Discrete wavelet based estimator for the Hurst parameter of multivariate fractional Brownian motion

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Abstract. In this paper, wavelets were used to study the multivariate fractional Brownian motion through the deviations of the random process to find an efficient estimation of Hurst exponent. The results of simulations experiments were shown that the performance of the proposed estimator was efficient. The estimation process was made by taking advantage of the detail coefficients stationarity from the wavelet transform, as the variance of this coefficient showed the power-law behavior. We use two wavelet filters (Haar and db5) to manage minimizing the mean square error of the model.

1. Introduction

In many areas such as hydrology, biology, telecommunications, and economics, the data available for analysis usually has a scaling behavior (long-memory) that needs to be discovered. The main point to detect scaling behavior is the estimation of the parameters of the models under study such as the Hurst parameter, which is used to determine self-similarity and long-range dependence. The study of these phenomena in the multidimensional case relies on fractal Gaussian noise, which is a first order incremental process

\[ G_H(t) = B_H(t + 1) - B_H(t) \] (1)

Where \( B_H(t) \) is the fractional Brownian motion and can be generalized to be of order \( n \) and written as \( n \)-FBM. This generalization introduced by Perrin et al (2001) through the derivation of the kernel in the continuous definition of FBM.

The fractional Brownian motion can be defined as a random process characterized by non-stationary self-similarity and a stationary continuous increasing and has been used to explain many random phenomena in many areas of research and although it has demonstrated in various applications, for example bone radiographs.

More recently, wavelet transformation has emerged as a powerful mathematical tool for signal processing by setting a timescale.

Wavelets give a new look to the statistical analysis of random processes, as wavelets rules in the time scale plan provide a suitable framework for representing functions with sharp spikes and analyzing them with good accuracy and allow studying basic characteristics such as long-memory, self-similarity and stationary increasing.

The objective of this paper is to develop a method that can be used to model and study nth order fractional Brownian motion through wavelet lens by estimating the Hurst parameter. The proposed procedure will rely on finding an unbiased estimate of the difference in the Gaussian noise of the
fractional Brownian motion using Haar and Daubechies 5 (db5) wavelet. The performance of this estimator was validated through a simulation study.

The paper is structured as follow: Section 2 we give an overview of discrete wavelet transform, in Section 3 the fractional Brownian motion properties will be presented, and in Section 4 the proposed method will be briefed, Section 5 the simulation study will be conducted and finally Section 6 the conclusions will be explain.

2. Discrete Wavelets Transform

The main problem in the early 1980s when working with wavelet was to create a multiresolution analysis where the scale function had compact and continuous support. In her paper that published in 1988, Daubechies found a way to build a multiresolution, continuous, orthonormal and compact support scale function that attracted the attention of researchers interested in the wavelet field at that time, and here we introduce the DWT beginning with wavelet function \( \psi(t) \), \( t \in \mathbb{R} \) such that [1]:

\[
\int \psi(t) \, dt = 0
\]  

(2)

Where this condition satisfy some integrability conditions, \( \psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \). Which require the wavelet to be bounded, centered around the origin, and have time and frequency support. Where it is finite or decreases very fast. Time and frequency concertation are restricted by the Gabor-Heisenberg uncertainty principal. A wavelet \( \psi \) have N zero moments (vanishing moment) if:

\[
\int t^k \psi(t) \, dt = 0, k = 0,1, ..., N - 1
\]  

(3)

Where \( N \) is an integer at least equal 1. The increase in the vanishing moments give an enlargement in the time support but bring smoothness, continuity, derivability and more concentration of the spectral in a given frequency \( v_0 \). An example of wavelet have derivative of standard normal density is the Haar wavelet and the Daubechies wavelet, where both constructed from multiresolution analysis (MRA), and also the Mexican hat wavelet which is the 2\(^{nd}\) derivative of the normal function and has 2 vanishing moments.

