LOCALIZATION FOR RANDOM WALKS IN RANDOM ENVIRONMENT IN DIMENSION TWO AND HIGHER

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ABSTRACT. In this paper, we introduce the notion of localization at the boundary for random walks in i.i.d. and uniformly elliptic random environment, in dimensions two and higher. Informally, this means that the walk spends a non-trivial amount of time at some point \( x \in \mathbb{Z}^d \) with \( \|x\|_1 = n \) at time \( n \), for \( n \) large enough. In dimensions two and three, we prove localization for (almost) all walks. In contrast, for \( d \geq 4 \) there is a phase-transition for environments of the form \( \omega_\epsilon(x, e) = q(e) + \epsilon \xi(x, e) \), where \( \{\xi(x)\}_{x \in \mathbb{Z}^d} \) is an i.i.d. sequence of random variables, and \( \epsilon \) represents the amount of disorder with respect to a simple random walk. The proofs involve a criterion that connects localization with the equality or difference between the quenched and annealed rate functions at the boundary.

1. INTRODUCTION AND BACKGROUND

Random walk in random environment (RWRE) is a fundamental model in probability that can be used as a prototype for a variety of phenomena. Examples of this include DNA chain replication [8], crystal growth [25], among others. This model was introduced in the ’70s to study motion in random media. In dimension \( d = 1 \), the model is well understood. Some of the known results include transience, recurrence, law of large numbers ([24], [1]), and large deviations ([15], [10]), among others. However, when \( d \geq 2 \), there are several open questions, including how to characterize precisely when the walk is transient or recurrent, or whether directional transience implies ballisticity. We refer to the reader the references [13] and [29] for a complete presentation of the model.

In this paper, we deal with the notion of localization. Informally, we say that the walk is localized if the asymptotic trajectory of the walk is confined to some region with positive probability. Otherwise we say that it is delocalized. For RWRE, this has been studied almost exclusively in the one-dimensional case (see for example the works of Sinai [23] and Golosov [14]). When the dimension is two or higher, the topic has been practically untouched (some exceptions are the works [6] and [12]). This paper aims to open the door to further research in this area. To motivate this notion, consider first a simple random walk (SRW) \( (S_n)_{n \in \mathbb{N}} \) on \( \mathbb{Z}^d \), conditioned to reach the boundary at time \( n \), that is, \( |S_n|_1 = n \) for each \( n \in \mathbb{N} \). This walk is an example of delocalization since it presents a diffusive behavior. A natural question is to ask if the same situation continues to happen if we perturb the walk in some (random) directions. It turns out that the introduction of a small disorder can change the typical paths of the walk in such a way that the perturbed walk has a favorite trajectory that it’s likely to visit. That is a good reason to study the localization/delocalization phenomena for the RWRE model since the disorder can be introduced naturally. In the previous example, we can consider environments of the type

\[
\omega_\epsilon(x, e) = \frac{1}{2d} + \epsilon \xi(x, e),
\]

(1.1)

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where \((\xi(x, \cdot))_{x \in \mathbb{Z}^d}\) is an i.i.d. family of mean-zero random variables. Under this setting, the question is whether there is localization or delocalization for a given \(\varepsilon\). As the case \(\varepsilon = 0\) corresponds to delocalization, one foresees that under a low disorder this will also be the case, and for large enough disorder the opposite will occur. Thus, we expect the existence of a phase transition in terms of the parameter \(\varepsilon\). In fact, we prove in Theorem 1.8 that this is true under a slightly generalization of (1.1). This dichotomy in terms of the disorder is not new in the field of random media, and it is plausible that new results in this direction can be obtained in the future, relating the disorder of the environment with properties of the walk.

Another factor that can play a crucial role in the localization/delocalization problem is the dimension. For example, in the SRW model it is well known the recurrence for dimension one and two, and the transience for dimensions three and higher. In general, if the dimension is low, the walk has less space to move, and this can help to create regions where the walk spends a non-trivial amount of time. We prove in Theorem 1.5 that for RWRE in dimensions two and three, we have localization for almost all the possible distributions of the environments. In fact, the walks that are delocalized in dimensions two and three correspond to random walks in space-time i.i.d. random environment (see for instance [27]). Moreover, these walks are delocalized at any dimension.

One of the main ingredients to prove Theorem 1.5 is a criterion that relates localization/delocalization with the equality or difference between the quenched and annealed rate functions of an RWRE at the boundary. Without being completely rigorous for now, consider a face \(F\) of the set \(D := \{x \in \mathbb{R}^d : |x| = 1\}\). If \(I_q\) and \(I_a\) are the quenched and annealed rate functions for a RWRE (cf. Eq. (2.11) for the definition), then in Theorem 2.2 we show that localization in the face \(F\) is equivalent to

\[
\inf_{x \in F} I_a(x) < \inf_{x \in F} I_q(x)
\]

and delocalization in the same face is equivalent to the equality in (1.2). For this reason we will include in the definition of localization (in the face \(F\)) the event where (1.2) holds. This criterion is one of the crucial results since the annealed rate function at the boundary can be computed explicitly (cf. Theorem 1 in [5]). Even though the quenched rate function has not an easy explicit formula (see for example Theorem 2 in [21]), one can obtain estimates for the quenched infimum in Eq. (1.2) that assures the strict inequality in the same equation. In Section 5.1 we show an example of when this happens. The connection between large deviations and the localization phenomena is an important discovery that we hope can be used to understand better the theory of large deviations for RWRE, and vice versa.

To finish this introduction, we mention that in the model of directed polymers in random environment the path localization of the walk has been studied for a while, and several remarkable results have been obtained in the last two decades (cf. [11], [2], [4], [3] to select a few of them). The lectures notes [9] contains an updated account of some of these articles. To which extent these results can be applied to an RWRE is a question that we would like to answer in later works; this is the first attempt to do it. Another issue is what additional features have in common these two models.

Now we proceed to introduce the basic definitions and notation of this work.

1.1. Definitions. Fix \(d \in \mathbb{N}\), the dimension where the walk moves. Define \(V := \{x \in \mathbb{Z}^d : |x|_1 = 1\} = \{\pm e_1, \ldots, \pm e_d\}\) the set of jumps of the walk (here \(e_i\) is the vector with zero coordinates excepting the one in the \(i\)th position). Next define \(\mathcal{P}\) as the set of probability vectors, that is,

\[
\mathcal{P} := \{p : V \to [0, 1] : \sum_{e \in V} p(e) = 1\}.
\]
Now we can define the environments. An environment is an element $\omega$ in the space
$$
\Omega := \{ \omega : \mathbb{Z}^d \times V \to [0,1] : \omega(x) \in \mathcal{P} \text{ for all } x \in \mathbb{Z}^d \} = \mathcal{P}^{\mathbb{Z}^d}.
$$

We usually write $\omega = \{ \omega(x,e) \}_{x \in \mathbb{Z}^d, e \in V}$. Finally, we can define a random walk in the random environment $\omega \in \Omega$ starting at a point $x \in \mathbb{Z}^d$ as the Markov chain $X = (X_n)_{n \in \mathbb{N}}$ with law $P_{x,\omega}$ that satisfies
$$
P_{x,\omega}(X_0 = x) = 1,
$$
$$
P_{x,\omega}(X_{n+1} = y + e | X_n = y) = \omega(y,e), \ n \geq 0, y \in \mathbb{Z}^d, e \in V.
$$

(1.3)

The measure $P_{x,\omega}$ in the literature is known as the quenched measure, in contrast to the annealed (or averaged) measure that we describe next.

Equip the space $\Omega$ with the Borel $\sigma$–algebra $\mathcal{B}(\Omega)$, and consider a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$. The annealed measure $P_x$ of the RWRE starting at $x \in \mathbb{Z}^d$ is defined as the measure on $\Omega \times (\mathbb{Z}^d)^\mathbb{N}$ that satisfies
$$
P_x(A \times B) = \int_A P_{x,\omega}(B) \, d\mathbb{P}
$$
for each $A \in \mathcal{B}(\Omega), B \in \mathcal{B}((\mathbb{Z}^d)^\mathbb{N})$, the Borel $\sigma$–algebras of $\Omega$ and $(\mathbb{Z}^d)^\mathbb{N}$ respectively. Expectations with respect to $P_{x,\omega}, P_x$ and $\mathbb{P}$ are denoted by $E_{x,\omega}, E_x$ and $\mathbb{E}$ respectively. We also define for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ its $\ell^1$ norm by $|x|_1 := \max(|x_1|, \cdots, |x_d|)$.

The basics assumptions in this work are the following:

**Assumption 1.1.**

(i) The random variables $\{\omega(x, \cdot)\}_{x \in \mathbb{Z}^d}$ are i.i.d under $\mathbb{P}$.

(ii) There exists a $\kappa > 0$ such that for every $x \in \mathbb{Z}^d$ and $e \in V$,

$$
\mathbb{P}(\omega(x,e) \geq \kappa) = 1.
$$

(1.5)

The two assumptions above are common in the literature. In particular, under assumption (i), we can define

$$
q(e) := \mathbb{E}[\omega(0,e)] = \mathbb{E}[\omega(x,e)], \ x \in \mathbb{Z}^d, e \in V.
$$

1.2. **Localization at the boundary.** Now we can make precise the statement about walks that reach the boundary. We will consider a RWRE $(X_n)_{n \in \mathbb{N}}$ such that $|X_0| = n$ for each $n$. Another point of interest is that such walks are related to the rate functions in the boundary of the set $\mathbb{D} := \{ x \in \mathbb{R}^d : |x|_1 = 1 \}$. More details about large deviations on the boundary can be found in [5].

We will be interested in the normalized quenched probability of reaching the boundary at time $n$, that is, if $x \in \mathbb{Z}^d$ satisfies $|x|_1 = n$,

$$
P_{0,\omega}(X_n = x | |X|_1 = n),
$$

(1.6)

and then study the long term behavior of these probabilities. Specifically, we are interested in knowing if for some sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^d$ such that $|x|_1 = n$ for all $n$, the quenched probability (1.6) is greater than some constant $c$, uniformly on $n$. In this case, the (conditioned) walk is "localized" around this path (the rigorous definition appears in Definition 1.2 below). There is a counterpart in the literature of directed polymers in random environment(see for example [9], page 88). In this model, there is a nice characterization of localization/delocalization depending on the disorder of the environment. For RWRE, the disorder measures how far is the environment $\omega(0,e)$ of its expectation $q(e)$. This allows us to obtain analogous results in our case.
At this point we proceed to define rigorously localization. We decompose \( \partial \mathbb{D} \) in faces \( \partial \mathbb{D}(s), s \in \{-1,1\}^d \), defined by
\[
\partial \mathbb{D}(s) := \{ x \in \partial \mathbb{D} : s_j x_j \geq 0, j = 1, \ldots, d \}.
\]
We also defined the s-allowed jumps by
\[
V(\mathbb{S}) := \{ s_j e_j : j = 1, \ldots, d \} \subseteq V.
\]
From now on, we fix the face \( \partial \mathbb{D}(\mathbb{S}) \), with \( \mathbb{S} = (1,1,\cdots,1) \), but the results in this paper can be applied to any face. Next, we consider the set
\[
\partial \mathbb{R}_n := n \partial \mathbb{D}(\mathbb{S}) = \{ x \in \mathbb{Z}^d : |x|_1 = n, \mathbb{S} x_j \geq 0 \text{ for all } j \in \{1, \ldots, d\} \}
\]
and define \( \mathbb{R}_n \) as the sets of all paths \( (z_0,z_1,\cdots,z_n) \in (\mathbb{Z}^d)^{n+1} \) for which \( z_0 = 0 \) and \( z_n \in \partial \mathbb{R}_n \). Note that this happens if and only if \( z_i - z_{i-1} \in V(\mathbb{S}) \) for each \( i = 1, \ldots, n \). An important observation is that
\[
z = (0,z_1,\cdots,z_n) \in \mathbb{R}_n \iff (0,\cdots,z_{n-1}) \in \mathbb{R}_{n-1} \text{ and } z_n - z_{n-1} \in V(\mathbb{S}).
\]
We also consider the sequence \( (J_n)_{n \in \mathbb{N}} \) defined by \( J_1 := 1 \), and for \( n \geq 2 \),
\[
J_n := \max_{x \in \partial \mathbb{R}_{n-1}} P_{0,\omega}(X_{n-1} = x|A_{n-1}),
\]
where \( A_n := \{ X_n \in \partial \mathbb{R}_n \} \) (if the walk starts and \( x \in \mathbb{Z}^d \), then \( A_n = \{ X_n \in x + \partial \mathbb{R}_n \} \)).

