Note

The definition of mass in asymptotically de Sitter space-times

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Abstract
The definition of invariant quantities due to Wald and collaborators is used to calculate the mass of a black-hole in asymptotically de Sitter space-time. The construction relies on the existence of a time-like Killing vector on a sphere surrounding the mass but does not require going to an asymptotic region, in particular the mass can be calculated exactly on a sphere inside the cosmological horizon. The formalism requires varying the background metric solution by a perturbation that satisfies the linearized equations of motion but need not share the Killing symmetry of the solution and is therefore applicable when a gravitational wave within the cosmological horizon perturbs a background metric with a time-like Killing vector.

Keywords: general relativity, mass, black holes, symmetries, asymptotically de Sitter space-time

1. Introduction
It is not straightforward to give an invariant definition of mass in an asymptotically de Sitter space-time containing mass in a compact region, due to problems arising from the existence of a cosmological event horizon. This is an old question that has been addressed by many authors, for example [1–6] among others. It is well known that the Komar mass [7] gives an ambiguous answer: if the Komar two-form associated with rotations is normalised so as to give the same angular momentum as that of the Arnowitt–Deser–Misner (ADM) formalism [8] then the two-form associated with time translations gives a value for the mass that differs from the ADM mass by a factor of two. This particular problem was resolved by Wald in [9] where he showed that there is an extra contribution to the Komar mass which rectifies this
A comprehensive review of quasi-local definitions of conserved quantities in general relativity is conspicuously silent about asymptotically de Sitter space-times [10].

An approximate definition, that works well provided any black-hole horizon $r_{BH}$ is very much smaller that than the cosmological horizon $r_C$, was given in [1]. In that work it is assumed that there is a region $r_{BH} \ll r < r_C$ in which the full de Sitter group $SO(1, 4)$ is an approximate symmetry and a time-like generator is used to define the energy density, from which a mass is obtained by integrating over a two-sphere of radius $r$. The resulting mass is a good candidate provided corrections of order $r_{BH}$, but still with $r < r_C$, can be ignored. Using the formalism of Wald and collaborators [9, 11, 12], which was developed further in [13], we give in the following an exact definition of mass in asymptotically de Sitter space-times which only requires the existence of one Killing vector at some $r$ with $r_{BH} < r < r_C$, but there is no approximation requiring $r \gg r_{BH}$ and the full $SO(1, 4)$ symmetry is not necessary—one time-like Killing vector is sufficient.

The basic problem is most easily appreciated by examining a static asymptotically de Sitter Schwarzschild black hole with line element

\[ ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \]  

(1)

for which the Killing vector $\partial_t$ is space-like for $r > r_C$, where $r_C$ is the cosmological horizon—the largest root of the cubic equation

\[ \Lambda r^3 - 3r + 6m = 0. \]  

(2)

The most widely accepted definition of mass in general relativity involves identifying an asymptotically time-like Killing vector at either spatial infinity [8, 14] or light-like infinity [15] and evaluating the energy over a two-sphere there, but there is no such time-like Killing vector for an asymptotically de Sitter black hole. The same difficulty applies to definitions using Yano tensors [16].

This is particular vexing as the cosmological constant is measured to be positive [17] so in principle we do not have a rigorous definition of mass for a black hole in our Universe (though the observed $\Lambda$ is so small that the definition in [1] should suffice for all practical purposes for any known astrophysical black hole). Nevertheless it would be gratifying to have a more mathematically rigorous definition.

