The functional dissipativity of certain systems of partial differential equations

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Abstract

In the present paper we consider the functional dissipativity of the Dirichlet problem for systems of partial differential operators of the form \( \partial_h(\mathcal{A}^h(x)\partial_k) \) \( (\mathcal{A}^h \text{ being } m \times m \text{ matrices with complex valued } L^1_{\text{loc}} \text{ entries}) \). In the particular case of the operator \( \partial_h(\mathcal{A}^h(x)\partial_k) \) (where \( \mathcal{A}^h \) are \( m \times m \) matrices) we obtain algebraic necessary and sufficient conditions. We give also three different notions of functional ellipticity and investigate the relations between them and the functional dissipativity for the operators in question.

1 Introduction

The concept of \( L^\Phi \)-dissipativity (or functional dissipativity) of a linear operator was recently introduced in [3]. In particular, if \( A \) is the scalar second order partial differential operator \( \nabla(\mathcal{A} \nabla) \), \( \mathcal{A} \) being a square matrix whose entries are complex valued \( L^\infty \text{ functions defined in a domain } \Omega \subset \mathbb{R}^n \), we say that \( A \) is \( L^\Phi \)-dissipative with respect to a given positive function \( \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) if

\[
\Re \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla(\varphi(|u|) u) \rangle \, dx \geq 0
\]

for any \( u \in \dot{H}^1(\Omega) \) such that \( \varphi(|u|) u \in \dot{H}^1(\Omega) \).

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We say that $A$ is strict $L^\Phi$-dissipative if there exists $\kappa > 0$ such that

$$\Re \int_\Omega \langle \mathcal{A} \nabla u, \nabla (\varphi(|u|) u) \rangle \, dx \geq \kappa \int_\Omega |\nabla (\sqrt{\varphi(|u|)} u)|^2 \, dx$$

for any $u \in \hat{H}^1(\Omega)$ such that $\varphi(|u|) u \in \hat{H}^1(\Omega)$.

The functional dissipativity is an extension of the concept of $L^p$-dissipativity, which is obtained taking $\varphi(t) = t^{p-2}$ ($1 < p < \infty$). In a series of papers [3, 4, 5, 7] we have studied the problem of characterizing the $L^p$-dissipativity of scalar and matrix partial differential operators. A recent survey can be found in [10]. In the monograph [6] these results are considered in the more general frame of the theory of semi-bounded operators.

Later the concept of $p$-ellipticity, which is closely related to the strict $L^p$-dissipativity, has been introduced. It has been considered in different papers by Carbonaro and Dragičević [1, 2], Dindoš and Pipher [12, 13, 14, 15], Egert [16]. It is worthwhile to remark that, if the partial differential operator has no lower order terms, the concepts of $p$-ellipticity and strict $L^p$-dissipativity coincide.

Recently Dindoš, Li and Pipher [11] have considered the $p$-ellipticity for linear systems of partial differential equations, giving three different definitions: the strong $p$-ellipticity, the integral $p$-ellipticity and the weak $p$-ellipticity, showing that strong $p$-ellipticity $\Rightarrow$ integral $p$-ellipticity $\Rightarrow$ weak $p$-ellipticity.

The aim of the present paper is to study the $L^\Phi$-dissipativity of the operator

$$Au = \partial_h (\mathcal{A}^h(x) \partial_h u)$$

where $\mathcal{A}^h(x) = \{a^h_{ij}(x)\}$ $(i, j = 1, \ldots, m)$ are matrices with complex locally integrable entries defined in a domain $\Omega \subset \mathbb{R}^n$ $(h = 1, \ldots, n)$. We give necessary and sufficient condition for the functional dissipativity of such an operator and also for its functional ellipticity, a concept which was introduced in [8] for scalar operators and generalizes the concept of $p$-ellipticity. As done in [11] for $p$-ellipticity, here we consider three kinds of functional ellipticity for the operator $A$: the strong, the integral and the weak functional ellipticity. We prove that all of them, but the strong one, are equivalent to the strict $L^\Phi$-dissipativity of $A$. We prove also that the integral functional ellipticity of this operator takes place if and only if there exists $\kappa > 0$ such that $A - \kappa \Delta$ is $L^\Phi$-dissipative. These results appear to be interesting even in the particular case of $p$-ellipticity.

The present paper is organized as follows. After some preliminaries in Section 2 we study the $L^\Phi$-dissipativity for systems of ordinary differential equations in
Section 3. The particular case of real operators is considered in Section 4, where several criteria are given in terms of eigenvalues of the coefficient matrices.

Section 5 is devoted to the $L^\Phi$-dissipativity of operator $A$. Finally, in the last Section 6 some concepts of functional ellipticity for systems are introduced and the relations between them are investigated for the operators considered in the paper.

2 Preliminaries

The positive function $\varphi$ is required to satisfy the following conditions

(i) $\varphi \in C^1((0, +\infty))$;

(ii) $(s \varphi(s))' > 0$ for any $s > 0$;

(iii) the range of the strictly increasing function $s \varphi(s)$ is $(0, +\infty)$;

(iv) there exist two positive constants $C_1, C_2$ and a real number $r > -1$ such that

$$C_1 s^r \leq (s \varphi(s))' \leq C_2 s^r, \quad s \in (0, s_0)$$

for a certain $s_0 > 0$. If $r = 0$ we require more restrictive conditions: there exists the finite limit $\lim_{s \to 0^+} \varphi(s) = \varphi_+(0) > 0$ and $\lim_{s \to 0^+} s \varphi'(s) = 0$.

(v) There exists $s_1 > s_0$ such that

$$\varphi'(s) \geq 0 \text{ or } \varphi'(s) \leq 0 \quad \forall s \geq s_1.$$  

We note that from condition (iv) it follows that, for any $r > -1$,

$$\varphi(s) \simeq s^r, \quad s \in (0, s_0).$$

Let us denote by $t \psi(t)$ the inverse function of $s \varphi(s)$. The functions

$$\Phi(s) = \int_0^s \sigma \varphi(\sigma) d\sigma, \quad \Psi(s) = \int_0^s \sigma \psi(\sigma) d\sigma$$

are conjugate Young functions. Moreover the function $\varphi$ satisfies conditions (i)-(v) if and only if the function $\psi$ satisfies the same conditions with $-r/(r + 1)$ instead of $r$ (see [8, Lemma 1]).
In the following we shall use the function \( \Lambda \) which is defined by the relation
\[
\Lambda \left( s \sqrt{\varphi(s)} \right) = -\frac{s \varphi'(s)}{s \varphi'(s) + 2 \varphi(s)}.
\]

Let us consider a general system of the form
\[
A = \partial_h (\omega^{hk}(x) \partial_k)
\]
where \( \partial_k = \partial/\partial x_k \) and \( \omega^{hk}(x) = \{a_{ij}^{hk}(x)\} \) are \( m \times m \) matrices whose elements are complex valued \( L^1_{\text{loc}} \)-functions defined in a domain \( \Omega \subset \mathbb{R}^n \) (1 \( \leq i, j \leq m, \ 1 \leq h, k \leq n \)). Here and in the sequel, we adopt the standard summation convention on repeated indices. We say that the operator (1) is \( L^\Phi \)-dissipative if
\[
\Re \int_{\Omega} \langle \omega^{hk} \partial_k u, \partial_h (\varphi(|u|) u) \rangle \ dx \geq 0
\]
for any \( u \in [\hat{C}^1(\Omega)]^m \) such that \( \varphi(|u|) u \in [\hat{C}^1(\Omega)]^m \).

We say that the operator (1) is strict \( L^\Phi \)-dissipative if there exists \( \kappa > 0 \) such that
\[
\Re \int_{\Omega} \langle \omega^{hk} \partial_k u, \partial_h (\varphi(|u|) u) \rangle \ dx \geq \kappa \int_{\Omega} |\nabla(\sqrt{\varphi(|u|)} u)|^2 \ dx
\]
for any \( u \in [\hat{C}^1(\Omega)]^m \) such that \( \varphi(|u|) u \in [\hat{C}^1(\Omega)]^m \).

**Remark 1** The coefficients of the operator (1) are supposed to be \( L^1_{\text{loc}} \)-functions and not \( L^\infty \)-functions, as in [9]. This is the reason why in definitions (2) and (3) we consider the space \([\hat{C}^1(\Omega)]^m\) instead of \([H^1(\Omega)]^m\), as done in [9].

