A transition of limiting distributions of large matchings in random graphs

Pu Gao\textsuperscript{a,1}, Cristiane M. Sato\textsuperscript{b,c,2}

\textsuperscript{a} University of Toronto, Canada
\textsuperscript{b} Universidade Federal do ABC, Brazil
\textsuperscript{c} University of Waterloo, Canada

\begin{abstract}
We study the asymptotic distribution of the number of matchings of size $\ell = \ell(n)$ in $\mathcal{G}(n,p)$ for a wide range of $p = p(n) \in (0,1)$ and for every $1 \leq \ell \leq \lfloor n/2 \rfloor$. We prove that this distribution changes from normal to log-normal as $\ell$ increases, and we determine the critical value of $\ell$, as a function of $n$ and $p$, at which the transition of the limiting distribution occurs.

© 2015 Elsevier Inc. All rights reserved.
\end{abstract}

\section{Introduction}

Let $\mathcal{G}(n,p)$ denote the probability space of random graphs on $n$ vertices, where each edge is included independently with probability $p$. A classical result by Ruciński \cite{18} shows that the distribution of the number of small subgraphs (meaning the number of subgraphs isomorphic to a graph with a fixed size) is asymptotically normal if its expected value goes to infinity as $n$ goes to infinity. This is naturally expected as this random variable can be expressed as a sum of indicator random variables such that

\begin{thebibliography}{99}
\end{thebibliography}
each variable is dependent only on a small proportion of the other variables. However, this intuitive explanation fails when the size of the subgraphs increases since then each indicator variable depends on more and more of the other variables. It has been shown by Janson [9] that the numbers of spanning trees, perfect matchings, and Hamilton cycles in $G(n,p)$ (when $p$ is in an appropriate range) are asymptotically log-normally distributed, which behave quite differently from variables with the normal distribution. The first author [5,6] recently proved that the numbers of $d$-factors (for $d$ not growing too fast), triangle-factors and triangle-free subgraphs also follow a log-normal distribution (when $p$ is in an appropriate range). Comparing the result by Ruciński [18] with that by Janson [9], we notice that the distribution of the number of $\ell$-matchings (matchings of size $\ell$) must undergo certain phases of transition, starting from normal and ending with log-normal, when $\ell$ increases from a constant size to $[n/2]$. This motivates our research in this paper. We study the asymptotic distribution of the number of matchings of size $\ell$ in $G(n,p)$, denoted by $X_{n,\ell}$, for every $1 \leq \ell \leq [n/2]$. In particular, we prove that $X_{n,\ell}$ is asymptotically normal if $\ell = o(n\sqrt{p})$ and is asymptotically log-normal if $\ell = \Omega(n\sqrt{p})$. This holds for all $p$ such that $1-p = \Omega(1)$ and $n^{1/8-\epsilon}p \to \infty$, where $\epsilon > 0$ is an arbitrarily small constant. To our best knowledge, this is the first paper that studies the distribution of the number of copies of a subgraph whose order is between constant and $n$ in $G(n,p)$.

This same phenomenon of the transition of limiting distributions of a certain subgraph count as the size of the subgraph increases has been observed and studied in another well-known random graph space: the random $d$-regular graphs. There is a classical result by Bollobás [3] and Wormald [19] stating that the distributions of the numbers of short cycles (cycles with constant sizes) in a random $d$-regular graph are asymptotically Poisson, known as the Poisson paradigm [1], whereas it was observed later by Robinson and Wormald [16,17] that the number of Hamilton cycles is determined by the numbers of short cycles. Janson [10] proved that the logarithm of the number of Hamilton cycles can be expressed as the linear combination of a sequence of independent Poisson variables, based on the results in [16,17]. Garmo [8] filled the gap and determined the distribution of all long cycles, whose sizes vary from constant to $n$ (i.e. the Hamiltonian cycles). His result also describes the critical point (of the size of the cycles), at which the distribution of the number of the cycles changes from a linear combination of independent Poisson variables to the exponential of that form, the same as what was described in [10].

Note that the proof of our main theorem is not just a generalisation of the proofs in [18,9,5,6]. In fact, we use very different approaches and new techniques. We do apply basic tools that also appear in [18,6] to show that a sequence of distributions converges to normal or log-normal. Our proof consists of three parts. In the first part, we study the subcritical case, where $\ell = o(n\sqrt{p})$. The second part deals with $\ell$ such that $\ell = \Omega(n\sqrt{p})$ but $\ell$ is not too close to $n/2$, whereas the last part focuses on the near-perfect matchings, where $\ell$ is very close to $n/2$ (i.e. $\ell = n/2 - O(n^\alpha)$ for some $0 < \alpha < 1$). The proof techniques and tools used in these three parts are different. In the first part, we will use the method of moments [11] to show that the distribution of $X_{n,\ell}$ is asymptotically
normal. This same method was also used in Ruciński’s proof for constant \( \ell \). However, the method of moments cannot be used to prove distributions that are not uniquely determined by its moments, for instance, the log-normal distribution. (For more details on the problem of moments, we refer the reader to [2].) For this reason, we will use another theorem from [6], known as the log-normal paradigm, as a basic tool to prove the second and third parts, equipped with the switching method (described below). The proof for the third part is a generalisation of the proof in [5] for the perfect matchings, whereas the switchings used in the second part are very different.

The switching method was first introduced by McKay [12] to enumerate (sparse) graphs with given degree sequences. In general, the method defines a set of switching operations that map graphs in a set \( A \) to graphs in another set \( B \). By computing the number of switchings from \( A \) to \( B \) and the inverse switchings from \( B \) to \( A \), we can estimate the ratio \( |A|/|B| \) in some cases quite precisely. This method has been widely used to estimate the probability that a multigraph generated by the configuration model [4] is simple (e.g. see [13,14,7]) or to estimate probabilities of certain events (e.g. see [15]).

Applying the log-normal paradigm in [6, Theorems 1 and 3] requires a close analysis of the set of ordered pairs of \( \ell \)-matchings \((M_1, M_2)\) that share exactly \( j \) edges, for \( j \) in a certain range. The analysis in [5] (for perfect matchings) is based on an argument using the switching method and that proof easily extends to our proof for the near-perfect matchings. Compared with the case of the near-perfect matchings, the difference in the proof for the second part (for large \( \ell \) but not near-perfect matchings) lies in the additional effort to analyse the typical values of the number of vertices incident to both matchings in the pair \((M_1, M_2)\).

In the proof for the subcritical case \( \ell = o(n\sqrt{p}) \), in order to apply the method of moments, we need to compute the \( k \)-th central moment for each integer \( k \geq 1 \), which requires a close study of the graph structure composed by the union of \( k \) (not necessarily distinct) \( \ell \)-matchings. With an unusual use of the switching method (unlike in [12–14] and most other work that uses the switching method, in which usually a small number of edges are switched, we may switch \( o(n) \) edges in a single step), we will characterise the graph structure that leads the contribution to the \( k \)-th central moment. As shown in Lemma 9 in Section 3.1, for each even \( k \), the leading structure is \( k/2 \) edge-disjoint kissing pairs; whereas for odd \( k \), the leading structure is \((k - 3)/2 \) edge-disjoint kissing pairs together with a chained triple or a flower with 3 petals. (The terminology of kissing pairs, chained triples and flowers are defined in Section 3.1 and an example is given in Fig. 1.) We think this is the first time that the switching method is used to determine certain graph structures. These leading structures were proved by Ruciński for constant \( \ell \) (with a different approach), but the use of the switchings allows us to derive a proof for all \( \ell = o(n\sqrt{p}) \).

In this paper, we assume that \( 1 - p = \Omega(1) \) and \( p \geq n^{-1/8+\epsilon} \) for some small constant \( \epsilon > 0 \). In fact, we only assume \( 1 - p = \Omega(1) \) and \( p = \omega(n^{-2}) \) for the subcritical case. The case where \( 1 - p \to 0 \) is less interesting as there is less “randomness”, and this condition is indeed necessary for the supercritical case since, otherwise, the limiting distribution
of the number of perfect matchings (assuming that \( n \) is even) will be normal instead of log-normal (see [5, Theorem 2.3]). The case \( p = O(n^{-2}) \) is also less interesting as in this case there will be bounded number of edges present, pairwise vertex-disjoint, with probability going to 1. The asymptotic distribution function of \( X_{n,\ell} \) can be explicitly formulated and it is easy to see that \( X_{n,\ell} \) is not Poisson convergent unless \( \ell = 1 \). This agrees with the result by Ruciński [18, Theorem 1]. In that sense, our result covers almost all interesting values of \( p \). For the supercritical case, we only use the condition \( p \geq n^{-1/8+\epsilon} \) for values of \( \ell = n/2 - O(n^{7/8+\epsilon}) \) (see Theorem 6 and the remark below that). In the proof for other values of \( \ell \), we only assume that \( p = \omega(n^{-1/2}) \). In fact, \( p = \omega(n^{-1/2}) \) is likely to be another necessary condition in the supercritical case since a result by Janson in [9] implies that the hypotheses in the tool [6, Theorems 1 and 3] that we use will no longer be satisfied (for \( \ell = n/2 \)). We conjecture that the condition \( p \geq n^{-1/8+\epsilon} \) in our main theorem can be weakened to \( p = \omega(n^{-1/2}) \). The distribution of \( X_{n,\ell} \) in the supercritical case for \( p = O(n^{-1/2}) \) remains open.

2. Main results

An \( \ell \)-matching is a matching with \( \ell \) edges. Let \( X = X_{n,\ell} \) denote the number of subgraphs of \( G(n, p) \) that are isomorphic to an \( \ell \)-matching. Throughout the paper, let \( N = \binom{n}{\ell} \) and define \( m!! \) to be \( \prod_{i=0}^{\lceil (m-1)/2 \rceil} (m-2i) \) for any real number \( m \geq 1 \). Then, the number of \( \ell \)-matchings in the complete graph \( K_n \) is

\[
\binom{n}{2\ell}(2\ell-1)!! = \binom{n}{2\ell}\frac{(2\ell)!}{2^{\ell!}}. \tag{2.1}
\]

Let \( \lambda_{n,\ell} := \mathbb{E}X_{n,\ell} \) and \( \sigma_{n,\ell} := \sqrt{\text{Var}(X_{n,\ell})} \). Then, obviously,

\[
\lambda_{n,\ell} = \binom{n}{2\ell}\frac{(2\ell)!}{2^{\ell!}}p^{\ell}. \]

Define

\[
\bar{\sigma} := \bar{\sigma}_{n,\ell} = \left( \ell \binom{n}{2\ell} \binom{n-2}{2\ell-2} (2\ell-1)!!(2\ell-3)!!(p^{2\ell-1} - p^{2\ell}) \right)^{1/2}. \tag{2.2}
\]

We will show that \( \sigma_{n,\ell} \sim \bar{\sigma}_{n,\ell} \). Moreover, we will prove the following theorem about the central moments of \( X_{n,\ell} \).

Theorem 1. Suppose that \( 1 - p = \Omega(1) \). Then, for every positive \( \ell = \ell(n) = o(n^{1/2}) \) and for every fixed integer \( k \geq 2 \),

\[
\mathbb{E}((X_{n,\ell} - \lambda_{n,\ell})^k) = \begin{cases} (1 + o(1))(k-1)!!\bar{\sigma}^k, & \text{if } k \text{ is even;} \\ o(\bar{\sigma}^k), & \text{if } k \text{ is odd.} \end{cases}
\]
By Theorem 1 and using the method of moments (Theorem 7 below), we immediately have the following theorem for the subcritical case.

**Theorem 2.** Suppose that $1 - p = \Omega(1)$. For every positive $\ell = \ell(n) = o(n\sqrt{p})$,

$$
\frac{X_{n,\ell} - \lambda_{n,\ell}}{\sigma_{n,\ell}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{as } n \to \infty,
$$

where $\mathcal{N}(0, 1)$ is the standard normal distribution.

**Remark.** Note that condition $p = o(n^{-2})$ is implicit in Theorems 1 and 2 so that $\ell = o(n\sqrt{p})$ can be satisfied by some positive integer $\ell$.

The following result gives the asymptotic distribution of $X_{n,\ell}$ in the supercritical case.

