On Wiener – Hopf factorization of scalar polynomials

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Abstract

In the work we propose an algorithm for a Wiener – Hopf factorization of scalar polynomials based on notions of indices and essential polynomials. The algorithm uses computations with finite Toeplitz matrices and permits to obtain coefficients of both factorization factors simultaneously. Computation aspects of the algorithm are considered. An a priori estimate for the condition number of the used Toeplitz matrices is obtained. Upper bounds for the accuracy of the factorization factors are established. All estimates are effective.

Keywords: Wiener – Hopf factorization, polynomial factorization, Toeplitz matrices

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1. Introduction

The Wiener – Hopf technique is a powerful method used in various areas of mathematics, mechanics and mathematical physics (see, e.g., [11], [14], [16]). The core of the method is the Wiener – Hopf factorization problem for matrix functions (or the Riemann boundary-value problem) [10, 14]. In the scalar case there is an explicit formula for a solution of the problem [12], but its use for numerical computations is quite difficult. Therefore to factorize a scalar function, it is usually approximated by a rational function (see, e.g., [15]). Thus the scalar factorization reduces to the polynomial factorization.

The matrix case is more complicated because there are no explicit formulas for the factorization factors and for important integer invariants of the problem such as partial indices. Moreover, it is difficult to develop approximate methods of the factorization since the matrix factorization is unstable. For this reason it is very important to find cases when the problem can be effectively or explicitly solved. Almost all cases of constructive factorization known to date are considered in the review [17]. In [2] an explicit method for the factorization of meromorphic matrix functions was proposed. In this paper the factorization
problem is called explicitly solved if it is reduced to the factorization of given scalar functions and to solving of finitely many finite systems of linear algebraic equations. In [4] some classes of matrix functions are listed for which the factorization problem can be explicitly reduced to the factorization of analytic matrix functions.

In this paper, we start a project to develop algorithms of the factorization and their implementations for these classes of matrix functions. We propose to consider the factorization of matrix polynomials, analytic and meromorphic matrix functions, triangular matrix functions and matrix functions with one non-meromorphic row. Some preliminary results in this direction were obtained in [5].

The first stage in solving of the factorization problem for the above-mentioned classes is the factorization of scalar functions that can be reduced to the polynomial factorization.

We consider coefficients of a polynomial $p(z)$ as initial data and coefficients of its factors as output data of the problem. The naive method of the polynomial factorization is the following way. First we compute roots of a polynomial $p(z)$ and perform their separation. Second we find the coefficients of the factors of $p(z)$ by their roots. It is well known that the roots of a polynomial are in general not well-condition functions of its coefficients (see, e.g., [13]), and coefficients of a polynomial are also not well-condition functions of its roots [18].

The latter means that, in general, we can not solve numerically the polynomial factorization problem by the naive way.

Nevertheless there exist numerical methods for solving of this problem. The basic works in this direction are cited in [8]. It should be especially pointed out the articles [8]–[7], where finding of the polynomial factorization was based on computations with Toeplitz matrices.

Our approach resembles the method proposed by D.A. Bini and A. Böttcher [8] in algorithm 3. As the authors we solve systems with finite Toeplitz matrices consisting of Laurent coefficients of the function $1/p(z)$. The main difference between the method of D.A. Bini and A. Böttcher and ours is that algorithm 3 permits to obtain only coefficients of the factor $p_-(z)$ for $p(z)$, whereas we find coefficients of both factors simultaneously. This is important because the polynomial division is, in general, not well-condition operation in numerical computations. Moreover, we use a different technique that can be extended to the factorization problem for analytic functions. In this case we can obtain all coefficients of the polynomial factor and a required number of Taylor coefficients of the analytic factor.

The paper is organized as follows. In section 2 we consider the setting of the polynomial factorization problem and formulate some results on used norms of polynomials and their estimates. Section 3 contains basic tools for solving of the problem. Here we introduce notions of indices and essential polynomials in terms of which the problem will be solved. In Section 4 we prove the basic results on the factorization of a polynomial $p(z)$. In Section 5 some computation aspects of the factorization problem are considered. Here we obtain an a priori estimate for the condition number of the used Toeplitz matrices and establish
upper bounds for the accuracy of the factorization factors. All estimates are effective. Section 6 contains an algorithm and numerical examples.

2. Preliminaries

Let \( p(z) = p_0 + p_1 z + \cdots + p_\nu z^\nu \) be a complex polynomial of degree \( \nu > 1 \) and \( p_0 \neq 0, p_\nu = 1 \). We suppose that \( p(z) \neq 0 \) on the unit circle \( T \), hence \( p(z) \neq 0 \) on a closed circular annulus \( K := \{ z \in \mathbb{C} : r \leq |z| \leq R \} \) for some \( 0 < r < 1 < R < \infty \). Denote by \( \kappa = \text{ind}_T p(z) \) the index of \( p(z) \) with respect to \( T \), i.e. the number of zeros of the polynomial inside the unit circle. Let \( \xi_j, j = 1, \ldots, \nu \), be the zeros of \( p(z) \) and

\[
0 < |\xi_1| \leq \ldots \leq |\xi_\kappa| < r < 1 < R < |\xi_{\kappa+1}| \leq \ldots \leq |\xi_\nu|.
\]

Here the zeros are counted according to their multiplicity.

In the work we will consider the following factorization of \( p(z) \)

\[
p(z) = p_1(z)p_2(z),
\]

where \( p_1(z) := (z - \xi_1) \cdots (z - \xi_\kappa), \quad p_2(z) := (z - \xi_{\kappa+1}) \cdots (z - \xi_\nu). \)

Denote \( p_-(z) = \frac{p_1(z)}{z^{\kappa}}, \quad p_+(z) = p_2(z). \) Then the representation

\[
p(z) = p_-(z)z^\kappa p_+(z), \quad |z| = 1,
\]

is the Wiener – Hopf factorization of \( p(z) \) normalized by the condition \( p_-(\infty) = 1 \).

Throughout this paper, \( \|x\| \) means the Hölder 1-norm

\[
\|x\| = |x_1| + \cdots + |x_k|,
\]

\( x = (x_1, \ldots, x_k)^T \in \mathbb{C}^k \). A norm \( \|A\| \) of a matrix \( A \in \mathbb{C}^{\ell \times k} \) is always the induced norm

\[
\|A\| = \max_{1 \leq j \leq k} \sum_{i=1}^{\ell} |A_{ij}|.
\]

Respectively, the norm of a polynomial \( p(z) = p_0 + p_1 z + \cdots + p_\nu z^\nu \) is the norm of the vector \( (p_0, p_1, \ldots, p_\nu)^T \). For \( p(z) \) we will also use the maximum norm

\[
\|p\|_C = \max_{z \in \mathbb{T}} |p(z)|
\]

on the unit circle \( \mathbb{T} \).

The norms \( \| \cdot \| \) and \( \| \cdot \|_C \) are equivalent. Clearly, \( \|p\|_C \leq \|p\| \). Since for \( p(z) \) it is fulfilled the equality

\[
\sum_{k=0}^{\nu} |p_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\varphi})|^2 d\varphi,
\]
we have \( \|p\| \leq \sqrt{\nu + 1} \|p\|_2 \leq \sqrt{\nu + 1} \|p\|_C. \) Thus,
\[
\|p\|_C \leq \|p\| \leq \sqrt{\nu + 1} \|p\|_C.
\]

In order to study stability of the factorization problem, we will need estimates for the norm of inverses of some Toeplitz matrices. Such estimates will be obtained in terms of \( \|p_1\| \|p_2\|, \) where \( p_1(z), p_2(z) \) are the factorization factors of \( p(z) \). To get effective estimates, it will be required to estimate \( \|p_1\| \|p_2\| \) via \( \|p\| \).

Let \( q(z) = q_1(z)q_2(z), \) where \( q_1(z), q_2(z) \) are arbitrary monic complex polynomials and \( \nu = \deg q \). It is obvious that
\[
\|q\| \leq \|q_1\| \|q_2\|.
\]

In the work [9], D.W. Boyd proved the following inequality
\[
\|q_1\|_C \|q_2\|_C \leq \delta^\nu \|q\|_C,
\]
where \( \delta = e^{2G/\pi} = 1.7916228120695934247 \ldots \) and \( G \) is Catalan’s constant. The inequality is asymptotically sharp as \( \nu \to \infty \).

Taking into account (3), in our case we obtain
\[
\|p\| \leq \|p_1\| \|p_2\| \leq \delta^\nu \sqrt{\nu + 1} \|p\|.
\]

However, the exponential factor \( \delta^\nu \) can overestimate the upper bound. In some special cases we can obtain more precise estimates. For example, this can be done for a so-called spectral factorization of polynomials.

**Proposition 2.1.** Let \( p(z) = \sum_{j=0}^{2m} p_j z^j \) be a complex polynomial of degree \( 2m \) such that \( p_{2m-j} = \bar{p}_j \) for \( j = 0, \ldots, m \), \( p_0 = 1 \). Suppose that \( p(z) \neq 0 \) on \( \mathbb{T} \).

If \( p(z) = p_1(z)p_2(z) \) is factorization (1), then
\[
\|p_1\| \|p_2\| \leq (m + 1) \|p\|.
\]

**Proof.** It is clear that if \( \xi_k \) is a root of \( p(z) \), then so is \( \frac{1}{\bar{\xi}_k} \). Hence \( m \) is the index of \( p(z) \).