**Remark 2** If \( r \geq 0 \) in condition (iv) \( u \in [\hat{C}^1(\Omega)]^m \) implies \( \varphi(|u|) u \in [\hat{C}^1(\Omega)]^m \) and then we can say that the operator (1) is \( L^\Phi \)-dissipative if and only if (2) holds for any \( u \in [\hat{C}^1(\Omega)]^m \). If \( r < 0 \), setting \( w = \varphi(|u|) u \), i.e. \( u = \psi(|w|) w \), we can write condition (2) as
\[
\Re \int_{\Omega} \langle (\omega^{kh})^* \partial_k w, \partial_h (\psi(|w|) w) \rangle \ dx \geq 0
\]
for any \( w \in [\hat{C}^1(\Omega)]^m \) such that \( \psi(|w|) w \in [\hat{C}^1(\Omega)]^m \). Thanks to [8, Lemma 1], the function \( \psi \) satisfies conditions (ii)(v) with \(-r/(r+1)\) instead of \( r \). The number \(-r/(r+1)\) being greater than 0, the operator (1) is \( L^\Phi \)-dissipative if and only if (4) holds for any \( w \in [\hat{C}^1(\Omega)]^m \). The same remark applies to (3), since \( \sqrt{\psi(|w|)} w = \sqrt{\varphi(|u|)} u \) (see [8, formula (43)]).
We end this Section by proving a Lemma similar to [9, Lemma 6].

**Lemma 1** Let $\Omega$ be a domain in $\mathbb{R}^n$. The operator (1) is $L^\Phi$-dissipative if and only if

\[
\Re \int_\Omega \left( \langle \mathcal{A}^h k \partial_x v, \partial_t v \rangle + \Lambda(|v|)|v|^{-2}(\langle \mathcal{A}^h k - (\mathcal{A}^h k)^* \rangle v, \partial_t v) \Re \langle v, \partial_k v \rangle \\
- \Lambda^2(|v|)|v|^{-4}(\langle \mathcal{A}^h k v, v \rangle \Re \langle v, \partial_k v \rangle \Re \langle v, \partial_t v \rangle \right) dx \geq 0
\]

for any $v \in [\tilde{C}^1(\Omega)]^m$. Here and in the sequel the integrand is extended by zero on the set where $v$ vanishes.

**Proof. Sufficiency.** First suppose $r \geq 0$ in condition (iv). Take $u \in [\tilde{C}^1(\Omega)]^m$ and define $v = \sqrt{\varphi(|u|)} u$. Also $v$ belongs to $[\tilde{C}^1(\Omega)]^m$ and, reasoning as in the proof of [9, Lemma 6], we find the identity

\[
\Re \langle \mathcal{A}^h k \partial_x u, \partial_t (\varphi(|u|) u) \rangle = \Re (\langle \mathcal{A}^h k \partial_x v, \partial_t v \rangle \\
+ \Lambda(|v|)|v|^{-2}(\langle \mathcal{A}^h k - (\mathcal{A}^h k)^* \rangle v, \partial_t v) \Re \langle v, \partial_k v \rangle \\
- \Lambda^2(|v|)|v|^{-4}(\langle \mathcal{A}^h k v, v \rangle \Re \langle v, \partial_k v \rangle \Re \langle v, \partial_t v \rangle \right)
\]

on the set $\{x \in \Omega \mid u(x) \neq 0\} = \{x \in \Omega \mid v(x) \neq 0\}$. Inequality (5) implies (2) and the sufficiency is proved when $r \geq 0$.

If $-1 < r < 0$, recalling Remark 2 we can say that operator (1) is $L^\Phi$-dissipative if and only if (4) holds for any $w \in [\tilde{C}^1(\Omega)]^m$. Therefore what we have already proved for $r \geq 0$ shows that (4) holds if

\[
\Re \int_\Omega \left( \langle \mathcal{A}^h k \partial_x v, \partial_t v \rangle + \tilde{\Lambda}(|v|)|v|^{-2}(\langle \mathcal{A}^h k - \mathcal{A}^h k \rangle v, \partial_t v) \Re \langle v, \partial_k v \rangle \\
- \tilde{\Lambda}^2(|v|)|v|^{-4}(\langle \mathcal{A}^h k v, v \rangle \Re \langle v, \partial_k v \rangle \Re \langle v, \partial_t v \rangle \right) dx \geq 0
\]

for any $v \in [\tilde{C}^1(\Omega)]^m$, where $\tilde{\Lambda}$ is defined by the relation

\[
\tilde{\Lambda} \left( s \sqrt{\psi(s)} \right) = -\frac{s \psi'(s)}{s \psi'(s) + 2 \psi(s)}.
\]

Since $\tilde{\Lambda}(|v|) = -\Lambda(|v|)$ (see [8, Lemma 2]), condition (6) coincides with (5) and the sufficiency is proved also for $-1 < r < 0$.

**Necessity.** If $r \geq 0$ we can repeat the first part of the proof of Necessity in [9, Lemma 6] to show that (2) implies (5) for any $v \in [\tilde{C}^1(\Omega)]^m$. If $r < 0$ we rewrite
(2) as (4). As before, what we have proved for $r \geq 0$ shows that (4) implies (6) for any $v \in [\mathring{C}^1(\Omega)]^m$ and this concludes the proof.

In the same way (see also [9, Lemma 7]) one can prove the next result

**Lemma 2** Let $\Omega$ be a domain in $\mathbb{R}^n$. The operator (1) is strict $L^\Phi$-dissipative if and only if there exists $\kappa > 0$ such that

$$\Re \int_{\Omega} \left( \langle \mathcal{A}^{hk} \partial_h v, \partial_h v \rangle + \Lambda(|v|) |v|^{-2} \langle (\mathcal{A}^{hk} - (\mathcal{A}^{kh})^*) v, \partial_h v \rangle \Re \langle v, \partial_h v \rangle \right. - \left. \Lambda^2(|v|) |v|^{-4} \langle \mathcal{A}^{hk} v, v \rangle \Re \langle v, \partial_h v \rangle \Re \langle v, \partial_h v \rangle \right) dx \geq \kappa \int_{\Omega} |\nabla v|^2 dx$$

for any $v \in [\mathring{C}^1(\Omega)]^m$.

### 3 Functional dissipativity for systems of ordinary differential equations

In this Section we are going to consider the operator $A$ defined as

$$Au = (\mathcal{A}(x) u')'$$

where $\mathcal{A}(x) = \{a_{ij}(x)\} (i, j = 1, \ldots, m)$ is a matrix with complex locally integrable entries defined in the bounded or unbounded interval $(a, b) \subset \mathbb{R}$.

**Lemma 3** The operator $A$ is $L^\Phi$-dissipative if and only if

$$\int_a^b \left( \Re \langle \mathcal{A} v', v' \rangle - \Lambda^2(|v|) |v|^{-4} \Re \langle \mathcal{A} v, v \rangle (\Re \langle v, v' \rangle)^2 \right. + \left. \Lambda(|v|) |v|^{-2} \Re \langle (\mathcal{A} v', v') - \langle \mathcal{A} v', v \rangle, \Re \langle v, v' \rangle \rangle \Re \langle v, v' \rangle \right) dx \geq 0,$$

for any $v \in [\mathring{C}^1((a, b))]^m$.

**Proof.** This is just a particular case of Lemma [1] \qed

As in [9], from now on we require also the following condition on the function $\varphi$:
(vi) the function
\[ |s \varphi'(s)/\varphi(s)| \]

is not decreasing.

This condition implies that the function \( \Lambda^2(t) \) is not decreasing on \((0, +\infty)\) (see [9, Lemma 8]).

The next Theorem provides an algebraic necessary and sufficient condition for the \( L^\Phi \)-dissipative of operator (8).

**Theorem 1** The operator \( A \) is \( L^\Phi \)-dissipative if and only if

\[
\begin{align*}
\Re\langle A(x)\lambda, \lambda \rangle &- \Lambda_\infty^2 \Re\langle A(x)\omega, \omega \rangle(\Re\lambda, \omega)^2 \\
+ \Lambda_\infty \Re\langle A(x)\omega, \lambda \rangle - \langle A(x)\lambda, \omega \rangle \Re\lambda, \omega &\geq 0
\end{align*}
\]

for almost every \( x \in (a, b) \) and for any \( \lambda, \omega \in \mathbb{C}^m, |\omega| = 1 \).

