Implementing conventional and unconventional nonadiabatic geometric quantum gates via SU(2) transformation

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A simple yet versatile protocol to inversely engineer time-dependent Hamiltonian is proposed. By utilizing SU(2) transformation, a given speedup goal of gate operation can be achieved with larger freedom to select the control parameters. As an application, this protocol is adopted to realize conventional and unconventional nonadiabatic geometric quantum gates with any desired evolution paths by controlling the pulses in the diamond nitrogen-vacancy (NV) center system. We show that the designed gate can realize geometric quantum computation with a more economical evolution time that decreases the influence of noise on gate operation.

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I. INTRODUCTION

Quantum computation has been shown to be more efficient than classical one in solving some problems, such as factoring large integers, searching big database and finding optimal solutions by quantum annealing [1]. However, it still faces great challenges both in theory and applications, especially, due to the inevitable decoherence or noise introduced by the interaction with environment, which destroys the coherence of the state which should be maintained in the parallel computation.

In order to overcome this challenge, geometric quantum computation (GQC) has been proposed. As geometric phase is solely related to the structure of an evolution path and independent of the middle details, quantum gate designed based on geometric phase is immune to local disturbances during the evolution [2-5]. However, the first scheme of geometric gate based on adiabatic Abelian and non-Abelian geometric phase [6,7] takes a slowly cyclic evolution. The lengthy gate operation time of adiabatic holonomic quantum computation is still vulnerable to the environment-induced decoherence. To relax the limit of evolution speed, non-adiabatic holonomic quantum computation (NHQC) based on nonadiabatic non-Abelian geometric phase was proposed by constructing driving Hamiltonians with time-independent eigenstates [8-22]. It has been proved that the implementation of high speed gates for quantum computation is plausible [23].

As the traditional geometric quantum computation should undergo cyclic evolutions and immune dynamical phases to keep its gauge invariance, the evolution paths were mainly restricted to special forms such as the former multiple loops and the newly orange-slice-shaped loops [24-27]. The multiple-loop scheme adopts several closed loops to cancel the dynamical phases and the orange-slice-shaped-loop scheme takes the geodesic path on Bloch sphere to eliminate the dynamical phases during the evolution. Although the paths in the orange-slice-shaped-loop scheme are generally shorter than those in the multiple-loop scheme, they still take longer times to realize geometric gates beyond decoherence time and do not integrate well with the experiments. Hence, how to optimize the evolution paths for realizing nonadiabatic geometric quantum computation becomes a topic with great interests.

In this paper, a novel optimization scheme for GQC is proposed in two-level system. By using universal SU(2) transformation to design the evolutionary path of the system, a given speedup goal of the system, a given speedup goal can be achieved with large freedom to select the control parameters. Due to the flexibility of this approach, the conventional and unconventional nonadiabatic geometric gates with any desired evolution paths can be designed. This approach is much more powerful to find better evolutionary paths and can be well integrated with the experiments. As an demonstration, we adopt NV center system to illustrate this approach. The nonadiabatic geometric gate can be realized by manipulating solid-state spins in NV center by appropriately controlling the amplitude, phase and frequency of the pulsing fields. Compared with the previous schemes, a faster evolution speed with a higher fidelity is achieved, in particular in matching the parameters with the experiment, which usually, is an important problem for different quantum computation platforms.

II. GENERAL FRAMEWORK BY TRANSFORMATION METHOD

Generally, a two-level (one-qubit) quantum system can be described by (ℏ = 1)

\[ \hat{H}_0(t) = h_x(t)\hat{\sigma}_x + h_y(t)\hat{\sigma}_y + h_z(t)\hat{\sigma}_z, \]

where \( h_k(t)(k = x, y, z) \) are arbitrary real functions of time to be designed, and \( \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \) are Pauli operators. The time evolution of the system state is given by

\[ |\Psi(t)\rangle = \hat{U}_0(t)|\Psi(0)\rangle, \]

where the initial state \( |\Psi(0)\rangle \) is reset by initialization and the evolution operator \( \hat{U}_0(t) \) can be properly designed. In order to engineer feasible Hamiltonians \( \hat{H}_0(t) \) that give desired dynamics, we adopt unitary transformations of \( \hat{R}(t) \) parameterized by [28]

