Anomaly-free multiple singularity enhancement in F-theory

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We study global Calabi–Yau realizations of multiple singularity enhancement relevant to family-unification model building in F-theory. We examine the conditions under which the generation of extra chiral matter at multiple singularities on 7-branes in 6D F-theory can be consistent with anomaly cancellation. It is shown that the generation of extra matter is consistent only if it is accompanied by simultaneous degenerations of loci of the leading polynomial of the discriminant so that the total amount of chiral matter does not change. We also show that the number of singlets expected to arise matches the decrease of the complex structure moduli for the restricted geometry.

1. Introduction

In Ref. [1], it was pointed out that multiple singularities on 7-branes in F-theory [2] may serve as the basis for a realization of the coset sigma model spectrum relevant for “family unification” [3–17]. The key observation was that, in six dimensions, the representation of chiral matter localized at an enhanced split-type singularity [18–20] is labeled by some homogeneous Kähler manifold, the reason for which was explained [21,22] by investigating the string junctions [23–33] near the singularity. Applying the same argument to the singularities where multiple matter branes simultaneously intersect the gauge 7-branes, it was argued that the chiral matter hypermultiplet spectrum at such a multiple singularity consists of those that form a homogeneous Kähler manifold with more than one $U(1)$ factor in the denominator of the coset.

The aim of the present paper is to clarify whether one can realize a Kähler coset of the form $G/(H \times U(1)^r)$ with $r \geq 2$ as a local matter spectrum without conflicting anomaly cancellation.\(^1\) In six dimensions the condition for anomaly cancellation imposes a severe restriction on the chiral matter spectrum [35–38].\(^2\) At first sight, it seems that the matter spectrum corresponding to a coset with multiple $U(1)$ factors ruins the balance of anomalies since it needs to be accompanied by the generation of extra chiral hypermultiplets. We will, however, show that such a coset spectrum is indeed possible. Rather, in some cases when the complex structure moduli take certain values, the absence of anomalies requires that there must occur such generation of extra matter at the

\(^1\) Singularity enhancement with rank more than one has been considered in an extensive study [34] on singularities and matter representations.

\(^2\) Anomaly analysis has also been useful for the study of 6D conformal field theories (see, e.g., Refs. [39–43]).
multiple singularity, thereby proving the conjecture of Ref. [1]. Although we work in the F-theory compactifications on elliptic Calabi–Yau threefolds over a Hirzebruch surface $F_n$ [18–20], the best understood example of an F-theory compactification, the mechanism that we find is local and will apply to other compactifications on elliptic Calabi–Yau manifolds.

We would like to emphasize that this paper is not intended, of course, to show the absence of anomalies at particular points in the moduli space of the F-theory compactification on a CY3 over $F_n$, nor does it aim to show that there can be an anomaly somewhere in the moduli space. Rather, assuming the absence of anomalies, we will use the anomaly analysis to investigate the local matter spectrum at a multiply enhanced singularity and to show that it coincides with some Kähler coset relevant to family-unification model building. We will consider in this paper the cases of enhancement from $SU(5)$ to $SO(12)$, $E_6$, $E_7$, and $E_8$, which are the most interesting cases for phenomenological applications including $E_7/(SU(5) \times U(1)^3)$ (the Kugo–Yanagida model) or $E_8/(SU(5) \times U(1)^4)$ relevant for family unification in the SU(5) GUT, and already reveal the essential features. The generalization to other groups will be presented in a separate publication.

In the next section we recall what representations of hypermultiplets are expected to arise at a multiple singularity. In Sect. 3 we review the anomaly cancellation mechanisms for $N = 1, D = 6$ supersymmetric theories. We will also see there that, in the case of 6D F-theory on an elliptic CY3 over $F_n$, which is known to be dual to an $E_8 \times E_8$ heterotic string on $K3$, no net increase of chiral matter is allowed either by the ordinary heterotic Green–Schwarz mechanism or by the generalized Green–Schwarz mechanism first applied by Sadov. In Sect. 4, we give examples of 7-brane configurations with some multiple singularities at which the coset spectrum is locally realized, but the total number of hypermultiplets in each representation does not change compared with the generic 7-brane configurations in the nearby moduli space, and hence the theory remains anomaly-free. As we will see, such a transition is possible if and only if it is accompanied by simultaneous degenerations of loci of the leading polynomial of the discriminant so that the necessary extra degrees of freedom at the singularity may be supplemented by the appropriate number of “extra-zero” loci joining there simultaneously. We will also show that the decrease of the dimensions of the moduli space for the special class of configurations matches the number of new singlets appearing at the multiple singularity, which is consistent with the anomaly cancellation.

2. Multiple singularity enhancement in F-theory in six dimensions

2.1. F-theory on an elliptic CY3 over $F_n$

Let us recall the basic setting of the F-theory compactification on an elliptically fibered Calabi–Yau over a Hirzebruch surface $F_n$ [18–20]. The threefold is defined by the Weierstrass equation:

\[ y^2 = x^3 + f(z, z') x + g(z, z'), \]  

\[ f(z, z') = \sum_{i=0}^{8} z^i f_{8+i}(z'), \]  

\[ g(z, z') = \sum_{i=0}^{12} z^i g_{12+i}(z'). \]

A Hirzebruch surface $F_n$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. $z$ and $z'$ are the coordinates of the fiber and the base, respectively. The coefficients $f_{8+i}(z')$ ($i = 0, \ldots, 6$) and $g_{12+i}(z')$ ($i = 0, \ldots, 12$)
are polynomials of \( z' \) of degrees specified by the subscripts. Both \( x \) and \( y \) are complex, so Eq. (1) determines some torus at each \((z, z')\). More precisely, \( x, y, f, \) and \( g \) are sections of \( \mathcal{L}^2, \mathcal{L}^3, \mathcal{L}^4, \) and \( \mathcal{L}^6 \), where \( \mathcal{L} \) is the anticanonical line bundle of the base \( F_n \). The total space is then an elliptic Calabi–Yau threefold, which is also a \( K3 \) fibration over the \( \mathbb{P}^1 \) parameterized by \( z' \).

In order to illustrate what kind of singularity we are interested in, let us first consider a concrete example. Suppose that the coefficient polynomials of the lower-order terms in the expansions of \( f(z, z') \) (2) and \( g(z, z') \) (3) take the particular forms

\[
\begin{align*}
 f_{4n+8} &= -3h_{n+2}^4, \\
 f_{3n+8} &= 12h_{n+2}^2H_{n+4}, \\
 f_{2n+8} &= 12(h_{n+2}q_{n+6} - H_{n+4}^2), \\
 g_{6n+12} &= 2h_{n+2}^6, \\
 g_{5n+12} &= -12h_{n+2}^4H_{n+4}, \\
 g_{4n+12} &= 12h_{n+2}^2(2H_{n+4}^2 - h_{n+2}q_{n+6}), \\
 g_{3n+12} &= -f_{n+8}h_{n+2}^2 + 24h_{n+2}H_{n+4}q_{n+6} - 16H_{n+4}^3, \\
 g_{2n+12} &= -f_{n+8}h_{n+2}^2 + 2f_{n+8}H_{n+4} + 12q_{n+6}^2
\end{align*}
\]  (4)

for some polynomials \( h_{n+2}, H_{n+4}, \) and \( q_{n+6} \); they are so arranged that the discriminant starts with the \( z^5 \) term to produce an \( I_5 = SU(5) \) Kodaira singularity \([44]\) along the line \( z = 0 \). For later convenience we present an explicit form of the lower-order expansions of this curve in Appendix A.

The independent polynomials preserving this particular singularity structure are

\[
h_{n+2}, H_{n+4}, q_{n+6}, f_{n+8}, \text{ and } g_{n+12}.
\]  (5)

The total degrees of freedom are thus

\[
(n + 3) + (n + 5) + (n + 7) + (n + 9) + (n + 13) - 1 = 5n + 36,
\]  (6)

which matches the number of \( SU(5) \) singlets computed by using the index theorem on the heterotic side.\(^3\)

Since the leading-order term of the discriminant \( \Delta \) is

\[
\Delta = 108z^5h_{n+2}^4P_{3n+16} + \cdots,
\]

\[
P_{3n+16} \equiv -2f_{n+8}h_{n+2}^2H_{n+4} - 2f_{n+8}h_{n+2}q_{n+6} + f_{n+2}h_{n+2}^4 + g_{n+12}h_{n+2}^2 - 24H_{n+4}q_{n+6}^2,
\]  (7)

the singularity gets enhanced to a higher one at the \( n + 2 \) zero loci of \( h_{n+2} \) and the \( 3n + 16 \) loci of \( P_{3n+16} \).\(^4\)

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\(^3\) Note that the “middle” coefficients \( f_8, g_{12} \) and the higher ones \( f_{8-n}; \ldots; g_{12-n}; \ldots \) are not counted here as the complex structure moduli that are to be compared with the singlets arising from this \( E_8 \) factor. This is because the middle ones \( f_8, g_{12} \) correspond to the geometric moduli of the elliptic \( K3 \) while the higher ones are taken into account in the similar analysis for the singularity at \( z = \infty \), corresponding to the other (partially broken) \( E_8 \) gauge factor.

