Equivalence between a topological and non-topological quantum dot - hybrid structures

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In this work, we demonstrate an intriguing equivalence on the single-electron transport properties between a topological quantum system and a (conventional) non-topological one. Our results predicts that the Fano resonances obtained in a T-shaped double quantum dot system coupled to two normal leads and one superconducting lead (QD–QD–SC) are identical to the obtained in a ring system composed of a quantum dot coupled to two Majorana bound states confined at the ends of a one dimensional topological superconductor nanowire (QD–MBSs). We have also found that the Fano resonance is a clear signature of the existence of Majorana modes in the QD-MBSs system. Furthermore, the Fano resonance in the QD–QD–SC system could give a sign of the role of the superconducting lead in the system. We believe that our results can motivate further theoretical and experimental works toward the understanding of transport properties of topological quantum hybrid structures from conventional non-topological quantum systems.

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I. INTRODUCTION

Andreev bound states (ABSs) are discrete entangled-particle-hole excitations confined to a quantum dot that is coupled to a conventional superconductor and with an energy in the superconducting gap. Since it was realized that Majorana fermions have profound technological applications for quantum information processing, there has been renewed interest in ABSs. This is due to that Majorana fermions are zero-energy ABSs (also known as Majorana zero modes or Majorana bound states - MBSs) that exist at the surface of a topological superconductor. Its been shown that Majorana fermions play a topologically protected role in quantum computing, which in contrast to ordinary quantum computation, Majorana fermions are immune to local noise by virtue of their nonlocal topological nature, hence their implementation as data carriers would not require any quantum error correction. In general, Majorana fermions take place in quantum systems with strong spin-orbit coupling, superconductivity, and broken time-reversal symmetry. The most promising platform, involves, topological superconductors realized in semiconductors. The main advantage of the semiconductor/superconductor proposal is its simplicity: it includes a conventional semiconductor with strong Rashba coupling such as InAs or InSb, a conventional superconductor such as Al or Nb, and an in-plane magnetic field.

The possible detection of MBSs have promoted the research on hybrid systems that involve quantum dots coupled to topological superconductors (QD–MBSs configurations), therefore explored intensively to verify the existence of MBSs via probing their transport conductance spectrum, wherein quantum interference as Fano (anti)resonances and current noise yield clear evidence of their existence. Fano resonances in transport have been known for a long time and are caused by the interference of two transport pathways, a resonant and a nonresonant one. This quantum mechanical interference yields a characteristic asymmetric lineshape that is featured by the Fano factor, which is a measure of the coupling strength between the continuum and the resonant state. In hybrid systems that involve QDs coupled to normal (N) leads and conventional superconducting (SC) leads show that the Fano (anti)resonances are an indication of the existence of ABSs. Likewise, the MBSs are detected by Fano (anti)resonances in hybrid nanostructures of QDs coupled to topological superconductor nanowires.

The peculiarity of characterizing ABSs and MBSs by Fano (anti)resonances have motivated several works. In this sense, we would like to address particular attention to the recent paper of Zeng et. al. In this paper the electron transport inside a topological configuration that consists of a ring system composed of a QD coupled to two MBSs confined at the ends of a 1D topological superconductor nanowire (TSNW) is investigated. It is found that these MBSs are strongly influenced by the threaded magnetic flux and that the Fano profile in the conductance spectrum can be used
to detect the Majorana zero-energy modes. The authors tried to find the equivalent non-topological configuration by comparing the TSNW to a conventional system of two coupled quantum dots (QD–QD system). However, the distinctive signals in the Fano resonances show that both systems are indistinguishable non-equivalent. The key issue in their comparison appeared when the threading magnetic flux through the ring is \( \phi = (2n + 1)\pi \). Under this condition, the conductance at Fano resonances in the QD–MBSs system is not suppressed to zero, while the conductance at Fano resonances in the QD–QD system do. It is an expected result if we consider that the drop to zero in Fano resonance of QD–QD systems is a well-known fact.

However, it is known from previous works20,26 that when a conventional (i.e., non-topological) superconducting lead is coupled to a QD–QD system, this will exhibit in its transport properties two Fano antiresonances due to the ABSs and MBSs, we found that these two systems are equivalent and that the Fano effect in each system is determined by their own structural parameters, i.e., QD–SC coupling, QD–normal leads coupling and the coupling between the Majorana fermions.

In order to demonstrate the equivalence between QD–QD–SC and QD–MBSs, this paper is organized as follows. In Sec. II we describe the model to study the QD–QD–SC and QD–MBSs, in Sec. III we discuss the numerical results and, finally, a brief summary is given.

