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Free Vibration Analysis of Nonlinear Structural-Acoustic System with Non-Rigid Boundaries Using the Elliptic Integral Approach

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Abstract: This study addresses the free vibration analysis of nonlinear structural-acoustic system with non-rigid boundaries. In practice, the boundaries of a panel–cavity system are usually imperfectly rigid. Therefore, this study examines the effect of cavity boundary on the resonant frequencies of the nonlinear system. It is the first work of employing the elliptic integral approach for solving this problem, which is involved with the nonlinear multi-mode governing equations of a large amplitude panel coupled with a cavity. The main advantage of this approach is that less nonlinear algebraic equations are generated in the solution steps. The present elliptic integral solution agrees reasonably well with the results obtained from a finite element harmonic balance method. The effects of other parameters such as vibration amplitude, cavity depth, aspect ratio, etc., are also investigated.

Keywords: nonlinear panel vibration; structural dynamics; wave equation; elliptical integral; structural-acoustic interaction

1. Introduction

In recent decades, a large amount of research articles about linear panel-cavity system has been published (e.g., [1–4]). There are also numerous research works about nonlinear panel vibration, but the research investigations about nonlinear structural acoustics are still limited. Almost all of these adopted an assumption of perfectly rigid acoustic boundary (e.g., [5–8]). In practice, the boundaries of a panel–cavity system are usually imperfectly rigid. That is the rationale why this paper aims to study the effect of acoustic boundary condition. Although the structural acoustic research work in [9] considered a non-rigid boundary, it focused on the forced vibration responses of a long duct. To the best of the author’s knowledge, it was the only one which considered the non-rigid boundary condition in structural acoustic research. On the other hand, the solution method employed in [9] was the harmonic balance method. In fact, there have been numerous nonlinear structural vibration and nonlinear oscillation problems solved by the harmonic balance, perturbation, and multiple time scale methods (e.g., [10,11]). In the process of solving the governing equation of nonlinear panel or nonlinear oscillation, a set of nonlinear algebraic equations is generated. Higher accuracy results require more harmonic or nonlinear terms in the solution forms. In other words, it is more time-consuming. In [12], the elliptical integral approach was modified to solve the governing equation of a large amplitude panel coupled with a cavity. The main advantage of the proposed approach is that a set of harmonic components can be embedded into one elliptic integral solution form. Hence, there are much less nonlinear algebraic equations generated in the process of solving the governing equation. It should be noted that in the solution process, a lot of time would be spent on manually doing various equation manipulations and substitutions and deriving the nonlinear algebraic equations. Therefore, more harmonic terms mean more time spent on the manual process.
2. Structural-Acoustic Formulation

Figure 1 shows the model considered in this study, which includes a large amplitude panel coupled with an acoustic cavity. The governing equation of the sound pressure field within the cavity is given by [7–9] (i.e., the well-known wave equation).

\[
\nabla^2 p^h - \frac{1}{C_a^2} \frac{\partial^2 p^h}{\partial t^2} = 0
\]

where \( p^h \) is the \( h \)th harmonic component of the acoustic pressure induced by the nonlinear panel; \( C_a \) is the speed of sound; \( t \) is time. The acoustic mode shape functions for perfectly rigid, fully open and non-rigid (or semi-open) cases are shown in Equations (2)–(4), respectively.

\[
\phi_{uv}(x, y) = \cos\left(\frac{u\pi}{a} x\right)\cos\left(\frac{v\pi}{b} y\right)
\]

\[
\phi_{uv}(x, y) = \sin\left(\frac{(u + 1)\pi}{a} x\right)\sin\left(\frac{(v + 1)\pi}{b} y\right)
\]

\[
\phi_{uv}(x, y) = \cos\left(\frac{(u + \eta)\pi}{a} x - \eta\frac{\pi}{2}\right)\cos\left(\frac{(v + \eta)\pi}{b} y - \eta\frac{\pi}{2}\right)
\]

where \( u \) and \( v \) are the mode numbers; \( a \) and \( b \) are the dimensions of the cavity; \( \eta = \frac{\theta}{(\pi/4)} \) (i.e., the phase shift parameter); \( \theta \) is the phase shift angle. In Equation (2), \( \frac{\partial \phi_{uv}}{\partial x} = \frac{\partial \phi_{uv}}{\partial y} = 0 \) for \( x = 0 \) or \( a \), and \( y = 0 \) or \( b \). It is implied that the velocities at the boundaries are zero because of the rigid boundary condition. In Equation (3), \( \phi_{uv} = 0 \) for \( x = 0 \) or \( a \) and \( y = 0 \) or \( b \). It is implied that pressure levels at the boundaries are zero because of the fully open boundary condition. In Equation (4), if \( \eta \) is set equal to zero, then the acoustic mode shape becomes that in the perfectly rigid case in Equation (2); if \( \eta \) is set equal to one, then the acoustic shape becomes that in the fully open case in Equation (3).

