SESHADRI CONSTANTS VIA LELONG NUMBERS

THOMAS ECKL

Abstract. One of Demailly’s characterization of Seshadri constants on ample line bundles works with Lelong numbers of certain positive singular hermitian metrics. In this note sections of multiples of the line bundle are used to produce such metrics and then to deduce another formula for Seshadri constants. It is applied to compute Seshadri constants on blown up products of curves, to disprove a conjectured characterization of maximal rationally connected quotients and to introduce a new approach to Nagata’s conjecture.

0. Introduction

In 1990 Demailly introduced Seshadri constants $\epsilon(L, x)$ for nef line bundles $L$ on projective complex manifolds $X$ [Dem92]:

$$\epsilon(L, x) := \inf_{C \ni x} \frac{L \cdot x}{\text{mult}_x C}$$

where the infimum is taken over all irreducible curves passing through $x$. They refine constants $\epsilon(L)$ appearing in Seshadri’s ampleness criterion [Laz04, Thm.1.4.13] and quantify how much of the positivity of an ample line bundle can be localized at a given point.

These constants gained immediately a lot of interest in algebraic geometry; for example lower bounds on Seshadri constants were used to produce sections in adjoint bundles [Laz97]. It also turned out that explicit calculations of Seshadri constants are difficult in almost every concrete situation (see for example the work of Garcia [Gar05] on ruled surfaces) and it is not easier to give (interesting) upper and lower bounds for them. From their very definition it seems easier to determine upper bounds (by showing that a curve with appropriate intersection number and multiplicity exists) than lower bounds (via the non-existence of such curves) [EKL95, Bau99].

On the other hand Demailly gave two more equivalent definitions of Seshadri constants [Dem92 Thm.6.4] or Prop. 4.2:

$$\epsilon(L, x) = \gamma(L, x) := \sup_{\gamma \in \mathbb{R}^+} \left\{ \exists \text{ singular metric } h \text{ on } L : i \Theta_h \geq 0, \begin{array}{l} x \text{ isolated pole of } \Theta_h, \\ \nu(\Theta_h, x) = \gamma \end{array} \right\},$$

where $\nu(\Theta_h, x)$ is the Lelong number of the curvature current $i \Theta_h$ in $x$, and

$$\epsilon(L, x) = \sigma(L, x) := \sup_{k \in \mathbb{N}} \frac{1}{k} s(kL, x)$$

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where
\[ s(kL, x) := \max_{s \in \mathbb{N}} \left\{ \text{the sections in } H^0(X, kL) \text{ generate all } s - \text{jets in } J^s_x kL = \mathcal{O}_X(kL)/m^{s+1}_x \right\}. \]

In the nef case we still have
\[ \epsilon(L, x) \geq \gamma(L, x) \geq \sigma(L, x). \]

The definitions of \( \sigma(L, x) \) and \( \gamma(L, x) \) allow to give lower bounds for Seshadri constants by constructing sections of \( kL \) with special properties. For \( \sigma(L, x) \) this is obvious whereas for \( \gamma(L, x) \) we first need a better understanding of singular hermitian metrics, their curvature currents and Lelong numbers.

1. Lelong Numbers and Seshadri Constants

Let us repeat the relevant definitions (consult [Dem00] for further properties and examples): A singular hermitian metric \( h \) on a holomorphic line bundle \( L \) is given in any trivialization \( \theta \):
\[ L|_{\Omega} \sim_{\rightarrow} = \Omega \times \mathbb{C} \]
by
\[ \|\xi\|_h = |\theta(\xi)|e^{-\phi_h(x)}, \quad x \in \Omega, \quad \xi \in L_x, \]
where the function \( \phi_h \in L^1_{\text{loc}}(\Omega) \) is called the weight of the metric \( h \) (w.r.t. the trivialization \( \theta \)).

The curvature current \( \Theta_h \) of \( L \) is given by the closed \((1,1)\)-current \( \Theta_h = i\pi \partial \overline{\partial} \phi_h \).

This current exists in the sense of distribution theory because of \( \phi_h \in L^1_{\text{loc}}(\Omega) \) and is independent of the chosen trivialization.

