Running coupling and Borel singularities at small $x$

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Abstract

Starting from the dipole representation of small-$x$ evolution we implement the running of the coupling in a self-consistent way. This results in an evolution equation for the dipole density in Borel ($b$) space. We show that the Borel image of the dipole density is analytic in the neighbourhood of $b = 0$ and that it is equal to the BFKL solution at $b = 0$. We study the Borel singularity structure of the dipole cascade emanating from a virtual photon at small $x$ and find a branch cut on the positive $b$-semiaxis starting at $b = 1/\beta_0$. This indicates the presence of $1/Q^2$ power corrections to the small-$x$ structure functions. Finally we present numerical results in the context of D.I.S.

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1 Introduction

The study of QCD in the high energy limit has a history spanning at least twenty years and is still actively pursued. Within perturbation theory (pQCD) an understanding has emerged concerning hadronic processes in the semihard regime, such as near-forward elastic and diffractive scattering, small-$x$ deeply inelastic scattering and rapidity gap events. The evolution of the partonic cascade stemming from the initial state hadrons is known to be described in the leading logarithmic approximation LLA($x$) by the BFKL equation either in its original form [1] or in its improved reformulations [2]. However, the picture is yet incomplete because of two major problems encountered in this regime. The first has to do with unitarity [3], which is violated by the LLA($x$) result. The second concerns the infrared sensitivity of the LLA($x$) result, which is infrared finite but does receive contributions from low momentum regions, where physical observables are sensitive to large non-perturbative corrections. It is the second problem that we shall consider in this paper.

Unlike the more conventional high momentum transfer ($Q$) processes with final state particles in the central rapidity region, the semihard kinematic regime is characterised by three ordered scales, $\sqrt{s} \gg Q \gg \Lambda_{\text{QCD}}$. Note that the use of pQCD is justified by the second of the previous inequalities, in the absence of which, Regge phenomenology of soft processes is the only available approach so far. The pQCD factorisation theorem, applicable for high-$Q$ processes, states that inclusive observables can be expressed as a product of parton distribution functions and a hard scattering cross section (matrix elements and coefficient function in OPE parlance). In the semihard regime this theorem needs amendment. Here, the product is not only a convolution in longitudinal momentum fractions of the partons but also in their transverse momentum. This is known as $k_T$-factorisation [2] and in impact parameter space it translates to a convolution in impact parameter of the radiated secondary partons and an off-shell hard amplitude or cross section. This leads us to the first of the two theoretical inputs of the present study, namely the use of the impact parameter or dipole emission formulation of semihard processes, as developed by Mueller [4] and independently by Nikolaev and Zakharov [5]. The merit of this formalism, apart from the simplicity of the final results, is that it organises the process in terms of sequential soft gluon emissions in the $s$-channel, which in the Coulomb gauge admit a clear physical interpretation. They are to be understood as initial state radiation occurring before the hard scattering. Final state interactions are not included in the evolution equation because they only lead to a unitarity rearrangement that does not affect inclusive observables.

To study the infrared sensitivity of a semihard process we are bound to consider a subset of next-
to-leading logarithmic corrections, specifically the ones that have to do with the running coupling. In the BFKL approach, the evolution kernel, although infrared finite, is sensitive to the running coupling because of diffusion towards the infrared of the transverse momentum along the exchanged gluon ladder. With specific assumptions about the infrared dynamics (i.e. long wavelength gluon propagation) and the scale of the running coupling, the effect on the Regge trajectories has been analyzed in refs. [3]. In the present study we take the point of view that in the dipole formalism, just like in timelike parton cascades, the scale of the running coupling is set by the virtuality of the emitted gluon. The problem of translating this into impact parameter space is resolved through the use of the Borel transformation. We require that the dipole evolution kernel has the same Borel singularity structure as the one obtained in momentum space. This condition is sufficient to determine the scale of the coupling as a function of the impact parameters of the emitted gluon. Therefore, the second theoretical input in our approach is renormalon analysis [7]. Once the dipole kernel with running coupling is constructed, we study the power corrections that are generated when this kernel is used to evolve the wave function of an initial hadronic state such as a virtual photon in small-$x$ D.I.S.

The structure of this paper is as follows. In section 2 we highlight the main concepts and results of the dipole formalism. Here we also review the derivation of the dipole evolution equation with running coupling. In section 3 we discuss the Borel singularity structure of the dipole density. We find a series of infrared renormalon singularities at $b/\beta_0 = n$, $n = 1, 2, 3, ...$ as well as a leading singularity at $b/\beta_0 = \gamma$, with $0 < \gamma < 1$ the anomalous dimension of the solution. We show that the Borel image of the dipole kernel is analytic in the neighbourhood of $b = 0$ and compute its action as a power series expansion in $b$. In the limit $b \to 0$ the BFKL result is recovered. Iterations of the kernel turn the renormalon poles into branch cuts but do not shift their positions. In section 4 we discuss small-$x$ D.I.S. in the dipole formalism using the results derived previously. The main conclusion is that the leading Borel singularity is at $b = 1/\beta_0$, which implies $1/Q^2$ power corrections consistent with the Wilson OPE expectation. The dipole evolution equation for fixed coupling can be solved by virtue of its scale invariance and leads to the same spectrum as that of the BFKL equation. For the Borel image this scale invariance is necessarily lost and the resulting evolution equation can only be solved in numerical approximation by successive iterations. In section 5 we present results for the first few iterations of the Borel transformed kernel in deeply inelastic scattering, and demonstrate that the singularity conforms with that discussed in section 4. We summarise in the final section.
2 The dipole kernel with running coupling

We consider small-x D.I.S. as a specific example of a semihard process and describe it in the rest frame of the nucleon target of mass $m_N$. Let $\psi^{(0)}(z, r)$ denote the wave function for $\gamma^*$ to fluctuate into a $q\bar{q}$ dipole of transverse size $r$ and with the quark carrying longitudinal momentum fraction $z$. Then, the virtual photoabsorption cross section $\sigma(x, Q^2)$ for given polarisation of $\gamma^*(T,L)$, can be written as

$$\sigma_{T,L}(x, Q^2) = \int_0^1 dz \int d^2r |\psi_{T,L}^{(0)}(z, r)|^2 \sigma_{dN}(Y, r),$$

where $\sigma_{dN}$ denotes the dipole-nucleon total cross section and $Y = \ln(z/x)$ is the large rapidity parameter to be resummed within pQCD. The physical interpretation of eq. (1) is that the transition $\gamma^* \to q\bar{q}$, with light cone time scale $\tau = O(1/xm_N)$, occurs much earlier than subsequent interactions with the nucleon target. The lowest order in $\alpha_s$ dipole cross section $\sigma_{dN}^{(0)}$ is

$$\sigma_{dN}^{(0)}(r) = \frac{1}{N_c} \int \frac{d^2l}{(l^2)^2} \alpha_s^2 V(l) \text{Re}(1 - e^{-i l \cdot r}),$$

with $V(l)$ proportional to the absorptive part of the gluon-gluon-nucleon-nucleon vertex function.

