RANDOM SUM-FREE SUBSETS OF ABELIAN GROUPS

JÓZSEF BALOGH, ROBERT MORRIS, AND WOJCIECH SAMOTIJ

Abstract. We characterize the structure of maximum-size sum-free subsets of a random subset of an Abelian group $G$. In particular, we determine the threshold above which, with high probability as $|G| \to \infty$, each such subset is contained in some maximum-size sum-free subset of $G$, whenever $q$ divides $|G|$ for some (fixed) prime $q$ with $q \equiv 2 \pmod{3}$. Moreover, in the special case $G = \mathbb{Z}_{2n}$, we determine the sharp threshold for the above property. The proof uses recent ‘transference’ theorems of Conlon and Gowers, together with stability theorems for sum-free subsets of Abelian groups.

1. Introduction

One of the most important developments in Combinatorics over the past twenty years has been the introduction and proof of various ‘random analogues’ of well-known theorems in Extremal Graph Theory, Ramsey Theory, and Additive Combinatorics. Such questions were first introduced for graphs by Babai, Simonovits, and Spencer [7] (see also the work of Frankl and Rödl [20]) and for additive structures by Kohayakawa, Luczak, and Rödl [33]. There has since been a tremendous interest in such problems (see, for example, [24, 37, 38, 39]). This extensive study has recently culminated in the remarkable results of Conlon and Gowers [16] and Schacht [44] (see also [8, 25, 41, 43]) in which a general theory was developed to attack such questions.

The main theorems in [16] and [44] resolved many long-standing open questions; however, they provide only asymptotic results. For example, they imply that, with high probability, the largest triangle-free subgraph of the random graph $G(n, p)$ has

$$\left(\frac{1}{2} + o(1)\right)p\left(\frac{n}{2}\right)$$

eq 1/\sqrt{n}$, which is best possible. A much more precise question asks the following:

For which functions $p = p(n)$ is, with high probability, the largest triangle-free subgraph of $G(n, p)$ bipartite? It was proved that this is true if $p = 1/2$ by Erdős, Kleitman, and Rothschild [19], if $p \geq 1/2 - \delta$ by Babai, Simonovits, and Spencer [7], and if $p \geq n^{-\varepsilon}$ by Brightwell, Panagiotou, and Steger [11]; here, $\delta$ and $\varepsilon$ are small positive constants. Finally, after this work had been completed, DeMarco and Kahn [17] showed that this also holds if

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Research supported in part by: (JB) NSF CAREER Grant DMS-0745185, UIUC Campus Research Board Grants 09072 and 11067, and OTKA Grant K76099; (RM) ERC Advanced grant DMMCA, and a Research Fellowship from Murray Edwards College, Cambridge; (WS) Parker Fellowship and Schark Fellowship (from UIUC Mathematics Department) and ERC Advanced Grant DMMCA.

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\footnote{We say that a sequence $(A_n)$ of events holds with high probability if the probability that $A_n$ holds tends to 1 as $n \to \infty$.}
A threshold function is sharp if a subset \( B \) of an Abelian group \( G \) is \( \delta \)-Roth if every subset \( A \subseteq B \) with \( |A| \geq \delta |B| \) contains a 3-term arithmetic progression. The main result of [33] is that for every \( \delta > 0 \), if \( p \geq C(\delta)/\sqrt{n} \), then the \( p \)-random subset of \( \mathbb{Z}_n \) is \( \delta \)-Roth with high probability. This is again best possible up to the constant \( C(\delta) \).

Given a sequence \((G_n)\) of Abelian groups with \( |G_n| = n \), we say that \( p_c = p_c(n) \) is a threshold function for a property \( \mathcal{P} \) if the \( p \)-random subset \( B \subseteq G_n \) has \( \mathcal{P} \) with high probability if \( p = p(n) \gg p_c(n) \) and \( B \) does not have \( \mathcal{P} \) with high probability if \( p \ll p_c \). A threshold function is sharp if moreover the above holds with \( p > (1+\varepsilon)p_c \) and \( p < (1-\varepsilon)p_c \) for any fixed \( \varepsilon > 0 \). Bollobás and Thomason [10] proved that every monotone property has a threshold function, and there exists a large body of work addressing the problem of the existence of sharp thresholds for monotone properties, see [22, 23, 31]. Non-monotone properties, such as the one which we shall be studying, are more complicated and in general they do not have such thresholds. We shall prove that such a threshold does exist in the setting of sum-free subsets of Abelian groups described below and, moreover, we shall determine it. For \( \mathbb{Z}_{2n} \), we shall prove much more: that there exists a sharp threshold.

A subset \( A \) of an Abelian group \( G \) is said to be sum-free if there is no solution to the equation \( x + y = z \) with \( x, y, z \in A \). Such forbidden triples \((x, y, z)\) are called Schur triples. (Note that we forbid some triples with \( x = y \), and also some with \( x = z \); the results and proofs in the case that such triples are allowed are identical.) In 1916, Schur [45] proved that, given any \( r \)-colouring of the integers, there exists a monochromatic Schur triple. Graham, Rödl, and Ruciński [24] studied the random version of Schur’s Theorem and proved that the threshold function for the existence of a 2-colouring of the \( p \)-random set \( B \subseteq \mathbb{Z}_n \) without a monochromatic Schur triple is \( 1/\sqrt{n} \). The extremal version of this question, that is, that of determining the size of the largest sum-free subset of the \( p \)-random subset of \( \mathbb{Z}_n \), was open for fifteen years, until it was recently resolved by Conlon and Gowers [16] and Schacht [44].

Sum-free sets have been extensively studied over the past 40 years (see, e.g., [1, 5, 13, 15, 21, 46]), mostly in the extremal setting. For example, in 1969 Diananda and Yap [18] determined the extremal density for a sum-free set in every Abelian group \( G \) such that \( |G| \) has a prime divisor \( q \) with \( q \not\equiv 1 \pmod{3} \). However, more than 30 years had passed until the classification was completed by Green and Ruzsa [30] (see Theorem [21, 11] below). Another well-studied problem was the Cameron-Érdős Conjecture (see [2, 12, 14, 29, 36]), which asked for the number of sum-free subsets of the set \( \{1, \ldots, n\} \) and was finally solved by Green [27] and, independently, by Sapozhenko [42]. A refinement of their theorem has recently been established in [4].

In this paper, we shall study the analogue of the Babai-Simonovits-Spencer problem for sum-free sets. Let \( G = \mathbb{Z}_{2n} \). It is not hard to see that the unique maximum-size sum-free set in \( G \) is the set \( O_{2n} \) of odd numbers. We shall prove a probabilistic version of this statement.
and determine the threshold $p_c$ for the property that the unique maximum-size sum-free subset of the $p$-random subset $B$ of $G$ is the set $B \cap O_{2n}$. In fact, we shall do better and determine a sharp threshold for this property.

For a set $B \subseteq G$, let $SF_0(B)$ denote the collection of maximum-size sum-free subsets of $B$. The following theorem is our main result.

**Theorem 1.1.** Let $C = C(n) \gg \log n/n$ and let $p = p(n)$ satisfy $p^2n = C \log n$.

Let $G_p$ be the $p$-random subset of $\mathbb{Z}_{2n}$. Then

$$\mathbb{P}\left( SF_0(G_p) = \{G_p \cap O_{2n}\} \right) \rightarrow \begin{cases} 0 & \text{if } \limsup_n C(n) < 1/3 \\ 1 & \text{if } \liminf_n C(n) > 1/3 \end{cases}$$

as $n \to \infty$. Moreover, if $C = 1/3$, then $\mathbb{P}\left( SF_0(G_p) = \{G_p \cap O_{2n}\} \right) \to 0$ as $n \to \infty$.

We remark that the threshold for the asymptotic versions of this statement, i.e., that (i) the largest sum-free subset of $G_p$ has $(1 + o(1))pn$ elements and (ii) that all sum-free subsets of $G_p$ with $(1 + o(1))pn$ elements contain only $o(pn)$ even numbers, determined by Conlon and Gowers [16] (both (i) and (ii)) and Schacht [44] (only (i)), is $1/\sqrt{n}$ and so it differs from our threshold by a factor of $\sqrt{\log n}$.

We shall also determine the threshold $p_c(n)$ for all Abelian groups of type I($q$) (see the definition below), i.e., those for which $|G| = n$ has a (fixed) prime divisor $q$ with $q \equiv 2$ (mod 3). Say that a set $B \subseteq G$ is sum-free good if every maximum-size sum-free subset of $B$ is of the form $B \cap A$ for some $A \in SF_0(G)$.

**Theorem 1.2.** Let $q$ be a prime number satisfying $q \equiv 2$ (mod 3). There exist positive constants $c_q$ and $C_q$ such that the following holds.

Let $G = (G_n)_{n \in \mathbb{N}}$ be a sequence of Abelian groups, with $|G_n| = n$, such that $q$ divides $|G|$ for every $G \in \mathcal{G}$. Let $C = C(n) \gg \log n/n$, let $p = p(n)$ satisfy $p^2n = C \log n$, and let $G_p$ be the $p$-random subset of $G = G_n$. Then

$$\mathbb{P}\left( G_p \text{ is sum-free good} \right) \rightarrow \begin{cases} 0 & \text{if } C < c_q \\ 1 & \text{if } C > C_q \end{cases}$$

as $n \to \infty$.

We remark that the conclusion of the theorem fails to hold when we do not assume that $|G|$ has a prime divisor $q$ with $q \equiv 2$ (mod 3). Indeed, we shall show (see Proposition 1.1) that if $G = \mathbb{Z}_{3q'}$ where $q$ is prime and $q \equiv 1$ (mod 3), then the probability that $G_p$ is sum-free good goes to zero (as $n = |G| \to \infty$) for all $p \ll (n \log n)^{-1/3}$. We shall also prove the same bound for the group $G = \mathbb{Z}_q$, where $q \equiv 2$ (mod 3) (see Proposition 4.2), which shows that the condition $n \gg q$ in Theorem 1.2 is also necessary.

We note also that, perhaps not surprisingly, the constant $C = 1/3$ in Theorem 1.1 is not the same for every Abelian group $G$ with $|G|$ even. Indeed, we shall show (see Proposition 4.3) that for the hypercube $G = \mathbb{Z}_2^n$ with $|G| = 2n$, the threshold $p_c$ is at least $\sqrt{\log n/(2n)}$.

Finally, we remark that, in a recent paper with Alon [33], we prove a “counting version” of Theorem 1.2. More precisely, we determine the threshold for $m$ above which a uniformly selected random $m$-element sum-free subset of an $n$-element group $G$ of type $I$ is contained
in some maximum size sum-free subset of $G$. The somewhat related problem of determining the size of the largest Sidon set\(^2\) in a $p$-random subset of the set $\{1, \ldots, n\}$ has also recently been addressed in [32], where sharp bounds were obtained for a large range of $p$.

The remainder of the paper is organized as follows. In Section 2, we collect some extremal results on Abelian groups and the probabilistic tools that will be needed later. In particular, we state our main tool, a theorem of Conlon and Gowers [16] (see also [11, 44]), which provides asymptotic versions of Theorems 1.1 and 1.2. Theorem 1.2 is proved in Section 3 and in Section 4 we prove the lower bounds for other Abelian groups described above. Finally, in Section 5, we prove Theorem 1.1.

