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Self organizing maps for the parametric analysis of COVID-19 SEIRS delayed model

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Abstract
Since 2019, entire world is facing the accelerating threat of Corona Virus, with its third wave on its way, although accompanied with several vaccination strategies made by world health organization. The control on the transmission of the virus is highly desired, even though several key measures have already been made, including masks, sanitizing and disinfecting measures. The ongoing research, though devoted to this pandemic, has certain flaws, due to which no permanent solution has been discovered. Currently different data based studies have emerged but unfortunately, the pandemic fate is still unrevealed. During this research, we have focused on a compartmental model, where delay is taken into account from one compartment to another. The model depicts the dynamics of the disease relative to time and constant delays in time. A deep learning technique called “Self Organizing Map” is used to extract the parametric values from the data repository of COVID-19. The input we used for SOM are the attributes on which, the variables are dependent. Different grouping/clustering of patients were achieved with 2- dimensional visualization of the input data (https://creativecommons.org/licenses/by/2.0/). Extensive stability analysis and numerical results are presented in this manuscript which can help in designing control measures.

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1. Introduction
The spread rate of SARS-2 infection is alarming currently due to its, mutated versions and the continuous waves [1,2] in different parts of the world, especially Europe and America. The new variants of the disease are emerging with the passage of time, increasing the pressure on “designing accurate control measures”.

Coronavirus (SARS-CoV2) belongs to the beta-coronavirus genus. The transmission mechanism focuses on the viral S protein (spike protein) that binds the virus to a reaction catalyzed by some enzymes located on the surface of the host cell.

Different control measure studies, at different levels, have recently emerged in the literature [3,4]. The daily based incidence data of the disease positive case-counts, was discussed by Clouston et al. [3]. They examined the spread and severity of the infectious disease with the aid of statistical analysis, by keeping in view different variables, such as age, gender and race. As a major outcome, the importance of social distancing was concluded.

Computational frameworks have been reported in the literature, to address the compartmental transmission dynamics of this infection, see [5–7] and the references therein. Some models were based on the data based studies, whereas others were evidence based studies. The work conducted by [6] raised questions linked with the pandemic and addressed the questions in a technical manner. It was reported that the modeling results showed great variation with respect to time and number of individuals suffer-
ing from the pandemic. Perhaps there was not much information and evidence available for the statistical inference at the beginning to the disease outbreak, especially before January 23 when Wuhan was quarantined and locked down, and that there was a lack of reliable data, except for the confirmed case data that could be used for model calibration.

It has been reported in the literature that delay played an important role in the dynamics of the disease transmission [8, 9] reported the delayed dynamics due to the quarantine strategy. Recently the research conducted by [10] reported that the pandemic was delayed due to the control measures practiced by different countries in different ways, including extensive testing, contact tracking, and quarantining; thus the widespread measures, enforcing “social isolation” were successful but not ideal for long term practice due to the socioeconomic pressure.

Mathematical models can help to understand the nonlinear dynamics in a cost effective manner [11–16]. In the field of computational biology, different methods have been used [17,18]. During this research, we have presented a delayed nonlinear model with demographic effects. The model takes into account the delays between compartments, thus making it more realistic. Complete stability analysis is provided in the manuscript to make it more authentic. Numerical results, leading to some proposals to control the pandemics are reported at the end.

2. Materials and methods

2.1. Proposed model

The "SEIRS" model (as shown in schematic (Fig. 1 and Table 1)) utilized during this research is given as:

\[
\begin{align*}
\frac{dV_1(t)}{dt} &= n\kappa - \kappa V_1(t) - \rho V_1(t) V_2(t) + V_3(t) + \psi V_4(t), \\
\frac{dV_2(t)}{dt} &= \rho V_1(t)(t - \tau_1) V_2(t - \tau_1) + V_3(t - \tau_1) - \kappa V_2(t) - \sigma V_2(t), \\
\frac{dV_3(t)}{dt} &= \sigma V_2(t - \tau_2) - \eta V_3(t) - \kappa V_3(t), \\
\frac{dV_4(t)}{dt} &= \eta V_3(t) - \kappa V_4(t) - \psi V_4(t),
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
V_1(\theta) &= \varphi_1(\theta) \geq 0, \\
V_2(\theta) &= \varphi_2(\theta) \geq 0, \\
V_3(\theta) &= \varphi_3(\theta) \geq 0, \\
V_4(\theta) &= \varphi_4(\theta) \geq 0, \\
\theta &= [\tau, 0], \\
\tau &= \max[\tau_1, \tau_2].
\end{align*}
\]

Here, \(\varphi_i(\theta)\) are the initial functions, where \(i = 1, 2, 3, 4\). All the parameters are non negative. Each equation in the above system (1) is linked with the parameters. Parametric values and their definitions are listed in table (2). If time delay \(\tau\) is negative all the nodes(population) will be negative which is not possible. The value of \(\tau\) can never be negative, for that reason.

2.2. Parametric analysis

Structure of SOM

The self organizing maps (SOM) are used to develop mapping for dimension reduction and for other multiple purposes in the fields of applied sciences. These maps are different from typical Artificial Neural Networks (ANN) in:

- architecture,
- algorithmic properties.

The algorithm of SOM is actually based on the competitive learning, whereas, the algorithm of ANN is based on the error correction learning. To preserve the topological properties of the given data, the SOM algorithm is linked with the neighborhood function.

There are two layers of the SOM-Kohonen Neural Networks. First layer is called the input layer, that is fully connected with the competitive layer of the processing neurons whereas the next layer is termed as the output layer.

For a network, we use an \(n \times 1\) input vector in general practice. The components of the vector are:

\[
\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]

The \(n\)-components of this input vector, are connected with each neurons in the array (the output layer of \(m \times m\) processing neurons).

The input layer is connected to the Kohonen layer by weight, \(w_i = (w_{i1}, w_{i2}, w_{in})\), where \(w_{ij}\) is the weight value associated with \(i^{th}\) component of input vector to the \(j^{th}\) neurons.

