BOREL SETS OF RADO GRAPHS AND RAMSEY’S THEOREM

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Abstract. The well-known Galvin-Prikry Theorem [10] states that Borel subsets of the Baire space are Ramsey: Given any Borel subset $\mathcal{X} \subseteq [\omega]^{\omega}$, where $[\omega]^{\omega}$ is endowed with the metric topology, each infinite subset $X \subseteq \omega$ contains an infinite subset $Y \subseteq X$ such that $[Y]^{\omega}$ is either contained in $\mathcal{X}$ or disjoint from $\mathcal{X}$. Kechris, Pestov, and Todorcevic point out in [12] the dearth of similar results for homogeneous structures. Such results are a necessary step to the larger goal of finding a correspondence between structures with infinite dimensional Ramsey properties and topological dynamics, extending their correspondence between the Ramsey property and extreme amenability.

In this article, we prove an analogue of the Galvin-Prikry theorem for the Rado graph. Any such infinite dimensional Ramsey theorem is subject to constraints following from work in [14]. The proof uses techniques developed for the author’s work on the Ramsey theory of the Henson graphs ([4] and [6]) as well as some new methods for fusion sequences, used to bypass the lack of a certain amalgamation property enjoyed by the Baire space.

1. Introduction

Ramsey theory was initiated by the following celebrated result.

Theorem 1.1 (Infinite Ramsey Theorem, [20]). Given positive integers $m$ and $\ell$, suppose the collection of all $m$-element subsets of $\omega$ is partitioned into $\ell$ pieces. Then there is an infinite subset $N \subseteq \omega$ such that all $m$-element subsets of $N$ are contained in the same piece of the partition.

In the arrow notation, this is written as follows:

\[ \forall m, j \geq 1, \omega \to ([\omega]^{m})^\ell. \]

One may ask whether analogues of this theorem exist when, instead of $m$-sized sets, one wants to partition the infinite sets of natural numbers into finitely many pieces. Using standard set-theoretic notation, $\omega$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$, $[\omega]^{\omega}$ denotes the set of all infinite subsets of $\omega$, and given $X \in [\omega]^{\omega}$, the collection of infinite subsets of $X$ is denoted by $[X]^{\omega}$. Erdős and Rado [8] showed that there is a partition of $[\omega]^{\omega}$ into two sets such that for each $X \in [\omega]^{\omega}$, the set $[X]^{\omega}$ intersects both pieces of the partition. However, this example is highly non-constructive, using the Axiom of choice to generate the partition, and Dana Scott suggested that all sufficiently definable sets might satisfy an infinite dimensional Ramsey analogue. This was proven to be the case, as we now review.

We hold to the convention that sets of natural numbers are enumerated in increasing order, and we write $s \sqsubseteq X$ exactly when $s$ is an initial segment of $X$.

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The collection of finite subsets of natural numbers is denoted by $[\omega]^\omega$. The Baire space is the set $[\omega]^\omega$ with the topology generated by basic open sets of the form \( \{ X \in [\omega]^\omega : s \subseteq X \} \), for $s \in [\omega]^\omega$. We call this the metric topology since it is the topology generated by the metric defined as follows: For distinct $X, Y \in [\omega]^\omega$, $\rho(X, Y) = 2^{-n}$, where $n$ is maximal such that $X$ and $Y$ have the same initial segment of cardinality $n$. A subset $X \subseteq [\omega]^\omega$ is called Ramsey if there is an $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq X$ or else $[X]^\omega \cap X = \emptyset$.

The first achievement in the line of infinite dimensional Ramsey theory is the result of Nash-Williams in [17] showing that clopen subsets of the Baire space are Ramsey. Three years later, Galvin stated in [9] that this generalizes to all open sets in the Baire space. Soon after, the following significant result was proved by Galvin and Prikry. In order to present their result, first a bit of terminology is introduced.

Given a finite set $s \in [\omega]^\omega$ and an infinite set $X \in [\omega]^\omega$, let
\[
(s, X) = \{ Y \in [X]^\omega : s \subseteq Y \}.
\]

A subset $X \subseteq [\omega]^\omega$ is called completely Ramsey if for each finite $s$ and infinite $X$ with $s \subseteq X$, there is a $Y \in [s, X]$ such that either $[s, Y] \subseteq X$ or else $[s, Y] \cap X = \emptyset$.

Theorem 1.2 (Galvin and Prikry, [10]). Every Borel subset of the Baire space is completely Ramsey.

It follows that Borel sets are Ramsey. This weaker statement is written as
\[
\omega^\text{Borel} \subseteq [\omega]^\omega.
\]

Shortly after this, Silver proved in [22] that analytic subsets of the Baire space are completely Ramsey. The apex of results on infinite dimensional Ramsey theory of the Baire space was attained by Ellentuck in [7]. He used the idea behind completely Ramsey sets to introduce a topology refining the metric topology on the Baire space. In current terminology, the topology generated by the basic open sets of the form $(s, X)$ in equation (2) is called the Ellentuck topology. Ellentuck used this topology to precisely characterize those subsets of $[\omega]^\omega$ which are completely Ramsey.

Theorem 1.3 (Ellentuck, [7]). A subset $X$ of $[\omega]^\omega$ is completely Ramsey if and only if $X$ has the property of Baire in the Ellentuck topology.

Remark 1.4. The notion of a completely Ramsey subset of the Baire space defined above is due to Galvin and Prikry and was used by Silver in [22]. The definition of completely Ramsey used in [10] is actually slightly stronger, but we use the form defined above, as it is the most widely known and provides the best analogy for our results.

Expanding now to the setting of structures, given a structure $\mathcal{B}$ and a substructure $\mathcal{A}$ of $\mathcal{B}$, let $\mathcal{B}(\mathcal{A})$ denote the set of all copies of $\mathcal{A}$ in $\mathcal{B}$. A Fraïssé class $\mathcal{K}$ has the Ramsey property if for any $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ with $\mathcal{A}$ embedding into $\mathcal{B}$, for any $\ell \in \omega$, there is some $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{B}$ embeds into $\mathcal{C}$ and for any coloring of $\mathcal{B}(\mathcal{A})$ into $\ell$ colors, there is some $\mathcal{B}' \in \mathcal{B}(\mathcal{A})$ all members of $\mathcal{B}(\mathcal{A})$ have the same color. In [12], Kechris, Pestov, and Todorcevic proved a beautiful correspondence between the Ramsey property and topological dynamics: The group of automorphisms of the Fraïssé limit $\mathcal{K}$ (also called a Fraïssé structure) of a Fraïssé order class $\mathcal{K}$ is extremely amenable if and only if $\mathcal{K}$ has the Ramsey property (Theorem 4.7). In
Problem 11.2, they ask for the topological dynamics analogue of a corresponding infinite Ramsey-theoretic result for several Fraïssé structures, in particular, the rationals, the Rado graph, and the Henson graphs. By an infinite Ramsey-theoretic result, they mean a result of the form

\[(4) \quad \forall \ell \in \omega, \ K \to^*_{\ell,t}(K)\]

where equation (4) reads: “For each \( \ell \in \omega \) and each partition of \((K)\) into \( \ell \) many definable subsets, there is a \( J \in (K) \) such that \( (J) \) is contained in no more than \( t \) of the pieces of the partition.” Here, one assumes a natural topology on \((K)\) and definable refers to any reasonable class of sets definable relative to the topology, for instance, open, Borel, analytic, or property of Baire. A sub-question implicit in Problem 11.2 in [12] is the following:

**Question 1.5.** For which ultrahomogeneous structures \( K \) is there some positive integer \( t \) such that for all \( \ell \in \omega \), \( K \to^*_{\ell,t}(K) \)?

This question makes sense in view of the natural topology on \((K)\) inherited as a subspace of the Baire space. Since the universe \( K \) of \( K \) is countable, we may assume that \( K = \omega \). Letting \( \iota : (K) \to [\omega]^{<\omega} \) be the map defined by \( \iota(J) = J \), the universe of \( J \), for each \( J \in (K) \), we see that the \( \iota \)-image of \((K)\) forms a subspace of the Baire space, \([\omega]^{<\omega} \). Kechris, Pestov, and Todorcevic point out that very little is known about Question 1.5, and immediately move on to discuss the problem of big Ramsey degrees of Fraïssé structures.

We give a brief word about big Ramsey degrees of Fraïssé structures as they present constraints towards answering Question 1.5. A Fraïssé limit \( K \) of a Fraïssé class \( K \) is said to have finite big Ramsey degrees if for each \( A \in K \), there is some positive integer \( t \) such that for each \( \ell \geq 2 \),

\[(5) \quad K \to (K)^A_{\ell,t}. \]

This is the structural analogue of the infinite Ramsey Theorem [14] as the copies of some finite structure are partitioned into finitely many pieces, and one wants a copy of the infinite structure which meets as few of the pieces as possible. When such a \( t \) exists for a given \( A \), using the notation and terminology from [12], we let \( T(A,K) \) denote the minimal such \( t \) and call this the big Ramsey degree of \( A \) in \( K \). In all known cases, the big Ramsey degree \( T(A,K) \) corresponds to a canonical partition of \((K)\) into \( T(A,K) \) many pieces each of which is persistent, meaning that for any member \( J \) of \((K)\), the set \( (J) \) meets every piece in the partition. Thus, it can be useful to think of the existence of finite big Ramsey degrees as a structural Ramsey theorem where one finds some \( J \in (K) \) so that \( (J) \) achieves one color for all copies of \( A \) in the same piece of the canonical partition.

Big Ramsey degrees for the rationals as a linear order were studied by Sierpiński, Galvin, and Laver, culminating in work of Devlin [2]. The Rado graph was shown to have finite big Ramsey degrees in [21] (extending prior work in [19] for edge colorings), exact degrees being generated in [14] and calculated in [15]. Finite big Ramsey degrees were proved for ultrahomogeneous Urysohn spaces in [18] and for rationals with finitely many equivalence relations in [13]. Zucker recently answered Question 11.2 of Kechris, Pestov, and Todorcevic in [25] in the context of big Ramsey degrees, finding a correspondence between big Ramsey structures.
(Fraïssé structures with big Ramsey degrees which cohere in a natural manner) and topological dynamics.

A fundamental constraint toward answering Question 1.5 for the Rado graph, which we denote by $\mathbb{R}$, comes from work of Sauer in [21] and its culmination in work by Laflamme, Sauer, and Vuksanovic in [14]. In those papers, they use antichains in the tree $S$ of all finite sequences of 0's and 1's to represent the Rado graph. Letting $G$ denote the Fraïssé class of finite graphs, it is proved in [14] that for a given finite graph $A$, its big Ramsey degree $T(A, G)$ equals the number of strong similarity types (see Definition 2.6) of (strongly diagonal) antichains representing $A$. In particular, these strong similarity types form a canonical partition of $\binom{\mathbb{R}}{A}$. The big Ramsey degrees $T(A, G)$ grow quickly as the number of vertices in the finite graph $A$ increase (see [15] for these numbers).

Furthermore, Laflamme, Sauer, and Vuksanovic prove in [14] that given any representation of $\mathbb{R}$ as an antichain in $S$ and given any member $\mathbb{R}' \in \binom{\mathbb{R}}{\mathbb{R}}$, every strong similarity type of a finite or even infinite (strongly diagonal) antichain in $S$ embeds into the set of nodes representing $\mathbb{R}'$. In particular, one can apply these results to construct a Borel coloring of $\binom{\mathbb{R}}{\mathbb{R}}$ with $\omega$ many colors each of which persists in any member of $\binom{\mathbb{R}}{\mathbb{R}}$. Therefore, any positive answer to Question 1.5 must restrict to a subspace of $\binom{\mathbb{R}}{\mathbb{R}}$ where all members have the same (induced) strong similarity type. (See Section 2 for more details.) Therefore, this is the tack we must take.

Given any Rado graph $\mathbb{R}$ with universe $\omega$, let $\mathcal{R}(\mathbb{R})$ denote the collection of all subgraphs $\mathbb{R}' \in \binom{\mathbb{R}}{\mathbb{R}}$ with the same (induced) strong similarity type as $\mathbb{R}$. (This space will be precisely defined at the end of Section 3.) Note that $\mathcal{R}(\mathbb{R})$ is a topological space, with the topology inherited from the $\varphi$-image of $\mathcal{R}(\mathbb{R})$ as a closed subspace of the Baire space. For $\mathbb{R}' \in \mathcal{R}(\mathbb{R})$, we let $\mathcal{R}(\mathbb{R})'$ denote the subspace of those $\mathbb{R}'' \in \mathcal{R}(\mathbb{R})$ which are subgraphs of $\mathbb{R}'$. This space also inherits the Ellentuck topology, and the notion of completely Ramsey makes sense in this setting (see Section 6). The following is the main theorem of the paper.

**Main Theorem.** Let $\mathbb{R} = (\omega, E)$ be the Rado graph. Then each Borel subset of $\mathcal{R}(\mathbb{R})$ is completely Ramsey. In particular, if $\mathcal{X} \subseteq \mathcal{R}(\mathbb{R})$ is Borel, then for each $\mathbb{R}' \in \mathcal{R}(\mathbb{R})$, there is a Rado graph $\mathbb{R}'' \in \mathcal{R}(\mathbb{R}')$ such that $\mathcal{R}(\mathbb{R}'')$ is either contained in $\mathcal{X}$, or else is disjoint from $\mathcal{X}$.

Investigations into big Ramsey degrees of Henson graphs set the stage for the work in this paper. In January 2019, the author finished writing the proof that the $k$-clique-free universal ultrahomogeneous graphs have finite big Ramsey degrees in [6], building on work for the triangle-free case in [4]. The constructions in these papers utilized ideas from Milliken’s topological space of strong trees [16] and ideas from Sauer’s work on the Rado graph in [21]. Developments unique to [4] and [6] include the introduction of distinguished nodes in the trees used to code specific vertices in a fixed graph to ensure that no $k$-cliques are ever introduced, and the expansion to the $k$-clique-free setting of a method of Harrington using forcing techniques to give an alternate ZFC proof of the Halpern-Läuchli Theorem.

Interestingly, these ideas turned out to be useful, and in fact necessary for a satisfactory infinite dimensional Ramsey theorem for Rado graphs, as we shall discuss in Section 2. In that section, strong trees, the Halpern-Läuchli and Milliken Theorems, and relevant ideas and results from [14] are presented to provide the reader with some intuition for the work in this paper. There, we will review how nodes in
trees can be used to represent graphs, the notion of strong similarity type, and how the work in [14] necessitates restricting to subspaces of \( (\mathbb{R}^\omega) \) in which all members have the same (induced) strong similarity type, as we do in the Main Theorem. We will also discuss why the classical Milliken Theorem is not sufficient to provide an answer to Question 1.5.

