Contextuality-by-default for behaviours in compatibility scenarios

Alisson Tezzin, Rafael Wagner, Barbara Amaral
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Compatibility-hypergraph approach to contextuality
Compatibility scenarios

A compatibility scenario is a triple \( S \equiv (X, C, O) \) where

- \( X \) is a finite set (set of measurements)
- \( C \) is a collection of contexts
- \( O \) is a finite set (set of outcomes)
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- $\mathcal{C}$ is a collection of contexts
- $O$ is a finite set (set of outcomes)
Behaviours

OCdenotes the set of all functions $C \rightarrow O$.

A behaviour $p$ for a scenario $(X, C, O)$ is a function which associates, to each context $C$, a probability distribution $p_C$ over $O$. 
Behaviours

- $O^C$ denotes the set of all functions $C \rightarrow O$
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- A **behaviour** $p$ for a scenario $(X, C, O)$ is a function which associates, to each context $C$, a probability distribution $p^C$ over $O^C$
From behaviours to random variables
“We label all measurements contextually: this means that a property $q$ is represented by different random variables $R^C_q$ depending on the context $C$.”

[1] J. V. Kujala, E. N. Dzhafarov, and J.-A. Larsson, “Necessary and sufficient conditions for an extended noncontextuality in a broad class of quantum mechanical systems,” Phys. Rev. Lett., vol. 115, p. 150401, Oct 2015
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From behaviours to random variables

- “Scenario + behaviour ⇒ system”?
From behaviours to random variables

- “Scenario + behaviour $\Rightarrow$ system”? 
- We do that using marginal distributions
From behaviours to random variables
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| System   | Behaviour |
|----------|-----------|
| consistent connected | maximally non-contextual |
| non-degenerate | non-contextual in the extended sense |
| System          | Behaviour        |
|-----------------|------------------|
| consistent      | connected        |
From behaviours to random variables

| System | Behaviour |
|--------|-----------|
| consistent connected | non-degenerate |
## From behaviours to random variables

| System                              | Behaviour                      |
|-------------------------------------|--------------------------------|
| consistent connected                | non-degenerate                 |
| maximally non-contextual description|                                |
| **System**                        | **Behaviour**                  |
|----------------------------------|--------------------------------|
| consistent connected             | non-degenerate                 |
| maximally non-contextual         | non-contextual in the extended sense |
Results and conclusions

$NC = \bullet$

$ND = \bullet \bullet$
Results and conclusions

\[ NC = \text{O} \]

\[ ND = \text{O} \quad \text{and} \quad \text{O} \]

\[ ND_{eg} = \text{O} + \text{O} + \text{O} \]
Results and conclusions

\[ NC = \circ \]

\[ ND = \circ \bullet \]

\[ ND_{eg} = \circ \bullet \circ \circ \]

\[ NC_{ext} = \circ \circ \circ \circ \]
Results and conclusions

$\text{NC}_{\text{ext}} = \bigcirc + \bigcirc$

$\text{ND}_{\text{eg}} = \bigcirc + \bullet + \bullet$

$\text{ND} = \bigcirc + \bullet$

$\text{NC} = \bigcirc$
Results and conclusions

- The idea behind contextuality-by-default is implicit in the compatibility-hypergraph approach to contextuality
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• Non-degeneracy (consistent connectedness) defines a polytope
Results and conclusions

- The idea behind contextuality-by-default is implicit in the compatibility-hypergraph approach to contextuality.
- Non-degeneracy (consistent connectedness) defines a polytope.
- We can relax the non-disturbance condition as a physical requirement.
Thank you
For a context $C$ we can associate a probability space $(\Omega^C, \Sigma^C, \mu^C)$ where

$$x_C : \Omega^C \rightarrow O$$

$$p^C(s) = \mu^C(\bigcap_{x \in C} x^{-1}_C(s_x))$$

$$p^C_x(o) = \mu^C(x^{-1}_C(o))$$
Classical behaviours

$p$ in $(\mathcal{X}, \mathcal{C}, O)$ is classical iff exists:

