On the switching control of the DC–DC zeta converter operating in continuous conduction mode

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Abstract
Here, a switching control mechanism for the stabilization of a DC–DC zeta converter operating in continuous conduction mode is proposed. The switching control algorithm is based on a control Lyapunov function and extends the method proposed for a two-dimensional boost converter model presented in the literature to a four-dimensional zeta converter model. The local asymptotical stability of the operating point is established using LaSalle’s invariance principle for differential inclusions. By applying spatial regularization, a modified switching control algorithm reduces the switching frequency and keeps the state-trajectory around a neighbourhood of the operating point. The method works well even if the operation point changes significantly and it is valid for both step-up and step-down operations. Furthermore, by approximating the state-trajectory near the operating point, an explicit relation between the modified switching algorithm and the switching frequency is obtained, which allows to choose systematically the desired switching frequency for the converter to operate. The effectiveness of the proposed method is illustrated with simulation results.

1 | INTRODUCTION

In energy harvesting systems, DC–DC converters are part of the power management system. Because of the uncertain nature of the ambient energy, for example, low or high irradiance of sun and fluctuation of the wind speed, the voltage generated by the energy harvester, which is connected to the input of the DC–DC converter, can be higher or lower than the output voltage. For this reason, a fourth-order DC–DC converter is a good candidate to be deployed, since it has step-up and step-down capability. There are a few topologies available, and the zeta topology is selected for our research due to two reasons: (1) positive output voltage, and low output voltage ripple [3], (2) natural DC input-to-output voltage isolation [4].

To control a DC–DC converter, the conventional fixed-frequency, average-based system control methods are commonly deployed such as proportional integral (PI) [5–9], optimal [11–16], sliding mode [17, 18], fuzzy [19, 20], model predictive [21, 22], adaptive [23], and fuzzy neural [24], to name a few. The PI control produces fast output voltage regulation, however, it suffers from high control duty-ratio effort [9] that can lead to PWM circuitry problem [12]. The conventional optimal linear quadratic regulator (LQR) control produces optimal compensation with minimal control effort, but lack of robustness if the parameter is uncertain [14]. While LMI-LQR control is robust, its control duty-ratio signal has quite a large ripple [15], which may produce non-linear effect if the ripple exceeded 20% [10]. As for the sliding mode, fuzzy, model predictive, adaptive, and fuzzy neural, because they use the so-called small-signal average model, when the duty ratio largely deviates from the nominal one, the small-signal average model is not a good approximation. As a result, the controller design is no longer valid, which in turn jeopardizes the system performance. On the other hand, non-average-based system control, typically known as hybrid control, is a variable switching frequency type of control where the switching frequency is initially low and it becomes arbitrarily fast at operating point. The hybrid control is more robust than average-based system control [2], due to the former’s ability to execute the switching mechanism online. Hybrid control has been implemented for the stabilization of the DC–DC converter [25–32]. In [25–28], the authors propose a switching algorithm by approximating the state-trajectory and restricting
the state-trajectory within a limits specified by the guard conditions. Even though the output voltage regulation is achieved and demonstrated, no theoretical work is presented to prove the stability of the system. In [29–32], the authors propose a Lyapunov-based hybrid control to stabilize the DC–DC converters. The study [29] basically proposes the switching rule that assigns the mode decreasing the value of the Lyapunov function most. When the trajectory reaches the switching boundary, it evolves as a sliding mode solution, which means that the switching interval becomes infinitesimally small. In [30, 31], the authors use sampled-data control to avoid sliding mode solutions. Though switching frequency is controlled by the sampling period, it tends to be small because the method is based on sufficient conditions. Hence the trajectory is close to the sliding solution. Though our paper uses the same switching mechanism as in [32], the stability analysis is fundamentally different.

A second-order boost converter model in [32] can be analysed by the standard Lyapunov approach by showing the decrease of the Lyapunov function along the trajectory. The derivative along the trajectory of the fourth-order zeta converter model becomes zero even if the point is not the operating point. Hence LaSalle’s invariance principle proved for this class of differential inclusion should be established. The preliminary and much shorter versions of our work were presented in the conference proceedings [1, 2].

The remainder of the paper is organized as follows. In Section 2, we establish a switching control mechanism for a two-mode system. The two-mode system is instrumental to model the DC–DC zeta converter operating in continuous conduction mode (CCM). In Section 3, we analyse the stability of the switching system in Section 2, and apply it to the zeta converter. In Section 4, the switching control mechanism discussed in Section 2 is modified to limit the switching frequency of the zeta converter. In addition, by using linear-line approximation of the trajectory, we show how to decide the switching frequency. Simulation results to show the effectiveness of our proposed method are presented in Section 5. Lastly, in Section 6, we conclude our work and state the plan for future work.

Notation: \( \mathbb{R} \) denote the set of real numbers. For \( \rho \in \mathbb{R}, \mathbb{R}_\rho, \mathbb{R}_{\leq \rho}, \mathbb{R}_{\geq \rho}, \mathbb{R}_{< \rho}, \mathbb{R}_{> \rho} \), and \( \mathbb{R}_{\leq \rho} \) denote the set of real numbers larger, smaller, larger than or equal to, and smaller than or equal to \( \rho \), respectively. The notation \( \text{conv} \) denotes the convex hull of a set. For a function: \( \mathbb{R}^n \rightarrow \mathbb{R} \), \( \alpha^{-1} \) denotes the inverse image of \( \alpha \). For a singleton \( \{c\}, c \in \mathbb{R} \), we use the simplified notation, \( \alpha^{-1}(c) = \alpha^{-1}(\{c\}) \). A set-valued map \( F \) is denoted as \( F: \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m} \) where for \( x \in \mathbb{R}^n, F(x) \subseteq \mathbb{R}^m \).

\[ \frac{dx}{dt} = A_1 x + B_1 u_1 \]  \quad (1) 
\[ \frac{dx}{dt} = A_2 x + B_2 u_1 \]  \quad (2)

where \( x(t) \in \mathbb{R}^n \) is the state and \( u(t) \in \mathbb{R} \) is the input. Fix \( u_0 \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \). Assume that 
\[ \lambda A_1 + (1 - \lambda) A_2 \]

is invertible. Define 
\[ x^* = -(\lambda A_1 + (1 - \lambda) A_2)^{-1} (\lambda B_1 + (1 - \lambda) B_2) u_0. \]  \quad (3)

Note that we do not assume the stability nor the non-singularity of the matrices \( A_1 \) and \( A_2 \). Let \( S_i (i = 1, 2) \) denote the set of stationary points of the systems (1) and (2); namely 
\[ S_1 = \{ x : A_1 x + B_1 u_0 = 0 \}, S_2 = \{ x : A_2 x + B_2 u_0 = 0 \}. \]  \quad (4)

If \( A_i \) is non-singular, \( S_i \) is a singleton; otherwise it may be empty or infinite. The following proposition is easy to derive, but useful in the subsequent discussions.

