NUMERICAL APPROXIMATION OF ELLIPTIC PROBLEMS WITH LOG-NORMAL RANDOM COEFFICIENTS

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In this work, we consider a non-standard preconditioning strategy for the numerical approximation of the classical elliptic equations with log-normal random coefficients. In [45], a Wick-type elliptic model was proposed by modeling the random flux through the Wick product. Due to the lower-triangular structure of the uncertainty propagator, this model can be approximated efficiently using the Wiener chaos expansion in the probability space. Such a Wick-type model provides, in general, a second-order approximation of the classical one in terms of the standard deviation of the underlying Gaussian process. Furthermore, when the correlation length of the underlying Gaussian process goes to infinity, the Wick-type model yields the same solution as the classical one. These observations imply that the Wick-type elliptic equation can provide an effective preconditioner for the classical random elliptic equation under appropriate conditions. We use the Wick-type elliptic model to accelerate the Monte Carlo method and the stochastic Galerkin finite element method. Numerical results are presented and discussed.

KEY WORDS: Wiener chaos expansion; Wick product; Stochastic elliptic PDE; Uncertainty quantification; log-normal random coefficient

1. INTRODUCTION

Numerical approximation of elliptic problems with log-normal random coefficients has received a lot of attention. We consider the following mathematical model

\[
\begin{align*}
-\nabla \cdot (a(x, \omega) \nabla u(x, \omega)) &= f(x), \quad x \in D, \\
u(x, \omega) &= 0, \quad x \in \partial D,
\end{align*}
\]

(1)

where \(\ln a(x, \omega)\) is a second-order homogeneous Gaussian random process, and the force term is assumed to be deterministic for simplicity. We call problem (1) model I in this paper. Theoretical difficulties of problem (1) are mainly related to the lack of uniform ellipticity, where the Lax-Milgram lemma is not applicable. The existence and uniqueness of the solution of problem (1) are usually established with respect to a weighted norm [11,20,29] or a weighted measure [24], or by using the Fernique theorem [7,33]. Considering the Wiener chaos approach and Galerkin
Depending on the properties of the random coefficient, we mainly consider the Monte Carlo method and the Wiener I since model II can be approximated efficiently and the difference between models I and II can be very small. The upper-triangular part will be weak if the truncation order in the M-R formula is relatively small. Doing so, the corresponding uncertainty propagator will be not lower-triangular any more, although the coupling in the product of two random variables, say $a$, the propagator of model II is also lower-triangular, which can be solved efficiently. Another way to approximate the flux underlying Gaussian process goes to infinity, model II has the same solution as model I. In addition, the uncertainty process. However, the latter choice provides a much smaller difference. Actually when the correlation length of the order approximation of the solution of model I in terms of the standard deviation ($\sigma$) which we call model II in this paper. In general, both fluxes $a \nabla u$ as $a \circ \nabla u$ with $\circ$ being the Wick product \cite{14,40,44}, motivated by the observations that the Wick product is consistent with Skorohod stochastic integral in a Hilbert space and can smooth the irregularity induced by white noise. Once the Wick product is adopted, the equations for the coefficients of Wiener chaos expansion are decoupled and can be solved one-by-one. Although this is a very nice property for numerical computation, the original equation is changed and the model difference becomes the main concern. In \cite{45,46}, a new Wick-type model was proposed by modeling the flux as $(a^{-1}) \circ (-1) \circ \nabla u$:

$$
\text{Model II:} \quad \begin{cases} 
-\nabla \cdot \left((a^{-1}) \circ (-1)(x, \omega) \circ \nabla u(x, \omega)\right) = f(x), & x \in D, \\
u(x, \omega) = 0, & x \in \partial D,
\end{cases}
$$

(2)

which we call model II in this paper. In general, both fluxes $a \circ \nabla u$ and $(a^{-1}) \circ (-1) \circ \nabla u$ will introduce a second order approximation of the solution of model I in terms of the standard deviation ($\sigma < 1$) of the underlying Gaussian process. However, the latter choice provides a much smaller difference. Actually when the correlation length of the underlying Gaussian process goes to infinity, model II has the same solution as model I. In addition, the uncertainty propagator of model II is also lower-triangular, which can be solved efficiently. Another way to approximate the flux $a \nabla u$ using the product is to employ the Mikulevicius-Rozovskii (M-R) formula \cite{23}, which shows that the product of two random variables, say $X$ and $Y$, has a Taylor-like expansion

$$XY = X \circ Y + \sum_{n=1}^{\infty} \frac{D^n X \circ D^n Y}{n!},$$

(3)

where $D$ indicates the Malliavin derivative \cite{27}. It is seen that $X \circ Y$ is the lowest-order term in this expansion. We can include more terms from the M-R formula to get a better approximation of $a \nabla u$ \cite{43,47}. It is shown in \cite{47} that with respect to the truncation order $Q$ of the Malliavin derivative and the standard deviation of the underlying Gaussian process such a strategy provides a difference of $O(\sigma^{2Q+1})$ from the solution of model I. However, upon doing so, the corresponding uncertainty propagator will be not lower-triangular any more, although the coupling in the upper-triangular part will be weak if the truncation order in the M-R formula is relatively small.

In this work, we will explore the possibility to use model II as a predictor to improve some algorithms for model I since model II can be approximated efficiently and the difference between models I and II can be very small. Depending on the properties of the random coefficient, we mainly consider the Monte Carlo method and the Wiener chaos approach with Galerkin projection for model I.

This paper is organized as follows. In section 2 we define the Wiener chaos space and the Wick product. Stochastic elliptic models are discussed in section 3 and the corresponding uncertainty propagators are given in section 4.
Numerical algorithms are proposed in section 5. We present numerical results in section 6, followed by a summary section.

2. WIENER CHAOS SPACE AND WICK PRODUCT

Since the underlying random variables of the model are i.i.d. Gaussian, whose corresponding stochastic orthogonal polynomials are Hermite. We first introduce basic properties of Hermite polynomials.

2.1 Hermite polynomials

The one-dimensional (probabilistic) Hermite polynomials of degree $n$ are defined as

$$H_n(\xi) := (-1)^n e^{\frac{1}{2} \xi^2} \frac{d^n}{d\xi^n} e^{-\frac{1}{2} \xi^2}.$$  

(4)

$H_n(\xi)$ are orthogonal with respect to the weight $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \xi^2}$, in the sense

$$\int_{-\infty}^{\infty} H_m(\xi)H_n(\xi) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \xi^2} d\xi = n! \delta_{nm}.$$  

(5)

The values of Hermite polynomials can be evaluated using the following three-term recurrence formula:

$$H_0(\xi) = 1, \quad H_1(\xi) = \xi, \quad H_{n+1}(\xi) = \xi H_n(\xi) - nH_{n-1}(\xi), \quad n \geq 2.$$  

Hermite polynomials satisfy a very simple derivative relation:

$$H_n'(\xi) = nH_n(\xi) - \xi H_{n-1}(\xi) \forall n \geq 0.$$  

(6)

We list below in Lemma 1 several properties of Hermite polynomials, which will be used later.

**Lemma 1.**

For one-dimensional Hermite polynomials, the following properties hold

$$\exp \left( s\xi - \frac{1}{2} s^2 \right) = \sum_{i=0}^{\infty} \frac{s^i}{i!} H_i(\xi),$$  

(7)

$$H_n(\xi + s) = \sum_{i=0}^{n} \binom{n}{i} s^{n-i} H_i(\xi),$$  

(8)

$$H_i(\xi)H_j(\xi) = \sum_{k \leq i \land j} \chi(i, j, k) H_{i+j-2k}(\xi).$$  

(9)

where $s \in \mathbb{R}, i \land j := \min\{i, j\}$ and

$$\chi(i, j, k) = \frac{i! j!}{k!(i-k)!(j-k)!}.$$  

2.2 Wick product

Now we list the definition and some basic properties of Wick product, which can be found in existing literature (e.g. [14], [15]).

The Wick product of a set of random variables with finite moments is defined recursively as follows:

$$\langle \emptyset \rangle = 1, \quad \frac{\partial}{\partial X_i} (X_1, \ldots, X_k) = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k), \quad k \geq 1,$$  

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together with the constraint that the average is zero

\[ \mathbb{E} \langle X_1, \ldots, X_k \rangle = 0, \quad k \geq 1. \]

It follows that

\[ \langle X \rangle = X - \mathbb{E}[X], \quad \langle X, Y \rangle = XY - \mathbb{E}[Y]X - \mathbb{E}[X]Y + 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[XY]. \]

If \( X, Y \) are independent, from about formula, we know

\[ \langle X, Y \rangle = \langle X \rangle \langle Y \rangle. \]

On the other hand, if \( Y = X \), we get

\[ \langle X, X \rangle = X^2 - 2\mathbb{E}[X]X + 2\mathbb{E}[X]^2 - \mathbb{E}[X^2]. \]

Define \( X \diamond Y := \langle X, Y \rangle \) and

\[ P_n(X) := X^{\diamond n} = \langle X, \ldots, X \rangle_n \]

then \( P_n'(x) = nP_{n-1}(x) \).

