LIE STRUCTURE OF TRUNCATED SYMMETRIC POISSON ALGEBRAS

ILANA Z. MONTEIRO ALVES AND VICTOR PETROGRADSKY

Abstract. The paper naturally continues series of works on identical relations of group rings, enveloping algebras, and other related algebraic structures. Let $L$ be a Lie algebra over a field of characteristic $p > 0$. Consider its symmetric algebra $S(L) = \bigoplus_{n=0}^{\infty} U_n/U_{n-1}$, which is isomorphic to a polynomial ring. It also has a structure of a Poisson algebra, where the Lie product is traditionally denoted by $\{\ ,\ \}$. This bracket naturally induces the structure of a Poisson algebra on the ring $s(L) = S(L)/(x^p \mid x \in L)$, which we call a truncated symmetric Poisson algebra. We study Lie identical relations of $s(L)$. Namely, we determine necessary and sufficient conditions for $L$ under which $s(L)$ is Lie nilpotent, strongly Lie nilpotent, solvable and strongly solvable, where we assume that $p > 2$ to specify the solvability. We compute the strong Lie nilpotency class of $s(L)$. Also, we prove that the Lie nilpotency class coincides with the strong Lie nilpotency class in case $p > 3$.

Shestakov proved that the symmetric algebra $S(L)$ of an arbitrary Lie algebra $L$ satisfies the identity $\{x, \{y, z\}\} \equiv 0$ if, and only if, $L$ is abelian. We extend this result for the (strong) Lie nilpotency and the (strong) solvability of $S(L)$. We show that the solvability of $s(L)$ and $S(L)$ in case $\text{char} \ K = 2$ is different to other characteristics, namely, we construct examples of such algebras which are solvable but not strongly solvable.

We use delta-sets for Lie algebras and the theory of identical relations of Poisson algebras. Also, we study filtrations in Poisson algebras and prove results on products of terms of the lower central series for Poisson algebras.

1. Introduction

The theory of associative PI-algebras, i.e. algebras satisfying nontrivial polynomial identities, is a classical area of the modern algebra [11]. This is an important instrument to study structure and properties of associative algebras. Now, there is an established theory of identical relations in Lie algebras [1]. It has many applications to group theory such as the solution of the Restricted Burnside Problem. Also, identical relations were applied to study another algebraic structures.

The first starting point for our research is the result of Passman on existence of identical relations in group rings [31] (Theorem 3.1). This paper caused an intensive research on different types of identical relations in group rings, such as Lie nilpotence, solvability, non-matrix identical relations, classes of Lie nilpotence, solvability lengths, etc. There are at least 50 papers published in this area.

Second, Latyshev [23] and Bahturin [2] started to study identical relations in universal enveloping algebras of Lie algebras. Passman [32] and Petrogradsky [33] specified existence of identical relations in restricted enveloping algebras (Theorem 3.4). There are many papers in this area studying different types of identical relations, such as Lie nilpotence, solvability, non-matrix identical relations, classes of Lie nilpotence, solvability lengths, etc. So, Riley and Shalev determined necessary and sufficient conditions for restricted Lie algebras under which the restricted enveloping algebra is Lie nilpotent or solvable [37]. The research was further extended to new objects, such as Lie superalgebras, color Lie superalgebras, smash products. These problems were studied in numerous papers by Bahturin, Bergen, Kochetov, Petrogradsky, Riley, Shalev, Siciliano, Spinelly, Usefi, et.al.

Poisson algebras appeared in works of Berezin [7] and Vergne [51]. Free Poisson algebras were introduced by Shestakov [40]. A basic theory of identical relations for Poisson algebras was developed by Farkas [13, 14]. Identical relations of symmetric Poisson algebras of Lie (super)algebras were studied by Kostant [21], Shestakov [40], and Farkas [14]. The third starting point for our research is the result of Giambruno and Petrogradsky [15] on existence of non-trivial multilinear Poisson identical relations in truncated symmetric algebras of restricted Lie algebras (Theorem 3.11).

2000 Mathematics Subject Classification. 17B63, 17B50, 17B01, 17B30, 17B65, 16R10.

Key words and phrases. Poisson algebras, identical relations, solvable Lie algebras, nilpotent Lie algebras, symmetric algebras, truncated symmetric algebras.

The author was partially supported by grants CNPq, FEMAT.
In Section 2 we supply basic facts on Poisson algebras and their identities. In Section 3 we collect principal results on identical relations of group rings, enveloping algebras and Poisson symmetric algebras that motivated our research. Our main results are formulated in Section 4 (Theorem 4.1 and Theorem 4.3). In Section 6 we study filtrations in (truncated) symmetric Poisson algebras. In Section 8 we prove technical results on products of terms of the lower central series of Poisson algebras, which are analogues of respective results for associative algebras.

By $K$ denote the ground field, as a rule of positive characteristic $p$. By $\langle S \rangle$ or $\langle S \rangle_K$ denote the linear span of a subset $S$ in a $K$-vector space. Let $L$ be a Lie algebra. The Lie brackets are left-normed: $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$, $n \geq 1$. Denote $\text{ad} : L \to L$ by $\text{ad}(x)(y) = [x, y]$, $x, y \in L$. One defines the lower central series: $\gamma_1(L) = L$, $\gamma_{n+1}(L) = [\gamma_n(L), L]$, $n \geq 1$. Also, $L^2 = [L, L] = \gamma_2(L)$ is the commutator subalgebra. By $U(L)$ denote the universal enveloping algebra and $S(L) = \bigoplus_{n=0}^{\infty} U_n/U_{n-1}$ the related symmetric algebra [1, 4, 10]. On restricted Lie algebras and restricted enveloping algebras see [1, 19]. Let us note that all our Lie algebras over a field of positive characteristic need not be restricted.

2. Poisson algebras and their identities

2.1. Poisson algebras. Poisson algebras naturally appear in different areas of algebra, topology and physics. Poisson algebras probably first were introduced in 1976 by Berezin [7], see also Vergne [51] (1969). Poisson algebras are used to study universal enveloping algebras of finite dimensional Lie algebras in zero characteristic [21, 28]. In particular, abelian subalgebras in symmetric Poisson algebras are used to study commutative subalgebras in universal enveloping algebras of finite-dimensional semi-simple Lie algebras in zero characteristic [48, 52]. Applying Poisson algebras, Shestakov and Umirbaev managed to solve a long-standing problem: they proved that the Nagata automorphism of the polynomial ring in three variables $\mathbb{C}[x, y, z]$ is wild [41]. Related algebraic properties of free Poisson algebras were studied by Makar-Limanov, Shestakov and Umirbaev [25, 26].

The free Poisson algebras were defined by Shestakov [40]. A basic theory of identical relations for Poisson algebras was developed by Farkas [13, 14]. See further developments on the theory of identical relations of Poisson algebras, in particular, theory of so called codimension growth in characteristic zero by Mishchenko, Petrogradsky, and Regev [27], and Ratseev [36].

Throughout this paper, as a rule, $K$ denotes a field of characteristic $p > 0$. Recall that a vector space $A$ is a Poisson algebra provided that, beside the addition, $A$ has two $K$-bilinear operations which are related by the Leibnitz rule. More precisely,

- $A$ is a commutative associative algebra with unit, denote the multiplication by $a \cdot b$ (or $ab$), where $a, b \in A$;
- $A$ is a Lie algebra which product is traditionally denoted by the Poisson bracket $\{a, b\}$, where $a, b \in A$;
- these two operations are related by the Leibnitz rule
  \[ \{a \cdot b, c\} = a \cdot \{b, c\} + b \cdot \{a, c\}, \quad a, b, c \in A. \]

2.2. Examples of Poisson algebras. Typical examples are as follows.

Example 1. Consider the polynomial ring $H_{2m} = K[X_1, \ldots, X_m, Y_1, \ldots, Y_m]$. Set $\{X_i, Y_j\} = \delta_{i,j}$ and extend this bracket by the Leibnitz rule. We obtain the Poisson bracket:

\[ \{f, g\} = \sum_{i=1}^{m} \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial Y_i} - \frac{\partial f}{\partial Y_i} \frac{\partial g}{\partial X_i} \right), \quad f, g \in H_{2m}. \]

The commutative product is the natural multiplication. We obtain the Hamiltonian Poisson algebra $H_{2m}$.

Example 2. Let $L$ be a Lie algebra over an arbitrary field $K$, $\{U_n \mid n \geq 0\}$ the natural filtration of its universal enveloping algebra $U(L)$. Consider the symmetric algebra $S(L) = \text{gr} U(L) = \bigoplus_{n=0}^{\infty} U_n/U_{n+1}$ (see [10]). Recall that $S(L)$ is identified with the polynomial ring $K[v_i \mid i \in I]$, where $\{v_i \mid i \in I\}$ is a $K$-basis of $L$. Define the Poisson bracket as follows. Set $\{v_i, v_j\} = [v_i, v_j]$ for all $i, j \in I$, and extend to the whole of $S(L)$ by linearity and using the Leibnitz rule. For example,

\[ \{v_i \cdot v_j, v_k\} = v_i \cdot \{v_j, v_k\} + v_j \cdot \{v_i, v_k\}, \quad i, j, k \in I. \]
Thus, $S(L)$ has a structure of a Poisson algebra, called the symmetric algebra of $L$.

**Example 3.** Let $L$ be a Lie algebra with a $K$-basis $\{v_i | i \in I\}$, where char $K = p > 0$. Consider a factor algebra of the symmetric (Poisson) algebra

$$s(L) = S(L)/(v^p | v \in L) \cong K[v_i | i \in I]/(v^p_i | i \in I),$$

we get an algebra of truncated polynomials. Observe that

$$\{v^p, u\} = pv^{p-1}\{v, u\} = 0, \quad v \in L, \ u \in s(L).$$

So, the Poisson bracket on $S(L)$ yields a Poisson bracket on $s(L)$. Thus, $s(L)$ is a Poisson algebra, we call it truncated symmetric algebra. Remark that $L$ need not be a restricted Lie algebra.

**Example 4.** Let $K$ be a field of positive characteristic $p$. We introduce the truncated Hamiltonian Poisson algebra as

$$h_{2m}(K) = K[X_1, \ldots, X_m, Y_1, \ldots, Y_m]/(X^p_i, Y^p_i | i = 1, \ldots, m),$$

where we define the bracket as in Example 1 using the observation of Example 3.

The Hamiltonian algebras $h_{2m}(K)$ and $H_{2m}(K)$ in the class of Poisson algebras play a role similar to that of the matrix algebras $M_n(K)$ for associative algebras.

2.3. Poisson identities. The objective of this subsection is to supply basic facts on polynomial identities of Poisson algebras.

Consider the free Lie algebra $L = L(X)$ generated by a set $X$ and its symmetric algebra $F(X) = S(L(X))$. Then, $F(X)$ is a free Poisson algebra in $X$, as was shown by Shestakov [40]. For example, let $L = L(x, y)$ be the free Lie algebra of rank 2. Consider its Hall basis [1]

$$L = \langle x, y, [y, x], [[y, x], x], [[y, x], y], [[[y, x], x], x], \ldots \rangle_K.$$

We obtain the free Poisson algebra $F(x, y) = S(L)$ of rank 2, which has a canonical basis as follows:

$$F(x, y) = \langle x^{n_1}y^{n_2}\{y, x\}^{n_3}\{\{y, x\}, x\}^{n_4} \cdots | n_i \geq 0 \rangle_K,$$

where only finitely many $n_i$, $i \geq 1$, are non-zero in the monomials above.

A definition of a Poisson PI-algebra is standard, identities being elements of the free Poisson algebra $F(X)$ of countable rank. Assume that basic facts on identical relations of linear algebras are known to the reader, (see, e.g., [1, 11]). Farkas introduced so called customary identities [13]:

$$\sum_{\sigma \in S_{2n}} \mu_\sigma (x_{\sigma(1)}, x_{\sigma(2)}) \cdots (x_{\sigma(2n-1)}, x_{\sigma(2n)}) \equiv 0, \quad \mu_\sigma \in K,$$

where $\mu_\tau = 1$, for the identity permutation. Denote by $T_{2n}$ the set of permutations $\tau \in S_{2n}$ appearing in the some above. The importance of customary identities is explained by the following fact.

**Theorem 2.1** ([13]). Suppose that $V$ is a nontrivial variety of Poisson algebras over a field $K$ of characteristic zero. Then $V$ satisfies a nontrivial customary identity.

Let us show the idea of the proof. Let a Poisson algebra $R$ satisfy the identity $f(X, Y, Z) = \{\{X, Y\}, Z\} \equiv 0$. Then, $R$ also satisfies the identity:

$$0 \equiv f(X_1 X_2, Y, Z) - X_1 f(X_2, Y, Z) - X_2 f(X_1, Y, Z)$$

$$= \{\{X_1 X_2, Y\}, Z\} - X_1\{\{X_2, Y\}, Z\} - X_2\{\{X_1, Y\}, Z\}$$

$$= \{X_1, Y\}\{X_2, Z\} + \{X_1, Z\}\{X_2, Y\},$$

which is customary. Farkas called this process a customarization [13], it is an analogue of the linearization process for associative algebras. The arguments of [13] actually prove the following.

**Theorem 2.2** ([13]). Suppose that a Poisson algebra $A$ over an arbitrary field satisfies a nontrivial multilinear Poisson identity. Then $A$ satisfies a nontrivial customary identity.
Let us explain why we need all polynomials to be multilinear in case of positive characteristic $p$. The linearization process is simply not working for Poisson algebras in positive characteristic as it does for associative and Lie algebras. For example, the Poisson identity $\{x, y\}^p \equiv 0$ is given by a nonzero element of the free Poisson algebra $F(X)$. Observe that its full linearization is trivial:

$$\sum_{\sigma, \pi \in S_p} \{x_{\sigma(1)}, y_{\pi(1)}\} \cdot \cdots \cdot \{x_{\sigma(p)}, y_{\pi(p)}\} = p! \sum_{\pi \in S_p} \{x_1, y_{\pi(1)}\} \cdot \cdots \cdot \{x_p, y_{\pi(p)}\} = 0.$$ 

Moreover, let us check that any truncated symmetric algebra $s(L)$ satisfies the identity $\{x, y\}^p \equiv 0$. Indeed, let $a, b \in s(L)$, then $\{a, b\}$ is a truncated polynomial without constant term, its $p$th power is zero by the Frobenius rule $(v + w)^p = v^p + w^p$. Thus, it does not make sense to study nonlinear Poisson identities for truncated symmetric algebras.

In the theory of Poisson PI-algebras, the analogue of the standard polynomial is ([13], [14]):

$$\text{St}_{2n} = \text{St}_{2n}(x_1, \ldots, x_{2n}) = \sum_{\sigma \in \Pi_{2n}} (-1)^{\sigma} \{x_{\sigma(1)}, x_{\sigma(2)}\} \cdot \cdots \cdot \{x_{\sigma(2n-1)}, x_{\sigma(2n)}\}.$$ 

This is a customary polynomial, skewsymmetric in all variables [13]. One has the following fact similar to the theory of associative algebras.

