On the complexity of convex inertial proximal algorithms

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Abstract

The inertial proximal gradient algorithm is efficient for the composite optimization problem. Recently, the convergence of a special inertial proximal gradient algorithm under strong convexity has been also studied. In this paper, we present more novel convergence complexity results, especially on the convergence rates of the function values. The non-ergodic $O(1/k)$ rate is proved for inertial proximal gradient algorithm with constant stepsize when the objective function is coercive. When the objective function fails to promise coercivity, we prove the sublinear rate with diminishing inertial parameters. When the function satisfies some condition (which is much weaker than the strong convexity), the linear convergence is proved with much larger and general stepsize than previous literature. We also extend our results to the multi-block version and present the computational complexity. Both cyclic and stochastic index selection strategies are considered.

Keywords: Heavy-ball method, convex inertial forward-backward splitting, complexity, convergence rates, coordinate descent

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1 Introduction

In this paper, we study the following composite optimization problem

$$\min_x \{F(x) := f(x) + g(x)\}, \quad (1.1)$$

where $f$ is differentiable and $\nabla f$ is Lipschitz continuous with $L$, and $g$ is proximable. The inertial proximal algorithm for the problem (iPiano) [13] can be described as

$$x^{k+1} = \text{prox}_{\gamma_k g}[x^k - \gamma_k \nabla f(x^k) + \beta_k (x^k - x^{k-1})], \quad (1.2)$$

where $\gamma_k$ is the stepsize and $\beta_k$ is the inertial parameter. The iPiano is closely related to two classical algorithms: the forward-backward splitting method [6] (when $\beta_k \equiv 0$) and heavy-ball method (when $g \equiv 0$) [16]. The iPiano is a combination of forward-backward splitting method and and heavy-ball method. However, different from forward-backward splitting, the sequence generated by iPiano is not Fejér monotone due to the inertial term $\beta_k (x^k - x^{k-1})$. This brings troubles in proving the convergence rates in the convex case. Note that the heavy-ball method is a special form of iPiano. The difficulty also exists in analyzing the complexity of heavy-ball method. In the existing literatures, the sublinear convergence rate of the heavy-ball was established only in the sense of ergodicity. In this paper, we propose a novel Lyapunov function to address this issue, and prove the non-ergodic convergence rates.

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1.1 Interpretation by dynamical systems

The discretization of the following dynamical system gives the heavy-ball method with $g \equiv 0$ \[1\]:

$$\ddot{x}(t) + \alpha \dot{x}(t) + \nabla f(x(t)) = 0$$

(1.3)

for some $\alpha > 0$. If further $\beta_k \equiv 0$, the heavy-ball method reduces to basic gradient descent, which results in the discretization of the following ODE

$$\alpha \dot{x}(t) + \nabla f(x(t)) = 0.$$ 

(1.4)

Studying the property of the above dynamical systems helps us to understand the algorithms. More importantly, it motivates us to construct the proper Lyapunov function. We notice that some important relation between $\ddot{x}(t)$ and $\dot{x}(t)$ is missing. It is thus natural to add the missing information back to (1.3).

In the discretization, $\ddot{x}(t)$ is replaced by $\frac{x^{k+1} - 2x^k + x^{k-1}}{h^2}$, where $h$ is stepsize for discretization. Then it holds that

$$\left\| \frac{x^{k+1} - 2x^k + x^{k-1}}{h^2} \right\| \leq \frac{1}{h} \left( \left\| \frac{x^{k+1} - x^k}{h} \right\| + \left\| \frac{x^k - x^{k-1}}{h} \right\| \right).$$

(1.5)

Note that both $\frac{x^{k+1} - x^k}{h}$ and $\frac{x^k - x^{k-1}}{h}$ can be viewed as the discretization of $\dot{x}(t)$. Motivated by this observation, we propose to modify (1.3) by adding the following constraint

$$\|\ddot{x}(t)\| \leq \theta \|\dot{x}(t)\|,$$

(1.6)

where $\theta > 0$. In Section 2, we study the system $(1.3) + (1.6)$. With the extra constraint (1.6), the sublinear asymptotical convergence rate can be established. The analysis enables the non-ergodic sublinear convergence rate for heavy-ball (inertial) algorithm.

1.2 Related works

The inertial term was first proposed in the heavy-ball algorithm \[16\]. When the objective function is twice continuously differentiable, strongly convex (almost quadratic), the Heavy-ball method is proved to converge linearly. Under weaker assumption that the gradient of the objective function is Lipschitz continuous, \[21\] proved the convergence to a critical point, yet without specifying the convergence rate. The smoothness of objective function is critical for the heavy-ball to converge. In fact, there is an example that the heavy-ball method diverges for a strongly convex but nonsmooth function \[9\]. Different from the classical gradient methods, heavy-ball algorithm fails to generate a Fejér monotone sequence. In general convex and smooth case, the only convergence rate result is ergodic $O(1/k)$ in terms of the function values \[7\].

The iPiano combines heavy-ball method with the proximal mapping as in forward-backward splitting. In the nonconvex case, convergence of the algorithm was thoroughly discussed \[13\]. The local linear convergence of iPiano and heavy-ball method has been proved in \[14\]. In the strongly convex case, the linear convergence was proved for iPiano with fixed $\beta_k$ \[12\]. In the paper \[15\], inertial Proximal Alternating Linearized Minimization (iPALM) was introduced as a variant of iPiano for solving two-block regularized problem. Without the inertial terms, this algorithm reduces to the Proximal Alternating Linearized Minimization (PALM) \[4\], being equivalent to the two-block case of the Coordinate Descent (CD) algorithm \[20\]. In the convex case, the two-block CD methods are also well studied \[2, 17, 3, 18\]. Recently, there is growing interests in studying CD method using the operators \[5, 14, 19, 8\].

1.3 Contribution and organization

In this paper, we present the first non-ergodic $O(1/k)$ convergence rate result for iPiano in general convex case. Compared with results in \[7\], our convergence is established with a much larger stepsize under the coercive assumption. If the function fails to be coercive, we can choose asymptotic stepsizes. We also present the linear convergence under an error bound condition without assuming strong convexity.
Similar to the coercive case, our results hold for relaxed stepsizes. In addition, we extend our result to the coordinate descent version of iPiano. Both cyclic and stochastic index selection strategies are considered. The contributions of this paper are summarized as follows:

1. **A novel dynamical interpretation:** We propose a modified dynamical system of the inertial algorithm, from which we derive the sublinear asymptotical convergence rate with a proper Lyapunov function.

2. **The non-ergodic sublinear convergence rate:** We are the first to prove the non-ergodic convergence rates of the inertial proximal gradient algorithm. The linear convergence rate is also proved for the objective function without strong convexity. The brief idea of proof is to bound the Lyapunov function, and connect this bound to the successive difference of the Lyapunov function.

3. **Better linear convergence:** Stronger linear convergence results are proved for inertial algorithms. Compared with that in the literature, we have relaxed stepsize and inertial parameters. The strong convexity assumption can be weaken. More importantly, we show that the stepsize can be chosen independent of the strong convexity constant.

4. **Extensions to multi-block version:** The convergence of multi-block versions of inertial methods is studied. Both cyclic and stochastic index selection strategies are considered. The sublinear and linear convergence rates are proved for both algorithms.

The rest of the paper is organized as follows. In Section 2, we study the modified dynamical system and present technical lemmas. In Section 3, we show the convergence rates for inertial proximal gradient methods. We extend the results to the multi-block version of iPiano in Section 4, and to the stochastic version in Section 5. Section 6 concludes this article.

## 2 Dynamical motivation and technical lemmas

In this part, we first analyze the performance of the modified dynamical system \((1.3)+(1.6)\). The existence of the system is beyond the scope of this paper and will not be discussed. And then, two lemmas are introduced for the sublinear convergence rates analysis.