**Definition 1.2.** Given RWRE \( (X_n)_{n \in \mathbb{N}} \), we say that it is localized at the boundary (in the face \( \partial \mathbb{D}(\mathbb{S}) \)) if
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n J_n > 0 \quad \mathbb{P} - \text{a.s.}
\]
Similarly, the RWRE is delocalized at the boundary if
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n J_n = 0 \quad \mathbb{P} - \text{a.s.}
\]

**Remark 1.3.** An alternative way to define localization/delocalization is considering the sequence \( J_n \) itself instead of its Césaro means, and studying the liminf. However, the localization given by (1.10) is stronger. Also note that a priori, the walk can be neither localized nor delocalized. However, in Theorem 2.2, we show that this cannot happen for walks that satisfy Assumption 1.1.

1.3. **Main results.** The main result of this paper is that localization holds for (almost) all uniformly elliptic and i.i.d environments in dimensions two and three, and for small disorder in dimensions \( d \geq 4 \).

The following condition will play a remarkable role in the results that follow.

**Assumption 1.4.** The measure \( \mathbb{P} \) satisfies
\[
\mathbb{P} \left( \sum_{e \in V(\mathbb{S})} \left[ \omega(0,e) - q(e) \right] = 0 \right) < 1.
\]

**Theorem 1.5.** Let \( (X_n)_{n \in \mathbb{N}} \) be a RWRE that satisfies Assumption 1.1, and \( d \in \{2,3\} \). If Assumption 1.4 holds, then there is localization. Otherwise, there is delocalization. The last statement also holds for any dimension \( d \geq 2 \).

**Remark 1.6.** This result suggest that a local limit theorem may not hold in dimensions two and three (compare it with Theorem 1.11 in [6]). Also, it can provide additional information of the invariant measure with respect to \( \mathbb{P} \) (if it exists, see for example Theorem 5 in [12]).
A related result in RWRE is found in the article [28] of Yilmaz and Zeitouni. They show that for walks that satisfy certain ballisticity condition\(^1\), there is a class of measures \(\mathcal{P}\), such that the quenched and annealed rate functions differ in a neighborhood of the LLN velocity (and a stronger result for RWRE in i.i.d space-time environments).

In the directed polymers model, Lacoin [18] proves localization in dimension \(1 + 2\) (one dimension for time, and two for space), and also gives a proof in dimension \(1 + 1\). This result was improved by Berger and Lacoin [7], where they give a precise asymptotic of the difference between the quenched and annealed free energies.

For \(d \geq 4\), we consider a certain family of environments, parameterized by \(\varepsilon\). This parameter represents how much the distribution of the jumps in a RWRE differs from the ones in a simple random walk.

**Definition 1.7.** Assume that \((X_n)_{n \in \mathbb{N}}\) is a RWRE that satisfies Assumption 1.1. For \(\varepsilon \in [0, \varepsilon_{\text{max}}]\), we say that the environments are in \(\varepsilon\)-low disorder (with respect to \(V(\mathfrak{s})\)) if and only if there exists a sequence of i.i.d (centered) random variables \(\eta(x, e)\) such that

\[
\omega(\varepsilon, x, e) := q(e) + \varepsilon \xi(x, e).
\]

Here, \(\varepsilon_{\text{max}}\) is defined by

\[
\varepsilon_{\text{max}} := \min_{e \in V(\mathfrak{s})} \left[ \frac{q(e)}{\xi(0, e) - \xi(0, e)_-} \right],
\]

and \(\xi(0, e)_-\) is the negative part of \(\xi(0, e)\).

The low disorder regime for RWRE has been studied in, [22], [19], and others, mostly in the ballistic case. Recently, it also has been considered in [5] to prove the monotonicity of the map

\[
\varepsilon \rightarrow I_a(x, \cdot) - I_q(x, \cdot),
\]

where \(I_q(x, \cdot), I_a(x, \cdot)\) are the quenched (respectively annealed) rate functions of a RWRE in the environment \(\omega(\varepsilon)\). For such class of environments we have a phase transition for localization/delocalization.

**Theorem 1.8.** Let \((X_n)_{n \in \mathbb{N}}\) be a RWRE whose environments are in the \(\varepsilon\)-low disorder with respect to \(V(\mathfrak{s})\). Then there is \(\tau \in [0, \varepsilon_{\text{max}}]\) such that if \(\varepsilon < \tau\) there is delocalization, and if \(\varepsilon_{\text{max}} \geq \varepsilon > \tau\) we have localization. Moreover we have the following:

(i) If Assumption 1.4 does not hold, then \(\tau = \varepsilon_{\text{max}}\). Otherwise,

(ii) if \(d = 2\) or \(3\), then \(\tau = 0\);

(iii) if \(d \geq 4\), \(\tau > 0\). Also, there are examples of walks that satisfies \(\tau < \varepsilon_{\text{max}}\).

The theorem above combines results from [5] together with Theorem 1.5 and the example from Section 5.1. Moreover, this result says that for walks in low disorder there is also a phase transition in the dimension of the walk. For dimensions two and three, the only possible critical values are the extreme points of the spectrum. If \(d \geq 4\), there are non-trivial points. Thus, we obtain a complete picture of whether localization or delocalization happens for all dimensions.

**Remark 1.9.** We expect that analog results of Theorems 1.5 and 1.8 hold for more general trajectories, that is, not only paths that reach the boundary at each time.

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\(^1\)In the aforementioned article, it’s used the so-called condition (T). This is equivalent to the ballisticity conditions (T') and \(\mathcal{P}_M\), as showed in [16].
1.4. Organization of the paper. The paper is divided into three main parts. In Section 2 we give an equivalent criterion of localization, that will be very useful to prove our results. This translates the problem of whether an inequality is strict or not. It turns out that this problem is an analog to the study of the difference between the quenched and annealed free energies of directed polymers. The second part is devoted to the proofs of localization in dimensions two and three. Section 3 is dedicated to Theorem 1.5 for $d = 2$, and Section 4 for $d = 3$. In the third part we address the phase transition. In Section 5 we prove Theorem 1.8. Finally, in Sections A.1 and A.2, we provide some auxiliary results that we use here, but their proofs do not affect the body of the text.

2. AN EQUIVALENT CRITERION FOR LOCALIZATION

In this section, we prove an equivalent criterion of localization/delocalization that will be useful to show our results. First, we need to define the following quantities.

Definition 2.1. Let $(X_n)_{n \in \mathbb{N}}$ be a RWRE. Define the limits

$$p(\omega) := \lim_{n \to \infty} \frac{1}{n} \log P_{0,\omega}(X_n \in \partial R_n),$$

$$\lambda := \lim_{n \to \infty} \frac{1}{n} \log P_0(X_n \in \partial R_n).$$

(2.1)

In the directed polymer literature, these limits are known as quenched and annealed free energy. We leave the proof of the existence of these limits to the end of the section (see Lemma 2.7). Moreover, we will show that the first limit does not depend on the environment (it is constant $\mathbb{P}$-a.s.). Assuming that, by Jensen’s inequality we deduce that

$$p \leq \lambda.$$  

(2.2)

Theorem 2.2. Let $(X_n)_{n \in \mathbb{N}}$ be a RWRE that satisfies Assumption 1.1.

(i) The RWRE is localized at the boundary if and only if $p < \lambda$.

(ii) The RWRE is delocalized at the boundary if and only if $p = \lambda$.

In particular, the walk is either localized or delocalized $\mathbb{P}$-almost surely.

2.1. Proof of Theorem 2.2. In order to prove the result, we need to introduce a couple of definitions. The first is a martingale that is related to $p$ and $\lambda$, and the second is a quantity related to $J_n$.

Definition 2.3. Given a RWRE $(X_n)_{n \in \mathbb{N}}$ that satisfies Assumption 1.1, define the random variable in $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$

$$W_n(\omega) := \frac{P_{0,\omega}(X_n \in \partial R_n)}{P_0(X_n \in \partial R_n)}, \quad n \in \mathbb{N}.$$  

(2.3)

Proposition 2.4. The process $(W_n)_{n \in \mathbb{N}}$ is a mean-one $\mathcal{F}_n$-martingale under the filtration $(\mathcal{F}_n)_{n \geq 0}$ given by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, and for $n \geq 1$, $\mathcal{F}_n := \{\omega(x, e) : |x|_1 < n, x \in \mathbb{Z}^d, e \in V(\overline{s})\}$. 
Proof. By definition, $E[W_n] = 1$ for all $n \in \mathbb{N}$. The martingale property follows from (1.8):

$$E[W_{n+1}|\mathcal{F}_n] = \sum_{x \in \mathbb{R}_n} \sum_{e \in V(\mathbb{R})} \frac{1}{(\sum_{e \in V(\mathbb{R})} q(e))^{n+1}} E\left[ P_{0,\omega}(X_n = x, X_{n+1} = x + e) \mid \mathcal{F}_n \right]$$

$$= \sum_{x \in \mathbb{R}_n} \sum_{e \in V(\mathbb{R})} \frac{1}{(\sum_{e \in V(\mathbb{R})} q(e))^{n+1}} E\left[ P_{0,\omega}(X_n = x) \omega(x, e) \mid \mathcal{F}_n \right]$$

$$= \sum_{x \in \mathbb{R}_n} \frac{1}{(\sum_{e \in V(\mathbb{R})} q(e))^{n}} P_{0,\omega}(X_n = x) = W_n.$$  

In the last line we used that $P_{0,\omega}(X_n = x)$ is $\mathcal{F}_n$-measurable and $\omega(x, e)$ is independent of this $\sigma$-algebra. 

The martingale convergence theorem implies that $W_\infty := \lim_{n \to \infty} W_n$ exists and is non-negative $P$-almost surely. Moreover, observe that

$$W_n(\omega) = \frac{\sum_{e \in V(\mathbb{R})} P_{0,\omega}(X_1 = e) P_{0,\omega}(X_n \in \mathbb{R}_n | X_1 = e)}{(\sum_{e \in V(\mathbb{R})} q(e))^{n}}$$

$$= \frac{1}{(\sum_{e \in V(\mathbb{R})} q(e))^{n}} \sum_{e \in V(\mathbb{R})} \omega(0, e) \times \frac{P_{0,T_\omega}(X_{n-1} \in \mathbb{R}_n | X_{n-1} = e)}{(\sum_{e \in V(\mathbb{R})} q(e))^{n-1}}$$

$$= \frac{1}{(\sum_{e \in V(\mathbb{R})} q(e))^{n}} \sum_{e \in V(\mathbb{R})} \omega(0, e) W_{n-1}(T_\omega),$$

where $T_\omega(x, e) := \omega(x + e, e)$. Letting $n \to \infty$ at both sides in the last display, we deduce that $W_\infty(\omega) = \frac{1}{(\sum_{e \in V(\mathbb{R})} q(e))^{\infty}} \sum_{e \in V(\mathbb{R})} \omega(0, e) W_\infty(T_\omega)$. Thus, the event $\{W_\infty = 0\}$ is $T_\omega$-invariant $P$-almost surely for each $e \in V(\mathbb{R})$. The ergodicity of $P$ implies that $P(W_\infty = 0) \in \{0,1\}$. This consequence will be useful in the proposition that we state below.

