Adapting to Misspecification in Contextual Bandits

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Abstract

A major research direction in contextual bandits is to develop algorithms that are computationally efficient, yet support flexible, general-purpose function approximation. Algorithms based on modeling rewards have shown strong empirical performance, but typically require a well-specified model, and can fail when this assumption does not hold. Can we design algorithms that are efficient and flexible, yet degrade gracefully in the face of model misspecification? We introduce a new family of oracle-efficient algorithms for $\varepsilon$-misspecified contextual bandits that adapt to unknown model misspecification—both for finite and infinite action settings. Given access to an online oracle for square loss regression, our algorithm attains optimal regret and—in particular—optimal dependence on the misspecification level, with no prior knowledge. Specializing to linear contextual bandits with infinite actions in $d$ dimensions, we obtain the first algorithm that achieves the optimal $\tilde{O}(d\sqrt{T} + \varepsilon\sqrt{dT})$ regret bound for unknown misspecification level $\varepsilon$.

On a conceptual level, our results are enabled by a new optimization-based perspective on the regression oracle reduction framework of Foster and Rakhlin (2020), which we anticipate will find broader use.

1 Introduction

The contextual bandit is a sequential decision making problem that is widely deployed in practice across applications including health services (Tewari and Murphy, 2017), online advertisement (Li et al., 2010; Abe et al., 2003), and recommendation systems (Agarwal et al., 2016). At each round, the learner observes a feature vector (“context”) and an action set, then selects an action and receives a loss for the action selected. To facilitate generalization across contexts, the learner has access to a family of policies (e.g., linear models or neural networks) that map contexts to actions. The objective of the learner is to achieve cumulative loss close to that of the optimal policy in hindsight.

To develop efficient, general purpose algorithms, a common approach to contextual bandits is to reduce the problem to supervised learning primitives such as classification and regression (Langford and Zhang, 2008; Dudik et al., 2011; Agarwal et al., 2012, 2014; Syrgkanis et al., 2016; Agarwal et al., 2016; Luo et al., 2018). Recently, Foster and Rakhlin (2020) introduced SquareCB, which provides the first optimal and efficient reduction from $K$-armed contextual bandits to square loss regression, and can be applied whenever the learner has access to a well-specified model for the loss function (“realizability”). In light of this result, a natural question is whether this approach can be generalized beyond the realizable setting and, more ambitiously, whether we can shift from realizable to misspecified models without prior knowledge of the amount of misspecification. A secondary question, which is relevant for practical applications, is whether the approach can be generalized to large or infinite action spaces. This is precisely the setting we study in the present paper, where the action set is large or infinite, but where the learner has a “good” feature representation available—that is, up to some unknown amount of misspecification.

Adequately handling misspecification has been a subject of intense recent interest even for the simple special case of linear contextual bandits. Du et al. (2019) questioned whether “good” is indeed enough, that is, whether we can learn efficiently even without realizability. Lattimore et al. (2020) gave a positive answer

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Our results. We provide an affirmative answer to all of the questions above. We generalize SquareCB to infinite action sets, and use this strategy to adapt to unknown misspecification by combining it with a bandit model selection procedure akin to that of Agarwal et al. (2017). Our algorithm is oracle-efficient, and adapts to misspecification efficiently and optimally whenever it has access to an online oracle for square loss regression. When specialized to linear contextual bandits, it resolves the question of Lattimore et al. (2020).

On the technical side, a conceptual contribution of our work is to show that one can view the action selection scheme used by SquareCB as an approximation to a log-barrier regularized optimization problem, which paves the way for a generalization to infinite action spaces. Another byproduct of our results is a generalization of the original CORRAL algorithm (Agarwal et al., 2017) for combining bandit algorithms, which is simple, flexible, and enjoys improved logarithmic factors.

1.1 Related Work

Contextual bandits are a well-studied problem, and misspecification in bandits and reinforcement learning has received much attention in recent years. Below we discuss a few results closely related to our own.

For linear bandits in $d$ dimensions, Lattimore et al. (2020) gave an algorithm with regret $O(d\sqrt{T} + \varepsilon \sqrt{dT})$, and left adapting to unknown misspecification for changing action sets as an open problem. Concurrent work of Pacchiano et al. (2020) addresses this problem for the special case where contexts and action sets are stochastic, and also leverages CORRAL-type aggregation of contextual bandit algorithms. Our results resolve this question in the general, fully adversarial setting.

Within the literature general-purpose contextual bandit algorithms, our approach builds on a recent line of research that provides reductions to (online/offline) square loss regression (Foster et al., 2018; Foster and Rakhlin, 2020; Simchi-Levi and Xu, 2020; Xu and Zeevi, 2020; Foster et al., 2020). In particular, our work builds on and provides a new perspective on the online regression reduction of Foster and Rakhlin (2020). The infinite-action setting we consider was introduced in Foster and Rakhlin (2020), but algorithms were only given for the special case where the action set is the sphere; our work extends this to arbitrary action sets. Concurrent work of Xu and Zeevi (2020) gives a reduction to offline oracles that also accommodates infinite action sets. These results are not strictly comparable: An online oracle can always be converted to an offline oracle through online-to-batch conversion and hence is a stronger assumption, but when an online oracle is available, our algorithm is more efficient. In addition, by working with online oracles, we support adversarially chosen contexts.

