An Inequality for the trace of matrix products, using absolute values

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Abstract

The absolute value of matrices is used in order to give inequalities for the trace of products. An application gives a very short proof of the tracial matrix Hölder inequality.

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1 The baby inequality and its application

1 THEOREM. Using absolute values: Consider two complex \( m \times n \) matrices \( A, B \) and their absolute values, \( |A| = (A^*A)^{1/2}, \ |A^*| = (AA^*)^{1/2} \). Then

\[
|\text{Tr} \ A^*B| \leq (\text{Tr} \ |A| \cdot |B|)^{1/2} \cdot (\text{Tr} \ |A^*| \cdot |B^*|)^{1/2}
\]  

(1)

Proof. Let \( e_i \) be an orthonormal basis made of normalized eigenvectors of \( A^*A \), with eigenvalues \( a_i^2 \geq 0 \). For \( a_i \neq 0, f_i = a_i^{-1} A e_i \) are normalized eigenvectors of \( AA^* \), obeying \( A^* f_i = a_i e_i \). Eventually, to make a full basis, this set has to be completed by introducing eigenvectors of \( AA^* \) with eigenvalue 0. Analogously, there are basis-sets of normalized vectors \( g_j \) and \( h_j \), such that \( B g_j = b_j h_j, B^* h_j = b_j g_j \), with \( b_j \geq 0 \). With these vectors we get

\[
|\text{Tr} \ A^*B| = \left| \sum_{i,j} a_i b_j \langle g_j, e_i \rangle \langle f_i, h_j \rangle \right|.
\]  

(2)

Applying the Cauchy-Schwarz inequality gives

\[
|\text{Tr} \ A^*B| \leq \left( \sum_{i,j} a_i b_j \langle e_i, g_j \rangle \langle f_j, e_i \rangle \right)^{1/2} \left( \sum_{i,j} a_i b_j \langle f_i, h_j \rangle \langle h_j, f_i \rangle \right)^{1/2}
\]  

(3)

\[
= \left( \text{Tr} \ |A| \cdot |B| \right)^{1/2} \left( \text{Tr} \ |A^*| \cdot |B^*| \right)^{1/2}
\]  

(4)

In the last step the identities \( |A| e_i = a_i e_i, |A^*| f_i = a_i f_i, |B| g_j = b_j g_j \), and \( |B^*| h_j = b_j h_j \) have been used. \( \square \)

We remark that the inequality is sharp. It becomes an equality in case both matrices \( A \) and \( B \) have rank one. This follows from the fact that the “sum” in \( \langle 2 \rangle \) consists of only one term, so the Schwarz inequality in \( \langle 3 \rangle \) becomes an equality. A simple example is given with \( 2 \times 2 \) matrices:

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},
\]  

(5)

with \( |B| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ |B^*| = \sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \);

so \( \text{Tr} \ A^*B = 1, \ |A| \cdot |B| = 1/\sqrt{2}, \ |A^*| \cdot |B^*| = \sqrt{2} \).

Other cases where \( \langle 3 \rangle \) becomes an equality are \( \langle e_i, g_j \rangle = \alpha \langle f_i, h_j \rangle \ \forall i,j \), for some constant \( \alpha \).

The example shows also that the distinction between the absolute values \( |B| \) and \( |B^*| \) is necessary. Only if both \( A \) and \( B \) are normal matrices the absolute values are equal, \( |A^*| = |A|, |B^*| = |B| \). In such a case equation \( \langle 1 \rangle \) becomes \( |\text{Tr} \ A^*B| \leq |\text{Tr} \ |A| \cdot |B| | \).