\(^4\) Although it contains \( f_8 \) and \( f_{8-n} \), they only affect the positions of the loci and do not affect the total number of loci.
At the zero loci of a generic $P_{3n+16}$ (so that $h_{n+2} \neq 0$, in particular), the order of $\Delta$ becomes $\geq 6$ while ord$f$ and ord$g$ remain zero. If ord$\Delta = 6$, the singularity is enhanced to $I_6 = SU(6)$ and a chiral matter in 5 appears at each zero locus of $P_{3n+16}$. On the other hand, at the $n+2$ loci of $h_{n+2}$, the first few terms of $f$ and $g$ simultaneously vanish so that $f$ starts with $z^2$ and $g$ with $z^3$, as long as $H_{n+4}$ does not vanish there. Also, the order of the discriminant becomes 7. This is the $I_7^* = SO(10)$ singularity, and the chiral matter is 10 at each zero of $h_{n+2}$ for generic $P_{3n+16}$. In all, the matter spectrum for the generic $SU(5)$ curve is

$$(n + 2)10, \ (3n + 16)5, \ (5n + 36)1. \quad (8)$$

Originally [20], the kind of charged matter that should appear at these enhanced “extra zeros” was determined by referring to the massless spectrum of the dual heterotic model [18,19], i.e., the $K3$ compactification of the $E_8 \times E_8$ heterotic string with instanton numbers $(12 - n, 12 + n)$. The relationship between the extra zeros of the discriminant and the massless charged matter was first explained by Katz and Vafa [21] by mapping the problem to that of deformations of the singularities of $K3$. Later it was proposed by one of the present authors [22] that the chiral matter spectrum is understood by investigating string junctions near the enhanced singularity.

2.2. Spectral cover, matter localization, and the Mordell–Weil group

One of the remarkable features of heterotic/F-theory duality is that a brane-like object naturally comes into play in heterotic theory through the construction of a vector bundle over the elliptic Calabi–Yau manifold [45]. Basically, the statement of heterotic/F-theory duality is made in a certain limit in the moduli space on both sides: F-theory is compactified on a $K3$-fibered Calabi–Yau where the $K3$ goes to a stable degeneration limit into two, themselves elliptically fibered, $dP_9$ intersecting along a two-torus $E$, and heterotic string theory is on an elliptically fibered Calabi–Yau whose fiber torus has a large volume and the same complex structure as $E$. The moduli space of the vector bundle over each torus is known as Looijenga’s weighted projective space; for an $SU(5)$ gauge group this is an ordinary projective space. The spectral cover is a polynomial equation of $x$ and $y$, the variables in the Weierstrass equation describing a heterotic torus fiber. The defining polynomial has five (for $SU(5)$) zero loci (which add up to zero) on the torus, each of which specifies a Wilson line of a Cartan generator and coordinatizes Looijenga’s projective space. In the $SU(5)$ case, the polynomial is explicitly [45]:

$$w = a_0 + a_2x + a_3y + a_4x^2 + a_5xy \quad (9)$$

for some coefficients $a_0, \ldots, a_5$.

On the other hand, we consider a pencil [46]

$$(y^2 + x^3 + a_1xy + a_2x^2v^2 + a_3yv^3 + a_4xy^4 + a_6v^6) + p(v,x,y)u = 0, \quad (10)$$

$$p(v,x,y) = a_0v^5 + a_2xv^3 + a_3yv^2 + a_4x^2v + a_5xy \quad (11)$$

in $WP^{3}_{(1,1,2,3)}$ with the equivalence relation $(u,v,x,y) \sim (\lambda u, \lambda v, \lambda^2 x, \lambda^3 y), \lambda \in \mathbb{C}$. Obviously, $p(v,x,y)$ (11) is the homogenization of $w$ (9). After blowing up $u = v = 0$, the pencil (10) becomes $dP_9$, which we regard as one of two $dP_9$ appearing in the stable degeneration limit on the F-theory side. Indeed, we can show that, if we set
\[a_5 = 2\sqrt{3}u^{-1}h_{n+2},\]
\[a_4 = u^{-1}(\sqrt{3}\alpha_1 h_{n+2} + 6H_{n+4}),\]
\[a_3 = -\sqrt{3}u^{-1}\left(\frac{1}{6}(\alpha_2^2 - 4\alpha_2)h_{n+2} + 4q_{n+6}\right),\]
\[a_2 = u^{-1}\left(\sqrt{3}\left(-\frac{1}{12}\alpha_3^3 + \frac{1}{3}\alpha_1\alpha_2 + \alpha_3\right)h_{n+2} + (-\alpha_1^2 + 4\alpha_2)H_{n+4} - 2\sqrt{3}\alpha_1 q_{n+6} + f_{n+8}\right),\]
\[a_0 = u^{-1}\left(\frac{\sqrt{3}}{12}\alpha_3 (4\alpha_2 - \alpha_1^2) h_{n+2} + (2\alpha_4 - \alpha_1\alpha_3)H_{n+4} - 2\sqrt{3}\alpha_3 q_{n+6} + \frac{1}{12}(4\alpha_2 - \alpha_1^2)f_{n+8} - g_{n+12}\right).\] 
\[(12)\]

The pencil (10) precisely reproduces the lower terms (up to the “middle” ones) of the SU(5) Weierstrass equation (1), (2), (3) with (4). Therefore, the polynomials (5) of Ref. [20] correspond to
\[h_{n+2} \sim a_5, \quad H_{n+4} \sim a_4, \quad q_{n+6} \sim a_3, \quad f_{n+8} \sim a_2, \quad g_{n+12} \sim a_0.\] 
\[(13)\]

Furthermore, it was also shown by using the Leray spectral sequence [47,48] that the matter is localized where some of the \(a_i\) vanish or satisfy a relation so that one of the sections of \(dP_0\) goes to infinity (the zero section). We note that this may be intuitively understood as a consequence of the structure theorem of the Mordell–Weil group [49–51]. Indeed, the equation \(p(v, x, y) = 0\) defines sections of \(dP_0\), and, since the structure theorem [52,53] states that the singularities and sections are orthogonal complements of each other in \(E_8\), the fewer sections we have, the more singularities we get instead.

The Mordell–Weil lattice was studied in detail in terms of string junctions in Ref. [29] using the isomorphism between the string junction algebra and the Picard lattice of a rational elliptic surface. For a recent F-theory phenomenological aspect of the Mordell–Weil group see Refs. [54,55].

2.3. Multiple singularity enhancement from SU(5) to \(E_6\)
We will now consider what happens if \(h_{n+2}\) and \(H_{n+4}\) simultaneously vanish. In this case, the \(z^2\) term of \(f\) and the \(z^3\) term of \(g\) vanish at these points, and the order of the discriminant increases to 8. This means that the singularity gets enhanced from \(I_5 = SU(5)\) to \(IV^* = E_6\) there. We summarize Kodaira’s classification in Table 1.

| Fiber type | ord(\(f\)) | ord(\(g\)) | ord(\(\Delta\)) | Singularity type | 7-branes | Brane type |
|------------|-------------|-------------|-----------------|-----------------|---------|------------|
| \(I_n\)    | 0           | 0           | \(n\)           | \(A_{n-1}\)     | \(A^n\) | \(A_{n-1}\) |
| \(II\)     | \(\geq 1\)  | 1           | 2               | \(A_0\)         | \(AC\)  | \(H_0\)    |
| \(III\)    | 1           | \(\geq 2\)  | 3               | \(A_1\)         | \(A^2C\) | \(H_1\)    |
| \(IV\)     | \(\geq 2\)  | 2           | 4               | \(A_2\)         | \(A^3C\) | \(H_2\)    |
| \(I_n^*\)  | 2           | \(\geq 3\)  | 6               | \(D_4\)         | \(A^4BC\) | \(D_4\)    |
| \(I_n^*\)  | 2           | \(\geq 3\)  | \(6 + n\)       | \(D_{n+4}\)     | \(A^{n+4}BC\) | \(D_{n+4}\) |
| \(II^*\)   | \(\geq 4\)  | 5           | 10              | \(E_6\)         | \(A^5BC^2\) | \(E_6\)    |
| \(III^*\)  | 3           | \(\geq 5\)  | 9               | \(E_7\)         | \(A^6BC^2\) | \(E_7\)    |
| \(IV^*\)   | \(\geq 3\)  | 4           | 8               | \(E_6\)         | \(A^5BC^2\) | \(E_6\)    |
Fig. 1. Left: $D_5$ singularity. Right: $E_6$ singularity.

Note that, if $h_{n+2} = 0$, that $H_{n+4}$ vanishes means that $P_{3n+16}$ also does. Thus this higher singularity can be viewed as a consequence of a collision of an $I_7^c = SO(10)$ singularity, occurring at a zero of $h_{n+2}$, and an $I_6 = SU(6)$ singularity, which corresponds to a zero of $P_{3n+16}$.