It bears mentioning that misspecification in contextual bandits can be formalized in many ways, some of which go beyond the setting we consider. One line of work reduces stochastic contextual bandits to oracles for cost-sensitive classification (Langford and Zhang, 2008; Dudik et al., 2011; Agarwal et al., 2012, 2014). These results are agnostic, meaning they make no assumption on the model, and in particular do not require realizability. However, in the presence of misspecification, this type of guarantee is somewhat different from what we provide here: rather than giving a bound on regret to the true optimal policy, these results give bounds on the regret to the best-in-class policy. Another line of work considers a model in which the feedback received by the learning algorithm at each round may be arbitrarily corrupted by an adaptive adversary (Seldin and Slivkins, 2014; Lykouris et al., 2018; Gupta et al., 2019; Bogunovic et al., 2020). Typical results for this setting incur additive error $O(C)$, where $C$ is the total number of corrupted rounds. While this model was originally considered in the context of non-contextual stochastic bandits, it has recently been extended to Gaussian process bandit optimization, which is closely related to the contextual bandit setting, though these results only tolerate $C \leq \sqrt{T}$. While these results are complementary to our own, we mention in passing that our notion of misspecification satisfies $\varepsilon \leq \sqrt{C/T}$, and hence our main theorem (Theorem 1) achieves additive error $\sqrt{C/T}$ for this corrupted setting (albeit, only with an oblivious adversary).
2 Problem Setting

We consider the following contextual bandit protocol: At each round \( t = 1, \ldots, T \), the learner first observes a context \( x_t \in \mathcal{X} \) and an action set \( \mathcal{A}_t \subseteq \mathcal{A} \), where \( \mathcal{A} \subseteq \mathbb{R}^d \) is a compact action space; for simplicity, we assume throughout that \( \mathcal{A} = \{ a \in \mathbb{R}^d : \|a\| \leq 1 \} \), but place no restriction on \( (\mathcal{A}_t)_{t=1}^T \). Given the context and action set, the learner chooses action \( a_t \in \mathcal{A}_t \), then observes a stochastic loss \( \ell_t \in [-1, +1] \) depending on the action selected. We assume that the sequence of context vectors \( x_t \) and the associated sequence of action sets \( \mathcal{A}_t \) are generated by an oblivious adversary.

Let \( \mu(a,x) := \mathbb{E}[\ell_t | x_t = x, a_t = a] \) denote the mean loss function, which we assume to be time-invariant, and which is unknown to the learner. We adopt a semi-parametric approach to modeling the losses, in which \( \mu(a,x) \) is modelled as (approximately) linear in the action \( a \), but can depend on the context \( x \) arbitrarily (Foster and Rakhlin, 2020; Xu and Zeevi, 2020; Chernozhukov et al., 2019). In particular, we assume the learner has access to a class of functions \( \mathcal{F} \subseteq \{ f : \mathcal{X} \rightarrow \mathbb{R}^d \} \) where for each \( f \in \mathcal{F} \), \( \langle a, f(x) \rangle \) is a prediction for the value of \( \mu(a,x) \). In the well-specified/realizable setting, one typically assumes that there exists some \( f^* \in \mathcal{F} \) such that \( \mu(a,x) = \langle a, f^*(x) \rangle \). In this paper, we make no such assumption, but the performance of our algorithms will depend on how far this is from being true. In particular, we measure performance of the learner in terms of pseudoregret \( \text{Reg}(T) \) against the best unconstrained policy:

\[
\text{Reg}(T) := \mathbb{E}\left[ \sum_{t=1}^{T} \mu(a_t, x_t) - \inf_{a \in \mathcal{A}_t} \mu(a, x_t) \right].
\]

Here, and throughout the paper, \( \mathbb{E}[\cdot] \) denotes expectation with respect to both the randomized choices of the learner and the stochastic realization of the losses \( \ell_t \).

This setup encompasses the finite-arm contextual bandit setting with \( K \) arms by taking \( \mathcal{A}_t = \{ e_1, \ldots, e_K \} \). Another important special case is the well-studied linear contextual bandit setting, where \( \mathcal{F} \) consists of constant vector-valued functions that do not depend on \( \mathcal{X} \). Specifically, for any \( \Theta \subseteq \mathbb{R}^d \), take \( \mathcal{F} = \{ x \mapsto \theta | \theta \in \Theta \} \). In this case, the prediction \( \langle a, f(x) \rangle \) simplifies to \( \langle a, \theta \rangle \), which a static linear function of the action. This special case recovers the most widely studied version of the linear contextual bandit problem (Abe and Long, 1999; Auer, 2002; Abbasi-Yadkori et al., 2011; Chu et al., 2011; Abbasi-Yadkori et al., 2012; Agrawal and Goyal, 2013; Crammer and Gentile, 2013), as well as Gaussian process extensions (Srinivas et al., 2010; Krause and Ong, 2011; Djolonga et al., 2013; Sui et al., 2015).

2.1 Misspecification

As mentioned above, contextual bandit algorithms based on modeling losses typically rely on the assumption of a well-specified model (or, “realizability”): That is, existence of a function \( f^* \in \mathcal{F} \) such that \( \mu(a, x) = \langle a, f^*(x) \rangle \) for all \( a \in \mathcal{A} \) and \( x \in \mathcal{X} \) (Chu et al., 2011; Abbasi-Yadkori et al., 2011; Agrawal et al., 2012; Foster et al., 2018). Since the assumption of exact realizability does not typically hold in practice, a more recent line of work has begun to investigate algorithms for misspecified models. In particular, Crammer and Gentile (2013); Ghosh et al. (2017); Lattimore et al. (2020); Foster and Rakhlin (2020); Zanette et al. (2020) consider a uniform \( \varepsilon \)-misspecified setting in which

\[
\inf_{f \in \mathcal{F}} \sup_{a \in \mathcal{A}, x \in \mathcal{X}} |\mu(a, x) - \langle a, f(x) \rangle| \leq \varepsilon,
\]

for some misspecification level \( \varepsilon > 0 \). Notably, Lattimore et al. (2020) show that for linear contextual bandits, regret must grow as \( \Omega(d\sqrt{T} + \varepsilon\sqrt{dT}) \). Since \( d\sqrt{T} \) is the optimal regret for a well-specified model, \( \varepsilon\sqrt{dT} \) may be thought of as the price of misspecification.

