Parametric phase transition in one dimension.

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PACS. 02.50.-r – Probability theory, stochastic processes, and statistics.
PACS. 64.60.Cn – Order disorder transformations; statistical mechanics of model systems.

Abstract. – We calculate analytically the phase boundary for a nonequilibrium phase transition in a one-dimensional array of coupled, overdamped parametric harmonic oscillators in the limit of strong and weak spatial coupling. Our results show that the transition is reentrant with respect to the spatial coupling in agreement with the prediction of the mean field theory.

Since the landmark paper of Onsager on the solution of the 2-dimensional Ising model \(^4\), the search for exactly solvable models displaying equilibrium phase transitions has become one of the hot pursuits in statistical physics. Over the last two decades, the interest has shifted from equilibrium to nonequilibrium phase transitions \(^2\). In many respects, the latter transitions are richer and more difficult to analyse. In particular, the analogue of the equilibrium distribution is typically not known. On the other hand, the arguments ruling out phase transitions in one dimension for a large class of equilibrium models no longer apply, and a number of exact solutions for nonequilibrium phase transitions in one dimension have been found \(^3\). Our focus here is on a nonequilibrium phase transition in a one-dimensional array of coupled overdamped parametric oscillators. This new transition was discovered \(^4\) in the line of work on phase transitions induced by multiplicative noise \(^1\), see also \(^4\). The phase transition occurs in all dimensions. Surprisingly, it was found that a mean field analysis predicts quite accurately the location of the phase boundary as obtained from numerical simulations for the one-dimensional array. Furthermore, the transition was found to be reentrant with respect to the coupling strength. In this letter, we derive the analytical form of the phase boundary in the limit of weak and strong spatial coupling. Our exact results allow for a detailed comparison with the the mean field result. In particular the reentrance of the transition is firmly established.

The system considered in \(^4\) is a linear chain of overdamped particles coupled harmonically and parametrically excited:

\[
\dot{x}_i = [-1 + A \cos(\omega t + \phi_i)] x_i - K(2x_i - x_{i+1} - x_{i-1}).
\]  

(1)

The disorder stems from a random choice of the phases \(\phi_i\) in \([0, 2\pi]\). This system exhibits a nonequilibrium phase transition from a quiescent phase to an exploding phase at a critical value of the perturbation amplitude \(A\). In the exploding phase the amplitude of all oscillators...
diverges with time. This is surprising because if one suppresses either the disorder (i.e., one replaces the \( \phi_i \)-s by a unique \( \phi \)) or the coupling (i.e., setting \( K = 0 \)), no destabilization occurs. Hence, the instability is entirely caused by the conjugate effects of coupling and disorder.

To analyse the model further, it is convenient to introduce a dimensionless time unit \( \tau = \omega t \) and scaled variables \( \alpha = A/\omega \) and \( \kappa = K/\omega \). Turning first to the weak coupling limit \( \kappa \ll 1 \), we switch to new variables \( y_i, x_i = y_i \times \exp[-(1/\omega + 2\kappa)\tau + \alpha \sin(\tau + \phi_i)] \), and note that the overall time evolution of these variables:

\[
\dot{y}_i = \kappa (e^{\alpha \sin(\tau + \phi_{i+1}) - \alpha \sin(\tau + \phi_i)} y_{i+1} + e^{\alpha \sin(\tau + \phi_{i-1}) - \alpha \sin(\tau + \phi_i)} y_{i-1})
\]

(2) is slow due to the assumption that the prefactor \( \kappa \) in the r.h.s. of (2) is small. On this slow time scale the periodic parametric perturbations are fast and one can replace in (2) the following quantities by their average over one period of the oscillations:

\[
e^{\alpha \sin(\tau + \phi_{i+1}) - \alpha \sin(\tau + \phi_i)} \sim \frac{1}{2\pi} \int_0^{2\pi} d\tau e^{\alpha \sin(\tau + \phi_{i+1}) - \alpha \sin(\tau + \phi_i)} = \frac{1}{2\pi} \int_0^{2\pi} d\tau e^{2\alpha \sin(\phi_{i+1} - \phi_i) \cos \tau} \equiv \lambda_i.
\]

(3)

This procedure is tantamount to the Bogoliubov-Mitropolski technique, familiar from the literature on oscillators \[8\]. With the replacement given in (3), (2) reduces to:

\[
\dot{y}_i = \kappa (\lambda_i y_{i+1} + \lambda_{i-1} y_{i-1}).
\]

(4)