In Section 3, we construct topological spaces \( T_\mathbb{R} \) of strong Rado coding trees. Fixing any Rado graph \( \mathbb{R} = (\omega, E) \) with universe \( \omega \), the prototype tree \( S_\mathbb{R} \) corresponding to \( \mathbb{R} = (\omega, E) \) is constructed by placing distinguished nodes \( \langle c_n : n < \omega \rangle \) in the tree \( S \), where \( c_n \) is the node with length \( n \) representing the \( n \)-th vertex of \( \mathbb{R} \). These distinguished nodes \( c_n \) are called coding nodes. The space \( T_\mathbb{R} \) consists of all subtrees of \( S_\mathbb{R} \) which are strongly similar to \( S_\mathbb{R} \) as trees with coding nodes (see Definition 3.3). There is a one-to-one correspondence between the members of \( T_\mathbb{R} \) and \( \mathbb{R}(\mathbb{R}) \), which is shown at the end of Section 3. The set \( T_\mathbb{R} \) will be endowed with the topology generated by basic open sets determined by finite initial subtrees of members of \( T_\mathbb{R} \), generating a Polish space. This corresponds in a simple manner to the topology on \( \mathbb{R}(\mathbb{R}) \).

Given a strong Rado coding tree \( T \in T_\mathbb{R} \), we let \( \mathcal{T}(T) \) denote the subspace of all members of \( T_\mathbb{R} \) which are all subtrees of \( T \). We say that a subset \( X \subseteq T_\mathbb{R} \) is Ramsey if for each \( T \in T_\mathbb{R} \), there is a subtree \( S \in \mathcal{T}(T) \) such that either \( \mathcal{T}(S) \subseteq X \) or else \( \mathcal{T}(S) \cap X = \emptyset \).

**Theorem 1.6.** For any Rado graph \( \mathbb{R} = (\omega, E) \), Borel subsets of the space \( T_\mathbb{R} \) are Ramsey.

The Main Theorem will be deduced from Theorem 1.6, via the homeomorphism between \( \mathbb{R}(\mathbb{R}) \) and \( T_\mathbb{R} \), discussed at the end of Section 3. In fact, we shall prove that Borel subsets of \( \mathbb{R}(\mathbb{R}) \) are completely Ramsey (and an even stronger property called \( \text{CR}^* \)) in Theorem 5.16, from which we deduce Theorem 1.6 and the Main Theorem.

The basic outline of the proof of Theorem 1.6 is simply to prove that the collection of subsets of \( T_\mathbb{R} \) which are Ramsey contains all open sets and is closed under complements and countable unions. Somewhat surprisingly, it is the containment of all open sets that presents the largest difficulty. The space \( T_\mathbb{R} \) satisfies almost all of the four axioms presented by Todorcevic in [23], but the axiom \( A.3(b) \) fails irreparably. (See Chapter 5, Section 1 of [23] for further details on topological Ramsey spaces and the four axioms.) Thus, we cannot simply apply the machinery of topological Ramsey spaces to conclude the Main Theorem. Here is where the ideas from [4] and [6] come into play.

In Section 4 we prove in Theorem 4.3 that colorings of level sets of strong Rado coding trees have the Ramsey property. Importantly, this is proved while preserving the width of some finite initial segment of a Rado coding tree, thus serving as a surrogate for the missing Axiom \( A.3(b) \).

In Section 5 we prove that Borel subsets of \( T_\mathbb{R} \) are completely Ramsey. We begin by noticing that open sets are in one-to-one correspondence with Nash-Williams families (Definition 5.1), and we prove in Theorem 5.4 that all open sets are completely Ramsey. The specific formulation of Theorem 4.3 enables us to do fusion arguments.

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1The proof of Theorem 5.4 included in this final version of the paper was developed for [5], fixing a glitch in our original proof given in 2019. Later, it was pointed out that a similar asymmetric
From there, we prove that the collection of sets which are completely Ramsey are closed under complementation and countable union, this last step also relying on how we set up Theorem 4.3 so that we can do fusion arguments without Axiom A.3(b). The proof actually achieves more, showing that all Borel subsets of $T_\mathcal{R}$ are CR* (see Definition 5.10). The translation back to $\mathcal{R}(\mathbb{R})$ in Section 6 concludes the proof of the main theorem.

An interesting quandary is whether the analogue of Ellentuck's theorem holds for the space of ordered Rado graphs $\mathcal{R}$. A discussion of this as well as the future aim for the ultimate answer to Question 1.5 appears in Section 7.

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2. Milliken’s Theorem and constraints on the infinite dimensional Ramsey theory of Rado graphs

Minimal background on strong trees, the Ramsey theorems for strong trees due to Halpern-Láuchli and Milliken, and topological Ramsey spaces are set forth in this section. These theorems provide some guidelines and intuition for our work. For a more general exposition of this area, the reader is referred to Chapter 6 in [23].

We use standard set-theoretic notation. The set of natural numbers \(\{0,1,2,\ldots\}\) is denoted by $\omega$. Each natural number $n$ is defined to be the set of natural numbers less than $n$. Thus, for $n \in \omega$, $n = \{0,\ldots,n-1\}$. We write $n < \omega$ to mean $n \in \omega$. For $n < \omega$, $2^n$ denotes the set of all functions from $n$ into $2$. Such functions may be thought of as sequences of 0’s and 1’s of length $n$, and we also write $s \in 2^n$ as

$$s = \langle s(0),\ldots,s(n-1) \rangle = \langle s(i) : i < n \rangle.$$  

Throughout, we shall let $S$ denote $\bigcup_{n<\omega} 2^n$; thus, $S$ is the set of all finite sequences of 0’s and 1’s. For $s \in S$, write $|s|$ to denote the domain of $s$, or equivalently, the length of $s$ as a sequence. For $m \leq |s|$, write $s \upharpoonright m$ to denote the truncation of the sequence to domain $m$. For $s,t \in S$, we write $s \subseteq t$ if and only if for some $m \leq |t|$, $s = t \upharpoonright m$. We write $s \subset t$ to denote that $s$ is a proper initial segment of $t$, meaning that $s = t \upharpoonright m$ for some $m < |t|$. The notion of tree we use is weaker than the usual definition, but is standard for this area.

**Definition 2.1.** A set of nodes $T \subseteq S$ is called a tree if there is a set of lengths $L \subseteq \omega$ such that $t \in T$ implies that $|t| \in L$ and also for each $l \in L$ less than $|t|$, $t \upharpoonright l \in T$. Thus, $T$ is closed under initial segments with lengths in $L$. We call $L$ the set of levels of $T$. 

version of combinatorial forcing was developed by Todorcevic in notes for a graduate course in Ramsey theory in 2022. However, those notes do not directly apply to sets of the form $[B,T]^*$, nor do they include the concluding argument in our proof after Lemma 5.9.
**Definition 2.2** (Strong Subtrees). A tree \( T \subseteq S \) is a **strong subtree** of \( S \) if for each \( t \in T, |t| \in L \) and for each \( l \in L \) with \( l \leq |t| \), there are nodes \( t_0, t_1 \) in \( T \) such that, letting \( s = t \upharpoonright l, t_0 \supseteq s \upharpoonright 0 \) and \( t_1 \supseteq s \upharpoonright 1 \). Given \( T \) a strong subtree of \( S \), we say that \( S \) is a **strong subtree** of \( T \) if \( S \) is a strong subtree of \( S \) and \( S \) is a subset of \( T \). We let \( S \) denote the set of all strong subtrees of \( S \). We define a partial order \( \leq \) on \( S \) by \( S \leq T \) if and only if \( S \) is a subtree of \( T \), for \( S, T \in S \).

For \( s, t \in S \), \( s \wedge t \), called the **meet** of \( s \) and \( t \), equals the sequence \( s \upharpoonright m = t \upharpoonright m \) where \( m \) is maximal such that \( s \upharpoonright m = t \upharpoonright m \). Given \( s, t \in S \), define \( s <_{\text{lex}} t \) if and only if \( s \) and \( t \) are incomparable under \( \subseteq \) and \( t(|s \wedge t|) < t(|s \wedge t|) \). We say that a bijection \( \varphi \) from a tree \( T \) to another tree \( S \) is a **tree isomorphism** if \( \varphi \) preserves the tree structure and the lexicographic order of the nodes. Given \( S, T \in S \), note that there is exactly one tree isomorphism between them; this is called the **strong tree isomorphism** between \( S \) and \( T \). Given \( T \in S \) and \( L \) its set of levels, let \( \{l_n : n < \omega \} \) be the increasing enumeration of \( L \). For \( n < \omega \), let \( T(n) \) denote the set \( \{t \in T : |t| = l_n \} \). A **level set** in \( T \) is a subset \( X \subseteq T \) such that each node in \( X \) has the same length; equivalently, \( X \subseteq T(n) \) for some \( n < \omega \). Any strong tree isomorphism takes level sets to level sets.

The Halpern-Läuchli Theorem is a Ramsey theorem for colorings of products of level sets of finitely many trees. We present the version restricted to \( S \), as this is all that is needed in this article.

**Theorem 2.3** (Halpern-Läuchli, [11]). Suppose \( d \geq 1 \) and \( T_i \in S \), for each \( i < d \). Let

\[
(7) \quad c : \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to 2
\]

be given. Then there are an infinite set \( N = \{n_k : k < \omega \} \subseteq \omega \) and strong subtrees \( S_i \subseteq T_i \) such that for each \( i < d \) and \( k < \omega \), \( S_i(k) \subseteq T_i(n_k) \), and \( c \) is monochromatic on

\[
(8) \quad \bigcup_{k<\omega} \prod_{i<d} S_i(k).
\]

The Halpern-Läuchli Theorem is used to obtain a space of strong trees with infinite dimensional Ramsey properties.

**Definition 2.4** (Milliken space). The **Milliken space** is the triple \( (S, \leq, r) \), where \( \leq \) is the partial ordering of subtree on \( S \) and \( r_k(T) = \bigcup_{n<k} T(n) \) is called the **k-th restriction** of \( T \).

Thus, \( r_k(T) \) is a finite tree with \( k \) many levels. We refer to such trees as finite strong trees, and we let \( AS \) denote the set of all finite strong trees. The letters \( A, B, C, \ldots \) will denote finite subsets of \( S \), as is the custom in Section 6.2 of [24] and in [3]. For \( A \in AS \) and \( T \in S \), let

\[
(9) \quad [A, T] = \{S \in S : \exists k (r_k(S) = A) \text{ and } S \leq T\}.
\]

The sets of the form \([A, S]\), \( A \in AS \), generate the **metric topology** on \( S \), similarly to the metric topology on the Baire space. The sets of the form \([A, T]\), \( A \in AS \) and \( T \in S \), generate a finer topology on \( S \), analogous to the Ellentuck topology on the Baire space; this is called the **Ellentuck topology** on \( S \). The Halpern-Läuchli Theorem is central to the proof of the next theorem.
Theorem 2.5 (Milliken, [16]). If \( X \subseteq S \) has the property of Baire in the Ellentuck topology on \( S \), then for each basic open set \([A, T]\), where \( A \in \mathcal{A}S \) and \( T \in S \), there is an \( S \in [A, T] \) such that either \([A, S] \subseteq X\) or else \([A, S] \cap X = \emptyset\).

This states that subsets of \( S \) with the property of Baire in the finer topology are completely Ramsey. In the current terminology set forth in [23], we say that Milliken’s space of strong trees forms a topological Ramsey space. As a special case, this implies

\[
\forall \ell \in \omega, \quad \mathcal{S} \xrightarrow{\text{Borel}} (\mathcal{S})^\ell_{\ell,1},
\]

where \((\mathcal{S})^\ell_{\ell,1}\) is \( S \).

Harrington came up with a novel proof of the Halpern-Läuchli Theorem which uses the method of forcing to achieve a ZFC result. The proof was known in certain circles, but not widely available until a version appeared in [24]; this proof utilizes a smaller uncountable cardinal, which benefits inquiries into minimal hypotheses, but at the expense of a more complex proof. A simpler version closer to Harrington’s original proof appears in [3], filling in an outline provided to the author by Laver in 2011, at which time she was unaware of the proof in [24]. Harrington’s “forcing proof” uses the language and machinery of forcing to prove the existence of finitely many finite level sets whose product is monochromatic. Since these objects are finite, they must be in the ground model. This is iterated infinitely many times to the strong subtrees in the conclusion of Theorem 4.3. These ideas were utilized in [4] and [6] and will be utilized again in Section 4. The difference is that we will be working with trees with special nodes to code vertices of graphs, as discussed in the next section, and so the forcing partial order must be tailored to this set-up.

2.1. Constraints on infinite dimensional Ramsey theory of Rado graphs.

In the work on big Ramsey degrees of the Rado graph in [21] and [14], the authors use an antichain of nodes in \( S \) to represent a copy of the Rado graph. The representation uses the idea of passing numbers to code the edge/non-edge relation. Given \( s, t \in S \) with \(|s| < |t|\), if the nodes \( s \) and \( t \) represent vertices in a graph, then they represent an edge between those vertices if and only if \( t(|s|) = 1 \). This number \( t(|s|) \) is called the passing number of \( t \) at \( s \). An antichain \( A \subseteq S \) is called strongly diagonal if its meet closure has no two nodes of the same length, and the passing number of any node at a splitting node is 0, except of course for extensions of the splitting node itself. Antichains are particularly useful for coding graphs, since all nodes having different lengths makes the passing numbers, and hence the graph being coded, clear.

Sauer introduced the notion of strong similarity map in [21], where he found upper bounds for the big Ramsey degrees of the Rado graph given by the number of strong similarity types of strongly diagonal antichains representing a given finite graph.

**Definition 2.6** (Strong similarity, [21]). Let \( S \) and \( T \) be meet-closed subsets of \( S \). A function \( f : S \rightarrow T \) is a strong similarity of \( S \) to \( T \) if \( f \) is a bijection and for all nodes \( s, t, u, v \in S \), the following hold:

1. \( f \) preserves lexicographic order: \( s <_{\text{lex}} t \) if and only if \( f(s) <_{\text{lex}} f(t) \).
2. \( f \) preserves initial segments: \( s \land t \subseteq u \land v \) if and only if \( f(s) \land f(t) \subseteq f(u) \land f(v) \).
3. \( f \) preserves meets: \( f(s \land t) = f(s) \land f(t) \).
(4) $f$ preserves relative lengths: $|s \wedge t| < |u \wedge v|$ if and only if $|f(s) \wedge f(t)| < |f(u) \wedge f(v)|$.

(5) $f$ preserves passing numbers: If $s, t \in S$ with $|s| < |t|$, then $f(t)(|f(s)|) = t(|s|)$.

We say that $S$ and $T$ are strongly similar and write $S \sim T$ exactly when there is a strong similarity between $S$ and $T$. We call the equivalence classes of strongly similar meet-closed sets strong similarity types.

The number of strong similarity types of strongly diagonal antichains representing a finite graph $A$ were proved to be the exact big Ramsey degree $T(A, G)$ in [14]. It follows from Theorem 4.1 in that paper that given any strongly diagonal antichain $U \subseteq S$ representing the Rado graph, each strong similarity type of any strongly diagonal antichain, finite or infinite, embeds into $U$. Using this result, one can construct an open coloring of all subcopies of the Rado graph ($\mathbb{R}_R$) as a topological subspace of $[\omega]^{\omega}$ into $\omega$ many colors such that for any $R' \in (\mathbb{R}_R)$, all of the colors appear in $\mathbb{R}_R$. Thus, any positive answer to Question 1.5 must restrict to a subspace of $(\mathbb{R}_R)$ in which all members have the same strong similarity type. This is the approach we present in the next section.

