Global Well-posedness for the Three Dimensional Simplified Inertial Ericksen-Leslie Systems Near Equilibrium

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Abstract

We study a simplified Ericksen-Leslie system with the inertial term for the nematic liquid crystal flow, which can be also viewed as a system coupling Navier-Stokes equations and wave map equations. We prove the global existence of classical solution with initial data near equilibrium.

1 Introduction

The Ericksen-Leslie system is a hydrodynamical theory for nematic liquid crystals which was established by Ericksen [4] and Leslie [22] in 1960’s. It has been successful to model various dynamical behavior for nematic liquid crystals. In this paper, we consider the simplified Ericksen-Leslie system [22] with inertial term:

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p &= \mu \Delta v - \nabla \cdot (\nabla d \otimes \nabla d), \\
\sigma_0 D_t^2 d + \sigma_1 D_t d - \Delta d &= (|\nabla d|^2 - \sigma_0 |D_t d|^2) d, \\
\nabla \cdot v &= 0.
\end{align*}
\]

on \( \mathbb{R}^n \times \mathbb{R}^+ \) (\( n \geq 2 \)). Here \( v(x, t) \in \mathbb{R}^n \) is the bulk velocity and \( d(x, t) \in \mathbb{S}^2 \) is the direction field representing the alignment of liquid crystal molecules. \( D_t = (\partial_t + v \cdot \nabla) \) denotes the material derivative. The term \( \sigma_0 D_t^2 d \) is called the inertial term, while \( \sigma_1 D_t d \) is called the damping term. If \( \sigma_0 = 0, \sigma_1 > 0 \), then (1.1) becomes the simplified (non-inertial) Ericksen-Leslie system which a parabolic type equation and has been widely studied in literatures since the work of Lin etc. [25, 27, 28, 29]. If \( \sigma_0 > 0, \sigma_1 = 0 \), then (1.1) becomes a hyperbolic one. For simplicity, the system considered here neglects complicated Leslie’s stress terms in the momentum equation and corresponding co-rotational and stretching terms in the angular momentum equation, however, the inertial term is kept. We refer to [22] or [5] for its full mathematical form.

There have been many works on the non-inertial Ericksen-Leslie system. For the simplified system without Leslie’s stress, Lin-Lin-Wang [26] and Hong [6] established the existence of a global weak solution in \( \mathbb{R}^2 \), see also Lin-Wang [30], Xu-Zhang [51], Hong-Xin [7] and Lei-Li-Zhang [19] for related results in two dimensional case. Recently, Lin-Wang [31] proved the global existence of weak solution for dimension three when the initial alignments \( d_0 \) locates in the upper half sphere. We also refer to [24, 9] for global existence of strong solution with small data for dimension three. For the full non-inertial system with Leslie’s stress, Wang-Zhang-Zhang [50] proved the local existence for general data and global existence for data near equilibrium of smooth solutions under optimal constraints on the Leslie coefficients for three dimensional case. Wang-Wang [48] extended these results to the general Oseen-Frank energy case. For 2D case, the existence of global weak solution was proved by Wang-Wang [48] and Huang-Lin-Wang [8], while the uniqueness of weak solutions was considered by Wang-Wang-Zhang [49] and Li-Titi-Xin [23].
To the best knowledge of the authors, the results considering the full Ericksen-Leslie system with inertial term are very few. There are some results on the one dimensional problem with general Oseen-Frank energy, however, the fluid coupling is neglected (i.e. $v = 0$). We refer to [2, 52, 53, 54] and references therein for examples. For the full inertial Ericksen-Leslie system, Jiang-Luo [10] studied the local wellposedness of classical solution of inertial Ericksen-Leslie system as well as the global wellposedness when the damping coefficients $\sigma_1 > 0$. In this work, we study the global wellposedness of classical solution to the simplified inertial Ericksen-Leslie system (1.1) in three dimensions for the case $\sigma_1 = 0$. Note that the term $\sigma_1 D_d d$ will bring an additional damping effect if $\sigma_1 > 0$. In addition, we assume $\sigma_0 = 1$, $\mu = 1$ without loss of generality. The main result of this paper is stated as follows (the notations will be explained in Section 2.2):

**Theorem 1.1.** Let $(v_0(x), d_0(x) - e) \in H^k_\Lambda$ with $k \geq 9$ where $e \in S^2$ is some constant director. Suppose

$$
\|(v_0, d_0 - e, d_1)\|_{eH^k_\Lambda}^2 \leq \epsilon.
$$

There exists a positive constant $\epsilon_0$ such that, if $\epsilon \leq \epsilon_0$, the system (1.1) with initial data

$$
v(x, 0) = v_0(x), \quad d(x, 0) = d_0(x), \quad \partial_t d(x, 0) = d_1(x)
$$

has a unique global classical solution such that

$$
E^d_{\kappa+1}(t) \leq C_0(\epsilon(t))^{\delta}, \quad E^\kappa_\kappa + E^d_{\kappa-1}(t) \leq C_0 \epsilon
$$

for some positive $C_0 > 0$, $0 < \delta < \frac{1}{2}$ depending on $\kappa$ and $\epsilon_0$ uniformly for all $0 \leq t < \infty$.

**Remark 1.1.** Following the recent work [3, 18, 20], we can apply the ghost weight energy method [1] to even prove a uniform bound for the highest-order energy. However, as the current estimate is sufficient to obtain the global existences of solutions, we will not pursue this direction in this paper.

**Remark 1.2.** The regularity index $\kappa$ may be lowered a little bit, however, restricted by vector field method, we can not expect it close to natural energy space. Hence, we do not aim to find the lowest regularity index here.

**Remark 1.3.** Inspired by the threshold global regularity on wave maps [36, 37], it might be natural to conjecture that the similar results also holds for the two dimensional inertial liquid crystal. This issue will be addressed in our future work.

We will perform the analysis under Eulerian coordinates, although Lagrangian coordinates may also work. Considering the wave nature of the equations for $d$, it is natural to use the vector field theory and the weighted energy method. However, the system has neither Lorentz invariance nor scaling invariance. We use the weighted $L^2$ norm introduced by Klainerman and Sideris [15] to overcome the difficulty from the lack of Lorentz invariance, while the lack of of scaling invariance is solved by directly applying the scaling operator and dealing with the commutators [11]. To close the energy estimate, it is necessary to obtain the subcritical decay for the solution. However, the interaction between velocity filed and the orientation field weakens the dissipative nature of the velocity field and the dispersive effect of the orientation field, and even worse, it is strengthened by the quasilinear nature of the system. These difficulties make the decay estimate very delicate. In addition, to close the higher-order energy estimate, we need to explore the symmetry structure of the system to deal with the quasilinear terms which may cause loss of derivatives at a first glance. This structure is not obvious in the Eulerian formulation.

When velocity equal zero and $\sigma_0 = 1$, $\sigma_1 = 0$, the inertial Ericksen-Leslie system reduces to wave maps where the target manifold is $S^2$. There are huge important progresses on the wave maps in the past two decades. To list a few of them, we refer to Shatah, Klainerman, Tao, Tataru etc. [12, 13, 14, 16, 17, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] and references therein.

The remaining part of this paper is organized as follows. In the next section, we will introduce the vector fields applied onto the system, ansatz for the method of continuity, and some preliminary
estimates. In Section 3, we will estimate of weighted $L^2$ norm. Section 4 is devoted to the decay estimate of velocity. In the last section, we give various energy estimates which are crucial steps to prove Theorem 1.

2 Preliminaries

2.1 Application of vector fields

To begin with, we rewrite the system (1.1) as

\[
\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v - \nabla \cdot (\nabla d \otimes \nabla d), \\
\partial_t d - \Delta d = (|\nabla d|^2 - |\partial_t d|^2 - |v \cdot \nabla d|^2 - 2 v \cdot \nabla d \cdot \partial_t d) d \\
\quad - (2 v \cdot \nabla \partial_t d + \partial_t v \cdot \nabla d + v \cdot \nabla (v \cdot \nabla d)), \\
\nabla \cdot v = 0.
\end{cases}
\]

(2.1)

We have seven nonlinear terms in the equations for the orientation field, which make the whole argument a little annoying.

Now let us take a look at the invariant groups of the system (2.1). Suppose that $(u(t, x), d(t, x))$ is a solution of (2.1), then one can check that $(Q^\top u(t, Qx), d(t, Qx))$ is also a solution of (2.1) for any orthogonal matrix $Q$. We choose $Q$ to be one parameter group generated by anti-symmetric matrices:

\[Q = e^{\lambda A_i}, \quad \forall \ 1 \leq i \leq 3,\]

where

\[A_1 = e_2 \otimes e_3 - e_3 \otimes e_2,\]
\[A_2 = e_3 \otimes e_1 - e_1 \otimes e_3,\]
\[A_3 = e_1 \otimes e_2 - e_2 \otimes e_1.\]

The perturbed angular momentum operators are defined as infinitesimal generators of the orthogonal groups:

\[
\begin{cases}
\tilde{\Omega}_i v = \Omega_i v + A_i v, \\
\tilde{\Omega}_i d = \Omega_i d,
\end{cases}
\]

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the rotational gradient operator defined by

\[\Omega = x \wedge \nabla.\]

Schematically, we have the commutation:

\[\left[ \partial, \Omega \right] = \partial, \quad \text{(2.2)}\]

where $\partial$ on the right hand side of (2.2) means the span of $\{\partial_t, \partial_1, \partial_2, \partial_3\}$. (2.2) and the following commutation (2.4) is often used implicitly.

Putting $(Q^\top u(t, Qx), d(t, Qx))$ into (2.1) and differentiate with respect to $\lambda$ and let $\lambda \to 0$, we...
have
\[
\begin{aligned}
\partial_t \tilde{\Omega} v + \tilde{\Omega} v \cdot \nabla v + v \cdot \nabla \tilde{\Omega} v + \nabla \tilde{\Omega} p \\
= \mu \Delta \tilde{\Omega} v - \nabla \cdot (\tilde{\Omega} d \otimes \nabla d) - \nabla \cdot (\nabla d \otimes \tilde{\Omega} d),
\end{aligned}
\]
\[
\partial_t \tilde{\Omega} d - \Delta \tilde{\Omega} d = 2(\nabla \tilde{\Omega} d \cdot \nabla d - \partial_t \tilde{\Omega} d \cdot \partial_t d)d \\
- (2\tilde{\Omega} v \cdot \nabla d \cdot \partial_t d + 2v \cdot \nabla \tilde{\Omega} d \cdot \partial_t d + 2v \cdot \nabla d \cdot \partial_t \tilde{\Omega} d)d \\
- 2[(\tilde{\Omega} v \cdot \nabla d) \cdot (v \cdot \nabla d) + (v \cdot \nabla) \tilde{\Omega} d \cdot (v \cdot \nabla d)]d \\
+ (|\nabla d|^2 - |\partial_t d|^2 - |v \cdot \nabla d|^2 - 2v \cdot \nabla d \cdot \partial_t d)\tilde{\Omega} d \\
- (2\tilde{\Omega} v \cdot \nabla d) d + 2v \cdot \nabla \partial_t \tilde{\Omega} d + \partial_t \tilde{\Omega} v \cdot \nabla d + \partial_t v \cdot \nabla \tilde{\Omega} d) \\
- \tilde{\Omega} v \cdot \nabla (v \cdot \nabla d) + v \cdot \nabla (\tilde{\Omega} v \cdot \nabla d) + v \cdot \nabla (v \cdot \nabla \tilde{\Omega} d) \\
\end{aligned}
\]
(2.3)
\[
\nabla \cdot \tilde{\Omega} v = 0.
\]

Next, we try to apply scaling operator onto the system. The scaling operator is defined as
\[
S = t^k + \alpha d.
\]

Unfortunately, the system \[2.4\] doesn’t have any scaling invariance (nor Lorentz invariance). Inspired by \[3.11\], we are still able to use the scaling operator under this circumstance.

Applying \(S + 1\) onto \[2.1\], \[2.3\], and applying \(S + 2\) onto \[2.1\], thanks to the commutation:
\[
(S + 1)\partial = \partial S, \quad (S + 2)\partial^2 = \partial^2 S,
\]
we can derive by directly calculation to get that
\[
\begin{aligned}
\partial_t S v + Sv \cdot \nabla v + v \cdot \nabla S v + \nabla S p \\
= \mu \Delta (S - 1) v - \nabla \cdot (\nabla (S - 1) d \otimes \nabla d) - \nabla \cdot (\nabla d \otimes \nabla (S - 1) d),
\end{aligned}
\]
\[
\partial_t S d - \Delta S d = 2(\nabla S d \cdot \nabla d - \partial_t S d \cdot \partial_t d)d \\
- (2S v \cdot \nabla d \cdot \partial_t d + 2v \cdot \nabla S d \cdot \partial_t d + 2v \cdot \nabla d \cdot \partial_t S d)d \\
- 2[(S v \cdot \nabla d) \cdot (v \cdot \nabla d) + (v \cdot \nabla) S d \cdot (v \cdot \nabla d)]d \\
+ (|\nabla d|^2 - |\partial_t d|^2 - |v \cdot \nabla d|^2 - 2v \cdot \nabla d \cdot \partial_t d)S d \\
- (2S v \cdot \nabla \partial_t d + 2v \cdot \nabla \partial_t S d + \partial_t S v \cdot \nabla d + \partial_t v \cdot \nabla S d) \\
- [S v \cdot \nabla (v \cdot \nabla d) + v \cdot \nabla (S v \cdot \nabla d) + v \cdot \nabla (v \cdot \nabla S d)].
\end{aligned}
\]
(2.5)
\[
\nabla \cdot S v = 0.
\]

Note that there are some commutators in the equation for velocity in \[2.4\] due to the appearance of viscosity. This is different from the application of rotation operator.

