HOMOTOPICAL STABLE RANKS FOR CERTAIN $C^*$-ALGEBRAS

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Abstract. We study the general and connected stable ranks for $C^*$-algebras. We estimate these ranks for pullbacks of $C^*$-algebras, and for tensor products by commutative $C^*$-algebras. Finally, we apply these results to determine these ranks for certain commutative $C^*$-algebras, non-commutative CW-complexes, and certain continuous fields of Kirchberg algebras.

Stable ranks for $C^*$-algebras were first introduced by Rieffel [20] in his study of the nonstable K-theory of noncommutative tori. A stable rank of a $C^*$-algebra $A$ is a number associated to the $C^*$-algebra, and is meant to generalize the notion of covering dimension for topological spaces. The first such notion introduced by Rieffel, called topological stable rank, has played an important role ever since. In particular, the structure of $C^*$-algebras having topological stable rank one is particularly well understood.

Since the foundational work of Rieffel, many other ranks have been introduced for $C^*$-algebras, including real rank, decomposition rank, nuclear dimension, etc. In this paper, we return to the original work of Rieffel, and consider two other stable ranks introduced by him: the connected stable rank and general stable rank. The general stable rank determines the stage at which stably free projective modules are forced to be free. The connected stable rank is a related notion, but its definition is less transparent. What links these two ranks, and differentiates them from the topological stable rank, is that they are homotopy invariant. This is particularly striking since both these ranks are purely ring-theoretic notions.

This was highlighted in a paper by Nica [17], where he emphasized the relationship between these two ranks, and how they differ from topological stable rank. Furthermore, in order to compute these ranks for various examples, he showed how they behave with respect to some basic constructions (matrix algebras, quotients, inductive limits, and extensions).

The goal of this paper is to extend these results by examining how the connected and general stable rank (together referred to as homotopical stable ranks) behave with respect to iterated pullbacks and tensor products with commutative $C^*$-algebras, and thereby calculate these ranks for some familiar $C^*$-algebras. We mention here that the homotopical stable ranks play an important role in K-theory. Although we do not dwell on that much, we believe that further investigations into these ranks will yield a much better understanding of nonstable phenomena in K-theory.

We now describe our results. Henceforth, we write $tsr, gsr$, and $csr$ to denote the topological, general and connected stable ranks respectively. To begin with, consider a pullback diagram of unital $C^*$-algebras

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \gamma & D \\
\end{array}
\]

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where either \( \gamma \) or \( \delta \) is surjective. Note that \( tsr(A) \) can be estimated by a theorem of Brown and Pedersen [1, Theorem 4.1]

\[
tsr(A) \leq \max\{tsr(B), tsr(C)\}
\]

However, simple examples (See Example 2.1) show that the analogous estimate for \( gsr \) and \( csr \) cannot hold. Instead, we show that

**Theorem 1.** Given a pullback diagram as above,

\[
\begin{align*}
sgsr(A) &\leq \max\{csr(B), csr(C), gsr(C(\mathbb{T}) \otimes D)\}, \\
ccsr(A) &\leq \max\{csr(B), csr(C), csr(C(\mathbb{T}) \otimes D)\}
\end{align*}
\]

Furthermore, if \( K_1(D) = 0 \), the first estimate may be improved to

\[
sgsr(A) \leq \max\{sgsr(B), gsr(C), gsr(C(\mathbb{T}) \otimes D)\}
\]

We should mention here that the precise estimates are slightly finer than those mentioned above (See Proposition 2.6), and they depend on specific information about the homotopy groups of the groups \( GL_n(D) \) of invertible matrices over \( D \).

We turn to tensor products by commutative \( C^* \)-algebras. If \( Y \) is a compact Hausdorff space, a projective module over \( C(Y) \) corresponds to a vector bundle over \( Y \). Furthermore, if \( Y = \Sigma X \), the reduced suspension of \( X \), then any vector bundle over \( Y \) of rank \( n \) corresponds to the homotopy class of a map from \( X \) to \( GL_n(\mathbb{C}) \). Building on work of Rieffel [22], we describe all projective modules over \( C^* \)-algebras of the form \( C(\Sigma X) \otimes A \) in an analogous fashion. We conclude that

**Theorem 2.** For a compact Hausdorff space \( X \) and a unital \( C^* \)-algebra \( A \),

\[
sgsr(C(\Sigma X) \otimes A) = \max\{sgsr(A), inj_X(A)\}
\]

where \( inj_X(A) \) denotes the least \( n \geq 1 \) such that the map \([X, GL_{m-1}(A)] \rightarrow [X, GL_m(A)]\) is injective for all \( m \geq n \).

We also obtain various estimates for \( csr(C(X) \otimes A) \) as well which, once again, depend on the homotopy groups of \( GL_n(A) \). These estimates are particularly sharp in the case where the natural map \( GL_{m-1}(A) \rightarrow GL_n(A) \) is a weak homotopy equivalence. In that case, we completely determine the ranks of \( C(X) \otimes A \) in terms of those of \( A \) (Theorem 3.10).

Finally, we apply these results to a variety of examples. In particular, using Theorem 2, we determine \( gsr(C(\mathbb{T}^d)) \) (Example 4.4), thus answering a question of Nica [17, Problem 5.8]. We also estimate the homotopical stable ranks for noncommutative CW-complexes (Theorem 4.5), which naturally fall into the ambit of this paper. Finally, using an approximation theorem due to Dadarlat [4], we determine these ranks for certain continuous fields of Kirchberg algebras.

**Theorem 3.** Let \( A \) be a separable continuous field of \( C^* \)-algebras over a compact metric space of finite covering dimension, such that each fiber of \( A \) is a semi-projective Kirchberg algebra with torsion-free \( K_0 \)-group. Then

\[
sgsr(A) = csr(A) = 2
\]

Furthermore, the natural map \( GL_1(A)/GL_1^0(A) \rightarrow K_1(A) \) is an isomorphism.

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1. Preliminaries

1.1. Stable Ranks. Let $A$ be a unital $C^*$-algebra, and $n \in \mathbb{N}$. A vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ is said to be unimodular if $A_1 + A_2 + \ldots + A_n = A$. Equivalently, $\mathbf{a}$ is unimodular if $\exists \mathbf{a}' = (a_1', a_2', \ldots, a_n') \in A^n$ such that $(\mathbf{a}', \mathbf{a}) := \sum_{i=1}^{n} a_i' a_i = 1_A$. We write $Lg_n(A)$ for the set of all unimodular vectors in $A^n$. Note that unimodular vectors in our setting are usually referred to in the literature as left unimodular vectors - however, since $A$ carries a continuous involution, we do not make a distinction between left and right unimodular vectors.

Unimodular vectors are related to projective modules by the following observation (See [20, Lemma 10.4]): If $\mathbf{a} \in Lg_n(A)$, then let $\mathbf{a}' \in A^n$ be such that $(\mathbf{a}', \mathbf{a}) = 1_A$. Then the map $f : A^n \to A\mathbf{a}$ given by $\mathbf{x} \mapsto (\mathbf{a}', \mathbf{x})\mathbf{a}$ induces an isomorphism

$$A^n \cong P \oplus A\mathbf{a}$$

where $P = \ker(f)$. Conversely, if $P$ is a projective module over $A$ such that $P \oplus A \cong A^n$, then there is a unimodular vector $\mathbf{a} \in Lg_n(A)$ such that Equation (1) holds. The most interesting fact in all this is that this unimodular vector $\mathbf{a}$ also determines when $P$ is itself a free module.

Let $GL_n(A)$ denote the set of invertible elements in $M_n(A)$. Note that $GL_n(A)$ acts on $Lg_n(A)$ by left multiplication: $(T, \mathbf{a}) \mapsto T(\mathbf{a})$. Let $e_n \in Lg_n(A)$ denote the vector $(0, 0, \ldots, 0, 1)$. Now, if $P$ is a projective module over $A$ such that Equation (1) holds, then $P \cong A^{n-1}$ if and only if $\exists T \in GL_n(A)$ such that $T(\mathbf{a}) = e_n$ (See, for instance, [13, Proposition 4.14]). This leads to the following definition

**Definition 1.1.** Let $A$ be a unital $C^*$-algebra, then the general stable rank (gsr) of $A$ is the least $n \geq 1$ such that either (and hence both) of the following hold:

- $GL_m(A)$ acts transitively on $Lg_m(A)$ for all $m \geq n$
- For all $m \geq n$, if $P$ is a projective module over $A$ such that $P \oplus A \cong A^m$, then $P \cong A^{m-1}$

If no such $n$ exists, we simply write $gsr(A) = +\infty$. To avoid repetition, we will adopt the same convention in the definitions of connected and topological stable rank below.

Recall that $A$ has the invariant basis number (IBN) property if $A^m \cong A^n$ implies that $m = n$. This is equivalent to requiring that [A] has infinite order in $K_0(A)$. Now a swindle argument (See [13, Corollary 4.22]) shows that if $A$ does not have the IBN property, then $gsr(A) = +\infty$. Thus, in this paper, we will be concerned only with $C^*$-algebras having this property.

Given a $C^*$-algebra $A$ that has the IBN property, any projective module $P$ satisfying the condition $P \oplus A^m \cong A^n$ (ie. stably free projective modules) may be assigned a rank $(n - m)$, which is independent of the isomorphism. Hence, the general stable rank of $A$ simply determines the least rank at which stably free projective modules are forced to be free.

Now, the first condition of Definition 1.1 leads to another observation: Any subgroup of $GL_n(A)$ also acts on $Lg_n(A)$. We will be simultaneously interested in two subgroups of $GL_n(A)$. Let $GL^n_0(A)$ denote the connected component of the identity in $GL_n(A)$, and let $El_n(A)$ denote the subgroup of $GL_n(A)$ generated by elementary matrices, ie. matrices which differ from the identity matrix by at most one off-diagonal entry. Note that $El_n(A)$ is a subgroup of $GL^n_0(A)$. It was proved by Rieffel [20, Corollary 8.10] that, for $n \geq 2$, $GL^n_0(A)$ acts transitively on $Lg_n(A)$ if and only if $El_n(A)$ acts transitively on $Lg_n(A)$. Furthermore, he observed that the least $n$ at which this occurs also has a topological interpretation [20, Corollary 8.5], given as the second condition below.
**Definition 1.2.** Let $A$ be a unital $C^*$-algebra, then the connected stable rank (csr) of $A$ is the least $n \geq 1$ such that either (and hence both) of the following hold:

- $GL_m^0(A)$ acts transitively on $Lg_m(A)$ for all $m \geq n$
- $Lg_m(A)$ is connected for all $m \geq n$.

For $n \geq 2$ this is equivalent to the condition

- $El_m(A)$ acts transitively on $Lg_m(A)$ for all $m \geq n$

We observe by Definition 1.1 and Definition 1.2, that both these stable ranks are purely ring-theoretic notions. We now turn to the topological notion of stable rank that has proved to be most useful in applications.

**Definition 1.3.** Let $A$ be a unital $C^*$-algebra. Then the topological stable rank (tsr) of $A$ is the least $n \geq 1$ such that $Lg_n(A)$ is dense in $A^n$.