**Theorem 2.3** ([27]). In case of zero characteristic, any Poisson PI-algebra satisfies an identity $(\text{St}_{2n})^m = 0$, for some integers $n, m$.

Another important fact on the standard identity is as follows.

**Lemma 2.4** ([13]). Let $A$ be a Poisson algebra over an arbitrary field $K$ and $A$ is $k$-generated as an associative algebra. Then it satisfies the standard identity $\text{St}_{2m} = 0$, whenever $2m > k$.

### 3. Identical relations of group rings and enveloping algebras

In this section we present a motivation for our research project. Namely, we shortly review results on existence of nontrivial polynomial identities in enveloping algebras and group rings.

#### 3.1. Identities of group rings

Passman obtained necessary and sufficient conditions for a group ring $K[G]$ to satisfy a nontrivial polynomial identity over a field $K$ of an arbitrary characteristic. A group $G$ is said $p$-abelian if $G$ is abelian in case $p = 0$ and, in case $p > 0$, $G'$, the commutator subgroup of $G$, is a finite $p$-group.

**Theorem 3.1** ([31]). The group algebra $K[G]$ of a group $G$ satisfies a nontrivial polynomial identity if and only if the following conditions are satisfied.

1. There exists a subgroup $A \subseteq G$ of finite index;
2. $A$ is $p$-abelian.

All our associative algebras are with unity. Recall the notions of a (strong) Lie nilpotence and (strong) solvability for associative algebras. Let $A$ be an associative algebra, and $A^{(\gamma)}$ the related Lie algebra. Consider its lower central series: $\gamma_1(A) = A$, $\gamma_{i+1}(A) = [\gamma_i(A), A]$, $i \geq 1$. Algebra $A$ is said Lie nilpotent of class $s$ iff $\gamma_{s+1}(A) = 0$ and $\gamma_s(A) \neq 0$. Also consider upper Lie powers: $A^{(0)} = A$ and $A^{(n+1)} = [A^{(n)}, A]$, $A$, $n \geq 0$ (we use the shifted enumeration in comparison with [30, 38] because $\{A^{(n)} \mid n \geq 0\}$ is a filtration, a proof is similar to that of Lemma 6.3). Now, $A$ is strongly Lie nilpotent of class $s$ iff $A^{(s)} = 0$ and $A^{(s-1)} \neq 0$. One defines the derived series: $\delta_0(A) = A$, $\delta_{i+1}(A) = [\delta_i(A), \delta_i(A)]$, $i \geq 0$. Algebra $A$ is solvable of length $s$ iff $\delta_s(A) = 0$ and $\delta_{s-1}(A) \neq 0$. Consider also the upper derived series: $\delta_0(A) = A$, $\delta_{i+1}(A) = [\delta_i(A), \delta_i(A)]A$, $i \geq 0$. Now, $A$ is strongly solvable of length $s$ iff $\delta_s(A) = 0$ and $\delta_{s-1}(A) \neq 0$.

Passi, Passman and Sehgal specified the Lie nilpotence and solvability of $K[G]$ [30].

**Theorem 3.2** ([30]). Let $K[G]$ be the group ring of a group $G$ over a field $K$, $\text{char} \ K = p \geq 0$. Then

1. $K[G]$ is Lie nilpotent if and only if $G$ is $p$-abelian and nilpotent;
2. $K[G]$ is solvable if and only if $G$ is $p$-abelian, for $p \neq 2$;
3. $K[G]$ is solvable if and only if $G$ has a 2-abelian subgroup of index at most 2, for $p = 2$. 


Using the upper Lie powers, one defines Lie dimension subgroups of a group (our enumeration is shifted) [29]:

\[ D_{(n)}(G) = G \cap (1 + K[G]^{[n]}), \quad n \geq 0. \]

See another description [9]:

\[ D_{(n)}(G) = \prod_{(i-1)p^k \geq n} \gamma_i(G)^{p^k}, \quad n \geq 0. \] (1)

There is a formula for the Lie nilpotency class of a modular group ring.

**Theorem 3.3** ([9]). Let \( G \) be a group, \( K \) a field of characteristic \( p > 3 \) such that the group ring \( K[G] \) is Lie nilpotent. The Lie nilpotency class of \( K[G] \) coincides with its strong Lie nilpotency class and is equal to

\[ 1 + (p - 1) \sum_{m \geq 1} m \log_p |D_{(m)}(G): D_{(m+1)}(G)|. \]

3.2. **Identities of enveloping algebras.** Latyshev proved that the universal enveloping algebra of a finite dimensional Lie algebra over a field of characteristic zero satisfies a nontrivial polynomial identity if and only if the Lie algebra is abelian [23]. Later Bahturin noticed that the condition of a finite dimensionality is inessential (see e.g. [1]).

Bahturin settled a similar problem on the existence of a nontrivial identity for the universal enveloping algebra over a field of positive characteristic [2]. Also, Bahturin found necessary and sufficient conditions for the universal enveloping algebra of a Lie superalgebra over a field of characteristic zero to satisfy a nontrivial polynomial identity [3].

Passman [32] and Petrogradsky [33] described restricted Lie algebras \( L \) whose restricted enveloping algebra \( u(L) \) satisfies a nontrivial polynomial identity.

**Theorem 3.4** ([32], [33]). Let \( L \) be a Lie \( p \)-algebra. The restricted enveloping algebra \( u(L) \) satisfies a nontrivial polynomial identity if and only if there exist restricted ideals \( Q \subseteq H \subseteq L \) such that

1) \( \dim L/H < \infty, \dim Q < \infty; \)
2) \( H/Q \) is abelian;
3) \( Q \) is abelian and has a nilpotent \( p \)-mapping.

Riley and Shalev determined when \( u(L) \) is Lie nilpotent, solvable (for \( p > 2 \)), or satisfies the Engel condition [37].

**Theorem 3.5** ([37]). Let \( u(L) \) be the restricted enveloping algebra of a restricted Lie algebra \( L \) over a field \( K \) of characteristic \( p > 0 \).

1) \( u(L) \) is Lie nilpotent if and only if \( L \) is nilpotent and \( L^2 \) is finite dimensional and \( p \)-nilpotent;
2) \( u(L) \) is \( n \)-Engel for some \( n \) if and only if \( L \) is nilpotent, \( L^2 \) is \( p \)-nilpotent, and \( L \) has a restricted ideal \( A \) such that both \( L/A \) and \( A^2 \) are finite dimensional.
3) \( u(L) \) is solvable if and only if \( L^2 \) is finite dimensional and \( p \)-nilpotent, for \( p \neq 2 \).

Let \( L \) be a restricted Lie algebra, char \( K = p > 0 \). Similar to the dimension subgroups, using upper Lie powers, Riley and Shalev defined Lie dimension subalgebras [38]:

\[ D_{(n)}(L) = L \cap u(L)^{(n)}, \quad n \geq 0. \]

(recall that our enumeration is shifted). They gave another description [38]:

\[ D_{(n)}(L) = \sum_{(i-1)p^k \geq n} \gamma_i(L)^{p^k}, \quad n \geq 0. \] (2)

Siciliano proved [42] that in case \( p > 2 \), the strong solvability of the restricted enveloping algebra \( u(L) \) is equivalent to its solvability. Moreover, the strong solvability in case \( p = 2 \) is described by the same conditions of Part 3 of Theorem 3.5. Also, in case \( p = 2 \) he observed an example of the restricted enveloping algebra \( u(L) \) that is solvable but not strongly solvable.

The following is an analogue of results on the Lie nilpotency classes of group rings (Theorem 3.3).

**Theorem 3.6** ([38]). Let \( L \) be a restricted Lie algebra over a field \( K \) of characteristic \( p \) such that \( u(L) \) is Lie nilpotent. Then
1) The strong Lie nilpotency class of $u(L)$ is equal to

$$1 + (p - 1) \sum_{m \geq 1} m \dim(D_m(L)/D_{m+1}(L));$$

2) In case $p > 3$, the Lie nilpotency class coincides with the strong Lie nilpotency class.

The solvability of restricted enveloping algebras in case of characteristic 2 was only recently settled in [46]. Lie nilpotence, solvability, and other non-matrix identities for (restricted) enveloping algebras of (restricted) Lie (super)algebras are studied in [8, 34, 43, 44, 49, 50]. More on derived lengths, Lie nilpotency classes for $u(L)$, or identities for symmetric elements of $u(L)$, etc., see the survey [45].

More general cases of (restricted) enveloping algebras for (color) Lie $p$-(super)algebras are treated in [4]. Further developments have been obtained for smash products $U(L)#K[G]$ and $u(L)#K[G]$, where a group $G$ acts by automorphisms on a (restricted) Lie algebra $L$ [5]. Identities of smash products $U(L)#K[G]$, where $L$ is a Lie superalgebra in characteristic zero were studied in [20]. The results on identities of smash products are of interest because they combine, as particular cases, both, the results on identities of group ring and enveloping algebras.

3.3. Identities of symmetric Poisson algebras. The following result is an analogue of the classical Amitsur-Levitzki theorem on identities of matrix algebras. Kostant used another terminology, but as observed Farkas [14], this is a result on identities of symmetric Poisson algebras.

**Theorem 3.7** ([21, 14]). Let $L$ be a finite dimensional Lie algebra over a field of characteristic zero. The symmetric algebra $S(L)$ satisfies the standard Poisson identity $S^d \equiv 0$ as soon as $2d$ exceeds the dimension of a maximal coadjoint orbit of $L$.

The Lie nilpotence of class 2 of symmetric algebras $S(L)$, where $L$ is a Lie superalgebra, was specified by Shestakov. The next statement follows from Theorem 4 and Theorem 5 of [40].

**Theorem 3.8** ([40]). The symmetric algebra $S(L)$ of a Lie algebra $L$, over a field $K$, satisfies the identity $\{x, \{y, z\}\} \equiv 0$ if and only if $L$ is abelian.

Farkas proved the following statement that generalizes Kostant’s Theorem 3.7.

**Theorem 3.9** ([14]). Let $L$ be a Lie algebra over a field of characteristic zero. The symmetric algebra $S(L)$ satisfies a nontrivial Poisson identity if and only if $L$ contains an abelian subalgebra of finite codimension.

Giambruno and Petrogradsky extended this result to an arbitrary characteristic [15].

**Theorem 3.10** ([15]). Let $L$ be a Lie algebra over an arbitrary field. The symmetric algebra $S(L)$ satisfies a nontrivial multilinear Poisson identity if and only if $L$ contains an abelian subalgebra of finite codimension.

The following result was obtained for the truncated symmetric algebra $s(L)$ of a restricted Lie algebra $L$ (see definitions [19]).

**Theorem 3.11** ([15]). Let $L$ be a restricted Lie algebra. The truncated symmetric algebra $s(L)$ satisfies a nontrivial multilinear Poisson identity if and only if there exists a restricted ideal $H \subseteq L$ such that

1) $\dim L/H < \infty$;
2) $\dim H^2 < \infty$;
3) $H$ is nilpotent of class 2.

The present paper is heavily based on this result. Remark that the identities of the (strong) Lie nilpotence and (strong) solvability are multilinear. We shall apply this result in a more precise form given by Theorem 5.6.

4. Main results: Lie identities of symmetric algebras

In this section we formulate our main results.
4.1. Lie nilpotence of truncated symmetric algebras \( s(L) \). Let \( R \) be an arbitrary Poisson algebra. Consider the lower central series of \( R \) as a Lie algebra, i.e., \( \gamma_1(R) = R \) and \( \gamma_{n+1}(R) = \{\gamma_n(R), R\}, n \geq 1 \). We say that \( R \) is Lie nilpotent of class \( s \) if and only if \( \gamma_{s+1}(R) = 0 \) but \( \gamma_s(R) \neq 0 \). Clearly, the condition \( \gamma_{s+1}(R) = 0 \) is equivalent to the identity of Lie nilpotence of class \( s \):

\[
\{\ldots\{X_0, X_1\}, X_2\}, \ldots, X_s\} = 0.
\]

Similar to the associative case, one defines upper Lie powers as follows. At each step we take ideals generated by commutators, namely, set \( R^{(0)} = R \) and \( R^{(n)} = \{R^{(n-1)}, R\} \cdot R \) for all \( n \geq 1 \) (our enumeration is shifted, a justification is that \( \{R^{(n)}|n \geq 0\} \) is a filtration, see Lemma 6.3). We call a Poisson algebra \( R \) strongly Lie nilpotent of class \( s \) if and only if \( R^{(s)} = 0 \) but \( R^{(s-1)} = 0 \). The condition \( R^{(s)} = 0 \) is equivalent to the identical relation of the strong Lie nilpotence of class \( s \):

\[
\{\ldots\{X_0, X_1\} \cdot Y_1, X_2\} \cdot Y_2, \ldots, X_{s-1}\} \cdot Y_{s-1}, X_s\} = 0.
\]

Observe that

\[
\gamma_n(R) \subseteq R^{(n-1)}, \quad n \geq 1.
\]

Thus, the strong Lie nilpotence of class \( s \) implies the Lie nilpotence of class at most \( s \). The Lie nilpotence of class 1 is equivalent to the strong Lie nilpotence of class 1 and equivalent to the fact that \( R \) is abelian as a Lie algebra.

The following is the first main result of the paper.

**Theorem 4.1.** Let \( L \) be a Lie algebra over a field of positive characteristic \( p \). Consider its truncated symmetric Poisson algebra \( s(L) \). The following conditions are equivalent:

1) \( s(L) \) is strongly Lie nilpotent;
2) \( s(L) \) is Lie nilpotent;
3) \( L \) is nilpotent and \( \dim L^2 < \infty \).

Let \( L \) be a Lie algebra over an arbitrary field \( K \) (\( \text{char} K = p \)). Using the upper Lie powers, define Poisson dimension subalgebras (truncated Poisson dimension subalgebras, respectively) of \( L \):

\[
D^S_{(n)}(L) = L \cap (S(L))^{(n)}, \quad n \geq 0;
\]

\[
D^s_{(n)}(L) = L \cap (s(L))^{(n)}, \quad n \geq 0.
\]

By Corollary 7.2 (Claim 3 of Lemma 7.1, respectively), we obtain a description of these subalgebras similar to that for the group rings (1) and the restricted enveloping algebras (2) (compare with the first term of that products):

\[
D^S_{(n)}(L) = \gamma_{n+1}(L), \quad n \geq 0;
\]

\[
D^s_{(n)}(L) = \gamma_{n+1}(L), \quad n \geq 0.
\]

We compute the classes of Lie nilpotence and strong Lie nilpotence. Our formula is an analogue of the formulas known for group rings (Theorem 3.3) and restricted enveloping algebras (Theorem 3.6). The analogy is better seen in terms of truncated Poisson dimension subalgebras.

**Theorem 4.2.** Let \( L \) be a Lie algebra over a field of positive characteristic \( p > 3 \), such that the truncated symmetric Poisson algebra \( s(L) \) is Lie nilpotent. The following numbers are equal:

1) the strong Lie nilpotency class of \( s(L) \);
2) the Lie nilpotency class of \( s(L) \);
3) \[ 1 + (p - 1) \sum_{n \geq 1} n \cdot \dim(\gamma_{n+1}(L)/\gamma_{n+2}(L)). \]

In cases \( p = 2, 3 \), the numbers 1) and 3) remain equal.