Now we are going to apply compositions of generalized operators. Let
\[
\Gamma \in \{\partial_1, \partial_2, \partial_3, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3\},
\]
and \(Z^a = S^{a_1} \Gamma^{a'}\), where \(a = (a_1, a_2, \ldots, a_k) = (a_1, a')\), \(\Gamma^{a'} = \Gamma^{a_2} \Gamma^{a_3} \ldots \Gamma^{a_k}\). Using reduction argument, we can use \[2.3\] and \[2.5\] to derive that
\[
\begin{aligned}
\partial_a Z^a v - \mu \Delta (S - 1)^{a_1} \Gamma^{a'} v &= f^1_a, \\
\partial_a Z^a d - \Delta Z^a d &= f^2_a, \\
\nabla \cdot Z^a v &= 0.
\end{aligned}
\]
(2.6)
The above commutation relation (2.6), (2.7) is the starting point of this paper.

2.2 Some notations

Throughout this paper, we use the generalized energy defined by

\[ E^w_\kappa(t) = \|Z^\kappa \nu(t, \cdot)\|_{L^2}^2, \quad E^d_{\kappa+1}(t) = \|\partial Z^\kappa d(t, \cdot)\|_{L^2}^2. \]

where \( Z^\kappa \nu = \{ Z^a \nu : |a| \leq \kappa \}, Z^\kappa d = \{ Z^a d : |a| \leq \kappa \}. \)

We also use the weighted energy norm of Klainerman-Sideris [15]:

\[ X^d_\kappa(t) = \| (r - t)^{\kappa - 2} Z^\kappa d \|_{L^2}^2, \]

in which \( \langle \sigma \rangle = \sqrt{1 + \sigma^2} \), for \( \kappa \geq 2 \).

To describe the space of initial data, we introduce (see [33])

\[ \Lambda = \{ \nabla, \Omega, r \partial_r \}, \]

and

\[ H^\kappa_\Lambda = \{ (f, g, h) : \sum_{|a| \leq \kappa} \| \Lambda^a f \|_{L^2} + \| \nabla \Lambda^a g \|_{L^2} + \| \Lambda^a h \|_{L^2} < \infty \}, \]

with the norm

\[ \|(f, g, h)\|_{H^\kappa_\Lambda} = \sum_{|a| \leq \kappa} \left( \| \Lambda^a f \|_{L^2} + \| \nabla \Lambda^a g \|_{L^2} + \| \Lambda^a h \|_{L^2} \right), \]

for vector of \( f, g \) and \( h \).

Throughout the whole paper, we will use \( A \lesssim B \) to denote \( A \leq CB \) for some positive absolute constant \( C \), whose meaning may change from line to line. We remark that, without specification, the constant depends only on \( \kappa \), but not on \( t \).

2.3 Ansatz for the method of continuity

The local well-posedness problem has been proved recently in [10]. To extend the local solutions to be global one, we need to show the uniform estimate in time.

We make the following ansatz for the general energy:

\[ E^w_\kappa \leq C_\tau, \quad E^d_{\kappa+1} \leq C\tau(\tau)^\delta, \quad E^d_{\kappa-1} \leq C\tau, \]
for $\kappa \geq 9$, which is $\Box 2.3$ in Theorem 2.3. Under the above a priori assumption, we first show that the weighted $L^2$ norm $X_\kappa$ can be controlled by the generalized energy:

$$X_{\kappa-1}^d \lesssim E_{\kappa-1}, \quad X_{\kappa+1}^d \lesssim E_{\kappa+1},$$

see Section 3. Next, we show the decay estimate for velocity in Section 4:

$$\begin{align*}
\| (t) \frac{1}{2} v \|_{L^\infty} \lesssim e^\frac{t}{2}, & \quad \| (t) \frac{1}{2} \nabla v \|_{L^\infty} \lesssim e^\frac{t}{2}, \\
\| (t) \frac{1}{2} \nabla^0 v \|_{L_\infty^\kappa} \lesssim e^\frac{t}{2}(\ln(t))^\frac{1}{2}, & \quad \forall 2 \leq |a| \leq \kappa - 3, \\
\| \partial_t v \|_{L^\infty} \lesssim \langle t \rangle^{-\frac{1}{2}}(\ln(t))^\frac{1}{2} e^\frac{t}{2}.
\end{align*}$$

(2.8)

Then, we can finally close the energy estimates by establishing the following inequalities:

$$E_\kappa^d(t) + \mu \| \nabla Z^k v(\cdot, \cdot) \|^2_{L_x^2 L_t^2} \lesssim E_\kappa^d(0) + \int_0^t \langle \tau \rangle^{-2} E_{\kappa+1}^d(\tau) E_{\kappa-1}^d(\tau) \, d\tau + \| \nabla Z^k v(\cdot, \cdot) \|^2_{L_x^2 L_t^2} E_\kappa^d(t).$$

(2.9)

$$\begin{align*}
\frac{d}{dt} E_{\kappa+1}^d & \lesssim \langle t \rangle^{-1}(E_\kappa^d + E_{\kappa-1}^d) E_{\kappa+1}^d + \| \nabla Z^k v(\cdot, \cdot) \|^2_{L_x^2 L_t^2} E_\kappa^d \\
& \quad + \langle t \rangle^{-1}\| \nabla Z^k v \|_{L^\infty} E_{\kappa+1}^d + \langle t \rangle^{-1}\| \partial_t v \|_{L^\infty} E_{\kappa+1}^d.
\end{align*}$$

(2.10)

$$\begin{align*}
\frac{d}{dt} E_{\kappa-1}^d & \lesssim \langle t \rangle^{-1}\| \nabla Z^k v \|_{L^2}^2 E_{\kappa}^d(E_{\kappa+1}^d + E_{\kappa-1}^d) \frac{1}{2} E_{\kappa-1}^d \\
& \quad + \langle t \rangle^{-\frac{1}{2}} E_{\kappa-1}^d(E_{\kappa+1}^d) \frac{1}{2} + \| \nabla Z^k v \|^2_{L_x^2 L_t^2} E_{\kappa-1}^d.
\end{align*}$$

(2.11)

This is the main topic in Section 5.

With the help of above estimates (2.8)-(2.11), we can prove the main theorem of the paper.

**Proof of Theorem 2.1.** It suffices to show that under the bootstrap assumption:

$$E_{\kappa+1}^d(t) \leq C_0 \epsilon(t)^\delta, \quad E_\kappa^d(t) + \frac{1}{2}\mu \| \nabla Z^k v(\cdot, \cdot) \|^2_{L_x^2 L_t^2} + E_{\kappa-1}^d(t) \leq C_0 \epsilon \ll 1,$$

(2.12)

for some positive $C_0 > 0$, $0 < \delta < \frac{1}{2}$ depending on $\kappa$ and $\epsilon_0$ uniformly for $0 \leq t < T$, we can derive a stronger estimate:

$$E_{\kappa+1}^d(t) \leq \frac{1}{2} C_0 \epsilon(t)^\delta, \quad E_\kappa^d(t) + \frac{1}{2}\mu \| \nabla Z^k v(\cdot, \cdot) \|^2_{L_x^2 L_t^2} + E_{\kappa-1}^d(t) \leq \frac{1}{2} C_0 \epsilon.$$

(2.13)

Then we can use continuity argument to extend life span of the solutions.

Firstly, under the assumption of (2.12), (2.9) becomes

$$\begin{align*}
E_\kappa^d(t) + \frac{1}{2}\mu \| \nabla Z^k v(\cdot, \cdot) \|^2_{L_x^2 L_t^2} \leq CE_\kappa^d(0) + C \int_0^t \langle \tau \rangle^{-2} E_{\kappa+1}^d(\tau) E_{\kappa-1}^d(\tau) \, d\tau \\
\leq CE_\kappa^d(0) + C \int_0^t \langle \tau \rangle^{-2} C_0^2 \epsilon^2 \, d\tau \\
\leq C\epsilon + CC_0^2 \epsilon^2.
\end{align*}$$

Let $C_0$ and $\epsilon$ such that

$$\max\{8, 8C\} \leq C_0, \quad 8CC_0 \epsilon \leq 1,$$

(2.14)

then $E_\kappa^d(t) + \frac{1}{2}\mu \| \nabla Z^k v(\cdot, \cdot) \|^2_{L_x^2 L_t^2} \leq \frac{1}{4} C_0 \epsilon.$

Secondly, Under the assumption of (2.12), (2.10) becomes

$$\frac{d}{dt} E_{\kappa+1}^d \leq C \langle t \rangle^{-1}(E_\kappa^d + E_{\kappa-1}^d) \frac{1}{2} + \| \nabla Z^k v \|^2_{L_x^2 L_t^2} + \langle t \rangle^{-1}\| \nabla Z^k v \|_{L^\infty} E_{\kappa+1}^d.$$

(6)
Then, Gronwall inequality gives us that
\[ E_{\kappa_1+1}^d(t) \leq E_{\kappa_1+1}^d(0) \exp \left( C \int_0^t (\tau)^{-1} (E_{\kappa_1}^d + E_{\kappa_1-1}^d)^{\frac{3}{2}} (\tau) + \|\nabla Z^{\kappa} v(\tau)\|_{L^2}^2 + (\tau)^{-1} \|\nabla Z^{\kappa} v(\tau)\|_{L^2} + \|\partial_\tau v\|_{L^\infty} d\tau \right) \]
\[ \leq \epsilon \exp \left( (C C_0 \epsilon)^{\frac{1}{2}} \ln(t) + C C_0 \epsilon + (C C_0 \epsilon)^{\frac{3}{2}} + C \epsilon^{\frac{3}{2}} \right) \]
\[ \leq \epsilon \exp ((C C_0 \epsilon)^{\frac{1}{2}}) \langle t \rangle^{CC_0 \epsilon^{\frac{9}{10}}} . \]
Taking \( \epsilon \) small enough such that
\[ \exp ((C C_0 \epsilon)^{\frac{1}{2}}) \leq \frac{1}{2} C_0, \quad CC_0 \epsilon^{\frac{9}{10}} \leq \delta \leq \frac{1}{2}, \tag{2.15} \]
we have \( E_{\kappa_1+1}^d(t) \leq \frac{1}{2} C_0 \epsilon \).

Finally, using Holder's inequality, we derive from (2.14) that
\[ \frac{d}{dt} E_{\kappa - 1}^d \leq C ((t)^{-1} \|\nabla Z^{\kappa} v\|_{L^2}^2 (E_{\kappa-1}^d)^{\frac{3}{2}} + \langle t \rangle^{-\frac{3}{2}} (E_{\kappa - 1}^d)^{\frac{1}{2}} + \|\nabla Z^{\kappa} v\|_{L^2}^2 (E_{\kappa - 1}^d)^{\frac{3}{2}} E_{\kappa - 1}^d + \|\nabla Z^{\kappa} v\|_{L^2}^2 (E_{\kappa - 1}^d)^{\frac{3}{2}} E_{\kappa - 1}^d) . \]
Then the Gronwall inequality gives us that
\[ E_{\kappa - 1}^d(t) \leq \left( E_{\kappa - 1}^d(0) + C \int_0^t (\tau)^{-1} \|\nabla Z^{\kappa} v(\tau)\|_{L^2}^2 (E_{\kappa - 1}^d(\tau))^{\frac{3}{2}} (E_{\kappa - 1}^d(\tau))^{\frac{1}{2}} + \|\nabla Z^{\kappa} v(\tau)\|_{L^2}^2 d\tau \right) \exp \left( (\epsilon + C C_0 \epsilon)^{\frac{1}{2}} \exp (CC_0 \epsilon + CC_0 \epsilon^{\frac{3}{2}}) \right) . \]
As \( c_0 > 8 \), we can choose \( \epsilon \) small enough such that
\[ (\epsilon + C C_0 \epsilon)^{\frac{1}{2}} \exp (CC_0 \epsilon + CC_0 \epsilon^{\frac{3}{2}}) \leq \frac{1}{4} C_0 \epsilon, \tag{2.16} \]
then we have \( E_{\kappa - 1}^d(t) \leq \frac{1}{4} C_0 \epsilon \).

