probabilistic finiteness properties for profinite groups

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Abstract. We introduce a probabilistic version of the $\text{FP}_n$ property for profinite groups, called $\text{PFP}_n$, using the notion of positively finitely generated modules. We further define and study related properties such as positive finite presentability. Finally we answer some questions from [14], describing the relation between PFG and PFR groups.

1. Introduction

1.1. Motivation. The study of finiteness properties of abstract groups has a long history; see [4] for some background. Analogously to abstract groups, a profinite group $G$ is said to be of type $\text{FP}_n$ over a profinite ring $R$ if there is a projective resolution $\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow R \rightarrow 0$ with $P_0, \ldots, P_n$ finitely generated profinite $R[[G]]$-modules.

Even the first steps into studying this property run into some difficulties for profinite groups that do not occur in the abstract case; for instance $\text{FP}_1$ and finite generation are not equivalent (see [8]). One reason is that we have to deal with topological generation in the category of profinite groups and this is more fragile than finite generation for abstract groups.

In this paper we consider an alternative notion of finite generation. The class of positively finitely generated groups (PFG groups for short) was introduced in [18] and it consists of those profinite groups $G$ where, for some $k$, $k$ Haar-random elements generate $G$ with positive probability (cf. Section 2.3). Positive finite relatedness was introduced in [14] in the spirit of PFG (see Section 2.3 for the definition).

Analogously, we set out to introduce a new family of higher finiteness properties for profinite groups via the concept of PFG modules and modules of type PFP$_n$. We will also define and study related properties, such as positively finitely presented profinite groups.

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1.2. Main results. We introduce in Section 3 the notion of positively finitely presented profinite groups, as a natural analogue of the definition of finitely presented groups— that is, groups that have ‘PFG presentation’ (cf. Definition 3.5). We show in Lemma 3.9 that this class is closed under extensions, and in Proposition 3.10 that a group is positively finitely presented if and only if the kernel of the universal Frattini cover $\hat{G} \to G$ of $G$ is positively normally finitely generated in $\hat{G}$. Moreover, in the class of PFG groups, we show that positive finite presentability is equivalent to the property of being ‘positively finitely related’ introduced in [14] (Corollary 3.8).

We can define positively finite generated (PFG) modules analogously to positively finitely generated groups.

**Definition 1.1.** A profinite group $G$ has type $PFP_n$ over a commutative profinite ring $R$ if there is a projective resolution $P_*$

\begin{equation}
\ldots \to P_n \to \ldots \to P_1 \to P_0 \to R \to 0
\end{equation}

with $P_0, \ldots, P_n$ PFG $R[G]$-modules.

The first crucial observation, using the theory of projective covers, is that a profinite group has type $PFP_0$ over $R$ if and only if $R$ is PFG; and it turns out that type $PFP_n$ coincides with type $FP_n$ in the class of prosolvable groups (cf. Example 5.6).

Our first main result is an equivalent cohomological characterisation of type $PFP_n$. For a profinite ring $R$ and $k \in \mathbb{N}$, denote by $S^R_k$ the set of simple $R$-modules of order $k$.

**Theorem A.** Let $G$ be a profinite group and $R$ a profinite ring. Then, $G$ has type $PFP_n$ over $R$ if and only if $\sum_{S \in S^R_k} (|H^m_R(G, S)| - 1)$ has polynomial growth in $k$ for all $m \leq n$.

This is shown in Theorem 4.15.

In Section 5.1 we study in more detail two natural modules associated to a profinite group $G$: the group ring $\hat{\mathbb{Z}}[G]$ and the augmentation ideal $I_2[\hat{\mathbb{Z}}]$. When $\hat{\mathbb{Z}}[G]$ is PFG (as a module for itself) we say that $G$ has UBERG (to maintain consistency with the terminology of [14]). $I_2[\hat{\mathbb{Z}}]$ being PFG is studied in [8], and we maintain consistency with the terminology there by saying that $G$ is APFG in this case. We show that if the $I_2[\hat{\mathbb{Z}}]$ is PFG, then $\hat{\mathbb{Z}}[G]$ is, and the converse is true if $G$ has type $FP_1$ over $\hat{\mathbb{Z}}$.

Next, we look at closure properties of the class of groups of type $PFP_n$. The proofs of the next propositions can be found in Proposition 5.8 and Proposition 5.19.

**Proposition B.** Let $G$ be a profinite group and let $H$ an open subgroup of $G$. Then $G$ is of type $PFP_n$ if and only if $H$ is.
Proposition C. Let $G_1$ and $G_2$ be profinite groups of type PFP$_n$ and PFP$_m$ over $R$, respectively. Then $G_1 \times G_2$ is of type PFP$_{\min(n,m)}$ over $R$.

In Section 5.2 we start examining property PFP$_1$ in more detail. Here we compare property PFP$_1$ with APFG, and we show that APFG implies PFP$_1$ (Lemma 5.9). Using this, we can give an example of group of type PFP$_1$ that is not PFG (see Example 5.11). Additionally we settle the second part of [14, Question 1.2]: we prove

Proposition D. Let $G$ be a finitely presented profinite group. If $G$ is PFG, then it is PFR.

In Section 5.3, we investigate the relation between positive finite presentability and type PFP$_2$. We show in Lemma 5.12 that positive finite presentability implies PFP$_2$, and in Proposition 5.13 that type PFP$_2$ can be detected by considering the minimal presentation of a group: we show that for $G$ a profinite group of type PFP$_1$, and $\tilde{G} \to G$ the universal Frattini cover of $G$, $G$ has type PFP$_2$ if and only if $R = \ker(\tilde{G} \to G)$ has polynomial maximal $\tilde{G}$-stable abelian subgroup growth.

In the case of modules for abstract groups, or the usual definition of finite generation for profinite modules, we have the following nice property: if $M$ is an $H$-module with $H \leq G$, $M$ is finitely generated if and only if Ind$_G^H M$ is. But it is not hard to show that an analogous property fails for positive finite generation. We confront this problem in Section 5.4.

Specifically, we define a relative version of type PFP$_n$. Given profinite groups $H \leq G$, we say that $H$ has relative type PFP$_n$ in $G$ if all PFG projective $R[[G/H]]$-modules have type PFP$_n$ over $R[[G]]$. Note that in the type FP$_n$ case, relative type FP$_n$ is equivalent to type FP$_n$.

Theorem E. Suppose that $H$ has relative type PFP$_m$ in $G$ over $R$.

(i) If $G$ has type PFP$_n$ over $R$, then $G/H$ has type PFP$_{\min(m+1,n)}$ over $R$.