In the standard 7-brane representation of the Kodaira singularity, the $SU(5)$ singularity is made of a collection of five $A$-branes, while the $SO(10)$ singularity is represented by $A^5BC$. Thus the zero loci of the polynomial $h_{n+2}$ are the places where $B$- and $C$-branes intersect the five $A$-branes lying on top of each other (Fig. 1, left). On the other hand, if $H_{n+4}$ happens to vanish at the same point, then the singularity becomes $E_6$, which is represented by $A^5BCC$. Therefore, this multiple singularity occurs when an extra $C$-brane simultaneously meets the five $A$-branes, in addition to the $B$- and $C$-branes (Fig. 1, right).

However, suppose that we move slightly away from this special point in the complex structure moduli space to another where $h_{n+2}$ and $H_{n+4}$ do not simultaneously vanish but the roots of $h_{n+2} = 0$ and $P_{3n+16} = 0$ are still close. (Note that, if $h_{n+2}$ is not zero, $H_{n+4} = 0$ does not mean that $P_{3n+16} = 0$.) This will correspond to the split of the multiple $E_6$ singularity into an $SO(10)$ singularity and an $SU(6)$ singularity. While it is fine for the pair of $B$- and $C$-branes to form the $D_5$ singularity, how can the remaining $C$-brane yield the $A_5$ singularity with the five $A$-branes?

This apparent contradiction can be explained as follows: We should first note that any isolated discriminant locus has monodromy

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and hence, locally, is identified as a location of an $A$-brane. However, when this discriminant locus merges with a $D_5$ singularity to form an $E_6$ singularity, a pair of zero loci of $g$ and $f$ also joins with the discriminant locus since the orders of $g$ and $f$ are respectively enhanced by one.

To understand why the $C$-brane can produce the $A_5$ singularity, we must know the monodromies around the zero loci of $f$ and $g$. The zero locus of $f$ is mapped, by the inverse $J$-function via the relation

$$J(\tau) = \frac{4f^3}{4f^3 + 27g^2}.$$
Fig. 2. An $E_6$ singularity is split into $D_5$ and $A_5$ singularities.

to (taking the starting point in $\text{Im} J > 0$ in the standard fundamental region of the modular group) $\tau = e^{2\pi i}$, near which $J(\tau)$ behaves like

$$J(\tau) = \left( \tau - e^{2\pi i} \right)^3 \left( 1 + O \left( \tau - e^{2\pi i} \right) \right).$$  \hspace{1cm} (16)

That is, if $J$ changes its value along a small closed path encircling 0 three times, $\tau$ goes around $e^{2\pi i}$ precisely once, back to the original fundamental region; this can be verified by tracing the value of the $J$ function [22]: Since $J \simeq \text{const.} f^3$ near $f = 0$, if one goes around the zero locus of $f$ once counterclockwise on the $z$ plane, the value of $J$ goes around zero three times counterclockwise. Therefore, the monodromy around the locus of $f$ is

$$(ST^{-1})^3 = -1$$

$$\simeq 1 \text{ in } PSL(2, \mathbb{Z}),$$  \hspace{1cm} (17)

and hence is the identity as a modular transformation. Here

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (18)

Likewise, the zero locus of $g$ is mapped to $\tau = i$, and the expansion of $J(\tau)$ is then

$$J(\tau) = 1 + (\tau - i)^2(1 + O(\tau - i)).$$  \hspace{1cm} (19)

So, if the value of $J$ goes around 1 twice, $\tau$ then goes around $i$ once, again back to the original fundamental region. Since $J - 1 \simeq \text{const.} g^2$ near $g = 0$, circling around the zero locus of $g$ once on the $z$ plane means that the corresponding $\tau$ circles around $i$ once. Thus the monodromy around the locus of $g$ reads

$$(S^{-1})^2 = -1$$

$$\simeq 1 \text{ in } PSL(2, \mathbb{Z}),$$  \hspace{1cm} (20)

and again is identity.

Now let us consider the effect of these loci of $f$ and $g$ to the monodromy of the other coalescing 7-branes. As we discussed above, any discriminant locus is locally an $A$-brane. However, when this
and the $A^5BC$-branes come close to merging, it turns out that there is also a locus of $g$ situated in between them. So, if the reference point of the monodromy is set near the $A^5BC$-branes, then one undergoes the $S^{-1}$ transformation when one passes by the locus of $g$. Therefore, the monodromy of the discriminant locus is

$$S^{-1}TS^{-1}$$  \hspace{1cm} (21)

(Fig. 3), which is equal to

$$-T^{-1}CT \simeq T^{-1}CT \quad \text{in } PSL(2, \mathbb{Z}),$$  \hspace{1cm} (22)

where $C$ is the monodromy matrix of $C$:

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}. \hspace{1cm} (23)$$

Since the monodromy matrix of $A^5BC$ is $=-T$, which is invariant under $T$ conjugation, we see that this discriminant locus can merge with the $D_5$ singularity as a $C$-brane. If, on the other hand, the $B$- and $C$-branes split off from the coalesced five $A$-branes and one “$C$-brane”, the locus of $g$ also splits off, and therefore what has been a $C$-brane in the $E_6$ singularity turns into an $A$-brane, yielding the $A_5$ singularity together with the five $A$-branes (Fig. 2).

As we saw in the previous section, if the $A$-brane alone meets the gauge 7-branes while the $B$- and $C$-branes are apart, then this singularity corresponds to one of the roots of $P_{3n+16} = 0$ and the charged matter 5 appears. These Bogomol’nyi–Prasad–Sommerfield (BPS) states are thought of as coming from string junctions connecting the $A$-brane and the gauge branes localized near the intersection point. Likewise, if only the $B$- and $C$-branes intersect while the $A$-brane is apart, the
**Table 2.** Singularities and string junctions for the unbroken \(SU(5)\) case.

| Singularity | 7-brane | String junction | \(SU(5)\) rep. |
|-------------|---------|----------------|-----------------|
| Generic \(z'\) | \(I_5\) | \(A^5\) | \(\pm(a_i - a_j) (i < j) + \text{Cartan}\) | 24 (gauge symmetry) |
| Locus of \(h_{n+2}\) | \(I_1^*\) | \(A^5BC\) | \(a_i + a_j - b - c (i < j)\) | 10 |
| Locus of \(P_{3n+16}\) | \(I_6\) | \(A^5A'\) | \(a_i - a' (i = 1, \ldots, 5)\) | 5 |
| Common locus of \(h_{n+2}\) and \(P_{3n+16}\) | \(IV^*\) | \(A^5BCC'\) | \(a_i + a_j - b - c (i < j)\) | 10 |
|  |  |  | \(a_i + a_j - b - c' (i < j)\) | 10 |
|  |  |  | \(\sum_{k=1}^5 a_k - a_i - 2b - c - c'\) | 5 |
|  |  |  | \(c - c'\) | 1 |

Singularity is of \(SO(10)\) and a 10 will arise due to the string junctions connecting the the B- and C-branes and the gauge branes; this happens at the loci of \(h_{n+2}\). Therefore, if \(P_{3n+16} = 0\) and \(h_{n+2}\) are simultaneously zero, then there will arise both 10 and 5 at that point. The former comes from the string junctions \(a_i + a_j - b - c (1 \leq i < j \leq 5)\) while the latter can be identified as those of the form \(\sum_{k=1}^5 a_k - a_i - 2b - c - c'\) (Table 2).

But we also notice that, at this \(E_6\) point, there are not only these two kinds of string junctions but still more BPS junctions: those of the form \(a_i + a_j - b - c' (1 \leq i < j \leq 5)\) and \(c - c'\). They are special BPS junctions that appear only at this higher singularity and are not present at generic points in the moduli space. Since they are BPS, these extra string junctions are also expected to give rise to chiral matter at the multiple singularity. This was the proposal of Ref. [1].

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\(^5\) For each matter locus there are in fact twice as many BPS junctions corresponding to overall \(\pm\) multiplications.
However, there is a puzzle about this naive expectation: Suppose that the theory is at some generic point in the moduli space of an elliptic Calabi–Yau threefold over a Hirzebruch surface where it is dual to some $E_8 \times E_8$ heterotic string compactification on $K3$. Of course, the theory is anomaly-free. Then suppose that the values of the moduli parameters are tuned to some special ones so that the 7-branes develop a multiple singularity. If it gives rise to more chiral matter hypermultiplets than those present at a generic point in the moduli space, does that not conflict with anomaly cancellation?

In the next section, we will see that the requirement of anomaly cancellation severely limits the conditions under which this phenomenon consistently occurs.

3. Anomaly cancellation in six dimensions

3.1. A review of anomaly cancellation mechanisms

The relevant anomaly eight-forms are given by

$$\tilde{I}^{(D=6)}_{3/2} = \frac{49}{36} Y_4 - \frac{43}{72} Y_2^2,$$

$$\tilde{I}^0_{1/2} = \frac{1}{180} Y_4 + \frac{1}{72} Y_2^2,$$

$$\tilde{I}^{L_i}_{1/2} = \dim L_i \left( \frac{1}{180} Y_4 + \frac{1}{72} Y_2^2 \right) + \frac{1}{16} \left( \frac{2}{3} \text{tr} L_i F^4 - \frac{1}{6} \text{tr} L_i F^2 \text{tr} R^2 \right),$$

$$\tilde{I}_A = \frac{7}{45} Y_4 - \frac{1}{9} Y_2^2,$$

where

$$Y_{2m} \equiv \frac{1}{2} \left( -\frac{1}{4} \right)^m \text{tr} R^{2m},$$

and $L_i$ denotes the representation of the unbroken gauge group $G$.