\[ \hat{R}(t) \equiv R(\theta, \varphi) = \begin{bmatrix} \cos \frac{\theta(t)}{2} & -e^{-i\varphi(t)} \sin \frac{\theta(t)}{2} \\ e^{i\varphi(t)} \sin \frac{\theta(t)}{2} & \cos \frac{\theta(t)}{2} \end{bmatrix}, \]

\[ e^{-i\varphi(t)} \sin \frac{\theta(t)}{2} & \cos \frac{\theta(t)}{2} \end{bmatrix}, \]

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which enables a representation transformation of $|\psi(t)\rangle = R^\dagger |\Psi(t)\rangle$. The corresponding Schrödinger equation becomes
\[ \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}_R(t) |\psi(t)\rangle, \] and the transformed Hamiltonian is
\[ \hat{H}_R(t) = R^\dagger \hat{H}_0(t) R + i \partial_t R^\dagger R, \]
where $R^\dagger \hat{H}_0(t) R$ is often called dynamical part which is related with dynamical phase and $i \partial_t R^\dagger R$ is the non-Abelian part which brings geometric phase \([30]\). Then the time-evolution operator in $R$-representation is
\[ \hat{U}_R(t) = \mathcal{T} \exp \left[ -i \int_0^t \hat{H}_R(t') dt' \right]. \]
In order to remove the time ordering operator $\mathcal{T}$ to calculate $\hat{U}_R(t)$, we can design a diagonal form of
\[ \hat{H}_R(t) = F(t) \hat{\sigma}_z \]
by opportunely choosing the transformation parameters $\theta$ and $\phi$. In this case, the time evolution operator becomes
\[ \hat{U}_R(t) = e^{-i \int_0^t H_R(t') dt'} = e^{-i \gamma(t) \hat{\sigma}_z}, \]
where $\gamma(t) = \int_0^t F(t') dt'$. The time-evolution operator in the former representation can be obtained by
\[ \hat{U}_0(t) = R(t) \hat{U}_R(t) R^\dagger (0), \]
In order to realize $\hat{H}_R(t)$ to retain only the diagonal part, a good choice is to make the non-diagonal parts of $R^\dagger \hat{H}_0(t) R$ and $i \partial_t R^\dagger R$ cancel out at any time. Then the diagonal matrices $K(t) = \text{diag}[R^\dagger \hat{H}_0(t) R]$ and $A(t) = \text{diag}[i \partial_t R^\dagger R]$ will safely lead to Eq.\([10]\). In order to confine the control freedom for a reliable design, we consider a special scheme that $K(t)$ is proportional to $A(t)$, i.e., $K(t) = \eta A(t)$, which gives $\gamma(t) = \int_0^t F(t') dt' = \int_0^t (1 + \eta) A(t') dt'$,
\[ \gamma(t) = \int_0^t F(t') dt' = \int_0^t (1 + \eta) A(t') dt', \]
where $\eta$ is a new introduced constant parameter and $\eta \neq -1$ in order to avoid a trivial case \([32]\). The physical meaning of $\eta$ can be seen if we set $\eta = 0$, the rotation phase, $\gamma(t) = \gamma_0(t) \equiv \int_0^t A(t') dt'$, reduces to a pure geometric phase without any dynamical component. In order to discriminate them, we can denote the total phase $\gamma(t)$ as $\gamma(t) = (1 + \eta) \gamma_0(t)$.

