q–NONLINEARITY, DEFORMATIONS
AND PLANCK DISTRIBUTION

V. I. Man’ko∗†, G. Marmo∗, and F. Zaccaria∗

∗ Dipartimento di Scienze Fisiche,
Università di Napoli “Federico II”
Istituto Nazionale di Fisica Nucleare, Sezione di Napoli
Mostra d’Oltremare, Pad.19 - 80125 Napoli, Italy

† Lebedev Physics Institute
53 Leninsky Prospekt, 117924 Moscow, Russia

INTRODUCTION

In this contribution we will review a new approach to some nonlinear
dynamical systems which is related to the q–deformation (or other types of
deformations) of linear classical and quantum systems considered in [1]. The
main idea of this approach is to replace constants like frequency or mass, etc.,
which are parameters of the linear systems with constants of the motion of the
system. This procedure produces from the initial linear system a nonlinear
one and as it was demonstrated in [1, 2] the q–oscillator of [3, 4] may be
considered as physical system with this specific nonlinearity which was called
q–nonlinearity. At the example of q–oscillator the constant parameter which
was replaced by the constant of the motion dependent on the amplitude of
the vibration was the frequency.

But this approach may be used for many dynamical systems. So, the
constant masses in Klein–Gordon and Dirac equations or the constant signal
velocity in the wave equation may be replaced by the dynamical variables but these variables are chosen to be the constant of the motion of the dynamical system under consideration. Conceptually we address the problem (which is not new) if the physical constant parameters like light velocity, Planck constant or gravitational constant are constant parameters in reality or they may change, for example, in the process of evolution? The same question may be put about such characteristics of elementary particle as electric charge or mass, say, of electron or fine structure constant which play the role of constant parameters in the dynamical equations of the theory. The example of the q–nonlinearity of [1] showed that a linear equation or system of equations may be considered as a limit of nonlinear system characterized by a nonlinearity parameter which is not important for small intensities but starts to play important role for the high field intensities or for large energy densities.

Taking into account such phenomenon naturally yields the deformation of the linear equations of the motion which are transformed due to the deformation into the nonlinear ones but of specific type. The corresponding nonlinear system of equations is simply a "reparametrized" initial linear system of equations in which constant parameters are replaced by the constants of the motion of the nonlinear system. This specific procedure of nonlinearization of the initial linear system of equations selects a broad enough class of integrable dynamical systems which are simple but could describe the mentioned above physical phenomena. Also such dynamical systems are appropriate to treat the q–deformations and other aspects of quantum groups [5], [6], [7] and quantum qeometry [8] as influence of different types of nonlinearities in the corresponding physical processes.

Of course, the formulated approach may be extended in the sense that one could deform not only initial linear system to make it the nonlinear one using method of replacing constant parameters by constants of the motion. In principle, one could deform a simple initial nonlinear system containing some constant parameters and to transform it into another nonlinear system by replacing the constants with the integrals of the motion of the nonlinear system. So, it is possible to extend the class of integrable nonlinear systems starting from simple integrable nonlinear systems and reparametrizing them, i. e. replacing the constant parameters by the constants of the motion. Such approach may be useful, for instance, in q–deforming general relativity or Yang–Mills type gauge theories. Up to our knowledge such classes of both linear and nonlinear deformed integrable dynamical systems have not
been studied till now. It should be noted that the method of obtaining the integrable nonlinear systems starting from the linear ones is close in spirit to the generalized reduction procedure of considering the integrable dynamical systems [9], [10].

The goal of our work is to present a general scheme for the described procedure of deforming the linear systems with finite number of degrees of freedom (next Section). In subsequent sections we also will give some examples of the deformation including infinite number of degrees of freedom (wave equation). Then the physical consequences of the q–nonlinearity as blue shift effect [4], deformation of Planck distribution formula [2], [11], and change of a charge form–factor [12] will be discussed. In the last two sections deformed Klein–Gordon equation and Maxwell equation in the vacuum will be considered.

**SPECIAL NONLINEAR SYSTEMS OBTAINED FROM LINEAR ONES**

We consider a carrier space \( \mathcal{R}^n \). With any matrix \( A \) we associate the vector field \( X_A = x_i A^i_j \partial / \partial x_j \). This association is a Lie algebra isomorphism. By using the associative product of matrices, we can associate powers with any matrix \( A \cdot A \cdot \cdots A = A^k \) for any integer \( k \). The number of independent ones is given by the degree of the minimal polynomial of \( A \). If we associate vector fields with any one of these powers we get

\[
X_k = (A^k)^m_n x_m \frac{\partial}{\partial x_n}
\]  

(1)

and

\[
[X_k, X_j] = 0,
\]

(2)

therefore we get mutually commuting symmetries for our initial dynamical system \( X_A \). We notice that for a generic matrix \( A \) there will be \( n \)–independent matrices and they are a basis of all infinitesimal symmetries for \( X_A \) with coefficients constants of the motion for \( X_A \).

For nonsingular matrices \( A^k \), i. e. \( \det(1 + A^k) \neq 0 \), we can define invertible transformations

\[
\Phi_k = \frac{1 - A_k}{1 + A_k}
\]

(3)
which are symmetries for $X_A$, i.e.
\[ \Phi_k^{-1} A \Phi_k = A, \] (4)
or we can consider $e^{sA^k} = \varphi_k$, which are also symmetries for $X_A$.

When $A$ is not generic, there are other infinitesimal symmetries which are not obtained from powers of $A$ and moreover the algebra of symmetries need not be Abelian. Again we can use either the Cayley map or the exponentiation to get finite symmetries out of the infinitesimal ones.

Now we can make the dynamics and the infinitesimal symmetries nonlinear by replacing the entries $A^i_j \in \mathcal{R}$ with constants of the motion for $X_A$. With this procedure, infinitesimal symmetries are no more linear and they do not close on a finite dimensional Lie algebra.

If $\mathcal{F}_A \in \mathcal{F}(\mathcal{R}^n)$ is the subring of constants of the motion for $X_A$, and $X_1, X_2, \ldots, X_S$ is a basis of linear infinitesimal symmetries for $X_A$, we get nonlinear infinitesimal symmetries $X_f = f^1 X_1 + f^2 X_2 + \cdots + f^S X_S$ for any choice of $f^1, f^2, \ldots, f^S \in \mathcal{F}_A$. If $X_A$ is ”reparametrized”, i.e. replaced with $f X_A$, $f \in \mathcal{F}_A$, previous symmetries are broken except for those which preserve $f$, i.e. $\mathcal{L}_{X_j} f = 0$.

To make contact with the quantum situation we shall consider previous procedure in the framework of Hamiltonian dynamics. If we consider the carrier space $\mathcal{R}^{2n}$, with the symplectic matrix
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] (5)
the dynamics represented by $X_A$ will be Hamiltonian if $^t A J + J A = 0$. It follows that $J A$ is a symmetric matrix and we can define a function $f_A(x) = ^t x J A x$. This function turns out to be the Hamiltonian function giving rise to $X_A$. In constructing infinitesimal symmetries for $X_A$ by taking powers of $A$, not all powers will give rise to matrices in the symplectic Lie algebra, we have to restrict to odd powers, i.e. $A^1, A^3, \ldots, A^{2k+1}, \ldots, A^{2n+1}$, indeed
\[ ^t (A^{2k+1}) J + J A^{2k+1} = 0 \] (6)
and we find functions
\[ f_k(x) = ^t x (J A^{2k+1}) x. \] (7)
From $[A^{2k+1}, A^{2j+1}] = 0$ and the Lie algebra isomorphism
\[ \{f_B, f_C\} = f_{[B,C]} \]
when \( f_B(x) = \langle x, Bx \rangle \), we get \( \{f_k, f_j\} = 0 \), therefore odd powers of \( A \) give rise to constants of the motion which are in involution. For a generic matrix \( A \), \( X_A \) turns out to be a completely integrable system in the Liouville sense.

