Three Weighted Residuals Methods for Solving the Nonlinear Thin Film Flow Problem

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Abstract. In this paper, the methods of weighted residuals: Collocation Method (CM), Least Squares Method (LSM) and Galerkin Method (GM) are used to solve the thin film flow (TFF) equation. The weighted residual methods were implemented to get an approximate solution to the TFF equation. The accuracy of the obtained results is checked by calculating the maximum error remainder functions (MER). Moreover, the outcomes were examined in comparison with the 4th-order Runge-Kutta method (RK4) and good agreements have been achieved. All the evaluations have been successfully implemented by using the computer system Mathematica®10.

1. Introduction

Physical phenomena and engineering problems are often formulated in the form of ordinary differential equations with initial or boundary conditions. Many researchers have studied the exact or approximate solution of these types of equations [1, 2].

Thin liquid films are important in many applications, for example, the non-Newtonian fluids are polymer, oils, and greases, melts, blood and drilling muds. Solving the nonlinear problems is still not an easy task, and the exact solution cannot be obtained for many problems, therefore, either approximate or numerical solutions can be achieved. Many researchers have studied and addressed non-Newtonian fluid behavior problems with a wide range of applications [3, 4].

During the past few years many researchers have been interested to study the non-Newtonian fluids, thin film equation and treated by using different analytic methods such as: He [5] has solved the TFF equation by using the variational iteration method (VIM), AL-Jawary [6] solved the TFF equation by Temimi and Ansari method (TAM), Al-Jawary et al. [7] applied the Banach contraction principle method (BCPM) to find the approximate solutions of the nonlinear TFF equation and made a comparison with the solutions obtained from the variational iteration method and the homotopy perturbation method. Moreover, Sajid and Hayat [8] applied the homotopy perturbation method to provide the analytic solutions to two TFF equations of third order. Also, Manafian and Sindi [9] have solved the TFF equation by the optimal homotopy asymptotic method (OHAM).

On the other hand, there are accurate and reliable approximate techniques used to solve differential equations: the Collocation method, least square method and Galerkin method. Arnau et al. [10] introduced and used a new method constructed on a polynomial collocation to solve the models of one-dimensional flow in the systems of intake and exhaust of interior combustion engines. Also, Aswhad [11] applied the collocation method with the Bernstein polynomials as a basis function to solve the first
order linear delay differential equations approximated. Moreover, Vaferi et al. [12] presented the transient pressure response study by using the method of orthogonal collocation to solve the diffusion equation in the system of radial transient flow, Raghad et al. [13] obtained the approximate solutions for nth-order retarded by using Collocation method with the aid of B-Spline functions and Weddle method, Ahuja [14] obtained the solution using the technique of normal modes and weighted residual Galerkin approximation. Also, Ganji and Hatami [15] have been applied the collocation, Galerkin and least square methods for solving Jeffery-Hamel Flow. Daşcioğlu and Acar [16] used the collocation method for approximate solutions of initial and boundary value problems. Furthermore, Al-Hawasy and Jaber [17] used the Galerkin method to demonstrate the existence and uniqueness theorem of the state vector solution, Al-Hawasy and Jawad [18] applied the Galerkin finite element method for the equations of the nonlinear parabolic boundary value problem, Tufekci et al.[19] studied the forced vibration characteristics of the rotating disk by using the Galerkin method.

This paper aims to apply the weighted residuals methods (WRMs): Collocation Method (CM), Least Squares Method (LSM) and Galerkin Method (GM) to find the approximate solutions of the TFF equation.

This paper is arranged as follows: In section 2, the nonlinear TFF problem will be presented. The basic concepts of the methods of weighted residuals are introduced in section 3. In section 4, solving the nonlinear TFF equation will be introduced by the weighted residuals methods. Finally, the conclusion is given in section 5.

