Abstract

In this paper we prove that if $X$ is a Banach space, then for every lower semicontinuous bounded below function $f$, there exists a $(\varphi_1, \varphi_2)$-convex function $g$, with arbitrarily small norm, such that $f + g$ attains its strong minimum on $X$. This result extends some of the well-known variational principles as that of Ekeland [18], that of Borwein-Preiss [6] and that of Deville-Godefroy-Zizler [14, 15].

Keywords: $(\varphi_1, \varphi_2)$-convex function, $(\varphi_1, \varphi_2)$-variational principle, Ekeland’s variational principle, smooth variational principle.

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1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space. Let $f$ be a real-valued function defined on $X$, lower semicontinuous and bounded below. Let $P$ be a class of functions in $X$ which serves as a source of possible perturbations for $f$. By a variational principle we mean an assertion ensuring the existence of at least one perturbation $g$ from $P$ such that $f + g$ attains its minimum on $X$.

The first variational principle, based on the Bishop-Phelps lemma [3, 27], was established by Ekeland [18]. It says that $P$ is a family of suitable translations of the norm.

If $g$ is required to be smooth, then we speak about the smooth variational principle. The first result of this type was shown by Stegall [31, 27], where $P$ is the elements of
the dual space $X^\ast$. He proved that if $X$ has the Radon-Nikodym property in particular, if $X$ is reflexive, then one can take for $g$ even a linear functional, with arbitrarily small norm. In [17], Deville-Maaden shown under the same hypotheses as Stegall that $X$ has the Radon-Nikodym property and the function $f$ is lower semicontinuous and superlinear with the set $P$ consisting of radial smooth functions. However, this principle does not cover some important Banach spaces. For example the space $c_0$ does not have the Radon-Nikodym property while it, in fact, admits a smooth norm [5]. In this direction Borwein-Preiss [6] proved a smooth variational principle imposing no additional conditions on the space, except, the presence of some equivalent smooth norm with the set $P$ being the family of smooth combinations of the norm in $X$. Haydon’s work [23], shows that there exists a Banach spaces with smooth bump function without an equivalent smooth norm (a function $b$ is bump if it has a non empty and bounded support). So, the variational principle of Borwein-Preiss is not applicable in these spaces. So that, Deville et al [14, 15] extended the Borwein-Preiss variational principle to spaces with smooth bump function, with $P$ equal to the family of Lipschitz smooth functions.

In an analytical approach we can often associate a geometrical approach to complete study of which or stimulates the analytical approach. From this perspective Browder [8] gives a geometrical result which bears at present the name of the Drop Theorem (see also [10]). Penot in [26, 21] showed that the drop theorem is a geometrical version of the Ekeland’s variational principle. After this, Maaden in [25, 22] introduced and studied the notion of the smooth drop which can be seen as a geometrical version of the smooth variational principle of Borwein-Preiss.

Those variational principles are a tools that have been very important in nonlinear analysis, in that they enjoyed a big deal of applications from the geometry of Banach spaces [3, 4, 7] to the optimization theory [18, 19, 30] and of generalized differential and sub-differential calculus [1, 2, 6, 11, 12, 13, 26], calculus of variations [9, 18] up to the nonlinear semi-groups theory [7, 18] and the viscosity solutions of Hamilton-Jacobi equations [12, 13, 15].

In [28, 29], Pini et al defined the notion of $(\varphi_1, \varphi_2)$-convex functions. They say that a real valued function $f$ defined on a non empty subset $D$ of $\mathbb{R}^n$ is $(\varphi_1, \varphi_2)$-convex if $f(\varphi_1(x, y, \lambda)) \leq \varphi_2(x, y, \lambda, f)$ for all $x, y \in D$ and for all $\lambda \in [0, 1]$. Where $\varphi_1$ is a
function from $D \times D \times [0,1]$ in $\mathbb{R}^n$ and $\varphi_2$ is a function from $D \times D \times [0,1] \times F$ in $\mathbb{R}$, with $F$ is a given vector space of real valued functions defined on the set $D$. In this paper we shall use the same definition of $(\varphi_1, \varphi_2)$-convex functions as above with using any Banach spaces instead of $\mathbb{R}^n$. In this way, we prove that under suitable choices of the functions $\varphi_1$ and $\varphi_2$ a new variational principle for the set of $(\varphi_1, \varphi_2)$-convex functions (see Theorem 3.1). This $(\varphi_1, \varphi_2)$-variational principle is providing a unified framework to deduce Ekeland’s, Borwein-Preiss’s and Deville’s variational principles.