The most important example for the rest of these notes is the possibly singular metric on \( L \) induced by non-zero holomorphic sections \( \sigma_1, \ldots, \sigma_N \) of \( L \) which is given in any trivialization \( \theta \) by
\[ \|\xi\|^2_h = \frac{|\theta(\xi)|^2}{|\theta(\sigma_1(x))|^2 + \ldots + |\theta(\sigma_N(x))|^2}. \]

Then the associated weight function is
\[ \phi_h(x) = \log \left( \sum_{1 \leq j \leq N} |\theta(\sigma_j(x))|^2 \right)^{\frac{1}{2}}, \]
which is a plurisubharmonic function, so \( \Theta_h \) is a (closed) positive current. The order of logarithmic poles of a plurisubharmonic function in a point \( x \in X \) is measured by the Lelong number
\[ \nu(\phi, x) := \liminf_{z \to x} \frac{\phi(z)}{\log |z - x|}. \]

**Lemma 1.1.** Let \( \phi(z) = \log(\sum_{1 \leq j \leq N} |\theta(\sigma_j(x))|^2)^{\frac{1}{2}} \) be the weight of the hermitian metric on \( L \) induced by the holomorphic sections \( \sigma_1, \ldots, \sigma_N \) w.r.t. a trivialization \( \theta \) on an open subset \( \Omega \subset X \). Then for every point \( x \in \Omega \):
\[ \nu(\phi, x) = \min_{1 \leq j \leq N} \{ \text{ord}_x \sigma_j \}. \]

**Proof.** In dimension 1 this is clear by definition. In higher dimensions restriction to sufficiently general lines shows at least
\[ \nu(\phi, x) \leq \min_{1 \leq j \leq N} \{ \text{ord}_x \sigma_j \}. \]
To get the other inequality set $x := 0$ and add enough monomials $z^\alpha$ with $|\alpha| = \nu$ until we get

$$\phi(z) \leq \phi'(z) = \log \left( \sum_{|\alpha| = \nu} |\alpha|! \cdot z^\alpha \cdot h_\alpha(z) |^2 \right)^{\frac{1}{2}}$$

for some holomorphic functions $h_\alpha$ defined around 0. These $h_\alpha$ are bounded in a neighborhood of 0 hence there exist constants $C, C' > 0$ such that

$$\phi'(z) \leq \log C + \log \left( \sum_{|\alpha| = \nu} |\alpha|! \cdot z^\alpha |^2 \right)^{\frac{1}{2}} \leq \log C' + \log(|z^\nu|^2)^{\frac{1}{2}}.$$ 

But then

$$\liminf_{z \to x} \frac{\log(|z^\nu|^2)^{\frac{1}{2}}}{\log |z|} = \nu.$$ 

Now we are able to calculate $\gamma(L, x)$ from multiplicities of divisors passing through $x$:

**Theorem 1.2.** Let $L$ be an ample line bundle on an $n$-dimensional projective complex manifold $X$. Then

$$\epsilon(L, x) = \sup_{k; D_1, \ldots, D_n \ni x} \left\{ \frac{\min_i \{\text{mult}_x D_i\}}{k} \right\}$$

where the supremum is taken over all divisors $D_1, \ldots, D_n \in |kL|$ such that $x$ is an isolated point of $D_1 \cap \ldots \cap D_n$.

**Proof.** We know that for ample line bundles

$$\epsilon(L, x) = \sigma(L, x) = \sup_k \frac{1}{k} s(kL, x).$$

so for every $\delta > 0$ there is a $k \gg 0$ such that

$$\epsilon(L, x) - \delta \leq \frac{1}{k} s(kL, x) \leq \epsilon(L, x).$$

Hence $H^0(X, kL)$ generates all $s$-jets in $x$ and we can find holomorphic sections $f_1, \ldots, f_n$ whose $s$-jets in $x$ are the monomials $z_i^s$, $i = 1, \ldots, n$. The weight of the associated metric on $L$ has an isolated pole of Lelong number $\frac{1}{k}$ in $x$ and in the limit we get

$$\epsilon(L, x) \leq \sup_{k; D_1, \ldots, D_n \ni x} \left\{ \frac{\min_i \{\text{mult}_x D_i\}}{k} \right\}.$$ 

On the other hand $n$ holomorphic sections $\sigma_1, \ldots, \sigma_n$ of $kL$ (which are the 0-divisors of $D_1, \ldots, D_n$) define a metric $h$ on $kL$. If we multiply $\phi_h$ by $\frac{1}{k}$ we get a metric on $L$ with isolated pole in $x$ hence the other inequality. \qed

**Remark 1.3.** An algebraic proof of the theorem may be deduced from an observation already used in the algebraic proof of the equality

$$\epsilon(L, x) = \sigma(L, x),$$
2. Seshadri constants on blown up products of two curves

We use Theorem 1.2 for computing Seshadri constants on products of two curves and some of their blow ups.