In general, the total anomaly polynomial is given by

$$\tilde{I}^\text{total}_8 = - \left( \tilde{I}^{(D=6)}_{3/2} + \tilde{I}_A \right) + n_T \left( \tilde{I}_A + \tilde{I}^0_{1/2} \right) - \sum_{\alpha} \tilde{I}^{\text{Ad}G_{\alpha}}_{1/2},$$

$$+ \sum_i n_H^{i} \tilde{I}^{L_i}_{1/2} + n_0^H \tilde{I}^0_{1/2},$$

where $n_T$ is the number of tensor multiplets, $n_H^i$ is the number of massless hypermultiplets in the representation $L_i$ of the unbroken gauge group, and $n_0^H$ is the number of other neutral hypermultiplets not counted in $n_H^i$ as singlets. We assume that the unbroken gauge group is a direct product $\prod_{\alpha} G_{\alpha}$. We write

$$n_V = \sum \dim G_{\alpha},$$

$$n_H = \sum_i n_H^i \dim L_i + n_0^H;$$

then, if they satisfy the well-known relation,

$$n_H - n_V = 273 - 29n_T,$$

---

The terms are $-16\pi^4$ times the ones given in Ref. [35].
the $\text{tr} R^4$ terms cancel out and we have [56]

$$
\hat{\mathcal{I}}_{8}^{\text{total}} = \frac{9 - n_{F}^{2}}{2} Y_2^2 - \frac{1}{12} Y_2 \sum_{\alpha} \left( \text{Tr}_{\alpha} F_{\alpha}^2 - \sum_{i} n_{H}^{a_{i} \alpha} \text{tr}_{L_{i}^{\alpha}} F_{\alpha}^2 \right) - \frac{1}{24} \sum_{\alpha} \left( \text{Tr}_{\alpha} F_{\alpha}^4 - \sum_{i} n_{H}^{a_{i} \alpha} \text{tr}_{L_{i}^{\alpha}} F_{\alpha}^4 \right) + \frac{1}{4} \sum_{\alpha < \beta} \sum_{i,j} n_{H}^{a_{i} \beta j} \text{tr}_{L_{i}^{\alpha}} F_{\alpha}^2 \text{tr}_{L_{j}^{\beta}} F_{\beta}^2, \tag{32}
$$

where, as usual, $\text{Tr}_{\alpha}$ denotes the trace taken in the adjoint representation of $G_{\alpha}$, $n_{H}^{a_{i} \alpha}$ is the number of hypermultiplets in the representation $L_{i}^{\alpha}$ of $G_{\alpha}$, and $n_{H}^{a_{i} \beta j}$ is the one in $L_{i}^{\alpha} \otimes L_{j}^{\beta}$ of $G_{\alpha} \times G_{\beta}$.

It is known [18–20] that F-theory compactified on an elliptic Calabi–Yau threefold over the Hirzebruch surface $F_{n}$ is dual to the K3 compactification of the $E_{8} \times E_{8}$ heterotic string with instanton numbers $(12 + n, 12 - n)$, so let us first recall the perturbative spectrum of the K3 compactifications of the $E_{8} \times E_{8}$ heterotic string.

Let $H_{(m)}^{(m)} (m = 1, 2)$ be the gauge group of the instanton in $E_{8}^{(m)}$ with instanton number $12 + (-1)^{m-1} n$, and $G^{(m)}$ be the maximal commutant in $E_{8}^{(m)}$. Let the decomposition of the adjoint of $E_{8}$ in the representations of $G^{(m)} \times H_{(m)}$ be

$$
248^{(m)} = \oplus_{i} (f_{i}^{(m)} \otimes c_{i}^{(m)}) \tag{33}
$$

for each $m = 1, 2$. Let $F_{0}^{(m)} (m = 1, 2)$ be the field strength of the instanton in $H_{(m)}$, and define $r_{i}^{(m)}$ as the ratio of the traces

$$
\text{tr}_{c_{i}^{(m)}} F_{0}^{(m)} = r_{i}^{(m)} \text{Tr}_{E_{8}} F_{0}^{(m)}^2. \tag{34}
$$

Then the number of hypermultiplets is given by the index theorem:

$$
- n_{H}^{0} = -21, \quad - n_{H}^{(m) a_{i}} = \dim c_{i}^{(m)} - \frac{1}{8 \pi^2} \int_{K3} \frac{1}{2} \text{tr}_{c_{i}^{(m)}} F_{0}^{(m)2}. \tag{35}
$$

Using these expressions in (32), one can show that [35]

$$
\hat{\mathcal{I}}_{8}^{\text{total}} = 4 \left( Y_2 + \frac{1}{8} (x^{(1)} + x^{(2)}) \right) \left( Y_2 + \frac{n}{16} (x^{(1)} - x^{(2)}) \right), \tag{36}
$$

where

$$
x^{(m)} = \frac{1}{30} \text{Tr}_{E_{8}} F_{0}^{(m)2} \quad (m = 1, 2). \tag{37}
$$

Thus the anomaly of the K3 compactification of the $E_{8} \times E_{8}$ heterotic string factorizes and hence can be canceled by the Green–Schwarz mechanism.

In F-theory, an alternative anomaly cancellation mechanism is known: The generalized Green–Schwarz mechanism assumes [56] that the same anomaly polynomial (32) can be written in a bilinear form

$$
\hat{\mathcal{I}}_{8}^{\text{total}} = \frac{1}{32} \Omega_{ij} X^{i} X^{j}, \tag{38}
$$

$$
X^{i} = \frac{1}{2} a^{i} \text{tr} R^2 + \sum_{\alpha} 2 b^{i}_{\alpha} \text{tr}_{\alpha} F_{\alpha}^2 \tag{39}
$$
for some metric $\Omega_{ij}$ on the space of $B$ fields and some constants $a^i$ and $b^j_\alpha$. Here the hatted indices $\hat{i}, \hat{j}$ are the ones for the space of $B$ fields and run from one through the total number of $B$ fields. The anomaly is then written as

$$\int \Omega_{ij} \omega^i_2 X^j$$

(40)

with

$$X^i = d\omega^i_3,$$

(41)

$$\delta \omega^i_3 = d\omega^i_2 (\Lambda),$$

(42)

which can be canceled by the contribution from the counterterm

$$\int \Omega_{ij} B^i X^j,$$

(43)

assuming that the anomalous transformations of the $B^i$ fields

$$\delta \omega^i_3 = -\omega^i_2 (\Lambda).$$

(44)

The conditions for the anomaly polynomial to be written in the form (38) are summarized by the following set of equations:

$$9 - n_T = \sum_{i,j} \Omega_{ij} a^i a^j,$$

(45)

$$\text{index}(\text{Ad}G_\alpha) - \sum_i n_H^{ij} \text{index}(L^i_\alpha) = 6 \sum_{i,j} \Omega_{ij} a^i a^j,$$

(46)

$$x_{\text{Ad}G_\alpha} - \sum_i n_H^{ij} x L^i_\alpha = 0,$$

(47)

$$y_{\text{Ad}G_\alpha} - \sum_i n_H^{ij} y L^i_\alpha = -3 \sum_{i,j} \Omega_{ij} b^i_\alpha b^j_\alpha,$$

(48)

$$\sum_{i,j} n_H^{ij} \text{index}(L^i_\alpha) \text{index}(L^j_\beta) = \sum_{i,j} \Omega_{ij} b^i_\alpha b^j_\beta,$$

(49)

where, following Ref. [56], we have defined

$$\text{tr}_{L^i_\alpha} F^2_\alpha = \text{index}L^i_\alpha \text{tr}_{L^i_\alpha} F^2_\alpha,$$

(50)

$$\text{tr}_{L^i_\alpha} F^4_\alpha = x_{L^i_\alpha} \text{tr}_{L^i_\alpha} F^4_\alpha + y_{L^i_\alpha} (\text{tr}_{L^i_\alpha} F^2_\alpha)^2,$$

(51)

for some trace $\text{tr}_\alpha$ taken in a preferred representation of $G_\alpha$. In the following we take the fundamental representation for this representation for $SU(N)$ or $SO(2N)$, 27 for $E_6$, 56 for $E_7$, and 248 for $E_8$.

