NONLINEAR MAGNETIC DIFFUSIVITY AND ALPHA TENSORS IN HELICAL TURBULENCE

A. Brandenburg\textsuperscript{1}, K.-H. Rädder\textsuperscript{2}, M. Rheinhardt\textsuperscript{2}, and K. Subramanian\textsuperscript{3}

ABSTRACT

The effect of a dynamo-generated magnetic field of Beltrami type on turbulent transport coefficients is studied. Although the turbulence becomes anisotropic and inhomogeneous, the dependence of the mean electromotive force on the mean field can still be expressed in terms of a pseudoscalar $\alpha$ and a scalar turbulent magnetic diffusivity $\eta_t$. Using the testfield method the dependence of $\alpha$ and $\eta_t$ on the magnetic Reynolds number $Re_M$ is determined for magnetic fields that are in equipartition with the velocity field. Increasing $Re_M$ from 2 to 600 reduces $\eta_t$ only by a factor of 2, suggesting that the quenching of $\eta_t$ is, in contrast to the 2-dimensional case, essentially independent of $Re_M$. Over the same range of $Re_M$, $\alpha$ is reduced by a factor of 8, but this can be explained by a corresponding increase of a magnetic contribution to the $\alpha$ effect with opposite sign. Within this framework, the corresponding kinetic contribution to the $\alpha$ effect turns out to be independent of $Re_M$ for $2 \leq Re_M \leq 600$. The level of fluctuations of $\alpha$ and $\eta_t$ is only 10% and 20% of the respective kinematic reference values.

Subject headings: MHD – turbulence

1. INTRODUCTION

Magnetic fields in stars and galaxies tend to display large scale spatial order, and in the case of the Sun also long term temporal order, as manifested by its 22 year cycle. The underlying dynamo responsible for this is generally believed to be a turbulent large-scale or mean-field dynamo. A dynamo which works with helical turbulence in the shear flow of an incompressible fluid is known as a helical dynamo. Of course, the assumption of unquenched values of $\alpha_K$ and $\eta_t$ is unrealistic (Kleeorin & Rogachevskii 1999) and some level of quenching of $\eta_t$ was found to be necessary to reproduce the simulations (see the review by Brandenburg & Subramanian 2005).

Since the original work of Vainshtein & Cattaneo (1992) a lot of effort has gone into determining the quenching of $\alpha$ and there is now no doubt that under certain conditions, when the mean field is defined as volume average over a periodic box, $\alpha$ is quenched like $Re_M^{-1}$ for mean fields of equipartition strength (Cattaneo & Hughes 1996, Brandenburg & Subramanian 2005). However, subsequent work showed that this is a particular consequence of the use of full volume averages, in which case the mean current density is zero (Blackman & Brandenburg 2002). This restriction will be relaxed in the following.

A final frontier of research in mean-field dynamos has been the quenching of $\eta_t$. A number of earlier studies have already tried to determine $\eta_t$. For example, in the two-dimensional case, $\eta_t$ is indeed catastrophically quenched (Cattaneo & Vainshtein 1991), but in three dimensions the quenching may depend just on $\overline{B^2}$ and not also on $Re_M$. This has already been found from the decay rate of a nonhelical large-scale magnetic field in driven small-scale turbulence (Yousef et al. 2003). Similar indications come also from fitting mean field models to corresponding simulations (Blackman & Brandenburg 2002). Quantifying more precisely the simultaneous quenching of $\alpha$ and $\eta_t$ is the goal of the present paper. In the following we allow both $\alpha$ effect and turbulent diffusivity to be tensors, denoted by $\alpha_{ij}$ and $\eta_{ij}$, respectively, and we calculate their components using the testfield method, as described in a number of earlier papers (Schrinner et al. 2007, Brandenburg et al. 2008, Sur et al. 2008). However, unlike some of the earlier work where the velocity field was a kinematic one, we now allow the velocity to be the result of the fully nonlinear hydromagnetic equations.