### 2.1 Performance of the modified dynamical system

Let us assume the existence of system \((1.3)+(1.6)\), and consider the Lyapunov function

\[
\xi(t) := f(x(t)) + \frac{1}{2} \|\dot{x}(t)\|^2 - \min f. \tag{2.1}
\]

With direct computation, it holds that

\[
\dot{\xi}(t) = \langle \nabla f(x(t)), \dot{x}(t) \rangle + \langle \dot{x}(t), \ddot{x}(t) \rangle = -\alpha \|\dot{x}(t)\|^2. \tag{2.2}
\]

Assume that \(f\) is coercive, noting \(\xi(t)\) is decreasing and nonnegative, \(x(t)\) must be bounded. With the continuity of \(\nabla f, \nabla f(x(t))\) is also bounded. That means \(\dot{x}(t) + \alpha \ddot{x}(t)\) is also bounded. If \(\alpha > \theta\), with the triangle inequality,

\[
\|\ddot{x}(t) + \alpha \ddot{x}(t)\| \geq \alpha \|\ddot{x}(t)\| - \|\dddot{x}(t)\| \geq (\alpha - \theta)\|\dddot{x}(t)\|. \tag{2.3}
\]

We then obtain the boundedness of \(\ddot{x}(t)\) and \(\dot{x}(t)\). Let \(x^* \in \arg \min f\), we have

\[
f(x(t)) - f(x^*) \leq \langle \nabla f(x(t)), x(t) - x^* \rangle \\
\leq \|\nabla f(x(t))\| \cdot \|x - x^*\| \\
\leq (\alpha + \theta)\|\dot{x}(t)\| \cdot \|x(t) - x^*\|. \tag{2.4}
\]

With the boundedness, denote that

\[
R := \sup_{t \geq 0} \left\{ \max\{\alpha + \theta \cdot \|x(t) - x^*\|, \frac{\|\dot{x}(t)\|}{2}\} \right\} < +\infty.
\]
Then, we can easily have
\[ \xi(t)^2 \leq R^2 \| \dot{x}(t) \|^2. \tag{2.5} \]

With \( \xi(t)^2 \leq -\frac{R^2}{\alpha} \xi(t) \),
\[ \frac{d}{dt} \frac{\alpha}{R^2} dt \leq \frac{d\xi}{\xi^2}. \tag{2.6} \]

Taking integrations of both sides, we then have
\[ f(x(t)) - f(x^*) \leq \xi(t) \leq 1 + \frac{1}{\frac{\alpha}{R^2} t + \xi(0)}. \]

**Remark 1.** If we just consider \( (1.3) \), only convergence can be proved without sublinear asymptotical rates. Obviously, \( (1.6) \) is crucial for the analysis.

### 2.2 Technical lemmas

This part contains two lemmas on nonnegative sequences: Lemma 1 is used to derive the convergence rate. It can be regarded as the discrete form of \( (2.6) \); Lemma 2 is developed to bound the sequence when inertial parameters are decreasing.

**Lemma 1** (Lemma 3.8, [2]). Let \( \{\alpha_k\}_{k \geq 1} \) be nonnegative sequence of real numbers satisfying
\[ \alpha_k - \alpha_{k+1} \geq \gamma \alpha_{k+1}^2. \]

Then, we have
\[ \alpha_k = O\left( \frac{1}{k} \right). \]

**Lemma 2.** Let \( (t_k)_{k \geq 0} \) be a nonnegative sequence and follow the condition
\[ t_{k+1} \leq (1 + \beta_k) t_k + \beta_k t_{k-1}. \tag{2.7} \]

If \( (\beta_k)_{k \geq 0} \) is descending and
\[ \sum_k \beta_k < +\infty, \]
\( (t_k)_{k \geq 0} \) is bounded.

**Proof.** Adding \( \beta_k t_k \) to both sides of \( (2.7) \),
\[ t_{k+1} + \beta_k t_k \leq (1 + \beta_k) t_k + \beta_k t_{k-1} + \beta_k t_k \leq (1 + 2\beta_k)(t_k + \beta_k t_{k-1}). \tag{2.8} \]

Noting the decent of \( (\beta_k)_{k \geq 0} \), \( (2.8) \) is actually
\[ t_{k+1} + \beta_k t_k \leq (1 + 2\beta_k)(t_k + \beta_{k-1} t_{k-1}). \]

Letting
\[ h_k := t_k + \beta_k t_{k-1}, \]
we then have
\[ h_{k+1} \leq (1 + 2\beta_k) h_k \leq e^{2\beta_k} h_k. \]

Thus, for any \( k \)
\[ h_{k+1} \leq e^{2 \sum_{i=1}^{k} \beta_i} h_1 < +\infty. \]

The boundedness of \( \{h_k\}_{k \geq 0} \) directly yields the boundedness of \( \{t_k\}_{k \geq 0} \).
3 Convergence rates

In this section, we prove convergence rates of iPiano. The core of the proof is to construct a proper Lyapunov function.

**Lemma 3.** Suppose $f$ is a convex function with $L$-Lipschitz gradient and $g$ is convex, and $\min F > -\infty$. Let $(x^k)_{k \geq 0}$ be generated by the inertial proximal gradient algorithm with non-increasing $(\beta_k)_{k \geq 0} \subseteq [0, 1)$. Choosing the step size

$$\gamma_k = \frac{2(1 - \beta_k)c}{L}$$

for arbitrary fixed $0 < c < 1$, we have

$$F(x^k) + \frac{\beta_k}{2\gamma_k}||x^k - x^{k-1}||^2 - F(x^{k+1}) - \frac{\beta_{k+1}}{2\gamma_{k+1}}||x^{k+1} - x^k||^2 \geq \left(\frac{1 - \beta_k}{\gamma_k} - \frac{L}{2}\right)||x^k - x^{k-1}||^2.$$  

(3.1)

**Proof.** Updating $x^{k+1}$ directly gives

$$\frac{x^k - x^{k+1}}{\gamma_k} - \nabla f(x^k) + \frac{\beta_k}{\gamma_k}(x^k - x^{k-1}) \in \partial g(x^{k+1}).$$

(3.2)

With the convexity of $g$, we have

$$g(x^{k+1}) - g(x^k) \leq \left(\frac{x^{k+1} - x^k}{\gamma_k} + \nabla f(x^k) + \frac{\beta_k}{\gamma_k}(x^{k-1} - x^k), x^k - x^{k+1}\right).$$

(3.3)

With Lipschitz continuity of $\nabla f$,

$$f(x^{k+1}) - f(x^k) \leq \langle -\nabla f(x^k), x^k - x^{k+1}\rangle + \frac{L}{2}||x^{k+1} - x^k||^2.$$  

(3.4)

Combining (3.3) and (3.4),

$$F(x^{k+1}) - F(x^k) \leq \frac{\beta_k}{\gamma_k}||x^k - x^{k-1}, x^{k+1} - x^k|| + \left(\frac{L}{2} - \frac{1}{\gamma_k}\right)||x^{k+1} - x^k||^2$$

$$\leq \frac{\beta_k}{2\gamma_k}||x^k - x^{k-1}||^2 + \left(\frac{L}{2} - \frac{1}{\gamma_k} + \frac{\beta_k}{2\gamma_k}\right)||x^{k+1} - x^k||^2.$$  

(3.5)

where $a$) uses the Schwarz inequality $\langle x^k - x^{k-1}, x^{k+1} - x^k\rangle \leq \frac{1}{2}||x^k - x^{k-1}||^2 + \frac{1}{2}||x^{k+1} - x^k||^2$. With direct calculations, we then obtain

$$\left[F(x^k) + \frac{\beta_k}{2\gamma_k}||x^k - x^{k-1}||^2\right] - \left[F(x^{k+1}) + \frac{\beta_k}{2\gamma_k}||x^{k+1} - x^k||^2\right]$$

$$\geq \left(\frac{1 - \beta_k}{\gamma_k} - \frac{L}{2}\right)||x^k - x^{k-1}||^2.$$  

(3.6)

With the non-increasing of $(\beta_k)_{k \geq 0}$, $(\frac{\beta_k}{2\gamma_k} = \frac{\beta_k}{\gamma_k} - \frac{L}{2})_{k \geq 0}$ is also non-increasing. Thus, we obtain the (3.1). □