The Daubechies wavelets constitute a family of wavelets indexed by their vanishing moments and give rise to the orthonormal basis. The Haar wavelet is a special case of both the Daubechies and spline families that has one vanishing moment and has finite time support. The wavelet coefficients are equivalent to the increments of the process, and can be defined beginning from the function:

\[
\psi_{j,k}(t) = 2^{j/2} \psi(2^{-j}t - k), \quad j, k \in \mathbb{Z}
\]  

(4)

Where \( j, k \) are dilation and translation of \( \psi \) respectively.

\[
\int \psi_{j,k}(t) \, dt = \int \psi^2(t) \, dt
\]  

(5)

It allows the \( L^2(\mathbb{R}) \) norm to be preserved. Using the functions \( \{\psi_{j,k}, j, k \in \mathbb{Z}\} \) as a set of filters we can now define the DWT of a function \( \{X(t), t \in \mathbb{R}\} \) as:

\[
d_{j,k} = \int X(t) \psi_{j,k}(t) \, dt
\]  

(6)

Where the \( d_{j,k} \) are the wavelet coefficients called details, as it encode the differential information between adjacent scales centered about \( 2^j \), and the time \( 2^j k \). The details sometimes written as \( d_X(j, k) \) to emphasize that they correspond to \( X \). The adjective discrete in DWT refers to the fact the indices \( j,k \) take discrete values in contrast to the CWT where they take real values.
Multiresolution analysis is a specific class of wavelets which is of particular interest because it has potentially stronger mathematical properties and because it gives birth to fast recursive pyramidal decomposition algorithms. Their key properties are:

i. The number of zero moments, regularity, time or frequency support, can be easily and flexibly tuned.
ii. Fast pyramidal algorithms to compute the wavelet coefficients.

The construction of such wavelets of which the Daubechies and spline families are famous examples is integral to the so called the multiresolution analysis theory. The dilation equation for the Daubechies scale function can be written as [1]:

\[ \phi(t) = 2^{-j/2} \sum_{k \in \mathbb{Z}} h_k \phi(2^{-j} t - k) \] (7)

Where \( h_k \) is a scale filter and also called the average block and now if we had a multiresolution analysis and had \( h_k \) real values from dilation equation then \( h_k \) have the following properties:

1. \( \sum_{k \in \mathbb{Z}} h_k = 2^{1/2} \) (8)
2. \( \sum_{k \in \mathbb{Z}} h_k^2 = 1 \)

There is a great benefit to this filter in calculating the projection coefficients according to the following equation:

\[ b_\ell = \sum_{k \in \mathbb{Z}} a_k h_{k-2\ell}, \text{where } k, \ell \in \mathbb{Z} \] (9)

Where \( a_k \) are the approximation coefficients. The dilation equation for the wavelet function is the same as the formula of the scale function’s dilation equation, but in terms of wavelet filter as follows:

\[ \psi(t) = 2^{-j/2} \sum_{k \in \mathbb{Z}} g_k \phi(2^{-j} t - k) \] (10)

It should also be noted that the value of the wavelet filter can be calculated through the scale filter.

\[ g_k = (-1)^k h_{k-1} \] (11)

3. Fractional Brownian Motion

Fractional Brownian motion (FBM) is a centered stochastic Gaussian process. Mandelbrot & Van Ness [2] called it Brownian, and to define it they use different fractional integral of white noise:

\[ B_H(t) = B_H(0) + \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^{0} [(t-s)^H - (-s)^H] dB_s + \int_{0}^{t} (t-s)^{H-1/2} dW_s \right\} \] (12)

The most important difference between Brownian motion and fractional Brownian motion is in the increments so in the classic process the increments are independent but in fractional Brownian motion (FBM) are not, when \( H > 1/2 \) there is positive autocorrelation on other hand if \( H < 1/2 \) the autocorrelation is negative.

Fractional Brownian motion has zero mean and variance covariance equal to [3]:

\[ E[B_H(t)B_H(s)] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \] (13)

Where \( H \) is a Hurst parameter which was defined by both Mandelbrot and Van Ness and its value ranges between [0, 1] and shows the extent of roughness of the movement, as its value increase softer the signal become so the type of the random process is related to the value of this parameter.
The existence of the fractional Brownian motion comes from the presence of the centered Gaussian process. If we assume that there is a real parameter $H > 0$, then there is a continuous central Gaussian process $B_H = (B_H^t)_{t \geq 0}$ with a known positive variance if and only if $H \leq 1$. [4]

The main properties of fractional Brownian motion is:

1. Self-similarity is visually seen as the same pattern repeating both seen up close and seen from a far. In other words, there are small versions of the larger pattern repeated inside larger patterns. This property of a random process is achieved if:

$$B_H(at) \sim |a|^H B_H(t) \quad (14)$$

This property is due to the fact that the covariance function is homogeneous of order $2H$, which gives the process a fractional character.