(ii) If $G/H$ has type PFP$_n$ over $R$, then $G$ has type PFP$_{\min(m,n)}$ over $R$.

This is shown in Corollary 5.17. Incidentally, the previous theorem specialises to the correspondent result for abstract groups and our proof that does not depend on the usual “spectral sequence argument”.

We finish with Section 6, where we produce some novel examples to distinguish some of the aforementioned classes of groups. First, for any prime $p$, we give examples of type PFP$_1$ groups $G$ over $\mathbb{Z}_p$ such that the group ring $\mathbb{Z}_p[[G]]$ is not PFG (see Proposition 6.1). We would like to give examples of this behaviour over $\hat{\mathbb{Z}}$, to distinguish between the classes of groups with UBERG and type PFP$_1$ over $\hat{\mathbb{Z}}$, but for
now the question remains open. Such examples cannot appear among
pronilpotent groups (see Proposition 6.5 and the related Question
about the prosolvable case).

Although, trivially, if a module has type PFP, it has type FP, we
show that the converse is not true, by showing in Section 6.1 that the
free profinite group on 3 generators has type FP but not type PFP
over \( \hat{\mathbb{Z}} \).

Finally, we answer the first part of [14, Question 1.2] negatively, as
we find a PFR group which is not PFG (see Section 6.2).

2. Preliminaries and notation

We state now some conventions which will be in force for the rest
of this article. All subgroups and submodules will be assumed to be
closed. Generation will always be intended in the topological sense.
All homomorphisms will be continuous. Modules will be assumed to
be left modules.

2.1. Homology theory for profinite groups. In the course of this
work, we will need the usual ‘homological lemmas’ for profinite groups,
such as snake lemma, horseshoe lemma, Schanuel’s lemma, Lyndon-
Hochschild-Serre spectral sequence, the long exact sequence in coho-
mology, and Ext groups – see for instance [20]. Other tools such as
the mapping cone construction, valid in all abelian categories, can be
found in [22] for example.

2.2. Haar measure. For a profinite group \( G \), we denote by \( \mu_G \) the
(left) Haar measure of \( G \), see [17, Chap. 11] for basic properties. We
will always consider the normalised Haar measure, in this way we can
turn a profinite group into a probability space. We will need the fol-
lowing basic lemma.

**Lemma 2.1** ([17, Lemma 11.1.1]). Let \( G \) be a profinite group, let \( K \)
be a closed normal subgroup of \( G \) and \( \pi : G \to G/K \) be the natural
projection. If \( X \) is a closed subset of \( G \), then \( \mu_{G/K}(\pi(X)) \geq \mu_G(X) \); if \( Y \)
is a closed subset of \( G/K \), then \( \mu_G(\pi^{-1}(Y)) = \mu_{G/K}(Y) \).

2.3. PFG, PFR and more. We say that a profinite group \( G \) is PFG
if there is a positive integer \( k \) such that the probability of \( k \) Haar-
random elements of \( G \) generating the whole group is positive. This
condition has been studied extensively and here we only mention the
Mann-Shalev theorem [19, Theorem 4]: a profinite group \( G \) is PFG if
and only if it has polynomial maximal subgroup growth.

In the spirit of the Mann-Shalev theorem, the authors of [14] study
a related property called PFR. We list below some of the conditions
considered there that we will need; the interested reader may check [14]
for more details.

A profinite group \( G \):
(i) is $PFR$ if it is finitely generated, and for every epimorphism $f : H \to G$ with $H$ finitely generated, the kernel of $f$ is positively finitely normally generated in $H$.

(ii) has PMEG, if the number of isomorphism classes of minimal extensions of $G$ of order $n$ grows polynomially in $n$.

(iii) has UBERG, if the number of simple $\hat{\mathbb{Z}}[G]$-modules of order $n$ grows polynomially in $n$.

**Proposition 2.2** ([14]). UBERG is equivalent to the group algebra $\hat{\mathbb{Z}}[G]$ being PFG, for all profinite groups $G$. PFR and PMEG are equivalent for $G$ finitely generated. All the conditions are equivalent for $G$ finitely presented.

Note that the equivalence of UBERG to $\hat{\mathbb{Z}}[G]$ being PFG is only stated in [14] for finitely generated groups, but the proof for general groups goes through without change.

We will see later that there are groups with UBERG which are not PFG; the question of whether there are non-finitely generated groups with UBERG remains open. But we do have the following result.

**Proposition 2.3.** If $G$ has UBERG, it is countably based.

*Proof.* Suppose $G$ is not countably based; we will show it has uncountably many (isomorphism classes of) simple modules.

Write $G$ as an inverse limit of profinite quotients $G_\alpha$, such that $G_0 = 1$, $\ker(G_\alpha \to G_{\alpha-1})$ is finite for $\alpha$ a successor, and $G_\alpha = \lim_{\beta < \alpha} G_\beta$ for $\alpha$ a limit. By [20, Theorem 2.6.4], such a sequence exists with $G = G_{w(G)}$, where $w(G)$ is the weight of $G$ (defined in [20, Section 2.6]). In particular, $w(G)$ is uncountable.

Now the set of successor ordinals less than $w(G)$ is also uncountable. Thus it suffices to show that for each successor $\alpha$, $G_\alpha$ has a simple module which is not the restriction of a $G_{\alpha-1}$-module. Pick an open normal subgroup $H$ of $G_\alpha$ which has trivial intersection with $\ker(G_\alpha \to G_{\alpha-1})$. Then it is enough to find a simple $G_\alpha/H$-module on which the $\ker(G_\alpha \to G_{\alpha-1})$-action is non-trivial; this exists by standard techniques of representation theory of finite groups. □

2.4. **Extension growth.** We can give another condition equivalent to PMEG. Recall that extensions $1 \to K \to E \to G$ and $1 \to K \to E' \to G \to 1$ are said to be equivalent if there is a commutative diagram

$$
\begin{array}{ccc}
K & \longrightarrow & E \\
\downarrow & & \downarrow \\
K & \longrightarrow & E'
\end{array}
\quad
\begin{array}{ccc}
E & \longrightarrow & G \\
\downarrow & & \downarrow \\
E' & \longrightarrow & G
\end{array}
$$
they are said to be isomorphic if there is a commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{\cong} & E \\
\downarrow & & \downarrow \\
K & \xrightarrow{\cong} & E'.
\end{array}
\]

So equivalent extensions are isomorphic. Moreover, isomorphic extensions induce a $G$-automorphism of $K$ (where $K$ is defined to be a not-necessarily-abelian $G$-module in an appropriate way), and simple $G$-modules are 2-generated by \[14\], Lemma 3.6], so the number of $G$-automorphisms is at most $|K|^2$. Therefore the number of equivalence classes of minimal extensions of degree $n$ in one isomorphism class is at most $n^2$.