2. THE METHOD

Following earlier work by Brandenburg (2001), we consider a compressible gas with an isothermal equation of state with sound speed $c_s$, but now we also solve a set of testfield equations, as was done in Brandenburg et al. (2008) for the
the electromotive force, $rac{\partial \mathbf{u}}{\partial t} = -\mathbf{U} \cdot \nabla \mathbf{u} - \nabla \ln \rho + \mathbf{J} \times \mathbf{B} + \nabla \times 2\rho \nu \mathbf{S}$, (1)

\[
\frac{\partial \ln \rho}{\partial t} = -\mathbf{U} \cdot \nabla \ln \rho - \nabla \cdot \mathbf{U},
\]

\[
\frac{\partial \mathbf{A}}{\partial t} = \mathbf{U} \times \mathbf{B} - \eta \mathbf{J},
\]

\[
\frac{\partial \mathbf{b}^{pq}}{\partial t} = \mathbf{U} \times \mathbf{b}^{pq} + \mathbf{u} \times \bar{\mathbf{B}}^{pq} + \mathbf{u} \times \mathbf{b}^{pq} - \mathbf{u} \times \mathbf{b}^{pq} - \eta j^{pq},
\]

where overbars denote horizontal (xy) averages, so $\bar{U}(z,t)$ is the mean velocity, $\mathbf{u} = \mathbf{U} - \mathbf{U}$ is the deviation from the average, and the superscripts $pq$ refer to four separate equations that are characterized by four different testfields $\bar{B}^{pq}$ that have a cos $kz$ or sin $kz$ dependence ($q = c, s$) in the $x$ and $y$ components ($p = 1, 2$). We employ the magnetic vector potential both for the actual magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and for the response to the testfields, $\mathbf{b}^{pq} = \nabla \times \mathbf{a}^{pq}$. Unlike the earlier kinematic work of Brandenburg et al. (2008), we allow here the velocity $\mathbf{U}$ to be affected by the magnetic field $\mathbf{B}$ through the Lorentz force, $\mathbf{J} \times \mathbf{B}$, where $\mathbf{J}$ is the mean current density.

The flow is driven by a random forcing function $\mathbf{f}$ consisting of circularly polarized plane waves with positive helicity and random direction (giving rise to a flow with maximal helicity), and $\mathbf{S}_{ij} = \frac{1}{2} (\mathbf{U}_{ij} + \mathbf{U}_{ji}) - \frac{1}{2} \eta \mathbf{J} \cdot \nabla \mathbf{U}$ is the traceless rate of strain tensor. The forcing function is chosen such that the magnitudes of the wavevectors, $|k|$, are in a narrow interval around an average value, which is denoted simply by $k_f$. The corresponding scale, $2\pi/k_f$, is also referred to as the energy-carrying scale of the turbulence.

Important control parameters are the magnetic Reynolds and Prandtl numbers, $\text{Re}_M = u_{rms}/(\eta k_f)$ and $\text{Pr}_M = \nu/\eta$, respectively. Here, $\text{Re}_M$ is defined with respect to the actual (magnetically affected) rms velocity, $u_{rms} = \langle \mathbf{u}^2 \rangle^{1/2}$, where angular brackets denote volume averages. The smallest possible wavenumber in a triply-periodic domain of size $L \times L \times L$ is $k_1 = 2\pi/L$. In order to achieve large values of $\text{Re}_M$, we keep the value of $k_f/k_1$ small, but still large enough to allow for a clear separation of scales between the system scale and the energy-carrying scale. We use $k_f/k_1 = 3$ as a compromise.

The solutions to the test field equations allow us to calculate the electromotive force, $\mathbf{E}^{pq} = \mathbf{u} \times \mathbf{b}^{pq}$ and the 4+4 components of $\alpha_{ij}$ and $\eta_{ij}$ in the expansion

\[
\mathbf{E}^{pq} = \alpha_{ij} \mathbf{b}^{ij} - \eta_{ij} \mathbf{J}^{ij}. \tag{5}
\]

For details we refer to Brandenburg et al. (2008), but note that now the resulting mean field, which is of Beltrami-type (see below), imposes a preferred direction to the turbulence and makes it anisotropic and, in general, also inhomogeneous. The resulting $\alpha_{ij}$ and $\eta_{ij}$ tensors can then be written in the form

\[
\alpha_{ij} (\mathbf{B}) = \alpha (\mathbf{B}) \delta_{ij} + \alpha (\mathbf{B}) \hat{B}_i \hat{B}_j,
\]

\[
\eta_{ij} (\mathbf{B}) = \eta (\mathbf{B}) \delta_{ij} + \eta (\mathbf{B}) \hat{B}_i \hat{B}_j. \tag{7}
\]

However, when inserting this into the general expression for the electromotive force, $\mathbf{E} = \alpha \mathbf{B} - \eta \mathbf{J}$, this reduces to

$\mathbf{E} = \alpha \mathbf{B} - \eta \mathbf{J}$, with non-tensorial coefficients

$\alpha = \alpha_1 + \alpha_2 - \eta_2 k_m$ and $\eta = \eta_1$, \tag{8}

where $k_m = k_m(z,t) \equiv \mathbf{J} \cdot \mathbf{B}/B_m^2$ is a pseudoscalar that quantifies the helicity of the large-scale field. It turns out that in the cases considered here $k_m$ is approximately constant and, since the helicity of the forcing is positive, the helicity of the large-scale field is negative, so $k_m/k_1 \approx -1$.

