CELLULAR HOMOLOGY OF REAL FLAG MANIFOLDS

LONARDO RABELO AND LUIZ A. B. SAN MARTIN

Abstract. Let $\mathbb{F}_\Theta = G/P_\Theta$ be a generalized flag manifold, where $G$ is a real noncompact semi-simple Lie group and $P_\Theta$ a parabolic subgroup. A classical result says the Schubert cells, which are the closure of the Bruhat cells, endow $\mathbb{F}_\Theta$ with a cellular CW structure. In this paper we exhibit explicit parametrizations of the Schubert cells by closed balls (cubes) in $\mathbb{R}^n$ and use them to compute the boundary operator $\partial$ for the cellular homology. We recover the result obtained by Kocherlakota [1995], in the setting of Morse Homology, that the coefficients of $\partial$ are 0 or $\pm 2$ (so that $\mathbb{Z}_2$-homology is freely generated by the cells). In particular, the formula given here is more refined in the sense that the ambiguity of signals in the Morse-Witten complex is solved.

AMS 2010 subject classification: 57T15, 14M15.

Key words and phrases: Flag manifolds, cellular homology, Schubert cells.

Introduction

Let $\mathbb{F}_\Theta = G/P_\Theta$ be a flag manifold of the non-compact semi-simple Lie group $G$ where $P_\Theta$ is a parabolic subgroup. A classical result says that a cellular structure of $\mathbb{F}_\Theta$ is given by the Schubert cells $S_w^\Theta$ which are the closure of the Bruhat cells, that is, the components of the Bruhat decomposition

$$\mathbb{F}_\Theta = \bigsqcup_{w \in W/\mathcal{W}_\Theta} N \cdot wb_\Theta,$$

where $N$ is the nilpotent component of the Iwasawa decomposition, $W$ is the Weyl group of the corresponding Lie algebra and $\mathcal{W}_\Theta$ is the subgroup of $W$ associated with $\Theta$.

In order to compute the cellular homology of $\mathbb{F}_\Theta$, our first task in this paper is to provide explicit parametrizations of the Schubert cells $S_w^\Theta$ by cubes $[0, \pi]^d \subseteq \mathbb{R}^d$ which are defined in terms of the reduced decompositions of $w$. This description turns out to be useful to get algebraic formulas for the boundary operator $\partial$ of the cellular homology.

Our strategy consists by working firstly in the maximal flag manifolds, denoted by $\mathbb{F}$, and then by projecting down the Schubert cells via the canonical map $\pi_\Theta : \mathbb{F} \to \mathbb{F}_\Theta$. To parametrize a Schubert cell $S_w$, $w \in W$, in the maximal flag manifold $\mathbb{F}$, we start with a minimal decomposition $w = r_1 \cdots r_n$ of $w$ as a product of reflections $r_i = r_{\alpha_i}$ with respect to the simple roots. Then, similar to the construction of Bott-Samelson desingularization, we see $S_w$ as a product $K_1 \cdots K_n \cdot b_0$, where $b_0 = P$ is the origin of $\mathbb{F}$ and $K_i$ are maximal compact subgroups of rank one Lie groups $G_i$ (see Section 1). This presents $S_w$ as successive

---

Supported by FAPESP grant number 08/04628-6.

Supported by CNPq grant n° 305513/2003-6 and FAPESP grant n° 07/06896-5.
fibrations by spheres $S^{d_i}$, where $d_i$ are the multiplicities of the roots $\alpha_i$ - which may be not equal to 1. Thus a parametrization $\Phi_w : B^d \to S_w$ of a cell of dimension $d = d_1 + \cdots + d_n$ is obtained by viewing $S^{d_i}$ as the ball $B^{d_i}$ whose boundary is collapsed to a point.

The case of interest for homology are the roots $\alpha_i$ with multiplicity $d_i \neq 1$. This is because the boundary operator $\partial$ for the cellular homology takes the form $\partial S_w = \sum c(w, w') S_{w'}$ with $w' = r_1 \cdots \hat{r}_i \cdots r_n$ and the index $i$ is such that $d_i = 1$. In this case, the characteristic map $\Phi_w$ is defined in $B^{d-1} \times \{0, \pi\}$ and the coefficient $c(w, w')$ is the sum of the degrees of the attaching maps, that is, the restrictions of $\Phi_w$ to $B^{d-1} \times \{0\}$ and $B^{d-1} \times \{\pi\}$ (see Section 2, in particular the example of $\text{SL}(3, \mathbb{R})$ in Subsection 2.2). This way we get that any coefficient $c(w, w')$ is 0 or ±2. In particular, the $\mathbb{Z}_2$-homology is the vector space with basis $S_w$, $w \in \mathcal{W}$.

Once the maximal flag manifold is worked out, we get the boundary operator $\partial^\Theta$ in a general flag manifold $F^{\Theta}$. Actually we can prove that for a cell $S^\Theta_w$ in $F^{\Theta}$ there exists a unique (minimal) cell $S_w$ in $F$ with $\pi^\Theta(S_w) = S^\Theta_w$. Then $\partial^\Theta$ is obtained directly from the $\partial$ applied to the minimal cells.

These results were already obtained by Kocherlakota [8] in the realm of Morse homology. In [8], Theorem 1.1.4, it is proved that the boundary operator for the Morse-Witten complex has coefficients 0 or ±2 as well. Clearly the cellular and the Morse-Witten complexes are intimately related since the Bruhat cells are the unstable manifolds of the gradient flow of a Morse function (see Duistermat-Kolk-Varadarajan [3]). Nevertheless the cellular point of view has the advantage of showing the geometry in a more evident way. For instance, in the Subsection 1.5, we provide a description of the flow lines of the gradient flow inside a Bruhat cell in terms of characteristic maps of the cellular decomposition. Also, the choice of minimal decompositions for the elements of $\mathcal{W}$ fix certain signs that are left ambiguous in the Morse-Witten complex.

The construction of cellular decompositions of group manifolds and homogeneous spaces is an old theme. For the classical compact Lie groups one can build cells using products of reflections via a method that goes back to Whitehead [16] and was later developed by Yokota [18], [19]. By projection the decomposition on group level induces decompositions on the Stiefel manifolds $V_{n,k}$, that were exploited by Miller [11] to get several homological properties of these manifolds. On the contrary the cellular decompositions of the group manifolds do not project, in general, to cells in the flag manifolds. Hence that method does not yield cellular decomposition of the flag manifolds.

On the other hand the Schubert cells are central objects in the study of (co) homological properties of the flag manifolds (see e.g. Bernstein-Gelfand-Gelfand [6] and references therein). In the complex case the cellular homology is computed trivially since the cells are all even dimensional hence boundary operator $\partial = 0$ and the homology groups are freely generated. We refer also to Casian-Stanton [1] for an approach through representation theory of algebraic reductive groups.

For the real flag manifolds $\partial$ is not, in general, trivial and its computation requires explicit expressions for the gluing maps between the cells as we provide in this paper. To the best of
our knowledge there is no systematic construction of the cellular decomposition of the flag manifolds (of arbitrary semi-simple Lie groups) through the Bruhat cells and their closures the generalized Schubert cells.

The cells constructed here appeared before (up to cells of dimension two) in Wiggerman [17], that uses them to get generators and relations for the fundamental groups of the flag manifolds. Also in Rabelo [12] and Rabelo-Silva [13] the method of this paper is used to compute the integral homology of the Real isotropic Grassmannians (those of type B, C and D).

The article is organized as follows: In Section 1 we construct the parametrizations of the Schubert cells on the maximal flag manifolds and analyze the attaching (gluing) maps. In particular, in the subsection 1.5 we look at some aspects of the gradient flow yielding Morse homology. Section 2 is devoted to the boundary operator $\partial$ on the maximal flag manifold. The partial flag manifolds are treated in Section 3.

In this point we would like to thank Lucas Seco for his comments on some proofs and for his interest in the problem suggesting interesting references related to this question.

**Notation.** Flag manifolds are defined as homogeneous spaces $G/P$ where $G$ is a noncompact semi-simple Lie group and $P$ is a parabolic subgroup of $G$.

Let $g$ be a noncompact real semi-simple Lie algebra. The flag manifolds for the several groups $G$ with Lie algebra $g$ are the same. With this in mind we take always $G$ to be the identity component of the automorphism group of $g$, which is centerless.

Take a Cartan decomposition $g = k \oplus a$ with $k$ the compactly embedded subalgebra and denote by $\theta$ the corresponding Cartan involution. Let $a$ be a maximal abelian subalgebra contained in $s$ and denote by $\Pi$ the set of roots of the pair $(g, a)$. Fix a simple system of roots $\Sigma \subset \Pi$. Denote by $\Pi^\pm$ the set of positive and negative roots respectively and by $a^+$ the Weyl chamber

$$a^+ = \{ H \in a : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma \}.$$  

Let $n = \sum_{\alpha \in \Pi^+} g_\alpha$ be the direct sum of root spaces corresponding to the positive roots. The Iwasawa decomposition of $g$ is given by $g = k \oplus a \oplus n$. The notations $K$, $A$ and $N$ are used to indicate the connected subgroups whose Lie algebras are $k$, $a$ and $n$ respectively.

A sub-algebra $h \subset g$ is said to be a Cartan sub-algebra if $h_C$ is a Cartan sub-algebra of $g_C$. If $h = a$ is a Cartan sub-algebra of $g$ we say that $g$ is a split real form of $g_C$.

A minimal parabolic subalgebra of $g$ is given by $g = m \oplus a \oplus n$ where $m$ is the centralizer of $a$ in $k$. Let $P$ be the minimal parabolic subgroup with Lie algebra $p$ which is the normalizer of $p$ in $G$. We call $\mathbb{F} = G/P$ the maximal flag manifold of $G$ and denote by $b_0$ the base point $1 \cdot P$ in $G/P$.

Associated to a subset of simple roots $\Theta \subset \Sigma$ there are several Lie algebras and groups. We write $g(\Theta)$ for the semi-simple Lie algebra generated by $g_{\pm \alpha}$, $\alpha \in \Theta$. Let $G(\Theta)$ be the connected group with Lie algebra $g(\Theta)$. Moreover, let $n_\Theta$ be the subalgebra generated by
the roots spaces $g_{-\alpha}, \alpha \in \Theta$ and put

$$p_{\Theta} = n_{\Theta} \oplus p.$$  

The normalizer $P_{\Theta}$ of $p_{\Theta}$ in $G$ is a standard parabolic subgroup which contains $P$. The corresponding flag manifold $\mathbb{F}_{\Theta}$ is called a partial flag manifold of $G$ or flag manifold of type $\Theta$. We denote by $b_{\Theta}$ the base point $1 \cdot P_{\Theta}$ in $G/P_{\Theta}$. Such a flag manifold can also be written as $\mathbb{F}_{\Theta} = K/K_{\Theta}$ where $K_{\Theta} = P_{\Theta} \cap K$.

The Weyl group $W$ associated to $a$ is the finite group generated by the reflections over the root hyperplanes $\alpha = 0$ contained in $a, \alpha \in \Sigma$, and can be alternatively given as the quotient $M^*/M$ where $M^*$ and $M$ are respectively the normalizer and the centralizer of $a$ in $K$ (the Lie algebra of $M$ is $m$). We use the same letter to denote a representative of $w$ in $M^*$.

For the subset $\Theta \subset \Sigma$, there exists the subgroup $W_{\Theta}$ which acts trivially on $a_{\Theta} = \{ H \in a : \alpha(H) = 0, \alpha \in \Theta \}$. Alternatively, $W_{\Theta}$ may be seen as the subgroup of the Weyl group generated by the reflections with respect to the roots $\alpha \in \Theta$.

Viewing the elements of $W$ as product of simple reflections, the length $\ell(w)$ of $w \in W$, is the number of simple reflections in any reduced expression of $w$ which is equal to the cardinality of $\Pi_w = \Pi^+ \cap w\Pi^-$, the set of positive roots sent to negative roots by $w^{-1}$. If $w = r_1 \cdots r_n$ is a reduced expression of $w$ then

$$\Pi_w = \{ \alpha_1, r_1\alpha_2, \ldots, r_1 \cdots r_{n-1} \alpha_n \}.$$

There are two equivalent definitions of order between elements in the Weyl group (see Humphreys [5]).

(1) First, two elements are connected, denoted $w_1 \rightarrow w_2$, if $\ell(w_1) < \ell(w_2)$ and there is a root $\alpha$ (not necessarily simple) such that $w_1r_{1\alpha} = w_2$. Now that $w_1 < w_2$ if there are $u_1, \ldots, u_k \in W$ with

$$w_1 \rightarrow u_1 \rightarrow \cdots u_k \rightarrow w_2.$$

It may happen that $w_1 \rightarrow w_2$ with $\ell(w_1) + 1 = \ell(w_2)$ but there is no simple root with $w_1r_{1\alpha} = w_2$.

The definition may be changed by multiplication in the left $r_{\alpha}w_1 = w_2$ because $r_{\alpha}w_1 = w_1(r_{\alpha}^{-1}r_{\alpha}w_1) = w_1r_{\beta}$ with $\beta = w^{-1}\alpha$.

(2) $w_1 \leq w_2$ if given a reduced expression $w_2 = r_1 \cdots r_{\ell(w_2)}$ then $w_1 = r_{i_1} \cdots r_{i_k}$ for some indices $i_1 < \cdots < i_k$.

There is a unique $w_0 \in W$ such that $w_0\Pi^+ = \Pi^-$ which we call the principal involution and is the maximal element in the Bruhat-Chevalley order.

A partial flag manifold is the base space for the natural equivariant fibration $\pi_{\Theta} : \mathbb{F} \rightarrow \mathbb{F}_{\Theta}$ whose fiber is $P_{\Theta}/P$. This fiber is a flag manifold of a semi-simple Lie group $M_{\Theta} \subset G$ whose rank is the order of $\Theta$. The Weyl group of $M_{\Theta}$ is the subgroup $W_{\Theta}$. Its orbit through $b_0$ is contained in the fiber $\pi_{\Theta}^{-1}(\pi_{\Theta}(b_0))$.