Wick product is closely related to Hermite polynomials. If \( \xi \) is a normally distributed variable with variance 1, then

\[ \xi^{\diamond n} = H_n(\xi). \]

(10)

and

\[ H_n(\xi) \diamond H_m(\xi) = H_{n+m}(\xi). \]

(11)

Using Taylor series, one can define the exponential function of Wick product as

\[ e^{\diamond X} := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}. \]

(12)

For a normally distributed variable \( \xi \), it can be checked that

\[ e^{\diamond [\sigma \xi]} = e^{\sigma \xi - \sigma^2/2}, \]

(13)

\[ e^{\diamond [\sigma \xi]} \diamond e^{\diamond [-\sigma \xi]} = 1, \]

(14)

and the following statistics hold

\[ \mathbb{E} \left[ e^{\diamond [\sigma \xi]} \right] = 1, \quad \text{Var} \left[ e^{\diamond [\sigma \xi]} \right] = e^{\sigma^2} - 1. \]

(15)

2.3 Wiener chaos space

We define \( \mathbb{F} := (\Omega, \mathcal{F}, P) \) as a complete probability space, where \( \mathcal{F} \) is the \( \sigma \)-algebra generated by the countably many i.i.d. Gaussian random variables \( \{ \xi_k \}_{k \geq 1} \). Define \( \xi := (\xi_1, \xi_2, \ldots) \). Let \( \mathcal{J} \) be the collection of multi-indices \( \alpha \) with \( \alpha = (\alpha_1, \alpha_2, \ldots) \) so that \( \alpha_k \in \mathbb{N}_0 \) and \( |\alpha| := \sum_{k \geq 1} \alpha_k < \infty \). For \( \alpha, \beta \in \mathcal{J} \), we define

\[ \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots), \quad \alpha! = \prod_{k \geq 1} \alpha_k!, \quad \binom{\alpha}{\beta} = \prod_{k \geq 1} \binom{\alpha_k}{\beta_k}. \]
We use \((0)\) to denote the multi-index with all zero entries: \((0)_k = 0\) for all \(k\). Define the collection of random variables \(\Xi\) as follows:
\[
\Xi := \{ h_{\alpha}, \alpha \in J \}, \quad h_{\alpha}(\xi) := \prod_{k \geq 1} \frac{1}{\sqrt{\alpha_k!}} H_{\alpha_k}(\xi_k),
\]
where \(H_{\alpha}(\xi)\) are the one-dimensional (probabilistic) Hermite polynomials. For convenience, we also define
\[
H_{\alpha}(\xi) := \prod_{k \geq 1} H_{\alpha_k}(\xi_k).
\]

For any fixed \(k\)-dimensional i.i.d. Gaussian random variable \(\xi\), the following relations hold
\[
\mathbb{E}[H_{\alpha}(\xi)H_{\beta}(\xi)] = \delta_{\alpha \beta} \alpha! \quad \text{and} \quad \mathbb{E}[h_{\alpha}(\xi)h_{\beta}(\xi)] = \delta_{\alpha \beta}.
\]
The set \(\Xi\) forms an orthonormal basis for \(L_2(\mathbb{F})\) \(9\), that is: if \(\eta \in L_2(\mathbb{F})\), then
\[
\eta = \sum_{\alpha \in J} \eta_{\alpha} h_{\alpha}, \quad \eta_{\alpha} = \mathbb{E}[\eta h_{\alpha}]
\]
and
\[
\mathbb{E}[\eta^2] = \sum_{\alpha \in J} \eta_{\alpha}^2.
\]
The Wick product of multi-dimensional stochastic Hermite polynomials are:
\[
H_{\alpha}(\xi) \diamond H_{\beta}(\xi) = H_{\alpha + \beta}(\xi), \quad h_{\alpha}(\xi) \diamond h_{\beta}(\xi) = \sqrt{\frac{(\alpha + \beta)!}{\alpha! \beta!}} h_{\alpha + \beta}(\xi).
\]

Note that if we consider the expansion of \(H_{\alpha}(\xi)H_{\beta}(\xi)\) using the base set \(\Xi\), it is obvious that there exist low-order terms in addition to \(H_{\alpha + \beta}(\xi)\); however, in the definition of Wick product, all these low-order terms are removed, cf. equation \(19\) and equation \(21\). Such a difference of the Wick product from the regular multiplication stems from the fact that the Wick product should be interpreted from the viewpoint of stochastic integral. The correspondence between the Wick product and the Ito-Skorokhod integral can be found in \([14,18,27,44]\).

For the numerical approximation, the number of Gaussian random variables and the polynomial order need to be truncated. We define
\[
J_{M,p} = \{ \alpha | \alpha = (\alpha_1, \ldots, \alpha_M), |\alpha| \leq p \},
\]
where \(p \in \mathbb{N}_0\) is the maximum total degree. (To reduce the number of stochastic bases, one can also consider the sparse grids or sparse spectral Galerkin method, see e.g. \([8,25,37,39]\), where the overall procedure is similar.) Correspondingly, \(\xi\) is split into two parts
\[
\xi = \xi_1 \oplus \xi_2 = (\xi_1, \ldots, \xi_M) \oplus (\xi_{M+1}, \ldots).
\]

For simplicity, we use \(\xi\) for both finite-dimensional and infinite-dimensional cases, and the dimensionality will be indicated by the set \(J\) or \(J_{M,p}\) for the index. Let \(N_{M,p}\) be the cardinality of \(J_{M,p}\). It is obvious that there exists a one-to-one correspondence between \(1 \leq i \leq N_{M,p}\) and \(\alpha \in J_{M,p}\). We use \(i(\alpha)\) or \(\alpha(i)\) to indicate such a one-to-one mapping whenever necessary.

Given a real separable Hilbert space \(X\), we denote by \(L_2(\mathbb{F}; X)\) the Hilbert space of square-integrable \(\mathcal{F}\)-measurable \(X\)-valued random elements \(f\). When \(X = \mathbb{R}\), we write \(L_2(\mathbb{F})\) instead of \(L_2(\mathbb{F}; \mathbb{R})\). Given a collection \(R = \{ r_{\alpha}, \alpha \in J \}\) of positive real numbers with an upper bound \(R\), i.e. \(r_{\alpha} < R\) for all \(\alpha\), we define the space \(RL_2(\mathbb{F}; X)\) as the closure of \(L_2(\mathbb{F}; X)\) in the norm
\[
\| u \|^2_{RL_2(\mathbb{F}; X)} = \sum_{\alpha \in J} r_{\alpha} \| u_{\alpha} \|_X^2,
\]
where \(u = \sum_{\alpha \in J} u_{\alpha} h_{\alpha}(\xi)\). The space \(RL_2(\mathbb{F}; X)\) is called a weighted chaos space, it is a natural norm for the stochastic space using Karhunen-Loève expansion. In this work, \(X\) is chosen as \(H_0^1(D)\) for elliptic problems with homogeneous boundary conditions.
3. STOCHASTIC ELLIPTIC MODELS

In this paper, we consider the following two stochastic elliptic models:

\begin{align*}
\text{Model I:} \quad - \nabla \cdot (a(x, \omega) \nabla u_1(x, \omega)) &= f(x), \\
\text{Model II:} \quad - \nabla \cdot \left( (a^{-1})^{\diamond(-1)}(x, \omega) \cdot \nabla u_{II}(x, \omega) \right) &= f(x),
\end{align*}

with boundary condition \( u(x, \omega) = 0 \) on \( \partial D \), where \( a^{-1}(x, \omega) \cdot \diamond ((a^{-1}(x, \omega)))^{\diamond(-1)} = 1 \). In particular, we assume that the force term \( f(x) \) is deterministic for simplicity and the random coefficient \( a(x, \omega) \) takes the following form

\[ a(x, \omega) = e^{\diamond(aG(x, \omega))} = e^{\alpha G(x, \omega) - \frac{1}{2} \sigma^2}, \]

where \( G(x, \omega) \) is a stationary Gaussian random process with zero mean and unit variance, subject to a normalized covariance kernel \( K(x_1, x_2) = K(|x_1 - x_2|) = E[G(x_1, \omega)G(x_2, \omega)] \). According to the Mercer theorem \([34]\), \( K(x_1, x_2) \) has an expansion as

\[ K(x_1, x_2) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x_1) \phi_i(x_2), \]

where \( \{\lambda_i, \phi_i(x)\}_{i=1}^{\infty} \) are eigen-pairs of \( K(x_1, x_2) \) satisfying

\[ \int_D K(x_1, x_2) \phi_i(x_1) \phi_i(x) dx_1 = \lambda_i \phi_i(x_1), \quad \int_D \phi_i(x) \phi_j(x) dx = \delta_{ij}. \]

Then \( G(x, \omega) \) has the following Karhunen-Loève (K-L) expansion

\[ G(x, \omega) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \phi_i(x) \xi_i, \]

where \( \xi_i \) are independent Gaussian random variables. Furthermore,

\[ \sum_{i=1}^{\infty} \lambda_i \phi_i^2(x) = K(x, x) = E[G^2(x, \omega)] = 1, \quad \forall x \in D. \]

Using equations (28), (29) and (7), we can obtain the Wiener chaos expansion of the log-normal random process \( a(x, \omega) \)

\[ a(x, \omega) = e^{\sum_{i=1}^{\infty}\sqrt{\lambda_i} \phi_i(x) \xi_i - \frac{\sigma^2}{2} \lambda_i \phi_i^2(x)} = \sum_{\alpha \in \mathcal{S}} \frac{\Phi^{\alpha}}{\alpha!} H_{\alpha}(\xi), \]

where \( \Phi(x) = (\sigma \sqrt{\lambda_1} \phi_1(x), \sigma \sqrt{\lambda_2} \phi_2(x), \ldots) \).

From equation (13), it can be easily derived that

\[ (a^{-1}(x, \omega))^{\diamond(-1)} = e^{-\sigma^2 e^{\diamond(aG(x, \omega))}}. \]

Hence, the difference between Wiener chaos expansions of \( (a(x, \omega)^{-1})^{\diamond(-1)} \) and \( a(x, \omega) \) is just a scaling factor \( e^{-\sigma^2} \).