### 2. Preliminaries and tools

#### 2.1. Extremal results on Abelian groups.

Let $G$ be a finite Abelian group. Given two subsets $A, B \subseteq G$, we let

$$A \pm B = \{a \pm b : a \in A \text{ and } b \in B\}.$$  

Note that a subset $A$ of $G$ is sum-free if $(A + A) \cap A = \emptyset$. We begin with an important definition in the study of sum-free subsets of finite Abelian groups.

**Definition.** Let $G$ be a finite Abelian group. We say that

1. $G$ is of type I if $|G|$ has at least one prime divisor $q$ with $q \equiv 2 \pmod{3}$.
2. $G$ is of type II if $|G|$ has no prime divisors $q$ with $q \equiv 2 \pmod{3}$, but $|G|$ is divisible by 3.
3. $G$ is of type III if every prime divisor $q$ of $|G|$ satisfies $q \equiv 1 \pmod{3}$.

Moreover, we say that $G$ is of type I($q$) if $G$ is of type I and $q$ is the smallest prime divisor of $|G|$ with $q \equiv 2 \pmod{3}$.

Given an Abelian group $G$, let $\mu(G)$ be the density of the largest sum-free subset of $G$ (so that this subset has $\mu(G)|G|$ elements). As noted in the Introduction, the problem of determining $\mu(G)$ for an arbitrary $G$ has been studied for more than 40 years, but only recently was it solved completely by Green and Ruzsa [30].

**Theorem 2.1** (Diananda and Yap [18], Green and Ruzsa [30]). Let $G$ be an arbitrary finite Abelian group. Then

$$\mu(G) = \begin{cases} \frac{1}{3} + \frac{1}{3q} & \text{if } G \text{ is of type I($q$)}, \\ \frac{1}{3} & \text{if } G \text{ is of type II}, \\ \frac{1}{3} - \frac{1}{3m} & \text{if } G \text{ is of type III}, \end{cases}$$

where $m$ is the exponent (the largest order of any element) of $G$.

Note that it follows immediately from Theorem 2.1 that

$$2/7 \leq \mu(G) \leq 1/2$$

for every finite Abelian group $G$.

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A set $A$ is called a Sidon set if all the sums $x + y$, where $x, y \in A$ with $x \leq y$ are distinct.
Recall that $\text{SF}_0(G)$ denotes the collection of all maximum-size sum-free subsets of $G$, i.e., those that have $\mu(G)|G|$ elements. As well as determining $\mu(G)$, Diananda and Yap [18] described $\text{SF}_0(G)$ for all groups of type I (see also [30, Lemma 5.6]).

**Theorem 2.2** (Diananda and Yap [18]). Let $G$ be a group of type $I(q)$ for some prime $q = 3k + 2$. For each $A \in \text{SF}_0(G)$, there exists a homomorphism $\varphi : G \to \mathbb{Z}_q$ such that $A = \varphi^{-1}(\{k + 1, \ldots, 2k + 1\})$.

In other words, every $A \in \text{SF}_0(G)$ is a union of cosets of some subgroup $H$ of $G$ of index $q$, $A/H$ is an arithmetic progression in $G/H$, and $A \cup (A + A) = G$.

We shall also need the following well-known bound on the number of homomorphisms from an arbitrary finite Abelian group to a cyclic group of prime order, which follows easily from Kronecker’s Decomposition Theorem.

**Proposition 2.3.** For every prime $q$, the number of homomorphisms from a finite Abelian group $G$ to the cyclic group $\mathbb{Z}_q$ is at most $|G|$.

Theorem 2.2 and Proposition 2.3 immediately imply the following.

**Corollary 2.4.** Let $G$ be an arbitrary group of type I. Then $|\text{SF}_0(G)| \leq |G|$.

We will also need the following corollary from the classification of maximum-size sum-free subsets of $\mathbb{Z}_{3q}$, where $q$ is a prime with $q \equiv 1 \pmod{3}$, due to Yap [47].

**Corollary 2.5.** If $q$ is a prime with $q \equiv 1 \pmod{3}$, then $|\text{SF}_0(\mathbb{Z}_{3q})| \leq 2q$.

In the proof of Theorem 1.2, we will use the following two lemmas, which were proved in [30] and [35] respectively. The first lemma establishes a strong stability property for large sum-free subsets of groups of type I.

**Lemma 2.6** (Green and Ruzsa [30]). Let $G$ be an Abelian group of type $I(q)$. If $A$ is a sum-free subset of $G$ and

$$|A| \geq \left( \mu(G) - \frac{1}{3q^2 + 3q} \right) |G|,$$

then $A$ is contained in some $A' \in \text{SF}_0(G)$.

The second lemma is a rather straightforward corollary of a much stronger result of Lev, Luczak, and Schoen [35].

**Lemma 2.7** (Lev, Luczak, and Schoen [35]). Let $\varepsilon > 0$, let $G$ be a finite Abelian group, and let $A \subseteq G$. If

$$|A| \geq \left( \frac{1}{3} + \varepsilon \right) |G|,$$

then one of the following holds:

(a) $|A \setminus A'| \leq \varepsilon |G|$ for some sum-free $A' \subseteq A$.

(b) $A$ contains at least $\varepsilon^3 |G|^2 / 27$ Schur triples.

We remark that a more general statement, the so-called Removal Lemma for groups, was proved by Green [28] (for Abelian groups) and Král, Serra, and Vena [34] (for arbitrary groups). Lemmas 2.6 and 2.7 immediately imply the following.
Corollary 2.8. Let $G$ be a group of type $I(q)$, where $q$ is a prime $q \equiv 2 \pmod{3}$, and let $0 < \varepsilon < \frac{1}{3q^2 + 3q}$. Let $A \subseteq G$ and suppose that 

$$|A| \geq (\mu(G) - \varepsilon)|G|.$$ 

Then one of the following holds:

(a) $|A \setminus A'| \leq \varepsilon|G|$ for some $A' \in \text{SF}_0(G)$.

(b) $A$ contains at least $\varepsilon^3|G|^2/27$ Schur triples.

Proof. Suppose first that $|A \setminus A''| > \varepsilon|G|$ for every sum-free $A'' \subseteq A$. By Theorem 2.1 and our choice of $\varepsilon$, we have $|A| \geq (1/3 + \varepsilon)|G|$. Hence, by Lemma 2.7, $A$ contains at least $\varepsilon^3|G|^2/27$ Schur triples, as required.

So assume that there exists a sum-free set $A'' \subseteq A$ with $|A \setminus A''| \leq \varepsilon|G|$. Then 

$$|A''| \geq |A| - \varepsilon|G| \geq (\mu(G) - 2\varepsilon)|G| > \left(\mu(G) - \frac{1}{3q^2 + 3q}\right)|G|,$$

and so, by Lemma 2.6, $A''$ is contained in some $A' \in \text{SF}_0(G)$. But then 

$$|A \setminus A'| \leq |A \setminus A''| \leq \varepsilon|G|,$$

and so we are done in this case as well. \qed

2.2. Probabilistic tools. We shall next recall three well-known probabilistic inequalities: the FKG inequality, Janson’s inequality, and Chernoff’s inequality.

Given an arbitrary set $X$ and a real number $p \in [0, 1]$, we denote by $X_p$ the $p$-random subset of $X$, i.e., the random subset of $X$, where each element is included with probability $p$ independently of all other elements. In the proof of our main results, we shall need several bounds on the probabilities of events of the form

$$\bigwedge_{i \in I} (B_i \not\subseteq X_p),$$

where $B_i$ are subsets of $X$. The first such estimate can be easily derived from the FKG Inequality (see, e.g., [6, Section 6.3]).

The FKG inequality. Suppose that $\{B_i\}_{i \in I}$ is a family of subsets of a finite set $X$ and let $p \in [0, 1]$. Then

$$\mathbb{P}(B_i \not\subseteq X_p \forall i \in I) \geq \prod_{i \in I} \mathbb{P}(B_i \not\subseteq X_p) = \prod_{i \in I} (1 - p^{|B_i|}). \quad (1)$$

The second result, due to Janson (see, e.g., [6, Section 8.1]), gives an upper bound on the probability in the left-hand side of (1) expressed in terms of the intersection pattern of the sets $B_i$.

Janson’s inequality. Suppose that $\{B_i\}_{i \in I}$ is a family of subsets of a finite set $X$ and let $p \in [0, 1]$. Let

$$M = \prod_{i \in I} (1 - p^{|B_i|}), \quad \mu = \sum_{i \in I} p^{|B_i|}, \quad \Delta = \sum_{i \sim j} p^{|B_i \cup B_j|},$$
where \( i \sim j \) denotes the fact that \( i \neq j \) and \( B_i \cap B_j \neq \emptyset \). Then,
\[
P(B_i \not\subseteq X_p \text{ for all } i \in I) \leq \min \left\{ Me^{\Delta/(2-2p)}, e^{-\mu+\Delta/2} \right\}.
\]
Furthermore, if \( \Delta \geq \mu \), then
\[
P(B_i \not\subseteq X_p \text{ for all } i \in I) \leq e^{-\mu^2/(2\Delta)}.
\]

Finally, we will need the following well-known concentration result for binomial random variables; see, e.g., [6, Appendix A].

Chernoff’s inequality. Let \( X \) be the binomial random variable with parameters \( n \) and \( p \). Then for every positive \( a \),
\[
P\left(X - pn > a\right) \leq \exp\left(-\frac{a^2}{2pn} + \frac{a^3}{2(pm)^2}\right) \quad \text{and} \quad P\left(X - pn < -a\right) \leq \exp\left(-\frac{a^2}{2pn}\right).
\]

2.3. The Conlon-Gowers Theorem. In order to prove the 1-statements in Theorems 1.1 and 1.2, i.e., that every maximum-size sum-free subset of \( G_p \) is of the form \( A \cap G_p \) for some \( A \in \text{SF}_0(G) \), provided that \( p \) is sufficiently large, we need the following approximate version of this statement, which can be derived from Corollary 2.8 using the transference theorems of Conlon and Gowers [16].

**Theorem 2.9.** For every \( \varepsilon > 0 \) and every prime \( q \) with \( q \equiv 2 \text{ (mod 3)} \), there exists a constant \( C = C(q, \varepsilon) > 0 \) such that the following holds. Let \( G \) be an arbitrary \( n \)-element group of type I(q). If
\[
p \geq C \sqrt{n},
\]
then, with high probability as \( n \to \infty \), for every sum-free subset of \( G_p \) with
\[
|B| \geq \left( \mu(G) - \frac{1}{40q^2 + 40q} \right) p|G|,
\]
there is an \( A \in \text{SF}_0(G) \) such that \( |B \setminus A| \leq \varepsilon pn \).

After this work had been completed a more transparent proof of Theorem 2.9 than the one originally included here was given in [41] using a refinement of the methods of [44]. Moreover, Theorem 2.9 can be quite easily deduced from the main results of [8] and [43]. Therefore, for the sake of brevity, we suppress the derivation of the theorem from the main results of [16] and refer the interested reader to the original preprint [9].

