Classical $r$-matrices of two and three dimensional Lie super-bialgebras and their Poisson-Lie supergroups

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January 12, 2010

Abstract

In this paper classical $r$-matrices of two and three dimensional Lie super-bialgebras (classified in arXiv:0901.4471 and arXiv:0911.1760) are obtained. In this manner, all two and three dimensional coboundary Lie super-bialgebras and their types (triangular, quasi-triangular or factorizable) are classified. Then, using Sklyanin superbracket, the super Poisson structures on the related Poisson-Lie supergroups are obtained.

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1 Introduction

From mathematical point of view, Lie bialgebras were first introduced by Drinfel’d as algebraic structures and classical limit of underlying quantized enveloping algebras (quantum groups) [1]. In particular, every deformation of a universal enveloping algebra induces a Lie bialgebra structure on the underlying Lie algebra. Conversely, as it has been shown in [2] that each Lie bialgebra admits quantization. So the classification of Lie bialgebras can be seen as the first step in the classification of quantum groups. On the other hand, from the physical point of view, the theory of classical integrable system naturally relates to the geometry and representation theory of Poisson-Lie groups and the corresponding Lie bialgebras and their classical r-matrices (see for example [3]). Furthermore, Lie bialgebras and Poisson-Lie groups have applications in the theory of Poisson-Lie T-dual sigma models [4] and N = 2 superconformal field theories [5]. In the same way, Lie super-bialgebras [6], as the underlying symmetry algebras, play an important role in the integrable structure of AdS/CFT correspondence [7], as well as in Toda models on Lie superalgebras [8]. Similarly, one can consider Poisson-Lie T-dual sigma models on Poisson-Lie supergroups [9]. In this way and by considering that there is a universal quantization for Lie super-bialgebras [10], the classification of Lie super-bialgebras (especially low dimensional Lie super-bialgebras) is important role from both physical and mathematical point of view. In [11] and [12] we classify all two and three dimensional Lie super-bialgrbras. In the present paper, following [11] and [12], we find the classical r-matrices of these Lie super-bialgebras and determine their types (triangular, quasi-triangular or factorizable). Furthermore, we obtain super Poisson structures on the related Poisson-Lie supergroups.

The paper is organized as follows. In section two, we give some basic definitions and notations on coboundary Lie super-bialgebras and Manin super triple. In section three, we list the decomposable and indecomposable Lie superalgebras and their related Lie super-bialgebras [11], [12]. In section four, using adjoint and matrix representation we calculate classical r-matrices and determine two and three dimensional coboundary (triangular, quasi-triangular or factorizable) Lie super-bialgebras. In section five, we first calculate invariant supervector fields and then using Sklyanin superbracket we obtain super Poisson structures on the related two and three dimensional Poisson-Lie supergroups.

2 Definitions and notations

In the present paper, we apply DeWitt notations for supervector spaces, supermatrices,...[13]. Let us recall some basic definitions and propositions on Lie super-bialgebras [6, 11, 14].

Definition: A Lie superalgebra \( g \) is a graded vector space \( g = g_B \oplus g_F \) with gradings \( \text{grade}(g_B) = 0 \) and \( \text{grade}(g_F) = 1 \) so that Lie bracket satisfies the super antisymmetric and super Jacobi identities, i.e. in the graded basis \( \{X_i\} \) of \( g \) we have\(^1\)

\[
[X_i, X_j] = f^k_{ij}X_k,
\]

and

\[
(-1)^{(i+k)} f^m_{ji} f^l_{jk} + f^m_{il} f^l_{jk} + (-1)^{(i+j)} f^m_{ki} f^l_{ij} = 0,
\]

so that

\[
f^k_{ij} = -(-1)^{ij} f^k_{ji}.
\]

Furthermore, we have

\[
f^k_{ij} = 0, \quad i f \quad \text{grade}(i) + \text{grade}(j) \neq \text{grade}(k) \pmod{2}.
\]

Let \( g \) be a finite-dimensional Lie superalgebra and \( g^* \) be its dual superspace with respect to a non-degenerate canonical pairing \( ( , ) \) on \( g^* \oplus g \).

Definition: A Lie super-bialgebra structure on a Lie superalgebra \( g \) is a super skew-symmetric linear map \( \delta : g \to g \otimes g \) (the super cocommutator) so that

\(^1\)Note that the bracket of one boson with one boson or one fermion is usual commutator, but for one fermion with one fermion is anticommutator. Furthermore, we identify grading of indices by the same indices in the power of \((-1)\), for example \( \text{grading}(i) \equiv i; \) this is the notation that DeWitt applied in [13].
1) $\delta$ is a super one-cocycle, i.e.

$$\delta([X,Y]) = (ad_X \otimes I + I \otimes ad_X)\delta(Y) - (ad_Y \otimes I + I \otimes ad_Y)\delta(X) \quad \forall X, Y \in \mathfrak{g},$$

(5)

2) the dual map $\delta^t: \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie super bracket on $\mathfrak{g}^*$, i.e.

$$(\xi \otimes \eta, \delta(X)) = (\delta^t(\xi \otimes \eta), X) = ([\xi, \eta]_s, X) \quad \forall X \in \mathfrak{g}; \ \xi, \eta \in \mathfrak{g}^*.$$

(6)

The Lie super-bialgebra defined in this way will be denoted by $(\mathfrak{g}, \mathfrak{g}^*)$ or $(\mathfrak{g}, \delta)$ [6] [11].

**Definition:** A Lie super-bialgebra is **coboundary** if the super cocommutator is a one-coboundary, i.e. if there exists an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta(X) = (I \otimes ad_X + ad_X \otimes I)r \quad \forall X \in \mathfrak{g}.$$  

(7)

**Proposition:** Two coboundary Lie super-bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{g}', \mathfrak{g}'^*)$ defined by $r \in \mathfrak{g} \otimes \mathfrak{g}$ and $r' \in \mathfrak{g}' \otimes \mathfrak{g}'$ are **isomorphic** if and only if there is an isomorphism of Lie superalgebras $\alpha: \mathfrak{g} \rightarrow \mathfrak{g}'$ such that $(\alpha \otimes \alpha)r - r'$ is $\mathfrak{g}'$ invariant [14] i.e.

$$(I \otimes ad_X + ad_X \otimes I)((\alpha \otimes \alpha)r - r') = 0 \quad \forall X \in \mathfrak{g}'.$$

(8)

**Definition:** Coboundary Lie super-bialgebras can be of two different types:

- **a)** if $r$ is a super skew-symmetric solution of the classical Yang-Baxter equation (CYBE)

$$[[r, r]] = 0,$$

(9)

then the coboundary Lie super-bialgebra is said to be **triangular**; wherein Schouten superbracket is defined by

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}],$$

(10)

and if we denote $r = r^{ij}X_i \otimes X_j$, then $r_{12} = r^{ij}X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij}X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij}1 \otimes X_i \otimes X_j$. A solution of the CYBE is often called a **classical r-matrix**.

- **b)** if $r$ is a solution of CYBE such that $r_{12} + r_{21}$ is a $\mathfrak{g}$ invariant element of $\mathfrak{g} \otimes \mathfrak{g}$, then the coboundary Lie super-bialgebra is said to be **quasi-triangular**. Moreover, if the super symmetric part of $r$ is invertible, then $r$ is called **factorizable**.

Sometimes condition **b)** can be replaced with the following:

- **b’)** if $r$ is a super skew-symmetric solution of the modified CYBE

$$[[r, r]] = \omega = \wedge^3 \mathfrak{g},$$

(11)

then the coboundary Lie super-bialgebra is said to be **quasi-triangular** [6].

Note that if $\mathfrak{g}$ is a Lie super-algebra then $\mathfrak{g}^*$ is also a Lie super-algebra [6], but this is not always true for the coboundary property.

**Definition:** Suppose $\mathfrak{g}$ be a coboundary Lie super-bialgebra with one-cocycle $\delta$ and $\mathfrak{g}^*$ be also coboundary Lie super-bialgebra with the one-cocycle $\delta^*$

$$\forall \xi \in \mathfrak{g}^* \quad \exists r^* \in \mathfrak{g}^* \otimes \mathfrak{g}^* \quad \delta^*(\xi) = (I \otimes ad_{\xi} + ad_{\xi} \otimes I)r^*,$$

(12)

where $\delta^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$. Then the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called a bi-r-matrix Lie super-bialgebra [14] if the Lie super bracket $[,]$ on $\mathfrak{g}$ defined by $\delta^{st}$

$$\delta^{st}(\xi, X \otimes Y) = (\xi, \delta^{st}(X \otimes Y)) = ([\xi, [X, Y]]^t) \quad \forall X, Y \in \mathfrak{g}, \ \xi \in \mathfrak{g}^*,$$

(13)

is equivalent to the original ones [14] i.e.

$$[X, Y]^t = S^{-1}[SX, SY] \quad \forall X, Y \in \mathfrak{g}, \quad S \in Aut(\mathfrak{g}).$$

(14)

\(^2\)Note that $r$ has even Grassmann parity and Grassmann parity of $r^{ij}$ comes from indices; for example, we have

$$[r_{12}, r_{13}] = (-1)^{(k+l)+jl} r^{ij}r^{kl} [X_i, X_k] \otimes X_j \otimes X_l.$$
**Definition:** A *Manin* super triple is a triple of Lie superalgebras \((\mathcal{D}, g, \tilde{g})\) together with a non-degenerate ad-invariant super symmetric bilinear form \(<, >\) on \(\mathcal{D}\) such that

1. \(g\) and \(\tilde{g}\) are Lie sub-superalgebras of \(\mathcal{D}\),
2. \(\mathcal{D} = g \oplus \tilde{g}\) as a supervector space,
3. \(g\) and \(\tilde{g}\) are isotropic with respect to \,<,>\, i.e.