To make the difference between models I and II clearer, we look at the following two linear systems

\begin{align*}
\text{I:} \quad \begin{cases} 
\nabla u_1 &= a^{-1} \ast F_1, \\
-\nabla \cdot F_1 &= f,
\end{cases} \quad \text{II:} \quad \begin{cases} 
\nabla u_{II} &= a^{-1} \diamond F_{II}, \\
-\nabla \cdot F_{II} &= f.
\end{cases}
\end{align*}
where * denotes the operation of the regular product. Thus model II is basically making the gradient “smoother” through the Wick product. Then the equation for $u_l - u_{II}$ can be obtained as

$$\begin{align*}
\nabla (u_l - u_{II}) &= a^{-1} \ast (F_l - F_{II}) + a^{-1} (\ast - \diamond) F_{II}, \\
-\nabla \cdot (F_l - F_{II}) &= 0,
\end{align*}$$

which corresponds to a second order elliptic equation for $u_l - u_{II}$ as

$$-\nabla \cdot (a \nabla (u_l - u_{II})) = -\nabla \cdot (a \ast (a^{-1}(\ast - \diamond) F_{II})).$$

Note that we express explicitly the regular products on the right-hand side since the regular and Wick products do not commute. It is seen that all chaos coefficients in equation (39) are coupled together, which means that they must be solved together. From the numerical point of view, a proper choice would be iterative methods. Before we look into the numerical algorithms, we now address the properties of the matrix $E [a(x, \omega)H_{\alpha}H_{\gamma}]$ for any $x \in D$.

**Theorem 2** (36). Let $F = -\nabla \cdot (a \ast (a^{-1}(\ast - \diamond) F_{II}))$, where * indicates the regular product. Assume that $F \in RL^2(\mathcal{F}; H^{-1}(D))$, where $D \in \mathbb{R}^d$, $d = 1, 2, 3$. Then there exists a set of weights $\tilde{\mathcal{R}} = \{\tilde{r}_\alpha, \alpha \in \mathcal{J}\}$, such that

$$\|u_l - u_{II}\|_{RL^2(\mathcal{F}; H^0_0(D))} = C(l_c) \sigma^2 = O(\sigma^2),$$

where $l_c$ is the correlation length. Furthermore, $C(l_c) \to 0$ as $l_c \to \infty$.

**Remark 1.** It can be shown theoretically that for one-dimensional cases $D \in \mathbb{R}^1$, $C(l_c) \to 0$ as $l_c \to 0$. For high-dimensional cases, according to the Landau-Lifshitz-Matheron conjecture [19][22] in the homogenization theory for log-normal random coefficients, when $l_c \to 0$, $C(l_c) \to \frac{1}{2}$ if $d = 2$, and $C(l_c) \to \frac{1}{3}$ if $d = 3$.

**Remark 2.** By noting the Mikulevicius-Rozovskii formula [23]

$$h_{\alpha}h_{\beta} = \sum_{n=0}^{\infty} \frac{D^n h_{\alpha} \diamond D^n h_{\beta}}{n!},$$

where $D^n$ denotes the $n$th-order Malliavin derivative, model I can be approximated arbitrarily well as

$$-\nabla \cdot \left( \sum_{n=0}^{\infty} \frac{D^n a(x, \omega) \diamond \nabla D^n u}{n!} \right) = f(x).$$

When $n = 0$, equation (37) recovers the Wick-type

$$-\nabla \cdot (a(x, \omega) \diamond \nabla u(x, \omega)) = f(x).$$

More discussions about the new Wick-type model given by equation (37) can be found in [47].

## 4. STOCHASTIC GALERKIN METHOD

### 4.1 Uncertainty propagators

We now look at the uncertainty propagator of model I. Substituting the Wiener chaos expansion

$$u_l(x, \omega) \approx \sum_{\alpha \in \mathcal{J}_{M,P}} u_{l,\alpha}(x)H_{\alpha}(\xi)$$

into equation (24a) and implementing Galerkin projection in the probability space, we obtain the uncertainty propagator for model I as

$$-\sum_{\alpha \in \mathcal{J}_{M,P}} \nabla \cdot (E [a(x, \omega)H_{\alpha}H_{\gamma}] \nabla u_{l,\alpha}(x)) = f(x)\delta_{(0),Y}, \ \forall Y \in \mathcal{J}_{M,P}.$$  

It is seen that all chaos coefficients in equation (39) are coupled together, which means that they must be solved together. From the numerical point of view, a proper choice would be iterative methods. Before we look into the numerical algorithms, we now address the properties of the matrix $E [a(x, \omega)H_{\alpha}H_{\gamma}]$ for any $x \in D$.  

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Lemma 3. For any given \( x \in D \), the matrix \( B_{ij}(x) = \mathbb{E}[a(x, \omega)H_{\alpha(i)}H_{\gamma(j)}] \) is symmetric and positive definite, where \( a(x, \omega) \) is a log-normal random process defined in equation (25) and \( \alpha, \gamma \in J_{M,p} \).

Proof. Apparently, the matrix \( B_{ij}(x) \) is symmetric for any \( x \in D \). For any nonzero vector \( c = (c_1, c_2, \ldots, c_{N_{M,p}}) \neq 0 \), the following inequality holds for any \( x \in D 
\begin{align*}
  c^T B_i(x) c &= \sum_{i,j=1}^{N_{M,p}} c_i c_j \mathbb{E}\left[e^{\sigma G(x, \omega)} H_{\alpha(i)} H_{\gamma(j)}\right] \\
  &= \mathbb{E}\left[ \sum_{i,j=1}^{N_{M,p}} c_i c_j e^{\sigma G(x, \omega)} H_{\alpha(i)} H_{\gamma(j)} \right] \\
  &= \mathbb{E}\left[ \left( \sum_{i=1}^{N_{M,p}} \left(e^{\sigma G(x, \omega)} \right)^{1/2} H_{\alpha(i)} c_i \right)^2 \right] \geq 0,
\end{align*}

In other words, \( B_i \) is non-negative definite.

We subsequently show that if \( c^T B_i(x) c = 0 \), then \( c = 0 \). Let \( b \in \mathbb{R}^M \). It is easy to generalize equation (8) to the high-dimensional case

\[
H_{\alpha}(\xi + b) = \prod_{k=1}^{M} H_{\alpha_k}(\xi_k + b_k) = \prod_{k=1}^{M} \sum_{i=0}^{\alpha_k} \alpha_k \binom{\alpha_k}{i} b_k^{\alpha_k - i} H_i(\xi_k) \\
= \sum_{\beta \leq \alpha} \left( \binom{\alpha}{\beta} b^{\alpha - \beta} H_{\beta}(\xi) \right). \quad (40)
\]

Let \( \Phi(x) = \Phi_1(x) \oplus \Phi_2(x) \), where

\[
\Phi_1(x) = (\sigma \sqrt{\lambda_1} \Phi_1(x), \ldots, \sigma \sqrt{\lambda_M} \Phi_M(x)) \quad \text{and} \quad \Phi_2(x) = (\sigma \sqrt{\lambda_{M+1}} \Phi_{M+1}(x), \sigma \sqrt{\lambda_{M+2}} \Phi_{M+2}(x), \ldots).
\]
Let $\hat{\xi} = (\xi_{M+1}, \xi_{M+2}, \ldots)$. We then have

$$c^T B_t(x)c = \mathbb{E} \left[ \left( \sum_{i=1}^{N_{M,p}} \left( e^{\frac{1}{2} \sigma G(\xi, \omega)} \right)^{1/2} H_{\alpha(i)} c_i \right)^2 \right]$$

$$= \mathbb{E} \left[ e^{\frac{1}{2} \Phi_1^T \xi + \frac{1}{2} \sigma^2} \left( \sum_{i=1}^{N_{M,p}} H_{\alpha(i)} c_i \right)^2 \right]$$

$$= e^{\frac{1}{2} \Phi_1^T \xi + \frac{1}{2} \sigma^2} \mathbb{E} \left[ e^{\frac{1}{2} \Phi_1^T \xi} \left( \sum_{i=1}^{N_{M,p}} H_{\alpha(i)} (\xi + \Phi_1) c_i \right)^2 \right]$$

$$= e^{\frac{1}{2} \Phi_1^T \xi} \mathbb{E} \left[ \left( \sum_{i=1}^{N_{M,p}} \sum_{\beta \leq \alpha(i)} \left( \begin{array}{c} \alpha(i) \\ \beta \end{array} \right) \Phi_1^{\alpha(i)-\beta} H_{\beta} (\xi) c_i \right)^2 \right]$$

$$= e^{\frac{1}{2} \Phi_1^T \xi} \mathbb{E} \left[ \left( \sum_{\beta \in J_{M,p}} \sum_{\alpha(i) \geq \beta} \left( \begin{array}{c} \alpha(i) \\ \beta \end{array} \right) \Phi_1^{\alpha(i)-\beta} c_i(\alpha) \right) \right]^2$$

If $c^T B_t(x)c = 0$, we have

$$\sum_{\alpha \geq \beta} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \Phi_1^{\alpha-\beta} c_i(\alpha) = 0, \quad \forall \beta \in J_{M,p}, \ x \in D.$$
Proof. First, equation (9) can be generalized straightforwardly to the multi-dimensional case as
\[ H_\alpha H_\beta = \sum_{\kappa \leq \alpha \land \beta} \chi(\alpha, \beta, \kappa) H_{\alpha + \beta - 2\kappa} \]
with
\[ \chi(\alpha, \beta, \kappa) = \frac{\alpha! \beta!}{\kappa!(\alpha - \kappa)!(\beta - \kappa)!}. \]
Using equation (30), we have
\[ E[a(x, \omega)H_\alpha H_\beta] = \sum_{\gamma \in J} \Phi_\gamma(x) E[H_\gamma H_\alpha H_\beta] \]
\[ = \sum_{\gamma \in J} \Phi_\gamma(x) \sum_{\kappa \leq \alpha \land \beta} \chi(\alpha, \beta, \kappa) E[H_{\alpha + \beta - 2\kappa} H_\gamma] \]
\[ = \sum_{\kappa \leq \alpha \land \beta} \chi(\alpha, \beta, \kappa) \Phi_{\alpha + \beta - 2\kappa}. \]

Remark 4. When \( \alpha = \beta \), we have
\[ E[a(x, \omega)H_\alpha^2] = \sum_{\kappa \leq \alpha} \chi(\alpha, \alpha, \kappa) \Phi_{2(\alpha - \kappa)}(x) \geq \chi(\alpha, \alpha, \alpha) = \alpha!. \]

Remark 5. Lemma 4 implies that to compute \( E[a(x, \omega)H_\beta(i)H_\gamma(j)] \) exactly, we require the coefficients of Wiener Chaos expansion of \( a(x, \omega) \) up to order \( 2\beta(N_{M,p}) \).