We mention here that if $Lg_n(A)$ is dense in $A^n$, then $Lg_m(A)$ is dense in $A^m$ for all $m \geq n$. However, the analogous statements are not true with respect to Definition 1.1 and Definition 1.2. Indeed, it is possible that $GL_n(A)$ acts transitively on $Lg_n(A)$, but $GL_{n+1}(A)$ does not act transitively on $Lg_{n+1}(A)$. For instance, if $A$ is a finite $C^*$-algebra, then $GL_1(A) = Lg_1(A)$, so $GL_1(A)$ clearly acts transitively on $Lg_1(A)$, but it is not true that $gsr(A) = 1$ when $A$ is finite.

**Remark 1.4.** If $A$ is a non-unital $C^*$-algebra, then the stable rank of $A$ is simply defined as the stable rank of $A^+$, the $C^*$-algebra obtained by adjoining a unit to $A$. We now enumerate some basic properties of these ranks that are known or are easily observed from the definitions. While the original proofs are scattered through the literature, [17] is an immediate reference for all these facts.

1. $gsr(A \oplus B) = \max\{gsr(A), gsr(B)\}$. Analogous statements hold for csr and tsr.
2. $gsr(A) \leq csr(A) \leq tsr(A) + 1$.
   Strict inequalities are possible in both cases. In fact, it is possible that $tsr(A) = +\infty$, while $csr(A) < \infty$.
3. For any $n \in \mathbb{N}$,
   $$csr(M_n(A)) \leq \left\lceil \frac{csr(A) - 1}{n} \right\rceil + 1, \quad \text{and} \quad gsr(M_n(A)) \leq \left\lceil \frac{gsr(A) - 1}{n} \right\rceil + 1$$
   Here, $[x]$ refers to the least integer greater than or equal to $x$.
4. If $\pi : A \to B$ is surjective, then
   $$csr(B) \leq \max\{csr(A), tsr(A)\}, \quad \text{and} \quad gsr(B) \leq \max\{gsr(A), tsr(A)\}$$
5. Furthermore, if $\pi : A \to B$ is a split epimorphism (i.e. there is a morphism $s : B \to A$ such that $\pi \circ s = id_B$), then
   $$csr(B) \leq csr(A), \quad \text{and} \quad gsr(B) \leq gsr(A)$$
6. If $0 \to J \to A \to B$ is an exact sequence of $C^*$-algebras, then
   $$csr(A) \leq \max\{csr(J), csr(B)\}, \quad \text{and} \quad gsr(A) \leq \max\{gsr(J), csr(B)\}$$
   It is worth mentioning here that when $J$ is an ideal of $A$, then there is, a priori, no relation between their homotopical stable ranks.
7. If $\{A_i : i \in J\}$ be an inductive system of $C^*$-algebras with $A := \varinjlim A_i$, then
   $$csr(A) \leq \liminf_i csr(A_i), \quad \text{and} \quad gsr(A) \leq \liminf_i gsr(A_i)$$
8. If $gsr(A) = 1$ (and hence if $csr(A) = 1$), then $A$ is stably finite. Conversely, if $gsr(A) \leq 2$ and $A$ is finite, then $gsr(A) = 1$. 4
Finally, we turn to the most interesting property shared by gsr and csr, viz. homotopy invariance. Two morphisms \( \phi_0, \phi_1 : A \to B \) are said to be homotopic if there is a path \( h : A \to C([0,1], B) \) such that \( \varphi_0 = p_0 \circ h \) and \( \varphi_1 = p_1 \circ h \), where \( p_t(\zeta) := \zeta(t) \). In symbols, we denote this by \( \varphi_0 \simeq \varphi_1 \). We say that \( A \) homotopically dominates \( B \) if there are morphisms \( \varphi : A \to B \) and \( \psi : B \to A \) such that \( \varphi \circ \psi \simeq \text{id}_B \). If, in addition, \( \psi \circ \varphi \simeq \text{id}_A \), then we say that \( A \) and \( B \) are homotopy equivalent (in symbols, \( A \simeq B \)). In the commutative case, \( C(X) \simeq C(Y) \) if and only if \( X \) and \( Y \) are homotopy equivalent as topological spaces (we once again, we write \( X \simeq Y \) if this happens). The following result is due to Nistor [18, Lemma 2.8] for the connected stable rank and Nica [17, Theorem 4.1] for the general stable rank.

**Theorem 1.5.** If \( A \) homotopically dominates \( B \), then
\[
\text{csr}(A) \geq \text{csr}(B), \quad \text{and} \quad \text{gsr}(A) \geq \text{gsr}(B)
\]
In particular, if \( A \simeq B \), then \( \text{csr}(A) = \text{csr}(B) \) and \( \text{gsr}(A) = \text{gsr}(B) \).

1.2. Axiomatic Homology. We now turn to the problem of computing these ranks. For the general stable rank, it would be ideal if one could describe all projective modules over a given \( C^* \)-algebra. We do so in some situations, but for the connected stable rank, we focus on the second condition of Definition 1.2 - that of determining when \( Lg_m(A) \) is connected.

An important tool in such an investigation is the following (See [21, Section 1]): For \( m \geq 2 \), the orbit of \( e_m \in Lg_m(A) \) under the action of \( GL_m(A) \) is called the space of last columns of \( A \), and is denoted by \( Lc_m(A) \). It was first proved by Corach and Larotonda [3] that the natural map \( t : GL_m(A) \to Lc_m(A) \) defines a principal, locally trivial fiber bundle on \( Lc_m(A) \), with structural group \( TL_m(A) \), the set of matrices of the form
\[
\begin{pmatrix}
x & 0 \\
c & 1
\end{pmatrix}
\]
where \( x \in GL_{m-1}(A) \) and \( c \in A^{m-1} \). Now, \( TL_m(A) \) is homotopy equivalent to \( GL_{m-1}(A) \), so the long exact sequence of homotopy groups arising from the fibration \( TL_m(A) \to GL_m(A) \to Lc_m(A) \) takes the form
\[
\ldots \to \pi_{n+1}(Lc_m(A)) \to \pi_n(GL_{m-1}(A)) \to \pi_n(GL_m(A)) \to \pi_n(Lc_m(A)) \to \ldots
\]
which ends in a sequence of pointed sets \( \pi_0(GL_{m-1}(A)) \to \pi_0(GL_m(A)) \to \pi_0(Lc_m(A)) \). Furthermore, it is clear that if \( m \geq \text{gsr}(A) \), then \( Lc_m(A) = Lg_m(A) \). These two observations led us to consider the work of Thomsen [24], where he develops two homology theories which will be of importance to us.

Recall [23] that a homology theory is a sequence \( \{ h_n \} \) of covariant functors from an admissible category \( \mathcal{D} \) of \( C^* \)-algebras to abelian groups which satisfies the following axioms:

- **Homotopy Axiom:** If \( \varphi_0, \varphi_1 : A \to B \) are homotopic morphisms (in the sense described above), then \( (\varphi_0)_* = (\varphi_1)_* : h_n(A) \to h_n(B) \) for all \( n \in \mathbb{N} \).
- **Exactness axiom:** Let \( 0 \to J \to A \to B \to 0 \) be a short exact sequence in \( \mathcal{D} \), then there is a map \( \partial : h_n(B) \to h_{n-1}(J) \) and a long exact sequence \( \ldots \to h_n(J) \to h_n(A) \to h_n(B) \to h_{n-1}(J) \to h_{n-1}(A) \to \ldots \). The map \( \partial \) is natural with respect to morphisms of short exact sequences.
These two axioms imply that any homology theory is additive: If $0 \to J \to A \to B$ is a split exact sequence in $\mathcal{D}$, then there is a natural isomorphism $h_n(A) \cong h_n(J) \oplus h_n(B)$ for all $n \in \mathbb{N}$.

We now review the work of Thomsen, where he extends the fibration described earlier to the non-unital setting. This allows him to define analogous objects for ideals of $C^*$-algebras, and thus develop a homology theory. To begin with, let $A$ be a $C^*$-algebra (not necessarily unital), $A^+$ the $C^*$-algebra obtained by adjoining a unit to $A$, and consider $A$ as an ideal of $A^+$. Define a composition on $A$ by

$$a \cdot b := a + b - ab$$

Now define $gl(A) := \{a \in A : \exists b \in A : a \cdot b = b \cdot a = 0\}$. If $A$ is unital, then the map $GL_1(A) \to gl(A)$ given by $v \mapsto (1 - v)$ is a homeomorphism. Furthermore, Thomsen defines

$$L_g(A) := \{(a_i) \in L_g(A^+) : a_1, a_2, \ldots, a_{n-1} \in A, a_n \equiv 1 \pmod{A}\}$$

$$GL_n(A) := \{x \in GL_n(A^+) : x \equiv 1 \pmod{M_n(A)}\}$$

Note that if $A$ is unital, this definition of $L_g(A)$ agrees with the earlier definition. Furthermore, $GL_n(A)$ is isomorphic to $gl(M_n(A))$. As before, there is an action of $GL_n(A^+)$ on $L_g(A)$. Let $L_c(A)$ denote the orbit of $e_n := (0,0,\ldots,1)$ under the action of $GL_n(A) \subset GL_n(A^+)$, and observe that $L_c(A) \subset L_g(A)$. Write $TL_n(A)$ for the stabilizer of $e_n$ in $GL_n(A)$, and note that $T\text{L}_n(A) \cong GL_{n-1}(A)$. This leads to the following

**Theorem 1.6.** [24, Theorem 2.5] For a fixed $m \in \mathbb{N}$, the functor

$$h_n(A) := \pi_n(GL_m(A))$$

defines a homology theory from the category of $C^*$-algebras to the category of groups.

Now consider the groups $\pi_n(Lc_m(A))$ for $m \in \mathbb{N}$ fixed. Thomsen proves [24, Lemma 3.8, 3.9], that if $0 \to J \to A \xrightarrow{\phi} B \to 0$ is a short exact sequence, then the induced map $q : Lc_m(A) \to q(Lc_m(B))$ is a fibration with fiber $Lc_n(J)$. Furthermore, he proves that the connected component on $q(Lc_m(A))$ is precisely $Lc_m(B)_0$, the connected component of $e_m \in Lc_m(B)$. Hence we obtain a long exact sequence of homotopy groups, which proves that

**Theorem 1.7.** For a fixed $m \in \mathbb{N}$, the functor

$$\tilde{h}_n(A) := \pi_n(Lc_m(A))$$

defines a homology theory from the category of $C^*$-algebras to the category of groups.