Proposition 2.1. Let \( X = C_1 \times C_2 \) be the product of two smooth projective curves, \( p_1 : X \to C_1, p_2 : X \to C_2 \) the projections and \( a \) a divisor of degree \( a > 0 \) on \( C_1 \), \( b \) a divisor of degree \( b > 0 \) on \( C_2 \). Let \( L = p_1^*a + p_2^*b \) be an ample divisor on \( X \) and \( x \) any point on \( X \). Then

\[
\epsilon(L, x) = \min(a, b).
\]

Proof. Let \( F_i \) be the numerical equivalence class of the fibers of \( p_i, i = 1, 2 \). Let \( E \) be the exceptional divisor of the blow up of \( X \) in \( x \). Then

\[
(aF_1 + bF_2 - mE)(F_1 - E) = b - m , \quad (aF_1 + bF_2 - mE)(F_2 - E) = a - m.
\]

Since \( F_1 - E, F_2 - E \) are the (numerical classes of the) strict transforms of the vertical resp. horizontal fiber through \( x \) a line bundle of numerical class \( aF_1 + bF_2 - mE \) is ample only if \( b - m > 0, a - m > 0 \) by Seshadri’s ampleness criterion. This implies

\[
\epsilon(L, x) \leq \min(a, b).
\]

On the other hand let \( P_1 = p_1(x) \in C_1 \) and \( P_2 = p_2(x) \in C_2 \). For \( a' = ak, k \gg 0 \), there exists an \( A_2 \in [a'P_1] \) such that \( A_2 - P_1 \) is not effective. Similarly for \( b' = bk \) there is a \( B_2 \in [b'P_2] \) such that \( B_2 - P_2 \) is not effective. Setting \( A_1 = a'P_1, B_1 = b'P_2 \) we conclude that \( x \) is an isolated point of \( (A_1 + B_2) \cap (A_2 + B_1) \) and

\[
\frac{\min(\text{mult}_x(A_1 + B_2), \text{mult}_x(A_2 + B_1))}{k} = \frac{\min(ak, bk)}{k} = \min(a, b).
\]

Now Seshadri constants are by definition numerical invariants. Since \( A_1 + B_2 \) and \( A_2 + B_1 \) belong to a line bundle with numerical class \( k(aF_1 + bF_2) \) the theorem follows from Theorem 1.2. \( \square \)

Let \( X = C_1 \times C_2, a, b, a, b \) and \( L \) be as in the last proposition.

Proposition 2.2. Let \( \pi_n : X^{(n)} \to X \) be the blow up of \( X \) in \( n \) points \( x_1, \ldots, x_n \) where no two of them lie on the same horizontal or vertical fiber. Then for

\[
L^{(n)} := \pi_n^*L - \sum_{i=1}^n m_i E_i, \quad 0 < m_i < \min(a, b), \quad \sum_{i=1}^n m_i \leq \max(a, b),
\]

\( E_i \) the exceptional divisor over \( x_i \), we have

\[
\epsilon(L^{(n)}, y) = \min(a, b)
\]

for every point \( y \) not lying on the same horizontal or vertical fiber as one of the \( x_i \).
Proof. Starting with Proposition 2.1 and using inductively the proposition together with Seshadri’s ampleness criterion we conclude that \( L^{(n)} \) is ample. As before we compute for the blow up \( X^{(n+1)} \) of \( X^{(n)} \) in \( y \) that \((E_{n+1} \) the exceptional divisor over \( y \))

\[
(\pi^{*}_{n+1}L - \sum_{i=1}^{n} m_i E_i - m E_{n+1})(F_1 - E_{n+1}) = b - m
\]

and analogously for \( F_2 - E_{n+1} \). Consequently

\[
\epsilon(L^{(n)}, y) \leq \min(a, b).
\]

It is enough to show the proposition for \( a = b \) since for two nef line bundles \( L, M \) it is by definition true that

\[
\epsilon(L + M, x) \geq \epsilon(L, x).
\]

Let \( P_i = p_1(x_i), P = p_1(y), Q_i = p_2(x_i), Q = p_2(y) \). We distinguish two cases:

1. \( a > \sum_i m_i \). Then there is \( q \) such that \(|q a P - q \sum m_i P_i| \) is a base point free linear system on \( C_1 \). Consequently we have a divisor \( P'_1 + \ldots + P'_{q(a-\sum m_i)} \) in this linear system such that all \( P'_j \neq P \). Similarly, for \( q \gg 0 \) the linear system \(|q Q - q \sum m_i Q_i| \) is base point free on \( C_2 \) and contains an element \( Q'_1 + \ldots + Q'_{q(\sum m_i)} \) with \( Q'_j \neq Q \). Setting

\[
A_1 = qa P, \quad A_2 = q \sum m_i P_i + \sum P'_j,
\]

\[
B_1 = q \sum m_i Q_i + \sum Q'_j,
\]

\[
B_2 = qa Q
\]

we can apply Theorem 1.2 on

\[
\pi^{*}_{n}L' - \sum m_i E_i \equiv q(\pi^{*}_{n}L - \sum m_i E_i)
\]

where \( L' = O(A_1 + B_1) = O(A_2 + B_2) \) and conclude

\[
\epsilon(L, y) \geq \frac{qa}{q} = a.
\]