Since Laflamme, Sauer, and Vuksanovic used Milliken’s Theorem to find the big Ramsey degrees of the Rado graph, it is natural to ask whether it can be used to answer Question 1.5 for the Rado graph. In fact, one can obtain a nominal infinite dimensional Ramsey theory for Rado graphs using Milliken’s Theorem, but nothing similar to the Galvin-Prikry Theorem, as we now review.

Each strong tree $T \in S$ codes a universal graph, say $U$, in the following way: Let each node in $T$ represent a vertex. Two nodes $s, t \in T$ code an edge between the vertices they represent if and only if $|s| \neq |t|$ and the longer node has passing number 1 at the shorter node. As was pointed out in [21], this $U$ is universal, so $U$ embeds into the Rado graph and the Rado graph embeds into $U$; however, they are not isomorphic. In this way, Milliken’s Theorem can be interpreted as an infinite-dimensional Ramsey theorem on subcopies of $U$ whose tree representations have the same strong similarity type as $S$. However, this is not the same as coloring copies of a Rado graph.

Now one can use Milliken’s Theorem to achieve the following sort of soft infinite dimensional Ramsey theorem on Rado graphs: Let $\mathbb{R}$ be the Rado graph with universe $\omega$. Let the nodes in $S$ represent a copy of $U$ as an induced subgraph of $\mathbb{R}$. Given a strong similarity type $\tau$ of a representation of the Rado graph inside of $S$, and given a (Baire measurable with respect to the Ellentuck topology on the Milliken space) coloring of the members of $\tau$ into finitely many colors, there is a strong subtree $S' \subseteq S$ such that all members of $\tau$ contained inside $S'$ have the same color. Indeed, this is a special case of Theorem 6.13 in [23]. Then one can pull out a representation of the Rado graph $R \subseteq S$, and all Rado graphs represented by subsets of $R$ with strong similarity type $\tau$ will have the same color. One can even repeat this process for finitely many different strong similarity types, and afterward, pull out a representation of the Rado graph so as to have one color on each of those strong similarity types.

This might seem at first like a satisfactory solution to Question 1.5. However, it is quite far from providing analogues of the Galvin-Prikry or Milliken Theorems, as it fails a “density” property, precluding any hope of proving that definable partitions
are completely Ramsey. By the “density” property, we mean that given a Rado graph $\mathcal{R}$, a strong similarity type $\tau$, and a definable coloring on $\binom{\mathcal{R}}{\omega}$, one ought to be able to fix any $\mathcal{R}' \in \binom{\mathcal{R}}{\omega}$ in $\tau$ and find a subgraph $\mathcal{R}'' \in \binom{\mathcal{R}'}{\omega}$, again in $\tau$, so that the members of $\binom{\mathcal{R}''}{\omega}$ in $\tau$ all have the same color. However, if one fixes a Rado subgraph $\mathcal{R}'$ of $\mathcal{R}$ represented by some nodes $(s_n : n < \omega)$ within $\mathcal{S}$, an application of Milliken’s Theorem may provide a strong subtree $S$ which does not contain any of those nodes $s_n$; so while there will be subsets of $S$ coding Rado graphs, none of them will represent a subgraph of $\mathcal{R}'$. For this same reason, one cannot conclude from Theorem 6.13 in [23] that Borel subsets of $\binom{\mathcal{R}}{\omega}$ (even restricting to those with the same strong similarity type) are completely Ramsey. These failures hinge on the fact that there is no hard-coded representation of the Rado graph inside strong trees.

Our way around these undesired issues is to use special distinguished nodes to represent the vertices in a fixed Rado graph. Then it is always clear which subgraph a strong subtree represents. This will allow us to prove the Main Theorem as well as the stronger Theorem 5.16, showing that Borel subsets of $\mathcal{R}(\mathcal{R})$ are CR* and hence, completely Ramsey.

2.2. Brief introduction to topological Ramsey spaces. The Ellentuck space, mentioned in the Introduction, is the prototype for all topological Ramsey spaces. After Ellentuck’s theorem, many spaces with similar properties were built, including the Milliken space ($\mathcal{S}$). After Ellentuck’s theorem, many spaces with similar properties were built, including the Milliken space ($\mathcal{S}$). These were first abstracted by Carlson and Simpson in [11], and their approach was refined by Todorcevic, who distilled the key properties of the Ellentuck space into the four axioms below. The rest of this subsection is taken almost verbatim from Section 5.1 in [23], with a few modifications and explanations tailored to this paper.

One assumes a triple $(\mathcal{R}, \leq, r)$ of objects with the following properties. $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \rightarrow \mathcal{AR}$ is a mapping giving us the sequence $(r_n(\cdot) = r(\cdot, n))$ of approximation mappings, where $\mathcal{AR}$ is the collection of all finite approximations to members of $\mathcal{R}$. For $a \in \mathcal{AR}$ and $X, Y \in \mathcal{R}$,

$$[a, Y] = \{ X \in \mathcal{R} : X \leq Y \text{ and } (\exists n) r_n(X) = a \}.$$

A.1 (a) $r_0(X) = \emptyset$ for all $X \in \mathcal{R}$.
(b) $X \neq Y$ implies $r_n(X) \neq r_n(Y)$ for some $n$.
(c) $r_n(X) = r_m(Y)$ implies $n = m$ and $r_k(X) = r_k(Y)$ for all $k < n$.

For $a \in \mathcal{AR}$, let $|a|$ denote the length of the sequence $a$. Thus, $|a|$ equals the integer $k$ for which $a = r_k(X)$ for some $X \in \mathcal{R}$. For $a, b \in \mathcal{AR}$, we write $a \sqsubseteq b$ if and only if there are $X \in \mathcal{R}$ and $m \leq n$ such that $a = r_m(X)$ and $b = r_n(X)$. We write $a \sqsubset b$ if and only if $a \sqsubseteq b$ and $a \neq b$.

A.2 There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that
(a) $\{ a \in \mathcal{AR} : a \leq_{\text{fin}} b \}$ is finite for all $b \in \mathcal{AR}$.
(b) $X \leq Y$ iff $\big( \forall n \big) \big( \exists m \big) r_n(X) \leq_{\text{fin}} r_m(Y)$.
(c) $\forall a, b, c \in \mathcal{AR} [a \sqsubseteq b \land b \leq_{\text{fin}} c \rightarrow \exists d \sqsubset c \ a \leq_{\text{fin}} d]$. 
The number depth$_Y(a)$ is the least $n$, if it exists, such that $a \leq_n r_n(Y)$. If such an $n$ does not exist, then we write depth$_Y(a) = \infty$. If depth$_Y(a) = n < \infty$, then $[\text{depth}_Y(a), Y]$ denotes $[r_n(Y), Y]$.

**A.3** (a) If depth$_Y(a) < \infty$ then $[a, X] \neq \emptyset$ for all $X \in [\text{depth}_Y(a), Y]$.

(b) $X \subseteq Y$ and $[a, X] \neq \emptyset$ imply that there is $X' \in [\text{depth}_Y(a), Y]$ such that $\emptyset \neq [a, X'] \subseteq [a, X]$.

For each $n < \omega$, $\mathcal{A}R_n = \{r_n(X) : X \in \mathcal{R}\}$. Given $X \in \mathcal{R}$, $\mathcal{A}R_n(X)$ denotes the set $\{r_n(Y) : Y \subseteq X\}$; that is, $\mathcal{A}R_n$ relativized to $X$. If $n > |a|$, then $r_n[a, X]$ denotes the collection of all $b \in \mathcal{A}R_n(X)$ such that $a \subseteq b$.

**A.4** If depth$_Y(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{A}R_{|a|+1}$, then there is $X \in [\text{depth}_Y(a), Y]$ such that $r_{|a|+1}[a, X] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, X] \subseteq \mathcal{O}^r$.

The Ellentuck topology on $\mathcal{R}$ is the topology generated by the basic open sets $[a, \mathcal{X}]$; it extends the usual metrizable topology on $\mathcal{R}$ when we consider $\mathcal{R}$ as a subspace of the Tychonoff cube $\mathcal{A}R^\omega$. Given the Ellentuck topology on $\mathcal{R}$, the notions of nowhere dense, and hence of meager are defined in the natural way. We say that a subset $\mathcal{X}$ of $\mathcal{R}$ has the property of Baire if and only if $\mathcal{X} = \mathcal{O} \cap \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$. A subset $\mathcal{X}$ of $\mathcal{R}$ is completely Ramsey if for every $\emptyset \neq [a, X]$, there is a $Y \in [a, X]$ such that $[a, Y] \subseteq \mathcal{X}$ or $[a, Y] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is completely Ramsey null if for every $\emptyset \neq [a, X]$, there is a $Y \in [a, X]$ such that $[a, Y] \cap \mathcal{X} = \emptyset$

**Remark 2.7.** In [23], Todorcevic omits the word “completely” and calls such sets simply Ramsey and Ramsey null. In this paper, we use the terminology of Galvin and Prikry in [10], as the main theorem of this paper provides an analogue of their theorem.

**Definition 2.8** ([23]). A triple $(\mathcal{R}, \leq, r)$ is a topological Ramsey space if every subset of $\mathcal{R}$ with the property of Baire is completely Ramsey and if every meager subset of $\mathcal{R}$ is completely Ramsey null.

The following result can be found as Theorem 5.4 in [23].

**Theorem 2.9** (Abstract Ellentuck Theorem). If $(\mathcal{R}, \leq, r)$ is closed (as a subspace of $\mathcal{A}R^\omega$) and satisfies axioms A.1, A.2, A.3, and A.4, then every subset of $\mathcal{R}$ with the property of Baire is completely Ramsey, and every meager subset is completely Ramsey null; in other words, the triple $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.

The Ellentuck space of course satisfies these four axioms with $\mathcal{R} = [\omega]^{<\omega}$, the partial ordering $\leq$ being $\subseteq$, and the $n$-th approximation to an infinite set $X$ of natural numbers being $r_n(X) = \{x_i : i < n\}$, where $\{x_i : i < \omega\}$ enumerates $X$ in increasing order. Here, $\leq_n$ is simply the partial order $\subseteq$.

Milliken’s space $(\mathcal{S}, \leq, r)$ of strong subtrees of $\mathcal{S}$ also forms a topological Ramsey space, by Theorem 2.7. The restriction map presented in Definition 2.4 yields that for $T \in \mathcal{S}$, $r_n(T)$ is the finite tree consisting of the first $n$ levels of $T$. For finite trees $A, B \in \mathcal{A}S$, we write $A \leq_n B$ if and only if $A$ is a subtree of $B$ and the maximal nodes in $A$ are also maximal in $B$. (Recall that we will be using $A, B, C, \ldots$ to denote finite subsets of $\mathcal{S}$.)

**Remark 2.10.** Topological Ramsey space theory, especially Milliken’s space, informs our approach to answering Question 1.5. However, Axiom A.3(b), an amalgamation
property, fails for our space of strong Rado coding trees, necessitating the work in Sections 3 and 5.

3. Strong Rado Coding Trees

Topological spaces of strong Rado coding trees are introduced in this section. Recall Definition 2.4 the slightly looser definition of tree which is appropriate to the setting of strong trees. The next two definitions are taken from [3], in which the author developed the notion of trees with coding nodes to prove that the triangle-free Henson graph has finite big Ramsey degrees. It turns out that these ideas are also useful for coding homogeneous structures without forbidden configurations, in particular, the Rado graph. Indeed, the designated coding nodes will help us achieve the “density” property, discussed at the end of Subsection 2.1 which is crucial for showing that Borel sets of strong Rado coding trees are completely Ramsey.

Definition 3.1 ([4]). A tree with coding nodes is a tree \( T \subseteq S \) along with a unary function \( c^T : N \to T \), where \( N \leq \omega \) and \( c^T : N \to T \) is an injective function such that \( m < n < N \) implies \( |c^T(m)| < |c^T(n)| \).

The \( n \)-th coding node in \( T \), \( c^T(n) \), will often be denoted as \( c^T_n \). We will use \( l^T_n \) to denote \( |c^T_n| \), the length of \( c^T_n \). The next definition shows how nodes in trees can be used to code a graph. This idea goes back to Erdős, Hajnal, and Posa, who noticed that the edge/non-edge relation induces the lexicographic order on any given ordered collection of vertices in a graph. The only difference here is that we distinguish from the outset certain nodes to code particular vertices.

Definition 3.2 ([4]). A graph \( \mathbb{G} = (G; E) \) with vertex set \( G \) enumerated as \( \langle v_n : n < N \rangle \) (\( N \leq \omega \)) is represented by a tree \( T \) with coding nodes \( \langle c^T_n : n < N \rangle \) if and only if for each pair \( m < n < N \), \( v_m \mathrel{E} v_n \iff c^T_m(l^T_m) = 1 \). We will often simply say that \( T \) codes \( \mathbb{G} \). The number \( c^T_m(l^T_m) \) is called the passing number of \( c^T_n \) at \( c^T_m \).

Before we define topological spaces of strong Rado coding trees, we extend the definition of strong similarity type to trees with coding nodes. The following appears as Definition 4.9 in [4]; it is simplified for the setting of this paper.

Definition 3.3. Let \( S, T \) be trees with coding nodes. A function \( f : S \to T \) is a strong similarity of \( S \) to \( T \) if \( f \) is a bijection and for all nodes \( s, t, u, v \in S \), (1)–(5) in Definition 2.6 hold as well as the following:

\[
(6) \text{ } f \text{ preserves coding nodes: } f \text{ maps the set of coding nodes in } S \text{ onto the set of coding nodes in } T.
\]

We say that \( S \) and \( T \) are strongly similar and write \( S \sim T \) exactly when there is a strong similarity between \( S \) and \( T \). A function \( f \) from \( S \) into itself is called a strong similarity embedding if \( f \) is a strong similarity from \( S \) to \( f[S] \).

It follows from (4) and (6) that if \( S \) and \( T \) are strongly similar, then they have the same number \( N \leq \omega \) of coding nodes, and for each \( n < N \), \( f(c^S_n) = c^T_n \). Note that \( \sim \) is an equivalence relation. We call the equivalence classes strong similarity types.

Definition 3.4 (Spaces of Strong Rado Coding Trees \((T_R, \leq, r)\)). Fix a Rado graph \( R = (\omega, E) \) so that its universe is \( \omega \). Use \( v_n \) to denote \( n \), the \( n \)-th vertex of \( R \), so that there is no ambiguity when we are referring to vertices. Let \( S_R \) be the tree \( S \)
with coding nodes \( \langle c_n^S : n < \omega \rangle \), where \( c_n^S \) is the node of length \( n \) representing \( v_n \). That is, \( c_n^S \) is the node in \( S \) of length \( n \) such that for all \( m < n \), \( c_m^S(m) = 1 \) if and only if \( v_m \in v_n \).