**Proof.** Necessity. Assume for the moment that the coefficients \( a_{ij} \) are constant and \((a, b) = \mathbb{R}\).

Let us fix \( \lambda \) and \( \omega \) in \( \mathbb{C}^m \), with \( |\omega| = 1 \), and choose \( v(x) = \eta(x/R) w(x) \) where

\[
w_j(x) = \begin{cases} 
\mu \omega_j & \text{if } x < 0' \\
\mu \omega_j + x^2(3 - 2x)\lambda_j & \text{if } 0 \leq x \leq 1, \\
\mu \omega_j + \lambda_j & \text{if } x > 1,
\end{cases}
\]

where \( \mu, R \in \mathbb{R}^+, \eta \in \mathcal{C}_\infty(\mathbb{R}), \text{spt}\eta \subset [-1, 1] \) and \( \eta(x) = 1 \) if \( |x| \leq 1/2 \).

Put this \( v \) in (9). By repeating the arguments used in [4, pp.250–251] and observing that

\[
\Lambda(|w(x)|) = \begin{cases} 
\Lambda(\mu) & \text{if } x < 0, \\
\Lambda(|\mu\omega + x^2(3 - 2x)\lambda|) & \text{if } 0 \leq x \leq 1, \\
\Lambda(|\mu\omega + \lambda|) & \text{if } x > 1,
\end{cases}
\]

implies \( \lim_{\mu \to \infty} \Lambda(|w(x)|) = \Lambda_\infty \), we find

\[
36 \int_0^1 \left( \Re\langle A\lambda, \lambda \rangle - \Lambda_\infty^2 \Re\langle A\omega, \omega \rangle(\Re\lambda, \omega)^2 \\
+ \Lambda_\infty \Re\langle A\omega, \lambda \rangle - \langle A\lambda, \omega \rangle \Re\lambda, \omega \right) x^2(1 - x)^2 dx \geq 0
\]
and (10) is proved.

If \( \{a_{hk}\} \) are defined on \((a, b)\) and are not necessarily constant, consider

\[
v(x) = \psi(\frac{x - x_0}{\varepsilon})
\]

where \( x_0 \) is a fixed point in \((a, b)\), \( \psi \in [\tilde{C}^1((-1, 1))]^m \) and \( 0 < \varepsilon < \min\{x_0 - a, b - x_0\} \). Putting \( v \) in (9) we get

\[
\int_{\mathbb{R}} \left( \mathcal{R} \mathcal{A}(x_0 + \varepsilon y)\psi', \psi' \right) - \Lambda_\infty^2 |\psi|^{-4} \mathcal{R} \mathcal{A}(x_0 + \varepsilon y)\psi \left( \mathcal{R} \mathcal{A}(x_0 + \varepsilon y)\psi \right)^2
\]

\[
+ \Lambda_\infty |\psi|^{-2} \mathcal{R} \left( \mathcal{A}(x_0 + \varepsilon y)\psi, \psi' \right) - \mathcal{A}(x_0 + \varepsilon y)\psi' \right) \mathcal{R} \mathcal{A}(x_0 + \varepsilon y)\psi \right)^2 \right) \right) dy \geq 0.
\]

Letting \( \varepsilon \to 0^+ \) we find for almost every \( x_0 \)

\[
\int_{\mathbb{R}} \left( \mathcal{R} \mathcal{A}(x_0)\psi', \psi' \right) - \Lambda_\infty^2 |\psi|^{-4} \mathcal{R} \mathcal{A}(x_0)\psi \left( \mathcal{R} \mathcal{A}(x_0)\psi \right)^2
\]

\[
+ \Lambda_\infty |\psi|^{-2} \mathcal{R} \left( \mathcal{A}(x_0)\psi, \psi' \right) - \mathcal{A}(x_0)\psi' \right) \mathcal{R} \mathcal{A}(x_0)\psi \right)^2 \right) \right) dy \geq 0.
\]

What we have obtained for constant coefficients gives the result.

**Sufficiency.** We start by observing that (10) implies

\[
\mathcal{R} \mathcal{A}(x)\lambda, \lambda \rangle \geq 0
\]

for almost every \( x \in (a, b) \) and for any \( \lambda \in \mathbb{C}^m \). Indeed, let us take \( x \in (a, b) \) such that (10) holds and \( \lambda \in \mathbb{C}^m \); we can choose \( \omega \in \mathbb{C}^m, |\omega| = 1 \), such that \( \langle \lambda, \omega \rangle = 0 \), and (11) is proved.

Let us fix \( x \in (a, b) \) and \( \lambda, \omega \in \mathbb{C}^m, |\omega| = 1 \), such that (10) holds. We can write this condition as

\[
\alpha \Lambda_\infty^2 - \beta \Lambda_\infty - \gamma \leq 0,
\]

where

\[
\alpha = \mathcal{R} \mathcal{A}(x)\omega, \omega \rangle \mathcal{R} \mathcal{A}(x)\lambda, \lambda \rangle, \quad \gamma = \mathcal{R} \mathcal{A}(x)\lambda, \lambda \rangle,
\]

\[
\beta = \mathcal{R} \mathcal{A}(x)\omega, \lambda \rangle - \mathcal{R} \mathcal{A}(x)\lambda, \omega \rangle \mathcal{R} \mathcal{A}(x)\lambda, \omega \rangle.
\]

Note that, in view of (11), \( \alpha \) and \( \gamma \) are non-negative. Then, if \( \alpha > 0 \), we have \( \Lambda_1 \Lambda_2 \leq 0 \), where \( \Lambda_1 \) and \( \Lambda_2 \) are the roots of the equation \( \alpha \Lambda^2 - \beta \Lambda - \gamma = 0 \). We claim that from (12) it follows

\[
\alpha \Lambda^2(t) - \beta \Lambda(t) - \gamma \leq 0
\]

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for any \( t \geq 0 \). Indeed, suppose \( \alpha > 0 \) and \( \Lambda_1 \leq 0 \leq \Lambda_2 \). Inequality (12) means
\[
\Lambda_1 \leq \Lambda_{\infty} \leq \Lambda_2.
\]
We recall that the function \( \Lambda(t) \) is monotone and does not change sign (see [9, Lemma 8]). If \( \Lambda(t) \geq 0 \), we have
\[
0 \leq \Lambda(t) \leq \Lambda_{\infty} \leq \Lambda_2,
\]
while if \( \Lambda(t) \leq 0 \), \[
\Lambda_1 \leq \Lambda_{\infty} \leq \Lambda(t) \leq 0.
\]
In any case, we find \( \Lambda_1 \leq \Lambda(t) \leq \Lambda_2 \) for any \( t \geq 0 \) and (13) is proved under the assumption \( \alpha > 0 \).

If \( \alpha = 0 \) and \( \beta = 0 \) (13) is trivial. If \( \alpha = 0 \) and \( \beta \neq 0 \), (12) becomes \( \beta \Lambda_{\infty} + \gamma \geq 0 \). By using again the fact that \( \Lambda(t) \) does not change sign, we get \( \beta \Lambda(t) + \gamma \geq 0 \) for any \( t \geq 0 \). We have then proved that
\[
\Re(\langle x \rangle \lambda, \lambda) - \Lambda^2(t) \Re(\langle x \rangle \omega, \omega)(\Re(\lambda, \omega))^2 \\
+ \Lambda(t) \Re(\langle x \rangle \omega, \lambda) - \langle x \rangle \lambda, \omega \rangle \Re(\lambda, \omega) \geq 0
\]
(14)
for almost every \( x \in (a, b) \) and for any \( t > 0, \lambda, \omega \in \mathbb{C}^m, |\omega| = 1 \). This shows that the integrand in (9) is non-negative almost everywhere and Lemma 3 gives the result.

\[\square\]

**Remark 3** In the proof of the Sufficiency we have shown that - assuming condition (15) - inequality (10) implies that (14) holds for any \( t > 0 \). The viceversa being obvious, we have that (14) for any \( t > 0 \) and (10) are equivalent.

**Corollary 1** If the operator \( A \) is \( L^\Phi \)-dissipative, then (11) holds for almost every \( x \in (a, b) \) and for any \( \lambda \in \mathbb{C}^m \).

**Proof.** In the proof of Theorem 1 we have already seen that inequality (10) implies (11).

\[\square\]

We have also
Lemma 4  The operator $A$ is strict $L^Φ$-dissipative if and only if there exists $κ > 0$ such that
\[
\int_a^b \left( \Re(\mathcal{A}v', v') - Λ^2(|v|) |v|^4 \Re(\mathcal{A}v, v)(\Re(v, v'))^2 
+ Λ(|v|) |v|^{-2} \Re((\mathcal{A}v, v') - (\mathcal{A}v', v)) \Re(v, v') \right) dx \geq κ \int_a^b |v'|^2 dx,
\]
for any $v \in [\hat{C}^1(Ω)]^m$.

