Two–loop Contributions of Flavor Changing Neutral Higgs Bosons

to $\mu \to e\gamma$

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The two–loop mechanism of Bjorken and Weinberg is used to constrain flavor changing neutral Higgs bosons. We calculate the complete set of two–loop diagrams for the rare decay $\mu \to e+\gamma$ induced by such neutral Higgs bosons, for arbitrary Higgs and top masses. The analytic result is used to set limits on Higgs masses for some recent models with specific ansatz about the flavor changing couplings. For example, in the Cheng–Sher scenario of multi-Higgs doublet models, all neutral Higgs bosons possess flavor changing ($f_i \leftrightarrow f_j$) couplings proportional to $\sqrt{m_im_j}$. We find that the present limit on $\mu \to e\gamma$ implies that, in such scheme, these neutral Higgs bosons should be heavier than 200 GeV.

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I. INTRODUCTION

In most theories beyond the Standard Model, neutral Higgs boson couplings are typically flavor changing unless special arrangements, such as imposing discrete symmetries, are made to eliminate them. The mass of such Higgs bosons can be estimated from its potential contribution to the $K_L - K_S$ mass difference, $\Delta m_K$. If one assumes that the flavor changing $sdH$ vertex has the same Yukawa coupling as that of the heaviest quark of the same type, that is, the $b$-quark in this case, then $\Delta m_K$ implies that the Higgs mass should be at least 150 TeV \[^{[1]}\]. This value is much larger than the electroweak symmetry breaking scale and immediately poses two potential problems: (1) the existence of an unnatural hierarchy in scale; (2) the Higgs sector would be strongly coupled and predictive power is lost.

To avoid such pitfalls, two options are often exercised. The first is to impose some discrete symmetry to achieve what is called “natural flavor conservation (NFC)” \[^{[2]}\], that is, to avoid tree level flavor changing neutral currents or couplings (FCNC). This is easily done by requiring that only one Higgs boson vacuum expectation value (VEV) contributes to each type of fermion mass \[^{[3]}\]. The second is to use some scheme such that tree level flavor changing neutral Higgs couplings $Y_{ij}$ are naturally suppressed in low energy processes. The latter has received revived attention in the literature recently \[^{[4],[5]}\], in part because it may have interesting consequences at high energies, such as the decay \[^{[6]}\] of the top quark into the charm quark and a light neutral Higgs boson.

For example, motivated by the Fritzsch ansatz \[^{[7]}\] of mass-mixing relations, Cheng and Sher pointed out \[^{[4]}\] that low energy FCNC constraints may in fact be evaded in multi-Higgs doublet models without invoking the NFC condition. Let the contribution of the $k$-th Higgs doublet to the fermion mass matrix be $M_{ij}^{(k)}$, it is not unreasonable to assume that $M_{ij}^{(k)} = X_{ij}^{(k)} \sqrt{m_im_j}$ for every $k$, where $X_{ij}^{(k)}$ is of order unity. Upon diagonalization, the fermion mass and mixing patterns can be roughly accounted for, but in general, neutral Higgs bosons in mass basis would all have flavor changing couplings. Their scheme can be summarized by assuming that the flavor changing couplings $\bar{f}_i (F_{ij}^{aL} P_L + F_{ij}^{aR} P_R) f_j H_a$ of the
The a’th neutral Higgs scalar $H_a$ have the following natural pattern

$$F_{ij}^{aL,R} = \frac{g}{g^2} m^{aL,R}_{ij} = \frac{g}{2} \Delta^{aL,R}_{ij} \frac{\sqrt{m_i m_j}}{M_W},$$

with $\Delta^{aL,R}_{ij}$ of order one. We shall refer to the lightest such scalar boson as $H_1$. Since low energy constraints typically involve lower generation fermions, they are evaded by the associated tiny Yukawa couplings in Eq. (1). For example, taking $\Delta^{1L}_{sd} = \pm \Delta^{1R}_{sd} \sim 1$, the $\Delta m_K$ constraint is weakened. If the lightest Higgs has pseudoscalar couplings, it is only required to be heavier than about 1 TeV, which is roughly the scale where the Higgs sector is expected to become strongly coupled. For scalar Higgs couplings, the bound is further lowered to roughly the symmetry breaking scale.

The Cheng-Sher scenario was not widely appreciated, and subsequent work concentrated on rare decays of $\tau$, $B$ and $D$, as well as the utility of the $\mu \rightarrow e\gamma$ process as a constraint. It was recently realized, however, that the progressive nature of FCNC Higgs couplings could have interesting implications for very heavy quarks. In particular, the top quark may possess non-trivial couplings to a neutral Higgs boson $H_1$ and a charm quark, of order $m_t/v \sim \sqrt{m_t m_c}/v$. This may result in an appreciable branching fraction for a new channel $t \rightarrow cH_1$ in the decay of the top quark. Alternatively, neutral Higgs bosons may have appreciable rates into $t\bar{c}$ type of final states. Because of the importance of top and Higgs physics, it is natural to ask whether we have exhausted low energy constraints on the Cheng-Sher scenario.

