MOTIVIC LANDWEBER EXACT THEORIES AND THEIR CONNECTIVE COVERS

MARC LEVINE

Abstract. Let \( k \) be a field of characteristic zero and let \( \Omega^* \) denote the universal oriented cohomology theory on \( \text{Sm}/k \), algebraic cobordism [11]. Let \((F, R)\) be a Landweber exact formal group law. We examine the Landweber exact \( T \)-spectra \( E := R \otimes L \text{MGL} \) and its effective cover \( f_0 E \rightarrow E \) with respect to Voevodsky’s slice tower. The coefficient ring \( R_0 \) of \( f_0 E \) is the subring of \( R \) consisting of elements of \( R \) of non-positive degree; the power series \( F \in R[[u, v]] \) has coefficients in \( R_0 \) although \((F, R_0)\) is not necessarily Landweber exact. In spite of this, we show that the geometric part of the theory \( f_0 E, f_0 E^*(X) := (f_0 E)^{2*,*}_*(X) \), is canonically isomorphic to the oriented cohomology theory \( X \mapsto R_0 \otimes L \Omega^*(X) \), assuming that \( k \) is a field of characteristic zero. This recovers results of Dai-Levine as the special case of algebraic \( K \)-theory and its connective cover, connective algebraic \( K \)-theory.

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INTRODUCTION

Let \( S \) be a fixed base-scheme, \( \text{Sm}/S \) the category of smooth quasi-projective \( S \)-schemes, and \( \mathcal{SH}(S) \) the motivic stable homotopy category of \( T \)-spectra. In this paper we consider two types of so-called oriented cohomology theories. The first type are those bi-graded theories on \( \text{Sm}/S \), \( X \mapsto E^{*,*}(X) \) represented by a (weak) commutative ring \( T \)-spectrum \( E \in \mathcal{SH}(S) \) with an orientation \( \vartheta \) in the reduced cohomology \( E^{2,1}(\mathbb{P}^\infty) \). The second are the oriented theories in the sense

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of \[11\], definition 1.1.1], that is contravariant functors \(X \rightarrow A^*(X)\) from \(\text{Sm}/S\) to commutative graded rings, together with pushforward maps \(f_*\) for projective morphisms \(f : Y \rightarrow X\), satisfying a number of functorialities and compatibilities. Assigning to \(E^{\ast\ast}\) the geometric part \(X \rightarrow E^{2\ast\ast}(X)\) gives the link between these two notions.

Among the bi-graded oriented theories the theory represented by Voevodsky’s algebraic cobordism spectrum \(\text{MGL}\) (see \[20\]) is the universal one \[20\], whereas for \(S = \text{Spec} k, k\) a field of characteristic zero, algebraic cobordism \(\Omega^*\) as defined in \[11\] is the universal theory. For a formal group law \((F, R), F(u, v) \in R[[u, v]]\), one can form the oriented theory (in the second sense) \(X \rightarrow R \otimes L \Omega^*(X)\), where \(L\) is the Lazard ring and \(L \rightarrow R\) the classifying homomorphism for \(F\). In the homotopy theory setting, the situation is more delicate, however, just as in the classical case, if \((F, R)\) is a Landweber exact formal group law, there is a corresponding oriented weak commutative ring spectrum \(\text{MGL}(R)\) with \(\text{MGL}(R)^{\ast\ast}(X) \cong R \otimes L \text{MGL}^{\ast\ast}(X)\) (see \[18\]). Using the universal property of \(\Omega^*\), there is a canonical natural transformation \(R \otimes L \Omega^* \rightarrow \text{MGL}(R)^{2\ast\ast}\) (for \(S = \text{Spec} k\) as above) and this is an isomorphism \[10\].

In the homotopy theory setting, one can go a bit farther by considering the connective cover \(f_0E \rightarrow E\) of a \(T\)-spectrum \(E \in \text{SH}(S)\). Here \(f_0\) is the truncation functor with respect to Voevodsky’s slice tower. \(f_0E\) is an analog of the classical -1 connective cover of a spectrum, and has many analogous properties. In particular, for \(E\) an oriented weak commutative ring \(T\)-spectrum, the connective cover \(f_0E\) inherits from \(E\) a canonical structure of an oriented weak commutative ring \(T\)-spectrum; the coefficient ring \(R_0 := (f_0E)^{2\ast\ast}(S)\) is just the degree \(\leq 0\) part of the coefficient ring \(R := E^{\ast\ast}(k)\) of \(E\). In particular, the coefficients of the group law \(F_E\) associated to \(E\) actually lie in \(R_0\), but even if \((F_E, R)\) is Landweber exact, it is usually the case that \((F_E, R_0)\) is not.

Besides these type of results on the connective theories associated to an oriented bi-graded theory, our main result is a comparison with algebraic cobordism. Let \(E = \text{MGL}(R)\) be the oriented weak commutative ring \(T\)-spectrum associated to a Landweber exact formal group law with coefficient ring \(R\), let \(E_0 = f_0E\) be the connective cover of \(E\), with the orientation induced from \(E\) and let \(R_0\) be the coefficient ring of \(E_0\). For \(k\) a characteristic zero field, \(S = \text{Spec} k\), the canonical natural transformations

\[
R \otimes L \Omega^* \rightarrow E^{2\ast\ast},
\]

\[
R_0 \otimes L \Omega^* \rightarrow (f_0E)^{2\ast\ast}.
\]

are isomorphisms of oriented cohomology theories on \(\text{Sm}/k\). We actually prove a stronger result (see corollary \[13\]) concerning the oriented Borel-Moore homology theories on \(\text{Sch}_k\) defined by \(E\) and \(f_0E\). The main idea for these results appears already in our treatment (with S. Daï) of the case of algebraic \(K\)-theory and its connective cover \[1\].

We begin by recalling some of the basic notions concerning oriented (weak) commutative ring spectra in the motivic stable homotopy category in \[1\] where we also recall the main results on the universality of \(\text{MGL}\). In \[2\] we recall basic facts about the slice tower in the motivic stable homotopy category and some of its basic properties; we discuss as well some issues of convergence of the slice spectral sequence. In \[3\] we introduce the connective cover of an oriented weak ring \(T\)-spectrum and show that it too defines an oriented weak commutative ring
homotopy category. We recall that a morphism \( E \Rightarrow F \) is a two-sided ideal, and the coefficient ring of the universal rank one commutative formal group law \( F_\mathbb{L} \in \mathbb{L}[[u,v]] \). We let \( \mathbb{L}^i \) denote \( \mathbb{L} \) with the grading determined by \( \deg a_{ij} = 1 - i - j \) if \( F_i(u,v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j \) and let \( \mathbb{L}_* \) denote \( \mathbb{L} \) with the opposite grading \( \mathbb{L}_n := \mathbb{L}^{-n} \).

1. Oriented ring \( T \)-spectra

We recall that a morphism \( f : \mathcal{E} \to \mathcal{F} \) in a compactly generated triangulated category \( \mathcal{T} \) is a phantom map if for each compact object \( A \) in \( \mathcal{T} \), the induced map \( f_* : \text{Hom}_\mathcal{T}(A, \mathcal{E}) \to \text{Hom}_\mathcal{T}(A, \mathcal{F}) \) is zero; it is enough to check on compact objects of the form \( \mathcal{X}[[n]] \), where \( \mathcal{X} \) is an element in a given set of compact generators. The subset of phantom maps \( \text{Hom}_\mathcal{T}(\mathcal{E}, \mathcal{F})_{\text{ph}} \subset \text{Hom}_\mathcal{T}(\mathcal{E}, \mathcal{F}) \) is clearly a two-sided ideal, so we may form the category \( \mathcal{T}/\text{ph} \) with the same objects as \( \mathcal{T} \) and morphisms \( \text{Hom}_\mathcal{T}/\text{ph}(\mathcal{E}, \mathcal{F}) := \text{Hom}_\mathcal{T}(\mathcal{E}, \mathcal{F})/\text{Hom}_\mathcal{T}(\mathcal{E}, \mathcal{F})_{\text{ph}} \).
We will be mainly interested in the case $T = S\mathcal{H}(S)$, in which the suspension spectra $\Sigma^n X_+, X \in \text{Sm}/S$, form a set of compact generators. Thus, two maps $f, g : E \to F$ are equal modulo phantom maps if and only if $f_* = g_*$, as maps of the associated bi-graded cohomology theories $f_*, g_* : E^{*,*} \to F^{*,*}$ on $\text{Sm}/S$.

**Definition 1.1.** A commutative ring $T$-spectrum is a $T$-spectrum $E \in \text{Spt}_T(S)$ together with maps $\mu : E \wedge E \to E$, $1 : S_0 \to E$ such that $(E, \mu, 1)$ is a commutative monoid in $S\mathcal{H}(S)$. A weak commutative ring $T$-spectrum is a $T$-spectrum $E \in \text{Spt}_T(S)$ together with maps $\mu : E \wedge E \to E$, $1 : S_0 \to E$ such that $(E, \mu, 1)$ is a commutative monoid in $S\mathcal{H}(S)/\text{ph}$.

For $E, F$ (weak) commutative ring $T$-spectra, a morphism $f : E \to F$ in $S\mathcal{H}(S)$ is a monoid map (resp. weak monoid map) if $f$ is a map of monoid objects in $S\mathcal{H}(S)$ (resp. in $S\mathcal{H}(S)/\text{ph}$).

**Definition 1.2.** An orientation on a weak commutative ring $T$-spectrum $(E, \mu, 1)$ is an element $c \in E^{2,1}((\mathbb{P})_\infty)$ such that $i^*c \in E^{2,1}(\mathbb{P}_1)$ corresponds to $1 \in E^{0,0}(S)$ under the suspension isomorphism $E^{0,0}(S) \cong E^{2,1}(T) \cong E^{2,1}(\mathbb{P}_1)$, where $i : \mathbb{P}_1 \to \mathbb{P}_\infty$ is the standard inclusion.

A pair $(E, c)$ consisting of a weak commutative ring $T$-spectrum $(E, \mu, 1)$ and an orientation $c$ is a weak oriented ring $T$-spectrum. We say that a weak commutative ring $T$-spectrum $(E, \mu, 1)$ is orientable if there is an orientation $c$ on $E$.

We refer to a pair $(E, c)$ as above as an oriented theory on $\text{Sm}/S$ and sometimes omit the explicit mention of the orientation $c$.

**Remark 1.3.** The algebraic cobordism spectrum $\text{MGL}$ has been studied in [20]. $\text{MGL}$ is the $T$-spectrum $(\text{MGL}_0, \text{MGL}_1, \ldots)$ with $\text{MGL}_n$ the Thom space $\text{Th}(E_n)$, with $E_n \to \text{BGL}_n$ the universal $n$-plane bundle. $\text{MGL}_\infty$ is a commutative ring $T$-spectrum in $S\mathcal{H}(S)$ with an orientation $c_{\text{MGL}} \in \text{MGL}^{2,1}(\mathbb{P}_\infty)$ given by the diagram

$$
\begin{array}{ccc}
E_1 & \to & \text{Th}(E_1) = \text{MGL} \\
\downarrow & & \\
\mathbb{P}_\infty
\end{array}
$$

noting that $E_1 \to \text{BGL}_1 = \mathbb{P}_\infty$ is an isomorphism in $\mathcal{H}(S)$.

For later use, we record the following result of Panin-Pimenov-Röndigs:

**Theorem 1.4 ([20] theorem 1.1).** For $E$ a commutative ring $T$-spectrum in $S\mathcal{H}(S)$, sending a weak monoid morphism $\varphi : \text{MGL} \to E$ to $\varphi(c_{\text{MGL}})$ gives a bijection of the set of monoid maps $\varphi$ with the set of orientations $c\in E^{2,1}(\mathbb{P}_\infty)$.

Given an oriented monoid $(E, c)$ in $S\mathcal{H}(k)$, we let

$$
\varphi_{E,c} : \text{MGL} \to E
$$

denote the corresponding morphism of commutative ring $T$-spectra.