**Proposition 1.** The point \( x^* \) is given by (3) if and only if it satisfies the following equation:
\[ \lambda (A_1 x^* + B_1 u_0) = -(1 - \lambda) (A_2 x^* + B_2 u_0). \]  \quad (5)

Furthermore, \( A_j x^* + B_j u_0 \neq 0 \) \( (i = 1, 2) \) if and only if 
\[ S_1 \cap S_2 = \emptyset. \]

**Proof.** Since 
\[ (\lambda A_1 + (1 - \lambda) A_2) x^* = (\lambda B_1 + (1 - \lambda) B_2) u_0, \]

the equivalence of (3) and (5) is immediate. If \( x^* \in S_1 \cap S_2 \), then 
\[ \lambda (A_1 x^* + B_1 u_0) = -(1 - \lambda) (A_2 x^* + B_2 u_0) = 0, \]

which implies \( x^* \in S_1 \cap S_2 \). Conversely, if \( A_1 x^* + B_1 u_0 = 0 \), then \( A_2 x^* + B_2 u_0 = 0 \) by (5). Thus, \( x^* \in S_1 \cap S_2 \). \( \square \)

We are interested in a switching control law that drives the state of the switching system with the modes (1) and (2) to \( x^* \) under \( u(t) \equiv u_0 \). For this, define a candidate Lyapunov function 
\[ V(x) = (x - x^*)^T P (x - x^*), \]  \quad (6)

where \( P \) is a positive definite matrix. Because \( P > 0 \), there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[ c_1 \| x - x^* \|^2 \leq V(\alpha) \leq c_2 \| x - x^* \|^2 \]  
(7)

holds. The derivatives of \( V(\alpha) \) along the trajectories of (1) and (2) are

\[
\alpha_1(\alpha) := \frac{dV}{dx}(x_1 + B_1 u_0) \\
= (x - x^*)^T (P A_1 + A_1^T P) (x - x^*) \\
+ 2(A_1 x^* + B_1 u_0)^T P (x - x^*) ,
\]

(8)

\[
\alpha_2(\alpha) := \frac{dV}{dx}(x_2 + B_2 u_0) \\
= (x - x^*)^T (P A_2 + A_2^T P) (x - x^*) \\
+ 2(A_2 x^* + B_2 u_0)^T P (x - x^*) ,
\]

(9)

respectively.

**Proposition 2.** Suppose that \( P A_1 + A_1^T P \leq 0 \) and \( P A_2 + A_2^T P \leq 0 \). Then,

\[
\alpha_2^{-1}(\mathbb{R}_{\geq 0}) \subset \alpha_1^{-1}(\mathbb{R}_{\leq 0}), \alpha_1^{-1}(\mathbb{R}_{\geq 0}) \subset \alpha_2^{-1}(\mathbb{R}_{\leq 0}), \]

(10)

\[
\alpha_1^{-1}(0) \cap \alpha_2^{-1}(0) = \{ x : x - x^* \in \ker (B_1 + A_1^T P) \} \\
\cap \ker (B_2 + A_2^T P) \cap \ker (A_1 x^* + B_1 u_0)^T P \}.
\]

(11)

The proof is based on the following observation.

**Lemma 1.** Let \( Q_1 \leq 0 \) and \( Q_2 \leq 0 \) be \( n \times n \) symmetric matrices. Let \( r_1 \in \mathbb{R}^n \) and \( r_2 \in \mathbb{R}^n \) satisfy \( \lambda r_1 + (1 - \lambda) r_2 = 0 \) for some \( 0 < \lambda < 1 \). Define

\[
p_1(\alpha) := x^T Q_1 x + r_1^T x, \quad p_2(\alpha) := x^T Q_2 x + r_2^T x.
\]

Then

\[
p_1^{-1}(\mathbb{R}_{\leq 0}) \subset p_2^{-1}(\mathbb{R}_{\geq 0}), \quad p_2^{-1}(\mathbb{R}_{\leq 0}) \subset p_1^{-1}(\mathbb{R}_{\geq 0}),
\]

\[
p_1^{-1}(0) \cap p_2^{-1}(0) = \ker Q_1 \cap \ker Q_2 \cap \ker r_1^T.
\]

**Proof.** If \( p_1(\alpha) > 0 \), then

\[
0 < \lambda p_1(\alpha) = \lambda x^T Q_1 x + \lambda r_1^T x \\
\leq \lambda r_1^T x = -(1 - \lambda) r_2^T x \]

\[
\leq -(1 - \lambda) x^T Q_2 x = -(1 - \lambda) p_2(\alpha)
\]

holds. Thus, \( p_2(\alpha) < 0 \). Because \( p_1^{-1}(\mathbb{R}_{\geq 0}) = \mathbb{R}^n \setminus p_2^{-1}(\mathbb{R}_{\geq 0}) \) and \( p_2^{-1}(\mathbb{R}_{\leq 0}) = \mathbb{R}^n \setminus p_1^{-1}(\mathbb{R}_{\leq 0}) \), By interchanging \( p_1 \) and \( p_2 \), in the argument, it follows that \( p_1^{-1}(\mathbb{R}_{\geq 0}) \subset p_2^{-1}(\mathbb{R}_{\leq 0}) \). If \( p_1(\alpha) = p_2(\alpha) = 0 \), then

\[
0 = \lambda p_1(\alpha) = \lambda x^T Q_1 x + \lambda r_1^T x \\
\leq \lambda r_1^T x = -(1 - \lambda) r_2^T x \\
\leq -(1 - \lambda) x^T Q_2 x = -(1 - \lambda) p_2(\alpha) = 0.
\]

Hence all the inequalities hold as equalities. This implies \( r_1^T x = 0, x^T Q_1 x = 0 \), and \( x^T Q_2 x = 0 \), which means \( x \in \ker Q_1 \cap \ker Q_2 \cap \ker r_1^T \). Conversely, if \( x \in \ker Q_1 \cap \ker Q_2 \cap \ker r_1^T \), then \( x \in \ker r_2^T \) and \( p_1(\alpha) = p_2(\alpha) = 0 \).

**Proof of Proposition 2.** Define

\[
Q_1 := P A_1 + A_1^T P, \quad Q_2 := P A_2 + A_2^T P \\
p_1 := 2P(A_1 x^* + B_1 u_0), \quad p_2 := 2P(A_2 x^* + B_2 u_0), \\
p_1(\alpha) := \alpha_1(x + x^*), \quad p_2(\alpha) := \alpha_2(x + x^*).
\]

Notice that the assumptions of Lemma 1 are satisfied since (5) holds. Then the proof is immediate from Lemma 1.

Based on Proposition 2, we propose the following switching control mechanism.

**Switching Mechanism A**

- If the system is operating in mode 1 and reaches \( \alpha_1^{-1}(0) \), then it switches to mode 2.
- If the system is operating in mode 2 and reaches \( \alpha_2^{-1}(0) \), then it switches to mode 1.

**Remark 1.** Switching Mechanism A is initially proposed for a boost converter model in [32]. Proposition 2 clarifies a condition that ensures that Switching Mechanism A is well defined.

To analyse the stability of the switching control law, we consider the differential inclusion

\[
\frac{dx}{dt} \in F(\alpha),
\]

(12)

where

\[
M_1 = \{ x : \alpha_1^{-1}(\mathbb{R}_{\leq 0}) \cap \alpha_1^{-1}(\mathbb{R}_{\geq 0}) = \alpha_1^{-1}(\mathbb{R}_{\leq 0}) \},
\]

\[
M_2 = \{ x : \alpha_1^{-1}(\mathbb{R}_{\leq 0}) \cap \alpha_2^{-1}(\mathbb{R}_{\leq 0}) = \alpha_2^{-1}(\mathbb{R}_{\leq 0}) \},
\]

\[
M_3 = \{ x : \alpha_1^{-1}(\mathbb{R}_{\leq 0}) \cap \alpha_2^{-1}(\mathbb{R}_{\leq 0}) = \alpha_2^{-1}(\mathbb{R}_{\leq 0}) \}.
\]

The set-valued map \( F : \mathbb{R}^n \to \mathbb{R}^n \) is upper semi-continuous, and its values are bounded closed convex sets. Solutions of (12)
include solutions of (1), (2) with the switching control mechanism. We shall analyse the stability of the operating point \( x^* \) of the differential inclusion (12).