In particular, the group $M_{\Theta}$ is of rank one if $\Theta$ is a singleton. For example, if $\alpha$ is a simple root, the fiber of $\mathbb{F} \rightarrow \mathbb{F}_\alpha = G/P_\alpha$ which is $P_\alpha/P$, coincides with the (unique) flag manifold
of the group $G(\alpha)$ whose Lie algebra is $\mathfrak{g}(\alpha)$, generated by $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_\alpha$. These rank one flag manifolds are spheres $S^m$, where $m = \dim(\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha})$.

The Bruhat decomposition presents the flag manifolds as a union of $N$-orbits (or one of its conjugates). It says that the $N$-orbits on a flag manifold $\mathbb{F}_\Theta$ is finite and coincide with the orbits that goes through the $A$-fixed points.

**Proposition 0.1.** Let $b_\Theta$ be the origin of $\mathbb{F}_\Theta$. Then the set $A$-fixed points coincides with the orbit $M^*b_\Theta$. This set is finite and is in bijection with $W/W_\Theta$.

Thus the Bruhat decomposition reads

$$\mathbb{F}_\Theta = \bigsqcup_{w \in W/W_\Theta} N \cdot wb_\Theta, \quad w \in M^*,$$

where $N \cdot w_1b_\Theta = N \cdot w_2b_\Theta$ if $w_2W_\Theta = w_1W_\Theta$. When there is an equivariant fibration $\mathbb{F}_{\Theta_1} \rightarrow \mathbb{F}_{\Theta_2}$ (in particular when $\mathbb{F}_{\Theta_1} = \mathbb{F}$) the $N$-orbits project onto $N$-orbits by equivariance, hence the fibration respects the Bruhat decompositions.

Each $N$-orbit through $w$ is diffeomorphic to an Euclidean space. Such an orbit $N \cdot wb_\Theta$ is called a Bruhat cell. Its dimension is given by the formula

$$\dim (N \cdot wb_\Theta) = \sum_{\alpha \in \Pi_w \setminus \langle \Theta \rangle} m_\alpha$$

where $m_\alpha$ is the multiplicity of the root space $\mathfrak{g}_\alpha$ and $\langle \Theta \rangle$ denotes the roots in $\Pi$ generated by $\Theta$ (see the Lemma 2.4 for the maximal flag case and Lemma 3.1 for the partial flag case). In particular, the Bruhat cell $N \cdot w_0b_\Theta$ is an open and dense orbit. The closure of the Bruhat cells are called (generalized) Schubert cells.

**Definition 0.2.** A Schubert cell is the closure of a Bruhat Cell:

$$S^\Theta_w = \text{cl}(N \cdot wb_\Theta).$$

The Schubert cells endow the flag manifolds with a cellular decomposition. For a maximal flag manifold we avoid the superscript $\Theta$ and write simply

$$S_w = \text{cl}(N \cdot wb_0)$$

We recall the following well known facts (see [3] or Warner [15]).

**Proposition 0.3.** $S^\Theta_{w_1} \subset S^\Theta_{w_2}$ if and only if $w_1 \leq w_2$.

**Proposition 0.4.** $S^\Theta_w = \cup_{u \leq w} N \cdot wb_\Theta$.

In the forthcoming sections we will look carefully at the cellular decompositions of the flag manifolds given by the Schubert cells. Before going into them we present examples showing that classical cell decompositions of compact groups are not well behaved with respect to projections to flag manifolds.

**Example:** In the cellular decomposition of $\text{SO}(3)$ of [16] and [11] there are 4 cells of dimensions 0, 1, 2 and 3. The 2-dimensional cell is given by the map $f : \mathbb{R}P^2 \rightarrow \text{SO}(3)$.
given by \( f([x]) = r_xd, \ x \in \mathbb{R}^3\setminus\{0\} \), where \( r_x \) is the reflection in \( \mathbb{R}^3 \) with respect to the plane orthogonal to \( x \) and \( d = \text{diag}\{1,1,-1\} \) needed to correct the determinant. This map is viewed as a two-cell \( B_2 \rightarrow SO(3) \) by taking the interior of the 2-ball \( B_2 \) as the set \( \{[x] \in \mathbb{RP}^2 : x_3 \neq 0\} \) with \( x = (x_1, x_2, x_3) \). The boundary of \( B_2 \) is mapped to the 1-dimensional cell which is the image under \( f \) of \( \mathbb{RP}^1 = \{(x_1, x_2, 0) \in \mathbb{RP}^2\} \). If \( \{e_1, e_2, e_3\} \) is the standard basis of \( \mathbb{R}^3 \) then \( f([e_1]) = \text{diag}\{-1,1,-1\}, f([e_2]) = \text{diag}\{1,-1,-1\} \) and \( f([e_3]) = \text{id} \). These three elements belong to the group \( M \) where \( \mathbb{F} = SO(3)/M \) is the maximal flag manifold of \( SL(3, \mathbb{R}) \). Hence the projection to \( \mathbb{F} \) of the 2-cell in \( SO(3) \) is not a cell in \( \mathbb{F} \) because \( f([e_i]), i = 1,2,3 \) are projected to the same point, namely the origin of \( \mathbb{F} \).

For further examples we recall the cellular decomposition of \( SU(n) \) given in [19], Theorem 7.2, where the positive dimensional cells have dimension \( \geq 3 \). Hence this construction does not yield, by projection \( SU(n) \rightarrow SU(n)/H \), a cellular decomposition of \( SU(n)/H \) if this homogeneous space has nontrivial homology at the levels 1 or 2. This happens, for instance, with the flag manifolds of \( SL(n, \mathbb{C}) \), that have nontrivial \( H_2 \). Also, the maximal compact subalgebra of the split real form of the exceptional type \( E_7 \) is \( \mathfrak{su}(8) \). However the maximal flag manifold of a split real form has nontrivial fundamental group (and hence \( H_1 \)) as follows by Johnson [7] and [17].

1. Schubert cells in maximal flag manifolds

In this section we give a detailed description of the Schubert cells in the maximal flag manifolds. This description includes a parametrization by compact groups (subsets of them) which allows explicit expressions for the gluing maps between the cells. The partial flag manifolds will be treated in the Section 3.

1.1. Schubert cells and product of compact subgroups. The main result here is a suitable parametrization for the Schubert cells which is the basis for the computation of the boundary operator for the cellular homology.

As before, \( \mathbb{F} = G/P \) is the maximal flag manifold. We denote by \( \mathbb{F}_i = G/P_i \) the partial flag manifolds where \( P_i = P_{(\alpha_i)} \), with \( \alpha_i \) a simple root. The canonical fibration is \( \pi_i : \mathbb{F} \rightarrow \mathbb{F}_i \).

The Schubert cells are firstly described by the “fiber-exhausting” map \( \gamma_i \) defined by

\[
\gamma_i(X) = \pi_i^{-1}\pi_i(X), \quad X \subset \mathbb{F},
\]

that is, \( \gamma_i(X) \) is the union of the fibers of \( \pi_i : \mathbb{F} \rightarrow \mathbb{F}_i \) crossing \( X \subset \mathbb{F} \). Notice that each \( \gamma_i \) is an equivariant map, i.e., \( g\gamma_i(X) = \gamma_i(gX) \), for all \( g \in G \) and \( X \subset \mathbb{F} \), since the projections \( \pi_i \) are equivariant maps.

For \( w \in \mathcal{W} \), put \( N^w = wNw^{-1} \).

Every Schubert cell is the image of some \( g \in G \) of \( \text{cl}(N^w b_0) \). The following result was proved in [9].

**Theorem 1.1.** Let \( w = r_1 \cdots r_n \) be a reduced expression of \( w \in \mathcal{W} \) as a product of reflections with respect to the simple roots. Then, for any \( k = 1, \ldots, n \), we have

\[
\text{cl}(N^w b_0) = \gamma_1 \cdots \gamma_k (\text{cl}(N^w r_1 \cdots r_kb_0)).
\]
In particular, for \( k = n \) we have

\[
\text{cl}(N^w b_0) = \gamma_1 \cdots \gamma_n \left( \text{cl} \left( Nw^{-1} wb_0 \right) \right) = \gamma_1 \cdots \gamma_n \{ wb_0 \}
\]

because \( Nb_0 = b_0 \). From this equality we get the following expression for the Schubert cell \( S_w \).

**Corollary 1.2.** Let \( w = r_1 \cdots r_n \) be a reduced expression as a product of reflections with respect to the simple roots in \( \Sigma \). Then,

\[
S_w = \gamma_n \cdots \gamma_1 \{ b_0 \}
\]

(\text{Note that the order of the indexes is reversed.})

**Proof.** We have \( \text{cl}(Nw \cdot b_0) = w \left( \text{cl}(N^{w^{-1}} b_0) \right) \), hence by (1) with \( w^{-1} = r_n \cdots r_1 \) instead of \( w \) we have

\[
S_w = w \gamma_n \cdots \gamma_1 (w^{-1} b_0) = \gamma_n \cdots \gamma_1 \{ b_0 \},
\]

where the last equality follows by equivariance. \( \square \)

Now we change slightly the above expression in terms of exhausting-fiber maps to get the Schubert cells as unions of successive orbits of the parabolic subgroups \( P_i \). This construction is in the same spirit as the Bott-Samelson dessingularization (see [3]).

It starts with the remark that the fiber \( \gamma_i \{ b_0 \} \) of \( \pi_i : F \to F_i \) through the origin is the orbit \( P_i \cdot b_0 \). In general, the fiber through \( g \cdot b_0 \in F \) is given by \( g \cdot \gamma_i \{ b_0 \} \) by equivariance of \( \gamma_i \). Now, if we have two iterations \( \gamma_2 \gamma_1 \), then by equivariance we get

\[
\gamma_2 \gamma_1 \{ b_0 \} = \gamma_2 \left( \bigcup_{g \in P_i} g \cdot b_0 \right) = \gamma_2 \left( \bigcup_{g \in P_i} g \cdot \gamma_2 (b_0) \right) = \left( \bigcup_{g \in P_i} g \cdot (P_2 b_0) \right) = P_1 P_2 \cdot b_0.
\]

Proceeding successively by induction, we obtain

\[
S_w = \gamma_n \cdots \gamma_1 \{ b_0 \} = P_1 \cdots P_n \cdot b_0,
\]

where the indexes of \( P_1 \cdots P_n \) is the same as those of minimal decomposition \( w = r_i \ldots r_n \in \mathcal{W} \).

The same expression still holds with the compact \( K_i = K \cap P_i \) instead of \( P_i \). In fact, \( K_i \cdot b_0 = P_i \cdot b_0 \) by the Langlands decomposition \( P_i = K_i AN \) and \( AN \cdot b_0 \). Hence the same arguments yield the following description of the Schubert cells.
**Proposition 1.3.** Let \( w = r_1 \cdots r_n \) be a reduced expression as a product of reflections with respect to the simple roots in \( \Sigma \). Then,

\[
S_w = K_1 \cdots K_n \cdot b_0.
\]

(Here, different from Corollary 1.2, the indexes of the \( r_i \)'s and \( K_i \)'s are in the same order).

**Remark:** In general, there is more than one reduced expression for \( w \in \mathcal{W} \), which provides distinct compact subgroups \( K_i \) and distinct parametrizations.

**Example:** Let \( G = \text{Sl}(n, \mathbb{R}) \) with \( g = \text{sl}(n, \mathbb{R}) \). The simple roots are given by \( \alpha_{i,i+1} = \lambda_i - \lambda_{i+1} \). The compact group \( K_i \) associated to the simple root \( \alpha_{i,i+1} \) is given by the rotations

\[
R_i^t = \exp(tA_{i,i+1}) = \begin{pmatrix}
1 & & \\
& \cos t & \sin t \\
& -\sin t & \cos t \\
& & \ddots
\end{pmatrix}
\]

where \( A_{i,i+1} = E_{i,i+1} - E_{i+1,i} \). In this case, a Schubert cell has the form

\[
S_w = R_{t_1}^{t_1} \cdots R_{t_m}^{t_m} \cdot b_0,
\]

that is, is the image of the map \((t_1, \ldots, t_m) \rightarrow R_{t_1}^{t_1} \cdots R_{t_m}^{t_m} \cdot b_0 \in \mathbb{F})\).

Continuing with the example, let \( n = 3 \), with \( \mathcal{W} \) the permutation group in three letters. The Schubert cell \( S_{(13)} \) is the whole flag \( \mathbb{F}^3_{1,2} \) since \( (13) \) is the principal involution. If we decompose \( (13) = (12)(23)(12) \), \( S_{(13)} \) may be parametrized as:

\[
\begin{pmatrix}
\cos t_1 & \sin t_1 \\
-\sin t_1 & \cos t_1 \\
& & 1
\end{pmatrix}
\begin{pmatrix}
1 & \cos t_2 & \sin t_2 \\
& -\sin t_2 & \cos t_2 \\
& & \ddots
\end{pmatrix}
\begin{pmatrix}
\cos t_3 & \sin t_3 \\
-\sin t_3 & \cos t_3 \\
& & 1
\end{pmatrix} \cdot b_0.
\]

If we choose to write \( (13) = (23)(12)(23) \) we parametrize \( S_{(13)} \) as:

\[
\begin{pmatrix}
1 & \cos t_1 & \sin t_1 \\
-\sin t_1 & \cos t_1 \\
& & 1
\end{pmatrix}
\begin{pmatrix}
\cos t_2 & \sin t_2 \\
-\sin t_2 & \cos t_2 \\
& & 1
\end{pmatrix}
\begin{pmatrix}
\cos t_3 & \sin t_3 \\
-\sin t_3 & \cos t_3 \\
& & 1
\end{pmatrix} \cdot b_0.
\]

In these examples the parameter \( t_i \) range in the interval \([0, \pi] \) because \( R_i^t \cdot b_0 = b_0 \) for any \( i \) (\( b_0 = (V_1 \subset V_2) \) where \( V_1 \) is the one dimensional subspace of \( \mathbb{R}^3 \) spanned by the first basic vector and \( V_2 \) is spanned by the first two basic vectors). This is a general feature since our cell maps will be defined in cubes \([0, \pi]^m \).
1.2. Bruhat cells inside the Schubert cell. The next results determine the points of a Schubert cell $S_w = K_1 \cdots K_n \cdot b_0$ which are in the corresponding Bruhat cell $N \cdot w b_0$.

