A new property of absorbed diffusions

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Abstract

We consider stochastic diffusion processes absorbed at the boundary of a domain. It is shown that there exist initial distributions which ensure a given decreasing of density of the absorbed process.

Key words: Diffusion processes, absorption, Kolmogorov’s parabolic equations

1 Introduction

The paper investigates distributions of diffusion processes with adsorption on the boundary of a domain. We consider the problem of controlled absorption, and we obtain a new property of absorbed process: we establish an existence of initial distributions which ensure a given decreasing of density of the absorbed process. More precisely, we show that, for any piecewise continuous function \( \gamma(x) \geq 0 \) such that \( \gamma(\cdot) \neq 0 \), there exists an initial probability density function \( p(\cdot, 0) \) and a number \( \alpha > 0 \) such that \( p(x, 0) \equiv p(x, T) + \alpha \gamma(x) \), where \( p(x, t) \) is the probability density function of a process \( y(t) \) absorbed at the boundary \( \partial D \). Thus, it can be concluded that decreasing of density of the absorbed process can be programmed.

It can be mentioned that some problems for controlled absorption were solved by author (1994, 1995) for another type of constraints, when \( v(x, T) \equiv \mu v(x, 0) \) with \( \mu \in (0, 1) \), where \( v(x, t) \) is a solution of a backward Kolmogorov equation.

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2 Definitions

Consider a $n$-dimensional Wiener process $w(t)$ with independent components on a complete probability space $(\Omega, \mathcal{F}, P)$, such that $w(0) = 0$.

Let $D \in \mathbb{R}^n$ be a bounded domain with $C^2$ - smooth boundary $\partial D$, and let $T > 0$ be a fixed number. Consider the following Itô’s stochastic differential equation:

$$
\begin{cases}
    dy(t) = f(y(t), t)dt + \beta(y(t), t)dw(t), & t \in [0, T], \\
y(0) = y_0.
\end{cases}
$$

(1)

Here $y_0$ is a random vector which is independent of $w(\cdot)$ and such that $y \in D$ a.s.

We assume that the function $f(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is measurable and bounded, the function $\beta(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is continuous, and that there exist bounded derivatives $\partial^k \beta(x, t)/\partial x^k_i$, where $i = 1, \ldots, n$, $k = 1, 2$. We assume also that $\beta(x, t)\beta(x, t)^T \geq \delta I_n$, where $\delta > 0$ is a constant, and $I_n$ is the unit matrix.

Under these assumptions, there exists the unique weak solution $y(t)$ of (1) (see e.g. Gihman and Skorohod (1979)). We consider the process $y(t)$ which is absorbed at $\partial D$, i.e. until first exit from $D$.

Spaces and classes of functions

For a Banach space $X$, we denote the norm by $\| \cdot \|_X$.

Let $H^0 \triangleq L_2(D)$ and $H^1 \triangleq W_2^1(D)$ be the standard Sobolev Hilbert spaces. Let $H^{-1}$ be the dual space to $H^1$, with the norm $\| \cdot \|_{H^{-1}}$ such that $\|u\|_{H^{-1}}$ for $u \in H^0$ is supremum of $(u, v)_{H^0}$ over all $v \in H^0$ such that $\|v\|_{H^1} \leq 1$; we have $H^1 \subset H^0 \subset H^{-1}$, assuming the standard embedding.

We shall denote the Lebesgue measure and the $\sigma$-algebra of Lebesgue sets in $\mathbb{R}^m$ by $\ell_m$ and $\mathcal{B}_m$, respectively.

Introduce the spaces

$$C^k(s, T) \triangleq C\left([s, T]; H^k\right), \quad \mathcal{W}^k(s, T) \triangleq L^2([s, T], \mathcal{B}_1, \ell_1; H^k), \quad k = 0, \pm 1,$$

and the space

$$\mathcal{V}^k(s, T) \triangleq \mathcal{W}^k(s, T) \cap C^{k-1}(s, T), \quad k > 0$$

with the norm $\|u\|_{\mathcal{V}^k(s, T)} \triangleq \|u\|_{\mathcal{W}^k(s, T)} + \|u\|_{C^{k-1}(s, T)}$.