2 Auxiliaries results

In this section we shall give some definitions and establish some auxiliaries results which we shall use to prove our main result in this paper.

Let $(X, \|\|)$ be a Banach space. For a continuous function $f : X \rightarrow \mathbb{R}$ we define

$$
\mu (f) = \sum_{n=1}^{\infty} \frac{\|f\|_n}{2^n},
$$

where

$$
\|f\|_n = \sup \{|f(x)|; x \in X, \|x\| \leq n\}.
$$

Let $M$ be the set of all continuous functions $f$ such that $\mu (f) < \infty$. It is routine to check that $(M, \mu)$ is a Banach space.

Let $\varphi_1 : X \times X \times [0,1] \rightarrow X$ and $\varphi_2 : X \times X \times [0,1] \times F \rightarrow \mathbb{R}$, two functions where $F$ is a given set of real functions on $X$. Define,

**Definition 2.1** A function $f : X \rightarrow \mathbb{R}$ is said to be $(\varphi_1, \varphi_2)$-convex if

$$
f (\varphi_1 (x, y, \lambda)) \leq \varphi_2 (x, y, \lambda, f), \forall x, y \in X, \forall \lambda \in [0,1].
$$

Remarking that under suitable assumptions on $\varphi_1$ and/or $\varphi_2$, the class of $(\varphi_1, \varphi_2)$-convex functions is a convex cone. For example:

1) If $\varphi_2$ is super-linear with respect to $f \in F$ (that $\varphi_2$ is super-additive and positively homogeneous), then the class of $(\varphi_1, \varphi_2)$-convex functions is a convex cone.

Indeed, let $f, g$ are two $(\varphi_1, \varphi_2)$-convex functions and $\alpha > 0$. Then, for $x, y \in X$ and $\lambda \in [0,1]$ we have

$$
(f + g) (\varphi_1 (x, y, \lambda)) \leq \varphi_2 (x, y, \lambda, f) + \varphi_2 (x, y, \lambda, g) \\
\leq \varphi_2 (x, y, \lambda, f + g)
$$

3
and

\[(\alpha f)(\varphi_1(x,y,\lambda)) = \alpha(f(\varphi_1(x,y,\lambda)))\]
\[\leq \alpha \varphi_2(x,y,\lambda, f) \]
\[= \varphi_2(x,y,\lambda, \alpha f).
\]

2) If \(\varphi_2(x,y,\lambda, f) = C((1-\lambda)f(x) + \lambda f(y))\) for some \(C > 0\), the set of \((\varphi_1, \varphi_2)\) - convex functions is a convex cone.

In all the sequel, we define the following sets:

\[\Phi = \{f \in M : f \text{ is } (\varphi_1, \varphi_2) \text{ - convex and } f \geq 0\},\]
\[F = \{f \in \Phi : f(x) \to +\infty \text{ as } ||x|| \to +\infty\}.
\]

The metric \(\rho\) on \(\Phi\) is defined as:

\[\rho(f,g) = \mu(f-g) = \sum_{n \geq 1} \frac{\|f-g\|_n}{2^n}\]
for all \(f, g \in \Phi\),

and it is easy to show that \((\Phi, \rho)\) is a complete metric space.

Throughout this paper, the functions \(\varphi_1\) and \(\varphi_2\) satisfies the following assumptions:

\((P_1)\) \(\varphi_1(x,x,0) = x; \forall x \in X\);
\((P_2)\) \(\varphi_1(x,y,\lambda) + \varphi_1(z,z,0) = \varphi_1(x+z,y+z,\lambda); \forall x,y,z \in X, \forall \lambda \in [0,1]\);
\((P_3)\) \(\exists C \geq 1, \text{ such that } \varphi_2(\lambda x,\lambda x,0,h) \leq C[1-\lambda) h(0) + \lambda h(x)]; \forall x \in X, \forall \lambda \in [0,1], \forall h \in \Phi;\)
\((P_4)\) For \(x_0 \in X, \varphi_2(x-x_0, y-x_0, \lambda, h) \leq \varphi_2(x,y,\lambda, h(. - x_0)); \forall x,y \in X, \forall \lambda \in [0,1]; \forall h \in \Phi;\)
\((P_5)\) The class of \((\varphi_1, \varphi_2)\) - convex functions is a convex cone.

