WEIGHTED COGROWTH FORMULA FOR FREE GROUPS

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Abstract. We investigate the relationship between geometric, analytic and probabilistic indices for quotients of the Cayley graph of the free group Cay($F_n$) by an arbitrary subgroup $G$ of $F_n$. Our main result, which generalizes Grigorchuk’s cogrowth formula to variable edge lengths, provides a formula relating the bottom of the spectrum of weighted Laplacian on $G\backslash$Cay($F_n$) to the Poincaré exponent of $G$. Our main tool is the Patterson-Sullivan theory for Cayley graphs with variable edge lengths.

1. Introduction and statement of results

Let $F_n = \langle a_1, \ldots, a_n \rangle$ denote the free group of rank $n \geq 2$ and let Cay($F_n$) denote its Cayley graph. For an arbitrary subgroup $G \subset F_n$, the action of $G$ on Cay($F_n$) defines the quotient graph $G \backslash$Cay($F_n$). In this paper, we compare fundamental indices of geometric, analytic and probabilistic nature associated with $G$ acting on Cay($F_n$). The geometric index is the Poincaré exponent $\delta_G$ given by the exponential growth rate of $G$-orbits

$$\delta_G = \limsup_{R \to \infty} \frac{\log \# \{ g \in G \mid d(id, g) \leq R \} }{R},$$

where $d$ denotes the metric on $F_n$ giving each edge of Cay($F_n$) the length one. The analytic index is the bottom of the spectrum of the Laplacian $\Delta = I - A$ on $L^2(G\backslash$Cay($F_n$)) denoted by $\lambda_0^G$. Here, $I$ denotes the identity matrix and $A$ the transition matrix of the simple random walk on Cay($F_n$), which is for each function $f$ on the vertex set of $G\backslash$Cay($F_n$) given by

$$(Af)(x) = \frac{1}{2n} \{ f(xa_1) + f(xa_1^{-1}) + \cdots + f(xa_n) + f(xa_n^{-1}) \} \quad (x \in G \backslash$Cay($F_n$)).$$

The two indices, geometric and analytic, are related by the following well-known formula. Note that the edge lengths of Cay($F_n$) and the weights of $A$ are constant.

**Theorem 1.1** (Grigorchuk’s cogrowth formula [Gri80, GdlH97]).

$$\lambda_0^G = \begin{cases} \frac{1}{2n} \left( 2n - 1 - e^{\delta_G} \right) \left( 1 - e^{-\delta_G} \right) & (\delta_G > \frac{1}{2} \log(2n - 1)) \\ 1 - \frac{\sqrt{2n - 1}}{2} & (\delta_G \leq \frac{1}{2} \log(2n - 1)) \end{cases}.$$**

That $\lambda_0 := \lambda_0^{\{id\}} = 1 - \sqrt{2n - 1}/2$ follows from earlier work of Kesten ([Kes59]) who proved that the spectral radius of $A$ is equal to the decay rate of the return probabilities of the simple random walk on Cay($F_n$). Also note that $\delta_{F_n} = \log(2n - 1)$, so that $\lambda_0$ is related
to $\delta_{F_n}/2$. Related results for discrete groups acting on hyperbolic space were obtained by Elstrodt, Patterson and Sullivan in [Sul87]. The case of pinched negative curvature was recently considered in [RT15].

In this paper, we consider the case of variable edge lengths of Cay$(F_n)$. For any $r = (r_1, \ldots, r_n)$ with $r_1 + \cdots + r_n = 1/2$ and $r_i > 0$ for all $i$, we define the length of the edge corresponding to the generator $a_i^\pm$ to be $-\log r_i$ for all $i$. The Cayley graph Cay$(F_n)$ equipped with this distance $d_r$ is denoted by $X_r$.

Any subgroup of $G \subset F_n$ acts on $X_r$ isometrically, properly discontinuously, and freely. The Poincaré exponent $\delta_G(r)$ of $G$ acting on $X_r$ is defined in the same manner. In our normalization of the edge length, the even length case with $r_i = 1/(2n)$ for all $i$ gives $\delta_{F_n}(r) = \log(2n-1)/\log(2n)$. Unlike the case of equal edge lengths, even in the special case $G = F_n$, the value of $\delta(r) := \delta_{F_n}(r)$ is unclear in the variable edge length setting, since it is not easy to count $\#\{g \in F_n \mid d_r(id, g) \leq R\}$ directly. We will consider the problem to compute $\delta(r)$ in Theorem 1.2 below.

We also consider variable weights for the discrete Laplacian. For every $p = (p_1, \ldots, p_n)$ with $p_1 + \cdots + p_n = 1/2$ and $p_i > 0$ for all $i$, the stochastic transition matrix $A_p = (p(x, y))_{x,y}$ for vertices $x, y \in F_n$ of Cay$(F_n)$ is given by $p(x, y) = p_i$ if $y = xa_i^\pm$. This defines an operator which is, for each function $f$ on the vertex set of Cay$(F_n)$, given by

$$(A_p f)(x) := \sum_{i=1}^{n} p_i (f(xa_i) + f(xa_i^{-1})).$$

The weighted Laplacian is then defined by $\Delta_p := I - A_p$.

For a subgroup $G \subset F_n$, the Laplacian $\Delta_p$ acts on $L^2(G\setminus \text{Cay}(F_n))$ as a bounded symmetric operator. The bottom of the spectrum of $\Delta_p$ is denoted by $\lambda_0^G(p)$. Since $A_p$ is also a bounded symmetric operator with non-negative entries, the spectral radius $\rho^G(p)$ of $A_p$ coincides with its operator norm, and this is also given by

$$\rho^G(p) = \sup \langle A_p f, f \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(G\setminus \text{Cay}(F_n))$ and the supremum is taken over all $f \in L^2(G\setminus \text{Cay}(F_n))$ with $\langle f, f \rangle = 1$. Then, we have that

$$\lambda_0^G(p) = 1 - \rho^G(p) = \inf \langle \Delta_p f, f \rangle.$$

It is easy to see that $\rho^{F_n}(p) = 1$ and $\lambda_0^{F_n}(p) = 0$ for every $p$.

Concerning $\rho(p) := \rho^{(id)}(p)$ and $\lambda_0(p) := \lambda_0^{(id)}(p) = 1 - \rho(p)$, the following formula is well-known:

$$\rho(p) = \min_{t>0} \frac{1}{t} \left\{ \sum_{i=1}^{n} \sqrt{1 + 4p_i^2 t^2} - (n-1) \right\}. \tag{1.1}$$

The formula (1.1) is a special case of [AO76]. The case $n = 2$ was considered in [Ger77]. Further references can be found in [Voe00]. See Section 9 and the Notes at the end of Chapter II of this book for details. We will also obtain this formula in the course of our
arguments. Moreover, we will express \( \rho(p) \) in a different way by using the Poincaré exponent of \( F_n \) (see Theorem 1.3 below).

We investigate the problems mentioned above for the variable parameters. Our method is to find the proper correspondence between the edge length parameter \( r \) and the weight \( p \) for the Laplacian. To obtain eigenfunctions of the Laplacian \( \Delta_p \), we use an integral representation by the Patterson measure instead of the integral of the Martin kernel. An idea of choosing weights of the Laplacian from Patterson measures can be found in [CP96].