Remark 2. The switching mechanism that selects the mode defined by

\[
\text{arg min}\{\alpha_i(x) : i = 1, 2\}
\]

is in line with the method used in [29], except that [29] assumes that

\[
P(\lambda A_1 + (1 - \lambda) A_2) + (\lambda A_1 + (1 - \lambda) A_2)^T P < 0
\]

for some \( \lambda \in (0, 1) \). This mechanism does not stabilize the operating point asymptotically when we only assume \( PA_1 + A_1^T P \leq 0 \) and \( PA_2 + A_2^T P \leq 0 \). The following simple example

\[
A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

results in the differential inclusion

\[
\frac{dx}{dt} \in F(x),
\]

\[
F(x) = \begin{cases} 
1 & \text{if } [0 \ 1] x \neq 0, \\
1,2 & \text{if } [0 \ 1] x = 0.
\end{cases}
\]

If \( [0 \ 1] x_0 = 0 \), then the differential inclusion has the unique solution \( x(t) \equiv x_0 \), which shows that it is not asymptotically stable. Note that this is not a counterexample of [29, Theorem 2]. Nevertheless, the example shows that the distinction of positive definiteness and positive semi-definiteness is meaningful.

3 | STABILITY OF SWITCHING SYSTEM

In this section, we analyse the stability of the switching mechanism proposed in the previous section and apply the method to a DC–DC zeta converter operating in the CCM.

3.1 | Stability analysis

Consider the differential inclusion (12) and the function \( V(x) \) in (6). Define \( \dot{V}(x) : \mathbb{R}^n \to \mathbb{R} \) by

\[
\dot{V}(x) = \frac{\partial V}{\partial x} F(x) = \left\{ \frac{\partial V}{\partial x} \omega : \omega \in F(x) \right\}.
\]

It is easy to verify that

\[
\dot{V}(x) = \begin{cases} 
\{\alpha_1(x)\} & \text{if } x \in M_1, \\
\{\alpha_2(x)\} & \text{if } x \in M_2, \\
\text{conv}\{\alpha_1(x), \alpha_2(x)\} & \text{if } x \in M_0.
\end{cases}
\]

The inverse image \( \dot{V}^{-1}(S) \), where \( S \subset \mathbb{R} \), is defined by

\[
\dot{V}^{-1}(S) = \left\{ y \in \mathbb{R}^n : \dot{V}(y) \cap S \neq \emptyset \right\}.
\]

When \( S = \{a\} \), we write \( \dot{V}^{-1}(a) := \dot{V}^{-1}\{a\} \).

Proposition 3. Consider the differential inclusion (12) and the Lyapunov function (6). Then,

\[
\dot{V}^{-1}(0) = \alpha_1^{-1}(0) \cup \alpha_2^{-1}(0).
\]

Proof. If \( x \in \alpha_1^{-1}(0) \), then \( \dot{V}(x) = \text{conv}\{0, \alpha_2(x)\} \not\supset 0 \). Similarly, we have \( 0 \in \dot{V}(x) \) if \( x \in \alpha_2^{-1}(0) \). Hence \( \dot{V}^{-1}(0) \supseteq \alpha_1^{-1}(0) \cup \alpha_2^{-1}(0) \). Conversely, if \( 0 \in \dot{V}(x) \), then \( x \in \alpha_1^{-1}(\mathbb{R}_{\leq 0}) \cap \alpha_2^{-1}(\mathbb{R}_{\leq 0}) \) and \( 0 \in \text{conv}\{\alpha_1(x), \alpha_2(x)\} \). Because \( \alpha_1(x) \leq 0 \) and \( \alpha_2(x) \leq 0 \), this implies either \( \alpha_1(x) = 0 \) or \( \alpha_2(x) = 0 \).

Proposition 4. Let \( x^*, S_1, \) and \( S_2 \) be defined by (3) and (4). Then \( \{x^*\} \cup S_1 \subset \alpha_1^{-1}(0) \) holds.

Proof. We have already shown that \( x^* \in \alpha_1^{-1}(0) \cap \alpha_2^{-1}(0) \) in Proposition 2. If \( x \in S_1 \), then \( \alpha_1(x) = 0 \) by (8). Similarly, if \( x \in S_2 \), then \( \alpha_2(x) = 0 \).

Proposition 5. Let \( x^* \in \{x^*\} \cup S_1 \cup S_2 \). Then, the differential inclusion (12) has a stationary solution \( \phi(t, x^*) \equiv x^* \).

Proof. If \( x^* \in S_1 \), then \( \alpha_1(x^*) = 0 \) by Proposition 4. Thus \( F(x^*) = \text{conv}\{0, A_2 x^* + B_2 \eta_0\} \not\supset 0 \). Similarly, \( x^* \in S_2 \) implies \( 0 \in F(x^*) \). By Proposition 4, \( \alpha_1(x^*) = \alpha_2(x^*) = 0 \). Consequently, \( F(x^*) = \text{conv}\{A_1 x^* + B_1 \eta_0, A_2 x^* + B_2 \eta_0\} \not\supset \lambda(A_1 x^* + B_1 \eta_0) + (1 - \lambda)(A_2 x^* + B_2 \eta_0) = 0 \).

By Proposition 5, the operation point \( x^* \) is not globally asymptotically stable if \( S_1 \cup S_2 \neq \emptyset \). We shall study the local asymptotic stability of \( x^* \). The next result shows that \( x^* \) is stable in this sense.

Theorem 1. Suppose that \( PA_1 + A_1^T P \leq 0 \) and \( PA_2 + A_2^T P \leq 0 \) hold. If \( \alpha_1^{-1}(0) \cap \alpha_2^{-1}(0) \) contains no solution of (12) except for \( x(t) \equiv x^* \), then \( x^* \) is locally asymptotically stable.

The proof of Theorem 1 hinges on a couple of Lemmas. The first one states that the operating point \( x^* \) is stable.

Lemma 2. Suppose that \( PA_1 + A_1^T P \leq 0 \) and \( PA_2 + A_2^T P \leq 0 \) hold. Then \( x^* \) is stable.