2.4. Notation. For the sake of brevity, we shall write \( y \) for the set \( \{y\} \). We shall also use \( \Delta \) to denote the “correlation measure” as in Janson’s inequality, above, and \( \Delta(G) \) for the maximum degree in a graph \( G \). We trust that neither of these will confuse the reader.
3. Abelian groups of Type I

In this section, we shall prove Theorem 1.2. For the sake of clarity of the argument, from now on, we will assume not only that \( q \) divides \(|G|\), but also that \( G \) is of type I\((q)\), i.e., that \( q \) is the smallest prime \( q' \) that divides \(|G|\) and satisfies \( q' \equiv 2 \pmod{3} \). Since for each \( q \), there are at most finitely many primes \( q' \) smaller than \( q \) that satisfy \( q' \equiv 2 \pmod{3} \), this assumption clearly does not affect the validity of our argument.

We begin with the 0-statement, i.e., that if \( \log \frac{n}{n} \ll p(n) \ll c_q \sqrt{\log \frac{n}{n}} \), then with high probability not all maximum-size sum-free subsets of \( G_p \) are of the form \( A \cap G_p \), with \( A \in SF_0(G) \). In fact, we shall prove that with high probability none of them have this form. The proof uses Janson's inequality and the second moment method.

Remark 3.1. If \( \log \frac{n}{n} \ll p(n) \ll n^{-2/3} \), then the 0-statement in Theorem 1.2 becomes almost trivial. Indeed, since \( G \) contains at most \( n^2 \) Schur triples, then with high probability the set \( G_p \) itself is sum-free and (by Chernoff's inequality) \( |A \cap G_p| < |G_p| \) for every \( A \in SF_0(G) \).

Proof of the 0-statement in Theorem 1.2. We wish to prove that, for each prime \( q \equiv 2 \pmod{3} \), if \( c_q \) is sufficiently small then the following holds with probability tending to 1 as \( n \to \infty \). Let \( G \) be an Abelian group of type I\((q)\) with \(|G| = n\), let

\[
\frac{\log n}{n} \ll p(n) \ll c_q \sqrt{\frac{\log n}{n}},
\]

then with high probability not all maximum-size sum-free subsets of \( G_p \) are of the form \( A \cap G_p \), with \( A \in SF_0(G) \). In fact, we shall prove that with high probability none of them have this form. The proof uses Janson's inequality and the second moment method.

Proof of the 0-statement in Theorem 1.2. We wish to prove that, for each prime \( q \equiv 2 \pmod{3} \), if \( c_q \) is sufficiently small then the following holds with probability tending to 1 as \( n \to \infty \). Let \( G \) be an Abelian group of type I\((q)\) with \(|G| = n\), let

\[
\frac{\log n}{n} \ll p \ll c_q \sqrt{\frac{\log n}{n}},
\]

and let \( G_p \) be a random \( p \)-subset of \( G \). Then, for any maximum-size sum-free subset \( B \subseteq G_p \), we have \( |B| > |A \cap G_p| \) for every \( A \in SF_0(G) \).

The proof will be by the second moment method. To be precise, we shall show that, given \( A \in SF_0(G) \), with high probability there exist at least \( 10 \sqrt{pn \log n} \) elements \( x \in G_p \), each chosen from a sum-free subset of a subgroup of \( G \) disjoint from \( A \), such that \( (A \cap G_p) \cup \{x\} \) is sum-free. It will be easy to bound the expected number of such elements using the FKG inequality; to bound the variance we shall need to calculate more carefully, using Janson’s inequality. The result then follows by Chernoff’s inequality, since the size of the sets \( \{ A \cap G_p : A \in SF_0(G) \} \) is highly concentrated.

To begin, observe that for any \( A \in SF_0(G) \), we have \( |A| \geq n/3 \) (by Theorem 2.1) and

\[
\mathbb{P}
\left(
|A \cap G_p| - p|A| > 4\sqrt{pn \log n}\right) \leq n^{-4},
\]

by Chernoff’s inequality (with \( a = 4\sqrt{pn \log n} \)) and by our lower bound on \( p \). By Corollary 2.4 it follows that, with high probability,

\[
|A \cap G_p| - p|A| \leq 4\sqrt{pn \log n} \quad \text{for every } A \in SF_0(G). \tag{2}
\]

Throughout the rest of the proof, let \( A \in SF_0(G) \) be fixed. By Theorem 2.2, there exists a subgroup \( H \) of \( G \) of index \( q \) such that \( A \) is a union of cosets of \( H \). Since \( H \) is not sum-free
(it is a subgroup), it follows that \( A \cap H = \emptyset \). Recall that \( \mu(H) \geq 2/7 \), by Theorem 2.1 and let \( E \in SF_0(H) \) be an arbitrary maximum-size sum-free subset of \( H \), so

\[
|E| = \mu(H)|H| \geq \frac{2|G|}{7q}.
\]

We shall find in \( E \) our elements \( x \) such that \((A \cap G_p) \cup \{x\}\) is sum-free. To this end, for each \( x \in E \) we define

\[
C_1(x) = \{ y \in A : x = y + y \},
\]

\[
C_2(x) = \{ y, z \in \binom{A}{2} : x = y + z \},
\]

\[
C_3(x) = \{ y, z \in \binom{A}{2} : x = y - z \},
\]

and let \( C(x) = C_1(x) \cup C_2(x) \cup C_3(x) \). Note that \( |C_2(x)| \leq n/2 \) and \( |C_3(x)| \leq n \) for every \( x \in E \).

We shall say that an element \( x \in E \) is \textit{safe} if no set in \( C(x) \) is fully contained in \( G_p \). Thus \( x \) is safe if and only if the set \((A \cap G_p) \cup \{x\}\) is sum-free. We shall show below that, with high probability, \( E \cap G_p \) contains more than \( 10^{\sqrt{pn} \log n} \) safe elements. Since \( E \) is sum-free, and \( H \) is a subgroup, we have \( E \pm E \subseteq H \setminus E \). Since \( A \cap H = \emptyset \), it follows that the set

\[
B := (A \cap G_p) \cup \{ x \in E \cap G_p : x \text{ is safe} \}
\]

is sum-free. By (2), it will follow that \( B \) is larger than \( A' \cap G_p \) for every \( A' \in SF_0(G) \).

We begin by giving a lower bound on the expected number of safe elements in \( E \cap G_p \). In fact, in order to simplify the calculation of the variance, below, we shall focus on a subset \( E'' \subseteq E \), defined as follows. First, let \( E' = \{ x \in E : |C_1(x)| < 7q \} \) and note that, since the \( C_1(x) \) are disjoint subsets of \( A \), we have

\[
|E \setminus E'| \cdot 7q \leq |A| \leq n,
\]

and so \( |E'| \geq |E| - n/(7q) \geq n/(7q) \). Now, let \( E'' \) be an arbitrary subset of \( E' \) of cardinality at least \( |E'|/2 \) such that there are no distinct \( x, y \in E'' \) with \( x = -y \). Finally, let \( S \) be the number of safe elements in \( E'' \).

For each \( x \in E \), denote by \( S_x \) the event that \( x \) is safe. By the FKG inequality, we have

\[
\mathbb{P}(S_x) \geq (1 - p)^{|C_1(x)|} (1 - p^2)^{|C_2(x)| \cup C_3(x)|}.
\] (3)

Thus, since \( p \leq c_q \sqrt{\log n / n} \), and using the bounds \( |E''| \geq n/(14q) \), \( |C_1(x)| \leq 7q \), \( |C_2(x)| \leq n/2 \), and \( |C_3(x)| \leq n \), we have

\[
\mathbb{E}[S] = \sum_{x \in E''} \mathbb{P}(S_x) \geq |E''|(1 - p)^{7q} (1 - p^2)^{3n/2} \geq \frac{n}{20q} e^{-2p^2 n} \geq \sqrt{n},
\] (4)

where the last inequality holds if \( c_q \) is sufficiently small.

The following bound on \( \text{Var}(S) \) will allow us to apply Chebychev’s inequality.

**Claim.** \( \text{Var}[S] = o(\mathbb{E}[S]^2) \).
Proof. Given distinct elements $x, y \in E$, define a graph $J(x, y)$ on the elements of $C(x) \cup C(y)$ as follows: let $B_1 \sim B_2$ if $B_1 \cap B_2 \neq \emptyset$ and $B_1 \neq B_2$. Now define
\[
\Delta_{x,y} = \sum_{B_1 \sim B_2} p^{|B_1 \cup B_2|},
\]
where the sum is taken over all edges of $J(x, y)$. For the sake of brevity, let $C_1(x, y) = C_1(x) \cup C_1(y)$ and let $C'(x, y) = C_2(x) \cup C_2(y) \cup C_3(x) \cup C_3(y)$. By Janson’s inequality,
\[
\mathbb{P}(S_x \land S_y) \leq (1 - p)^{|C_1(x, y)|} (1 - p^2)^{|C'(x, y)|} e^{\Delta_{x,y}}. \tag{5}
\]
We claim that for each $x, y \in E''$ with $x \neq y$,
\[
\Delta_{x,y} \leq |C_1(x, y)| \cdot 6p^2 + |C'(x, y)| \cdot 12p^3 = O(np^2 + np^3). \tag{6}
\]
Indeed, if $B_1 \subseteq C_1(x, y) \cup C'(x, y)$ and $B_2 \subseteq C'(x, y)$ are such that $B_1 \cap B_2 = \{z\}$, then $B_2 = \{z, z'\}$, where $z' \in \{x - z, z - x, z + x, y - z, z - y, z + y\}$.

Thus, given $B_1 \ni z$, there are at most six sets $B_2 \in C'(x, y)$ such that $B_1 \cap B_2 = \{z\}$. It follows that for every $B_1 \in C_1(x, y)$, there are at most six sets $B_2 \in C'(x, y)$ with $B_1 \cap B_2 \neq \emptyset$ and for every $B_1 \in C'(x, y)$, there are at most twelve sets $B_2 \in C'(x, y)$ with $B_1 \cap B_2 \neq \emptyset$.

The second inequality follows since $|C_1(x, y)| = |C_1(x)| + |C_1(y)| < 14q$ and $|C'(x, y)| \leq 3n$.

Combining (3), (5), and (6), we obtain
\[
\mathbb{P}(S_x \land S_y) \leq \mathbb{P}(S_x)\mathbb{P}(S_y)(1 - p^2)^{C_1(x,y)} e^{O(np^2 + np^3)}, \tag{7}
\]
where $C_1(x, y) = |C'(x, y)| - |C_2(x) \cup C_3(x)| - |C_2(y) \cup C_3(y)|$. Hence, it only remains to bound $C_1(x, y)$ from below.

Recall that $E''$ does not contain any pairs $\{x, y\}$ with $x = y$. Hence $C_3(x) \cap C_3(y) = \emptyset$, and trivially $C_2(x) \cap C_2(y) = \emptyset$. It follows by elementary manipulation that
\[
|C'(x, y)| \geq |C_2(x) \cup C_3(x)| + |C_2(y) \cup C_3(y)| - |C_2(x) \cap C_3(y)| - |C_2(y) \cap C_3(x)|, \tag{8}
\]
i.e., $C_1(x, y) \geq -|C_2(x) \cap C_3(y)| - |C_2(y) \cap C_3(x)|$.