Now we consider the quantum geometric gates in a cyclic evolution for recycling transformation $R(\tau) = R(0)$, i.e., $R(\theta_\tau, \phi_\tau) = R(\theta_0, \phi_0)$, where the parameters are labeled by $\theta(t) \equiv \theta_0$, $\phi(t) \equiv \phi_0$ for convenience, then the former evolution operator will be
\[ \hat{U}_0(\tau) = R(0) \hat{U}_R(\tau) R^\dagger (0) = e^{-i \gamma(0) \hat{\sigma}_z}, \]
where $\gamma(0) \equiv (\sin \theta_0 \cos \phi_0, \sin \theta_0 \sin \phi_0, \cos \theta_0)$ is a unit vector and $\hat{U}_0(\tau)$ conducts a rotation around $\hat{\sigma}_z$ by an angle $2\gamma(\tau) = 2(1 + \eta) \gamma_0(\tau)$, from which an arbitrary geometric gate for a single-qubit can be designed with the original system $\hat{H}_0(t)$. In the spherical parametric space of $(1, \theta, \phi)$, the curve traces a closed path $C$ of a cyclic evolution during $t = 0$ to $\tau$ and $\gamma_0(\tau)$ represents a half of the solid angle enclosed by path $C$. This implies that the geometric phase $\gamma_0(\tau)$ is only determined by the evolution path of the parameters $\theta(t)$ and $\phi(t)$ and independent of the evolutionary details, which is robust against the control errors and depends only on the topological aspects of the evolution path. As a matter of fact, $\gamma_0(\tau)$ is invariant as long as the area enclosed by the path does not change \([33]\).

Based on the above discussion, we propose a scheme of nonadiabatic geometric quantum computation with unconventional geometric phases that the dynamical phases do not need to be avoided. Although the total phase $\gamma(t)$ accumulated in the designed gate operation contains dynamical component, it still relies on global geometric feature and the corresponding gate is also a kind of geometric one in a general sense. When $\eta = 0$ the dynamic phase is totally removed, our scheme will reduce to the conventional nonadiabatic scheme.

### III. THE DESIGNED HAMILTONIAN

Bashed on the above method, we adopt unitary transformation on the Hamiltonian Eq.\([1]\) and obtain the explicit form of $R^\dagger \hat{H}_0(t) R$ and $i \partial_t R^\dagger R$ as follows:
\[ R^\dagger \hat{H}_0(t) R = f_x(t) \hat{\sigma}_x + f_y(t) \hat{\sigma}_y + f_z(t) \hat{\sigma}_z, \]
where
\[ f_x(\theta, \phi) = \begin{cases} \cos \theta \cos^2 \phi + \sin^2 \phi & h_x(t) \\ -\sin^2 \theta \sin 2\phi h_y(t) & \\ -\sin \theta \cos \phi h_z(t) & \end{cases} \]
\[ f_y(\theta, \phi) = \begin{cases} -\sin^2 \theta & \sin 2\phi h_x(t) \\ + \cos \theta \sin^2 \phi + \cos^2 \phi & h_y(t) \\ -\sin \theta \sin h_z(t) & \end{cases} \]
\[ f_z(\theta, \phi) = \begin{cases} \sin \theta \cos \phi h_x(t) \\ + \sin \theta \sin h_y(t) + \cos \theta h_z(t) \end{cases} \]
and
\[ i \partial_t R^\dagger R = g_x(t) \hat{\sigma}_x + g_y(t) \hat{\sigma}_y + g_z(t) \hat{\sigma}_z, \]
where
\[ g_x(\theta, \phi) = \frac{\dot{\theta}}{2} \sin \phi + \frac{\dot{\phi}}{2} \sin \theta \cos \phi, \]
\[ g_y(\theta, \phi) = -\frac{\dot{\theta}}{2} \cos \phi + \frac{\dot{\phi}}{2} \sin \theta \sin \phi, \]
\[ g_z(\theta, \phi) = \frac{\dot{\theta}}{2} \sin \phi - \frac{\dot{\phi}}{2} \sin \theta \cos \phi, \]
In order to diagonalize \( \hat{H}_R(t) \) according to Eq. (4), the non-diagonal components of \( R \hat{H}_R(t) R \) and \( i \partial_t R R^\dagger \) should cancel out leading to \( f_x(t) = -g_x(t) \) and \( f_y(t) = -g_y(t) \). The diagonal components should be proportional to each other: \( f_z(t) = \eta g_z(t) \). We can naturally arrive the familiar geometric phase

\[
\gamma_g(\tau) = \frac{1}{2} \int_0^\tau [1 - \cos \theta(t)] \phi(t) dt. \tag{11}
\]

Putting the above integral in the parametric space of unit sphere, it can be recasted as \( \gamma_g(\tau) = \frac{1}{4} \oint C (1 - \cos \theta) d\varphi \) and the total rotation phase reads \( \gamma(\tau) = (1 + \eta) \gamma_g \), which clearly possesses a global geometric feature.