Let \( |\xi_k| < 1 \) for \( k = 1, \ldots, m \) and
\[
p_1(z) = (z - \xi_1) \cdots (z - \xi_m), \quad p_2(z) = (z - 1/\xi_1) \cdots (z - 1/\bar{\xi}_m)
\]
are the factorization factors of \( p(z) \). On the unit circle we have
\[
\overline{p_1(t)} = (-1)^m (\xi_1 \cdots \xi_m) t^{-m} p_2(t) = p_1(0) t^{-m} p_2(t), \quad |t| = 1.
\]
It follows from this that \( \|p_1\|_C = \|p_1(0)\| \|p_2\|_C \) and
\[
p(t) = \frac{t^m}{p_1(0)} \overline{p_1(t) p_1(t)}.
\]
Thus we have
\[
\|p\|_C = \frac{1}{\|p_1(0)\|} \|p_1\|_C^2 = \|p_1\| \|p_2\|_C.
\]
Applying inequality (3), we arrive at the assertion. \( \square \)
It is possible that the equality $\|p\| = \|p_1\| \|p_2\|$ holds.

**Proposition 2.2.** Let $p(z) = \sum_{j=0}^{2m} p_j z^j$ be a real polynomial of degree $2m$ and $p_{2m-j} = p_j$ for $j = 0, \ldots, m$, $p_0 = 1$. Suppose that $p(z) \neq 0$ on $\mathbb{T}$ and all roots of $p(z)$ have negative real parts.

If $p(z) = p_1(z)p_2(z)$ is factorization (1), then

$$\|p\| = \|p_1\| \|p_2\|.$$

**Proof.** In this case, if $\xi_k$ is a root of $p(z)$, then so are $\bar{\xi}_k$ and $1/\xi_k$.

Let $|\xi_k| < 1$ for $k = 1, \ldots, m$. Then

$$p_1(z) = (z - \xi_1) \cdots (z - \xi_m), \quad p_2(z) = (z - 1/\xi_1) \cdots (z - 1/\xi_m).$$

If $z = \xi$ is a complex root of $p(z)$, then $(z - \xi)(z - \bar{\xi})$ has positive coefficients. Obviously, $z - \xi$ has also positive coefficients for a real root $\xi$. Then $p_1(z)$ has positive coefficients as a product of such polynomials. Similar statement holds for $p_2(z)$ and $p(z)$.

Thus we have $\|p\| = p(1) = p_1(1)p_2(1) = \|p_1\| \|p_2\|$. \qed

Since for a given polynomial $p(z)$ more precise estimates can exist, we will use the inequalities

$$\|p\| \leq \|p_1\| \|p_2\| \leq \delta_0 \|p\|, \quad 1 \leq \delta_0 \leq \delta^{\nu} \sqrt{(\xi + 1)(\nu - \xi + 1)},$$

instead of (4).

**3. Basic tools**

Let $M, N$ be integers, $M < N$, and $c^N_M = (c_M, c_{M+1}, \ldots, c_N)$ a nonzero sequence of complex numbers. In this section we introduce notions of indices and essential polynomials for the sequence $c^N_M$. These notions were given in more general setting in the paper [3]. Here we will consider the scalar case only. The proofs of all statements of this section can be found in [3].

Let us form the family of all Toeplitz matrices

$$T_k(c^N_M) = \begin{pmatrix} c_k & c_{k+1} & \cdots & c_M \\ c_k & \cdots & c_{M+1} \\ \vdots & \vdots & \vdots \\ c_N & c_{N-1} & \cdots & c_{N+M-k} \end{pmatrix}, \quad M \leq k \leq N,$$

which can be constructed with the help of the sequence $c^N_M$. We will used the short designation $T_k$ in place of $T_k(c^N_M)$ if there is not the possibility of misinterpretation.

Our nearest aim is to describe a structure of the kernels $\ker T_k$. It is more convenient to deal not with vectors $Q = (q_0, q_1, \ldots, q_{k-M})^T \in \ker T_k$ but with their generating polynomials $Q(z) = q_0 + q_1 z + \cdots + q_{k-M} z^{k-M}$. We will use
the spaces $N_k$ of the generating polynomials instead of the spaces $\ker T_k$. The generating function $\sum_{j=M}^{N} c_k z^k$ of the sequence $c_M^N(z)$ will be denoted by $c_M^N(z)$.

Let us introduce a linear functional $\sigma$ by the formula:

$$\sigma\{z^j\} = c_{-j}, \quad -N \leq j \leq -M.$$  

The functional is defined on the space of rational functions of the form

$$Q(z) = \sum_{j=-N}^{-M} q_j z^j.$$  

Besides this algebraic definition of $\sigma$ we will use the following analytic definition

$$\sigma\{Q(z)\} = \frac{1}{2\pi i} \int_{\Gamma} t^{-1} c_M^N(t) Q(t) \, dt.$$  

(7)

Here $\Gamma$ is any closed contour around the point $z = 0$.

Denote by $N_k$ ($M \leq k \leq N$) the space of polynomials $Q(z)$ with the formal degree $k - M$ satisfying the orthogonality conditions:

$$\sigma\{z^{-i} Q(z)\} = 0, \quad i = k, k+1, \ldots, N.$$  

(8)

It is easily seen that $N_k$ is the space of generating polynomials of vectors in $\ker T_k$. For convenience, we put $N_{M-1} = 0$ and denote by $N_{N+1}$ the $(N-M+2)$-dimensional space of all polynomials with the formal degree $N-M+1$. If necessary, the more detailed notation $N_k(c_M^N)$ instead of $N_k$ is used.

Let $d_k$ be the dimension of the space $N_k$ and $\Delta_k = d_k - d_{k-1}$ ($M \leq k \leq N+1$). The following proposition is crucial for the further considerations.

Proposition 3.1. For any non-zero sequence $c_M^N$ the following inequalities

$$0 = \Delta_M \leq \Delta_{M+1} \leq \ldots \leq \Delta_N \leq \Delta_{N+1} = 2$$  

(9)

are fulfilled. $\square$

It follows from the inequalities (9) that there exist integers $\mu_1 \leq \mu_2$ such that

$$\Delta_M = \ldots = \Delta_{\mu_1} = 0,$$

$$\Delta_{\mu_1+1} = \ldots = \Delta_{\mu_2} = 1,$$

$$\Delta_{\mu_2+1} = \ldots = \Delta_{N+1} = 2.$$  

(10)

If the second row in these relations is absent, we assume $\mu_1 = \mu_2$.

Definition 3.1. The integers $\mu_1$, $\mu_2$ defined in (10) will be called the essential indices (briefly, indices) of the sequence $c_M^N$.

The following proposition gives formulas for the indices.
Proposition 3.2. Let $\pi = \text{rank} T_{[N+M]}$, where $[N+M]$ is the integral part of $N+M/2$. Then the indices $\mu_1$, $\mu_2$ are found by the formulas:

$$\mu_1 = M + \pi - 1, \quad \mu_2 = N - \pi + 1. \quad \square$$

It follows from the definition of $N_{k+1}$ that $N_k$ and $zN_k$ are subspaces of $N_{k+1}$, $M - 1 \leq k \leq N$. Let $h_{k+1}$ be the dimension of any complement $\mathcal{H}_{k+1}$ of the subspace $N_k + zN_k$ in the whole space $N_{k+1}$.

From (10) we see that $h_{k+1} \neq 0$ iff $k = \mu_j$ ($j = 1, 2$), $h_{k+1} = 1$ if $\mu_1 < \mu_2$, and $h_{k+1} = 2$ for $\mu_1 = \mu_2$. Therefore, for $k \neq \mu_j$

$$N_{k+1} = N_k + zN_k, \quad (11)$$

and for $k = \mu_j$

$$N_{k+1} = (N_k + zN_k) \oplus \mathcal{H}_{k+1}. \quad (12)$$

Definition 3.2. Let $\mu_1 = \mu_2$. Any polynomials $Q_1(z), Q_2(z)$ that form a basis for the two-dimensional space $N_{\mu_1+1}$ are called the essential polynomials of the sequence $c^N_M$ corresponding to the index $\mu_1 = \mu_2$.

If $\mu_1 < \mu_2$, then any polynomial $Q_j(z)$ that is a basis for an one-dimensional complement $\mathcal{H}_{\mu_j+1}$ is said to be the essential polynomial of the sequence corresponding to the index $\mu_j$, $j = 1, 2$.

It follows from Theorem 4.1 of the work [3] that in the scalar case the following criterion of essentialness is fulfilled.

Proposition 3.3. Integers $\mu_1, \mu_2$, $\mu_1 + \mu_2 = M + N$, are the indices and polynomials $Q_1(z) \in N_{\mu_1+1}$, $Q_2(z) \in N_{\mu_2+1}$ are the essential polynomials of the sequence $c^N_M$ iff

$$\sigma_0 := \sigma \{z^{-\mu_1}Q_{2,\mu_2-M+1}Q_1(z) - z^{-\mu_2}Q_{1,\mu_1-M+1}Q_2(z)\} \neq 0. \quad (13)$$

Here $Q_{j,\mu_j-M+1}$ is the coefficient of $z^{\mu_j-M+1}$ in the polynomial $Q_j(z). \quad \square$

Now we can describe the structure of the kernels of the matrices $T_k$ in terms of the indices and the essential polynomials.

Proposition 3.4. Let $\mu_1, \mu_2$ be the indices and $Q_1(z), Q_2(z)$ the essential polynomials of the sequence $c^N_M$.