If $C_2(x) \cap C_3(y) \neq \emptyset$, then there exist $a, b \in A$ such that $x = a + b$ and $y = a - b$, and thus $2a = x + y$ and $2b = x - y$. We split into two cases.

Case 1: $|G|$ is odd.

In this case the equations $2a = x + y$ and $2b = x - y$ have at most one solution $(a, b)$, and so $|C_2(x) \cap C_3(y)| \leq 1$. Hence $C_1(x, y) \geq -2$, and so, by (7),
\[
\mathbb{P}(S_x \land S_y) \leq \mathbb{P}(S_x)\mathbb{P}(S_y)(1 - p^2)^{-2} e^{O(np^2 + np^3)}.
\]
Thus, using the bounds $p \ll n^{-1/3}$ and $\mathbb{E}[S] \gg 1$,
\[
\text{Var}[S] = \sum_{x, y \in E''} \left( \mathbb{P}(S_x \land S_y) - \mathbb{P}(S_x)\mathbb{P}(S_y) \right) \leq \mathbb{E}[S] + \mathbb{E}[S]^2 \left( (1 - p^2)^{-2} e^{O(np^2 + np^3)} - 1 \right)
\leq \mathbb{E}[S] + O(np^2 + np^3)\mathbb{E}[S]^2 = \mathbb{E}[S] + o(\mathbb{E}[S]^2) = o(\mathbb{E}[S]^2),
\]
as required.

Case 2: $|G|$ is even.
Since $|G|$ is even, it follows (by Theorem 2.2) that $H$ is of index 2, and so $A = G \setminus H$. Let
\[ I := \{ h \in H : h + h = 0 \}, \]
note that $I$ is a subgroup of $H$, and let $i = |I|$. We claim that
\[ |C_2(x) \cap C_3(y)| = 0 \text{ or } i/2 \leq |C_2(x) \cap C_3(y)| \leq i. \]
To see this, simply observe that if $\{a, b\} \in C_2(x) \cap C_3(y)$ and $h \in I$, then $\{a + h, b + h\} \in C_2(x) \cap C_3(y)$. Indeed, if $x = a + b$ and $y = a - b$ for some $a, b \in A$, then $x = (a + h) + (b + h)$ and $y = (a + h) - (b + h)$, and $a + h, b + h \in A$ since $A = G \setminus H$. Conversely, if $x = a + b = a' + b'$ and $y = a - b = a' - b'$ for some $a, a', b, b' \in A$, then $a' - a = b' - b \in I$. Hence, there are precisely $i$ ordered pairs $(a, b)$ with $\{a, b\} \in C_2(x) \cap C_3(y)$.

Next, note that there are at most $n^2/(2i)$ pairs $(x, y) \in (E''/2)$ with $|C_2(x) \cap C_3(y)| + |C_2(y) \cap C_3(x)| > 0$. Indeed, there are at most $n^2/4$ quadruples $(a, b, x, y)$ in $A^2 \times (E'')^2$ with $x = a + b$ and $y = a - b$, and there are at least $i/2$ such quadruples for each pair $(x, y)$ as above. Moreover, for every $x \in E''$, either $|C_3(x)| = 0$ or $|C_3(x)| \geq n/4$, since any solution $(a, b) \in A^2$ of the equation $x = a - b$ may be shifted by an arbitrary element of $H$. Hence, for such an $x$,
\[ \mathbb{P}(S_x \wedge S_y) - \mathbb{P}(S_x)\mathbb{P}(S_y) \leq \mathbb{P}(S_x) \leq (1 - p^2)^{|C_2(x)|/3} = (1 - p^2)^{n/12} \leq e^{-p^2n/12}, \]
where the second inequality follows because there exists a matching in $C_3(x)$ of size $|C_3(x)|/3$. (This follows because the graph $C_3(x)$ has maximum degree 2; the worst case is a disjoint union of triangles.)

Finally, we divide once again into two cases: $i \leq \sqrt{n}$ and $i > \sqrt{n}$. In the former case we have
\[ C^*(x, y) \geq -|C_2(x) \cap C_3(y)| - |C_2(y) \cap C_3(x)| \geq -2\sqrt{n}, \]
and so, by (7),
\[ \mathbb{P}(S_x \wedge S_y) \leq \mathbb{P}(S_x)\mathbb{P}(S_y)(1 - p^2)^{-2\sqrt{n}}e^{O(qp^2 + np^3)} = (1 + o(1))\mathbb{P}(S_x)\mathbb{P}(S_y), \]
since $p^2\sqrt{n} + p^3n = o(1)$. Hence $\text{Var}[S] = o(\mathbb{E}[S]^2)$, as required.

If $i > \sqrt{n}$, we partition the sum $\sum_{x, y \in E''} (\mathbb{P}(S_x \wedge S_y) - \mathbb{P}(S_x)\mathbb{P}(S_y))$ into two parts, according to whether or not $|C_2(x) \cap C_3(y)| - |C_2(y) \cap C_3(x)| > 0$. By (9), we obtain
\[ \sum_{x, y \in E''} (\mathbb{P}(S_x \wedge S_y) - \mathbb{P}(S_x)\mathbb{P}(S_y)) \leq \mathbb{E}[S] + \mathbb{E}[S]^2(e^{O(qp^2 + np^3)} - 1) + \frac{n^2}{2i} \cdot e^{-p^2n/12}. \]
Now, recalling from (4) that $\mathbb{E}[S] \geq \frac{n}{20q}e^{-2p^2n}$ and noting that $\frac{1}{\sqrt{n}}e^{-p^2n/12} \ll e^{-4p^2n}$ if $c_q$ is sufficiently small, we deduce that
\[ \text{Var}[S] = o\left(\mathbb{E}[S]^2 + n^2e^{-4p^2n}\right) = o(\mathbb{E}[S]^2), \]
as required. \qed

Since $\text{Var}[S] = o(\mathbb{E}[S]^2)$, the number of safe elements in $E''$ satisfies
\[ S \geq \frac{\mathbb{E}[S]}{2} \geq \frac{n}{40q} \cdot e^{-2p^2n}, \]
with high probability, by Chebyshev’s inequality. Finally, note that the events \( \{S_x\}_{x \in E''} \) depend only on the set \( A \cap G_p \), and recall that \( A \cap E'' = \emptyset \). Hence the events \( \{ x \in G_p \}_{x \in E''} \) are independent of the events \( \{S_x\}_{x \in E''} \), and so, by Chernoff’s inequality, the number of safe elements in \( E'' \cap G_p \) is at least \( pS/2 \) with high probability.

Hence
\[
\left| \{x \in E'' \cap G_p : x \text { is safe}\} \right| \geq \frac{pm}{80q} \cdot e^{-2p^2n} > 10\sqrt{pn \log n},
\]

where the last inequality follows from the bounds \( \log n \ll p < c_q \sqrt{\log n} \), provided that \( c_q \) is sufficiently small and \( n \) is sufficiently large. Thus, by (2),
\[
B := (A \cap G_p) \cup \{x \in E'' \cap G_p : x \text { is safe}\}
\]
is larger than \( A' \cap G_p \) for every \( A' \in SF_0(G) \), and is sum-free, as required. \( \square \)

We now turn to the 1-statement in Theorem 1.2. The proof uses Theorem 2.9 together with Janson’s inequality.

**Proof of the 1-statement in Theorem 1.2.** Let \( q \) be a prime with \( q \equiv 2 \pmod{3} \). We shall prove that if \( C_q \) is sufficiently large, then the following holds with high probability as \( n \to \infty \).

Let \( G \) be an Abelian group of type I(q) with \( |G| = n \), let
\[
p \geq C_q \sqrt{\frac{\log n}{n}}, \tag{10}
\]
and let \( G_p \) be a random \( p \)-subset of \( G \). Then every maximum-size sum-free subset \( B \subseteq G_p \) is of the form \( A \cap G_p \) for some \( A \in SF_0(G) \).

The proof will be roughly as follows. If a maximum-size sum-free subset \( B \subseteq G_p \) is not of the form \( A \cap G_p \), where \( A \in SF_0(G) \), then there must exist a nonempty set \( S \subseteq G_p \setminus A \) and a set \( T \subseteq A \cap G_p \) satisfying \( |S| \geq |T| \) such that \( B = S \cup (A \cap G_p) \setminus T \). We shall show that the expected number of such pairs \((S, T)\) is small when \( |S| \leq \varepsilon pn \), using Janson’s inequality. The case \( |B \setminus A| > \varepsilon pn \) for every \( A \in SF_0(G) \) is dealt with using Theorem 2.9.

Let \( \varepsilon > 0 \) be sufficiently small and let \( B \) be a maximum-size sum-free subset of \( G_p \). Note that \( |B| \geq |A \cap G_p| \) for any \( A \in SF_0(G) \) and so, by Chernoff’s inequality,
\[
|B| \geq \left( \mu(G) - \frac{\varepsilon}{40q^2 + 40q} \right) p|G|
\]
with high probability as \( n \to \infty \). Hence, by Theorem 2.9, we may assume that \( |B \setminus A| \leq \varepsilon pn \) for some \( A \in SF_0(G) \). We shall prove that in fact, with high probability, \( B = A \cap G_p \).

We shall say that a pair of sets \((S, T)\) is *bad* for a set \( A \in SF_0(G) \) if the following conditions hold:

(a) \( S \subseteq G_p \setminus A \) and \( T \subseteq A \cap G_p \),
(b) \( 0 < |S| = |T| \leq \varepsilon pn \),
(c) \( S \cup (A \cap G_p) \setminus T \) is sum-free.

We shall prove that, for every \( A \in SF_0(G) \), the probability that there exists a bad pair \((S, T)\) is \( o(1/n) \) as \( n \to \infty \). It will follow (by Corollary 2.4) that with high probability no such pair exists for any \( A \in SF_0(G) \). By the above discussion, it will imply that \( G_p \) is sum-free good.
with high probability. We remark that a bound of the form $o(1)$ on the probability of the existence of a bad pair does not suffice since the events “$|B \setminus A| \leq \varepsilon p n$” and “there exists a pair $(S, T)$ which is bad for $A$” are not independent of one another.

Fix some $A \in \text{SF}_0(G)$. As in the proof of the lower bound, for every $x \in G \setminus A$, define

$$C_1(x) = \left\{ y \in A : x = y + y \right\} \quad \text{and} \quad C_2(x) = \left\{ \{y, z\} \in \binom{A}{2} : x = y + z \right\}.$$ 

We begin by proving two easy properties of the sets $C_1(x)$ and $C_2(x)$, which will be useful in what follows.

**Claim 1.** For every $x \in G \setminus A$, we have $\max \{|C_1(x)|, |C_2(x)|\} \geq n/(3q)$.

*Proof.* By Theorem 2.2, there exists a subgroup $H$ of $G$ of index $q$ such that $A$ is a union of cosets of $H$ and $A \cup (A + A) = G$. It follows that $x = y + z$ for some $y, z \in A$, and that $y + h, z - h \in A$ for every $h \in H$. Thus $\{y + h, z - h\} \subseteq C_1(x) \cup C_2(x)$ for every $h \in H$, and hence $|C_1(x)| + 2|C_2(x)| \geq |H| = n/q$. \hfill $\Box$

Let $A^* = \{x \in G \setminus A : |C_1(x)| \geq n/(3q)\}$. We now show that we can immediately disregard bad pairs that contain an element of $A^*$.