For cases \( p = 2, 3 \), the number above yields an upper bound for the Lie nilpotency class. Also, we have a lower bound for the Lie nilpotency class, \( L \) being non-abelian (Lemma 9.2):

\[ 2 + (p - 1) \sum_{n \geq 2} (n - 1) \cdot \dim(\gamma_{n+1}(L)/\gamma_{n+2}(L)). \]
4.2. Solvability of truncated symmetric algebras \( s(L) \). Let \( R \) be a Poisson algebra. Consider its derived series as a Lie algebra: \( \delta_0(R) = R \), \( \delta_{n+1}(R) = \{\delta_n(R), \delta_n(R)\} \), \( n \geq 0 \). Polynomials of solvability are defined by recursion: \( \delta_1(X_1, X_2) = \{X_1, X_2\} \) and 
\[
\delta_{n+1}(X_1, X_2, \ldots, X_{2^n+1}) = \{\delta_n(X_1, \ldots, X_{2^n}), \delta_n(X_{2^n+1}, \ldots, X_{2^{n+1}})\}, \quad n \geq 1.
\]

A Poisson algebra \( R \) is solvable of length \( s \) if, and only if, \( \delta_s(R) = 0 \) and \( \delta_{s-1}(R) \neq 0 \), equivalently, that \( R \) satisfies the above identity of Lie solvability \( \delta_s(\ldots) \equiv 0 \), where \( s \) is minimal.

Define the upper derived series: \( \tilde{\delta}_0(R) = R \) and \( \tilde{\delta}_{n+1}(R) = \{\tilde{\delta}_n(R), \tilde{\delta}_n(R)\} \cdot R \), \( n \geq 0 \). Define polynomials of the strong solvability by \( \tilde{\delta}_1(X_1, X_2, Y_1) = \{X_1, X_2\} \cdot Y_1 \), and 
\[
\tilde{\delta}_{n+1}(X_1, \ldots, X_{2^n+1}, Y_1, \ldots, Y_{2^n+1-1})
\]
\[
= \left\{ \tilde{\delta}_n(X_1, \ldots, X_{2^n}, Y_1, \ldots, Y_{2^n-1}),
\tilde{\delta}_n(X_{2^n+1}, \ldots, X_{2^{n+1}}, Y_{2^n}, \ldots, Y_{2^{n+1}-2}) \right\} \cdot Y_{2^n+1-1}, \quad n \geq 1.
\]

A Poisson algebra \( R \) is called strongly solvable of length \( s \) if and only if \( \tilde{\delta}_s(R) = 0 \) and \( \tilde{\delta}_{s-1}(R) \neq 0 \), or equivalently that \( R \) satisfies the above identity \( \tilde{\delta}_s(\ldots) \equiv 0 \), where \( s \) is minimal. Observe that 
\[
\delta_s(R) \subseteq \tilde{\delta}_s(R), \quad s \geq 0.
\]

Thus, the strong solvability of length \( s \) implies the solvability of length at most \( s \). The solvability of length 1 is equivalent to the strong solvability of length 1 and equivalent to the fact that \( R \) is abelian as a Lie algebra.

The following is the second main result of the paper.

**Theorem 4.3.** Let \( L \) be a Lie algebra over a field of positive characteristic \( p \geq 3 \). Consider its truncated symmetric Poisson algebra \( s(L) \). The following conditions are equivalent:

1) \( s(L) \) is strongly solvable;
2) \( s(L) \) is solvable;
3) \( L \) is solvable and \( \dim L^2 < \infty \).

In case \( p = 2 \), conditions 1) and 3) remain equivalent.

Observe that, the description of solvable group rings in characteristic 2 looks very nice (Theorem 3.2). But the answer to a similar question for the restricted enveloping algebras is rather complicated and was obtained only recently [46].

The problem of solvability of \( s(L) \) in case \( \text{char } K = 2 \) is open. As a first approach, we show that the situation is different to other characteristics. Namely, in case \( \text{char } K = 2 \), we obtain two examples of truncated symmetric Poisson algebras that are solvable but not strongly solvable, see Lemma 11.1 and Lemma 11.2. A close fact is that the Hamiltonian algebras \( \mathfrak{h}_2(K) \) and \( \mathfrak{h}_2(K) \) are solvable but not strongly solvable in case \( \text{char } K = 2 \) (Lemma 11.3). This is an analogue of a well-known fact that the matrix ring \( M_2(K) \) of \( 2 \times 2 \) matrices over a field \( K \), \( \text{char } K = 2 \), is solvable but not strongly solvable.

4.3. Nilpotency and solvability of symmetric algebras \( S(L) \). Finally, we prove the following extension of the result of Shestakov (Theorem 3.8).

**Theorem 4.4.** Let \( L \) be a Lie algebra over a field \( K \), and \( S(L) \) its symmetric Poisson algebra. The following conditions are equivalent:

1) \( L \) is abelian;
2) \( S(L) \) is strongly Lie nilpotent;
3) \( S(L) \) is Lie nilpotent;
4) \( S(L) \) is strongly solvable;
5) \( S(L) \) is solvable (here assume that \( \text{char } K \neq 2 \)).

In case \( \text{char } K = 2 \), the solvability of the symmetric Poisson algebra \( S(L) \) is an open question. Two examples of Lie algebras mentioned above also yield solvable symmetric algebras which are not strongly solvable (Lemma 12.1 and Lemma 12.2).

**Conjecture.** Let \( L \) be a Lie algebra over a field \( K \), \( \text{char } K = 2 \). The symmetric algebra \( S(L) \) (probably, the truncated symmetric algebra \( s(L) \) as well) is solvable if and only if \( L = \langle x \rangle_K \oplus A \), where \( A \) is an abelian ideal on which \( \text{ad } x \) acts algebraically.
Remark. Formally, our statements on ordinary Lie nilpotency and solvability are concerned only with the Lie structure of Poisson algebras $s(L)$ and $S(L)$. But our proof heavily relies on Theorem 5.6 of [15], which in turn uses the existence of a nontrivial customary identity given by Theorem 2.2 (Farlás [13]). In this way, we need the Poisson structure of our algebras to prove our results. We do not see ways to prove them using the theory of Lie identical relations only.

5. Delta-sets and multilinear Poisson identical relations

The goal of this section is to present a reduction step given by Theorem 5.6. Delta-sets in groups were introduced by Passman to study identities in the group rings [31]. In case of Lie algebras, the delta-sets were introduced by Bahturin to study identical relations of the universal enveloping algebras [2]. Let $L$ be a Lie algebra, one defines the delta-sets as sets of elements of finite width as follows:

$$\Delta_n(L) = \{ x \in L \mid \dim[L, x] \leq n \}, \quad n \geq 0;$$

$$\Delta(L) = \bigcup_{n=0}^{\infty} \Delta_n(L) = \{ x \in L \mid \dim[L, x] < \infty \}.$$

Note that $\Delta_n(L)$, $n \geq 0$, is not a subalgebra or even a subspace in a general case. The basic properties of the delta-sets are as follows.

Lemma 5.1 ([4, 34]). Let $L$ be a (restricted) Lie algebra, $n, m \geq 0$.

1) $\Delta_n(L)$ is invariant under scalar multiplication;

2) let $x \in \Delta_n(L)$, $y \in \Delta_m(L)$, then $\alpha x + \beta y \in \Delta_{n+m}(L)$, where $\alpha, \beta \in K$;

3) let $x \in \Delta_n(L)$, $y \in L$, then $[x, y] \in \Delta_{2n}(L)$;

4) let $x \in \Delta_n(L)$ and $L$ a restricted Lie algebra, then $x^p \in \Delta_n(L)$;

5) $\Delta(L)$ is a (restricted) ideal of $L$.

Lemma 5.2 ([37]). Let $L$ be a Lie algebra.

1) if $I$ is a finite dimensional ideal of $L$, then $\Delta(L/I) = (\Delta(L) + I)/I$;

2) if $H$ is a subalgebra of finite codimension in $L$, then $\Delta(H) = \Delta(L) \cap H$.

Suppose that $W$ is a subset in a $K$-vector space $V$. We say that $W$ has finite codimension in $V$ if there exist $v_1, \ldots, v_m \in V$ such that $V = \{ w + \lambda_1 v_1 + \cdots + \lambda_m v_m \mid w \in W, \lambda_1, \ldots, \lambda_m \in K \}$. Let $m$ be the minimum integer with such property, then we denote $\dim V/W = m$. We also introduce a notation $m = W = \{ w_1 + \cdots + w_m \mid w_i \in W \}$, where $m \in \mathbb{N}$.

Lemma 5.3 ([5, Lemma 6.3]). Let $V$ be a $K$-vector space. Suppose that a subset $T \subseteq V$ is stable under multiplication by scalars and $\dim V/T \leq n$. Then one obtains its linear span as follows: $\langle T \rangle_K = 4^n \cdot T$.

We need a result on bilinear maps.

Theorem 5.4 (P.M. Neumann, [1]). Let $U, V, W$ be vector spaces over a field $K$ and $\varphi : U \times V \to W$ a bilinear map. Suppose that for all $u \in U$ and $v \in V$, $\dim \varphi(u, V) \leq m$ and $\dim \varphi(U, v) \leq l$. Then $\dim \langle \varphi(U, V) \rangle_K \leq ml$.

The following crucial result was proved in case of restricted Lie algebras [15], but its proof remains valid for truncated symmetric algebras as well.

Theorem 5.5 ([15]). Let $L$ be a Lie algebra. Suppose that the symmetric algebra $S(L)$ (or the truncated symmetric algebra $s(L)$) satisfies a multilinear Poisson identity. There exist integers $n, N$ such that $\dim L/\Delta_N(L) < n$.

It yields the following reduction step, which is actually contained in proofs of [15].

Theorem 5.6 ([15]). Let $L$ be a Lie algebra such that the symmetric algebra $S(L)$ (or the truncated symmetric algebra $s(L)$) satisfies a multilinear Poisson identity. Denote $\Delta = \Delta(L)$. There exist integers $n, M$ such that

1) $\Delta = \Delta_M(L)$;

2) $\dim L/\Delta < n$;

3) $\dim \Delta^2 \leq M^2$.

Proof. By Theorem 5.5, we have $\dim L/\Delta_N(L) < n$. By Lemma 5.1 and Lemma 5.3, to obtain the linear span $\langle \Delta_N(L) \rangle_K$ it is sufficient to consider $4^n$-fold sums. Applying Lemma 5.1, we get $\langle \Delta_N(L) \rangle_K \subseteq \Delta_{4^n N}(L)$. By virtue of finite codimension, we get $\Delta = \Delta_M(L)$ for some integer $M$. Finally, $\dim[\Delta, \Delta] \leq M^2$ by Theorem 5.4. □
6. Filtrations of Lie algebras and symmetric algebras

Let \( L \) be a Lie algebra over a field \( K \) of positive characteristic \( p \). A filtration of the truncated symmetric Poisson algebra \( s(L) \) is a descending sequence of subspaces

\[
\begin{align*}
    s(L) &= E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_n \supsetneq \cdots, \\
    E_i \cdot E_j &\subseteq E_{i+j}, \quad \{E_i, E_j\} \subseteq E_{i+j}, \quad i, j \geq 0.
\end{align*}
\]

By setting \( L_n = L \cap E_n, n \geq 0, \) one gets naturally a sequence of subspaces:

\[
    L = L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_n \supsetneq \cdots, \quad \bigcap_{n=0}^{\infty} L_n = 0, \quad \{L_i, L_j\} \subseteq L_{i+j}, \quad i, j \geq 0.
\]

The last inclusion follows by definition of the product \( \{ , \} \) in \( s(L) \). We call a sequence of subspaces above a filtration of \( L \). We have just seen that a filtration of \( s(L) \) gives rise to a filtration of \( L \). One has a converse statement.

**Lemma 6.1.** Let \( \{L_n \mid n \geq 0 \} \) be a filtration of a Lie algebra \( L \) over a field \( K \) of characteristic \( p > 0 \). For each \( 0 \neq x \in L \) define the height of \( x \), denoted by \( \nu(x) \), as the largest subscript \( n \) such that \( x \in L_n \). Define

\[
    E_n = \{ y_1 y_2 \cdots y_l \mid y_i \in L, \, \nu(y_1) + \nu(y_2) + \cdots + \nu(y_l) \geq n \} \quad (5)
\]

Then, \( \{E_n \mid n \geq 0 \} \) is a filtration of \( s(L) \).

**Proof.** The inclusion \( E_n \cdot E_m \subseteq E_{n+m}, n, m \geq 0, \) is clear. Let us prove that \( \{E_n, E_m\} \subseteq E_{n+m}, n, m \geq 0, \). Let \( a = x_1 \cdots x_i \in E_n \), and \( b = y_1 \cdots y_k \in E_m \). By the Leibnitz rule, \( \{a, b\} = \sum_{j} x_1 \cdots \hat{x}_i \cdots x_l \cdot y_1 \cdots \hat{y}_j \cdots y_k \{x_i, y_t\} \). Since \( \nu\{x_i, y_j\} \geq \nu(x_i) + \nu(y_j) \), we conclude that \( \{a, b\} \in E_{n+m} \).

We call \( \{E_n \mid n \geq 0 \} \) the induced filtration. A close relationship between these filtrations is demonstrated by the next result. A similar relationship was used in [35].

**Lemma 6.2.** Let \( \{L_n \mid n \geq 0 \} \) be a filtration of a Lie algebra \( L \) over a field \( K \) of characteristic \( p > 0 \), \( \{x_i \mid i \in I \} \) an ordered basis of \( L \) chosen so that \( L_n = \langle x_i \mid \nu(x_i) \geq n \rangle \), \( n \geq 0 \). Let \( \{E_n \mid n \geq 0 \} \) be the induced filtration of \( s(L) \). We have the following statements for \( n \geq 0 \):

1. \( E_n = \langle \eta = x_{i_1} a_{i_1} \cdots x_{i_l} a_{i_l} \mid \nu(\eta) = \sum_{j} a_i \nu(x_i) \geq n, \, i_1 < \cdots < i_l, \, 0 \leq a_j < p, \, l \geq 0 \rangle \);  
2. \( \langle \eta \mid \nu(\eta) = n \rangle \) is a basis of \( E_n \) modulo \( E_{n+1} \);
3. the set of all such monomials \( \eta \) forms a basis of \( s(L) \);
4. \( L_n = L \cap E_n \) (i.e., we return to the initial filtration).

**Proof.** Clearly, all monomials \( \eta \) above belong to \( E_n \). Consider \( 0 \neq y \in L_m, \, m \geq 0, \) present it as a finite sum \( y = \sum_{i, j} \lambda_i x_i \lambda_j K \). Apply such expansions for all \( y \) in product (5) and reorder multiplicands of the resulting monomials. In this way we present products (5) as linear combinations of monomials \( \{\eta \mid \nu(\eta) \geq n \}, \) the latter being linearly independent as a canonical basis of \( s(L) \). The first claim is proved. The remaining claims follow.