In this paper, we give a careful analysis of the effect of these flavor changing neutral Higgs bosons on the $\mu \rightarrow e\gamma$ process up to the two–loop level. We parametrize our calculation in a way that is as model independent as possible so that our result can be applicable to any models with flavor changing neutral Higgs bosons. Our one–loop result differs from previous calculation [8]. Our two–loop study not only improves on previous rough estimates [9,10], but also uncovers some interesting characteristics that were overlooked before. In particular, the two–loop contribution can be larger than the one–loop result, as argued some time ago by Bjorken and Weinberg, and the heavy–Higgs–boson effect is not decoupled.
The contributions of the two-loop graphs that contain a heavy gauge boson loop diverge logarithmically as the Higgs mass is taken to infinity. However, due to a unitarity condition associated with flavor changing neutral Higgs bosons, if all such Higgs bosons are degenerate in mass, their contributions mutually cancel in such diagrams. This cancellation mechanism introduces model-dependence, reducing one’s capability to make strict predictions. We present our results such that anyone can extract constraints on his favorite model. In this paper, we do not consider the charged Higgs boson, which gives an independent contribution with its own parameters.

II. BJORKEN–WEINBERG MECHANISM

One flavor changing mode of particular interest is the celebrated $\mu \to e\gamma$ mode. The existing limit, at $4.9 \times 10^{-11}$ [1], is one of the most impressive. It cannot occur at tree level, and it involves lepton number violation. In the Cheng–Sher scenario, the leading one-loop contribution has the neutral scalar and the $\tau$ in the loop, with the photon radiated from the internal $\tau$ line. The usual $\mu–\mu–e$ sequence is much more suppressed. The $\tau$ contribution (Fig. 1) to the 1–loop branching fraction of $\mu \to e\gamma$ is

$$
\text{BR}_{1\text{-loop}}(\mu \to e\gamma) = \frac{3 \alpha m_e}{4 \pi m_\mu} \left| \sum_a \Delta_{e\tau}^a \Delta_{\mu\tau}^a \frac{m_e^2}{M_{H_a}^2} \left( \ln \frac{M_{H_a}^2}{m_\mu^2} + \frac{3}{2} \right) \right|^2, \quad (2)
$$

where $M_{H_a}$ stands for the the mass of the $a$’th flavor changing Higgs boson $H_a$. Clearly the contribution from the lightest neutral scalar boson dominates in general. This result holds if the flavor changing couplings are purely scalar ($\Delta_{ij}^{aL} = \Delta_{ij}^{aR}$) or purely pseudoscalar ($\Delta_{ij}^{aL} = - \Delta_{ij}^{aR}$). Also, our result is quite different from previous estimates [8,9]. Assuming lightest scalar dominance, we have

$$
\text{BR}_{1\text{-loop}}(\mu \to e\gamma) = 5 \times 10^{-11} |\Delta_{e\tau}^1 \Delta_{\mu\tau}^1|^2 \left( \frac{91\text{GeV}}{M_{H_1}} \right)^4 \left( 1 - 0.31 \ln \frac{91\text{GeV}}{M_{H_1}} \right)^2, \quad (3)
$$

which implies that, if one takes $\Delta_{e\tau}^1 \Delta_{\mu\tau}^1 = 1$ as in Ref. [8] for the lightest scalar, any mass above 91 GeV is still phenomenologically allowed. This limit is more stringent than that
obtained in Ref. [8] because of the extra large $\ln(m_\tau^2/m_{H_1}^2)$ term. The strong $M_{H_1}$ dependence in Eq.(3) means that the bound on $M_{H_1}$ will improve rather slowly with improvements on the BR($\mu \to e\gamma$) limit.

Here we wish to draw attention to an observation [9] made by Bjorken and Weinberg 15 years ago, that certain two–loop graphs may in fact dominate over the one–loop contribution. The mechanism is as follows. Dipole transitions demand a chirality flip between the initial state and the final state of the fermion. For the one–loop graph involving virtual scalars, three chirality flips are involved: twice in the Yukawa couplings, and once in the fermion propagator. This fact is indeed just an accident at the one–loop level, but can be avoided at higher orders. Clearly, at the one–loop level, the $\mu–\mu–e$ sequence is extremely suppressed, while even for the $\mu–\tau–e$ sequence discussed here, one pays the price of a suppression factor $\sqrt{m_e m_\mu m_\tau^2}/v^3 = \mathcal{O}(10^{-9})$. Going to two–loop order, one pays the typical price of $g^2/16\pi^2$, but if one could avoid two of the extra chirality flips, one may still gain enormously against the one–loop graph. In a set of two–loop graphs found by Bjorken and Weinberg, the virtual scalar boson couples only once to the lepton line, inducing the needed chirality flip. Through some heavy–particle (e.g. W or top) loop, the boson is then converted into two photons, one of which is reabsorbed by the lepton line. Assuming $M_{H_1}^2 \ll M_W^2$, Bjorken and Weinberg estimated the branching fraction due to the leading two–loop graph from the W contribution (Fig. 2a) to be