Then

$$N_k = \left\{ \begin{array}{ll}
\{0\}, & M \leq k \leq \mu_1, \\
\{q_1(z)Q_1(z)\}, & \mu_1 + 1 \leq k \leq \mu_2, \\
\{q_1(z)Q_1(z) + q_2(z)Q_2(z)\}, & \mu_2 + 1 \leq k \leq N,
\end{array} \right. \quad \square$$

where $q_j(z)$ is an arbitrary polynomial of the formal degree $k - \mu_j - 1$, $j = 1, 2$. 7
4. Construction of Wiener – Hopf factorization of scalar polynomials

In this section we propose a method for solving the problem of the polynomial factorization in terms of indices and essential polynomials of some sequence.

Let \( p(z) = p_0 + p_1 z + \cdots + p_\nu z^\nu \) be a polynomial of degree \( \nu > 1 \), \( p_0 \neq 0 \). We suppose that \( p(z) \neq 0 \) on the unit circle \( T \), hence \( p(z) \neq 0 \) on a closed annulus \( K := \{ z \in \mathbb{C} : r \leq |z| \leq R \} \) for some \( 0 < r < 1 < R < \infty \). Let \( \kappa = \text{ind}_T p(z) \) be the index of \( p(z) \), i.e. the number of zeros of the polynomial inside the unit circle. We can assume that \( 0 < \kappa < \nu \), otherwise the factorizations are trivial. Put \( n_0 = \max \{ \kappa, \nu - \kappa \} \).

We will find the factorization of \( p(z) \) in the form

\[
p(z) = p_1(z)p_2(z),
\]

where \( \deg p_1 = \kappa \), \( \deg p_2 = \nu - \kappa \), the polynomial \( p_1(z) \) is monic, and all zeros of \( p_1(z) \) (respectively \( p_2(z) \)) lie in the domain \( \{ z \in \mathbb{C} : |z| < r \} \) (respectively in \( \{ z \in \mathbb{C} : |z| > R \} \)). This means that

\[
p(z) = p_-(z) z^\kappa p_+(z), \quad z \in T,
\]
is the Wiener – Hopf factorization of \( p(t) \) normalized by the condition \( p_-(\infty) = 1 \). Here \( p_-(z) = \frac{p_1(z)}{z^\kappa} \), \( p_+(z) = p_2(z) \). Let

\[
p_1(z) = \sum_{k=-\infty}^{\infty} c_k z^k
\]

be the Laurent series for analytic function \( p_1(z) \) in the annulus \( K \). Here

\[
c_k = \frac{1}{2\pi i} \int_{|z|=\rho} t^{-k-1} p_1(t) \, dt, \quad k \in \mathbb{Z}, \ \rho \leq \rho \leq R.
\]

Note that \( c_k \) are the Fourier coefficients of the function \( p_1(t) \), \( t \in T \).

Form the sequence \( c_{n-\kappa} = (c_{n-\kappa}, \ldots, c_{\kappa}, \ldots, c_{n-\kappa}) \) for the given \( n \geq n_0 \). The generating function \( c_{n-\kappa}(z) \) of the sequence is a partial sum of the Laurent series of \( p_1(z) \) and in formula (7) we can replace \( c_{n-\kappa}(z) \) by \( p_1(z) \).

**Theorem 4.1.** For any \( n \geq n_0 \) the sequence \( c_{n-\kappa} \) is non-zero; the integers \( -\kappa, -\kappa \) are the indices; and \( Q_1(z) = z^{\nu-\kappa+1} p_1(z), Q_2(z) = p_2(z) \) are essential polynomials of the sequence.

**Proof.** Let us prove the first statement of the theorem. Since \( Q_1(z) = z^{\nu+1} + \alpha_{\nu} z^{\nu} + \cdots + \alpha_{\kappa+1} z^{\kappa+1} \), then

\[
\sigma(z^{n+\kappa-1} Q_1(z)) = c_{-\nu} + c_{-\nu+1} \alpha_{\nu} + \cdots + c_0 \alpha_{\nu+1}.
\]

Otherwise, by analytic definition (7) of \( \sigma \) we have

\[
\sigma(z^{n+\kappa-1} Q_1(z)) = \frac{1}{2\pi i} \int_T t^{-1} p_1(t) p_1(t) \, dt = \frac{1}{2\pi i} \int_T t^{-1} p_2(t) p_2(t) \, dt = p_2^{-1}(0) \neq 0.
\]
Therefore,
\[ c_{\kappa} + c_{\kappa+1}p_{-1} + \cdots + c_0p_{-\kappa} \neq 0, \]
and not all of the numbers \( c_{\kappa}, c_{\kappa+1}, \ldots, c_0 \) are zero. For \( n \geq n_0 \) the sequence \( \{c_{\kappa}, c_{\kappa+1}, \ldots, c_0\} \) is a subsequence of \( c_{-n-\kappa} \). Hence, \( c_{-n-\kappa} \) is also non-zero.

Now we prove that \( Q_1(z), Q_2(z) \) belong to the space \( \mathcal{N}_{-\kappa+1} \). A formal degree of polynomials from this space should be \( n + 1 \). In our case we have \( \deg Q_1(z) = n + 1 \) and \( \deg Q_2(z) = \nu - \kappa < n + 1 \). Verify orthogonality condition (8). We have
\[
\sigma\{z^{-j}Q_1(z)\} = \frac{1}{2\pi i} \int_{\mathbb{T}} t^{-\kappa-j}p_2^{-1}(t) \, dt = 0
\]
for \( n - \kappa - j \geq 0 \). Here we use analytic definition (7) of \( \sigma \) and the Wiener–Hopf factorization (14). In particular,
\[
\sigma\{z^{-j}Q_1(z)\} = 0, \quad j = -\kappa + 1, \ldots, n - \kappa.
\]
Hence \( Q_1(z) \in \mathcal{N}_{-\kappa+1} \).

Similarly,
\[
\sigma\{z^{-j}Q_2(z)\} = \frac{1}{2\pi i} \int_{\mathbb{T}} t^{-\kappa-j}p_1^{-1}(t) \, dt = \frac{1}{2\pi i} \int_{\mathbb{T}} t^{-\kappa-j-1}p_1^{-1}(t) \, dt = 0
\]
for \( j = -\kappa + 1, \ldots, n - \kappa \), and \( Q_2(z) \in \mathcal{N}_{-\kappa+1} \) since \( p_-(z) = z^{-\kappa}p_1(z) \) is analytic in the domain \( \{z \in \mathbb{C} : |z| > 1\} \).

Moreover,
\[
\sigma\{z^{\kappa}Q_2(z)\} = p_1^{-1}(\infty) = 1,
\]
and \( Q_2(0) = p_2(0) \neq 0 \). Now we apply Proposition 3.3. Let us find the test-number \( \sigma_0 \) from formula (13):
\[
\sigma_0 = \sigma\{z^{\kappa}Q_{2,n+1}Q_1(z) - z^{\kappa}Q_{1,n+1}Q_2(z)\} = -\sigma\{z^{\kappa}Q_2(z)\} = -1 \neq 0.
\]
Hence by Proposition 3.3, the integers \( -\kappa, -\kappa \) are the indices; and \( Q_1(z), Q_2(z) \) are the essential polynomials of the sequence \( c_{-n-\kappa} \). \( \square \)

In this theorem we have proved that there exist the essential polynomials \( Q_1(z), Q_2(z) \) of the sequence \( c_{-n-\kappa} \), such that the following additional properties are fulfilled:

(i) \( \deg Q_1(z) = n + 1 \), \quad \( Q_1(0) = 0 \), \quad \( Q_{1,n+1} = 1 \).

(ii) \( \deg Q_2(z) < n + 1 \), \quad \( Q_2(0) \neq 0 \), \quad \( \sigma\{z^{\kappa}Q_2(z)\} = 1 \).

Vice versa, if \( Q_1(z), Q_2(z) \in \mathcal{N}_{-\kappa+1} \) and satisfy conditions (i)-(ii), then \( \sigma_0 = -1 \) and \( Q_1(z), Q_2(z) \) are the essential polynomials of the sequence \( c_{-n-\kappa} \).

**Definition 4.1.** Let \( n \geq n_0 \). Polynomials \( Q_1(z), Q_2(z) \in \mathcal{N}_{-\kappa+1} (c_{-n-\kappa}) \) satisfying conditions (i)-(ii) will be called the factorization essential polynomials of \( c_{-n-\kappa} \).
**Theorem 4.2.** The factorization essential polynomials are uniquely determined by conditions (i)–(ii). Let \( n \geq n_0 + 1 \) and \( R_1(z), R_2(z) \) are any essential polynomials of the sequence \( c_{-n-\kappa} \). Then the factorization essential polynomials of \( c_{-n-\kappa} \) can be found by the formulas

\[
Q_1(z) = -\frac{1}{\sigma_1}(R_{2,0}R_1(z) - R_{1,0}R_2(z)),
\]

\[
Q_2(z) = \frac{1}{\sigma_0}(R_{2,n+1}R_1(z) - R_{1,n+1}R_2(z)).
\]

(16)

Here \( R_{j,0} = R_j(0), R_{j,n+1} \) is the coefficient of \( z^{n+1} \) in the polynomial \( R_j(z) \), \( j = 1, 2 \), and

\[
\sigma_1 = \begin{vmatrix} R_{1,0} & R_{2,0} \\ R_{1,n+1} & R_{2,n+1} \end{vmatrix} \neq 0, \quad \sigma_0 = \sigma\{z^\kappa(R_{2,n+1}R_1(z) - R_{1,n+1}R_2(z))\} \neq 0.
\]

**Proof.** Let \( Q_1(z), Q_2(z) \) and \( \tilde{Q}_1(z), \tilde{Q}_2(z) \) be any couples of the factorization essential polynomials. Since \( \{Q_1(z), Q_2(z)\} \) is a basis of the space \( \mathcal{N}_{-\kappa+1} \), we have

\[
\tilde{Q}_1(z) = \alpha_1 Q_1(z) + \alpha_2 Q_2(z), \quad \tilde{Q}_2(z) = \beta_1 Q_1(z) + \beta_2 Q_2(z).
\]

Then \( 0 = \tilde{Q}_1(0) = \alpha_1 Q_1(0) + \alpha_2 Q_2(0) = \alpha_2 Q_2(0) \). Hence \( \alpha_2 = 0 \). It follows from the conditions \( Q_{1,n+1} = Q_{2,n+1} = 1 \) that \( \alpha_1 = 1 \), and \( \tilde{Q}_1(z) = Q_1(z) \). In similar manner we can prove \( Q_2(z) = Q_2(z) \).

