On $p$-nilpotency of finite groups*

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Abstract

Let $H$ be a subgroup of a group $G$. $H$ is said satisfying Π-property in $G$, if $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number for any chief factor $L/K$ of $G$, and, if there is a subnormal supplement $T$ of $H$ in $G$ such that $H \cap T \leq I \leq H$ for some subgroup $I$ satisfying Π-property in $G$, then $H$ is said Π-normal in $G$. By these properties of some subgroups, we obtain some new criterions of $p$-nilpotency of finite groups.

1 Introduction

Throughout this paper, all groups considered are finite. Let $G$ be a group. $\pi(G)$ denotes the set of all prime divisors of $|G|$ and $\pi$ denotes a subset of $\mathbb{P}$, the set of all primes, and $\pi'$ is the complement of $\pi$ in $\mathbb{P}$. An integer $n$ is called a $\pi$-number if all its prime divisors belong to $\pi$.

Recall that a subgroup $A$ of a group $G$ is said to permute with a subgroup $B$ if $AB = BA$. It is known that $AB$ is a subgroup of $G$ if and only if $A$ permutes with $B$. Thus the permutability of subgroups is very important and is researched extensively. In the literature, seminormal subgroups [4, 15], S-quasinormally embedded subgroups [1], X-permutable subgroups [8], c-normal subgroups [18], weakly s-permutable subgroups [14], and so on have been studied by many mathematicians. To discuss on some essential properties of these generalized permutabilities of subgroups, the following definitions were arisen in [12].

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Definition 1.1 Let $H$ be a subgroup of $G$. We call that $H$ has Π-property in $G$ if for any $G$-chief factor $L/K$, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number.

Definition 1.2 Let $H$ be a subgroup of $G$. If there is a subgroup $T$ of $G$ such that $HT = G$ and $H \cap T \leq I \leq H$, where $I$ has Π-property in $G$, then $H$ is called Π-supplemented in $G$. If, furthermore, $T$ is subnormal in $G$, then $H$ is called Π-normal in $G$.

By the work in [12], we known that Π-property is an essential property of many generalized permutabilities of subgroups and many known results were uniformed by using Π-property and Π-normality of subgroups. In this paper, we shall do some further research and give some new criterions of $p$-nilpotency of groups by the Π-normality of primary subgroups of given order (primary subgroups of given order $|D|$ was first studied by A. Skiba (cf[14])). The main results in this paper is

Theorem A. Let $p$ be an odd prime and $N$ a normal subgroup of $G$ with $p$-nilpotent quotient. Assume that $P$ is a Sylow $p$-subgroup of $N$ and $N_G(P)$ is $p$-nilpotent. If there is an integer $m$ with $1 < p^m < |P|$ such that every subgroup of $P$ of order $p^m$ not having $p$-nilpotent supplement in $G$ is Π-normal in $G$, then $G$ is $p$-nilpotent.

Theorem B. Let be $G$ be a group and $p$ a prime with $(|G|, p − 1) = 1$. Assume that $N$ is a normal subgroup of $G$ with $p$-nilpotent quotient and $P$ a Sylow $p$-subgroup of $N$. Suppose that there is an integer $m$ with $1 < p^m < |P|$ such that every subgroup of $P$ of order $p^m$ not having $p$-nilpotent supplement in $G$ is Π-normal in $G$, then $G$ is $p$-nilpotent if one of the following conditions holds:
(i) $m \geq 2$; 
(ii) $P$ is abelian or $p > 2$ is an odd prime; 
(iii) every cyclic subgroup of $P$ of order 4 not having $p$-nilpotent supplement in $G$ is Π-normal in $G$; 
(iv) $N$ is soluble and $P$ is quaternion-free.

2 Preliminaries

In this section, we list some lemmas which should be used in the proofs of our main results, and introduce some notions and terminologies. For notations and terminologies not given, the reader is referred to the texts of W. Guo [7] or B. Huppert [10].

Lemma 2.1 ([12 Proposition 2.1]) Let $H$ be a subgroup of $G$ and $N$ a normal subgroup of $G$.
(1) If $H$ has Π-property in $G$, then $HN/N$ has Π-property in $G/N$.
(2) If $H$ has Π-property in $G$, then $H$ is both Π-normal and Π-supplemented in $G$.
(3) If $H$ is Π-normal (Π-supplemented) in $G$, then $HN/N$ is Π-normal (Π-supplemented) in $G/N$ when $N \leq H$ or $(|H|, |N|) = 1$.

Lemma 2.2 Let $p$ be a prime and $N$ a normal subgroup of $G$ with a $p$-nilpotent quotient. Assume that $P$ is a Sylow $p$-subgroup of $N$ and $N_G(P) = C_G(P)$. Then $G$ is $p$-nilpotent.
Assume that $O_p(M) = N \cap C_G(P) = C_N(P)$, $N$ is $p$-nilpotent by Burnside Theorem (cf. [5 Ch 7, Theorem 4.3]). Let $Q$ be the normal $p$-complement of $N$. Then the hypotheses still hold on $G/Q$. If $Q \neq 1$ then $G/Q$ is $p$-nilpotent by induction on $|G|$ and so $G$ is $p$-nilpotent. Assume $Q = 1$. Then $N = P$ and $G = N_G(P) = C_G(P)$, that is $P \leq Z(G)$. Since $G/N$ is $p$-nilpotent, we see that $G$ is $p$-nilpotent and the lemma holds.

Recall, a generalized Fitting subgroup of a group $G$, denoted by $F^*(G)$, is the maximal quasinilpotent normal subgroup of $G$ and $F^*(G) = F(G)$ if and only if $F^*(G)$ is soluble (cf. [11 Chapter X , §13]). The supersoluble-hypercentral, $Z^\omega(G)$, of a group $G$ is the maximal normal subgroup of $G$ with all $G$-chief factor lying in it is cyclic (it is also called the supersoluble embedded subgroup of $G$ in [13]).

**Lemma 2.3** ([12 Lemma 2.17]) Let $G$ be a group and $E$ a normal subgroup of $G$. If $F^*(E) \subseteq Z^\omega(G)$, then $E \subseteq Z^\omega(G)$.

**Lemma 2.4** ([12 Proposition 2.7]) Let $H$ be a $p$-subgroup of $G$ and $N$ a minimal normal subgroup of $G$. Assume that $H$ is $\Pi$-normal in $G$. If there is a Sylow $p$-subgroup $G_p$ of $G$ such that $H \cap N \unlhd G_p$, then $H \cap N = 1$.

**Lemma 2.5** ([12 Proposition 2.8]) Let $H$ be a $p$-subgroup of $G$ for some prime divisor $p$ of $|G|$ and $L$ a minimal normal subgroup of $G$. Assume that $H$ is $\Pi$-normal in $G$. Then $L$ is a $p$-group if $H \cap L \neq 1$.

**Lemma 2.6** Let $p$ be an odd prime and $L$ a normal subgroup of $G$ with $G/L$ is a $p$-group. Suppose $P$ is a Sylow $p$-subgroup of $G$ and $N_G(P)$ is $p$-nilpotent. If every maximal subgroup of $P$ either has a $p$-nilpotent supplement in $G$ or satisfies $P_1 \cap L = 1$, then $G$ is $p$-nilpotent.

