Light quark masses and condensates in QCD

Jan Stern

Division de Physique Théorique, Institut de Physique Nucléaire, Université Paris-Sud, 91406 Orsay, France

Abstract

We review some theoretical and phenomenological aspects of the scenario in which the spontaneous breaking of chiral symmetry is not triggered by a formation of a large condensate $\langle \bar{q}q \rangle$. Emphasis is put on the resulting pattern of light quark masses, on the constraints arising from QCD sum rules and on forthcoming experimental tests.

IPNO/TH 97-30

---

1Invited plenary talk given at the Workshop on Chiral Dynamics 1997, Mainz, Germany, Sept. 1-5, 1997
2Unité de Recherche des Universités Paris XI et Paris VI, associée au CNRS
Light quark masses and condensates in QCD

Jan Stern

ABSTRACT We review some theoretical and phenomenological aspects of the scenario in which the spontaneous breaking of chiral symmetry is not triggered by a formation of a large condensate $<\bar{q}q>$. Emphasis is put on the resulting pattern of light quark masses, on the constraints arising from QCD sum rules and on forthcoming experimental tests.

1 Introduction

In the presence of $N_f$ massless flavours, the QCD Lagrangian exhibits the chiral symmetry $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$. Two theoretical facts can be inferred from first principles.

**Theorem 1** If $N_c \geq 3$, $N_f \geq 3$, and provided quarks are confined (no coloured physical states), the chiral symmetry $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$ is necessarily broken down to its diagonal subgroup $U_V(N_f)$ generated by vector currents.

This theorem follows from the analysis of constraints imposed by anomalous Ward identities on the spectrum of massless physical states [t'H 80], [FSBY81], [CG82] and from some non-trivial properties of vector-like gauge theories, such as the so called "persistent mass condition" [PW81]: a bound state can be massless only if all its constituents are massless [VW84a, VW84b]. The proof of Theorem 1 gives a hint about the actual content of the statement of spontaneous breaking of chiral symmetry ($SB\chi S$): the statement merely concerns the existence of massless Goldstone boson states (pions) coupled to conserved axial currents:

$$<0|A_i^\mu|\pi^j\vec{p}> = i\delta^{ij}F_0p_\mu.$$  \hspace{1cm} (1.1)

One proves that in QCD with $N_f \geq 3$ and no coloured states, it would be impossible to satisfy all anomalous Ward identities if $F_0$ would have to vanish. This brings us to the second theoretical fact.

**Theorem 2** A necessary and sufficient criterion of $SB\chi S$ is a non-zero value of the left-right correlation function

$$\lim_{m \to 0} i \int d^4x <\Omega|TL^i_\mu(x)R^j_\nu(0)|\Omega>= -\frac{1}{4}\eta_{\mu\nu}\delta^{ij}F_0^2, \hspace{1cm} (1.2)$$
where $L_\mu = \frac{1}{2}(V_\mu - A_\mu)$, $R_\mu = \frac{1}{2}(V_\mu + A_\mu)$ are Noether currents generating the left and right chiral rotations respectively.

The correlator (1.2) encompasses both essential features of $SB\chi S$. i) The asymmetry of the vacuum: if the vacuum would be symmetric, $F_0^2$ should vanish. ii) The existence of massless Goldstone bosons: the limit (1.2) is non-vanishing if and only if the correlator (1.2) contains a massless pion-pole. $F_0^2$ is an order parameter, whose non-zero value is necessary and sufficient for $SB\chi S$ and Goldstone bosons to occur. There are, of course, many other order parameters, such as local quark condensates $\langle \bar{q}q \rangle$, $\langle \bar{q} \sigma_{\mu\nu} F^{\mu\nu} q \rangle$...

A non-zero value of each of them by itself implies $SB\chi S$, but the converse is not true. $SB\chi S$ can take place (i.e. $F_0 \neq 0$), even if some of these condensates vanishes. In particular, there is no proof available as in the case of $F_0^2$, showing that $\langle \bar{q}q \rangle \neq 0$ is a necessary consequence of $SB\chi S$. The $\bar{q}q$ condensate plays an analogous role as the spontaneous magnetization $\langle \vec{M} \rangle$ of spin systems with broken rotation symmetry. Although there is no symmetry reason for the latter to vanish, its actual value depends on the nature of the magnetic order in the ground state: $\langle \vec{M} \rangle \neq 0$ for a ferromagnet, whereas $\langle \vec{M} \rangle = 0$ for an antiferromagnet.

In the next section, I shall briefly illustrate how $SB\chi S$ without $\langle \bar{q}q \rangle$ condensation could naturally arise in QCD. The existence of such a theoretical possibility means that one should remain open-minded and precautionous concerning the value of $\langle \bar{q}q \rangle$, especially, since the latter is not yet under an experimental control. I shall mainly review the pattern of light quark masses as they would look like if the condensate was considerably smaller than usually believed, adding some comments on the constraints imposed on quark masses and condensates by QCD-sum rules. Finally, I shall briefly mention few forthcoming experimental tests. Some of them are discussed in details in other contributions to this Workshop [Eck97], [Mei97], [Sai97], [Kne97], [K197], [Low97], [Di97].

2 QCD vacuum as a disordered system

Upon evaluating non-perturbative quantities such as the correlator (1.2) or the $\langle \bar{q}q \rangle$ condensate, it is useful to consider the theory in an Euclidean space-time box $L \times L \times L \times L$ with periodic boundary conditions and to integrate over quark fields first. This leads to a quantum mechanical problem of a single quark in a random gluonic background $G_\mu^a(x)$, which is defined by the hermitean Hamiltonian

$$H = \gamma_\mu (\partial_\mu + i G_\mu^a t^a) .$$  (1.3)

The result of this integration over quarks may be formally expressed in terms of eigenvalues $\lambda_n$ and of (orthonormal) eigenvectors $\phi_n(x)$ of the Dirac Hamiltonian $H$. The spectrum is symmetric around 0: $\gamma_5 \phi_n = \phi_{-n}$,
\[ \lambda_{-n} = -\lambda_n. \]

Subsequently, the resulting expression has to be averaged over all gluon configurations:

\[
\langle\langle X[G] \rangle\rangle = \int d[G] \exp(-S_Y M[G]) \prod_{\lambda_n \geq 0} (m^2 + \lambda_n^2)^N J X[G].
\] (1.4)

The fact that the integral (1.4) involves a positive probability measure suggests a possible analogy with disordered systems.

