The Ginibre ensemble and Gaussian analytic functions

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Abstract

We show that as $n$ changes, the characteristic polynomial of the $n \times n$ random matrix with i.i.d. complex Gaussian entries can be described recursively through a process analogous to Pólya’s urn scheme. As a result, we get a random analytic function in the limit, which is given by a mixture of Gaussian analytic functions. This gives another reason why the zeros of Gaussian analytic functions and the Ginibre ensemble exhibit similar local repulsion, but different global behavior. Our approach gives new explicit formulas for the limiting analytic function.

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In studies of random point sets in the complex plane, two canonical examples have emerged, sharing some features but differing in others.

The first is the infinite Ginibre ensemble, attained as the limit of the finite Ginibre ensembles, i.e. the set of eigenvalues of \( n \times n \) random matrices \( A_n \) filled with independent standard complex Gaussian entries. It can be thought of as the stationary distribution of a system of particles performing planar Brownian motions but repelling each other through a drift given by their inverse distance.

The second is the zero set of the random power series \( \sum a_n z^n / \sqrt{n!} \) for independent standard complex Gaussians \( a_n \). It is an example of a random analytic function whose
values are jointly centered complex Gaussian, a **Gaussian analytic function** or **GAF**, for short. Sodin (2000) has shown that the intensity measure (under which the measure of a Borel subset of $\mathbb{C}$ is the expected number of zeros that fall in the set) of the zeros of such a function in fact determines the entire distribution of zeros. The above power series, called **planar GAF**, has intensity measure given by a constant multiple of the Lebesgue measure. Sodin’s theorem implies that this is the only Gaussian analytic function whose zero set has a translation-invariant distribution on $\mathbb{C}$.

Beyond translation invariance, these two processes share some local properties. For example, in both cases we have

$$P(\text{two points in a fixed disk of radius } \varepsilon) \asymp \varepsilon^6$$

where four of the six in the exponent come from the square of the area of the disk and the extra two comes from quadratic repulsion. Contrast this with Poisson process which only has $\varepsilon^4$. This behavior is shared by all non-degenerate GAFs, see the work of Nazarov and Sodin (2010) on universality.

On the global scale, however, the two models are very different. For smooth, compactly supported $\varphi : \mathbb{C} \to \mathbb{R}$ with total integral zero we have two central limit theorems

- for Ginibre: $\sum_z \varphi(z/n) \Rightarrow N(0, \frac{1}{4\pi} \|\nabla \varphi\|^2)$,
- for planar GAF: $n \sum_z \varphi(z/n) \Rightarrow N(0, c\|\Delta \varphi\|^2)$,

where the sum is over all points of the processes. These results are due to Rider and Virág (2007) and Sodin and Tsirelson (2004), respectively.

The strikingly different central limit theorems show that the global behavior of the two random point processes are very different.

The goal of this paper is to prove a theorem which clarifies why this phenomenon happens. We study the distributional limit of the characteristic polynomial $Q_n(z)$ of $A_n$. It is known (see Girko (1990) or Kostlan (1992)) that $|Q_n(0)|$, the absolute value of the determinant, has the same distribution as the product of independent Gamma$(i, 1)$ random variables with $i = 1, \ldots, n$. Since $\log(\Gamma(i, 1))$ has variance asymptotic to $1/i$, we see that $\log |Q_n(0)|$, centered and divided by $\sqrt{\log n}$ converges in distribution to a standard normal random variable.

A simple consequence of this fact is that there are no constants $a_n, b_n$ so that the random variable $(Q_n(0) - a_n)/b_n$ converges in law to a non-degenerate limit. In contrast, we prove the following. Note that the convergence in distribution here is with respect to the topology of uniform convergence on compact subsets of the complex plane.
Theorem 1. There exist positive random variables $A_n$ so that the normalized characteristic polynomial $Q_n(z)/A_n$ converges in distribution to a random analytic function $Q(z)$.

Moreover, there exists a random positive definite Hermitian function $K(z, w)$ so that given $K$ the function $Q(z)$ is a Gaussian field with covariance kernel $K$. Further, $K$ is analytic in $z$ and anti-analytic in $w$, hence $Q$ is a GAF conditional on $K$.

Thus, the limit is in fact a randomized Gaussian analytic function. Theorem 1 thus gives a novel link between the world of random matrices and Gaussian analytic functions. In physics, certain connections between determinantal and Gaussian structures are referred to as super-symmetry; Theorem 1 is a very specific, mathematically precise instance of a connection between these two structures.

A Gaussian analytic function (GAF) $f$ is a complex Gaussian field on $\mathbb{C}$ which is almost surely analytic. By the theorem of Sodin (2000) referred to earlier, the distribution of the zeros of a Gaussian analytic function is determined by their intensity measure
\begin{equation}
    d\mu(z) = \frac{1}{\pi} \Delta \log K(z, z) dL(z),
\end{equation}
where $K(z, w) = \mathbb{E}[f(z)f(w)]$ is the covariance kernel and $L$ is Lebesgue measure. Formula 1 is a special case of the well-known Kac-Rice formulas (see sec. 2.4 of Hough, Krishnapur, Peres and Virág (2009)). In our setting this is just an averaged version of Green’s formula in complex analysis.

A direct consequence is a connection between the infinite Ginibre points and Gaussian analytic zero points, depicted in the front-page figure.

Corollary 2. The infinite Ginibre point process has the same distribution as Gaussian analytic zero points having a randomly chosen intensity measure $\mu$.

Note that this does not make the infinite Ginibre point process the zero set of a true Gaussian analytic function, only one with a randomized covariance kernel. This randomization does not change the qualitative local behavior, but changes the global one, which explains the phenomenon discussed above.

We conclude our paper by computing the first moment (everywhere) and the second moment (at zero) of the limiting covariance kernel normalized so that these moments exist.

Real Ginibre matrices. With our methods, we also prove an analogous theorem for characteristic polynomials of real Ginibre matrices (having i.i.d. real standard normal entries). However there are a few modifications as the characteristic polynomials
are complex-valued but the values are not complex Gaussian (as complex Gaussians are isotropic, by definition, for us).

**Theorem 3.** Let $A_n$ be the real Ginibre matrix with i.i.d. $N(0,1)$ entries. There exist positive random variables $A_n$ so that the normalized characteristic polynomial $Q_n(z)/A_n$ converges in distribution to a random analytic function $Q(z)$.

Moreover, there exist functions $K(z, w), \tilde{K}(z, w)$ so that given $K, \tilde{K}$, the function $Q(\cdot)$ is a Gaussian field with $E[Q(z)Q^*(w)] = K(z, w)$ and $E[Q(z)Q(w)] = \tilde{K}(z, w)$. Further, $K$ is analytic in $z$ and anti-analytic in $w$ while $\tilde{K}$ is analytic in both $z$ and $w$. Therefore, $Q$ is a random analytic function whose real and imaginary parts are jointly Gaussian fields.

For later purposes, we introduce the following notation.