Proof. It is a particular case of Lemma 2.

Corollary 2  Suppose
\[
Λ_∞^2 = \sup_{t>0} Λ^2(t) < 1.
\]
The operator $A$ is strict $L^Φ$-dissipative if and only if there exists $κ > 0$ such that $A - κI(d^2/dx^2)$ is $L^Φ$-dissipative.

Proof. It is a particular case of [9, Corollary 1].

Following the ideas of [4] we prove

Theorem 2  Let us assume condition (15). There exists $κ > 0$ such that $A - κI(d^2/dx^2)$ is $L^Φ$-dissipative if and only if
\[
\text{ess inf}_{(x, λ, ω) ∈ (a, b) × C^m × C^m, |ω| = 1} P(x, λ, ω) > 0,
\]
where
\[
P(x, λ, ω) = \Re(\mathcal{A}(x)λ, λ) - Λ_∞^2 \Re(\mathcal{A}(x)ω, ω)(\Re(λ, ω))^2
+ Λ_∞ \Re((\mathcal{A}(x)ω, λ) - (\mathcal{A}(x)λ, ω)) \Re(λ, ω).
\]

Proof. Thanks to Theorem 1 $A - κI(d^2/dx^2)$ is $L^Φ$-dissipative if and only if
\[
P(x, λ, ω) - κ(|λ|^2 - Λ_∞^2(\Re(λ, ω))^2) \geq 0
\]
for almost every $x ∈ (a, b)$ and for any $λ, ω ∈ C^m$, $|ω| = 1$. Since
\[
|λ|^2 - Λ_∞^2(\Re(λ, ω))^2 \geq (1 - Λ_∞^2)|λ|^2 > 0
\]
for any \( \lambda \neq 0 \), there exists \( \kappa > 0 \) such that (17) holds if and only if
\[
\text{ess inf}_{(x, \lambda, \omega) \in (a, b) \times \mathbb{C}^m \times \mathbb{C}^m \atop |\lambda| = |\omega| = 1} \frac{P(x, \lambda, \omega)}{1 - \Lambda^2_\infty (\Re \langle \lambda, \omega \rangle)^2} > 0.
\] (19)

On the other hand inequality (18) leads to
\[
P(x, \lambda, \omega) \leq \frac{P(x, \lambda, \omega)}{1 - \Lambda^2_\infty (\Re \langle \lambda, \omega \rangle)^2} \leq (1 - \Lambda^2_\infty)^{-1} P(x, \lambda, \omega)
\]
for almost every \( x \in (a, b) \) and for any \( \lambda, \omega \in \mathbb{C}^m, |\lambda| = |\omega| = 1 \). This shows that (16) and (19) are equivalent and this concludes the proof.

**Corollary 3** Let us assume condition (15). The operator \( A \) is strict \( L^\Phi \)-dissipative if and only if there exists \( \kappa > 0 \) such that
\[
\Re \langle A(x)\lambda, \lambda \rangle - \Lambda^2_\infty \Re \langle A(x)\omega, \omega \rangle (\Re \langle \lambda, \omega \rangle)^2 + \Lambda_\infty \Re (\langle A(x)\omega, \lambda \rangle - \langle A(x)\lambda, \omega \rangle) \Re \langle \lambda, \omega \rangle \geq \kappa |\lambda|^2
\]
for almost every \( x \in (a, b) \) and for any \( \lambda, \omega \in \mathbb{C}^m, |\omega| = 1 \).

**Proof.** It follows immediately from Corollary 2 and Theorem 2, because (16) means that there exists \( \kappa > 0 \) such that
\[
P(x, \lambda, \omega) \geq \kappa |\lambda|^2
\]
for any \( \lambda, \omega \in \mathbb{C}^m, |\omega| = 1 \) and for almost every \( x \in \Omega \). \[ \square \]

### 4 Real coefficient operators

This section is devoted to real coefficient operators. We shall give different necessary and sufficient conditions for the functional dissipative which are expressed in terms of the eigenvalues of the matrix \( \mathcal{A} \).

**Theorem 3** Let \( \mathcal{A} \) be a real matrix \( \{a_{hk}\} \) with \( h, k = 1, \ldots, m \). Let us suppose \( \mathcal{A} = \mathcal{A}^t \) and \( \mathcal{A} > 0 \) (in the sense \( \langle \mathcal{A}(x)\xi, \xi \rangle > 0 \), for almost every \( x \in (a, b) \) and for any \( \xi \in \mathbb{R}^m \setminus \{0\} \)). The operator \( A \) is \( L^\Phi \)-dissipative if and only if
\[
\Lambda^2_\infty (\mu_1(x) + \mu_m(x))^2 \leq 4 \mu_1(x)\mu_m(x)
\]
almost everywhere, where \( \mu_1(x) \) and \( \mu_m(x) \) are the smallest and the largest eigenvalues of the matrix \( \mathcal{A}(x) \) respectively. In the particular case \( m = 2 \), this condition is equivalent to

\[
\Lambda_\infty^2 (\text{tr} \mathcal{A}(x))^2 \leq 4 \det \mathcal{A}(x)
\]

almost everywhere.

**Proof.** In view of Theorem 1 we have the \( L^\Phi \)-dissipativity of \( A \) if and only if (10) holds for almost every \( x \in (a, b) \) and for any \( \lambda, \omega \in \mathbb{C}^m, |\omega| = 1 \). Reasoning as in the proof of [4, Theorem 5, p.255], we see that in the present case this condition is equivalent to

\[
\langle \mathcal{A}(x) \xi, \xi \rangle - \Lambda_\infty^2 \langle \mathcal{A}(x) \omega, \omega \rangle (\langle \xi, \omega \rangle)^2 \geq 0
\]

for almost every \( x \in (a, b) \) and for any \( \xi, \omega \in \mathbb{R}^m, |\omega| = 1 \). As in [4, Theorem 5, p.255], this inequality is satisfied if and only if

\[
\Lambda_\infty^2 (\mu_1 \omega^2) (\mu_m^{-1} \omega_k^2) \leq 1
\]

for any \( \omega \in \mathbb{R}^m, |\omega| = 1 \), i.e., if and only if (see [4, Lemma 4, p.253])

\[
\Lambda_\infty^2 \frac{(\mu_1 + \mu_m)^2}{4 \mu_1 \mu_m} \leq 1.
\]

The result for \( m = 2 \) follows from the identities

\[
\mu_1(x) \mu_2(x) = \det \mathcal{A}(x), \quad \mu_1(x) + \mu_2(x) = \text{tr} \mathcal{A}(x).
\]

\[(20)\]

In the rest of the Section we assume condition (15).

**Corollary 4** Let \( \mathcal{A} \) be a real and symmetric matrix. Denote by \( \mu_1(x) \) and \( \mu_m(x) \) the smallest and the largest eigenvalues of \( \mathcal{A}(x) \) respectively. There exists \( \kappa > 0 \) such that \( A - \kappa I (d^2/dx^2) \) is \( L^\Phi \)-dissipative if and only if

\[
\text{ess inf}_{x \in (a, b)} \left[ \left( 1 + \sqrt{1 - \Lambda_\infty^2} \right) \mu_1(x) - \left( 1 - \sqrt{1 - \Lambda_\infty^2} \right) \mu_m(x) \right] > 0.
\]

\[(21)\]

In the particular case \( m = 2 \), condition (21) is equivalent to

\[
\text{ess inf}_{x \in (a, b)} \left[ \sqrt{1 - \Lambda_\infty^2} \text{tr} \mathcal{A}(x) - \sqrt{(\text{tr} \mathcal{A}(x))^2 - 4 \det \mathcal{A}(x)} \right] > 0.
\]

\[(22)\]
Proof. Necessity. From inequality (17) we deduce that $A - \kappa' I (d^2/dx^2)$ is $L^0$-dissipative for any $0 < \kappa' < \kappa$. Moreover, by Corollary 1, $\mathcal{A}(x) - \kappa' I > 0$ almost everywhere for any $0 < \kappa' < \kappa$. By Theorem 3 we find that

$$\Lambda_\infty^2(\mu_1(x) + \mu_m(x) - 2\kappa')^2 \leq 4 (\mu_1(x) - \kappa')(\mu_m(x) - \kappa')$$ (23)

almost everywhere. By observing that $4\mu_1\mu_m = (\mu_1 + \mu_m)^2 - (\mu_1 - \mu_m)^2$, the last inequality can be written as

$$(1 - \Lambda_\infty^2)(\mu_1(x) + \mu_m(x) - 2\kappa')^2 - (\mu_1(x) - \mu_m(x))^2 \geq 0$$ (24)

almost everywhere. Since this holds for any $\kappa' < \kappa$, we have that $\kappa$ is less than or equal to the smallest root of the left hand side of (24), i.e.

$$\kappa \leq \frac{1}{2} \left(\left(1 + 1/\sqrt{1 - \Lambda_\infty^2}\right) \mu_1(x) + \left(1 - 1/\sqrt{1 - \Lambda_\infty^2}\right) \mu_m(x)\right)$$ (25)

almost everywhere and (21) is proved.