II. DESCRIPTION OF THE MODEL

In this paper, we consider a system composed by a T-shaped double quantum dot (with a single-level in each quantum dot) that is coupled to two normal metallic leads and to a superconductor lead, as shown in Fig. 1 (a). The double quantum dot is modeled by a two impurity Anderson Hamiltonian and the Hamiltonian for the whole system can be written as:

\[
H = H_{L(R)} + H_S + H_{dot} + H_T ,
\]

where \( H_{L(R)} \) is the Hamiltonian for the left (right) normal lead, which is given by

\[
H_{L(R)} = \sum_{k\sigma} c_{k\sigma L}^\dagger c_{k\sigma R} \quad \text{and} \quad C_{k\sigma L(R)}^\dagger c_{k\sigma L(R)} \sigma \quad \text{being} \quad C_{k\sigma L(R)}^\dagger \text{creation and annihilation operator for electrons with momentum } k_{L(R)} \text{ and spin } \sigma \text{ in the metallic lead } L(R), \]

metallic lead \( L(R) \), respectively. The standard BCS Hamiltonian for the superconductor lead is

\[
H_S = \sum_{k\sigma} \epsilon_{k\sigma} C_{k\sigma L}^\dagger C_{k\sigma R} + \sum_{k\Delta} \Delta \left( C_{k\sigma L}^\dagger C_{-k\sigma R} + h.c. \right) ,
\]

where \( C_{k\sigma L(R)}^\dagger \) and \( C_{k\sigma L(R)} \) are the creation and annihilation operators for electrons in the superconducting lead, while \( \Delta \) is the superconducting gap function which is assumed to be s-wave, i.e., \( k \)-independent and real (\( \Delta^\dagger = \Delta \)). The Hamiltonian for the double quantum dot is given by

\[
H_{dot} = \sum_{\ell\sigma} \epsilon_{\ell\sigma} d_{\ell\sigma}^\dagger d_{\ell\sigma} + \sum_{\ell\sigma} \left( V_{\ell\sigma}^L C_{k\sigma L}^\dagger d_{\ell\sigma} + h.c. \right) + \sum_{\ell\sigma} \left( V_{\ell\sigma}^R C_{k\sigma R}^\dagger d_{\ell\sigma} + h.c. \right) ,
\]

where the embedded quantum dot (QD1) is coupled to the side-coupled quantum dot (QD2) via the interdot coupling \( t \), which is taken as being a real parameter.

In general, the current from the lead \( \alpha \) (\( \alpha = L \) to \( R \)) is given by

\[
I_\alpha = e \langle N_\alpha \rangle = \frac{e}{\hbar} \langle [H, N_\alpha] \rangle
\]

from which, we can get the zero-temperature conductance is

\[
I_L = -\frac{2e^2}{\hbar} \int d\omega \left[ f_L (\omega) - f_R (\omega) \right] \frac{\Gamma_L \Gamma_R}{\Gamma_L + \Gamma_R} \Im \left[ G_{1,11}^L (\omega) \right] ,
\]

Then, the Green function of the embedded quantum dot \( G_{1,11}^L (\omega) \) in the far subgap regime \( |\omega| \ll \Delta \) (in which only the off-diagonal terms of the superconductor’s self-
energy matrix are preserved tending to the static value \( \Gamma_S/2 \) is

\[
G_{r,11}^\tau (\omega) = \frac{1}{D(\omega)} \left( (\omega + \epsilon_2) + i \frac{\Gamma}{2} - N(\omega)(\omega - \epsilon_1) \right),
\]

where \( D = \mathcal{F}_+(\omega)\mathcal{F}_-(\omega) - ((\Gamma/2)N(\omega))^2 \), \( \mathcal{F}_\pm(\omega) = \omega \mp \epsilon_2 + i \frac{\Gamma}{2} - N(\omega)(\omega \pm \epsilon_1) \), and \( N(\omega) = t^2(\omega - \epsilon_1)(\omega + \epsilon_1) - (\frac{\Gamma}{2})^2 \). The retarded Green function of the QD for the system QD-TSNW (Fig. 2(b)) has the following form

\[
G_{d,d1}^\tau (\omega) = \left[ \omega - \epsilon_d + i \frac{\Gamma}{2} - A(\omega) - B(\omega) \right]^{-1},
\]