We now look at the uncertainty propagators of model II. Let \( \hat{a}(x, \omega) = (a^{-1})^{o(-1)} \). Using equations (30) and (31), the Wiener chaos expansion of \( \hat{a}(x, \omega) \) can be explicitly derived as
\[ \hat{a}(x, \omega) = \sum_{\alpha \in J} \hat{a}_\alpha(x) H_\alpha(\xi) = \sum_{\alpha \in J} e^{-\sigma^2} \frac{\Phi_{\alpha}}{\alpha!} H_\alpha(\xi). \quad (42) \]
Following the same procedure for model I, we can obtain the uncertainty propagator of model II as
\[ -\sum_{\alpha \leq \gamma} \nabla \cdot (\hat{a}_{\gamma - \alpha}(x) \nabla u_{II,\alpha}(x)) = f(x) \delta(0, \gamma), \quad \forall \gamma \in J_{M,p}. \quad (43) \]
It is seen that \( u_{II,\gamma} \) only depends on the chaos coefficients \( u_{II,\alpha} \) with \( \alpha < \gamma \), which introduces a lower-triangular structure into the matrix \( B_{II,ij}(x) = \hat{a}_{\gamma(j) - \alpha(i)}(x) \). In other words, the deterministic PDEs for \( u_{II,\gamma} \) are naturally decoupled and can be solved one by one. Furthermore, equation (43) can be rewritten as
\[ -\nabla \cdot (\hat{a}_{(0)}(x) \nabla u_{II,\gamma}(x)) = \sum_{\alpha < \gamma} \nabla \cdot (\hat{a}_{\gamma - \alpha}(x) \nabla u_{II,\alpha}(x)) + f(x) \delta(0, \gamma). \]
Thus, if we employ finite element method to solve the PDE system (43), the bilinear form remains the same for all chaos coefficients \( u_{II,\gamma} \), which only depends on \( \hat{a}_{(0)}(x) \).
4.2 Finite element discretization of uncertainty propagators

We now look at the finite element discretization of uncertainty propagators of models I and II. Let $\mathcal{T}_h$ be a family of triangulations of $D$ with straight edges and $h$ the maximum size of the elements in $\mathcal{T}_h$. We assume that the family is regular, in other words, the minimal angle of all the elements is bounded from below by a positive constant. We define the finite element space as

$$ V_{h,q}^K = \left\{ v \mid v \circ F_K^{-1} \in \mathcal{P}_q(R) \right\}, \quad V_{h,q} = \left\{ v \in H_0^1(D) \mid v|_K \in V_{h,q}^K, \ K \in \mathcal{T}_h \right\}, $$

where $F_K$ is the mapping function for the element $K$ which maps the reference element $R$ (for example, an equilateral triangle or an isosceles right triangle) to the element $K$ and $\mathcal{P}_q(R)$ denotes the set of polynomials of degree at most $q$ on $R$. We assume that $v|_{\partial D} = 0$ for any $v \in V_{h,q}$. Thus, $V_{h,q}$ is an approximation of $H_0^1(D)$ by piece-wise polynomial functions. There exist many choices of basis functions on the reference elements, such as $h$-type finite elements [9], spectral/hp elements [17,36], etc. Let

$$ V_{h,q} = \text{span} \{ \theta_1(x), \theta_2(x), \ldots, \theta_{N_x}(x) \} \subset H_0^1(D), $$

where $N_x$ is the total number of basis functions in the finite element space $V_{h,q}$.

The truncated Wiener chaos space $W_{M,p}$ is defined as

$$ W_{M,p} = \left\{ \sum_{\alpha \in J_{M,p}} c_{\alpha} H_{\alpha}(\xi) \mid c_{\alpha} \in \mathbb{R} \right\}, $$

(44)

The stochastic finite element method for model I can be formulated as follows: Find $u_{i,h} \in V_{h,q} \otimes W_{M,p}$, such that for all $v \in V_{h,q} \otimes W_{M,p}$

$$ \mathcal{B}_I(u_{i,h}, v) = \mathcal{L}(v), $$

(45)

where the bilinear form is

$$ \mathcal{B}_I(v_1, v_2) = \int_D \mathbb{E} \left[ a(x, \omega) \nabla v_1 \cdot \nabla v_2 \right] dx, $$

(46)

and the linear form

$$ \mathcal{L}(v) = \int_D \mathbb{E}[fv] dx. $$

(47)

Lemma 5.

The stiffness matrix for the stochastic finite element method of model I is symmetric and positive definite.

Proof. Consider the approximation

$$ u_{i,h}(x, \xi) = \sum_{\alpha \in J_{M,p}} u_{i,h,\alpha} H_{\alpha}(\xi) = \sum_{1 \leq i \leq N_x} u_{i,h,\alpha,i} \theta_i(x) H_{\alpha}(\xi), $$

(48)

where $u_{i,h,\alpha,i} \neq 0$ for some $i$ and $\alpha$. We have

$$ \mathcal{B}_I(u_{i,h}, u_{i,h}) = \sum_{1 \leq i \leq N_x} \sum_{\alpha \in J_{M,p}} \int_D u_{i,h,\alpha,i} u_{i,h,\beta,j} \mathbb{E} \left[ a(x, \omega) H_{\alpha} H_{\beta} \right] \nabla \theta_i(x) \cdot \nabla \theta_j(x) dx $$

\[
= \int_D \sum_{\alpha, \beta \in J_{M,p}} \mathbb{E} \left[ a(x, \omega) H_{\alpha} H_{\beta} \right] \nabla u_{i,h,\alpha} \cdot \nabla u_{i,h,\beta} dx \\
= \int_D \left( \sum_{j=1}^d \partial_{x_j} \left( \hat{u}_i(x) \right)^\top B_I(x) \partial_{x_j} \hat{u}_i(x) \right) dx,
\]

where the vector $\hat{u}_i(x)$ is defined as $\left( \hat{u}_i(x) \right)_k = u_{i,h,\alpha(k)}(x)$, $k = 1, \ldots, N_{M,p}$. Due to the homogeneous boundary conditions, a nonzero constant mode does not exist in the space $V_{h,q}$. Using Lemma 3 we know that $\mathcal{B}_I(u_{i,h}, u_{i,h}) > 0$, and the conclusion follows. \qed

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4.3 Structures of stiffness matrices of the sFEM

Based on equation (48), we define some matrix notations:

\[
\begin{align*}
\mathbf{u}_i &= \begin{bmatrix} u_{i,1}^1 \\ u_{i,2}^1 \\ \vdots \\ u_{i,N_{M,p}}^1 \end{bmatrix}, \\
\mathbf{u}^{ij} &= \begin{bmatrix} u_{ij,1}^1 \\ u_{ij,2}^1 \\ \vdots \\ u_{ij,N_{M,p}}^1 \end{bmatrix}, \quad i = 1, \ldots, N_{M,p}.
\end{align*}
\]  

(49)

Obviously, the total number of unknowns is \( N_x \times N_{M,p} \). The weak form (45) leads to the linear system \( A_{1}\mathbf{u}_i = \mathbf{f} \) with the block structure

\[
A_{1} = \begin{pmatrix}
A_{1,11} & A_{1,12} & \cdots & A_{1,1N_{M,p}} \\
A_{1,21} & A_{1,22} & \cdots & A_{1,2N_{M,p}} \\
\vdots & \vdots & \ddots & \vdots \\
A_{1,N_{M,p}1} & A_{1,N_{M,p}2} & \cdots & A_{1,N_{M,p}N_{M,p}}
\end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N_{M,p}} \end{pmatrix}.
\]  

(50)

Considering the approximation of \( a(x, \omega) \) as (see equation (30))

\[
a^{M,\tilde{p}}(x, \xi) = \frac{\Phi(x)}{\alpha!} \sum_{\alpha \in J_{M,\tilde{p}}} \alpha_{\alpha} H_{\alpha}(\xi),
\]  

(51)

where \( \tilde{p} \) is the polynomial order of the Wiener chaos expansion. Then the blocks \( A_{1,ij} \) can be expressed as

\[
A_{1,ij} = \sum_{\alpha \in J_{M,\tilde{p}}} \mathbb{E} \left[ H_{\alpha} H_{\beta(i)} H_{\gamma(j)} \right] S_{\alpha}, \quad i, j = 1, \ldots, N_{M,p}
\]  

(52)

where

\[
(S_{\alpha})_{ij} = \int_{D} a^{M,\tilde{p}}(x) \nabla \theta_i(x) \cdot \nabla \theta_j(x) dx.
\]  

(53)

Define matrix \( C_{\alpha} \) as

\[
(C_{\alpha})_{ij} = \mathbb{E} \left[ H_{\alpha} H_{\beta(i)} H_{\gamma(j)} \right].
\]  

(54)

Then the matrix \( A_{1} \) can be rewritten in the tensor-product form as

\[
A_{1} = \sum_{\alpha \in J_{M,\tilde{p}}} C_{\alpha} \otimes S_{\alpha}.
\]  

(55)

Then the matrix-vector multiplication of \( A_{1}\mathbf{u}_i \) can be computed in a relatively efficient way. We rewrite the vector \( A_{1}\mathbf{u}_i \) of length \( N_x N_{M,p} \) to an \( N_{M,p} \)-by-\( N_x \) matrix and denote such a matrix as \( [A_{1}\mathbf{u}_i] \). Then we have

\[
[A_{1}\mathbf{u}_i] = \sum_{\alpha \in J_{M,\tilde{p}}} \left[ S_{\alpha} \mathbf{u}_{1}^{1} S_{\alpha} \mathbf{u}_{1}^{2} \ldots S_{\alpha} \mathbf{u}_{N_{M,p}}^{1} \mathbf{C}_{\alpha}^{T} \right],
\]  

(56)

where \( S_{\alpha} \mathbf{u}_{1}^{1} \) is the \( i \)-th column vector of an \( N_{M,p} \)-by-\( N_x \) matrix.