Let \( R_1(z), R_2(z) \) be any essential polynomials of the sequence \( c_{-n-\kappa} \) for \( n \geq n_0 + 1 \). Then \( \sigma_0 \) is the test-number of the essential polynomials \( R_1(z), R_2(z) \) and \( \sigma_0 \neq 0 \).

Suppose that \( \sigma_1 = \begin{vmatrix} R_{1,0} & R_{2,0} \\ R_{1,n+1} & R_{2,n+1} \end{vmatrix} = 0 \). Then

\[
\lambda_1 \begin{pmatrix} R_{1,0} \\ R_{1,n+1} \end{pmatrix} + \lambda_2 \begin{pmatrix} R_{2,0} \\ R_{2,n+1} \end{pmatrix} = 0.
\]

Define \( Q(z) = \lambda_1 R_1(z) + \lambda_2 R_2(z) \). By definition, \( Q(0) = 0 \) and \( \deg Q \leq n \). Put \( \tilde{Q}(z) = zQ(z) \), where \( \deg \tilde{Q}(z) \leq n - 1 \). Since \( Q(z) \in \mathcal{N}_{-\kappa+1}(c_{-n-\kappa}), \) we have \( \tilde{Q}(z) \in \mathcal{N}_{-\kappa}(c_{-n-\kappa-1}). \) However, if \( n \geq n_0 + 1 \), then the sequence \( c_{-n-\kappa-1} \) has indices \( -\kappa, -\kappa \), and, by Proposition 3.4, \( \mathcal{N}_{-\kappa}(c_{-n-\kappa-1}) = \{0\} \). Thus, \( Q(z) \equiv 0 \), and the polynomials \( R_1(z), R_2(z) \) are linearly dependent. But it is impossible and the contradiction proves the inequality \( \sigma_1 \neq 0 \).

The polynomials \( Q_1(z), Q_2(z) \), defined by (16), belong to \( \mathcal{N}_{-\kappa+1}(c_{-n-\kappa}) \) and satisfy conditions (i)–(ii). Hence \( Q_1(z), Q_2(z) \) are the factorization essential polynomials. The theorem is proved. \( \Box \)

Now we can construct the Wiener–Hopf factorization of a polynomial with the help of the factorization essential polynomials. Note that existence and uniqueness of the factorization essential polynomials were proved under condition \( n \geq n_0 \).
Theorem 4.3. Let $n \geq n_0$ and let $Q_1(z), Q_2(z)$ be the factorization essential polynomials of the sequence $c^{-\nu}_{n-x}$. Then the Wiener–Hopf factorization of the polynomial $p(z)$ can be constructed by the formula

$$p(z) = p_-(z)z^{\nu}p_+(z),$$

where

$$p_-(z) = z^{-n-1}Q_1(z), \quad p_+(z) = Q_2(z). \quad (17)$$

Proof. In Theorem 4.1 we prove that $Q_1(z) = z^{n-x+1}p_1(z)$, $Q_2(z) = p_2(z)$ are the factorization essential polynomials. By Theorem 4.2, the factorization essential polynomials are determined uniquely. Thus formulas (17) hold.

Now we can obtain the main result of the section about an explicit formulas for the factors of the Wiener–Hopf factorization of a polynomial $p(z)$.

Theorem 4.4. The matrices $T_{-\nu}(c^{-\nu}_{n-x})$ are invertible for all $n \geq n_0$.

Let $n \geq n_0 + 1$. Denote by $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $\beta = (\beta_0, \ldots, \beta_n)^T$ the solutions of the systems

$$T_{-\nu}(c^{-\nu-1}_{n-x+1}) \alpha = -(c^{-\nu-1}_{n-x})^T, \quad T_{-\nu}(c^{-\nu}_{n-x}) \beta = e_1, \quad (18)$$

respectively. Here $e_1 = (1, 0, \ldots, 0)^T$.

Then $\alpha_1 = \ldots = \alpha_{n-x} = 0$, $\beta_0 \neq 0$, $\beta_{n-x+1} = \ldots = \beta_n = 0$, and the factors from the Wiener–Hopf factorization of $p(z)$ are found by the formulas

$$p_-(z) = z^{-\nu}(\alpha_{n-x+1} + \cdots + \alpha_n z^{-1} + z^\nu), \quad p_+(z) = \beta_0 + \beta_1 z + \cdots + \beta_n z^{\nu-\nu}. \quad$$

Proof. For the first index of the sequence $c^{-\nu}_{n-x}$ by Proposition 3.2 we have

$$\mu_1 = -n - \nu + \pi - 1,$$

where $\pi = \text{rank} T_{-\nu}(c^{-\nu}_{n-x})$. By Theorem 4.1, we have $\mu_1 = -\nu$ for $n \geq n_0$. Hence rank of the $(n+1) \times (n+1)$ matrix $T_{-\nu}(c^{-\nu}_{n-x})$ is equal to $n+1$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ satisfies the equation $T_{-\nu}(c^{-\nu-1}_{n-x+1}) \alpha = -(c^{-\nu-1}_{n-x})^T$, then the vector $(0, \alpha_1, \ldots, \alpha_n, 1)^T$ belongs to the space $\ker T_{-\nu+1}(c^{-\nu}_{n-x})$, i.e. $Q_1(t) = \alpha_1 z + \cdots + \alpha_2 z^2 + z^{n+1} \in N_{-\nu+1}$.

Similarly, $(\beta_0, \ldots, \beta_n, 0)^T \in \ker T_{-\nu+1}(c^{-\nu}_{n-x})$, and $Q_2(z) = \beta_0 + \beta_1 z + \cdots + \beta_n z^n \in N_{-\nu+1}$. Moreover, $\sigma\{z^\nu Q_2(z)\} = c_{-\nu}\beta_0 + \cdots + c_{-\nu}\beta_n = 1$.

Let us prove $\beta_0 \neq 0$. If we suppose $\beta_0 = 0$, then the non-zero vector $(\beta_1, \ldots, \beta_n, 0)^T$ belongs to $\ker T_{-\nu}(c^{-\nu-1}_{n-x+1})$. This is impossible since the matrix $T_{-\nu}(c^{-\nu-1}_{n-x+1})$ is invertible for $n \geq n_0 + 1$.

Therefore the polynomials $Q_1(z), Q_2(z)$ satisfy conditions (i)–(ii) and they are the factorization essential polynomials of $c^{-\nu}_{n-x}$. By Theorem 4.3,

$$Q_1(z) = z^{n+1}p_-(z), \quad Q_2(z) = p_+(z).$$

Hence,

$$\alpha_1 = \ldots = \alpha_{n-x} = 0, \quad \beta_{n-x+1} = \ldots = \beta_n = 0.$$

The theorem is proved. □
5. Some computational aspects of the factorization problem

5.1. Computation of the index

It is well known that the index $\kappa$ of a scalar function is stable under small perturbations of the function. In particular, the following statement holds.

**Proposition 5.1.** If $p(z) \neq 0$ on the unit circle $T$ and a polynomial $\tilde{p}(z)$ satisfies the inequality

$$\|p - \tilde{p}\| < \min_{|z|=1} |p(z)|,$$

then $\tilde{p}(z) \neq 0$ on $T$ and $\text{ind}_T p(z) = \text{ind}_T \tilde{p}(z)$.

The proof is carried out using standard arguments. □

There are many ways to calculate the index. Numerical experiments show that it is more convenient to use formula (12.6) from [12]. Let $p(e^{i\phi}) = \xi(\phi) + i \eta(\phi), \phi \in [0, 2\pi]$, where $\xi(\phi)$ and $\eta(\phi)$ are real continuously differentiable functions on $[0, 2\pi]$. Then

$$\kappa = \frac{1}{2\pi} \int_0^{2\pi} \frac{\xi(\phi)\eta'(\phi) - \xi'(\phi)\eta(\phi)}{\xi^2(\phi) + \eta^2(\phi)} d\phi. \quad (19)$$

The integral can be computed numerically by the Gaussian quadrature method. Since $\kappa$ is integer, the result is rounded up to the nearest integer.

5.2. An a priori estimate of the condition number for the factorization problem

Theorem 4.4 shows that solving of the factorization problem is equivalent to solving of linear systems with the invertible matrix $T^{-\kappa}(c_{n-\kappa}^{-n})$, $n \geq n_0$.

Here we obtain an upper bound for the condition number $k(T^{-\kappa}(c_{n-\kappa}^{-n})) = \|T^{-\kappa}(c_{n-\kappa}^{-n})\| \|T^{-1}(c_{n-\kappa}^{-n})\|$ in terms of the given polynomial $p(z)$.

Recall that $n_0 = \max\{\kappa, \nu - \kappa\}$, $m_K = \min_{z \in K} |p(z)|$, $\rho = \max\{r, 1/R\}$.