We will also assume that \(\varphi_1\) is continuous with respect to \(\lambda\).

We present now three preliminaries lemmas, which are useful for the proof of our principal result of this paper. In the first, we use \((P_1)\) and \((P_3)\) to prove the following:

**Lemma 2.2** Let \(h \in \Phi\) and let \(y = \lambda x, \lambda > 1\). Then, \(h(y) - h(0) \geq \frac{\lambda}{C}(h(x) - Ch(0)).\)
Proof:
Let \( \mu = 1/\lambda \). Then \( x = \mu y \). By using \((P_1)\) and \((P_3)\) we obtain that:

\[
\begin{align*}
h(x) &= h(\mu y) \\
    &= h(\varphi_1(\mu y, \mu y, 0)) \\
    &\leq \varphi_2(\mu y, \mu y, 0, h) \\
    &\leq C((1 - \mu)h(0) + \mu h(y)).
\end{align*}
\]

Consequently, we get

\[
h(x) - Ch(0) \leq C\mu(h(y) - h(0)).
\]

Since \( c > 0 \) and \( \mu > 0 \), we deduce that

\[
h(y) - h(0) \geq \frac{1}{C\mu}(h(x) - Ch(0)) = \frac{\lambda}{C}(h(x) - Ch(0))
\]

and the proof is complete. \( \blacksquare \)

We have now all tools to confirm that \((F, \rho)\) is a Baire space. For this, it suffices to show that

**Lemma 2.3** \((F, \rho)\) is open in \( \Phi \).

Proof:
Let \( f \in F \). Let \( N > Cf(0) \) (where \( C \) is given by \((P_3)\)) and let \( \varepsilon > 0 \) be such that

\[
0 < \varepsilon < \frac{N - Cf(0)}{2C + 1}.
\]

Since \( f \in F; f \to +\infty \) as \( \|x\| \to +\infty \); there exists \( n \in \mathbb{N} \) such that

\[
f(x) > N \text{ whenever } \|x\| \geq n.
\]

Let \( g \in \Phi \) such that \( \rho(f, g) < \frac{\varepsilon}{2^n} \). Then

\[
\frac{\|f - g\|_n}{2^n} < \frac{\varepsilon}{2^n}.
\]

Hence,

\[
|f(x) - g(x)| < \varepsilon \quad \text{whenever } \|x\| \leq n.
\]
In particular, we have

\[ |f(0) - g(0)| < \varepsilon. \]  \hspace{1cm} (4)

Combining (2) and (3) we obtain for \( \|x\| = n \),

\[ g(x) > N - \varepsilon. \]  \hspace{1cm} (5)

On the first, for \( y \in X \) such that \( \|y\| \to +\infty \), there exist \( x \in X \) with \( \|x\| = n \) and \( \lambda > 1 \) such that \( y = \lambda x \), and we have \( \lambda \to +\infty \). Therefore, combining Lemma 2.2, (5), (4) and (1) we obtain

\[
g(y) - g(0) \geq \frac{\lambda}{C} (g(x) - Cg(0)) \\
> \frac{\lambda}{C} (N - \varepsilon - Cg(0)) \\
\geq \frac{\lambda}{C} (N - \varepsilon - Cf(0) - C\varepsilon) \\
> \lambda \varepsilon,
\]

which implies that

\[ g(y) > \lambda \varepsilon + g(0) \geq \lambda \varepsilon > 0. \]

On the other hand, we know that \( \lambda \to +\infty \) as \( \|y\| \to +\infty \). So, we get that \( g(y) \to +\infty \) as \( \|y\| \to +\infty \). So that, \( g \in F \) and \( F \) is open. \( \blacksquare \)

Next, by using \( (P_1), (P_2) \) and \( (P_4) \) we obtain the following:

**Lemma 2.4** Let \( \theta \) be a \((\varphi_1, \varphi_2)\)-convex function and let \( h(x) = \theta(x - x_0) \). Then, \( h \) is a \((\varphi_1, \varphi_2)\)-convex function.