**Theorem 3.** Let $0 < \epsilon < 1/8$ be an arbitrarily small constant. Suppose that $1 - p = \Omega(1)$ and $p \geq n^{-1/8+\epsilon}$. Then, for every positive $\ell = \ell(n) = \Omega(n\sqrt{p})$,

$$
\frac{\ln(e^{\beta_{n,\ell}^2/2}X_{n,\ell}/\lambda_{n,\ell})}{\beta_{n,\ell}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{as } n \to \infty,
$$

where $\beta_{n,\ell} = \ell\sqrt{(1 - p)/pN}$.

Immediately, we have the following corollary of Theorems 2 and 3.

**Corollary 4.** Let $0 < \epsilon < 1/8$ be fixed. Suppose that $1 - p = \Omega(1)$ and $p \geq n^{-1/8+\epsilon}$. For any integer $\ell = \ell(n) \in [1, n/2]$,

(i) if $\ell = o(n\sqrt{p})$, then $X_{n,\ell}$ is asymptotically normally distributed;

(ii) if $\ell = \Omega(n\sqrt{p})$, then $X_{n,\ell}$ is asymptotically log-normally distributed.

**Theorem 5.** Let $\alpha \in (7/8, 1)$ be fixed and suppose that $np \to \infty$ and $1 - p = \Omega(1)$. Then, for every positive $\ell = \ell(n) = \Omega(n\sqrt{p})$ such that $\ell \leq n/2 - n^\alpha$ and $\ell^3 = o(n^4 p^2)$, we have

$$
\frac{\ln(e^{\beta_{n,\ell}^2/2}X_{n,\ell}/\lambda_{n,\ell})}{\beta_{n,\ell}} \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } n \to \infty,
$$

where $\beta_{n,\ell} = \ell\sqrt{(1 - p)/pN}$ and $\mathcal{N}(0, 1)$ is the standard normal distribution.
Theorem 6. Let \( \alpha \in (1/2, 1) \) be fixed. Suppose that \( pn^{1-\alpha} \to \infty \) and \( 1 - p = \Omega(1) \). Then, for every positive \( \ell = \ell(n) = n/2 - O(n^\alpha) \),

\[
\frac{\ln(e^{\beta_n,\ell/2}X_{n,\ell}/\lambda_{n,\ell})}{\beta_n,\ell} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty,
\]

where \( \beta_{n,\ell} = \ell\sqrt{(1-p)/pN} \) and \( N(0,1) \) is the standard normal distribution.

Remark. Theorems 5 and 6 deal with the case \( \ell = \Omega(n\sqrt{p}) \). Note that the condition \( \ell^3 = o(n^4p^2) \) in Theorem 5 is weaker than the condition \( p^2n \to \infty \). Indeed, assuming \( p^2n \to \infty \), we have that \( \ell^3 \leq n^3 = o(n^3 \cdot p^2n) \). Thus, the condition \( p \geq n^{-1/8+\epsilon} \) in Theorem 3 is only used while applying Theorem 6 (by taking \( \alpha = 7/8 + \epsilon \)) and it is likely that it may be weakened to \( p^2n \to \infty \), as we conjectured in the introduction.

3. Proof of Theorems 1 and 2

Theorem 2 follows from Theorem 1 and the following theorem, known as the method of moments.

Theorem 7. (See Corollary 6.3 in \([11]\).) If \( Z_1, Z_2, \ldots \) are random variables with finite moments and \( a_n \) are positive numbers such that, for fixed integer \( k \geq 2 \), as \( n \to \infty \),

\[
\mathbb{E}((Z_n - \mathbb{E}Z_n)^k) = \begin{cases} (k-1)!!a_n^k + o(a_n^k), & \text{if } k \text{ is even;} \\ o(a_n^k), & \text{if } k \text{ is odd}; \end{cases}
\]

then \( (Z_n - \mathbb{E}Z_n)/\sqrt{\text{Var}(Z_n)} \xrightarrow{d} N(0,1) \).

We proceed to prove Theorem 1. Let \( \mathcal{M} = \{M_1, \ldots, M_s\} \) be the set of \( \ell \)-matchings in \( K_{[n]} \), where \( K_{[n]} \) is the complete graph on \([n]\). Thus, by (2.1), \( s = \binom{n}{2\ell}/(2\ell - 1)!! \). Let \( H_\mathcal{M} \) denote the graph on \( \mathcal{M} \) such that a matching \( M \) is adjacent to another matching \( M' \) if and only if \( M \cap M' \neq \emptyset \). Let \( k \geq 1 \) be any fixed integer. For any \( (i_1, \ldots, i_k) \in [s]^k \), let \( H_\mathcal{M}(i_1, \ldots, i_k) \) be the subgraph of \( H_\mathcal{M} \) induced by \( \{M_{i_1}, \ldots, M_{i_k}\} \) and let \( C(i_1, \ldots, i_k) \) denote the set of components of \( H_\mathcal{M}(i_1, \ldots, i_k) \). For \( C \in C(i_1, \ldots, i_k) \), let \( \tilde{n}_C = |\{j : M_{i_j} \in V(C)\}| \) and \( \tilde{m}_C = |\bigcup_{M \in C} M| \). That is, \( \tilde{n}_C \) is the number of matchings in \( C \) (counting repetitions), whereas \( \tilde{m}_C \) counts edges in the union of the matchings in \( C \). We will use \( i \) to denote \( (i_1, \ldots, i_k) \) and \( C(i) \) to denote \( C(i_1, \ldots, i_k) \).

Two matchings are called a kissing pair if they share exactly one edge. An ordered triple of matchings \( \{M_1, M_2, M_3\} \) is called a chained triple if \( |M_1 \cap M_2| = 1 \) and \( |M_2 \cap M_3| = 1 \) and \( |M_1 \cap M_3| = 0 \). A set of matchings \( \{M_1, \ldots, M_t\} \) of size \( t \) is called a flower with \( t \) petals if there exists an edge \( e \) such that \( M_i \cap M_j = \{e\} \) for any distinct \( i, j \in [k] \). Hence, a flower with two petals is a kissing pair.

Let \( \mathcal{K} \) denote the subset of \([s]^k\) such that \( i \in \mathcal{K} \) if each component \( C \in C(i) \) satisfies \( \tilde{n}_C \geq 2 \). Let \( \mathcal{K}' \) be the subset of \( \mathcal{K} \) such that \( i \in \mathcal{K}' \) if...
Fig. 1. Kissing pair, chained triple and flower with 3 petals. Each dot represents an edge and each ellipse represents a matching.

(a) $|\mathcal{C}(i)| = \lfloor k/2 \rfloor$ (and so $\tilde{n}_C \in \{2, 3\}$ for every $C \in \mathcal{C}(i)$);
(b) if $\tilde{n}_C = 2$ for some $C \in \mathcal{C}(i)$, then $C$ is a kissing pair;
(c) if $\tilde{n}_C = 3$ for some $C \in \mathcal{C}(i)$, then $C$ is a chained triple or it is a flower with 3 petals.

Remark. By the definition of $\mathcal{K}'$, if $k$ is even, then every component of $\mathcal{C}(i)$ for $i \in \mathcal{K}'$ is a kissing pair; whereas if $k$ is odd, every component in $\mathcal{C}(i)$ is a kissing pair except one, which is a chained triple or a flower with 3 petals.

For $i \in [s]$, let $X_i$ be the indicator variable for the event that $M_i \subseteq \mathcal{G}(n, p)$ and let $Y_i = X_i - \mathbf{E}X_i$. Note that $\mathbf{E}X_i = p^\ell$ for all $i \in [s]$. We first estimate $\mathbf{E}(\prod_{j=1}^k Y_{i_j})$, for $i \in \mathcal{K}'$.

Proposition 8. For any $i = (i_1, \ldots, i_k) \in \mathcal{K}'$, if $k$ is even,

$$\mathbf{E}\left(\prod_{j=1}^k Y_{i_j}\right) = (p^{2\ell - 1} - p^{2\ell})^{k/2};$$

if $k$ is odd,

$$\left|\mathbf{E}\left(\prod_{j=1}^k Y_{i_j}\right)\right| \leq (p^{2\ell - 1} - p^{2\ell})^{(k-3)/2} p^{3\ell-2}.$$

Proof. We only give a detailed proof for the case that $k$ is even and the proof for the other case is analogous. By the definition of $\mathcal{K}'$, $\mathcal{C}(i)$ contains $k/2$ components, each of
which is a kissing pair. Without loss of generality, we may assume that \( \{M_{i_{2j-1}}, M_{i_{2j}}\} \), 1 \( \leq j \leq k/2 \) are kissing pairs. Since each kissing pair does not share any edge with other kissing pairs, \( Y_{i_{2j-1}} \) is independent with all other \( Y_{i'} \), \( i' \in i \), except for \( Y_{i_{2j}} \). Hence,

\[
E\left( \prod_{j=1}^{k} Y_{i_j} \right) = \prod_{j=1}^{k/2} E\left( Y_{i_{2j-1}}Y_{i_{2j}} \right) = E\left( Y_{i_1}Y_{i_2} \right)^{k/2}.
\]

Since \( M_{i_1} \) and \( M_{i_2} \) share exactly one edge, we have \( |M_1 \cap M_2| = 2\ell - 1 \) and so

\[
E(Y_{i_1}Y_{i_2}) = E(X_{i_1}X_{i_2}) - p^{2\ell} = p^{2\ell-1} - p^{2\ell}.
\]

This completes the proof for the case that \( k \) is even. For odd \( k \), we can easily show that, if the component with three matchings is a chained triple, then

\[
E\left( \prod_{j=1}^{k} Y_{i_j} \right) = (p^{2\ell-1} - p^{2\ell})^{(k-3)/2}p^{3\ell-2}(1-p)^2,
\]

and, if the component with three matchings is a flower with 3 petals, then

\[
E\left( \prod_{j=1}^{k} Y_{i_j} \right) = (p^{2\ell-1} - p^{2\ell})^{(k-3)/2}p^{3\ell-2}(1-3p + 2p^2).
\]

Since \( 0 \leq (1-p)^2 \leq 1 \) and \( |1-3p + 2p^2| \leq 1 \), the inequality in the lemma follows. \( \square \)

The following lemma shows that the leading contribution to the \( k \)-th central moment of \( X_{n,\ell} \) is from graph structures in \( K' \).

**Lemma 9.** Suppose that \( 1-p = \Omega(1) \) and \( \ell^2 = o(n^2p) \). For every fixed even \( k \in \mathbb{N} \) with \( k \geq 2 \),

\[
E\left( (X - EX)^k \right) = \sum_{i \in K} \sum_{j=1}^{k} E\left( \prod_{j=1}^{k} Y_{i_j} \right) \sim \sum_{i \in K'} \sum_{j=1}^{k} E\left( \prod_{j=1}^{k} Y_{i_j} \right) = |K'| (p^{2\ell-1} - p^{2\ell})^{k/2}; \quad (3.1)
\]

and, for every odd \( k \in \mathbb{N} \) with \( k \geq 3 \),

\[
|E\left( (X - EX)^k \right)| \leq \sum_{i \in K} \left| E\left( \prod_{j=1}^{k} Y_{i_j} \right) \right| \leq (1+o(1))|K'| (p^{2\ell-1} - p^{2\ell})^{(k-3)/2}p^{3\ell-2}. \quad (3.2)
\]

**Remark.** Note that \( (p^{2\ell-1} - p^{2\ell})^{(k-3)/2}p^{3\ell-2} \) in (3.2) is an upper bound of \( |E\left( \prod_{j=1}^{k} Y_{i_j} \right)| \) for any \( i \in K' \) by Proposition 8. In fact, it is possible to strengthen (3.2) to
\[ |E((X - EX)^k)| \leq (1 + o(1)) \sum_{i \in \mathcal{C}'} |E(\prod_{j=1}^k Y_{ij})| \] with a slightly more delicate proof. But, an upper bound as in (3.2) is sufficient and it allows us to present a slightly simpler proof.

We leave Lemma 9 to be proved in Section 3.1. Now we complete the proof of Theorem 1 by assuming Lemma 9.