Proof. Note that the set-valued function \( F(x) \) in (12) is defined for all \( x \in \mathbb{R}^n \) from (10) in Proposition 2. Let \( \varepsilon > 0 \), and choose \( \delta = \frac{\varepsilon_1}{2} > 0 \). If \( \|x_0 - x^*\| < \delta \), it follows from (7) that \( V(x_0) \leq c_2 \delta = \varepsilon_1 \). Along the trajectory \( \phi(t, x_0) \) of
It follows that $\dot{\alpha}_1(x)$ is a continuous function; moreover,

$$\dot{\alpha}_1(x) = (A_1 x + B_1 u_0)' P (A_1 x + B_1 u_0) > 0,$$

and from the continuity of $\dot{\alpha}_1(x)$, $\dot{\alpha}_1(x) > 0$ in some neighbourhood of $x^*$, say $N = \{x : V(x) < r\}$ for some $r > 0$. Let $\omega \in (\alpha_1^{-1}(0) \setminus \alpha_2^{-1}(0)) \cap N$. Since $\alpha_1(\omega) = 0$, $\alpha_2(\omega) < 0$ by (10). We can take $\tau > 0$ small enough, so $\alpha_2(\phi(t, \omega)) < 0$ for all $t \in [0, \tau]$. If $V(\phi(t, \omega)) = V(\omega)$ for all $t \in [0, \tau]$, then

$$\frac{d}{dt} V(\phi(t, \omega)) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = 0,$$

for almost all $t$. Let $T = \{t : \alpha_1(\phi(t, \omega)) > 0\}$. Note that $T$ is an open set. If $t \in T$, then by (12) $F(\phi(t, \omega)) = \{A_2 \phi(t, \omega) + B_2 u_0\}$, and hence $\frac{\partial V}{\partial x} \frac{dx}{dt} = \alpha_2(\phi(t, \omega)) < 0$. Hence $T = \emptyset$. Consequently, $F(\phi(t, \omega)) = \{A_1 \phi(t, \omega) + B_1 u_0, A_1 \phi(t, \omega) + B_2 u_0\}$, and $\frac{\partial V}{\partial x} \frac{dx}{dt} = \alpha_2(\phi(t, \omega)) < 0$ implies $\frac{d}{dt} \phi(t, \omega) = A_1 \phi(t, \omega) + B_1 u_0$ for almost all $t$. Hence $\phi(t, \omega)$ is the solution of the differential equation

$$\frac{dx}{dt} = A_1 x + B_1 u_0, \quad x(0) = \omega, \quad 0 \leq t \leq \tau,$$

and $V(\phi(t, \omega))$ is twice continuously differentiable, and

$$\frac{d^2}{dt^2} V(\phi(t, \omega)) = \dot{\alpha}_1(\phi(t, \omega)) > 0, \quad t \in [0, \tau].$$

This implies that $\alpha(\phi(t, \omega)) > \alpha(\phi(0, \omega)) = 0$ for some $t$. But $T = \emptyset$, and this is not possible. Hence, $V(\phi(t, \omega)) \leq V(\phi(\tau, \omega)) < V(\omega)$ for some $t \in [0, \tau]$. The proof for $\omega \in (\alpha_2^{-1}(0) \setminus \alpha_1^{-1}(0)) \cap N$ is similar. \hfill $\Box$

**Proof of Theorem 1.** Since $\phi(t, x_0)$ is bounded, its limit set $\Omega$ is an invariant set by Lemma 3. Since $V(x)$ is bounded from below and $V(\phi(t, x_0))$ is monotonically non-increasing for every sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, $c := \lim V(\phi(t_k, x_0))$ exists. If $\omega \in \Omega$, then there exists a sequence $\{t_k\}$ such that $\omega = \lim \phi(t_k, x_0)$. This means $V(\omega) = V(\lim \phi(t_k, x_0)) = \lim V(\phi(t_k, x_0)) = c$. Because $\Omega$ is an invariant set, $0 \in V(\omega)$ for any $\omega \in \Omega$. From Proposition 3, $\omega \in \alpha_1^{-1}(0) \cup \alpha_2^{-1}(0)$. Take $r > 0$ and $N = \{x : V(x) < r\}$ as in Lemma 4. If $\omega \in \{\alpha_2^{-1}(0) \setminus \alpha_1^{-1}(0) \cup \alpha_2^{-1}(0) \setminus \alpha_1^{-1}(0)\} \cap N$, then $\omega$ is not a limit point by Lemma 4. Thus, $\omega \in \alpha_2^{-1}(0) \cup \alpha_2^{-1}(0)$. Hence, if $V(x_0) < r$, then $\phi(t, x_0)$ does not have a limit point except $x^*$. \hfill $\Box$

**Remark 3.** Theorem 1 is a consequence of LaSalle’s invariance principle proved for the differential inclusion (12). This is a useful tool to prove the stability of the switching control applied to a DC–DC zeta converter in Section 3.2.
Consider the DC–DC zeta converter circuit shown in Figure 1. The circuit consists of two inductors \(L_1\) and \(L_2\), two capacitors \(C_1\) and \(C_2\), an ideal diode \(d\) and \(v_g\), a DC voltage source, a resistive load \(R\), and an ideal switch \(S\). Denote the currents of \(i_{L1}\) and \(i_{L2}\), the voltages of \(v_{C1}\) and \(v_{C2}\) as \(i_{L1}\) and \(v_{C1}\), respectively.

The converter is in CCM if the diode \(d\) is open when the switch \(S\) is on and it is shorted when the switch is off. When the switch is closed (mode 1), the converter is equivalent to the circuit shown in Figure 2, and when the switch is open (mode 2), the converter is equivalent to the circuit shown in Figure 3. With the state vector \(x = [i_{L1} \ i_{L2} \ v_{C1} \ v_{C2}]^T\) and the input \(u = v_g\), the matrices for the two modes are given by

\[
A_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{L_2} & -\frac{1}{L_2} \\
0 & -\frac{1}{C_1} & 0 & 0 \\
0 & \frac{1}{C_2} & 0 & -\frac{1}{RC_2}
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
\frac{1}{L_1} \\
0 \\
-\frac{1}{L_2} \\
0
\end{bmatrix},
\]

If \(m_0 > 0\), then mode 1 has no stationary solution, and mode 2 has a unique stationary solution \(x^* = [0 \ 0 \ 0 \ 0]^T\).

For \(\lambda \in (0, 1)\),

\[
x^* = -(\lambda A_1 + (1-\lambda) A_2)^{-1}(\lambda B_1 + (1-\lambda) B_2) u_0
\]

\[
= \begin{bmatrix}
0 & 0 & \frac{1-\lambda}{L_1} & 0 \\
0 & 0 & \lambda & -\frac{1}{L_2} \\
-\frac{1-\lambda}{C_1} & -\frac{\lambda}{C_2} & 0 & 0 \\
0 & \frac{1}{C_2} & 0 & -\frac{1}{RC_2}
\end{bmatrix} \begin{bmatrix}
\frac{1}{L_1} \\
\frac{1}{L_2} \\
\alpha \\
\beta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
v_1 \\
v_2 \\
v_r \\
v_r
\end{bmatrix} = \begin{bmatrix}
v_{C1}^{\star} \\
v_{C2}^{\star}
\end{bmatrix},
\]

where \(v_1 := u_0\) and \(v_r := \frac{\lambda u_0}{1-\lambda}\). Based on the energy stored in the zeta converter, define

\[
P := \begin{bmatrix}
\frac{L_1}{2} & 0 & 0 & 0 \\
0 & \frac{L_2}{2} & 0 & 0 \\
0 & 0 & \frac{C_1}{2} & 0 \\
0 & 0 & 0 & \frac{C_2}{2}
\end{bmatrix},
\]

Then

\[
P A_1 + A_1^T P = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{R}
\end{bmatrix} \leq 0,
\]
$PA_2 + A_2^T P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R} \end{bmatrix} \leq 0,$

$\langle A_1 x^* + B_1 v_0 \rangle^T P = \begin{bmatrix} \frac{v_1}{2} & \frac{v_2}{2} & -\frac{v}{2R} & 0 \end{bmatrix},$

$\langle A_2 x^* + B_2 v_0 \rangle^T P = \begin{bmatrix} -\frac{v_1}{2} & -\frac{v_2}{2} & -\frac{v^2}{2R} & 0 \end{bmatrix}.$ (19)

From (19), $\ker (PA_1 + A_1^T P) \cap \ker (PA_2 + A_2^T P) \cap \ker \langle A_1 x^* + B_1 v_0 \rangle^T = \text{span} \{d_1, d_2\}$ where

$d_1 = \begin{bmatrix} \frac{v_1}{R} \\ 0 \\ \frac{v_2}{R} \\ 0 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 0 \\ \frac{v_1}{R} \\ \frac{v_2}{R} \\ 0 \end{bmatrix}. \quad \text{(20)}$

The function $V(x)$ decreases along the trajectory as long as $x \in \alpha^{-1}_1(0) \cap \alpha^{-1}_2(0)$. It remains to see what happens when the trajectory reaches $\alpha^{-1}_1(0) \cap \alpha^{-1}_2(0)$.