The following definition will be useful.
**Definition 1** A function \( \gamma(\cdot) : D \to \mathbb{R} \) is said to be piecewise continuous if there exists an integer \( N > 0 \) and a set of open domains \( \{D_i\}_{i=1}^N \) such that the following holds:

- \( D = \bigcup_{i=1}^N D_i, \ D_i \cap D_j = \emptyset \) for \( i \neq j \);
- for any \( i \in \{1, \ldots, N\} \), the function \( \gamma|_{D_i} \) is continuous and can be continued to a continuous function \( \gamma_i : D_i \cup \partial D_i \to \mathbb{R} \);
- for any \( x \in \bigcup_{i=1}^N \partial D_i \), there exists \( j \in \{1, \ldots, N\} \) such that \( x \in \partial D_j \) and \( \gamma_j(x) = \gamma(x) \).

**3 The result**

**Theorem 1** Let \( \gamma(\cdot) : \mathbb{R}^n \to \mathbb{R} \) be a piecewise continuous function such that \( \gamma(x) \geq 0 \) (\( \forall x \)) and \( \gamma(\cdot) \neq 0 \). Then there exists a random vector \( y_0 \) which is independent of \( w(\cdot) \) and such that the following holds:

(i) the process \( y(t) \), defined by (1) and absorbed at \( \partial D \), has the probability density function \( p(x, t) \in V^1(0, T) \); and
(ii) there exists a constant \( \alpha > 0 \) such that \( p(x, 0) \equiv p(x, T) + \alpha \gamma(x) \).

**Proof.** Let \( y_0 \) have a probability density function \( \rho(x) \in L_2(D) \). In that case, the probability density function \( p(x, t) \) of the process \( y(t) \) absorbed at \( \partial D \) satisfies the forward Kolmogorov equation

\[
\begin{align*}
\frac{\partial p}{\partial t} &= A p, \quad t > 0, \\
p(x, t)|_{x \in \partial D} &= 0, \quad p(\cdot, 0) = \rho(\cdot),
\end{align*}
\]

where

\[
A p \triangleq \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x, t)p(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_i(x, t)p(x))
\]

and \( a(x, t) \triangleq \{a_{ij}(x, t)\} = \frac{1}{2} \beta(x, t)\beta(x, t)^\top \).

For \( s \in [0, T] \) and \( \xi \in H^0 \), consider the following auxiliary boundary value problem:

\[
\begin{align*}
\frac{dv}{dt} &= A v, \quad t > s, \\
v(x, t)|_{x \in \partial D} &= 0, \quad v(\cdot, s) = \xi(\cdot),
\end{align*}
\]

From the classic theory of parabolic equations, it follows that

\[
\|v(\cdot, T)\|_{H^1} \leq C_1\|v(\cdot, s)\|_{H^1},
\]
where \( C_1 > 0 \) is a constant which does not depend on \( \xi \) and \( s \) (see e.g. Ladyzenskaya et al (1968)).

Introduce operators \( \mathcal{L}_s : H^0 \to \mathcal{V}^1(s, T) \), such that \( \mathcal{L}_s \xi = v \), where \( v \) is the solution in \( \mathcal{V}^1(s, T) \) of the problem \( \mathbb{8} \). These linear operators are continuous (see e.g. Ladyzenskaya et al (1968)). Introduce an operator \( \mathcal{Q} : H^0 \to H^0 \), such that \( \mathcal{Q} \xi = v(\cdot, T) \), where \( v = \mathcal{L}_0 \xi \).

Clearly, this operator is linear and continuous.

Lemma 1 (i) The operator \( \mathcal{Q} : H^0 \to H^0 \) is compact;

(ii) If the equation \( \mathcal{Q} \xi = \xi \) has the only solution \( \xi = 0 \) in \( H^0 \), then the operator \( (I - \mathcal{Q})^{-1} : H^0 \to H^0 \) is continuous.