We consider a weaker notion of misspecification. Given a sequence \( S = (x_1, A_1), \ldots, (x_T, A_T) \) of context-action set pairs, we define the average misspecification level \( \varepsilon_T(S) \) as

\[
\varepsilon_T(S) := \inf_{f \in \mathcal{F}} \left( \frac{1}{T} \sum_{t=1}^{T} \sup_{a \in \mathcal{A}_t} (\langle a, f(x_t) \rangle - \mu(a, x_t))^2 \right)^{1/2}.
\]

This quantity measures the misspecification level for the specific sequence \( S \) at hand, and hence offers tighter guarantees than uniform misspecification. In particular, the uniform bound in Eq. (1) directly implies \( \varepsilon_T(S) \leq \varepsilon \) for all \( S \) in Eq. (2), and \( \varepsilon_T(S) = 0 \) whenever the model is well-specified.
We provide regret bounds that optimally adapt to $\varepsilon_T(S)$ for any given realization of the sequence $S$, with no prior knowledge of the misspecification level. The issue of adapting to unknown misspecification has not been addressed even for the stronger uniform notion (1). Indeed, previous efforts typically use prior knowledge of $\varepsilon$ to encourage conservative exploration when misspecification is large; see Lattimore et al. (2020, Appendix E), Foster and Rakhlin (2020, Section 5.1), Crammer and Gentile (2013, Section 4.2), and Zanette et al. (2020) for examples. Naively adapting such schemes using, e.g., doubling tricks, presents difficulties because the quantities in Eq. (1) and Eq. (2) do not appear to be estimable without knowledge of $\mu$.

2.2 Regression Oracles

Following Foster and Rakhlin (2020), we assume access to an online regression oracle $\text{Alg}_{\text{Sq}}$, which is simply an algorithm for sequential prediction with the square loss, using $\mathcal{F}$ as a benchmark class. Concretely, the oracle operates under the following protocol. At each round $t \in [T]$, the algorithm receives a context $x_t \in \mathcal{X}$, outputs a prediction $\hat{y}_t \in \mathbb{R}^d$ (in particular, we interpret $\langle a, \hat{y}_t \rangle$ as the predicted loss for action $a$), then observes an action $a_t \in \mathcal{A}$ and loss $\ell_t \in [-1, +1]$ and incurs error $(\langle a_t, \hat{y}_t \rangle - \ell_t)^2$. Formally, we make the following assumption.\footnote{As in Foster and Rakhlin (2020), the square loss itself does not play a crucial role, and can be replaced by any loss that is strongly convex with respect to the learner's predictions.}

**Assumption 1.** The regression oracle $\text{Alg}_{\text{Sq}}$ guarantees that for any (potentially adaptively chosen) sequence $\{(x_t, a_t, \ell_t)\}_{t=1}^T$,
\[
\sum_{t=1}^T (\langle a_t, \hat{y}_t \rangle - \ell_t)^2 - \inf_{f \in \mathcal{F}} \sum_{t=1}^T (\langle a_t, f(x_t) \rangle - \ell_t)^2 \leq \text{Reg}_{\text{Sq}}(T),
\]
for some (non-data-dependent) function $\text{Reg}_{\text{Sq}}(T)$.

For the finite-action setting, this definition coincides with that of Foster and Rakhlin (2020). To simplify the presentation of our results, we assume throughout the paper that $\text{Reg}_{\text{Sq}}(T)$ is a non-decreasing function of $T$.

While this type of oracle suffices for all of our results, our algorithms are stated more naturally in terms of a stronger notion of oracle that supports weighted online regression. In this model, we follow the same protocol as in Assumption 1, except that at each time $t$, the regression oracle observes a weight $w_t \geq 0$ at the same time as the context $x_t$, and the error incurred is given by $w_t \cdot ((\langle a_t, \hat{y}_t \rangle - \ell_t)^2$. For technical reasons, we allow the oracle for this model to be randomized. We make the following assumption.

**Assumption 2.** The weighted regression oracle $\text{Alg}_{\text{Sq}}$ guarantees that for any (potentially adaptively chosen) sequence $\{(w_t, x_t, a_t, \ell_t)\}_{t=1}^T$,
\[
\mathbb{E} \left[ \sum_{t=1}^T w_t ((\langle a_t, \hat{y}_t \rangle - \ell_t)^2 - \inf_{f \in \mathcal{F}} \sum_{t=1}^T w_t ((\langle a_t, f(x_t) \rangle - \ell_t)^2) \right] \leq \mathbb{E} \left[ \max_{t \in [T]} w_t \right] \cdot \text{Reg}_{\text{Sq}}(T),
\]
for some (non-data-dependent) function $\text{Reg}_{\text{Sq}}(T)$, where the expectation is taken with respect to the oracle's randomization.

We show in Appendix A (Algorithm 5) that any unweighted regression oracle satisfying Assumption 1 can be transformed into a randomized oracle for weighted regression that satisfies Assumption 2, with no overhead in runtime. Hence, to simplify exposition, for the remainder of the paper we state our results in terms of weighted regression oracles satisfying Assumption 2.

Online regression is a well-studied problem, and efficient algorithms are known for many standard function classes. One example, which is important for our applications, is the case where $\mathcal{F}$ is linear.

**Example 1** (Linear Models). Suppose $\mathcal{F} = \{ x \mapsto \theta \mid \theta \in \Theta \}$, where $\Theta \subseteq \mathbb{R}^d$ is a convex set with $\| \theta \| \leq 1$. Then the online Newton step algorithm (Hazan et al., 2007) satisfies Assumption 1 with $\text{Reg}_{\text{Sq}}(T) = O(d \log(T))$ and—via reduction (Algorithm 5)—can be augmented to satisfy Assumption 2.

