Multi-partite Quantum Entanglement versus Randomization: Fair and Unbiased Leader Election in Networks

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Keywords: multi-partite quantum entanglement, leader election, randomization, bias, fairness, referee, agreement.

Abstract

In this paper we show that sufficient multi-partite quantum entanglement helps in fair and unbiased election of a leader in a distributed network of processors with only linear classical communication complexity. We show that a total of $O(\log n)$ distinct multi-partite maximally entanglement sets (ebits) are capable of supporting such a protocol in the presence of nodes that may lie and thus be biased. Here, $n$ is the number of nodes in the network. We also demonstrate the difficulty of performing unbiased and fair election of a leader with linear classical communication complexity in the absence of quantum entanglement even if all nodes have perfect random bit generators. We show that the presence of a sufficient number $O(n/\log n)$ of biased agents leads to a non-zero limiting probability of biased election of the leader, whereas, the presence of a smaller number
$O(\log n)$ of biased agents matters little. We define two new related complexity classes motivated by the our leader election problem and discuss a few open questions.

1 Introduction

Quantum entanglement is one of the most fascinating aspects of quantum physics. Two particles in an entangled state behave as if possessing a common state, even if they are physically separated over a great distance. Two maximally entangled qubits can be in the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Each of the qubits here is in a mixed state with probability $\frac{1}{2}$ in state $|0\rangle$ and with probability $\frac{1}{2}$ in state $|1\rangle$. However, both together form a pure state $\rho$. Such a pair of qubits is called an EPR pair (after Einstein, Podolsky and Rosen) [6, 8]. Such entangled qubits can exhibit what physicists call nonlocal effects. Nonlocal effects cannot occur between particles as per laws of classical physics unless communication is permitted between the physically separated particles, in which case communication would have to be at a speed exceeding that of light. Such behaviour was referred to as spooky actions at a distance in [5] and was earlier mentioned in the celebrated 1935 paper by Einstein, Podolsky and Rosen [6]. Nonlocal effects however do not involve any communication and therefore the question of faster than light communication does not arise. Even if two agents Alice and Bob share entangled pairs, there is no way for Alice to manipulate her own particles to convey any information, she has, to Bob; information must be sent explicitly from Alice to Bob. In this context, it is necessary to investigate whether quantum computing resources help Alice and Bob solve a problem together with lesser communication complexity than that required when no quantum resource is used.

In Yao’s model [16], Alice and Bob are allowed to communicate using qubits. Kremer’s result in [10] does not show superiority of quantum communication over classical communication for the INNER PRODUCT function $f(x, y) = x \cdot y$ where $x$ and $y$ are $n$-bit vectors each, $x$ given only to Alice as input and $y$ given only to Bob as input; the lower bound on qubits required for computing INNER PRODUCT is $\Omega(n)$. However, the result of Buhrman, Cleve and Wigderson [3] requires only $O(\sqrt{n} \log n)$ qubit communication cost for the DISJOINT function, whose classical probabilistic communication complexity is $\Omega(n)$ [9, 13]. The DISJOINT(x,y) function gives 0 if any bit of $n$-bit vector $x$ is equal to the corresponding bit of the $n$-bit vector $y$; otherwise, it returns a 1.

Just as two particles at a distance could be quantum entangled forming an EPR pair, it is also possible to entangle multiple mutually physically separated particles. One example is due to Greenberger, Horne and Zeilinger [7, 11]; here, three particles are quantum entangled. Singh, Kumar and Pal [14] develop two protocols that entangle three nodes $A, B$ and $C$ in a network in the GHZ state $(|0_A0_B0_C\rangle + |1_A1_B1_C\rangle)/\sqrt{2}$ if two pairs (say $(A, B)$ and $(B, C)$) were earlier entangled as two EPR pairs. The first protocol uses the standard teleportation circuit [12] and the second one uses a new circuit developed in [14]. Both
protocols use only classical communication. Buhrman, Cleve and van Dam [2], make use of three-party entanglement and demonstrate the existence of a function whose computation requires strictly lesser classical communication complexity compared to the scenario where no quantum entanglement is used.

