Weakly Smooth Structures in Gromov-Witten Theory

Preliminary Version

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1 Introduction

In order to establish Fredholm theory on stratified topological Banach manifolds in Gromov-Witten theory, we have introduced flat structures on such manifolds in [L4]. Such a structure is obtained from local flat coordinate charts. The transformations between these charts are only continuous in general. The purpose of this paper is two-fold: firstly to show that on the topological Banach manifolds and bundles appeared in GW theory, there are enough smooth functions and sections viewed in any admissible charts and trivializations; secondly to demonstrate some finer aspects about the weakly smooth sections. As far as the Fredholm theory in [L4] is concerned, the existence of sufficiently many smooth functions and sections makes these topological Banach manifolds and bundles behave as if they are the smooth ones. The motivation of this work and [L4] is to overcome the analytic difficulty in Gromov-Witten and Floer type theories, the lack of differentiability of the orbit spaces of $L^p_k$ stable maps. These orbit spaces are quotient spaces of parametrized stable $L^p_k$-maps by the actions of the reparametrization groups. For our purpose, they are the primary examples of stratified topological Banach manifolds. In GW and Floer type theories, each of them comes along with a stratified topological Banach bundle with a proper Fredholm section. As mentioned in [L4], the lack of differentiability shows up at a few different levels: (A) Even there is only one stratum, the transition functions between different coordinate charts are not smooth due to the non-smoothness of the actions of the reparametrization group. But at least, the specified Fredholm section, the $\bar{\partial}_J$-section, is smooth viewed in any chart due to the equivariancy of such a section. (B) When there are at least two strata, the pre-gluing joining different strata introduces coordinate charts near the "ends" of the higher strata, which correspond to the deformations and
degenerations of the domains of the stable maps. These charts together with the related bundles are fibrations over the Deligne-Mumford moduli space of stable curves locally. Even within a fixed stratum, locally the $\bar{\partial}_J$-section should be considered as a family of smooth sections on the "central fiber" parametrized by the Deligne-Mumford moduli space. Only in this sense, the $\bar{\partial}_J$-section is a stratified smooth section. (C) Unlike the usual stratified spaces, there is a pathological phenomenon in the stratum structure of the space of stable $L^p_k$ maps: each lower stratum viewed as end of a higher stratum appears to have "more" dimensions than the higher stratum has. In other words, the family of the stable $L^p_k$-maps parametrized by the gluing parameters is not "flat". Therefore, it does not even make sense to talk about smoothness of the $\bar{\partial}_J$-section at a point in the lower stratum along the normal directions pointing to the higher stratum simply because there is no local "product structure" near the end.

In this paper, we mainly deal with difficulty (A) for topological Banach manifolds and bundles in GW theory. This means that we only deal with part of the ambient spaces in GW theory, which has only one stratum with trivial isotropy groups. Of course, a general ambient space is decomposed into its strata, and the theory of this paper is applicable to each stratum with a "fixed domain". However, in the general case, each stratum may have "ends" defined by the pre-gluing and moving double points. Some modifications are needed in order to apply the work here to these cases. These modifications and other cases and difficulties for establishing a Fredholm theory on stratified Banach manifolds are treated in [L2] and [L3]. At the end of this section, we will outline the construction of the flat chart so that readers can have a general idea on how the difficulties in (B) and (C) are resolved in [L2] and [L3].

Large part of this paper is to use the ambient space of $L^p_k$-stable maps as an example to illustrate on how to use its weakly smooth structure to establish the $C^{m_0}$-smoothness of the moduli spaces of perturbed $J$-holomorphic maps. Here $m_0 = k - \frac{2}{p} > 1$ is the Sobolev smoothness of an $L^p_k$-map.

As far as the smoothness of the moduli space is concerned, the main result of this paper is the following theorem.

**Theorem 1.1** Let $s : B_{k,p}(A) \to L_{k-1,p}$ be the $\bar{\partial}_J$-section of the bundle $L_{k-1,p}$ over the space of stable $L^p_k$-maps of class $A \in H_2(M, \mathbb{Z})$ from $\Sigma = S^2$ to a compact symplectic manifold $(M, \omega)$ with an $\omega$-compatible almost complex structure $J$, where the fiber of $L_{k-1,p}$ at $(f : \Sigma \to M)$ is $L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$. Assume that $s$ is proper in the sense that the moduli space of $J$-holomorphic sphere of class $A$, $\mathcal{M}(J, A) = s^{-1}(0)$ is compact. Assume further that all isotropy groups are trivial and that the virtual dimension of $\mathcal{M}(J, A)$ is less than $m_0$. Then there are generic small perturbations $\nu = \{\nu_i, i \in I\}$, which are compatible sections of the local bundles $L_i \to W_i$ of class $C^{m_0}$ defined on the local uniformizers $W_i, i \in I$, such that the perturbed moduli space $\mathcal{M}^\nu(J, A) = \cup_{i \in I}(s + \nu_i)^{-1}(0)$ is a compact manifold of class $C^{m_0}$.

Note that the main point of this theorem is (i) the degree of the smoothness
of the moduli space, namely it has at least the same smoothness as the Sobolev
differentiability of a generic element in ambient space has; (ii) the smoothness is
achieved by the coordinate transformations between the given "natural" charts.

In view of (i) and (ii) above, this is the best one can get presumably from
the analytic set-up here. Of course, this implies that if all geometric data are
of class $C^\infty$, we get $C^r$-smoothness for the perturbed moduli space $\mathcal{M}^r(J,A)$
with $r$ arbitrarily large. Using the comments below on regularity of perturbed
$J$-holomorphic maps, it is possible to show that in this case there is a common
perturbation independent of $r$ so that the moduli space is in fact $C^\infty$-smooth.
On the other hand, the theorem is still true even when the geometric data is
not $C^\infty$-smooth, but only sufficient smooth comparing to $m_0$. The exact degree
of the required smoothness for the geometric data can be determined from the
proof of the theorem. Presumably $2k$ should be sufficient.

The existence of such a smooth or stratified smooth structure on the moduli
space have many applications in $GW$ and Floer type theories. Note that if we
only want some weaker results such as the existence of a topological manifold
structure on the perturbed moduli space, it is possible to avoid part of the
discussion on the finer aspect of weakly smoothness in this section. In fact, this
topological manifold structure on the moduli space is already sufficient for most
of applications. For instance, it was used in [LT] to prove the Arnold conjecture.
However, when [LT] was written, the issue of lacking smoothness was not yet
addressed. Nevertheless, the matter was resolved by the author during the
preparation for his seminar courses. We now describe what modifications to
the global perturbation method of [LT] are needed to get a topological manifold
structure on extended moduli spaces. We will mainly restrict ourself to the
above case as it is well-known that this basic case already captures the main
issue of the lack of differentiability. The main steps for generalization will be
only outlined. Since the $\bar{\partial}$-section itself is smooth ( stratified smooth for the
general case) viewed in any local uniformizer (which is called a local slice in
this paper), we only need to make sure that the perturbation coming from the
cokernel defined on one local slice is at least $C^1$-smooth viewed in any other
local slice. Those local perturbation are obtained by extending the elements $\xi$
of the cokernel $K_f$ of linearization of the $\bar{\partial}$-section at a $J$-holomorphic map $f$.

The extension is obtained by extending $\xi$ to a "constant" section $\tilde{\xi}$ over a
local slice $W_f$ first by using the trivialization of the local bundle $\mathcal{L}(f) \to W_f$
induced by the parallel transport of $M$, then multiplying $\tilde{\xi}$ by a cut-off function
supported on $W_f$.

Therefore, it is sufficient to show that both cut-off function and $\tilde{\xi}$ are at
least of class $C^1$ viewed in any slice. Any such functions or sections are called
weakly smooth ones of class $C^1$ in this paper.

• Existence of the $C^{m_0}$-Smooth Cut-off Functions:

It is well-known that when $p$ is a positive even integer, the $p$-th power of the
$L^p_k$-norm of the Banach space $L^p_k(\Sigma, f^*TM)$ is smooth. This smoothness of $L^p_k$-norm of the Banach space and the related cut-off functions were already used in [LT]. In order to get the desired cut-off function on the space of unparametrize stable $L^p_k$-maps, we need to show that the above $p$-th power of norm function is still smooth after composed with the action map by the Lie group $G$ of the reparametrizations of the domain. Denote the composed function by $\Psi_p$. In the case that $\Sigma = \mathbb{R}^1 \times S^1$ and $G$ is the group of translations, we have $\Psi_p \circ g = \Psi_p$ for any $g \in G$. Hence $\Psi_p$ is $G$-invariant despite of the fact that $G$-action is only continuous generically. Because the simplest case above is supposed to capture the main difficulty of lack of differentiability already, this immediately suggests that in the general case $\Psi_p$ should be smooth as well and how this can be proved. Indeed the only difference of this special case with the general case is that $G$-action here preserves the volume form on $\Sigma$, and the smoothness in $G$-direction for the general case can be proved by a simple variable change formula in calculus. For completeness, we include the proofs of the above well-known facts in the last section of this paper.

Since open sets in $L^p_k(\Sigma, f^*TM)$ serve as coordinate charts of the space of parametrized stable $L^p_k$-maps, the smoothness of the $L^p_k$-norm with respect to the $G$-actions should imply that the push-forward of this function to a local slice should be $C^{m_0}$-smooth viewed in any other slice. Whether or not this is true is not completely obvious.

We proceed slightly differently. First embed the symplectic manifold $M$ with the induced metric isometrically into some $\mathbb{R}^d$. Then consider the space of parametrized stable $L^p_k$-maps in $M$ as a closed Banach submanifold of the corresponding space of the maps in $\mathbb{R}^d$. This later space is a Banach space so that the result above is applicable. Note that here we have used the fact that when $m_0$ is large enough, the two Sobolev metrics on the space of $L^p_k$-maps from $\Sigma$ to $M$ are equivalent. As a result of this method, in addition to the cut-off function that we are looking for, we also obtain a large collection of $G$-smooth functions on the space of parametrized $L^p_k$-maps in $M$ by using pulling-backs. They give rise the corresponding weakly $C^{m_0}$-smooth functions defined on the space of unparametrized stable $L^p_k$-maps of $M$.

Note that in general, even above composed function $\Psi_p$ is of class $C^\infty$ and the cut-off function is of class $C^\infty$ in a given slice, we still can not conclude that the cut-off function is $C^\infty$-smooth viewed in any slices (see the proof in section 3 and the discussion below for the reason of this). In other words, it is weakly $C^{m_0}$-smooth but not weakly $C^\infty$-smooth. Therefore we get the desired cut-off functions.

• $C^1$-Smooth Perturbations:

As for the weakly smoothness of $\tilde{\xi}$, without using the discussion below on some finer aspect of weakly smooth sections, we can only prove that it is of class $C^1$. Of course, there is no need to assume that $\xi$ is in the cokernel. But we do need to assume that it is an element of the fiber of $(\mathcal{E}_{k-1,p}(f))_f$ of class at least
Let $\tilde{B}$ be the space of parametrized stable $L^p_k$-maps and $\tilde{L} \to \tilde{B}$ be the corresponding bundles. Denote a small neighbourhood of $f$ in $\tilde{B}$ by $W_f$ and the corresponding local bundle by $\tilde{L}(f)$. We have already denoted its restriction to the slice $W_f$ by $L(f)$. Assume that $W_f$ is part of the $G$-orbit of $W_f$ near $f$. Then by abusing the notation, we have a homeomorphism $\tilde{W}_f \sim G \cdot W_f$ of two open sets in a Banach space, and two trivializations of $\tilde{L}$ on $\tilde{W}_f$: one is given by using parallel transport of the central fiber; the other is obtained by using parallel transport of the central fiber over the slice $W_f$ first, then using the pullbacks of the $G$-actions to bring the fibers over $G \cdot W_f$. The two trivializations are only topological equivalent. Use the first trivialization, the "standard" one, we get a "constant" section over $\tilde{W}_f$ from $\xi$ in the central fiber, denoted by $\xi_1$, which is smooth but not $G$-equivariant. To get a $G$-equivariant extension, we use the second trivialization to extend $\xi$, the restriction of $\xi_1$ to the slice $W_f$, over $\tilde{W}_f \sim G \cdot W_f$. Clearly, this latter extended section, denote by $\xi_2$ is smooth with respect to the second trivialization. The question is about the degree of the smoothness of the section $\tilde{\Phi}_{1,2} \circ \xi_2$, where $\tilde{\Phi}_{1,2}$ is the topological identification from the second trivialization to the first one. In other words, we want to know the smoothness of $\xi_2$ viewed in the standard trivialization.

To this end, we first sketch a argument using $sc$-smoothness in polyfold theory. Note that by letting $k$ varying, all identifications above are of class $sc^{\infty}$. Therefore, $\tilde{\Phi}_{1,2} \circ \xi_2$ gives rise a $sc^{\infty}$-section since $\xi_2$ is such a section. Under the trivializations, above sections become maps from the two domains to the central fiber, $L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$, which give rise the corresponding $sc^{\infty}$- equivalence. Note that while the map $\xi_2$ can be lifted further into higher Sobolev space by our assumption, $\tilde{\Phi}_{1,2} \circ \xi_2$ can not be. Roughly speaking the reason for this is that the $sc$-regularity for the transition map between the two trivializations is of degree $k$=the degree of a generic element of the above two domains.
To start the direct proof, denote $\hat{\Phi}_{1,2} \circ \xi_2$ by $[\xi_1]$ for short. We start with the two Banach coordinates on $\tilde{W}_f \sim G \cdot W_f$. We have the standard coordinate given by $(w, t)$ with $w$ in the local slice $W_f$ and $t$ in $\hat{H}^*$, where $\hat{H}^*$ is the complement space of $W_f$ in $\tilde{W}_f$ which can be identified with the direct sum of the orthogonal complements, denoted by $\hat{H}^*$ of the corresponding local hypersurfaces used to define the local slice above (see Sec. 3 or [LT] for the more details). Note that dimension of $\hat{H}^*$ is equal to dimension of $G$. The second coordinate from the above identification is denoted by $(u, g)$ with $u$ in $W_f$ and $g$ in $G_e$. Here $G_e$ is a small neighbourhood of $e$ in $G$. Note that according to Sec. 3, the coordinate $(w, t)$ for $\tilde{W}_f = W_f \times \hat{H}^*$ with respect to the splitting is $C^\infty$-compatible with the "standard" exponential coordinate for $\tilde{W}_f$. In particular, different choices of the "base point" $h \in W_f$ (= origin), splitting factor $\hat{H}^*$ give $C^\infty$-equivalent coordinates and related trivializations. Moreover since for a fixed $g \in G$, its action $\Psi_g$ is a smooth differomorphism from $\tilde{W}_f$ to its image and the action lifts to a smooth isomorphism between the two local bundles via pull-backs. It follows from these considerations together with the fact that $\xi_2$ is $G$-equivariant implies that we only need to compute $D[\xi_1]|_h$ with $h \in W_f$ using the standard coordinate $(w, t)$ with $h$ as base point. Since $[\xi_1]|_{W_f} = \hat{\xi}$ is a "constant" section with respect to the trivialization and hence smooth, we have that along $w$-direction, $D[\xi_1]|_h$ exists and equals to zero.

To compute the partial derivatives along $\hat{H}^*$ or $t$-direction, consider a point $h^* \in \hat{H}^*$ with coordinate $(w, t) = (0, h^*)$. Then there is a unique $g = g(h^*) \in G$, such that $h^* = g(h^*)$ is in $W_f$. Hence the $(u, g)$-coordinate of $h^*$ is $(u, g) = (h^* \circ g(h^*), (g(h^*))^{-1})$. We now show that $h^* \to g(h^*)$ as a map form $\hat{H}^*$ to $G$ is of class $C^m_{\infty}$. To this end, note that the identification $\hat{H}^* \to \tilde{H}^*$ is the restriction to $\hat{H}^*$ of the $C^\infty$-smooth map $\tilde{\pi}_{\hat{H}^*} \circ ev^l : \tilde{W}_f \to \tilde{H}^*$. Here $ev^l : \tilde{W}_f \to M^l$ is the $l$-fold evaluation map used to define the local slice $W_f$ with $2l = \dim(G)$, $\tilde{\pi}_{\hat{H}^*}$ is the projection from a neighbourhood of $h(x) \in M^l$ to $\tilde{H}^*$. By definition, $(\tilde{\pi}_{\hat{H}^*} \circ ev^l)(h^*)$ is just the $l$-coordinate of $h^*$, and it has been denoted by $\hat{h}^*$. Now we have the local identification of class $C^m_{\infty}$ given by $h : D^l(x) \to \hat{H}^* \subset M^l$. Here $D^l(x)$ is the $l$-fold product of small discs centred at the (minimal number of) marked points of the free components. Note that $h(x)$ is the "origin" of $\tilde{H}^*$. Therefore $\hat{h}^* \to h^{-1}(\hat{h}^*)$ as a local map from $\tilde{H}^*$ to $D^l(x)$ is of class $C^m_{\infty}$.

Clearly $g(h^*)$ is determined by the location of $h^{-1}(\hat{h}^*)$ in $D^l(x)$, and smoothly depends on it. Put this together, we have proved that $g(h^*)$ depends on $h^*$ with $C^m_{\infty}$-smoothness. Now let $h^* = h_s$ be a smooth curve in $\hat{H}^*$ such that $h_0 = h \in W_f$. Then it has $(w(s), t(s))$-coordinate $(0, h_s^*)$ as smooth functions of $s$. The corresponding $(u, g)$-coordinate is $(u(s), g(s)) = (h_s \circ g(h_s), (g(h_s))^{-1})$. It is of class $C^m_{\infty}$ in $s$. Here we have abused notation by writing $h_s$ for its coordinate. Let $v = \frac{\partial h_s}{\partial s}|_{s=0}$. Then the partial derivative along $\hat{H}^*$ is given by

$$D[\xi_1]|_h(v) = \left. \frac{\partial[\xi_1](u(s), g(s))}{\partial s} \right|_{s=0}$$
is of class \(L\) explicit formula for \([\bar{\xi}_1]\) fiberation over a local chart (uniformizer) of \(\bar{C}\) class \(W\) here is to note that near the end of a higher stratum, a local slice to show that the same is true in each stratum. As mentioned above, main point one stratum. The general case is not really much harder. In this case, we want to leave it to the readers to prove its continuity with respect to \(h\). Since \(\bar{\xi}_1\) is of class \(L\), \(d\) differomorphisms are of class \(L\) too. Therefore, we only need to consider the case that \(h\) is in \(W_f\). In this case the derivatives along \(u\)-directions are still equal to zero since \([\bar{\xi}_1]\) is a constant on \(W_f\).

Therefore, we only need to show existence of the partial derivatives of \([\bar{\xi}_1]\) as a continuous function in \((u, g)\). We will only prove the existence of the partial derivatives and leave it to the readers to prove its continuity.

For the same reason as above, we only need to consider the case that \(h\) is in \(W_f\). In this case the derivatives along \(u\)-directions are still equal to zero since \([\bar{\xi}_1]\) is a constant on \(W_f\).

Therefore, we only need to show that \((\partial [\bar{\xi}_1]/\partial g)_{(u, g)}\) exists.

To this end, assume that \(\partial [\bar{\xi}_1]/\partial g|_{s=0} = 0\). Here \(h_s = h \circ g_s : \Sigma \to M\) with \(s \in (-\epsilon, \epsilon)\) is a curve in \(W_f\) with \(h_0 = h\) representing \(\partial [\bar{\xi}_1]/\partial g|_{h}\) and defined by the corresponding curve \(g_s\) in \(G\).

Then we have \((\partial [\bar{\xi}_1]/\partial g)_{h}|_{s=0} = (\partial [\bar{\xi}_1]/\partial g_{(h \circ g_s)})_{s=0}\). Now being considered as \(s\)-dependent sections on the fixed bundle \(\Lambda^{0,1}(f^*TM) \to \Sigma\),

\[
[\bar{\xi}_1](h_s) = [\bar{\xi}_1](h \circ g_s) = \Pi_{h_s}^{-1}((g_s)^{-1})^*(\Pi_h(\xi).
\]

Here these \(\Pi_{h_s}\) are bundle morphisms from \(\Lambda^{0,1}(\Sigma, f^*TM) \to \Sigma\) to \(\Lambda^{0,1}(\Sigma, h^*_sTM) \to \Sigma\).

Since \(h_s\) is of class \(L^p_k\), so is the bundle \(\Lambda^{0,1}(\Sigma, h^*_sTM) \to \Sigma\). Therefore these differomorphisms are of class \(L^p_k\) too. Note that the section \(\Pi_h(\xi)\) of the bundle \(\Lambda^{0,1}(h^*TM) \to \Sigma\) can only be of class \(L^p_k\) even \(\xi\) is of class \(C^{\infty}\). From this explicit formula for \([\bar{\xi}_1](h_s)\), one concludes that its derivative with respect to \(s\) is of class \(L^p_{k-1}\). This proves the existence of \(D[\bar{\xi}_1]|_{h}\) at any point \(h\). Again we leave it to the readers to prove its continuity with respect to \(h\) so that \([\bar{\xi}_1]\) is of class \(C^1\).

This concludes the necessary modifications to \([LT]\) for the case there is only one stratum. The general case is not really much harder. In this case, we want to show that the same is true in each stratum. As mentioned above, main point here is to note that near the end of a higher stratum, a local slice \(W_f\) is a fiberation over a local chart (uniformizer) of \(\mathcal{M}_{0,k}\) with \(\Sigma_f\) in a lowest stratum.
of \(M_{0,k}\). Hence within a stratum, it splits locally as a product of \(W^\alpha_f\) and a neighbourhood \(\Lambda_s(\alpha_0)\) of \([\Sigma_{\alpha_0}]\) in the corresponding stratum of \(M_{0,k}\). Here \(\alpha\) is the local parameter of \(M_{0,k}\) near \(\alpha_0\) and \(\Sigma_{\alpha_0} = \Sigma_{f_{\alpha_0}}\), where \(f_{\alpha_0}\) is one of the "base" points in the given stratum obtained from \(f\) by moving its double points first, then make the corresponding pre-gluing. There is also a corresponding (only \(C^\infty\)) product structure for the universal curve \(\mathcal{U}|_{\Lambda_s(\alpha_0)} \simeq \Sigma_{\alpha_0} \times \Lambda_s(\alpha_0)\) so that we may view \(\Sigma_\alpha\) with \(\alpha \in \Lambda_s(\alpha_0)\) as a family of complex structures defined on the fixed \(\Sigma_{\alpha_0}\) parametrized by \(\alpha\). Using this product structure, the \(\partial_J\)-section on \(W_f\) is translated into a family of sections \(\partial^J_i\) defined on the fixed fiber \(W^\alpha_{f_0}\). Similar interpretations are applicable for those perturbations. The metric on the domains used to define the space of \(L^2\) maps and related bundles become a family of metrics on a fixed domain. Moreover, the action of the automorphism group acting on the free components of \(\Sigma_f\) extends naturally to the product \(\Sigma_{\alpha_0} \times \Lambda_s(\alpha_0)\), which, in turn, induces an action on \(W^\alpha_{f_0} \times \Lambda_s(\alpha_0)\). This essentially put us in similar situation as above, and what one needs to do is show the corresponding statements accordingly. At this point, the proofs of these statements are the straightforward generalizations of what we have done above. The details will be given in a subsequent paper.

Therefore, with the supplement here, for the case that all isotropy groups being trivial, what proved by the argument of [LT] is the following theorem.

**Theorem 1.2** Let \(s\) be the \(\partial_J\)-section. Assume that all isotropy groups are trivial. Let \(B(r) = \oplus_{i \in I} B(r_i)\) be the ball of "radius" \(r\) in \(R = \oplus_{i \in I} R_i = \text{collection of all compatible perturbations } \nu = \{\nu_i, i \in I\}\). Here \(\nu_i \in r_i\) is a stratified weakly \(C^1\)-section supported on the local uniformizers \(W_i, i \in I\) obtained from the cokernel as above with \(\|\nu_i\| < B(r_i)\), and \(\{W_i, i \in I\}\) covers \(\mathcal{M}(A, J)\) in \(B\). Then for \(r\) small enough, each extended local moduli space \(E\mathcal{M}^i(J, A) = (s+ev(1,B(r)))^{-1}(0)\) is a stratified submanifold of class \(C^1\) in \(R \times W_i\). Moreover, these local extended moduli spaces patch together to form a stratified topological manifold \(E\mathcal{M}(J, A) = \cup_{i \in I} E\mathcal{M}^i(J, A)\), the total space of the obstruction sheaf.

See the discuss later in this section on how to obtain the perturbed moduli space from the extended moduli space.

**Remark 1.1** To define the boundary operators in symplectic Floer homology, one only needs to use the moduli spaces of virtual dimension equal to zero. One possible way to define \(GW\)-invariants and quantum cohomology described in [L0] only uses such moduli spaces. Therefore, for the purpose here, we only need to consider the case that the projection map

\[
\pi : E\mathcal{M}(J, A) = \cup_{i \in I} E\mathcal{M}^i(J, A) \to B(r) = \oplus_{i \in I} B(r_i)
\]

has relative dimension equal to zero. The restriction of \(\pi\) to each \(E\mathcal{M}^i(J, A)\), denoted by \(\pi_i\), is a map of class \(C^1\). The Sard’s theorem is applicable (and just applicable) to \(\pi_i\). We conclude that for a generic choice of \(\nu = \oplus_{i \in I} \nu_i\) in \(B(r)\),
the perturbed moduli space \( \mathcal{M}^r(J,A) = (s + \nu)^{-1}(0) \) consists of finitely many points with signs. Here we have used the facts that (I) for sufficiently small \( r \), \( \mathcal{M}^r(J,A) \) is compact. This is proved in [LT] and Sec. 3. (II) There is an orientation on \( \mathcal{M}^r(J,A) \) discussed in [F] for instance. As for the the proof of the invariancy of the GW-invariants so defined, we need to apply a parametrized version of the above discussion. In this case, the relative dimension is one, and the corresponding local \( C^1 \)-smoothness is not sufficient for applying the Sard’s theorem (for manifolds with boundary) to this case. Similarly, to show that \( \partial^2 = 0 \) in Floer homology, we also need to consider the moduli space of virtual dimension equal to one. However, in this particular case, one can avoid part of the discussion in this section, the local Fredholm theory. Roughly speaking, the reason for this is that the virtual dimension of the lower stratum caused by bubbling is equal to negative one. We may assume that transversality is already achieved for such lower strata. Therefore, we are essentially in the case that there is only one stratum, the top stratum, together with the stratum of virtual dimension equal to zero as the boundary of the top stratum. The extended moduli space for the all strata can be obtained by using sufficiently small further perturbations so that there are still no lower strata other than the boundary components. Then equipped with the weakly smooth structure described in Sec. 2 and sec. 3, the extended moduli space is not only a stratified \( C^1 \)-manifold, but also a true \( C^1 \)-manifold with boundary. Now we are in the position to apply the discussion at the end of this section. By deforming all \( C^1 \) objects here slightly into \( C^\infty \) ones, we can prove the desired invariancy for GW-invariants or \( \partial^2 = 0 \) in Floer homology in the usual manner.

This concludes the proof of the following theorem and fills in the gap in [LT].

**Theorem 1.3** With the supplement above, the Floer homology in [LT] is well-defined. Similarly using the method in [L0], the work in [LT] establishes the existence of GW-invariants and quantum cohomology for a general symplectic manifold.

- "Geometric" \( C^{m_0} \)-Perturbation:
  
  As mentioned before, the best we can get for extending an element \( \xi \) in the central fiber \( L(f) \) to a constant section \( \tilde{\xi} \) over \( W_f \) is to get a weakly smooth section of class \( C^1 \) even we start with a \( C^\infty \) element \( \xi \) like ones in the cokernel.

  In order to get the desired \( C^{m_0} \) smoothness for perturbations used in the main theorem of this paper, we proceed differently.

  Note that by linearity we only need to extend each element in a basis of the cokernel (or any prescribed finite dimensional space of the central fiber).

  The key observation then is that instead of extending each element in the basis using the standard process above, we decompose each element \( \eta \) into a finite sum of elements, each of them is localised near a point of the domain \( \Sigma \). Of course, these new elements are not in the cokernel anymore. In the
case that \( f \) and \( \eta \) are smooth, each of these new elements localized in a disc \( D_\delta(x_0) \subset \Sigma \) can be written as a finite sum the terms of the form \( \phi \otimes (\iota_i, \iota_j) \xi \). Here \( \phi \) is a smooth \((0,1)\)-form on \( \Sigma \) supported in \( D_\delta(x_0) \) and \( \xi \) is a restriction of a smooth global vector field \( \xi \) on \( M \). Therefore, we only need to extend the elements of the above form. This can be done for \( \phi \) and \( \xi \) separately in the corresponding bundles. More specifically, let \( \mathcal{L} \to \mathcal{B} \) be the bundle used to define \( \partial \)-section for parametrized stable \( L_k^p \)-maps. Then the fibre of \( \mathcal{L} = \mathcal{L}_{k-1,p} \) at \( f \), \( L^p_{k-1}(\Sigma, \Lambda^{0,1}(\Sigma) \otimes f^*(TM)) \), is linearly homeomorphic to \( L^p_{k-1}(\Sigma, f^*(TM)) \otimes L^p_{k-1}(\Lambda^{0,1}(\Sigma)) \) when \( k \) is large enough. This gives rise a bundle isomorphism \( \mathcal{L} \simeq \tilde{T} \otimes \tilde{\Omega}^1 \). Here the fibre of \( \tilde{T} = \tilde{T}_{k-1,p} \) at \( f \) is \( L^p_{k-1}(\Sigma, f^*(TM)) \), and \( \tilde{\Omega}^1 \) is the trivial bundle whose fiber is \( L^p_{k-1}(\Lambda^{0,1}(\Sigma)) \).

Note that the \( G \)-actions on \( \tilde{T} \) and \( \tilde{\Omega}^1 \) are compatible with the one on \( \mathcal{L} \). So are the local trivializations and \( G \)-equivariant local trivializations induced by the ones on a local slice. This implies that we really can deal with the two components of \( \eta \) separately.

Clearly, the vector field \( \xi \) induces a global section on \( \tilde{T} \), denoted by \( \xi_0 \) and defined by \( \xi_0(g) = \xi(g) \) for any \( g \in \mathcal{B} \). Since \( G \) only acts on the domain, the section \( \xi_0 \) is clearly \( G \)-equivariant. It is easy to see that it is also smooth (see Sec. 4 for the proof). This completes the desired extension for \( \xi \).

To extend \( \phi \), use the trivialization \( \tilde{\Omega}^1 \simeq \mathcal{B} \times L^p_{k-1}(\Lambda^{0,1}(\Sigma)) \). We get a smooth global section \( \tilde{\phi} \) from \( \phi \) and its restriction to a local slice \( W_1(f) \), denoted by \( \tilde{\phi}_1 \). As above, the question is if it is still smooth viewed in the other slices. We follow the same idea, using the \( G \)-action to obtain a \( G \)-equivariant section, denoted by \( \phi_1 \) over the open set \( G \times W_1(f) \), then deciding if \( \phi_1 \) is smooth on \( G \times W_1(f) \).

Since the situation here is much better than the general extension problem we discussed before, in stead of getting \( C^1 \)-smoothness, the similar computation there shows that \( \phi_1 \) is of class \( C^{m_0} \). In Sec. 3, we give a direct proof that the extension \( \phi_1 \) on \( W_1(f) \) is \( C^{m_0} \)-smooth viewed in any other local slices.

Put this together, we get the desired extensions of the elements in the cokernel over a local slice, which are \( C^{m_0} \)-smooth viewed in any local slices. We will show in Sec. 3 that these local extensions can be used to achieve local transversality. Since these extensions are of geometrical nature, we will refer the sections and perturbations so obtained as geometric sections and perturbations though they are not the usual geometric perturbations in the sense used in GW theory.