The solutions to the anomaly cancellation conditions (45)–(49) were given by Sadov [56]: By investigating a possible form of the counterterms from a reduction of the 10D anomalous couplings, it was shown that the anomalies are canceled if the anomaly 8-form is written as (here the overall normalization is different from Ref. [56] by a factor of 16)

$$\hat{I}^\text{total}_8 = \frac{1}{32} \left( \frac{1}{2} K \text{tr} R^2 + \sum_\alpha D_\alpha \text{tr}_{L^i_\alpha} F^2_\alpha \right)^2,$$

(52)
where $K$ is the canonical divisor of the base 2-fold $B_2$ (= a Hirzebruch surface $F_n$ in the dual case) of the elliptic fibration of the 6D F-theory, and $D_\alpha$ is the divisor representing the locus of the stack of gauge 7-branes of the gauge group $G_\alpha$. The square is understood as the intersection product of the divisors, which means that the metric $\Omega_{ij}$ has been chosen to be that of harmonic forms on $B_2$. Indeed, the index theorem shows that

$$K \cdot K = 10 - \text{(the total number of } B \text{ fields)} = 9 - n_T, \quad (53)$$

which means that (45) is satisfied. Also, Eqs. (46)–(49) read

$$\text{index}(\text{Ad} G_\alpha) - \sum_i n_H^{\alpha i} \text{index}(L_i^\alpha) = 6K \cdot D_\alpha, \quad (54)$$

$$x_{\text{Ad} G_\alpha} - \sum_i n_H^{\alpha i} x_{L_i^\alpha} = 0, \quad (55)$$

$$y_{\text{Ad} G_\alpha} - \sum_i n_H^{\alpha i} y_{L_i^\alpha} = -3D_\alpha \cdot D_\alpha, \quad (56)$$

$$\sum_{i,j} n_H^{\alpha i;\beta j} \text{index}(L_i^\alpha) \text{index}(L_j^\beta) = D_\alpha \cdot D_\beta, \quad (57)$$

which are solved for the numbers of hypermultiplets $n_H^{\alpha i}$ and $n_H^{\alpha i;\beta j}$ in terms of intersections of divisors.

In Ref. [56], various examples of the matter content of F-theory have been presented, including the fundamental, adjoint, and other representations for $SU(N)$, $Sp(k)$, and $SO(n)$ gauge groups. For example, let $N_A, N_F$, and $N_{AS}$ be the numbers of hypermultiplets in the adjoint, fundamental, and antisymmetric tensor representations. Then, for $G_\alpha = SU(N)$ for $N > 3$ (assuming no 20 representations for $SU(6)$), these are solved as

$$N_A = \frac{1}{2} (K + D_\alpha) \cdot D_\alpha + 1 = g,$$

$$N_F = (-8K - ND_\alpha) \cdot D_\alpha,$$

$$N_{AS} = -K \cdot D_\alpha, \quad (58)$$

where $g$ is the genus of the divisor $D_\alpha$. In Table A2 we have summarized the expressions of divisors corresponding to representations of various $G_\alpha$, including $E_6$ and $E_7$, whose intersections to $D_\alpha(\equiv D_u)$ give the number of multiplets.

Going back to F-theory on a Hirzebruch surface $F_n$, let $D_u, D_v$ be the divisors of the sections $z = 0, \infty$, respectively. Then

$$K = -(2 + n)D_s - D_v,$$

$$D_u = nD_s + D_v, \quad (59)$$

where $D_s$ is the divisor of the ($\mathbb{P}^1$) fiber of $F_n$ with an intersection matrix

$$\Omega_{ij} = \begin{pmatrix} D_s D_s & D_s D_v \\ D_s D_v & D_v D_v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}. \quad (60)$$

Setting $D_\alpha = D_u$ and $N = 5$, (58) becomes

$$N_A = 0, \quad N_F = N_5 = 3n + 16, \quad N_{AS} = N_{10} = n + 2, \quad (61)$$

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which agrees with (8). More generally, one can check that the divisors presented in the last column of Table A2 precisely reproduce, as the intersection numbers with the divisor $D_\mu$, the numbers of charged matter multiplets obtained by the index theorem in the dual heterotic theory compactified on $K3$ for the gauge group with the instanton number $12 + n$. Also, the numbers of multiplets for the other gauge group with the instanton number $12 - n$ can be obtained by replacing $D_\mu$ with $D_\nu$.

In fact, with two divisors $D_\mu$ and $D_\nu$ in $F_n$, one can show that the anomaly 8-form can be written as

$$\hat{\mathcal{I}}_{\text{total}}^8 = \frac{1}{32} \left( \frac{1}{2} K \text{tr} R^2 + D_\mu x^{(1)} + D_\nu x^{(2)} \right)^2$$

(62)

with $x^{(1)}$ and $x^{(2)}$ given by (37), which coincides with (36), as it should for the heterotic/F-theory duality. Thus F-theory on an elliptic Calabi–Yau over $F_n$, which shares the same matter spectrum as that of an $E_8 \times E_8$ heterotic string on $K3$ with instanton numbers $(12 + n, 12 - n)$, can also be anomaly-free by this anomaly cancellation mechanism.

It should be emphasized here that the numbers of hypermultiplets are totally determined by the intersections of divisors. Due to their topological nature, we see that the respective numbers of hypermultiplets in various representations independently remain unchanged under any smooth deformation of complex structure moduli parameters.

### 3.2. Multiple singularities and anomalies

As we have already mentioned, the F-theory compactification on an elliptic Calabi–Yau over $F_n$ is dual to the $E_8 \times E_8$ heterotic string compactification on $K3$ with instanton numbers $(12 + n, 12 - n)$. More precisely, suppose that $G \times H$ is a direct product maximal subgroup of, say, the first factor of $E_8$, such that (1) $G$ is simple and simply laced, and (2) $H$ is semisimple. We assume that $H$ has $12 + n$ instantons so the unbroken gauge group from this $E_8$ is $G$. For these cases the massless spectra of heterotic string theory and the dual geometries of F-theory are summarized in Appendix B. As is shown there, for each such pair $(G, H)$, one can find a specialized Weierstrass form of the elliptic fibration such that

1. The total number of deformation parameters of the curve that preserve the particular singularity structure is precisely equal to the number of neutral hypermultiplets (arising from this $E_8$) computed by the index theorem in heterotic string theory.
2. The leading-order term in $z$ of the discriminant of the Weierstrass form factorizes, and the degree in $z'$ of each factor again coincides with the number of charged hypermultiplets obtained by the index theorem.

Therefore, at least for this class of F-theory compactifications, there is no anomaly since they share the same massless matter contents as those of heterotic strings on $K3$. This is the situation where both theories are at “generic” points in the moduli spaces. On the other hand, suppose that the F-theory curve is deformed in such a way that more than one factor of the leading-order term of the discriminant comes to share a common zero locus, at which the singularity is more enhanced than the ones occurring at the ordinary matter loci at generic points in the moduli space. As we discussed at the end of the previous section, such a multiple singularity supports more BPS string junctions than when the discriminant loci are split apart. Since, for all the cases listed in Table A2, half of the localized string junctions at the enhanced point precisely correspond to the matter representations predicted from the heterotic string analysis, one may also expect that the additional string junctions appearing at the multiple point will also play a role in the generation of massless matter in F-theory.
However, as we discussed in the previous section, the anomaly cancellation condition forbids such net increase of chiral matter. For the Green–Schwarz mechanism to work, the total anomaly polynomial must factorize into the form (36) or (38). This requires the absence of the $Y_4$ term in $\hat{r}_\text{total}$, and that imposes the constraint

$$n_H - n_V = 273 - 29n_T,$$

as we saw previously. Thus, as long as the number of tensor multiplets is one, as it is for the smooth heterotic compactifications, the only possible change in the number of hypermultiplets is one associated with the simultaneous change in the number of vector multiplets, i.e., the Higgs mechanism. In the present case, however, there is no gauge symmetry enhancement to expect at the multiple singularity, so this does not happen.

Also, even if one allows the number of tensor multiplets to change, the anomalies from the net increase of the hypermultiplets cannot be canceled. This is because the coefficient of $n_T$ in (31) is minus 29, and hence an increase of $n_T$ means a decrease of the hypermultiplets.

The total amount of chiral matter is also constrained by geometry. In the generalized Green–Schwarz mechanism, a change in the number of tensor hypermultiplets means a change in the self-intersection number of the canonical class $K$ of the base manifold of the elliptic fibration; see (45). In the present case, the base is a Hirzebruch surface. The canonical class can change if the surface is blown up at some points. Suppose that the Hirzebruch surface is blown up at a point; the canonical class is changed to

$$K = -(2 + n)D_s - 2D_v$$

$$\rightarrow -(2 + n)D_s - 2D_v + e_1,$$

where $e_1$ is the exceptional divisor that has arisen due to the blow up. Since its intersection pairing is

$$e_1 \cdot e_1 = -1, \quad D_s \cdot e_1 = D_v \cdot e_1 = 0,$$

the self-intersection $K \cdot K$ decreases from eight to seven, which also implies that there arise more tensor multiplets and fewer hypermultiplets are allowed to exist.\(^7\)

Therefore, in any case, any net change of the total amount of chiral matter is inconsistent with anomaly cancellation. Is there any transition of geometry without any change of the total number of hypermultiplets before and after the transition? In fact, an example of such a transition to special points in the moduli space has already been found in Ref. [22], where the branes have some multiple singularities and at the same time the theory remains anomaly-free. We will discuss this in the next section.