4.4 Comments on the bilinear form \( B_{1} \)

Using the log-normal random coefficient \( a(x, \omega) \), we have shown that the bilinear form \( B_{1}((\cdot, \cdot)) \) is positive definite. However, we do not have the ellipticity here because \( a(x, \omega) \) is not strictly positive. Instead of using the Lax-Milgram lemma, the existence and uniqueness of a solution \( u(x, \omega) \in L_{2}(H_{0}^{1}(D)) \) can be established by the Fernique theorem with appropriate regularity assumptions for the covariance function of the underlying Gaussian field \( \mathbf{u} \). The key
observation is that the random variable $a^{-1}_{\text{min}}(\omega) = \min_{x \in D} a(x, \omega) \in L_p(\mathbb{F}), p > 0$. From the theoretical point of view, an inf-sup condition can be established for the continuous bilinear form $B_1(v_1, v_2)$, where $v_1 \in L_2(\mathbb{F}; H^1_0(D))$ and $v_2 \in L_2(\mathbb{F} = (\Omega, \mathcal{F}, a^{-1}_{\text{min}}(\omega))P(d\omega); H^1_0(D))$ \cite{2}. Note here that the measure of the probability space for test functions $v_2$ is weighted by the random variable $a^{-2}_{\text{min}}(\omega)$. According to theoretical observations, one choice for the test functions can be

$$\left\{ \frac{v}{a_{\text{min}}(\omega)} : v \in L_2(\mathbb{F}; H^1_0(D)) \right\}.$$ 

However, it is not clear how to deal with $a_{\text{min}}(\omega)$ numerically. For numerical studies of model I with the Galerkin projection, we usually choose test functions from $\mathcal{V}$ and the following ansatz of $u$.

$$u \in L_2(\mathbb{F}; H^1_0(D)) \text{ is symmetric and positive definite, the existence and uniqueness of solution } u_i \text{ is guaranteed. No divergence of the solution with respect to } L_2(\mathbb{F}; H^1_0(D)) \text{ norm has been observed for such a procedure.}$$

5. NUMERICAL ALGORITHMS

Based on the properties of Wick product and the assumptions of Theorem \cite{2}, we have the following asymptotic results \cite{46} for equation (34) satisfied by $u_i - u_{II}$. With respect to $\sigma$, we have the following power series

$$-\nabla \cdot (a^{-1}(\ast - \phi) f) = \sigma^2 \tilde{f}_2(x, \xi) + \sigma^2 \tilde{f}_3(x, \xi) + \ldots.$$ 

Substituting

$$a(x, \omega) = a_0(x) + \sigma a_1(x, \omega) + \sigma^2 a_2(x, \omega) + \ldots$$

and the following ansatz of $u_i - u_{II}$

$$u_i - u_{II} = \tilde{u}_0(x) + \sigma \tilde{u}_1(x, \xi) + \sigma^2 \tilde{u}_2(x, \xi) + \ldots$$

into equation (34) and comparing the coefficients of $\sigma^i$, we obtain

$$-\nabla \cdot (a_0 \nabla \tilde{u}_0) = 0,$$

$$-\nabla \cdot (a_0 \nabla \tilde{u}_1) = \nabla \cdot (a_1 \nabla \tilde{u}_0),$$

$$-\nabla \cdot (a_0 \nabla \tilde{u}_2) = \nabla \cdot (a_2 \nabla \tilde{u}_0) + \nabla \cdot (a_1 \nabla \tilde{u}_1) + \tilde{f}_2(x, \xi),$$

$$\ldots .$$

which results in

$$\tilde{u}_0(x) = \tilde{u}_1(x, \xi) = 0, \quad \tilde{u}_i(x, \xi) \neq 0, i = 2, 3, \ldots$$

Thus, $u_i - u_{II}$ has the following power series expansion with respect to $\sigma$

$$u_i - u_{II} = \sigma^2 \tilde{u}_2(x, \xi) + \sigma^3 \tilde{u}_3(x, \xi) + \ldots ;$$

which holds for any $x \in D$. Then both the mean and standard deviation of $u_i - u_{II}$ are of $O(\sigma^2)$ if they exist.

When $l_\xi \to \infty$, the random coefficient becomes

$$a(x, \omega) = e^{\sigma \xi - \frac{1}{2} \sigma^2},$$

where $\xi \sim N(0, 1)$. In other words, the noise is spatially independent. Model II becomes

$$-\nabla \cdot \left((a^{-1})^{\phi(-1)} \nabla u \right) = -(a^{-1})^{\phi(-1)} \Delta u = f(x),$$

which is equivalent to model I, since

$$-\Delta u = a^{-1} f(x) = a^{-1} f(x).$$
We now consider a perturbation of the coefficient given in equation (58)

\[ a(x, \omega) = e^{\sigma(1 + \epsilon \phi(x)) - \frac{1}{2} \sigma^2}, \] (61)

where \( \epsilon \) is a small positive number. When \( \epsilon \to 0 \), \( u_{\|} \to u_{I} \). We use the random coefficient (61) to mimic the case that \( l_c \to \infty \).

**Example 1.** Consider a one-dimensional exponential covariance kernel on \( x \in [0, 1] \)

\[ K(x_1, x_2) = e^{-\frac{|x_1 - x_2|}{lc}}. \]

Its eigenvalues satisfy

\[ w^2 = \frac{2\epsilon - \epsilon^2 \lambda_i}{\lambda_i}, \quad (w^2 - \epsilon^2) \tan(w) - 2\epsilon w = 0, \] (62)

where \( \epsilon = 1/l_c \). Its eigenfunctions are

\[ \phi_i(x) = \frac{w \cos(wx) + \epsilon \sin(wx)}{\sqrt{\frac{1}{2}(\epsilon^2 + w^2) + (w^2 - \epsilon^2) \frac{\sin(2w)}{4w} + \frac{\epsilon}{2}(1 - \cos(2w))}}. \] (63)

It can be shown that as \( \epsilon \to 0 \), \( w \sim \sqrt{2} \epsilon^{1/2} \), which results in that \( \lambda_1 = 1 + O(\epsilon) \) and \( \phi_1(x) = 1 + O(\epsilon) \). Thus it is reasonable to consider a perturbation given in equation (61) with \( \epsilon = 1/l_c \).

We here use a one-dimensional elliptic problem to examine the random coefficient (61) and present a numerical study of the convergence behavior of \( u_{\|} \to u_{I} \) as \( \epsilon \to 0 \). In figure [1] we plot the relative difference between \( u_{I} \) and \( u_{\|} \) defined as

\[ \epsilon_r = \frac{\|u_{I} - u_{\|}\|_{L^2(\Omega; H^1_0(D))}}{\|u_{I}\|_{L^2(\Omega; H^1_0(D))}} \]

with respect to \( \sigma \) and \( \epsilon \). It is seen that the dominant error takes a form

\[ \log(\epsilon_r) = \log(\epsilon) + 2 \log(\sigma) + C, \] (64)

i.e.,

\[ \epsilon_r \sim C \epsilon \sigma^2, \] (65)

where \( C \) is a general constant. This suggests that although model II provides a general second-order approximation of model I, the constant before \( \sigma^2 \) goes to zero linearly with respect to \( 1/l_c \) as \( l_c \) goes to infinity.

To accelerate the numerical algorithms for model I, such as Monte Carlo method and Galerkin projection method, we take the advantage of the small difference between \( u_{I} \) and \( u_{\|} \) either when \( \sigma \) is relatively small or the correlation length is relatively large such that the constant \( C(l_c) \) is close to 0, and the fact that \( u_{\|} \) can be obtained effectively. Based on this idea, we use the solution \( u_{\|} \) as a predictor of \( u_{I} \), or the stiffness matrix \( A_{\|} \) of model II as a preconditioner of \( A_{I} \).