Then we immediately get:

**Proposition 2.4.** Polynomial minimal extension growth is equivalent to the number of equivalence classes of minimal extensions by non-isomorphic $G$-modules of order $n$ growing polynomially in $n$.

### 3. Positively finitely presented groups

Before getting further into the module theory, we introduce a new condition on profinite groups.

For a profinite group $G$, the direct power $G^k$ can be viewed as a profinite group, and as such it supports a Haar measure — which we will denote by $\mu_{G^k}$. For $g_1, \ldots, g_k \in G$, we denote by $\langle g_1, \ldots, g_k \rangle$ the closed subgroup of $G$ they generate.

Now we define the set

\[ X(G, k) = \{ (g_1, \ldots, g_k) \in G^k \mid \langle g_1, \ldots, g_k \rangle = G \}, \]

so that $P(G, k) = \mu_{G^k}(X(G, k))$ is non-zero for some $k$ if and only if $G$ is PFG.

We can immediately deduce an equivalent definition for modules to be PFG.

**Lemma 3.1.** Let $F_n$ be the free profinite group on $n$ generators. Then $F_R(G, n) > 0$ if and only if

\[ \mu_{\text{Hom}(F_n, G)}(\{ \varphi \in \text{Hom}(F_n, G) \mid \varphi \text{ surjective} \}) > 0. \]

**Proof.** Fix a basis $x_1, \ldots, x_n$ for $F_n$. Under the obvious isomorphism $\text{Hom}(F_n, M) \cong M^n$, surjections $f : F_n \to M$ correspond to subsets $\{ f(x_1), \ldots, f(x_n) \}$ which generate $M$. \qed

### 3.1. Frattini covers and PFG.

For a profinite group $G$, write $\Phi(G)$ for the Frattini subgroup of $G$: that is, the intersection of all the maximal open subgroups of $G$. An epimorphism $f : H \to G$ is called a Frattini cover of $G$ if $\ker(f) \leq \Phi(H)$. The Frattini covers of $G$ form an inverse system whose inverse limit, called the universal Frattini cover
of $G$, is again a Frattini cover of $G$ and is a projective profinite group. See [9, Chapter 22] for background on this.

**Lemma 3.3.** A Frattini cover $H$ of a profinite group $G$ is PFG if and only if $G$ is.

**Proof.** Without loss of generality, we can assume that both $H$ and $G$ are finitely generated. In fact, we just have to observe that, for any generating set $S$ of $G$, any lift of $S$ to $H$ generates $H$, since the kernel is contained in the Frattini subgroup of $H$.

Indeed, we have a homomorphism $\text{Hom}(F_n, H) \rightarrow \text{Hom}(F_n, G)$. By the definition of projective cover, the preimage of the set of surjective maps in $\text{Hom}(F_n, G)$ is exactly the set of surjective maps in $\text{Hom}(F_n, H)$. The claim follows by Lemma 2.1 and Lemma 3.1.

Therefore, every PFG group admits a short exact sequence of the form

(3.4) $1 \rightarrow R \rightarrow P \rightarrow G \rightarrow 1$

with $P$ a PFG projective profinite group. In the next subsection, we will think of such sequences as presentations for $G$.

### 3.2. Positively finitely presented groups.

Given that the idea of PFR is an higher analogue of PFG, an alternative condition would require that $G$ has a ‘PFG presentation’.

**Definition 3.5.** A profinite group $G$ is said to be **positively finitely presented** if $G$ is PFG and for every short exact sequence (3.4) with $P$ a PFG projective profinite group, $R$ is PFG as a normal subgroup of $P$.

The kernel $R$ being PFG as a normal subgroup of $P$ is equivalent to $R$ having polynomial maximal $P$-stable subgroup growth by the same argument as [14, Section 3.3]. We will see in this section that for PFG groups, PFR is equivalent to positive finite presentation, but that not all PFR groups are PFG, answering the first part of [14, Open Question 1.2].

First we justify our use of the term positively finitely presented by showing that groups satisfying this condition are finitely presented.

We will use here a result whose proof we defer until Section 5 after discussing the APFG condition on profinite groups.

**Proposition 3.6.** Let $G$ be a finitely presented profinite group. If $G$ is PFG, then it is PFR.

**Proposition 3.7.** Positively finitely presented groups are finitely presented. Finitely presented PFG groups are positively finitely presented.

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1We choose to avoid abbreviating this to PFP because of potential clashes with some future paper about a positively type FP condition.
Proof. Given a positively finitely presented group $G$, fix a presentation $3.4$ with $P$ PFG projective. Finitely generated projective profinite groups are finitely presented by [16, Proposition 1.1], so this exhibits $G$ as a quotient of a finitely presented group by a normally finitely generated group: it is standard that such groups are finitely generated.

PFG and finitely presented implies PFR by Proposition 3.6 and then PFR plus PFG imply positively finitely presented by [14, Lemma 3.4].

In particular, PFG projective groups are positively finitely presented.

**Corollary 3.8.** For PFG groups, positive finite presentation is equivalent to PFR.

Proof. PFR groups are finitely presented, which with PFG implies positive finite presentation by Proposition 3.7. Positively finitely presented groups are PFG and finitely presented, which we have seen implies PFR by Proposition 3.6. □

The next two results show that the class of positively finitely presented profinite groups is well behaved:

**Lemma 3.9.** For $N$ a normal subgroup of $G$, if $N$ and $G/N$ are positively finitely presented, so is $G$.

Proof. $N$ and $G/N$ are finitely presented and PFG. Both these properties are closed under extensions so $G$ is finitely presented and PFG, so it is PFR by Proposition 3.6, and hence positively finitely presented by Corollary 3.8. □

Compare this to the class of PFR groups: it remains an open question whether this is closed under extensions ([14, p.3]).

We conclude this section by showing that $G$ being positively finitely presented is witnessed by its universal Frattini cover. Compare this to the class of PFR groups: in general minimal presentations are not sufficient to determine whether a group is PFR ([14, Section 7]).

**Proposition 3.10.** Let $G$ be a PFG profinite group and let $f : \bar{G} \to G$ be the universal Frattini cover of $G$. Write $R$ for the kernel of this map. If $R$ is positively normally finitely generated in $\bar{G}$, $G$ is positively finitely presented.