**Proof of Theorem 1.** Suppose that \( k \) is even. By Lemma 9, it suffices to show that \( |\mathcal{C}'|(p^{2\ell - 1} - p^{2\ell})^{k/2} \sim (k - 1)!|\mathcal{T}|. \) Recall that \( i \in \mathcal{C}' \) if \( |C(i)| = k/2 \) and each \( C \in C(i) \) is a kissing pair. First we evaluate the size of the set \( T \) of \((k/2)\)-uples \( ((i'_1, i''_1), \ldots, (i'_{k-1}, i''_{k-1})) \in ([s] \times [s])^{k/2} \) such that, for each pair \((i'_j, i''_{j+1})\) with odd \( j \in [k - 1], \) we have that \( M_{i'_j} \) and \( M_{i''_{j+1}} \) are a kissing pair and \(|M_{i'_j} \cap M_{i''_{j+1}}| = 0 \) and \(|M_{i''_{j+1}} \cap M_{i'_j}| = 0 \) for all \( t < j. \) Given any \( i' \in T \) and any perfect matching \( P \) of \([k], \) we obtain a \( k \)-tuple \( i \in \mathcal{C}' \) as follows. Order the edges in \( P \) as \((u_1v_1, \ldots, u_{k/2}v_{k/2})\) in a way such that, for every \( j \in [k/2], u_j < v_j \) and \( u_j < u_{j'} \) for any \( j' > j. \) Then set \( i_{uv} = i''_{2j-1} \) and \( i_{vu} = i'_{2j} \) for all \( 1 \leq j \leq k/2. \) Each \( i \in \mathcal{C}' \) is generated by a unique pair \((i', P), \) where \( i' \in T \) and \( P \) is a perfect matching on \([k]. \) Since there are exactly \((k - 1)! \) such matchings, we have that

\[ |\mathcal{C}'| = (k - 1)!|\mathcal{T}|. \] (3.3)

Note that \( |V(H_M)| = s. \) Recall from (2.1) that \( s = \binom{n}{2l}(2l - 1)!! \). Note also that \( H_M \) is a regular graph. Let \( D \) denote the degree of any vertex in \( H_M \) and let \( d \) denote the number of \( \ell \)-matchings with exactly one edge in common with a given fixed \( \ell \)-matching. Let \( \Delta_r \) denote the number of \( \ell \)-matchings containing a given \( r \)-matching. We have that

\[ D \leq \ell \Delta_1 = \ell \left( \frac{n - 2}{2l - 2} \right) (2l - 3)!! = O \left( s \frac{\ell^2}{n^2} \right) = o(s) \] (3.4)

since \( \ell^2/n^2 = o(1). \) Moreover, by the Inclusion–Exclusion Principle,

\[ d \leq \ell \Delta_1 \quad \text{and} \quad d \geq \ell \Delta_1 - \left( \frac{\ell}{2} \right) \Delta_2 = \ell \left( \frac{n - 2}{2l - 2} \right) (2l - 3)!! - \left( \frac{\ell}{2} \right) \left( \frac{n - 4}{2l - 4} \right) (2l - 5)!! \sim \ell \Delta_1 \] (3.5)

since \( \left( \frac{\ell}{2} \right) \Delta_2/(\ell \Delta_1) = O(\ell^2/n^2) = o(1). \) Thus, we have shown that \( d \sim \ell \Delta_1. \) Suppose we already chose \((i'_1, i''_1), \ldots, (i'_{j-2}, i''_{j-2}), (i'_{j-1}, i''_{j-1}) \). We compute the number of choices for \((i'_j, i''_{j+1})\). We have at most \( s \) choices for \( i'_j \) and, using (3.4), at least \( s - kD \sim s \) choices. Next, we estimate the number of choices for \( i''_{j+1}. \) The matching \( M_{i''_{j+1}} \) must be chosen among the ones that have exactly one edge in common with \( M_{i'_j} \). Hence, the number of choices is at most \( d. \) On the other hand, a matching containing exactly one edge \( e_1 \) in common with \( M_{i'_j} \) cannot be chosen as \( M_{i''_{j+1}} \) only if it has another edge \( e_2 \) in common with some matching \( M_{i'_t} \) with \( t < j. \) There are \( \ell \) choices for \( e_1 \) and at most \( k\ell \) choices for \( e_2. \) The
number of $\ell$-matchings containing $e_1$ and $e_2$ is at most $\Delta_2$. Thus, the number of choices for $M_{i_j+1}$ is at least $d - k\ell^2 \Delta_2$. By (3.5), $d - k\ell^2 \Delta_2 \sim d$ and so the number of choices for $i'_{j+1}$ is asymptotically $d$. Thus, $|T| \sim (sd)^{k/2}$, and we are done by (3.4), (3.5) and (3.3), finishing the proof for even $k$.

Now suppose that $k$ is odd. By Lemma 9, (2.2), and (3.5), it suffices to show that
\[
|K'| \left( p^{2\ell - 1} - p^{2\ell} \right)^{(k-3)/2} p^{3\ell - 2} = o \left( (sd(p^{2\ell - 1} - p^{2\ell}))^{k/2} \right). \tag{3.6}
\]
Recall that $i \in K'$ if $C(i)$ contains $(k - 1)/2$ components, in which $(k - 3)/2$ are kissing pairs and the other one is a chained triple or a flower with 3 petals. Similarly to the previous case when $k$ is even, we consider the set $T$ of $[k/2]$-tuples $((i'_1, i'_2), \ldots, (i'_{k-2}, i'_{k-3}), (i'_{k-2}, i'_{k-1}, i'_k))$ with every $i'_j \in [s]$, such that all $\{M_{i'_{2j-1}}, M_{i'_{2j}}\}$, $1 \leq j \leq (k-3)/2$, are kissing pairs and $(M_{i'_{k-2}}, M_{i'_{k-1}}, M_{i'_k})$ is a chained triple or a flower with 3 petals and all kissing pairs and chained triples (or the flower) are edge disjoint. Similar to the previous argument, we have $|K'| = O(|T|)$ since $k$ is fixed. The number of choices for the first $(k - 3)/2$ pairs in $T \in T$ is at most $(sd)^{(k-3)/2}$ and the number of choices for the triple $(i'_{k-2}, i'_{k-1}, i'_k)$ is at most $2sd^2$ (at most $sd^2$ choices for a chained triple and at most $sd^2$ choices for a flower with 3 petals). Thus, we have that $|K'| = O\left( (sd)^{k/2} \cdot d / \sqrt{sd} \right)$ and, in order to prove (3.6), it suffices to show that $dp^{3\ell - 2} = o\left( \sqrt{sd(p^{2\ell - 1} - p^{2\ell})^3} \right)$. Indeed, using (3.4) and (3.5),
\[
\frac{d^2 p^{6\ell - 4}}{(p^{2\ell - 1} - p^{2\ell})^3 sd} = O \left( \frac{dp^{6\ell - 4}}{sp^{6\ell - 3}(1 - p)^3} \right) = O \left( \frac{\ell^2}{n^2 p(1 - p)^3} \right),
\]
which goes to zero since $\ell^2/(n^2 p) = o(1)$ and $1 - p = \Omega(1)$.

3.1. Proof of Lemma 9

In this section, we assume $\ell = o(n \sqrt{p})$ and $1 - p = \Omega(1)$ and $k \in \mathbb{N}$ is fixed, which are the hypotheses of Lemma 9. Recall that for $C \in C(i)$ we defined $\tilde{n}_C = |\{j : M_{ij} \in V(C)\}|$ and $\tilde{m}_C = |\bigcup_{M \in C} M|$. For each $C \in C(i)$ where $i \in [s]^k$, define
\[
Y_C = \prod_{j : M_{ij} \in C} Y_{ij},
\]
and define
\[
\tilde{p}_C = \begin{cases} 
p^{\tilde{m}_C} - p^{\tilde{n} C \ell}, & \text{if } \tilde{n}_C \leq 2; \\
p^{\tilde{m}_C}, & \text{otherwise};
\end{cases}
\]
define then
\[
\tilde{p}(i) = \prod_{C \in C(i)} \tilde{p}_C.
\]
Lemma 10. For every \( i \in [s]^k \) and any \( C \in C(i) \),

\[ |EY_C| \leq 2^{\mathbf{n}_C} \mathbf{p}_C. \]

Moreover, if \( \mathbf{n}_C \leq 2 \), then \( EY_C = \mathbf{p}_C \). In particular, \( \mathbf{p}_C = 0 \) if \( \mathbf{n}_C = 1 \).

Proof. This follows from the fact that, for any subset \( I \in [k] \), we have that

\[
E \left( \prod_{j=1}^{k} Y_{ij} \right) = E \left( \prod_{j \in I} (X_i - p^\ell) \right) = \sum_{I' \subseteq I} (-1)^{|I \setminus I'|} p^{|I \setminus I'|} p^{|\cup_{j \in I'} M_i|}
\]

\[
\leq 2^{|I|} p^{|\cup_{i \in I} M_i|}
\]

since there are \( 2^{|I|} \) choices for \( I' \) and, for each \( I' \), we have

\[
|\cup_{i \in I} M_i| \leq |\cup_{i \in I'} M_i| + |\cup_{i \in I \setminus I'} M_i| \leq |\cup_{i \in I'} M_i| + \ell |I \setminus I'|.
\]

The conclusion for the case where \( \mathbf{n}_C \leq 2 \) follows directly from the definition of \( \mathbf{p}_C \). \( \square \)

Proof of Lemma 9. For any \( i \in [s]^k \),

\[
E \left( \prod_{j=1}^{k} Y_{ij} \right) = \prod_{C \in C(i)} EY_C,
\]

(3.7)

since \( Y_C, C \in C(i) \) are independent random variables. If \( C(i) \) has a component \( C \) with \( \mathbf{n}_C = 1 \), then it follows directly from (3.7) and Lemma 10 that \( E \left( \prod_{j=1}^{k} Y_{ij} \right) = 0 \). Thus, we have \( E((X - E(X))^k) = \sum_{i \in \mathcal{K}} E \left( \prod_{j=1}^{k} Y_{ij} \right) \). By Proposition 8 and the definition of \( \mathbf{p}_C \), we have that

\[
E \left( \prod_{j=1}^{k} Y_{ij} \right) = \mathbf{p}(i) \text{ if } k \text{ is even; } E \left( \prod_{j=1}^{k} Y_{ij} \right) \leq \mathbf{p}(i) \text{ if } k \text{ is odd}.
\]

(3.8)

Next, we prove that

\[
\sum_{i \in \mathcal{K}} \mathbf{p}(i) \sim \sum_{i \in \mathcal{K}'} \mathbf{p}(i).
\]

(3.9)
Note that Lemma 9 then follows from the above and the definition of \( \hat{p}(i) \). Indeed, for even \( k \), we have that
\[
\sum_{i \in \mathcal{K}} E\left( \prod_{j=1}^{k} Y_{i,j} \right) = \sum_{i \in \mathcal{K}'} E\left( \prod_{j=1}^{k} Y_{i,j} \right) + \sum_{i \in \mathcal{K}\setminus\mathcal{K}'} E\left( \prod_{j=1}^{k} Y_{i,j} \right) = \sum_{i \in \mathcal{K}'} \hat{p}(i) + O\left( \sum_{i \in \mathcal{K}\setminus\mathcal{K}'} \hat{p}(i) \right) \sim \sum_{i \in \mathcal{K}'} \hat{p}(i),
\]
where the second equality follows by (3.8) and Lemma 10, and the last relation holds by (3.9). Similarly, for odd \( k \),
\[
\left| \sum_{i \in \mathcal{K}} E\left( \prod_{j=1}^{k} Y_{i,j} \right) \right| \leq \sum_{i \in \mathcal{K}'} E\left( \prod_{j=1}^{k} Y_{i,j} \right) + \sum_{i \in \mathcal{K}\setminus\mathcal{K}'} E\left( \prod_{j=1}^{k} Y_{i,j} \right) \leq \sum_{i \in \mathcal{K}'} \hat{p}(i) + O\left( \sum_{i \in \mathcal{K}\setminus\mathcal{K}'} \hat{p}(i) \right) \sim \sum_{i \in \mathcal{K}'} \hat{p}(i).
\]

Thus, it suffices to show (3.9). Let \( \mathcal{K}(x_1, \ldots, x_k) \) be the restriction of \( \mathcal{K} \setminus \mathcal{K}' \) to the tuples \((i_1, \ldots, i_k)\) such that, for every \( 1 \leq j \leq k \), the number of edges \( e \) with \(|\{j' : e \in M_{i,j'}\}| = j \) is \( x_j \), that is, the number of edges that appear in exactly \( j \) matchings is \( x_j \). For any such tuple \((x_1, \ldots, x_k)\), we have \( \sum_{i=1}^{k} ix_i = k\ell \). Since \( i \in \mathcal{K} \), the number of edges contained only in \( M_{i,j} \) is at most \( \ell - 1 \) for every \( 1 \leq j \leq k \). It follows immediately that \( x_1 \leq k(\ell - 1) \) and so we must have \( \sum_{i \geq 2} ix_i \geq k \). Let \( \mathcal{X} = \{(x_1, \ldots, x_k) \in \mathbb{N}_k : \sum_{i \geq 1} ix_i = k\ell, \sum_{i \geq 2} ix_i \geq k \} \). In order to prove (3.9), we will define switchings from \( \mathcal{K}(x_1, \ldots, x_k) \) to \( \mathcal{K}' \) for every \((x_1, \ldots, x_k) \in \mathcal{X} \) and thereby we prove that the contribution to (3.9) from \( \mathcal{K} \) is dominated by the contribution from \( \mathcal{K}' \). We discuss the cases when \( k \) is odd and even separately.