**Lemma 6.3.** Let \( R \) be a Poisson algebra. The upper Lie powers \( \{R^{(n)} \mid n \geq 0 \} \) form a filtration.

**Proof.** First, we prove that \( R^{(i)} \cdot R^{(j)} \subseteq R^{(i+j)} \) for all \( i, j \geq 0 \) by induction on \( j \). Let \( j = 0 \), then \( R^{(0)} = R \) and there is nothing to prove. Suppose that the statement is valid for \( j - 1 \geq 0 \). We use the Leibnitz rule in form \( a[b, c] = \{ab, c\} - b\{a, c\} \) and apply the inductive assumption:

\[
    R^{(i)} \cdot R^{(j)} = R^{(i)} \{R^{(j-1)}, R\} R \subseteq \{R^{(i)}, R^{(j-1)}, R\} R + R^{(j-1)} \{R^{(i)}, R\} R \subseteq \{R^{(i+j-1)}, R\} R + R^{(j-1)} R^{(i-1)} \subseteq R^{(i+j)}.
\]

Second, we prove that \( \{R^{(i)}, R^{(j)}\} \subseteq R^{(i+j)} \) for all \( i, j \geq 0 \), by induction on \( j \). Let \( j = 0 \), then \( \{R^{(i)}, R^{(0)}\} = \{R^{(i)}, R\} \subseteq R^{(i+1)} \subseteq R^{(i)} \). Suppose that the statement is valid for \( j - 1 \geq 0 \). We use the Leibnitz rule, the Jacobi identity, and the inclusion above:

\[
    \{R^{(i)}, R^{(j)}\} = \{R^{(i)}, \{R^{(j-1)}, R\} \cdot R\} \subseteq \{R^{(j-1)}, R\} \cdot \{R^{(i)}, R\} + \{R^{(i)}, \{R^{(j-1)}, R\}\} \cdot R \subseteq \{R^{(i)}, R^{(i+j-1)}\} = \left(\{R^{(i+j-1)}, R\} + \{R^{(j-1)}, R^{(i)}\}\right) \cdot R \subseteq R^{(i+j)} + \{R^{(i+j)}, R\} \subseteq R^{(i+j)}.
\]
7. Lie nilpotence of truncated symmetric algebras $s(L)$

The goal of this section to prove the first main result. Now, consider our Poisson algebra $R = s(L)$. We shift the enumeration of the lower central series terms, namely we consider:

$$L = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_n \supseteq \cdots, \quad \text{where} \quad L_n = \gamma_{n+1}(L), \, n \geq 0.$$  

We show that this shifted filtration $\{L_n \mid n \geq 0\}$ induces the filtration of $s(L)$ by the upper Lie powers $\{s(L)^{(n)} \mid n \geq 0\}$, see Lemma 6.3.

**Lemma 7.1.** Let $L$ be a Lie algebra over a field of positive characteristic $p$.  

1) the upper Lie powers of the truncated symmetric algebra $s(L)$ are as follows

$$s(L)^{(n)} = \sum_{1 \leq m_1 \leq m_2 \leq \cdots \leq m_s} \gamma_{m_1}(L) \gamma_{m_2}(L) \cdots \gamma_{m_s}(L), \quad n \geq 0;$$

2) $\{s(L)^{(n)} \mid n \geq 0\}$ is induced by the shifted filtration $\{L_i = \gamma_{i+1}(L) \mid i \geq 0\};$

3) $L \cap s(L)^{(n)} = \gamma_{n+1}(L), \, n \geq 0$ (these are truncated Poisson dimension subalgebras).

**Proof.** Consider a finite number of integers $m_j \geq 1$ such that $\sum_j (m_j - 1) \geq n$. By (3), we have $\gamma_{m}(L) \subseteq s(L) \subseteq s(L)^{(m-1)}, \, m \geq 1$. We use Lemma 6.3

$$\gamma_{m_1}(L) \cdots \gamma_{m_s}(L) \subseteq s(L)^{(m_1-1)} \cdots s(L)^{(m_s-1)} \subseteq s(L)^{(m_1-1) + \cdots + (m_s-1)} \subseteq s(L)^{(n)},$$

implying one inclusion of the first claim. We prove the reverse inclusion by induction on $n$. The case $n = 0$ is trivial. Let $a \in \gamma_{m_i}(L), \, m_i \geq 1$, and $w = b_1 \cdots b_t \in s(L), \, b_j \in L, \, j = 1, \ldots, t$, then

$$\{a, w\} = \sum_{j=1}^{t} b_1 \cdots \hat{b}_j \cdots b_t \{a, b_j\}, \quad \{a, b_j\} = [a, b_j] \in \gamma_{m_j+1}(L), \quad b_k \in L = \gamma_1(L), \, k \neq j. \quad (6)$$

Consider $n \geq 1$. We apply (6) to prove the inductive step

$$s(L)^{(n)} = \{s(L)^{(n-1)}, s(L)\} s(L) \subseteq \left\{ \sum_{1 \leq m_1 \leq m_2 \leq \cdots \leq m_s} \gamma_{m_1}(L) \cdots \gamma_{m_s}(L), s(L) \right\} s(L) \subseteq \sum_{1 \leq m_1 \leq m_2 \leq \cdots \leq m_s} \gamma_{m_1}(L) \cdots \gamma_{m_s}(L).$$

Now Claim 2 follows by Claim 1 and definitions of the shifted filtration and the induced filtration. Claim 3 follows by Claim 4 of Lemma 6.2. □

**Corollary 7.2.** Let $L$ be a Lie algebra. Consider the upper Lie powers of the symmetric algebra $S(L)^{(n)}, \, n \geq 0$. Define Poisson dimension subalgebras: $D_S^{(n)}(L) = L \cap (S(L))^{(n)}, \, n \geq 0$. Then $D_S^{(n)}(L) = \gamma_{n+1}(L), \, n \geq 0$.

**Proof.** Follows by the same arguments. □

**Corollary 7.3.** $s(L)^{(n)}$ modulo $s(L)^{(n+1)}$ has the following basis for all $n \geq 0$:

$$\langle \eta = \sum_{0 \leq i_0 < i_1 < \cdots < i_m} e^{i_0 a_{i_0}} \cdots e_{i_m} a_{i_m} | \nu(\eta) = \sum_{i=0}^{m} \sum_{j=1}^{s} \alpha_{ij} = n; \ 0 \leq \alpha_{ij} \leq p-1, \ s \geq 0 \rangle. \quad (7)$$

**Proof.** Follows by Claim 2 of Lemma 6.2. □

Let $\{e_{ij} \mid j \in J_i, i \geq 0\}$ be an ordered basis of $L$ chosen so that $\{e_{ij} \mid j \in J_i\}$ is a base for $L_i = \gamma_{i+1}(L)$ modulo $L_{i+1} = \gamma_{i+2}(L)$, for all $i \geq 0$. We have the height function $\nu(e_{ij}) = i, \ j \in J_i, \ i \geq 0$.

**Proof of Theorem 4.1.** Implication 1) ⇒ 2) follows by (3).

Implication 3) ⇒ 1). Suppose that $L$ is nilpotent and $\text{dim } L^2 < \infty$. Then $d_i = \text{dim } \gamma_{i+1}(L)/\gamma_{i+2}(L) < \infty$ for $i \geq 1$. Let $\{e_{i1}, e_{i2}, \ldots, e_{id_i}\}$ be a base for $\gamma_{i+1}(L)$ modulo $\gamma_{i+2}(L)$, for all $i \geq 1$. Consider a basis element $\eta$ of the truncated symmetric algebra (7). Put $N = (p-1) \sum_{i \geq 1} 1/d_i$. We have $\nu(\eta) = \sum_{i=1}^{s} \sum_{j=1}^{d_i} \alpha_{ij} \leq (p-1) \sum_{i \geq 1} 1/d_i = N$. Hence, $s(L)^{(N+1)} = 0$. On the other hand, consider $v = e_{i_1}^{p-1} \cdots e_{i_m}^{p-1}$ with $\nu(v) = (p-1) \sum_{i \geq 1} 1/d_i = N$, so $0 \neq v \in s(L)^{(N)}$, thus $s(L)^{(N)} \neq 0$. We proved that $s(L)$ is strongly Lie nilpotent of class $N+1$. 

Finally, let us prove implication 2) $\Rightarrow$ 3). Since $L$ is clearly nilpotent, it remains to prove that $L^2$ is finite dimensional. By Theorem 5.6, there exist integers $n, M$ such that $\dim L/\Delta < n$, $\Delta = \Delta(L) = \Delta_M(L)$, and $\dim \Delta^2 \leq M^2$. It suffices to prove that $L = \Delta$. Let us assume the opposite, that $L \neq \Delta$.

Consider the factor algebra $\bar{L} = L/\Delta^2$. Since $\dim \Delta^2 < \infty$, by Lemma 5.2, $\Delta(\bar{L}) = \Delta/\Delta^2$. Hence, $\Delta(L)$ is abelian and $\bar{L} = \Delta(L)$. Replacing $L$ by $\bar{L}$ we may assume that $\Delta$ is abelian.

Fix $x \in L \setminus \Delta$ and consider $H = \langle x \rangle_K \oplus \Delta$. By Lemma 5.2 we have $\Delta(H) = \Delta$. By nilpotency of $s(L)$ there exists an integer $k$ such that $(ad x)^k(H) = 0$. Let us reduce our arguments to the case $(ad x)^2(H) = 0$. Put $D_0 = \Delta$ and $D_{m+1} = [x, D_m]$, $m \geq 0$. We obtain a chain of $H$-ideals:

$$\Delta = D_0 \supseteq D_1 = [x, \Delta] = H^2 \supseteq D_2 \supseteq \cdots \supseteq D_k = 0. \quad (8)$$

Since $\dim D_1 = \infty$, there exists $i$ (where $1 \leq i < k$) such that $\dim D_i/D_{i+1} = \infty$. Consider $H = ((x)_K \oplus D_{i-1})/D_{i+1}$, let $\bar{x}$ be the image of $x$ in $H$. We have $\dim[\bar{x}, H] = \dim([x, D_{i-1}] + D_{i+1})/D_{i+1} = \dim D_i/D_{i+1} = \infty$. By construction, $(ad \bar{x})^2(H) = 0$.

Now our arguments are reduced to the case $H = \langle x \rangle_K \oplus \Delta$, where $\Delta$ is an abelian ideal, $(ad x)^2\Delta = 0$, and $\dim[x, \Delta] = \infty$. We find elements $\{y_i | i \in \mathbb{N}\} \subseteq \Delta$ such that the elements $z_i = [x, y_i]$, where $i \in \mathbb{N}$, are linearly independent. One checks that $\{y_i, z_i | i \in \mathbb{N}\}$ is a linearly independent set. We claim that $\{x, xy_1, xy_2, \ldots, xy_n\} = xz_1z_2 \cdots z_n$ for all $n \geq 1$. The case $i = 1$ follows by the Leibnitz rule. Suppose that the statement is valid for $n - 1$. Observe that $z_1, z_2, \ldots$ are central and we get

$$\{x, xy_1, xy_2, \ldots, xy_n\} = \{x, xy_1, xy_2, \ldots, xy_{n-1}, xy_n\} = \{xz_1 \cdots z_{n-1}, xy_n\} = xz_1 \cdots z_{n-1}z_n, \quad n \geq 1.$$ 

The right hand side is non-zero by PBW-theorem for all $n \geq 1$, a contradiction with our assumption that $s(H)$ is Lie nilpotent. The contradiction proves that $L = \Delta$. \hfill \Box

Above we also proved the following part of Theorem 4.2.

**Corollary 7.4.** Let $L$ be a Lie algebra over a field of positive characteristic $p$. Assume that $L$ is nilpotent and $\dim L^2 < \infty$. Then $s(L)$ is strongly Lie nilpotent of class

$$1 + (p-1) \sum_{n \geq 1} n \cdot \dim(\gamma_{n+1}(L)/\gamma_{n+2}(L)).$$

8. **Products of commutators in Poisson algebras**

The goal of this section is to prove a technical result on products of commutators in Poisson algebras (Theorem 8.1) that is used to get a lower bound on the Lie nilpotency class of $s(L)$.

Products of terms of the lower central series for *associative algebras* appear in works of many mathematicians, the results being reproved without knowing the earlier works. We do not pretend to make a complete survey here. Probably, the first observations on products of commutators in associative algebras were made by Latyshev in 1965 [24] and Volichenko in 1978 [53]. There are further works e.g. [6, 12, 16, 17, 22].

In case of associative algebras, Claim 1 of Theorem 8.1, probably, first was established by Sharma-Shrivastava in 1990, [39, Theorem 2.8]. As was remarked in [38], the proof of the associative version of Claim 2 of Theorem 8.1 is implicitly contained in [39], where it is proved for group rings. A weaker statement (the associative version of Lemma 8.7) is established by Gupta and Levin [18, Theorem 3.2].

The following statement is a Poisson version of the respective results for associative algebras. The validity of it is not automatically clear. We follow a neat approach due to Krasnikov [22].

**Theorem 8.1.** Let $R$ be an arbitrary Poisson algebra over a field $K$, char $K \neq 2, 3$.

1) suppose that one of integers $n, m \geq 1$ is odd, then

$$\gamma_n(R) \cdot \gamma_m(R) \subseteq \gamma_{n+m-1}(R)R;$$

2) for all $x_1, \ldots, x_n \in R$, $n, m \geq 1$ we have

$$\{x_1, \ldots, x_n\}^m \in \gamma_{(n-1)m+1}(R)R.$$

We proceed by steps.

**Lemma 8.2.** Let $R$ be an arbitrary Poisson algebra.

1) for any $a_1, a_2, a_3, a_4, a_5 \in R$, where one of $a_1, a_2, a_5$ belongs to $\gamma_m(R)$, $m \geq 1$, we have

$$\{a_1, a_2, a_3\} \{a_4, a_5\} + \{a_1, a_2, a_4\}\{a_3, a_5\} \in \gamma_{m+3}(R)R;$$
Thus, by properties of Lie commutators, let Lemma 8.4.