3. RESULTS

Throughout this paper we focus on the case $\text{Pr}_M = 1$ and vary $\text{Re}_M$ between 2 and 600. For large values of $\text{Re}_M$, the turbulence becomes more fully developed and a broader range of scales is excited, as can be seen in spectra of kinetic and magnetic energy, $E_K$ and $E_M$, respectively. Such spectra, summarized here by $k^{-3/2}$, are shown in Fig. [1]. In the range $4 < k/k_1 < 30$ both spectra are comparable to a $k^{-3/2}$ spectrum, as has been seen earlier for nonhelical turbulence (Haugen et al. 2003). Spectra of kinetic and magnetic helicity, $H_K$ and $H_M$, respectively, are also shown for comparison, summarized by $k^{1.2}$ and $k^{3.2}$.

For $\text{Re}_M < 2$ there is no dynamo action, but in all other cases a large scale magnetic field is maintained (Fig. [2], just as in Brandenburg (2001), except that here $k_f/k_1 = 3$ instead
of 5 or larger. The large scale field is a Beltrami field,
\[ \mathbf{B}(z, t) = \mathbf{B}(t) (\cos \theta, \sin \theta, 0), \quad \text{where} \quad \theta = k_1 z + \phi, \]
with phase \( \phi \). We use this field also as initial condition with \( \phi = 0 \). However, during the course of the run \( \phi \) can drift slightly, so we determine \( \phi(t) \) by taking the volume average of the dot product of the actual field with the initial field at negative argument, \(-k_1 z\). This yields equations for \( \cos \phi \) and \( \sin \phi \) and hence
\[ \tan \phi(t) = \frac{\langle B_z \cos k_1 z \rangle - \langle B_z \sin k_1 z \rangle}{\langle B_z \sin k_1 z \rangle + \langle B_z \cos k_1 z \rangle}. \]

The coefficients \( \alpha_1 \) and \( \alpha_2 \) are determined by taking suitably defined weighted averages,
\[ \alpha_1 + \alpha_2 = (\langle \alpha_{11} \cos^2 \theta \rangle + \langle \alpha_{22} \sin^2 \theta \rangle) \]
\[ + (\langle \alpha_{12} + \alpha_{21} \rangle \cos \theta \sin \theta) \]
\[ \alpha_2 = 4 (\langle \alpha_{12} + \alpha_{21} \rangle \cos \theta \sin \theta) \]
and analogously for \( \eta_1 \) and \( \eta_2 \). We then calculate \( \alpha \) and \( \eta \) using equation (8).

In order to avoid the complications associated with the possibility of time derivatives in the \( \alpha \) quenching formula (e.g., Klecorin & Ruzmaikin 1982, Blackman & Brandenburg 2002), we focus on statistically steady solutions. However, the \( \alpha^2 \) dynamo has only one such solution, giving access to only one value of \( \mathbf{B}^2 \). Therefore we restrict ourselves to steady solutions and consider the results for \( \alpha_i(\mathbf{B}) \) and \( \eta_i(\mathbf{B}) \) for just one value of \( \mathbf{B}^2 \), but different values of \( \Re_M \). In Table 1 we represent the results in nondimensional form, indicated by a tilde. Again, we use the rms velocity in the nonlinear saturated state, \( u_{ms} \). We express the rms value of the resulting mean field and the rms value of the fluctuations via
\[ \mathbf{B}^2 = \mathbf{B}^2 / B_{eq}^2 \quad \text{and} \quad \mathbf{b}^2 = b^2 / B_{eq}^2, \]
respectively. Here, \( B_{eq}^2 = \mu_0 u_{rms}(\mathbf{B}) \) and \( \mathbf{b} \) is the mean density. We also introduce
\[ \bar{\eta}_1 = \eta_1 / \eta_0, \quad \bar{\eta}_2 = \eta_2 / \eta_0, \quad \bar{\eta} = \eta / \eta_0, \]
\[ \bar{\alpha}_1 = \alpha_1 / \alpha_0, \quad \bar{\alpha}_2 = \alpha_2 / \alpha_0, \]
where \( \eta_0 = 1 / \mu_0 u_{ms}(\mathbf{B}) k_f \) and \( \alpha_0 = 1 / \mu_0 u_{ms}(\mathbf{B}) \) are reference values that are already magnetically affected. (Note that the kinematic values of \( u_{ms} \) would not be available for all runs, because we use initial conditions where the field strength is already close to the final value.) This normalization implies that in the kinematic case \( \bar{\alpha}_1 = \bar{\alpha}_1 = 1 \) (Sut et al. 2008), while \( \bar{\alpha}_2 = \bar{\eta}_2 = 0 \). As in earlier work, error bars are calculated based on the maximum departure obtained from the 3 time series taken over 1/3 of the full sequence.