We first prove (3.9) for even \( k \). Let \((x_1, \ldots, x_k) \in \mathcal{X} \). Let \( i = (i_1, \ldots, i_k) \in \mathcal{K}(x_1, \ldots, x_k) \). We define the following switching from \( i \) to \( k \)-tuples in \( \mathcal{K}' \) (see Fig. 2). For each \( j \), let \( I_j \) denote the set of edges that \( M_{i,j} \) shares with \( \cup_{j' \neq j} M_{i,j'} \).

1. (Delete the shared edges) For each \( j \in [k] \), let \( M'_{i,j} := M_{i,j} \setminus I_j \).
2. (Obtain pairwise disjoint partial matchings each of size \( \ell - 1 \)) For each \( j \in [k] \), choose edges \( a_1^{(j)}, \ldots, a_{|I_j| - 1}^{(j)} \), one after the other, so that \( M'_{i,j} \cup \{a_1^{(j)}, \ldots, a_{|I_j| - 1}^{(j)}\} \) is an \((\ell - 1)\)-matching and, for every \( r \), the edge \( a_r^{(j)} \) is not in \( \left( \bigcup_{j' = 1}^{k} M'_{i,j'} \right) \cup \left( \bigcup_{j'<j} a_1^{(j')}, \ldots, a_{|I_{j'}|-1}^{(j')} \right) \). Let \( M''_{i,j} := M'_{i,j} \cup \{a_1^{(j)}, \ldots, a_{|I_j| - 1}^{(j)}\} \).
3. (Build kissing pairs) Choose a perfect matching \( P \) in \([k]\). (Here we are choosing the pairs of matchings that will form a kissing pair in \( \mathcal{K}' \).)
4. (Choose shared edges in kissing pairs) Let \( e_1, \ldots, e_{k/2} \) be an enumeration of the edges in \( P \). For each \( r \in [k/2] \), let \( j \) and \( j' \) denote the ends of \( e_r \) and choose an

\[
\sum_{i \in \mathcal{K}} E\left( \prod_{j=1}^{k} Y_{i,j} \right) = \sum_{i \in \mathcal{K}'} E\left( \prod_{j=1}^{k} Y_{i,j} \right) + \sum_{i \in \mathcal{K}\setminus\mathcal{K}'} E\left( \prod_{j=1}^{k} Y_{i,j} \right) = \sum_{i \in \mathcal{K}'} \hat{p}(i) + O\left( \sum_{i \in \mathcal{K}\setminus\mathcal{K}'} \hat{p}(i) \right) \sim \sum_{i \in \mathcal{K}'} \hat{p}(i).
\]
edge \( f_r \) such that \( f_r \notin \left( \bigcup_{a=1}^{k} M''_{i_a} \right) \cup \left( \bigcup_{b=1}^{r-1} f_b \right) \) and both \( M'''_{i_j} := M''_{i_j} \cup \{ f_r \} \) and \( M''''_{i_j} := M'''_{i_j} \cup \{ f_r \} \) are matchings. (Note that \( M'''_{i_j} \) and \( M''''_{i_j} \) are each of size \( \ell \) and they form a kissing pair.)

5. \( \text{(Update the indices)} \) Let \( i''' \) be the new tuple such that for every \( 1 \leq j \leq k \), \( M_{i_j}''' = M_{i_j}'''' \).

Let

\[
L(x_1, \ldots, x_k) = \left( \frac{1}{2} \left( \frac{n}{2} \right) \right)^{k\ell - x_1 - k/2}.
\]

(3.10)

Now we show that, for every \( (x_1, \ldots, x_k) \in \mathcal{X} \) and for every \( i \in \mathcal{K}(x_1, \ldots, x_k) \), the number of applicable switchings defined above for \( i \) is at least \( L(x_1, \ldots, x_k) \), regardless of the choice of \( i \), for all sufficiently large \( n \). In Step 2, given \( r \in [|I_j| - 1] \), we have that any edge with no ends in the set of vertices induced by the set of edges \( (\bigcup_{j'=1}^{k} M'_{i_{j'}}) \cup (\bigcup_{j' < j} \{ a_{i_{j'}}^{(j')} \}, \ldots, a_{|I_{j'}| - 1}^{(j')} \}) \cup (\bigcup_{r < r'} a_r^{(j)}) \) is a possible choice for \( a_r^{(j)} \). Since this set has at most \( k\ell \) edges, it induces at most \( 2k\ell \) vertices and so there are at least \( \left( \binom{n}{2} - 2k\ell n \right)^{|I_j| - 1} \) choices for \( a_r^{(j)} \) in Step 2 and thus we have at least \( \left( \binom{n}{2} - 2k\ell n \right)^{|I_j| - 1} \) choices in Step 2. There are \( k! / ((k/2)!2^{k/2}) \geq 1 \) choices for \( P \) in Step 3. With the same argument as before, given the choice of \( P \) in Step 3, there are at least \( \left( \binom{n}{2} - 2k\ell n \right) \) choices for \( f \) for each pair in \( P \) in Step 4. Since there are \( k/2 \) pairs in \( P \) in total, it follows then that the number of
applicable switchings is at least
\[
\left( \binom{n}{2} - 2kln \right)^{k/2 + \sum_{j=1}^k (|I_j|-1)} = \left( \binom{n}{2} - 2kln \right)^{-k/2 + \sum_{j=1}^k |I_j|} \geq L(x_1, \ldots, x_k), \tag{3.11}
\]
for all sufficiently large \( n \), where we used the fact that \(\ell^2 = o(n^2p)\) and that \(\sum_j |I_j| = k\ell - x_1\) in the last inequality.

Now we describe the inverse switching, which converts \(i = (i_1, \ldots, i_k) \in K'\) to some \(k\)-tuples in \(K(x_1, \ldots, x_k)\).

1. (Choose the number of edges shared by each matching) Choose an integral vector \(r = (r_1, \ldots, r_k)\) so that \(\sum_{j=1}^k r_j = k\ell - x_1\) such that \(r_j \geq 1\) for every \(j\). (In the following steps, we will convert \(i\) to some \(i''\) so that \(M_i''\) contains \(r_j\) shared edges)

2. (Make room for shared edges) For each \(j \in [k]\), let \(f_j\) be the unique edge in \(M_{i_j} \cap (\bigcup_{j' \neq j} M_{i_{j'}})\). Choose \(r_j - 1\) edges one by one without repetition in \(M_{i_j} - f_j\). Let \(M_i'\) be the \((\ell - r_j)\)-matching obtained from \(M_{i_j}\) by deleting these edges and \(f_j\).

3. (Choose shared edges) Choose a set of edges \(X\) of size \(\sum_{i=1}^k x_i\) such that no edge in \(X\) is contained in \(\bigcup_{j=1}^k M_i'\). Partition \(X\) into sets \(X_2, \ldots, X_k\) such that \(|X_i| = x_i\) for every \(i\).

4. (Assign shared edges) Construct a bipartite graph \(Q\) with bipartition \((X, [k])\) so that the degree of each \(e \in X_i\) is \(i\) and the degree of each \(j \in [k]\) is \(r_j\). Let \(M_i''\) be obtained from \(M_i'\) by including the edges in \(X\) that are adjacent to \(j\) in \(Q\).

5. (Update the indices) If each \(M_i''\) is an \(\ell\)-matching, let \(i''\) be the new tuple such that for every \(1 \leq j \leq k\), \(M_i'' = M_{i_j'}\).

Let
\[
U(x_1, \ldots, x_k) = (k\ell - x_1)^{k-1} \ell^{k\ell - x_1 - k} \binom{n}{2} \sum_{r \geq 2} x_r \beta^{k\ell - x_1}, \tag{3.12}
\]
for some constant \(\beta\) to be determined later. We prove that, for every \(k\)-tuple \((x_1, \ldots, x_k) \in X\) and for every \(i = (i_1, \ldots, i_k) \in K'\), the number of inverse switchings that can convert \(i\) to some \(k\)-tuples in \(K(x_1, \ldots, x_k)\) is at most \(U(x_1, \ldots, x_k)\). Note that it is possible for some choices made in Steps 3 and 4, \(M_i''\) may not be an \(\ell\)-matching. But we only need an upper bound for the number of inverse switchings in our case. The number of integer compositions of \(k\ell - x_1\) into \(k\) positive parts is \(\binom{k\ell - x_1 - 1}{k-1}\), and so we have \(\binom{k\ell - x_1 - 1}{k-1}\) choices for the vector \(r\) in Step 1. The number of choices in Step 2 is at most \(\ell^{\sum_{j=1}^k (r_j-1)}\). In Step 3, we have at most \(\binom{n}{|X|}\) choices for \(X\) and \(\binom{|X|}{x_2, x_3, \ldots, x_{k-1}}\) choices for the partition of \(X\). Now we bound the number of choices for \(Q\) in Step 4. This number equals the number of (simple) bipartite graphs \((X, [k])\) with the degrees as in Step 4 (note that the sum of the degrees of vertices in \(X\) is \(k\ell - x_1\)). One can obtain
a bipartite multigraph with the degrees as in Step 4 by the following procedure: replace each vertex in $Q$ with a set of points of size equal to its degree; add a perfect matching between the points arising from vertices in $X$ to points arising from vertices in $[k]$; for each vertex, contract the set of points arising from it. Note that each simple bipartite graph corresponds to $\prod_{i=2}^{k} (d!)^{x_i} \prod_{j=1}^{k} r_j !$ matchings. Moreover, there are $(k\ell - x_1)!$ choices for perfect matchings in the procedure. Thus, after restricting to counting only perfect matchings corresponding to simple bipartite graphs, the number of choices for $Q$ is at most

$$\frac{(k\ell - x_1)!}{\prod_{i=2}^{k} (d!)^{x_i} \prod_{j=1}^{k} r_j !} \leq \frac{(k\ell - x_1)!}{\prod_{j=1}^{k} r_j !} \leq \frac{(k\ell - x_1)!}{k!} \leq \beta^{k\ell - x_1},$$

for a constant $\beta$ (depending only on $k$) by Stirling’s approximation. Thus, the number of inverse switchings applicable on each $i \in \mathcal{K}'$ is at most

$$\binom{k\ell - x_1 - 1}{k - 1} \ell^{\sum_{1 \leq j \leq k}(r_j - 1)} \binom{n}{2} \frac{|X|}{x_2, x_3, \ldots, x_k-1} \beta^{k\ell - x_1}$$

$$\leq (k\ell - x_1)^{k-1} \ell^{k\ell - x_1 - k} \binom{n}{2} \sum_{r \geq 2} x_r \beta^{k\ell - x_1} = U(x_1, \ldots, x_k),$$

where we used that $\sum_{1 \leq j \leq k} r_j = k\ell - x_1$. We will now proceed to bound the ratio

$$\frac{\sum_{i \in \mathcal{K}(x_1, \ldots, x_k)} \hat{p}(i)}{\sum_{i' \in \mathcal{K}'} \hat{p}(i')}$$

for $(x_1, \ldots, x_k) \in \mathcal{X}$. Construct a bipartite multigraph $R$ with bipartition $(\mathcal{K}(x_1, \ldots, x_k), \mathcal{K}')$ such that $i \in \mathcal{K}(x_1, \ldots, x_k)$ and $i' \in \mathcal{K}'$ are adjacent if there is a switching mapping $i$ to $i'$ and the number of edges joining them is the number of such switchings. By (3.11), the degree of any vertex in $\mathcal{K}(x_1, \ldots, x_k)$ is at least $L(x_1, \ldots, x_k)$, and, on the other hand, by (3.13), the degree of any vertex in $\mathcal{K}'$ is at most $U(x_1, \ldots, x_k)$. Moreover, for any $i \in \mathcal{K}(x_1, \ldots, x_k)$ and $i' \in \mathcal{K}'$, we have that

$$\frac{\hat{p}(i)}{\hat{p}(i')} \leq \frac{\prod_{C \in \mathcal{C}(i)} p_{\mathcal{m}}^{|C|}}{(p^{2\ell} - 1 - p^{2\ell})^{k/2}} \leq \frac{p^{\sum_{r=1}^{x_r} x_r}}{p^{k\ell - k/2} (1 - p)^{k/2}} =: p(x_1, \ldots, x_k).$$