2.3. Data source

In this analysis, we accessed the data repository: https://creativecommons.org/licenses/by/2.0/ (an open source data repository). The list of confirmed cases, reported cases and deaths, based on several countries regular counts. Data from 21 march 2020 can
be accessed as time series. We have obtained and checked data from the COVID-19 database.

2.4. Basic consequences

The dynamical analysis of the mathematical models is of great significance in the field of biopharmaceuticals [20–22]. The positivity of solutions expresses existence of population, whereas the boundedness gives explanation of natural control of the growth because of restriction of the resources. We arrived at following Theorems.

**Theorem 2.1.** Each solution of a model (1) corresponding to the initial conditions (2) will remain a positive for all values of the $t \geq 0$, if initial-history function is positive.

**Proof.** As all parameters have positive values so in first equation $n\kappa > 0$,

$$\frac{dV_1(t)}{dt} \geq \psi V_4(t) - \kappa V_1(t) - \rho V_1(t)(V_2(t) + V_3(t)).$$

Consequently, by the separating variables and then integrating

$$V_1(t) \geq V_1(0) \exp \left\{ \left( -V_1(t) + V_2(t) \right) - \kappa \right\} dt,$$

$$\left( -V_1(t) + V_2(t) \right) \int \psi V_4(t) \geq V_1(0) \int \psi V_4(t) \exp \left\{ \left( -V_1(t) + V_2(t) \right) - \kappa \right\} dt.$$ (3)

It shows that $V_1(t)$ will remain positive as the $V_1(0)$ is a positive and thus it is a property of positive invariant. □

To prove that $V_2(t)$ is positive we use (1) in the $[0, \tau]$, the second equation is

$$\frac{dV_2(t)}{dt} \geq -\kappa V_2(t) - \sigma V_2(t),$$

since the parametric values are non negative. By separating and integrating both side it gives

$$V_2(t) \geq V_2(0) \exp \left\{ -\left( \kappa + \sigma \right) t \right\}. \quad (4)$$

Similarly, from 3rd and 4th equations of model (1) and by using (2) we have;

$$\frac{dV_3(t)}{dt} \geq -\eta V_3(t) - \kappa V_3(t),$$

$$\frac{dV_4(t)}{dt} \geq -\kappa V_4(t) - \psi V_4(t).$$

By separating $V_3(t)$ and $V_4(t)$ receptively and integrating both sides; it gives

$$V_3(t) \geq V_3(0) \exp \left\{ -\eta \kappa t \right\}, \quad (5)$$

$$V_4(t) \geq V_4(0) \exp \left\{ -\kappa \psi t \right\}. \quad (6)$$

From preceding analysis it is conclude that the positivity of $V_1(t)$, $V_2(t)$, $V_3(t)$ and $V_4(t)$ depends on the initial-history functions; $V_1(0), V_2(0), V_3(0)$ and $V_4(0)$ which are positive. Similarly, in the interval $[\tau, 2\tau]$ for equation 2nd 3rd and 4th equations respectively we have

$$V_2(t) \geq V_2(\tau) \exp \left\{ -t \left( \kappa + \sigma \right) \right\}, \quad (7)$$

$$V_3(t) \geq V_3(\tau) \exp \left\{ -\eta t \kappa \right\}, \quad (8)$$

$$V_4(t) \geq V_4(\tau) \exp \left\{ -\kappa \psi t \right\}. \quad (9)$$

By this approach, the above method can be generalized to any finite interval $[0, t]$ and this prove that $V_2(t), V_3(t) \text{ and } V_4(t)$ will always positive $V t \geq 0$.

**Lemma 1.** Given the 1st order differential inequality:

$$\frac{dX}{dt} + \alpha X \leq \beta X(t) = X_0.$$ (10)

the solution of this inequality will satisfy that

$$X(t) \leq \frac{\beta}{\alpha} (1 - e^{-\alpha t}) + X_0 e^{-\alpha t}.$$ (11)

and hence

$$\limsup_{t \to \infty} X(t) \leq \frac{\beta}{\alpha}.$$ (12)

**Proof.** Suppose we have 1st order differential equation;

$$\frac{dX}{dt} + \alpha X = \beta$$

there is an integration factor $e^{\alpha t}$. By multiplying both sides of the equation (10) with the integrating factor $e^{\alpha t}$ gives an exact differential equation

$$\frac{d}{dt}(Xe^{\alpha t}) \leq \beta e^{\alpha t}, X(0) = X_0.$$ (13)

Integrating over $[0, t]$

$$X(t)e^{\alpha t} - X_0 \leq \frac{\beta}{\alpha} (e^{\alpha t} - 1)$$

after manipulation we obtain the required inequality. The final term of solution is obtained by taking the limit $t \to \infty$. □

**Theorem 2.2.** If the solution of a model (1) is positive invariant, then it follows that all solutions of the model (1) are ultimately bounded in the following domain

$$\sum \left\{ \left\{ || V_1 || \leq n, || V_2 + V_3 \right\} \right\} \leq \frac{\eta}{m + \kappa} \cdot || V_3 + V_4 || \leq \kappa. \quad (14)$$

**Proof.** By solving 1st equation of model (1)

$$\frac{dV_1}{dt} = n \kappa - \kappa V_1 - \rho V_1 (V_2 + V_3) + \psi V_4 \leq -\rho V_1 (V_2 + V_3).$$

For that reason, the positivity of $V_1, V_2$ and $V_3$ in the above equation implies that the maximum value that $V_1$ can attain is $n$, since parameter $\frac{dV_1}{dt} < 0$ and which implies that

$$\limsup_{t \to \infty} V_1 \leq n. \quad (15)$$

Again by adding the 2nd and 3rd equations of the model (1) we obtain

$$\frac{d}{dt} (V_2 + V_3) = -\eta V_3 - \kappa V_2 - \kappa V_2 - \sigma V_2 \leq -\rho V_1 (V_2 + V_3) + \psi V_4 \leq -\rho V_1 (V_2 + V_3).$$

