Quantum games via search algorithms

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Abstract

We build new quantum games, similar to the spin flip game, where as a novelty the players perform measurements on a quantum system associated to a continuous time search algorithm. The measurements collapse the wave function into one of the two possible states. These games are characterized by a continuous space of strategies and the selection of a particular strategy is determined by the moments when the players measure.

Key words: Quantum game; Quantum algorithm

1 Introduction

In the field of quantum computation, discoveries like the algorithms of Shor and Grover [1,2] showed a superior efficiency with regard to their classical equivalents. The Shor algorithm factors efficiently any large number and the Grover algorithm locates a marked item in a disordered list of $N$ elements in a number of steps proportional to $\sqrt{N}$, instead of the $O(N)$ of its classical counterpart. This search algorithm has also a continuous time version [3] that has been described as the analog analogue of the original Grover algorithm. We have recently developed a new quantum search algorithm with continuous time [4], that finds a discrete eigenstate of a given Hamiltonian $H_0$, if its eigenenergy is given. The essence of our algorithm, consists on producing a resonance between the initial state and the looked for state. This resonant algorithm behaves like Grover’s, and its efficiency depends on the spectral density of the Hamiltonian $H_0$. On the other hand, in the last years, the theory of games has been used to explore the nature of quantum information. Initially quantum games were proposed as the quantum generalization of the
classical games but, due to the principles of quantum mechanics, new game possibilities arose. These studies showed the surprising result that the quantum strategies could be more efficient than classical ones. The simplest quantum game, described originally by D. Meyer [5], is the PQ penny flip. It has the virtue of allowing an easy connection with quantum algorithms. In this game, two players apply unitary operators alternatingly on the same qubit. The result of the game depends on the final projection of the qubit over its basic states. In this context, we propose new quantum games built with our quantum search algorithm. These games are similar to the PQ penny flip but we have incorporated to them a continuous time dynamics together with a quantum measurement process. The strategies of the players will be determined by the moments when the players make measurements on the system.

In the following section we review our quantum search algorithm. In section 3 the measurement process is introduced and the dynamical equation of the system are obtained. In the section 4 we propose different two person zero sum games and we analyze the phase space of strategies. Conclusions are drawn in the last section.

2 Search Algorithm

The algorithm is built on a Hamiltonian $H_0$ with normalized eigenstates $\{|n\rangle\}$ and eigenvalues $\{\varepsilon_n\}$. Consider a subset $N$ of $\{|n\rangle\}$ formed by $N$ states. Let us call $|s\rangle$ the unknown searched state in $N$ whose energy $\varepsilon_s$ is given and $|j\rangle$ the known initial state, whose eigenvalue $\varepsilon_j$ is also known. But this initial state does not belong to the search set $N$. So knowing $\varepsilon_s$ is equivalent to “marking” the searched state in Grover’s algorithm. To build the quantum search algorithm a potential $V(t)$ is necessary that produces the coupling between the initial state and the searched state. Our proposal [4] is the following potential

$$
V(t) = |p\rangle \langle j| \exp (i\omega_{sj}t) + |j\rangle \langle p| \exp (-i\omega_{sj}t),
$$

(1)

where $|p\rangle \equiv \frac{1}{\sqrt{N}} \sum_{n \in N} |n\rangle$ is an unitary vector which can be interpreted as the average of the set of vectors in $N$, and $\omega_{sj} \equiv \varepsilon_j - \varepsilon_s$. This proposal assures that the interaction potential is hermitian, that the transition probabilities $W_{nj} \equiv |\langle n|V(t)|j\rangle|^2 = \frac{1}{N}$, from state $|j\rangle$ to any state of the set $N$ are all equal, and finally that the sum of the transition probabilities verifies $\sum_{n \in N} W_{nj} = 1$.

The objective of the algorithm is to find the eigenvector $|s\rangle$ whose transition energy from the initial state $|j\rangle$ is the Bohr frequency $\omega_{sj} = \varepsilon_s - \varepsilon_j$, with $\hbar = 1$. In order to perform this task the Schrödinger equation, with the Hamiltonian $H = H_0 + V(t)$, is solved. The wave-function, $|\Psi(t)\rangle$, is expressed as an expansion in the eigenstates $\{|n\rangle\}$ of $H_0$, $|\Psi(t)\rangle = \sum_m a_m(t) \exp (-i\varepsilon_m t) |m\rangle$. The
time dependent coefficients \(a_m(t)\) have initial conditions \(a_j(0) = 1, a_m(0) = 0\) for all \(m \neq j\). After solving the Schrödinger equation the following coefficients are obtained: \(a_j(t) \simeq \cos(\Omega t), a_s(t) \simeq \sin(\Omega t), a_n(t) \simeq 0,\) for \(n \neq j\) and \(n \neq s\) with \(\Omega = \frac{1}{\sqrt{N}}\). Then the state-probabilities are

\[
\begin{align*}
P_j(t) & \simeq \cos^2(\Omega t), \\
P_s(t) & \simeq \sin^2(\Omega t), \\
P_n(t) & \simeq 0, \ n \neq j \text{ and } n \neq s.
\end{align*}
\]

From these equations it is clear that a measurement made at the time \(t = \tau \equiv \frac{\pi}{2\Omega}\) has a probability very close to one of yielding the searched state. This approach is valid as long as all the Bohr frequencies satisfy \(\omega_{nm} \gg \Omega\) and, in this case, our algorithm behaves qualitatively like Grover’s.