$$\text{BR}_{\text{BJW}}(\mu \to e\gamma) \simeq \frac{147}{16} \left(\frac{\alpha}{\pi}\right)^3 \frac{m_e}{m_\mu} \sum_{a>1} \cos \phi_a \Delta_{\epsilon\mu}^a \ln \left| \frac{M_{H_a}^2}{M_{H_1}^2} \right| ^2,$$

$$\simeq 56 \times 10^{-11} \sum_{a>1} \cos \phi_a \Delta_{\epsilon\mu}^a \ln \left| \frac{M_{H_a}^2}{M_{H_1}^2} \right| ^2.$$  (4)

Here the neutral Higgs boson $H_a$ couples to the $W$ boson at a relative strength $\cos \phi_a$ with respect to that of the Higgs boson in the Standard Model. It is reasonable to assume that $\cos \phi_a$ is of order one, if the neutral Higgs boson originated from a Higgs multiplet that contributes significantly to $SU_L(2)$ breaking. Since this estimate works only when $M_H \ll M_W$, the present experimental limit implies that the Cheng-Sher scenario is probably not viable for light Higgs bosons, unless $\Delta_{\epsilon\mu}^a$ is significantly smaller than one, or the amplitudes
from different Higgs bosons cancel each other by accident, which could happen when the relevant Higgs bosons are degenerate in mass as discussed before.

In the context of studying the electric dipole moment (edm) of the electron within neutral Higgs models of CP violation, Barr and Zee [12] made independent observations that are analogous to that of Bjorken and Weinberg. Without knowing the work of Cheng and Sher, recently Barr [10] estimated the two–loop contributions to $\mu \rightarrow e\gamma$ for the case of very heavy Higgs bosons. Assuming only the $W$ loop in the effective $H\gamma\gamma$ coupling and assuming $M_H^2 \gg M_W^2$, Barr estimated the branching fraction from the two–loop effect to be

$$BR_{\text{Barr}}(\mu \rightarrow e\gamma) \simeq \frac{3}{4} \left( \frac{\alpha}{\pi} \right)^3 \left( \frac{35}{8} \right)^2 (\cos \phi_1 \Delta_{1e\mu})^2 \frac{m_e}{m_\mu} \left( \frac{M_W}{M_{H_1}} \right)^4 \ln \left| \frac{M_W^2}{M_{H_1}^2} \right|^4.$$

(5)

Note that there is $M_H^{-4}$ suppression as in the one–loop case. Taking this result seriously and assuming $\cos \phi_1 \Delta_{1e\mu} \simeq 1$ as before, one find that the present experimental limit requires $M_H \gtrsim 730$ GeV in the Cheng–Sher scenario.

Given the two estimates quoted above, where the lightest Higgs boson with flavor changing couplings is either very light or very heavy compared to $M_W$, one may naturally be curious about the situation for Higgs masses in between. In this article, we report on a detailed calculation [13] of the complete set of two–loop diagrams where Higgs and top masses are kept arbitrary.

III. COMPLETE TWO–LOOP RESULTS

The diagrams needed for the calculation of the transition dipole moment in our case are analogous to those for the electric dipole moments of quarks [14] and electrons [12,15,16]. The most detailed depiction of these graphs can be found in Ref. [16], where one of the external electrons should be replaced by a muon.

The two–loop graphs of interest can be classified into three sets, A, B and C, that are separately gauge invariant. Set A contains a heavy fermion loop. The fermion in the loop is most likely the heaviest one which is the top quark. They can be further classified into two
gauge invariant subsets. The first one involves an internal photon line in Fig. 2a, while the second is obtained by replacing the internal photon line with a $Z$ boson line. According to Furry’s theorem, only the vector coupling of $Z$ boson contributes to the fermion loop. On the other hand, for both electron electric and magnetic dipole moments, the corresponding operators of the moments, $\sigma_{\mu\nu}$ and $\sigma_{\mu\nu}\gamma_5$, are odd under charge conjugation, $C$, just as the vector coupling of $Z$. Therefore it is not hard to see that only the vector coupling of $Z$, not the axial one, to the external leptons can contribute to these operators. This argument can be carried over diagrammatically to the case of the transitional moments in the process $\mu \to e\gamma$. Since the vector coupling of the $Z$ boson is known to be relatively suppressed, the second group of diagrams can be ignored in the first approximation.