**Notation.** We write $Y \sim \text{Normal}[m(\lambda), L(\lambda, \mu), \hat{L}(\lambda, \mu)]$ to mean that the real and imaginary parts of $Y$ are jointly Gaussian fields, $E[Y(\lambda)] = m(\lambda)$ and $E[Y(\lambda)Y^*(\mu)] = L(\lambda, \mu)$ and $E[Y(\lambda)Y(\mu)] = \hat{L}(\lambda, \mu)$.

Observe that $Y$ is a random analytic function if and only if $m(\lambda)$ is analytic in $\lambda$, $L(\lambda, \mu)$ is analytic in $\lambda$ and anti-analytic in $\mu$, and $\hat{L}$ is analytic in both $\lambda$ and $\mu$. Further, $Y$ is a complex Gaussian field if and only if $\hat{L} = 0$. In particular, for a GAF the third argument is identically zero.

**A Pólya’s urn scheme for characteristic polynomials.** We construct the random covariance kernel is via a version of Pólya’s urn scheme. By similarity transformations, we first convert the matrix into a Hessenberg (lower triangular plus an extra off-diagonal) form. Then, we consider characteristic polynomials of the successive minors. It turns out that these polynomials develop in a version of Pólya’s urn scheme, which we recall here briefly. An urn containing a black and a white ball is given, and at each time an extra ball is added to the urn whose color is black or white with probabilities proportional to number of the balls of the same color already in the urn. In short, we may write

$$X_1 = 0, \quad X_2 = 1, \quad X_{k+1} = \text{Bernoulli} \left[ \frac{X_1 + \ldots + X_k}{k} \right], \quad k \geq 3$$

For us, the essential part of Pólya’s urn is that the $X_{k+1}$ given the events up to time $k$ is a random variable whose mean is close to the average of the previous $X_k$. The conclusion is that this average converges almost surely to a random limit.

It turns out that a similar recursion this also holds for the random constant multiples $X_n = X_n(\lambda)$ of the characteristic polynomials $Q_n$. We shall see that
\[ X_1 = 1, \quad X_{k+1} \mid_{X_1, \ldots, X_k} = \text{Normal} \left[ m_k(\lambda), M_k(\lambda, \mu), \tilde{M}_k(\lambda, \mu) \right] \]

where

\[ m_k(\lambda) = \frac{\lambda X_k(\lambda)}{b_k}, \quad M_k(\lambda, \mu) = \frac{X_1(\lambda)X_1^*(\mu) + \ldots + X_k(\lambda)X_k^*(\mu)}{b_k^2} \]

\[ \tilde{M}_k(\lambda, \mu) = \begin{cases} b_k^{-2} (X_1(\lambda)X_1(\mu) + \ldots + X_k(\lambda)X_k(\mu)) & \text{if } \beta = 1. \\ 0 & \text{if } \beta = 2. \end{cases} \]

with \( b_k^2/k \to 1 \).

The main feature of Pólya’s urn is that the parameter of the Bernoulli distribution converges to a random limit almost surely, and the samples are asymptotically independent from this random distribution. Similarly, in our case, the variance parameter converges almost surely, and given the limit, the samples are asymptotically independent. In particular, as we shall show, in the limit they behave like a Gaussian analytic function with a random covariance kernel.

## 2 The recursion for the characteristic polynomial

Start with the real (\( \beta = 1 \)) or complex (\( \beta = 2 \)) Ginibre matrix, having i.i.d real or complex Gaussian entries. For the purposes of this paper, a standard complex Gaussian random variable will mean \( X + iY \) where \( X, Y \) are i.i.d. \( N(0, 1/2) \).

Following the randomized application of the Lanczos algorithm as pioneered by Trotter (1984), we conjugate the matrix \( A \) by an orthogonal/unitary block matrix as follows. Let

\[
A = \begin{pmatrix}
a_{11} & \ldots & b & \ldots \\
\vdots & & \ddots & \vdots \\
c & & A_{11} & \\
\vdots & & & \\
\end{pmatrix}, \quad O = \begin{pmatrix}
1 & \ldots & 0 & \ldots \\
\vdots & & \ddots & \vdots \\
0 & & O_{11} & \\
\vdots & & & \\
\end{pmatrix}
\]

so that we get

\[
OAO^* = \begin{pmatrix}
a_{11} & \ldots & bO_{11}^* & \ldots \\
\vdots & & \ddots & \vdots \\
o_{11}c & & O_{11}A_{11}O_{11}^* & \\
\vdots & & & \\
\end{pmatrix}.
\]
If $O_{11}$ is chosen depending only on $b$ so that it rotates $b$ into the first coordinate vector, then we get a matrix of the form

$$O^* = \begin{pmatrix} \mathcal{N} & \frac{1}{\sqrt{\beta}} \chi(n-1) & 0 & \cdots & 0 \\ \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} \end{pmatrix}$$

where all the entries are independent, and $\mathcal{N}$ indicates that the entry has Normal (real or complex) distribution and $\chi_n$ indicates that the entry has the distribution of the length of the vector with independent standard real Gaussian entries in $n$ dimensions. This is because the normal vector $c$ has a distribution that is invariant under rotation and the matrix $A_{11}$ has a distribution that is invariant under conjugation by a rotation. Repeated application of this procedure (where the rotation matrices are block diagonal with an identity matrix of increasing dimension at the top) brings $M$ to a form

$$\begin{pmatrix} \mathcal{N} & \frac{1}{\sqrt{\beta}} \chi(n-1) & 0 \\ \mathcal{N} & \mathcal{N} & \frac{1}{\sqrt{\beta}} \chi(n-2) \\ \vdots & \vdots & \ddots \\ \mathcal{N} & \cdots & \frac{1}{\sqrt{\beta}} \chi_1 \\ \mathcal{N} & \cdots & \mathcal{N} \end{pmatrix}$$

We conjugate this matrix by the reverse permutation matrix and transpose it. The eigenvalue equation of the resulting matrix reads

$$X = \lambda X.$$ (2)

Remark 4. This reduction is analogous to the tridiagonal form of the GUE matrix obtained by Trotter (1984) and that has been of much use in studying the scaling limits of eigenvalues in recent years (for example, see Dumitriu and Edelman (2002), Ramírez, Rider and Virág (2011) and Valkó and Virág (2009)).

The matrix on the left of (2) is the one whose eigenvalues we will study. Indeed, let $\mathcal{N}_k$ denote the $k$th row of this matrix, with the $\chi$ variable removed. Then the $k$th row of
the eigenvalue equation is

\[ \overrightarrow{X}_k \cdot \mathcal{N}_k + \frac{1}{\sqrt{\beta}} \chi_{\beta k} X_{k+1} = \lambda X_k \]  

(3)

where \( \overrightarrow{X}_k = (X_1, \ldots, X_k) \). For any \( \lambda \in \mathbb{C} \), let \( X_1(\lambda) = 1 \), and define \( X_{k+1}(\lambda) \) recursively as the solution to the equation given above for \( k < n \). Then, \( \lambda \) is an eigenvalue of the matrix if and only if it satisfies the last equation \( \overrightarrow{X}_n(\lambda) \cdot \mathcal{N}_n = \lambda X_n(\lambda) \) or equivalently, if we solve for \( k = n \) also and get \( X_{n+1}(\lambda) = 0 \).

