WEIGHTED INEQUALITIES FOR THE FRACTIONAL LAPLACIAN AND THE EXISTENCE OF EXTREMALS

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Abstract. In this article we obtain improved versions of Stein-Weiss and Caffarelli-Kohn-Nirenberg inequalities, involving Besov norms of negative smoothness. As an application of the former, we derive the existence of extremals of the Stein-Weiss inequality in certain cases, some of which are not contained in the celebrated theorem of E. Lieb [14].

1. Introduction

In the Euclidean space $\mathbb{R}^n$, it is well-known that negative powers of the Laplacian admit the integral representation in terms of the Riesz potential or fractional integral operator:

$$(−Δ)^{-s/2}f(x) = I_s(f) = c(n, s)\int_{\mathbb{R}^n} \frac{f(y)}{|x−y|^{n−s}}\,dy \quad 0 < s < n.$$  

One basic result for this operator is the Stein-Weiss inequality, which gives its behaviour in Lebesgue spaces with power weights:

Theorem 1.1. [17, Theorem $B^s$] Let $n \geq 1$, $0 < s < n$, $1 < p \leq r < \infty$, $\alpha < \frac{n}{p'}$, $\gamma > -\frac{n}{p'}$, $\alpha \geq \gamma$, and

$$(1) \quad \frac{1}{r} = \frac{1}{p} + \frac{\alpha - \gamma - s}{n}.$$  

Then,

$$(2) \quad \| |x|^\gamma(−Δ)^{-s/2}f\|_{L^r} \leq C \| |x|^\alpha f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^n, |x|^\alpha p).$$

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Equivalently, we can rewrite this result as fractional Sobolev inequality, namely,

\[ \| |x|^{\gamma} u \|_{L^r} \leq C \| |x|^{\alpha} (-\Delta)^{s/2} u \|_{L^p} \quad \forall u \in \dot{H}^{s,p}_{\alpha}(\mathbb{R}^n) \]

meaning that we have a continuous embedding

\[ \dot{H}^{s,p}_{\alpha}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n, |x|^{\gamma r}) \]

where \( \dot{H}^{s,p}_{\alpha}(\mathbb{R}^n) = \{ u = (-\Delta)^{-s/2} f : f \in L^p(\mathbb{R}^n, |x|^{\alpha p}) \} \)

is the weighted homogeneous Sobolev space of potential type, which is a Banach space with the norm \( \| u \|_{\dot{H}^{s,p}_{\alpha}} = \| f |x|^{\alpha} \|_{L^p} \).

We remark that this embedding is not compact due to the scaling invariance of the Stein-Weiss inequality. In other words, the scaling condition \( p \) means that \( r \) plays the role of the critical Sobolev exponent in the weighted setting.

Our first aim in this work is to obtain an improved version of \( (2) \) and \( (3) \). More precisely, for suitable values of the parameters we will prove that there holds:

\[ \| |x|^{\gamma} (-\Delta)^{-s/2} f \|_{L^r} \leq C \| |x|^{\alpha} f \|_{L^p}^{\theta} \| f \|_{\dot{B}^{-\mu-s}_{\infty,\infty}}^{1-\theta} \]

for every \( f \in L^p(\mathbb{R}^n, |x|^{\alpha p}) \cap \dot{B}^{-\mu-s}_{\infty,\infty} \), or, equivalently,

\[ \| |x|^{\gamma} u \|_{L^r} \leq C \| |x|^{\alpha} (-\Delta)^{s/2} u \|_{L^p}^{\theta} \| u \|_{\dot{B}^{-\mu}_{\infty,\infty}}^{1-\theta} \]

for every \( u \in \dot{H}^{s,p}_{\alpha}(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{\infty,\infty} \), where the Besov norm of negative smoothness is defined in terms of the heat kernel (see Section 2 for a precise definition).

The reader will observe that inequality \( (6) \) is reminiscent of the well-known Caffarelli-Kohn-Nirenberg first order interpolation inequality:

**Theorem 1.2.** \( [3] \) Assume

\[ p, q \geq 1, \quad r > 0, \quad 0 \leq \theta \leq 1, \quad \frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0, \]

where

\[ \gamma = \theta \sigma + (1 - \theta) \beta. \]

Then, there exists a positive constant \( C \) such that the following inequality holds for all \( u \in C^\infty_0(\mathbb{R}^n) \)

\[ \| |x|^{\gamma} u \|_{L^r} \leq C \| |x|^{\alpha} \nabla u \|_{L^p}^{\theta} \| |x|^{\beta} u \|_{L^q}^{1-\theta} \]
if and only if the following relations hold:

\[
\frac{1}{r} + \frac{\gamma}{n} = \theta \left( \frac{1}{p} + \frac{\alpha - 1}{n} \right) + (1 - \theta) \left( \frac{1}{q} + \frac{\beta}{n} \right)
\]

\[0 \leq \alpha - \sigma \quad \text{if } \theta > 0,
\]

and

\[\alpha - \sigma \leq 1 \quad \text{if } \theta > 0 \quad \text{and} \quad \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}.
\]

Indeed, in the local case \(s = 1\) we will also obtain an improvement of this inequality in some cases, namely, that

\[(8) \quad \|x^\gamma u\|_{L^r} \leq C \|x^\alpha \nabla u\|_{L^p}^{\theta} \|u\|_{1 - \theta}^{\theta} \dot{B}^{-\mu}_{\infty,\infty}
\]

holds for every \(u \in \dot{H}^{1,p}(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{\infty,\infty}\) for an appropriate range of parameters.

Our second aim in this paper is to prove the existence of extremals of inequality (2) by means of a rearrangement-free technique, that allows us to obtain some previously unknown cases. Let us recall that, by definition, the best constant \(S\) in (2) is

\[(9) \quad S = \sup \frac{\|x^\gamma (-\Delta)^{-s/2} f\|_{L^r}}{\|x^\alpha f\|_{L^p}}
\]

where the supremum is taken over all the non vanishing functions \(f \in L^p(\mathbb{R}^n, |x|^{ap})\).