The space \( T_R \) consists of all images of strong similarity embeddings of \( S_R \) into itself. Thus, each \( T \in T_R \) is a strong coding subtree of \( S_R \) such that \( T \approx S_R \). The members of \( T_R \) are called strong Rado coding trees, and abbreviated as \( sRc \) trees.

We partially order \( T_R \) by inclusion. Thus, for \( S, T \in T_R \), we write \( S \leq T \) if and only if \( S \) is a subtree of \( T \). Define the restriction map \( r \) as follows: Given \( T \in T_R \) and \( k < \omega \), \( r_k(T) \) is the finite subtree of \( T \) consisting of all nodes in \( T \) with length less than \( l_k^T \). Define

\[
(12) \quad AT_k = \{ r_k(T) : T \in T_R \},
\]

the set of all \( k \)-th restrictions of members of \( T_R \). Let

\[
(13) \quad AT = \bigcup_{k<\omega} AT_k,
\]

the set of all finite approximations to members of \( T_R \).

For \( A, B \in AT \) we write \( A \subseteq B \) if and only if there is some \( T \in T_R \) and some \( j \leq k \) such that \( A = r_j(T) \) and \( B = r_k(T) \). In this case, \( A \) is called an initial segment of \( B \); we also say that \( B \) end-extends \( A \). If \( A \subseteq B \) and \( A \neq B \), then we say that \( A \) is a proper initial segment of \( B \) and write \( A \subset B \). Furthermore, when a \( j \) exists such that \( A = r_j(T) \), we shall also write \( A \sqsubseteq T \) and call \( A \) an initial segment of \( T \).

The metric topology on \( T_R \) is the topology induced by basic open cones of the form

\[
(14) \quad [A, S_R] = \{ S \in T_R : \exists k (r_k(S) = A) \},
\]

for \( A \in AT \). The Ellentuck topology on \( T_R \) is induced by basic open sets of the form

\[
(15) \quad [A, T] = \{ S \in T_R : \exists k (r_k(S) = A) \text{ and } S \leq T \},
\]

where \( A \in AT \) and \( T \in T_R \). Thus, the Ellentuck topology refines the metric topology.

Given \( A \in AT \), let \( \max(A) \) denote the set of all maximal nodes in \( A \). Define the partial ordering \( \leq fin \) on \( AT \) as follows: For \( A, B \in AT \), write \( A \leq fin B \) if and only if \( A \) is a subtree of \( B \) and \( \max(A) \subseteq \max(B) \). Define depth\( T(A) \) to equal the \( k \) such that \( A \leq fin r_k(T) \), if it exists; otherwise, define depth\( T(A) = \infty \). Lastly, given \( j < k < \omega \), \( A \in AT_j \) and \( T \in T_R \), define

\[
(16) \quad r_k[A, T] = \{ r_k(S) : S \in [A, T] \}.
\]

**Notation 3.5.** Coding nodes in \( S_R \) will be notated simply as \( \langle c_n : n < \omega \rangle \). The length of \( c_n \) will be denoted as \( l_n \).

In the definition of \( S_R \), we specified that \( l_n = n \). In order to avoid confusion, we shall write \( l_n \) when we are referring to the length of the \( n \)-th coding node in \( S_R \), so as to be consistent with the usage of \( l_n^T \) for the length of the \( n \)-th coding node in a tree \( T \in T_R \).

The following are immediate consequences of the above definitions. For any \( S, T \in T_R \), \( S \) and \( T \) are strongly similar, and the strong similarity map from \( S \) to \( T \) takes \( c_n^S \) to \( c_n^T \), for each \( n < \omega \). Note that for any \( T \in T_R \), \( r_0(T) \) is the empty set.
Therefore, the graphs $G$ subgraphs

\[ \exists \text{Remark} \]

Given a Rado graph $\mathcal{G}$ its subgraphs $\mathcal{G}_T$ for each $T \in \mathcal{T}_R$, the subgraph $\mathcal{G}_T$ of $\mathcal{R}$ represented by $T$ is order-isomorphic to \( \mathcal{R} \).

**Proof.** By definition, $\mathcal{G} = (\omega, E)$ is a Rado graph. Given a Rado coding tree $T \in \mathcal{T}_R$, $T$ is the image of a strong similarity embedding from $\mathcal{R}$. Hence, $T \sim \mathcal{S}_R$, so for each pair $m < n < \omega$,

\[ c_n^T(l_m^T) = 1 \iff c_n(l_m) = 1. \]

Therefore, the graphs $\mathcal{G}_T$ and $\mathcal{G}_{\mathcal{S}_R} = \mathcal{R}$ are order-isomorphic. \( \square \)

**Definition 3.8.** Given a Rado graph $\mathcal{R} = (\omega, E)$, let $\mathcal{R}(\mathcal{R})$ denote the set of subgraphs $\mathcal{G}_T \subseteq \mathcal{R}$ such that $T \in \mathcal{T}_R$.

**Remark 3.9.** Note that $\mathcal{R}(\mathcal{R})$ is a subset of $\binom{\mathcal{R}}{\omega}$, and each member of $\mathcal{R}(\mathcal{R})$ is order-isomorphic to $\mathcal{R} = (\omega, E)$. However, there are order-isomorphic copies of $\binom{\mathcal{R}}{\omega}$ which are not members of $\mathcal{R}(\mathcal{R})$, because there are many different strong similarity types of trees with coding nodes which represent order-isomorphic copies of $\mathcal{R}$.

**Definition 3.10.** Given a subgraph $\mathcal{G} \subseteq \mathcal{R} = (\omega, E)$, the universe $G$ of $\mathcal{G}$ is a set of natural numbers. Let $T_G$ denote the subtree of $\mathcal{S}_R$ induced by the coding nodes $\{c_n : n \in G\}$; thus, the set of levels of $T_G$ is $L_G = \{l_n : n \in G\}$, and $T_G$ is the tree produced by taking all meets of coding nodes in $\{c_n : n \in G\}$ and then taking restrictions of the nodes in this meet-closed set to the levels in $L_G$.

By Definition 3.8, the mapping $T \mapsto \mathcal{G}_T$ produces a one-to-one correspondence between members of $\mathcal{T}_R$ and $\mathcal{R}(\mathcal{R})$. The mapping is easily reversible: Given $\mathcal{R}' \in \mathcal{R}(\mathcal{R})$, note that $T_{\mathcal{R}'} \in \mathcal{T}_R$, and that $\mathcal{G}_{T_{\mathcal{R}'}} = \mathcal{R}'$. Conversely, given $\mathcal{G} \in \mathcal{T}_R$, $G_S$ is a member of $\mathcal{R}(\mathcal{R})$, and $T_{\mathcal{G}_S} = S$.

The space of all infinite subgraphs of $\mathcal{R} = (\omega, E)$ corresponds to the Baire space $\omega^\omega$ by associating a subgraph of $\mathcal{G} \subseteq \mathcal{R}$ with its universe $G$, which is a subset of $\omega$. In particular, the map $\iota : \mathcal{R}(\mathcal{R}) \to [\omega]^\omega$ defined by $\iota(G) = G$ identifies $\mathcal{R}(\mathcal{R})$ with its $\iota$-image, which is a closed subspace of $[\omega]^\omega$. In what follows, we identify $\mathcal{R}(\mathcal{R})$ with $[\mathcal{R}(\mathcal{R})]$. The basic open sets of $\mathcal{R}(\mathcal{R})$ are those of the form $\text{Cone}(s) \cap \mathcal{R}(\mathcal{R})$, where $\text{Cone}(s) = \{X \in [\omega]^\omega : s \subseteq X\}$ for $s \in [\omega]^\omega$. The map $\theta : \mathcal{R}(\mathcal{R}) \to \mathcal{T}_R$, defined by $\theta(G) = T_G$, is a bijection, with the further property that $G \subseteq \mathcal{H}$ if and only if $T_G \subseteq \mathcal{T}_R$. Moreover, $\theta$ is a homeomorphism, since for each $s \in [\omega]^\omega$ which is an initial segment of a member of $\mathcal{R}(\mathcal{R})$, $\theta(\text{Cone}(s))$ is open in $\mathcal{T}_R$. Furthermore, for each $A \in \mathcal{A}_T$, $\theta^{-1}(\{A, \mathcal{S}_R\})$ is a union of basic open sets in $\mathcal{R}(\mathcal{R})$. Thus, results about Borel subsets of $\mathcal{T}_R$ correspond to results about Borel subsets of $\mathcal{R}(\mathcal{R})$. This will be revisited at the end of Section 5.
4. A Halpern-Läuchli-style Theorem for Strong Rado Coding Trees

Fix a Rado graph $\mathbb{R} = (\omega, E)$. We shall usually drop subscripts and let $S$ denote $S_\mathbb{R}$ and $T$ denote $T_\mathbb{R}$. The topological space $T$ of strong Rado coding trees defined in the previous section turns out to satisfy all but half of one of the four axioms of Todorcevic in [23] guaranteeing a topological Ramsey space. The first two axioms are easily shown to hold, and the pigeonhole principle (Axiom A.4) is a consequence of work in this section (see Corollary 4.7). However, the amalgamation principle (Axiom A.3) fails for $T$, so we cannot simply apply Todorcevic’s axioms and invoke his Abstract Ellentuck Theorem to deduce infinite dimensional Ramsey theory on $T$. It is this failure of outright amalgamation that presents the interesting challenge to proving that Borel subsets in $T$ have the Ramsey property.

Our approach is to build the infinite dimensional Ramsey theory on $T$ in a similar manner as Galvin and Prikry did for the Baire space in [10]. However, even that approach is not exactly replicable in $T$, again due to lack of amalgamation. In this section, we prove a Ramsey theorem for colorings of level sets, namely Theorem 4.3. This theorem will yield an enhanced version of Axiom A.4 strong enough to replace some uses of Axiom A.3(b) in the Galvin-Prikry proof, providing alternate means for proving that Borel subsets of $T$ are Ramsey in the next section.

Here, we mention a theorem of Erdős and Rado which will be used in the proof of Theorem 4.3. This theorem guarantees cardinals large enough to have the Ramsey property for colorings with infinitely many colors.

**Theorem 4.1 (Erdős-Rado).** For $r < \omega$ and $\mu$ an infinite cardinal,

$$\beth_r(\mu^+) \to (\mu^+)_{\mu+} r^+.$$  

We begin setting up notation needed for Theorem 4.3. Recall that for $t \in S$ and $l \leq |t|$, $t \upharpoonright l$ denotes the initial segment of $t$ with domain $l$. For any subset $U$ of $S$, finite or infinite, we let

$$\{ t \upharpoonright l: t \in U \text{ and } l \leq |t| \},$$

the tree of all initial segments of members of $U$. For a finite subset $A \subseteq S$, define

$$l_A = \max \{|t|: t \in A\},$$

the maximum of the lengths of nodes in $A$. For $l \leq l_A$, let

$$A \upharpoonright l = \{ t \upharpoonright l: t \in A \text{ and } |t| \geq l \}$$

and let

$$A \upharpoonright l = \{ t \in A: |t| < l \} \cup A \upharpoonright l.$$  

Thus, $A \upharpoonright l$ is a level set, while $A \upharpoonright l$ is the set of nodes in $A$ with length less than $l$ along with the truncation to $l$ of the nodes in $A$ of length at least $l$. In particular, $A \upharpoonright l = \emptyset$ for $l > l_A$, and $A \upharpoonright l = A$ for $l \geq l_A$. If $l$ is not the length of any node in $A$, then $A \upharpoonright l$ will not be a subset of $A$, but it is of course a subset of $A$. Let

$$\hat{\mathcal{AT}} = \{ A \upharpoonright l: A \in \mathcal{AT} \text{ and } l \leq l_A \}.$$  

Given $T \in \mathcal{T}$, let $\mathcal{AT}(T)$ denote the members of $\mathcal{AT}$ which are contained in $T$. For $k < \omega$, let $\mathcal{AT}_k(T)$ denote the set of those $A \in \mathcal{AT}_k$ such that $A$ is a subtree of $T$. Let $L_T = \{|t|: t \in T\}$ and define

$$\hat{\mathcal{AT}}(T) = \{ A \upharpoonright l: A \in \mathcal{AT}(T) \text{ and } l \in L_T \}.$$
It is important that the maximal nodes in any member of $\AT(T)$ have length in $L_T$ and therefore split in $T$. However, there are members of $\AT(T)$ which are not strongly similar to $r_n(S)$ for any $n$, and hence are not members of $\AT(T)$. These notions can be relativized to any $B \in \AT$ in place of $T$.

Parts of the following definition will be used in this section, and the rest will be used in the next section.

**Definition 4.2.** Given $T \in \mathcal{T}$ and $B \in \AT(T)$, letting $m$ be the least integer for which there exists $C \in \AT_m$ such that $\max(C) \supseteq \max(B)$, define
\begin{equation}
[B, T]^* = \{ S \in \mathcal{T} : \max(r_m(S)) \supseteq \max(B) \text{ and } S \subseteq T \}.
\end{equation}

For $n \geq m$, define
\begin{equation}
r_n[B, T]^* = \{ r_n(S) : S \in [B, T]^* \},
\end{equation}
and let
\begin{equation}
r[B, T]^* = \bigcup_{n \geq m} r_n[B, T]^*.
\end{equation}

We point out that $r_n[B, T]^*$ defined in equation (26) is equal to $\{ C \in \AT_n(T) : \max(C) \supseteq \max(B) \}$.

**Hypotheses for Theorem 4.3.** Let $T \in \mathcal{T}$ be fixed. Suppose $D = r_n(T)$ for some $n < \omega$. Given $A \in \AT(T)$ with $\max(A) \subseteq \max(D)$, let $A^+$ denote the union of $A$ with the set of immediate successors in $\hat{T}$ of the members of $\max(A)$; thus,
\begin{equation}
A^+ = A \cup \{ s^i : s \in \max(A) \text{ and } i \in \{ 0, 1 \} \}
\end{equation}
and $\max(A^+)$ is a level set of nodes of length $l_A + 1$. Let $B$ denote the subset of $A^+$ which will be end-extended to members of $\AT(T)$ which are colored. We consider two cases for triples $(A, B, k)$, where $B \in \AT(T)$, $A \subseteq B$, and $\max(B) \subseteq \max(A^+)$:

Case (a). $k \geq 1$, $A \in \AT_k(T)$, and $B = A^+$.

Case (b). $A$ has at least one node, and each member of $\max(A)$ has exactly one extension in $B$. Let $k$ be the integer satisfying $2^k = \card(\max(A))$.

In both cases, we will be working with $r_{k+1}[B, T]^*$. It can be useful to notice that this set $r_{k+1}[B, T]^*$ is equal to $\{ C \in \AT_{k+1}(T) : \max(C) \supseteq \max(B) \}$.

In Case (a), $A$ is actually a member of $\AT_k(T)$. In Case (b), $A$ may or may not be a member of $\AT$; it is possible that there are several different truncations of $A$ which are members of $\AT$.