2. Stationary increments which satisfied when:

$$B_H(t) - B_H(s) \sim B_H(t - s) \quad (15)$$

3. Long-range dependence or the long memory is a phenomenon that appears in the analysis of time-related data and expresses the slow exponential decay of the autocorrelation coefficient and its appearance clearly in fBm when $H > 1/2$.

$$\sum_{n=1}^{\infty} E \left[ B_H(1) \left( B_H(n + 1) - B_H(n) \right) \right] = \infty \quad (16)$$

4. Regularity where the sample-paths are non-derivative at almost every point, i.e., all trajectories are locally Holder continuous for any function have d-dimensional Euclidian space is Holder continuous when there is a positive constant $C$ and $\alpha > 0$ such that $|f(x) - f(y)| \leq C|x - y|^{\alpha}$ That is, for each trajectory path, for each $T > 0$, and for each $\epsilon > 0$, there is a constant value of $C$ so that:

$$|B_H(t) - B_H(s)| \leq C|t - s|^{\alpha - \epsilon} \quad (17)$$

5. Semi-Martingale in the theory of probability, Martingale is a series of random variables (any random process) so that at any time the conditional expectation of the present value excluding all previous values is equal to the first value.

$$E[X_{n+1} | X_1, ..., X_n] = X_n \quad (18)$$

This property occurs in many random processes such as classic Brownian motion i.e. when $H = \frac{1}{2}$. It’s so obvious so it should be that this property is not realized in the case of fractional Brownian motion because the covariance of this type not equal to zero and many researchers have demonstrated this property when $H > 1/2$ and when $H < 1/2$.

**4. Second Moment of Gaussian Noise using Wavelet (SMGNW)**

The analysis of the fractional Brownian motion depends on the estimation of the Hurst parameter, which is sometimes called the parameter of roughness. The difficulty in the estimation process lies in the nonstationary and the fact that this type of random process is locally Holder Continuous, that is, it does not have a derivative at almost every point.

This topic has raised the interest of many researchers, and here in this paper, we find a more efficient way to conduct this work. It was relied on finding an unbiased estimate of the difference in the Gaussian noise of the fractional Brownian motion using wavelets, which proved effective in improving the estimator’s efficiency and its approach to the real value too close and the error is relatively too low, here we will be discuss in detail. Beginning with the definition of wavelet function.
\[ \psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j} t - k), \quad j, k \in \mathbb{Z} \]  \hspace{1cm} (19)

Thus, the details coefficients for the transformation can be defined as

\[ d_{j,k} = \int_{\mathbb{R}} X(t) \psi_{j,k}(t) dt \]  \hspace{1cm} (20)

Where \( d_{j,k} \) can be calculated from \( d_{j-1,k} \) as follow [13]

\[ d_{j,k} = \int_{\mathbb{R}} X(t) 2^{-j/2} \psi(2^{-j} t - k) dt \]
\[ = \int_{\mathbb{R}} X(t) 2^{-j/2} \sqrt{2} \sum_{n} v_{n} \psi(2(2^{-j} t - k) - n) dt \]
\[ = \sum_{n} v_{n} \int_{\mathbb{R}} X(t) 2^{-j+1/2} \psi(2^{-j+1} t - 2k - n) dt \]
\[ d_{j,k} = \sum_{n} v_{n} d_{j-1,2k+n} \]  \hspace{1cm} (21)

These coefficients are characterized by not being affected by the polynomials due to the vanishing moments and their stationary, and this is evident when fix the scale. We will notice that the translation is not affected by the increases

At the scale \( s \) and the assumed increase of translation by the value of \( r \) [5]

\[ d_{s,k+r} = \int_{\mathbb{R}} X(t) \psi(t - k - r) dt \]