Best constants and the existence of minimizers/maximizers for the Stein-Weiss, Sobolev and related inequalities have been studied extensively in the literature, and it would be impossible to cite all the references, but we mention some of them that are closely related to our work. In a celebrated paper, E. Lieb proved [14, Theorem 5.1] the existence of minimizers of the Stein-Weiss inequality (in an equivalent formulation), under the extra assumptions

\[p < r, \alpha \geq 0, \gamma \leq 0.
\]

The sign restriction on the exponents in his result comes from the fact that his argument is based on a symmetrization (rearrangement) technique.

Recently, new rearrangement-free techniques for dealing with these inequalities in the unweighted case have been introduced in [7, 8, 16]. Avoiding the use of rearrangements could be useful to extend the results to settings where this technique is not available, for instance when mixed norms are considered (see, for instance, [5]), or in the setting of
stratified Lie groups like in [4, 8]. Indeed, our results can be extended
to the latter setting without essential modifications (replacing $n$ by the
homogeneous dimension of the group, and the Euclidean norm by an
homogeneous norm in the group). However, we have chosen to work
in the Euclidean space $\mathbb{R}^n$, in order to make our paper accessible to a
broader audience.

Improved Sobolev inequalities play an important role for the proofs
of existence of maximizers via concentration-compactness arguments.
We can mention, for instance, [8, Lemma 4.4] where R. L. Frank and E.
Lieb obtain an (unweighted) improved inequality with a Besov norm
in the context of the Heisenberg group, which they use derive sharp
constants for analogues to the Hardy-Littlewood-Sobolev inequality in
that group. More recently, the work of G. Palatucci and A. Pisante
[16], deals with the existence of maximizers in the unweighted case ($\alpha =
\gamma = 0$) in $\mathbb{R}^n$, also using and improved Sobolev inequality involving a
Morrey norm [16, Theorem 1.1], of which they give two different proofs.
One of them, related to our work, is based on the refined Sobolev
inequality of P. Gérard, Y. Meyer and F. Oru [9] involving a Besov
norm of negative smoothness, and an embedding result between Besov
and Morrey spaces [16, Lemma 3.4].

Along these lines, the existence of maximizers of (9) in the case
$\alpha = 0$ was considered in [20]. However, we believe that the argument
in that paper is not correct. Indeed, we could not check the validity of
inequality (3.2) in [20], as the application of the invoked rearrangement
inequality would require a decreasing function, that is, a negative ex-
ponent in the previous inequality. Hence, we don’t know whether it is
possible to perform the argument using the refined Sobolev
inequality with the Morrey norm in the presence of weights. For this reason, we
choose to work directly with the Besov norm and exploit some prop-
erties of the heat semigroup. With our improved inequality (Theorem
3.1), a weighted compactness result (Proposition 2.6), and the so-called
“method of missing mass” (invented by E. Lieb in [14]), we can prove
the existence of minimizers of the Stein-Weiss inequality only in the
case $p = 2$ but, in turn, we can have any $\gamma$ in the range $-\frac{n}{r} < \gamma < \alpha$,
thus extending the range $\gamma \geq 0$ of [14].

It should be mentioned that there is increasing literature devoted to
the study of improved versions of the Sobolev-Gagliardo-Nirenberg and
related inequalities for their own sake. Besides the above mentioned
articles [9] and [16], we can mention [11, 12, 13, 19], among others.
In the proof of our improved inequality, instead of using the Littlewood-Paley characterization of the Besov space (as in [9] and [11]), we use a simpler approach inspired by [4], which is based on the thermal definition of the Besov spaces and the representation of the negative powers of the Laplacian in terms of the heat semigroup. Besides that, we use the boundedness of the Hardy-Littlewood maximal function with Muckenhoupt weights, and the Stein-Weiss inequality. Moreover, our method does not involve truncations (as in [13]), a technique which seems not to work in our context due to the non-local character of the fractional Laplacian; and makes no use of rearrangements (as in [12]).

The rest of the paper is organized as follows: in Section 2 we recall the definition of the Besov spaces of negative smoothness and some results on the heat semigroup that will be used in the rest of the paper; in Section 3 we obtain the improved Stein-Weiss inequality (1) and its rewritten form (6), as well as the improved Caffarelli-Kohn-Nirenberg inequality (8); in Section 4 we prove that the embedding given by (6) is locally compact; and finally in Section 5 we use the method of missing mass and the results of the previous sections to prove the existence of extremals of the Stein-Weiss inequality.

2. Weighted Estimates for the Heat Semigroup

In this section we recall the definition of the Besov spaces of negative smoothness and collect some auxiliary results for the heat semigroup, which will play a central role in our approach.

We recall that the heat semigroup $e^{t\Delta}$ in $\mathbb{R}^n$ is given by

$$e^{t\Delta}f(x_0) = f * h_t(x_0) = \int_{\mathbb{R}^n} f(x)h_t(x_0 - x) \, dx$$

where

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp \left\{ -\frac{|x|^2}{4t} \right\}$$

is the heat kernel.

We shall use the following thermic definition of the Besov spaces, which goes back to the work of T. Flett [6]:

**Definition 2.1.** For any real $\delta > 0$ one can define the homogeneous Besov space $\dot{B}_{\infty,\infty}^{-\delta}$ as the space of tempered distributions $u$ on $\mathbb{R}^n$ (possibly modulo polynomials) for which the following norm

$$\|u\|_{\dot{B}_{\infty,\infty}^{-\delta}} := \sup_{t > 0} t^{\delta/2}\|e^{\Delta t}f\|_{L^\infty}$$

is finite.
We shall also need the following result:

**Proposition 2.2.** [15, Proposition 3.2] Let \( n \geq 2, \ 1 < p \leq q < +\infty \).
Assume further that \( \alpha, \beta \) satisfy the set of conditions

\[
\begin{align*}
\beta &> -\frac{n}{q}, \\
\alpha &< \frac{n}{p'}, \\
\alpha &\geq \beta.
\end{align*}
\]

Then the following estimates hold:

\[
\| x^\beta \partial^n e^{t\Delta} u \|_{L^q} \leq \frac{c_\eta}{t^{(|\eta|^p_\frac{n}{p}+\frac{n}{q}+\alpha-\beta)/2}} \| x^\alpha u \|_{L^p}, \quad t > 0,
\]

provided that \( |\eta|^p + \frac{n}{p} - \frac{n}{q} + \alpha - \beta \geq 0 \), for each multi-index \( \eta \). The range of admissible \( p, q \) indices can be relaxed to \( 1 \leq p \leq q \leq +\infty \) provided that \( \alpha > \beta \).