Since \( \text{Cay}(F_n) \) is a tree, \( X_r = (\text{Cay}(F_n), d_r) \) is a Gromov 0-hyperbolic space. Given a boundary point \( \xi \in \partial X_r \), we define \( j_r(x, \xi) = \exp(-b_\xi(x)) \) for every vertex \( x \in X_r \), where \( b_\xi \) is the Busemann function with respect to the geodesic ray \( \beta_\xi : [0, \infty) \to X_r \) from the base point \( o = \beta_\xi(0) \) to \( \xi = \beta_\xi(\infty) \) given by

\[
b_\xi(x) = \lim_{t \to \infty} (t - d_r(x, \beta_\xi(t))).
\]

For the Laplacian \( \Delta_p \) of weight \( p \), the eigenrelation

\[
\Delta_p j_r(x, \xi)^s = \lambda j_r(x, \xi)^s \quad (\forall \xi \in \partial X_r)
\]

with \( \lambda \in \mathbb{R} \) and \( s \in (0, 1) \) gives the correspondence between \( r \) and \( p \). This can be explicitly given in the following way.

We set the spaces of parameters

\[
\mathcal{R} := \{ r = (r_1, \ldots, r_n) \mid r_1 + \cdots + r_n = 1/2, \ r_i > 0 \ (\forall i) \} ;
\]

\[
\mathcal{P} := \{ p = (p_1, \ldots, p_n) \mid p_1 + \cdots + p_n = 1/2, \ p_i > 0 \ (\forall i) \}.
\]

We also define a diffeomorphism \( H : \mathcal{R} \times (0, \infty) \to (0, 1)^n \) by \( H(r, s) = u := (u_1, \ldots, u_n), \ u_i = r_i^s \). Under this transformation, relation (1.2) turns out to be

\[
\lambda = 1 - 2 \sum_{k=1}^n u_k p_k - (u_i^{-1} - u_i) p_i \quad (i = 1, \ldots, n).
\]

Solving these equations for unknown variables \( p = (p_1, \ldots, p_n) \) and \( \lambda \) by linear algebra, we have functions \( p(u) \) and \( \lambda(u) \) if the determinant is not zero. On the other hand, given \( p \in \mathcal{P} \) and \( \lambda \geq 0 \), we can obtain a solution \( u \in (0, 1)^n \) by using the Green function of the random walk on \( \text{Cay}(F_n) \) if \( \lambda \leq \lambda_0(p) \).

The following theorem, which will be proved in Section 2, allows us to compute the Poincaré exponent.

**Theorem 1.2.** For every \( r = (r_1, \ldots, r_n) \in \mathcal{R} \), the Poincaré exponent \( \delta(r) \) of \( F_n \) satisfies the equation \( \lambda \circ H(r, s) = 0 \) for \( s = \delta(r) \). More precisely, \( \delta(r) \) is the unique solution \( s \in (0, 1) \) of the equation

\[
\sum_i r_i^s + 3 \sum_{(i_1, i_2)} (r_{i_1} r_{i_2})^s + 5 \sum_{(i_1, i_2, i_3)} (r_{i_1} r_{i_2} r_{i_3})^s + \cdots + (2n - 1)(r_1 \cdots r_n)^s = 1,
\]

where the subscript \( (i_1, \ldots, i_m) \) represents taking all indices satisfying \( i_1 < \cdots < i_m \).
Theorem 1.3. To each $p \in P$, there corresponds a unique $r \in R$ such that the bottom of the spectrum of $\lambda_0(p)$ of the Laplacian $\Delta_p$ on $\text{Cay}(F_n)$ is given by $\lambda \circ H(r, \delta(r)/2)$.

From this theorem, we can expect that the appropriate weight $p_*(r, s)$ for the Laplacian is given by

$$p_*(r, s) := \begin{cases} p \circ H(r, s) & (s > \delta(r)/2) \\ p \circ H(r, \delta(r)/2) & (s \leq \delta(r)/2). \end{cases}$$

In Section 4, we generalize Grigorchuk’s cogrowth formula in the following form. This is our main result of this paper.

Theorem 1.4. For any subgroup $G \subseteq F_n$ and for any $r \in R$, the bottom of the spectrum $\lambda_0^G(p_*(r, \delta_G(r)))$ of the Laplacian $\Delta_{p_*(r, \delta_G(r))}$ on the quotient graph $G\text{Cay}(F_n)$ is given by

$$\lambda_0^G(p_*(r, \delta_G(r))) = \begin{cases} \lambda \circ H(r, \delta_G(r)) & (\delta_G(r) > \delta(r)/2) \\ \lambda \circ H(r, \delta(r)/2) & (\delta_G(r) \leq \delta(r)/2). \end{cases}$$

We recall from [Kes99, Kes99a] that if $N$ is a normal subgroup of $F_n$ then $\lambda_0^N(p)$ is equal to zero for any $p$ if and only if $F_n/N$ is amenable. Combining this characterization with Theorem 1.4 applied to $p = p \circ H(r, \delta(r))$, we obtain the following amenability criterion. The corollary below was proved in [Jae14] using different methods. In the case of equal edge lengths, the corollary is Grigorchuk’s amenability criterion. An alternative proof is given in [OW07]. For Kleinian groups a related result is due to Brooks ([Bro85]).

Corollary 1.5 (Weighted cogrowth criterion for amenability). Let $N$ be a normal subgroup of $F_n$. Then the weighted cogrowth $\delta_N(r)/\delta_{F_n}(r)$ is equal to one if and only if $F_n/N$ is amenable.

2. A computation of the Poincaré exponent

The Cayley graph $\text{Cay}(F_n)$ of the free group $F_n = \langle a_1, a_2, \ldots, a_n \rangle$ is the regular tree of valency $2n$. For any positive real numbers $r_1, r_2, \ldots, r_n > 0$ with the normalization $r_1 + r_2 + \cdots + r_n = 1/2$, we assign length $-\log r_i$ to the edges of labels $a_i$ and $a_i^{-1}$ in $\text{Cay}(F_n)$ for $i = 1, 2, \ldots, n$. We regard this proper metric space as a Gromov hyperbolic space and represent it by $X_r$ with the distance $d_r$ for every

$$r \in R := \{ r = (r_1, \ldots, r_n) \mid r_1 + \cdots + r_n = 1/2, \ r_i > 0 \ (\forall i) \}.$$
dimension $s$ is a family of positive finite Borel measures $\{\mu_x\}_{x \in \mathbb{X}_r}$ on $\partial \mathbb{X}_r$ such that

$$\frac{d\mu_x}{d\mu_y}(\xi) = \left(\frac{j(x, \xi)}{j(y, \xi)}\right)^s$$

for any vertices $x, y \in \mathbb{X}_r$. For a subgroup $G \subset \mathbb{F}_n$, the conformal measure $\{\mu_x\}_{x \in \mathbb{X}_r}$ is $G$-invariant if $\mu_{g(x)}(g(E)) = \mu_x(E)$ for every vertex $x \in \mathbb{X}_r$ and for every Borel subset $E \subset \partial \mathbb{X}_r$. For any $G$-invariant conformal measure $\mu = \{\mu_x\}_{x \in \mathbb{X}_r}$ of dimension $s$, the total mass function

$$\varphi_\mu(x) = \int_{\partial \mathbb{X}_r} d\mu_x = \int_{\partial \mathbb{X}_r} j(x, \xi)^s d\mu_0(\xi)$$

is $G$-invariant.

For any subgroup $G \subset \mathbb{F}_n$, the exponent of convergence is defined by

$$\delta_G(r) = \limsup_{R \to \infty} \frac{\log \#\{g \in G \mid d_r(o, g(o)) \leq R\}}{R}.$$ 

A $G$-invariant conformal measure of dimension $\delta_G(r)$ is called a Patterson measure for $G$.