Proof of Lemma \( \mathbb{1} \). Let \( \xi \in H^0 \) and \( v \triangleq \mathcal{L}_0 \xi \), i.e. \( v \) is the solution of the problem \( \mathbb{8} \). We have that \( v|_{t \in [s, T]} = \mathcal{L}_s v(\cdot, s) \) for all \( s \in [0, T] \), hence

\[
\|v(\cdot, T)\|_{H^1} \leq C_1 \inf_{t \in [0, T]} \|v(\cdot, t)\|_{H^1} \\
\leq \frac{C_1}{\sqrt{T}} \left( \int_0^T \|v(\cdot, t)\|_{H^1}^2 \, dt \right)^{1/2} \leq \frac{C_2}{\sqrt{T}} \|v\|_{\mathcal{V}^1} \leq \frac{C^2}{\sqrt{T}} \|\xi\|_{H^0}
\]

for constants \( C_i > 0 \) which do not depend on \( \xi \). Hence the operator \( \mathcal{Q} : H^0 \to H^1 \) is continuous. The embedding \( H^1 \to H^0 \) is a compact operator (see e.g. Yosida (1965), Ch.10.3). Then (i) follows. Further, (ii) follows from Fredholm Theorem. This completes the proof of Lemma \( \mathbb{1} \). \( \square \)

Lemma 2 For any \( \gamma \in H^0 \), there exists the unique solution \( u \in \mathcal{V}^1 \) of the following problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= Au, \quad t > 0, \\
u(x, t)|_{x \in \partial D} &= 0, \quad u(x, 0) = \gamma(x).
\end{aligned}
\tag{5}
\]

Proof of Lemma \( \mathbb{2} \). First, we shall show that if \( \gamma(\cdot) = 0 \) then the unique solution of \( \mathbb{8} \) in \( \mathcal{V}^1 \) is \( u(\cdot) = 0 \).

Let \( u \in \mathcal{V}^1 \) solve \( \mathbb{8} \) with \( \gamma(\cdot) = 0 \). Clearly, \( u = \mathcal{L}_0 u(\cdot, 0) \). Denote \( \zeta^+(x) \triangleq \max(0, u(x, 0)) \) and \( \zeta^-(x) \triangleq \max(0, -u(x, 0)) \). Denote \( u^- \triangleq \mathcal{L}_0 \zeta^- \) and \( u^+ \triangleq \mathcal{L}_0 \zeta^+ \). We have that \( u^+ \geq 0 \) and \( u^- \geq 0 \) a.e., \( u = u^+ - u^- \) and \( u(x, 0) \equiv \zeta^+(x) - \zeta^-(x) \).

If \( u(\cdot, 0) \neq 0 \) then either \( \zeta^+(\cdot, 0) \neq 0 \) or \( \zeta^-(\cdot, 0) \neq 0 \). It follows from absorption at \( \partial D \) that if \( \zeta^+(\cdot, 0) \neq 0 \) then

\[
\int_D u^+(x, T) \, dx < \int_D \zeta^+(x) \, dx.
\]

(It suffices to note that the process \( q \triangleq \nu^{-1} u^+ \) is a probability density function of a process
with absorption on $\delta D$, where $\nu \triangleq \int_D \zeta^+(x)dx$. Similarly, if $\zeta^-(\cdot,0) \neq 0$ then

$$\int_D u^-(x,T)dx < \int_D \zeta^-(x)dx.$$  

Hence

$$\int_D |u(x,T)|dx \leq \int_D u^+(x,T)dx + \int_D u^-(x,T)dx \leq \int_D \zeta^+(x)dx + \int_D \zeta^-(x)dx = \int_D |u(x,0)|dx,$$

i.e. $\int_D |u(x,T)|dx < \int_D |u(x,0)|dx$, and the condition $u(x,0) \equiv u(x,T)$ fails to be satisfied for $u(\cdot) \neq 0$. Thus, the unique solution of (5) for $\gamma(\cdot) = 0$ is $u(\cdot) = 0$.

