A MODULI SCHEME PARAMETRIZING BLOWUPS OF SMOOTH PROJECTIVE SURFACES

MONICA MARINESCU

Abstract. We construct a moduli scheme $F[n]$ that parametrizes tuples $(S_1, S_2, \ldots, S_{n+1}, p_1, p_2, \ldots, p_n)$ in which $S_1$ is a fixed smooth surface over $\text{Spec} \ R$ and $S_{i+1}$ is the blowup of $S_i$ at a point $p_i$, $\forall 1 \leq i \leq n$. We show that this moduli scheme is smooth and projective. We prove that $F[n]$ has smooth divisors $D_{i,j}$, $\forall 1 \leq i < j \leq n$, which correspond to tuples that map $p_j \mapsto p_i$ under the projection morphism $S_j \to S_i$. When $R = k$ is an algebraically closed field, we demonstrate that the Chow ring $\mathbb{A}^*(F[n])$ is generated by these divisors over $\mathbb{A}^*(S_1^r)$. We end by giving a precise description of $\mathbb{A}^*(F[n])$ when $S_1$ is a complex rational surface.

1. Definition of the moduli problem

Let $S_1$ be a fixed smooth projective surface over a ring $R$. Our goal is to construct a space that parametrizes surfaces obtained through a series of $n$ ordered blowups of the base surface $S_1$. More specifically, we focus our attention on the following objects:

$$(S_1, S_2, \ldots, S_{n+1}, p_1, p_2, \ldots, p_n),$$

where $p_i \in S_i$ and $S_{i+1}$ is the blowup of $S_i$ at $p_i$, for all $1 \leq i \leq n$. We define formally the functor $\mathcal{F}[S_1, n]$ as follows:

**Definition 1.1.** Let $S_1$ be a smooth projective surface over a ring $R$ and $n \geq 0$ an integer. Consider the contravariant functor:

$$\mathcal{F}[n] = \mathcal{F}[S_1, n] : \text{Sch}(R) \to \text{Sets}$$

defined as follows:

- For any $R$-scheme $B$, an object in $\mathcal{F}[S_1, n](B)$ is a tower of morphisms:

$$\Sigma_{B,n+1} \xrightarrow{\pi_{n+1}} \Sigma_{B,n} \xrightarrow{\pi_n} \Sigma_{B,n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} \Sigma_{B,1} = B \times S_1$$

such that the following conditions are satisfied:

1. $\pi_1 = \text{pr}_1$ is the projection onto the first factor;
2. for each $1 \leq i \leq n$, the morphism $p_i : B \to \Sigma_{B,i}$ is a section of the composed map $\Sigma_{B,i} \to B$;
(3) for each \(1 \leq i \leq n\), the morphism \(\pi_{i+1} : \Sigma_{B,i+1} \to \Sigma_{B,i}\) is the blowup of \(\Sigma_{B,i}\) along the locus \(p_i(B)\).

Shortened notations for such a family are \(\Sigma_{B, \leq n+1} \to B\) and \((\Sigma_{B,i}, \pi_i, p_i)_{i=1}^{n}\). Notice that although \(\Sigma_{n+1}\) is not included in the latter notation, this scheme is uniquely defined by the data given.

- For every \(R\)-scheme \(B\), \(\mathcal{F}[n](B)\) is the set of all families over \(B\), up to isomorphism. An isomorphism between two families is defined as:

\[
\Theta = (\theta_i)_{i=1}^{n} : (\Sigma_{B,i}, \pi_i, p_i)_{i=1}^{n} \sim (\Sigma'_{B,i}, \pi'_i, p'_i)_{i=1}^{n},
\]

where, \(\forall 1 \leq i \leq n\), \(\theta_i : \Sigma_i \to \Sigma'_i\) is a scheme isomorphism that commutes with the maps of the two families.

- Let \(B_1, B_2\) be \(R\)-schemes and \(f : B_1 \to B_2\) an \(R\)-morphism between the two schemes. There exists a natural contravariant map \(\mathcal{F}(f) : \mathcal{F}[n](B_2) \to \mathcal{F}[n](B_1)\) that assigns to each family over \(B_2\) a corresponding family over \(B_1\) by pulling back the schemes \(\Sigma_{B,i}\) and the sections \(p_i\) along \(f\). The resulting tower of morphisms over \(B_1\) is a valid object in \(\mathcal{F}[n](B_1)\) (this is an immediate application of Lemma 2.2 from Section 2).

- The identity map \(\text{id} : B \to B\) on any \(R\)-scheme corresponds to the identity map on sets \(\mathcal{F}(\text{id}) = \text{id} : \mathcal{F}[n](B) \to \mathcal{F}[n](B)\).

- Let \(B_1, B_2, B_3\) be \(R\)-schemes and \(g : B_2 \to B_3\) be \(R\)-morphisms. Then \(\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)\). This follows from the uniqueness of pullbacks (up to isomorphism).

**Remark 1.2.** Let \(B\) be a \(R\)-scheme and \(\Sigma_{B, \leq n+1} \to B\) a family in \(\mathcal{F}[n](B)\). For any point \(x \in B\), the fiber over \(x\) is a sequence \(S_{n+1} \to \cdots \to S_1\), where \(S_{i+1}\) is the blowup of \(S_i\) at some point \(p_i\), \(\forall 1 \leq i \leq n\). Therefore, this fiber corresponds to a tuple \((S_1, \ldots, S_{n+1}, p_1, \ldots, p_n)\) as the ones introduced in the beginning of the section.

**Remark 1.3.** For every integer \(n \geq 0\), there exists a natural forgetful map \(\mathcal{F}[n+1] \to \mathcal{F}[n]\) which sends families \(\Sigma_{B, \leq n+2} \to B\) in \(\mathcal{F}[n+1](B)\) to families \(\Sigma_{B, \leq n+1} \to B\) in \(\mathcal{F}[n](B)\), for all \(R\)-schemes \(B\).

**Remark 1.4.** For every integer \(n > 0\), there exists a natural transformation of functors \(\mathcal{F}[n] \to S_1^n\) which, for any \(R\)-scheme \(B\), maps a family \((\Sigma_{B,i}, \pi_i, p_i)_{i=1}^{n}\) over \(B\) to \((\overline{p_1}, \overline{p_2}, \ldots, \overline{p_n})\), where \(\overline{p_i} \in S_1(B)\) is the composition map \(B \xrightarrow{p_i} \Sigma_{B,i} \to \cdots \to \Sigma_{B,1} = B \times S_1 \xrightarrow{p_1} S_1\).

2. Construction of the moduli scheme

In this section we prove that the functor \(\mathcal{F}[n]\) has a fine moduli scheme, which we denote by \(F[n]\). We show that \(F[n]\) and all the schemes in its universal family are smooth and projective over \(\text{Spec } R\).
Before we start working towards our main result, note that we reserve the following notation for the universal family over $F[n]$: 

$$
\sum_{n+1,n+1} \xrightarrow{\pi_{n+1,n+1}} \sum_{n+1,n} \xrightarrow{\pi_{n+1,n}} \ldots \xrightarrow{\pi_{n+1,2}} \sum_{n+1,1} = F[n] \times S_1
$$

**Example 2.1.** We start by constructing $F[0], F[1]$ (which represent the functors $\mathcal{F}[0]$ and $\mathcal{F}[1]$, respectively), and their universal families. It is easy to see that $F[0] = \text{Spec } R$, since every scheme $B \in \text{Sch}(R)$ comes equipped with a structure morphism to $\text{Spec } R$. By definition, $\Sigma_{1,1} = \text{Spec } R \times S_1$: 

$$
\Sigma_{B,1} \xrightarrow{pr_1} \Sigma_{1,1} \quad B \times S_1 \xrightarrow{pr_1} S_1
$$

Next, we show that $F[1] \cong S_1$. Intuitively, every object in this moduli space is a triple $(S_1, S_2, p_1)$ where $S_2 = \text{Bl}_{p_1} S_1$, so it is uniquely identified by the point $p_1 \in S_1$. More concretely, we need to show that $\mathcal{F}[1](B) \cong \text{Mor}(B, S_1)$. The equivalence goes as follows: for a family over $B$ like in the figure below, the corresponding map $B \to S_1$ is $f = pr_2 \circ p_1$. Conversely, given a morphism $f : B \to S_1$, we obtain a section $p_1 = \text{id} \times f : B \to \Sigma_{B,1} = S_1 \times B$ and $\Sigma_{B,2}$ is the blowup of $\Sigma_{B,1}$ along this section:

$$
S_1 \xrightarrow{pr_2} \Sigma_{B,1} = B \times S_1
$$

The top scheme in the universal family over $F[1]$ is $\Sigma_{2,2} = \text{Bl}_{\Delta} (S_1 \times S_1)$. This follows immediately from the figure above, considering the special case where $B = F[1] \cong S_1$ and $f = \text{id} : S_1 \to S_1$:

$$
\Sigma_{2,2} \xrightarrow{\Delta} \text{Bl}_{\Delta} (S_1 \times S_1) \xrightarrow{\text{id}} S_1
$$

Before we give the general construction of the moduli scheme corresponding to the functor $\mathcal{F}[n]$, we state and prove the following lemmas, which will be the key ingredients in our proof:
Lemma 2.2. Let $A, B, \Sigma_{B,1}$ be schemes over $R$. Let $\pi : \Sigma_{B,1} \to B$ be a smooth morphism with a section $\sigma : B \to \Sigma_{B,1}$, and let $\Sigma_{B,2}$ be the blowup of $\Sigma_{B,1}$ along the locus $\sigma(B)$. Given $f : A \to B$ an $R$-morphism, let $\Sigma_{A,1}$ and $\sigma^*$ be the pullbacks along $f$ of $\Sigma_{B,1}$ and $\sigma$, respectively. The following statements hold:

(i) The composed morphism $\Sigma_{B,2} \to B$ is smooth.

(ii) The blowup of $\Sigma_{A,1}$ along the locus $\sigma^*(A)$, denoted by $\Sigma_{A,2}$, is the pullback of $\Sigma_{B,2}$ along $f$.

\[
\begin{array}{ccc}
\Sigma_{A,2} & \longrightarrow & \Sigma_{B,2} \\
\downarrow & & \downarrow \\
\Sigma_{A,1} & \longrightarrow & \Sigma_{B,1} \\
\sigma^* \leftarrow \pi^* & \leftarrow & \pi^* \\
A & \longrightarrow & B
\end{array}
\]

Proof. (i) We show that the morphism $\Sigma_{B,2} \to B$ is smooth by proving that it is flat, locally of finite presentation, and has smooth fibers (see [6], Tag 02K5). Affine locally, on the level of rings, we are given a smooth morphism $\pi : T \to T'$ and $\sigma : T' \to T$ a section of $\pi$. Let $I = \ker(\sigma) \subset T'$. The blowup of $\text{Spec } T'$ along $I$ is $\text{Bl}_I(\text{Spec } T') = \text{Proj}(T' \oplus I \oplus I^2 \oplus \ldots)$.