Clearly, for almost all initial conditions, the asymptotic time dependence of \( y_i \) will be dominated by the largest eigenvalue \( \lambda_+ \) of the symmetric tri-diagonal random matrix \( \Lambda_{i,i+1} = \Lambda_{i+1,i} = \lambda_i \), \( \forall i \), the remaining elements \( \Lambda_{i,j} \) being zero. The phase boundary for the transition to explosion is thus given by \((1/\omega + 2\kappa)/\kappa = \lambda_+\). In the limit of an infinite system, the value of \( \lambda_+ \) is expected to be self-averaging. To determine this value, we note that the \( \lambda_i \)'s are statistically independent, hence our problem belongs to the traditional theory of random tri-diagonal symmetric matrices with independent random elements. Since in our case the diagonal elements of the matrix \( \Lambda \) are zero - unlike the famous cases of matrices coming from disordered models of linear chains with randomly chosen spring constants and/or masses - our matrix belongs to the “type I” disorder in Dyson’s terminology \[3,4\]. Furthermore the support of \( \lambda_+ \) is bounded, \( \lambda_+ \in [\lambda_{\min}, \lambda_{\max}] \), and a theorem due to Hori and Matsuda \[10,11\] states that \( \lambda_+ \) is bounded from above by \( 2\lambda_{\max} \). The latter bound is realized for the choice \( \lambda_i = \lambda_{\max} \), \( \forall i \). We are however concerned with the typical value of \( \lambda_+ \) as it is observed with probability one in the thermodynamic limit. A numerical evaluation of the spectrum of a large \( \Lambda \) matrix in fact points to a value of \( \lambda_+ \) significantly lower than the "naive" estimation given by the upperbound (cf. fig. 1). A detailed analytic analysis (cf. appendix for the major steps of the proof) however reveals that \( 2\lambda_{\max} \) is in fact the typical value of \( \lambda_+ \) and the observed deviations in the numerics are entirely the result of strong finite-size effects.

We conclude that in the limit of small \( \kappa \) the phase boundary is given by:

\[
\frac{1/\omega + 2\kappa}{\kappa} = 2\lambda_{\max} = \frac{2}{\pi} \int_0^{\pi} d\tau e^{2\alpha \cos \tau} = 2I_0(2\alpha).
\]

(5)

This result has to be compared with the mean-field prediction \[4\]:

\[
\int_0^{\infty} d\tau e^{-(1/\omega + \kappa)\tau} I_0(2\alpha \sin(\tau/2)) = \frac{1}{\kappa}.
\]

(6)
which, in the region of $\kappa$ small and $\alpha$ large, reduces to:

$$\frac{I_0(2\alpha)}{\sqrt{\alpha}} = \frac{\sinh[\pi(1 + 1/\omega)]}{\sqrt{\pi \kappa}}.$$  \hspace{1cm} (7)

The two asymptotic formulae are not identical, but the leading term of the asymptotic branch is the same, namely $\alpha \sim -\frac{1}{2} \log \kappa$. This partly explains the observed similarity of the two instability curves in this region. We close the discussion of the small $\kappa$ limit by noting that the above method can unfortunately not be generalized to higher dimensions: the arguments are very specific to dimension 1, while the variables $\lambda_i$ are no longer independent in higher dimensions.

Next we turn to the description of the large $\kappa$ regime. In this case it is convenient to transform away the $-x_i$ term by setting $x_i = y_i \exp(-\tau/\omega)$, and to introduce the discrete Fourier transforms:

$$y_q = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} y_j e^{2\pi i j q / N}, \quad q = 0, \ldots, N - 1.$$  \hspace{1cm} (8)

The evolution equation for these Fourier modes reads:

$$\dot{y}_q = \alpha \sum_{q' = 0}^{N-1} y_{q'} C_{q-q'}(\tau) - \left(4\kappa \sin^2 \frac{\pi q}{N}\right) y_q,$$  \hspace{1cm} (9)

with $C_{q}(\tau) = \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i j q / N} \cos(\tau + \phi_j)$.