**Claim 2.** With probability $1 - o(1/n)$, there is no bad pair $(S, T)$ with $S \cap A^* \neq \emptyset$.

*Proof.* Let $x \in A^*$, so $x \in G \setminus A$ and $|C_1(x)| \geq n/(3q)$. If $(S, T)$ is a bad pair for $A$ with $x \in S$, then $C_1(x) \cap G_p \subseteq T$, since $S \cup (A \cap G_p) \setminus T$ is sum-free, and $|T| \leq \varepsilon p n$. But by Chernoff’s inequality, we have

$$|C_1(x) \cap G_p| \geq \frac{pn}{6q} > \epsilon p n \geq |T|$$

with probability at least $1 - \exp \left( - \frac{pn}{24q} \right) \geq 1 - n^{-3}$. By the union bound, with probability $1 - o(1/n)$ this holds for every $x \in A^*$, and hence with probability $1 - o(1/n)$ there is no bad pair $(S, T)$ with $S \cap A^* \neq \emptyset$. \hfill $\Box$

We now arrive at an important definition. Given a set $S \subseteq G \setminus (A \cup A^*)$ of size $k$, let $G_S$ denote the graph with vertex set $A$ and edge set $\bigcup_{x \in S} C_2(x)$. The key observation about $G_S$ is the following: If $(S, T)$ is a bad pair for $A$, then $A \cap G_p \setminus T$ is an independent set in $G_S$. This follows because the endpoints of each edge of $G_S$ sum to an element of $S$.

By Claim 1 and the definition of $A^*$, we have that $|C_2(x)| \geq n/(3q)$ for every $x \in S$. Since, by definition, $C_2(x) \cap C_2(x') = \emptyset$ if $x \neq x'$, we have $e(G_S) \geq kn/(3q)$. Moreover, each $C_2(x)$ is a matching (i.e., no two edges share an endpoint) and hence $\Delta(G_S) \leq |S| = k$.

Given a subset $T \subseteq A$ of size $k$, let $G_{ST} = G_S[A \setminus T]$. Note that $e(G_{ST}) \geq kn/(3q) - k^2$ and $\Delta(G_{ST}) \leq k$ for every such $T$. Also, if $(S, T)$ is a bad pair for $A$, then $e(G_{ST}[G_p]) = 0$. Crucially, since the sets $S, T, A \setminus T$ are pairwise disjoint, the events $S \subseteq G_p, T \subseteq G_p, e(G_{ST}[G_p]) = 0$ are independent. Hence, the expected number of bad pairs for $A$ is at most

$$\sum_{S,T} \mathbb{P}(S \subseteq G_p) \mathbb{P}(T \subseteq G_p) \mathbb{P}(e(G_{ST}[G_p]) = 0),$$

where the summation ranges over all $S \subseteq G \setminus (A \cup A^*)$ and $T \subseteq A$ with $1 \leq |S| = |T| \leq \varepsilon p n$. 

\[\text{(11)}\]
Fix a $k$ with $1 \leq k \leq \varepsilon pn$ and let $S \subseteq G \setminus (A \cup A^*)$ and $T \subseteq A$ be sets of size $k$. Let

$$\mu = \mathbb{E}[e(G_{S,T}[G_p])]$$

and observe that

$$\mu = p^2e(G_{S,T}) \geq p^2 \left( \frac{kn}{3q} - k^2 \right) \geq \frac{p^2 kn}{6q}, \tag{12}$$

since $k \leq \varepsilon n$ and $\varepsilon$ is sufficiently small. Furthermore, let $\Delta = \sum_{B_1\sim B_2} p|B_1 \cup B_2|$, where the summation ranges over all $B_1, B_2 \in E(G_{S,T})$ such that $B_1 \neq B_2$ and $B_1 \cap B_2 \neq \emptyset$ (in other words, $B_1$ and $B_2$ are two edges of $G_{S,T}$ that share an endpoint), and note that

$$\Delta \leq n \left( \frac{\Delta(G_{S,T})}{2} \right) p^3 \leq p^3 k^2 n. \tag{13}$$

By Janson’s inequality, if $\Delta \leq \mu$, then (12) and (10) imply that

$$\mathbb{P}(e(G_{S,T}[G_p]) = 0) \leq e^{-\mu/2} \leq \left( e^{-\varepsilon n/(12q)} \right)^k \leq n^{-5k} \leq n^{-3k}.$$

and if $\Delta \geq \mu$, then (13) implies that

$$\mathbb{P}(e(G_{S,T}[G_p]) = 0) \leq e^{-\mu^2/(2\Delta)} \leq e^{-pm/(72q^2)}.$$

Hence, using (11), for those $k$ such that $\Delta \leq \mu$, the expected number of bad pairs $(S, T)$ with $|S| = |T| = k$ is at most

$$\left( \frac{n}{k} \right)^2 p^{2k} n^{-5k} \leq n^{-3k} \leq n^{-3}.$$

Also, recalling that $k \leq \varepsilon pn$, for those $k$ such that $\Delta > \mu$, the expected number of bad pairs $(S, T)$ with $|S| = |T| = k$ is at most

$$\left( \frac{n}{k} \right)^2 p^{2k} e^{-\frac{\mu n}{72q^2}} \leq \left( \frac{\varepsilon pn}{k} \right)^2 e^{-\frac{\mu n}{72q^2}} \leq \left[ \left( \frac{e - 1}{\varepsilon} \right)^2 e^{-\frac{1}{72q^2}} \right]^{pn} \leq e^{-\sqrt{n}}$$

whenever $\varepsilon$ is sufficiently small and $n$ is sufficiently large (depending only on $q$). It follows that $\varepsilon pn \cdot \max\{n^{-3}, e^{-\sqrt{n}}\}$ is an upper bound on the probability that there exists a bad pair $(S, T)$ for $A$. Since $|SF_0(G)| \leq n$, the expected number of pairs $(S, T)$ which are bad for some $A \in SF_0(G)$ tends to 0 as $n \to \infty$. This completes the proof. \qed

4. Lower bounds for other Abelian groups

In this section we shall prove the following three propositions, which show that the threshold for some groups can be much larger than that determined in Theorems 1.1 and 1.2. The first of these results shows that for certain groups of Type II, the threshold is at least $(n \log n)^{-1/3}$. Recall that $G_p$ denotes the $p$-random subset of $G$.

**Proposition 4.1.** Let $q$ be a prime with $q \equiv 1 \pmod{3}$, let $n = 3q$, and let $G = \mathbb{Z}_n$. If

$$\frac{\log n}{n} \ll p(n) \ll \left( \frac{1}{n \log n} \right)^{1/3},$$

then, with high probability as $n \to \infty$, there exists a sum-free subset of $G_p$ which is larger than $A \cap G_p$ for every $A \in SF_0(G)$. 
The second result gives the same bounds for groups of Type I of prime order.

**Proposition 4.2.** Let $q$ be a prime with $q \equiv 2 \pmod{3}$, let $n = q$, and let $G = \mathbb{Z}_n$. If
\[
\frac{\log n}{n} \ll p(n) \ll \left(\frac{1}{n \log n}\right)^{1/3},
\]
then, with high probability as $n \to \infty$, there exists a sum-free subset of $G_p$ which is larger than $A \cap G_p$ for every $A \in \text{SF}_0(G)$.

The third proposition shows that for the hypercube $\{0,1\}^k$ on $2^n$ vertices, the threshold is different from the threshold for $\mathbb{Z}_2^n$.

**Proposition 4.3.** Let $C < 1/2$ and let $k,n \in \mathbb{N}$ satisfy $n = 2^{k-1}$, and let $G = \mathbb{Z}_2^k$. If
\[
\frac{\log n}{n} \ll p \ll \sqrt{C \log n},
\]
then, with high probability as $n \to \infty$, there exists a sum-free subset of $G_p$ which is larger than $A \cap G_p$ for every $A \in \text{SF}_0(G)$.

The proofs of Propositions 4.1 and 4.2 are almost identical and are based on the following general statement providing a lower bound for the size of a largest sum-free subset of the $p$-random subset of $\mathbb{Z}_n$.

**Lemma 4.4.** Let $G = \mathbb{Z}_n$ and let $m = \min\{n,p^{-2}\}/100$. If
\[
\frac{\log n}{n} \ll p(n) \ll 1,
\]
then, with high probability as $n \to \infty$, the largest sum-free subset of $G_p$ has at least
\[
\frac{pn}{3} + \frac{pm}{4} - 4\sqrt{pn \log n}
\]
elements.

**Proof.** The idea is as follows: First, we construct a sum-free set $A \subseteq G$ of size about $n/3 - 2m$; then we observe that $A \cap G_p$ typically has at least $pn/3 - 2pm - 4\sqrt{pn \log n}$ elements; finally, we show that, with high probability, we can add $9pm/4$ elements to $A \cap G_p$ while still remaining sum-free.

We begin by letting $A = \{\ell, \ldots, r\}$, where $\ell = \lceil n/3 \rceil + \lceil 4m \rceil + 1$ and $r = \lceil 2n/3 \rceil + \lceil 2m \rceil$. Observe that $A$ is sum-free, since $2\ell > r$ and $2r - n < \ell$, and that
\[
\frac{n}{4} \leq \frac{n}{3} - 2m - 3 \leq |A| \leq \frac{n}{3} - 2m.
\]
By Chernoff’s inequality (applied with $a = 3\sqrt{pn \log n}$) and our lower bound on $p$, we have
\[
\left||A \cap G_p| - \left(\frac{pn}{3} - 2pm\right)\right| \leq 4\sqrt{pn \log n} \tag{14}
\]
with high probability.

Next, let $A' = \{\ell', \ldots, \ell - 1\}$ and let $A'' = \{r', \ldots, r\} \subseteq A$, where $\ell' = \lceil n/3 \rceil + \lceil m \rceil + 1$ and $r' = \lceil 2n/3 \rceil - \lfloor m \rfloor$. Since $A$ is sum-free, $2\ell' > r$ and $(r' - 1) + r - n < \ell'$, it follows that every Schur triple $(x,y,z)$ in $A \cup A'$ satisfies $x,y \in A''$ and $z \in A'$.
For every $x \in A'$, let
\[ C_1(x) = \left\{ y \in A'' : x = y + y \right\}, \quad C_2(x) = \left\{ \{y, z\} \in \binom{A''}{2} : x = y + z \right\}, \]
and $C(x) = C_1(x) \cup C_2(x)$. Moreover, note that $|C_1(x)| \leq 1$ and $|C_2(x)| \leq |A''|/2 \leq 2m$ for every $x \in A'$. Call an $x \in A'$ safe if no $B \in C(x)$ is fully contained in $G_p$. By the above observation about the Schur triples in $A \cup A'$, the set
\[ (A \cap G_p) \cup \left\{ x \in A' : x \text{ is safe} \right\} \]
is sum-free.

In the remainder of the proof, we will show that a.a.s. $A' \cap G_p$ contains at least $9pm/4$ safe elements. Together with (14), this will imply the existence of a sum-free subset of $G_p$ with at least $pn/3 + pm/4 - 4\sqrt{pm \log n}$ elements.