2. The Nonlinear TFF Problem

In this section, the flow of thin film of the non-Newtonian fluid will be presented on a moving belt [20]. The flow is fixed, laminar and uniform. The thickness of the film is uniform as well as the flow. We can introduce the governing equations of this problem as follows

\[
\frac{d^2 u}{dx^2} + \frac{6(k_2+k_3)}{\mu}\left(\frac{du}{dx}\right)^2 - \frac{\rho g}{\mu} = 0, \quad (1)
\]

\[
u(0) = k, \quad \text{and} \quad \frac{du}{dx} = 0, \quad \text{at} \quad x = \delta, \quad (2)
\]

where \( u \) is the velocity, \( k_2 \) and \( k_3 \) are the third degree fluid materiality constants, \( \mu \) symbolize the dynamic viscosity, \( \rho \) present a density, \( g \) is gravitational acceleration, \( \delta \) describes the thickness of the film and \( k \) is the belt speed.

We can write the dimensionless variables as below

\[
\tilde{x} = \frac{x}{\delta}, \quad \tilde{u} = \frac{u}{k}, \quad \gamma = \frac{(k_2+k_3)k^2}{\mu\delta^2}, \quad \text{and} \quad \sigma = \frac{\rho g \delta^2}{\mu k} \quad (3)
\]

By removing (˘) from equation (3) and substituting equation (3) in equations (1) and (2), the equation of dimensionless form for the nonlinear boundary value problem is

\[
\frac{d^2 u}{dx^2} + 6\gamma\left(\frac{du}{dx}\right)^2 - \sigma = 0, \quad (4)
\]

\[
u(0) = 1, \quad \text{and} \quad \frac{du}{dx} = 0, \quad x = 1. \quad (5)
\]

Equation (4) is considered to be a well-posed problem because it is a second order nonlinear ODE and it has two boundary conditions.

By integrating equation (4) twice and using the boundary conditions given in equation (5), we get:

\[
\frac{du}{dx} + 2\gamma\left(\frac{du}{dx}\right)^3 - \sigma x = a, \quad (6)
\]
where \( a \) is the constant of integration process. To calculate the constant in equation (6), we used the 2nd condition given in equation (5), and then the value of \( a = -\sigma \). Thus, we can write the nonlinear system in equations (4) and (5) as follows

\[
\frac{du}{dx} + 2\gamma \left( \frac{du}{dx} \right)^3 - \sigma (x - 1) = 0, \text{ where } u(0) = 1.
\]  

(7)

3. Weighted Residuals Methods (WRMs) [12, 15]

The basic idea of the weighted residuals method is to find the approximate solution in the form of a polynomial to the differential equation of the formula

\[
D[u(x)] = \varepsilon(x).
\]  

(8)

Let us assume that \( \tilde{u} \) is approximate to \( u \), and \( \tilde{u} \) is a linear combination of basis functions selected by a linearly independent set that must satisfy the boundary conditions, that is:

\[
u \cong \tilde{u} = \sum_{i=1}^{n} c_i \varphi_i.
\]  

(9)

Substituting equation (9) in equation (8), when the differential operator \( D \) is linear or nonlinear, the result of the operation is generally \( \varepsilon(x) \). So, there will be an error or residual:

\[
R(x) = D(\tilde{u}(x)) - \varepsilon(x) \neq 0.
\]  

(10)

The idea in the WRMs is to impose the error to zero. That is:

\[
\int_{X} R(x) W_i(x) dx = 0, \quad i = 1, 2, ..., n
\]  

(11)

The number of weight functions \( W_i \) is similar to the number of unknown constants \( c_i \) in \( \tilde{u} \). The results will be the set of \( n \) algebraic equations of \( c_i \). Three methods of WRMs will be presented in the next subsections to solve these equations and get the values of unknown constants.

3.1. Collocation Method (CM)

This method is based on the weighting functions, which depend on the Dirac \( \delta \) functions in the domain. Such that

\[
\delta(x - x_i) = \begin{cases} 
1 & x = x_i \\
0 & \text{otherwise}
\end{cases}
\]  

(12)

Thus, the integral of the weighted residual results in the residual equals zero at some selected points in the domain. That is, insert equation (12) in equation (11), results in \( R(x_i) = 0 \).