Specializing the above result for \( q = \infty \) and \( \beta = 0 \) we obtain the following corollaries:

**Corollary 2.3.** We have an embedding

\[
L^p(\mathbb{R}^n, |x|^\alpha p) \subset \dot{B}^{-\mu-s}_{\infty, \infty}
\]

with

\[
\mu = \frac{n}{p} + \alpha - s
\]

provided that \( \mu > 0 \) and \( 0 < \alpha < \frac{n}{p'} \). By the lifting property of Besov spaces (see [15, Section 5.2.3, Theorem 1]), this implies that

\[
H^{s,p}_\alpha(\mathbb{R}^n) \subset \dot{B}^{-\mu}_{\infty, \infty}.
\]

It follows that, with this choice of \( \mu \), our inequality (3) is indeed a refinement of the weighted fractional Sobolev inequality (3) and that (3) is in some cases a refinement of the Caffarelli-Kohn-Nirenberg inequality (7), in the sense of [16].

**Corollary 2.4.** Assume that

\[
0 < \alpha < \frac{n}{p'}
\]

and fix \( t > 0 \). Then

\[
|e^{t\Delta} f(x_0)| \leq C t^{-\frac{1}{2}(\frac{n}{p}+\alpha)} \| x^\alpha f \|_{L^p},
\]

and

\[
|\partial_{x_j} e^{t\Delta} f(x_0)| \leq C t^{-\frac{1}{2}(\frac{n}{p}+\alpha+1)} \| x^\alpha f \|_{L^p} \quad j = 1, 2, \ldots, n
\]

for any \( x_0 \) and any \( f \in L^p(\mathbb{R}^n, |x|^\alpha p) \).
Proposition 2.5. Let \( n \geq 2, 1 \leq p \leq +\infty \). Then, for any fixed \( t > 0 \), the operator \( e^{t\Delta} \) is compact from \( L^p(\mathbb{R}^n, |x|^\alpha) \) to \( L^\infty(\mathbb{R}^n) \) provided that

\[
0 < \alpha < \frac{n}{p'}.
\]

Proof. Let \( \{u_j\}_{j \in \mathbb{N}} \) be a bounded sequence in \( L^p(\mathbb{R}^n, |x|^\alpha) \), so that

\[
\| |x|^{\alpha} u_j \|_{L^p} \leq C.
\]

Let \( v_j = e^{t\Delta} u_j \). Then by Corollary 2.4, \( \{v_j\}_{j \in \mathbb{N}} \) is bounded in \( L^\infty(\mathbb{R}^n) \).

For each \( k \in \mathbb{N} \), let us consider the compact set

\[
C_k = \{ x \in \mathbb{R}^n : 0 \leq |x| \leq k \}.
\]

The estimates of Corollary 2.4 also imply that \( \{v_j\}_{j \in \mathbb{N}} \) is equibounded in \( C_k \), and so are their first order derivatives, hence \( \{v_j\}_{j \in \mathbb{N}} \) is also equicontinuous in \( C_k \). Using the Arzelà-Ascoli theorem and Cantor’s diagonal argument, we conclude that passing again to a subsequence we may assume that

\[
v_j \to v \text{ uniformly in each } C_k.
\]

Since \( \alpha > 0 \) we can choose \( \delta > 0 \) such that \( 0 < \delta < \alpha \). Then, by Proposition 2.2 we get

\[
\sup_{|x| > k} |v_j| \leq \sup_{|x| > k} \left( \frac{|x|}{k} \right)^\delta |v_j| \\
\leq \frac{1}{k^\delta} \| |x|^\alpha v_j \|_{L^\infty} \leq \frac{C}{k^\delta} \| |x|^\alpha u_j \|_{L^p} \leq \frac{C_1}{k^\delta}
\]

which tends to 0 as \( k \to \infty \), uniformly in \( j \). A standard argument gives that \( v_j \to v \) strongly in \( L^\infty(\mathbb{R}^n) \).

\[\blacksquare\]

3. Improved Inequalities

This section is devoted to establishing the improved Stein-Weiss and Caffarelli-Kohn-Nirenberg inequalities.

Theorem 3.1. Let \( n \geq 2, 0 < s < n, 1 < p \leq r, \alpha < \frac{n}{p}, -\gamma < \frac{n}{r}, \alpha - \frac{\gamma}{\theta} \geq 0, \mu > 0, \max\{\frac{p}{r}, \frac{\mu + s}{\mu + s - \alpha}\} \leq \theta \leq 1, \) and

\[
\gamma + \frac{n}{r} = \theta \left( \alpha + \frac{n}{p} - s \right) + (1 - \theta)\mu.
\]

Then, for every \( f \in L^p(\mathbb{R}^n, |x|^\alpha) \cap \dot{B}^{-\mu-s}_{\infty,\infty} \) there holds:

\[
\| |x|^\gamma (-\Delta)^{-s/2} f \|_{L^r} \leq C \| |x|^\alpha f \|_{L^p}^\theta \| f \|_{\dot{B}^{-\mu-s}_{\infty,\infty}}^{1-\theta}
\]
or, equivalently, for every $u \in \dot{H}^{s,p}(\mathbb{R}^n) \cap \dot{B}^{-\mu}_{\infty,\infty}$,
\[
\| |x|^\gamma u \|_{L^r} \leq C \| |x|^\alpha (-\Delta)^{s/2} u \|^{\theta}_{L^p} \| u \|^{1-\theta}_{\dot{B}^{-\mu}_{\infty,\infty}}.
\]

**Proof.** Notice that the case $\theta = 1$ corresponds to Theorem 1.1 so we may restrict ourselves to the case $\theta < 1$.

Let $u := (-\Delta)^{-s/2} f$, where $f \in L^p(\mathbb{R}^n, |x|^{\alpha p} dx)$. Hence, we write
\[
u = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{t\Delta} f \, dt
\]
and, for fixed $T > 0$ to be chosen later, we split the above integral in high and low frequencies, setting
\[
Hf(x) := \frac{1}{\Gamma(s/2)} \int_T^\infty t^{s/2-1} e^{t\Delta} f \, dt
\]
and
\[
Lf(x) := \frac{1}{\Gamma(s/2)} \int_0^T t^{s/2-1} e^{t\Delta} f \, dt.
\]

We will obtain pointwise bounds for $Lf$ and $Hf$.