Further examples include kernels (Valko et al., 2013), generalized linear models (Kakade et al., 2011), and standard nonparametric classes (Gaillard and Gerchinovitz, 2015). We refer to Foster and Rakhlin (2020) for a more comprehensive discussion.
Additional notation. For a set $X$, we let $\Delta(X)$ denote the set of all probability distributions over $X$. If $X$ is continuous, we restrict $\Delta(X)$ to distributions with countable support. We let $\|x\|$ denote the euclidean norm for $x \in \mathbb{R}^d$. For any positive definite matrix $H \in \mathbb{R}^{d \times d}$, we denote the induced norm on $x \in \mathbb{R}^d$ by $\|x\|_H^2 = \langle x, Hx \rangle$. For functions $f, g : X \rightarrow \mathbb{R}_+$, we write $f = O(g)$ if there exists some constant $C \geq 0$ such that $f(x) \leq Cg(x)$ for all $x \in X$. We write $f = \tilde{O}(g)$ if $f = O(g \max\{1, \text{polylog}(g)\})$, and define $\Omega(\cdot)$ analogously.

For each $f \in \mathcal{F}$, we let $\pi_f(\cdot, \cdot)$ denote the induced policy, whose action at time $t$ is given by $\pi_f(x_t, A_t) := \text{argmin}_{a \in A_t} \{a, f(x_t)\}$.

3 Adapting to Misspecification: An Oracle-Efficient Algorithm

We now present our main result: an efficient reduction from contextual bandits to online regression that adapts to unknown misspecification $\varepsilon_T(S)$ and supports infinite action sets. Our main theorem is as follows.

Theorem 1. There exists an efficient reduction which, given access to a weighted regression oracle $\text{Alg}_{\text{Sq}}$ satisfying Assumption 2, guarantees that for all sequences $S = (x_1, A_1), \ldots, (x_T, A_T)$,

$$\text{Reg}(T) = O\left( \sqrt{dT \text{Reg}_{\text{Sq}}(T) \log(T)} + \varepsilon_T(S) \sqrt{dT} \right).$$

The algorithm has two main building blocks: First, we extend the reduction of Foster and Rakhlin (2020) to infinite action sets via a new optimization-based perspective and—in particular—show that the resulting algorithm has favorable dependence on misspecification level when it is known in advance. Then, we combine this reduction with a scheme that aggregates multiple instances of the algorithm to adapt to unknown misspecification. When the time required for a single query to $\text{Alg}_{\text{Sq}}$ is $T_{\text{Alg}_{\text{Sq}}}$, the per-step runtime of our algorithm is $\tilde{O}\left( T_{\text{Alg}_{\text{Sq}}} + |A_t| \cdot \text{poly}(d) \right)$.

As an application, we solve an open problem recently posed by Lattimore et al. (2020): we exhibit an efficient algorithm for infinite-action linear contextual bandits which optimally adapts to unknown misspecification.

Corollary 1. Let $\mathcal{F} = \{x \mapsto \theta \mid \theta \in \mathbb{R}^d, \|\theta\| \leq 1\}$. Then there exists an efficient algorithm that, for any sequence $S = (x_1, A_1), \ldots, (x_T, A_T)$, satisfies

$$\text{Reg}(T) = O\left( d\sqrt{T \log(T)} + \varepsilon_T(S) \sqrt{dT} \right).$$

This result immediately follows from Theorem 1 by invoking the online Newton step algorithm as the regression oracle, as in Example 1. Modulo logarithmic factors, this bound coincides with the one achieved by Lattimore et al. (2020) for the simpler non-contextual linear bandit problem, for which the authors present a matching lower bound.

The remainder of this section is dedicated to proving Theorem 1. The roadmap is as follows. First, we revisit the reduction from $K$-armed contextual bandits to online regression by Foster and Rakhlin (2020) and provide a new optimization-based perspective. This new viewpoint leads to a natural generalization from the $K$-armed case to the infinite action case. We then provide an aggregation-type procedure which combines multiple instances of this algorithm to adapt to unknown misspecification, and finally put all the pieces together to prove the main result. As an extension, we also give a variant of the algorithm which enjoys improved bounds when the action sets $A_t$ lie in low-dimensional subspaces of $\mathbb{R}^d$.

Going forward, we abbreviate $\varepsilon_T(S)$ to $\varepsilon_T$ whenever the sequence $S$ is clear from context.

3.1 Oracle Reductions with Finite Actions: An Optimization-Based Perspective

A canonical special case of our setting is the finite-arm contextual bandit problem, where $A_t = \mathcal{K} := \{e_1, \ldots, e_K\}$. For this setting, Foster and Rakhlin (2020) proposed an efficient and optimal reduction called SquareCB, which is displayed in Algorithm 1. At each step, the algorithm queries the oracle $\text{Alg}_{\text{Sq}}$ with the current context $x_t$ and receives a loss predictor $\hat{\theta}_t \in \mathbb{R}^K$, where $(\hat{\theta}_i)_i$ predicts the loss of action $i$. The algorithm then samples an action using an inverse gap weighting (IGW) scheme introduced by Abe and Long (1999).
Specifically for parameter $\theta \in \mathbb{R}^K$ and learning rate $\gamma > 0$, we define $\text{IGW}(\theta, \gamma)$ as the distribution $p \in \Delta([K])$ obtained by first selecting any $i^* \in \arg\min_{i \in [K]} \theta_i$, then defining

$$p_i = \begin{cases} 1/(K+\gamma(\theta_i - a_i^*)), & \text{if } i \neq i^*, \\ 1 - \sum_{i' \neq i} p_{i'}, & \text{otherwise}. \end{cases}$$

(3)

By choosing $\gamma \propto \sqrt{KT}/(\text{Reg}_{\text{Sq}}(T) + \varepsilon T)$, one can show that this algorithm guarantees

$$\text{Reg}(T) \leq O\left(\sqrt{KT\text{Reg}_{\text{Sq}}(T) + \varepsilon T \sqrt{KT}}\right).$$