Let $N(i)$ denote the set of neighbours of $i$ in the bipartite multigraph $R$. For each $i \in \mathcal{K}(x_1, \ldots, x_k)$, we have then

$$\hat{p}(i) \leq \frac{p(x_1, \ldots, x_k) \sum_{i' \in N(i)} \hat{p}(i')}{|N(i)|} \leq \frac{p(x_1, \ldots, x_k) \sum_{i' \in N(i)} \hat{p}(i')}{L(x_1, \ldots, x_k)}.$$
and so
\[
\sum_{i \in \mathcal{K}(x_1, \ldots, x_k)} \hat{p}(i) \leq \frac{p(x_1, \ldots, x_k)}{L(x_1, \ldots, x_k)} \sum_{i \in \mathcal{K}(x_1, \ldots, x_k)} \sum_{i' \in N(i)} \hat{p}(i') \\
= \frac{p(x_1, \ldots, x_k)}{L(x_1, \ldots, x_k)} \sum_{i' \in K'} |N(i')| \hat{p}(i') \\
\leq \frac{p(x_1, \ldots, x_k)U(x_1, \ldots, x_k)}{L(x_1, \ldots, x_k)} \sum_{i' \in K'} \hat{p}(i').
\]

Recall from (3.10) and (3.12) that
\[
L(x_1, \ldots, x_k) = \left(\frac{1}{2} \binom{n}{2}\right)^{k\ell - x_1 - k/2},
\]
\[
U(x_1, \ldots, x_k) = (k\ell - x_1)^{k-1} \beta^{k\ell - x_1} \binom{n}{2}^{\sum_{r \geq 2} x_r} \beta^{k\ell - x_1}.
\]

Thus,
\[
\sum_{i \in \mathcal{K}(x_1, \ldots, x_k)} \hat{p}(i) \leq \frac{U(x_1, \ldots, x_k)}{L(x_1, \ldots, x_k)} \cdot p(x_1, \ldots, x_k) \\
\leq \frac{\ell^{k\ell - k - x_1}(k\ell - x_1)^{k-1} \beta^{k\ell - x_1} \binom{n}{2}^{\sum_{r \geq 2} x_r}}{(\frac{1}{2} \binom{n}{2})^{k\ell - x_1 - k/2}} \cdot \frac{p^{2n} \sum_{r=1}^n x_r}{(1 - p)^{k/2}} \\
= \frac{\ell^{k\ell - k - x_1}(k\ell - x_1)^{k-1} \beta^{k\ell - x_1} \binom{n}{2}^{\sum_{r \geq 2} x_r}}{(\frac{1}{2} \binom{n}{2} p)^{k\ell - x_1 - k/2}} \cdot \frac{1}{(1 - p)^{k/2}} \\
= O\left(\frac{(2\beta \ell)^{k\ell - k - x_1}(k\ell - x_1)^{k-1}}{\binom{n}{2} p^{k\ell - k - 2 \sum_{i=1}^n x_i}}\right),
\]

where the last equation holds because $1 - p = \Omega(1)$ and $k$ is fixed. Now we will bound
\[
\sum_{(x_1, \ldots, x_k) \in \mathcal{X}} \frac{\sum_{i \in \mathcal{K}(x_1, \ldots, x_k)} \hat{p}(i)}{\sum_{i' \in K'} \hat{p}(i')}.
\]

We partition $\mathcal{X}$ into two sets $\mathcal{X}_1$ and $\mathcal{X}_2$. Let $\mathcal{X}_1$ be subset of $\mathcal{X}$ such that $(x_1, \ldots, x_k) \in \mathcal{X}_1$ if $x_1 \leq k\ell - k - 1$ and let $\mathcal{X}_2$ be subset of $\mathcal{X}$ such that $(x_1, \ldots, x_k) \in \mathcal{X}_2$ if $x_1 = k\ell - k$. Note that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ since $x_1 \leq k\ell - k$ for all $(x_1, \ldots, x_k) \in \mathcal{X}$.

Recall that $\sum_{i=1}^k x_i = k\ell$ for all $(x_1, \ldots, x_k) \in \mathcal{X}$. Obviously, given the value of $x_1$, the number of nonnegative integral vectors $(x_2, \ldots, x_k)$ such that $\sum_{r=1}^k x_r = k\ell - x_1$ is
\[ O(k^{(k\ell-x_1)/2}) \text{, as } \sum_{r=2}^{k} x_r \leq (k\ell - x_1)/2. \] Together with (3.15), this implies that

\[
\sum_{(x_1, \ldots, x_k) \in \mathcal{X}_1} \frac{\sum_{i \in \mathcal{K}(x_1, \ldots, x_k)} \hat{p}(i)}{\sum_{j \in \mathcal{K}'} \hat{p}(j)} = O \left( \sum_{x_1=0}^{k-1} \frac{(2k\beta\ell)^{k-x_1}(k\ell - x_1)^{-1}}{((n/2)p)^{(k\ell-x_1-k)/2}} \right) = O \left( \sum_{x_1=0}^{k-1} \frac{(2k\beta\ell)^{k-x_1}(k\ell - x_1)^{-1}}{((n/2)p)^{(k\ell-x_1-k)/2}} \right).
\]

Let

\[ g(x_1) = \frac{(2k\beta\ell)^{k-x_1}(k\ell - x_1)^{-1}}{((n/2)p)^{(k\ell-x_1)/2-k/2}}. \]

For \( x_1 = k\ell - k - 1 \),

\[ g(x_1) = \frac{(2k\beta\ell)(k+1)^{-1}}{((n/2)p)^{1/2}} = O \left( \frac{\ell}{n\sqrt{p}} \right) = o(1), \]

since \( \ell^2 = o(n^2p) \). Moreover, for \( x_1 < k\ell - k - 1 \), using \( \ell^2 = o(n^2p) \) and \( k\ell - x_1 > k + 1 \),

\[ \frac{g(x_1)}{g(x_1 + 1)} = \frac{(2k\beta\ell)}{((n/2)p)^{1/2}} \cdot \left( \frac{k\ell - x_1}{k\ell - x_1 - 1} \right)^{k-1} \leq \frac{(2k\beta\ell)}{((n/2)p)^{1/2}} \cdot 2^{k-1} = O \left( \frac{\ell}{n\sqrt{p}} \right) = o(1), \]

since \( \ell^2 = o(n^2p) \) and both \( k \) and \( \beta \) are fixed constants. This shows that

\[
\sum_{(x_1, \ldots, x_k) \in \mathcal{X}_1} \frac{\sum_{i \in \mathcal{K}(x_1, \ldots, x_k)} \hat{p}(i)}{\sum_{j \in \mathcal{K}'} \hat{p}(j)} = o(1). \quad (3.16)
\]

Now we deal with the case \( (x_1, \ldots, x_k) \in \mathcal{X}_2 \), i.e., \( x_1 = k\ell - k \). We have that \( \sum_{i=2}^{k} i x_i = k \). This implies \( \sum_{i=2}^{k} x_i \leq k/2 \). But note that the only way \( \sum_{i=2}^{k} x_i = k/2 \), would be \( x_2 = k/2 \) and \( x_1 = 0 \) for \( i \geq 3 \) and \( \mathcal{K}(x_1, \ldots, x_k) \) would be empty (since all such \( i \) are in \( \mathcal{K}' \)) and there is nothing to prove in this case. Thus, we can assume \( \sum_{i=2}^{k} x_i \leq k/2 - 1 \). Using this fact together with that the number of choices for \( (x_2, \ldots, x_k) \in \mathbb{N} \) such that
\[ \sum_{i=2}^{k} ix_i = k \] is \( O(1) \) and (3.15),

\[
\sum_{(x_1, \ldots, x_k) \in X_k} \frac{\sum_{i \in K(k(\ell-k), x_1, \ldots, x_k)} \hat{p}(i)}{\sum_{i' \in K'} \hat{p}(i')} = O \left( \frac{(2\beta \ell)^{k-\ell} - 1}{(\frac{n}{2})^{k-\ell} - \sum_{i=1}^{k} x_i / 2} \right) \]

(3.17)

\[
= O \left( \frac{1}{\binom{n}{2} p} \right) = o(1),
\]

where in the last equality we use the fact that \( n^2 p \to \infty \).

Equation (3.16) and Equation (3.17) imply

\[
\sum_{(x_1, \ldots, x_k) \in K} \frac{\sum_{i \in K(x_1, \ldots, x_k)} \hat{p}(i)}{\sum_{i' \in K'} \hat{p}(i')} = o(1).
\]

Now we have completed the proof of Lemma 9 for even integers \( k \geq 2 \). The proof for odd integers \( k \) is analogous to that for even \( k \), with slightly more complication due to the treatment of the single component that is a chained triple or a flower with 3 petals that would appear in \( C(i) \), where \( i \in K' \) for odd \( k \). Thus, we will only give a sketch of the proof. Slightly different from the case where \( k \) is even, we split \( K \setminus K' \) into \( K_0 \) and \( K_1 \), where \( K_1 \) contains all \( K(x_1, \ldots, x_k) \) with \( x_1 \leq k(\ell - 1) - 1 \), whereas \( K_0 \) corresponds to \( x_1 = k(\ell - 1) \). We also partition \( K' \) into \( K'_0 \) and \( K'_1 \) such that \( K'_0 \) contains all \( i \in K' \) for which the only component in \( C(i) \) having three matchings is a flower with 3 petals (and thus \( K'_1 \) is with respect to the structure of a chained triple). Obviously, if \( i \in K(x_1, \ldots, x_k) \) where \( x_1 = k(\ell - 1) \), then each component \( C \) in \( \cup_{1 \leq j \leq k} M_{i,j} \) is a flower and at least one flower has at least 3 petals because \( k \) is odd. It is not difficult to verify (use switchings to remove all but one flower with 3 petals and form the other partial matchings into kissing pairs; the analysis for these switchings is similar but simpler compared with the above analysis for even \( k \)) that \( \sum_{i \in K_0} \hat{p}(i) = o\left( \sum_{i \in K'_0} \hat{p}(i) \right) \). Next we show that

\[
\sum_{i \in K_1} \hat{p}(i) = o \left( \sum_{i \in K'_1} \hat{p}(i) \right),
\]

(3.18)

which will complete the proof of Lemma 9 for odd \( k \).