To bound $Lf$, we proceed as in [4], using the thermal definition of Besov spaces (Definition 2.1) to deduce that
\[
|Lf(x)| \leq C \int_T^\infty t^{-\mu/2-1} \|f\|_{\dot{B}^{-\mu}_{\infty,\infty}} \, dt = CT^{-\mu/2} \|f\|_{\dot{B}^{-\mu}_{\infty,\infty}}.
\]

To bound $Hf$, we have to consider different cases, according to whether $\theta = \frac{\mu}{\mu+s}$ or $\theta > \frac{\mu}{\mu+s}$.

**First case:** $\theta = \frac{\mu}{\mu+s}$.

Observe that in this case, we must also have $\theta = \frac{p}{r}$. Indeed, replacing $\theta = \frac{\mu}{\mu+s}$ in (16) and rearranging terms we have
\[
\frac{\mu + s}{r} = \frac{\mu}{p} = \frac{\alpha p}{n} = \gamma(\mu + s) \, n \geq 0
\]
where the last inequality follows from the condition $\alpha - \frac{\gamma}{\theta} \geq 0$ and the fact that $\mu > 0$. This immediately implies that $\frac{p}{r} \geq \frac{\mu}{\mu+s}$ but, since the reverse inequality holds by hypothesis, we obtain $\frac{p}{r} = \frac{\mu}{\mu+s} = \theta$.

Now, replacing $\theta = \frac{\mu}{\mu+s} = \frac{p}{r}$ in (16), we obtain $\alpha p = \gamma r$. This will be useful in what follows.

Having established the relations between the parameters, we remark that this case is contained in [4, Annexe C], but we outline the result here for the sake of completeness. Following [4], we obtain
\[
|Hf(x)| \leq CT^{s/2} Mf(x),
\]
where $Mf$ is the Hardy-Littlewood maximal function.
Now we choose \( T \) to optimize the sum of (17) and (18), namely
\[
T = \left( \frac{\|f\|_{\dot{B}^{-\mu-s}_{\infty,\infty}}}{Mf} \right)^{\frac{2}{\mu+s}},
\]
and we arrive at the pointwise bound
\[
|(-\Delta)^{-s/2}f| \leq C(Mf)^{\theta}f\|_{\dot{B}^{-\mu-s}_{\infty,\infty}}^{1-\theta}f
\]
This implies
\[
\|(-\Delta)^{-s/2}f\|_L^r \leq C\|(Mf)^{\theta}|x|^r\|_L^r \|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}}
\]
\[
= C\|Mf|x|^\gamma/p\|_L^p \|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}}
\]
\[
\leq C\|f|x|^\alpha\|_L^p \|f\|^{1-\theta}_{\dot{B}^{-\mu-s}_{\infty,\infty}}
\]
where we have used the relations between the parameters and the fact that \(|x|^\alpha = |x|^\gamma/r\) belongs to the Muckenhoupt class \(A_p\) since \(-n < \gamma r = \alpha p < n(p-1)\) by hypothesis, and hence the Hardy-Littlewood maximal function is continuous in \(L^p\) with that weight.

**Second case: \(\theta > \frac{\mu}{\mu+s}\)**

In this case, observe that
\[
Hf(x) = (K_{s,T} * f)(x)
\]
where
\[
K_{s,T}(x) = \frac{1}{\Gamma(s/2)} \int_0^T t^{s/2-1} h_t(x) \, dt
\]
and
\[
h_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp \left\{ -\frac{|x|^2}{4t} \right\}
\]
is the heat kernel.

Now, setting \(2\varepsilon = \mu/\theta - \mu > 0\) and noting that \((n-s)/2 + \varepsilon > 0\), we have that
\[
e^{-x} \leq \frac{C}{x^{(n-s)/2+\varepsilon}} \quad \text{for } x > 0
\]
whence
\[
0 \leq K_{s,T}(x) \leq C \int_0^T t^{(s-n)/2-1} \left( \frac{4t}{|x|^2} \right)^{(n-s)/2+\varepsilon} \, dt
\]
\[
\leq C \frac{1}{|x|^{n-s+2\varepsilon}} \int_0^T t^{-1+\varepsilon} \, dt
\]
\[
\leq C \frac{1}{|x|^{n-s+2\varepsilon}} T^\varepsilon.
\]
Hence,

\[ |H f(x)| \leq C T^\varepsilon I_{s-2\varepsilon} f(x) \]

and to optimize the sum of (17) and (19) we have choose

\[ T = \left( \frac{\|f\|_{B^{-\mu}_{\infty, \infty}}}{I_{s-2\varepsilon} f(x)} \right)^{1/(\varepsilon + \mu/2)} \]

Hence, in this case we have the pointwise bound

\[ |(-\Delta)^{s/2} f(x)| \leq CI_{s-2\varepsilon} f(x)^{\theta} \|f\|^{1-\theta}_{B^{-\mu}_{\infty, \infty}}. \]

Setting \( \tilde{r} = \theta r \), taking \( r \)-norm and using Theorem 1.1, we have

\[ \||x|^\gamma (-\Delta)^{s/2} f\|_{L^p} \leq C \||x|^\gamma I_{s-2\varepsilon} f\|^{\theta}_{L^p} \|f\|^{1-\theta}_{B^{-\mu}_{\infty, \infty}} \]

\[(20)\]

It remains to check the conditions of Theorem 1.1: \( \alpha < \frac{n}{p'} \) and \( \alpha - \frac{n}{\theta} \geq 0 \) are immediate, while \( -\frac{n}{\theta} < \frac{n}{r}, p \leq \tilde{r} \) and \( \frac{1}{\tilde{r}} = \frac{1}{p} + \frac{\alpha - \gamma/\theta - (s-2\varepsilon)}{n} \) follow by our choice of \( \tilde{r} \) and \( \varepsilon \) and the hypotheses of our theorem.