Since this approach is the starting point for our results, it will be useful to sketch the proof. For $p \in \Delta(A)$, let $H_p := E_{a \sim p}[aa^\top]$ be the second moment matrix, and $a_p := E_{a \sim p}[a]$ be the mean action. Let the sequence $S$ be fixed, and let $f^* \in F$ be any regression function that attains the value of $\varepsilon T$ in Eq. (2). With $a_i^* := \pi_f(x_t, A_t)$ and $\theta_i^* := f^*(x_t)$, we have

$$E\left[\sum_{t=1}^T \mu(a_t, x_t) - \inf_{a \in A_t} \mu(a, x_t)\right]$$

$$\leq E\left[\sum_{t=1}^T (a_t - a_t^*, \theta_t^*)\right] + 2\varepsilon T$$

$$= E\left[\sum_{t=1}^T (\tilde{a}_p - a_t^*, \theta^*) - \frac{\gamma}{4}\|\theta^* - \hat{\theta}_t\|_{H_p}^2\right] + E\left[\sum_{t=1}^T \frac{\gamma}{4}\|\theta^* - \hat{\theta}_t\|_{H_p}^2\right] + 2\varepsilon T.$$

The first expectation term above is bounded by $O(KT/\gamma)$, which is established by showing that $\text{IGW}(\hat{\theta}, \gamma)$ is an approximate solution to the per-round minimax problem

$$\min_{p \in \Delta(K)} \max_{\theta \in \mathbb{R}^K} \max_{a^* \in K} \langle \tilde{a}_p - a^*, \theta - \hat{\theta}_t\rangle - \frac{\gamma}{4}\|\theta - \hat{\theta}_t\|_{H_p}^2,$$

(4)

with $O(K/\gamma)$. The second expectation term is bounded by $O(\gamma \cdot (\text{Reg}_{\text{Sq}}(T) + \varepsilon T))$, which follows readily from the definition of the square loss regret in Assumption 1 (see the proof of Theorem 3 for details). Choosing $\gamma$ to balance the terms leads to the result.

As a first step toward generalizing this result to infinite actions, we propose a new distribution that exactly solves the minimax problem (4). This distribution is the solution to a dual optimization problem based on log-barrier regularization, and provides a principled approach to deriving contextual bandit reductions.

**Lemma 1.** For any $\theta \in \mathbb{R}^K$ and $\gamma > 0$, the unique minimizer of Eq. (4) coincides with the unique minimizer of the $\text{log-barrier}(\theta, \gamma)$ optimization problem defined by

$$\text{log-barrier}(\theta, \gamma) = \arg\min_{p \in \Delta([K])} \left\{ \langle p, \theta \rangle - \frac{1}{\gamma} \sum_{a \in [K]} \log(p_a) \right\} = \left(\frac{1}{\gamma(a + \gamma \theta_i)}\right)_{i=1}^K,$$

(5)

where $\lambda$ is the unique value that ensures that the weights on the right-hand side above sum to one.

The $\text{IGW}$ distribution is closely related to the $\text{log-barrier}$ distribution: Rather than finding the optimal Lagrange multiplier $\lambda$ that solves the $\text{log-barrier}$ problem, the $\text{IGW}$ strategy simply plugs in $\lambda = K - \gamma \min_{i \in [K]} \theta_i$, then shifts weight to $p_i$ to ensure the distribution is normalized. Since the $\text{log-barrier}$ strategy solves the minimax problem Eq. (4) exactly, plugging it into the results of Foster and Rakhlin (2020) and Simchi-Levi and Xu (2020) in place of $\text{IGW}$ leads to slightly improved constants. More importantly, this new perspective leads to a principled way to extend these reductions to infinite actions.

### 3.2 Moving to Infinite Action Sets: The Log-Determinant Barrier

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**Algorithm 1: SquareCB**

(Foster and Rakhlin, 2020)

**Input:** Learning rate $\gamma$, horizon $T$.

**Initialize** Regression oracle $\text{Alg}_{\text{Sq}}$.

for $t = 1, \ldots, T$ do

Receive context $x_t$.

Let $\hat{\theta}_t$ be the oracle’s prediction for $x_t$.

Sample $I_t \sim \text{IGW}(\hat{\theta}_t, \gamma)$.

Play $a_t = e_{I_t}$ and observe loss $\ell_t$.

Update $\text{Alg}_{\text{Sq}}$ with $(x_t, a_t, \ell_t)$.

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2If the infimum is not obtained, it suffices to apply the argument that follows with a limit sequence.
To support infinite action sets, we replace the log-barrier distribution with a generalization based on the log-determinant function. In order to state the result, let $\dim(A)$ denote the dimension of the smallest affine linear subspace that contains $A$. When $\dim(A) < d$, we adopt the convention that the determinant function $\det(\cdot)$ takes the product of only the first $\dim(A)$ eigenvalues of the matrix in its argument. We define the logdet-barrier distribution as follows.

**Definition 1.** For parameter $\theta \in \mathbb{R}^d$, action set $A \subset \mathbb{R}^d$, and learning rate $\gamma > 0$, logdet-barrier($\theta, \gamma; A$) is defined as the set of solutions to

$$
\arg\min_{p \in A} \{ (\bar{a}_p, \theta) - \gamma^{-1} \log \det(H_p - \bar{a}_p\bar{a}_p^T) \}. \quad (6)
$$

In general, Eq. (6) does not admit a unique solution; all of our results apply to any minimizer. Our key result is that the logdet-barrier distribution also solves a minimax problem analogous to that of Eq. (4).

**Lemma 2.** Any solution to logdet-barrier($\hat{\theta}, \gamma; A$) satisfies

$$
\max_{p \in \mathbb{R}^d} \max_{a^* \in A} (\bar{a}_p - a^*, \theta) - \frac{\gamma}{2} \|\hat{\theta} - \theta\|^2_{H_p} \leq \gamma^{-1} \dim(A).
$$

By replacing the IGW distribution with the logdet-barrier distribution in Algorithm 1, we obtain an optimal reduction for infinite action sets. This algorithm, which we call SquareCB.Lin, is displayed in Algorithm 2.