2. \( a = \sum_i m_i \). Then for all \( q \in \mathbb{N} \) there exists a \( q' \) such that

\[
| - (q' qa - q')P + q' q \sum m_i P_i| \quad \text{and} \quad |q' q \sum m_i Q_i - (q' qa - q')Q|
\]

are base point free linear systems on \( C_1 \) and \( C_2 \). Consequently we have divisors \( P''_1 + \ldots + P''_{q} \) and \( Q''_1 + \ldots + Q''_{q} \) in these linear systems such that \( P''_j \neq P \) and \( Q''_j \neq Q \). Setting

\[
A_1 = (q' qa - q')P + \sum P''_j, \quad A_2 = q' q \sum m_i P_i,
\]

\[
B_1 = q' q \sum m_i Q_i
\]

\[
B_2 = (q' qa - q')Q + q' q \sum l_i Q_i + \sum Q''_j
\]

we can again apply Theorem 1.2 on

\[
\pi^{*}_{n}L' - \sum m_i E_i \equiv q' q(\pi^{*}_{n}L - \sum m_i E_i)
\]

where \( L' = O(A_1 + B_1) = O(A_2 + B_2) \) and conclude

\[
\epsilon(L, y) \geq \frac{q' qa - q'}{q' q} = a - \frac{1}{q}.
\]
But since \( q \) can be chosen arbitrarily big this implies
\[
\epsilon(L, y) \geq a.
\]

Of course we can similarly calculate Seshadri constants of (some) ample line bundles on the the blown up variety when some of the \( x_i \) lie on the same horizontal or vertical fiber or on exceptional divisors. Let us instead use Prop. 2.2 to study

3. Fibrations with rationally connected fibers

In [Miy86] Miyaoka presented a method to construct fibrations with rational connected fibers (see also Shepherd-Barrons account of this work in [SB92] and Bogomolov-McQuillan’s approach [BM01], further explained in [KTS05]):

Let \( X \) be an \( n \)-dimensional projective complex manifold and fix an ample line bundle \( A \) on \( X \) (a polarization of \( X \)). Consider sufficiently general complete intersection curves \( C \in [k_1A \cap \ldots \cap k_{n-1}A], k_i \gg 0 \) on \( X \). Miyaoka observed that the leaves of a foliation \( F \subset T_X \) (i.e. a saturated sheaf closed under the Lie bracket) are rationally connected fibers of a (rational) fibration if \( F|_C \) is ample.

To get such foliations we can use the properties of the Harder-Narasimhan filtration:

For every torsion-free sheaf \( E \) on \( X \) there is a unique filtration, the so-called Harder-Narasimhan filtration of \( E \),
\[
0 = E_0 \subset E_1 \subset \ldots \subset E_k = E
\]
such that the \( G_i = E_i/E_{i-1} \) are semi-stable torsion free sheaves whose slopes satisfy
\[
\mu_{\max}(E) := \mu(G_1) = \frac{\deg(G_1)}{rk(G_1)} > \ldots > \mu(G_k) =: \mu_{\min}(E).
\]

Furthermore the \( E_i \) are saturated in \( E \). Finally, by Mehta-Ramanathans theorem the restriction of the Harder-Narasimhan filtration of \( E \) on a general complete intersection curve \( C \) is again the Harder-Narasimhan filtration of the vector bundle \( E|_C \). See [Ses82, Siu87] for an introduction, proofs and further properties of the Harder-Narasimhan filtration and slopes.

For our purposes we need the following proposition explicitly shown in [KTS05]:

**Proposition 3.1.** Let
\[
0 = F_0 \subset F_1 \subset \ldots \subset F_k = T_X
\]
be the Harder-Narasimhan filtration of the tangent bundle of \( X \) with respect to the polarization \( A \). Assume that \( \mu(F_1) > 0 \) and set
\[
j := \max\{i : \mu(F_i/F_{i-1})\}.
\]

For any \( 0 < i \leq j \) the sheaf \( F_i \) is a foliation on \( X \), and \( F_i|_C \) is ample. If \( F_C \subset T_X|_C \) is any ample subbundle, then \( F_C \) is contained in \( F_j|_C \).