For each $x \in A'$, denote by $S_x$ the event that $x$ is safe, and let $S$ be the number of safe elements in $A'$. As in the proof of the lower bound in Theorem 1.2 by the FKG inequality we have
\[ \mathbb{E}[S] = \sum_{x \in A'} \mathbb{P}(S_x) \geq \sum_{x \in A'} (1 - p) |C_1(x)| (1 - p^2) |C_2(x)| \geq |A'| e^{-3p^2 m} \geq \frac{11m}{4}. \]

Here we used the bounds $|C_1(x)| \leq 1$, $|C_2(x)| \leq 2m$, $|A'| \geq 3m - 2$ and $p^2 m \leq 1/100$. Similarly, using Janson’s inequality, we obtain
\[ \text{Var}[S] = \sum_{x, y \in A'} \left( \mathbb{P}(S_x \land S_y) - \mathbb{P}(S_x) \mathbb{P}(S_y) \right) \leq \mathbb{E}[S] + \mathbb{E}[S]^2 \left( e^{O(p^2 + p^3 m)} - 1 \right) \]
\[ \leq \mathbb{E}[S] + O(p^2 + p^3 m) \cdot \mathbb{E}[S]^2 = o(\mathbb{E}[S]^2). \]

To see this, observe that (5), (6), and (7) still hold (with $n$ replaced by $2m$) and that $C^*(x, y) = 0$. The last inequality follows from the fact that $p^2 + p^3 m \ll 1$.

By Chebyshev’s inequality,
\[ \mathbb{P} \left( \left| S - \mathbb{E}[S] \right| > \frac{\mathbb{E}[S]}{11} \right) \leq \frac{121 \text{Var}[S]}{\mathbb{E}[S]^2} = o(1), \]
and it follows that, with high probability, the number of safe elements in $A'$ satisfies
\[ S \geq \frac{10 \mathbb{E}[S]}{11} \geq \frac{5m}{2}. \]

Finally, note that for every $x \in A'$, the event $S_x$ is independent of the events $\{y \in G_p : y \in A' \}$. Hence, by Chernoff’s inequality, with high probability the number of safe elements in $A' \cap G_p$ satisfies
\[ \left| \left\{ x \in A' \cap G_p : x \text{ is safe} \right\} \right| \geq \frac{9pS}{10} \geq \frac{9pm}{4}, \]
as required.

Propositions 4.1 and 4.2 both follow easily from Lemma 4.4 since the proofs are almost identical, we shall prove only the former and leave the details of the latter to the reader.
Proof of Proposition 4.1: Fix an $A \in \text{SF}_0(G)$ and recall that $|A| = n/3$ by Theorem 2.1. (Under the assumptions of Proposition 4.2, we would have that $|A| = \lfloor n/3 \rfloor$, again by Theorem 2.1.) Thus, by Chernoff’s inequality (with $a = 4\sqrt{pn \log n}$) and our lower bound on $p$, we have

$$\mathbb{P}\left(\left| |A \cap G_p| - \frac{pm}{3} \right| > 4\sqrt{pm \log n} \right) \leq \frac{1}{n^2}.$$  

Hence, by Corollary 2.5 (Corollary 2.4 in the proof of Proposition 4.2), with high probability,

$$|A \cap G_p| \leq \frac{pm}{3} + 4\sqrt{pm \log n} \quad \text{for every } A \in \text{SF}_0(G). \quad (15)$$

Now, by Theorem 4.4, with high probability $G_p$ contains a sum-free subset with at least

$$\frac{pn}{3} + \frac{pn}{4} - 4\sqrt{pn \log n}$$

elements, where $m = \min\{n, p^{-2}\}/100$.

Finally, observe that $pm \gg \sqrt{pm \log n}$, since if $m = n/100$, then this is equivalent to $pn \gg \log n$ and if $m = 1/(100p^2)$, then it is equivalent to $p^3n \log n \ll 1$. Hence, by (15), there exists a sum-free subset of $G_p$ that is larger than $A \cap G_p$ for every $A \in \text{SF}_0(G)$. \hfill \Box

We shall now prove Proposition 4.3. Note that for the hypercube, the conditions $x = y + z$ and $x = y - z$ are the same and so we have fewer restrictions for a vertex to be safe. This allows us to show that the threshold is different than in the case $G = \mathbb{Z}_{2n}$.

Proof of Proposition 4.3: With each non-zero element $a \in G = \mathbb{Z}_{2k}^2$, we can associate the subgroup $E(a)$ of elements that are orthogonal to $a$ (when viewing $\mathbb{Z}_{2k}^2$ as a linear space over the 2-element field); note that $E(a)$ has index 2 and let $O(a) = G \setminus E(a)$ to be its non-zero coset. Note that $n = |E(a)| = |O(a)|$, that $O(a)$ is sum-free, and that, by Theorem 2.2, every element of $\text{SF}_0(G)$ is of the form $O(a)$ for some non-zero $a \in G$. By Chernoff’s inequality, for every $a \in G \setminus \{0\}$,

$$\mathbb{P}\left(\left| |O(a) \cap G_p| - pn \right| > 4\sqrt{pn \log n} \right) \leq \frac{1}{n^2},$$

and hence, with high probability,

$$|O(a) \cap G_p| - pn \leq 4\sqrt{pn \log n} \quad \text{for every } a \in G \setminus \{0\}.$$  

Let $1 \in G$ be the all-ones vector. To simplify the notation, let $E = E(1)$ and $O = O(1)$. Since $E$ is isomorphic to $\mathbb{Z}_{2k-1}^2$, $E$ contains a sum-free subset of size $2^{k-2}$. Choose one such subset and denote it $E'$. Note that $0 \not\in E'$.

For each $x \in E'$, let

$$C(x) = \left\{ (y, z) \in \begin{pmatrix} O \\ 2 \end{pmatrix} : x = y + z \right\},$$

and note that $|C(x)| = n/2$. Call an $x \in E'$ safe if no $B \in C(x)$ is fully contained in $G_p$, and for every $x \in E'$, denote by $S_x$ the event that $x$ is safe. Let $\varepsilon = 1/4 - C/2 > 0$, let

$$\mu := \frac{np^2}{2} \leq C \log n = \left( \frac{1}{4} - \varepsilon \right) \log n,$$
and observe that, by the FKG inequality,
\[ \mathbb{P}(S_x) \geq (1 - p^2)^{n/2} \geq \exp \left( - \mu - O(np^4) \right) \geq \frac{n^{-1/4+\epsilon}}{4}, \]

since \( np^4 \ll 1 \). Let \( S \) be the number of safe elements in \( E' \) and note that
\[ \mathbb{E}[S] = \sum_{x \in E'} \mathbb{P}(S_x) \geq \frac{n^{3/4+\epsilon}}{4}. \]

Since for two distinct \( x, y \in E' \), the sets \( C(x) \) and \( C(y) \) are disjoint, it follows from Janson’s inequality (as in the proof of Theorem 1.2) that
\[ \mathbb{P}(S_x \land S_y) \leq (1 - p^2)^{|C(x) \cup C(y)|} e^{np^3} = (1 + o(1))\mathbb{P}(S_x)\mathbb{P}(S_y) \]

and hence \( \text{Var}[S] = o(\mathbb{E}[S]^2) \). By the inequalities of Chebyshev and Chernoff (as in the proof of Lemma 4.4), with high probability the set
\[ \left( \mathcal{O} \cap G_p \right) \cup \left\{ x \in E' \cap G_p : x \text{ is safe} \right\} \]
is sum-free and larger than \( O(a) \cap G_p \) for every non-zero \( a \in G \).

5. The group \( \mathbb{Z}_{2n} \)

Let \( E_{2n} \) and \( O_{2n} \) denote the sets of even and odd elements of \( \mathbb{Z}_{2n} \), respectively, and recall that \( O_{2n} \) is the unique maximum-size sum-free subset of \( \mathbb{Z}_{2n} \). Let \( G = \mathbb{Z}_{2n} \) and recall that \( G_p \) denotes the \( p \)-random subset of \( G \). We shall prove Theorem 1.1 which gives a sharp threshold for the property that \( \text{SF}_0(G_p) = \{ G_p \cap O_{2n} \} \).

For each \( x \in E_{2n} \cap [n - 1] \), let
\[ C_1(x) = \left\{ y \in O_{2n} : x = y + y \right\}, \]
\[ C_2(x) = \left\{ \{y, z\} \in \binom{O_{2n}}{2} : x = y + z \right\}, \]
\[ C_3(x) = \left\{ \{y, z\} \in \binom{O_{2n}}{2} : x = y - z \right\}, \]

and let \( C(x) = C_1(x) \cup C_2(x) \cup C_3(x) \). Call such an element \( x \) safe if no \( B \in C(x) \) is fully contained in \( G_p \) and observe that \( x \) is safe if and only if \( (G_p \cap O_{2n}) \cup \{ x \} \) is sum-free.

The following heuristic argument explains where the constant \( 1/3 \) in the statement of Theorem 1.1 comes from. To establish the 0-statement, it is enough to prove that, with high probability, \( G_p \) contains some safe element. Since \( |C_1(x)| \leq 2 \) and \( |C_2(x) \cup C_3(x)| \approx 3n/2 \) for all \( x \in E_{2n} \cap [n - 1] \), we have that
\[ \mathbb{E}[\# \text{safe elements in } G_p] \approx p \cdot (1 - p^2)^{3n/2} \cdot |E_{2n} \cap [n - 1]| \approx pe^{-3p^2n/2}/2. \]  

Now, note that the right-hand side of (16) grows with \( n \) precisely when \( p^2n \leq \frac{1}{3} \log n \), so, at least in expectation, the sharp threshold for the property of containing a safe element is at \( p = \sqrt{\log n/(3n)} \). To establish the 1-statement, one needs to prove that, with high probability, there are no \( S \subseteq G_p \cap E_{2n} \) and \( T \subseteq G_p \cap O_{2n} \) with \( |T| \leq |S| \) and \( |S| \geq 1 \) such that \( (G_p \setminus T) \cup S \) is sum-free. Theorem 2.9 rules out the possibility that such sets exists.
with $|S| = \Omega(pn)$. Otherwise, it turns out that the ‘worst’ case is already when $|S| = 1$ and $|T| = 0$; showing this is the most difficult and technical part of the proof. In other words, the threshold for the property of being sum-free good is the same as the threshold for the property of (not) containing a safe element.

We now turn to the rigorous proof of Theorem 1.2 and begin by proving the 0-statement. Even though we compute the precise value of the threshold, the proof is somewhat simpler than in the general case.

**Proof of the 0-statement in Theorem 1.1** Assume that
\[
\frac{\log n}{n} \ll p \ll \sqrt{\frac{\log n}{3n}}
\]
and let $G_p$ be the $p$-random subset of $G = \mathbb{Z}_{2n}$. We shall prove that, with high probability, $G_p \cap E_{2n}$ contains a safe element and hence there exists a sum-free subset $B \subseteq G_p$ that is larger than $G_p \cap O_{2n}$.