The results on the Patterson measure for a discrete group acting on a Gromov hyperbolic space can be summarized as follows in our particular situation.

**Theorem 2.1** (Coornaert [Coo93]). For every subgroup $G \subset \mathbb{F}_n$, there exists a $G$-invariant conformal measure $\mu$ of dimension $\delta_G(r)$. If $G$ is finitely generated, then it is unique up to constant multiples.

**Remark.** It is well known that $G$ is convex cocompact if and only if $G$ is finitely generated ([HH97, Sho91, Swe01]).

For every $p = (p_1, p_2, \ldots, p_n)$ with $p_1 + p_2 + \cdots + p_n = 1/2$ and $p_i > 0$ for all $i$, we define a transition matrix $A_p = (p(x, y))_{x,y}$ on the vertices of Cay($\mathbb{F}_n$) by $p(x, y) = p_i$ if $y = xa_i$ or $y = xa_i^{-1}$ for $i = 1, 2, \ldots, n$. The discrete Laplacian on Cay($\mathbb{F}_n$) of weight

$$p \in \mathcal{P} := \{p = (p_1, \ldots, p_n) \mid p_1 + \cdots + p_n = 1/2, \ p_i > 0 \ (\forall i)\}$$

is defined by $\Delta_p = I - A_p$.

**Proposition 2.2.** Let $\mu = \{\mu_x\}_{x \in \mathbb{X}_r}$ be the Patterson measure for $\mathbb{F}_n$ on $\partial \mathbb{X}_r$. Then

$$\int_{\partial \mathbb{X}_r} \Delta_p j(x, \xi)^{\delta(r)} d\mu_0(\xi) = 0$$

for every $p \in \mathcal{P}$, where $\delta(r) = \delta_{\mathbb{F}_n}(r)$.

**Proof.** Since $\mathbb{F}_n$ acts transitively on the vertices of Cay($\mathbb{F}_n$), the $\mathbb{F}_n$-invariant function $\varphi_\mu(x) = \int_{\partial \mathbb{X}_r} d\mu_x$ is constant. Hence, for every $p$,

$$\Delta_p \varphi_\mu(x) = \int_{\partial \mathbb{X}_r} \Delta_p j(x, \xi)^{\delta(r)} d\mu_0(\xi) = 0.$$ 

\qed
We compute $\Delta_p j(x, \xi)^s$ and obtain the following: if $\xi \in \partial X_r$ is in the direction of $a_i$ or $a_i^{-1}$ starting from a vertex $x \in X_r$ for $i = 1, 2, \ldots, n$, then

$$\frac{\Delta_p j(x, \xi)^s}{j(x, \xi)^s} = 1 - r_i^{-s}p_i - r_i^s p_i - 2 \sum_{j \neq i} r_j^s p_j =: c_i(r, s, p).$$

**Proposition 2.3.** The functions $c_i(r, s, p)$ ($i = 1, 2, \ldots, n$) of $s \in [0, \infty)$ satisfies the following properties for any fixed $r \in \mathcal{R}$ and $p \in \mathcal{P}$:

1. $c_i(r, 0, p) = 0$ and $\frac{\partial}{\partial s} c_i(r, s, p)|_{s=0} > 0$;
2. $\frac{\partial^2}{\partial s^2} c_i(r, s, p) < 0$;
3. $\lim_{s \to \infty} c_i(r, s, p) = -\infty$.

Hence, each $c_i(r, s, p)$ has a unique zero $s_i = s_i(r, p) \neq 0$, and satisfies $\frac{\partial}{\partial s} c_i(r, s, p)|_{s=s_i} < 0$.

**Proof.** The second assertion in (1) follows from the fact that

$$\frac{\partial}{\partial s} c_i(r, s, p)|_{s=0} = [\log(r_i)r_i^{-s}p_i - \log(r_i)r_i^sp_i - 2 \sum_{j \neq i} \log(r_j)r_j^sp_j]|_{s=0} = -2 \sum_{j \neq i} \log(r_j)p_j > 0.$$

The statement in (2) follows from

$$\frac{\partial^2}{\partial s^2} c_i(r, s, p) = \frac{\partial}{\partial s} [\log(r_i)r_i^{-s}p_i - \log(r_i)r_i^sp_i - 2 \sum_{j \neq i} \log(r_j)r_j^sp_j]$$

$$= - \log^2(r_i)r_i^{-s}p_i - \log^2(r_i)r_i^sp_i - 2 \sum_{j \neq i} \log^2(r_j)r_j^sp_j < 0.$$

The proofs of the remaining assertions are straightforward.

**Lemma 2.4.** $\delta(r)$ lies between $\min_{1 \leq i \leq n} s_i(r, p)$ and $\max_{1 \leq i \leq n} s_i(r, p)$.

**Proof.** Proposition 2.2 implies that, for every vertex $x \in X_r$,

$$\sum_{i=1}^n \int_{\partial X_i(x)} c_i(r, \delta(r), p)d\mu_x(\xi) = 0,$$

where $\partial X_i(x)$ is the portion of $\partial X_r$ whose points $\xi$ are in $a_i^{\pm 1}$ directions from $x$. It follows that the $c_i(r, \delta(r), p)$ cannot have the same sign. By Proposition 2.3, we see that $c_i(r, s, p)$ changes signs from positive to negative at $s_i(r, p)$, for each $i$. Therefore, $\delta(r)$ must lie between $\min_{1 \leq i \leq n} s_i(r, p)$ and $\max_{1 \leq i \leq n} s_i(r, p)$.

By this lemma, if we have $s_1(r, p) = \cdots = s_n(r, p) \neq 0$ for some weight $p \in \mathcal{P}$, then this value coincides with $\delta(r)$. Hence, we consider simultaneous equations

$$c_1(r, s, p) = \cdots = c_n(r, s, p) = 0$$

for a given $r = (r_1, \ldots, r_n)$. First, we solve $c_1(r, s, p) = \cdots = c_n(r, s, p)$ as a system of equations of $p$. 

We change the variables from \((r, s)\) to \(u = (u_1, \ldots, u_n)\) by \(u_i = r_i^s\) for \(i = 1, \ldots, n\). This correspondence defines a diffeomorphism

\[ H : \mathcal{R} \times (0, \infty) \to (0, 1)^n. \]

We also set \(c_i(r, s, p) = c_i(u, p)\) \((i = 1, \ldots, n)\) (by the same notation) under this correspondence. Namely,

\[
c_i(u, p) = 1 - 2 \sum_{k \neq i} u_k p_k - u_i^{-1} p_i - u_i p_i = 1 - 2 \sum_{k=1}^n u_k p_k - (u_i^{-1} - u_i) p_i.
\]

**Lemma 2.5.** Given \(u = (u_1, \ldots, u_n) \in (0, \infty)^n\), we consider the system of linear equations

\[ c_1(u, p) = \cdots = c_n(u, p) \]

for \(p = (p_1, \ldots, p_n)\) with \(p_1 + \cdots + p_n = 1/2\) and let

\[ D = D(u) := \sum_j \prod_{k \neq j} (u_k^{-1} - u_k). \]

1. If \(D \neq 0\) then there exists a unique solution \(p = p(u)\) given by

\[ p_i = p_i(u) = \frac{\prod_{k \neq i} (u_k^{-1} - u_k)}{2 \sum_j \prod_{k \neq j} (u_k^{-1} - u_k)} \quad (i = 1, \ldots, n). \]

The common value \(\lambda = \lambda(u) := c_1(u, p) = \cdots = c_n(u, p)\) is given by

\[
\lambda = D^{-1} \left( \sum_j (1 - u_j) \prod_{k \neq j} (u_k^{-1} - u_k) - \frac{1}{2} \prod_{\ell} (u_{\ell}^{-1} - u_{\ell}) \right) = D^{-1} \prod_{\ell} (u_{\ell}^{-1} - u_{\ell}) \left( \sum_j \left( \frac{u_j}{1 + u_j} \right) - \frac{1}{2} \right).
\]

Moreover, there exists at most one \(j\) such that \(u_j = 1\), and in that case we have that the solution is given by \(p_j = 1/2\) and \(p_i = 0\) for all \(i \neq j\) with \(\lambda = 0\).

