x^T \Gamma^{-1} x)$ for some univariate function $f(\cdot)$ [19]. Examples of elliptical distributions include:

1. **Multivariate Gaussian.** The density function is $p(x) \propto \exp(-\beta x^T \Gamma^{-1} x/2)$, where the constant $\beta$ is defined in (3). Proposition 3.1 implies that $\beta^{-1} \Gamma$ is the Gaussian covariance matrix.

2. **Multivariate Laplacian.** The density function is $p(x) \propto \exp(-\sqrt{\beta} x^T \Gamma^{-1} x)$, where the constant $\beta$ is determined through the canonical representation. The covariance matrix has formula $(p + 1) \beta^{-1} \Gamma$.

3. **Multivariate t.** The density function is $p(x) \propto (1 + \beta x^T \Gamma^{-1} x/d)^{-\frac{d+p}{2}}$, where $d$ is the degree of freedom. The constant $\beta$ is determined through the canonical representation. When $d > 2$, the covariance matrix is $\frac{d}{d-2} \beta^{-1} \Gamma$. Otherwise, the covariance does not exist.

4. **Multivariate Cauchy.** This is a special case of multivariate t distribution when $d = 1$. The density function is $p(x) \propto (1 + \beta x^T \Gamma^{-1} x)^{-\frac{p+1}{2}}$.

Proposition 5.3. For all the four examples above, $\beta$ is an absolute constant independent of $p$. Moreover, the condition (13) holds with absolute constants $\tau, \alpha, \kappa$ independent of $p$. 

13
Let us proceed to consider estimating the scatter matrix $\Gamma$ under the $\epsilon$-contamination model $\mathbb{P}_{(\epsilon, \Gamma, Q)} = (1 - \epsilon)P_\Gamma + \epsilon Q$. This requires the estimator to be robust in two senses. First, it should be resistant to the outliers. Second, it should be adaptive to the distribution.

Using the property of the scatter matrix spelled out in Proposition 5.2, we show that the depth-induced estimator (5) enjoys good statistical properties.

**Theorem 5.1.** Consider the estimator $\hat{\Gamma}$ defined in (7). Assume $\epsilon < \tau/3$, $p/n < c$ for some sufficiently small constant $c$ and the distribution $P_\Gamma = EC(0, \Gamma, \eta)$ satisfies (13). Then, we have

$$||\hat{\Gamma} - \Gamma||_\text{op}^2 \leq C\kappa \left( \frac{p}{n} \vee \epsilon^2 \right),$$

with $\mathbb{P}_{(\epsilon, \Gamma, Q)}$-probability at least $1 - \exp(-C'(p + n\epsilon^2))$ uniformly over all $Q$ and $\Gamma \in \mathcal{F}(M)$, where $C, C' > 0$ are some absolute constants.

**Theorem 5.2.** Consider the estimator $\hat{\Gamma}$ defined in (8). Assume that $\epsilon < \tau/3$, $(k + \log p)/n < c$ for some sufficiently small constant $c$ and the distribution $P_\Gamma = EC(0, \Gamma, \eta)$ satisfies (13). Then, we have

$$||\hat{\Gamma} - \Gamma||_\text{op}^2 \leq C\kappa \left( \frac{k + \log p}{n} \vee \epsilon^2 \right),$$

with $\mathbb{P}_{(\epsilon, \Gamma, Q)}$-probability at least $1 - \exp(-C'(k + \log p + n\epsilon^2))$ uniformly over all $Q$ and $\Gamma \in \mathcal{F}_k(M)$, where $C, C' > 0$ are some absolute constants.

**Theorem 5.3.** Consider the estimator $\hat{\Gamma}$ defined in (8) with $k = \lceil n^{\frac{1}{2\alpha + 1}} \rceil$. Assume that $\epsilon < \tau/3$, $n^{-\frac{2\alpha}{2\alpha + 1}} + \frac{\log p}{n} < c$ for some sufficiently small constant $c$ and the distribution $P_\Gamma = EC(0, \Gamma, \eta)$ satisfies (13). Then, we have

$$||\hat{\Gamma} - \Gamma||_\text{op}^2 \leq C\kappa \left( \min \left\{ n^{-\frac{2\alpha}{2\alpha + 1}} + \frac{\log p}{n}, \frac{p}{n} \right\} \vee \epsilon^2 \right),$$

with $\mathbb{P}_{(\epsilon, \Gamma, Q)}$-probability at least $1 - \exp(-C'(\min \left\{ n^{-\frac{2\alpha}{2\alpha + 1}} + \log p, p \right\} + n\epsilon^2))$ uniformly over all $Q$ and $\Gamma \in \mathcal{F}_a(M, M_0)$, where $C, C' > 0$ are some absolute constants.

**Theorem 5.4.** Consider the estimator $\hat{\Gamma}$ defined in (9). Assume that $\epsilon < \tau/3$, $s \log(ep/s)/n < c$ for some sufficiently small constant $c$ and the distribution $P_\Gamma = EC(0, \Gamma, \eta)$ satisfies (13). Then, we have

$$||\hat{\Gamma} - \Gamma||_\text{op}^2 \leq C\kappa \left( \frac{s \log ep}{n} \vee \epsilon^2 \right),$$

with $\mathbb{P}_{(\epsilon, \Gamma, Q)}$-probability at least $1 - \exp(-C'(s \log(ep/s) + n\epsilon^2))$ uniformly over all $Q$ and $\Gamma \in \mathcal{F}_s(M)$, where $C, C' > 0$ are some absolute constants.

**Theorem 5.5.** Consider the leading eigen-matrix with rank $r$ of $\hat{\Gamma}$ defined in (9), denoted by $\hat{V}$. Assume that $\kappa \left( \frac{s \log ep}{n\lambda^2} \vee \frac{\epsilon^2}{\lambda^2} \right) \leq c$ for some sufficiently small constant $c$, $r \leq C_1$ for some constant $C_1 > 0$ and the distribution $P_\Gamma = EC(0, \Gamma, \eta)$ satisfies (13). Then, we have

$$||\hat{V}^2 - VV^T||_F^2 \leq C\kappa \left( \frac{s \log ep}{n\lambda^2} \vee \frac{\epsilon^2}{\lambda^2} \right).$$
with \(P(\epsilon, \Gamma, Q)\)-probability at least \(1 - \exp(-C'(s \log(ep/s) + ne^2))\) uniformly over all \(Q\) and \(\Gamma \in \mathcal{F}_{s, \lambda}(M, r)\), where \(C, C' > 0\) are some absolute constants.

**Remark 5.1.** Theorem 5.5 requires the scatter matrix \(\Gamma\) to belong to \(\mathcal{F}_{s, \lambda}(M, r)\), which means that \(\Gamma = V \Lambda V^T + I_p\). While the \(I_p\) part has a clear meaning for covariance matrix, it may not be a suitable way of modeling the scatter matrix. However, we may consider a more general space which contains \(\Gamma = V \Lambda V^T + \sigma^2 I_p\) for some constant \(\sigma^2\) bounded in some interval \([M^{-1}, M]\). Then, the result of Theorem 5.5 still holds.

**Remark 5.2.** The problem of finding the leading principal subspace for \(EC(0, \Gamma, \eta)\) was coined as elliptical component analysis by [21]. While [21] extended sparse principal component analysis to the elliptical distributions, the influence of outliers was not investigated. In comparison, we show that our estimator is robust to both heavy-tailed distributions and the presence of outliers.

To close this section, we remark that the estimators via matrix depth function does not require the knowledge of the exact elliptical distribution. They are adaptive to all \(EC(0, \Gamma, \eta)\) that satisfy the condition (13). Since the class of elliptical distributions includes multivariate Gaussian as a special case, the lower bounds in Section 3 imply that the convergence rates obtained in this section are optimal.

### 6 Discussion

#### 6.1 Impact of Contamination on Convergence Rates

For all the problems we consider in this paper, the minimax rate under the \(\epsilon\)-contamination model has the expression \(M(\epsilon) \asymp M(0) \lor \omega(\epsilon, \Theta)\). Define

\[
\epsilon^* = \sup \{\epsilon : \omega(\epsilon, \Theta) \leq M(0)\}.
\]

Then, \(\epsilon^*\) is the maximal proportion of outliers under which the minimax rate obtained without outliers can still be preserved. Thus, \(n \epsilon^*\) is the maximal expected number of outliers for an optimal procedure to achieve the minimax rate as if there is no contamination.

Compared to the minimax rate, consistency is easier to achieve. Suppose \(M(0) = o(1)\), then the necessary and sufficient condition for consistency is \(\omega(\epsilon, \Theta) = o(1)\). In most cases where \(\omega(\epsilon, \Theta) \asymp \epsilon^2\), the condition reduces to \(\epsilon = o(1)\), meaning that as long as the expected number of outliers is at a smaller order of \(n\), the optimal procedure is consistent under the \(\epsilon\)-contamination model.