Indeed, for each $x \in E_{2n} \cap [n - 1]$, denote by $S_x$ the event that $x$ is safe and let $S$ denote the number of safe elements in $E_{2n} \cap [n - 1]$. Note that $|C_1(x)| \leq 2, \frac{n}{2} - 1 \leq |C_2(x)| \leq \frac{n}{2}$, and $|C_3(x)| = n$ for every $x \in E_{2n} \cap [n - 1]$, where we used the fact that $x \neq -x$ for every such $x$. By the FKG inequality,
\[
\mathbb{P}(S_x) \geq (1 - p)^{|C_1(x)|}(1 - p^2)^{|C_2(x)| + |C_3(x)|} \geq \exp\left(-\frac{3p^2n}{2} + O(p + p^4n)\right),
\]
and so, since $3p^2n \leq \log n$,
\[
\mathbb{E}[S] \geq \sum_{x \in E_{2n} \cap [n - 1]} \mathbb{P}(S_x) \geq \frac{n}{3} \cdot e^{-3p^2n/2} \geq \frac{1}{3} \sqrt{n}. \quad (17)
\]
The calculation of the variance is similar to those in the proofs of Theorem 1.2 and Lemma 4.4 and we omit the details, noting only that $C_2(x) \cap C_2(y) = C_3(x) \cap C_3(y) = \emptyset$ and $|C_2(x) \cap C_3(y)| \leq 4$ if $x \neq -y$. Using the fact that $E_{2n} \cap [n - 1]$ and $(E_{2n} \cap [n - 1])$ are disjoint, we obtain
\[
\text{Var}[S] = \sum_{x, y \in E_{2n} \cap [n - 1]} \left(\mathbb{P}(S_x \land S_y) - \mathbb{P}(S_x)\mathbb{P}(S_y)\right) \leq \mathbb{E}[S] + \mathbb{E}[S]^2\left(e^{O(p^2 + np^3)} - 1\right) \leq \mathbb{E}[S] + O(p^3 \log n)\mathbb{E}[S]^2 = o(\mathbb{E}[S]^2),
\]
since $\mathbb{E}[S] \gg 1$, by (17), and $p^3n = o(1)$. Hence, by Chebyshev’s inequality, with high probability the number of safe elements satisfies
\[
S \geq \frac{\mathbb{E}[S]}{2} \geq \frac{n}{6} \cdot e^{-3p^2n/2}.
\]
Finally, for each $x \in E_{2n}$, the event $x \in G_p$ is independent of all the events $\{S_x\}_{x \in E_{2n} \cap [n - 1]}$, so the probability that no safe element belongs to $G_p$ (given $|S| \geq \mathbb{E}[S]/2$) is at most
\[
(1 - p)^{\mathbb{E}[S]/2} \leq \exp\left(-\frac{pn}{6} \cdot e^{-3p^2n/2}\right) \leq \exp\left(-\frac{\log n}{20}\right) \leq o(1),
\]
as required. \qed
In order to obtain the 1-statement, we shall have to work much harder. To show that all sum-free sets $B$ with at least $\varepsilon pn$ even numbers are smaller than $G_p \cap O_{2n}$, we shall use the result of Conlon and Gowers (Theorem 2.9), as in the proof of Theorem 1.2. When $|B \cap E_{2n}| = o(pn)$, however, we shall need to study carefully the structure of the graph $G_S$ for each $S \subseteq E_{2n}$, where $V(G_S) = O_{2n}$ and $E(G_S)$ consists of all pairs $\{a, b\}$ such that either $a + b \in S$, or $a - b \in S$.

Proof of the 1-statement in Theorem 1.1. Let $\delta > 0$, let $C = C(n) \geq 1/3 + \delta$, let

$$p = p(n) = \sqrt{\frac{C \log n}{n}},$$

and let $G_p$ be the $p$-random subset of $G = \mathbb{Z}_{2n}$. We shall prove that, with high probability, $G_p \cap O_{2n}$ is the unique maximum-size sum-free subset of $G_p$.

Let $\varepsilon > 0$ be sufficiently small and let $B$ be a maximum-size sum-free subset of $G_p$. Note that $|B| \geq |G_p \cap O_{2n}|$ and so, by Chernoff’s inequality,

$$|B| \geq \left(\frac{1}{2} - \varepsilon^2\right) p|G|$$

with high probability as $n \to \infty$. Hence, by Theorem 2.9 we have $|B \setminus O_{2n}| \leq \varepsilon pn$ with high probability.

Let $S = B \cap E_{2n}$ and suppose that $|S| = k$ for some positive $k \leq \varepsilon pn$. Since $B$ is at least as large as $G_p \cap O_{2n}$, there must exist a set $T \subseteq G_p \cap O_{2n}$, with $|T| \leq k$, such that

$$B = S \cup (G_p \cap O_{2n}) \setminus T.$$ 

We shall bound the expected number of such pairs $(S, T)$, with $|S| = k$ and $T$ minimal.

For each set $S \subseteq E_{2n}$, define $G_S$ to be the graph with vertex set $O_{2n}$ whose edges are all pairs $\{a, b\} \in \binom{O_{2n}}{2}$, such that either $a + b \in S$ or $a - b \in S$. Note that we ignore loops. We say that a pair of sets $(S, T)$ is good if the following conditions hold:

(i) $S \subseteq G_p \cap E_{2n}$, with $|S| = k$,

(ii) $T \subseteq G_p \cap O_{2n}$, with $|T| \leq k$,

(iii) $G_p \setminus T$ is an independent set in $G_S$,

(iv) $G_p \setminus T'$ is not independent for every $T' \subseteq T$.

It is a simple (but key) observation that if $B = S \cup (G_p \cap O_{2n}) \setminus T$ is a maximum-size sum-free subset of $G_p$ with $|T| \leq |S| = k$, then $(S, T')$ is a good pair for some $T' \subseteq T$. Indeed, every edge of $G_S[G_p \cap O_{2n}]$ must have an endpoint in $T$ since $S \cup (G_p \cap O_{2n}) \setminus T$ is sum-free. Now take $T' \subseteq T$ to be minimal such that this holds.

Let $m$ denote the number of pairs $\{x, -x\} \subseteq S$, where $x \in E_{2n} \cap [n - 1]$, and let $1_S(x)$ denote the indicator function of the event $x \in S$.

Claim 1. $e(G_S) \geq \left(\frac{3k - 1}{2} - m\right) n - O(k^2)$ and $\Delta(G_S) \leq 3k$.

---

3 Hence we prove a slightly stronger result – even if we allow Schur triples of the form $(a, a, 2a)$ to be contained in a sum-free set, the odd numbers are still the best.
Proof. Each $x \in S$ (other than $x = n$, which contributes $n/2$) contributes $n$ edges $\{a, b\}$ with $a - b = x$ to $G_S$, with two edges incident to each vertex of $O_{2n}$. The edge sets corresponding to different members of $S$ are disjoint, except for those corresponding to $x$ and $-x$, which are identical. Thus the pairs $\{a, b\}$ with $a - b \in S$ contribute at least $(k - m - \frac{1}{2}1_S(n))n$ edges to $G_S$, and there are at most $2(k - m) \leq 2k$ such edges incident to each vertex.

Now consider the pairs $\{a, b\}$ with $a + b \in S$. Each $x \in S$ contributes at least $(n - 2)/2$ such edges (since there are at most two loops, at $x/2$ and $n + x/2$) and these sets of edges are disjoint for different members of $S$. Moreover, each vertex is incident to at most $k$ such edges, and so $\Delta(G_S) \leq 3k$.

Finally, if for some $\{a, b\}$, both $a + b$ and $a - b$ are in $S$, then $\{a, b\} = \{x + y, x - y\}$ or $\{a, b\} = \{n + x + y, n + x - y\}$, where $2x, 2y \in S$. There are at most $k^2$ such pairs $\{2x, 2y\}$, and so the number of pairs $\{a, b\}$ with $a + b \in S$ and $a - b \notin S$ is at least $nk/2 - 3k^2$. Hence the total number of edges is at least

$$
\left( k - m - \frac{1}{2}1_S(n) \right)n + \frac{nk}{2} - 3k^2,
$$

as required. \hfill \Box

The following idea is key:

Claim 2. If a pair $(S, T)$ is good, then there exists a subset $U \subseteq (G_p \cap O_{2n}) \setminus T$ with $|U| \leq |T|$, such that $T \subseteq N_{G_S}(U)$ and in $G_S$ there is a matching of size $|U|$ from $U$ to $T$.

Proof. To see this, we simply take a maximal matching $M$ from $T$ to $G_p \setminus T$ in $G_S$ and let $U$ be the set of vertices in $G_p \setminus T$ that are incident to $M$. By construction, $|U| = |M| \leq |T|$ and $M$ is a matching of size $|U|$ from $U$ to $T$.

It remains to prove that $T \subseteq N_{G_S}(U)$. Suppose not, i.e., assume that there is a vertex $a \in T \setminus N_{G_S}(U)$. Since $G_p \setminus T$ is an independent set in $G_S$ and $T$ is minimal, it follows that $a$ has a neighbor $b \in G_p \setminus T$. But $a \notin N_{G_S}(U)$, so $b \notin U$, and thus $M \cup \{a, b\}$ is a matching from $T$ to $G_p \setminus T$ which is larger than $M$. This contradicts the choice of $M$. It follows that $T \subseteq N_{G_S}(U)$, as claimed. \hfill \Box

The plan for the rest of the proof is the following: Say that a triple $(S, T, U)$ is good if $(S, T)$ is good, $U \subseteq (G_p \cap O_{2n}) \setminus T$ and $T \subseteq N_{G_S}(U)$. Let $Z' = Z'(k, \ell, m, j)$ denote the number of good triples $(S, T, U)$ with $|S| = k$, $|T| = \ell$, $|U| = j$, and with $m$ pairs $\{x, -x\}$ in $S$, and let

$$
Z := \sum_{k=1}^{zp(n/2)} \sum_{m=0}^{k/2} \sum_{\ell=0}^{k} \sum_{j=0}^{(k/2)} Z'(k, \ell, m, j).
$$

By Claim 2 and the discussion above, if there exists a maximum-size sum-free set $B \neq G_p \cap O_{2n}$, then $Z \geq 1$, and hence

$$
P\left( SF_0(G_p) \neq \{G_p \cap O_{2n}\} \right) \leq E[Z] = \sum_{k,\ell,m,j} E[Z'(k, \ell, m, j)]. \quad (18)
$$

It thus will suffice to bound $E[Z'(k, \ell, m, j)]$ for each $k$, $\ell$, $m$, and $j$. We shall prove that $E[Z'(k, \ell, m, j)] \leq n^{-3}$ if $kp \leq 1$, and $E[Z'(k, \ell, m, j)] \leq e^{-\sqrt{n}}$ otherwise.
Let us fix $k, m, \ell, j,$ and count the triples $(S, T, U)$ which contribute to $Z'(k, \ell, m, j)$. There are at most $3^k \binom{n}{k-m}$ choices for the set $S$ if $n \not\in S$ (since $S$ intersects $k-m$ of the pairs \{x, -x\}), and similarly there are at most $3^k \binom{n}{k-m-1}$ choices for $S$ if $n \in S$. Also, regardless of whether $n \in S$ or $n \not\in S$, there are at most $\binom{n}{k}$ choices for $S$ with $|S| = k$.