But these equations are consistent as \( n \) changes, so we may define the \( X_k(\lambda) \) for all \( k \geq 0 \) through the infinite version of the above matrix. Clearly \( X_k \) is a polynomial of degree \( k - 1 \) and hence, \( X_{k+1}(\lambda) \) is a random constant times the characteristic polynomial of the top \( k \times k \) submatrix for every \( k \).

We introduce the random functions

\[ M_k(\lambda, \mu) = \overrightarrow{X}_k(\lambda) \cdot \overrightarrow{X}_k^*(\mu) = \sum_{j=1}^{k} X_j(\lambda) X_j^*(\mu), \]

\[ \hat{M}_k(\lambda, \mu) = \begin{cases} \overrightarrow{X}_k(\lambda) \cdot \overrightarrow{X}_k^*(\mu) = \sum_{j=1}^{k} X_j(\lambda) X_j(\mu) & \text{if } \beta = 1, \\ 0 & \text{if } \beta = 2. \end{cases} \]

where we use both \( z^* \) and \( \bar{z} \) to denote the complex conjugate of \( z \).

Let \( \mathcal{F}_k \) denote the \( \sigma \)-field generated by the first \( k - 1 \) rows of the matrix together with \( \chi_{\beta k} \). From (3) note that \( X_{k+1} \) given \( \mathcal{F}_k \) is a Gaussian field with mean \( \lambda \sqrt{\beta} X_k / \chi_{\beta k} \) covariance structure given by \( \beta M_k / \chi_{\beta k}^2 \) and \( \beta \hat{M}_k / \chi_{\beta k}^2 \). Moreover, we have

\[ M_{k+1} - M_k = X_{k+1}(\lambda) X_{k+1}(\mu), \quad \hat{M}_{k+1} - \hat{M}_k = \begin{cases} X_{k+1}(\lambda) X_{k+1}(\mu) & \text{if } \beta = 1, \\ 0 & \text{if } \beta = 2. \end{cases} \]

Thus the evolution of \( X_k \) can be summarized as a randomized recursion with \( X_1 = 1 \) and

\[ X_{k+1} \mid \mathcal{F}_k = \begin{cases} \text{Normal } \left[ \frac{\lambda X_k}{\sqrt{\beta} \chi_{\beta k}}, \frac{X_1 X_1^* + \ldots + X_k X_k^*}{\beta \chi_{\beta k}^2} \right] & \text{if } \beta = 2, \\ \text{Normal } \left[ \frac{\lambda X_k}{\sqrt{\beta} \chi_{\beta k}}, \frac{X_1(\lambda) X_1(\mu) + \ldots + X_k(\lambda) X_k(\mu)}{\beta \chi_{\beta k}^2}, \frac{X_1(\lambda) X_1(\mu) + \ldots + X_k(\lambda) X_k(\mu)}{\beta \chi_{\beta k}^2} \right] & \text{if } \beta = 1. \end{cases} \]

for \( k \geq 1 \).

This is the recursion, analogous to Pólya’s urn discussed in the introduction. Next, we establish a framework for the asymptotic analysis of such recursions.
3 Pólya’s urn in Hilbert space – theorem and examples

The main tool for the analysis of Pólya’s urn schemes will be the following theorem.

**Theorem 5** (Pólya’s urn in Hilbert space). Let $\mathcal{H}$ be a Hilbert space, and let $R_1, R_2, \ldots, R_{k_0-1}$ be deterministic elements of $\mathcal{H}$. Let $\mathcal{F}_k$ be a filtration in some probability space and for each $k \geq k_0$ assume that the $\mathcal{H}$-valued random variables $R_k, M_k \in \mathcal{F}_k$ satisfy

$$
M_k = \frac{R_1 + \ldots + R_k}{k}
$$

$$
\|E[R_{k+1}|\mathcal{F}_k] - M_k\| = (1 + \|M_k\|)O(\varepsilon_k) + \|R_k\|O(1/k)
$$

$$
E[\|R_{k+1}\|^2|\mathcal{F}_k] = O(1 + \|M_{k-1}\|^2 + \|M_k\|^2)
$$

for some positive sequence $\varepsilon_k$ such that $\varepsilon_k k^{-1}$ is summable.

Then $M_k$ converges to some limit a.s. and in $L^2$.

The right hand sides of (4) and (5) should simply read as “small”. The specific error terms here are tailored to the application at hand.

Note that without the error term, (4) says that the mean of $R_{k+1}$ given $\mathcal{F}_k$ is equal to $M_k = (R_1 + \ldots + R_k)/k$. This is the setting of Pólya’s urn scheme.

### 3.1 Example: Classical Pólya’s urn

Let $\mathcal{H} = \mathbb{R}$ and $R_1 = 0, R_2 = 1$. Given $R_1, \ldots, R_k$ let

$$
R_{k+1} \sim \text{Bernoulli}(M_k), \quad \text{with} \quad M_k = \frac{R_1 + \ldots + R_k}{k}.
$$

Then (4) holds without error terms and $R_{k+1}^2$ is bounded by 1. So $M_k$ converges to a limit $M$ almost surely.

It is well-known that $M$ has a Beta distribution, and the limiting distribution of $R_k$ is that of $\text{Bernoulli}(M)$ (i.e., sample $M$ and sample from $\text{Bernoulli}(M)$). In fact, for each $k$, the distribution of $R_k$ given $M$ is $\text{Bernoulli}(M)$ but in our setting this is a special phenomenon owing to the fact that error terms vanish.

### 3.2 Example: a semi-random Hessenberg matrix

For the next example, let $D \subset \mathbb{C}$ be a closed disk. For our next example, we will consider the Hilbert space of 2-variable functions from $D^2 \to \mathbb{C}$, with the following inner product

$$
\langle f, g \rangle = \int_{D^2} f(x, y)\bar{g}(x, y) \, dx \, dy + \int_D f(x, x)\bar{g}(x, x) \, dx
$$

(6)
or, in other words, consider the usual $L^2$ inner product with respect to the sum of the Lebesgue measure on $D^2$ and Lebesgue measure on the diagonal of $D^2$. More precisely, we define this inner product on smooth functions and then take the completion to get a Hilbert space. We will denote the corresponding norm simply by $\| \cdot \|$. The following simple Lemma is needed.

**Lemma 6.** Let $X \sim \text{Normal}[0, M, \hat{M}]$ and let $m : D \to \mathbb{C}$ be a function. Let $R = (X + m) \otimes (\bar{X} + \bar{m})$ and $\hat{R} = (X + m) \otimes (X + m)$. Then

$$\max\{E\|R\|^2, E\|\hat{R}\|^2\} \leq 8(\|m\|^4 + \|m^2\|^2) + 24(1 + |D|) \left(\|M\|^2 + \|\hat{M}\|^2\right).$$

Here $|D|$ is the area of $D$ and the norms on $m, m^2$ are $L^2(D)$.