Next, we introduce a second random variable,

$$I_n(\omega) := \sum_{z \in \mathbb{R}_{n-1}} P_{0,\omega}(X_{n-1} = z | A_{n-1})^2. \quad (2.7)$$

This random variable is $\mathcal{F}_{n-1}$-measurable. Observe that

$$J_n^2 \leq I_n \leq J_n. \quad (2.8)$$

These inequalities imply that both $\frac{1}{n} \sum_{k=1}^{n} J_k$ and $\frac{1}{n} \sum_{k=1}^{n} I_k$ have the same asymptotics as $n$ goes to infinity.

The main ingredient in the proof of Theorem 2.2 is the next proposition that relates $W_n$ and $I_n$.

**Proposition 2.5.** Given a RWRE $(X_n)_{n \in \mathbb{N}}$ that satisfies both Assumption 1.1 and Assumption 1.4, the following is true:

$$A := \{W_\infty = 0\} = B := \left\{ \sum_{n=1}^{\infty} I_n = \infty \right\} \quad P - \text{a.s.} \quad (2.9)$$

Furthermore, if $P(W_\infty = 0) = 1$, there exists $c_1(P), c_2(P) \in (0, \infty)$ for which the following happens $P$-almost surely:

$$c_1 \sum_{k=1}^{n} I_k \leq - \log W_n \leq c_2 \sum_{k=1}^{n} I_k \quad \text{for } n \text{ large enough.} \quad (2.10)$$
Analogously, we say that the position of the walk satisfies an annealed large deviation principle if there is a lower semicontinuous function $I_\mathbf{1}$ there is a lower semicontinuous function $I_\mathbf{1}$ there is a lower semicontinuous function $I_\mathbf{1}$ there is a lower semicontinuous function $I_\mathbf{1}$ there is a lower semicontinuous function $I_\mathbf{1}$.

Also, by Jensen’s inequality and Fatou’s lemma, $I_\mathbf{1}$ there is a lower semicontinuous function $I_\mathbf{1}$.

Next, we characterize the rate functions at $\varnothing\mathbb{D}(\bar{\mathbb{R}})$ (cf., (1.7)).

The proof of Theorem 2.1 in [11] can be adapted to show Proposition 2.5. It is based on the Doob’s decomposition of the submartingale $-\log W_n$.

Proof of Theorem 2.2.

First recall that due to (2.8), we have

$$\left(\frac{1}{n} \sum_{k=1}^{n} J_k \right)^2 \leq \frac{1}{n} \sum_{k=1}^{n} J_k^2 \leq \frac{1}{n} \sum_{k=1}^{n} I_k \leq \frac{1}{n} \sum_{k=1}^{n} J_k.$$  

Thus, the liminf of the sequences $(\frac{1}{n} \sum_{k=1}^{n} I_k)_n$ and $(\frac{1}{n} \sum_{k=1}^{n} J_k)_n$ are of the same nature, that is, both are positive $\mathbb{P}$-a.s. or zero $\mathbb{P}$-a.s.

If $p < \lambda$, note that $W_\infty = 0 \mathbb{P}$-a.s. To check this, observe that if $W_\infty > 0$ then $\lim \log W_n / n \to 0$, but at the same time

$$\frac{\log W_n}{n} \to p - \lambda = 0.$$  

So, if $p < \lambda$, then $W_\infty = 0 \mathbb{P}$-a.s. and this implies by (2.9) that $\sum_n I_n = \infty$ a.s. and $-\log W_n = \Theta(\sum_{k=1}^{n} I_k)$. In particular, $\lim \inf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} I_n > 0$, so the RWRE is localized at the boundary. Reciprocally, if the walk is localized, $\sum_{k=1}^{n} I_k = \infty$, so by (2.9), $-\log W_n = \Theta(\sum_{k=1}^{n} I_k)$ and then

$$\lim \log W_n / n \to p - \lambda > 0.$$  

This proves i), and the proof of ii) is analogous. \(\square\)

2.2. Existence of the limits $p, \lambda$. In order to justify the existence of the limits in (2.1), we relate these quantities to the rate functions of large deviations for random walks in random environment. First, we recall some standard notation.

We say that the position of the walk satisfies a quenched large deviation principle if there is a lower semicontinuous function $I_\mathbf{1}: \mathbb{R}^d \to [0, \infty]$ such that for each Borel set $G \subset \mathbb{R}^d$

$$- \inf_{x \in G^c} I_\mathbf{1}(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P_{0,\omega}(X_n / n \in G) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_{0,\omega}(X_n / n \in G) \leq - \inf_{x \in G} I_\mathbf{1}(x).$$  

(2.11)

Here $G^c, \bar{G}$ are the interior and closure of $G$ respectively.

Analogously, we say that the position of the walk satisfies an annealed large deviation principle if there is a lower semicontinuous function $I_\mathbf{1}: \mathbb{R}^d \to [0, \infty]$ such that for every Borel set $G \subset \mathbb{R}^d$, (2.11) holds with $P_0$ instead of $P_{0,\omega}$. Informally, this means that

$$\lim_{n \to \infty} \frac{1}{n} \log P_{0,\omega} \left( \frac{X_n}{n} \approx x \right) \approx -I_\mathbf{1}(x) \ \mathbb{P} - \text{a.s.,}$$  

$$\lim_{n \to \infty} \frac{1}{n} \log P_0 \left( \frac{X_n}{n} \approx x \right) \approx -I_\mathbf{1}(x)$$  

It’s well known that the domain of both functions (that is, when $I_\mathbf{1}, I_\mathbf{1} \leq \infty$) is the set

$$\mathbb{D} := \{ x \in \mathbb{R}^d : \| x \|_1 \leq 1 \}.$$  

(2.12)

Also, by Jensen’s inequality and Fatou’s lemma, $I_\mathbf{1} \leq I_\mathbf{1}$.

Moreover, Varadhan proved in [26] that both functions exists under ergodic and uniform elliptic environments, and $I_\mathbf{1}$ is deterministic (i.e., it does not depend on $\omega$).

Next, we characterize the rate functions at $\varnothing\mathbb{D}(\bar{\mathbb{R}})$ (cf., (1.7)).
Lemma 2.6. Under Assumption 1.1, for any \( x \in \partial D(\mathbf{\Sigma}) \) there is a sequence \((x_n)_{n \in \mathbb{N}}\) such that for all \( n, x_n \in \mathbb{Z}^d, \|x_n\|_1 = n \), and

\[
I_q(x) = - \lim_{n \to \infty} \frac{1}{n} \log P_{0,\omega}(X_n = x_n), \\
I_a(x) = - \lim_{n \to \infty} \frac{1}{n} \log P_0(X_n = x_n).
\] (2.13)

This result is Lemma 9 in [5].

Finally, the existence of \( p \) and \( \lambda \) is consequence of the lemma below.

Lemma 2.7. For a RWRE that satisfies Assumption 1.1, the following identities hold:

\[
p = - \inf_{x \in \partial D(\mathbf{\Sigma})} I_q(x), \\
\lambda = - \inf_{x \in \partial D(\mathbf{\Sigma})} I_a(x).
\] (2.14)

In particular, \( p \) is not random (since \( I_q \) is deterministic).

The proof of this lemma is provided in Section A.1. As a corollary, we obtain the characterization of localization/delocalization with respect to the difference between the infima of the quenched and annealed rate functions:

Corollary 2.8. For a RWRE that satisfies Assumption 1.1, we have localization at the boundary if and only if

\[
\inf_{x \in \partial D(\mathbf{\Sigma})} I_a(x) < \inf_{x \in \partial D(\mathbf{\Sigma})} I_q(x)
\]

Remark 2.9. In the article [28] for random walks in space-time i.i.d random environment, it is proven that if condition (T) holds, then in dimension two \( I_q(x) = I_a(x) \) only at \( x = \xi_c \), the LLN velocity. By the last corollary, and recalling that these walks are delocalized, even though the rate functions are different at the boundary, their infima (over any face of the boundary) are equal.

2.3. Preliminaries for the proof of Theorem 1.5. First, note that Theorem 2.2 implies immediately delocalization when (1.12) does not hold. Indeed, in that case the martingale \( W_n(\omega) \equiv 1 \), so in particular \( W_\infty(\omega) > 0 \) a.s. This implies that \( p = \lambda \), because

\[
p = \lim_{n \to \infty} \frac{1}{n} \log P_{0,\omega}(X_n \in \partial R_n) = \lim_{n \to \infty} \frac{W_n(\omega)}{n} + \lambda = \lambda.
\]

When (1.12) holds, the idea is the following: under the annealed measure, conditioned on \( A_n \), the random walk \( X_n \) has the same distribution as a random walk \( \tilde{X}_n \) with increments given by

\[
\tilde{P}(\tilde{X}_1 = e_i) := \frac{q(e_i)}{\sum_{i=1}^d q(e_i)}.
\]

The advantage of this walk is that we don’t need to condition on an event that depends on \( n \), and

\[
P_0(X_n = n_1 e_1 + \cdots + n_d e_d | A_n) = \tilde{P}(\tilde{X}_n = n_1 e_1 + \cdots + n_d e_d)
\] (2.15)

We define also

\[
\mu := \tilde{E}(\tilde{X}_1), \quad \sigma^2 := \text{Var}_{\tilde{P}}[\tilde{X}_1].
\] (2.16)

Consider \( N = nm \) with \( n \) fixed (but large enough) and \( m \to \infty \). Recall that

\[
W_N(\omega) = \left( \frac{P_{0,\omega}(X_N \in \partial R_N)}{\sum_{e \in V(\mathbf{\Sigma})} q(e)} \right)^N.
\]
We define, for $y \in \mathbb{Z}^d$,\[ J_y := \left( (y - \frac{1}{2}) \sqrt{n}, (y + \frac{1}{2}) \sqrt{n} \right) \subset \mathbb{R}^d. \] (2.17)

Given $Y = (y_1, \cdots, y_m) \in (\mathbb{Z}^d)^m$, we decompose\[ W_N(\omega) = \sum_{Y} W_N(\omega, Y), \] (2.18)

where\[ W_N(\omega, Y) := \frac{1}{(\sum_{e \in V(\tau)} q(e))^N} P_{0,\omega}(X_{y_1} - jn\mu \in J_{y_j}, \forall j \leq m, X_N \in \partial\mathbb{R}_N) \]

The decomposition in (2.18) is valid, since $\mathbb{Z}^d \subset \bigcup_{y \in A} J_y$. By the inequality $(\sum a_i)^{1/2} \leq \sum a_i^{1/2}$, valid for countable indices, we obtain\[ \mathbb{E}[W_N(\omega)]^{1/2} \leq \sum_{Y} \mathbb{E}[W_N(\omega, Y)]^{1/2}. \]

This inequality gives us\[ p - \lambda = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[W_N] \leq \liminf_{N \to \infty} \frac{2}{N} \log \mathbb{E}[W_N]^{1/2} \leq \liminf_{N \to \infty} \frac{1}{N} \log \left( \sum_{Y} \mathbb{E}[W_N(\omega, Y)]^{1/2} \right). \] (2.19)

Now we estimate each expectation $\mathbb{E}[W_N(\omega, Y)]^{1/2}$, applying the change of measure. The plan is the following (recall that $N = mn$ with fixed $n$):

For $j \in \{1, \cdots, m\}$ fixed and $n$ a square integer, we define\[ B_j := \{(z, i) \in \mathbb{Z}^d \times \mathbb{N} : (j-1)n \leq i < jn, |z - i\mu - y_j\sqrt{n}| \leq C_1 \sqrt{n}\}, \] (2.20)

where $C_1$ is a constant to determine and $y_0 := 0$. An important observation is that if $x, y \in \mathbb{Z}^d$ such that $|x|_1 = i + j = |y|_1$ and $(x, i) \in B_i, (y, j) \in B_j$, then $\omega(x), \omega(y)$ are independent. This simple remark will be used several times later.