We have already known that the "constant" extension induced by the standard parallel transport can only be of class \( C^1 \). If we allow the parallel transport used here depending on \( f \), then the "constant extension" of a smooth element in the central fiber over a local slice is still \( C^{m_0} \)-smooth viewed in any slices.

- \( C^{m_0} \)-smoothness of constant extensions defined by \( f \)-dependent connections:

Assume that a point-section \( \eta \in C^\infty(\Sigma, \Lambda^{0,1}(f^*(TM))) \subset L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM))) \)
over \( f : \Sigma \to M \) satisfies the condition that (i) \( \eta \) is supported in an open disc \( D_\delta(x_0) \) in \( \Sigma \); (ii) \( f \) is an embedding on \( D_\delta(x_0) \); (iii) over \( D_\delta(x_0) \), \( \eta \) can be written as \( \eta = \phi \otimes (i,J) \xi \), where \( \phi \) is a smooth \((0,1)\)-form on \( \Sigma \) supported in \( D_\delta(x_0) \) and \( \xi \) is the restriction of a smooth vector field \( \tilde{\xi} \) on an open set \( U \) of \( M \) containing \( f(D_\delta(x_0)) \) with the property that \( \tilde{\xi} \) is a covariant constant vector field with respect to a \( J \)-invariant connection on \( U \).

The similar discussion as above (see Sec. 3) shows that in this case the extension of \( \eta \) to a local slice containing \( f \) by parallel transport is \( C^{m_0} \)-smooth viewed in any other slices.

On the other hand, one can show that by using a local parallel frame of the complex bundle \((TM,J)\) over \( U \), the condition (i) and (ii) implies that the condition (iii) can be arranged. Of course, for a symplectic manifold \((M,\omega)\) with \( \omega \)-compatible almost complex structure \( J \), the standard \((J,\omega)\)-invariant connection with Nijenhuis tensor as its torsion does not have such a local \( J \)-flat frame unless \( J \) is integrable over \( U \). However, if we regard \((TM,J)\) as an abstract complex vector bundle rather than as the tangent bundle of \( M \), any local complex trivialization of \( TM \) over \( U \) gives rise such a local \( J \)-flat frame over \( U \). This means that we have to use a family of \((f,x_0)\)-dependent connections on \( M \) to give the local trivializations of the bundle \((\mathcal{L} \to B)\) at least for \( f \) in the moduli space of \( J \)-holomorphic maps.

We conclude that if \( f \) satisfies the condition (i) and (ii), by using a \((f,x_0)\)-dependent connection on \( M \), the constant extension of \( \eta \) with respect to the connection is \( C^{m_0} \)-smooth.

Note that the point-sections to be extended in [LT] are obtained from the cokernel by multiplying all elements in the cokernel by a fixed cut-off function supported in an open set of \( \Sigma \) away from its double points (see page 29 of [LT], under the assumption that the evaluation map at double points is transversal to the multi-diagonal there at the \( J \)-holomorphic map \( f \)). By the unique continuation principle used in [LT], the same argument there implies that the cut-off function can be chosen to be supported in a disc \( D_\delta(x_0) \) for each component of the domain \( \Sigma \). Since \( f \) in this case is in the moduli space of \( J \)-holomorphic curves, it is automatically smooth and has some point \( x_0 \) on each component of \( \Sigma \) such that \( f \) is a local embedding near \( x_0 \). In fact such points are open and dense in \( \Sigma \). Therefore as long as we choose the center of each \( D_\delta(x_0) \) to be one of the above "good" points, both (i) and (ii) are automatically true. Consequently, with above modification, the extension used in [LT] is \( C^{m_0} \)-smooth viewed in any slices if we are willing to use a family of \( f \)-dependent connections to obtain "constant" extensions. Note that since the moduli space is compact, for the purpose of the constructions in [LT], there are only finitely many such connections are involved.

- **Regularity Assumption:**
  We have mentioned that it is possible to establish the \( C^{m_0} \)-smoothness of the perturbed moduli space without assuming that all geometric data are \( C^\infty \).
smooth but only assuming that they are only, for instance, $C^{2k}$-smooth.

This seemingly very technical point concerns the general philosophy on how to deal with the main difficulty of lack of differentiability in the current research. The reader might have been aware of that in the polyfold theory, the Frechet manifold of smooth stable maps lying inside the sc Banach manifold of stable maps plays a fundamental role in the formulations of various notions and constructions of the theory. The basic requirement that the resulting moduli space should lie inside the Frechet manifold is behind many considerations in the theory.

This motivates our efforts in this and subsequent papers to explore the possibilities: (i) to get the main construction of virtual moduli space without using the $C^\infty$ regularity results in the case of geometric data are smooth; or (ii) to get the same conclusion even without assuming the geometric data are smooth. As the proofs in this paper show, under the assumption that all geometric data are $C^{2k}$-smooth, one can eliminate the role played by the Frechet manifold and still obtain the extended moduli space, the total space of the obstruction bundle (sheaf), with $C^{m_0}$-smoothness, the same regularity as its elements.

As this stage, to get the perturbed moduli space, one has to use Smale-Sard theorem. Therefore one may simply require that the virtual dimension of the moduli space is less than $m_0$. The other possible way to deal with this last problem is to use Whitney’s theorem to give the extended moduli space a compatible $C^\infty$-smooth structure first, then to deform the ”projection” map from the extended moduli space to the linear space of the perturbations into a smooth one. This last method works without the assumption on the virtual dimension, but it requires the true $C^{m_0}$-smoothness of the extended moduli spaces (in [L3]) rather than just the stratified smoothness in the general case.

There is a related question about the regularity of individual perturbed $J$-homomorphic maps. This was used from the main theorem about $C^{m_0}$-smoothness of the moduli space to infer its $C^\infty$-smoothness under the assumption that geometric data are smooth. It has been suspected that the solution of a perturbed $\bar{\partial}_J$-equation may not be a $C^\infty$ smooth map any more but only $C^{m_0}$-smooth if we use $L^p_k$-maps as the ambient space to work with. In other words, as far as the regularity of each individual map is concerned, it may happen that nothing is special about a map being a solution of the perturbed $\bar{\partial}_J$-equation. Here of course our assumption is that all geometric data are of class $C^\infty$.

To see the smoothness of the solution of the perturbed $\bar{\partial}_J$-equation, we may use local coordinate charts centred at smooth curves since such curves are dense in the space of $L^p_k$-curves. In the case of using ”constant” perturbations, each element of the perturbations is obtained by the parallel transporting a smooth section at the center over a local slice first, then multiplying the cut-off function to make the constant section localized. In other words in the local chart and local trivialization, the the perturbed $\bar{\partial}_J$-equation takes the following form. $[\bar{\partial}]_J \xi + \gamma(\xi) \nu(\xi) = 0$. Here $[\bar{\partial}]_J : L^p_k(\Sigma, f^*(TM)) \to L^p_k(\Sigma, A^{0,1}(f^*(TM)))$ is
the $\bar{\partial}_J$-operator written in the local chart and trivialization centered at $f$. $\gamma$ is the cut-off function supporter in a small neighbourhood of $f$ and $\nu(\xi) = \nu(0)$ is a constant section. Therefore, if $\xi$ is a solution of this perturbed $\bar{\partial}_J$-equation, it is the solution of the equation for $\eta$: $[\bar{\partial}_J J + \gamma(\xi)\nu(0) = 0$. This is just an inhomogeneous equation for the usual quasi-linear elliptic operator, $\bar{\partial}_J$, with smooth data. Note that in this equation, $\xi$ is fixed, $\gamma(\xi)$ is a fixed real number. Therefore, the solution is smooth by the elliptic estimate for $\bar{\partial}_J$-operator. In our case of using localized geometric perturbations, we get similar equation with the term $\gamma(\xi)\nu(\xi)$ being replaced by linear combinations of the terms $\gamma(\xi)\omega \cdot [X(\xi)]$, where $\omega$ is a smooth $(0,1)$-form on $\Sigma$, $X$ is a smooth vector field on $M$, $X(\xi)$ is the pull-back of $X$ by the map $Exp_1 f$ and $[X(\xi)]$ is $X(\xi)$ written in the local trivialization at $f$. Since a generic $\xi$ is of class $L^p_k$, $X(\xi)$, and hence $\omega \cdot [X(\xi)]$ is of class $L^p_k$. The same argument above implies that the solution is in $L^p_{k+1}(\Sigma, f^*(TM))$. Therefore, we get the same conclusion by bootstrapping.

As mentioned above, the method in [LT] gives a stratified topological manifold structure on the extended moduli space. On the other hand in this paper and its sequels [L2] and [L3], we go further to establish the $C^m_0$-smoothness of the perturbed moduli spaces. This will be done by using the induced weakly smooth structures from the ambient space.

The details on how this can be done is given in Sec. 3. Here we just mention briefly how to get a $C^m_0$-smooth moduli space from a compatible collection of perturbed moduli space defined on local slice by using the weakly smooth structure defined in Sec. 2.

• $C^m_0$-Smooth Moduli Space and Smoothness of Evaluation Maps on the Ambient space:

In Sec. 2 and 3, we have defined a function or section on the ambient space of unparametrized stable $L^p_k$-maps to be weakly smooth if it is smooth with respect to some covering slices. The germs of such functions give rise the weakly smooth structure on the ambient space. At first sight, this obvious notion is just a convenient way to talk about smoothness in the present situation if we do not want to put any further structure like sc-smoothness on the ambient space but only regard it as a topological Banach manifold.

What is not so obvious is that in the case of GW and Floer type theories, the weakly smooth functions defined on the ambient spaces induce honest smooth structures on any finite dimensional topological manifolds embedded in the ambient spaces as weakly smooth submanifolds. In particular, the perturbed moduli space is one of such topological manifolds by the construction in sec. 3. In fact, the smooth structure on the moduli space is determined by the obvious weakly smooth maps, the evaluation maps, on the local slice of the ambient space.

More specifically, using the "global" geometric perturbations in Sec. 3, one obtains a collection of compatible local moduli spaces for a generic perturbation. Each of such local moduli spaces is lying inside a local slice and is a $C^m_0$-smooth
submanifold inside the local slice. They are compatible in the sense that on the overlaps of the images of theses local slices regarded as charts in the ambient space of the unparametrized stable \( L^p_k \)-maps, these local moduli spaces are mapping to each other under the ambient transition maps of the local slices. Since these transition functions are continuous, it follows that the "global" moduli space is at least a topological manifold. This together with some of related properties is sufficient for most of all applications. At first sight, it seems that this the best one can get since these transition functions are only continuous. However, the global moduli space above is an example of a finite dimensional weakly smooth submanifold inside the ambient space. We proved in Sec. 3 that in this case, the ambient space is a weakly smooth Banach manifold which is effective with respect to the equivalence class of the "covering data". This means that there are sufficiently many weakly \( C^m \)-smooth functions defined on the ambient space so that they give any finite dimensional weakly smooth submanifold as above an honest \( C^m \)-smooth structure. In fact, it is easy to see that the \( l \)-fold evaluation map \( ev_x : \tilde{B} \rightarrow M^l \) from the space \( \tilde{B} \) of parametrized \( L^p_k \) maps to \( M^l \) given by \( ev_x(f) = (f(x_1), \ldots, f(x_l)) \) is \( C^\infty \) smooth. Here \( x = (x_1, \ldots, x_l) \) with \( x_i \neq x_j \) for \( i \neq j \). Of course, the restrictions of \( ev_x \) to any local slice is still \( C^\infty \) smooth. It is proved in Sec. 3 that it is \( C^m \)-smooth viewed in any other slices. Therefore we get a collection of weakly smooth functions of class \( C^m \) on open sets of \( B \). By choosing \( l \) and \( x \) properly, the collection of such functions serves as "coordinates" for these local perturbed moduli spaces. In other words, in the case of the ambient topological Banach manifold appeared in GW and Floer type theories, these evaluation maps on the slices of the ambient space are already sufficient to detect the honest smoothness of an embedded finite dimensional object.

• Possible Abstraction:

The discussions above make it possible to define a "meaningful" notion of \( sc \)-manifold of finite length as we explain now. As mentioned above, most of notions in polyfold theory are formulated based on the assumption that there is a Frechet manifold \( B_\infty \) of smooth elements lying inside each \( B_k \) of the \( sc^\infty \)-manifold \( B : B_0 \supset B_1 \supset \cdots \supset B_k \cdots \). The thinking behind this, of course, is that the Frechet manifold \( B_\infty \) is the one that we are really interested. Its extensions \( B_k \) via different norm closures make it possible to apply standard or generalized analytic tools such as implicit function theorem, while the fact that the moduli space is inside a (\( C^\infty \)-smooth) Frechet manifold automatically implies that those local moduli space as above are transformed in a \( C^\infty \) manner.

Of course, one can define a \( sc \)-manifold of length \( l \) in the obvious way. Clearly the formal part of the standard theory about \( sc \)-manifold in the work of Hofer, Wysocki and Zehnder is still applicable. The question is if this obvious generalization is meaningful. One thing is clear that since we do not have the Frechet manifold \( B_\infty \) any more, for the applications such as the smoothness for the moduli spaces, we need certain weakly smooth structure, a graded ver-
sion of above that can detect the smoothness of a finite dimensional embedded object. In the category of \(sc\)-manifolds of finite length, such weakly smooth structures come naturally as follows. Let \(\phi : E_0 \to \mathbb{R}^1\) be continuous linear functionals defined on the degree zero piece of the \(sc\)-Banach space of length \(l\), \(E : E_0 \supset E_1 \supset \cdots \supset E_l\). Here \(E\) is the local linear model for a \(sc\)-manifold \(B\) of length \(l\). By pulling back \(\phi\) to a local chart \(U : V_0 \supset V_1 \supset \cdots \supset V_l\), we get a smooth function on \(V_0\), still denoted by \(\phi\). By definition, viewed in any other \(sc\)-chart \(V\), \(\phi\) is of class \(C^1\) on \(V_1\). Inductively, we have that \(\phi\) is of class \(C^k\) on \(V_k\) for \(1 \leq k \leq l\).

Any function \(\phi\) defined locally on \(B\) with the same property as above with respect to a admissible \(sc\)-covering of \(B\) will be called a local (graded) weakly smooth function with respect to the covering. The germs of such functions gives the weakly smooth structure. This is the formulation of Sec. 2 adapted in the \(sc\)-setting. A consequence of this is that in a \(sc\)-manifold of finite length, if \(M\) is a finite dimensional embedded \(C^1\)-submanifold lying in \(B_l\) in the sense that locally it is an embedded submanifold of class \(C^1\) in any admissible covering chart of \(B_l\), then it is in fact a honest \(C^1\)-smooth manifold. In other words, there are enough weakly smooth functions on \(sc\)-manifolds of finite length to detect the smoothness for any finite dimensional objects.

This is the starting point to have an abstract theory based on the notion of \(sc\) manifolds of finite length together with their weakly smooth functions. It also explains the necessity of some of the discussions above in this section. The details on the discussion of the graded weakly smooth structure in the setting of \(sc\)-manifolds of finite length and its generalizations will be given in [L5] and its sequel.

- Special Properties for \(B\):

Let \(B\) be the space of unparametrized stable \(L^p_k\)-maps with fixed domain \(S^2\) and trivial isotropy group. Recall that for each \([f] \in B\) with a representative \(f \in [f]\), there is a local uniformizer \(U_i(f, H)\) defined as follows. Let \(x = (x_1, x_2, x_3)\) be the standard markings on \(S^2\) and \(H = (H_1, H_2, H_3)\) be three local hypersurfaces of codimension 2 in \(M\) such that \(H_i, i = 1, 2, 3\) is passing through \(f(x_i)\) and transversal to \(f\) locally at \(x_i\). Then \(U_i(f, H)\) consists of all parametrized stable \(L^p_k\)-maps \(h\) such that \(h(x) \in H\) and \(\|h - f\|_{k,p} < \epsilon\). We will abuse the notation by identifying \(U_i(f, H)\) with is image \([U_i(f, H)]\) in \(B\).

Now cover \(B\) by a collection of such uniformizers \(U = \{U_i, i \in I\}\) with transition functions \(\Phi = \{\phi_{ji} : U_{ji,j} \to U_{ij,j}\}\), where \(U_{ij,i} = (\pi_i)^{-1}([U_i] \cap [U_j])\) and \(\pi_i : U_i \to B\) is the projection map send any \(L^p_k\)-map \(h\) to its class \([h]\). With respect to the covering data \((U, \Phi)\), \(B\) is a topological Banach manifold.

The discussion of above subsection shows that as a weakly smooth Banach manifold coming from a \(Sc\)-smooth structure, the "structure sheaf" \(\mathcal{O}(U_i, r)\) of the germs of weakly \(C^r\)-smooth functions over \(U_i\) contains all restrictions of continuous linear functionals defined on the ambient Banach space \(L^p_k(S^2, \mathcal{F}^k_r(TM))\) which can be extended to \(L^p_{k-r}(S^2, \mathcal{F}^k_r(TM))\) with \(k-r-2/p > 0\). In partic-

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ular, it contains all evaluation maps. We already mentioned that in this case the weakly smooth structure is effective in the sense that the collection of the weakly smooth functions, the structure sheaf $O_B$ can be used to detect the honest smoothness of any finite dimensional weakly smooth submanifold in $B$.

In general the notion of effectiveness of a weakly smooth structure given by the structure sheaf $O_B$ associated to covering data $(U, \Phi)$ on a topological Banach manifold $B$ can be defined as follows.

For any member $U \subset E$ in the cover $\mathcal{U}$ with the model Banach space $E$, the tangent space $T_xU = E_x \simeq E$. Let $O_L(U, x)$ be the sub-sheaf of $O_L(U)$ whose global sections form the linear space $E^*_x(U)$ consisting of linear functionals on $E_x$ which are weakly smooth as functions on $U$. The weakly smooth structure $O_B$ is said to be effective if for any $x \in U_i \subset E_i$ and any finite dimensional linear subspace $K_x \subset (E_i)_x$, the restriction map $E^*_x(U) \rightarrow K^*_x$ is surjective.

Now come back to the case that $B$ is the space of unparametrized stable $L^p$-maps. Clearly $O_B$ only captures the finite dimensional part of the "coordinate ring" of $B$. It restores the smoothness in the finite dimensional directions along which the transition maps $\phi_{ji}$ are not smooth. On the other hand, a special feature of $B$ is that for generic choices of the local slices $H_i$, on each "overlap" $U_{ij,i}$ there is a foliation (or fibration) by smooth Banach sub-manifolds with finite co-dimension such that $\phi_{ji} : U_{ij,i} \rightarrow U_{ij,j}$ is leaf-wisely smooth so that $O_B$ is almost smooth upto a finite dimensional deficiency in the sense above. The main part of the "coordinate ring" of $B$ are determined by these smooth leaves.

To formalize the discussion here, we introduce the notation of $F$-smoothness (=foliated smoothness with finite dimensional defect) on any topological Banach manifold $B$ with the covering data $(U, \Phi)$ is said to be $F$-smooth (strict $F$-smooth) if all transition functions $\phi_{ji} : V_i \rightarrow V_j = U_{ij,i} \rightarrow U_{ij,j}$ is $F$-smooth (strict $F$-smooth).

Here a $F$-smooth map (strict $F$-smooth) $\phi : V_1 \rightarrow V_2$ between two open $F$-sets $V_1$ and $V_2$ is defined as follows.

(A) An open set $V$ in a Banach space $E$ is said to be an open $F$-set, if there is a smooth diffeomorphism $\tau : V \rightarrow L \times W$ where $L \subset E_1$ is an open set of a Banach space $E_1$, and $W \subset E_2$ is an open set of a finite dimensional vector space $E_2$. The $F$-structure of an open $F$-set $(V, \tau)$ is specified by the structure map $\tau$ that gives $V$ a smooth (trivial) fiber bundle structure over $W$ with smooth fibers $V_w = \tau^{-1}(w) \simeq L, w \in W$. For our purpose, we will think $V$ being foliated by these finite co-dimensional smooth submanifolds $V_w \simeq L$. Each $V_w$ will be called a leaf of $V$. The special case that $W = pt$ and $V = L$ will be called the trivial $F$-structure.

A continuous map between two open $F$-sets in Banach spaces, $\phi : (V_1, \tau_1) \rightarrow (V_2, \tau_2)$ is said to be strictly $F$-smooth if the restriction of $\phi$ to each leaf of $V_1$ smoothly maps into a leaf of $V_2$. In other words, $\phi$ is just a fiber-wisely smooth bundle map. Clearly the composition of two such maps is still of the same type.

It seem that the notion of strictly smooth map is too restrictive. Note
that the intuitive notion of a continuous map between two open sets in Banach spaces, \( \phi : V_1 \to V_2 \) being almost smooth up to a finite dimensional defect can be formulated by requiring that there exists an \( F \)-structure \( \tau_1 \) on \( V_1 \) such that \( \phi : (V_1, \tau_1) \to (V_2, \tau_0^2)^0 \) is strict \( F \)-smooth. Here \( \tau^0_2 \) is the trivial \( F \)-structure. Such a map will be called \( F \)-smooth.

Given two such maps \( \phi : (V_1, \tau_1) \to (V_2, \tau_2^0) \) and \( \psi : (V_2, \tau_2) \to (V_3, \tau_3^0) \), assume that \( \phi : (V_1, \tau_1) \to (V_2, \tau_2) \) is not a fiber-preserving map, then in general the composition \( \psi \circ \phi : (V_1, \tau_1) \to (V_3, \tau_3^0) \) is not a \( F \)-smooth map any more. In the case that leaves in \( V_1 \) is infinite dimensional, since the leaves in \( (V_2, \tau_2) \) has finite codimension, for a "generic" \( \phi \), its restriction to any leaf of \( V_1 \) is transversal to all the leaves in \( (V_2, \tau_2) \). Hence the pull-backs \( \phi^{-1}_{(L_1)_{w_1}}( (L_2)_{w_2} ) \), \( w_1 \in W_1 \), \( w_2 \in W_2 \) of the leaves in \( V_2 \) are the leaves of a new \( F \)-smooth structure on \( V_1 \), denoted by \( \phi^{-1}(\tau_2) : V_1 \to \phi^{-1}_{(L_1)_{w_1}}( (L_2)_{w} ) \times W_2 \times W_1 \). Here \( (L_1)_{w_1} \) and \( (L_2)_{w} \) are typical leaves of \( V_1 \) and \( V_2 \) respectively. Now the new \( F \)-structure \( \phi^{-1}(\tau_2) \) on \( V_1 \) is a refinement of the old one.

A refinement of an open \( F \)-set \( (V, \tau_V) \) as above is defined by specifying an \( F \)-structure for \( L, \tau_L : L \to L_1 \times W_1 \) so that the leaf-type of the refinement is \( L_1 \). Hence each leaf \( V_{w'} \) of the refinement with \( w' \in W \times W_1 \) is still a smooth sub-manifold of \( V \) but with co-dimension = \( \dim(W) + \dim(W_1) \).

Therefore, at least in generic case, we can compose two \( F \)-smooth maps by passing to a refinement of the \( F \)-structures involved.

(B) In the case that the transversality condition in (A) is not satisfied. First stabilize \( \tau_1 : V_1 \to L_1 \times W_1 \) to get a new open \( F \)-set \( \tau_1^* : V_1^* \to (L_1 \times W_2^* ) \times W_1 \) with the new leaf-type \( L_1 \times (W_2^* ) \). Here \( W_2 \) is the space of leaves for \( \tau_2 : V_2 \to L_2 \times W_2 \), and \( W_2^* \) is a finite dimensional vector space equipped with a linear surjective map \( p_{W_2} : W_2^* \to W_2 \). Then stabilize the \( F \)-smooth map \( \phi \) to get a "generic" \( F \)-smooth map \( \phi \oplus p_{W_2} : V_1^* \to V_2 \).

In term of \( F \)-smoothness, the properties of the space \( B \) of unparametrized stable \( L^k_2 \)-maps and its associated bundle \( \mathcal{L} \to B \) used to define the \( \partial_J \)-section can be formulated as the following.

**Theorem 1.4** Let \( B \) be the space of unparametrized stable \( L^k_2 \)-maps. There is a consistent choice of the local uniformizers \( U_{i_1}, i \in I \) obtained by generic \( H_1 \) such that (a) each overlap \( U_{i_1} \) acquires a natural \( F \)-smooth structure; (b) each transition map \( \phi_{i_1} : U_{i_1} \to U_{i_2} \) is a strict \( F \)-smooth map. The bundle \( \mathcal{L} \to B \) has a natural \( F \)-structure such that the \( \partial_J \)-section \( s \) is \( F \)-smooth. Moreover, all the maps involved are weakly smooth and all weakly smooth objects involved are effective.

Consider a general bundle \( \mathcal{L} \to B \) with \( F \)-smooth and effectively weakly-smooth structures. In the case that \( s : B \to \mathcal{L} \) is an \( F \)-smooth and weakly smooth section whose leaf-wise linearization at a point \( h \in Z = s^{-1}(0) \) is surjective, then locally near \( h \), \( Z \) is a \( F \)-smooth and weakly smooth submanifold of \( B \). Here a section \( s \) of the bundle \( \mathcal{L} \to B \) is said to be \( F \)-smooth near a point
The corresponding map $[s]$ is $F$-smooth where $[s]$ is the map form a Banach coordinate chart near $h$ to the Banach space, the fiber $L_h$ obtained from a local trivialization. Consequently, if $s$ is an $F$-transversal section in the sense that for any $h \in Z$ the surjectivity for the linearization of $s$ at $h$ holds, then $Z$ is an $F$-smooth and weakly smooth submanifold of $B$. In other words, the implicit function theorem still holds in this case. Furthermore, assume that an $F$-smooth and weakly smooth section $s : B \to \mathbb{L}$ is leaf-wise Fredholm, then there are enough generic $F$-smooth and weakly smooth sections $\nu$ such that $s + \nu$ is an $F$-transversal section. Hence the perturbed moduli space, $Z_\nu = (s + \nu)^{-1}(0)$ is a finite dimensional manifold with expected dimension.

**Proof:**

We only indicate the idea on how to construct a consistent $F$-structure on the overlaps. To this end, note that although the transition map $\phi_{ji} : U_{ij,i} \to U_{ij,j}$ is only continuous, it is given by $\phi_{ji}(\xi) = \xi \circ t_{ij}(\xi)$ for any $\xi \in U_{ij,i}$. Here $t_{ij} : U_{ij,i} \to SL(2, \mathbb{C})$ is a smooth map of class $m_0$. If $t_{ij}$ is a constant map so that $\phi_{ji}$ is obtained by a fixed reparametrization, then it is a smooth map, and there is no lack of smoothness for this transition function $\phi_{ji}$. Of course, this is only an extreme case. However, the map $t_{ij} : U_{ij,i} \to SL(2, \mathbb{C})$ for this good case is the exceptional case. One can show that for generic choice of $H_j$ used to define the slice $U_j$, $t_{ij} : U_{ij,i} \to SL(2, \mathbb{C})$ is a submersion to its image so that the inverse images of each point in the image gives a leaf of a $F$-structure on $U_{ij,i}$. Clearly the restriction of $\phi_{ji}$ to any such leaf is smooth of class $C^{m_0}$ and the images of these leaves give the corresponding leaves for an $F$ structure on $U_{ij,j}$ so that $\phi_{ji}$ becomes a strict $F$-smooth map. Using the idea in section 4 of [LT], cover $B$ first by $\{[U_J] \mid J = (j_1, j_2, \ldots, j_l)\}$, $U_J = U_{j_1,i} \cap U_{j_2} \cap \cdots \cap U_{j_l}$, then replace the covering by $\{[V_J]\}$ constructed from $\{[U_J]\}$ in such a way that there are no overlaps between any two $[V_J]$ and $[V_{J'}]$ at the same level ($[J] = [J']$). Using these $\{V_J\}$ as the covering of $B$, one can inductively show that there is a consistent choice for generic $H_J$ so that all the transition functions on the overlaps are strict $F$-smooth.

**Flat Charts**

A local flat chart inside a local slice $W(f) = W_\epsilon(f, H)$ of the space of stable $L^k_\epsilon$-maps is the collect all $L^k_\epsilon$-maps in $W_\epsilon(f)$ which are $J$-holomorphic on a collection of prescribed disc neighbourhoods of the double points of the domain $\Sigma$ of $f$ and the corresponding prescribed annulus of the domain $\Sigma_\alpha$ of the pre-gluing $f_\alpha$.

To give a more precise definition, let $W^{D_1}_\epsilon(f)$ be one of the strata of $W(f)$. As above, by considering $W(f)$ as a local fibration over the Deligne-Mumford moduli space, we decompose $W(f)$ as the union of slices (fibers) with fixed gluing parameters, $W(f) = \cup_{\alpha \in \Lambda(D_1)} W^\alpha(f_\alpha)$, where $f_\alpha$ is a pre-gluing of $f$ and $\Lambda(D_1)$ is the set of the ”gluing” parameters within the stratum $D_1$. Note
that here gluing parameter $\alpha = (\alpha_t, \alpha_n)$ such that $f_{\alpha_n}$ is the pre-gluing along "normal" direction while the parameter $\alpha_t$ describes the local deformations of the domain $\Sigma f$ by the motions of its double points. Following [LT], collection the all pre-gluing maps $f_{\alpha}, \alpha \in \Lambda_\delta$ obtained from $f$ will be called "base" maps.

To define the flat charts and the related bundle, we start with one of the lowest strata, denoted by $D$.

Then $W^*_D(f)$ corresponds to the collection of above $W^*_\alpha(f_{\alpha_n})$ with $\alpha_n = 0$.

On this lowest stratum, the local "bundle" $\mathcal{L}(f)$ restricted to $W^*_D(f)$, denoted by $\mathcal{L}_D(f)$ has a trivialization so that each fiber can be identified with $(\tilde{L}(f))_f = L^p_k(\Sigma_0, \Lambda^{0,1}(f^*(TM)))$. More specifically, one each $W^*_\alpha(f_{\alpha_n})$ with a fixed $\alpha \in \Lambda(D_t)$, the domain $\Sigma_\alpha$ is fixed, the standard trivialization induced by parallel transport in this paper with some modification is still applicable (see [LT] for instance). On the other hand, the "standard" differomorphisms in [LT] to identify the domain $\Sigma f = \Sigma_0$ with $\Sigma_\alpha$ induce the identifications of the corresponding fibers. These differomorphisms are used to define the local $C^\infty$ product structure of the universal curve over Deligne-Mumford moduli space mentioned before. They can be realized as time 1-maps of the corresponding flows generated by vector fields supported on small annuli around double points. Note that the vector fields are not holomorphic on these annuli. Consequently, this last identification is only over $\mathbb{R}$. Though this does not affect our definition below, it seems better to use a fiberwise version define the objects for each fixed $\alpha$.

In any case, combing these together we get the trivialization of the local bundle over the lowest stratum $D$.

Consider the following sub-bundles: $\mathcal{L}_{D,\text{loc},\delta}(f)$ of $\mathcal{L}_D(f)$ defined by

$$\langle \mathcal{L}_{D,\text{loc},\delta}(f) \rangle_g = \{ \xi | \xi \in (\mathcal{L}_D(f))_g, \xi = 0 \text{ on each } \delta-\text{disc around a double point} \}.$$ 

Let $W^*_{D,\text{loc},\delta}(f) = \partial^{-1}_f(\mathcal{L}_{D,\text{loc},\delta}(f))$. This is one of the local flat chart in $W(f)$ restricted to the its lowest stratum. Hence any element in $W^*_{D,\text{loc},\delta}(f)$ has the property that it is $J$-holomorphic on the $\delta-\text{discs}$ above.