For the chiral condensate one gets

\[
\langle\langle \bar{q}q \rangle\rangle = -\lim_{m \to 0} \lim_{L \to \infty} \frac{1}{L^4} \langle\langle \sum_n \frac{m}{m^2 + \lambda_n^2} \rangle\rangle
\] (1.5)

where \( m \) is the quark mass. Similarly, the correlator (1.2) is given by the formula

\[
F_0^2 = \lim_{m \to 0} \lim_{L \to \infty} \frac{1}{L^4} \langle\langle \sum_{kn} \frac{m}{m^2 + \lambda_k^2} \frac{m}{m^2 + \lambda_n^2} J_{kn} \rangle\rangle,
\] (1.6)

where

\[
J_{kn} = \frac{1}{4} \sum_{\mu} \int dx \phi_{kn}^\dagger(x) \gamma_{\mu} \phi_{kn}(x)^2.
\] (1.7)

It is seen that both order parameters (1.5) and (1.6) are merely sensitive to the infrared end of the Dirac spectrum, \(|\lambda_n| < \epsilon\).

Consider now the Dirac Hamiltonian (1.3) as a generator of evolution in a fictitious time \( t \) added to the 4 Euclidean space-coordinates \( x_\mu \). In this 4+1 dimensional space-time one may switch on a homogenous color singlet electric field, adding to \( H \) a time dependent perturbation

\[
\delta H = i \gamma_\mu \xi_\mu \sin \omega t
\] (1.8)

with \( \xi_\mu \) constant. \( J_{kn} \) is then proportional to the \(|k \to |n \rangle\rangle\rangle\) transition probability triggered by the perturbation (1.8) with \( \lambda_k - \lambda_n = \pm \omega \).

This suggests that from the point of view of the fictitious 4+1 dimensional space-time, there is an analogy between \( SB\chi S \) and the electrically induced transport properties of massless quarks in a random medium which is characterized by a coloured magnetic type (static) disorder \( G_0^\mu(x) \) with a probability distribution given by eq. (1.4).

Indeed, the formula (1.6) can be rewritten as

\[
F_0^2 = \pi^2 \lim_{\epsilon \to 0} \lim_{L \to \infty} L^4 \tilde{J}(\epsilon, L)[\rho(\epsilon, L)]^2,
\] (1.9)

where \( \tilde{J}(\epsilon, L) \) is the transition probability (1.7) averaged over all initial and final states with "energy" \(|\lambda| < \epsilon\) and, of course, over the disorder, whereas
\( \rho(\epsilon, L) \) stands for the density of states, i.e. the number of states per unit energy \( \epsilon \) and volume \( L^4 \). The latter quantity defines the chiral condensate [BC80]:

\[ < \bar{q}q > = -\pi \lim_{\epsilon \to 0} \lim_{L \to \infty} \rho(\epsilon, L). \tag{1.10} \]

Eq. 1.9 can be viewed as an ultrarelativistic \((m = 0)\) version of the Kubo-Greenwood formula for electric conductivity (see e.g. [Mot70], [Tho74]). It shows that \( SB\chi S \) results from a conjunction of an appropriate "quark mobility" \( \bar{J}(\epsilon, L) \) and a density of states \( \rho(\epsilon, L) \). On the other hand, quark condensation is an exclusive affair of the density of states.

The pattern of \( SB\chi S \) depends on the degree of accumulation of eigenvalues \( \lambda_n \) near zero, as \( L \to \infty \). Suppose that the lowest eigenvalues averaged over the disorder behave as \( << \lambda_n >> \sim L^{-\kappa} \), where \( \kappa \geq 1 \) as shown by [VW84b]. Then for \( \epsilon \to 0, L \to \infty \)

\[ \rho(\epsilon, L) = \mu^3 \left( \frac{2\epsilon}{\mu} \right)^{\frac{4}{\kappa} - 1}, \tag{1.11} \]

where \( \mu \) is a mass scale. Consequently, \( \bar{q}q \) pairs condense if and only if \( \kappa = 4 \) [LS92]. The case \( \kappa > 4 \) represents a too strong infrared singularity, which can be excluded: \( < \bar{q}q > \) cannot explode [Gas97a]. On the other hand, \( SB\chi S \), i.e. a non-zero value of \( F_0^2 \) given by Eq. 1.9 can be shown to require \( \kappa \geq 2 \). Hence, one is faced to two extreme alternatives of \( SB\chi S \):

i) \( \kappa = 2 \): The density of states and \( < \bar{q}q > \) vanish as \( \epsilon \to 0 \), but still \( F_0^2 \neq 0 \), i.e. \( SB\chi S \) takes place and Goldstone bosons are formed, due to a large mobility of "low-energy" quarks, \( \bar{J} \sim \epsilon^{-2}L^{-4} \). This behavior occurs naturally provided quark states are delocalized.

ii) \( \kappa = 4 \): The density of states remains non-zero as \( \epsilon \to 0 \), i.e. \( < \bar{q}q > \neq 0 \), whereas the mobility \( \bar{J} \) must be suppressed by a factor \( L^{-4} \). Such a suppression of mobility could be naturally understood if the Euclidean quark states were in a sense localized.

The intermediate case \( 2 < \kappa < 4 \) cannot be excluded on general grounds. One can show, however, that an effective low energy theory characterized by an effective Lagrangian analytic in quark masses can only exist, provided \( 4/\kappa = \) integer. This, together with the condition \( \kappa \geq 2 \), selects the cases \( \kappa = 2 \) and \( \kappa = 4 \) as two distinct possibilities of realizing \( SB\chi S \) in QCD.

In Nature, both types of states belonging to the \( \kappa = 2 \) and \( \kappa = 4 \) bands can coexist and contribute to the \( SB\chi S \). Since only the \( \kappa = 4 \) band contributes to the chiral condensate, the actual value of \( < \bar{q}q > \) can hardly be guessed in advance. What matters in practice, is the size of the parameter

\[ B_0 = -\frac{1}{F_0^2} < \bar{q}q > . \tag{1.12} \]
One will have to distinguish on phenomenological grounds between a large condensate, typically $B_0 (1 \text{ GeV}) \sim 2 \text{ GeV}$ which seems to be suggested by lattice simulations and a small condensate, say $B_0 \sim 100 \text{ MeV}$ resulting from an attempt to variationally extend the QCD perturbation theory [AGKN97] and to calculate non-perturbative quantities such as $F_0$ and $\langle \bar{q}q \rangle$ [Kne96].

