Analytic continuation and resonance-free regions for Sturm-Liouville potentials with power decay

B.M. Brown, M.S.P. Eastham
Department of Computer Science, University of Cardiff, Cardiff, CF24 3XF, U.K.

February 4, 2022

1 Introduction

We consider the Sturm-Liouville equation

\[ y''(x) + \{\lambda - q(x)\}y(x) = 0 \quad (0 \leq x < \infty) \quad (1.1) \]

with a boundary condition

\[ y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad (1.2) \]

\( \lambda \) being the complex spectral parameter. As usual, \( \alpha \) is real and the potential \( q \) is real-valued and locally integrable on \([0, \infty)\). We further assume throughout the paper that \( q \) decays as \( x \to \infty \) in the sense that

\[ q \in L(0, \infty). \quad (1.3) \]

Let us write \( \lambda = z^2 \), where \( 0 \leq \arg z < \pi \) when \( 0 \leq \arg \lambda < 2\pi \). Then (1.3) implies that there is a solution \( \psi(x, z) \) of (1.1) such that

\[ \psi(x, z) \sim \exp(izx), \quad \psi'(x, z) \sim iz \exp(izx) \quad (1.4) \]

as \( x \to \infty \), and \( \psi(x, z) \) is analytic in \( z \) for \( \text{im} z > 0 \) \[13, \text{Theorem 1.9.1}\]. Then \( \psi(x, z) \) is the Weyl \( L^2(0, \infty) \) solution of (1.1) when \( \lambda \) is non-real and it forms the basis of the Weyl-Titchmarsh spectral theory of (1.1) \[5, \text{Chapter 9}, \ 17, \ 18\]. A central result of this spectral theory is the existence of a spectral function \( \rho_\alpha(\mu) \) \((-\infty < \mu < \infty)\) which is piecewise constant in \((-\infty, 0)\) and locally absolutely continuous in \([0, \infty)\) with \( \rho_\alpha'(\mu) > 0 \) \[17, \text{section 5.7}, \ 18, \text{p. 264}\]. In particular, (1.4) leads to the Kodaira formula

\[ \pi \rho_\alpha'(\mu) = \mu^{1/2} / |\Psi(\mu^{1/2})|^2 \quad (\mu > 0) \quad (1.5) \]

where

\[ \Psi(z) = \psi(0, z) \cos \alpha + \psi'(0, z) \sin \alpha \quad (1.6) \]

\[ 13, \text{p. 940}\]. Since the only possible eigenvalues of the problem (1.1)-(1.3) lie on the negative real \( \lambda \)-axis, \( \Psi(z) \) has no zeros for \( 0 \leq \arg z < \pi \) expect possibly when \( \arg z = \frac{1}{2} \pi \).
In addition to (1.5), the Weyl-Titchmarsh function \(m_\alpha(\lambda)\) [5, Chapter 9], [17, Chapter 2] also involves \(\Psi\) in the form

\[m_\alpha(\lambda) = \{\psi'(0, z) \cos \alpha - \psi(0, z) \sin \alpha\}/\Psi(z)\]  \hspace{1cm} (1.7)

again with \(0 \leq \arg z < \pi\). Now \(m_\alpha(\lambda)\) is related to the Green’s function and to the resolvent operator of \(1.1\)-(1.2) in the Hilbert space \(L^2(0, \infty)\), and the question arises whether these three spectral objects have analytic continuations into the so-called unphysical sheet \(\pi \leq \arg z < 2\pi\). As far as the Green’s function and resolvent are concerned, this question can be posed, not only for \((1.1)-(1.2)\), but also for the corresponding Schrödinger equation in two or more dimensions. However, in the case of \((1.1)\) itself, it is a question of the analytic continuation of \(\psi(0, z)\) and \(\psi'(0, z)\) in \((1.6)\) and \((1.7)\).

Analytic continuation into the strip \(-\frac{1}{2}a < \text{im} z < 0\) was established in [6] subject to a strengthening of (1.3) to

\[q(x) = O(e^{-ax}) \quad (x \to \infty)\]  \hspace{1cm} (1.8)

for some \(a > 0\) (see also [4, section 2.2]), and we refer again to [3] for a description of earlier work in this direction. Allowing \(a\) to be arbitrarily large in \((1.8)\) leads to the class of super-exponentially decaying potentials for which

\[q(x) = O(e^{-xf(x)}) \quad (x \to \infty)\]  \hspace{1cm} (1.9)

with some \(f(x) \to \infty\), and then we have analytic continuation into the whole of \(\text{im} z < 0\). [10], [12]. Two other specialisations of \((1.8)\) where again there is analytic continuation into the whole of \(\text{im} z < 0\) are

\[q(x) = e^{-ax} p(x)\]

with \(p(x)\) periodic and

\[q(x) = (\text{const.}) x^N e^{-ax}\]

where \(N(\geq 1)\) is an integer [3].

Once analytic continuation has been effected, the possibility is opened up of \(\Psi(z)\) having zeros in the unphysical sheet. Such zeros are called resonances and, by [12,7], they are singular spectral points associated with the Green’s function and resolvent operator. For potentials of the class \((1.8)\), the asymptotic distribution of resonances was obtained in [10] and, by another method, also in [12] along with other results on the location of resonances. In particular [12, Theorem 3.8], there is a resonance-free strip \(-b \leq \text{im} z < 0\) subject to \(q\) having a suitably small norm.