The method presented here has been used by Lacoin [18], Berger and Lacoin [7] in the directed polymers model, and by Yilmaz and Zeitouni [28] for random walks in random environment. First we focus in dimension $d = 2$.

### 3. Proof in the Case $d = 2$

The idea is to define a function that depends on the different blocks $B_j$. We define\[ D(B_j) := \sum_{y : (y, |y|_1) \in B_j} \tilde{\omega}(y), \quad \tilde{\omega}(y) := \sum_{e \in V(\tau)} [\omega(y, e) - q(e)]. \] (3.1)

In particular, $\mathbb{E}[D(B_j)] = 0$, and they form an independent family of random variables. It’s important to observe that (1.4) guarantees that $\tilde{\omega}$ and $D(B_j)$ are non-degenerate random variables. We also define $\delta_n := C_1^{-1/2} n^{-3/4}$. Observe that $\delta_n^2 |D(B_1)| = O(1)$. Finally, for $K > 0$ large enough (to determine), define\[ f_K(u) := -K 1_{\{u \geq eK^2\}}, \quad g(\omega, Y) := e^{\sum_{j=1}^m f_K(\delta_n D(B_j))}. \] (3.2)

By Cauchy-Schwarz inequality,
\[
\mathbb{E}[W_N(\omega, Y)^{1/2}] = \mathbb{E}[W_N(\omega, Y)^{1/2} g(\omega, Y)^{1/2} g(\omega, Y)^{-1/2}] \leq \mathbb{E}[W_N(\omega, Y) g(\omega, Y)]^{1/2} \mathbb{E}[g(\omega, Y)^{-1}]^{1/2}.
\] (3.4)
The expectation $E[g(\omega, Y)^{-1}]$ is easier to estimate. In fact, by independence

$$E[g(\omega, Y)^{-1}] = \prod_{j=1}^{m} E[e^{-f_{K}(\delta_n D(B_j))}] = \left( E[e^{-f_{K}(\delta_n D(B_1))}] \right)^m \leq \left( 1 + e^{K} P(\delta_n D(B_1) \geq e^{K^2}) \right)^m$$

By Chebyshev’s inequality,

$$e^{K} P(\delta_n D(B_1) \geq e^{K^2}) \leq e^{K - 2K^2} \delta_n^2 \mathbb{E}[D(B_1)^2] = e^{K - 2K^2} \delta_n^2 \sum_{y : (y, y_1) \in B_1} \mathbb{E}[\tilde{\omega}^2(y)] \leq 2C_1 e^{K - 2K^2} \delta_n^2 n^{3/2} \mathbb{E}[\tilde{\omega}^2(0)].$$

As $C_1 \delta_n^2 n^{3/2} = 1$, if $K$ is large enough we have the bound

$$E[g(\omega, Y)^{-1}]^{1/2} \leq 2^m.$$  \hspace{1cm}  (3.5)

Next we estimate $E[W_N(\omega, Y)g(\omega, Y)]$. We write $V(\pi) = \{e_1, e_2\}$ First observe that by the Markov property,

$$W_N(\omega, Y) = \frac{1}{(q(e_1) + q(e_2))^N} P_{0, \omega}(X_{n_j} - j \mu \in J_{y_j} \forall j \leq m, X_i \in \partial R_i \forall i \leq n)$$

$$= \sum_{x_0=0, x_1, \cdots , x_m : x_i \in \partial R_i + in \mu} \frac{1}{(q(e_1) + q(e_2))^N} \prod_{j=1}^{m} P_{x_{j-1}+(j-1)n,j} \omega(X_n - j \mu = x_j \in J_{y_j})$$

$$= \sum_{x_0=0, x_1, \cdots , x_m : x_i \in \partial R_i + in \mu} \frac{1}{(q(e_1) + q(e_2))^N} \prod_{j=1}^{m} P_{x_{j-1} - \sqrt{n}y_{j-1}, \sqrt{n}y_{j-1}} \omega(X_n - n \mu = x_j - \sqrt{n}y_{j-1} \in J_{y_j} - \sqrt{n}y_{j-1}).$$

Thus, by the i.i.d property on the environments,

$$E[W_N(\omega, Y)g(\omega, Y)]$$

$$= \sum_{x_0=0, x_1, \cdots , x_m : x_i \in \partial R_i + in \mu} \frac{1}{(q(e_1) + q(e_2))^N} \prod_{j=1}^{m} E_{x_{j-1} - \sqrt{n}y_{j-1}, \sqrt{n}y_{j-1}} \left( e^{f_{K}(\delta_n D(B_1))}, X_n - n \mu = x_j - \sqrt{n}y_{j-1} \in J_{y_j} \right)$$

$$= \sum_{x_0=0, x_1, \cdots , x_m : x_i \in \partial R_i + in \mu} \frac{1}{(q(e_1) + q(e_2))^N} \prod_{j=1}^{m} E_{x_{j-1} - \sqrt{n}y_{j-1}} \left( e^{f_{K}(\delta_n D(B_1))}, X_n - n \mu = x_j - \sqrt{n}y_{j-1} \in J_{y_j} - \sqrt{n}y_{j-1} \right).$$
Observe that
\[
\sum_{x_1}^2 \prod_{j=1}^{\infty} E_{x_{j-1}-\sqrt{n}y_{j-1}} \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu = x_j - \sqrt{n}y_{j-1} \in J_{y_j} - \sqrt{n}y_{j-1} \right)
\]
\[
\leq \max_{x_1 \in J_{y_1}} E_{x_1-\sqrt{n}y_1} \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu = x_2 - \sqrt{n}y_1 \in J_{y_2} - \sqrt{n}y_1 \right) \times E_{0} \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu \in J_{y_1} \right).
\]
We iterate this inequality over \(x_2, \ldots, x_m\) to bound \(\mathbb{E}[W_N(\omega, Y)g(\omega, Y)]\) by
\[
\frac{1}{(q(e_1) + q(e_2))^N} \prod_{j=1}^{\infty} \max_{x_{j-1} \in J_{y_{j-1}}} E_{x_{j-1}-\sqrt{n}y_{j-1}} \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu \in J_{y_j} - \sqrt{n}y_{j-1} \right).
\]
Note that \(J_{y_j} - \sqrt{n}y_{j-1} = J_{y_j-y_{j-1}}\). Using that, we can write the display above as
\[
\prod_{j=1}^{m} \max_{x_{j-1} \in J_{y_{j-1}}} E_{x_j} \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu \in J_{y_j-y_{j-1}} \mid A_n \right).
\]
By this inequality and (3.5) we conclude that
\[
\mathbb{E}[W_N(\omega, Y)^{1/2}] \leq \left( 2 \sum_{x \in J_0} \max_{x \in J_0} E_x \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu \in J_{y} \mid A_n \right) \right)^m.
\]
The bound (2.19) tell us that \(p - \lambda < 0\) once we are able to prove the following:

**Lemma 3.1.** For \(n, K\) and \(C_1\) large enough,
\[
\sum_{y \in \mathbb{Z}^d} \max_{x \in J_0} E_x \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu \in J_y \mid A_n \right) < 1/2.
\]

**Proof.** We decompose the sum \(\sum_{y} (\cdots) = \sum_{|y| > R} (\cdots) + \sum_{|y| \leq R} (\cdots)\) for some \(R > 0\) to determine. As \(e^{f_k(\cdot)} \leq 1\), we have
\[
\sum_{|y| > R} \max_{x \in J_0} E_x \left( e^{f_k(\delta_n D(B_1))}; X_n - \eta \mu \in J_y \mid A_n \right)
\]
\[
\leq \sum_{|y| > R} P_0 \left( X_n - \eta \mu \in J_y \mid A_n \right)
\]
\[
= \sum_{|y| > R} P_0 \left( |X_n - \eta \mu - \sqrt{n}y| \leq \sqrt{n}/2 \mid A_n \right)
\]
\[
= \sum_{|y| > R} P_0 \left( |X_n - \eta \mu - \sqrt{n}y| / \sqrt{n} \leq 1/2 \mid A_n \right).
\]
Note that
\[
\frac{|X_n - \eta \mu - \sqrt{n}y|}{\sqrt{n}} \leq 1/2 \text{ implies } \frac{|X_n - \eta \mu|}{\sqrt{n}} \geq |y| - \frac{1}{2} \text{ (recall (2.15))}
\]
So, (3.7) is bounded by
\[
\sum_{|y| > R} P_0 \left( \frac{|X_n - \eta \mu|}{\sqrt{n}} \geq |y| - \frac{1}{2} \mid A_n \right) = \sum_{|y| > R} \hat{P} \left( \frac{\hat{X}_n - \eta \mu}{\sqrt{n}} \geq |y| - \frac{1}{2} \right).
\]
Choosing $R$ sufficiently large, we obtain (for all $n$ large enough) that this sum is as small as we want.

Next we proceed to bound the sum over $|y| \leq R$. Clearly we have

$$\sum_{|y| \leq R} \max_{x \in \mathcal{J}_0} \mathbb{E}_x \left( e^{f_K(\delta_n D(B_1))}; X_n - n\mu \in J_y | A_n \right) \leq (2R + 1) \max_{x \in \mathcal{J}_0} \mathbb{E}_x \left( e^{f_K(\delta_n D(B_1))} | A_n \right) = (2R + 1) \max_{x \in \mathcal{J}_0} \mathbb{E}_0 \left( e^{f_K(\delta_n D(B_1 - x))} | A_n \right).$$

Thus, it’s enough to check that the expectation above is small, uniformly on $x \in J_0$.