Where $m = \max \{\eta, \sigma\}$. According to the lemma (10) we obtain

$$\limsup (V_2 + V_3) \leq \frac{\eta}{m + \kappa}. \quad (16)$$

Adding the 3rd and 4th equation
\[
\begin{align*}
\frac{d}{dt}(V_3 + V_4) &= -\psi V_4(t) - \kappa V_3 - \kappa V_4 \\
&\leq -\kappa (V_3 + V_4)
\end{align*}
\]
by applying lemma (10) we obtain
\[
\limsup_{t \to \infty} (V_3 + V_4) \leq \kappa. \tag{17}
\]
By combining (15), (16) and (17) it is established that its solution will ultimately be bounded in that domain $\Sigma$. □

### 2.5. Mathematical analysis

The model (1) contain two equilibrium points. The infection free equilibrium point is
\[
E_0 = (0, 0, 0, 0). \tag{18}
\]
The endemic equilibrium point exist if $\rho > \kappa$, otherwise nodes (exposed, infectious and recovered) will become zero.
\[
E^* = (V_1^*, V_2^*, V_3^*, V_4^*), \tag{19}
\]
where
\[
\begin{align*}
V_1^* &= \frac{(\eta + \kappa)(\kappa + \sigma)}{\rho(\eta + \kappa + \sigma)}, \\
V_2^* &= A^*(\kappa + \psi), \\
V_3^* &= A^*(\kappa + \psi), \\
V_4^* &= A^*(\kappa + \psi).
\end{align*}
\]
The coefficient $A^*$ is as follows:
\[
A^* = \frac{(\rho - \kappa)(\kappa + \sigma) - \eta(\kappa - \rho + \sigma)}{\rho(\eta + \kappa + \sigma)(\kappa + \psi)(\kappa + \psi)}. \tag{21}
\]

#### 2.5.1. Stability of $E_0$

The characteristic equation of Jacobian matrix, at the equilibrium point $E_0$ is as follows
\[
\begin{align*}
&(-\kappa - \lambda)(-\kappa - \lambda - \psi) \\
&\left( e^{\lambda t} (\eta + 2(\kappa + \lambda) + \sigma) \right) \\
&- \sqrt{e^{\lambda t}} (\eta - \sigma + \kappa \rho e^{\lambda t}) + 4n\rho\sigma e^{\lambda t} \\
&\left( e^{\lambda t} (\eta - \sigma + \kappa \rho e^{\lambda t}) + 4n\rho\sigma e^{\lambda t} \\
&+ \sqrt{e^{\lambda t}} (\eta + 2(\kappa + \lambda) + \sigma) \right) = 0.
\end{align*}
\]
Here we have two eigenvalues as $\lambda_1 = -\kappa$ and $\lambda_2 = -\kappa - \psi$ that are the negative, the model is stable if following two equation gives us two negative real values
\[
\begin{align*}
&\left( e^{\lambda t} (\eta + 2(\kappa + \lambda) + \sigma) \right) \\
&- \sqrt{e^{\lambda t}} (\eta - \sigma + \kappa \rho e^{\lambda t}) + 4n\rho\sigma e^{\lambda t} = 0, \tag{23}
\end{align*}
\]
\[
\begin{align*}
&\sqrt{e^{\lambda t}} (\eta - \sigma + \kappa \rho e^{\lambda t}) + 4n\rho\sigma e^{\lambda t} \\
&+ \sqrt{e^{\lambda t}} (\eta + 2(\kappa + \lambda) + \sigma) = 0. \tag{24}
\end{align*}
\]
The stability can be proved by the following theorem.

**Theorem 2.3.** If $\sqrt{((\eta - \sigma) + n\rho)^2 + 4n\rho\sigma} > n\rho - (\eta + 2\kappa + \sigma)$, then the equilibrium point $E_0$ values of delay $\tau_i$ $(i = 1 : 2)$ is always the locally stable.

**Proof.** Consider the equilibrium point $E_0$ is stable for $\tau_1 = \tau_2 = 0$ its mean equation (23) as has all roots to be negative
\[
\lambda < \sqrt{((\eta - \sigma) + n\rho)^2 + 4n\rho\sigma} - (n\rho - (\eta + 2\kappa + \sigma)) < 0 \tag{25}
\]
That implies $\sqrt{((\eta - \sigma) + n\rho)^2 + 4n\rho\sigma} > n\rho - (\eta + 2\kappa + \sigma)$. Now, suppose that $\tau_i = 0$ varies continuously in a positive direction such that there is a $\tau_i^*$ which give one imaginary eigenvalue pair by putting $\lambda = i\eta, \eta > 0$, Hence substituting $\lambda = i\eta$ in (23), after simplifying we obtain
\[
\frac{((\eta + \kappa)^2 + \eta^2)(\kappa + \sigma)^2 + \eta^2}{n^2((\eta + \kappa - \sigma)^2 + \eta^2)} = \rho^2. \tag{26}
\]
□

The equation (26) shows that $\rho^2 < 0$ if $\sqrt{((\eta - \sigma) + n\rho)^2 + 4n\rho\sigma} > n\rho - (\eta + 2\kappa + \sigma)$.

For $\tau_i = 0$ we have
\[
((\eta + \kappa)(\kappa + \sigma) + n\rho(-\eta - \kappa) - \eta^2)^2 + (\eta(-\eta - 2\kappa - \sigma) + n\rho\eta)^2 = (n\rho\eta)^2. \tag{27}
\]
This equation shows that $\rho^2 < 0$ if $\sqrt{((\eta - \sigma) + n\rho)^2 + 4n\rho\sigma} > n\rho - (\eta + 2\kappa + \sigma)$.

**Theorem 2.4.** If $\eta + 2\kappa - n\rho + \sigma > \sqrt{((\eta - \sigma) + n\rho)^2 + 4n\rho\sigma}$, then equilibrium point $E_0$ is stable $\forall$ values of $\tau_i$ where $i \in [1, 2]$.