3 Quark mass ratios

We consider the 3 light quark masses $m_u(\mu)$, $m_d(\mu)$ and $m_s(\mu)$ renormalized in the $\overline{MS}$ scheme at the running scale $\mu$. In this section we will be mostly concerned with the relation between the two quark-mass ratios

$$r = \frac{m_s}{\hat{m}}, \quad R = \frac{m_s - \hat{m}}{m_d - m_u}, \quad \hat{m} = \frac{1}{2}(m_u + m_d) \quad (1.13)$$

and the masses of unmixed Goldstone bosons $\pi^+$, $K^+$ and $K^0$.

3.1 The standard picture

The generally accepted picture of the ratios [1.13] goes back to [Wei77] and has been further elaborated by Gasser and Leutwyler [GL82], [GL85], [Leu90], [Leu96]. One writes

$$M_{\pi^+}^2 = (m_u + m_d)B_0 + \Delta_{\pi^+}$$
$$M_{K^+}^2 = (m_u + m_s)B_0 + \Delta_{K^+}$$
$$M_{K^0}^2 = (m_d + m_s)B_0 + \Delta_{K^0} \quad (1.14)$$

and one assumes that

$$\epsilon_P = \frac{\Delta_P}{M_P^2} \ll 1 \quad (1.15)$$

It follows [Wei77]

$$r = r_2 \{1 + O(\epsilon_P)\}, \quad r_2 = 2 \frac{M_K^2}{M_\pi^2} - 1 \approx 25.9 \quad (1.16)$$

and

$$R \frac{\Delta M_K^2}{M_K^2 - M_\pi^2} = 1 + O(\epsilon_P), \quad \Delta M_K^2 = (M_{K^0}^2 - M_{K^+}^2)_{\text{QCD}} \quad (1.17)$$

Gasser and Leutwyler have shown [GL85] that the $O(\epsilon_P)$ corrections in Eqs. [1.16] and [1.17] are related. Eliminating them one gets a hyperbolic relation

$$R \frac{\Delta M_K^2}{M_K^2 - M_\pi^2} = \frac{r_2 + 1}{r + 1} \{1 + O(\epsilon_P^2)\} \quad (1.18)$$
which is (almost exactly) equivalent to the Leutwyler’s ellipse \[\text{Leu90}\]

\[
\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2} \left(\frac{m_s}{m_d}\right)^2 = 1 + O(\epsilon_P^2), \quad Q^2 = \frac{M_K^2 - M^2}{\Delta M^2} \frac{M^2}{M^2}. \tag{1.19}
\]

This constitutes what is usually called ”a current-algebra picture of quark mass ratios” despite the fact that the assumption 1.15 follows neither from current algebra nor from QCD. None of the conclusions 1.16, 1.17, 1.18 or 1.19 holds if \(\epsilon_P \sim 1\), in particular, if \(B_0 \sim 0\).

### 3.2 Expansion of Goldstone boson masses

\(\epsilon_P \sim 1\) does not mean that the expansion of \(M_P^2\) in powers of \(m_q\) breaks down. One can actually argue that expansion of \(\Delta_P\) proceeds by powers of \(m_q/\Lambda_H\), where \(\Lambda_H \sim 1\) GeV is the characteristic mass-scale of first massive bound states. (The same parameter \(m_q/4\pi F_0\) characterizes the coefficients of chiral logs, \(4\pi F_0 \sim \Lambda_H\)).

Writing the expansion of \(\Delta_P\) as

\[
\Delta_P = \frac{1}{F_0^2} \sum_{n=2}^{\infty} a_n m_q^n + \text{(Chiral log’s)}, \tag{1.20}
\]

the coefficients \(a_n\) represent connected n-point functions of scalar and pseudoscalar quark bilinears with all Goldstone boson poles subtracted at vanishing external momenta. They are dominated by exchanges of bound states of mass \(\sim \Lambda_H \sim 1\) GeV and this fact allows one to roughly estimate their order of magnitude \[\text{Geo93}]:

\[
a_n = c_n F_0^2 \Lambda_H^{2-n}, \quad c_n \sim 1. \tag{1.21}
\]

The coefficients \(a_n\) are genuine QCD quantities independent of the Goldstone boson sector of the theory and the estimate 1.21 should hold independently of the size of the chiral condensate. On the other hand, the parameters \(\epsilon_P\) measure the relative importance of the condensate and \(\Delta_P\) contributions to \(M_P^2\). Since \(a_2\) is of the order \(F_0^2\), one can expect \(\epsilon_P \sim m_q/(m_0+m_q)\), where \(m_0 \sim B_0\). If \(B_0\) happened to be as small as, say, \(m_n\), the assumption \(\epsilon_K \ll 1\) would break down. Hence, according to the size of \(B_0\) as compared to \(\Lambda_H\), one should distinguish two expansion schemes of the effective Lagrangian and of \(M_P^2\):

**i) Standard Chiral Perturbation Theory - \(S\chi PT\)** \[\text{GL84, GL85, Gas97b}\] is a simultaneous expansion in powers of \(m_q/\Lambda_H\) and \(m_q/m_0\). It is defined by the chiral counting \(m_q = O(p^2), \ B_0 \sim m_0 \sim \Lambda_H = O(1)\).

**ii) Generalized Chiral Perturbation Theory - \(G\chi PT\)** \[\text{FSS91, SSF93, KS95}\] is an expansion in \(m_q/\Lambda_H\) alone. \((m_q \ll m_0\) is not assumed). This can be realized systematically, through a modified chiral counting: \(m_q = O(p), \ m_0 \sim B_0 = O(p)\). At any given order \(O(p^n)\), \(S\chi PT\) appears as a special case of \(G\chi PT\).
3.3 The ratio $r = m_s/\hat{m}$ in GχPT

Let us put, for a moment, $m_u = m_d = \hat{m}$ and write the expansion of $F^2_\pi M^2_\pi$ and $F^2_K M^2_K$ in the form

$$\frac{1}{F^2_\pi} F^2_\pi M^2_\pi = 2\hat{m} B(\mu) + 4\hat{m}^2 A_0(\mu) + \frac{1}{F^2_\pi} F^2_\pi \delta M^2_\pi$$

$$\frac{1}{F^2_K} F^2_K M^2_K = (m_s + \hat{m}) B(\mu) + (m_s + \hat{m})^2 A_0(\mu) + \frac{1}{F^2_K} F^2_K \delta M^2_K$$

which, as it stands, holds independently of the size of

$$B(\mu) = B_0 + 2(m_s + 2\hat{m}) Z_{0s}(\mu).$$

$A_0(\mu)$ and $Z_{0s}(\mu)$ are related to two-point functions of scalar and pseudoscalar currents. $Z_{0s}(\mu)$ violates the Zweig rule (it is suppressed as $N_c \to \infty$) and one may expect $B(\mu) \simeq B_0$. Both $A_0(\mu)$ and $Z_{0s}(\mu)$ are constants of the effective Lagrangian renormalized at the scale $\mu$. $\delta M^2_\pi$ involves chiral logs and higher order terms. They are relatively small: $\delta M^2_\pi < 0.1 M^2_\pi$ independently of the magnitude of $B$.