This proves our first inequality. To prove the equivalence with the second one we need to use the lifting property of Besov spaces

\[ \|f\|_{B^{-\mu}_{\infty, \infty}} = \|(-\Delta)^{s/2} u\|_{B^{-\mu}_{\infty, \infty}} = \|u\|_{B^{-\mu}_{\infty, \infty}} \]

and we arrive at the desired inequality.

\[\square\]

As announced, an analogous result can also be obtained in the local case \( s = 1 \), with the gradient instead of the fractional Laplacian. Indeed, Theorem 3.1 implies the following refined weighted Sobolev inequality, which is an improvement of the Caffarelli-Kohn-Nirenberg inequalities \([3]\) in some cases:

**Theorem 3.2.** Let \( n > 1 \), \( 1 < p \leq r \), \( \alpha < \frac{n}{p} \), \( -\frac{n}{r} < \gamma < \frac{n}{r} \), \( \alpha - \frac{\gamma}{\theta} \geq 0 \), \( \mu > 0 \), \( \max\{\frac{p}{r}, \frac{\mu}{\mu+1}\} \leq \theta \leq 1 \), and

\[ \gamma + \frac{n}{r} = \theta \left( \alpha + \frac{n}{p} - 1 \right) + (1 - \theta)\mu. \]

Then, for every \( u \in \dot{H}^{1, p}_{\alpha} (\mathbb{R}^n) \cap B^{-\mu}_{\infty, \infty} \)

\[ \||x|^\gamma u\|_{L^p} \leq C \||x|^\alpha \nabla u\|_{L^p} \|u\|^{1-\theta}_{B^{-\mu}_{\infty, \infty}}. \]
Proof. We consider the classical Riesz transforms $R_j$,

$$R_j u = \frac{\partial}{\partial x_j} \left(-\Delta\right)^{-1/2} u.$$  

Since $R_j$ is a Calderón-Zygmund operator, it is bounded in $L^r(\mathbb{R}^n, |x|^\gamma)$, because the weight $|x|^\gamma$ belongs to the Muckenhoupt class $A_r$ by hypothesis. Moreover, using the Fourier transform, it is easy to check that

$$\sum_{j=1}^n R_j^2 = -I.$$  

Then,

$$\|\|x|^\gamma u\|_{L^r} \leq \sum_{j=1}^n \|\|x|^\gamma R_j^2 u\|_{L^r} \leq C \sum_{j=1}^n \|\|x|^\gamma R_j u\|_{L^r}.$$  

We apply Theorem 3.1 (with $s = 1$) in order to obtain

$$\|\|x|^\gamma R_j u\|_{L^r} \leq C \|\|x|^\alpha (-\Delta)^{1/2} R_j u\|_{L^p}^\theta \|R_j u\|_{\dot{B}^{-\mu}_{\infty,\infty}}^{1-\theta}$$  

$$\leq C \|\|x|^\alpha \frac{\partial u}{\partial x_j}\|_{L^p}^\theta \|R_j u\|_{\dot{B}^{-\mu}_{\infty,\infty}}^{1-\theta}$$  

$$\leq C \|\|x|^\alpha \frac{\partial u}{\partial x_j}\|_{L^p}^\theta \|u\|_{\dot{B}^{-\mu}_{\infty,\infty}}^{1-\theta}$$  

since $R_j$ is a bounded operator in the Besov space $\dot{B}^{-\mu}_{\infty,\infty}$ (see [10]). Hence, we conclude that

$$\|\|x|^\gamma u\|_{L^r} \leq C \sum_{j=1}^n \|\|x|^\alpha \frac{\partial u}{\partial x_j}\|_{L^p}^\theta \|u\|_{\dot{B}^{-\mu}_{\infty,\infty}}^{1-\theta}$$  

$$\leq C \|\|x|^\alpha \nabla u\|_{L^p}^\theta \|u\|_{\dot{B}^{-\mu}_{\infty,\infty}}^{1-\theta}$$  

as announced.  

□

4. LOCAL COMPACTNESS OF THE EMBEDDING

In this section, we prove a version of the Rellich-Kondrachov theorem for the weighted homogeneous Sobolev space $\dot{H}^{\alpha,p}_{\gamma}(\mathbb{R}^n)$. We shall need it in the proof of Theorem 5.1 but we believe that it could be of independent interest for the study of other fractional elliptic problems. It is worth noting that this result does not seem to follow directly from the standard unweighted version of the compactness theorem, since it is not easy to perform truncation arguments due to the non-local nature of the fractional Laplacian operator in the definition (1) of the space.
Instead, we work directly with the definition of the weighted space.

**Theorem 4.1.** Let \( n \geq 1, 0 < s < n, 1 < p \leq q < \infty \). Assume further that \( \alpha, \beta \) and \( s \) satisfy the set of conditions

\[
\beta > -\frac{n}{q}, \quad \alpha < \frac{n}{p}, \quad \alpha \geq \beta
\]

and

\[
\alpha + \frac{n}{p} > s > \frac{n}{p} - \frac{n}{q} + \alpha - \beta > 0.
\]

Then, for any compact set \( K \subset \mathbb{R}^n \), we have the compact embedding

\[
\dot{H}^{s,p}_\alpha(\mathbb{R}^n) \subset L^q(K, |x|^{\beta q}).
\]

We observe that (22) is a subcriticality condition. Indeed, under the hypotheses of the theorem

\[
s > \frac{n}{p} - \frac{n}{q}
\]

and if \( s < \frac{n}{p} \), this is equivalent to

\[
q < p^* = \frac{np}{n - sp}.
\]

We first prove the continuity of the embedding (23). Let \( u \in \dot{H}^{s,p}_\alpha(\mathbb{R}^n) \). Then

\[
u = (-\Delta)^{-s/2} f \quad \text{with} \quad f \in L^p(\mathbb{R}^n, |x|^{\alpha p})
\]

and

\[
\|u\|_{\dot{H}^{s,p}_\alpha} = \| |x|^\alpha f \|_{L^p(\mathbb{R}^n)}.
\]

We define a new exponent \( \tilde{q} \) satisfying the Stein-Weiss scaling condition

\[
\frac{n}{\tilde{q}} := \frac{n}{p} + \alpha - \beta - s.
\]