#### 6.2 Connection with Breakdown Point

The notion of breakdown point [20] has been widely used to quantify the influence of outliers for a given estimator. In this section, we discuss the connection between the breakdown point and Huber’s \(\epsilon\)-contamination model. Let us start by the definition given in [12, 15, 14].
Consider the observations \( \{X_i\}_{i=1}^n \) that consist of two parts \( \{Y_i\}_{i=1}^{n_1} \) and \( \{Z_i\}_{i=1}^{n_2} \) with \( n_1 + n_2 = n \). We view \( \{Z_i\}_{i=1}^{n_2} \) as the outliers. Then, a robust estimator \( \hat{\theta}(\cdot) \) should not be influenced much by the outliers if the proportion \( n_2/(n_1 + n_2) \) is small. The breakdown point of \( \hat{\theta} \) with respect to \( \mathcal{Y} \) is defined as

\[
\epsilon(\hat{\theta}, \mathcal{Y}) = \min \left\{ \frac{n_2}{n_1 + n_2} : \sup_{\{Y_i\}_{i=1}^{n_1} \in \mathcal{Y}} \sup_{\{Z_i\}_{i=1}^{n_2}} \left\| \hat{\theta}(\{Y_i\}_{i=1}^{n_1}) - \hat{\theta}(\{X_i\}_{i=1}^{n}) \right\| = \infty \right\}, \tag{14}
\]

where \( \|\cdot\| \) is some norm. The breakdown point \( \epsilon(\hat{\theta}, \mathcal{Y}) \) measures the smallest proportion of outliers that an estimator \( \hat{\theta} \) can be totally ruined. In its original form, the supreme of \( \{Y_i\}_{i=1}^{n_1} \) over \( \mathcal{Y} \) does not appear in the definition. However, \( \{Y_i\}_{i=1}^{n_1} \) are usually assumed to be in a general position or follow some distribution. Thus, it is natural to apply this modification. Now let us consider the \( \epsilon \)-contamination model \( \mathbb{P}_{(\epsilon, \hat{\theta}, Q)} = (1 - \epsilon)\mathbb{P}_{\theta} + \epsilon Q \). For i.i.d. observations \( X_1, ..., X_n \sim \mathbb{P}_{(\epsilon, \hat{\theta}, Q)} \), it can be decomposed into two parts \( \{Y_i\}_{i=1}^{n_1} \) and \( \{Z_i\}_{i=1}^{n_2} \), where \( n_2 \sim \text{Binomial}(n, \epsilon) \) and \( n_1 = n - n_2 \). Conditioning on \( n_1, Y_1, ..., Y_{n_1} \sim \mathbb{P}_{\theta} \) and \( Z_1, ..., Z_{n_2} \sim Q \). Observe that \( \frac{n_2}{n_1 + n_2} \approx \epsilon \), which means the \( \epsilon \) in the contamination model plays a similar role to the ratio \( \frac{n_2}{n_1 + n_2} \) in (14). Motivated by this fact, we introduce a population counterpart of (14). Given an estimator \( \hat{\theta} \), its \( \delta \)-breakdown point with respect to some parameter space \( \Theta \) is defined as

\[
\epsilon(\hat{\theta}, \Theta, \delta) = \min \left\{ \epsilon : \sup_{\theta \in \Theta} \sup_{Q} \mathbb{P}_{(\epsilon, \hat{\theta}, Q)} \left( L(\hat{\theta}(\{Y_i\}_{i=1}^{n_1}), \hat{\theta}(\{X_i\}_{i=1}^{n})) > \delta \right) > c \right\}, \tag{15}
\]

where \( L(\cdot, \cdot) \) is some loss function, and \( c \in (0, 1) \) is some small constant. We may view (15) as the population counterpart of (14) because \( \sup_{Q} \) corresponds to \( \sup_{\{Z_i\}_{i=1}^{n_2}} \), \( \sup_{\theta \in \Theta} \) corresponds to \( \sup_{\{Y_i\}_{i=1}^{n_1} \in \mathcal{Y}} \) and \( \epsilon \) corresponds to \( \frac{n_2}{n_1 + n_2} \). We allow \( \delta \) to be a sequence of \( n \) instead of \( \infty \) because \( L(\cdot, \cdot) \) can be a bounded loss. Then, the \( \delta \)-breakdown point means the minimal \( \epsilon \) such that an estimator \( \hat{\theta} \) is influenced at least by the level of \( \delta \) under the \( \epsilon \)-contamination model. In fact, \( \epsilon(\hat{\theta}, \Theta, \delta) \) is a quantity directly related to the lower bound of the convergence rate of \( \hat{\theta} \) under the \( \epsilon \)-contamination model. This is rigorously stated in the following theorem.

**Theorem 6.1.** Assume the loss function is symmetric and satisfies

\[
L(\theta_1, \theta_2) \leq A(L(\theta_1, \theta_3) + L(\theta_2, \theta_3)) \quad \forall \theta_1, \theta_2, \theta_3 \in \Theta \text{ with some } A > 0, \tag{16}
\]

\[
\sup_{\theta \in \Theta} P_{\theta}^{n} \left\{ L(\hat{\theta}, \theta) > \frac{1}{2}c_1A^{-1}\delta \right\} \geq \sup_{\theta \in \Theta} P_{\theta}^{n'} \left\{ L(\hat{\theta}, \theta) > \frac{1}{2}A^{-1}\delta \right\} \quad \forall n' \geq \frac{n}{3}, \tag{17}
\]

with some constant \( c_1 \in (0, 1) \). Then, for \( \epsilon = \epsilon(\hat{\theta}, \Theta, \delta) < \frac{1}{2} \), we have

\[
\sup_{\theta \in \Theta} \sup_{Q} \mathbb{P}_{(\epsilon, \hat{\theta}, Q)} \left( L(\hat{\theta}, \theta) > \frac{1}{2}c_1A^{-1}\delta \right) > \frac{1}{3}c,
\]

for some \( c > 0 \) in (15) and sufficiently large \( n \).
Before discussing the implications of Theorem 6.1, we remark on the assumption (17). The notation $P^n_\theta$ means the estimator $\hat{\theta}(\cdot)$ takes a random argument $\hat{\theta}(\{Y_i\}_{i=1}^n)$ with distribution $Y_1, \ldots, Y_n \sim P_\theta$. Thus, the assumption (17) simply means when the sample sizes $n, n'$ are at the same order, the lower bounds remain at the same order. In most cases including all the examples considered in this paper, (17) automatically holds.

Theorem 6.1 gives a general lower bound based on the notion of $\delta$-breakdown point. Given an estimator $\hat{\theta}$ and an $\epsilon$-contamination model, the solution $\delta$ to the equation
\[ \epsilon(\hat{\theta}, \Theta, \delta) = \epsilon \]
lower bounds its rate of convergence. When $\hat{\theta}$ is a minimax optimal estimator with rate $M(\epsilon)$, we obtain $M(\epsilon) \gtrsim \delta$. In other words, the convergence rate $\delta$ under the $\epsilon$-contamination model automatically implies a $\delta$-breakdown point with the same $\epsilon$.

6.3 A Unified Framework of Robustness and Efficiency

Huber’s $\epsilon$-contamination model allows a simultaneous joint study of robustness and efficiency of an estimator. This is our major reason to develop the theory of robust covariance matrix estimation under this framework. We illustrate the importance of this view by re-visiting the coordinate median studied in Section 2. Without contamination, the coordinate median is a location estimator with minimax rate under Gaussian distribution. It is also robust because of its high breakdown point [14]. However, Proposition 2.1 shows that its performance under the presence of contamination is not optimal. In contrast, Tukey’s multivariate median shows its advantage over the coordinate median by obtaining optimality under the $\epsilon$-contamination model. This example suggests that the statistical efficiency and the robust property of an estimator should be studied together rather than separately.

Recently, Donoho and Montanari [17] have studied Huber’s M-estimator under the $\epsilon$-contamination model in a regression setting where $p/n$ converges to a constant. They find a critical $\epsilon^*$ that determines the variance breakdown point. The setting of $\epsilon$-contamination model plays a critical role in their work to illustrate both efficiency and robustness of Huber’s M-estimator in a unified way.

7 Proofs

This section provides proofs for the results in Section 3 and Section 4. The proofs of the remaining results are given in the supplementary material.

7.1 Auxiliary Lemmas

For i.i.d. data $\{X_i\}_{i=1}^n$ from a contaminated distribution $(1 - \epsilon)P + \epsilon Q$, it can be written as $\{Y_i\}_{i=1}^{n_1} \cup \{Z_i\}_{i=1}^{n_2}$. Marginally, we have $n_2 \sim \text{Binomial}(n, \epsilon)$ and $n_1 = n - n_2$. Conditioning on $n_1$ and $n_2$, $\{Y_i\}_{i=1}^{n_1}$ are i.i.d. from $P$ and $\{Z_i\}_{i=1}^{n_2}$ are i.i.d. from $Q$. The following lemma controls the ratio $n_2/n_1$. Its proof is given in the supplementary material.
Lemma 7.1. Assume $\epsilon < 1/2$. For any $\delta > 0$ satisfying $n^{-1} \log(1/\delta) < c$ for some sufficiently small constant $c$, we have
\[
\frac{n_2}{n_1} \leq \frac{\epsilon}{1 - \epsilon} + C \sqrt{\frac{\log(1/\delta)}{n}},
\tag{19}
\]
with probability at least $1 - \delta$, where $C > 0$ is an absolute constant. Moreover, assume $\epsilon^2 > 1/n$, and then we have
\[
\frac{n_2}{n_1} > c' \epsilon,
\tag{20}
\]
with probability at least $1/2$ for some constant $c' > 0$.

The following lemma characterizes an important property of the $\epsilon$-contamination model. Its proof is given in the supplementary material.

Lemma 7.2. Consider any parametric family $\{P_{\theta} : \theta \in \Theta\}$, and we have the relation
\[
\{(1 - \epsilon_1)P_{\theta} + \epsilon_1 Q : \theta \in \Theta, Q \} \subset \{(1 - \epsilon_2)P_{\theta} + \epsilon_2 Q : \theta \in \Theta, Q \},
\]
for any $0 \leq \epsilon_1 < \epsilon_2 \leq 1$.

Finally, we need the following DKW inequality due to [35].

Lemma 7.3. For i.i.d. real-valued data $X_1, \ldots, X_n \sim P$, we have for any $t > 0$,
\[
\mathbb{P}\left( \sup_{x \in \mathbb{R}} |\mathbb{P}_n(X \leq x) - \mathbb{P}(X \leq x)| > t \right) \leq 2e^{-2nt^2},
\]
where $\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$ is the empirical distribution.

7.2 Proofs of upper bounds in Section 3

We first prove the following master theorem.

Theorem 7.1. Consider the estimator $\hat{\Sigma}$ defined in (6). Assume $\epsilon < 1/4$ and $n^{-1} \log |\mathcal{U}|$ is sufficiently small. Then for any $\delta > 0$ such that $n^{-1} \log(1/\delta)$ is sufficiently small, we have
\[
\sup_{u \in \mathcal{U}} \left| u^T \hat{\Sigma} u - u^T \Sigma u \right| \leq C \left( \epsilon + \sqrt{\frac{\log |\mathcal{U}| + \log(1/\delta)}{n}} \right),
\]
$\mathbb{P}(\epsilon, \Sigma, Q)$-probability at least $1 - 2\delta$ uniformly over all $Q$ and $\Sigma \in \mathcal{F} \cap \mathcal{F}(M)$, where $C > 0$ is some absolute constant.

