Codimension one spheres which are null homotopic.

Laurence R. Taylor

Grove and Halperin [3] introduced a notion of taut immersions. Terng and Thorbergsson [5] give a slightly different definition and showed that taut immersions are a simultaneous generalization of taut immersions of manifolds into Euclidean spaces or spheres, and some interesting embeddings constructed by Bott and Samelson [1]. They go on to prove many theorems about such immersions. One particularly intriguing result, Theorem 6.25, concerned codimension one, null homotopic, tautly embedded spheres. Using a result of Ruberman, [4], they proved in many cases that this sphere had to be a distance sphere, that is, the image of a standard sphere under the exponential map, a generalization of a theorem of Chern and Lashof [2]. Here we observe that the methods of Terng–Thorbergsson and Ruberman suffice to classify tautly immersed, null homotopic, codimension one spheres. Informally one may say that the examples produced by Terng and Thorbergsson in [5] are all that there are. Precisely, we have

**Theorem 1.** Let $N^n$ be a complete Riemannian manifold and let $\phi: S \to N$ be a tautly immersed, codimension one sphere which is null homotopic. Then there exists a taut point $q \in N$ and a real number $l$, $0 < l < r_{\text{conj}}(q)$, such that $S$ is the image of the sphere of radius $l$ in $T_qN$ under the exponential map. Here $r_{\text{conj}}(q)$ denotes the conjugate radius at the point $q \in N$.

**Remark.** If we fix $q \in N$ and a real number $l$, $0 < l < r_{\text{conj}}(q)$, then the image of the sphere of radius $l$ under the exponential map from $T_qN$ is a taut immersion if and only if $q$ is taut. This is Theorem 6.23, p. 219, of Terng and Thorbergsson [5]. It follows from the definition of conjugate radius that $exp_q: T_qN \to N$ is an immersion when restricted to the ball of radius $l$ and from Theorem 6.23 [5] it follows that the spheres parallel to the boundary are all taut.

To explain the result of Ruberman eluded to above, we introduce some notation. A locally flat, codimension one sphere in $M$ which disconnects $M$ displays $M$ as a connected sum, $M = M_1 \# M_2$. We call this sphere a connected sum sphere. If we assume a codimension one, locally flat sphere is null homotopic, it must be a connected sum sphere. One obvious way to get null homotopic, connected sum spheres is for at least one of $M_1$ or $M_2$ to be a homotopy sphere. It is natural to wonder if there are any others.

Ruberman [4] gives examples of null homotopic connected sum spheres for which neither $M_1$ nor $M_2$ is a homotopy sphere. He also gives a nearly necessary and sufficient criterion to have a null homotopic, connected sum sphere, [4, Thm. 1]. We observe that a more vigorous use of Ruberman’s main lemma provides a complete characterization. This

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characterization shows that Ruberman’s list of examples is complete. The characterization
is given by the following theorem.

**Theorem 2.** A connected sum sphere in $M$, a connected, paracompact, Hausdorff, $m$–
dimensional topological manifold, with or without boundary, is null homotopic if and only
if either (1) or (2) below holds, where $M = M_1 \# M_2$.

1. At least one of $M_1$ or $M_2$ is a homotopy $m$–sphere.
2. Neither $M_1$ nor $M_2$ is a homotopy sphere, but there exist relatively prime integers
   $k_1$ and $k_2$, both greater than 1, so that each $M_i$ is a simply connected $\mathbb{Z}[\frac{1}{k_i}]$–homology
   $m$–sphere.

**Remarks.** We repeat Ruberman’s remark that case (2) can not hold if $m \leq 4$, but there
are infinitely many examples of (2) in each dimension $\geq 5$. We emphasize that case (1)
must occur unless $M$ is a compact, simply connected manifold without boundary which
is a rational homology $m$–sphere with torsion in its homology containing at least two distinct
primes. We also remark that Ruberman already proved Theorem 2 in the case
$M$ is simply connected and this is the only case we use in the proof of Theorem 1.

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sations.