**Proof.** For $h : D \to \mathbb{C}$, the norm (6) gives

$$\|h \otimes h^*\|^2 = \|h \otimes h\|^2 = \|h\|^4 + \|h^2\|^2$$

with the norms on the right hand side are for the usual $L^2(D)$. Using this, and triangle inequality we get

$$\|R\|^2 = \|X + m\|^4 + \|(X + m)^2\|^2 \leq 8\|X\|^4 + 8\|m\|^4 + (\|X^2\| + \|2Xm\| + \|m^2\|)^2$$

Since $\|2Xm\|^2 \leq \|X^2\| + \|m^2\|$, we get the bound

$$\|R\|^2 \leq 8(\|X\|^4 + \|X^2\|^2) + 8(\|m\|^4 + \|m^2\|^2)$$

By Cauchy-Schwarz

$$\|X\|^2 \leq \|1\|\|X^2\| = \sqrt{|D|} \|X^2\|,$$

and

$$E\|X^2\|^2 = \int_D E|X(z)|^4\,dz = \eta_{4,\beta} \left(\int_D M^2(z,z)\,dz + \int_D |\hat{M}|^2(z,z)\,dz\right) \leq 3 \left(\|M\|^2 + \|\hat{M}\|^2\right)$$

where $\eta_{4,\beta} = E|N|^4$ for a standard $\beta$-Gaussian $N$ (i.e. 3,2 for $\beta = 1, 2$ respectively). Putting all these together, we get

$$E\|R\|^2 \leq 8(\|m\|^4 + \|m^2\|^2) + 24(1 + |D|) \left(\|M\|^2 + \|\hat{M}\|^2\right). \square$$
Let \( b_k \) be deterministic positive numbers such that \( \epsilon_k := \frac{|b^2_k - k|}{k} \) satisfy \( \sum k^{-1} \epsilon_k < \infty \).

One example is to take \( b_k = \sqrt{k} \).

Consider the nested matrices

\[
\begin{pmatrix}
N & b_1 & 0 \\
N & N & b_2 \\
& \ddots & \ddots \\
& & b_{n-1} \\
N & \cdots & N
\end{pmatrix}
\]

where the \( N \) refer to different i.i.d. standard real (\( \beta = 1 \)) or complex (\( \beta = 2 \)) normal random variables.

Just as in section 2 for \( \lambda \in \mathbb{C} \), let \( X_1(\lambda) = 1 \), and define \( X_{k+1}(\lambda) \) recursively as the solution of the eigenvalue equation given by the \( k \)th row of the matrix above (for \( k < n \)).

In other words, let \( \overrightarrow{N}_k \) denote the vector formed by the first \( k \) entries of the \( k \)th row of the nested matrices, write \( \overrightarrow{X}_k(\lambda) = (X_1(\lambda), \ldots, X_k(\lambda)) \), and recursively define \( X_{k+1}(\lambda) \) as the solution to the \( k \)th row of the eigenvalue equation

\[
\overrightarrow{N}_k \cdot \overrightarrow{X}_k(\lambda) + b_k X_{k+1}(\lambda) = \lambda X_k(\lambda).
\]

Let \( \mathcal{F}_k \) be the sigma-field generated by the first \( k - 1 \) rows of the matrix. Define

\[
R_k(\lambda, \mu) = (X_k \otimes X_k^*)(\lambda, \mu) = X_k(\lambda)X_k^*(\mu), \quad \hat{R}_k(\lambda, \mu) = (X_k \otimes X_k)(\lambda, \mu) = X_k(\lambda)X_k(\mu).
\]

Then, given \( \mathcal{F}_k \) we have

\[
X_{k+1} \sim \text{Normal} \left[ \frac{\lambda}{b_k} X_k, \frac{k}{b^2_k} M_k, \frac{k}{b^2_k} \hat{M}_k \right], \quad \text{where} \quad (8)
\]

\[
M_k = \frac{\hat{R}_1 + \ldots + \hat{R}_k}{k}, \quad \hat{M}_k = \begin{cases} 
(\hat{R}_1 + \ldots + \hat{R}_k)/k & \text{if } \beta = 1, \\
0 & \text{if } \beta = 2.
\end{cases}
\]

This means that conditionally on \( \mathcal{F}_k \) the random variable \( X_{k+1} \) is a Gaussian field with the given mean function and covariance structure. For \( \beta = 2 \) it is a Gaussian analytic function, while for \( \beta = 1 \) it is a random analytic function with jointly Gaussian real and imaginary parts.

In order to set up a Hilbert space \( \mathcal{H} \), we first fix a closed disk \( D \subset \mathbb{C} \), and consider the norm (8). Let \( \mathcal{H}_\beta = \mathcal{H} \) for \( \beta = 2 \) and \( \mathcal{H}_\beta = \mathcal{H} \times \mathcal{H} \) for \( \beta = 1 \) (the inner product on \( \mathcal{H} \times \mathcal{H} \) is of course \( \langle (u, v), (u', v') \rangle = \langle u, u' \rangle \langle v, v' \rangle \)).
First consider the case $\beta = 2$ where we can forget $\hat{R}_k, \hat{M}_k$. Regard $R_k, M_k$ as random variables taking values in $\mathcal{H}$, and $X_k$ as an $L^2(D)$-valued random variable. Then

$$E[R_{k+1}(\lambda, \mu) | \mathcal{F}_k] = \frac{k}{b_k^2} M_k(\lambda, \mu) + \frac{\lambda \mu}{b_k^2} X_k(\lambda) X_k(\mu) = \frac{k}{b_k^2} M_k(\lambda, \mu) + \frac{\lambda \mu}{b_k^2} R_k(\lambda, \mu).$$

Thus, condition (4) follows because

$$\|E[R_{k+1} | \mathcal{F}_k] - M_k\| \leq \frac{|b_k^2 - k|}{b_k^2} \|M_k\| + \|R_k\| O(1/k) = O(\varepsilon_k) \|M_k\| + \|R_k\| O(1/k)$$

To check condition (5), we condition on $\mathcal{F}_k$ and use Lemma 6. Note the conditional mean and variance (14) of the Gaussian field $X_{k+1}$. We get

$$E[\|R_{k+1}\|^2 | \mathcal{F}_k] \leq \frac{24k^2}{b_k^2} (1 + |D|) \|M_k\|^2 + 8 \frac{1}{b_k^2} (\|\lambda X_k\|^2 + \|\lambda X_k\|^4)$$

$$\leq c \|M_k\|^2 + \frac{c}{k^2} (\|X_k^2\|^2 + \|X_k^4\|)$$

$$= c \|M_k\|^2 + \frac{c}{k^2} \|R_k\|^2,$$

where $c$ depends on the sequence $b_k$ and $D$ only, and the last equality follows from (7). Writing $R_k$ as $kM_k - (k-1)M_{k-1}$ we get the required upper bound $c'(\|M_k\|^2 + \|M_{k-1}\|^2)$.

