On relativization of the Sommerfeld-Gamow-Sakharov factor

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Motivation

- The problem of *non-perturbative* corrections and their matching to perturbative ones . . .
- Relativistic two-particle problem . . .
- Increasing experimental precision in observation of different $e^+e^-$ annihilation channels
- Radiative return method allows scrutinizing the threshold region
- Intriguing experimental results on threshold enhancement in $e^+e^- \rightarrow \Lambda\bar{\Lambda}$
- SGS-like factor in QCD?
- Permanent interest of community and discussions in literature
- How to treat the factor within general-purpose computer codes?
The SGS factor features (I)

It’s known for a long time that the re-scattering correction close to the threshold of a charged-particle-pair production is proportional to $|\Psi(0)|^2$ [A. Sommerfeld, *Atmobaun und Spektralinien* (1921); J. Schwinger, *Particles, Sources, and Fields*, Vol.3.]

G. Gamow found the corresponding factor for the Coulomb barrier in nuclear interactions (1928)

A. Sakharov considered just *Interaction of an electron and positron in pair production* (1948)

The Sommerfeld-Gamow-Sakharov (SGS) factor in the nonrelativistic approximation reads

$$T = \frac{\eta}{1 - e^{-\eta}}, \quad \eta = -Q_1 Q_2 \frac{2\pi \alpha}{v}$$

where $v$ is the relative velocity (*by construction*) of the particles with charges $Q_{1,2}$. 
The SGS factor features (II)
The SGS factor features (III)

Perturbative expansion in $\alpha$

$$T \approx 1 + \frac{\pi \alpha}{v} + \frac{\pi^2 \alpha^2}{3v^2} + \mathcal{O}\left(\frac{\alpha^3}{v^3}\right)$$

but for $v \to 0$ it breaks down

For opposite charges at very small $v$ the factor behaves as

$$T \bigg|_{Q_1 Q_2 = -1} \xrightarrow{v \to 0} \frac{2\pi \alpha}{v}$$

For equal charges at very small $v$ the factor vanishes

$$T \bigg|_{Q_1 Q_2 = 1} \xrightarrow{v \to 0} 0$$
Ad hoc: $v \rightarrow 2\sqrt{1 - \frac{m^2}{E^2}} = 2\beta$
Approaches to relativization of SGS factor (I)

- **Ad hoc**: \( v \rightarrow 2\sqrt{1 - \frac{m^2}{E^2}} = 2\beta \)

- (Quasi)-relativistic quasi-potential equations

  It is possible to solve the two-particle problem with relativistic kinematics and interaction, but . . .
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- Bethe-Salpeter equations (?)
Ad hoc: $\nu \rightarrow 2\sqrt{1 - \frac{m^2}{E^2}} = 2\beta$

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Resummation of (ladder) Feynman diagrams
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- Resummation of (ladder) Feynman diagrams

- Extrapolation of (one-loop) perturbative calculations
Conditions on a relativized factor:

- Non-relativistic limit (must have)
Approaches to relativization of SGS factor (II)

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- Matching with perturbative calculations (?)
There are many results in the literature on (quasi)relativistic two-particle eqs. Just look at $|\Psi(0)|^2$. Questions to these approaches always remain, but we can play there and understand the SGS factor better. In particular, in [A.A., Nuovo Cim.'1994] the case of scalar particles with arbitrary masses was evaluated

$$\frac{1}{i} \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \psi = \left( \sqrt{\vec{p}_1^2 + m_1^2} + \sqrt{\vec{p}_2^2 + m_2^2} \right) \psi$$

Use equal velocity reference frame: $\vec{p}_2 = -\vec{p}_1$ $m_2/m_1 \Leftrightarrow \vec{v}_1 = -\vec{v}_2$.

Minimal substitution $p_i^{\mu} \to p_i^{\mu} - eA_i^{\mu}$ gives

$$\left[ \frac{1}{\rho} \frac{\partial^2}{\partial \rho^2} \rho + 1 - \frac{2\alpha}{\nu \rho} - \frac{l(l+1)}{\rho^2} + \frac{\alpha^2}{4\rho^2} (-1 + \vec{u}^2) \right] R_l(\rho) = 0$$
For pure Coulomb interaction \((A^0)\) we get

\[
v_C = 2\sqrt{s - 4m^2} = 2\beta
\]

Taking into account \(\vec{A}\) leads to the relativistic relative velocity

\[
v = \sqrt{\frac{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]}{s - m_1^2 - m_2^2}}
\]

The same result for \(|\Psi(0)|^2\) was obtained also by I.T. Todorov [PRD’1971], H.W. Crater et al. [Ann.Phys.(NY)’1983, PRD’1992]

Obviously the limiting cases \(m_1 \ll m_2, m_1 \gg m_2\) where we have exact solutions of the Klein-Gordon (and Dirac) equations are reproduced
Jin-Hee Yoon and Cheuk-Yin Wong, “Relativistic modification of the Gamow factor” [PRC’2000, JPG’2005]:

\[ T(v) \rightarrow K(v) = T(v) \cdot \kappa(v) \]

where \( \kappa \) depends on the type of particles etc.