**Proof.** See [KTS05, Prop.29,30].

On the other hand Campana [Cam81, Cam94] and Kollár, Miyaoka and Mori [KMM92] constructed the maximal rationally connected (MRC) quotient of a projective manifold \( X \). If we call the sheaf \( F_j \) from the proposition above the maximal A-ample part of \( T_X \) it is tempting to ask...
Question 3.2. Does the MRC quotient induce a foliation $\mathcal{F}$ such that $\mathcal{F}$ is the maximal $A$-ample part of $T_X$?

Unfortunately the answer to this question is negative already on surfaces: Let $X = C \times \mathbb{P}^1$ be the ruled product surface over an elliptic curve $C$ with projections $p_1 : X \to C$, $p_2 : X \to \mathbb{P}^1$. Let $x_1, x_2, x_3$ be 3 points on $X$ such that no two of them lie on the same horizontal or vertical fiber. Let $\pi : \hat{X} \to X$ be the blow up of $X$ in $x_1, x_2, x_3$ and let

$$L = p_1^*a + p_2^*b, \quad \deg_C a = 3, \quad \deg_{\mathbb{P}^1} b = 4.$$  

$L$ is an ample line bundle on $X$ and it follows from Prop. 2.2 that

$$L' = \pi^*L - 2E_1 - 2E_2 - 2E_3$$

is ample, too. (Here the $E_i$ are the exceptional divisors on $\hat{X}$ over $x_i$.) On the other hand the MRC quotient of $\hat{X}$ is just the blown-up ruling of $X$ (that is a simple consequence of surface classification). The induced foliation $\mathcal{F}$ has tangent sheaf

$$T_F = p_2^*\mathcal{O}(2) - E_1 - E_2 - E_3$$

as a local computation around the blown up points show (see [Bru04]). We have

$$L'.T_F = 0$$

and hence no complete intersection curve $C \in |kL|$ intersects $T_F$ positively.

Of course the situation improves if we add to $L'$ some fibers of the projection onto $C$. Hence we may change Question 3.2 and ask for the existence of a polarization $A$ with the desired properties.

4. Seshadri constants on sets of points

Finally we turn to Seshadri constants associated to sets of points.

**Definition 4.1.** Let $L$ be an ample line bundle on an irreducible projective variety $X$ and let $x_1, \ldots, x_r$ be arbitrary points on $X$. Then we define

$$\epsilon(L; x_1, \ldots, x_r) = \max \{ \epsilon \geq 0 : \pi^*L - \epsilon \cdot \sum_{i=1}^r E_i \text{ is nef} \}$$

where $\pi : \hat{X} \to X$ is the blow up of $X$ in $x_1, \ldots, x_r$ and $E_i \subset \hat{X}$ is the exceptional divisor over $x_i$.

As for a single point there are other possibilities to compute $\epsilon(L; x_1, \ldots, x_r)$. For simplicity of notation we only consider the surface case:

**Proposition 4.2.** Let $L$ be an ample line bundle on a surface $X$ and $x_1, \ldots, x_r \in X$ be arbitrary points. Then $\epsilon(L; x_1, \ldots, x_r)$ equals

$$\inf_C \frac{L.C}{\sum_{i=1}^r \text{mult}_x C},$$

where the infimum is taken over all irreducible curves $C \subset X$,

$$\sup_k \frac{s(kL; x_1, \ldots, x_r)}{k}.$$
where \( s(kL; x_1, \ldots, x_r) \) is defined as the maximal \( s \) such that \( H^0(X, kL) \to \bigoplus_{i=1}^r \mathcal{O}_{X,x_i}/m_{x_i}^{s+1} \) is surjective and
\[
\sup_{k; D_1, D_2} \min_j \min_{i=1,2} (\text{mult}_{x_i} D_i) \frac{\min_j \min_{i=1,2} (\text{mult}_{x_i} D_i)}{k}
\]
where the supremum is taken over all pairs of divisors \( D_1, D_2 \in |kL| \) such that \( x_1, \ldots, x_r \) are isolated points in \( D_1 \cap D_2 \).

Proof. We slightly generalize the arguments in [Dem92, Thm.6.4]: The first equality is a direct consequence of Seshadri’s ampleness criterion. Next suppose that there are two divisors \( D_1, D_2 \in |kL| \) such that \( x_1, \ldots, x_r \) are isolated points in \( D_1 \cap D_2 \).