Set B contains a $W$ boson (and associated unphysical scalar) loop, and can be further divided into two gauge invariant subsets. For the first group, $B_I$, the $W$ boson loop induces a $H\gamma\gamma$ vertex as in Fig. 2. For the second group, $B_{II}$, the internal photon line is replaced by a $Z$ line. Just like set A, set $B_{II}$ is suppressed compared to set $B_I$ because of the small vector coupling of $Z$ to charged leptons. In addition, if one assumes CP invariance, then since only the scalar components of the Higgs bosons couple to the $W$ boson at the tree level, one expects that only the CP–even Higgs bosons will be relevant in this case. In general, from experience with the analysis of electric dipole moments in Refs. [12,14–16], one expects set B to dominate over set A also.

Set C involves graphs that have a different topological structure. They can be further divided into two gauge invariant groups $C_I$ and $C_{II}$. They correspond to graphs without or with a $Z$ boson line as in Figs. 3, 4 of Ref. [16], respectively. Again, the second group is small compared to the first due to the small $Z$ coupling. Numerical results of Ref. [16] indicate that, for the case of $edm$, the contribution of set C is in general much smaller than sets A and B. This conclusion should also be applicable to the transition dipole moment.

We shall consider sets A and B first. For flavor changing leptonic processes, the internal gauge boson line can be either the photon or $Z$ boson. However, if one is interested in flavor changing processes involving light quarks, a similar graph with both gauge bosons replaced
by gluons (i.e. $Hgg$ rather than $H\gamma\gamma$ or $H\gamma Z$) can also be important.

The calculational strategy is to first calculate the one–loop effective vertex with one
gauge boson, one photon and a neutral Higgs boson in the external lines. This has been
done many times before [17], and a recent calculation can be found in Ref. [18]. pseudoscalar
amplitudes Higgs–gluon–gluon vertices Higgs boson are assumed to be on shell in Ref. [18],
it is easy to extract the result with off-shell Higgs boson as long as one can tell which factor
of Higgs mass comes from the loop momentum and which one is due to the vertex. The
result of Ref. [18] is consistent with the recent calculation of electron $edm$ [16], where the
Higgs boson was kept explicitly off–shell, but only $H\gamma\gamma$ contribution was given.

Here we shall concentrate on leptonic FCNC. In that case the graphs with internal $Z$
boson are suppressed relative to the ones with internal photon line by a factor of $(1 –
4\sin^2 \theta_W)/4\sin^2 \theta_W$, which is about 0.087 for $\sin^2 \theta_W = 0.23$. Therefore one could ignore
these contributions even though they can be easily incorporated into the analysis.

We shall parametrize the relevant couplings as

$$
\mathcal{L} = - \frac{m_t}{v} \bar{t} (\Delta^a_{tt} P_L + \Delta^*_{tt} P_R) t H_a
- \frac{\sqrt{m_\mu m_e}}{v} \bar{e} (\Delta^a_{e\mu} P_L + \Delta^{aR}_{e\mu} P_R) e \mu H_a + g M_W \cos \phi_a W^+ W^- H_a + \cdots .
$$

(6)

Here $v = (\sqrt{2} G_F)^{-\frac{1}{2}} \simeq 246$ GeV. If one imposes CP conservation then $\text{Im} (\Delta^a_{tt})^2 = 0$ and
$\text{Im} (\Delta^a_{e\mu} \Delta^{aR}_{e\mu}) = 0$. Note also that in case of CP conservation, $\cos \phi_a$ is nonzero only for
those CP–even scalar Higgs bosons. (For the Higgs boson in Standard Model, $\Delta_{tt} = 1$)

To simplify long expressions, we define the reduced amplitude $A$, which is dimensionless,
for the transition $\mu \rightarrow e\gamma (\epsilon, k)$ as follows:

$$
i\mathcal{M} = \frac{e\sqrt{2} G_F \alpha}{16\pi^3} \sqrt{m_\mu m_e} \epsilon^\mu k^\alpha \sigma_{\mu\alpha} (A_L P_L + A_R P_R) ,
$$

(7)

and the branching fraction is

$$
\text{BR}(\mu \rightarrow e\gamma) = \frac{3}{4} \left( \frac{\alpha}{\pi} \right)^3 \frac{m_e}{m_\mu} \left( \frac{1}{2} |A_L|^2 + \frac{1}{2} |A_R|^2 \right) = 4.5 \times 10^{-11} \left( \frac{1}{2} |A_L|^2 + \frac{1}{2} |A_R|^2 \right) .
$$

(8)

Note that CP conservation will require $\text{Im} (A_L A_R^*) = 0$. For set $A$ with the top–quark
loop, the $H\gamma\gamma$ or $H\gamma Z$ vertices already contain one power of external photon momentum.
Therefore, we can set the virtual photon momentum and the Higgs boson momentum to be equal and the two–loop result can be easily produced. The $H\gamma\gamma$ contribution gives

$$A_{t\text{-loop}}^{L,R} H\gamma\gamma (\mu \rightarrow e + \gamma) = 3Q_t^2 \sum_a \Delta_{e\mu}^{a,L,R} 2 [ \text{Re} \Delta_{tt}^a f(z_{ta}) - i\lambda_{5}^{L,R} \text{Im} \Delta_{tt}^a g(z_{ta})] .$$

(9)

where $z_{ta} = m_t^2/M_a^2$. The chirality factors are defined as $\lambda_5^L = -1$ and $\lambda_5^R = 1$. The scalar $H\bar{t}t$ Yukawa coupling $\text{Re} \Delta_{tt}^a$ is associated with the following function,

$$f(z) = \frac{1}{2} \int_0^1 dx \ln \frac{x(1-x)}{z(1-x)}.$$  

(10)

The pseudoscalar coupling $\text{Im} \Delta_{tt}^a$ is associated with

$$g(z) = \frac{1}{2} \int_0^1 dx \ln \frac{x(1-x)}{z(1-x)}.$$  

(11)

We have closely followed the notations of Ref. [15].