Proof. By Lemma 7.1, we decompose the data $\{X_i\}_{i=1}^{n} = \{Y_i\}_{i=1}^{n_1} \cup \{Z_i\}_{i=1}^{n_2}$. The following analysis is conditioning on the set of $(n_1, n_2)$ that satisfies (19) with probability at least $1 - \delta$. To facilitate the proof, define
\[
\mathcal{D}_u(\Gamma, P_{\Sigma}) = \min \left\{ P_{\Sigma}(|u^T Y|^2 \leq u^T \Gamma u), P_{\Sigma}(|u^T Y|^2 > u^T \Gamma u) \right\},
\]
\[
\mathcal{D}_u(\Gamma, \{Y_i\}_{i=1}^{n_1}) = \min \left\{ \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{I}\{|u^T Y_i|^2 \leq u^T \Gamma u\}, \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{I}\{|u^T Y_i|^2 > u^T \Gamma u\} \right\},
\]
and
for each $u \in S^{p-1}$. Then, we have $D_u(\Gamma, P_\Sigma) = \min_{u \in U} D_u(\Gamma, P_\Sigma)$ and $D_u(\Gamma, \{Y_i\}_{i=1}^{n_1}) = \min_{u \in U} D_u(\Gamma, \{Y_i\}_{i=1}^{n_1})$. Observe that
\[
\sup_{\Gamma \in \mathcal{F}} |D_u(\Gamma, P_\Sigma) - D_u(\Gamma, \{Y_i\}_{i=1}^{n_1})| \\
\leq \sup_{\Gamma \in \mathcal{F}} \max_{u \in U} |D_u(\Gamma, P_\Sigma) - D_u(\Gamma, \{Y_i\}_{i=1}^{n_1})| \\
= \max_{u \in U} \sup_{t \in \mathbb{R}} \left| \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{I}\{|u^T Y_i|^2 \leq t\} - \mathbb{P}_\Sigma(|u^T Y|^2 \leq t) \right|.
\]
Applying Lemma 7.3 and union bound, we get
\[
\sup_{\Gamma \in \mathcal{F}} |D_u(\Gamma, P_\Sigma) - D_u(\Gamma, \{Y_i\}_{i=1}^{n_1})| \leq \sqrt{\frac{\log |U| + \log(2/\delta)}{2n_1}}, \tag{21}
\]
with probability at least $1 - \delta$. We lower bound $D_u(\hat{\Gamma}, P_\Sigma)$ by
\[
D_u(\hat{\Gamma}, P_\Sigma) \geq D_u(\hat{\Gamma}, \{Y_i\}_{i=1}^{n_1}) - \sqrt{\frac{\log |U| + \log(2/\delta)}{2n_1}} \tag{22}
\]
\[
\geq \frac{n}{n_1} D_u(\hat{\Gamma}, \{X_i\}_{i=1}^{n}) - \frac{n_2}{n_1} - \sqrt{\frac{\log |U| + \log(2/\delta)}{2n_1}} \tag{23}
\]
\[
\geq \frac{n}{n_1} D_u(\beta \Sigma, \{X_i\}_{i=1}^{n}) - \frac{n_2}{n_1} - \sqrt{\frac{\log |U| + \log(2/\delta)}{2n_1}} \tag{24}
\]
\[
\geq D_u(\beta \Sigma, \{Y_i\}_{i=1}^{n_1}) - \frac{n_2}{n_1} - \sqrt{\frac{\log |U| + \log(2/\delta)}{2n_1}} \tag{25}
\]
\[
\geq \frac{1}{2} - \frac{n_2}{n_1} - \sqrt{\frac{\log |U| + \log(2/\delta)}{2n_1}} \tag{26}
\]
The inequalities (22) and (26) are by (21). The inequalities (23) and (25) are due to the property of depth function that
\[
n_1 D_u(\Gamma, \{Y_i\}_{i=1}^{n_1}) \geq n D_u(\Gamma, \{X_i\}_{i=1}^{n}) - n_2 \geq n_1 D_u(\Gamma, \{Y_i\}_{i=1}^{n_1}) - n_2,
\]
for any $\Gamma \in \mathcal{F}$. The inequality (24) is by the definition of $\hat{\Gamma}$. Finally, the equality (27) is due to Proposition 3.1. Now let us use Lemma 7.1 so that the right hand side of (27) can be lower bounded by
\[
\frac{1}{2} - \frac{\epsilon}{1-\epsilon} - C_1 \sqrt{\frac{\log |U| + \log(2/\delta)}{n}} ,
\]
for some absolute constant $C_1 > 0$ with probability at least $1 - \delta$. Using the property that $D_u(\hat{\Gamma}, P_\Sigma) \geq D_u(\hat{\Gamma}, P_\Sigma)$ for each $u \in U$, we have shown that
\[
D_u(\hat{\Gamma}, P_\Sigma) \geq \frac{1}{2} - \frac{\epsilon}{1-\epsilon} - C_1 \sqrt{\frac{\log |U| + \log(2/\delta)}{n}} , \text{ for all } u \in U, \tag{28}
\]
with probability at least $1 - 2\delta$. By Proposition 3.1 and the fact that $\frac{1}{2} - \min(x, 1 - x) = |x - 1/2|$ for all $x \in [0, 1]$, we get

$$
\frac{1}{2} - D_u(\tilde{\Gamma}, P_\Sigma) = 2 \Phi(\sqrt{\beta}) - \Phi \left( \frac{u^T \tilde{\Gamma} u}{u^T \Sigma u} \right).
$$

Combining with (28), we have

$$
2 \sup_{u \in \mathcal{U}} \left| \Phi(\sqrt{\beta}) - \Phi \left( \frac{u^T \tilde{\Gamma} u}{u^T \Sigma u} \right) \right| \leq \frac{\epsilon}{1 - \epsilon} + C_1 \frac{\log |\mathcal{U}| + \log(2/\delta)}{n},
$$

with probability at least $1 - 2\delta$. Under the assumption that $\epsilon < 1/4$ and $\sqrt{\log |\mathcal{U}| + \log(2/\delta)/n}$ is sufficiently small, we have

$$
\sup_{u \in \mathcal{U}} \left| \sqrt{\beta} - \frac{u^T \tilde{\Gamma} u}{u^T \Sigma u} \right| \leq C_2 \left( \epsilon + \sqrt{\log |\mathcal{U}| + \log(2/\delta)/n} \right),
$$

with probability at least $1 - 2\delta$. Finally, note that $\Sigma$ has bounded spectra, which implies

$$
\sup_{u \in \mathcal{U}} \left| u^T \tilde{\Gamma} u/\beta - u^T \Sigma u \right| \leq C \left( \epsilon + \sqrt{\log |\mathcal{U}| + \log(2/\delta)/n} \right),
$$

with probability at least $1 - 2\delta$. Thus, the proof is complete.

**Proof of Theorem 3.1.** Since $\mathcal{U}_p$ is taken to be the $(1/4)$-net of $S^{p-1}$. By Lemma 5.4 of [45], we have $|\mathcal{U}| \leq 9^p$ and $\|\Sigma - \tilde{\Sigma}\|_{op} \leq 2 \max_{u \in \mathcal{U}_p} |u^T (\tilde{\Sigma} - \Sigma) u|$. The conclusion follows the result of Theorem 7.1 by setting $2\delta = \exp(-C'(p + n\epsilon^2))$ for some $C' > 0$.

**Proof of Theorem 3.3.** Consider the weights

$$
w_{ij} = k^{-1} ((2k - |i - j|)_+ - (k - |i - j|)_+).$$

Since $\tilde{\Sigma} - \Sigma = (\tilde{\sigma}_{ij} - \sigma_{ij}) \in \mathcal{F}_k$, we have $(\tilde{\sigma}_{ij} - \sigma_{ij}) = ((\tilde{\sigma}_{ij} - \sigma_{ij})w_{ij})$. This means $\tilde{\Sigma} - \Sigma$ can also be viewed as a tapered matrix. Then, Lemma 2 of [6] implies that $\|\Sigma - \tilde{\Sigma}\|_{op} \leq C \max_{u \in \mathcal{U}_k} |u^T (\tilde{\Sigma} - \Sigma) u|$ for some constant $C > 0$. Using Theorem 7.1 the fact that $\log |\mathcal{U}_k| \leq C_1 (k + \log p)$ for some constant $C_1 > 0$, the proof is complete by setting $2\delta = \exp(-C'(k + \log p + n\epsilon^2))$ for some $C' > 0$.

**Proof of Theorem 3.4.** The main point of the proof is bias-variance tradeoff. For $\Sigma = (\sigma_{ij}) \in \mathcal{F}_k$, we have $(\sigma_{ij}) = ((\sigma_{ij})w_{ij})$. This means $\Sigma$ can also be viewed as a tapered matrix. Then, Lemma 2 of [6] implies that $\|\Sigma - \tilde{\Sigma}\|_{op} \leq C \max_{u \in \mathcal{U}_k} |u^T (\tilde{\Sigma} - \Sigma) u|$ for some constant $C > 0$. Using Theorem 7.1 the fact that $\log |\mathcal{U}_k| \leq C_1 (k + \log p)$ for some constant $C_1 > 0$, the proof is complete by setting $2\delta = \exp(-C'(k + \log p + n\epsilon^2))$ for some $C' > 0$.
$\mathcal{F}_a(M, M_0)$, define $\Sigma_k = (\sigma_{ij} I\{|i-j| \leq k\})$. Then

$$\begin{align*}
|D_{u_k}(\beta \Sigma, P_\Sigma) - D_{u_k}(\beta \Sigma_k, P_\Sigma)| &\leq \max_{u \in U_k} |D_u(\beta \Sigma, P_\Sigma) - D_u(\beta \Sigma_k, P_\Sigma)| \\
&\leq 2 \max_{u \in U_k} \left| \Phi(\sqrt{\beta}) - \Phi\left(\sqrt{\beta \frac{u^T \Sigma_k u}{u^T \Sigma u}}\right) \right| \\
&\leq \sqrt{\frac{2\beta}{\pi}} \max_{u \in U_k} \left| 1 - \frac{u^T \Sigma_k u}{u^T \Sigma u} \right| \\
&\leq \sqrt{\frac{2\beta}{\pi}} \left\| \Sigma_k - \Sigma \right\|_{\text{op}}.
\end{align*}$$