Same as in the case of even \( k \), we will define a switching from \( K(x_1, \ldots, x_k) \) to \( K'_1 \) where \( x_1 \leq k(\ell - 1) - 1 \). As the analysis is almost the same to the previous case, we omit the calculations and just describe the switching operation and its inverse. Let \( (i_1, \ldots, i_k) \in K(x_1, \ldots, x_k) \). Similarly to the even case, for each \( j \), let \( I_j \) denote the set of edges that \( M_{i,j} \) shares with \( \cup_{j' \neq j} M_{i,j'} \). Note that, since \( x_1 \leq k(\ell - 1) - 1 \), there exists a matching \( M_{i,j} \) such that \( |I_j| \geq 2 \).
1. (Delete shared edges) For each $j \in [k]$, let $M'_{ij} := M_{ij} \setminus I_j$.
2. (Choose a matching that will share two edges in the chained triple) Choose $t \in [k]$ such that $|M'_{it}| \leq \ell - 2$.
3. (Obtain pairwise disjoint partial matchings each of size $(\ell - 1)$ but one of size $(\ell - 2)$) For each $j \in [k]$, let $h_j = |I_j| - 1$ if $j \neq t$ and $|I_j| - 2$ if $j = t$. For each $j \in [k]$, choose edges $a^{(j)}_1, \ldots, a^{(j)}_{h_j}$, one after the other, so that $M'_{ij} \cup \{a^{(j)}_1, \ldots, a^{(j)}_{h_j}\}$ is an $(\ell - 1)$-matching (or $(\ell - 2)$-matching if $j = t$) and $a^{(j)}_r \notin \bigcup_{j' = 0}^{k} M'_{ij'}$ for every $r$. Let $M''_{ij} := M'_{ij} \cup \{a^{(j)}_1, \ldots, a^{(j)}_{h_j}\}$.
4. (Build kissing pairs and chained triple) Choose a graph $P$ on $[k]$ such that the degree of $t$ is 2 and the degree of any other vertex is 1.
5. (Choose shared edges in kissing pairs and chained triple) Let $e_1, \ldots, e_{(k+1)/2}$ be an enumeration of the edges in $P$. Let $t_1 < t_2$ be such that $e_{t_1}$ and $e_{t_2}$ are the edges incident to $t$. For each $r \in [(k+1)/2] \setminus \{t_2\}$, let $j$ and $j'$ denote the ends of $e_r$ and choose an edge $f_r$ such that $M''_{ij} := M''_{ij} \cup \{f_r\}$ and $M''_{ij'} := M''_{ij'} \cup \{f_r\}$ are matchings and $f_r \notin \bigcup_{a=1}^{k} M''_{ia} \cup \bigcup_{b=1}^{t_2-1} f_b$. For $r = t_2$, let $j$ denote the end of $e_r$ other than $t$ and choose an edge $f_r$ such that $M''_{ij} := M''_{ij} \cup \{f_r\}$ and $M''_{ij'} \cup \{f_r\}$ are matchings and $f_r \notin \bigcup_{a=1}^{k} M''_{ia} \cup \bigcup_{b=t_2}^{n} f_b$. Redefine $M''_{ij}$ by including $f_r$.
6. (Update the indices) Let $i'''$ be the new tuple such that for every $1 \leq j \leq k$, $M''_{ij} = M''_{i''j}$.

Similarly to the even case, the number of applicable switchings for any $i \in \mathcal{K}(x_1, \ldots, x_k)$ is at least

$$\left(\frac{1}{2}\right)^{\sum_j h_j + \frac{k-1}{2} + 1} = \left(\frac{1}{2}\right)^{\sum_i x_i - \frac{k+1}{2}}.$$

Now we describe the inverse switching. Let $(i_1, \ldots, i_k) \in \mathcal{K}'_1$.

1. (Find the index of the matching in the chained triple that shares two edges) Let $t \in [k]$ such that $|M_{it} \cap \bigcup_{j \neq t} M_{ij}| = 2$.
2. (Choose the number of edges shared by each matching) Choose an integral vector $r = (r_1, \ldots, r_k)$ so that $\sum_{j=1}^{k} r_j = k\ell - x_1$ and $r_j \geq 1$ for every $j$ and $r_t \geq 2$.
3. (Make room for shared edges) For each $j \in [k]$, let $h_j = r_j - 1$ if $j \neq t$ and $h_t = r_t - 2$. Choose $h_j$ edges one by one without repetition in $M_{ij} \setminus I_j$. Let $M'_{ij}$ be $(\ell - r_j)$-matching obtained from $M_{ij}$ by deleting these edges and the edges in $I_j$.
4. (Choose shared edges) Choose a set $X$ of $\sum_{i=1}^{k} x_i$ edges that are not in $\bigcup_{j=1}^{k} M'_{ij}$. Partition $X$ into sets $X_1, \ldots, X_k$ such that $|X_i| = x_i$ for every $i$.
5. (Assign shared edges) Construct a bipartite graph $Q$ with bipartition $(X, [k])$ so that the degree of each $e \in X_i$ is $i$ and the degree of each $j \in [k]$ is $r_j$. Let $M''_{ij}$ be obtained from $M'_{ij}$ by including the edges in $X$ that are adjacent to $j$ in $Q$. 

6. (Update indices) If each $M''_{i,j}$ is an $\ell$-matching, let $Y''$ be the new tuple such that for every $1 \leq j \leq k$, $M''_{i,j} = M''_{i,j}$. Similarly to the even case, the number of inverse switchings applicable to any $i \in K_1'$ is at most

$$(k\ell - x_1)^{k-1}{\ell}^{k\ell-x_1-k-1}\left(\frac{n}{2}\right)^{\sum_{i=2}^{x_1} x_i} \beta^{k\ell-x_1},$$

for the same constant $\beta$ that is defined before. The same calculations as in the case of even $k$ (also by splitting the analysis into two cases $x_1 = k(\ell-1)-1$ and $x_1 \leq k(\ell-1)-2$) give verification of (3.18). This completes the proof of Lemma 9. $\square$

4. Proof of Theorem 5 and Theorem 6

4.1. The log-normal paradigm

Let $\mathcal{S}$ be a set of graphs on vertex set $[n]$ such that each graph in $\mathcal{S}$ has $h$ edges (e.g. $\mathcal{S}$ is the set of $\ell$-matchings in $K_{[n]}$). Let $s = |\mathcal{S}|$ and $X_n$ denote the number of graphs in $\mathcal{S}$ that are contained in a random graph ($\mathcal{G}(n, p)$ or $\mathcal{G}(n, m)$) as a subgraph. Define

$$\mu_n = s \left( \frac{N-h}{m-h} \right) / \left( \frac{N}{m} \right), \quad \lambda_n = sp^h. \quad (4.1)$$

Immediately we have

$$E_{\mathcal{G}(n, m)} X_n = \mu_n, \quad E_{\mathcal{G}(n, p)} X_n = \lambda_n.$$ 

When $\limsup_{n \to \infty} h/m < 1$ and $h^2 = \Omega(m)$, we can further simplify $\mu_n$ and obtain

$$\mu_n = s \cdot \frac{[m]^h}{[N]^h} = s(m/N)^h \exp \left( - \frac{N-m}{mN} \frac{h^2}{2} + O(h^3/m^2) \right). \quad (4.2)$$

Given $i \geq 0$, let $F(i) = \{(G_1, G_2) \in \mathcal{S}^2, |E(G_1) \cap E(G_2)| = i\}$ and let $f_{i} = |F(i)|$.

A slight generalisation of [6, Theorem 1], with almost the same proof (with only $f_j$ replaced by $f_j'$ and some equalities replaced by asymptotic equalities), gives the following theorem.

**Theorem 11.** Let $\mu_n$ be as in (4.1). Suppose there is a sequence $(f_j')_{j=0}^h$ such that $f_j \sim f_j'$ uniformly for all $j \geq 0$. Let $r_j = f_j'/f_{j-1}'$ for all $1 \leq j \leq h$. Assume that $h^3 = o(m^2)$, $h^2 = \Omega(m)$, and, for $\rho(n) = h^2/m$ and some function $\gamma(n)$, the following conditions hold:
(a) for all $K > 0$ and for all $1 \leq j \leq K\rho(n)$,

$$r_j = \frac{h^2}{N^j} \left(1 + o\left(\frac{m}{h^2}\right)\right);$$

(b) $r_j \leq m/2N$ for all $4\rho(n) \leq j \leq \gamma(n)$;

(c) $t(n) := \sum_{j>\gamma(n)} f_j = o(\mu_n|S|);$ 

Then, in $\mathcal{G}(n,m)$,

$$X_n/E_{\mathcal{G}(n,m)}(X_n) \xrightarrow{p} 1$$

as $n \to \infty$.

The following theorem can also be found in [6, Theorem 3].

**Theorem 12.** Assume $h^3 = o(p^2n^4)$. Let $\beta_n = h\sqrt{(1-p)/pN}$ and $\lambda_n = E_{\mathcal{G}(n,p)}X_n$. Assume further that $\liminf_{n \to \infty} \beta_n > 0$. If, for all $m = pN + O(\sqrt{pN})$, we have that $X_n/E_{\mathcal{G}(n,m)}(X_n) \xrightarrow{p} 1$, then, in $\mathcal{G}(n,p)$,

$$\frac{\ln(e^{\beta_n^2/2X_n}/\lambda_n)}{\beta_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty,$$

where $\mathcal{N}(0,1)$ is the standard normal distribution.

We will apply Theorems 11 and 12 with $S = M = \{M_1, \ldots, M_s\}$, the set of all $\ell$-matchings of $K_{[n]}$. For $0 \leq i \leq \ell$, let $F(i)$ be the set of pairs of $\ell$-matchings $(M, M')$ in $K_{[n]}$ such that $|M \cap M'| = i$ and let $f_i = |F(i)|$. For any element $g = (M, M')$, let $n_0 = n_0(g)$ denote the number of vertices that are incident with neither $M$ nor $M'$; $n_1 = n_1(g)$ the number of vertices incident with exactly one of $M$ and $M'$; $n_2 = n_2(g)$ the number of vertices incident with both $M$ and $M'$. Then we immediately have that $n = n_0 + n_1 + n_2$ and $4\ell = 2n_2 + n_1$. This implies that

$$n_1 = 4\ell - 2n_2, \quad \text{and} \quad n_0 = n - 4\ell + n_2. \quad (4.3)$$

We will constantly use the relation (4.3) in the following proofs. Now we close this section by proving a lemma that will be used to verify condition (c) of Theorem 11 in the proofs of Theorems 5 and 6.

**Lemma 13.** Let $m = \omega(n)$. Suppose that $\ell^3 = o(m^2)$ and $\ell^2 = \Omega(m)$. For $\delta > 4/5$, we have that $\sum_{i \geq \delta \ell} f_i = o(s\mu_n)$.

**Proof.** Let $I = [\delta \ell]$ and $P = m/N$. We bound the number of ways to choose a pair of matchings $(M, M')$ sharing at least $I$ edges. There are $s$ choices for $M$. The matching $M'$
has to share at least $I$ edges with $M$ and the other edges of $M'$ cannot intersect these $I$ edges. Thus, we have at most $\binom{\ell}{I}\left(\frac{n-2I}{2\ell-2I}\right)\left(\frac{2\ell-2I}{2\ell-I}\right)!\left(\frac{2(\ell-I)}{(\ell-I)!}\right)$ choices for $M'$. Thus, by (4.2),

$$\sum_{i \geq I} \frac{f_i}{s^i \mu_n} \leq \left(\frac{\ell}{I}\right)\left(\frac{n-2I}{2\ell-2I}\right)\left(\frac{2\ell-2I}{2\ell-I}\right)!\left(\frac{2(\ell-I)}{(\ell-I)!}\right) \exp \left(\frac{(1-P)^2}{2m} + O\left(\frac{\ell^3}{m^2}\right)\right).$$

Using Stirling’s approximation for $(2I)!$ and $I!$ and the fact that $\left(\frac{\ell}{I}\right) \leq \left(\frac{\sqrt{\ell}}{I}\right)^I$, we have

$$\left(\frac{\ell}{I}\right)\left(\frac{n-2I}{2\ell-2I}\right)\left(\frac{2\ell-2I}{2\ell-I}\right)!\left(\frac{2(\ell-I)}{(\ell-I)!}\right) = \left(\frac{\ell^2}{I!\left[n\right]_{2I}}\right)^I = \left(\frac{1}{\sqrt{\ell}P^{\ell/I-1}}\right) = \left(\frac{\ell^3}{n^4P^2}\right)^{1/2}.$$ 