If CP is invariant, the $a$'th Higgs boson, when it couples to the top quark, is either a scalar ($\text{Im} \Delta_{tt}^a = 0$), or a pseudoscalar ($\text{Re} \Delta_{tt}^a = 0$). However, we do not have relations between $\Delta_{e\mu}^{a,L}$ and $\Delta_{e\mu}^{a,R}$, except that they are relatively real.

For the $Z$–mediated diagrams,

$$A_{t\text{-loop}}^{L,R} HZ\gamma (\mu \rightarrow e + \gamma) = \frac{(1 - 4 \sin^2 \theta_W)(1 - 4Q_t \sin^2 \theta_W)}{16 \sin^2 \theta_W \cos^2 \theta_W}$$

$$\times 3Q_t \sum_a \Delta_{e\mu}^{a,L,R} 2 [\text{Re} \Delta_{tt}^a \tilde{f}(z_{ta}, z_{tZ}) - i\lambda_{5}^{L,R} \text{Im} \Delta_{tt}^a \tilde{g}(z_{ta}, z_{tZ})] .$$

(12)

Here $\tilde{f}(x, y) = yf(x)/(y - x) + xf(y)/(x - y)$ and similarly $\tilde{g}(x, y) = yg(x)/(y - x) + xg(y)/(x - y)$. We have also extended the previous definition to denote $z_{tZ} = m_t^2/M_Z^2$. Note that, in this Bjorken–Weinberg mechanism, there is only one power of light quark mass suppression [3], which has been explicitly written out in Eq.(7).

To derive the contribution of the bosonic loops, we shall classify the graphs into two gauge invariant types. The first type does not depend on Higgs mass in their couplings while the second set does. As a result, the first set is power suppressed by the Higgs mass while the second set is logarithmically increasing when the Higgs mass becomes very large, which is a very intriguing situation. For Higgs mass larger than a certain value the second type dominates. We shall only present the combined contribution of the two types.
The $H_aWW$ vertex in Eq.(6) is parametrized as $gM_W g^{\mu\nu} \cos \phi_a$, where $\cos \phi_a$ is a Higgs mixing angle. For the Standard Model Higgs boson, $\cos \phi = 1$. Before we proceed with our results, it is important to point out a unitarity constraint on the flavor changing neutral couplings. One can always make linear combinations of the scalar doublet fields such that only one doublet is responsible for symmetry breaking. It is the scalar component of this doublet that couples to $W$ boson pairs. Since this combination is also responsible for generating masses to the fermions, its Yukawa couplings should be automatically flavor conserving. Upon diagonalization of the Higgs boson mass matrix, all neutral Higgs bosons should in general possess flavor changing couplings. However, the above observation leads to a unitarity condition

$$\sum_a \cos \phi_a \Delta^a_{ij} L,R = 0 \ (\text{for } i \neq j),$$

which basically reflects the fact that the scalar doublets that mediate flavor violation must have zero vacuum expectation value at tree level. One important consequence is that, for the graphs in set B, terms that are independent of Higgs mass are cancelled away.

For the $H \gamma \gamma$ case, one obtains the two–loop amplitude

$$A_{W}^{L,R H \gamma \gamma} (\mu \rightarrow e + \gamma) = -\sum_a \cos \phi_a \Delta^a_{e\mu} L,R \left[3f(z_a) + 5g(z_a) + \frac{3}{4}g(z_a) + \frac{3}{4}h(z_a)\right],$$

with $z_a = M_W^2/M_{H_a}^2$. The function $h(z)$ is defined as

$$h(z) = z^2 \frac{\partial}{\partial z} \left( \frac{g(z)}{z} \right) = \frac{z}{2} \int_0^1 \frac{dx}{z-x(1-x)} \left[1 + \frac{z}{z-x(1-x)} \ln \frac{x(1-x)}{z} \right].$$

It is straightforward to see that $A_{W}^{L,R H \gamma \gamma} (\mu \rightarrow e + \gamma)$ for this $W$–loop amplitude and other purely bosonic loop contributions, unlike the situation for the $t$–loop with the pseudoscalar coupling $\text{Im}\Delta^a_{e\mu}$. Therefore we shall drop the chirality label for amplitudes from sets B and C for brevity. Note that if CP is conserved, $\cos \phi_a = 0$ unless the $H_a$ is a scalar. Therefore the flavor changing pseudoscalar Higgs boson does not have contribution of this class in such case.