**Proof** If $O_p'(G) \neq 1$ then one can obtained by induction on $|G|$ that $G/O_p'(G)$ is $p$-nilpotent and so is $G$. Assume that $O_p'(G) = 1$. Let $M$ be a proper subgroup of $G$ containing $P$. Then $M/M \cap L$ is a $p$-group and $N_M(P) = N_G(P) \cap M$ is $p$-nilpotent. Let $P_1$ be a maximal subgroup of $P$ with $P_1 \cap M \cap L \neq 1$. Then $P_1 \cap L \neq 1$ and so $P_1$ has a $p$-nilpotent supplement in $G$. It follows that $P_1$ has a $p$-nilpotent supplement in $M$. Thus the hypotheses hold on $M$ and hence we can have that $M$ is $p$-nilpotent by induction on $|G|$.

If for any characteristic subgroup $X$ of $P$, $N_G(X)$ is $p$-nilpotent, then $G$ is $p$-nilpotent by [13 Corollary]. Assume that there is a characteristic subgroup $X$ of $P$ with $N_G(X)$ is not $p$-nilpotent. Since $N_G(P)$ is $p$-nilpotent, we can choose $X$ to be with maximal order. Since $P \subseteq N_G(X)$, if $N_G(X) < G$ then $N_G(X)$ is $p$-nilpotent by above argument. So $N_G(X) = G$ and $X \unlhd G$. The maximality of $X$ also implies that for any characteristic subgroup $Y/X$ of $P/X$, $N_{G/X}(P/X)$ is $p$-nilpotent. Again by [13 Corollary], $G/X$ is $p$-nilpotent. In particular, $G$ is $p$-soluble.

Let $R$ be a minimal normal subgroup of $G$. Then $R$ is a $p$-group since $G$ is $p$-soluble and $O_p'(G) = 1$. If $P_1/R$ is a maximal subgroup of $P/R$ with $P_1/R \cap LR/R \neq 1$ then $P_1 \cap LR = (P_1 \cap L)R \not\subseteq R$ and hence $P_1 \cap L \neq 1$. Then, one can see that the hypotheses still hold on $G/R$ and by induction on $|G|$, we have that $G/R$ is $p$-nilpotent. Since the class of all $p$-nilpotent group is a saturated formation, $R$ is the only minimal subgroup of $G$ and $R \cap \Phi(G) = 1$. Thus there is a $p$-nilpotent complement $T$ of $R$ in $G$. Let $Q$ be the normal $p$-complement of $T$. Then $Q \leq T$. If $Q \leq G$ then $Q \subseteq O_p'(G) = 1$ and so $G$ is a $p$-group.
Assume that $Q$ is not normal in $G$. Then $T = N_G(Q)$. Let $P_1$ be a maximal subgroup of $P$ containing $P \cap T$. Since $R(P \cap T) = P \cap RT = P$, $P_1 = P_1 \cap R(P \cap T)$ and $R \cap P_1 < R$. If $|R| \neq p$, then $R \cap P_1 \neq 1$ and hence $P_1 \cap L \neq 1$. Thus $P_1$ has a $p$-nilpotent supplement $U$ in $G$. Since $G$ is $p$-soluble, we can assume that $Q \subseteq U$ and hence $U \subseteq N_G(Q) = T$. It follows that $G = P_1U = P_1T = (R \cap P_1)T$. So $|G| = |(R \cap P_1)T| \leq |(R \cap P_1)||T| < |R||T| = |G|$, a contradiction. Assume that $R$ is of order $p$. Since $R$ is the only minimal normal subgroup of $G$ and $R \not\subseteq \Phi(G)$, $R = C_G(R)$. Thus $G/R = G/C_G(R)$ is isomorphic to some subgroup of $\text{Aut}(R)$ which is a group of order $p - 1$. So $G/R$ is a $p'$-group and $R = P$. But this implies that $G = N_G(P)$ is $p$-nilpotent and the lemma holds.

**Lemma 2.7** Let $L$ be a normal subgroup of $G$ with $G/L$ is a 2-group. Suppose $P$ is a Sylow 2-subgroup of $G$. If every maximal subgroup $P_1$ of $P$ is either has a 2-nilpotent supplement in $G$ or satisfies $P_1 \cap L = 1$, then $G$ is 2-nilpotent.

**Proof** If $P \cap L$ is of order 2 then $L$ is of order $2n$, where $n$ is an odd number, and hence is 2-nilpotent. Let $R$ be the normal Hall 2′-subgroup of $L$. Then $G/R$ is a 2-group and hence $G$ is 2-nilpotent. Assume $|P \cap L| > 2$. Let $P_1$ be any maximal subgroup of $P$. Then $P_1 \cap L$ is maximal in $P \cap L$ or $P_1 \cap L = P \cap L$. Thus $P_1 \cap L \neq 1$. It follows that every maximal subgroup of $P$ has a 2-nilpotent supplement in $G$. Let $P_1$ be a maximal subgroup of $P$ and $T_1$ a 2-nilpotent supplement of $P_1$ in $G$ with maximal order. Let $Q$ be the Hall 2′-subgroup of $T_1$. Then $Q$ is also a Hall 2′-subgroup of $G$ and $Q \trianglelefteq T_1$. Thus $N_G(Q)$ is 2-nilpotent and $T_1 \subseteq N_G(Q)$. The maximality of $T_1$ shows that $T_1 = N_G(Q)$. If $P \cap T_1 = P$ then $G = T_1$ is 2-nilpotent. Assume that $P \cap T_1 \neq P$ and let $P_2$ be a maximal subgroup of $P$ with $P \cap T_1 \leq P_2$. Then $P_2$ has a 2-nilpotent supplement $T_2$ in $G$. Clearly, $T_2$ contains a Hall 2′-subgroup of $G$. By [6, Theorem A], $T_2$ contains some conjugate of $Q$ and without loss of generality, we can assume $Q \subseteq T_2$. It follows that $Q$ is also the normal Hall 2′-subgroup of $T_2$ and so $T_2 \subseteq N_G(Q) = T_1$. Thus $G = P_2T_2 = P_2T_1 = P_2Q < PQ = G$, a contradiction and the lemma holds.

**Lemma 2.8** Let $P$ be a normal $p$-subgroup of $G$, where $p$ is a prime dividing $|G|$. If every subgroups of $P$ of order $p$ or 4 (when $P$ is a nonabelian 2-group) not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$, then $P \subseteq Z_{\infty}(G)$.

**Proof** Let $R$ be the $\mathfrak{A}_{p-1}$-residual of $N$, where $\mathfrak{A}_{p-1}$ is the class of all abelian group of exponent dividing $p - 1$. Then by Lemma [12, Lemma 2.15], $P \subseteq Z_{\infty}(G)$ if and only if $P \subseteq Z_{\infty}(R)$. Hence if the lemma is not true then the set $\Gamma = \{L \leq P \mid L \trianglelefteq G, L \not\subseteq Z_{\infty}(R)\}$ is non-empty. Choose $L$ to be an element in the set of minimal order and let $L/K$ be a chief factor of $G$. Then $K \subseteq Z_{\infty}(R)$ and $L/K$ is not central in $R$. Since $G/C_G(L/K)$ is isomorphic to some subgroup of $\text{Aut}(L/K)$, if $L/K$ is cyclic then $G/C_G(L/K)$ is abelian of exponent dividing $p - 1$ and it follows that $C_G(L/K) \subseteq R$. This implies that $L \subseteq Z_{\infty}(R)$, a contradiction. So $L/K$ is noncyclic. Let $C = C_R(K)$. Then $O^p(R) \subseteq C$ by [2, A,(12.3)]. Clearly, $C$ is normal in $G$. Since $L/K$ is a $G$-chief factor and $L \cap KC = (L \cap C)K \leq G$, $(L \cap C)K = K$ or $L$. If $(L \cap C)K = K$ then $L \cap C \subseteq K$ and hence $[L,C] \leq K$. Thus $O^p(G) \leq C \leq C_R(L/K)$ and so $R/C_R(L/K)$ is a $p$-number. Then by [17, Lemma 1.7.11], we see that every $R$-chief factor between $L$ and $K$ is $R$-central, so $L \subseteq Z_{\infty}(R)$ since $K \subseteq Z_{\infty}(R)$, a contradiction. Assume that $(L \cap C)K = L$. Then
\[(L \cap C)/(K \cap C) = (L \cap C)/(L \cap C \cap K) \cong (L \cap C)K/K = L/K \text{ is a noncyclic } G\text{-chief factor.} \] Thus \(L \cap C \in \Gamma\). The minimality of \(L\) shows that \(L \subseteq C\) and hence \(K \subseteq Z(L)\).