The even powers of \( A \) give rise to symmetries which are not canonical. We can turn them into nonlinear ones by using the procedure we have already illustrated previously. To deal with nonlinear transformations it is better to work in a geometrical framework. Therefore on \( \mathcal{R}^{2n} \) we have the dynamics \( X_A = x_i A_j \partial / \partial x_j \), the simplectic structure \( \omega = \sum_i dx_i \wedge dx_{i+n} \) and canonical symmetries \( X_1, X_2, \ldots, X_n \) associated with odd powers of \( A \) and functions \( f_1, f_2, \ldots, f_n \), while \( Y_1, Y_2, \ldots, Y_n \) associated with even powers of \( A \) and are noncanonical. By exponentiating the noncanonical infinitesimal symmetries we get invertible transformations from \( \mathcal{R}^{2n} \) into \( \mathcal{R}^{2n} \) which takes us from one Hamiltonian description for \( X_A \) to a different Hamiltonian description.

As a final comment from linear algebra we recall that any matrix \( A \) can be decomposed into the sum
\[
A = S + N
\] (8)
where \( S \) is semisimple and \( N \) is nilpotent, moreover \([S, N] = 0\). From here it follows that the flow associated with \( A \) can be written as
\[
e^{tA} = e^{tS} e^{tN}.
\] (9)
This formula shows that if we want (non trivial) bounded motions associated with one Hamiltonian vector field \( A \), we have to impose \( N = 0 \) and \( S \) should have purely imaginary (non zero) eigenvalues. Therefore a generic linear Hamiltonian system with bounded motions must be a collection of noninteracting harmonic oscillators. To be more specific we turn to consider a two–dimensional harmonic oscillator.

### The Harmonic Oscillator

On \( \mathcal{R}^4 \) with coordinates \((\bar{x}, \bar{v})\) we consider the equations of motion
\[
\frac{d\bar{x}_k}{dt} = \bar{v}_k, \quad \frac{d\bar{v}_k}{dt} = -\omega_k^2 \bar{x}_k, \quad k = 1, 2.
\] (10)
We introduce new coordinates
\[ q_k = \tilde{x}_k, \quad p_k = \frac{\tilde{v}_k}{\omega_k}, \quad k = 1, 2 \] (11)
and get
\[ \frac{dq_k}{dt} = \omega_k p_k, \quad \frac{dp_k}{dt} = -\omega_k q_k \] (12)
associated with the matrix
\[ A = \begin{pmatrix}
0 & \omega_1 & 0 & 0 \\
-\omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_2 \\
0 & 0 & -\omega_2 & 0
\end{pmatrix}. \] (13)

We find the commuting symmetries
\[ A^0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \] (14)
\[ A^2 = \begin{pmatrix}
-\omega_1^2 & 0 & 0 & 0 \\
0 & -\omega_1^2 & 0 & 0 \\
0 & 0 & -\omega_2^2 & 0 \\
0 & 0 & 0 & -\omega_2^2
\end{pmatrix}, \] (15)
and
\[ A^3 = \begin{pmatrix}
0 & -\omega_1^3 & 0 & 0 \\
\omega_1^3 & 0 & 0 & 0 \\
0 & 0 & 0 & -\omega_2^3 \\
0 & 0 & \omega_2^3 & 0
\end{pmatrix}. \] (16)

A more convenient basis of symmetries gives rise to the vector fields
\[ X_1 = q_1 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial q_1}, \quad X_2 = q_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q_2}, \]
\[ Y_1 = q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1}, \quad Y_2 = q_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial q_2}. \] (17)

Associated functions are
\[ h_1 = p_1^2 + q_1^2, \quad h_1 = p_2^2 + q_2^2. \]
By using $Y_1$ and $Y_2$ made into nonlinear transformations, we can consider

$$x_k = f_k(h_k)p_k, \quad y_k = f_k(h_k)q_k.$$  \hspace{1cm} (18)

This nonlinear transformation is a symmetry for the dynamics but is non-canonical, it means

$$\sum_k dx_k \wedge dy_k = \sum_k d(f_k p_k) \wedge d(f_k q_k)$$  \hspace{1cm} (19)

which is different from $\sum_k dp_k \wedge dq_k$ giving rise to standard Hamiltonian description of our harmonic oscillator with Hamiltonian

$$H = \sum_k \frac{\omega_k}{2} (p_k^2 + q_k^2).$$

The picture we are presented with is the following: $X_A$ has Hamiltonian description given by $(\omega, H)$ or $(\tilde{\omega}, \tilde{H})$, i.e.

$$i_{X_A} \omega = -dH, \quad i_{X_A} \tilde{\omega} = -d\tilde{H}.$$  \hspace{1cm} (20)

What happens if we associate a vector field with $\tilde{H}$ by using $\omega$? i.e. we consider

$$i_X \omega = -d\tilde{H},$$  \hspace{1cm} (21)

it turns out that $X$ is a reparametrization of $X_A$ by a constant of the motion.

At this point we can consider the coordinate system where $\omega$ has the standard form (Heisenberg coordinates) and $\tilde{H}$ is a deformation of the quadratic expression or we can use a coordinate system where $\tilde{H}$ has the standard quadratic expression but $\omega$ is ”deformed” (via the nonlinear noncanonical transformation). As $\omega$ gives rise to Poisson brackets (”commutation relations”) this second choice can be interpreted via a deformation of the commutation relations and is the starting point for ”$q$-deformed oscillators.”

Introducing complex coordinates in $\mathbb{R}^4$ we consider complex coordinates

$$\alpha_k = x_k + iy_k, \quad \alpha_k^* = x_k - iy_k.$$  \hspace{1cm} (22)
The complex structure in $\mathbb{R}^4$ is given by the matrix

$$J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}$$

satisfying $J^2 = -1$. This matrix defines a complex structure commuting with the dynamical evolution

$$\Gamma = i(\alpha_k \frac{\partial}{\partial \alpha_k} - \alpha_k^* \frac{\partial}{\partial \alpha_k^*}).$$

(23)

In the $(p, q)$ coordinates we have new complex coordinates

$$\xi_k = p_k + iq_k, \quad \xi_k = p_k - iq_k,$$

(24)

and, setting

$$n_k = \alpha_k \alpha_k^*,$$

(25)

$$\xi_k = f_k(n_k) \alpha_k, \quad \xi_k^* = f_k(n_k) \alpha_k^*.$$  

(26)