#### 2.6. Reproductive number and sensitivity analysis

For the model (1) the basic reproduction number $R_0$ is as follows
\[
R_0 = \frac{n\rho(\eta + \kappa + \sigma)}{(\eta + \kappa)(\kappa + \sigma)}. \tag{28}
\]
The sensitivity if the reproductive number is analyzed by taking the partial derivative with respect to the parameter.

\[
\frac{dR}{dn} = \frac{\rho(\kappa + \sigma)}{\eta + \kappa + \sigma} > 0,
\]
\[
\frac{dR}{d\rho} = \frac{\eta + \kappa}{\eta + \kappa + \sigma} > 0,
\]
\[
\frac{dR}{d\eta} = -\frac{n\rho\sigma}{(\kappa + \sigma)^2} < 0,
\]
\[
\frac{dR}{d\kappa} = -\frac{\eta\rho\sigma}{(\kappa + \sigma)^2} < 0.
\]
Reproductive number is increased with increments in $n$ and infection rate while decreases with inclusion or drop-out rate, the recovery rate and the outbreak rate.

**Stability criteria**

**Theorem 2.5.** The infection free equilibrium point $E_0$ is stable asymptotically $\forall$ values of delay $\tau$ if $R_0 < 1$. The endemic equilibrium point $E^*$ exits if $R_0 > 1$.

**Proof.** It is obvious from definition and by Theorem (2.3) that if $\frac{n\rho(\kappa + \sigma)}{(\eta + \kappa)(\kappa + \sigma)} = 1$, then all the conditions will remain suitable. Furthermore, $\frac{n\rho(\kappa + \sigma)}{(\eta + \kappa)(\kappa + \sigma)} > 1$ is one of conditions for the existence of the positive equilibrium point. The remaining conditions for the existence from $R_0 > 1$. □
2.7. Stability of $E^*$

For the stability of endemic equilibrium point we will suppose that $R_0 > 1$. The Jacobian matrix at endemic equilibrium point

$$J_E = \begin{pmatrix} -A - \kappa - \lambda & Ce^{-\lambda \tau_1} & Be^{-\lambda \tau_1} & \psi \\ 0 & e^{\lambda \tau_1} & 0 & 0 \\ 0 & 0 & -\kappa - \lambda & 0 \\
\end{pmatrix}.$$ 

where as
\begin{align*}
A &= \frac{(\kappa + \psi)(\kappa + \sigma)(\kappa \rho - \kappa) - \eta(\kappa - \eta \rho + \sigma)}{\eta(\kappa + \sigma + \psi) + (\kappa + \sigma)(\kappa + \psi)}), \\
B &= \frac{(\eta + \kappa)(\kappa + \sigma)}{\eta + \kappa + \sigma}.
\end{align*}

The characteristic equation for the equi point $E^*$ is
\begin{equation}
\lambda^4 + (a_1 \lambda^2 + a_2 \lambda^2 + a_3 \lambda + a_4 + e^{-\lambda \tau_1} b_1 \lambda^3 + b_2 \lambda^2 + b_3 \lambda + b_4) + e^{-\lambda \tau_1} \eta c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + c_4 = 0,
\end{equation}

where the constants are defined as
\begin{align*}
a_1 &= A + \eta + 4\kappa + \sigma + \psi, \\
a_2 &= A(\eta + 3\kappa + \sigma + \psi) + \eta(3\kappa + \sigma + \psi) + \psi(3\kappa + \sigma + \psi) + 3(2\kappa + \sigma), \\
a_3 &= A(\eta(2\kappa + \sigma + \psi) + 3\kappa + \sigma + \psi) + \eta(3\kappa + \sigma + \psi) + \psi(3\kappa + \sigma + \psi) + 3(2\kappa + \sigma), \\
b_1 &= -B, \\
b_2 &= -B(\eta + 3\kappa + \psi), \\
b_3 &= -B(\eta(2\kappa + \psi) + \kappa(3\kappa + 2\psi)), \\
b_4 &= -Bc(\eta + \kappa + \psi), \\
c_1 &= a, \\
c_2 &= -c_1, \\
c_3 &= -b_3, \\
c_4 &= -b_4c(\kappa + \psi).
\end{align*}

Where $\sigma = 0$.

To gain insight regarding the endemic equilibrium point $E^*$ we will discuss stability of endemic equilibrium point and conditions of Hopf bifurcation of threshold parameters like $r_1$ and $r_2$ by considering the following cases.

**Case 1.** When $r_1 = 0$ and $r_2 = 0$ then equation (28) will become
\begin{equation}
\lambda^4 + \lambda^3(a_1 + b_1 + c_1) + \lambda^2(a_2 + b_2 + c_2) + \lambda(a_3 + b_3 + c_3) + (a_4 + b_4 + c_4) = 0.
\end{equation}

Therefore, the endemic equilibrium point $E^*$ is asymptotically stable with following conditions if $(R_1) (a_1 + b_1 + c_1) > 0$, $(a_4 + b_4 + c_4) > 0$ and $(a_1 + b_1 + c_1)(a_2 + b_2 + c_2)(a_3 + b_3 + c_3) - (a_1 + b_1 + c_3)^2 - (a_1 + b_1 + c_3^2)(a_4 + b_4 + c_4) < 0$ holds. Thus, according to the criteria of Routh-hurwitz all roots (29) have real negative values.

**Case 2.** When $r_1 = 0$ and $r_2 > 0$, (28) will become
\begin{equation}
\lambda^4 + \lambda^3(a_1 + b_1 + c_1) + \lambda^2(a_2 + b_2 + c_2) + \lambda(a_3 + b_3 + c_3) + (a_4 + b_4 + c_4) = 0.
\end{equation}

By assuming that for some values of $r_2 > 0$, there exist a real $\xi$ such that $\lambda = i\xi$ by putting $\lambda = i\xi$ after simplifying
\begin{align*}
\xi^4 - \xi^2(a_2 + b_2) + (a_4 + b_4) &= c_1 \xi^3 \sin(\xi \tau_2) + c_2 \xi^2 \cos(\xi \tau_2) \\
- c_3 \xi^3 \sin(\xi \tau_2) - c_4 \cos(\xi \tau_2), \\
(\xi(a_3 + b_3) - \xi^3(a_1 + b_1 + c_1) &= c_1 \xi^2 \cos(\xi \tau_2) - c_2 \xi^2 \sin(\xi \tau_2) \\
- c_3 \xi \cos(\xi \tau_2) + c_4 \sin(\xi \tau_2),
\end{align*}

squaring and adding both equations in (32)
\begin{equation}
v^4 + e_1 v^3 + e_2 v^2 - e_2 v + e_4 = 0, \xi^2 = v.
\end{equation}