The LLA(x) radiative corrections are generated by emission of soft gluons by the initial $q\bar{q}$ dipole with longitudinal momentum fractions strongly ordered as

$$z \gg z_1 \gg z_2 \gg \ldots \gg z_n.$$

The light cone time scale for the $k$-th gluon emission is $\tau_k = O(z_k Q/k_k^2)$, therefore it is emitted independently of the previous ones ($\tau_k \gg \tau_{k+1}$). The $\gamma^*$ wave function $\psi^{(n)}(z, z_i, r, r_i)$ for the emission of $n$ soft gluons with longitudinal momentum fractions $z_i$ and at transverse positions $r_i$ can be calculated order by order in pQCD and the inclusive probability distribution for $\gamma^*$ to fluctuate into $q\bar{q}$ plus soft radiation, $\Phi(z, r)$, is obtained by the square of $\psi^{(n)}$ integrated over the phase space of the emitted gluons and summed and averaged over their polarisations $\{\lambda_i\}$ and colours $\{a_i\}$.

$$\Phi^{(n)}(z, r) = \sum_{\lambda_i, a_i} \prod_{i=1}^n \frac{d^2r_i}{2\pi} \int dz_{i-1} \frac{dz_i}{2z_i} |\psi^{(n)}(a_i)_{\lambda_i}(z, z_1, \ldots, z_n, r, r_1, \ldots, r_n)|^2,$$

$$\Phi^{(0)}(z, r) = |\psi^{(0)}(z, r)|^2.$$

Here and until section 4 we suppress the $\gamma^*$ polarisation indices $(T,L)$ for brevity. Indeed it is the projection of these polarisation states for the virtual photon that is responsible for the fact that $\Phi^{(0)}(z, r)$ is taken to be a function of $r$ only.
In the Coulomb gauge the soft radiation can be viewed as a cascade of colour dipoles emanating from the initial $q$-$\overline{q}$ dipole. It is useful to introduce the dipole density $n(Y, r, \rho)$, for emission from the initial $q$-$\overline{q}$ dipole of size $r$ of a dipole of size $\rho$ and with the smallest longitudinal momentum fraction in the emitted dipole bounded from below by $e^{-Y}$. The dipole density is independent of the projectile bound state properties. When convoluted with the $\gamma^*$ wave function squared $\Phi^{(0)}(0)$ it generates the number density $N(Y, \rho)$ of dipoles of size $\rho$ and rapidity larger than $Y$ as

$$N(Y, \rho) = \int_0^1 dz \int d^2 r \Phi^{(0)}(z, r) n(Y, r, \rho).$$  \hspace{1cm} (5)$$

The whole dipole cascade can be constructed from the repeated action of a kernel $K$ on the initial density $n_0(Y, r, \rho)$ through the dipole evolution equation

$$n(Y, r, \rho) = n_0(r, \rho) + \int_0^Y dy \int_0^\infty dr' K(r, r') n(y, r', \rho).$$  \hspace{1cm} (6)$$

with driving term

$$n_0(r, \rho) = n(Y = 0, r, \rho) = r \delta(r - \rho).$$  \hspace{1cm} (7)$$

The evolution kernel $K$ can be calculated from the one soft gluon emission probability $\Phi^{(1)}(z, r)$. The result is

$$\Phi^{(1)}(z, r) = \Phi^{(0)}(z, r) \int^z \frac{dz_1}{z_1} \int d^2 r_1 \frac{\alpha_s C_F}{\pi^2} \frac{r^2}{r_1^2 r_1^2},$$  \hspace{1cm} (8)$$

where hatted vectors will always denote the distance from the second parent emitter, $\hat{r}_1 := r_1 - r$. The full kernel $K$ is

$$K(r, r') = -\frac{\alpha_s C_F}{\pi^2} \delta(r - r') \int d^2 r'' \frac{r^2}{r'^2 r''^2} + 2 \frac{\alpha_s C_F}{\pi^2} \int_0^\infty dr'' J(r, r', r'') \frac{r^2}{r'^2 r''^2}. $$  \hspace{1cm} (9)$$

The first term of $K$ comes from the one loop virtual corrections to the initial $q$-$\overline{q}$ dipole of size $r$. The $\delta$-function denotes the absence of gluon radiated into the final state. The second term corresponds to real emission and can be read off from eq. (8) taking into account eq. (6). A factor of 2 is included here to account for the fact that after one dipole splits into two, each one of the offsprings can act as parent for subsequent emissions. The change of integration variables

$$d^2 r' = J(r, r', \hat{r}') dr' d\hat{r}'$$  \hspace{1cm} (10)$$

generates the (triangle) Jacobian

$$J(r, r', \hat{r}') = 2\pi r' \hat{r}' \int_0^\infty d\kappa \kappa J_0(\kappa r) J_0(\kappa r') J_0(\kappa \hat{r}').$$  \hspace{1cm} (11)$$
Unlike the BFKL kernel, both virtual and real parts of the dipole kernel $K$ are separately IR finite in the limit $r', \hat{r}' \to \infty$. For fixed $\alpha_s$ and in the limit $N_c \to \infty$, for which $2C_F \to N_c$, $K$ is conformally invariant and has the same spectrum as the BFKL kernel, i.e.

$$\int_0^\infty dr' K(r, r') (r'^2)^\gamma = \frac{\alpha_s N_c}{\pi} \chi(\gamma) (r^2)^\gamma,$$

(12)

where

$$\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1 - \gamma),$$

(13)

is the BFKL spectral function. Indeed, the two approaches lead to the same phenomenological results for inclusive observables. The equivalence between the dipole and BFKL formalisms at the level of light cone perturbation theory diagrams has been shown explicitly in ref. [11].

After the introduction of the dipole density $n$, the dipole cross section $\sigma_{dN}$ of eq. (1) takes the form

$$\sigma_{dN}(Y, r) = \int \frac{d^2 \rho}{2\pi \rho^2} n(Y, r, \rho) \sigma^{(0)}_{dN}(\rho).$$

(14)

Eqs. (1, 14) provide the factorised form of small-$x$ D.I.S. in impact parameter space. We note that dependence on target mass scale $m_N$ enters through $\sigma^{(0)}_{dN}$ and that the universal part of the photoabsorption cross section $\sigma(x, Q^2)$ is indeed the dipole density $n$, which is independent of projectile or target.