**Theorem 2.** Given a regression oracle $Alg_A$ that satisfies Assumption 1, SquareCB.Lin with learning rate $\gamma \propto \sqrt{\frac{dT}{\Reg(S_q(T) + \varepsilon)}}$ guarantees that for all sequences $S$ with $\varepsilon_T(S) \leq \varepsilon$,

$$
\Reg(T) = O \left( \sqrt{dT \Reg(S_q(T)) + \varepsilon \sqrt{dT}} \right).
$$

The logdet-barrier optimization problem is closely related to the D-optimal experimental design problem, as well as the John ellipsoid problem (Khachiyan and Todd, 1990; Todd and Yıldırım, 2007); the latter corresponds to the case where $\theta = 0$ in Eq. (6) (Kumar and Yıldırım, 2005). By adapting specialized optimization algorithms for these problems (in particular, a Frank-Wolfe-type scheme), we can efficiently solve the logdet-barrier problem.

**Proposition 1.** An approximate minimizer for (6) that achieves the same regret bound up to a constant factor can be computed in time $\tilde{O}(|A| \cdot \text{poly}(d))$ and memory $\tilde{O}(\log|A| \cdot \text{poly}(d))$ per round.

The minimization algorithm, along with a full analysis for runtime and memory complexity and impact on the regret, is provided in Appendix D.

### 3.3 Adapting to Misspecification: Algorithmic Framework

The regret bound for SquareCB.Lin in Theorem 2 achieves optimal dependence on the dimension and misspecification level, but requires an a-priori upper bound on $\varepsilon_T(S)$ to set the learning rate. We now turn our attention to adapting to this parameter.

At a high level, our approach is to run multiple instances of SquareCB.Lin, each tuned to a different level of misspecification, then run an aggregation procedure on top to learn the best instance. Specifically, we initialize a collection of $M := \lceil \log(T) \rceil$ instances of Algorithm 2 in which the learning rate for instance $m$ is tuned for misspecification level $\varepsilon_m := \exp(-m)$ (that is, we follow a geometric grid). It is straightforward to show that there exists $m^* \in [M]$ such that the $m^*$th instance would enjoy optimal regret if we were to run it on the sequence $S$. Of course, $m^*$ is not known a-priori, so we run an aggregation (or, “Corralling”) procedure to select the best instance (Agarwal et al., 2017). This approach is, in general, not suitable for model selection, since it typically requires prior knowledge of the optimal regret bound to tune certain parameters appropriately (Foster et al., 2019). Our conceptual insight is to show that adaptation to misspecification is an exception to this rule, and offers a simple setting where model selection for contextual bandits is possible.
We use the aggregation scheme in Algorithm 3, which is a generalization of the CORRAL algorithm of Agarwal et al. (2017).

The algorithm is initialized with $M$ base algorithms, and uses a multi-armed bandit algorithm with $M$ arms as a master algorithm whose role is to choose the base algorithm to follow at each round.

In more detail, the master algorithm maintains a distribution $q_t \in \Delta([M])$ over the base algorithms. At each round $t$, it samples an algorithm $A_t \sim q_t$ and passes the current context $x_t$ into this algorithm, as well as the sampling probability $q_{t,A_t}$, and an importance weight $\rho_{t,A_t}$, where we define $\rho_{t,m} := 1/\min_{s \leq t} q_{s,m}$ for each $m$. At this point, the base algorithm $A_t$ selected by the master executes a standard contextual bandit round: Given the context $x_t$, it selects an arm $a_t$, receives the loss $\ell_{t,a_t}$, and updates its internal state. Finally, the master updates its state with the action-loss pair $(A_t, \ell_{t,A_t})$, where $\ell_{t,A_t} := \ell_{t} + 1$; for technical reasons related to our choice of master algorithm, it is useful to shift the loss by 1 to ensure non-negativity.

Define the importance-weighted regret for base algorithm $m$ as

$$\text{Reg}_{\text{Imp}}^m(T) := \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I} \{ A_t = m \} \left( \mu(a_t, x_t) - \inf_{a \in A_t} \mu(a, x_t) \right) \right],$$

which is simply the pseudoregret incurred in the rounds where Algorithm 3 follows this base algorithm, weighted inversely proportional to the probability that this occurs. It is straightforward to show that for any choice for the master and base algorithms, this scheme guarantees that

$$\text{Reg}(T) = \mathbb{E} \left[ \sum_{t=1}^{T} \ell_{t,A_t} - \hat{\ell}_{t,m} \right] + \text{Reg}_{\text{Imp}}^m(T),$$

where $\ell_{t,m}$ denotes the loss that the algorithm would have suffered at round $t$ if the master algorithm had chosen $A_t = m$. In other words, the regret of Algorithm 3 is equal to the regret $\text{Reg}_{\text{M}}(T) := \mathbb{E} \left[ \sum_{t=1}^{T} \ell_{t,A_t} - \hat{\ell}_{t,m} \right]$ of the master algorithm, plus the importance-weighted regret of the optimal base algorithm $m^*$.

The difficulty in instantiating this general scheme lies in the fact that the importance-weighted regret $\text{Reg}_{\text{Imp}}^m(T)$ of the optimal base algorithm typically scales with $\mathbb{E}[\rho_{t,m}^\alpha] \cdot \text{Reg}_{\text{Unw}}^m(T)$, where $\alpha \in [\frac{1}{2}, 1]$ is an algorithm-dependent parameter and $\text{Reg}_{\text{Unw}}^m(T) := \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I} \{ A_t = m \} \left( \mu(a_t, x_t) - \inf_{a \in A_t} \mu(a, x_t) \right) \right]$ denotes the unweighted regret of algorithm $m$. A-priori, the $\mathbb{E}[\rho_{t,m}^\alpha]$ can be unbounded, leading to large regret. The key to the analysis of Agarwal et al. (2017), and the approach we follow here, is to use a master algorithm with negative regret proportional to $\mathbb{E} \left[ \rho_{t,m}^\alpha \right]$, allowing to cancel this factor.