In fact, this result extends directly to the setting of oriented weak commutative ring spectra, replacing “monoid map” with “weak monoid map”. Indeed, the proof in [20] reduces to proving some identities in $E^{*,*}(\text{MGL}(n))$ or $E^{*,*}(\text{MGL}(n) \wedge \text{MGL}(n))$. It is shown in [20] that the canonical map

$$
E^{*,*}(\text{MGL}(n)) \to \lim_{\leftarrow} E^{*,*}(\text{Th}(T(n,m)))
$$
is an isomorphism, where $T(n, m) \to \text{Grass}(n, n + m)$ is the universal bundle, and $\text{Th}(\cdot)$ is the Thom space. Similarly

$$\mathcal{E}^{\ast, \ast}(\text{MGL}(n) \wedge \text{MGL}(n)) \cong \lim_{\rightarrow} \mathcal{E}^{\ast, \ast}(\text{Th}(T(n, m)) \wedge \text{Th}(T(n, m)))$$

Their proof relies on showing that the maps on $\mathcal{E}$-cohomology in the inverse system are surjective, hence the proofs only require the knowledge of $\mathcal{E}$-cohomology on objects in $\text{Sm}/S$, and therefore the proofs work for oriented weak commutative ring spectra without change.

In case $S = \text{Spec } k$, a field, Vezzosi’s proof of the universality of $\text{MGL}$ [24, theorem 4.3] also is based on obtaining identities in $\mathcal{E}^{\ast, \ast}(\text{MGL}(n))$ or $\mathcal{E}^{\ast, \ast}(\text{MGL}(n) \wedge \text{MGL}(n))$ and is therefore also adaptable to the setting of oriented weak commutative ring $T$-spectra.

Remark 1.5. Let $\mathcal{E} \in \mathcal{SH}(S)$ be a weak commutative ring $T$-spectrum. Let $t_\mathcal{E} \in \mathcal{E}^{1, 1}(\mathbb{G}_m)$ be the element corresponding to the unit $1 \in \mathcal{E}^{0, 0}(S)$ under the suspension isomorphism. By functoriality, $t_\mathcal{E}$ gives a map of pointed sets

$$t_\mathcal{E}(X) : O_X^f(X) \to \mathcal{E}^{1, 1}(X);$$

if $\mathcal{E}$ admits an orientation $c_\mathcal{E} \in \mathcal{E}^{2, 1}(\Sigma_2^\infty)$ (which we will from now on assume), then $t_\mathcal{E}(X)$ is a group homomorphism. Using the $\mathcal{E}^{\ast, \ast}(S)$-module structure on $\mathcal{E}^{\ast, \ast}(X)$, $t_\mathcal{E}(X)$ extends to a map of $\mathcal{E}^{\ast, \ast}(S)$-modules

$$t_\mathcal{E}(X) : \mathcal{E}^{2\ast, \ast}(S) \otimes_{\mathbb{Z}} O_X^f(X) \to \mathcal{E}^{2\ast, \ast+1}(X).$$

2. THE SLICE SPECTRAL SEQUENCE

We refer the reader to [25] and [7, 8] for an introduction to Voevodsky’s slice tower.

We consider a $T$-spectrum $\mathcal{E} \in \mathcal{SH}(S)$ and the slice tower

$$\ldots \to f_{p+1}E \to f_pE \to \ldots \to \mathcal{E}$$

Recall that the endofunctor $f_p$ on $\mathcal{SH}(S)$ is defined as the composition $i_p \circ r_p$, where $i_p : \Sigma_p^0 \mathcal{SH}^{eff}(k) \to \mathcal{SH}(k)$ is the inclusion of the localizing subcategory $\Sigma_p^0 \mathcal{SH}^{eff}(k)$ of $\mathcal{SH}(k)$ generated by objects $\Sigma_p^0 X$, with $X \in \text{Sm}/k$ and $n \geq p$. The functor $r_p : \mathcal{SH}(k) \to \Sigma_p^0 \mathcal{SH}^{eff}(k)$ is the right adjoint to $i_p$ which exists thanks to results of Neeman [15] on compactly generated triangulated categories.

The slice functor $s_p : \mathcal{SH}(S) \to \mathcal{SH}(S)$ is characterized up to unique isomorphism by natural transformations $f_p \to s_p \to f_{p+1}[1]$ so that for each $\mathcal{E} \in \mathcal{SH}(k)$, the triangle

$$f_{p+1}\mathcal{E} \to f_p\mathcal{E} \to s_p\mathcal{E} \to f_{p+1}\mathcal{E}[1]$$

is distinguished.

For $Y \in \text{Sm}/S$, we have the associated slice spectral sequence

$$E_2^{p, q}(Y; n) = (s_{-q}\mathcal{E})^{p+q, n}(Y) \Rightarrow \mathcal{E}^{p+q, n}(Y).$$

We may rewrite the $E_2$-term in (2.1) as

$$E_2^{p, q}(Y; n) = \text{Hom}_{\mathcal{SH}(S)}(\Sigma_p^\infty Y, \Sigma^{p+q, n}s_{-q}\mathcal{E})$$

1. Letting $S$ denote the sphere spectrum and writing $[a] := t_\mathcal{E}(a)$, this follows from the identity $[ab] = [a] + [b] + H[a][b]$ ($H : S \wedge \mathbb{G}_m \to S$ the stable Hopf map) and the fact that $H$ goes to zero in any oriented theory $\mathcal{E}$. Both these facts are proven by Morel in [13, 40].
The spectral sequence will be strongly convergent if for each fixed $m_0 \in \mathbb{Z}$, there is an integer $q(m_0)$ such that

$$\text{Hom}_{\text{SH}(S)}(\Sigma^\infty T Y_+, \Sigma^{m,n} f_q \mathcal{E}) = 0$$

for all $q \geq q(m_0)$ and all $m \geq -m_0$.

One can phrase this in another way: We have

$$\text{Hom}_{\text{SH}(S)}(\Sigma^\infty T Y_+, \Sigma^{m,n} f_q \mathcal{E}) = \text{Hom}_{\text{SH}_{\geq 1}(S)}(\Sigma^{m-1} \Sigma^\infty Y_+, \Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E})$$

and

$$\text{Hom}_{\text{SH}_{\geq 1}(S)}(\Sigma^{m-1} \Sigma^\infty Y_+, \Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E}) = \pi_{-m}(\Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E}(Y)).$$

Thus, the above condition is just saying that, given an integer $m_0$, the spectrum $\Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E}(Y)$ is $m_0$-connected for all $q \geq q(m_0)$, this being the usual condition for the strong convergence of the spectral sequence associated to the tower of spectra

$$\ldots \rightarrow \Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_{q+1} \mathcal{E}(Y) \rightarrow \Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E}(Y) \rightarrow \ldots \rightarrow \Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} \mathcal{E}(Y).$$

We conclude this section with a convergence criterion for the spectral sequence (2.1).

**Lemma 2.1.** Suppose that $S = \text{Spec } k$, $k$ a perfect field. Take $\mathcal{E} \in \text{SH}(S)$. Suppose that there is a non-decreasing function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ with $\lim_{n \rightarrow \infty} f(n) = \infty$, such that $\pi_{a+b,b} \mathcal{E} = 0$ for $a \leq f(b)$. Then the for all $Y$, the spectral sequence (2.1) is strongly convergent. More precisely, for an integer $m_0$, let $q(m_0)$ be chosen so that $f(q(m_0)) \geq m_0 + \text{dim } Y + n$. Then the spectrum $\Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E}(Y)$ is $m_0$-connected for $q \geq q(m_0)$.

**Proof.** We recall the 2-variable Postnikov tower $f_{a,b}$ defined in [7, §3], with $f_b = f_{-\infty,b}$. We also recall that

1. $\pi_{m+r,r} f_{a,b} F = 0$ for $m < a, b \in \mathbb{Z}$ [7, lemma 4.4],
2. the canonical map $\rho_{a,b} : f_{a,b} F \rightarrow f_b F$ induces an isomorphism on $\pi_{m+r,r}$ for all $m \geq a, r \geq b$,
3. $\rho_{a,b} : f_{a,b} F \rightarrow f_b F$ is an isomorphism if $\rho$ induces an isomorphism on $\pi_{m+r,r}$ for all $m \in \mathbb{Z}$ and $r \geq a$ [7, lemma 4.6].

The universal property of $f_{q+n}$ gives the isomorphism for all $r \geq q + n$

$$\pi_{m+r,r} f_{q+n} \Sigma^{0,n} \mathcal{E} \cong \pi_{m+r,r} \Sigma^{0,n} \mathcal{E} \cong \pi_{m+r,r-\text{dim } Y} \Sigma^{0,n} \mathcal{E}. $$

By assumption on $\mathcal{E}$, $\pi_{m+r,r-\text{dim } Y} \Sigma^{0,n} \mathcal{E} = 0$ for $m + n \leq f(r - n)$, and thus

$$\pi_{m+r,r} f_{q+n} (\Sigma^{0,n} \mathcal{E}) = 0$$

for $m \leq f(r - n) - n$ as well (assuming $r \geq q + n$).

Now take $q(m_0) \geq 0$ to be an integer such that $f(q(m_0)) \geq m_0 + \text{dim } Y + n$, and take $q \geq q(m_0)$. As $f$ is non-decreasing, we have $f(r - n) - n \geq f(q(m_0)) - n \geq m_0 + \text{dim } Y$ for all $r \geq q + n$, hence

$$\pi_{m+r,r} f_{q+n} (\Sigma^{0,n} \mathcal{E}) = 0$$

for $m \leq m_0 + \text{dim } Y, r \geq q + n$. But by (i)-(iii), this implies that the map

$$f_{m_0+\text{dim } Y+q+n} (\Sigma^{0,n} \mathcal{E}) \rightarrow f_{q+n} (\Sigma^{0,n} \mathcal{E})$$

is an isomorphism, and thus by (i), $\pi_{a+b,b} f_{q+n} (\Sigma^{0,n} \mathcal{E}) = 0$ for all $a \leq m_0 + \text{dim } Y, b \in \mathbb{Z}$. Translating back to $\Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E}$ via the isomorphisms

$$\pi_{a+b,b} f_{q+n} (\Sigma^{0,n} \mathcal{E}) \cong \pi_{a+b,b} \Sigma^{0,n} f_q \mathcal{E} \cong \pi_{a+b,b} \Omega^\infty_{\mathcal{G}_m} \Sigma^{0,n} f_q \mathcal{E}$$

for all $q \geq q(m_0)$ and all $m \geq -m_0$. Take
for \( b \geq 0, a \in \mathbb{Z} \), we have
\[
\pi_{a+b,b} \Omega_{\mathbb{S}_0}^{\infty} \Sigma^{0,n} f_q \mathcal{E} = 0
\]
for all \( a \leq m_0 + \dim Y, b \geq 0 \), in particular, taking \( b = 0 \),
\[
\pi_{a} \Omega_{\mathbb{S}_0}^{\infty} \Sigma^{0,n} f_q \mathcal{E} = 0
\]
for all \( a \leq m_0 + \dim Y, q \geq q(m_0) \).

We now apply the (strongly convergent) Brown-Gersten spectral sequence
\[
E_2^{a,b} = H^a(Y_{\mathbb{Z}_0}, \pi_{-b} \Omega_{\mathbb{S}_0}^{\infty} \Sigma^{0,n} f_q \mathcal{E}) \Rightarrow \pi_{-a-b} (\Omega_{\mathbb{S}_0}^{\infty} \Sigma^{0,n} f_q \mathcal{E}(Y)).
\]
As \( \pi_{-3} \Omega_{\mathbb{S}_0}^{\infty} \Sigma^{0,n} f_q \mathcal{E} = 0 \) for \( -b \leq m_0 + \dim Y \), and \( H^a(Y_{\mathbb{Z}_0}, -) = 0 \) for \( a > \dim Y \),
it follows that \( \Omega_{\mathbb{S}_0}^{\infty} \Sigma^{0,n} f_q \mathcal{E}(Y) \) is \( m_0 \)-connected for \( q \geq q(m_0) \). \( \square \)

3. Oriented theories and their \( T \)-connective covers

3.1. The \( T \)-connective cover.

**Proposition 3.1.** 1. Let \((\mathcal{E}, \mu, 1)\) be a weak commutative ring \( T \)-spectrum. Then \( f_0 \mathcal{E} \) has a unique structure of a weak commutative ring \( T \)-spectrum such that the canonical map \( \rho : f_0 \mathcal{E} \to \mathcal{E} \) is a weak monoid morphism.