**Proposition 5.2.** For $n \geq n_0$

$$\|T^{-\kappa}(c_{n-\kappa}^{-n})\| < \frac{1}{m_K} \frac{1 + \rho}{(1 - \rho)}. \quad (20)$$

**Proof.** From the structure of the matrix it is obvious that

$$\|T^{-\kappa}(c_{n-\kappa}^{-n})\| \leq \|c_{n-\kappa}^{-n}\| = \sum_{j=-n-\kappa}^{n-\kappa} |c_j| + \sum_{j=0}^{n-\kappa} |c_j|,$$

where

$$c_j = \frac{1}{2\pi i} \int_{|z|=\bar{\rho}} t^{-j-1} p^{-1}(t) dt, \quad r \leq \bar{\rho} \leq R, \quad j \in \mathbb{Z}.$$
For \( j \geq 0 \) we choose \( \bar{\rho} = R \). Then
\[
|c_j| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|dt|}{R^{j+1}|p(t)|} \leq \frac{1}{R} \min_{|z|=R} |p(z)| \leq \frac{\rho^j}{m_K}.
\]
In the same way, if \( j < 0 \) we have
\[
|c_j| \leq \min_{|z|=r} \frac{p^{(j)}|}{|p(z)|} \leq \frac{\rho^|j|}{m_K}.
\]
Therefore,
\[
\|T_{-\varkappa}(c_{-n,-\varkappa})\| \leq \|e_{-n,-\varkappa}\| \leq \frac{1}{m_K} \left( \sum_{j=1}^{n+\varkappa} \rho^j + \sum_{j=0}^{n-\varkappa} \rho^j \right) \leq \frac{1}{m_K} \frac{1 + \rho}{1 - \rho}.
\]
\( \square \)

**Remark 5.1.** It is clear that \( |c_j| \leq \frac{1}{m_1} \), where \( m_1 = \min_{|z|=1} |p(z)| \). Hence
\[
\|T_{-\varkappa}(c_{-n,-\varkappa})\| \leq \|e_{-n,-\varkappa}\| \leq \frac{2n + 1}{m_1}.
\] (21)
This rough estimate can be more precise for a polynomial of small degree.

Further, we will also use the estimates
\[
\|c_{-\varkappa-1}^{-1}\| \leq \|e_{-n,-\varkappa}\| < \frac{1}{m_K} \frac{1 + \rho}{1 - \rho}
\] (22)
or \( \|c_{-\varkappa-1}^{-1}\| \leq \frac{n}{m_1} \).

To obtain bounds for \( \|T_{-\varkappa}^{-1}(c_{-n,-\varkappa})\| \) we will get a formula for the inverse \( T_{-\varkappa}^{-1}(c_{-n,-\varkappa}) \) in terms of the factorization \( p(z) = p_1(z)p_2(z) \).

If \( p_1(z) = z^{\varkappa + 1} + \cdots + p_{1,0} \) and \( p_2(z) = z^{\varkappa - \nu} + p_{2,0} + \cdots + p_{2,0} \), then we denote
\[
p^{(k)}_1(z) = z^k + p_{1,\varkappa - 1} z^{k-1} + \cdots + p_{1,\varkappa - k}, \quad 0 \leq k \leq \varkappa,
\]
and
\[
p^{(k)}_2(z) = p_{2,k} z^k + p_{2,k-1} z^{k-1} + \cdots + p_{2,0}, \quad 0 \leq k \leq \nu - \varkappa.
\]

Let \( T_{-\varkappa}^{-1}(c_{-n,-\varkappa}) = (b_{ij})_{i=0}^{n} \) and \( B_j(z) = \sum_{i=0}^{n} b_{ij} z^i \) be the generating polynomial of the \( j \)-th column \( B_j \) of the inverse.

**Theorem 5.1.** For \( n \geq \nu \) we have
\[
B_j(z) = \begin{cases} 
  p^{(j)}_1(z)p_2(z), & 0 \leq j \leq \varkappa, \\
  z^{j-\nu} p(z), & \varkappa + 1 \leq j \leq n - \nu + \varkappa, \\
  z^{j-\nu} p_1(z)p^{(n-j)}_2(z), & n - \nu + \varkappa + 1 \leq j \leq n.
\end{cases}
\] (23)

Here the second row is absent if \( n = \nu \).
Proof. If \( n \geq \nu \), then \( T_{- \nu}(c_{-n-\nu}) \) is invertible, and \( Q_1(z) = z^{n-\nu+1}p_1(z) \), \( Q_2(z) = p_2(z) \) are the factorization essential polynomials of the sequence \( c_{-n-\nu} \) with the test-number \( \sigma_0 = -1 \). Let \( B(t, s) = \sum_{i,j=0}^n b_{ij} t^i s^{-j} \) be the generating function of the inverse matrix \( T_{- \nu}^{-1}(c_{-n-\nu}) = (b_{ij})_{i,j=0}^n \). By Theorem 2.2 from [1], we have

\[
B(t, s) = -s^{-n-1} \frac{Q_1(t)Q_2(s) - Q_1(s)Q_2(t)}{1-ts^{-1}}.
\]

Hence

\[
(t-s) \sum_{j=0}^n B_j(t)s^{n-j} = t^{n-\nu+1}p_1(t)p_2(s) - s^{n-\nu+1}p_1(s)p_2(t).
\]

Equating the coefficients of \( s^j \), we arrive to relation (23). \( \square \)

Corollary 5.1. For \( n \geq \nu \), we have

\[
\|p\| \leq \|T_{- \nu}^{-1}(c_{-n-\nu})\| \leq \|p_1\| \|p_2\| \leq \delta_0 \|p\|.
\]  

Proof. It follows from (23) that

\[
\|B_j\| = \begin{cases}
\|p_1(j)\| p_2, & 0 \leq j \leq \nu, \\
\|p_1\| p_2^{(n-j)}, & \nu + 1 \leq j \leq n - \nu + \nu,
\end{cases}
\]

Hence, \( \|T_{- \nu}^{-1}(c_{-n-\nu})\| = \max_{0 \leq j \leq n} \|B_j\| \geq \|p\| \). Moreover, \( \|p(j)\| \|p_2\| \leq \|p_1\| \|p_2\| \) and \( \|p_1 p_2^{(n-j)}\| \leq \|p_1\| \|p_2^{(n-j)}\| \leq \|p_1\| \|p_2\| \). Thus the second inequality also holds. \( \square \)

Both inequality (24) are sharp for \( p(z) = p_1(z)p_2(z) \), where \( p_1(z) \), \( p_2(z) \) are polynomials with real nonnegative coefficients. In this case \( \|p\| = p(1) = \|p_1\| \|p_2\| \) (see also Proposition 2.1).

Now, using (20), (21), (24), and (5) we get the final result

Corollary 5.2. For \( n \geq \nu \)

\[
k(T_{- \nu}(c_{-n-\nu})) \leq \frac{\delta_0}{m_1} \frac{1 + \rho}{(1 - \rho)} \|p\|,
\]  

and

\[
k(T_{- \nu}(c_{-n-\nu})) \leq \frac{(2n + 1) \delta_0}{m_1 \|p\|}.
\]  

5.3. Computation of the Laurent coefficients of analytic functions

To realize the factorization method proposed in Section 4 we must calculate the Laurent coefficients \( c_{-n-\nu}, \ldots, c_{-\nu}, \ldots, c_{-n}, \) of the function \( p^{-1}(z) \) for \( n \geq n_0 = \max\{\nu, \nu - \nu\} \).
In general, the coefficients can be found only approximately. In order to
do this, we will apply the method suggested by D.A. Bini and A. Böttcher
(see Theorem 3.3 in [8]). For future applications we will consider more
general situation than in this work. Moreover, our proof of inequality (27) differs from
the proof in the above mentioned theorem.

Let \( f(z) \) be a function that analytic in the annulus \( K = \{ z \in \mathbb{C} : r \leq |z| \leq R \}, \) \( 0 < r < 1 < R < \infty \). By \( f_k \) denote the Laurent coefficients of \( f(z) \):

\[
f_k = \frac{1}{2\pi i} \int_{|t|=\rho} t^{-k-1} f(t) \, dt, \quad r \leq \rho \leq R.
\]

For \( \ell, k \in \mathbb{Z}, \) \( \ell \geq 2, \) define

\[
\tilde{f}_k^{(\ell)} = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \frac{f(\omega_j)}{\omega_j^k},
\]

where \( \omega_j = e^{2\pi i j}, j = 0, \ldots, \ell - 1, \) are the zeros of the polynomial \( z^\ell - 1. \)

**Theorem 5.2.** Let \( M_K = \max_{z \in K} |f(z)|, \rho = \max\{r, 1/R\}, \) and \( \ell \) be an even positive integer. Then

\[
|f_k - \tilde{f}_k^{(\ell)}| < \frac{2M_K}{(1 - \rho^\ell)} \rho^{\ell/2}
\]

for \( k = -\ell/2, \ldots, 0, \ldots, \ell/2. \)

**Proof.** Define

\[
I = \frac{1}{2\pi i} \int_{|t|=\rho} f(t) \frac{t^{-k-1}}{t^\ell - 1} \, dt.
\]

Calculate the integral by residue theorem. The integrand is analytic in the
annulus \( K \) except simple poles at the points \( \omega_0, \ldots, \omega_{\ell-1}. \) Since

\[
\text{res}_{z=\omega_j} \frac{f(z)z^{\ell-k-1}}{z^\ell - 1} = f(\omega_j) \frac{1}{\ell \omega_j^k},
\]

we have \( I = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \frac{f(\omega_j)}{\omega_j^k} = \tilde{f}_k^{(\ell)}. \)