Proof. Since $\bar{G}$ is a projective cover of $G$, if $1 \to S \to Q \to G \to 1$ is another presentation of $G$ with $Q$ PFG, then $Q \to G$ factors into an epimorphism $Q \to \bar{G}$ and $f$. The diagram

$$
\begin{array}{ccc}
S & \longrightarrow & Q \\
\downarrow & & \downarrow \\
R & \longrightarrow & \bar{G}
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \downarrow & \downarrow \\
& G & G
\end{array}
$$
has exact rows; writing $T$ for the kernel of $Q \to \hat{G}$, we get $S/T \cong R$ by the Nine Lemma. Since $R$ is positively finitely normally generated in $\hat{G}$, it has polynomial maximal $\hat{G}$-stable subgroup growth. $\hat{G}$ is positively finitely generated and projective, hence positively finitely presented, so $T$ has polynomial maximal $Q$-stable subgroup growth. It follows that $S$, as an extension of $T$ by $R$, has polynomial $Q$-stable subgroup growth, and hence is positively normally finitely generated in $Q$, as required. \hfill \Box

4. Modules of type $\text{PFP}_n$

Let $R$ be a profinite ring. In the rest of this section, all modules will be profinite $R$-modules.

4.1. PFG modules. For a module $M$, as for a profinite group, the direct power $M^k$ can be viewed as an abelian profinite group, and supports a Haar measure which we will denote by $\mu_{M^k}$. For $m_1, \ldots, m_k \in M$, we denote by $\langle m_1, \ldots, m_k \rangle_R$ the closed submodule of $M$ they generate.

As for groups, for a positive integer $k$, we define the set

$$X_R(M, k) = \{ (m_1, \ldots, m_k) \in M^k \mid \langle m_1, \ldots, m_k \rangle_R = M \}.$$  

and $P_R(M, k) = \mu_{M^k}(X_R(M, k))$.

**Definition 4.1.** A profinite module $M$ is said to be $\text{PFG}$ if there is some $k \in \mathbb{N}$ such that $P_R(M, k) > 0$.

In the same way as Lemma 3.1, we have:

**Lemma 4.2.** Let $M$ be an $R$-module. Then $P_R(M, k) > 0$ if and only if

$$\mu_{\text{Hom}_R(R^k, M)}(\{ \varphi \in \text{Hom}_R(R^k, M) \mid \varphi \text{ surjective} \}) > 0.$$  

In [17, Proposition 11.2.1] it is shown that the class of PFG groups is closed under quotients and extensions. The same is true for PFG module with an analogous proof which we omit.

**Lemma 4.4.** PFG modules are closed under quotients and extensions.

4.2. Projective covers of PFG modules. Let $M$ be an $R$-module. A submodule $N$ of a module $M$ is superfluous if, for any submodule $H$ of $M$, $H + N = M$ implies $H = M$.

**Definition 4.5.** A homomorphism $P \to M$, with $P$ projective, is said to be a projective cover of $M$ if its kernel is a superfluous submodule of $P$.

It is easy to see that, if $P_1 \to M$ and $P_2 \to M$ are two projective covers of $M$, then $P_1 \cong P_2$. So we may abuse terminology by
referring to $P$ itself as the projective cover of $M$, instead of the homomorphism $P \to M$. Profinite modules have projective covers by \[21\] Remark 3.4.3(i)].

**Lemma 4.6.** $M$ is PFG if and only if its projective cover $P$ is.

*Proof.* This is the same argument as Lemma 3.3: we can assume that both $M$ and $P$ are finitely generated and observe that, for any generating set $S$ of $M$, any lift of $S$ to $P$ generates $P$, since the kernel is superfluous. □

### 4.3. Modules of type PFP$_n$.

The previous lemma suggests the following definition.

**Definition 4.7.** An $R$-module $M$ has type PFP$_n$ if it has a projective resolution $P^\ast$

\[
\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0
\]

with $P_0, \ldots, P_n$ PFG $R$-modules. The module $M$ has type PFP$_\infty$ if it has a projective resolution $P_\ast$ as in (4.8) with $P_n$ PFG for all $n$.

**Remark 4.9.** By Lemma 4.6, an $R$-module has type PFP$_0$ if and only if it is PFG.

**Proposition 4.10.** Suppose we have two partial resolutions $P^\ast$ of $M$ with each $P_i$ and $Q_i$ PFG projective. Then ker($P_{n-1} \to P_{n-2}$) is PFG if and only if ker($Q_{n-1} \to Q_{n-2}$) is.

*Proof.* Schanuel’s lemma. □

It follows that $M$ has type PFP$_\infty$ if and only if it has type PFP$_n$ for all $n$.

**Remark 4.11.** Note that, if $R$ is PFG as an $R$-module, then all finitely generated $R$-modules are PFG. Thus, type PFP$_n$ coincides with type FP$_n$ for PFG rings.

We will now show that the properties defined above behave well with respect to short exact sequences. See \[22\] for more detail on the constructions used.

**Proposition 4.12.** Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of profinite $R$-modules.

(i) If $A$ has type PFP$_{n-1}$ and $B$ has type PFP$_n$, $C$ has type PFP$_n$.

(ii) If $B$ has type PFP$_{n-1}$ and $C$ has type PFP$_n$, $A$ has type PFP$_{n-1}$.

(iii) If $A$ and $C$ have type PFP$_n$, so does $B$.

*Proof.* (i) Take a type PFP$_{n-1}$ resolution $P'_\ast$ of $A$ and a type PFP$_n$ resolution $P_\ast$ of $B$. There is a map $P'_\ast \to P_\ast$ extending $A \to B$. The mapping cone of $P'_\ast \to P_\ast$ is a type PFP$_n$ resolution of $C$. 


(ii) Fix a map $Q \xrightarrow{q} B$ with $Q$ PFG projective. Note that $Q$ has type $\text{PFP}_\infty$. We have a diagram

$$
\begin{array}{ccc}
\ker(q) & \longrightarrow & Q \\
\downarrow & & \downarrow g \\
\ker(g \circ q) & \longrightarrow & Q \\
\end{array}
\begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow C;
\end{array}
$$

with exact rows. By (i) applied to each of these rows, $\ker(q)$ is of type $\text{PFP}_{n-2}$ and $\ker(g \circ q)$ is of type $\text{PFP}_{n-1}$. By the snake lemma, $0 \to \ker(q) \to \ker(g \circ q) \to A \to 0$ is exact. Again by (i), $A$ is of type $\text{PFP}_{n-1}$.