Using this bound and modifying the arguments (22)-(28), we obtain

$$D_{u}(\hat{\Gamma}, P_\Sigma) \geq \frac{1}{2} - \frac{\epsilon}{1 - \epsilon} - C_1 \sqrt{\frac{\log |U_k| + \log(2/\delta)}{n}} - \sqrt{\frac{2\beta}{\pi}} M \left\| \Sigma_k - \Sigma \right\|_{\text{op}},$$

for all $u \in U_k$ with probability at least $1 - 2\delta$. Repeating the subsequent argument in the proof of Theorem 7.1, we have

$$\sup_{u \in U_k} \left| u^T \hat{\Gamma} u - u^T \Sigma u \right| \leq C_2 \left( \epsilon + \sqrt{\frac{\log |U_k| + \log(2/\delta)}{n}} + \left\| \Sigma_k - \Sigma \right\|_{\text{op}} \right).$$

A triangle inequality implies

$$\sup_{u \in U_k} \left| u^T \hat{\Sigma} u - u^T \Sigma_k u \right| \leq C_3 \left( \epsilon + \sqrt{\frac{\log |U_k| + \log(2/\delta)}{n}} + \left\| \Sigma_k - \Sigma \right\|_{\text{op}} \right).$$

Using the argument in the proof of Theorem 3.3, we get

$$\left\| \hat{\Sigma} - \Sigma_k \right\|_{\text{op}} \leq C_4 \left( \epsilon + \sqrt{\frac{\log |U_k| + \log(2/\delta)}{n}} + \left\| \Sigma_k - \Sigma \right\|_{\text{op}} \right).$$

Using triangle inequality again, we have

$$\left\| \hat{\Sigma} - \Sigma \right\|_{\text{op}} \leq C \left( \epsilon + \sqrt{\frac{\log |U_k| + \log(2/\delta)}{n}} + \left\| \Sigma_k - \Sigma \right\|_{\text{op}} \right).$$

A bias argument in [6] implies that $\left\| \Sigma_k - \Sigma \right\|_{\text{op}} \leq C_5 k^{-\alpha}$. Choosing $k = \lceil n^{-\frac{1}{2\alpha + 1}} \rceil$ and $2\delta = \exp\left( -C' \min\left\{ n^{-\frac{1}{2\alpha + 1}} + \log p, p \right\} + n\epsilon^2 \right)$, the proof is complete by observing that $\log |U_k| \lesssim n^{-\frac{1}{2\alpha + 1}} \wedge p$. \qed
Proof of Theorem 3.6. Note that \( \Sigma - \Sigma \in \mathcal{F}_{2s} \), and thus \( \| \Sigma - \Sigma \|_{op} = \max_{|S|=2s} \| (\Sigma - \Sigma)_{SS} \|_{op} \). Applying Lemma 5.4 of [45], we obtain that \( \| \Sigma - \Sigma \|_{op} \leq C \max_{u \in \mathcal{U}_s} |u^T (\Sigma - \Sigma) u| \) for some constant \( C > 0 \). Using Theorem 7.1 the fact that \( |\mathcal{U}_s| \leq 2^{2s}(\frac{1}{\delta}) \leq \exp \left( C_1 s \log \frac{n}{s} \right) \) for some constant \( C_1 > 0 \), the proof is complete by setting \( 2\delta = \exp \left( -C'(s \log(ep/s) + n\epsilon^2) \right) \) for some \( C' > 0 \).

Proof of Theorem 3.8. Since \( \mathcal{F}_{s,\lambda}(M, r) \subset \mathcal{F}_s(M + 1) \), the result of Theorem 3.6 applies and we get

\[
\| \hat{\Gamma}/\beta - \Sigma \|_{op}^2 \leq C \left( \frac{s \log \frac{ep}{s}}{n} \vee \epsilon^2 \right),
\]

with probability at least \( 1 - \exp \left( -C'(s \log(ep/s) + n\epsilon^2) \right) \). Weyl’s inequality implies \( |s_{k+1}(\hat{\Gamma}/\beta) - 1| \leq \| \hat{\Gamma}/\beta - \Sigma \|_{op} \). Under the assumption that \( \frac{s \log \frac{2n}{s}}{n} \vee \frac{n\epsilon^2}{s} \) is sufficiently small, we have \( s_k(\Sigma) - s_{k+1}(\hat{\Gamma}/\beta) > c\lambda \) for some constant \( c > 0 \). By Davis-Kahan theorem [11], we have \( \| \hat{\Gamma} \hat{V}^T - V V^T \|_F \leq C'\| \hat{\Gamma}/\beta - \Sigma \|_{op}/\lambda \), and the proof is complete.

7.3 Proofs of Lower Bounds

Proof of Theorem 4.1. When \( \mathcal{M}(0) \geq \omega(\epsilon, \Theta) \), we have \( \mathcal{M}(0) = \mathcal{M}(0) \lor \omega(\epsilon, \Theta) \). Thus,

\[
\inf_{\hat{\theta} \in \Theta} \sup_{\theta \in \Theta} \sup_Q P_{(\epsilon, \theta, Q)} \left\{ L(\hat{\theta}, \theta) \geq \mathcal{M}(0) \lor \omega(\epsilon, \Theta) \right\} = \inf_{\hat{\theta} \in \Theta} \sup_{\theta \in \Theta} \sup_Q P_{(\epsilon, \theta, Q)} \left\{ L(\hat{\theta}, \theta) \geq \mathcal{M}(0) \right\} \geq c.
\]

It is sufficient to prove when \( \mathcal{M}(0) < \omega(\epsilon, \Theta) \), we have

\[
\inf_{\hat{\theta} \in \Theta} \sup_{\theta \in \Theta} \sup_Q P_{(\epsilon, \theta, Q)} \left\{ L(\hat{\theta}, \theta) \geq \omega(\epsilon, \Theta) \right\} \geq c. \tag{29}
\]

Let us pick \( \theta_1, \theta_2 \) that are solution of the following program

\[
\max_{\theta_1, \theta_2 \in \Theta} L(\theta_1, \theta_2) \quad \text{s.t.} \quad TV(P_{\theta_1}, P_{\theta_2}) \leq \epsilon/(1 - \epsilon).
\]

Then, there exists \( \epsilon' \leq \epsilon \) such that

\[
L(\theta_1, \theta_2) = \omega(\epsilon, \Theta) \quad \text{and} \quad TV(P_{\theta_1}, P_{\theta_2}) = \frac{\epsilon'}{1 - \epsilon'}.
\]

For these \( \theta_1, \theta_2 \in \Theta \), let us define density functions

\[
p_{\theta_1} = \frac{dP_{\theta_1}}{d(P_{\theta_1} + P_{\theta_2})}, \quad p_{\theta_2} = \frac{dP_{\theta_2}}{d(P_{\theta_1} + P_{\theta_2})},
\]

Define \( Q_1 \) and \( Q_2 \) by their density functions

\[
\frac{dQ_1}{d(P_{\theta_1} + P_{\theta_2})} = \frac{(p_{\theta_2} - p_{\theta_1}) \mathbb{1}\{p_{\theta_2} \geq p_{\theta_1}\}}{TV(P_{\theta_1}, P_{\theta_2})}, \quad \frac{dQ_2}{d(P_{\theta_1} + P_{\theta_2})} = \frac{(p_{\theta_1} - p_{\theta_2}) \mathbb{1}\{p_{\theta_1} \geq p_{\theta_2}\}}{TV(P_{\theta_1}, P_{\theta_2})}.
\]
Let us first check that $Q_1$ and $Q_2$ are probability measures. Since
\[
\int (p_{\theta_2} - p_{\theta_1}) \mathbb{1}\{p_{\theta_2} \geq p_{\theta_1}\} = 1 - \int p_{\theta_1} \wedge p_{\theta_2} = \int (p_{\theta_1} - p_{\theta_2}) \mathbb{1}\{p_{\theta_1} \geq p_{\theta_2}\},
\]
and
\[
\int (p_{\theta_2} - p_{\theta_1}) \mathbb{1}\{p_{\theta_2} \geq p_{\theta_1}\} + \int (p_{\theta_1} - p_{\theta_2}) \mathbb{1}\{p_{\theta_1} \geq p_{\theta_2}\} = 2 \text{TV}(P_{\theta_1}, P_{\theta_2}),
\]
we have
\[
\int (p_{\theta_2} - p_{\theta_1}) \mathbb{1}\{p_{\theta_2} \geq p_{\theta_1}\} = \int (p_{\theta_1} - p_{\theta_2}) \mathbb{1}\{p_{\theta_1} \geq p_{\theta_2}\} = \text{TV}(P_{\theta_1}, P_{\theta_2}),
\]
which implies
\[
\int \frac{dQ_1}{d(P_{\theta_1} + P_{\theta_2})} d(P_{\theta_1} + P_{\theta_2}) = \int \frac{dQ_2}{d(P_{\theta_1} + P_{\theta_2})} d(P_{\theta_1} + P_{\theta_2}) = 1.
\]
Thus, $Q_1$ and $Q_2$ are well-defined probability measures. The least favorable pair in the parameter space is
\[
\mathbb{P}_1 = (1 - \epsilon') P_{\theta_1} + \epsilon' Q_1, \quad \mathbb{P}_2 = (1 - \epsilon') P_{\theta_2} + \epsilon' Q_2.
\]
By Lemma 7.2,
\[
\mathbb{P}_1, \mathbb{P}_2 \in \{(1 - \epsilon') P_\theta + \epsilon' Q : \theta \in \Theta, Q\} \subset \{(1 - \epsilon) P_\theta + \epsilon Q : \theta \in \Theta, Q\}.
\]
Direct calculation gives
\[
\frac{d\mathbb{P}_1}{d(P_{\theta_1} + P_{\theta_2})} = (1 - \epsilon') p_{\theta_1} + \epsilon' \frac{(p_{\theta_2} - p_{\theta_1}) \mathbb{1}\{p_{\theta_2} \geq p_{\theta_1}\}}{\epsilon'(1 - \epsilon)}
\]
\[
= (1 - \epsilon') (p_{\theta_1} + (p_{\theta_2} - p_{\theta_1}) \mathbb{1}\{p_{\theta_2} \geq p_{\theta_1}\})
\]
\[
= (1 - \epsilon') (p_{\theta_2} + (p_{\theta_1} - p_{\theta_2}) \mathbb{1}\{p_{\theta_1} \geq p_{\theta_2}\})
\]
\[
= (1 - \epsilon') p_{\theta_2} + \epsilon' \frac{(p_{\theta_1} - p_{\theta_2}) \mathbb{1}\{p_{\theta_1} \geq p_{\theta_2}\}}{\epsilon'(1 - \epsilon)}
\]
\[
= \frac{d\mathbb{P}_2}{d(P_{\theta_1} + P_{\theta_2})}.
\]
Hence, $\mathbb{P}_1 = \mathbb{P}_2$, which implies the corresponding $\theta_1$ and $\theta_2$ are not identifiable from the model, and their distance under $L(\theta_1, \theta_2)$ can be as far as $\omega(\epsilon, \Theta)$. A standard application of Le Cam’s two point testing method [51] leads to (29) and the proof is complete. \qed