Consider the original product, the first, and the last term obtained. By properties of Lie commutators, \( \{a_1, a_2, a_3, a_4, a_5\} \in \gamma_{m+3}(R) \), \( \{a_1, a_2, a_4, a_5\} \in \gamma_{m+3}(R) \), \( \{a_1, a_2, a_3, a_5\} \in \gamma_{m+3}(R) \). Two remaining middle terms exactly give the first claim

\[
\{a_1, a_2, a_3\}\{a_4, a_5\} + \{a_1, a_2, a_4\}\{a_3, a_5\} \in \gamma_{m+3}(R)R.
\]

Consider the second claim. By the Leibnitz rule:

\[
\{a_5, a_2 a_4, a_1, a_3\} = \{a_2 a_5, a_4\} + \{a_5, a_2\} a_4, a_1, a_3\}
\]

The initial product, the first and last terms belong to \( \gamma_{m+3}(R)R \). We apply the first claim to the sums in the second and forth lines. Each of them belongs to \( \gamma_{m+3}(R)R \), because we permute \( a_3, a_1 \) in both cases. Finally, the sum of two elements in the third line yields:

\[
\{a_2, a_1, a_3\}\{a_5, a_4\} + \{a_5, a_2\}\{a_4, a_1, a_3\}
\]

\[
= \{a_1, a_2, a_3\}\{a_4, a_5\} + \{a_1, a_4, a_3\}\{a_2, a_5\} \in \gamma_{m+3}(R)R. \quad \Box
\]

**Lemma 8.3.** Let \( R \) be a Poisson algebra. For any \( a_5 \in \gamma_m(R) \), \( m \geq 1 \), and \( a_1, a_2, a_3, a_4 \in R \) we have

\[
3\{a_1, a_2, a_3\}\{a_4, a_5\} \in \gamma_{m+3}(R)R.
\]

**Proof.** Consider \( W = \{a_1, a_2, a_3\}\{a_4, a_5\} \). Modulo \( \gamma_{m+3}(R)R \), it is skewsymmetric in \( a_3, a_4 \) (Lemma 8.2, first claim) and skewsymmetric in \( a_2, a_1 \) (Lemma 8.2, second claim). Therefore, it is skewsymmetric in \( a_2, a_3 \) modulo \( \gamma_{m+3}(R)R \). The skewsymmetry in \( a_1, a_2 \) follows by anticommutativity. Therefore, \( W \) is skewsymmetric in \( a_1, a_2, a_3 \). Consider even permutations

\[
\{a_1, a_2, a_3\}\{a_4, a_5\} + \{a_2, a_3, a_1\}\{a_4, a_5\} + \{a_3, a_1, a_2\}\{a_4, a_5\}
\]

\[
= 3\{a_1, a_2, a_3\}\{a_4, a_5\} \mod \gamma_{m+3}(R)R.
\]

On the other hand, the left hand side is equal to zero by the Jacobi identity. The result follows. \( \Box \)

**Lemma 8.4.** Let \( R \) be a Poisson algebra over a field \( K \), \( \text{char } K \neq 2, 3 \). Then

\[
\gamma_m(R)R, R, R \subseteq \gamma_{m+2}(R)R, \quad m \geq 2.
\]

**Proof.** Let \( c \in \gamma_m(R) \) and \( x, y, z \in R \). Consider the product

\[
\{c x, y, z\} = c\{x, y, z\} + \{c, y\}\{x, z\} + \{c, z\}\{x, y\} + \{c, y, z\}x.
\]

By properties of Lie commutators, \( \{c, y, z\} x \in \gamma_{m+3}(R)R \).

Since \( c \in \gamma_m(R) \), \( m \geq 2 \), we can consider that \( c = \{a, b\} \), where \( a \in R, b \in \gamma_{m-1}(R) \). Applying Lemma 8.3, we get \( c\{x, y, z\} \in \gamma_{m+2}(R)R \).

Consider two remaining middle terms in (9)

\[
\{c, y\}\{x, z\} + \{c, z\}\{x, y\} = \{z, x\}\{y, c\} + \{y, x\}\{z, c\}
\]

\[
= \{xy, x, c\} - z\{y, x, c\} - \{z, x, c\}y \in \gamma_{m+2}(R)R.
\]

Thus, \( \{c x, y, z\} \in \gamma_{m+2}(R)R \), yielding the result. \( \Box \)

Now we prove the first claim of Theorem 8.1 as a separate statement.
Lemma 8.5. Let $R$ be a Poisson algebra over a field $K$, char $K \neq 2, 3$. Suppose that one of integers $n, m \geq 1$ is odd, then $\gamma_n(R) \gamma_m(R) \subseteq \gamma_{n+m-1}(R) R$.

Proof. We assume that $n$ is odd and proceed by induction on $n$. The case $n = 1$ is trivial. The base of induction $n = 3$ is proved in Lemma 8.3.

Let $n \geq 5$ be odd. Take $a \in \gamma_n(R)$, $b \in \gamma_m(R)$. Since Lie products are linear combinations of left-normed Lie products, without loss of generality assume that $a = \{c, x, y\}$, $c \in \gamma_{n-2}(R)$, $x, y \in R$.

Then

\[ a \cdot b = \{c, x, y\}b = \{cb, x, y\} - \{c, x\}\{b, y\} - \{c, y\}\{b, x\} - c\{b, x, y\}. \quad (10) \]

Consider the first term in (10). By the inductive assumption, $\{cb, x, y\} \in \gamma_{n+m-3}(R) R$, and by Lemma 8.4, we get $\{cb, x, y\} \in \gamma_{n+m-1}(R) R$.

Consider the last term in (10). Clearly, $\{b, x, y\} \in \gamma_{n+2}(R)$, recall that $c \in \gamma_{n-2}(R)$ and $n - 2$ is odd. By the inductive assumption, $c\{b, x, y\} \in \gamma_{n+m-1}(R) R$.

Consider two remaining middle terms in (10).

\[ \{c, x\}\{b, y\} + \{c, y\}\{b, x\} = (y, b)\{x, c\} + (x, b)\{y, c\} = (yx, b, c) - (y, b, c)x - y\{x, b, c\}. \]

In the Lie brackets above, $c \in \gamma_{n-2}(R)$, $b \in \gamma_m(R)$, and the remaining element belongs to $R$. Thus, all three products above belong to $\gamma_{n+m-1}(R) R$. Lemma is proved.

We prove the second claim of Theorem 8.1 as a separate statement.

Lemma 8.6. Let $R$ be a Poisson algebra over a field $K$, char $K \neq 2, 3$. Let $x_1, \ldots, x_n \in R$, $n, m \geq 1$. Then $\{x_1, \ldots, x_n\}^m \in \gamma_{(n-1)m+1}(R) R$.

Proof. Case 1: $n$ is odd. The statement follows by consecutive application of Claim 1 of Theorem 8.1.

Case 2: $n$ is even and $m = 2$. Denote $a = \{x_1, \ldots, x_{n-1}\}$, $x = x_n$. Then $a \in \gamma_{n-1}(R)$ and $\{x_1, \ldots, x_n\} = \{a, x\}$. Consider

\[ \{(x^2, a), a\} = 2\{x, a\}\{x, a\} + 2x\{x, a, a\}; \]

\[ \{x, a\}^2 = 2\{x^2, a\}, a\} - x\{x, a, a\} \in \gamma_{2n-1}(R) R. \]

Case 3: $n$ is even and $m = 2q$, $q > 1$. We apply the previous cases.

\[ \{x_1, \ldots, x_n\}^{2q} = \{(x_1, \ldots, x_n)^2\}^q \in \gamma_{(2n-1)(R) R} \]

\[ = \gamma_{2n-1}(R) R \subseteq \gamma_{(2n-2)q+1}(R) R = \gamma_{(n-1)m+1}(R) R. \]

Case 4: $n$ is even and $m = 2q + 1$, $q \geq 1$ (the case $m = 1$ is trivial). We use the previous cases.

\[ \{x_1, \ldots, x_n\}^m \in \gamma_{(n-1)2q+1}(R) \gamma_{n}(R) \subseteq \gamma_{(n-1)(m-1)+n}(R) R = \gamma_{(n-1)m+1}(R) R. \]

The following is an analogue of the respective fact for associative algebras, see [18, Theorem 3.2]. It is weaker than Claim 1 of Theorem 8.1, but it is valid for an arbitrary characteristic.

Lemma 8.7. Let $R$ be a Poisson algebra over an arbitrary field $K$. Then

\[ \gamma_m(R) \gamma_n(R) \subseteq \gamma_{m+n-2}(R) R, \quad n, m \geq 2. \]

Proof. We proceed by induction on $n \geq 2$ for an arbitrary $m \geq 2$. In case $n = 2$ there is nothing to prove. Assume that $n > 2$. To study $\gamma_m(R) \gamma_n(R)$ it is sufficient to consider products of the form $u = \{r, a\}\{x, y, z_1, \ldots, z_{n-2}\}$, where $r \in \gamma_{m-1}(R)$ and all other elements belong to $R$. Consider $w = \{a\{x, y\}, r, z_1, \ldots, z_{n-2}\}$, we get $w \in \gamma_{m+n-2}(R)$. On the other hand

\[ w = \{a\{x, y\}, r, z_1, \ldots, z_{n-2}\} = \{a, r\}\{x, y, z_1, \ldots, z_{n-2}\} \]

\[ + \sum_{s=1}^{n-2} \sum_{i,j} \{a, r, z_{i_1}, \ldots, z_{i_s}\}\{x, y, z_{j_1}, \ldots, z_{j_{n-2-s}}\} \]

\[ + \sum_{s=0}^{n-2} \sum_{i,j} \{a, z_{i_1}, \ldots, z_{i_s}\}\{x, y, r, z_{j_1}, \ldots, z_{j_{n-2-s}}\}, \]

where the sums above correspond to all partitions of the set $\{1, \ldots, n - 2\}$ in two subsets of indices such that $i_1 < \cdots < i_s$, $j_1 < \cdots < j_{n-2-s}$. The first term yields the desired product $-u$. Products
of the first sums belong to $\gamma_{m+s}(R)\gamma_{n-s}(R) \in \gamma_{n+m-2}(R)R$ by the inductive assumption because $s > 1$. Products of the second sums belong to $\gamma_{s+1}(R)\gamma_{n+m-s-1}(R) \in \gamma_{n+m-2}(R)R$ by the inductive assumption because $s + 1 < n$. Lemma is proved. \hfill\Box

9. Lie nilpotency classes of truncated symmetric algebras $s(L)$

Recall that the strong Lie nilpotency class is an upper bound for the ordinary Lie nilpotency class. Actually, below we prove that these classes coincide for many examples of Poisson algebras that we are studying. We essentially apply the results of the previous section on products of commutators in Poisson algebras. The following result and Corollary 7.4 yield Theorem 4.2.

**Theorem 9.1.** Let $L$ be a Lie algebra over a field of positive characteristic $p > 3$. Assume that $L$ is nilpotent and $\dim L^2 < \infty$. The Lie nilpotency class of $s(L)$ coincides with the strong Lie nilpotency class and is equal to

$$1 + (p-1) \sum_{n \geq 1} n \cdot \dim(\gamma_{n+1}(L)/\gamma_{n+2}(L)).$$

**Proof.** Corollary 7.4 yields the required value for the strong Lie nilpotency class. We have finite numbers $d_i = \dim \gamma_{i+1}(L)/\gamma_{i+2}(L)$ for $i \geq 1$. Let $d_s \neq 0$ but $d_{s+1} = d_{s+2} = \ldots = 0$, for some $s$. Let $\{e_{ij} \mid 1 \leq j \leq d_i\}$ be a basis for $\gamma_{i+1}(L)$ modulo $\gamma_{i+2}(L)$, where $1 \leq i \leq s$. Consider

$$0 \neq v = e_{p-1,j_1} \cdots e_{p-1,j_d} \cdots e_{p-1,j_1} \cdots e_{p-1,j_i}.$$  

(11)

Observe that the basis elements above can be chosen as commutators, namely $e_{ij} = \{y_1, \ldots, y_{i+1}\}$, where $y_k \in L$. By Claim 2 of Theorem 8.1, $e_{ij}^{p-1} \in \gamma_{i(p-1)+1}(R)R$, for all $j = 1, \ldots, d_i$, $i = 1, \ldots, s$. Since $i(p-1) + 1$ is odd, we can apply Claim 1 of Theorem 8.1 and obtain that $e_{i(p-1)+1,j_i}^{p-1} \in \gamma_{d_i,i(p-1)+1}(R)R$. Similarly, the total product (11) enjoys the property $v \in \gamma_{N+1}(R)R$, where $N = (p-1)\sum_{i=1}^s d_i$. Therefore, $\gamma_{N+1}(R) \neq 0$. On the other hand, $\gamma_{N+2}(R) \subseteq R^{(N+1)} = 0$ by (3) and Corollary 7.4. We proved that $s(L)$ is Lie nilpotent of the required class $N + 1$. \hfill\Box

Now, let us establish bounds for the ordinary Lie nilpotency class in case $\text{char} K = 2, 3$.

**Lemma 9.2.** Let $L$ be a non-abelian Lie algebra over a field of characteristic $p = 2, 3$. Assume that $L$ is nilpotent and $\dim L^2 < \infty$. The Lie nilpotency class of $s(L)$ is bounded from above by the strong Lie nilpotency class and bounded from below by the following number

$$2 + (p-1) \sum_{n \geq 1} (n-1) \cdot \dim(\gamma_{n+1}(L)/\gamma_{n+2}(L)).$$

**Proof.** Let bases elements $e_{ij} \in \gamma_{i+1}(L)$ be chosen as in the proof above. Applying Lemma 8.7, we get

$$e_{ij}^{p-1} \in \gamma_{i(p-1)+2}(R)R,$$

where $1 \leq j \leq d_i$, $1 \leq i \leq s$. (Since $L$ is non-abelian, there exists at least one basis element $e_{11} \in \gamma_2(L)$, and product (11) is non-trivial.) Applying Lemma 8.7 to the whole of product (11), we get $0 \neq v \in \gamma_{M}(R)R$, where $M = (p-1)\sum_{i=1}^s (i-1)d_i + 2$. Thus, $\gamma_M(R) \neq 0$. Therefore, $M$ is a lower bound for the Lie nilpotency class of $s(L)$. \hfill\Box

10. Solvability of truncated symmetric algebras $s(L)$

In this section we prove Theorem 4.3 in case $p \neq 2$.

Implication 1) $\Rightarrow$ 2) follows by inclusion (4).

Now, we prove implication 3) $\Rightarrow$ 1). Suppose that $\dim L^2 < \infty$ and $L$ is solvable of length $k \geq 1$. We proceed by induction on $k$. The base of induction $k = 1$ is trivial because $L$ is abelian and $s(L)$ is abelian as well.

Assume that $k > 1$. We have $\delta_k(L) = 0$ and $\delta_{k-1}(L) \neq 0$. Put $\bar{L} = L/\delta_{k-1}(L)$ and consider the induced map $\varphi : s(L) \to s(\bar{L})$. By the induction hypothesis, $s(\bar{L})$ is strongly solvable, then there exists $m$ such that $\bar{\delta}_m(s(\bar{L})) = 0$, thus $\delta_m(s(L)) \subseteq J = s(L)\delta_{k-1}(L)$, the latter being the kernel of $\varphi$.

Let $\{b_1, \ldots, b_l\}$ be a basis of $\delta_{k-1}(L)$ and $\{a_i \in L \mid i \in I\}$ a basis of $L$ modulo $\delta_{k-1}(L)$. Consider a PBW-basis monomial $w \in J$, it has a form $w = a^\alpha b^\beta$, where $a^\alpha = a_1^\alpha_1 a_2^\alpha_2 \cdots a_n^\alpha_n$, $0 \leq \alpha_i < p$ and $b^\beta = b_1^\beta_1 b_2^\beta_2 \cdots b_l^\beta_l \in s(\delta_{k-1}(L))$, $0 \leq \beta_j \leq p-1$, where at least one $\beta_j$ is nonzero. Denote $|\beta| = \beta_1 + \cdots + \beta_l$. We have $|\beta| \geq 1$.