Let \(a, b\) be elements of order \(p\) in \(L\). Suppose \(p \geq 2\) or \(P\) is abelian. Then \((ab)^p = a^p b^p [b, a]^{p(p-1)} = 1\). Hence the product of elements of order \(p\) is of order \(p\) or 1 and so \(\Omega = \{a \in L \mid a^p = 1\}\) is a subgroup of \(L\). If \(\Omega \subseteq K\), then all elements of \(C\) with \(p\)-order act trivially on every element of \(L\) of order \(p\) since they act trivially on \(K\). It follows from [10] IV, Satz 5.12 that all elements in \(C\) of \(p\)-order act trivially on \(L\). Thus \(O^p(C) = O^p(R) \subseteq C_R(L/K)\) and, as above argument, \(L \subseteq Z_\infty(R)\), a contradiction. If \(\Omega \nsubseteq K\), then \(L = \Omega K\).

Choose an element \(a\) in \(\Omega \setminus K\) such that \(\langle a \rangle K/K \subseteq L/K \cap Z(G_p/K)\). Let \(H = \langle a \rangle\). If \(H\) has a \(p\)-nilpotent supplement \(U \subseteq G\), then \(HK/K\) has a \(p\)-nilpotent supplement \(UK/K\) in \(G/K\). Thus \(G/K = (HK/K)(UK/K) = (L/K)(UK/K)\). Since \(L/K\) is minimal normal in \(G/K\) and is abelian, \(L/K \cap UK/K = 1\) or \(L/K \subseteq UK/K\) and \(UK/K = G/K\). If \(L/K \cap UK/K = 1\), then \(|L/K| = |G/K : UK/K| = |HUK/K : UK/K| \leq |H| = p\). It follows that \(L/K\) is cyclic of order \(p\), which contradicts to the choice of \(L/K\). If \(L/K \subseteq UK/K = G/K\), then \(L/K\) is cyclic since \(L/K\) is minimal normal in \(G/K\) and \(G/K = UK/K \cong U/U \cap K\) is \(p\)-nilpotent. Hence \(H\) has no \(p\)-nilpotent supplement in \(G\). Since \(a\) is of order \(p\), by the hypotheses, \(H\) is \(\Pi\)-normal in \(G\) and so there is a subnormal subgroup \(T\) of \(G\) such that \(G = HT\) and \(H \cap T \leq I \leq H\), where \(I\) has \(\Pi\)-property in \(G\). Since \(H\) is of prime order, \(H \cap T = H\) or 1. If \(H \cap T = H\), then \(H = I\) has \(\Pi\)-property in \(G\). By Proposition 2.1 (1), \(HK/K\) has \(\Pi\)-property in \(G/K\) and hence is \(\Pi\)-normal in \(G/K\). It follows from Lemma 2.1 that \(L/K = HK/K \cap L/K = HK/K\) is cyclic, a contradiction. Assume that \(H \cap T = 1\). Then \(|G : T| = p\) and, since \(T\) is subnormal in \(G\), we see that \(T\) is normal in \(G\). Clearly \(G = LT\) and therefore \(L/L \cap T \cong G/T\) is cyclic of order \(p\). Thus \(G/C_G(L/L \cap T)\) is a group of exponent dividing \(p - 1\). It follows that \(L/L \cap T\) is a central factor of \(R\). Since \(|L \cap T| < |L|\), \(L \cap T \subseteq Z_\infty(R)\) by the choice of \(L\). This induce that \(L \subseteq Z_\infty(R)\), a contradiction. This contradiction shows that \(\Gamma\) is empty and the lemma holds.

**Lemma 2.9** Let \(p\) be an odd prime and \(P\) a Sylow \(p\)-subgroup of \(G\). Assume that \(N_G(P)\) is \(p\)-nilpotent. If any minimal subgroup of \(P \cap O^p(G)\) either is contained in \(Z_\infty(G)\) or has a \(p\)-nilpotent supplements in \(G\). Then \(G\) is \(p\)-nilpotent.

**Proof** Let \(M\) be a proper subgroup of \(G\) containing \(P\). Then \(O^p(M) = \langle x \in M \mid [x] \text{ is not divided by } p \rangle \subseteq \langle x \in G \mid [x] \text{ is not divided by } p \rangle = O^p(G)\). So any minimal subgroup of \(P \cap O^p(M)\) either is contained in \(Z_\infty(G) \cap M \subseteq Z_\infty(M)\) or has a \(p\)-nilpotent supplements in \(G\) and so is in \(M\). Since \(N_M(P) = M \cap N_G(P)\) is \(p\)-nilpotent, the hypotheses hold on \(M\) and \(M\) is \(p\)-nilpotent by induction on \(|G|\). Hence, as argument in Lemma 2.6, we have that \(G\) is \(p\)-soluble and \(O_{p'}(G) = 1\).

By [12] Proposition 2.3, if a subgroup is contained in the hypercentral of \(G\) then it has \(\Pi\)-property in \(G\). So \(O_{p'}(O^p(G)) \subseteq Z^m_\infty(G)\) by Lemma 2.8. Since \(O_{p'}(O^p(G)) \leq O_{p'}(G) = 1\), \(O_p(O^p(G)) = F((O^p(G))) = F_p((O^p(G))) = F^*(O^p(G))\). By Lemma 2.3, \(O^p(G) \subseteq Z^m_\infty(G)\) and hence \(G\) is \(p\)-supersoluble. Again by \(O_{p'}(G) = 1\), we have that \(G\) is \(p\)-closed and \(G = N_G(P)\) is \(p\)-nilpotent. The lemma holds.
3 Proofs of Theorems A and B

Proof of Theorem A Assume that the theorem is not true and let $G$ be a counter example of minimal order. We divide the proof into several steps.

(1) it $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow $p$-subgroup of $G/O_{p'}(G)$. For any subgroup $M/O_{p'}(G)$ of $PO_{p'}(G)/O_{p'}(G)$ of order $p^m$, let $H = P \cap M$. Then $M = HO_{p'}(G)$ and $H$ is of order $p^m$. If $H$ has a $p$-nilpotent supplement $T$ in $G$, then $M/O_{p'}(G)$ has a $p$-nilpotent supplement $TO_{p'}(G)/O_{p'}(G)$ in $G/O_{p'}(G)$, so if $M/O_{p'}(G)$ has no $p$-nilpotent supplement in $G/O_{p'}(G)$ then $H$ has no $p$-nilpotent supplement in $G$ and hence is II-normal in $G$ by the hypotheses. Therefore, by Lemma 2.3, we see that $M/O_{p'}(G)$ is II-normal in $G/O_{p'}(G)$. Clearly, $N_G/O_{p'}(G)(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is still $p$-nilpotent. It follows that the hypotheses hold on $G/O_{p'}(G)$. Hence $G/O_{p'}(G)$ is $p$-nilpotent by the choice of $G$. It follows that $G$ is $p$-nilpotent, a contradiction. So $O_{p'}(G) = 1$.