1.3. Notational Conventions. We fix some notation we will use repeatedly throughout the paper: We write $S^n$ for the $n$-dimensional sphere, $D^n$ for the $n$-dimensional disk, $\mathbb{R}^k$ for the $k$-fold product of the unit interval $I$, and $\mathbb{T}^k$ for the $k$-fold product of the circle $\mathbb{T}$. Given a $C^*$-algebra $A$ and a compact Hausdorff space $X$, we identify $C(X) \otimes A$ with $C(X, A)$, the space of continuous functions taking values in $A$. If $X = \mathbb{T}^k$, we simply write $\mathbb{T}^kA$ for $C(\mathbb{T}^k, A)$. We write $\theta_A^n$ for the map $GL_{n-1}(A) \to GL_n(A)$ given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

If there is no ambiguity, we simply write $\theta_A$ for this map. Given a unital *-homomorphism $\varphi : A \to B$, we write $\varphi_n$ for the induced maps in a variety of situations, such as $M_n(A) \to M_n(B), GL_n(A) \to GL_n(B), L_g(A) \to L_g(B)$, etc. Furthermore, when there is no ambiguity, we once again drop the subscript and denote the map by $\varphi$. Finally, when dealing with modules over a $C^*$-algebra, we will implicitly be referring to finitely generated right modules.
2. Homotopical Stable Ranks of Pullbacks

Given unital $\ast$-homomorphisms $\gamma : C \to D$ and $\delta : B \to D$, we consider the pullback

$$A := B \oplus_D C = \{(b, c) \in B \oplus C : \delta(b) = \gamma(c)\}$$

As usual, $A$ is best described by a pullback diagram, which we fix throughout the section

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow{\beta} & & \downarrow{\delta} \\
C & \xrightarrow{\gamma} & D
\end{array}$$

(3)

where $\alpha$ and $\beta$ are the projection maps. Furthermore, we assume that either $\gamma$ or $\delta$ is surjective. As pointed out in the example following [1, Theorem 4.1], this is quite a natural assumption when considering stable ranks. The goal then is to determine $\text{gsr}(A)$ and $\text{csr}(A)$ in terms of those of $B$ and $C$. To put things in perspective, we recall that the topological stable rank of $A$ may be estimated by [1, Theorem 4.1]

$$\text{tsr}(A) \leq \max\{\text{tsr}(B), \text{tsr}(C)\}$$

Note that the corresponding estimate for homotopical stable ranks cannot hold.

Example 2.1. Consider the pullback diagram

$$\begin{array}{ccc}
C(\mathbb{S}^n) & \xrightarrow{\gamma} & C(\mathbb{D}^n) \\
\downarrow{\delta} & & \downarrow{\gamma} \\
C(\mathbb{D}^n) & \xrightarrow{\delta} & C(\mathbb{S}^{n-1})
\end{array}$$

where $\gamma$ and $\delta$ are the natural restriction maps. Since $\mathbb{D}^n$ is contractible, $\text{gsr}(C(\mathbb{D}^n)) = 1$, but $\text{gsr}(C(\mathbb{S}^n)) > 1$ if $n \geq 5$ (See Proposition 3.1)

2.1. General Stable Rank. To determine $\text{gsr}(A)$, we now describe a recipe due to Milnor [15] to construct projective modules over $A$. Given a unital ring homomorphism $f : R \to S$ and a right $R$-module $M$, we write $f_\#(M)$ for the right $S$-module $S \otimes_R M$, and denote by $f_\ast$ the canonical map $M \to f_\#(M)$ given by $m \mapsto 1 \otimes_R m$. Note that if $M$ is a free $R$-module with basis $\{b_\alpha\}$, then $f_\#(M)$ is a free $S$-module with basis $\{f_\ast(b_\alpha)\}$.

Now consider a pullback diagram as above. Let $P$ and $Q$ be projective modules over $B$ and $C$ respectively, and suppose we are given a $D$-isomorphism

$$h : \delta_\#(P) \to \gamma_\#(Q)$$

Then consider

$$M := \{(p, q) \in P \oplus Q : h \circ \delta_\ast(p) = \gamma_\ast(q)\}$$

$M$ has a natural right $A$-module structure given by $(p, q) \cdot a := (p \cdot \alpha(a), q \cdot \beta(a))$. We denote this module by $M(P, Q, h)$. Milnor now proves the following

Theorem 2.2. [15, Theorem 2.1-2.3]

1. The module $M = M(P, Q, h)$ is projective over $A$. Furthermore, if $P$ and $Q$ are finitely generated over $B$ and $C$ respectively, then $M$ is finitely generated over $A$.

2. Every projective $A$-module is isomorphic to $M(P, Q, h)$ for some suitably chosen $P, Q$ and $h$.

3. The modules $P$ and $Q$ are naturally isomorphic to $\alpha_\#(M)$ and $\beta_\#(M)$ respectively.
Furthermore, one has the following result. Recall that, for our purposes, we are only interested in $C^*$-algebras that have the IBN property.

**Proposition 2.3.** [13, Corollary 13.11] Given a pullback diagram as above, suppose $B$ or $C$ has the IBN property. Let $h_1, h_2 \in GL_n(D)$, then $M(B^n, C^m, h_1) \cong M(B^n, C^m, h_2)$ if and only if $h_1 = \delta(S_1)h_2\gamma(S_2)$ for some $S_1 \in GL_n(B), S_2 \in GL_n(C)$.

For $h_1, h_2 \in GL_n(D)$, we write $h_1 \sim h_2$ if and only if $\exists S_1 \in GL_n(B), S_2 \in GL_n(C)$ such that $h_1 = \delta(S_1)h_2\gamma(S_2)$. Note that this is an equivalence relation, whose equivalence classes are the double cosets

$$\delta(GL_n(B))\setminus GL_n(D)/\gamma(GL_n(C)).$$

The following observation now determines how we proceed.

**Lemma 2.4.** Consider the pullback diagram as above, and suppose $h_1, h_2 \in GL_n(D)$ be such that $h_1 \sim h_2$. Then $M(B^n, C^m, h_1) \cong M(B^n, C^m, h_2)$

**Proof.** Without loss of generality, assume that $\gamma$ is surjective. Since $h_2^{-1}h_1 \in GL_n^0(D), \exists S_2 \in GL_n(C)$ such that $h_2^{-1}h_1 = \gamma(S_2)$, and so $h_1 = \delta(I_C^n)h_2\gamma(S_2)$. The result follows by Proposition 2.3. $\square$

Consider the sequence of groups

$$\{1_D\} = GL_0(D) \hookrightarrow D^\times = GL_1(D) \hookrightarrow GL_2(D) \hookrightarrow \cdots$$

For any compact Hausdorff space $X$, this induces a sequence of homotopy groups (of maps based at the identity)

$$[X, GL_0(D)] \rightarrow [X, GL_1(D)] \rightarrow [X, GL_2(D)] \rightarrow \cdots$$

We define

- $\text{inj}_X(D)$ to be the least $n \geq 1$ such that the map $(\theta_D)_*: [X, GL_{m-1}(D)] \rightarrow [X, GL_m(D)]$ is injective for all $m \geq n$.
- $\text{surj}_X(D)$ to be the least $n \geq 1$ such that $(\theta_D)_*: [X, GL_{m-1}(D)] \rightarrow [X, GL_m(D)]$ is surjective for all $m \geq n$.
- For $X = S^n$, we write $\text{inj}_n(D)$ and $\text{surj}_n(D)$ for $\text{inj}_X(D)$ and $\text{surj}_X(D)$ respectively.

**Remark 2.5.** Let $A$ be a $C^*$-algebra, and $X$ a compact Hausdorff space. Then, note that the natural map $Lg_m(C(X) \otimes A) \hookrightarrow C(X, Lg_m(A))$ given by evaluation is a homeomorphism (See [21, Lemma 2.3]). It follows that

$$\pi_0(Lg_m(C(X) \otimes A)) = [X, Lg_m(A)]$$

Furthermore, evaluation at a point gives a split epimorphism $C(X) \otimes A \rightarrow A$. Hence, it follows from Remark 1.4, (5) that

$$\text{csr}(C(X) \otimes A) \geq \text{csr}(A), \quad \text{and} \quad \text{gsr}(C(X) \otimes A) \geq \text{gsr}(A)$$

We will be most interested in the following quantities, which appear in the estimates for the homotopical stable rank of a pullback.

**Proposition 2.6.** For a unital $C^*$-algebra $D$,

$$\text{surj}_0(D) \leq \text{csr}(D), \quad \text{inj}_0(D) \leq \text{gsr}(\mathbb{T}D), \quad \text{and} \quad \max\{\text{inj}_0(D), \text{surj}_1(D)\} \leq \text{csr}(\mathbb{T}D)$$
Hence, the map $\delta_m$.

For the first inequality, suppose $m \geq csr(D)$, then $m \geq gsr(D)$, so $L_{cm}(D) = L_{gm}(D)$ is connected. Hence, the map $\pi_0(GL_{m-1}(D)) \to \pi_0(GL_m(D)) \to \pi_0(L_{cm}(D))$.

For the second inequality: If $m \geq gsr(\mathbb{T}D)$, then the map $t : GL_m(\mathbb{T}D) \to L_{cm}(\mathbb{T}D)$ is surjective. However, $m \geq gsr(D)$ by Remark 2.5, which implies $L_{cm}(D) = L_{gm}(D)$, so we may identify $L_{cm}(\mathbb{T}D) = L_{gm}(\mathbb{T}D)$ with $C(\mathbb{T}, L_{gm}(D))$. Also identifying $GL_m(\mathbb{T}D) = C(\mathbb{T}, GL_m(D))$, we observe that the map $t : \pi_1(GL_m(D)) \to \pi_1(L_{cm}(D))$ is surjective. It follows by exactness of the above sequence that the map $\pi_0(GL_{m-1}(D)) \to \pi_0(GL_m(D))$ is injective, as required.

For the third inequality, if $m \geq csr(\mathbb{T}D)$, then $m \geq csr(D) \geq gsr(D)$, so $L_{cm}(D) = L_{gm}(D)$. Furthermore, $L_{gm}(\mathbb{T}D) = C(\mathbb{T}, L_{gm}(D))$, so we see that

$$0 = \pi_0(L_{gm}(C(\mathbb{T}, D))) = \pi_0(C(\mathbb{T}, L_{gm}(D))) = \pi_1(L_{gm}(D))$$

Once again, the result follows from the exactness of the sequence above.

We are now ready to prove an estimate for the general stable rank of the pullback as in Equation (3)

**Theorem 2.7.** Given a pullback diagram as above with either $\gamma$ or $\delta$ surjective,

$$gsr(A) \leq \max\{csr(B), csr(C), inj_0(D)\}$$

Furthermore, if $K_1(D) = 0$, then

$$gsr(A) \leq \max\{gsr(B), gsr(C), inj_0(D)\}$$

**Proof.** Let $m \geq \max\{csr(B), csr(C), inj_0(D)\}$, and $M$ be a projective $A$-module such that $M \oplus A \cong A^m$. Then write $M = M(P, Q, h)$ for some $P, Q$ and $h$ as in Theorem 2.2. Then

$$M(P \oplus B, Q \oplus C, h \oplus I_D) \cong A^m = M(B^m, C^m, I_D^m)$$

Hence,

$$P \oplus B \cong \alpha_\#(M) \cong \alpha_\#(A^m) \cong B^m$$

Since $m \geq csr(B) \geq gsr(B)$, it follows that $P \cong B^{m-1}$. Similarly, $Q \cong C^{m-1}$. Hence, we may think of $h \in GL_{m-1}(D)$. Now consider the diagram

$$\begin{array}{ccc}
GL_{m-1}(B) \cong GL_{m-1}(D) & \xrightarrow{\delta_{m-1}} & GL_{m-1}(C) \\
\theta_B & & \theta_D & & \theta_C \\
GL_m(B) & \xrightarrow{\delta_m} & GL_m(D) & \xrightarrow{\gamma_m} & GL_m(C)
\end{array}$$

By Proposition 2.3, $\theta_D(h) \sim I_D^m$, so $\exists b \in GL_m(B)$, and $c \in GL_m(C)$ such that $\theta_D(h) = \delta_m(b)\gamma_m(c)$. Since $m \geq csr(B)$, Proposition 2.6 implies that $m \geq surj_0(B)$, so $\exists b' \in GL_{m-1}(B)$ such that $b \sim h \theta_B(b')$. Hence,

$$\delta_{m}(b) \sim h \delta_{m}(\theta_B(b')) = \theta_D(\delta_{m-1}(b'))$$

Similarly, $\exists c' \in GL_{m-1}(C)$ such that $\gamma_{m}(c) \sim h \theta_D(\gamma_{m-1}(c'))$ so that

$$\theta_D(h) \sim h \theta_D(\delta_{m-1}(b')\gamma_{m-1}(c'))$$
Since \( m \geq \text{inj}_0(D) \), \( h \sim_{h} \delta_{m-1}(b')\gamma_{m-1}(c') \) and so by Lemma 2.4 and Proposition 2.3, we have
\[
M \cong M(B^{m-1}, C^{m-1}, h) \cong A^{m-1}
\]
Hence, \( m \geq \text{gsr}(A) \) as required.