By definition of the upper derived series, elements of $\bar{\delta}_{m+1}(s(\bar{L}))$ are expressed via $\{w_1, w_2\}v$ where $w_1, w_2 \in J$, which we write as above, and $v \in s(L)$. Compute using the Leibnitz rule $\{w_1, w_2\}v = \cdots$
\[ \sum_i u_j, \text{where we get a sum of products containing one factor of type } \{a_i, a_j\} \text{ or } \{a_i, b_j\} \in \delta_{k-1}(L) \text{ (as } \delta_{k-1}(L) \text{ is an ideal of } L), \text{ recall that } \{b_i, b_j\} = 0 \text{ because } \delta_{k-1}(L) \text{ is abelian. In all cases we obtain monomials with } |\beta''| \geq |\beta| + |\beta'| \geq 2. \text{ Thus, the PBW-monomials of } \delta_{m+1}(s(L)) \text{ contain at least two elements } b_j. \text{ Continuing this process, we conclude that monomials of } \delta_{m+N}(s(L)) \text{ contain at least } 2^N \text{ such elements. Recall that } \dim \delta_{k-1}(L) = l \text{ and } \dim s(\delta_{k-1}(L)) = p'. \text{ As soon as } 2^N > p', \text{ we get } \delta_{m+N}(s(L)) = 0. \text{ We proved that } s(L) \text{ is strongly solvable.} \]

The rest of the section is devoted to implication 2) \(\Rightarrow\) 3). We assume that \(s(L)\) is solvable. Since \(L \subseteq s(L)\), clearly \(L\) is solvable, it remains to prove that \(\dim L^2 < \infty\). By Theorem 5.6, there exist integers \(n, M\) such that \(\dim L/\Delta < n\), where \(\Delta = \Delta(L) = \Delta_M(L)\) and \(\dim \Delta^2 \leq M^2\). So, it suffices to prove that \(L = \Delta\). Let us assume the opposite, that \(L \neq \Delta\), in the rest of the section we prove that this assumption leads to a contradiction. Consider \(\bar{L} = L/\Delta^2\). Since \(\dim \Delta^2 < \infty\), by Lemma 5.2, \(\Delta(\bar{L}) = \Delta/\Delta^2\). Hence, \(\Delta(\bar{L})\) is abelian and \(\bar{L} \neq \Delta(\bar{L})\). Replacing \(L\) by \(\bar{L}\) we may assume that \(\Delta\) is abelian. Since our identity is multilinear, we can assume that \(K\) is algebraically closed.

**Lemma 10.1.** Suppose that \(L\) is a Lie algebra over a field \(K\), \(x \in L \setminus \Delta\) and \(\Delta\) an abelian ideal. Let \(s(L)\) (or symmetric algebra \(S(L)\)) be solvable of length \(s\). Then \(ax\) acts on \(\Delta\) algebraically, of degree bounded by \(M(s) = (s + 1)^2 + s\).

**Proof.** By assumption, \(s(L)\) satisfies the identity \(\delta_i(x_1, x_2, \ldots, x_{2^r}) \equiv 0\). Fix \(x \in L \setminus \Delta\). Consider an element \(0 \neq v \in \Delta\). Denote \(v_i = (ax)^iv\) for \(i \geq 0\). We want to prove that \(\{v_i \mid 0 \leq i \leq M\}\) is a linearly dependent set, where \(M = M(s)\). By way of contradiction, assume that \(\{v_i \mid 0 \leq i \leq M\}\) is linearly independent. We shall show that, choosing \(N\) sufficiently large, \(\delta_i(xv_N, xv_2N, \ldots, xv_{2^N}N) \neq 0\).

We compare elements of the same length \(u = xv_{\alpha_1}v_{\alpha_2} \cdots v_{\alpha_k}, w = xv_{\beta_1}v_{\beta_2} \cdots v_{\beta_k}\) lexicographically starting with the senior indices. Namely, assume that \(\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k, \beta_1 \leq \beta_2 \leq \cdots \leq \beta_k\). Then \(u < w\) if and only if \(\alpha_k < \beta_k\), or \(\alpha_k = \beta_k\), but \(\alpha_k-1 < \beta_k-1\), etc.

Let \(k = 1\), consider the following evaluation \(x_1 = xv_N, x_2 = xv_2N:\)

\[ \delta_1(x_1, x_2) = \{x_1, x_2\} = \{xv_N, xv_N\} = xv_N(v_{2N}v_{2N+1} - v_{2N+1}v_{2N}), \]

where the elements \(xv_{N2N+1}\) and \(xv_{N+1}v_{2N}\) are nonzero and different.

Let \(k = 2\), consider the evaluation by \(x_1 = xv_N, x_2 = xv_2N, x_3 = xv_3N, x_4 = xv_4N:\)

\[ \delta_2(xv_N, xv_2N, xv_3N, xv_4N) = \{xv_N, xv_2N, xv_3N, xv_4N\} = \{xv_{N2N+1} - v_{N+1}v_{2N}, xv_{3N}v_{4N}v_{3N+1} - v_{3N+1}v_{4N}\} = x(v_{N2N+1} - v_{N+1}v_{2N})(v_{3N}v_{4N}v_{3N+2}-v_{3N+2}v_{4N}) \]

\[ + x(v_{N+2}v_{2N} - v_{N}v_{2N+2})(v_{4N}v_{4N+1} - v_{4N+1}v_{4N+1}), \]

where the element \(xv_{N2N+1}v_{3N}v_{4N+2}\) is the unique maximal with respect to the lexicographic order.

In general, let \(k \geq 1\), making a similar evaluation \(x_i = xv_iN\), for all \(i = 1, 2, \ldots, 2^k\), we present \(\delta_k(x_1, x_2, \ldots, x_{2^k})\) as a sum

\[ \sum_{\epsilon_j} \theta_{\epsilon_j}xv_{N+\epsilon_1}v_{2N+\epsilon_2}v_{3N+\epsilon_3} \cdots v_{kN+\epsilon_k}, \quad \theta_{\epsilon} \in K; \]

\[ \epsilon_1 + \epsilon_2 + \cdots + \epsilon_k = 2^k - 1, \quad 0 \leq \epsilon_i \leq k, \quad i = 1, 2, \ldots, 2^k. \]

The sum contains a unique largest product having \(\theta_{\epsilon} = \pm 1\). Indeed, let for \(k\) we have a unique maximal element \((12)\) given by a tuple \((\alpha_1, \alpha_2, \ldots, \alpha_{2^k})\), then the unique maximal element for \(k + 1\) is given by the tuple \((\alpha_1, \ldots, \alpha_{2^k}, \alpha_1, \ldots, \alpha_{2^k} - 1, \alpha_{2^k} + 1)\). Also, we check by induction that \(\epsilon_i \leq k\).

Now, we consider \(k = s\) and fix \(N = s + 1\) to guarantee that the indices of \(v_k\) in \((12)\) do not overlap. Observe that the highest index in \((12)\) is bounded by \(M(s) = (s + 1)^2 + s\). The unique maximal tuple yields a non-zero product \(xv_{N+\alpha_1}v_{2N+\alpha_2}v_{3N+\alpha_3} \cdots v_{kN+\alpha_k}\) in \((12)\). A contradiction to the fact that \(\delta_k(x_1, x_2, \ldots, x_{2^k}) = 0\) in \(s(L)\).

The contradiction proves that for each \(v \in \Delta\) there exists a polynomial \(h_v(t)\), such that \(h_v(ad(x)v) = 0\) and \(\deg h_v(t) \leq M\). Therefore, \(\Delta\) is a periodic \(K[t]\)-module under the action \(t \circ w = (ad(x)w, w \in \Delta)\).

Consider the respective Jordan decomposition

\[ \Delta = \bigoplus_{i \in I} V(\lambda_i), \quad V(\lambda_i) = \{y \in \Delta \mid (ad(x - \lambda_i E))^m y = 0\}, \lambda_i \in K, \; i \in I, \]

where \(m_i\) is minimal. The minimal polynomial of the action of \(ad(x)\) on \(\Delta\) is \(h(t) = \prod_{i \in I}(t - \lambda_i)^{m_i}\). Indeed, it is sufficient to consider the action on elements \(v = \sum_{i \in I} v_i\), where \(0 \neq v_i \in V(\lambda_i)\) correspond.
to minimal degrees \( m_i, i \in I \). Hence, \( \text{deg} h(t) = \sum_{i \in I} m_i \leq M(s) \). Thus, \( a \cdot x \) acts algebraically on \( \Delta \) of degree bounded by \( M(s) \).

**Corollary 10.2.** Let \( x \in L \setminus \Delta \) act on \( \Delta \) as above. Then

1. The number of different eigenvalues \( \lambda_i, i \in I \), is bounded by \( M(s) \).
2. Let \( V_{\lambda_i}, V_{(\lambda_i)} \) be the eigenspace and respective Jordan subspace corresponding to an eigenvalue \( \lambda_i \). Then \( \dim V_{\lambda_i} \leq \dim V_{\lambda_i} \setminus M(s) \), \( i \in I \).
3. There exists an eigenvalue \( \lambda \) with \( \dim [x, V_{(\lambda)}] = \infty \).
4. Let \( \lambda \) be as above, then \( \dim V_{\lambda} = \infty \).

**Proof.** The first claim follows by the bound on the degree of the minimal polynomial \( h(t) \) found above. The second claim follows by the fact that the size of each Jordan matrix block corresponding to a given \( \lambda_i \) is bounded by \( M(s) \).

Consider \( H = \langle x \rangle_K \oplus \Delta \). By Lemma 5.2, \( \Delta(H) = \Delta \), thus \( \dim[x, \Delta] = \infty \). Since \( [x, \Delta] = \sum_{i \in I} [x, V_{(\lambda_i)}] \), the third claim follows.

Claim 4 follows from Claims 2 and 3. □

Now, by Corollary 10.2, our proof is reduced to the following situation. We have \( L = \langle x \rangle \oplus V_{(\lambda)} \) with \( \dim[x, V_{(\lambda)}] = \infty \) and \( \dim V_{\lambda} = \infty \). In our proof we have two cases: A) \( \lambda = 0 \), and B) \( \lambda \neq 0 \).

**The case A:** \( \lambda = 0 \). Now \( V_{(0)} = \{ y \mid (\text{ad } x)^m y = 0 \} \), where \( m \geq 2 \) is minimal. We can assume that \( (\text{ad } x)^2 V_{(0)} = 0 \). This reduction is made as in the proof on Lie nilpotence (see application of the chain (8)). The case is reduced to the following statement, in which proof we follow approach of [37].

**Proposition 10.3.** Let \( L \) be a Lie algebra over a field of odd characteristic such that

1. \( L = \langle x \rangle \oplus \Delta \), where \( \Delta \) is an abelian ideal;
2. \( (\text{ad } x)^2 \Delta = 0, \dim[x, \Delta] = \infty \).

Then \( s(L) \) is not solvable.

We adopt the following notations. Denote \( R = s(L) \) and \( a' = \{ x, a \} \), for any \( a \in s(\Delta) \). Denote \( s(x) = \langle x^i \mid 0 \leq i \leq p - 1 \rangle \). We have \( R = s(x)s(\Delta) \). Let \( S = \{ a \in s(\Delta) \mid a'' = 0 \} \). Note that \( S \) is a linear subspace of \( s(\Delta) \), but it need not be a subring. However, we do have the following result.

**Lemma 10.4.** Let \( a, b \in S \). Then \( a'b \in S \) and \( a'b + ab' \in S \).

**Proof.** By the Leibnitz rule, \( (a'b)''' = \{ x, \{ x, a'b \} \} = \{ x, a'b' \} = 0 \). □

**Lemma 10.5.** Let \( a, b \in S \). For all \( 0 \leq i, j < p \) we have:

1. \( \{ x^i, a \} = ix^{i-1}a' \);
2. \( \{ x^i a, x^j b \} = x^{i+j-1}(iab' - ja'b) \).

**Proof.** By induction. □

**Lemma 10.6.** Let \( a, b \in S \) such that \( s(x)a, s(x)b \subseteq \delta_{n-1}(R) \). Then \( s(x)(a'b + ab') \subseteq \delta_n(R) \).

**Proof.** By Lemma 10.5, \( \delta_n(R) \ni \{ x^i a, b \} = ix^{i-1}ab', i = 1, \ldots, p - 1 \). Hence, \( x^i ab' \in \delta_n(R) \), and similarly \( x^i a'b \in \delta_n(R) \), for \( i = 0, 1, \ldots, p - 2 \). It remains to settle the case \( i = p - 1 \). By Lemma 10.5, \( x^{p-1}(ab' + a'b) = x^{p-1}(ab' - (p - 1)a'b) = \{ x, x^{p-1}b \} \in \delta_n(R) \).

We supply \( S \) with a new bilinear operation \((a, b) \mapsto a \circ b = a'b + ab', a, b \in S \). We define a new “derived series” of \( S \) by \( S[0] = S \) and \( S[n+1] = S[[n+1]] \circ S[[n-1]] \), the linear space generated by the elements \( a \circ b \) where \( a, b \in S[[n-1]] \), where \( n \geq 1 \).

**Lemma 10.7.** \( s(x)s[[n]] \subseteq \delta_n(R) \) for all \( n \geq 0 \).

**Proof.** Follows from Lemma 10.6 by induction on \( n \). □

For all \( a_1, a_2, \ldots, a_l \in S \), where \( l \geq 1 \), let us define a new product:

\[
[[a_1, a_2, \ldots, a_l]] = a_1' a_2' \cdots a_{l-1}' a_l + a_1' a_2' \cdots a_{l-2}' a_{l-1}' + \cdots + a_1 a_2' \cdots a_l'.
\]

(13)

**Lemma 10.8.** \{ \{[a_1, a_2, \ldots, a_{2^n}] | a_1, a_2, \ldots, a_{2^n} \in S \} \subseteq S[[n]] \} for all \( n \geq 0 \).
Proof. We proceed by induction on \( n \). Consider \( n = 0 \), we have \( \{a_1\} = a_1 \in S = S^{[0]} \). Consider \( n = 1 \), we have \( \{a_1, a_2\} = a_1a_2' + a_2a_1' = a_1 \circ a_2 \in S \circ S = S^{[1]} \).

Assume that the assertion is valid for \( n \geq 1 \) and consider \( n+1 \). Note that, since \( a_1, a_2, \ldots, a_{2n} \in S \), we have \( \{a_1, a_2, \ldots, a_{2n}\}' = 2^na_1'a_2' \cdots a_{2n}' \). Hence,

\[
\{a_1, a_2, \ldots, a_{2n}\} \circ \{b_1, b_2, \ldots, b_{2n}\} \]

\[
= 2^n \left( a_1'a_2' \cdots a_{2n}' \{b_1, b_2, \ldots, b_{2n}\} + a_1a_2, \ldots, a_{2n}\} b_1b_2' \cdots b_{2n}' \right)
\]

By induction hypothesis, \( \{a_1, a_2, \ldots, a_{2n}\}, \{b_1, b_2, \ldots, b_{2n}\} \in S^{[n]} \). Since \( p \neq 2 \), we get

\[
\{a_1, a_2, \ldots, a_{2n}, b_1, b_2, \ldots, b_{2n}\} \in S^{[n]} \circ S^{[n]} = S^{[n+1]}. \]

**Lemma 10.9.** For all \( a_1, a_2, \ldots, a_{2n} \in S \), we have \( \mathfrak{s}(x) \{a_1, a_2, \ldots, a_{2n}\} \subseteq \delta_n(R) \), where \( n \geq 0 \).