Where the constants are as follows
\begin{align*}
e_1 &= (a_1 + b_1)(a_1 + b_1 + 2c_1) - 2(a_2 + b_2), \\
e_2 &= -2(a_1 + b_1) + (a_1 + b_1 + c_1) + (a_2 + b_2)^2 + 2(a_4 + b_4) - c_2^2 + 2c_1c_3, \\
e_3 &= (a_3 + b_3)^2 - 2(a_2 + b_2) + (a_4 + b_4) + c_1 + 2c_2c_4, \\
e_4 &= (a_4 + b_4)^2 - c_2^2.
\end{align*}

By rule of signs of Descartes (32) has at least on real positive root if $(D_2) c_1 > 0$ and $(a_4 + b_4)^2 < c_4^2$ holds.

Eliminating $\sin(\xi \tau_2)$ form the equations (31) we have
\begin{equation}
\tau_{2,j} = \frac{1}{\xi_0} \cos^{-1} \left( \frac{(E + \xi_0^2(c_2(a_4 + b_4) - c_3(a_3 + b_3) + c_4(a_2 + b_2))) - c_4(a_4 + b_4))}{c_1^2 \xi_0^2 + c_2^2 \xi_0^4 - 2c_1c_3 \xi_0^2 + c_1^2 \xi_0^2 - 2c_2c_4 \xi_0^2 + c_4^2} \right) + \frac{2j\pi}{\xi_0}.
\end{equation}
where \( E = \xi_0^4(c_2 - c_1(a_1 + b_1 + c_1)) + \xi_0^4(c_2(-a_2 + b_2)) + c_3(a_1 + b_1) + c_1(a_3 + b_3 + c_3) - c_4 \) and \( j = 0, 1, 2, \ldots \)

By differentiating (32) with respect to the \( \tau_2 \), transversality will be obtained as \( \tau_{2,j} = \tau_2 \) and \( \xi = \xi_0 \) that is

\[
\text{Re}(\frac{d\lambda}{d\tau_2})^{-1} = \frac{a_3 + b_3 + b_4 + c_1 - A_1 \xi_0^4 + A_2 \xi_0^2 + A_3 \xi_0^4}{\xi_0^4((\xi_0^2(-b_2 + c_2)) + b_4 + c_1^2) - \xi_0^4((\xi_0^2(b_1 + c_1) + b_3 + c_3)^2)},
\]

(34)

where:

\[
A_1 = 2(b_4 + c_1^2)(a_2 + b_2 + c_2),
\]

\[
A_2 = 2(b_2 + c_2)(a_2 + b_2 + c_2) + 3(b_1 + c_1)(a_1 + b_1 + c_1),
\]

\[
A_3 = 3(b_1 + c_1)(a_1 + b_1 + c_1).
\]

If \( \text{Re}(\frac{d\lambda}{d\tau_2})^{-1} = 0 \) a Hopf bifurcation will occur for delay \( \tau_2 \) we reached following theorems.

**Theorem 2.6.** Suppose that (R1) and (D1) is hold with delay \( \tau_1 = 0 \) and there exist a \( \tau_2 > 0 \) such that \( E^* \) remain stable for \( \tau_2 < \tau_2^* \) and unstable for \( \tau_2 > \tau_2^* \), where \( \tau_2^* = \text{min} [\tau_{2,j}] \) (33). Furthermore, model (1) undergoes the hopf bifurcation at point \( E^* \) when \( \tau_2 = \tau_2^* \).

**Case 3.** When \( \tau_2 > 0 \) and \( \tau_1 > 0 \) then equation (28) will be

\[
\lambda^4 + a_1 \lambda^2 + a_2 \lambda^2 + a_3 \lambda + a_4 + e^{-\tau \xi_1} \lambda^2(b_1 + c_1) + \lambda^2(b_2 + c_2) + \lambda(b_3 + c_3) + b_4 + c_4 = 0.
\]

(35)

We suppose for some values of \( \tau_1 \) the \( \lambda = i\xi \) we have two equations

\[
\xi^4 - a_2 \xi^2 + a_4 = \xi^2(1 - j \xi_1) + \xi^2(1 + j \xi_1) \cos(\xi \tau_1),
\]

\[
\cos(\xi \tau_1) = \frac{1}{\xi_0^4((\xi_0^2(-b_2 + c_2)) + b_4 + c_1^2) - \xi_0^4((\xi_0^2(b_1 + c_1) + b_3 + c_3)^2)},
\]

(36)

Squaring and adding both equations

\[
u^4 + f_1 \nu^2 + f_2 \nu^2 + f_3 \nu + f_4 = 0, \xi^2 = \nu.
\]

(37)

Where the coefficients are:

\[
f_1 = a_1^2 - 2a_2 - (b_1 + c_1)^2,
\]

\[
f_2 = a_1^2 - 2a_3 a_3 + 2a_4 - (b_2 + c_2)^2 + (b_1 + c_1)(b_3 + c_3),
\]

\[
f_3 = a_3^2 + 2a_2 a_4 - (b_3 + c_3)^2,
\]

\[
f_4 = a_2^2 - (b_4 + c_4)^2.
\]

Similarly to previous case, we arrived at the following theorem.