All these existing results require exponential decay of the potential \(q\). In this paper, we allow \(q\) to have only power decay \(O(x^{-\gamma}) \quad (x \to \infty)\) for some \(\gamma > 1\) and, under further conditions on the analyticity of \(q\), we establish analytic continuation of \(\Psi(z)\) into a sector \(2\pi - \theta_0 < \arg z < 2\pi\) of the unphysical sheet. The necessary construction is given in section 2. Then in sections 3 and 4 we show that our methods lead to certain resonance-free regions which are adjacent to part of the real \(z\)-axis. Finally, in section 5, we discuss the numerical computation of resonances lying in the complement of our resonance-free regions.

2 Analytic continuation

The method which we develop in this section for continuing \(\psi(x, z)\) and \(\psi'(x, z)\) analytically into \(\text{im} z < 0\) is based on the integral equation by means of which \((1.4)\) is proved [5, sections 1.3 and 1.9]. Thus we begin by writing \((1.4)\)
1) (with \( \lambda = z^2 \neq 0 \) and \( y = \psi \)) as a first-order system in a standard way by defining

\[
W = \frac{1}{2} e^{-i x z} \left( \begin{array}{cc}
1 & -i/z \\
1 & i/z \\
\end{array} \right) \left( \begin{array}{c}
\psi \\
\psi' \\
\end{array} \right).
\]  

(2.1)

Then

\[
W' = \left\{ \left( \begin{array}{cc}
0 & 0 \\
0 & -2i z \\
\end{array} \right) + Q \left( \begin{array}{cc}
-1 & -1 \\
1 & 1 \\
\end{array} \right) \right\} W,
\]  

(2.2)

where

\[
Q = \frac{1}{2} i q/z,
\]  

(2.3)

and the corresponding integral equation is

\[
W(x, z) = e_1 + \int_x^\infty Q(t) K(t - x, z) W(t, z) dt,
\]  

(2.4)

where

\[
e_1 = \left( \begin{array}{c}
1 \\
0 \\
\end{array} \right), \quad K(s, z) = \left( \begin{array}{cc}
1 & 1 \\
-e^{2i sz} & -e^{2i sz} \\
\end{array} \right).
\]  

(2.5)

Iteration of (2.4) gives

\[
W(x, z) = e_1 + \sum_{n=1}^\infty W_n(x, z),
\]  

(2.6)

where

\[
W_n(x, z) = \int_x^\infty Q(t) K(t - x, z) W_{n-1}(t, z) dt 
\]  

(2.7)

and \( W_0(x, z) = e_1 \), provided of course that the infinite integral converges. We note that, in terms of the components of \( W_n \) and \( W_{n-1} \), (2.7) is

\[
\left( \begin{array}{c}
u_n \\
v_n \\
\end{array} \right)(x, z) = \int_x^\infty Q(t) \left\{ u_{n-1}(t, z) + v_{n-1}(t, z) \right\} \left( \begin{array}{c}1 \\
-e^{2i(t-x)z} \\
\end{array} \right) dt.
\]  

(2.8)

Also, the transformation (2.1) back to (1.1) gives

\[
\psi = e^{i x z} \left( 1 + \sum_{n=1}^\infty \{ u_n(x, z) + v_n(x, z) \} \right),
\]  

(2.9)

\[
\psi' = i z e^{i x z} \left( 1 + \sum_{n=1}^\infty \{ u_n(x, z) - v_n(x, z) \} \right).
\]  

(2.10)

In what follows, we write

\[
|W_n| = |u_n| + |v_n|.
\]  

(2.11)

We now introduce the more detailed conditions on \( q \) that we require.
Condition 2.1 We suppose that the real-valued function \( q(x) \) can be extended into a sector \( S \) of the complex \( \xi \)-plane as an analytic function \( q(\xi) \) as follows.

1. \( q(\xi) \) is regular in a sector \( S \) defined by \( -\theta_0 < \arg \xi < \theta_0 \) and \( \xi \neq 0 \), with some \( \theta_0 \) such that \( 0 < \theta_0 < \pi \).

2. There are constants \( \gamma \) (> 1) and \( k \) such that
   \[ |q(\xi)| \leq k |\xi|^{-\gamma} \]  
   as \( |\xi| \to \infty \) and \( \xi \in S \).

Our method involves extending also the definition of \( W(x,z) \) into the complex \( \xi \)-plane. To do this, we write \( t = x + s \) in (2.7) and consider the iterative definition
   \[ W_n(\xi,z) = \int_0^\infty Q(\xi+s)K(s,z)W_{n-1}(\xi+s,z)ds \]  
   for \( \xi \in S \), with \( W_0(\xi,z) = e_1 \). In the following lemma we give a simple estimate for the size of \( W_n \) in order to deal with the convergence of the infinite integral in (2.13).

Lemma 2.2 Let \( q \) satisfy Condition 2.1. Let \( \xi \in S \) and let \( \text{Im} z > 0 \) in (2.5) and (2.13). Then for \( n \geq 0 \)
   \[ |W_n(\xi,z)| \leq \frac{1}{n!} \left( 2 \int_0^\infty |Q(\xi+s)|ds \right)^n, \]  
   and \( W_n(\xi,z) \) is a regular function of \( \xi \) in \( S \).

Proof. We note that the infinite integral in (2.14) converges because of (2.12). The lemma is clearly true when \( n = 0 \) and, proceeding by induction on \( n \), we use the form of (2.8) which corresponds to (2.13). By (2.11) and (2.14) (with \( n = 1 \)), this gives
   \[ |W_n(\xi,z)| \leq 2^n \int_0^\infty |Q(\xi+s)| \left( \frac{1}{(n-1)!} \left( \int_0^\infty |Q(\xi+s+\sigma)|d\sigma \right)^{n-1}ds \right. \]
   \[ = \frac{2^n}{(n-1)!} \int_0^\infty |Q(\xi+s)| \left( \int_s^\infty |Q(\xi+\sigma)|d\sigma \right)^{n-1}ds, \]
   from which (2.14) follows.