$S\chi PT$ considers the first term in Eq. (1.22) as the leading $O(p^2)$ contribution and the remaining two terms as a small $O(p^4)$, $O(p^6)$... perturbation. $G\chi PT$ admits that the first two terms could equally contribute to the leading $O(p^2)$ order, (because $B$ is small) and it treats $\delta M^2_\pi$ as a small $O(p^3)$, $O(p^4)$... perturbation. It is clear that $S\chi PT$ can be applied only provided

$$\hat{m} \ll m_0 = \frac{B_0}{2A_0}, \quad m_s \ll 2m_0.$$  (1.24)

If the condition (1.24) is not satisfied, Eq. (1.22) does not allow to determine the ratio $r = m_s/\hat{m}$ as in Eq. (1.16). At most can one relate the magnitude of $r$ to the amount of violation of the GOR relation $^{GMOR68}$, $M^2_\pi = 2\hat{m} B_0$.

Eqs. (1.22) yield the exact formula

$$X_{GOR} = \frac{2\hat{m} F^2_\pi B}{F^2_\pi M^2_\pi} = \frac{1}{r^2 - 1} \{r - r^*_1(r)\} \{r + r^*_1(r) + 2\} \left(\frac{M^*_\pi}{M^*_\pi}\right)^2$$

where

$$r^*_1(r) = \frac{2F_K}{F_\pi} \frac{M^*_K(r)}{M^*_\pi(r)} - 1, \quad [M^*_p(r)]^2 = M^2_p - \delta M^2_p(r).$$

The dependence of the GOR-ratio $X_{GOR}$ on the quark mass ratio $r$ is displayed in Fig. 1, together with the uncertainties attached with our estimates of higher order corrections. Eq. (1.25) meets a zero at $r = r_{crit}$, defined by

$$r_{crit} = r^*_1(r_{crit}) = \frac{2M^*_K}{M^*_\pi} - 1 + O(m) = 6.3 + O(m).$$

(1.27)
Including higher order effects, one gets
\[ r_{\text{crit}} = 8.22 \pm 0.64 \, . \] (1.28)

Since vacuum stability requires \( B \geq 0 \), one must have \( r \geq r_{\text{crit}} \). Smaller \( B \), closer the ratio \( r \) to its critical value 1.28.

3.4 The ratio \( R = m_s - \frac{\hat{m}}{m_d - m_u} \) in G\( \chi \)PT: dismiss of the ellipse

We now turn to the isospin breaking effects due to \( m_d \neq m_u \). There is a "magic" linear combination of the three unmixed Goldstone boson masses \cite{FSS91, KMSF95}

\[ R\Delta M_K^2 - (M_K^2 - M_\pi^2) - \left( M_K^2 - \frac{r+1}{2} M_\pi^2 \right) \frac{r-1}{r+1} = O \left( \frac{m_3^3}{\Lambda_H} \right) \] (1.29)

in which all \( O(m) \), \( O(m^2) \) and even \( O(m^2 \ln m) \) terms cancel, independently of the size of \( \langle \bar{q}q \rangle \). In order to see the connection with Gasser-Leutwyler's hyperbola \cite{Gasser:1987rj}, alias Leutwyler's ellipse \cite{Leutwyler:1995qg}, let us rewrite Eq. 1.29, including the \( O(m_3^3) \) corrections, as

\[ \frac{R\Delta M_K^2}{M_K^2 - M_\pi^2} = \left( \frac{r+1}{r+1} \right)^2 \frac{r_2-r}{r_2+1} - \left\{ \left( \frac{F_2^2}{F_2} - 1 \right) \frac{r_2-r}{r_2+1} + \frac{m_3^3 \rho_2}{M_K^2 - M_\pi^2} \frac{(r-1)^2(r+1)}{r^3} + \cdots \right\} \] (1.30)

where \( r_2 = 2M_K^2/M_\pi^2 - 1 \simeq 25.9 \) and \( |\rho_2| \lesssim 1/\Lambda_H \) is a constant contained in the \( \mathcal{L}_{\text{eff}} \) component of \( \mathcal{L}_{\text{eff}} \) \cite{Kaplan:1994xi, KMSF95}. Keeping only the first term on the right hand side of Eq. 1.30 corresponds to the Gasser-Leutwyler hyperbola (shown as the dashed curve on Fig. 2). Including also the second term (which in \( S\chi PT \) would be \( O(m^2) \)), one obtains the leading order \( G\chi PT \) expression, which differs from the standard case to the extent \( r \) differs from \( r_2 \), (see the solid line on Fig. 2). Finally, the curly bracket
1. Light quark masses and condensates in QCD

represents the NLO $G\chi PT$ correction, which is included in the long dashed curve on Fig. 2 for the case $\rho_2 = 0$. Fig. 3 shows the details of the whole expression 1.30, including the uncertainty in $\rho_2$.

The leading order $S\chi PT$ formula 1.17 implies $R \simeq 43$, assuming the validity of the Dashen theorem in evaluating $\Delta M^2_K$, [GL82]. For lower $r$, say $r \sim 10$, $G\chi PT$ suggests a similar value, typically $R \simeq 40 \div 45$, since the 10-20% increase of the ratio 1.30 is compensated by a similar increase of $\Delta M^2_K$, due to the violation of the Dashen theorem, [Bij93], [Mon97].

Fig. 2: Relation between the two quark mass ratio $R$ and $r$, (see the text after Eq. 1.30)

Fig. 3: Sensitivity of the relation between $R$ and $r$ to the NLO parameter $\rho_2$. $\rho_2 = 0$ solid curve, $\rho_2 = \pm \frac{1}{\sqrt{2}}$ dashed and long-dashed curves.