Singh, Kumar and Pal [14] show how $n$ nodes connected in a network can be entangled together in a maximally entangled $n$-partite state ($|0_10_2...0_n> + |1_11_2...1_n>)\sqrt{2}$ using linear classical communication complexity provided the nodes were earlier entangled in $(n-1)$ EPR pairs. The $(n-1)$ EPR pairs given initially are required to be shared between agents forming edges of a spanning tree in the network; such a spanning tree is called the *spanning EPR tree* [14]. Brassard et al. show in [1] that prior multi-partite entanglement can be used by $n$ agents in a *perfect* quantum protocol (not using any form of communication between the $n$ agents), to solve a multi-party distributed problem, whereas, no classical deterministic protocol succeeds in solving the same problem with probability away from $\frac{1}{2}$ by a fraction that is larger than an inverse exponential in the number of agents. Multi-partite entanglement is also used by Buhrman, van Dam, Hoyer and Tapp [4] where an $n$-party problem was shown to have quantum communication complexity $n$ qubits, whereas the classical communication complexity was shown to be $\Omega(n \log n)$ cbits.

In this paper we show that a sufficient amount of multi-partite quantum entanglement helps in electing a leader unambiguously in a distributed network of agents where it is not possible for any agent to introduce any statistical bias in the election. We show in our Protocol I that a total of $O(\log n)$ distinct multi-partite entanglement states are capable of supporting a protocol with linear classical communication complexity for leader election even in the presence of nodes that may lie or be biased. Here, $n$ is the number of nodes in the network. Unbiased election is enabled by the sharing of multiple multi-partite entangled states. Using one honest referee, we also ensure that all honest nodes agree on the elected leader. We also demonstrate the difficulty of performing unbiased and fair election of a leader with linear classical communication complexity in the absence of quantum entanglement, even if all nodes have perfect random bit generators. We demonstrate this through our Protocol II where we show the following separation result: too many biased liars would prevent the referee from conducting a fair and unbiased election, whereas, a small number of liars would matter little. This separation result has two aspects. Firstly, it shows as expected that the probability of the election being biased is small and indeed vanishes as the number of agents grows asymptotically when there is a small number of biased agents. So, we observe that it is possible to conduct fair and unbiased election in our randomized Protocol II with small error probability with linear communication cost. Protocol I performs fair and unbiased election but needs $O(\log n)$ multi-partite maximally entangled states or ebits. Note however that even though the number of biased voters ($\log n$) is small, it is an increasing function of $n$. So, with small but growing number of liars, the randomized Protocol II achieves vanishingly small biased election probability as $n$ grows asymptotically; this demonstrates partial tractability of our leader election problem with pure randomization and no quantum resource. Secondly, the finite limiting probability $e^{-C}, C > 0$ ($C$ is a constant), of fair and unbiased election with
the number of biased voters as high as \( n/\log n \) (Protocol II) shows that randomization cannot handle too many cheating agents who are essentially biased in a statistical sense. Even here, we keep the large number of biased agents \( o(n) \) by choosing \( n/\log n \) biased agents.

Further, we define two complexity classes (i) \( QECC(g, n) \), the class of problems solvable using \( g \) ebits shared amongst \( n \) agents with \( O(n) \) cbit communication complexity, and, (ii) \( RCC(r, n) \), the class of problems solvable using \( r \) random bits shared between \( n \) agents, with \( O(n) \) cbit communication complexity. We show that \( RCC(\log n, n) \subseteq QECC(\log n, n) \). The problem of determining whether this is a proper containment is an interesting fundamental open problem concerning the power of quantum entanglement as a computational and communication resource.

Throughout our analysis, we follow a uniform statistical notion of fairness and bias, as we define shortly below. In short, an unbiased agent is supposed to always vote a 0 or a 1 with equal probability as per an ideal random bit generator that the agent possesses. So, any biased agent, using differing probabilities for 0 and 1 votes, would cause some bias towards some agent addresses in the election of the leader. Fairness is ensured by giving agents equal opportunity in voting for determination of the leader.

On the one hand, the successful completion of a protocol would require all agents in the network to agree on an elected leader. On the other hand, agents may be biased in voting for a leader. It is therefore also a matter of concern that one agent amongst all in the network must act as a referee, trying to conduct a fair and unbiased election process. Since election essentially requires voting by all agents in a fair setup and, agreement by all agents about the elected leader, we observe that linear classical communication complexity is essentially a lower bound in our model. In this sense, our model differs from the models of Yao [15, 16] and also those of [2, 4].