We employ the following Lyapunov function

$$\xi_k := F(x^k) + \delta_k||x^k - x^{k-1}||^2 - \min F,$$  

(3.7)

where

$$\delta_k := \frac{\beta_k}{2\gamma_k} + \frac{1}{2}\left(\frac{1 - \beta_k}{\gamma_k} - \frac{L}{2}\right).$$

(3.8)

Function (3.7) can be regarded as the discretization of (2.1). We present a very useful technique lemma which is the key to results.
Lemma 4. Suppose the conditions of Lemma 3 hold. Let $\bar{x}^k$ denote the projection of $x^k$ onto $\arg\min F$, assumed to exist, and define

$$\varepsilon_k := \frac{4c\eta_k^2 + 4c}{(1-c)\tilde{L} + (1-c)L\gamma_k^2}. \quad (3.9)$$

Then it holds

$$(\xi_{k+1})^2 \leq \varepsilon_k (\xi_k - \xi_{k+1}) \cdot (2\|x^{k+1} - x^{k+1}\|^2 + \|x^{k+1} - x^{k}\|^2). \quad (3.10)$$

Proof. With direct computation and Lemma 3, we have

$$\xi_k - \xi_{k+1} \geq \frac{1}{2} \left( \frac{1}{\gamma_k} - \frac{L}{2} \right) \cdot (\|x^{k+1} - x^k\|^2 + \|x^k - x^{k-1}\|^2)$$

$$= \frac{L}{4} \left( \frac{1}{c} - 1 \right) \cdot (\|x^{k+1} - x^k\|^2 + \|x^k - x^{k-1}\|^2). \quad (3.11)$$

The convexity of $g$ yields

$$g(x^{k+1}) - g(\bar{x}^{k+1}) \leq \langle \nabla g(x^{k+1}), x^{k+1} - \bar{x}^{k+1} \rangle,$$

where $\nabla g(x^{k+1}) \in \partial g(x^{k+1})$. By (3.2), we then have

$$g(x^{k+1}) - g(\bar{x}^{k+1}) \leq \langle \frac{x^{k+1} - x^k}{\gamma_k} - \nabla f(x^k) + \frac{\beta_k}{\gamma_k} (x^k - x^{k-1}), x^{k+1} - \bar{x}^{k+1} \rangle. \quad (3.12)$$

Similarly, we have

$$f(x^{k+1}) - f(\bar{x}^{k+1}) \leq \langle \nabla f(x^{k+1}), x^{k+1} - \bar{x}^{k+1} \rangle. \quad (3.13)$$

Summing (3.12) and (3.13) yields

$$F(x^{k+1}) - F(\bar{x}^{k+1}) \leq \beta_k \langle x^{k+1} - x^k, x^{k+1} - \bar{x}^{k+1} \rangle + \left( \frac{x^k - x^{k+1}}{\gamma_k}, x^{k+1} - \bar{x}^{k+1} \right)$$

$$a \leq \frac{\beta_k}{\gamma_k} \|x^k - x^{k-1}\| \cdot \|x^{k+1} - \bar{x}^{k+1}\| + \frac{1}{\gamma_k} \|x^k - x^{k+1}\| \cdot \|x^{k+1} - \bar{x}^{k+1}\|$$

$$b \leq \frac{1}{\gamma_k} \left( \|x^k - x^{k+1}\| + \|x^k - x^{k-1}\| \right) \cdot \|x^{k+1} - \bar{x}^{k+1}\|, \quad (3.14)$$

where $a$ is due to the Schwarz inequalities, $b$ depends on the fact $0 \leq \beta_k < 1$. With (3.7) and (3.14), we have

$$\xi_{k+1} \leq \frac{1}{\gamma_k} \left( \|x^k - x^{k+1}\| + \|x^k - x^{k-1}\| \right) \cdot \|x^{k+1} - \bar{x}^{k+1}\| + \delta_{k+1} \|x^{k+1} - x^k\|^2.$$

Let

$$a^k := \begin{bmatrix} \frac{1}{\gamma_k} \|x^k - x^{k+1}\| \\ \delta_{k+1} \|x^{k+1} - x^k\| \end{bmatrix}, \quad b^k := \begin{bmatrix} \|x^{k+1} - x^{k+1}\| \\ \|x^{k+1} - x^k\| \end{bmatrix}. \quad (3.15)$$

Using this and the definition of $\xi_{k+1}$ (3.7), we have:

$$(\xi_{k+1})^2 \leq |\langle a^k, b^k \rangle|^2 \leq |a^k|^2 \cdot |b^k|^2. \quad (3.16)$$

Direct calculation yields

$$|a^k|^2 \leq (\delta_{k+1}^2 + \frac{1}{\gamma_k}) \cdot (\|x^k - x^{k+1}\|^2 + \|x^k - x^{k-1}\|^2).$$
and
\[ \|b^k\|^2 \leq 2\|x^{k+1} - x^{k+1}\|^2 + \|x^{k+1} - x^k\|^2. \]

Thus, we derive
\[ (\xi_{k+1})^2 \leq (\delta_{k+1}^2 + \frac{1}{\gamma_k}) \cdot (\|x^k - x^{k+1}\|^2 + \|x^{k-1} - x^k\|^2) \cdot (2\|x^{k+1} - x^{k+1}\|^2 + \|x^{k+1} - x^k\|^2). \tag{3.17} \]

Combining (3.11) and (3.17), we then prove the result.

### 3.1 Sublinear convergence rate under weak convexity

In this subsection, we present the sublinear of the convex iPiano. The coercivity of the function is critical for the analysis. If \( F \) is coercive, the parameter \( \beta_k \) can be bounded from 0; however, if \( F \) fails to be promised to be coercive, \( \beta_k \) must be descending to zero. Thus, this subsection will be divided into two parts in term of the coercivity.

#### 3.1.1 \( F \) is convercive

First, we present the non-ergodic \( O\left(\frac{1}{k}\right) \) convergence rate of the function value. The rate can be derived if \((\beta_k)_k \geq 0 \) is bounded from 0 and 1.

**Theorem 1.** Assume the conditions of Lemma 3 hold, and
\[ 0 < \inf_k \beta_k \leq \beta_k \leq \beta_0 < 1. \]

Then we have
\[ F(x^k) - \min F = O\left(\frac{1}{k}\right). \tag{3.18} \]

**Proof.** By Lemma 4, \( \sup_k \{\xi_k\} < +\infty \), thus, \( \sup_k \{F(x^k)\} < +\infty \) and \( \sup_k \{\|x^k - x^{k-1}\|^2\} < +\infty \). Noting the coercivity of \( F \), sequences \((x^k)_k \geq 0 \) and \((\bar{x}^k)_k \geq 0 \) are bounded. With the assumptions on \( \gamma_k \) and \( \beta_k \), \( \sup_k \{\varepsilon_k\} < +\infty \). Thus, \( \varepsilon_k (2\|x^{k+1} - x^{k+1}\|^2 + \|x^{k-1} - x^k\|^2) \) is bounded; and we assume the bound is \( R \), i.e.,
\[ \varepsilon_k (2\|x^{k+1} - x^{k+1}\|^2 + \|x^{k-1} - x^k\|^2) \leq R. \]

By Lemma 4, we then have
\[ \xi_{k+1}^2 \leq R(\xi_k - \xi_{k+1}). \]

By Lemma 4
\[ \xi_k = O\left(\frac{1}{k}\right). \]

Using the fact \( F(x^k) - \min F \leq \xi_k \), we then prove the result.

To the best of our knowledge, this is the first time to prove the non-ergodic \( O\left(\frac{1}{k}\right) \) convergence rate in the perspective of function values for iPiano and heavy-ball method in the convex case.

#### 3.1.2 \( F \) fails to be coercive

In this case, to obtain the boundedness of the sequence \((x^k)_k \geq 0 \), we must employ diminishing \( \beta_k \), i.e., \( \lim_k \beta_k = 0 \). The following lemma can derive the needed boundedness.

