A New Scale-Dependent Cosmology with the Generalized
Robertson–Walker Metric and Einstein Equation

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Abstract

Based on the observed increase of $\Omega_0$ as a function of cosmic scale, the
Robertson–Walker metric and the Einstein equation are generalized so that
$\Omega_0$, $H_0$, and the age of the Universe, $t_0$, all become functions of cosmic scales
at which we observe them. The calculated local (within our galaxy) age of
the Universe is about 18 Gyr, consistent with the ages of the oldest stars and
globular clusters in our galaxy, while the ages at distant scales are shorter
than the local age, solving the age puzzle. It is also shown that $H_0$ increases
as scale increases, qualitatively consistent with the recent observations.

95.30.-k, 95.30.Sf, 98.80.-k

Typeset using REVTeX

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In this Letter, we propose a new scale-dependent cosmology in which the Hubble constant \(H_0\), \(\Omega_0(\equiv \rho_0/\rho_c)\), and the age of the Universe \(t_0\), all become scale-dependent. The need of such a model was recently discussed by one of us (CWK \[1\]) in an attempt to resolve the recent two cosmological conflicts by incorporating the running (scale-dependent) gravitational constant as suggested by the Asymptotically-Free Higher-Derivative (AFHD) quantum gravity \[2\] into cosmology. The first conflict refers to the disagreement between the two recent measurements \[3,4\] of \(H_0\), the second being the discrepancy between the calculated age of the Universe of about 8 Gyr, based on the larger of the two observed \(H_0\), and the measured ages of 14 to 18 Gyr for the oldest stars and globular clusters in our galaxy \[5\].

In \[1\], the treatment was entirely relied on the result of the AFHD quantum gravity. Here, we present a completely different approach which is purely classical without resort to quantum gravity and is motivated by the observed increase of \(\Omega_0\) as a function of cosmic scale \[6\]. Our model is based on the following two ansatzs.

- The recent epoch of the matter-dominated Universe is described by the metric

\[
dr^2 = dt^2 - R^2(t, r)[dr^2 + r^2d\Omega^2],
\]

where \(R(t, r)\) is the generalized scale factor which depends on both \(t\) and \(r\). This metric manifestly violates the Cosmological Principle.

- The equation of gravity is given by a generalized version of the Einstein equation, i.e.,

\[
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi GT^{\mu\nu},
\]

with a constraint \([GT^{\mu\nu}]_{;\nu} = 0\), where a semicolon denotes a covariant derivative. An immediate consequence is that a product of the new gravitational constant \(G\) and the energy density \(\rho\) is a function of \(r\) as well as \(t\). \(G\) becomes the Newton’s constant, \(G_N\), when \(r = 0\) at present epoch.
The first ansatz is motivated by the observation that the present Universe with large scale structures and with changing $\Omega_0$ as a function of scale cannot be isotropic and homogeneous as implied by the Cosmological Principle. Rather, the present Universe is locally inhomogeneous but still appears approximately isotropic to us. Since two observers at different locations observe different isotropic $\rho(r)$ (after some angular average and ignoring local variations), the Universe cannot be reduced to be homogeneous. (Recall that for a Universe with perfect fluid, isotropy implies homogeneity.) In this metric, as in the Schwartzschild metric, there is, strictly speaking, a special observer to whom the Universe is perfectly isotropic. We may or may not be this special observer. However, since the observed cosmic microwave background radiation and $\Omega_0(r)$ look almost isotropic to us, we cannot be too far from this observer. Since the left-hand side of the generalized Einstein equation has $r$ dependence through the Ricci tensors, the second ansatz is necessary in order to have the $r$ dependence on the right-hand side to counter that of the left-hand side.

Having generalized both the Robertson–Walker metric and the Einstein equation, we now proceed to discuss their consequences. When the non-vanishing elements of Ricci tensor calculated from Eq.(1) are substituted into the generalized Einstein equation in Eq.(2), we obtain the following equations for $tt$, $rr$, and $\theta\theta$ components:

\[
\begin{align*}
3 \frac{\dddot{R}(t, r)}{R(t, r)} - 2 \frac{\ddot{R}(t, r)}{R(t, r)} + \frac{\dot{R}^2(t, r)}{R^3(t, r)} + \frac{R''(t, r)}{R^4(t, r)} - 4 \frac{R'(t, r)}{rR^3(t, r)} &= 8\pi [Gp](t, r) \\
2 \frac{\dddot{R}(t, r)}{R(t, r)} + \frac{\ddot{R}^2(t, r)}{R^2(t, r)} - \frac{\dot{R}^2(t, r)}{R^3(t, r)} - 2 \frac{R'(t, r)}{rR^3(t, r)} &= -8\pi [Gp_\phi](t, r) \\
2 \frac{\dddot{R}(t, r)}{R(t, r)} + \frac{\ddot{R}^2(t, r)}{R^2(t, r)} - \frac{\dot{R}^2(t, r)}{R^3(t, r)} + \frac{R^2(t, r)}{R^4(t, r)} - \frac{R'(t, r)}{rR^3(t, r)} &= -8\pi [Gp_\theta](t, r),
\end{align*}
\]

where dots and primes denote, respectively, derivatives with respect to $t$ and $r$, and $p$ is the pressure. The equation containing $p_\phi$ is identical to Eq.(5) with $p_\theta$ replaced by $p_\phi$. One more non-vanishing Ricci tensor $R_{01}$ is proportional to momentum density $T_{01}$ which we assume to be small but finite. This nonvanishing $T_{01}$ has been, in the matter-dominated era, responsible for allowing matter to flow from its initially homogeneous distribution to the present inhomogeneous distribution. In addition, it also prevents $R(r, t)$ from being
factored out as \( a(t)S(r) \), which is the case of no interest to us. As designed, when the \( r \) dependence is simply dropped (i.e., \( R' = R'' = 0 \)) and \( p_r = p_\theta = p_\phi = p \), we recover the Friedman equation with \( k=0 \). With \( p_r = p_\theta = p_\phi = p \neq 0 \), we obtain, from Eqs.(4) and (5), a constraint on \( R(t, r) \) given by

\[
R''(t, r) - 2 \frac{R'(t, r)^2}{R(t, r)} - \frac{R'(t, r)}{r} = 0.
\] (6)

The constraint in Eq.(6) is valid for \( r \neq 0 \), because of its singular behavior at \( r = 0 \). In the following, \( r \simeq 0 \) implies very small \( r \), but excluding \( r = 0 \), corresponding to our local neighborhood of 10 \( \sim \) 50 Kpc. The above constraint can easily be solved, yielding

\[
R(t, r) = \frac{a(t)}{1 - \left[ \frac{ra(t)}{2A(t)} \right]^2},
\] (7)

where \( a(t) \) and \( A(t) \) are unknown functions of \( t \) alone. We have chosen a negative sign in Eq.(7) in order to have a locally open Universe (see Eq.(8) below). Different \( t \) dependences in \( A(t) \) and \( a(t) \) are protected by \( p \neq 0 \) and \( T_{01} \neq 0 \). The \( R(t, r) \) in Eq.(7) has a singularity at \( r = 2A(t)/a(t) \). This suggests that our Universe appears to be inside a cavity or bubble surrounded by a wall with infinite density. The observed approximate isotropy of the matter distribution and the cosmic microwave background radiation indicates that we are not near the wall. The wall may be much farther than our horizon. Such a model was previously discussed [7] in an entirely different context.

When Eq.(7) is substituted into the last three terms in Eq.(3), Eq.(3) becomes

\[
\left[ \frac{\dot{R}(t, r)}{R(t, r)} \right]^2 = \frac{8\pi G \rho}{3} + \frac{1}{A(t)^2}.
\] (8)

(When \( A(t)^2 \) is constant, the last term in Eq.(8) is nothing but a Cosmological Constant. Thus the Cosmological Constant is generated, even though we started with a generalized Einstein equation without it.) Since we are interested in the matter-dominated era, we do not consider equations with \( p \) and \( T_{01} \), assuming they are small, but Eq.(8) is exact. In any gravitational experiment or phenomenon, a relevant quantity is always a product of \( G \) and \( \rho \) which cannot be separated. We may define \( G(r) \equiv G_N[1 + \delta(r)] \) as an effective gravitational
constant, or interpret $\rho$ as $\rho(t, r) \simeq \mathcal{P}_0[R(t_0, r)/R(t, r)]^2[1 + \delta(r)]$, approximately valid in the matter-dominated era, with the present local density $\mathcal{P}_0$. Although both interpretations yield the same phenomenology, their physical implications are different. The former is the running of the gravitational constant, the latter implying that the dark matter content increases as scale increases. The truth may be some combination of the two. Since our formulation is phenomenological, we cannot distinguish between the two, but it is extremely important to separate them, for the difference reveals the true content of dark matter.