![Figure 1. Nonlinear panel–cavity system with non-rigid boundaries.](image)

The boundary conditions at \( z = c \) and 0 are given in Equations (5) and (6), respectively:

\[
\frac{\partial p^h}{\partial z} = 0
\]
\[
\frac{\partial p_h}{\partial z} = -\rho \frac{\partial^2 A_{mn}^h}{\partial t^2} \Phi_{mn}(x, y) \tag{6}
\]

where \(A_{mn}^h\) is the \(h\)th harmonic component of the nonlinear panel vibration; \(\Phi_{mn}(x, y)\) is a double sine function (i.e., the panel mode shape); \(m\) and \(n\) are the panel mode numbers; \(\rho\) is panel surface density.

According to \([2,12]\), the general solution form of the wave equation is given by

\[
P^h(x, y, z, t) = \sum_{u=1}^{U} \sum_{v=1}^{V} \left[ \mu_{uv}^{h} \sinh(\mu_{uv}^{h} z) + N_{uv}^{h} \cosh(\mu_{uv}^{h} z) \right] \varphi_{uv}(x, y) \cos(\omega t) \tag{7}
\]

where \(\mu_{uv}^{h} = \sqrt{(u\pi/a)^2 + (v\pi/b)^2 - (\omega t/c_o)^2}\); \(U\) and \(V\) are the indices of modes used; \(L_{uv}^{h}\) and \(N_{uv}^{h}\) are constants dependent on the boundary conditions in Equations (5) and (6).

By applying the boundary conditions in Equations (5) and (6) and putting Equation (7) to Equation (1), the two constants \(L_{uv}^{h}\) and \(N_{uv}^{h}\) can be found and the acoustic pressure at \(z = c\) can be given by

\[
P^h_c(x, y, t) = \rho_a(h, \omega)^2 \sum_{u=1}^{U} \sum_{v=1}^{V} \coth\left(\frac{\mu_{uv}^{h} c}{\mu_{uv}^{h}}\right) \frac{A_{mn}^{h}}{A_{mn}^{h}} \frac{\alpha_{mn}^{uv}}{\alpha_{mn}^{uv}} \varphi_{uv}(x, y) \cos(\omega t) \tag{8}
\]

where \(\alpha_{mn}^{uv} = \int_0^b \int_0^a \varphi_{uv} \Phi_{mn} dxdy; \rho_a\) is air density.

By taking integration over the surface of \(z = c\), the modal acoustic pressure is given by

\[
P^h_c(t) = \rho_a(h, \omega)^2 \sum_{u=1}^{U} \sum_{v=1}^{V} \coth\left(\frac{\mu_{uv}^{h} c}{\mu_{uv}^{h}}\right) \frac{A_{mn}^{h}}{A_{mn}^{h}} \frac{\alpha_{mn}^{uv}}{\alpha_{mn}^{uv}} \varphi_{uv}(x, y) \cos(\omega t) \tag{9}
\]

where \(\alpha_{mn}^{uv} = \int_0^b \int_0^a \varphi_{uv} \Phi_{mn} dxdy; \alpha_{mn}^{uv} = \int_0^b \int_0^a \Phi_{mn} \Phi_{mn} dxdy\).

Hence, the total modal acoustic pressure, which contains all harmonic components, is given by

\[
F_c = \sum_{h=1,3,5,...}^H P^h_c \tag{10}
\]

where \(H\) is the number of harmonic components considered.