Let \( u = t - r \) then \( t = u + r \)

\[ = \int_{\mathbb{R}} X(u + r) \psi(u - k) du \]
\[ = \int_{\mathbb{R}} [X(u + r) - X(r)] \psi(u - k) du \]

At \( r = 0 \)

\[ \approx \int_{\mathbb{R}} [X(u) - X(0)] \psi(u - k) du \]
\[ = \int_{\mathbb{R}} X(u) \psi(u - k) du = d_{s,k} \]  \hspace{1cm} (22)

The third step was achieved through one of the main characteristics of the wavelets, which is \( \int_{\mathbb{R}} \psi(u) du = 0 \). The fourth step was achieved by approximate the integration by sum and taking advantage of the stationary increasing. These characteristics of the detail parameters make it able to give accurate estimates even in signals of self-similar and stationary increases by finding an unbiased value for the variance of these coefficients, and that the value of this variation can be calculated using the definition of variance of the process knowing that fBm has zero mean.
\[
\text{var}(d_{j,k}) = E(d_{j,k})^2 = \frac{1}{n} \sum_{k=1}^{n} |d_{j,k}|^2 \quad (23)
\]

Istas [6] and Abry [5] have indicated a bias in this amount. However, it is possible to reduce it through the wavelet transformation through its previously mentioned characteristics.

The problem is the choosing of a wavelet with a sufficient number of vanishing moments to smooth the signal without increasing the estimator bias, but enough so does not increase the variance of the estimate.

For this reason, we have found that using two wavelets would be the best way to adjust the amount of variance without increasing the estimate bias or the variance.

The db 5 wavelet and the Haar wavelet were used separately to perform the transformation of the series after taking the differences to it and find the Gaussian noise in the series as follows:

\[
G_H(t_j) = B_H(t_j) - B_H(t_{j-1}) \quad (24)
\]

Now filtration will be made to the differences using the wavelet filter

\[
d_1 = [c \ast (g_1, g_2, g_3, \ldots, g_p)], \quad p = \text{length of } c
\]

\[
d_2 = [c \ast (g_2, g_3, g_4, \ldots, g_{p+1})]
\]

\[
d_3 = [c \ast (g_3, g_4, g_5, \ldots, g_{p+2})]
\]

\[
d_n = [c \ast (g_{n-p+1}, g_{n-p+2}, g_{n-p+3}, \ldots, g_n)], \quad n = \text{length of } G \quad (25)
\]

The transformation has been made, Flandrin [7] has discussed and appointed out that the approximation coefficients are strongly correlated, but on the other hand, there is independency in the detail coefficient values that can be adopted in calculating an unbiased estimate of the Hurst parameter, and so the value of variance is as follows:

\[
E(d_{j,k})^2 = E(d_{0,0})^2 2^{j(2H+1)} \quad (26)
\]

Where \(E(d_{0,0})^2\) is a constant, as shown by Houdre [8] the logarithms of the variance will give an estimate for the Hurst parameter

\[
\log_2 \left( \frac{E(d_{j,k})^2}{E(d_{0,0})^2} \right) = (2H + 1)j + \log_2 (E(d_{0,0})^2) \quad (27)
\]

Where \(j\) will be a linear function (\(j\) represents the courser level) and the slope of this function will be an estimate of \(H\), but it will be a biased estimate. For eliminating this bias, Istas [6] find out using quadratic variation that dividing two filtered variances by using different bands will give an unbiased estimate for the variance, for this purpose we use Haar and db 5 wavelet to find an unbiased estimate for the variance of detail coefficients

\[
V_1 = \frac{\text{var}(d_{j,k})_{\text{half band Haar}}}{\text{var}(d_{j,k})_{\text{Haar}}} \quad \text{and} \quad V_2 = \frac{\text{var}(d_{j,k})_{\text{half band db 5}}}{\text{var}(d_{j,k})_{\text{db 5}}}
\]

\[
V = \frac{1}{2} (V_1 + V_2) \quad (28)
\]