Therefore,

\[
f_k - \tilde{f}_k^{(\ell)} = \frac{1}{2\pi i} \int_{|t|=R} f(t) \left[ \frac{1}{t^k} \left( 1 - \frac{1}{t^\ell} \right) \right] dt + \frac{1}{2\pi i} \int_{|t|=r} \frac{f(t)t^{\ell-k-1}}{t^\ell - 1} \, dt =
\]

\[
- \frac{1}{2\pi i} \int_{|t|=R} f(t) \left[ \frac{1}{t^{k+1}} \left( t^\ell - 1 \right) \right] dt + \frac{1}{2\pi i} \int_{|t|=r} \frac{f(t)t^{\ell-k-1}}{t^\ell - 1} \, dt.
\]

Then

\[
|f_k - \tilde{f}_k^{(\ell)}| \leq \frac{1}{2\pi} \int_{|t|=R} \frac{|f(t)|}{R^{k+1}|t^\ell - 1|} \, |dt| + \frac{1}{2\pi} \int_{|t|=r} \frac{|f(t)|t^{\ell-k-1}}{|t^\ell - 1|} \, |dt| \leq
\]
\[ \mathcal{M} = \max_{|t|=R} |f(t)|, \quad \mathcal{M}_r = \max_{|t|=r} |f(t)|. \]

Here \( \mathcal{M} = \max_{|t|=R} |t^\ell - 1| \), \( \mathcal{M}_r = \max_{|t|=r} |t^\ell - 1| \). It is easily seen that
\[ \min_{|t|=R} |t^\ell - 1| = R^\ell - 1, \quad \min_{|t|=r} |t^\ell - 1| = 1 - r^\ell. \]

Hence,
\[ |f_k - \tilde{f}_k(\ell)| \leq \frac{\mathcal{M} R}{R^\ell (R^\ell - 1)} + \frac{\mathcal{M}_r r^{\ell-k}}{(1 - r^\ell)}. \]

Now, from the definitions of \( M_R \) and \( \rho \), it follows that
\[ |f_k - \tilde{f}_k(\ell)| \leq \frac{M_K}{(1 - \rho^\ell)} [\rho^{\ell+k} + \rho^{\ell-k}]. \]

If \( k = -\ell/2, \ldots, \ell/2 \), then \( \rho^{\ell+k} \leq \rho^{\ell/2} \) and \( \rho^{\ell-k} < \rho^{\ell/2} \). Estimate (27) has been obtained. \( \square \)

By the theorem, in order to compute every element of the sequence \( f_M, f_{M+1}, \ldots, f_N \) with the given accuracy, we have to select an appropriate number \( \ell \).

5.4. Stability of the factors \( p_1(z), p_2(z) \)

Recall that we consider the coefficients of the polynomial \( p(z) \) as the initial data of the factorization problem. First of all we must study the sensitivity of the factors \( p_1(z), p_2(z) \) with respect to variations in these data. Moreover, we compute approximately the Laurent coefficients \( c_k \) of the function \( f(z) = p^{-1}(z) \) by the formula
\[ c_k \approx \tilde{c}_k = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \frac{1}{p(\omega_j) \omega_j^k}. \]  

We can not consider these variations as a perturbation of \( p(z) \). Hence we must study the sensitivity of the factors \( p_1(z), p_2(z) \) with respect to change in the Laurent coefficients separately.

For this reason, we first study the behavior of the factorization essential polynomials \( Q_1(z), Q_2(z) \) of the sequence \( e^{-n-\kappa} \) under small perturbations. Recall that the sequence consists of the Laurent coefficients of \( p^{-1}(z) \) and has indices \(-\kappa, -\kappa\), where \( \kappa = \text{ind}_p p(z) \) and \( n \geq n_0 \).

We will need some modification of a well known result on the absolute error for the solutions of linear systems. The statement can be proved by the standard method.

**Lemma 5.1.** Let \( A \) be an invertible matrix and \( x = A^{-1}b \). If \( \|A - \tilde{A}\| \leq \frac{q}{\|A^{-1}\|} \) for some \( 0 < q < 1 \), then \( \tilde{A} \) is invertible and for \( \tilde{x} = \tilde{A}^{-1}\tilde{b} \) we have
\[ \|x - \tilde{x}\| \leq \frac{\|A^{-1}\|}{1 - q} \left[ \|A^{-1}\| \|A - \tilde{A}\| \|b\| + \|b - \tilde{b}\|\right]. \] \( \square \)
Theorem 5.3. Let \( n \geq n_0 \) and let \( \tilde{c}_n^{\kappa,x} \) be a sequence such that
\[
\|c_n^{\kappa,x} - \tilde{c}_n^{\kappa,x}\| \leq \frac{q}{\delta_0\|p\|} \tag{30}
\]
for some \( 0 < q < 1 \). Then

(i) The indices of the sequence \( \tilde{c}_n^{\kappa,x} \) are \(-\kappa, -\kappa\).
(ii) The sequence has the factorization essential polynomials \( \tilde{Q}_1(z), \tilde{Q}_2(z) \).
(iii) For \( n \geq n_0 + 1 \) the following estimates are fulfilled:
\[
\|Q_1 - \tilde{Q}_1\| \leq \frac{\delta_0\|p\|}{1-q} \left( \frac{1+\rho}{n\kappa} \right) \|c_n^{\kappa,x} - \tilde{c}_n^{\kappa,x}\| - 1
\]
\[
\|Q_2 - \tilde{Q}_2\| \leq \frac{\delta_0\|p\|}{1-q} \|c_n^{\kappa,x} - \tilde{c}_n^{\kappa,x}\|.
\]

Proof. (i). Form the matrix \( T_{-\kappa}(\tilde{c}_n^{\kappa,x}) \). If
\[
\|T_{-\kappa}(c_n^{\kappa,x}) - T_{-\kappa}(\tilde{c}_n^{\kappa,x})\| \leq \frac{q}{\|T_{-\kappa}(\tilde{c}_n^{\kappa,x})\|}, \tag{31}
\]
then \( T_{-\kappa}(\tilde{c}_n^{\kappa,x}) \) is invertible. This means that the sequence \( \tilde{c}_n^{\kappa,x} \) has the indices \(-\kappa, -\kappa\). Let us find the condition that guarantees the fulfillment of inequality (31).

From inequalities (24) we have
\[
\frac{q}{\delta_0\|p\|} \leq \frac{q}{\|p_1\| \|p_2\|} \leq \frac{q}{\|T_{-\kappa}(\tilde{c}_n^{\kappa,x})\|}.
\]
Since \( \|T_{-\kappa}(c_n^{\kappa,x}) - T_{-\kappa}(\tilde{c}_n^{\kappa,x})\| \leq \|c_n^{\kappa,x} - \tilde{c}_n^{\kappa,x}\| \), then from inequality (30) it follows (31). Hence the indices of \( \tilde{c}_n^{\kappa,x} \) are \(-\kappa, -\kappa\).

(ii). Let \( \tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n), \tilde{\beta} = (\tilde{\beta}_0, \ldots, \tilde{\beta}_n) \) be the solutions of systems (18) for the perturbed sequence \( \tilde{c}_n^{\kappa,x} \). Then \( \tilde{Q}_1(z) = \tilde{\alpha}_1 z + \cdots + \tilde{\alpha}_n z^n + z^{n+1} \) and \( \tilde{Q}_2(z) = \tilde{\beta}_0 + \cdots + \tilde{\beta}_n z^n \) belong to \( \mathcal{N}_{-\kappa+1}(\tilde{c}_n^{\kappa,x}) \) and satisfy the conditions of Definition 4.1. Thus \( \tilde{Q}_1(z), \tilde{Q}_2(z) \) are the factorization essential polynomials of \( \tilde{c}_n^{\kappa,x} \).

(iii). For \( n \geq n_0 + 1 \) the inequality \( \|c_n^{\kappa,x-1} - \tilde{c}_n^{\kappa,x-1}\| \leq \frac{q}{\delta_0\|p\|} \) is valid, condition (31) is fulfilled, and, by Lemma 5.1, we have
\[
\|Q_1 - \tilde{Q}_1\| = \|\alpha - \tilde{\alpha}\| \leq \frac{\|T_{-\kappa}(c_{n-\kappa}^{\kappa,x-1})\|}{1-q} 
\]
\[
\left[ \|T_{-\kappa}(c_{n-\kappa}^{\kappa,x-1})\| \|T_{-\kappa}(\tilde{c}_{n-\kappa}^{\kappa,x-1}) - T_{-\kappa}(\tilde{c}_n^{\kappa,x})\| \|c_{n-\kappa}^{\kappa,x-1}\| + \|c_n^{\kappa,x} - \tilde{c}_n^{\kappa,x}\| \right].
\]
Taking into account estimates (24) and (22) we arrive at the desired inequality for \( \|Q_1 - \tilde{Q}_1\| \). The estimate for \( \|Q_2 - \tilde{Q}_2\| \) can be proved analogously. \( \square \)
Now we can study the behavior of the factors $p_1, p_2$ under small perturbations of $p(z)$. Let $m_1 = \min_{|z|=1} |p(z)|$. By Proposition 5.1, if $\|p - \bar{p}\| < m_1$, then $\bar{p}(z) \neq 0$ on $T$ and $\text{ind}_T p(z) = \text{ind}_T \bar{p}(z)$. Let $\bar{p}(z) = \bar{p}_1(z)\bar{p}_2(z)$ be the factorization of $\bar{p}(z)$. By $\bar{c}_j$ we denote the Laurent coefficients of $\bar{p}^{-1}(z)$.