Sufficiency. Let $\kappa'$ be such that

$$0 < \kappa' < \text{ess inf}_{x \in (a,b)} \frac{1}{2} \left(\left(1 + 1/\sqrt{1 - \Lambda_\infty^2}\right) \mu_1(x) + \left(1 - 1/\sqrt{1 - \Lambda_\infty^2}\right) \mu_m(x)\right).$$ (26)

Since $(1 - 1/\sqrt{1 - \Lambda_\infty^2})\mu_m(x) \leq (1 - 1/\sqrt{1 - \Lambda_\infty^2})\mu_1(x)$ and then

$$\left(1 + 1/\sqrt{1 - \Lambda_\infty^2}\right) \mu_1(x) + \left(1 - 1/\sqrt{1 - \Lambda_\infty^2}\right) \mu_m(x) \leq 2\mu_1(x),$$

condition (26) shows that $\mathcal{A}(x) - \kappa' I > 0$ almost everywhere. Moreover inequality (26) implies that $\kappa'$ satisfies (23). The result follows from Theorem 3.

If $m = 2$ the equivalence between (21) and (22) follows from the identities (20).

Under an additional assumption on the matrix $\mathcal{A}$ we have also

Corollary 5 Let $\mathcal{A}$ be a real and symmetric matrix. Suppose $\mathcal{A} > 0$ almost everywhere. Denote by $\mu_1(x)$ and $\mu_m(x)$ the smallest and the largest eigenvalues of $\mathcal{A}(x)$ respectively. If there exists $\kappa > 0$ such that $A - \kappa I (d^2/dx^2)$ is $L^0$-dissipative, then

$$\text{ess inf}_{x \in (a,b)} \left[ \mu_1(x)\mu_m(x) - \frac{\Lambda_\infty^2}{2}(\mu_1(x) + \mu_m(x))^2 \right] > 0.$$ (27)
If, in addition, there exists $C$ such that

$$\langle A(x)\xi, \xi \rangle \leq C|\xi|^2$$

for almost every $x \in (a, b)$ and for any $\xi \in \mathbb{R}^m$, the converse is also true. In the particular case $m = 2$ condition (27) is equivalent to

$$\text{ess inf}_{x \in (a,b)} \left[ \det \mathcal{A}(x) - \frac{\Lambda^2}{2} (\text{tr} \mathcal{A}(x))^2 \right] > 0.$$  

**Proof.** Necessity. By the proof of Corollary 4, inequality (25) holds. On the other hand we have

$$\left( \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_1(x) + \left( 1 - 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x) \right)$$

$$\leq \left( \left( 1 - 1/\sqrt{1 - \Lambda^2} \right) \mu_1(x) + \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x) \right)$$

and then

$$4k^2 \leq \left( \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_1(x) + \left( 1 - 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x) \right)$$

$$\times \left( \left( 1 - 1/\sqrt{1 - \Lambda^2} \right) \mu_1(x) + \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x) \right)$$

almost everywhere. This inequality can be written as

$$(1 - \Lambda^2)k^2 \leq \mu_1(x)\mu_m(x) - \frac{\Lambda^2}{2} (\mu_1(x) + \mu_m(x))^2$$

and (27) is proved.

Sufficiency. Let $h > 0$ such that

$$(1 - \Lambda^2)h \leq \mu_1(x)\mu_m(x) - \frac{\Lambda^2}{2} (\mu_1(x) + \mu_m(x))^2$$

almost everywhere, i.e.

$$4h \leq \left( \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_1(x) + \left( 1 - 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x) \right)$$

$$\times \left( \left( 1 - 1/\sqrt{1 - \Lambda^2} \right) \mu_1(x) + \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x) \right)$$

almost everywhere. Since $\mu_1(x) > 0$, we have also

$$\left( 1 - 1/\sqrt{1 - \Lambda^2} \right) \mu_1(x) + \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x)$$

$$\leq \left( 1 + 1/\sqrt{1 - \Lambda^2} \right) \mu_m(x)$$

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and then
\[ 4h \leq \left(1 + 1/\sqrt{1 - \Lambda_\infty^2}\right)^2 \mu_1(x) + \left(1 - 1/(1 - \Lambda_\infty^2)\right) \mu_m(x) \]

ess sup \( \mu_m(y) \)
almost everywhere. By (28), ess sup \( \mu_m \) is finite and by (27) it is greater than zero. Then (21) holds and Corollary 4 gives the result.

**Remark 4** As showed in [4, Remark 2, p.258], condition (28) cannot be omitted.

## 5 Functional dissipativity for some systems of partial differential equations

In this Section we consider the particular class of matrix operators defined as

\[ Au = \partial_h (\mathscr{A}^h(x) \partial_h u) \quad (29) \]

where \( \mathscr{A}^h(x) = \{a^h_{ij}(x)\} \) are matrices with complex locally integrable entries defined in a domain \( \Omega \subset \mathbb{R}^n \). We give necessary and sufficient condition for its functional dissipativity.

We start by considering the strict \( L^{\Phi} \)-dissipativity of operator (29) and by explicitly writing the next result, which is particular case of Lemma 2.

**Lemma 5** The operator (29) is strict \( L^{\Phi} \)-dissipative if and only if there exists \( \kappa > 0 \) such that

\[ \text{Re} \int_\Omega \left( (\mathscr{A}^h \partial_h v, \partial_h v) - \Lambda^2 |v|^{-4} (\mathscr{A}^h v, v) (\text{Re} \langle v, \partial_h v \rangle)^2 ight) \\
+ \Lambda(|v|) |v|^{-2} (\mathscr{A}^h (\mathscr{A}^h)^* v, \partial_h v) \text{Re} \langle v, \partial_h v \rangle \right) dx \geq \kappa \int_\Omega |\nabla v|^2 dx \quad (30) \]

for any \( v \in \tilde{C}^1(\Omega) \).

By \( y_h \) we denote the \((n - 1)\)-dimensional vector \((x_1, \ldots, x_{h-1}, x_{h+1}, \ldots, x_n)\) and we set \( \omega(y_h) = \{x_h \in \mathbb{R} | x \in \Omega\} \).

**Lemma 6** Let us assume condition (15). The operator (29) is strict \( L^{\Phi} \)-dissipative if and only if the ordinary differential operators (37) are uniformly strict \( L^{\Phi} \)-dissipative in \( \omega(y_h) \) for almost every \( y_h \in \mathbb{R}^{n-1} \) \((h = 1, \ldots, n)\). This condition is void if \( \omega(y_h) = \emptyset \).
Proof. Sufficiency. Suppose $r \geq 0$. The strict $L^\Phi$-dissipative can be written as

$$\Re \sum_{h=1}^{n} \int_{\mathbb{R}^{n-1}} dy_h \int_{\omega(y_h)} \langle A^h \partial_h u, \partial_h (\varphi(|u|) u) \rangle dx_h \geq \kappa \sum_{h=1}^{n} \int_{\mathbb{R}^{n-1}} dy_h \int_{\omega(y_h)} |\partial_h (\sqrt{|u|} u)|^2 dx_h$$

for any $u \in (\hat{C}^1(\Omega))^m$. By assumption, there exists $\kappa > 0$ such that

$$\Re \int_{\omega(y_h)} \langle A^h(x)v'(x_h), (\varphi(|v(x_h)|)v(x_h))' \rangle dx_h \geq \kappa \int_{\omega(y_h)} |(\sqrt{\varphi(|v(x_h)|)} v(x_h))'|^2 dx_h$$

for almost every $y_h \in \mathbb{R}^{n-1}$ and for any $v \in (\hat{C}^1(\omega(y_h)))^m$, provided $\omega(y_h) \neq \emptyset$ ($h = 1, \ldots, n$). This implies

$$\Re \sum_{h=1}^{n} \int_{\Omega} \langle A^h(x)\partial_h u, \partial_h (\varphi(|u|) u) \rangle dx \geq \kappa \int_{\Omega} |\nabla (\sqrt{\varphi(|u|)} u)|^2 dx \quad (31)$$

for any $u \in (\hat{C}^1(\Omega))^m$, and the strict $L^\Phi$-dissipativity is proved. The proof for $-1 < r < 0$ runs in the same way. We have just to replace (31) by

$$\Re \sum_{h=1}^{n} \int_{\Omega} \langle (A^h)^* \partial_h w, \partial_h (\psi(|w|) w) \rangle dx \geq \kappa \int_{\Omega} |\nabla (\sqrt{\psi(|w|)} w)|^2 dx \quad (32)$$

for any $w \in (\hat{C}^1(\Omega))^m$.