where \( A(\omega) = K \left( |\lambda_1|^2 + |\lambda_2|^2 + 2\frac{\epsilon_M}{\omega} |\lambda_1||\lambda_2| \cos \frac{\phi}{2} \right) \) and \( B(\omega) = -\frac{K^2(1 + 2|\lambda_1|^2 - 2|\lambda_2|^2 \cos \phi)}{\frac{\epsilon_M}{\omega} - A(\omega)} \), with \( K \) and \( \Gamma \) being defined as \( K = \frac{\omega - \epsilon_d - \epsilon_1 - \epsilon_2}{\epsilon_M} \) and \( \Gamma = \Gamma_L + \Gamma_R \). The parameter \( \epsilon_M \) is the overlap of the Majorana fermions \( \gamma_1 \) and \( \gamma_1 \) and \( \epsilon_d \) is the energy of the QD.

### III. RESULTS AND DISCUSSION

In this section we discuss the transport properties at zero temperature \( T = 0 \). The dot-lead couplings is set to be symmetric \( \Gamma_L = \Gamma_R \). \( \Gamma = \Gamma_L + \Gamma_R \) will be considered as the energy unit and \( E_F = 0 \).

In Fig. 2 we show the conductance (in units of \( e^2/\hbar \)) in function of the bias voltage \( eV/\Gamma \) for the QD–QD–SC system. We can observe how the conductance changes for several values of the coupling with the superconducting lead, \( \Gamma_S \). When the superconducting lead is not connected to the 2QDs system, that is, \( \Gamma_S = 0 \) (see the panel (a)) we can notice a symmetric Fano line shape that vanishes at \( eV/\Gamma = 0 \). Once the superconductor is weakly coupled we can observe two small antiresonances.

As the coupling with the superconducting lead \( \Gamma_S \) increases, the antiresonances become into two accentuated Fano antiresonances, whose minimal do not fall to zero, that is, the antiresonances have a complex Fano factor \( q_S \). These Fano antiresonances originate from the interference between electrons traversing in different paths (a resonant and a non resonant one) when they propagate from the left to the right leads. The antiresonances, located around \( \pm \Gamma_S/2 \) have identical shape but an opposite sign of the parameter \( q_S \) and are due to the Andreev Bound States (ABS).

As mentioned in the paper of Zeng et. al, Fano profile in the conductance can be used to detect the Majorana zero-energy mode in the topological QD–MBSs. Authors further argue that the QD–MBSs can be mapped into a QD–QD, except when the threading magnetic flux through the ring is \( \phi = (2n + 1) \pi \). At this condition the QD–MBSs and the QD–QD are not equivalent to each other, since the conductance at Fano resonances in the QD–MBSs is not suppressed to zero, while the conductance at Fano resonances in the QD–QD does. Motivated by these results, we realized an alternative non-topological QD–QD system that is fully equivalent to the topological QD–MBSs configuration.

Our system denoted as QD–QD–SC consists of a su-
perconducting lead connected to the QD–QD system. As we display in Fig. 2 when a superconducting lead is connected, the transport properties exhibit two Fano resonances that do not drop to zero. In order to compare the QD–QD–SC and the QD–MBSs to each other, we show in Fig. 3 the conductance as a function of the bias voltage for both systems. In the right panel we show the conductance as a function of the bias voltage eV/Γ for the QD–QD–SC for several values of Γ_S: (a) Γ_S = 1Γ and (b) Γ_S = 1.5Γ. In this figure we observe two Fano resonances located approximately in ± Γ_S/2, whose values of the Fano factor increase with increasing the coupling with the superconductor Γ_S. These two Fano resonances move away as we increase the coupling with the superconductor. The conductance formula in the region eV ≈ Γ_S/2 can be fitted to a Fano function as

\[ G_{SC} \approx A_S \left( \frac{eV}{\Gamma_S} + q_S \right)^2 \left( \frac{eV}{\Gamma_S} + 1 \right)^{-1}, \]

where

\[ A_S = -2t^2/\Gamma_S^2, \quad B_S = \left( 8t^4/\Gamma_S^2 - 4t^2/\Gamma_S^2 + \Gamma_S^2/\Gamma \right)/\left( 2\Gamma_S^2 \Gamma \right), \quad C_S = 2t^2/(\Gamma_S^2 + 4t^2)/\Gamma_S^2, \quad \Gamma_S = \Gamma^2 + \Gamma^2, \]

and the absolute value of the Fano factor \( q_S \) is given by

\[ |q_S| = \frac{1}{\sqrt{2}} + \frac{\Gamma_S}{\Gamma} \left( \frac{eV}{\Gamma} \right)^2 + \frac{4t^2}{\Gamma^2 + \Gamma_S^2 - 4t^2}. \]  \( \text{(7)} \)