**Proof of Theorem 2.2.** In this case $L(\theta_1, \theta_2) = \|\theta_1 - \theta_2\|^2$. Therefore,
\[
\omega(\epsilon, \Theta) = \sup \{\|\theta_1 - \theta_2\|^2 : \text{TV}(N(\theta_1, I_\theta), N(\theta_2, I_\theta)) = \epsilon/(1 - \epsilon); \theta_1, \theta_2 \in \Theta\}
\]
\[
\geq \sup \{\|\theta_1 - \theta_2\|^2 : \|\theta_1 - \theta_2\|^2/4 \leq \epsilon^2; \theta_1, \theta_2 \in \Theta\}
\]
\[
= 4\epsilon^2.
\]
Moreover the rate $\mathcal{M}(0) \asymp \frac{p}{n}$ is classical (see, for example, [33]). Hence, we have $\mathcal{M}(\epsilon) \asymp (p/n) \vee \epsilon^2$ by Theorem 4.1. \qed
**Proofs of Theorem 3.2, Theorem 3.5 and Theorem 3.7.** Consider $\Sigma_1 = I_p$ and $\Sigma_2 = I_p + \epsilon E_{11}$, where $E_{11}$ is a matrix with 1 in the $(1,1)$-entry and 0 elsewhere. Note that both $\Sigma_1$ and $\Sigma_2$ are in all matrix classes considered in Section 3. Then,

$$TV(N(0, \Sigma_1), N(0, \Sigma_2))^2 \leq \frac{1}{2}D(N(0, \Sigma_1)||N(0, \Sigma_2)) \leq \frac{1}{8}\|\Sigma_1 - \Sigma_2\|_F^2 = \frac{\epsilon^2}{8},$$

and $L(\Sigma_1, \Sigma_2) = \|\Sigma_1 - \Sigma_2\|_F^2 = \epsilon^2$. Therefore, $\omega(\epsilon, \Theta) \geq \epsilon^2$. For the space $\mathcal{F}(M)$, we have $\mathcal{M}(0) \asymp \frac{k}{n}$ according to Theorem 6 of [33]. For $\mathcal{F}_k(M)$ and $\mathcal{F}_\alpha(M, M_0)$, we have $\mathcal{M}(0) \asymp \frac{k\log p}{n}$ and $\mathcal{M}(0) \asymp n^{-\frac{2\alpha}{2\alpha + 1}} + \frac{\log p}{n}$, respectively, which are implied by Theorem 3 of [6]. Finally, for $\mathcal{F}_s(M)$, $\mathcal{M}(0) \asymp \frac{s \log(ep/s)}{n}$ by Theorem 4 of [10]. By Theorem 4.1, we obtain the desired lower bound.  

**Proof of Theorem 3.9.** Consider $\Sigma_1 = \lambda \theta_1^T + I_p$ and $\Sigma_2 = \lambda \theta_2^T + I_p$. It is obvious that $\Sigma_1, \Sigma_2 \in \mathcal{F}_{s, \lambda}(M, 1)$. For $r \leq 2$, we may consider some $V \in O(p, r - 1)$ with $\text{supp}(V) \subset \{3, 4, \ldots, p\}$. Then, let $\Sigma_1 = \lambda \theta_1^T + \lambda V V^T + I_p$ and $\Sigma_2 = \lambda \theta_2^T + \lambda V V^T + I_p$. For both cases, we have $\Sigma_1, \Sigma_2 \in \mathcal{F}_{s, \lambda}(M, r)$. Since $TV(N(0, \Sigma_1), N(0, \Sigma_2))^2 \leq \frac{\lambda^2}{8}\|\theta_1^T - \theta_2^T\|_F^2$ and $L(\Sigma_1, \Sigma_2) = \|\theta_1^T - \theta_2^T\|_F^2$, we have

$$\omega(\epsilon, \Theta) \geq \sup \left\{\|\theta_1^T - \theta_2^T\|_F^2 : \frac{\lambda^2}{8}\|\theta_1^T - \theta_2^T\|_F^2 \leq \epsilon^2\right\} \geq \frac{\epsilon^2}{\lambda^2} \wedge c,$$

for some constant $c > 0$. The reason we need $c$ in the above inequality is because $\|\theta_1^T - \theta_2^T\|_F^2$ is a bounded loss. By Theorem 3 of [7], $\mathcal{M}(0) \gtrsim \frac{s \log(ep/s)}{n \lambda^2}$. By Theorem 4.1, we obtain the desired lower bound.  

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**References**

[1] Peter L Bartlett and Shahar Mendelson. Rademacher and Gaussian complexities: Risk bounds and structural results. *The Journal of Machine Learning Research*, 3(3):463–482, 2003.

[2] Peter J Bickel and Elizaveta Levina. Covariance regularization by thresholding. *The Annals of Statistics*, 36(6):2577–2604, 2008.

[3] Peter J Bickel and Elizaveta Levina. Regularized estimation of large covariance matrices. *The Annals of Statistics*, 36(1):199–227, 2008.

[4] Aharon Birnbaum, Iain M Johnstone, Boaz Nadler, and Debashis Paul. Minimax bounds for sparse PCA with noisy high-dimensional data. *The Annals of Statistics*, 41(3):1055–1084, 2013.
[5] T Tony Cai and Harrison H Zhou. Optimal rates of convergence for sparse covariance matrix estimation. *The Annals of Statistics*, 40(5):2389–2420, 2012.

[6] T Tony Cai, Cun-Hui Zhang, and Harrison H Zhou. Optimal rates of convergence for covariance matrix estimation. *The Annals of Statistics*, 38(4):2118–2144, 2010.

[7] T Tony Cai, Zongming Ma, and Yihong Wu. Sparse PCA: Optimal rates and adaptive estimation. *The Annals of Statistics*, 41(6):3074–3110, 2013.

[8] T Tony Cai, Zhao Ren, and Harrison H Zhou. Optimal rates of convergence for estimating Toeplitz covariance matrices. *Probability Theory and Related Fields*, 156(1-2):101–143, 2013.

[9] T Tony Cai, Zhao Ren, and Harrison H Zhou. Estimating structured high-dimensional covariance and precision matrices: Optimal rates and adaptive estimation. Technical report, University of Pennsylvania. URL http://www-stat.wharton.upenn.edu/~tcai/paper/html/Covariance-Survey.html, 2014.

[10] T. Tony Cai, Zongming Ma, and Yihong Wu. Optimal estimation and rank detection for sparse spiked covariance matrices. *Probability Theory and Related Fields*, 161(3-4):781–815, 2015.

[11] Chandler Davis and William Morton Kahan. The rotation of eigenvectors by a perturbation. iii. *SIAM Journal on Numerical Analysis*, 7(1):1–46, 1970.

[12] David L Donoho. Breakdown properties of multivariate location estimators. Technical report, Harvard University, Boston. URL http://www-stat.stanford.edu/~donoho/Reports/Oldies/BPMLE.pdf, 1982.

[13] David L Donoho. Statistical estimation and optimal recovery. *The Annals of Statistics*, 22(1):238–270, 1994.

[14] David L Donoho and Miriam Gasko. Breakdown properties of location estimates based on halfspace depth and projected outlyingness. *The Annals of Statistics*, 20(4):1803–1827, 1992.

[15] David L Donoho and Peter J Huber. The notion of breakdown point. *A Festschrift for Erich L. Lehmann*, pages 157–184, 1983.

[16] David L Donoho and Richard C Liu. Geometrizing rates of convergence, iii. *The Annals of Statistics*, 19(2):668–701, 1991.

[17] David L Donoho and Andrea Montanari. High dimensional robust M-estimation: Asymptotic variance via approximate message passing. *arXiv preprint arXiv:1310.7320*, 2013.

[18] Jianqing Fan, Fang Han, and Han Liu. PAGE: Robust pattern guided estimation of large covariance matrix. Technical report, Princeton University, 2014.
[19] Kai-Tai Fang, Samuel Kotz, and Kai Wang Ng. *Symmetric Multivariate and Related Distributions*. Chapman and Hall, 1990.

[20] Frank R Hampel. A general qualitative definition of robustness. *The Annals of Mathematical Statistics*, 42(6):1887–1896, 1971.

[21] Fang Han and Han Liu. ECA: High dimensional elliptical component analysis in non-Gaussian distributions. *arXiv preprint arXiv:1310.3561*, 2013.

[22] Fang Han and Han Liu. Optimal rates of convergence for latent generalized correlation matrix estimation in transelliptical distribution. *arXiv preprint arXiv:1305.6916*, 2013.

[23] Fang Han and Han Liu. Scale-invariant sparse PCA on high-dimensional meta-elliptical data. *Journal of the American Statistical Association*, 109(505):275–287, 2014.

[24] Fang Han, Junwei Lu, and Han Liu. Robust scatter matrix estimation for high dimensional distributions with heavy tails. Technical report, Princeton University, 2014.

[25] Joseph L Hodges Jr. Efficiency in normal samples and tolerance of extreme values for some estimates of location. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pages 163–186, 1967.

[26] Peter J Huber. Robust estimation of a location parameter. *The Annals of Mathematical Statistics*, 35(1):73–101, 1964.