To deal with the regularity of the \( W_n \), we note that (2.12) and (2.14) imply that the infinite integral in (2.13) converges uniformly with respect to \( \xi \) in any closed bounded region \( S_1 \subset S \). Thus the regularity in \( S \) of \( W_n \) follows from that of \( W_{n-1} \), and the lemma is proved.

The next step is to re-write (2.13) in a form which does not require \( \text{Im} z > 0 \) and which therefore provides the analytic continuation of \( W_n(\xi,z) \) (as a function of \( z \)) into the lower half of the \( z \)-plane. At this stage we restrict \( \xi \) so that \( \text{re} \xi \geq 0 \), the reason being given in the proof of the following theorem. Ultimately we specialise \( \xi \) to be the positive real variable \( x \).
Theorem 2.3 Let \( q \) satisfy Condition 2.1. Then, for all \( \xi \) and \( z \) in \( S \) with \( \text{re} \, \xi \geq 0 \),

\[
W_n(\xi, z) = \frac{1}{z} \int_0^\infty Q(\xi + \frac{s}{z}) K(s, 1) W_{n-1}(\xi + \frac{s}{z}, z) ds,
\]

and the series

\[
W(\xi, z) = e_1 + \sum_{n=1}^\infty W_n(\xi, z)
\]

defines a regular function of \( z \) in \( S \) which, when \( \xi = x \), continues to satisfy the differential equation (2. 2).

Proof. We suppose first that \( \text{im} \, z > 0 \), so that (2. 13) holds. We consider the contour integral

\[
\int_C Q(\xi + \eta \frac{z}{\bar{z}}) K(\eta, 1) W_{n-1}(\xi + \eta \frac{z}{\bar{z}}, z) d\eta
\]

where \( C \) is the closed contour in the complex plane formed by the positive real axis, the line through \( z \) from 0 to \( \infty \), and the smaller part of the circle \(| \eta | = R \). The assumption that \( \text{re} \, \xi \geq 0 \) guarantees that the point \( \xi + \eta \frac{z}{\bar{z}} \) lies in \( S \), and therefore the integrand in (2. 17) is defined as a regular function of \( \eta \) within and on \( C \). Then, by Cauchy’s Theorem, the value of (2. 17) is zero. Thus (2. 15) follows from (2. 13) when \( R \to \infty \), provided that the contribution to (2. 17) from \(| \eta | = R \) tends to zero.

By (2. 3), (2. 5) and (2. 14), this contribution does not exceed in modulus

\[
\text{(const.)} R \int_0^{\arg z} |\xi + \frac{\eta}{z}|^{\gamma-1} \left( \int_0^\infty |\xi + \frac{\eta}{z} + s|^{-\gamma} ds \right)^{n-1} d\theta
\]

in which \( \eta = \text{Re} \, e^{i\theta} \). Since \( 0 < \arg z < \theta_0 \ (< \pi) \), it is easy to check that

\[
|X + iY + \frac{\eta}{z}|^2 \geq \frac{1}{2} (1 - |\cos \theta_0|) (X^2 + R^2 / |z|^2) - \kappa Y^2
\]

where \( X = \text{re} \, \xi + s \), \( Y = \text{im} \, \xi \) and \( \kappa = (1 + 3 \cos^2 \theta_0) / ((1 - |\cos \theta_0|) (1 + 3 |\cos \theta_0|))) \). Then, since \( \gamma > 1 \), the \( \theta \)-integral in (2. 18) is \( O(R^{-\gamma(n-1)(\gamma-1)}) \), and hence (2. 18) tends to zero as \( R \to \infty \) for all \( n \geq 1 \). This proves (2. 13) for \( \text{im} \, z > 0 \).

We turn now to \( \text{im} \, z \leq 0 \) and we show that (2. 13) continues to provide an iterative definition of the \( W_n(\xi, z) \) as regular functions of \( z \). We note that, when \( \text{re} \, \xi \geq 0 \) and \( \text{im} \, z \leq 0 \), the point \( \xi + s/z \) in (2. 13) continues to lie in \( S \). An induction argument similar to that used for (2. 14) shows that

\[
|W_n(\xi, z)| \leq \frac{1}{n!} \left( \frac{2}{|z|} \int_0^\infty |Q(\xi + \frac{s}{z})| ds \right)^n.
\]

Again, as for (2. 13), the infinite integral in (2. 13) converges uniformly with respect to \( z \) in any closed bounded region \( S_1 \subset S \), by (2. 12). Hence each \( W_n(\xi, z) \) is a regular function of \( z \) in \( S \). Further, (2. 19) also guarantees the uniform convergence of the series (2. 16) with respect to \( z \) in \( S_1 \), and hence \( W(\xi, z) \) is also a regular function of \( z \) in \( S \).
Finally, we show that \( W(\xi,z) \) satisfies \( (2.2) \) in the more general form with \( \xi \) in place of \( x \). In \( (2.15) \), we sum for \( n \) going from \( 1 \) to \( \infty \) and we write \( s = z(t - \xi) \) to obtain

\[
W(\xi,z) = e_1 + \int_\xi^\infty Q(t)K(z(t - \xi),1)W(t,z)dt,
\tag{2.20}
\]

where \( \infty \) denotes the point at infinity on the line through \( \xi \) in the direction of the vector \( 1/z \). The interchange of integration and summation involved in \( (2.20) \) is justified by means of \( (2.19) \). Differentiation of \( (2.20) \) with respect to \( \xi \) now recovers \( (2.2) \) with \( \xi \) in place of \( x \), and the proof of the theorem is complete.