To define the corresponding objects on a higher stratum, consider the restriction of the bundle $\mathcal{L}_{D,\text{loc},\delta}(f)$ to the base maps $f_{\alpha_t}$ with $\alpha_t$ in $D$. According to what we have done for a fixed stratum, we only need to know how to extend the fiber $(\mathcal{L}_{D,\text{loc},\delta}(f))_{f_{\alpha_t}}$ of the bundle above into the fiber ( of the bundle to be defined) over $f_\alpha$. Here $\alpha = (\alpha_t, \alpha_n)$ and $f_{\alpha_t}$ is obtained from $f_{\alpha_n}$ by the normal pre-gluing with the gluing parameter $\alpha_n$. The key point is that when $\alpha_t$ and $\delta$ are fixed, for $|\alpha_n| < < \delta$, the pre-gluing from the domain $\Sigma_{\alpha_t}$ to $\Sigma = \Sigma_{(\alpha_t, \alpha_n)}$ identifies $\Sigma_{\alpha_t} \{ \delta-\text{discs at double points} \}$ with $\Sigma_{\alpha} \{ "\delta-\text{annuli}" \text{ around double points} \}$. Here for simplicity, we have assumed that non of the component of $\alpha_n$ is equal to zero so that $\alpha$ is in the top stratum.

This completes the construction of the bundle over the base maps $f_{\alpha}, \alpha \in \Lambda_\delta$. As above by parallel transporting these "central fiber" over $f_{\alpha}$, we get the subbundle over $W^*_\alpha(f)$, denoted by $\mathcal{L}_{\alpha,\text{loc},\delta}(f_{\alpha})$.
We define \( W^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha) = \overline{\partial}_J^{-1}(\mathcal{L}_{\alpha, \text{Loc}, \delta}(f_\alpha)) \). Then in the general case any element in \( W^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha) \) is a \( J \)-holomorphic on the \( \delta \)-discs or "\( \delta \)-annuli" of \( \Sigma_\alpha \).

The union of these \( W^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha) \), denoted by \( W_{\epsilon, \text{Loc}, \delta}(f) \) is one of the flat charts inside the local slice \( W(f) \).

Of course, when \( \delta \) is getting smaller, so is the size of \( \Lambda \). Since a transformation of a flat chart in one slice into another slice is induced by transformations on the domains, the size and "shapes" of these charts given by parameters "\( \delta \)" may change under the transformations. Since for our purpose here, there are only finitely many fixed local uniformizers are used to covering \( M(A, J) \), by allowing different shapes of \( \delta \)-discs in the obvious sense in this situation, the collection of all such flat charts are closed under coordinate transformations between uniformizers. They define the flat structure that we are looking for.

Let \( W^0_\epsilon(f) = \cup_\alpha W_{\epsilon, \text{Loc}, \delta}(f) = \cup_\alpha, \delta W^\alpha_{\epsilon, \text{Loc}, \delta}(f) \) be the union of these local flat charts and \( \mathcal{L}^0 = \cup_\delta \mathcal{L}_{\text{Loc}, \delta}(f) = \cup_\alpha, \delta \mathcal{L}_{\alpha, \text{Loc}, \delta}(f_\alpha) \).

Here \( \mathcal{L}_{\text{Loc}, \delta}(f) = \cup_\alpha \mathcal{L}_{\alpha, \text{Loc}, \delta}(f_\alpha) \).

Note: Part of the above construction, the definition of \( W^D_{\epsilon, \text{Loc}, \delta}(f) \) was already introduced in [LT], page 25. However, the key part of the construction on flat charts above, the definition of \( W^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha) \) dealing with the norm gluing parameters was missing in [LT].

**Properties of the flat charts:**

(I) The indices of the \( \overline{\partial}_J \)-section on any strata of a local slice are the same as the indices of the section restricted to the corresponding strata of a flat chart considered as a section of the corresponding sub-bundle.

(II) The part of moduli space of stable \( J \)-holomorphic maps inside a given slice near an "end" is contained in any of these flat charts.

(III) For a fixed \( \alpha \) with \( ||\alpha|| < \delta \), each flat chart \( W^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha) \) of "size" \( \delta \) is a Banach manifold.

(IV) For a fixed \( \delta \) small enough, all \( W^\alpha_{\epsilon, \text{Loc}, \delta}(f) \) fit together to form a Banach manifold.

The property (III) follows from the following facts: (i) the cokernel \( K_{f_\alpha} \) of \( D\overline{\partial}_J f_\alpha \) is of finite dimensional; (ii) elements in \( K_{f_\alpha} \) satisfy the unique continuation principle. In deed, (i) and (ii) imply that \( \overline{\partial}_J : W^\alpha(f) \to \mathcal{L}(f) \) is transversal to the sub-bundle \( \mathcal{L}_{\epsilon, \text{Loc}, \delta}(f_\alpha) \) so that its inverse image a Banach manifold. Note that here we have used the fact that under the local trivializations of the two bundles, the central fiber of the subbundle is a closed subspace of the corresponding central fiber. This is true under our assumption that \( m_\alpha > 1 \). The property (I) can be derived, for instance, by using (III).

**Gluing and Shape of flat charts:** It follows from (III) that in the standard local coordinate chart \( Exp_{f_\alpha} : V^\alpha \subset I^\epsilon_k(\Sigma_\alpha, f^\alpha_\alpha(TM), h) \to W^\alpha(f) \) of the local slice, the flat chart \( Exp^{-1}_{f_\alpha}(W^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha)) \), denoted by \( V^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha) \) in \( V^\alpha \) is realized as a graph of a map from the open ball \( B(r_\alpha) \) in the tangent space of \( V^\alpha_{\epsilon, \text{Loc}, \delta}(f_\alpha) \).
at $f_\alpha$ to its orthogonal complement (assuming that $p = 2$ so that the $L^p_k$-space is a Hilbert space). Here $r_\alpha$ is the radius of the ball depending on $\alpha$. Therefore the key point to prove (IV) is to prove that there is a positive lower bound for $r_\alpha$ independent of $\alpha$.

Note that it follows from a parametrized version of implicit function theorem that when $\alpha$ is moving within same stratum slightly there is such a bound. Therefore, the real question is about the motion form $\alpha_0 = (\alpha_t, 0)$ to $\alpha = (\alpha_t, \alpha_n)$ with normal gluing parameter $\alpha_n$.

In this case, we need to show that the radius $r_\alpha$ does not goes to zero as $\|\alpha_n\|$ goes to zero so that the size and shape of each $V^\alpha_{\epsilon, \text{Loc}, \delta}(f)$ is comparable with the fixed $V^\alpha_{\epsilon, \text{Loc}, \delta}(f)$. This is exactly the situation that the so called gluing technique in [L0] and [LT] can be used to deal with. The main estimate in [LT], page 32-41, with some modifications implies that (IV) above holds. The detail of the proof of (IV) is given in [L4]. We will give a outline of its proof below. As a by-product of the proof, we have also obtained two different new proofs for the gluing of $J$-holomorphic maps.

The following theorem summarizes these properties of the flat charts and their consequences.

**Theorem 1.5** For any fixed $\delta$, the local flat chart $W_{\epsilon, \text{Loc}, \delta}(f) = \bigcup_{\|\alpha\| < \delta} W^\alpha_{\epsilon, \text{Loc}, \delta}(f)$ is a Banach manifold ("ball") and $L_{\text{Loc}, \delta}(f) = \bigcup_{\alpha} L_{\alpha, \text{Loc}, \delta}(f_\alpha) \to W_{\epsilon, \text{Loc}, \delta}(f)$ is a Banach bundle. The restriction of the $\bar{\partial}_J$-section $s$ to the flat chart, still denoted by $s$ is a smooth Fredholm section, whose linearizations have the same indices as the ones for the "old" $\bar{\partial}_J$-section along all strata.

The zero sets $\bar{\partial}_J^{-1}(0)$ in $W_{\epsilon, \text{Loc}, \delta}(f)$, when projected to $B$ is just $M(J, A) \cap \{W\}_{\epsilon}(f; H)$.

Furthermore, the $L^p_k$-topology and "weak"-topology on any flat chart above are equivalent.

- Exponentially weighted $L^p_k$-norm and further fibration:

Note that in the last statement of the above theorem, the $L^p_k$-topology here is measured in term of cylindrical coordinates rather than the "spherical" ones. The same is applied to the discussions on property (IV). The necessity of using cylindrical coordinates on the $\delta$- discs $D_\delta^\pm(\pm d)$ centred at each double point $d = \pm d$ of the domain $\Sigma = \Sigma_{\alpha_0} = \Sigma_f$ is clear since as $f_\alpha : \Sigma_{\alpha} \to M$ degenerates into $f : \Sigma_{\alpha_0} \to M$, in the "spherical" coordinate the injective radius of $\Sigma_{\alpha}$ goes to zero, which is not adequate for using general $L^p_k$-norms for analytic set-up here unless $k = 1$ that implies that $m_0 = 0$.

In any lowest stratum $W_{\epsilon, \text{Loc}}(f; H)$, consider one of its fibers, for instance, for simplicity, the central fiber $W^{\alpha_0}_{\epsilon}(f; H)$. Then all the elements have the same domain $\Sigma = \Sigma_{\alpha_0} = \Sigma_f$. 

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Now we identify each pair of $\delta$-discs at a double point $d = \pm d$ of \( \Sigma \) with a pair of half cylinders \( C_\pm \simeq \mathbf{R}^+ \times S^1 \) of infinite length with \( \{ \pm \infty \} \times S^1 \) corresponding to the double point. Let \( w = \pm w \) and \((s,t) = \pm (s,t)\) be the corresponding spherical and cylindrical coordinates. Then \( w = \exp(-(s+it)) \). Clearly, any \( h = \exp \frac{1}{\kappa} \xi \) in \( W^\alpha(f; \mathbf{H}) \) satisfying the condition that \( h(d) = f(d) \) satisfies the exponential decay estimate that \( \| D^i \xi \| \sim O(\exp(-s)) \) for \( i \geq 0 \). Moreover, if \( f \) and \( h \) are smooth, then for any \( 0 < \kappa < 1 \), \( \| \xi \|_{k,p} = \| e^{k \xi} \|_{k,p} < \infty \). This means that we should introduce a new fibration, \( ev^l : W^\alpha_w(f; \mathbf{H}) \to M^l \) given by the evaluation map at all double points of \( \Sigma \). Here \( l \) is the number of these fixed double point. One can show that both above \( ev^l \) and its restriction to any of flat charts inside the stratum, are in deed fibrations over a neighbourhood of \( f(d) \) in \( M^l \). Therefore, for any \( m \in M^l \) near \( f(d) \) we get the corresponding fibers, \( W^\alpha_{\epsilon,m}(f; \mathbf{H}) \) and \( W^\alpha_{\epsilon,l,\text{Loc,} \delta}(f) \). As mentioned above, all elements in \( W^\alpha_{\epsilon,m}(f; \mathbf{H}) \) are exponential decay and have a finite \( L^p_\kappa \)-norm with exponential weight \( \kappa \) for any \( 0 < \kappa < 1 \). One can show that at least for elements \( h = \exp \frac{1}{\kappa} \xi \) in \( W^\alpha_{\epsilon,m}(f) \), we have that the usual \( L^p_\kappa \)-norm dominates the exponential \( L^p_{k,\kappa} \)-norm. Moreover, the indices of the linearization of \( \partial_f \)-operator, \( D_{\kappa}h \) are the same for any \( h \) in \( W^\alpha_{\epsilon,m}(f) \) with the property that if it is surjective with respect to the usual \( L^p_\kappa \)-norm, it is still surjective with respect to the exponential norm. Therefore, switching to the exponential norm all the desired properties are preserved. In the rest of the discussion of this section on flat chart, we will assume that the \( L^p_\kappa \)-maps are measured by exponentially weighted \( L^p_\kappa \)-norms.

The corresponding (enlarged) spaces will be denoted by \( W^\alpha_{\epsilon,m}(f) \), etc.

We now extend above discussion to higher stratum. Note that by using normal coordinate at \( f(d) \) to identify its neighbourhood in \( M \) with an open ball in \( T_{f(d)} M \) and replacing the double point \( d \) by the corresponding \( S^1 \) at infinity in the cylindrical coordinates, the evaluation map \( ev^l(h) \) can be calculated by the integral \( \int_{\Sigma} \xi \). The point is that this later formula for the fibration given by \( ex^l \) is sill applicable to the higher strata \( W^\alpha_{\epsilon,m}(f; \mathbf{H}) \). For simplicity, we assume that \( \alpha = (0, \alpha_n) \) with \( \alpha_0 = (0, 0) \). Then we get a corresponding fibration, denoted by \( ev^l_{\alpha} : W^\alpha_{\epsilon,m}(f; \mathbf{H}) \to M^l \) given by the "evaluation map" at all middle circle \( S^1_{\alpha} \) of \( \Sigma_{\alpha} \). Here \( S^1_{\alpha} \) is the middle circle of the \( \delta \)-neck in \( \Sigma_{\alpha} \) obtained from the pre-gluing, and \( ev^l_{\alpha} \) is defined by the similar integral formula above by replacing \( S^1 \) by \( S^1_{\alpha} \). Again, one can show that \( ev^l_{\alpha} \) are fibrations for the both cases.

Now using the corresponding cylindrical coordinates of finite length \( -\log \| \alpha \| \) defined on \( \Sigma_{\alpha} \) to define the \( L^p_\kappa \)-norm \( \kappa \), we obtain the corresponding fibers \( W^\alpha_{\epsilon,m}(f) \) and \( W^\alpha_{\epsilon,l,\text{Loc,} \delta}(f) \).

Let \( \exp_{f^\alpha} : W^\alpha_{\epsilon,m} \subset L^p_{k,\kappa}(\Sigma_{\alpha}, f^\alpha_{\alpha}(TM); h, S^1_{\alpha}) \to W^\alpha_{\epsilon,m}(f) \) be the local coordinate of the local slice with fixed parameter \((\alpha, m)\). Here the new constrain for an element \( \xi \) in \( L^p_{k,\kappa}(\Sigma_{\alpha}, f^\alpha_{\alpha}(TM); h, S^1_{\alpha}) \) is that the integral \( \int_{S^1_{\alpha}} \xi = 0 \).
Let $[s]^{(\alpha,m)} : \alpha V^{(\alpha,m)} \subset L^p_{(k;\kappa)}(\Sigma_\alpha, f^*_\alpha(TM); h, S^1_\alpha \to L^p_{(k;\kappa)}(\Sigma_\alpha, \Lambda^{0,1}(f^*_\alpha(TM)))$ be the $\bar{\partial}_J$-operator written in the standard local coordinate chart and trivialization. Note that here we have assumed that $\alpha = (0, \alpha_n)$ is the normal gluing parameter.

We now make two assumptions on $D[s]^{(\alpha,m)}(\alpha_0, m) : L^p_{(k;\kappa)}(\Sigma_{\alpha_0}, f^*_{\alpha_0}(TM); h, S^1_{\alpha_0}) \to L^p_{(k;\kappa)}(\Sigma_{\alpha_0}, \Lambda^{0,1}(f^*_{\alpha_0}(TM))$

- • (A1): It is surjective.

- • (A2): It is injective.

The second assumption can be achieved locally by introducing more marked points $y$ located on the 'fixed part' of $\Sigma_\alpha$ and requiring that $h(y)$ lies on some prescribed local hypersurfaces of $M$ for $h$ in $W^1_{(\alpha_0,m)}(f)$. Here as before, the 'fixed part' $\Sigma_0^0 = \Sigma \setminus \{ \text{all } \delta\text{-annuli around double points} \}$, which can be identified with the 'fixed part' $\Sigma^0_0 = \Sigma \setminus \{ \text{all } \delta\text{-discs around double points} \}$ by the pregluing. Then the key to prove (IV) is the following proposition

**Proposition 1.1** There exists a fixed $r_0 > 0$ independent of $\alpha$ such that for any $0 < r \leq r_0$, when $\|\alpha\|$ is small enough, $[s]^{(\alpha,m)} : B^1_r(\alpha,m) \subset L^p_{(k;\kappa)}(\Sigma_\alpha, f^*_\alpha(TM); h, S^1_\alpha) \to L^p_{(k-1;\kappa)}(\Sigma_\alpha, \Lambda^{0,1}(f^*_\alpha(TM)))$ is a diffeomorphism from a ball of radius $r$ to its image.

Note that for $\alpha = \alpha_n$ with all non-zero entry, all such balls $B^1_r(\alpha,m)$ can be identified each other. However, their dimension are "infinitely smaller" than the one for $\alpha = (0, 0)$.

By applying Picard method, the proof of this proposition follows from the following "main estimate" in the gluing technique mentioned above.

**Proposition 1.2** Consider a family of linear operators

$$D[s]^{(\alpha,m)}_r : L^p_{(k;\kappa)}(\Sigma_\alpha, f^*_\alpha(TM); h, S^1_\alpha) \to L^p_{(k-1;\kappa)}(\Sigma_\alpha, \Lambda^{0,1}(f^*_\alpha(TM))).$$

There exists a fixed $c_0 > 0$ independent of $\alpha$ such that for $\|\alpha\|$ small enough, $\|D[s]^{(\alpha,m)}_r\| > c_0$.

**Proof:**

The easy half proof of the main estimate in [LT], page 38-41 or the half of the proof in [L0], page 276-277 implies the Proposition.
This is the first "new" proof of the gluing we mentioned above. This proof still goes along the same line as the old proof. But because the fibration above, we have eliminated the extra terms in the exponential weighted $L^p_\alpha$-norms used in [LT] and [L0] coming from the integral along $S_\alpha^1$ above. This simplifies the argument there and make the proof here almost as simple as Floer’s gluing for non-degenerate broken connection orbits in [F], page 599.

Heuristic reasoning for the second proof of the gluing:

For our purpose, it is sufficient to establish a weaker form of the first proposition above for the flat charts.

Let $Exp_{f_\alpha}(\kappa V^{(\alpha,m)}_{\epsilon, \text{Loc}, \delta}(f))$, denoted by $\kappa V^{(\alpha,m)}_{\text{Loc}, \delta}$ be the inverse image of flat chart in $\kappa V^{(\alpha,m)}$.

Denote the restriction of $[s]^{(\alpha,m)}$ to $\kappa V^{(\alpha,m)}_{\text{Loc}, \delta}$ by the same notation, $[s]^{(\alpha,m)} : \kappa V^{(\alpha,m)}_{\text{Loc}, \delta} \to L^p_{(k;\kappa;\text{Loc}, \delta)}(\Sigma_\alpha, (f_\alpha)(TM))$. Here the right hand side is the fiber at $f_\alpha$ of the local bundle used to define the flat chart.

Then the assumptions (A1) and (A2) imply the two corresponding statements for this new $D[s]^{(\alpha,m)}$.

The weaker form of the first proposition above is the following proposition.

**Proposition 1.3** There exists a fixed $r_0 > 0$ independent $\alpha$ such that for any $0 < r \leq r_0$, when $\|\alpha\|$ is small enough, $[s]^{(\alpha,m)} : \kappa V^{(\alpha,m)}_{\text{Loc}, \delta} \cap B_r^{(\alpha,m)} \to L^p_{(k-1;\kappa;\text{Loc}, \delta)}(\Sigma_\alpha, (f_\alpha)(TM))$ is a diffeomorphism form a ball of radius $r$ to its image. Here $B_r^{(\alpha,m)}$ is the ball of radius $r$ in $L^p_{(k;\kappa)}(\Sigma_\alpha, (f_\alpha)(TM))$ and $S_\alpha^1$.

Moreover, $\kappa V^{(\alpha,m)}_{\text{Loc}, \delta} \cap B_r^{(\alpha,m)}$ can be realized as the graph of a smooth function from the ball of radius $r'$, denoted by $B_r^{\alpha}$, in the tangent space of $V^{(\alpha,m)}_{\text{Loc}, \delta}$ at $f_\alpha$, denoted by $T_\alpha$, to its orthogonal complement (assuming that $p = 2$) in $L^p_{(k;\kappa)}(\Sigma_\alpha, (f_\alpha)(TM))$, denoted by $O_\alpha$. Consequently, its image in

$$L^p_{(k-1;\kappa;\text{Loc}, \delta)}(\Sigma_\alpha, (f_\alpha)(TM))$$

contains a ball $B_r^{\alpha}$ of fixed radius $r''$ independent of $\alpha$. Note that by our construction, these images can be thought inside the same space

$$L^p_{(k-1;\kappa;\text{Loc}, \delta)}(\Sigma_\alpha, (f_\alpha)(TM))$$

the central fiber denoted by $L_\delta$ for short.

**Proof:**

(Sketch):

Let $G_\alpha : \hat{B}_r^{\alpha} \subset T_\alpha \to O_\alpha$ be the map representing $\kappa V^{(\alpha,m)}_{\text{Loc}, \delta} \cap B_r^{(\alpha,m)}$. Denote its graph

$$id \oplus G_\alpha : \hat{B}_r^{\alpha} \to \hat{B}_r^{\alpha} \oplus O_\alpha$$
by $F_\alpha$. Let $S^\alpha = [s]^\alpha \circ F_\alpha : \hat{B}^\alpha_{\alpha} \to \mathcal{L}_\delta$. We need to show that there is a positive $r' \leq r'_\alpha$ independent of $\alpha$ such that $S^\alpha$ restricted to $\hat{B}^\alpha_{\alpha}$ is a diffeomorphism to its image for all $\alpha$. Clearly we may assume that $S^\alpha$ does so.

It is sufficient to prove the following.

(I) $DS^\alpha_{f_0} : T_\alpha \to \mathcal{L}_\delta$ has an uniform inverse.

This is the most difficult step for the usual gluing estimate in [10] and [LT]. But it is almost trivial here simply because the sections $\xi$ in $T_\alpha$ restricted to the $\delta$-annulus $A_{\delta,\alpha} \subset \Sigma_\alpha$ satisfy the linearised $\partial \bar{J}$-equation at $f_\alpha$. In cylindrical coordinates, they are exponential decay in the sense that their norms satisfy

$$\|\xi|_{A_{\delta,\alpha}}\|_{k,p,\kappa} \leq C exp\left(-(-ln\delta)\right)\|\xi|_{C_{\kappa,\alpha}}\|_{k,p,\kappa} = C\delta\|\xi|_{C_{\kappa,\alpha}}\|_{k,p,\kappa}.$$

Here $C$ is a constant independent of $\alpha$, $C_{\delta,\alpha}$ is the two cylinders of length one (in cylindrical coordinates) at the ends of $A_{\delta,\alpha}$. Note that all $C_{\delta,\alpha}$ can be identified with $C_{\delta,\alpha}$ in $\Sigma_\alpha$. Therefore, $\|\xi|_{A_{\delta,\alpha}}\|_{k,p,\kappa}$ are ignorable for $\delta$ small enough. For such small $\delta$ we may assume that for all $\alpha$, $f_\alpha|_{\Sigma^0_{\delta,\alpha}}$ is equal to the restriction of $f_\alpha$ to $\Sigma^0_{\delta,\alpha}$ by the construction the pre-gluing. Here $\Sigma^0_{\delta,\alpha} = \Sigma_\alpha \setminus A_{\delta,\alpha}$ which can be identified with $\Sigma^0_{\delta,\alpha}$. Therefore, the operator norm

$$\|(DS^\alpha_{f_0})^{-1}\| \sim \|(DS^\alpha_{f_0}|_{\Sigma^0_{\delta,\alpha}})^{-1}\|$$

$$\sim \|(DS^\alpha_{f_0}|_{\Sigma^0_{\delta,\alpha}})^{-1}\| \sim \|(DS^\alpha_{f_0})^{-1}\|.$$

(II) Assume that the $r' > 0$ exists first. It will be proved later.

(III) Second order estimate: Since the desired second order estimate for $[s]^{\alpha,m}$ for Picard method is well-known by Floer’s work and is independent of $\alpha$, it easy the see that we only need to show that there is a constant $C$ independent of $\alpha$ such that $\|F_\alpha(\xi)\| \leq C\|\xi\|$ for any $\xi \in \hat{B}^\alpha_{\alpha} \subset \hat{B}^\alpha_{\alpha}$. For any $\xi_\alpha \in \hat{B}^\alpha_{\alpha}$, by using cut-off functions one can construct a corresponding $\xi_{\alpha,0} \in \hat{B}^\alpha_{r'}$ such that an “uniform ignorable” exponential decay factor $\|F_\alpha(\xi_\alpha)\| \sim \|F_{\alpha,\alpha}(\xi_{\alpha,0})\|$. The proof of this last statement is straight forward but takes quite a few steps. It says that the shape of the graph of $F_\alpha$ is similar to the fixed graph of $f_\alpha$. Note that here we need to use the surjectivity of $D[s]_f$ stated in (A1) not just the corresponding one in (I) above. In other words, we can not give a complete ”intrinsic” proof for this proposition.

(II) follows form (III): Note that in the proof of (III) above, we may replace $r'$ by $r'_\alpha$. We only need to show that $r'_\alpha$ is bounded below by a positive number. In other words, the graph of $F_\alpha$ does not get off $V^{(\alpha,\alpha)}$ too fast. This is indeed the case since by (III) it has the “same” increasing rate as the fixed $F_{\alpha,0}$.

$\blacksquare$
This second proof is not necessarily shorter in details. But it is conceptually more elementary and almost trivialize the gluing analysis. It says that in the set-up above one essentially can freely manipulate the contributions from the neck areas by multiplying them with cut-off function without affecting the any related estimates, a conclusion that one tried to get for the usual gluing estimate with quite effort.

- Local Fredholm theory and tautological coordinate on flat charts:

In this setting to establish the Fredholm for the local flat charts and their related bundles essentially amounts to know how to take derivatives along “normal” direction $\alpha = (\alpha_0, \alpha_n)$ at a point $h$ in the $\alpha_0$-stratum $\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta}$ for $[s]^{(\alpha, m)}$:

$$\kappa V^{(\alpha, m)}_{\text{Loc}, \delta} \cap B^p_{(\kappa-1; \text{Loc}, \delta)}(\Sigma_\alpha, \Lambda_0^{0,1}(f_\alpha^*(T M)))$$

and related sections used to obtain local perturbed moduli space. Denote $\kappa V^{(\alpha, m)}_{\text{Loc}, \delta}$ for short.

Still work under the assumptions (A1) and (A2). The key point is to introduce a product structure near the “end” $\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta}$. That is that we need to identify $\kappa V^{(\alpha, m)}_{\text{Loc}, \delta}$ with all entries $\alpha_n \neq 0$ with $\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta}$. By the above proposition, for these flat charts, we have much better chance to have such identifications. However, the identification is still not immediately since we are not in the finite dimensional situation. On the other hand by definition when $\alpha_t = \alpha_0$ is fixed, for all $\alpha$ we have

$$\int_{(\kappa-1; \text{Loc}, \delta)}^p(\Sigma_\alpha, \Lambda_0^{0,1}(f_\alpha^*(T M))) = \int_{(\kappa-1; \text{Loc}, \delta)}^p(\Sigma_{\alpha_0}, \Lambda_0^{0,1}(f_{\alpha_0}^*(T M))).$$

Therefore, by the above proposition, we may assume all $\kappa V^{(\alpha, m)}_{\text{Loc}, \delta}$ can be identified with $\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta}$ since we have the identifications $\kappa V^{(\alpha, m)}_{\text{Loc}, \delta}$ with $\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta}$.

Here

$$\kappa V^{(\alpha, m)}_{\text{Loc}, \delta} = [s]^{(\alpha, m)}(\kappa V^{(\alpha, m)}_{\text{Loc}, \delta})$$

and

$$\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta} = [s]^{(\alpha, m)}(\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta})$$

are the ”balls” of almost the same sizes in the fixed target space. Under these identification, we get the product structure for

$$\bigcup_{\alpha} \kappa V^{(\alpha, m)}_{\text{Loc}, \delta} \sim \kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta} \times \Lambda(\alpha_0)$$

where $\Lambda(\alpha_0) = \{\alpha\}$ is the collection of the local normal gluing parameters.

Any element in

$$\kappa V^{(\alpha_0, m)}_{\text{Loc}, \delta} \times \Lambda(\alpha_0)$$

called a ”tautological coordinate”. Therefore we get a new coordinate system for a slice of the flat chart. Clearly, it is tautologically true that with respect to
this coordinate chart, $[s]$ is automatically smooth along $\alpha$-direction. In fact, it is constant along $\alpha$-direction.

This resolves the difficulty (C) for $[s]$ in a "strange" way and under the assumptions (A1) and (A2). We already know how to remove (A2). We will not give a complete argument here on how to deal with (A1) (in [L4]). For our purpose to establish the smoothness of extended moduli spaces by Fredholm theory here, it is sufficient to know that in the general case, the tautological coordinate chart together with the evaluation map at $y$ introduced above to deal with (A2) gives us the desired coordinate on the product of the flat chart with the cokernel of $D[s]$ at $f_{\alpha_0}$. It follows from this that at least in this fixed flat chart, the perturbations obtained from the cokernel is also smooth at point $f$ along normal direction. Here we may assume that any element in the "cokernel" has the property that it is equal to zero on the $\rho_1$-discs on $\Sigma_{\alpha_0}$ around double points by analytic continuation principle. In order to show that the same is true for these perturbations viewed in any other flat charts, we assume that $\rho_1$ and $\delta$ are in the same order with $\delta << \rho_1$. Of course, to obtain a global section from the local perturbations here, we need to have the cut-off function which is obtained from the corresponding $L_{k,\kappa}^p$-norm. In this setting above, the $L_{k,\kappa}^p$-norm is the measured in the target space of $[s]$.

• • Kronecker’s principle and the compatibility with other charts:

At the first sight, the way to deal with the main difficulty (C) here is not very conventional. However, the general idea here can be traced back at least to Kronecker’s work. While Gauss’s algebraic proof of the fundamental theorem of algebra was truly ingenious at his time, the Kronecker’s work on field extensions can be viewed as a way to find a consistent system to accommodate the declaration that the symbol "$x$" is the solution of the polynomial equation $f(x) = 0$. In the same vein, in order to show that $[s]$ is smooth along normal direction for a point $f$ in a lower stratum, we simply declare that $[s]$ is essentially just an identity map. This immediately suggest that we need to check the smoothness of the sections and functions used to define local extended moduli spaces, defined in term of the tautological charts here, is compatible with the one defined by using the "natural" and "admissible" (in the obvious sense) charts before. Of course, the compatibility here just refers to all admissible charts within a fixed flat chart. It has nothing to do with difficulty (A).

Perhaps the easiest way to establish the compatibility of the notion of smoothness defined above with the "natural" one is to introduce a functorial system of "higher level" coordinates charts. Such a system of Banach coordinate charts is also useful for one of the methods below to establish Fredholm theory directly only using "natural" chart.

Before we discuss these "higher level" coordinates charts, we need to make a modification on the definition of flat charts in the following remark.

**Remark 1.2** In the definition of flat chart, we only require that (i) the cen-
ter $f$ of the chart is $J$-holomorphic on all prescribed small $\delta$-discs around its double points; (ii) all the $L^p$-maps in the flat chart have similar properties. These requirements are sufficient for establishing the local Fredholm theory on the flat ends. However, to study the functorial properties of the flat charts, it is more convenient to require that the center $f$ has an addition property that it is also holomorphic on small $\delta$-discs around all its marked points, so are all the elements in the flat chart. More generally for any uniformizer $U_\epsilon(f, H)$ of $B$ of stable $L^p$-maps centred at any $f$ with smooth components, even $f$ is lying on the top stratum, we introduce the corresponding flat charts $W_{\epsilon,loc,\delta}(f)$ inside an uniformizer $U_\epsilon(f, H)$, by the three requirements above. Note that within the same stratum $D$, using the Cech type construction described in [LT, Sec 4] one can construct a functorial system from a fixed covering of $B$ of uniformizers, $U^D = \{ U^{D, I} = \cap_{i,j=1,\ldots,n} W^{D, i,j}_\epsilon(f_i, H_j), I = (i_1, i_2, \ldots, i_n) \in N \}$. Here $N$ is the nerve of the fixed covering and the intersection $\cap$ here is interpreted as the properly defined fiber product (See [LT] for the detail). By adjusting the parameters $\delta$, we get a corresponding functorial subsystem from the flat charts contained in the above uniformizers, denoted by $W^D = \{ W^{D, I} = \cap_{i,j=1,\ldots,n} W^{D, i,j}_{\epsilon,loc,\delta_j}(f_i), I = (i_1, i_2, \ldots, i_n) \in N \}$.