2. If there exists a solution \(p \in \mathcal{P}\) (i.e., \(p_i > 0\) for all \(i\)), then either \(u_i = 1\) for all \(i\), \(u_i > 1\) for all \(i\), or \(u_i < 1\) for all \(i\). In the first case, \(D = 0\) and every \(p\) is a solution with \(\lambda = 0\). In the second case, \(D \neq 0\) and the above formulas hold with \(\lambda < 0\). In the third case, \(D \neq 0\) and the above formulas hold but the sign of \(\lambda\) is indefinite.

**Proof.** If \(D \neq 0\), then existence and uniqueness of solutions follows by verifying that \(D\) is the determinant of the system of equations. More explicitly, we can solve these equations as follows. We first note that \(c_1(u, p) = \cdots = c_n(u, p)\) is equivalent to

\[
(u_1^{-1} - u_1)p_1 = (u_2^{-1} - u_2)p_2 = \cdots = (u_n^{-1} - u_n)p_n.
\]
Proposition 2.6. For every $p, l$, we will find

Proof. We set this common value as $\tau$. If $\tau \neq 0$, then we have $p_i = \tau/(u_i^{-1} - u_i)$ for all $i$. Since $\sum_j p_j = 1/2$, it follows that $\tau \sum_j (u_j^{-1} - u_j)^{-1} = 1/2$. Hence,

$$p_i = \frac{(u_i^{-1} - u_i)^{-1}}{2 \sum_j (u_j^{-1} - u_j)^{-1}} = \frac{\prod_{k \neq i} (u_k^{-1} - u_k)}{2 \sum_j \prod_{k \neq j} (u_k^{-1} - u_k)^{-1}}.$$

Since $D \neq 0$, it is clear that there exists at most one $j$ with $u_j = 1$. If so, then $\tau = 0$, $p_j = 1/2$ and $p_i = 0$ for all $i \neq j$, which also satisfies the above formulas for $p_i$. The common value $\lambda$ is obtained by substituting these solutions to any of $c_i(u, \mathbf{p})$. To prove (2), suppose that there exists a solution $\mathbf{p}$ with $p_i > 0$. Then, according to the value $\tau$ of (2.1), we have that $u_i = 1$, $u_i > 1$, or $u_i < 1$ for all $i$ simultaneously. The other assertions follow from the representations of $D$ and $\lambda$.

By this lemma, the original problem to obtain $\delta(\mathbf{r})$ is reduced to finding a system of solutions of the equation $\lambda(\mathbf{u}) = 0$ concerning $\mathbf{u} \in (0, 1)^n$. Then by $H^{-1}\mathbf{u}$, we have a system of equations for $\mathbf{r}$ and $s$. From this, for a given $\mathbf{r} \in \mathcal{R}$, we can obtain the exponent $s \in (0, \infty)$ which is equal to $\delta(\mathbf{r})$.

For this purpose, we give another representation of $\lambda(\mathbf{u})$ obtained in Lemma 2.5 as follows:

$$\lambda(\mathbf{u}) = D(\mathbf{u})^{-1} \left( \sum_j (1 - u_j) \prod_{k \neq j} (u_k^{-1} - u_k) - \frac{1}{2} \prod_{\ell} (u_\ell^{-1} - u_\ell) \right)$$

$$= \frac{\prod_i u_i \cdot \left( \sum_j (1 - u_j) \prod_{k \neq j} (u_k^{-1} - u_k) - \frac{1}{2} \prod_{\ell} (u_\ell^{-1} - u_\ell) \right)}{\prod_i u_i \cdot D(\mathbf{u})}$$

$$= \frac{\left( \sum_j (1 - u_j) u_j \prod_{k \neq j} (1 - u_k^2) - \frac{1}{2} \prod_{\ell} (1 - u_\ell^2) \right)}{\prod_i u_i \cdot D(\mathbf{u})}$$

$$= \frac{\prod_j (1 - u_j) \left( 2 \sum_j u_j \prod_{k \neq j} (1 + u_k) - \prod_{\ell} (1 + u_\ell) \right)}{\prod_i u_i \cdot D(\mathbf{u})}.$$

Here, we define

$$l(\mathbf{u}) := 2 \sum_j u_j \prod_{k \neq j} (1 + u_k) - \prod_{\ell} (1 + u_\ell).$$

Then, $\lambda(\mathbf{u}) = 0$ is equivalent to $l(\mathbf{u}) = 0$ for $\mathbf{u} \in (0, 1)^n$.

**Proposition 2.6.** For every $\mathbf{r} \in \mathcal{R}$, $\delta(\mathbf{r})$ is the unique $s > 0$ such that $\mathbf{u} = H(\mathbf{r}, s)$ satisfies $l(\mathbf{u}) = 0$.

Proof. We will find $s > 0$ such that $l(H(\mathbf{r}, s)) = 0$. For a fixed $\mathbf{r}$, it is easy to see that $l \circ H(\mathbf{r}, s)$ is a strictly decreasing continuous function such that $\lim_{s \to 0} l \circ H(\mathbf{r}, s) > 0$ and $\lim_{s \to \infty} l \circ H(\mathbf{r}, s) = -1$. Hence, such an $s$ uniquely exists. That $s = \delta(\mathbf{r})$ follows from Lemma 2.4. \qed
Proof of Theorem 1.2. By expanding \( l(u) \), we have
\[
l(u) = -1 + \sum_i u_i + 3 \sum_{(i,i_2)} u_{i_1} u_{i_2} + 5 \sum_{(i,i_2,i_3)} u_{i_1} u_{i_2} u_{i_3} + \cdots + (2n - 1) u_1 \cdots u_n.
\]
Then, the statement follows from Proposition 2.6. \( \square \)

3. \( \lambda_0 \) IN TERMS OF \( \delta \) ON \( \text{Cay}(F_n) \)

In this section, we will prove Theorem 1.3. To this end, we consider the maximal value of \( \lambda(u) \) for \( u \in (0,1)^n \) under a constraint condition \( p(u) = p_0 \) for some fixed \( p_0 = (p_1, \ldots, p_n) \in P \). We note that the condition \( p(u) = p_0 \) is equivalent to \( c_1(u,p_0) = \cdots = c_n(u,p_0) \), where
\[
\lambda(u) = (u_1^2 - u_1) p_1 = (u_2^2 - u_2) p_2 = \cdots = (u_n^2 - u_n) p_n
\]
for \( u = (u_1, \ldots, u_n) \) by (2.1).