**Lemma 1.4.** Let $w = r_1 \cdots r_{n-1} r_n$ a minimal decomposition. Define $v = w r_n = r_1 \cdots r_{n-1}$. Let the parabolic subgroup $P_n = P_{\{a_n\}}$ with $r_n$ the reflection with respect to $a_n$ and $F_n = G / P_n$. Let $\pi_n : F \to F_n$ be the canonical projection and denote by $b_0$ the origin of $G / P_n$. Then we have the disjoint union

\[ \pi_n^{-1}(N \cdot w b_0) = (N \cdot v b_0) \cup (N \cdot v b_0) \]

**Proof.** The fiber $\pi_n^{-1}(w b_0)$ is the flag manifold of a rank one group. Its Bruhat decomposition reads

\[ \pi_n^{-1}(w b_0) = \{v b_0\} \cup (\pi_n^{-1}(w b_0) \cap (N \cdot v b_0)) \]

Indeed, in $\pi_n^{-1}(w b_0)$ there are a 0-cell which is $\{v b_0\}$ and an open cell. This latter one is $\pi_n^{-1}(w b_0) \cap (N \cdot v b_0)$ because it is contained in the Bruhat cell $N \cdot w b_0$ and $v b_0 \notin N \cdot w b_0$.

The result follows by acting $N$. In fact, $w b_0 \in \pi_n^{-1}(w b_0) \cap (N \cdot v b_0)$, hence $N \cdot w b_0 = N (\pi_n^{-1}(w b_0) \cap (N \cdot v b_0))$. Also, by equivariance of $\pi_n$ we get $N \pi_n^{-1}(w b_0) = \pi_n^{-1}(N \cdot w b_0)$.

Then,

\[ \pi_n^{-1}(N \cdot w b_0) = N \left( \{v b_0\} \cup (\pi_n^{-1}(w b_0) \cap (N \cdot v b_0)) \right) = (N \cdot v b_0) \cup (N \cdot v b_0). \]

We notice that Equation (3) is equivalent to

\[ \pi_n^{-1}(N \cdot w b_0) = S_v \cup (N \cdot v b_0) \]

because $\pi_n^{-1}(N \cdot w b_0) \cap S_v = N \cdot v b_0$ and $\pi_n^{-1}(N \cdot w b_0) = \pi_n^{-1}(N \cdot v b_0)$ since $w b_0 = v b_0$ in $F_n$.

**Proposition 1.5.** Write $S_w = K_1 \cdots K_n \cdot b_0$. Take $b = u_1 \cdots u_n \cdot b_0$, with $u_i \in K_i$. Then $b \in S_w \setminus N \cdot w b_0$ if and only if if $u_i \in M$ for some $i = 1, \ldots, n$.

In other words, an element $b \in S_w$ is inside the Bruhat cell $N \cdot w b_0$ if and only if there is no $u_i \in M$.

**Proof.** Suppose that $u = u_i \in M$ for some $i$. Then $u \in K_j$ for all $j$, since $M \subset K_j$, so that $v_j = u u_j u^{-1} \in K_j$. Hence $b$ can be rewritten as $b = u_1 \cdots u_{i-1} v_{i+1} \cdots v_n u \cdot b_0$. Since $u b_0 = b_0$, it follows that $b \in S_v$, with $v = r_1 \cdots r_i \cdots r_n$, which implies that $b \notin N \cdot w b_0$ since $v < w$ and $S_w \setminus N \cdot w b_0 = \cup_{u \ll w} S_u$.

For the converse we use induction on the length of $w$. If $w = r_1$ has length one, then the Schubert cell is $S_{r_1} = b_0 \cup (N \cdot u_1 b_0)$. So if $u_1 \notin M$, then $u_1 b_0 \neq b_0$ and hence $u_1 b_0 \in N \cdot u_1 b_0$.

For $n > 1$, let $b = u_1 \cdots u_n \cdot b_0$ with $u_i \notin M$. We must show that $b \notin N \cdot w b_0$. Put $x = u_1 \cdots u_{n-1} \cdot b_0$. Note that $b \neq x$ for otherwise $u_n b_0 = b_0$ which gives $u_n \in M$, contradicting the assumption.

The induction hypothesis says that $x \in N \cdot v b_0$, $v = r_1 \cdots r_{n-1}$. Moreover, $\pi_n(b_0) = \pi_n(u_n b_0)$ which implies that $\pi_n(x) = \pi_n(b)$, that is, $x$ and $b$ are in the same fiber of $\pi_n$. Hence $\pi_n(b) \in \pi_n(N \cdot w b_0)$, so that by Lemma 1.4 $b \in (N \cdot v b_0) \cup (N \cdot v b_0)$. 


Now \( b \notin N \cdot vb_0 \) for otherwise \( b = x \). In fact, as \( \pi_n(b) = \pi_n(x) = zb_n \), for some \( z \in N \), we have \( b \in \pi_n^{-1}(zb_n) \cap N \cdot vb_0 \). Since this intersection reduces to \( zb_0 \) we have \( x = zb_0 \), because \( x \in N \cdot vb_0 \). Hence \( b \in N \cdot wb_0 \), concluding the proof. 

### 1.3. Parametrization of subsets of compact subgroups.

The next step is to find subsets of the subgroups \( K_i \) that cover \( S_w = K_1 \cdots K_n \cdot b_0 \) and thus find parametrizations of the cells.

The fiber \( P_i/P \) of the projection \( \mathbb{F} \to \mathbb{F}_i \) is the flag of the rank one Lie group \( G(\alpha) \) whose Lie algebra is \( \mathfrak{g}(\alpha) \), generated by \( \mathfrak{g}_{\alpha} \). The flag manifold \( \mathbb{F}_\alpha \) of \( G(\alpha) \) is a sphere \( S^m \) with dimension \( m = \dim \mathfrak{s}_\alpha - 1 \) where \( \mathfrak{s}_\alpha \) is the symmetric part of the Cartan decomposition of \( \mathfrak{g}(\alpha) \). If \( \{1, w\} \) is the Weyl group of \( G(\alpha) \) and \( b_0 \) is the origin of \( \mathbb{F} \) then \( b_0 \) and \( wb_0 \) are antipodal points in \( S^m \). The parametrization we seek is provided by the following lemma whose general proof is only sketched below. In the sequel we write down the details for the case when \( \dim \mathfrak{g}_\alpha = 1 \) and \( \mathfrak{g}_{2\alpha} = \{0\} \) so that \( \mathfrak{g}(\alpha) \approx \mathfrak{sl}(2, \mathbb{R}) \).

**Lemma 1.6.** Let \( G = G(\alpha) \) be a real one rank group of rank with maximal compact subgroup \( K = K_\alpha \) and the corresponding flag manifold \( \mathbb{F} = S^m \) with origin \( b_0 \). Let \( B^m \) be the closed ball in \( \mathbb{R}^m \). Then, there exists a continuous map \( \psi : B^m \to K \) such that

- \( \psi(S^{m-1}) \subset M \) and hence \( \psi(S^{m-1}) \cdot b_0 = b_0 \).
- If \( x \in B^m \setminus S^{m-1} \), then \( \psi(x) \cdot wb_0 \) is a diffeomorphism onto the Bruhat cell which is the complement of \( b_0 \).

For the proof of the lemma recall that the following list exhaust the rank one Lie algebras (see [15], pages 30-32). In the list \( d_\alpha = \dim \mathfrak{g}_\alpha \) and \( d_{2\alpha} = \dim \mathfrak{g}_{2\alpha} \).

- \( \mathfrak{so}(1, n) \); \( d_\alpha = n - 1, d_{2\alpha} = 0 \); \( \dim \mathfrak{s} = n \). (This class includes \( \mathfrak{sl}(2, \mathbb{R}) \approx \mathfrak{sp}(1, \mathbb{R}) \approx \mathfrak{so}(1, 2), \mathfrak{sl}(2, \mathbb{C}) \approx \mathfrak{so}(1, 3) \) and \( \mathfrak{su}^*(4) \approx \mathfrak{so}(1, 5) \).)
- \( \mathfrak{su}(1, n) \); \( d_\alpha = 2(n - 1), d_{2\alpha} = 1 \); \( \dim \mathfrak{s} = 2n \). (This class includes \( \mathfrak{so}^*(6) \approx \mathfrak{su}(1, 3) \).)
- \( \mathfrak{sp}(1, n) \); \( d_\alpha = 4(n - 1), d_{2\alpha} = 3 \); \( \dim \mathfrak{s} = 4n \).
- A real form of the exceptional Lie algebra \( F_4 \); \( d_\alpha = 8, d_{2\alpha} = 7 \); \( \dim \mathfrak{s} = 16 \).

The exceptional algebra \( F_4 \) does not appear as a \( \mathfrak{g}(\alpha) \) in any Lie algebra different from itself because apart from \( F_4 \) the multiplicities \( d_{2\alpha} \) are at most 3 (see [15], pages 30-32). Hence, we can discard it.

On the other hand the classical groups \( \text{SO}(1, n), \text{SU}(1, n) \) and \( \text{Sp}(1, n) \) contain the compact subgroups \( \text{SO}(n), \text{SU}(n) \) and \( \text{Sp}(n) \) whose actions on the respective flag manifolds \( S^{n-1}, S^{2n-1} \) and \( S^{4n-2} \) are the standard ones coming from the linear actions in \( \mathbb{R}^n, \mathbb{C}^n \) and \( \mathbb{H}^n \), respectively. In each case the origin \( b_0 \) of the flag is the first basic vector \( e_1 \) while \( wb_0 = -e_1 \).

Now, take matrices

\[
A_\gamma = \begin{pmatrix}
0 & -\gamma^T \\
\gamma & 0
\end{pmatrix}
\]

with \( \gamma \) in \( \mathbb{R}^{n-1}, \mathbb{C}^{n-1} \) and \( \mathbb{H}^{n-1} \) respectively, such that \( \|\gamma\| = 1 \). If \( m = n - 1, 2n - 1 \) or \( 4n - 1 \) then \( U \) is one of the groups \( \text{SO}(n), \text{SU}(n) \) and \( \text{Sp}(n) \). If \( U = \text{SO}(n) \) then the map \( \psi : S^{m-1} \times [0, 2\pi] \to U \) given by \( \psi(\gamma, t) = e^{tA_\gamma} \) satisfies the requirements of Lemma 1.6.
because $e^{tA_\gamma} \cdot e_1 = \cos te_1 + \sin t\tilde{\gamma}$ where $\tilde{\gamma} = (0, \gamma)$. The complex and quaternionic cases are made similarly with slight modifications. If $U = SU(n)$, let $B^{2n-2} = \{ t\gamma : \| \gamma \| = 1, t \in [0, \pi] \}$ and define the map $\psi : B^{2n-2} \times [-\pi, \pi] \rightarrow SU(n) \subset K$ by $\psi(t\gamma, \theta) = e^{tA_\gamma}e^{D_\theta}$ where

$$D_\theta = \text{diag}\left\{i\theta, -\frac{i}{n-1}\theta, \ldots, -\frac{i}{n-1}\theta\right\}.$$ 

It follows that $\psi$ is the desired parametrization. If $U = Sp(n)$, the map $\psi : B^{4n-4} \times B^3 \rightarrow Sp(n) \subset K$ that realizes the parametrization is defined by $\psi(t\gamma, q) = e^{tA_\gamma}e^{D_q}$, where $B^{4n-4} = \{ t\gamma \in \mathbb{H}^{n-1} : \| \gamma \| = 1, t \in [0, \pi] \}$, $B^3 = \{ q \in i\mathbb{H} : \| q \| \leq \pi \}$ and

$$D_q = \text{diag}\left\{ q, -\frac{1}{n-1}q, \ldots, -\frac{1}{n-1}q \right\}.$$

From now on we consider the case when $\dim \mathfrak{g}_\alpha = 1$ and $\mathfrak{g}_{2\alpha} = \{0\}$, so that $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{R})$ and compact Lie algebra of $K_\alpha$ is $\mathfrak{so}(2)$. This is the only relevant case to the computation of homology of the flag manifolds (c.f. Proposition 1.10).

Let $\theta$ be the Cartan involution. Take $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha = \theta(X_\alpha) \in \mathfrak{g}_{-\alpha}$ such that $\langle X_\alpha, Y_\alpha \rangle = \frac{2}{(\alpha, \alpha)}$. Hence, $[X_\alpha, Y_\alpha] = H_\alpha^* = \frac{2H_\alpha}{(\alpha, \alpha)}$. Denote by $A_\alpha = X_\alpha + Y_\alpha \in \mathfrak{f}$. The Lie algebra $\mathfrak{g}(\alpha) = \mathfrak{g}_{-\alpha} \oplus \langle H_\alpha^* \rangle \oplus \mathfrak{a}_\alpha$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Explicitly, write $\rho : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}(\alpha)$, with $\rho(H) = H_\alpha^*, \rho(X) = X_\alpha$ and $\rho(Y) = Y_\alpha$ where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This homomorphism extends to a homomorphism $\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}_C(\alpha)$. Note that $\text{ad} \circ \phi$ is a representation of $\mathfrak{sl}(2, \mathbb{C})$ in $\mathfrak{g}_C$. As $\text{Sl}(2, \mathbb{C})$ is simply connected, this representation extends to a representation $\Phi$ of $\text{Sl}(2, \mathbb{C})$ in $\mathfrak{g}_C$ and they are related by $e^{\text{ad} \circ \phi(X)} = \Phi(\exp(X))$ for any $X \in \mathfrak{sl}(2, \mathbb{C})$. We have

$$e^{\text{ad}(\pi A_\alpha)} = e^{\text{ad} \circ \phi(A)} = \Phi(\exp(\pi A)),$$

where $A = X + Y$. But in $\text{Sl}(2, \mathbb{C})$ we have

$$\exp(\pi A_\alpha) = \exp \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \exp \begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix} = \exp(i\pi H).$$

Therefore,

$$e^{\text{ad}(\pi A_\alpha)} = \Phi(\exp(i\pi H)) = e^{\text{ad} \circ \phi(i\pi H)} = e^{\text{ad}(i\pi H_\alpha^*)}. \quad (4)$$

Put

$$m_\alpha = \exp(\pi i H_\alpha^*) = \exp(\pi A_\alpha).$$

Then $m_\alpha$ centralizes $A$ ($m_\alpha = \exp(\pi i H_\alpha^*)$) and belongs to $K$ ($m_\alpha = \exp(\pi A_\alpha)$). Hence $m_\alpha \in M = Z_K(\mathfrak{a})$. 
Now consider the curve \( \gamma(t) = \exp(tA_\alpha) \cdot b_0 \) in the fiber of \( \mathbb{F} \to G/P_\alpha \) through the origin. Since \( m_\alpha \in M, \gamma(\pi) = m_\alpha b_0 = b_0 \). Actually \( \gamma(t) \) covers the fiber in the interval \([0, \pi]\).