Therefore, if we choose appropriate \( C_0 \) and small \( \epsilon \) such that (2.14), (2.15) and (2.16) holds, then a better estimates (2.13) can be obtained. Thus the theorem is proved. \( \square \)

### 2.4 Preliminary Weighted Estimates

In this section, we list a few weighted estimates, which will be frequently used throughout this paper.

First, we give two weighted \( L^\infty - L^2 \) estimates of the unknown near the light cone. They are essentially due to Klainerman and Sideris [15].

**Lemma 2.1.** Let \( u \in H^2(\mathbb{R}^n) \), then there hold
\[ \langle r \rangle^{1/2} |u(x)| \lesssim \sum_{|\alpha| \leq 1} \|\nabla \Omega^\alpha u\|_{L^2}, \tag{2.17} \]
\[ \langle r \rangle |u(x)| \lesssim \sum_{|\alpha| \leq 1} \|\partial_\tau \Omega^\alpha u\|_{L^2}^{1/2} \cdot \sum_{|\alpha| \leq 2} \|\Omega^\alpha u\|_{L^2}^{1/2}, \tag{2.18} \]
provided the right hand side is finite.
Proof. For (2.17), see Lemma 4.2 in [15]. For (2.18), see Lemma 3.3 in [35]. □

Next, we present some weighted $L^\infty - L^2$ estimate away from the light cone.

Lemma 2.2. Let $u \in H^2(\mathbb{R}^3)$, then there hold
\[
\langle t \rangle \| u(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} \lesssim \| u \|_{L^2} + \| \langle t-r \rangle \nabla u \|_{L^2} + \| \langle t-r \rangle \nabla^2 u \|_{L^2}, \tag{2.19}
\]
\[
\langle t \rangle \| u(t, \cdot) \|_{L^6(\mathbb{R}^3)} \lesssim \| u \|_{L^2} + \| \langle t-r \rangle \nabla u \|_{L^2}, \tag{2.20}
\]
\[
\langle t \rangle^{\frac{1}{2}} \| u(t, \cdot) \|_{L^{3\infty}(\mathbb{R}^3)} \lesssim \| u \|_{L^2(\mathbb{R}^3)} \| \langle r-t \rangle \nabla u \|_{L^2(\mathbb{R}^3)} + \| u \|_{L^2(\mathbb{R}^3)} \langle t \rangle^{\frac{1}{2}}, \tag{2.21}
\]
provided the right hand side is finite.

The first inequality (2.19) comes from (20) of Lemma 4.3.

Remark 2.1. This lemma depends on the spatial dimension is three or higher. In two dimensional case, the conclusion would be much weak.

Proof. These three inequalities follow from the following Sobolev imbedding respectively: $\| u \|_{L^\infty(\mathbb{R}^3)} \lesssim \| \nabla u \|_{L^2(\mathbb{R}^3)} \| \nabla^2 u \|_{L^2(\mathbb{R}^3)}$, $\| u \|_{L^6(\mathbb{R}^3)} \lesssim \| \nabla u \|_{L^2(\mathbb{R}^3)}$, and $\| u \|_{L^{3\infty}(\mathbb{R}^3)} \lesssim \| u \|_{L^2(\mathbb{R}^3)} \| \nabla u \|_{L^2(\mathbb{R}^3)}$, providing $u \in L^2(\mathbb{R}^3)$. Due to the similarity of these inequalities, we only present a detailed proof for the third one.

Choose a radial cut-off function $\phi \in C^\infty(\mathbb{R}^3)$ which satisfies
\[
\phi(x) = \begin{cases}
1, & \text{if } r \leq \frac{t}{2} \ , \\ 0, & \text{if } r \geq \frac{2}{3}
\end{cases}, \quad |\nabla \phi| \lesssim 1.
\]

For each fixed $t \geq 1$, let $\phi^t(x) = \phi(x/t)$. Clearly, one has
\[
\phi^t(x) \equiv 1 \quad \text{for } r \leq \frac{t}{2}, \quad \phi^t(x) \equiv 0 \quad \text{for } r \geq \frac{2t}{3}
\]
and
\[
|\nabla \phi^t(x)| \lesssim (t)^{-1}.
\]

Consequently,
\[
\| u \|_{L^3(\mathbb{R}^3)} \lesssim \| \phi^t u \|_{L^3(\mathbb{R}^3)}
\]
\[
\lesssim \| \phi^t u \|_{L^2(\mathbb{R}^3)} \| \langle r-t \rangle \|_{L^2(\mathbb{R}^3)} + \| u \|_{L^2(\mathbb{R}^3)} \langle t \rangle^{-\frac{1}{2}} \| \nabla u \|_{L^2(\mathbb{R}^3)} + \| u \|_{L^2(\mathbb{R}^3)} \langle t \rangle^{\frac{1}{2}},
\]
which yields (2.21). □

Now we state two lemmas of weighted estimates using the structure of wave type equations, which can be found in [15] and [18]. We only state them without giving the details of the proof.

Lemma 2.3. There holds
\[
X_d^d \lesssim E_d^d + \langle t + r \rangle \langle \partial_r^2 - \Delta \rangle d \| L^2, \n\]
provided the right hand side is finite.

Proof. See Lemma 2.3 and Lemma 3.1 in [15]. □

Near the light cone, the good unknown $((\partial_t + \partial_r)d$ has better decay. The following lemma comes from [18] where the two dimension case was proved. Indeed, it holds for all dimension $n \geq 2$.

Lemma 2.4. For $\frac{t}{2} \leq r$, there holds
\[
\langle t \rangle \langle \partial_t + \partial_r \rangle d \| \lesssim |\nabla d| + |\nabla Z d| + t \langle \partial_r^2 - \Delta \rangle d |.\]

Proof. See Lemma 3.4 in [18]. The proof given in [18] is for $\frac{t}{2} \leq r \leq \frac{5t}{2}$ with space dimension $n = 2$. However, one can easily check that the proof is valid for $r \geq \frac{t}{2}$ and $n \geq 2$. □
3 Estimates of the weighted $L^2$ norm

This section is devoted to the estimate of weighted $L^2$ norm $X_\kappa$. To this end, we need to estimate the $L^2$ norm of $f^2_a$ with some weights.

**Lemma 3.1.** For all multi-index $a$, there holds

$$
\| (t + r)f^2_a \|_{L^2}^2 \lesssim E^d_{[a]+1}(X_{[a]/2} + E^d_{[a]/2} + 1)(E^d_{[a]+2} + X_{[a]+2}) + E^d_{[a]}(1 + E^d_{[a]/2}/X_{[a]}) + (E^d_{[a]} + E^d_{[a]+2} + 1)(E^d_{[a]+3} + 1),
$$

provided the right hand side is finite.

**Proof.** Recalling the definition of $f^2_a$ in (2.7), we write

$$
\sum_{b+c+e=a} C^{b,c}(t + r)(\nabla Z^b \cdot \nabla Z^e d - \partial_t Z^b \cdot \partial_t Z^e d)Z^e d \|_{L^2}^2

+ \sum_{b+c=a} C^b|(t + r)(2Z^b v \cdot \nabla \partial_t Z^e d)|_{L^2}^2

+ \sum_{b+c=a} C^b|(t + r)(\partial_t Z^e v \cdot \nabla Z^e d)|_{L^2}^2

+ \sum_{b+c+e+f=a} C^{b,c,e,f}(t + r)(Z^b v \cdot \nabla Z^e d \cdot ((Z^e v \cdot \nabla Z^e ) Z^f d))Z^f d \|_{L^2}^2

+ \sum_{b+c+e+f=a} C^{b,c,e,f}(t + r)(2(Z^b v \cdot \nabla Z^e d \cdot \partial_t Z^e d)Z^f d \|_{L^2}^2

= H_1 + H_2 + H_3 + H_4 + H_5 + H_6.

In the sequel, we will focus our mind on $H_1$, $H_2$ and $H_3$, since these terms contain quadratic terms. The remain terms $H_4, H_5, H_6$ are all cubic or higher order ones whose estimates are similar and easier.

We first estimate $H_1$:

$$
H_1 \lesssim \|(t + r)\partial Z^{[a]} d\|_{L^2}^2 \|(t + r)\partial Z^{[a]/2} d\|_{L^2}^2 \| Z^{[a]/2} d \|_{L^2}^2

+ \sum_{|a/2| \leq |b| \leq |a|} \|(t + r)\partial Z^{[a/2]} d \|_{L^2}^2 \| Z^b d \|_{L^2}^2.
$$

(3.1)

If $r \geq (t/2)$, by (3.18), the right hand side of (3.1) can be controlled by

$$
\| \partial Z^{[a]} d \|_{L^2}^2 \| r \partial Z^{[a]/2} d \|_{L^r \geq (t/2)} \| Z^{[a]/2} d \|_{L^2}^2

+ \sum_{|a/2| \leq |b| \leq |a|} \| r \partial Z^{[a/2]} d \|_{L^r \geq (t/2)} \| Z^{[a/2]} d \|_{L^2}^2 \| Z^b d \|_{L^2}^2

\lesssim E^d_{[a]+1} E^d_{[a]/2} + 3(1 + E^d_{[a]/2} + 3).

Otherwise, if $r \leq (t/2)$, by (3.19) and (3.20), the right hand side of (3.1) can be controlled by

$$
\| \partial Z^{[a]} d \|_{L^2}^2 \| (t) \partial Z^{[a/2]} d \|_{L^r \leq (t/2)} \| Z^{[a/2]} d \|_{L^2}^2

\lesssim E^d_{[a]+1} E^d_{[a]/2} + 3(1 + E^d_{[a]/2} + 3).

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Thus we conclude that

\[ H_1 \lesssim E_{[a]+1}^d(X_{[a]/2}^d + E_{[a]/2}^d)(E_{[a]/2}^d + 1). \]

For \( H_2 \), we have:

\[ H_2 = 4 \sum_{b+c=a} C_b^c \| (t + r) Z^b v \cdot \nabla \partial_t Z^c d \|_{L_2}^2. \]  \hfill (3.2)

When \( r \geq \langle t \rangle / 2 \), by (2.18), we can estimate the right hand side of (3.2) as

\[ \| r Z^{[a]/2} v \|_{L_\infty(\mathbb{R}^{\langle t \rangle / 2})} \| \nabla \partial_t Z^{[a]} d \|_{L_2}^2 + \| Z^{[a]} v \|_{L_2} \| r \nabla \partial_t Z^{[a]/2} d \|_{L_\infty(\mathbb{R}^{\langle t \rangle / 2})} \lesssim E_{[a]/2}^d E_{[a]+2}^d + E_{[a]}^u E_{[a]/2}^d + 4. \]

When \( r \leq \langle t \rangle / 2 \), using (2.19), the right hand side of (3.2) can be controlled by

\[ \| Z^{[a]/2} v \|_{L_\infty(\mathbb{R}^{\langle t \rangle / 2})} \| (t - r) \nabla \partial_t Z^{[a]} d \|_{L_2}^2 + \| Z^{[a]} v \|_{L_2} \| (t) \nabla \partial_t Z^{[a]/2} d \|_{L_\infty(\mathbb{R}^{\langle t \rangle / 2})} \lesssim E_{[a]/2}^d X_{[a]+2}^d + E_{[a]}^u X_{[a]/2}^d + 4 E_{[a]}^u E_{[a]/2}^d + 2. \]

Hence we conclude that

\[ H_2 \lesssim E_{[a]/2}^d E_{[a]+2}^d + E_{[a]}^u E_{[a]/2}^d + 4 + E_{[a]}^u X_{[a]/2}^d + 4 + E_{[a]}^u X_{[a]+2}^d + 4 E_{[a]}^u E_{[a]+1}. \]

Then consider

\[ H_3 \lesssim \| (t + r) \partial_t Z^{[a]} v \|_{L_2} \| \nabla Z^{[a]/2} d \|_{L_2}^2 + \| (t + r) \partial_t Z^{[a]/2} v \|_{L_1} \| \nabla Z^{[a]} d \|_{L_2}^2. \]  \hfill (3.3)

When \( r \geq \langle t \rangle / 2 \), similar to the estimate of (3.2), the right hand side of (3.3) can be bounded by

\[ E_{[a]+1}^d E_{[a]/2}^d + E_{[a]}^u E_{[a]/2}^d + 3 E_{[a]+1}. \]

For the case of \( r \leq \langle t \rangle / 2 \), with the help of (2.19) and (2.20), we can estimate the right hand side of (3.3) as

\[ \| \partial_t Z^{[a]} v \|_{L_2} \| \nabla Z^{[a]/2} d \|_{L_2}^2 + \| \partial_t Z^{[a]} v \|_{L_1} \| \nabla Z^{[a]} d \|_{L_2}^2 \lesssim E_{[a]+1}^u (X_{[a]/2} + E_{[a]/2}^d) + E_{[a]/2}^d (X_{[a]+2} + E_{[a]+1}). \]

Therefore, we have

\[ H_3 \lesssim E_{[a]+1}^u (X_{[a]/2}^d + E_{[a]/2}^d) + E_{[a]/2}^d (X_{[a]+2} + E_{[a]+1}). \]

For \( H_4 \), \( H_5 \) and \( H_6 \), along the same line, we can obtain that

\[ H_4 \lesssim E_{[a]+1}^u (E_{[a]/2}^d + X_{[a]/2}^d + E_{[a]/2}^d)(E_{[a]+2}^d + X_{[a]+2}^d + E_{[a]+2}^d), \]

\[ H_5 \lesssim (E_{[a]}^u E_{[a]/2}^d + E_{[a]+2}^d)(E_{[a]/2}^d + X_{[a]/2}^d + E_{[a]/2}^d), \]

\[ H_6 \lesssim (E_{[a]}^u E_{[a]/2}^d + E_{[a]+2}^d)(E_{[a]/2}^d + X_{[a]/2}^d + E_{[a]+2}^d). \]

Combining the estimates of \( H_1 \), ..., \( H_6 \), gives the lemma.