First, define the auxiliary set

$$\mathcal{B}_1 := \{(z, i) \in \mathbb{Z}^d \times \mathbb{N} : 0 \leq i < n; |z - i\mu| \leq (C_1 - 1/2)\sqrt{n}\} \subset B_1 - x \forall x \in J_0.$$

Recall the definition of $f_K$ (cf., (3.2)) to decompose the expectation above as (for fixed $x \in J_0$)

$$\mathbb{E}_0 \left( e^{f_K(\delta_n D(B_1 - x))} | A_n \right) = e^{-K} \mathbb{P}_0 \left( \delta_n D(B_1 - x) \geq e^{K^2} | A_n \right) + \mathbb{P}_0 \left( \delta_n D(B_1 - x) < e^{K^2} | A_n \right) \leq e^{-K} + \mathbb{P}_0 \left( \delta_n D(B_1 - x) < e^{K^2} | A_n \right). \quad (3.8)$$

The first term is small enough if $K$ is sufficiently large. It remains to bound the second term. To do this, we bound the this term by

$$\mathbb{P}_0 \left( \{X_i, i : 0 \leq i < n\} \subseteq \mathcal{B}_1 | A_n \right) + \mathbb{P}_0 \left( \delta_n D(B_1 - x) < e^{K^2}; \{X_i, i : 0 \leq i < n\} \subset \mathcal{B}_1 | A_n \right) = \mathbb{P} \left( \{X_i, i : 0 \leq i < n\} \subseteq \mathcal{B}_1 | A_n \right) + \mathbb{P}_0 \left( \delta_n D(B_1 - x) < e^{K^2}; \{X_i, i : 0 \leq i < n\} \subset \mathcal{B}_1 | A_n \right)$$

By Donsker’s invariance principle (applied to the random walk $X_n$), the first probability is uniformly small on $n$ if $C_1$ is large enough. To handle the second one, recall that $\delta_n := C_1^{-1/2} n^{-3/4}$ and the definition of $D(B_1 - x)$ (cf., (3.1)). Define $A_n := n^{1/5}$. Also, we write $(y, |y|) = (y, i)$, where $i = |y|$. Then we decompose the second term as

$$\mathbb{P}_0 \left( \delta_n D(B_1 - x) < e^{K^2}; \{X_i, i : 0 \leq i < n\} \subset \mathcal{B}_1 | A_n \right) \leq \mathbb{P}_0 \left( \delta_n \sum_{y : (y, i) \in \mathcal{B}_1 - x} \omega(y) < -A_n; \{X_i, i : 0 \leq i < n\} \subset \mathcal{B}_1 | A_n \right) +$$

$$\mathbb{P}_0 \left( \delta_n \sum_{y : (y, i) \in \mathcal{B}_1 - x, y = X_i} \omega(y) < e^{K^2} + A_n; \{X_i, i : 0 \leq i < n\} \subset \mathcal{B}_1 | A_n \right). \quad (3.9)$$
The first probability can be written as

\[
\frac{1}{(q(e_1) + q(e_2))^n} \sum_{x_i \in \mathcal{R}_1, \forall i \leq n: (x_i, i) \in B_1} \mathbb{E} \left[ P_0, \omega (X_i = x_i \forall i \leq n); \delta_n \sum_{(y, l) \in B_1 - x} \omega(y) < -A_n \right]
\]

\[
= \frac{1}{(q(e_1) + q(e_2))^n} \sum_{x_i \in \partial \mathcal{R}_1, \forall i \leq n: (x_i, i) \in B_1} \mathbb{P} \left( \delta_n \sum_{(y, l) \in B_1 - x} \omega(y) < -A_n \right) P_0(X_i = x_i \forall i \leq n)
\]

(3.10)

\[
\leq \max_{x_i \in \partial \mathcal{R}_1, \forall i \leq n: (x_i, i) \in B_1} \mathbb{P} \left( - \sum_{(y, l) \in B_1 - x} \omega(y) > A_n / \delta_n \right)
\]

(3.11)

\[
\leq \frac{\delta_n^2}{A_n^2} \max_{x_i \in \partial \mathcal{R}_1, \forall i \leq n: (x_i, i) \in B_1} \mathbb{E} \left[ \left( \sum_{(y, l) \in B_1 - x} \omega(y) \right)^2 \right] \leq 2C_1 n^{3/2} \frac{\delta_n^2}{A_n^2} \mathbb{E}[\omega^2(0)].
\]

(3.12)

In (3.10) we used independence. In (3.11) we used that

\[
\sum_{x_i \in \partial \mathcal{R}_1, \forall i \leq n: (x_i, i) \in B_1} P_0(X_i = x_i \forall i \leq n) = (q(e_1) + q(e_1))^n.
\]

Finally, in (3.12) we used Chebyshev’s inequality, the fact that the \(\omega(y)\)'s are independent and centered, and \(|B(1)| \leq 2C_1 n^{3/2}\). As the last expression in (3.12) goes to zero as \(n \to \infty\), we conclude that the first term in (3.9) is arbitrarily small for large enough \(n\). Finally, we bound the second probability in (3.9). Indeed, if \(\alpha > 0\), we can write this quantity as

\[
P_0 \left( \delta_n \sum_{(y, l) \in B_1 - x, y = X_i} [\omega(y) - \alpha] < e^{K^2} + A_n - \alpha n \delta_n; \{(X_i, i) : 0 \leq i < n\} \subset \mathcal{B}_1 \mid A_n \right)
\]

\[
\leq P_0 \left( \delta_n \sum_{(X_i, i) \in B_1 - x} [\omega(X_i) - \alpha] < e^{K^2} + A_n - \alpha n \delta_n \mid A_n \right)
\]

\[
\leq P_0 \left( - \sum_{i=0}^{n-1} [\omega(X_i) - \alpha] > (\alpha n \delta_n - A_n - e^{K^2}) / \delta_n \mid A_n \right)
\]

\[
\leq \frac{\delta_n^2}{(\alpha n \delta_n - A_n - e^{K^2})^2} E_0 \left( \left( \sum_{i=0}^{n-1} [\omega(X_i) - \alpha] \right)^2 \mid A_n \right).
\]
This last inequality holds for $n$ large enough, because for such $n$, $\alpha n \delta_n - A_n - e^{K^2} > 0$. We decompose the expectation

$$E_0 \left( \left( \sum_{i=0}^{n-1} (\bar{\omega}(X_i) - \alpha) \right)^2 \right| A_n \right) = \sum_{i=0}^{n-1} E_0 \left( (\bar{\omega}(X_i) - \alpha)^2 \right| A_n \right) + \sum_{i \neq j} E_0 \left( (\bar{\omega}(X_i) - \alpha)(\bar{\omega}(X_j) - \alpha) \right| A_n \right). \quad (3.13)

The first term is $nE \left[ (\bar{\omega}(0) - \alpha)^2 \right]$. As $c_n := \frac{\delta_n^2}{(\mu n \delta_n - A_n - e^{K^2})^2}$, this term vanishes as $n \to \infty$. On the other hand, if we choose

$$\alpha := \frac{\sum_e E[\omega(0, e)^2] - (\sum_e q(e))^2}{\sum_e q(e)} > 0 \text{ by (1.4)},$$

then by independence we have for $i \neq j$,

$$E_0 \left( (\bar{\omega}(X_i) - \alpha)(\bar{\omega}(X_j) - \alpha) \right| A_n \right) = 0. \quad (3.14)$$

Indeed, note that $\alpha$ satisfies for all $x \in \mathbb{Z}^d$

$$\sum_e E[\omega(x, e)(\bar{\omega}(x) - \alpha)] = 0. \quad (3.15)$$

And by definition we have for $i < j$ (here $\Delta x_m := x_m - x_{m-1}$, and the product below is 1 if $i + 2 < j$)

$$E_0 \left( (\bar{\omega}(X_i) - \alpha)(\bar{\omega}(X_j) - \alpha) \right| A_j \right) = \frac{1}{(\sum_e q(e))^j} \sum_{(x_0, x_1, \ldots, x_i \in R_j)} E \left[ \left( \prod_{m=1}^j \omega(x_{m-1}, \Delta x_m) \right)(\bar{\omega}(x_i) - \alpha)(\bar{\omega}(x_j) - \alpha) \right] = 0.$$

Combining the previous results, such election of constants help us to deduce that Lemma 3.1 is true, and therefore $p - \lambda < 0$.

\[\square\]

4. PROOF IN CASE $d = 3$

First, we argue why the method in $d = 2$ fails if we try to apply the same functions and bounds. Note that when $d = 2$, the choice of $\delta_n$ implies that $\delta_n^2 |D(B_1)| = O(1)$, and also $n \delta_n \to \infty$ as $n \to \infty$. The first estimate is used in (3.11), and the second one is used to bound the second probability in (3.9). Thus, to replicate that in $d = 3$ we need $\delta_n = O(n^{-1})$, but in this case $n \delta_n$ is bounded. This motivates the definition below.

We will use the same denominations of $B_1$ (cf., (2.20)). Following [18] and [28], we define $D(B_j)$ as

$$D(B_j) := \sum_{y, z}(y, z) V(y, z)\bar{\omega}(y)\bar{\omega}(z), \quad (4.1)$$
Lemma 4.2. For $y \in \mathbb{Z}^3$ such that $|y|_1 \in \{0, \ldots, n - 1\}$ and $x_0, \ldots, x_{n-1} \in \mathbb{Z}^3$ with $|x_i|_1 = 1$, we have
\begin{equation}
\sum_{k=0}^{n-1} V(x_k, y) \leq 2 \log n. \tag{4.3}
\end{equation}
for some constant $C_2$ to determine. We use as before the convention of writing $(y, i)$, $(z, j)$, where $i = |y|_1$, $j = |z|_1$. The inequalities below will be useful later, and their proof are immediate.

Lemma 4.1. For $y \in \mathbb{Z}^3$ such that $|y|_1 \in \{0, \ldots, n - 1\}$ and $x_0, \ldots, x_{n-1} \in \mathbb{Z}^3$ with $|x_i|_1 = 1$, we have
\begin{equation}
\sum_{k=0}^{n-1} V(x_k, y) \leq \sum_{0 \leq k \neq i < n} \frac{1}{|n-i|} (2C_2 \sqrt{|n-i|}) \leq 4C_2^2 n^2. \tag{4.4}
\end{equation}
Recall that $\bar{\omega}(y) \leq 4$ for all $y$ and $\mathbb{E}[|D(B_1)|] = 0$, so
\begin{equation}
\mathbb{E}[|D(B_1)|^2] = \sum_{y: (y, i) \in B_1} V(y, z)^2 \bar{\omega}(y)^2 \bar{\omega}(z)^2 \leq 16 \times 32C_1^2 C_2^2 n^2 \log n = 512C_1^2 C_2^2 n^2 \log n, \tag{4.6}
\end{equation}
by (4.6). In particular, if $\delta_n := n^{-1} (\log n)^{-1/2}$, then $\text{Var}(\delta_n D(B_1)) = O(1)$. The arguments can be repeated as in the $d = 2$ case up to (3.8). Thus, we need to estimate
\begin{equation}
P_0 \left( \delta_n D(B_1 - x) \leq e^{k^2} \left| A_n \right. \right). \tag{4.7}
\end{equation}
We consider only the case $x = 0$; the argument is the same for all $x \in J_0$. We define
\begin{equation}
\nu(n, X) := \sum_{0 \leq i, j < n} V(X_i, X_j). \tag{4.8}
\end{equation}

Lemma 4.2. For any $\delta > 0$, there is $C_2$ large enough so that
\begin{equation}
P_0 \left( \nu(n, X) < \frac{n}{2} \log(n - 1) \left| A_n \right. \right) \leq \delta. \tag{4.9}
\end{equation}
Proof. Observe first that $\nu(n, X) \leq H_n := \sum_{0 \leq i, j < n; i \neq j} \frac{1}{|i-j|}$. We have
\begin{align*}
\mathbb{E}_0(\nu(n, X)|A_n) &= \sum_{0 \leq i, j < n} \mathbb{E}_0(\nu(X_i, X_j)|A_n) \\
&= \sum_{0 \leq i, j < n; i \neq j} \frac{1}{|i-j|} P_0 \left( |X_i - X_j - (i-j)| \mu < C_2 \sqrt{|i-j|} |A_n| \right) \\
&= \sum_{0 \leq i, j < n; i \neq j} \frac{1}{|i-j|} \mathbb{P}_0 \left( |\tilde{X}_i - \tilde{X}_j - (i-j)| \mu < C_2 \sqrt{|i-j|} \right).
\end{align*}
By the Central Limit Theorem, the probability above is greater or equal than \(1 - \frac{\delta}{2}\) for all \(i \neq j\) if \(C_2\) is big enough. Then we have

\[
E_0(\nu(n,X)|A_n) \geq \left(1 - \frac{\delta}{2}\right) \sum_{0 \leq i < j < n} \frac{1}{|i - j|} = \left(1 - \frac{\delta}{2}\right) H_n
\]  

(4.9)

Note also that

\[
E_0(\nu(n,X)|A_n) \geq \left(1 - \frac{\delta}{2}\right) H(n) \Leftrightarrow E_0(H(n) - \nu(n,X)|A_n) \leq \frac{\delta}{2} H(n).
\]