Now consider the special case where \( K_1(D) = 0 \): We follow the proof of the first part of the theorem until we obtain \( h \in GL_{m-1}(D) \). Note that, up until this point, we only used the fact that \( m \geq \max\{\text{gsr}(B), \text{gsr}(C)\} \). Since \( m \geq \text{inj}_0(D) \) and \( K_1(D) = 0 \), it follows that \( h \sim_{h} I_{D^{m-1}} \), so that \( M \cong M(B^{m-1}, C^{m-1}, h) \cong A^{m-1} \) by Lemma 2.4. We conclude that \( m \geq \text{gsr}(A) \) as required. \( \square \)

Note that Example 2.1 shows that the term \( \text{inj}_0(D) \) on the right hand side cannot be dropped, and furthermore, that equality can hold in the above estimate, since \( \text{inj}_0(C(S^4)) = \text{gsr}(C(S^5)) = 4 \).

2.2. Connected Stable Rank. Given a pullback diagram \( A = B \oplus_D C \) as before, we wish to estimate \( \text{csr}(A) \) in terms of \( \text{csr}(B) \) and \( \text{csr}(C) \). To do this, we exploit the homotopy theory \( A \mapsto \pi_n(L_c(A)) \) from Theorem 1.7, and the following result due to Schochet.

**Theorem 2.8.** [23, Theorem 4.5] Suppose that \( \{h_n\} \) is a homology theory for \( C^* \)-algebras in the sense described above. Given a pullback diagram \( A = B \oplus_D C \) as before with either \( \gamma \) or \( \delta \) surjective, there is a long exact sequence
\[
\ldots \to h_{n+1}(B) \oplus h_{n+1}(C) \to h_n(D) \to h_n(A) \xrightarrow{(\alpha, \beta)} h_n(B) \oplus h_n(C) \to \ldots
\]
From this, the following crucial estimate immediately follows

**Theorem 2.9.** Given a pullback diagram as above, with either \( \gamma \) or \( \delta \) surjective
\[
\text{csr}(A) \leq \max\{\text{csr}(B), \text{csr}(C), \text{inj}_0(D), \text{surj}_1(D)\}
\]
Together with Proposition 2.6 and Theorem 2.7, this theorem completes the proof of Theorem 1.

**Proof.** Applying the above theorem to the homology theory \( A \mapsto \pi_n(L_c(A)) \) gives a long exact sequence of homotopy groups that ends as a sequence of pointed sets
\[
\pi_1(L_c(B)) \oplus \pi_1(L_c(C)) \to \pi_1(L_c(D)) \to \pi_0(L_c(A)) \to \pi_0(L_c(B)) \oplus \pi_0(L_c(C)) \to \pi_0(L_c(D))
\]
If \( m \geq \max\{\text{csr}(B), \text{csr}(C), \text{inj}_0(D), \text{surj}_1(D)\} \), then we wish to show that \( \pi_0(L_g(A)) = 0 \). Since \( m \geq \text{csr}(B), \pi_0(L_g(B)) = 0 \). But since \( m \geq \text{gsr}(B) \), \( L_g(B) = L_c(B) \), whence \( \pi_0(L_c(B)) = 0 \).

Similarly, \( \pi_0(L_c(C)) = 0 \).

Now consider the long exact sequence of homotopy groups arising from the fibration \( TL_m(D) \to GL_m(D) \to L_c(D) \) (Equation (2)),
\[
\ldots \to \pi_1(GL_{m-1}(D)) \to \pi_1(GL_m(D)) \to \pi_1(L_c(D)) \to \pi_0(GL_{m-1}(D)) \to \pi_0(GL_m(D)) \to \ldots
\]
Since \( m \geq \max\{\text{inj}_0(D), \text{surj}_1(D)\} \), exactness implies that \( \pi_1(L_c(D)) = 0 \). Returning to the long exact sequence above, we conclude that \( \pi_0(L_c(A)) = 0 \). However, by Theorem 2.7, \( m \geq \text{gsr}(A) \), so \( L_c(A) = L_g(A) \). Hence, \( \pi_0(L_g(A)) = 0 \), proving that \( m \geq \text{csr}(A) \) as required. \( \square \)

3. Tensor products by commutative \( C^* \)-algebras

We now wish to calculate the homotopical stable ranks for algebras of the form \( C(X) \otimes A \). Once again, we consider the general and connected stable ranks separately.
3.1. General Stable Rank. To compute \( gsr(C(X) \otimes A) \), we wish to describe all projective modules over \( C(X) \otimes A \). If \( A = \mathbb{C} \), by the Serre-Swan theorem, this amounts to describing all vector bundles over \( X \). This is prohibitively difficult, of course, so we consider the potentially simpler situation, when \( X \) is itself a suspension.

Let \( X \) be a compact Hausdorff space, and \( x_0 \in X \) be a fixed base point. The reduced suspension of \( X \) is \( \Sigma X = (X \times \mathbb{I})/\sim \) where

\[
(0, x) \sim (0, x_0), (1, x) \sim (1, x_0), \text{ and } (s, x_0) \sim (0, x_0) \quad \forall x \in X, s \in \mathbb{I}
\]

Now we observe that vector bundles of rank \( n \) over \( \Sigma X \) correspond to homotopy classes of maps from \( X \) into \( GL_n(\mathbb{C}) \). Hence, \( gsr(C(\Sigma X)) \) is the least \( n \geq 1 \) such that the map \( [X, GL_{m-1}(\mathbb{C})] \to [X, GL_m(\mathbb{C})] \) induced by \( \theta_C \) is injective for all \( m \geq n \). In our notation, this simply gives

\[
gsr(C(\Sigma X)) = \text{inj}_X(\mathbb{C})
\]

This is precisely the observation used by Nica to give the first non-trivial calculation of the general stable rank.

**Proposition 3.1.** [17, Proposition 5.5]

\[
gsr(C(\mathbb{S}^d)) = \begin{cases} \frac{d}{2} + 1 & : \text{if } d > 4, \text{ and } d \notin 4\mathbb{Z} \\ \frac{d}{2} & : \text{if } d > 4, \text{ and } d \in 4\mathbb{Z} \\ 1 & : d \leq 4 \end{cases}
\]

The goal of this section is to expand on this idea, by explicitly describing projective modules over \( C(\Sigma X) \otimes A \), where \( A \) is a unital \( C^* \)-algebra, which allows us to prove an analog of Equation (4).

To begin with, we fix a unital \( C^* \)-algebra \( A \), and we identify functions \( f : \Sigma X \to A \) with functions \( f : \mathbb{I} \times X \to A \) such that

\[
f(0, x) = f(1, x) = f(s, x_0) \quad \forall x \in X, s \in \mathbb{I}
\]

We now follow [22] to construct projective modules over \( C(\Sigma X) \otimes A \). Given a projective right \( A \)-module \( V \), \( \text{Aut}_A(V) \) is equipped with the point-norm topology. Let \( C_{x_0}(X, \text{Aut}_A(V)) \) be the space of continuous functions from \( u : X \to \text{Aut}_A(V) \) such that \( u(x_0) = \text{id}_V \). Given a projective right \( A \)-module \( V \) and \( u \in C_{x_0}(X, \text{Aut}_A(V)) \), we define

\[
X(u) = \{ \varphi : \mathbb{I} \times X \to V : \varphi(0, x) = \varphi(s, x_0) \text{ and } \varphi(1, x) = u(x)\varphi(0, x) \quad \forall x \in X, \quad \forall s \in \mathbb{I} \}
\]

Note that \( X(u) \) is a right \( C(\Sigma X) \otimes A \)-module with the action given by

\[
(\varphi \cdot f)(t, x) := \varphi(t, x)f(t, x)
\]

**Lemma 3.2.** If \( u_0, u_1 \) are path connected in \( C_{x_0}(X, \text{Aut}_A(V)) \), then \( X(u_0) \cong X(u_1) \)

**Proof.** Let \( H : \mathbb{I} \to C_{x_0}(X, \text{Aut}_A(V)) \) be a path such that \( H(0) = u_0, H(1) = u_1 \), then define \( F : X(u_0) \to X(u_1) \) by

\[
F(\varphi)(s, x) := H(s, x)u_0(x)^{-1}\varphi(s, x)
\]

Since \( H \) is a path in \( C_{x_0}(X, \text{Aut}_A(V)) \), we may think of it as a map \( H : \mathbb{I} \times X \to \text{Aut}_A(V) \). So \( F(\varphi)(1, x) = H(1, x)u_0(x)^{-1}\varphi(1, x) = u_1(x)\varphi(0, x) \) and \( F(\varphi)(0, x) = \varphi(0, x) \). Hence, \( F \) is well-defined. Also, \( F \) is clearly a module homomorphism because the action of \( C(\Sigma X) \otimes A \) is on the right. To show that \( F \) is an isomorphism, we define \( G : X(u_1) \to X(u_0) \) by

\[
G(\psi)(s, x) := u_0(x)H(s, x)^{-1}\psi(s, x)
\]

Then \( G \) is a well-defined module homomorphism such that \( G \circ F = \text{id}_{X(u_0)} \) and \( F \circ G = \text{id}_{X(u_1)} \). 