In \( \mathfrak{sl}(2, \mathbb{R}) \) we have that

\[
\text{Ad}(e^{tA})H = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.
\]

That is, \( \text{Ad}(e^{tA})H = -\sin 2tX + \cos 2tH + \sin 2tY \). Applying the formula

\[
\rho(\text{Ad}(e^{tA})H) = \text{Ad}(e^{tA})H^\vee
\]

we get \( \text{Ad}(e^{tA})H^\vee = -\sin 2tX_\alpha + \cos 2tH_\alpha^\vee + \sin 2tY_\alpha \). This shows that \( e^{tA} \) centralizes \( H_\alpha^\vee \) if and only if \( t = n\pi \). In particular, \( e^{tA} \in M \) if and only if \( t = n\pi \). Hence, the period of \( \gamma \) is exactly \( \pi \). Summarizing,

**Lemma 1.7.** The one-dimensional version of the Lemma 1.6 is realized by

\[
\psi : [0, \pi] \to K_\alpha, t \mapsto \exp(tA_\alpha).
\]

In particular, \( \psi(0) = 1 \) and \( \psi(\pi) = m_\alpha = \exp(\pi A_\alpha) \).

Moreover, if \( X \in \mathfrak{g}_\beta \), then:

\[
\text{Ad}(m_\alpha)(X) = \text{Ad}(\exp(\pi i H_\alpha^\vee))(X) = e^{\text{ad}(\pi i H_\alpha^\vee)}(X) = e^{\pi i \epsilon(\alpha, \beta)}(X),
\]

where \( \epsilon(\alpha, \beta) = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \) is the Killing number. This implies that

**Lemma 1.8.** The root spaces \( \mathfrak{g}_\beta \) are invariant by the action of \( \text{Ad}(m_\alpha) \) and

\[
\text{Ad}(m_\alpha)|_{\mathfrak{g}_\beta} = (-1)^{\epsilon(\alpha, \beta)} \text{id}.
\]

1.4. **Gluing cells.** A Schubert cell \( S_w \) is obtained from smaller cells \( S_v, v < w, \) by gluing a cell of dimension \( \dim(N \cdot wb_0) \). Once this process is done for each \( w \in \mathcal{W} \), we get a cellular decomposition for \( \mathbb{F} \) which is explicitly given by characteristic maps and attaching maps. (We follow the terminology of Hatcher [1]; the characteristic map is defined in a closed ball while the attaching map is the restriction characteristic map to the boundary of the ball.)

In order to define a characteristic map for \( S_w, w \in \mathcal{W} \), we must choose a reduced expression

\[
w = r_1 \cdots r_n
\]

as product of simple reflections \( r_i = r_{\alpha_i} \). We know that \( S_w = K_1 \cdots K_n \cdot b_0 \). By Lemma 1.6 for each \( i \), there exists \( \psi_i : B^{d_i} \to K_i \), where \( d_i \) is the dimension of the fiber of \( \mathbb{F} \to \mathbb{F}_i \), that is, the dimension of the flag of \( G(\alpha_i) \).

Let \( B_w = B^{d_1} \times \cdots \times B^{d_n} \) be the ball with dimension \( d = d_1 + \cdots + d_n \). Then the characteristic map \( \Phi_w : B_w \to \mathbb{F} \) is defined by

\[
\Phi_w(t_1, \ldots, t_n) = \psi_1(t_1) \cdots \psi_n(t_n) \cdot b_0.
\]
Proof. The first condition holds by construction since \( x \), that is, \( \psi \) and \( t \mapsto \psi(t) \) is a characteristic map for \( S_w \) that is, \( \psi(t) \) is the map defined above and take \( t = (t_1, \ldots, t_n) \in B_w \). Then, \( \Phi_w \) is a characteristic map for \( S_w \), that is,

1. \( \Phi_w(B_w) \subset S_w \).
2. \( \Phi_w(t) \in S_w \backslash N \cdot wb_0 \) if and only if \( t \in \partial B_w = S^{d-1} \).
3. \( \Phi|_{B_w^o} : B_w^o \to N \cdot wb_0 \) is a diffeomorphism \( (B_w^o \text{ is the interior of } B_w) \).

Proof. The first condition holds by construction since \( \psi_i(t_i) \in K_i \) and hence \( \Phi_w(t_1, \ldots, t_n) = \psi_1(t_1) \cdots \psi_n(t_n) \cdot b_0 \in K_1 \cdots K_n \cdot b_0 = S_w \).

The second statement follows as a consequence of the Proposition 1.5 by which we have that \( \psi_1(t_1) \cdots \psi_n(t_n) \) is not in \( N \cdot wb_0 \) if and only if some \( \psi_i(t_i) \in M \). By Lemma 1.4, this implies that \( t_i \in \{0, \pi\} = \partial S^{d-1} \), that is, \( t \in \partial S^{d-1} \).

Finally, we already have that \( \Phi|_{B_w^o} \) is a surjective map. Let us prove its injectivity by induction on the length \( l(w) \) of \( w \). If \( l(w) = 1 \), this is Lemma 1.7. For \( l(w) > 1 \), suppose that \( \Phi_w(t) = \Phi_w(s) \) with \( t = (t_1, \ldots, t_n) \) and \( s = (s_1, \ldots, s_n) \) in \( B_w^o \). Then we claim that \( x = y \) where

\[
\begin{align*}
x & = \psi_1(t_1) \cdots \psi_{n-1}(t_{n-1}) \cdot b_0 \\
y & = \psi_1(s_1) \cdots \psi_{n-1}(s_{n-1}) \cdot b_0.
\end{align*}
\]

In fact, the elements \( \psi_i(t_i) \) and \( \psi_i(s_i) \) are not in \( M \), hence by Proposition 1.5 both \( x, y \in N \cdot vb_0 \), \( v = r_1 \cdots r_{n-1} \). Also, \( \pi_n(x) = \pi_n(\Phi_w(t)) = \pi_n(\Phi_w(s)) = \pi_n(x) \), that is, \( x \) and \( y \) belong to the same fiber of \( \pi_n : F \to \mathbb{F}_n \). It follows by Lemma 1.4 that \( x = y \) since \( N \cdot vb_0 \) meets each fiber of \( \pi_n \) at a unique point. By the induction hypothesis \( (t_1, \ldots, t_{n-1}) = (s_1, \ldots, s_{n-1}) \), so that \( \psi_1(t_1) \cdots \psi_{n-1}(t_{n-1}) = \psi_1(s_1) \cdots \psi_{n-1}(s_{n-1}) \). Applying this to the equality \( \Phi_w(t) = \Phi_w(s) \) we conclude that \( \psi_n(t_n) \cdot b_0 = \psi_n(s_n) \cdot b_0 \), which in turn implies that \( t_n = s_n \), \( l(r_n) = 1 \). Therefore, \( \Phi_w \) is a closed continuous and bijective map, hence it is a homeomorphism (differentiability comes from the construction of the maps \( \psi_i \)). \( \square \)

As a consequence of the last item of the above proposition, we have the following construction. Let \( d = \dim S_w = \dim N \cdot wb_0 \). The sphere \( S^d \) is the quotient \( B_w/\partial(B_w) \) where the boundary is collapsed to a point. We can do the same with the Schubert cell \( S_w \). Define \( \sigma_w = S_w/(S_w \backslash N \cdot wb_0) \), i.e., the space obtained by identifying the complement of the Bruhat cell \( S_w \backslash N \cdot wb_0 \) in \( S_w \) to a point. As \( \Phi_w(\partial(B_w)) \subset S_w \backslash N \cdot wb_0 \), it follows that \( \Phi_w \) induces a map \( S^d \to \sigma_w \) which is a homeomorphism. The inverse of this homeomorphism will be denoted by

\[
(5) \quad \Phi_w^{-1} : \sigma_w \to S^d
\]

(although this is not the same as the inverse of \( \Phi_w \)).
A very useful data is the determination of pairs \( w, w' \in \mathcal{W} \) for which \( w' \leq w \) and \( \dim S_w - \dim S_{w'} = 1 \).

**Proposition 1.10.** Let \( w, w' \in \mathcal{W} \). The following statements are equivalent.

1. \( S_{w'} \subset S_w \) and \( \dim S_w - \dim S_{w'} = 1 \).
2. If \( w = r_1 \cdots r_n \) is a reduced expression of \( w \in \mathcal{W} \) as a product of simple reflections, then
   
   \( (i) \) \( w' = r_1 \cdots \hat{r}_i \cdots r_n \) is a reduced expression.
   
   \( (ii) \) If \( r_i = r_\alpha \), then \( g(\alpha_i) \cong sl(2, \mathbb{R}) \). This is the same as saying the fiber of \( F \to F_i \) has dimension 1.

**Proof.** In fact, \( S_{w'} \subset S_w \) if and only if \( w' < w \) in the Bruhat-Chavvalley order. In this case if \( w = r_1 \cdots r_n \) and \( w' = r_{i_1} \cdots r_{i_j} \) are reduced expressions then \( \dim S_w \) is \( \dim S_{w'} \) plus the sum of the multiplicities of the roots missing in the reduced expression for \( w' \). Hence \( \dim S_w - \dim S_{w'} = 1 \) if and only if \( w' = r_1 \cdots \hat{r}_i \cdots r_n \) and \( \alpha_i \) has multiplicity 1, that is, \( g(\alpha_i) \cong sl(2, \mathbb{R}) \). \( \square \)

**Remark:** Given \( w' \) as above, the decomposition \( w' = r_1 \cdots r_i \cdots r_n \) is unique. In fact, if \( w = r_1 \cdots r_i \cdots r_j \cdots r_n \) and \( w' = r_1 \cdots r_i \cdots \hat{r}_j \cdots r_n \) then \( r_{i+1} \cdots r_j = r_i \cdots r_{j-1} \) which cannot happen (see [10], Chapter 9).

1.5. **Gradient flow.** It is known that the vector field \( \tilde{H} \) with flow \( \exp tH \) induced on a flag manifold \( F_\Theta \) by a regular element \( H \in a^+ \) is the gradient of a Morse function. For this flow, the singularities are \( wb_0, w \in \mathcal{W} \), whose unstable and stable manifolds are Bruhat cells \( W^u(wb_0) = N \cdot wb_0 \) and \( W^s(wb_0) = N^- \cdot ub_0 \) (see [3], for details).

Below we describe these orbits in terms of characteristic maps of the cellular decomposition constructed above.

Take \( w = r_1 \cdots r_n \in \mathcal{W} \) with the characteristic map \( \Phi_w \) and let \( w' = r_1 \cdots \hat{r}_i \cdots r_n \) such that \( S_{w'} \subset S_w \) and \( \dim S_{w'} = \dim S_w - 1 \), by Proposition 1.10.

Note that by construction \( wb_0 = \Phi_w(\pi/2, \ldots, \pi/2, \ldots, \pi/2) \), that is, \( wb_0 \) is the image of the center of the cube. Now, consider the path

\[
\phi_i(t) = \Phi_w(\pi/2, \ldots, t, \ldots, \pi/2) \quad , \quad t \in [0, \pi],
\]

where \( t \) is in the \( i \)-th position. Then \( \phi(\pi/2) = w \cdot b_0 \), \( \phi(0) = w' \cdot b_0 \) comes from the 0-face and \( \phi(\pi) = w' \cdot b_0 \) comes from the \( \pi \)-face. Below we prove that the two pieces of \( \phi_i(t) \), from \( \pi/2 \) to \( \pi \) and from 0 to \( \pi/2 \) (in reversed direction) are the two gradient lines joining the singularities \( w \cdot b_0 \) and \( w' \cdot b_0 \).

In what follows, we write \( w' = r_1 \cdots \hat{r}_i \cdots r_n = u \cdot v \), i.e., \( u = r_1 \cdots r_{i-1} \) and \( v = r_{i+1} \cdots r_n \). Put \( X_\beta = \text{Ad}(u)X_\alpha, \quad Y_\beta = \theta(X_\beta) \) and \( A_\beta = X_\beta + Y_\beta \).

**Lemma 1.11.** With the above notation, we have that

\[
\phi_i(t) = \exp(sA_\beta)wb_0
\]
where \( s = t + \pi/2 \in [-\pi/2, \pi/2] \) if \( t \in [0, \pi] \).

**Proof.** The result is a consequence of the following computation.

\[
\phi_i(t) = u \exp(tA_i)v_{b_0} = u \exp(tA_i)r_{r_i}v_{b_0} = u \exp(tA_i) \cdot \exp((\pi/2)A_i)rv_{b_0} = \exp((t + \pi/2)\text{Ad}(u)A_i)v_{b_0} = \exp(sA_\beta)v_{b_0}
\]

where \( \beta = u\alpha_i \) implies that \( A_\beta = \text{Ad}(u)A_i \).

\( \Box \)

Let us consider \( \phi_i(s) = \exp(sA_\beta)v_{b_0} \). It follows that \( \phi_i(0) = v_{b_0}, \phi_i(\pm\pi/2) = w'b_0 \).