The problem of how to include the running coupling in the dipole evolution kernel $K(r, r')$ has been studied in ref. [12]. The form of the kernel with running coupling is fixed by the following two requirements. The first is that in the emission probability of a soft gluon with transverse momentum $k$ the coupling runs as $\alpha_s(k^2)$. The second is that the emission probability in impact parameter space should generate the same singularities in Borel space as the Borel transformed emission probability in momentum space. Then the form of the one-dipole emission probability $\Phi^{(1)}(z, r)$ is completely determined. Explicitly it is

$$\Phi^{(1)}(z, r) = \frac{C_F}{\pi^3} \Phi^{(0)}(z, r) \int^z \frac{dz_1}{z_1} \int d^2 r_1 \int_0^\infty d\tau d\hat{\tau} J_1(\tau) J_1(\hat{\tau}) \int_0^1 \frac{d\omega}{\omega^{1/2} (1 - \omega)^{1/2}} \times \left\{ \alpha_s \left( \frac{\lambda^2}{R^2_1} \right) r_1^2 + \left[ \alpha_s \left( \frac{\lambda^2}{R^2_1} \right) - \alpha_s \left( \frac{\lambda^2}{R^2_1} \right) \right] \hat{r}_1^2 \right\}.$$

(15)

where

$$\lambda^2(\tau, \hat{\tau}, \omega) = (\tau^2 \omega (\hat{\tau}^2)^{(1 - \omega)}), \quad R^2_1(\omega) = (r_1^2 \omega (\hat{r}_1^2)^{(1 - \omega)}).$$

(16)
In the limit of fixed $\alpha_s$ the above equation reproduces the expression for the one-dipole emission $\Phi^{(1)}$ of eq. (8). It turns out that in impact parameter space the scale of the coupling is not a simple function of the distances $r_1$, $\hat{r}_1$ but is weighted with three further parameters $\tau$, $\hat{\tau}$ and $\omega$.

The Borel image $\tilde{\Phi}^{(1)}(z, r; b)$ is defined, as usual, by the transformation

$$\Phi^{(1)}(z, r) = \int_0^\infty db \tilde{\Phi}^{(1)}(z, r; b) e^{-b/\alpha_s(Q^2)} .$$

(17)

This transformation can be easily inverted when the one-loop running coupling is used. Then the Borel image $\tilde{\alpha}_s(k^2/Q^2; b)$ of $\alpha_s(k^2)$ is

$$\tilde{\alpha}_s(k^2/Q^2; b) = \left( \frac{k^2}{Q^2} \right)^{-\beta_0} , \quad \beta_0 = \frac{1}{4\pi} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right) ,$$

(18)

and from eqs. (13, 17) we obtain

$$\tilde{\Phi}^{(1)}(z, r; b) = \frac{C_F}{\pi} \Phi^{(0)}(z, r) \ln(z) \int d^2 r_1 \int_0^\infty d\tau d\hat{\tau} J_1(\tau) J_1(\hat{\tau}) \int_0^1 \frac{d\omega}{\omega^{1/2} (1 - \omega)^{1/2}}$$

$$\times \frac{1}{\tau^2} \frac{1}{\hat{\tau}^2} \frac{(1 - \omega) b_0}{r_1^2 \hat{r}_1^2}$$

$$\times \left[ (r_1^2 + \hat{r}_1^2) + (\hat{r}_1^2 + \hat{r}_1^2) r_1^2 + 2 r_1 \cdot \hat{r}_1 (r_1^2 + \hat{r}_1^2 + (1 - \omega) b_0) \right] .$$

(19)

The integrals over $z_1$, $r_1$, $\tau$, $\hat{\tau}$ and $\omega$ can be performed and the result is

$$\tilde{\Phi}^{(1)}(z, r; b) = -\frac{2C_F}{\pi} \ln(z) \left( \frac{Q^2 r^2}{4} \right)^{b_0} \Phi^{(0)}(z, r) .$$

(20)

From eqs. (13, 20) we can construct the dipole evolution kernel in Borel space $\tilde{K}(r, r'; b)$. It consists of two pieces. The virtual contribution is the coefficient of the $\Phi^{(0)}$ in eq. (20) with the sign inverted as required by unitarity [3]. The real contribution is obtained by the coefficient of $\Phi^{(0)}$ in eq. (19). After renaming $r_1 \rightarrow r'$, changing integration variables as in eq. (10) and performing the integration over $\hat{\tau}'$ we obtain the explicit form of the kernel. It is

$$\tilde{K}(r, r'; b) = \frac{N_c}{\pi} \left( \frac{Q^2 r^2}{4} \right)^{b_0} \frac{\Gamma(-b_0)}{\Gamma(1 + b_0)} \delta(r - r')$$

$$+ 2 \frac{N_c}{\pi^2} \frac{1}{r'} \left( \frac{Q^2 r'^2}{4} \right)^{b_0} \int_0^1 \frac{d\omega}{\omega^{1/2} (1 - \omega)^{1/2}}$$

$$\times \left\{ \left( \frac{r'^2}{r^2} \right)^{b_0 - 1} 2F_1 \left( 1 - b_0, 1 - b_0; 1; \frac{r'^2}{r^2} \right) \right\} .$$

(21)

The factor of 2 in front of the second term of this equation has been inadvertently omitted in eq. (16) of ref. [3].
where \( r_\leq = \min(r, r') \) and \( r_\geq = \max(r, r') \). Finally, the evolution equation for the dipole density in Borel space reads

\[
\frac{\partial}{\partial Y} \tilde{n}(Y, r, \rho; b) = \int_0^\infty dr' \int_b^b d\omega' \tilde{K}(r, r'; \omega') \tilde{n}(Y, r', \rho; b - b'),
\]

with boundary condition

\[
\tilde{n}(Y = 0, r, \rho; b) = r \delta(r - \rho) \delta(b).
\]

### 3 The Borel singularity structure of the dipole cascade

To study the Borel singularity structure generated by the dipole kernel \( \tilde{K} \) we consider its action on a test function of the form \((r^2)^\gamma\). Defining the function \( \chi(\gamma, b) \) by

\[
\int_0^\infty dr' \tilde{K}(r, r'; b)(r'^2)^\gamma = \frac{N_c}{\pi} \chi(\gamma, b) \left( \frac{Q^2 r^2}{4} \right)^{b\beta_0} (r^2)^\gamma,
\]

we obtain the expression of \( \chi(\gamma, b) \) from eq. (21),

\[
\chi(\gamma, b) = \frac{\Gamma(-b\beta_0)}{\Gamma(1+b\beta_0)} + \frac{\Gamma(-\gamma-b\beta_0)}{\Gamma(1+\gamma+b\beta_0)} \frac{1}{\pi} \int_0^1 d\omega \frac{\Gamma(1-\omega b\beta_0)}{\Gamma(1+(1-\omega)b\beta_0)} \frac{\Gamma(1-(1-\omega)b\beta_0)}{\Gamma(1+\omega b\beta_0)}
\times \frac{\Gamma(1+\gamma)}{\Gamma(-\gamma)} \frac{\Gamma(b\beta_0)}{\Gamma(1-b\beta_0)} - 2 \frac{\Gamma(1+\gamma+(1-\omega)b\beta_0)}{\Gamma(1-\gamma-(1-\omega)b\beta_0)} \frac{\Gamma(1+\omega b\beta_0)}{\Gamma(1-\omega b\beta_0)}
\]