**Proof:**

Let \( x, y \in X \) and \( \lambda \in [0, 1] \). By using \( (P_1), (P_2) \) and \( (P_4) \) we get

\[
h(\varphi_1(x, y, \lambda)) = \theta(\varphi_1(x, y, \lambda) - x_0) \\
= \theta(\varphi_1(x, y, \lambda) + \varphi_1(-x_0, -x_0, 0)) \\
= \theta(\varphi_1(x - x_0, y - x_0, \lambda)) \\
\leq \varphi_2(x - x_0, y - x_0, \lambda, \theta) \\
\leq \varphi_2(x, y, \lambda, \theta(x_0)) \\
= \varphi_2(x, y, \lambda, h),
\]

which shows that \( h \) is a \((\varphi_1, \varphi_2)\)-convex function. \( \blacksquare \)
Corollary 2.5 Let $\theta$ be a $(\varphi_1, \varphi_2)$–convex function in $F$ then the function $h(x) = \theta(x - x_0)$ is in $F$.

3 The main result

In this section we shall establish a $(\varphi_1, \varphi_2)$–variational principle. We show that the set $P$; which is a source of perturbation for $f$; is a class of $(\varphi_1, \varphi_2)$–convex functions. Furthermore we can take them of $C^\infty$ in smooth Banach spaces.

In the mathematical field of topology, a $G_\delta$ set is a subset of a topological space that is a countable intersection of open sets. In a complete metric space, a countable union of nowhere dense sets is said to be meagre; the complement of such a set is a residual set.

An element $y$ of a Banach space $X$ is said a strong minimum for a real function $f$ defined on the space $X$, if $f(y)$ is the infimum of $f$ and any minimizing sequence for $f$ converges to $y$.

The aim result in this paper is the following variational principle:

**Theorem 3.1** Let $X$ be a Banach space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function bounded from below. Let $Y$ be a subset of $F$ such that:

i) the metric $\rho_Y$ in $Y$ is such that $\rho_Y (f, g) = \mu_Y (f - g) \geq \mu (f - g)$, for all $f, g \in Y$.

ii) $(Y, \rho_Y)$ is a Baire space.

iii) there exists $\theta \in Y$ such that $\mu_Y (\theta) < +\infty, \theta (0) = 0$, there is $k \in ]0, 1[$ such that for every $\|x\| \geq k$ we have $\theta (x) \geq k^2$ and $\mu_Y (\theta (. - x_0)) \leq \mu_Y (\theta) + \|\theta\|_{\|x_0\|}$.

Then the set

$$\{g \in Y : f + g \text{ attains its strong minimum on } X\}$$

is residual in $Y$.

Next, we shall show that Theorem 3.1 is providing a unified framework to deduce Ekeland’s variational principle [18], Borwein-Preiss’s [6] variational principle and Deville-Godefroy-Zizler’s Varitional principle [15].

**Application 1.** As a first application we get the Ekeland’s variational principle [18].
Let \((X, \|\|)\) be a Banach space. Assume that \(\varphi_1(x, y, \lambda) = \lambda x + (1 - \lambda) y\) and \(\varphi_2(x, y, \lambda, f) = \lambda f(x) + (1 - \lambda) f(y)\). Then \(\varphi_1\) and \(\varphi_2\) satisfies \((P_1), (P_2), (P_3)\) and \((P_4)\). Let

\[ Y = \{ f : X \to \mathbb{R} : f \text{ convex, Lipschitz, } \geq 0, f \to +\infty \text{ as } \|x\| \to +\infty \}. \]

We define on \(Y\) the metric \(\rho_Y\) such that for \(f, g \in Y\),

\[ \rho_Y(f, g) = \mu_Y (f - g) = \sum_{n \geq 1} \frac{|f - g|_n}{2^n} + \sup \left\{ \frac{|(f - g)(x) - (f - g)(y)|}{\|x - y\|} ; x \neq y \right\}. \]

It is clear that \((Y, \rho_Y)\) satisfies \((P_5)\) and the conditions \((i)\) and \((ii)\) of Theorem 3.1. Also, the function \(\theta = \|x\|\) satisfies the assertion \((iii)\) of Theorem 3.1. Consequently we have the following:

**Corollary 3.2** Let \((X, \|\|)\) be a Banach space, consider a lower semi-continuous bounded below function \(f : X \to \mathbb{R} \cup \{+\infty\}\). Then for each \(\varepsilon > 0\), there exists \(x_0 \in X\) such that

\[ f(x) + \varepsilon \|x - x_0\| \geq f(x_0). \]