If \( f_{1j}, f_{2j} \) are local equations of \( D_1, D_2 \) w.r.t. a trivialization \( \theta_j \) in a neighborhood \( \Omega_j \) of \( x_j \),
\[
|\xi|_h := |\theta_j(\xi)|e^{-\phi_j}
\]
with weight function \( \phi_j = \frac{1}{k} \log(|f_{1j}|^2 + |f_{2j}|^2) \) defines a positive singular hermitian metric on \( L \) with isolated poles at \( x_j \) and Lelong numbers
\[
\gamma_j := \nu(\phi_j, x_j) = \min\{\text{mult}_{x_j} D_1, \text{mult}_{x_j} D_2\}.
\]
Set \( c(L) = \frac{i}{2} \partial \bar{\partial} \phi \) on \( \Omega_j \) and let \( C \) be an irreducible curve. Then
\[
L.C = \int_C c(L) \geq \sum_{j=1}^r \int_{C \cap \Omega_j} \frac{i}{\pi} \partial \bar{\partial} \phi \geq \sum_{j=1}^r \gamma_j \nu(C, x_j)
\]
\[
= \sum_{j=1}^r \gamma_j \text{mult}_{x_j} C \geq \min_j \min_{i=1,2} (\text{mult}_{x_i} D_i) \frac{\min_j \min_{i=1,2} (\text{mult}_{x_i} D_i)}{k} \sum_{j=1}^r \text{mult}_{x_j} C
\]
because the last integral is larger than the Lelong number of the integration current \( [C] \) w.r.t. the weight \( \phi_j \) and we may apply the comparison theorem with the ordinary Lelong number associated to the weight \( \log |z - x| \) (see [Dem00] 2.B for more details). Therefore
\[
\epsilon(L; x_1, \ldots, x_r) \geq \sup_{k; D_1, D_2} \min_j \min_{i=1,2} (\text{mult}_{x_i} D_i) \frac{\min_j \min_{i=1,2} (\text{mult}_{x_i} D_i)}{k}.
\]

Now fix a coordinate system \( (z_1, z_2) \) on \( \Omega_j \) centered at \( x_j \). For \( s := s(kL; x_1, \ldots, x_r) \) the sections in \( H^0(X, kL) \) generate all combinations of \( s \)-jets at all points \( x_j \) and we can find two holomorphic sections \( f_1, f_2 \) whose \( s \)-jets are \( z_j^1, z_j^2 \) at every \( x_j \). We define a global positive singular metric by
\[
|\xi|_h := |\theta_j(\xi)|e^{-\phi_j}
\]
with weight function \( \phi_j = \frac{1}{k} \log(|\theta_j(f_1)|^2 + |\theta_j(f_2)|^2) \). Then \( \phi_j \) has isolated poles with Lelong number \( \frac{s}{k} \) at \( x_j \) thus
\[
\sup_{k; D_1, D_2} \min_j \min_{i=1,2} (\text{mult}_{x_i} D_i) \frac{\min_j \min_{i=1,2} (\text{mult}_{x_i} D_i)}{k} \geq \frac{s}{k}.
\]
Finally by using Kodaira’s vanishing theorem for \( \mathcal{O}(p\pi^*L - q \sum_{j=1}^r E_j) \) with
\[
\frac{q}{p} < \epsilon(L; x_1, \ldots, x_r)
\]
we can show that
\[
\sup_k \frac{s(kL; x_1, \ldots, x_r)}{k} \geq \epsilon(L; x_1, \ldots, x_r).
\]
5. The Nagata conjecture

A celebrated conjecture of Nagata may be seen as a statement on Seshadri constants associated to sets of points in $\mathbb{P}^2$: Let

$$\pi : \widetilde{X} = \text{Bl}_r(\mathbb{P}^2) \to \mathbb{P}^2$$

be the blow up of the projective plane in a finite set $S \subset \mathbb{P}^2$ consisting of $r$ very general points. Denote by $H$ the pull back of a line to $\widetilde{X}$, and let $E_1, \ldots, E_r$ be the exceptional divisors over the $x_j$. Nagata [Nag59] conjectured in effect that the $\mathbb{R}$-divisor

$$H - \frac{1}{r} \sum_{j=1}^{r} E_j$$

is nef on $\widetilde{X}$ provided that $r \geq 9$.

It is well known that Nagata’s conjecture is implied by another conjecture of Harbourne and Hirschowitz about spaces $L_d(n)$ of plane curves of given degree $d$ and multiplicity at least $m$ at $n$ general points [Mir99, CM01]. This conjecture tries to detect those of the spaces $L_d(n)$ which do not have the expected dimension

$$\max(-1, \frac{d(d+3)}{2} - n \cdot \frac{m(m+1)}{2})$$

In particular it implies that the $L_d(n)$ do have expected dimension if $d \geq 3m$ [CM00].