**Lemma 5.** Suppose the conditions of Lemma 3 hold, and
\[ \beta_k = \frac{1}{(k+1)^\theta}, \]
where \( \theta > 1 \). Let \((x^k)_k \geq 0 \) be generated by the inertial proximal gradient algorithm algorithm, then, \((x^k)_k \geq 0 \) is bounded.
Proof. First, we prove that $\cdot - \gamma_k \nabla f(\cdot)$ is a contractive operator. For any $x, y$,

$$
\|x - \gamma_k \nabla f(x) - y + \gamma_k \nabla f(y)\|^2 \\
= \|x - y\|^2 - 2\gamma_k \langle \nabla f(x) - \nabla f(y), x - y \rangle + \gamma_k^2 \|\nabla f(x) - \nabla f(y)\|^2 \\
\leq \|x - y\|^2 - \left(\frac{2\gamma_k}{L} - \gamma_k^2\right) \|\nabla f(x) - \nabla f(y)\|^2 \\
\leq \|x - y\|^2,
$$

where the first inequality depends on the fact $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{L} \|\nabla f(x) - \nabla f(y)\|^2$, and the second one is due to $0 < \gamma_k \leq \frac{1}{L}$.

Let $x^*$ be a minimizer of $F$. Obviously, it holds

$$
x^* = \text{prox}_{\gamma_k g}[x^* - \gamma_k \nabla f(x^*)].
$$

Noting $\text{prox}_{\gamma_k g}(\cdot)$ is contractive,

$$
\|x^{k+1} - x^*\| = \|\text{prox}_{\gamma_k g}[x^k - \gamma_k \nabla f(x^k) + \beta_k(x^k - x^{k-1})] - \text{prox}_{\gamma_k g}[x^* - \gamma_k \nabla f(x^*)]\| \\
\leq \|\{x^k - \gamma_k \nabla f(x^k) + \beta_k(x^k - x^{k-1})\} - \{x^* - \gamma_k \nabla f(x^*)\}\| \\
\leq \|x^k - x^*\| + \beta_k \|x^k - x^*\| + \beta_k \|x^{k-1} - x^*\|.
$$

With Lemma 2 we then prove the result. \hfill \Box

Now, we are prepared to present the $O(1/k)$ rate of the function values when $F$ is not coercive.

**Theorem 2.** Suppose the conditions of Lemma 3 hold. Let $(x^k)_{k \geq 0}$ be generated by the inertial proximal gradient algorithm, then we have

$$
F(x^k) - \min F = O\left(\frac{1}{k}\right).
$$

Proof. With Lemma 3 the sequence is bounded. And it is easy to verify the boundedness of $\varepsilon_k$. Thus, $\varepsilon_k (2\|x^k + \frac{x^k}{k+1}\|^2 + \|x^{k-1} - x^k\|^2)$ is bounded. With almost the same proofs in Theorem 1 we then prove the result. \hfill \Box

### 3.2 linear convergence and sublinear convergence under optimal strong convexity condition

We say that the function $F$ satisfies the optimal strong convexity condition, if

$$
F(x) - \min F \geq \nu \|x - \overline{x}\|^2,
$$

where $\overline{x}$ is the projection of $x$ onto the set arg min $F$, and $\nu > 0$. This condition is much weaker than the strong convexity.

**Theorem 3.** Suppose the conditions of Theorem 1 hold, and $F$ satisfies (3.19). Then we have

$$
F(x^k) - \min F \sim O(\omega^k),
$$

for some $0 < \omega < 1$.

Proof. With (3.19), we have

$$
2\|x^{k+1} - \frac{x^k}{k+1}\|^2 \leq \frac{2}{\nu}(F(x^{k+1}) - \min F) \leq \frac{2}{\nu}\xi_{k+1} \leq \frac{2}{\nu}\xi_k.
$$
On the other hand, from the definition of (3.7),
\[ \|x^k - x^{k-1}\|^2 \leq \frac{1}{\delta_k} \xi_k. \]

With Lemma 4 we then derive
\[ \xi^2 \leq \varepsilon_k \left( \frac{1}{\delta_k} + \frac{2}{\nu} \right) (\xi_k - \xi_{k+1}) \cdot \xi_k. \]

With the assumption, \( \sup_k \{\varepsilon_k (\frac{1}{\delta_k} + \frac{2}{\nu})\} < +\infty \), and the bound is assumed as \( \ell > 0 \). And then, we have the following result,
\[ \xi^2 \leq \ell (\xi_k - \xi_{k+1}) \cdot \xi_k. \]

If \( \xi_k = 0 \), we have \( 0 = \xi_{k+1} = \xi_{k+2} = \ldots \). The result certainly holds. If \( \xi_k \neq 0 \),
\[ \frac{(\xi_{k+1})^2 + \ell (\frac{\xi_{k+1}}{\xi_k}) - \ell}{\xi_k} \leq \frac{2\ell}{\sqrt{\nu^2 + 4\ell + \ell^2}}. \]

By defining \( \omega = \frac{2\ell}{\sqrt{\nu^2 + 4\ell + \ell^2}} \), we then prove the result.

Remark 2. Compared with previous linear convergence result presented in [12]. Our result enjoys three advantages: 1. The strongly convex assumption is weaken to (3.19). 2. More general parameters setting can be used. 3. The stepsizes and inertial parameters are independent with the strongly convex constants.

4 Cyclic coordinate descent inertial algorithm

This part analyzes the cyclic coordinate inertial proximal algorithm. The two-block version is proposed in [15], which focuses on the nonconvex case. Here, we consider the multi-block version and prove its convergence rate under convexity assumption. The minimization problem can be described as
\[ \min_{x_1, x_2, \ldots, x_m} \{ D(x_1, x_2, \ldots, x_m) := H(x_1, x_2, \ldots, x_m) + \sum_{i=1}^m g_i(x_i) \}, \tag{4.1} \]

where \( H \) and \( f_i \) \((i = 1, \ldots, m)\) are all convex. We use the notation
\[ \nabla_i^k H := \nabla_i H(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i^{k}, \ldots, x_m^{k}), \quad x^k := (x_1^k, x_2^k, \ldots, x_m^k). \]

The cyclic coordinate descent inertial algorithm runs as: for \( i \) from 1 to \( m \),
\[ x_i^{k+1} = \text{prox}_{\gamma_{k,i} g_i} [x_i^k - \gamma_{k,i} \nabla_i^k H + \beta_{k,i} (x_i^k - x_i^{k-1})], \tag{4.2} \]

where \( \gamma_{k,i}, \beta_{k,i} > 0 \). The iPALM can be regarded as the two-block case of this algorithm. The function \( H \) is assumed to satisfy
\[ \|\nabla_i H(x_1, x_2, \ldots, x_i^1, \ldots, x_m) - \nabla_i H(x_1, x_2, \ldots, x_i^2, \ldots, x_m)\| \leq L_i x_i^1 - x_i^2 \| \tag{4.3} \]

for any \( x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m \) and \( x_i^1, x_i^2 \), and \( i \in [1, 2, \ldots, m] \). With (4.3), we can easily obtain
\[ H(x_1, x_2, x_i^1, \ldots, x_m) \leq \nabla_i H(x_1, x_2, x_i^2, \ldots, x_m) + \frac{L_i}{2} x_i^1 - x_i^2 \|^2. \tag{4.4} \]

The proof is similar to [Lemma 1.2.3, [13]] and will not be reproduced. In the following part of this paper, we use the following assumption

**A1:** for any \( i \in [1, 2, \ldots, m] \), the sequence \((\beta_{k,i})_{k \geq 0} \subseteq [0, 1)\) is non-increasing.
Lemma 6. Let $H$ be a convex function satisfying (4.3) and $g_i$ is convex ($i \in [1, 2, \ldots, m]$), and finite $\min D$. Let $(x^k)_{k \geq 0}$ be generated by scheme (4.2) and assumption A1 is satisfied. Choose the step size

$$\gamma_{k,i} = \frac{2(1 - \beta_{k,i})c}{L_i}, \quad i \in [1, 2, \ldots, m]$$

for arbitrary fixed $0 < c < 1$. Then, we can obtain

$$\left[ D(x^k) + \sum_{i=1}^{m} \frac{\beta_{k,i}}{2\gamma_{k,i}} ||x^k_i - x^{k-1}_i||^2 \right] - \left[ D(x^{k+1}) + \sum_{i=1}^{m} \frac{\beta_{k+1,i}}{2\gamma_{k+1,i}} ||x^{k+1}_i - x^k_i||^2 \right] \geq \frac{(1 - c)L}{2c} ||x^{k+1} - x^k||^2,$$

where $L = \min_{i \in [1, 2, \ldots, m]} \{L_i\}$.