Therefore, by Chebyshev’s inequality,

\[
P_0 \left( \nu(n,X) < \frac{H(n)}{2} \middle| A_n \right) = P_0 \left( H(n) - \nu(n,X) > \frac{H(n)}{2} \middle| A_n \right) \leq \delta.
\]

As \(H(n) \geq n \log(n - 1)\), this completes the proof. \(\square\)

We estimate the probability (4.7) as follows:

\[
P_0 \left( \delta_n D(B_1 - x) < e^{K^2} \middle| A_n \right) \leq P_0 \left( \delta_n D(B_1) < e^{K^2}, \nu(n,X) \geq \frac{n}{2} \log(n - 1) \middle| A_n \right)
\]

\[
+ P_0 \left( \nu(n,X) < \frac{n}{2} \log(n - 1) \middle| A_n \right).
\]

The second term in the sum above is less than \(\delta\) if \(C_2\) is large enough. We only need to estimate the first term. Recall the definition of \(\alpha\) (cf.(3.14)), and note that if \(\nu \geq \frac{n}{2}\log(n - 1)\), then eventually \(e^{K^2} - \alpha \delta_n \nu(n,X) < 0\). For such \(n\), we apply Chebyshev’s inequality to deduce

\[
P_0 \left( \delta_n D(B_1) < e^{K^2}, \nu(n,X) \geq \frac{n}{2} \log(n - 1) \middle| A_n \right)
\]

\[
= P_0 \left( (\alpha^2 \nu(n,X) - D(B_1)) > \frac{\alpha^2 \delta_n \nu(n,X) - e^{K^2}}{\delta_n}, \nu(n,X) \geq \frac{n}{2} \log(n - 1) \middle| A_n \right)
\]

\[
\leq \left( \frac{\delta_n}{\alpha^2 \delta_n \nu(n,X) - e^{K^2}} \right)^2 E_0 \left( \left[ D(B_1) - \alpha^2 \nu(n,X) \right]^2 \middle| A_n \right).
\]

As \(\left( \frac{\delta_n}{\alpha^2 \delta_n \nu(n,X) - e^{K^2}} \right)^2 = O((n \log n)^{-2})\), it’s sufficient to show the following:

**Lemma 4.3.**

\[
E_0 \left( \left[ D(B_1) - \alpha^2 \nu(n,X) \right]^2 \middle| A_n \right) = O(n^2 \log n).
\]

**Proof.** We decompose

\[
D(B_1) - \alpha^2 \nu(n,X) = 2\alpha \sum_{k=0}^{n-1} \sum_{y: (y,i) \in B_1} V(X_k,y) [\hat{\omega}(y) - \alpha \mathbb{1}_{\{X_i = y\}}] + \sum_{(y,z),(y,j) \in B_1, (z,j) \in B_1} V(y,z) [\hat{\omega}(y) - \alpha \mathbb{1}_{\{X_i = y\}}] [\hat{\omega}(z) - \alpha \mathbb{1}_{\{X_j = z\}}]
\]
By the inequality \((a + b)^2 \leq 2(a^2 + b^2)\), we get

\[
E_0 \left( \left[ D(B_1) - \alpha^2 \nu(n, X) \right]^2 \right| A_n 
\right)
\leq 8\alpha^2 E_0 \left( \sum_{k=0}^{n-1} \sum_{y, i \in B_1} V(X_k, y) [\bar{\omega}(y) - \alpha \mathbb{1}_{X_i = y}] \right)^2 \left| A_n \right| + 
2E_0 \left( \sum_{y, z : (y, i) \in B_1, (z, j) \in B_1} V(y, z) [\bar{\omega}(y) - \alpha \mathbb{1}_{X_i = y}] [\bar{\omega}(z) - \alpha \mathbb{1}_{X_j = z}] \right)^2 \left| A_n \right|
\]
\[= 8\alpha^2 a_1 + 2a_2.\]

We proceed to estimate first \(a_1\) and then \(a_2\).

\[
a_1 = \sum_{y, y' : (y, i) \in B_1, (y', i') \in B_1} \sum_{k, k' = 0}^{n-1} E_0 \left( V(X_k, y) [\bar{\omega}(y) - \alpha \mathbb{1}_{X_i = y}] V(X_{k'}, y') [\bar{\omega}(y') - \alpha \mathbb{1}_{X_i' = y'}] \right| A_n \].
\]

We split the sum whether \(y = y'\) or not. In the former case we obtain

\[
E_0 \left( \sum_{y : (y, i) \in B_1} \left( \sum_{k = 0}^{n-1} V(X_k, y) [\bar{\omega}(y) - \alpha \mathbb{1}_{X_i = y}] \right)^2 \right| A_n 
\right)
\leq (2 + \alpha)^2 E_0 \left( \sum_{y : (y, i) \in B_1} \left( \sum_{k = 0}^{n-1} V(X_k, y) \right)^2 \right| A_n 
\right)
\leq (2 + \alpha)^2 \times 8C^2 n^2 \log n,
\]

where the last inequality is by (4.5).

Next we consider the case \(y \neq y'\). In this scenario, the expectation is zero unless \(X_i = y, X_i' = y'\) (by independence). Therefore, the sum over \(y \neq y'\) is

\[
\sum_{k, k', i, i' : i \neq i'} E_0 \left( V(X_k, X_i) V(X_{k'}, X_{i'}) [\bar{\omega}(X_i) - \alpha] [\bar{\omega}(X_{i'}) - \alpha] \right| A_n \].
\]

(4.10)
First we consider the case where $i' > \max(i, k, k')$. In these cases the expectation is zero. Indeed, if we define $q := q(e_1) + q(e_2) + q(e_3)$, then by independence

$$E_0 \left[ V(X_{k'}, X_{i'}) V(X_{k}, X_{i}) | \tilde{\omega}(X_{i'}) - \alpha |, A_n \right]$$

$$= \frac{1}{q^{n-i'+1}} \sum_{x_1, \ldots, x_{i'}} E_0 \left[ V(x_{k'}, x_{i'}) V(x_{k}, x_{i}) | \tilde{\omega}(x_{i'}) - \alpha |, X_m = x_m \forall m \leq i' \right]$$

$$= \frac{1}{q^{n-i'+1}} \sum_{x_1, \ldots, x_{i'}} V(x_{k'}, x_{i'}) V(x_{k}, x_{i}) \mathbb{E} \left[ [\tilde{\omega}(x_{i'}) - \alpha |, X_m = x_m \forall m \leq i', X_{i'+1} = x_{i'} + e] \times \mathbb{E} \left[ V(X_k, x_i) V(X_{k'}, x_{i'}) | X_{i'+1} = x_{i'} + e, A_n \right] \right]$$

$$= \sum_{x_1, \ldots, x_{i'}} \mathbb{E} \left[ [\tilde{\omega}(x_{i'}) - \alpha |, X_m = x_m \forall m \leq i', X_{i'+1} = x_{i'} + e] \times \mathbb{E} \left[ V(X_k, x_i) V(X_{k'}, x_{i'}) | X_{i'+1} = x_{i'} + e, A_n \right] \right]$$

The second expectation depends on $e$. However, the difference is small among different elections of $e$. Indeed, for $e, e' \in V(\mathcal{F})$ we have (using again $\hat{P}$, and recall the definition of $V$ in (4.2))

$$\mathbb{E} \left[ V(X_k, x_i) V(X_{k'}, x_{i'}) | X_{i'+1} = x_{i'} + e \right] - \mathbb{E} \left[ V(X_k, x_i) V(X_{k'}, x_{i'}) | X_{i'+1} = x_{i'} + e' \right] \leq \sum_{x_{k'}, \forall x_{k'}: V(x_{k'}, x_{i'}) > 0} \frac{1}{k' - i'} |\hat{P}(X_{k'} = x_{k'} | X_{i'+1} = x_{i'} + e' - \hat{P}(X_{k'} = x_{k'} | X_{i'+1} = x_{i'} + e')| \leq 4C_2 \left( \frac{k' - i'}{(k' - i')(k - i)} \right) \sup_{x_{k'}} |\hat{P}(X_{k'} = x_{k'} | X_{i'+1} = x_{i'} + e - \hat{P}(X_{k'} = x_{k'} | X_{i'+1} = x_{i'} + e')| \tag{4.11}$$

The supremum above is $O((k' - i')^{-1})$ and it is uniform on the path $x_{i}, \ldots, x_{i'+1}$ with $|x_i| = i$. This fact is proved in Section A.2. Thus, the difference between any pair of expectations as above is $O((k - i)^{-1} (k' - i')^{-1}) \hat{A} \hat{u}$ Then, we can write for some $e_0 \in V(\mathcal{F})$ fixed

$$E_0 (V(X_k, x_i) V(X_{k'}, x_{i'}) | X_{i'+1} = x_{i'} + e, A_n) = E_0 (V(X_k, x_i) V(X_{k'}, x_{i'}) | X_{i'+1} = x_{i'} + e_0, A_n)$$

$$+ d(k, k', i, i', e),$$
and \(|d(k, k', i, i', e)| \leq C \frac{1}{(k-i)(k'-i')}\) uniformly on \(e \in V(\mathcal{S})\). Therefore,

\[
E_0 \left[ V(X_{k}, X_i) V(X_{k'}, X_{i'}) \left| \tilde{\omega}(X_i) - \alpha \right| \left| \tilde{\omega}(X_{i'}) - \alpha \right| A_n \right]
\]

\[
= \sum_{x_1, \ldots, x_n} \sum_{i, i' \in V(\mathcal{S})} E_0 \left[ \left| \tilde{\omega}(X_i) - \alpha \right| \left| \tilde{\omega}(X_{i'}) - \alpha \right|, X_m = x_m, m \leq i', X_{i'+1} = x_{i'} + e A_n \right] \times \left[ E_0 \left[ V(X_k, X_i) V(X_{k'}, X_{i'}) \left| X_{i'+1} = x_{i'} + e_0, A_n \right| + d(k, k', i, i', e) \right] \right].
\]

The first sum is zero, because

\[
E_0 \left[ \left| \tilde{\omega}(X_i) - \alpha \right| \left| \tilde{\omega}(X_{i'}) - \alpha \right|, X_m = x_m, m \leq i', X_{i'+1} = x_{i'} + e A_n \right] = 0
\]

by the definition of \(\alpha\). Finally, the second sum can be bounded by

\[
\left| d(k, k', i, i') \sum_{x_1, \ldots, x_n} \sum_{i, i' \in V(\mathcal{S})} E_0 \left[ \left| \tilde{\omega}(X_i) - \alpha \right| \left| \tilde{\omega}(X_{i'}) - \alpha \right|, X_m = x_m, m \leq i', X_{i'+1} = x_{i'} + e A_n \right] \leq (2 + \alpha)^2 C \frac{1}{(k-i)(k'-i')},
\]

and then

\[
\sum_{0 \leq i \leq v \leq k' < k \leq n} E_0 \left[ V(X_k, X_i) V(X_{k'}, X_{i'}) \left| \tilde{\omega}(X_i) - \alpha \right| \left| \tilde{\omega}(X_{i'}) - \alpha \right| A_n \right] \leq (2 + \alpha)^2 C \sum_{0 \leq i < k' < k \leq n} \frac{1}{(k-i)(k'-i')} = O(n^2 \log n).
\]

To see why the last estimate holds, note that the summands only depend on the differences \(k - i, k' - i'\). For fixed \(i < k, k - i \in \{1, \ldots, n\}\) and for each \(1 \leq j \leq n\), there are \(n - j\) elections of \(i < k\) such that \(k - i = j\). Therefore,

\[
\sum_{0 \leq i < k' < k \leq n} \frac{1}{(k-i)(k'-i')} = \sum_{j=1}^{n} \frac{1}{j} \# \{ 0 \leq i < k \leq n : k - i = j \} \sum_{0 < i' < k'} \frac{1}{k'-i'}
\]

\[
= \sum_{j=1}^{n} \frac{n-j}{j} \sum_{0 < i' < k'} \frac{1}{k'-i'}
\]

Similarly, for each \(j, k' - i' \in \{1, \ldots, j - 2\}\), and for each \(1 \leq m \leq j - 2\), there are \(j - m - 1\) choices of \(0 < i' < k' < j\) with \(k' - i' = m\). Then, the last sum becomes

\[
\sum_{j=1}^{n} \frac{n-j}{j} \sum_{m=1}^{j-1} \frac{j-m-1}{m} \leq \sum_{j=1}^{n} \frac{n-j}{j} \sum_{m=1}^{j-1} \frac{j-m-1}{m} = \sum_{j=1}^{n} \frac{n-j}{j} (j-1)[H_{j-1} - 1], \quad (4.12)
\]

where \(H_0 := 1, H_n := \sum_{k=1}^{n} \frac{1}{k}\) for \(n \geq 1\). As \(H_{j-1} - 1 \leq H_n\), the sum in (4.12) can be bounded by

\[
H_n \sum_{j=1}^{n} \frac{n-j}{j} (j-1) \leq H_n \sum_{j=1}^{n} (n-j) = O(n^2 \log(n)).
\]
The other cases are similar, so we conclude that \( a_1 = O(n^2 \log n) \).