Thus transformation is not "analytic" (we notice that analyticity depends on the complex structure and here we have two alternative complex structures compatible with the dynamics). In these complex coordinates, if we take the new point, stemming from quantum mechanics, which takes the Hamiltonian as a primitive concept for the dynamics, we are naturally led to consider the following two Hamiltonians

$$H_1 = \frac{1}{2} \sum_k n_k$$

(27)

in the $\alpha$-coordinates and

$$H_2 = \frac{1}{2} \sum_k \xi_k \xi_k^*$$

(28)

in the $\xi$-coordinates. To compare, we express them in the same variables to find

$$H_1 = \frac{1}{2} \sum_k n_k$$

(29)
and

\[ H_2 = \frac{1}{2} \sum_k f_k^2(n_k) n_k. \] (30)

We now use the same bracket for them, say

\[ \{ \alpha_k, \alpha_k^* \} = i. \] (31)

We obtain two different dynamical systems, where the one associated with \( H_2 \) is even not necessarily isotropic. The evolution goes from periodic orbit to orbit whose closure is a 2-dimensional torus, i.e. the associated systems are completely different. Of course, there is no contradiction. Indeed to have the same dynamics we should use different Poisson Brackets, namely, the one we get transforming Eq. (31). It is now clear that under quantization these complex coordinates go into creation and annihilation operators. Therefore for the corresponding commutators we can repeat what we have said for the Poisson Bracket.

As a final remark, we notice that the notion of coherent states is a kinematical notion. The coherent states are discussed in quantum mechanics from the point of view of the states the closest to classical one. According to this physical property the coherent states of the electromagnetic field harmonic oscillator were introduced in [13]. The coherent states (classical Gaussian packets) of a charge moving in magnetic field were introduced in [14], [15]. Coherent states of spin were introduced in [16].

We start with a Poisson bracket compatible with a complex structure, we consider a canonical chart to be anyone such that

\[ \{ \chi_k, \chi_j^* \} = i\delta_{kj}, \] (32)

then we associate creation and annihilation operators with these variables and define the associated coherent states. It is clear therefore that the ingredients needed for coherent states are:

1) a symplectic structure
2) a compatible complex structure

and together they imply the existence of an Hermitian structure. When there are various structures compatible with a given dynamical evolution, we find various coherent states (i.e. associated with various symplectic structures) preserved by this evolution. The discussed complex structure has been used to construct Jordan–Schwinger map for some functional groups in [17].
CLASSICAL q–OSCILLATOR AS "REPARAMETRIZATION"

In previous section we described the general scheme of deformations. Now we give in detail the example of q–oscillator. Also, the q–oscillator mathematical properties and its modifications have been used to associate with some physical phenomena [18]–[26]. The q–oscillators are interesting from the point of view of mathematical structure [27]–[32]. Let us start from the considering the one–dimensional linear harmonic oscillator with frequency \( \omega \) and corresponding nonlinear q–oscillator following the approach of Ref. [1]. The linear second order equation for the coordinate of this oscillator \( x \) is

\[
\ddot{x} + \omega^2 x = 0. \tag{33}
\]

In the form of the first order equations for the position and momentum this harmonic oscillator is described by the system

\[
\begin{align*}
\dot{p} &= -\omega^2 x, \\
\dot{x} &= p. \tag{34}
\end{align*}
\]

The variables \( x \) and \( p \) are real ones. If we introduce complex variables

\[
\begin{align*}
\alpha &= \frac{1}{\sqrt{2}}(\sqrt{\omega}x + i\sqrt{\omega}p), \\
\alpha^* &= \frac{1}{\sqrt{2}}(\sqrt{\omega}x - i\sqrt{\omega}p) \tag{35}
\end{align*}
\]

the harmonic oscillator is described by the first order differential equations

\[
\begin{align*}
\dot{\alpha} &= -i\omega \alpha, \\
\dot{\alpha}^* &= i\omega \alpha^*. \tag{36}
\end{align*}
\]

Both systems (34) and (35) may be rewritten in matrix form of Ref. [1]

\[
\dot{X} = AX. \tag{37}
\]
The real two–vector $\mathbf{X} \in \mathbb{R}^2$ and the matrix $A$ for the system (34) are
\[
\mathbf{X} = \begin{pmatrix} p \\ x \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\omega^2 \\ 1 & 0 \end{pmatrix}.
\tag{38}
\]

The complex vector $\mathbf{X}$ and matrix $A$ for the system (35) are
\[
\mathbf{X} = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}, \quad A = -i\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{39}
\]

According to our general procedure let us replace the constant matrix elements $A_{ij}$ of matrix $A$ by functions of the constants of the motion. The system obtained is nonlinear and it may be integrated by exponentiation as easily as the initial linear system. We illustrate this procedure again on the harmonic oscillator equations of motion (39) since in this case the matrix $A$ is diagonal one. Thus the constant frequency $\omega$ should be replaced by the constant of the motion $\tilde{\omega}(\alpha, \alpha^*)$ and we have new system for one and the same variable $\alpha$
\[
\dot{\alpha} = -i\tilde{\omega}(\alpha, \alpha^*)\alpha, \\
\dot{\alpha}^* = i\tilde{\omega}(\alpha, \alpha^*)\alpha^*.
\tag{40}
\]

If one takes the constant of the motion $\tilde{\omega}(\alpha, \alpha^*)$ to be integral of motion independent explicitly on time we have the condition for this function
\[
\frac{d}{dt}\tilde{\omega}(\alpha, \alpha^*) = \frac{\partial \tilde{\omega}}{\partial \alpha} \dot{\alpha} + \frac{\partial \tilde{\omega}}{\partial \alpha^*} \dot{\alpha}^* = 0
\tag{41}
\]
which after using the equation of motion (40) may be integrated and the solution for this equation is
\[
\tilde{\omega}(\alpha, \alpha^*) = \Omega(\alpha\alpha^*),
\tag{42}
\]
i. e. any function $\Omega$ of the modulus of the amplitude is the frequency which is the constant of the motion. We will take this function in the form
\[
\Omega = \omega f_q(\alpha\alpha^*).
\tag{43}
\]
The function $f_q(\alpha\alpha^*)$ will specify the q–deformation of the harmonic oscillator and we will take it in the form
\[
f_q(z) = \frac{\lambda}{\sinh \lambda} \cosh \lambda z, \quad q = e^\lambda,
\tag{44}
\]
where $\lambda$ is the parameter of $q$-nonlinearity of the vibrations. In case $\lambda \to 0$ the function $f_q(z) \to 1$. The relation to classical $q$-oscillator (and its quantum counter part) consists of the existence the nonlinear noncanonical transform in the phase space of the oscillator which in the variables $\alpha$ looks [4] as

$$\alpha_q = \sqrt{\frac{\sinh \lambda \alpha^* \alpha}{\alpha^* \sinh \lambda}} \alpha, \quad \alpha^*_q = \sqrt{\frac{\sinh \lambda \alpha^* \alpha}{\alpha^* \sinh \lambda}} \alpha^*.$$  \hspace{1cm} (45)