It is useful to know the shapes of these functions $f$, $g$ and $h$. Numerically, $f(1)$ is about 0.8, while $g(1)$ is about 1.2. The general $z$ dependence of these functions are given in Fig. 3.
Unless \( z \) is very small or very large, these functions are of order unity. For very large or very small \( z \) \[12,14,15\],

\[
\begin{align*}
    f(z \gg 1) &\sim \frac{1}{3} \ln z + \frac{13}{18}, &
    g(z \gg 1) &\sim \frac{1}{3} \ln z + 1, &
    h(z \gg 1) &\sim -\frac{1}{2} (\ln z + 1), \\
    f(z \ll 1) &\sim \frac{1}{2} (\ln z)^2, &
    g(z \ll 1) &\sim \frac{1}{2} (\ln z)^2, &
    h(z \ll 1) &\sim z \ln z.
\end{align*}
\] (16)

For the large \( z \) asymptotic forms, we obtain Eq.(4) in the light–Higgs limit from Eq.(14).

It is tempting to use also Eq.(14) to find the heavy–Higgs \( z \ll 1 \) limit, which will produce the estimate Eq.(5) given in ref. [10]. However, this estimate clearly overlooks other non–decoupling contributions in Fig. 2c,d that we will discuss.

For the \( HZ\gamma \) case, one has

\[
A^{HZ\gamma}_{W\text{-loop}}(\mu \to e+\gamma) = -\frac{1 - 4 \sin^2 \theta_W}{4 \sin^2 \theta_W} \sum_a \cos \phi_a \Delta^a_{\epsilon \mu} \\
\times \left[ \frac{1}{2}(5 - \tan^2 \theta_W)f(z_a, z_Z) + \frac{1}{2}(7 - 3 \tan^2 \theta_W)g(z_a, z_Z) \\
+ \frac{3}{4}h(z_a) \right].
\] (17)

with \( z_Z = M^2_W/M^2_Z \). The suppression factor of \((1 - 4 \sin^2 \theta_W)/4 \sin^2 \theta_W \) comes from the vector part of the \( eeZ \) coupling. The \( W\text{-loop} \) contribution of \( HZ\gamma \) is about 10% of that of \( H\gamma\gamma \) and they have the same sign.

If we assume that CP is conserved and the Higgs boson is a CP–odd pseudoscalar, it does not couple to the \( W \) boson and there will be no contribution from set B and C. However, there will be \( \text{Im}\Delta^a_{tt} \) contributions in Eq.(3) from set A.

The type of diagrams that involve the Higgs mass squared in the coupling are shown in Fig. 2b,c,d, which are Fig. 2k,l,m in Ref. [10] respectively. They contain the coupling of the physical Higgs boson \( H \) to the unphysical Higgs pair \( G^+G^- \). An important exception is the contribution related to Fig. 2b. This diagram has been grouped into Fig. 2a with the bosonic inner loop because they are gauge related. The contribution proportional the Higgs mass squared, from the \( HG^+G^- \) vertex, can be combined with part of Fig. 2a to form a vertex that is proportional to the inverse Higgs propagator which then cancels with the Higgs
propagator in the outer loop. Thus the resulting contribution for each $H_a$ is independent of $M_{H_a}$ and therefore cancels each other completely because of the unitarity condition Eq. (13).

The contribution from Fig. 2c,d gives

$$A_{H\gamma\gamma \text{-loop}}^{\text{loop}}(\mu \rightarrow e + \gamma) = -\sum_a \cos \phi_a \Delta_{\epsilon\mu}^a \frac{1}{2z_a} \left[ f(z_a) - g(z_a) \right],$$

and

$$A_{H\gamma\gamma \text{-loop}}^{\text{loop}}(\mu \rightarrow e + \gamma) = -\frac{1 - 4 \sin^2 \theta_W}{8 \sin^2 \theta_W}(1 - \tan^2 \theta_W) \times \sum_a \cos \phi_a \Delta_{\epsilon\mu}^a \frac{1}{2z_a} \left[ \tilde{f}(z_a, z) - \tilde{g}(z_a, z) \right].$$

Note that $f(1) - g(1) = -0.4$, while for small $z$, which corresponds to the case of very large Higgs mass, $f(z) - g(z) \sim z (\ln z + 2)$. This leads to the peculiar situation where the contribution increases logarithmically with the Higgs boson mass. The coefficients are small enough that these contributions are not so significant as compared to the $W$-loop contribution discussed earlier, except for the case of very heavy Higgs boson. Of course, one may not trust the perturbative estimate if the Higgs mass becomes too heavy and the Higgs self-coupling becomes nonperturbative.