It remains to bound \( a_2 \). We can write it as

\[
\sum_{y, y', z, z' : (y_i, z_j) \in B_1} E_0 \left( V(y, z) V(y', z') | \tilde{w}(y) - \alpha \mathbb{1}_{\{X_i = y\}} | \tilde{w}(z) - \alpha \mathbb{1}_{\{X_j = z\}} \right) \times \]

\[
[ \tilde{w}(z') - \alpha \mathbb{1}_{\{X_{i'} = z'\}} ] A_n. \]

By symmetry, it’s only necessary to consider the next three situations:

(i) \( y = z \) and \( y' = z' \),

(ii) \( y = y' \) and \( z = z' \), and

(iii) \( y \neq y' \) and \( z \neq z' \).

In the first case we have

\[
\sum_{y, z : (y_i, z_j) \in B_1} E_0 \left( \left[ V(y, z) | \tilde{w}(y) - \alpha \mathbb{1}_{\{X_i = y\}} | \tilde{w}(z) - \alpha \mathbb{1}_{\{X_j = z\}} \right]^2 \right) | A_n \leq 16 \sum_{y, z : (y_i, z_j) \in B_1} E_0 \left( V(y, z)^2 | A_n \right) = O(n^2 \log n) \text{ by (4.6).} \]

In the second one, that is, \( y = y' \) and \( z \neq z' \), by the same argument as in \( a_1 \), we only need to consider the case for which \( X_i = y, X_j = z, X_{i'} = z' \) (otherwise, the expectation vanishes). Then, the sum can be written as

\[
\sum_{i, j : i' \neq j'} E_0 \left( V(X_i, X_j) V(X_{i'}, X_{j'}) | \tilde{w}(X_i) - \alpha | \tilde{w}(X_j) - \alpha | \tilde{w}(X_{i'}) - \alpha | \tilde{w}(X_{j'}) - \alpha \right) | A_n \]

This sum can be handled in a similar way as in (4.10). Similarly, the sum over \( y \neq y' \), \( z \neq z' \) can be written as

\[
\sum_{i, i' : i' \neq i} E_0 \left( V(X_i, X_j) V(X_{i'}, X_{j'}) | \tilde{w}(X_i) - \alpha | \tilde{w}(X_j) - \alpha | \tilde{w}(X_{i'}) - \alpha | \tilde{w}(X_{j'}) - \alpha \right) | A_n \]

and this sum also can be controlled as in (4.10). The estimates on \( a_1, a_2 \) allow us to conclude the proof of Lemma 4.3. \( \square \)

5. Phase Transition

In dimension \( d \geq 4 \) we study a phase transition the parameter \( \varepsilon \in [0, \varepsilon_{\max}] \) (cf., (1.13)). So, we consider a family of environments \( (\omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_{\max})} \) once we fix the family of i.i.d random variables \( \tilde{\xi}(x, \cdot) \). Therefore, we refer to \( p(\varepsilon), \lambda(\varepsilon) \), etc. when we refer to the environment \( \omega_\varepsilon \).

The first part of Theorem 1.8 is consequence of the lemma below:

**Lemma 5.1.** For each \( n \in \mathbb{N} \), the map

\[
\varepsilon \in [0, \varepsilon_{\max}] \rightarrow \frac{1}{n} \left[ E \log P_{0, \omega_\varepsilon}(X_n \in \partial R_n) - \log P_0(X_n \in \partial R_n) \right] \text{ is non-increasing.}
\]

This is an easy adaptation of Lemma 16 in [5]. If we let \( n \) to infinity, then we deduce that \( p(\varepsilon) - \lambda(\varepsilon) \) is non-increasing. To finish the proof, define

\[
\overline{\varepsilon} := \inf \{ \varepsilon \in (0, \varepsilon_{\max}) : p(\varepsilon) - \lambda(\varepsilon) < 0 \},
\]
with the convention that $\inf \emptyset = \epsilon_{\max}$.

The rest of the section is devoted to prove (i), (ii) of Theorem 1.8. We’ve already proved (i) as a consequence of Theorem 1.5. The main ingredient to show the first part in (ii) is the next lemma.

**Lemma 5.2.** If $\epsilon > 0$ is small enough, then $\sup_n \|W_n^2\|_2 < \infty$.

**Proof of Lemma 5.2.** This is a particular case of Lemma 5 with $x = 0 = \theta$ in [5]. □

Recall the following:

$$W_\infty(\epsilon) := W_\infty(\omega_\epsilon) > 0 \to p(\epsilon) = \lambda(\epsilon) \leftrightarrow \text{delocalization}.$$  

Indeed, if $W_\infty > 0$, then $\log(W_\infty) = \lim_{n \to \infty} \log(W_n) < \infty$, so

$$p(\epsilon) = \lim_{n \to \infty} \frac{1}{n} \log P_{0,\omega_\epsilon}(X_n \in \partial R_n) = \lim_{n \to \infty} \frac{W_n(\omega_\epsilon)}{n} + \lambda(\epsilon) = \lambda(\epsilon).$$

Now pick $\epsilon > 0$ small enough such that $\sup_n \|W_n^2\|_2 < \infty$ as in Lemma 5.2, and call it $\epsilon^*$. By the martingale convergence theorem, $W_n(\epsilon^*) \to W_\infty(\epsilon^*)$ a.s. and in $L^2$. As $\|W_n\|_2 = 1$ for all $n$, then we necessarily have $W_\infty(\epsilon^*) > 0$, and therefore $p(\epsilon^*) = \lambda(\epsilon^*)$. But the map $\epsilon \to p(\epsilon) - \lambda(\epsilon)$ is non-increasing, so $p = \lambda$ on $[0, \epsilon^*)$, and thus $\tau \geq \epsilon^* > 0$.

### 5.1. An example on which $\tau < \epsilon_{\max}$

By Eq. (2.14), it’s enough to find conditions so that for large $\epsilon$ we have $\inf_{x \in \mathbb{D}(\tau)} I_a < \inf_{x \in \mathbb{D}(\tau)} I_q(x)$. This is equivalent to proving that

$$\sup_{x \in \mathbb{D}(\tau)} \lim_{n \to \infty} \frac{1}{n} \log P_{0,\omega_\epsilon}(X_n \in \partial R_n) < \sup_{x \in \mathbb{D}(\tau)} \lim_{n \to \infty} \frac{1}{n} \log P_0(X_n \in \partial R_n) = -I_a(\overline{\tau}).$$

where $\overline{\tau} := \sum_{e \in \mathbb{V}(\tau)} q(e)e$, and $\overline{q}(\epsilon) := \sum_{e \in \mathbb{V}(\tau)} q(e)$. Now, if $x \neq \overline{\tau}$,

$$\lim_{n \to \infty} \frac{1}{n} \log P_{0,\omega_\epsilon}(X_n \in \partial R_n) = \lim_{x_n/n \to x} \frac{1}{n} \mathbb{E} \log \left( \sum_{y_1, \ldots, y_n = x} \prod_{i=1}^{n} q(\Delta y_i) \right),$$

where $(x_n) \subset \mathbb{Z}^d$ is some sequence with $|x_n|_1 = n$ and $x_n/n \to x$ (recall that the quenched limit is also deterministic). The last expression is bounded by above by

$$\lim_{x_n/n \to x} \frac{1}{n} \sum_{y_1, \ldots, y_n = x_n} \log \left( \prod_{i=1}^{n} q(\Delta y_i) \right) + \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log \left( \frac{\sum_{y_1, \ldots, y_n = x_n} \prod_{i=1}^{n} q(\Delta y_i) \prod_{i=1}^{n} (1 + \epsilon q(y_{i-1}, \Delta y_i))}{\sum_{y_1, \ldots, y_n = x_n} \prod_{i=1}^{n} q(\Delta y_i)} \right) = -I_a(x) + \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log \left( \frac{\sum_{y_1, \ldots, y_n = x_n} \prod_{i=1}^{n} q(\Delta y_i) \prod_{i=1}^{n} (1 + \epsilon q(y_{i-1}, \Delta y_i))}{\sum_{y_1, \ldots, y_n = x_n} \prod_{i=1}^{n} q(\Delta y_i)} \right) < -I_a(\overline{\tau}),$$

and the last inequality holds because $-I_a(x) < -I_a(\overline{\tau})$ for $x \neq \overline{\tau}$, and the inequality $\log(x) \leq 1 - x$ implies

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log \left( \frac{\sum_{y_1, \ldots, y_n = x_n} \prod_{i=1}^{n} q(\Delta y_i) \prod_{i=1}^{n} (1 + \epsilon q(y_{i-1}, \Delta y_i))}{\sum_{y_1, \ldots, y_n = x_n} \prod_{i=1}^{n} q(\Delta y_i)} \right) < 0$$
because the expectation inside the log is 1. Thus, we only need to study the case $x = \bar{x}$. We proceed in a similar way of the previous case to get

$$
\lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{y_1, \ldots, y_n = \bar{x}} \prod_{i=1}^{n} \{ q(\Delta y_i) + \varepsilon \xi_i(y_{i-1}, \Delta y_i) \} \right) 
\leq -I_d(\bar{x}) + \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{\sum_{y_1, \ldots, y_n = \bar{x}} \prod_{i=1}^{n} q(\Delta y_i) \prod_{i=1}^{n} (1 + \varepsilon \xi_i(y_{i-1}, \Delta y_i))}{\sum_{y_1, \ldots, y_n = \bar{x}} \prod_{i=1}^{n} q(\Delta y_i)} \right).
$$

Therefore, we only need to check conditions so that the lim sup above is less than 0 for large enough $\varepsilon$. For fixed $n$, we have a rough bound

$$
\frac{1}{n} \log \left( \frac{\sum_{y_1, \ldots, y_n = \bar{x}} \prod_{i=1}^{n} q(\Delta y_i) \prod_{i=1}^{n} (1 + \varepsilon \xi_i(y_{i-1}, \Delta y_i))}{\sum_{y_1, \ldots, y_n = \bar{x}} \prod_{i=1}^{n} q(\Delta y_i)} \right) 
\leq \max_{y_1, \ldots, y_n = \bar{x}} \sum_{i=1}^{n} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right).
$$