**Lemma 1.12.** \( \exp(tY_\beta)v_{b_0} = v_{b_0} \).

**Proof.** The main idea is translate to the origin. That is

\[
\exp(tX_{-\beta})v_{b_0} = w(w^{-1}\exp(tX_{-\beta}))v_{b_0} = w \exp(t\text{Ad}(w^{-1}X_{-\beta}))v_{b_0}.
\]

The root \( \beta \) is positive while \( w^{-1}\beta \) is negative. As \( u(g_\alpha) = g_{u\alpha} \) we have that \( \text{Ad}(w^{-1}X_{-\beta}) \in g_{-w^{-1}\beta} \). But since \( w^{-1}\beta \) is a negative root, it follows that \( -w^{-1}\beta \) is positive. Hence, it belongs to \( n^+ \subset \mathfrak{p} \) and therefore \( \exp(t\text{Ad}(w^{-1}X_{-\beta}))v_{b_0} = v_{b_0} \) for all \( t \in \mathbb{R} \). \( \Box \)

**Lemma 1.13** (\[N\] Lemma 2.4.1). Let \( t = \tan(s), r = -\sin(s)\cos(s), \lambda = \cos(s)^{-1} \). Hence

\[
\phi_i(s) = e^{tx_{\beta}}e^{cy_{\beta}}e^{\log(\lambda)h_\beta}v_{b_0}.
\]

Note however that both matrices \( e^{\log(\lambda)h_\beta} \) and \( e^{cy_{\beta}} \) fix \( v_{b_0} \). This gives that

\[
\phi_i(s) = e^{\tan(s)x_{\beta}}v_{b_0} , \ s \in (-\pi/2, \pi/2).
\]

Finally, we have that

\[
\lim_{t \to \pm\infty} \exp(tX_{\beta})v_{b_0} = \phi_i(\pm\pi/2) = w'b_0.
\]

From \( \ref{3} \) we get the behaviour of the gradient flow \( h^t = \exp tH \) with \( H \in \mathfrak{a}^+ \).

Let \( s \neq 0 \). It is easy to see that \( h^t(\phi_i(s)) = \exp(\tan(s)e^{t\beta(H)x_{\beta}})v_{b_0} \). This may be written as \( h^t(\phi_i(s)) = \phi_i(s') \) with \( s' = \arctan(\tan(s)e^{t\beta(H)}) \). Hence, we conclude that the gradient flow leaves the path \( \phi_i \) invariant.

Observe that \( \beta \) is a positive root. So \( e^{t\beta(H)x_{\beta}} \to 0 \) as \( t \to -\infty \) and hence \( s' \to 0 \), i.e.,

\[
\lim_{t \to -\infty} h^t\phi_i(s) = \phi_i(0) = v_{b_0}.
\]

When \( t \to +\infty \) it follows that \( \tan(s)e^{t\beta(H)x_{\beta}} \to \pm\infty \) depending only in the sign of \( \tan(s) \). Hence \( s' \to \pm\pi/2 \), i.e.,

\[
\lim_{t \to +\infty} h^t\phi_i(s) = \phi_i(\pm\pi/2) = w'b_0.
\]

Thus we get the desired result.

**Proposition 1.14.** \( \phi_i(s) \) give the two gradient flow lines between \( v_{b_0} \) and \( w'b_0 \). One of them belongs to the interval \( s \in (-\pi/2, 0) \) while the other belongs to the interval \( s \in (0, \pi/2) \).
2. Homology of maximal flag manifolds

The cellular homology of a CW complex is defined from a cellular decomposition of the complex and is isomorphic to the singular homology of the space. It means that the homology group does not depend on the choice of the cellular decomposition, although the boundary operator may change according to the choice of the cellular decomposition, i.e., the way the cells are glued.

In view of that, we fix once and for all reduced expressions

$$w = r_1 \cdots r_n$$

as a product of simple reflections, for each $w \in \mathcal{W}$. After making these choices we define as before the characteristic map $\Phi_w$, $w \in \mathcal{W}$, that glues the ball $B_w$ in the union of Schubert cells $S_u$ with $u < w$.

2.1. The boundary map. We recall (in our context) the definition of the cellular boundary maps giving the homology with coefficients in a ring $R$ (see [4]). Let $\mathcal{C}$ be the $R$-module freely generated by $S_w$, $w \in \mathcal{W}$. The boundary maps $\partial : \mathcal{C} \to \mathcal{C}$ are defined by

$$\partial S_w = \sum_{w'} c(w, w') S_{w'}$$

where the coefficients $c(w, w') \in R$ satisfy the properties:

1. $c(w, w') = 0$ in case $\dim S_w - \dim S_{w'} \neq 1$.
2. If $\dim S_w - \dim S_{w'} = 1$ then $c(w, w') = \deg \left( \phi_{w,w'} : S_{w'}^{d-1} \to S_w^{d-1} \right)$, where $\phi_{w,w'}$ is the composition of the following maps:
   a. The attaching map: $\Phi_w|_{B_d^w} : S_{w'}^{d-1} = \partial(B_d^w) \to S_w \setminus N \cdot w b_0 = \cup_{u < w} S_u = X^{d-1}$, where $X^{d-1}$ denotes the $(d-1)$-skeleton of $S_w$.
   b. The quotient map $X^{d-1} \to X^{d-1}/(X^{d-1}\setminus S_{w'})$ where we take the cell $S_{w'}$ inside $X^{d-1}$ and identify its complement in $S_{w'}$ to a point.
   c. The identification: $X^{d-1}/(X^{d-1}\setminus S_{w'}) \cong S_{w'}/(S_{w'} \setminus N \cdot w' b_0)$ which are in the same space. This last one is $S_{w'}/(S_{w'} \setminus N \cdot w' b_0) = \sigma_{w'}$ by definition.
   d. $\Phi_{w'}^{-1} : \sigma_{w'} \to S_{w'}^{d-1}$. This is the map defined in (5).

Remark: There is a subtlety which must be emphasized: $\phi_{w,w'}$ is a map $S^{d-1}_w \to S^{d-1}$ whose domain is the boundary of a ball in some $\mathbb{R}^N$ (the ball $B_w$) and, hence, it is a canonically defined sphere. However, the codomain is the space $\sigma_{w'}$ which is homeomorphic to $S^{d-1}$.

To get the boundary map a homeomorphism $\sigma_{w'} \to S^{d-1}$ must be fixed beforehand, since distinct homeomorphisms may yield maps with distinct degrees. Here is where it is needed to choose in advance the reduced expressions of $w \in \mathcal{W}$.

To compute the the degree $c(w, w') = \deg \left( \phi_{w,w'} : S_{w'}^{d-1} \to S_w^{d-1} \right)$ when $w = r_1 \cdots r_n$ and $w' = r_1 \cdots r_i \cdots r_n$ are minimal decompositions we proceed with the following steps.
Step 1: Domain and codomain spheres. First we identify the spheres $S_{w}^{d-1}$ in the domain and $S_{w}^{d-1}$ in the codomain.

Remember that $B_{w} = B^{d_{1}} \times \cdots \times B^{d_{n}}$ where $B^{d_{i}}$ is the 1-dimensional chosen to be the interval $[0, \pi]$, as in the construction of Lemma 1.7. The dimension of $B_{w}$ is $d = d_{1} + \cdots + d_{n}$ and the domain of $\phi_{w,w'}$ is

$$S_{w}^{d-1} = \partial(B_{w}) = \{(t_{1}, \ldots, t_{n}) : \exists j, t_{j} \in \partial B^{d_{j}}\}$$

the union of “faces” of $B_{w}$.

On the other hand let $B_{w'} = B^{d_{1}} \times \cdots \times \hat{B}^{d_{i}} \times \cdots \times B^{d_{n}}$. Then codomain is the sphere $S_{w'}^{d-1}$ obtained by collapsing to a point the boundary of $B_{w'}$. This is seen by items (c) and (d) in the above definition of $\partial$.

Step 2: $\sigma_{w'}$ in the image $\Phi_{w}(S_{w}^{d-1})$. The second step is to see how $\sigma_{w'}$ sits inside the image $\Phi_{w}(S_{w}^{d-1})$. The following lemma says how is the pre-image of $N \cdot w' b_{0}$ under $\Phi_{w}$.

Lemma 2.1. $\Phi_{w}(t_{1}, \ldots, t_{n}) \in N \cdot w' b_{0}$ if and only if $t_{j} \in (B^{d_{i}})^{\circ}, j \neq i$ and $t_{i} \in \partial B^{d_{i}}$, that is, $t_{i} = 0$ or $\pi$.

Proof. If $t_{i} \in \partial B^{d_{i}}$ then $\psi_{i}(t_{i}) \in M$ by Lemma 1.7. This implies that

$$\Phi_{w}(t_{1}, \ldots, t_{n}) = \psi_{1}(t_{1}) \cdots \psi_{n}(t_{n}) \cdot b_{0} \in K_{1} \cdots K_{i} \cdots K_{n} = S_{w}$$

since $M \subset K_{s}$ for all sub-index $s$. By Proposition 1.5 we see that $\Phi_{w}(t_{1}, \ldots, t_{i}, \ldots, t_{n}) \in N \cdot w' b_{0}$ if and only if $\psi_{j}(t_{j}) \notin M$ for $j \neq i$, which in turn is equivalent to $t_{j} \in (B^{d_{j}})^{\circ}, i \neq j$, by Lemma 1.7.

In other words the pre-image $\Phi_{w}^{-1}(N \cdot w' b_{0}) \subset B_{w}$ is the union of the interior of the two faces corresponding to the $i$-th coordinate, that is, the faces where $t_{i} = 0$ and $t_{i} = \pi$, respectively.

In the quotient $\sigma_{w'} = S_{w'}/(S_{w'} \setminus N \cdot w' b_{0})$ the faces of $\partial B_{w}$ corresponding to the $j$-th coordinates, $j \neq i$ are collapsed to a point.

Step 3: Degrees. The degree of $\phi_{w,w'}$ is the sum of the degree of two maps, namely the maps obtaining by restricting to the faces

$$F^{0}_{i} = \{(t_{1}, \ldots, 0, \ldots, t_{n})\} \quad \text{and} \quad F^{i}_{\pi} = \{(t_{1}, \ldots, \pi, \ldots, t_{n})\}.$$

The values of $\phi_{w,w'}$ in these faces are given by

$$f^{0}_{i}(t) = \Phi_{w'}^{-1}(\psi_{1}(t_{1}) \cdots \psi_{i}(0) \cdots \psi_{n}(t_{n}) \cdot b_{0})$$

$$f^{\pi}_{i}(t) = \Phi_{w'}^{-1}(\psi_{1}(t_{1}) \cdots \psi_{i}(\pi) \cdots \psi_{n}(t_{n}) \cdot b_{0})$$

where $t = (t_{1}, \ldots, \hat{t}_{i}, \ldots, t_{n})$ and $\Phi_{w'}$ is given by a choice of a reduced expression $w' = s_{1} \cdots s_{m}$ (chosen in advance) which may be different from the reduced expression $w = t_{1} \cdots \hat{t}_{i} \cdots t_{n}$.
The degree of $\phi_{w,w'}$ is the sum of the degrees of $f_i^0$ and $f_i^\pi$ which may be considered as maps $S^{d-1} \to S^{d-1}$ by collapsing the boundary to points of the faces.

Now, the degree of a map $\varphi$ can be computed as the sum of the local degrees in the inverse image of $\varphi^{-1}(\xi)$ which has a finite number of points (see [4], Proposition 2.30).

In the case of our map $\phi_{w,w'}$, the maps $f_i^0$ and $f_i^\pi$ are homeomorphisms so that pre-image $\phi_{w,w'}^{-1}(\xi)$ of a generic point has two points. Namely a point $x_1$ in the face $\mathcal{F}_0^i$ and another one $x_2$ in the face $\mathcal{F}_{\pi}^i$. The local degree at $x_1$ is the degree of $f_i^0$ since $f_i^0$ is a homeomorphism. The same the local degree at $x_2$ is the degree of $f_i^\pi$.

Finally the degrees of $f_i^0$ and $f_i^\pi$ are $\pm 1$ since they are homeomorphisms.

**Summarizing:** To get the degree of $\phi_{w,w'}$ we must restrict $\Phi_{w,w'}^{-1} \circ \Phi_w$ to the faces $\mathcal{F}_0^i$ and $\mathcal{F}_{\pi}^i$ and view these faces as spheres (with the boundaries collapsed to points). The sum of the degrees of these two restrictions is the degree of $\phi_{w,w'}$.

The restrictions of $\Phi_{w,w'}^{-1} \circ \Phi_w$ to the faces $\mathcal{F}_0^i$ and $\mathcal{F}_{\pi}^i$ are homeomorphisms and hence have degree $\pm 1$. It follows that the total degree of $\phi_{w,w'}$ is $0$ or $\pm 2$. This is one of the main results on the homology of flag manifolds.

**Theorem 2.2.** The coefficient $c(w,w') = \deg(f_i^0) + \deg(f_i^\pi) = 0 \text{ or } \pm 2$, for any $w,w' \in \mathcal{W}$.

In particular, in the case of $\mathbb{Z}_2$ coefficients all boundary maps vanish.

**Corollary 2.3.** The homology of $\mathcal{F}$ over $\mathbb{Z}_2$ is a vector space of dimension $|\mathcal{W}|$.

**Remark:** The above computations are particularly interesting when the simple root $\alpha_i$ has multiplicity $\dim \mathfrak{g}_{\alpha_i} = 1$. If all the simple roots have multiplicity $\geq 2$ then the boundary operator $\partial$ is identically zero and homology is freely generated by the Schubert cells. This happens in the classical case of the complex Lie algebras, where any root has (real) multiplicity two. An example of a real Lie algebra where the simple roots have multiplicity $\geq 2$ is the real form of $\mathfrak{sl}(n,\mathbb{C})$ whose Satake diagram is

```
●--------●
```

In this case the simple roots are complex and hence their multiplicities are $\geq 2$.

2.2. Illustration. In order to illustrate the above description of the boundary operator $\partial$ we consider here the maximal flag manifold $\mathcal{F}$ of the split real form of $\mathfrak{sl}(3,\mathbb{R})$. In this case, the Weyl group is $S_3$, the permutation group in three elements. The simple reflections are $(12) = r_{\alpha_{1,2}} = r_1$ and $(23) = r_{\alpha_{2,3}} = r_2$. Only $(13)$ has two reduced expressions: $(13) = (12)(23)(12)$ and $(13) = (23)(12)(23)$. We fix the following minimal decompositions

1, $(12), (23), (123) = (12)(23), (132) = (23)(12), (13) = (12)(23)(12)$.