Theorem 5 implies that $M_k$ converges (and uniformly on $D$) almost surely to a limit $M$. Since local $L^2$ convergence for analytic functions implies sup-norm convergence, the limit $M$ is analytic in its two variables. Also

$$\frac{R_k}{k} = \frac{kM_k - (k-1)M_{k-1}}{k} \to 0$$

and so $X_k/b_k \to 0$. The conditional law of $X_{k+1}$ given $\mathcal{F}_k$ converges to Normal$[0, M, 0]$. As each $M_k(\lambda, \mu)$ is analytic in the first variable and anti-analytic in the second, the same holds for $M$ and it follows that Normal$[0, M, 0]$ is in fact a Gaussian analytic function, conditional upon $M$.

For $\beta = 1$, we consider the $(R_k, \hat{R}_k), (M_k, \hat{M}_k)$ as elements of $\mathcal{H}_1 = \mathcal{H} \times \mathcal{H}$. The estimates obtained above for $R_k$ hold also for $\hat{R}_k$ with the obvious changes, and hence conditions (4), (5) are easily verified for $(R_k, \hat{R}_k)$. Consequently Theorem 5 assures the existence of $M = \lim M_k$ and $\hat{M} = \lim \hat{M}_k$ and that the conditional distribution of $X_{k+1}$ given $\mathcal{F}_k$ converges to Normal$[0, M, \hat{M}]$. 


3.3 Example: Hessenberg matrices with independent entries

Combining the previous arguments with the central limit theorem, we can show that randomly scaled characteristic functions of the following Hessenberg matrices converge almost surely to a random analytic function which is a mixture of Gaussian analytic functions.

\[
\begin{pmatrix}
X & b_1 & 0 \\
X & X & b_2 \\
\vdots & \ddots & \ddots \\
X & \ldots & b_{n-1} \\
X & \ldots & X
\end{pmatrix}
\]

where \( X \) are independent, identically distributed mean zero and finite fourth moment, and \( b_k = \sqrt{k} + O(k^{1/2-\varepsilon}) \). This condition could be significantly weakened (as long as the rate of growth of \( b_k \)'s is not too slow and not too fast); but this is not the main topic in the paper.

We have universality in the sense that the limit is a randomized Gaussian analytic function, even when the \( X \) are not Gaussian. In particular, the local behavior of the zeros is universal; we expect that the probability of two zeros in an disk of radius \( \varepsilon \) decays like \( \varepsilon^6 \).

The Central Limit Theorem holds because the actual conditional covariance matrix has to be scaled by \( k \) to converge – in particular, a given characteristic function value has to use more and more of the independent \( X \)-es. Therefore it must be asymptotically normal; we omit the details.

4 Pólya’s urn in Hilbert space – proof

The goal of this section is to prove Theorem 5.

We recall a few facts about Hilbert-space valued random variables and martingales.

**Fact 7.** In the following, \( X_n \) will be a Hilbert-space valued random sequence.

(i) If \( X_n \) is a martingale, and \( \sum_{n=1}^{\infty} \mathbb{E}\|X_{n+1} - X_n\|^2 \) is finite, then there exists a limit \( X \) so that \( \|X_n - X\| \to 0 \) almost surely and in \( L^2(\Omega) \).

(ii) If \( \sum_n \mathbb{E}\|X_n - X\|^2 \) is finite, then \( \|X_n - X\| \to 0 \) almost surely and in \( L^2(\Omega) \).
(iii). If \( \sum (\mathbb{E}\|X_n\|^2)^{1/2} \) is finite, then there exists a limit \( S \) so that \( S - \sum_{k=1}^{n} X_k \to 0 \) almost surely and in \( L^2(\Omega) \).

Proof. \( \square \) The claim implies that \( \sum_n \|X_n\|^2 \) is finite almost surely (since its expected value is), so \( \|X_n\| \to 0 \) almost surely.

By the triangle inequality, we see that \( X_n \) is a Cauchy sequence in \( L^2(\Omega) \) and therefore it converges to some \( S \). Further,

\[
\|S - \sum_{k=1}^{n} X_k\| \leq \sum_{k=n+1}^{\infty} \|X_k\|
\]

the latter being a monotone sequence that converges to 0 in expectation, hence almost surely. \( \square \)

We are now ready for the main proof.

Proof of Theorem \( \square \) Let

\[
M_{k+1} - M_k = A_k + B_k, \quad A_k = M_{k+1} - \mathbb{E}[M_{k+1}\mid\mathcal{F}_k], \quad B_k = \mathbb{E}[M_{k+1}\mid\mathcal{F}_k] - M_k
\]

be the Doob decomposition of the process \( M_k \); the \( A_k \) are martingale increments and the \( B_k \) are increments of a predictable process. It suffices to show that \( \sum A_k \) and \( \sum B_k \) converge. We have

\[
A_k = \frac{R_{k+1} - \mathbb{E}[R_{k+1}\mid\mathcal{F}_k]}{k+1}
\]

which gives

\[
\mathbb{E}[\|A_k\|^2\mid\mathcal{F}_k] = \frac{1}{(k+1)^2} \left( \mathbb{E}[\|R_{k+1}\|^2\mid\mathcal{F}_k] - \mathbb{E}[\|R_{k+1}\mid\mathcal{F}_k]\|^2 \right)
\]

\[
\leq \frac{1}{(k+1)^2} \mathbb{E}[\|R_{k+1}\|^2\mid\mathcal{F}_k] = \frac{1}{k^2} O(1 + \|M_{k-1}\|^2 + \|M_k\|^2)
\]

(9)

and

\[
B_k = \frac{\mathbb{E}[R_{k+1} - M_k\mid\mathcal{F}_k]}{k+1}, \quad \|B_k\| \leq (1 + \|M_k\|)O(\varepsilon_k/k) + \|R_k\|O(1/k^2)
\]

(10)

First we will show that \( \mathbb{E}\|M_k\|^2 \) is bounded. Then it will follow from Fact \( \square \) that \( \sum A_k \) converges in \( L^2 \) and a.s., because it is an \( L^2 \)-bounded martingale. Then we will show that \( \sum (\mathbb{E}\|B_k\|^2)^{1/2} \) is finite. Fact \( \square \) then implies that \( B_1 + \ldots + B_k \) is Cauchy in \( L^2 \) and \( \mathbb{E}\sum\|B_k\| \) is finite. Thus \( \sum B_k \) converges a.s. and in \( L^2 \). We write

\[
\mathbb{E}\|M_{k+1}\|^2 = \mathbb{E}\|M_k\|^2 + \mathbb{E}\|A_k\|^2 + \mathbb{E}\|B_k\|^2 + 2\Re \{\mathbb{E}\langle M_k, B_k \rangle\} + 2\mathbb{E}\langle M_k + B_k, A_k \rangle
\]

(11)
the last term vanishes, because $E|A_k|F_k|k = 0$ and $M_k, B_k \in F_k$. By (10) we have

$$E\|B_k\|^2 \leq (1 + E\|M_k\|^2)O(\varepsilon_k^2/k^2) + E\|R_k\|^2O(1/k^4)$$

$$\leq (1 + E\|M_{k-1}\|^2 + E\|M_k\|^2)O(\varepsilon_k^2/k^2 + 1/k^4) \quad (12)$$

The last inequality follows from the bound (5). Formula (12) and Cauchy-Schwarz imply that if $E\|M_k\|^2$ is bounded then $\sum E\|B_k\|$ is finite.