(2) Let $L$ be a minimal normal subgroup of $G$. Then $L$ is an abelian $p$-group.

If $L$ does not contain in $N$ then $L \cong LN/N$ is a minimal normal subgroup of a $p$-nilpotent group $G/N$. Thus $L$ is either an abelian $p$-group or a $p'$-group. But if $L$ is a $p'$-group then $L \subseteq O_{p'}(G) = 1$, a contradiction. Hence $L$ is abelian. Assume that $L \subseteq N$. Since $O_{p'}(G) = 1$, $L$ is a $p$-Sylow subgroup. Let $H$ be any subgroup of $P$ of order $p^m$ with $H \cap L \neq 1$. If $H$ has no $p$-nilpotent supplement in $G$, then $H$ is II-normal in $G$. It follows from Lemma 2.3 that $L$ is an abelian $p$-group. Assume that any such subgroup $H$ has a $p$-nilpotent supplement in $G$ then so does in $LP$. Let $P_1$ be any maximal subgroup of $P$ with $P_1 \cap L \neq 1$. Then $P_1$ must contain some subgroup $H$ of order $p^m$ and $H \cap L \neq 1$ and so $P_1$ has a supplement $p$-nilpotent in $LP$. Applying Lemma 2.4 we see that $LP$ is $p$-nilpotent and so is $L$. Since $O_{p'}(L) \subseteq O_{p'}(G) = 1$, $L$ is an abelian $p$-group and (2) holds.

In the following, $L$ is always a minimal normal subgroup of $G$ contained in $N$.

(3) $|L| = p^m$

Let $G_p$ be a Sylow $p$-subgroup of $G$ containing $P$. If $|L| > p^m$, then $L$ has a proper subgroup $H$ of order $p^m$ with $L \leq G_p$. If $H$ is II-normal in $G$, then by Lemma 2.3 $H = L$ or 1. This is nonsense by the choice of $H$. If $H$ has a $p$-nilpotent supplement $T$ in $G$. Then $G = HT = LT$ and since $L$ is minimal normal in $G$, $L \subseteq T$ or $L \cap T = 1$. If $L \cap T = 1$ then $[G] = [HT] = [H][T] < [L][T] = [LT] = [G]$, a contradiction. Thus $L \subseteq T$ and so $G = LT = T$ is $p$-nilpotent. This is contrary to the choice of $G$. Therefore, $|L| \leq p^m$.

Assume that $|L| \neq p^m$ then $|L| < p^m$. Let $|L| = p^l$. By a same argument as in (1), we can obtain that every subgroup of order $p^{m-l}$ of $P/L$ having no $p$-nilpotent supplement in $G$ is II-normal in $G$ and, clearly, $N_{G/L}(P/L)$ is $p$-nilpotent. Thus $G/L$ is $p$-nilpotent by the choice of $G$. Since the class of all $p$-nilpotent subgroup is a saturated formation, we see that $L$ is the only minimal normal subgroup of $G$ contained in $N$ and $\Phi(G) \cap N = 1$. Thus there is a maximal subgroup $M$ of $G$ such that $G = L \rtimes M$ and $M \cong G/L$ is $p$-nilpotent. Since $\Phi(N) \subseteq \Phi(G) \cap N$, $\Phi(N) = 1$ and so $F(N)$ is abelian. It follows that $F(N) \cap M \leq (F(N), M) = G$. But $L$ is the only minimal normal subgroup of $G$ contained in $N$ and $L \cap M = 1$, so $F(N) \cap M = 1$. Hence $F(N) = F(N) \cap LM = L(F(N) \cap M) = L$ and therefore,
\( L = O_p(N) = F(N) = C_N(L) \).

Since \( G_p = G_p \cap LM = L \rtimes (G_p \cap M) \) is a Sylow \( p \)-subgroup of \( G \) containing \( P \), \( G_p \cap M \) is a Sylow \( p \)-subgroup of \( M \). Let \( L_1 \) be a maximal subgroup of \( L \) with \( L_1 \unlhd G_p \) and \( H_1 \) be a subgroup of \( G_p \cap M \cap P \) of order \( p^{n+1}/|L| \). Then \( H = L_1H_1 \) is a subgroup of \( P \) of order \( p^n \). If \( H \) is \( \Pi \)-normal in \( G \), then \( L_1 = H \cap L = L \) or 1 by Lemma 2.1. But \( L_1 \) is proper in \( L \), so \( L_1 = 1 \) and \( L \) is of order \( p \). It follows that \( N/L = N/C_N(L) \) is isomorphic to some subgroup of \( \text{Aut}(L) \) which is a group of order \( p - 1 \), so \( N/L \) is a \( p' \)-group. Hence \( L = P \) is the Sylow \( p \)-subgroup of \( N \) and \( G = N_G(L) \) is \( p \)-nilpotent, contrary to the choice of \( G \). Thus by hypotheses, \( H \) has a \( p \)-nilpotent supplement \( U \) in \( G \). Let \( Q \) be a Hall \( p' \)-subgroup of \( U \). Then \( Q \) is also a \( p' \)-subgroup of \( G \). Since \( G = LM \) is \( p \)-soluble, we can assume that \( Q \) is also a Hall \( p' \)-subgroup of \( M \). Since \( M \) is \( p \)-nilpotent, \( M \subseteq N_G(Q) \). If \( N_G(Q) = G \) then \( Q \subseteq O_p'(G) = 1 \) by (1) and so \( G \) is a \( p \)-group, a contradiction. Thus, \( M = N_G(Q) \) by the maximality of \( M \). The \( p \)-nilpotency of \( U \) implies that \( U \leq N_G(Q) \), so \( U \leq M \). Hence \( G = HU = HM = L_1H_1M = L_1M \) and \( |G| \leq |L_1||M| < |L||M| = |LM| = |G| \), a contradiction. This contradiction shows that (3) is true.

(4) \( m = 1 \)

Since, by (3), \( |L| = p^m \), we need only to prove that \( L \) is of order \( p \). Assume that \( L \) is noncyclic. We claim that all minimal subgroup of \( N/L \) of order \( p \) having no \( p \)-nilpotent supplement in \( G \) is \( \Pi \)-normal in \( G \). Assume \( A/L \) is of order \( p \) and \( A/L \subseteq N/L \). Clearly, \( A \) is noncyclic since \( L \) is noncyclic. Thus there is a maximal subgroup \( H \) of \( A \) different from \( L \). Therefore, \( A/L = HL/L \) and \( |H| = |L| = p^m \). If \( A/L \) has no \( p \)-nilpotent supplement in \( G/L \), then \( H \) has no \( p \)-nilpotent supplement in \( G \). Hence, by hypotheses, \( H \) is \( \Pi \)-normal in \( G \). Choose \( T \) to be a subnormal subgroup of \( G \) with \( G = HT \) and \( H \cap T \leq I \), where \( I \) is a subgroup of \( H \) having \( \Pi \)-property in \( G \). Since \( |G:T| \) is a \( p \)-number, \( Op(G) \subseteq T \). If \( L \not\subseteq T \), then \( L \not\subseteq Op(G) \). It follows that \( L \cong LOp(G)/Op(G) \) is a cyclic chief factor of \( G \). This is nonsense because \( L \) is noncyclic. Thus \( L \subseteq T \). It follows that \( G/L = (A/L)(T/L) \) and \( A/L \cap T/L = HL/L \cap T/L = (H \cap T)L/L \leq IL/L \). Since, By Lemma 2.1 \( IL/L \) has \( \Pi \)-property in \( G/L \), \( A/L \) is \( \Pi \)-normal in \( G \) and our claim holds. It is easy to see that \( N_{G/L}(P/L) = N_G(P)/L \) is \( p \)-nilpotent. If \( P/L \) is cyclic of order \( p \), then \( N_{N/L}(P/L) = C_{N/L}(P/L) \) since \( N_{N/L}(P/L) = N/L \cap N_{G/L}(P/L) \) is \( p \)-nilpotent. It follows from Lemma 2.2 that \( G/L \) is \( p \)-nilpotent. If \( P/L \) is noncyclic, then the hypotheses hold on \( G/L \). Thereby, \( G/L \) is \( p \)-nilpotent by the choice of \( G \). Now, by a similar argument as in (3), one can prove that \( G \) is \( p \)-nilpotent. This contradicts to the choice of \( G \) and hence \( L \) is cyclic of order \( p \).