Based on the above diagonalization conditions, our idea to design Eq. (11) for the general geometric gates can be realized by opportunistically choosing the parameters \( h_{x,y,z}(t) \) as follows:

\[
h_x(t) = \frac{\dot{\varphi}}{2} \left[ \eta - (1 + \eta) \cos \theta \right] \sin \theta \cos \varphi - \frac{\dot{\theta}}{2} \sin \varphi,
\]

\[
h_y(t) = \frac{\dot{\varphi}}{2} \left[ \eta - (1 + \eta) \cos \theta \right] \sin \theta \sin \varphi + \frac{\dot{\theta}}{2} \cos \varphi,
\]

\[
h_z(t) = \frac{\dot{\varphi}}{2} \left[ \sin^2 \theta + 2\eta \cos \theta \sin^2 \frac{\theta}{2} \right].
\]

The above designed Hamiltonian \( \hat{H}_0(t) \) is related to a general two-level model of

\[
\hat{H}_0(t) = \frac{1}{2} \begin{bmatrix} \Delta(t) & \Omega_R(t)e^{-i\phi(t)} \\ \Omega_R(t)e^{i\phi(t)} & -\Delta(t) \end{bmatrix}, \tag{12}
\]

where

\[
\Omega_R(t) = \sqrt{\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \left[ \eta - (1 + \eta) \cos \theta \right]^2},
\]

\[
\phi(t) = \arctan\left( \frac{(\eta - 1 - \eta) \sin \theta \varphi \sin \varphi + \dot{\theta} \cos \varphi}{(\eta - 1 + \eta) \sin \theta \varphi \cos \varphi - \dot{\theta} \sin \varphi} \right),
\]

\[
\Delta(t) = \varphi \left[ \sin^2 \theta + 2\eta \cos \theta \sin^2 \frac{\theta}{2} \right].
\]

It is well-known that Eq. (12) can be fulfilled by the laser-driven atomic system, the quantum dot spin or Josephson junction system \([33,38]\) as well as the diamond nitrogen-vacancy (NV) center controlled by tailoring the parameters of microwave or laser fields \([39,43]\). This method is generally beyond the adiabatic dynamics without any confinement for the slowly varying parameters.

IV. GATE IMPLEMENTATION

To demonstrate our approach, we design geometric rotation gates by controlling the light pulses in the diamond NV center system \([44]\). A given geometric rotation can be realized by tailoring \( \Delta(t) \) and \( \Omega_R(t) \) of the microwave pulses along a desired evolution path in the parametric space of \( (1, \theta, \varphi) \). In the following, we take \( \hat{U}_Z(\tau) = e^{-i \frac{\pi}{2} \hat{\sigma}_z} \) as a target example.

As shown in Fig. 1, the NV center has a spin-triplet ground state and the nearby nuclear spins \( ^{13}N \) and \( ^{13}C \) are polarized by a magnetic field of about 500G along the NV axis. We use two lower Zeeman levels \( |m_s = 0 \rangle \equiv |0 \rangle \) and \( |m_s = -1 \rangle \equiv |1 \rangle \) of NV center to encode the qubit and the nuclear spins of \( ^{13}C \) atom for further controls. The qubit is manipulated by a microwave pulse whose spectrum, intensity, and phase can be adjusted by a hybrid waveform generator. The pulse parameters used here are \( \Omega_0 = 20MHz \) (the maximal Rabi frequency), \( \Delta_0 = 20MHz \) (the maximal detuning) and the \( \pi \)-pulse control time \( \tau \) (in unit of \( \tau_0 = \pi/\Omega_0 \)) \([45]\).