The map $\sigma$ is a section of the smooth morphism $\pi$, which means that $I$ is a regular ideal. Since $I$ is regular, then $\text{Bl}_I(\text{Spec } T')$ is of finite presentation over $\text{Spec } T$ (see [6], Tag 0BIQ).

To show that $\Sigma_{B,2} \to B$ is flat, it is enough to prove that $(T' \oplus I \oplus I^2 \oplus \ldots)$ is flat over $T$. We know $T'$ is flat over $T$ because $T \to T'$ is smooth. We show inductively that $I^n$ is flat over $T$. By the following short exact sequence, it is enough to prove that $T'/I^n$ is $T$-flat:

\[0 \to I^n \to T' \to T'/I^n \to 0.\]

When $n = 1$, $T'/I \cong T$, so the claim is true. For the inductive step, consider another short exact sequence:

\[0 \to I^{n-1}/I^n \to T'/I^n \to T'/I^{n-1} \to 0.\]

Since $I$ is a regular ideal, then $I^{n-1}/I^n$ is a locally free finite $T$-module, hence it is flat. By the induction hypothesis, $T'/I^{n-1}$ is flat over $T$. Putting these two facts together, we conclude that $T'/I^n$ is flat over $T$, completing the induction step.

Lastly, we show that the morphism $\Sigma_{B,2} \to B$ has smooth fibers. By part (ii) below, the fiber over every point $x \in B$ is:

\[V' = \text{Bl}_xV \longrightarrow V \longrightarrow \text{Spec } k(x),\]

where $k(x)$ is the residue field of $x$. The scheme $V$ is smooth over $k(x)$, hence $V' = \text{Bl}_xV$ is also smooth over $k(x)$. With this, we conclude that the morphism $\Sigma_{B,2} \to B$ is smooth.
(ii) We work affine locally, on the level of rings, where we have the following figure:

\[ S' = S \otimes_T T' \leftarrow T' \]

\[ S \leftarrow T \]

Let \( \pi_S \) and \( \sigma_S \) be the pullbacks of \( \pi \) and \( \sigma \), respectively. Let \( I_S = \ker(\sigma_S) \subset S' \). The blowup of \( \text{Spec} \ S' \) along \( I_S \) is \( \text{Bl}_{I_S} \text{Spec} \ S' = \text{Proj}(S' \oplus I_S \oplus I_S^2 \oplus \ldots) \). Our claim boils down to showing that \( I_S \) is flat over \( T \), so tensoring the top short exact sequence with \( S \) preserves the exactness of the resulting sequence:

\[
\begin{array}{cccccc}
0 & \rightarrow & I_S \otimes_T S & \rightarrow & T' \otimes_T S & \rightarrow & T'/I_S \otimes_T S & \rightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \rightarrow & I_S^g & \rightarrow & S' & \rightarrow & S'/I_S^g & \rightarrow & 0.
\end{array}
\]

\[ \square \]

Lemma 2.3. Let \( A, B, C \) be \( R \)-schemes. Let \( A \xrightarrow{h} C \) be an \( R \)-morphism that factors as \( A \xrightarrow{f} B \xrightarrow{g} C \). Given any \( R \)-morphism \( V_C \xrightarrow{v_C} C \), let \( V_B \xrightarrow{v_B} B \) be its pullback along \( g \), and \( V_A \xrightarrow{v_A} A \) be its pullback along \( h \). There exists a unique map \( f' : V_A \rightarrow V_B \) which makes the top triangle commutative and the left square cartesian.

\[
\begin{array}{cccccc}
V_A & \xrightarrow{h'} & V_C \\
\downarrow v_A & & \downarrow v_C \\
A & \xrightarrow{f} & B
\end{array}
\]

\[
\begin{array}{cccccc}
V_A & \xrightarrow{v_A} & V_B \\
\downarrow & & \downarrow \\
A & \xrightarrow{h} & C
\end{array}
\]

\[
\begin{array}{cccccc}
V_B & \xrightarrow{g'} & V_C \\
\downarrow v_B & & \downarrow v_C \\
B & \xrightarrow{g} & C
\end{array}
\]

Proof. Let \( V'_A = V_B \times_B A \) with \( f' : V'_A \rightarrow V_B \) the associated map. Since \( V_B = V_C \times_C B \) and \( h = g \circ f \), then \( V'_A = V_C \times_C A \). By the uniqueness of pullbacks, we conclude that \( V_A = V'_A \) and \( f' \) is the unique map in question. \( \square \)

Theorem 2.4. The functor \( F[n] \) has a fine moduli scheme \( F[n] \). Moreover, the following is true:

(a) For every \( n \geq 0 \), \( F[n] \) and all the schemes \( \Sigma_{n+1,1}, \ldots, \Sigma_{n+1,n+1} \) in its universal family are smooth and projective over \( \text{Spec} \ R \);

(b) For every \( n \geq 1 \), the top scheme \( \Sigma_{n+1,n+1} \) in the universal family over \( F[n] \) can be identified as:

\[
\Sigma_{n+1,n+1} \cong \text{Bl}_{\phi}(F[n] \times_{F[n-1]} F[n]),
\]

where the cartesian product is induced by the forgetful map \( F[n] \rightarrow F[n-1] \);
(c) For every $n \geq 0$, the top scheme $\Sigma_{n+1,n+1}$ in the universal family of $F[n]$ is the moduli scheme representing the functor $\mathcal{F}[n+1]$. Under this identification, the map

$$\Pi_{n+1} = \pi_{n+1} \circ \cdots \circ \pi_{n+1,n+1} : \Sigma_{n+1,n+1} \cong F[n+1] \to F[n]$$

corresponds to the forgetful functor $\mathcal{F}[n+1] \to \mathcal{F}[n]$.

Proof. We prove these statements inductively over $n$. The base case is covered in Example 2.1. For the inductive step, assume that the moduli scheme $F[k]$ exists, for all $k < n$. Assume $F[k], \Sigma_{k+1,1}, \ldots, \Sigma_{k+1,k+1}$ are all smooth and projective. Here are the steps in our proof:

(i) Let $W = \Sigma_{n,n}$ be the top scheme in the universal family over $F[n-1]$. We construct a family over $W$ in $\mathcal{F}[n](W)$, then show that $W$ is the fine moduli scheme corresponding to the functor $\mathcal{F}[n]$ and that the family we just defined over $W$ is the universal family.

(ii) We show that $F[n], \Sigma_{n+1,1}, \ldots, \Sigma_{n+1,n+1}$ are smooth and projective, and that $\Sigma_{n+1,n+1}$ is isomorphic to $\text{Bl}_\delta(\text{trunc}(F[n] \times_{F[n-1]} F[n]))$.

(iii) We prove that the map $\Pi_n : \Sigma_{n,n} \cong F[n] \to F[n-1]$ corresponds to the forgetful functor $\mathcal{F}[n] \to \mathcal{F}[n-1]$.

The first step is to construct the family over $W$ in $\mathcal{F}[n](W)$. Let $\Pi_n = \pi_{n,1} \circ \cdots \circ \pi_{n,n} : W = \Sigma_{n,n} \to F[n-1]$ be the composed morphism and $\Sigma_{W,\leq n} \to W$ be the pullback of the universal family $\Sigma_{n,\leq n} \to F[n-1]$ along this map $\Pi_n$ (see figure below). In particular, notice that $\Sigma_{W,n} \cong W \times_{F[n-1]} W$. Let $p_{W,n} := \Delta : W \to \Sigma_{W,n} = W \times_{F[n-1]} W$ be the diagonal embedding and $\Sigma_{W,n+1} = \text{Bl}_\delta(W \times_{F[n-1]} W)$:

$$
\begin{array}{ccc}
\Sigma_{W,n+1} = \text{Bl}_\delta(W \times_{F[n-1]} W) & \downarrow \\
\Sigma_{W,n} = W \times_{F[n-1]} W & \longrightarrow & \Sigma_{n,n} = W \\
\downarrow & & \downarrow \\
p_{W,n} = \Delta : \Sigma_{W,\leq n-1} & \longrightarrow & \Sigma_{n,\leq n-1} \\
\downarrow & & \downarrow \\
W & \longrightarrow & F[n-1].
\end{array}
$$

The resulting tower of morphisms $\Sigma_{W,\leq n+1} \to W$ is indeed a family in $\mathcal{F}[n](W)$, as a result of Lemma 2.2. We claim that $W$ is the fine moduli space for $\mathcal{F}[n]$, and that the family constructed above is the universal family over $F[n]$. We prove this statement by using this family over $W$ to build the correspondence $\mathcal{F}[n](B) \cong \text{Mor}(B,W)$, for any $R$-scheme $B$.

We start by constructing a functor map:

$$C_1 : \mathcal{F}[n] \to \text{Mor}(-,W).$$

Let $B$ be an $R$-scheme and $\Sigma_{B,\leq n+1} \to B$ a family in $\mathcal{F}[n](B)$. We need to associate a morphism $B \to W$ to this family. The truncated family $\Sigma_{B,\leq n} \to B$ is an element of $\mathcal{F}[n-1](B)$, so it corresponds uniquely to a morphism $f_{n-1} : B \to F[n-1]$ that gives the figure below. Now, recall that the family $\Sigma_{B,\leq n+1} \to B$ comes equipped
with a section $p_n : B \to \Sigma_n$, so we obtain the desired map $f_W : B \to W$ by composing $B \xrightarrow{p_n} \Sigma_{B,n} \xrightarrow{\text{pr}_2} W$:

\[
\begin{array}{c}
\Sigma_{B,n} \xrightarrow{\text{pr}_2} \Sigma_{n,n} = W \\
\downarrow \\
\Sigma_{B,\leq n-1} \xrightarrow{r} \Sigma_{n,\leq n-1} \\
\downarrow \\
B \xrightarrow{f_{n-1}} F[n-1].
\end{array}
\]

Second, we construct a functor map:

$C_2 : \text{Mor}(\cdot, W) \to \mathcal{F}[n].$

Let $B$ be an $R$-scheme and $f : B \to W$ an $R$-morphism. We obtain a corresponding family in $\mathcal{F}[n](B)$ by pulling back the family $\Sigma_{W,\leq n+1} \to W$ along $f$. The resulting tower of morphisms is indeed a family in $\mathcal{F}[n](B)$, as a result of Lemma 2.2:

$\Sigma_{B,\leq n+1} \to \Sigma_{W,\leq n+1} \xrightarrow{\text{pr}_*} B \xrightarrow{f} W.$

Now we want to show that the maps $C_1$ and $C_2$ are inverses of each other. Say we start with a morphism $f : B \to W$. We construct a family over $B$ by pulling back $\Sigma_{W,\leq n+1} \to W$ along $f$, then use this family to obtain a corresponding morphism $f_W : B \to W$:

$\Sigma_{B,n+1} \to \Sigma_{W,n+1} \xrightarrow{\text{pr}_*} B \xrightarrow{f} W.$

We prove that $f = f_W$, which concludes that $C_1 \circ C_2 = \text{id}$:

$[B \xrightarrow{f_W} W] = [B \xrightarrow{p_n} \Sigma_{B,n} \to \Sigma_{W,n} \to \Sigma_{n,n}]$

$= [B \xrightarrow{f} W \xrightarrow{\text{pr}_*} \Sigma_{W,n} \to \Sigma_{n,n}]$

$= [B \xrightarrow{f} W \to W \times_{F[n-1]} W \xrightarrow{\text{pr}_2} W]$  

$= [B \xrightarrow{f} W].$

Say we start with a family over $B$ in $\mathcal{F}[n](B)$, denoted as $\Sigma_{B,\leq n+1} \to B$. As before, we obtain a corresponding morphism $f_W : B \to W$. We want to show that if we pull back $\Sigma_{W,\leq n+1} \to W$ along $f_W$, we recover the same family we started with. Since the
truncated family $\Sigma_{B,\leq n} \to B$ is an object in $\mathcal{F}[n-1](B)$, it corresponds uniquely to a morphism $f : B \to F[n-1]$ which gives this figure:

\[
\begin{array}{ccc}
\Sigma_{B,n+1} & \to & \Sigma_{n,n} \\
\downarrow & & \downarrow \\
\Sigma_{B,n} & \xrightarrow{pr_2} & \Sigma_{n,n} \\
\downarrow & & \downarrow \\
p_n \Sigma_{B,\leq n-1} & \xrightarrow{r} & \Sigma_{n,\leq n-1} \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & F[n-1].
\end{array}
\]

We first claim that the map $B \xrightarrow{f} F[n-1]$ factors as $B \xrightarrow{f_W} W \xrightarrow{\Pi_n} F[n-1]$. This is true because of the following equivalence of maps:

\[
[B \xrightarrow{f} F[n-1]] = [B \xrightarrow{id} B \xrightarrow{f} F[n-1]] \\
= [B \xrightarrow{p_n} \Sigma_{B,n} \to B \xrightarrow{f} F[n-1]] \\
= [B \xrightarrow{p_n} \Sigma_{B,n} \xrightarrow{pr_2} \Sigma_{n,n} \xrightarrow{\Pi_n} F[n-1]] \\
= [B \xrightarrow{f_W} W \xrightarrow{\Pi_n} F[n-1]].
\]

Since the morphism $B \xrightarrow{f} F[n-1]$ factors as $B \xrightarrow{f_W} W \xrightarrow{\Pi_n} F[n-1]$, we can use Lemma [2.3] repeatedly, from the bottom up, to recover the maps $\Sigma_i \to \Sigma_{W,i}$, for all $i \leq n$, which make every rectangle in the figure below cartesian, and every triangle commutative:

\[
\begin{array}{ccc}
\Sigma_{B,n+1} & \to & \Sigma_{W,n+1} \\
\downarrow & & \downarrow \\
\Sigma_{B,\leq n} & \xrightarrow{f_W} & \Sigma_{n,\leq n} \\
\downarrow & & \downarrow \\
\Sigma_{W,\leq n} & \to & W \\
\downarrow & & \downarrow \\
B & \xrightarrow{f_W} & F[n-1] \\
\downarrow & & \downarrow \\
W & \xrightarrow{\Pi_n} & F[n-1].
\end{array}
\]

Lastly, we need to show that the section $p_n : B \to \Sigma_{B,n}$ is the pullback along $f_W$ of the diagonal embedding $p_{W,n} = \Delta : W \to \Sigma_{W,n} = W \times_{F[n-1]} W$. This follows from the figure below: by definition, $f_W$ is the composition $B \xrightarrow{p_n} \Sigma_{B,n} \to \Sigma_{W,n} = W \times_{F[n-1]} W \to \Sigma_{n,n} = W$, so all maps in the figure commute as expected, and $p_n$ is indeed the pullback of $p_{W,n} = \Delta$. By Lemma [2.2] we obtain that $\Sigma_{B,n+1}$ is the pullback of $\Sigma_{W,n+1}$, and the proof of (i) is complete:
For part (ii) we show inductively that $F[n]$ and all the schemes in its universal family are smooth and projective over $	ext{Spec } R$. As the base case, recall that $F[0] \cong \text{Spec } R$ and $\Sigma_{1,1} \cong S_1$ have this property. Inductively, assume that $F[n-1]$ and all the schemes in its universal family are smooth and projective over $	ext{Spec } R$. From the arguments above, $F[n]$ can be identified with the top scheme $\Sigma_{n,n}$ over $F[n-1]$, hence it is also smooth and projective. The first scheme in the universal family over $F[n]$ is, by definition, $\Sigma_{n+1,1} \cong F[n] \times S_1$. By construction, all the maps $\Sigma_{n+1,i} \to F[n]$ are smooth (this is an application of Lemma 2.2), which means the corresponding sections $\sigma_{n+1,i} : F[n] \to \Sigma_{n+1,i}$ are regular embeddings. Going up in the tower of morphisms, we can conclude step by step that $\Sigma_{n+1,2}, \ldots, \Sigma_{n+1,n+1}$ are smooth and projective, since each of them is obtained by blowing up a smooth projective scheme along a smooth projective subscheme. This concludes claim (ii).

Lastly, we need to show that $\Pi_n : \Sigma_{n,n} \cong F[n] \to F[n-1]$ corresponds to the forgetful functor $F[n] \to F[n-1]$. For any $R$-scheme $B$ and any morphism $f : B \to F[n]$, let $\Sigma_{B,\leq n+1} \to B$ be its corresponding family in $F[n](B)$. The map $\Pi_n \circ f : B \to F[n-1]$ corresponds to the truncated family $\Sigma_{B,\leq n} \to B$ in $F[n-1](B)$, concluding part (iii):

\[
\begin{array}{cccc}
\Sigma_{n+1} & \to & \Sigma_{n+1,n+1} \\
\downarrow & & \downarrow \\
\Sigma_{\leq n} & \to & \Sigma_{n+1,\leq n} & \to & \Sigma_{n,\leq n} \\
\downarrow & & \downarrow & & \downarrow \\
B & \to & F[n] & \xrightarrow{\Pi_n} & F[n-1].
\end{array}
\]

\[\square\]

Remark 2.5. As a consequence of the construction in the proof of Theorem 2.4, we obtain the following ascending ladder, where each square is cartesian:
\[ \ldots \rightarrow \Sigma_{5,5} \]  
\[ \pi_{5,5} \downarrow \]  
\[ \ldots \rightarrow \Sigma_{5,4} \rightarrow \Sigma_{4,4} \]  
\[ \pi_{5,4} \downarrow \pi_{4,4} \]  
\[ \ldots \rightarrow \Sigma_{5,3} \rightarrow \Sigma_{4,3} \rightarrow \Sigma_{3,3} \]  
\[ \pi_{5,3} \downarrow \pi_{4,3} \downarrow \pi_{3,3} \]  
\[ \ldots \rightarrow \Sigma_{5,2} \rightarrow \Sigma_{4,2} \rightarrow \Sigma_{3,2} \rightarrow \Sigma_{2,2} \]  
\[ \pi_{5,2} \downarrow \pi_{4,2} \downarrow \pi_{3,2} \downarrow \pi_{2,2} \]  
\[ \ldots \rightarrow \Sigma_{5,1} \rightarrow \Sigma_{4,1} \rightarrow \Sigma_{3,1} \rightarrow \Sigma_{2,1} \rightarrow \Sigma_{1,1} \]  
\[ \pi_{5,1} \downarrow \pi_{4,1} \downarrow \pi_{3,1} \downarrow \pi_{2,1} \downarrow \pi_{1,1} \]  
\[ \ldots \rightarrow F[4] \rightarrow F[3] \rightarrow F[2] \rightarrow F[1] \rightarrow F[0]. \]

In particular, since \( \Sigma_{n,n} \cong F[n] \), for all \( n \geq 1 \), we obtain the following identification:

(1) \[ \Sigma_{n+1,i} \cong F[n] \times_{F[i-1]} F[i], \forall 1 \leq i \leq n + 1. \]

In light of Equation (1), we look back to the universal family over \( F[n] \) and give another description of the projection maps \( \pi_{n+1,*} \) and the sections \( \sigma_{n+1,*} \). Before we do so, we need to establish some notation:

**Notation 2.6.** Let \( B \) be an \( R \)-scheme. A family \((\Sigma_{B,i}, \pi_i, p_i)_{i=1}^n\) in \( F[n](B) \) will simply be denoted as \((p_1, \ldots, p_n)\). Similarly, a \( B \)-point of \( \Sigma_{n+1,i} \cong F[n] \times_{F[i-1]} F[i] \) will be denoted by \((p_1, \ldots, p_n; \mathbf{p}_i)\), with the understanding that \((p_1, \ldots, p_n)\) is the corresponding point in \( F[n] \) and \((p_1, \ldots, p_{i-1}, p_i')\) is the one in \( F[i] \).

**Proposition 2.7.** Let \( B \) be a \( R \)-scheme and \((\Sigma_{n+1,i}, \pi_{n+1,i}, \sigma_{n+1,i})_{i=1}^n\) the universal family over the moduli scheme \( F[n] \). Using Notation 2.6 above, the morphisms \( \pi_{n+1,*} \) and \( \sigma_{n+1,*} \) map \( B \)-points as follows:

(a) \( \sigma_{n+1,i} : F[n](B) \rightarrow \Sigma_{n+1,i}(B) \) maps \((p_1, \ldots, p_n)\) to \((p_1, \ldots, p_n; p_i)\);

(b) \( \pi_{n+1,i} : \Sigma_{n+1,i}(B) \rightarrow \Sigma_{n+1,i-1}(B) \) maps \((p_1, \ldots, p_n; p_i')\) to \((p_1, \ldots, p_n; \mathbf{p}_i')\), where \( \mathbf{p}_i' : B \rightarrow \Sigma_{B,i-1} \) is the image of \( p_i' \) under the projection map \( \Sigma_{B,i} \rightarrow \Sigma_{B,i-1} \).
Proof. (a) Let $f : F[n] \to F[i]$ be the projection morphism which maps $B$-points $(p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_i)$. We have the following diagram:

$$
\begin{array}{ccc}
F[n] & \xrightarrow{\sigma_{n+1,i}} & \Sigma_{n+1,i} = F[n] \times_{F[i-1]} F[i] \\
\downarrow \Delta \circ f & & \downarrow f \\
F[n] & \xrightarrow{\text{id}} & F[i],
\end{array}
$$

where the section $\sigma_{n+1,i} : F[n] \to \Sigma_{n+1,i}$ is defined to be the unique morphism making the two triangles of the diagram commute. It is therefore clear that $\sigma_{n+1,i}$ maps a $B$-point $(p_1, \ldots, p_n)$ of $F[n]$ to $(p_1, \ldots, p_i) \in \Sigma_{n+1,i}(B)$.