(5) \( O_p(N) \subseteq Z^d_p(G) \).

By (2), \( O_p(N) \neq 1 \). If \( O_p(N) = p \), then (5) is clear true. If \( |O_p(N)| \neq p \), then (5) holds from (4) and Lemma 2.3.

(6) \( N \) is not \( p \)-soluble.

Assume that \( N \) is \( p \)-soluble. Then since \( O_p'(N) \subseteq O_p'(G) = 1 \), \( F^*(N) = F(N) = F_p(N) = O_p(N) \subseteq Z^d_p(G) \). By Lemma 2.8 \( N \) is supersoluble. Again by \( O_p(N) = 1 \), we have that \( P \) is normal in \( N \) and hence is normal in \( G \) for \( P \) is the Sylow \( p \)-subgroup of \( N \). It follows that \( G = N_G(P) \) is \( p \)-nilpotent.

(7) The final contradiction

Let \( R \) be the \( \mathfrak{A}_{p-1} \)-residual of \( N \), where \( \mathfrak{A}_{p-1} \) is the class of all abelian group of exponent dividing 7.
$p - 1$. Then $R$ is not $p$-soluble since $N$ is not. For any $N$-chief factor $U/V$ with $U \subseteq O_p(N)$, $[U/V] = p$ by (5). Hence $N/C_N(U/V)$ is an abelian group of order dividing $p - 1$ and so $C_N(U/V) \subseteq R$, it follows that $O_p(N) \subseteq Z_\infty(R)$. Let $O = O_p(N) \cap O^p(N)$. Then $N \subseteq Z_\infty(R)$. Since $R$ is not $p$-soluble, $O < R \cap O^p(N)$. Let $Q/O$ be a minimal normal subgroup of $G/O$ with $Q \subseteq R \cap O^p(N)$. Then $Q$ is not $p$-soluble. Otherwise, the minimality of $Q/O$ implies $Q/O$ is a $p'$-group or a $p$-group. It follows that $Q$ is $p$-nilpotent since $O \subseteq Z_\infty(R)$ and so is $G$ supplement in $N$ with $p$ normal in $H$. Then $H$ is $\Pi$-normal in $p$. Let $<H/O/O> \subseteq H/O/O$ as in proof of Theorem A, $H/O/O$ is abelian by Lemma 2.5, a contradiction. Thereby, we have that for any subgroup $H$ of order $p$, if $H$ is not contained in $O$, $H$ has a $p$-nilpotent supplement in $G$. Let $X = QP$. Then $X \subseteq R$ and it follows from $O \subseteq Z_\infty(R)$ that $O \subseteq Z_\infty(X)$. Since $X/Q$ is a $p'$-group, $O^p(X) = O^p(Q) \subseteq Q$. Thus, by above, every minimal subgroup of order $p$ of $P \cap O^p(X)$ either is contained in $Z_\infty(X)$ or has a $p$-nilpotent supplement in $X$ and therefore, $X = QP$ is $p$-nilpotent by Lemma 2.4 so $Q$ is $p$-soluble, a contradiction. This is the final contradiction and the theorem holds.

**Proof of Theorem B** We shall prove the theorem by induction on the order of $G$. By a similar argument as in proof of Theorem A, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Hence, if $O_{p'}(G) \neq 1$ then $G/O_{p'}(G)$ is $p$-nilpotent by induction and so is $G$. Now assume that $O_{p'}(G) = 1$. Then

1. Let $L$ be a minimal normal subgroup of $G$. Then $L$ is an abelian $p$-group.

   If $L \nsubseteq N$ then $L \supseteq LN/N \leq G/N$. But $G/N$ is $p$-nilpotent, and hence $L$ is. Since $O_{p'}(G) = 1$, $L$ is a $p$-group and is abelian. Assume $L \subseteq N$. If $P$ possesses a subgroup $H$ of order $p^m$ with $H \cap L \neq 1$ and $H$ is $\Pi$-normal in $G$. Then by Lemma 2.5 $L$ is a $p$-group and is abelian. Assume that every subgroup $H$ of $P$ of order $p^m$ with $H \cap L \neq 1$ has a $p$-nilpotent supplement in $G$. Let $P_1$ be any maximal subgroup of $P$ with $P_1 \cap L \neq 1$. Then $P_1$ must contain a subgroup $H$ of order $p^m$ with $H \cap L \neq 1$. It follows that $P_1$ has a $p$-nilpotent supplement in $G$ and so does in $LP$. Therefore, if $P = 2$ then $LP$ is $p$-nilpotent by Lemma 2.7 and so is $L$. It follows from $O_{p'}(G) = 1$ that $L$ is a $p$-group and is abelian. Assume $p > 2$. Then $G$ is of odd order since $(|G|, p - 1) = 1$. Thus $G$ is soluble and hence $L$ is abelian and is a $p$-group by $O_{p'}(G) = 1$.

2. If $p = 2$ and every cyclic subgroup of order 2 or 4 (when $P$ is nonabelian) of $P$ having no $p$-nilpotent supplement in $G$ is $\Pi$-normal, then $G$ is $p$-nilpotent.

   $O_p(N) > 1$ by (1). If $O_p(N)$ is of order 2 then $O_p(N) \subseteq Z(G) \subseteq Z_\infty(G)$. If $O_p(N) > 2$ then $O_p(N) \subseteq Z_\infty(G)$ holds by Lemma 2.8. Since $O_{p'}(G) = 1$, $F(N) = O_p(N) \subseteq Z_\infty(G)$. If $F'(N) = F(N)$ then $N \subseteq Z_\infty(G)$ and hence $G$ is $p$-supersoluble since $G/N$ is. But since $(|G|, p - 1) = 1$, that $G$ is $p$-supersoluble implies that it is $p$-nilpotent. Assume that $F'(N) \neq F(N)$ and let $R/F(N)$ be a $G$-chief factor with $R \subseteq F'(N)$. Then $R$ is not soluble, otherwise $R$ is nilpotent since $F(N) = O_p(N) \subseteq Z_\infty(G)$ and so $R \subseteq F(N)$, a contradiction. Let $O = O^p(R)$. Then $R = OF(N)$ and $O$ is not soluble. If there is an element $a \in O \setminus F(N)$ of order 2, then $H = \langle a \rangle$ does not avoid $R/F(N)$. If $H$ has a $p$-nilpotent
supplement $T$ in $G$, then since $p = 2$, $T$ is soluble and since $|H| = 2$, $T = G$ or $|G : T| = 2$. But in both case we have $G$ is soluble and so $F^*(N) = F(N)$, a contradiction. Assume that $H$ is $\Pi$-normal in $G$. Let $T$ be a subnormal supplement of $H$ in $G$. Then $O \subseteq O^p(G) \subseteq T$ since $|G : T| = |H| = p$. Thus $H$ has $\Pi$-property in $G$ and so $HF(N)/F(N)$ has $\Pi$-property in $G/F(N)$ by Lemma 2.1. Since $1 \neq HF(N)/F(N) = HF(N)/F(N) \cap R/F(N)$, $R/F(N)$ is a $p$-group and is abelian by Lemma 2.5. Thus all elements of order 2 of $O$ are lying $F(N)$. If there is an element $a \in O \setminus F(N)$ of order 4, then $a^2 \in F(N)$ since $|a^2| = 2$. By a similar argument, one can find a contradiction. Thus all elements of order 2 or 4 of $O$ are lying $F(N)$. Since $O$ is not soluble, $O$ has a minimal non-$2$-nilpotent subgroup $X$. Then $X = A \times B$, where $A$ is a $p$-group of exponent $p$ or 4 (when $A$ is nonabelian) and $B$ is a $p'$-group. But all elements of order 2 or 4 of $O$ is in $F(N)$, so $A \subseteq F(N) \subseteq Z_\infty(G)$. It follows that $B$ acts trivially on $A$, a contradiction. Thus $F^*(N) = F(N)$ and (2) holds.