### 4. Anomaly-free multiple singularities

So far we have seen that the singularity enhancement of the type shown in the bottom row of Table 2 does not occur simply as a coalescence of two ordinary rank-1 enhanced singularities because the numbers of chiral matter multiplets are different before and after the coalescence; singularity enhancement is hence forbidden by anomaly cancellation. To achieve locally such a set of chiral matter forming a homogeneous Kähler manifold with more than one $U(1)$ factor in the denominator

\(^7\) Such a transition was first considered in Ref. [58]. Colliding singularities in F-theory on a blown-up Hirzebruch were studied in Ref. [59].
We see that the coefficient of the leading-order term of $D$ term. This is a $h$ from $q_n$ of the coset, we follow the idea first explored in Ref. [21] and further investigated in Refs. [34,60] and tune complex structure moduli parameters in such a way that the necessary number of singularity loci join simultaneously so that no extra matter may appear or disappear.

To illustrate the idea, we begin with an example.

### 4.1. Enhancement from $SU(5)$ to $SO(12)$

The curve found in Ref. [22] is one that has an $SU(5) = I_5$ singularity at $z = 0$, and also is parameterized by (4), except that $g_{n+6}$ is further specialized to the form

$$q_{n+6} = h_{n+2}q_4$$

(66)

for some fourth-order polynomial $q_4$ in $z'$.\(^8\) This means that all the roots of the equation $h_{n+2}$ are also ones of $q_{n+6}$. In this particular case we have

\[
\begin{align*}
  f(z,z') &= -3h_{n+2}^4 + 12zh_{n+2}^2H_{n+4} + z^2(12q_{n+6}h_{n+2}^2 - 12H_{n+4}^2) + z^3f_{n+8} + \cdots, \\
  g(z,z') &= 2h_{n+2}^6 - 12zh_{n+2}^4H_{n+4} + z^2h_{n+2}^2(24H_{n+4}^2 - 12q_{n+6}^2h_{n+2}^2) \\
                  &\quad + z^3(-f_{n+8}h_{n+2}^2 + 24q_{n+6}h_{n+2}^2H_{n+4} - 16H_{n+4}^3) + z^4(2f_{n+8}H_{n+4} + 12q_{n+6}^2h_{n+2}^2) \\
                  &\quad + z^5g_{n+12} + \cdots, \\
  \Delta &= 108z^5h_{n+2}^6(-2q_{n+8} + g_{n+12} - 24q_{n+6}^2H_{n+4}) \\
                  &\quad - 96h_{n+2}^4(-96q_{n+8}H_{n+4} + f_{n+8}^2 + 72g_{n+12}H_{n+4} + 96q_{n+6}^2h_{n+2}^2 - 1152q_{n+8}^2H_{n+4}^2) \\
                  &\quad + 36z^2h_{n+2}^2(30q_{n+8}^2h_{n+2}^2 - 24q_{n+6}H_{n+4}) + f_{n+8}^2H_{n+4} + 18q_{n+8}g_{n+12}h_{n+2}^2 \\
                  &\quad + 36g_{n+12}H_{n+4}^2 + 432q_{n+6}^2h_{n+2}^2H_{n+4} - 288q_{n+8}^2H_{n+4}^3) \\
                  &\quad - 18z^8(3f_{n+8}g_{n+12}h_{n+2}^2 - 72q_{n+8}^2h_{n+2}^2H_{n+4} - 8q_{n+8}^2h_{n+2}^2 + 2f_{n+8}H_{n+4}^2 \\
                  &\quad - 72q_{n+8}g_{n+12}h_{n+2}^2H_{n+4} + 48g_{n+12}H_{n+4}^2 - 216q_{n+8}^4h_{n+2}^2) + \cdots. 
\end{align*}
\]

(67)

We see that the coefficient of the leading-order term of $\Delta$ has been changed to the form

\[
\begin{align*}
  \Delta &= 108z^5h_{n+2}^6P_{n+12} + \cdots, \\
  P_{n+12} &= -2q_{n+8} + g_{n+12} - 24q_{n+6}^2H_{n+4} 
\end{align*}
\]

(68)

from $h_{n+2}^4P_{3n+16}$ (7) for the generic $SU(5)$ curve. If $h_{n+2}$ vanishes, then $f$ and $g$ start from $O(z^2)$ and $O(z^3)$, respectively, and $\Delta$ vanishes all the way up to $O(z^7)$ with $O(z^8)$ being the first nonvanishing term. This is a $D_6 = I_5^7$ singularity, which means that the curve has $n + 2$ points with multiple singularity enhancement $SU(5) \to SO(12)$.

Note that this is another case of a collision of the loci of $h_{n+2}$ and $P_{3n+16}$ discussed in Sect. 2. To see this we set $h_{n+2} = 0$ in $P_{3n+16}$ (7) to find that

$$P_{3n+16} \sim -24H_{n+4}q_{n+6}^2.$$  

(69)

---

\(^8\) Examples of multiple singularity enhancement from $SU(5)$ to $SO(12)$, $E_6$ or $E_7$ have been more recently considered in Ref. [34].
Thus if either of $H_{n+4}$ or $q_{n+6}$ vanishes, there occurs a collision.\(^9\) The former case was discussed in Sect. 2, where the singularity was enhanced to $E_6$.\(^10\)

In the present case, the BPS junctions are the ones corresponding to the homogeneous Kähler manifold $SO(12)/(SU(5) \times U(1)^2)$:

$$10(SO(10)) \oplus 10(SU(5)) = 5 \oplus \tilde{5} \oplus 10$$

(70)

plus one $1$ from the extra Cartan subalgebra. Since $5$ and $\tilde{5}$ are indistinguishable in six dimensions, we have

$$5 \oplus 5 \oplus 10 \oplus 1$$

(71)

residing at each zero of $h_{n+2}$ ($SO(10)$ point). Thus the hypermultiplets at the brane intersections are

$$(n + 2) (5 \oplus 5 \oplus 10 \oplus 1) \oplus (n + 12)5 = (n + 2)10 \oplus (3n + 16)5 \oplus (n + 2)1,$$  

(72)

where the $(n + 12)5$ on the left-hand side come from the zeros of $P_{n+12}$ (68). In addition, there are singlets from the complex structure moduli; their number is determined by the degrees of freedom of the polynomials

$$h_{n+2}, H_{n+4}, q_4, f_{n+8} \text{ and } g_{n+12},$$

(73)

which yield

$$(n + 3) + (n + 5) + 5 + (n + 9) + (n + 13) - 1 = 4n + 34$$

(74)

more $1$, and hence $5n + 36$ singlets in all. Thus the matter spectrum coincides with (8) and hence is unchanged from that for the generic unbroken $SU(5)$ curve that we saw in Sect. 2, and therefore the theory remains anomaly-free!

How can this happen despite the extra $5$ at each zero locus of $h_{n+2}$? We can see this by noticing that the degree of the other factor of the leading term of the discriminant is changed to $n + 12$ from $3n + 16$ for the generic case. That is, $2n + 4$ of $3n + 16$ loci of $5$ have pairwise degenerated into $n + 2$ pairs and simultaneously coalesced with the locus of $10$ (Fig. 4(a))! Thus the total amount of charged matter is unchanged. It is also remarkable that the balance of the neutral hypermultiplets is maintained before and after the multiple singularity enhancement; the emergence of the $n + 2$ extra singlets at the singularity is precisely compensated by the decreased amount of complex structure moduli for the restricted geometry.\(^11\)

---

\(^9\) This collision is not the kind of one that needs a blow up on the base, unlike the cases discussed in Ref. [59]. An extra tensor multiplet would make the theory anomalous in the present case, as we saw at the end of the previous section.

\(^10\) As we will see below, the fact that $q_{n+6}$ is being squared is important since it means a simultaneous degeneration of two loci of $P_{3n+16}$.

\(^11\) A similar anomaly cancellation can be seen for $SU(3)$ curves with multiple singularities. The generic $SU(3)$ curve (see Table A1) has the discriminant $\Delta = h_{n+2}^3 P_{6n+32} z^3 + \cdots$. At each root of $P_{6n+18}$, the enhancement $I_1 \rightarrow I_4$ ($SU(3) \rightarrow SU(4)$) occurs and a $3$ appears, giving in all $(6n + 18)3$. At a root of $h_{n+2}$, the fiber type changes as $I_1 \rightarrow IV$ ($SU(3) \rightarrow SU(3)$), where a $B$-brane intersects the three $A$-branes. At this point, no extra BPS string junction can exist and hence no hypermultiplet appears. Specializing the generic curve in Table A1 to $H_{2n+6} = h_{n+2} q_{n+4}$, we obtain multiple singularities [22]. The discriminant changes to $\Delta = h_{n+2}^6 P_{3n+12} z^3 + \cdots$. This means that, among the $6n + 18$ roots of $P_{6n+18}, 3n + 6$ roots triply degenerate
(a) $SU(5) \rightarrow SO(12)$

(b) $SU(5) \rightarrow E_6$

Fig. 4. Anomaly-free multiple singularities.