The following application was at the origin of my investigations, searching for a simple proof of the Hölder inequality for matrices and operators. (For other proofs, see f.e. \[MBR72, RS75, RB97, EC09\])
2 THEOREM. **Matrix Hölder Inequality:** Consider two \( m \times m \) matrices \( A, B \) and their absolute values, then

\[
|\text{Tr} A^*B| \leq (\text{Tr} |A|^p)^{1/p} \cdot (\text{Tr} |B|^q)^{1/q}, \quad 1 \leq p, q \leq \infty, \quad p^{-1} + q^{-1} = 1 \quad (6)
\]

**Proof.** Using the same notation as in the proof of Theorem 1, we note that

\[
\text{Tr} |A|^p = \text{Tr} |A^*|^p = \sum_i a_i^p. \quad \text{So, the Hölder Inequality for the left hand side of (1) is proven, if it holds for each factor on the right hand side. There we have normal operators. For these we can use the classical Hölder Inequality for weighted sums, followed by using completeness of the basis sets } \{ e_i \} \quad \text{and} \quad \{ g_j \}:
\]

\[
\text{Tr} |A| \cdot |B| = \sum_{i,j} a_i b_j \langle e_i, g_j \rangle^2 \leq \left( \sum_{i,j} a_i^p \langle e_i, g_j \rangle^2 \right)^{1/p} \cdot \left( \sum_{i,j} b_j^q \langle e_i, g_j \rangle^2 \right)^{1/q} = \left( \sum_i a_i^p \right)^{1/p} \cdot \left( \sum_j b_j^q \right)^{1/q} = (\text{Tr} |A|^p)^{1/p} \cdot (\text{Tr} |B|^q)^{1/q}.
\]

Analogously for \( \text{Tr} |A^*| \cdot |B^*| \). \qed

If one is interested in characterizing cases of equality for the matrix Hölder inequality, one has to check whether \( \langle e_i, g_j \rangle = \alpha \langle f_i, h_j \rangle \), as stated above, and also whether the classical Hölder inequality used in the proof becomes an equality.

2 Generalizations

There are possibilities to generalize the baby inequality. It can grow by: Inserting extra matrices, one \( m \times m \) another one \( n \times n \); using different exponents for the different absolute values; going into vector spaces with infinite dimension.

One can insert extra matrices \( M \) and \( N \):

3 THEOREM. **Inequality with intermediate matrices:**

\[
|\text{Tr} MA^*NB| \leq (\text{Tr} M |A| M^* |B|)^{1/2} \cdot (\text{Tr} N^* |A^*| N |B^*|)^{1/2} \quad (7)
\]

**Proof.** Just insert the extra matrices in the right places in the inner products appearing in (2) and (3): Replace \( \langle g_j, e_i \rangle \) by \( \langle g_j, Me_i \rangle \) and \( \langle f_i, h_j \rangle \) by \( \langle f_i, Nh_j \rangle \). \qed

There is the possibility to consider different exponents:

4 THEOREM. **Inequality with exponents:**

\[
|\text{Tr} A^*B| \leq (\text{Tr} |A|^\alpha |B|^\beta)^{1/2} \cdot (\text{Tr} |A^*|^{2-\alpha} |B^*|^{2-\beta})^{1/2}, \quad 0 \leq \alpha, \beta \leq 2, \quad (8)
\]

with \( 0^0 = 0 \), so that \( |A|^0 = \text{projector onto range}(|A|) \).
Proof. Modify (2) and (3) as

\[ |\text{Tr} A^* B| = \left| \sum_{i,j} \varphi_{i,j} \cdot \psi_{i,j} \right| \leq \left( \sum_{i,j} |\varphi_{i,j}|^2 \right)^{1/2} \cdot \left( \sum_{i,j} |\psi_{i,j}|^2 \right)^{1/2}, \tag{9} \]

with \( \varphi_{i,j} = a_i^{\alpha/2} b_j^{\beta/2} (e_i, g_j) \), \( \psi_{i,j} = a_i^{1-\alpha/2} b_j^{1-\beta/2} (h_i, f_i) \). Observe, that for \( a_i = 0 \) the matrix elements involving \( e_i \) or \( f_i \) are just absent. The same holds for \( b_j, g_j, h_j \).