[27] Iain M Johnstone and Arthur Yu Lu. On consistency and sparsity for principal components analysis in high dimensions. *Journal of the American Statistical Association*, 104(486):682–693, 2009.

[28] Clifford Lam and Jianqing Fan. Sparsistency and rates of convergence in large covariance matrix estimation. *The Annals of Statistics*, 37(6B):4254–4278, 2009.

[29] Regina Y Liu. On a notion of data depth based on random simplices. *The Annals of Statistics*, 18(1):405–414, 1990.

[30] Regina Y Liu, Jesse M Parelius, and Kesar Singh. Multivariate analysis by data depth: descriptive statistics, graphics and inference,(with discussion and a rejoinder by Liu and Singh). *The Annals of Statistics*, 27(3):783–858, 1999.

[31] Hendrik P Lopuhaa and Peter J Rousseeuw. Breakdown points of affine equivariant estimators of multivariate location and covariance matrices. *The Annals of Statistics*, 19(1):229–248, 1991.

[32] Zongming Ma. Sparse principal component analysis and iterative thresholding. *The Annals of Statistics*, 41(2):772–801, 2013.
[33] Zongming Ma and Yihong Wu. Volume ratio, sparsity, and minimaxity under unitarily invariant norms. In *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*, pages 1027–1031. IEEE, 2013.

[34] Ricardo Antonio Maronna. Robust M-estimators of multivariate location and scatter. *The Annals of Statistics*, 4(1):51–67, 1976.

[35] Pascal Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *The Annals of Probability*, 18(3):1269–1283, 1990.

[36] Ritwik Mitra and Cun-Hui Zhang. Multivariate analysis of nonparametric estimates of large correlation matrices. *arXiv preprint arXiv:1403.6195*, 2014.

[37] Hannu Oja. Descriptive statistics for multivariate distributions. *Statistics & Probability Letters*, 1(6):327–332, 1983.

[38] Peter J Rousseeuw and Mia Hubert. Regression depth. *Journal of the American Statistical Association*, 94(446):388–402, 1999.

[39] John W Tukey. T6: Order statistics, in mimeographed notes for Statistics 411. *Department of Statistics, Princeton University*, 1974.

[40] John W Tukey. Mathematics and the picturing of data. In *Proceedings of the International Congress of Mathematicians*, volume 2, pages 523–531, 1975.

[41] John W Tukey. Exploratory data analysis. *Addison-Wesley Series in Behavioral Science: Quantitative Methods, Reading, Mass.*, 1, 1977.

[42] David E Tyler. A distribution-free M-estimator of multivariate scatter. *The Annals of Statistics*, 15(1):234–251, 1987.

[43] Vladimir N Vapnik and A Ya Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability & Its Applications*, 16(2):264–280, 1971.

[44] Yehuda Vardi and Cun-Hui Zhang. The multivariate $\ell_1$-median and associated data depth. *Proceedings of the National Academy of Sciences*, 97(4):1423–1426, 2000.

[45] Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*, 2010.

[46] Vincent Q Vu and Jing Lei. Minimax sparse principal subspace estimation in high dimensions. *The Annals of Statistics*, 41(6):2905–2947, 2013.

[47] Marten Wegkamp and Yue Zhao. Adaptive estimation of the copula correlation matrix for semiparametric elliptical copulas. *arXiv preprint arXiv:1305.6526*, 2013.
[48] Lingzhou Xue and Hui Zou. Optimal estimation of sparse correlation matrices of semi-parametric gaussian copulas. *Manuscript, submitted*, 2011.

[49] Lingzhou Xue and Hui Zou. Rank-based tapering estimation of bandable correlation matrices. *Statistica Sinica*, 24:83–100, 2014.

[50] Victor J Yohai. High breakdown-point and high efficiency robust estimates for regression. *The Annals of Statistics*, 15(2):642–656, 1987.

[51] Bin Yu. Assouad, Fano, and Le Cam. In *Festschrift for Lucien Le Cam*, pages 423–435. Springer, 1997.

[52] Yijun Zuo and Robert Serfling. General notions of statistical depth function. *The Annals of Statistics*, 28(2):461–482, 2000.

[53] Yijun Zuo and Robert Serfling. Nonparametric notions of multivariate “scatter measure” and “more scattered” based on statistical depth functions. *Journal of Multivariate Analysis*, 75(1):62–78, 2000.
Proof of Theorem 2.1. Since our estimator is affine invariant, without loss of generality we consider the case where \( \theta = 0 \). By Lemma 7.1, we decompose the data \( \{X_i\}_{i=1}^{n_1} = \{Y_i\}_{i=1}^{n_1} \cup \{Z_i\}_{i=1}^{n_2} \). The following analysis is conditioning on the set of \((n_1, n_2)\) that satisfies (19). Define half space \( H_{u, \eta} = \{y : u^T y \leq u^T \eta\} \). Recall that the Tukey’s depth of \( \eta \) with respect to \( P_\theta \) and its empirical counterpart are

\[
D(\eta, P_\theta) = \inf_{u \in S_{p-1}} P_\theta(H_{u, \eta}) = \inf_{u \in S_{p-1}} P_\theta\{u^T Y \leq u^T \eta\},
\]

\[
D(\eta, \{Y_i\}_{i=1}^{n_1}) = \inf_{u \in S_{p-1}} P_{n_1}(H_{u, \eta}) = \min_{u \in S_{p-1}} \frac{1}{n_1} \sum_{i=1}^{n_1} I\{u^T Y_i \leq u^T \eta\},
\]

where \( P_{n_1} \) denotes the empirical distribution of \( \{Y_i\}_{i=1}^{n_1} \). The class of set functions \( \{I_{H_{u, \eta}} : u \in S_{p-1}, \eta \in \mathbb{R}^p\} \) consists of all half spaces in \( \mathbb{R}^p \) and hence has VC dimension \( p + 1 \) [43]. Then standard empirical processes theory (see, for instance, [1, Theorems 5,6]) implies that for any \( \delta > 0 \), there exists some absolute constant \( C_1 \) such that with probability at least \( 1 - \delta \), we have

\[
\sup_{u, \eta} |P_\theta(H_{u, \eta}) - P_{n_1}(H_{u, \eta})| \leq C_1 \sqrt{\frac{p}{n_1}} + \sqrt{\frac{\log(1/\delta)}{2n_1}}.
\]

As an immediate consequence, we have with probability at least \( 1 - \delta \),

\[
\sup_{\eta} |D(\eta, P_\theta) - D(\eta, \{Y_i\}_{i=1}^{n_1})| \leq C_1 \sqrt{\frac{p}{n_1}} + \sqrt{\frac{\log(1/\delta)}{2n_1}}. \tag{30}
\]
We lower bound $\mathcal{D}(\hat{\theta}, P_0)$ by

$$\mathcal{D}(\hat{\theta}, P_0) \geq \mathcal{D}(\hat{\theta}, \{Y_i\}_{i=1}^{n_1}) - C_1 \sqrt{\frac{p}{n_1}} - \frac{\log(1/\delta)}{2n_1}$$

(31)

$$\geq \frac{n}{n_1} \mathcal{D}(\hat{\theta}, \{X_i\}_{i=1}^{n}) - \frac{n_2}{n_1} - C_1 \sqrt{\frac{p}{n_1}} - \frac{\log(1/\delta)}{2n_1}$$

(32)

$$\geq \frac{n}{n_1} \mathcal{D}(\hat{\theta}, \{X_i\}_{i=1}^{n}) - \frac{n_2}{n_1} - C_1 \sqrt{\frac{p}{n_1}} - \frac{\log(1/\delta)}{2n_1}$$

(33)

$$\geq \mathcal{D}(\theta, \{X_i\}_{i=1}^{n_1}) - \frac{n_2}{n_1} - C_1 \sqrt{\frac{p}{n_1}} - \frac{\log(1/\delta)}{2n_1}$$

(34)

$$\geq \mathcal{D}(\theta, P_0) - \frac{n_2}{n_1} - 2C_1 \sqrt{\frac{p}{n_1}} - \frac{2\log(1/\delta)}{n_1}$$

(35)

$$= \frac{1}{2} - \frac{n_2}{n_1} - 2C_1 \sqrt{\frac{p}{n_1}} - \frac{2\log(1/\delta)}{n_1}.$$  

(36)

The inequalities (31) and (35) are by (30). The inequalities (32) and (34) are due to the property of depth function that

$$n_1 \mathcal{D}(\eta, \{Y_i\}_{i=1}^{n_1}) \geq n \mathcal{D}(\eta, \{X_i\}_{i=1}^{n}) - n_2 \geq n_1 \mathcal{D}(\eta, \{Y_i\}_{i=1}^{n_1}) - n_2,$$

for any $\eta \in \mathbb{R}^p$. The inequality (33) is by the definition of $\hat{\theta}$. Finally, the equality (36) is because $P_0 = N(\theta, I_p) = N(0, I_p)$, so that

$$\mathcal{D}(\eta, P_0) = 1 - \Phi(\|\eta\|),$$

(37)

for any $\eta \in \mathbb{R}^p$, where $\Phi(\cdot)$ is the cumulative distribution function of $N(0, 1)$. Combining (36) with (37) and (19), we have

$$\Phi(\|\hat{\theta}\|) \leq \frac{1}{2} + C_3 \sqrt{\frac{p}{n}} + C_2 \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\epsilon}{n - \epsilon},$$

with probability at least $1 - 2\delta$, where $C_2, C_3$ are absolute constants. Note that $\Phi(\|\hat{\theta}\|) - 1/2 = \int_0^{\|\hat{\theta}\|} (2\pi)^{-1/2} e^{-t^2/2} dt$ and $e^{-t^2/2}$ is bounded away from 0 in the neighborhood of $t = 0$. Thus, under the assumption that $\epsilon < 1/4$ and $\sqrt{p/n} + \sqrt{\log(1/\delta)/n}$ are sufficiently small, we obtain the bound $\|\hat{\theta}\| \leq C \left( \sqrt{\frac{p}{n}} + \epsilon + \frac{\sqrt{\log(1/\delta)/n}}{n} \right)$ with probability at least $1 - 2\delta$. Choosing $2\delta = \exp(-C'(p + n\epsilon^2))$ for some constant $C' > 0$, the proof is complete. 