Now we show that \( X_{[a]}^d \) can be controlled by the generalized energy under certain small energy assumption.
Lemma 3.2. Suppose $\kappa \geq 9$, $E^\kappa \lesssim \epsilon$, $E^d_{\kappa-1} \lesssim \epsilon$, then there hold that

$$X^d_{\kappa-1} \lesssim E^d_{\kappa-1}, \quad X^d_{\kappa+1} \lesssim E^d_{\kappa+1}.$$ 

Proof. Let $\kappa \geq 9$, $|a| + 2 \leq \kappa + 1$, then one has $|a/2| + 4 \leq \kappa - 1$. Before proving the lemma, we first show that under the assumption of $E^\kappa \lesssim \epsilon$, $E^d_{\kappa-1} \lesssim \epsilon$, there holds

$$X^d_{|a|+2} \lesssim E^d_{|a|+2} + E^d_{|a|+1}(X^d_{|a|/2} + E^d_{|a|/2} + 3)$$
$$+ E^d_{|a|+1}(E^d_{|a|/2} + X^d_{|a|/2}) + (E^d_{|a|+2} + X^d_{|a|+2})E^w_{|a|/2} + 3.$$  \hspace{1cm} (3.4)

Actually, by Lemma 2.3, one has

$$X^d_{|a|+2} \lesssim E^d_{|a|+2} + \sum_{|b| \leq |a|} \|t + r\|f^2_b \|L^2\|.$$ 

On the other hand, thanks to Lemma 3.1 and the assumption $E^\kappa \lesssim \epsilon$, $E^d_{\kappa-1} \lesssim \epsilon$, one easily check that

$$\|t + r\|f^2_a \|L^2\| \lesssim E^d_{|a|+1}(X^d_{|a|/2} + E^d_{|a|/2} + 3)$$
$$+ E^d_{|a|+1}(E^d_{|a|/2} + X^d_{|a|/2}) + (E^d_{|a|+2} + X^d_{|a|+2})E^w_{|a|/2} + 3.$$ 

Thus (3.4) can be deduced directly.

Now we turn the proof of the lemma. Let $|a| + 2 \leq \kappa - 1$, one can get by (3.4) that

$$X^d_{\kappa-1} \lesssim E^d_{\kappa-1} + E^d_{\kappa-1}(E^d_{\kappa-1} + E^w_{\kappa-1}) + (E^d_{\kappa-1} + E^w_{\kappa-1})X^d_{\kappa-1}.$$ 

Using the assumption of $E^\kappa \lesssim \epsilon$ and $E^d_{\kappa-1} \lesssim \epsilon$ yields

$$X^d_{\kappa-1} \lesssim E^d_{\kappa-1}.$$ 

Furthermore, for $|a| \leq \kappa - 1$, we get from (3.4) that

$$X^d_{\kappa+1} \lesssim E^d_{\kappa+1} + E^d_{\kappa+1}(E^d_{\kappa+1} + E^w_{\kappa+1}) + (E^d_{\kappa+1} + E^w_{\kappa+1})X^d_{\kappa+1},$$ 

from which together with the assumption implies

$$X^d_{\kappa+1} \lesssim E^d_{\kappa+1}.$$ 

Thus the lemma is proved. \hspace{1cm} \Box

An immediate consequence of the weighted $L^2$ norm estimate is that we can gain more decay for the good unknowns of the orientation field.

Lemma 3.3. Suppose $\kappa \geq 9$, $E^\kappa \lesssim \epsilon$, $E^d_{\kappa-1} \lesssim \epsilon$, then there hold

$$\|t\|\partial Z^a_{\partial t} \|L^2\| \lesssim (E^d_{\kappa})^{1/2}, \quad \forall |a| \leq \kappa - 3,$$
$$\|t\|\partial Z^a_{\partial t} \|L^2\| \lesssim (E^d_{\kappa+1})^{1/2}, \quad \forall |a| \leq \kappa - 1.$$ 

Proof. It is a direct consequence of Lemma 2.4 Lemma 3.1 and Lemma 3.2 \hspace{1cm} \Box

Another consequence is that we can obtain the decay for $L^\infty$ norm of $\partial Z^a_d$.

Lemma 3.4. Suppose $\kappa \geq 9$, $E^\kappa \lesssim \epsilon$, $E^d_{\kappa-1} \lesssim \epsilon$, then there hold

$$\langle t \rangle \|\partial Z^a d\| \lesssim (E^d_{\kappa-1})^{1/2}, \quad \forall |a| \leq \kappa - 4,$$
$$\langle t \rangle \|\partial Z^a d\| \lesssim (E^d_{\kappa+1})^{1/2}, \quad \forall |a| \leq \kappa - 2.$$ 

Proof. By (2.18) and (2.19), one has

$$\|\partial Z^a d\| \lesssim (t)^{-1}(\|r\partial Z^a d\|_{L^\infty(r \geq (t)/2)} + \|t\|\partial Z^a d\|_{L^\infty(r \leq (t)/2)})$$
$$\lesssim (t)^{-1}(E^d_{|a|+3}^{1/2} + X^d_{|a|+3}^{1/2}).$$ 

Using Lemma 3.2 we can obtain the result. \hspace{1cm} \Box
4 Decay estimates for the velocity

This section is devoted to the decay estimate under the a priori estimate assumption of the generalized energy. The estimate for the weighted $L^2$ norm in Section 3 is also applicable.

Lemma 4.1. Suppose $E_\kappa + \sum_{1 \leq |a| \leq \kappa} \| \nabla^a v \|_{L^2_{t,x}} + E_{\kappa-1}^d \lesssim \epsilon$ with $\kappa \geq 9$, then there hold

$$\| (t)^{\frac{3}{2}} v \|_{L^\infty_t} \lesssim \epsilon^\frac{1}{3},$$

$$\| (t)^{\frac{3}{2}} \nabla v \|_{L^\infty_t} \lesssim \epsilon^\frac{1}{3},$$

$$\| (t)^{\frac{5}{2}} \nabla^a v \|_{L^\infty_t} \lesssim \epsilon^\frac{1}{3} (\ln (t))^{\frac{1}{3}}, \quad \forall \ 2 \leq |a| \leq \kappa - 3. \quad (4.3)$$

Remark 4.1. The first two decay estimates (4.1) and (4.2) are sharp in the sense that the decay rate is the same as the linear heat equation if the initial data lies in energy space. The restricted decay rate for higher order derivatives (4.3) is due to the Ericksen stress.

Proof. Thanks to to the local well-posedness \[10\], one easily has an uniform bound on the life span of lower-bound of $\delta^{-1}$ where $\delta$ is the size of the initial perturbation around equilibrium. Thus in the following argument, we always assume $t \geq 1$. Correspondingly, the $L^p_t$ norm denotes $L^p([1,t])$ for simplicity, where the integral time interval is $[1,t]$.

We begin by writing down the expression for velocity:

$$v(t,x) = e^{t\Delta} v(1) - \int_1^t e^{(t-s)\Delta} \mathcal{P} [v \cdot \nabla v + \nabla \cdot (\nabla d \otimes \nabla d)](s) ds, \quad (4.4)$$

where $\mathcal{P}$ is the Leray projector. The inhomogeneous term can be rewritten as

$$\int_1^t e^{(t-s)\Delta} \mathcal{P} [v \cdot \nabla v + \nabla \cdot (\nabla d \otimes \nabla d)](s) ds$$

$$= \int_1^t \int_{\mathbb{R}^3} \nabla_y H(t-s,x-y)(v \otimes v + \nabla d \otimes \nabla d)(s,y) ds dy,$$

where $H(t,x)$ is a function of the three dimensional heat kernel $C t^{-\frac{3}{2}} \exp(-|x|^2/t)$ convoluting the Leray projection operator. Moreover, $H(t,x)$ behaves like (see for instance in [21], Proposition 11.1):

$$t^{-\frac{3}{2}} h(|x|/\sqrt{t}),$$

where

$$|h(y)| \lesssim 1/\langle y \rangle^3, \quad |\nabla_y h(y)| \lesssim 1/\langle y \rangle^4, \quad |\nabla^2_y h(y)| \lesssim 1/\langle y \rangle^5.$$

Now we are ready to show the lemma. We first prove (4.1). To this end, we write

$$t^{\frac{3}{2}} v = J_0 + J_1 + J_2 + J_3,$$

where

$$J_0 = t^{\frac{3}{2}} e^{t\Delta} v(1),$$

$$J_1 = t^{\frac{3}{2}} \int_1^t \int_{\mathbb{R}^3} \nabla_y H(t-s,x-y)(v \otimes v + \nabla d \otimes \nabla d)(s,y) ds dy,$$

$$J_2 = t^{\frac{3}{2}} \int_1^t \int_{\mathbb{R}^3} \nabla_y H(t-s,x-y)(v \otimes v)(s,y) ds dy,$$

$$J_3 = t^{\frac{3}{2}} \int_1^t \int_{\mathbb{R}^3} \nabla_y H(t-s,x-y)(\nabla d \otimes \nabla d)(s,y) ds dy.$$
For $J_0$, one gets by Young’s inequality that

$$J_0 \lesssim \|v(1, \cdot)\|_{L^2_x} \lesssim (E^u_\kappa)^{\frac{1}{2}}.$$  

For $J_1$, one has $t/2 \leq t - s \leq t$ from $1 \leq s \leq t/2$. Hence applying Young’s inequality and the Hardy-Littlewood-Sobolev inequality yields

$$J_1 \leq 2 \int_1^t (t - s)^{\frac{3}{2}} \|\nabla H(t - s, \cdot)\|_{L^2_x} \|(|v|^2 + |\nabla d|^2)(s, \cdot)\|_{L^2_x} ds$$

$$\lesssim \int_1^t (t - s)^{-\frac{1}{2}} \|(|v|^2 + |\nabla d|^2)(s, \cdot)\|_{L^2_x} ds$$

$$\lesssim \|\nabla|v|^2 + |\nabla d|^2\|_{L^2_x}.$$  

Then one can obtain the estimate for $J_1$ by using Lemma 3.4

$$J_1 \lesssim \|\nabla v\|_{L^2_{t,x}}(E^u_\kappa)^{\frac{1}{2}} + \|\delta(t)^{-1}\|_{L^2_t} E^d_\kappa - 1 \lesssim \|\nabla v\|_{L^2_{t,x}}^2 + E^u_\kappa + E^d_\kappa - 1.$$  

For the term $J_2$, one has $t/2 \leq s \leq t$. Hence, by Young’s inequality and Hardy-Littlewood inequality, we can derive that

$$J_2 \leq \int_0^t s^\frac{3}{2} \|\nabla H(t - s)\| \ast |(v \otimes v)(s)| \, ds$$

$$\lesssim \int_0^t \|\nabla H(t - s)\|_{L^2_x} \|v(s)\|_{L^\infty_x} \|s^\frac{3}{2} v(s)\|_{L^\infty_x} \, ds$$

$$\lesssim \int_0^t (t - s)^{-\frac{3}{2}} \|v(s)\|_{L^\infty_x} \|s^\frac{3}{2} v(s)\|_{L^\infty_x} \, ds$$

$$\lesssim \|v(s)\|_{L^2_t L^\infty_x} \|s^\frac{3}{2} v(s)\|_{L^\infty_x}$$

$$\lesssim \sum_{1 \leq |a| \leq 2} \|\nabla^a v\|_{L^2_t L^\infty_x} \|s^\frac{3}{2} v(s)\|_{L^\infty_x}.$$  

For $J_3$, similar to the estimate of $J_2$, one has

$$J_3 \lesssim \|s^\frac{3}{2} \nabla d(s)\|_{L^\infty_x L^\infty_t} \|\nabla d(s)\|_{L^2_t L^\infty_x} \lesssim E^d_\kappa - 1,$$

where we used Lemma 3.4 in the last estimate. Gathering the estimates for $J_0, J_1, J_2, J_3$, we conclude

$$t^\frac{3}{2} v(t) \lesssim (E^u_\kappa)^{\frac{1}{2}} + E^u_\kappa + E^d_\kappa - 1 + \|\nabla v\|_{L^2_{t,x}}^2 + \sum_{1 \leq |a| \leq 2} \|\nabla^a v\|_{L^2_t L^2_x} \|s^\frac{3}{2} v(s)\|_{L^\infty_x}.$$  

Absorbing the last term yields (4.1).