On the other hand there is a complete classification of (non) special systems $L_d(n)$ for $n \leq 9$:

**Theorem 5.1** ([CM00, Thm.2.4]). For $n \leq 9$ the special linear systems $L_d(n)$ are

- $L_d(2m)$ with $m \leq d \leq 2m - 2$
- $L_d(3m)$ with $3m/2 \leq d \leq 2m - 2$
- $L_d(5m)$ with $2m \leq d \leq (5m - 2)/2$
- $L_d(6m)$ with $12m/5 \leq d \leq (5m - 2)/2$
- $L_d(7m)$ with $21m/8 \leq d \leq (8m - 2)/3$
- $L_d(8m)$ with $48m/17 \leq d \leq (17m - 2)/6$.

We show now that it is not necessary to prove the Harbourne–Hirschowitz conjecture for every triple $(d, n, m)$ to get Nagata’s conjecture or at least a lower bound for the Seshadri constant of $H$ and $r$ general points on $\mathbb{P}^2$:

**Theorem 5.2.** Let $r > 9$ be an integer and $(d_i, m_i)$ a sequence of pairs of positive integers such that $d_i^2 \to \infty$, $\frac{1}{m_i} \to 1$ and the space $L_{d_i}(r^{m_i}+1)$ has expected dimension $\geq 0$. Then

$$H - a \cdot \sqrt{\frac{1}{r} \sum_{j=1}^{r} E_j}$$

is nef on $\widetilde{X}$. In particular, Nagata’s conjecture is true for $r$ general points in $\mathbb{P}^2$, if $a = 1$. 

Note furthermore that the assumptions of the theorem on the spaces $\mathcal{L}_d(rm_{i}+1)$ follow from the Harbourne–Hirschowitz conjecture since $\frac{d}{m_{i}} \to \sqrt{r}$ implies $d_{i} \geq 3m_{i}$ for $i$ big enough if $r > 9$.

**Proof of the Theorem.** Note first that if we define $m(d,r)$ as the maximal $m$ such that

$$\frac{d(d+3)}{2} - r \cdot \frac{m(m+1)}{2} \geq 0$$

we get for fixed $r$ that

$$\lim_{d \to \infty} \frac{m(d,r)}{d} = \frac{1}{\sqrt{r}}.$$ 

Next we observe the following fact: Since $\mathcal{L}_d(r^1)$ has also the expected dimension (this is the Multiplicity One Theorem in [CM98]) the intermediate spaces $\mathcal{L}_d(rm_{i})$ and

$$\mathcal{L}_d(km_{i}, (r-k)m_{i}+1), \; k=0, \ldots, r$$

(denoting the space of curves having multiplicity $m_{i}$ in $k$ points and multiplicity $m_{i}+1$ in $r-k$ points) have expected dimension.

Now let $\pi : \tilde{X} = Bl_{r}(\mathbb{P}^2) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ in $r$ general points $x_{1}, \ldots, x_{r}$. We choose $m_{i}$ points $y_{ij}, \ldots, y_{m_{i}j}$ on every exceptional divisor $E_{j}$ over $x_{j}$ and consider the space of curves $C \subset \tilde{X}$ passing through the points $y_{kj_{0}}, \; k=1, \ldots, m_{i}$ for some fixed $j_{0}$ such that

(i) $\deg \pi(C) = d_{i}$ and

(ii) $\pi(C)$ has multiplicity $m_{i}+1$ in the points $x_{1}, \ldots, x_{j_{0}-1}, x_{j_{0}+1}, \ldots, x_{r}$.

The expected dimension of this space is

$$\dim \mathcal{L}_{d_{i}}(1^{m_{i}}, (r-1)^{m_{i}+1}) - m_{i} > \dim \mathcal{L}_{d_{i}}(1^{m_{i}}, (r-1)^{m_{i}+1}) - m_{i} - 1 = \mathcal{L}_{d_{i}}(rm_{i}+1).$$

Hence there exists a curve $C \subset \mathbb{P}^2$ in this space having multiplicity exactly $m_{i}$ in $x_{j_{0}}$ and whose tangent cone is given by the points $y_{ij_{0}}, \ldots, y_{m_{i}j_{0}}$.

The $r$-tuples of pairwise disjoint points in $\mathbb{P}^2$ with one marked point is parametrized by the subvariety

$$W \xrightarrow{pr} ((\mathbb{P}^2)^{r} - D) \times \mathbb{P}^2 \xrightarrow{pr} \mathbb{P}^2$$

where $D$ collects all $r$-tuples in $(\mathbb{P}^2)^{r}$ containing one point (at least) twice. A fiber of $W$ over an $r$-tuple in $(\mathbb{P}^2)^{r} - D$ is just the scheme of the $r$ points in the tuple. Since the fibers of the other projection are Zariski-open and hence irreducible subsets of $(\mathbb{P}^2)^{r-1}$ the algebraic subset $W$ is irreducible.