In the limit of $\kappa$ large, the system becomes very "stiff" and any initial short wavelength perturbation is expected to be damped out very rapidly, cf. second term in the r.h.s. of (8).
We therefore assume that, as can be verified a posteriori, the long-time behaviour of the $q \neq 0$ modes is completely determined by their coupling to the slow zero-th mode $q = 0$, i.e. by the "reinjection" of energy through the $q' = 0$ contribution in the first term of the r.h.s. of (10). Dropping furthermore the irrelevant contributions due to initial conditions, one concludes that for $q \neq 0$:

$$y_q = \alpha \int_0^t du e^{-[4\kappa \sin^2 \pi q/N](t-u)} y_0(u) C_q(u).$$  \hspace{1cm} (10)$$

Inserting this result in the equation for $q = 0$ leads to:

$$\dot{y}_0 = \alpha^2 \sum_{q=0}^{N-1} \int_0^t du e^{-[4\kappa \sin^2 \pi q/N](t-u)} y_0(u) C_q(u) C_{-q}(t).$$  \hspace{1cm} (11)$$

One expects that the quantity $\sum_{q=0}^{N-1} Ne^{-[4\kappa \sin^2 \pi q/N](t-u)} C_q(u) C_{-q}(t)$ is self-averaging in thermodynamic limit \cite{13}, hence one can perform an average over the phase disorder directly in the dynamical equation:

$$\dot{y}_0 = \frac{\alpha^2}{2} \int_0^t du \left( \frac{1}{N} \sum_{q=1}^N e^{-[4\kappa \sin^2 \pi q/N](t-u)} \right) \cos(t-u) y_0(u)$$

$$= \frac{\alpha^2}{2} \int_0^t du \left( \int_0^1 dq e^{-[4\kappa \sin^2 \pi q](t-u)} \right) \cos(t-u) y_0(u).$$  \hspace{1cm} (12)$$

where the thermodynamic limit has been taken in the second equality. The resulting equation can be readily solved for the Laplace transform $\hat{y}_0(z)$ of $y_0$ (assuming $y_0(t=0) = 1$):

$$\hat{y}_0(z) = \frac{1}{z - \frac{\alpha^2}{2} \hat{g}(z)},$$  \hspace{1cm} (14)$$

where $\hat{g}(z)$, the Laplace transform of $g(t) = \cos t \times \int_0^1 dq e^{-[4\kappa \sin^2 \pi q]t}$, is found to be

$$\hat{g}(z) = \left( \frac{\sqrt{(z^2 + 1)((z + 4\kappa)^2 + 1) + z(z + 4\kappa) - 1}}{2(z^2 + 1)((z + 4\kappa)^2 + 1)} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (15)$$

The long-time asymptotic behaviour of $y_0(t)$ turns out to be governed by the single real positive pole $\gamma$ of $\hat{y}_0(z)$. The phase transition to explosion takes place when $\gamma = 1/\omega$, or more explicitly when

$$\frac{1}{\omega} = \frac{\alpha^2}{2} \hat{g}(1/\omega) \approx \frac{\alpha^2}{4\sqrt{2\kappa}} \left( \frac{\sqrt{\omega^{-2} + 1} + \omega^{-1}}{\omega^{-2} + 1} \right)^{\frac{1}{2}},$$  \hspace{1cm} (16)$$

where the large $\kappa$ limit is invoked in the second equality. The dependence $\alpha \propto \kappa^{1/4}$ explains the flat shape of the reentrance curve. In comparison the mean-field model predicts a behaviour $\alpha \sim \sqrt{2\kappa/\omega}$, consistent with the fact that the reentrance is expected to occur at a smaller value of $\kappa$ (for a given value of $\alpha$) in the mean-field system, since the connectivity of the system enhances its effective stiffness.

We finally turn to a comparison of the above derived asymptotic results with the mean field and numerical results, cf. fig. 2. The differences between the asymptotic 1D results and the mean-field value are rather small. More revealing is that the 1D system is less
stable than the mean-field model for small and (very) large $\kappa$ values (cf. the respective asymptotics $\kappa^{1/4} \ll \kappa^{1/2}$), while the opposite seems to be the case for intermediate values: if the asymptotic evaluation for $\kappa$ large is reliable for $\kappa \sim 4$, we can see from fig. 2 that there exists a range of coupling values for which the 1D system is more stable than the mean-field model. This inversion seems to be confirmed by the numerical simulations of the 1D model in [4] and highlights the subtleties of the destabilisation process taking place in the system.

We conclude that the analytic theory presented here describes the salient features of the phase boundary for the nonequilibrium phase transition occurring in a one-dimensional array of coupled, overdamped parametric harmonic oscillators with quenched random phases. The fact that the deviations from the mean field results are small raises the question of the critical dimension. Interestingly, the above presented procedure to calculate the phase boundary for large $\kappa$ can also be applied to higher dimensions. One finds a mean field behaviour for a spatial dimension larger or equal to 3, with the reentrance curve given by $\alpha \propto \sqrt{\kappa}$. The dimension 2 is “critical”, characterized by logarithmic corrections $\alpha \propto \sqrt{\kappa} / \log \kappa$. This strongly suggests that the critical dimension of the phase transition is in fact equal to two.