The proof of the next two lemmas is entirely obvious from the definition.
Lemma 3.3. If \( V_1 \) and \( V_2 \) are projective, right \( A \)-modules and \( u_1 \in C_{x_0}(X, \text{Aut}_A(V_1)), u_2 \in C_{x_0}(X, \text{Aut}_A(V_2)), \) then
\[
X(u_1 \oplus u_2) \cong X(u_1) \oplus X(u_2)
\]

Lemma 3.4. If \( \iota_{\mathbb{A}^n} \in C_{x_0}(X, \text{GL}_n(A)) \) denotes the identity automorphism on \( \mathbb{A}^n \), then
\[
X(\iota_{\mathbb{A}^n}) \cong (C(\Sigma X) \otimes A)^n
\]

Lemma 3.5. Let \( u, v \in C_{x_0}(X, \text{Aut}_A(V)) \) such that \( X(u) \cong X(v) \), then \( \exists w \in C(X, \text{Aut}_A(V)) \) such that \( v \sim_{h} wuw^{-1} \) in \( C_{x_0}(X, \text{Aut}_A(V)) \)

Proof. Note that \( X(u) \) and \( X(v) \) are both section algebras of locally trivial bundles over \( \Sigma X \) with fibers \( V \), so if \( X(u) \cong X(v) \), then the isomorphism is implemented by a map \( \tilde{g} : \Sigma X \to \text{Aut}_A(V) \). As in Equation (5), we identify \( \tilde{g} \) with a function \( g : \mathbb{I} \times X \to \text{Aut}_A(V) \) such that
\[
g(0, x) = g(1, x) = g(s, x_0) = \text{id}_V \quad \forall x \in X, \quad \forall s \in \mathbb{I}
\]

Then note that for any \( \varphi \in X(u) \),
\[
v(x)g(0, x)\varphi(0, x) = v(x)(g(\varphi))(0, x) = g(\varphi)(1, x) = g(1, x)\varphi(1, x) = g(1, x)u(x)\varphi(0, x)
\]

Hence, it follows that \( v(x)g(0, x) = g(1, x)u(x) \) for all \( x \in X \), so that
\[
v(x) = g(1, x)u(x)g(0, x)^{-1} \quad \forall x \in X
\]

Let \( w \in C(X, \text{Aut}_A(V)) \) be given by \( w(x) := g(0, x) \), and let \( H : \mathbb{I} \times X \to C_{x_0}(X, \text{Aut}_A(V)) \) be given by \( (t, x) \mapsto g(t, x)u(x)g(0, x)^{-1} \), then
\[
H(0, x) = w(x)u(x)w(x)^{-1} \quad \text{and} \quad H(1, x) = v(x)
\]

Furthermore, \( H(s, x_0) = g(s, x_0)u(x_0)g(0, x_0)^{-1} = u(x_0) = \text{id}_V \) for all \( s \in \mathbb{I} \). Hence \( H \) implements a homotopy \( v \sim_{h} wuw^{-1} \) in \( C_{x_0}(X, \text{Aut}_A(V)) \). \( \square \)

Lemma 3.6. Every projective \( C(\Sigma X) \otimes A \)-module is isomorphic to some \( X(u) \) for some projective \( A \)-module \( V \) and some \( u \in C_{x_0}(X, \text{Aut}_A(V)) \).

Proof. A projective \( C(\Sigma X) \otimes A \)-module \( M \) is isomorphic to \( P((C(\Sigma X) \otimes A)^n) \) for some projection \( P \in M_n(C(\Sigma X) \otimes A) \). We identify \( P \) with a map \( P : \mathbb{I} \times X \to M_n(A) \) satisfying Equation (5). Let \( p := P(0, x_0) \) and \( V := p(A^n) \). If we think of \( P \) as a path \( P : \mathbb{I} \to C(X) \otimes M_n(A) \), then there is a path of unitaries \( U : \mathbb{I} \to GL(C(X) \otimes M_n(A)) \) such that
\[
P(t, x) = U(t, x)^{-1}P(0, x)U(t, x) = U(t, x)^{-1}pU(t, x)
\]

Furthermore, we have \( U(0, x) = \text{id}_{A^n} = U(s, x_0) \quad \forall x \in X, \ s \in \mathbb{I} \). Hence,
\[
U(1, x)^{-1}pU(1, x) = U(1, x)^{-1}P(0, x)U(1, x) = P(1, x) = P(0, x) = p
\]

so \( U(1, x)p = pU(1, x) \), so we define \( u(x) := U(1, x)p \in \text{Aut}_A(V) \) and \( u \in C_{x_0}(X, \text{Aut}_A(V)) \). Finally, if \( f \in P(C(\Sigma X) \otimes A)^n) \), then we think of \( f \) as a function \( f : \mathbb{I} 
\times X \to A^n \) satisfying Equation (5) and \( P(t, x)f(t, x) = f(t, x) \). Hence, we may define \( \varphi : \mathbb{I} \times X \to V \) by
\[
\varphi(t, x) := U(t, x)f(t, x)
\]

and this is well-defined because
\[
p\varphi(t, x) = P(0, x)U(t, x)f(t, x) = U(t, x)p(t, x)f(t, x) = U(t, x)f(t, x) = \varphi(t, x)
\]

Furthermore, \( \varphi(1, x) = U(1, x)f(1, x) = U(1, x)f(0, x) \) and \( \varphi(0, x) = f(0, x) \). Hence, \( \varphi \in X(u) \). It is then easy to check that the map that sends \( f \) to \( \varphi \) is an isomorphism from \( P(C(\Sigma X) \otimes A)^n) \) to \( X(u) \). \( \square \)

We are now ready to prove the main theorem of this section. Recall that \( \text{inj}_X(A) \) is the least \( n \geq 1 \) such that the map \( (\theta_A)_* : [X, GL_{m-1}(A)] \to [X, GL_m(A)] \) is injective for all \( m \geq n \).
Theorem 3.7.  

\[ gsr(C(\Sigma X) \otimes A) = \max\{gsr(A), inj_X(A)\} \]

**Proof.** For simplicity of notation, write \( B := C(\Sigma X) \otimes A \). Let \( n \geq \max\{gsr(A), inj_X(A)\} \), and let \( P \) be a projective module over \( B \) such that \( P \otimes B \cong B^n \). By Lemma 3.6, there exists a projective \( A \)-module \( V \) and a map \( u \in C_{x_0}(X, \text{Aut}_A(V)) \) such that \( P \cong X(u) \). The map \( \pi : B \to A \) given by evaluation at \([0, x_0]\) \( \in \Sigma X \) is a ring homomorphism, so

\[ \pi_!(P) \oplus A \cong A^n \]

But \( \pi_!(P) \cong V \) and \( gsr(A) \leq n \) so \( V \cong A^{n-1} \). Hence, we may think of \( u \in C_{x_0}(X, GL_{n-1}(A)) \).

Now note that

\[ X(u \oplus \iota_A) \cong X(\iota_{A^n}) \]

so by Lemma 3.5 and Lemma 3.4, \( u \oplus \iota_A \sim_h \iota_{A^n} \) in \( C_{x_0}(X, GL_n(A)) \). Since \( n \geq inj_X(A) \), it follows that \( u \sim_h \iota_{A^n-1} \), whence \( P \cong B^{n-1} \) by Lemma 3.2. Hence, \( gsr(B) \leq n \) as required.

For the reverse inequality, let \( n \geq gsr(B) \), then by Remark 2.5, \( gsr(A) \leq n \). Now suppose \( u \in C_{x_0}(X, GL_{n-1}(A)) \) is such that \( u \oplus \iota_A \sim_h \iota_{A^n} \), then let \( P = X(u) \). By Lemma 3.3 and Lemma 3.4,

\[ P \oplus B \cong X(u \oplus \iota_A) \cong X(\iota_{A^n}) \cong B^n \]

By hypothesis, \( P \cong B^{n-1} \cong X(\iota_{A^n-1}) \). By Lemma 3.5, \( u \sim_h \iota_{A^n-1} \) in \( C_{x_0}(X, GL_{n-1}(A)) \), and so \( inj_X(A) \leq n \), completing the proof.

3.2. Connected Stable Rank. Let \( A \) be a \( C^* \)-algebra, and \( X \) a compact Hausdorff space, then we wish to determine estimates for \( csr(C(X) \otimes A) \) in terms of \( \dim(X) \) and other parameters that depend on \( A \). To this end, we notice that if \( X \) is a CW-complex of dimension \( \leq n \), we may write \( X = X_0 \cup \varphi \mathbb{D}^n \), where \( X_0 \) is a CW-complex of dimension \( \leq n \) and \( \varphi : \mathbb{S}^n \to X_0 \) is the attaching map. By [16, Lemma 1.4], we have a pullback diagram

\[
\begin{array}{ccc}
C(X) \otimes A & \to & C(X_0) \otimes A \\
\downarrow & & \downarrow \varphi^* \\
C(\mathbb{D}^n) \otimes A & \gamma & C(\mathbb{S}^{n-1}) \otimes A
\end{array}
\]

where \( \gamma \) is the restriction map. By Theorem 2.9, we get

\[
csr(C(X) \otimes A) \leq \max\{csr(C(X_0) \otimes A), csr(A), inj_0(C(S^{n-1}) \otimes A), surj_1(C(S^{n-1}) \otimes A)\}
\]

**Lemma 3.8.** Let \( D = C(S^{n-1}) \otimes A \), then

\[ \max\{inj_0(D), surj_1(D)\} \leq \max\{surj_1(A), inj_0(A), surj_n(A), inj_{n-1}(A)\} \]

**Proof.** Note that the homology theory \( A \mapsto \pi_n(GL(A)) \) is additive (See the discussion in Section 1.2). If \( A \) is any unital \( C^* \)-algebra, we have a split exact sequence

\[ 0 \to C_0(\mathbb{R}^{n-1}) \otimes A \to C(S^{n-1}) \otimes A \to A \to 0 \]

Furthermore, by [24, Lemma 2.3],

\[ \pi_n(GL(A)) \cong \pi_0(GL(C_0(\mathbb{R}^{n-1}) \otimes A)) \]

Hence, there is a natural isomorphism

\[
\pi_k(GL(C(S^{n-1}) \otimes A)) \cong \pi_{n+k-1}(GL(A)) \oplus \pi_k(GL(A))
\]
Applied to \( k = 1 \), it follows that if \( m \geq \max\{\text{surj}_1(A), \text{inj}_0(A), \text{surj}_n(A), \text{inj}_{n-1}(A)\} \), then

\[
\begin{align*}
\pi_1(GL_{m-1}(D)) & \to \pi_1(GL_m(D)) \text{ is surjective, and} \\
\pi_0(GL_{n-1}(D)) & \to \pi_0(GL_n(D)) \text{ is injective}
\end{align*}
\]

proving that \( m \geq \max\{\text{inj}_0(D), \text{surj}_1(D)\} \) as required. \( \square \)

This leads to the following estimate.

**Theorem 3.9.** Let \( X \) be a compact Hausdorff space of dimension \( \leq n \), then

\[
\text{csr}(C(X) \otimes A) \leq \max\{\text{csr}(A), \text{inj}_0(A), \text{surj}_1(A), \text{surj}_k(A), \text{inj}_{k-1}(A) : 1 \leq k \leq n\}
\]

**Proof.** If \( X \) is a compact Hausdorff space of dimension \( \leq n \), then \( X \) is an inverse limit of compact metric spaces \((X_i)\) such that \( \dim(X_i) \leq n \) [14]. Since \( C(X) \otimes A \cong \lim C(X_i) \otimes A \), it follows from Remark 1.4, (7) that \( \text{csr}(C(X) \otimes A) \leq \lim \inf \text{csr}(C(X_i) \otimes A) \). Furthermore, if \( X \) is a compact metric space of dimension \( \leq n \), then \( X \) is an inverse limit of finite CW-complexes \((Y_i)\), each of which has dimension \( \leq n \) [7]. Once again, \( \text{csr}(C(X) \otimes A) \leq \lim \inf \text{csr}(Y_i) \otimes A \). Hence, it suffices to assume that \( X \) is itself a finite CW-complex of dimension \( \leq n \).