A. Conventional geometric gate with “orange-slice” path

To realize gate \( U_Z(\tau) = e^{-i \frac{\pi}{2} \hat{\sigma}_z} \), a usual evolution path is selected in parametric space as shown in Fig. 2. The parameters \( (\theta(t), \varphi(t)) \) start from the north pole \((0, \varphi_0)\) to the south pole \((\pi, \varphi_0)\) along the great circle \( \varphi(t) = \varphi_0 \), then return back to the north pole from the south pole along another great circle \( \varphi(t) = \varphi_0 + \frac{\pi}{2} \). This path is the so-called resonant \( (\Delta = 0) \) orange-slice-shaped loop widely used in the control schemes of nonadiabatic geometric quantum computation \([46]\). To this end, the non-diagonal terms in Eq. (12) are

\[
\begin{align*}
\end{align*}
\]

FIG. 1: Two Zeeman levels \( |m_s = 0 \rangle \) and \( |m_s = -1 \rangle \) of the NV spin-triplet ground state are encoded as the qubit states \( |0 \rangle \) and \( |1 \rangle \).

FIG. 2: (Left) The orange-slice-shaped-loop path for the realization of \( U_Z(\tau) \) gate; (Right) The evolutions of state populations (solid lines) and the fidelities (dashed lines) along two separate paths.
\[ \Omega_R(t)e^{-i\phi(t)} = \begin{cases} 
\dot{\theta}(t)e^{-i(\varphi_0 - \frac{\pi}{2})}, & 0 \leq t \leq \frac{\pi}{2}, \\
\frac{\sqrt{3}}{3}\dot{\varphi}(t)e^{-i\varphi(t)}, & \frac{\pi}{2} < t \leq \tau_1, \\
\frac{\sqrt{3}}{3}\dot{\varphi}(t)e^{-i(\varphi_0 + \frac{\pi}{3})}, & \tau_1 < t \leq \tau_2, \\
\dot{\theta}(t)e^{-i(\varphi_0 + \frac{\pi}{3})}, & \tau_2 < t \leq \tau. 
\end{cases} \] (13)

Here, the pulse areas of the Rabi frequencies at their respective time intervals satisfy

\[ \int_0^{\frac{\pi}{2}} \Omega_R(t)dt = \pi, \int_{\frac{\pi}{2}}^{\tau_1} \Omega_R(t)dt = -\pi. \] (14)

If a square-wave pulse is used to do the calculation, the operation time is \( \tau = 2\tau_0 = 2\pi/\Omega_0 \). The geometric phase can be calculated from Eq. (11), \( \gamma_0 = \pi/2 \), which is obtained by the salutation of \( \phi(t) \) at the moment of \( t = \frac{\pi}{2} \) at the south pole. That is how the conventional non-adiabatic geometric quantum gate realized via the “orange slice” as shown above.

### B. Conventional geometric gate beyond “orange-slice” path

We can choose an alternative evolution path to realize \( U_Z \) gate to avoid the singular point at the pole without salutation of \( \phi \). As shown in Fig. 3, the parameters \( (\theta(t), \varphi(t)) \) start from the north pole \( (0, \varphi_0) \) to the point \( (\frac{\pi}{2}, \varphi_0) \) along the great circle \( \varphi(t) = \varphi_0 \). Then the parameters evolve from \( (\frac{\pi}{2}, \varphi_0) \) to \( (\frac{3\pi}{2}, \varphi_0 + \frac{\pi}{3}) \) along the arc \( \theta(t) = \frac{2\pi}{3} \), and finally return back to the north pole along the great circle \( \varphi(t) = \varphi_0 + \frac{2\pi}{3} \).

**FIG. 3:** (Left) The larger triangular path for the realization of \( U_Z(\tau) \) gate; (Right) The corresponding evolutions of state populations (solid lines) and the fidelities (dashed lines) along tree separate paths.