Necessity. Let $r \geq 0$. First suppose that $A^h$ are constant matrices and $\Omega = \mathbb{R}^n$. Fix $1 \leq k \leq n$. Let $\alpha \in (\hat{C}^1(\mathbb{R}))^m$ and $\beta$ be a real valued scalar function in $\hat{C}^1(\mathbb{R}^{n-1})$. Consider

$$u_\varepsilon(x) = \alpha(x_k/\varepsilon) \beta(y_k).$$

By assumption, there exists $\kappa > 0$ such that

$$\Re \sum_{h=1}^{n} \int_{\Omega} \langle A^h \partial_h u_\varepsilon, \partial_h (\varphi(|u_\varepsilon|) u_\varepsilon) \rangle dx \geq \kappa \int_{\Omega} |\nabla (\sqrt{\varphi(|u_\varepsilon|)} u_\varepsilon)|^2 dx. \quad (33)$$
Keeping in mind that $\beta$ is a scalar real valued function, if $h \neq k$ we have
\[
\partial_h(\phi(|u_\varepsilon|) u_\varepsilon) = g(|u_\varepsilon|) \partial_h u_\varepsilon,
\]
where $g(s) = (s\phi(s))'$, and
\[
\partial_h(\sqrt{\phi(|u_\varepsilon|)} u_\varepsilon) = \tilde{g}(|u_\varepsilon|) \partial_h u_\varepsilon,
\]
where $\tilde{g}(s) = (s\sqrt{\phi(s)})'$.

If $h = k$ we can write
\[
\partial_k(\phi(|u_\varepsilon|) u_\varepsilon) = \varepsilon^{-1} \partial_t \gamma(x_k/\varepsilon, \beta(y_k)) \beta(y_k),
\]
where $\gamma(t, s) = \phi(\alpha(t) s) \alpha(t)$, and
\[
\partial_k(\sqrt{\phi(|u_\varepsilon|)} u_\varepsilon) = \varepsilon^{-1} \partial_t \tilde{\gamma}(x_k/\varepsilon, \beta(y_k)) \beta(y_k),
\]
where $\tilde{\gamma}(t, s) = \sqrt{\phi(|\alpha(t) s|) \alpha(t)}$.

Therefore inequality (33) can be written as
\[
\varepsilon^{-2} \Re \int_{\mathbb{R}^{n-1}} \beta^2(y_k) dy_k \int_{\mathbb{R}} \langle \phi^k \alpha'(x_k/\varepsilon), \partial_t \gamma(x_k/\varepsilon, \beta(y_k)) \rangle dx_k
\]
\[
+ \sum_{h=1, h \neq k}^{n} \Re \int_{\mathbb{R}^{n-1}} (\partial_h \beta(y_k))^2 dy_k \int_{\mathbb{R}} \langle \phi^h \alpha(x_k/\varepsilon), \alpha(x_k/\varepsilon) \rangle g[|\alpha(x_k/\varepsilon) \beta(y_k)|] dx_k
\]
\[
\geq \kappa \varepsilon^{-2} \int_{\mathbb{R}^{n-1}} \beta^2(y_k) dy_k \int_{\mathbb{R}} |\partial_t \tilde{\gamma}(x_k/\varepsilon, \beta(y_k))|^2 dx_k
\]
\[
+ \kappa \sum_{h=1, h \neq k}^{n} \int_{\mathbb{R}^{n-1}} (\partial_h \beta(y_k))^2 dy_k \int_{\mathbb{R}} |\alpha(x_k/\varepsilon)|^2 (\tilde{g}(|\alpha(x_k/\varepsilon) \beta(y_k)|))^2 dx_k,
\]
i.e.
\[
\varepsilon^{-1} \Re \int_{\mathbb{R}^{n-1}} \beta^2(y_k) dy_k \int_{\mathbb{R}} \langle \phi^k \alpha'(t), \partial_t \gamma(t, \beta(y_k)) \rangle dt
\]
\[
+ \varepsilon \sum_{h=1, h \neq k}^{n} \Re \int_{\mathbb{R}^{n-1}} (\partial_h \beta(y_k))^2 dy_k \int_{\mathbb{R}} \Re \langle \phi^h \alpha(t), \alpha(t) \rangle g[|\alpha(t) \beta(y_k)|] dt
\]
\[
\geq \kappa \varepsilon^{-1} \int_{\mathbb{R}^{n-1}} \beta^2(y_k) dy_k \int_{\mathbb{R}} |\partial_t \tilde{\gamma}(t, \beta(y_k))|^2 dt
\]
\[
+ \kappa \varepsilon \sum_{h=1, h \neq k}^{n} \int_{\mathbb{R}^{n-1}} (\partial_h \beta(y_k))^2 dy_k \int_{\mathbb{R}} |\alpha(t)|^2 (\tilde{g}(|\alpha(t) \beta(y_k)|))^2 dt.
\]
Letting $\varepsilon \to 0^+$ we get
\[ \Re \int_{\mathbb{R}^{n-1}} \beta^2(y_k) dy_k \int_{\mathbb{R}} \langle \mathcal{A}^k \alpha'(t), \partial_t \gamma(t, \beta(y_k)) \rangle \, dt \geq \kappa \int_{\mathbb{R}^{n-1}} \beta^2(y_k) dy_k \int_{\mathbb{R}} |\partial_t \tilde{\gamma}(t, \beta(y_k))|^2 \, dt. \tag{34} \]

We claim that from the arbitrariness of $\beta$ it follows
\[ \Re \int_{\mathbb{R}} \langle \mathcal{A}^k \alpha'(t), \partial_t \gamma(t, 1) \rangle \, dt \geq \kappa \int_{\mathbb{R}} |\partial_t \tilde{\gamma}(t, 1)|^2 \, dt. \tag{35} \]

We prove (35) by contradiction. Set
\[ \Phi(s) = \int_{\mathbb{R}} [\Re \langle \mathcal{A}^k \alpha'(t), \partial_t \gamma(t, s) \rangle - \kappa |\partial_t \tilde{\gamma}(t, s)|^2] \, dt \]
and suppose $\Phi(1) < 0$. Take $\beta_\varepsilon \in \dot{C}^1(\mathbb{R}^{n-1})$ such that $0 \leq \beta_\varepsilon \leq 1$, $\beta_\varepsilon(y_k) = 1$ on the unit ball $B_1(0)$ and $\text{spt} \beta_\varepsilon \subset B_{1+\varepsilon}(0)$. Since $\Phi(1) < 0$ and keeping in mind (34), we can write
\[ 0 \leq \int_{\mathbb{R}^{n-1}} \beta^2_\varepsilon(y_k) \Phi(\beta_\varepsilon(y_k)) dy_k \]
\[ = \int_{\mathbb{R}^{n-1}} \beta^2_\varepsilon(y_k) [\Phi(\beta_\varepsilon(y_k)) - \Phi(1)] dy_k + \Phi(1) \int_{\mathbb{R}^{n-1}} \beta^2_\varepsilon(y_k) dy_k \]
\[ \leq \int_{B_{1+\varepsilon}(0) \setminus B_1(0)} \beta^2_\varepsilon(y_k) [\Phi(\beta_\varepsilon(y_k)) - \Phi(1)] dy_k + \Phi(1) |B_1(0)|. \]

Letting $\varepsilon \to 0^+$, we get $0 \leq \Phi(1) |B_1(0)| < 0$. This is a contradiction and (35) is proved. We then have
\[ \Re \int_{\mathbb{R}} \langle \mathcal{A}^k \alpha'(t), (|\varphi(\alpha(t))| \alpha(t))' \rangle \, dt \geq \kappa \int_{\mathbb{R}} |(\sqrt{\varphi(\alpha(t))} \alpha(t))'|^2 \, dt \]
for any $\alpha \in (\dot{C}^1(\mathbb{R}))^m$. This means that $A(y_k)$ is strict $L^\Phi$-dissipative.