By induction, we may assume that \( X = X_0 \cup \varphi \mathbb{D}^n \) where \( X_0 \) is a finite CW-complex of dimension \( \leq (n-1) \) and \( \varphi : S^{n-1} \to X_0 \) is a continuous function. As mentioned at the start of this section, it follows that

\[
\text{csr}(C(X) \otimes A) \leq \max\{\text{csr}(C(X_0) \otimes A), \text{csr}(C(\mathbb{D}^n) \otimes A), \text{inj}_0(C(S^{n-1}) \otimes A), \text{surj}_1(C(S^{n-1}) \otimes A)\}
\]

By homotopy invariance, \( \text{csr}(C(\mathbb{D}^n) \otimes A) = \text{csr}(A) \), so the result now follows by induction and Lemma 3.8. \( \square \)

We now turn our attention to a particularly tractable class of \( C^* \)-algebras. Let \( \mathcal{F} \) be the class of \( C^* \)-algebras \( A \) such that the map \( \theta_A : GL_{m-1}(A) \to GL_m(A) \) induces a weak homotopy equivalence for all \( m \geq 2 \). The following algebras are known to be in \( \mathcal{F} \).

- [9] If \( Z \) denotes the Jiang-Su algebra, then \( A \otimes Z \in \mathcal{F} \) for any \( C^* \)-algebra \( A \). In particular, if \( A \) is a separable, approximately divisible \( C^* \)-algebra, then \( A \cong A \otimes Z \), so \( A \in \mathcal{F} \).
- [21] If \( A \) is a non-commutative irrational torus, then \( A \in \mathcal{F} \).
- [24] If \( O_n \) denotes the Cuntz algebra, then \( A \otimes O_n \in \mathcal{F} \) for any \( C^* \)-algebra \( A \).
- [24] If \( A \) is an infinite dimensional simple \( AF \)-algebra, then \( A \otimes B \in \mathcal{F} \) for any \( C^* \)-algebra \( B \).
- [27] If \( A \) is a purely infinite simple \( C^* \)-algebra, and \( p \) any non-zero projection of \( A \), then \( pA p \in \mathcal{F} \).

Note that \( A \in \mathcal{F} \) if and only if \( \pi_n(Lc_m(A)) = 0 \) for all \( n \geq 1 \). Furthermore, if \( A \in \mathcal{F} \), then

\[
\pi_n(GL_m(A)) \cong \begin{cases} K_1(A) & : n \text{ even} \\ K_0(A) & : n \text{ odd} \end{cases}
\]

In particular, the natural map \( GL_1(A)/GL_0(A) \to K_1(A) \) is an isomorphism.

**Theorem 3.10.** Let \( A \in \mathcal{F} \), and let \( X \) be a compact Hausdorff space, then

\[
\begin{align*}
gsr(C(X) \otimes A) & = gsr(A) \\
csr(C(X) \otimes A) & = \begin{cases} \text{csr}(A) & : \text{csr}(A) \geq 2 \\ 1 \text{ or } 2 & : \text{csr}(A) = 1 \end{cases}
\end{align*}
\]
Proof. We first consider the connected stable rank: \( GL_k(A) \) is an open subset of a normed linear space, and so has the homotopy type of a CW-complex [12, Chapter IV, Corollary 5.5]. Since \( \theta_A \) is a weak homotopy equivalence, it follows from Whitehead’s theorem that it must be a homotopy equivalence. In particular, \([X, GL_{m-1}(A)] \to [X, GL_m(A)]\) is bijective for all \( m \in \mathbb{N} \).

For \( m \geq \max\{2, gsr(A)\} \), consider the fibration \( TL_m(A) \to GL_m(A) \to Lc_m(A) \). Since \( X \) is compact, \( \theta \) has the homotopy lifting property for \( X \) by a Theorem of Dold [6, Theorem 4.8]. Since \( TL_m(A) \simeq GL_{m-1}(A) \), we get a long exact sequence as in Equation (2)

\[
\ldots \to [X, GL_{m-1}(A)] \to [X, GL_m(A)] \to [X, Lc_m(A)] \to \ldots
\]

Since the map \([X, GL_{m-1}(A)] \to [X, GL_m(A)]\) is bijective, \([X, Lc_m(A)] = \{0\}\). However, since \( m \geq gsr(A), Lg_m(A) = Lc_m(A) \).

Furthermore, identifying \( Lg_m(C(X) \otimes A) \) with \( C(X, Lg_m(A)) \), we get

\[
\pi_0(Lg_m(C(X) \otimes A)) = [X, Lg_m(A)] = [X, Lc_m(A)] = \{0\}
\]

whence \( csr(C(X) \otimes A) \leq m \).

Hence,

\[
csr(C(X) \otimes A) \leq \max\{2, gsr(A)\}
\]

Now the result follows from the fact that \( gsr(A) \leq csr(A) \leq csr(C(X) \otimes A) \) (See Remark 2.5).

Now for the general stable rank: By the first part of the argument, we have

\[
gsr(A) \leq gsr(C(X) \otimes A) \leq \max\{2, gsr(A)\}
\]

If \( gsr(A) \geq 2 \), we have nothing to show. If \( gsr(A) = 1 \), then \( A \) must be stably finite, and hence \( C(X) \otimes A \) is finite. Since \( gsr(C(X) \otimes A) \leq 2 \), it must be that \( gsr(C(X) \otimes A) = 1 \) by Remark 1.4, (8). This completes the proof. \( \square \)

If \( X \) has finite covering dimension, then Theorem 3.10 follows from Theorem 3.9. However, the value of this proof is that we don’t need \( X \) to be finite dimensional.

Example 3.11. Some examples illustrate our results:

1. If \( A \in \mathcal{F} \) and \( csr(A) = 1 \), then it is possible that \( csr(C(X) \otimes A) = 2 \), depending on \( X \). For instance, if \( A \) is a simple, infinite dimensional, unital \( AF \)-algebra, then \( csr(A) = 1 \). Taking \( X = \mathbb{T} \), we see that \( K_1(\mathbb{T}A) \cong K_0(A) \oplus K_1(A) \neq 0 \) because \( A \) is stably finite. Hence, \( csr(\mathbb{T}A) = 2 \) by Remark 1.4, (9).

2. If \( A \) is a non-commutative torus, then \( sr(A) = 1 \) and \( K_1(A) \neq 0 \), so \( gsr(A) = 1 \) and \( csr(A) = 2 \) by Remark 1.4, (9) and (10). It follows that

\[
gsr(C(X) \otimes A) = 1 \quad \text{and} \quad csr(C(X) \otimes A) = 2
\]

for any compact Hausdorff space \( X \). This was proved by Rieffel [20, Proposition 2.5, 2.7] in the case where \( X = \mathbb{T}^k \). In fact, these were crucial in proving that \( A \in \mathcal{F} \).

3. If \( A \) is a Kirchberg algebra, then it was proved by Xue [25] that \( gsr(A) = csr(A) = 2 \) if and only if \( A \) has the IBN property (Otherwise \( gsr(A) = csr(A) = +\infty \)). Furthermore, as mentioned above, \( A \in \mathcal{F} \) by [27]. Hence, we see that

\[
gsr(C(X) \otimes A) = csr(C(X) \otimes A) = 2
\]

for any compact Hausdorff \( X \). In particular, this is true for \( A = \mathcal{O}_\infty \).

4. If \( A \) is a \( C^\ast \)-algebra of real rank zero, then it has been proved in [10, Lemma 2.2] that \( inj^0(A) = 1 \). Hence, it follows from Theorem 3.7 that \( gsr(\mathbb{T}A) = gsr(A) \). This is precisely the argument in [26, Proposition 3.1].
4. Examples and Calculations

We now turn to a few examples that have informed this investigation.

4.1. Commutative $C^*$-algebras. If $X$ and $Y$ are two compact Hausdorff spaces and $X \vee Y$ denotes their wedge sum, then $C(X \vee Y) \cong C(X) \oplus C(Y)$ where the maps $C(X) \to \mathbb{C}$ and $C(Y) \to \mathbb{C}$ are the evaluation maps at the common base point. Hence, we get the following satisfying corollary to Theorems 2.7 and 2.9.

**Corollary 4.1.** For any two compact Hausdorff spaces $X$ and $Y$,

$$gsr(C(X \vee Y)) = \max\{ggr(C(X)), ggsr(C(Y))\}, \text{ and}$$

$$cgr(C(X \vee Y)) = \max\{cgr(C(X)), cgsr(C(Y))\}$$

**Proof.** For the general stable rank: The inclusion map $i : X \hookrightarrow X \vee Y$ induces a surjection $i^* : C(X \vee Y) \to C(X)$. Furthermore, the ‘pinching’ map $P : X \vee Y \to X$ have the property that $i^* \circ P^* = \text{id}_{C(X)}$. So it follows from Remark 1.4, (5) that $ggsr(C(X \vee Y)) \geq ggr(C(X))$. By symmetry, the same true for $Y$, so

$$ggr(C(X \vee Y)) \geq \max\{ggr(C(X)), ggr(C(Y))\}$$

Now observe that $K_1(\mathbb{C}) = 0, \text{inj}_{0,0}(\mathbb{C}) = 1$ so the result follows from Theorem 2.7.

For the connected stable rank: The same argument as above shows that

$$\max\{cgr(C(X)), cgsr(C(Y))\} \leq cgr(C(X \vee Y)) \leq \max\{cgr(C(X)), cgsr(C(Y)), 2\}$$

where the second inequality follows from Theorem 2.9 and the fact that $\text{surj}_1(\mathbb{C}) = 2$. Thus, if $\max\{cgr(C(X)), cgsr(C(Y))\} \geq 2$, then the conclusion follows. Suppose $cgr(C(X)) = cgsr(C(Y)) = 1$. We must conclude that $cgr(C(X \vee Y)) = 1$. By the above inequality, we know that $cgr(C(X \vee Y)) \leq 2$. Hence, it suffices to show that $L_{g1}(C(X \vee Y))$ is connected. However,

$$\pi_0(L_{g1}(C(X \vee Y))) = \pi_0(C(X \vee Y, L_{g1}(\mathbb{C})) \cong [X \vee Y, \mathbb{T}]$$

Since $cgr(C(X)) = cgsr(C(Y)) = 1$, we know that $[X, \mathbb{T}]$ and $[Y, \mathbb{T}]$ are both trivial. If $f : X \vee Y \to \mathbb{T}$ is a map then $f \circ c : X \to \mathbb{T}$ must be null-homotopic. Similarly, if $j : Y \hookrightarrow X \vee Y$ denotes the inclusion map, then $f \circ j$ is also null-homotopic. Assuming all maps and homotopies preserve the common base point, we may paste the two homotopies together to conclude that $f$ is null-homotopic. Hence, $L_{g1}(C(X \vee Y))$ is connected, whence $cgr(C(X \vee Y)) = 1$ as required.