Of course, moving from a lower stratum to a higher one, the number of marked points are different for these flat charts. However, by a proper modification of the above construction, we still get a functorial subsystem from the flat charts allowing the changing of the strata. Denote the resulting functorial system of general flat charts by $W = \{ W^I = \cap_{i,j=1,\ldots,n} W^{I, i,j}_{\epsilon,loc,\delta_j}(f_i), I = (i_1, i_2, \ldots, i_n) \in N \}$.

The main point of the modified definition of flat charts above and its associated functorial system is the following theorem.

**Theorem 1.6** The functorial system constructed from a fixed "covering" flat charts, $W = \{ W^I = \cap_{i,j=1,\ldots,n} W^{I, i,j}_{\epsilon,loc,\delta_j}(f_i), I = (i_1, i_2, \ldots, i_n) \in N \}$ is a $C^{m_0}$-smooth system. The corresponding functorial system constructed from the bundles $L_{loc,\delta_j} \to W^{I, i,j}_{\epsilon,loc,\delta_j}(f_i)$ over the flat charts $W^{I, i,j}_{\epsilon,loc,\delta_j}(f_i)$, denoted by $L_{Loc}$, is a $C^{m_0-1}$-smooth functorial bundle system over $W$. The $\partial f$-section can be interpreted a section of class at least $C^{m_0-1}$.

This theorem can be proved by using the higher level charts introduced below, where we will indicate one of the main steps of the proof of the above theorem. Note that having such a $C^{m_0-1}$-smooth functorial bundle system over a $C^{m_0}$-smooth functorial system of covering flat charts together with the fact that there are sufficiently many $C^{m_0-1}$-smooth generic perturbation to the $\partial f$-section is sufficient to establish the $C^{m_0-1}$-smoothness of the moduli space of the perturbed $J$-holomorphic curves. The details of the proof of the theorem will appear somewhere else.

Here we will only describe a key observation that leads to the existence of above smooth functorial system of the flat charts. To simplify our presentation,
we will only consider the case that there is only one stratum in $\mathcal{B}$ and that all the isotropy groups of the elements in $\mathcal{B}$ are trivial. In this case, each representative $f \in [f] \in \mathcal{B}$ is a $L^p_f$-map $f : S^2 \to M$. Let $\mathcal{U} = \{U_i, i \in I\}$ be a collection of the coordinate chart (uniformizers in the general case) covering $\mathcal{B}$, where $U_i = U_i(f, H_i)$ with the center $f_i$ being of class $C^\infty$.

To see how far away the general case from the case considered here, we note that (i) when there are several strata, the most difficult and extra step to prove the theorem will be indicated in next subsection on higher charts; (ii) for the case there is only one stratum, but the automorphism group $\Gamma_f$ is non-trivial. The following lemma can be proved.

**Lemma 1.1** Let $\tilde{\mathcal{U}} = \tilde{U}_i(f)$ be the collection of parametrized $L^p_f$-stable maps $S^2 \to M$ near $f$. For any $g \in \Gamma_f$ and $h \in \tilde{U}_i(f)$ define the right action of $\Gamma_f$ on $\tilde{U}$ by $h * g = h \circ g$, assuming that $\tilde{U}$ is $\Gamma_f$-invariant. Then it is a smooth action. Let $x = (x_1, x_2, x_3)$ be the standard markings of $S^2$ at which $f$ is a local embedding so that $f$ (hence each element in $\tilde{U}$) is transversal to $H_f \subset M^3$ near $f(x)$. Locally near $f(x)$, $M^3 \simeq H_f \oplus C^3$. Consider $\pi_2 \circ ev_x: \tilde{U} \to C^3$, where $\pi_2: H_f \oplus C^3 \to C^3$ is the projection map and $ev_x(h) = h(x)$. Let $(\pi_2 \circ ev_x)^{\Gamma_f}: \tilde{U} \to C^3$ be the $\Gamma_f$-invariant function obtained by taking $\Gamma_f$-average of $\pi_2 \circ ev_x$. Then for $\epsilon$ small enough, $H^{\Gamma_f} = ((\pi_2 \circ ev_x)^{\Gamma_f})^{-1}(0)$ is a smooth and $\Gamma_f$-invariant slice which is close to the "standard" slice $H_f = (\pi_2 \circ ev_x)^{-1}(0)$.

**Proof:**

The key facts used here are (1) $\Gamma_f$ is a finite group; (2) for any fixed $g \in SL(2, \mathbb{C})$, the action map $\Psi_g : \tilde{U} \to \tilde{U}$ given by $h \to h \circ g$ is smooth.

Note that the induced $\Gamma_f$-action on $H_f$ depends on $h \in H_f$, hence is only continuous. If we view the action of each element $g \in \Gamma_f$ as a coordinate transformation from $H_f$ to itself, then these coordinate transformations are not smooth even locally as charts for $\mathcal{B}$. On the other hand, on the new slice $H_f^{\Gamma_f}$, these coordinate self-transformations become smooth. This resolves the difficulty of the lack of smoothness for the above coordinate transformations coming from the actions of $\Gamma_f$. In the view of the discussion of next subsection together with the general constructions in [LT, Sec 4], it is sufficient to only consider the special case below.

To describe the key observation, we need to introduce more notations. Let $\pi_i : U_i \to [U_i] \subset \mathcal{B}$ be the projection map sending $h$ to its class $[h]$. For each $(i, j)$ with $[U_{ij}] = [U_i] \cap [U_j]$ non-empty, let $U_{ij, i} = \pi_i^{-1}([U_{ij}])$ and $U_{ij, j} = \pi_j^{-1}([U_{ij}])$. Fix a center $[f_{ij}]$ in $[U_{ij}]$. Let $f_{ij, i} = \pi_i^{-1}([f_{ij}])$ and $f_{ij, j} = \pi_j^{-1}([f_{ij}])$ be the corresponding "centers" of $U_{ij, i}$ and $U_{ij, j}$ respectively. Then under our assumption that all isotropy groups are trivial, there is an unique $\phi_{ji} \in SL(2, \mathbb{C})$ such that $f_{ij, j} = f_{ij, i} \circ \phi_{ji}$. Here $\phi_{ji}$ and its related map $\tilde{\phi}_{ij} : U_{ij, i} \to SL(2, \mathbb{C})$ is defined as follows. Let $x_1 = 0$, $x_2 = 1$ and $x_3 = \infty$ be the standard markings on
the fixed domain \(S^2 \simeq \Sigma_i\) of any element \(g\) in \(U_i\). As before, denote \((x_1, x_2, x_3)\) by \(x\). Then for any \(g \in U_i\), \(g(x)\) is in \(H_i\) and \(g\) is transversal to \(H_i\) at \(x\). When \(g\) is in \(U_{ij,i}\), there exists an unique triple \(y^g = (y_1^g, y_2^g, y_3^g)\) in \(\Sigma_i\) such that \(g(y^g) \in H_j\) and that \(g\) is transversal to \(H_j\) at \(y^g\). Then \(\hat{\phi}_{ji}(g) : \Sigma_i \to \Sigma_i\), as an element in \(SL(2, \mathbb{C})\), is the biholomorphic map \(x \to y^g = \hat{\phi}_{ji}(g)(x)\). It follows from the Proposition ? of Section 3 that \(\hat{\phi}_{ij} : U_{ij,i} \to SL(2, \mathbb{C})\) is a \(C^{m_0}\)-smooth function In the case that \(g = f_{ij,i}\), the center of \(U_{ij,i}\), we denote \(y^g\) by \(y\) and \(\hat{\phi}_{ji}(g)\) by \(\phi_{ji}\).

For any \(g \in U_{ij,i}\), the coordinate transformation \(\Phi_{ji} : U_{ij,i} \to U_{ij,j}\) is given by \(\Phi_{ji}(g) = g \circ \hat{\phi}_{ij}^g\). Here \(\hat{\phi}_{ij}^g = \hat{\phi}_{ij}(g)\). In other words, the coordinate transformation \(\Phi_{ji}\) is induced by \(g\)-dependent reparametrizations of the domain \(S^2\), the elements \(\hat{\phi}_{ij}^g\). Because of this \(g\)-dependence of \(\hat{\phi}_{ij}\), the transformation \(\Phi_{ji}\) can only be continuous in general so that \(B\) is a topological Banach manifold only with respect to the coordinate charts \(\mathcal{U} = \{U_{i,i} \in I\}\) above. Let \(\mathcal{W}\) be the system associated to the above coordinate system \(U\) consisting of the corresponding flat charts. Then even in this associated system \(W\) of flat charts, the transition functions are still only continuous. Since these flat charts can be used to establish the smoothness of the \(\bar{\partial}\)-section locally even along the gluing directions, it is natural to ask that if it is possible to modifying the transition functions between the flat charts together with the corresponding bundles such that (1) the modified bundle system over the flat charts becomes smooth; (2) the \(\bar{\partial}\)-section is still smooth and Fredholm in the new bundle system. In other words, the question is if it is possible to get a smoothing of the bundle system over the flat charts together with the specified Fredholm section \(\bar{\partial}\). It turn out that in order to have the desired smoothing, even in the simplest case that \(B = Map_{k,p}(S^2, M)\), where all elements in the space of stable \(L^2_{k}\)-maps \(B\) have a fixed domain \((S^2, j)\), we have to use a 3-parameter (complex) family of deformations of the complex structure \(j\) coming from the pull-backs of \(j\) by the corresponding family of differomorphisms of \(S^2\) supported near the small discs centered at the three standard marked points of \(S^2\). Furthermore, we have to use such family of complex structures rather than just the fixed \(j\) to define the \(\bar{\partial}\)-section.

The desired smoothing is constructed inductively that we describe in the following. First we fix the following data: (a) a chart \(U_{i_0}\) as base element of the desired coordinate system; (b) for each \([U_{ij}\], the centers \(f_{ij,i} \in U_{ij,i}\) and \(f_{ij,j} \in U_{ij,j}\); (c) the centers in (b) determines the biholomorphic map \(\phi_{ji} : (S^2; x_i) \to (S^2; x_j)\) so that \(f_{ij,j} = f_{ij,i} \circ \phi_{ji}\). Here we have used \(x_i\) and \(x_j\) to denote \(x\) and \(y\) defined before.

Now fix \(i = i_0\) and consider \(U_i\) as the staring point of the induction, and let \(j\) varying. We want to modify the coordinate transformation \(\Phi_{ji} : U_{ij,i} = U_{ij,j}(f_{ij,i}, H_i) \to U_{ij,j} = U_{ij,j}(f_{ij,j}, H_j)\) as follows.

There is a corresponding coordinate transformation \(\Phi_{ji} : U_{ij,i} \to U'_{ij,i} \subset \bar{U}_j\)
given by $\Phi_{ji}(g) = g \circ \phi_{ji}$ for any $g \in U_{ij,i}$. Here we have used $\bar{U}_j$ to denote $\bar{U}_{\epsilon_j}(f_{ji})$ consisting of all $L^p_k$ maps $g : S^2 \to M$ such that $||g - f_{ji}||_{L^p_k} < \epsilon_j$, and $U'_{ij,i}$ is the image $\Phi_{ji}(U_{ij,i})$ in $\bar{U}_j$. Note that unlike $\bar{\Phi}_{ij}$, $\Phi_{ji} : U_{ij,i} \to \bar{U}_j$ is smooth. Note that as subsets of $\bar{U}_j$ with “center” $f_{ij,j}$, $U'_{ij,j}$ is “close” to $U_{ij,j}$.

In fact, the coordinate transformation $\Phi_{ji} : U_{ij,i} = U_{ij,i}(f_{ij,i}, H_j) \to U_{ij,j} = U_{ij,j}(f_{ij,j}, H_j)$ can be written as $\Phi_{ji} = \Phi'_{ji} \circ \Phi_{ji}$ with $\Phi'_{ji} : U'_{ij,i} \to U_{ij,j} \subset \bar{U}_j$ defined as follows.

For each $h \in U'_{ij,i}$, let $g = h \circ (\phi_{ji})^{-1}$. Then $\Phi'_{ji}(h) = g \circ \phi_{ji}^{-1} = h \circ ((\phi_{ji})^{-1} \circ \tilde{\phi}_{ji}^{-1})$. Recall that $\tilde{\phi}_{ji} = \tilde{\phi}_{ij}(g)$. Denote $((\phi_{ji})^{-1} \circ \tilde{\phi}_{ji}^{-1})$ with $g = h \circ (\phi_{ji})^{-1}$ by $\tilde{\phi}_{ji}^{-1}$. Then a similar argument to the proof of Lemma 7 shows that as elements in $\text{SL}(2, \mathbb{C})$, $(\tilde{\phi}_{ji}^{-1})^h$ depend on $h$ smoothly. Moreover when $\epsilon_i$ is small enough, $(\tilde{\phi}_{ji}^{-1})^h$ is close to the identity map of $S^2$. Let $(\tilde{\psi})^h = (\tilde{\phi}_{ji}^{-1})^h(x)$. Then $\Phi'_{ji}$ is induced by the $h$ dependent reparametrizations $(\tilde{\phi}_{ji}^{-1})^h$ of $S^2$ sending $x$ to $(\tilde{\psi})^h$.

Since $(\tilde{\phi}_{ji}^{-1})^h$ is close to identity map, $(\tilde{\psi})^h$ is close to $x$.

The key step is to replace the biholomorphic map $(\tilde{\phi}_{ji}^{-1})^h$ of $S^2$ above by a smooth differomorphism of $S^2$ which is equal to identity map outside the three small prescribed discs $D_h(x_i)$, $i = 1, 2, 3$, and is equal to the obvious “rotations” on the smaller discs $D_h(x_i)$ that bring $x_1$ to $(\tilde{y})^h$. One can show that the differomorphism can be obtained as the time-1 flow of a smooth vector field $X_h$ of $S^2$ supported on $D_h(x_1)$ and smoothly depending on $(\tilde{y})^h$. Denote the resulting differomorphism by $(\tilde{\phi}_{ji}^{-1})^h$. Now in the definition of $\tilde{\Phi}_{ji} : U'_{ij,i} \to U_{ij,j} \subset \bar{U}_j$ above, replacing $(\tilde{\phi}_{ji}^{-1})^h$ by $(\tilde{\phi}_{ji}^{-1})^h$, we obtain a map $\tilde{\Phi}_{ji} : U'_{ij,i} \to \bar{U}_j$. By the construction each element $h$ in the image $\tilde{\Phi}_{ji}''(U'_{ij,i})$ satisfies the condition that $h(x)$ is in $H_j$, the same condition that is satisfied by each element in $U_{ij,j}$.

One can show that $\tilde{\Phi}_{ji}''$ is a “coordinate transformation” in the sense that as a continuous map $U'_{ij,i} \to \bar{U}_j$, it is still injective and that the image $\tilde{\Phi}_{ji}''(U'_{ij,i})$ is almost the same as $U_{ij,j}$. Hence by slightly modifying the size of $\bar{U}_j$, the image $\tilde{\Phi}_{ji}''(U'_{ij,i})$ in $U_j$ can be regarded as the intersection $[U_i] \cap [U_j]$ represented inside $U_j$. Denote the resulting coordinate transformation $\tilde{\Phi}_{ji}'' \circ \Phi_{ji} : U_{ij,i} \to U_{ij,j}$ by $\Psi_{ji}$. Then $\Psi_{ji}$ is induced by the $h$-dependent differomorphisms $(\tilde{\phi}_{ji}^{-1})^h$ of $S^2$. Of course, each $(\tilde{\phi}_{ji}^{-1})^h$ is not holomorphic with respect to the fixed “standard” complex structure $j$ on $S^2$. Let $j_h = (\tilde{\phi}_{ji}^{-1})^h(j)$ be the $h$ dependent (in fact $(\tilde{y})^h$-dependent) complex structure. Then $(\tilde{\phi}_{ji}^{-1})^h : (S^2, j) \to (S^2, j_h)$ is biholomorphic. Therefore, each element $h$ in $\tilde{\Phi}_{ji}''(U'_{ij,i})$ has a domain $S^2$ with a $h$-dependent (in fact only $(\tilde{y})^h$-dependent) complex structure $j_h$. Here we have considered $\tilde{\Phi}_{ji}''(U'_{ij,i})$ as a subset of $U_j$. We may assume that the function $h \to (\tilde{\phi}_{ji}^{-1})^h(j)$ is defined on a small neighbourhood $N(\tilde{\Phi}_{ji}''(U'_{ij,i}))$ of $\tilde{\Phi}_{ji}''(U'_{ij,i})$ so that $h \to j_h$ is well-defined on the neighbourhood. Since for any $h$ in the
neighbourhood, \((\tilde{\phi}'_{ji})^h\), as a diffeomorphism of \(S^2\), is close to identity, one can extend the function \(h \rightarrow (\tilde{\phi}'_{ji})^h\) to all \(h\) in \(U_j\) such that any element in the complement of \(N(\tilde{\Phi}_{ji}'(U'_{ij,i}))\) maps to the identity map of \(S^2\). Consequently, for any \(h\) in \(U_j\), we have associated its domain \(S^2\) with a \(h\)-dependent complex structure \(j_h\) smoothly depending on \(h\). Therefore, using these \(h\)-dependent complex structures on the domain \(S^2\) for the elements in \(U_j\), the new coordinate transformation \(\Psi_{ji}\) is induced by biholomorphic maps of \(S^2\) with the above complex structures. Note that the \(\partial\jmath\)-section will be defined on each local chart using these complex structure on the domain \(S^2\) of each element in the chart.

Of course, the transition function \(\Psi_{ji}\) is still only continuous. However, if we restrict \(\Psi_{ji}\) to the corresponding flat charts whose elements are \(J\)-holomorphic on the small prescribed discs \(D_h(x_i), i = 1, 2, 3\), where the vector field \(X_h\) of \(S^2\) is supported, then it is smooth by the construction. Moreover, the \(\partial\jmath\)-section is smooth on these flat charts. It is in this sense, we have proved the theorem for the case that \(\mathcal{B}\) is covered by two charts \([U_i]\) and \([U_j]\).

This finishes the first step of the induction. To proceed further, note the following. (A) We may assume that the triple of \(\tilde{U}_i\), denoted by \(3\tilde{U}_j\), is well-defined for the each \(U_i\) of the covering of \(\mathcal{B}\). (B) The formula for coordinate transformation \(\Phi_{ji}\) given by \(\Phi_{ji}(g) = g \circ \phi_{ji}\) defined on \(U_{ij,i}\) before still makes sense for \(g\) in whole \(U_i\) as long as we can make sure that \(g \circ \phi_{ji}\) lies in a properly enlarged space of \(\tilde{U}_j\). We may assume that \(\Phi_{ji} : U_i \rightarrow 3\tilde{U}_j\) is well defined. (C) Let \(U'_{ij} = \Phi_{ji}(U_i)\) inside \(3\tilde{U}_j\). Recall that we have assumed that \(\tilde{U}_j\) is a ball of radius \(\epsilon_j\) centere at \(f_j\). Let \(\beta\) be a smooth cut-off function defined on \(3\tilde{U}_j\) which is equal to 1 on \((1 - \delta_j)\tilde{U}_j\) and 0 outside \((1 + \delta_j)\tilde{U}_j\). Here we assume that \(\delta_j << \epsilon_j\). The coordinate transformation \(\Psi_{ji}\) gives rise an map from the intersection \(U'_{ij,i}\) as a subset of \(U'_{ij}\) to the corresponding \(U_{ij,i}\) inside \(U_j\). Using the cut-off function \(\beta\) and \(1 - \beta\) "join" the two subsets \(U'_{ij}\) and \(U_{ij}\) in \(3\tilde{U}_j\) along the "line segments" connecting the points identified by \(\Psi_{ji}\). Note that two elements \(h'\) and \(h\) identified by \(\Psi_{ji}\) differ by a \((h', h)\)-dependent diffeomorphism \(\phi_{h, h'}\) of \(S^2\) whose vector field is localized near \(x\). The above "line segment" connecting \(h'\) and \(h\) is obtained by connection \(\phi_{h, h'}\) and the identity map in the pseudo-group of the above type of diffeomorphisms. In this way, we "glue" \(U'_{ij}\) and \(U_j\) together inside \(3\tilde{U}_j\) which is homeomorphic to \([U_i] \cup [U_j]\). In other words, we get a global topological embedding of \([U_i] \cup [U_j]\) inside \(3\tilde{U}_j\) as a topological (locally closed ) submanifold. Denote the resulting submanifold as \((U_i \cup U_j)_j\). Note that although the submanifold \((U_i \cup U_j)_j\) can not be smooth, the corresponding objects coming from the flat charts \(W_i\) and \(W_j\), denoted by \((W_i \cup W_j)_j\) is a smooth "object".

Now we are almost in the same position as the beginning of the induction. Namely, if there is a third chart \(U_k\), the "global" object \((U_i \cup U_j)_j\) plays the role of \(U_i\), the starting chart of the induction before. We can proceed as before for the new pair \((U_i \cup U_j)_j\) and \(U_k\). However, there is an asymmetry here. While \(U_i\) is a
smooth submanifold of $\tilde{U}_i$, $(U_i \cup U_j)_j$ is only topological in its ambient space. The key observation now is that (a) the corresponding $(W_i \cup W_j)_j$ is a smooth object; (b) it still remains so under that map $\Phi_{kj} : (W_i \cup W_j)_j \cap (\Phi_{kj})^{-1}(3\tilde{U}_k) \to 3\tilde{U}_k$.

This smoothness is sufficient for our purpose. Hence we are indeed in the same position as the start point of the induction.

Before closing this discussion, we mention two technical points. Note that in general after we have finished $n$-steps of the induction and obtained a consistent coordinate system $\{U_i, i = 1, \ldots, n\}$, we cannot expect to get a globalization $(\bigcup_{i=1}^n U_i)^n$ as a topological submanifold of $3\tilde{U}_n$. However, for our purpose of induction, we only need to look at those $U_i$ such that $\tilde{U}_i \cap [U_{n+1}]$ is not empty. Using the fact that $\mathcal{X}$ is paracompact, one can show that for a proper refinement of the original covering $U$, the "union" of these $U_i$ subject to above condition can be globalized as a a topological submanifold inside $3\tilde{U}_n$.

Another technical point used in above construction is that the above pseudo-group is a normal sub pseudo-group in the group of all differentomorphisms of $S^2$. This implies that we can freely transfer any construction obtained by using such pseudo-group in one chart to other charts by using a functorial system of transformations induced by $h$-independent reparametrizations such as the system $\Phi_{ji}$ used before.

- • • Higher level natural coordinate charts and push-forward of bundles:

Still work locally on a fixed flat chart. We have the "natural" chart $\kappa V^{(\alpha,m)}_{\text{Loc,}\delta}$ for a slice of the flat chart which is realized as a graph of a map from a ball of $T_\alpha$ to its orthogonal complete $O_\alpha$ in $L^p(k;\kappa)$.

In particular, any elements in $\kappa V^{(\alpha,m)}_{\text{Loc,}\delta}$ or $\kappa V^{(\alpha,m)}_{\text{Loc,}\delta}$ are $J$-holomorphic on the the $\delta$-discs or $\delta$-annuli around double points. Therefore, any such elements are determined by their restrictions on the fixed part $\Sigma^0(\alpha) = \Sigma^0(\alpha_0)$. Recall that $\Sigma^0(\alpha)$ and $\Sigma^0(\alpha_0)$ are defined by removing the $\delta$-annuli or $\delta$-discs from $\Sigma_\alpha$ and $\Sigma_{\alpha_0}$ respectively, and are identified by the pre-gluing. Of course, the surface $\Sigma^0(\alpha) = \Sigma^0(\alpha_0)$ depends on $\delta$. We denote it by $\Sigma^0(\delta)$.

Now fix a $\delta$ and consider $\Sigma^0(\gamma)$ for all $0 < \gamma << \delta$ as a trivial family of open curves inside the universal curve $\mathcal{X}$ locally over $\Lambda(\alpha_0)$. In fact, we should consider all the open surfaces "generated" by these $\Sigma^0(\gamma)$. But we will suppress this point in the discussion here.

As before by using exponential coordinate charts at $f_{\alpha_0}$ and $f_\alpha$ and abusing notations, each elements in

$L^p_{(k;\kappa)}(\Sigma^0(\gamma); f_\alpha|\Sigma^0(\gamma)(TM); h)$

or

$L^p_{(k;\kappa)}(\Sigma^0(\gamma), f_{\alpha_0}|\Sigma^0(\gamma)(TM); h)$

can be thought as a $L^p_k$-map from the fixed open Riemann surface $\Sigma^0(\gamma)$ to $M$. As before for any fixed $\gamma$, with this interpretation here we get a corresponding
space of \( L^p_k \)-maps, denoted by \( \hat{W}_\gamma(f_\alpha) = \hat{W}_\gamma(f_{\alpha_0}) \) near the restriction of \( f_\alpha \) to \( \Sigma^0(\gamma) \) which is the same as the restriction of \( f_{\alpha_0} \). As usual, there is a corresponding bundles over \( \hat{W}_\gamma(f_\alpha) \) whose fibers are the corresponding \( L^p_{k-1} \)-sections and a smooth \( \partial_J \)-section \( \hat{s} \). By varying \( \gamma \), we get a functorial system of bundles together with specified sections. In general, this system here is not very useful since \( \hat{s} \) is not Fredholm any more.

However, since any element in a flat chart \( \kappa V^{(\alpha,m)}_{\text{Loc},\delta} \) is determined by its values on any \( \Sigma^0(\gamma) \), we get an obvious embedding of \( \kappa V^{(\alpha,m)}_{\text{Loc},\delta} \) into \( \hat{W}_\gamma(f_\alpha) \) for any \( \gamma << \delta \). Moreover, the bundles used to define the flat charts are push-forwarded into the corresponding bundles such that the pull-back of \( \hat{s} \) is \( s \). Similarly, the section used to define perturbations can be lifted into sections of these higher bundles. In any case, for the flat charts and related bundles and sections, above seemingly useless system serves as a systems of ambient coordinate charts so that one can talk about smoothness in the similar way as the usual definition of the smoothness for a function defined on a "bad" set inside \( \mathbb{R}^n \) described, for instance at the beginning of Milnor’s book [M]. Note that in this setting, we have no difficulty to take derivatives along the normal directions.

This immediately brings us a new set of problems to verify the compatibility of the smoothness of an object like moduli space or a section obtained from various different "coordinate" systems.

Before we close this discuss here, just mention one technical point. Since we use open Riemann surfaces here, the \( L^p_k \)-maps usually have some pathology behaviour near the boundary, it is better to introduce a modifier, a fixed cut-off function \( \beta_\delta,\gamma \) on \( \Sigma^0(\gamma) \) which is compact supported on \( \Sigma^0(\delta) \) and equals to 1 on \( \Sigma^0(\delta) \). Then for any \( L^p_k \) map \( \xi : \Sigma^0(\gamma) \to M \), the pseudo-norm \( \|\xi\|_{k,p,k;\beta_\delta,\gamma} \) is defined to be \( \|\beta_\delta,\gamma \xi\|_{k,p,k} \) and becomes a real norm on the flat chart. This gives a family of induced norms on the flat chart.

• • • Functorial system of higher level flat coordinate charts:

First note that the above higher level coordinate charts and the related bundles can be made into a functorial system as we did before for a covering of the uniformizers. Then the above discussions suggest that we should require that each element in a higher chart in the functorial system is holomorphic on the prescribed small discs at all its marked points. We will call such a functorial system of higher charts with this additional requirement a functorial system of higher flat charts. Denote such a functorial system of higher flat charts by \( \tilde{W}_{\text{Loc}} \) and its associated bundle system by \( \tilde{L}_{\text{Loc}} \to \tilde{W}_{\text{Loc}} \). The main consequence of making this additional requirement is that the functorial system of higher flat charts is \( C^{m_0-1} \)-smooth and its associated bundle system is \( C^{m_0-1} \)-smooth. The \( \partial \)-sections as well as all the perturbations used to regularize the moduli space can be interpreted as sufficiently smooth sections of the bundle system. One can also construct sufficiently smooth cut-off functions to localize any desired constructions. Since all these desired properties can be easily established for
\( \mathcal{W}_{\text{Loc}} \), it can serve as an ambient space for establishing the smoothness of the objects described in the Theorem 1.5.

As a change of notation, denote the functorial system of the flat charts and its associated bundle system constructed before by the new notation \( \mathcal{L}_{\text{Loc}} \to \mathcal{W}_{\text{Loc}} \). Note that for each flat chart in the system \( \mathcal{W}_{\text{Loc}} \), its local Banach manifold structure near any of its ends is given by the so-called tautological smooth structure before. Therefore the key step to prove the Theorem 1.5 is to show that the system \( \mathcal{W}_{\text{Loc}} \) is embedded into \( \mathcal{W}_{\text{Loc}} \) "nicely" in the sense that the embedding on each flat chart is smooth. This can be done easily away from lower strata since here the tautological local smooth structure is the same as the natural one. The general case will follow if one can show that all the derivatives of the embedding at the points of the top stratum can be extended continuously over the lower strata of the flat chart. This can be achieved by establishing required uniform estimate for the derivatives of the embedding at the points on the top stratum as they approach to the lower strata.

Next we outline two other slight different constructions of flat charts.

- Local Fredholm theory in the natural coordinate chart of a flat end:

It still remains to directly establish the Fredholm theory in the natural coordinate chart and trivialization for the bundle on the flat end. As mentioned above, despite of the fact that each element in the flat chart is rigid along the neck, it is still not immediate to give the required product structure for the local Fredholm theory. In stead of trying to define a product structure directly, we construct a new chart for the end which automatically has a product structure and is "close" to the given flat chart. Then carry out all relevant constructions for this new chart as we did for the old flat chart.

To this end, recall that the flat chart near the end at \( f \) is denoted by \( W^0(f) = \bigcup_{\delta} W_{\epsilon, \text{Loc}, \delta}(f) = \bigcup_{\alpha, \delta} W_{\epsilon, \text{Loc}, \delta}(f) \). It comes with a bundle \( \mathcal{L}^0 = \bigcup_{\delta} \mathcal{L}_{\text{Loc}, \delta}(f) = \bigcup_{\alpha, \delta} \mathcal{L}_{\alpha, \text{Loc}, \delta}(f) \) and a section \( s^0 : W^0(f) \to \mathcal{L}^0 \) that is the restriction of \( \partial J \)-section \( s \). We defined a new "product" chart, denoted by \( (W')^0(f) \), by the same formula but denoting all objects involved by the same letters with a "prime".