(b) We start by showing that the morphism $\pi_{n+1,n+1} : \Sigma_{n+1,n+1} \to \Sigma_{n+1,n}$ maps $B$-points $(p_1, \ldots, p_n, p_{n+1}) \mapsto (p_1, \ldots, p_{n-1}, p_{n+1})$, where $p_{n+1}$ is the image of $p_{n+1}$ under the projection map $\Sigma_{B,n+1} \to \Sigma_{B,n}$. Since $\Sigma_{n+1,n} = F[n] \times_{F[n-1]} F[n]$, it is enough to show that:

$$
\begin{align*}
\text{pr}_1 \circ \pi_{n+1,n+1} : (p_1, \ldots, p_{n+1}) & \mapsto (p_1, \ldots, p_{n-1}, p_n), \\
\text{pr}_2 \circ \pi_{n+1,n+1} : (p_1, \ldots, p_{n+1}) & \mapsto (p_1, \ldots, p_{n-1}, p_{n+1}).
\end{align*}
$$

The map $\text{pr}_1 \circ \pi_{n+1,n+1} = \Pi_{n+1} : \Sigma_{n+1,n+1} \to F[n]$ corresponds to the forgetful functor $F[n+1] \to F[n]$ (see Theorem 2.4), so the first claim above is true. To show the second identity, let $f_{n+1} : B \to F[n+1]$ be such a point and $f_n = \Pi_{n+1} \circ f_{n+1} : B \to F[n]$ be the ‘truncated’ point. The maps $f_{n+1}$ and $f_n$ produce the following diagram, where $p_{n+1}$ is the pullback of $f_{n+1}$, and $p_1, \ldots, p_n$ are the pullbacks of $\sigma_{n+1,1}, \ldots, \sigma_{n+1,n}$, respectively:
Notice the following maps are equivalent:

\[
\begin{align*}
[B \xrightarrow{f_{n+1}} \Sigma_{n+1,n+1} & \xrightarrow{\pi_{n+1,n+1}} \Sigma_{n+1,n} \xrightarrow{pr_2} \Sigma_{n,n}] = \\
& = [B \xrightarrow{p_{n+1}} \Sigma_{B,n+1} \to \Sigma_{n+1,n+1} \to \Sigma_{n+1,n} \to \Sigma_{n,n}] \\
& = [B \xrightarrow{p_{n+1}} \Sigma_{B,n+1} \to \Sigma_n \to \Sigma_{n+1,n} \to \Sigma_{n,n}] \\
& = [B \xrightarrow{p_{n+1}} \Sigma_{B,n} \to \Sigma_{n+1,n} \to \Sigma_{n,n}].
\end{align*}
\]

By the equivalence above, \( pr_2 \circ \pi_{n+1,n+1} \) maps a \( B \)-point \((p_1, \ldots, p_{n+1})\) in \( \Sigma_{n+1,n+1} \) to \((p_1, \ldots, p_{n-1}, \overline{p_{n+1}})\) in \( \Sigma_{n,n} \), and this concludes the proof of our initial statement.

The behavior of the more general map \( \pi_{n+1,i} \) becomes clear from the following cartesian square, in which the bottom horizontal maps is induced by the forgetful map \( F[n] \to F[i-1] \):

\[
\begin{array}{ccc}
\Sigma_{n+1,i} & \cong & F[n] \times_{F[i-1]} F[i] \\
\downarrow \pi_{n+1,i} & & \downarrow \pi_{i,i} \\
\Sigma_{n,i-1} & \cong & F[n] \times_{F[i-2]} F[i-1] \to F[i-1] \times_{F[i-2]} F[i-1].
\end{array}
\]

\[\square\]

**Notation 2.8.** As a result of Proposition 2.7, we change notation and denote the section \( \sigma_{n+1,i} \) as \( \Delta_{i,n+1} \). We will use this notation throughout the rest of the paper.

### 3. Divisors of the moduli scheme

**Definition 3.1.** The moduli scheme \( F[n] \) comes equipped with divisors \( D_{i,j}^{(n)} \), \( \forall 1 \leq i < j \leq n \), which arise naturally from the construction outlined in the previous section. We start by defining the divisors \( D_{1,n}^{(n)}, \ldots, D_{n-1,n}^{(n)} \). To do so, recall that for all \( 1 \leq i \leq n-1 \), \( \Sigma_{n,i} \) is obtained by blowing up the previous variety \( \Sigma_{n,i} \) along the locus \( \Delta_{i,n} \) \( (F[n-1]) \). We define \( D_{i,n}^{(n)} \) on \( \Sigma_{n,i+1} \) to be the exceptional divisor of this blowup:

\[
D_{i,n}^{(n)} \subset \text{Bl}_{\Delta_{i,n}} \Sigma_{n,i} = \Sigma_{n,i+1} \longrightarrow \Sigma_{n,i} \longrightarrow F[n-1].
\]

By abuse of notation, the divisor \( D_{i,n}^{(n)} \) on \( F[n] = \Sigma_{n,n} \) is the strict transform of the exceptional divisor coming from \( \Sigma_{n,i+1} \) in the tower of blowups \( \Sigma_{n,n} \to \Sigma_{n,n-1} \to \cdots \to \Sigma_{n,i+1} \).

We define the other divisors inductively as follows: given \( D_{i,j}^{(n-1)} \) on \( F[n-1] \), where \( j \leq n-1 \), let \( D_{i,j}^{(n)} = D_{i,j}^{(n-1)} \times S_1 \) be the divisor on \( \Sigma_{n,1} = F[n-1] \times S_1 \). By abuse of notation, \( D_{i,j}^{(n)} \subset F[n] = \Sigma_{n,n} \) is defined as the strict transform of this divisor in the tower of blowups \( \Sigma_{n,n} \to \Sigma_{n,n-1} \to \cdots \to \Sigma_{n,1} \).
Proposition 3.2. For every $n \geq 1$, the natural projection map $F[n] \to S^n_1$ is a composition of blowups. Under this map, every divisor $D^{(n)}_{i,j}$ is mapped surjectively onto the diagonal $\Delta_{i,j}$.

Proof. In the previous section, we constructed a tower of blowups:

$$F[n] = \Sigma_{n,n} \xrightarrow{\pi_{n,n}} \Sigma_{n,n-1} \xrightarrow{\pi_{n,n-1}} \cdots \xrightarrow{\pi_{n,2}} \Sigma_{n,1} = F[n-1] \times S_1.$$  

Inductively, using that $F[0] \cong \text{Spec } R$, we obtain a morphism $F[n] \to S^n_1$ that decomposes as a series of blowups. Given the behavior of the maps $\pi_{n,i}$, outlined in Proposition 2.7, this map coincides with the natural projection morphism.

We show inductively on $n$ that the projection morphism maps surjectively any divisor $D^{(n)}_{i,j}$ onto the diagonal $\Delta_{i,j}$. As the base case, recall that $F[2]$ is isomorphic to $\text{Bl}_{\Delta_{1,2}}(S_1 \times S_1)$, and $D^{(2)}_{1,2}$ is the exceptional divisor of this blowup. For the induction step, consider a divisor $D^{(n)}_{i,j}$ of the moduli space $F[n]$. If $j < n$, then, by construction, the morphism $\Sigma_{n,n} \to \Sigma_{n,1} = F[n-1] \times S_1$ maps $D^{(n)}_{i,j} \to D^{(n-1)}_{i,j} \times S_1$, so the conclusion follows. If $j = n$, the divisor $D^{(n)}_{i,n}$ on $F[n] = \Sigma_{n,n}$ is the strict transform of the exceptional divisor:

$$D^{(n)}_{i,n} \subset \text{Bl}_{\Delta_{i,n}} \Sigma_{n,i} = \Sigma_{n,i+1} \xrightarrow{\pi_{n,i}} \Sigma_{n,i} \xrightarrow{\Delta_{i,n}} F[n-1].$$

Given the behavior of the maps $\pi_{n,i}$, outlined in Proposition 2.7, the projection morphism $F[n] = \Sigma_{n,n} \to \Sigma_{n,i} \to S^n_1$ maps $D^{(n)}_{i,n} \subset \Sigma_{n,n}$ onto the blowup locus $\Delta_{i,n} (F[n-1]) \subset \Sigma_{n,i}$, which is then mapped onto the diagonal $\Delta_{i,n} \subset S^n_1$, and the proof is complete. \hfill $\square$

Proposition 3.3. (a) Let $1 \leq i < j < n$, $1 \leq k \leq n$. The divisor $D^{(n)}_{i,j} \subset \Sigma_{n,k}$ is the inverse image of $D^{(n-1)}_{i,j} \subset F[n-1]$ under the projection map $\Sigma_{n,k} \to F[n-1]$.

(b) Let $1 \leq i < k \leq n$. The divisor $D^{(n)}_{i,n} \subset \Sigma_{n,k}$ is the inverse image of the exceptional divisor $D^{(n)}_{i,n} \subset \Sigma_{n,i+1}$ under the projection map $\Sigma_{n,k} \to \Sigma_{n,i+1}$.

Proof. (a) When $k = 1$, we defined the divisor on $\Sigma_{n,1} = F[n-1] \times S_1$ to be $D^{(n)}_{i,j} = D^{(n-1)}_{i,j} \times S_1$, so the conclusion holds. When $k > 1$, recall that we have the following figure:

$$\Sigma_{n,k+1} = \text{Bl}_{\Delta_{k,n}} \Sigma_{n,k} \xrightarrow{\pi_{n,k+1}} \Sigma_{n,k} \xrightarrow{\Delta_{k,n}} F[n-1].$$

The divisor $D^{(n)}_{i,j} \subset \Sigma_{n,k+1}$ is the strict transform of the divisor with the same name coming from $\Sigma_{n,k}$. We claim that the strict transform coincides with the total transform. By construction, the moduli space $F[n-1]$ is irreducible and the map $\Delta_{k,n}$ is a regular embedding, so the exceptional divisor $D^{(n)}_{k,n}$ of the blowup $\Sigma_{n,k+1} \to \Sigma_{n,k}$ is also irreducible. Hence, the only way statement (a) could fail is if the exceptional
divisor $D^{(n)}_{k,n} \subset \Sigma_{n,k+1}$ is completely contained in the inverse image $\pi^{-1}_{n,k+1}(D^{(n)}_{i,j})$, which could happen only if the blowup locus $\Delta_{k,n}$ $(F[n-1]) \subset \Sigma_{n,k}$ was completely contained in $D^{(n)}_{i,j}$. However, the projection $\Sigma_{n,k} \to F[n-1]$ maps $\Delta_{k,n}$ $(F[n-1]) \to F[n-1]$ and $D^{(n)}_{i,j} \to D^{(n-1)}_{i,j}$, so the induction step is complete and claim (a) is correct.