**Theorem 4.3.** Let $T$, $D$, $A$, $B$, $k$ be as in one of Cases (a) or (b) in the Hypotheses above. Let $h : r_{k+1}[B, T]^* \to 2$ be a coloring. Then there is a Rado tree $S \in [D, T]$ such that $h$ is monochromatic on $r_{k+1}[B, S]^*$.

**Proof.** Assume the hypotheses. Given $U \in \AT \cup \mathcal{T}$ with $U \subseteq T$, define
\begin{equation}
\Ext(U)(B) = \{ \max(C) : C \in r_{k+1}[B, T]^* \text{ and } C \subseteq U \}.
\end{equation}
The coloring $h$ induces a coloring $h' : \Ext_T(B) \to 2$ by defining $h'(X) = h(C_X)$, where $C_X$ is the member of $\AT$ induced by the meet-closure of $X$.

Let $d + 1$ be the number of nodes in $\max(B)$, and fix an enumeration $s_0, \ldots, s_d$ of the nodes in $\max(B)$ with the property that for any $X \in \Ext_T(B)$, the coding node in $X$ extends $s_d$. Note that in both Cases (a) and (b), $d + 1 = 2^k$, as any
$C \in \mathcal{AT}_{k+1}$ has $2^k$ maximal nodes. Let $L$ denote the collection of all $l < \omega$ for which there is a member of $\text{Ext}_{T}(B)$ with nodes of length $l$.

For $i \leq d$, let $T_i = \{ t \in T : t \supseteq s_i \}$. Let $\kappa = \beth_{2d}$, so that the partition relation $\kappa \rightarrow (\aleph_1)_{2d}^{\omega}$ holds by the Erdős-Rado Theorem. The following forcing notion $\mathbb{P}$ adds $\kappa$ many paths through each $T_i$, $i < d$, and one path through $T_d$. However, as our goal is to find a tree $S \subseteq [D,T]$ for which $h$ is monochromatic on $r_{k+1}[B,S]$, the forcing will be applied in finite increments to construct $S$, without ever moving to a generic extension.

Define $\mathbb{P}$ to consist of the set of finite functions $p$ of the form

$$p : (d \times \delta_p) \cup \{ d \} \rightarrow \bigcup_{i \leq d} T_i \upharpoonright l_p,$$

where $\delta_p \in [\kappa]^{<\omega}$, $l_p \in L$, $\{ p(i, \delta) : \delta \in \delta_p \} \subseteq T_i \upharpoonright l_p$ for each $i < d$, and $p(d)$ is the coding node in $T \upharpoonright l_p$ extending $s_d$. The partial ordering on $\mathbb{P}$ is defined as follows:

$q \leq p$ if and only if $l_q \geq l_p$, $\delta_q \supseteq \delta_p$, $q(d) \supseteq p(d)$, and $q(i, \delta) \supseteq p(i, \delta)$ for each $(i, \delta) \in d \times \delta_p$.

Given $p \in \mathbb{P}$, the range of $p$ is

$$\text{ran}(p) = \{ p(i, \delta) : (i, \delta) \in d \times \delta_p \} \cup \{ p(d) \}.$$  

If $q \in \mathbb{P}$ and $\delta_p \subseteq \delta_q$, define

$$\text{ran}(q \upharpoonright \delta_p) = \{ q(i, \delta) : (i, \delta) \in d \times \delta_p \} \cup \{ q(d) \}.$$  

Thus, $q \leq p$ if and only if $\delta_q \supseteq \delta_p$ and $\text{ran}(q \upharpoonright \delta_p)$ end-extends $\text{ran}(p)$.

For $(i, \alpha) \in d \times \kappa$, let

$$\hat{b}_{i, \alpha} = \{ p(i, \alpha), p : p \in \mathbb{P} \text{ and } \alpha \in \delta_p \},$$

a $\mathbb{P}$-name for the $\alpha$-th generic branch through $T_i$. Let

$$\hat{b}_d = \{ (p(d), p) : p \in \mathbb{P} \},$$

a $\mathbb{P}$-name for the generic branch through $T_d$. Given a generic filter $G \subseteq \mathbb{P}$, notice that $\hat{b}_d^G = \{ p(d) : p \in G \}$, which is a cofinal path of coding nodes in $T_d$. Let $\hat{L}_d$ be a $\mathbb{P}$-name for the set of lengths of coding nodes in $\hat{b}_d$, and note that $\mathbb{P}$ forces that $\hat{L}_d \subseteq L$. Let $\mathcal{U}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\hat{L}_d$. Given $p \in \mathbb{P}$, notice that

$$p \Vdash \forall (i, \alpha) \in d \times \delta_p (\hat{b}_{i, \alpha} \upharpoonright l_p = p(i, \alpha)) \land (\hat{b}_d \upharpoonright l_p = p(d)).$$

We will write sets $\{ \alpha_i : i < d \}$ in $[\kappa]^d$ as vectors $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle$ in strictly increasing order. For $\vec{\alpha} \in [\kappa]^d$, let

$$\hat{b}_{\vec{\alpha}} = \langle \hat{b}_{0, \alpha_0}, \ldots, \hat{b}_{d-1, \alpha_{d-1}}, \hat{b}_d \rangle.$$  

For $l < \omega$, let

$$\hat{b}_{\vec{\alpha}} \upharpoonright l = \langle \hat{b}_{0, \alpha_0} \upharpoonright l, \ldots, \hat{b}_{d-1, \alpha_{d-1}} \upharpoonright l, \hat{b}_d \upharpoonright l \rangle.$$  

Using these abbreviations, one sees that $h'$ is a coloring on level sets of the form $\hat{b}_{\vec{\alpha}} \upharpoonright l$ whenever this is forced to be a member of $\text{Ext}_{T}(B)$. Given $\vec{\alpha} \in [\kappa]^d$ and $p \in \mathbb{P}$ with $\vec{\alpha} \subseteq \delta_p$, let

$$X(p, \vec{\alpha}) = \{ p(i, \alpha_i) : i < d \} \cup \{ p(d) \}.$$  

Notice that $X(p, \vec{\alpha})$ is a member of $\text{Ext}_{T}(B)$.  

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**BOREL SETS OF RADO GRAPHS AND RAMSEY’S THEOREM**

17
For each $\vec{\alpha} \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ satisfying the following:

1. $\vec{\alpha} \subseteq \delta_{p_{\vec{\alpha}}}$.
2. There is an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash \text{“} h'(\dot{b}_{\vec{\alpha}} | l) = \varepsilon_{\vec{\alpha}} \text{ for } U \text{ many } l \in \dot{L}_d \text{”}. $ 
3. $h'(X(p_{\vec{\alpha}}, \vec{\alpha})) = \varepsilon_{\vec{\alpha}}$.

Such conditions can be found as follows: Fix some $\bar{X} \in \text{Ext}_T(B)$ and let $t_i$ denote the node in $\bar{X}$ extending $s_i$, for each $i \leq d$. For $\vec{\alpha} \in [\kappa]^d$, define

$$p_{\vec{\alpha}}^0 = \{(i, \delta, t_i) : i < d, \delta \in \vec{\alpha}\} \cup \{(d, t_d)\}.$$ 

Then (1) will hold for all $p \leq p_{\vec{\alpha}}^0$, since $\delta_{p_{\vec{\alpha}}^0} = \vec{\alpha}$. Next, let $p_{\vec{\alpha}}^1$ be a condition below $p_{\vec{\alpha}}^0$ which forces $h'(\dot{b}_{\vec{\alpha}} | l) = \varepsilon_{\vec{\alpha}}$ for $U$ many $l \in \dot{L}_d$. Extend this to some condition $p_{\vec{\alpha}}^2 \leq p_{\vec{\alpha}}^1$ which decides a value $\varepsilon_{\vec{\alpha}} \in 2$ so that $p_{\vec{\alpha}}^2$ forces $h'(\dot{b}_{\vec{\alpha}} | l) = \varepsilon_{\vec{\alpha}}$ for $U$ many $l \in \dot{L}_d$. Then (2) holds for all $p \leq p_{\vec{\alpha}}^2$. If $p_{\vec{\alpha}}^2$ satisfies (3), then let $p_{\vec{\alpha}} = p_{\vec{\alpha}}^2$. Otherwise, take some $p_{\vec{\alpha}}^3 \leq p_{\vec{\alpha}}^2$ which forces $h'(\dot{b}_{\vec{\alpha}} | l) = \varepsilon_{\vec{\alpha}}$ for some $l \in \dot{L}$ with $l_{p_{\vec{\alpha}}^3} < l \leq l_{p_{\vec{\alpha}}^2}$. Since $p_{\vec{\alpha}}^3$ forces that $\dot{b}_{\vec{\alpha}} | l$ equals $\{p_{\vec{\alpha}}^3(i, \alpha_i) | l : i < d\} \cup \{p_{\vec{\alpha}}^3(d) \downarrow | l \}$, which is exactly $X(p_{\vec{\alpha}}^3 | l, \vec{\alpha})$, and this level set is in the ground model, it follows that $h'(X(p_{\vec{\alpha}}^3 | l, \vec{\alpha})) = \varepsilon_{\vec{\alpha}}$. Let $p_{\vec{\alpha}} = p_{\vec{\alpha}}^3 | l$. Then $p_{\vec{\alpha}}$ satisfies (1)-(3).

Let $\mathcal{I}$ denote the collection of all functions $\iota : 2d \to 2d$ such that for each $i < d$, \{$(2i), (2i + 1)$\} $\subseteq \{2i, 2i + 1\}$. For $\vec{\theta} = (\theta_0, \ldots, \theta_{2d - 1}) \in [\kappa]^{2d}$, $\iota(\vec{\theta})$ determines the pair of sequences of ordinals $(\iota_2(\vec{\theta}), \iota_3(\vec{\theta}))$, where

$$\iota_2(\vec{\theta}) = (\theta_{i(0)}, \theta_{i(2)}, \ldots, \theta_{i(2d - 2)}),$$
$$\iota_3(\vec{\theta}) = (\theta_{i(1)}, \theta_{i(3)}, \ldots, \theta_{i(2d - 1)}).$$

(36)

We now proceed to define a coloring $f$ on $[\kappa]^{2d}$ into countably many colors. Let $\delta_{\vec{\alpha}}$ denote $\delta_{p_{\vec{\alpha}}}$, $k_{\vec{\alpha}}$ denote $|\delta_{\vec{\alpha}}|$, $l_{\vec{\alpha}}$ denote $l_{p_{\vec{\alpha}}}$, and let $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}} \rangle$ denote the enumeration of $\delta_{\vec{\alpha}}$ in increasing order. Given $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, to reduce subscripts let $\bar{\alpha}$ denote $\iota_2(\vec{\theta})$ and $\bar{\beta}$ denote $\iota_3(\vec{\theta})$, and define

$$f(\iota, \vec{\theta}) = (i, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p_{\vec{\alpha}}(d), \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d),$$
$$\langle (i, j) : i < d, j < k_{\vec{\alpha}}, \text{ and } \delta_{\vec{\alpha}}(j) = \alpha_i \rangle,$$

(37)

$$\langle (j, k) : j < k_{\vec{\alpha}}, k < k_{\vec{\beta}}, \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle.$$ 

Fix some ordering of $\mathcal{I}$ and define

$$f(\vec{\theta}) = (f(\iota, \vec{\theta}) : \iota \in \mathcal{I}).$$

(38)

By the Erdős-Rado Theorem [11], there is a subset $K \subseteq \kappa$ of cardinality $\aleph_1$ which is homogeneous for $f$. Take $K' \subseteq K$ so that between each two members of $K'$ there is a member of $K$. Given sets of ordinals $I$ and $J$, we write $I < J$ to mean that every member of $I$ is less than every member of $J$. Take $K_i \subseteq K'$ be countably infinite subsets satisfying $K_0 < \cdots < K_{d - 1}$. The next four lemmas are almost verbatim the Claims 3 and 4 and Lemma 5.3 in [11], with small necessary changes being made. The proofs are included here for the reader’s convenience.

Fix some $\vec{\gamma} \in \prod_{i < d} K_i$, and define

$$\varepsilon^* = \varepsilon_{\vec{\gamma}}, \quad k^* = k_{\vec{\gamma}}, \quad t_d = p_{\vec{\gamma}}(d),$$
$$t_{i,j} = p_{\vec{\gamma}}(i, \delta_{\vec{\alpha}}(j)) \text{ for } i < d, \quad j < k^*.$$ 

(39)

We show that the values in equation (39) are the same for any choice of $\vec{\gamma}$. 
Lemma 4.4. For all \( \vec{\alpha} \in \prod_{i < d} K_i \), \( \varepsilon_{\vec{\alpha}} = \varepsilon^* \), \( k_{\vec{\alpha}} = k^* \), \( p_\vec{\alpha}(d) = t_d \), and \( \langle p_\vec{\alpha}(i, \delta_\vec{\alpha}(j)) : j < k_{\vec{\alpha}} \rangle = \langle t_{i,j} : j < k^* \rangle \) for each \( i < d \).

Proof. Let \( \vec{\alpha} \) be any member of \( \prod_{i < d} K_i \), and let \( \vec{\gamma} \) be the set of ordinals fixed above. Take \( i \in I \) to be the identity function on \( 2d \). Then there are \( \vec{\vartheta}, \vec{\vartheta}' \in [K]^{2d} \) such that \( \vec{\alpha} = \iota_\varepsilon(\vec{\vartheta}) \) and \( \vec{\gamma} = \iota_\varepsilon(\vec{\vartheta}') \). Since \( f(i, \vec{\vartheta}) = f(i, \vec{\vartheta}') \), it follows that \( \varepsilon_{\vec{\alpha}} = \varepsilon_{\vec{\gamma}} \), \( k_{\vec{\alpha}} = k_{\vec{\gamma}} \), \( p_{\vec{\alpha}}(d) = p_{\vec{\gamma}}(d) \), and \( \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle = \langle p_{\vec{\gamma}}(i, \delta_{\vec{\gamma}}(j)) : j < k_{\vec{\gamma}} \rangle : i < d \rangle \). \( \square \)

Let \( l^* \) denote the length of the node \( t_d \), and notice that the node \( t_{i,j} \) also has length \( l^* \), for each \( (i, j) \in d \times k^* \).

Lemma 4.5. Given any \( \vec{\alpha}, \vec{\beta} \in \prod_{i < d} K_i \), if \( j, k < k^* \) and \( \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \), then \( j = k \).