Proof. For any $i \in [1, 2, \ldots, m]$,

$$\frac{x^k_i - x^{k-1}_i}{\gamma_{k,i}} - \nabla_i^k H + \frac{\beta_{k,i}}{\gamma_{k,i}} (x^k_i - x^{k-1}_i) \in \partial g_i(x^{k+1}_i).$$

With the convexity of $g_i$, we have

$$g_i(x^{k+1}_i) - g_i(x^k_i) \leq \left( \frac{x^{k+1}_i - x^k_i}{\gamma_{k,i}} + \nabla_i^k H + \frac{\beta_{k,i}}{\gamma_{k,i}} (x^k_i - x^{k-1}_i), x^k_i - x^{k+1}_i \right) \leq \langle -\nabla_i^k H, x^k_i - x^{k+1}_i \rangle + \frac{L}{2} ||x^{k+1} - x^k||^2.$$ (4.6)

With (4.4), we can have

$$H(x^k_1, \ldots, x^k_{i-1}, x^{k+1}_i, x^k_{i+1}, \ldots, x^k_m) - H(x^k_1, \ldots, x^{k+1}_i, x^k_{i+1}, \ldots, x^k_m) \leq \langle -\nabla_i^k H, x^k_i - x^{k+1}_i \rangle + \frac{L}{2} ||x^{k+1} - x^k||^2.$$ (4.7)

Combining (4.7) and (4.8),

$$(H(x^k_1, \ldots, x^k_{i-1}, x^{k+1}_i, x^k_{i+1}, \ldots, x^k_m) + g_i(x^{k+1}_i)) - (H(x^k_1, \ldots, x^{k+1}_i, x^k_{i+1}, \ldots, x^k_m) + g_i(x^k_i)) \leq \frac{\beta_{k,i}}{\gamma_{k,i}} ||x^k_i - x^{k-1}_i||^2 + \left( \frac{L}{2} - \frac{1}{\gamma_{k,i}} \right) ||x^{k+1} - x^k||^2$$

$$\leq \frac{\beta_{k,i}}{2\gamma_{k,i}} ||x^k_i - x^{k-1}_i||^2 + \left( \frac{L}{2} - \frac{1}{\gamma_{k,i}} \right) ||x^{k+1} - x^k||^2.$$ (4.9)

where a) uses the Schwarz inequality $\langle x^k_i - x^{k-1}_i, x^{k+1}_i - x^k_i \rangle \leq \frac{1}{2} ||x^k_i - x^{k-1}_i||^2 + \frac{1}{2} ||x^{k+1}_i - x^k_i||^2$. Summing (4.9) from $i = 1$ to $m$,

$$D(x^{k+1}) - D(x^k) \leq \sum_{i=1}^{m} \frac{\beta_{k,i}}{2\gamma_{k,i}} ||x^k_i - x^{k-1}_i||^2 + \left( \frac{L}{2} - \frac{1}{\gamma_{k,i}} \right) ||x^{k+1} - x^k||^2.$$ (4.10)

With direct calculations and the non-increasing of $(\beta_{k,i})_{k \geq 0}$, we then obtain (3.1).}

The following Lyapunov function is used for cyclic coordinate inertial proximal algorithm

$$\hat{\xi}_k := D(x^k) + \sum_{i=1}^{m} \delta_{k,i} ||x^k_i - x^{k-1}_i||^2 - \min D,$$

where

$$\delta_{k,i} := \frac{\beta_{k,i}}{2\gamma_{k,i}} + \frac{1}{2} \left( \frac{1 - \beta_{k,i}}{\gamma_{k,i}} - \frac{L_i}{2} \right).$$ (4.12)

With this Lyapunov function, we can present the following lemma.
Lemma 7. Suppose the conditions Lemma 6 hold. Let $\mathbf{x}^k$ denote the projection of $x^k$ onto $\arg \min D$, assumed to exist, and define

$$\hat{\xi}_k := \max \left\{ \frac{4c}{(1 - c)L} \sum_{i=1}^{m} \left( \delta_{k+1,i}^2 + L_i^2 \right), \frac{4c}{(1 - c)L} \sum_{i=1}^{m} \frac{1}{\gamma_{k,i}} \right\}. \quad (4.13)$$

Then, it holds that:

$$\hat{\xi}_{k+1}^2 \leq \hat{\xi}_k (\hat{\xi}_k - \hat{\xi}_{k+1}) \cdot (3\|x^{k+1} - \mathbf{x}^{k+1}\|^2 + \|x^k - x^k\|^2). \quad (4.14)$$

Proof. With Lemma 6, direct computing yields

$$\hat{\xi}_k - \hat{\xi}_{k+1} \geq \sum_{i=1}^{m} \frac{1}{2} \left( \frac{1 - \beta_{k,i}}{\gamma_{k,i}} - \frac{L_i}{2} \right) \cdot (\|x_i^{k+1} - x_i^k\|^2 + \|x_i^k - x_i^{k-1}\|^2)
= \frac{L_i}{4} \left( \frac{1}{c} - 1 \right) \cdot (\|x^{k+1} - x^k\|^2 + \|x^k - x^{k-1}\|^2). \quad (4.15)$$

For any $i \in [1, 2, \ldots, m]$, the convexity of $g_i$ gives

$$g_i(x_i^{k+1}) - g_i\left( \left[ x_i^{k+1} \right]_i \right) \leq \langle \nabla g_i(x_i^{k+1}), x_i^{k+1} - \left[ x_i^{k+1} \right]_i \rangle, \quad (4.16)$$

where $\nabla g_i(x_i^{k+1}) \in \partial g_i(x_i^{k+1})$. With (4.6), we then have

$$g_i(x_i^{k+1}) - g_i\left( \left[ x_i^{k+1} \right]_i \right) \leq \langle x_i^k - x_i^{k-1}, x_i^{k+1} - \left[ x_i^{k+1} \right]_i \rangle, \quad (4.17)$$

Summing (4.17) with respect to $i$ from 1 to $m$, and

$$H(x^{k+1}) - H(x^{k+1}) \leq \langle \nabla H(x^{k+1}), x^{k+1} - x^{k+1} \rangle, \quad (4.18)$$

we then have

$$D(x^{k+1}) - D(x^{k+1}) \leq \sum_{i=1}^{m} \frac{\beta_{k,i}}{\gamma_{k,i}} \langle x_i^k - x_i^{k-1}, x_i^{k+1} - \left[ x_i^{k+1} \right]_i \rangle + \sum_{i=1}^{m} \frac{1}{\gamma_{k,i}} \|x_i^k - x_i^{k-1}\| \cdot \|x_i^{k+1} - \left[ x_i^{k+1} \right]_i \|
+ \sum_{i=1}^{m} \left( \frac{\|x_i^{k+1} - x_i^k\|}{\gamma_{k,i}} + \frac{\|x_i^k - x_i^{k-1}\|}{\gamma_{k,i}} + L_i \|x_i^{k+1} - x^k\| \right) \cdot \|x_i^{k+1} - \left[ x_i^{k+1} \right]_i \|, \quad (4.19)$$

where $a)$ is due to the Schwarz inequalities and the smooth assumption A1, $b)$ depends on the fact $0 \leq \beta_{k,i} < 1$. With (5.3) and (5.14), we have

$$\hat{\xi}_{k+1} \leq \sum_{i=1}^{m} \left( \frac{\|x_i^{k+1} - x_i^k\|}{\gamma_{k,i}} + \frac{\|x_i^k - x_i^{k-1}\|}{\gamma_{k,i}} + L_i \|x_i^{k+1} - x^k\| \right) \cdot \|x_i^{k+1} - \left[ x_i^{k+1} \right]_i \|
+ \sum_{i=1}^{m} \delta_{k+1,i} \|x_i^{k+1} - x_i^k\|^2.$$
Let

\[
\begin{align*}
\hat{a}^k := & \begin{pmatrix}
\frac{1}{\gamma_{k,1}} \|x_1^{k+1} - x_1^k\| \\
\frac{1}{\gamma_{k,2}} \|x_2^{k+1} - x_2^k\| \\
\vdots \\
\frac{1}{\gamma_{k,m}} \|x_m^{k+1} - x_m^k\| \\
\end{pmatrix} \\
\hat{b}^k := & \begin{pmatrix}
\|x_1^{k+1} - x_1^k\| \\
\|x_2^{k+1} - x_2^k\| \\
\vdots \\
\|x_m^{k+1} - x_m^k\| \\
\end{pmatrix}
\end{align*}
\]