Proof. Follows form Lemma 10.7 and Lemma 10.8.

**Proof. of Proposition 10.3.** Since \( \dim[x, \Delta] = \infty \), we can choose \( \{a_n | n \in \mathbb{N}\} \subseteq \Delta \) such that \( \{a_n' | n \in \mathbb{N}\} \subseteq \Delta \) is linearly independent. One checks that \( \{a_n, a_n' | n \in \mathbb{N}\} \) is linearly independent as well. Since \( \mathfrak{s}(\Delta) \) is a truncated polynomial ring, monomials in (13) are linearly independent. Therefore, \( \{a_1, a_2, \ldots, a_{2n}\} \neq 0 \) and \( \delta_n(R) \neq 0 \) for \( n \geq 0 \) by Lemma 10.9. So, \( \mathfrak{s}(L) \) is not solvable.

**The case B:** \( \lambda \neq 0 \). By Corollary 10.2, we have the eigenspace \( V_\lambda \) with \( \dim V_\lambda = \infty \). Consider a subalgebra \( \langle x \rangle \oplus V_\lambda \subseteq L \), by taking \( \tilde{x} = \frac{1}{\lambda} x \), we can assume that \( \lambda = 1 \). The case is reduced to the following statement.

**Proposition 10.10.** Let \( L \) be a Lie algebra over a field of odd characteristic such that

1) \( L = \langle x \rangle \oplus \Delta \), where \( \Delta \) is an abelian ideal;

2) \( [x, v] = v \), for all \( v \in \Delta \), and \( \dim \Delta = \infty \);

Then \( \mathfrak{s}(L) \) is not solvable.

Proof. We choose a linearly independent set \( \Xi = \{v_i | i \in \mathbb{N}\} \subseteq \Delta \). Next, we construct sets of multilinear monomials in \( \Xi \) of type \( v = v_{i_1} \cdots v_{i_k} \in \mathfrak{s}(\Delta) \), denote their lengths as \( |v| = k \). Observe that \( v' = \langle x, v \rangle = |v|v \). We start with \( e_1 = v_{i_1}, h_1 = v_{i_1}v_{i_2}, f_1 = v_{i_1}v_{i_2}v_{i_3}, i_1 < i_2 < i_3 \), where these symbols denote sets of all multilinear monomials in \( \Xi \) of lengths 1, 2, 3, respectively. Next, define recursively sets of multilinear monomials in \( \Xi \):

\[
ed_{n+1} = e_nh_n, \quad h_{n+1} = h_nf_n, \quad f_{n+1} = h_nf_n, \quad n \geq 1.
\]

where we include these products in our new sets only when they are multilinear. One checks by induction that

\[
|e_n| = 2^n - 1, \quad |h_n| = 2^n, \quad |f_n| = 2^n + 1, \quad n \geq 1.
\]

Thus, \( e_n, h_n, f_n \) consist of all multilinear monomials in \( \Xi \) of the respective lengths. Let us prove by induction on \( n \) that \( xe_n, xh_n, xf_n \in \delta_{n-1}(\mathfrak{s}(L)) \), \( n \geq 0 \). The base of induction \( n = 0 \) is trivial. Assume the claim for \( k = n \). We have the following relations

\[
\{xe_n, xh_n\} = x(e_nh_n' - e_n'h_n) = x(|h_n| - |e_n|)e_nh_n = xe_nh_n;
\]

\[
\{xh_n, xf_n\} = x(h_nf_n' - h_n'f_n) = x(|f_n| - |h_n|)h_nf_n = xh_nf_n;
\]

\[
\{xe_n, xf_n\} = x(e_nf_n' - e_n'f_n) = x(|f_n| - |e_n|)e_nf_n = 2xe_nf_n.
\]

Considering only multilinear products above, we get the inductive step: \( xe_{n+1}, xh_{n+1}, xf_{n+1} \in \delta_n(\mathfrak{s}(L)) \). Therefore, \( \delta_n(\mathfrak{s}(L)) \neq 0 \) for all \( n \geq 0 \). We proved that \( \mathfrak{s}(L) \) is not solvable.

Proof. of Theorem 4.3 in case \( p \neq 2 \). In this section, the proof was reduced to two cases settled in Proposition 10.3 and Proposition 10.10. They lead to a contradiction, which implies that \( L = \Delta \) and \( \dim L^2 = \dim \Delta^2 < \infty \).
11. Solvability of truncated symmetric algebras $s(L)$, char $K = 2$

In this section we consider the case char $K = 2$. We finish the proof of a part of Theorem 4.3 on the strong solvability of $s(L)$ and show that the question of the solvability of $s(L)$ is different to other characteristics. Namely, we obtain two examples of the truncated symmetric algebras that are solvable but not strongly solvable. These examples also show that Proposition 10.3 and Proposition 10.10 are no longer valid in case char $K = 2$.

**Lemma 11.1.** Let $L = \langle x, y_i \mid [x, y_i] = y_i, i \in \mathbb{N} \rangle_K$, char $K = 2$, the remaining commutators being trivial. Then

1) $L^2 = \Delta(L) = \Delta_1(L) = \langle y_i \mid i \in \mathbb{N} \rangle$;
2) $s(L)$ is solvable of length 3;
3) $s(L)$ is not strongly solvable.

*Proof.* The first claim is checked directly.

Denote $H = \langle y_i \mid i \in \mathbb{N} \rangle$. The basis elements of $s(L)$ are of the first type $y_i, \ldots y_k \in s(H)$ and the second type $xy_i, \ldots y_k$, where $i_1 < \cdots < i_k$, and $k \geq 0$ will be referred to as the length. Now, $s(H)$ is abelian and the remaining commutators of the basis monomials are as follows:

$$\begin{align*}
\{xy_1, \ldots, y_n, y_{j_1}, \ldots, y_{j_k}\} &= ky_1, \ldots, y_n, y_{j_1}, \ldots, y_{j_k};
\{xy_1, \ldots, y_n, x_{j_1}, \ldots, y_{j_k}\} &= (n-m)xy_1, \ldots, y_n, y_{j_1}, \ldots, y_{j_k}.
\end{align*}$$

(14)

Commutators (14) are non-zero only when $m, n$ have different parities. Thus, $\delta_1(s(L)) = \{s(L), s(L)\}$ contains monomials of the first type and of the second type of odd length. Similarly, $\delta_2(s(L))$ contains monomials of the first type only. Finally, we get $\delta_3(s(L)) = 0$.

To prove the last claim, one checks by induction on $k$ that $\delta_k(s(L))$, $k \geq 0$, contain infinitely many monomials of the second type for both, even and odd, lengths. Indeed, use (14) and the fact that we can additionally multiply the commutators by $y_n$ to make the resulting length even. \qed

**Lemma 11.2.** Let $L = \langle x, y_i, z_i \mid [x, y_i] = z_i, i \in \mathbb{N} \rangle_K$, char $K = 2$, the remaining commutators being trivial. Then

1) $\Delta(L) = \Delta_1(L) = \langle y_i, z_i \mid i \in \mathbb{N} \rangle$ and $L^2 = \langle z_i \mid i \in \mathbb{N} \rangle$;
2) $s(L)$ is solvable of length 3;
3) $s(L)$ is not strongly solvable.

*Proof.* The first claim is checked directly.

Put $H = \langle y_i, z_i \mid i \in \mathbb{N} \rangle$. For any $a \in s(H)$ write $a' = \{x, a\}$. Consider a basis monomial $w = v_1 \cdots v_n \in s(H)$, where $v_i$ denote elements from the basis of $H$. Then

$$w'' = (v_1 \cdots v_n)'' = \sum_{i=1}^{n} v_1 \cdots v_i'' \cdots v_n + 2 \sum_{1 \leq i < j \leq n} v_1 \cdots v_i' \cdots v_j' \cdots v_n = 0. \quad (15)$$

A basis monomial of $s(L)$ without $x$ is of the first type, otherwise, of the second type. Let $xa, xb, xc, xd \in s(L)$ be all of the second type, where $a, b, c, d \in s(H)$. Using (15),

$$\delta_2(xa, xb, xc, xd) = \{xa, xb\}, \{xc, xd\} = \{x(a'b + ab'), x(c'd + cd')\}$$

$$= x(a''b + 2a'b' + 2ab')(c'd + cd') + x(a'b + ab')(c''d + 2c'd' + cd'') = 0.$$

Let $w_1, w_2, w_3, w_4 \in s(L)$ be basis monomials where at least one is of the first type. One checks that $\delta_2(w_1, w_2, w_3, w_4)$ can be of the first type only. We proved that $\delta_2(s(L)) \subseteq s(H)$. Finally, $\delta_3(s(L)) = 0$.

Let us prove the last claim. The following observation simplifies our computations below

$$\langle y_1 z_1 + z_1 y_1 \rangle' = y_1' z_1 + y_1 z_1' + z_1' y_2 + z_1 y_2 = 2z_1 z_2 = 0.$$

We use possibility to multiply additionally in the identity of the strong solvability

$$\{xy_1, xy_2\} = x(y_1 z_2 + z_1 y_2);$$

$$\{xy_1, xy_2\} y_5, \{xy_3, xy_4\} y_6 = \{x(y_1 z_2 + z_1 y_2), y_5, x(y_3 z_4 + z_3 y_4) y_6\}$$

$$= x(y_1 z_2 + z_1 y_2)(y_3 z_4 + z_3 y_4)(y_5 z_6 + z_5 y_6).$$

We continue computations, each time inserting new variables, like $y_5, y_6$ above, and conclude that $\delta_n(s(L)) \neq 0$ for all $n \geq 0$. Thus, $s(L)$ is not strongly solvable. \qed
Proof. of Theorem 4.3 in case $p = 2$. It remains to prove that the strong solvability implies that \( \dim L^2 < \infty \). Recall that Lemma 10.1 and its Corollary 10.2 remain valid in case $p = 2$. They reduced our proof above to the fact that the Lie algebras of Proposition 10.3 and Proposition 10.10 are not allowed. The arguments further lead to Lie algebras of Lemma 11.1 and Lemma 11.2, which are not strongly solvable in case char $K = 2$.

Two examples above are closely related to the following observation.

**Lemma 11.3.** Consider the truncated Hamiltonian Poisson algebra $P = \mathfrak{h}_2(K)$ (or the Hamiltonian Poisson algebra $P = \mathbf{H}_2(K)$), char $K = 2$. Then

1) $P$ is solvable of length 3.

2) $P$ is not strongly solvable.

**Proof.** Let $P = \mathfrak{h}_2(K) = K[X,Y]/(X^2,Y^2) = \langle 1, x, y, xy \rangle_K$, where $x, y$ denote the images of $X, Y$. We have $\delta_1(P) = \{P, P\} = \langle 1, x, y \rangle_K$, $\delta_2(P) = \{1\}_K$, and $\delta_3(P) = 0$. As above, one checks that $P$ is not strongly solvable.

Let $P = \mathbf{H}_2(K) = K[X,Y]$. The Poisson brackets of monomials $X^nY^m$, $n, m \geq 0$ depend on parities of $n, m$ of multiplicants. For simplicity, denote by $X^\alpha Y^\beta$ all monomials $X^\alpha Y^\beta \in K[X,Y]$ such that $\alpha$ is even and $\beta$ odd, etc. We get non-zero products only in the cases:

\[
\{X^1Y^0, X^0Y^1\} = X^0Y^0; \\
\{X^1Y^1, X^1Y^0\} = X^1Y^0; \\
\{X^1Y^1, X^0Y^1\} = X^0Y^1.
\]

Thus, $\delta_1(P)$ is spanned by monomials of three types obtained above. Consider their commutators, the first line yields that $\delta_2(P)$ is spanned by monomials of type $Y^0Y^0$. Finally, $\delta_3(P) = 0$.

Thus, the Poisson algebras $\mathfrak{h}_2(K)$, $\mathbf{H}_2(K)$ in characteristic 2 behave similarly to the associative algebra $M_2(K)$ of $2 \times 2$ matrices in case char $K = 2$.

12. Nilpotency and solvability of symmetric algebras $S(L)$

Finally, in this section we prove Theorem 4.4 that generalizes the result of Shestakov (Theorem 3.8).

**Proof. of Theorem 4.4.** Implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 5) and 1) $\Rightarrow$ 4) $\Rightarrow$ 5) are trivial.

It is sufficient to prove one nontrivial implication 5) $\Rightarrow$ 1). (In case $p = 2$ we prove 4) $\Rightarrow$ 1)). Consider the case char $K \neq 0$, then we can take the truncated symmetric algebra $S(L) = S(L)/(v^p \mid v \in L)$, which is solvable (strongly solvable for $p = 2$). Applying Theorem 4.3, we get $\dim L^2 < \infty$. By way of contradiction suppose that $L$ is not abelian. Then there exist $x, y \in L$ such that $[x, y] \neq 0$. We get a non-abelian finite dimensional subalgebra $\bar{L} = \langle x, y \rangle_K + L^2 \subseteq L$.

Consider the case char $K = 0$. By Theorem 3.9, $L$ has an abelian ideal $A$ of finite codimension. Take $x \in L \setminus A$, suppose that there exists $v \in A$ such that $[x, v] \neq 0$. Consider a subalgebra $\bar{L} = \langle x \rangle \oplus \langle (ad x)^n v \mid n \geq 0 \rangle \subseteq L$. By Lemma 10.1, the action of ad $x$ is algebraic, thus $\bar{L}$ is a finite dimensional non-abelian subalgebra. The remaining case is that for any $x \in L \setminus A$ we have $[x, A] = 0$. Now one has $\dim L^2 \leq \dim L/A + (\dim L/A)^2 < \infty$ and the arguments above again yield a non-abelian finite dimensional subalgebra.

Thus, without loss of generality, we assume that $L$ is a finite dimensional non-abelian Lie algebra. Since our identities are multilinear, we also can consider that $K$ is algebraically closed. It is well known that a finite dimensional non-abelian Lie algebra over an algebraically closed field either contains the two-dimensional solvable Lie algebra or the three-dimensional nilpotent Lie algebra (see [19], [1, 6.7.1]).