The geometry considered in this section is one with a maximal number of multiple enhanced points from $SU(5)$ to $SO(12)$; one may equally well consider the case where, for an arbitrary integer $r$ ($0 \leq r \leq n+2$), $2r$ of $3n+16$ loci of 5 pairwise merge with $r$ 10 loci while the rest of the 5 loci remain as they are. It is also easy to see in this case that the amounts of charged and neutral matter do not change before and after the coalescence of the singularities.

Conversely, if the extra matter did not arise at the multiple singularity enhancement with the simultaneous degeneration of matter loci as above, the balance of the matter multiplets (31) would be lost and the theory would become anomalous. Thus the absence of anomalies requires here the generation of extra matter at this multiple singularity.

into $n+2$ sets and coalesce with zeros of $h_{n+2}$, yielding the multiple singularities. At the remaining $3n+12$ roots of $P_{3n+12}$, $(3n+12)\mathbf{3}$ appear. The decrease of $(3n+6)\mathbf{3}$ is precisely compensated by the $\mathbf{3}$ at the multiple singularities of $h_{n+2}$. In fact, at each root of $h_{n+2}$, enhancement $I_3 \rightarrow I_3^*$ ($SU(3) \rightarrow SO(8)$) occurs and hypermultiplets in $\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1}$ appear. Note that this last $\mathbf{1}$ ($(n+2)\mathbf{1}$ in all) just compensates the decrease of $n+2$ neutral hypermultiplets due to the decrease of the complex structure moduli $H_{2n+6} \rightarrow q_{n+4}$ via the specialization.
4.2. Enhancement from SU(5) to E₆

Having understood how an anomaly-free multiple singularity enhancement can be realized, we can now find curves with other types of multiple singularity enhancement. Let us reexamine in this section the singularity enhancement \( SU(5) \rightarrow E_6 \) considered in Sect. 2. As we saw there, this happens when \( h_{n+2} \) and \( H_{n+4} \) have a common zero locus.

We first examine the case when \( H_{n+4} \) takes the form

\[
H_{n+4} = h_{n+2}P_{2n+14}z^7 + \cdots, \tag{75}
\]

so, if there were two 10 at each such \( E_6 \) multiple singularity locus as in Table 1, the total number of 10 would be \( 2(n+2) \), which is too large to cancel anomalies (see (8), (31)). Thus we conclude that in this case the naively expected set of chiral matter shown in Table 1 does not all arise at that point [34].

Therefore, to keep the number of 10 unchanged, we instead set

\[
h_{n+2} = h_{n+2}^2, \\
H_{n+4} = h_{n+4}^2H_{n+6}
\tag{77}
\]

for some \( h_{n+2} \), where \( n+2 \) is assumed to be divisible by two. In fact, it has been known for some time that such a complex structure tuning in which the degree of zero changes affects the matter spectrum [21,34,60], and we use this idea here. With (77) the discriminant reads

\[
\Delta = h_{n+2}^9P_{5n+30}z^7 + \cdots. \tag{78}
\]

We see that the \( n+2 \) roots of the equation \( h_{n+2} = 0 \) pairwise degenerate into \( \frac{n+2}{2} \) double roots, each of which merges with a root of \( P_{3n+16} \) (Fig. 4(b)).

The relevant homogeneous Kähler manifold in this case is \( E_6/(SU(5) \times U(1)^2) \) whose \( SU(5) \) representations are

\[
16(SO(10)) \oplus 10(SU(5)) = 10 \oplus \bar{5} \oplus 1 \oplus 10, \tag{79}
\]

and there is another 1 from the Cartan subalgebra. Thus the hypermultiplets coming from the brane intersections are

\[
n+2 \left( 2 \cdot 10 \oplus 5 \oplus 2 \cdot 1 \right) \oplus \frac{5n+30}{2} \bar{5} = (n+2)10 \oplus (3n+16)\bar{5} \oplus (n+2)1. \tag{80}
\]

On the other hand, the decrease of the degrees of freedom of the polynomials is \( \frac{n+2}{2} \) from \( h_{n+2} \) to \( h_{n+2}^2 \) and \( \frac{n+2}{2} \) from \( H_{n+4} \) to \( H_{n+6} \), in total \( n+2 \) again. This compensates the extra \( n+2 \) singlets in (80), and the theory after this transition also remains anomaly-free.

Although we have for simplicity considered the case with the maximally possible multiple \( E_6 \) points with \( n+2 \) even, we may similarly consider the case with fewer \( E_6 \) points and/or \( n+2 \) odd. All that we require is that a pair of degenerating loci of \( h_{n+2} \) and a single locus of \( P_{3n+16} \) merge together.
per $E_6$ multiple singularity. In this way the number of charged hypermultiplets is conserved, but, again, the fact that the number of new singlets also matches the decrease of the complex structure moduli is rather nontrivial.

4.3. Enhancement from $SU(5)$ to $E_7$

Let us now consider a singularity enhancement in which the rank of the Lie algebra characterizing the singularity jumps up by more than two. The enhancement from $SU(5)$ to $E_7$ is of particular interest because it is relevant to the F-theory realization [1] of the Kugo–Yanagida $E_7/(SU(5) \times U(1)^3)$ family-unification model [9].

The Kodaira classification tells us that the $E_7$ singularity occurs when $\text{ord} f = 3$, $\text{ord} g \geq 5$, and $\text{ord} \Delta = 9$. We can see from Appendix A that this happens when $h_{n+2}$, $H_{n+4}$, and $q_{n+6}$ all simultaneously vanish. The homogeneous Kähler manifold for this singularity is $E_7/(SU(5) \times U(1)^3)$ with the following $SU(5)$ representations:

$$27(E_6) \oplus 16(SO(10)) \oplus 10(SU(5)) = (16(SO(10) \oplus 10(SO(10)) \oplus 1) \oplus 16(SO(10)) \oplus 10(SU(5))$$

$$= 3 \cdot 10 + 4 \cdot 5 + 3 \cdot 1,$$  \hspace{1cm} (81)

where in the last line we have made no distinction between 5 and $\bar{5}$. In addition, we have, this time, two 1 from the Cartan subalgebra. In all, three 10, four 5, and five singlets are supposed to arise at each multiple $E_7$ singularity. Thus, in order for the anomalies to cancel, we need to have three loci of $h_{n+2}$ and four loci of $P_{3n+16}$ to simultaneously degenerate and join together per $E_7$ singularity. This is achieved, again for the maximal case, at the special points in the moduli space as follows:

$$h_{n+2} = h_{n+2}^3,$$
$$H_{n+4} = h_{n+2}^2 H_{n+8},$$
$$q_{n+6} = h_{n+2} q_{2n+16}/3$$  \hspace{1cm} (82)

for some $h_{n+2}$, $H_{n+8}$, and $q_{2n+16}$, where $n + 2$ is assumed to be divisible by three in this case. The nonmaximal case and/or the case in which $n + 2$ is not 0 mod 3 are treated similarly. With (82) the discriminant becomes

$$\Delta = h_{n+2}^{16} P_{3n+40} z^5 + \cdots$$  \hspace{1cm} (83)

The total number of 5 is thus

$$4 \times \frac{n+2}{3} + \frac{5n+40}{3} = 3n+16,$$  \hspace{1cm} (84)

which is a correct value. Also, the decrease of the degrees of freedom of the polynomials is

$$2 \times \frac{n+2}{3} + 2 \times \frac{n+2}{3} + 1 \times \frac{n+2}{3} = 5 \times \frac{n+2}{3},$$  \hspace{1cm} (85)

which match the five singlets residing at each of the $\frac{n+2}{3}$ $E_7$ points.

\[12\] In Ref. [34], it was observed in a detailed study of the rank-one enhancement $A_5 \to E_6$ that, depending on the order of zeros in the parameters, the singularity can either be incompletely or completely resolved, yielding different matter spectra. It would be interesting to explore the differences of the blowing-up patterns in all the cases that we have dealt with in this paper.
4.4. Enhancement from $SU(5)$ to $E_8$

The final example of anomaly-free singularity enhancement that we consider in this paper is the one from $SU(5)$ to $E_8$. This type of multiple singularity may also be used for particle physics model building because the $D = 4$ supersymmetric nonlinear sigma model with $E_8/(SU(5) \times U(1)^4)$ as the target also yields net three chiral generations. Furthermore, it has been pointed out [61] that this coset may also give rise to three sets of nonchiral singlet pairs needed in a scenario proposed by Sato and Yanagida [62] explaining the Yukawa hierarchies and large lepton-flavor mixings by the Froggatt–Nielsen mechanism.\(^{13}\)

The spectrum of $E_8/(SU(5) \times U(1)^4)$ is

$$5 \cdot 10 \oplus 10 \cdot 5 \oplus 10 \cdot 1.$$  \hspace{1cm} (86)

With the additional three singlets from the Cartan subalgebra, in all

$$5 \cdot 10 \oplus 10 \cdot 5 \oplus 13 \cdot 1$$  \hspace{1cm} (87)

reside at each $E_8$ point.\(^{14}\)

We can also find an anomaly-free curve with these $E_8$ multiple singularities. We again only present the case where $n + 2$ is divisible by five and all the $h_{n+2}$ loci turn into $E_8$ singularities:

$$h_{n+2} = h_{n+2}^5,$$

$$H_{n+4} = h_{n+2}^4 H_{n+12},$$

$$q_{n+6} = h_{n+2}^3 q_{2n+24},$$

$$f_{n+8} = h_{n+2}^2 f_{3n+36},$$  \hspace{1cm} (88)

for some $h_{n+2}, H_{n+12}, q_{2n+24},$ and $f_{3n+36}$. Then the discriminant reads

$$\Delta = h_{n+2}^{30} P_{n+122}^5 + \cdots.$$  \hspace{1cm} (89)

We can similarly verify that the numbers of both charged and neutral hypermultiplets are the same as those at generic points in the moduli space. Therefore, in this case too, the theory is anomaly-free.