The contribution of the two-loop graphs in set C can be easily translated from the calculation of Ref. [16]. The result is

$$A_C(\mu \rightarrow e + \gamma) = -\frac{1}{4 \sin^2 \theta_W} \sum_a \cos \phi_a \Delta_{\epsilon\mu}^a \left[ D_e^{(3a)}(z_a) + D_e^{(3b)}(z_a) + D_e^{(3c)}(z_a) \right. + D_e^{(3d)}(z_a) + D_e^{(3e)}(z_a) + D_e^{(3f)}(z_a) + D_e^{(3g)}(z_a) + D_e^{(3h)}(z_a) \right],$$

where the functions $D_e^{(3a,b,c,d,e)}(z)$ and $D_e^{(4a,c)}(z)$ are given in Appendix B of Ref. [16], $zz_a = M_Z^2/M_{H_a}^2$ and $D_e^{(4b)}(z) = 4 \sin^2 \theta_W \tan^2 \theta_W D_e^{(3c)}(z)$. Note that the terms with functions $D_e^{(3a,b,c,d,e)}(z_H)$ belong to the group $C_1$, while the rest belong to the second group $C_2$. As commented earlier the second group is suppressed relative to the first group. The reason one can easily translate the calculation of $edm$ from these graphs into contributions to the transitional magnetic moment is because the Higgs line is always attached to one of the external fermion lines, and because the Higgs boson only has scalar couplings to gauge
bosons at tree level. Therefore, in the case of edm the Higgs coupling to fermions is always pseudoscalar, while its coupling for transitional magnetic moment is always scalar.

Just as the calculation of edm in Ref. [16], the contribution of the graphs in set C to the transitional magnetic moment is also small. We therefore do not include them in our numerical analysis.

IV. DISCUSSION AND CONCLUSION

In our numerical analysis, we shall ignore CP violation and take $\Delta_{tt}^a$ to be real. We also assume that $\Delta_{e\mu}^L = \Delta_{e\mu}^R = \Delta_{e\mu}^a$, i.e., scalar Higgs couplings. Under this condition, the reduced amplitudes are simplified $A^L = A^R = A$. Comparison with experiment is given in Fig. 4. The result in general depends on three parameters, in addition to the unknown top and Higgs masses. They are the parameters $\Delta_{tt}$ and $\Delta_{e\mu}$ (Eq.(9)) which parametrize the $ttH$ and $e\mu H$ couplings, and $\cos\phi_a$, which parametrizes the $H_aWW$ coupling. We set these parameters to one in our figures as a reference point. At this moment, we pretend that the contributions from different Higgs bosons do not strongly cancel each other. The results due to the contributions from a single Higgs boson are shown in Fig. 4. Numerically the contribution from the top–quark loop is generally smaller than that from the $W$–boson loop. Also, the contributions due to the $Z$ boson can be ignored although we have included them in our numerical analysis.

It is instructive to look at the numerical results at the amplitude level. In Fig. 5, we show the reduced amplitudes $A$’s from various sources. The error bar of the data point indicates the experimental bound on $\mu \rightarrow e\gamma$, $|A| \lesssim 1$. The data point is used to guide the reader’s eyes. It is purposely located at the lower bound of the Higgs mass at 91 GeV from the one–loop result. Because the two–loop $W$ contribution does not vanish in the heavy–Higgs limit, one may need to sum up contributions from different Higgs bosons. For the case of the two–doublet model, the unitarity condition implies that the $W$ amplitude is just the difference between those from the Higgs bosons at two separate masses. It is understood
that when the Higgs bosons are degenerate in mass, the \(W\)-loop contributions cancel each other.

Some interesting features deserve special attention:

1. As \(M_H \to \infty\), the amplitude in Eq. (18) does not go to zero, unlike Barr’s estimate in Eq. (5). In fact it goes to infinity as \(\ln M_H\). This non-decoupling behavior is curious but can be easily understood. In Feynman gauge, the non-decoupling graphs involve neutral Higgs coupling to unphysical charged scalars, which is proportional to Higgs mass squared. Such couplings are dictated by the gauge symmetry and its breaking. At tree level, the Higgs field that appears in the \(W^+W^-H\) coupling is the scalar component of the Higgs doublet that generated the symmetry breaking (the other three components are precisely the unphysical scalar bosons). This component is in general not a mass eigenstate of course. It can be expressed as the sum of the scalar components of the Higgs fields that participate in the breaking of \(SU_L(2)\), with coefficients proportional to their respective VEV’s. Therefore any Higgs boson that couples to the \(W\) pair has to originate from some weak multiplet that contributes significantly to the breaking of \(SU_L(2)\). Hence the natural scale for such Higgs bosons should be the \(SU_L(2)\) breaking scale, \(v\). If their masses are artificially pushed much higher, say, by fine tuning, their physical consequences would not decouple. For similar reasons one can also conclude \(^{19}\) that, in Standard Model, the Higgs contribution to \(g-2\) of charged leptons would also not decouple. For very large Higgs mass, it should diverge as \(\ln M_H\).