By Lemma [1], it follows that the operator $(I - Q)^{-1} : H^0 \to H^0$ is continuous. Let $\gamma(\cdot) \in H^0$ be arbitrary. Then there exists $\zeta = (I - Q)^{-1} \gamma \in H^0$, and this $\zeta$ is unique. Let $u \triangleq L_0 \zeta$. By the definitions of $L_0$ and $Q$, it follows that $u(\cdot,T) = Qu(\cdot,0)$. We have that $u(\cdot,0) - Qu(\cdot,0) = \gamma(\cdot)$, i.e. $u(\cdot,0) - u(\cdot,T) = \gamma(\cdot)$. Thus, $u \triangleq L_0 \zeta = L_0(I - Q)^{-1} \gamma$ is the unique solution of (6) for any $\gamma(\cdot) \in H^0 = L_2(D)$. This completes the proof of Lemma [2].

Let us continue the proof of the theorem. Let $\gamma(x) \geq 0$ be such as in the assumptions of the theorem, and let $u \triangleq L_0(I - Q)^{-1} \gamma$ solve the problem (6).

It is easy to see that if $u(\cdot,0) = 0$ then $u(\cdot,T) = 0$ and $\gamma(\cdot) = 0$. By the assumptions, $\gamma(\cdot) \neq 0$, hence $u(\cdot,0) \neq 0$ and $u(\cdot) \neq 0$.

We remind that $u = L_0 \zeta$, where $\zeta = u(\cdot,0) \in H^0$. By Theorem 9.1 from Ch.IV of Ladyzenskaya et al (1968) applied for smooth functions which approximate $u(\cdot,0)$ in $H^0$, and by Theorem 8.1 from Ch.III of this book, it follows that there exists a representative $\overline{u}(\cdot,T)$ of the corresponding element of $H^0$ which is continuous in $x \in \overline{D}$, where $\overline{D} = D \cup \partial D$. (Note that, by the definition, an element of $H^0 = L_2(D)$ is a class of $T_n^2$-equivalent functions). We have that $u(\cdot,0) = u(\cdot,T) + \gamma(\cdot)$, hence there exists a piecewise continuous representative $u'(\cdot,0)$ of $u(\cdot,0) \in H^0$. Note that such $u'(\cdot,0)$ is not unique, because the boundary values at $\partial D_i$ in the Definition [1] can be choosen differently.

Let us show that $u(x,0) \geq 0$ for a.e. $x$. Suppose that

$$\exists x \in D : \quad u'(x,0) < 0. \quad (6)$$

If (6) holds, then there exists a piecewise continuous representative $\overline{u}(\cdot,0)$ such that there exists $\hat{x} \in D$ such that

$$\overline{u}(\hat{x},0) < 0, \quad \overline{u}(\hat{x},0) \leq \overline{u}(x,0) \quad \text{for a.e. } x \in D.$$
We have that
\[ \overline{u}(x, T) = \int_D G(x, y, T, 0) u(y, 0) dy, \]
where \( G(x, y, T, 0) > 0 \) is the corresponding Green’s function for the problem (1). Clearly,
\[ \Gamma \triangleq \int_D G(\hat{x}, y, T, 0) dy \in (0, 1). \]
Then
\[ \overline{u}(\hat{x}, T) = \left( \int_D G(\hat{x}, y, T, 0) dy \right) \overline{u}(\hat{x}, 0) + \int_D G(\hat{x}, y, T, 0) (\overline{u}(y, 0) - \overline{u}(\hat{x}, 0)) dy \geq \Gamma \overline{u}(\hat{x}, 0) > \overline{u}(\hat{x}, 0). \]

It follows that if (3) holds then \( u(x, t) \) does not satisfy (3). Thus, \( u(x, 0) \geq 0 \) a.e.

Now, let \( \alpha \triangleq (\int_D u(x, 0) dx)^{-1} \) and \( \rho(x) \triangleq \alpha \overline{u}(x, 0) \). We have that \( \rho(x) \geq 0 \) and \( \int_D \rho(x) dx = 1 \). Then there exists a random vector \( y_0 \) such that \( y_0 \) is independent of \( w(\cdot) \) and has the probability density function \( \rho \). Let \( p \triangleq \mathcal{L}_0 \rho \), i.e. \( p \) solve (2). Clearly, \( p(x, t) \) is the probability density function of the process \( y(t) \) satisfying (1). By linearity of (5), it follows that \( p = \alpha u \) and \( p(\cdot, 0) - p(\cdot, T) = \alpha \gamma(\cdot) \). This completes the proof of Theorem 1. \( \square \)

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