We now examine the physical meaning of Eq.(8). If we naively set $r = 0$ in Eq.(8), we would end up with an equation which is not only different from the Friedman equation with $k = 0$ but also simply wrong. Equation (8) in fact approaches the Friedman equation with $k = 0$ as $r$ increases. As cosmic scale grows at any fixed time, the $G\rho$ term also grows, whereas the $1/A(t)^2$ term remains fixed, making the curvature-like term relatively smaller compared with the $G\rho$ term. This is the way we recover a flat Friedman Universe. (This is of course based on the assumption that the inflationary scenario is correct and we have formulated accordingly.)

In this cosmology, we have two expansion rates; $H(t, r) \equiv \dot{D}(t, r)/D(t, r)$ and $\mathcal{H}(t, r) \equiv \dot{R}(t, r)/R(t, r)$. The former is the proper expansion rate with $D(t, r) \equiv \int_0^r R(t, r')dr'$ and the latter is the one that appears in Eq.(8), which for distinction we call the scale expansion rate. We adapt the notation $\overline{Q}$ for a quantity $Q(r)$ evaluated at our local neighborhood $r \simeq 0$. With this notation, we have $\overline{H}_0 = \overline{H}_0$. Evaluated at $r \simeq 0$ and at present epoch, Eq.(8) becomes,

$$\overline{H}_0^2 = \frac{8\pi G N\overline{\mathcal{P}}_0}{3} + \frac{1}{A_0^2},$$

which relates the unknown $A_0$ to $\overline{H}_0$ and $\overline{\Omega}_0$.

The $\Omega_0(r)$ is, from Eq.(8), $\Omega_0(r) \equiv (G\rho)_0/(G\rho)_{c,0} = 1 - 1/\mathcal{H}_0^2A_0^2$, where $(G\rho)_{c,0}$ is defined by $\mathcal{H}(t_0, r)^2 \equiv 8\pi(G\rho)_{c,0}/3$. Its local version (i.e., evaluated at $r \simeq 0$) is, with Eq.(9),
\[ \Omega_0 = 1 - \frac{1}{\mathcal{H}_0^2 A_0^2} = \frac{1}{\mathcal{H}_0^2} \frac{8\pi G_N \rho_0}{3}. \]  

(10)

From Eqs.(8), (9), (10) and expression of \( G\rho \) below Eq.(8), we obtain

\[ \mathcal{H}_0(r) = \mathcal{H}_0 \sqrt{1 + \Omega_0 \delta(r)}. \]  

(11)

Similarly, we find

\[ \Omega_0(r) = \Omega_0 \frac{1 + \delta(r)}{1 + \Omega_0 \delta(r)}. \]  

(12)

We note that this \( \Omega_0 \) is different from the standard one defined with a constant \( \rho_c \). Both \( \mathcal{H}_0(r) \) and \( \Omega_0(r) \) are increasing functions of \( r \) because \( \delta(r) \) also is, as expected from the behavior of the observed \( \Omega_0(r) \). \( \Omega_0(r) \) approaches unity, but can never be greater than one. (The possibility of running \( H_0 \) was previously mentioned in [8].) The expressions given in Eqs.(11) and (12) are identical to those in [1].

Now let us calculate the present age of the Universe, \( t_0 \). In the age equation, the main \( r \)-dependent contribution comes from the \( 8\pi G\rho/3 \) term. Since \( R(t, r) \) is a very slowly varying function of \( r \) up to scales of our interest, 15 Mpc, defining \( x \equiv R(t, r)/R(t_0, r) \), and using Eqs.(8), (9), and (10), and the approximate expression for \( \rho(t, r) \) mentioned below Eq.(8), we obtain the following age equation

\[ t_0(r) \simeq \frac{1}{\mathcal{H}_0} \int_0^1 \frac{\sqrt{2} dx}{\sqrt{\Omega_0[1 + \delta(r)] + x[1 - \Omega_0]}}. \]  

(13)

Equation (13) is identical to that of [1], with exception of \( \delta(r) \) being replaced by \( \delta_G \) due to the running \( G(r) \) alone. It is clear that \( t_0(r) \) is now a decreasing function of \( r \), with the oldest age at \( r \simeq 0 \) (or \( \delta(r) \simeq 0 \))

\[ t_0 \simeq \frac{0.9}{\mathcal{H}_0} = 18 \text{ Gyr} \quad \text{for} \quad \mathcal{H}_0 \simeq 50 \text{ Km/secMpc}. \]  

(14)

This age can easily accommodate the observed ages, 14 \( \sim 18 \) Gyr, of the oldest stars and globular clusters in our galaxy.