3 Repeated measurements in the algorithm

Our search has an oscillatory transition between the initial state and the sought state, the other states are negligibly populated. The wave function behaves as a time dependent qubit where the coefficients of the eigenstates \(|j\rangle\) and \(|s\rangle\) are alternating in time, that is

\[
|\Psi(t)\rangle \approx a_j(t) \ |j\rangle + a_s(t) |s\rangle \approx \cos(\Omega t) \ |j\rangle + \sin(\Omega t) \ |s\rangle,
\]

see Fig. 1. Then the measurement of any observable of the system produces

![Fig. 1. Coefficients of the searched \(a_s\) and the initial \(a_j\) states as a function of the time in a circle of unit radius.](image)

the collapse of the wave function in the state \(|j\rangle\) or \(|s\rangle\). Supposing that the measurement does not modify the Hamiltonian \(H_0\) and the potential \(V(t)\) then, after the measurement, the resonance algorithm is not affected and continues to work but starting now from a new initial state, \(|j\rangle\) or \(|s\rangle\). Thus, any measurement process will leave the system, with high probability, in one of these two states. The probabilities associated with the states \(|s\rangle\) and \(|j\rangle\) initially evolve according to the map of eq.(2). If at time \(t_1\) the state is measured, the probabilities that the wave function collapses into either \(|s\rangle\) or \(|j\rangle\) are given
by the eq. (2) evaluated at \( t = t_1 \). Immediately after this first measurement, the system is in state \(|s\rangle \) (or \(|j\rangle\)) and it has an unitary evolution until the time \( t_2 = t_1 + \Delta t_1 \) when a second measurement is made. The probabilities of the states \(|s\rangle\) and \(|j\rangle\) after the second measurement at the time \( t_2 \), are given by

\[
P_s(t_2) \approx \cos^2(\Omega \Delta t_1),
\]
\[
P_j(t_2) \approx \sin^2(\Omega \Delta t_1),
\] (4)

(or the same equations exchanging \( s \) and \( j \)). Therefore the system undergoes an unitary evolution for arbitrary intervals \( \Delta t_i = t_{i+1} - t_i \) between consecutive measurements and the probabilities of the states \(|s\rangle\) and \(|j\rangle\) satisfy the matrix equation

\[
\begin{pmatrix}
P_s(t_{i+1}) \\
P_j(t_{i+1})
\end{pmatrix}
=
\begin{pmatrix}
p_i & q_i \\
q_i & p_i
\end{pmatrix}
\begin{pmatrix}
P_s(t_i) \\
P_j(t_i)
\end{pmatrix},
\]

(5)

where \( p_i = \cos^2(\Omega \Delta t_i) \) and \( q_i = 1 - p_i \) are transition probabilities, always within the approximation \( \omega_{nm} \gg \Omega \). This last equation looks like a master equation, which suggests that the global evolution of the system could be a Markov process. Markovian process have the property that for any set of successive times \( (t_1, t_2, t_3, ..., t_n) \) the conditional probability at \( t_n \) is uniquely determined by the value of stochastic variables at \( t_{n-1} \) and is not affected by any knowledge of the values at earlier times. In other words, the system depends only on the current state and not on the path of the process. In our case, the measurement of the states of the system is a simple but extreme form of introducing decoherence that produces a loss of long range memory. But in eq. (5), the matrix of conditional probabilities is time-interval dependent, then this equation does not represent a Markovian process. Anyway, a general solution of the previous equation, for any sequence of measurements, is obtained.

\[
\begin{pmatrix}
P_s(t_m) \\
P_j(t_m)
\end{pmatrix}
=
\begin{pmatrix}
\alpha_m & \beta_m \\
\beta_m & \alpha_m
\end{pmatrix}
\begin{pmatrix}
P_s(0) \\
P_j(0)
\end{pmatrix},
\]

(6)

where \( P_s(0) = 0 \), \( P_j(0) = 1 \) and

\[
\alpha_m = \frac{1}{2} \left[ 1 + \prod_{i=0}^{m} [2p_i - 1] \right],
\]
\[
\beta_m = \frac{1}{2} \left[ 1 - \prod_{i=0}^{m} [2p_i - 1] \right].
\] (7)