**Proof:**

From Theorem 3.1, for each \(\varepsilon > 0\), there exits \(g \in Y\) such that \(\mu_Y (g) < \varepsilon\) and \(f + g\) attains a strong minimum at \(x_0\). Therefore, for all \(x \in X\),

\[ f(x) + g(x) \geq f(x_0) + g(x_0) \quad \text{and} \quad \sum_{n \geq 1} \frac{|g|_n}{2^n} + \sup \left\{ \frac{|g(x) - g(y)|}{\|x - y\|} ; x \neq y \right\} < \varepsilon, \]

which implies that

\[ f(x) \geq f(x_0) + g(x_0) - g(x) \]
\[ \geq f(x_0) - \varepsilon \|x - x_0\|. \]

**Application 2.** Let \((X, \|\|)\) be a Banach space with smooth norm. Assume that \(\varphi_1(x, y, \lambda) = \lambda x + (1 - \lambda) y\) and \(\varphi_2(x, y, \lambda, f) = \lambda f(x) + (1 - \lambda) f(y)\). Then \(\varphi_1\) and \(\varphi_2\) satisfies \((P_1), (P_2), (P_3)\), and \((P_4)\). Let

\[ Y = \{ f \text{ is } C^1\text{-smooth, convex, } \geq 0 \text{ and } f \to +\infty \text{ as } \|x\| \to +\infty \}. \]
We define the metric $\rho_Y$ in $Y$ by:

$$
\rho_Y(f, g) = \mu_Y(f - g) = \sum_{n \geq 1} \frac{||f - g||_n}{2^n} + ||(f - g)'||_\infty \text{ for all } f, g \in Y
$$

where $||f'||_\infty := \sup_{||x|| \leq 1} ||f'(x)||_X$, and the space $(Y, \rho_Y)$ satisfies (i) and (ii) of Theorem 3.1 and so also $(P_5)$.

Let

$$
h : [0, +\infty[ \longrightarrow [0, +\infty[ \text{ and } t \mapsto \begin{cases} t^2 & \text{if } 0 \leq t \leq 1 \\ 2t - 1 & \text{if } t > 1. \end{cases}
$$

The function $\theta(x) = h(||x||) \in Y$ satisfies the assertion (iii) of Theorem 3.1.

Therefore we have the Borwein-Preiss's variational principle [6, 27]:

**Corollary 3.3** Let $(X, ||.||)$ be a Banach space with a smooth norm and consider a lower semi-continuous function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ bounded from below. Then the set

$$
\{g \in Y : f + g \text{ attains its strong minimum on } X\}
$$

is residual in $Y$.

**Application 3.** Let $X$ be a Banach space admitting Lipschitz $C^1$–smooth bump function. According to a construction of Leduc [24], there exists a Lipschitz function $d : X \longrightarrow \mathbb{R}$ which is $C^1$–smooth on $X \setminus \{0\}$ and satisfies:

i) $d(\lambda x) = \lambda d(x)$ for all $\lambda > 0$ and for all $x \in X$,

ii) there exists $C \geq 1$ such that $||x|| \leq d(x) \leq C ||x||$ for all $x \in X$.

Moreover the function $d^2$ is $C^1$–smooth on all the space $X$.

Let $\varphi_1(x, y, \lambda) = \lambda x + (1 - \lambda) y$ and $\varphi_2(x, y, \lambda, f) = C^2[\lambda f(x) + (1 - \lambda) f(y)]$. Then $\varphi_1$ and $\varphi_2$ satisfies $(P_1), (P_2), (P_3)$ and $(P_4)$. Let $\theta(x) = d^2(x)$.

We have

$$
d^2(\lambda x + (1 - \lambda) y) \leq C^2 \|\lambda x + (1 - \lambda) y\|^2.
$$

Since the function $||.||^2$ is convex we deduce that

$$
d^2(\lambda x + (1 - \lambda) y) \leq C^2 \left(\lambda d^2(x) + (1 - \lambda) d^2(y)\right).
$$
That is the function $d^2$ is a $(\varphi_1, \varphi_2)$-convex function.

Let

$$Y = \left\{ f \ a (\varphi_1, \varphi_2) - \text{convex}, C^1 - \text{Lipschitz}, \geq 0 \ and \ f \to +\infty \ as \ ||x|| \to +\infty \right\}$$

and so the set $Y$ satisfies $(P_5)$.