Now we estimate $\|p_1 - \bar{p}_1\|, \|p_2 - \bar{p}_2\|$ via $\|p - \bar{p}\|$.

**Theorem 5.4.** Let $n \geq n_0 + 1$. If $\|p - \bar{p}\| \leq \min \left\{ qm_1, \frac{q(1-q)m_1^2}{(2n+1)\delta_0 \|p\|} \right\}$, then

$$\|p_1 - \bar{p}_1\| < \frac{(2n+1)\delta_0 \|p\|}{(1-q)^2 m_1^2} \left[ \frac{\delta_0 \|p\|}{m_\xi} \frac{1 + \rho}{1 - \rho} + 1 \right] \|p - \bar{p}\|, \quad (32)$$

and

$$\|p_2 - \bar{p}_2\| < \frac{(2n+1)\delta_0 \|p\|^2}{(1-q)^2 m_1^2} \|p - \bar{p}\|. \quad (33)$$

**Proof.** Let us apply Theorem 5.3. To do this we must estimate $\|c^{n-\xi} - \bar{c}^{n-\xi}\|_{\mathfrak{A}}$ via $\|p - \bar{p}\|$. We have

$$\|c^{n-\xi} - \bar{c}^{n-\xi}\| = \sum_{j=-n-\xi}^{n-\xi} |c_j - \bar{c}_j|,$$

and

$$|c_j - \bar{c}_j| \leq \frac{1}{2\pi} \int_{|z|=1} |p^{-1}(t) - \bar{p}^{-1}(t)| |dt| \leq \|p^{-1} - \bar{p}^{-1}\|_C.$$

Hence,

$$\|c^{n-\xi} - \bar{c}^{n-\xi}\| \leq (2n+1)\|p^{-1} - \bar{p}^{-1}\|_C.$$

For any Banach algebra $\mathfrak{A}$ the following inequality

$$\|a^{-1} - \bar{a}^{-1}\|_\mathfrak{A} \leq \frac{\|a^{-1}\|_\mathfrak{A}^2}{1 - q} \|a - \bar{a}\|_\mathfrak{A}$$

holds if $\|a - \bar{a}\|_\mathfrak{A} \leq \frac{q}{\|a^{-1}\|_\mathfrak{A}}$ for some $0 < q < 1$.

In our case $\mathfrak{A} = C(T)$, $\|p^{-1}\|_C = \frac{1}{m_1}$, and $\|p - \bar{p}\|_C \leq \|p - \bar{p}\|$. Hence, if $\|p - \bar{p}\| \leq q m_1$, then $\|p^{-1} - \bar{p}^{-1}\|_C \leq \frac{1}{(1-q)m_1^2} \|p - \bar{p}\|^2 \leq \frac{q(1-q)m_1^2}{(1-q)m_1^2} \|p - \bar{p}\|$ and

$$\|c^{n-\xi} - \bar{c}^{n-\xi}\| \leq \frac{(2n+1)}{(1-q)m_1^2} \|p - \bar{p}\|.$$

If $\|p - \bar{p}\| \leq \min \{qm_1, \frac{q(1-q)m_1^2}{(2n+1)\delta_0 \|p\|} \}$, then condition (30) of Theorem 5.3 is fulfilled. Applying this theorem we arrive desired statement. \qed

Now we consider perturbations of the polynomials $p_1(z), p_2(z)$ caused by the approximation of the sequence $c^{n-\xi}$ by $\bar{c}^{n-\xi}$, where $\bar{c}_k = \frac{1}{\pi} \sum_{j=0}^{\ell} \frac{1}{p(z_j) - \bar{p}(z_j)}$.

Let $\bar{p}_1(z), \bar{p}_2(z)$ be polynomials that define by Eq. (18) for the sequence $\bar{c}^{n-\xi}$.
Theorem 5.5. Let $n \geq n_0 + 1$, $\ell$ is an even integer such that $\ell \geq 2(n + \kappa)$, and

$$\frac{\rho^{\ell/2}}{1 - \rho^{\ell}} < \frac{q m_K}{(4n + 2)\delta_0\|p\|}$$

(34)

Then

$$\|p_1 - \tilde{p}_1\| < \frac{(4n - 2)\delta_0\|p\|}{(1 - q)m_K} \left(\frac{\delta_0(1 + \rho)\|p\|}{(1 - q)(1 - \rho)m_K} + 1\right) \frac{\rho^{\ell/2}}{1 - \rho^{\ell}},$$

$$\|p_2 - \tilde{p}_2\| < \frac{(4n + 2)\delta_0^2\|p\|^2}{(1 - q)m_K} \frac{\rho^{\ell/2}}{1 - \rho^{\ell}}.$$

PROOF. By formula (27), we have $|c_k - \tilde{c}_k| < \frac{2}{m_K} \frac{\rho^{\ell/2}}{1 - \rho^{\ell}}$. Therefore,

$$\|c_{-n-\kappa}^{n-\kappa} - \tilde{c}_{-n-\kappa}^{n-\kappa}\| < \frac{4n - 2}{m_K} \frac{\rho^{\ell/2}}{1 - \rho^{\ell}}, \quad \|c_{-n-\kappa}^{n-\kappa} - \tilde{c}_{-n-\kappa}^{n-\kappa}\| < \frac{4n + 2}{m_K} \frac{\rho^{\ell/2}}{1 - \rho^{\ell}}.$$

If condition (34) is fulfilled, then $\|c_{-n-\kappa}^{n-\kappa} - \tilde{c}_{-n-\kappa}^{n-\kappa}\| < \frac{q}{\delta_0\|p\|}$ and in order to obtain the required estimates it is sufficient to apply Theorem 5.3. □

From this theorem it is easy to obtain

Corollary 5.3. Let $\varepsilon < \frac{q}{1 - q}$ and

$$\alpha = \varepsilon \frac{(1 - q)m_K}{\delta_0\|p\|} \min \left\{ (4n - 2) \left(1 + \frac{\delta_0\|p\|}{m_K(1 - \rho)}\right), (4n + 2)\delta_0\|p\| \right\}.$$

If $\ell$ is an even integer such that

$$\ell > 2 \max \left\{ n + \kappa, \frac{\log \left(\sqrt{1 + \frac{1}{4\alpha^2} + \frac{1}{2\alpha}}\right)}{\log \rho}\right\},$$

(35)

then $\|p_1 - \tilde{p}_1\| < \varepsilon$, $\|p_2 - \tilde{p}_2\| < \varepsilon$. □

6. Algorithm and numerical examples

The above results can be summarized in the form of the following algorithm.

Algorithm. Wiener – Hopf factorization of a scalar polynomial

INPUT. The coefficients of the polynomial $p(z)$, the parameter $\rho$ of the annulus $K$, $m_1 = \min_{|z|=1} |p(z)|$, $m_K = \min_{z \in K} |p(z)|$, the given accuracy $\Delta$ for the coefficients of $p(z)$: $\|p - \tilde{p}\| < \Delta$.

COMPUTATION.
1. Compute the index $\kappa$ of $p(z)$ by formula (19). The result is rounded up to the nearest integer.
2. Compute $\|p\|$, choose $n > \nu = \deg p$. For the sake of simplicity, put $q = 1/2$.
3. Choose $\delta_0 = 1$, or $\delta_0 = \kappa + 1$, or $\delta_0 = \delta\nu \sqrt{\kappa + 1}(\nu - \kappa + 1)$. Here $\delta = e^{G/\pi}$ is Boyd’s constant.
4. Find accuracy $\varepsilon_1, \varepsilon_2$ for $p_1, p_2$ by formulas (32), (33). Compute the theoretically guaranteed accuracy $\varepsilon := 10^{-d} < \max\{\varepsilon_1, \varepsilon_2\}$.
5. Estimate the condition number $k \leq 10^d$ by formula (25) or (26). Put $\bar{\varepsilon} := 10^{-d-\delta}$.
6. Find an even integer $\ell$ satisfying inequality (35), where $\varepsilon := \bar{\varepsilon}$.
7. Form the sequence $c^{-n-\kappa}$ by formula (28).
8. Form the Toeplitz matrix $T_{-\kappa+1}(c^{-n-\kappa})$ and find a basis $\{R_1, R_2\}$ of its kernel. The last operation can be done with the help of SVD.
9. Find the factorization essential polynomials $Q_1(z) = \alpha_1 z + \cdots + \alpha_n z^n + z^{n+1}, Q_2(z) = \beta_0 + \beta_1 z + \cdots + \beta_n z^n$ by (16).
10. Verify that the absolute values of the coefficients $\alpha_1, \ldots, \alpha_{n-\kappa}, \beta_{\nu-\kappa+1}, \ldots, \beta_n$ are less than $\varepsilon$ and delete these coefficients (see Theorem 4.4).
11. $\hat{p}_1(z) := z^{-n+\kappa-1}Q_1(z), \hat{p}_2(z) := Q_2(z)$.
12. end

OUTPUT. The coefficients $\hat{p}_1^k, \hat{p}_2^k$ of the factors $\hat{p}_1(z), \hat{p}_2(z)$ with the guaranteed accuracy $\varepsilon$.

In the following examples we use the Maple computer algebra system. All computations were performed on a desktop.

The polynomial $p(z)$ in Example 6.1 satisfies the conditions of Proposition 2.2 and its Wiener – Hopf factorization is actually the spectral factorization.

Example 6.1. Let $p(z) = (z+1/2)(z+1/3) \cdots (z+1/12)(z+2)(z+3) \cdots (z+12)$. Taking into account the values of the coefficients of $p(z)$, we choose the precision Digits := 20. Assume that the accuracy of the input data $\Delta$ is equal to $10^{-15}$. We may take $p := 0.51$. Then $m_1 = 3.326340 \times 10^6, m_K = 30.448076$.