Our next goal is determining $ggr(C(\mathbb{T}^d))$. Observe that $C(\mathbb{T} \times X) = \mathbb{T}A$ where $A = C(X)$. From the discussion preceding Proposition 3.1, $ggr(C(\Sigma X)) = \text{inj}_X(\mathbb{C}) = \text{inj}_0(C(X))$, thus the next corollary follows directly from Theorem 3.7.

**Corollary 4.2.** If $X$ is a compact Hausdorff space,

$$ggr(C(\mathbb{T} \times X)) = \max\{ggr(C(X)), ggsr(C(X))\}$$

Recall that a space $X$ is said to homotopically dominate $Y$ if there is a map $P : X \to Y$ which has a homotopy right inverse. This is equivalent to saying that $C(X)$ homotopically dominates $C(Y)$, so it follows from Theorem 1.5 that

$$ggr(C(X)) \geq ggr(C(Y))$$

if $X$ homotopically dominates $Y$. In order to calculate $ggr(C(\mathbb{T}^d))$, we need

**Lemma 4.3.** If $X = \prod_{i=1}^k S^{n_i}$, then $\Sigma X$ homotopically dominates $S^{n+1}$ where $n = \sum_{i=1}^k n_i$. In particular, $\Sigma \mathbb{T}^n$ homotopically dominates $S^{n+1}$
Proof. We claim that
\[ \Sigma X \simeq \mathbb{S}^{n+1} \lor M \]
for some manifold \( M \) of dimension \( \leq n \). To see this, we proceed by induction on \( k \). It is clearly true if \( k = 1 \), so let \( Y = \prod_{i=1}^{k-1} \mathbb{S}^{n_i} \) and assume \( \Sigma Y \simeq \mathbb{S}^{\ell+1} \lor N \), where \( \ell = \sum_{i=1}^{k-1} n_i \) and \( N \) is a manifold of dimension \( \leq \ell \). Then by [8, Proposition 4I.1],
\[
\Sigma X = \Sigma(Y \times \mathbb{S}^{n_k}) \simeq \Sigma Y \lor \mathbb{S}^{n_k+1} \lor \Sigma(Y \lor \mathbb{S}^{n_k}) \\
\lor \mathbb{S}^{\ell+1} \lor N \lor \mathbb{S}^{n_k} \lor \Sigma(n_k(\mathbb{S}^{\ell+1} \lor N)) \\
\lor \mathbb{S}^{\ell+1} \lor N \lor \mathbb{S}^{n_k+1} \lor \Sigma(n_k(N) \lor \mathbb{S}^{\ell+n_k+1} \lor \Sigma \lor \mathbb{S}^{n+1} \lor \mathbb{S}^{n+1}
\]
where \( M = \mathbb{S}^{\ell+1} \lor N \lor \mathbb{S}^{n_k+1} \lor \Sigma n_k(N) \). Note that
\[
\dim(M) \leq \max\{\ell + 1, \ell, n_k + 1, n_k + \ell\} \leq n_k + \ell = n
\]
This proves the claim. So we get a map \( P : \Sigma X \rightarrow M \lor \mathbb{S}^{n+1} \lor \mathbb{S}^{n_k} \lor \Sigma X \) by composing the homotopy equivalence with the natural map \( \mathbb{S}^{n+1} \rightarrow \mathbb{S}^{n+1} \lor M \). Note that \( P_* : H_{n+1}(\Sigma X) \rightarrow H_{n+1}(\mathbb{S}^{n+1}) \) is an isomorphism because \( \dim(M) \leq n \), and \( f_* : H_{n+1}(\mathbb{S}^{n+1}) \rightarrow H_{n+1}(\Sigma X) \) is also an isomorphism. Hence,
\[
(P \circ f)_* : H_{n+1}(\mathbb{S}^{n+1}) \rightarrow H_{n+1}(\mathbb{S}^{n+1})
\]
is an isomorphism. Since both \( P \) and \( f \) are orientation-preserving, it follows that \( P \circ f \) has degree 1, and so \( P \circ f \simeq \text{id}_{\mathbb{S}^{n+1}} \) as required. \( \square \)

The following is an answer to a question posed by Nica [17, Problem 5.8]. Before we begin, we observe that if \( X \) is a compact Hausdorff space whose covering dimension is \( \leq 4 \), then Nica has shown [17, Proposition 5.5] that \( gsr(C(X)) = 1 \). The point of this next example, thus, is using the previous lemma to compare \( gsr(C(\mathbb{T}^d)) \) and \( gsr(C(\mathbb{S}^d)) \) for \( d \geq 5 \).

To put this in perspective, if \( X \) is a compact Hausdorff space of covering dimension \( \leq n \), then \( csr(C(X)) \leq \lceil n/2 \rceil + 1 \) by [18, Corollary 2.5]. Furthermore, Nica has shown [17, Theorem 5.3] that this upper bound is attained provided the top cohomology group \( H^\text{odd}(X) \) is non-vanishing. In particular, this implies that, for all \( d \geq 1 \),
\[
csr(C(\mathbb{T}^d)) = \left\lceil \frac{d}{2} \right\rceil + 1
\]

Example 4.4.

\[
gsr(C(\mathbb{T}^d)) = \begin{cases} 1 & : d \leq 4 \\ \left\lceil \frac{d}{2} \right\rceil + 1 & : d > 4 \end{cases}
\]

Proof. For \( d \leq 4 \), the result follows from the preceding discussion. For \( d \geq 5 \), we know that
\[
gsr(C(\mathbb{T}^d)) \leq csr(C(\mathbb{T}^d)) \leq \left\lceil \frac{d}{2} \right\rceil + 1
\]
so it suffices to prove the reverse inequality. We proceed by induction on \( d \). For \( d = 5 \), by Corollary 4.2 and Lemma 4.3,
\[
gsr(C(\mathbb{T}^5)) \geq gsr(C(\Sigma \mathbb{T}^4)) \geq gsr(C(\mathbb{S}^5))
\]
and \( gsr(C(\mathbb{S}^5)) = 4 \) by Proposition 3.1. For \( d \geq 6 \), by induction
\[
gsr(C(\mathbb{T}^d)) = \max\{gsr(C(\mathbb{T}^{d-1})), gsr(C(\Sigma \mathbb{T}^{d-1}))\} \geq \max\left\{\left\lceil \frac{d-1}{2} \right\rceil + 1, gsr(C(\mathbb{S}^d))\right\}
\]
Once again the result follows from Proposition 3.1. \( \square \)
4.2. **NonCommutative CW-Complexes.** As observed in Section 3.2, a commutative $C^*$-algebras whose spectrum is a finite CW-complex can be expressed as an (iterated) pullback. Noncommutative CW-complexes (NCCW complexes), first studied by Pedersen [19], are meant to generalize this idea: A NCCW complex $A_0$ of dimension 0 is a finite dimensional $C^*$-algebra. A NCCW complex $A_k$ of dimension $k$ is described by a pullback

$$
A_k \xrightarrow{\gamma} A_{k-1} \\
C(\mathbb{D}^k) \otimes F_k \xrightarrow{\pi} C(\mathbb{S}^{k-1}) \otimes F_k
$$

where $F_k$ is a finite dimensional $C^*$-algebra, $A_{k-1}$ is an NCCW complex of dimension $(k-1)$, and $\gamma$ is the restriction map. If $F$ is a finite dimensional $C^*$-algebra, then it follows from Remark 1.4 that $\text{csr}(F) = 1$. Hence, $\text{csr}(A_0) = 1$ and $\text{csr}(C(\mathbb{D}^k) \otimes F_k)) = \text{csr}(F_k) = 1$ by homotopy invariance. If $D = C(\mathbb{S}^{k-1}) \otimes F_k$, then by Lemma 3.8,

$$\max\{\text{inj}_0(D), \text{surj}_1(D)\} \leq \max\{\text{surj}_1(F_k), \text{inj}_0(F_k), \text{surj}_k(F_k), \text{inj}_{k-1}(F_k)\}$$

Write $F_k = \bigoplus_{i=1}^{n_k} M_{\ell_i}(\mathbb{C})$, then $\text{inj}_0(F_k) = 1$ and $\text{surj}_1(F_k) = 2$, so computing the right hand side boils down to asking whether, for all $1 \leq i \leq n_k$, the map

$$\pi_k(GL_{\ell_i(m-1)}(\mathbb{C})) \to \pi_k(GL_{\ell_i,m}(\mathbb{C}))$$

is surjective, and

$$\pi_{k-1}(GL_{\ell_i,m-1}(\mathbb{C})) \to \pi_{k-1}(GL_{\ell_i,m}(\mathbb{C}))$$

is injective

By Bott periodicity, these maps are isomorphisms if $k \leq 2\ell_i(m-1) - 1$ (See, for instance, [11, Page 251-254]). Furthermore, if $k = 2\ell_i(m-1)$, then both conditions are satisfied because the second map is an isomorphism, and $\pi_k(GL_{\ell_i,m}(\mathbb{C})) = 0$. So if $d_k = \min\{\ell_i : 1 \leq i \leq j_k\}$, then we have

$$\max\{\text{inj}_0(D), \text{surj}_1(D)\} \leq \left\lfloor \frac{k}{2d_k} \right\rfloor + 1$$

The following estimate is thus a corollary of Theorem 2.9

**Theorem 4.5.** Let $A_n$ be an NCCW complex of topological dimension atmost $n$ whose structure can be described as above. Then

$$\text{csr}(A_n) \leq \max_{1 \leq k \leq n} \left\{ \left\lfloor \frac{k}{2d_k} \right\rfloor + 1 \right\} \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$$

A special case of this theorem is that of a commutative $C^*$-algebra whose spectrum is a finite CW-complex. As in the proof of Theorem 3.9, by passing to inductive limits we obtain yet another proof of a well-known result.

**Theorem 4.6.** [18, Corollary 2.5] If $X$ is a compact Hausdorff space of dimension $\leq n$, then $\text{csr}(C(X)) \leq \lfloor n/2 \rfloor + 1$

4.3. **Continuous Fields of Kirchberg algebras.** The goal of this section is to apply our earlier results to understand the homotopical stable ranks and nonstable K-theory of certain continuous fields that are particularly amenable to our methods. Throughout this section, let $\mathcal{C}$ denote a class of $C^*$-algebras consisting of finite direct sums of semiprojective Kirchberg algebras with the IBN property. By Example 3.11(2), for any compact Hausdorff space $Y$, and $D \in \mathcal{C}$,

$$\text{gsr}(C(Y) \otimes D) = \text{csr}(C(Y) \otimes D) = 2$$

We wish to extend this result to separable continuous fields of $C^*$-algebras over a finite dimensional, compact metric space, each of whose fibers are in $\mathcal{C}$. 

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In order to prove this, we need an approximation theorem due to Dadarlat [4, Theorem 5.2], which says that any such continuous field can be expressed as an inductive limit of certain iterated pullbacks. Before we describe the result, we need to briefly recall the basic definitions about continuous fields of $C^*$-algebras.