Further, two applications of Cauchy-Schwarz give

$$|E\langle M_k, B_k \rangle| \leq E\|M_k\|\|B_k\|$$

$$\leq (E\|M_k\|^2E\|B_k\|^2)^{1/2}$$

$$\leq (1 + E\|M_k\|^2 + E\|M_{k-1}\|^2)O(\varepsilon_k/k + 1/k^2) \quad (13)$$

where the last inequality follows from (12). Using (9), (12) and (13) in (11) we finally get

$$E\|M_{k+1}\|^2 \leq E\|M_k\|^2 + (1 + E\|M_k\|^2 + E\|M_{k-1}\|^2)O(\varepsilon_k/k + 1/k^2),$$

so with the notation

$$y_k = \max_{1 \leq \ell \leq k} E\|M_{\ell}\|^2$$

we have

$$y_{k+1} \leq y_k + (1 + y_k + y_{k-1})O(\varepsilon_k/k + 1/k^2)$$

$$\leq y_k(1 + O(\varepsilon_k/k + 1/k^2))$$

which shows

$$y_k \leq y_1 \prod_{\ell=2}^{k} (1 + O(\varepsilon_k/\ell + 1/\ell^2))$$

in particular $E\|M_k\|^2$ is bounded, completing the proof. \qed

5 Analysis of the Ginibre ensembles

In this section we prove Theorem 1 and Theorem 3. By the Lanczos algorithm as explained in Section 2, it suffices to show that the characteristic polynomials of the Hessenberg matrices on the left side of (2) converge in distribution to a random analytic function $Q$, and that there exists a random positive definite Hermitian function $K(z, w)$ so that given $K$ the function $Q(z)$ is a Gaussian field with covariance kernel $K$.  

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If we condition on all $b_k^2 := \frac{1}{\beta} \chi_{3k}^2$ variables on the first super diagonal, then the matrix reduces to the semi-random Hessenberg matrix of Example 3.2. We leave it to the reader to check that $k^{-2}|b_k^2 - k|$ is summable, a.s., so that the conclusions of Example 3.2 apply. Thus, conditional on all $b_k^2$, the limit $Q$ of $Q_n$ (in distribution) exists, and is a mixture of Gaussian analytic functions. This implies that unconditionally also, $Q$ is a mixture of Gaussian analytic functions.

But we want to be able to study the properties of the random covariance kernel. Hence we directly analyze the Hessenberg matrix with the chi-squared variables in place, and get the same conclusions.

**Proof of Theorem**

Consider the nested matrices

$$
\begin{pmatrix}
N & b_1 & 0 \\
N & N & b_2 \\
& & \ddots \\
& & & b_{n-1} \\
N & \cdots & N
\end{pmatrix}
$$

where the $N$ refer to different real ($\beta = 1$) or complex ($\beta = 2$) normal random variables and $b_k^2$ to Gamma $(k\beta/2, \beta/2)$ random variables, and all entries are independent. Note that Gamma $(k\beta/2, \beta/2)$ is the distribution of the length-squared of a $k$-dimensional standard real or complex Gaussian vector, for $\beta = 1, 2$ respectively.

Define $\vec{N}_k$ as the vector formed by the first $k$ entries of the $k$th row of the nested matrices. As in Example 3.2, for $\lambda \in \mathbb{C}$, let $X_1(\lambda) = 1$, and for $k < n$, recursively define $X_{k+1}(\lambda)$ as the solution to

$$
\vec{N}_k \cdot \vec{X}_k(\lambda) + b_k X_{k+1}(\lambda) = \lambda X_k(\lambda)
$$

where

$$
\vec{X}_k(\lambda) = (X_1(\lambda), \ldots, X_k(\lambda)).
$$

Then $X_{k+1}$ is the characteristic function of the top $k \times k$ principal submatrix.

We note that

$$
\mathbb{E} b_k^{-2} = \begin{cases}
\frac{1}{k-1}, & \beta = 2, \ k \geq 2 \\
\frac{1}{k-2}, & \beta = 1, \ k \geq 3
\end{cases}
$$

and

$$
\mathbb{E} b_k^{-4} = \begin{cases}
\frac{1}{(k-1)(k-2)}, & \beta = 2, \ k \geq 3 \\
\frac{1}{(k-2)(k-4)}, & \beta = 1, \ k \geq 5
\end{cases}
$$
And in light of this we will set \( k_0 = 1 + 4/\beta \). Let \( \mathcal{F}_k \) be the sigma-field generated by the first \( k \) rows of the matrix. Define

\[
R_k(\lambda, \mu) = (X_k \otimes X_k^*)(\lambda, \mu) := X_k(\lambda)X_k(\mu), \quad \hat{R}_k(\lambda, \mu) = (X_k \otimes X_k)(\lambda, \mu) := X_k(\lambda)X_k(\mu).
\]

Then, given \( \mathcal{F}_k \) and \( b_k \) we have

\[
X_{k+1} \sim \text{Normal}
\begin{bmatrix}
\frac{\lambda}{b_k}X_k, \frac{k}{b_k^2}M_k, \frac{k}{b_k^2}\hat{M}_k
\end{bmatrix},
\]

where

\[
M_k = \frac{R_1 + \ldots + R_k}{k}, \quad \hat{M}_k = \begin{cases} (\hat{R}_1 + \ldots + \hat{R}_k)/k & \text{if } \beta = 1, \\ 0 & \text{if } \beta = 2. \end{cases}
\]

This means that conditionally on \( \mathcal{F}_k \) and \( b_k \) the random variable \( X_{k+1} \) is a Gaussian field with the given mean and covariance.

In order to set up a Hilbert space, we first fix a closed disk \( D \subset \mathbb{C} \). Recall the Hilbert space \( \mathcal{H} \) defined in \( \mathcal{O} \) and the spaces \( \mathcal{H}_2 = \mathcal{H} \) and \( \mathcal{H}_1 = \mathcal{H} \times \mathcal{H} \) that were introduced in section \( 3.2 \). Regard \( R_k, M_k, \hat{R}_k, \hat{M}_k \) as \( \mathcal{H} \)-valued random variables, and \( X_k \) as an \( L^2(D) \)-valued random variable.