**Theorem 2.7.** Suppose that (R1) and (D2) is holds with delay \( \tau_2 = 0 \) and there exists a \( \tau_1 > 0 \) such that \( E^* \) is locally asymptotically stable for \( \tau_1 < \tau_1^* \) and unstable for \( \tau_1 > \tau_1^* \), where \( \tau_1^* = \text{min} [\tau_{1,j}] \). Furthermore, model (1) undergoes the hopf bifurcation which occur at point \( E^* \) when \( \tau_1 = \tau_1^* \).

\[
\tau_{1,j} = \frac{1}{\xi_0} \arccos\left[ \frac{B_1 - (-a_2 \xi^2 + a_4 + \xi^4)((\xi^2(b_2 + c_2) + b_4 + c_4)}{\xi^2((-b_2 + c_2)^2) + \xi^4((b_1 + c_1) - (b_2 + c_2))^2 + (b_4 + c_4)^2} \right] + \frac{2\pi j}{\xi_0}.
\]

(38)

Where \( B_1 = a_3 \xi_1^2 (\xi_0^2(b_1 + c_1) - b_2 - c_2^2) + a_4 \xi_1^4 \xi_0^2(\xi_0^2(b_1 + c_1) + b_3 + c_3) \) and \( j = 0, 1, 2, \ldots \)

**Case 4.** When \( \tau_1 > 0 \) and \( \tau_2 > 0 \) the equation (28) will be

\[
\lambda^4 + \lambda^2(a_1 + b_1 e^{-\tau_2 \xi_1} + c_1 e^{-\tau \xi_1}) + \lambda^2(a_3 + b_3 e^{-2\tau_2 \xi_1} + c_3 e^{-2\tau \xi_1}) + \lambda^2(a_2 + b_2 e^{-3\tau_2 \xi_1} + c_2 e^{-3\tau \xi_1}) + \lambda^2(a_1 + b_1 e^{-\tau \xi_1} + c_1 e^{-\tau \xi_1}) + \lambda^2(a_3 + b_3 e^{-2\tau \xi_1} + c_3 e^{-2\tau \xi_1}) + \lambda^2(a_2 + b_2 e^{-3\tau \xi_1} + c_2 e^{-3\tau \xi_1}) = 0.
\]

(39)

We suppose for some values of \( \tau_1 \) and \( \tau_2 \) there is a real number \( \xi \) we have two equations of real and imaginary values at \( \lambda = i\xi \).

\[
-a_2 \xi^2 + a_4 + \xi^4 = (b_1 \xi^2 - b_2 \xi) \sin(\xi \tau_1) - (b_4 - b_2 \xi)^2 \cos(\xi \tau_1)
\]

\[
+(c_1 \xi^3 - c_3 \xi) \sin(\xi \tau_1) + (c_1 \xi^3 - c_3 \xi) \sin(\xi \tau_1) - (c_4 - c_2 \xi^2) \cos(\xi (\tau_1 + \tau_2)) + (a_3 \xi - a_1 \xi^3) \cos(\xi (\tau_1 + \tau_2)) + (a_4 + b_4 e^{-\tau \xi_1} + c_4 e^{-\tau \xi_1}) \sin(\xi (\tau_1 + \tau_2))
\]

(40)

By squaring and adding both equations in (41). Applying Rouche’s Theorem we have

\[
u^4 + g_1 \nu^2 + g_2 \nu^2 + g_3 \nu + g_4 = 0, \xi^2 = \nu.
\]

(41)

Where the coefficients are:

\[
g_1 = a_1^2 - 2a_2 - 2b_1 c_1 \cos(\xi \tau_2) - b_1^2 - c_1^2,
\]

\[
g_2 = a_1^2 - 2a_3 a_3 + 2a_4 + 2b_2(Ac_1 + b_1^2) - 2b_2 c_1 \cos(\xi \tau_2)
\]

\[
+2c_1(b_1 \cos(\xi \tau_2) + c_1) - b_2 - c_2^2,
\]

\[
g_3 = a_3^2 - 2a_2 a_4 + 2b_2(2c_1 \cos(\xi \tau_2) + b_2) - 2b_2 c_1 \cos(\xi \tau_2) + 2c_4(b_2 \cos(\xi \tau_2) + c_2) - b_3 - c_3^2,
\]

\[
g_4 = a_2^2 - 2b_4 c_4 \cos(\xi \tau_2) - b_4^2 - c_4^2.
\]

By rule of signs of Descartes equation (41) has at least one positive real root if (D3) \( a_1^2 - 2a_2 > 2b_1 c_1 \cos(\xi \tau_2) + b_1^2 + c_1^2 \) and \( a_3^2 - 2b_4 c_4 \cos(\xi \tau_2) + b_4^2 + c_4^2 > 0 \) hold. By eliminating \( \sin(\xi \tau_1) \) we have

\[
\tau_{1,j} = \frac{1}{\xi_0} \cos^{-1}\left( \frac{G_2(a_1 \xi_1^3 - a_3 \xi_0^2 + G_1(-a_2 \xi_0^2 + a_4 + \xi_0^4)}{G_3 G_1 - G_2^2} \right) + \frac{2\pi j}{\xi_0}, \quad j = 0, 1, 2, \ldots
\]

(42)
Fig. 2. Left column $\eta > 0.5$, right column $\eta < 0.5$, delay $\tau_1 = 0, \tau_2 = 0$. Dynamics for $\sigma = 0.3$. 
Fig. 3. Left column $\eta > 0.5$, right column $\eta < 0.5$, delay $\tau_1 = 0$, $\tau_2 = 0$. Dynamics for $\sigma = 0.6$. 

$\psi = 0.09$, $\rho = 0.035$, $\eta = 0.6$

$\psi = 0.9$, $\rho = 0.035$, $\eta = 0.6$

$\psi = 0.09$, $\rho = 0.035$, $\eta = 0.3$

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Where
\[ G_1 = \xi_0 (c_1 \cos(\xi_0 t_1) - \xi_0 (c_2 \cos(\xi_0 t_2) + b_2 + c_4 \sin(\xi_0 t_2)) + c_4 \cos(\xi_0 t_2) + b_4). \]
\[ G_2 = \xi_0 (c_1 \xi_0 \cos(\xi_0 t_2) + b_1 \xi_0 - c_2 \sin(\xi_0 t_2)) - c_1 \cos(\xi_0 t_2) - b_1 + c_4 \sin(\xi_0 t_2). \]
\[ G_3 = (c_4 - c_2 \xi_0^2) \cos(\xi_0 t_2) + b_2 \xi_0^2 - b_4 + B_\xi_0(c_1 - c_2 \xi_0^2) \sin(\xi_0 t_2). \]