We note that, in terms of the components of \( W_n \) and \( W_{n-1} \), \( (2.15) \) is

\[
\left( \begin{array}{c} u_n \\ v_n \end{array} \right)(\xi,z) = \frac{1}{2}iz^{-2} \int_0^\infty q(\xi + \frac{s}{z})(u_{n-1} + v_{n-1})(\xi + \frac{s}{z},z) \left( \frac{1}{e^{2is}} \right) ds
\tag{2.21}
\]
corresponding to \( (2.8) \), and we have used \( (2.3) \). Here \( (2.21) \) is valid for all \( \xi \) and \( z \) in \( S \) with \( \text{re} \, \xi \geq 0 \), and then \( (2.9) \) and \( (2.10) \) provide the desired analytic continuation of \( \psi(x,z) \) and \( \psi'(x,z) \) into the lower half of the sector \( S \).

3 Resonance-free regions

The basic result on non-resonance which follows from \( (2.9), (2.10) \) and \( (2.21) \) is given in the next theorem.

**Theorem 3.1**Let \( z \in S \) with \( \text{im} \, z < 0 \) and \( z \neq i \cot \alpha \). Let

\[
\int_0^\infty |q(s/z)| \, ds < |z|^2 \log(1 + \delta^{-1}),
\tag{3.1}
\]

where

\[
\delta = \int \frac{1}{|\cos \alpha - iz \sin \alpha|} \left/ \frac{1}{|\cos \alpha + iz \sin \alpha|} \right. \begin{cases} 1, & (0 \leq \alpha \leq \pi/2) \\ \frac{\pi/2 - \alpha}{\alpha}, & (\pi/2 < \alpha < \pi). \end{cases}
\tag{3.2}
\]

Then \( \Psi(0,z) \neq 0 \) and \( z \) is not a resonance.

**Proof.** By \( (1.6), (2.3) \) and \( (2.10) \), we have

\[
\Psi(z) = (\cos \alpha + iz \sin \alpha) \left( 1 + \sum_{1}^{\infty} \{ u_n(0,z) + Zv_n(0,z) \} \right),
\tag{3.3}
\]

where \( Z = (\cos \alpha - iz \sin \alpha)/(\cos \alpha + iz \sin \alpha) \). It is easy to check that, since \( \text{im} \, z < 0 \), \( |Z| \leq 1 \) for \( 0 \leq \alpha \leq \pi/2 \) and \( |Z| > 1 \) for \( \pi/2 < \alpha < \pi \). Hence, with \( \delta \) as in \( (3.2) \),

\[
\sum_{1}^{\infty} \{ u_n(0,z) + Zv_n(0,z) \} \leq \delta \sum_{1}^{\infty} (|u_n(0,z)| + |v_n(0,z)|)
\leq \delta \left\{ \exp \left( |z|^{-2} \int_{0}^{\infty} |q(s/z)| \, ds \right) - 1 \right\}
\]

6
by (2.19) and (2.21). It now follows from (3.3) that $\Psi(z)$ is non-zero if $z \neq i \cot \alpha$ and
\[
\exp\left(|z|^{-2} \int_0^\infty |q(s/z)| \, ds\right) - 1 < \delta^{-1},
\]
and the latter is guaranteed by (3.1).

Let us note that, with the change of variable $s = |z| t$, (3.1) can be written as
\[
\int_0^\infty |q(te^{i\theta})| \, dt < |z| \log(1 + \delta^{-1})
\]
where $\theta = 2\pi - \arg z > 0$. The condition (3.4) defines a region of the complex plane within which there are no resonances, and the nature of this resonance-free region depends on the nature of $q$. Before we turn to detailed examples, we give one general property of resonance-free regions which is a consequence of (3.4).

**Corollary 3.2** There are real numbers $R_1 (> 0)$ and $\theta_1 (0 < \theta_1 < \pi)$ such that the sectorial region $|z| \geq R_1$, $2\pi - \theta_1 < \arg z < 2\pi$ is resonance-free.

**Proof.** Suppose first that $0 \leq \alpha \leq \pi/2$, so that $\delta = 1$ in (3.2). We choose $R_1$ so that
\[
R_1 > (\log 2)^{-1} \int_0^\infty |q(t)| \, dt.
\]
Then, by continuity in $\theta$, we have
\[
\int_0^\infty |q(te^{i\theta})| \, dt < R_1 \log 2
\]
for $\theta$ in some range $(0, \theta_1)$ with $\theta_1 > 0$. Hence (3.4) holds for $|z| \geq R_1$, and the corollary is proved for this range of $\alpha$.

Next suppose that $\pi/2 < \alpha < \pi$. Then, with $\delta$ as in (3.2) and $\cot \alpha < 0$, it is easy to check that
\[
\delta \leq (1 + \sin \theta) / |\cos \theta|
\]
where again $\arg z = 2\pi - \theta$. We choose $R_1$ as in (3.3) but, in place of (3.4), we can say that
\[
\int_0^\infty |q(te^{i\theta})| \, dt < R_1 \log \left(1 + \frac{|\cos \theta|}{1 + \sin \theta}\right) < R_1 \log(1 + \delta^{-1})
\]
for $\theta$ in some range $(0, \theta_1)$ with $\theta_1 > 0$. Hence (3.4) again holds for $|z| \geq R_1$, as required.