Proof. Let \( \vec{\alpha}, \vec{\beta} \) be members of \( \prod_{i < d} K_i \) and suppose that \( \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \) for some \( j, k < k^* \). For \( i < d \), let \( \rho_i \) be the relation from among \( \{<, =, >\} \) such that \( \alpha_i \rho_i \beta_i \). Let \( i \) be the member of \( I \) such that for each \( \vec{\vartheta} \in [K]^{2d} \) and each \( i < d \), \( \theta_{i(2i)} \rho_i \theta_{i(2i+1)} \). Fix some \( \vec{\vartheta} \in [K']^{2d} \) such that \( \iota_\varepsilon(\vec{\vartheta}) = \vec{\alpha} \) and \( \iota_\varepsilon(\vec{\vartheta}) = \vec{\beta} \). Since between any two members of \( K' \) there is a member of \( K \), there is a \( \vec{\zeta} \in [K]'^{2d} \) such that for each \( i < d \), \( \alpha_i \rho_i \zeta_i \) and \( \zeta_i \rho_i \beta_i \). Let \( \vec{\mu}, \vec{\nu} \) be members of \( [K]^{2d} \) such that \( \iota_\varepsilon(\vec{\mu}) = \vec{\alpha} \) and \( \iota_\varepsilon(\vec{\nu}) = \vec{\beta} \). Since \( \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \), the pair \( (j, k) \) is in the last sequence in \( f(i, \vec{\vartheta}) \). Since \( f(i, \vec{\mu}) = f(i, \vec{\nu}) = f(i, \vec{\vartheta}) \), also \( (j, k) \) is in the last sequence in \( f(i, \vec{\mu}) \) and \( f(i, \vec{\nu}) \). It follows that \( \delta_{\vec{\alpha}}(j) = \delta_{\vec{\zeta}}(k) \) and \( \delta_{\vec{\beta}}(j) = \delta_{\vec{\zeta}}(k) \). Hence, \( \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \), and therefore \( j \) must equal \( k \). \( \square \)

By the second line of equation (37), there is a strictly increasing sequence \( \langle j_i : i < d \rangle \) of members of \( k^* \) such that \( \delta_{\vec{\alpha}}(j_i) = \alpha_i \). By homogeneity of \( f \), this sequence \( \langle j_i : i < d \rangle \) is the same for all members of \( \prod_{i < d} K_i \). Then letting \( t_{i,j}^* \) denote \( t_{i,j} \), one sees that

\[
(40) \quad p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} = t_{i,j_i}^*.
\]

Let \( t_d^* \) denote \( t_d \).

Lemma 4.6. For any finite subset \( \vec{J} \subseteq \prod_{i < d} K_i \), \( p_{\vec{J}} := \bigcup \{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J} \} \) is a member of \( \mathcal{P} \) which is below each \( p_{\vec{\alpha}} \), \( \vec{\alpha} \in \vec{J} \).

Proof. Given \( \vec{\alpha}, \vec{\beta} \in \vec{J} \), if \( j, k < k^* \) and \( \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \), then \( j \) and \( k \) must be equal, by Lemma 4.5. Then Lemma 4.4 implies that for each \( i < d \),

\[
(41) \quad p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j} = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(k)).
\]

Hence, for all \( \delta \in \delta_{\vec{\alpha}} \cap \delta_{\vec{\beta}} \) and \( i < d \), \( p_{\vec{\alpha}}(i, \delta) = p_{\vec{\beta}}(i, \delta) \). Thus, \( p_{\vec{J}} := \bigcup \{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J} \} \) is a function with domain \( \delta_{\vec{J}} \cup \{d\} \), where \( \delta_{\vec{J}} = \bigcup \{\delta_{\vec{\alpha}} : \vec{\alpha} \in \vec{J} \} \). Thus, \( p_{\vec{J}} \) is a member of \( \mathcal{P} \). Since for each \( \vec{\alpha} \in \vec{J} \), \( \text{ran}(p_{\vec{J}} \upharpoonright \delta_{\vec{\alpha}}) = \text{ran}(p_{\vec{\alpha}}) \), it follows that \( p_{\vec{J}} \leq p_{\vec{\alpha}} \) for each \( \vec{\alpha} \in \vec{J} \). \( \square \)

Now we build an sRc tree \( S \in [D, T] \) so that the coloring \( h \) will be monochromatic on \( r_{k+1}(B, S)^* \). Recall that \( D = r_\varepsilon(T) \). Let \( M = \{m_j : j < \omega \} \) be the strictly increasing enumeration of those integers \( m > n \) such that for each \( F \in r_m[D, T] \),
the coding node in \(\max(F)\) extends \(s_d\). The integers in \(M\) represent the stages at which we will use the forcing to find the next level of \(S\) so that the members of \(r_{k+1}[B,S]^*\) will have the same \(h\)-color.

For each \(i \leq d\), extend the node \(s_i \in B\) to the node \(t_i^*\). Then extend each node in \(\max(D^+) \setminus B\) to a node in \(T \upharpoonright t^*\). If one wishes to be concrete, take the leftmost extensions in \(T\); how the nodes in \(\max(D^+) \setminus B\) are extended makes no difference to the conclusion of the theorem. Set

\[
U^* = \{t_i^* : i \leq d\} \cup \{u^* : u \in D^+ \setminus B\}.
\]

\(U^*\) end-extends \(\max(D^+)\). If \(m_0 = n + 1\), then \(D \cup U^*\) is a member of \(r_{m_0}[D,T]\). In this case, let \(U_{m_0} = D \cup U^*\), and let \(U_{m_1-1}\) be any member of \(r_{m_1-1}[U_{m_0}, T]\). Notice that \(U^*\) is the only member of \(\Ext_{U_{m_1-1}}(B)\), and it has \(h'\)-color \(\varepsilon^*\).

Otherwise, \(m_0 > n + 1\). In this case, take some \(U_{m_0-1} \in r_{m_0-1}[D,T]\) such that \(\max(U_{m+1})\) end-extends \(U^*\), and notice that \(\Ext_{U_{m_0-1}}(B)\) is empty. Now assume that \(j < \omega\) and we have constructed \(U_{m_j-1} \in r_{m_j-1}[D,T]\) so that every member of \(\Ext_{U_{m_j-1}}(B)\) has \(h'\)-color \(\varepsilon^*\). Fix some \(V \in r_{m_j}[U_{m_j-1}, T]\) and let \(Z = \max(V)\). We will extend the nodes in \(Z\) to construct \(U_{m_j} \in r_{m_j}[U_{m_j-1}, T]\) which is homogeneous for \(h'\) in value \(\varepsilon^*\). This is done by constructing the condition \(q\), below, and then extending it to some \(r \leq q\) which decides all members of \(\Ext_T(B)\) coming from the nodes in \(\operatorname{ran}(r)\) have \(h'\)-color \(\varepsilon^*\).

Let \(q(d)\) denote the coding node in \(Z\) and let \(l_q = |q(d)|\). For each \(i < d\), let \(Z_i\) denote the set of nodes in \(Z \cap T_i\); this set has \(2^{m_j-1}\) many nodes. For each \(i < d\), take a set \(J_i \subseteq K_i\) of cardinality \(2^{m_j-1}\) and label the members of \(Z_i\) as \(\{z_\alpha : \alpha \in J_i]\). Let \(\vec{J}\) denote \(\prod_{i<d} \vec{J}_i\). By Lemma 1.6 the set \(\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}\) is compatible, as evidenced by the fact that \(p_{\vec{J}} := \bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}\) is a condition in \(P\).

Let \(\vec{\delta}_q = \bigcup \{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}\). For \(i < d\) and \(\vec{\alpha} \in \vec{J}\) and \(i < d\),

\[
q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{J}}(i, \alpha_i),
\]

and

\[
q(d) \supseteq t_d^* = p_{\vec{\alpha}}(d) = p_{\vec{J}}(d).
\]

For \(i < d\) and \(\delta \in \vec{\delta}_q \setminus J_i\), let \(q(i, \delta)\) be the leftmost extension of \(p_{\vec{J}}(i, \delta)\) in \(T\) of length \(l_q\). Define

\[
q = \{q(d)\} \cup \{(i, \delta, q(i, \delta)) : i < d, \delta \in \vec{\delta}_q\}.
\]

This \(q\) is a condition in \(P\), and \(q \leq p_{\vec{J}}\).

To construct \(U_{m_j}\), take an \(r \leq q\) in \(P\) which decides some \(l_j\) in \(\mathcal{L}_d\) for which \(h'(\vec{b}_{\vec{\alpha}} \upharpoonright l_j) = \varepsilon^*\), for all \(\vec{\alpha} \in \vec{J}\). This is possible since for all \(\vec{\alpha} \in \vec{J}\), \(p_{\vec{\alpha}}\) forces \(h'(\vec{b}_{\vec{\alpha}} \upharpoonright l) = \varepsilon^*\) for \(\mathcal{L}\) many \(l \in L_d\). By the same argument as in creating the conditions \(p_{\vec{\alpha}}\) to satisfy (3), we may assume that the nodes in the image of \(r\) have length \(l_j\). Since \(r\) forces \(\vec{b}_{\vec{\alpha}} \upharpoonright l_j = X(r, \vec{\alpha})\) for each \(\vec{\alpha} \in \vec{J}\), and since the coloring \(h'\) is defined in the ground model, it follows that \(h'(X(r, \vec{\alpha})) = \varepsilon^*\) for each \(\vec{\alpha} \in \vec{J}\). Let \(Y\) be the level set consisting of the nodes \(\{r(d)\} \cup \{r(i, \alpha) : i < d, \alpha \in J_i\}\) along with a unique node \(v_z\) in \(T \setminus l_j\) extending \(z\), for each \(z \in Z \setminus \{r(d)\} \cup \{r(i, \alpha) : i < d, \alpha \in J_i\}\). Then \(Y\) end-extends \(Z\). Letting \(U_{m_j} = U_{m_j-1} \cup Y\), we see that \(U_{m_j}\) is a member of \(r_{m_j}[U_{m_j-1}, T]\) such that \(h'\) has value \(\varepsilon^*\) on \(\Ext_{U_{m_j}}(B)\). Let \(U_{m_{j+1}-1}\) be any member of \(r_{m_{j+1}-1}[U_{m_j}, T]\). This completes the inductive construction.
Let \( S = \bigcup_{j < \omega} U_m \). Then \( S \) is a member of \([D, T]\) and for each \( X \in \text{Ext}_S(B)\), \( h'(X) = \varepsilon^*\). Thus, \( S \) satisfies the theorem.

Corollary 4.7 follows immediately from Theorem 4.3. Case (a) handles \( k \geq 1 \) and Case (b) handles \( k = 0 \). For the reader interested in topological Ramsey spaces, we point out that this corollary states that Axiom A.4 of Todorcevic’s Axioms in Chapter 5, Section 1 of [23] holds for the space \( \mathcal{T} \) of strong Rado coding trees.

**Corollary 4.7.** Let \( k < \omega \), \( A \in \mathcal{AT}_k \), and \( T \in \mathcal{T} \) be given with \( A \subseteq T \), and let \( n = \text{depth}_T(A) \). For any subset \( \mathcal{O} \subseteq r_{k+1}[A, T] \), there is an \( S \in [r_n(T), T] \) such that either \( \mathcal{O} \subseteq r_{k+1}[A, S] \) or else \( \mathcal{O} \cap r_{k+1}[A, S] = \emptyset \).

5. **Borel sets of strong Rado coding trees are completely Ramsey**

In this section we prove Theorem 5.0. Given a Rado graph \( R = (\omega, E) \), Borel subsets of the space \( \mathcal{T}_R \) of strong Rado coding trees are completely Ramsey (see Definition 5.10). The proof entails showing that the collection of completely Ramsey subsets of \( \mathcal{T}_R \) is a \( \sigma \)-algebra containing all open sets. Showing that open sets are completely Ramsey is accomplished by induction on the rank of Nash-Williams families, collections of finite sets which determine basic open sets. However, since the space \( \mathcal{T}_R \) does not possess the same amalgamation property that the Baire space has, we will need to do induction on open sets extending members of \( \mathcal{AT} \). This is the reason the broader statement in Theorem 4.3 was proved, rather than simply Corollary 4.7. We will apply Theorem 1.6 in the next section to prove the Main Theorem.

As usual, fix a Rado graph \( R = (\omega, E) \), and let \( \mathcal{S} \) denote \( \mathcal{S}_R \) and \( \mathcal{T} \) denote \( \mathcal{T}_R \). Recall the definitions of \( \mathcal{AT} \) and \( \mathcal{AT}(T) \), in equations (23) and (24), respectively, as well as Definition 4.2. Given \( B \in \mathcal{AT} \) and \( T \in \mathcal{T} \) an important property of the set \([B, T]^*\) is that it is open in the Ellentuck topology on \( T \): If \( B \in \mathcal{AT}_k \) for some \( k \), then the set \([B, T]^*\) is the union of \([B, T]\) along with all \([C, T]\), where \( C \in \mathcal{AT}_k \) and \( \max(C) \) end-extends \( \max(B) \). If \( B \) is in \( \mathcal{AT} \) but not in \( \mathcal{AT}_k \), then letting \( k \) be the least integer for which there is some \( C \in \mathcal{AT}_k \) with \( \max(C) \supseteq \max(B) \), we see that \([B, T]^*\) equals the union of all \([C, T]\), where \( C \in \mathcal{AT}_k \) and \( C \supseteq B \). For the same reasons, the set \([B, S]^*\) is open in the metric topology on \( T \).

**Definition 5.1.** A subset \( \mathcal{F} \subseteq \mathcal{AT} \) is said to have the Nash-Williams property if for any two distinct members \( F, G \in \mathcal{F} \), neither is an initial segment of the other.

A Nash-Williams family \( \mathcal{F} \) determines the metrically open set
\[
O_{\mathcal{F}} = \bigcup_{F \in \mathcal{F}} [F, S].
\]

Conversely, to each open set \( O \subseteq \mathcal{T} \) in the metric topology there corresponds a Nash-Williams family \( \mathcal{F}(O) \) by defining \( F \in \mathcal{AT} \) to be a member of \( \mathcal{F}(O) \) if and only if \([F, S] \subseteq O \) and if \( G \) is any proper initial segment of \( F \), then \([G, S] \not\subseteq O \).

Given \( \mathcal{F} \subseteq \mathcal{AT} \) and \( B \in \mathcal{AT} \), define
\[
\mathcal{F}_B = \{ F \in \mathcal{F} : \exists k (\max(r_k(F)) \supseteq \max(B)) \}.
\]

In particular, if \( \mathcal{F} \subseteq r[B, S]^* \), then \( \mathcal{F}_B = \mathcal{F} \). If \( \mathcal{F} \) is a Nash-Williams family, then \( B \in \mathcal{F} \) if and only if \( \mathcal{F}_B = \{ B \} \). Given \( T \in \mathcal{T} \), let
\[
\mathcal{F}|T = \{ F \in \mathcal{F} : F \in \mathcal{AT}(T) \}.
\]
With this notation, notice that \( \mathcal{F}_B | T = \mathcal{F} \cap r[B, T]^* \), for any \( B \in \hat{\mathcal{A}} T \).

For \( F \in \mathcal{A} T \), recall from Subsection 2.2 that \( |F| \) denotes the \( k \) for which \( F \in \mathcal{A} T_k \). Given a set \( \mathcal{F} \subseteq \mathcal{A} T \), let

\[
F \subseteq \mathcal{A} T \text{ accepts } \quad \text{given} \quad \text{Theorem 5.4.}
\]

\[
(49) \quad \hat{\mathcal{F}} = \{ r_k(F) : F \in \mathcal{F} \text{ and } k \leq |F| \},
\]

and note that \( \hat{\mathcal{F}} \subseteq \mathcal{A} T \). If \( \mathcal{F} \) is a Nash-Williams family, then \( \mathcal{F} \) consists of the \( \subseteq \)-maximal members of \( \hat{\mathcal{F}} \).