Start with the definition of a lowest stratum, \( W''_{\epsilon, \text{Loc}, \delta}(f) \). One way is to define it be the same old corresponding object. A more convenient way is to define it to be the collection of all elements \( g' \). Each \( g' \) is obtained from an element \( g \) in \( W_{\epsilon, \text{Loc}, \delta}(f) \) defined as following: (i) on each small disc of radius \( \rho \ll \delta \) centred at one of the double points, \( g' \) is a constant function equal to the value of \( g \) at the double point; (ii) away from these discs of radius \( 2\rho \), \( g' = g \); (iii) on the neck areas, \( g' \) is obtained by joining the constant function with \( g \) using a fixed cut-off function supported in the neck area. Note that \( \rho = \rho(f) \) needs to be chosen sufficiently small so that \( g' \) is still in the neighbourhood of \( f \) required in the main gluing estimate above (or see [LT]). For a normal gluing parameter \( \alpha \) with \( |\alpha| \ll \rho \), \( (W')^0_{\epsilon, \text{Loc}, \delta}(f) \) is defined to be the collection of maps \( g'_{\alpha} : \Sigma \to M \), each coming from a map \( g' : \Sigma_{\alpha} \to M \) in \( W''_{\epsilon, \text{Loc}, \delta}(f) \). More specifically, since
$g'$ is constant on those $\rho$-discs centred at double points, for $|\alpha| < \rho/2$, $g'_\alpha$ is defined to be $g'$ on the part of $\Sigma_\alpha$ that corresponds to $\Sigma_{\alpha_0} \setminus \{\rho/2$-discs $\}$ and constants on the rest. To define the bundle $(\mathcal{L}')^0$, note that the old bundle $(\mathcal{L})^0$ is the restriction to $W_\epsilon^0(f)$ of the corresponding bundle over the original ambient space $W_\epsilon(f)$. Hence the bundle $(\mathcal{L}')^0$ is defined to be the same as before by taking restriction. Of course the values of $s^0$ may not be in the bundle $(\mathcal{L}')^0$ any more. Using the inner product structure for $L^2$-spaces, we defined $(s')^0$ to be the projection of $s^0$ to $(\mathcal{L}')^0$. Since $s^0(g'_\alpha)$ restricted to the neck areas with $s > -\log \rho + 2$ is equal to zero, one can show that $s^0$ considered as a section of the original bundle $\mathcal{L}$ is smooth even along $\alpha$ direction for any points in $(W')^0_\epsilon(f)$. Since the projection is linear and only decreases the norm, the section $(s')^0$ is still smooth in the same way as $(s^0)$. To spill out a bit more on this, note that although the projection is a global operation, $s^0(g'_\alpha)$ is only supported on the "fixed part" $\Sigma_\alpha$. Hence the computations of the derivatives of $(s')^0$ can be thought effectively as the ones carried out only on the fixed part of $\Sigma_{\alpha_0} \setminus \{\rho/2$-discs $\}$ near double points so that the well-known difficulty of lack of $C^1$-differentiability for taking derivatives in $\alpha$-direction disappears.

In this way we get a local smooth Fredholm section $(s')^0 : W_\epsilon^0(f) \to (\mathcal{L}')^0$ in the usual sense. After a perturbing by generic "geometric" perturbation or a finite dimensional collection of such perturbations, its zero locus, denoted by $\mathcal{M}'$ or $EM'$ can be thought as local regularized moduli space or local extended moduli space.

Before we indicate the main steps to globalize the objects here, we need to embed part of $W_\epsilon^\alpha_{e,Loc,\delta}(f)$ with $\alpha \neq 0$ into the corresponding part of the original $W_\epsilon^\alpha(f)$ with the desired properties described below. For simplicity and without lost generality, we may only consider the part of $W_\epsilon^\alpha_{e,Loc,\delta}(f)$ with $\alpha$ in the top stratum and near a fixed normal gluing parameter $\beta_0$. In this case, there is a product structure

$$W_\epsilon^{\beta_0}(f) \times \Lambda_r(\beta_0) \simeq \cup_{\alpha \in \Lambda_r(\beta_0)} W_\epsilon^\alpha(f),$$

where $\Lambda_r(\beta_0)$ is the collection of $\alpha$ with $|\alpha - \beta_0| < r$. The product $W_\epsilon^{\beta_0}(f) \times \Lambda_r(\beta_0)$ is one of the topological coordinate charts of the top stratum of $\mathcal{B}$. Unlike the coordinate chart without involving gluing parameters, the $\overline{\partial}$-operator $s$ is not even $C^1$ along $\alpha$-direction. On the other hand, by our construction, when restricted to $\cup_{\alpha \in \Lambda_r(\beta_0)} W_\epsilon^{\alpha}(f)$ or $\cup_{\alpha \in \Lambda_r(\beta_0)} W_\epsilon^{\alpha,Loc,\delta}(f)$, $s$ is automatically smooth along any directions. Note that both of above spaces are contained in $\cup_{\alpha \in \Lambda_r(\beta_0)} W_\epsilon^{\alpha}(f)$ as closed Hilbert submanifolds. Therefore by using a smooth cut-off function we can interpolate them to form a new Hilbert manifold $\cup_{\alpha \in \Lambda_r(\beta_0)} W_\epsilon^{\alpha,Loc,\delta}(f)$ which is equal to the object with "prime" near $\beta_0$ and is equal to the other near the boundary of $\Lambda_r(\beta_0)$. One can do the interpolation for all relevant objects to obtain: (a) $\mathcal{M}'$ and $EM'$; (b) $(s'')^0$ whose zero locus after perturbations are $\mathcal{M}'$ and $EM'$; (c) $s''$ such that after composing with
an obvious projection to the orthogonal complement of Hilbert subbundle $L^0$, the zero locus of the composed section is $\bigcup_{\alpha \in \Lambda^r(\beta_0)} W^\alpha_{\epsilon, \text{Loc}, \delta}(f)$. Note that we require that sections $s''$ and $(s'')^0$ are Fredholm in the corresponding bundles. Since Fredholmness is an open property, this can be done if all data involved are sufficiently close to fixed ones. This completes the desired process to extend $W^0_{\epsilon, \text{Loc}, \delta}(f)$ or $W^0_{\epsilon, \text{Loc}, \delta}(f)$ of the lowest strata to the top strata such that the extension is equal to $W^\alpha_{\epsilon, \text{Loc}, \delta}(f)$ near the boundary at each stratum. Note that while $W^\alpha_{\epsilon, \text{Loc}, \delta}(f)$ is "intrinsic", $W^\alpha_{\epsilon, \text{Loc}, \delta}(f)$ is more convenient for the "normal" extension. Our interpolating above makes the normally extended $W^\alpha_{\epsilon, \text{Loc}, \delta}(f)$ back into an intrinsic object.

• Global Fredholm theory:
  
  The desired global regularized moduli space with $C^{m_0}$-smoothness can be obtained as a consequence of (i) having a global Fredholm theory on the global ambient space made out of these local $W^\epsilon_{\text{Loc}, \delta}(f)$ and their local ambient spaces and (ii) extending the notion of weak smoothness in section 2 to such global ambient space. Here we will only indicate the main steps for (i) as it appears as the main difficulty.

  The first step is to have a global "normal" gluing process. To this end, Cover $W^D$ by a locally finite covering $(W^D_i(f_i), i \in I)$. Here $W^D$ is the lowest strata of the collection of local slices of $B$.

  Using local finiteness of the covering, the key covering lemma, Lemma 4.3 in [LT] can be generalized to this situation. Using this lemma, the normal gluing for the general case can be done essentially in the same way as the following special case: $W^D = W^D_{\epsilon_1}(f_1) \cup W^D_{\epsilon_2}(f_2) \cup W^D_{\epsilon_{12}}(f_{12})$ with the requirement as follows. Denote $W^D_i(f_i), i = 1, 2$ and $W^D_{\epsilon_{12}}(f_{12})$ by $W^D_1$ and $W^D_2$. Similar notations will be used for other strata. The requirement is that $W^D_1$ and $W^D_2$, the two covering sets in the same level with respect to the nerve of the covering, do not intersect.

  For a map $g_i \in W^D_1, \ldots, i = 1, 2$ with $g_i$ being considered as an unparametrized map lying in $W^D_{\epsilon_1} \cap \cdots \cap W^D_{\epsilon_2}$, the $\delta_i$-discs at double points of its domain transform to distorted discs depending on $g$ at the corresponding double points. By taking $\delta_{12} << \rho_i$, and defining $(W^0)^D_1$ and $(W^0)^D_2$ in the same way as we did before, one can show that these new objects are compatible with the corresponding ones in $W^D_i, i = 1, 2$ in the sense that the old objects are embedded into the new ones. Moreover their defining sections, the sections $s_i', i = 1, 2$ and $s_{12}'$ as well as the resulting local moduli spaces are compatible also. Hence we get a globally defined $(W^0)^D$ as well as all related objects at least in the lowest strata $D$. Their further extension to $(W^0)$, etc. along higher strata can be done similarly as before without much difficulty as long as the normal gluing parameter $\alpha$ is relatively small comparing to $\rho_{12}$.

  The next step is to globalize the local interpolating process. This is done in [L3] and [L4].

• Another method for Fredholm theory:
One of the disadvantages of the method above for Fredholm theory is that in general the unperturbed moduli space of stable $J$-holomorphic curves is not in $(W'_0, Loc, \delta, f)$ locally. This happens even in the "good" case that $s$ is already a transversal section along all strata so that the unperturbed moduli space of stable $J$-holomorphic curves is stratified smooth. Note that having regularized moduli space consisting of compatible zero loci of the "generically perturbed" $\bar{\partial}$-sections of the smooth local Fredholm systems described above is sufficient for our purpose. Despite of this, it is still desirable that in the "good" case above, our method should imply that the unperturbed moduli space is in fact $C^\infty$-smooth not just stratified smooth. In fact, a weaker version of our method above implies that it is possible to only use $W'_0, Loc, \delta, \rho, f$ to carry out the above process and form a compatible topological local Fredholm systems. Each such local Fredholm system is smooth for a fixed gluing parameter. With extra effort, one can show that the generically perturbed moduli space is stratified smooth. Then we have to show that the two compatible local Fredholm systems are topological equivalent so that the topological system that contains the unperturbed moduli space acquires a smooth structure by the equivalence. This will be done in [L3] and [L4].

On the other hand, there is a different way to construct a Fredholm theory only using $W'_0, Loc, \delta, \rho, f$ as intrinsic objects but treating $W'_0, Loc, \delta, \rho, f$ as one of its coordinate charts. This method will automatically have the above equivalence of the above two local systems. It also avoids the interpolating process above. We now indicate the main step of this construction.

Note that by a similar construction as above using the nerve of the locally finite covering, one can globalize these local $W'^0, Loc, \delta, \rho, f$, $i \in I$ together with the bundle $L'^0, Loc, \delta, \rho, f$, $i \in I$ near the end of each lowest strata in such a way that the resulting objects are embedded into the original corresponding objects in a compatible manner. Therefore, we only need to show that for a fixed $W'^0, Loc, \delta, \rho, f$, using $W'^0, Loc, \delta, \rho, f$ as a coordinate chart, with a proper choice of the trivialization of the bundle $L'^0, Loc, \delta, \rho, f$, $s'^0$ is a smooth section even along the directions of gluing parameters.

Denote $W'_0, Loc, \delta, \rho, f$ and $L'^0, Loc, \delta, \rho, f$ by $W'^0, Loc, \delta, \rho, f$ and $L'^0, Loc, \delta, \rho, f$ respectively. Embed $W'^0, Loc, \delta, \rho, f$ into $W^0, Loc, \delta, \rho, f$ with $\delta_1 << \rho << \delta$. Then $W'^0, Loc, \delta, \rho, f$ is also embedded into $W^0, Loc, \delta, \rho, f$. Hence we have used the better behaved space $W'^0, Loc, \delta, \rho, f$ and bundle $L'^0, Loc, \delta, \rho, f$ to replace the original local ambient space and its associated bundle. By adding the cokernel of the linearisation of $s'^0$, we may assume that $s'^0 = s'^0, \delta, \rho, f$ is already a transversal section that is smooth along the slice with fixed gluing parameter. Then by the implicit function theorem, $W'^0, Loc, \delta, \rho, f$ can be realized as a graph of a function from $W'^0, Loc, \delta, \rho, f$ to its orthogonal complement in $W'^0, Loc, \delta, \rho, f$. Here we have considered $W'^0, Loc, \delta, \rho, f$ as an open ball in a Banach space and $W^0, Loc, \delta, \rho, f$ is contained in a closed subspace. More precisely, they are two families of such objects over the gluing parameters. Therefore, $W'^0, Loc, \delta, \rho, f$ acquires a product structure from $W'^0, Loc, \delta, \rho, f$. With respect to this product structure, we need to show that $s'^0$ is smooth even along the directions.
of gluing parameters. This can be done. Its detail will be given somewhere else. Here we proceed differently by combining the idea here with the tautological chart before.

To this end, we give $W_{\delta_1}^0$ a different coordinate chart. Roughly speaking, $W_{\delta_1}^0$ can be realized as a disc bundle over $W_{\delta}^0$ with each fiber being one of copies of the orthogonal complement space above. We give the tautological chart introduced before for each fiber. Hence, the desired coordinate chart for $W_{\delta_1}^0$ is a family of tautological charts over $W_{\delta}^0$. Note that the base $W_{\delta}^0$ has a "natural" product structure, so does each fiber tautologically. Let us spell out a bit more on the new fiberwise tautological chart. It is defined by using the map $\Phi : W_{\delta_1}^0 \to W_{\delta}^0 \times \mathcal{L}_{\delta_1,f}^0$ defined below. The domain of the chart is the open subset $\Phi^{-1}(B\mathcal{L}_{\delta_1,f})$ in $W_{\delta_1}^0$, where $(B\mathcal{L}_{\delta_1,f})$ is the ball of radius $r$ centred at zero of the fiber of $\mathcal{L}_{\delta_1}$ at the center $f$. As before, in the local chart $W_{\delta_1}$ and local trivialization of $\mathcal{L}_{\delta_1}^0 \simeq W_{\delta_1}^0 \times \mathcal{L}_{\delta_1,f}^0$, the $\bar{\partial}$-section, $s_{\delta_1}$, becomes a map $[s_{\delta_1}^0] : W_{\delta_1}^0 \to \mathcal{L}_{\delta_1,f}^0$. Under the assumption that $s^0$ is transversal to zero section at $f$, $\Phi$ is defined as $f$

The tautological chart is defined by using the map $\phi : W_{\delta_1}^0 \to E\mathcal{M}_{\delta_1}^0 \times \mathcal{L}_{\delta_1,f}^0$. Here for any $g \in W_{\delta_1}^0$, $\phi(g) = (\pi_{E\mathcal{M}}(g), \pi_{\mathcal{L}}^* [s_{\delta_1}^0](g))$, where $E\mathcal{M}_{\delta_1}^0$ is obtained from $E\mathcal{M}_{\delta_1}$ by the above taking "prime" process, and $E\mathcal{M}_{\delta_1}^0$ is the enlarged local moduli space centred at $f$ given by the inverse image $[s_{\delta_1}^0]^{-1}(R(f))$. Here (a) $R(f)$ is the cokernel $(Ds^0_{\delta_1})_f$; (b) $R(f)^\perp$ is the orthogonal complement of $R(f)$ in $\mathcal{L}_{\delta_1}^0,f$; (c) $\pi_{\mathcal{L}}^* : \mathcal{L}_{\delta_1,f}^0 \to R(f)^\perp$ is the projection map; (d) $\pi_{E\mathcal{M}} : \phi : W_{\delta_1}^0 \to E\mathcal{M}_{\delta_1}^0$ is the "orthogonal projection".

More precisely, to define the projection map in (d), we have identified $W_{\delta_1}^0$ with its coordinate chart at $f$, $(\text{Exp}_f)^{-1}W_{\delta_1}^0$.

This gives a product structure for $W_{\delta_1}^0$. One can show that $s_{\delta_1}^0$ is smooth at any point and along any direction in this new tautological coordinate chart for $W_{\delta_1}^0$. Moreover it is a Fredholm and transversal (by our assumption) section. Therefore $W_{\delta_1}^0$ is also realized as a graph over $W_{\delta}^0$ in this new setting. Note that this time the graph is a closed smooth Banach submanifold in $W_{\delta_1}^0$ rather than the one that is only smooth for each fixed gluing parameter. Clearly, the desired smoothness of $s^0 = s_{\delta_1}^0 : W_{\delta_1}^0 \to \mathcal{L}_{\delta_1,f}^0$ follows by taking restriction.

Note that all our effort here is to define a product structure for $W_{\delta_1}^0$ at the "center" $f$ along the normal gluing directions at which $f$ changes it topological type. Within a fixed stratum, $s^0 = s_{\delta_1}^0 : W_{\delta_1}^0 \to \mathcal{L}_{\delta_1,f}^0$ is already smooth with respect to the original local product structure, the natural one mentioned before, even in the normal gluing directions. The proof of the smoothness of $s_{\delta_1}^0$ implies that the local product structure here is smoothly compatible the new one for the new ambient space $W_{\delta_1}^0$. Using this one can show that within a fixed stratum the two local product structures on for $W_{\delta_1}^0$ are compatible in term of weak smoothness.
Weakly Smooth Structure on the ambient space with flat ends:

Using above construction, we have obtained a new ambient space with flat ends, denoted by $W$. It is constructed inductively starting with the lowest strata. At each stage, a small neighbourhood of the (relatively) lowest strata is replaced by the flat ends consisting of a compatible system of objects made out of these $W_0$ via a Cech type of construction. On the overlaps with higher strata, these $W_0$ and their related objects are natural embedded into the corresponding objects obtained from the original locally finite covering for the higher strata. Note that the coverings here are the ones for open strata without any gluing parameter involved and the embedding is only topological. After removing these overlaps from the higher strata, we complete one step of induction, and continue in this fashion. Therefore, to complete the construction of $C^{m_0}$-smooth moduli space of perturbed $J$-holomorphic curves, it remain to show that (i) there are still enough "geometric" perturbations that are weakly smooth to achieve the transversality; (ii) there are still enough weakly smooth functions in the "structure sheaf" of $W$ to endure a genuine $C^{m_0}$-smooth structure for the compatible collection of zero loci of the perturbed local $\bar{\partial}$-sections; (iii) there still exists $C^{m_0}$-smooth cut-off functions to localise all relevant sections.

By lifting the relevant object into the higher level coordinate chart introduced before, these properties can be established in a relatively simple manner. We indicate the main idea here.

Clearly, the evaluation maps at the points away from $\gamma$-discs and its associated $\gamma$-necks with $\gamma >> \delta$ can be lifted into the corresponding higher chart of the form $\hat{W}_\gamma(f)$. Since these charts are insensitive to gluing parameter. We conclude that the "structure sheaf" of $W$ at least still contains all such evaluation maps. This will be sufficient for (ii).

To establish (i), note that the "geometric" perturbations $\nu$ defined in this paper can be generalized to including the case considered here in such a way that the value $\nu(f)$ at a $L^p_k$-map $f : \Sigma \to M$ has a support outside.

One of the key steps to establish these properties is to understand when a smooth function or section defined on a local flat chart can be transformed into a $C^{m_0}$-smooth function or section in another local flat chart such that when the later flat chart is embeded into the original chart higher strata.

We will generalize the work on weak smoothness in Sec.2 and Sec.3 to the cases used above for constructing Fredholm theory in [L2] and [L3]. Combining these together we get the following results on the smoothness of the moduli spaces in [L2] and [L3].

**Theorem 1.7** Let $s$ be the $\bar{\partial}_J$-section. Assume that all isotropy groups are trivial. Let $B(r) = \oplus_{i \in I} B(r_i)$ be the ball of "radius" $r$ in $R = \oplus_{i \in I} R_i$=collection of all compatible perturbations $\nu = \{\nu_i, i \in I\}$. Here $\nu_i \in B(r_i)$ is a section of the bundle over a local flat chart $W_i = W_{i,loc,\delta}(f_i)$, supported on $W_i, i \in I$ and $\{W_i, i \in I\}$ covers $\mathcal{M}(A,J)$. These sections $\nu = \{\nu_i, i \in I\}$ are obtained from the cokernel either (i) as "constant" extensions given by [LT], or (ii) as
"geometric" extensions given by this paper. In the first case, they are weakly smooth of class \( C^1 \), while in the second case, they are of class \( C^{m_0} \).

Then for \( r \) small enough, each extended local moduli space \( EM'(J, A) = (s + ev(i, B(r)))^{-1}(0) \) is either (i) a submanifold of class \( C^1 \) in \( R \times W_i \), or (ii) a submanifold of class \( C^{m_0} \).

Moreover, these local extended moduli spaces patch together to form a manifold, denoted by \( EM(J, A) \) of class \( C^1 \) or \( C^{m_0} \) respectively.

As mentioned before to get the required smoothness for the perturbed moduli space, we have to deal with the above two cases separately.

• Case (i): In this case, Sard theorem is not applicable, we give \( EM(J, A) \) a compatible \( C^\infty \) structure first, then deform the \( C^1 \) projection map \( \pi' : EM(J, A) \to R \) slightly into a \( C^\infty \)-map \( \pi \).

Then we have the following theorem for case (i).

**Theorem 1.8** For generic choice of \( \nu \) in \( B(r) \) with \( \|\nu\| \) small enough, the perturbed moduli space \( M^{\nu,\pi}(J, A) = (\pi)^{-1}(\nu) \) is a compact \( C^\infty \)-manifold whose cobordism class is well defined independent of all the choices made.

The proof of this is in \([L2]\).

• Case (ii): We may assume that \( m_0 > \) the index of \( Ds_f \). Then Sard’s theorem is applicable to \( \pi' \). We have the following theorem in \([L3]\).

**Theorem 1.9** For generic choice of \( \nu \) in \( B(r) \) with \( \|\nu\| \) small enough, the perturbed moduli space \( M^{\nu}(J, A) = (s + \nu)^{-1}(0) \) is a compact \( C^{m_0} \)-manifold.

In this paper we will also discuss in detail a few topics that may be considered as "side" issues since they are not directly related to the main theme, the lack of differentiability discussed here. These include the proof of the compactness of the perturbed moduli space; the existence of a positive lower bound for \( \|\bar{\partial}J\|_{k-1,p} \) on the boundary of a \( L^p_k \)-neighbourhood of the moduli space inside the ambient space; the sufficiency for transversality of the extended and perturbed moduli space inside the \( L^p_k \)-neighbourhood when the space of perturbations is only of finite dimensional coming from the localized geometric sections derived from the cokernels. In particular, for the last issue, unlike polyfold theory where there are much more abstract perturbations that can be used to achieve transversality, there is a possibility that in our case, we may get into Zeno type paradox. Perhaps all these issues are obvious to the experts. However, as these "side" issues are quite of general nature, we give a complete treatment on them in this paper. Of course, there are overlaps of the discussions here with \([LT]\), and our treatment here is not necessarily simpler mainly because of the last
side issue. Note that while most of the results in Sec. 3 are only applicable for the moduli space in GW and Floer type theories with only one stratum, our treatment for these side issues in Sec. 3 works for the general cases.

This paper is organized as following.

Sec. 2 introduces the notion of weak smoothness as a general framework to overcome the difficulty (A).

Sec. 3 establishes the $C^{m_0}$-smoothness of the perturbed moduli space stated in the main theorem.

The argument in this section with some modifications is applicable to the other cases in GW and Floer type theories. In fact, after a quick review on some of notations in GW theory, we immediately start to use "generic" terminologies and notations in GW and Floer theories. Therefore, almost all the arguments and results are applicable to the part of moduli spaces in GW and Floer theories that have a fixed stratum with trivial isotropy groups.

In Sec. 4 collects some well-known analytic facts. In particular, we include an elementary proof of the smoothness of the $p$-th power of $L^p_k$-norms of the Sobolev spaces when $p$ is an even positive integer.

This paper and subsequent ones are written based on author’s notes. The electronic files of part of these notes are listed as [C] and [L1] in the reference. All ideas and almost all results in this paper are already written in those notes.

2 Weakly Smooth Banach Manifolds and Bundles

In this section, we introduce the notion of weakly smoothness structure associated with a covering on a topological Banach manifold. We start with a topological Banach manifold $B$ and Banach bundle $L \to B$.

Fix a covering $\mathcal{U} = \{U_i, i \in I\}$ of $B$ and the collection of locally trivialized bundles $\{L_i \to U_i, i \in I\}$. They will be called a collection of admissible charts and trivializations. We define a function on an open set of $B$ to be weakly smooth with respect to $\mathcal{U}$ if it is smooth viewed in any admissible coordinate charts of $\mathcal{U}$. The germs of such weakly smooth functions gives rise the weakly smooth structure $\mathcal{O}$ on $B$. The weakly smooth sections of $L$ can be defined similarly by using the local bundles $L_i$.

The real question is if this obvious notion of weak smoothness is useful.

We will show in this and subsequent papers that in the case of GW and Floer type theories, the stratified topological manifolds appeared there do have enough smooth functions and section so that they behave as if they are honest smooth manifolds. Moreover, any finite dimensional weakly submanifold in such a manifold with induced weakly smooth structure is in fact a honest smooth one. This gives a way to have perturbed $C^{m_0}$-smooth moduli spaces in GW and Floer
theories that do not lie inside the space of smooth stable maps, as we mentioned in the introduction.

There are several immediate questions on the above definition of weakly smooth functions and sections associated with the covering $U$: (I) To what extent does this notion depend on covering $U$? Since we only have a topological Banach manifold, we expect that there are not too many weakly smooth functions with respect to the fixed covering. It is desirable to have more weakly smooth functions. On the other hand, replacing $U$ by a compatible refinement does give rise more smooth functions with respect to new covering. This suggests that at least one should incorporate the effect of compatible refinement in order to formulate the notion of weakly smoothness. (II) What is the functorial behaviour of this notion? Is it possible to form a category of weakly smooth Banach manifolds so that the usual functorial constructions in smooth category can be performed?

The answer to these and related question motivates the definition below.

- Weakly smooth Banach manifolds and bundles:

We mimic the usual way to use ringed space to specify analytic or algebraic-geometric structures on topological spaces by gluing the corresponding local structures. We start with the formal definition for the weak smooth structure on $B$. Then we define the notion of weakly smooth sections of subbundles of $L$. Those subbundles will be used as obstruction bundles to perturb the Fredholm section $s$ to achieve the transversality.

Let $B$ be a paracompact topological Banach manifold locally modelled on a separable Banach space $E$. Assume that it is covered by open sets $B = \cup_i U_i$ with coordinate chart $\phi_i : U_i \to W_i$, where $W_i$ is a open set of $E$.

The collection $(U, \Phi) = (\{U_i, i \in I\}, \{\phi_i : U_i \to W_i, i \in I\})$ forms the fixed covering data for $B$.

On each $U_i, i \in I$, we consider the "structure sheaf" $\mathcal{O}_i$ which is "admissible" by the "induced" covering data of $(U, \Phi)$ on $U_i$. More specifically, each $\mathcal{O}_i, i \in I$ is a sheaf of subrings consisting of germs of smooth functions on $U_i$ with respect to the "induced" covering data. Here the "induced" covering data on $U_i$, denoted by $(U, \Phi)(U_i)$ is just the restriction of $(U, \Phi)$ to $U_i$. As usual, the choices of the subrings depend on (or give) the "structures" that we want to define associated to the covering. For our purpose here, the obvious canonical choice for $\mathcal{O}_i$ is the one consisting of germs of all consistent smooth functions on $U_i$. More precisely, any element in $\mathcal{O}_i$ defined on an open subset $W_i$ of $U_i$ is weakly smooth if and only if it is smooth viewed on any open subset $W_i \cap U_j, j \in I$ after composing with the corresponding transition function. Strictly speaking, we should write the structure sheaf $\mathcal{O}_i$ on $U_i$ as $\mathcal{O}_i((U, \Phi))(U_i))$ to indicate its dependence on the induced covering data. Once each structure sheaf $\mathcal{O}_i$ is obtained, the structure sheaf on $B$ associated to the covering data $(U, \Phi)$, denoted by $\mathcal{O}_U$, can be obtained by the "gluing" these $\mathcal{O}_i, i \in I$ together in the usual manner.

Now assume that $U'$ is a "compatible" refinement of the covering $U$ given by
$i : \mathcal{U}' \to \mathcal{U}$ in the sense that each coordinate chart $\phi'_j : U'_j \to B$ is the restriction of $\phi_{i(j)} : U_{i(j)} \to B$. Clearly the obvious pull-back map $i^* : \mathcal{O}_\mathcal{U} \to \mathcal{O}_{\mathcal{U}'}$ makes $\mathcal{O}_\mathcal{U}$ a subsheaf of $\mathcal{O}_{\mathcal{U}'}$, and it is a strict subsheaf in general. In other other words, there are more weakly smooth functions with respect to $\mathcal{U}'$ in general.

The collection of all possible coverings of $B$ that are compatible refinements of a fixed $\mathcal{U}$ with respect to the partial order given by inclusion form a "directed" set and the collection of locally finite ones are cofinal. We denote the collection by $[\mathcal{U}]$. We define the structure sheaf $\mathcal{O}_{[\mathcal{U}]}$ of the weakly smooth structure on $B$ to be the inverse limit of $\mathcal{O}_\mathcal{U}$ with respect to all $\mathcal{U}$-compatible locally finite coverings on $B$. In summary, an element in $\mathcal{O}_{[\mathcal{U}]}$ is represented by a function $f$ defined on an open set of $B$ such that there is a locally finite covering which is a compatible refinement of $\mathcal{U}$ so that $f$ is smooth after pulling back to each admissible coordinate chart of the covering.

The weakly smooth structure $\mathcal{O}$ on $B$ so defined are obtained by selecting all possible compatible smooth sections of the trivial bundle $B \times \mathbb{R}^1 \to B$. We define the notion of weakly smooth sections for a topological Banach bundle $V \to B$ in a similar way. Here we assume that locally over each $U_i, i \in I$, $V_i = V|_{U_i}$ has a fixed trivialization so that it makes sense to talk about local smooth sections of $V$ over $U_i, i \in I$ with respect to the covering $\mathcal{U}$ and local trivializations. Let $V_{\mathcal{U}_i}$ be the maximum of compatible $\mathcal{O}_i$-modules $V_i, i \in I$, where each $V_i$ is a sheaf of submodules over $\mathcal{O}_i$ consisting of the germs of smooth sections of $V_i$. Any global section of $V$ over an open set $W$ is called weakly smooth if it is a smooth section viewed in any "admissible" local bundles $V_i, i \in I$ for some compatible covering in $[\mathcal{U}]$.

As before, strictly speaking in the above definition of the $\mathcal{O}_B$-module, $V$, in addition to the covering data on $B$, we should also specify more carefully on the data for bundles.

Here is another reason that we need to introduce the compatible refinements to a covering.

A continuous map $F$ between $(\mathcal{B}', \mathcal{U}', \Phi')$ and $(\mathcal{B}, \mathcal{U}, \Phi)$ is said to be weakly smooth if $F^*$ maps $\mathcal{O}_{[\mathcal{U}, \Phi]}$ into the corresponding one with respect to the compatible refinement of $(\mathcal{U}', \Phi')$ by intersecting it with $F^{-1}\mathcal{U}$. Clearly, $F$ induces a map on the inverse limits. Therefore, the collection $(\mathcal{B}, \mathcal{O}_{[\mathcal{U}, \Phi]})$ with the morphism just defined form a category of weakly smooth Banach manifold with a specified compatible weakly smooth charts $[\mathcal{U}, \Phi]$. The equivalence between two such objects are defined in the obvious way. In particular, each $\mathcal{O}_{[\mathcal{U}, \Phi]}$ determine an equivalent class, the weakly smooth structure associated to the covering data $[\mathcal{U}, \Phi]$. In other words, when the covering data $[\mathcal{U}, \Phi]$ of $B$ is fixed, the effect of equivalent class is to collecting all possible "admissible" coordinate charts not just the ones that are compatible refinements of the given covering data.

This can be done similarly for the $(V \to B)$ to get the corresponding category of $\mathcal{O}$-modules. The functorial nature of these constructions makes it possible to define various familiar notions. We mention a few relevant ones.

There is a well defined cotangent functor and hence, a corresponding tangent
functor, in the category weakly smooth Banach manifolds. On any $U_i$ of the admissible cover of $B$, the cotangent sheaf $T^*U_i$ is simply defined to be the $\mathcal{O}|_{U_i}$-modules generated by the derivatives $d\phi$, for a weakly smooth function $\phi \in \mathcal{O}(W)$, where $W$ is an open subset of $U_i$. Unlike the category of smooth Banach manifolds, in which the cotangent functor gives rise Banach bundles, in the case of weakly smooth Banach manifolds, the cotangent functor only gives a sheaf of modules over $\mathcal{O}_B$ in the sense defined above in general.