(3) Let $L$ be a normal subgroup of $G$ contained in $N$. If $|L| = p$ and $p = m = 2$ then $G$ is $p$-nilpotent.

Let $H$ be a cyclic subgroup of order 4. Then, since $p = m = 2$, $H$ has a $p$-nilpotent supplement in $G$ or is $\Pi$-normal in $G$. Now let $A = \langle a \rangle$ be a subgroup of $P$ of order 2. Then $LA$ is of order 4 and hence either has a $p$-nilpotent supplement in $G$ or is $\Pi$-normal in $G$. Assume $LA$ has a $p$-nilpotent supplement $T$ in $G$. Since $|L| = p = 2$, $L \subseteq Z(G)$. It follows that $LT$ is still $p$-nilpotent and we can assume that $T = LT$ and so $G = AT$. Thereby $|G : T| = |AT : T| \leq 2$. This means that $T$ is normal in $G$ and the normal Hall $p'$-subgroup of $T$ is also the normal Hall $p'$-subgroup of $G$. Thus $G$ is $p$-nilpotent.

Assume that $LA$ is $\Pi$-normal in $G$. Then there is a subnormal subgroup $T$ of $G$ with $LA \cap T \leq I \leq LA$ and $I$ has $\Pi$-property in $G$. We claim that $A$ is $\Pi$-normal in $G$. Assume that $I = LA$ and then $T = G$. Let $U/V$ be any $G$-chief factor. If $LAV \leq U$ then $LV/V \leq U/V$. Since $LV \leq G$ and $U/V$ is a $G$-chief factor, $LV = U$ or $V$. If $LV = U$ then $U/V$ is cyclic. Hence $AV/V \cap U/V = U/V$ or 1. So $AV/V \cap U/V \leq G/V$ and $|G/V : N_{G/V}(AV/V \cap U/V)| = 1$. If $LV = V$ then $L \subseteq V$ and $LAV \cap U = AV \cap U$. Thus $|G/V : N_{G/V}(AV/V \cap U/V)| = |G/V : N_{G/V}(LAV/V \cap U/V)|$ is a 2-number since $LA = I$ has $\Pi$-property in $G$. Assume that $LAV \nsubseteq U$ then $ALV/V \cap U/V < ALV/V$ and so $ALV/V \cap U/V$ is of order 1 or 2 since $|ALV/V| \leq |LA| = 4$. It follows that $AV/V \cap U/V = ALV/V \cap U/V$ or 1. If $AV/V \cap U/V = 1$ then $|G/V : N_{G/V}(AV/V \cap U/V)| = 1$. If $AV/V \cap U/V = ALV/V \cap U/V$, then $|G/V : N_{G/V}(AV/V \cap U/V)| = |G/V : N_{G/V}(LAV/V \cap U/V)|$ is a 2-number since $LA = I$ has $\Pi$-property in $G$. Thus in this case we have that $A$ has $\Pi$-property in $G$ and so is $\Pi$-normal in $G$. Assume that $I \nsubseteq LA$. Then $|I| = 1$ or 2. If $|I| = 1$ then $LA \cap T = 1$. It follows that $|G : T| = 4$ and $LT < G$ and so $A \cap LT = 1$. Since $G = LAT = A(LT)$ and $LT$ is normal in $G$, $A$ is $\Pi$-normal in $G$. Assume $|I| = 2$. Then $|G : T| = 2$. If $A \nsubseteq T$ then $G = AT$ and $A \cap T = 1$. This implies that $A$ is $\Pi$-normal in $G$. If $A \subseteq T$ then $A = LA \cap T = I$ is $\Pi$-property in $G$ and hence is $\Pi$-normal in $G$. Thus our claim holds.

By (2), we see that $G/L$ is $p$-nilpotent and so is $G$ since $|L| = p = 2$. Hence (3) holds.

(4) Let $L$ be a normal subgroup of $G$ contained in $N$. If $|L| \neq p^m$, then $G$ is $p$-nilpotent.

If $|L| > p^m$ then $L$ has a proper subgroup $H$ of order $p^m$ with $H \leq G_p$, where $G_p$ is some Sylow $p$-subgroup containing $P$. Since $L$ is minimal normal in $G$ and $H < L$, $G$ is the only supplement of $H$ in $G$. So, if $H$ has a $p$-nilpotent supplement in $G$, then $G$ is $p$-nilpotent. Assume that $H$ is $\Pi$-normal in $G$.  

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Then by Lemma 2.3, \( H = H \cap L = L \) or 1. This is nonsense and so we can assume that \( |L| \leq p^m \).

Suppose \(|L| < p^m\). We claim that the hypotheses still hold on \( G/L \). Let \(|L| = p^l\). For any subgroup \( H/L \) of order \( p^{m-l} \) of \( P/H \), \( H \) is a subgroup of order \( p^m \) of \( P \). Thus \( H \) is \( \Pi \)-normal in \( G \) if it has no \( p \)-nilpotent supplement in \( G \). Clearly if \( H/L \) has no \( p \)-nilpotent supplement in \( G/L \) then \( H \) has no \( p \)-nilpotent supplement in \( G \), so, by Lemma 2.1(3), if \( H/L \) has no \( p \)-nilpotent supplement in \( G/L \) then it is \( \Pi \)-normal in \( G/L \), that is, every subgroup of order \( p^{m-l} \) of \( P/L \) having no \( p \)-nilpotent supplement is \( \Pi \)-normal in \( G \).

If \( p > 2 \) or \( P \) is abelian or \( m - l \neq 1 \) then condition (i) or (ii) holds on \( G/L \) and our claim is true.