The Poisson Brackets of the variables $\alpha_q$ reproduce the structure of the annihilation and creation operators and the commutation relations of the $q$-oscillator [3] (see, below). The dynamics of the physical variables $\alpha$ due to the Hamiltonian

$$H = \omega \alpha^*_q \alpha_q,$$ \hspace{1cm} (46)

which is taken in form to be the same as the Hamiltonian of the harmonic oscillator

$$H = \omega \alpha^* \alpha,$$ \hspace{1cm} (47)

is the nonlinear vibration corresponding to Eq. (40). The linear dynamics of the variable $\alpha$ corresponding to the Hamiltonian (47) is given by the solution to the Eq. (36). It is necessary to point out that in the suggested approach the physical meaning of the variables $\alpha$ as related to physical position $x$ and physical momentum $p$ in the sense of the measurement procedure by the formula (35) is preserved for nonlinearized dynamics. It means that the dynamics of the coordinate $x$ and momentum $p$ of the nonlinear $q$-oscillator is described by the system of equations

$$\dot{x} = f_q(\alpha^*)p,$$

$$\dot{p} = -\omega^2 f_q(\alpha^*)x$$ \hspace{1cm} (48)

where

$$\alpha^* = \frac{1}{2}(\omega x^2 + \frac{1}{\omega} p^2).$$ \hspace{1cm} (49)

For the deformed equations of motion of $q$-oscillator as the formula (48) shows the momentum is the function of the velocity and position. This function may be obtained as the solution to the functional equation

$$p(x, \dot{x}) = \frac{\sinh \lambda}{\lambda} \frac{\dot{x}}{\cosh[\frac{1}{2}(\omega x^2 + \frac{1}{\omega} p^2(x, \dot{x}))]}$$ \hspace{1cm} (50)
considered as the implicit formula for the giving momentum as the function of the position \( x \) and velocity \( \dot{x} \). The solution to q–oscillator equation of motion

\[
\dot{\alpha} = -i\omega\alpha \frac{\lambda}{\sinh \lambda} \cosh \lambda \alpha^* \tag{51}
\]
is

\[
\alpha(t) = \alpha_0 \exp[-i\omega t \frac{\lambda}{\sinh \lambda} \cosh \lambda \alpha^*_0] \tag{52}
\]
where

\[
\alpha_0 = \alpha(t = 0)
\]
is the initial complex amplitude of the nonlinear q–oscillator. The solution to the equation of motion for the coordinate \( x \) of the nonlinear q–oscillator

\[
\ddot{x} + \omega^2 \frac{\lambda^2}{\sinh^2 \lambda} \cosh^2 \{\frac{\lambda}{2\omega} [x_0^2 \omega^2 + p^2(x, \dot{x})]\} = 0 \tag{53}
\]
where the function \( p(x, \dot{x}) \) is given explicitly by the relation (50), may be written in the form

\[
x(t) = \frac{x_0}{2} \{\exp[i\lambda \omega t \sinh \lambda \cosh \{\frac{\lambda}{2\omega} [x_0^2 \omega^2 + p^2(x_0, \dot{x}_0)]\}] + \exp[-i\lambda \omega t \sinh \lambda \cosh \{\frac{\lambda}{2\omega} [x_0^2 \omega^2 + p^2(x_0, \dot{x}_0)]\}] + \frac{x_0 \sinh \lambda}{2i\lambda \omega} \cosh^{-1} \{\frac{\lambda}{2\omega} [x_0^2 \omega^2 + p^2(x_0, \dot{x}_0)]\} \times \{\exp[i\lambda \omega t \sinh \lambda \cosh \{\frac{\lambda}{2\omega} [x_0^2 \omega^2 + p^2(x_0, \dot{x}_0)]\}] - \exp[-i\lambda \omega t \sinh \lambda \cosh \{\frac{\lambda}{2\omega} [x_0^2 \omega^2 + p^2(x_0, \dot{x}_0)]\}]. \tag{54}
\]

Here \( x_0 = x(t = 0) \) and \( \dot{x}_0 = \dot{x}(t = 0) \) are the initial position and velocity of the nonlinear q–oscillator. In the limit \( \lambda \to 0 \) we have the usual solution for the linear harmonic oscillator.

One can find for small nonlinearity \( \lambda \ll 1 \) the approximate expression for the momentum solving the equation (50) by iteration method. We have

\[
p = \dot{x}[1 + \frac{\lambda^2}{6} - \frac{\lambda^2}{8} (\omega x^2 + \frac{\dot{x}^2}{\omega})^2]. \tag{55}
\]
This formula may be interpreted as the negative shift of the mass of the oscillator by the factor depending quadratically on the energy of the oscillations.

**TWO–DIMENSIONAL CLASSICAL q–OSCILLATOR**

The dynamics of generically deformed two–dimensional oscillator is described in previous sections. In this Section we will give the formula for specific q–deformation of this oscillator [1]. If one considers two degrees of freedom, the amplitudes of the first harmonic oscillator $\alpha_+$ and of the second oscillator $\alpha_-$ may be q–deformed by two different methods. One is to have frequency of the first oscillator to be dependent only on the energy of this oscillator $n_+ = |\alpha_+|^2$ and the frequency of the second oscillator to be dependent only on its energy $n_- = |\alpha_-|^2$. Another method of deformation is to make the frequencies of both oscillators to be dependent on the full energy of vibrations

$$n = n_+ + n_-.$$ 

We now will describe the q–deformed variables $\alpha_{q\pm}$ in the case of q–deformation when the frequency of vibrations depends on full energy. Then we have the following non–zero Poisson Brackets

$$\{\alpha_{q\pm}, \alpha_{q\mp}^*\} = i\alpha_+ \alpha_-^* \frac{(\lambda n) \cosh n\lambda - \sinh n\lambda}{n^2 \sinh \lambda}, \quad (56)$$

and

$$\{\alpha_{q\pm}, \alpha_{q\mp}^*\} = \frac{i}{n \sinh \lambda} [(1 - \frac{n_+}{n}) \sinh n\lambda + \lambda n_\pm \cosh n\lambda]. \quad (57)$$

Here

$$\alpha_{q\pm} = \alpha_\pm F(n) \quad (58)$$

where

$$F(n) = \sqrt{\frac{\sinh \lambda n}{n \sinh \lambda}}. \quad (59)$$

The introduced deformation of two harmonic oscillators implies the interaction of these oscillators which exists due to q–nonlinearity. Thus we have not only selfinteraction of the different modes but the mutual influence of the motion of one oscillator onto the other.
DEFORMED WAVE EQUATION

Now we consider the case of the system with infinite number of degrees of freedom. An example of such a system is the wave equation with the constant parameter which is wave velocity. This constant parameter we take to be unity. Thus we start from the wave equation of the form

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \varphi(x, t) = 0. \tag{60}
\]

In order to clarify the deformation procedure we represent this equation as system of equations for decoupled oscillators. To do this we rewrite this equation in momentum representation

\[
\ddot{\varphi}(k, t) + k^2 \varphi(k, t) = 0 \tag{61}
\]

where the complex Fourier amplitude

\[
\varphi(k, t) = \frac{1}{2\pi} \int \varphi(x, t) \exp(-ikx) \, dx, \tag{62}
\]

plays the role of new coordinate. Since \( \varphi(x, t) = \varphi^*(x, t) \) we have \( \varphi(k, t) = \varphi^*(-k, t) \). Eq. (61) describes a two–dimensional oscillator with equal frequencies for both modes labelled by \( k \) and \( -k \). Writing the Eq. (61) in the form

\[
\dot{\varphi}(k, t) = \pi(k, t), \\
\dot{\pi}(k, t) = -k^2 \varphi(k, t) \tag{63}
\]

we have the equations of the form (38) in which frequency \( \omega^2 = k^2 \), \( p \to \pi(k, t) \) and \( x \to \varphi(k, t) \). Due to this we can deform this linear system taking the integral of the motion

\[
\mu = \int dk \left\{ \frac{1}{2|k|} [k^2 |\varphi|^2(k, t) + |F_k|^2] \right\} \tag{64}
\]

in which the function \( F_k \) playing the role of complex momentum of \( k \)–field mode is solution to the infinite system of equations

\[
\dot{\varphi}(k, t) = F_k f_q \left\{ \int dk' \frac{1}{2|k'|} [k'^2 |\varphi|^2(k', t) + |F_{k'}|^2] \right\}. \tag{65}
\]
The function $f_q$ is given by the Eq. (44). Then the parameter $\mu$ plays the role of the initial number of vibrations corresponding to given Cauchy initial conditions.