Corollary 3.2 provides theoretical support for an observation by Aslanyan and Davies [2] concerning the numerical computation of resonances for the potential
\[
q(x) = x^2 \exp(-\epsilon x^2),
\]
where $\epsilon (> 0)$ is a small parameter. In [3, p. 16 and Table 10] it is noted that there are resonances very close to the positive real axis but, at a certain point, they turn sharply away into the lower half plane. Now (3.7) satisfies Condition 2.1 with $\theta_0 < \pi/4$ (see also Example 4.6 below), and the existence of the sectorial region in Corollary 3.2
precludes as a general feature the occurrence of resonances close to the positive real axis beyond a certain distance from the origin.

Corollary 3.2 can also be related to [3, Theorem 1] concerning the localization of spectral concentration points to a bounded interval on the real spectral axis (see also [3, section 2]). Insofar as spectral concentration is associated with resonances located near to the real axis, Corollary 3.2 provides another proof that spectral concentration points are confined to a bounded interval for a class of potentials satisfying (1.3).

In the Dirichlet case $\alpha = 0$ of (1.2), there are additional non-resonance results like Theorem 3.1 and Corollary 3.2 but with, in the corollary, the vertex of the sector at the origin.

**Theorem 3.3** Let $\alpha = 0$ in (1.2) and let $q$ satisfy Condition 2.1 with

$$\gamma > 2 \quad (3.8)$$

in (2.13). Let $z \in S$ with $\text{im} \ z < 0$, and let

$$\int_0^\infty s \ | q(s/z) \ | \ ds < | z |^2 \log 2 \quad (3.9)$$

Then $\psi(0, z) \neq 0$ and $z$ is not a resonance.

**Proof.** We note that (3.8) guarantees the convergence of the integral in (3.9). In (2.21), we use the inequality $|1 - e^{2is}| \leq 2s$ to obtain

$$| (u_n + v_n)(\xi, z) | \leq | z |^{-2} \int_0^\infty s \ | q(\xi + s/z) \ | (u_{n-1} + v_{n-1})(\xi + s/z, z) \ | \ ds.$$ 

Then, as for (2.11), an induction argument gives

$$| (u_n + v_n)(\xi, z) | \leq \frac{1}{n!} \left( \frac{1}{| z |^2} \int_0^\infty s \ | q(\xi + s/z) \ | \ ds \right)^n.$$ 

This inequality is used in (2.9) and (1.6) (with $\alpha = 0$), and the theorem follows from (3.9) in the same way as Theorem 3.1 followed from (3.1).

As for (3.4), the change of variable $s = | z | \ t$ in (3.9) leads to

$$\int_0^\infty t \ | q(te^{i\theta}) \ | \ dt < \log 2 \quad (3.10)$$

and this in turn leads immediately to the next corollary.

**Corollary 3.4** Let $\alpha = 0$ in (1.2) and, in addition to (3.8), let

$$\int_0^\infty t \ | q(t) \ | \ dt < \log 2 \quad (3.11)$$

Then there is a real number $\theta_1 \ (0 < \theta_1 < \pi)$ such that the sector $| z | > 0, 2\pi - \theta_1 < \text{arg} \ z < 2\pi$ is resonance-free.
The condition \((3.11)\) can be related to the condition
\[
\int_0^\infty t \left| q(t) \right| dt < 0.1735
\] \((3.12)\)
which is shown in \([4\), (2.16)] to imply the absence of any spectral concentration points on the positive spectral axis \((0, \infty)\). The smaller the value of the integral in \((3.11)\), the larger \(\theta_1\) can be, and the further away from the real axis are any resonances pushed. Thus, in the case of \((3.12)\), any resonances are too far from the real axis to produce spectral concentration \([4, \text{section 3(iv)}]\).

4 Examples

We consider now some examples of \(q\) which show in more detail the type of region that arises from \((3.4)\). We keep to the case \(0 \leq \alpha \leq \pi/2\) for which \(\delta = 1\) in \((3.2)\); in the other case, \(\delta \to 1\) as \(|z| \to \infty\) and the regions are asymptotically similar for large \(|z|\).

4.1 Example \(q(x) = c(x + a)^{-\gamma}\)

where \(\gamma > 1, c\) and \(a\) are real and \(a > 0\). In Condition 2.1, we take \(q(\xi) = c(\xi + a)^{-\gamma}\) with, if \(\gamma\) is not an integer, a cut in the \(\xi\)-plane from \(-a\) to \(-\infty\) along the real axis. Thus we can take \(\theta_0 = \pi\). The integral in \((3.4)\) is now
\[
|c| \int_0^\infty |t + ae^{-i\theta}|^{-\gamma} dt = |c| \int_0^\infty (t^2 + 2at \cos \theta + a^2)^{-\gamma/2} dt = I(\theta)
\] \((4.1)\)
say. Hence \(I(\theta)\) increases from \(|c| a^{-\gamma+1}/(\gamma - 1)\) to \(\infty\) as \(\theta\) increases from 0 to \(\pi\), and \((3.4)\) becomes
\[
|z| > I(\theta)/\log 2.
\] \((4.2)\)

Thus we have a resonance-free region which lies in the lower half of the the complex plane, bounded by a curve which starts at the point \(|c| a^{-\gamma+1}/(\gamma - 1)\) on the real axis and recedes from the origin as \(\theta = -\arg z\) increases from 0 to \(\pi\). The region is of course on the side of the curve remote from the origin. When \(\gamma = 2\) in particular, the integration in \((4.1)\) can be performed and \((4.2)\) becomes
\[
|z| > \frac{|c| \theta}{a \log 2 \sin \theta}, \quad (\theta = -\arg z).
\]

Thus the boundary curve in this case is asymptotic from above to the line \(\text{im} z = -|c| \pi/(a \log 2)\) as \(\theta \to \pi\).