(b) When $k > i + 1$, we have a similar picture as before:

$$\Sigma_{n,k+1} = \text{Bl}_{\Delta_{k,n}} \Sigma_{n,k} \xrightarrow{\pi_{n,k+1}} \Sigma_{n,k} \xrightarrow{\pi_{n,k}} F[n-1].$$

The divisor $D^{(n)}_{i,n} \subset \Sigma_{n,k+1}$ is defined to be the strict transform of the divisor with the same name coming from $\Sigma_{n,k}$. Using the same argument as in (a), it is enough to prove that the blowup locus $\Delta_{k,n}$ $(F[n-1])$ is not fully contained in $D^{(n)}_{i,n}$. We prove this inductively over $k = i + 1, \ldots, n$.

When $k = i + 1$, a $B$-point of $D^{(n)}_{i,n} \subset \Sigma_{n,i+1}$ is of the form $(p_1, \ldots, p_{n-1}; p'_{i+1})$, where $p'_{i+1} \in \Sigma_{i+1}$ maps to $p_i$ under the projection map $\pi_{i,n+1} : \Sigma_{B,i+1} \to \Sigma_{B,i}$, and a $B$-point of $\Delta_{i+1,n}$ $(F[n-1]) \subset \Sigma_{n,i+1}$ is of the form $(p_1, \ldots, p_{n-1}; p_{i+1})$. Clearly $\Delta_{i+1,n}$ $(F[n-1])$ is not fully contained in $D^{(n)}_{i,n}$. In the induction step, we know that $D^{(n)}_{i,n} \subset \Sigma_{n,k}$ is the inverse image of the exceptional divisor of the blowup $\Sigma_{n,i+1} \to \Sigma_{n,i}$, hence a $B$-point of $D^{(n)}_{i,n} \subset \Sigma_{n,k}$ can be summarized as $(p_1, \ldots, p_{n-1}; p_k')$, where $p_k' \in \Sigma_{B,k}$ maps to $p_k$ under the projection map $\pi_{B,k} : \Sigma_{B,k} \to \Sigma_{B,i}$. We conclude $\Delta_{k,n}$ $(F[n-1]) \notin D^{(n)}_{i,n}$ and the proof is complete.

**Proposition 3.4.** Let $1 \leq i < j \leq n$ and $D^{(n)}_{i,j}$ be a divisor of $F[n]$ as defined above. A $B$-point of $F[n]$, denoted by a tuple $(p_1, p_2, \ldots, p_n)$ consistent with Notation 2.6, is a point of $D^{(n)}_{i,j}$ if and only if the projection $\pi_{B,j} : \Sigma_{B,j} \to \Sigma_{B,i}$ maps the point $p_j$ to the point $p_i$.

**Proof.** When $j < n$, $D^{(n)}_{i,j}$ is the inverse image of $D^{(n-1)}_{i,j}$ under the forgetful map $F[n] \to F[n-1]$. Hence we can assume without loss of generality that $j = n$. By Proposition 3.3 the divisor $D^{(n)}_{i,n} \subset F[n] = \Sigma_{n,n}$ is the inverse image of the blowup locus $\Delta_{i,n}$ $(F[n-1]) \subset \Sigma_{n,i}$ under the projection map $\pi_{n,n} : \Sigma_{n,n} \to \Sigma_{n,i}$:

$$D^{(n)}_{i,n} = \text{pr}^{-1}(\Delta_{i,n} (F[n-1])) \subset \Sigma_{n,n} \xrightarrow{\text{pr}} \Sigma_{n,i} \xrightarrow{\pi_{n,i}} F[n-1].$$

By Proposition 2.7 the morphism $\pi_{n,n} : \Sigma_{n,n} \to \Sigma_{n,i}$ maps points $(p_1, \ldots, p_{n-1}; p_n) \mapsto (p_1, \ldots, p_{n-1}; \pi_n)$, where $\pi_n \in \Sigma_{B,i}$ is the projection of $p_n$ under the map $\pi_{B,n} : \Sigma_{B,n} \to \Sigma_{B,i}$. On the other hand, the section $\Delta_{i,n}$: $F[n-1] \to \Sigma_{n,i}$ maps points $(p_1, \ldots, p_{n-1}; p_n)$ to $(p_1, \ldots, p_{n-1}; p_i)$. In conclusion, a $B$-point $F[n]$ is a point of $D^{(n)}_{i,n}$ if and only if $p_n \to p_i$ under the map $\pi_{B,n} : \Sigma_{B,n} \to \Sigma_{B,i}$.

$\square$
**Proposition 3.5.** Let $1 \leq i < j \leq n$. The divisor $D_{i,j}^{(n)} \subset F[n]$ is smooth over $\text{Spec } R$.

**Proof.** We prove this statement inductively over $n$. When $n = 2$, we have only one divisor of this form, namely $D_{1,2}^{(2)}$, which is the exceptional divisor of the blowup of $S_1 \times S_1$ along the diagonal, so it is clearly smooth. Inductively, we need to analyze two cases: either $j < n$ or $j = n$.

Assume $j < n$. We show inductively over $k$ that $D_{i,j}^{(n)} \subset \Sigma_{n,k}$ is smooth. As the base case $k = 1$, recall that the divisor $D_{i,j}^{(n)} \subset \Sigma_{n,1} = F[n-1] \times S_1$ is defined to be $D_{i,j}^{(n-1)} \times S_1$, hence it is smooth by the induction hypothesis. Now assume $D_{i,j}^{(n)} \subset \Sigma_{n,k}$ is smooth. We know that $\Sigma_{n,k+1} = \text{Bl}_{\Delta_{k,n}} \Sigma_{n,k}$ and $D_{i,j}^{(n)} \subset \Sigma_{n,k+1}$ is the strict transform of the divisor with the same name coming from $\Sigma_{n,k}$. We claim that smoothness is preserved because the intersection of this divisor with the blowup locus $D_{i,j}^{(n)} \cap \Delta_{k,n} (F[n-1])$ inside $\Sigma_{n,k}$ is itself smooth. In fact, we show it satisfies the following isomorphism:

$$D_{i,j}^{(n)} \cap \Delta_{k,n} (F[n-1]) \cong D_{i,j}^{(n-1)}.$$ 

As a result of Proposition 2.7 and Proposition 3.3, a $B$-point of $D_{i,j}^{(n)} \subset \Sigma_{n,k}$ has the form $(p_1, \ldots, p_{n-1}; p'_k)$, where $p_j \to p_i$ under the projection map $\Sigma_{B,j} \to \Sigma_{B,i}$, and a $B$-point of $\Delta_{k,n} (F[n-1]) \subset \Sigma_{n,k}$ has the form $(p_1, \ldots, p_{n-1}; p_k)$. In conclusion, a $B$-point of $D_{i,j}^{(n)} \cap \Delta_{k,n} (F[n-1])$ can be summarized as $(p_1, \ldots, p_{n-1}; p_k)$, where $p_j \to p_i$ under the projection map $\Sigma_{B,j} \to \Sigma_{B,i}$. This means the functors of points of $D_{i,j}^{(n)} \cap \Delta_{k,n} (F[n-1])$ and $D_{i,j}^{(n-1)}$ are isomorphic, hence the two schemes are isomorphic, proving our claim.

Now assume that $j = n$. We show inductively that $D_{i,n}^{(n)} \subset \Sigma_{n,k}$ is smooth, for all $i + 1 \leq k \leq n$. When $k = i + 1$, $D_{i,n}^{(n)} \subset \Sigma_{n,i+1}$ is the exceptional divisor of the blowup of $\Sigma_{n,i}$ along the locus $\Delta_{i,n} (F[n-1])$. By Theorem 2.4, $\Sigma_{n,i}$ is smooth and the blowup locus $\Delta_{i,n} (F[n-1])$ is regularly embedded, so this exceptional divisor $D_{i,n}^{(n)} \subset \Sigma_{n,i+1}$ is smooth. For the induction step we show, as before, that the intersection $D_{i,n}^{(n)} \cap \Delta_{k,n} (F[n-1]) \subset \Sigma_{n,k}$ satisfies the following isomorphism, and hence it is smooth:

$$D_{i,n}^{(n)} \cap \Delta_{k,n} (F[n-1]) \cong D_{i,k}^{(n-1)}.$$ 

As a result of Proposition 2.7 and Proposition 3.3, a $B$-point of $D_{i,n}^{(n)} \subset \Sigma_{n,k}$ has the form $(p_1, \ldots, p_{n-1}; p'_k)$, where $p'_k \to p_i$ under the projection map $\Sigma_{B,k} \to \Sigma_{B,i}$, and a $B$-point of $\Delta_{k,n} (F[n-1]) \subset \Sigma_{n,k}$ has the form $(p_1, \ldots, p_{n-1}; p_k)$. In conclusion, a $B$-point of $D_{i,n}^{(n)} \cap \Delta_{k,n} (F[n-1])$ can be identified as $(p_1, \ldots, p_{n-1}; p_k)$, where $p_k \to p_i$ under the projection map $\Sigma_{B,k} \to \Sigma_{B,i}$. This means the functors of points of $D_{i,n}^{(n)} \cap \Delta_{k,n} (F[n-1])$ and $D_{i,k}^{(n-1)}$ are isomorphic, hence the two schemes are isomorphic. □
4. The Chow ring of the moduli scheme

In this section we assume that the underlying ring $R$ is an algebraically closed field $k$. In this new setup, the moduli space $F[n]$ and all the schemes in its universal family are smooth projective varieties.

The main result here is that the Chow ring of the moduli space $F[n]$ is generated by the classes of the divisors $\{D_{i,j}^{(n)}\}_{1 \leq i < j \leq n}$ over the Chow ring $\mathbb{A}^*(S^n)$. We conclude the section by proving certain key relations among these classes in the Chow ring $\mathbb{A}^*(F[n])$. In the next and final section, we prove that these relations are sufficient to give a precise description of the Chow ring $\mathbb{A}^*(F[n])$ in the special case where $S_1$ is a rational surface and the base field is the complex numbers $\mathbb{C}$.