(iii) Given a type $\text{PFP}_n$ resolution for $A$ and another for $C$, the resolution for $B$ constructed using the horseshoe lemma has type $\text{PFP}_n$, since PFG is closed under extensions by [17, Proposition 11.2.1].

4.4. Growth conditions. The famous Mann-Shalev Theorem in [19] characterises PFG groups algebraically as those profinite groups with “few” open maximal subgroups. We would like to mimic this theorem as well as producing a cohomological criterion for when modules have type $\text{PFP}_n$.

**Definition 4.13.** For $R$ a profinite ring and $M$ a profinite $R$-module, let $m^R_k(M)$ be the number of maximal (open) submodules of $M$ of index $k$. We say that $M$ has polynomial maximal submodule growth, or PMSMG for short, if there is some constant $c > 0$ such that $m^R_k(M) \leq k^c$ for all $k$.

For a profinite ring $R$ and $k \in \mathbb{N}$, denote by $S^R_k$ the set of simple $R$-modules of order $k$.

**Proposition 4.14.** Let $M$ be a profinite $R$-module. Then the following conditions are equivalent:

1. $M$ is PFG.
2. $M$ has PMSMG.
3. $\sum_{S \in S^R_k} ([\text{Hom}_R(M, S)] - 1)$ has polynomial growth in $k$.

**Proof.** (1) $\Leftrightarrow$ (2): Imitate [14, Proposition 6.1 (2) $\Rightarrow$ (3)] and [14, Proposition 3.5].

We will now show that (2) is equivalent to (3). First, note that for each maximal submodule in $M$ we get a quotient map to a simple module, so we have an injection from the set of maximal submodules of index $k$ to the set of surjective (or equivalently, non-trivial) maps to simple modules of order $k$. Hence,

$$m^R_k(M) \leq \sum_{S \in S^R_k} ([\text{Hom}_R(M, S)] - 1).$$
Conversely, if $M$ has PMSMG, it is $d$-generated for some $d$. Hence $|\text{Hom}_R(M, S)| \leq |S|^d$, so

$$
\sum_{S \in S_k} (|\text{Hom}_R(M, S)| - 1) \leq k^d m_R^k(M).
$$

\[\square\]

Since $\text{Hom}(M, S)$ is just $\text{Ext}^0_R(M, S)$, we can now apply the proposition to give conditions equivalent to a module having type $\text{PFP}_n$.

**Theorem 4.15.** Let $M$ be an $R$-module. Then, $M$ has type $\text{PFP}_n$ if and only if $\sum_{S \in S_k} (|\text{Ext}^m_R(M, S)| - 1)$ has polynomial growth in $k$ for all $m \leq n$.

**Proof.** For an $R$-module $M$, in the course of the proof we will write $f_M^m(k) = \sum_{S \in S_k} (|\text{Ext}^m_R(M, S)| - 1)$ for conciseness.

The case $n = 0$ is just Proposition 4.14. Now, suppose that $n \geq 1$ and the theorem is true for every $m \leq n - 1$. Let $M$ be an $R$-module of type $\text{PFP}_n$. By hypothesis, $f_M^m(k)$ is polynomial in $k$ for $m \leq n - 1$; it remains to check that $f_n^m(k)$ is polynomial in $k$. By Lemma 4.6, we have a short exact sequence

$$
0 \to K \to P \to M \to 0
$$

with $P$ PFG projective. $K$ has type $\text{PFP}_{n-1}$, so by hypothesis $f_{n-1}^K(k)$ has polynomial growth in $k$; since $\text{Ext}^n_R(P, -) = 0$ for $n \geq 1$, we have that $f_n^P(k) = 0$. Using the long exact sequence in cohomology, we see that $f_n^m(k) \leq f_{n-1}^m(k) + f_P^m(k)$, and we are done. \[\square\]

Note that $S_k^R$ may be infinite, and the sum $\sum_{S \in S_k} (|\text{Ext}^m_R(M, S)| - 1)$ may nonetheless be finite. We will see in Section 6 that, for the profinite group $G = \text{Alt}(5)^N$, there are infinitely many values of $k$ such that

$$
\sum_{S \in S_k^2[G]} (|H_1^Z(G, S)| - 1) = 0
$$

even though $S_k^2[G]$ is infinite.

5. **Groups of type $\text{PFP}_n$**

5.1. **APFG.** It will be useful here to compare our conditions on profinite groups to another condition introduced in $\text{[S]}$. Recall that the augmentation map $\varepsilon : \hat{Z}[G] \to \hat{Z}$ is induced by $\varepsilon(g) = 1$, for $g \in G$. Define the augmentation ideal as $I_\varepsilon[G] = \ker \varepsilon$, so we have a short exact sequence

$$
0 \to I_\varepsilon[G] \to \hat{Z}[G] \to \hat{Z} \to 0.
$$

$$
(5.1)
$$
Note that the group $G$ has type FP$_1$ if and only if its augmentation ideal $I_{\hat{\mathbb{Z}}}[G]$ is finitely generated. The following definition is quite natural at this point.

**Definition 5.2.** A profinite group $G$ is said to be APFG if the augmentation ideal $I_{\hat{\mathbb{Z}}}[G]$ is PFG as a $\hat{\mathbb{Z}}[G]$-module.

**Remark 5.3.** There is an error in the statement of [8, Corollary 2.5] (which does not affect the rest of [8]): it should say “$I_{\hat{\mathbb{Z}}}[G]$ is finitely generated as a $\hat{\mathbb{Z}}[G]$-module if and only if there exists $d \in \mathbb{N}$ such that $\delta_G(M)/r_G(M) \leq d$ for any $M \in \text{Irr}(\mathbb{F}_p[G])$ and $p \in \pi(G)$”.

It is shown in [8, Theorem 3] that, if $G$ is countably based and has type FP$_1$ over $\hat{\mathbb{Z}}$, $G$ is APFG if and only if the number of simple $\hat{\mathbb{Z}}[G]$-modules has polynomial growth (and, hence, if and only if $\hat{\mathbb{Z}}[G]$ is PFG). Here we may generalise this theorem by dropping the countably based requirement.

**Proposition 5.4.** Let $G$ be a profinite group. If $G$ is APFG, then $G$ has UBERG. If $G$ has UBERG and has type FP$_1$, then $G$ is APFG.

**Proof.** Suppose that the augmentation ideal $I_{\hat{\mathbb{Z}}}[G]$ of $G$ is PFG. Then the group ring $\hat{\mathbb{Z}}[G]$ fits into the exact sequence (5.1) and it is PFG as an extension of two PFG modules.