2. If \((\mathcal{E}, c)\) is an oriented weak ring \( T \)-spectrum, then there is a unique element \( c_0 \in (f_0 \mathcal{E})^{2,1}(\mathbb{P}^\infty) \) with \( \mu_*(c_0) = c \), and \( c_0 \) defines an orientation on the weak commutative ring \( T \)-spectrum \( f_0 \mathcal{E} \).

3. We have \( t_{\mathcal{E}} = \rho_*(f_0 \mathcal{E}) \) in \( E^{1,1}(\mathbb{S}_0) \).

**Proof.** Write \( \mathcal{E}_0 \) for \( f_0 \mathcal{E} \). We first lift the multiplication. We note that \( SH^{eff}(S) \) is closed under smash product, as the generators are, so \( \mathcal{E}_0 \wedge \mathcal{E}_0 \) is in \( SH^{eff}(S) \).

Thus, the composition
\[
\mathcal{E}_0 \wedge \mathcal{E}_0 \xrightarrow{\rho \wedge \rho} \mathcal{E} \wedge \mathcal{E} \xrightarrow{\mu} \mathcal{E}
\]
admits a unique lifting \( \mu_0 : \mathcal{E}_0 \wedge \mathcal{E}_0 \to \mathcal{E}_0 \) making
\[
\begin{array}{ccc}
\mathcal{E}_0 \wedge \mathcal{E}_0 & \xrightarrow{\mu_0} & \mathcal{E}_0 \\
\rho \wedge \rho & \downarrow & \downarrow \rho \\
\mathcal{E} \wedge \mathcal{E} & \xrightarrow{\mu} & \mathcal{E}
\end{array}
\]
commute.

We claim that \( \mu_0 \) is associative modulo phantom maps. Indeed, let
\[
\varphi : \mathcal{E}_0 \wedge \mathcal{E}_0 \wedge \mathcal{E}_0 \to \mathcal{E}_0
\]
denote the difference \( \mu_0 \circ (\mu_0 \wedge \text{id}) - \mu_0 \circ (\text{id} \wedge \mu_0) \). Choosing a cofibrant model for \( \mathcal{E}_0 \wedge \mathcal{E}_0 \wedge \mathcal{E}_0 \) in \( Spt_T(S) \) (and calling this also \( \mathcal{E}_0 \wedge \mathcal{E}_0 \wedge \mathcal{E}_0 \)), we can write \( \mathcal{E}_0 \wedge \mathcal{E}_0 \wedge \mathcal{E}_0 \) as a cell complex, with cells given by compact objects mapping to \( SH^{eff}(S) \). Thus, for \( A \) a compact object of \( SH(S) \) and \( h : A \to \mathcal{E}_0 \wedge \mathcal{E}_0 \wedge \mathcal{E}_0 \) a morphism in \( SH(S) \), there is a finite subcomplex \( i : F_0 \to \mathcal{E}_0 \wedge \mathcal{E}_0 \wedge \mathcal{E}_0 \) of \( \mathcal{E}_0 \wedge \mathcal{E}_0 \wedge \mathcal{E}_0 \) and a factorization of \( h \) through \( h_0 : A \to F_0 \). But as \( F_0 \) is compact, the composition \( \rho \circ \varphi \circ i \) in \( SH(S) \) is zero, and since \( F_0 \) maps to \( SH^{eff}(k) \), the universal property of \( \rho : \mathcal{E}_0 \to \mathcal{E} \) implies that \( \varphi \circ i = 0 \) in \( SH^{eff}(S) \). Thus \( \varphi \circ h = \varphi \circ i \circ h_0 = 0 \).

The unit map \( 1 : S_k \to \mathcal{E} \) factors uniquely through \( \rho \), since \( S_k \) is in \( SH^{eff}(k) \), giving the unit map \( 1_0 : S_k \to \mathcal{E}_0 \); the identities
\[
\mu_0 \circ (\text{id}_{\mathcal{E}_0} \wedge 1_0) = \text{id}_{\mathcal{E}_0} = \mu_0 \circ (1_0 \wedge \text{id}_{\mathcal{E}_0})
\]
in \(SH(S)/\phi \) and the commutativity of \(\mu_0\) modulo phantom maps follow as for the proof of associativity, as does the uniqueness of \(\mu_0\) modulo phantom maps (as \(S_k\) is compact, the lifting of the unit map is unique in \(SH(S)\)).

For (2), recall that \(\star = (1 : 0, \ldots, 0) \in \mathbb{P}^n\). Let \(\mathbb{A}^n(\star) \subset \mathbb{P}^n\) be the affine open subset \(X_0 \neq 0\). Then the quotient map \(\mathbb{P}^n_\star \to \mathbb{P}^n/\mathbb{A}^n(\star)\) is an isomorphism in \(\mathcal{H}_*(k)\), giving the isomorphism \(\mathbb{P}^\infty_\star \cong \varprojlim_n \mathbb{P}^n/\mathbb{A}^n(\star)\). In particular, both \(\mathbb{P}^n_\star\) and \(\mathbb{P}^\infty_\star\) are in \(\Sigma^1_2 SH^{eff}(k)\).

We have the isomorphism \(\Sigma_T f_0 \mathcal{E} \cong f_1 \Sigma_T \mathcal{E}\), and under this isomorphism \(\Sigma_T \rho : \Sigma_T f_0 \mathcal{E} \to \Sigma_T \mathcal{E}\) goes over to the universal map \(\rho_1 : f_1 \Sigma_T \mathcal{E} \to \Sigma_T \mathcal{E}\). Thus \(c : \mathbb{P}^\infty_\star \to \Sigma_T \mathcal{E}\) factors uniquely through \(\Sigma_T \rho\) via a map \(c_0 : \mathbb{P}^\infty_\star \to \Sigma_T f_0 \mathcal{E}\). The restriction of \(c_0\) to \(\mathbb{P}^1_\star\) is the unique lifting of \(c_{|_{\mathbb{P}^1}}\), which shows that \(c_{|_{\mathbb{P}^1}} \in (f_0 \mathcal{E})^{2,1}(\mathbb{P}^1_\star)\) corresponds to \(1_0 \in (f_0 \mathcal{E})^{0,0}(k)\) under the suspension isomorphism. Thus \(c_0 \in (f_0 \mathcal{E})^{2,1}(\mathbb{P}^\infty_\star)\) is an orientation for \(f_0 \mathcal{E}\).

The proof of (3) is similar, noting that \(\Sigma_G_{m_*}(f_0 \mathcal{E}) \cong f_1 \Sigma_G_{n,*} \mathcal{E}\) and that \(G_m\) is in \(\Sigma_T \Sigma^1_2 SH^{eff}(k)\).

3.2. **Landweber exact theories.** We specialize to the setting \(S = \text{Spec } k\), \(k\) a field of characteristic zero.

We recall the Landweber exactness conditions for a formal group law \((F, R)\): Let \(\varphi : \mathbb{L}^* \to R\) be the classifying map for \((F, R)\). Choose homogeneous polynomial generators for \(\mathbb{L}^*\) over \(\mathbb{Z}\), \(\mathbb{L}^* = \mathbb{Z}[x_1, x_2, \ldots]\), with deg\(x_i = -i\). Then for each prime \(p\), the sequence \(p, x_{p-1}, \ldots, x_{p^i-1}, \ldots\) is a regular sequence on \(R\).

For an oriented theory \((\mathcal{E}, c)\), let \(R^*_c\) be the coefficient ring, that is, \(R^*_c := \mathcal{E}^{2_+,*}(k)\). Let \(F_{\mathcal{E}, c}(u, v) \subset R^*_c[[[u, v]]\) be the formal group law of the oriented theory \((\mathcal{E}, c)\).

**Remark 3.2.** By the Hopkins-Morel-Hoyois theorem [8], the map \(\rho_{\mathcal{MGL}} : \mathbb{L}^* \to \mathbb{MGL}^{2_+,*}(k)\) classifying the formal group law \(F_{\mathcal{MGL}}\) is an isomorphism; we henceforth identify \(\mathbb{L}^*\) and \(\mathbb{MGL}^{2_+,*}(k)\) via \(\rho_{\mathcal{MGL}}\).

**Definition 3.3.** An oriented weak commutative ring \(T\)-spectrum \((\mathcal{E}, c)\) is said to be **Landweber exact** if
1. The classifying map \(\varphi_{\mathcal{E}, c} : \mathbb{L}^* \to R^*_c\) for the formal group law \(F_{\mathcal{E}, c}\) satisfies the Landweber exactness conditions
2. For all finite spectra \(\mathcal{F}\), the map \(R^*_c \otimes_{\mathbb{L}^*} \mathbb{MGL}^{2_+,*}(\mathcal{F}) \to \mathcal{E}^{2_+,*}(\mathcal{F})\) induced by the classifying map \(\varphi_{\mathcal{E}, c}\) and the product map \(\mathcal{E}^{2n,n}(k) \otimes \mathcal{E}^{a,b}(\mathcal{F}) \to \mathcal{E}^{a+2n,b+n}(\mathcal{F})\) is an isomorphism.

**Remark 3.4.** Naumann-Spitzweck-Ostvær [18 theorem 8.7] show that, for each Landweber exact \(\mathbb{L}^*\)-algebra \(R^*\), the bigraded functor from finite spectra to bigraded algebras, \(\mathcal{F} \mapsto R^*_c \otimes_{\mathbb{L}^*} \mathbb{MGL}^{2_+,*}(\mathcal{F})\), is represented by an object \(\mathbb{MGL}(R^*)\) in \(\mathcal{SH}(k)\) with morphisms \(\mu : \mathbb{MGL}(R^*) \wedge \mathbb{MGL}(R^*) \to \mathbb{MGL}(R^*)\), \(1 : S_k \to \mathbb{MGL}(R^*)\) defining a (oriented) weak commutative ring \(T\)-spectrum. It does not seem to be known if one can give \(\mathbb{MGL}(R^*)\) the structure of an oriented commutative ring \(T\)-spectrum.

In fact, Naumann-Spitzweck-Ostvær work in the setting of homology theories on \(SH(S)\) rather than cohomology theories on finite spectra, and they prove their results for \(S\) a regular, noetherian separated scheme of finite Krull dimension. However, since the base-scheme is \(Spec k, k\) a field of characteristic zero, all finite spectra
in $\mathcal{SH}(k)$ are strongly dualizable \cite[theorem 4.9]{22}, so one may easily pass from homology theories on $\mathcal{SH}(k)$ to cohomology theories on finite spectra in $\mathcal{SH}(k)$.

Lemma 3.5. Suppose that $(\mathcal{E}, c)$ is Landweber exact. Then the classifying map $\varphi_{\mathcal{E},c} : \text{MGL} \to \mathcal{E}$ and the product map $R^z \otimes_{\mathbb{Z}} E_{*,*}(F) \to E_{*,*}(F)$ induces an isomorphism of homology theories on $\mathcal{SH}(k)$

$$R^z \otimes_{L_*} \text{MGL}_{*,*}(F) \to E_{*,*}(F).$$

Proof. Both functors $R^z \otimes_{L_*} \text{MGL}_{*,*}(-)$ and $E_{*,*}(-)$ are cohomological functors on $\mathcal{SH}(k)$, compatible with coproducts, hence the full subcategory of objects $F$ for which the lemma holds is a localizing subcategory of $\mathcal{SH}(k)$. As $\mathcal{SH}(k)$ is generated as a localizing category by $\mathcal{SH}(k)_{\text{fin}}$, this reduces us to the case $F \in \mathcal{SH}(k)_{\text{fin}}$.