\[<X_i, X_j> = <\bar{X}^i, \bar{X}^j> = 0, \quad \delta^j_i = <X_i, \bar{X}^j> = (-1)^{ij} <\bar{X}^j, X_i> = (-1)^{ij} \delta^j_i, \quad (15)\]

where \(\{X_i\}\) and \(\{\bar{X}^i\}\) are basis of Lie superalgebras \(g\) and \(\tilde{g}\) respectively \([6, 11]\). Note that in the above relation \(\delta^j_i\) is the ordinary delta function. There is a one-to-one correspondence between Lie super-bialgebra \((g, g^*)\) and Manin super triple \((\mathcal{D}, g, \tilde{g})\) with \(\tilde{g} = g^*\). If we choose the structure constants of Lie superalgebras \(g\) and \(\tilde{g}\) as follows:

\[[X_i, X_j] = f^{k}_{ij} X_k, \quad [\bar{X}^i, \bar{X}^j] = \bar{f}^{ij}_{k} \bar{X}^k, \quad (16)\]

then ad-invariance of the bilinear form \,<,>\, on \(\mathcal{D} = g \oplus \tilde{g}\) implies that \([11]\)

\[[X_i, \bar{X}^j] = (-1)^{ij} \bar{f}^{jk} X_k + (-1)^i f^j_{ki} \bar{X}^k. \quad (17)\]

Clearly, using the equations (15), (16) and (6) we have\(^3\)

\[\delta(X_i) = (-1)^{jk} \bar{f}^{jk} X_j \otimes X_k. \quad (18)\]

As a result of applying this relation to the super one-cocycle condition (5), the super Jacobi identities (2) for the dual Lie superalgabra and the following mixed super Jacobi identities are obtained\(^4\)

\[f^{mk}_{j} \bar{f}^{ml}_{j} = f^{mk}_{j} \bar{f}^{ml}_{j} + f^{j}_{jm} \bar{f}^{im}_{k} + (-1)^{jl} f^{i}_{jm} \bar{f}^{ml}_{k} + (-1)^{ik} f^{j}_{mk} \bar{f}^{im}_{j}. \quad (19)\]

### 3 Two and three dimensional Lie superalgebras and Lie super-bialgebras

In \([11]\) we find and classify all two and three dimensional Lie super-bialgebras for all two and three dimensional indecomposable Lie superalgebras. The method of classification is new and indeed it is improvement and generalization of the method of \([16]\)\(^5\) to the Lie superalgebras. In this method by use of adjoint representation of super Jacobi and mixed super Jacobi identities (2) and (19) we find dual Lie superalgebras by direct calculation; then by use of automorphism Lie supergroups of Lie superalgebras we classify all non isomorphic two and three dimensional Lie super-bialgebras \([11]\) and \([12]\). Here for presentation of the notations that we will use and for self consistence of the paper, the list of two and three dimensional indecomposable and decomposable Lie superalgebras\(^6\) \([15]\) and their related Lie super-bialgebras \([11]\) and \([12]\) are given in tables 1–4 respectively.

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\(^3\)Note that the appearance of \((-1)^{jk}\) in this relation is due to the definition of the bilinear form as:

\[<X_i \otimes \bar{X}^j, X_k \otimes \bar{X}^l> = (-1)^{jk} \delta^l_i \delta^j_k.\]

\(^4\)This relation can also be obtained from super Jacobi identity of \(\mathcal{D}\).

\(^5\)Note that in \([17]\) unfortunately there is not standard and logical method for obtaining of low dimensional Lie bialgebras.

\(^6\)Note that as we use DeWitt notation and standard basis here, the structure constants \(C^F_{p,F}\) must be pure imaginary.
Table 1: Two and three dimensional decomposable\(^7\) and indecomposable Lie superalgebras\(^8\)

| Type   | \(\mathfrak{g}\) | Bosonic basis | Fermionic basis | Non zero (Anti) Commutation relations | Comments                      |
|--------|------------------|---------------|-----------------|--------------------------------------|-------------------------------|
| (1,1)  | \(B\)           | \(X_1\)       | \(X_2\)         | \([X_1, X_2] = X_2\)                 | Trivial                       |
|        | \((A_{1,1} + A)\) | \(X_1\)       | \(X_2\)         | \([X_2, X_2] = X_1\)                 | Nontrivial                    |
| (2,1)  | \((B + A_{1,1})\) | \(X_1, X_2\)  | \(X_3\)         | \([X_1, X_3] = X_3\)                 | Solvable, Trivial             |
|        | \((2A_{1,1} + A)\) | \(X_1, X_2\)  | \(X_3\)         | \([X_1, X_3] = X_1\)                 | Nilpotent, Nontrivial         |
|        | \(C^1_0\)       | \(X_1, X_2\)  | \(X_3\)         | \([X_1, X_2] = X_2\)                 | Solvable, Trivial             |
|        | \(C^2_p\)       | \(X_1, X_2\)  | \(X_3\)         | \([X_1, X_2] = X_2, \ [X_1, X_3] = pX_3\) | \(p \neq 0\), Trivial        |
|        | \(C^4_{\frac{3}{2}}\) | \(X_1, X_2\)  | \(X_3\)         | \([X_1, X_2] = X_2, \ [X_1, X_3] = \frac{3}{2}X_3\), \(\{X_3, X_3\} = X_2\) | Nontrivial                    |
| (1,2)  | \(C^2_0\)       | \(X_1\)       | \(X_2, X_3\)    | \([X_1, X_2] = X_2\)                 | Solvable, Trivial             |
|        | \((A_{1,1} + 2A)^0\) | \(X_1\)       | \(X_2, X_3\)    | \([X_2, X_2] = X_1\)                 | Nilpotent, Nontrivial         |
|        | \(C^2_p\)       | \(X_1\)       | \(X_2, X_3\)    | \([X_1, X_2] = X_2, \ [X_1, X_3] = pX_3\) | \(0 < |p| < 1\), Trivial       |
|        | \(C^4\)         | \(X_1\)       | \(X_2, X_3\)    | \([X_1, X_3] = X_2\)                 | Nilpotent, Trivial            |
|        | \(C^4\)         | \(X_1\)       | \(X_2, X_3\)    | \([X_1, X_2] = X_2, \ [X_1, X_3] = X_2 + X_3\) | Trivial                       |
|        | \((A_{1,1} + 2A)^1\) | \(X_1\)       | \(X_2, X_3\)    | \([X_2, X_2] = X_1, \ {X_2, X_3} = X_1\) | Nilpotent, Nontrivial         |
|        | \((A_{1,1} + 2A)^2\) | \(X_1\)       | \(X_2, X_3\)    | \([X_2, X_2] = X_1, \ {X_2, X_3} = -X_1\) | Nilpotent, Nontrivial         |

Table 2: Three dimensional Lie super-bialgebras of type \((2,1)\)

| \(\mathfrak{g}\) | \(\mathfrak{g}\) | Non zero (anti) commutation relations of \(\mathfrak{g}\) | Comments |
|------------------|------------------|-----------------------------------------------|----------|
| \((2A_{1,1} + A)\) | \(I_{(2,1)}\) | \([\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3\) | \(p \in \mathbb{R}\) |
| \((B + A_{1,1})\) | \(I_{(2,1)}\) | \([\tilde{X}^3, \tilde{X}^3] = \tilde{X}^1\) | \(p \in \mathbb{R}\) |
| \((2A_{1,1} + A)\) | \(I_{(2,1)}\) | \([\tilde{X}^3, \tilde{X}^3] = -\tilde{X}^3\) | \(p \in \mathbb{R}\) |
| \(C^1_p\) | \(I_{(2,1)}\) | \([\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, \ [\tilde{X}^2, \tilde{X}^3] = \frac{1}{p}\tilde{X}^3\) | \(p \in \mathbb{R}\) |
| \(C^1_{\frac{3}{2}}\) | \(I_{(2,1)}\) | \([\tilde{X}^1, \tilde{X}^2] = \frac{3}{2}\tilde{X}^1, \ [\tilde{X}^2, \tilde{X}^3] = -\frac{1}{2}\tilde{X}^3\) | \(p \in \mathbb{R}\) |
| \(C^1_{\frac{1}{2}}\) | \(I_{(2,1)}\) | \([\tilde{X}^1, \tilde{X}^2] = \frac{1}{2}\tilde{X}^1, \ [\tilde{X}^2, \tilde{X}^3] = \frac{1}{2}\tilde{X}^3\) | \(p \in \mathbb{R}\) |
| \(C^1_{\frac{1}{2}}\) | \(I_{(2,1)}\) | \([\tilde{X}^1, \tilde{X}^2] = -k\tilde{X}^2, \ [\tilde{X}^1, \tilde{X}^3] = -\frac{k}{2}\tilde{X}^3\) | \(k \in \mathbb{R} \setminus \{0\}\) |

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\(^7\) Note that decomposable Lie superalgebras are as follows:
\(B + A_{1,1} = B \oplus A_{1,1}, \ (2A_{1,1} + A) = (A_{1,1} + A) \oplus A_{1,0}, C^1_0 = C^1_{p=0}, C^2_0 = C^2_{p=0} = B \oplus A_{0,1}\) and \((A_{1,1} + 2A)^0 = (A_{1,1} + A) \oplus A_{0,1}\).