**Lemma 5.** Let $x \in \alpha^{-1}_1(0) \cap \alpha^{-1}_2(0)$. Then the following properties hold:

a. $0 \in F(x)$ if and only if $x = x^*$.

b. If $x \neq x^* \notin \text{span} \{d_1\}$, then $F(x) \cap (PA_1 + A_1^T P) \cap \ker (PA_2 + A_2^T P) \neq \emptyset$.

c. If $x \neq x^* \notin \text{span} \{d_1\}$, and $x \neq x^*$, then $F(x) \cap \text{span} \{d_1\} = \emptyset$.

**Proof.** It follows from (17) that

$A_1 x^* + B_1 v_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{v_1}{L_2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{v_2}{L_2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{v_1}{L_1} \\ \frac{v_2}{L_2} \\ \frac{v_1}{L_2} \\ \frac{v_2}{L_2} \end{bmatrix}, \quad \text{(21)}$

$A_2 x^* + B_2 v_2 = \begin{bmatrix} -\frac{v_1}{L_2} \\ -\frac{v_2}{L_2} \\ \frac{v_1}{L_2} \\ \frac{v_2}{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{v_1}{L_1} \\ -\frac{v_2}{L_2} \\ \frac{v_1}{L_2} \\ \frac{v_2}{L_2} \end{bmatrix}. \quad \text{(22)}$

Note that $x \in \alpha^{-1}_1(0) \cap \alpha^{-1}_2(0)$ if and only if $x = x^* + \Delta x$ with

$\Delta x = \delta_1 d_1 + \delta_2 d_2 = \begin{bmatrix} \delta_1 \frac{v_1}{L_1} \\ \delta_2 \frac{v_2}{L_2} \\ \delta_1 \frac{v_1}{L_2} \\ \delta_2 \frac{v_2}{L_2} \end{bmatrix}. \quad \text{(23)}$

From this, it follows that

$A_1 \Delta x = \begin{bmatrix} 0 \\ \delta_1 \frac{v_1}{L_2} \\ \delta_2 \frac{v_1}{L_2} \\ \delta_2 \frac{v_2}{L_2} \end{bmatrix}, \quad A_2 \Delta x = \begin{bmatrix} -\delta_2 \frac{v_2}{L_2} \\ \delta_1 \frac{v_1}{L_2} \\ \delta_2 \frac{v_1}{L_2} \\ \delta_2 \frac{v_2}{L_2} \end{bmatrix}. \quad \text{(24)}$

Hence if $x \in \alpha^{-1}_1(0) \cap \alpha^{-1}_2(0)$, then

$A_1 x + B_1 v_2 = A_1 x^* + B_1 v_2 + A_1 \Delta x$

$= \begin{bmatrix} 0 \\ \frac{v_1}{L_2} \\ \frac{v_2}{L_2} \\ \frac{v_1}{L_2} \\ \frac{v_2}{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_1 \frac{v_1}{L_2} \\ \delta_2 \frac{v_1}{L_2} \\ \delta_2 \frac{v_2}{L_2} \end{bmatrix}$

$A_2 x + B_2 v_2 = A_2 x^* + B_2 v_2 + A_2 \Delta x$

$= \begin{bmatrix} -\frac{v_1}{L_2} \\ -\frac{v_2}{L_2} \\ \frac{v_1}{L_2} \\ \frac{v_2}{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ -\delta_2 \frac{v_2}{L_2} \\ \delta_1 \frac{v_1}{L_2} \\ \delta_2 \frac{v_2}{L_2} \end{bmatrix}. \quad \text{(25)}$

Hence if $\omega \in F(x)$ for $x \in \alpha^{-1}_1(0) \cap \alpha^{-1}_2(0) \setminus \text{span} \{d_1\}$, then

$[0 \ 0 \ 0 \ 1] \omega = \delta_2 \frac{v_2}{L_2} \neq 0,$

which shows $F(x) \cap \ker (PA_1 + A_1^T P) \cap \ker (PA_2 + A_2^T P) \neq \emptyset$ and $0 \notin \text{conv} \{A_1 x + B_1 v_2, A_2 x + B_2 v_2\}$. Suppose $x \neq x^* \in \text{span} \{d_1\}$, then

$\text{rank} [ (A_1 x + B_1 v_2) \ (A_2 x + B_2 v_2) ]$
Finally, note that $\nabla \in \mathbb{R}^n$ is independent, and hence $0 = \nabla$. Because $x = x^* \in \mathbb{R}^n$, $A_1 x + B_1 v_g$ and $A_2 x + B_2 v_g$ are linearly independent, and hence $0 \notin F(x)$. If $\delta_1 = -\frac{\nu + \nu_g}{\nu_1}$, then

$$x = x^* \quad \Delta x = \begin{bmatrix} \frac{\nu}{L_1} & -\frac{\nu}{L_2} & -\frac{\nu}{L_4} & -\frac{\nu}{L_1} \\ \frac{\nu}{L_2} & -\frac{\nu}{L_2} & -\frac{\nu}{L_2} & -\frac{\nu}{L_4} \\ -\frac{\nu}{C_R} & -\frac{\nu}{C_R} & -\frac{\nu}{C_R} & -\frac{\nu}{C_R} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nu \\ -\nu_1 \\ -\nu_2 \\ -\nu_4 \end{bmatrix}.$$ 

$$A_1 x + B_1 v_g = \begin{bmatrix} \frac{\nu}{L_1} \\ \frac{\nu}{L_2} \\ -\frac{\nu}{C_R} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\nu + \nu_g}{\nu_1} \\ \frac{\nu + \nu_g}{\nu_1} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\nu}{L_1} \\ \frac{\nu}{L_2} \\ \frac{\nu}{C_R} \\ 0 \end{bmatrix},$$

$$A_2 x + B_2 v_g = \begin{bmatrix} \frac{\nu}{L_1} \\ \frac{\nu}{L_2} \\ -\frac{\nu}{C_R} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\nu + \nu_g}{\nu_1} \\ \frac{\nu + \nu_g}{\nu_1} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\nu}{L_1} \\ \frac{\nu}{L_2} \\ \frac{\nu}{C_R} \\ 0 \end{bmatrix}.$$ 

Because $A_1 x + B_1 v_g = A_2 x + B_2 v_g$, we have $0 \notin F(x)$. Finally, note that $\nabla x^T P (A_1 x + B_1 v_g)$

Let $x - x^* \in \text{span}\{d_1\}$ and $x \notin x^*$. Then for every $\omega \in F(x)$, $d_1^T P \omega = 0$. Since $0 \notin F(x)$, it follows that $F(x) \cap \text{span}\{d_1\} = \emptyset$.