4.2 Example \(q(x) = c(x^n + a^n)^{-\gamma}\)

where \(n \geq 2\) is an integer, \(n\gamma > 1, c\) and \(a\) are real and \(a > 0\). This is similar to Example 3.1 but now \(\theta_0 = \pi/n\). The integrand in \((4.1)\) is replaced by \((t^{2n} + 2a^n t^n \cos n\theta + a^{2n})^{-\gamma/2}\), and \(I(\theta)\) increases to \(\infty\) as \(\theta \to \pi/n\).

In the case when \(n = 2\) and \(\gamma = 2\), we find that
\[
I(\theta) = (\pi |c| /4a^3) \sec \theta,
\]

and thus the boundary curve in this case is asymptotic from above to the line \(\text{im} z = -|c| \pi/(4a^3)\) as \(\theta \to \pi\).
and the resonance-free region \((4.2)\) is the quadrant
\[
\text{re } z > \pi \quad | c | / (4a^3 \log 2), \quad \text{im } z < 0.
\]

Also in this case, the Dirichlet condition \((3.10)\) gives
\[
2\theta / \sin 2\theta < 2(a^2 / | c |) \log 2.
\]

Thus, on the assumption that \(| c | < 2a^2 \log 2\), this being \((3.11)\), the value of \(\theta_1\) in Corollary 3.4 is the solution of
\[
2\theta_1 / \sin 2\theta_1 = 2(a^2 / | c |) \log 2.
\]

4.3 Example \(q(x) = c \{ (x - w)(x - \bar{w}) \}^{-\gamma}\)

where \(2\gamma > 1\), \(w \neq 0\) and \(0 < \arg w < \pi\). This again is similar. Here \(\theta_0 = \arg w \quad (= \phi, \text{ say }\), and the integrand in \((4.1)\) is replaced by
\[
\{(t^2 - | w |^2)^2 - 4 | w | t(t - | w |)^2 \cos \phi \cos \theta + 4 | w |^2 t^2(\cos \theta - \cos \phi)^2\}^{-\gamma/2}.
\]

Again \(I(\theta) \to \infty\) as \(\theta \to \phi\) because we approach a singularity at \(t = | w |\) in \(\{ ... \}^{-\gamma/2}\). However, \(I(\theta)\) is not necessarily monotonic unless \(\cos \phi \leq 0\).

We conclude this group of examples by noting that similar remarks apply when \(q\) is a product of terms already considered with differing values of \(a\), \(w\), and \(\gamma\) and, indeed, when \(q\) is a ratio of two such products. We give one example of this more general type for future reference in Section 5.

4.4 Example \(q(x) = c(x - 1)/(x + 1)^4\).

Here \(I(\theta)\) in \((4.1)\) and \((4.2)\) is replaced by
\[
I(\theta) = | c | \int_0^\infty | te^{i\theta} - 1 | / (t^2 + 2t \cos \theta + 1)^2 dt.
\]

Now the boundary curve of the resonance-free region \((4.2)\) starts at the point \(0.36 | c |\) on the real axis and, since \(I(\theta) \sim (\text{const.})(\sin \theta)^{-3}\) when \(\theta \to \pi\), the curve is asymptotically like \(| z |^2 = (\text{const.}) | \text{im } z |^3\) (see also Figure 1 below).

Next, we turn to examples with exponential decay which are also covered by Condition 2.1.

4.5 Example \(q(x) = 2e^{-ax} \sin x \quad (a > 0)\).

In Condition 2.1, we take
\[
q(\xi) = i \left( e^{-(a+i)\xi} - e^{-(a-i)\xi} \right).
\]

Since \(e^{-(a+i)\xi} = e^{-a \xi} e^{i \xi} \cdot (2.12)\) is certainly satisfied if
\[
\theta_0 < \tan^{-1} a.
\]
By (4.4), the left-hand side of (3.4) does not exceed

\[ 2 \int_0^\infty \exp \{ (-a \cos \theta + \sin \theta) t \} \, dt = 2(a \cos \theta - \sin \theta)^{-1}. \]

Hence (3.4) holds if \( |z| (a \cos \theta - \sin \theta) > 2/\log 2 \), or

\[ a(\text{re } z) + (\text{im } z) > 2/\log 2. \]

Since \( \theta_0 \) can be arbitrarily near to \( \tan^{-1} a \) in (4.3), we therefore have a resonance-free region in the lower half of the complex plane lying to the right of the line through the point \( 2/(a \log 2) \) on the real axis and with gradient \(-a\).

We observe that independent support for this gradient \(-a\) is provided by the quite different analytic continuation method developed in [4, Prop. 2.1]. This latter method constructs the analytic continuation of \( \Psi(z) \) into the whole of \( \text{im } z < 0 \) except for poles at the points

\[ -\frac{1}{2} (\nu + m) \quad (|\nu| \leq m, \ m = 1, 2, \ldots), \]

\( \nu \) being an integer. Thus the line through the origin with the same gradient \(-a\) delineates a pole-free region for \( \Psi(z) \) within which \( \Psi(z) \) is regular. The methods in [4] do not however lead readily to resonance-free regions.