1. **The proof of Theorem 2.**

   To state Ruberman’s main lemma requires further notation and conventions. To
form the connected sum requires embeddings $i_i: B^m \to M_i$. Then $M_1 \# M_2$ is formed by
removing the interiors of $i_i(B^m)$ and identifying the boundaries. If $M_1 \vee M_2$ denotes
the wedge $M_1$ and $M_2$ then there is a natural map $M_1 \# M_2 \to M_1 \vee M_2$ obtained by
pinching $i(S^{m-1})$ to a point and identifying $M_i/i_i(B^m)$ with $M_i$ via an inverse to the
map $M_i \to M_i/i_i(B^m)$ which is the identity outside a small neighborhood of $B^m$. Let
$\hat{B}_i \subset M_i$ be a larger ball containing $i_i(B^m)$ so that $\hat{B}_i - i_i(B_i) = S^{m-1} \times [0,1]$. There
are maps $M_i \to S^m$ which pinch all of $M_i - \hat{B}_i$ to a point. Composition yields a map
$M_1 \# M_2 \to M_1 \vee M_2 \to S^m \vee S^m$. Orient $S^{m-1} = \partial B^m$ and use the inward normal last
of $\partial B^m$ to orient the balls in $M_1$ and $M_2$. This orients the two $S^m$ as well.

   Now suppose the connected sum sphere is null homotopic. This means that $i$ extends
to a map of the disk $I: D^m \to M_1 \# M_2$. The map $I$ yields two maps $I_i: S^m \to M_i$.
Define the degree of the map $I_i$ as the degree of the composite $S^m \xrightarrow{i_i} M_i \to S^m$ where we
orient the first $S^m$ from $D^m$ using the inward normal last from the orientation on $S^{m-1}$.
Ruberman’s main lemma then becomes

   **Lemma 3.** (Ruberman [4,Lemma 3]). With notation and conventions as above,

   $$\deg I_1 + \deg I_2 = 1.$$  

   For completeness we give a proof. The following diagram commutes

   $\begin{array}{ccc}
   M_1 \# M_2 & \to & M_1 \vee M_2 \\
   \downarrow & & \downarrow \\
   S^m \# S^m & \to & S^m \vee S^m
   \end{array}$
It follows that it suffices to prove the result for the case \( M_1 = M_2 = S^m \). Orient \( S^m \# S^m \) so that the map \( \mathbb{Z} \simeq \tilde{H}_m(S^m \# S^m) \to \tilde{H}_m(S^m \vee S^m) \simeq \mathbb{Z} \oplus \mathbb{Z} \) sends 1 to \((1, -1)\). Given two null homotopies, \( I, J : D^m \to S^m \# S^m \) of the connected sum sphere, they differ by an element in \( \pi_m(S^m \# S^m) \) and the map \( \mathbb{Z} \simeq \pi_m(S^m \# S^m) \to \tilde{H}_m(S^m \vee S^m ; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \) sends 1 to \((1, -1)\). Hence the sum of the degrees is constant. The evident null homotopy which collapses the connected sum sphere in \( S^m \# S^m \) using the first \( S^m \) has degrees \((1, 0)\), so that difference is 1.

Two further lemmas will be required.

**Lemma 4.** If \( X \) is a simply connected, \( \mathbb{Z}[\frac{1}{k}] \)-acyclic space, then \( \pi_{m-1}(X) \) is \( k \)-torsion, \( m \geq 2 \).

As Ruberman observes, this is an elementary consequence of Serre’s mod \( C \) theory. He also gives a proof of this next lemma as his Lemma 2.

**Lemma 5.** If \( f : S^m \to M^m, m > 1 \), has degree \( k \), \( k \neq 0 \), then \( \pi_1(M) \) is finite of order dividing \( k \). It further follows that \( M \) is a \( \mathbb{Z}[\frac{1}{k}] \)-homology \( m \)-sphere.

**Remark 6.** Note that a simply connected homology sphere is a homotopy sphere. If \( k = \pm 1 \) in Lemma 5, \( M \) is a homotopy sphere even if \( m = 1 \). If \( M \) is not closed, compact and orientable, then automatically \( k = 0 \).