On the other hand, if $\hat{\mathbb{Z}}[G]$ is PFG, then modules are PFG if and only if they are finitely generated; for $G$ of type FP$_1$, the augmentation ideal is finitely generated and hence PFG.

Note that profinite groups of type FP$_1$ over $\hat{\mathbb{Z}}$ which are not countably based exist, by [6, Example 7.1], but they do not have UBERG by Proposition 2.3.

For the rest of this section $R$ will be a commutative profinite ring. We can now introduce the main novelty of our investigation.

**Definition 5.5.** A profinite group $G$ has type PFP$_n$ over $R$ if $R$ has type PFP$_n$ as a $R[G]$-module.

**Example 5.6 (PFP$_n$ for prosolvable groups).** By [14, Proposition 6.1], $\hat{\mathbb{Z}}[G]$ is PFG if and only if $G$ has UBERG. As mentioned earlier, [14, Proposition 6.1] states the hypothesis that $G$ is finitely generated, but this is not required anywhere in the proof. This holds for all finitely generated prosoluble groups by [14, Corollary 6.12]. It fails for non-abelian free profinite groups by [14, Proposition 6.14].

In particular, the ring $R = \hat{\mathbb{Z}}[G]$ is PFG whenever $G$ is PFG. This implies that PFP$_n$ and FP$_n$ coincide for finitely generated prosoluble groups (cf. Conjecture 6.6 below).

Every group is of type FP$_0$ over every ring. This is false for type PFP$_0$. 

Lemma 5.7. A profinite group $G$ has type PFP$_0$ over $R$ if and only if $R$ is PFG as an $R$-module.

Proof. Any subset of $R$ generates it as an $R$-module if and only if it generates it as an $R[[G]]$-module, since the $G$-action is trivial. □

Next we show that the class of groups of type PFP$_n$ is closed under commensurability. Recall that for an $R[[G]]$-module $M$, we denote by $\text{Res}_G^H M$ the $R[[H]]$-module with the same underlying set as $M$ and restricting the action of $G$ to $H$.

Proposition 5.8. Let $G$ be a profinite group and let $H$ be an open subgroup of $G$. Then, $H$ has type PFP$_n$ over $R$ if and only if $G$ does.

Proof. We first claim that an $R[[G]]$-module $M$ is PFG if and only if $\text{Res}_G^H M$ is PFG. Indeed, clearly any set of generators for $\text{Res}_G^H M$ generates $M$. Conversely, say $|G:H| = c$. A set of $t$ generators for $M$ generates an open submodule of $\text{Res}_G^H M$ of index at most $c^t$, so if $P_{R[[G]]}(M, t) > 0$ for some $t$, $P_{R[[H]]}(\text{Res}_G^H M, t + c^t) > 0$.

It now follows by the same techniques as for abstract modules (see [4, VIII, Proposition 5.1]) that $M$ has type PFP$_n$ if and only if $\text{Res}_G^H M$ does, and in particular this holds for $M = R$. □

5.2. APFG and PFP$_1$. As the reader might guess, APFG and PFP$_1$ are related. From Proposition 5.3 we deduce:

Lemma 5.9. If $G$ is APFG, then it has type PFP$_1$ over $\hat{\mathbb{Z}}$.

Proof. By Proposition 5.4, $\hat{\mathbb{Z}}[G]$ is PFG. So the exact sequence (5.1) of PFG modules shows that $G$ has type PFP$_1$. □

Corollary 5.10. If $G$ is PFG, then it has type PFP$_1$ over $\hat{\mathbb{Z}}$.

Proof. By [8, Theorem 4.4], $G$ being PFG implies it is APFG. The result follows by the previous lemma. □

Using the relation between APFG and PFP$_1$, we can give an example of group of type PFP$_1$ that is not PFG.

Example 5.11. In [8] Example 4.5 it is shown that, for $N$ large enough, the group $\prod_{n \geq N} \text{Alt}(n)^{2^n}$ is APFG but not PFG. This example is therefore PFP$_1$ over $\hat{\mathbb{Z}}$, and has UBERG, but is not PFG.

Using the above ideas we can also answer the second part of [14, Quest. 1.2], by finally giving the promised proof of Proposition 3.6.

Proof of Proposition 3.6. By [8, Theorem 4.4], $G$ is APFG. Hence $\hat{\mathbb{Z}}[G]$ fits into (5.1) and it is PFG. Now [14, Theorem A] shows that $G$ is PFR. □
5.3. **Positive finite presentation and** PFP$_2$. It is shown in [14, Theorem 3.9] that for finitely generated profinite groups, PFR is equivalent to PMEG (see Section 2.3). We may compare this to our condition for groups of type PFP$_1$ to have type PFP$_2$: that the number of equivalence classes of minimal abelian extensions of degree $n$ grows polynomially in $n$, by Proposition 4.15.

Recall that in Proposition 2.4 we showed that polynomial minimal extension growth is equivalent to the number of isomorphism classes of extensions growing polynomially. Among finitely generated groups, this gives another equivalent characterisation of being PFR. The same is true if we restrict to equivalence/isomorphism classes of minimal abelian extensions.

**Lemma 5.12.** Positively finitely presented implies PFP$_2$.

**Proof.** A positively finitely presented group $G$ is PFR and PFG, so has type PFP$_1$ by Corollary 5.10 and by the previous proposition, the number of equivalence classes of minimal extensions of $G$ of degree $n$ grows polynomially in $n$. In particular the number of equivalence classes of minimal abelian extensions of $G$ of degree $n$ grows polynomially in $n$, so $G$ has type PFP$_2$ by Proposition 4.15. □

Alternatively, we can get this result by observing that a positively finitely presented group $G$ is PFG, so it has UBERG, and hence for $G$ having type PFP$_2$ is equivalent to having type FP$_2$. Since $G$ is finitely presented, it has type FP$_2$.

We also observe, in contrast to [14, Section 7], that we can check the PFP$_2$ property by considering the minimal presentation of a group.

Imitating [14, Section 3], we say that a presentation (3.4) of $G$ has polynomial maximal $\tilde{G}$-stable abelian subgroup growth if the number of maximal $\tilde{G}$-stable subgroups $S$ of $R$ of index $n$ with $R/S$ abelian grows polynomially in $n$.

We can now state an equivalent formulation of property PFP$_2$.

**Proposition 5.13.** Let $G$ be a profinite group of type PFP$_1$ and let $\tilde{G} \to G$ be the universal Frattini cover of $G$. Then $G$ has type PFP$_2$ if and only if $R = \ker(\tilde{G} \to G)$ has polynomial maximal $\tilde{G}$-stable abelian subgroup growth.