O.P. Solovtsova and Yu.D. Chernichenko, “Threshold resummation S-factor in QCD: the case of unequal masses”, [Yad.Fiz.’2010]
V.N. Baier and V.S. Fadin, [ZhETF’1969] have shown that resummation of ladder-type Feynman diagrams for $e^+e^-$ pair production leads to the factor in the form:

\[ T_{\text{resum.}} = \frac{\pi \alpha / \beta}{1 - e^{-\pi \alpha / \beta}} = T(2\beta) \]

Omitting diagrams with crossed photon lines corresponds to keeping only the **Coulomb interactions**. In this case we have an agreement with the quasi-potential picture.

Resummation of non-ladder diagrams is difficult . . .
The SGS factor has a non-perturbative nature. But its expansion in $\alpha/\nu$ for $\alpha \ll \nu \ll 1$ makes sense.

What goes on there in direct perturbative calculations?

Let’s look at $O(\alpha)$ FSR corrections to $e^+e^- \to \mu^+\mu^-$ with exact muon mass dependence, see e.g. A.A., D.Bardin, A.Leike [MPLA’1992], A.A. et al. [JHEP’2007]

The expansion in $\beta$ starts from $\pi\alpha/\beta$ which has the correct non-relativistic limit, i.e. it agrees with the SGS factor expansion.

What is the source of the Coulomb singularity in perturbative calculation?

What appears in the further expansion over $\beta$?
A. Hoang [PRD’1997]: one-loop contributions to moduli squared electric and magnetic form factors above the threshold:

\[
\left( \frac{\alpha}{\pi} \right) g^{(1)}_e(s) \xrightarrow{\beta \to 0} \frac{\alpha\pi}{2\beta} - 4\frac{\alpha}{\pi} + \frac{\alpha\pi\beta}{2} - \frac{4\alpha}{3\pi} \left[ \ln \frac{m^2}{\lambda^2} + \frac{2}{3} \right] \beta^2 + O(\beta^3)
\]

\[
\left( \frac{\alpha}{\pi} \right) g^{(1)}_m(s) \xrightarrow{\beta \to 0} \frac{\alpha\pi}{2\beta} - 4\frac{\alpha}{\pi} + \frac{\alpha\pi\beta}{2} - \frac{\alpha}{3\pi} \left[ 4 \ln \frac{m^2}{\lambda^2} - \frac{1}{3} \right] \beta^2 + O(\beta^3)
\]

The first and the third terms agree with the expansion of the SGS factor if \( v = 2\beta/(1 + \beta^2) \) i.e. the true relativistic relative velocity.

The second term comes from short distance (\( \sim 1/m \)) interactions. Factorization then gives

\[
T(v) \cdot \left( 1 - 4\frac{\alpha}{\pi} \right)
\]

N.B. The picture is reproduced with the \( O(\alpha^2) \) form factors [R.Barbieri, J.A.Mignaco, E.Remiddi, Nuovo Cim.’1972]
After adding contributions of soft and hard photons the picture persists: nontrivial additional terms start to appear in $\mathcal{O}(\beta^3)$.

The source of the singularity is the scalar triangular loop diagram:

$$
\delta \sigma^{1-\text{loop}} = \sigma^{\text{Born}} \frac{\alpha}{2\pi} (s - m_1^2 - m_2^2) \cdot C_0(m_1^2, m_2^2, s, m_1^2, m_2^2, m_1^2, m_2^2)
$$

The pre-factor $(s - m_1^2 - m_2^2)$ and $C_0(\ldots)$ are universal for spinor, scalar and vector particles and for all partial waves involved in $\sigma^{\text{Born}}$.

Direct calculations of $C_0$ for arbitrary masses gives an agreement to the expansion of the SGS factor with the proper relative relativistic velocity

$$
\nu = \frac{\sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]}}{s - m_1^2 - m_2^2}
$$
Matching with perturbative calculations

We should **match** perturbative and non-perturbative results:

\[
\sigma^{\text{Corr.}} = \sigma^{\text{Born}} \left( T(\nu) - \frac{\pi \alpha}{\nu} - \frac{\pi^2 \alpha^2}{3 \nu^2} - \ldots \right) \\
+ \Delta \sigma^{1-\text{loop}} + \Delta \sigma^{2-\text{loop}} + \ldots
\]

For a cross check the matching should be always verified **analytically** by looking at the threshold behavior of the perturbative corrections.
In the case of production of unstable particles, e.g. $t\bar{t}$ or $W^+W^-$ one can not rely upon bound state effect ($|\Psi(0)|^2$) (multiple photon exchange) since the lifetime is comparable with the b.s. formation time, see e.g. V.S. Fadin and V.A. Khoze [Yad.Fiz.’1988], V.S. Fadin, V.A. Khoze and T. Sjostrand [Z.Phys.C’1990]

Nevertheless, direct perturbative calculations show the presence of the Coulomb singularity. See e.g. $z, \gamma^* \rightarrow W^+W^-$ in [D.Y.Bardin, W.Beenakker, A.Denner, PLB’1993].

N.B. Authors of this paper got

$$\beta = \sqrt{\frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2}}$$

But if we restore the factor being omitted there

$$\frac{s}{2(s - m_1^2 - m_2^2)} \approx 1$$

we get exactly the relativistic relative velocity (one half)
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