Putting the common value of these equations as \( \tau \in (0,\infty) \), we can solve \( u_i \in (0,1) \) for each \( i \) as
\[
(3.1) \quad u_i = u_i(\tau) = \frac{1}{2}(\sqrt{\tau^2 p_i^2 - 4} - \tau p_i^{-1}).
\]
Then, we have a smooth curve \( \gamma_{p_0}(\tau) := (u_1(\tau), \ldots, u_n(\tau)) \) (\( 0 < \tau < \infty \)) in \( (0,1)^n \) such that
\[
\{ \gamma_{p_0}(\tau) \mid \tau \in (0,\infty) \} = \{ u \mid p(u) = p_0 \}.
\]
Moreover, \( \lim_{\tau \to 0} \gamma_{p_0}(\tau) = (1, \ldots, 1) \) and \( \lim_{\tau \to \infty} \gamma_{p_0}(\tau) = (0, \ldots, 0) \).

Proposition 3.1. For every \( p_0 = (p_1, \ldots, p_n) \in P \), the function \( \lambda \circ \gamma_{p_0}(\tau) \) on \( (0,\infty) \) takes the unique maximum at \( \tau_0 \) where the derivative
\[
(\lambda \circ \gamma_{p_0})'(\tau) = -\sum_i \frac{\tau}{\sqrt{\tau^2 + 4p_i^2}} + (n - 1)
\]
vanishes. Moreover, \( \lambda \circ \gamma_{p_0}(\tau_0) > 0 \).

Proof. By substituting (3.1) to \( \lambda(u) = c_i(u,p_0) \), we have that
\[
\lambda \circ \gamma_{p_0}(\tau) = 1 - 2 \sum_i \frac{1}{2}(\sqrt{\tau^2 p_i^{-2} - 4} - \tau p_i^{-1}) p_i - \tau
\]
\[
= 1 - \sum_i \sqrt{\tau^2 + 4p_i^2} + (n - 1)\tau.
\]
Then \( \lim_{\tau \to 0} \lambda \circ \gamma_{p_0}(\tau) = 0 \) and \( \lim_{\tau \to \infty} \lambda \circ \gamma_{p_0}(\tau) = -\infty \).

Moreover, the derivative of \( \lambda \circ \gamma_{p_0}(\tau) \) is
\[
(\lambda \circ \gamma_{p_0})'(\tau) = -\sum_i \frac{\tau}{\sqrt{\tau^2 + 4p_i^2}} + (n - 1).
\]
This is a strictly decreasing continuous function from a positive \( \lim_{\tau \to 0}(\lambda \circ \gamma_{p_0})'(\tau) = n - 1 \) to a negative \( \lim_{\tau \to \infty}(\lambda \circ \gamma_{p_0})'(\tau) = -1 \). The statement then follows easily. \( \square \)
The following claim shows the way of choosing \( r \in \mathcal{R} \) corresponding to \( p \in \mathcal{P} \).

**Lemma 3.2.** For every \( p_0 \in \mathcal{P} \), assume that the function \( \lambda \circ \gamma_{p_0}(\tau) \) for \( \tau \in (0, \infty) \) takes the unique maximum at \( \tau_0 \). Then, there exists a unique \( r_0 \in \mathcal{R} \) such that \( H(r_0, \delta(r_0)/2) = \gamma_{p_0}(\tau_0) \).

**Proof.** Set \( \gamma_{p_0}(\tau_0) = u_0 = (u_1, \ldots, u_n) \). Then we have

\[
(u_1^{-1} - u_1)p_1 = (u_2^{-1} - u_2)p_2 = \cdots = (u_n^{-1} - u_n)p_n = \tau_0
\]

for \( p_0 = (p_1, \ldots, p_n) \). Since \( \tau_0^{-1}p_i = u_i/(1 - u_i^2) \) for all \( i \), which follows from the above equations, we have that

\[
(\lambda \circ \gamma_{p_0})'(\tau_0) = -\sum_{i} \frac{1}{\sqrt{1 + 4(\tau_0^{-1}p_i)^2}} + (n-1)
\]

\[
= -\sum_{i} \frac{1 - u_i^2}{\sqrt{(1-u_i^2)^2 + 4u_i^2}} + (n-1)
\]

\[
= -\sum_{i} \frac{1 - u_i^2}{1 + u_i^2} + (n-1)
\]

\[
= -\sum_{i} (1 - u_i^2) \prod_{k \neq i} (1 + u_k^2) + (n-1) \prod_k (1 + u_k^2)
\]

\[
= \frac{-\sum_{i} (1 - u_i^2) \prod_{k \neq i} (1 + u_k^2) + (n-1) \prod_k (1 + u_k^2)}{\prod_k (1 + u_k^2)}.
\]

Since \( (\lambda \circ \gamma_{p_0})'(\tau_0) = 0 \), we have that the numerator

\[
-\sum_{i} (1 - u_i^2) \prod_{k \neq i} (1 + u_k^2) + (n-1) \prod_k (1 + u_k^2)
\]

\[
= 2 \sum_i u_i \prod_{k \neq i} (1 + u_k^2) - \prod_k (1 + u_k^2)
\]

\[
+ \left( -\sum_i \prod_{k \neq i} (1 + u_k^2) - \sum_i u_i^2 \prod_{k \neq i} (1 + u_k^2) + n \prod_k (1 + u_k^2) \right)
\]

\[
= 2 \sum_i u_i^2 \prod_{k \neq i} (1 + u_k^2) - \prod_k (1 + u_k^2).
\]

is equal to zero. We define \( (r_0, s_0) \) to be \( H^{-1}(u_0) \) and set \( r_0 = (r_1, \ldots, r_n) \). By the definition of the function \( l \), we have that \( l \circ H(r_0, 2s_0) = 0 \). This implies that \( 2s_0 = \delta(r_0) \) by Proposition 2.6. Hence, \( u_0 = H(r_0, \delta(r_0)/2) \).

**Remark.** The above proof also implies that if \( u_0 \) is given by \( H(r_0, \delta(r_0)) \) for any \( r_0 \in \mathcal{R} \), then \( \lambda(u) \) takes the maximum at \( u_0 \) under the constraint condition \( p(u) = p_0 := p(u_0) \). The fact that \( u_0 \) is the critical point of \( \lambda(u) \) is also verified by the method of Lagrange multiplier without using \( \lambda \circ \gamma_{p_0}(\tau) \). We note that since \( p = (p_1, \ldots, p_n) \) satisfies \( p_1 + \cdots + p_n = 1/2 \),
the constraint condition can be determined only by \( p_1, \ldots, p_{n-1} \). If \( \lambda(u) \) attains a constraint local maximum or minimum at \( u_0 \), then \( u_0 = (u_1, \ldots, u_n) \) must satisfy
\[
\det \begin{bmatrix}
\frac{\partial p_1}{\partial u_1}(u) & \cdots & \frac{\partial p_{n-1}}{\partial u_1}(u) & \frac{\partial \lambda}{\partial u_1}(u) \\
\frac{\partial p_1}{\partial u_2}(u) & \cdots & \frac{\partial p_{n-1}}{\partial u_2}(u) & \frac{\partial \lambda}{\partial u_2}(u) \\
\vdots & \cdots & \vdots & \vdots \\
\frac{\partial p_1}{\partial u_n}(u) & \cdots & \frac{\partial p_{n-1}}{\partial u_n}(u) & \frac{\partial \lambda}{\partial u_n}(u)
\end{bmatrix} = 0.
\]

By Mathematica, we can check that this is equivalent to \( l(u_1^2, \ldots, u_n^2) = 0 \).

If we start from the edge length parameter \( r \in \mathcal{R} \), our main result in this section can be alternatively expressed as follows. This will be discussed again in the next section.