Now, observe that for fixed $y_1, \ldots, y_n$, the random variables $\{ \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right) \}_{i=1, \ldots, n}$ are independent and live in the interval $[\log \kappa, -\log \kappa]$ (cf., (1.5)). Therefore, by the Hoeffding inequality (Theorem 2 in [17]) we deduce for $\alpha > 0$

$$
\sum_{n=1}^{\infty} \mathbb{P} \left( \max_{y_1, \ldots, y_n = \bar{x}} \left[ \sum_{i=1}^{n} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right) - \sum_{i=1}^{n} \mathbb{E} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right) \right] > n\alpha \right) 
\leq \sum_{n=1}^{\infty} d^n \max_{y_1, \ldots, y_n = \bar{x}} \mathbb{P} \left( \sum_{i=1}^{n} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right) - \sum_{i=1}^{n} \mathbb{E} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right) > n\alpha \right) 
\leq \sum_{n=1}^{\infty} d^n \exp \left( -\frac{n\alpha^2}{\log(1/\kappa^2)} \right) < \infty
$$

if $\alpha > \sqrt{\log(d) \log(1/\kappa^2)}$. For such $\alpha$, we have by Borel-Cantelli’s lemma

$$
\frac{1}{n} \max_{y_1, \ldots, y_n = \bar{x}} \left[ \sum_{i=1}^{n} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right) - \sum_{i=1}^{n} \mathbb{E} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right) \right] \leq \alpha \text{ a- s.}
$$

Thus, there is a constant $C = C(d, \kappa)$ such that

$$
\limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{\sum_{y_1, \ldots, y_n = \bar{x}} \prod_{i=1}^{n} q(\Delta y_i) \prod_{i=1}^{n} (1 + \varepsilon \xi_i(y_{i-1}, \Delta y_i))}{\sum_{y_1, \ldots, y_n = \bar{x}} \prod_{i=1}^{n} q(\Delta y_i)} \right) 
\leq C + \limsup_{n \to \infty} \max_{y_1, \ldots, y_n = \bar{x}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \log \left( 1 + \frac{\varepsilon \xi_i(y_{i-1}, \Delta y_i)}{q(\Delta y_i)} \right).
$$

Note that for large $n$, there are approximately $n\sqrt{\mathbb{E}[\xi]}$ jumps in the direction $e_i$. As the expectations above only depends on the jumps $\Delta y_i$ (and they are non-positive), we only need to check that for $e \in V(\bar{x})$ such that $\varepsilon_{\text{max}} = \left( \frac{q(e)}{q(0,e)} \right)_{|e|_{\bar{x}}} \infty$ we have

$$
\mathbb{E}[\xi] \mathbb{E} \left[ \log \left( 1 + \frac{\varepsilon \xi(0,e)}{q(e)} \right) \right] < -C \text{ for } \varepsilon < \varepsilon_{\text{max}} \text{ large enough.}
$$
Suppose that $\xi(0, e) \in [-b, b]$ for some $b > 0$. In this case we have $\varepsilon_{\max} = \frac{q(\varepsilon)}{b}$. In general, we can check that (5.1) holds if $\xi$ is close to $-b$ with positive probability. For example, assume that $\xi = -b(1 - \delta)$ with probability $\frac{1}{2}$ and $b(1 - \delta)$ with probability $\frac{1}{2}$, for some $\delta > 0$ (so that $E[\xi] = 0$). Then

$$E \left[ \log \left( 1 + \frac{\varepsilon \xi(0, e)}{q(\varepsilon)} \right) \right] = \frac{1}{2} \left( \log \left( 1 - \frac{\varepsilon b(1 - \delta)}{q(\varepsilon)} \right) + \log \left( 1 + \frac{\varepsilon b(1 - \delta)}{q(\varepsilon)} \right) \right).$$

Taking $\varepsilon \to \varepsilon_{\max}$ the expression above is equal to

$$\frac{1}{2} \log(2) \to -\infty \text{ as } \delta \to 0.$$

This implies that (5.1) is true if $\delta > 0$ is small enough.

It’s not difficult to see that the same holds more generally for random variables that are close to $b$ (more generally, close to $-|\xi(0, e)|_\infty$) with a positive probability. How close $\xi$ has to be to this value depends on the constant $C(d, k)$.

A. APPENDIX

A.1. Proof of Lemma 2.7. We argue for the quenched case. The ideas comes from the proof of Lemma 16.12 in [20]. We provide the proof here by completeness.

By definition we have

$$\frac{1}{n} \log P_{0, \omega}(X_n \in \partial R_n) \geq \frac{1}{n} \log P_{0, \omega}(X_n = x_n) \forall x_n \in \partial R_n.$$  

Then $\liminf_{n \to \infty} \frac{1}{n} \log P_{0, \omega}(X_n \in \partial R_n) \geq \sup_{x \in \partial(D(\pi))} -I_q(x) = -\inf_{x \in \partial(D(\pi))} I_q(x)$. For the upper bound, we write

$$\frac{1}{n} \log P_{0, \omega}(X_n \in \partial R_n) \leq \max_{x_n \in \partial R_n} \frac{1}{n} \log P_{0, \omega}(X_n = x_n) + \frac{C \log n}{n}.$$  

for some constant $C > 0$. We would like to have a maximum over a fixed amount of elements, for all $n \in \mathbb{N}$. To do this, fix $\varepsilon > 0$ small enough, and a positive integer $k \geq \frac{d + 1(1 + \varepsilon)}{\varepsilon}$. Consider the set

$$A := \left\{ z = \left( \frac{i_1}{k}, \ldots, \frac{i_d}{k} \right), i_j \geq 0, \sum_{j=1}^d i_j = 1 \right\}.$$  

This is a finite set. We will show that for fixed $n$ and $x_n \in \partial R_n$, there is some $z \in A, n_1 \in \mathbb{N}$ such that $n_1 z \in \partial R_n$ and a path of length $n_1 - n \geq 0$ from $x$ to $n_1 z$.

Write $x_n = \sum_{i=1}^d a_i e_i, a_i \geq 0, \sum_i a_i = n$. Define for $i = 1, \ldots, d$,

$$b_i := \left\lfloor \frac{ka_i}{1 + \varepsilon n} \right\rfloor$$  

and

$$m_n := \left\lfloor \frac{(1 + \varepsilon) n}{k} \right\rfloor$$  

Observe that $\frac{m_n k - n}{n} \to \varepsilon$ as $n \to \infty$. By definition, $m_n b_i - a_i \geq 0$. On the other hand

$$\sum_{i=1}^{d-1} (m_n b_i - a_i) \leq \frac{m_n k}{1 + \varepsilon} + m_n (d - 1) - n \leq m_n k - n.$$  

The last inequality comes from the bound $k \geq \frac{(d - 1)(1 + \varepsilon)}{\varepsilon}$. Thus, if we define

$$z = z(x_n) := \frac{1}{k} \sum_{i=1}^{d-1} b_i e_i + \left( 1 - \frac{1}{k} \sum_{i=1}^{d-1} b_i \right) e_d,$$  

we have $z \in A$, $m_n z = m_n x_n$, and $d_z(x, n_1 z) \geq 0$.

Thus, $\liminf_{n \to \infty} \frac{1}{n} \log P_{0, \omega}(X_n \in \partial R_n) \geq \inf_{z \in A} \frac{1}{n} \log P_{0, \omega}(X_n = x_n) = \inf_{z \in A} \frac{1}{n} \log P_{0, \omega}(Z_n = z) = \inf_{z \in A} I_{q}(z).$
then $n_1 = m_n k$ satisfies the required properties. Continuing with (A.1),
\[
\frac{1}{n} \log P_{0,\omega}(X_n \in \partial R_n) \leq \frac{1}{n} \log P_{0,\omega}(X_n = x_n) + C \frac{\log n}{n}
\]
\[
\leq \max_{x_n \in \partial R_n} \frac{1}{n} \log P_{0,\omega}(X_n = x_n, X_{n_1} = n_1 z(x_n)) + \frac{n_1 - n}{n} x (-\log \kappa) + C \frac{\log n}{n}
\]
\[
\leq \max_{z \in A} \frac{1}{n} \log P_{0,\omega}(X_{n_1} = n_1 z) + \frac{n_1 - n}{n} x (-\log \kappa) + C \frac{\log n}{n}.
\]
In (*) we used the Markov property and uniform ellipticity:
\[
P_{0,\omega}(X_n = x_n, X_{n_1} = n_1 z) \geq P_{0,\omega}(X_n = x_n) P_{0,\omega}(X_{n_1} = n_1 z) \geq P_{0,\omega}(X_n = x_n) \kappa^{n_1 - n}.
\]
Letting $n \to \infty$ yields
\[
\limsup_{n \to \infty} \frac{1}{n} \log P_{0,\omega}(X_n \in \partial R_n) \leq \max_{z \in A} \left(-I_q(z)(1 + \varepsilon) - \varepsilon \log \kappa \leq \max_{x \in \mathbb{Z}^d \setminus \{0\}} (-I_q(x))(1 + \varepsilon) - \varepsilon \log \kappa. \right.
\]
Finally, take $\varepsilon \to 0$ to conclude the proof in the quenched case. The annealed case can be seen as a particular case of the quenched one (with fixed environment) so the analog result also applies.
\[\square\]

A.2. Some estimates of the decay of the annealed probability. We use the following to estimate the supremum in (4.11):

**Lemma A.1.** For every $e, e' \in V(\mathbb{Z})$ we have
\[
\sup_{x \in \mathbb{Z}^d} |P_0(X_n = x + e | A_n) - P_0(X_n = x + e') | A_n| = O(n^{-1}) \text{ as } n \to \infty.
\]

**Proof.** Clearly we only need to focus on $x$ such that $|x|_1 = n - 1$. Observe that we have
\[
P_0(X_n = x + e | A_n) = P_0(X_{n-1} = x | A_{n-1}) \frac{q(e)}{q},
\]
where we recall that $q = \sum_{e' \in V(\mathbb{Z})} q(e')$. From here, we deduce that it’s enough to check that there is a constant $c_1$ such that for large $n$,
\[
\sup_{x \in \mathbb{Z}^d} P_0(X_n = x | A_n) \leq \frac{c_1}{1 + n}. \tag{A.2}
\]
Recall that $X_n \in \partial R_n$ means that for each $i \in \{1, \cdots n\}$ it’s true that $X_i - X_{i-1} \in \{e_1, e_2, e_3\}$. Then, if $x = x_n = n_1 e_1 + n_2 e_2 + n_3 e_3$,
\[
(n + 1) P_0(X_n = x | A_n) = \frac{(n + 1)!}{n_1! n_2! n_3!} q(e_1)^{n_1} q(e_2)^{n_2} q(e_3)^{n_3}, \tag{A.3}
\]
where we assume that $x = n_1 e_1 + n_2 e_2 + n_3 e_3$, and $n_1 + n_2 + n_3 = n$. This expression is maximized when
\[
n_1 \approx n \frac{q(e_i)}{q(e_1) + q(e_2) + q(e_3)}, \quad i \in \{1, 2, 3\}.
\]
But in this case, applying Stirling’s formula for large enough n, the expression in (A.3) is approximately
\[
\frac{(q(e_1) + q(e_2) + q(e_3))^{3/2}}{2n \sqrt{q(e_1) q(e_2) q(e_3)}} \left(1 + \frac{1}{n}\right)^n \approx \frac{(q(e_1) + q(e_2) + q(e_3))^{3/2}}{2n \sqrt{q(e_1) q(e_2) q(e_3)}}
\]
Therefore, for large enough $n$, we can bound uniformly $(n + 1) P_0(X_n = x | A_n)$ by a constant that only depends on $q(e_1), q(e_2), q(e_3)$.
\[\square\]
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