\(^8\) The Lie superalgebra \(A\) is one dimensional Abelian Lie superalgebra with one fermionic generator where Lie superalgebra \(A_{1,1}\) is its bosonization. Furthermore, \(C^1_{\frac{1}{2}}\) is different from \(C^2_p\) and we show the latter by \(C^1_{\frac{1}{2}}\).
Table 3: Two dimensional Lie super-bialgebras of the type $(1,1)$

| $\mathfrak{g}$       | $\tilde{\mathfrak{g}}$ | Non zero (anti) commutation relations of $\tilde{\mathfrak{g}}$ |
|-----------------------|--------------------------|---------------------------------------------------------------|
| $(A_{1,1} + A)$       | $I_{(1,1)}$              |                                                              |
| $B$                   |                          |                                                              |
| $(A_{1,1} + A)$       | $\tilde{I}_{(1,1)}$     | $\{\tilde{X}^2, \tilde{X}^2\} = \tilde{X}^1$              |
| $(A_{1,1} + A)_i$     |                          | $\{\tilde{X}^2, \tilde{X}^2\} = -\tilde{X}^1$            |

Table 4: Three dimensional Lie super-bialgebras of the type $(1,2)$

| $\mathfrak{g}$ | $\tilde{\mathfrak{g}}$ | Comments |
|----------------|--------------------------|----------|
| $I_{(1,2)}$    |                          |          |
| $(A_{1,1} + 2A)_{0,0,0}^0$, $(A_{1,1} + 2A)_{0,0,1}^0$ | $-1 < k < 1$ |
| $(A_{1,1} + 2A)_{1,0,0}^0$, $(A_{1,1} + 2A)_{0,1,0}^0$ | $0 < k$, $s < 0$ |
| $(A_{1,1} + 2A)_{k,0,0}^0$, $(A_{1,1} + 2A)_{s,0,1}^0$ | $k < 0$, $0 < s$ |
| $C^3$          |                          |          |
| $(A_{1,1} + 2A)_{0,0,0}^1$, $(A_{1,1} + 2A)_{0,0,1}^1$ | $-1 < k < 1$ |
| $(A_{1,1} + 2A)_{1,0,0}^1$, $(A_{1,1} + 2A)_{0,1,0}^1$ | $0 < k$, $s < 0$ |
| $(A_{1,1} + 2A)_{k,0,0}^1$, $(A_{1,1} + 2A)_{s,0,1}^1$ | $k < 0$, $0 < s$ |
| $C^4$          |                          |          |
| $(A_{1,1} + 2A)_{0,0,0}^2$, $(A_{1,1} + 2A)_{0,0,1}^2$ | $-1 < k < 1$ |
| $(A_{1,1} + 2A)_{1,0,0}^2$, $(A_{1,1} + 2A)_{0,1,0}^2$ | $0 < k$, $s < 0$ |
| $(A_{1,1} + 2A)_{k,0,0}^2$, $(A_{1,1} + 2A)_{s,0,1}^2$ | $k < 0$, $0 < s$ |
| $C^5$          |                          |          |
| $(A_{1,1} + 2A)_{0,0,0}^3$, $(A_{1,1} + 2A)_{0,0,1}^3$ | $-1 < k < 1$ |
| $(A_{1,1} + 2A)_{1,0,0}^3$, $(A_{1,1} + 2A)_{0,1,0}^3$ | $0 < k$, $s < 0$ |
| $(A_{1,1} + 2A)_{k,0,0}^3$, $(A_{1,1} + 2A)_{s,0,1}^3$ | $k < 0$, $0 < s$ |

Where in the above table $\epsilon = \pm 1$.

For three dimensional dual Lie superalgebras $(A_{1,1} + 2A)_{\alpha,\beta,\gamma}^0$, $(A_{1,1} + 2A)_{\alpha,\beta,\gamma}^1$, $(A_{1,1} + 2A)_{\alpha,\beta,\gamma}^2$, which are isomorphic with $(A_{1,1} + 2A)_{\alpha,\beta,\gamma}^0$, $(A_{1,1} + 2A)_{\alpha,\beta,\gamma}^1$, $(A_{1,1} + 2A)_{\alpha,\beta,\gamma}^2$ respectively, we have the following anti commutation relations:

\[
\{\tilde{X}^2, \tilde{X}^2\} = \alpha \tilde{X}^1, \quad \{\tilde{X}^2, \tilde{X}^3\} = \beta \tilde{X}^1, \quad \{\tilde{X}^3, \tilde{X}^3\} = \gamma \tilde{X}^1, \quad \alpha, \beta, \gamma \in \mathbb{R}. \tag{20}
\]

Note that these Lie superalgebras are non isomorphic and they differ in the bound of their parameters.
4 Two and three dimensional coboundary Lie super-bialgebras

In this section, we determine how many of 74 of two and three dimensional Lie super-bialgebras of tables 2–4 are coboundary? Note that here we work in nonstandard bases, so we omit coefficient $i = \sqrt{-1}$ from all commutation relations for Lie superalgebras and Lie super-bialgebras of [11] and [12]. In this way, we must find $r = r^{il} X_i \otimes X_l \in g \otimes g$ such that the super cocommutator of Lie super-bialgebras can be written as (7). Using (7), (16) and (18), we have

$$\tilde{Y}_i = X_i^{st} r + (-1)^i r X_i,$$

where $(X_i)_l^k = -f_{il}^{jk}$ are adjoint representations for the basis $g$ and $l$ is the row of $X_i$ matrix. Now using the above relations, we can find the $r$-matrix of the Lie super-bialgebra. In this manner, we determine which of presented Lie super-bialgebras in tables 2–4 are coboundary and obtain their $r$-matrices. We also perform this work for the dual Lie super-bialgebras $(\tilde{g}, g)$ using the following equations as (21)

$$\tilde{Y}^i = (\tilde{X}^i)^{st} \tilde{r} + (-1)^i \tilde{r} \tilde{X}^i,$$

where as above, $(\tilde{X}^i)_l^k = -\tilde{f}_{il}^{jk}$ are the adjoint representations of the basis of the Lie superalgebra $\tilde{g}$. The results are summarized in tables 5–7. Note that for determining Schouten superbrackets on Poisson-Lie supergroups, information on the type of Lie super-bialgebras (triangular or quasi-triangular) is important. As a result, we classify all types of two and three dimensional coboundary Lie super-bialgebras. Two points should be highlighted concerning these tables. First, we have listed coboundary Lie super-bialgebras $(g, \tilde{g})$ with coboundary duals $(\tilde{g}, g)$ in all tables. Since such structures can be specified (up to automorphism) by pairs of r-matrices, then it is natural to call them bi-r-matrix super-bialgebras (b-r-sb) [14]. Here, we give complete list of two and three dimensional coboundary and b-r-sb. Secondly, as it is clearly seen, we have considered super skew-symmetric r-matrix solutions in tables 5–7. Of course there are other solutions for some Lie super-bialgebras of these tables. We have also listed these solutions in those tables.

**Table 5:** Two dimensional coboundary and bi-r-matrix Lie super-bialgebra

| $(g, \tilde{g})$ | Comments |
|-----------------|----------|
| $(A_{1,1} + A), I_{(1,1)}$ | $a X_1 \otimes X_1$, $a \in \mathbb{R}$ |
| $[r, r]$ | 0 |
| $(B, (A_{1,1} + A)$ | $-\frac{1}{2} X_2 \wedge X_2$ |
| $[r, r]$ | 0 |
| $\tilde{r}$ | $a \tilde{X}^1 \otimes \tilde{X}^1 + \frac{1}{2} \tilde{X}^2 \wedge \tilde{X}^2$, $a \in \mathbb{R}$ |
| $[[r, r]]$ | $-\frac{1}{2} \tilde{X}^1 \wedge \tilde{X}^2 \wedge \tilde{X}^2$ |
| $(B, (A_{1,1} + A), i)$ | $\frac{1}{2} X_2 \otimes X_2$ |
| $[r, r]$ | 0 |
| $\tilde{r}$ | $a \tilde{X}^1 \otimes \tilde{X}^1 - \frac{1}{2} \tilde{X}^2 \wedge \tilde{X}^2$, $a \in \mathbb{R}$ |
| $[[r, r]]$ | $\frac{1}{2} \tilde{X}^1 \wedge \tilde{X}^2 \wedge \tilde{X}^2$ |

Note that as shown in table 5 we have two bi-r-matrix Lie super-bialgebras such that the Lie super-bialgebras $(B, (A_{1,1} + A))$ and $(B, (A_{1,1} + A), i)$ are triangular while their duals i.e. the Lie super-bialgebras $((A_{1,1} + A), B)$ and $((A_{1,1} + A), i), B)$ are quasi-triangular. Furthermore to obtaining super skew-symmetric Poisson superbrackets for $((A_{1,1} + A), I_{(1,1)})$ we must put $a = 0$ i.e. $r = 0$, in this case we have trivial solution.

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9 Here superscript $st$ stands for supertranspose.

10 Note that in general we have

$$X_i \wedge X_j \wedge X_k = X_i \otimes X_j \otimes X_k + (-1)^{(j+k)} X_j \otimes X_k \otimes X_i + (-1)^{(k+i)} X_k \otimes X_i \otimes X_j$$

$$-(-1)^k X_i \otimes X_k \otimes X_j - (-1)^i X_j \otimes X_i \otimes X_k - (-1)^{i+k+j} X_k \otimes X_j \otimes X_i.$$
Table 6: Three dimensional coboundary and bi-r-matrix Lie super-bialgebras of the type (2, 1)