**Theorem 3.3.** For any \( r_0 \in \mathcal{R} \), the bottom of the spectrum \( \lambda_0(p_0) \) of the Laplacian \( \Delta_{p_0} \) for \( p_0 = p \circ H(r_0, \delta(r_0)/2) \) on \( X_{r_0} \) coincides with \( \lambda \circ H(r_0, \delta(r_0)/2) \).

**Proof.** It is well known that (see e.g. Lemmas 7.2 and 7.6 in Woess [Woe00]) the spectral radius \( \rho(p_0) \) of the Markov chain determined by \( A_{p_0} \) is the minimum of eigenvalues for positive eigenfunctions \( h \) on \( \text{Cay}(F_n) \). Since \( \Delta_{p_0} = I - A_{p_0} \) and \( \lambda(p_0) = 1 - \rho(p_0) \), we see that
\[
\lambda_0(p_0) = \max \{ \lambda \mid \exists h \geq 0, \quad \Delta_{p_0} h = \lambda h \}.
\]

Let \( H(r_0, \delta(r_0)/2) = u_0 \). By the definition of the function \( \lambda \), the positive function \( h(x) = j_{r_0}(x, \xi)^{\delta(r_0)/2} \) for any \( \xi \in \partial X_{r_0} \) satisfies \( \Delta_{p_0} h = \lambda(u_0) h \). From this, we have \( \lambda(u_0) \leq \lambda_0(p_0) \). Hence, the problem is to show the converse inequality.

By Lemma 2.3, we see that if some \( u \neq (1, \ldots, 1) \) satisfies the simultaneous equations \( \lambda = c_i(u, p) (i = 1, \ldots, n) \) for given \( p = p_0 \) and \( \lambda = \lambda_0 \), then \( p_0 \) and \( \lambda_0 \) are represented as \( p(u) \) and \( \lambda(u) \), respectively. Theorem 3.2 below asserts that for \( \lambda_0 = \lambda_0(p_0) \) there exists some \( u_1 \in (0, 1)^n \) that satisfies these equations. By the fact mentioned above, \( \lambda_0(p_0) \) is represented as \( \lambda(u_1) \) by using this \( u_1 \), which also satisfies the condition \( p(u_1) = p_0 \).

We consider the function \( \lambda(u) \) of variables \( u \in (0, 1)^n \) under the constraint \( p(u) = p_0 \). Then, by the proof of Lemma 3.2 (see also the remark after the proof), we have \( \lambda(u) \leq \lambda(u_0) \). This yields the desired inequality \( \lambda_0(p_0) \leq \lambda(u_0) \), which completes the proof. \( \square \)

The arguments above imply formula (1.1). We also note that if we assume (1.1), then we can prove Theorem 3.3 without showing Theorem 3.4. To prove (1.1) we proceed as follows. By Theorem 3.3, \( \rho(p_0) \) is given by \( 1 - \lambda(u_0) \), where \( \lambda(u_0) \) is the maximal value of \( \lambda(u) \) under the constraint condition \( p(u) = p_0 \) by Lemma 3.2. Proposition 3.1 implies that this constraint maximum coincides with \( \max_{\tau \in (0, \infty)} \lambda(p_0)(\tau) \). Hence, we have
\[
\rho(p_0) = 1 - \max_{\tau \in (0, \infty)} \lambda(p_0)(\tau) = \min_{\tau \in (0, \infty)} \frac{(1 - \lambda(p_0)(\tau))\tau^{-1}}{\tau^{-1}},
\]
and by a short calculation using the formula
\[
\lambda(\gamma(p_0)(\tau)) = 1 - \sum_i \sqrt{\tau^2 + 4p_i^2 + (n - 1)\tau},
\]
the desired formula (1.1) follows.

We construct a solution \( u \in (0, 1)^n \) of the equations \( \lambda = c_i(u, p) \) \((i = 1, \ldots, n)\) for given \( p \in \mathcal{P} \) and \( \lambda \geq 0 \) in the following way.

**Theorem 3.4.** For a given \( p \in \mathcal{P} \), if \( 0 \leq \lambda \leq \lambda_0(p) \), then the simultaneous equations \( \lambda = c_i(u, p) \) \((i = 1, \ldots, n)\) for given \( p \in \mathcal{P} \) and \( \lambda \geq 0 \) in the following way.

**Proof.** By \( \lambda_0(p) = 1 - \rho(p) \), the condition \( 0 \leq \lambda \leq \lambda_0(p) \) is equivalent to \( \rho(p) \leq t \leq 1 \) for \( t := 1 - \lambda \). Since \( \rho(p) = \|A_p\| = \lim_{m \to \infty} \|A_p^m\|^{1/m} > 0 \)

for the operator norm \( \|A_p\| \) of the transition matrix \( A_p = (p(x, y))_{x,y} \) of the Markov chain acting on \( L^2(\text{Cay}(F_n)) \), we have that the Green function

\[
G_t(x, w) := \sum_{m=0}^{\infty} p^m(x, w)t^{-m}
\]

covers every \( t > \rho(p) \) for all vertices \( x, w \in \text{Cay}(F_n) \). Here, \( p^m(x, y) \) denotes the entry of \( A^m \). In fact, it is known that also \( G_t < \infty \) if \( t = \rho(p) \) because the random walk determined by \( A_p \) on \( \text{Cay}(F_n) \) is \( \rho(p) \)-transient (see Theorem 7.8 in [Woe00]).

We denote by \( f^{(m)}(e, g) \) the probability that the random walk, starting at the group identity \( e \), hits the element \( g \) after \( m \) steps for the first time. Since \( f^{(m)}(e, g) \leq p^m(e, g) \) and \( G_t < \infty \), we can define

\[
u_i := \sum_{m=0}^{\infty} f^{(m)}(e, a_i)t^{-m} > 0 \quad (i = 1, \ldots, n).
\]

Note that \( f^{(0)}(e, a_i) = 0 \).

We first prove that \( u = (u_1, \ldots, u_n) \) defined as above satisfies

\[
1 - t = c_i(u, p) \quad (i = 1, \ldots, n).
\]

We write \( u_ic_i(u, p) \) for \( i = 1, \ldots, n \) as

\[
p_i + u_i(p_iu_i + 2 \sum_{k \neq i} p_ku_k) = t^{-1}p_i + u_i(t^{-1}p_iu_i + 2t^{-1}\sum_{k \neq i} p_ku_k).
\]

Then, it suffices to show that

\[
u_i = t^{-1}p_i + u_i(t^{-1}p_iu_i + 2t^{-1}\sum_{k \neq i} p_ku_k).
\]
Decomposition of the event of ever hitting $a_i$ gives
\[ u_i = \sum_{m=1}^{\infty} f^{(m)}(e, a_i) t^{-m} = p(e, a_i) t^{-1} + p(e, a_i^{-1}) t^{-1} \sum_{m=1}^{\infty} f^{(m)}(a_i^{-1}, e) t^{-m} \sum_{m=1}^{\infty} f^{(m)}(e, a_i) t^{-m} + \sum_{k \neq i} p(e, a_k) t^{-1} \sum_{m=1}^{\infty} f^{(m)}(a_k, e) t^{-m} \sum_{m=1}^{\infty} f^{(m)}(e, a_i) t^{-m} + \sum_{k \neq i} p(e, a_k^{-1}) t^{-1} \sum_{m=1}^{\infty} f^{(m)}(a_k^{-1}, e) t^{-m} \sum_{m=1}^{\infty} f^{(m)}(e, a_i) t^{-m}. \]

It follows that
\[ u_i = p_it^{-1} + p_i t^{-1} u_i + 2 \sum_{k \neq i} p_k t^{-1} u_k u_i \]
for each $i$.