where the first term comes from virtual correction and the second from real emission. The virtual contribution to \( \chi(\gamma, b) \) contains a series of poles at \( b\beta_0 = 1, 2, 3... \) which are identified with the IR renormalons and correspond to power corrections of \( \mathcal{O}((m^2_N/Q^2)^n) \), \( n = 1, 2,... \). Note that these poles are independent of the specific form of the test function. This set of poles, resulting from the exponentiation of soft radiation, has also been derived in the context of the Drell-Yan process in ref. [13]. In addition, we observe the presence of a series of poles at \( b\beta_0 = n - \gamma, n = 0, 1, 2... \) generated by the \( \Gamma(-\gamma-b\beta_0) \) dependence of the real contribution to \( \chi(\gamma, b) \). For \( \text{Re}(\gamma) \geq m \) these poles correspond to IR renormalons for \( n > m \). Their IR origin is established by observing that these singularities arise from the \( r' > r \) integration region of eq. (22), where the offspring dipole is
emitted with size larger than the parent dipole. Taken at face value the $\gamma$-dependent poles indicate the presence of $\mathcal{O}\left(\frac{m_N^2}{Q^2}\right)^{n-\gamma}$ power corrections. For the moment it suffices to note that in this section we are studying the dipole cascade independently of its embedding in a particular physical process. The interpretation of the $\gamma$-dependent power corrections in terms of OPE matrix elements is not an issue here.

Inspecting the evolution equation (22) we now examine whether the kernel introduces singularity at the lower end of the $b'$-integration. The answer is that for fixed $r$ the limit $b \to 0$ of $\tilde{K}(r, r', b)$ is not singular. The $b = 0$ pole appearing in the virtual contribution to $\tilde{K}$ is of UV origin and cancels against a corresponding singularity in the real contribution. The presence of such a singularity can be seen already in the fixed coupling case. The integral in eq. (8) is logarithmically divergent in the region $r_1 \to 0$. One way of tracing the cancellation of the UV singularities is to impose a lower cutoff in the impact parameter integration as in ref. [4]. Alternatively, in the case of running coupling we can use the Borel parameter $b$ as a dimensional regulator. Defining

$$\zeta = \frac{r_2^2}{r_2^2}$$

we note that the UV divergences in the real part of $\tilde{K}$ arise from the region $\zeta \to 1$. This is the region where the gluon is emitted at impact parameter almost equal to that of the parent dipole. In the limit $b \to 0$ the only UV divergent contribution comes from the first term inside the curly brackets of eq. (21). It is due to the divergence of the hypergeometric function $2F_1(1 - b\beta_0, 1 - b\beta_0; 1; \zeta = 1)$ at $b \to 0$. To parametrise this divergence we use the linear transformation [14]

$$2F_1(1 - b\beta_0, 1 - b\beta_0; 1; \zeta) = (1 - \zeta)^{-1+2b\beta_0} 2F_1(b\beta_0, b\beta_0; 1; \zeta).$$

(27)

Then, in the limit $b \to 0$ the kernel takes the form

$$\tilde{K}(r, r'; b) = \frac{N_c}{\pi} \left( \frac{Q^2}{2} \right)^{b\beta_0} \left\{ \frac{1}{(-b\beta_0)} \delta(r - r') + \left[ \frac{1}{\zeta} \right]_+ \right\}. \quad (28)$$

The singular factor $(1 - \zeta)^{-1+2b\beta_0}$ can be expressed in terms of distributions as

$$(1 - \zeta)^{-1+2b\beta_0} = \frac{1}{2b\beta_0} \delta(1 - \zeta) + \left[ \frac{1}{1 - \zeta} \right]_+ + ...$$

(29)

Substituting this into eq. (28) it follows that the UV singular term in the virtual part of $\tilde{K}$ cancels against the one in the real part. Since the kernel is regular around $b = 0$ we compute the power series expansion in $b$ of $\chi(\gamma, b)$. From eq. (25) we obtain

$$\chi(\gamma, b) = \chi(\gamma) + b\beta_0 \chi^{(1)}(\gamma) + \mathcal{O}(b^2 \beta_0^2).$$

(30)
where
\[
\chi^{(1)}(\gamma) = -\frac{1}{\gamma} \chi(\gamma) - 2\Psi(1)\chi(\gamma) + \frac{1}{2} \chi(\gamma)^2 + \frac{1}{2} \chi'(\gamma)
\] (31)

The $O(1/b)$ singular terms have cancelled as anticipated and the $O(b^0)$ term is the BFKL spectral function $\chi(\gamma)$, eq. [13].

Returning to the dipole evolution equation (22) we introduce the usual Laplace transform with respect to the rapidity $Y$
\[
\tilde{n}_\omega(r, \rho; b) = \int_0^\infty dY e^{-\omega Y} \tilde{n}(Y, r, \rho; b),
\] (32)

and the anomalous dimension $\gamma$ via the Mellin transform
\[
\tilde{n}_{\omega,\gamma}(b) = \int_0^\infty \frac{dr^2}{r^2} \left( \frac{r^2}{\rho^2} \right)^{-\gamma} \tilde{n}_\omega(r, \rho; b).
\] (33)

Then eq. (22) with the boundary condition (23) through the use of (24) becomes an equation for the spectral amplitudes $\tilde{n}_{\omega,\gamma}(b)$,
\[
\omega \tilde{n}_{\omega,\gamma}(b) = 2\delta(b) + \frac{N_c}{\pi} \int_0^b db' \chi(\gamma - b'\beta_0, b') \left( \frac{Q^2\rho^2}{4} \right)^{b'\beta_0} \tilde{n}_{\omega,\gamma-b'\beta_0}(b-b').
\] (34)

This equation is universal for all semihard processes. The scale $Q^2$ is only constrained by the requirement $Q^2 \gg \Lambda^2_{QCD}$ and its specific value is otherwise arbitrary. It is identified with the $\gamma^*$ virtuality here because we consider the specific case of small-$x$ D.I.S. The dipole density is reconstructed from the spectral amplitudes as
\[
\tilde{n}(Y, r, \rho; b) = \int \frac{d\omega}{2\pi i} e^{\omega Y} \int \frac{d\gamma}{2\pi i} \left( \frac{r^2}{\rho^2} \right)^{\gamma} \tilde{n}_{\omega,\gamma}(b).
\] (35)