Theorem \( 5 \) will be applied to the \( \mathcal{H}_2 \)-valued random variables \( R_k \) for \( \beta = 2 \) and to the \( \mathcal{H}_1 \)-valued random variables \( (R_k, \hat{R}_k) \) for \( \beta = 1 \). Then (4) holds as

\[
\mathbb{E}[R_{k+1} | \mathcal{F}_k] = \mathbb{E}[b_k^{-2}] (kM_k + \lambda \mu X_kX_k^*) = \frac{k}{k - \frac{2}{\beta}} M_k + \frac{\lambda \mu}{k - \frac{2}{\beta}} R_k
\]

(15)

where \( \lambda, \mu \) are the arguments of \( R_{k+1} \). Thus

\[
\|\mathbb{E}[R_{k+1} | \mathcal{F}_k] - M_k\| = \|M_k\|O(1/k) + \|R_k\|O(1/k)
\]

which proves condition (4) for \( \beta = 2 \). Similarly, \( \|\mathbb{E}[\hat{R}_{k+1} | \mathcal{F}_k] - \hat{M}_k\| = \|\hat{M}_k\|O(1/k) + \|\hat{R}_k\|O(1/k) \) from which we get

\[
\|\mathbb{E}[(R_{k+1}, \hat{R}_{k+1}) | \mathcal{F}_k] - (M_k, \hat{M}_k)\| = \|(M_k, \hat{M}_k)\|O(1/k) + \|(R_k, \hat{R}_k)\|O(1/k)
\]

which proves condition (4) for \( \beta = 1 \).

We now proceed to check condition (5). First, by Lemma \( 6 \) we have

\[
\mathbb{E}[\|R_{k+1}\|^2 | \mathcal{F}_k, b_k] \leq \frac{24k^2}{b_k^4} (1 + |D|) \|M_k\|^2 + \frac{8}{b_k^4} (\|X_k\|^2 + \|X_k^2\| + \|X_k^4\|)
\]

\[
\leq \frac{c}{b_k^4} (k^2 \|M_k\|^2 + \|X_k^2\|^2 + \|X_k^4\|)
\]

\[
= \frac{c}{b_k^4} (k^2 \|M_k\|^2 + \|R_k\|^2),
\]
where $c$ depends on $D$ only, and the last equality follows from (7). Writing $R_k$ as $kM_k - (k - 1)M_{k-1}$ we get the upper bound
\[
E[\|R_{k+1}\|^2 | F_k, b_k] \leq \frac{c'k^2}{b_k^4}(\|M_k\|^2 + \|M_{k-1}\|^2).
\]
we complete checking condition (5) for $\beta = 2$ by noting that
\[
E[\|R_{k+1}\| | F_k] = E[E[\|X_{k+1}\|^4 | F_k, b_k] | F_k] \\
\leq c'k^2 E[b_k^{-4}(\|M_k\|^2 + \|M_{k-1}\|^2)] \\
= O(\|M_k\|^2 + \|M_{k-1}\|^2)
\]
For $\beta = 1$, we make similar computation for $\hat{R}_{k+1}$ and combine it with the above to verify condition (5).

Thus, Theorem 5 implies that $(M_k, \hat{M}_k)$ converges almost surely to a limit $(M, \hat{M})$. Of course $\hat{M} = 0$ for $\beta = 2$. Since local $L^2$ convergence for analytic functions implies sup-norm convergence, the limit $M$ is analytic in the first variable and anti-analytic in the second variable while $\hat{M}$ is analytic in both its arguments. Also
\[
\frac{R_k}{k} = \frac{kM_k - (k - 1)M_{k-1}}{k} \to 0
\]
and so $X_k/b_k \to 0$. Thus, the conditional distribution of $X_{k+1}$ given $F_k$ converges to Normal$[0, M, \hat{M}]$. For $\beta = 2$ this is a Gaussian analytic function with random covariance kernel $M$ while for $\beta = 1$ it is a random analytic function with jointly Gaussian real and imaginary parts.

\section{The mean of the random covariance kernel}

We have shown that the limit of characteristic polynomials is a centered Gaussian analytic function with random covariance kernels $M, \hat{M}$. Now we try to understand the distribution of $M$ itself. We are able to calculate the first two moments. We show the calculation for $M$ only, which is sufficient for $\beta = 2$.

The expectation of $M(\lambda, \mu)$: From equation (15) we have
\[
E[R_{k+1} | F_k] = E[b_k^{-2}(R_1 + \ldots + R_k + \lambda \mu R_k)]
\]
so we would like to set $r_k$ to be the $E R_k$ to get the recursion
\[
\frac{r_{k+1}}{k} = \frac{r_1 + \ldots + r_k + \lambda \mu r_k}{k - \frac{2}{\beta}}
\]

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but this is invalid, because $E[R_k] = \infty$ simply because $E[b_k^2] = \infty$ for $k \leq 2/\beta$. Therefore we set

$$r_k = E[b_k^2 R_k] \quad \text{for } \beta = 1, \quad r_k = E[b_k^2] \quad \text{for } \beta = 2.$$ 

These $r_k$ are finite and the recursion (16) above holds for these quantities, for $k > 2/\beta$.

Fix $z = \lambda \mu$ and use (16) to write

$$r_1 + \ldots + r_k = (k - 2/\beta) r_{k+1} - z r_k \quad r_1 + \ldots + r_{k-1} = (k - 1 - 2/\beta) r_k - z r_{k-1}$$

Taking differences we get

$$r_{k+1} - r_k = \frac{z}{k - \frac{2}{\beta}} (r_k - r_{k-1}) = \begin{cases} (r_3 - r_2) \frac{z^{k-2}}{(k-2)!} & \text{when } \beta = 1 \\ (r_2 - r_1) \frac{z^{k-1}}{(k-1)!} & \text{when } \beta = 2 \end{cases}$$

Summing these we get

$$\beta = 2 : \quad r_{k+1} - r_1 = (r_2 - r_1) \sum_{j=0}^{k-1} z^j / j! \quad \beta = 1 : \quad r_{k+1} - r_2 = (r_3 - r_2) \sum_{j=0}^{k-2} z^j / j! . \quad (17)$$

By direct computation we have the initial values

$$r_1 = \begin{cases} 1 & \text{if } \beta = 2. \\ 2 & \text{if } \beta = 1. \end{cases} \quad r_2 = \begin{cases} 1 + \lambda \mu & \text{if } \beta = 2. \\ 2 + 2 \lambda \mu & \text{if } \beta = 1. \end{cases} \quad r_3 = 2 + 2 \lambda \mu + (\lambda \mu)^2 \quad \text{for } \beta = 1.$$ 

Plugging these into (17), we get

$$r_{k+1} = \begin{cases} 1 + z \sum_{j=0}^{k-1} \frac{z^j}{j!} \to 1 + ze^z & \text{for } \beta = 2. \\ 2 + 2z + z^2 \sum_{j=0}^{k-2} \frac{z^j}{j!} \to 2 + 2z + z^2 e^z & \text{for } \beta = 1. \end{cases}$$

as $k \to \infty$. Thus the limiting covariance kernel satisfies

$$E[M(\lambda, \mu)] = \begin{cases} 1 + \lambda \mu e^{\lambda \mu} & \text{for } \beta = 2. \\ 2 + 2 \lambda \mu + (\lambda \mu)^2 e^{\lambda \mu} & \text{for } \beta = 1. \end{cases}$$

Contrast this with the planar Gaussian analytic function $f(z) := \sum_n a_n \frac{z^n}{\sqrt{n!}}$ which has covariance kernel $e^{z \overline{w}}$, for both real and complex i.i.d Gaussian coefficients $a_n$ (lest this sound like a contradiction, one must again consider the direct second moment $E[f(z)f(w)]$ which is again equal to $e^{z \overline{w}}$ for the real case but vanishes identically in the complex case!).
7 Second moment of the covariance kernel

In this section we consider the second moment of the covariance kernel. Just like the first moment case, we have to multiply by random constants for this moment to exist.