In this work we apply Wald and collaborators’ definition of mass to a black hole in de Sitter space-time that does not rely on taking $r \to \infty$, all that is necessary is that at some value of $r < r_C$ there is a time-like Killing vector in a region of space-time containing a two-sphere surrounding the mass. The definition is in essence very like Gauss’ law in electrostatics, though in detail it is a lot more involved. The mass can be calculated by integrating over a sphere of any radius as long as it completely surrounds the mass and is in a region with a time-like Killing vector—this is a quasi-local definition in the same spirit as the Komar two-form approach but, as mentioned earlier, Wald’s definitions had an extra term that rectifies the 50% deficit of the Komar approach. This observation has been made before and the simplest asymptotically de Sitter example is Schwarzschild–de Sitter space-time, where the calculation is easy can be done analytically for any radius $r$ and explicitly shown to be independent of $r$. A more complete calculation for a rotating black-hole in asymptotically de Sitter space-time is much more involved and is presented here for the Kerr–de Sitter metric for the first time. For technical reasons the calculation can only be done analytically as $r \to \infty$, but the general formalism ensures that the same answer would be obtained for finite $r < r_C$ were it possible to push it through analytically.
The construction relies on the work of Wald and collaborators [9, 11, 12] in which a Noether form associated with a Killing vector was identified which can be used to give a Noether charge associated to the symmetry generated by the Killing vector. The definition of the charge involves perturbing the background metric and the metric perturbation need not share the Killing symmetry of the background metric, it could for example correspond to a gravitational wave in a region where the background metric has a time-like Killing vector. For realistic values of the cosmological constant numerical calculations could be performed in a region where \( r \) is large enough for \( \frac{\delta g}{r^2} \) to be small but still \( r < r_C \), where \( \delta g \) is the metric variation. It was shown in [18] that Wald’s formalism leads to the canonically accepted Henneaux–Teitelboim mass for the asymptotically anti-de Sitter–Kerr metric and the analysis here can be succinctly summarized in the statement that the Wald mass of the asymptotically de Sitter–Kerr metric is simply the analytic continuation of [18] from negative to positive \( \Lambda \).

A definition of mass for black-holes in asymptotically de Sitter space-time was given in [19], where a regularised action with boundary counter-terms is used to define the mass. This definition lies withing the Iyer–Wald scheme though in space-time dimensions greater than 4 it requires an extra surface term over and above the usual formulation of general relativity that introduces the intrinsic curvature of the boundary into the action as well as the more usual extrinsic curvature.

2. The Iyer–Wald invariant mass

For a 4D space-time \( \mathcal{M} \) with metric \( g_{\mu\nu} \) and co-ordinates \( x^\alpha \) we foliate \( \mathcal{M} \) with constant time hypersurfaces and let \( x^\alpha = (t, x^\alpha) \) where \( \alpha = 1, 2, 3 \) and \( t \) is a time co-ordinate. We use the standard ADM decomposition: \( t = \text{const} \) are space-like hypersurfaces, \( \Sigma_t \), and we denote the induced metric on \( \Sigma_t \) by \( h_{\alpha\beta}(t) \). The 4D line element decomposes as

\[
\begin{align*}
\text{ds}^2 &= g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu = g_{tt}^2 + 2g_{t\alpha} \text{d}t \text{d}x^\alpha + g_{\alpha\beta} \text{d}x^\alpha \text{d}x^\beta \\
&= -N^2 \text{d}t^2 + h_{\alpha\beta}(\text{d}x^\alpha + N^\alpha \text{d}t)(\text{d}x^\beta + N^\beta \text{d}t),
\end{align*}
\]

(3)

where \( g_{tt} = -N^2 + h_{\alpha\beta}N^\alpha N^\beta \), \( g_{t\alpha} = g_{\alpha\beta}N^\beta \) and \( h_{\alpha\beta} = g_{\alpha\beta} \).

We shall employ differential form notation using orthonormal one-forms \( e^a \) for the metric \( g \), which can be expressed in a co-ordinate basis as

\[
e^\mu = e^\mu_\alpha \text{d}x^\alpha.
\]

The connection one-forms are determined by the torsion-free condition

\[
De^a = de^a + \omega^a_b \wedge e^b = 0
\]

and the curvature two-forms are

\[
R_{ab} = \frac{1}{2}R_{abcd}e^c \wedge e^d = d\omega_{ab} + \omega_a^c \wedge \omega_c^b
\]

where orthonormal indices are raised and lowered using \( \eta^{ab} = \eta_{ab} = \text{diag}(-1, +1, +1, +1) \).

Under the above foliation denote orthonormal one-forms for \( h_{\alpha\beta} \) by

\[
e^\alpha = e^\alpha_i \text{d}x^i,
\]

with \( i = 1, 2, 3 \), so \( e_{\alpha} = e^{\alpha}_i \) with
\[ e^0 = Nd t \quad \text{and} \quad e^i = \tilde{e}^i + \frac{N_i}{N} e^0, \]

and \( N^i = e^\alpha N^\alpha \) the orthonormal components of the shift vector. The connection one-forms associated with \( \tilde{e}^\alpha \) on \( \Sigma_t \) are defined using the zero torsion condition

\[ \tilde{d} \tilde{e}^i + \tilde{\omega}^i_j \wedge \tilde{e}^j = 0 \]

with \( \tilde{d} = \tilde{e}^i \partial_i \) the exterior derivative on \( \Sigma_t \) at constant \( t \).