Where the half band coefficients can be defined as follow

\[
(d_{j,k})_{\text{half band}} = [d_1, 0, d_2, 0, d_3, 0, \ldots \ldots] \quad (29)
\]
From the mentioned relations the estimate can be found using the following expression

\[ \hat{H} = \frac{1}{2} \log_2(V) \]  

(30)

5. Simulation study

In this section, we will conduct simulation experiments of multivariate fractional Brownian motion, where wavelet synthesis method proposed by Sellan, Mayer and Abry [9] [10] will be used for generating the mentioned random process, and then we will use Second moment of Gaussian noise using wavelet (SMGNW). The length of the series to be generated will be as \( n = 100,200 \) and the number of variables \( m = 4,8,12 \) we will repeat the process with \( rep = 500 \) for the sake of increasing the accuracy in the estimation process. As for the wavelet used for the generation, \( s = 2^{10},2^{12} \). Taking into account that the estimation process for the Hurst parameter will be for all levels \( H = [0.1:0.9] \). The mean square error and the bias of the estimator will be calculated as follows

\[ MSE(\hat{H}) = Var(\hat{H}) + Bias^2(\hat{H}) \]

\[ = \left( \frac{1}{4} \sum_j \sigma_j^2 \theta_j^2 \right) + (\hat{H} - H)^2 \]

Where \( \theta_j = \frac{\sum(S_j^i - S_j) \sigma_j^2}{s \sum s_j^2 - s_j^2} \), and \( S = \sum \frac{1}{\sigma_j^2}, S_j = \sum \frac{j^2}{\sigma_j^2}, S_{jj} = \sum j^2 \)

\[ Var(y_j) = \sigma_j^2 = \zeta(\frac{N_j}{2}) \frac{s_n}{ln^2(2)}, E(y_j) = 2(H + 1) + constant \]  

(31)

Where \( \zeta(\frac{N_j}{2}) \) is a generalized Riemann Zeta function [11].

Also, a new random series will be generated each time before the estimation in order to get the best view of the performance of the method used and at different levels of the random process.

| Table 1 of Hurst est. bias and MSE using SMNGW |
|-----------------------------------------------|
| Var. no. | \( H \) | \( s = 2^{10} \) | n=100 | Bias | MSE | Bias | MSE | Bias | MSE | n=200 | Bias | MSE |
|----------|--------|----------------|-------|------|-----|------|-----|------|-----|-------|------|-----|
| m=4      |        |                |       |      |     |      |     |      |     |       |      |     |
| 3.1      | 0.0250 | 0.0099         | -0.0011 | 0.0002 | 0.0066 | 0.0079 | -0.0113 | 0.0011 |
| 3.2      | 0.0088 | 0.0187         | 0.0045 | 0.0010 | 0.0042 | 0.0130 | -0.0354 | 0.0028 |
| 3.3      | 0.0045 | 0.0300         | 0.0163 | 0.0025 | 0.0072 | 0.0165 | 0.0023 | 0.0035 |
| 3.4      | -0.0147 | 0.0378         | 0.0111 | 0.0033 | 0.0153 | 0.0196 | -0.0160 | 0.0052 |
| 3.5      | 0.0355 | 0.0314         | -0.0028 | 0.0117 | -0.0096 | 0.0251 | -0.0169 | 0.0020 |
| 3.6      | 0.0056 | 0.0578         | 0.0191 | 0.0033 | -0.0024 | 0.0208 | -0.0232 | 0.0107 |
| 3.7      | -0.0142 | 0.0574         | 0.0071 | 0.0021 | -0.0049 | 0.0209 | -0.0932 | 0.0195 |
| 3.8      | 0.0380 | 0.0634         | 0.0133 | 0.0033 | -0.0029 | 0.0192 | -0.0168 | 0.0181 |
| 3.9      | 0.0238 | 0.0476         | 0.0012 | 0.0109 | -0.0179 | 0.0235 | 0.0193 | 0.0270 |