In Section 2 we introduce our own definitions and notions of fairness, bias, referee and agreement. We essentially propose two leader election protocols, both use only linear amounts of classical communication. The first one (Protocol I in Section 3) assumes the presence of a sufficient number of \( n \)-partite maximally entangled states, and, the second one (Protocol II in Section 4) assumes no such entanglement but only perfect random bit generation capability in each agent. We show a separation result by demonstrating that too many biased liars would prevent the referee from conducting a fair and unbiased election, whereas, a small number of liars would matter little. In Section 5 we define two new complexity classes related to the problem of unbiased election of a leader and pose a few open problems.

## 2 The Communication Framework

We wish to have a complete protocol with a starting signal going to each agent, signifying initiation. This requires communicating a constant number of bits of classical information (cbits) from one special agent to the rest. (There may be a constant number of special tasks
like the starting and ending of protocols. So a constant number of bits are sufficient for such signals.) We fix one arbitrary node to initiate the protocol by sending this initial message to all other agents. This node is also set as the referee node; the referee is supposed to be conducting the protocol and collecting agreements from all agents, including itself, about the elected leader. If an agent does not accept the elected leader, he is simply left out; further computation can be carried out by only those agents who accept the election of the leader. In other words, we require total agreement over the elected leader by all the agents if all of them wish to continue computations together, after leader election. Such an agreement comes in the form of a cbit from each agent after the leader is elected. The referee first informs the leader about his election and the leader then sends a constant number of cbits to each agent, seeking agreement or approval.

Identifying a leader amounts to fixing $\lceil \log n \rceil$ bits to uniquely address an agent out of $n$ agents. Any protocol determining a leader must therefore come up with these bits. We assume without loss of generality that $n$ is a power of 2 and therefore we need to determine exactly $\log n$ bits.

Now we state the communication and message passing model for classical communication in our protocols. We assume that units of information (cbits) are communicated from a source agent $S$ to a destination agent $T$. The destination can store as many addresses of senders of incoming messages so that in future it may choose to reply to any of them. We also assume that the cost of communicating a single bit of information between any source $S$ and any destination $T$ is the same and is counted as one cbit, independent of the network connecting the agents. We are now in a position to derive our protocols.

We define bias as follows. If an agent $A_i$ can somehow influence the election of the leader in a way that is statistically different from the way another agent $A_j$ influences election, then we say that there is a bias in the election process. In other words, the determination of the address of the leader must be equally influenced by each participating agent.

The notion of fairness is subtly distinct from that of bias. Here, fairness means ensuring that all agents get a fair deal in participating in the election process, in a statistical sense. These notions of statistical bias and fairness will become more explicit when we analyse our protocols in the next two sections.

### 3 Fair and Unbiased Leader Election Using $n$-partite Entanglement

We assume that the $n$ agents $A_1, A_2, ..., A_n$ are in multiple $n$-partite maximally entangled states presented in the computational bases. Such states are called ebots [8]. Each agent houses his own qubits in these entangled states. We also assume that there is a sufficiently