If $\mathcal{A}^k$ are not necessarily constant and defined in $\Omega$, we observe that, thanks to Lemma 5, $A$ is strict $L^\Phi$-dissipative if and only if condition (30) holds. Consider
\[ v(x) = w((x - x_0)/\varepsilon) \]
where $x_0 \in \Omega$ is a fixed point, $w \in (\dot{C}^1(\mathbb{R}^n))^m$, $spt \, w \subset B_1(0)$ and $0 < \varepsilon < \text{dist}(x_0, \partial \Omega)$. Putting this particular $v$ in (30) we get

$$
\text{Re} \int_{\Omega} \left( \langle \mathcal{A}^h(x) \partial_h w((x-x_0)/\varepsilon), \partial_h w((x-x_0)/\varepsilon) \rangle \right.
- \Lambda^2(\|w((x-x_0)/\varepsilon)\|) \|w((x-x_0)/\varepsilon)\|^{-4} \langle \mathcal{A}^h(x) w((x-x_0)/\varepsilon), w((x-x_0)/\varepsilon) \rangle \times (\text{Re} \langle w((x-x_0)/\varepsilon), \partial_h w((x-x_0)/\varepsilon) \rangle^2
+ \Lambda(\|w((x-x_0)/\varepsilon)\|) \|w((x-x_0)/\varepsilon)\|^{-2}
\times \langle \mathcal{A}^h(x) - (\mathcal{A}^h)^*(x) \rangle w((x-x_0)/\varepsilon), \partial_h w((x-x_0)/\varepsilon) \rangle
\times \text{Re} \langle w((x-x_0)/\varepsilon), \partial_h w((x-x_0)/\varepsilon) \rangle \right) dx
\geq \kappa \int_{\Omega} |\nabla w((x-x_0)/\varepsilon)|^2 dx
$$

and then

$$
\text{Re} \int_{\Omega} \left( \langle \mathcal{A}^h(x_0 + \varepsilon y) \partial_h w(y), \partial_h w(y) \rangle
- \Lambda^2(\|w(y)\|) \|w(y)\|^{-4} \langle \mathcal{A}^h(x_0 + \varepsilon y) w(y), w(y) \rangle \langle \text{Re} \langle w(y), \partial_h w(y) \rangle \rangle^2 \right.
\times \text{Re} \langle w(y), \partial_h w(y) \rangle dy \geq \kappa \int_{\Omega} |\nabla w(y)|^2 dy.
$$

Letting $\varepsilon \to 0^+$, we obtain

$$
\text{Re} \int_{\mathbb{R}^n} \left( \langle \mathcal{A}^h(x_0) \partial_h w, \partial_h w \rangle - \Lambda^2(\|w\|) \|w\|^{-4} \langle \mathcal{A}^h(x_0) w, w \rangle \langle \text{Re} \langle w, \partial_h w \rangle \rangle^2
+ \Lambda^2(\|w\|) \|w\|^{-2} \langle \mathcal{A}^h(x_0) w, \partial_h w \rangle - \langle \mathcal{A}^h(x_0) \partial_h w, w \rangle \right) \text{Re} \langle w, \partial_h w \rangle dy
\geq \kappa \int_{\Omega} |\nabla w(y)|^2 dy
$$

for almost every $x_0 \in \Omega$.

Because of the arbitrariness of $w \in (\dot{C}^1(\mathbb{R}^n))^m$, Lemma 5 shows that the constant coefficient operator $\partial_h(\mathcal{A}^h(x_0) \partial_h)$ is strict $L^\Phi$-dissipative. From what has already been proved, the ordinary differential operators $(\mathcal{A}^h(x_0)v')'$ are strict $L^\Phi$-dissipative ($h = 1, \ldots, n$).

By Corollary 5 we have

$$
\text{Re} \langle \mathcal{A}^h(x_0) \lambda, \lambda \rangle - \Lambda^2 \text{Re} \langle \mathcal{A}^h(x_0) \omega, \omega \rangle \langle \text{Re} \langle \lambda, \omega \rangle \rangle^2
+ \Lambda \text{Re} \langle (\mathcal{A}^h(x_0) \omega, \lambda) - \langle \mathcal{A}^h(x_0) \lambda, \omega \rangle \rangle \text{Re} \langle \lambda, \omega \rangle \geq \kappa' |\lambda|^2
$$

(36)
for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \ldots, n$, where $\kappa' = (1 - \Lambda_\infty^2)\kappa$.

Reasoning as in [4, p.261] we find that, for almost every $y_h \in \mathbb{R}^{n-1}$, we have

$$\Re(\langle A_h(x)\lambda, \lambda \rangle) - \Lambda_\infty^2 \Re(\langle A_h(x)\omega, \omega \rangle)(\Re(\lambda, \omega))^2 + \Lambda_\infty \Re(\langle A_h(x)\omega, \lambda \rangle - \langle A_h(x)\lambda, \omega \rangle) \Re(\lambda, \omega) \geq \kappa' |\lambda|^2$$

for almost every $x_h \in \omega(y_h)$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, provided $\omega(y_h) \neq \emptyset$. The conclusion follows from Corollary [3].

If $-1 < r < 0$ the proof runs in the same way, recalling (32).

We are now in a position to prove the necessary and sufficient algebraic condition for the strict $L^\Phi$-dissipativity of $A$.

**Theorem 4** Let us assume condition (15). The operator (29) is strict $L^\Phi$-dissipative if and only if there exists $\kappa' > 0$ such that (36) holds for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \ldots, n$.

**Proof.** Necessity. This has been already proved in the necessity part of Lemma [6].

Sufficiency. We know that if (36) holds for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, the ordinary differential operators $A(y_h)$ are uniformly strict $L^\Phi$-dissipative for almost every $y_h \in \mathbb{R}^{n-1}$, provided $\omega(y_h) \neq \emptyset$ ($h = 1, \ldots, n$).

By Lemma [6] $A$ is strict $L^\Phi$-dissipative.

Similar results hold for the $L^\Phi$-dissipativity.

**Lemma 7** The operator (29) is $L^\Phi$-dissipative if and only if the ordinary differential operators

$$A(y_h)[u(x_h)] = d(A_h(x)du/dx_h)/dx_h$$

are $L^\Phi$-dissipative in $\omega(y_h)$ for almost every $y_h \in \mathbb{R}^{n-1}$ ($h = 1, \ldots, n$). This condition is void if $\omega(y_h) = \emptyset$.

**Proof.** The proof runs as the proof of Lemma [6] in which we set $\kappa = 0$. We note that here we do not need to assume condition (15).

**Theorem 5** The operator (29) is $L^\Phi$-dissipative if and only if

$$\Re(\langle A(x)\lambda, \lambda \rangle) - \Lambda_\infty^2 \Re(\langle A(x)\omega, \omega \rangle)(\Re(\lambda, \omega))^2 + \Lambda_\infty \Re(\langle A(x)\omega, \lambda \rangle - \langle A(x)\lambda, \omega \rangle) \Re(\lambda, \omega) \geq 0$$

holds for almost every $x_0 \in \Omega$ and for any $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \ldots, n$. 20
Proof. The proof is similar to the proof of Theorem 4.

In the case of a real coefficient operator (29), we have also

**Theorem 6** Let \( A \) be the operator (29), where \( \mathcal{A}^h \) are real matrices \( \{a_{ij}^h\} \) with \( i,j = 1, \ldots, m \). Let us suppose \( \mathcal{A}^h = (\mathcal{A}^h)^t \) and \( \mathcal{A}^h > 0 \) (\( h = 1, \ldots, n \)). The operator \( A \) is \( L^\Phi \)-dissipative if and only if

\[
\Lambda^2_{\infty}(\mu^1_h(x) + \mu^m_h(x))^2 \leq 4 \mu^1_h(x) \mu^m_h(x)
\]

(38)

for almost every \( x \in \Omega, h = 1, \ldots, n \), where \( \mu^1_h(x) \) and \( \mu^m_h(x) \) are the smallest and the largest eigenvalues of the matrix \( \mathcal{A}^h(x) \) respectively. In the particular case \( m = 2 \) this condition is equivalent to

\[
\Lambda^2_{\infty}(\text{tr} \mathcal{A}^h(x))^2 \leq 4 \det \mathcal{A}^h(x)
\]

for almost every \( x \in \Omega, h = 1, \ldots, n \).