(4.20)

Using this and the definition of $\xi_{k+1}$ (3.7), we have:

\[
(\xi_{k+1})^2 = \left| \langle \hat{a}^k, \hat{b}^k \rangle \right|^2 \leq \|\hat{a}^k\|^2 \cdot \|\hat{b}^k\|^2.
\]

Direct calculation yields

\[
\|\hat{a}^k\|^2 \leq \max \left\{ \sum_{i=1}^{m} \left( \delta_{k+1,i}^2 + L_i^2 \right) \cdot \frac{1}{\gamma_{k,i}} \right\} \cdot \left( \|x^k - x^{k+1}\|^2 + \|x^{k-1} - x^k\|^2 \right)
\]

and

\[
\|\hat{b}^k\|^2 \leq 3\|x^{k+1} - x^{k+1}\|^2 + \|x^{k+1} - x^k\|^2.
\]

Thus, we derive

\[
(\xi_{k+1})^2 \leq \max \left\{ \sum_{i=1}^{m} \left( \delta_{k+1,i}^2 + L_i^2 \right) \cdot \frac{1}{\gamma_{k,i}} \right\} \times \left( \|x^k - x^{k+1}\|^2 + \|x^{k-1} - x^k\|^2 \right)
\]

\[
\times (3\|x^{k+1} - x^{k+1}\|^2 + \|x^{k+1} - x^k\|^2).
\]

(4.21)

Combining (4.15) and (4.21), we then prove the result. \qed

### 4.1 Sublinear convergence rates of cyclic coordinate descent inertial algorithm

This part proves the sublinear convergence rates of cyclic coordinate descent inertial algorithm. In multi-block case, it is always to assume that the objective function is coercive. Like the previous section, we obtain the $O(1/k)$ convergence rate of the algorithm if $D$ is coercive.

**Theorem 4.** Suppose the conditions of Lemma 2 hold, $D$ is coercive, and

\[
0 < \inf_k \beta_k \leq \beta_{k,i} \leq \beta_0 < 1, \ i \in [1, 2, \ldots, m].
\]

Then we have

\[
D(x^k) - \min D = O\left(\frac{1}{k}\right).
\]

(4.22)
Proof. With Lemma 7, \( \sup_k \{ \hat{\xi}_k \} < +\infty \), thus, \( \sup_k \{ D(x^k) \} < +\infty \) and \( \sup_k \{ \| x^k - x^{k-1} \|^2 \} < +\infty \). Noting the coercivity of \( D \), sequences \( (x^k)_{k \geq 0} \) and \( (\hat{\xi}_k)_{k \geq 0} \) are bounded. With the assumptions on \( \gamma_{k,i} \) and \( \beta_{k,i} \), \( \sup_k \{ \hat{\xi}_k \} < +\infty \). Thus, \( \hat{\xi}_k (3\| x^{k+1} - x^{k+1} \|^2 + \| x^{k-1} - x^k \|^2) \) is bounded; and we assume the bound is \( R \), i.e.,

\[
\hat{\xi}_k (3\| x^{k+1} - x^{k+1} \|^2 + \| x^{k-1} - x^k \|^2) \leq R.
\]

With Lemma 7, we then have \( \hat{\xi}_{k+1}^2 \leq R(\hat{\xi}_k - \hat{\xi}_{k+1}) \).

From Lemma 11

\[
\hat{\xi}_k \sim O(\frac{1}{k}).
\]

Using the fact \( D(x^k) - \min D \leq \hat{\xi}_k \), we then obtain the result.

\[ \square \]

4.2 Linear convergence rate of cyclic coordinate descent inertial algorithm

Theorem 5. Suppose the conditions of Lemma 2 hold, \( D \) satisfies (3.19), and

\[
0 < \inf_k \beta_k \leq \beta_{k,i} \leq \beta_0 < 1, \quad i \in [1, 2, \ldots, m].
\]

Then we have

\[
D(x^k) - \min D = O(\omega^k)
\]

for some \( 0 < \omega < 1 \).

Proof. With the optimal strong convexity condition, we have

\[
3\| x^{k+1} - x^k \|^2 \leq \frac{3}{\nu} \hat{\xi}_{k+1} \leq \frac{3}{\nu} \hat{\xi}_k.
\]

The direct computing yields

\[
\| x^{k-1} - x^k \|^2 \leq \frac{\hat{\xi}_k}{\min_{i} \{ \delta_{k,i} \}}
\]

With Lemma 7

\[
(\hat{\xi}_{k+1})^2 \leq \left( \hat{\xi}_k + \frac{3}{\nu} + \frac{1}{\min_{i} \{ \delta_{k,i} \}} \right) (\hat{\xi}_k - \hat{\xi}_{k+1}) \hat{\xi}_k.
\]

It is easy to see that \( \hat{\xi}_k + \frac{3}{\nu} + \frac{1}{\min_{i} \{ \delta_{k,i} \}} \) is bounded by some positive constant \( \ell > 0 \). Then, we have

\[
D(x^k) - \min D \leq \hat{\xi}_k \sim O(\frac{2\ell}{\sqrt{\ell^2 + 4\ell + \ell}})^k).
\]

Letting \( \omega = \frac{2\ell}{\sqrt{\ell^2 + 4\ell + \ell}} \), we then prove the result.

\[ \square \]

5 Stochastic coordinate descent inertial algorithm

We still aim to minimizing problem (4.1) but using the stochastic index selection strategy. In the \( k \)-th iteration, pick \( i_k \) uniformly from \( [1, 2, \ldots, m] \), and then

\[
x_{i_k}^{k+1} = \text{prox}_{\gamma_k g_{i_k}} [x_{i_k}^{k} - \gamma_k \nabla_{i_k} H(x^k) + \beta_k (x_{i_k}^k - x_{i_k}^{k-1})].
\]

The sub-algebra \( \chi^k \) is defined as

\[
\chi^k := \sigma(x^0, x^1, \ldots, x^k).
\]
In this section, we use the following assumption following assumption

\textbf{A2:} the sequence \((\beta_k)_{k \geq 0} \subseteq [0, 1)\) is non-increasing.

Denote that

\[ S_\gamma(x) = x - \text{prox}_{\gamma \nabla H}(x). \]  \hfill (5.3)

It is easy to see

\[ S_\gamma(x^*) = 0 \iff x^* \text{ minimize } D \]  \hfill (5.4)

for any \(\gamma > 0\).

### 5.1 Sublinear convergence rates of stochastic coordinate descent inertial algorithm

This part proves the sublinear convergence rates of cyclic coordinate descent inertial algorithm. The \(O(1/k)\) convergence rate of the algorithm is proved for the successive difference of the points.