**Proof.** We may use the argument of [14, Proposition 7.1]: the maximal $\tilde{G}$-stable subgroups of $R$ of degree $n$ correspond precisely to the isomorphism classes of non-split minimal abelian extensions of $G$ of degree $n$ by [12, (3.2)]; the number of these grows polynomially in $n$ if and only if the number of equivalence classes of non-split minimal abelian extensions of $G$ of degree $n$ does, just as in Proposition 2.4, since $G$ has type PFP$_1$, this condition is equivalent to having type PFP$_2$ by Proposition 4.15. □
5.4. Relative type $\text{PFP}_n$.

**Definition 5.14.** A closed subgroup $H$ of a profinite group $G$ has relative type $\text{PFP}_n$ in $G$ over $R$ if all PFG projective $R[G/H]$-modules have type $\text{PFP}_n$ over $R[G]$.

**Remark 5.15.** Notice that in the $\text{FP}_n$ case, $H$ has relative type $\text{FP}_n$ in all groups if and only if $H$ has type $\text{FP}_n$, whereas when $R[G]$ is not PFG, the trivial subgroup does not have relative type $\text{PFP}_0$ in $G$, so we need the two different definitions.

**Theorem 5.16.** Let $G$ be a profinite group, $H$ normal in $G$ and let $M$ be a profinite $R[G/H]$-module. Suppose $H$ has relative type $\text{PFP}_m$ in $G$ over $R$.

(i) If $M$ has type $\text{PFP}_n$ over $R[G]$ (via restriction), then it has type $\text{PFP}_{\min(m+1,n)}$ over $R[G/H]$.

(ii) If $M$ has type $\text{PFP}_n$ over $R[G/H]$, then it has type $\text{PFP}_{\min(m,n)}$ over $R[G]$.

**Proof.** First, note that $M$ is PFG as an $R[G]$-module if and only if it is PFG as an $R[G/H]$-module, because the action is by restriction.

(i) Use induction on $n$. When $n = 0$ we are done. Take a PFG projective $R[G/H]$-module $P$ and an epimorphism $P \to M$ with kernel $K$. The module $K$ has type $\text{PFP}_{\min(m,n-1)}$ over $R[G]$ by Proposition 4.12, so by hypothesis it has type $\min(m+1, \min(m,n-1)) = \min(m,n-1)$ over $R[G/H]$. Therefore $M$ has type $\min(m,n-1) + 1 = k$ over $R[G/H]$.

(ii) Use induction on $n$. When $n = 0$ we are done. Take a PFG projective $R[G/H]$-module $P$ and an epimorphism $P \to M$ with kernel $K$. Now $K$ has type $\text{PFP}_{n-1}$ over $R[G/H]$-module by Proposition 4.12, so by hypothesis it has type $\text{PFP}_{\min(m,n-1)}$ over $R[G]$. Also $P$ has type $\text{PFP}_m$ over $R[G]$, so by Proposition 4.12 $M$ has type $\text{PFP}_{\min(m,n)}$ over $R[G]$.

\[ \square \]

In particular this holds for $M = R$.

**Corollary 5.17.** Suppose that $H$ has relative type $\text{PFP}_m$ in $G$ over $R$.

(i) If $G$ has type $\text{PFP}_n$ over $R$, then $G/H$ has type $\text{PFP}_{\min(m+1,n)}$ over $R$.

(ii) If $G/H$ has type $\text{PFP}_n$ over $R$, then $G$ has type $\text{PFP}_{\min(m,n)}$ over $R$.

**Remark 5.18.** Exactly the same approach gives the analogous classical result (see [3, Proposition 2.7] for example) that relates type $\text{FP}_n$ conditions for extensions and quotients of abstract groups. As far as
we know, this is the first proof of this result which avoids the use of spectral sequences; those unfamiliar with the mysteries of spectral sequences may find this new perspective enlightening.

At the moment the property of relative type PFP, and thus the behaviour of type PFP under extensions, remains mysterious. But we have the following result:

**Proposition 5.19.** Suppose $G$ and $H$ are profinite groups. Let $M$ be an $R[G]$-module and $N$ an $R[H]$-module. Suppose $M$ has type PFP, and $N$ has type PFP. Then $M \otimes_R N$ has type PFP as an $R[G \times H]$-module.

**Proof.** We will show that if $P$ is a PFG projective $R[G]$-module and $Q$ a PFG projective $R[H]$-module, then $P \otimes_R Q$ is PFG as an $R[G \times H]$-module. The result follows by taking tensor products of partial PFG projective resolutions for $M$ and $N$.

Each maximal ideal of $P$ gives an epimorphism onto a simple $k[G]$-module for some field $k$ which is a quotient of $R$. Now define the function $f^G_n(P,k')$ over all finite extensions $k'$ of fields $k$ which appear as quotients of $R$, as follows. Let $S_n(G,k')$ be the set of absolutely irreducible representations of $G$ of dimension $n$ which are defined over $k'$; we think of elements of $S_n(G,k')$ as $R[G]$-modules via restriction along $R \to k'$. [14 Lemma 6.7] gives a bijection $\Phi_k$ from Galois orbits of simple $k[G]$-modules to simple $k[G]$-modules, where $\bar{k}$ is the algebraic closure of $k$, and we identify $k'$ with a subfield of $\bar{k}$. Then $f^G_n(P,k') = \sum_{M \in S_n(G,k')}(|\text{Hom}_{R[G]}(P,\Phi_k(M))| - 1)$. Exactly the same approach as [14 Lemma 6.8] shows that $P$ is PFG if and only if there is some $b$ such that $f^G_n(P,k') \leq |k'|^{bn}$ for all $n$ and all $k'$ where it is defined.

As in the proof of [14 Theorem 6.4], the absolutely irreducible representations of $G \times H$ over $k'$ are precisely the tensor products of absolutely irreducible representations of $G$ and $H$ over $k'$. As there, we deduce that if $f^G_n(P,k') \leq |k'|^{bn}$ for some $b$, and similarly for $f^H_n(Q,k')$, there exists a $c$ such that $f^G_{n \times H}(P \otimes_R Q,k') \leq |k'|^{cn}$, and $P \otimes_R Q$ is PFG, as required.

**Corollary 5.20.** If $G$ has type PFP over $R$ and $H$ has type PFP over $R$, then $G \times H$ has type PFP over $R$.

Compare this to the result in [14 Theorem 6.4] that UBERG is preserved by (finite) direct products.