We have $\nu = 22, \kappa = 11, \|p\| = 20237817600$. Put $n = \nu + 1 = 23$. By Proposition 2.2, $\delta_0 = 1$. The computation of the theoretically guaranteed accuracy $\varepsilon$ gives the following result $\varepsilon = 0.695883 \times 10^{-5}$. By formula (26), we obtain the following estimate $k(T_{-\kappa}(c_{n-\kappa}^{-n})) \leq 2.859480 \times 10^5$. It follows from this that $\bar{\varepsilon} = 10^{-22}$ and we get $\ell = 136$.

In this example the exact output is known

$p_1(z) = (z + 1/2)(z + 1/3) \cdots (z + 1/12), \ p_2(z) = (z + 2)(z + 3) \cdots (z + 12)$.

Table 1 shows the results of computations of the factors $\hat{p}_1(z), \hat{p}_2(z)$. It contains coefficients $\hat{p}_1^k, \hat{p}_2^k$, absolute errors $|\hat{p}_1^k - \hat{p}_1^k|, |\hat{p}_2^k - \hat{p}_2^k|$ for the coefficients
Table 1: Coefficients $\tilde{p}_1^k, \tilde{p}_2^k$

| $k$ | $\tilde{p}_1^k$ | $|\tilde{p}_1^k - p_1^k|$ | $\tilde{p}_2^k$ | $|\tilde{p}_2^k - p_2^k|$ |
|-----|----------------|--------------------------|----------------|--------------------------|
| 0   | 0              | 2.087675e-9              | 479001600.0000 | 1.04000e-9               |
| 1   | 0              | 1.60751e-7              | 1007441280.0000 | 1.26000e-8               |
| 2   | 0              | 0.55114e-5              | 924118272.0000  | 3.78100e-8               |
| 3   | 0.00011        | 1.97436e-18             | 489896616.0000  | 5.78000e-8               |
| 4   | 0.00145        | 4.98731e-17             | 167310220.0000  | 5.46800e-8               |
| 5   | 0.01300        | 9.54140e-17             | 38759930.0000   | 3.62700e-8               |
| 6   | 0.08091        | 1.20571e-16             | 6230301.0000    | 2.20830e-8               |
| 7   | 0.34928        | 1.18000e-16             | 696333.0000     | 1.81875e-8               |
| 8   | 1.02274        | 8.87000e-17             | 53130.0000      | 1.72958e-8               |
| 9   | 1.92925        | 4.76000e-17             | 2640.0000       | 1.24380e-8               |
| 10  | 2.10321        | 1.40000e-17             | 77.0000         | 2.80480e-9               |
| 11  | 1.00000        | 0                        | 0.999999        | 9.23655e-9               |

$\|\tilde{p}_1^k - p_1^k\|$ = 0.56743e-5, $\|\tilde{p}_2^k - p_2^k\|$ = 2.82246e-7

$p_1^k$, $p_2^k$, and $\|\tilde{p}_1 - p_1\|$, $\|\tilde{p}_2 - p_2\|$. For $\tilde{p}_1^k$, $\tilde{p}_2^k$ the number of decimal places obtained accurately is shown.

Thus $\|\tilde{p}_1 - p_1\| = 0.56743 \times 10^{-5} < 0.695884 \times 10^{-5} = \varepsilon$, and $\|\tilde{p}_2 - p_2\| = 2.82246 \times 10^{-7} < 0.695884 \times 10^{-5} = \varepsilon$. We obtain $p_1(z), p_2(z)$ with the desired accuracy.

The following example was taken from [8]. Since $p(z)$ has real coefficients $p_j$ and $p_{\nu-j} = p_j$, we can use Proposition 2.1 and the factorization of $p(z)$ is also the spectral factorization.

Example 6.2. Let $p(z) = \sum_{i=0}^{10} z^i + 4z^5$, Digits := 20, $\Delta = 10^{-12}$. We may take $\rho = 0.83$. Now $m_1 = 1.542464$, $m_K = 0.062855$.

In this example $\nu = 10$, $\kappa = 5$, $\|p\| = 15$, $n = \nu + 1 = 11$. By Proposition 2.1, $\delta_0 = \kappa + 1 = 6$. For the accuracy $\varepsilon$ we obtain $\varepsilon = 0.536458 \times 10^{-4}$. From formula (26) it follows the following estimate $k(T - \nu(e^{-\kappa - \nu})) \leq 1342.008991$. This yields $\varepsilon = 10^{-17}$ and $t = 418$.

The computed coefficients of the factors $\tilde{p}_1(z)$, $\tilde{p}_2(z)$ are given by Table 2. We only indicate 5 decimal places here.

Table 2: Coefficients $\tilde{p}_1^k, \tilde{p}_2^k$

| $k$ | 0   | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|-----|
| $\tilde{p}_1^k$ | 0.23193 | 0.20715 | 0.17674 | 0.14253 | 0.10685 | 1.00000 |
| $\tilde{p}_2^k$ | 4.31154 | 0.46071 | 0.61452 | 0.76203 | 0.89314 | 1.00000 |

In order to verify the computation correctness of $p_2(z)$, we can use the following relation between the factors $p_1(z)$ and $p_2(z)$ in the spectral factorization:
\[ p_2(z) = z^n p_1(1/z)/p_1(0). \] For our example we have \( \|p_2 - z^n \hat{p}_1(1/z)/\hat{p}_1(0)\| = 5.78 \times 10^{-18}. \) Moreover, the residual error is \( \| \hat{p}_1 \hat{p}_2 - p \| = 8.1 \times 10^{-18}. \)

In the next example the random polynomial \( p(z) \) was generated with the help of package Random Tools.

**Example 6.3.** Let

\[
p = z^{11} - \frac{17}{30} z^{10} + \frac{13}{10} z^9 + \left( \frac{223}{60} + \frac{848}{135} i \right) z^8 + \left( \frac{28}{15} + \frac{514}{135} i \right) z^7 + \\
\left( -\frac{43}{60} + \frac{106}{135} i \right) z^6 + \left( \frac{43}{60} + \frac{764}{135} i \right) z^5 + \left( -\frac{31}{6} + \frac{68}{135} i \right) z^4 + \left( \frac{7}{3} - \frac{2}{3} i \right) z^3 + \\
\left( -1 + \frac{814}{135} i \right) z^2 + \left( \frac{39}{10} + \frac{58}{15} i \right) z + \left( -\frac{61}{60} + \frac{16}{9} i \right).
\]

Digits := 20, \( \Delta = 10^{-18}. \) Calculations show that \( \rho = 0.943396, \ m_1 = 2.293009, \ m_K = 0.241435. \)

For the polynomial we have \( \nu = 11, \ z = 3, \ n = \nu + 1 = 12, \|p\| = 42.442968. \) Since \( p(z) \) is a polynomial of general type, \( \delta_0 = 3663.225630 \) is found by the formula \( \delta_0 = \delta' \sqrt{(z + 1)(\nu - z + 1)}. \) In fact, the use of this value of \( \delta_0 \) in the estimate \( \|p_1\| \|p_2\| \leq \delta_0 \|p\| \) makes it approximately 1,000 times worse. By these reasons, we are forced to use more accurate input data. Then the guaranteed accuracy in the output is \( \varepsilon = 0.653797 \times 10^{-4}. \)

From formula (26) we obtain the estimate

\[ k(T_{-\nu}(c_{n+1-z}^{-1})) \leq 1.695132 \times 10^6. \]

Hence \( \bar{\varepsilon} = 10^{-26} \) and \( \ell = 1994. \)

The computed coefficients of the factors \( \hat{p}_1(z), \hat{p}_2(z) \) are given by Table 3.

| \( k \) | \( \hat{p}_1(z) \) | \( \hat{p}_2(z) \) |
|------|----------------|----------------|
| 0    | -0.099841 - 0.150475i | -5.090491 - 10.133912i |
| 1    | -0.236722 + 0.118527i | -14.129949 + 0.552043i |
| 2    | -0.385402 - 0.732498i | -4.543939 + 4.838437i |
| 3    | 0.99999999 | -7.958489 + 1.840704i |
| 4    |                  | -5.515909 + 9.645327i |
| 5    |                  | 4.196252 + 7.320240i |
| 6    |                  | 0.930308 + 0.031004i |
| 7    |                  | -0.181264 + 0.732498i |
| 8    |                  | 1.000000 |

The residual error is \( \| \hat{p}_1 \hat{p}_2 - p \| = 2.638787 \times 10^{-17}. \)

Let \( \hat{p}_1(z), \hat{p}_2(z) \) be the factorization factors of \( p(z) \) obtained by the naive method (via the roots of \( p(z) \)). Then

\[ \| \hat{p}_1 - p \| = 1.3 \times 10^{-10}, \| \hat{p}_2 - p \| = 2.053051 \times 10^{-8}. \]
7. Conclusion

We have considered the Wiener – Hopf factorization problem for scalar polynomials with a numerical point of view. An algorithm has been given that is closed to algorithm 3 of D.A. Bini and A. Böttcher [8]. However, in contrast to algorithm 3 our method permits to find coefficients of both factors $p_1(z), p_2(z)$ simultaneously. Moreover, effective estimates of $\|p_1 - \tilde{p}_1\|, \|p_2 - \tilde{p}_2\|$ via $\|p - \tilde{p}\|$ are obtained. These estimates allow to find $p_1(z), p_2(z)$ with the guaranteed accuracy depending from the accuracy of the input data. We have illustrated this by examples.

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