Take $F$ is in $\mathcal{SH}(k)_{\text{fin}}$. Then $F$ is strongly dualizable, with $F^D \in \mathcal{SH}(k)_{\text{fin}}$, and we have $G_{*,*}(F) \cong G^{-*,*}(F^D)$ for all $G \in \mathcal{SH}(k)$. The fact that $\varphi_{\mathcal{E},c}$ induces an isomorphism

$$R^z \otimes_{L_*} \text{MGL}_{*,*}(F^D) \to E_{*,*}(F^D).$$

shows that the lemma is true for $F \in \mathcal{SH}(k)_{\text{fin}}$. $\square$

Lemma 3.6. Let $(\mathcal{E}, c)$ be a Landweber exact oriented weak commutative ring $T$-spectrum, $R^z := R^z_k$. Then there is an isomorphism $\psi : \text{MGL}(R^z) \to \mathcal{E}$ in $\mathcal{SH}(k)$.

Proof. Let $f : \text{Spec } k \to \text{Spec } \mathbb{Z}$ be the canonical morphism of schemes, giving the adjoint functors $L^f : \mathcal{SH}(\mathbb{Z}) \longrightarrow \mathcal{SH}(k) : Rf_*$. Consider the object $Rf_*\mathcal{E} \in \mathcal{SH}(\mathbb{Z})$ and take $F \in \mathcal{SH}(\mathbb{Z})$. Since $Rf_* (\mathcal{E} \land L^f F) \cong Rf_* \mathcal{E} \land F$, we have

$$(Rf_*\mathcal{E})_{*,*}(F) = \mathcal{E}_{*,*}(L^f F) \cong R^z \otimes_{L_*} \text{MGL}_{Z*,*}(L^f F).$$

that is, $Rf_*\mathcal{E}$ represents the homology theory $F \mapsto R^z \otimes_{L_*} \text{MGL}_{Z*,*}(L^f F)$ on $\mathcal{SH}(\mathbb{Z})$. The spectrum $\text{MGL}_Z(R^z_k)$ similarly represents the homology theory $F \mapsto R^z \otimes_{L_*} \text{MGL}_{Z*,*}(F)$. As $\mathcal{SH}(\mathbb{Z})$ is a Brown triangulated category (see \cite[theorem 1]{17} and \cite[proposition 4.11 and theorem 5.1]{15}) the inclusion $\mathcal{SH}(\mathbb{Z}) \to \mathcal{SH}(\mathbb{Z})$ satisfies Brown representability, in the sense of \cite[definition 3.1]{15}. Thus, the commutative triangle of homology theories

$$\text{MGL}_Z \quad \text{MGL}_Z(R^z_k) \quad \text{MGL}_Z(R^z_k)$$

induces the diagram in $\mathcal{SH}(\mathbb{Z})$

$$\text{MGL}_Z \quad \text{MGL}_Z(R^z_k) \quad \text{MGL}_Z(R^z_k)$$

\begin{tikzcd}
\text{MGL}_Z \arrow[d, \alpha] & \text{MGL}_Z(R^z_k) \arrow[r, \psi] & Rf_*\mathcal{E} \arrow[l, \beta, phantom, near start, shift={(0.5,0.5)}]
\end{tikzcd}
which is commutative up to a phantom map. Using the adjoint property of $Lf^*$ and $Rf_*$, this gives us the diagram in $\mathcal{SH}(k)$

\[
\begin{array}{ccc}
\text{MGL} & \xrightarrow{\alpha} & \text{MGL}(R^*_F) \\
\downarrow \beta & & \downarrow \psi \\
\text{E} & & \\
\end{array}
\]

which we claim commutes up to a phantom map. Indeed, MGL is a filtered colimit of finite subspectra $F_0 \text{MGL} \to F_1 \text{MGL} \to \ldots \to \text{MGL}$ with $F_n \text{MGL} \cong Lf^* F_n \text{MGL}_{\mathbb{Z}}$ and with each of the maps $F_n \text{MGL}_{\mathbb{Z}} \to F_{n+1} \text{MGL}_{\mathbb{Z}}$ a cofibration of cofibrant objects in $\text{Spt}_{T}(\mathbb{Z})$. Using adjointness again, we see that $\psi \circ \alpha = \beta$ after restriction to $F_n \text{MGL}$. As $\mathcal{SH}(k)_{\text{fin}}$ consists of compact objects, this shows that $\psi \circ \alpha = \beta$ as maps of cohomology theories on $\mathcal{SH}(k)_{\text{fin}}$, and hence maps of homotopy theories on $\mathcal{SH}(k)$, using the dualizability of $\mathcal{SH}(k)_{\text{fin}}$ and the denseness of $\mathcal{SH}(k)_{\text{fin}}$ in $\mathcal{SH}(k)$, as in the proof of lemma 3.3. This verifies our claim.

Next, we claim that $\beta = \varphi_{E,c}$, up to a phantom map. The proof is similar to the one we just used: the restriction of $\tilde{\beta}$ to $F_n \text{MGL}_{\mathbb{Z}}$ is the image of $1 \otimes Lf^*(i_n^\mathbb{Z})$ under the isomorphism $\text{id} \otimes \varphi$, where $i^\mathbb{Z}_n : F_n \text{MGL}_{\mathbb{Z}} \to \text{MGL}_{\mathbb{Z}}$ is the inclusion. But $Lf^*(i_n^\mathbb{Z})$ is the inclusion $i_n$ of $F_n \text{MGL}$ in MGL, and clearly $\varphi \circ i_n = (\text{id} \otimes \varphi)(1 \otimes i_n)$. Thus $\beta \circ i_n = \varphi \circ i_n$, which shows that $\beta = \varphi_{E,c}$, up to a phantom map. In particular, $\beta$ induces the composition

\[
\text{MGL}^*_{\mathbb{Z}}(-) \xrightarrow{1 \otimes \text{id}} R^*_E \otimes_L \text{MGL}^*_{\mathbb{Z}}(-) \xrightarrow{\text{id} \otimes \varphi_{E,c}} \mathcal{E}^*_{\mathbb{Z}}(-)
\]

on the underlying cohomology theories.

In addition, the map $\alpha$ induces the map on cohomology theories on $\mathcal{SH}(k)_{\text{fin}}$

\[
1 \otimes \text{id} : \text{MGL}^*_{\mathbb{Z}}(-) \to R^*_E \otimes_L \text{MGL}^*_{\mathbb{Z}}(-)
\]

(this is proven in [13], proposition 8.5, theorem 8.7). By construction, the map $\tilde{\psi}_*: \text{MGL}(R^*)_{2,+} \to (Rf_* \mathcal{E})_{2,+}$ is the identity map $R^*_{+} \to R^*_{+}$, hence by adjointness, the same is the case for $\psi_*$. Thus $\psi$ is an $R^*_{E} = R^*_{\text{MGL}(R^*)}$ module map, from which it follows that $\psi$ induces the isomorphism of cohomology theories

\[
R^*_{E} \otimes_L \text{MGL}^*_{\mathbb{Z}}(-) \xrightarrow{\text{id} \otimes \varphi_{E,c}} \mathcal{E}^*_{\mathbb{Z}}(-)
\]

As this shows that $\psi$ induces an isomorphism on the homotopy sheaves, it follows that $\psi$ is an isomorphism in $\mathcal{SH}(k)$.

3.3. Geometrically Landweber exact theories. As above, let $(\mathcal{E}, c)$ be a oriented weak commutative ring $T$-spectrum in $\mathcal{SH}(k)$.

As we have noted in remark 1.5, we have the element $t_\mathcal{E} \in \mathcal{E}^{1,1}(\mathbb{G}_m)$ corresponding to the unit $1 \in \mathcal{E}^{0,0}(k)$ under the suspension isomorphism, inducing the group homomorphism

\[
t^\mathcal{E}_k(X) : \mathcal{O}_X^\mathcal{E}(X) \to \mathcal{E}^{1,1}(X);
\]

Using the $\mathcal{E}^{*,-}(k)$-module structure on $\mathcal{E}^{*,-}(X)$, $t^\mathcal{E}_k(X)$ extends to a map of $\mathcal{E}^{*,-}(k)$-modules

\[
t^\mathcal{E}_k(X) : \mathcal{E}^{2s,-}(k) \otimes_{\mathbb{Z}} \mathcal{O}_X^\mathcal{E}(X) \to \mathcal{E}^{2s+1,-1}(X).
\]

Let $(\mathcal{E}, c)$ be an oriented weak commutative ring $T$-spectrum in $\mathcal{SH}(k)$. The classifying map $\varphi_{\mathcal{E},c} : \text{MGL} \to \mathcal{E}$ induces the map $\varphi_{\mathcal{E},c} : \mathbb{L}^* \to R^*_E$; combining these
with the product in $\mathcal{E}$-cohomology gives rise to the homomorphism

$$\varphi_X : R^\mathcal{E}_* \otimes_{R_{\text{MGL}}} \text{MGL}^{2s-c,s}(X) \to \mathcal{E}^{2s-c,s}(X)$$

for $X \in \text{Sm}/k$, natural with respect to morphisms in $\text{Sm}/k$. If $\eta$ is a generic point of $X$, we may pass to the limit over Zariski open neighborhoods of $\eta$, giving the homomorphism

$$\varphi_\eta : R^\mathcal{E}_* \otimes_{R_{\text{MGL}}} \text{MGL}^{2s-c,s}(\eta) \to \mathcal{E}^{2s-c,s}(\eta)$$

**Definition 3.7.** We say that an oriented weak commutative ring $\text{MGL}$ in the map $E_{\text{SH}} \otimes \mathbb{L}$ all terms $L^m,(c)$ arise from the classifying map $p_\eta : E_{n,n}(k) \to \text{MGL}^{2n}$. For $Y$ for $Y \in \text{Sm}/k, n \in \mathbb{Z}$, $\text{MGL}^{2n+a,n}(Y) = 0$ for all $a > 0$.

**Proposition 3.8.** 1. The classifying map $\varphi_{\text{MGL}} : \mathbb{L}^* \to \text{MGL}^{2s-c,*}(k)$ is an isomorphism. In particular, $\text{MGL}^{2n,n}(k) = 0$ for $n > 0$.

2. For $\eta \in X \in \text{Sm}/k$, let $p_\eta : \eta \to \text{Spec} k$ be the structure morphism. Then $p_\eta^* : \text{MGL}^{2s-c,*}(k) \to \text{MGL}^{2s-c,*}(\eta)$ is an isomorphism

3. For $\eta \in X \in \text{Sm}/k, the map $t^1_{\text{MGL}}(\eta) : k(\eta)^\times \to \text{MGL}^{1,1}(\eta)$ is an isomorphism and the map $t^1_{\text{MGL}}(\eta) : \text{MGL}^{2s-c,*}(k) \otimes_{\mathbb{Z}} k(\eta)^\times \to \text{MGL}^{2s-c+1,*+1}(\eta)$ is a surjection.

4. For $Y \in \text{Sm}/k, n \in \mathbb{Z}$, $\text{MGL}^{2n+a,n}(Y) = 0$ for all $a > 0$.

**Proof.** We use the (strongly convergent) Hopkins-Morel spectral sequence

$$(1) \quad E_2^{p,q}(n) := \mathbb{L}^q \otimes H^{p-q}(Y, \mathbb{Z}(n-q)) \implies \text{MGL}^{p+q,n}(Y).$$

Recall that for $Y \in \text{Sm}/k, H^q(Y, \mathbb{Z}(b)) = 0$ for $b < 0$ or $a > 2b$, and that $\mathbb{L}^q = 0$ for $q > 0$. As $p + q > 2n$ implies $p - q > 2(n-q)$, we see that $E_2^{p,q}(n) = 0$ for $p + q > 2n$, hence $\text{MGL}^{m,n}(Y) = 0$ for $m > 2n$, $Y \in \text{Sm}/k$, proving (4).