| (B, K) | r | c | Comments |
|--------|---|---|----------|
| (2A1,1 + A), I(2,1) | r | aX₁ ⊗ X₁ + bX₁ ⊗ X₂ + cX₂ ⊗ X₁ + dX₂ ⊗ X₂ | a, b, c, d ∈ ℜ |
| [r, r] | 0 | | |
| (B + A,1), I(2,1) | r | aX₂ ⊗ X₂ | a ∈ ℜ |
| [r, r] | 0 | | |
| (B + A,1), (B + A,1), i | r | aX₂ ⊗ X₂ + X₁ ∧ X₂ | a ∈ ℜ |
| [r, r] | 0 | | |
| (B + A,1), (B + A,1) | r | aX₂ ⊗ X₂ − \(\frac{1}{4}X₃ ∧ X₃\) | a ∈ ℜ |
| [r, r] | 0 | | |
| (B + A,1), (2A1,1 + A) | r | aX₂ ⊗ X₂ + \(\frac{1}{4}X₃ ∧ X₃\) | a ∈ ℜ |
| [r, r] | 0 | | |
| (B + A,1), (2A1,1 + A), i | r | b\(\tilde{X}^1 \otimes \tilde{X}^1 - \tilde{X}^1 \wedge \tilde{X}^2\) | b ∈ ℜ |
| [r, r] | 0 | | |
| C⁺ₜ (2A1,1 + A) | r₁ | \(-\frac{1}{4p}X₃ ∧ X₃\) | p ∈ ℜ − {0, 1} |
| [r₁, r₁] | 0 | | |
| r₂ | aX₁ ⊗ X₃ + bX₃ ⊗ X₂ − \(\frac{1}{4}X₃ ∧ X₃\) | p = 1, a, b ∈ ℜ |
| [r₂, r₂] | 0 | | |
| C⁺ₜ (2A1,1 + A), i | r₁ | \(\frac{1}{8p}X₃ ∧ X₃\) | p ∈ ℜ − {0, 1} |
| [r₁, r₁] | 0 | | |
| r₂ | aX₁ ⊗ X₃ + bX₃ ⊗ X₂ + \(\frac{1}{4}X₃ ∧ X₃\) | p = 1, a, b ∈ ℜ |
| [r₂, r₂] | 0 | | |
| r₃ | X₁ ∧ X₂ | p ∈ ℜ − {0, 1} |
| [r₃, r₃] | 0 | | |
| C⁺ₜ (C⁻₁ₚ, i) | \(\tilde{r}_1\) | -\(\tilde{X}^1 \wedge \tilde{X}^2 + \frac{2}{3} \tilde{X}^3 \wedge \tilde{X}^3\) | p = 0, a ∈ ℜ |
| [\(\tilde{r}_1, \tilde{r}_1\)] | 0 | | |
| \(\tilde{r}_2\) | -\(\tilde{X}^1 \wedge \tilde{X}^2 + b\tilde{X}^1 \wedge \tilde{X}^3\) | p = -1, b ∈ ℜ |
| [\(\tilde{r}_2, \tilde{r}_2\)] | 0 | | |
| \(\tilde{r}_3\) | -\(\tilde{X}^1 \wedge \tilde{X}^2\) | p ∈ ℜ − {0, -1} |
| [\(\tilde{r}_3, \tilde{r}_3\)] | 0 | | |
In Table 6 \((2A_{1,1} + A), I_{(2,1)}\) is triangular Lie super-bialgebras such that as shown in the next section to obtaining super skew-symmetric Poisson superbrackets we must put \(a = d = 0\), \(b = -c\) so that its r-matrix has the form \(r = X_1 \wedge X_2\).

In the same way for triangular Lie super-bialgebra \((B + A_{1,1}), I_{(2,1)}\) we have \(r = 0\) and for bi-r-matrix Lie super-bialgebra \((B + A_{1,1}), (B + A_{1,1}), i\) with triangular type we have \(\tilde{r} = -\tilde{X}_1 \wedge \tilde{X}_2\) (i.e. \(b = 0\)) and for bi-r-matrix Lie super-bialgebras \((B + A_{1,1}), (2A_{1,1} + A))\) and \((B + A_{1,1}), (2A_{1,1} + A)).i\) with triangular type from one hand and quasi-triangular type for their duals, we have \(r = -\frac{1}{3} X_3 \wedge X_3\) (i.e. \(a = 0\)) and \(r = \frac{1}{3} X_3 \wedge X_3\) (i.e. \(a = 0\)) respectively.

Note that the Lie super-bialgebras \((C_p^{1}, (2A_{1,1} + A))\) and \((C_p^{1}, (2A_{1,1} + A)).i\) are triangular and have not coboundary duals. Furthermore to obtaining super skew-symmetric Poisson superbrackets we must put \(r_2 = aX_2 \otimes X_3 \pm \frac{1}{3} X_3 \otimes X_3\) (for \(p = 1\)) i.e. \(b = -a\). The Lie super-bialgebras \((C_p^{1}, C_p^{1}).i\) and their duals for value \(p = 0\) (\(r_1\) and \(r_2\)), \(p = 1\) (\(r_2\) and \(\tilde{r}_{3_{p=1}}\)), \(p = -1\) (\(\tilde{r}_{3_{p=-1}}\) and \(\tilde{r}_2\)) and \(p \in \mathbb{R} \setminus \{-1, 0, 1\}\) (\(r_{3}\) and \(\tilde{r}_{3}\)) are bi-r-matrix Lie super-bialgebras with triangular type.

The Lie super-bialgebras \((C_{p}^{1}, I_{(2,1)}\)) are triangular for values \(p = 0\) and \(p = -1\), furthermore to obtaining super skew-symmetric super Poisson brackets we must put \(r_2 = aX_2 \otimes X_3\) i.e. \(b = -a\). The \((C_p^{1}, C_p^{1}).i\) and \((C_p^{1}, C_p^{1}).i\) are bi-r-matrix Lie super-bialgebras with triangular type while their duals are quasi-triangular.
Table 7: Three dimensional coboundary and bi-r-matrix Lie superbialgebras of the type (1, 2)

| (g, ĝ) | r₁ | Comments |
|--------|----|----------|
|        | -\(\frac{\alpha}{2}\)X₂ \land X₂ - \beta X₂ \land X₃ + \frac{\beta}{2}X₃ \land X₃ | p = 0; \(\gamma = 0; b \in \mathbb{R}\) |
| [r₁, r₁] | 0 | |
| r₂ | -\(\frac{\alpha}{2}\)X₂ \land X₂ + bX₂ \land X₃ + cX₃ \land X₂ + \frac{\gamma}{2}X₃ \land X₃ | p = -1; \(\beta = 0; b, c \in \mathbb{R}\) |
| [r₂, r₂] | 0 | |
| r₃ | -\(\frac{\alpha}{2}\)X₂ \land X₂ - \(\frac{\beta}{2}\)X₂ \land X₃ - \(\frac{\gamma}{2}\)X₃ \land X₃ | p \in (-1, 1) - \{0\} |
| [r₃, r₃] | 0 | |

\(\mathcal{C}_{p, \tilde{G}_{01}}\)

| \(|p| \leq 1\) | \(\tilde{r}_1\) | b\(\tilde{X}^{1} \land \tilde{X}^{1} + c\tilde{X}^{1} \land \tilde{X}^{3} + d\tilde{X}^{3} \land \tilde{X}^{1} + \frac{2\alpha}{32\gamma} \tilde{X}^{2} \land \tilde{X}^{2} + \frac{\beta}{2} \tilde{X}^{3} \land \tilde{X}^{3}\) | p = 0; \(\beta = \gamma = 0; \alpha \neq 0\) |
| \([\tilde{r}_1, \tilde{r}_1]\) | -\(\frac{\alpha}{2}\)\(\tilde{X}^{1} \land \tilde{X}^{2} \land \tilde{X}^{2}\) | b, c, d, e \in \mathbb{R} |

\(\mathcal{C}_3, I_{(1,2)}\)

| \(r\) | bX₁ \land X₂ + \(\frac{\gamma}{2}\)X₂ \land X₃ + d(X₂ \land X₃ - X₃ \land X₂) | b, c, d \in \mathbb{R} |
| \([r, r]\) | 0 | |

\(\mathcal{C}_3, (A_{1,1} + 2A)_{1,0,0,0}^1\)

| \(\tilde{r}\) | m\(\tilde{X}^{1} \land \tilde{X}^{1} + n(\tilde{X}^{1} \land \tilde{X}^{3} + \tilde{X}^{3} \land \tilde{X}^{1}) + \tilde{X}^{2} \land \tilde{X}^{2} + \frac{2\alpha}{32\gamma} \tilde{X}^{3} \land \tilde{X}^{3}\) | m, n, d \in \mathbb{R} |
| \([\tilde{r}, \tilde{r}]\) | -\(\frac{\alpha}{2}\)\(\tilde{X}^{1} \land \tilde{X}^{3} \land \tilde{X}^{3}\) | |

\(\mathcal{C}_3, (A_{1,1} + 2A)_{0,1,0,0}^2\)

| \(\tilde{r}\) | m\(\tilde{X}^{1} \land \tilde{X}^{1} + n(\tilde{X}^{1} \land \tilde{X}^{3} + \tilde{X}^{3} \land \tilde{X}^{1}) + \frac{2\alpha}{32\gamma} \tilde{X}^{3} \land \tilde{X}^{3}\) | m, n \in \mathbb{R} |
| \([\tilde{r}, \tilde{r}]\) | 0 | |

\(\mathcal{C}_4, \tilde{G}_{0\alpha}\)

| \(\tilde{r}\) | \(\frac{2(\beta - \alpha) - \gamma}{8}X₂ \land X₂ + \frac{\gamma - \delta}{4}X₂ \land X₃ = \frac{\gamma}{4}X₃ \land X₃\) | α = \(\gamma = 0, \beta \neq 0; \alpha \in \mathbb{R}\) |
| \([\tilde{r}, \tilde{r}]\) | 0 | |