4 The value of the strange quark mass

Recently, the first experimental determination of $m_s(\mu)$ has been reported [AL97], based on a precise measurement by ALEPH collaboration of inclusive branching ratio’s $R^S=0$ and $R^S=1$ of $\tau$-lepton into $\nu_\tau$ and hadronic final states with total strangeness 0 and 1 respectively [Dav96]. The determination follows the method of Braaten, Narison and Pich [BNP92] which, using the QCD OPE at the scale $M^2_\tau$, gives an expression of $R^S_\tau$ in terms of $\alpha_s(M_\tau)$, $m_s(M_\tau)$ and various (less important) non-perturbative parameters. The preliminary result reads [AL97]

$$m_s(M_\tau) = (172^{+26}_{-31}) \text{ MeV}, \; m_s(1 \text{ GeV}) = (235^{+35}_{-42}) \text{ MeV}.$$ (1.31)
This value is somewhat higher than expected from recent lattice [ea97] and some sum rule [Jam97] estimates. It is, however, compatible with a former determination by [Nar95], which is based on a similar method, but uses less accurate $e^+e^−$ data as input.

4.1 Consequences for parameters $B$ and $A₀$

It has been already shown that Eqs. 1.22 lead to the expression of $\hat{m}B(\mu)$, Eq. 1.25, as a function of $r = m_s/\hat{m}$. A similar expression can be obtained for $m^2A₀$:

$$\frac{4\hat{m}^2F^2_0A₀}{(F_\pi M^*_\pi)^2} = 2\frac{r^*_2(r) - r}{r^2 - 1},$$  \hspace{1cm} (1.32)

where

$$r^*_2(r) = 2\left[\frac{F_KM^*_K(r)}{F_\pi M^*_\pi(r)}\right]^2 - 1.$$  \hspace{1cm} (1.33)

Using the central value $m_s$ (1 GeV) = 235 MeV, one can obtain the condensate parameter $B(\mu)$ and the quadratic slope parameter $A₀(\mu)$ as functions of $r$. The results are displayed on Figs. 4 and 5. They correspond to the QCD scale $\nu = 1$ GeV and to the $\chi$PT scale $\mu = M_\eta$. One observes that for lower values of $r$, the critical mass scale $m₀ = B₀/2A₀$ can indeed be rather small: $m₀ \sim (20-25)$ MeV for $r \sim 10$. Under these circumstances, even the non-strange quark mass $\hat{m} \sim 235$ MeV/\(r\) would be comparable to $m₀$, invalidating the use of the $S\chi PT$ even in the non-strange sector.
4.2 Natural size of $A_0$

Let us consider the two-point function of (octet) scalar and pseudoscalar quark currents $S^a(x)$ and $P^a(x)$ respectively. In the chiral limit $m_s = \tilde{m} = 0$, one has for small $q^2$

$$\prod S_P(q^2)\delta^{ab} = \frac{i}{q^2} \int dx e^{iqx} <0|\{S^a(x)S^b(0) - P^a(x)P^b(0)\}|0> = \begin{pmatrix} B_0 \frac{B_0^2}{q^2} + \frac{3}{32\pi^2 B_0^2} \ln \frac{\mu^2}{q^2} + 1 \end{pmatrix} + A_0(\mu) + O(q^2).$$  \hspace{1cm} (1.34)

The first and second terms on the right-hand side represent the GB pole and loop respectively, whereas the constant $A_0(\mu)$ receives contributions from exchanges of massive $O^{++}$ and $O^{-+}$ resonances: $f_0, \pi', \ldots$. This fact suggests that the constant $A_0(\mu)$ should be rather insensitive to the GB sector of the theory, in particular, to the size of $B_0$. $A_0(\mu)$ is related to the renormalized $S\chi PT, O(p^4)$ low-energy constants $L_8^r(\mu)$,

$$L_8^r(\mu) = \frac{F_0^2}{16B_0^2}A_0(\mu),$$  \hspace{1cm} (1.35)

which has been estimated [GL85] to be $L_8^r(M_\eta) = (1.1 \pm 0.3) \times 10^{-3}$. Taking the standard value of the condensate, $B_0 \simeq 1.5$ GeV (at QCD scale $\nu = 1$ GeV), this corresponds to

$$A_0(M_\eta)_{S\chi PT} = 4.5 \pm 1.2.$$  \hspace{1cm} (1.36)

Notice that this estimate, which is obtained assuming a large value of the condensate, is comparable to the values displayed on Fig. 5 for the case of lower $r$ ($r \sim 10$), for which the condensate $B_0 \sim B$ is about ten times smaller, c.f. Fig. 4. We believe that this is not accidental: on the one hand, there are quantities like $L_8$, or the matrix element $<0|P^a|\pi^b>$, which are rather sensitive to the size of $B_0$, because they either diverge or vanish as $B_0 \to 0$. On the other hand, couplings and masses of non-Goldstone particles ($f_0, \pi', \ldots$) and, consequently, the constant $A_0$ show a moderate dependence on the chiral condensate $\bar{q}q$. 

---

Fig. 5: The parameter $A_0$ (at $\mu = M_\eta$) as a function of $r$. The NLO uncertainty is shown as in Figure 3.
5 QCD sum rules

If \( m_s \approx 235 \text{ MeV} \) and \( r \approx 10 \), the non-strange mass \( \hat{m} \) should be \( 3 \div 4 \) times larger than the typical output of existing standard QCD sum rule analysis \([\text{DdR87}], \text{[BPdR95]}\). In this section it will be argued that whereas the method of QCD sum rules is by itself perfectly adapted for a determination of quark masses and condensates, the crucial experimental data needed as input are still missing. Instead, in existing determinations of \( \hat{m} \), data are replaced by models which implicitly assume a large value of \( < \bar{q}q > \). Consequently, the resulting analysis represents a consistency check of the large condensate scenario, rather than an independent determination of \( \hat{m} \).