The consistency of the results for \( \alpha \) and \( \eta \) with the assumption of a steady state can be assessed by calculating the linear growth rate, \( \lambda \), of the associated mean field dynamos, i.e.
\[ \lambda = \alpha K_m - \eta k_m^2, \]
where \( \eta_T = \eta + \gamma \) is the sum of turbulent and molecular magnetic diffusivities. In the quenched steady state \( \lambda \) should vanish. Again, we present \( \lambda \) in nondimensional form, here in terms of the global turbulent decay rate,
\[ \lambda = \lambda / (\eta_0 k_f^2), \]
\[ = \bar{\alpha} k_f \bar{K}_m - \bar{\eta} \bar{k}_m^2. \]

Within error bars, the value of \( \lambda \) is consistent with zero, thus supporting the consistency with the assumed steady state; see Table 1 (An exception is Run A, because it is subcritical and so \( \lambda < 0 \)). This consistency gives evidence that the testfield method is indeed applicable also to the nonlinear case. Another measure of the reliability of the averages is the length of the time series in turnover times, \( \delta t = u_{ms} k_f (t_{max} - t_{min}) \). Our results presented in Table 1 show a moderate decline of \( \tilde{\alpha} \) and mild decline of \( \tilde{\eta} \) as \( \Re_M \) increases.

Note also that there are random fluctuations of \( \tilde{\alpha} \) and \( \tilde{\eta} \), denoted here by their rms values, \( \bar{\alpha}_{rms} = \alpha_{rms} / \alpha_0 \) and \( \bar{\eta}_{rms} = \eta_{rms} / \eta_0 \), but even for large values of \( \Re_M \) these nondimensional values remain around 0.1 and 0.2, respectively. This is less than in the kinematic case (Brandenburg et al. 2008b), but can be comparable to the respective mean values. If shear were included, this could then give rise to an additional incoherent alpha–shear dynamo (Vishniac & Brandenburg 1997, Proctor 2007, Brandenburg et al. 2008b).

4. DISCUSSION

Let us now put our results in relation to earlier work, most of which was done with uniform fields using mean fields defined as full volume averages. In that case \( \alpha \) was quenched all the way to zero like \( \Re_M^0 \) as \( \Re_M \) increases. This result can be understood in terms of a mutual cancelation of kinetic and magnetic contributions to the \( \alpha \) effect (Pouquet et al. 1976),
\[ \alpha = \alpha_K + \alpha_M, \quad \text{where} \quad \alpha_M = \frac{1}{\tau} \mathbf{f} \cdot \mathbf{B}, \]
with \( \tau = St / u_{ms} k_f \). Following earlier work (Brandenburg & Subramanian 2005), who found \( St = 1 \) for uniform fields, we assume \( St = 1 \) in the present work as well. The resulting values of \( \alpha_M \) are also listed in Table 1. We can then calculate \( \alpha_K = \alpha - \alpha_M \), which is plotted in Fig. 3 as a function of \( \Re_M \). Surprisingly, it turns out that \( \alpha_K \) is roughly independent of \( \Re_M \) within the range from 2 to 600. The kinematic value of \( \bar{\alpha}_K \) is 1, so the reduction to 0.8 is practically negligible, which is surprising given that \( \mathbf{B} / B_{eq}^2 \) is of order unity. On the other hand, \( \bar{\eta} \) shows a mild decline from 0.67 to 0.24 as \( \Re_M \) increases by 2.5 orders of magnitude from 2 to 600, although this is much weaker than the decrease in the two-dimensional case where \( \eta_1 \) decreases like \( \Re_M^{-1} \). Again, however, for large values of \( \Re_M \), \( \bar{\alpha} \) is only quenched to values around \( \bar{\eta}_T / k_f \), as is necessary for a steady state. For the present case with \( k_f = 3 \), we note that \( \bar{\eta}_T / k_f \) decreases from 0.2 to 0.1 as \( \Re_M \) increases from 2 to 600 and this limits how much \( \bar{\alpha} \) could be quenched down to. This can be clearly seen in Fig. 3 where we also plot \( \bar{\eta}_T / k_f \) as a function of \( \Re_M \). The constancy of \( \alpha_K \) with \( \Re_M \) is appealing and this simplicity may also be suggestive of the validity of equation (18) for \( \alpha_M \). We should emphasize, however, that this result is based on closure calculations, whereas the results for \( \alpha \) and \( \eta \) are obtained completely independently from the actual simulations data.