First we prove that the connected sum sphere in case (1) or in case (2) is null homotopic. The proof presented is essentially Ruberman’s. In case (1), the connected sum sphere is null homotopic because \( M_i - D^m \) is a homotopy disk if \( M_i \) is a homotopy sphere. In case (2), recall that \( m \geq 5 \). Note that \( M_i - D^m \) is \( \mathbb{Z}[\frac{1}{k_i}] \)-acyclic and the map \( S^{m-1} \to M_1 \# M_2 \) factors through each \( M_i - D^m \). By Lemma 4, the class in \( \pi_{m-1}(M_1 \# M_2) \) represented by the connected sum sphere is both \( k_1 \) torsion and \( k_2 \) torsion. Since \( k_1 \) and \( k_2 \) are relatively prime, \( i \) is null homotopic.

Now suppose the connected sum sphere is null homotopic and that neither side is a homotopy sphere. We must show case (2) holds.

**Step 1.** We have \( m > 1 \) and there exist relatively prime numbers \( k_1 \) and \( k_2 \), both greater than 1 such that each \( M_i \) is a \( \mathbb{Z}[\frac{1}{k_i}] \)-homology \( m \)-sphere. The fundamental group of \( M_i \) is finite of order dividing \( k_i \).

**Proof:** From Lemma 5, Remark 6 and our assumption that neither side is a homotopy sphere, it follows that neither \( \deg I_i \) can be \( \pm 1 \). It follows from Lemma 3 that neither \( \deg I_i \) can be 0 either. It follows from Remark 6 that \( M_i \) is closed, compact, orientable. If \( m = 1 \), \( M_i = S^1 \) which is a contradiction. Hence \( m > 1 \). Let \( k_i \) be the absolute value of \( \deg I_i \) and note that each \( k_i \) is greater than 1. It follows from Lemma 3 that the \( k_i \) are relatively prime. It follows from Lemma 5 that each \( M_i \) has finite fundamental group of order dividing \( k_i \) and is a \( \mathbb{Z}[\frac{1}{k_i}] \)-homology \( m \)-sphere.

If \( M \) is a rational homology sphere with finite fundamental group, let \( r(M) \) denote the order of \( \pi_1(M) \) and let \( \ell(M) \) denote the order of the direct sum of the torsion subgroups of the homology. Step 1 can be summarized as
Summary. If the connected sum sphere in $M^m = M_1 \# M_2$ is null homotopic with $M$ connected and neither $M_i$ a homotopy sphere, then each $M_i$ is a rational homology $m$–sphere, $(\ell(M_1), \ell(M_2)) = 1$, and $(r(M_1), r(M_2)) = 1$. Since neither $M_i$ is a homotopy sphere, $m > 1$ and $\ell(M_i) \cdot r(M_i) > 1$.

Consider the following construction. Let $\tilde{M}_i$ denote the universal cover of $M_i$ and let $Y_i$ be the connected sum of $\tilde{M}_i$ and $(r(M_i) - 1)$ copies of $M_{3-i}$.

Step 2. $Y_i$ is not a homotopy sphere.

Proof: Since $M_{3-i}$ is not a homotopy sphere, neither is $Y_i$ if $r(M_i) > 1$. But if $r(M_i) = 1$, $Y_i = M_i$ is not a homotopy sphere either. ■

The next two formulae follow from the Mayer–Vietoris and the van–Kampen theorems.

\begin{align}
\ell(Y_i) &= \ell(\tilde{M}_i) \cdot \ell(M_{3-i})^{r(M_i)-1} \\
\pi_1(Y_i) &= \pi_1(M_{3-i})^{r(M_i) - 1} \times \cdots \times \pi_1(M_{3-i})^{r(M_i) - 1} \quad r(M_i) - 1 \text{ times.}
\end{align}

Step 3. The connected sum sphere in $Y_i \# M_{3-i}$ is null homotopic.