If we now consider that the measurement processes are performed at regular time intervals, \( t_n = n \Delta t \), eq. (7) becomes

\[
\alpha_m = \frac{1}{2} \left[ 1 + (\cos(2\Omega \Delta t))^m \right],
\]
\[
\beta_m = \frac{1}{2} \left[ 1 - (\cos(2\Omega \Delta t))^m \right].
\] (8)
In this case all the $\Delta t_i$ are equal and the probability distribution satisfies a master equation then the global evolution, in a time involving many measurement events, can be described as a Markovian process. The system has an unitary evolution only between consecutive measurements. At first sight, for a sufficiently large number of measurements, the eq.\((8)\) imply that both $P_s$ and $P_j$ tend to $1/2$ independently of the interval between measurements $\Delta t$ and the initial conditions. However if the considered total time is fixed the situation is different. When $m$ measurements are performed in a total time $\tau = \frac{\pi}{2\Omega}$, we have $\Delta t = \tau/m$ and the coefficients are

$$\alpha_m = \frac{1}{2} \left[ 1 + \left( \cos \frac{\pi}{m} \right)^m \right]$$

$$\beta_m = \frac{1}{2} \left[ 1 - \left( \cos \frac{\pi}{m} \right)^m \right], \quad (9)$$

then in this case $P_s \simeq 0$ and $P_j \simeq 1$ when $m \to \infty$. This simply means that the more frequently the wave function collapses, the harder it becomes for the algorithm to significantly depart from the initial state. Therefore, in this case the algorithm behaves as an example of the quantum Zeno effect, where the a high frequency of measurements hinders the departure of the system from its initial state \([6,7]\).

4 The search games

In above theoretical framework, let us consider a simple quantum state flip game played between Silvia and Juan. Initially the system is prepared in the state $|j\rangle$ and the dynamics develops according to the unitary operator $U(t)$ associated to the Hamiltonian $H(t)$ of the search algorithm. At the time $T_1 \in [0, \tau)$, Juan measures the state of the system and afterwards it evolves again with $U(t)$. Silvia knows the time at which Juan measured but does not know the result of his measurement. She measures the system at the time $T_2 \in [T_1, \tau]$ and then the game concludes. The result of the last measurement determines who wins the game, if the state is $|s\rangle$ Silvia wins $1$ (Juan loses $1$) and if the state is $|j\rangle$ Juan wins $1$ (Silvia loses $1$). This is a two-person, zero-sum game, then the payoff to Silvia is the exact opposite of that to Juan. The players must make their measurements obeying the condition $0 \leq T_1 \leq T_2 \leq \tau$, but the precise moment when they measure remains their decision and this determines their strategies. For example, one of the worst strategies for Juan is to measure at $T_1 \sim \tau$, because independently of the time Silvia measures, he loses almost always. To understand this note that $\tau$ is the optimal time to obtain the searched state; Silvia’s measurement is very close to Juan’s, then the wave function does not evolve (Zeno effect) and the last state is $|s\rangle$ with high probability. The probability of the searched state as a function of
Fig. 2. Searched state probability as a function of $T_1$ and $T_2$. The time is expressed in units where $\tau = 1$.

the times $T_1$ and $T_2$ at which Silvia and Juan carry out their measurements is presented in Fig. 2 where we used the eq. (6) with $m = 2$; its values lie above the straight line $T_1 = T_2$ due to the condition $T_1 \leq T_2$ and the different tonalities of gray indicate different probability intervals. Looking at this figure, the players can plan their strategies and quickly conclude that it is not an equitable game. Silvia has many winning strategies, Juan has no winning strategy, but he has only one strategy that allows him to tie the game. Juan’s optimal strategy is to carry out his measurement at $T_1 = 0.5\, \tau$, because in this case both players have same probability to win the game, independently from Silvia’s measurement. This could be an extreme case of the Nash equilibrium [8] because any change in the strategy of Juan could worsen his results and any change of strategy of Silvia would be indifferent for her. The existence of this equilibrium adds to the game another element of interest and in a certain way can surprise us, because we have used quantum rules in the game (the wave function collapse). But remembering that the global evolution of this game is a Markovian process, its existence is a consequence of the mathematical similarity with classical games [9].

An interesting variant of the previous game is obtained when a third measurement at $t = T_3 = \tau$ is incorporated. In this new game, after Silvia’s measurement at $t = T_2$, the system evolves unitarily until anyone of the players makes a third measurement at time $t = \tau$. Again Silvia wins $\$1$ and Juan loses $\$1$, if the result of the last measurement is $|s\rangle$, independently of the result of the previous measurements, otherwise Juan wins $\$1$ and Silvia loses $\$1$. Fig. 3 shows the probability of the searched state as function of the times $T_1$ and $T_2$ for the new game. In this figure a new area with high probability appears when the players make their measurements at the beginning of the
game. It is easy to understand this result using the previous ideas of Zeno effect and optimal time $\tau$ to make the measurement. The Nash equilibrium is also present for the same strategy as that used in the previous game.