For this path, the Rabi frequencies of the laser pulse read

\[ \Omega_R(t)e^{-i\phi(t)} = \begin{cases} 
\dot{\theta}(t)e^{-i(\varphi_0 - \frac{\pi}{3})}, & 0 \leq t \leq \tau_1, \\
\frac{\sqrt{3}}{3}\dot{\varphi}(t)e^{-i\varphi(t)}, & \tau_1 < t \leq \tau_2, \\
\dot{\theta}(t)e^{-i(\varphi_0 + \frac{\pi}{3})}, & \tau_2 < t \leq \tau. 
\end{cases} \] (15)

Here, their respective pulse areas and detunings are

\[ \int_0^{\tau_1} \Omega_R(t)dt = \frac{2\pi}{3}, \Delta(t) = 0, \]
\[ \int_{\tau_1}^{\tau_2} \Omega_R(t)dt = \frac{\sqrt{3}\pi}{6}, \Delta(t) = \frac{3}{4}\dot{\phi}(t), \]
\[ \int_{\tau_2}^{\tau} \Omega_R(t)dt = -\frac{2\pi}{3}, \Delta(t) = 0. \] (16)

The total evolution time of this triangular path by square-wave pulse is \( \frac{\pi}{2}\tau_0 + \frac{\pi}{2}\tau_0 + \frac{\pi}{2}\tau_0 \approx 1.833\tau_0 \), which is shorter than that of the orange-slice-shaped loop. Further, a shorter evolution path to realize this gate can also be designed by this method if the Rabi frequency \( \Omega(t) \) and detuning \( \Delta(t) \) can reach the experimental maximum at the same time, and the optimal evolution time will be about \( 1.792\tau_0 \).

### C. Unconventional geometric gate

Although the above triangular path evolves faster than the orange-slice one in conventional geometric gate, the requirement of zero dynamic phase imposes stringent constraints on the driving Hamiltonian. However, our method relaxes the experimental conditions and combines geometric phase control with non-adiabatic method to validate the dynamical phase (\( \eta \neq 0 \) in the gate design). Therefore, we can provide better evolution paths with more relaxed experimental conditions to design \( \Omega(t) \) and \( \Delta(t) \).

For example, we can choose the path in this way (see Fig. 4): the parameters \( (\theta(t), \varphi(t)) \) start from the north pole \( (0, \varphi_0) \) to the point \( (\frac{\pi}{2}, \varphi_0) \) along the great circle \( \varphi(t) = \varphi_0 \), then evolve along the equator to \( (\frac{\pi}{2}, \varphi_0 + \frac{\pi}{3}) \), and finally return back to the north pole along the great circle \( \varphi_0 + \frac{2\pi}{3} \). Along this path, we can set \( \eta = 1 \) to make Rabi frequency \( \Omega(t) \) and detuning \( \Delta(t) \) both reaching maximum at the same time. Therefore, the Rabi frequencies of the control pulses are

**FIG. 4:** (Left) The unconventional triangular path for the realization \( U_Z(\tau) \) gate; (Right) The corresponding evolutions of state populations (solid lines) and the fidelities (dashed lines) along three separate paths.
and their respective pulse areas and detunings satisfy
\[ \Omega_R(t)e^{-i\phi(t)} = \begin{cases} \hat{\theta}(t)e^{-i\varphi(t)}, & 0 \leq t \leq \tau_1 \\ \hat{\phi}(t)e^{-i\psi(t)}, & \tau_1 < t \leq \tau_2 \\ \hat{\theta}(t)e^{-i\phi_0}, & \tau_2 < t \leq \tau \end{cases} \] \tag{17}
and their respective pulse areas and detunings satisfy
\[ \int_0^{\tau_1} \Omega_R(t)dt = \frac{\pi}{2}, \Delta(t) = 0, \]
\[ \int_{\tau_1}^{\tau_2} \Omega_R(t)dt = \frac{\pi}{2}, \Delta(t) = \phi(t), \]
\[ \int_{\tau_2}^{\tau} \Omega_R(t)dt = -\frac{\pi}{2}, \Delta(t) = 0. \]

The total operation time along this path by square-wave pulses is \( \frac{1}{3}\tau_0 + \frac{1}{3}\tau_0 + \frac{1}{3}\tau_0 = 1.5\tau_0 \), which clearly demonstrates that the unconventional QGC owns the shortest gate time comparing with the conventional ones. More importantly, the freely adjusting parameter \( \eta \) in our theory can propose more optimal strategy to do GQC under relaxing experimental conditions.