Let $X$ be a locally compact Hausdorff space. Recall that a $C(X)$-algebra is a $C^*$-algebra endowed with a $*$-homomorphism $\theta$ from $C_0(X)$ to $ZM(A)$, the center of the multiplier algebra of $A$ such that $C_0(X)A$ is dense in $A$. We write $fa$ for $\theta(f)a$ for any $f \in C_0(X)$ and $a \in A$. If $Y \subset X$ is a closed subspace, let $C_0(X,Y)$ denote the ideal of $C_0(X)$ consisting of functions vanishing on $Y$. Then $C_0(X,Y)A$ is a closed two-sided ideal of $A$ (by the Cohen factorization theorem [2, Theorem 4.6.4]). The quotient of $A$ by this ideal is denoted by $A(Y)$, and the quotient map is denoted $\pi_Y : A \rightarrow A(Y)$. If $Y = \{x\}$ is a point, then write $A(x)$ for $A(\{x\})$, called the fiber of $A$ at $x$, and we write $\pi_x$ for $\pi_{\{x\}}$. Thus, for any $a \in A$, we obtain a function $X \rightarrow \mathbb{R}$ by $x \mapsto \|\pi_x(a)\|$. If this map is continuous for each $a \in A$, then we say that $A$ is a continuous field of $C^*$-algebras over $X$.

We now describe the iterated pullbacks mentioned earlier. From here on, we will be interested in continuous fields whose fibers lie in the class $\mathcal{C}$ mentioned earlier.

**Definition 4.7.** [4, Definition 2.7] Let $Z$ be a compact Hausdorff space. A $C(Z)$-algebra $E$ is called $\mathcal{C}$-elementary if there is a finite partition $\{Z_1, Z_2, \ldots, Z_n\}$ of $Z$ into closed sets and $C^*$-algebras $D_1, D_2, \ldots, D_n \in \mathcal{C}$ such that

$$E \cong \bigoplus_{i=1}^{n} C(Z_i, D_i)$$

The category of a $C(X)$-algebra with respect to a class $\mathcal{C}$ is defined inductively as follows: If $A$ is $\mathcal{C}$-elementary, then $\text{cat}_\mathcal{C}(A) = 0$; $\text{cat}_\mathcal{C}(A) \leq n$ if there are closed subsets $Y$ and $Z$ of $X$ with $X = Y \cup Z$ and there is a $C(Y)$-algebra such that $\text{cat}_\mathcal{C}(B) \leq n-1$, and a $\mathcal{C}$-elementary $C(Z)$-algebra $E$, and a $*$-monomorphism of $C(Y \cap Z)$-algebras $\gamma : E(Y \cap Z) \rightarrow B(Y \cap Z)$ such that $A$ is isomorphic to the pullback

$$B \oplus_{B(Y \cap Z)} E := \{(b, d) \in B \oplus E : \pi_Y^Y \cap Z(b) = \gamma \pi_Y^Z \cap Z(d)\}$$

By definition, $\text{cat}_\mathcal{C}(A)$ is $n$ if $n$ is the smallest number such that $\text{cat}_\mathcal{C}(A) \leq n$. If not such $n$ exists, then $\text{cat}_\mathcal{C}(A) = +\infty$.

The following is the approximation theorem mentioned earlier.

**Theorem 4.8.** [4, Theorem 5.2] Let $\mathcal{C}$ be a class consisting of finite direct sums of semiprojective simple $C^*$-algebras. Let $X$ be a finite dimensional compact metrizable space and let $A$ be a separable continuous field of $C^*$-algebras over $X$ such that all its fibers admit exhaustive sequences consisting of $C^*$-algebras isomorphic to $C^*$-algebras in $\mathcal{C}$. Then $A$ is isomorphic to the inductive limit of a sequence $A_k$ of continuous fields of $C^*$-algebras over $X$ such that $\text{cat}_\mathcal{C}(A_k) \leq \dim(X)$.

**Lemma 4.9.** If $\text{cat}_\mathcal{C}(A) \leq n$ as a $C(X)$-algebra and $Y \subset X$ is closed, then $\text{cat}_\mathcal{C}(A(Y)) \leq n$ as a $C(Y)$-algebra.

**Proof.** We induct on $n := \text{cat}_\mathcal{C}(A)$. If $n = 0$, then $A = \bigoplus_{i=1}^{n} C(Z_i, D_i)$ and $A(Y) \cong \bigoplus_{i=1}^{n} C(Y \cap Z_i, D_i)$ so that $\text{cat}_\mathcal{C}(A(Y)) = 0$ as well. Now suppose the theorem is true for $k \leq n-1$, and suppose $\text{cat}_\mathcal{C}(A) = n$. Then write $A = B \oplus_D E$, where $X = Y_1 \cup Y_2$. $B$ is a $C(Y_1)$-algebra such that $\text{cat}_\mathcal{C}(B) \leq n-1$ and $E$ is a $C(Y_2)$-algebra such that $\text{cat}_\mathcal{C}(E) = 0$, and $D = B(Y_1 \cap Y_2)$. By induction, $\text{cat}_\mathcal{C}(B(Y)) \leq n-1$ and $\text{cat}_\mathcal{C}(E(Y)) = 0$, so it suffices to prove that

$$A(Y) \cong B(Y) \oplus_{D(Y)} E(Y)$$
The goal is to apply [19, Proposition 9.2], so for simplicity, we adopt the same notation. Let \( U = X \setminus Y \), and consider the commutative diagram whose rows are extensions:

\[
\begin{array}{ccc}
0 & \longrightarrow & E(U) \longrightarrow E \longrightarrow E(Y) \longrightarrow 0 \\
\alpha & & \pi & & \alpha \\
0 & \longrightarrow & D(U) \longrightarrow D \longrightarrow D(Y) \longrightarrow 0 \\
\beta & & \beta & & \beta \\
0 & \longrightarrow & B(U) \longrightarrow B \longrightarrow B(Y) \longrightarrow 0
\end{array}
\]

and consider the induced diagram

\[
\begin{array}{ccc}
A(U) & \longrightarrow & A \longrightarrow A(Y) \\
\overline{\gamma} & & \gamma \\
B(U) & \longrightarrow & B \longrightarrow B(Y) \\
\overline{\beta} & & \delta \\
E(U) & \longrightarrow & E \longrightarrow E(Y) \\
\overline{\pi} & & \overline{\pi} \\
D(U) & \longrightarrow & D \longrightarrow D(Y)
\end{array}
\]

Note that the vertical square in the middle is a pullback, so in order to apply [19, Proposition 9.2], we need to verify two things:

1. \( \ker(\overline{\gamma}) \cap \ker(\delta) = \{0\} \): To see this, fix \( x \in \ker(\overline{\gamma}) \cap \ker(\delta) \), so \( x = \rho(a) \) for some \( a \in A \). Now \( \overline{\gamma} \circ \rho = \pi \circ \overline{\gamma} \), so \( \overline{\gamma}(a) \in \ker(\pi_B) = B(U) \). Similarly, \( \overline{\delta}(a) \in E(U) \). However, since \( C_0(U) \) is nuclear, we have that \( A(U) \cong B(U) \oplus D(U) E(U) \) by [19, Theorem 3.9]. Hence, \( a \in A(U) \), so that \( x = \rho(a) = 0 \).

2. \( \overline{\pi}(E) \cap \overline{\beta}(B) \cap D(U) \subseteq \alpha(E(U)) + \beta(B(U)) \): To see this, fix \( x \in D(U), e \in E, b \in B \) such that \( x = \overline{\pi}(e) = \overline{\beta}(b) \). For any \( f \in C_0(U) \), we have \( fe \in C_0(U) E = E(U) \), so

\[
f x = \overline{\pi}(fe) = \overline{\pi}(E(U)) = \alpha(E(U))
\]

However, since \( D(U) = C_0(U)D \), if \( \{f_\lambda\} \) is an approximate unit in \( C_0(U) \), then \( f_\lambda x \to x \). Since \( \alpha(E(U)) \) is closed, it follows that \( x \in \alpha(E(U)) \subseteq \alpha(E(U)) + \beta(B(U)) \) as required.

\[\square\]

**Theorem 4.10.** Let \( \mathcal{C} \) denote a class of \( C^* \)-algebras consisting of finite direct sums of semi-projective, Kirchberg algebras with the IBN property. Let \( X \) be a finite dimensional compact metrizable space, and let \( A \) be a separable, continuous field of \( C^* \)-algebras over \( X \) such that all its fibers admit exhaustive sequences consisting of \( C^* \)-algebras isomorphic to \( C^* \)-algebras in \( \mathcal{C} \), then

\[
gsr(C(W) \otimes A) = csr(C(W) \otimes A) = 2
\]

for any compact Hausdorff space \( W \).

**Proof.** First note that if \( A \) satisfies the hypotheses of theorem, then each fiber of \( A \) is purely infinite since it absorbs \( \mathcal{O}_\infty \) tensorially. Hence, \( A \) is infinite by [5, Theorem 7.4], so \( gsr(A) > 1 \). Since \( gsr(A) \leq gsr(C(W) \otimes A) \leq csr(C(W) \otimes A) \), it suffices to show that \( csr(C(W) \otimes A) \leq 2 \).
By Theorem 4.8, $A$ can be expressed as an inductive limit of continuous fields of $C^*$-algebras $A_k$ satisfying $\text{cat}_C(A_k) \leq \dim(X)$. Since $C(W) \otimes A \cong \lim \text{dim}(C(W) \otimes A_k)$, it suffices to assume that $\text{cat}_C(A) \leq \dim(X)$, by Remark 1.4 (7). We now induct on $n := \text{cat}_C(A)$. If $n = 0$, then $A \cong \bigoplus_{i=1}^n C(Z_i, D_i)$, so by Remark 1.4 (1)

$$\text{csr}(C(W) \otimes A) = \max\{\text{csr}(C(W \times Z_i) \otimes D_i) : 1 \leq i \leq n\} = 2$$

If $n > 0$, assume that the result is true for $k \leq n - 1$, and write $A = B \oplus D E$, where $\text{cat}_C(B) \leq n - 1, \text{cat}_C(E) = 0$ and $D = B(Y \cap Z)$. By the previous lemma, $\text{cat}_C(D) \leq n - 1$ as a $C(Y \cap Z)$-algebra, so by induction hypothesis,

$$\max\{\text{csr}(C(W) \otimes B), \text{csr}(C(W) \otimes E), \text{csr}(C(W \times T) \otimes D)\} \leq 2$$

Since $C(W)$ is nuclear, [19, Theorem 3.9] implies that $C(W) \otimes A \cong C(W) \otimes B \oplus_{C(W) \otimes D} C(W) \otimes E$ so it follows from Theorem 1 that $\text{csr}(C(W) \otimes A) \leq 2$ as required.

In particular, if $A$ is as in the previous theorem, then $\text{csr}(\mathbb{T}^k A) = 2$ for all $k \in \mathbb{N}$. As proved in [21, Theorem 3.2], this implies that $\pi_n(Lc_m(A)) = 0$ for all $m \geq 2$, and $n \geq 0$. Hence, we obtain the following corollary.

**Corollary 4.11.** If $A$ is a continuous field of $C^*$-algebras satisfying the conditions of the previous theorem, then

$$\pi_n(GL_m(A)) \cong \begin{cases} K_1(A) & : n \text{ even} \\ K_0(A) & : n \text{ odd} \end{cases}$$

In particular, the natural map $GL_1(A)/GL_1^0(A) \to K_1(A)$ is an isomorphism.

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