In this gauge

\[ e^\mu_a = \left( \begin{array}{cc} N & 0 \\ N^i & \tilde{e}^i \beta \end{array} \right), \quad (e^{-1})^\alpha_a = \left( \begin{array}{cc} \frac{1}{N^\alpha} & 0 \\ -N e^{\alpha} & (\tilde{e}^{-1})^\alpha \end{array} \right) \]

and the unit vector normal to \( \Sigma_t, \tilde{n} \), has orthonormal components \( n^\mu = (1,0,0,0) \) so the metric dual one-form is \( n = n_a e^a = -\delta^0 \).

Metric variations are described by variations in the tetrad \( \delta e^\mu = \delta e^\mu_a d\kappa^a \) which can be written into the square matrix

\[ \Delta^a_b = (\delta e^\mu_a) (e^{-1})^\mu_b = \left( \begin{array}{cc} \frac{\Delta N}{N} & 0 \\ \tilde{\Delta}^i_j \end{array} \right), \]

with \( \Delta^i_j = (\delta \tilde{e}^\alpha_a) (\tilde{e}^{-1})^\alpha_j \). \( \Delta^i_j \) can be decomposed into symmetric and anti-symmetric parts

\[ S_{ij} = \Delta_{\{ij\}} = \frac{1}{2} (\Delta_{ij} + \Delta_{ji}), \quad A_{ij} = \Delta_{[ij]} = \frac{1}{2} (\Delta_{ij} - \Delta_{ji}). \]

Let \( S = S_f \) be the trace of \( S_{ij} \) and \( \kappa_{ij} \) be the extrinsic curvature\(^1\) of \( \Sigma_t \) and \( \kappa = \kappa_i \) its trace. For the Einstein action with a cosmological constant,

\[ S = \int_\mathcal{M} (R_{ab} \wedge \ast (e^a \wedge e^b) - 2 \Lambda \ast 1), \]

it was shown in [20] that, if \( \vec{K} = \frac{\tilde{n}}{n} \) is Killing and \( \Sigma_t \) can be foliated into 2D spheres \( S^2 \mid_{\Sigma_t} \) parameterized by \( r \), then the variation of Wald’s charge is

\[ \delta Q[\vec{K}] = \frac{1}{8\pi} \int_{S^2 \mid_{\Sigma_t}} \left\{ N (\tilde{D}_j S_i - \partial_i S) + (\partial_i N) S - (\partial_j N) S_j^l + X_{ij} N^l - N_l (\kappa_{jk} S^k + \kappa S) \right\} \ast e^0, \]

where \( \tilde{D}_j \) is the co-variant derivative associated with the orthonormal one-forms \( \tilde{e}^i \) (here

\[ X_{ij} = \delta \kappa_{ij} + [\kappa, \Delta]_{ij} + \kappa \Delta S \]

and \( [\kappa, \Delta]_{ij} \) is the commutator of the matrices \( \kappa_{ij} \) and \( \Delta_{ij} \)). \( \delta Q[\vec{K}] \) is guaranteed to be independent of \( r \) and \( \tilde{Q}[\vec{K}] \) is the mass contained within \( S^2 \mid_{\Sigma_t} \). For asymptotically flat space-times this corresponds to the ADM mass when \( r \to \infty \) with \( t \) fixed [12] and it gives the Bondi mass when \( r \to \infty \) and \( t \to \infty \) with \( (t-r) \) fixed [20].

A crucial observation is that it is not necessary to take the asymptotic limit as long as the perturbation satisfies Einstein’s equations and the two-surface \( S^2 \mid_{\Sigma_t} \) lies in a region where \( \vec{K} \) is Killing [18, 20]—it is not even necessary that the perturbation has the Killing symmetry.