In contrast, in panels (c) and (d) of Fig. 3, we observe the conductance spectra for the QD–MBSs for several values of the coupling between Majorana fermions \( \epsilon_M \). We observe two Fano resonances around eV = ± \( \epsilon_M \) whose minima do not fall to zero. These Fano resonances originate from the interference between those electrons going through the QD without going in the nanowire and those going into the nanowire. The conductance formula in the region eV ≈ \( \epsilon_M \) can be fitted to a Fano function as

\[ G_M \approx A_M \left( \frac{eV}{\epsilon_M} + q_M \right)^2 \left( \frac{eV}{\epsilon_M} + 1 \right)^{-1}, \]

where

\[ A_M = -\frac{1}{2} \left( \frac{1}{\epsilon^2_M + (\frac{eV}{\epsilon_M})^2} \right), \quad B_M = \left( 1 + \frac{2|\lambda_1|^2}{\epsilon^2_M + (\frac{eV}{\epsilon_M})^2} \right) \epsilon_M, \]

\[ C_M = \sqrt{\frac{4|\lambda_1|^4}{\epsilon^2_M + (\frac{eV}{\epsilon_M})^2} \left( 1 - \frac{\epsilon^2_M + (\frac{eV}{\epsilon_M})^2}{\epsilon^2 + (\frac{eV}{\epsilon_M})^2} \right)}, \]

and the absolute value of the Fano factor \( q_M \) is given by

\[ |q_M| = \sqrt{\frac{1}{2} + \frac{C_M}{\epsilon_M}}. \]  \( \text{(8)} \)

The absolute value of the Fano factor increase as we increase the coupling \( \epsilon_M \) between the Majorana fermions. Increasing (decreasing) \( \epsilon_M \) also increases (decreases) the distance between the Fano anti-resonances. At \( \epsilon_M = 0 \) the two Fano resonances merge into one resonance, hence the two MBSs do not overlap, which in terms of conductance means that the peak value of the dot conductance at zero bias (when the dot is symmetrically coupled to the leads) is \( e^2/2h \). At this point, it is relevant to mention that when the nanowire is in its topological phase, the zero temperature value of the dot conductance at \( \epsilon_M = 0 \), is predicted to be \( e^2/2h \). In contrast, when the wire is in its trivial phase, the conductance peak value is zero whenever a regular fermionic zero mode occurs on the wire ends, as occurs in Fig. 2 (a) for \( \Gamma_S = 0 \).

From Fig. 5 we can observe that the QD–QD–SC can be mapped to a QD–MBSs, when \( \phi = (2n + 1) \pi \); that is, when the nanowire is in its topological phase (the one with Majorana zero modes at the end of the nanowire). This equivalence arises from the fact that in the QD–QD–SC system the Fano resonances are mediated by the ABS, which are particle-hole excitations of the SC whose field operators, written as \( \gamma^1 = (\gamma_1 + i\gamma_2)/\sqrt{2} \) and \( f = (\gamma_1 - i\gamma_2)/\sqrt{2} \), can be decomposed into a pair of Majorana operators \( \gamma_1 = \gamma_1^\dagger, \gamma_2 = \gamma_2^\dagger \). The equivalence can also be explained from the QD–MBSs side. When the nanowire is in its topological non-trivial phase the Majorana zero modes appear at the ends of the TSNW (which can also be seen as zero energy ABSs), which can be represented by \( \gamma_1 = (f^1 + f)/\sqrt{2} \) and \( \gamma_2 = i(f^1 - f)/\sqrt{2} \). Since these fields are the self-adjoint \( \gamma_1 = \gamma_1^\dagger, \gamma_2 = \gamma_2^\dagger \), therefore, they represent mixtures of particle-hole states (ABBS). In consequence, we can conclude that the QD–QD–SC and QD–MBS systems are equivalent.