V. PERFORMANCE OF UNCONVENTIONAL GATES

Now we check the reliability of the quantum geometric gates designed by our method in an open system. The performance of a \( U \) gate in this case can be simulated by using Lindblad master equation as
\[ \dot{\rho}(t) = i \left[ \rho(t), \hat{H}(t) \right] + \frac{1}{2} \left[ \gamma_1 \mathcal{L}(\hat{\sigma}^+) + \gamma_2 \mathcal{L}(\hat{\sigma}_z) \right], \tag{19} \]
where \( \rho(t) \) is the density matrix of the designed system, and \( \mathcal{L}(\hat{A}) = 2\hat{A}\rho\hat{A}^\dagger - \hat{A}^\dagger \hat{A} \rho - \rho \hat{A}^\dagger \hat{A} \) is the Lindbladian of operator \( \hat{A} : \hat{\sigma}^+ \equiv |1\rangle\langle 0|, \hat{\sigma}_z \equiv |1\rangle\langle 1| - |0\rangle\langle 0|. \) The decoherent effects of the environment are considered by the damping rates \( \gamma_1 \) and \( \gamma_2 \), respectively. In our simulations, the decay and dephasing rates are set \( \gamma_1 = \gamma_2 = 4 \times 10^4 \text{ Hz} \). Suppose that the qubit is initially prepared in the state \( |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \), the time-dependence of the state populations and the state fidelities \( F = ||\psi(t)||^2 \) of the \( U_Z \) gates in different paths are shown in Fig.2, Fig.3 and Fig.4 respectively. We can see the final fidelities of 99.35% for conventional geometric gate with an “orange-slice” path, 99.57% for conventional geometric gate in the triangular path, and 99.67% for the unconventional geometric gate.

Moreover, a conditional two-qubit gate will be realized if we use two different pairs of orthogonal cyclic states of the target qubit, conditioned on the state of another control qubit. The target qubit is exploited by the electron spin of NV center and one nearby \( ^1\text{C} \) nuclear spin as the control qubit. In this case, a product one-qubit basis \( \{ |0\rangle, |1\rangle \} \otimes \{ |\downarrow\rangle, |\uparrow\rangle \} \) serves as two-qubit computational basis, which are coupled by different state-selective pulses and radio-frequency fields. Under the parametric controls of the pulses, the effective Hamiltonian of this two-qubit system has an extensible form
\[ \hat{H}_Z = \hat{H}_+ + \hat{H}_-, \tag{20} \]
in the two-qubit basis. In their separate subspaces \( \{|1\uparrow\rangle, |0\uparrow\rangle\} \) and \( \{|1\downarrow\rangle, |0\downarrow\rangle\} \), \( H_+ \) and \( H_- \) can selectively satisfy Eq.6 if the pulse frequency is on resonance with the computational states \( |1, \uparrow\rangle \) and \( |0, \uparrow\rangle \), and far detuned from the computational states \( |1, \downarrow\rangle \) and \( |0, \downarrow\rangle \) with a detuning \( \delta \omega = \Delta_+ - \Delta_- \). The unwanted mixing caused by the coupling with the subspace of nuclear spin pointing downward can be neglected when \( \Delta_+ < \delta \omega \) is satisfied for geometric gate [45, 48, 49]. With the same routine design as that of a single-qubit gate, we can achieve the nontrivial geometric two-qubit gate as
\[ \hat{U}_{tq} = |\uparrow\rangle\langle \uparrow| \otimes \hat{U}_{sq} + |\downarrow\rangle\langle \downarrow| \otimes \hat{I}. \tag{21} \]

VI. CONCLUSION

In conclusion, we have proposed an approach to realize conventional and unconventional nonadiabatic geometric quantum computations under the framework of SU(2) transformation. Our approach relaxes the constraints imposed for the driving Hamiltonian in the approach of NHQC, and we can use the designed Hamiltonians to realize nonadiabatic geometric gates with any desired evolutionary paths. Our approach is able to minimize the operation time needed for high-fidelity geometric gates which is better combined with experimental techniques. To show its potential applications, we simulate the performance of the geometric gates with optimal parametric paths in the NV center platform. A shorter evolution time and higher gate fidelity than that in the previous schemes are clearly demonstrated.

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