The proper expansion rate is given by, with Eqs.(7), (9), and (11),

\[ \mathcal{H}(t_0) = \frac{0.9}{t_0} = 50 \text{ Km/secMpc}. \]  

(15)
\[ H_0(r) = \frac{\int_0^r \sqrt{1 + \Omega_0 \delta(r') (1 - \beta r'^2)^{-1}} dr'}{\int_0^r (1 - \beta r'^2)^{-1} dr'}, \tag{15} \]

where \( \beta \equiv \left(\frac{a_0}{A_0}\right)^2/4 \) is an unknown constant. \( H_0(r) \) is a weighted average of \( H_0(r) \) with weight \( R(t_0, r) \), and is more slowly increasing function of \( r \) than that of Eq.(11) but approaches that of Eq.(11) for large \( r \).

Before presenting numerical results, we mention that because of its specific form, \( R(t, r) \) dictates the form of the \( r \) dependence of cosmological quantities. This leads to

\[ \delta(r) = \frac{1}{\Omega_0} \left[ (1 + \frac{\alpha r^2}{1 - \beta r^2})^2 - 1 \right], \tag{16} \]

where \( \alpha \) and \( \beta \) are positive parameters to be determined. By fitting the (very uncertain) data of \( \Omega_0[1 + \delta(r)] \) in [6], we obtained \( \alpha \sim 10^{-10} \) and \( \beta \leq 10^{-10} \) with \( r \) converted into a proper distance in units of Kpc. (Unfortunately, it is not possible to determine \( \beta \) because of the poor data quality.)

Using Eq.(13) and the \( \delta(r) \) as described above, we obtain the age at 15 Mpc, where one of the new measurements of \( H_0 \) has recently been made, as \( t_0(15 \text{ Mpc}) \sim 17 \text{ Gyr} \). Although this value agrees with the local ages, it should not, in principle, be compared with the local ages, as in the case of values of the fine structure constant at different energies. We find, from Eq.(15), \( H_0(15 \text{ Mpc}) \simeq 52 \text{ Km/secMpc} \) with \( \Omega_0 = 50 \text{ Km/secMpc} \). Although, the calculated \( H_0 \) shows an increasing trend, the increase is not fast enough to explain the observed value, \( H_{0, \text{expt}}(15 \text{ Mpc}) = (87 \pm 7) \text{ Km/secMpc} \) [3]. The other measurements of \( H_0 \) based on the type Ia supernovae are \( H_0(B) = 52 \pm 8 \text{Km/secMpc} \) and \( H_0(V) = 55 \pm 8 \text{Km/secMpc} \) [4]. These values were obtained from the Hubble diagrams in B and V for the supernovae Ia with distances ranging from \( \sim 20 \text{ Mpc} \) to \( \sim 100 \text{ Mpc} \). In this case, our calculated values of \( H_0 \) are \( 55 \sim 70 \text{ Km/secMpc} \). It remains to be seen whether this discrepancy will persist in the future observation. Comparison of the observed \( \Omega_0 \) and the one predicted purely by quantum gravity can reveal the true content of dark matter. Such a study and derivations of new (complex) relationships among distance, redshift, and the two expansion rates in this new cosmology, which may shed some light on the discrepancy in \( H_0 \), are given elsewhere.
An important prediction of this cosmology is a sharp rise of $\rho(r)$ beyond 100 Mpc.

**ACKNOWLEDGMENTS**

The authors would like to thank B. Bosco, A. Bottino, R. Casalbuoni, D. Dominici, G. Grunberg, M. Im, L. Lusanna, T.N. Pham and T.N. Truong for helpful discussions. Special thanks are due to A. Chakrabarti and G. Feldman who pointed out the importance of nonvanishing $p$ and $T_{01}$. In particular, without help of G. Feldman, this paper could not have been in its present form. One of the authors (A.S.) would like to thank the late R. Pierluigi for very interesting discussions. This work was supported in part by the National Science Foundation, U.S.A.
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