\(^1\)In the gauge (6) \( \kappa_q = \frac{1}{2} (D_i n_j + D_j n_i) \).
2.1. Asymptotically de Sitter stationary black holes

The line element outside a rotating black hole in de Sitter space-time is [21]

\[
\frac{ds^2}{\rho^2} = -\frac{\Delta}{\rho^2} \left( dt - \frac{a \sin^2 \tilde{\vartheta}}{\Xi} d\tilde{\varphi} \right)^2 + \rho^2 \left( \frac{dr^2}{\Delta} + \frac{d\tilde{\varphi}^2}{\Xi} \right) + \frac{\Xi \tilde{\rho}^2}{\rho^2} \left( \frac{\tilde{r}^2 + a^2}{\Xi} d\tilde{\varphi} - adt \right)^2 \tag{10}
\]

with

\[
\Delta = \frac{(\tilde{r}^2 + a^2)(L^2 - \tilde{r}^2)}{L^2} - 2m\tilde{\rho}, \quad \Xi = 1 + \frac{a^2}{L^2} \cos^2 \tilde{\vartheta},
\]

\[
\tilde{\rho}^2 = \tilde{r}^2 + a^2 \cos^2 \tilde{\vartheta} \quad \text{and} \quad \Xi = 1 + \frac{a^2}{L^2}. \tag{11}
\]

This can be decomposed into a pure de Sitter part and a part that vanishes when \( m = 0 \),

\[
ds^2 = ds^2_{\text{dS}} + ds^2_m
\]

where

\[
ds^2_{\text{dS}} = -\left( 1 - \frac{\tilde{r}^2 + a^2 \sin^2 \tilde{\vartheta}}{L^2} \right) dt^2 - 2a \left( \frac{\tilde{r}^2 + a^2}{L^2} \right) \frac{\sin^2 \tilde{\vartheta}}{\Xi} d\tilde{\varphi} d\tilde{t}
+ \frac{L^2 \tilde{\rho}^2}{(\tilde{r}^2 + a^2)(L^2 - \tilde{r}^2)} dt^2 + \frac{\tilde{\rho}^2 d\tilde{\varphi}^2}{\Xi} + (\tilde{r}^2 + a^2) \frac{\sin^2 \tilde{\vartheta}}{\Xi} d\tilde{\varphi}^2, \tag{12}
\]

\[
ds^2_m = \frac{2m\tilde{\rho}}{\rho^2} \left( \left( \frac{\tilde{r}^2 + a^2}{L^2} \right) \sin^2 \tilde{\vartheta} d\tilde{t} d\tilde{\vartheta} + \frac{\sin^4 \tilde{\vartheta}}{\Xi^2} d\tilde{\varphi}^2 \right)
+ \frac{2m\tilde{\rho}^2 L^4}{(\tilde{r}^2 + a^2)(L^2 - \tilde{r}^2)(\tilde{r}^2 + a^2)(L^2 - \tilde{r}^2) + 2m\tilde{\rho} L^2} d\tilde{r}^2. \tag{13}
\]

Despite appearances (12) is the dS Sitter metric, but not in standard co-ordinates. The co-ordinate transformation that puts it into a more standard form was given in [22]: with

\[
t = \tilde{t}, \quad \varphi = \tilde{\varphi} - \frac{a\tilde{t}}{L}, \quad r^2 \cos^2 \vartheta = \tilde{r}^2 \cos^2 \tilde{\vartheta}, \quad r^2 \sin^2 \vartheta = (\tilde{r}^2 + a^2) \frac{\sin^2 \tilde{\vartheta}}{\Xi}.
\]

Some useful relations are

\[
(L^2 - \tilde{r}^2) \Xi = (L^2 - r^2) \Xi \tag{14}
\]

\[
\left( \frac{\tilde{r}^2 + a^2}{\tilde{r}^2} \right) \tan^2 \tilde{\vartheta} = \Xi \tan^2 \vartheta. \tag{15}
\]

One finds that (12) is the more familiar
\[ ds^2 = - \left(1 - \frac{r^2}{L^2}\right) dt^2 + \frac{1}{1 - \frac{r^2}{L^2}} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \]