### 3.3.1 Choosing the Base Algorithm

As the first step towards instantiating the aggregation scheme above, we specify the base algorithm. We use a modification to SquareCB\textnormal{Lin} (denoted by SquareCB\textnormal{Lin+}) based on importance weighting, which is designed to ensure that the importance-weighted regret in Eq. (8) is bounded. Pseudocode for the $m$th base algorithm is given in Algorithm 4.

SquareCB\textnormal{Lin+} proceeds as follows. Let the instance $m$ be fixed, and let $Z_{t,m} = \mathbb{I} \{ A_t = m \}$ indicate the event that this instance is chosen to select an arm; note that we have $Z_{t,m} \sim \text{Ber}(q_{t,m})$ marginally. When $Z_{t,m} = 1$, instance $m$ receives $q_{t,m}$ and $\rho_{t,m} = \max_{s \leq t} q_{s,m}^{-1}$ from the master algorithm. The instance then follows the same update scheme as

#### Algorithm 3: Corralling (Agarwal et al., 2017)

**Input:** Master algorithm Master, $T$

**Initialize**: (Base$_{m}^M$)$_{m=1}^M$

**for** $t = 1, \ldots, T$ **do**

- Receive context $x_t$.
- Receive $A_t, q_{t,A_t}$ from Master.
- Pass $(x_t, A_t, q_{t,A_t}, \rho_{t,A_t})$ to Base$_{A_t}$.
- Base$_{A_t}$ plays $a_t$ and observes $\ell_t$.
- Update Master with $\hat{\ell}_{t,A_t} = (\ell_t + 1)$.

#### Algorithm 4: SquareCB.Lin+ (for base alg. $m$)

**Input**: $T$, $\text{Reg}_{\text{Sq}}^m(T)$

**Initialize**: Weighted regression oracle Alg$_{\text{Sq}}$.

**for** $t = (\tau_1, \tau_2, \ldots) \subset [T]$ **do**

- Receive context $x_t$ and $(q_{t,m}, \rho_{t,m})$.
- Set $\gamma_{t,m} = \min \left\{ \frac{\sqrt{\alpha}}{\rho_{t,m}}, \sqrt{dT/\rho_{t,m} \text{Reg}_{\text{Sq}}(T)} \right\}$.
- Set $w_t = \gamma_{t,m}/q_{t,m}$.
- Compute oracle’s prediction $\hat{y}_t$ for $x_t, w_t$.
- Sample $a_t \sim \text{logdet-barrier}(\hat{y}_t, \gamma_{t,m}; A_t)$.
- Play $a_t$ and observe loss $\ell_t$.
- Update Alg$_{\text{Sq}}$ with $(w_t, x_t, a_t, \ell_t)$.
in the vanilla version of \textbf{SquareCB-Lin}, except that i) it uses an adaptive learning rate $\gamma_{t,m}$, which is tuned based on $\rho_{t,m}$, and ii) it uses a weighted square loss regression oracle (as in Assumption 2), with the weight $w_t$ set as a function of $\gamma_{t,m}$ and $q_{t,m}$.

The importance weighted regret $\Reg_{\text{Imp}}^m(T)$ for this scheme is bounded as follows.

**Theorem 3.** When invoked within Algorithm 3 using a weighted regression oracle satisfying Assumption 2, the importance-weighted regret for each instance $m \in [M]$ of Algorithm 4 satisfies

$$
\Reg_{\text{Imp}}^m(T) \leq \frac{2}{3} \mathbb{E}[\sqrt{\rho T}] \sqrt{d T \Reg_{\text{Sq}}(T)} + \left( \left( \frac{\varepsilon_m}{\varepsilon_T} + \frac{\varepsilon'_m}{\varepsilon_T} \right) \sqrt{d} + 2 \right) \varepsilon_T T. \tag{9}
$$

The key feature of this regret bound is that only the leading term involving $\Reg_{\text{Sq}}(T)$ depends on the importance weights, not the second misspecification term. This means that the optimal tuning for the master algorithm will depend on $d$, $T$, and $\Reg_{\text{Sq}}(T)$, but not on $\varepsilon_T$, which is critical to adapt without prior knowledge of the misspecification. Another important feature is that as long as $\varepsilon'_m$ is within a constant factor of $\varepsilon_T$, the second term simplifies to $O(\varepsilon_T \sqrt{dT})$ as desired.

### 3.3.2 Improved Master Algorithms for Combining Bandit Algorithms

It remains to provide a master algorithm for use within Algorithm 3. While it turns out the master algorithm proposed in Agarwal et al. (2017) suffices for this task, we go a step further and propose a new master algorithm called $(\alpha, R)$–hedged FTRL, which is simpler and enjoys slightly improved regret, removing logarithmic factors. While this is not the focus of the paper, we find it to be a useful secondary contribution because it provides a new approach to designing master algorithms for bandit aggregation. We hope it will find use more broadly.

The $(\alpha, R)$–hedged FTRL algorithm is parameterized by a regularizer and two scale parameters $\alpha \in (0, 1)$ and $R > 0$. We defer a precise definition and analysis to Appendix C, and state only the relevant result for our aggregation setup here. We consider a special case of the $(\alpha, R)$–hedged FTRL algorithm that we call $(\alpha, R)$–hedged Tsallis-INF, which instantiates the framework using the Tsallis entropy as a regularizer (Audibert and Bubeck, 2009; Abernethy et al., 2015; Zimmert and Seldin, 2019). The key property of the algorithm is that the regret with respect to a policy playing a fixed arm $m$ contains a negative contribution proportional to $\rho_{T,m}^2 R$. The following result is a corollary of a more general theorem, Theorem 6 (Appendix C).