To justify our definition, we will call a weakly smooth structure on $B$ effective if any finite dimensional topological submanifold of $B$ with the induced weakly smooth structure from $B$ is a honest smooth manifold. In other words, if a weakly smooth structure is effective, then there are enough smooth functions on $B$ to detect a finite dimensional object in $B$. In next section, we will show that the weakly smooth structures appeared in GW are effective in the context of this paper.

Therefore, if the weakly smooth structure on $B$ is effective, then for any finite dimensional topological, hence smooth submanifold $M$ of $B$, the restriction the cotangent bundle to $M$ is just the usual cotangent bundle $T^*M$. Of course, as usual, once $T^*B$ is defined, the sheaf of differential forms $\Omega^*(B)$ is well defined.

It is again a functor on the category of weakly smooth Banach manifolds. Consequently, integration of differential forms over a finite dimensional submanifold in $B$ is well defined.

\section{Weakly Smooth Structure on the Ambient Space of the Moduli Space of $J$-holomorphic Maps.}

Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$ with $\omega$ compatible almost complex structure $J$. Denote the associated metric by $g_J = \omega(-, J-).$ Assume that all geometric data above are of class $C^\infty$. Fix an effective second homology class $A \in H_2(M, \mathbb{Z})$. Let $(\Sigma, i) = \mathbb{P}^1$ with its standard complex structure and metric.

Recall that a map $f: (\Sigma, i) \to (M, J)$ is said to be $J$-holomorphic of class $A$ if (1) the homology class of $f$, $[f]$ is equal to $A$; (2) the map $f$ satisfies $\bar{\partial} J f = 0$. Here the operator $\bar{\partial}$ is defined by $\bar{\partial} u = du + J(u) \circ du \circ i$.

Let $\mathcal{M}(A)$ be the collection all such $J$-holomorphic maps. The virtual dimension of $\mathcal{M}(A)$ can be calculated by the index of the linearization $D\bar{\partial}J|_u$ at any $u \in \mathcal{M}(A)$ and is equal to $2c_1(A) + 2n$. The group $G = \text{PSL}(2, \mathbb{C})$ acts on $\mathcal{M}(A)$ as the group of reparametrizations. Under the assumption that $A$ is effective, each element in $\mathcal{M}(A)$ is stable in the sense that it has no infinitesimal automorphism. Consequently, in this case, the action $G$ is proper with finite isotropy group. For the purpose of this paper, we assume that the isotropy groups are trivial for all elements in $\mathcal{M}(A)$. Denote the quotient space of the unparametrized $J$-holomorphic maps of class $A$ by $\mathcal{M}'(A)$. Assume that it is
compact. This is the moduli space that we want to regularize by perturbing the defining section \( \partial_J \). We will outline a proof in this section that in this situation, the resulting perturbed moduli space is a \( C^{m_0} \)-smooth manifold, where \( m_0 = k - \frac{2}{p} \) is the Sobolev differentiability of the ambient space \( B(A) \) of stable \( L^p_k \)-maps of class \( A \).

Our next goal is to describe the weakly smooth structure on \( B(A) \) and to show that there are enough weakly smooth functions to make it effective. To this end, we need to give more details on \( B(A) \) first. To simplify our notation, from now on, we will drop the "\((A)\)" in all the notations and write \( B(A) \) as \( B \), etc.

Note that each \( u \) in \( \tilde{M} \) has some point \( x_0 \in \Sigma \) such that \( u \) is a local embedding near \( x_0 \). This actually implies that \( u \) has no infinitesimal automorphisms and hence stable. It also implies that we can find a local slice for the \( G \)-action near \( u \) by using a local hypersurface \( H_u \) of codimension 2 transversal to the local image of \( u \) near \( x_0 \).

Now embed \( \tilde{M} \) into \( \tilde{B} \), the space of parametrized stable \( L^p_k \)-maps of class \( A \). Here \( \tilde{B} = \tilde{B}_{k,p}(A) \) is defined to be the collection of all maps \( u : \Sigma \to M \) of class \( L^p_k \) measured by the metrics \( g_J \) on \( M \) and the standard one on \( \Sigma \) such that (i) \([u] = A\); (ii) \( u \) is stable. In [L1], the stability for \( L^p_k \)-maps is formulated in more general setting. Here for simplicity, we assume that each element \( u \in \tilde{B} \), like \( J \)-holomorphic ones, has a local embedding point \( x_0 \). Again, we will assume that the stabilizer of \( u \) is trivial. These assumptions imply that the \( G \)-action on \( \tilde{B} \) has local slices.

Here we assume that the \( m_0 = k - \frac{2}{p} > 1 \) so that each element in \( \tilde{B} \) is at least of class \( C^1 \). Another assumption that we use in rest of the paper is that \( p \) is a positive even integer. This assumption implies that the \( p \)-th power of the \( L^p_k \)-norm is a smooth function on \( L^p_k(\Sigma, f^* (TM)) \).

It is well-know that when the Sobolev index \( m_0 = k - \frac{2}{p} \) is greater than zero, the notion of an element \( f \) being in \( L^p_{k,loc} (\mathbb{R}^m, \mathbb{R}^m) \) is invariant with respect the local differomorphisms of both the domain and target. This implies that \( \tilde{B} \) is well-defined. To specify the topological and smooth structures on \( \tilde{B} \), we introduce local coordinate charts for \( \tilde{B} \).

• The local coordinate of \( \tilde{B} \):

For any \( f \) in \( \tilde{B} \), when \( \epsilon \) is small enough, we have a local coordinate chart \( Exp_f : U_\epsilon(f) \to \tilde{B} \) for an \( \epsilon \)-ball \( U_\epsilon(f) \) in \( L^p_k(\Sigma, f^* (TM)) \). Here \( Exp_f \) is defined to be \( Exp_f (\xi) (x) = exp_f (x) \xi(x) \) for \( \xi \in U_\epsilon(f) \) and \( x \in \Sigma \), and \( exp_y : T_y M \to M \) is the exponential map of \( M \). It is well-know in Gromov-Witten and Floer theories that smoothness of exponential map in \( M \) implies that the transition functions between these coordinate charts are \( C^\infty \) even although the elements in \( \mathcal{M}_{k,p}(\Sigma, M) \) are only in \( L^p_k \). This makes \( \tilde{B} = \tilde{B}^0_k(\Sigma, M) \) a \( C^\infty \) Banach manifold.

Next Lemma is well-known in Gromov-Witten theory. We include the simple proof here for completeness.
Lemma 3.1 For any fixed $x = (x_1, \ldots, x_l)$ in $\Sigma^l$ with $x_i \neq x_j$ for $i \neq j$, define the evaluation map $e_x : \tilde{B} \to M^l$ by $e_x(f) = (f(x_1), \ldots, f(x_l))$ for any $f \in \mathcal{M}$. Then $e_x$ is a $C^\infty$ submersion.

Proof: Recall that for any $f \in \tilde{B}$, we have the coordinate chart $\text{Exp}_f : U_r(f) \to \tilde{B}$ for an $\epsilon$-ball $U_r(f)$ in $L^p(\Sigma, f^*(TM))$, and $\text{Exp}_f(\xi)(x) = \exp_f(\xi(x))$ for $\xi \in T_x U_r(f)$ and $x \in \Sigma$. Similarly we introduce the coordinate chart $\text{exp}_y : U^i_r(y) \to M^l$ for an $\epsilon$-neighbourhood $U^i_r(y)$ of the zero in $T_y M^l$. It is defined by $\text{exp}_y(\eta) = (\exp_{y_1}(\eta_1), \ldots, \exp_{y_l}(\eta_l))$ for any $\eta = (\eta_1, \ldots, \eta_l)$ in $T_y M^l$ with $y = (y_1, \ldots, y_l)$ in $M^l$. With respect to these coordinate charts of $\tilde{B}$ and $M^l$, with $y = f(x)$, the evaluation map $e_x$ has the form $\text{Exp}_f^{-1}(\text{exp}_y)(\xi(x_1), \ldots, \text{exp}_f(\xi(x_l))) = (\xi(x_1), \ldots, \xi(x_l))$. It is induced from a linear map from $L^p(\Sigma, f^*(TM))$ to $T_{f(x)} M^l$. Under the assumption that $m_0 > 0$, it is continuous linear map, hence smooth. The surjectivity of $\text{Exp}_x$ at $f$ follows from the fact that for any $\gamma = (\gamma_1, \ldots, \gamma_l)$ with $\gamma_i \in T_{f(x_i)}$ for $i = 1, \ldots, l$, there exits a $\xi$ in $L^p(\Sigma, f^*(TM))$ such that $\xi(x_i) = \gamma_i$ for $i = 1, \ldots, l$. \hfill $\square$

Lemma 3.2 For any fixed $g \in G$, $\Psi_g : \tilde{B} \to \tilde{B}$ is smooth. Here $\Psi_g$ is the action map for a fixed $g \in G$ and defined by $\Psi_g(f) = h \circ g$.

Proof: Let $h = \text{Exp}_f(\xi)$ for $\xi$ in $U_r(f) \subset L^p(\Sigma, f^*(TM))$. Then $\Psi_g(h) = h \circ g = \text{Exp}_{f \circ g}(\xi \circ g) = \text{Exp}_{f \circ g}^g(\xi)$ in $U_r(f \circ g) \subset L^p(\Sigma, (f \circ g)^*(TM))$. Therefore, in the local coordinate charts $U_r(f)$ and $U_r(f \circ g)$, $\Psi_g$ can be identified with $\Psi_g : L^p(\Sigma, f^*(TM)) \to L^p(\Sigma, (f \circ g)^*(TM))$ defined by $\Psi_g(\xi) = g^*(\xi)$, which is just the pull-back on sections. Clearly $\Psi_g$ is linear and continuous, and hence smooth. \hfill $\square$

Proposition 3.1 The action map $\Psi : G \times \tilde{B} \to \tilde{B}$ composed with the evaluation map $e_x$, denoted by $\Phi_x : G \times \tilde{B} \to M^l$, is of class $C^{m_0}$.

Proof: This follows from the $sc^k$-smoothness of the action map. A direct proof was given in [L1] and will be given in [L3].

Note that it follows from above two Lemmas that $\Phi_x$ is $C^\infty$ along $\tilde{B}$-direction and is at least of class $C^{m_0}$ along $G$ direction by Sobolev embedding theorem. The only question is about the smoothness of the mixed derivatives. This can be reduced as follows. Note that $\Phi_x : G \times \tilde{B} \to M^l$ is the composition of the follow tow maps: $\Psi_x : G \times \tilde{B} \to \Sigma^l \times \tilde{B}$ given by $\Psi_x : (g, f) = (g(x), f)$ and

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$ev^i : \Sigma^i \times \tilde{B} \to M^i$ given by $ev^i(y,f) = f(y)$. The first map is obvious smooth. The total evaluation map $ev^i$ is $C^\infty$-smooth along $\tilde{B}$-direction, and it is $C^m$-smooth along $\Sigma^i$-direction by Sobolev embedding theorem. Again the question is about the mixed derivatives.

\[ \square \]

- The local slice of the $G$ action on $\tilde{B}$:

The argument here is only outlined. The more details for more general cases were in [L] and will be given in [L3]. For simplicity, we only consider the case that $\mathcal{B}$ is a small neighbourhood of the moduli space $\mathcal{M}$ of stable $J$-holomorphic maps. Given a stable $J$-holomorphic map $f$ in $\tilde{B}$ with trivial isotropy group, consider its $G$-orbit $O_f$ as an injective map $O_f : G \to \tilde{B}$. The following Lemma was proved in [L] and the detail will be given in [L3].

**Lemma 3.3** The condition above implies that there are open neighbourhoods $V$ of $e$ in $G$ and $W(f)$ of $f$ in $\tilde{B}$ such that for any point $g$ in $G \setminus V$, $O_f(g)$ is not in $W(f)$. Here we require that $V$ is pre-compact.

Moreover, when $W(f)$ is small enough, for any $h$ in $W(f)$, the same is true for the orbit map $O_h$ with a fixed $V$ independent of $h$.

The lemma implies that we can replace the $G$-space $\tilde{B}$ by $W(f)$, and we only need to get local slice for the action from the pre-compact set $V$ on $W(f)$.

To this end, fix a point $p = (p_1, p_2, p_3)$ on $\Sigma^3$ such that $f$ is a local $C^1$ embedding near $p_i$, $i = 1, 2, 3$. We assume further that each $p_i$ is an injective point. Let $H = H_f = (H_1, H_2, H_3)$ and $H_i$ be a local hypersurface of codimension 2 at $f(p_i)$ transversal to $f$ locally. The assumption implies that when $H$ and $W(f)$ are small enough, there are open discs $D_i$ centred at $p_i$ such that for any $h$ in $W(f)$, (i) $h(\Sigma \setminus D_i)$ does not intersect $H_i$; (ii) the restriction of $f$ to each $D_i$ intersects with $H_i$ transversally only at $f(p_i)$.

To define the local slice, note that since $ev_p : \tilde{B} \to M$ is a $C^\infty$ surjective map, it follows from the implicit function theorem that when $\epsilon$-small enough, the inverse image $W_\epsilon(f,H) = W_\epsilon(f) \cap (ev_p)^{-1}(H)$ is a $C^\infty$-submanifold of the open set $W_\epsilon(f) = \text{Exp}_f(U_\epsilon(f))$. Since the action of each element $g$ in $V \subset G$ on $h$ is determined by $h \circ g(p_i)$, the above conditions imply that $W_\epsilon(f,H)$ is indeed a slice of the $G$-action on $W_\epsilon(f)$. Another way to get a local slice is the following. Note that although the $G$-actions are only continuous, the map $\Phi_p : G \times W_\epsilon(f) \to M$ is at least of class $C^1$ and is transversal to $H$ at $(e,f)$ even restricting to $G \times \{f\}$ by the local injectivity assumption on $p$. The same is true for the restriction of $\Phi_p$ to $G \times W_\epsilon(f,H)$. By implicit function theorem, this at least implies that $W_\epsilon(f,H)$ is a slice for the action of a small neighbourhood of $G$ on $W_\epsilon(f)$ when $\epsilon$ is small enough. The fact that the stabilizer of $f$ is trivial implies that it is a slice for the $G$-action on $W_\epsilon(f)$.

Therefore, the quotient space of $\tilde{B}$ by the $G$-actions, denoted by $\mathcal{B}$, locally is modelled on an open ball $U_\epsilon(f,h)$ in the Banach $L^0_k(\Sigma, f^*(TM), h)$. Here
$L^p_k(\Sigma, f^*(TM), h)$ is the closed subspace of $L^p_k(\Sigma, f^*(TM))$, whose element $\xi$ is subject to the condition that $\xi(p)$ is in $h$, which is the tangent space of $H$ at $f(p)$.

More precisely, it follows from the implicit function theorem that in the local coordinate $U_\epsilon(f)$, the inverse image of the local slice $Exp_f^{-1}(W_\epsilon(f, H))$ is realized as a graph of a function from the open ball $U_\epsilon(f, h)$ in $L^p_k(\Sigma, f^*(TM), h)$ to the $L^2$ orthogonal complement of $L^p_k(\Sigma, f^*(TM), h)$ in $L^p_k(\Sigma, f^*(TM))$. Note that $L^p_k(\Sigma, f^*(TM), h)$ is the tangent space of $W_\epsilon(f, H)$ at $f$. We may assume that $H$ are geodesic submanifolds so that $Exp_f^{-1}(W_\epsilon(f, H))$ becomes the open ball $U_\epsilon(f, h)$ in $L^p_k(\Sigma, f^*(TM), h)$ and $Exp_f : U_\epsilon(f, h) \rightarrow W_\epsilon(f, H))$ is a local coordinate chart (slice) for $B$.

To see that $B$ is a topological Banach manifold, we need to find the transition function between two local slices $W_\epsilon(f_1, H_1)$ and $W_\epsilon(f_2, H_2)$. We will abuse notation by writing the transition function as $t_{21} : W_\epsilon(f_1, H_1) \rightarrow W_\epsilon(f_2, H_2)$.

**Lemma 3.4** There is a $C^{m_0}$-smooth function $T_{21} : W_\epsilon(f_1, H_1) \rightarrow G$ such that $t_{21}(f) = f \circ T_{21}(f)$.

**Proof:**

For simplicity and without lose the generality, we may assume that $f_1$ and $f_2$ are in the same $G$-orbit with $f_2 = f_1 \circ g_0$. In fact, one can reduce further by assuming that $g_0 = e$ so that we are looking at the same neighbourhood $W_\epsilon(f)$ with two different slices centered at $f = f_1 = f_2$. Let $O(W_\epsilon(f))$ be the open set of the orbit of $W_\epsilon(f)$. Then $\Psi : G \times W_\epsilon(f_1, H_1) \rightarrow O(W_\epsilon(f))$ is a homeomorphism, which induces another (incompatible) smooth structure on $W_\epsilon(f)$. By the Lemma above, $\Phi_{p_2} = ev_{p_2} \circ \Psi : G \times W_\epsilon(f_1, H_1) \rightarrow M$ is of class $C^{m_0}$. With respect this new smooth structure, $\Phi_{p_2}$ is transversal to $H_2$ along $G$-direction at the point $(f_1, e)$. It follows from the implicit function theorem that the inverse image $\Phi_{p_2}^{-1}(H_2)$ is a $C^{m_0}$-smooth submanifold of $G \times W_\epsilon(f_1, H_1)$ and it can be realized as the graph $\{(T_{21}(\xi), \xi) \in W_\epsilon(f_1, H_1)\}$ for a function $T_{21} : W_\epsilon(f_1, H_1) \rightarrow G$. Clearly, $T_{21}$ is the function that we are looking for, which is of class $C^{m_0}$.

\[\square\]

Having given $B$ a topological Banach manifold structure, we now specify a weakly smooth structure on this space.

- **Weakly Smooth Structure on $B$:**

  We will take the covering $U$ of $B$ to be the collection of all possible local slices $W_\epsilon(f, H)$, and define $O = O(U)$ to be generated by the germs all functions which are smooth viewed in any $W_\epsilon(f, H)$. Clearly, for any $x = (x_1, \cdots, x_l)$ in $\Sigma^l$, the evaluation map $e_x : W_1 = W_\epsilon_1(f_1, H_1) \rightarrow M^l$ is smooth. Next lemma shows that it is $C^{m_0}$-smooth viewed in any slice $W_2 = W_\epsilon_2(f_2, H_2)$. Therefore, its composition with any smooth function on $M^l$ is a weakly smooth function of
class $C^{m_o}$. In particular by using local coordinate functions on $M^l$, we obtain
a collection of such smooth functions associated to each $e_x$. For our purpose, it
is more convenient to regard $e_x$ as a $M^l$-valued weakly smooth function.

**Proposition 3.2** The evaluation map $e_x : W_1 = W_{e_1}(f_1, H_1) \rightarrow M^l$ is $C^{m_o}$-smooth viewed in any other slices.

**Proof:**

Let $W_2 = W_{e_2}(f_2, H_2)$ be another slice and $t_{12} : W_2 \rightarrow W_1$ be the coordinate
transformation. Then $e_x \circ t_{12} : W_2 \rightarrow M^l$ can be written as $e_x \circ t_{12} = e_x \circ \Psi \circ (T_{12}, Id)$, where $\Psi : G \times \mathcal{B} \rightarrow \mathcal{B}$ is the action map. We have already proved that $e_x \circ \Psi$ is of class $C^{m_o}$. Since $T_{12} : W_2 \rightarrow G$ is of class $C^{m_o}$, so is $e_x \circ t_{12}$. □

Recall that in local coordinates $Exp_f : U_\epsilon(f) \rightarrow W_\epsilon(f)$ and $exp_x : T_x M^l \rightarrow M^l$, the evaluation map $e_x$ has the form $exp_{f(x)}^{-1}( e_x \circ Exp_f(\xi) = (\xi(x_1), \cdots, \xi(x_l))$. That is a just a collection of linear functionals. In particular the derivative $D(e_x)|_f$ at $f$ is given by $D(e_x)|_f(\xi) = (\xi(x_1), \cdots, \xi(x_l)) = 0$.

We now prove one of the main results of this section.

**Proposition 3.3** The weakly smooth structure so defined on $\mathcal{B}$ is effective. That is any finite dimensional weakly smooth submanifold $N$ in $\mathcal{B}$ with respect to any subcovering $U'$ is in fact a $C^{m_o}$-smooth manifold with respect to the induced weakly smooth structure.

**Proof:**

We only give the proof for the full covering $U$. Our assumption implies that for any point $f$ in $N$, there is neighbourhood $V(f) = N \cap W_\epsilon(f, H)$ such that $V(f)$ is a closed smooth submanifold of $W_\epsilon(f, H)$. Then $N$ is covered such
neighbourhoods. Assume that dimension $N = n$. Since any weakly smooth function
on $\mathcal{B}$ is smooth when it is restricted to any of such $V(f)$, the restrictions of
these weakly smooth functions to $N$ give it a smooth structure if we can show
that on each $V(f)$ there are enough such functions to give $V(f)$ a local coordinate
system for sufficient small $\epsilon$. Using those $M^l$-valued evaluation map,
we only need to show that when $l$ is large enough, $e_x : V(f) \rightarrow M^l$ is a local
embedding for suitable choice of $x$. To this end, we only need to show that $D(e_x)|_f : T_f V(f) \rightarrow T_{f(x)} M^l$ is injective. Now $T_f V(f)$ is a $n$-dimensional linear
subspace of $T_f W_\epsilon(f, H) = L_n^\epsilon(\Sigma, f^*TM, h)$ and the derivative $D(e_x)|_f$ is the re-
striction of the corresponding one of for the evaluation map on the ambient space
$W_\epsilon(f, H)$, which is given by the formula $D(e_x)|_f(\xi) = (\xi(x_1), \cdots, \xi(x_l))$. When
$l = 1, x = x_1$ is just a point on $\Sigma$. For proper choices of $x_1$, the condition
that $\xi(x_1) = 0$ will cut the dimension of $T_f V(f)$ at least by one. Therefore for
the proper choices of $x$ with $l \geq n$, the equation $D(e_x)|_f(\xi) = (\xi(x_1), \cdots, \xi(x_l)) = 0$ can only have the trivial solution. This implies that $D(e_x)|_f$ is injective. □
Note:

(i) In applications, the submanifold $N$ is only of class $C^{m_{0}}$ viewed in each admissible charts of $\mathcal{B}$, the conclusion and the proof above remain the same.

(ii) Consider the case that in a given stratum of $\mathcal{B}$, there are coordinate charts of the form $\cup_{\alpha \in \Lambda} W_{\epsilon}(f_{\alpha}, H)$ where $\Lambda$ is the collection of the gluing parameters near a lower stratum. Then there is a similar statement, which includes the case of the regularized moduli space. This implies that the moduli space is a stratified cornered manifold. A proof of this is given in [L2]. In [L2] and [L3], we will show that there is a Fredholm theory for the stratified Banach manifolds appeared in GW and Floer theories so that the induced weakly smooth structure on the perturbed moduli space is in fact $C^{m_{0}}$-smooth rather than just stratified smooth.

- More weakly smooth functions on $\mathcal{B}$:

  Let $M$ be a compact Riemannian manifold and $i : M \to \mathbb{R}^{d}$ be an isometric embedding. Consider the induced embedding $i_{\ast} : \mathcal{B}(M) = \mathcal{B}_{k}^{p}(M) \to \mathcal{B}(\mathbb{R}^{d}) = \mathcal{B}_{k}^{p}(\mathbb{R}^{d})$, where $\mathcal{B}(M)$ and $\mathcal{B}(\mathbb{R}^{d})$ are the spaces of the parametrized $L_{k}^{p}$-stable maps from the domain $\Sigma$ to $M$ and $\mathbb{R}^{d}$ respectively. It is well known that the induced Sobolev metric on $\mathcal{B}(M)$ from the embedding is equivalent to the intrinsic one when $m_{0} = k - \frac{2}{p}$ is large. Note that $\mathcal{B}(\mathbb{R}^{d})$ is an open set of the Banach space $L_{k}^{p}(\Sigma, \mathbb{R}^{d})$. For any $x$ in $\Sigma$, the evaluation map $e_{x} : \mathcal{B}(\mathbb{R}^{d}) \to \mathbb{R}^{d}$ is linear and $M$ is closed in $\mathbb{R}^{d}$. This implies that $e_{x}^{-1}(M)$ is closed for each $x$, and that $\mathcal{B}(M)$ as the intersection of all such inverse images is also closed in $\mathcal{B}(\mathbb{R}^{d})$. Clearly any smooth function $\phi$ on $L_{k}^{p}(\Sigma, \mathbb{R}^{d})$ pulls back to a smooth function function $i_{\ast} \phi$. We are particular interested in those smooth $\phi$ with property that the composition $\phi \circ \Phi : \Sigma \times L_{k}^{p}(\Sigma, \mathbb{R}^{d}) \to \mathbb{R}^{1}$ is still smooth. Here $\Phi : \Sigma \times L_{k}^{p}(\Sigma, \mathbb{R}^{d}) \to L_{k}^{p}(\Sigma, \mathbb{R}^{d})$ is the action map. Any such $\phi$ will be called G-smooth. Since the G-actions are compatible with $i_{\ast}$, if $\phi$ is G-smooth, so is $i_{\ast} \phi$.

- Weakly smooth cut-off functions on $\mathcal{B}$: In the last section, we give an elementary proof of the well-known fact that for $p$ is a positive even integer, the $p$-th power of the $L_{k}^{p}$-norm on $L_{k}^{p}(\Sigma, \mathbb{R}^{d})$, denoted by $\phi_{k,p}$ is a G-smooth function. Composed with a bump-off function $\beta$, we get a cut of function $\beta \circ \phi_{k,p}$ on $L_{k}^{p}(\Sigma, \mathbb{R}^{d})$ denoted by $\psi_{k,p}$. Let $\psi_{k,p}$ be its pull-back to $\mathcal{B}$. Then for any fixed local slice $W_{\epsilon}(f, H)$, the restriction of $\psi_{k,p}$ to the slice is smooth. Using the explicit form of the transition function $T_{f,g} : W_{\epsilon}(g, H) \to W_{\epsilon}(f, H)$ defined before, it is easy to see that $\psi_{k,p} \circ T_{f,g} : W_{\epsilon}(g, H) \to \mathbb{R}^{1}$ is of class $C^{m_{0}}$. This implies the existence of $C^{m_{0}}$-smooth cut-off function on $\mathcal{B}$. Then the standard argument in Lang’s book implies that $\mathcal{B}$ has partition of unit subordinated to any locally finite covering.

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• Weakly smooth functions on $\mathcal{B}$ induced from $G$-linear functionals on $L^q_{-k}(\Sigma, \mathbb{R}^d)$:

As usual, any $\mathbb{R}^d$-valued smooth function $\phi : \Sigma \to \mathbb{R}^d$ induces a continuous linear functional $T_\phi$ on $L^q_{-k}(\Sigma, \mathbb{R}^d)$ by $L^2$-paring. The same elementary argument in last section show that $T_\phi$ is $G$-smooth on $L^q_{-k}(\Sigma, \mathbb{R}^d)$. Therefore its pull-back to $\mathcal{B}$ defines a weakly $C^{m_0}$-smooth function. By completing all such $T_\phi$ in $L^q_{-k}(\Sigma, \mathbb{R}^d)$ and pulling them back to $\mathcal{B}$, we get the corresponding weakly smooth functions of certain regularity induced from $L^q_{-k}(\Sigma, \mathbb{R}^d)$. For instance, when $q' = -k + 1$, the regularity is of class $C^1$. This gives a large supplies of basic $C^r$-smooth function. One can use these functions to prove the $C^{m_0}$-effective smoothness of $\mathcal{B}$ instead of using those evaluation maps.

• The local trivializations of the bundle $(\mathcal{L} \to \mathcal{B})$:

Recall that in GW theory discussed here, the fiber of the bundle $\mathcal{L}_f$ is defined to be $L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$.

To give $\mathcal{L}$ a Banach bundle structure rather than just a family of Banach spaces, we recall one of the standard ways to give it a local trivialization by using the $(J, G_f)$-invariant connection $\nabla = \nabla^M$ of $M$ defined before. Given two points $x$ and $y$ in $M$ sufficient close to each other so that they can be jointed by a unique short geodesic, we will denote the parallel transport of $TM$ from $x$ to $y$ along the connecting geodesic by $\pi_{xy}$. Then a local $C^\infty$-trivialization of $\mathcal{L}$ near a point $f$ can be given by $\Phi : U_r(f) \times L^p_{k-1}(\Sigma, \Lambda^{1}(f^*(TM))) \to \mathcal{L}$. $\Phi(\xi, \eta)(x) = \pi_{f(x)\exp_{f(x)}(\xi(x))}(\eta(x))$ for any $(\xi, \eta) \in U_r(f) \times L^p_{k-1}(\Sigma, \Lambda^{1}(f^*(TM)))$. Note that the parallel transport only acts on the part of $\eta(x)$ involving its "value" in $T_{f(x)}M$ and has no effect on the 1-form part.

By restricting $\Phi$ to the local slices $W(f, H)$, we get local trivializations for the local bundle $\mathcal{L}|_{W(f, H)}$. Note that if two of these local slices $W(f, H)$, being considered as coordinate charts for $\mathcal{B}$, have non-empty intersections, they may not intersect in $\mathcal{B}$ in general. They are only related by some (non-constant) actions of $G$. To define a locally trivial bundle structure for the quotient bundle $\mathcal{L} \to \mathcal{B}$, it is necessary to compare the above local trivializations over these slices that are only related by $G$-actions. To this end, we introduce different coordinate charts and local trivializations for $\mathcal{L} \to \mathcal{B}$, which are $G$-equivariant but only continuous.

• $G$-equivariant local trivialization for the bundle $\mathcal{L} \to \mathcal{B}$.

Note that the local trivialization for $\mathcal{L}$ above is given by parallel transport that has no effect on 1-forms.

This is not adequate for some part of our discussion. We now introduce another system of local chart and local trivialization which behave better with respect to the $G$-action.

We cover $\mathcal{B}$ by the charts $W(f, H) \times G$, and consider it as a topological Banach manifold. With respect to each of these coordinate charts $W(f, H) \times G$,
we define the trivialization of $\tilde{L}$ in the obvious way as follows. Along $W(f, H)$ direction, we trivialize $\tilde{L}$ as before, but along $G$-direction, we simply use the pull-backs induced by the $G$-actions as reparametrizations of the domain to identifying fibres along a $G$-orbit. This gives rise $G$-invariant coordinate charts and $G$-equivariant local trivializations for the bundle $\tilde{L} \to \tilde{B}$. Of course, the $G$-equivariant transition functions between charts and trivializations so defined are only continuous.

By restricting $\tilde{L}$ to local slices $W(f, H)$, or to its $G$-orbits $G \times W(f, H)$, we get a collection of trivial bundles. As we mentioned before, these local bundles together define a bundle $\tilde{L}$ that can be thought as a topological Banach bundle on $\tilde{B}$.

Recall that the transition functions $t_{21}$ between two local slices $W(f_1, H_1)$ and $W(f_1, H_1)$ have been given by using (non-constant) $G$-actions obtained from the $C^m_0$-map $T_{21} : W(f_1, H_1) \to G$. Then transition functions between any two local bundles over these local slices are induced by pull-backs from the actions. This gives the local trivializations for the topological Banach bundle $\tilde{L}$ over $\tilde{B}$.

• Weakly smooth sections of the topological Banach bundle $(\tilde{L}, \tilde{B})$.