Thus we made the deformation by method used in the previous Section for two–dimensional oscillator introducing the interaction of modes through the parameter $\mu$. The deformed wave equation may be written as

$$\ddot{\varphi}(x, t) = - \int k^2 f^2_q(\mu) \varphi(k, t) e^{ikx} dk.$$  \hspace{1cm} (66)

This equation may be rewritten in the form for which only scalar field in time–space coordinate $\varphi(x, t)$ is present. For this we replace in the integrand the Fourier components $\varphi(k, t)$ by its expression in terms of $\varphi(x, t)$ as well as in the integral of motion $\mu$ which, in principle, may depend on the momentum vector $k$ through the dependence on the Fourier components of the initial field $\varphi(x, 0)$ and $\dot{\varphi}(x, 0)$. Thus we have the deformed wave equation

$$\ddot{\varphi}(x, t) = - \frac{1}{2\pi} \int \int k^2 \exp\{i k(x - x')\} [\varphi(x', t) f^2_q(\mu)] dk dx', \hspace{1cm} (67)$$

We have pointed out here that the integral of motion $\mu$ is the functional of the field $\varphi(x, t)$ of the special form. Being the integral of motion it depends only on initial values of field and field velocities.

Since the parameter $\mu$ is chosen as the common integral of motion for all field oscillators it does not depend on wave vector $k$, and the function $f^2_q(\mu)$ can be considered as a common factor. Then the wave equation may be rewritten in the form

$$\ddot{\varphi}(x, t) = f^2_q(\mu) \frac{\partial^2}{\partial x^2} \varphi(x, t). \hspace{1cm} (68)$$

We have the differential–functional equation which looks like usual wave equation with the wave velocity $f_q(\mu)$ which is constant of the motion. Thus the procedure of deformation yields us the nonlinear equation for which the velocity of wave propagation depends on the initial configuration of the field and its time derivative. This equation may be appropriate to describe a field behaviour in a media with strong nonlinear response of its properties to the presence of the quanta of the field.

Now we will obtain the solutions to nonlinear deformed wave equation (68). The parameter $\mu$ behaves as constant for any choice of the initial
conditions \( \varphi(x, t = 0), \dot{\varphi}(x, t = 0) \). Due to this the solution to the nonlinear mode–vibration equation (61) is

\[
\varphi(k, t) = \frac{1}{2} \varphi(k, 0) \{ \exp[i|k|f_q(\mu)]t + \exp[-i|k|f_q(\mu)]t \} \\
+ \frac{1}{2i} \dot{\varphi}(k, 0) \{ \exp[i|k|f_q(\mu)]t - \exp[-i|k|f_q(\mu)]t \} \frac{1}{i|k|f_q(\mu)},
\]

(69)

and

\[
\varphi(k, 0) = \frac{1}{2\pi} \int \varphi(x, 0) \exp(-ikx) \, dx, \\
\dot{\varphi}(k, 0) = \frac{1}{2\pi} \int \dot{\varphi}(x, 0) \exp(-ikx) \, dx.
\]

(70)

It is multimode generalization of the solution (54) of the one–mode nonlinear oscillator. Thus given initial condition \( \varphi(x, 0), \dot{\varphi}(x, 0) \) implies that \( \varphi(k, 0) \) and \( \dot{\varphi}(k, 0) \), and \( \mu \) are given, too. In terms of these values we have the \( k \)-th mode solution \( \varphi(k, t) \) and the solution to nonlinear q–deformed wave equation (68) are given by Eqs. (62), (69). It is easy to prove that the q–deformed wave equation (68) has the soliton–like solutions

\[
\varphi_\pm(x, t) = \Phi(x \pm f_q(\mu)t)
\]

(71)

where \( \Phi \) is an arbitrary function. In fact, discussed q–deformation implies the existence of nonlinear interaction among the modes. The generalization to the case of three space coordinates may be performed following the same reparametrization procedure. The q-deformed Klein–Gordon equation is considered by this method in [33]. From the point of view of deformed Poincare symmetry group the relativistic equations are discussed in [34], [35].

**QUANTUM q–OSCILLATOR**

To make more clear the relation of the suggested approach to classical equations of motion with standard quantum q–oscillator formalism of [3] we review the description of the oscillator given in [1]. Let us introduce the usual creation and annihilation oscillator operators \( a \) and \( a^\dagger \) obeying bosonic commutation relations

\[
[a, a^\dagger] = 1.
\]

(72)
Below we assume the classical dynamical variables to which $a$ and $a^\dagger$ correspond to oscillate with a frequency $\omega = 1$. It is known that the operators $a$, $a^\dagger$, 1 form the Lie algebra of Heisenberg–Weyl group. So, the linear harmonic oscillator may be connected with the generators of pure Heisenberg–Weyl Lie group. In view of the commutation relation (72) the usual scheme for generating the states of the harmonic oscillator is based on the properties of the Hermitean number operator $\hat{n} = a^\dagger a$

$$[a, \hat{n}] = a, \quad [a^\dagger, \hat{n}] = -a^\dagger.$$  

Thus constructing the vacuum state $|0\rangle$ obeying the equation

$$a|0\rangle = 0,$$

and the excited states

$$|n\rangle = \frac{a^\dagger^n}{\sqrt{n!}}|0\rangle$$

which are eigenstates of the number operator $\hat{n}$

$$\hat{n}|n\rangle = n|n\rangle, \ n \in \mathbb{Z}^+$$

the matrix representation of the operators $a$ and $a^\dagger$ in the basis (75) have the known expressions

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & \ldots \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ \sqrt{1} & 0 & 0 & \ldots \\ 0 & \sqrt{2} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

while the number operator $\hat{n}$ is described by the matrix

$$\hat{n} = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 2 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}.$$
The Hamiltonian for such a system is defined as

$$H = \frac{a^a a + a a^a}{2}.$$  \hfill (79)

The q–oscillators may be introduced by generalizing the matrices (77) and (78) with the help of the q–integer numbers $n_q$,

$$n_q = \frac{\sinh n \lambda}{\sinh \lambda}, \quad q = e^\lambda.$$ \hfill (80)