### 5.1 Variance reduction for the Monte Carlo method

When the correlation length \( l_c \) is relatively small, eigenvalues of the covariance kernel decay slowly implying that a relatively large number of Gaussian random variables need to be kept for a good approximation of the log-normal random coefficient. For such a case, the Monte Carlo method can be more efficient than the Wiener chaos expansion. We then propose the following two-step methodology:

(i) **Predictor given by** \( u_{\|,h} \): We first consider Wiener chaos expansion of model II to obtain the numerical solution \( u_{\|,h} \). Its mean will be just the zeroth order coefficient \( u_{\|,h,(0)} \).
Numerical approximation of elliptic problems with log-normal random coefficients

(iii) A predictor-corrector method: Using the solution \( u_{II,h} \) as a control variate for variance reduction, we further refine the Monte Carlo simulations of \( u_{I,h} \) in the following way:

\[
\tilde{u}_{I,h}(x, \xi) := u_{II,h}(x; \xi) + (u_{I,h}(x; \xi) - u_{II,h}(x; \xi)),
\]

\[
E_{IS}[u_{I,h}](x) := E_{mc}[\tilde{u}_{I,h}](x) := \frac{1}{N_{mc}} \sum_{i=1}^{N_{mc}} \tilde{u}_{I,h}(x; \xi^{(i)}),
\]

where \( N_{mc} \) indicates the number of samples of \( \xi \) and \( \xi^{(i)} \) the \( i \)-th sample.

Based on equation (57), we have the following lemma:

**Lemma 6.**

We have the following error estimate

\[
\| E_{IS}[u_{I,h}] - E[u_{I,h}] \|_{L^2(D; H^1_0(D))} = \int_D \text{Var}(E_{IS}[u_{I,h}]) (x) dx = O(\sigma^4) N_{mc}^{-1}.
\]

**Proof.** Firstly, it is easy to check that \( E[ E_{IS}[u_{I,h}]] = E[u_{I,h}] \), so the first equal sign holds. Secondly,

\[
\text{Var}(E_{IS}[u_{I,h}]) = N_{mc}^{-1} \text{Var}(\tilde{u}_{I,h}) = N_{mc}^{-1} \text{Var}(u_{I,h} - u_{II,h})
\]

\[
= N_{mc}^{-1} \left[ \text{Var}(u_{I,h} - u_{II,h}) \right] - \text{Ev}^2[ u_{I,h} - u_{II,h} ]
\]

\[
\leq N_{mc}^{-1} \text{E} \left[ (u_{I,h} - u_{II,h})^2 \right]
\]

\[
= N_{mc}^{-1} \int (u_{I,h} - u_{II,h})^2 p(\xi) d\xi = O(\sigma^4) N_{mc}^{-1},
\]

where the last step is obtained using (57). Then the second equal sign of (68) is obtained by taking integration of the above equation with respect to spatial variable \( x \).

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From (68), we have
\[ \| E_{1S}[u_{1,h}] - E[u_{1,h}] \|_{L_2(F;H_1(D))} = O(\sigma)^N_{mc}^{-1/2}. \] (69)

Since a direct Monte Carlo method to calculate \( E[u_{1,h}] \) has an error \( O(1)N_{mc}^{-1} \), so the standard deviation reduction is quadratic with respect to \( \sigma \).

We now look at the computation cost. For the brute-force Monte Carlo method, the cost is \( O((\tau_1 + \tau_2)N_{mc}) \), where \( \tau_1 \) is the time for construction of the stiffness matrix and \( \tau_2 \) the time for solving a linear system. For the proposed strategy, the cost is \( O((\tau_1 + \tau_2 + \tau_3)N_{mc} + \tau_4) \), where \( \tau_3 \) is the time for the evaluation of \( u_{1,h}(x; \xi_i^{(1)}) \), which is much smaller than \( \tau_1 + \tau_2 \), and \( \tau_4 \) is the time to obtain \( u_{1,h} \). To obtain \( u_{1,h} \), only one stiffness matrix is needed. Since the uncertainty propagator is decoupled, \( \tau_4 \approx \tau_1 + N_{M,p} + \tau_1 \). Thus if a low-order Wiener chaos solution \( u_{1,h} \) serves as an effective control variate, the proposed strategy can be much more efficient than the brute-force Monte Carlo method, since \( N_{mc} \) can be much smaller than \( N_{mc} \) for the same accuracy.

**Remark 6.** Consider
\[ \tilde{u}_{1,h}(\alpha; x, \xi) = u_{1,h}(x; \xi) - \alpha(u_{1,h}(x; \xi) - u_{1,h}(0)(x)), \] (70)
where \( \alpha \) is a real number. It is well known that for all \( \alpha \in (-\infty, \infty) \), \( \tilde{u}_{1,h}(\alpha) \) provides an unbiased estimator of \( E[u_{1,h}] \) through
\[ E_{mc}[\tilde{u}_{1,h}] = \frac{1}{N_{mc}} \sum_{i=1}^{N_{mc}} \tilde{u}_{1,h}(\alpha; x, \xi^{(i)}), \] (71)
which holds for any \( x \in D \). For a fixed \( x \in D \), we know that if we choose \( \alpha^* = \frac{\sigma_{1,H}}{\sigma^2} \) with
\[ \sigma_i = E[(u_{1,h} - \bar{u}_{1,h})^2]^{1/2}, \quad i = 1, 2 \quad \text{and} \quad \sigma_{1,H} = E[(u_{1,h} - \bar{u}_{1,h})(u_{1,h} - \bar{u}_{1,h})], \]
the variance of \( \tilde{u}_{1,h} \) is minimized with respect to \( \alpha \) such that
\[ \text{Var}(\tilde{u}_{1,h})(\alpha^*) = \sigma_i^2(1 - \rho_{1,H})^2, \]

where \( \rho_{1,H} = \sigma_{1,H}/(\sigma_i \sigma_{1,H}) \) is the autocorrelation function of \( u_{1,h} \) and \( u_{1,H} \). Due to the fact given by equation (57) and theorem 2, \( \rho_{1,H} \approx 1 \) for small \( \sigma \) or large \( l_c \), when \( u_{1,h} \) and \( u_{1,H} \) are almost linear corresponding to \( \alpha^* \approx 1 \) (see more numerical experiments in [46]). This is the reason we choose \( \alpha = 1 \) in equation (67).

**Algorithm 1:** Variance reduction for Monte Carlo simulations

```python
Solve model II to obtain the Wiener chaos expansion of \( u_{1,h}(x, \xi) \).
for \( i = 1, 2, \ldots, N_{mc} \) do
    Sample model I to obtain \( u_{1,h}(x, \xi^{(i)}) \);
    Sample the solution of model II to obtain \( u_{1,H}(x, \xi^{(i)}) \);
    Update the statistics using an unbiased estimator as equation (67).
end
```

### 5.2 Stochastic Galerkin projection method

Due to the large number of unknowns and the strong coupling between the chaos coefficients \( u_{1,\alpha} \), iterative numerical methods are more appropriate for solving the linear system given by the finite element discretization of uncertainty propagator (39) of model I. In other words, an effective preconditioner is required. Consider the linear system
\[ A_1 u_1 = f. \] (72)
Let $u_{II}$ be a vector consisting of unknowns from the discretization of $u_{II,h}$ based on the same basis as that for $u_{I,h}$. Define $A_{II}$ as the stiffness matrix corresponding to the discretization of uncertainty propagator of model II. Then the stochastic finite element method for model II has the following matrix form

$$A_{II}u_{II} = f. \quad (73)$$

Based on structure of the uncertainty propagator of model II, we know that $A_{II}$ is a block lower triangular matrix

$$A_{II} = \begin{pmatrix}
A_{II,11} & 0 & \cdots & 0 \\
A_{II,21} & A_{II,22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{II,N_{M,p}1} & A_{II,N_{M,p}2} & \cdots & A_{II,N_{M,p},N_{M,p}}
\end{pmatrix}, \quad (74)$$

where the blocks $A_{II,ij}$ is defined as

$$A_{II,ij} = S_{\gamma(i)} - \alpha(j), \quad i \geq j. \quad (75)$$

with

$$(S_{\gamma(i)} - \alpha(j))_{m,n} = \int_D \hat{a}_{\gamma(i)} - \alpha(j)(x)\nabla \theta_m(x) \cdot \nabla \theta_n(x) \, dx. \quad (76)$$

Note that

$$A_{II,11} = A_{II,22} = \ldots = A_{II,N_{M,p},N_{M,p}} = S_{(0)}. \quad (77)$$

**Lemma 7.**

Consider the stiffness matrices $A_I$ and $A_{II}$. We have that the condition number

$$\kappa (A_{II}^{-1}A_I) \leq 1 + O(\sigma^2). \quad (78)$$

**Proof.** Since the difference between $u_I$ and $u_{II}$ is of $O(\sigma^2)$, we have in the matrix form

$$\|u_I - u_{II}\| = \|A_I^{-1}f - A_{II}^{-1}f\| = O(\sigma^2), \quad (79)$$

which holds for any $f$. Hence

$$\|A_I^{-1} - A_{II}^{-1}\| = O(\sigma^2). \quad (80)$$

Then the condition number of $A_{II}^{-1}A_I$ is

$$\kappa = \|A_{II}^{-1}A_I\| \|A_{II}^{-1}A_{II}\|$$

$$= \| (A_{II}^{-1} - A_I^{-1} + A_I^{-1})A_I \| \| (A_I^{-1} - A_{II}^{-1} + A_{II}^{-1})A_{II} \|$$

$$= \| I + (A_{II}^{-1} - A_I^{-1})A_I \| \| I + (A_I^{-1} - A_{II}^{-1})A_{II} \|$$

$$\leq 1 + \| A_I \| \| A_{II} \| (O(\sigma^2) + O(\sigma^4)).$$

$$\quad (81)$$

**Remark 7.** When $\sigma$ is relatively small, we expect that $A_{II}$ can provide a good preconditioner for linear system $73$. Instead of solving equation $73$, we can solve

$$A_{II}^{-1}A_Iu_I = A_{II}^{-1}f. \quad (82)$$
5.2.1 Preconditioned Richardson’s iteration