Recall that we use the "full" covering $U$ of the collection of all local slices $W(f, H)$ to define the weakly smooth structure on $\tilde{B}$. We now use the same covering to define the $O_B$-module $L$ associated to the bundle $(\tilde{L}, \tilde{B})$. We first describe a way to construct weakly smooth section of $L$. Consider a smooth and $G$-equivariant section $s : \tilde{B} \to \tilde{L}$. Then $s$ restricted to any slice $S(f, H)$ is still smooth, and any two such restrictions are related by a transition function for the bundle $\tilde{L}$. In other words, $s$ can be thought as a section of the quotient bundle $\tilde{L}$, which is smooth viewed in any admissible local trivialization despite of the fact that $G$-action is only continuous. Recall that such a section $s$ was called an weakly smooth section of $\tilde{L}$.

Here are some details on above construction in term of local charts and trivializations.

Let $W_1$ and $W_2$ are two local slices of $\tilde{B}$. The $C^m_0$-coordinate transformation $t_{21} : W_1 \to W_2$ is given by $t_{21}(\xi) = \xi \circ T_{21}(\xi)$. Assume that $s$ is smooth and $G$-equivariant. Then for any $g \in G$, $g^* (s) = (g^{-1})^* (s \circ g) = s$. Let $s_i = s|_{W_i}, i = 1, 2$. After composed with the transition functions between the two coordinate charts and trivializations, the section $s_1$ is transformed into a section in $W_2$, denoted by $s_1'$. Then for any $\eta$ in $W_2$, we have

$$s_1'(\eta) = (T_{21})^* (\eta) s_1(T_{12}(\eta) \cdot (\eta))$$

$$= (g^{-1})^* (s|_{W_1} \circ g) = (g^{-1})^* (s \circ g)|_{g^{-1}(W_1)} = s|_{W_1}.$$

Here $g = T_{12}$. Therefore, $s_1' = s_2$ and hence is of class $C^\infty$.

Note that $T_{12}$ is only of class $C^m_0$. 

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We already mentioned before, in GW and Floer theory, we need to decide if a smooth section defined on a local slice $W_f$ is still sufficient to achieve the local transversality for the perturbed $s$-cut-off function. We will show in this section that the extension $\tilde{\partial}_f: \tilde{B} \to \tilde{L}$ used to define the Floer trajectories, or $\partial_f$-section used to define $J$-holomorphic maps.

- The section $\tilde{\partial}_f: \tilde{B} \to \tilde{L}$.

It is well-know that $\tilde{\partial}_f$ is a $G$-equivariant $C^\infty$-section. Therefore, we get a weakly smooth section of the bundle $L \to B$, denoted by $s$, which is a smooth Fredholm section viewed in any local slice.

The question is if the assumed Fredholm section $s: B \to L$ has enough weakly smooth perturbations to achieve transversality for the moduli space. The lack of the transversality of $s$ at a point $f$ with $f$ in the moduli space $M = s^{-1}(0)$, is measured by the co-kernel of the derivative of $s$ at $f$. Here the derivative is a bounded linear map $Dsf: L^p_k(\Sigma, f^*TM), h) \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(\Sigma) \otimes f^*(TM))$.

The elliptic regularity for the non-linear $\tilde{\partial}_{f,H}$-operator or $\tilde{\partial}_f$-operator implies that $f$ is smooth and the co-kernel $K_f$ is a finite dimensional subspace of $L^p_{k-1}(\Sigma, \Lambda^{0,1}(\Sigma) \otimes f^*(TM))$ consisting of smooth elements. However, as mentioned in the introduction, our construct below does not use the smoothness of $f$.

Each element $\eta$ in $K_f$ can be thought as a section of $L$ over the single point $f$. We need to extend $\eta$ into a weakly $C^{m_0}$-smooth section over $\mathcal{B}$. These extension satisfy the conditions described below.

(I) Let $GE : K_f \to GE(K_f)$ be the desired extension. We require that it is a linear map (in fact it is a linear isomorphism). Therefore, we only need to describe the extension for finitely many elements that form a basis of $K_f$.

(II) Let $W_f$ be a local slice and $[s_f]: W_f \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(\Sigma) \otimes f^*(TM))$ be the section $s$ written in the local slice and trivialization. Assume that $GE(K_f)$ is the extension over $W_f$. Then it gives the evaluation map written in the local trivialization, $[ev]: W_f \times GE(K_f) \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(\Sigma) \otimes f^*(TM))$. We require that $D[s_f]_g \oplus [ev]_g: L^p_k(\Sigma, f^*(TM), h) \otimes GE(K_f) \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(\Sigma) \otimes f^*(TM))$ is "quantitatively" surjective over $W_f$. The exact meaning of the "quantitative" surjectivity will be explained later in this section.

(III) We require that $GE(K_f)$ is localized near $f$. In other words, the support of each element of $GE(K_f)$ is contained in $W_f$. This can be done by multiplying a weakly $C^{m_0}$-smooth cut-off function supported in $W_f$ and equal to one on $W_f' \subset W_f$. Still denote the resulting extension by $EG(K_f)$. Then the transversality condition in (II) is still satisfied over $W_f$.

Note that in (III) we have used the existence of the weakly $C^{m_0}$-smooth cut-off function. We will show in this section that the extension $GE(K_f)$ above is sufficient to achieve the local transversality for the perturbed $s$-section.
As mentioned in the introduction, the desired extension can not be obtained by using the "standard" method by regarding the elements in $K_f$ as "constant" sections over $W_f$, then moving them over the orbit of $W_f$ by pull-backs.

To get the desired extension, fix a $C^\infty$ partition of unit $1 = \beta_i, i \in I$, subordinated to a finite by sufficiently fine (depending on $f$) covering of $\Sigma = \cup_{i \in I} D_\delta_i(x_i)$. Then the cokernel $K_f$ is contained the finite sum $\Sigma_{i \in I} K_i$ as finite dimensional spaces of $L^{p}_{k-1}(\Sigma, \wedge^{0,1}(\Sigma) \otimes f^*(TM))$. Here $K_i = \beta_i K_f$ and $D_\delta_i(x_i)$ is the disc of radius $\delta_i$ on $\Sigma$ centered at $x_i$. We will assume that the covering is fine enough and $W_f$ is small enough such that for any $i \in I$, $g \in W_f$ and $x \in D_\delta_i$, $|g(x) - f(x)| < \epsilon_i$. Since $\|f - g\|_{C^0}$ is bounded by the corresponding $L^p_k$-norm. The assumption can be satisfied. Here $\epsilon_i$ is chosen such that there is a fixed trivialization of $TM$ over the ball of radius $2\epsilon_i$ centered at $f(x_i)$. Note that the image $g(D_\delta_i(x_i))$ is in this ball for all $g \in W_f$. Clearly, we only need to construct the extension for the elements in a basis of $K_i$ $i \in I$.

Now we change the notation. Simply use $K_f$ to denote one of these $K_i$.

Clearly, it is sufficient to show that for each $\eta$ in the basis of $K_f$, there exists an approximated extension $GE(\eta)$ over $W_f$ such that the $L^2$-norm of $(GE(\eta)(f) - \eta)$ is less than a prescribed positive number. This is what we are going to do next. After this we will show that it is possible to give a true extension rather than just an approximate one under the assumption that all the initial data are $C^\infty$-smooth.

Let $\eta$ be a element in the basis of $K_f$. Then it is a finite linear combinations of the elements of the form $\phi \otimes \xi$, where $\phi$ is a smooth $(0, 1)$-form supported in $D_\delta_i(x_i)$ and $\xi$ is a local $L^p_k$-section of the bundle $(f^*(TM) \rightarrow \Sigma)$.

We now show that $GE(\eta)$ can be obtained by combining two kinds of $G$-equivariant sections. Both are "geometric" nature.

- The "geometric" perturbations of the $\bar{\partial}_f$-section:
  
  Note that the fibre of $\tilde{\mathcal{L}}_{k,p}$ at $f$, $L^p_k(\Sigma, \wedge^{0,1}(\Sigma) \otimes f^*(TM))$, is linearly homeomorphic to $L^p_k(\Sigma, f^*(TM)) \otimes L^p_k(\wedge^{0,1}(\Sigma))$ when $k$ is large enough. This gives rise a bundle isomorphism $\tilde{\mathcal{L}}_{k,p} \simeq \tilde{\mathcal{W}}_{k,p} \otimes \Omega^1$. Here the fibre of $\tilde{\mathcal{W}}_{k,p}$ at $f$ is $L^p_k(\Sigma, f^*(TM))$, and $\Omega^1$ is the trivial bundle whose fiber is $L^p_k(\Sigma)$. We will show that certain smooth section of $\Omega^1$ on a local slice is in fact $C^{m_0}$-smooth viewed in other slices after composed with the transition functions between the slices. To this end, we restrict the $G$-bundle $\Omega^1$ to the orbit $G \times W_1(f,H)$ of a local slice $W_1(f,H)$.

  Now let $\phi$ be a $C^\infty (0, 1)$-form on $\Sigma$ considered as a element of the fiber of $\Omega^1$. Since the bundle is trivial, we get a constant section $\psi_1$ on the slice $W_1(f)$ defined by $\psi_1(\xi) = \phi$. It extends to a $G$-equivariant section over $G \times W_1(f)$ by pull-backs. Denote the extended section by $\tilde{\psi}_1$. Even assume that $\phi$ is a $C^\infty (0, 1)$-form, it is still not immediately clear that this extended $G$-section is smooth on the open set $G \times W_1(f)$. On each $G$-slice, the extended section $\tilde{\psi}_1 : G \times \{h_0\} \rightarrow L^p_k(\wedge^{0,1}(\Sigma))$ has the form $\tilde{\psi}_1(g, h_0) = g^*(\phi)$, which is smooth.
as a function on $G$. Presumably, this should imply that the extended section is smooth as usually the lack of differentiability comes from $G$-direction. However, as mentioned in the introduction, the best one can get is its $C^{m_0}$-smoothness.

Instead of repeating a similar argument, we directly prove the weaker statement that the restriction of the extended section to another slice $W_2(f)$ is of class $C^{m_0}$.

Recall that the coordinate transformation between the two slices $t_{1,2} : W_2(f) \rightarrow W_1(f \circ g_0)$ is defined to be $t_{1,2}(\xi) = \xi \circ T_{1,2}(\xi)$. Here $T_{1,2} : W_2(f) \rightarrow G$ is of class $C^{m_0}$ and $G$ is the group of reparametrisations of $\Sigma$. Then $\psi_2(\xi) = \psi_1 \circ t_{1,2}(\xi) = \psi_1(\xi \circ T_{1,2}(\xi)) = (T_{1,2}(\xi))^* (\psi_1)(\xi) = (T_{1,2}(\xi))^* (\phi)$. Here since $\tilde{\Omega}^1$ is a trivial bundle, the sections $\psi_i, i = 1, 2$ are considered to be maps $W_i : \rightarrow L^\infty_\xi (\Lambda^{0,1}(\Sigma))$.

In other words, $\psi_2 = O_\phi \circ T_{1,2}$ where the orbit map $O_\phi : G \rightarrow L^\infty_\xi (\Lambda^{0,1}(\Sigma))$ is defined by $O_\phi(g) = g^* (\phi)$. Note that since $\phi$ is $C^\infty$, $O_\phi$ is of class $C^\infty$. Therefore, $\psi_2$ is $C^{m_0}$-smooth. Note that there are two more general cases that the argument above implies the same conclusion: (i) the constant map $\psi_1 : W_1(f) \rightarrow L^\infty_\xi (\Lambda^{0,1}(\Sigma))$ can be replaced by any smooth map with image lying in $C^\infty(\Lambda^{0,1}(\Sigma))$; (ii) $\phi$ is only of class $C^{2m_0}$.

What we did above gives a way to extend the part of the section of $\phi \otimes \xi$, $\phi$, to a smooth section $\psi_1$ over $W_1(f)$ which is of class $C^{m_0}$ viewed in any other slices.

As for the $\xi$, we do not extend it, but approximate it first under the assumption that the covering is very fine so that we can approximate $\xi$ by the constant section $\tilde{\xi}_0$ on $D_{\xi}(x_i)$ defined by $\tilde{\xi}_0(x) = \xi(x_i)$ for $x \in D_{\xi}(x_i)$. Note that the constant section can be approximated by the pull-back of $f|_{D_{\xi}(x_i)}(\tilde{\xi}_0)$. Here $\tilde{\xi}_0$ is the local constant section of the bundle $TM \rightarrow M$ in a neighbourhood of $f(x_i)$ given by transporting the value $\xi_0(x_i)$ in $T_{f(x_i)}M$ to the neighbourhood using $D\exp_{f(x_i)}$.

Clearly $\tilde{\xi}_0$ is smooth and we may assume that it is the restriction of a global smooth section, still denoted by $\tilde{\xi}_0$ of the bundle $TM \rightarrow M$. Therefore, the section $\xi$ can be approximated by the pull-back of a global section $\tilde{\xi}_0$ of $TM$ by $f|_{D_{\xi}(x_i)}$.

Now $\xi$ can be approximately extended to a global section $\tilde{\xi}$ over $\tilde{B}$ defined by $\tilde{\xi}(g) = g^* (\tilde{\xi}_0)$. Since $\tilde{\xi}$ comes from the pull-backs of the smooth "geometric" section of $TM \rightarrow M$, it is automatically $G$-equivariant and smooth.

More specifically, since the $G$-actions only act on the domain, the $G$-equivariancy of $\xi$ is clear. On the other hand, using the embedding $\iota : M \rightarrow \mathbb{R}^d$, the proof of its smoothness, which is supposed to be well-known, can be reduced further to the case that $M$ is just $\mathbb{R}^d$. In next section, we include the elementary proof of smoothness for this latter case for completeness.

At this point, we need to make sure that after putting all these extended section together, we get the extended section $GE(\eta)$ of $\eta$ such that $\|GE(\eta)(f) - \eta\|_{0,2} < \delta$ for a prescribed $\delta > 0$. To this end, we write everything in local charts of $\Sigma$ given by the covering $\Sigma = \bigcup_{i \in I} D_{\delta_i}(x_i)$ and fixed local trivializations of
Then $\eta = \sum_{i \in I} \beta_i \eta_i$. Let $\phi_i$ be the $C^\infty$ local frame of the bundle $\Lambda^{0,1} \to \Sigma$ over $D_{\delta_i}(x_i)$ with $\|\phi_i\|_{C^0} = 1$. Denote $\eta|_{D_{\delta_i}(x_i)}$ by $\eta_i$. Then $\eta_i = \phi_i \cdot \xi_i$ for some $\xi_i \in L^2_0(\Sigma, f^*(TM), h)$. In fact, we only need each $\xi_i$ to be defined over $D_{\delta_i}(x_i)$. Using the local trivializations of $TM$ near $T_f(x_i)$, we define the corresponding local constant section $\xi_i^{x_i}$ over $D_{\delta_i}(x_i)$ given by $\xi_i^{x_i}(x) = \xi_i(x_i)$.

Then

$$\eta = \sum_{i \in I} (\beta_i \eta_i) = \sum_{i \in I} (\beta_i \phi_i \cdot \xi_i).$$

We define a $K^p_k$-section $\eta'$ which is $C^0$-close to $\eta$ as follows,

$$\eta' = \sum_{i \in I} (\beta_i \phi_i \cdot \xi_i^{x_i}).$$

Note that $\eta'$ behaves like a $C^\infty$ section: $\beta_i \phi_i$ is smooth on $\sigma$ and $\xi_i^{x_i}$ is obtained by pulling back a smooth vector field on $M$ by $f|_{D_{\delta_i}(x_i)}$.

Then there constants $C$ and $C'$ such that when the covering is fine enough, for any prescribed $\epsilon'$,

$$\|\eta - \eta'\|_{0,2} \leq C\|\eta - \eta'\|_{C^0} \leq C \max \{\|\beta_i \phi_i \cdot (\xi_i - \xi_i^{x_i})\|_{C^0}\} \leq C'Max \{\|\xi_i - \xi_i^{x_i}(x)|_{D_{\delta_i}(x_i)}\|_{C^0}\} \leq \epsilon'.$$

Using this $\eta'$ to replace $\eta$, the argument before gives the desired approximate extension $GE(\eta)$.

In above argument, we only assume that $f$ and $\eta$ is of class $L^p_k$ and $L^p_{k-1}$ respectively. If we assume that they are of class $C^\infty$ or of class $C^{2m_0}$, we get better result. We will only give the proof for $C^\infty$ case.

To this end, we change notation and denote $\beta_i \eta_i$ by $\eta_i$. Then each $\eta_i = \sum_{j \in J} \phi_{ij} \cdot e_j^i(f_i)$. Here $f_i = f|_{D_{\delta_i}(x_i)}$, $\phi_{ij}$ is a $C^\infty$-smooth $(0,1)$-form on $\Sigma$ supported on $D_{\delta_i}(x_i)$ and the collection of $e_j^i, j \in J$ is a local $C^\infty$-frame of $TM$ near $T_f(x_i)$. From this local expression, one immediately see that each $e_j^i(f_i) = f_i^*e_j^i$, and hence it extends to a smooth section $\tilde{e}_j^i$ on $W_f$ defined by $\tilde{e}_j^i(g) = g^*e_j^i$, where $g_i = g|_{D_{\delta_i}(x_i)}$. We can extend each $\phi_j^i$ into $\phi_j^{x_i}$ as before. Let $GE(\eta) = \sum_{i \in I} GE(\eta_i) = \sum_{i \in I, j} \phi_j^{x_i} \cdot \tilde{e}_j^i$. Then $GE(\eta)$ so defined is a true extension of $\eta$ rather than just an approximate extension.

In this way, we extend each element $\eta$ in the basis of $K_f$ to a smooth section over $W_1(f)$, which is $C^{m_0}$-smooth viewed in any other slices. The results, so far in this section are sufficient to resolve the difficulty of lacking of differentiability and to establish local transversality for the case that there is only one stratum.

In the rest of this section, we will finish the construction of perturbed moduli space and prove that it is a smooth compact manifold with the expected dimension.
The argument below for regularizing a compact moduli space $\mathcal{M}$ works quite generally under the assumptions that $\mathcal{M}$ has only one stratum, all isotropy groups for elements in $\mathcal{B}$ are trivial and the Fredholm section $s : \mathcal{B} \to \mathcal{L}$ is proper.

- Regularization of the moduli space $\mathcal{M}$:

Let $\mathcal{M} = s^{-1}(0)$ be the moduli space of unparametrized stable $J$-holomorphic spheres. By our assumption $\mathcal{M}$ is compact. For each $f$ in $\mathcal{M}$, let $W_f = W_c(f, H_f)$ be a local slice regarded as a local chart of $\mathcal{B}$ containing $f$. In the local chart $W_f$ and the trivialization of $\mathcal{L}$ over $W_f$, we denote the Fredholm section $s$ by $[s] = [s_f]$.

The derivative of $s$ at point $f$ written as local trivialization over $W_f$, $D[s_f]|_f : L^p_k(\Sigma, f^*(TM), h) \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$, is given by the well-known formula, $D[s_f]|_f(\xi) = \nabla \xi + J(f) \nabla \xi \circ i + N(\partial f, \xi)$. Here $s$ is the $\partial J$-section, $\nabla$ is the unique $J$-invariant connection preserving the $g_J$-metric whose torsion is equal to the torsion of $J$, $N = N_J$ and $g_J$ is defined by $g_J = \omega(\cdot, J\cdot)$. There is a similar formula for $\partial_{J,H}$-section. Although, without introducing a connection it only makes sense to take derivatives of a section invariantly at its zeros, the above formula is still applicable at a general point $g$ in $W_f$ if we use the local trivialization above. In particular, from these explicit formulas, we have that when $\epsilon$ is small enough, for any $g$ in $W_{f,\epsilon}$, $D[s_f]|_g$ and $D[s_f]|_f$ are close to each other with respect to the operator norm.

- Choice of $\epsilon = \epsilon_f$ for $W_{f,\epsilon}$:

We need to make choice of $\epsilon = \epsilon_f$ for $W_{f,\epsilon}$ such that the perturbed sections of $\partial J$ by certain collection of sections related to the cokernels achieve the "controlled" transversality.

Let $K_f$ be the cokernel of $D_f = D[s_f]|_f$, $C_f$ be its kernel. We denote the $L^2$ orthogonal complement of $C_f$ in $L^p_k(\Sigma, f^*(TM), h)$ by $N_f$, and the $L^2$ orthogonal complement of $K_f$ in $L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$ by $N_f$. Then $D[s_f]|_f : N_f \to N_f$ is an isomorphism between the two Banach spaces.

When $\epsilon = \epsilon_f$ and $\delta = \delta_f$ are small enough, for any $g$ in $W_{f,\epsilon}$ and bounded linear operator $T : L^p_k(\Sigma, f^*(TM), h) \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$ with operator norm less than $\delta$, $D[s_f]|_g + T : N_f \to N_f$ is still an isomorphism. This $T$ corresponds to the derivatives of perturbation sections along $W_f$ directions. Since the perturbations will be made small enough so that all relevant derivatives along $W_f$ directions are ignorable, we let $T$ to be zero first.

By definition, we have

$$D[s_f]|_f \oplus I_{K_f} : L^p_k(\Sigma, f^*(TM), h) \oplus K_f \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$$

is surjective with the same kernel $C_f$. The right inverse of this map $G_{f,K_f} : L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM))) \to N_f \subset L^p_k(\Sigma, f^*(TM))$ is a linear bounded operator with operator norm $||G_{f,K_f}||$. Here $I_{K_f} : K_f \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))$ is the inclusion map.
Fix a basis \((\eta_1, \cdots, \eta_{n_f})\) of \(K_f\) as finite dimensional subspace of 
\[ L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM))) \]

Let \(G(K_f)\) be linear space spanned by approximate sections \((G(\eta_1), \cdots, G(\eta_{n_f}))\) of \(L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))\). Assume that \(\|\eta_i - G(\eta_i)\|_{k-1,p} < \delta\) for all \(i\). When \(\delta\) and \(\epsilon\) are small enough, for any \(g\) in \(W_{\epsilon,f}\) and \(\|G(\eta_i) - \eta_i\|_{k-1,p} < \delta\), \(i = 1, \cdots, n_f\), the linear map \(D[s_f]_g \oplus I_{G(K_f)} : L^p_k(\Sigma, f^*(TM), \mathfrak{h}) \oplus G(K_f) \to L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))\) is still surjective. Moreover, its right inverse, denoted by \(G_{g,G(K_f)}\), has the operator norm which is almost the same as the fixed one, \(\|G_{f,K_f}\|\).

Now we fix such \(\epsilon = \epsilon_f\) and \(\delta = \delta_f\) temporarily, and assume that for each point \(f\) in \(\mathcal{M}\) such a \(W_f\) is already chosen.

- The space \(G(E(K_f))\) of geometric perturbations derived from \(K_f\):

  First fix a cut-off function \(\gamma_f\) supported in \(W_f\), that is equal to one on a smaller neighbourhood \(W'_f \subset W_f\).

  Recall that in the usual construction of abstract perturbation, one considers each element of \(\eta\) of \(K_f\) as a section of \(L \to B\) at the point \(f\) and extend it over the slice \(W_f\) by parallel transport to get a constant section, denoted by \(E(\eta)\). Let \(LE(\eta) = \gamma \cdot E(\eta)\) be the corresponding localized section. Then in the local chart and trivialization over \(W_f\), \(LE(\eta)(\xi) = \gamma E(\eta)(\xi)\) for all \(\xi\) so that \(LE(\eta)\) is still a constant section. Let \(LE(K_f)\) to be the collection of all such \(LE(\eta)\). Note that since the two operations used in extending \(K_f\) to \(LE(K_f)\) are linear, \(LE(K_f)\) is a finite dimensional vector space inside the space of smooth local sections of \(L\) over \(W_f\), which is isomorphic to \(K_f\) and hence has the same dimension as that of \(K_f\). Because of linearity, \(K_f\) is obtained from the extensions of \(\eta_i, i = 1, \cdots, n_f\) in a basis of \(K_f\). Note that since on \(W'_f\), any section \(LE(\eta)\) is a constant section in the local trivialization, the collection of all evaluations of the sections in \(LE(K_f)\) at any point \(g\) in \(W'(f)\) is just the cokernel \(K_f\), which is independent of \(g\).

  We have already proved that for the fixed basis \((\eta_1 \cdots \eta_{n_f})\), each element \(LE(\eta_i)\) of \(LE(K_f)\) can be approximated by a localized geometric section over \(W_f\), denote by \(G(E(\eta_i))\). Recall that the approximation is obtained by approximating the corresponding constant section first when multiplying the resulting section by the cut-off function.

  Let \(G(E(K_f))\) be the linear space spanned by the geometric section \(G(E(\eta_i)), i = 1, \cdots, n_f\). The main reason to switch to \(G(E(K_f))\) is that its elements are not only smooth over \(W_f\) but also \(C^{m_0}\)-smooth viewed in other local slices. This may not be true for the elements in \(LE(K_f)\).

  Now we bring the fixed positive constant \(\delta_f\) above into the discussion. Once \(\delta = \delta_f\) is fixed, for sufficiently small \(\epsilon_f\), we may assume that each approximated section \(G(E(\eta_i))\) satisfies that for any \(g\) in \(W_f\), \(\|G(E(\eta_i))(g) - LE(\eta_i)(g)\|_{k-1,p} < \delta\), \(i = 1, \cdots, n_f\). Here we have considered \(G(E(\eta_i))(g)\) and \(LE(\eta_i)(g)\) as elements in \(L^p_{k-1}(\Sigma, \Lambda^{0,1}(f^*(TM)))\) by using the trivialization. Note that since on \(W'_f\),
\( LE(η_1)(g) = η_1 \) in the local trivialization. The choice of \( δ \) implies that for any \( g \) in \( W_f' \), \((GE(η_1)(g), \cdots, (GE(η_n)(g))\) is a basis of the corresponding linear subspace in \( L^p_{k-1}(Σ, \Lambda^{0,1}(f^*(TM))) \), which is very close to \( K_f \). Consequently, with the choices of \( ε \) and \( GE(η_1) \), for any point \( g \) in \( W_f' \), the linear map \( D[s_f]_g ⊕ [ev]_g : L^p_{k}(Σ, f^*(TM), h) ⊕ GE(K_f) \rightarrow L^p_{k-1}(Σ, \Lambda^{0,1}(f^*(TM))) \) is surjective. Moreover, its right inverse, denoted by \( G_{g,GE(K_f)}(g) \), has almost the same operator norm as the fixed one \( ||G_{f,G(K_f)}|| \). Here \( [ev]_g : GE(K_f) \rightarrow L^p_{k-1}(Σ, \Lambda^{0,1}(f^*(TM))) \) is the obvious evaluation map at \( g \) written in the local trivialization of \( L \) over \( W_f \).

- Size of local perturbation space:

  Our next task is to decide the "size" of each local perturbation space so that a quantitative version of implicit function theorem is applicable to the perturbed \( \partial J_f \)-operators. To this end, we need (i) to find the derivatives of the perturbed \( \partial J_f \)-section over each local slice, and (ii) to understand how these derivatives are transformed between different slices.

  Consider the local perturbed section \( ps = s ⊕ ev : W_f × GE(K_f) \rightarrow L|_{W_f} \), defined by \( s ⊕ ev(g, λ) = s(g) + λ(g) \). Let \( [s_f] ⊕ [ev] \) be the corresponding map written in the local trivialization. Then its derivative at \( (g, λ) \), \( D([s_f] ⊕ [ev]) = (D[s_f])_g + (D[ev])_g, λ \). The partial derivative of \( [ev] \) at \( (g, λ) \) along \( GE(K_f) \)-directions is just the linear inclusion map \( [ev]_g : GE(K_f) \rightarrow L^p_{k-1}(Σ, \Lambda^{0,1}(f^*(TM))) \) introduced before, which is independent of \( λ \). We already know that \( (D[s_f])_g ⊕ [ev]_g \) is surjective with right inverse whose norm is bounded by \( ||G_{f,G(K_f)}|| \).

  Now denote the partial derivatives of \( [ev] \) along \( W_f \)-directions by \( \partial^W [ev] \). Then for any fixed \( g \) in \( W_f \), \( λ \) and \( c \) in \( GE(K_f) \) for some positive constant \( c \), we have \( \partial^W [ev]_g, c, λ(ξ) = c \partial^W [ev]_g, λ(ξ) \) for any \( ξ \) in the tangent space of \( W_f \) at \( g \). Consequently the operator norm \( ||\partial^W [ev]_g, c, λ|| = c||\partial^W [ev]_g, λ|| \). In other words, the operator norm of \( \partial^W [ev]_g, λ \) gets rescaled by \( c \) if the size of \( λ \) is rescaled by \( c \). This is the crucial fact that we need to get local transversality for the section \( s ⊕ ev \). However, in order to get desired global perturbation of \( s \), we need to apply Picard method to get more quantitative information on the local extended moduli spaces. To this end, we need to compute higher derivatives. Clearly, higher partial derivatives of \( [ev] \) along \( W_f \)-directions also rescale in the same way as the first derivative does. This is better than what we need. To find second partial derivatives of \( [ev] \) along the other directions, note that \( [ev]_g, s, λ \) is already linear along \( GE(K_f) \)-directions, therefore we only need to find the mixed partial derivatives of \( [ev] \). Since

\[
\left( \frac{∂^2}{∂W∂GE(K_f)} [ev] \right)_g, (η, ̃η) = (\partial^W ̃η)|_g(ξ).
\]

We have

\[
||\left( \frac{∂^2}{∂W∂GE(K_f)} [ev] \right)_g, (ξ, ̃η)||_{k-1,p} \leq (max_{g ∈ W_f} (Σ_{i ∈ I}|| (\partial^W (GE(η_i))_g ||) · || ̃η(g)||_{k-1,p} · ||ξ||_{k,p})
\]
Here \(||(\partial W(GE(\eta_i)))_g||\) is the operator norm of the partial derivative at \(g\) and each \(GE(\eta_i)\) is the approximated extension of the element \(\eta_i, i \in I\) in the fixed basis of \(K_f\).

It follows from this that

(I) when \(\rho = \rho_f\) is small enough, for all \(\lambda\) in the small \(\rho\)-ball \(Bl(K_f, \rho)\) of \(GE(K_f)\) and \(g\) in \(W_f\), the operator norm of \(||\partial W(ev)_g,\lambda||\) is less than \(\delta_f\) specified before. Consequently, in the local trivialization, the derivative the section \(ps = s + ev\), \((D[s_f]g \oplus (D[ev])_g,\lambda) : W'_f \times Bl(K_f, \rho) \rightarrow L|_{W'_f}\) is a surjective map at any point. This solves our problem to achieve the local transversality by using perturbation form \(Bl(K_f, \rho)\) only. In other words, the solution set of the local equation \([s_f] \oplus [ev](X) = 0\) is a smooth submanifold in \(W'_f \times Bl(K_f, \rho)\).

(II) The right inverse of the derivative \((D[s_f]g \oplus (D[ev])_g,\lambda),\) denote by \(G_{g,\lambda}\), still has almost the same operator norm as the fixed one \(||G_{f,G}(K_f)||\).

(III) Let \(N\) be the non-linear term appeared in the Taylor expansion of \([s_f] \oplus [ev]\) at \((f,0)\), then it satisfies the condition on \(N\) required by the following Lemma.

**Lemma 3.5 Picard method**

Assume that a smooth map \(F : E \rightarrow L\) from Banach spaces \((E, ||\cdot||)\) to \(L\) has a Taylor expansion

\[F(\xi) = F(0) + DF(0)\xi + N(\xi)\]

such that \(DF(0)\) has a finite dimensional kernel and a right inverse \(G\) satisfying

\[||GN(\xi) - GN(\eta)|| \leq C(||\xi|| + ||\eta||)||\xi - \eta||\]

for some constant \(C\). Let \(\delta_1 = \frac{1}{8C}.\) If \(||G \circ F(0)|| \leq \frac{1}{4}\), then the zero set of \(F\) in \(Bl_{\delta_1} = \{\xi, ||\xi|| < \delta_1\}\) is a smooth manifold of dimension equal to the dimension of \(kerDF(0)\). In fact, if

\[K_{\delta_1} = \{\xi : ||\xi|| < \delta_1\}\]

and \(K^\perp = G(L)\), then there exists a smooth function

\[\phi : K_{\delta_1} \rightarrow K^\perp\]

such that \(F(\xi + \phi(\xi)) = 0\) and all zeros of \(F\) in \(Bl_{\delta_1}\) are of the form \(\xi + \phi(\xi)\).