From (22) it follows that \( \tilde{q} > q \). We define:

\[
r := \frac{\tilde{q}}{q} > 1, \quad \tilde{\beta} := \frac{\beta q}{\tilde{q}} = \frac{\beta}{r}
\]

so that

\[
\beta > -\frac{n}{q} \Leftrightarrow \tilde{\beta} > -\frac{n}{\tilde{q}}
\]
and apply Hölder’s inequality with exponents \( r \) and \( r' \) to obtain

\[
\int_{K} |u|^{q}|x|^\beta q \, dx = \int_{K} |u|^{q}|x|^{\frac{\beta q}{r'}} \, dx \\
\leq \left( \int_{K} |u|^{q} |x|^{\tilde{\beta} q} \, dx \right)^{1/r} \left( \int_{K} |x|^{\beta q} \, dx \right)^{1/r'} \\
\leq C_{K} \left( \int_{\mathbb{R}^n} |u|^{q} |x|^{\tilde{\beta} q} \, dx \right)^{1/r}
\]

since \( \beta > -\frac{n}{q} \). Then, using Theorem 1.1

\[
\left( \int_{K} |u|^{q}|x|^\beta q \, dx \right)^{1/q} \leq C_{K}^{1/q} \left( \int_{\mathbb{R}^n} |u|^{\tilde{q}} |x|^{\tilde{\beta} q} \, dx \right)^{1/\tilde{q}} \\
\leq C \left( \int_{\mathbb{R}^n} |f|^{p}|x|^\alpha p \, dx \right)^{1/p} \\
= C \|u\|_{\dot{H}^{\alpha,p}}.
\]

This shows that (23) is a continuous embedding.

The main difficulty in the proof of the local compactness is that the kernel

\[ K^t_s(x) = C(n, s) |x|^{-(n-s)} \]

of the Riesz potential is not in the dual space \( L^{p'}(\mathbb{R}^n, |x|^{-\alpha p'}) \). For this reason, we introduce for \( t > 0 \) the truncated kernels

\[ K^t_s(x) = C(n, s) |x|^{-(n-s)} \chi_{\{|x|>t\}}. \]

The following lemma gives a kind of pseudo-Poincaré inequality using these kernels.

**Lemma 4.2.** Under the conditions of Theorem 4.1 set

\[ \delta = s - \left( \frac{n}{p} - \frac{n}{q} + \alpha - \beta \right). \]

Then, for any function, \( u \in L^{p}(\mathbb{R}^n, |x|^{\alpha p}) \) and any \( t > 0 \), we have that

\[ \| (K^t_s * u - K_s * u) |x|^\beta \|_{L^q} \leq Ct^\delta \| |x|^\alpha u \|_{L^p}. \]

**Proof.** Notice that

\[ K^t_s * f - K_s * f(x) = C(n, s) \int_{|x-y| \leq t} \frac{f(y)}{|x-y|^{n-s}} \, dy. \]
Hence, since \( \delta > 0 \) by (22),
\[
|K^t_s \ast f - K_s \ast f(x)| \leq C(n, s) \int_{|x-y| \leq t} \frac{|f(y)|}{|x-y|^{n-s}} \, dy \\
\leq C(n, s) \int_{|x-y| \leq t} \frac{|f(y)|}{|x-y|^{n-s}} \left( \frac{t}{|x-y|} \right)^\delta \, dy \\
\leq C(n, s) t^\delta \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-(s-\delta)}} \, dy
\]
and the lemma follows from the Stein-Weiss inequality since \( s - \delta > 0 \) (the definition of \( \delta \) means that the required scaling-condition holds with \( s - \delta \) in place of \( s \)). \( \square \)

Now we are ready to prove the compactness of the embedding (23): let \((u_k)\) be a bounded sequence \( \dot{H}^{s,p}_\alpha(\mathbb{R}^n) \). We may write it as
\[
u_k = (-\Delta)^{-s/2} f_k = K_s \ast f_k
\]
where \( f_k \) is a bounded sequence in \( L^p(\mathbb{R}^n, |x|^{\alpha p}) \). By reflexivity, passing to a subsequence, we may assume that
\[
f_k \rightharpoonup f \text{ weakly in } L^p(\mathbb{R}^n, |x|^{\alpha p}).
\]
We consider the functions
\[
u^i_k = K^i_s \ast f_k \\
\nu^t = K^t_s \ast f
\]
and we write
\[
\|(u_k-u)|x|^\beta\|_{L^q(\mathcal{K})} \leq \|(u_k-u^i_k)|x|^\beta\|_{L^q(\mathcal{K})} + \|(u_k-u^t)|x|^\beta\|_{L^q(\mathcal{K})} + \|(u^t-u)|x|^\beta\|_{L^q(\mathcal{K})}.
\]
We observe that
\[
\|(u_k-u^i_k)|x|^\beta\|_{L^q(\mathcal{K})} \leq t^\delta \|f_k|x|^\alpha\|_{L^p} \leq C t^\delta
\]
and that
\[
\|(u^t-u)|x|^\beta\|_{L^q(\mathcal{K})} \leq t^\delta \|f|x|^\alpha\|_{L^p} \leq C t^\delta.
\]
Hence, given \( \varepsilon > 0 \) we can make this two terms less than \( \varepsilon t^\delta \) for all \( k \), provided that we fix \( t \) small enough.