Here $\lambda$ and $q$ are dimensionless $c$-numbers, which appear at this purely mathematical level. When $\lambda = 0$, $q = 1$ and the q–integer $n_q$ coincides with $n$. Then, replacing the integers in (77) and (78) by q–integers we obtain matrices which define the annihilation and creation operators of the quantum q-oscillator,

\[
\begin{align*}
a_q &= \begin{pmatrix}
0 & \sqrt{1_q} & 0 & \ldots \\
0 & 0 & \sqrt{2_q} & \ldots \\
0 & 0 & 0 & \sqrt{3_q} \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}, \\
a^a_q &= \begin{pmatrix}
0 & 0 & 0 & \ldots \\
\sqrt{1_q} & 0 & 0 & \ldots \\
0 & \sqrt{2_q} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}, \\
\hat{n}_q &= \begin{pmatrix}
0 & 0 & 0 & \ldots \\
0 & 1_q & 0 & \ldots \\
0 & 0 & 2_q & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}, \quad (81)
\end{align*}
\]

since the action of $\hat{n}_q$ on eigenstates $|n>$ is given by

$$\hat{n}_q |n> = \frac{\sinh n \lambda}{\sinh \lambda} |n>.$$ \hfill (82)

The above matrices obey the commutation relation

$$[a_q, \hat{n}] = a_q, \quad [a^a_q, \hat{n}] = -a^a_q.$$ \hfill (83)
but the commutation relations of the operators $a_q$ and $a_q^\dagger$ do not coincide with the boson commutation relations. Eq. (72) is replaced by

$$[a_q, a_q^\dagger] = F(\hat{n})$$

(84)

where the function $F(\hat{n})$ has the form

$$F(\hat{n}) = \frac{\sinh \lambda (\hat{n} + 1) - \sinh \lambda \hat{n}}{\sinh \lambda}.$$  

(85)

For $\lambda = 0$ (84) reduces to (72). In addition to the above commutation relation there exists the reordering relation

$$a_q a_q^\dagger - qa_q^\dagger a_q = q^{-\hat{n}}$$

(86)

which usually is taken as the definition of q–oscillators.

It is worthy noting that the operators $a_q$ and $a_q^\dagger$ can be expressed in terms of the operators $a$ and $a^\dagger$ (see, for example, [1])

$$a_q = af(\hat{n}), \quad a_q^\dagger = f(\hat{n})a^\dagger$$

(87)

where

$$f(\hat{n}) = \sqrt{\frac{\hat{n}q}{\hat{n}}}.$$  

(88)

The comparison of formulae (45) and (87) shows the complete analogy of the quantum q–oscillator and the classical q–oscillator discussed in previous sections. We have also

$$\hat{n}_q = a_q^\dagger a_q$$

(89)

and

$$[a_q, \hat{n}_q] = F(\hat{n})a_q, \quad [a_q^\dagger, \hat{n}_q] = -a_q F(\hat{n}).$$

(90)

In the Schrödinger representation the evolution operator of the harmonic oscillator

$$U(t) = \exp \left[ -i\omega \frac{(a^\dagger a + aa^\dagger)}{2} t \right]$$

(91)

gives the possibility to find out explicitly linear integrals of motion which depend on time

$$A(t) = U(t)aU^{-1}(t) = e^{i\omega t}a, \quad A^\dagger(t) = U(t)a^\dagger U^{-1}(t) = e^{-i\omega t}a^\dagger.$$  

(92)
The matrices of the integrals of motion (92) in Fock basis may be obtained from the equations

\[
A(t)|n\rangle = e^{i\omega t}\sqrt{n}|n-1\rangle,
\]
\[
A^\dagger(t)|n\rangle = e^{-i\omega t}\sqrt{n+1}|n+1\rangle.
\]

Let us now introduce the Hamiltonian

\[
\hat{H} = \omega \frac{a_q a_q^\dagger + a_q^\dagger a_q}{2}
\]

for which the evolution operator takes the form

\[
U_q(t) = \exp\left[-i\omega t\left(a_q a_q^\dagger + a_q^\dagger a_q\right)\right].
\]

We have for the integrals of motion

\[
A_q(t) = U_q(t)a_q U_q^{-1}(t), \quad A_q^\dagger(t) = U_q(t)a_q^\dagger U_q^{-1}(t)
\]

the following explicit matrix expressions

\[
A_q(t) = \begin{pmatrix}
0 & \sqrt{q}e^{i(1_q-0_q)\omega t} & 0 & \cdots \\
0 & 0 & \sqrt{2q}e^{i(2_q-1_q)\omega t} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

\[
A_q^\dagger(t) = \begin{pmatrix}
\sqrt{q}e^{-i(1_q-0_q)\omega t} & 0 & 0 & \cdots \\
0 & 0 & \sqrt{2q}e^{-i(2_q-1_q)\omega t} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}.
\]

These operators are the generalizations of the linear integrals of motion (92) to the case of nonlinear Hamiltonian. This result is a generalization for q–oscillators of such integrals of motion for usual and parametric quantum oscillator which have been found in \[36\] and discussed for constructing coherent states in \[37\] and \[38\].

ANALOGY OF CLASSICAL AND QUANTUM DEFORMATIONS
By using the example of the oscillator we clarify now the analogy which exists in the suggested approach to deformation of classical systems and quantum ones. Now we recall what has been done in the previous Section for q–deformed quantum oscillator to make clear the connection with the procedure of deforming the classical oscillator of first sections.

Equations of motion for the harmonic oscillator amplitude $a$ we rewrite for another operator

$$ A = h(a^{\dagger}a)a $$

where $h$ is a real function, and its hermite conjugate $A^{\dagger} = a^{\dagger} h(a^{\dagger}a)$ in the same form

$$ \dot{A} = -i\omega A, \quad \dot{A}^{\dagger} = i\omega A^{\dagger}. $$

We have in our Hilbert space the vacuum state $\Psi_0$ which satisfies

$$ a\Psi_0 = 0, \quad A\Psi_0 = 0. $$

Thus we could construct two bases in the vector space. One is the standard basis

$$ \Psi_n = \frac{a^{\dagger n}}{\sqrt{n!}}\Psi_0 $$

which is orthonormal in standard scalar product

$$ \langle \Psi_n | \Psi_m \rangle = \delta_{nm}. $$

Another basis is constructed using the operator $A^{\dagger}$

$$ \tilde{\Psi}_n = \frac{(A^{\dagger})^n}{\sqrt{n!}}\Psi_0. $$

We define new scalar product in the same vector space which is given by

$$ \langle \tilde{\Psi}_n | \tilde{\Psi}_m \rangle = \delta_{nm}. $$

The adjoint with respect to this new scalar product need not coincide with the old one. We can define the operators

$$ b^{\dagger}\tilde{\Psi}_n = \sqrt{n+1}\tilde{\Psi}_{n+1}, \quad b\tilde{\Psi}_n = \sqrt{n}\tilde{\Psi}_{n-1} $$

where $^{\dagger}$ means the adjoint in the new scalar product. These operators satisfy the commutation relations $[b, b^{\dagger}] = 1$. Taking the Hamiltonian $H = \omega b^{\dagger}b$ we
have for the operators $b, b^*$ the equation of motion of the harmonic oscillator. Thus for one and the same vector space we have possibility to introduce two Hilbert space structures. As for the dynamics we have, like in the classical case, two different descriptions. Very much as we did for the classical case we can use the new Hamiltonian and the old commutator relations to get a "deformed" dynamics. As for the partition function, similarly to the classical case, we can use the trace defined via the two different scalar products to get the same result, i. e. the partition function depends only on the dynamics and not on the particular Hamiltonian description we use. Either we change the Hamiltonian and for the old scalar product we obtain new dynamics. Or changing Hamiltonian and simultaneously the scalar product we obtain the same dynamics. For partition function in such case we have the same value that was for nondeformed oscillator.