For $Y = \text{Spec} F$, $F$ a field, we have in addition $H^q(Y, \mathbb{Z}(b)) = 0$ for $a > b$. For $p + q = 2n$ we have $p - q = 2(n-q)$, hence $E_2^{2n-q,q}(n) = 0$ if $q \neq n$. Similarly, for $p + q = 2n - 1$, we have $p - q = 2(n-q) - 1$, so the only possible non-zero terms $E_2^{2n-q-1,q}(n)$ are $E_2^{2n-1,n}(n)$ and $E_2^{n,n-1}(n)$. As $H^{-1}(F, \mathbb{Z}(0)) = 0$, we have $E_2^{2n-q-1,q}(n) = 0$ if $q \neq n - 1$. The remaining non-zero terms contributing to $\text{MGL}^{2n,n}(Y)$ and $\text{MGL}^{2n-1,n}(Y)$ are

$$E_2^{n,n}(n) = \mathbb{L}^n, \quad E_2^{n,n-1}(n) = \mathbb{L}^{n-1} \otimes F^\times$$

since $H^1(\text{Spec} F, \mathbb{Z}(1)) = F^\times$. This also shows that $d_1^{n,n}, d_1^{n-r,n+r-1}$ and $d_1^{n,n-1}$ are all zero for all $r \geq 2$, giving an isomorphism $\mathbb{L}^n \cong \text{MGL}^{2n,n}(\eta)$ and a surjection $\mathbb{L}^{n-1} \otimes k(\eta)^\times \to \text{MGL}^{2n-1,n}(\eta)$ for all $n$. Finally, taking $n = 1$, $E_2^{0,0}(1) = \mathbb{L}^q \otimes H^{-2q}(Y, \mathbb{Z}(1))$; as $H^m(Y, \mathbb{Z}(1)) = 0$ for $m \neq 1$, it follows that $E_2^{0,0}(1) = 0$ for all $q$, so all differentials entering $E_2^{1,0}(1)$ are zero and hence $\text{MGL}^{1,1}(\eta) = k(\eta)^\times$.

As the spectral sequence is natural in $Y$ and the computation of the $E_2^{n,n}(n)$-terms are independent of $\eta$, this shows that $p_\eta^* : \text{MGL}^{2s-c,*}(k) \to \text{MGL}^{2s-c,*}(\eta)$ is an isomorphism, proving (2).

To complete the proof, we must check that the isomorphisms $\mathbb{L}^q \to \text{MGL}^{2q,q}(k)$ arise from the classifying map $\mathbb{L}^* \to \text{MGL}^{2s-c,*}(k)$, and also that the surjections $\mathbb{L}^q \otimes k(\eta)^\times \to \text{MGL}^{2q-1,q}(\eta)$ are induced by $t^1_{\text{MGL}}(\eta)$ and the isomorphism $\mathbb{L}^q \to \text{MGL}^{2q,q}(k)$. 
The fact that the isomorphisms $\mathbb{L}^q \to \text{MGL}^{2q,q}(k)$ are just the maps coming from the homomorphism $\mathbb{L}^* \to \text{MGL}^{2*,*}(k)$ classifying the formal group law for MGL follows directly from construction of the spectral sequence in \([3]\); the spectral sequence arises from a choice of polynomial generators $x_1, x_2, \ldots$ for $\mathbb{L}^*$, which are then considered as maps $x_n : S_k \to \Sigma^{2n,n}\text{MGL}$ via the classifying map, and these maps are in turn used to construct the tower which gives rise to the spectral sequence (which is then identified with the slice tower for MGL).

Let us now check that the map $t_1^\text{MGL}(\eta) : k(\eta)^* \to \text{MGL}^{1,1}(\eta)$ is the isomorphism given by the spectral sequence. Since $f_0\text{MGL} = \text{MGL}$, we have the distinguished triangle $f_!\text{MGL} \to \text{MGL} \xrightarrow{\eta_0} s_0\text{MGL} \to f_1\text{MGL}[1]$. Via the isomorphism $s_0\text{MGL} \cong \text{MZ}$, the unit for MGL goes to the unit in MZ, and thus MGL $\to$ MZ induces a commutative diagram

\[
\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{t_0^\text{MGL}} & \Sigma\mathbb{G}_m\text{MGL} \\
\downarrow{t_\text{MZ}} & & \downarrow{\Sigma\text{MZ}} \\
\Sigma\mathbb{G}_m\text{MZ} & & \\
\end{array}
\]

But the spectral sequence $E(1)$ arises by taking the slice tower for MGL and applying $\Sigma\mathbb{G}_m$. The spectral sequence computation we have just map shows that the isomorphism $\text{MGL}^{1,1}(\eta) \to H^1(\eta, \mathbb{Z}(1))$ arises from the edge homomorphism associated to the map $\Sigma\mathbb{G}_m\eta_0$, which is split by composing the inverse of the isomorphism $t_1^\text{MGL} : k(\eta)^* \to H^1(\eta, \mathbb{Z}(1))$ with $t_1^\text{MGL}$. This verifies our assertion for $t_1^\text{MGL}(\eta)$.

The remaining assertion follows from this and the product structure on the slice spectral sequence (see \([21]\)).

\textbf{Theorem 3.9.} Let $(\mathcal{E}, c)$ be an oriented weak commutative ring $T$-spectrum. Suppose that $(\mathcal{E}, c)$ is Landweber exact. Then

1. Both $(\mathcal{E}, c)$ and the connective cover $(f_0\mathcal{E}, c_0)$ are geometrically Landweber exact.
2. The coefficient ring of $f_0\mathcal{E}$ is the graded subring of $R^*_c$ given by truncation:

\[R^p_{f_0\mathcal{E}} = \begin{cases} 
R^p_c & \text{if } n \leq 0 \\
0 & \text{if } n > 0.
\end{cases}\]

3. The slice spectral sequences for $\mathcal{E}^{*,*}(Y)$ and for $f_0\mathcal{E}^{*,*}(Y)$ are strongly convergent for all $Y \in \text{Sm}/k$.

\textbf{Proof.} We first show (3). As $f_q(f_0\mathcal{E}) = f_q(\mathcal{E})$ for $q \geq 0$, it suffices to show that slice spectral sequences for $\mathcal{E}^{*,*}(Y)$ is strongly convergent for all $Y \in \text{Sm}/k$.

We have already seen that $\text{MGL}^{n+a,n}(Y) = 0$ for all $Y \in \text{Sm}/k$, $a > 0$, $n \in \mathbb{Z}$. Since $\mathcal{E}$ is Landweber exact, this shows that $\mathcal{E}^{2n+a,n}(Y) = 0$ for all $Y \in \text{Sm}/k$, $a > 0$, $n \in \mathbb{Z}$, and thus $\pi_{m+r,r}(\mathcal{E}) = 0$ for all $m \leq r-1$, $r \in \mathbb{Z}$. We may therefore apply lemma \([2.1]\) with $f$ the function $f(r) = r-1$, giving the convergence.

We now prove (1) and (2) by comparing the slice spectral sequences for $\mathcal{E}$ and $f_0\mathcal{E}$. For any spectrum $F$, and $Y \in \text{Sm}/k$, the $E_2$-term is given by

\[E_2^{p,q}(n)(F, Y) = [\Sigma^{\infty}_Y Y_+, \Sigma^{p+q,n}s_{-q}F]_{\text{SH}(k)}.
\]

Since $s_{-q}f_0\mathcal{E} = s_{-q}\mathcal{E}$ for $q \leq 0$ and is zero for $q > 0$, we have

\[E_2^{p,q}(n)(f_0\mathcal{E}, Y) = \begin{cases} 
E_2^{p,q}(n)(\mathcal{E}, Y) & \text{for } q \leq 0 \\
0 & \text{for } q > 0.
\end{cases}\]


Since $\mathcal{E} \cong \text{MGL}(R^*_F)$, it follows from [23] theorem 6.1\footnote{The assumption (SIMGL) in the statement of this result is fulfilled, by the work of Hoyois [5]} that

$$E^p,q_2(n)(\mathcal{E}, Y) = H^{p-q}(Y, R^*_F(n-q)).$$

The computation is now essentially the same as for $\mathcal{E} = \text{MGL}$, given in proposition 3.3. Take $Y = \text{Spec } F$ for some finitely generated field extension of $k$. We have

$$H^a(Y, R^*_F(b)) = 0$$

for $a > b$, $H^0(Y, R^*_F(0)) = R^*_F$ and $H^1(Y, R^*_F(1)) = R^*_F \otimes F^\times$. In addition, $H^a(Y, R^*_F(0)) = 0$ for $a \neq 0$, $H^0(Y, R^*_F(1)) = 0$ for $a \neq 0, 1$. This gives

$$E_2^{n,n}(n) = R^*_F, \quad E_2^{n,n-1}(n) = R^*_F \otimes F^\times$$

as the only $E_2$ terms contributing to $E^{2*,*}(k)$ and $(f_0E)^{2*,*}(k)$, resp. $E^{2*,1-*}(k)$ and $(f_0E)^{2*,1-*}(k)$. Just as for MGL, there are no differentials entering or leaving the $E_r^{n,n}(n)$ term or leaving the $E_r^{n,n-1}(n)$ term for all $r \geq 2$. Thus $E^{2n,n}(\eta) = R^*_F$, there is a surjective edge homomorphism $R^{n-1} \otimes k(\eta)^\times \to E^{2n-1,n}(\eta)$, and for $(p, q) = (2n, n)$ or $(p, q) = (2n-1, n)$, we have

$$(f_0E)^{p,q}(\eta) = \begin{cases} E^{p,q}(\eta) & \text{for } q \leq 0 \\ 0 & \text{for } q > 0. \end{cases}$$

Arguing as in the proof of proposition 3.3 result follows. \hfill \Box

4. ORIENTED DUALITY THEORIES

For the remainder of the paper, we will take $S = \text{Spec } k$, $k$ a field of characteristic zero.

Recall from [3] \S 1 the category $\text{SP}/k$ of smooth pairs over $k$, with objects $(M, X)$, $M \in \text{Sm}/k$ and $X \subset M$ a closed subset (not necessarily smooth); a morphism

$$f : (M, X) \to (N, Y)$$

is a morphism $f : M \to N$ in $\text{Sm}/k$ such that $f^{-1}(Y) \subset X$. For a full subcategory $\mathcal{V}$ of $\text{Sch}/k$, let $\mathcal{V}'$ be the subcategory of $\mathcal{V}$ with the same objects as $\mathcal{V}$, but with morphisms the projective morphisms in $\text{Sch}/k$.