According to \([5,12]\), the large amplitude panel vibration is governed by the von Kármán plate theory. The governing equation of large amplitude panel with cubic nonlinear stiffness is given by

\[
\rho \frac{d^2 A_{mn}}{dt^2} + \rho \omega_c^2 A_{mn} + \beta A_{mn}^3 + F_c = 0 \tag{11}
\]

where \(A_{mn}(t)\) is the modal structural displacement response; \(\omega_c\) is the linear resonant frequency of the panel; \(\rho\) is panel density; \(\gamma\) is aspect ratio; \(\tau\) is panel thickness; \(E\) is Young’s modulus; \(\nu\) is Poisson’s ratio. Note that \(F_c\) is the excitation acoustic force within the cavity acting on the panel surface at \(z = c\). The nonlinear stiffness term is given by

\[
\beta = \frac{E\tau}{4(1-\nu^2)} \left( \frac{m\pi}{a} \right)^4 \left( \frac{n}{m} \gamma \right)^4 \left( \frac{1}{4} - \frac{v^2}{4} \right) + \nu \left( \frac{n}{m} \gamma \right)^2 \tag{12}
\]

### 3. Elliptic Integral Approach

By introducing a new variable \(\overline{A}\) to replace \(A_{mn}\), Equation (11) is rewritten as

\[
\rho \frac{d^2 \overline{A}}{dt^2} + \rho \omega_c^2 \overline{A} + \beta \overline{A}^3 + F_c = R \tag{13}
\]
where $R$ is the residual induced in the equation. It is because $\overline{A}$ is not the exact solution. $\overline{A}$ is the solution of the following equation

$$
\rho \frac{d^2\overline{A}}{dt^2} + (\rho a_0^2 + K_\alpha)\overline{A} + \beta \overline{A}^3 = 0
$$

(14)

where $\overline{A} = A_\alpha Cn(\Phi(x))$; $\Phi$ is the elliptic integral; $\kappa = \frac{\beta A_\omega^2}{2(\rho a_0^2 + K_\alpha + \beta A_\omega^2)}$ is the modulus of $\Phi$; $Cn$ is the elliptic cosine; $A_\alpha$ is the initial modal displacement; $K_\alpha$ is a constant to be determined.

By putting Equation (14) into Equation (13), the following equation can be given

$$
F_c - K_\alpha \overline{A} = R
$$

(15)

Note that according to Equation (10), $F_c = \sum_{h=1,3,5} H_c^h$. $H_c^h$ depends on $\overline{A}^h$, which is the $h$th harmonic component of $\overline{A}$. According to Equation (13), $\overline{A}$ depends on $K_\alpha$. By minimizing the residual $R$ in Equation (15), the value of $K_\alpha$ can be obtained. Eventually, the resonant frequency of the nonlinear system can be given by [12,13]

$$
\omega = \frac{2\pi}{T}
$$

(16)

where $T = \frac{4}{(\omega_\alpha^2 + K_\alpha + \beta A_\omega^2)^{1/2}} \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2 \sin^2(\psi)}} d\psi$ (i.e., the period of the elliptic integral function).

4. Harmonic Balance Method

According to the classical harmonic balance method [14], the solution form of Equation (11) is given by

$$
\overline{A} = A_1 \cos(\omega t) + A_3 \cos(3\omega t) + A_5 \cos(5\omega t)
$$

(17)

where $\overline{A}$ is the approximate solution; $A_1$, $A_3$, and $A_5$ are the amplitudes of the 1st, 2nd, and 3rd harmonic components. Note that the three harmonic term formulation is adopted for the purpose of demonstration.

By putting Equation (17) into Equation (11) and taking harmonic balances of $\cos(\omega t)$, $\cos(3\omega t)$, and $\cos(5\omega t)$, the following equation can be given by

$$
\int_0^{2\pi} \left( \rho \frac{d^2 \overline{A}}{dt^2} + \rho a_0^2 \overline{A} + \beta \overline{A}^3 + F_c(\overline{A}) \right) \cos(\omega t) dt = 0
$$

(18)

$$
(\rho a_0^2 + \alpha_1) A_1 + \beta \left( \frac{3}{2}(A_1)^3 + \frac{1}{2}(A_1)^2 A_3 + \frac{1}{2}(A_3)^2 A_1 + \frac{1}{2}(A_3)^3 A_5 + \frac{3}{2}A_1 A_3 A_5 \right) = 0
$$

(19)

$$
(\rho a_0^2 + \alpha_3) A_3 + \beta \left( \frac{3}{2}(A_1)^3 + \frac{1}{2}(A_1)^2 A_3 + \frac{1}{2}(A_3)^2 A_1 + \frac{1}{2}(A_3)^3 A_5 + \frac{3}{2}A_1 A_3 A_5 \right) = 0
$$

(20)

where $\alpha_h = \rho_\alpha (h, \omega)^2 \sum_{n_1} \sum_{n_2} \sum_{n_3} \frac{\cosh \left( \frac{\mu \omega}{\lambda} \right)}{\rho_\alpha \omega^2} \frac{(\mu \omega)^2}{(\lambda \omega)^2}, h = 1, 2, 3$.