DEFORMED PLANCK DISTRIBUTION

In this Section we will discuss what physical consequences may be found if the considered $q$–nonlinearity influences the vibrations of the real field mode oscillators like, for example, electromagnetic fields ones or the oscillations of the nuclei in polyatomic molecules. First of all this nonlinearity changes the specific heat behaviour. To show this we have to find the partition function for a single $q$–oscillator corresponding to the Hamiltonian $H = \hat{n}_q$

$$Z(T) = \sum_{n=0}^{\infty} \exp\left(-\beta n_q\right)$$  \hspace{1cm} (98)

where the variable $\beta$ is the function of the temperature $T^{-1}$. The evaluation of the quantum partition function of the $q$–oscillator yields for the specific heat that it decreases for $T \to \infty$ as

$$C \propto \frac{1}{\ln T}.$$  \hspace{1cm} (99)

Thus the behaviour of the specific heat of the $q$–oscillator found in [1] for $\lambda \ll 1$ is different from the behaviour of the usual oscillator in the high temperature limit. This property may serve for an experimental check of the existence of vibrational nonlinearity of the $q$-oscillator fields.
q–Deformed Bose distribution can be obtained by the same method starting from the Hamiltonian $H = \frac{1}{2}\{a_q^+, a_q\}$, and one obtains

$$< n > = \bar{n}_0 - \beta \frac{\lambda^2}{6} \left[ \frac{1}{2}((\bar{n}^2)_0 - (\bar{n})^2_0) + \frac{3}{2}((\bar{n}^3)_0 - \bar{n}_0(\bar{n}^2)_0) + (\bar{n}^4)_0 - \bar{n}_0(\bar{n}^3)_0 \right]$$

in which $\bar{n}_0$ is the usual Bose distribution function and

$$\bar{n}^k_0 = 2 \sinh \beta \sum_{n=0}^{\infty} n^k e^{-\frac{\beta}{2}(n+1/2)}.$$  \hspace{1cm} (100)

Calculating the partition function for small q–nonlinearity parameter we have also the following q–deformed Planck distribution formula

$$< n > = \frac{1}{e^{\hbar \omega/kT} - 1} - \lambda^2 \frac{\hbar \omega e^{3\hbar \omega/kT}}{kT} + \frac{4e^{2\hbar \omega/kT} + e^{\hbar \omega/kT}}{(e^{\hbar \omega/kT} - 1)^4}.$$ \hspace{1cm} (102)

It means that q–nonlinearity deforms the black body radiation formula [4].

One can write down the high and low temperature approximations for the deformed Planck distribution formula [11]. For small temperature the behaviour of the deformed Planck distribution differs from the usual one

$$< n > - \bar{n}_0 = -\lambda^2 \frac{\hbar \omega}{kT} e^{-\hbar \omega/kT}.$$ \hspace{1cm} (103)

For the high temperature the nonlinear correction to the usual Planck distribution also depends on temperature

$$< n > - \bar{n}_0 = -6\lambda^2 \left( \frac{\hbar \omega}{kT} \right)^{-3}.$$ \hspace{1cm} (104)

As it was seen, the discussed q-nonlinearity produces a correction to Planck distribution formula and also this may be subjected to an experimental test.

As it was suggested in [2] the q–nonlinearity of the field vibrations produces blue shift effect which is the effect of the frequency increase with the field intensity. For small nonlinearity parameter $\lambda$ and for large quantity of photons $n$ in a given mode the relative shift of the light frequency is

$$\frac{\delta \omega}{\omega} = \frac{\lambda^2}{2} \left( n - \frac{1}{3} \right).$$
This phenomenon of possible existence of the q–nonlinearity may be essential for the models of the early stage of the Universe.

Another possible phenomenon related to the q–nonlinearity was considered in [12] where it was shown that if one deforms the electrostatics equation using the method of deformed creation and annihilation operators the form-factor of a point charge appears due to q–nonlinearity.

**NONLINEAR KLEIN–GORDON EQUATION**

To demonstrate how the q–nonlinearity may appear in Klein–Gordon equation we start from the consideration of usual Klein–Gordon equation with mass equal to zero ( \( c = 1 \))

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta \right) \phi(x, t) = 0. \tag{105}
\]

Let us take the plane wave solutions of the equation, i.e., we represent the field \( \phi(x, t) \) in the form

\[
\phi(x, t) = \int \phi(k, t) \exp(ikx) \, dk, \tag{106}
\]

where Fourier amplitude

\[
\phi(k, t) = \frac{1}{(2\pi)^3} \int \phi(x, t) \exp(-ikx) \, dx, \tag{107}
\]

plays the role of new coordinate. It satisfies the integral equation

\[
\int \dddot{\phi}(k, t) \exp(ikx) \, dk = \int (-k^2) \phi(k, t) \exp(ikx) \, dk. \tag{108}
\]

This integral equation is equivalent to the differential one

\[
\dddot{\phi}(k, t) + k^2 \phi(k, t) = 0, \tag{109}
\]

which is an infinite system of decoupled oscillators with frequencies \( \omega^2 = k^2 \).

According to suggested procedure we replace this equation by

\[
\dddot{\phi}(k, t) + k^2 f_q^2(\mu) \phi(k, t) = 0. \tag{110}
\]
Here the new frequency of \( k \)-th mode appeared
\[
\omega^2 = k^2 f^2_q(\mu),
\] (111)
where the choice of parameter \( \mu \) determines the deformation. According to our ideology it is possible to take it to be an integral of motion of the Klein–Gordon equation. There exists a common integral of motion which is the full number of the scalar field quanta, and we take it to be equal \( \mu \). As well as for the wave equation case the parameter \( \mu \) behaves as constant for any choice of the initial conditions \( \varphi(x, t = 0) \), \( \dot{\varphi}(x, t = 0) \). Due to this the solution to the nonlinear mode–vibration equation is the following
\[
\varphi(k, t) = \frac{1}{2} \varphi(k, 0) \{\exp[i|k|f_q(\mu)t] + \exp[-i|k|f_q(\mu)t]\}
\]
\[
+ \frac{1}{2i} \dot{\varphi}(k, 0) \{\exp[i|k|f_q(\mu)t] - \exp[-i|k|f_q(\mu)t]\} \frac{1}{i|k|f_q(\mu)}
\]
(112)
Here
\[
\varphi(k, 0) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \varphi(x, 0) \exp(-ikx) \, dx,
\]
\[
\dot{\varphi}(k, 0) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \dot{\varphi}(x, 0) \exp(-ikx) \, dx.
\] (113)
Thus the given initial conditions \( \varphi(x, 0) \), \( \dot{\varphi}(x, 0) \) determine also \( \varphi(k, 0) \), \( \dot{\varphi}(k, 0) \), and \( \mu \). In terms of these values we have the \( k \)-th mode solution \( \varphi(k, t) \) and the solution to nonlinear \( q \)-deformed Klein–Gordon equation
\[
\left( \frac{\partial^2}{\partial t^2} - f^2_q(\mu) \Delta \right) \varphi(x, t) = 0.
\] (114)
The constant of motion \( f_q(\mu) \) plays the role of signal velocity.