**Proof.** By Theorem 5, \( A \) is \( L^\Phi \)-dissipative if and only if

\[
\langle \mathcal{A}^h(x) \lambda, \lambda \rangle - \Lambda^2_{\infty}(\mathcal{A}^h(x) \omega, \omega) (\Re\langle \lambda, \omega \rangle)^2 \geq 0
\]

for almost every \( x \in \Omega \), for any \( \lambda, \omega \in \mathbb{C}^m, |\omega| = 1, h = 1, \ldots, n \). The proof of Theorem 3 shows that these conditions are equivalent to (38).

\[\square\]

## 6 Functional ellipticity

In [11] Dindoș, Li and Pipher have introduced different notions of \( p \)-ellipticity, establishing relationships between them. Here we extend these definitions in the frame of functional ellipticity, introduced in [8] for scalar operators.

Coming back to the general operator (11), we say that the tensor \( \{a_{ij}^{hk}(x)\} \) satisfies the strong \( \Phi \)-ellipticity condition if there exists \( \kappa > 0 \) such that

\[
\Re\langle \mathcal{A}^{hk}(x) \xi_k, \xi_h \rangle - \Lambda^2(t) \Re\langle (\mathcal{A}^{hk}(x) - (\mathcal{A}^{kk})^*(x)) \omega, \xi_h \rangle \Re\langle \omega, \xi_k \rangle \\
+ \Lambda(t) \Re\langle \mathcal{A}^{hk}(x) \omega, \omega \rangle \Re\langle \omega, \xi_k \rangle \Re\langle \omega, \xi_h \rangle \geq \kappa |\xi|^2
\]

(39)

for any \( \xi, \omega \in \mathbb{C}^m, |\omega| = 1, t > 0 \) and for almost every \( x \in \Omega \).
We say that the tensor \( \{a_{ij}^{hk}(x)\} \) satisfies the integral \( \Phi \)-ellipticity condition if there exists \( \kappa > 0 \) such that condition (7) holds for any \( v \in [\mathcal{C}^1(\Omega)]^m \). We note that if there are no lower order terms, as in the case we are considering here, the concepts of strong dissipativity and integral ellipticity are equivalent, thanks to Lemma 2. This is not the case if there are lower order terms. The \( \Phi \)-ellipticity is still given by (7), while the formula for \( \Phi \)-dissipativity has to be changed, taking into account the lower order terms.

It is trivial that strong \( \Phi \)-ellipticity implies integral \( \Phi \)-ellipticity. Indeed, (39) implies that the integrand in (7) is non-negative almost everywhere.

We say that the tensor \( \{a_{ij}^{hk}(x)\} \) satisfies the Legendre-Hadamard \( \Phi \)-ellipticity condition (or weak \( \Phi \)-ellipticity condition) if there exists \( \kappa > 0 \) such that

\[
\Re\langle (A^{hk}(x)q_hq_k)\lambda, \lambda \rangle - \Lambda^2(t) \Re\langle (A^{hk}(x)q_hq_k)\omega, \omega \rangle (\Re\langle \lambda, \omega \rangle)^2
\]

\[
+\Lambda(t) \Re\langle (A^{hk}(x)q_hq_k)\omega, \lambda \rangle - \Re\langle (A^{hk}(x)q_hq_k)\lambda, \omega \rangle \Re\langle \lambda, \omega \rangle > \kappa |q|^2 |\lambda|^2
\]

for any \( q \in \mathbb{R}^n, \lambda, \omega \in \mathbb{C}^m, |\omega| = 1, t > 0 \) and for almost every \( x \in \Omega \).

If \( \varphi(t) = t^{p-2} \) and then \( \Lambda(t) = -(1 - 2/p) \), conditions (39), (7) and (40) coincide with (17), (20) and (31) of [11], respectively.

We remark that, if condition (15) is satisfied, then inequalities (39) and (40) for any \( t > 0 \) are equivalent to

\[
\Re\langle A^{hk}(x)\xi_k, \xi_h \rangle - \Lambda^2_\infty \Re\langle (A^{hk}(x) - (A^{hk})^*(x))\omega, \xi_h \rangle \Re\langle \omega, \xi_k \rangle
\]

\[
+\Lambda_\infty \Re\langle (A^{hk}(x)\omega, \omega) \Re\langle \omega, \xi_k \rangle \Re\langle \omega, \xi_h \rangle \geq \kappa |\xi|^2
\]

and

\[
\Re\langle (A^{hk}(x)q_hq_k)\lambda, \lambda \rangle - \Lambda^2_\infty \Re\langle (A^{hk}(x)q_hq_k)\omega, \omega \rangle (\Re\langle \lambda, \omega \rangle)^2
\]

\[
+\Lambda_\infty \Re\langle (A^{hk}(x)q_hq_k)\omega, \lambda \rangle - \Re\langle (A^{hk}(x)q_hq_k)\lambda, \omega \rangle \Re\langle \lambda, \omega \rangle > \kappa |q|^2 |\lambda|^2,
\]

respectively (see Remark 3).

We can now rephrase some of our results as follows.

**Theorem 7** Let \( n = 1 \) and \( A \) be the operator (8). Assume (15) holds. The following statements are equivalent:

(a) the operator \( A \) is strict \( L^\Phi \)-dissipative;
(b) there exists $\kappa > 0$ such that $A - kI(d^2/dx^2)$ is $L^\Phi$-dissipative;

(c) the matrix $\{a_{ij}(x)\}$ satisfies the strong $\Phi$-ellipticity condition;

(d) the matrix $\{a_{ij}(x)\}$ satisfies the integral $\Phi$-ellipticity condition;

(e) the matrix $\{a_{ij}(x)\}$ satisfies the weak $\Phi$-ellipticity condition.

**Proof.** The equivalence between [a] and (b) is given by Corollary 2. The equivalence between [a] and (e) was proved in Corollary 3. It is also clear that, when $n = 1$, (39) and (40) coincide and then (c) and (e) are equivalent. Finally, we have already remarked that, if the operator has no lower order terms, conditions [a] and (d) are equivalent.

An analogous result holds in any dimension for the operator (29).

**Theorem 8** Let $n \geq 2$ and $A$ be the operator (29). Assume (15) holds. The following statements are equivalent:

(a) the operator $A$ is strict $L^\Phi$-dissipative;

(b) there exists $\kappa > 0$ such that $A - k\Delta$ is $L^\Phi$-dissipative;

(c) the matrix $\{a_{ij}(x)\}$ satisfies the integral $\Phi$-ellipticity condition;

(d) the matrix $\{a_{ij}(x)\}$ satisfies the weak $\Phi$-ellipticity condition.

Moreover, if the matrix $\{a_{ij}(x)\}$ satisfies the strong $\Phi$-ellipticity condition, then [a]-[d] hold.

**Proof.** The equivalence between [a] and (b) was proved in [9, Corollary 1] for the more general operator (1). The equivalence between [a] and (c) is given by Lemma 5. To prove the equivalence between [a] and (d), we first note that condition (41) in the present case is

\[
\sum_{h=1}^{n} \left( \text{Re}(\langle \phi^h(x)q_h^2 \rangle \lambda, \lambda) - \Lambda_\infty^2 \text{Re}(\langle \phi^h(x)q_h^2 \rangle \omega, \omega) \text{Re}(\lambda, \omega)^2 \right)
+ \Lambda_\infty \text{Re}(\langle \phi^h(x)q_h^2 \rangle \omega, \lambda) - \langle \phi^h(x)q_h^2 \rangle \lambda, \omega) \text{Re}(\lambda, \omega) \right) \geq \kappa |q|^2 |\lambda|^2,
\]

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for any \( q \in \mathbb{R}^n, \lambda, \omega \in \mathbb{C}^m, |\omega| = 1, h > 0 \) and for almost every \( x \in \Omega \). Thanks to the arbitrariness of the vector \( q \), this is equivalent to the \( n \) inequalities
\[
\operatorname{Re} \langle A^h(x)\lambda, \lambda \rangle - \Lambda_\infty^2 \operatorname{Re} \langle A^h(x)\omega, \omega \rangle (\operatorname{Re} \langle \lambda, \omega \rangle)^2 \\
+ \Lambda_\infty \operatorname{Re}(\langle A^h(x)\omega, \lambda \rangle - \langle A^h(x)\lambda, \omega \rangle) \operatorname{Re} \langle \lambda, \omega \rangle \geq \kappa |\lambda|^2
\]
for any \( \lambda, \omega \in \mathbb{C}^m, |\omega| = 1, h = 1, \ldots, n \). Therefore Theorem 4 gives the equivalence between [a] and [d]. Finally, we know that the strong \( \Phi \)-ellipticity implies the integral \( \Phi \)-ellipticity and the proof is complete.

\[
\square
\]

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