\(\mathcal{C}_{p, \tilde{G}_{0\alpha\gamma}}, \ p \geq 0\)

| \(r₁\) | \(\frac{\alpha}{2}\)X₂ \land X₂ + \(\frac{\beta}{2}\)X₃ \land X₃ + bX₂ \land X₃ + (\(\gamma - b\))X₃ \land X₂ | p = 0; \(\alpha + \gamma = 0; a, b \in \mathbb{R}\) |
| \([r₁, r₁]\) | 0 | |
| \(r₂\) | -\(\frac{2q^2 + \alpha + \gamma}{8(1 + p)}\)X₂ \land X₂ + \(\frac{\beta - \alpha}{8(1 + p)}\)X₂ \land X₃ - \(\frac{2q^2 + \alpha + \gamma}{8(1 + p)}\)X₃ \land X₃ | p > 0 |
| \([r₂, r₂]\) | 0 | |
| \(r₃\) | \(\frac{\alpha \tilde{X}^{1} \land \tilde{X}^{1} - \frac{\beta}{2\gamma}(\tilde{X}^{2} \land \tilde{X}^{2} - \tilde{X}^{3} \land \tilde{X}^{3}) - \frac{\gamma}{2\gamma} \tilde{X}^{2} \land \tilde{X}^{3}\) | p ≥ 0; \(\alpha + \gamma = 0; a \in \mathbb{R}\) |
| \([r₃, r₃]\) | -\(\frac{\alpha}{2\gamma}\)\(\tilde{X}^{1} \land \tilde{X}^{2} \land \tilde{X}^{2} - \tilde{X}^{1} \land \tilde{X}^{3} \land \tilde{X}^{3}\) + \(\frac{\beta}{2\gamma}\)\(\tilde{X}^{1} \land \tilde{X}^{2} \land \tilde{X}^{3}\) | γ ≠ 0 |
Table 7-Continued

| Condition                      | Description                                                                 |
|--------------------------------|-----------------------------------------------------------------------------|
| \( r \) \( aX_1 \otimes X_1 + bX_2 \otimes X_2 + cX_3 \otimes X_3 + \frac{d}{2}X_3 \wedge X_3 \) | \( a, b, c, d \in \mathbb{R} \) |
| \( ([r, r]) \) \( 0 \)                                                      |                                                                             |
| \( r \) \( aX_1 \otimes X_1 \) \( a \in \mathbb{R} \)                       |                                                                             |
| \( (A_{1,1} + 2A)^0, I_{(1,2)} \)                                          |                                                                             |
| \( (A_{1,1} + 2A)^1, I_{(1,2)} \)                                          |                                                                             |
| \( (A_{1,1} + 2A)^2, I_{(1,2)} \)                                          |                                                                             |

For three-dimensional Lie super-bialgebras with the form \((g, \tilde{G}_{\alpha\beta\gamma})\) where \(g = C_p^2, C_3^3, C_4^4, C_5^5\) and \(\tilde{G}_{\alpha\beta\gamma}\) is one of the dual Lie superalgebras of table 4, we have table 7.

We see that for \(p = 0\) the Lie super-bialgebras \((C_p^2, \tilde{G}_{\alpha\beta\gamma})\) are bi-r-matrix only for \(\beta = \gamma = 0\) and \(\alpha \neq 0\) (i.e. for \(\tilde{G}_{0,0,0} = (A_{1,1} + 2A)^0_{0,0,0}\) of table 4). Note that \((C_p^2, I_{(1,2)})\) are triangular while their duals are quasi-triangular. Furthermore Lie super-bialgebras \((C_p^0, \tilde{G}_{0,0,0})\) (i.e. \(\tilde{G}_{0,0,0} = I_{(1,2)}, (A_{1,1} + 2A)^0_{0,1,0}, (A_{1,1} + 2A)^2_{0,1,0}\) are triangular. For \(p = -1\), the Lie super-bialgebras \((C_p^2, \tilde{G}_{\alpha\beta\gamma})\) with \(\beta = 0\) and \(\alpha, \gamma \neq 0\) (i.e. \(\tilde{G}_{\alpha\neq0,\beta=0,\gamma\neq0} = (A_{1,1} + 2A)^3_{0,0,1}\) are triangular while their duals are triangular. Note that for \(p = 0\) the Lie super-bialgebras \((C_p^0, \tilde{G}_{0,0,0})\) triangular. For \(b = c = 1\) in \(r_2\). For \(p \in \{-1, 0\}\) with \(\beta = 0\) and \(\alpha, \gamma \neq 0\) the Lie super-bialgebras \((C_p^2, (A_{1,1} + 2A)^1_{0,0,0})\) and \((C_p^2, (A_{1,1} + 2A)^2_{0,0,0})\) are triangular while their duals are triangular. Further to obtaining super skew-symmetric Poisson super-brackets for \((C_p^3, (A_{1,1} + 2A)^0_{0,0,0})\) we must put \(c = -1\) in \(r\) and for \((C_p^3, (A_{1,1} + 2A)^2_{0,0,0})\) we must put \(m = n = 0\) in \(r\).

Note that the Lie super-bialgebras \((C_p^3, \tilde{G}_{\alpha\beta\gamma})\) for \(\alpha = \gamma = 0\) and \(\beta \neq 0\) (i.e. \(\tilde{G}_{\alpha=0,\beta=0,\gamma=0} = (A_{1,1} + 2A)^2_{0,0,0}\)) are triangular. For \(\beta = 0\), \((C_p^3, \tilde{G}_{\alpha\beta\gamma})\) (i.e. \(\tilde{G}_{\alpha=0,\beta=0,\gamma=0} = I_{(1,2)}, (A_{1,1} + 2A)^0_{0,0,0}, (A_{1,1} + 2A)^0_{0,0,0}, (A_{1,1} + 2A)^1_{k,0,0,1}, (A_{1,1} + 2A)^2_{k,0,0,1}, (A_{1,1} + 2A)^2_{k,0,0,1}\) are triangular. For \(p > 0\), \(\alpha + \gamma = 0\) and \(\gamma \neq 0\) the Lie super-bialgebras \((C_p^0, (A_{1,1} + 2A)^2_{0,0,1})\) are triangular while their duals are quasi-triangular. For \(p > 0\) and \(\alpha, \beta, \gamma \in \mathbb{R}\) (i.e. \(\tilde{G}_{\alpha\beta\gamma} = I_{(1,2)}, (A_{1,1} + 2A)^0_{0,0,0}, (A_{1,1} + 2A)^1_{k,0,0,1}, (A_{1,1} + 2A)^2_{k,0,0,1}\) are triangular.

Finally the Lie super-bialgebras \((A_{1,1} + 2A)^0, I_{(1,2)}\), \((A_{1,1} + 2A)^1, I_{(1,2)}\) and \((A_{1,1} + 2A)^2, I_{(1,2)}\) are triangular. Note that to obtaining super skew-symmetric Poisson superbrackets we must put \(a = 0, b = -c\) for Lie super-bialgebra \((A_{1,1} + 2A)^0, I_{(1,2)}\) i.e. \(r = bX_1 \wedge X_3 + \frac{d}{2}X_3 \wedge X_3\) and for \((A_{1,1} + 2A)^1, I_{(1,2)}\) we must put \(a = 0\) (i.e. \(r = 0\)) and for \((A_{1,1} + 2A)^2, I_{(1,2)}\) we must put \(a = b = 0\) (i.e. \(r = 0\)); where in the two later cases we have trivial solutions.
Note that all these coboundary Lie super-bialgebras are non-isomorphic. In the previous section, we mentioned the conditions (8) under which the coboundary Lie super-bialgebras are isomorphic. Here, we consider these conditions in a more exact and non-formal way. Using the matrix form of the isomorphism map \( \alpha : g \rightarrow g' \)

\[
\alpha(X_i) = (-1)^j \alpha_i^j X'_j, \tag{23}
\]

relation (8) can be rewritten as

\[
\left[ X_{i}^{st} \left( (-1)^{j+k+j} \alpha_{st} r_{st} \alpha - (-1)^{j+kn} r' \right) \right]_{st} = -X_{i}^{st} \left[ (-1)^{n+j+kn} r_{st} \alpha - (-1)^{j+nn} r' \right], \tag{24}
\]

where in the left hand side the indices \( k \) and \( j \) are row and column of matrix \( r_{st} \) respectively and \( l \) corresponds to the column of matrix \( \alpha \). In the right hand side the indices \( j \) and \( k \) are row and column of matrix \( r' \) respectively and \( n \) denotes the column of matrix \( \alpha \). If the above matrix relations are satisfied then the two coboundary Lie super-bialgebras \((g, \tilde{g})\) and \((g', \tilde{g}')\) are isomorphic. In this way, one can investigate these relations for all coboundary Lie super-bialgebras of tables and see that all of them are non-isomorphic.

5 Calculation of super Poisson structures by Sklyanin superbracket

For the triangular and quasitriangular Lie super-bialgebras one can obtain the corresponding Poisson-Lie supergroups by means of Sklyanin superbracket provided by a given super skew-symmetric \( r \)-matrix \( r = r^{ij} X_i \otimes X_j \) [6]

\[
\{f, h\} = \int \frac{\partial}{\partial x^\mu} \mu X_i^{(L, r)} r^{ij} j X_j^{(L, r)} \frac{\partial}{\partial x^\nu} h - \int \frac{\partial}{\partial x^\mu} \mu X_i^{(R, r)} r^{ij} j X_j^{(R, r)} \frac{\partial}{\partial x^\nu} h, \quad \forall f, h \in C^\infty(G) \tag{25}
\]

where \( j X_i^{(L, r)} X_j^{(L, r)} \) and \( j X_i^{(R, r)} X_j^{(R, r)} \) are left and right invariant supervector fields with left (right) derivations on the Poisson-Lie supergroup \( G \). If \( r \) is a solution of (CYBE), the following superbrackets are also super Poisson structures on the supergroup \( G \) [6]

\[
\{f, h\}^L = \int \frac{\partial}{\partial x^\mu} \mu X_i^{(L, r)} r^{ij} j X_j^{(L, r)} \frac{\partial}{\partial x^\nu} h, \tag{26}
\]

\[
\{f, h\}^R = \int \frac{\partial}{\partial x^\mu} \nu X_i^{(R, r)} r^{ij} j X_j^{(R, r)} \frac{\partial}{\partial x^\nu} h. \tag{27}
\]