Building on work of Mocanasu [12] and Panin [19], we have defined in [3] definition 3.1 the notion of a bi-graded oriented duality theory $(H, A)$ on $\text{Sch}/k$. Here $A$ is a bi-graded oriented cohomology theory on $\text{SP}/k$, $(M, X) \to A^{*,*}_X(M)$, and $H$ is a functor from $\text{Sch}/k$ to bi-graded abelian groups. The oriented cohomology theory $A$ satisfies the axioms listed in [3] definitions 1.2, 1.5]. In particular, $(M, X) \to A^{*,*}_X(M)$ admits a long exact sequence

$$\ldots \to A^{*,*}_X(M) \to A^{*,*}(M) \to A^{*,*}(M \setminus X) \xrightarrow{\partial} A^{*,1-*}_X(M) \to \ldots$$

where for instance $A^{*,*}(M) := A^{*,*}_X(M)$ and the boundary map $\partial$ is part of the data. In addition, there is an excision property and a homotopy invariance property. The ring structure is given by external products and pull-back by the diagonal. The orientation is given by a collection of isomorphisms $\text{Th}^X : A^*_X(M) \to A^*_X(E)$, for $(M, X) \in \text{SP}/k$ and $E \to M$ a vector bundle, satisfying the axioms of [19] def. 3.1.1.\footnote{We extend some of the results of [19] in [3] theorem 1.12, corollary 1.13 to show that the data of an orientation is equivalent to giving well-behaved push-forward maps $f_* : A^*_X(M) \to A^*_Y(N)$ for $(M, X), (N, Y) \in \text{SP}/k$, with the meaning of “well-behaved” detailed in [3] \S 1].}

The homology theory $H$ comes with restriction maps $j^* : H_{*,*}(X) \to H_{*,*}(U)$ for each open immersion $j : U \to X$ in $\text{Sch}/k$, external products $\times : H_{*,*}(X) \otimes H_{*,*}(Y) \to H_{*,*}(X \times Y)$, boundary maps $\partial_{X,Y} : H_{*,*}(X \setminus Y) \to H_{*,*}(X \setminus Y)$ for each...
closed subset $Y \subseteq X$, isomorphisms $\alpha_{M,X} : H_{*,*}(X) \to A_X^{2m-*,-*}(M)$ for each $(M,X) \in \text{SP}/k$, $m = \dim_k M$, and finally cap product maps

$$f^*(-) \cap : A_X^{a,b}(M) \otimes H_{*,*}(Y) \to H_{*,a-b}(Y \cap f^{-1}(X))$$

for $(M,X) \in \text{SP}/k$, $f : Y \to X$ a morphism in $\text{Sch}/k$. These satisfy a number of axioms and compatibilities (see [9] §3 for details), which essentially say that a structure for $A_X^{a,*}(M)$ is compatible with the corresponding structure for $H_{*,*}(X)$ via the isomorphism $\alpha_{M,X}$. Roughly speaking, this is saying that a particular structure for $A_X^{a,*}(M)$ depends only on $X$ and not the choice of embedding $X \to M$.

**Remark 4.1.** Let $L \to Y$ be a line bundle on some $Y \in \text{Sm}/k$ with 0-section $0 : Y \to L$. For an oriented cohomology theory $A$ one has the element

$$c_1^A(L) := 0^*(0_*(1^A_1)),$$

where $1^A_1 \in A^0(Y)$ is the unit element. As pointed out in [19] corollary 3.3.8, or as noted in [9] remark 1.17, for line bundles $L, M$ on some $Y \in \text{Sm}/k$, the elements $c_1(L), c_1(M) \in A^1(Y)$ are nilpotent, and commute with one another, hence for each power series $F(u,v) \in A^*(k)[[u,v]]$ the evaluation $F(c_1(L), c_1(M))$ gives a well-defined element of $A^*(Y)$. In addition, the cohomology theory $A$ has a unique associated formal group law characteristic $F_A(u,v) \in A^*(k)[[u,v]]$ with

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M)$$

for all line bundles $L, M$ on $Y \in \text{Sm}/k$.

For an $X \in \text{Sch}/k$ with line bundle $L \to X$, the quasi-projectivity of $X$ implies that $X$ admits a closed immersion $i : X \to M$ for some $M \in \text{Sm}/k$ such that $L$ extends to a line bundle $L \to M$. One can then define

$$\tilde{c}_1(L) : H_*(X) \to H_{*-1}(X)$$

via the product

$$(\cdot) \cdot c_1(L) : A_X^{*}(M) \to A_X^{*+1}(M)$$

and the isomorphisms $H_*(X) \cong A_X^{2m-*}(M)$. One shows that this is independent of the choice of $(M,L)$, giving the well-defined operator $\tilde{c}_1(L)$.

The main example of oriented duality theory $(H,A)$ is given by an weak oriented commutative ring $T$-spectrum $E$ in $\text{SH}(k)$, assuming $k$ is a field admitting resolution of singularities (e.g., characteristic zero), defined by taking

$$E_X^{a,b}(M) := \text{Hom}_{\text{SH}(k)}(\Sigma^\infty_-(M/M \setminus X), \Sigma^{a,b}E),$$

i.e., the usual bi-graded cohomology with supports. For each $X \in \text{Sch}/k$, choose a closed immersion of $X$ into a smooth $M$ and set $E_X^{a,b}(X) := E_X^{2m-a,m-b}(M)$, where $m = \dim_k M$. The fact that $(M,X) \to E_X^{*,*}(M)$ defines an oriented bi-graded ring cohomology theory is proved just as in the case of $E = \text{MGL}$, which was discussed in [9] §4; the main point is Panin’s theorem [14] theorem 3.7.4, which says that an orientation for $E$ (in the sense of definition [1.2]) defines an orientation in the sense of ring cohomology theories for the bi-graded $E$-cohomology with supports. We note that the orientation for the ring cohomology theory $E^{*,*}$ depends only on the class of a given orientation $\vartheta \in E^{2,1}(E^\infty_*)$ modulo phantom maps.
Remark 4.2. The results of [9] were proven in the setting of an oriented commutative ring $T$-spectrum $E$, not that of a weak oriented commutative ring $T$-spectrum. However, the constructions and proofs only use values of $E$-cohomology on finite diagrams of smooth $k$-schemes, and thus only rely on identities modulo phantom maps. The arguments thus remain valid in the larger context of weak oriented commutative ring $T$-spectra. We will henceforth make use of the results of [9] in this wider context without further comment.

The fact that the formula given above for the homology theory $E'_{*,*}$ is well-defined and extends to make $(E'_{*,*}(-), E_{*,*}(-))$ a bi-graded oriented duality theory is [9] theorem 3.4. The essential point is to show that the cohomology with support $E_X^{2d_* - 1, 2s_* - 1}(M)$, for $X \to Y$ a closed immersion of some $X$ in a smooth $M$ of dimension $d$, depends (up to canonical isomorphism) only on $X$, and similarly, given a projective morphism $f : Y \to X$ in $\text{Sch}/k$, there are smooth pairs $(M, X), (N, Y)$, an extension of $F$ to a projective morphism $F : N \to M$ and the map $F_* : E_Y^{2d_* - 1, 2s_* - 1}(N) \to E_X^{2d_* - 1, 2s_* - 1}(M)$ is independent (via the canonical isomorphisms $E_{*,*}^*(Y) \cong E_Y^{2d_* - 1, 2s_* - 1}(N), E_{*,*}^*(X) \cong E_X^{2d_* - 1, 2s_* - 1}(M)$) of the choices. The other structures for $E'_{*,*}(-)$ are defined similarly via the $E$-cohomology with supports, and one has the corresponding independence of any choices.

It follows directly from the construction of $E'$ that the assignment $(E, c_E) \mapsto (E', E)$ is functorial in the oriented cohomology theory $(E, c_E)$. In particular, let $\text{ch} : (\text{MGL}, c_{\text{MGL}}) \to (E, c_E)$ be a morphism of oriented cohomology theories, that is, $\text{ch} : \text{MGL} \to E$ is a monoid morphism in $\text{SH}(k)$ with $\text{ch}(c_{\text{MGL}}) = c_E$. Then we have an extension of $\text{ch}$ to a natural transformation of oriented duality theories

$$(\text{ch}', \text{ch}) : (\text{MGL}', \text{MGL}) \to (E', E)$$

5. Algebraic cobordism and oriented duality theories

We recall the theory of algebraic cobordism $X \mapsto \Omega_*(X), X \in \text{Sch}/k$ [11]. For each $X \in \text{Sch}/k$, $\Omega_n(X)$ is an abelian group with generators $(f : Y \to X), Y \in \text{Sm}/k$ irreducible of dimension $n$ over $k$ and $f : Y \to X$ a projective morphism [11] lemma 2.5.11]. $\Omega_*$ is the universal oriented Borel-Moore homology theory on $\text{Sch}/k$ [11] theorem 7.1.1], where an oriented Borel-Moore homology theory on $\text{Sch}/k$ consists of the data of a functor from $\text{Sch}/k'$ to graded abelian groups, external products, first Chern class operators $c_1(L) : \Omega_*(X) \to \Omega_{*+1}(X)$ for $L \to X$ a line bundle, and pull-back maps $g^* : \Omega_*(X) \to \Omega_{*+d}(Y)$ for each l.c.i. morphism $g : Y \to X$ of relative dimension $d$. These of course satisfy a number of compatibilities and additional axioms, see [11] §5.1 for details.

For an oriented duality theory $(H, A)$ on $\text{Sch}/k$ and $Y$ in $\text{Sm}/k$ of dimension $d$ over $k$, the fundamental class $[Y]_{H, A} \in H_d(Y)$ is the image of the unit $1_Y \in A^0(Y)$ under the inverse of the isomorphism $\alpha_Y : H_d(Y) \to A^0(Y)$. For an oriented Borel-Moore homology theory $B$ on $\text{Sch}/k$, we similarly have the fundamental class $[Y]_B \in B_d(Y)$ defined by $[Y]_B := p^*(1),$ where $1 \in B_0(\text{Spec}k)$ is the unit and $p : Y \to \text{Spec}k$ the structure morphism.

We recall the following result from [9].

Proposition 5.1 ([9] propositions 4.2, 4.4, 4.5]). Let $k$ be a field admitting resolution of singularities and let $(H, A)$ be a $\mathbb{Z}$-graded oriented duality theory on $\text{Sch}/k$.

1. There is a unique natural transformation $\partial_H : \Omega_* \to H_*$ of functors $\text{Sch}/k' \to$
GrAb, such that \( \vartheta_H(Y) \) is compatible with fundamental classes for \( Y \in \text{Sm}/k \).

In addition, \( \vartheta_H \) is compatible with pull-back maps for open immersions in \( \text{Sch}/k \), with 1st Chern class operators, with external products and with cap products.

2. For \( Y \in \text{Sm}/k \), the map \( \vartheta^A(Y) : \Omega^*(Y) \to A^*(Y) \) induced by \( \vartheta_H \), the identity \( \Omega^*(Y) = \Omega_{\dim Y - *}(Y) \) and the isomorphism \( \alpha_Y : H_{\dim Y - *}(Y) \to A^*(Y) \) is a ring homomorphism and is compatible with pull-back maps for arbitrary morphisms in \( \text{Sm}/k \). Finally, one has

\[
\vartheta^A(Y)(c_1^0(L)) = c_1^A(L)
\]

for each line bundle \( L \to Y \).

Remark 5.2. We have already noted that one has a formal group law \( F_A(u, v) \in A^*(k)[[u, v]] \) associated to the oriented cohomology theory \( A \). Similarly, for each oriented Borel-Moore homology theory \( B \) on \( \text{Sch}/k \), there is an associated formal group law \( F_B(u, v) \in B^*(k)[[u, v]] \), characterized by the identity \( F_B(c_1(L), c_1(M)) = c_1(L \otimes M) \) for each pair of line bundles \( L, M \) on some \( Y \in \text{Sm}/k \) (this follows from [11, corollary 4.1.8, proposition 5.2.1, proposition 5.2.6]). Letting \( \varphi_A : L^* \to A^*(k) \), \( \varphi_B : L^* \to B^*(k) \) denote the classifying maps associated to \( F_A \), \( F_B \), respectively, suppose that \( A \) extends to an oriented duality theory \( (H, A) \). Then

\[
\vartheta^A(F_A) = F_A.
\]

Indeed, \( F_A \) is characterized by identity \( F_A(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M) \) for each pair of line bundles \( L, M \) on some \( Y \in \text{Sm}/k \), and since \( \vartheta^A(c_1^A(N)) = c_1^A(N) \) for each line bundle \( N \to Z, Z \in \text{Sm}/k \), the fact that \( \vartheta^A(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M) \) combined with proposition [11, 2] yields the identity (5.1).

Via the universal property of the Lazard ring, the relation (5.1) is equivalent to the identity

\[
\vartheta^A \circ \varphi_{\Omega} = \varphi_A.
\]

Finally, we recall that the classifying map \( \varphi_{\Omega} : \mathbb{L}^* \to \Omega_*(k) \) is an isomorphism [11, theorem 1.2.7].