We start with a theorem by Keel, which is the key ingredient in our proof:

**Theorem 4.1.** Let $Y$ be a variety and let $i : X \hookrightarrow Y$ be a regularly embedded subvariety. Let $\tilde{Y}$ be the blowup of $Y$ along $X$. Suppose the map of bivariate rings $i^* : A^*(Y)[T] \to A^*(X)$ is surjective. Then:

$$A^*(\tilde{Y}) \cong \frac{A^*(Y)[T]}{(P(T), T \cdot \ker(i^*))},$$

where $P(T) \in A^*(Y)[T]$ is any polynomial whose constant term is $[X]$ and whose restriction to $A^*(X)$ is the Chern polynomial of the normal bundle $N = N_X Y$, i.e.:

$$i^*P(T) = T^d + T^{d-1}c_1(N) + \cdots + c_{d-1}(N)T + c_d(N),$$

where $d = \text{codim}(X,Y)$. This isomorphism is induced by

$$\pi^* : A^*(Y)[T] \to A^*(\tilde{Y}),$$

and by sending $-T$ to the class of the exceptional divisor.

**Proof.** See [4], Appendix, Theorem 1. □

**Remark 4.2.** The moduli space $F[n]$ is a smooth projective variety, for any $n \geq 1$, hence its bivariate ring $\mathbb{A}^*(F[n])$ is isomorphic to its Chow ring $\mathbb{H}^*(F[n])$ (see [3], Ch. 17).

**Corollary 4.3.** Let $1 \leq i < n$. Let $\Sigma_{n,i+1}$ and $\Sigma_{n,i}$ be two of the varieties in the universal family over the moduli variety $F[n-1]$. The Chow ring of $\Sigma_{n,i+1}$ has the following description:

$$A^*(\Sigma_{n,i+1}) \cong \frac{A^*(\Sigma_{n,i})[D_{i,n}^{(n)}]}{\langle P_{i,n}(-D_{i,n}^{(n)}), D_{i,n}^{(n)} \cdot \ker(\Delta_{i,n}^*) \rangle},$$

where $P_{i,n}$ is a quadratic polynomial with coefficients in $\mathbb{A}^*(\Sigma_{n,i})$.

**Proof.** In the universal family over the moduli variety $F[n-1]$, $\Sigma_{n,i+1}$ is the blowup of $\Sigma_{n,i}$ along the locus $\Delta_{i,n}$ ($F[n-1]$), and the corresponding exceptional divisor is $D_{i,n}^{(n)}$. By Theorem 2.4, $\Sigma_{n,i}$ is a variety and $\Delta_{i,n} : F[n-1] \hookrightarrow \Sigma_{n,i}$ is a regularly embedded subvariety. Additionally, $\Delta_{i,n}$ is a section of the projection $\Sigma_{n,i} \to F[n-1]$, hence
the corresponding map on Chow rings $\Delta_{i,n}^* : \mathbb{A}^*(\Sigma_{n,i}) \to \mathbb{A}^*(F[n-1])$ is surjective. The conclusion follows immediately as an application of Theorem 4.1.

**Remark 4.4.** By Proposition 3.3 the induced map on Chow rings: $\pi_{n,i+1}^* : \mathbb{A}^*(\Sigma_{n,i}) \to \mathbb{A}^*(\Sigma_{n,i+1})$ sends the class of any divisor $D_{j,k}^{(n)}$ to the class of the divisor with the same name inside $\mathbb{A}^*(\Sigma_{n,i})$. This means that the notation remains consistent in all statements of this section.

**Theorem 4.5.** With notation as in Corollary 4.3 the Chow ring of the moduli space $F[n]$ is as follows:

$$\mathbb{A}^*(F[n]) \cong \frac{\mathbb{A}^*(S_1^n)[D_{i,j}^{(n)}]_{1 \leq i < j \leq n}}{\langle D_{i,j}^{(n)} \cdot \ker(\Delta_{i,j}^*), P_{i,j}(-D_{i,j}^{(n)})_{1 \leq i < j \leq n} \rangle}.$$ 

**Proof.** To generalize Corollary 4.3 we first need to give an alternative way of defining the divisors $D_{i,j}^{(n)} \subset F[n]$. Recall that the natural projection map $F[n] \to S_1^n$ decomposes as a series of blowups; more specifically, we encounter the following situation:

$$F[n] = \Sigma_{n,n} \to \ldots \to \Sigma_{j,i+1} \times S_1^{n-j} \to \Sigma_{j,i} \times S_1^{n-j} \to \ldots \to F[j] \times S_1^{n-j} \to \ldots \to S_1^n.$$ 

The scheme $\Sigma_{j,i+1} \times (S_1)^{n-j}$ is the blowup of $\Sigma_{j,i} \times (S_1)^{n-j}$ along the locus $\Delta_{i,j} \times \text{id}$. We define $D_{i,j}^{(n)} \subset \Sigma_{j,i+1} \times (S_1)^{n-j}$ to be the exceptional divisor of this blowup. By abuse of notation, we define $D_{i,j}^{(n)} \subset F[n] = \Sigma_{n,n}$ to be the strict transform of this exceptional divisor in the tower of blowups. Following an identical argument as the one in Proposition 3.3, more is true: $D_{i,j}^{(n)} \subset F[n]$ is actually the full inverse image of the exceptional divisor in the tower of blowups. Therefore, by Theorem 4.1, we conclude that:

$$\mathbb{A}^*(\Sigma_{j,i+1} \times S_1^{n-j}) \cong \frac{\mathbb{A}^*(\Sigma_{j,i} \times S_1^{n-j})[D_{i,j}^{(n)}]}{\langle P_{i,j}(-D_{i,j}^{(n)}), D_{i,j}^{(n)} \cdot \ker(\Delta_{i,j}^*) \rangle},$$

where $P_{i,j}$ is a quadratic polynomial with coefficients in $\mathbb{A}^*(\Sigma_{j,i} \times (S_1)^{n-j})$.

Applying this procedure step by step from $S_1^n$ all the way up to $\Sigma_{n,n} = F[n]$, we immediately obtain the desired formula for the Chow ring of the moduli variety $F[n]$.

**Proposition 4.6.** Let $d$ be a divisor class of $S_1$ and $d_i^* \in \mathbb{A}^*(F[n])$ the image of $d$ under the composed projection $F[n] \to S_1^n \stackrel{pr_1}{\to} S_1$. The following relations hold in the Chow ring $\mathbb{A}^*(F[n])$:

(i) $D_{i,j}^{(n)}d_i^* = D_{i,j}^{(n)}d_j^*$, $\forall 1 \leq i < j \leq n$;

(ii) $D_{i,j}^{(n)}D_{i,k}^{(n)} = D_{i,k}^{(n)}D_{i,j}^{(n)}$, $\forall 1 \leq i < j < k \leq n$.

**Proof.** (i) This is true because the natural projection $F[n] \to S_1^n$ maps $D_{i,j}^{(n)}$ surjectively onto the diagonal $\Delta_{ij}$, for all $1 \leq i < j \leq n$. 


(ii) Intuitively, the relation holds because the left-hand side parametrizes tuples \((p_1, \ldots, p_n)\) in which \(p_j \mapsto p_i\) and \(p_k \mapsto p_j\), while the right-hand side parametrizes tuples in which \(p_k \mapsto p_i\) and \(p_k \mapsto p_j\).

We first reduce the proof to the case where \(k = n\). If \(k < n\), we can assume inductively that a similar relation holds in the Chow ring of \(F[n-1]\):

\[
D_{i,j}^{(n-1)} D_{j,k}^{(n-1)} = D_{i,k}^{(n-1)} D_{j,k}^{(n-1)}.
\]

Now recall that inside the variety \(\Sigma_{n,1} = F[n-1] \times S_1\), the divisors \(D_{i,j}^{(n)}, D_{i,k}^{(n)}, D_{j,k}^{(n)}\) are defined to be \(D_{i,j}^{(n-1)} \times S_1, D_{i,k}^{(n-1)} \times S_1, D_{j,k}^{(n-1)} \times S_1\), respectively, so relation (ii) holds inside the Chow ring \(\mathbb{A}^*(\Sigma_{n,1})\). By Remark 4.3, the same relation lifts to the Chow ring \(\mathbb{A}^*(F[S_1, n])\).

We are left to show that (ii) holds when \(k = n\). By Remark 4.4, it suffices to show it holds inside the Chow ring of \(\Sigma_{n,j+1}\). The Chow ring of \(\Sigma_{n,j+1}\) is given as follows:

\[
\mathbb{A}^*(\Sigma_{n,j+1}) \cong \frac{\mathbb{A}^*(\Sigma_{n,j}) [D_{j,n}^{(n)}]}{\langle P_{j,n}(-D_{j,n}^{(n)}), D_{j,n}^{(n)}, \ker(\Delta_{j,n}^*) \rangle},
\]

hence we need to show that:

\[
D_{i,j}^{(n)} - D_{i,n}^{(n)} \in \ker(\Delta_{j,n}^*; \mathbb{A}^*(\Sigma_{n,j}) \to \mathbb{A}^*(F[n-1])).
\]

Consider first the following diagram:

\[
D_{i,n}^{(n)} \subset \Sigma_{n,i+1} \xrightarrow{r} \Sigma_{j,i+1} \supset D_{i,j}^{(j)}
\]

\[
\Sigma_{n,i} \xrightarrow{r} \Sigma_{j,i} \quad \Delta_{i,n}
\]

\[
F[n-1] \longrightarrow F[j-1].
\]

The bottom square is cartesian and the section \(\Delta_{i,n}\) is the pullback of \(\Delta_{i,j}\) along the projection map \(F[n-1] \to F[j-1]\), hence the exceptional divisor \(D_{i,n}^{(n)} \subset \Sigma_{n,i+1}\) is the pullback of the exceptional divisor \(D_{i,j}^{(j)} \subset \Sigma_{j,i+1}\). In other words, \(D_{i,n}^{(n)} \cong F[n-1] \times_{F[j-1]} D_{i,j}^{(j)}\) inside \(\Sigma_{n,i+1}\). This relation lifts to \(\Sigma_{n,j}\). On the other hand, we have that \(\Sigma_{n,j} \cong F[n-1] \times_{F[j-1]} F[j]\). By Proposition 3.3, under this isomorphism, \(D_{i,j}^{(n)} \cong D_{i,j}^{(n-1)} \times_{F[j-1]} F[j]\).