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1 All logarithms in this paper are to the base 2.
large number of such states. In particular, we choose to have log $n$ such ebit states $S_1, ..., S_{\log n}$, with each agent $A_i, 1 \leq i \leq n$ housing log $n$ qubits $q_{u_i,j}, 1 \leq j \leq \log n$, one for each of the log $n$ states. The state $S_k$ is therefore $(|0_{1,k}0_{2,k}...0_{n,k} \rangle + |1_{1,k}1_{2,k}...1_{n,k} \rangle)/\sqrt{2}$ for $1 \leq k \leq \log n$, where the subscript $k, i$ denotes the $k$-th qubit of the $i$-th agent. Clearly, a total of $n \log n$ qubits are used. It is possible to create such a maximally entangled multi-partite (ebit) state using a protocol in [14] that requires linear classical communication complexity and uses up $(n-1)$ EPR pairs generated apriori in the form of an EPR spanning tree of the network. Our Protocol I is initiated by the referee who sends one cbit to all agents signalling the start of the protocol. On getting this trigger, each node $A_i, 1 \leq i \leq n$ performs measurements on its log $n$ qubits $q_{u_i,j}, 1 \leq j \leq \log n$. These measurements are performed in the respective computational bases for each of the log $n$ entanglement sets. Whoever measures a qubit, gets a value zero or one, with probability $\frac{1}{2}$ for each of 0 and 1 values. Indeed, this is a perfect random bit generator. Also, no matter in what order $A_i$ and $A_j, 1 \leq j < i \leq n$ measure their $k$-th qubits, both of them get the same answer. This follows from the postulates of quantum mechanics and the property of maximal entanglement for state $S_k$. The referee can now use these log $n$ measured bits to produce an address or index $m$ for the agent $A_m, 1 \leq m \leq n$ and accept it as an unbiased leader. Note that such a leader is perfectly random and therefore totally unbiased. There is no hiding of the measurements made by the agents about their $k$-th qubits since the referee can always check the measured value of the $k$-th qubit of the $k$-th entangled state $S_k$ by reading its own $k$-th qubit. Even if the referee seeks this value from another agent and the agent lies, the referee can simply ignore any false information provided because it knows the true value of the measured $k$-th qubit. Now it is the turn of all agents to send one cbit to the referee and express their agreement over the elected leader. If an agent agrees on the leader by sending a cbit 1, it is fine and the agent henceforth sticks to the commitment of having agreed on the leader. Otherwise, we say that the agent is discarded and only agreeing agents are considered to continue meaningful computations any further. We also assume that the elected leader is committed to act as leader henceforth. We summarize the lack of bias and the guarantee of fairness in the election of the leader in the above Protocol I in the following theorem.

**Theorem 3.1** It is possible to conduct an unbiased and fair election of a leader amongst $n$ agents who share log $n$ maximally entangled $n$-partite states with $O(n)$ bits of classical communication.

### 4 Randomized Election

We believe that randomization alone is not capable of supporting distributed protocols that can achieve fair and unbiased leader election provided only a linear number of cbits are exchanged in the protocol. Although multi-partite entanglement helps us perform fair and unbiased leader election with only linear classical communication complexity (as we see in...
Protocol I in the previous section), randomization alone, coupled with only linear number of cbits, may lead to fair and unbiased election of a leader only in certain special and restricted cases. Recall that the leader selected in the case of Protocol I is perfectly random, irrespective of the order in which the agents make measurements on the entangled qubits; also, each agent is equally likely to become the leader. In this section we attempt to mimic similar unbiased and fair election and we propose a Protocol II which succeeds in certain cases in a randomized sense. To this effect, we design our Protocol II by making the agents together play the role of an adversary to the referee; the leader declared by the referee may be biased if an agent so chooses. The evil designs of the agents may however not work always and therefore a finite probability of fair and unbiased election may result.

What really is fairness and bias in this scenario, respecting the overall definition we gave earlier in Section 2? Firstly, fairness forces us to design protocols that treat all agents equally, providing them with equal opportunity in the election process, statistically speaking. This must be ensured at the referee’s end. Secondly, bias can be introduced in the choice presented by an agent so that individual agents have unequal probabilities of becoming elected as leader. So, for unbiased election, we may expect agents to honestly use their ideal random bit generating capability. A biased agent (adversary) may not use its perfect random bit generator and may instead bias all his outcomes in a protocol. Providing such biased bits to the referee would lead to a biased election where all the agents may not have equal probability of getting elected. So, we propose our Protocol II where each agent sends just one bit (biased or perfectly ideal random bit) to the referee. Being honest and fair, the referee collects all these $n$ bits from the $n$ agents and then selects a perfectly random sample of $\log n$ bits from the $n$ collected bits to compute the index $m$ of the leader $A_m$. Initiation of the protocol and the agreement step are as in Protocol I. This completes the description of Protocol II.

We now show that biased adversary agents can not fool the referee in an asymptotic and statistical sense, if there are $k = \log n$ biased agents. We also show that a larger number of biased agents, proportional to $n / \log n$, can turn the table the other way; in this case the probability that the referee can not vanquish the evil designs of the biased agents is a nonzero constant. We state this as a theorem.

**Theorem 4.1** It is possible to design a randomized protocol that uses no quantum entanglement, but uses perfect random bit generating capability in each of the $n$ agents and only $O(n)$ bits of classical communication so that as $n$ grows asymptotically, (i) fair and unbiased election of a leader is guaranteed with probability approaching 1 if there are only $O(\log n)$ biased agents, and (ii) unbiased election is possible only with a nonzero finite limiting probability if the number of biased agents is $O(n / \log n)$.