**Theorem 2.** Consider the differential inclusion (12) defined by the system matrices (16) and the operating point $x^*$ in (17). Then, the operating point $x^*$ is locally asymptotically stable.

Proof. Assume that $\phi(t, x_0)$ is a solution of the differential inclusion satisfying $\phi(t, x_0) \in \alpha^{-1}(0) \cap \alpha^{-1}(0)$ for $t \geq 0$ and $x_0 \notin x^*$. If $x_0 - x^* \notin \text{span}\{d_1\}$, then $\frac{d}{dt} \phi(t, x_0) = \text{ker}(PA_1 + A_1^T P) \cap \text{ker}(PA_2 + A_2^T P)$ by Lemma 5, but this contradicts the assumption that $\phi(t, x_0) \in \alpha^{-1}(0) \cap \alpha^{-1}(0)$ for $t \geq 0$. If $x_0 - x^* \notin \text{span}\{d_1\}$, then from Lemma 5, there exists $t_1$ such that $x_1 := \phi(t_1, x_0)$ satisfies $x_1 - x^* \notin \text{span}\{d_1\}$. Then the trajectory $\phi(t, x_1)$ cannot stay in $\alpha^{-1}(0) \cap \alpha^{-1}(0)$ just as we have proved before. This completes the proof.

**Remark 4.** The stability of switching control of a boost converter is proved in [32]. The state space of the boost converter model is two dimensional, and the set $\alpha^{-1}(0) \cap \alpha^{-1}(0)$ is a singleton consisting of $x^*$. The state dimension of the zeta converter model is four, and the set $\alpha^{-1}(0) \cap \alpha^{-1}(0)$ includes the two-dimensional affine set spanned by $d_1$ and $d_2$ in (20). The stability of the operating point is a consequence of Theorem 1, which is a differential-inclusion version of LaSalle’s invariance principle.

**Remark 5.** The switching mechanisms proposed in [29, 30, 31] basically pick up the mode which nearly decreases the Lyapunov function most while our method retains the mode as long as it decreases the Lyapunov function. The trajectory of [29] evolves as a sliding mode solution when it approaches the sliding boundary, which means that the switching interval becomes
infinitesimally small. The sampled-data control approach in [30] and [31] can reduce the switching frequency by adjusting
the sampling period. However, the period depends on the feasibility of matrix inequality, which is a sufficient condition and hence incurs conservativeness. Our method approaches a sliding mode solution only when the trajectory is near the operating point. Further reduction of switching frequency
to the predetermined level is possible by using the modified switching mechanism stated in the next section.

4 LIMITING THE SWITCHING FREQUENCY

The switching control proposed in Section 2 requires unbounded number of switching as a solution approaches the operating point \( x^* \). In this section, the switching control mechanism discussed in Section 2 is modified to limit the switching frequency of a zeta converter.

4.1 Modified switching mechanism

The switching mechanism considered in Section 2 is based on the signs of the derivatives along the trajectories (8) and (9). Let \( \rho_1, \rho_2 > 0 \) and \( \bar{\alpha}_1 \) and \( \bar{\alpha}_2 \) be modified switching functions which satisfy the following:

\[
\bar{\alpha}_1 (x) \geq \alpha_1 (x), \bar{\alpha}_2 (x) \geq \alpha_2 (x),
\]

\[
\alpha_2^{-1} (\mathbb{R}_{\geq 0}) \subset \bar{\alpha}_1^{-1} (\mathbb{R}_{\leq \rho_1}), \alpha_1^{-1} (\mathbb{R}_{\geq 0}) \subset \bar{\alpha}_2^{-1} (\mathbb{R}_{\leq \rho_2}).
\]  

(26)

Notice that (27) is equivalent to

\[
\bar{\alpha}_1^{-1} (\mathbb{R}_{\geq \rho_1}) \subset \alpha_2^{-1} (\mathbb{R}_{\geq 0}), \bar{\alpha}_2^{-1} (\mathbb{R}_{\geq \rho_2}) \subset \alpha_1^{-1} (\mathbb{R}_{\leq 0}).
\]  

(27)

Based on (26) and (27), we propose the following modified switching control mechanism:

**Switching Mechanism B**

- If the system is operating at mode 1 and reaches \( \bar{\alpha}_1^{-1} (\rho_1) \), then it switches to mode 2.
- If the system is operating at mode 2 and reaches \( \bar{\alpha}_2^{-1} (\rho_2) \), then it switches to mode 1.

The differential inclusion (12) is modified accordingly.

\[
\frac{dx}{dt} \in \bar{F} (x),
\]

\[
\bar{F} (x) := \begin{cases} 
\{ A_1 x + B_1 u_0 \} & \text{if } x \in \bar{M}_1, \\
\{ A_2 x + B_2 u_0 \} & \text{if } x \in \bar{M}_2, \\
\text{conv} \{ A_1 x + B_1 u_0, A_2 x + B_2 u_0 \} & \text{if } x \in \bar{M}_0,
\end{cases}
\]

where

\[
\bar{M}_1 = \{ x : \alpha_1^{-1} (\mathbb{R}_{< \rho_1}) \cap \bar{\alpha}_2^{-1} (\mathbb{R}_{\geq \rho_2}) = \bar{\alpha}_2^{-1} (\mathbb{R}_{\geq \rho_2}) \},
\]

\[
\bar{M}_2 = \{ x : \bar{\alpha}_1^{-1} (\mathbb{R}_{\geq \rho_1}) \cap \alpha_2^{-1} (\mathbb{R}_{< \rho_1}) = \alpha_2^{-1} (\mathbb{R}_{\geq \rho_1}) \},
\]

\[
\bar{M}_0 = \{ x : \bar{\alpha}_1^{-1} (\mathbb{R}_{\geq \rho_1}) \cap \bar{\alpha}_2^{-1} (\mathbb{R}_{\leq \rho_2}) \}.
\]

**Assumption 1.** The sets \( \bar{\alpha}_1^{-1} (\mathbb{R}_{\geq \rho_1}) \cap \alpha_1^{-1} (\mathbb{R}_{0}) \) and \( \bar{\alpha}_2^{-1} (\mathbb{R}_{\geq \rho_1}) \cap \alpha_2^{-1} (\mathbb{R}_{0}) \) are bounded.

**Proposition 6.** Suppose Assumption 1 holds. Let \( \epsilon > 0 \) satisfy

\[
\epsilon > \sup \{ V (x) : x \in (\bar{\alpha}_1^{-1} (\mathbb{R}_{\geq \rho_1}) \cap \alpha_1^{-1} (\mathbb{R}_{0})) \cup (\bar{\alpha}_2^{-1} (\mathbb{R}_{\geq \rho_1}) \cap \alpha_2^{-1} (\mathbb{R}_{0})) \}.
\]

Then for any solution \( \bar{\phi} (t, x_0) \) of (28), there exists \( T > 0 \) such that \( V (t, x_0) \in \{ x : V (x) < \epsilon \} \) for \( T > t \).