Let $A = E_{1,2} - E_{2,1}$ and $B = E_{2,3} - E_{3,2}$ be the matrices whose exponentials provide parametrizations for the compact groups $K_1$ and $K_2$ respectively. With these choices, the characteristic maps are
Then we obtain expressions for 

(1) \( \Phi_1(0) = b_0. \)
(2) \( \Phi_{(12)}(t) = e^{tA} \cdot b_0, \ t \in [0, \pi]. \)
(3) \( \Phi_{(23)}(t) = e^{tB} \cdot b_0, \ t \in [0, \pi]. \)
(4) \( \Phi_{(13)}(t, s) = e^{tA} e^{sB} \cdot b_0, \ (t, s) \in [0, \pi]^2. \)
(5) \( \Phi_{(12)}(t, s) = e^{tB} e^{sA} \cdot b_0, \ (t, s) \in [0, \pi]^2. \)
(6) \( \Phi_{(13)}(t, s, z) = e^{tA} e^{sB} e^{sA} \cdot b_0, \ (t, s, z) \in [0, \pi]^3. \)

In fact, the boundary of the cube \([0, \pi]^3\) is \(S^2\) oriented with the normal vector pointing outwards. The face \((t, s, 0)\) (in this order) when viewed in the domain is negatively oriented while in the codomain the orientation agrees with the \(S^2\) orientation. Hence the degree is \(-1\).

In this case the face \((t, s, \pi)\) agrees with the positive orientation.

(6) \( c((13), (132)) = 0. \) Note that \((132) = (12)(23)(12) \). So we consider the maps \( f_3^0, f_3^\pi : S^2 \to S^2. \)

(a) \( f_3^0(t, s, 0) = e^{tA} e^{sB} e^{0A} \cdot b_0 = e^{tA} e^{sB} \cdot b_0, \) and we have

\[ \deg f_3^0 = -1. \]

(b) \( f_3^\pi(t, s, \pi) = e^{tA} e^{sB} e^{sA} \cdot b_0 = e^{tA} e^{sB} \cdot b_0 \) with

\[ \deg f_3^\pi = 1. \]
(a) \( f_1^0(t, s) = e^{0A}e^{tB}e^{sA} \cdot b_0 = e^{tB}e^{sA} \cdot b_0 \) with
\[
\deg f_3^0 = -1.
\]
In this case the face \((0, t, s)\) (in this order) in the domain has a negative orientation while in the codomain the orientation agrees with the positive one. Hence the degree is \(-1\).

(b) \( f_1^\tau(t, s) = e^{\pi A}e^{tB}e^{sA} \cdot b_0 = \exp(-tB)e^{sA} \cdot b_0 \) since \( Ad(e^{\pi A}B) = -B \) and \( Ad(e^{\pi A}A) = A \). We want to describe this map with a domain in \([0, \pi]^2\). So, first \( \exp(-tB)e^{sA} \cdot b_0 = \exp((\pi - t)B)e^{\pi B}e^{sA} \cdot b_0 \). Finally, since \( Ad(e^{\pi B}A) = -A \) we get
\[
\exp(-sB)e^{sA} \cdot b_0 = e^{(\pi - s)B}e^{(\pi - s)A} \cdot b_0.
\]
Hence the degree of \( f_1^\tau \) is the degree of \((t, s) \mapsto (\pi - t, \pi - s)\). This degree is \(+1\) since it preserves the orientation.

Summarizing, the boundary operator is given by
- \( \partial_3S_{(13)} = 0; \)
- \( \partial_2S_{(123)} = -2S_{(23)} \) and \( \partial_2S_{(132)} = -2S_{(12)}; \)
- \( \partial_1S_{(12)} = \partial S_{(23)} = 0. \)

Hence the integer homology groups are
- \( H_3(\mathbb{F}, \mathbb{Z}) = \mathbb{Z} \) generated by \( S_{(13)} \).
- \( H_2(\mathbb{F}, \mathbb{Z}) = 0 \) (ker \( \partial_2 = 0 \)).
- \( H_1(\mathbb{F}, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) (ker \( \partial_1 \) is \( \mathbb{Z} \oplus \mathbb{Z} \) and the image of \( \partial_2 \) is \( 2\mathbb{Z} \cdot S_{(12)} \oplus 2\mathbb{Z} \cdot S_{(23)} \)).

### 2.3. Algebraic expressions for the degrees.

Here we compute the coefficients \( c(w, w') \) in terms of the roots by finding the degrees of the maps involved.

For a diffeomorphism \( \varphi \) of the sphere its degree is local degree at a point \( x \) which in turn is the sign of the determinant \( \det(d\varphi_x) \) with respect to a volume form of \( S^d \). Let us apply this in our context.

We let \( w = r_1 \cdots r_n \) and \( w' = r_1' \cdots r_n' \) be reduced expressions, with \( r_i = r_{\alpha_i} \) and assume throughout that the simple root \( \alpha_i \) has multiplicity \( d_i = d_{\alpha_i} = 1 \).

We must find the degrees of \( f_i^0 \) and \( f_i^\tau \) defined by
\[
(1) \quad f_i^0(t_1, \ldots, 0, \ldots, t_n) = \Phi_{w'}^{-1}(\psi_1(t_1) \cdots \psi_n(t_n) \cdot b_0).
(2) \quad f_i^\tau(t_1, \ldots, \pi, \ldots, t_n) = \Phi_{w'}^{-1}(\psi_1(t_1) \cdots m_{\alpha_i} \cdots \psi_n(t_n) \cdot b_0).
\]

In these expressions \( \Phi_{w'}^{-1} \) is defined by a previously chosen reduced expression \( w' = s_1 \cdots s_m \) of \( w' \) which may be distinct of \( w' = r_1 \cdots r_n \). On the other hand \( w' = r_1' \cdots r_n' \) can be used to define another characteristic map, which will be denoted by \( \Psi_{w'} \). This new characteristic map define new functions
\[
(1) \quad p_i^0(t_1, \ldots, 0, \ldots, t_n) = \Psi_{w'}^{-1}(\psi_1(t_1) \cdots \psi_n(t_n) \cdot b_0) \text{ and }
(2) \quad p_i^\tau(t_1, \ldots, \pi, \ldots, t_n) = \Psi_{w'}^{-1}(\psi_1(t_1) \cdots m_{\alpha_i} \cdots \psi_n(t_n) \cdot b_0).
\]

The two pair of functions are related by
\[
f_i^\epsilon = (\Phi_{w'}^{-1} \circ \Psi_{w'}) \circ p_i^\epsilon \quad \epsilon = 0, \pi.
\]
The composition $\Phi_w^{-1} \circ \Psi_w$ (also understood as a map between spheres in which the boundary are collapsed to points) are homeomorphisms of spheres and, hence, have degree $\pm 1$. Hence we can concentrate on the computation of degrees of the $p_i^\epsilon$’s since the total degree will be multiplied by $\pm 1$.

Before getting these degrees we make the following discussion on the orientation of the faces of the cube $[-1,1]^d$, centered at the origin of $\mathbb{R}^d$, which is given with the basis $\{e_1, \ldots, e_d\}$.

Starting with the $(d-1)$-dimensional sphere $S^{d-1}$ we orient the tangent space at $x \in S^{d-1}$ by a basis $\{f_2, \ldots, f_d\}$ such that $\{x, f_2, \ldots, f_d\}$ is positively oriented. The faces of $[-1,1]^d$ are oriented accordingly: Given a base vector $e_j$, we let $F_j^-$ be the face perpendicular to $e_j$ that contains $-e_j$ and $F_j^+$ the one that contains $e_j$. Then $F_j^-$ has the same orientation as the basis $e_1, \ldots, e_j, \ldots, e_d$ if $j$ is even ($-e_j, e_1, \ldots, e_j, \ldots, e_d$ is positively oriented in $\mathbb{R}^d$) and opposite orientation if $j$ is odd. Therefore the orientation of $F_j^-$ is $(-1)^j$ the orientation of $e_1, \ldots, \hat{e}_j, \ldots, e_d$. Analogously, the orientation of $F_j^+$ is $(-1)^{j+1}$ the orientation of $e_1, \ldots, \hat{e}_j, \ldots, e_d$.

The following facts about the action of an element $m \in M$ will be used below in the computation of the degrees.

**Lemma 2.4.** For a root $\alpha$ consider the action on $F$ of $m = m_\alpha = \exp(\pi A_\alpha) \in M$. Then

1. $mw_0 = w_0$ and $mN_m^{-1} = N$. Therefore $m$ leaves invariant any Bruhat cell and hence any Schubert cell $S_w$.
2. The restriction of $m$ to $N \cdot w_0$ is a diffeomorphism.
3. The differential $dm_{w_0}$ identifies to $\text{Ad}(m)$ restricted to the subspace

$$\sum_{\beta \in \Pi_w} g_\beta.$$

**Proof.** Since $\text{Ad}(m_\alpha)g_\beta = g_\beta$, $\beta \in \Pi$ (cf. Lemma [8]), the first and second statements follow easily.

For the third statement we use the notation $X \cdot x = d/dt (\exp(tX))_{t=0}$, $x \in F$ and $X \in g$. Also, for $A \subset g$ write $A \cdot x = \{X \cdot x : X \in A\}$.

Note that $N \cdot w_0 = w(w^{-1}Nw) \cdot b_0$, and the tangent space to $(w^{-1}Nw) \cdot b_0$ at $b_0$ is spanned by $g_\alpha \cdot b_0$ with $\alpha < 0$ such that $\alpha = w^{-1}\beta$ and $\beta > 0$, that is, $w \cdot \alpha > 0$. Since $(dw)(g_\alpha \cdot b_0) = g_w \cdot \alpha \cdot b_0$, it follows that $T_{w_0}(N \cdot w_0)$ is spanned by $g_\beta \cdot b_0$ with $\beta = w \cdot \alpha > 0$ such that $w^{-1} \cdot \beta = \alpha < 0$. Hence the result. $\square$

The next statement computes the degree of $p_i^\epsilon$’s in terms of Killing numbers.

**Proposition 2.5.** $\deg(p_i^0) = (-1)^I$ and $\deg(p_i^\epsilon) = (-1)^{I+1+\sigma}$, where

$$\sigma = \sigma(w, w') = \sum_{\beta \in \Pi_u} \frac{2\langle \alpha_i, \beta \rangle}{\langle \alpha_i, \alpha_i \rangle} \dim g_\beta, \quad \Pi_u = \Pi^+ \cap u\Pi^-, \quad u = r_{i+1} \cdots r_n,$$

and $I$ is the sum of the multiplicities of the roots $\alpha_j$ with $j \leq i$. 

Proof. The map $p_i^0$ is the projection of the face of a $d$-dimensional cube onto the face of a $(d - 1)$-dimensional cube, i.e., in coordinates

$$(t_1, \ldots, 0, \ldots, t_n) \mapsto (t_1, \ldots, t_i, \ldots, t_n).$$

Note that with respect to the basis $e_1, \ldots, e_d$ the $t_i$-coordinate appears in the $I$-th position. Hence, by the orientation of the cube, discussed above, the projection preserves or reverses orientation if $I$ is even or odd, respectively. Therefore, $\deg(p_i^0) = (-1)^I$.

To get $\deg(p_i^\pi)$ write $m_i = m_{\alpha_i}$ for the element of $M$ appearing in the expression of $p_i^\pi$. Its action on $\mathbb{F}$ leaves invariant any Bruhat cell $N \cdot wb_0$ (because $m_i N m_i^{-1} = N$ and $m_i wb_0 = wb_0$), and hence any Schubert cell. Moreover, the restriction of $m_i$ to $N \cdot wb_0$ is a diffeomorphism (given by the conjugation $y \in N \mapsto m_i y m_i^{-1}$).

In particular, we restrict the action of $m_i$ to the cell $\mathcal{S}_u$, $u = r_{i+1} \cdots r_n$. Using the parametrization of this cell by the cube $B_u$ we get

$$m_i \psi_{i+1}(t_{i+1}) \cdots \psi_n(t_n) \cdot b_0 = \psi_{i+1}(s_{i+1}) \cdots \psi_n(s_n) \cdot b_0,$$

with $(s_{i+1}, \ldots, s_n) = \overline{m}_i(t_{i+1}, \ldots, t_n)$ with $\overline{m}_i : B_u \to B_u$ continuous and a diffeomorphism of the interior of $B_u$.

Hence, $p_i^\pi(t_1, \ldots, \pi, \ldots, t_n)$ becomes the projection of the $i - 1$ first coordinates and the composition of $\overline{m}_i$ with the projection of the last $j$-coordinates, $j = i + 1, \ldots, n$. From the choice of the orientation of $B_w = [0, \pi]^d$, the face $(t_1, \ldots, \pi, \ldots, t_n)$ of $B_u$ has orientation $(-1)^{I+1}$ with respect to the orientation of the coordinates $(t_1, \ldots, \hat{t}_i, \ldots, t_n)$. Hence, after collapsing the boundary to a point, we get the degree

$$\deg p_i^\pi = (-1)^{I+1} \deg \overline{m}_i.$$

The degree of $\overline{m}_i$ equals its local degree at one point which in turn is sign of the determinant of the differential $d(m_i)_{ub_0}$ restricted to the tangent space to Bruhat cell $N \cdot ub_0$ at $ub_0$:

$$\deg(p_i^\pi) = (-1)^{I+1} \text{sgn} \left[ \det (d(m_i)_{ub_0} | T_{ub_0}(N \cdot ub_0)) \right].$$

By the third statement in the Lemma 2.3 $T_{ub_0}(N \cdot ub_0)$ identifies to $\sum_{\beta \in \Pi_w} g_\beta$.