Now assume that \( p = 2 \), \( m - l = 1 \). If \( l = 1 \) then \( G \) is \( p \)-nilpotent by (3). Assume that \( l > 1 \). Let \( H/L \) be any cyclic subgroup of \( P/L \) of order 4. Then there is an element \( a \in H \) with \( H = L\langle a \rangle \). Since \( a^4 \in L \), \(|a| = 4 \text{ or } 8\). Let \( L_1 \) be a maximal subgroup of \( L \). Since \( m = l + 1 \), if \(|a| = 4 \), then \( \langle a \rangle L_1 \) is of order \( p^m \). Assume \(|a| = 8 \). Since \( l > 1 \), \( L_1 \neq 1 \) and we can choose a maximal subgroup \( L_2 \) of \( L_1 \). Then either \( \langle a \rangle L_1 \) or \( \langle a \rangle L_2 \) is of order \( p^m \). Thus in any case, \( H \) has a subgroup \( H_1 \) of order \( p^m \) containing \( a \). By the hypotheses, if \( H_1 \) has no \( p \)-nilpotent supplement in \( G \) then \( H \) is \( \Pi \)-normal in \( G \). If \( H_1 \) has a \( p \)-nilpotent supplement in \( G \) then so does \( H \) and therefore, \( H/L \) has a \( p \)-nilpotent supplement in \( G/L \). Assume that \( H_1 \) is \( \Pi \)-normal in \( G \). Then there is a subnormal subgroup \( T \) of \( G \) with \( H_1 T = G \) and \( H_1 \cap T = T \leq H_1 \), where \( I \) has \( \Pi \)-property in \( G \). Since \(|G : T| \) is a \( p \)-number, \( O^p(G) \subseteq T \). If \( L \not\subseteq T \) then \( L \not\subseteq O^p(G) \) and \( L \not\subseteq L \). \( L \) is a chief factor of \( G \) and hence is cyclic. This is contrary to \( l > 1 \). So \( L \subseteq T \) and hence \( H/L \cap T/L = (H_1/L) \cap T/L = (H_1 \cap T)/L/L \subseteq H/L \). By Lemma 2.1, \( L/L \) has \( \Pi \)-property in \( G \), so \( H/L \) is \( \Pi \)-normal in \( G \). Thus condition (iii) holds on \( G/L \). So, in any case, hypotheses still hold on \( G/L \) and our claim is true.

By Induction, we have \( G/L \) is \( p \)-nilpotent. If \( L \in \Phi(G) \) then \( G \) is \( p \)-nilpotent by [7, Lemma 1.8.1]. Assume \( L \not\subseteq \Phi(G) \). Then \( G \) has a maximal subgroup \( M \) with \( G = L \rtimes M \). It follows that \( M \cong G/L \) is \( p \)-nilpotent. Let \( Q \) be the Hall \( p' \)-subgroup. Then \( Q \subseteq M \). Since \( M \) is maximal in \( G \) and \( O_{p'}(G) = 1 \), \( M = N_G(Q) \). If \( L \) is cyclic, then \( L \subseteq Z(G) \) since \(|G|, p - 1| = 1 \) and then \( G \) is \( p \)-nilpotent by \( G/L \) is. Assume that \( L \) is noncyclic. Then \( L \) has a maximal subgroup \( L_1 \) with \( L_1 \neq 1 \) and \( L_1 \leq \Phi(G) \), where \( G_p \) is a Sylow \( p \)-subgroup of \( G \). Let \( H_1 \) be a subgroup of \( P \cap M \) of order \( p^{m-1+1} \). Then \( H = L_1 H_1 \) is of order \( p^m \). If \( H \) is \( \Pi \)-normal in \( G \). Then \( L_1 = L_1 \cap H = L \) or 1 by Lemma 2.3. This is nonsense and so \( H \) has a \( p \)-nilpotent supplement in \( G \). Let \( T \) be a \( p \)-nilpotent supplement of \( H \) in \( G \) and \( Q_1 \) is the Hall \( p' \)-subgroup of \( T \). If \( p > 2 \) then \( G \) is of odd order by \(|G|, p - 1| = 1 \) and is solvable. Thus \( Q \) is conjugate with \( Q_1 \). If \( p = 2 \) then by [4, Theorem A], \( Q \) is conjugate with \( Q_1 \). Since any conjugate of \( T \) is also a \( p \)-nilpotent supplement of \( H \) in \( G \), we can assume \( T \subseteq G \). It follows that \( T \subseteq N_G(Q) = M \) and so \( G = HT = HM = L_1 M < LM = G \). This is nonsense and so (4) holds.

(5) Let \( L \) be a minimal normal subgroup of \( G \) contained in \( N \). Then \(|L| = p \).

Assume that \(|L| > p \) and let \( X/L \) be a minimal subgroup of \( P/L \). Then \( X = L(x) \) for some \( x \in X \). Let \( \bar{U} = \langle a^p \mid a \in X \rangle \cap L \). Then \( \bar{U} \subseteq \Phi(X) \subseteq L \) since \( L \) is maximal in \( X \). If \( \bar{U} = L \) then \( \bar{U} = \Phi(X) = L \) and hence \( X \) is cyclic because \( X/L = X/\Phi(X) \) is. This imply that \( L \) is cyclic, a contradiction. Thus \( \bar{U} < L \). Let \( L_1 \) be a maximal subgroup of \( L \) with \( \bar{U} \subseteq L_1 \). Then \( x^p \in \bar{U} \subseteq L_1 \) and \( x \notin L_1 \). It follows that
In Theorem A, if the hypothesis “$H$ is $\Pi$-normal in $G$” is deleted, we can prove similarly that $G$ is $p$-nilpotent when $G$ is $p$-soluble. Otherwise, we have the following example.

**Example 4.1** Let $G = A_5 \times Z_5$, where $A_5$ is the alternating group of degree 5 and $Z_5$ is a cyclic group of order 5. Then $Z_5$ is the only subgroup of $G$ of order 5 which has no 5-nilpotent supplement in $G$. Clearly $Z_5$ is normal in $G$ but $G$ is not 5-nilpotent.

2. By choose $m$ to be some special number, one can obtain some corollaries of Theorem A. For example, when $\frac{|P|}{p^m} = p$, or in the same, subgroups of order $p^m$ of $P$ are maximal in $P$, we have the following corollary.
Corollary 4.2. Let $p$ be an odd prime and $N$ a normal subgroup of $G$ with $p$-nilpotent quotient. Assume that $P$ is a Sylow $p$-subgroup of $N$ and $N_G(P)$ is $p$-nilpotent. If every maximal subgroup of $P$ not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$, then $G$ is $p$-nilpotent.

Proof. Let $P_1$ be a maximal subgroup of $P$ and assume $|P_1| = p^m$. If $P$ is not of order $p$, then $1 < p^m < |P|$. Since all maximal subgroups of $P$ are of same order, the hypotheses of Theorem A hold. Thus $G$ is $p$-nilpotent. If $|P| = p$, then $P \subseteq C_G(P)$. It follows from $N_G(P)$ is $p$-nilpotent that $N_G(P) = C_G(P)$. By Lemma 2.1 we see that $G$ is $p$-nilpotent.

Assume subgroup of order $p^m$ is 2-maximal in $P$, that is, it is a maximal subgroup of a maximal subgroup of $P$. Then we have

Corollary 4.3. Let $p$ be an odd prime and $N$ a normal subgroup of $G$ with $p$-nilpotent quotient. Assume that $P$ is a Sylow $p$-subgroup of $N$ and $N_G(P)$ is $p$-nilpotent. If every 2-maximal subgroup of $P$ not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$, then $G$ is $p$-nilpotent.

Proof. Let $P_1$ be a 2-maximal subgroup of $P$ and assume $|P_1| = p^m$. Then $|P : P_1| = p^2$. If the order of $P$ is greater than $p^2$, then the hypotheses of Theorem A hold. Thus $G$ is $p$-nilpotent. If $|P| = p^2$ or $p$, then since any group of order $p^2$ or $p$ is abelian, it can be proved that $G$ is $p$-nilpotent by argument as in Corollary 1.2.

Similarly, if $m = 1$ or 2, we can obtain

Corollary 4.4. Let $p$ be an odd prime and $N$ a normal subgroup of $G$ with $p$-nilpotent quotient. Assume that $P$ is a Sylow $p$-subgroup of $N$ and $N_G(P)$ is $p$-nilpotent. If every minimal subgroup of $P$ not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$, then $G$ is $p$-nilpotent.

Corollary 4.5. Let $p$ be an odd prime and $N$ a normal subgroup of $G$ with $p$-nilpotent quotient. Assume that $P$ is a Sylow $p$-subgroup of $N$ and $N_G(P)$ is $p$-nilpotent. If every 2-minimal subgroup of $P$ not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$, then $G$ is $p$-nilpotent.