The proof of this Lemma is an elementary application of Banach’s fixed point theorem.

Now applying this lemma to our case with the obvious interpretations of the notations, we conclude that the perturbed section \(s \oplus ev\) defined on \(W'_f \times Bl(K_f, \rho_f)\) is transversal to the zero section so that the zero locus, the local extended moduli space \((s \oplus ev)^{-1}(0)\) in \(W'_f \times Bl(K_f, \rho_f)\), is a smooth manifold. Denote this extended moduli space by \(\mathcal{E}M^{f, ev}\). The key point is that it is
realized as a graph over a disc of radius $\delta_{1,f}$ in the kernel of $D[s]_f$ in the local trivialization. Therefore by shrinking $\mathbf{E}\mathcal{M}^{f,\rho_f}$ a little bit corresponding to taking $\delta_{1,f}$ to be a smaller $\delta_{2,f}$, we get a corresponding spaces $\mathbf{E}\mathcal{M}^{f,\rho_f}_2$. Denote the original larger space as $\mathbf{E}\mathcal{M}^{f,\rho_f}_1$. Then closure of $\mathbf{E}\mathcal{M}^{f,\rho_f}_2$ is compact. We may assume that $\mathbf{E}\mathcal{M}^{f,\rho_f}_2$ is the corresponding solution space of the equation $[s] \oplus [ev](X) = 0$ in the smaller space $W''_f \times \text{Bl}(K_f, \rho_f)$ for some $W''_f \subset W'_f$.

Note that each element $\lambda_f$ of the perturbation space $GE(K_f)$ has support inside a local slice $W_f$, hence can be considered as a global section of $\mathcal{L} \to \mathcal{B}$. In particular, $\lambda_f$ can be viewed as a local section over another local slice $W'_f$. We will cover $\mathcal{M}$ by finitely many such local slices $W_{f,i}, i \in I$ and consider the corresponding $\text{Bl}(K_{f,i}, \rho_{f,i})$ and the resulting global perturbation space $\oplus_{i \in I} \text{Bl}(K_{f,i}, \rho_{f,i})$. Our goal is to show that for proper choice of the covering and small enough generic $\nu$ in $\oplus_{i \in I} \text{Bl}(K_{f,i}, \rho_{f,i})$, the perturbed moduli space $\mathcal{M}'$ is a compact $C^m$-manifold.

To this end, we need to modify the above discussion to incorporate the effect of the perturbations from $\text{Bl}(K_{f,i}, \rho_{f,i})$ with $j \neq i$ on the slice $W_f = W_{f,i}$.

For that purpose, we give a more general discussion first.

Consider a ball of radius $r$ centered at origin in a finite dimensional vector space, denoted by $\text{Bl}(r)$, which is linearly mapping into the space of "global" weakly smooth sections of the bundle $\mathcal{L} \to \mathcal{B}$. Here a "global" section of the bundle is simply a collection of compatible local sections with respect a covering data that we will specified in a moment. This $\text{Bl}(r)$ will play the role of the space of global perturbations. Assume that the $W_f$ is one of the local slice, and $U_f \subset W''_f$ is a small open neighbourhood of $f$. Denote corresponding sections of $\text{Bl}(r)$ over $U_f$ by $\text{Bl}(f, r)$.

Consider the solution space of the equation about $(g, \eta)$ in $U_f \times \text{Bl}(f, r)$ given by $[s](g) + \eta(g) = 0$. Clearly the above solution space is covered by the solution space about $(g, \xi, \eta)$ with $(g, \xi)$ in $U_f \times \text{Bl}(K_f, \rho_f)$ and $\eta$ in $\text{Bl}(f, r)$ given by $[s](g) + \eta(g) = -\eta(g)$. By the implicit function theorem above and the way we define $[s] \oplus [ev]$, there is a sufficient small positive $r_f$ such that when $r < r_f$, for any fixed $\eta$ in $\text{Bl}(f, r_f)$, all solutions of the above equation about $(g, \xi)$ in $U_f \times \text{Bl}(K_f, \rho_f)$ is homeomorphic to a finite dimensional disc, still denoted by $K_{\delta_f}$, the same notation used in the Picard method. Therefore, the closure of the solution space is contained in a compact set in $W''_f \times \text{cl}(\text{Bl}(K_f, \rho_f)) \times \text{cl}(\text{Bl}(f, r_f))$ which is homeomorphic to $\text{cl}(K_{\delta_f}) \times \text{cl}(\text{Bl}(f, r_f))$.

To summary what we have done here, consider the solution space of the equation about $(g, \eta)$ in $U_f \times \text{Bl}(f, r_f)$ given by $[s](g) + \eta(g) = 0$, and denote it by $\mathbf{E}\mathcal{M}(\text{Bl}(f, r_f))$. Then $(1) \text{cl}(\mathbf{E}\mathcal{M}(\text{Bl}(f, r_f))) \subset \pi_{W_f}(\text{cl}(\mathbf{E}\mathcal{M}^{f,\rho_f}_2)) \times \text{cl}(\text{Bl}(f, r_f))$ which is compact in $W''_f \times \text{cl}(\text{Bl}(f, r_f))$. Here $\pi_{W_f} : W_f \times \text{Bl}(K_f, \rho_f) \to W_f$ is the projection to $W_f$. (2) For any fixed $\nu_f$ in $\text{Bl}(f, r_f)$, Let $\mathcal{M}'$ be the solution space of the equation $[s](g) + \nu(g) = 0$. Then $\text{cl}(\mathcal{M}'')$ is compact in $W''_f$. Since $W'_f$ is a local slice for $\mathcal{B}$, we conclude that the closure in $\mathcal{B}$, $\text{cl}_B(\mathcal{M}'')$ is compact.
Since $\mathcal{M}$ is compact, we can select a finite covering $\mathcal{W} = \{W_i = W_{f_i}, i \in N\}$ such that $\mathcal{M}$ is already covered by the corresponding open subsets $U_i \subset \subset W''_i, i \in N$.

To finish the construction of the perturbed moduli space, we need to make two assumptions:

**A1**: Let $U = \bigcup_{i \in I} U_i$ be the open subset of $B$. Then the boundary of $U$, denoted by $\text{Bd}(U)$, has no intersection with $\mathcal{M}$. Here $\text{Bd}(U) = \text{cl}_B(U) \setminus U$.

**A2**: There is a positive constant $C_0$, such that for any $g$ on $\text{Bd}(U)$, $||s(g)|| > C_0$.

We will prove that these two assumptions can be achieved latter in this section.

Let $B_i = BL(K_{f_i}, r_{f_i})$ be the ball of radius $r_i = \rho_{f_i}$ in the linear space $GE(K_{f_i})$.

Then each element of $GE(K_{f_i})$ as a section of $L$ over $W_i$ is supported in a closed subset of $W_i$ and hence can be regarded as a global section of the bundle $L \rightarrow B$.

Set $r = \sum_{i \in I} r_i$, and let $B(r) = \oplus_i B_i$.

Consider the global perturbation map $s \oplus ev : U \times B(r) \rightarrow L$. Here we still use $L$ to denote the pull-back of the bundle over $U \times B(r)$. Clearly, by our construction, $s \oplus ev$ is $C^{m_0}$-smooth when it is viewed in any admissible local charts $U_i \times B(r), i \in I$, and trivializations. We will show that

- **C1**: when $r$ is small enough, $s \oplus ev : U \times B(r) \rightarrow L$ is transversal to the zero section on each $U_i \times B(r), i \in I$.

Assume that C1 is true. Let $E_M^{B(r)}$ be the collection of the zero loci of the section inside $U_i \times B(r), i \in I$. Then it is a weakly $C^{m_0}$-smooth submanifold in $B \times B(r)$ in the sense defined before in this paper. Therefore, with the induced smooth structure, it is a $C^{m_0}$-manifold with expected dimension $\text{ind}(D_{s_f}) + \text{dim}(B(r))$. Let $\pi : E_M^{B(r)} \rightarrow B(r)$ be the projection map between the two finite dimensional manifolds, which is of class $C^{m_0}$. Then the "index" of $D\pi$ is equal to the $\text{ind}(s)$ which is fixed on each stratum. Assume that $m_0 > \text{ind}(s)$. Then for generic choice of $\nu$ in $B(r)$, perturbed moduli space $M^\nu = (s^\nu)^{-1}(0)$ is a $C^{m_0}$-submanifold of $E_M^{B(r)}$ with dimension equal to $\text{ind}(s)$. Here $\nu = \oplus_{i \in I} \nu_i$ in $B(r)$, and $s^\nu = s + \nu$ is the $\nu$-perturbed section of $s$ over $U$.

To finish the construction, we need to show that $M^\nu$ is compact for $r$ small enough.

To this end, assume that $r << C_0$. Then $s^\nu(X) = 0$ has no solution on $\text{Bd}(U)$. On the other hand, since $s^\nu$ is continuous on $B$, any point in $\text{cl}_B(M^\nu)$ is still a solution of $s^\nu(X) = 0$. Since $\text{cl}_B(U) = U \cup \text{Bd}(U)$, this implies that $\text{cl}_B(M^\nu)$ is inside $U$. Therefore, $\text{cl}_B(M^\nu) = M^\nu$.

Now $M^\nu = \bigcup_{i \in I} M^\nu_i$ and $\text{cl}_B(M^\nu) = \bigcup_{i \in I} \text{cl}_B(M^\nu_i)$. We already proved that each $\text{cl}_B(M^\nu_i)$ is compact in a more general setting as an application of Picard method. Therefore, $M^\nu = \text{cl}_B(M^\nu)$ is compact for any $\nu$ in $B(r)$.
This finishes the construction of perturbed moduli space and proves the following theorem under the assumption that $\mathcal{B}$ has only one stratum and all isotropy groups are trivial.

**Theorem 3.1** When $r$ is sufficiently small, for any $\nu$ in $B(r)$, the perturbed moduli space $\mathcal{M}'$ is compact. For a generic choice of $\nu$, $\mathcal{M}'$ is a finite dimensional compact topological manifold. Moreover, in the latter case, as a topological submanifold of $\mathcal{B}$ with induced weakly smooth structure with respect to the equivalent class of the covering data $[U]$, $\mathcal{M}'$ is in fact an honest smooth manifold of class $C^{m_0}$.

- Proof of $C1$: Fix a local slice $U_i \subset \subset W''_i \subset \subset W_i = W_{i,f}$. Let $[s_i] \oplus [ev] : U_i \times B(r) \to L^p_k (\Sigma, \Lambda^{0,1}(f^*_i TM))$ be the section $s \oplus ev : U \times B(r) \to L$ written in the local trivialization over $U_i \times B(r)$. In the local trivialization, each $\nu$ in $B(r)$ still takes a form $\nu = (\nu_1, \cdots, \nu_l)$, where $l$ is the cardinality of the index set $I$. But each $\nu_j$ with $j \neq i$ is obtained from the corresponding one composed with transition functions between the two slices and trivializations. Since there are only finitely many such $\nu_j$, for a fixed $i$, there exists a constant $C_i > 0$, such that for each such $\nu_j$, $\|\nu_j(g)\|_{k-1,p}$ is bounded by $C_i \cdot r$ for all $g \in U_i$.

To prove $C1$, we only need to show the stronger statement that when $r$ is small enough, $D([s_i] \oplus [ev])_{g,\lambda} : L^p_k (\Sigma, f^*_i TM, h_i) \times T_\lambda B(r) \to L^p_k (\Sigma, \Lambda^{0,1}(f^*_i TM))$ is surjective at any point $(g, \lambda)$ in $U \times B(r)$.

We already calculated the this derivative and gave the related estimate for quantitative transversality before. We recall the computation.

$$D([s_i] \oplus [ev])_{g,\lambda} (\xi, \eta) = D [s_i]_{g} (\xi) + [ev]_{g} (\eta) + (\partial^{W,\lambda})_{g} (\xi).$$

Now it was prove that

$$D [s_i]_{g} + [ev]_{g} : L^p_k (\Sigma, f^*_i TM, h_i) \times \oplus_{i \in I} GE(K_{f_i}) \to L^p_k (\Sigma, \Lambda^{0,1}(f^*_i TM))$$

is surjective. In fact, even

$$D [s_i]_{g} + [ev]_{g} : L^p_k (\Sigma, f^*_i TM, h_i) \times GE(K_{f_i}) \to L^p_k (\Sigma, \Lambda^{0,1}(f^*_i TM))$$

is already surjective with a right inverse whose operator norm is bounded above for all $g$ in $U_i$. Therefore, there is a positive constant $\delta_i$ such that for any bounded linear operator $T_g : T_g U_f = L^p_k (\Sigma, f^*_i TM, h_i) \to L^p_k (\Sigma, \Lambda^{0,1}(f^*_i TM))$, if the operator norm $\|T_g\| < \delta_i$,

$$D [s_i]_{g} + T_g + [ev]_{g} : L^p_k (\Sigma, f^*_i TM, h_i) \times GE(K_{f_i}) \to L^p_k (\Sigma, \Lambda^{0,1}(f^*_i TM))$$

is still surjective.
We already proved before that when r is small enough for \( \| \lambda \| < r \), the operator norm of the partial derivative \( \partial^{W_1} \lambda \) acting on \( L^p_k(\Sigma, \mathcal{A}^{0,1}(f^*TM)) \) is less than the prescribed value \( \delta_i \) for any \( g \) in \( U_i \).

Put this together, we have proved that for sufficiently small \( r \), \( D([s_i] \oplus [ev])g,\lambda \) is surjective for any \( g, \lambda \) in \( U_i \times B(r) \).

It remains to prove that the two assumptions can be arranged.

• Proof of A1:

We assume that \( \mathcal{M} \) is already covered by \( \mathcal{U}' = \cup_{i \in I} U_i' \), where \( U_i' \subset \subset U_i \) is a smaller ball inside the ball \( U_i \) with the same center \( f_i \). Here we have already assumed that the radius of each \( U_i \) is much smaller than the injective radius of \( M \) so that each \( U_i \) is really identified with a ball in the model space, and \( U_i' \) is a strictly smaller ball.

By our assumption, \( \mathcal{M} \) is contained in \( \cup_{i \in I} \mathcal{M}'_i \), where \( \mathcal{M}'_i = \mathcal{M} \cap U_i' \). To prove A1, we only need to show that the intersection of each \( \mathcal{M}'_i \) with \( Bd(U) \) is empty.

Recall that by definition \( Bd(U) = cl_B(U) \setminus U \). Then \( cl_B(U) = \cup_{i \in I} cl_B(U_i) \), and each \( cl_B(U_i) \) is just the corresponding closed ball. Clearly, \( Bd(U) = \cup_{i \in I} Bd_i \) where \( Bd_i = Bd(U_i) \setminus (\cup_{j \in I} U_j) \). Now consider \( \mathcal{M}'_i \) with \( i \) being fixed.

Since the intersection of \( \mathcal{M}'_i \) with \( Bd(U_i) \) is empty, so is the intersection of \( \mathcal{M}'_i \) with \( Bd_i \). On the other hand, for any \( j \neq i \), \( Bd_j \) has no intersection with \( U_i \).

Since \( \mathcal{M}'_i \) is inside \( U_i' \), we still have that the intersection of \( \mathcal{M}'_i \) with \( Bd(U) \) is empty. Therefore, the intersection of each \( \mathcal{M}'_i \) with \( Bd(U) \) is empty.

• Proof of A2:

A equivalent form of A2 is the statement that if \( \{g_k\}_{k=1}^\infty \) is a sequence in \( W' \) such that \( \lim_{k \to \infty} ||s(g_k)||_{k-1,p} = 0 \), then there is a subsequence, still denoted by \( \{g_k\}_{k=1}^\infty \), such that it is \( L^p_k \)-convergent to an element in \( \mathcal{M} \).

Assume this is true, and that A2 is not true. Then there is a sequence \( \{g_k\}_{k=1}^\infty \) in \( Bd(U) \) such that \( \lim_{k \to \infty} ||s(g_k)||_{k-1,p} = 0 \). Then we have that \( L^p_k \)-limit \( \lim_{k \to \infty} g_k = g_\infty \) exits and is in \( \mathcal{M} \).

Since \( Bd(U) \) is closed in \( \mathcal{B} = \mathcal{B}_{k,p} \), \( g_\infty \) is also in \( Bd(U) \). Therefore, \( \mathcal{M} \) and \( Bd(U) \) has non-empty intersection, which contradicts to A1.

The equivalent form of A2 for \( s = \partial_J \) on a fixed stratum was proved by Floer in [F1] based on local elliptic estimate for the non-linear \( \partial_J \)-operator near each point on the domain \( \Sigma \). The method there is the ”standard” one.

Here we give a more ”global” proof based on the Taylor expansion for \( \partial_J \)-operator used in Picard method. This proof works for general Fredholm sections as long as those reasonable conditions specified in the proof are satisfied.

• Proof of the equivalent statement of A2:

Since there only finitely many \( W_i \)'s, we only need to look at those \( W_i' \) which contains infinitely many \( g_k \)'s. So we may assume that the sequence \( \{g_k\}_{k=1}^\infty \) is
contained in $W'_i \subset W_i = W_{f_i, \epsilon_i}$ for a fixed $i$. However, we need to assume that $\epsilon_i$ is small enough so that some conditions to be specified can be satisfied.

We remark that once $\mathcal{M}$ is given, for each $f$ in $\mathcal{M}$, we need to choose these $\epsilon_f$’s satisfying the conditions below together with the conditions before. We then define the corresponding $W_f, U_f$, etc. and select the finite covering of $\mathcal{M}$ by $W_i$ and $U_i$ with $i \in I$ as before.

Now we work with the local slice, $W'_f \subset W_f$ with $f$ in $\mathcal{M}$. Identify them with two open balls of radius $\delta_2 < \delta_1$ in $L^p_k(\Sigma, f^*(TM), h)$ with radius $\delta_2 < \delta_1$. Then $f$ has local coordinate $f = 0$. In the local trivialization of $L$ over $W_f$, we have the Taylor expansion at $f$ of $[s] = [s_f]$, the $\partial_f$-section written in the trivialization as follows. For any $\xi$ in $W_f$,

$$[s](\xi) = [s](0) + D[s]f\xi + N(\xi) = D[s]f\xi + N(\xi),$$

since $[s](0) = 0$. Moreover, there is a constant $C$ depending on $f$, $\delta_2$ and geometric data of $M$ only such that $N$ satisfying

$$\|N(\xi) - N(\eta)\|_{k-1,p} \leq C(\|\xi\|_{k,p} + \|\eta\|_{k,p})\|\xi - \eta\|_{k,p}.$$ 

For our purpose, we introduce an equivalent metric on $E = L^p_k(\Sigma, f^*(TM), h)$. Let $K = ker Ds_f$ and $K^\perp$ be its $L^2$ orthogonal complement. Since $K$ is finite dimensional, we have the decomposition $E = K \oplus K^\perp$.

For any $\xi = (\gamma, \eta)$ in $E$, we define a weaker but equivalent norm, $\|\xi\|_{k,p}^w = \|\gamma\|_{0,2} + \|\eta\|_{k,p}$. By elliptic estimate for the linear elliptic operator $Ds_f$, we know that there exists a constant of similar nature, still denoted by $C$, such that $\|\gamma\|_{k,p} \leq C\|\gamma\|_{0,2}$. Therefore, we get the same estimate for $N$ as above by replacing all $L^p_k$-norms on the right hand side by the new weaker norms and changing the constant accordingly.

The upper semi-continuity of the dimension of the $ker Ds_f$ with respect to the variable $f$ moving in $\mathcal{M}$ together the fact that $\mathcal{M}$ is compact, implies that the constants above are bonded above, hence independent of $f$ any more if we work on $W = \cup_{i \in I} W_i$.

Go back to $W_f$. In local charts, the sequence in $W'_f$ has the form $g_k = (\gamma_k, \eta_k)$ with respect to the decomposition above.

Note that $\gamma_k$ is in $K_{\delta_2} \subset K_{\delta_1}$. Here $K_{\delta_2}$ and $K_{\delta_1}$ are the two discs in $K$ of radii $\delta_2$ and $\delta_1$ respectively measured in $L^p_k$-norm. Therefore, after taking a subsequence of $\{g_k\}_{k=1}^\infty$, we may assume that $\{\gamma_k\}_{k=1}^\infty$ is already convergent in $L^p_k$ and hence $L^2$-norm. In particular, $\|\gamma_k - \gamma\|_{0,2}$ goes to zero when $k$ and $l$ go to infinity.

Now by elliptic estimate for $Ds_f$, there is a (uniform) constant $C_1$ such that

$$\|\eta_k - \eta\|_{k,p} \leq C_1\|D[s]f(\eta_k - \eta)\|_{k-1,p} = C_1\|D[s]f(\gamma_k, \eta_k) - D[s]f(\gamma_l, \eta_l)\|_{k-1,p} = C_1\|D[s]f(g_k) - D[s]f(g_l)\|_{k-1,p} \leq C_1\|N(g_k) - N(g_l)\|_{k-1,p} \leq C_1 \cdot C(\|g_k\|_{k,p} + \|g_l\|_{k,p})\|g_k - g_l\|_{k,p}^w.$$
\[= C_1 \cdot C (\|g_k\|_{k,p} + \|g_l\|_{k,p})(\|\eta_k - \eta_l\|_{k,p} + \|\gamma_k - \gamma_l\|_{0,2}).\]

We now assume that \(\delta_2\) and \(\delta_1\) are chosen in such a way that \(2C_1 \cdot C \delta_1 \leq \frac{1}{2}\).

Since \(g_k\) is in \(W^f\) which is a ball of radius equal to \(\delta_2\), we have \(C_1 \cdot C (\|g_k\|_{k,p} + \|g_l\|_{k,p}) \leq \frac{1}{2}\). Therefore, we have

\[\|\eta_k - \eta_l\|_{k,p} \leq \frac{1}{2}\|\eta_k - \eta_l\|_{k,p} + \frac{1}{2}\|\gamma_k - \gamma_l\|_{0,2}.\]

This implies that

\[\|\eta_k - \eta_l\|_{k,p} \leq \|\gamma_k - \gamma_l\|_{0,2}.\]

Consequently, after taking a subsequence, \(\{g_k\}_{k=1}^\infty\) is a Cauchy sequence with respect to \(L^p_k\) or its equivalent norm. Therefore \(\lim_{k \to \infty} g_k = g_\infty\) exists and satisfies \(s(g_\infty) = 0\) by continuity.

### 4 Smoothness of Banach Norm

In this section we collect some results related to the smoothness of \(L^p_k\)-norms. Almost all the results are well-known and mentioned in various books on geometric analysis. Despite of the elementary nature of these results, it seems that not all of the proofs are widely known. We outline the key ideas of the proofs of these results here.

**Lemma 4.1** Let \(\pi : V \to \Sigma\) be a \(C^\infty\)-smooth metric vector bundle on a \(C^\infty\)-smooth Riemannian manifold \(\Sigma\), and \(G\) be a Lie sub-group of the \(C^\infty\)-diffeomorphisms of \(\Sigma\). The action of \(G\) on \(\Sigma\) induces a action on \(L^p_k(\Sigma, V)\). Let \(\Psi : G \times L^p_k(\Sigma, V) \to L^p_k(\Sigma, V)\) be the induced action map. Denote composition of \(\Psi\) with the \(p\)-th power of \(L^p_k\)-norm by \(F : G \times L^p_k(\Sigma, V) \to \mathbb{R}^1\). Then \(F\) is smooth if \(p\) is an even non negative integer. Here we have assumed that \(\Sigma\) is oriented and \(G\)-action preserves the orientation.

**Proof:**

The main issue here is the smoothness along \(G\)-direction. In view of the lack of smoothness of the action map \(\Psi\) along this direction, it seems that this part of the lemma can not be true.

To argue that this is plausible, observe that in the case that \(\Sigma\) is \(\mathbb{R}^n\) or \(T^n\) and \(G\) is the group of translations, for a fixed section \(\xi \in L^p_k(\Sigma, V)\), \(F_\xi(g) = F(g, \xi) : G \to \mathbb{R}^1\) is a constant map, hence trivially smooth.

This simple observation immediately suggests that the general case follows from the changing variable formula for integrations in calculus. We only give the formula for the case that \(k = 1\). The case of general \(k\) can be proved similarly with more complicated notations.

We will write \(F = F_0 + F_1\), and deal with each term separately.
Then,

$$F_0(g, \xi) = \int_{\Sigma} ||g \cdot \xi||^p dx = \int_{\Sigma} ||\xi(g(x))||^p dx$$

$$= \int_{g(\Sigma)} ||\xi(y)||^p det(g^{-1}) dy = \int_{\Sigma} ||\xi(y)||^p det(J_{g^{-1}}(y)) dy.$$

Here \(x\) and \(y\) are two coordinate systems on \(\Sigma\) such that \(y = g(x)\) and \(J_{g^{-1}}(y)\) is the Jacobian matrix of the differomorphism \(g^{-1} : \Sigma \rightarrow \Sigma\).

Clearly, the last identity show that \(F_0\) is smooth in \(g\).

$$F_1(g, \xi) = \int_{\Sigma} \Sigma_i ||\partial_x, (\xi(g(x)))||^p dx$$

$$= \int_{\Sigma} \Sigma_i ||\partial_y, (g^{-1}(y))\partial_y, (\xi(y)))||^p det(J_{g^{-1}}(y)) dy$$

$$= \int_{\Sigma} \Sigma_i ||\partial_y, (J_g)^i_j (g^{-1}(y))\partial_y, (\xi(y)))||^p det(J_{g^{-1}}(y)) dy$$

Here \(\partial_x,\) is the covariant derivatives on \(V\) along the \(x^i\)-direction and \((J_g)^i_j\) is the \((j, i)\)-entry of the Jacobian of the orientation preserving \(C^\infty\) differomorphism \(g : \Sigma \rightarrow \Sigma\).

Again, the last identity shows the smoothness of \(F_1\) in \(g\) when \(p\) is an even non-negative integer.

Note that the role of above changing variable formulas is to switch the \(G\)-action on \(\xi\), which is only \(L^p_1\) hence may "lose" derivative under \(G\)-action, to the terms like \((J_g)^i_j (g^{-1}(y))\). Since \(g\) is a \(C^\infty\)-differomorphism so that terms like \(J_g\) have \(C^\infty\) smooth entries, it is well known that the lack of derivatives does not happen in this case.

The \(C^\infty\)-smoothness along the \(\xi\)-direction can be deduced from the following well-known fact:

**Lemma 4.2** The \(p\)-th power of the \(L^p_\infty\)-norm on \(L^p_\infty(\Sigma, V)\) is smooth for even \(p\). When for \(p\) is positive but not even, it is at least \([p - 1]\)-smooth.

**Proof:**

Note that the \(L^p_\infty\)-norm that we have used here is \(||\xi||_{k,p} = (\Sigma \leq k \int_{\Sigma} |\nabla^i \xi|^p dx)^{1/p}\), not the one obtained as the summation of the \(k\) semi-norms. Clearly it is sufficient to prove this only for \(k = 0\). Assume that \(p = 2l\). Then \(F(\xi) = \int_{\Sigma} <\xi, \xi >^l dx\) as a function from \(L^p_\infty(\Sigma, V)\) to \(\mathbb{R}\). If the derivative of \(F\) exists at \(\xi\), it has the form \(DF(\xi)(\eta) = 2l \int_{\Sigma} <\xi, \xi >^{l-1} <\xi, \eta > dx\). One needs to show that this formal derivative is the real one. To proof this, one of key ingredients is to show that it is a bounded linearly map from \(L^p_\infty(\Sigma, V)\) to \(\mathbb{R}\). This follows form Horder inequality as following:

$$|DF(\xi)(\eta)| \leq 2l \int_{\Sigma} |\xi|^{2(l-1)+1} |\eta| dx$$

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\[
\leq 2l \left( \int_{\Sigma} |\xi|^{(2l-1)q} dx \right)^{1/q} \cdot \left( \int_{\Sigma} |\eta|^p dx \right)^{1/p} \\
= 2l \left( \int_{\Sigma} |\xi|^p dx \right)^{1/q} \cdot \left( \int_{\Sigma} |\eta|^p dx \right)^{1/p} \\
= 2l \|\xi\|^{p/q} \cdot \|\eta\|_p.
\]

Here we have used the identity \((p-1)q = p\) which follows from \(1/p + 1/q = 1\).

\[\blacksquare\]

There are several well-known corollaries of the above lemmas that we will use.

**Corollary 4.1** Let \( E = L^p_k(\Sigma, V) \) with \( p \) being even and positive. There exists smooth cut-off function \( \phi : E \to [0, 1] \) supported in the unit ball of \( E \) such that \( \phi(x) = 1 \) on the ball of radius half.

**Proof:**

Let \( \beta : \mathbb{R} \to [0, 1] \) be a “standard” smooth bump function supported on \([-1, 1]\) and equal to 1 on \([-1/2, 1/2]\). Then \( \phi = \beta \circ F \) does the job, where \( F \) is the \( p \)-th power of the \( L^p_k \)-norm.

\[\blacksquare\]

**Corollary 4.2** Let \( B \) be a paracompact Banach manifold of class \( C^m \) modelled on \( E = L^p_k(\Sigma, V) \) with \( p \) even and positive. Then \( B \) admits \( C^m \) partition of unit.

**Proof:**

The proof in Lang’s book for Hilbert manifold works equally well for this case. The key fact used in Lang’s proof is that in Hilbertian case the square of Hilbert norm is smooth.

\[\blacksquare\]

**Lemma 4.3** Let \( B = L^p_k(\Sigma, V) \) be the Banach space of all \( P^k \)-maps from \( \Sigma \) to \( V = \mathbb{R}^d \), \( f : V \to \mathbb{R}^d \) is a \( C^\infty \) function, regarded as a smooth vector field on \( V \). Assume that the Sobolev weight \( m_0 = k - \frac{n}{p} \) is large enough so that \( L^p_k(\Sigma, \mathbb{R}^1) \) is a Banach algebra. Then the pull back of \( f \), \( \Psi_f : B = L^p_k(\Sigma, V) \to L^p_k(\Sigma, \mathbb{R}^d) \) is also smooth. Here \( \Psi_f(h) = f \circ h \).

**Proof:**

The detailed proof of this lemma is lengthy and will be given somewhere else. Here we only compute the derivatives formally.

We only need to prove the case that \( d = 1 \) and \( f : V \to \mathbb{R}^1 \). Denote \( \Psi_f \) by \( \Psi \). Then

\[
(D\Psi)_h(\xi) = \left. \frac{d}{dt} \Psi(h + t\xi) \right|_{t=0} = \left. \frac{d}{dt} f \circ (h + t\xi) \right|_{t=0}
\]
\[ \nabla f \cdot \xi = \sum_i \xi_i \frac{\partial f}{\partial x_i}. \]

The second derivative is
\[
(D^2 \Psi)_h(\xi, \eta) = \frac{\partial^2}{\partial t \partial s} \Psi(h + (t \xi + s \eta))|_{t=0, s=0} = \frac{\partial}{\partial s} \left( \sum_i \xi_i \frac{\partial f}{\partial x_i}(h + s \eta) \right)|_{s=0} = \sum_{i,j} \xi_i \eta_j \frac{\partial^2 f}{\partial x_i \partial x_j}(h).
\]

The higher derivatives can be calculated similarly.

It remains to show that all the formal derivatives calculated above are the genuine ones.

\[ \square \]

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