We have that $\ell^3/(n^4P^2) = o(1)$ since $\ell^3 = o(m^2)$ and

$$\sqrt{\ell}P^{\ell/I-1} = \Omega(m^{1/4})P^{d-1} = \Omega(n^{1/4}P^{1/4}) \to \infty,$$

where the first equality holds because $\ell^2 = \Omega(m)$ and the second equality because $\delta > 4/5$, and the last asymptotics holds because $nP \to \infty$ (since $m = \omega(n)$). Hence,

$$\sum_{i \geq I} \frac{f_i}{s^i \mu_n} = \exp \left(\frac{(1-P)^2}{2m} + O\left(\frac{\ell^3}{m^2}\right) - \omega(I)\right) = o(1)$$

since $\ell^3/m^2 = o(1)$ and $I = \Omega(\ell) = \Omega(\ell^2/m)$. \qed

### 4.2. Proof of Theorem 6

In this section, we apply Theorem 11 and Theorem 12 to prove Theorem 6, which deals with near-perfect matchings.
Lemma 14. Let $0 < \alpha < 1$ be fixed and assume $n - 2\ell = O(n^\alpha)$. For any fixed $0 < \delta < 1$ and any $1 \leq i \leq \delta \ell$,

$$
\frac{f_i}{f_{i-1}} = \frac{n^2}{8i \ell^2} (1 + O(i/n + n^{\alpha-1})).
$$

Proof. We define the following switching (see Fig. 3). For any $g = (M, M') \in F(i)$, pick an edge $x \in M \cap M'$ and label the end vertices of $x$ by 1 and 2. Then pick edges $y \in M \setminus M'$ and $z \in M' \setminus M$ such that $y$ and $z$ are disjoint and label the end vertices of $y$ and $z$ by 3, 4 and 5, 6 respectively. Replace $x$ and $y$ by $\{1, 3\}$ and $\{2, 4\}$ in $M$ and replace $x$ and $z$ by $\{1, 5\}$ and $\{2, 6\}$ in $M'$. This operation results in $g' \in F(i-1)$. The number of ways to perform such a switching is $2i \cdot (\ell - i) \cdot (\ell - i + O(1))$, since the numbers of ways to choose $x$ and $y$ are $i$ and $\ell - i$ respectively, and the number of ways to choose $z$ is $\ell - i + O(1)$ where $O(1)$ accounts for the choices of $z$ such that $z$ and $y$ are not disjoint, and for each edge there are two ways to label its end vertices. The inverse switching can be described as follows. For any $g' = (Q, Q') \in F(i-1)$, pick a 2-path in $Q \cup Q'$ and label the vertices as 3, 1, 5 such that $\{3, 1\} \in Q$ and $\{1, 5\} \in Q'$. Pick another 2-path in $Q \cup Q'$ and label the vertices as 4, 2, 6 such that $\{4, 2\} \in Q$, $\{2, 6\} \in Q'$ and $\{3, 4\} \notin Q$, $\{5, 6\} \notin Q'$. Replace $\{3, 1\}$ and $\{4, 2\}$ by $\{3, 4\}$ and $\{1, 2\}$ in $Q$ and replace $\{1, 5\}$ and $\{2, 6\}$ by $\{1, 2\}$ and $\{5, 6\}$ in $Q'$. This operation is applicable if and only if all six vertices $i$, $1 \leq i \leq 6$, are distinct. Recall that $n_2 = n_2(g')$ denotes the number of vertices incident with both $Q$ and $Q'$. By (4.3) and the assumption that $n - 2\ell = O(n^\alpha)$, it follows immediately that $n_2 = n - O(n^\alpha)$. There are $n_2 - 2(i - 1)$ ways to choose vertex 1 and then the vertices 3 and 5 are determined by the choice of vertex 1. The number of ways to choose 4, 2, 6 is $n_2 - 2(i - 1) - O(1)$, where $2(i - 1)$ counts the number of vertices incident to edges in $Q \cap Q'$ and there are $O(1)$ ways to choose vertex 2 so that either the six vertices are not all distinct, or $\{3, 4\} \notin Q'$, or $\{5, 6\} \notin Q$. Hence, the number of applicable inverse switchings for any $g' \in F(i-1)$ is $(n_2 - O(i))^2 = n_2(1 + O(i/n + n^{\alpha-1}))$. Hence, for any $1 \leq i \leq \delta \ell$,

$$
\frac{|F(i)|}{|F(i-1)|} = \frac{n^2}{8i \ell - i)^2} (1 + O(i/n + n^{\alpha-1})) = \frac{n^2}{8i \ell^2} (1 + O(i/n + n^{\alpha-1})).
$$

Proof of Theorem 6. $m = pN + O(\sqrt{pN})$, consider $\mathcal{G}(n, m)$. Apply Theorem 11 with $h = \ell$ and $f'_j = f_j$ for all $0 \leq j \leq \ell$. By our assumption, $pn^{1-\alpha} \to \infty$ as $n \to \infty$ and $\ell = n/2 - O(n^\alpha)$, where $\alpha > 1/2$, which imply $p^2n \to \infty$. Hence, we have $\ell^3 = o(m^2)$ and $\ell^2 = \Omega(m)$. Let $\gamma(n) = 9\ell/10$. Conditions (a) and (b) are satisfied by Lemma 14 (by...
taking $\delta = 9/10$ and the assumption that $pn^{1-\alpha} \to \infty$ as $n \to \infty$, and condition (c) is satisfied by Lemma 13. Hence, for all $m = pN + O(\sqrt{pN})$, we have $X_{n,\ell}/E_{g(n,m)}(X_{n,\ell}) \overset{p}{\to} 1$ as $n \to \infty$. We also have $1 - p = \Omega(1)$ and $\ell = \Omega(n)$ by assumption, and so $\beta_{n,\ell} = \ell\sqrt{(1 - p)/pN} = \Omega(1)$. Then Theorem 6 follows by Theorem 12. \qed

4.3. Proof of Theorem 5

In this section, we apply Theorem 11 and Theorem 12 to prove Theorem 5. We assume the hypotheses in Theorem 5: $np \to \infty$, $1 - p = \Omega(1)$, $\ell = \Omega(n\sqrt{p})$, $\ell \leq n/2 - n^\alpha$ with $\alpha \in (7/8, 1)$ being fixed, and $\ell^3 = o(n^4p^2)$.

For $0 \leq i \leq \ell$ and $0 \leq n_2 \leq 2\ell$, let $F(i, n_2)$ be the set of pairs $(M, M') \in F(i)$ such that $|V(M) \cap V(M')| = n_2$ and let $f(i, n_2) = |F(i, n_2)|$. Let $\delta = 9/10$. For $0 \leq i \leq \delta\ell$, define

$$z(i) = \frac{4(\ell - i)^2}{n - 2i},$$

$$f'_i = \sqrt{\pi} \left(\frac{1}{2z(i)} + \frac{1}{2\ell - z(i) - 2i} + \frac{1}{2(n - 4\ell + z(i) + 2i)}\right)^{-1/2} f(i, z(i) + 2i).$$ (4.4)

For $\delta\ell < i \leq \ell$, let $f'_i = f_i$.

First we prove that $f_i$ and $f'_i$ are asymptotically equal. In many places, we ignore the floor sign if a certain variable is required to be integral (e.g. the number of edges) but the error caused by ignoring it in the analysis is negligible.

**Lemma 15.** For $i \leq \delta\ell$, we have that $f_i = f'_i(1 + o(1))$, uniformly for $i$. Moreover, $f(i, z(i) + 2i + k)/f(i, z(i) + 2i) = 1 + O(1/z(i) + n/(\ell(n - 2\ell)) + n/(n - 2\ell)^2)$ for $k = O(1)$, uniformly for $i$.

**Proof.** We define the following switching (see Fig. 4). Given a pair of matchings $(M, M') \in F(i, n_2)$, choose a vertex $v$ saturated by both $M, M'$ with distinct edges, say $av \in M, bv \in M'$, choose a vertex $u$ not saturated by neither matching, delete $av$ from $M$ and add $au$ to $M$. The new pair of matchings is in $F(i, n_2 - 1)$. Note that there are $(n_2 - 2i)n_0$ ways of performing this switching.

The inverse switching is described as follows. Given a pair of matchings $(M, M') \in F(i, n_2 - 1)$, choose vertices $u$ covered by $M$ but not by $M'$ and $v$ covered by $M'$ but not by $M$ such that the edge $au \in M$ and the edge $bv \in M'$ satisfy $a \neq b$. Delete $au$ from $M$ and add $av$ to $M$. The number of vertices that are saturated only by $M$ is $n_1/2$ and so is the number of vertices that are saturated only by $M'$. Hence, there are
\( n_1(n_1 - O(1))/4 \) ways of doing the switching, where the \( O(1) \) accounts for the choices of \( v \) such that \( a = b \), given the choice of \( u \).

By (4.3),

\[
\frac{f(i, n_2)}{f(i, n_2 - 1)} = \frac{n_1(n_1 - O(1))/4}{n_2 - 2i}n_0 = \frac{(2\ell - n_2)^2}{(n_2 - 2i)(n - 4\ell + n_2)} \left( 1 + O \left( \frac{1}{2\ell - n_2} \right) \right).
\]

For \( n_2 = z(i) + 2i \), we have that this ratio (ignoring the error term) is \( 1 \). Thus, for \( k < z(i)^{\alpha'/3} \) with \( \alpha' \in (1.5, 2) \) satisfying \( 2\alpha > 1 + \alpha'/2 \),

\[
\frac{f(i, z(i) + 2i + k)}{f(i, z(i) + 2i)} = \prod_{j=1}^{k} \frac{f(i, z(i) + 2i + j)}{f(i, z(i) + 2i + j - 1)}
\]

\[
= \prod_{j=1}^{k} \frac{(2\ell - z(i) - j - 2i)^2}{(n - 4\ell + z(i) + j + 2i)} \left( 1 + O \left( \frac{1}{2\ell - z(i) - j - 2i} \right) \right)
\]

\[
= \prod_{j=1}^{k} \frac{(2\ell - z(i) - 2i)^2}{z(i)(n - 4\ell + z(i) + 2i)} \left( 1 - \frac{j}{2\ell - z(i) - 2i} \right)^2 \left( 1 + O \left( \frac{1}{n - 4\ell + z(i) + 2i} \right) \right)
\]

\[
= \prod_{j=1}^{k} \exp \left( - \frac{2j}{2\ell - z(i) - 2i} - \frac{j}{z(i)} - \frac{j}{n - 4\ell + z(i) + 2i} \right)
\]

\[
+ O \left( \frac{j^2}{(2\ell - z(i) - 2i)^2} + \frac{j^2}{z(i)^2} + \frac{j^2}{(n - 4\ell + z(i) + 2i)^2} + \frac{1}{2\ell - z(i) - j - 2i} \right)
\]

\[
= \exp \left( - \frac{k^2}{2\ell - z(i) - 2i} - \frac{k^2}{2z(i)} - \frac{k^2}{2(n - 4\ell + z(i) + 2i)} \right)
\]

\[
+ O \left( \frac{k}{z(i)} + \frac{k}{n - 4\ell + z(i) + 2i} \right)
\]

\[
+ O \left( \frac{k^3}{(2\ell - z(i) - 2i)^2} + \frac{k^3}{z(i)^2} + \frac{k^3}{(n - 4\ell + z(i) + 2i)^2} + \frac{k}{2\ell - z(i) - k - 2i} \right).
\]