Finally, we verify that $u$ is in $(0, 1)^n$. In the case when $t = 1$, we have that
\[ u_i := \sum_{m=0}^{\infty} f^{(m)}(e, a_i) < 1 \quad (i = 1, \ldots, n) \]
since the random walk is transient. If $t < 1$, then we consider the original equations $\lambda = c_i(u, p)$ ($i = 1, \ldots, n$) for $\lambda > 0$. By Lemma 2.5 (2), we see that $u$ satisfies $u_i < 1$ for all $i$. Thus, we have $u \in (0, 1)^n$ in any case.

**Remark.** Ledrappier [Led01, Lemma 2.2] considered a solution of equivalent equations to the above in the case when $t = 1$.

By Lemma 3.2 and Theorem 3.3, we obtain the theorem mentioned in the introduction.

**Proof of Theorem 1.3.** For a given $p \in P$, choose $r \in R$ as in Lemma 3.2. Then, Theorem 3.3 yields the assertion. \(\square\)

## 4. Generalization of the cogrowth formula

We investigate the relationship between the Poincaré exponent and the bottom of the spectrum of the Laplacian for a subgroup $G \subset F_n$. For an edge length parameter $r \in R$, we denote by $X_r$ the Cayley graph $\text{Cay}(F_n)$ with the distance $d_r$ as before. Since $G$ acts on $X_r$ isometrically, discontinuously and freely, we obtain the quotient graph $G \backslash \text{Cay}(F_n)$ endowed with the metric induced by $d_r$. We use an appropriate weight $p \in P$ to consider the Laplacian $\Delta_p$ on $G \backslash \text{Cay}(F_n)$. By the facts shown in the previous sections, we see that the weight $p$ can be given not only in terms of $r$ but also depending on the dimension $s = \delta_G(r)$ of a subgroup $G \subset F_n$.

We will prove Theorem 1.4 by dividing it into two cases according to formula (1.4). The first case follows from the following claim, which is the main part of the cogrowth formula.
Theorem 4.1. For any subgroup $G \subset F_n$ and any $r \in \mathcal{R}$, if $\delta_G(r) > \delta(r)/2$, then the bottom of the spectrum $\lambda_0^G(p)$ of the Laplacian $\Delta_p$ for $p = p \circ H(r, \delta_G(r))$ on the quotient graph $G \setminus \text{Cay}(F_n)$ coincides with $\lambda \circ H(r, \delta_G(r))$.

Proof. Let $\mu = \{\mu_x\}_{x \in X_r}$ be a Patterson measure for $G$. Consider the positive $G$-invariant total mass function

$$\varphi_\mu(x) = \int_{\partial X_r} d\mu_x = \int_{\partial X_r} j(x, \xi)^{\delta_G(r)} d\mu_\circ(\xi) \quad (x \in X_r).$$

For $p = p \circ H(r, \delta_G(r))$, we have

$$\Delta_p \varphi_\mu(x) = \int_{\partial X_r} \Delta_p j(x, \xi)^{\delta_G(r)} d\mu_\circ(\xi)
= \int_{\partial X_r} \lambda \circ H(r, \delta_G(r)) j(x, \xi)^{\delta_G(r)} d\mu_\circ(\xi) = \lambda \circ H(r, \delta_G(r)) \varphi_\mu(x).$$

Therefore, $\varphi_\mu$ descends to a positive eigenfunction of $\Delta_p$ on $G \setminus X_r$ with the eigenvalue $\lambda \circ H(r, \delta_G(r))$. Since $\lambda_0^G(p)$ is known to be the maximum of such eigenvalues, we conclude that $\lambda \circ H(r, \delta_G(r)) \leq \lambda_0^G(p)$.

For the converse inequality, we first assume that $G$ is finitely generated and show that $\varphi_\mu \in L^2(G \setminus \text{Cay}(F_n))$. Since $G$ is convex cocompact, the quotient graph $G \setminus \text{Cay}(F_n)$ consists of a finite core graph $C_G$ to which a finite number of rooted regular trees $(T_i, \hat{x}_i) (i = 1, \ldots, m)$ of valency $2n$ (the valency at $\hat{x}_i$ is 1) are attached. Let $(\tilde{T}_i, x_i)$ be a connected component of the inverse image of $(T_i, \hat{x}_i)$ under the quotient map $X_r \to G \setminus \text{Cay}(F_n)$. We note that the restriction of the quotient map to $(\tilde{T}_i, x_i)$ is an isometry onto $(T_i, \hat{x}_i)$. It suffices to show that $\varphi_\mu$ is square integrable on each $\tilde{T}_i$.

We estimate $j(x, \xi)$ for $x \in \tilde{T}_i$ by representing it as

$$j(x, \xi) = \exp(-b_t(x)) = \exp\{2d_r(o, y_i) - d_r(o, x)\},$$

where $y_i$ is the nearest point from $x$ to the geodesic ray $[o, \xi)$. We may assume that the projection of the base point $o$ is in $C_G$. If $\xi$ is a limit point of $G$, then the projection of the geodesic ray $[o, \xi)$ is in $C_G$, from which we see that $y_i$ is on the geodesic segment $[o, x_i]$. In particular, there is some $C_i > 0$ such that $\exp(2d_r(o, y_i)) \leq C_i$ for every limit point $\xi \in \partial X_r$ and for every $x \in \tilde{T}_i$.

The above estimate of $j(x, \xi)$ implies that

$$\varphi_\mu(x) \leq C_i e^{-\delta_G(r) d_r(x_i, x)} \quad (x \in \tilde{T}_i)$$

for each $i$. Then, we obtain that

$$\sum_{x \in \tilde{T}_i} \varphi_\mu(x)^2 \leq C_i^2 \lim_{R \to \infty} \int_0^R e^{-2\delta_G(r) t} dN(t) \quad (i = 1, \ldots, m)$$

for $N(t) := \#\{x \in \tilde{T}_i \mid d_r(x_i, x) \leq t\}$.
We choose some \( \varepsilon > 0 \) such that \( 2\delta_G(\mathbf{r}) \geq \delta(\mathbf{r}) + 2\varepsilon \). Since \( N(t) \leq D e^{(\delta(\mathbf{r})+\varepsilon)t} \) for some constant \( D > 0 \), we see that
\[
\int_0^R e^{-2\delta_G(\mathbf{r})t} dN(t) = e^{-2\delta_G(\mathbf{r})R} N(R) + 2\delta_G(\mathbf{r}) \int_0^R e^{-2\delta_G(\mathbf{r})t} N(t) dt \\
\leq D \left( e^{-\varepsilon R} + 2\delta_G(\mathbf{r}) \int_0^R e^{-\varepsilon t} dt \right),
\]
which has a finite limit as \( R \to \infty \). This implies that \( \varphi_{\mu} \) is square integrable on \( G\backslash \text{Cay}(F_n) \), and hence the eigenvalue \( \lambda \circ H(\mathbf{r}, \delta_G(\mathbf{r})) \) for \( \Delta_p \) is not less than \( \lambda_0^G(\mathbf{p}) \). Thus, we obtain that \( \lambda \circ H(\mathbf{r}, \delta_G(\mathbf{r})) = \lambda_0^G(\mathbf{p}) \) for a finitely generated subgroup \( G \subset F_n \).