To study Hopf bifurcation we would fix \( \tau_2 \) in the stable interval and take derivative with respect to \( \tau_1 \) of equation (40) while using substitutions of \( \xi = \xi_0 \tau_1 = \tau_{1,0} \)

\[ P_1 \left( \frac{d(\tau_1)}{d\tau_1} \right) |_{\tau_1 = \tau_{1,0}} + P_2 \left( \frac{d\xi}{d\tau_1} \right) |_{\tau_1 = \tau_{1,0}} = P_3, \]
\[ -P_2 \left( \frac{d\xi}{d\tau_1} \right) |_{\tau_1 = \tau_{1,0}} + P_1 \left( \frac{d\xi}{d\tau_1} \right) |_{\tau_1 = \tau_{1,0}} = P_4. \]

Where
\[ C_1 = \cos(\xi_0 t_1). \]
\[ C_2 = \cos(\xi_0 t_2). \]
\[ S_1 = \sin(\xi_0 t_1). \]
\[ S_2 = \sin(\xi_0 t_2). \]
\[ P_1 = 2a_2 \xi_0 + C_1 (3b_2 \xi_0^2 + C_2 \tau_2 (c_1 \xi_0^2 - c_2 \xi_0)) - C_2 (-2c_2 \xi_0) + S_2 \tau_2 (c_4 - c_2 \xi_0^2) + S_1 (3c_1 \xi_0^2 - c_1)) \]
\[ + S_1 (3b_2 \xi_0^2 - b_2 + C_2 \tau_2 (c_1 \xi_0^2 - c_2 \xi_0)) + C_2 (3c_1 \xi_0^2 - c_1) - S_2 \tau_2 (c_1 \xi_0^2 - c_2 \xi_0) - S_2 (2c_2 \xi_0^2) - 4 \xi_0^3. \]
\[ P_2 = 3a_2 \xi_0^2 - a_3 + C_1 (3b_2 \xi_0^2 - b_3 + C_2 \tau_2 (c_4 - c_2 \xi_0^2) - C_2 (c_1 - 2c_2 \xi_0^2) + S_2 \tau_2 (c_1 \xi_0^2 - c_2 \xi_0^2) - S_2 (2c_2 \xi_0^2)) \]
\[ + S_1 (-2b_2 \xi_0^2 + C_2 \tau_2 (c_1 \xi_0^2 - c_2 \xi_0^2)) - S_2 \tau_2 (c_1 \xi_0^2 - c_2 \xi_0^2) + S_1 (3c_1 \xi_0^2 - c_1)) \]
\[ + S_1 (3b_2 \xi_0^2 - b_2 + C_2 \tau_2 (c_1 \xi_0^2 - c_2 \xi_0^2)) + S_2 (c_4 - c_2 \xi_0^2) \]
\[ -C_1 \xi_0 (b_2 (-\xi_0^2) + b_3 + C_2 (c_4 - c_2 \xi_0^2)) + S_2 (c_1 \xi_0^2 - c_2 \xi_0^2) \]
\[ -C_2 \xi_0 (b_2 (-\xi_0^2) + b_3 + C_2 (c_4 - c_2 \xi_0^2)) + S_2 (c_1 \xi_0^2 - c_2 \xi_0^2) \]
\[ -C_1 \xi_0 (b_2 (-\xi_0^2) + b_3 + C_2 (c_4 - c_2 \xi_0^2)) + S_2 (c_1 \xi_0^2 - c_2 \xi_0^2). \]

We obtain
\[ \frac{d(\lambda)}{d\tau_1} = P_3 P_2 - P_3 P_4. \]

Hopf bifurcation will occur for \( \tau_1 = \tau_{1,0} \) if \( \frac{d(\lambda)}{d\tau_1} > 0 \).

**Theorem 2.8.** If \( E^* \) is existent, such that \( R_1 \) and \( D_1 \) hold, with \( \tau_1 = 0 \) and \( \tau_1 \in (0, \tau_2) \), there is an existent positive parameter \( \tau_1^* \) such that endemic equilibrium point is locally stable for \( \tau_2 < \tau_2^* \) and unstable for \( \tau_2 > \tau_2^* \), where \( \tau_2^* = \min\{\tau_2, j\} \) as in (42). Moreover, model (1) will undergoes hopf bifurcation at point \( E^* \) when \( \tau_2 = \tau_2^* \).

**Note:** Similarly, For \( \tau_2 \in (0, \tau_2^*) \), there is exists threshold parameter \( \tau_1^* \) such that endemic equi point is locally asymptotically a stable for \( \tau_1 < \tau_1^* \) and unstable if \( \tau_1 > \tau_1^* \). Moreover, hopf bifurcation occur for model (1) as \( \tau_1 = \tau_1^* \), where \( \tau_1^* = \min\{\tau_1, j\} \) that is

\[ \tau_{2,j} = \frac{1}{\xi_2} \cos^{-1} \left( \frac{\alpha_1 (c_2 \xi_2^2 - c_4) + \alpha_1 \xi_2 (c_1 \xi_2^2 - c_1)}{\xi_2^2 (c_1 \xi_2^2 - c_1) + (c_2 \xi_2^2 - c_4)^2} \right) + \frac{2j\pi}{\xi_2}, \quad j = 0, 1, 2, \ldots \]

Where
\[ \alpha_1 = \xi_2 (a_1 \cos(\xi_2 \tau_1) - b_1 \xi_2^2 + b_3) - a_1 \xi_2^2 \cos(\xi_2 \tau_1) + (a_4 + \xi_2^2) \sin(\xi_2 \tau_1) - a_2 \xi_2^2 \sin(\xi_2 \tau_1). \]
\[ \alpha_2 = b_4 - \xi_2 (a_1 s_2 + b_2 \xi_2) + (a_4 + \xi_2^2) \cos(\xi_2 \tau_1) - a_2 \xi_2^2 \cos(\xi_2 \tau_1) + a_1 \xi_2^2 \sin(\xi_2 \tau_1). \]
We thus emphasize on the fact that it is not only important to control the interaction of infected people with the susceptible, but it is really necessary to discover a drug, which will reduce the probability of cyclic outbreak of the disease, where the recovered individuals will have fewer chances of becoming infectious again.