Corollary 5.3. Let \((\mathcal{E}, c_{\mathcal{E}})\) be pair consisting of a weak commutative ring \( T \)-spectrum \( \mathcal{E} \in S\mathcal{H}(k) \) with orientation \( c \), and let \((\mathcal{E}'_{*,*}, \mathcal{E}^{2*,*})\) be the corresponding bi-graded oriented duality theory. There is a unique natural transformation

\[
\vartheta_{(\mathcal{E}, c_{\mathcal{E}})} : \Omega_* \to \mathcal{E}'_{2*,*}
\]

of functors \( \text{Sch}/k' \to \text{GrAb} \), such that \( \vartheta_{(\mathcal{E}, c_{\mathcal{E}})}(Y) \) is compatible with fundamental classes for \( Y \in \text{Sm}/k \). In addition, \( \vartheta_{(\mathcal{E}, c_{\mathcal{E}})} \) is compatible with pull-back maps for open immersions in \( \text{Sch}/k \), 1st Chern class operators, external products and cap products. For \( Y \in \text{Sm}/k \), the map \( \vartheta^E(Y) : \Omega^*(Y) \to \mathcal{E}^{2*,*}(Y) \) induced by \( \vartheta_{(\mathcal{E}, c_{\mathcal{E}})} \) is a ring homomorphism and is compatible with pull-back maps for arbitrary morphisms in \( \text{Sm}/k \), and satisfies

\[
\vartheta_{(\mathcal{E}, c_{\mathcal{E}})}(Y)(c_1^0(L)) = c_1^E(L)
\]

for each line bundle \( L \to Y \).

Remark 5.4. By [11, lemma 2.5.11], \( \Omega_*(X) \) is generated as an abelian group by the cobordism cycles \( (f : Y \to X) \), \( Y \in \text{Sm}/k \) irreducible, \( f : Y \to X \) a projective
Furthermore, the identity \((f : Y \to X) = f_*([Y]_\Omega)\) holds in \(\Omega_{\dim Y}(X)\). Thus \(\vartheta_{(\mathcal{E}, \mathcal{E}')}(f : Y \to X)\) is characterized by the formula
\[
\vartheta_{(\mathcal{E}, \mathcal{E}')}(f : Y \to X) := f^*\left(\vartheta_{\mathcal{E}, \mathcal{E}'}\right).
\]

We may apply corollary 5.3 in the universal case: \(\mathcal{E} = \text{MGL}\) with its canonical orientation. This gives us the natural transformation
\[
(\vartheta_{\text{MGL}})_{\Omega_0} : \Omega_0 \to \text{MGL}^2_{2^*, \ast}.
\]

**Theorem 5.5 (\cite{10} theorem 3.1).** If \(k\) is a field of characteristic zero, then the natural transformation (5.3) is an isomorphism.

**Remark 5.6.** This result relies on the Hopkins-Morel spectral sequence, see \cite{4, 5}.

In the course of the proof of theorem 5.5, we proved another result which we will be using here.

Let \(X\) be in \textbf{Sch}/\(k\) and let \(d = d_X := \max_X \dim_X X'\), as \(X'\) runs over the irreducible components of \(X\). We define \(\text{MGL}^{(1)}_{2^*, \ast}(X)\) by
\[
\text{MGL}^{(1)}_{2^*, \ast}(X) := \lim_{\overset{W}{\longrightarrow}} \text{MGL}^2_{2^*, \ast}(W)
\]
as \(W\) runs over all (reduced) closed subschemes of \(X\) which contain no dimension \(d\) generic point of \(X\); \(\Omega^{(1)}_0(X)\) is defined similarly. The natural transformation \(\vartheta_{\text{MGL}}\) gives rise to the commutative diagram
\[
\begin{array}{ccc}
\Omega^{(1)}_0(X) & \xrightarrow{\vartheta^{(1)}} & \Omega_0(X) \\
\downarrow^{\vartheta(\mathcal{E})} & & \downarrow^{\vartheta(\mathcal{E})} \\
\text{MGL}^{(1)}_{2^*, \ast}(X) & \xrightarrow{\vartheta(\mathcal{E})} & \oplus_{\eta \in X(d)} \text{MGL}^2_{2^*, \ast}(k(\eta)) \\
\end{array}
\]
with exact rows and with all vertical arrows isomorphisms. As \((\text{MGL}', \text{MGL})\) is an oriented duality theory, the bottom line extends to the long exact sequence
\[
\ldots \to \oplus_{\eta \in X(d)} \text{MGL}^2_{2^*, \ast+1}(k(\eta)) \xrightarrow{\vartheta^{(1)}} \text{MGL}^{(1)}_{2^*, \ast}(X) \\
\xrightarrow{\vartheta(\mathcal{E})} \text{MGL}^2_{2^*, \ast}(X) \xrightarrow{\vartheta(\mathcal{E})} \oplus_{\eta \in X(d)} \text{MGL}^2_{2^*, \ast}(k(\eta)) \to 0.
\]
Furthermore, the Hopkins-Morel spectral sequence \cite{4, 5}
\[
E_2^{p, q} := L^{-q}_{-p} \otimes H^{p-q}(Y, \mathbb{Z}(n + q)) \Rightarrow \text{MGL}_{p+q,n}^{(1)}(Y)
\]
gives a surjection for each \(\eta \in X(d)\)
\[
t_{\text{MGL}}(\eta) : \mathbb{L}_{-d+1} \otimes k(\eta)^\times \to \text{MGL}^2_{2^*, \ast+1}(k(\eta))
\]
(see proposition 3.3). We have constructed in \cite{10, 6} a group homomorphism
\[
\text{Div} : \mathbb{L}_{-d+1} \otimes \oplus_{\eta \in X(d)} \mathbb{Z}[k(\eta)^\times] \to \Omega^{(1)}_0(X)
\]
with \(\vartheta^{(1)} \circ \text{Div} = \vartheta \circ \oplus_{\eta \in X(d)} t_{\text{MGL}}(\eta)\). Since the maps \(\vartheta(\mathcal{E})\) and \(\vartheta(\mathcal{E}')\) are isomorphisms, the map \(\text{Div}\) factors through the surjection
\[
\mathbb{L}_{-d+1} \otimes \oplus_{\eta \in X(d)} \mathbb{Z}[k(\eta)^\times] \to \mathbb{L}_{-d+1} \otimes \oplus_{\eta \in X(d)} k(\eta)^\times,
\]
we have the exact sequence
\[(5.5) \oplus_{\eta \in X(c)} \mathbb{L}^{s-d+1} \otimes k(\eta) \div \Omega^i_s(X) \xrightarrow{i^*} \Omega^i_s(X) \xrightarrow{j^*} \oplus_{\eta \in X(c)} \Omega^i_s(k(\eta)) \to 0\]
and the extension of diagram (5.4) to the commutative diagram
\[(5.6) \]
\[
\begin{array}{c}
\oplus_{\eta} \mathbb{L}^{s-d+1} \otimes k(\eta) \div \Omega^i_s(X) \xrightarrow{i^*} \Omega^i_s(X) \xrightarrow{j^*} \oplus_{\eta} \Omega^i_s(k(\eta)) \to 0 \\
\phi \downarrow \partial \downarrow \phi \downarrow \\
\oplus_{\eta} \text{MGL}^{i-1}(X) \xrightarrow{\partial(X)} \text{MGL}^i(X) \xrightarrow{j^*} \oplus_{\eta} \text{MGL}^i_k(\eta) \to 0
\end{array}
\]
with exact rows and vertical arrows isomorphisms. Here \(\text{div}_{\text{MGL}} := \partial \circ \eta \text{MGL}(\eta)\).

6. The comparison map

Let \((E, c)\) be a weak commutative ring \(T\)-spectrum in \(SH(k)\) with orientation \(c_E\), and let \((E'_2, c_2)\) be the corresponding bi-graded oriented duality theory. We have the natural transformation
\[\vartheta_{(E, c_E)} : \Omega \to E'_2,\]
given by corollary 5.3. In case \(E = \text{MGL}\) with its canonical orientation, \(\vartheta\) is an isomorphism. The map \(\vartheta_{(E, c_E)}(k)\) makes \(R^E_+\) an \(\Omega_+\)-algebra; we let \(\Omega^E_+\) be the oriented Borel-Moore homology theory \(\Omega^E_+ := R^E_+ \otimes_{\Omega_+} \Omega_+\).

As the external products make \(E'_2, c_2\) an \(R^E_+\)-module and the maps \(f_*, \tilde{c}_1(L)\) are \(R^E_+\)-module maps, we see that \(\vartheta_{(E, c_E)}\) descends to a natural transformation
\[\tilde{\vartheta}_{(E, c_E)} : \Omega^E_+ \to E'_2,\]
\[\begin{lemma}
Suppose that the oriented weak commutative ring \(T\)-spectrum \((E, c)\) is geometrically Landweber exact. Then
1. For \(X \in \mathbf{Sch}/k\) and \(\eta \in X\) a point, the map \(\tilde{\vartheta}_E : \Omega^E_+(\eta) \to E'_2(\eta)\) is an isomorphism.
2. Take \(X \in \mathbf{Sch}/k\) and let \(j_i : \eta_i \to X, i = 1, \ldots, r\) be all the generic points of \(X\). Then the restriction map \(\tilde{\vartheta}_E^{(j_i)} : E'_2(\eta_i) \to \oplus_{i=1}^r E'_2(\eta_i)\) is surjective.
3. For each generic point \(\eta\) of \(X\), the map \(t_{\tilde{\vartheta}_E}(\eta) : E'_2(\eta) \otimes \mathbb{Z} k(\eta)^{E_2} \to E'_2(X, \eta)\) is surjective.
\end{lemma}

\[\begin{proof}
For (1), we note that \(\tilde{\vartheta}_E = \varphi_{E} \circ \vartheta_{\text{MGL}}\). In addition \(\vartheta_{\text{MGL}}\) is an isomorphism, so \(\tilde{\vartheta}_E(X)\) is up to isomorphism the same as the map \(\text{MGL}^i_2(\eta) \otimes_{L_+} R^E_+ \to E'_2(\eta)\). (1) then follows from the hypothesis on \(E\).

For (2), we have the commutative diagram
\[
\begin{array}{c}
\Omega_+(X) \xrightarrow{j^*} \oplus_{\eta \in X(c)} \Omega_+(\eta) \\
\vartheta \downarrow \vartheta \downarrow \vartheta \downarrow \\
E'_2(\eta) \xrightarrow{j^*} \oplus_{\eta \in X(c)} E'_2(\eta)
\end{array}
\]
By (1), \(\vartheta(\eta) : \Omega_+(\eta) \to E'_2(\eta)\) is surjective. The map \(j^*_\Omega\) is also surjective, using the right exact localization sequence for \(\Omega_+\). Thus, the map \(j^*_E\) is also surjective.
For (3), we can rewrite the map $t_{\mathcal{E}}(\eta)$ as the composition

$$
R^s_{\mathcal{E}} \otimes_{\mathbb{L}} \mathbb{MGL}^{2s,*}(\eta) \otimes \mathbb{MGL}^{1,1}(\eta) \overset{1}{\longrightarrow} R^s_{\mathcal{E}} \otimes_{\mathbb{L}} \mathbb{MGL}^{2s+1,*,+1}(\eta)
$$

As the map $\mathbb{MGL}^{2s,*}(\eta) \otimes \mathbb{MGL}^{1,1}(\eta) \rightarrow \mathbb{MGL}^{2s+1,*,+1}(\eta)$ is surjective, and the map $\mathbb{MGL}^{2s+1,*,+1}(\eta) \rightarrow \mathbb{MGL}^{2s+1,*,+1}(\eta)$ is surjective by hypothesis, it follows that $t_{\mathcal{E}}(\eta)$ is also surjective.

Remark 6.2. By [11, corollary 4.4.3], $\Omega^{*}$ is generically constant in the sense of [11, definition 4.4.1], that is, for $F$ a finitely generated field over $k$, the pull-back map $\Omega^{*}(k) \rightarrow \Omega^{*}(F)$ is an isomorphism.