Proof: The manifold $Y_i \# M_{3-i}$ is the total space of a cover of $M_i \# M_{3-i}$: $\pi: Y_i \# M_{3-i} \to M_i \# M_{3-i}$ denotes the covering map. Check that $\pi$ of the connected sum sphere in $Y_i \# M_{3-i}$ is the connected sum sphere in $M_i \# M_{3-i}$. Since $m \geq 2$, the induced map $\pi_*: \pi_{m-1}(Y_i \# M_{3-i}) \to \pi_{m-1}(M_i \# M_{3-i})$ is an injection. ■

The Final Step: $\pi_1(M_i) = 0$: i.e. $r(M_i) = 1$.

Proof: Suppose $r(M_i) > 1$. By Steps 2 and 3, the Summary applies to $Y_i \# M_{3-i}$. Hence $\ell(Y_i)$ and $\ell(M_{3-i})$ are relatively prime, so $\ell(M_{3-i}) = 1$ by (5). It follows from the Summary then that $r(M_{3-i}) > 1$. Since $\pi_1(Y_i)$ is finite, (6) implies $r(M_i) = 2$ and hence $r(Y_i) = r(M_{3-i})$. This is a contradiction since $(r(Y_i), r(M_{3-i})) = 1$. Hence $r(M_i) = 1$. ■

2. The proof of Theorem 1.

We begin with three results we need in the proof. First observe that more mileage is available from the proof of Theorem 2.5, p. 188 of [5].

Theorem 7. Let $\phi: M \to N$ be a taut immersion and fix a point $x \in M$. Assume $M$ is connected. Let $\pi: \tilde{N} \to N$ be the cover for which there is a choice of $y \in \tilde{N}$ such that the subgroup $\pi_*\left(\pi_1(N, y)\right) \subset \pi_1(N, \phi(x))$ equals the subgroup $\phi_*\left(\pi_1(M, x)\right) \subset \pi_1(N, \phi(x))$. Let $\tilde{\phi}: M \to \tilde{N}$ be the unique lift with $\tilde{\phi}(x) = y$. Then $\tilde{\phi}$ is an embedding.

Proof: Note that $P(\tilde{N}, \tilde{\phi} \times q)$ is connected since $M$ is and $\tilde{\phi}$ induces an isomorphism on $\pi_1$. Next note that $P(\tilde{N}, \tilde{\phi} \times q)$ is a component of $P(N, \phi \times \pi(q))$. If we choose $q \in \tilde{N}$ as Terng and Thorbergsson do on page 188, then the energy function is a perfect Morse function on $P(\tilde{N}, \tilde{\phi} \times q)$. The result follows just as in [5]. ■
Proposition 8. Let \( \phi: M \to N \) be a map, let \( \pi: \tilde{N} \to N \) be a cover, and let \( \tilde{\phi}: M \to \tilde{N} \) be a map covering \( \phi \). Then \( \phi \) is a taut immersion if and only if \( \tilde{\phi} \) is.

Proof: It follows from the structure of covering spaces that \( \phi \) is an immersion if and only if \( \tilde{\phi} \) is. Next note that an immersion \( \psi: X \to Y \) is taut if and only if the energy function on the path space \( P(Y, \psi \times q) \) is a perfect Morse–Bott function for all points \( q \in Y \). One direction is Theorem 2.9, p. 193, in [5]. The other direction follows from the remark on page 183 of [5] that the energy function on the path space is Morse at all non–focal points, plus the observation that a proper Morse–Bott function which is Morse is a proper Morse function.

Now the path space for \( \phi \) is the disjoint union of various path spaces

\[
P(N, \phi \times p) = \coprod_{p_\alpha \in \pi^{-1}(p)} P(\tilde{N}, \tilde{\phi} \times p_\alpha).
\]

Since the energy function on each \( P(\tilde{N}, \tilde{\phi} \times p_\alpha) \) is just the restriction of the energy function on \( P(N, \phi \times p) \), the result is immediate. ■

Remark. G. Thorbergsson has shown the author an elementary proof of Proposition 8 that does not need Theorem 2.9.

Finally we need a generalization of Theorem 6.25 [5].