4.6 Example \( q(x) = cx^m \exp(-x^n) \)

where \( m \) and \( n \) are positive integers. In Condition 2.1, we take

\[ q(\xi) = c\xi^m \exp(-\xi^n), \]

and (2.12) is certainly satisfied if

\[ \theta_0 < \pi/2n. \] (4.6)

Now the left-hand side of (3.4) is

\[ |c| \int_0^\infty t^m \exp (-t^n \cos n\theta) \, dt = |c| (\cos n\theta)^{-(m+1)/n} I, \]

where \( I = \int_0^\infty u^m \exp(-u^n) \, du \). Hence (3.4) holds if

\[ |z| (\cos n\theta)^{(m+1)/n} > |c| \log 2. \] (4.7)

In (4.6), \( \theta_0 \) can be arbitrarily near to \( \pi/2n \) and hence, in (4.7), we can let \( \theta \) increase from 0 to \( \pi/2n \). Thus (4.7) defines a region in the lower half plane whose boundary starts at the point \( |c| \log 2 \) on the real axis and recedes to infinity as \( \theta \to \pi/2n \).

A typical example of (4.7) is when \( m = 0 \) and \( n = 2 \), in which case the boundary is the part of the rectangular hyperbola \( X^2 - Y^2 = (|c| \log 2)^2 \) which lies in the fourth quadrant of the \((X,Y)\)-plane and \( z = X + iY \). Again the resonance-free region lies on the side of the hyperbola remote from the origin. Independent support for the nature of this boundary is provided by the findings of Siedentop [13] and Froese [10]. In [13] (where \( c = -1 \)), the first few resonances found computationally are already near to, but below, the line \( \arg z = -\pi/4 \) (see also [4, Example 6.4]) while, in [10], the resonances are shown to be asymptotically near to this same line.
5 Computational resonance-finding

We turn now to the numerical computation of resonances for explicit $q$, such as those in section 4, which satisfy Condition 2.1. One possible direct method is to compute the $u_n$ and $v_n$ $(1 \leq n \leq N)$ recursively in (2.21) and substitute the results into (2.9) and (2.10), the infinite series being truncated at $N$ with an error term. Then a zero-finding algorithm would be applied to the resulting approximation to $\Psi$ in (1.6). A similar procedure was applied successfully to the formulae for $u_n$ and $v_n$ in [4] when $q$ has exponential decay. However, in our present situation of power decay, it has proved difficult to use (2.21) when $N \geq 2$, repeated integration being involved, and in addition the error term for $N = 1$ is not small.

Instead, we have computed resonances by the method of complex scaling. We refer to Simon [16] for a discussion of this method in relation to resonances and to Agmon [1] for a recent definitive account in a very general setting. The method of complex scaling is closely associated with (2.20) and (2.9) and, in fact, our approach in section 2 provides an independent justification of the validity of this method for (1.1), as we now describe.

The transformation of (2.20) back to $\psi(\xi,z)$ via (2.1) (with $\xi$ in place of $x$) gives

$$d^2\psi/d\xi^2 + \{z^2 - q(\xi)\} \psi = 0$$

corresponding to (1.1). With $\xi$ in polar form $\xi = r \exp(i\phi)$ $(0 < \phi < \theta_0)$, we therefore have

$$e^{-2i\phi}d^2\psi/dr^2 + \{z^2 - q(re^{i\phi})\} \psi = 0 \quad (5.1)$$

and, by (2.9) and (2.19),

$$|\psi(re^{i\phi},z)| = [\exp(-r | z | \sin(\phi + \arg z))]|1 + o(1)|,$n

where $o(1)$ refers to $r \to \infty$. It follows that $\psi(re^{i\phi},z)$ is an $L^2(0,\infty)$ solution of (5.1) if

$$2\pi - \phi < \arg z < 2\pi. \quad (5.2)$$

Thus the zeros of $\Psi(z)$ in (1.4) provide the eigenvalues $z^2$ of (5.1) on $0 \leq r < \infty$ with the boundary condition

$$\psi(0,z) \cos \alpha + \psi'(0,z) \sin \alpha = 0$$

at $r = 0$. Here (5.1) is said to be obtained from (1.4) by complex scaling, the scaling factor being $e^{i\phi}$ [2, section 5] [10, section 3].

We have therefore applied a computational eigenvalue finder [11] to (5.1) with a suitable value of $\phi$. This locates eigenvalues $z^2$ and hence resonances in the sector (5.2). We focus the discussion of our computational findings now on Examples 4.1-4.4 since it is potentials with only power decay which are the main object of this paper.

We consider first

$$q(x) = c(x^2 + 1)^{-2}, \quad (5.3)$$

being the case $n = \gamma = 2$, $a = 1$ of Example 4.2. Here $\theta_0 = \pi/2$ but, if $\phi$ in (5.1) is close to $\pi/2$, the code in [11] reports unreliable results due to the sharp (but non-singular) maximum of the scaled $|q(re^{i\phi})|$ near to $r = 1$. Accordingly we have chosen $\phi = 1.5$. We have found no resonances satisfying (5.2) within the disk $|z| < 10$ when $c$ has the range of values $-1, -5, -10, -15, -20$. This is certainly consistent with the resonance-free quadrant.
re \( z > \pi \mid c \mid / (4 \log 2) \), \( \text{im } z < 0 \) in Example 4.2, but there remains the open question whether resonances occur elsewhere in \( \text{im } z < 0 \).

A similar example, but with a higher singularity located nearer to the real axis in the complex plane, is

\[
q(x) = c(x^6 + 1)^{-20}.
\]

Despite this extra feature, this example also produces no spectral concentration and no resonances. Here \( \theta_0 = \pi/6 \) and we have chosen \( \phi = 0.5 \). The values of \( c \) investigated were \(-1, -5, -15, -20, -25, -30 \). The reason for choosing \( c \) negative in (5.3) and (5.4) is to give \( q(x) \) a negative minimum, a property which in exponentially decaying examples is often associated with spectral concentration and resonances [4, Section 6].