(a) a measurable space $(\Omega, \Sigma)$
(b) a function $\pi : \mathcal{X} \to MF(\Omega, O)$
(c) A probability measure $\mu$ in $(\Omega, \Sigma)$
satisfying

- For any $C \in \mathcal{C}$ and any $s \in O^C$,

$$p^C(s) = \mu\left(\bigcap_{x \in C} \pi(x)^{-1}(s_x)\right)$$
$p$ is classical iff exists a distribution $\bar{p} : O^X \to [0, 1]$ satisfying, for each context $C$

$$\bar{p}_C = p^c$$
Quantum behaviours

$p$ in $(\mathcal{X}, \mathcal{C}, O)$ is a quantum behaviour iff exists

(a) A Hilbert space $H$
(b) A function $\theta : \mathcal{X} \to \mathcal{B}(H)^\mathbb{R}$
(c) A density operator $\rho \in \mathcal{B}(H)$

satisfying

- For any $C \in \mathcal{C}$,
  \[ [\theta(x), \theta(y)] = 0 \quad \forall x, y \in C \]

- For any $C \in \mathcal{C}$ and $s \in O^C$
  \[ p^C(s) = \text{Tr}(\rho \prod_{x \in C} P^{(x)}_{ss}) \]
Classical behaviours
Classical behaviours
Quantum behaviours
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Quantum behaviours
Quantum behaviours

- The joint spectrum of $A_1, \ldots, A_n \in \mathcal{B}(H)^\mathbb{R}$ is

  $$\Omega^C \equiv \sigma(A) \equiv \{(\lambda_1, \ldots, \lambda_n) \in \sigma(A_1) \times \ldots \times \sigma(A_n); \prod_{i=1}^{n} P_{\lambda_i}^{A_i} \neq 0\}.$$  

- $\hat{A}_i : \sigma(A) \to \sigma(A_i)$ is the projection

  $$\sigma(A) \ni (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_i \in \sigma(A_i).$$  

- Consequently

  $$(\lambda_1, \ldots, \lambda_n) = \bigcap_{i=1}^{n} \hat{A}_i^{-1}(\lambda_i).$$
Quantum behaviours

- A state $\rho$ defines a probability measure $\mu^\rho_A$ in $\sigma(A)$ by means of the Born rule:

$$
\mu^\rho_A(\bigcap_{i=1}^n \hat{A}_i^{-1}(\lambda_i)) = \mu^\rho_A((\lambda_1, ..., \lambda_n)) = \text{Tr}(\rho \prod_{i=1}^n P^A_{\lambda_i})
$$

- We also have

$$
\mu^\rho_A(\hat{A}_i^{-1}(\lambda)) = \text{Tr}(\rho P^A_{\lambda})
$$
Quantum behaviours

\((\sigma(A), \mathcal{P}(\sigma(A)), \mu^\rho_A)\) “satisfies”

\[ x_C : \Omega^C \rightarrow O \]  \hspace{1cm} (4)

\[ p^C(s) = \mu^C(\bigcap_{x \in C} x_C^{-1}(s_x)) \]  \hspace{1cm} (5)

\[ p^C_x(o) = \mu^C(x_C^{-1}(o)) \]  \hspace{1cm} (6)
• For any $x \in \mathcal{X}$ and $C \in C_x$ we define $p^C_x : O \rightarrow [0, 1]$ by

$$p^C_x(o) \doteq p^C_{\{x\}}(o) = \sum_{s \in O^C \mid s_x = o} p^C(s)$$
From behaviours to random variables

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$$p^C_x(o) \doteq p^C_{\{x\}}(o) = \sum_{s \in \mathcal{O}^C_x, s_x = o} p^C(s)$$

- $p^C_x, C \in C_x$
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- $p^C_x$, $C \in C_x$
- $x_C$, $C \in C_x$
For any \( x \in \mathcal{X} \) and \( C \in \mathcal{C}_x \) we define \( p^C_x : O \to [0, 1] \) by

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- \( p^C_x \), \( C \in \mathcal{C}_x \)
- \( x_C \), \( C \in \mathcal{C}_x \)
- A behaviour defines a system
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- $p^C_x$, $C \in \mathcal{C}_x$
- $x_C$, $C \in \mathcal{C}_x$
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