**Definition 5.2.** Suppose \( T \in \mathcal{T} \) and \( B \in \hat{\mathcal{A}} T(T) \). We say that a family \( \mathcal{F} \subseteq r[B, T]^* \) is a front on \( [B, T]^* \) if \( \mathcal{F} \) is a Nash-Williams family and for each \( S \in [B, T]^* \), there is some \( C \in \mathcal{F} \) such that \( C \subseteq S \).

Notice that a front \( \mathcal{F} \) on \( [B, T]^* \) determines a collection of disjoint (Ellentuck) basic open sets \([C, T] , C \in \mathcal{F} \) whose union is exactly \( [B, T]^* \).

**Assumption 5.3.** Let \( T \in \mathcal{T} \) be fixed, and let \( D = r_d(T) \) for some \( d < \omega \). Given \( A \in \hat{\mathcal{A}} T(T) \) with \( \max(A) \subseteq \max(D) \). Recall that \( A^+ \) denotes the union of \( A \) with the set of immediate extensions in \( \hat{T} \) of the members of \( \max(A) \). Let \( B \) be a member of \( \hat{\mathcal{A}} T \) such that \( \max(A) \subseteq \max(B) \subseteq \max(A^+) \). We consider two cases for triples \( (A, B, k) \):

**Case (a).** \( k \geq 1, A \in \mathcal{A} T_k(T) \), and \( B = A^+ \).

**Case (b).** \( A \) has at least one node, and each member of \( \max(A) \) has exactly one extension in \( B \). Let \( k \) be the integer satisfying \( 2^k = \text{card}(\max(A)) \).

**Theorem 5.4.** Given \( T \in \mathcal{D}, (A, B, k), \ d = \text{depth}_T(A) \), and \( D = r_d(T) \) as in Assumption 5.3, let \( \mathcal{F} \subseteq r[B, T]^* \) be a Nash-Williams family. Then there is an \( S \in [D, T] \) such that either \( \mathcal{F}|S \) is a front on \( [B, S]^* \) or else \( \mathcal{F}|S = \emptyset \).

**Proof.** Recall that a front on \( [B, S]^* \) determines a collection of disjoint (Ellentuck) basic open sets \([C, T] , C \in \mathcal{F} \) whose union is exactly \( [B, S]^* \).

**Fact 5.5.** If \( S \) accepts \( C \), then so does each \( P \leq S \) with \( C \in \mathcal{A} D(P) \). If \( S \) w-rejects \( C \), then either \( C \notin r[B, S]^* \) and every \( P \leq S \) also w-rejects \( C \), or else \( C \in r[B, S]^* \) and every \( P \in [\text{depth}_S(C), S] \) w-rejects \( C \).

**Proof.** Suppose \( S \) accepts \( C \) and \( P \leq S \) with \( C \in \mathcal{A} D(P) \). Since \( \mathcal{F}_C|S \) is a front on \([C, S] \), it follows that \( \mathcal{F}_C|P \) is a front on \([C, P] \). Hence \( P \) accepts \( C \).

Suppose \( S \) w-rejects \( C \). If \( C \notin r[B, S]^* \), then also for each \( P \leq S \), \( C \notin r[B, S]^* \) and hence \( P \) w-rejects \( C \). Otherwise, \( C \in r[B, S]^* \). Let \( n = \text{depth}_S(C) \) and suppose \( P \in [n, S] \). Since \( S \) w-rejects \( C \), for each \( Q \in [n, S] \) there is an \( R \in [n, Q] \) such that for all \( m, r_m(R) \notin \mathcal{F} \). Note that \( P \in [n, S] \) implies \( [n, P] \subseteq [n, S] \); so for each \( Q \in [n, P] \) there is an \( R \in [a, Q] \) such that for all \( m, r_m(Q) \notin \mathcal{F} \). Therefore, \( P \) w-rejects \( C \).

**Lemma 5.6.** Given \( C \in r[B, S]^* \) and \( n = \text{depth}_S(C) \), either \( \exists P \in [n, S] \) which w-rejects \( C \), or else \( \forall P \in [n, S] \) \( \exists Q \in [n, P] \) which accepts \( C \).
Proof. Suppose there is no $P \in [n, S]$ which w-rejects $C$. Then $\forall P \in [n, S]$,
\begin{equation}
\exists Q \in [n, P] \forall X \in [C, Q] \exists m(r_m(X) \in \mathcal{F}).
\end{equation}
Thus, for all $P \in [n, S]$ there is a $Q \in [n, P]$ such that $\mathcal{F}_C|Q$ is a front on $[C, Q]$; that is, $Q$ accepts $C$. \hfill \Box

**Fact 5.7.** (a) For each $C \in r[B, T]^*$, there is an $S \in [\text{depth}_T(C), T]$ which decides $C$.

(b) If $C \in r[B, T]^*$, then $S \in [B, T]^*$ with $C \in \mathcal{AD}(S)$ accepts $C$ if and only if $S$ accepts each $F \in r|C|+1[C, S]$.

Proof. For (a), let $n = \text{depth}_S(C)$. By Lemma 5.6 either there is an $S \in [n, T]$ which w-rejects $C$, or else there is an $S \in [n, T]$ which accepts $C$.

For (b), given the hypotheses, $S$ accepts $C$ if $\mathcal{F}_C|S$ is a front on $[C, S]$ iff for each $F \in r|C|+1[C, S]$, $\mathcal{F}_C|S$ is a front on $[F, S]$ iff $S$ accepts each $F \in r|C|+1[C, S]$. \hfill \Box

Recall that $B \in \mathcal{AD}$, but is not necessarily a member of $\mathcal{AD}$. We shall say that $S$ accepts $B$ if $S$ accepts $F$ for all $F \in r_{k+1}[B, S]^*$.

**Fact 5.8.** If $S \in [B, T]^*$ accepts $B$, then $\mathcal{F}_B|S$ is a front on $[B, S]^*$.

Proof. For each $C \in r_{k+1}[B, S]^*$, $S$ accepts $C$ implies that $\mathcal{F}_C|S$ is a front on $[C, S]$. Since $[B, S]^* = \bigcup\{[C, S] : C \in r_{k+1}[B, S]^*\}$, it follows that $\mathcal{F}|S = \bigcup\{\mathcal{F}_C|S : C \in r_{k+1}[B, S]^*\}$, which is a front on $[B, S]^*$. \hfill \Box

**Lemma 5.9.** There is an $S \in [d, T]$ which decides each $C$ in $r[B, S]^*$.

Proof. By finitely many applications of Fact 5.7 we obtain a $T_1 \in [d+1, T]$ such that $T_1$ decides each $C \in r[B, T]^*$ with $C \subseteq r_{d+1}(T)$. Given $T_1$, by finitely many applications of Fact 5.7 we obtain a $T_{i+1} \in [d+i+1, T_i]$ such that $T_{i+1}$ decides each $C \in r[B, T_i]^*$ with $C \subseteq r_{d+i+1}(T_i)$. Let $S = \bigcup_{i=1}^\infty r_{d+i}(T_i)$, which is the same as $\bigcup_{i=1}^\infty r_{d+i+1}(T_i)$. Then $S \in [d, T]$ (in fact, $S \in [d+1, T]$) and for $C \in r[B, S]^*$, $T_i$ decides $C$, where $i$ is the least index satisfying $C \subseteq r_{d+i}(T_i)$. Since $S \in [d+i, T_i]$, it follows that $S$ decides $C$ in the same way that $T_i$ does. Thus, $S$ decides every $C \in r[B, S]^*$. \hfill \Box

Now we finish the proof of the theorem. Take $S$ as in Lemma 5.9 and define a coloring $f : r[B, S]^* \to 2$ by $f(C) = 0$ if $S$ accepts $C$ and $f(C) = 1$ if $S$ w-rejects $C$.

By the Extended Pigeonhole Principle, Theorem 4.3, there is a $P \in [d, S]$ for which $f$ is monochromatic on $r_{k+1}[B, P]^*$. Now if $f$ has color 0 on this set, then $P$ accepts $B$ and by Fact 5.8, $\mathcal{F}|P$ is a front on $[B, P]^*$.

Otherwise, $f$ has color 1 on $r_{k+1}[B, P]^*$ so $P$ w-rejects each member of $r_{k+1}[B, P]^*$. Let $P_0 = P$. Apply Theorem 4.3 finitely many (possibly 0) times, to obtain some $P_i \in [d+i, P_0]$ such that for each $C \in r[B, P_i]$ with $C \subseteq r_{d+i}(P_0)$, all members of $r|C|+1[C, P_i]^*$ have the same $f$-color. Since such a $C$ is necessarily in $r_{k+1}[B, P_0]^*$ and $P_i$ w-rejects $C$, Fact 5.7 implies that this $f$-color must be 1.

For $i \geq 1$, we have the following the induction hypothesis: $P_i \in [d+i, P_{i-1}]$ and for each $C \in r[B, P_{i-1}]$ with $C \subseteq r_{d+i}(P_{i-1})$, $P_i$ w-rejects all members of $r|C|+1[C, P_i]$. Apply Theorem 4.3 finitely many times to obtain a $P_{i+1} \in [d+i+1, P_i]$ such that $f$ is monochromatic on $r|C|+1[C, P_{i+1}]$ for each $C \in r[B, P_i]^*$ with $C \subseteq r_{d+i+1}(P_i)$. Fix a $C \in r[B, P_i]^*$ with $C \subseteq r_{d+i+1}(P_i)$. If $|C| = k + 1$ then $P_{i+1}$ w-rejects $C$, since $C \in r_{k+1}[B, P_i]^*$ and $P_{i+1} \in [B, P_i]^*$. Suppose now that $|C| > k + 1$. By the induction hypothesis, $P_i$ w-rejects $C$ since $C \in r|F|+1[F, P_i]$.\hfill \Box
where \( F = r_{[C|-1]}(C) \subseteq r_{a+i}(P_{i-1}) \). Now if the \( f \)-color on \( r_{[C|-1]}(C, P_{i+1}) \) is 0, then \( P_{i+1} \) accepts \( C \) by Fact [5.7] a contradiction. Hence, \( f \) has color 1 on \( r_{[C|-1]}(C, P_{i+1}) \); in particular, \( P_{i+1} \) w-rejects each member of \( r_{[C|-1]}(C, P_{i+1}) \).

Let \( Q = \bigcup_{i=1}^{\infty} r_{d+i}(P_i) \). Then \( Q \) w-rejects each member of \( r[B, Q]^{*} \). By definition of w-rejects, for each \( C \in r[B, Q]^{*} \),

\[
\forall R \in [\text{depth}_{Q}(C), Q] \exists X \in [C, R] \forall n(r_n(X) \notin F)
\]

Suppose toward a contradiction that there is an \( C \in F \setminus Q \). Then for all \( X \in [C, Q] \), \( r_{[C|]}(X) = C \in F \). So \( Q \in [\text{depth}_{Q}(C), Q] \) and for all \( X \in [C, Q] \), \( \exists n(r_n(X) \in F) \). But this contradicts (51). Thus \( F \setminus Q \) must be empty. \( \square \)

**Definition 5.10.** Let \( \mathcal{X} \) be a subset of \( \mathcal{T} \). We say that \( \mathcal{X} \) is Ramsey if for each \( T \in \mathcal{T} \) there is an \( S \leq T \) such that either \( \mathcal{X} \subseteq [\emptyset, S] \) or else \( \mathcal{X} \cap [\emptyset, S] = \emptyset \). \( \mathcal{X} \) is said to be completely Ramsey (CR) if for each \( C \in \mathcal{AT} \) and each \( T \in \mathcal{T} \), there is an \( S \in [C, T] \) such that either \([C, S] \subseteq \mathcal{X}\) or else \([C, S] \cap \mathcal{X} = \emptyset\). For this article, we introduce additional terminology: \( \mathcal{X} \) is CR\(^{*}\) if for each quadruple \( T, A, B, D \) as in Assumption 5.3, there is an \( S \in [D, T] \) such that either \([B, S]^{*} \subseteq \mathcal{X} \) or else \([B, S]^{*} \cap \mathcal{X} = \emptyset \).

**Remark 5.11.** Theorem [5.4] shows that metrically open sets are completely Ramsey. Importantly, it proves the stronger statement that metrically open sets are CR\(^{*}\). This stronger statement will be used to get around the lack of amalgamation (Todorcevic’s Axiom A.3(b)) for \((\mathcal{T}, \leq, r)\), to prove that Borel sets are completely Ramsey, and in fact, even CR\(^{*}\).

**Lemma 5.12.** Complements of CR\(^{*}\) sets are CR\(^{*}\).

**Proof.** Suppose \( \mathcal{X} \subseteq \mathcal{T} \) is CR\(^{*}\). Given \( T, B, D \) as in Assumption 5.3, by definition of CR\(^{*}\), there is an \( S \in [D, T] \) such that either \([B, S]^{*} \subseteq \mathcal{X} \) or else \([B, S]^{*} \cap \mathcal{X} = \emptyset \).

Letting \( \mathcal{Y} = \mathcal{T} \setminus \mathcal{T} \), the complement of \( \mathcal{X} \), we see that either \([B, S]^{*} \cap \mathcal{Y} = \emptyset \) or else \([B, S]^{*} \subseteq \mathcal{Y} \). \( \square \)

In the rest of this section, given \( T \in \mathcal{T} \), assume that \([\emptyset, T] \) inherits the subspace topology from \( T \) with the metric topology. Thus, the basic metrically open sets of \([\emptyset, T] \) are of the form \([C, T] \), where \( C \in \mathcal{T}(T) \). The next two lemmas set up for Lemma 5.15 that countable unions of CR\(^{*}\) sets are CR\(^{*}\).

**Lemma 5.13.** Suppose \( \mathcal{X} \subseteq \mathcal{T} \) is CR\(^{*}\). Then for each \( T \in \mathcal{T} \) and each \( D \in \mathcal{AT}(T) \), there is an \( S \in [D, T] \) such that \( \mathcal{X} \cap [\emptyset, S] \) is metrically open in \([\emptyset, S] \).

**Proof.** Fix \( T \in \mathcal{T} \) and \( D \in \mathcal{AT}(T) \). Let \( \langle (A_j, B_j) : j < \tilde{j} \rangle \) be an enumeration of the pairs \((A_j, B_j)\) satisfying Assumption 5.3 for \( T \) and \( D \). Notice that \( \bigcup_{j<\tilde{j}} [B_j, T]^* = [\emptyset, T] \). Let \( T_{-1} = T \). For \( j < \tilde{j} \), given \( T_{j-1} \), by the definition of CR\(^{*}\) we may take some \( T_j \in [D, T_{j-1}] \) such that either \([B_j, T_j]^* \subseteq \mathcal{X} \) or else \( \mathcal{X} \cap [B_j, T_j]^* = \emptyset \).