3.2. Least Squares Method (LSM)

This method depends on the minimization of the continuous summation of all the squared residuals. That is, a minimum of

\[
S = \int_{X} R(x) R(x) dx = \int_{X} R^2(x) dx,
\]  

(13)

The derivatives of equation (13) must be zero with respect to all the unknown parameters, in order to obtain a minimum of a scalar function. That is,

\[
\frac{\partial S}{\partial c_i} = 2 \int_{X} R(x) \frac{\partial R}{\partial c_i} dx = 0.
\]  

(14)

Then, compared equation (14) to equation (11), the weight functions can be
\[ W_i = 2 \frac{\partial R}{\partial c_i} \]  

(15)

However, the "2" will be canceled from the equation (15), we can neglect it. Then, the LSM weight functions are just the residual derivatives for the unknown constants:

\[ W_i = \frac{\partial R}{\partial c_i} \]  

(16)

3.3. Galerkin Method (GM)

This method can be considered as a modification of the LSM. The derivative of the approximate function is used instead of the residual derivative for the unknown \( c_i \). In other words, if the function is approximated as in equation (9), then the weight functions are given by

\[ W_i = \frac{\partial \bar{u}}{\partial c_i} \]  

(17)

\[ \int_X W_i R(x) dx = 0, \ i = 1,2, ..., 6. \]  

(18)

By substituting equation (17) in equation (18) and putting the weighted residual integration equal to zero, then the unknown coefficients in the approximate solution will be determined.

4. Solving The Nonlinear TFF Equation by The Method of Three Weighted Residuals

Before solving the nonlinear TFF equation, we recall the Weierstrass theorem [21]:

Theorem 4.1: If \( u: [a, b] \rightarrow \mathbb{C} \) is continuous and \( \epsilon > 0 \) then there exists a polynomial \( p \) such that \( \| u(x) - p(x) \| < \epsilon \), for all \( x \in [a, b] \).

4.1. CM for Solving TFF Equation

The trial function is considered as [22]:

\[ \bar{u}(x) = 1 + \sum_{i=1}^{n} c_i x_i^n \quad n = 6. \]  

(19)

Where, equation (19) satisfies the initial condition. By replacing equation (19) in equation (10) and using equation (7), then we get the residual function:

\[ R(x) = -\sigma(-1 + x) + c_1 + 2xc_2 + 3x^2c_3 + 4x^3c_4 + 5x^4c_5 + 6x^5c_6 + 2\gamma(c_1 + 2xc_2 + 3x^2c_3 + 4x^3c_4 + 5x^4c_5 + 6x^5c_6)^3, \]  

(20)

Furthermore, the residual function should be tend to zero, to reach this goal, we first collocate equation (20) at \( n-1 \) suitable points and choosing by [23]:

\[ x_i = \left( \frac{i}{2^n} \right) \left( \cos \left( \frac{i\pi}{n} \right) + 1 \right), \ i = 1,2, ..., n \]  

(21)

By substituting equation (21) and choosing the values of \( \gamma = 0.5 \) and \( \sigma = 0.3 \) as proposed in [24] in equation (20), then a set of six equations and six unknown coefficients were obtained by replacement these points into the residual function \( R(x) \) with \( n = 6 \). After solving these unknown parameters \( c_1, c_2, c_3, c_4, c_5 \) and \( c_6 \), the formula of the \( \bar{u}(x) \) will be determined

\[ \bar{u}(x) = 1 -0.2784179903218099x + 0.12170283596956218x^2 + 0.013323650734029828x^3 + 0.000641634296380529x^4 + 0.0012608938213647149x^5 - 0.00020578092642158x^6. \]  