Theorem 6.3. Suppose $\mathcal{E}$ is geometrically Landweber exact. Then the natural transformation $\tilde{\vartheta}_{(\mathcal{E},c)} : \Omega_{*}^{c} \rightarrow \mathcal{E}'_{2s,*}$ is a natural isomorphism.

Proof. We write $\tilde{\vartheta}$ for $\tilde{\vartheta}_{(\mathbb{E},c)}$. For $\eta$ a dimension $d$ generic point of $X$, the map $\tilde{\vartheta}(\eta) : \Omega_{*}^{c} \rightarrow \mathcal{E}'_{2s,*}$ is an isomorphism by lemma 6.1(1). In particular, if $X$ has dimension zero over $k$, then $\tilde{\vartheta}(X)$ is an isomorphism.

We proceed by induction on the maximum $d$ of the dimensions of the components of $X$; we may assume that $X$ is reduced. We use the constructions and notations from theorem 6.3 and the discussion following that theorem. We let $\mathcal{E}_{2s,*}^{(1)}(X)$ be the inductive limit

$$
\mathcal{E}_{2s,*}^{(1)}(X) := \lim_{\longrightarrow W} \mathcal{E}_{2s,*}'(W)
$$

as $W$ runs over all (reduced) closed subschemes of $X$ which contain no dimension $d$ generic point of $X$. This, together with the map $\text{Div}$ defined following theorem 6.5 and the localization exact sequence for $\mathcal{E}'_{2s,*}$ gives us the commutative diagram with exact rows

$$
\begin{array}{rcc}
\oplus_{\eta \in X(d)} & \mathcal{E}_{2s+d,*,*}^{(1)}(k(\eta)) & \oplus_{\eta \in X(d)} \mathcal{E}_{2s+d,*,*}^{(1)}(X) \\
\longrightarrow & \oplus_{\eta \in X(d)} \mathcal{E}_{2s+d,*,*}^{(1)}(\eta) & \oplus_{\eta \in X(d)} \mathcal{E}_{2s+d,*,*}^{(1)}(X(\eta)) \\
\longrightarrow & \oplus_{\eta \in X(d)} \mathcal{E}_{2s+d,*,*}^{(1)}(\eta) & \oplus_{\eta \in X(d)} \mathcal{E}_{2s+d,*,*}^{(1)}(X(\eta)) \\
\end{array}
$$

the surjectivity in the bottom row comes from lemma 6.1(2).

We apply $R^s_{\mathcal{E}} \otimes_{\omega_{*}(k)} (-)$ to the top row in (6.1). As noted at the beginning of this section, the vertical maps in (6.1) descend to give the commutative diagram

$$
\begin{array}{rcc}
\oplus_{\eta \in X(d)} & R^s_{\mathcal{E}} \otimes_{\omega_{*}(k)} \mathcal{E}_{2s+d,*,*}^{(1)}(k(\eta)) & \oplus_{\eta \in X(d)} R^s_{\mathcal{E}} \otimes_{\omega_{*}(k)} \mathcal{E}_{2s+d,*,*}^{(1)}(X(\eta)) \\
\longrightarrow & \oplus_{\eta \in X(d)} R^s_{\mathcal{E}} \otimes_{\omega_{*}(k)} \mathcal{E}_{2s+d,*,*}^{(1)}(\eta) & \oplus_{\eta \in X(d)} R^s_{\mathcal{E}} \otimes_{\omega_{*}(k)} \mathcal{E}_{2s+d,*,*}^{(1)}(X(\eta)) \\
\longrightarrow & \oplus_{\eta \in X(d)} R^s_{\mathcal{E}} \otimes_{\omega_{*}(k)} \mathcal{E}_{2s+d,*,*}^{(1)}(\eta) & \oplus_{\eta \in X(d)} R^s_{\mathcal{E}} \otimes_{\omega_{*}(k)} \mathcal{E}_{2s+d,*,*}^{(1)}(X(\eta)) \\
\end{array}
$$

By induction on $d$, the map $\vartheta^{(1)}$ is an isomorphism; we have already seen that $\tilde{\vartheta}$ is an isomorphism. We note that the bottom row is a sequence of $R^s_{\mathcal{E}}$-modules via the the $\mathcal{E}_{2s,*}'(k(\eta))$-module structure given by external products.
Take \( \eta \in X_{(d)} \). By lemma 6.3, we have the surjection of \( R^\ell \)-modules

\[
t_\varepsilon : R^\ell_{s-d+1} \otimes k(\eta)^\times \to \mathcal{E}'_{2s+1,s}(\eta).
\]

Putting this into the diagram (6.2) gives us the commutative diagram

\[
\begin{array}{ccc}
\oplus_{\eta \in X_{(d)}} R^\ell_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{\partial} & \oplus_{\eta \in X_{(d)}} \Omega^2_{s} (\eta) \\
\downarrow \delta^{(1)} & & \downarrow \delta(X) \\
\oplus_{\eta \in X_{(d)}} R^\ell_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{\div_\varepsilon} & \mathcal{E}'_{2s+1,s}(X) \\
\downarrow \partial & & \downarrow \partial \\
\end{array}
\]

with the bottom row exact and the top row a complex.

We claim the identity map on \( R^\ell_{s-d+1} \otimes k(\eta)^\times \) fills in the diagram (6.3) to a (up to sign) commutative diagram. Assuming this claim, it follows by a diagram chase that the top row is exact and the map \( \delta(X) \) is an isomorphism.

To prove the claim, the orientation \( c \) for \( \varepsilon \) and the universal property of MGL gives the canonical map of oriented cohomology theories

\[
\varphi_\varepsilon : (\text{MGL}, c_{\text{MGL}}) \to (\varepsilon_c)
\]

which in turn gives us the map of bi-graded oriented duality theories

\[
\varphi_\varepsilon : (\text{MGL}'_{s,*}, \text{MGL}^{*,*}) \to (\varepsilon'_{s,*}, \varepsilon^{*,*}).
\]

It follows from the characterization of \( \partial_{\text{MGL}} \), \( \vartheta_\varepsilon \) given in remark 5.4 that

\[
\vartheta_\varepsilon = \varphi_\varepsilon \circ \partial_{\text{MGL}}.
\]

As discussed at the beginning of 3.3, the orientations \( c_{\text{MGL}} \), \( c \) give rise to canonical elements

\[
t_{\text{MGL}} \in \text{MGL}^{1,1}(\mathbb{G}_m), \ t_\varepsilon \in \varepsilon^{1,1}(\mathbb{G}_m).
\]

These in turn give by functoriality canonical homomorphisms for each \( X \in \text{Sm}/k \)

\[
t_{\text{MGL}}(X) : \mathcal{O}_X^\times (X) \to \text{MGL}^{1,1}(X), \ t_\varepsilon(X) : \mathcal{O}_X^\times (X) \to \varepsilon^{1,1}(X)
\]

with \( t_\varepsilon(X) = \rho_\varepsilon(X) \circ t_\varepsilon(X) \). We extend \( t_{\text{MGL}} \) to

\[
t_{\text{MGL}} : \mathcal{O}_X^\times (X) \otimes \mathbb{L}^* \to \text{MGL}^{2s+1,*,*+1}(X)
\]

using the \( \mathbb{L}^* \)-module structure, and similarly have the extension of \( t_\varepsilon(X) \) to

\[
t_\varepsilon : \mathcal{O}_X^\times (X) \otimes R^\ell_{s} \to \varepsilon^{2s+1,*,*+1}(X).
\]

The map \( k(\eta)^\times \to \text{MGL}^{1,1}(k(\eta)) \) arising in the Hopkins-Morel spectral sequence is the map \( t_{\text{MGL}}(\eta) \). As \( \varphi_\varepsilon \) is a map of \( \mathbb{L}^*-R^\ell \) modules, we have the commutative diagram

\[
\begin{array}{ccc}
\oplus_{\eta \in X_{(d)}} \mathbb{L}_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{t_{\text{MGL}}} & \oplus_{\eta \in X_{(d)}} \text{MGL}'_{2s+1,s} (k(\eta)) \\
\pi & & \varphi_\varepsilon \\
\oplus_{\eta \in X_{(d)}} R^\ell_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{t_\varepsilon} & \mathcal{E}'_{2s+1,s} (X) \\
\downarrow \partial & & \downarrow \partial \\
\end{array}
\]

where \( \pi \) is induced by the classifying map \( \mathbb{L}_{s} \to R^\ell_{s} \). The map \( \div_{\text{MGL}} \) in diagram (5.0) is the composition \( \partial \circ t_{\text{MGL}} \) in the diagram above. Defining

\[
\div_\varepsilon : \oplus_{\eta \in X_{(d)}} R^\ell_{s-d+1} \otimes k(\eta)^\times \to \mathcal{E}'_{2s+1,s} (X)
\]
as $\text{div}_E := \partial \circ t_E$ gives us the commutative diagram
\[
\begin{array}{ccc}
\oplus_{\eta \in X_{(d)}} L_{s-d+1} \otimes k(\eta)^{\times} & \xrightarrow{\text{div}_E \otimes L} & \Omega^{(1)}_{2s,*}(X) \\
\downarrow & & \downarrow \\
\oplus_{\eta \in X_{(d)}} R^E_{s-d+1} \otimes k(\eta)^{\times} & \xrightarrow{\text{div}_E} & E'^{(1)}_{2s,*}(X)
\end{array}
\]
patching in the left-hand square in the commutative diagram (5.6) yields the commutative diagram (6.4)
\[
\begin{array}{ccc}
\oplus_{\eta \in X_{(d)}} L_{s-d+1} \otimes k(\eta)^{\times} & \xrightarrow{\text{Div}} & \Omega^{(1)}_{s}(X) \\
\downarrow & & \downarrow \\
\oplus_{\eta \in X_{(d)}} R^E_{s-d+1} \otimes k(\eta)^{\times} & \xrightarrow{\text{div}_E \otimes L} & E'^{(1)}_{2s,*}(X)
\end{array}
\]
As $\text{Div}_E : R^E_s \otimes k(\eta)^{\times} \to \Omega^{E^{(1)}_s}(X)$ is just the map formed by applying the functor $R^E_s \otimes L (\cdot)$ to $\text{Div} : L \otimes k(\eta)^{\times} \to \Omega^{(1)}_{s}(X)$, the desired commutativity follows from the commutativity of (6.4).

**Corollary 6.4.** Let $(E, c)$ be a Landweber exact oriented weak ring $T$-spectrum in $SH(k)$, $k$ a field of characteristic zero and let $(f_0 E, c_0)$ be the connective cover of $(E, c)$. Then the canonical natural transformations of oriented Borel-Moore homology theories on $\text{Sch}_k$
\[
d_{E,c} : R^E_s \otimes L \Omega_s \to E'_{2s,*},
d_{f_0 E, c_0} : R^{f_0 E}_s \otimes L \Omega_s \to f_0 E'_{2s,*}
\]
are isomorphisms. Moreover, the canonical natural transformations of oriented cohomology theories on $\text{Sm}/k$
\[
d_{E,c} : R^E_s \otimes L \Omega_s^* \to E^{2s,*},
d_{f_0 E, c_0} : R^{f_0 E}_s \otimes L \Omega_s^* \to f_0 E^{2s,*}
\]
are isomorphisms.

**Proof.** By theorem 3.9 both $(E, c)$ and $(f_0 E, c_0)$ are geometrically Landweber exact. We then apply theorem 6.3 to yield the desired isomorphisms of oriented Borel-Moore homology theories.

The statement about the oriented cohomology theories on $\text{Sm}/k$ follows by restriction from $\text{Sch}_k$ to $\text{Sm}/k$, using the equivalence of oriented Borel-Moore homology theories and oriented cohomology theories on $\text{Sm}/k$ [11 proposition 5.2.1].
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Universität Duisburg-Essen, Fakultät Mathematik, Thea-Leymann-Strasse 9, 45127 Essen, Germany

E-mail address: marc.levine@uni-due.de