**Lemma 8.** Let \(H\) be a convex function whose gradient is Lipschitz continuous with \(L\), and \(g_i\) is convex \((i \in [1, 2, \ldots, m])\), and finite \(\min D\). Let \((x^k)_{k \geq 0}\) be generated by scheme (5.1) and assumption \textbf{A2} is satisfied. Choose the step size

\[ \gamma_k = \frac{2(1 - \beta_k/\sqrt{m})c}{L} \]

for arbitrary fixed \(0 < c < 1\). Then, we can obtain

\[
\left[ E D(x^k) + \frac{\beta_k}{2\sqrt{m}\gamma_k} E \|x^k - x^{k-1}\|^2 \right] - \left[ E D(x^{k+1}) + \frac{\beta_{k+1}}{2\sqrt{m}\gamma_{k+1}} E \|x^{k+1} - x^k\|^2 \right] \\
\geq \left( 1 - \frac{\beta_k / \sqrt{m}}{\gamma_k} \right) - \frac{L}{2} E \|x^{k+1} - x^k\|^2. \tag{5.5}
\]

And,

\[
\min_{0 \leq i \leq k} E \|x^{k+1} - x^k\|^2 = o\left(\frac{1}{k}\right). \tag{5.6}
\]

**Proof.** In the \(k\)-th iteration,

\[
\frac{x^k - x^{k+1}}{\gamma_k} - \nabla_{i_k} H(x^k) + \frac{\beta_k}{\gamma_k} (x^k - x^{k-1}) \in \partial g_{i_k}(x^{k+1}_{i_k}).
\]

With the convexity of \(g_{i_k}\), we have

\[
g_{i_k}(x^{k+1}_{i_k}) - g_{i_k}(x^{k}_{i_k}) \leq (\frac{x^{k+1}_{i_k} - x^k_{i_k}}{\gamma_k}) + \nabla_{i_k} H(x^k) + \frac{\beta_k}{\gamma_k} (x^{k-1}_{i_k} - x^k_{i_k}, x^k_{i_k} - x^{k+1}_{i_k}). \tag{5.7}
\]

With the Lipschitz of \(\nabla H\), we can have

\[
H(x^{k+1}) - H(x^k) \leq \langle -\nabla_{i_k} H(x^k), x^k_{i_k} - x^{k+1}_{i_k} \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2, \tag{5.8}
\]

where we used the fact \(\langle -\nabla_{i_k} H(x^k), x^k_{i_k} - x^{k+1}_{i_k} \rangle = \langle -\nabla H(x^k), x^k - x^{k+1} \rangle\). Combining (5.7) and (5.8),

\[
D(x^{k+1}) - D(x^k) \leq \left[ H(x^{k+1}) + g_{i_k}(x^{k+1}_{i_k}) \right] - \left[ H(x^k) + g_{i_k}(x^{k}_{i_k}) \right] \\
\leq \beta_k \left( x^{k+1}_{i_k} - x^{k-1}_{i_k}, x^{k+1}_{i_k} - x^{k}_{i_k} \right) + \left( \frac{L}{2} - \frac{1}{\gamma_k} \right) \|x^{k+1} - x^k\|^2 \\
\leq \sqrt{m} \beta_k \|x^{k+1} - x^{k-1}\|^2 + \left( \frac{L}{2} - \frac{1}{\gamma_k} + \frac{\beta_k}{2\sqrt{m}\gamma_k} \right) \|x^{k+1} - x^k\|^2. \tag{5.9}
\]
where \(a\) uses the Schwarz inequality \(\langle x^k_i - x^{k-1}_i, x^{k+1}_i - x^k_i \rangle \leq \frac{m}{2} \|x^k_i - x^{k-1}_i\|^2 + \frac{1}{2m} \|x^{k+1}_i - x^k_i\|^2\) and the fact \(\|x^{k+1} - x^k\|^2 = \|x^{k+1}_i - x^k_i\|^2\). Taking conditional expectations of (5.9) on \(\chi^k\),

\[
\mathbb{E}[D(x^{k+1}) \mid \chi^k] - D(x^k) \leq \frac{\beta_k}{2\sqrt{m} \gamma_k} \|x^k - x^{k-1}\|^2 + \left(\frac{L}{2} - \frac{1}{\gamma_k} + \frac{\beta_k}{2\sqrt{m} \gamma_k}\right) \mathbb{E}(\|x^{k+1} - x^k\|^2 \mid \chi^k).
\]  

(5.10)

Taking total expectations on (5.10), and using \(\mathbb{E}(\cdot \mid \chi^k) = \mathbb{E}(\cdot)\),

\[
\mathbb{E}D(x^{k+1}) - \mathbb{E}D(x^k) \leq \frac{\beta_k}{2\sqrt{m} \gamma_k} \mathbb{E}\|x^k - x^{k-1}\|^2 + \left(\frac{L}{2} - \frac{1}{\gamma_k} + \frac{\beta_k}{2\sqrt{m} \gamma_k}\right) \mathbb{E}\|x^{k+1} - x^k\|^2.
\]

Thus, we have

\[
\left[\mathbb{E}D(x^k) + \frac{\beta_k}{2\sqrt{m} \gamma_k} \mathbb{E}\|x^k - x^{k-1}\|^2\right] - \left[\mathbb{E}D(x^{k+1}) + \frac{\beta_k}{2\sqrt{m} \gamma_k} \mathbb{E}\|x^{k+1} - x^k\|^2\right] \\
\geq \left(1 - \frac{\beta_k}{2\sqrt{m} \gamma_k} \frac{L}{2}\right) \mathbb{E}\|x^{k+1} - x^k\|^2.
\]

With the non-increasity of \((\beta_k)_{k \geq 0}\), \(\frac{\beta_k}{2\sqrt{m} \gamma_k} = \frac{\beta_k L}{4(1 - \beta_k/\sqrt{m} \gamma_k)}\) \(k \geq 0\) is also non-increasing, and then we prove the result.

\[\blacksquare\]

**Theorem 6.** Suppose that the conditions of Lemma [8] hold, and \(0 \leq \beta < \sqrt{m}\). Then we have

\[
\min_{0 \leq i \leq k} \mathbb{E}\|S_\gamma(x^k)\|^2 = o\left(\frac{1}{k}\right).
\]

(5.11)

**Proof.** Direct calculations yield

\[
\|S_\gamma(x^k)\|^2 = m \cdot \mathbb{E}(\|x^k_i - \text{prox}_{\gamma g_{ik}}[x^k_i - \gamma \nabla_{ik} H(x^k)]\|^2 \mid \chi^k) \\
= m \cdot \mathbb{E}(\|x^k_i - \text{prox}_{\gamma g_{ik}}[x^k_i - \gamma \nabla_{ik} H(x^k)] + \beta(x^k_i - x^{k-1}_i)\|^2) \\
+ \text{prox}_{\gamma g_{ik}}[x^k_i - \gamma \nabla_{ik} H(x^k)] - \text{prox}_{\gamma g_{ik}}[x^{k-1}_i - \gamma \nabla_{ik} H(x^{k-1})]\|^2} \mid \chi^k) \\
\leq 2m \cdot \mathbb{E}(\|x^{k+1} - x^k\|^2 \mid \chi^k) + 2m \cdot \mathbb{E}(\|x^k - x^{k-1}\|^2 \mid \chi^k) \\
= 2m \cdot \mathbb{E}(\|x^{k+1} - x^k\|^2 \mid \chi^k) + 2\|x^k - x^{k-1}\|^2.
\]

Taking expectations of both sides,

\[
\mathbb{E}\|S_\gamma(x^k)\|^2 \leq 2m\mathbb{E}\|x^{k+1} - x^k\|^2 + 2\mathbb{E}\|x^k - x^{k-1}\|^2.
\]

From Lemma [8]

\[
\sum_k \mathbb{E}\|S_\gamma(x^k)\|^2 < +\infty.
\]

\[\square\]

### 5.2 Linear convergence rates of stochastic coordinate descent inertial algoirthm

**Lemma 9.** Assume the function satisfy the optimal strong convexity condition \([9,10]\), and \((x^k)_{k \geq 0}\) is generated by the scheme (5.1). If

\[
\beta = \frac{\nu}{4m}
\]

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we have

\[
\begin{align*}
\mathbb{E}[\|x^k - \bar{x}^k\|^2 + 2\gamma(D(x^k) - D^*)] \\
\leq \mathbb{E}[\|x^k - \bar{x}^k\|^2 + 2\gamma(D(x^k) - D^*\|D^*)] - \mathbb{E}[\|x^{k+1} - \bar{x}^{k+1}\|^2 + 2\gamma(D(x^{k+1}) - D^*)] \\
- (1 - \gamma L + 2\beta)\mathbb{E}\|x^{k+1} - x^k\|^2 + \frac{\beta}{m}\mathbb{E}\|x^k - x^{k-1}\|^2
\end{align*}
\]

(5.12)

where \( \ell := \frac{\gamma}{2\beta} \) and \( \bar{x}^k \) is also the projection from \( x^k \) to \( \arg\min D \).