Eq. (34) is an integral equation of the Volterra type. Such equations are known not to have eigenfunctions for bounded kernels. One way of constructing solutions is by iteration. In our case, the iterative solution is given by the formal expression
\[
\tilde{n}_{\omega,\gamma}(b) = \sum_{k=0}^\infty \tilde{n}^{(k)}_{\omega,\gamma}(b),
\] (36)

with
\[
\tilde{n}^{(0)}_{\omega,\gamma}(b) = \frac{2}{\omega} \delta(b), \quad \tilde{n}^{(1)}_{\omega,\gamma}(b) = \frac{2}{\omega} \left( \frac{N_c}{\pi \omega} \right) \left( \frac{Q^2\rho^2}{4} \right)^{b\beta_0} \chi(\gamma - b\beta_0, b),
\] (37)

and
\[
\tilde{n}^{(k)}_{\omega,\gamma}(b) = \frac{2}{\omega} \left( \frac{N_c}{\pi \omega} \right)^k \left( \frac{Q^2\rho^2}{4} \right)^{b\beta_0} \int_0^b db_{k-1} \chi(\gamma - (b - b_{k-1})\beta_0, b - b_{k-1})
\]
\[ \times \int_0^{b_{k-1}} db_{k-2} \chi((\gamma - (b - b_{k-2})\beta_0, b_{k-1} - b_{k-2}) \ldots \]
\[ \times \int_0^{b_3} db_2 \chi((\gamma - (b - b_2)\beta_0, b_3 - b_2) \]
\[ \times \int_0^{b_2} db_1 \chi((\gamma - (b - b_1)\beta_0, b_2 - b_1) \chi(\gamma - b\beta_0, b_1), \quad (38) \]

for \( k \geq 2 \). The index \( k \) counts the number of iterations of the kernel in eq. (34).

Having identified the IR renormalon poles of \( \chi(\gamma, b) \), eq. (23), we can study the Borel singularities of the spectral amplitude \( \tilde{n}_{\omega,\gamma}(b) \) from eq. (38). The contribution to \( \tilde{n}_{\omega,\gamma}^{(k)}(b) \) from the virtual (\( \gamma \)-independent) terms only is

\[ \frac{2}{\omega} \left( \frac{N_c}{\omega \pi} \right)^k \left( \frac{Q^2 \rho^2}{4} \right)^{b\beta_0} \times \int_0^b \frac{db_{k-1}}{\Gamma(1 + (b - b_{k-1})\beta_0)} \int_0^{b_{k-1}} \frac{db_{k-2}}{\Gamma(1 + (b_{k-2} - b_{k-1})\beta_0)} \ldots \]
\[ \times \int_0^{b_2} \frac{db_1 \Gamma((b_2 - b_1)\beta_0)}{\Gamma(1 + (b_2 + b_1)\beta_0)} \Gamma(1 + b_1\beta_0). \quad (39) \]

Because the \( b \)-integrals are nested, for \( b < 1/\beta_0 \) no new singularity is introduced in the solution from these virtual corrections. For \( b > 1/\beta_0 \) some of the \( \Gamma \)-function poles will be encountered on the \( b \)-integration paths turning the leading \( b = 1/\beta_0 \) pole into a branch cut. Real contributions introduce singularities on the positive \( b \)-semiaxis for real \( \gamma \), and for \( 0 < \gamma < 1 \), the leading one is found from eqs. (37, 38) to be at \( b = \gamma/\beta_0 \). We note that for \( b\beta_0 < \gamma \) the real contributions to \( \tilde{n}_{\omega,\gamma}^{(k)}(b) \) will not generate singularities because none of the poles of the \( \chi \)-functions in eq. (38) are encountered along the nested \( b \)-integration paths. For \( b\beta_0 > \gamma \), as in the case of virtual contributions, poles are found along some of the \( b \)-paths turning the leading \( b = \gamma/\beta_0 \) pole into a branch cut. From eqs. (23, 38) it follows that there are also singularities for negative values of \( b \). This means that \( b = \gamma/\beta_0 \) is the position of the leading singularity for positive \( b \) only when \( \gamma \) is in the range \( 0 < \gamma < 1 \). If we were to take \( \gamma \) in the range \( n < \gamma < n + 1, n \in \mathbb{Z} \), then the leading IR singularity would be at \( \gamma - n \), and the singularities for negative \( b \) would be displaced further to the left on the negative \( b \)-semiaxis. Although these singularities do not affect the estimate of the power corrections from the non-perturbative effects, they reflect the fact that as \( \gamma \) increases the dipole densities generated from these test functions become increasingly IR divergent.

The above argument can be repeated for the contributions to the solution from cross products of virtual terms for some of the \( \chi \)-functions and real terms for the rest. The net result of this analysis is that the region of analyticity in Borel space of \( \tilde{n}_{\omega,\gamma}(b) \) with \( 0 < \gamma < 1 \) contains the interval \( 0 < b < \gamma/\beta_0 \) and the leading singularity is a branch cut at \( b = \gamma/\beta_0 \). In the asymptotic limit \( Y \to \infty \) the anomalous dimension is \( \gamma = 1/2 + iv \) with \( v \) the spectral parameter to be integrated.
over in eq. (35). Such a \( \text{Re}(\gamma) \) implies the presence of power corrections of \( O(m_N/Q) \) for the dipole density \( n(Y, r, \rho) \). These \( 1/Q \) power corrections have also been reported in the context of conventional BFKL approach in ref. \[13\].

The evolution equation (22) is not conformally invariant because the action of the dipole kernel results in a shift of \( \gamma \) by \( b' \beta_0 \) in eq. (34). However, having established above a region of analyticity in Borel space that contains \( b = 0 \) we can use the \( b \)-expansion as a measure of deviation from the conformal limit. From eqs. (31, 38) the \( b \)-expansion of the spectral amplitude generated after \( k \) iterations of the kernel is

\[
\tilde{n}_{\omega, \gamma}^{(k)}(b) = \frac{2}{\omega} \left( \frac{N_c}{\pi \omega} \right)^k \left\{ \frac{b^{k-1}}{(k-1)!} \chi(\gamma)^k + \frac{b^k}{(k-1)!} \left[ \beta_0 \ln \left( \frac{Q^2 \rho^2}{4} \right) \chi(\gamma)^k + \beta_0 \chi(1)(\gamma) \chi(\gamma)^{k-1} \right] \\
- \frac{b^k}{2(k-1)!} \beta_0 \chi(\gamma)^{k-1} \frac{d \chi(\gamma)}{d \gamma} + O(b^{k+1}) \right\}.
\] (40)

The full spectral amplitude \( n_{\omega, \gamma}(b) \), defined in eq. (36), can be constructed in the small-\( b \) region from summing the powers of \( b \). Summation of the leading powers of \( b \), i.e. retaining only the \( O(b^{k-1}) \) terms in eq. (40) is required in the region

\[
b \ll 1, \quad \frac{N_c \chi(\gamma)}{\pi \omega} b \sim 1.
\] (41)

This ‘resummation’ of the \( O(N_c \chi(\gamma) b/(\omega \pi)) \) terms is well defined if the \( O(b^k) \) terms are small relative to the \( O(b^{k-1}) \) terms in eq. (40). For the \( O(b^k \ln(Q^2 \rho^2/4)) \) term this translates to the requirement

\[
b \beta_0 \ln \frac{Q^2 \rho^2}{4} \ll 1 \iff b \left( \frac{1}{\alpha_s(Q^2)} - \frac{1}{\alpha_s(1/\rho^2)} \right) \ll 1.
\] (42)