**Proof.** First, we shall prove that \( \bar{F} (x) \subset F (x) \) if \( x \notin \Xi_\rho := (\bar{\alpha}_1^{-1} (\mathbb{R}_{\leq \rho_1}) \cap \alpha_1^{-1} (\mathbb{R}_{0})) \cup (\bar{\alpha}_2^{-1} (\mathbb{R}_{\leq \rho_2}) \cap \alpha_2^{-1} (\mathbb{R}_{0}) \). From (27), \( \bar{\alpha}_1^{-1} (\mathbb{R}_{\geq \rho_1}) \subset \alpha_1^{-1} (\mathbb{R}_{0}) \). So, if \( \bar{\alpha}_2 (x) > \rho_2 \), then

\[
\bar{F} (x) = \{ A_1 x + B_1 u_0 \} = F (x), \alpha_2 (x) > 0,
\]

\[
\bar{F} (x) = \{ A_2 x + B_2 u_0 \} = \text{conv} \{ A_1 x + B_1 u_0, A_2 x + B_2 u_0 \} = F (x), \alpha_2 (x) \leq 0.
\]

From Assumption 1, the number \( \epsilon > 0 \) exists. If \( V (x_0) > \epsilon \), then a solution \( \bar{\phi} (t, x_0) \) of (28) satisfies \( \bar{\phi} (t, x_0) = \bar{\phi} (t, x_0) \) as long as \( \bar{\phi} (t, x_0) \notin \Xi_\rho \). This implies that \( \bar{\phi} (t, x_0) = \bar{\phi} (t, x_0) \) and \( V (\bar{\phi} (t, x_0)) \geq \epsilon \) for \( 0 \leq t \leq T \). Furthermore, \( V (\bar{\phi} (t, x_0)) \) is non-increasing when \( \bar{\phi} (t, x_0) \notin \Xi_\rho \). Therefore, \( V (\bar{\phi} (t, x_0)) < \epsilon \) for \( T > t \).

4.2 Example for the DC–DC zeta converter

From (19),

\[
\alpha_1 (x) = (x - x^*)^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/R \end{bmatrix} (x - x^*) + \frac{v_F}{R} (i_{L1} - i^*_{L1})
\]

\[
+ \frac{v_F}{R} (i_{L2} - i^*_{L2}) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/R \end{bmatrix} (v_{C1} - v^*_{C1}),
\]

\[
\alpha_2 (x) = (x - x^*)^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/R \end{bmatrix} (x - x^*) - \frac{v_F}{R} (i_{L1} - i^*_{L1})
\]

\[
+ \frac{v_F}{R} (i_{L2} - i^*_{L2}) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/R \end{bmatrix} (v_{C1} - v^*_{C1}).
\]
Define

\[ d_3 := \begin{bmatrix} \frac{v_{RC}}{I_1} \\ \frac{v_{RC}}{I_2} \\ -v_r \\ 0 \end{bmatrix}, \quad d_4 := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{31} \]

Then \( \{d_1, d_2, d_3, d_4\} \) with \( d_1 \) and \( d_2 \) in (20) is a basis of \( \mathbb{R}^4 \), and thus any \( x \in \mathbb{R}^4 \) can be written as

\[ x - x^* = \Delta x = \delta_1 d_1 + \delta_2 d_2 + \delta_3 d_3 + \delta_4 d_4 . \tag{32} \]

The modified functions \( \tilde{\alpha}_1(x) \) and \( \tilde{\alpha}_2(x) \) can be defined as

\[ \tilde{\alpha}_1(x) := \alpha_1(x) + k_1 \delta^2_1 + \beta (\epsilon_1 \delta_1, \delta_2) , \tag{33} \]

\[ \tilde{\alpha}_2(x) := \alpha_2(x) + k_2 \delta^2_1 + \beta (\epsilon_2 \delta_1, \delta_2) , \tag{34} \]

where \( \| \delta_1, \delta_2 \| \) is any norm in \( \mathbb{R}^2 \), and \( \beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is a monotone non-decreasing function satisfying

\[ \beta (0) = 0, \quad \beta (\varepsilon) = \rho, \text{ if } \varepsilon \geq \rho , \]

\[ 0 < k_1 \left( \frac{\rho^2}{R} , \frac{\rho^2}{R} , c_1 \right) c_2 , \quad 0 , \quad \rho , \quad 0 \).

**Proposition 7.** The functions \( \tilde{x}_1 \) and \( \tilde{x}_2 \) defined by (33) and (34) satisfy (26), (27), and Assumption 1.

**Proof.** It is obvious that (26) holds. Suppose \( \alpha_2(x) \geq 0 \). Define \( \rho_1(x - x^*) := \alpha_1(x) + k_1 \delta^2_1 \) and \( \rho_2(x - x^*) := \alpha_2(x) + k_2 \delta^2_1 \). Then the quadratic terms of \( \rho_1 \) and \( \rho_2 \) are non-positive, and hence by Lemma 1, we assert that \( \alpha_1(x) + k_1 \delta^2_1 \geq 0 \). Because \( \beta (\| \delta_1, \delta_2 \|) \leq \rho_1 \), we obtain \( \tilde{\alpha}_1(x) \leq \rho_1 \). Similarly, \( \alpha_1(x) \geq 0 \) implies \( \tilde{\alpha}_2(x) \leq \rho_2 \). To show that \( \tilde{\alpha}_1^{-1} (\mathbb{R}_{\geq 0}) \cap \tilde{\alpha}_2^{-1} (\mathbb{R}_{\leq 0}) \) is bounded, we use the representation (30) and show that the set \( \{ (d_1, d_2, d_3, d_4) : x \in \tilde{\alpha}_1^{-1} (\mathbb{R}_{\geq 0}) \cap \tilde{\alpha}_2^{-1} (\mathbb{R}_{\leq 0}) \} \) is bounded. If \( \alpha_1(x) > 0 \) and \( \delta_1(x) \leq \rho_1 \), then

\[ \rho_1 > \tilde{\alpha}_1(x) - \alpha_1(x) = k_1 \delta^2_1 + \beta (\epsilon_1 \delta_1, \delta_2) \geq \left\{ \begin{array}{c} \beta (\epsilon_1 \delta_1, \delta_2) , \\ k_1 \delta^2_1 , \\ \end{array} \right\} . \tag{35} \]

From (35), it follows that \( \| \delta_1, \delta_2 \| \leq \rho_1 \) and \( \| \delta_4 \| \leq \sqrt{\frac{\rho_1}{k_1}} \).

Let

\[ M := \sup \left\{ \gamma (\delta_1, \delta_2, \delta_4) : \delta_1, \delta_2 < \frac{\rho_1}{c_1}, | \delta_4 | < \sqrt{\frac{\rho_1}{k_1}} \right\} , \]

\[ m := \inf \left\{ \gamma (\delta_1, \delta_2, \delta_4) : \delta_1, \delta_2 < \frac{\rho_1}{c_1}, | \delta_4 | < \sqrt{\frac{\rho_1}{k_1}} \right\} . \]

Then,

\[ 0 < \alpha_1 (x) = k \delta_3 + \gamma (\delta_1, \delta_2, \delta_4) \leq k \delta_3 + M , \]

\[ \alpha_1 > \tilde{\alpha}_1 (x) = k \delta_3 + \gamma (\delta_1, \delta_2, \delta_4) \geq k \delta_3 + m , \]

and it follows that \( -\frac{M}{k} < \delta_1 \leq \frac{\rho_1 - m}{k} \). The boundedness of the set \( \tilde{\alpha}_1^{-1} (\mathbb{R}_{\geq 0}) \cap \tilde{\alpha}_2^{-1} (\mathbb{R}_{\leq 0}) \) can be proved similarly.

**Remark 6.** From Proposition 6 and 7, we conclude that any solution of (28) converges to the set \( \{ \delta : V' (\delta) < \delta \} \). The spatial regularization was studied for the boost converter in [29] to reduce the high rate of switching. The method discussed in this section extends the idea to the zeta converter by adding extra terms in (33) and (34) to cope with the four-dimensional state space.