By setting $A_1$ as a known constant, there are three unknowns in Equations (18)–(20) (i.e., $\omega$, $A_3$ and $A_5$). The resonant frequency can be found by solving the three equations. The comparison between the solution equations from the elliptic integral and harmonic balance
methods (i.e., Equations (15), (18)–(20)) clearly shows that there are less nonlinear equations in the former method.

5. Results and Discussion

In this case study, the configurations and material properties are: 0.3048 m × 0.3048 m × 1.2192 mm aluminum panel, Young’s modulus equal to 7 × 10¹⁰ N/m², Poisson’s ratio equal to 0.3, panel mass density equal to 2700 kg/m³, cavity depth equal to 0.0508 m and air density equal to 1.2 kg/m³. The frequency ratio is defined as $\omega_n/\omega_o$. Table 1 shows the mode convergences for the first three resonant frequencies (i.e., the (1,1), (2,3) and (3,3) panel mode resonant frequencies). In the convergence study, two harmonic terms are adopted. It can be seen that the nine acoustic mode approach is good enough for four-digit accuracy.

| Number of Acoustic Modes Used | 1st Resonance | 2nd Resonance | 3rd Resonance |
|-------------------------------|---------------|---------------|---------------|
| 1                             | 2.6828        | 1.9179        | 1.6975        |
| 4                             | 2.6774        | 1.9029        | 1.6862        |
| 9                             | 2.6773        | 1.9011        | 1.6835        |
| 16                            | 2.6773        | 1.9010        | 1.6833        |

Figure 2 shows the dimensionless amplitude plotted against the first resonant frequency ratio for various phase shift parameters. Table 2 is a summary about the observations in Figures 2–4. Note that the first resonant frequency stands for the resonant frequency of the (1,1) mode. In the comparison between the lines and crosses (which are obtained from [14]), it can be seen that they are quite close (within 2%), and the present elliptical integral method can generate reasonable and trustable results. The cases of $\eta$ equal to 0 and 100% represent the perfectly rigid and fully open boundaries, respectively. Obviously, the cavity stiffness is the strongest when the acoustic boundaries are perfectly rigid, and it is the weakest when the acoustic boundaries are fully open. That is why the frequency ratio of $\eta$ equal to 0 is the highest. The frequency ratio of $\eta$ equal to 15% (not the one of 100%) is the lowest one because of the negative cavity stiffness. Figures 3 and 4 show the dimensionless amplitudes plotted against the second and third resonant frequency ratios for various phase shift parameters, respectively. In each of these two figures, the three curves are very close (less than 1% difference). It is because the panel stiffness of the (1,3) mode or (3,3) mode is much higher. Relatively, the effect of the cavity stiffness on the higher mode resonant frequencies would be much smaller. Therefore, the frequency–amplitude curves are so close. Figure 5 shows the first resonant frequency ratio plotted against the phase shift parameter for various cavity depths. Obviously, around $\eta$ equal to 10%, there is an abrupt jump at each curve. It is due to the phenomenon of cavity stiffness turning from very positive to very negative (i.e., the term $\mu_{\eta}^{h}$ in Equation (9) is turning from complex into real). For $\eta$ less than 10%, the frequency ratio of each curve is monotonically increasing, and the frequency ratio of the shortest cavity depth (i.e., 0.0508 m) is the highest. The frequency ratio is minimum around 11% and gradually converging to the particular value in the fully open case (i.e., $\eta$ equal to 100%). There is no detectable difference between the frequency ratios of the three fully open cases (i.e., cavity depth equal to 0.0508, 0.1524, and 0.6096 m).
Figure 2. Amplitude ratio versus 1st resonant frequency ratio for various phase shift parameters.