It is not illuminating to write \( ds^2 \) in \((t, r, \vartheta, \varphi)\) co-ordinates in general but we shall need its asymptotic form for \( \tilde{r} \gg L \) (and \( r \gg L \)). Let
\[ \Sigma^2 = 1 + \frac{a^2}{L^2} \sin^2 \vartheta \]
then some useful formulae for the deriving the asymptotic form of \( ds^2 \) in \((t, r, \vartheta, \varphi)\) co-ordinates are
\[ \Xi_{\tilde{r} \vartheta} = \left( \frac{L^2 - r^2}{L^2 - \tilde{r}^2} \right) \Xi = \Xi \Sigma^2 + O \left( \frac{1}{r^2} \right) \]
\[ \tan^2 \tilde{\vartheta} = \Xi \tan^2 \vartheta + O \left( \frac{1}{r^2} \right) \]
\[ \tilde{r}^2 = \Sigma^2 r^2 + O(1) \]
\[ \cos^2 \tilde{\vartheta} = \frac{\cos^2 \vartheta}{\Sigma^2} + O \left( \frac{1}{r^2} \right) \]
\[ \sin^2 \tilde{\vartheta} = \Xi \left( \frac{\sin^2 \vartheta}{\Sigma^2} \right) + O \left( \frac{1}{r^2} \right) \].

Using these one finds
\[ \left( \frac{dr}{r d\tilde{r}} \right) = \left( \frac{\Sigma_{\tilde{r} \vartheta} a^2 \cos \vartheta \sin \vartheta}{0 \ rac{L^2 \Sigma_{\tilde{r} \vartheta}}{\Sigma^2} \right) \left( \frac{dr}{rd\vartheta} \right) + O \left( \frac{1}{r^2} \right) \]
and the leading terms in \( ds^2 \) are
\[ ds^2 = \frac{2m}{r \Sigma^2} (dr - a \sin^2 \vartheta d\varphi)^2 + \frac{2L^4}{r^5 \Sigma^2} dr^2 + \cdots. \] (16)

In the time-gauge the vierbeins are of the form (6) with leading order terms
\[
e^a_\mu = \begin{pmatrix}
  \tilde{r} \sqrt{1 - \frac{r^2}{L^2} - \frac{ma}{r \Sigma^2}} & 0 & 0 & 0 \\
  0 & \frac{1}{r} \sqrt{1 - \frac{r^2}{L^2} + \frac{ma}{r \Sigma^2}} & 0 & 0 \\
  0 & 0 & r + \frac{ma^2 L^2 \sin^2 \vartheta}{r^3 \Sigma^2} & 0 \\
 -\frac{2ma \sin \vartheta}{r \Sigma^2} & 0 & 0 & \left(1 + \frac{ma^2 \sin^2 \vartheta}{r^3 \Sigma^2} \right) r \sin \vartheta \\
\end{pmatrix} + \cdots.
\] (17)

In particular
\[ N = \frac{r}{L} \sqrt{1 - \frac{L^2}{r^2} - \frac{mL}{r^2 \Sigma^2}} + O \left( \frac{1}{r^3} \right). \]

The \( mlr^4 \) term in \( e^2_r \) is retained because we define \( f(r, \vartheta) \) via
\[ e^1_r = \frac{1}{f}. \]
with
\[ f = \frac{r}{L} \sqrt{1 - \frac{L^2}{r^2}} - \frac{mL}{r^3 \Sigma_0} + O \left( \frac{1}{r^2} \right), \]
while the \( \frac{m}{r^4} \) term in \( e^2 \phi \) does not affect the subsequent analysis and can be discarded.

To order \( \frac{1}{r} \) the connection one-forms are\(^2\)

\[
\begin{align*}
\omega_{01} &= -\frac{\partial N}{N} f e^0, \\
\omega_{02} &= -\frac{\partial N}{r} e^0, \\
\omega_{12} &= -\frac{\partial f}{r} e^1 - f e^2, \\
\omega_{23} &= -\frac{\cot \theta}{r} e^3.
\end{align*}
\]

Hence asymptotically\(^3\)
\[
N \to \frac{L}{r}, \quad f \to \frac{r}{L}, \\
\omega_{01,0} \to -\frac{1}{L}, \quad \omega_{12,2} \to -\frac{1}{L}, \quad \omega_{13,3} \to -\frac{1}{L}.
\]