For calculation of the left and right invariant supervector fields on the supergroup \( G \), it is enough to determine the left and right invariant one forms. For \( g \in G \) we have

\[
g^{-1} dg = L^i X_i = (-1)^i L^{(r)i} \frac{\partial}{\partial x^\mu} X_i^{(L, r)} X_i = (-1)^i L^{(r)i} \frac{\partial}{\partial x^\mu} X_i, \tag{28}
\]

\[
dg g^{-1} = R^i X_i = (-1)^i R^{(r)i} \frac{\partial}{\partial x^\mu} X_i^{(R, r)} X_i = (-1)^i R^{(r)i} \frac{\partial}{\partial x^\mu} X_i, \tag{29}
\]

where \( x^\mu \) are coordinates on the supergroup. Now using [13]

\[
< \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} > = -< \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} > \quad < \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} > = \frac{\partial}{\partial x^i} \quad < \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j} > = \frac{\partial}{\partial x^j}, \tag{30}
\]

in the following relations

\[
< j X_i^{(L, r)i}, L(R)^{(r)i} > = j \delta^i, \quad < L(R)^{(r)i}, j X_j^{(L, r)j} > = j \delta_j, \tag{31}
\]

where

\[
j X_i^{(L, r)i} := j X_i^{(L, r)i \mu} \frac{\partial}{\partial x^\mu}, \quad \quad X_j^{(L, r)j} := \frac{\partial}{\partial x^\nu} \nu X_j^{(L, r), r} \tag{32}
\]

we obtain the following results

\[
j X_i^{(L, r)i} = j L^{(r)i} \frac{\partial}{\partial x^i}, \quad \quad \eta X_j^{(L, r)j} = \eta L^{(r)j} \frac{\partial}{\partial x^j}, \tag{33}
\]
To calculate the above matrices, we assume the following parameterization of the two dimensional supergroup \( G \):

\[
g = e^{x_1 X_1} e^{y X_2},
\]

in the same way, for three dimensional supergroups (with two bosonic and one fermionic generators) we assume

\[
g = e^{x_1 X_1} e^{y X_2} e^{\psi X_3},
\]

and for three dimensional supergroups (with two fermionic and one bosonic generators) we use the following parameterization

\[
g = e^{x_1 X_1} e^{y X_2} e^{\chi X_3}.
\]

The results are summarized in tables 8–10. Note that to obtaining these results for dual Lie supergroups \( \tilde{G} \) we must use the following relations. The reason why we cannot use the above formulas (28)-(33) for dual Lie supergroup \( \tilde{G} \) is that for dual Lie superalgebras we use basis with upper indices and in DeWitt notations these are different to lower indices for Lie superalgebras

\[
\tilde{g}^{-1} \tilde{d} \tilde{g} = \tilde{L}_i \tilde{X}^i = \tilde{L}_i^{(r)} \frac{\partial}{\partial \tilde{x}^\mu} \tilde{X}^i = \frac{\partial}{\partial \tilde{x}^\mu} \tilde{L}_i^{(l)} \partial \tilde{X}^i,
\]

\[
d \tilde{g}^{-1} = \tilde{R}_i \tilde{X}^i = \tilde{R}_i^{(r)} \frac{\partial}{\partial \tilde{x}^\nu} \tilde{X}^i = \frac{\partial}{\partial \tilde{x}^\nu} \tilde{R}_i^{(l)} \partial \tilde{X}^i,
\]

\[
< j \tilde{X}^{(L, r)} , \tilde{L}_i^{(l)} >= \delta_{i j}, \quad < \tilde{L}_i^{(r)} , \tilde{X}^{j(L, l)} >= \delta_{i j},
\]

\[
j \tilde{X}^{(L, l)} := j \tilde{X}^{(L, l)} \mu \frac{\partial}{\partial \tilde{x}^\mu}, \quad \tilde{X}^{j(L, r)} := \frac{\partial}{\partial \tilde{x}^\nu} \mu \tilde{X}^{j(L, r)},
\]

\[
j \tilde{X}^{(L, l)} \mu = j \tilde{L}_i^{(l)} \mu,
\]

Note that the Lie supergroup parameterizations for \( \tilde{G} \) are similar with \( G \) (i.e. the parameters \( x, y, \psi, \chi \) are replaced with \( \tilde{x}, \tilde{y}, \tilde{\psi}, \tilde{\chi} \)).

| \( g \) | \( B \) | \( g \) | \( (A_{1,1} + A) \) | \( (A_{1,1} + A) \) |
|---|---|---|---|---|
| \( 1 \tilde{X}^{(L, l)} \) | \( \frac{\partial}{\partial x} - \psi \frac{\partial}{\partial \psi} \) | \( 1 \tilde{X}^{(L, l)} \) | \( \frac{\partial}{\partial x} \) | \( -\frac{\partial}{\partial \psi} \) |
| \( 2 \tilde{X}^{(L, l)} \) | \( -\frac{\partial}{\partial \psi} \) | \( 2 \tilde{X}^{(L, l)} \) | \( \frac{\partial}{\partial x} \) | \( -\frac{\partial}{\partial \psi} \) |
| \( X_1^{(L, r)} \) | \( \frac{\partial}{\partial x} - \psi \frac{\partial}{\partial \psi} \) | \( X_1^{(L, r)} \) | \( \frac{\partial}{\partial x} \) | \( -\frac{\partial}{\partial \psi} \) |
| \( X_2^{(L, r)} \) | \( \frac{\partial}{\partial \psi} \) | \( X_2^{(L, r)} \) | \( \frac{\partial}{\partial x} \) | \( -\frac{\partial}{\partial \psi} \) |
| \( 1 \tilde{X}^{(R, l)} \) | \( -e^{-x} \frac{\partial}{\partial \psi} \) | \( 1 \tilde{X}^{(R, l)} \) | \( -\frac{\partial}{\partial x} - \psi \frac{\partial}{\partial \psi} \) | \( \frac{\partial}{\partial \psi} \) |
| \( 2 \tilde{X}^{(R, l)} \) | \( \frac{\partial}{\partial \psi} \) | \( 2 \tilde{X}^{(R, l)} \) | \( -\frac{\partial}{\partial x} - \psi \frac{\partial}{\partial \psi} \) | \( \frac{\partial}{\partial \psi} \) |
| \( X_1^{(R, r)} \) | \( \frac{\partial}{\partial \psi} \) | \( X_1^{(R, r)} \) | \( -\frac{\partial}{\partial x} - \psi \frac{\partial}{\partial \psi} \) | \( \frac{\partial}{\partial \psi} \) |
| \( X_2^{(R, r)} \) | \( -e^{-x} \frac{\partial}{\partial \psi} \) | \( X_2^{(R, r)} \) | \( -\frac{\partial}{\partial x} - \psi \frac{\partial}{\partial \psi} \) | \( \frac{\partial}{\partial \psi} \) |
Table 9: Left and right invariant supervector fields over three dimensional Lie supergroups

| \( g \)          | \( (2A_{1,1} + A) \) | \( (B + A_{1,1}) \) | \( C_1^p (p \in \mathbb{R}) \) | \( C_1^{-p} \) |
|---------------|-------------------|-------------------|-----------------------------|-----------------|
| \( X^{(L)} \)  | \( \frac{\partial}{\partial x} - \psi \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - p\psi \frac{\partial}{\partial \psi} \) | \( \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} \) | \( \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} \) |
| \( X^{(R)} \)  | \( \psi \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( -\psi \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( -\psi \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( -\psi \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) |
| \( X_1^{(L)} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) |
| \( X_2^{(L)} \) | \( -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |
| \( X_3^{(L)} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |

\( g \) \( (A_{1,1} + 2A)^0 \) \( (A_{1,1} + 2A)^1 \) \( (A_{1,1} + 2A)^2 \) \( C_2^p (|p| \leq 1) \)

| \( X^{(L)} \)  | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) |
| \( X^{(R)} \)  | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |
| \( X_1^{(R)} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) |
| \( X_2^{(R)} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |
| \( X_3^{(R)} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |

| \( g \)          | \( C_3^p \)     | \( C_4^p \)     | \( C_6^p (p \geq 0) \) |
|---------------|----------------|----------------|-----------------------------|
| \( X^{(L)} \)  | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) |
| \( X^{(R)} \)  | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |
| \( X_1^{(R)} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \) |
| \( X_2^{(R)} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |
| \( X_3^{(R)} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) | \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) |
Table 10: Left and right invariant supervector fields over three dimensional Lie supergroups

| \( g \) | \( (B + A_{1,1})_i \) | \( (2A_{1,1} + A) \) | \( (2A_{1,1} + A)_i \) | \( C^{1}_{p,i} \) | \( p \in \mathbb{R} \) |
|---|---|---|---|---|---|
| \( 1 \tilde{X}(L, l) \) | \( \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |
| \( 2 \tilde{X}(L, l) \) | \( \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) |
| \( 3 \tilde{X}(L, l) \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^1(L, r) \) | \( \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^2(L, r) \) | \( \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^3(L, r) \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^1(R, l) \) | \( \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^2(R, l) \) | \( \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^3(R, l) \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^1(R, r) \) | \( \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^2(R, r) \) | \( \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) | \( \frac{\partial^2}{\partial y \partial y} + \frac{\partial}{\partial y} \psi \) |
| \( \tilde{X}^3(R, r) \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) | \( \frac{\partial}{\partial y} \psi \) |

Now using these results we can calculate the super Poisson structures over the Lie supergroup \( G \) and \( \tilde{G} \). We can simplify rewrite relation (25) in the following matrix form

\[
\{ f, g \} = \left( \begin{array}{ccc} f_{X_1} & f_{X_2} & f_{X_3} \end{array} \right) \begin{pmatrix} \frac{\partial^2}{\partial y \partial y} & \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi \\ \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi \\ \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi \end{pmatrix} \left( \begin{array}{ccc} \frac{\partial^2}{\partial y \partial y} & \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi \\ \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi \\ \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi & \frac{\partial}{\partial y} \psi \end{array} \right) \left( \begin{array}{c} f_{X_1} \cr f_{X_2} \cr f_{X_3} \end{array} \right),
\]

and we can similarly rewrite (26) and (27)\footnote{Note that for dual Lie supergroup \( \tilde{G} \) we use the following Sklyanin superbracket

\[
\{ \tilde{f}, \tilde{h} \} = (-1)^i \left( \frac{\partial^2}{\partial \tilde{y} \partial \tilde{y}} \tilde{f} \tilde{X}^{(L)} \tilde{h} \right) \left( \frac{\partial}{\partial \tilde{y}} \tilde{X}^{(L)} \right) \tilde{f} \tilde{h} - \left( \frac{\partial^2}{\partial \tilde{y} \partial \tilde{y}} \tilde{f} \tilde{h} \right) \left( \frac{\partial}{\partial \tilde{y}} \tilde{X}^{(L)} \right) \tilde{f} \tilde{h}, \quad \forall \tilde{f}, \tilde{h} \in \mathcal{C}(\tilde{G}).
\]}