**q–DEFORMED ELECTRODYNAMICS**

We will consider the system of usual linear Maxwell equations in vacuum for the field \( E(x, t) \), \( H(x, t) \) expressed in terms of the potentials \( A(x, t) \), \( \varphi(x, t) \)
\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} - \frac{\partial \varphi}{\partial x},
\]
\[
H = \text{rot } A,
\] (115)
The suggested approach for scalar field of previous Section may be applied if one reexpresses the fields \( \varphi(x, t) \), \( A(x, t) \) in Fourier basis

\[
\varphi(x, t) = \int \varphi(k, t) \exp(ikx) \, dk, \\
A(x, t) = \int A(k, t) \exp(ikx) \, dk.
\]  

Then we have

\[
\ddot{\varphi}(k, t) + k^2 \varphi(k, t) = 0, \\
\ddot{A}(k, t) + k^2 A(k, t) = 0
\]  

(119) together with gauge constraint

\[
\dot{\varphi}(k, t) + i k A(k, t) = 0.
\]  

(120) The approach is based on the taking into account that the equations describing the linear forced oscillators of the electromagnetic field are deformed due to dependence of the frequency of the oscillations on the energy of the electromagnetic vibrations. As in the case of the scalar field we will introduce the total "number of quanta" of the deformed electromagnetic field \( \mu \) which is integral of motion. Then in this case we have the deformed nonlinear equations of vibrations

\[
\ddot{\varphi}(k, t) + f_q^2(\mu)k^2 \varphi(k, t) = 0, \\
\ddot{A}(k, t) + f_q^2(\mu)k^2 A(k, t) = 0
\]  

(121) The sense of the function \( f_q(\mu) \) is the signal velocity. Due to this we have to reparametrize the gauge condition to become

\[
f_q^{-1}(\mu)\varphi(k, t) + ikA(k, t) = 0.
\]  

(122)
After this we could reconstruct Maxwell equations in space–time analogously to the wave equation case. It should be noted that parameter $\mu$ in the Klein–Gordon and Maxwell equations is given by Eq. (64). The limit of electrostatics in the suggested type of $q$–deformation coincides with the usual electrostatics described by the Laplace equation (compare with [12]). It would be interesting to take into account interaction of the deformed electromagnetic field with the sources, but it will be taken up elsewhere.

References

[1] V. I. Man’ko, G. Marmo, S. Solimeno, and F. Zaccaria, Int. J. Mod. Phys. A 8, 3577 (1993).

[2] V. I. Man’ko, G. Marmo, S. Solimeno, and F. Zaccaria, Phys. Lett. A 176, 173 (1993).

[3] L. C. Biedenharn, J. Phys. A 22, L873 (1989).

[4] A. J. Macfarlane, J. Phys. A 22, 4581 (1989).

[5] V. G. Drinfeld, in: Quantum Groups, Proc. Int. Conf. of Math. (MSRI, Berkeley, CA., 1986) p. 798.

[6] M. Jimbo, Int. J. Mod. Phys. A 4, 3759 (1989).

[7] S. Woronowicz, Comm. Math. Phys. 111, 615 (1987).

[8] Y. Manin, Comm. Math. Phys. 123, 163 (1989).

[9] V. I. Man’ko and G. Marmo, Mod. Phys. Lett. A 7, 3411 (1992).

[10] J. Grabowski, G. Landi, G. Marmo, and G. Vilasi, Fortschr. Phys. 42, 393 (1994).

[11] V. I. Man’ko, G. Marmo, S. Solimeno, and F. Zaccaria, ”Q–nonlinearity of Electromagnetic Field and Deformed Planck Distribution,” in: Technical Digests of EQEC’93-EQUAP’93, Firenze, September 10-13, 1993, eds. P. De Natale, R. Meucci, and S. Pelli, vol. 2 (1993).

[12] V. I. Man’ko, G. Marmo, and F. Zaccaria, Phys. Lett. A 191, 13 (1994).
[13] R. J. Glauber, Phys. Rev. Lett. 10, 84 (1963).

[14] I. A. Malkin and V. I. Man’ko, Sov. Phys. JETP 28, 527 (1969).

[15] A. Feldman and A. H. Kahn, Phys. Rev. B 1, 4584 (1970).

[16] J. M. Radcliffe, J. Phys. A 4, 313 (1971).

[17] V. I. Man’ko, G. Marmo, P. Vitale, and F. Zaccaria, Int. J. Mod. Phys. A 9, 5541 (1994).

[18] K. N. Ilinski and V. M. Uzdin, Phys. Lett. A 174, 179 (1993).

[19] S. V. Shabanov, Quantum and Classical Mechanics and q-deformed Systems, Preprint BUTP 92/24 (1992).

[20] E. Celeghini, M. Rasetti, and G. Vitiello, Phys. Rev. Lett. 66, 2056 (1991).

[21] G. Su and M. Ge, Phys. Lett. A 173, 17 (1993).

[22] P. P. Kulish and E. V. Damashinsky, J. Phys. A 23, L415 (1990).

[23] R. N. Alvarez, D. Bonatsos, and Yu. F. Smirnov, Phys. Rev. A 250, 1088 (1994).

[24] P. Shanta, S. Chaturvedi, V. Srinivasan, and R. Jagannathan, J. Phys. A 27, 6433 (1994).

[25] G. Brodimas, A Jannusis, and R. Mignani, J. Phys. A 25, L323 (1992).

[26] M. Chaichian, D. Ellinas, and P. Kulish, Phys. Rev. Lett. 65, 980 (1990).

[27] D. B. Fairlie and C. K. Zachos, Phys. Lett. B 256, 43 (1991).

[28] D. Bonatsos and C. Daskaloyannis, Phys. Lett. B 307, 100 (1993).

[29] C. Daskaloyannis, J. Phys. A 24, L789 (1991); 25, 2261 (1992).

[30] C. Quesne, Phys. Lett. A 193, 245 (1994).

[31] A. P. Polychronakos, Mod. Phys. Lett. A 5, 2335 (1990).
[32] R. Floreanini, V. P. Spiridonov, and L. Vinet, in: *Group Theoretical Methods in Physics, Proc. Inter. Colloquium, Moscow, 4-9 June, 1990*, eds. V. V. Dodonov and V. I. Man’ko, Lecture Notes in Physics **382** (Springer Verlag, 1991).

[33] V. I. Man’ko, G. Marmo, and F. Zaccaria, Phys. Lett. A, (1994).

[34] J. Lukierski, A. Nowicki, and H. Ruegg, Phys. Lett. B **293**, 344 (1992).

[35] M. Pillin, J. Math. Phys. **35**, 2804 (1994).

[36] I. A. Malkin and V. I. Man’ko, Phys. Lett. A **32**, 243 (1970).

[37] I. A. Malkin and V. I. Man’ko *Dynamical Symmetries and Coherent States of Quantum Systems* (Nauka Publishers, Moscow, 1979) [in Russian].

[38] V. V. Dodonov and V. I. Man’ko, *Invariants and Evolution of Nonstationary Quantum Systems*, *Proc. of Lebedev Physics Institute* **183**, ed. M. A. Markov (Nova Science, Commack, N. Y., 1989).