2. In set B, the contribution from Eqs. (18,19) that gives rise to the non-decoupling behavior has a different sign compared to the other part from Eqs. (14,17). At a low Higgs mass below 200 GeV, the non-decoupling contribution are small. This is the region where Barr’s estimate applies \(^{11}\). However, around 600 GeV a perfect cancellation occurs as shown in the solid curve Fig. 5. Even if the contribution from the top–quark loop is included, it will only shift the position of the amplitude zero by a small amount, depending on the value of \(m_t\). If the mass of the flavor changing neutral Higgs boson happens to lie in this region, \(BR(\mu \to e\gamma)\) would be very suppressed and further improvements of the experimental limits will not be
very constraining. Such cancellation behavior can potentially become a crucial issue in the future.

(3) The present experimental limit requires $M_H \gtrsim 200$ GeV under the same simplifying assumptions made earlier. This is a factor of at least 2 better than the one–loop limit. Our conservative limit is substantially lower than the bound 730 GeV from Eq.(5), where only some of the $W$–loop contributions are included. It turns out that other contributions from the non–decoupling term and the $t$–loop diagram reduce the bound substantially.

(4) The result is only mildly sensitive to the top quark mass. However, there exist models in which the flavor changing neutral Higgs bosons have very small couplings to the $W$ boson. That is, $\cos \phi$ may be very small. In that case the top quark contribution dominates at two–loop and is still larger than the one–loop result. Conversely, there are some other models in which the Higgs boson that couples to leptons is different from the one that couples to up–type quarks. In that case, the parameter $\Delta_{tt}$ would be zero and the top–loop contribution should be ignored.

(5) If the Higgs boson couples also to the down type quarks as in the general Cheng–Sher scheme, the constraint from $K_L - K_S$ mass difference in general would dominate over those from $\mu \to e\gamma$, although they are modulated by different $\Delta_{ij}$ factors.

(6) The limit on $\mu \to e\gamma$ of course will improve in the future. However, note that most of the severe low energy FCNC constraints such as $K_L - K_S$ mass difference in general would dominate over those from $\mu \to e\gamma$, although they are modulated by different $\Delta_{ij}$ factors. FCNC constraints involving up–type quarks (e.g. $D^0 - \bar{D}^0$ mixing) are rather weak. As pointed out in Ref. (6), it is in fact easy to avoid constraints from $K$, $B$ and $\mu$ systems completely, by assuming NFC for down–type quarks and charged leptons. Although the ansatz may seem a bit artificial, it does, however, permit tantalizing phenomenological consequences for the top quark (6), despite the depressed $\mu \to e\gamma$ transitions.

To conclude, we have derived the result for the two–loop contribution of flavor changing neutral Higgs bosons to the celebrated rare decay $\mu \to e\gamma$, for arbitrary Higgs and top masses. This is one of the rare situations in which higher order contributions actually dominate over
lower order ones. The numerical consequences depend, of course, on the model. For the
generic case in the scheme of Cheng and Sher [4], the result shown in Fig. 4 improves the
one-loop bound by more than a factor of two. The curious behavior of non-decoupling of
very heavy Higgs boson effects at two-loop is emphasized.

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FIGURES

FIG. 1. A one–loop Feynman diagram for $\mu \rightarrow e + \gamma$ through the $\tau$ lepton as the intermediate fermion.

FIG. 2. Feynman diagrams for $\mu \rightarrow e + \gamma$. The generic inner–loop in (a) involves the $t$–quark, the $W$ boson and its ghost. For purely bosonic contributions, diagram (a) includes sea-gull graphs and other gauge related graphs except that we separate out those with vertices $G^+G^-H \propto M_H^2$ in different diagrams (b), (c) and (d). We also do not show conjugate diagrams with lines of the neutral gauge boson and the Higgs boson exchanged.

FIG. 3. Numerical values for the functions $f(z), g(z)$ and $h(z)$.

FIG. 4. Numerical estimate of the $\mu \rightarrow e + \gamma$. The dash–dotted line is the one–loop result via the intermediate $\tau$ lepton. The two–loop result, assuming coming from one single neutral Higgs boson at $M_H$, is given by the solid (dashed) curve for the case $m_t = 100$ (200) GeV and $\cos \phi_a \Delta^{a}_{\mu} = 1, \Delta_H = 1$.

FIG. 5. Reduced amplitudes $A$’s for the process $\mu \rightarrow e + \gamma$. The dotted line is the one–loop result via the intermediate $\tau$ lepton. The data point is located at the $M_H$ lower bound due to the one–loop result. The error bar indicates the experimental bound on the reduced amplitude. The two–loop $t$ contribution is given by the dashed (dash–dotted) curve for the case $m_t = 100$ (200) GeV and $\Delta_H = 1$. The two–loop $W$ contribution, for the case $\cos \phi_a \Delta^{a}_{\mu} = 1$, is given by the solid curve, which does not vanish in the heavy–Higgs limit.