The metric $\rho_Y$ on $Y$ is such that, for $f, g \in Y$

$$\rho_Y(f, g) = \mu_Y(f - g) = \sum_{n \geq 1} \frac{||f - g||_n}{2^n} + \sum_{n \geq 1} \frac{||(f - g)'||_n}{2^n}$$

where $\|f'\|_n = \sup_{||x|| \leq n} \|f'(x)||_{X^*}$.

In the other hand, let $\theta(x) = d^2(x)$. So that,

i) $\theta(0) = 0$

ii) $\mu_Y(\theta) < \infty$

iii) let $0 < k < 1$. Hence, for all $x \in X$ such that $||x|| \geq k$ we have $d^2(x) \geq ||x||^2 \geq k^2$.

Therefore the function $\theta \in Y$ and satisfies $(iii)$ of Theorem 3.1.

Thus we have the following variational principle (for unbounded functions) of Deville-Godefroy-Zizler [14, 15, 16, 20]:

**Corollary 3.4** Let $(X, ||.||)$ be a Banach space admitting a $C^1$-Lipschitz bump function and consider a lower semi-continuous bounded below function $f : X \to \mathbb{R} \cup \{+\infty\}$. Then the set

$$\{ g \in Y : f + g \text{ attains its strong minimum on } X \}$$

is residual in $Y$.

Now, we are ready to give the proof of Theorem 3.1.

**Proof of Theorem 3.1**

Following the method of [15, 20], for $n \in \mathbb{N} \setminus \{0\}$, we let

$$G_n = \{ g \in Y : \exists x_0 \in X, (f + g)(x_0) < \inf \{(f + g)(x) : ||x - x_0|| \geq 1/n\} \}.$$
Claim 1. We claim that $G_n$ is open for each $n$. Indeed, let $n \in \mathbb{N}$ and $g \in G_n$. So that there is $x_0$ in $X$ such that

$$(f + g)(x_0) < \inf \{(f + g)(x) : \|x - x_0\| \geq 1/n\}.$$ 

Let $0 < \varepsilon < 1$ such that

$$(f + g)(x_0) + 2\varepsilon < \inf \{(f + g)(x) : \|x - x_0\| \geq 1/n\}.$$  \hspace{1cm} (1)

Let $A = C(f + g)(x_0) + C(g(0) - \inf (f)) + (2C + 3)\varepsilon$, where $C$ is given by $(P_3)$. Since $g \in Y$, $g$ goes to $+\infty$ as $\|x\|$ goes to $+\infty$. This means that, there is $k$ in $\mathbb{N}$ such that $k > \|x_0\|$ and $g(x) > A$ whenever $\|x\| \geq k$. This is equivalent to say that

$$g(x) > C(f + g)(x_0) + C(g(0) - \inf (f)) + (2C + 3)\varepsilon \quad \text{whenever} \quad \|x\| \geq k. \quad (2)$$

Let $h \in Y$ such that $\rho_Y(h, g) < \varepsilon/2^k$. We have

$$\sum_{n \geq 1} \frac{\|h - \rho_n\|_n}{2^n} \leq \rho_Y(h, g) = \mu_Y(h - g) < \frac{\varepsilon}{2^k}.$$ 

Thus

$$\frac{\|h - g\|_k}{2^k} < \frac{\varepsilon}{2^k}.$$ 

So that

$$|h(x) - g(x)| < \varepsilon \quad \text{whenever} \quad \|x\| \leq k, \quad (3)$$

in particular

$$|h(x_0) - g(x_0)| < \varepsilon. \quad (4)$$

Combining (2) with (3) we obtain that

$$h(x) > C(f + g)(x_0) + C(g(0) - \inf (f)) + (2C + 2)\varepsilon > 0 \quad \text{whenever} \quad \|x\| = k.$$ 

Since $C \geq 1$ and $h \geq 0$, we deduce that for $\|x\| = k$

$$h(x) \geq \frac{h(x)}{C} > (f + g)(x_0) + (g(0) - \inf (f)) + (2 + (2/C))\varepsilon. \quad (5)$$

In the first hand, let $y \in X$ such that $\|y\| > k$. Then, there exist $\lambda > 1$ and $x \in X$ with $\|x\| = k$, such that $y = \lambda x$. By using Lemma 2.2, we deduce that

$$h(y) - h(0) \geq \frac{\lambda}{C} (h(x) - C h(0)) \geq \frac{1}{C} (h(x) - C h(0)) = \frac{h(x)}{C} - h(0).$$
Combining this with (5) we show for \( \|y\| \geq k \) that,

\[
h(y) - h(0) > (f + g)(x_0) + g(0) - \inf f + \left(2 + \frac{2}{C}\right)\varepsilon - h(0).
\]