Table 2. Effect of phase shift parameters for various resonant amplitude curves.

| Phase Shift Parameter | $\eta = 0$ | $\eta = 0.15$ | $\eta = 1$ |
|-----------------------|------------|---------------|------------|
| Amp. curve            |            |               |            |
| First resonance       | Fully close cavity, strongest cavity stiffness | Partially open cavity, negative cavity stiffness | Fully open cavity, almost zero effect |
| Second resonance      | Almost no difference |               |            |
| Third resonance       | Almost no difference |               |            |

Figure 3. Amplitude ratio versus 2nd resonant frequency ratio for various phase shift parameters.
Figure 4. Amplitude ratio versus 3rd resonant frequency ratio for various phase shift parameters.

Figure 5. Resonant frequency ratio versus phase shift parameter for various cavity depths.

Figure 6 shows the first resonant frequency ratio plotted against the aspect ratio for various phase shift parameters. Table 3 is a summary about the observations in Figure 6. Similar to that in Figure 5, there is an abrupt jump at each curve. At the curve of $\eta$ equal to 0, the jump magnitude is the biggest at the aspect ratio equal to 0.55. It is because the cavity stiffness is the strongest. Unlike that in Figure 5, before their abrupt jumps, the three curves are almost the same for the aspect ratio less than 0.3; after their abrupt jumps, each frequency ratio is generally and slightly decreasing and converging to different values. By referring to the low frequency value in the case of $\eta$ equal to 15% and cavity depth equal to 0.0508m in Figure 5, it can be seen why the curve of $\eta$ equal to 15% in Figure 6 is always the lowest after the abrupt jump.
Figure 6. Frequency ratio versus aspect ratio for various phase shift parameters.

Table 3. Effect of phase shift parameters for various aspect ratios.

| Phase Shift Parameter | =0 | =0.15 | =1 |
|-----------------------|----|-------|----|
| Aspect ratio           |    |       |    |
| Short 0 to 0.25        | Almost no difference |
| Medium 0.25 to 0.6     | Biggest abrupt change at \(a/b = 0.55\) |
|                       | Big abrupt change at \(a/b = 0.36\) |
|                       | Middle-sized abrupt change at \(a/b = 0.32\) |
| High 0.6 to 1          | Lowest frequency ratio |
|                       | Middle frequency ratio |
|                       | Highest frequency ratio |

Figure 7 shows the first resonant frequency ratio plotted against the cavity depth for various phase shift parameters. Table 4 is a summary about the observations in Figure 7. Obviously, the curve of \(\eta = 0\) equal to 100% is almost a horizontal line. In this case, it is implied that the cavity boundaries are fully open. The cavity stiffness is so weak and close to zero, and thus inert to the cavity depth change. In the case of \(\eta\) equal to 0%, the frequency ratio goes to a positive infinity when the cavity depth approaches zero and corresponding cavity stiffness is close to infinity. The frequency ratio is monotonically deceasing and getting close to that of \(\eta\) equal to 100%, when the cavity depth is long or its stiffness is weak. Contrary to that in the case of \(\eta\) equal to 0%, the frequency ratio curve of \(\eta\) equal to 15% is far below the other frequency ratio curve of \(\eta\) equal to 100%, when the cavity depth approaches zero. Additionally, the frequency ratio is monotonically increasing with the cavity depth and getting close to that of \(\eta\) equal to 100%. From the above observations, it is implied that the cavity stiffness is always positive in the case of \(\eta\) equal to 0%, while it is always negative in the case of \(\eta\) equal to 15%. That is why one is above the curve of \(\eta\) equal to 100%, and the other one is below it.
6. Conclusions

This study has employed the elliptic integral approach to conduct the free vibration analysis of nonlinear structural-acoustic system with non-rigid boundaries. The proposed elliptic integral solution form is applied to this nonlinear structural acoustic problem. The nonlinear modal formulation has been developed from the governing equation of large amplitude panel vibration, which is coupled with the homogenous wave equation for non-rigid boundaries. The results obtained from the proposed method and finite element harmonic balance method are generally consistent. The results show that (1) the imperfectly rigid boundaries can induce a negative cavity stiffness to make the first resonant frequency much lower; (2) the effect of boundary condition on the second and third resonant frequencies are very minimal because the modal stiffnesses are much higher than the cavity stiffness; (3) the cavity stiffness is negligible in the fully open case and thus it is indifferent to the cavity depth.

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