Proposition 9. Let \( \psi: S \to M \) be a taut embedding and assume \( \psi(S) \) bounds a simply connected, \( \mathbb{Z}/2\mathbb{Z} \) homology ball \( B \subset M \). Then \( B \) is a ball and there exists a taut point \( p \in B \subset M \) and a real number \( l > 0 \) such that \( \psi(S) \) is the image of the sphere of radius \( l \) in \( T_pM \) and \( l \) is less than \( r_{\text{conj}}(p) \).

Proof: We indicate the changes needed from the proof of Theorem 6.25 on pages 219 and 220 of [5]. Following Terng and Thorbergsson, pick \( p \in B \subset M \) so that there is a geodesic perpendicular to \( \psi(S) \) with first focal point \( p \) at a distance \( l \) from \( \psi(S) \). We further assume \( l \) is minimal with this property. Observe that the space \( P(B, \psi \times p) \) fits into a fibration sequence

\[
\Omega B \to P(B, \psi \times p) \xrightarrow{\pi} S.
\]

Since \( B \) is simply connected and mod 2 acyclic, it follows from the Serre spectral sequence that \( \Omega B \) is also mod 2 acyclic. It then follows from the Serre spectral sequence that \( \pi_* \) induces an isomorphism in mod 2 homology. Hence \( P(B, \psi \times p) \) has no homology below dimension \( n - 1 \). Argue just as in Terng and Thorbergsson that the multiplicity of the focal point \( p \) is \( n - 1 \), so all the geodesics starting perpendicularly to \( \psi(S) \) meet at a distance \( l \) in the point \( p \). Furthermore, all of these geodesics minimize distance between \( p \) and \( \psi(S) \). Hence the conjugate radius at \( p \) is bigger then \( l \). Furthermore no two of these geodesics intersect in \( B \) since they minimize distance, so \( B \) is the ball of radius \( l \) centered at \( p \). It follows from [5,Thm. 6.23, p. 219] that \( p \) is taut. ■
there is a simply connected, $\mathbb{Z}/2\mathbb{Z}$ homology ball, $B \subset \tilde{N}$, with boundary $\partial B = \tilde{\phi}(S)$. The embedding $\tilde{\phi}$ is taut by Proposition 8. Applying Proposition 9, we now have a taut point $\tilde{q} \in \tilde{N}$ and a real number $l$ such that $\tilde{\phi}(S)$ is the distance sphere of radius $l$ centered at $\tilde{q}$ with $l < r_{\text{conj}}(\tilde{q}) = r_{\text{conj}}(q)$, where $q$ denotes the image of $\tilde{q}$ in $N$. It follows that $\phi(S)$ is the image of the sphere of radius $l$ in $T_qN$ under the exponential map. Finally apply Proposition 8 again: since $\tilde{q} \in \tilde{N}$ is taut, $q$ is taut in $N$ as required by Theorem 1.

**Remark.** The hypothesis in Theorem 1 that the immersion $\phi$ is null homotopic is sometimes unnecessary. By [5,Thm. 2.5], if $\pi_1(N) = 0$ then $\phi$ is an embedding and in many cases, the $\mathbb{Z}/2\mathbb{Z}$ cohomology ring will force one of the pieces in the connected sum decomposition to be a simply connected $\mathbb{Z}/2\mathbb{Z}$ homology ball. Then Proposition 9 will show that $\phi$ is null homotopic. Examples include all simply connected manifolds with the $\mathbb{Z}/2\mathbb{Z}$ cohomology of the rank 1 symmetric spaces, $H^*(S^n;\mathbb{Z}/2\mathbb{Z})$, $H^*(CP^n;\mathbb{Z}/2\mathbb{Z})$, $H^*(HP^n;\mathbb{Z}/2\mathbb{Z})$ and the $\mathbb{Z}/2\mathbb{Z}$ cohomology of the Cayley projective plane. Products of such spaces are also examples.

**Remark.** One can generalize Ruberman's theorem to show that if the order of $\phi \in \pi_{n-1}(N)$ is odd, then the embedding $\tilde{\phi}$ bounds a simply connected $\mathbb{Z}/2\mathbb{Z}$ homology ball, so again Proposition 9 will show that $\phi$ is null homotopic.

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