We have that the derivative of \( z(i) \) with respect to \( i \) is \(-8(\ell - i)(n - \ell - i)/(n - 2i)^2\). Since \( n - \ell - i \geq n - 2\ell \geq 0 \), this implies that \( z(i) \geq z(\lfloor \delta \ell \rfloor) \) for all \( i \leq \delta \ell \). Moreover, \( z(\lfloor \delta \ell \rfloor) = \Omega(\ell^2/n) = \omega(1) \) because \( \ell = \omega(\sqrt{n}) \) (this holds since \( \ell^2 = \Omega(n^2p) \) and \( np = \omega(1) \)). Thus, using \( k < z(i)^{\alpha'/3} \) and \( \alpha' < 2 \), we have that

\[
\frac{k^3}{z(i)^2} < \frac{1}{z(i)^{2-\alpha'}} \leq \frac{1}{z(\lfloor \delta \ell \rfloor)^{2-\alpha'}} = o(1),
\]

and so \( k/z(i) = o(1) \) as well.
We have that $2\ell - z(i) - 2i = \Omega(\ell(n - 2\ell)/n)$ and $n - 4\ell + z(i) + 2i = \Omega((n - 2\ell)^2/n)$. This implies that $f(i, z(i) + 2i + k)/f(i, z(i) + 2i) = 1 + O(1/z(i) + n/(\ell(n - 2\ell) + n/(n - 2\ell)^2)$ for $k = O(1)$.
Note that the ratio between consecutive terms is decreasing as $n_2$ increases (moreover we can ignore the error in the ratio because we only need an upper bound now). If $k' = z^{\alpha''/3}$ with $1.5 < \alpha'' < \alpha'$, then

$$\sum_{j \geq k'} f(i, z(i) + 2i + j) \leq (1 + o(1)) \sum_{j \geq k'} \exp\left(-\frac{j^2}{2z(i)}\right)$$

$$\leq (1 + o(1)) \sum_{j \geq k'} \exp\left(-\frac{jk'}{2z(i)}\right)$$

$$\leq (1 + o(1)) \frac{\exp\left(-\frac{(k')^2}{2z(i)}\right)}{1 - \exp\left(-\frac{k'}{2z(i)}\right)}$$

$$\sim \exp\left(-\frac{(k')^2}{2z(i)} - \ln\left(z(i)\right)\right)$$

$$= o(1),$$

since $z(i)^{1/2} < k' < z(i)^{2/3}$ and $z(i) = o(1)$. Thus, we can ignore the terms with $z(i) + 2i + j$ with $j > k'$. By similar computations, we have

$$f_i \sim \sum_{k = -z(i)}^{z(i)} f(i, z(i) + 2i) \times$$

$$\times \exp\left(-\frac{k^2}{2\ell - z(i) - 2i} - \frac{k^2}{2z(i)} - \frac{k^2}{2(n - 4\ell + z(i) + 2i)}\right)$$

$$\sim f(i, z(i) + 2i)\sqrt{n} \times$$

$$\times \int_{y = -\infty}^{\infty} \exp\left(-y^2\left(\frac{n}{2\ell - z(i) - 2i} + \frac{n}{2z(i)} + \frac{n}{2(n - 4\ell + z(i) + 2i)}\right)\right) dy$$

$$\sim \sqrt{\pi}\left(\frac{1}{2z(i)} + \frac{1}{2\ell - z(i) - 2i} + \frac{1}{2(n - 4\ell + z(i) + 2i)}\right)^{-1/2} f(i, z(i) + 2i)$$

$$= f_i'. \quad \Box$$

In the next lemma, we compute the ratio $f'_i / f'_{i-1}$. 
Lemma 16. Suppose that \( i \leq \delta \ell \). Then
\[
\frac{f_i}{f'_{i-1}} = \frac{z(i)^2}{8i(\ell - i)^2} \left( 1 + O\left( \frac{n}{\ell(n - 2\ell)} \right) + O\left( \frac{n}{(n - 2\ell)^2} \right) + O\left( \frac{1}{z(i)} \right) \right).
\]

Proof. By (4.4), for \( i \leq \delta \ell \), we have
\[
\frac{f_i}{f'_{i-1}} = \sqrt{\frac{1}{2z(i - 1)} + \frac{1}{2\ell - z(i - 1) - 2i + 2} + \frac{1}{2(n - 4\ell + z(i - 1) + 2i - 2)}} \cdot \frac{f(i - 1, z(i) + 2i)}{f(i - 1, z(i - 1) + 2i - 2)} \cdot \frac{f(i, z(i) + 2i)}{f(i - 1, z(i) + 2i)}.
\]

We will analyse each of these three ratios separately. The square of the first ratio (the expression can be easily simplified using Maple) equals
\[
\frac{(\ell - i)^2(n - 2i + 2)^3}{(\ell - i + 1)^2(n - 2i)^3} = 1 + O\left( \frac{1}{\ell - i} \right) + O\left( \frac{1}{n - 2i} \right).
\]

We have that \( 1/(\ell - i) = O(1/\ell) \) since \( i \leq \delta \ell \) with \( \delta < 1 \) and similarly, \( 1/(n - 2i) = O(1/\ell) \). Thus, the first ratio is \( 1 + O(1/\ell) \). Next, we analyse the second ratio. Using that \( \ell^2/n \to \infty \), it follows easily that \( z(i - 1) - z(i) = O(1) \). Thus, we have that the second ratio is \( 1 + O(1/z(i)) + n/(\ell(n - 2\ell)) + n/(n - 2\ell)^2 \) by Lemma 15.

Finally, we analyse the last ratio. We use the same switching in the proof of Lemma 14 to analyse the ratio \( f(i, n_2)/f(i - 1, n_2) \), where \( n_2 = z(i) + 2i \). As it was shown that the number of ways to perform a switching is \( 8i((\ell - i)/(\ell - i + O(1))) = 8i(\ell - i)^2(1 + O(1/\ell)) \), and the number of ways to perform an inverse switching is \( (n_2 - 2(i - 1))(n_2 - 2(i - 1) + O(1)) = (z(i - 1)^2(1 + O(1/z(i)))) \). Thus, we have
\[
\frac{f(i, z(i) + 2i)}{f(i - 1, z(i) + 2i)} = \frac{z(i)^2}{8i(\ell - i)^2} \left( 1 + O\left( \frac{1}{\ell} \right) + O\left( \frac{1}{z(i)} \right) \right). \quad \Box
\]

Proof of Theorem 5. For any \( m = pN + O(\sqrt{Np}) \), consider \( G(n, m) \). Apply Theorem 11 with \( h = \ell \) and \( (f_j')_{j=1}^\ell \). By Lemma 15, we have that \( f_j \sim f_j' \) for all \( 1 \leq j \leq \ell \). By our assumptions on \( p \) and \( \ell \), it is straightforward to verify that \( \ell^3 = o(m^2) \) and \( \ell^2 = \Omega(m) \).

Next, we show that the conditions (a)–(c) in Theorem 11 hold for \( \gamma(n) := \delta \ell = 9\ell/10 \).

By Lemma 16, for \( 1 \leq j \leq K\ell^2/m \), we have that
\[
r_j = \frac{z(j)^2}{8j(\ell - j)^2} \left( 1 + O\left( \frac{n}{\ell(n - 2\ell)} \right) + O\left( \frac{n}{(n - 2\ell)^2} \right) + O\left( \frac{1}{z(j)} \right) \right)
= \frac{\ell^2}{Nj} \left( 1 + O\left( \frac{j}{\ell} \right) + O\left( \frac{n}{\ell(n - 2\ell)} \right) + O\left( \frac{n}{(n - 2\ell)^2} \right) + O\left( \frac{1}{z(j)} \right) \right).
\]
We have that \( \ell = \Omega(n^{1/4}) = \Omega(\sqrt{n} \cdot \sqrt{np}) = \omega(\sqrt{n}) \) since \( np \to \infty \). Using this together with \( \ell^3 = o(n^2p^2) \), we obtain

\[
n^{1+\alpha}p = \omega \left( \frac{n^{1+\alpha} \ell^{3/2}}{n^2} \right) = \ell \omega \left( \frac{\ell^{1/2}}{n^{1-\alpha}} \right) = \ell \omega(n^{1/8})
\]

and

\[
n^{1+2\alpha}p = \omega \left( \frac{n^{1+2\alpha} \ell^{3/2}}{n^2} \right) = \ell^2 \omega \left( \frac{n^{-1+2\alpha} \ell^{1/2}}{n^{1/2}} \right) = \ell^2 \omega(n^{1/4}).
\]

This implies \( \ell = o(n^{1+\alpha}p) \) and \( \ell^2 = o(n^{1+2\alpha}p) \). Thus, we have that

\[
\frac{j}{\ell} \leq \frac{K\ell}{m} = O \left( \frac{\ell}{n^2p} \right) = O \left( \frac{n^2p}{\ell^2} \right) \cdot O \left( \frac{\ell^3}{n^4p^2} \right) = o \left( \frac{n^2p}{\ell^2} \right);
\]

and

\[
\frac{n}{\ell(n-2\ell)} = O \left( \frac{n^{1-\alpha}}{\ell} \right) = O \left( \frac{n^2p}{\ell^2} \right) \cdot O \left( \frac{\ell}{n^{1+\alpha}p} \right) = o \left( \frac{n^2p}{\ell^2} \right);
\]

and

\[
\frac{n}{(n-2\ell)^2} = O \left( \frac{n^{1-2\alpha}}{\ell^2} \right) = O \left( \frac{n^2p}{\ell^2} \right) \cdot O \left( \frac{\ell^2}{n^{1+2a}p} \right) = o \left( \frac{n^2p}{\ell^2} \right);
\]

and

\[
\frac{1}{z(j)} = O \left( \frac{n}{\ell^2} \right) = O \left( \frac{n^2p}{\ell^2} \right) \cdot O \left( \frac{1}{np} \right) = o \left( \frac{n^2p}{\ell^2} \right).
\]

Thus, condition (a) holds. Now we will check condition (b). We have that for, \( 4\ell^2/m \leq j \leq \delta \ell \),

\[
\frac{2(j-\ell)^2}{j(n-2j)^2} = \frac{2(j-\ell)^2}{j(n-2j)^2}.
\]

By computing the derivative of the RHS with respect to \( j \) and using \( n \geq 2\ell \), it is easy to see that the derivative is negative. At \( j = 4\ell^2/m \), using \( \ell^2/(n^3p) = o(1) \),

\[
\frac{2(j-\ell)^2}{j(n-2j)^2} \leq \frac{m}{2(n-2j)^2} = \frac{m}{2n^2(1+o(1))} \leq \frac{m}{4N(1+o(1))}.
\]

So condition (b) holds. Condition (c) holds by Lemma 13 (with \( \delta = 9/10 \)). Hence, we have that \( X_{n,\ell}/E_{G(n,m)}(X_{n,\ell}) \nrightarrow 1 \) by Theorem 11. Since \( 1 - p = \Omega(1) \) by assumption, we have \( \beta_{n,\ell} = \Omega(\ell/\sqrt{pn}) = \Omega(1) \). Then Theorem 5 follows by Theorem 12. \( \Box \)
References

[1] N. Alon, J. Spencer, The Probabilistic Method, third edition, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, Inc., Hoboken, NJ, 2008, xviii+352 pp.
[2] P. Billingsley, Probability and Measure, third edition, Wiley Ser. Probab. Math. Statist., A Wiley–Interscience Publication, John Wiley & Sons, Inc., New York, 1995.
[3] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, European J. Combin. 1 (1980) 311–316.
[4] B. Bollobás, Random Graphs, second edition, Cambridge Stud. Adv. Math., vol. 73, Cambridge University Press, Cambridge, 2001, xviii+498 pp.
[5] P. Gao, Distribution of the number of spanning regular subgraphs in random graphs, Random Structures Algorithms 43 (3) (2013) 265–397.
[6] P. Gao, Distributions of sparse spanning subgraphs in random graphs, SIAM J. Discrete Math. 27 (1) (2013) 386–401.
[7] P. Gao, Y. Su, N. Wormald, Induced subgraphs in sparse random graphs with given degree sequences, European J. Combin. 33 (6) (2012) 1142–1166.
[8] H. Garmo, The asymptotic distribution of long cycles in random regular graphs, Random Structures Algorithms 15 (1) (1999) 43–92.
[9] S. Janson, The numbers of spanning trees, Hamilton cycles and perfect matchings in a random graph, Combin. Probab. Comput. 3 (1994) 97–126.
[10] S. Janson, Random regular graphs: asymptotic distributions and contiguity, Combin. Probab. Comput. 4 (4) (1995) 369–405.
[11] S. Janson, T. Łuczak, A. Ruciński, Random Graphs, Wiley-Intersci. Ser. Discrete Math. Optim., Wiley–Interscience, New York, 2000, xii+333 pp.
[12] B.D. McKay, Asymptotics for symmetric 0–1 matrices with prescribed row sums, Ars Combin. 19A (1985) 15–25.
[13] B.D. McKay, N.C. Wormald, Uniform generation of random regular graphs of moderate degree, J. Algorithms 11 (1990) 52–67.
[14] B.D. McKay, N.C. Wormald, Asymptotic enumeration by degree sequence of graphs with degrees $o(\sqrt{n})$, Combinatorica 11 (1991) 369–382.
[15] G. Perarnau, G. Petridis, Matchings in random biregular bipartite graphs, Electron. J. Combin. 20 (1) (2013) P60.
[16] R.W. Robinson, N.C. Wormald, Almost all cubic graphs are Hamiltonian, Random Structures Algorithms 3 (2) (1992) 117–125.
[17] R.W. Robinson, N.C. Wormald, Almost all regular graphs are Hamiltonian, Random Structures Algorithms 5 (2) (1994) 363–374.
[18] A. Ruciński, When are small subgraphs of a random graph normally distributed?, Probab. Theory Related Fields 78 (1) (1988) 1–10.
[19] N.C. Wormald, The asymptotic distribution of short cycles in random regular graphs, J. Combin. Theory Ser. B 31 (1981) 168–182.