5. CONCLUSIONS

For the first time it has been possible to determine both \( \alpha_{ij} \) and \( \eta_{ij} \) in the magnetically quenched case. The quenched turbulent coefficients are described by the non-tensorial values \( \alpha \) and \( \eta \). Our results are compatible with the general idea of catastrophic quenching in that \( \alpha \) is quenched by \( -\alpha_M \) approaching \( \alpha_K \) for finite field strengths and large \( \Re_M \). Generally, this \( \alpha \) will be quenched to whatever is the value of \( \eta k_m \) (Blackman & Brandenburg 2002). However, up until now we
had no idea how big is the quenched value of $\eta_i$. There was the possibility that $\eta_i$ was quenched to very small values, just like $\eta$ can be determined either by knowledge of the transport coefficients for runs in the range $2 \leq Re_M \leq 600$ at saturation field strengths.

Concerning astrophysical dynamos, we expect that the cycle frequency of oscillatory dynamos in the saturated state is prolonged by a certain factor, depending on the reduction of $\eta_i$. In the present work this factor was 4, but it could well be larger. Furthermore, the amount of small scale magnetic helicity production will be important, and this reduces $\alpha$ by a somewhat larger factor (here by a factor of 8). For closed domains, this inevitably leads to a resistively slow saturation phase, but this tendency will be offset in an open domain if there are magnetic helicity fluxes.

We thank Eric Blackman and Alexander Hubbard for useful discussions at an early phase of this work when it was still unclear that the testfield method used here would actually work in the nonlinear case. We acknowledge Nordita for providing a stimulating atmosphere during the program “Turbulence and Dynamos”, when the essential pieces of this paper were conceived. We are also grateful for computing resources provided by the Swedish National Allocations Committee at the National Supercomputer Centre in Linköping.

REFERENCES

Blackman, E. G., & Brandenburg, A. 2002, ApJ, 579, 359
Brandenburg, A. 2001, ApJ, 550, 824
Brandenburg, A., & Subramanian, K. 2005, Phys. Rep., 417, 1
Brandenburg, A., Rädler, K.-H., Rheinhardt, M., & Käpylä, P. J. 2008, ApJ, 676, 740
Cattaneo, F., & Vainshtein, S. I. 1991, ApJ, 376, L21
Cattaneo, F., & Hughes, D. W. 1996, Phys. Rev. E, 54, R4532
Field, G. B., & Blackman, E. G. 2002, ApJ, 572, 685
Haugen, N. E. L., Brandenburg, A., & Dobler, W. 2003, ApJ, 597, L141
Kleiner, N., & Rogachevskii, I. 1999, Phys. Rev. E, 59, 6724
Kleiner, N. I., & Ruzmaikin, A. A. 1982, Magnetohydrodynamics, 18, 116
Krause, F., & Rädler, K.-H. 1980, Mean-field magnetohydrodynamics and dynamo theory (Pergamon Press, Oxford)
Moffatt, H. K. 1978, Magnetic field generation in electrically conducting fluids (Cambridge University Press, Cambridge)
Pouquet, A., Frisch, U., & Léorat, J. 1976, J. Fluid Mech., 77, 321
Proctor, M. R. E. 2007, MNRAS, 382, L39
Scherrer, M., Rädler, K.-H., Schmitt, D., Rheinhardt, M., & Christensen, U. R. 2007, Geophys. Astrophys. Fluid Dyn., 101, 81
Subramanian, K. 2002, Bull. Astr. Soc. India, 30, 715
Subramanian, K., & Brandenburg, A. 2004, Phys. Rev. Lett., 93, 205001
Subramanian, K., & Brandenburg, A. 2006, ApJ, 648, L71
Sur, S., Brandenburg, A., & Subramanian, K. 2008, MNRAS, 385, L15
Vainshtein, S. I., & Cattaneo, F. 1992, ApJ, 393, 165
Vishniac, E. T., & Brandenburg, A. 1997, ApJ, 475, 263
Vishniac, E. T., & Cho, J. 2001, ApJ, 550, 752
Yousef, T. A., Brandenburg, A., & Rüdiger, G. 2003, A&A, 411, 321