In conclusion, inside \(\Sigma_{n,j} \cong F[n-1] \times_{F[j-1]} F[j]\), we have \(D_{i,j}^{(n)} \cong D_{i,j}^{(n-1)} \times_{F[j-1]} F[j]\) and \(D_{i,n}^{(n)} \cong F[n-1] \times_{F[j-1]} D_{i,j}^{(j)}\). Additionally, the morphism \(\Delta_{j,n}: F[n-1] \to \Sigma_{n,j} = F[n-1] \times_{F[j-1]} F[j]\) acts like a ‘truncated’ diagonal embedding, so the proof is complete:

\[
D_{i,j}^{(n)} - D_{i,n}^{(n)} \in \ker(\Delta_{j,n}^*; \mathbb{A}^*(\Sigma_{n,j}) \to \mathbb{A}^*(F[n-1])).
\]
5. The Chow ring of the moduli scheme for rational surfaces

As a special case of the theory developed above, we give a precise description of the Chow ring $\mathbb{A}^*(F[n])$ when the base surface $S_1$ is a smooth projective rational surface over the complex numbers ($\text{Spec } R = \text{Spec } \mathbb{C}$). The result relies on a few key ideas. First, the canonical map $\text{cl}: \mathbb{A}^*(F[n]) \to \mathbb{H}^{2*}(F[n])$ is an isomorphism. Second, for some prime $p$ and any $q = p^l$, where $l \gg 0$, we can define the moduli space $F[n] \otimes \mathbb{F}_q$ over the finite field $\mathbb{F}_q$. The number of $\mathbb{F}_q$-points on $F[n] \otimes \mathbb{F}_q$ is given by a polynomial $R_n(q)$ that coincides with the Poincare polynomial of $F[n]$. We use this fact to derive precise formulas for the Betti numbers of $F[n]$; using these formulas, we show that the relations in Proposition 4.6 are enough to give a complete description of $\mathbb{A}^*(F[n])$.

Definition 5.1. A scheme $X$ of characteristic zero is called an HI (for Homology Isomorphism) scheme if the canonical map from the Chow groups of $X$ to the homology groups is an isomorphism:

$$\mathbb{A}^*_*(X) \xrightarrow{\text{cl}} \mathbb{H}^{2*}(X).$$

Proposition 5.2. Let $Y$ be a variety and $i : X \to Y$ a regularly embedded subvariety. Let $\tilde{Y}$ be the blowup of $Y$ along $X$. If $X$ and $Y$ are HI schemes, then so is $\tilde{Y}$.

Proof. See [4], Appendix, Theorem 2. □

Proposition 5.3. Let $S_i$ be a smooth projective variety over an algebraically closed field $k$ of characteristic zero. If $S_i^n$ is an HI scheme, for all $n \geq 0$, then so is $F[n]$.

Proof. We prove inductively over $n$ something stronger: $\forall n, j \geq 0$, the variety $F[n] \times S_j^i$ is an HI scheme. The base cases follow immediately, since $F[0] \cong \text{Spec } k$ and $F[1] \cong S_1$.

For the induction step, assume that $F[n] \times S_j^i$ is an HI scheme, for all $j \geq 0$. Recall that we can obtain $F[n + 1]$ from $F[n] \times S_1$ as a series of blowups:

$$F[n + 1] = \Sigma_{n+1,n+1} \to \Sigma_{n+1,n} \to \ldots \to \Sigma_{n+1,2} \to \Sigma_{n+1,1} = F[n] \times S_1$$

By the induction hypothesis, $F[n]$ and $F[n] \times S_1$ are HI varieties. By Theorem 2.4, the blowup locus $\Delta_i,n+1$: $F[n] \to \Sigma_{n+1,i}$ is a regular embedding, for all $1 \leq i \leq n$. Applying Proposition 5.2 repeatedly, we conclude step by step that $\Sigma_{n+1,2}, \Sigma_{n+1,3}, \ldots, \Sigma_{n+1,n+1} \cong F[n + 1]$ are all HI varieties. More generally, we conclude that $F[n + 1] \times S_j^i$ is an HI variety by considering a similar figure to the one above. □

Proposition 5.4. Let $k \geq 1$ and $S_1, \ldots, S_k$ be complex smooth projective rational surfaces. Then $\prod_{i=1}^k S_i$ is an HI variety.
Proof. We prove this statement inductively over $k$. Let $S$ be a complex smooth projective rational surface. By the Enriques-Kodaira classification of complex surfaces (see [1], Part VI), there exist smooth projective surfaces $S_{n-1}, \ldots, S_1, S_0$, and morphisms $S = S_n \to S_{n-1} \to \cdots \to S_1 \to S_0$, such that each $S_{i+1} \to S_i$ is the contraction of a $(-1)$-curve and $S_0$ is a minimal rational surface (either $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_a$, for $a = 0$ or $a \geq 2$). Both $\mathbb{P}^2$ and $\mathbb{F}_a$ have algebraic cell decompositions (since they are toric), which means they are HI varieties. We apply Proposition 5.2 repeatedly, obtaining step by step that $S_1, S_2, \ldots, S_n$ are HI varieties, since each of them is obtained by blowing up a smooth HI surface at a smooth point.

Inductively, let $S_1, \ldots, S_k$ be complex smooth projective rational surfaces. As before, each surface $S_i$ is obtained by blowing up a minimal rational surface $S_{i,0}$. The product $\prod_{i=1}^k S_i$ is thus obtained through series of blowups of the base product $\prod_{i=1}^k S_{i,0}$, which is an HI variety because it admits a cell decomposition. Every blowup locus is, by the induction hypothesis, an HI variety. We apply again Proposition 5.2 repeatedly and conclude that every variety in the sequence of blowups is HI, finishing the proof. \hfill \Box

Corollary 5.5. Let $S_1$ be a complex smooth projective rational surface and $F[S_1, n] = F[n]$ its associated moduli variety. There exists a canonical isomorphism:

$$\mathbb{A}^*(F[n]) \xrightarrow{\sim} \mathbb{H}^{2*}(F[n]).$$

Proof. This is an immediate result of Proposition 5.3 and Proposition 5.4. \hfill \Box

Setup 5.6. Let $S$ be a complex surface. There exists $R \subset \mathbb{C}$ a finitely generated $\mathbb{Z}$-algebra such that $S$ is defined over $R$, i.e. there exists a surface $S_R$ over $\text{Spec } R$ such that the following square is cartesian:

$$\begin{array}{ccc}
S & \xrightarrow{r} & S_R \\
\downarrow & & \downarrow \\
\text{Spec } \mathbb{C} & \xrightarrow{\gamma} & \text{Spec } R.
\end{array}$$

For any $m \subset R$ maximal ideal, the field $\kappa(m) = R/m$ is finite, so there exists some prime $p$ and $q = p^l$, where $l \gg 0$, such that $\kappa(m) \subseteq \mathbb{F}_q$. We obtain the following figure:

$$\begin{array}{ccc}
S & \xrightarrow{r} & S_R & \xleftarrow{\gamma} & S_{\kappa(m)} & \xleftarrow{\gamma} & S_{\mathbb{F}_q} & \xleftarrow{\gamma} & S_{\mathbb{F}_q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Spec } \mathbb{C} & \xrightarrow{\gamma} & \text{Spec } R & \xleftarrow{\gamma} & \text{Spec } \kappa(m) & \xleftarrow{\gamma} & \text{Spec } \mathbb{F}_q & \xleftarrow{\gamma} & \text{Spec } \mathbb{F}_q.
\end{array}$$

For the rest of this section, when we say a complex surface $S$ can be defined as a surface $S_{\mathbb{F}_q}$ over a finite field $\mathbb{F}_q$, it means we do a procedure as above.

Proposition 5.7. Let $S$ be a complex smooth projective rational surface. There exists a prime integer $p$ and $q = p^l$, for some $l \gg 0$, such that $S$ can be defined as a smooth
Let us consider a projective rational surface $S_{F_q}$ over $\text{Spec } F_q$. For this choice of $p$ and $q$, there exists a quadratic polynomial $r(t)$ with the property that, for any $a \geq 1$ and $q' = q^a$, the number of $F_{q'}$-points on $S_{F_{q'}}$ equals $r(q')$.

**Proof.** Let $S$ be a complex smooth projective rational surface. By the Enriques-Kodaira classification of surfaces, there exist complex smooth projective surfaces $S_{n-1}, \ldots, S_1, S_0$, and a sequence of morphisms $S = S_n \rightarrow S_{n-1} \rightarrow \cdots \rightarrow S_1 \rightarrow S_0 \rightarrow \text{Spec } \mathbb{C}$, such that each $S_{i+1} \rightarrow S_i$ is the contraction of a $(-1)$-curve and $S_0$ is a minimal rational surface. As in the Setup 5.6 above, we can find $R \subset \mathbb{C}$ a finitely generated $\mathbb{Z}$-algebra such that all the surfaces $S_i$ are defined over $\text{Spec } R$. Moreover, we can pick $R$ in such a way that, if $S_0$ is $\mathbb{P}^2$ or $\mathbb{F}_a$, then the surface $S_{0,R}$ is either $\mathbb{P}_R^2$ or $\mathbb{F}_{a,R}$, respectively:

$$
\begin{align*}
S_n & \xrightarrow{\gamma} S_{n,R} \xleftarrow{\gamma} S_{n,\kappa(m)} \xleftarrow{\gamma} S_{n,F_q} \xleftarrow{\gamma} S_{n,F_{q'}} \\
\downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow \\
\vdots & \vdots \vdots \vdots \vdots \vdots \\
S_1 & \xrightarrow{\gamma} S_{1,R} \xleftarrow{\gamma} S_{1,\kappa(m)} \xleftarrow{\gamma} S_{1,F_q} \xleftarrow{\gamma} S_{1,F_{q'}} \\
\downarrow & \downarrow \downarrow \downarrow \downarrow \\
S_0 & \xrightarrow{\gamma} S_{0,R} \xleftarrow{\gamma} S_{0,\kappa(m)} \xleftarrow{\gamma} S_{0,F_q} \xleftarrow{\gamma} S_{0,F_{q'}} \\
\downarrow & \downarrow \downarrow \downarrow \downarrow \\
\text{Spec } \mathbb{C} & \rightarrow \text{Spec } R \leftarrow \text{Spec } \kappa(m) \leftarrow \text{Spec } F_q \leftarrow \text{Spec } F_{q'}.
\end{align*}
$$

In every blowup $S_{i+1} \rightarrow S_i$, we replace one smooth point of $S_i$ with a copy of $\mathbb{P}^1$, so the number of $F_{q'}$-points on $S_{F_{q'}}$ is given by a polynomial $r(q')$ that satisfies:

$$
\begin{align*}
r(q') &= \begin{cases} 
(q')^2 + (n + 1)q' + 1, & \text{if } S_0 = \mathbb{P}^2, \\
(q')^2 + (n + 2)q' + 1, & \text{if } S_0 = \mathbb{F}_a. 
\end{cases}
\end{align*}
$$

**Proposition 5.8.** Let $S_1$ be a complex smooth projective rational surface. Let $p$, $q$, and $r(t)$ be defined as in Proposition 5.7. For any $a \geq 1$ and $q' = q^a$, the number of $F_{q'}$-points on the moduli space $F[S_{1,F_{q'}}] = F[S_{1,F_{q'}}, n]$ is given by the following polynomial $R_n(q')$:

$$
R_n(q') = \prod_{i=0}^{n-1}(r(q') + iq').
$$

**Proof.** We prove the statement inductively, using the fact that the number of $F_{q'}$-points of $S_{1,F_{q'}}$, blown up at $n$ points equals $r(q') + nq'$. When $n = 1$, $F[1]_{F_{q'}} \cong S_{1,F_{q'}}$ has exactly $r(q')$ points over $F_{q'}$. For the induction step we need to show that:

$$
R_{n+1}(q') = R_n(q')(r(q') + nq').
$$
To see this, recall that we have a forgetful map $F[n + 1]_{F_{q'}} \to F[n]_{F_{q'}}$. Under this map, the fiber of any point $x = (p_1, \ldots, p_n) \in F[n]_{F_{q'}}$ is isomorphic to the blown up surface $S_{n+1,F_{q'}}$, so it has $r(q') + nq'$ rational points.