Hence, leading \( b \)-power resummation is valid in the conformal limit of fixed coupling. In this case the Borel transform can be readily inverted in terms of a fixed \( \alpha_s \) to yield the well known answer for the BFKL spectral amplitude

\[
n_{\omega, \gamma} = \sum_{k=0}^{\infty} \frac{2}{\omega} \left[ 1 + \frac{N_c \alpha_s \chi(\gamma)}{\pi \omega} + \left( \frac{N_c \alpha_s \chi(\gamma)}{\pi \omega} \right)^2 + \ldots \right] = \frac{2}{\omega - N_c \alpha_s \chi(\gamma)/\pi},
\] (43)

and via eq. (35) the asymptotic solution for the dipole density is obtained \[10\]

\[
n(r, \rho, Y) = \frac{1}{2} \frac{r}{\rho} \frac{\exp[(\alpha P - 1)Y]}{\sqrt{(7/2) \alpha_s N_c \zeta(3) Y}} \exp \left( -\frac{\pi \ln^2(r/\rho)}{14 \alpha_s N_c \zeta(3) Y} \right),
\] (44)

with \( \alpha P - 1 = 4(\alpha_s N_c/\pi) \ln 2. \)
4 Small-\(x\) D.I.S. and power corrections

The structure functions in the small-\(x\) region can be written in the following form

\[
F_{T,L}(x,Q^2) = \frac{Q^2}{4\pi\alpha_{em}} \int_0^1 dz \int d^2r \Phi_{T,L}^{(0)}(z,r) \sigma_{d,N}(Y = \ln(z/x), r). \tag{45}
\]

Using the definition of the dipole-nucleon cross section, eq. (14), the above expression becomes

\[
F_{T,L}(x,Q^2) = \frac{Q^2}{4\pi\alpha_{em}} \int_0^1 dz \int d^2r \Phi_{T,L}^{(0)}(z,r) \int \frac{d^2\rho}{2\pi\rho^2} n(Y,r,\rho) \sigma_0(\rho, m_N) \\
= \frac{Q^2}{4\pi\alpha_{em}} \int \frac{d^2\rho}{2\pi\rho^2} N_{T,L}(Y,\rho) \sigma_0(\rho, m_N). \tag{46}
\]

This equation is the factorisation theorem for the structure function in the small-\(x\) regime. As mentioned in the introduction, it contains in addition to convolution in the longitudinal momentum fraction a convolution in impact parameter. \(\Phi^{(0)}\) is the lowest order transition probability for \(\gamma^* \rightarrow q\bar{q}\). (This is the lowest order impact factor of ref. [17].) It can be calculated reliably in perturbation theory in the limit \(Q^2 \gg \Lambda_{QCD}^2\). The dipole density \(n(Y,r,\rho)\) contains the universal evolution of the initial \(q\bar{q}\) dipole into a cascade of dipoles with ordered rapidities. This is also calculable in perturbation theory as we saw in the previous section. Finally \(\sigma_0(\rho, m_N)\) is the cross section for the absorption of a dipole at impact parameter \(\rho\) by the nuclear target. This is a non-perturbative quantity that normalises the structure function and introduces dependence on the mass scale characteristic of the nucleon target.

So far we have derived the evolution equation that determines the dipole density \(n(Y,r,\rho)\). For completeness let us briefly review the calculation of \(\Phi^{(0)}\) defined in eq. (5). In momentum space and to \(\mathcal{O}(\alpha_s^0)\) the \(\gamma^*\) wave function is [16]

\[
\psi^{(0)}_{T,L}(z,k) = \frac{1}{2q^+\sqrt{z(1-z)}} \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \overline{u}\gamma^+G_{\mu}(k,q)\gamma^+v \epsilon^\mu_{T,L}(q), \tag{47}
\]

where \(u\) and \(v\) are the spinors for the outgoing fermions, \(\epsilon^\mu_{T,L}(q)\) is the polarisation vector of \(\gamma^*\) and \(G_{\mu}\) is the \(\gamma^*q\overline{q}\) Green’s function

\[
G_{\mu}(k,q) = ieZ_q \frac{k + m_q}{k^2 - m_q^2 + i\varepsilon} \gamma^\mu \frac{(k-q)^2 + m_q^2}{(k-q)^2 - m_q^2 + i\varepsilon}, \tag{48}
\]

with \(Z_q\) the electric charge of the quark. Performing the numerator algebra and the \(k^-\) integration we obtain

\[
\psi^{(0)}_T(z,k) = -\frac{eZ_q}{2q^+\sqrt{z(1-z)}} \overline{u}(z \not\! k - (1-z) \not\! k - m_q \not\! k)\gamma^+v, \tag{49}
\]

\[
\psi^{(0)}_L(z,k) = -\frac{eZ_q}{2q^+\sqrt{z(1-z)}} z(1-z)Q \overline{u}\gamma^+v \frac{k^2 + \mu^2}{k^2 + \mu^2}. \tag{50}
\]
where
\[ \mu^2 = z(1 - z)Q^2 + m_q^2. \] (51)

After taking the impact parameter transformation defined by
\[ \psi^{(0)}_{T,L}(z, r) = \int \frac{d^2 k}{(2\pi)^2} e^{ikr} \psi^{(0)}_{T,L}(z, k), \] (52)
and squaring as in eq. (5) we find
\[ \Phi^{(0)}_{T,L}(z, r) = 2N_c e^2 Z^2 q (2\pi)^2 \left\{ [z^2 + (1 - z)^2] \mu^2 K_1(\mu r)^2 + m_q^2 K_0(\mu r)^2 \right\}. \] (53)

These expressions have also been derived in ref. [8] from the imaginary part of the \( \gamma^*g \) forward amplitude.

The factorised expression for the structure functions \( F_{T,L}, \) eq. (46), allows us to study the effects of soft radiation in the small-\( x \) region beyond the leading power of \( Q \). To this end we shall follow the standard lore of renormalon analysis [7]. This means that we shall study the singularity structure of the Borel image \( \tilde{F}_{T,L}(x, Q^2; b) \) defined in the usual way,
\[ F_{T,L}(x, Q^2) = \int_0^\infty db \tilde{F}(x, Q^2; b) e^{-b/\alpha_s(Q^2)}. \] (54)

We emphasise that even in the leading power in \( Q \) analysis the structure functions always contain a non-perturbative component, denoted by \( \sigma_0 \) here. However, we are looking for the Borel singularities that are generated in the subasymptotic \( Q \) regime by the components that are calculable in perturbation theory, \( \Phi^{(0)} \) and \( n \) in our case. This is how perturbation theory is assumed to signal the presence of power corrections. Inspecting eq. (46) we see that the source of Borel singularities is the dipole density \( n \). \( \Phi^{(0)} \) has no \( \alpha_s \) dependence and therefore it stays unaffected by the Borel transformation. It affects the number density of the produced dipoles \( N(Y, r, \rho) \) by determining the distribution of the initial \( q\bar{q} \) dipole in impact parameter (transverse size) space. \( \Phi^{(0)} \) contains infrared regulator set by the scale \( \mu \), eq. (51). This follows from eqs. (53) and the asymptotic behaviour of the Bessel-\( K \) functions for \( \mu r \to \infty \),
\[ K_\nu(\mu r) \to \sqrt{\frac{\pi}{2\mu r}} e^{-\mu r} \left( 1 + \mathcal{O}\left( \frac{1}{\mu r} \right) \right). \] (55)