One commonly used iterative method for the uncertainty propagator \([39]\) of model I is the block Gauss-Seidel method, which can be expressed as

\[
- \nabla \cdot \left( \mathbb{E} \left[ a(\mathbf{x}, \omega)H_2^\gamma \right] \nabla u_{I,n+1}^\gamma(\mathbf{x}) \right) \\
= \sum_{i=1}^{k(\gamma)-1} \nabla \cdot \left( \mathbb{E} \left[ a(\mathbf{x}, \omega)H_{\alpha(i)}^\gamma \right] \nabla u_{\alpha(i),n+1}^\gamma(\mathbf{x}) \right) \\
+ \sum_{i=k(\gamma)+1}^{N_{M,p}} \nabla \cdot \left( \mathbb{E} \left[ a(\mathbf{x}, \omega)H_{\alpha(i)}^\gamma \right] \nabla u_{\alpha(i),n}^\gamma(\mathbf{x}) \right) + f(\mathbf{x})\delta(0), \forall \mathbf{y} \in \mathcal{J}_{M,p},
\]

where the superscript \(n\) indicates the iteration step. It is shown in Lemma 4 that \(\mathbb{E} \left[ a(\mathbf{x}, \omega)H_2^\gamma \right]\) is strictly positive. We know that the block Gauss-Seidel method corresponds to a fixed point iteration on a preconditioned system \(M^{-1}A_1 u = M^{-1}f\),

where \(M\) is the lower-triangular part of matrix \(A_1\). Based on the comparability of models I and II, we can construct the following preconditioned Richardson’s iterative method \([35]\):

\[
u_{I,k+1} = u_{I,k} + \gamma A_1^{-1} (A_1 u_{I,k} - f),\]

where \(\gamma\) is the non-negative acceleration parameter. We know that the Richardson’s iterative method converges when \(\gamma < 2/\rho(A_1^{-1} A_1)\), where \(\rho(\cdot)\) indicates the spectral radius of a matrix. Based on the relation between \(A_1\) and \(A_{II}\), we expect that \(\rho((A_{II})^{-1} A_1)\) is close to 1 when \(\sigma\) is relatively small.

5.2.2 Preconditioned GMRES method

We also consider Krylov subspace methods. Since \(A_1\) is symmetric and positive definite, a common choice to solve the linear system is preconditioned Conjugate Gradient (CG) method. We here consider to use \(A_{II}\) as a preconditioner, which is not symmetric. Hence we use a preconditioned GMRES method \([35]\) instead of CG method.

6. NUMERICAL RESULTS

We consider both one-dimensional and two-dimensional \((D = [-1, 1]^d, d = 1, 2)\) elliptic problem with random coefficient subject to a non-zero force term

\[
f(\mathbf{x}) = \prod_{i=1}^{d} (x_i^2 + 4x_i + 1)e^{x_i}
\]

and homogeneous boundary conditions. Assume the underlying Gaussian random field of the log-normal coefficient \(a(\mathbf{x}, \omega) = e^{G(\mathbf{x}, \omega) - \frac{1}{2} \sigma^2}\), with \(G\)’s correlation function is given by:

\[
K(\mathbf{x}_1, \mathbf{x}_2) = e^{-\frac{|x_1 - x_2|^2}{2\ell_c^2}},
\]

or

\[
K(\mathbf{x}_1, \mathbf{x}_2) = e^{-\frac{|x_1 - x_2|^2}{r^2}},
\]

where \(\ell_c\) being the correlation length and \(\sigma\) the standard deviation. Due to the analyticity of the Gaussian kernel, the eigenvalues decay exponentially \([12]\). The decay rate is determined by the value of the correlation length, where a larger \(\ell_c\) corresponds to a faster decay rate. The physical discretization is given by 25 uniform finite element with
order $q = 4$ for the one-dimensional case, and $32 \times 32$ uniform quadratic finite elements for the two-dimensional cases. We test the parameters $\sigma = 0.2, 0.6, 1$ and $l_c = 20, 2, 0.2$. The solution differences of model I and model II is similar to the results in [46] and [47]. So we only sketch the results for two-dimensional case here.

The results for 2-dimensional case with Gaussian type kernel are given in Fig 2, 3, 4 for $l_c = 20, 2, 0.2$, respectively. The results for 2-dimensional exponential kernel with $l_c = 20, 2, 0.2$ are given in Fig 5, 6, 7, respectively. The truncation errors of the K-L expansion for the Gaussian kernel and exponential kernel are set to be $2 \times 10^{-3}$ and $3 \times 10^{-2}$, respectively. For Model I, if the dimension of the stochastic space $M$ is less than 20, we use stochastic Galerkin method, otherwise we use Monte Carlo method. From these figures, we say for small $\sigma$ values, the results of Model II agree very well with the results of Model I. A larger correlation length $\ell_c$ also makes a better agreement between the results of Model I and Model II. This is consistent to the theoretical results.

6.1 Using $u_{II,h}$ as a control variate

When the correlation length is relatively small, a large number of random variables are required to represent the random coefficient and the Monte Carlo method would be a better choice for computation. The mean and variance are given by the following unbiased estimators, respectively:

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FIG. 4: The average (left) and standard deviation (right) of model I and II at the horizontal line $y = 0$: Gaussian kernel with $\ell_c = 0.2$, $M = 94$, and $p = 1$ are used for the stochastic Galerkin approximation of Model II. $M = 94$ and $N_{mc} = 10000$ are used for the Monte Carlo method of model I.

FIG. 5: The average (left) and standard deviation (right) of model I and II at the horizontal line $y = 0$: exponential kernel with $\ell_c = 20$, $M = 3$, and $p = 8$ are used for the stochastic Galerkin approximation of both Model I and Model II.

FIG. 6: The average (left) and standard deviation (right) of model I and II at the horizontal line $y = 0$: exponential kernel with $\ell_c = 2$, $M = 28$, and $p = 2$ are used for the stochastic Galerkin approximation of Model II. $M = 28$ and $N_{mc} = 10000$ are used for the Monte Carlo method of model I.
For almost all the cases, the preconditioned Richardson iteration and GMRES are both better than the commonly-used Gauss-Seidel iteration, especially for large $l_c$ or small $\sigma$. The iteration numbers of Richardson method and GMRES are much smaller than Gauss-Seidel method, meanwhile their increases with respect to the standard deviation parameter $\sigma$ are also slower, except for the cases with $p = 1$. For large variance, the preconditioned GMRES method behaves much better than Gauss-Seidel and Richardson methods. Note that we use the solution of model II as initial values for Richardson and GMRES iterations, so in the cases that model II is a very good approximation of model I, the corresponding iteration numbers are 0.

According to our understanding of $u_{II}$, the worst scenario for the proposed preconditioners is when $l_c$ is small and $\sigma$ is large. In a very few cases (e.g. $l_c = 0.2$ and $\sigma = 0.6, 1$ in Table 3 and 4), the preconditioned Richardson iteration requires more iterations to converge than Gauss-Seidel, this probably because a first order Wiener Chaos approximation is used, the big approximation error together with the big modeling error deteriorate the performance of the preconditioning and the parameter $\omega$ in the Richardson method is not optimal.

6.2 Using $A_{II}$ as a preconditioner

The results of using model II to precondition model I is given in Table 1 and 2 (for 1-d cases) and Table 3 and 4 (for 2-d cases). We set default relaxation parameter in Richardson iteration to $\gamma = 1/(1 + 3\sigma^2)$.

As for all the cases, the preconditioned Richardson iteration and GMRES are both better than the commonly-used Gauss-Seidel iteration, especially for large $l_c$ or small $\sigma$. The iteration numbers of Richardson method and GMRES are much smaller than Gauss-Seidel method, meanwhile their increases with respect to the standard deviation parameter $\sigma$ are also slower, except for the cases with $p = 1$. For large variance, the preconditioned GMRES method behaves much better than Gauss-Seidel and Richardson methods. Note that we use the solution of model II as initial values for Richardson and GMRES iterations, so in the cases that model II is a very good approximation of model I, the corresponding iteration numbers are 0.

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FIG. 7: The average (left) and standard deviation (right) of model I and II at the horizontal line $y = 0$: exponential kernel with $\ell_c = 0.2$, $M = 86$ and $p = 1$ are used for the stochastic Galerkin approximation of Model II. $M = 86$ and $N_{mc} = 10000$ are used for the Monte Carlo method of model I.
FIG. 8: The mean and standard deviation of the Monte Carlo method for model I with and without important sampling in 1-dimensional case. The exponential kernel with correlation length \( l_c = 1 \) is used. \( M = 12, p = 4 \) for the stochastic Galerkin approximation of Model II. \( M = 12, N_{mc} = 10000 \) for the Monte Carlo method. Note that log scale is used for the standard deviation.

FIG. 9: The variance reduction for the 1-dimensional case with exponential kernel having different correlation length and different values of \( \sigma \). The y-axes are \( \| \var{\tilde{u}_{I,h}} \|_{H^1(D)} / \| \var{u_{I,h}} \|_{H^1(D)} \). \( N_{mc} = 10000 \) samples are used for the Monte Carlo method. The tolerance of K-L expansion is set to \( 3 \times 10^{-2} \). The values of \( M, p \) corresponding to the stochastic Galerkin approximation of Model II with \( l_c = 8, 4, 2, 1, 0.5, 0.25 \) are \((3, 6), (4, 6), (7, 5), (12, 4), (19, 3), (27, 3)\), respectively. Note that log scales are used for both \( x \) and \( y \) axes.
FIG. 10: The mean and standard deviation of the Monte Carlo method for model I with and without important sampling in 2-dimensional case. The exponential kernel with correlation length $l_c = 2$ is used. $M = 19, p = 2$ for the stochastic Galerkin approximation of Model II. $M = 19, N_{mc} = 1000$ for the Monte Carlo method. Note that log scale is used for the standard deviation.