Let \( S = T_{j-1} \). Then \( S \in [D, T] \), and for each \( j < \tilde{j} \), \([B_j, S]^* \subseteq [B_j, T_j]^* \). Notice that \( [\emptyset, S] = \bigcup_{j<\tilde{j}} [B_j, S]^* \). (If \( B_j \not\subseteq S \), then \([B_j, S]^* = \emptyset \).) Hence,

\[
\mathcal{X} \cap [\emptyset, S] = \bigcup_{j<\tilde{j}} (\mathcal{X} \cap [B_j, S]^*).
\]
For $j < \tilde{j}$, if $[B_j, T_j]^* \subseteq \mathcal{X}$ then $\mathcal{X} \cap [B_j, S]^* = [B_j, S]^*$; and if $\mathcal{X} \cap [B_j, T_j]^* = \emptyset$ then $\mathcal{X} \cap [B_j, S]^* = \emptyset$. Thus, 

\[(53) \quad \mathcal{X} \cap [0, S] = \bigcup_{j \in J} [B_j, S]^*,\]

where $J = \{ j < \tilde{j} : [B_j, T_j]^* \subseteq \mathcal{X} \}$. As each $[B_j, S]^*$ is metrically open in the subspace $[0, S]$, $\mathcal{X} \cap [0, S]$ is also metrically open in the subspace $[0, S]$. \hfill \Box

**Lemma 5.14.** Suppose $\mathcal{X}_n$, $n < \omega$, are CR* sets. Then for each $T \in \mathcal{T}$ and each $D \in \mathcal{AT}(T)$, there is an $S \in [D, T]$ such that for each $n < \omega$, $\mathcal{X}_n \cap [0, S]$ is metrically open in $[0, S]$.

**Proof.** Suppose $\mathcal{X}_n$, $n < \omega$, are CR* sets. Since $\mathcal{X}_0$ is CR*, Lemma 5.13 implies that there is an $S_0 \in [r_d(T), T]$ such that $\mathcal{X}_0 \cap [0, S_0]$ is metrically open in the subspace topology on $[0, S_0]$. Let $\mathcal{O}_0 \subseteq \mathcal{T}$ be a metrically open set satisfying $\mathcal{X}_0 \cap [0, S_0] = \mathcal{O}_0 \cap [0, S_0]$. In general, given $i < \omega$ and $S_i \in [r_{d+i}(T), T]$, by Lemma 5.13 there is some $S_{i+1} \in [r_{d+i+1}(S_i), S_i]$ and some metrically open $\mathcal{O}_i \subseteq \mathcal{T}$ such that $\mathcal{X}_i \cap [0, S_i] = \mathcal{O}_i \cap [0, S_i]$. Let $S = \bigcup_{i < \omega} r_{d+i}(S_i)$. Then $S$ is a member of $[r_d(T), T]$. Letting $S_{i-1} = T$, note that $S \in [r_{d+i}(S_i), S_{i-1})$ for each $i < \omega$. It follows that $\mathcal{X}_i \cap [0, S] = \mathcal{O}_i \cap [0, S]$; hence $\mathcal{X}_i \cap [0, S]$ is metrically open in $[0, S]$. \hfill \Box

**Lemma 5.15.** Countable unions of CR* sets are CR*.

**Proof.** Suppose $\mathcal{X}_n$, $n < \omega$, are CR* subsets of $\mathcal{T}$, and let $\mathcal{X} = \bigcup_{n < \omega} \mathcal{X}_n$. Let $T, B, D, d, k$ be as in Assumption 5.3. We claim that there is some $U \in [D, T]^*$ such that $[B, U]^* \subseteq \mathcal{X}$ or $[B, U]^* \cap \mathcal{X} = \emptyset$.

By Lemma 5.14 there is an $S \in [D, T]$ such that for each $n < \omega$, $\mathcal{X}_n \cap [0, S]$ is metrically open in $[0, S]$. Thus, $\mathcal{X} \cap [0, S]$ is metrically open in $[0, S]$, so $\mathcal{X} \cap [0, S] = \mathcal{O} \cap [0, S]$ for some metrically open set $\mathcal{O} \subseteq \mathcal{T}$. Relativizing to $[0, S]$, Theorem 5.3 implies that $\mathcal{O}$ is CR* (in $[0, S]$). Since $r_d(S) = D$, by definition of CR* there is some $U \in [D, S]$ such that either $[B, U]^* \subseteq \mathcal{O}$ or else $[B, U]^* \cap \mathcal{O} = \emptyset$. Therefore, either 

\[(54) \quad [B, U]^* = [B, U]^* \cap [0, S] \subseteq \mathcal{O} \cap [0, S] = \mathcal{X} \cap [0, S],\]

or else

\[(55) \quad [B, U]^* \cap \mathcal{X} = [B, U]^* \cap [0, S] \cap \mathcal{X} \subseteq [B, U]^* \cap [0, S] \cap \mathcal{O} \subseteq [B, U]^* \cap \mathcal{O} = \emptyset.\]

Thus, $\mathcal{X}$ is CR*.

\hfill \Box

**Theorem 5.16.** The collection of CR* subsets of $\mathcal{T}$ contains all Borel subsets of $\mathcal{T}$. In particular, Borel subsets of the space $\mathcal{T}$ of strong Rado coding trees are completely Ramsey.

**Proof.** This follows from Theorem 5.4 and Lemmas 5.12 and 5.15. \hfill \Box

**Remark 5.17.** We end this section with a remark about the infinite dimensional Ramsey theory of the rationals. Extend now the lexicographic order on $\mathbb{S}$ as follows: For $s, t \in \mathbb{S}$, define $s <_{\text{lex}} t$ exactly when one of the following hold: (a) $s$ and $t$ are incomparable and $s(\lfloor s \wedge t \rfloor) < t(\lfloor s \wedge t \rfloor)$; (b) $s \subseteq t$ and $t(\lfloor s \rfloor) = 1$; or (c) $t \subset s$...
and \( s(|t|) = 0 \). This defines a total order on \( \mathcal{S} \) so that \( (\mathcal{S}, \leq_{\text{lex}}) \) is order isomorphic to \((\mathbb{Q}, <)\). Using this isomorphism, say \( \varphi : (\mathcal{S}, \leq_{\text{lex}}) \to (\mathbb{Q}, <) \), Milliken’s Theorem provides infinite dimensional Ramsey theorem for the space of all subsets of \( \mathbb{Q} \) which are \( \varphi \)-images of strong subtrees of \( \mathcal{S} \). This handles only one strong similarity type of subcopies of \( \mathbb{Q} \), though.

It is useful to note that since the coding nodes in the trees in \( \mathcal{T} \) are dense, they also represent the rationals, using the extended lexicographic order \( \leq_{\text{lex}} \) just defined. Thus, Theorem 5.10 provides an infinite dimensional Ramsey theorem for the rationals. Similarly to the constraints provided by the work in [14] for the Rado graph, the work of Devlin [2] on the big Ramsey degrees of the rationals provides constraints, again in terms of strong similarity types, for the infinite dimensional Ramsey theory of the rationals.

6. The Main Theorem

We now prove the Main Theorem. Recall the homeomorphism \( \theta : \mathcal{R}(\mathbb{R}) \to \mathcal{T}_\mathbb{R} \) defined at the end of Section 3 by \( \theta([\mathbb{R}']) = T_{\mathbb{R}'} \), for \( \mathbb{R}' \in \mathcal{R}(\mathbb{R}) \). Note that given a Borel subset \( \mathcal{X} \subseteq \mathcal{R}(\mathbb{R}) \), the set \( \theta[\mathcal{X}] \) is a Borel subset of \( \mathcal{T}_\mathbb{R} \). The subspace \( \mathcal{R}(\mathbb{R}) \) of the Baire space inherits the Ellentuck topology, refining the metric topology on \([\omega]^\omega\); given \( \mathbb{R}' \in \mathcal{R}(\mathbb{R}) \) and \( n \in \omega \), define \( r_n(\mathbb{R}') \) to be the subgraph of \( \mathbb{R}' \) induced on the first \( n - 1 \) vertices of \( \mathbb{R}' \). Let

\[
AR = \{r_n(\mathbb{R}') : \mathbb{R}' \in \mathcal{R}(\mathbb{R}) \text{ and } n < \omega\}.
\]

For \( F \in AR \) and \( \mathbb{R}' \in \mathcal{R}(\mathbb{R}) \), write \( F \sqsubseteq \mathbb{R}' \) if and only if \( F = r_n(\mathbb{R}') \) for some \( n \). Define

\[
[F, \mathbb{R}'] = \{\mathbb{R}'' : \mathbb{R}'' \in \mathcal{R}(\mathbb{R}) : F \sqsubseteq \mathbb{R}''\}.
\]

We say that a set \( \mathcal{X} \subseteq \mathcal{R}(\mathbb{R}) \) is completely Ramsey if for any \( F \in AR \) and \( \mathbb{R}' \in \mathcal{R}(\mathbb{R}) \), there is some \( \mathbb{R}'' \in [F, \mathbb{R}'] \) such that either \([F, \mathbb{R}''] \subseteq \mathcal{X}\) or else \([F, \mathbb{R}''] \cap \mathcal{X} = \emptyset\).

**Main Theorem.** Let \( \mathbb{R} = (\omega, E) \) be the Rado graph. Then every Borel subset \( \mathcal{X} \subseteq \mathcal{R}(\mathbb{R}) \) is completely Ramsey. In particular, if \( \mathcal{X} \subseteq \mathcal{R}(\mathbb{R}) \) is Borel, then for each \( \mathbb{R}' \in \mathcal{R}(\mathbb{R}) \), there is a Rado graph \( \mathbb{R}'' \in \mathcal{R}(\mathbb{R}') \) such that \( \mathcal{R}(\mathbb{R}'') \) is either contained in \( \mathcal{X} \), or else is disjoint from \( \mathcal{X} \).

**Proof.** Let \( \mathcal{X} \) be a Borel subset of \( \mathcal{R}(\mathbb{R}) \), and suppose \( F \in AR \) and \( \mathbb{R}' \in \mathcal{R}(\mathbb{R}) \). If \([F, \mathbb{R}'] = \emptyset\) then we are done, so assume now that there is some \( \mathbb{R}'' \in [F, \mathbb{R}'] \). Then \( F \) is an initial segment of \( \mathbb{R}'' \). Let \( n \) be the integer such that \( F = r_n(\mathbb{R}'') \), and let \( A = r_n(T_{\mathbb{R}''}) \). Since \( F \) is an initial segment of \( \mathbb{R}'' \), it follows that \([A, T_{\mathbb{R}'''}] \) is the \( \theta \)-image of \([F, \mathbb{R}''\])

Let \( \mathcal{Y} = \theta[\mathcal{X}] \) and apply Theorem 5.10 to obtain an \( S \in [A, T_{\mathbb{R}'''}] \) such that either \([A, S] \subseteq \mathcal{Y}\) or else \([A, S] \cap \mathcal{Y} = \emptyset\). Since \( \theta^{-1} \) is a homeomorphism from \( \mathcal{T}_\mathbb{R} \) to \( \mathcal{R}(\mathbb{R}) \), we have that either \( \theta^{-1}[A, S] \subseteq \mathcal{X} \) or else \( \theta^{-1}[A, S] \cap \mathcal{X} = \emptyset\). Notice that

\[
\theta^{-1}[A, S] = \{\theta^{-1}(S') : S' \in [A, S]\}
\]

\[
= \{G_{S'} : S' \in [A, S]\}
\]

\[
= [G_A, G_S]
\]

\[
= [F, G_S].
\]
Thus, we have found a $G_S \in [\mathcal{F}, \mathcal{R}']$ such that either $[\mathcal{F}, G_S] \subseteq \mathcal{X}$ or else $[\mathcal{F}, G_S] \cap \mathcal{X} = \emptyset$. The case where $\mathcal{F}$ is the empty graph (no vertices) yields the second half of the theorem. 

7. Concluding remarks and further directions

In this paper, we have proved that Borel subsets of certain closed spaces of Rado graphs are completely Ramsey. This is a Rado graph analogue of the Galvin-Prikry Theorem 1.2 for the Baire space. As pointed out in Section 2, the work of Laflamme, Sauer, and Vuksanovic in [14] necessitate that we restrict our spaces to collections of Rado graphs with the same strong similarity type. We now point out three areas for improvement on the results of this paper.

Firstly, we would like to have an analogue of Ellentuck’s Theorem; that is, we would like to further obtain a result showing that all subsets with the property of Baire in the Ellentuck topology on $T_R$ are completely Ramsey. There is a serious breaking point when trying to adjust the methods of Ellentuck to the setting of strong Rado coding trees, which seems very closely tied with not having an amalgamation property like the Baire space does, this property being made concrete in the axiom $A.3(b)$ of Todorcevic. It does not seem possible to develop the ‘combinatorial forcing’ method used by Nash-Williams, Galvin and Prikry, and finally Ellentuck without having this strong amalgamation property. Thus, either new methods will be necessary, or it may be that the spaces $T_R$ might have a subset with the property of Baire in the Ellentuck topology which is not Ramsey. We leave this as an open problem.

**Question 7.1.** Is every subset of $T_R$ with the property of Baire with respect to the Ellentuck topology Ramsey?

If the answer is no, then this would decide the following fundamental question in the positive.

**Question 7.2.** Is there a topological Ramsey space which does not satisfy Todorcevic’s Axiom $A.3(b)$?

Secondly, we would like to extend the Main Theorem to apply to any strong similarity type of a Rado graph. The work in [21] and [14] on finding the big Ramsey degrees for the Rado graph has the important feature that, at the end of the applications of Milliken’s Theorem, they take a strongly diagonal antichain of nodes in $S$ which codes the Rado graph. It is within this antichain that they prove that the number of strong similarity types representing a given finite graph is the upper bound for the big Ramsey degree (see [21]) as well as the lower bound (see [14]). By virtue of how we defined $S_R$, given a Rado graph $\mathbb{R} = (\omega, E)$, one sees that the coding nodes in $S_R$, and hence in any member of $T_R$, are dense in the tree. This aids in the proofs, especially with the forcing arguments. However, we would prefer a theorem of the following sort: Given a strongly diagonal antichain $A$ of nodes coding the Rado graph, the space of all sub-antichains with the same strong similarity type form a space in which Borel sets are Ramsey. This seems achievable and is the subject of ongoing work.

In tandem with this, we come to the third area for improvement, which Todorcevic mentioned to the author at the 2019 Luminy Workshop in Set Theory: namely, that the “correct” infinite dimensional Ramsey theorem should recover the big
Ramsey degrees for the Rado graph. The results in the present paper recover upper bounds for the big Ramsey degrees by using initial segments of the Rado coding trees as envelopes and applying Main Theorem. However, this approach does not recover the lower bounds proved by Laflamme, Sauer, and Vuksanovic in [14]. So, this paper may be thought of as providing a reasonable answer to the stated question in [12], but not recovering everything in the intended question. We hope that the work in this paper will pave the way.

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