| m=8      |        |                |       |      |     |      |     |      |     |       |      |     |
| 7.1      | -0.0279 | 0.0038         | 0.0002 | 0.0023 | -0.0036 | 0.0058 | -0.0040 | 0.0036 |
| 7.2      | -0.0226 | 0.0111         | 0.0148 | 0.0060 | -0.0077 | 0.0118 | -0.0130 | 0.0085 |
| 7.3      | -0.0015 | 0.0203         | 0.0015 | 0.0065 | -0.0123 | 0.0159 | -0.0212 | 0.0121 |
| 7.4      | 0.0353 | 0.0342         | 0.0169 | 0.0076 | 0.0159 | 0.0226 | 0.0017 | 0.0148 |
| 7.5      | 0.0181 | 0.0152         | 0.0180 | 0.0107 | -0.0154 | 0.0201 | 0.0211 | 0.0111 |
| 7.6      | -0.0216 | 0.0408         | 0.0034 | 0.0078 | -0.0008 | 0.0281 | 0.0321 | 0.0132 |
| 7.7      | 0.0161 | 0.0544         | 0.0044 | 0.0130 | -0.0051 | 0.0328 | 0.0228 | 0.0120 |
| 7.8      | 0.0102 | 0.0625         | 0.0050 | 0.0227 | -0.0009 | 0.0378 | -0.0013 | 0.0187 |
| 7.9      | 0.0206 | 0.0759         | -0.0035 | 0.0302 | -0.0125 | 0.0385 | 0.0336 | 0.0415 |

| m=12     |        |                |       |      |     |      |     |      |     |       |      |     |
| 11.1     | -0.0106 | 0.0059         | 0.0028 | 0.0037 | 0.0047 | 0.0097 | 0.0002 | 0.0025 |
|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 11.2 | -0.0224 | 0.0125 | -0.0084 | 0.0073 | -0.0071 | 0.0186 | -0.0273 | 0.0055 |
| 11.3 | -0.0012 | 0.0209 | -0.0273 | 0.0096 | -0.0134 | 0.0208 | -0.0151 | 0.0086 |
| 11.4 | -0.0031 | 0.0263 | -0.0181 | 0.0112 | 0.0177 | 0.0286 | 0.0108 | 0.0113 |
| 11.5 | 0.0015 | 0.0203 | 0.0155 | 0.0098 | 0.0015 | 0.0171 | 0.0074 | 0.0096 |
| 11.6 | 0.0061 | 0.0406 | 0.0082 | 0.0134 | 0.0052 | 0.0323 | -0.0334 | 0.0103 |
| 11.7 | 0.0153 | 0.0495 | 0.0069 | 0.0169 | -0.0057 | 0.0368 | 0.0127 | 0.0113 |
| 11.8 | 0.0146 | 0.0578 | -0.0155 | 0.0244 | 0.0118 | 0.0470 | 0.0001 | 0.0163 |
| 11.9 | -0.0097 | 0.0699 | -0.0076 | 0.0348 | -0.0027 | 0.0555 | 0.0167 | 0.0334 |

(a) 12-variate self-similarity analysis at sample size n=100 and FBM synthesis $s = 2^{10}$ with Hurst exponent 0.5
(b) 12-variate self-similarity analysis at sample size n=200 and FBM synthesis $s = 2^{10}$ with Hurst exponent 0.9
(c) 12-variate self-similarity analysis at sample size n=100 and FBM synthesis $s = 2^{12}$ with Hurst exponent 0.9
(d) 12-variate self-similarity analysis at sample size n=200 and FBM synthesis $s = 2^{12}$ with Hurst exponent 0.1

6. Conclusions
The estimation methods for signals with long memory, non-stationary often suffer from bias in the model variance that usually used for estimation, which in turn affects the error in the model estimation. In the SMGNW method used above, we consider reducing the amount of bias by truncating a part equal to the length of the filter used for the transformation, which led to the reduction of this bias to lower levels. The results of this truncation affected the estimation process positively. In this method, we also adopted the half-band method, which contributed in elimination of unwanted noise to obtain an efficient
estimator. Also, by relying on wavelet filters to perform the estimation process, the linear effect of the model was excluded from the model through the vanishing moments, which is one of the most important properties of the wavelet. The uses of Haar and db5 wavelet filters made our estimator better than other methods that’s because we rely on denoising twice which make the signal smoother. This procedure made the proposed estimation process an efficient method for all levels of estimation through mean square error and bias as is evident from the results obtained.

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