Fig. 3. Searched state probability as a function of $T_1$ and $T_2$. The time is expressed in units where $T_3 = \tau = 1$.

In this game the zone $P_s \leq 0.3$ has disappeared as a result of the increase of asymmetry in the players’ strategies, then Silvia’s probability to win is bigger than Juan’s. With the aim to quantify the inequity of these games we now introduce the mean payoff of the game for each player. In the first place, note that the probabilities of the eq.(6), with $m = 2$, have a parametric dependencies on $T_1$ and $T_2$. Then, using eq.(6) for the first game, we define the win density for Silvia $\sigma_s$ and Juan $\sigma_j$ as

$$
\begin{align*}
\left( \frac{\sigma_s(T_1, T_2)}{\sigma_j(T_1, T_2)} \right) &\equiv \frac{2}{\tau^2} \left( \frac{\beta_2(T_1, T_2)}{\alpha_2(T_1, T_2)} \right), \\
\end{align*}
$$

(10)

where

$$
\begin{align*}
\alpha_2 &= \frac{1}{2} \left\{ 1 + (2 \cos^2(\Omega(T_2 - T_1)) - 1)(2 \cos^2(\Omega T_1) - 1) \right\}, \\
\beta_2 &= \frac{1}{2} \left\{ 1 - (2 \cos^2(\Omega(T_2 - T_1)) - 1)(2 \cos^2(\Omega T_1) - 1) \right\}. \\
\end{align*}
$$

(11)

If the players have as strategy to choose $T_1$ and $T_2$ at random, but obeying the constraints, then the probabilities of winning the game are
\[ \pi_s = \int_0^\tau dT_1 \int_0^{T_1} \sigma_s(T_1, T_2) \, dT_2 = 0.5 \quad (12) \]
\[ \pi_j = \int_0^\tau dT_1 \int_0^{T_1} \sigma_j(T_1, T_2) \, dT_2 = 1 - \pi_s = 0.5, \quad (13) \]

for Silvia and Juan respectively. The expected payoff of Silvia is \( \bar{\pi}_s \equiv \$1\pi_s - \$1\pi_j \) and the expected payoff of Juan is \( \bar{\pi}_j \equiv \$1\pi_j - \$1\pi_s \), in this game \( \bar{\pi}_s = \bar{\pi}_j = \$0 \). In the second game, we define the win densities as in first game but substituting \( \alpha_2 \) and \( \beta_2 \) by

\[ \alpha_3 = \frac{1}{2} \left\{ 1 + (2\cos^2(\Omega(\tau - T_2)) - 1) \right\}, \]
\[ \beta_3 = \frac{1}{2} \left\{ 1 - (2\cos^2(\Omega(\tau - T_2)) - 1) \right\}. \quad (14) \]

Now, \( \pi_s = \frac{7}{8}, \pi_j = \frac{3}{8} \) and the expected payoff of Silvia is \( \bar{\pi}_s = \$0.5 \) and the expected payoff of Juan is \( \bar{\pi}_j = -\$0.5 \). Then, they tie in the first game but Silvia wins a little more in average in the second game, although they have the same strategy.

Changing the number of measurements allowed in the game and the time intervals between them it is possible to favor anyone of the players. In the second game Silvia was the winner but, for example, in games where the players make many measurements at regular time intervals the eq.(9) tell us that Juan will be the winner. Then it is possible to modify the proposed games or to introduce others using the equations of the previous section.

5 Conclusions

A quantum game, as a quantum algorithm, may be seen as a definite sequence of unitary transformations acting over a quantum state, in some Hilbert space. Concepts like interference phenomena, quantum measurements, resonances, quantum parallelism, amplification techniques, entanglement, etc should be employed in the new field of quantum games. We have developed a new kind of quantum game for which there is no classical analogue, it is simple enough and shows the importance of measurement as a fundamental element in the development of quantum games. These games may be a tool to study quantum algorithms subjected to external decoherence, as in the extreme case of measurement \[10\].
These games are inspired by the quantum search algorithm but they are very interesting by themselves. They may be thought, as time dependent games where to win or to lose is determined by the collapse of the qubit in its basic states. The games strategies are developed by the players choosing the times of measurement.

Recent experimental advances allow to obtain and preserve the quantum states for a system of atoms [11]. This opens interesting possibilities to trap a quantum system with only two energy levels, that allow the experimental realization of the quantum games proposed in the paper. Finally we should point out that in this work entanglement is absent, the game is developed with only one qubit; then a challenge for the future is to introduce measurements in quantum games with entanglement [12], surely in these cases new interesting behaviors will be obtained.

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