In this manner, we calculate the fundamental Poisson superbrackets of all triangular and quasitriangular Lie super-bialgebras. The results are given in tables 11–14. Note that for triangular Lie super-bialgebras we have calculated all super Poisson structures (25)–(27) and listed them in separate tables.
Table 11: Poisson superbrackets related to the bi-r-matrix, triangular and quasi-triangular three dimensional Lie super-bialgebras of the type \((2, 1)\)

| \((\mathfrak{g}, \tilde{\mathfrak{g}})\) | \((C_{p, -1}^{1}, C_{1-p}^{1})\)_{p=0} | \((C_{p, -1}^{1}, C_{1-p}^{1})\)_{p=1} | \((C_{p, -1}^{1}, C_{1-p}^{1})\)_{p=-1} | \((B + A_{1, 1}), (B + A_{1, 1})_{i}\) |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| \{x, y\}^{L}\hspace{1cm} 1\hspace{1cm} 1\hspace{1cm} 1\hspace{1cm} 1\hspace{1cm} 1 \{x, ψ\}^{L}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{y, ψ\}^{L}\hspace{1cm} 0\hspace{1cm} ψ - b\hspace{1cm} -ψ\hspace{1cm} pψ\hspace{1cm} ψ \{ψ, ψ\}^{L}\hspace{1cm} 0\hspace{1cm} a\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{x, y\}^{R}\hspace{1cm} e^{-x}\hspace{1cm} e^{-x}\hspace{1cm} e^{-x}\hspace{1cm} e^{-x}\hspace{1cm} 1 \{x, ψ\}^{R}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{y, ψ\}^{R}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} -be^{-2x}\hspace{1cm} 0\hspace{1cm} 0 \{ψ, ψ\}^{R}\hspace{1cm} 0\hspace{1cm} a\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{x, y\}\hspace{1cm} 1 - e^{-x}\hspace{1cm} 1 - e^{-x}\hspace{1cm} 1 - e^{-x}\hspace{1cm} 1 - e^{-x}\hspace{1cm} 0 \{x, ψ\}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{y, ψ\}\hspace{1cm} 0\hspace{1cm} ψ - b(1 - e^{-2x})\hspace{1cm} -ψ\hspace{1cm} pψ\hspace{1cm} ψ \{ψ, ψ\}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{x}, \tilde{y}\}^{L}\hspace{1cm} -e^{-\tilde{y}}\hspace{1cm} -e^{-\tilde{y}}\hspace{1cm} -e^{-\tilde{y}}\hspace{1cm} -e^{-\tilde{y}}\hspace{1cm} -1 \{\tilde{x}, \tilde{ψ}\}^{L}\hspace{1cm} 0\hspace{1cm} \tilde{ψ}e^{-\tilde{y}}\hspace{1cm} (b - \tilde{ψ})e^{-\tilde{y}}\hspace{1cm} p\tilde{ψ}e^{-\tilde{y}}\hspace{1cm} \tilde{ψ} \{\tilde{y}, \tilde{ψ}\}^{L}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{ψ}, \tilde{ψ}\}^{L}\hspace{1cm} 0\hspace{1cm} a\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{x}, \tilde{y}\}^{R}\hspace{1cm} -1\hspace{1cm} -1\hspace{1cm} -1\hspace{1cm} -1\hspace{1cm} -1 \{\tilde{x}, \tilde{ψ}\}^{R}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} be^{\tilde{y}}\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{y}, \tilde{ψ}\}^{R}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{ψ}, \tilde{ψ}\}^{R}\hspace{1cm} 0\hspace{1cm} a\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{x}, \tilde{y}\}\hspace{1cm} 1 - e^{-\tilde{y}}\hspace{1cm} 1 - e^{-\tilde{y}}\hspace{1cm} 1 - e^{-\tilde{y}}\hspace{1cm} 1 - e^{-\tilde{y}}\hspace{1cm} 0 \{\tilde{x}, \tilde{ψ}\}\hspace{1cm} 0\hspace{1cm} \tilde{ψ}e^{-\tilde{y}}\hspace{1cm} -\tilde{ψ}e^{-\tilde{y}} - 2b \sinh \tilde{y}\hspace{1cm} p\tilde{ψ}e^{-\tilde{y}}\hspace{1cm} \tilde{ψ} \{\tilde{y}, \tilde{ψ}\}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{ψ}, \tilde{ψ}\}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0

| \((\mathfrak{g}, \tilde{\mathfrak{g}})\) | \((C_{p, -1}^{1}, C_{1-p}^{1})\)_{p=0} | \((C_{p, -1}^{1}, C_{1-p}^{1})\)_{p=1} | \((B + A_{1, 1}), (2A_{1, 1} + A)\) | \((B + A_{1, 1}), (2A_{1, 1} + A)_{i}\) |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| \{x, y\}^{L}\hspace{1cm} 1\hspace{1cm} -1\hspace{1cm} 0\hspace{1cm} 0 \{x, ψ\}^{L}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{y, ψ\}^{L}\hspace{1cm} 0\hspace{1cm} -\psi\hspace{1cm} 0\hspace{1cm} 0 \{ψ, ψ\}^{L}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} -\frac{1}{2} \frac{1}{2} \{x, y\}^{R}\hspace{1cm} e^{-x}\hspace{1cm} e^{-x}\hspace{1cm} 0\hspace{1cm} 0 \{x, ψ\}^{R}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{y, ψ\}^{R}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{ψ, ψ\}^{R}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} -\frac{1}{2} e^{-2x}\hspace{1cm} \frac{1}{2} e^{-2x} \{x, y\}\hspace{1cm} 1 - e^{-x}\hspace{1cm} e^{-x} - 1\hspace{1cm} 0\hspace{1cm} 0 \{x, ψ\}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{y, ψ\}\hspace{1cm} -\psi\hspace{1cm} 0\hspace{1cm} 0 \{ψ, ψ\}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} -\frac{1}{2}(1 - e^{-2x})\hspace{1cm} \frac{1}{2}(1 - e^{-2x}) \{\tilde{x}, \tilde{y}\}^{L}\hspace{1cm} 1 - e^{-\tilde{y}}\hspace{1cm} e^{\tilde{y}} - 1\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{x}, \tilde{ψ}\}^{L}\hspace{1cm} \frac{1}{2} e^{\tilde{y}}\hspace{1cm} \tilde{ψ}\hspace{1cm} -\tilde{ψ}\hspace{1cm} -\tilde{ψ} \{\tilde{y}, \tilde{ψ}\}^{L}\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0\hspace{1cm} 0 \{\tilde{ψ}, \tilde{ψ}\}^{L}\hspace{1cm} e^{\tilde{y}} - 1\hspace{1cm} 1 - e^{\tilde{y}}\hspace{1cm} 0\hspace{1cm} 0
Table 12: Poisson superbrackets related to the triangular and quasi-triangular three dimensional Lie super-bialgebras of the type (2, 1)

| (g, ˜g)         | \((2A_{1,1} + A).I(2,1)\) | \(C_{p=0}^1.I(2,1)\) | \(C_{p=1}^1.I(2,1)\) | \(C_{p=1}^1.(2A_{1,1} + A)\) |
|-----------------|----------------------------|------------------------|------------------------|-------------------------------|
| \{x, y\}       | \(b\)                      | 0                      | 0                      | 0                             |
| \{x, ψ\}       | 0                          | 0                      | 0                      | 0                             |
| \{y, ψ\}       | 0                          | 0                      | \(-a\)                 | \(-a\)                       |
| \{ψ, ψ\}       | \(-\frac{1}{2}\)          | 0                      | 0                      | 0                             |
| \{x, y\}       | \(b\)                      | 0                      | 0                      | 0                             |
| \{x, ψ\}       | 0                          | 0                      | 0                      | 0                             |
| \{y, ψ\}       | 0                          | 0                      | \(-ae^{-2x}\)          | \(-ae^{-2x}\)                |
| \{ψ, ψ\}       | \(-\frac{1}{2}\)          | 0                      | 0                      | 0                             |

| (g, ˜g)         | \((C_{p=1}^1.(2A_{1,1} + A)\) | \(C_{p=1}^1.(2A_{1,1} + A)\) | \(C_{p=1}^1.(2A_{1,1} + A)\) | \(C_{p=1}^1.(2A_{1,1} + A)\) |
|-----------------|----------------------------|------------------------|------------------------|-------------------------------|
| \{x, ψ\}       | 0                          | 0                      | 0                      | 0                             |
| \{x, ψ\}       | 0                          | 0                      | 0                      | 0                             |
| \{y, ψ\}       | \(-\frac{1}{2}\)          | \(-\frac{1}{2}\)      | \(-\frac{1}{2}\)      | \(-\frac{1}{2}\)          |
| \{ψ, ψ\}       | \(-\frac{1}{2}\)          | \(-\frac{1}{2}\)      | \(-\frac{1}{2}\)      | \(-\frac{1}{2}\)          |