Secondly, we treat (4.2). Similar to the estimate of (4.1), we write

$$t^\frac{3}{2} \nabla v(t, x) = J_{01} + J_{11} + J_{12} + J_{13},$$

where

$$J_{01} = t^\frac{3}{2} \nabla v_{\theta \Delta v}(1),$$

$$J_{11} = t^\frac{3}{2} \int_1^t \int_{R^3} \nabla_y^2 H(t - s, x - y)(v \otimes v + \nabla d \otimes \nabla d)(s, y) \, ds dy,$$

$$J_{12} = t^\frac{3}{2} \int_1^t \int_{R^3} \nabla_y H(t - s, x - y) \nabla (v \otimes v)(s, y) \, ds dy.$$  

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\[ J_0^3 = t^\frac{5}{2} \int_{\mathbb{R}^3} \nabla_y H(t - s, x - y) \nabla(\nabla d \otimes \nabla d)(s, y) \, ds \, dy. \]

We remark that the formulation of \( J_1^1 \) is different from \( J_1 \), while the other terms are similar. For \( J_0^1 \), we get by Young’s inequality that
\[ J_0^1 \lesssim \|v(1, \cdot)\|_{L^2_{t,x}} \lesssim (E^v_\kappa)^{\frac{1}{2}}. \]

Applying Young’s inequality and the Hardy-Littlewood-Sobolev inequality, it yields that
\[ J_1^1 \leq 2 \int_1^t (t - s)^{\frac{5}{2}} \| \nabla^2 H(t - s, \cdot) \|_{L^2_t} \left( \|v\|^2 + \|\nabla d\|^2 \right)(s, \cdot) \|_{L^2_{t,x}} \, ds \lesssim \|v\|^2 + \|\nabla d\|^2 \|_{L^2_{t,x}}^2 + E^v_\kappa + E^d_\kappa, \]
where the relation \( t/2 \leq t - s \leq t \) has been used. For \( J_2^1 \), we derive that
\[ J_2^1 \lesssim \sum_{1 \leq |a| \leq 2} \| \nabla^a v \|_{L^2_{t,x}} \| \hat{s}^{\frac{5}{2}} v(s) \|_{L^\infty_{t,x}}. \]

The estimate for \( J_3^1 \) is similar to \( J_2 \) with slight modifications. By Young’s inequality, Hardy-Littlewood inequality and Lemma 3.4, one derives that
\[ J_3^1 \lesssim \int_1^t \| \nabla H(t - s) \|_{L^2_t} \| \hat{s}^{\frac{5}{2}} \nabla^2 d(s) \|_{L^\infty_t} \|s \nabla d(s)\|_{L^\infty_{t,x}} \, ds \lesssim \|s^{\frac{5}{2}} \|_{L^2_t} \|s \nabla^2 d(s)\|_{L^\infty_t} \|s \nabla d(s)\|_{L^\infty_{t,x}} \lesssim E^d_\kappa - 1. \]

Gathering the estimate for \( J_0^1, J_1^1, J_2^1, J_3^1 \), we conclude
\[ t^\frac{5}{2} v(t) \lesssim \left( E^v_\kappa \right)^{\frac{1}{2}} + E^v_\kappa + E^d_\kappa - 1 + \| \nabla v \|^2_{L^2_{t,x}} + \sum_{1 \leq |a| \leq 2} \| \nabla^a v \|_{L^2_{t,x}} \| \hat{s}^{\frac{5}{2}} v(s) \|_{L^\infty_{t,x}}. \]

Absorbing the last term yields (4.2).

Finally, we treat (4.3). For simplicity of presentation, we only show the case for \( |a| = 2 \). The higher-order case can be estimated in the same but lengthier argument.

Similar to the estimate of (4.2), we write:
\[ t^\frac{5}{2} \nabla^2 v(t, x) = J_0^2 + J_1^2 + J_2^2 + J_3^2, \]
where
\[ J_0^2 = t^\frac{5}{2} e^{t\Delta} \nabla^2 v(1), \]
\[ J_1^2 = t^\frac{5}{2} \int_1^t \int_{\mathbb{R}^3} \nabla_y H(t - s, x - y) \nabla(v \otimes v + \nabla d \otimes \nabla d)(s, y) \, ds \, dy, \]
\[ J_2^2 = t^\frac{5}{2} \int_1^t \int_{\mathbb{R}^3} \nabla_y H(t - s, x - y) \nabla^2(v \otimes v)(s, y) \, ds \, dy, \]
\[ J_3^2 = t^\frac{5}{2} \int_1^t \int_{\mathbb{R}^3} \nabla_y H(t - s, x - y) \nabla^2(\nabla d \otimes \nabla d)(s, y) \, ds \, dy. \]

As the estimates of \( J_0^2, J_1^2 \) and \( J_2^2 \) are similar those of \( J_0^1, J_1^1 \) and \( J_2^1 \) respectively, we only sketch them here. By Young’s inequality, the Hardy-Littlewood inequality and Lemma 3.4, one has
\[ J_0^2 \lesssim t^{-\frac{3}{2}} \|v(1, \cdot)\|_{L^2_{t,x}} \lesssim (E^v_\kappa)^{\frac{3}{2}}. \]
Lemma 4.2. Under the assumption of Lemma 4.1, there holds

\[ J_2^2 \lesssim \| \nabla (v \otimes v + \nabla d \otimes \nabla d) \|_{L_t^4 L_x^2} \lesssim \sum_{1 \leq |a| \leq \kappa} \| \nabla^a v \|_{L_t^2 L_x^2}^2 + E_{\kappa}^v + E_{\kappa-1}^d, \]

\[ J_2^2 \lesssim \| s^{\frac{5}{2}} \nabla^2 (v \otimes v) \|_{L_t^4 L_x^\infty} \]

\[ \lesssim \sum_{1 \leq |a| \leq 2} \| \nabla^a v \|_{L_t^2 L_x^2} \| s^{\frac{5}{2}} \nabla^2 v \|_{L_t^\infty L_x^\infty} + \| s^{\frac{5}{2}} \nabla v \|_{L_t^2 L_x^\infty} \| s^{-1} \|_{L_t^2}. \]

The term \( J_3^2 \) is estimated in a different way. By Young’s inequality, Hardy-Littlewood inequality, one derives that

\[ J_3^2 \lesssim \int_t^t \| \nabla_y H(t-s) \|_{L_{t,x}^\infty} \sum_{2 \leq |a| \leq 3} \| s^{\frac{5}{2}} \nabla^a d(s) \|_{L_t^p} \sum_{1 \leq |a| \leq 2} \| s \nabla^a d(s) \|_{L_t^\infty} \| s \|_{L_t^\infty} ds \]

\[ \lesssim \int_t^t (t-s)^{-\frac{3}{2} - \frac{1}{2p}} \sum_{2 \leq |a| \leq 3} \| s^{\frac{5}{2}} \nabla^a d(s) \|_{L_t^p} \sum_{1 \leq |a| \leq 2} \| s \nabla^a d(s) \|_{L_t^\infty} \| s \|_{L_t^\infty} ds \]

\[ \lesssim \sum_{2 \leq |a| \leq 3} \| s^{\frac{5}{2}} \nabla^a d(s) \|_{L_t^{\frac{3}{2}} L_x^\infty} \sum_{1 \leq |a| \leq 2} \| s \nabla^a d(s) \|_{L_t^\infty L_x^\infty}, \]

where \( p \) and \( p' \) are dual index, \( 3 < p \leq \infty \). Here, the constraint \( 3 < p \) is due to the application of the Hardy-Littlewood inequality. To earn the maximum decay, we take \( p = \infty \). Consequently,

\[ J_3^2 \lesssim \sum_{1 \leq |a| \leq 3} \| s \nabla^a d(s) \|_{L_t^\infty L_x^\infty} \| s^{-\frac{1}{2}} \|_{L_t^2} \lesssim (\ln(t))^{\frac{1}{2}} E_{\kappa-1}^d. \]

Consequently, one concludes that

\[ t^{\frac{5}{2}} \nabla^2 v(t) \lesssim (E_{\kappa}^v)^{\frac{1}{2}} + E_{\kappa}^v + (\ln(t))^{\frac{1}{2}} E_{\kappa-1}^d + \sum_{1 \leq |a| \leq \kappa} \| \nabla^a v \|_{L_t^2 L_x^2}^2 + \| s^{\frac{5}{2}} \nabla v(s) \|_{L_t^\infty L_x^\infty}^2 \]

\[ + \sum_{1 \leq |a| \leq 2} \| \nabla^a v \|_{L_t^2 L_x^2} \| s^{\frac{5}{2}} \nabla^2 v \|_{L_t^\infty L_x^\infty}. \]

Absorbing the last term, (4.3) can be inferred from (4.2).

An immediate consequence of the above lemma is the decay estimate for \( \partial_t v \).

**Lemma 4.2.** Under the assumption of Lemma 4.1, there holds

\[ \| \partial_t v \|_{L_t^\infty} \lesssim e^{\frac{1}{2} t} (t)^{-\frac{1}{2}} (\ln(t))^\frac{1}{2}. \]

**Proof.** We first show that (this decay rate is not optimal, however it is enough for our purpose)

\[ \| \nabla p \|_{L_t^\infty} \lesssim e^{t} \lesssim e^{-\frac{3}{2}}. \]

(4.5)

Due the incompressible condition for \( v \), we can use the Leray projector to write the pressure explicitly:

\[ \nabla p = \mathcal{P}(v \nabla v + \nabla \cdot (\nabla d \otimes \nabla d)). \]

Then by Sobolev imbedding, one deduces that

\[ \| \nabla p \|_{L_t^\infty} \lesssim \| \nabla p \|_{L_t^6} \| \nabla^2 p \|_{L_t^6}^{\frac{1}{2}} \]

\[ \lesssim \sum_{|a| \leq 1} \| \nabla^a (v \nabla v + \nabla \cdot (\nabla d \otimes \nabla d)) \|_{L_t^6} \]
Recalling the expression for \( f \) due to the symmetry between the operator and the viscosity terms. Fortunately, we can take the approach borrowed from [3, 11].

In this subsection, we estimate the generalized energy for the velocity, which turns out to be uniformly bounded in time. This section is devoted to the energy estimates, which corresponds to generalized energy estimate for the velocity, the higher-order and the lower-order generalized energy estimates for the orientation field.

### 5.1 Generalized energy estimate for velocity

In this subsection, we estimate the generalized energy for the velocity, which turns out to be uniformly bounded in time. The main trouble in the estimate of \( E^v_\kappa \) is due to commutators between the scaling operator and the viscosity terms. Fortunately, we can take the approach borrowed from [3, 11].

Let \( \kappa \geq 9 \), \( 0 \leq |a| \leq \kappa \). Taking the \( L^2 \) inner product of (2.6) with \( Z^a v \), we have

\[
\int_{\mathbb{R}^n} \frac{1}{2} \frac{d}{dt} |Z^a v|^2 \, dx - \int_{\mathbb{R}^n} \mu \Delta (S - 1)^{a_1} \Gamma^a \cdot v \cdot Z^a v \, dx = \int_{\mathbb{R}^n} f^1_a \cdot Z^a v \, dx. \tag{5.1}
\]

Recalling the expression for \( f^1_a \) in (2.7), one has

\[
\int_{\mathbb{R}^n} f^1_a \cdot Z^a v \, dx \lesssim \sum_{|b| \leq |a|} \|\nabla Z^a v\|_{L^2} \left( \|\nabla Z^b d\|_{L^2} \|\nabla Z^c d\|_{L^2} + \|Z^b v\|_{L^2} \right).
\]

Due to the symmetry between \( b \) and \( c \), we assume \( |b| \leq |c| \) without loss of generality. Thus one has \( |c| \leq |a|/2 \leq \kappa - 4 \). Consequently, by Lemma 3.3, the above can be further bounded by

\[
\|\nabla Z^a v\|_{L^2} \left( \|\nabla Z^a d\|_{L^2} \|\nabla Z^{a/2} d\|_{L^\infty} + \|Z^a v\|_{L^2} \right) \lesssim \|\nabla Z^a v\|_{L^2} (t)^{-1} (E^d_{\kappa+1} E^{d\kappa}_{\kappa-1})^{\frac{1}{4}} + \|\nabla Z^a v\|_{L^2} \|Z^a v\|_{L^2}.
\]

Next, we estimate the diffusion terms with indefinite sign. To this end, we need a technical lemma followed from [3].