Suppose that $L$ contains the two-dimensional solvable Lie algebra $H = \langle x, y \mid [x, y] = 0 \rangle_K$. Introduce sequences of elements: $e_k = xy^{2k-1}$, $h_k = xy^{2k}$, $f_k = xy^{2k+1}$ for all $k \geq 1$. Consider the case char $K \neq 2$. We check by induction on $k \geq 1$ that $e_k, h_k, f_k \in \delta_{k-1}(S(H))$. The base of induction $k = 1$ is trivial. Let $k > 1$, the inductive step follows by relations

\[
\delta_k(S(H)) \ni \{e_k, h_k\} = \{xy^{2k-1}, xy^{2k}\} = xy^{2k+1} = e_{k+1}; \\
\delta_k(S(H)) \ni \{h_k, f_k\} = \{xy^{2k}, xy^{2k+1}\} = xy^{2k+1+1} = f_{k+1}; \\
\delta_k(S(H)) \ni \{e_k, f_k\} = \{xy^{2k-1}, xy^{2k+1}\} = 2xy^{2k+1} = 2h_{k+1}.
\]
Thus, $\delta_k(S(H)) \neq 0$ for all $k \geq 1$, a contradiction with solvability of $S(L)$. In case char $K = 2$, we check by induction that $e_k, h_k, f_k \in \delta_{k-1}(S(H))$, $k \geq 1$, using relations above and

$$\delta_k(S(H)) \ni \{e_k, h_k\} y = \{xy^{k+1}, xy'\} y = xy^{k+1} = h_{k+1}.$$ 

Thus, $S(H)$ is not strongly solvable, a contradiction.

Suppose that $L$ contains the three-dimensional nilpotent Lie algebra $H = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 0 \rangle_K$. Introduce sequences of elements: $e_k = xy^{k+1} z_k, h_k = xy^{k+1} z^{k+1}$, $f_k = xy^{k+2} z^{k+1}$ for all $k \geq 1$. Consider the case char $K \neq 2$. We check by induction on $k \geq 1$ that $e_k, h_k, f_k \in \delta_{k-1}(S(H))$. The base of induction $k = 1$ is trivial. Let $k > 1$, the inductive step follows by relations

$$\delta_k(S(H)) \ni \{e_k, h_k\} \{x^{k+1}, z^{k+1} = e_{k+1};$$
$$\delta_k(S(H)) \ni \{h_k, f_k\} \{x^{k+1} z^{k+1}, xy^{k+2} z^{k+1} = f_{k+1};$$
$$\delta_k(S(H)) \ni \{e_k, f_k\} \{x^{k+1} z^{k+1}, xy^{k+2} z^{k+1} = 2x^{k+1} z^{k+1} = 2h_{k+1}.$$ 

In case char $K = 2$, we check by induction that $e_k, h_k, f_k \in \delta_{k-1}(S(H))$, $k \geq 1$, using relations above and

$$\delta_k(S(H)) \ni \{e_k, h_k\} y = \{xy^{k+1}, xy^{k+2} z^{k+1} = xy^{k+1} z^{k+1} = h_{k+1}.$$ 

Thus, $S(H)$ is not strongly solvable, a contradiction, which proves that $L$ is abelian. $\square$

The question of the solvability of the symmetric algebra $S(L)$ in case char $K = 2$ is more complicated as shown below. The same Lie algebras of Lemma 11.1 and Lemma 11.2, also yield solvable symmetric algebras, of course, they are not strongly solvable by that Lemmas.

**Lemma 12.1.** Let $L = \langle x, y_i \mid [x, y_i] = y_i, i \in \mathbb{N} \rangle_K$, other commutators being trivial, char $K = 2$. The symmetric Poisson algebra $S(L)$ is solvable of length 3 and not strongly solvable.

**Proof.** Denote $H = \langle y_i \mid i \in \mathbb{N} \rangle_K$. For a monomial $v = y_1 y_2 \cdots y_k \in S(H)$ introduce its length $|v| = k$. Then $v' = \{x, v\} = |v|v$. A basis of $S(L)$ is formed by $x^\alpha v$, $\alpha \geq 0$, where $v \in S(H)$ are respective basis monomials. Consider products:

$$\{x^\alpha v, x^\beta w\} = x^{\alpha+\beta-1}(\alpha|v| + \beta|v|)vw.$$ 

(16)

These products depend on parities of $\alpha, \beta, |v|, |w|$. For simplicity, denote by $x^\beta v$ all monomials $x^\beta v \in S(L)$ such that $\alpha$ is even and $|v|$ is odd, etc. The only non-zero products (16) are of types:

$$\{x^1 v^0, x^0 v^1\} = x^0 v^1;$$
$$\{x^1 v^1, x^0 v^0\} = x^1 v^0;$$
$$\{x^0 v^1, x^0 v^0\} = x^0 v^0.$$

Thus, $\delta_1(S(L))$ is spanned by monomials of three types obtained above. Consider their commutators, the last line yields that $\delta_2(S(L))$ is spanned by monomials of type $x^0 v^0$. Finally, $\delta_3(S(L)) = 0$. $\square$

**Lemma 12.2.** Let $L = \langle x, y_i, z_i \mid [x, y_i] = z_i, i \in \mathbb{N} \rangle_K$, other commutators being trivial, char $K = 2$. The symmetric Poisson algebra $S(L)$ is solvable of length 3 and not strongly solvable.

**Proof.** Set $H = \langle y_i, z_i \mid i \in \mathbb{N} \rangle_K$. As above, denote $a' = \{x, a\}$ for $a \in S(H)$. Consider basis elements $x^\alpha a, x^\beta b, x^\gamma c, x^\delta d \in S(L)$, where $a, b, c, d \in S(H)$ are basis monomials and $\alpha, \beta, \gamma, \delta \geq 0$. Then

$$\{x^\alpha a, x^\beta b\} = x^{\alpha+\beta-1}(\alpha a' + \beta a'b);$$
$$\{x^\gamma c, x^\delta d\} = x^{\gamma+\delta-1}(\gamma cd' + \delta cd').$$ 

(17)

We use these computations and the observation above (15), that $a'' = 0$ for any $a \in S(H)$.

$$\delta_2(x^\alpha a, x^\beta b, x^\gamma c, x^\delta d) = \left\{x^{\alpha+\beta-1}(\alpha a'b + \beta a'b), x^{\gamma+\delta-1}(\gamma cd' + \delta cd')\right\}$$
$$= x^{\alpha+\beta+\gamma+\delta-3}\left((\alpha + \beta - 1)(\gamma + \delta)(\alpha a'b + \beta a'b)c'd' + (\gamma + \delta - 1)(\alpha + \beta)a'b'((\gamma cd' + \delta cd').$$

The first summand is nonzero only in the case $\alpha + \beta$ is even and $\gamma + \delta$ odd, while the second one is non-zero in the opposite case. Both cases yield that $\alpha + \beta + \gamma + \delta - 3$ is even. Thus, $\delta_2(S(L))$ is spanned by monomials of type $x^\alpha a, a \in S(H)$, and $\alpha$ even. Finally, by (17), $\delta_3(S(L)) = 0$. $\square$
Acknowledgments. The authors are grateful to Raimundo Bastos, Alexei Krasilnikov, Plamen Koshlukov, and Ivan Shestakov for useful discussions. The authors are especially grateful to Alexei Krasilnikov for explaining the mystery of products of commutators in associative algebras.

References

[1] Bahturin Yu. A., Identities in Lie algebras. VNU Science Press, Utrecht, 1987.
[2] Bahturin Yu., Identities in the universal envelopes of Lie algebras. J. Austral. Math. Soc. 18 (1974), 10–21.
[3] Bakhourin Yu.A., Petrogradsky V.M. and Zaicev M.V., Infinite dimensional Lie superalgebras. de Gruyter Exp. Math. 7. de Gruyter, Berlin, 1992.
[4] Bakhourin Yu.A., Petrogradsky V., Polynomial identities in smash products. J. Lie Theory 12 (2002), no. 2, 369–395.
[5] Bakhturin Yu., Petrogradsky V., Polynomial identities of order two. J. Sov. Math. 42 (1988), 63–80.
[6] Bakhturin Yu., Petrogradsky V., Polynomial identities of order two. Mosc. Math. J. 11 (2011), no. 4, 683–730.
[7] Bakhturin Yu.A., Petrogradsky V.M. and Zaicev M.V., Infinite dimensional Lie algebras. de Gruyter Exp. Math. 7. de Gruyter, Berlin, 1992.
[8] Bhandari, A., Passi, I. B. S., Lie-nilpotency indices of group algebras. Bull. London Math. Soc. 24 (1992), no. 1, 68–70.
[9] Dixmier J., Enveloping algebras. AMS, Rhode Island, 1996.
[10] Drensky V., Free algebras and PI-algebras. Graduate course in algebra. Springer-Verlag Singapore, Singapore, 2000.
[11] Etingof, P., Klim, J., Ma, On universal Lie nilpotent associative algebras. J. Algebra 321 (2009), No. 2, 697–703.
[12] Farkas D. R., Poisson polynomial identities. Comm. Algebra 26 (1998), no. 2, 401–416.
[13] Farkas D. R., Poisson polynomial identities. II. Arch. Math. (Basel) 72 (1999), no. 4, 252–260.
[14] Giambruno A., Petrogradsky V., Poisson identities of enveloping algebras. Arch. Math. 87 (2006), 505–515.
[15] Gordanenko, A.S., Codimension of commutators of length 4. Russ. Math. Surv. 62 (2007), No. 1, 187–199; translation from Usp. Mat. Nauk 62 (2007), No. 1, 191–192.
[16] Grishin A. V., Pchelintsev S. V., On the centers of relatively free associative algebras with the identity of Lie nilpotency. Mat. Sb. 206 (2015), no. 11, 113–130; translation in Sb. Math. 206 (2015), no. 11, 113–130.
[17] Gupta, N.; Levin, F., On the Lie ideals of a ring. J. Algebra 81 (1983), no. 1, 225–231.
[18] Johnson, N., Lie algebras, Interscience, New York. 1962.
[19] Kochetov, M. V. Identities in the smash product of the universal enveloping superalgebra of a Lie superalgebra and a group algebra. Mat. Sb. 194 (2003), no. 1, 87–102; translation in Sb. Math. 194, no. 1–2, 89–103.
[20] Kostant, B., A Lie algebra generalization of the Amitsur-Levitski theorem. Adv. Math. 40 (1981) 155–175.
[21] Krasilnikov, A., The additive group of a Lie nilpotent associative ring. J. Algebra 392 (2013), 10–22.
[22] Latyshev V.N., Codimension of commutators of length 4. Russ. Math. Surv. 62 (2007), No. 1, 187–199; translation from Usp. Mat. Nauk 62 (2007), No. 1, 191–192.
[23] Makar-Limanov, L., Sheshkov, I., Polynomial and Poisson dependence in free Poisson algebras and free Poisson fields. J. Algebra 349 (2012), No. 1, 372–379.
[24] Makar-Limanov, L., Umirbaev, U., The Freiheitssatz for Poisson algebras. J. Algebra 328 (2011), No. 1, 495–503.
[25] Mishchenko S.P., Petrogradsky V.M. and Regev A., Poisson PI algebras, Trans. Amer. Math. Soc., 359 (2007), no. 10, 4669–4694.
[26] Ono A. I., The Poisson center and polynomial, maximal Poisson commutative subalgebras, especially for nilpotent Lie algebras of dimension at most seven. J. Algebra 365 (2012), 83–113.
[27] Passi, I.B.S. Group rings and their augmentation ideals. Lecture Notes in Mathematics. 715, Springer-Verlag, 1979.
[28] Passi I. B. S., Passman D. S., Sehgal S. K., Lie solvable group rings. Can. J. Math. 25 (1973), no. 4, 748–757.
[29] Passman D.S., Group rings satisfying a polynomial identity. J. Algebra 20 (1972), 103–117.
[30] Passman D.S., Enveloping algebras satisfying a polynomial identity. J. Algebra 134 (1990), no. 2, 469–490.
[31] Petrogradsky V.M., The existence of an identity in a restricted envelope, Mat. Zametki 49 (1991), no. 1, 84–93; translation in Math. Notes 49 (1991), no. 1-2, 60–66.
[32] Petrogradsky V.M., Identities in the enveloping algebras for modular Lie superalgebras, J. Algebra 145 (1992), no. 1, 1–21.
[33] Petrogradsky V.M., Codimension growth of strong Lie nilpotent associative algebras, Comm. Algebra, 39 (2011), no. 3, 918–928.
[34] Ratseev, S. M. Correlation of Poisson algebras and Lie algebras in the language of identities. Math. Notes 96 (2014), no. 3–4, 538–547; Translation of Mat. Zametki 96 (2014), no. 4, 567–577.
[35] Ratseev, S. M., Petrogradsky V.M., The free associative algebra and Lie algebras. J. Algebra 162 (1993), 46–61.
[36] Riley D.M., Shalev A., Restricted Lie algebras and their envelopes, Can. J. Math. 47 (1995), 146–164.
[37] Sharma R. K., Srivastava J. B., Lie ideals in group rings. J. Pure Appl. Algebra 63 (1990), 63–80.
[38] Shestakov, I., Umirbaev, U.U., The tame and the wild automorphisms of polynomial rings in three variables. J. Am. Math. Soc. 17, (2004) No. 1, 197–227.
[39] Siciliano, S., Lie derived lengths of restricted universal enveloping algebras. Publ. Math. 68, (2006) No. 3-4, 503–513.
[40] Siciliano, S., Spinelli E., Lie nilpotency indices of restricted universal enveloping algebras. Commun. Algebra 34, (2006) No. 1, 151–157.
[44] Siciliano S., Usefi H., Lie identities on symmetric elements of restricted enveloping algebras. Israel J. Math. 195 (2013), no. 2, 999–1012.

[45] Siciliano S., Usefi H., Lie properties of restricted enveloping algebras. Lie algebras and related topics. Contemp. Math., Amer. Math. Soc., Providence, RI 652 (2015), 141–152.

[46] Siciliano S., Usefi H., Lie solvable enveloping algebra of characteristic two. J. Algebra 382 (2013), 314–331.

[47] Siciliano S., Usefi H., Engel condition on enveloping algebras of Lie superalgebras. J. Pure Appl. Algebra 219 (2015), no. 12, 5631–5636.

[48] Tarasov A. A., On some commutative subalgebras in the universal enveloping algebra of the Lie algebra $gl_n(C)$. Mat. Sb. 191 (2000), no. 9, 115–122; translation in Sb. Math. 191 (2000), no. 9-10.

[49] Usefi H., Lie identities on enveloping algebras of restricted Lie superalgebras. J. Algebra 393 (2013), 120–131.

[50] Usefi H., Non-matrix polynomial identity enveloping algebras. J. Pure Appl. Algebra 217 (2013), no. 11, 2050–2055.

[51] Vergne M., La structure de Poisson sur l’algèbre symétrique d’une algèbre de Lie nilpotente. C. R. Acad. Sc. Paris 269 (1969), Série A, 950–952.

[52] Vinberg, E. B. Some commutative subalgebras of a universal enveloping algebra. Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 1, 3–25, 221; translation in Math. USSR-Izv. 36 (1991), no. 1, 1–22.

[53] Volichenko I.B., The T-ideal generated by the element $[x_1, x_2, x_3, x_4]$, Preprint no. 22, Inst. of Mathematics of the Academy of Sciences of BSSR, Minsk 1978, 13 pp. (Russian).

Department of Mathematics, Federal University of Amazonas, Humaitá, Amazonas, Brazil
E-mail address: calmariezuila@gmail.com

Department of Mathematics, University of Brasilia, 70910-900 Brasilia DF, Brazil
E-mail address: petrogradsky@rambler.ru