### 6. Examples

In this section we will construct some examples of groups of type PFP over $R$ although the group ring $R[G]$ is not PFG. We also give examples of groups of type FP over $\mathbb{Z}$ which do not have type PFP, and groups which are PFR but not PFG.
The examples studied in [8] suggest a strategy of looking at products of finite simple groups.

**Proposition 6.1.** Let $S$ be a finite non-abelian simple group and let $G$ be an infinite product of copies of $S$. For any prime $p$ not dividing the order of $S$, $\mathbb{Z}_p[[G]]$ is not PFG, but $G$ has type PFP$_1$ over $\mathbb{Z}_p$.

**Proof.** The group $G$ has infinitely many irreducible representations of dimension $\leq |S|$ over $\mathbb{F}_p$, so $\mathbb{Z}_p[[G]]$ is not PFG.

On the other hand, [13] Reduction Theorem] shows that $H^1(G, M) = 0$ is trivial for all simple $\mathbb{F}_p[G]$-modules $M$ with $p \notin \pi$. Therefore, by Theorem 4.15, we can see that $G$ is of type PFP$_1$ over $\mathbb{Z}_p$. $\square$

**Remark 6.2.** Similarly, if we denote by $\pi = \pi(S)$ the set of prime divisors of the order of $S$, the group $S^\mathbb{N}$ has type PFP$_1$ over $\mathbb{Z}_{\pi'} = \prod_{p \notin \pi} \mathbb{Z}_p$.

By varying the simple group $S$, we get examples of this behaviour over $\mathbb{Z}_p$ for all odd primes $p$. We do not know of any such examples over $\hat{\mathbb{Z}}$, and leave it as a question.

**Open Question 6.3.** Are there groups of type PFP$_1$ over $\hat{\mathbb{Z}}$ which do not have UBERG?

[8] constructs groups with UBERG which are not PFG, but another remaining question is whether such groups must be finitely generated.

**Open Question 6.4.** Are there groups with UBERG which are not finitely generated?

On the other hand, examples like the above cannot appear among pronilpotent groups. In future work with S. Kionke, we show:

**Proposition 6.5.** Let $G$ be a pronilpotent group. Then $G$ has UBERG if and only if $G$ is finitely generated.

The class of prosoluble groups appears often in these contexts as groups where pathological behaviour cannot occur. For example, finitely generated prosoluble groups are PFG, and prosoluble groups have type PFP$_1$ if and only if they are finitely generated by Corollary 5.10 and [8] Remark 3.5(a)]. So it is very natural to ask:

**Open Question 6.6.** Are all prosoluble groups with UBERG finitely generated?

6.1. **Type FP$_1$ and type PFP$_1$**. Clearly type PFP$_1$ (over any $R$) implies type FP$_1$ over $R$; we show the converse does not hold. Consider the free profinite group $F_3$ on three generators, which we think of as the profinite free product $\hat{\mathbb{Z}} * F_2$, for some fixed copy of $\hat{\mathbb{Z}}$. We use the Mayer-Vietoris sequence of [20, Proposition 9.2.13]. Note that our profinite free product is proper by [20 Exercise 9.2.6], so the Mayer-Vietoris sequence applies.
Now $F_3$ has type $FP_1$ because it is finitely generated. To show it does not have type $PFP_1$ we use the cohomological characterisation: we will show $\sum_{S_3 \in S_3^k} |H^1(G, S)| - 1$ grows faster than polynomially in $k$. By the Mayer-Vietoris sequence, it suffices to show $\sum_{S_3 \in S_3^k} |H^1(\hat{\mathbb{Z}}, S)| - 1$ grows faster than polynomially in $k$.

Let $T_{k, F_3}$ be the set of simple $F_3$-modules of order $k$ on which restriction to $\hat{\mathbb{Z}}$ gives the trivial action. By the universal property of free products, we can identify this with $S_{F_2}^k$. The sequence $|S_{F_2}^k|$ grows faster than polynomially in $k$ by [14, Lemma 6.16].

6.2. $PFR \neq PFG$. We finish by giving an example of a PFR group which is not PFG, answering the first half of [14, Quest. 1.2].

Consider the group $G = \prod_{n \geq N} Alt(n)^{2n}$. In [8, Example 4.5], it is shown that for large $N$ this group is finitely generated but not PFG, and that (thanks to Proposition 5.4) the group ring $\hat{\mathbb{Z}}[G]$ is positively finitely generated. By [14, Theorem A], if $G$ is finitely presented, it is PFR.

We will show $G$ is finitely presented using the equivalent condition given in [16, Theorem 0.3]: a finitely generated profinite group is finitely presented if and only if there exists a positive constant $C$ such that, for every prime $p$ and every finite simple $\mathbb{F}_p[G]$-module $M$, $\dim H^2(G, M) \leq C \dim M$.

**Theorem 6.7.** The profinite group $G = \prod_{n \geq N} Alt(n)^{2n}$ is finitely presented, and hence PFR.

**Proof.** The essential tool for computing the second cohomology of $G$ is [10, Theorem C]: if $H$ is a finite group, $F$ a field and $M$ is a simple faithful $FH$-module, then $\dim H^2(H, M) \leq (18.5) \dim M$.

Now suppose $M$ is a finite simple $\mathbb{F}_p[G]$-module, then $M$ is a simple faithful module for some finite quotient $G/K$ of $G$. The kernel $K$ is a normal subgroup of $G$, so by a standard argument it is a product of all but finitely many of the alternating groups in the product $G$. So $G/K$ is isomorphic to the product $L$ of the remaining alternating groups, and $G \cong K \times L$. In particular, $L$ is normal in $G$. Now, by the Lyndon-Hochschild-Serre spectral sequence we know

$$\dim H^2(G, M) \leq \dim H^2(K, M^L) + \dim H^1(K, H^1(L, M)) + \dim H^2(L, M)^K.$$
Since $M$ is simple and faithful as an $L$-module, $M^L = 0$. Since the actions of $K$ on $L$ and $M$ are trivial, the action of $K$ on $H^1(L, M)$ is trivial. Moreover, since $H^1(L, M)$ is abelian and $K$ is a product of non-abelian simple groups, we have $H^1(K, H^1(L, M)) = 0$. Therefore
\[
\dim H^2(G, M) \leq \dim H^2(L, M)^K \leq \dim H^2(L, M) \leq (18.5) \dim M.
\]

\[\Box\]

Remark 6.8. The above argument shows more generally that any finitely generated product of finite non-abelian simple groups is finitely presented. Note that in this particular case, the constant 18.5 can be improved to 47/12 by [10, Theorem 5.3, Theorem 6.2], and this is “likely quite far from best possible” (cf. [10]).

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