### 4.3 Estimating the switching frequency

Although the modified switching mechanism is able to limit the switching frequency, the value of the switching frequency itself, however, is controlled by the parameters in Switching Mechanism B. In this subsection, we will show how to decide such parameters based on a linear-line approximation of the trajectory.

From Section 4.2, the switching occurs when

\[ \rho_1 := \tilde{\alpha}_1 (x^* + \Delta x_1) , \]

\[ \rho_2 := \tilde{\alpha}_2 (x^* + \Delta x_2) , \]

where \( \Delta x_1 := [\Delta i_{L_{\alpha 1}} \Delta i_{L_{\alpha 2}} \Delta v_{RC_{1 \alpha}} 0]^T \) and \( \Delta x_2 := [\Delta i_{L_{\alpha 1}} \Delta i_{L_{\alpha 2}} \Delta v_{RC_{1 \alpha}} 0]^T \) are the difference of the approximated state-trajectory from the operating point at their respective switching instants as shown in Figure 4.

Observing Figure 4 and from (21) and (22), the gradient of the state-trajectory at the operating point is given by

\[ \frac{2\Delta x_1}{\Delta T_{sw}} = \begin{bmatrix} \frac{\rho_1}{L_1} \\ \frac{\rho_2}{L_2} \\ -\frac{v_r}{c_1 R} \\ \frac{v_r^2}{C_1 R} \end{bmatrix}, \quad \frac{2\Delta x_2}{\Delta T_{sw}} = \begin{bmatrix} -\frac{\rho_1}{L_1} \\ \frac{\rho_2}{L_2} \\ \frac{v_r}{c_1 R} \\ \frac{v_r^2}{C_1 R} \end{bmatrix}. \]
where \( T_{sw} = \frac{1}{f} \) is the period of the switching frequency \( f \). With

\[
\lambda = \frac{\nu_r}{\nu_r + \sigma_1}
\]

(from (17)) the above expressions can be rewritten as

\[
\Delta x_1 = \begin{bmatrix}
\frac{-\nu_r}{2fL_1(r_1+\nu_r)} \\
\frac{-\nu_r}{2fL_2(r_1+\nu_r)} \\
\frac{-\nu_r}{2fC_1R(r_1+\nu_r)}
\end{bmatrix}, \quad \Delta x_2 = \begin{bmatrix}
\frac{-\nu_r}{2fL_1(r_2+\nu_r)} \\
\frac{-\nu_r}{2fL_2(r_2+\nu_r)} \\
\frac{-\nu_r}{2fC_1R(r_2+\nu_r)}
\end{bmatrix}.
\]

Define penalty functions \( \sigma_1 := k_1\delta_1^2 + \beta(\sigma_1, \delta_1, \delta_2) \) and \( \sigma_2 := k_2\delta_2^2 + \beta(\sigma_1, \delta_1, \delta_2) \) and assume the state-trajectory near the operating point. Therefore, the penalty functions are close to 0 such that \( \sigma_1 \approx 0 \) and \( \sigma_2 \approx 0 \), consequently, \( \rho_1 \approx \alpha_1(x^r + \Delta x_1) \) and \( \rho_2 \approx \alpha_2(x^r + \Delta x_2) \). Nevertheless, the effect of \( \sigma_1 > 0 \) and \( \sigma_2 > 0 \) will be investigated and illustrated graphically later in Section 5. Therefore, with (17) and (36), and from (29) and (30), we have

\[
\rho_1 \approx \frac{\nu_r}{2fC_1L_1R_2^2} \left( v_r + \nu_r \right), \tag{37}
\]

\[
\rho_2 \approx \frac{\nu_r^2}{2fC_1L_1R_2^2} \left( v_r + \nu_r \right). \tag{38}
\]

From (37) and (38), we observe how the desired switching frequency \( f \) is related to the thresholds \( \rho_1 \) and \( \rho_2 \). Therefore, the DC–DC zeta converter will operate at the prescribed switching frequency under the modified switching rule. Though the expressions of \( \rho_1 \) and \( \rho_2 \) look complex, they are straightforwardly processed beforehand (offline). Nowadays, considering the capability of the high-speed processors like in the DSP, FPGA, or even (maybe) microcontroller, there should be no performance issue in executing the switching mechanism.

### Table 1: The DC–DC zeta converter parameters

| Parameter   | Value      |
|-------------|------------|
| \( v_r \)   | 18 V       |
| \( v_{C1} \) | 5 V        |
| \( R \)     | 2.5 \( \Omega \) |
| \( L_1 \)   | 100 \( \mu H \) |
| \( L_2 \)   | 100 \( \mu H \) |
| \( C_1 \)   | 100 \( \mu F \) |
| \( C_2 \)   | 220 \( \mu F \) |
| \( f \)     | 100 kHz    |

### 5. Simulation Results

The simulations are carried out using the circuit simulation software PSIM® with the parameters shown in Table 1. With the input voltage \( v_{io} \), the capacitor voltage \( v_{C2} \), and the load current \( i_2 \) are the variables that are sensed in the circuit in Figure 1. As can be seen, with the nominal \( v_r = 18 \) V and \( i_r = 2 \) A, no overshoot for the output voltage \( v_e \) is observed at the start-up, and the settling time is approximately 10 ms. At \( t = 20 \) ms, \( v_e \) drops to 9 V and \( i_r \) reduces to 1 A. Despite the large input voltage drops, the overshoot at the output voltage is considerably small with some oscillations can be seen before it settles down at approximately \( t = 30 \) ms. Afterwards, at \( t = 40 \) ms, the input voltage drops further to 3 V and \( i_r = 0.33 \) A. Similarly, although more oscillation and longer settling time are observed, nonetheless the output voltage is able to return to its operating point. Moreover, the converter is now operating in step-up mode (instead of step-down mode for the first two perturbations), thus proving the effectiveness of the switching mechanism.
control in regulating the output voltage at both operation modes. Finally, at $t = 80$ ms, the input voltage returns to its nominal value of 18 V. Although the increment is very significant (+500%), the switching control can regulate the output voltage well with minimum overshoot (approximately 10%) and considerably fast settling time (approximately 8 ms). On the other hand, the steady-state switching waveforms in close view for the three different input voltage perturbations are illustrated in Figure 6. As shown, the switching control algorithm is able to produce the desired switching frequency of approximately 100 kHz for all three instances.

In the next simulation, the effect of introducing the penalty functions $\sigma_1 > 0$ and $\sigma_2 > 0$, defined in Section 4, are shown in Figure 7. As can be observed, the introduction of $\sigma_1$ and $\sigma_2$ does not have much effect on the response of the output voltage. Increasing $\sigma_1$ and $\sigma_2$, however, increases the switching frequency as shown in Figure 8. These observations are expected: (37) and (38) are no longer valid, since $\sigma_1$ and $\sigma_2$ are not approximately zero. As $\sigma_1$ and $\sigma_2$ reach $\rho_1$ and $\rho_2$, respectively, the number of switching becomes unbounded, which is identical for the case of the switching control mechanism in Section 2.
state error. Although the approximate state waveforms are used to find $\rho_1$ and $\rho_2$, the close agreement between the theoretical and simulation results of the desired switching frequency shows that the approximation is indeed justified. In future, we plan to add the internal resistances in the zeta converter model, consider the effect of interference, and most importantly validate the findings with experimental results.

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