(22)
4.2. **LSM for Solving TFF Equation**

The trial function in equation (19) must satisfy the initial condition, and the residual will be defined by equation (20). By replacing the residual function \( R(x) \) given in equation (14) with taking into account equation (16), a set of six equations will be obtained. We will solve this system of equations to find the values of coefficients \( c_1, c_2, c_3, c_4, c_5 \) and \( c_6 \), we get:

\[
\tilde{u}(x) = 1 - 0.2784200109488398x + 0.12174092145386706x^2 + 0.013114374195112015x^3 + 0.001173100601081297x^4 - 0.001736399714055255x^5 - 0.000032540026460812244x^6. \tag{23}
\]

4.3. **GM for Solving TFF Equation**

In this method, the approximate function \( \tilde{u}(x) \) is derived with respect to the unknown coefficients \( c_i \) to obtain the weight function \( W_i \) as mentioned in the equations (17) and (18), the weight functions will be:

\[
W_1 = x, W_2 = x^2, W_3 = x^3, W_4 = x^4, W_5 = x^5, W_6 = x^6. \tag{24}
\]

By substituting equation (24) into equation (25), and applying equation (14), we getting a system of algebraic equations that will be solved to get the values of unknown coefficients, and substituting these values into equation (19), we have:

\[
\tilde{u}(x) = 1 - 0.2784269835482562x + 0.12180368394378813x^2 + 0.012886759804023607x^3 + 0.0015069962962708115x^4 - 0.00205157568378486x^5 + 0.00006458861736018161x^6. \tag{26}
\]

To check the accuracy of the approximate solutions and since the exact solution is unknown, we define the following maximal error remainder function

\[
MER_n = |ER_n|, \tag{27}
\]

where

\[
ER_n(x) = \frac{d}{dx} \left( \tilde{u}(x) \right) + 2\gamma \left( \frac{d}{dx} \left( \tilde{u}(x) \right) \right)^3 - \sigma(x - 1) = 0. \tag{28}
\]

**Figure 1.** Comparison between \( \tilde{u}(x) \) obtained by the WRMs and the RK4 for \( 0 \leq x \leq 1 \) when \( \gamma = 0.5 \) and \( \sigma = 0.3 \).
Figure 2. The logarithmic plots for the curves of the MER_{Collocation}, MER_{Galerkin} and MER_{LeastSquares} when $\gamma = 0.5$ and $\sigma = 0.3$.

Figure 3. (a) MER_{Collocation} with various value of $\gamma$ when $\sigma = 0.3$ (b) MER_{Collocation} with various values of $\sigma$ when $\gamma = 0.5$.

Figure 4. (a) MER_{Galerkin} with various values of $\gamma$ when $\sigma = 0.3$ (b) MER_{Galerkin} with various values of $\sigma$ when $\gamma = 0.5$.

Figure 5. (a) MER_{LeastSquares} with various values of $\gamma$ when $\sigma = 0.3$ (b) MER_{LeastSquares} with various values of $\sigma$ when $\gamma = 0.5$. 
Figure 1 shows good agreements have been obtained between the approximate solution from the use of WRMs as shown in the equations (22), (23) and (26) and the numerical solution obtained by the RK4 method. Also, Figure 2 shows the approximate solution obtained by the LSM has a lower error and good accuracy compared to the other methods.

Moreover, figures 3, 4 and 5 are present the effects of $\gamma$ and $\sigma$ values on the error of $MER_n$ of the WRMs of the CM, GM and LSM, respectively that calculated according to the equations (27) and (28). An additional numerical examination can be performed by studying the influence of the non-Newtonian parameter $\gamma$ and the value $\sigma$ on the velocity given in equation (7). The differences are shown in figures 6(a), 7(a), and 8(a), when the $\gamma$ value is stationary, the velocity tends to the Newtonian state when $\sigma$ values increase. On the other side, figures. 6(b), 7(b), and 8(b) show that the non-Newtonian parameter $\gamma$ is decreased when $\sigma$ is kept fixed; the solution approaching the Newtonian case.

![Figure 6](image_url)

**Figure 6.** (a) Impact of $\sigma$ on $\bar{u}(x)$ when $\gamma = 0.5$. (b) Impact of $\gamma$ on $\bar{u}(x)$ when $\sigma = 0.3$.