Fix $\beta = 2$ and let $\alpha_k = \mathbb{E}[b_1^4 b_2^4 R_k^2]$, $\beta_k = \mathbb{E}[b_1^4 b_2^4 S_k]$, where $S_k = \sum_{j=1}^{k-1} R_j R_k$. We shall get recursions for these quantities, by first evaluating conditional expectations given $\mathcal{F}_k$. To this end, set $z = |\lambda|^2$ and observe that

$$
\mathbb{E}[R_{k+1}^2 | \mathcal{F}_k] = \mathbb{E}[b_k^{-4}] \left( z^2 R_k^2 + 2k^2 M_k^2 + 4z k M_k R_k \right)
= \frac{z^2 R_k^2 + 2k^2 M_k^2 + 4z k M_k R_k}{(k-1)(k-2)}
= \frac{z^2 R_k^2 + 2 \sum_{j=1}^{k} R_j^2 + 4z S_k + 4z R_k^2 + 4 \sum_{j=1}^{k} S_j}{(k-1)(k-2)}.
$$

$$
\mathbb{E}[S_{k+1} | \mathcal{F}_k] = \sum_{i=1}^{k} R_i \mathbb{E}[R_{k+1} | \mathcal{F}_k]
= \left( \sum_{j=1}^{k} R_j \right) \frac{1}{k-1} (k M_k + z R_k)
= \frac{1}{k-1} \left[ \left( \sum_{j=1}^{k} R_j \right)^2 + z S_k + z R_k^2 \right]
= \frac{\sum_{j=1}^{k} R_j^2 + 2 \sum_{j=1}^{k} S_j + z S_k + z R_k^2}{k-1}.
$$

Multiplying by $b_1^4 b_2^4$ and taking expectations, we get that for $k \geq 3$

$$
\alpha_{k+1} = \frac{z(z+4) \alpha_k + 2 \sum_{j=1}^{k} \alpha_j + 4z \beta_k + 4 \sum_{j=1}^{k} \beta_j}{(k-1)(k-2)}.
$$

$$
\beta_{k+1} = \frac{\sum_{j=1}^{k} \alpha_j + 2 \sum_{j=1}^{k} \beta_j + z \beta_k + z \alpha_k}{k-1}.
$$

These can be rephrased as

$$
2 \sum_{j=1}^{k} \alpha_j + 4 \sum_{j=1}^{k} \beta_j = (k-1)(k-2) \alpha_{k+1} - z(z+4) \alpha_k - 4z \beta_k.
$$

$$
\sum_{j=1}^{k} \alpha_j + 2 \sum_{j=1}^{k} \beta_j = (k-1) \beta_{k+1} - z \beta_k - z \alpha_k.
$$
For smaller $k$, it is possible to compute manually the following values

| $k$ | $\alpha_k$ | $\beta_k$ |
|-----|-------------|-------------|
| 1   | 12          | 0           |
| 2   | $6 (2 + 4z + z^2)$ | $6 (1 + z)$ |
| 3   | $12 + 24z + 24z^2 + 8z^3 + z^4$ | $2 (6 + 9z + 6z^2 + z^3)$ |
| 4   | $72 + 144z + 144z^2 + 96z^3 + 33z^4 + 6z^5 + \frac{1}{2}z^6$ | $36 + 60z + 48z^2 + 24z^3 + \frac{11}{2}z^4 + \frac{1}{2}z^5$ |

The $k = 4$ cases in fact follow from the recursions above. For $k \geq 4$, by writing the same equation for $k-1$ and taking the difference we get the first two of the following equations. The last one follows by taking the difference of the above two.

\[
2\alpha_k + 4\beta_k = (k-1)(k-2)\alpha_{k+1} - (k-2)(k-3)\alpha_k - z(z+4)\alpha_k + z(z+4)\alpha_{k-1} - 4z\beta_k + 4z\beta_{k-1}.
\]

\[
\alpha_k + 2\beta_k = (k-1)\beta_{k+1} - (k-2)\beta_k - z\beta_k + z\beta_{k-1} - z(\alpha_k - \alpha_{k-1}).
\]

\[
2(k-1)\beta_{k+1} - 2z\beta_k = (k-1)(k-2)\alpha_{k+1} - z(z+4)\alpha_k - 4z\beta_k + 2z\alpha_k.
\]

Using symbolic computations, we found that the coefficients of $z^j$ is the same in all $a_k$ for $k \geq j + 3$. Similarly, the coefficients of $b_k - b_{k-1}$ stabilize and they are exactly half the corresponding stable coefficients of $a_k$. The first few coefficients are

\[
a_\infty(z) = 72 + 192z + \frac{802z^2}{3} + \frac{776z^3}{3} + \frac{3799z^4}{20} + \frac{9967z^5}{90} + \frac{666847z^6}{12600} + \frac{11161z^7}{525} + \frac{474659z^8}{64800} + \ldots
\]

however, we have not been able to find a closed-form formula for the coefficients or $a_\infty(z)$. The final answer is

\[
\lim_{k \to \infty} E M_k(\lambda, \lambda)^2 = a_\infty(|\lambda|^2)/2.
\]

**Special case** $z = 0$: The equations become

\[
2\alpha_k + 4\beta_k = (k-1)(k-2)\alpha_{k+1} - (k-2)(k-3)\alpha_k.
\]

\[
\alpha_k + 2\beta_k = (k-1)\beta_{k+1} - (k-2)\beta_k.
\]

\[
2\beta_{k+1} = (k-2)\alpha_{k+1}.
\]

Substituting the last one into the second, we get

\[
(k-1)(k-2)\alpha_{k+1} = \alpha_k(2 + 2(k-3) + (k-2)(k-3))
\]

which implies $\alpha_{k+1} = \alpha_k$. Then $\beta_k = \frac{k-3}{2} \alpha_3$ for all $k \geq 3$. Thus

\[
E M_k^2 = \frac{1}{k^2} \sum_{j=1}^{k} \alpha_j^2 + 2\beta_j \to \alpha_3.
\]
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