Appendix. In this appendix, we review some spectral properties of the random matrix $\Lambda$. First, one notices that its spectrum is always symmetric with respect to zero, for if $\{y_n\}$ is an eigenvector associated with the eigenvalue $\lambda$, then $\{-1\}^n y_n$ is another eigenvector with the eigenvalue $-\lambda$. We can thus restrict our attention to the negative eigenvalues $-\lambda$, $\lambda > 0$. The eigenvector equation reads $-\lambda y_n = \lambda_n y_{n+1} + \lambda_{n-1} y_{n-1}$ and can be recast into the form $z_{n+1} = -\mu_n/(\lambda + z_n)$, with $z_n = \lambda_{n-1} y_{n-1}/y_n$, $\mu_n = \lambda_n^2$. One can now reproduce step by step the demonstration performed by Schmidt [12], whose conclusions are that the proportion of negative eigenvalues comprised in $[-\lambda, 0]$ is equal in the thermodynamic limit to the proportion of positive $z_n$ in the sequence $z_0, z_1, \ldots$. Turning first to the case where the $\mu_n$ are constant (= $\mu$) two different situations arise. For $\lambda^2 < 4\mu$, the mapping from $z_n$ to $z_{n+1}$ has no (real) fixed points, and as a result the positive $z_n$ are substantially visited (i.e. the proportion of positive $z_n$ is non zero when $N \to \infty$). For $\lambda^2 > 4\mu$ the appearance of a globally attractive negative fixed point $z_+ = (-\lambda + \sqrt{\lambda^2 - 4\mu})/2$ implies that the sequence will ultimately end up in the $\mathbb{R}^-$ region. Hence, $-\lambda = -2\sqrt{\mu}$ is the lower bound of the spectrum, since beyond
this value, the proportion of negative $z_n$-values is 1.

We next turn to the more complicated case of $\lambda_n$ being independent random variables sampled from a distribution with support $[\lambda_{\min}, \lambda_{\max}]$ (with $\lambda_{\min} \geq 0$). Since the $\mu \in [\mu_{\min}, \mu_{\max}]$ are also independent random variables, the sequence $z_n$ becomes a Markov process. In continuation of the arguments given above, the following conclusions can be drawn. For $\lambda > 2\lambda_{\max}$ it follows that $\lambda^2 > 4\mu_n$ is true for all allowed values of $\mu_n$, and the $z_n$ are attracted to a region $[z_+ (\mu_{\max}), z_+ (\mu_{\min})]$ which lies entirely on the negative real axis. One concludes that the spectrum, assumed to be self-averaging in the limit $N \to \infty$, cannot extend beyond the value $2\lambda_{\max}$ (a result also demonstrated in [4]). Considering the situation where $\lambda$ is slightly less than $2\lambda_{\max}$, the $z_n$ will still more often than not be attracted to the region $[-\lambda/2, z_+ (\mu_{\min})]$. Exceptionally however, when $\mu_n$ is sampled outside $[\mu_{\min}, \lambda^2/4]$, the $z_n$ will make an excursion toward the region $[-\infty, -\lambda]$. If a long enough such series of successive $\mu_n$ is sampled, $z_n$ will eventually cross the $-\lambda$ value and next become positive. Even though unlikely, the probability for these events is not zero in thermodynamic limit, for their occurrence only requires a finite number of steps in the sequence of $z_n$. As a result, one concludes that the effective boundary for the spectrum is indeed $\pm 2\lambda_{\max}$. Moreover, one can understand qualitatively why the spectra of figure 3 do not seem to saturate this bound as follows. The number of successive steps with $\mu_n \in I = [\lambda^2/4, \mu_{\max}]$, required to reach $\mathbb{R}^+$ can be considered more or less as constant (with respect to $\lambda$), say $n_0$. Then the probability for observing a positive value of $z_n$ is of order $p^{n_0}$, where $p$ is the probability to sample $\mu_n$ in $I$. In the example of figure 3, $p$ is simply given by $(\lambda_{\max} - \lambda/2)/(\lambda_{\max} - \lambda_{\min}) = 2 - \lambda/2$. Moreover, a rough evaluation of $n_0$ gives at least 4 steps. Then the proportion of eigenmodes greater than 3.5 can be (very generously!) estimated to be $(2 - 3.5/2)^4 = 0.3\ldots$, explaining the apparent absence of these eigenvalues in figure 3.

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this includes the assumption that no macroscopic correlations appear between this quantity and $y_0$. 