Once we have the generators $g_\beta \cdot ub_0$, $\beta \in \Pi_w$ for $T_{ub_0}(N \cdot ub_0)$ together with the action of $\text{Ad}(m_i)$ over $g_\beta$ given by the Lemma 18 $\text{Ad}(m_\alpha)g_\beta = (-1)^{\langle \alpha, \beta \rangle} g_\beta$ we conclude that the signal of $\det (d(m_i)_{ub_0} | T_{ub_0}(N \cdot ub_0)) = (-1)^\sigma$ where

$$\sigma = \sum_{\beta \in \Pi_w} \frac{2\langle \alpha_i, \beta \rangle}{\langle \alpha_i, \alpha_i \rangle} \dim g_\beta.$$

Summarizing, we have the following algebraic expression for the coefficient $c(w, w')$.

**Theorem 2.6.** Let be $\sigma(w, w')$ be defined as in (7). Then

$$c(w, w') = \deg \left( \Phi_w^{-1} \circ \Psi_w \right) (-1)^I (1 - (-1)^{\sigma(w, w')}).$$

We will now derive another formula for $\sigma(w, w')$ that does not depend on the reduced expressions of $w$ and $w'$. This formula is the same one given by Theorem A of 8.
For \( w \in \mathcal{W} \), let
\[
\phi(w) = \sum_{\beta \in \Pi_w} \dim g_{\beta} \cdot \beta
\]
be the sum of roots in \( \Pi_w = \Pi^+ \cap w\Pi^- \) counted with their multiplicity.

As before let \( w = r_1 \cdots r_n \) and \( w' = r_1 \cdots \hat{r}_i \cdots r_n \) be reduced expressions.

**Proposition 2.7.** Let \( \beta \) be the unique root (not necessarily simple) such that \( w = r_\beta w' \), that is, \( \beta = r_1 \cdots r_{i-1} \alpha_i \). Then
\[
\phi(w) - \phi(w') = (1 - \sigma) \beta
\]
where \( \sigma = \sigma(w, w') \) is the sum \( \deg \).

**Proof.** By the reduced expressions \( w^{-1} = r_n \cdots r_1 \) and \( w'^{-1} = r_n \cdots \hat{r}_i \cdots r_1 \) and \( u^{-1} = r_n \cdots r_{i+1} \) we obtain the sets

1. \( \Pi_w = \{ \alpha_1, r_1 \alpha_2, \ldots, r_1 \cdots r_{i-1} \alpha_i, r_1 \cdots r_i \alpha_{i+1}, \ldots, r_1 \cdots r_{n-1} \alpha_n \} \).
2. \( \Pi_{w'} = \{ \alpha_1, r_1 \alpha_2, \ldots, r_1 \cdots r_{i-1} \alpha_i, r_1 \cdots \hat{r}_i \cdots r_{n-1} \alpha_n \} \).
3. \( \Pi_u = \{ \alpha_{i+1}, r_{i+1} \alpha_{i+2}, \ldots, r_n \cdots r_{n-1} \alpha_n \} \).

The first \((i-1)\) roots of \( \Pi_w \) and \( \Pi_{w'} \) coincide. The remaining ones are related by the equalities
\[
(r_1 \cdots r_{i-1})r_i \cdots r_j \alpha_{j+1} = r_\beta (r_1 \cdots r_{i-1})r_{i+1} \cdots r_j \alpha_{j+1}, \quad j = i, \ldots, n-1,
\]
because \( (r_1 \cdots r_{i-1})r_i (r_1 \cdots r_{i-1})^{-1} = r_{i-1} \cdots \hat{r}_i \cdots r_1 \alpha_i = r_\beta \). It follows that the remaining roots \( r_1 \cdots r_j \alpha_{j+1} \) and the roots \( r_1 \cdots \hat{r}_i \cdots r_j \alpha_{j+1} \) have the same multiplicity \( d_j, \quad j = i, \ldots, n-1 \). Write \( \gamma_j = r_{i+1} \cdots r_j \alpha_{j+1} \), so that \( \Pi_u = \{ \gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1} \} \). Then
\[
\phi(w) - \phi(w') = \beta + \sum_{j=i}^{n-1} d_j (r_1 \cdots r_{i-1}) (r_1 (\gamma_j) - \gamma_j)
\]
because \( \beta = r_1 \cdots r_{i-1} \alpha_i \) has multiplicity 1 as \( \alpha_i \).

Since \( r_1 (\gamma_j) - \gamma_j = -\frac{2 }{ \langle \alpha_i, \gamma_j \rangle } \alpha_i \) we rewrite \( \phi(w) \) as
\[
\phi(w) - \phi(w') = \left( 1 - \sum_{j=i}^{n-1} d_j \frac{2 }{ \langle \alpha_i, \gamma_j \rangle } \right) \beta
\]
\[
= (1 - \sigma) \beta
\]
concluding the proof. \( \square \)

Combining the above proposition with Theorem 2.6 we get immediately the following formula for \( c(w, w') \) (cf. [8], Theorem A).

**Theorem 2.8.**
\[
c(w, w') = \deg \left( \Phi^{-1}_{w'} \circ \Psi_{w'} \right) (-1)^t (1 + (-1)^{\kappa(w, w')})
\]
where \( \kappa(w, w') \) is the integer defined by \( \phi(w) - \phi(w') = \kappa(w, w') \cdot \beta \) and \( \beta \) is the unique root such that \( w = r_\beta w' \).
Remark: If \( w = r_1 \cdots r_n \) and \( w' = r_1 \cdots r_{i-1} \) then \( c(w, w') = 0 \) because \( m_{\alpha_n} \) does not affect the computations of the degrees (see Proposition 2.5).

Example of \( \text{Sl}(3, \mathbb{R}) \): Let us use Formula (10) to rederive the homology of the maximal flag of \( \text{Sl}(3, \mathbb{R}) \), the split real form of the algebra whose Dynkin diagram is \( A_2 \). Let fix the same reduced expressions for elements in \( \mathcal{W} \) and note that the unique element which has more than one reduced expression is (13) which implies that the factor \( (\Phi_{w'}^{-1} \circ \Psi_{w'}) \) is 1 in all cases. In this case, we have the following table 2.3 which determines completely the coefficients \( c(w, w') \), as in Subsection 2.2.

| \( \mathcal{W} \) | \( \Pi_w \) | \( \phi(w) \) |
|---|---|---|
| 1 | \( \emptyset \) | 0 |
| (12) | \( \alpha_1 \) | \( \alpha_1 \) |
| (23) | \( \alpha_2 \) | \( \alpha_2 \) |
| (123) | \( \alpha_1, \alpha_1 + \alpha_2 \) | \( 2\alpha_1 + \alpha_2 \) |
| (132) | \( \alpha_2, \alpha_1 + \alpha_2 \) | \( \alpha_1 + 2\alpha_2 \) |
| (13) | \( \Pi^+ \) | \( 2\alpha_1 + 2\alpha_2 \) |

Table 1. Homology of the Maximal Flag of \( A_2 \)

For instance, let us compute \( \partial_3 S_{(13)} = 0 \). According to the table 2.3, \( \sigma((13), (123)) = 2(\alpha_1 + \alpha_2) - (2\alpha_1 + \alpha_2) = \alpha_2 \) and \( \sigma((13), (132)) = 2(\alpha_1 + \alpha_2) - (\alpha_1 + 2\alpha_2) = \alpha_1 \). It implies that \( \kappa((13), (123)) = \kappa((13), (132)) = 1 \) by which we conclude that \( c((13), (123)) = c((13), (132)) = 0 \).

Example of \( G_2 \): Let us apply the results above for the two groups with Dynkin diagram \( G_2 \), namely the complex group and the split real form. In the complex case, we already have \( \partial = 0 \). Now we proceed to the real case. Let \( \Sigma = \{\alpha_1, \alpha_2\} \) be the simple roots. The set \( \Pi^+ \setminus \Sigma = \{\alpha_3 = \alpha_2 + \alpha_1, \alpha_4 = \alpha_1 + 2\alpha_2, \alpha_5 = \alpha_1 + 3\alpha_2, \alpha_6 = 2\alpha_1 + 3\alpha_2\} \) contains the remaining positive roots. The Weyl group with the respective fixed reduced expressions is \( \mathcal{W} = \{1, r_1, r_2, s_1 = r_1 r_2, s_2 = r_2 r_1, r_1 s_2, r_2 s_1, s_1^2, s_2^2, r_1 s_2, r_2 s_1, s_3^3\} \), where \( r_i = r_{\alpha_i} \) are the simple reflections and \( s_3^3 = s_3^2 \) is the unique element with two different minimal decompositions.

The next table 2.3 presents the data useful to compute the homology coefficients.

By (10) the boundary operator is given as

- \( \partial_3 S_{s_3^2} = 0 \);
- \( \partial_3 S_{r_1 s_2^2} = -2S_{s_2^2} \) and \( \partial_0 S_{r_2 s_1^2} = -2S_{s_1^2} \);
- \( \partial_3 S_{s_1^2} = \partial_3 S_{s_2^2} = 0 \);
- \( \partial_3 S_{r_1 s_2} = \partial_3 S_{r_2 s_1} = 0 \);
- \( \partial_2 S_{s_1} = -2S_{r_2} \) and \( \partial_2 S_{s_2} = -2S_{r_1} \);
- \( \partial_1 S_{r_1} = \partial_1 S_{r_2} = 0 \).
Table 2. Homology of the maximal flag of $G_2$

| $\mathcal{W}$ | $\Pi_w$ | $\phi(w)$ |
|---------------|---------|-----------|
| $1$           | $\emptyset$ | $0$       |
| $r_1$         | $\alpha_1$ | $\alpha_1$ |
| $r_2$         | $\alpha_2$ | $\alpha_2$ |
| $s_1$         | $\alpha_1, \alpha_3$ | $2\alpha_1 + \alpha_2$ |
| $s_2$         | $\alpha_2, \alpha_5$ | $\alpha_1 + 4\alpha_2$ |
| $r_1s_2$      | $\alpha_1, \alpha_3, \alpha_6$ | $4\alpha_1 + 4\alpha_2$ |
| $r_2s_1$      | $\alpha_2, \alpha_5, \alpha_4$ | $2\alpha_1 + 6\alpha_2$ |
| $s_1^2$       | $\alpha_1, \alpha_3, \alpha_6, \alpha_4$ | $5\alpha_1 + 6\alpha_2$ |
| $s_2^2$       | $\alpha_2, \alpha_5, \alpha_4, \alpha_6$ | $4\alpha_1 + 9\alpha_2$ |
| $r_1s_2^2$    | $\alpha_1, \alpha_3, \alpha_6, \alpha_4, \alpha_5$ | $6\alpha_1 + 9\alpha_2$ |
| $r_2s_1^2$    | $\alpha_2, \alpha_5, \alpha_4, \alpha_6, \alpha_3$ | $5\alpha_1 + 10\alpha_2$ |
| $s_1^3$       | $\Pi^+$ | $6\alpha_1 + 10\alpha_2$ |

Hence

- $H_6(\mathbb{F}, \mathbb{Z}) = \mathbb{Z}$.
- $H_5(\mathbb{F}, \mathbb{Z}) = 0$.
- $H_4(\mathbb{F}, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- $H_3(\mathbb{F}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.
- $H_2(\mathbb{F}, \mathbb{Z}) = 0$.
- $H_1(\mathbb{F}, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3. Partial flag manifolds

In this section we project down the constructions made for the maximal flag manifolds, via the canonical map $\pi _\Theta : F \to F_\Theta$, to obtain analogous results for the homology of a partial flag manifold. In $F_\Theta$ the Schubert cells are $S^\Theta_w$, $w \in \mathcal{W}/\mathcal{W}_\Theta$, with $S^\Theta_w = S^\Theta_{w_1}$ if $w\mathcal{W}_\Theta = w_1\mathcal{W}_\Theta$. The next lemma chooses a special representative in $w\mathcal{W}_\Theta$ for $S^\Theta_w$.

**Lemma 3.1.** There exists an element $w_1 = wu$ of the coset $w\mathcal{W}_\Theta$ such that

$$\dim S^\Theta_w = \dim S^\Theta_{w_1}.$$  

This element is unique and minimal with respect to the Bruhat-Chevalley order.

**Proof.** By Proposition 1.1.2.13 of [15] any $v \in \mathcal{W}$ can be written uniquely as

$$v = v_s v_u$$

with $v_s \in \mathcal{W}_\Theta$ and $v_u$ satisfying $\Pi^+ \cap v_u \Pi^- \cap \angle(\Theta) = \emptyset$, that is, no positive root in $\angle(\Theta)$ is mapped to a negative root by $v_u^{-1}$. Note that the condition for $v_u$ is equivalent to $\Pi^- \cap v_u \Pi^+ \cap \angle(\Theta) = \emptyset$, since a root $\alpha > 0$ belongs to $\Pi^+ \cap v_u \Pi^- \cap \angle(\Theta)$ if and only if $-\alpha \in \Pi^- \cap v_u \Pi^+ \cap \angle(\Theta)$. 


Let $w^{-1} = v_s v_u$ be the decomposition for $w^{-1}$ so that $w = v_u^{-1} v_s^{-1}$. Then $w_1 = v_u^{-1} \in \mathcal{W}_\Theta$ is the required element.

In fact $\Pi^- \cap w_1^{-1} \Pi^+ \cap \langle \Theta \rangle = \emptyset$, and hence $\Pi^+ \cap w_1 \Pi^- \cap w_1 \langle \Theta \rangle = \emptyset$.

Now the tangent space $T_{w_1 b_0}(N \cdot w_1 b_0)$ is

$$\langle g_\beta \cdot w_1 b_0 : \beta \in \Pi^+ \cap w_1 \Pi^- \rangle$$

(cf. Lemma 2.4). On the other hand the tangent space to the fiber $\pi_\Theta^{-1}(w_1 b_0)$ is the translation under $w_1$ of the tangent space to fiber at origin. Hence, $T_{w_1 b_0} \pi_\Theta^{-1}(w_1 b_0)$ is spanned by $w_1(g_\alpha \cdot b_0)$, with $\alpha \in \langle \Theta \rangle$ and $\alpha < 0$. By the translation formula, we have $w_1(g_\alpha \cdot b_0) = g_{w_1 \alpha} \cdot w_1 b_0$. Therefore, by writting $\gamma = w_1 \alpha$ we conclude that $T_{w_1 b_0} \pi_\Theta^{-1}(w_1 b_0)$ is spanned by $g_\gamma \cdot w_1 b_0$ with $w_1^{-1} \gamma \in \langle \Theta \rangle$ and $w_1^{-1} \gamma < 0$, that is, with $w_1^{-1} \gamma \in \Pi^- \cap \langle \Theta \rangle$. So that

$$T_{w_1 b_0}(\pi_\Theta^{-1}(w_1 b_0)) = \langle g_\gamma \cdot w_1 b_0 : \gamma \in w_1 \Pi^- \cap w_1 \langle \Theta \rangle \rangle.$$ 

Since $\Pi^+ \cap w_1 \Pi^- \cap w_1 \langle \Theta \rangle = \emptyset$, it follows that none of roots $\gamma$ spanning $T_{w_1 b_0}(\pi_\Theta^{-1}(w_1 b_0))$ can be positive.