5.1 Two-point function of \( \partial^\mu A_\mu \)

I will mainly concentrate on the most elaborated and relevant example of the two point function of \( D^{\bar{u}d}_5 \equiv \partial^\mu [\bar{u} \gamma_\mu \gamma_5 d] = 2\hat{m} \bar{u} i \gamma_5 d \):

\[
\psi^{\bar{u}d}_5(q^2) = i \int dx e^{i q x} < \Omega | T D^{\bar{u}d}_5(x) D^{\bar{u}d}_5(0) | \Omega > . \quad (1.37)
\]

The spectral function \( \rho(t) \) is in principle measurable in tau decays, but so far, neither its normalization nor its shape are known: \( \rho(t) \) is proportional to \( \hat{m}^2 \) and it is precisely \( \hat{m} \) we want to determine. Three informations are available:

i) For large \( t \), the QCD asymptotics takes over,

\[
\rho(t) \rightarrow \frac{3}{2\pi^2} [\hat{m}(t)]^2 t \left\{ 1 + \frac{17}{3} \frac{\alpha_s(t)}{\pi} + \ldots \right\} . \quad (1.39)
\]

ii) In the intermediate energy region, there are two resonances \( \pi' \,(1300) \) and \( \pi'' \,(1770) \) with the couplings to the axial current \( F_{\pi'} \) and \( F_{\pi''} \approx \hat{m} \) and unknown.

iii) Finally, for \( t \sim 0 \), the dominant component \( \rho_{3\pi}(t) \) is given by \( G\chi PT \): to the leading order one gets \([\text{SFK94}]\)

\[
\rho_{3\pi}(t) = \frac{F^2_\pi}{768\pi^4} \left( \frac{M_\pi}{F_\pi} \right)^4 t \left\{ 1 + 10 \frac{r_2 - r}{r^2 - 1} + 30 \left( \frac{r_2 - r}{r^2 - 1} \right)^2 \right\} + \ldots \quad (1.40)
\]

as a function of \( r = m_s/\hat{m} \). The special case of \( S\chi PT \), i.e. of the large condensate, corresponds to \( r = r_2 \) and it has been suggested to pin down the normalization of \( \rho \) using this information. This then yields the well known result \( \hat{m} \,(1 \text{ GeV}) \sim (6-7) \text{ MeV}, \) \([\text{DdR87}], \text{[BPdR93]}\). The problem is that for \( B_0 \sim 0 \), one has \( r \approx r_1 = 2 \frac{M_K}{M_\pi} - 1 \) and the curly bracket in Eq.
It introduces an enhancement factor $1 + 10 \times \frac{1}{2} + 30 \times \frac{1}{4} = 13.5$. This is compatible with the expectation that for $B_0 \sim 0$, the normalization of $\rho(t)$ should increase proportionally to the increase of $\hat{m}^2$.

It is worth noting that the positivity of the spectral function $\rho(t)$ alone implies interesting lower bounds for $\hat{m}$ which are independent of the size of $B_0$. For a recent discussion of these bounds and for a comparison with lattice determinations of $\hat{m}$ and $m_s$, see [LdRT97].

5.2 QCD - Hadron duality

It has been suggested [BPdR95], [Pra97] that the principle of QCD - hadron duality could help in fixing the normalization of $\rho$. One considers the ratio

$$R_{Had}(s) = \frac{3}{2s} \int_0^s dt \frac{t \text{Im}\psi_5(t)}{\text{Im}\psi_5(t)} = 3 \frac{\int_0^s dt \, t \rho(t)}{2s \int_0^s dt \, \rho(t) + \frac{F^2 M^4}{2s} \int_0^s dt \, \rho(t)}$$

(1.41)

and one requires that in a suitable interval of $s$ it coincides with a similar ratio $R_{QCD}(s) = 1 + \text{OPE corrections}$, calculated in QCD perturbation theory. Indeed, within the $S\chi PT$, one expects both the pion and the continuum contributions on RHS of Eq. 1.41 to be of a comparable size $O(p^4)$. Under these circumstances, one could expect that $R_{Had}(s)$ will be sensitive to the normalization of $\rho$, whereas $R_{QCD}(s)$ should be almost independent of $B_0$ and $\hat{m}$. However, for large enough $\rho$, such that in the denominator of 1.41 the continuum dominates over the pion contribution, $R_{Had}(s)$ becomes independent of the normalization. This is what is expected to happen as a consequence of the $G\chi PT$ chiral counting: the $\rho$-contributions, both in the numerator and denominator of Eq. 1.41 are $O(m^2) = O(p^4)$, whereas the pion term is $O(p^4)$. Hence, within the $G\chi PT$ scenario, the duality criterion at most constrains the shape of $\rho(t)$ and it should be satisfied without including the pion contribution. After all, in the scalar channel there is no pion pole and $R_{QCD}(s)$ is about the same both in the $O^{++}$ and $O^{-+}$ channels. An explicit counter example to the statement that duality constrains the normalization of $\rho$ is shown in Figs. 6 and 7. The model of the spectral function in the intermediate energy region is displayed in Fig. 6. The corresponding ratio $R_{Had}(s)$ with no pion contribution included is compared in Fig. 7 with $R_{QCD}(s)$. Notice that due to the (unknown) contribution of direct instantons, one should not expect in this channel the onset of QCD(OPE)-hadron duality at too low $s$. 
5.3 Relation between quark masses and condensates

Even if today, QCD sum rules for $\psi_{5}^{\bar{u}d}$ do not yet allow to determine $\hat{m}$, they can be combined with the Ward identity

$$\psi_{5}^{\bar{u}d}(0) = -(m_u + m_d) < \Omega | \bar{u}u + \bar{d}d | \Omega >,$$  \hspace{1cm} (1.42)

and this additional information can be used to eliminate the unknown normalization of the spectral function. In this way, one can investigate the variation of the condensate 1.42 with $\hat{m}$, keeping fixed $F_\pi^2 M_\pi^2$ at its physical value. Notice that 1.42 does not directly involve the condensate parameter $B_0$ which is defined in the chiral limit. For small $\hat{m}$, one has, however

$$\frac{1}{F_\pi^2} \psi_{5}^{\bar{u}d}(0) = 4\hat{m}B_0 + 2\hat{m}^2C + \ldots,$$  \hspace{1cm} (1.43)

where $C$ is an ultraviolet counterterm depending on the way $\psi_{5}(0)$ is renormalized. One may now consider the Laplace sum rule

$$\tilde{u}\tilde{\phi}_{5}(u) + \frac{1}{\pi} \int \frac{dx}{x} \exp(-\frac{x}{u}) \text{Im} \psi_{5}^{\bar{u}d}(x) = \psi_{5}^{\bar{u}d}(0),$$  \hspace{1cm} (1.44)