Proof. The optimization condition of the iteration yields

\[
g_{ik}(\bar{x}^k_{ik}) - g_{ik}(x^{k+1}_{ik}) + \langle (\bar{x}^k - x^{k+1})_{ik}, \nabla_i H(x^k) + \frac{\beta}{\gamma}(x^k_{ik} - x^{k-1}_{ik}) \rangle
\geq \frac{1}{\gamma}\langle (x^k - x^{k+1})_{ik}, x^k_{ik} - x^{k+1}_{ik} \rangle.
\]

(5.13)

With the scheme of the algorithm,

\[
\|x^{k+1} - \bar{x}^{k+1}\|^2 \leq \|x^{k+1} - \bar{x}^k\|^2
\]

\[
= \|x^k - \bar{x}^k\|^2 + \|x^{k+1} - x^k\|^2 - 2\langle (x^k - \bar{x}^k)_{ik}, (x^k - x^{k+1})_{ik} \rangle
\]

\[
= \|x^k - \bar{x}^k\|^2 - \|x^{k+1} - x^k\|^2 + 2\langle (\bar{x}^k - x^{k+1})_{ik}, (x^k - x^{k+1})_{ik} \rangle
\]

(5.13)

\[
\leq \|x^k - \bar{x}^k\|^2 - \|x^{k+1} - x^k\|^2 + 2\beta\langle (\bar{x}^k - x^{k+1})_{ik}, (x^k - x^{k-1})_{ik} \rangle
\]

\[
+ 2\gamma\langle H(x^k) - H(x^{k+1}) + g_{ik}(\bar{x}^k_{ik}) - g_{ik}(x^{k+1}_{ik}) \rangle
\]

\[
+ 2\beta\langle (x^k - \bar{x}^k)_{ik}, (x^k - x^{k-1})_{ik} \rangle
\]

In the following, we bound the expectations of I, II, III:

I : \( 2\gamma\mathbb{E}(\langle (x^k - x^k)_{ik}, \nabla_i H(x^k) \rangle | x^k) \leq \frac{2\gamma}{m}\mathbb{E}(x^k - x^k, \nabla H(x^k)) \leq \frac{2\gamma}{m}(H(x^k) - H(x^k)) \).

and

II : \( \mathbb{E}(H(x^k) - H(x^{k+1}) + g_{ik}(\bar{x}^k_{ik}) - g_{ik}(x^{k+1}_{ik}) | x^k) \)

\[
= H(x^k) - \mathbb{E}(H(x^{k+1}) | x^k) + \frac{1}{m}g(x^k) - \mathbb{E}(g(x^{k+1}) | x^k) + \frac{m - 1}{m}g(x^k).
\]

and

III : \(2\beta\mathbb{E}(\langle (x^k - x^{k+1})_{ik}, (x^k - x^{k-1})_{ik} \rangle | x^k) \)

\[
\leq \beta\mathbb{E}(\|x^k - x^{k+1}\|^2 | x^k) + \frac{\beta}{m}\mathbb{E}(\|x^k - x^{k-1}\|^2 | x^k)
\]

\[
\leq 2\beta\|\bar{x}^k - x^k\|^2 + 2\beta\mathbb{E}(\|x^k - x^{k+1}\|^2 | x^k) + \frac{\beta}{m}\|x^k - x^{k-1}\|^2.
\]

Combining the inequalities I, II and III,

\[
\begin{align*}
\mathbb{E}(\|x^{k+1} - \bar{x}^{k+1}\|^2 | x^k) & \leq (1 + 2\beta)\|x^k - \bar{x}^k\|^2 - (1 - \gamma L + 2\beta)\mathbb{E}(\|x^{k+1} - x^k\|^2 | x^k) \\
& + \frac{2\gamma}{m}(D^* - D(x^k)) + 2\gamma(D(x^k) - \mathbb{E}(D(x^{k+1}) | x^k)) + \frac{\beta}{m}\|x^k - x^{k-1}\|^2.
\end{align*}
\]

(5.14)
Taking expectations, then,

\[
\frac{2\gamma}{m} \mathbb{E}(D(x^k) - D^*) - 2\beta \mathbb{E}\|x^k - \overline{x}\|^2 \leq \mathbb{E}\|x^k - \overline{x}\|^2 + 2\gamma D(x^k) - \mathbb{E}\|x^{k+1} - \overline{x}^{k+1}\|^2 + 2\gamma D(x^{k+1})
\]

\[- (1 - \gamma L + 2\beta) \mathbb{E}\|x^{k+1} - x^k\|^2 + \frac{\beta}{m} \mathbb{E}\|x^k - x^{k-1}\|^2.
\]

(5.15)

We can further obtain

\[
\frac{2\gamma}{m} (D(x^k) - D^*) - 2\beta \mathbb{E}\|x^k - \overline{x}\|^2
\]

\[
= \frac{\gamma}{m} (D(x^k) - D^*) + \frac{\gamma}{m} (D(x^k) - D^*) - 2\beta \mathbb{E}\|x^k - \overline{x}\|^2
\]

\[
= \frac{\gamma}{m} (D(x^k) - D^*) + \left(\frac{\gamma^2}{m} - 2\beta\right) \mathbb{E}\|x^k - \overline{x}\|^2
\]

\[
\geq \mathbb{E}\left[\frac{\gamma}{m} (D(x^k) - D^*) + \mathbb{E}\|x^k - \overline{x}\|^2\right].
\]

(5.16)

With (5.15) and (5.16), we derive the result.

\[\square\]

**Theorem 7.** Suppose the conditions of Lemma 9 hold. If

\[
\gamma = \frac{1 + [2 - \frac{1 - \ell}{m}]}{L}, \beta = \frac{\gamma \nu}{4m}
\]

we have

\[
\mathbb{E}[D(x^k) - D^*] = O((1 - \frac{\gamma \nu}{2m})^k).
\]

(5.17)

**Proof.** In this setting,

\[
1 - \gamma L + 2\beta = (1 - \ell) \frac{\beta}{m}.
\]

(5.18)

Thus, from Lemma 9

\[
\mathbb{E}[\|x^k - \overline{x}\|^2 + 2\gamma (D(x^k) - D^*) + \frac{\beta}{m} \mathbb{E}\|x^k - x^{k-1}\|^2]
\]

\[
\leq \mathbb{E}[\|x^k - \overline{x}\|^2 + 2\gamma (D(x^k) - D^*) + \frac{\beta}{m} \|x^k - x^{k-1}\|^2]
\]

\[- \mathbb{E}[\|x^{k+1} - \overline{x}^{k+1}\|^2 + 2\gamma (D(x^{k+1}) - D^*) + \frac{\beta}{m} \|x^{k+1} - x^k\|^2].
\]

(5.19)

That is also

\[
\mathbb{E}[\|x^{k+1} - \overline{x}^{k+1}\|^2 + 2\gamma (D(x^{k+1}) - D^*) + \frac{\beta}{m} \|x^{k+1} - x^k\|^2]
\]

\[
\leq (1 - \ell) \mathbb{E}[\|x^k - \overline{x}\|^2 + 2\gamma (D(x^k) - D^*) + \frac{\beta}{m} \|x^k - x^{k-1}\|^2].
\]

(5.20)

Then, we have

\[
\mathbb{E}[2\gamma (D(x^{k+1}) - D^*)] \leq \mathbb{E}[\|x^{k+1} - \overline{x}^{k+1}\|^2 + 2\gamma (D(x^{k+1}) - D^*) + \frac{\beta}{m} \|x^{k+1} - x^k\|^2]
\]

\[
= O((1 - \frac{\gamma \nu}{2m})^k).
\]

(5.21)

\[\square\]
6 Conclusion

In this paper, we focus on the complexity of the inertial methods in the convex setting. We prove the sublinear convergence rate of the algorithm and linear convergence rate under larger stepsize. We extend our results to the multi-block inertial algorithm. For both cyclic and stochastic index selection strategies, the convergence rates are proved.

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