Consequently no Zariski-closed subset of $W$ can dominate $(\mathbb{P}^2)^{r} - D$ and we can find a tuple $(x_{1}, \ldots, x_{r})$ such that for every $j_{0} = 1, \ldots, r$ there are curves $C$ as described above. Considering these curves as divisors of degree $d_{i}$ and adding them gives a divisor $D$ which has exactly multiplicity $m_{i}$ in every point $x_{i}$ and whose tangent cone in $x_{i}$ is described by the $y_{ij}$, $j = 1, \ldots, m_{i}$.

Choosing points $y'_{ij}$ such that

$$\{y_{i1}, \ldots, y_{im_{i}}\} \cap \{y'_{i1}, \ldots, y'_{im_{i}}\} = \emptyset, \; i = 1, \ldots, r$$
we get in the same way another divisor $D'$ such that $D \cap D'$ contains $x_1, \ldots, x_r$ as isolated points and $\text{mult}_{x_i} D = \text{mult}_{x_i} D' = m_i$. The two divisors $D$ and $D'$ may be used to derive from Prop. 4.2

$$\frac{m_i}{d_i} \leq \epsilon(H; x_1, \ldots, x_r).$$

On the other hand computing the self intersection of $\pi^* H - \frac{1}{\sqrt{r}} \sum E_i$ shows that

$$\epsilon(H; x_1, \ldots, x_r) \leq \frac{1}{\sqrt{r}}$$

hence the last part of the theorem on the Nagata conjecture. □

The reason behind this approach to Nagata’s conjecture is that recent attacks on the Harbourne–Hirschowitz conjecture work with recursions which unfortunately have some gaps. But it might be worth checking if around these gaps there are sequences of triples $(r, d_i, m_i)$ as in the theorem. We carry out this program with the Ciliberto-Miranda recursion \cite{CM98, CM00}. It relies on a deformation argument splitting linear systems on $\mathbb{P}^2$ in another one on $\mathbb{P}^2$ and one on $\mathbb{P}^2$ blown up in one point (which can immediately be transformed into a linear system on $\mathbb{P}^2$ again).

Studying their intersection Ciliberto and Miranda were able to prove

Proposition 5.3 \cite[(Cor.3.4)]{CM98}. Fix $d, n$ and $m$. Suppose that positive integers $k$ and $b$ exist, with $0 < k < d$ and $0 < b < n$ such that $\mathcal{L}_{d-k-1}((n-b)m)$ and $\mathcal{L}_d(1^{d-k+1}, b^m)$ are non-empty and non-special linear systems. Then $\mathcal{L}_d(m^n)$ is non-empty and non-special.

Here, $\mathcal{L}_d(1^{d-k+1}, b^m)$ is the linear system of degree $d$ curves on $\mathbb{P}^2$ having multiplicity $d - k + 1$ in one general point and $m$ in $b$ other general points. To deal with such quasi-homogeneous systems Ciliberto and Miranda used the quadratic Cremona transformation to show

Proposition 5.4 \cite[(Prop.6.2)]{CM98}. Let $\mathcal{L} = \mathcal{L}_d(1^{d-m}, n^m)$ with $2 \leq m \leq d$. Write $d = qm + \mu$ with $0 \leq \mu \leq m - 1$ and $n = 2h + e$ with $e \in \{0, 1\}$. If $q > h$ then $\mathcal{L}$ is non-empty and non-special.

Corollary 5.5 \cite[(Cor.6.3)]{CM98}. Let $\mathcal{L} = \mathcal{L}_d(1^{d-m+1}, n^m)$ and $\mathcal{L}' = \mathcal{L}_{d-n}(1^{d-n-m+1}, n^m)$. Then $\mathcal{L}$ is non-empty and non-special if $\mathcal{L}'$ is non-empty and non-special.

Combining these results with the classification of non-special systems for $r \leq 9$ points in Theorem 5.1 S. Barkowski \cite{Bar06} was able to construct sequences as in Theorem 5.2 recursively on the number of points $r$. Using the notation of Theorem 5.2 their properties can be summarized as follows:

| $r$ | $s^2 + 1$ | $s^2 + 2$ | $s^2 + 3$ | $s^2 + 4$ |
|-----|----------|----------|----------|----------|
| $\sqrt{r}/a$ | $s + 1$ | $s + 1$ | $s + 1$ | $s + 1$ |

Unfortunately, for $s^2 + k$, $5 \leq s \leq 2s + 1$ the best lower bound is still $\frac{1}{\sqrt{r}}$, which can already be deduced from Nagata’s solution for a square number $r = (s+1)^2$ of points. Even worse, it seems not be possible to improve these lower bounds with the Ciliberto-Miranda recursion.
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