where $\tilde{\phi}_{5}(u)$ is the Borel-transform of the function $\psi_{5}(-Q^2)/Q^2$, together with its u-derivatives. $\tilde{\phi}_{5}(u)$ is assumed to be given by the QCD OPE for
\( u \geq 2 \text{ GeV}^2 \). Using inside these two sum rules a simple model for the spectral function
\[
\rho(t) = (m_u + m_d)^2 \text{GeV} \left\{ g^2 \delta(t - M_1^2) + \kappa \delta(t - M_2^2) \right\} + \theta(t - t_0) \gamma_{as}(t),
\]
(1.45)
where \( M_1 = 1300 \text{ MeV}, M_2 = 1770 \text{ MeV}, \kappa \sim 1 \) and \( \gamma_{as}(t) \) is given by QCD asymptotics, one can eliminate the unknown constant \( g^2 \) and for each \( \hat{m} \) (1 GeV), infer a value of the ratio \( \psi_{ud}^d(0)/2F_\pi^2 M_\pi^2 \). In order to control the sum rule stability, individual output values of the condensate ratio are displayed in Fig. 8 as a function of the Borel-transform variable \( u \). Fig. 9 contains a compilation of the dependence of the condensate ratio on \( \hat{m} \) indicating the uncertainty arising from the weak dependence on the Borel-transform variable.

The shape of this curve can be understood within \( G\chi PT \) combined with the expansion (1.43). One obtains
\[
\psi_{ud}^d(0) = 1 - 4 \frac{A_0 - \frac{1}{4} C}{M_\pi^2} \hat{m}^2 + \ldots.
\]
(1.46)
Comparing with the curve on Fig. 9 one concludes that \( A_0 - \frac{1}{4} C = 4.7 \pm 0.7 \).
and remains unaffected if one varies $\hat{m}$ and keeps $M^2_\pi$ fixed. This number should be confronted with the estimates of $A_0$ discussed in sect. 4.

The same analysis may now be performed in the $O^-$ strange channel: it suffices to replace $\psi_{ud}^\text{ad}$ by $\psi_{us}^\text{qs}$, $\hat{m}$ by $\frac{1}{2}(m_u + m_s)$, $F_\pi^2 M_\pi^2$ by $F_K^2 M_K^2$ and change the position of resonances in the model $\cite{4.45}$ for the spectral function ($M_1 = 1460$ MeV, $M_2 = 1830$ MeV). The stability of the output for various values of $m_u + m_s$ is shown on Fig. 10 and the final dependence of $\psi_{us}^\text{qs}(0)/2F_K^2 M_K^2$ on $m_u + m_s$ is collected in Fig. 11. Notice that $\psi_{us}^\text{qs}(0) = 0$ for $m_u + m_s \simeq 250$ MeV to be compared with the experimental value of $m_s \cite{1.31}$.

6 Experimental tests

Experimental signatures of the chiral condensate are not easy to identify: $<\bar{q}q>$ enters observable quantities multiplied by quark masses and it only manifests itself through tiny symmetry breaking effects. $G\chi PT$ provides a theoretical basis for a systematic search of possible experimental tests of
the importance of \( q \bar{q} \)-condensation. In this way, the best manner to control the GOR ratio \( X_{\text{GOR}} = \frac{2\hat{m}B_0}{M^2} \) and the quark mass ratio \( r = m_s/\hat{m} \) experimentally, has been found so far in the low-energy \( \pi - \pi \) scattering \cite{FSS91,SSF93}. The corresponding scattering amplitude at the leading \cite{Wei66} and one loop \cite{GLS83} level has to be corrected including the two-loop contribution, in order to achieve the theoretical accuracy needed for a decisive test. This has been done both in the standard \cite{BCE+96,BCE+97} and the generalized \cite{KMSF95,KMSF96} frameworks.

![Fig. 12: \( \pi - \pi \) phases](image)

Fig. 12: \( \pi - \pi \) phases \( \delta_0^0 - \delta_1^1 \) as a function of c.m. energy. a) \( G_{\chi PT} \) convergence: \( O(p^2) \), \( O(p^4) \) and \( O(p^6) \) orders are compared. b) \( G_{\chi PT} \) predictions: \( X_{\text{GOR}} \approx 1 \) (dotted), \( X_{\text{GOR}} = 0.6 \) (dashed), \( X_{\text{GOR}} = 0.2 \) (dash-dotted). Data are from \cite{ea77}. The typical observable, the phase shift difference \( \delta_0^0(E) - \delta_1^1(E) \) measurable is high statistics \( K_{l\Lambda} \) decays is shown and compared with existing data points \cite{ea77} on Fig. 12. Fig. 12a illustrates the convergence of \( G_{\chi PT} \), comparing the tree, one-loop and two-loop results. Fig. 12b shows the predictions for \( X_{\text{GOR}} \approx 1 \) (dotted curve), \( X_{\text{GOR}} \approx 0.2 \) (dash-dotted curve) as well as the best fit obtained for \( X_{\text{GOR}} = 0.6 \pm 0.4 \).
A need for more precise data is obvious. A similar situation is observed with \(s\)-wave scattering lengths: the experimental value \(a_0 = 0.26 \pm 0.05\) has to be compared with (two-loop) \(S\chi PT\) prediction \(a_0 = 0.21\) and a value \(a_0 = 0.27\) corresponding to the case \(r = m_s/\bar{m} \simeq 10\). Results of new high precision experiments are awaited: more precise \(K^+\) data should come from BNL \[Low97\] and from the new \(\phi\)-factory Da\phi\ne \[Kl97\]. The experiment "Dirac" at CERN aims at a determination of \(a_0 - a_5\) to 5\%, measuring and correctly interpreting the lifetime of \(\pi^+\pi^-\)-atoms \[Di97\]. It is conceivable that in a few years the experimental answer to the question of the size of \(<\bar{q}q>\) will be known.

Low energy \(\pi\pi\) scattering is not the only possible source of the missing information: i) The size of the spectral function \(\rho_{2\pi}\), \[38\], can be directly controlled, measuring the tiny \textbf{azimuthal asymmetries in the decay} \(\tau \to 3\pi + \nu\), \[SFK94\]. ii) The quark mass ratio \(r = m_s/\bar{m}\) can be extracted, comparing the \textbf{deviations from the Goldberger-Treiman relation} in three different channels \[FSS90\]. This test requires an accurate determination of strong coupling constants \(g_{\pi NN}, g_{KAN}, g_{K\Sigma N}\). iii) Additional tests in \(K\) and \(\eta\)-decays are possible.