Now, for each $S \subseteq E_{2n}$ and $\ell, j \in \mathbb{N}$, let $W(S, \ell, j)$ denote the number of pairs $(T, U)$ such that $T, U \subseteq G_p \cap O_{2n}$, $T \cap U = \emptyset$ and $T \subseteq N_{G_S}(U)$, with $|T| = \ell$ and $|U| = j$.

**Claim 3.** If $|S| = k$ and $0 \leq j \leq \ell \leq k \leq \varepsilon pn$, then
\[
\mathbb{E}[W(S, \ell, j)] \leq (3e^2p^2n)^k \ll (C \log n)^{2k}.
\tag{19}
\]

**Proof.** We have at most $\binom{n}{j}$ choices for $U$ and, given $S$ and $U$, there are at most $3^{kj} \binom{3kj}{\ell}$ choices for $T$ since $T \subseteq N_{G_S}(U)$ and $\Delta(G_S) \leq 3k$. Since $T$ and $U$ are disjoint, the probability that $T$ and $U$ are contained in $G_p$ is $p^{\ell+j}$.

To simplify the computation, note that for fixed $\ell$ and $k$ with $0 \leq \ell \leq k \leq \varepsilon pn$, the functions $\binom{n}{j}p^j$ and $3^{kj} \binom{3kj}{\ell}$ are increasing in $j$ if $0 \leq j \leq \ell$. Therefore,
\[
\mathbb{E}[W(S, \ell, j)] \leq p^{\ell+j} \binom{n}{j} \binom{3kj}{\ell} \leq p^{2\ell} \binom{n}{\ell} \binom{3k\ell}{\ell} \leq \left(\frac{3e^2p^2nk}{\ell}\right)^{\ell}.
\tag{20}
\]

Since, for any $a > 0$, the function $x \mapsto \left(\frac{a}{x}\right)^x$ is increasing for $0 \leq x \leq \frac{a}{e}$ and since $3e^2p^2n \gg 1$ and $0 \leq \ell \leq k$, the quantity in the right-hand side of (20) is maximized when $\ell = k$. This yields (19).

Finally, recall that if $(S, T, U)$ is good, then no edge of the graph
\[
G_{S,T,U} := G_{S}[O_{2n} \setminus (T \cup U)]
\]
has both its endpoints in $G_p$. Let $S(k, m)$ denote the set of $S \subseteq E_{2n}$ with $|S| = k$ and with $m$ pairs \{x, -x\} in $S$. Since the vertex set of $G_{S,T,U}$ is disjoint from $S \cup T \cup U$, it follows that the events $e(G_{S,T,U}[G_p]) = 0$ and $S, T, U \subseteq G_p$ are all independent. Therefore,
\[
\mathbb{E}[Z'(k, \ell, m, j)] \leq \sum_{S \subseteq S(k, m)} \mathbb{P}(S \subseteq G_p) \cdot \mathbb{E}[W(S, \ell, j)] \cdot \max_{T,U} \left\{\mathbb{P}\left(e(G_{S,T,U}[G_p]) = 0\right)\right\},
\tag{21}
\]
where the maximum is taken over all pairs $(T, U)$ as in the definition of $W(S, \ell, j)$. Hence, to complete the proof, it only remains to give a uniform bound on the probability that $e(G_{S,T,U}[G_p]) = 0$. We shall do so using Janson’s inequality.

Note first that, by Claim 1, $\Delta(G_{S,T,U}) \leq \Delta(G_S) \leq 3k$ and
\[
e(G_{S,T,U}) \geq e(G_S) - (|T| + |U|)\Delta(G_S) = \left(\frac{3k - 2m - 1_S(n)}{2}\right)n - O(k^2) \geq \frac{3k}{2} n,
\tag{22}
\]
where the final inequality follows since $2m \leq k \ll n$. Set
\[
\mu = \frac{3k}{2} p^2 n \quad \text{and} \quad \mu' = p^2 \cdot e(G_{S,T,U})
\]

...
and observe that $\mu/3 \leq \mu' \leq \mu$, by (22), and that

\[ k^2 p^3 n \leq \Delta' := \sum_{B_1 \sim B_2} p^{|B_1 \cup B_2|} \leq \left( \frac{3k}{2} \right)^3 n, \tag{23} \]

where the sum is over pairs of edges of $G_{S,T,U}$ that share an endpoint. There are two cases.

**Case 1.** $\Delta' \leq \mu'$.

In this case, we shall show that $\mathbb{E}[Z'(k,\ell,m,j)] \leq n^{-k}$. By Janson's inequality and (22), we have

\[ \mathbb{P}\left( e(G_{S,T,U}[G_p]) = 0 \right) \leq e^{-\mu' + \Delta'/2} \leq e^{-\mu + \Delta'/2} \exp \left( \frac{2m + 1s(n)}{2} p^2 n + O(p^2 k^2) \right). \tag{24} \]

Suppose first that $\Delta' \leq \varepsilon \mu$. Then $kp \leq 1$, by (23), and so

\[ -\mu + \frac{\Delta'}{2} + O(p^2 k^2) \leq -(1-\varepsilon)\mu = -(1-\varepsilon)\frac{3C}{2} kp \leq -\left( \frac{1}{2} + 2\varepsilon \right) k \log n, \tag{25} \]

since $p^2 n = C \log n$ and $C > 1/3 + 2\varepsilon$, provided that $\varepsilon > 0$ is sufficiently small. Moreover,

\[ \sum_{S \in S(k,m)} \exp \left( \frac{2m + 1s(n)}{2} p^2 n \right) \leq 3^k \left[ \left( \frac{n}{k-m} \right) e^{p^2 n(m+1)} \right] \leq 2^{O(k)n^{Cm} \left( \frac{n}{k} \right)^{k-m}} \left( 1 + kn^{C-1} \right) \tag{26} \]

by the discussion above Claim 3, and since $2m \leq k$ and $e^{p^2 n} = n^C$.

Hence, combining Claim 3 with (21) and (24)–(26), we obtain

\[ \mathbb{E}[Z'(k,\ell,m,j)] \leq p^k \cdot (C \log n)^{2k} \sum_{S \in S(k,m)} \max_{T,U} \left\{ \mathbb{P}\left( e(G_{S,T,U}[G_p]) = 0 \right) \right\} \leq \left( O(1) \cdot p \cdot (C \log n)^2 \cdot \frac{n^{-1/2 - 2\varepsilon}}{n} \right)^k \left( kn^{C-1} \right)^m \left( 1 + kn^{C-1} \right) \ll n^{-\varepsilon k} \tag{27} \]

as $n \to \infty$, assuming that $C < 1/2$ (which implies that $kn^{C-1} < 1$, since $kp \leq 1$). If $C \geq 1/2$, then a similar calculation works, but we do not need to estimate so precisely, as we can obtain a much stronger bound in (25), which allows us to replace the $n^{-1/2 - 2\varepsilon}$ term in (27) with $n^{-4C/3}$; moreover, we recall that $m \leq k/2$ and if $C \leq 4$, then $pm \leq 2\sqrt{n} \log n$. We leave the details to the reader.

The case $\varepsilon \mu < \Delta' \leq \mu'$ is similar, so we shall skip some of the details. Note that $3kp > \varepsilon$, by (23), and hence $pn/k < (3/\varepsilon)p^2 n = (3C/\varepsilon) \log n$. Observe also that (26) still holds and that

\[ -\mu + \frac{\Delta'}{2} + O(p^2 k^2) \leq -\frac{2\mu}{5} = -\frac{3C}{5} k \log n. \]

\[ ^4 \text{Here we used the usual bound } \binom{n}{k} \leq \left( \frac{en}{k} \right)^k, \text{ except when } k - m = 1, \text{ in which case } \binom{n}{k-m-1} = 1. \]
since $\Delta' \leq \mu' \leq \mu$ and $k \ll n$. Thus (27) becomes in this case
\[
E[Z'(k, \ell, m, j)] \leq \left( O(1) \cdot p \cdot (C \log n)^2 \cdot n^{-3C/5} \cdot \frac{n^k}{k^m} \right) (kn^{-1})^m (1 + kn^{-1}) \ll n^{-\varepsilon k},
\]
since $m \leq k/2$. It follows that if $\Delta' \leq \mu'$, then $E[Z'(k, \ell, m, j)] \leq n^{-\varepsilon k}$, as claimed.

**Case 2.** $\Delta' \geq \mu'$.

In this case, we shall show that $E[Z'(k, \ell, m, j)] \leq e^{-\sqrt{n}}$. Indeed, by Janson’s inequality,
\[
P\left( e(G_{S,T,U}[G_p]) = 0 \right) \leq \exp\left( -\frac{(\mu')^2}{2\Delta'} \right) \leq \exp\left( -\frac{\mu^2}{18\Delta'} \right) \leq e^{-pn/36},
\]
by (23), and since $\mu' \geq \frac{\varepsilon \mu}{3}$. We shall also need an improved version of Claim 3 when $k$ is large, since the upper bound ($3kj$) on the number of choices for $T$ becomes very bad. Fortunately, the following bound is trivial:
\[
E[W(S, \ell, j)] \leq \binom{n}{\ell} \binom{n}{j} p^{\ell+j} \leq \binom{n}{k}^2 p^{2k},
\]
where the second inequality follows since $j \leq \ell \leq k \leq \varepsilon pn$. Finally, recall that we have at most $\binom{n}{k}$ choices for $S$, and that $pn \gg \sqrt{n}$. By (21), it follows that
\[
E[Z'(k, \ell, m, j)] \leq p^k \cdot \binom{n}{k}^2 p^{2k} \sum_{S \in S(k,m)} \max_{T,U} \{ P\left( e(G_{S,T,U}[G_p]) = 0 \right) \}
\leq \binom{n}{k}^3 p^{3k} e^{-n^2/(18\Delta')} \leq \left( \frac{epn}{k} \right)^{3k} e^{-pn/36} \leq e^{-\sqrt{n}},
\]
since $k \leq \varepsilon pn$, as claimed.

Having bounded $E[Z']$ in both cases, the result now follows easily by summing over $k, \ell, m,$ and $j$. Indeed, by (18), together with the application of Theorem 2.9 noted at the start of the proof, we have
\[
P\left( SF_0(G_p) \neq \{G_p \cap O_{2n}\} \right) \leq \sum_{k=1}^{\varepsilon \mu n} \sum_{m=0}^{k/2} \sum_{\ell=0}^{k} \sum_{j=0}^{\ell} E[Z'(k, \ell, m, j)] + o(1)
\leq \sum_{k=1}^{\varepsilon \mu n} (k + 1)^3 \max \left\{ n^{-\varepsilon k}, e^{-\sqrt{n}} \right\} + o(1) \to 0
\]
as $n \to \infty$, as required. This completes the proof of Theorem 1.1.

**Acknowledgment.** The bulk of this work was done during the IPAM Long Program, Combinatorics: Methods and Applications in Mathematics and Computer Science. We would like to thank the organizers of this program, the members of staff at IPAM, and all its participants for creating a fantastic research environment. We would also like to thank David Conlon, Gonzalo Fiz Pontiveros, Simon Griffiths, Tomasz Luczak, and Vera Sós for helpful and stimulating discussions.
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