For an infinitely generated subgroup \( G \), we choose an exhaustion by a sequence of finitely generated subgroups \( G_k \) such that
\[
G_1 \subset G_2 \subset \cdots \subset \bigcup_k G_k = G.
\]
In this case, clearly \( \delta_{G_1}(\mathbf{r}) \leq \delta_{G_2}(\mathbf{r}) \leq \cdots \leq \delta_G(\mathbf{r}) \leq \delta_G(\mathbf{r}) \), from which we can verify that \( \lim_{k \to \infty} \delta_{G_k}(\mathbf{r}) = \delta_G(\mathbf{r}) \). Indeed, we take the Patterson measure \( \mu_k \) for \( G_k \) with the normalization \( \mu_k(\partial X_r) = 1 \). Then, \( \{\mu_k\} \) has a subsequence that converges to a probability measure \( \mu \) on \( \partial X_r \) in the weak-* sense. Note that \( \delta := \lim_{k \to \infty} \delta_{G_k}(\mathbf{r}) \) exists, which is bounded from above by \( \delta_G(\mathbf{r}) \). It is easy to see that \( \mu \) is a \( G \)-invariant conformal measure of dimension \( \delta \). Since the dimension of any \( G \)-invariant conformal measure is not less than \( \delta_G(\mathbf{r}) \) (see [Coo93]), we have that \( \delta \geq \delta_G(\mathbf{r}) \). Hence \( \delta = \delta_G(\mathbf{r}) \).

Since \( \lambda \circ H(\mathbf{r}, \cdot) \) is continuous, we have that \( \lim_{k \to \infty} \lambda \circ H(\mathbf{r}, \delta_{G_k}(\mathbf{r})) = \lambda \circ H(\mathbf{r}, \delta_G(\mathbf{r})) \). Similarly, \( \mathbf{p}_k := \mathbf{p} \circ H(\mathbf{r}, \delta_{G_k}(\mathbf{r})) \) converges to \( \mathbf{p} = \mathbf{p} \circ H(\mathbf{r}, \delta_G(\mathbf{r})) \) by the continuity of \( \mathbf{p} \circ H(\mathbf{r}, \cdot) \). Moreover, if the weights \( \mathbf{p}_k \) of the Laplacian on the graph \( G\backslash X_r \) converge to \( \mathbf{p} \), then the bottom of the spectra \( \lambda_0^G(\mathbf{p}_k) \) converge to \( \lambda_0^G(\mathbf{p}) \) as \( k \to \infty \). Indeed, for the inner product \( \langle \cdot, \cdot \rangle \) on \( L^2(G\backslash \text{Cay}(F_n)) \), we have that \( \langle \Delta_{\mathbf{p}_k} f, f \rangle \) converges to \( \langle \Delta_{\mathbf{p}} f, f \rangle \) uniformly for all \( f \in L^2(G\backslash \text{Cay}(F_n)) \) with \( \langle f, f \rangle = 1 \). On the other hand, by lifting positive eigenfunctions on \( G\backslash \text{Cay}(F_n) \) to \( G_n \backslash \text{Cay}(F_n) \), we easily see that \( \lambda_0^G(\mathbf{p}_k) \geq \lambda_0^G(\mathbf{p}_k) \). Since \( \lambda_0^G(\mathbf{p}_k) = \lambda \circ H(\mathbf{r}, \delta_{G_k}(\mathbf{r})) \) by the above arguments for finitely generated subgroups \( G_k \), we conclude that \( \lambda \circ H(\mathbf{r}, \delta_G(\mathbf{r})) \geq \lambda_0^G(\mathbf{p}) \).

On the other hand, by Theorem 3.3 obtained in the previous section, we can say that the proper weight of the Laplacian \( \Delta_p \) on \( \text{Cay}(F_n) \) in the case of \( G = \{\text{id}\} \) is \( \mathbf{p} = \mathbf{p} \circ H(\mathbf{r}, \delta(\mathbf{r})/2) \). In the next theorem, we show that this result can be generalized for any \( G \) with \( \delta_G(\mathbf{r}) \leq \delta(\mathbf{r})/2 \).

**Theorem 4.2.** For any subgroup \( G \subset F_n \) and any \( \mathbf{r} \in \mathcal{R} \), if \( \delta_G(\mathbf{r}) \leq \delta(\mathbf{r})/2 \), then the bottom of the spectrum \( \lambda_0^G(\mathbf{p}) \) of the Laplacian \( \Delta_p \) for \( \mathbf{p} = \mathbf{p} \circ H(\mathbf{r}, \delta(\mathbf{r})/2) \) on the quotient graph \( G\backslash \text{Cay}(F_n) \) coincides with \( \lambda \circ H(\mathbf{r}, \delta(\mathbf{r})/2) \).

**Proof.** We take a \( G \)-invariant conformal measure \( \mu = \{\mu_x\}_{x \in X_r} \) of dimension \( \delta(\mathbf{r})/2 \), which is not less than \( \delta_G(\mathbf{r}) \) by assumption. In the case where \( \delta(\mathbf{r})/2 = \delta_G(\mathbf{r}) \), we just take a
Patterson measure $\mu$ for $G$ by Theorem 2.1. In the case where $\delta(r)/2 > \delta_G(r)$, the existence of such a measure $\mu$ can be seen as follows. We consider the sum of weighted Dirac masses

$$\mu_{x,y} = \sum_{g \in G} e^{-s d_g(x,y)} \sum_{g \in G} e^{-s d_g(x,y)} \mathbf{1}_{gy}$$

for any vertices $x, y \in X_r$ and for $s = \delta(r)/2$. Note that the Poincaré series $\sum_{g \in G} e^{-s d_g(x,y)}$ converges if $s > \delta_G(r)$. Since $G$ is not cocompact, we can choose a sequence $y_k \in X_r$ within a fundamental domain of $G$ that converges to a point at infinity. Then, a subsequence of $\{\mu_{x,y_k}\}$ converges to some $G$-invariant conformal measure $\{\mu_x\}$ of dimension $s$ in the weak-* sense. This is a modification of the construction of ending measures for Kleinian groups by Anderson, Falk and Tukia [AFT07].

We consider the positive $G$-invariant total mass function

$$\varphi_\mu(x) = \int_{\partial X_r} d\mu_x = \int_{\partial X_r} j(x, \xi) \delta(r/2) d\mu_0(\xi) \quad (x \in X_r).$$

For $p = p \circ H(r, \delta(r)/2)$, this satisfies $\Delta_p \varphi_\mu = \lambda \circ H(r, \delta(r)/2) \varphi_\mu$. Thus, we obtain a positive eigenfunction function for $\Delta_p$ on $G \backslash \text{Cay}(F_n)$ with eigenvalue $\lambda \circ H(r, \delta(r)/2)$. This implies that $\lambda^G_0(p) \geq \lambda \circ H(r, \delta(r)/2)$.

On the other hand, $\lambda_0(p) = \lambda \circ H(r, \delta(r)/2)$ by Theorem 1.3. Since any positive eigenfunction for $\Delta_p$ on $G \backslash \text{Cay}(F_n)$ can be lifted to $\text{Cay}(F_n)$, we see that $\lambda_0(p) \geq \lambda^G_0(p)$. This concludes that $\lambda^G_0(p) = \lambda \circ H(r, \delta(r)/2)$.

**Proof of Theorem 1.4.** By Theorems 1.1 and 1.2 with the definition of the appropriate weight $p_*(r, s)$ for the Laplacian, we immediately have the result. 

We note that Theorem 1.4 for the case of $r = (1/2n, \ldots, 1/2n)$ implies the original Grigorchuk cogrowth formula. In other words, Theorem 1.1 can be obtained as a corollary to Theorem 1.4.

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