Acknowledgements

I am indebted to Norman Fuchs, Marc Knecht and Bachir Moussallam for a collaboration during a preparation of this talk. Discussions with Juerg Gasser, Heiri Leutwyler and Andrei Smilga have been extremely helpful. Aron Bernstein, Dieter Drechsel and Thomas Walcher should be acknowledged for a perfect organization of the workshop.

7 References

[AGKN97] G. Arvanitis, F. Geniet, J.L. Kneur, and A. Neveu. \textit{Phys. Lett. B}, 390:385, 1997.

[(AL97] S. Chen (ALEPH). In \textit{Workshop QCD 97}, Montpellier, France, July 1997. (To be published).

[BC80] T. Banks and A. Casher. \textit{Nucl. Phys. B}, 168:103, 1980.

[BCE+96] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. Sainio. \textit{Phys. Lett. B}, 374:210, 1996.

[BCE+97] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. Sainio. Technical report, 1997. \textit{hep-ph/9707291}.

[Bij93] J. Bijnens. \textit{Phys. Lett. B}, 306:343, 1993.
1. Light quark masses and condensates in QCD

[BNP92] E. Braaten, S. Narison, and A. Pich. *Nucl Phys. B*, 373:581, 1992.

[BPdR95] J. Bijnens, J. Prades, and E. de Rafael. *Phys. Lett. B*, 348:226, 1995.

[CG82] S. Coleman and B. Grossman. *Nucl. Phys. B*, 203:205, 1982.

[Dav96] M. Davier. In *Fourth International Conference on Tau lepton Physics*, Colorado, September, 16-19 1996.

[DdR87] C.A. Dominguez and E. de Rafael. *Ann. Phys.*, 174:372, 1987.

[(Di97] J. Schacher (Dirac). Working group 2. In *These proceedings*, 1997.

[ea77] L. Rosselet et al. *Phys. Rev. D*, 15:574, 1977.

[ea97] N. Eicker et al. Sesam collaboration. *Phys. Lett. B*, 407:581, 1997.

[Eck97] G. Ecker. In *These Proceedings*, 1997.

[FSBY81] Y. Frishman, A. Schwimmer, T. Banks, and S. Yankielowicz. *Nucl. Phys. B*, 177:157, 1981.

[FSS90] N.H. Fuchs, H. Sazdjian, and J. Stern. *Phys. Lett. B*, 238:380, 1990.

[FSS91] N.H. Fuchs, H. Sazdjian, and J. Stern. *Phys. Lett. B*, 269:183, 1991.

[Gas97a] J. Gasser. Private communication, 1997.

[Gas97b] J. Gasser. In *These proceedings*, 1997.

[Geo93] H. Georgi. *Phys. Lett. B*, 298:187, 1993.

[GL82] J. Gasser and H. Leutwyler. *Phys. Rep.*, 87:77, 1982.

[GL83] J. Gasser and H. Leutwyler. *Phys. Lett. B*, 125:325, 1983.

[GL84] J. Gasser and H. Leutwyler. *Ann. Phys.*, 158:142, 1984.

[GL85] J. Gasser and H. Leutwyler. *Nucl. Phys. B*, 250:465, 1985.

[GMOR68] M. Gell-Mann, R.J. Oakes, and B. Renner. *Phys. Rev.*, 175:2195, 1968.

[Jam97] M. Jamin. In *Workshop QCD 97*, Montpellier, France, July 1997. hep-th/9709495 and references therein.
[KL97] J. Lee-Franzini (Kloe). Working group 4. In These proceedings, 1997.

[KMSF95] M. Knecht, B. Moussallam, J. Stern, and N.H. Fuchs. Nucl. Phys B, 457:513, 1995.

[KMSF96] M. Knecht, B. Moussallam, J. Stern, and N.H. Fuchs. Nucl. Phys. B, 471:445, 1996.

[Kne96] J.L. Kneur. Technical report, 1996. hep-ph/9609265.

[Kne97] M. Knecht. ππ-working group. In These Proceedings, 1997.

[KS95] M. Knecht and J. Stern. In The Second Daphne Physics Handbook. Eds. L. Maiani, N. Paver, G. Pancheri INFN-LNF-publication, 1995.

[LdRT97] L. Lellouch, E. de Rafael, and J. Taron. Technical report, 1997. Preprint CPT-97-P-3519, hep-ph/9707523.

[Leu90] H. Leutwyler. Nucl. Phys. B, 337:108, 1990.

[Leu96] H. Leutwyler. Phys. Lett. B, 378:313, 1996.

[Low97] J. Lowe. ππ-working group. In These proceedings, 1997. BNL-E865.

[LS92] H. Leutwyler and A. Smilga. Phys. Rev. D, 46:5607, 1992.

[Mei97] G. Meissener. In These Proceedings, 1997.

[Mot70] N.F. Mott. Phil. Mag., 22:7, 1970.

[Mou97] B. Moussallam. Nucl. Phys. B, 504:381, 1997.

[Nar95] S. Narison. Phys. Lett. B, 358:113, 1995.

[Pra97] J. Pradèes. In Proceedings of the workshop QCD 97, Montpellier, France, July 1997. hep-ph 9708395.

[PW81] J. Preskill and S. Weinberg. Phys. Rev. D, 24:1059, 1981.

[Sai97] M. Sainio. ππ-working group. In These Proceedings, 1997.

[SFK94] J. Stern, N.H. Fuchs, and M. Knecht. In Proceedings of the Third workshop on the Tau-Charm Factory, Marbella, Spain, 1994. Eds. J. Kirkby and R. Kirkby, Editions Frontières.

[SSF93] J. Stern, H. Sazdjian, and N.H. Fuchs. Phys. Rev. D, 47:3814, 1993.
1. Light quark masses and condensates in QCD

| Reference | Author(s) | Journal/Book | Year |
|-----------|-----------|--------------|------|
| [t’H80]   | G. t’Hooft | Recent developments in gauge theories | 1980 |
| [Tho74]   | D.J. Thouless | Phys. Reports | 1974 |
| [VW84a]   | C. Vafa and E. Witten | Nucl. Phys. B | 1984 |
| [VW84b]   | C. Vafa and E. Witten | Comm. Math. Phys. | 1984 |
| [Wei66]   | S. Weinberg | Phys. Rev. Lett. | 1966 |
| [Wei77]   | S. Weinberg | A Festschrift for I.I. Rabi | 1977 |