Proof of Proposition 2.1. Given the conclusion of Lemma 7.2, it is sufficient to consider the case $\epsilon \leq 1/4$. Recall the constant $c' > 0$ in Lemma 7.1. When $\epsilon^2 \leq \left( \frac{64\log(12)}{(c')^2} \right) \vee 1) n^{-1}$, the classical minimax lower bound implies

$$\sup_{\theta, Q} P_{(\epsilon, \theta, Q)} \left\{ \|\hat{\theta} - \theta\|^2 \geq C' p/n \right\} > c,$$

30
for some small constants $0 < c < 1/3$ and $C' > 0$, by considering the case $Q = P_\theta$. Since $1/n \geq 1/n \lor \epsilon^2$, we have
\[
\sup_{\theta, Q} \mathbb{P}_{P(e, \theta, Q)} \left\{ \| \hat{\theta} - \theta \|^2 \geq C p(n^{-1} \lor \epsilon^2) \right\} > c,
\]
where $C = C'(1 \lor \frac{(c')^2}{64 \log(12)})$. Hence, it is sufficient to consider the case $\epsilon^2 > (\frac{64 \log(12)}{(c')^2}) n^{-1}$.

Let us consider the distribution $\mathbb{P} = \mathbb{P}_{P(e,0,Q)}$ with $Q\{ Z = (1,1,\ldots,1)^T \} = 1$. Decompose the observations into $\{ X_i \}_{i=1}^{n_1} = \{ Y_i \}_{i=1}^{n_1} \cup \{ Z_i \}_{i=1}^{n_2}$ as in Lemma 7.1. Then for each $j \in [p]$, define the event
\[
E_j = \left\{ \sup_{\eta} \left| \frac{1}{n_1} \sum_{i=1}^{n_1} I\{Y_{ij} \leq \eta\} - \Phi(\eta) \right| \leq \sqrt{\frac{\log(12)}{n}} \right\},
\]
and
\[
E = \left\{ c' \epsilon < \frac{n_2}{n_1} < \frac{1}{2} \right\},
\]
with $c' > 0$ specified in Lemma 7.1. We claim that
\[
\mathbb{P} \left\{ \sum_{j=1}^{p} I E_j \geq c_1 p, E \right\} \geq \frac{1}{3}, \tag{38}
\]
for some small constant $c_1 > 0$. We will establish the conclusion of Proposition 2.1 by assuming (38) holds. The inequality (38) will be proved in the end. Let us first show that $E \cap E_j \subset \{ \hat{\theta}_j^2 \geq c_2 \epsilon^2 \}$, where the absolute constant $0 < c_2 < 1$ depends on $c'$ only. For each $\eta < 1$, we have
\[
n D(\eta,\{ X_{ij} \}_{i=1}^{n}) = \left( \sum_{i=1}^{n_1} I\{Y_{ij} > \eta\} + \sum_{i=1}^{n_2} I\{Z_{ij} > \eta\} \right) \land \left( \sum_{i=1}^{n_1} I\{Y_{ij} \leq \eta\} + \sum_{i=1}^{n_2} I\{Z_{ij} \leq \eta\} \right) = \left( \sum_{i=1}^{n_1} I\{Y_{ij} > \eta\} + n_2 \right) \land \sum_{i=1}^{n_1} I\{Y_{ij} \leq \eta\}.
\]
Therefore, the event $E_j$ implies
\[
\sup_{\eta < 1} \left| \frac{n}{n_1} D(\eta,\{ X_{ij} \}_{i=1}^{n}) - \{ (1 - \Phi(\eta) + n_2/n_1) \land \Phi(\eta) \} \right| \leq \sqrt{\frac{\log(12)}{n}}. \tag{39}
\]
When $\hat{\theta}_j < 1$, we have
\[
\Phi(\hat{\theta}_j) \geq (1 - \Phi(\hat{\theta}_j) + n_2/n_1) \land \Phi(\hat{\theta}_j)
\geq \frac{n}{n_1} D(\hat{\theta}_j,\{ X_{ij} \}_{i=1}^{n}) - \sqrt{\frac{\log(12)}{n}} \tag{40}
\geq \frac{n}{n_1} D(\hat{\theta}_j^*,\{ X_{ij} \}_{i=1}^{n}) - \sqrt{\frac{\log(12)}{n}} \tag{41}
\geq (1 - \Phi(\hat{\theta}_j^*) + n_2/n_1) \land \Phi(\hat{\theta}_j^*) - 2 \sqrt{\frac{\log(12)}{n}} \tag{42}
\geq \frac{1}{2} + \frac{n_2}{2n_1} - 2 \sqrt{\frac{\log(12)}{n}},
\]
where $\theta^*$ is defined by the equation $(1 - \Phi(\theta^*_j) + n_2/n_1) \land \Phi(\theta^*_j) = \frac{1}{2} + \frac{c_1 \epsilon}{2n_1}$, which is guaranteed to have a solution when $n_2/n_1 < 1/2$ under the event $E$. The inequalities (40) and (42) are due to (39) and the inequality (41) is by the definition of $\hat{\theta}$. By $\epsilon^2 > \frac{64 \log(12)}{(c')^2} n^{-1}$ and the event $E$, we have

$$\Phi(\hat{\theta}_j) \geq \frac{1}{2} + \frac{1}{4} \epsilon,$$

which implies $\hat{\theta}_j^2 \geq c_2 \epsilon^2$. When $\hat{\theta}_j < 1$ does not hold, we have $\hat{\theta}_j^2 \geq 1 \geq \epsilon^2$. This establishes $E \cap E_j \subset \{ \hat{\theta}_j^2 \geq c_2 \epsilon^2 \}$, noting $c_2 < 1$. Hence, when $\epsilon^2 > \frac{64 \log(12)}{(c')^2} n^{-1}$, we can pick small enough constant $C > 1$ such that

$$\sup_{\theta, Q} \mathbb{P}(\epsilon, \theta, Q) \{ \| \hat{\theta} - \theta \|_2 \geq C p(\epsilon^2 \lor n^{-1}) \} \geq \sup_{\theta, Q} \mathbb{P}(\epsilon, \theta, Q) \{ \| \hat{\theta} - \theta \|_2 \geq c_1 c_2 p \epsilon^2 \} \geq \frac{1}{3},$$

where recall $\mathbb{P} = \mathbb{P}(\epsilon, 0, Q)$ in the above argument. This gives the desired conclusion, noting $c < 1/3$.

Finally, let us prove (38) to close proof. First, we have $\mathbb{P}(E) \geq 2/5$ by Lemma 7.1 and the assumptions $\epsilon < 1/4$ and $\epsilon^2 > n^{-1}$. Moreover,

$$\mathbb{P}(E_j) \geq \mathbb{P} \{ \sup_{\eta} \left| \frac{1}{n_1} \sum_{i=1}^{n_1} I\{Y_{ij} \leq \eta\} - \Phi(\eta) \right| \leq \sqrt{\frac{\log(12)}{n}} |E| \} \mathbb{P}(E) \geq \frac{2}{5} \min_{m \geq n/2} \mathbb{P} \{ \sup_{\eta} \left| \frac{1}{m} \sum_{i=1}^{m} I\{Y_{ij} \leq \eta\} - \Phi(\eta) \right| \leq \sqrt{\frac{\log(12)}{n}} \} \geq \frac{1}{3},$$

by Lemma 7.3. Therefore, by Hoeffding’s inequality,

$$\mathbb{P} \left\{ \sum_{j=1}^{p} I_{E_j} \geq c_1 p \right\} \geq 0.99,$$

for some small $c_1$ when $p$ is sufficiently large. Hence,

$$\mathbb{P} \left\{ \sum_{j=1}^{p} I_{E_j} \geq c_1 p, E \right\} \geq 1 - \mathbb{P} \left\{ \sum_{j=1}^{p} I_{E_j} < c_1 p \right\} - \mathbb{P}(E^c) > \frac{1}{3},$$

and the proof is complete.

### B Proofs in Section 5

Note that the proofs in Section 3 all depend on Theorem 7.1. Similarly, all results in Section 5 are consequences of the following result analogous to Theorem 7.1.
Theorem B.1. Consider the estimator $\hat{\Gamma}$ defined in (5). Assume $\epsilon < \tau/3$, $n^{-1}\log |U|$ is sufficiently small and the distribution $P_{\hat{\Gamma}} = ECP_\tau(0, \Gamma, \eta)$ satisfies (13). Then for any $\delta > 0$ such that $n^{-1}\log(1/\delta)$ is sufficiently small, we have
\[
\sup_{u \in U} \left| u^T(\hat{\Gamma} - \Gamma)u \right| \leq C\kappa^{1/2} \left( \epsilon + \sqrt{\frac{\log |U| + \log(2/\delta)}{n}} \right),
\]
with $\mathbb{P}_{(\epsilon, \Gamma, Q)}$-probability at least $1 - 2\delta$ uniformly over all $Q$ and $\Gamma \in F(M) \cap \mathcal{F}$, where $C > 0$ is some absolute constant.

Proof. The proof is similar to that of Theorem 7.1. We focus on the difference and omit the overlapping content. In particular, the inequality (28) can be derived by using the same argument under the elliptical distribution $P_{\hat{\Gamma}} = ECP_\tau(0, \Gamma, \eta)$. That is,
\[
D_u(\hat{\Gamma}, P_{\Gamma}) \geq \frac{1}{2} - \frac{\epsilon}{1 - \epsilon} - C_1 \sqrt{\frac{\log |U| + \log(2/\delta)}{n}}, \quad \text{for all } u \in U,
\]
with probability at least $1 - 2\delta$. By Proposition 5.2 and the definition of $G(t)$ in (12), we have
\[
\frac{1}{2} - D_u(\hat{\Gamma}, P_{\Gamma}) = \left| G(1) - G\left( \frac{u^T\hat{\Gamma}u}{u^T\Gamma u} \right) \right|.
\]
Combining this fact with (43), we have
\[
\sup_{u \in U} \left| G(1) - G\left( \frac{u^T\hat{\Gamma}u}{u^T\Gamma u} \right) \right| \leq \frac{\epsilon}{1 - \epsilon} + C_1 \sqrt{\frac{\log |U| + \log(2/\delta)}{n}},
\]
with probability at least $1 - 2\delta$. As long as $\epsilon < \tau/3$ and $\sqrt{\frac{\log |U| + \log(2/\delta)}{n}}$ is sufficiently small, we have $\frac{\epsilon}{1 - \epsilon} + C_1 \sqrt{\frac{\log |U| + \log(2/\delta)}{n}} < \tau$. By the assumption (13), we must have $\left| 1 - \frac{u^T\hat{\Gamma}u}{u^T\Gamma u} \right| \leq \alpha$, which further implies
\[
\sup_{u \in U} \left| 1 - \frac{u^T\hat{\Gamma}u}{u^T\Gamma u} \right| \leq C_2\kappa^{1/2} \left( \epsilon + \sqrt{\frac{\log |U| + \log(2/\delta)}{n}} \right),
\]
with probability at least $1 - 2\delta$. By the fact that $\Gamma \in F(M)$, we have
\[
\sup_{u \in U} \left| u^T(\hat{\Gamma} - \Gamma)u \right| \leq C_3\kappa^{1/2} \left( \epsilon + \sqrt{\frac{\log |U| + \log(2/\delta)}{n}} \right),
\]
with probability at least $1 - 2\delta$, which completes the proof.