We will now evaluate (8) using this asymptotic behaviour. We have \( \partial_i = f \frac{\partial}{\partial r} + O \left( \frac{1}{r^2} \right) \) and, although \( \partial_i N \sim 1/L \), for transverse derivatives \( \partial_i N \sim O \left( \frac{1}{r} \right) \) and \( \omega_{1,1} \sim O \left( \frac{1}{r^3} \right) \). One finds
\[
\int_{S_2} \left\{ N \left( \bar{D}_i S_i' \right) - \partial_i (S_i') \right\} + \partial_i N \left( n^i S_j' - n^i S_j \right) \right\} r^2 \hat{e}^{23}
\]
\[
= \int_{S^2} \left\{ Nf \left( \frac{2}{r} S_{11} - \frac{1}{r} S_\perp - (S_\perp)' \right) + N' S_\perp \right\} r^2 \hat{e}^{23} + O \left( \frac{1}{r} \right)
\]
where \( S_\perp = S_{22} + S_{33} \) is the transverse trace of \( S_{ij} \) and
\[
\hat{e}^{23} = \sin \vartheta d\vartheta \wedge d\phi
\]
is the volume form on the unit sphere. Now
\[
Nf = -\frac{r^2}{L^2} + 1 - \frac{m}{r} \left( \frac{1}{\Sigma_0} + \frac{1}{\Sigma_0} \right) + O \left( \frac{1}{r^2} \right),
\]
\[
N'f = -\frac{r}{L^2} + O \left( \frac{1}{r^2} \right)
\]
so
\[
\int_{S^2} \left\{ Nf \left( \frac{2}{r} S_{11} - \frac{1}{r} S_\perp - (S_\perp)' \right) + N' S_\perp \right\} r^2 \hat{e}^{23}
\]
\[
= \int_{S^2} \left\{ \left( 1 - \frac{r^2}{L^2} \right) \frac{2}{r} S_{11} - (S_\perp)' \right\} r^2 \hat{e}^{23} + O \left( \frac{1}{r} \right)
\]
Now we can expand \( S_{11} \) and \( S_\perp \) in inverse powers of \( r \) as

\(^2\)We stress that, as long as \( r \) lies outside of the region containing the mass, Wald’s formalism ensures that the answer is independent of \( r \). The \( r \to \infty \) limit is only being used here as a technical device to push the calculation through analytically.
\[ S_\perp = \sum_{n=1}^{\infty} \frac{b_n(\theta, \varphi)}{r^n}, \]
\[ S_{11} = \sum_{n=1}^{\infty} \frac{c_n(\theta, \varphi)}{r^n} \]

then
\[
\int_S \left\{N_f \left( \frac{2}{r} S_{11} - \frac{1}{r} S_\perp - (S_\perp')' \right) + N_r' S_\perp \right\} r^2 \varepsilon^{23} = - \int_S \left\{ \frac{(b_1 + 2c_1)}{L^2} + \frac{2(b_2 + c_2)}{L^2 r} - \frac{2c_1}{r^2} + \frac{(3b_3 + 2c_3)}{L^2 r^2} \right\} r^2 \varepsilon^{23} + O \left( \frac{1}{r^7} \right). \tag{18}
\]

Averaging over the sphere let \( \bar{b}_1 = \frac{1}{4\pi} \int_{S^2} b_1(\theta, \varphi) \sin \vartheta d\vartheta d\varphi \) and \( \bar{c}_1 = \frac{1}{4\pi} \int_{S^2} c_1(\theta, \varphi) \sin \vartheta d\vartheta d\varphi \) then, for a finite expression in (18) as \( r \to \infty \), we must demand that
\[ \bar{b}_1 + 2\bar{c}_1 = 0 \quad \text{and} \quad \bar{b}_2 + \bar{c}_2 = 0. \]

If we wish the deformation of the area of the sphere at infinity to remain finite we must further demand that \( \bar{b}_1 = 0 \), so \( \bar{c}_1 = 0 \) (if we wish the area to be invariant we impose the stronger restriction \( \bar{b}_1 = \bar{b}_2 = 0 \Rightarrow \bar{c}_1 = \bar{c}_2 = 0 \)).