In order to reinforce this statement, in Fig. 6 (a) we show the contour plot of the Fano factor \( |q_S| \) as a function of \( t/\Gamma \) and \( \Gamma_S/\Gamma \). On the one hand, when \( \Gamma_S \gg t \) we observe that the dependence of \( |q_S| \) with the parameter \( t \) is irrelevant. We also observe a monotonous growth of \( |q_S| \) with \( \Gamma_S \) for any value of \( t \). On the other hand, when the previous condition is not fulfilled (when \( \Gamma_S \lesssim t \) or \( \Gamma_S \approx t \)), the contour plot indicates that the dependence of \( |q_S| \) with the parameters \( t \) and \( \Gamma_S \) is equally important and in consequence the growth of \( |q_S| \) with \( \Gamma_S \) is not monotonous as for the case \( \Gamma_S \gg t \). However, in order to underpin our previous analysis about the equivalent between both systems as shown in Fig. 3, in this paper we focus our attention to the limit \( \Gamma_S \gg t \). Indeed, a one-to-one relation between \( \epsilon_M \) and \( \Gamma_S \) can be obtained for several values of the parameter \( t \). For this, we assumed that \( |q_S| = |q_M| \) then from equations \( 7 \) and \( 8 \).
we obtain that

$$\epsilon_M = \sqrt{\left(\frac{\Gamma_S}{2t}\right)^2 + \frac{4t^2}{\Gamma^2 + \sqrt{\Gamma^2 - 4t^2}}}$$  \hspace{1cm} (9)$$

From this equation we observe a linear relation between $\epsilon_M$ and $\Gamma_S$ with a proportional term of $1/2$ at the limit $\Gamma_s \gg t$. This can be verified in Fig. 4 (b) and Fig. 4 (c) where the linear and derivative relation between $\epsilon_M$ and $\Gamma_S$ are displayed, respectively. It is worth noticing here that the assumption $|q_S| = |q_M|$ is more straight at our limit of interest. Indeed, by taking $\Gamma_s \gg t$ in the equation (7) we obtain $|q_S| \approx \sqrt{\frac{1}{2} + 2 \left(\frac{\Gamma_M}{\Gamma_S}\right)^2}$, which coincide in an identical mathematical aspect to the results obtained for $|q_M|$ in equation 8 as far as we map $\epsilon_M$ to $\Gamma_S/2$. Accordingly, we can conclude that the systems QD–QD–SC and QD–MBS are equivalent to each other from their transport properties point of view, furthermore, that the Fano effect can be determined by the structural parameters of the system, i.e., $\Gamma_S$, $t$, $\epsilon_M$ and $\Gamma$. 

Finally, it is worth to mentioning that a complex Fano factor $q$ is an indication of the broken time reversal symmetry, which can be introduced, for instance, with the application of a magnetic field. This is the case of the QD–MBS where the Fano antiresonances do not fall to zero (as shown in Fig. 3 (c) and (d)) due to the applied magnetic field. This notable feature can be used to detect the Majorana zero energy modes in the system. In contrast, in the QD–QD–SC we observe the same nonzero Fano antiresonances as shown in Fig. 3 (a) and (b). This is an non-trivial results worth of being highlighted given that the ground state of a superconductor does not conserve the number of particles and maintains the time reversal symmetry. One possible explanation to this behaviour could be as follows. Despite the attractive picture of Andreev reflected hole as the time reverse of the incident electron (i.e., the time reverse property is maintained), we can realize that this scheme breaks down upon closer inspection29. There is a phase shift acquired upon Andreev reflection that spoils the time-reversing properties; therefore we could deduce that the superconducting lead has the role of introducing this phase and consequently, breaking the time-reversal symmetry in the system QD–QD–SC.

**SUMMARY**

In this work, we demonstrated an interesting case where the electronic transport properties of a topological and a non-topological quantum dot hybrid structures are identical to each other. Specifically, we studied the case of (1) a T-shaped double quantum dot system coupled to two normal leads and one superconducting lead (QD–QD–SC) and (2) a ring system composed of a QD coupled to two Majorana bound states confined at the ends of a 1D topological superconductor nanowire (QD–MBSs). Our analysis is focused on the physics of the Fano effect originated in both systems due to the interference among different transmission paths. We demonstrated a regime of parameters where the Fano factor $q_S$ of the QD–QD–SC is equivalent to the Fano factor $q_M$ of the QD-MBSs. Furthermore, we found the functional dependence of the Fano effect in terms of the structural parameters of the system, i.e., $\Gamma_S$, $t$, $\epsilon_M$ and $\Gamma$. We have also found that the complex Fano factor $q_M$ is a clear signature of the existence of Majorana modes in the QD-MBSs. Analogously, the complex Fano factor $q_S$ is a signal of the fact that the superconducting lead is introducing a phase in the QD–QD–SC. We argue that the equivalence between both systems is, indeed, due to this induced phase. We believe that our results can motivate further works (theoretical and experimental) toward the understanding of transport properties of topological quantum hybrid structures from conventional non-topological quantum systems.

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