Dynamics with delay

Next, we have run the numerical experiments, based on the stability analysis. We can see from Figs. 4 and 5 that, different dynamics were revealed relative to different combinations of the delays.

For lower and equal delays in different compartments, there was less delay in the onset of infection, whereas, for higher values of delay, there were higher intervals of delay in the onset of infection.

4. Conclusions

Over the past year, different mathematical models have been proposed in literature to highlight the importance of social distancing, as a precautionary tool, to control the spread of the novel corona virus. In this manuscript, important agent based (each individual was treated as an agent), strategy is adopted and the machine learning tool of self organizing maps is used to explore the parameters of the resulting mathematical model.

In this research, we conclude that it is not only important to emphasize on the isolation of susceptible individuals in a population, it is also necessary to control the interaction among the infected, recovered and asymptomatic individuals. We have verified this hypothesis with the aid of numerical simulations.

The outcome of isolation and lockdown, can delay and reduce the disease transmission, but it is not the permanent solution to the problem, since the social distancing, isolation and lockdown have a great influence on the world’s economy and is becoming a cause of the socioeconomic crisis, of the third world countries.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.chaos.2021.111202.

CRediT authorship contribution statement

Zhenhua Yu: Conceptualization, Data curation. Robia Arif: Formal analysis. Mohamed Abdelsalboum Fahmy: Conceptualization. Ayesha Sohail: Conceptualization, Supervision.

References

[1] Dan JM, Mateus J, Kato Y, Hastie KM, Yu ED, Faliti CE, Grifoni A, Ramirez SI, Haupt S, Frazier A, et al. Immunological memory to SARS-cov-2 assessed for up to 8 months after infection. Science 2021.
[2] Ozono S, Zhang Y, Ode H, Sano K, Tan TS, Imai K, Miyoshi K, Kishigami S, Ueno T, Iwata Y, et al. Sars-cov-2 d614g spike mutation increases entry efficiency with enhanced ace2-binding affinity. Nature Commun 2021;12(1):1–9.
[3] Clouston SAP, Natale G, Link BG. Socioeconomic inequalities in the spread of coronavirus-19 in the united states: a examination of the emergence of social inequalities. Soc Sci Med 2021;268:113554.
[4] Meyers C, Robison R, Milici J, Alam S, Quillen D, Goldenberg D, Kass R. Lowering the transmission and spread of human coronavirus. J Med Virol 2021;93(3):1605–12.
Yu, R. Arif, M.A. Faheem et al. \(\text{Chaos, Solitons and Fractals 150 (2021) 111202}\)

[5] Mahmoudi MR, Baleanu D, Mansor Z, Tuan BA, Pho K-H. Fuzzy clustering method to compare the spread rate of covid-19 in the high risks countries. Chaos Soliton Fract 2020;140:110230.

[6] Roda WC, Varughese MB, Han D, Li MY. Why is it difficult to accurately predict the COVID-19 epidemic? Infect Dis Model 2020.

[7] Din A, Khan A, Baleanu D. Stationary distribution and extinction of stochastic coronavirus (COVID-19) epidemic model. Chaos Soliton Fractal 2020;139:110036.

[8] Sohail A, Nutini A. Forecasting the timeframe of coronavirus and human cells interaction with reverse engineering. Progr Biophys Mol Biol 2020.

[9] Awaerter PG. Coronavirus COVID-19 (SARS-2-cov). Johns Hopkins AHEX Guide 2020.

[10] Nikolich-Zugich J, Knox KS, Rios CT, Natt B, Bhattacharya D, Fain MJ. Sars-cov-2 and covid-19 in older adults: what we may expect regarding pathogenesis, immune responses, and outcomes. Gerontology 2020;1–10.

[11] Chen W-C. Nonlinear dynamics and chaos in a fractional-order financial system. Chaos Soliton Fractal 2008;36(5):1305–14.

[12] Baleanu D, Jajarmi A, Mohammadi H, Rezapour S. A new study on the mathematical modelling of human liver with caputo–fabrizio fractional derivative. Chaos Solitons Fractal 2020;134:109705.

[13] Du H, Zeng Q, Wang C. Modified function projective synchronization of chaotic system. Chaos Soliton Fractal 2009;42(4):2399–404.

[14] Holden AV, Biktashev VN. Computational biology of propagation in excitable media models of cardiac tissue. Chaos Soliton Fractal 2002;13(8):1643–58.

[15] Iftikhar M, Iftikhar S, Sohail A, Javed S. Ai-modelling of molecular identification and feminization of wobbachia infected aedes aegypti. Progr Biophys Mol Biol 2020;150:104–11.

[16] Baleanu D, Mohammadi H, Rezapour S. A fractional differential equation model for the COVID-19 transmission by using the caputo–fabrizio derivative. Adv Diff Eqs 2020;2020(1):1–27.

[17] Baleanu D, Mohammadi H, Rezapour S. A mathematical theoretical study of a particular system of caputo–fabrizio fractional differential equations for the rubella disease model. Adv Diff Eqs 2020;2020(1):1–17.

[18] Baleanu D, Mohammadi H, Rezapour S. Analysis of the model of HIV-1 infection of CD4+ CD4\(^+\) t-cell with a new approach of fractional derivative. Adv Diff Eqs 2020;2020(1):1–17.

[19] Blackwood JC, Childs LM. An introduction to compartmental modeling for the budding infectious disease modeler. Lett Biomath 2018;5(1):195–221.

[20] Yu Zhenhuan, et al. Journal of Molecular Liquids 2021. https://www.sciencedirect.com/science/article/pii/S0167732220371051?casa_token=NCU80gz18p8AAAAA:m7HBFWhCrK-iC1nLNVTN1p4aHYiHIq8idUK74VCciZAPRxPvv2FLrnXst8XnhdBp7s58txq2w. doi:10.1016/j.molliq.2020.114863. In press.

[21] Yu Zhenhuan, et al. Front. Mol. Biosci 2021. https://www.frontiersin.org/articles/10.3389/fmolb.2020.585245/full.

[22] Al-Utaibi KA, et al. results in physics 2021. https://www.sciencedirect.com/science/article/pii/S2211379721004174.