(6)

Combining the fact that \( h \geq 0 \), (6), (3) and (4) we obtain for all \( x \in X \) such that \( \|x\| \geq k \):

\[
(f + h)(x) \geq \inf (f) + h(x)
\]
\[
\geq \inf (f) + h(x) - h(0)
\]
\[
> \inf (f + (f + g)(x_0) + g(0) - \inf f + \left(2 + \frac{2}{C}\right)\varepsilon - h(0)
\]
\[
> (f + g)(x_0) + \left(1 + \frac{2}{C}\right)\varepsilon
\]
\[
> (f + h)(x_0) + \frac{2}{C}\varepsilon
\]
\[
> (f + h)(x_0).
\]

Therefore for all \( x \in X \) such that \( \|x\| \geq k \), we have

\[
(f + h)(x) > (f + h)(x_0).
\]

In other hand, if \( \|x\| \leq k \) and \( \|x - x_0\| \geq 1/n \), and combining (4), (1) and (3) we obtain that

\[
(f + h)(x_0) < (f + g)(x_0) + \varepsilon
\]
\[
\leq \inf \{(f + g)(x) : \|x - x_0\| \geq 1/n\} - 2\varepsilon + \varepsilon
\]
\[
\leq (f + g)(x) - \varepsilon
\]
\[
< (f + h)(x).
\]

Then for all \( x \) such that \( \|x - x_0\| \geq 1/n \) we have

\[
(f + h)(x_0) < (f + h)(x).
\]

Hence \( h \in G_n \) and \( G_n \) is open.

Claim 2. We confirm that \( G_n \) is dense in \( Y \). Indeed, let \( g \in Y \) and \( 0 < \varepsilon < 1 \). Let \( c > 0 \) be such that

\[
(f + g)(x) > \inf (f + g) + 1 \text{ whenever } \|x\| > c.
\]
Let $1 > \delta > 0$ be such that $\delta (\mu_Y (\theta) + \|\theta\|_c) < \varepsilon$. Let $x_0 \in X$ be such that
\[
(f + g) (x_0) < \inf (f + g) + \frac{\delta}{n^2}.
\]
(6)
Since $\frac{\delta}{n^2} < 1$, we deduce that
\[
\|x_0\| \leq c.
\]
(7)
Let $h (x) = \delta \theta (x - x_0)$. Now Corollary 2.5 ensure that $h$ is a $(\varphi_1, \varphi_2)$ - convex function in $F$.

From the hypothesis $(iii)$ of Theorem 3.1 and (7), we get
\[
\rho_Y (h, 0) = \mu_Y (h) = \delta \mu_Y (\theta (\cdot - x_0)) \leq \delta \mu_Y (\theta) + \delta \|\theta\|\|x_0\| \leq \delta (\mu_Y (\theta) + \|\theta\|_c) < \varepsilon.
\]
Now if $\|x - x_0\| \geq 1/n$, and by $(iii)$ of Theorem 3.1 we deduce that,
\[
h (x) = \delta \theta (x - x_0) \geq \frac{\delta}{n^2}.
\]
By using (6), we get
\[
\inf \{f + g + h : \|x - x_0\| \geq 1/n\} \geq \inf \{f + g : \|x - x_0\| \geq 1/n\} + \frac{\delta}{n^2}
\]
\[
\geq \inf \{f + g\} + \frac{\delta}{n^2}
\]
\[
> (f + g) (x_0) - \frac{\delta}{n^2} + \frac{\delta}{n^2}.
\]
Moreover $h (x_0) = \delta \theta (0) = 0$, then
\[
\inf \{f + g + h : \|x - x_0\| \geq 1/n\} > (f + g) (x_0) = (f + g + h) (x_0).
\]
Thus $(g + h) \in G_n$ and $G_n$ is a dense subset in $Y$.

Therefore the set $\bigcap_{n \geq 1} G_n$ is residual in $Y$. Following the proof of [15], we can show $f + g$ attains its strong minimum for each $g \in \bigcap_{n \geq 1} G_n$. $\blacksquare$

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