**Lemma 5.1.** [3] (Iteration lemma) Let \( \{f_l\}, \{g_l\}, \{F_l\} \) be three nonnegative sequences, where \( 0 \leq l \leq \kappa \). Suppose that

\[
f_0 + g_0 \lesssim F_0,
\]

and for all \( 1 \leq l \leq \kappa \),

\[
f_l + g_l - g_{l-1} \lesssim F_l.
\]

Then there holds

\[
\sum_{0 \leq m \leq l} (f_m + g_m) \lesssim \sum_{0 \leq m \leq l} F_m,
\]

for all \( 0 \leq l \leq \kappa \).
Now we are ready to estimate the diffusion terms as follows:

\[- \mu \int_{\mathbb{R}^n} \Delta (S - 1)^a \Gamma^a v \cdot S^a \Gamma^a v \, dx \]

\[= - \mu \sum_{l \leq a_1} C^l_{a_1} (1 - 1)^{a_1 - 1} \int_{\mathbb{R}^n} \Delta S^l \Gamma^a v \cdot S^a \Gamma^a v \, dx \]

\[\geq - \frac{1}{2} \mu \|\nabla S^l \Gamma^a v\|_{L^2}^2 - \mu C \sum_{l \leq a_1 - 1} \|\nabla S^l \Gamma^a v\|_{L^2}^2.\]

Inserting the above into (5.1), together with the estimate of the nonlinearities yield

\[\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |Z^a v|^2 \, dx + \frac{1}{2} \mu \|\nabla Z^a v\|_{L^2}^2 - \mu C \sum_{l \leq a_1 - 1} \|\nabla S^l \Gamma^a v\|_{L^2}^2 \]

\[\leq C\|\nabla Z^\kappa v\|_{L^2}^2 \|Z^\kappa v\|_{L^2} + C(t)^{-2} E^d_{\kappa+1} E^d_{\kappa+1}.\]

Then integrating in time over [0, t] on both sides of the above inequality gives

\[\|Z^a v(t, \cdot)\|_{L^2}^2 + \mu \|\nabla Z^a v\|_{L^2}^2 - \mu C \sum_{l \leq a_1 - 1} \|\nabla S^l \Gamma^a v\|_{L^2}^2 \]

\[\leq \|Z^a v(0, \cdot)\|_{L^2}^2 + \int_0^t \langle \tau \rangle^{-2} E^d_{\kappa+1}(\tau) E^d_{\kappa+1}(\tau) \, d\tau + \|\nabla Z^\kappa v\|_{L^2}^2 \|Z^\kappa v\|_{L^2}.\]

Now we can use Lemma 5.1 to absorb the lower order diffusion terms to derive that

\[\|Z^a v(t, \cdot)\|_{L^2}^2 + \mu \|\nabla Z^a v\|_{L^2}^2 \]

\[\leq \|Z^a v(0, \cdot)\|_{L^2}^2 + \int_0^t \langle \tau \rangle^{-2} E^d_{\kappa+1}(\tau) E^d_{\kappa+1}(\tau) \, d\tau + \|\nabla Z^\kappa v(\cdot, \cdot)\|_{L^2}^2 \|Z^\kappa v(\cdot, \cdot)\|_{L^2}^2.\]

This gives the a priori estimate (24).

5.2 Higher-order energy estimate for the orientation field

In this subsection, we estimate the higher-order energy for the orientation field \(d\), which will exhibit some polynomial growth in time. The main difficulty is the potential derivative loss due to the quasilinear effect. Fortunately, this difficulty can be overcome by a symmetry structure of the system.

See the estimate of I_{12} below.

Let \(\kappa \geq 9, 0 \leq |a| \leq \kappa\). Taking the \(L^2\) inner product of (24) with \(d^a Z^a d\) gives

\[\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\partial Z^a d|^2 \, dx = \int_{\mathbb{R}^n} f^a_{\partial} \cdot \partial Z^a d \, dx.\]

Thanks to (27), the right hand side above can be divided into \(I_1\) and \(I_2\), where

\[I_1 = - \int_{\mathbb{R}^n} \left[ v \cdot \nabla (v \cdot \nabla Z^a d) + 2 v \cdot \nabla \partial_t Z^a d \right] \cdot \partial_t Z^a d \, dx,\]

\[- \int_{\mathbb{R}^n} (\partial_t Z^a v \cdot \nabla) d \cdot \partial_t Z^a d \, dx\]

\[\triangleq I_{11} + I_{12}\]

contains the highest order terms which may lose one derivative at first glance, and \(I_2\) refers to the lower order ones:

\[I_2 = - \sum_{b + c = a \atop c \neq a} C^b_a \int_{\mathbb{R}^n} 2 Z^b v \cdot \nabla \partial_t Z^c d \cdot \partial_t Z^a d \, dx\]
If \( |c| \leq |b| \), by Lemma 3.2, the above can be bounded by
\[
\sum_{b+c=e, \ a \neq b} \| Z^b v \|_{L^2} \| \nabla \partial_t Z^c d \|_{L^\infty} \| \partial_t Z^a d \|_{L^2} \lesssim \langle t \rangle^{-1} (E^d_{\kappa} E^d_{\kappa+1} E^d_{\kappa-1})^{\frac{1}{2}}.
\]
Otherwise, if \(|b| \leq |c|\), by (2.18) and Lemma 3.2, one has
\[
\sum_{b+c=e, \ c \neq a} \int_{|t| \leq (|b|/2) - 1} (|Z^b v| |\nabla \partial_t Z^c d| |\partial_t Z^a d|) \text{ dx} \lesssim \langle t \rangle^{-1} \| r Z^{|a|/2} v \|_{L^\infty} (\langle t \rangle / 2) \| \nabla \partial_t Z^{|a|-1} d \|_{L^2} \| \partial_t Z^a d \|_{L^2} \lesssim \langle t \rangle^{-1} (E^d_{\kappa} E^d_{\kappa+1})^{\frac{1}{2}}.
\]

Estimate of \( I_{22} \):
\[
I_{22} = - \sum_{b+c=e, \ b \neq a} \int_{\mathbb{R}^3} \partial_t Z^b v \cdot \nabla Z^c d \cdot \partial_t Z^a d \text{ dx}.
\]

First for the case of \( c = a \), the above quantity becomes
\[
- \int_{\mathbb{R}^3} (\partial_t v \cdot \nabla) Z^a d \cdot \partial_t Z^a d \text{ dx} \lesssim \| \partial_t v \|_{L^\infty} \| \partial Z^a d \|_{L^2}^2.
\]
Next, if \(|b| \leq |c| \leq |a| - 1\), by (2.20), Lemma 3.2 and (2.18), one has
\[
- \sum_{b+c=e, \ |b| \leq |c| \leq |a| - 1} \int_{|t| \leq (|b|/2) - 1} \partial_t Z^b v \cdot \nabla Z^c d \cdot \partial_t Z^a d \text{ dx}.
\]
\[
\lesssim (t)^{-1} \sum_{b+c+d=a} \|\partial_t Z^b v\|_{L^\infty} \| (t-r) \nabla Z^c d\|_{L^\infty(r \leq (t)/2)} \| \partial_t Z^a d\|_{L^2} \\
+ (t)^{-1} \sum_{b+c+d=a} \| r \partial_t Z^b v\|_{L^\infty(r \geq (t)/2)} \| \nabla Z^c d\|_{L^2} \| \partial_t Z^a d\|_{L^2} \\
\lesssim (t)^{-1} (E_k^{d})^{\frac{1}{2}} E_{k+1}^{d}.
\]

Otherwise, if \(|c| \leq |b|\), by Lemma 3.4 we have the control of

\[
\| \partial_t Z^a d\|_{L^2} \| \nabla Z^b]\|_{L^\infty} \lesssim (t)^{-1} (E_k^{d})^{\frac{1}{2}} E_{k+1}^{d}.
\]

**Estimate of** \(I_{23}\):

\[
I_{23} = \int_{\mathbb{R}^3} \sum_{b+c+d=a} C_b^{c,d} (\nabla Z^b v \cdot \nabla Z^c Z^d - \partial_t Z^b \cdot \partial_t Z^c Z^d)\| \partial_t Z^a d\|_{L^2} dx
\]

By Lemma 3.4 the above can be bounded by

\[
\sum_{b+c+d=a} \int_{\mathbb{R}^3} |\partial Z^b| \| \partial Z^c| \| |Z^d| \| \partial_t Z^a| dx.
\]

If \(|f| \geq |a/2|\), by Sobolev inequalities and Lemma 3.4 the above can be controlled by

\[
\lesssim \int_{\mathbb{R}^3} \sum_{b+c+d+f=a} \|Z^b v\|_{L^\infty} \|\partial Z^c| \| |Z^d| \| \partial_t Z^a| dx.
\]

Otherwise, if \(|f| \leq |a/2|\), by Lemma 3.4 \(I_{34}\) can be controlled by

\[
\lesssim (t)^{-1} E_{k+1}^{d} (E_k^{d} E_{k-1}^{d})^{\frac{1}{2}} [1 + (E_{k-1}^{d})^{\frac{1}{2}}].
\]
Estimate of $I_{25}$: By Lemma 3.3, we get

$$I_{25} = - \sum_{b+c+f+g=a} C_{a}^{b,c,e,f} \int_{\mathbb{R}^{3}} \left[ (Z^{b} \cdot \nabla) Z^{c} \cdot ((Z^{d} \cdot \nabla) Z^{e} d) \right] Z^{f} \cdot \partial_{t} Z^{g} d \, dx \leq \sum_{|a|/2 \leq |g| \leq |a|} \|Z^{[a/2]} v\|_{L^{2}} \|Z^{[a/2]} d\|_{L^{\infty}} \|Z^{a} d\|_{L^{2}} \|\partial_{t} Z^{a} d\|_{L^{2}}$$

Estimate of $I_{26}$: By Hölder inequality, one has

$$I_{26} = - \sum_{b+c+f+e=a} C_{a}^{b,c,e} \int_{\mathbb{R}^{3}} Z^{b} \cdot \nabla (Z^{c} \cdot \nabla Z^{e} d) \cdot \partial_{t} Z^{a} d \, dx \leq \|\nabla Z^{a} v\|_{L^{2}}^{2} E_{k+1}^{d}.$$

5.2.2 Estimate of $I_{1}$

We estimate $I_{1}$ in this part.

**Estimate of $I_{11}$**: Employing integration by parts, one has

$$I_{11} = - \int_{\mathbb{R}^{n}} [v \cdot \nabla (v \cdot \nabla Z^{a} d)] \cdot \partial_{t} Z^{a} d \, dx$$

$$= - \int_{\mathbb{R}^{n}} (\partial_{t} v \cdot \nabla) Z^{a} d \cdot (v \cdot \nabla) Z^{a} d \, dx + \frac{1}{2} \partial_{t} \int_{\mathbb{R}^{n}} |(v \cdot \nabla) Z^{a} d|^{2} \, dx$$

$$\leq \|\nabla Z^{k-1} v\|_{L^{2}}^{2} E_{k+1}^{d} + \frac{1}{2} \partial_{t} \int_{\mathbb{R}^{n}} |(v \cdot \nabla) Z^{a} d|^{2} \, dx.$$

**Estimate of $I_{12}$**: The difficulty in estimating $I_{12}$ lies on the possible derivative loss problem. At first glance, one loses the symmetry of the system. However, this difficulty can be bypassed by using the symmetry structure of the system.