Extensions into infinite dimensions can be done in different ways. I present the following result:

5 THEOREM. Inequality for two Hilbert Schmidt class operators:
Let \( A \) and \( B \) be operators from Hilbert space \( \mathcal{H} \) to the Hilbert space \( \mathcal{K} \), with the properties \( \text{Tr}_\mathcal{H} A^* A = \text{Tr}_\mathcal{K} A \cdot A^* < \infty \) and \( \text{Tr}_\mathcal{H} B^* B = \text{Tr}_\mathcal{K} B \cdot B^* < \infty \). Then

\[ |\text{Tr}_\mathcal{H} A^* B| \leq (\text{Tr}_\mathcal{H} |A| |B|)^{1/2} \cdot (\text{Tr}_\mathcal{K} |A^*| |B^*|)^{1/2} \] \tag{10} \]

Proof. As in the proof of Theorem 1 we use the singular values \( a_i \) and basis sets \( e_i, f_i \), here \( e_i \in \mathcal{H}, f_i \in \mathcal{K} \), such that \( A e_i = a_i f_i \), and analogously \( B g_i = b_i h_i \). In Dirac’s notation \( A = \sum_i |f_i\rangle a_i \langle e_i| \) and \( B = \sum_i |h_i\rangle b_i \langle g_i| \). \( A \) being in the Hilbert-Schmidt class means

\[ \text{Tr}_\mathcal{H} A^* A = \text{Tr}_\mathcal{K} A \cdot A^* = \text{Tr}_\mathcal{H} |A|^2 = \text{Tr}_\mathcal{K} |A^*|^2 = \left( \sum_i a_i^2 \right)^{1/2} < \infty, \]

and the analogue for \( B \). Introducing the operators with finite rank

\[ A_N = \sum_{i}^N |f_i\rangle a_i \langle e_i| \quad \text{and} \quad B_N = \sum_{i}^N |h_i\rangle b_i \langle g_i| \], \tag{11} \]

one can apply Theorem 1 to the matrices which represent these operators, giving the inequality (10) for \( A_N \) and \( B_N \). One observes the convergences in norm:

\[ \| A - A_N \| = \| A^* - A_N^* \| = \| |A| - |A_N| \| = \| |A^*| - |A_N^*| \| = \left( \sum_{N+1}^\infty a_i^2 \right)^{1/2} \rightarrow_{N \rightarrow \infty} 0, \]

and the same for \( B \). The Hilbert-Schmidt inner products are jointly norm-continuous in both factors, so each side of the inequality (10) for \( A_N \) and \( B_N \) converges as \( N \rightarrow \infty \), giving the same inequality without the \( N \) as an index. \( \square \)

Applications are new proofs for Hölder type inequalities used in mathematical physics. They will be discussed in a following article.
3 Comparison with another use of absolute values

The product $A^* \cdot B$ can be represented as

$$A^* \cdot B = U \cdot |A^*| \cdot |B^*| \cdot V^*, \tag{12}$$

by extending the $m \times n$ matrices to $N \times N$ matrices, where $N = \max\{m,n\}$, and using the polar decompositions $A^* = U \cdot |A^*|$, $B^* = V \cdot |B^*|$ with unitary operators $U$, $V$. (Equivalently, one can stay with the $m \times n$ matrices and use isometries $U$ and $V$ instead of unitary operators.) This equality implies that the set of singular values of $A^* \cdot B$ is identical to that of $|A^*| \cdot |B^*|$. (Eventually, when staying with $m \neq n$, the numbers of zeroes are different.) So, all the unitarily invariant norms, see [RB97], are identical. One identity is for the operator norm

$$\|A^* \cdot B\| = \| |A^*| \cdot |B^*| \|, \tag{13}$$

another one gives

$$\text{Tr} \, |A^* \cdot B| = \text{Tr} \, (|A^*| \cdot |B^*|). \tag{14}$$

Together with $|\text{Tr} M| \leq \text{Tr} |M|$, which holds for each matrix, we get

$$|\text{Tr} A^* \cdot B| \leq \text{Tr} \, (|A^*| \cdot |B^*|). \tag{15}$$

This inequality is not as sharp as the baby inequality given in Theorem 1. As an example use the same matrices as in equation (5). They give 1 on the l.h.s. but $\sqrt{2}$ on the r.h.s. of (15).

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