3. Corollaries of Theorem B.

Corollary 4.6. Let be $G$ be a group and $p$ a prime with $([G], p - 1) = 1$. Assume that $N$ is a normal subgroup of $G$ with $p$-nilpotent quotient and $P$ a Sylow $p$-subgroup of $N$. Suppose that every maximal subgroup of $P$ not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$, then $G$ is $p$-nilpotent.

Corollary 4.7. Let be $G$ be a group and $p$ a prime with $([G], p - 1) = 1$. Assume that $N$ is a normal subgroup of $G$ with $p$-nilpotent quotient and $P$ a Sylow $p$-subgroup of $N$. If every subgroup of $P$ of order $p$ or $4$ (when $P$ is nonabelian 2-group) not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$ then $G$ is $p$-nilpotent.

Corollary 4.8. Let $N$ be a soluble normal subgroup of $G$ with $2p$-nilpotent quotient and $P$ a Sylow 2-subgroup of $N$. If $P$ is quaternion-free and every subgroup of $P$ of order 2 not having $p$-nilpotent supplement in $G$ is $\Pi$-normal in $G$, then $G$ is 2-nilpotent.
Corollary 4.9 Let be G be a group and p a prime with $([G], p^2 - 1) = 1$. Assume that N is a normal subgroup of G with p-nilpotent quotient and P a Sylow p-subgroup of N. If every 2-maximal subgroup of P not having p-nilpotent supplement in G is II-normal in G, then G is p-nilpotent.

Corollary 4.10 Let be G be a group and p a prime with $([G], p^2 - 1) = 1$. Assume that N is a normal subgroup of G with p-nilpotent quotient and P a Sylow p-subgroup of N. Suppose that every 2-minimal subgroup of P not having p-nilpotent supplement in G is II-normal in G, then G is p-nilpotent.

In Corollaries 4.9 and 4.10 the hypothesis “$([G], p^2 - 1) = 1$” can not be replaced by “$([G], p - 1) = 1$”, for example $G = N = A_4$ and $p = 2$. Moreover, counter example also exists when p is odd.

Example 4.11 Let $G = \langle a, b \rangle \times \langle x \rangle$, where $a^2 = b^5 = x^3 = 1$, $ab = ba$ and $a^x = ab, b^x = a^2b^3$. Then G is not 5-nilpotent. Let $P = \langle a, b \rangle$. Then P is the Sylow 5-subgroup of G. The 2-minimal subgroup is $P$ and 2-maximal subgroup of $P$ is 1, which are both normal in G.

The following lemma shows that if p is minimal and is odd, then the hypothesis “$([G], p^2 - 1) = 1$” can be delete, and when $p = 2$, this hypothesis can be replaced by $G$ is $A_4$-free.

Lemma 4.12 ([13] Lemma 2.8) Let $p$ be the minimal prime divisor of the order of a group $G$. Assume that $G$ is $A_4$-free and $L$ is a normal subgroup of $G$. If $G/L$ is p-nilpotent and $p^3 \mid |L|$, then $G$ is p-nilpotent.

4. In the literature one can also find many special cases of Theorems A and B, for example:

Corollary 4.13 ([17] Theorem 3.1) Let $p$ be an odd prime dividing the order of $G$ and $P$ a Sylow p-subgroup of $G$. Suppose $N_G(P)$ is p-nilpotent and there exists a subgroup $D$ of $P$ with $1 < |D| < |P|$ such that every subgroup $H$ of $P$ with order $|D|$ is s-semipermutable in $G$. Then $G$ is p-nilpotent.

Corollary 4.14 ([17] Theorem 3.2) Let $p$ be a priming dividing the order of $G$ satisfying $([G], p - 1) = 1$ and $P$ be a Sylow p-group of $G$. Suppose there exists a nontrivial subgroup $D$ of $P$ such that $1 < |D| < |P|$ and every subgroup $H$ with order $|D|$ and $2|D|$ (if $P$ is a non-abelian 2-group and $|P : D| > 2$) is $s$-semipermutable in $G$, then $G$ is a p-nilpotent group.

Corollary 4.15 ([17] Theorem 3.1) Let $p$ be an odd prime dividing the order of $G$ and $P$ a Sylow p-subgroup of $G$. If $N_G(P)$ is p-nilpotent and every maximal subgroup of $P$ is s-semipermutable in $G$. Then $G$ is p-nilpotent.

Corollary 4.16 ([17] Theorem 3.3) Let $p$ be the smallest prime number dividing the order of $G$ and $P$ a Sylow p-subgroup of $G$. If every maximal subgroup of $P$ is s-semipermutable in $G$. Then $G$ is p-nilpotent.

Corollary 4.17 ([17] Theorem 3.5) Let $p$ be the smallest prime number dividing the order of $G$ and $P$ a Sylow p-subgroup of $G$. If every 2-maximal subgroup of $P$ is s-semipermutable in $G$ and $G$ is $A_4$-free. Then $G$ is p-nilpotent.
References

[1] A. Ballester-Bolinches and M. C. Pedraza-Aguilera. Sufficient conditions for supersolubility of finite groups. *J. Pure Appl. Algebra*, 127(2):113–118, 1998.

[2] K. Doerk and T. Hawkes. *Finite soluble groups*. Walter de Gruyter & Co., Berlin, 1992.

[3] L. Dornhoff. *M*-groups and 2-groups. *Math. Z.*, 100:226–256, 1967.

[4] T. Foguel. On seminormal subgroups. *J. Algebra*, 165(3):633–635, 1994.

[5] D. Gorenstein. *Finite groups*. Chelsea Publishing Co., New York, second edition, 1980.

[6] F. Gross. Conjugacy of odd order Hall subgroups. *Bull. London Math. Soc.*, 19(4):311–319, 1987.

[7] W. Guo. *The theory of classes of groups*. Kluwer Academic Publishers Group, Dordrecht, 2000.

[8] W. Guo, K. P. Shum, and A. N. Skiba. X-permutable maximal subgroups of Sylow subgroups of finite groups. *Ukraїn. Mat. Zh.*, 58(10):1299–1309, 2006.

[9] Z. Han. On s-semipermutable subgroups of finite groups and p-nilpotency. *Proc. Indian Acad. Sci. Math. Sci.*, 120(2):141–148, 2010.

[10] B. Huppert. *Endliche Gruppen. I*. Springer-Verlag, Berlin, 1967.

[11] B. Huppert and N. Blackburn. *Finite groups. III*. Springer-Verlag, Berlin, 1982.

[12] B. Li. On II-property and II-normality of subgroups of finite groups. *J. Algebra*, 334:321–337, 2011.

[13] B. Li and A. Skiba. New characterizations of finite supersoluble groups. *Sci. China Ser. A*, 51(5):827–841, 2008.

[14] A. Skiba. On weakly s-permutable subgroups of finite groups. *J. Algebra*, 315(1):192–209, 2007.

[15] X. Su. Seminormal subgroups of finite groups. *J. Math. (Wuhan)*, 8(1):5–10, 1988.

[16] J. G. Thompson. Normal p-complements for finite groups. *Math. Z.*, 72:332–354, 1959/1960.

[17] L. Wang and Y. Wang. On s-semipermutable maximal and minimal subgroups of Sylow p-subgroups of finite groups. *Comm. Algebra*, 34(1):143–149, 2006.

[18] Y. Wang. c-normality of groups and its properties. *J. Algebra*, 180(3):954–965, 1996.

[19] M. Weinstein(ed.). *Between nilpotent and solvable*. Polygonal Publ. House, Washington, N. J., 1982.