**Proof.** The referee collects the single bit votes sent by the agents, each agent sending one bit vote. To be fair to all, the sincere referee picks up a random sample of $\log n$ bits from the $n$ votes and arranges the bits in random order to create the $\log n$ bit address of the leader who is thus deemed elected. Note here that as long as each agent is honestly sending a vote
which is a perfect random bit, the address generated for the leader will be fair and statistically uniform over all agents. So, as long as the agents themselves are honest, leader election is fair even about who becomes the leader. The problem arises when the bit sent by a dishonest or biased agent is not perfectly random. If the referee happens to select any such biased bit in his random sample for the leader’s address, then the leader elected is statistically biased, not uniform over all agents. We note that the probability that the random sample would be able to avoid biased bits is \( p(n, k) = C(n - k, \log n)/C(n, \log n) \) if there are \( k < n - \log n \) biased bits. We show in Appendix that \( p(n, \log n) \) approaches 1 as \( n \) grows asymptotically. Thus, we conclude that as long as there are only a small number of liars amongst the agents who attempt to bias leader election by casting biased votes, the referee succeeds in deciding upon a leader without bias with very high probability approaching unity. We also show in Appendix that \( p(n, n/\log n) \) approaches a non-zero constant as \( n \) grows asymptotically, showing that there is a non-zero probability that the referee fails to deliver unbiased election when the number of liars biasing their votes is as large as \( O(n/\log n) \).

\[ \square \]

## 5 New Complexity Classes

In the context of our results (Theorems 3.1 and 4.1), we define two new complexity classes where the \( n \) agents are required to use at most linear classical communication but may use some quantum resource or randomized resource. We say that a problem belongs the class \( QECC(g, n) \) if the agents use \( O(g) \) ebits and \( O(n) \) cbits. An ebit is equivalent to an \( n \)-partite maximally entangled state shared between the agents as we have used in Section 3. We saw in our Theorem 3.1 how the fair and unbiased election of a leader is possible in \( QECC(\log n, n) \). We define another class \( RCC(r, n) \), of problems that can be solved using \( n \) agents that share an \( O(r) \)-bit random string and use only linear classical communication complexity. It is easy to observe that \( RCC(r, n) \subseteq QECC(r, n) \). This is because by measuring his own qubits, one qubit in each of the ebits shared with other agents, an agent can generate as many ideal random bits shared between all the agents. Whether this relationship is a proper containment is an interesting fundamental open problem concerning the power of quantum entanglement as a computational and communication resource.

## 6 Concluding Remarks

It would be interesting to design other classical and randomized protocols for fair and unbiased leader election that work for any number of liars, possibly using superlinear communication complexity. Other fundamental problems in distributed computing related to multicasting, broadcasting or consensus formation may also be studied to see if quantum entanglement can reduce classical communication requirement in asymptotically randomized protocols.
Appendix

Let $C(n, r)$ be the number of combinations of $n$ things taken $r$ at a time without repetitions. We note that

$$C(n - k, \log n) = \frac{(n - k)!/(\log n)!}{n!/(\log n)!}$$

Cancelling the common terms and applying Sterling’s approximation to the four terms for large $n >> k$, we have the estimate

$$\frac{(n - k)(n - \log n)^{n - \log n + \frac{1}{2}}}{(n - k - \log n)^{n - k - \log n + \frac{1}{2}} n^{n + \frac{1}{2}}}$$

$$= \frac{(n - k + \frac{1}{2})(n - \log n)^{n - \log n + \frac{1}{2}}}{(n - \log n)^{n - k - \log n + \frac{1}{2}} (1 - \frac{k}{n - \log n})^{n - k - \log n + \frac{1}{2}} n^{n + \frac{1}{2}}}$$

$$= \frac{(1 - \frac{k}{n})^{n - k + \frac{1}{2}}}{(1 - \frac{k}{n - \log n})^{n - k - \log n + \frac{1}{2}}}$$

Now note that putting $k = \log n$, and using Euler’s formula with $n$ increasing asymptotically, the above estimate approaches unity. On the other hand, putting $k = Cn/\log n$, for a constant $C > 0$ takes the estimate to $e^{-C}$ as $n$ blows up. Here $e$ is Euler’s constant.

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