Employing integration by parts, we have

$$I_{12} = - \int_{\mathbb{R}^{n}} (\partial_{t} Z^{a} v \cdot \nabla) d \cdot \partial_{t} Z^{a} d \, dx$$

$$= - \partial_{t} \int_{\mathbb{R}^{n}} (Z^{a} v \cdot \nabla) d \cdot \partial_{t} Z^{a} d \, dx + \int_{\mathbb{R}^{n}} (Z^{a} v \cdot \nabla) \partial_{t} d \cdot \partial_{t} Z^{a} d \, dx$$

$$+ \int_{\mathbb{R}^{n}} (Z^{a} v \cdot \nabla) d \cdot \partial_{t} Z^{a} d \, dx. \quad (5.2)$$

By Lemma 3.3, the second term on the right hand side of (5.2) is controlled by

$$\|Z^{a} v\|_{L^{2}} \|\partial_{t} Z^{a} d\|_{L^{2}} \|\nabla \partial_{t} d\|_{L^{\infty}} \leq (t)^{-1} (E_{k+1}^{d} E_{k-1}^{d})^{2}.$$

For the last term of $(5.2)$, we are going to insert the equations for orientation field (2.1) into this expression to show the symmetry.

$$\int_{\mathbb{R}^{n}} (Z^{a} v \cdot \nabla) d \cdot \partial_{t} Z^{a} d \, dx$$

$$= \int_{\mathbb{R}^{n}} (Z^{a} v \cdot \nabla) d \cdot \Delta Z^{a} d \, dx + \int_{\mathbb{R}^{n}} (Z^{a} v \cdot \nabla) d \cdot f_{a}^{2} \, dx. \quad (5.3)$$
Thanks to Lemma 4.5, we get by integration by parts that
\[
\int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot \Delta Z^a d \ dx \leq \|\nabla Z^a d\|_{L^2}^2 (\|\nabla Z^a v\|_{L^2} \|\nabla d\|_{L^\infty} + \|Z^a v\|_{L^2} \|\nabla^2 d\|_{L^\infty})
\]
\[
\lesssim (t)^{-1} (E_{\kappa+1}^d E_{\kappa}^d)^{\frac{1}{2}} (\|\nabla Z^a v\|_{L^2} + (E_{\kappa}^a)^{\frac{1}{2}}). 
\]

On the other hand, it holds that

**Lemma 5.2.**
\[
\int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot f_a^2 \ dx \leq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \|d\|^2 \ dx + (t)^{-1} (E_n^v d + E_{\kappa+1}^d)^{\frac{1}{2}} E_{\kappa+1}^d
\]
\[
+ \|\nabla Z^a v\|_{L^2}^2 E_{\kappa+1}^d + (t)^{-1} \|\nabla Z^a v\|_{L^2} E_{\kappa+1}^d + \|\partial_t v\|_{L^\infty} E_{\kappa+1}^d.
\]

**Proof.** Inserting the expression of \(f_a^2\) in (2.11) into \(\int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot f_a^2 \ dx,\) we get
\[
\int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot f_a^2 \ dx
\]
\[
= - \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot \left[(v \cdot \nabla)(v \cdot \nabla Z^a d + 2\partial_t Z^a d) + \partial_t Z^a v \cdot \nabla d\right] \ dx + L,
\]
where
\[
L = \sum_{b+c+c=a} C^{b,c} \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot (\nabla Z^b d \cdot \nabla Z^c d - \partial_t Z^b d \cdot \partial_t Z^c d) Z^d d \ dx
\]
\[
- \sum_{b+c+c+f=a} C^{b,c,e} \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot [(Z^b v \cdot \nabla) Z^c d \cdot \partial_t Z^d] Z^f d \ dx
\]
\[
- \sum_{b+c+c+f+g=a} C^{b,c,e} \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot [(Z^b v \cdot \nabla) Z^c d \cdot (Z^e v \cdot \nabla) Z^d] Z^f d \ dx
\]
\[
- \sum_{b+c+c+e=a} C^{b,c} \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot Z^b v \cdot \nabla (Z^c v \cdot \nabla Z^e) d \ dx
\]
\[
- \sum_{b+c+c+e=a} C^{b} \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot (2Z^b v \cdot \nabla \partial_t Z^c d + \partial_t Z^c v \cdot \nabla Z^b d) \ dx
\]
\[
\triangleq K_1 + K_2 + K_3 + K_4 + K_5.
\]

The first term on the right hand side of (5.4) refers to the highest order term, while the remaining terms denoted by \(L\) refers to the lower order ones. The estimate of these lower order term \(L,\) on one hand, is similar to the estimate of \(I_2,\) on the other hand, is much easier than \(I_2.\) That’s because \(I_2\) contain quadratic terms, while \(L\) contains only cubic terms or higher.

Now let us estimate the right hand side of (5.4) one by one. The first part of the higher order terms can be bounded by
\[
- \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot \left[(v \cdot \nabla)(v \cdot \nabla Z^a d + 2\partial_t Z^a d)\right] \ dx
\]
\[
\lesssim \|\nabla Z^a d\|_{L^2}^2 (\|\nabla Z^a v\|_{L^2} \|\nabla d\|_{L^\infty} + \|Z^a v\|_{L^2} \|\nabla^2 d\|_{L^\infty})
\]
\[
+ \|\partial_t Z^a d\|_{L^2} \|\nabla Z^a v\|_{L^2} \|\nabla d\|_{L^\infty} + \|\partial_t Z^a d\|_{L^\infty} \|\nabla Z^a v\|_{L^2} \|\nabla^2 d\|_{L^6}
\]
\[
\lesssim E_{\kappa+1}^d ((E_{\kappa}^a)^{\frac{1}{2}} + 1) \|\nabla Z^a v\|_{L^2}. 
\]
For the second part of the higher order terms, now one can see the symmetry is present. Thus, employing integration by parts, one has

\[- \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d\cdot (\partial_t Z^a v \cdot \nabla d) \, dx \]

\[= -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |(Z^a v \cdot \nabla) d|^2 \, dx + \int_{\mathbb{R}^n} (Z^a v \cdot \partial_t \nabla) d \cdot (Z^a v \cdot \nabla d) \, dx \]

\[\leq -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |(Z^a v \cdot \nabla) d|^2 \, dx + \|\nabla Z^a v\|^2_{L^2} E_{\kappa+1}^d. \]

Now we show the estimate for $L$ in (3.14). The estimate is similar to the estimate of $I_1$ to $I_6$ in the higher-order energy estimate for liquid crystal molecule. Hence we only sketch the argument. Due to Hölder inequality, (2.19), (2.19), Lemma 3.2 and Lemma 3.4, one deduces that

\[K_1 \lesssim \|Z^a v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]} d\|_{L^2} \|\partial Z^{[a]/2} d\|_{L^3} \|Z^{[a]/2} d\|_{L^6} \]

\[+ \|\nabla Z^{[a]} v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]/2} d\|_{L^2} \|\partial Z^{[a]/2} d\|_{L^3} \|Z^{[a]} d\|_{L^6} \]

\[\lesssim \langle t \rangle^{-1} \|\nabla Z^a v\|^2_{L^2} (E_{\kappa+1}^d)^\frac{1}{2} E_{\kappa-1}^d (1 + (E_{\kappa-1}^d)^\frac{1}{2}), \]

and

\[K_2 \lesssim \|Z^a v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]} d\|_{L^2} \|\partial Z^{[a]/2} d\|_{L^3} \|Z^{[a]/2} d\|_{L^6} \]

\[+ \|\nabla Z^{[a]/2} v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]} d\|_{L^2} \|\partial Z^{[a]/2} d\|_{L^3} \|Z^{[a]} d\|_{L^6} \]

\[\lesssim \langle t \rangle^{-1} \|\nabla Z^a v\|^2_{L^2} (E_{\kappa+1}^d)^\frac{1}{2} E_{\kappa-1}^d (1 + (E_{\kappa-1}^d)^\frac{1}{2}), \]

and

\[K_3 \lesssim \|Z^a v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]} d\|_{L^2} \|\partial Z^{[a]/2} d\|_{L^3} \|Z^{[a]/2} d\|_{L^6} \]

\[+ \|\nabla Z^{[a]} v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]/2} d\|_{L^2} \|\partial Z^{[a]/2} d\|_{L^3} \|Z^{[a]} d\|_{L^6} \]

\[\lesssim \langle t \rangle^{-1} \|\nabla Z^a v\|^2_{L^2} (E_{\kappa+1}^d E_n^\nu)^\frac{1}{2} E_{\kappa-1}^d (1 + (E_{\kappa-1}^d)^\frac{1}{2}), \]

and

\[K_4 \lesssim \|Z^a v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]} d\|_{L^2} \|\partial Z^{[a]/2} v\|_{L^6} \|\nabla Z^{[a]/2} d\|_{L^2} \]

\[+ \|Z^a v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]/2} v\|_{L^3} \|\nabla Z^{[a]} d\|_{L^2} \|\nabla Z^{[a]} d\|_{L^3} \]

\[\lesssim \langle t \rangle^{-1} \|\nabla Z^a v\|^2_{L^2} (E_n^\nu E_{\kappa-1}^d E_{\kappa+1}^d)^\frac{1}{2}, \]

and

\[K_5 \lesssim \|Z^a v\|_{L^6} \|\nabla d\|_{L^\infty} \|\partial Z^{[a]} d\|_{L^2} \|\partial Z^{[a]} d\|_{L^6} \|\nabla Z^{[a]} d\|_{L^2} \lesssim \|\nabla Z^a v\|^2_{L^2} E_{\kappa+1}^d. \]

Combining all the estimate in this lemma and note $E_n^\nu \lesssim \epsilon$ and $E_{\kappa-1}^d \lesssim \epsilon$, we get the lemma. \(\square\)

### 5.2.3 Completing the estimates

Combining all the above estimates in this subsection, one has

\[\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\partial Z^a d|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |(v \cdot \nabla) Z^a d|^2 \, dx \]

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\[ + \frac{d}{dt} \int_{\mathbb{R}^n} (Z^a \cdot \nabla) d \cdot \partial_t Z^a d \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |(Z^a \cdot \nabla)d|^2 \, dx \]
\[ \lesssim \langle t \rangle^{-1} (E_{\kappa}^v + E_{\kappa-1}^d)^{\frac{1}{2}} E_{\kappa+1}^d + \| \nabla Z^a v \|_{L^2} E_{\kappa+1}^d \]
\[ + \langle t \rangle^{-1} \| \nabla Z^a v \|_{L^2} E_{\kappa+1}^d + \| \partial_t v \|_{L^\infty} E_{\kappa+1}^d. \]

Summing over \(|a| \leq \kappa\), and noting that
\[ \sum_{|a| \leq \kappa} \int_{\mathbb{R}^n} |(Z^a \cdot \nabla)d|^2 \, dx \lesssim E_{\kappa}^v E_{\kappa-1}^d \lesssim \varepsilon E_{\kappa-1}^d, \]
we can deduce
\[ \sum_{|a| \leq \kappa} \left( \frac{1}{2} \int_{\mathbb{R}^n} |\partial Z^a d|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^n} |(v \cdot \nabla)Z^a d|^2 \, dx \right) \]
\[ + \int_{\mathbb{R}^n} (Z^a v \cdot \nabla) d \cdot \partial_t Z^a d \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |(Z^a v \cdot \nabla)d|^2 \, dx \]
\[ \sim \frac{1}{2} \sum_{|a| \leq \kappa} \int_{\mathbb{R}^n} |\partial Z^a d|^2 \, dx = \frac{1}{2} E_{\kappa+1}^d. \]

Here we have used the assumption \(E_{\kappa}^v \lesssim \varepsilon\) and \(E_{\kappa-1}^d \lesssim \varepsilon\) again. This gives (2.11).

### 5.3 Lower-order energy estimate for the orientation field

This subsection is devoted to the lower-order energy estimate for the orientation field \(d\). It turns out that the lower-order energy is uniformly bounded. To this end, we need to obtain subcritical decay for the nonlinearities or say, \(L^1\) integrability in time.

Let \(\kappa \geq 9\), \(0 \leq |a| \leq \kappa - 2\). Taking the \(L^2\) inner product of (2.6) with \(\partial_t Z^a d\), we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\partial Z^a d|^2 \, dx = \int_{\mathbb{R}^n} f_a^2 \cdot \partial_t Z^a d \, dx. \]

Recalling the expression of \(f_a^2\) in (2.7), we will rewrite the right hand side of the above equality as
\[ - \sum_{b+c=c=a} C_a^b \int_{\mathbb{R}^3} (2Z^b v \cdot \nabla \partial_t Z^c d + \partial_t Z^b v \cdot \nabla Z^c d) \cdot \partial_t Z^a d \, dx \]
\[ + \int_{\mathbb{R}^3} \sum_{b+c+c=a} C_a^{b,c} (\nabla Z^b d \cdot \nabla Z^c d - \partial_t Z^b d \cdot \partial_t Z^c d) Z^c d \cdot \partial_t Z^a d \, dx \]
\[ - \sum_{b+c+c+a} C_a^{b,c} \int_{\mathbb{R}^3} Z^b v \cdot \nabla (Z^c v \cdot \nabla Z^c d) \cdot \partial_t Z^a d \, dx \]
\[ - \sum_{b+c+c+f=a} C_a^{b,c,f} \int_{\mathbb{R}^3} \left[ 2(Z^b v \cdot \nabla) Z^c d \cdot \partial_t Z^f d \right] Z^f d \cdot \partial_t Z^a d \, dx \]
\[ - \sum_{b+c+c+f+g=a} C_a^{b,c,f} \int_{\mathbb{R}^3} \left[ (Z^b v \cdot \nabla) Z^c d \cdot ((Z^f v \cdot \nabla) Z^f d) \right] Z^g d \cdot \partial_t Z^a d \, dx \]
\[ = M_1 + M_2 + M_3 + M_4 + M_5. \]

In the sequel, we will estimate \(M_1\) to \(M_5\) one by one.