![Figure 7](image_url)

**Figure 7.** (a) Impact of $\sigma$ on $\bar{u}(x)$ when $\gamma = 0.5$. (b) Impact of $\gamma$ on $\bar{u}(x)$ when $\sigma = 0.3$.

![Figure 8](image_url)

**Figure 8.** (a) Impact of $\sigma$ on $\bar{u}(x)$ when $\gamma = 0.5$. (b) Impact of $\gamma$ on $\bar{u}(x)$ when $\sigma = 0.3$. 
Table 1 presents the values of $\tilde{u}(x)$ when $\gamma = 0.5$ and $\sigma = 0.3$ which are obtained by applying the CM, LSM, GM and RK4 method, a good agreement between the approximate solution obtained by the proposed methods and RK4 method. Furthermore, table 2 presents the numerical values of the MER functions for the three proposed methods, and the errors will be reduced when the values of $n$ are increased, and the errors of the LSM are less than others which state that the LSM provides the best approximate solution.

Table 1. The numerical values of $\tilde{u}(x)$ of the WRMs and the RK4 when $0 \leq x \leq 1$

| $x$ | Collocation method | Least squares method | Galerkin method | RK4 method |
|-----|--------------------|----------------------|----------------|------------|
| 0   | 0.973388604326959  | 0.9733886168283186  | 0.973388355492877 | 0.97338860199038 |
| 0.1 | 0.949291714539164  | 0.949291779627536   | 0.949291603550059 | 0.94929173866873 |
| 0.2 | 0.927789579962055  | 0.9277895747879666  | 0.927789447431362 | 0.92778960206722 |
| 0.3 | 0.908960642679965   | 0.9089605521220884  | 0.908960384164320 | 0.90896060771800 |
| 0.4 | 0.892879654057870   | 0.892879582613215   | 0.892879538912867 | 0.89287957663844 |
| 0.5 | 0.879615643100685   | 0.879615692734783   | 0.879615465849836 | 0.87961562345976 |
| 0.6 | 0.869229736651382   | 0.869229785341048   | 0.869229695533261 | 0.86922984013857 |
| 0.7 | 0.861772831421431   | 0.8617728771289226  | 0.8617727287864787 | 0.86177292659036 |
| 0.8 | 0.8572831117877873  | 0.8572829626709952  | 0.8572827770820347 | 0.85728297544108 |
| 0.9 | 0.855783455930375   | 0.8557836550197314  | 0.855783649249016 | 0.85578364908701 |

Table 2. The numerical values of MER of the WRMs when $m = 0.3$ and $\beta = 0.5$

| $x$ | $\text{MER}_{\text{Collocation}}$ | $\text{MER}_{\text{Least squares}}$ | $\text{MER}_{\text{Galerkin}}$ |
|-----|---------------------------------|---------------------------------|----------------|
| 0   | 0.29999999999999999            | 0.15                            | 0.2         |
| 1   | 0.015253981352280344           | 0.00640565921938471             | 0.0118177211239134 |
| 2   | 0.00252086249197119            | 0.0008507902995215852           | 0.00165764422136968 |
| 3   | 0.0005682077594781135          | 0.0001412537654080243           | 0.00042630589671925623 |
| 4   | 0.000027156623291654805       | 0.000034378328659685            | 0.0000015185839074067 |
| 5   | 0.000013983660434798394       | 0.000003043944378259069         | 0.00001108466640066876 |

5. CONCLUSION

In this paper, we obtained the approximate solution to the TFF equation by applying the WRMs which are the CM, LSM, and GM. The approximate solutions obtained that provided the WRMs are reliable and effective methods. A good agreement has been achieved by comparing the numerical solution obtained by using the 4th-order Runge-Kutta method and the proposed methods. It can be concluded from the figures and tables, that the maximal error remainder will be reduced when the values of polynomial power $n$ are increased. Moreover, the LSM provided the best approximate solution with less error. Finally, we found that selecting the parameters had an influence on convergence as well.

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