It is interesting to notice that the powers of $h_{n+2}$ factors in (88) are precisely the exponents of $SU(5)$, i.e., the powers of the canonical class projectivized in the weighted projective bundle [45], of which (13) are sections. Perhaps this coincidence may be interpreted in terms of spectral covers of the dual heterotic string theory.

5. Conclusions

We have shown that the Kähler coset of the form $G/(H \times U(1)^r)$ with $r \geq 2$ is indeed realized as a local matter spectrum at a multiply enhanced singularity in 6D F-theory without causing an

\(^{13}\) In fact, the $E_8$ curve given in the original version of Ref. [61] did not take account of the simultaneous degenerations of loci and hence was anomalous as a 6D theory. A revised version is in preparation.

\(^{14}\) At first sight it seems that the $D = 4, N = 1$ supersymmetric nonlinear sigma model with this target space may have five generations. However, one can show [15,16] that it is not possible to choose a so-called “Y-charge”, a $U(1)$ charge that determines the complex structure of the coset space, in such a way that all the five “flavors” may have the same chirality. See Refs. [15,16] for more details.
imbalance of anomalies. We have considered concrete examples in F-theory compactifications on an elliptically fibered Calabi–Yau over a Hirzebruch surface $F_n$. Anomaly cancellation requires that there should be no net change in the numbers of hypermultiplets after the coalescence of matter loci. We have presented such particular points in the moduli space in the case of the unbroken $SU(5)$ gauge group, where the singularity is multiply enhanced to $SO(12)$, $E_6$, $E_7$, or $E_8$. In this paper we have concentrated on the multiple enhancement from $SU(5)$ because this is the most interesting case for phenomenological applications including the Kugo–Yanagida coset, and it already exhibits the essential features. It is also natural to expect that similar phenomena occur in other singularities, and this will be reported elsewhere.

The original motivation to consider multiple singularity enhancement in F-theory was to construct “family-unification” particle physics models in string theory. But if the amount of chiral matter does not change after the coalescence of singularities, what is the use of the multiple singularities in string phenomenology model building?

We believe that consideration of the multiple singularity enhancement in F-theory has at least three virtues:

1. In general, a special point in the moduli space can be an endpoint of whatever flow in the moduli space after the supersymmetry is broken and potentials are generated; if it is not a special point, there is no reason for the flow to stop at that point.
2. The multiple singularity may occur, in principle, in any elliptic Calabi–Yau manifold. Since the structure is universal, it may offer a potential ubiquitous mechanism for generating three generations of flavors in the framework of F-theory.
3. Last but not least, the homogeneous Kähler structure of the spectrum of the multiple singularity is naturally endowed with conserved $U(1)$ charges. This may also be useful for particle physics model building.

One of the recent advances in 4D F-theory model building is the recognition that the Yukawa couplings are localized at the intersections of matter curves [46,63,64]. The relevant singularities are thus the very type that we have been discussing in this paper (see, e.g., Ref. [65]); it would be interesting to extend the analysis done in this paper to four dimensions.

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Appendix A. $SU(5)$ curve

\[
\begin{align*}
  f(z,z') &= -3h_{n+2}^4 + 12z h_{n+2}^2 H_{n+4} + z^2 (12h_{n+2} q_{n+6} - 12H_{n+4}^2) + z^3 f_{n+8} \\
  &\quad + f_8 z^4 + z^5 f_{8-n} + O(z^6), \\
  g(z,z') &= 2h_{n+2}^6 - 12z h_{n+2}^4 H_{n+4} + 12z^2 H_{n+4}^2 (2H_{n+4}^2 - h_{n+2} q_{n+6}) \\
  &\quad + z^3 (-f_{n+8} h_{n+2}^2 + 24h_{n+2} H_{n+4} q_{n+6} - 16H_{n+4}^3).
\end{align*}
\]

(A1)
Appendix B. Summary of 6D heterotic/F-theory duality

In this appendix we summarize the details of the correspondence [20] between massless matter spectra of $E_8 \times E_8$ heterotic string theory compactified on $K3$ and geometric data of the elliptically fibered Calabi–Yau threefold over Hirzebruch surfaces on which F-theory is compactified.

Let $E_8^{(1)} (E_8^{(2)})$ be the first (second) factor of $E_8 \times E_8$ and $G^{(m)} \times H^{(m)}$ ($m = 1, 2$) be a direct product maximal subgroup of $E_8^{(m)}$ ($m = 1, 2$). We assume that $H^{(1)} (H^{(2)})$ has $12 + n$ ($12 - n$) instantons. We restrict ourselves to the cases where (1) $G^{(m)}$ is simple and simply laced, and (2) $H^{(m)}$ is semisimple. The massless spectrum of the heterotic string can be computed [35] by the index theorem (35).

Table A1 shows the neutral matter spectrum of the heterotic string arising from $E_8^{(1)}$, the corresponding Weierstrass form of the F-theory curve, and the independent polynomials that parameterize the curve. The subscripts denote the degrees of the polynomials in $z^i$. For each pair of $G = G^{(1)}$ and $H = H^{(1)}$, the sum of the numbers of the coefficients of the independent polynomials, minus...
one, which takes account of the overall rescaling, always coincides with the number of heterotic singlets obtained by the index theorem, as was verified in (6) in Sect. 2. A similar result holds for the neutral matter from $E_8^{(2)}$ and the coefficients of the Weierstrass form $\sum_{i=5}^{8} z^i f_{8+i+4-i} (z')$ and $\sum_{i=5}^{12} z^i g_{12+i-6-3i} (z')$, which determine the singularity at $z = \infty$.

Table A2 shows the spectrum of the charged hypermultiplets. For each $(G, H)$, the leading-order term in $z$ of the discriminant of the Weierstrass form factorizes, and the degree in $z'$ of each factor coincides with the number of charged hypermultiplets obtained by the index theorem. The representation that occurs is related to the pattern of the singularity enhancement, as explained in the text.
### Table A2. Heterotic/F-theory duality: Charged hypermultiplets.

| $G$   | $H$               | Heterotic/charged matter | Matter locus | Singularity enhancement | Divisor               |
|-------|-------------------|--------------------------|--------------|--------------------------|-----------------------|
| $E_7$ | $SU(2)$           | $\frac{n+2}{2} 56$       | $f_{8+n}$    | $E_7 \rightarrow E_8$   | $-2K - \frac{1}{2}D_a$|
| $E_6$ | $SU(3)$           | $(n + 6) 27$             | $q_{n+6}$    | $E_6 \rightarrow E_7$   | $-3K - 2D_u$          |
| $SO(12)$ | $SO(4)$    | $(n + 8) 12$             | $h_{n+4}$    | $SO(12) \rightarrow SO(14)$ | $-4K - 3D_u$          |
| $SO(10)$ | $SU(4)$    | $(n + 4) 16$             | $H_{n+4}$    | $SO(10) \rightarrow E_6$ | $-2K - D_u$           |
| $SO(8)$ | $SO(8)$         | $(n + 4) 8$              | $k_{n+4}$    | $SO(8) \rightarrow SO(10)$ | $-2K - D_u$           |
| $SU(6)$ | $SU(3) \times SU(2)$ | $(n - r + 2) 15$       | $t_r$        | $SU(6) \rightarrow E_6$ | $-\frac{1}{2}K - \frac{3}{2}D_u$ |
| $SU(5)$ | $SU(5)$         | $(n + 2) 10$             | $h_{n+2}$    | $SU(5) \rightarrow SO(10)$ | $-K$                 |
| $SU(4)$ | $SO(10)$        | $(n + 2) 6$              | $h_{n+2}$    | $SU(4) \rightarrow SO(8)$ | $-K$                 |
| $SU(3)$ | $E_6$           | $(6n + 18) 3$            | $P_{6n+18}$  | $SU(3) \rightarrow SU(4)$ | $-9K - 3D_u$          |
| $SU(2)$ | $E_7$           | $(6n + 16) 2$            | $P_{6n+16}$  | $SU(2) \rightarrow SU(3)$ | $-8K - 2D_u$          |

We have also shown in the last column the corresponding divisor whose intersection number with $D_a$ (the divisor for the $z = 0$ section), determined by the anomaly cancellation conditions (45)–(49), gives the number of charged hypermultiplets.

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