We first estimate \(M_1\). There are two terms inside the expression for \(M_1\). In order to estimate them in a uniform way, we write by integration by parts that
\[ \sum_{b+c=a} C_a^b \int_{\mathbb{R}^3} (\partial_t Z^b v \cdot \nabla Z^c d) \cdot \partial_t Z^a d \, dx = \sum_{b+c=a} C_a^b \partial_t \int_{\mathbb{R}^3} (Z^b v \cdot \nabla Z^c d) \cdot \partial_t Z^a d \, dx \]
\[ - \sum_{b+c+c+a} C_a^{b,c} \int_{\mathbb{R}^3} Z^b v \cdot \nabla (Z^c v \cdot \nabla Z^c d) \cdot \partial_t Z^a d \, dx \]
\[ - \sum_{b+c+c+f=a} C_a^{b,c,f} \int_{\mathbb{R}^3} \left[ 2(Z^b v \cdot \nabla) Z^c d \cdot \partial_t Z^f d \right] Z^f d \cdot \partial_t Z^a d \, dx \]
\[ - \sum_{b+c+c+f+g=a} C_a^{b,c,f} \int_{\mathbb{R}^3} \left[ (Z^b v \cdot \nabla) Z^c d \cdot ((Z^f v \cdot \nabla) Z^f d) \right] Z^g d \cdot \partial_t Z^a d \, dx \]
\[ = M_1 + M_2 + M_3 + M_4 + M_5. \]
Thus $M_1$ can be bounded by

\[- \sum_{b+c=a} C_a^b \int_{\mathbb{R}^3} (Z^b v \cdot \nabla \partial_t Z^a d) \cdot \partial_t Z^a d \, dx - \sum_{b+c=a} C_a^b \int_{\mathbb{R}^3} (Z^b v \cdot \nabla Z^a d) \cdot \partial_t Z^a d \, dx .\]

Now we estimate the last term in the above expression. For the integral domain of $\{ r \leq (t)/2 \}$, one deduces from Lemma 3.2 and the Sobolev inequality $\| u \|_{L^\infty} \lesssim \| \nabla u \|_{L^2}^\frac{3}{2} \| \nabla^2 u \|_{L^2}^\frac{1}{2}$ that

\[
\int_{r \leq (t)/2} |Z^{[a]} v| \| \partial Z^{[a]} d\| \| \partial^2 Z^{[a]} d\| \, dx \\
\lesssim \langle t \rangle^{-1} \| Z^{[a]} v \|_{L^\infty} \| \partial Z^{[a]} d\|_{L^2} \langle t - r \rangle \| \partial^2 Z^{[a]} d\|_{L^2(r \leq (t)/2)} \\
\lesssim \langle t \rangle^{-1} \| \nabla Z^{[a]} v \|_{L^2}^\frac{3}{2} (E_k^{d_{\alpha - 1}})^{\frac{1}{2}} (E_{k+1}^{d_{\alpha + 1}})^{\frac{1}{2}} .
\]

For the integral region of $\{ r \geq (t)/2 \}$, we have from (2.18) that

\[
\int_{r \geq (t)/2} |Z^{[a]} v| \| \partial Z^{[a]} d\| \| \partial^2 Z^{[a]} d\| \, dx \\
\lesssim \langle t \rangle^{-1} \| r Z^{[a]} v \|_{L^\infty(r \geq (t)/2)} \| \partial Z^{[a]} d\|_{L^2} \| \partial^2 Z^{[a]} d\|_{L^2} \\
\lesssim \langle t \rangle^{-1} \| \nabla Z^{[a]} v \|_{L^2}^\frac{3}{2} (E_k^{d_{\alpha - 1}})^{\frac{1}{2}} (E_{k+1}^{d_{\alpha + 1}})^{\frac{1}{2}} .
\]

Next, we write $M_2$ as:

\[
M_2 = \int_{\mathbb{R}^3} \sum_{b+c+e=a} C_a^{b,c}(\nabla Z^b d \cdot \nabla Z^c d - \partial_t Z^b d \cdot \partial_t Z^c d) Z^e d \cdot \partial_t Z^a d \, dx .
\]

In the integral domain of $\{ r \leq (t)/2 \}$, employing (2.20), (2.21) and Lemma 3.2 yields

\[
\int_{r \leq (t)/2} \sum_{b+c+e=a} C_a^{b,c}(\nabla Z^b d \cdot \nabla Z^c d - \partial_t Z^b d \cdot \partial_t Z^c d) Z^e d \cdot \partial_t Z^a d \, dx \\
\lesssim \sum_{|e| \geq |a|/2} \| \partial Z^{[e]}/2 d\|_{L^\infty(r \leq (t)/2)} \| \partial Z^{[e]}/2 d\|_{L^2(r \leq (t)/2)} \| Z^a d\|_{L^2} \| \partial_t Z^a d\|_{L^2} \\
+ \sum_{|e| \leq |a|/2} \| \partial Z^{[e]}/2 d\|_{L^\infty(r \leq (t)/2)} \| \partial Z^{[e]}/2 d\|_{L^2(r \leq (t)/2)} \| Z^a d\|_{L^\infty} \| \partial_t Z^a d\|_{L^2} \\
\lesssim \langle t \rangle^{-\frac{3}{2}} E_{k-1}^{d_{\alpha - 1}} (E_{k+1}^{d_{\alpha + 1}})^{\frac{1}{2}} (1 + (E_{k-1}^{d_{\alpha - 1}})^{\frac{1}{2}}).
\]

To estimate the integral domain of $\{ r \geq (t)/2 \}$, we need to use the null condition. To this end, we first write

\[
\sum_{b+c+e=a} C_a^{b,c} \int_{r \geq (t)/2} (\nabla Z^b d \cdot \nabla Z^c d - \partial_t Z^b d \cdot \partial_t Z^c d) Z^e d \cdot \partial_t Z^a d \, dx \\
= \sum_{b+c+e=a} C_a^{b,c} \int_{r \geq (t)/2} (\omega_i \partial_t + \nabla_i) Z^b d \cdot (\omega_i \partial_t - \nabla_i) Z^c d Z^e d \cdot \partial_t Z^a d \, dx \\
\lesssim \sum_{b+c+e=a} \int_{r \geq (t)/2} |(\omega_i \partial_t + \nabla_i) Z^b d| \| \partial Z^c d\| \| Z^e d\| \| \partial_t Z^a d\| \, dx ,
\]

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where \( \omega = x/r \). Consequently, employing (2.17) and Lemma 3.4, the above can be further bounded by
\[
\sum_{|e| \leq |a|/2} \| (\omega \partial_t + \nabla_i) Z^{[a]} d \|_{L^\infty(r \geq (t)/2)} \| \partial Z^{[a]} d \|_{L^2} \| Z^e d \|_{L^\infty} \| \partial_t Z^a d \|_{L^2} \\
+ \sum_{|e| \geq |a|/2} \| (\omega \partial_t + \nabla_i) Z^{[a]/2} d \|_{L^\infty(r \geq (t)/2)} \| \partial Z^{[a]/2} d \|_{L^2} \| Z^e d \|_{L^6} \| \partial_t Z^a d \|_{L^2} \\
\lesssim (t)^{+ \frac{2}{7}} (E_{k+1}^d)^{\frac{4}{7}} E_{k-1}^d (1 + E_{k-1}^d)^{\frac{1}{7}}.
\]

Here we have used the spatial decomposition along radial and reverse direction:
\[
\nabla = \frac{x}{r} \partial_r - \frac{\omega}{r} \wedge \Omega.
\]

Now we estimate \( M_3, M_4 \) and \( M_5 \) which contain cubic, quartic and quintic terms but no quadratic term. Thanks to (2.17), (2.19) and Lemma 3.2, one gets
\[
M_3 = - \sum_{b+c+e+f=a} C_{b,c} \int_{\mathbb{R}^3} Z^b v \cdot \nabla (Z^c v \cdot \nabla Z^d) \cdot \partial_t Z^a d \ dx
\]
\[
\lesssim \left( \int_{r \geq (t)/2} + \int_{r \geq (t)/2} \right) \| Z^{[a]} v \| \| Z^{[a]+1} v \| \| \nabla Z^{[a]+1} d \| \| \partial_t Z^a d \| \ dx
\]
\[
\lesssim \| Z^{[a]} v \|_{L^\infty(r \geq (t)/2)} \| Z^{[a]+1} v \|_{L^\infty} \| \nabla Z^{[a]+1} d \|_{L^\infty} \| \partial_t Z^a d \|_{L^2}
\]
\[
+ \| Z^{[a]+1} v \|_{L^2} \| \nabla Z^{[a]+1} d \|_{L^6(r \geq (t)/2)} \| \partial_t Z^a d \|_{L^2}
\]
\[
\lesssim (t)^{-1} \| \nabla Z^a v \|_{L^2} \| Z^e d \|_{L^2} (E_{k+1}^d)^{\frac{1}{7}} (E_{k-1}^d)^{\frac{4}{7}}.
\]

Moreover, we have
\[
M_4 = - \sum_{b+c+e+f=a} C_{b,c} \int_{\mathbb{R}^3} (2(Z^b v \cdot \nabla) Z^c d \cdot \partial_t Z^d) Z^f d \cdot \partial_t Z^a d \ dx
\]
\[
\lesssim \sum_{b+c+e+f=a} \int_{\mathbb{R}^3} |Z^b v| \| \partial Z^c d \|_{L^2} \| Z^f d \|_{L^2} \| \partial_t Z^a d \| \ dx.
\]

By Sobolev inequalities and Lemma 3.4, the above can be further bounded by
\[
\sum_{|e| \leq |f| \leq |a|} \| Z^{[a]} v \|_{L^6} \| \partial Z^{[a]} d \|_{L^6} \| \partial Z^{[a]/2} d \|_{L^2} \| Z^e d \|_{L^\infty} \| Z^f d \|_{L^\infty} \| \partial_t Z^a d \|_{L^2}
\]
\[
+ \sum_{|e| \leq |f| \leq |a|} \| Z^{[a]} v \|_{L^\infty} \| \partial Z^{[a]} d \|_{L^2} \| \partial Z^{[a]/2} d \|_{L_\infty} \| Z^e d \|_{L^\infty} \| Z^f d \|_{L^\infty} \| \partial_t Z^a d \|_{L^2}
\]
\[
\lesssim (t)^{-1} \| \nabla Z^a v \|_{L^2} \| Z^e d \|_{L^2} (E_{k+1}^d)^{\frac{1}{7}} [((E_{k+1}^d)^{\frac{4}{7}} + 1].
\]

The term \( M_5 \) can be estimated as
\[
M_5 = - \sum_{b+c+e+f+g=a} C_{a} \int_{\mathbb{R}^3} [(Z^b v \cdot \nabla) Z^c d \cdot ((Z^e v \cdot \nabla) Z^f d)] Z^g d \cdot \partial_t Z^a d \ dx
\]
\[
\lesssim \sum_{|e|+|f|+|g| \leq |a|} \int_{\mathbb{R}^3} |Z^{[a]} v|^2 |\nabla Z^c d| |\nabla Z^f d| |Z^g d| \| \partial_t Z^a d \| \ dx
\]
\[
\lesssim \sum_{|g| \leq |a|} |Z^{[a]} v|^2 |\partial Z^{[a]} d|_{L^2} \| \partial Z^{[a]/2} d|_{L_\infty} \| Z^e d \|_{L^\infty} \| \partial_t Z^a d \|_{L^2}.
\]
\[ + \sum_{|\alpha| \leq |\beta| \leq |\alpha|} \|Z^{\alpha}v\|_{L^2}^2 \|\partial Z^{\alpha}d\|_{L^2} \|\partial Z^{\alpha/2}d\|_{L^\infty} \|\partial Z^{\alpha}d\|_{L^\infty} \|\partial \partial Z^{\alpha}d\|_{L^2}. \]

\[ \lesssim \|\nabla Z^\kappa v\|_{L^2}^2 (E^{d}_{\kappa-1})^{\frac{1}{2}} (1 + E^{d}_{\kappa-1})^{\frac{1}{2}}. \]

Combining all the estimates of \(M_1\) to \(M_5\) leads

\[ \frac{d}{dt} \int_{\mathbb{R}^n} \left[ \frac{1}{2} |\partial Z^\alpha d|^2 + \sum_{b+c=a} C^b_{\kappa}(Z^b v \cdot \nabla Z^c d) \cdot \partial_t Z^\alpha d \right] \, dx \]

\[ \lesssim (t)^{-1} \|\nabla Z^\kappa v\|_{L^2}^2 (E^{d}_{\kappa-1} E^{d}_{\kappa+1})^{\frac{1}{2}} + (t)^{- \frac{1}{2}} E^{d}_{\kappa-1} (E^{d}_{\kappa+1})^{\frac{1}{2}} + \|\nabla Z^\kappa v\|_{L^2}^2 E^{d}_{\kappa-1}. \]

Summing over \(|\alpha| \leq \kappa - 2\), and note that

\[ \sum_{|\alpha| \leq \kappa - 2} \frac{1}{2} \int_{\mathbb{R}^n} |\partial Z^\alpha d|^2 + \sum_{b+c=a} C^b_{\kappa}(Z^b v \cdot \nabla Z^c d) \cdot \partial_t Z^\alpha d \, dx \]

\[ \sim \sum_{|\alpha| \leq \kappa - 2} \frac{1}{2} \int_{\mathbb{R}^n} |\partial Z^\alpha d|^2 \, dx = \frac{1}{2} E^{d}_{\kappa-1}. \]

Thus we finish the proof of (2.11).

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