**Definition 5.9.** Let $X$ be a smooth, irreducible complex algebraic variety. The Poincaré polynomial of $X$ is:

$$P_X(q) = \sum_{i=1}^{2 \dim X} b_i q^i,$$

where $b_i$ is the rank of the $i$th singular homology group $H^i(X, \mathbb{Z})$.

**Lemma 5.10.** Let $S_1$ be a complex smooth projective rational surface. Let $p$ be a prime integer and $q = p'$, as in Proposition 5.11. The Poincaré polynomial of $F[n] = F[S_1, n]$, denoted by $P_n(q)$, coincides with the polynomial $R_n(q)$ which gives the number of $\mathbb{F}_q$-points on the moduli space $F[n]_{\mathbb{F}_q}$.

**Proof.** Let $X = F[n] = F[S_1, n]$ be the moduli variety corresponding to $S_1$ over $\text{Spec} \, \mathbb{C}$. We can regard $S_1$ as a rational surface $S_{1,F_q}$ over $\mathbb{F}_q$, as in Proposition 5.7 above. Let $X_{\mathbb{F}_q}$ be the moduli space associated to $S_{1,F_q}$. Since $X$ is smooth and projective, the Betti numbers corresponding to the $l$-adic cohomology (where $l \neq 0 \mod p$) are independent of $l$, and coincide with the Betti numbers corresponding to the ordinary (integral) cohomology of the topological space $X$ (see [5]):

$$b_i = \text{rk } H^i(X, \mathbb{Z}) = \text{rk } H^i(X, \mathbb{Q}) = \text{rk } H^i_{et}(X_{\mathbb{F}_q}, \mathbb{Q}_l).$$

One the other hand, recall the Grothendieck-Lefschetz Trace Formula (see [5], Thm. 13.4, p. 292), which states the following:

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2n} (-1)^i \text{tr}(\text{Frob}_q | H^i_c(X_{\mathbb{F}_q}, \mathbb{Q}_l)).$$

Since $X$ is proper, $H^i_c(X_{\mathbb{F}_q}, \mathbb{Q}_l) = H^i(X_{\mathbb{F}_q}, \mathbb{Q}_l)$. Moreover, as a consequence of the Weil conjectures (see [2]), we have:

$$\text{tr}(\text{Frob}_q | H^i(X_{\mathbb{F}_q}, \mathbb{Q}_l)) = z_{i,1} + \cdots + z_{i,b_i},$$

where $z_{i,1}, \ldots, z_{i,b_i}$ are the eigenvalues of the Frobenius map, and they satisfy $|z_{i,j}| = q^{i/2}$, for all $j$. Now, if we replace the field $\mathbb{F}_p$ by $\mathbb{F}_{q'}$, where $q' = q^a$, then:

$$\text{tr}(\text{Frob}_{q'} | H^i(X_{\mathbb{F}_{q'}}, \mathbb{Q}_l)) = z_{i,1}^a + \cdots + z_{i,b_i}^a.$$

In conclusion, for all $a \geq 1$, $R_n(q^a) = \sum_{i=1}^{2n} (-1)^i \sum_{j=0}^{b_j} z_{i,j}^a$, where $|z_{i,j}| = q^{i/2}, \forall i, j$. It follows immediately that $H^{2i+1}(X_{\mathbb{F}_{q'}}, \mathbb{Q}_l) = 0$, for all $i$ and $l \neq 0 \mod p$, and $z_{2i,j} = q^i$, for all $i, j$. With this, we conclude our statement:

$$R_n(q) = P_n(q) = \sum_{i=1}^{n} b_{2i} q^i.$$

□
Let \( r(q) = q^2 + kq + 1 \) be the Poincaré polynomial of \( S_1 \). This means that \( \mathbb{A}^*(S_1, \mathbb{Z}) \) is generated in degree 1 by \( k \) classes \( d_1, \ldots, d_k \), and by one class in degree 2.

**Theorem 5.11.** Let \( S_1 \) be a complex smooth projective rational surface. Let \( \Pi : F[n] \to S^n_1 \) be the natural projection map and \( pr_i : S^n_1 \to S_1 \) the projection onto the \( i \)-th copy, \( \forall 1 \leq i \leq n \). Let \( pr^n_1 \circ \Pi^* : \mathbb{A}^*(S_1) \to \mathbb{A}(F[n]) \) be the induced map on Chow rings and \( d_{i1}, \ldots, d_{ik} \) the images of the classes \( d_1, \ldots, d_k \), respectively. The Chow ring of the moduli space \( A \) is generated in degree 1 by

\[ \mathbb{A}^*(F[n]) \cong \frac{\langle \mathbb{A}^*(S_1) \rangle^{\otimes n}[D_{ij}^{(n)}]_{1 \leq i < j \leq n} \rangle}{\langle D_{ij}^{(n)}(D_{ik}^{(n)} - D_{ij}^{(n)}), D_{jk}^{(n)}(d_{ij} - d_{ik}), P_{ij}(-D_{ij}^{(n)}) \rangle} \]

**Proof.** Let \( R = \frac{\langle \mathbb{A}^*(S_1) \rangle^{\otimes n}[D_{ij}^{(n)}]_{1 \leq i < j \leq n} \rangle}{\langle D_{ij}^{(n)}(D_{ik}^{(n)} - D_{ij}^{(n)}), D_{jk}^{(n)}(d_{ij} - d_{ik}), P_{ij}(-D_{ij}^{(n)}) \rangle} \). We claim the following composition of morphisms is an isomorphism, after tensoring by \( \mathbb{Q} \):

\[ R \to \mathbb{A}^*(F[n], \mathbb{Z}) \cong H^*(F[n], \mathbb{Z}) \to \mathbb{H}^*(F[n], \mathbb{Q}). \]

By Theorem 5.5 we know that \( \mathbb{A}^*(F[n]) \) is generated over \( \mathbb{A}^*(S^n_1) \) by the classes of the divisors \( \{D_{ij}^{(n)}\}_{1 \leq i < j \leq n} \). Moreover, by Proposition 4.6 we know that the following relations hold in the Chow ring \( \mathbb{A}^*(F[n]) \):

\[
\begin{align*}
D_{j,k}^{(n)}(d_{ij} - d_{ik}) &= 0 \\
D_{j,k}^{(n)}(D_{ij}^{(n)} - D_{ik}^{(n)}) &= 0 \\
P_{ij}(-D_{ij}^{(n)}) &= 0.
\end{align*}
\]

We show that the relations above are sufficient by looking at the Betti numbers of the moduli space \( F[n] \). By definition, the \( j \)-th Betti number of \( F[n] \), denoted by \( b_{n,j} \), represents the number of codimension \( j \) linearly independent generators of \( \mathbb{A}^*(F[n]) \) as a \( \mathbb{Z} \)-module. We obtain the following recursive relation from Proposition 5.8

\[ b_{n+1,j} = b_{n,j} + (n + k)b_{n,j-1} + b_{n,j-2}. \]

We give the following interpretation to the relation above: consider the map on Chow rings \( \Pi_{n+1}^* : \mathbb{A}^*(F[n]) \to \mathbb{A}^*(F[n+1]) \) corresponding to the forgetful functor. Compared to the moduli space \( F[n] \), the space \( F[n+1] \) has \( n + k \) extra divisors: \( d_{n+1,1}, \ldots, d_{n+1,k}, D_{1,n+1}^{(n+1)}, \ldots, D_{n,n+1}^{(n+1)} \). A generator in \( \mathbb{A}^j(F[n+1]) \) is either a class inherited from \( \mathbb{A}^j(F[n]) \) under the map \( \Pi_{n+1}^* \) (this accounts for \( b_{n,j} \) generators), or it is a product between a generator class coming from \( \mathbb{A}^{j-1}(F[n]) \) and one of the \( n + k \) new divisor classes (this accounts for \( (n + k)b_{n,j-1} \) generators), or it is a product between a generator class coming from \( \mathbb{A}^{j-2}(F[n]) \) and the one generator class coming from \( \mathbb{A}^2(S_1) \) under the projection map \( pr_{n+1} : S_1^{n+1} \to S_1 \) (this accounts for \( b_{n,j-2} \) generators). It is easy to see that these are the only generators, since the divisors \( D_{ij}^{(n)} \) satisfy the identities of Equation 2. \( \square \)
**Corollary 5.12.** When $S_1 = \mathbb{P}^2_C$, the Chow ring of the moduli space $A^*(F[\mathbb{P}^2, n])$ is:

$$A^*(F[\mathbb{P}^2, n]) \cong \mathbb{Z}[H_1^*, D_{j,k}^{(n)}]_{1 \leq i \leq n, 1 \leq j < k \leq n} \langle D_{j,k}^{(n)}(D_{i,j}^{(n)} - D_{i,k}^{(n)}), D_{j,k}^{(n)}(H_j^* - H_k^*), H_i^3, P_{i,j}(-D_{i,j}^{(n)}) \rangle,$$

where, $\forall 1 \leq i \leq n$, $H_i^*$ is the image of the hyperplane class $H \in A^*(\mathbb{P}^2)$ under the composition $F[\mathbb{P}^2, n] \to (\mathbb{P}^2)^n \overset{pr_i}{\to} \mathbb{P}^2$.

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