Next we consider Example 4.4

\[
q(x) = c(x - 1)/(x + 1)^4,
\]

this time with \( c > 0 \) to give the negative minimum (at \( x = 0 \)). Here of course \( \theta_0 = \pi \) and we have chosen \( \phi = 3.0 \). There is one real point of spectral concentration when \( c = 35 \) located at \( \lambda = 0.26 \), for which \( \sqrt{\lambda} = 0.51 \), and we have tracked the corresponding resonance for a range of values down to \( c = 0.5 \). The resonance broadly recedes from the real \( z \)-axis as \( c \) decreases, and we give a selection of these findings in Table 1. For small values of \( c \), \( \text{re } z \)

| \( c \) | \( \text{rez} \) | \( \text{im } z \) |
|---|---|---|
| 35 | 0.50 | -0.06 |
| 25 | 0.65 | -0.23 |
| 15 | 0.57 | -0.44 |
| 10 | 0.42 | -0.55 |
| 4  | 0.06 | -0.68 |
| 3  | -0.05 | -0.68 |
| 2  | -0.21 | -0.66 |
| 1  | -0.45 | -0.58 |
| 0.5| -0.67 | -0.45 |

Table 1: Resonance \( z \)

appears to increase rapidly in the negative direction, but \( \text{arg } z \) becomes too close to \( \pi \) for the code [11] to produce reliable values.

In order to gain an idea of how Table 1 relates to the resonance-free region given by (4.2) and (4.3), we note that \( I(\theta) \) contains a factor \( |c| \). Accordingly, we have applied a scaling factor \( |c|^{-1} \) to both \( I(\theta) \) and the values in Table 1. The result is Figure 1, in which the diamonds denote the scaled resonances from Table 1, and the dotted curve denotes the boundary curve scaled to \( c = 1 \). Figure 1 confirms the general nature of our theoretical result (3.4).

We also mention that there are two additional similar strings of resonances: when \( c = 10 \) for example, there are resonances at \( z = -1.27 - 1.39i \) and \( z = -5.05 - 5.31i \) in addition to the value in Table 1. These additional resonances, however, lie further from the resonance-free region than the resonance-string shown in Figure 1.

Finally, we have also considered the example

\[
q(x) = c(x - 1)/(x^4 + 1)
\]
for which $\theta_0 = \pi/4$ and we have taken $\phi = 0.75$. There is one real point of spectral concentration when $c = 11$ located at $\lambda = 0.15$, for which $\sqrt{\lambda} = 0.39$. We have tracked the corresponding resonance from $z = 0.39 - 0.03i$ when $c = 11$ as far as $z = 0.73 - 0.59i$ when $c = 3.2$. For smaller $c$, the code [1] again flags unreliability, but there is a corresponding picture to Figure 1 to similarly confirm the theoretical result (3. 4).

References

[1] S. Agmon. A perturbation theory of resonances. *Comm. Pure Appl. Math.*, 51(11-12):1255–1309, 1998.

[2] A. Aslanyan and E. B. Davies. Spectral instability for some Schrödinger operators, *to appear*.

[3] B. M. Brown and M. S. P. Eastham. Spectral concentration for perturbed equations of harmonic oscillator type. *Submitted*, 1999.

[4] B. M. Brown, M. S. P. Eastham, and D. K. R. McCormack. Resonances and analytic continuation for exponentially decaying Sturm-Liouville potentials. *J. Comp. Appl. Math.*, to appear.

[5] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill: New York, 1955.

[6] C. L. Dolph, J. B. McLeod, and D. Thoe. The analytic continuation of the resolvent kernel and scattering operator associated with the Schroedinger operator. *J. Math. Anal. Appl.*, 16:311–332, 1966.

[7] M. S. P. Eastham. *The asymptotic solution of linear differential systems*. London Math. Soc. Monographs 4. Clarendon Press, Oxford, 1989.
[8] M. S. P. Eastham. On the location of spectral concentration for Sturm-Liouville problems with rapidly decaying potential. Mathematika, 45:23–36, 1998.

[9] M. S. P. Eastham. The convexity of the spectral function in Sturm-Liouville problems. Mathematika, to appear.

[10] R. Froese. Asymptotic distribution of resonances in one dimension. J. Differential Equations, 137(2):251–272, 1997.

[11] L. Greenberg and M. Marletta. Numerical solutions of nonselfadjoint Sturm-Liouville problems and related systems. submitted.

[12] M. Hitrik. Bounds on scattering poles in one dimension. Commun. Math. Phys., to appear.

[13] K. Kodaira. The eigenvalue problem for ordinary differential equations of the second order, and Heisenberg’s theory of S-matrices. Amer. J. Math., 71:921–945, 1949.

[14] H. Siedentop. On the localization of resonances. Internat. J. Quantum Chem., 31:795–821, 1987.

[15] H. Siedentop. A generalization of Rouché’s theorem with application to resonances. In Resonances (Lertorpet, 1987), pages 77–85. Springer, Berlin, 1989.

[16] B. Simon. Resonances and complex scaling: A rigorous overview. Internat. J. Quantum Chem., 14:529–542, 1978.

[17] E. C. Titchmarsh. Eigenfunction Expansions Associated with Second Order Differential Equations, Part I. (2nd ed). Clarendon Press, Oxford, 1962.

[18] H. Weyl. Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungsn willkürlicher Funktionen. Math. Annln., 68:220–269, 1910.