We check that \( K^i_s(x_0 - \cdot) \) is in \( \dot{L}^p(\mathbb{R}^n, |x|^{-\alpha p'}) \). For that, we consider the integral.
\[
I(x) := \int_{\mathbb{R}^n} |K^i_s(x - y)|^{p'} |y|^{-\alpha p'} \, dy \\
= C(n, s)^{p'} \int_{|x-y| > t} \frac{1}{|x-y|^{(n-s)p'}} |y|^{-\alpha p'} \, dy.
\]
The integrability condition at zero is
\[-\alpha' + n > 0 \iff \alpha < \frac{n}{p'}\]
and at infinity is
\[-\alpha' - (n-s)p' + n < 0 \iff s < \alpha + \frac{n}{p'}.
Hence \(I(x)\) is finite. We conclude that, that for any fixed \(t > 0\), and any fixed \(x_0\) we have that
\[u^t_k(x_0) = K^t_s * f_k(x_0) \to K^t_s * f(x_0) = u^t(x_0).
Moreover, we have that for any \(x_0\) in the compact set \(K\)
\[
\int_{\mathbb{R}^n} |K^t_s(x_0 - x)|^{p'} |x|^{-\alpha'} \, dx \leq C_K.
\]
Indeed, if we write
\[I(x) = I_0(x) + I_\infty(x)\]
where \(K \subset B(0, R)\) and
\[I_0(x) = C(n, s)^{p'} \int_{|x-y| > t, |y| \leq 2R} \frac{1}{|x-y|(n-s)p'} |y|^{-\alpha'} \, dy,
I_\infty(x) = C(n, s)^{p'} \int_{|x-y| > t, |y| > 2R} \frac{1}{|x-y|(n-s)p'} |y|^{-\alpha'} \, dy,
\]
then,
\[I_0(x) \leq C(n, s)^{p'} \int_{|y| \leq 2R} \frac{1}{(n-s)p'} |y|^{-\alpha'} \, dy = C(n, s)^{p'}\frac{1}{l(n-s)p'} (2R)^{-\alpha' + n}.
\]
On the other hand, when \(x \in K\) and \(y\) is in the integration region of \(I_\infty\)
\[|x| \leq R < \frac{1}{2}|y|\]
and
\[|x-y| \geq |y| - |x| \geq |y| - \frac{1}{2}|y| = \frac{1}{2}|y|.
\]
Hence,
\[I_\infty(x) \leq C(n, s)^{p'} \int_{|y| > 2R} \frac{1}{(\frac{1}{2}|y|)^{(n-s)p'}} |y|^{-\alpha'} \, dy
= \tilde{C}(s, n, p) R^{-(n-s)p' - \alpha' + n}
\]
which implies (24). Thus, \((u^t_k)\) is uniformly bounded on \(K\), and by the bounded convergence theorem,
\[\|(u^t_k - u^t)|x|^\beta\|_{L^q(C)} \to 0\]
as $k \to \infty$ (since the condition $\beta > -\frac{n}{q}$ means that the weight $|x|^\beta q$ is integrable on $K$). Therefore, we can make it less than $\varepsilon/3$ for $k \geq k_0(\varepsilon)$.

We conclude that $u_k \to u$ strongly in $L^q(\mathbb{R}^n, |x|^\beta q)$ as we wanted.

5. Existence of maximizers of the Stein-Weiss Inequality

In this section we prove our main theorem, which extends the result of Lieb [14, Theorem 5.1] to some previously unknown cases when $p = 2$.

The proof uses a well-known strategy, but the results are new thanks to our improved Stein-Weiss inequality (Theorem 3.1), and the weighted compactness results (Proposition 2.5 and Theorem 4.1). First, we show that from any maximizing sequence we can extract -after a suitable rescaling- a subsequence with a non-zero weak limit. In the second part, we use the so-called “method of missing mass” (invented by E. Lieb in [14]) to prove that such a limit is actually an optimizer.

**Theorem 5.1.** Assume that $n \geq 2$, $0 < s < \frac{n}{2}$, $2 < r < \infty$, $0 < \alpha < \frac{n}{2}$, $-\frac{n}{r} < \gamma < \alpha$ and that the relation

$$\frac{1}{r} - \frac{1}{2} = \frac{\alpha - \gamma - s}{n}$$

holds. Then, there exists a maximizer for $S$.

**Remark 5.2.** Notice that condition $0 < s < \frac{n}{2}$ does not appear explicitly in [14, Theorem 5.1] but is implied by the other conditions on the parameters. Indeed, since $\alpha \geq \gamma$, we have that $1 < r \leq \frac{2n}{n-2s}$ and, in particular, there must hold $n-2s > 0$.

**Proof of Theorem 5.1.** Let $\{f_k\}_{k \in \mathbb{N}}$ be a maximizing sequence of $S$, that is,

$$\| |x|^\alpha f_k \|_{L^2} = 1 \quad \text{and} \quad \| |x|^\gamma (-\Delta)^{-s/2} f_k \|_{L^r} \to S.$$  \hspace{1cm} (25)

By Corollary 2.3 if we set

$$\mu = \frac{n}{2} + \alpha - s,$$  \hspace{1cm} (26)

it holds that

$$\| f_k \|_{B^{-\mu-s}_{\infty,\infty}} \leq C \| |x|^\alpha f_k \|_2 = C.$$  \hspace{1cm} On the other hand, Theorem 3.1 gives that

$$\| f_k \|_{B^{-\mu-s}_{\infty,\infty}} \geq C > 0$$  \hspace{1cm} provided we choose $\theta$ such that

$$\max\left\{ \frac{p}{r}, \frac{\mu}{\mu + s}, \frac{\gamma}{\alpha} \right\} < \theta < 1$$
which is possible by the hypotheses of the Theorem. This means that
\[ \sup_{t > 0} t^{\frac{n + s}{2}} \| e^{t \Delta} f_k \|_{L^\infty} \geq C > 0. \]
Consequently, for each \( k \in \mathbb{N} \) we can find \( t_k > 0 \) such that
\[ t_k^{\frac{n + s}{2}} \| e^{t_k \Delta} f_k \|_{L^\infty} \geq \frac{C}{2} > 0. \]
Now we set
\[ (27) \ \tilde{f}_k(x) = t_k^{\frac{1}{2}(\frac{n}{2} + \alpha)} f_k(t_k^{\frac{1}{2}} x), \]
and observe that, by parabolic scaling,
\[ e^{1.\Delta} \tilde{f}_k(x) = t_k^{\frac{1}{2}(\frac{n}{2} + \alpha)} e^{t_k \Delta} f_k(t_k^{\frac{1}{2}} x). \]
Then,
\[ \| e^{1.\Delta} \tilde{f}_k \|_{L^\infty} = t_k^{\frac{1}{2}(\frac{n}{2} + \alpha)} \| e^{t_k \Delta} f_k(t_k^{\frac{1}{2}} x) \|_{L^\infty} \]
\[ = t_k^{\frac{n + s}{2}} \| e^{t_k \Delta} f_k \|_{L^\infty} \geq \frac{C}{2} > 0 \]
since relation \( \text{(26)} \) holds.