In any case we finally arrive at
\[
\lim_{r \to \infty} \delta Q[e^\alpha, \mathcal{L}_R e^\alpha, \delta e^\alpha] = 2\bar{c}_1 - \frac{1}{2L^2}(3\bar{b}_3 + 2\bar{c}_3). \tag{19}
\]

For example if we vary the parameters \( m \to m + \delta m \) and \( a \to \delta a \) in the original metric (17), keeping \( L \) fixed, then \( e^1 = \frac{1}{r} dr \) and
\[
S_{11} = -\frac{\delta f}{f} = -\frac{1}{2} \frac{\delta f^2}{f^2} = \frac{L^2}{r^2} \frac{m}{r \Sigma^3_0} + O \left( \frac{1}{r^7} \right) = -\frac{L^2}{r^2} \delta \left( \frac{m}{\Sigma^3_0} \right) + O \left( \frac{1}{r^7} \right),
\]
with
\[
S_\perp = \frac{1}{r \sin \theta} \delta \left( \frac{ma^2 \sin^2 \theta}{r^2 \Sigma^3_0} \right) + O \left( \frac{1}{r^7} \right) = \frac{1}{r} \delta \left( \frac{ma^2}{\Sigma^5_0} \right) + O \left( \frac{1}{r^7} \right)
\]
so
\[
c_1 = 0, \quad b_3 = \sin^2 \theta \delta \left( \frac{ma^2}{\Sigma^5_0} \right) \quad \text{and} \quad c_3 = -L^2 \delta \left( \frac{m}{\Sigma^3_0} \right).
\]

(19) is therefore
\[
\lim_{r \to \infty} Q[e^\alpha, \mathcal{L}_R e^\alpha, \delta e^\alpha] = \frac{1}{4\pi} \int_{S^2} \left\{ 2 \delta \left( \frac{m}{\Sigma^3_0} \right) - \frac{3 \sin^2 \theta}{L^2} \delta \left( \frac{ma^2}{\Sigma^5_0} \right) \right\} \sin \vartheta d\vartheta d\varphi \]
\[
= \frac{1}{4\pi} \delta \left\{ \int_{S^2} m \left( \frac{2}{\Sigma^3_0} - \frac{3a^2 \sin^2 \theta}{L^2 \Sigma^5_0} \right) \sin \vartheta d\vartheta d\varphi \right\} \]
\[
= \delta \left\{ \frac{m}{4} \int_0^{2\pi} \left( \frac{2}{\Sigma^3_0} - \frac{3a^2 \sin^2 \theta}{L^2 \Sigma^5_0} \right) \sin \vartheta d\theta \right\}.
\]

The integrals are elementary.
\[ \int_0^{2\pi} \frac{\sin \vartheta d\vartheta}{\Sigma_\vartheta^3} = \frac{2}{(1 + \frac{\vartheta}{\varphi})^2}, \quad \int_0^{2\pi} \frac{\sin^3 \vartheta d\vartheta}{\Sigma_\vartheta^3} = \frac{4}{3(1 + \frac{\vartheta}{\varphi})^2}, \quad (20) \]

giving
\[ \delta Q[e^\mu, \mathcal{L}_{K^\alpha} e^\mu, \delta e^\mu] = \delta M \]

with
\[ M = \frac{m}{\left(1 + \frac{\varphi}{\vartheta}\right)^2}. \quad (21) \]

This is actually the analytic continuation of the mass determined in [22] (indeed it is presumably no coincidence that the integrals (20) are precisely the ones that appear in equation (B.7) of that reference).

Finally a note on normalization. For asymptotically flat space-time the normalization of the time-like killing vector \( \vec{K} = \partial_t \) is chosen that \( \vec{K} \) has unit norm at \( r \to \infty \). This criterion cannot be used in asymptotically de Sitter space-time. For asymptotically de Sitter space-time the normalization can be fixed by using the natural normalization of the generators of the de Sitter group \( \text{SO}(1, 4) \) at \( r \to \infty \), in which case the natural normalization is \( \vec{K} = L \frac{\partial}{\partial t} \) and the invariant quantity obtained from this normalization is \( mL \).

3. Conclusions

We have shown that, for 4D Kerr–de Sitter space-time with rotational parameter \( a \) [21], the formalism of Wald and collaborators gives
\[ Q[\vec{K}] = \frac{m}{\left(1 + \frac{\varphi}{\vartheta}\right)^2}. \quad (22) \]

An important aspect of the formulation is that the charge can be calculated exactly by integrating over any sphere in a region of space where the Killing symmetry holds, it is not necessary to go to an asymptotic region. For a time-like Killing vector this allows masses to be calculated in asymptotic de Sitter space-times, provided there is a region inside the cosmological horizon where the Killing symmetry holds.

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