Therefore, $T_{w_1 b_0}(N \cdot w_1 b_0) \cap T_{w_1 b_0}(\pi_\Theta^{-1}(w_1 b_0)) = \{0\}$. This implies that the differential of $\pi_\Theta$ maps $T_{w_1 b_0}(N \cdot w_1 b_0)$ onto the tangent space of $\pi_\Theta(N \cdot w_1 b_0) = N \cdot w_1 b_0$. Hence the two Bruhat cells have the same dimension.

Finally, $N \cdot w_1 b_0$ has the minimum possible dimension among the cells $N \cdot w b_0$, $w \in w_1 \mathcal{W}$, because all of them project onto $N \cdot w_1 b_0$. Hence $w_1$ has minimal length in $w_1 \mathcal{W}$ which is known to be unique and minimal with respect to the Bruhat-Chevalley order as well (see Deodhar [2]).

We will denote by $\mathcal{W}_\Theta^{\min}$ the set of minimal elements of the cosets in $\mathcal{W} / \mathcal{W}_\Theta$.

Now we construct a cellular decomposition for $\mathcal{F}_\Theta$ with the aid of the minimal elements $w \in \mathcal{W}_\Theta^{\min}$ in their cosets $w \mathcal{W}_\Theta$, satisfying $\dim \mathcal{S}_w^\Theta = \dim \mathcal{S}_w$. Using a reduced decomposition of such minimal element $w$ we have new functions $\Phi_w^\Theta$ defined in the same way, but replacing the origin $b_0$ of $\mathcal{F}$ by the origin $b_\Theta$ of $\mathcal{F}_\Theta$, that is,

$$\Phi_w^\Theta(t_1, \ldots, t_n) = \psi_1(t_1) \cdots \psi_n(t_n) \cdot b_\Theta.$$

By equivariance, $\Phi_w^\Theta = \pi_\Theta \circ \Phi_w$. This function satisfies the required properties to be a characteristic map for the Schubert cells $\mathcal{S}_w^\Theta$.

**Proposition 3.2.** Take $w \in \mathcal{W}_\Theta^{\min}$ so that $\dim \mathcal{S}_w^\Theta = \dim \mathcal{S}_w$ and let $w = r_1 \cdots r_n$ be a reduced expression as a product of simple reflections. Let $\Phi_w^\Theta : B_w \to \mathcal{F}_\Theta$ be defined by $\Phi_w^\Theta = \pi_\Theta \circ \Phi_w$ and take $t = (t_1, \ldots, t_n) \in B_w$. Then $\Phi_w^\Theta$ is a characteristic map for $\mathcal{S}_w^\Theta$, that is, satisfies the following properties:

1. $\Phi_w^\Theta(B_w) = \mathcal{S}_w^\Theta$.
2. $\Phi_w^\Theta(t) \in \mathcal{S}_w^\Theta \setminus N \cdot w b_\Theta$ if and only if $t \in \partial B_w = \mathcal{S}_w^{d-1}$.
3. $\Phi_w^\Theta B_w^\circ : B_w^\circ \to N \cdot w b_\Theta$ is a homeomorphism, where $B_w^\circ$ is the interior of $B_w$.

**Proof.** This is the Proposition 1.9 in this generalized flag context. The first item follows by equivariance of $\pi_\Theta$. The second assertion is true because $\pi_\Theta(\mathcal{S}_w \setminus N \cdot w b_0) = \mathcal{S}_w^\Theta \setminus N \cdot w b_\Theta$ and
\[ \Phi_w^\Theta = \pi_\Theta \circ \Phi_w. \] The last item is a consequence of the equality of the dimension of Bruhat cells \( N \cdot wb_0 \) and \( N \cdot wb_\Theta. \)

Now we can find out the boundary maps \( \partial^\Theta \) with coefficients \( c^\Theta([w],[w']) \), where \([w]\) denotes the the coset \( w\mathcal{W}_\Theta \). We have \( c^\Theta([w],[w']) = 0 \), unless

1. \( \dim S_w^\Theta = \dim S_{w'}^\Theta + 1 \)
2. \( S_{w'}^\Theta \subset S_w^\Theta \).

Here the inclusions among the Schubert cells are also given by the Bruhat-Chevalley order (cf. Proposition 3.3), namely \( S_{w'}^\Theta \subset S_w^\Theta \) if and only if there is \( u \in w'\mathcal{W}_\Theta \) such that \( u < w \). (This follows immediately from the projections \( \pi_\Theta S_w = S_{w'}^\Theta \).) Actually, we have the following complement to Lemma 3.1.

**Lemma 3.3.** Let \( w \in \mathcal{W}_\Theta^{\text{min}} \) minimal in its coset and suppose that there exists \( u \in w'\mathcal{W}_\Theta \) with \( u < w \) and \( \dim S_w^\Theta = \dim S_{w'}^\Theta + 1 \). Then \( u \) is minimal in \( w'\mathcal{W}_\Theta \).

**Proof.** We have \( \dim S_w = \dim S_{w'} + \dim S_{w'}^\Theta + 1 \leq \dim S_u + 1 \). But if \( u < w \) then \( \dim S_u \leq \dim S_{w'} - 1 \), so that \( \dim S_w \leq \dim S_u + 1 \leq \dim S_{w'} \), implying that

\[ \dim S_u = \dim S_{w'}^\Theta - 1 = \dim S_w^\Theta. \]

Hence \( u \) is minimal in its coset. \( \square \)

**Remark:** The assumption \( \dim S_w^\Theta = \dim S_{w'}^\Theta + 1 \) in Lemma 3.3 is essential. Without it there may be \( u \in w'\mathcal{W}_\Theta \) which is not minimal although \( u < w \) and \( w \) is minimal. Geometrically this happens when \( \dim S_w = \dim S_{w'} + 1 \) but \( \dim S_w^\Theta > \dim S_{w'}^\Theta + 1 \), which may give \( c(w,u) \neq 0 \) and \( c^\Theta([w],[u]) = 0 \). The following example illustrates this situation.

**Example:** In the Weyl group \( S_4 \) of \( A_3 \) take \( w = (12)(23)(34) \) and \( \Theta = \{\alpha_{23}\} \). Then \( w \) is minimal in the coset \( w\mathcal{W}_{\{\alpha_{23}\}} \). The roots \( \alpha_{12}, (12)\alpha_{23} = \alpha_{12} + \alpha_{23} \) and \( (12)(23)\alpha_{34} = \alpha_{12} + \alpha_{23} + \alpha_{34} \) are positive roots mapped to negative roots by \( w^{-1} \) and none of these roots lie in \( \langle \Theta \rangle \). However, \( w' = (12)(23) = (123) \) is not minimal in its coset since \( (12) < (12)(23) \) and \( (123) \) is the same as the attaching map between \( \delta_{12}^\Theta \) and \( \delta_{12}^\Theta (12) \) is the same as the attaching map between \( S_w^\Theta \) and \( S_{w'}^\Theta \). Hence the coefficients for \( \delta^\Theta \) and \( \delta \) are the same, that is,

\[ c^\Theta([w],[w']) = c(w,w'). \]

Hence the computation of \( c^\Theta([w],[w']) \) reduces to a computation on \( \mathbb{F} \).

**Theorem 3.4.** The cellular homology of \( \mathbb{P}_\Theta \) is isomorphic to the homology of \( \delta^\Theta_{\text{min}} \) which is the boundary map of the free module \( \mathcal{A}^\Theta_{\text{min}} \) generated by \( S_w \), \( w \in \mathcal{W}_\Theta^{\text{min}} \), obtained by restricting \( \partial \) and projecting onto \( \mathcal{A}^\Theta_{\text{min}} \).
Corollary 3.5. $c^\Theta ([w], [w']) = 0$ or $\pm 2$. In particular taking coefficients in $\mathbb{Z}_2$, $\partial_1^\Theta = 0$ and the $\mathbb{Z}_2$-homology of $F^\Theta$ is freely generated by $S_{[w]}^\Theta$, $[w] \in \mathcal{W}/\mathcal{W}_\Theta$.

**Remark:** Let $w$ be minimal in its coset $w\mathcal{W}_\Theta$ and suppose that $u < w$ is of the form $w = ur_\beta$, with $\beta$ a simple root and $\Theta = \{\beta\}$. Hence $u$ is minimal in its coset. In fact, this conditions imply that $w\beta < 0$. In fact, $w\beta = ur_\beta(\beta) = -u\beta$ and $u\beta \in \Pi_u$. So, if $u$ is not minimal in its coset, there is $\gamma > 0$ such that $u^{-1}\gamma < 0$ and $\gamma \in \langle \Theta \rangle$. As $w$ is minimal, by the same fact we know that $w^{-1}\gamma > 0$ (otherwise we would have $\Pi^+ \cap w\Pi^- \cap \langle \Theta \rangle \neq \emptyset$). Hence $\gamma = -u\beta = ur_\beta(\beta) = w\beta < 0$. This implies that $r_\beta(u^{-1}\gamma) > 0$ and hence $u^{-1}\gamma = -\beta$. Hence $\gamma = -u\beta = ur_\beta(\beta) = w\beta < 0$. This is a contradiction because $\gamma > 0$.

**Example:** Let us consider the example of $G_2$ with $\Theta = \{\alpha_1\}$. We have the following cosets

$$\mathcal{W} = \{1, r_1\}, \{r_2, s_2\}, \{s_1, r_1s_2\}, \{r_2s_1, s_2^2\}, \{s_1^2, r_1s_2^2\}, \{r_2s_1^2, s_1^3\}.$$

The boundary maps for the minimal element in each coset is computed using the table 2.3

- $\partial_5 S_{r_2s_1^2} = -2S_{s_1^2}$
- $\partial_4 S_{s_1^2} = 0$
- $\partial_3 S_{r_2s_1} = 0$
- $\partial_2 S_{s_1} = -2S_{r_2}$
- $\partial_1 S_{r_2} = 0$

Hence

- $H_5(F_{\alpha_1}, \mathbb{Z}) = 0$ (in particular $F_{\{\alpha_1\}}$ is not orientable).
- $H_4(F_{\alpha_1}, \mathbb{Z}) = \mathbb{Z}_2$.
- $H_3(F_{\alpha_1}, \mathbb{Z}) = \mathbb{Z}$.
- $H_2(F_{\alpha_1}, \mathbb{Z}) = 0$.
- $H_1(F_{\alpha_1}, \mathbb{Z}) = \mathbb{Z}_2$.

As another source of examples, we refer to the papers [12] and [14] which computes the coefficients of the isotropic and orthogonal Grassmannians.

**References**

1. L. Casian and R. J. Stanton, *Schubert cells and representation theory*, Invent. Math. 137 (1999), no. 3, 461–539.
2. V.V. Deodhar, *Some characterizations of bruhat ordering on a coxeter group and determination of the relative mobius function*, Inventiones mathematicae 39 (1977), 187–198.
3. J. J. Duistermat, J. A. C. Kolk, and V. S. Varadarajan, *Functions, flows and oscilatory integral on flag manifolds*, Compos. Math. 49 (1983), 309–393.
4. A.T. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.
5. J.E. Humphreys, *Cambridge university press*, vol. 29, Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Mathematics, 1990.
6. I. M. Gel’fand I. N. Bernstein and S.I. Gel’fand, *Schubert cells and cohomology of the spaces g/p*, Russian Math. Surveys 28 (1973), no. 3, 1 – 26.
7. K. D. Johnson, *The structure of parabolic subgroups*, J. Lie Theory 14 (2004), 287 – 316.
8. R. R. Kocherlakota, *Integral homology of real flag manifolds and loop spaces of symmetric spaces*, Adv. Math. **110** (1995), 1–46.

9. L.A.B. San Martin, *Order and domains of attraction of control sets in flag manifolds*, J. Lie Theory **8** (1998), no. 2, 335–350.

10. ———, *Lie algebras*, Ed. Unicamp, 2010.

11. C. E. Miller, *The topology of rotation groups*, Annals of Mathematics **57** (1953), no. 1, 90–114.

12. L. Rabelo, *Cellular Homology of Real Maximal Isotropic Grassmannians*, Adv. in Geometry **16** (2016), no. 3, 361–380.

13. L. Rabelo and J. Lambert, *Covering relations of k-grassmannian permutations of type b*, arXiv:1803.03282.

14. ———, *Homology of isotropic and orthogonal grassmannians: a combinatorial approach*, 2018.

15. G. Warner, *Harmonic Analysis on Semisimple Lie Groups I*, Springer-Verlag, 1972.

16. J.H.C. Whitehead, *On the groups $\pi_r(v_{n,m})$ and spheres bundles*, Proc. London Math. Soc. **48** (1944), 243–291.

17. M. Wiggerman, *The fundamental group of real flag manifolds*, Indag. Mathem. **1** (1998), 141–153.

18. I. Yokota, *On the cell structures of SU(n) and Sp(n)*, Jour. Inst. Poly. Osaka City Univ. **6** (1955), 673–677.

19. ———, *On the cellular decompositions of unitary groups*, Jour. Inst. Poly. Osaka City Univ. **7** (1956), 39–49.

Department of Mathematics, Federal University of Juiz de Fora, Juiz de Fora 36036-900, Minas Gerais, Brazil

E-mail address: lonardo@ice.ufjf.br

IMECC - UNICAMP, Departamento de Matemática. Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeférrino Vaz. 13083-859 Campinas São Paulo, Brazil

E-mail address: smartin@ime.unicamp.br