Table 13: Poisson superbrackets related to the triangular and quasi-triangular two dimensional Lie super-bialgebras of the type (1, 1)

| (g, ˜g) | \(B.(A_{1,1} + A)\) | \(B.(A_{1,1} + A)\) |
|---------|------------------------|------------------------|
| \{x, ψ\} | 0                      | 0                      |
| \{ψ, ψ\} | \(-\frac{1}{2}\)  | \(-\frac{1}{2}\)  |
| \{x, ψ\} | 0                      | 0                      |
| \{ψ, ψ\} | \(-\frac{1}{2}(1 - e^{-2x})\) | \(-\frac{1}{2}(1 - e^{-2x})\) |
| \{φ, ψ\} | \(-\bar{φ}\)          | \(-\bar{φ}\)          |
| \{φ, φ\} | 0                      | 0                      |
| (g, δ) | (C², G_{α, β, 0}) | Comments | (C², G_{α, β, γ}) | Comments | (C², G_{α, β, γ})_{p ∈ (-1, 1) − {0}} | Comments |
| --- | --- | --- | --- | --- | --- | --- |
| {x, ψ}³ | 0 | 0 | 0 | 0 | 0 | 0 |
| {x, χ}³ | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ}³ | −β | b | −β | −β | −β | −β |
| {ψ, χ}³ | −ψ | −ψ | −ψ | −ψ | −ψ | −ψ |
| {χ, χ}³ | b | b | b | b | b | b |
| {x, χ}² | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ}² | −βe⁻ˣ | b | −βe⁻ˣ | −βe⁻ˣ | −βe⁻ˣ | −βe⁻ˣ |
| {ψ, χ}² | −ψe⁻²ˣ | −ψe⁻²ˣ | −ψe⁻²ˣ | −ψe⁻²ˣ | −ψe⁻²ˣ | −ψe⁻²ˣ |
| {χ, χ}² | b | b | b | b | b | b |
| {x, ψ} | 0 | 0 | 0 | 0 | 0 | 0 |
| {x, χ} | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ} | β(e⁻ˣ − 1) | 0 | β(e⁻ˣ − 1) | β(e⁻ˣ − 1) | β(e⁻ˣ − 1) | β(e⁻ˣ − 1) |
| {ψ, χ} | 1/2(e⁻²ˣ − 1) | 1/2(e⁻²ˣ − 1) | 1/2(e⁻²ˣ − 1) | 1/2(e⁻²ˣ − 1) | 1/2(e⁻²ˣ − 1) | 1/2(e⁻²ˣ − 1) |
| {χ, χ} | 0 | 1/2(1 − e⁻²ˣ) | 1/2(1 − e⁻²ˣ) | 1/2(1 − e⁻²ˣ) | 1/2(1 − e⁻²ˣ) | 1/2(1 − e⁻²ˣ) |

| {x, ψ}³ | 0 | 0 | 0 | 0 | 0 | 0 |
| {x, χ}³ | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ}³ | b | b | b | b | b | b |
| {ψ, χ}³ | b | b | b | b | b | b |
| {χ, χ}³ | b | b | b | b | b | b |
| {x, χ}² | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ}² | e⁻²ˣ[2γx + γ − 2β] | e⁻²ˣ[2γx + γ − 2β] | e⁻²ˣ[2γx + γ − 2β] | e⁻²ˣ[2γx + γ − 2β] | e⁻²ˣ[2γx + γ − 2β] | e⁻²ˣ[2γx + γ − 2β] |
| {ψ, χ}² | e⁻²ˣ[−γx² + (2β − γ)x + β − γx] | e⁻²ˣ[−γx² + (2β − γ)x + β − γx] | e⁻²ˣ[−γx² + (2β − γ)x + β − γx] | e⁻²ˣ[−γx² + (2β − γ)x + β − γx] | e⁻²ˣ[−γx² + (2β − γ)x + β − γx] | e⁻²ˣ[−γx² + (2β − γ)x + β − γx] |
| {χ, χ}² | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ} | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ} | 0 | 0 | 0 | 0 | 0 | 0 |
| {χ, χ} | 0 | 0 | 0 | 0 | 0 | 0 |

Table 14-Continued

| (g, δ) | (C², G_{1/2}) | (C², A_{1,1} + 2A_{0,0}) | Comments |
| --- | --- | --- | --- |
| {x, ψ}³ | −b | 0 | 0 |
| {x, χ}³ | 0 | 0 | 0 |
| {ψ, χ}³ | −γ/4 | 2(β − γ) − γ | −γ/4 | 2(β − γ) − γ | −γ/4 | 2(β − γ) − γ |
| {ψ, χ}³ | γ/2 | γ/2 | γ/2 | γ/2 | γ/2 | γ/2 |
| {χ, χ}³ | 0 | 0 | 0 | 0 | 0 | 0 |
| {x, χ}² | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ}² | 0 | 0 | 0 | 0 | 0 | 0 |
| {ψ, χ}² | 0 | 0 | 0 | 0 | 0 | 0 |
| {χ, χ}² | 0 | 0 | 0 | 0 | 0 | 0 |

Note that Poisson superbrackets exist if the conditions on the comments satisfied.
Table 14-Continued

| (g, ȓ) | (C^g_{p=0}, G_{-0,0,γ}) | Comments | (C^g_{p>0}, G_{0,0,γ}) | Comments |
|--------|--------------------------|----------|--------------------------|----------|
| {x, ψ}^L | 0 | 0 | | 0 |
| {x, χ}^L | 0 | 0 | | 0 |
| {ψ, χ}^L | \(\frac{−α+γ}{4p(1+p^2)}\) | | | |
| {ψ, ψ}^L | \(\frac{−α+γ+2p^2}{4p(1+p^2)}\) | | | |
| {x, ψ}^R | 0 | 0 | | 0 |
| {x, χ}^R | 0 | 0 | | 0 |
| {ψ, χ}^R | \(\frac{−α+γ}{4p(1+p^2)}\) | | | |
| {ψ, ψ}^R | \(\frac{−α+γ+2p^2}{4p(1+p^2)}\) | | | |
| {ψ, χ}^R | \(\frac{−α+γ}{4p(1+p^2)}\) | | | |
| {ψ, χ}^R | \(\frac{−α+γ+2p^2}{4p(1+p^2)}\) | | | |
| {ψ, χ}^R | \(\frac{−α+γ}{4p(1+p^2)}\) | | | |
| {ψ, χ}^R | \(\frac{−α+γ+2p^2}{4p(1+p^2)}\) | | | |
| \(\tilde{ψ}, \tilde{χ}\) | 0 | 0 | | 0 |
| \(\tilde{x}, \tilde{ψ}\) | \(\tilde{χ} − p\tilde{ψ}\) | \(\tilde{χ} \neq 0\) | | \(\tilde{χ} \neq 0\) |
| \(\tilde{x}, \tilde{χ}\) | \(\tilde{ψ} \neq 0\) | \(\tilde{ψ} − p\tilde{x}\) | | \(\tilde{ψ} \neq 0\) |
| \(\tilde{x}, \tilde{χ}\) | 0 | 0 | | \(\tilde{χ} \neq 0\) |
| \(\tilde{x}, \tilde{χ}\) | 0 | 0 | | \(\tilde{χ} \neq 0\) |

Table 14-Continued

| (g, ȓ) | (C^{A_{1,1} + 2A}_{0,0}) | (A_{1,1} + 2A)^{0}_{0} | (A_{1,1} + 2A)^{1}_{1}| |
|--------|--------------------------|--------------------------|--------------------------|
| {x, ψ} | 0 | \(\{x, ψ\}^L\) | 0 | 0 |
| {x, χ} | 0 | \(\{x, χ\}^L\) | 0 | \(−b\) |
| {ψ, χ} | \(−ex\) | \(\{ψ, χ\}^L\) | 0 | 0 |
| {ψ, ψ} | \(ex^2\) | \(\{ψ, ψ\}^L\) | 0 | \(\{x, ψ\}^L\) |
| {x, χ} | 0 | \(\{x, χ\}^L\) | 0 | \(\{x, χ\}^L\) |
| {x, ψ} | 0 | \(\{x, ψ\}^R\) | 0 | \(\{x, ψ\}^R\) |
| {ψ, χ} | 0 | \(\{ψ, χ\}^R\) | 0 | \(\{ψ, χ\}^R\) |
| {ψ, ψ} | 0 | \(\{ψ, ψ\}^R\) | 0 | \(\{ψ, ψ\}^R\) |
| {x, χ} | 0 | \(\{x, χ\}^R\) | 0 | \(\{x, χ\}^R\) |
| {x, ψ} | 0 | \(\{x, ψ\}^R\) | 0 | \(\{x, ψ\}^R\) |
| {ψ, χ} | 0 | \(\{ψ, χ\}^R\) | 0 | \(\{ψ, χ\}^R\) |
| {ψ, ψ} | 0 | \(\{ψ, ψ\}^R\) | 0 | \(\{ψ, ψ\}^R\) |
| {x, χ} | 0 | \(\{x, χ\}^R\) | 0 | \(\{x, χ\}^R\) |
| {x, ψ} | 0 | \(\{x, ψ\}^R\) | 0 | \(\{x, ψ\}^R\) |
| {ψ, χ} | 0 | \(\{ψ, χ\}^R\) | 0 | \(\{ψ, χ\}^R\) |
| {ψ, ψ} | 0 | \(\{ψ, ψ\}^R\) | 0 | \(\{ψ, ψ\}^R\) |
| {x, χ} | 0 | \(\{x, χ\}^R\) | 0 | \(\{x, χ\}^R\) |
6 Conclusion

Having determined the types (triangular, quasi-triangular or factorizable) of two and three dimensional Lie superbialgebras and obtained their r-matrices and super Poisson structures we are now in a position to perform the quantization of these Lie super-bialgebras. Furthermore, one can now investigate integrability under super Poisson-Lie T duality by studying the super Poisson-Lie T dual sigma models [9] over bi-r-matrix super-bialgebras.

Acknowledgments

This work has supported by research vice chancellor of Azarbaijan University of Tarbiat Moallem. We would like to thank S. Moghadassi for carefully reading the manuscript and useful comments.

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