Proofs of Theorems 5.1-5.5. The result of these results follow Theorem B.1 and the arguments in the proofs of Theorem 3.1, Theorem 3.3, Theorem 3.4, Theorem 3.6 and Theorem 3.8.
C Proofs in Section 6

Proof of Theorem 6.1. Let us shorthand \( P_{\epsilon, \Theta, Q}, \{X_i\}_{i=1}^n \) and \( \{Y_i\}_{i=1}^{n_1} \) by \( P, X \) and \( Y \). For any estimator \( \hat{\theta}(\cdot) \), we have

\[
\mathbb{P} \left\{ L(\hat{\theta}(X), \theta) > \frac{1}{2} A^{-1} \delta \right\} \\
\geq \mathbb{P} \left\{ L(\hat{\theta}(X), \hat{\theta}(Y)) > \delta, L(\hat{\theta}(Y), \theta) \leq \frac{1}{2} A^{-1} \delta \right\} \\
\geq \mathbb{P} \left\{ L(\hat{\theta}(X), \hat{\theta}(Y)) > \delta \right\} - \mathbb{P} \left\{ L(\hat{\theta}(Y), \theta) > \frac{1}{2} A^{-1} \delta \right\},
\]

where the inequality (44) is due to (16) and the inequality (45) is union bound. The identity \( \epsilon = \epsilon(\hat{\theta}, \Theta, \delta) \) means that \( \sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(X), \hat{\theta}(Y)) > \delta \right\} > c \). Hence, by (45), we have

\[
\sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(X), \theta) > \frac{1}{2} A^{-1} \delta \right\} + \sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(Y), \theta) > \frac{1}{2} A^{-1} \delta \right\} \geq c.
\]

Let us upper bound \( \sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(Y), \theta) > \frac{1}{2} A^{-1} \delta \right\} \) by

\[
\sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(Y), \theta) > \frac{1}{2} A^{-1} \delta \right\} \\
= \sup_{\theta \in \Theta} \mathbb{E} P_{\theta}^{n_1} \left\{ L(\hat{\theta}, \theta) > \frac{1}{2} A^{-1} \delta \right\} \\
\leq \sup_{\theta \in \Theta} \max_{n_1 \geq n/3} \sup_{\theta \in \Theta} P_{\theta}^{n_1} \left\{ L(\hat{\theta}, \theta) > \frac{1}{2} A^{-1} \delta \right\} + \mathbb{P} \left\{ n_1 < \frac{n}{3} \right\} \\
\leq \max_{n_1 \geq n/3} \sup_{\theta \in \Theta} P_{\theta}^{n_1} \left\{ L(\hat{\theta}, \theta) > \frac{1}{2} c_1 A^{-1} \delta \right\} + \exp \left( -\frac{n}{18} \right) \\
\leq \sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(X), \theta) > \frac{1}{2} c_1 A^{-1} \delta \right\} + \exp \left( -\frac{n}{18} \right).
\]

In the equality (47), the expectation operator \( \mathbb{E} \) is associated with the probability \( n_1 \sim \text{Binomial}(n, 1 - \epsilon) \). The inequality (48) is by Hoeffding’s inequality and the assumption \( \epsilon < 1/2 \). The inequality (49) is by the assumption (17). Finally, the inequality (50) is due to the relation \( \{ P_{\theta} : \theta \in \Theta \} \subset \{(1 - \epsilon) P_{\theta} + \epsilon Q : \theta \in \Theta, Q \} \). Combining the above argument with (46) and the inequality \( \sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(X), \theta) > \frac{1}{2} c_1 A^{-1} \delta \right\} \geq \sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(X), \theta) > \frac{1}{2} A^{-1} \delta \right\} \), we get

\[
2 \sup_{\theta \in \Theta} \sup_Q \mathbb{P} \left\{ L(\hat{\theta}(X), \theta) > \frac{1}{2} c_1 A^{-1} \delta \right\} \geq c - \exp \left( -\frac{n}{18} \right),
\]

which leads to the desired conclusion for a sufficiently large \( n \).\[\square\]
D Proofs of Propositions and Lemmas

Proof of Proposition 3.1. For any $u \in S^{p-1}$, we have $\mathcal{D}_u(\beta \Sigma, P_\Sigma) = P_\Sigma \left( |u^T X| < \beta u^T \Sigma u \right)$ and $P_\Sigma \left( |u^T X| > \beta u^T \Sigma u \right)$, which equals $(2\Phi(\sqrt{\beta}) - 1) \wedge (1 - 2\Phi(\sqrt{\beta}))$ because $u^T X/\sqrt{u^T \Sigma u} \sim N(0,1)$. Since $\Phi(\sqrt{\beta}) = 3/4$, we have $(2\Phi(\sqrt{\beta}) - 1) \wedge (1 - 2\Phi(\sqrt{\beta})) = 1/2$ and thus $\mathcal{D}_u(\beta \Sigma, P_\Sigma) = \inf_{u \in U} \mathcal{D}_u(\beta \Sigma, P_\Sigma) = 1/2$. □

Proof of Proposition 5.1. The existence is guaranteed by the continuity. Suppose there are two canonical representation, then equivalently, $G(t) = \frac{1}{t}$ will have another solution besides $t = 1$. However, $G(t) = G(1)$ for some $t \neq 1$ contradicts (13). This completes the proof. □

Proof of Proposition 5.2. For any $u \in S^{p-1}$, we have $\mathcal{D}_u(\Gamma, P_\Gamma) = P_\Gamma \left( |u^T X| < u^T \Gamma u \right) \wedge P_\Gamma \left( |u^T X| > u^T \Gamma u \right)$, which equals $G(1) \wedge (1 - G(1^-))$ by the definition of $G(t)$. According to the definition of canonical representation, $G(1) = 1/2$ and thus the proof is complete. □

Proof of Proposition 5.3. For $X \sim EC_p(0, \Gamma, \eta)$, its characteristic function must be in the form $E \exp(\sqrt{-1} t^T X) = \phi(t^T \Gamma t)$ with some univariate function $\phi(\cdot)$ called characteristic generator. The characteristic generator $\phi(\cdot)$ is completely determined by the distribution of $\eta$ [19]. For multivariate Gaussian, $\phi(v) = \exp(-v/(2\beta))$. For multivariate Laplacian, $\phi(v) = 1/(1 + \beta v/2)$. For multivariate $t$, $\phi(v) = 2^{(\beta \nu)/4} K_{d/2}(\sqrt{\beta \nu})$, where $K_{d/2}(\cdot)$ is the modified Bessel of the second kind. Note that for all the examples considered, $\phi(\cdot)$ does not depend on $\Gamma$ or the dimension $p$, which means the distribution of $\eta$ does not depend on $p$, either. Since the distribution of $\eta$ fully determines the function $G(\cdot)$ defined by (12), the equation $G(1) = 1/2$ is satisfied for some constant $\beta$ independent of $p$. Finally, the condition (13) is satisfied for some constants $\tau, \alpha, \kappa$ independent of $p$ because of the smoothness of the derivative of $G(\cdot)$. □

Proof of Lemma 7.1. Note that $n_2 \sim \text{Binomial}(n, \epsilon)$. By Hoeffding’s inequality, $P(n_2 > ne + t) \leq \exp(-2t^2/n)$ for any $t > 0$. Thus, $n_2 \leq ne + \sqrt{\frac{2}{n} \log(1/\delta)}$ with probability at least $1 - \delta$. This implies $n_1 \geq n(1 - \epsilon) - \sqrt{\frac{1}{2n} \log(1/\delta)}$ and therefore,

$$\frac{n_2}{n_1} \leq \frac{\epsilon + \sqrt{\frac{1}{2n} \log(1/\delta)}}{1 - \epsilon - \sqrt{\frac{1}{2n} \log(1/\delta)}}$$

with probability at least $1 - \delta$. Under the assumption that $\epsilon < 1/2$ and $n^{-1} \log(1/\delta)$ sufficiently small, we have $n_2/n_1 \leq \epsilon/(1 - \epsilon) + C \sqrt{n^{-1} \log(1/\delta)}$ with probability at least $1 - \delta$. This proves (19). A symmetric argument leads to

$$\frac{n_2}{n_1} \geq \frac{\epsilon - \sqrt{\frac{1}{2n} \log(1/\delta)}}{1 - \epsilon + \sqrt{\frac{1}{2n} \log(1/\delta)}}$$

with probability at least $1 - \delta$. For $\delta = 1/2$ and $\epsilon^2 > 1/n$, we have $n_2/n_1 \geq \epsilon \epsilon$ with probability at least $1/2$. Thus, the proof is complete. □
Proof of Lemma 7.2. For any \( \theta \in \Theta \) and \( Q \), define

\[
Q' = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2} Q + \frac{\epsilon_1}{\epsilon_2} P_\theta.
\]

It is easy to see that \( Q' \) is a probability measure, and it satisfies

\[
(1 - \epsilon_1) P_\theta + \epsilon_1 Q = (1 - \epsilon_2) P_\theta + \epsilon_2 Q'.
\]

This immediately gives the desired conclusion. \( \square \)