Note that the leading term in this asymptotic expansion is independent of the \( \gamma^* \) polarisation. Since \( \Phi^{(0)}(z, r) \) is dominated by values of \( z \) away from the end points \( z = 0 \) and \( z = 1 \), we can think of the initial \( \gamma^* \to q\bar{q} \) fluctuation as having size of \( \mathcal{O}(1/Q) \). The object that regulates the
emission of dipoles of large sizes at the end of the cascade is the dipole-nucleon cross section \( \sigma_0 \).

Indeed, dipoles of size \( \rho \gg R_N \), with \( R_N = O(1/m_N) \) the nucleon size, will not couple to the target. From this discussion we can formulate the problem of the Borel singularities of the small-\( x \) structure functions as follows. The Borel image of the structure functions is given by

\[
F_{T,L}(x, Q^2; b) = \frac{Q^2}{4\pi \alpha_{em}} \int_0^1 dz \int d^2r \Phi_{T,L}(z, r) \tilde{\sigma}_{dN}(Y = \ln(z/x), r; b)
\]

(56)

where \( \tilde{\sigma}_{dN} \) is a solution to the evolution equation

\[
\frac{\partial}{\partial Y} \tilde{\sigma}_{dN}(Y, r; b) = \int_0^\infty dr' \int_0^b db' \tilde{K}(r, r'; b') \tilde{\sigma}_{dN}(Y, r'; b - b')
\]

(57)

with kernel given in eq. (21) and boundary condition

\[
\tilde{\sigma}_{dN}(Y = 0, r) = \delta(b) \sigma_0(r, m_N).
\]

In the previous section we studied the action of the kernel \( \tilde{K} \) on the test functions \((r^2)^\gamma\), see eqs. (24, 25). Then these functions were used as a basis for decomposing the dipole density in terms of the spectral amplitudes, eq. (33), and we observed that this leads to \( \gamma \)-dependent IR renormalons. If this procedure were applicable for constructing the structure functions \( \tilde{F}_{T,L} \) it would yield a leading IR renormalon at \( b\beta_0 = \gamma \), which could be to the left of unity for some \( 0 < \gamma < 1 \), corresponding to power corrections of \( O(1/Q^p) \) with \( p < 2 \). This would be in contradiction with the standard OPE expansion of the structure functions, which predicts leading power corrections of \( O(1/Q^2) \), coming from the next-to-leading twist operators.

In the case at hand such a contradiction does not arise for the following reason. We have noted that IR renormalon singularities come from the integration region \( r' \gg r \) in eq. (22), i.e. from emission at large impact parameters. This means that in eq. (57) we are sampling large values of \( r' \). Although \( \sigma_0(r', m_N) \) cannot be calculated in pQCD it is controlled by a nucleon wavefunction which is suppressed at large \( r' \), indicating that the probability to find a sufficiently large primary dipole inside a nucleon is negligible. We can model \( \sigma_0 \) (as we shall do in the next section) by assuming that it has the same asymptotic impact parameter dependence as that of a virtual photon and mass scale \( \mu \rightarrow m_N \). Such a functional form, see eq. (57), cannot be decomposed in the \((r^2)^\gamma\) basis. We emphasise that this does not mean that IR renormalons do not arise from real emission. It means that the leading IR renormalon is anticipated at \( b\beta_0 = 1 \) leading to \( O(1/Q^2) \) corrections consistent with the Wilson OPE expectation.

We conclude this section with the following remarks. The conventional (or Wilson) OPE expansion for large \( Q \) cannot be used for small-\( x \) resummation, as we noted in the introduction.

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Semihard processes involve two large scales and a generalisation of the Wilson OPE is required. Such an expansion would not only separate soft from hard particles but also high rapidity from low rapidity ones. This type of formalism has been brought to a considerable degree of maturity by Balitsky in ref. [17], although it is still far from the point of being a computational algorithm, like Wilson’s OPE. In the high energy OPE, evolution equations are non-linear beyond the LLA(\(\alpha\)) approximation, whereas eq. (22) is linear. This is because our evolution equation does not resum the full set of the next-to-leading logarithms of \(1/x\) but only a subset that involves the running of the coupling. Hence, our solution is not expected to be unitary but it will parametrise sensitivity with respect to low transverse scales.

5 Numerical results

In this section we demonstrate by numerical means that the Borel transform of a deeply inelastic scattering structure function is regular in the region \(0 < b\beta_0 < 1\) and has a singularity at \(b\beta_0 = 1\).

In perturbation theory we cannot calculate the cross section \(\sigma_0\) of eq. (58) for electron-proton scattering. Nevertheless we know that it must be controlled by the nucleon wave function, which vanishes rapidly (assumed exponentially) for sufficiently large impact parameter. A reasonable model, therefore, is to take the impact factor dependence calculated for the virtual photon, eq. (53). For the realistic case of D.I.S. this is indeed a model, whereas if we were considering the scattering of two virtual photons (onium-onium scattering) then this would be exact, up to a factor arising from the integration over the longitudinal momentum fraction, \(z\), in the wavefunction of the target photon. We require that the model nucleon wave function be normalisable and this leads us to choose the case of a longitudinal virtual photon since for the transverse photon the behaviour of the wave function for small impact parameter gives rise to an ultraviolet divergence associated with the photon wave function renormalisation. The scale \(\mu\) in eq. (53) is set to a typical hadron scale of \(\mu = m_N = 1\) GeV. To this end we set the Borel transform of \(\sigma_{dN}^{(0)}(\rho)\) in eqs. (14, 58) to

\[
\tilde{\sigma}_{dN}^{(0)}(\rho; b) = \rho^2 K_0(m_N \rho)^2 \delta(b).
\]  

(59)

We consider this modeling of the dipole-nucleon cross section to be appropriate for studying the IR effects arising from emission and subsequent interaction of large size dipoles. Note that, as stated in the previous section, the large \(r\) asymptotics of the \(\gamma^*\) wave function, which simulates the nucleon target here, is independent of polarisation. So our choice of \(K_0\) Bessel function in the previous equation is convenient and plausible.