FIG. 11: The variance reduction for the 2-dimensional case with exponential kernel having different correlation lengths and different values of $\sigma$. The y-axes are $\|\text{Var}(\hat{u}_{1,h})\|_{H_0^1(D)}/\|\text{Var}(u_{1,h})\|_{H_0^1(D)}$. $N_{mc} = 1000$ samples are used for the Monte Carlo method. The tolerance of K-L expansion is set to $3 \times 10^{-2}$. The values of $M, p$ corresponding to the stochastic Galerkin approximation of Model II with $l_c = 8, 4, 2, 1, 0.5, 0.25$ are $(5, 5), (11, 4), (19, 3), (28, 3), (35, 2), (40, 2)$, respectively. Note that log scales are used for both x and y axes.
TABLE 1: Preconditioning results of 1-dimensional problem with Gaussian kernel. $n_{GS}, n_\gamma, n_{GMRES}$ means the iteration number of Gauss-Seidel, Richardson and GMRES, respectively. We take $\gamma = 1/(1+3\sigma^2)$ for the Richardson method. The tolerance of K-L expansion is set to $2 \times 10^{-3}$. The relative tolerance for the iteration solvers is set to $10^{-3}$.

| $\ell_c$ | $\sigma$ | $M$ | $p$ | $N_{M,p}$ | $n_{GS}$ | $n_\gamma$ | $n_{GMRES}$ |
|---------|----------|-----|-----|-----------|-----------|------------|------------|
| 20      | 0.2      | 1   | 10  | 11        | 3         | 0          | 0          |
| 20      | 0.6      | 1   | 10  | 11        | 27        | 0          | 0          |
| 20      | 1        | 1   | 10  | 11        | >100      | 22         | 5          |
| 2       | 0.2      | 3   | 1   | 286       | 3         | 1          | 1          |
| 2       | 0.6      | 3   | 1   | 286       | 22        | 3          | 1          |
| 2       | 1        | 3   | 10  | 286       | >100      | 19         | 9          |
| 0.2     | 0.2      | 11  | 3   | 364       | 3         | 1          | 1          |
| 0.2     | 0.6      | 11  | 3   | 364       | 10        | 5          | 5          |
| 0.2     | 1        | 11  | 3   | 364       | 29        | 12         | 9          |

TABLE 2: Preconditioning results of 1-dimensional problem with exponential kernel. $n_{GS}, n_\gamma, n_{GMRES}$ means the iteration number of Gauss-Seidel, Richardson and GMRES, respectively. We take $\gamma = 1/(1+3\sigma^2)$ for the Richardson method. The tolerance of K-L expansion is set to $3 \times 10^{-2}$. The relative tolerance for the iteration solvers is set to $10^{-3}$.

| $\ell_c$ | $\sigma$ | $M$ | $p$ | $N_{M,p}$ | $n_{GS}$ | $n_\gamma$ | $n_{GMRES}$ |
|---------|----------|-----|-----|-----------|-----------|------------|------------|
| 20      | 0.2      | 2   | 10  | 66        | 3         | 0          | 0          |
| 20      | 0.6      | 2   | 10  | 66        | 24        | 2          | 1          |
| 20      | 1        | 2   | 10  | 66        | >100      | 16         | 9          |
| 2       | 0.2      | 8   | 5   | 1287      | 3         | 1          | 1          |
| 2       | 0.6      | 8   | 5   | 1287      | 17        | 4          | 3          |
| 2       | 1        | 8   | 5   | 1287      | >100      | 9          | 9          |
| 0.2     | 0.2      | 51  | 2   | 1378      | 3         | 1          | 1          |
| 0.2     | 0.6      | 51  | 2   | 1378      | 7         | 5          | 3          |
| 0.2     | 1        | 51  | 2   | 1378      | 15        | 7          | 6          |

Based on the above observations, we advocate to use GMRES with model II as a preconditioner for solving the model I.

In the end, we compare our approach with some existing methods by solving a test problem studied in [32]. The physical domain is set to $[0, 1]^2$, and the force term $f(x) = 1$. The underlying Gaussian field of the log-normal coefficient $a(x, \omega)$ has a correlation function $K(x_1, x_2) = \sigma^2 rK_1(r)$, where $r = \|x_1 - x_2\|_2$ and $K_1$ is the modified Bessel function of the second kind with order one. Set $M = 5$ in the K-L expansion, such that 97% of the Gaussian field’s total variance is captured. The iteration numbers of Richardson and GMRES method for the stochastic Galerkin method of model I with model II as preconditioner for different $\sigma$ and $p$ are given in Table 5. From the table, we see that both Richardson and GMRES methods are efficient. As $p$ increases, the iteration numbers increase slowly. As $\sigma$ increases, the iteration numbers also increase slowly. The preconditioning effects are still very good for the cases with $\sigma = 1$. These results are very competitive comparing to the algebraic preconditioners studied in [32] for this test example.

7. SUMMARY

In this work, we consider the Wick approximation of two stochastic elliptic problems with log-normal random coefficients, where Model II is a second order approximation of model I with respect to $\sigma$. Model II can be used as a
Numerical approximation of elliptic problems with log-normal random coefficients

| $l_c$ | $\sigma$ | $M$ | $p$ | $N_{M,p}$ | $n_{GS}$ | $n_{\gamma}$ | $n_{GMRES}$ |
|-------|-----------|-----|-----|-----------|----------|-------------|-------------|
| 20    | 0.2       | 1   | 16  | 17        | 3        | 0           | 0           |
| 20    | 0.6       | 1   | 16  | 17        | 25       | 0           | 0           |
| 20    | 1         | 1   | 16  | 17        | 29       | 1           | 1           |
| 2     | 0.2       | 4   | 5   | 126       | 3        | 0           | 0           |
| 2     | 0.6       | 4   | 5   | 126       | 17       | 5           | 4           |
| 2     | 1         | 4   | 5   | 126       | 48       | 14          | 7           |
| 0.2   | 0.2       | 80  | 1   | 81        | 2        | 1           | 1           |
| 0.2   | 0.6       | 80  | 1   | 81        | 3        | 4           | 2           |
| 0.2   | 1         | 80  | 1   | 81        | 4        | 7           | 3           |

**TABLE 3:** Preconditioning results of 2-dimensional problem with Gaussian kernel. $n_{GS}$, $n_{\gamma}$, $n_{GMRES}$ means the iteration number of Gauss-Seidel, Richardson and GMRES, respectively. We take $\gamma = 1/(1 + 3\sigma^2)$ for the Richardson method. The tolerance of K-L expansion is set to $10^{-2}$. The relative tolerance for the iteration solvers is set to $10^{-3}$.

| $l_c$ | $\sigma$ | $M$ | $p$ | $N_{M,p}$ | $n_{GS}$ | $n_{\gamma}$ | $n_{GMRES}$ |
|-------|-----------|-----|-----|-----------|----------|-------------|-------------|
| 20    | 0.2       | 3   | 8   | 165       | 3        | 0           | 0           |
| 20    | 0.6       | 3   | 8   | 165       | 12       | 1           | 1           |
| 20    | 1         | 3   | 8   | 165       | 41       | 14          | 10          |
| 2     | 0.2       | 28  | 2   | 435       | 3        | 1           | 1           |
| 2     | 0.6       | 28  | 2   | 435       | 4        | 3           | 3           |
| 2     | 1         | 28  | 2   | 435       | 10       | 9           | 4           |
| 0.2   | 0.2       | 86  | 1   | 87        | 2        | 1           | 1           |
| 0.2   | 0.6       | 86  | 1   | 87        | 2        | 3           | 2           |
| 0.2   | 1         | 86  | 1   | 87        | 4        | 7           | 3           |

**TABLE 4:** Preconditioning results of 2-dimensional problem with exponential kernel. $n_{GS}$, $n_{\gamma}$, $n_{GMRES}$ means the iteration number of Gauss-Seidel, Richardson and GMRES, respectively. We take $\gamma = 1/(1 + 3\sigma^2)$ for the Richardson method. The tolerance of K-L expansion is set to $3 \times 10^{-2}$. The relative tolerance for the iteration solvers is set to $10^{-3}$.

| $\sigma$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |
|----------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.2      | 5       | 6       | 5       | 6       | 6       | 3       | 3       | 4       | 4       | 4       |
| 0.4      | 10      | 10      | 11      | 10      | 10      | 3       | 4       | 5       | 6       | 7       |
| 0.6      | 14      | 16      | 17      | 18      | 19      | 4       | 5       | 6       | 7       | 8       |
| 0.8      | 16      | 19      | 21      | 23      | 25      | 5       | 6       | 7       | 8       | 9       |
| 1.0      | 16      | 19      | 21      | 24      | 26      | 5       | 7       | 8       | 9       | 11      |

**TABLE 5:** The iteration numbers of Richardson and GMRES method solving the 2-dimensional problem with Matern-tye kernel studied in [32]. We take $\gamma = 1/(1 + 3\sigma^2)$ for the Richardson method. The relative tolerance for the iteration solvers is set to $10^{-8}$. $M = 5$. 
precondition for model I in a stochastic Galerkin method. The numerical results show that the preconditioned Richardson iteration is better than commonly used Gauss-Seidel method when $\sigma$ is small or $l_c$ is large. Meanwhile, the former method have a parameter to tune. The preconditioned GMRES method works very well for all the values of $\sigma$ and $l_c$ tested using defaults parameters. The model II can also be used as an efficient important sampling process for model I to reduce the variance of a Monte Carlo approach when the stochastic dimension in a Karhunen-Loève expansion is very high.

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