Observe that, in view of the scaling invariance of the \( L^2(\mathbb{R}^n, |x|^{2\alpha}) \) norm, the sequence \( \{ \tilde{f}_k \}_k \) is bounded in \( L^2(\mathbb{R}^n, |x|^{2\alpha}) \) and, that the maximization problem \( \text{(9)} \) is invariant under the rescaling given by \( \tilde{f}_k \) as long as
\[ \gamma = \frac{n}{2} - \frac{n}{r} + \alpha - s, \]
which holds by our assumptions. Indeed,
\[ \| |x|^{\alpha} \tilde{f}_k \|_{L^2} = \| |x|^{\alpha} f_k \|_{L^2} = 1. \]
Consequently, \( \{ \tilde{f}_k \}_k \) is also a minimizing sequence of \( S \), that is,
\[ \| |x|^{\alpha} \tilde{f}_k \|_{L^2} = 1 \quad \text{and} \quad \| |x|^{\gamma} (-\Delta)^{-s/2} \tilde{f}_k \|_{L^r} \to S. \]
It remains to show that there exists \( g \neq 0 \) such that \( \tilde{f}_k \to g \) strongly in \( L^2(\mathbb{R}^n, |x|^{2\alpha}) \). The last requirement will allow us to deduce that \( g \) is also a minimizer for \( S \), i.e.,
\[ \| |x|^{\alpha} g \|_{L^2} = 1 \quad \text{and} \quad \| |x|^{\gamma} (-\Delta)^{-s/2} g \|_{L^r} = S. \]
By reflexivity, from \( \text{(29)} \) there exists \( g \in L^2(\mathbb{R}^n, |x|^{2\alpha}) \) and a subsequence still denoted by \( \tilde{f}_k \) such that
\[ \| |x|^{\alpha} \tilde{f}_k \|_{L^2} = 1 \quad \text{and} \quad \| |x|^{\gamma} (-\Delta)^{-s/2} \tilde{f}_k \|_{L^r} \to S. \]
We set
\[ u_k := (-\Delta)^{-s/2} \tilde{f}_k, \quad w := (-\Delta)^{-s/2} g. \]
From Proposition 2.5, the compactness of the operator $e^{1 \Delta}$ implies that

$$e^{1 \Delta} \tilde{f}_k \to e^{1 \Delta} g$$ strongly in $L^\infty(\mathbb{R}^n)$

and then $g \neq 0$, since by (28) we have

$$\|e^{1 \Delta} g\|_{L^\infty} \geq \frac{C}{2} > 0.$$

By Theorem 4.1 with $2 < q < r$ and $\alpha = \beta$, for any compact set $K$ we have the compact embedding

$$\dot{H}^{s,2}(\mathbb{R}^n) \subset L^q(K)$$

which implies that passing to a subsequence we may assume that

$$u_k \to w$$ strongly in $L^q(K)$

and, therefore, up to a subsequence, $u_k \to w$ a.e. in $K$. By using a diagonal argument we obtain that, again, up to a subsequence,

$$u_k \to w \quad \text{a.e.} \mathbb{R}^n. \quad (33)$$

Let us prove that (31) holds. Since $u_k \to w$ a.e. $\mathbb{R}^n$, the Brezis-Lieb Lemma ([14, Lemma 2.6] and [2]) claims that

$$\lim_{k \to \infty} \left( \int_{\mathbb{R}^n} |u_k|^r |x|^{\gamma} \ dx - \int_{\mathbb{R}^n} |u_k - w|^r |x|^{\gamma} \ dx \right) = \int_{\mathbb{R}^n} |w|^r |x|^{\gamma} \ dx,$$

but, from (30),

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} |u_k|^r |x|^{\gamma} \ dx = \int_{\mathbb{R}^n} |w|^r |x|^{\gamma} \ dx + \lim_{k \to \infty} \int_{\mathbb{R}^n} |u_k - w|^r |x|^{\gamma} \ dx. \quad (34)$$

Combining (34) with (30) and the elementary inequality

$$a^\frac{r}{s} + b^\frac{r}{s} \leq (a + b)^\frac{r}{s}$$

for $a, b \geq 0$ and $r > 2$, we have

$$S^r = \lim_{k \to \infty} \int_{\mathbb{R}^n} |u_k|^r |x|^{\gamma} \ dx = \int_{\mathbb{R}^n} |w|^r |x|^{\gamma} \ dx + \lim_{k \to \infty} \int_{\mathbb{R}^n} |u_k - w|^r |x|^{\gamma} \ dx$$

$$\leq S^r \left( \int_{\mathbb{R}^n} |g|^2 |x|^{2\alpha} \ dx \right)^\frac{r}{2} + S^r \left( \limsup_{k \to \infty} \int_{\mathbb{R}^n} |\tilde{f}_k - g|^2 |x|^{2\alpha} \ dx \right)^\frac{r}{2}$$

$$\leq S^r \left( \int_{\mathbb{R}^n} |g|^2 |x|^{2\alpha} \ dx + \limsup_{k \to \infty} \int_{\mathbb{R}^n} |\tilde{f}_k - g|^2 |x|^{2\alpha} \ dx \right)^\frac{r}{2} = S^r,$$
where we have used that
\[
\int_{\mathbb{R}^n} |g|^2 |x|^{2\alpha} \, dx + \lim\sup_{k \to \infty} \int_{\mathbb{R}^n} |\tilde{f}_k - g|^2 |x|^{2\alpha} \, dx
\]
(36)
\[
= \lim\sup_{k \to \infty} \int_{\mathbb{R}^n} |\tilde{f}_k|^2 |x|^{2\alpha} \, dx = 1
\]
since \(\tilde{f}_k \rightharpoonup g\) weakly in \(L^2(\mathbb{R}^n, |x|^{2\alpha})\).

Observe that (35) is a strict inequality unless \(a = 0\) or \(b = 0\). Hence, since all the previous inequalities are in fact equalities, we obtain that
\[
\| |x|^a g \|_{L^2} = 1 \quad \text{and} \quad \tilde{f} \to g \text{ strongly in } L^2(\mathbb{R}^n, |x|^{2\alpha}).
\]

By Theorem 1.1, \((-\Delta)^{-s/2} \) is a continuous operator from \(L^2(\mathbb{R}^n, |x|^{2\alpha})\) into \(L^r(\mathbb{R}^n, |x|^{\gamma r})\) and then
\[
u_k \to w \quad \text{strongly in } L^r(\mathbb{R}^n, |x|^{\gamma r})
\]
from where (31) follows. The proof is now complete.

\[\square\]

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