The form of the input functions prevent an analytic solution to eq. (57) from being found so we
perform the integrations over $\omega, r'$ and the convolution in $b'$ numerically using standard quadrature methods. More precisely we expand

$$\tilde{\sigma}_{dN}(Y, r; b) = \sum Y^n \frac{1}{n!} \tilde{\sigma}^{(n)}_{dN}(r; b),$$

so as to bring the evolution equation in the form

$$\tilde{\sigma}^{(n)}_{dN}(r; b) = \int dr' \int_0^b db' \tilde{K}(r, r'; b - b') \tilde{\sigma}^{(n-1)}_{dN}(r'; b').$$

The first convolution of the kernel with the longitudinal input function, $\tilde{\sigma}^{(0)}_{dN}(r; b)$ exhibits the structure we expect from section 4, see fig. (1). We observe that the leftmost singularity in the Borel plane appears at $b\beta_0 = 1$ and we identify this as the leading IR renormalon. There is no evidence of any singular behaviour in the region $0 \leq b\beta_0 \leq 1$ which would have been in contradiction with the OPE expectation. No singularity emerges as $b \to 0$ either, due to the cancellation between real and virtual parts of the kernel as discussed in section 3. On the other hand, the expected leading singularity at $b\beta_0 = 1$ can be seen clearly from the right hand graph of fig. (1). For the first iteration of the Borel transformed kernel, this singularity occurs as a pole and it is generated entirely by the virtual correction contribution to $\tilde{K}$.

There is a distinct difference between the asymptotic behaviour of $\tilde{\sigma}^{(0)}_{dN}(r; b)$ and $\tilde{\sigma}^{(1)}_{dN}(r; b)$ as $\mu r \to \infty$, fig. (2). Whereas $\tilde{\sigma}^{(0)}_{dN}(r; b)$ decays exponentially at large $\mu r$, eq. (55), the asymptotic behaviour of $\tilde{\sigma}^{(1)}_{dN}(r; b)$ is:

$$\tilde{\sigma}^{(1)}_{dN}(r; b) \to (\mu r)^{2(b\beta_0 - 1)}.$$  

Examination of the kernel reveals that $\tilde{K}$ scales as $(\mu r)^{2b\beta_0 - 1}$ at large $\mu r'$ and this is the scaling behaviour which dominates in the convolution to determine the behaviour of $\tilde{\sigma}^{(1)}_{dN}$. For intermediate values of $\mu r$, the data obtained can be fitted accurately if we include logarithmic correction terms and we find

$$\tilde{\sigma}^{(1)}_{dN}(r; b) \simeq (\mu r)^{2(b\beta_0 - 1)}(a_1 + a_2 \ln(\mu r) + a_3 \ln(\mu r)^2),$$

where $a_1, a_2$ and $a_3$ are fit parameters. This fit is exhibited in fig. (2).

In section 4, we demonstrated that the exponential nature of $\sigma^{(0)}$ at large $\mu r$ was responsible for the Borel singularity structure observed in the first convolution. The power like behaviour of $\tilde{\sigma}^{(1)}_{dN}(r; b)$ at large $\mu r$ does not imply that new singularities will appear in the Borel plane to the left of $b\beta_0 = 1$. It is the action of the convolution in $b'$ in each subsequent iteration which guarantees this. For large $r'$, the convolution integral scales as $r'^{(2b\beta_0 - 1)}$ from the kernel, $r'^{2(b-b')\beta_0 - 1}$ from the behaviour of $\tilde{\sigma}^{(1)}_{dN}(r'; b)$ and the integration introduces a factor of $r$; the result being that the
second convolution also exhibits a power dependence of the form given in eq. (62) for large $r$. Indeed this large $r$ behaviour is sufficient to predict the position of the leading Borel plane singularity for subsequent iterations. Since the infrared singularities are determined by the infrared (large $r$) behaviour of the function on which the kernel acts, we can consider

$$\int_0^b db' \int d r' \tilde{K}(r, r', b - b')(r'^2)^{b_0 - 1} = \frac{N_c \pi}{\beta_0} \int_0^b db' \chi(b' \beta_0 - 1, b - b') \left( \frac{Q^2 r^2}{\pi} \right)^{b_0} (r'^2)^{b_0 - 1}. \quad (64)$$

Then, from eq. (63) we see that $\chi(b' \beta_0 - 1, b - b')$ has a leading singularity at $b_0 = 1$, which is where the leading singularity will be found for all subsequent iterations. This is demonstrated in fig. (3). We also note that the convolution in $b$ leads to a linear fall off as $b_0 \rightarrow 0$. For further iterations of the kernel it is the intermediate $b_0$ range which will be of most significance. Although the leading singularity occurs again at $b_0 = 1$ as expected from the discussion above, the shape of the distribution is broader. This is because for the second and subsequent iterations the leading singularity is converted into a cut with branch point at $b_0 = 1$ and receives singular contributions from both the virtual correction and real emission part of the kernel.

The generation of each iteration is a numerically intensive task. After generating a sufficient number of iterations, we will be in a position to solve eq. (61) and construct the structure functions numerically. As explained above, we do not expect the position of the leading singularity to shift. However, the behaviour of the wave function in the vicinity of the singularity contains important information about the contribution to structure functions from the infrared regions of transverse momentum space. We will report on findings for the structure functions in a future publication.

6 Summary

Although there exist numerous parametrisations of experimental data for semihard processes that include non-perturbative effects in plausible ways, here we have attempted to study in a systematic fashion the emergence of non-perturbative corrections as signaled by perturbation theory itself. For small-$x$ D.I.S. we have worked within the dipole cascade formalism and introduced the running coupling in a self-consistent way. To study the resulting loss of scale invariance we considered the dipole evolution equation in Borel ($b$) space. The variable $b$ provides a measure of deviation from the conformal limit $b = 0$, where we have shown that the evolution equation reproduces the well known BFKL result.

We have identified the Borel singularities of the dipole density and have studied how these singularities change once the dipole density is convoluted with the dipole-nucleon cross section. In
this case the evolution equation generates leading singularity which is a branch cut at $b\beta_0 = 1$.
This was established by numerical calculation for D.I.S. at small $x$. Such a singularity indicates the presence of $1/Q^2$ power corrections to the small-$x$ structure functions coming from the emission of dipoles of large transverse size. These correspond to ‘ladder gluons’ of small transverse momentum in the BFKL formalism. Hence our approach is suitable for studying the region of diffusion of transverse momentum towards the infrared. Moreover, it would be interesting to see how the scale dependence of the interactions in the dipole cascade modify the dependence of the structure functions on $1/x$.

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Figure 1: The Borel structure of $\tilde{\sigma}^{(1)}_{dN}(r; b)$, for $\mu r = 1$. The graphs show intermediate $b\beta_0$ and $b\beta_0 \to 1$ behaviour.

Figure 2: The behaviour of $\tilde{\sigma}^{(1)}_{dN}(r; b)$, in the intermediate $\mu r$ range and at $b\beta_0 = 0.25$, using the fit in eq. (63).

Figure 3: The Borel structure of $\tilde{\sigma}^{(2)}_{dN}(r; b)$, for $\mu r = 1$. The graphs show intermediate $b\beta_0$ and $b\beta_0 \to 1$ behaviour.