**Corollary 2.** Consider the adversarial multi-armed bandit problem with $M$ arms and losses $\hat{\ell}_{t,m} \in [0, 2]$. For any $\alpha \in (0, 1)$ and $R > 0$, the $(\alpha, R)$–hedged Tsallis-INF algorithm with learning rate $\eta = \sqrt{1/(2T)}$ guarantees that for all $m^* \in [M],$

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \hat{\ell}_{t,A_t} - \hat{\ell}_{t,m^*} \right] \leq 4\sqrt{2MT} + \mathbb{E} \left[ \min \left\{ \frac{1}{1-\alpha}, 2 \log(\rho_{T,m^*}) \right\} M^\alpha - \rho_{T,m^*}^\alpha \right] \cdot R. \tag{10}
$$

### 3.4 Putting Everything Together

When invoked within Algorithm 3, $(\alpha, R)$–hedged Tsallis-INF has a negative contribution to the cumulative regret which, for sufficiently large $R$ and appropriate $\alpha$, can be used to offset the regret incurred from importance-weighting the base algorithms. In particular, $\left( \frac{1}{T}, \frac{3}{2} \sqrt{d T \Reg_{\text{Sq}}(T)} \right)$–hedged Tsallis-INF has exactly the negative regret contribution needed to cancel the importance weighting term in Eq. (9) if we use \textbf{SquareCB-Lin} as the base algorithm. In more detail, we prove Theorem 1 by combining the regret bounds for the master and base algorithms as follows.

**Proof sketch for Theorem 1.** Using Eq. (8), it suffices to bound the regret of the bandit master $\Reg_M(T)$ and the importance-weighted regret $\Reg_{\text{Imp}}^m(T)$ for the optimal instance $m^*$. By Corollary 2, using $\left( \frac{1}{T}, \frac{3}{2} \sqrt{d T \Reg_{\text{Sq}}(T)} \right)$–hedged Tsallis-INF as the master algorithm gives

$$
\Reg_M(T) \leq O \left( \sqrt{d T \Reg_{\text{Sq}}(T) \log(T)} \right) - \frac{2}{3} \mathbb{E}[\sqrt{\rho T} \log(T)] \sqrt{d T \Reg_{\text{Sq}}(T)}.
$$
Whenever the misspecification level is not trivially small, the geometric grid ensures that there exists $m^* \in [M]$ such that $e^{-1} \varepsilon_T \leq e^m \leq \varepsilon_T$. For this instance, Theorem 3 yields

$$\text{Reg}^*_\text{imp}(T) \leq \frac{1}{2} E[\sqrt{\nu_{T,m^*}}] \sqrt{dT \text{Reg}^*_\text{sq}(T)} + O(\varepsilon_T \sqrt{dT}).$$

Summing the two bounds using Eq. (8) completes the proof.

### 3.5 Extension: Adapting to the Average Dimension

A well-known application for linear contextual bandits is the problem of online news article recommendation, where the context $x_t$ is taken to be a feature vector containing information about the user, and each action $a \in A_t$ is the concatenation of $x_t$ with a feature representation for a candidate article (e.g., Li et al. (2010)). In this and other similar applications, it is often the case that while examples lie in a high-dimensional space, the true dimensionality $\dim(A_t)$ of the action set is small, so that $d_{\text{avg}} := \frac{1}{T} \sum_{t=1}^{T} \dim(A_t) \ll d$.

If we have prior knowledge of $d_{\text{avg}}$ (or an upper bound thereof), we can exploit this low dimensionality for tighter regret. In fact, following the proof of Theorem 3 and Theorem 1, and bounding $\sum_{t=1}^{T} \dim(A_t)$ by $d_{\text{avg}}T$ instead of $dT$, it is fairly immediate to show that Algorithm 3 enjoys improved regret $\text{Reg}(T) = O(\sqrt{d_{\text{avg}}T \text{Reg}^*_\text{sq}(T)} \log(T) + \varepsilon_T \sqrt{d_{\text{avg}}T})$, so long as $d_{\text{avg}}$ is replaced by $d$ in the algorithm’s various parameter settings. Our final result shows that it is possible to adapt to unknown $d_{\text{avg}}$ and unknown misspecification simultaneously. The key idea to apply a doubling trick on top of Algorithm 3.

**Theorem 4.** There exists an algorithm that, under the same conditions as Theorem 1, satisfies $\text{Reg}(T) = O\left(\sqrt{d_{\text{avg}}T \text{Reg}^*_\text{sq}(T)} \log(T) + \varepsilon_T \sqrt{d_{\text{avg}}T}\right)$ without prior knowledge of $d_{\text{avg}}$ or $\varepsilon_T$.

We remark that while the bound in Theorem 4 replaces the $d$ factor in the reduction with the data-dependent quantity $d_{\text{avg}}$, the oracle’s regret $\text{Reg}^*_\text{sq}(T)$ may itself still depend on $d$ unless a sufficiently sophisticated algorithm is used.

### 4 Discussion

We have given the first general-purpose, oracle-efficient algorithms that adapt to unknown model misspecification in contextual bandits. For infinite-action linear contextual bandits, our results yield the first optimal algorithms that adapt to unknown misspecification with changing action sets. Our results suggest a number of interesting questions:

- Can our optimization-based perspective lead to new oracle-based algorithms for more rich types of infinite action sets? Examples include nonparametric action sets and structured (e.g., sparse) linear action sets.

- Can our reduction-based techniques be lifted to more sophisticated interactive learning settings such as reinforcement learning?

On the technical side, we anticipate that our new approach to reductions will find broader use; natural extensions include reductions for offline oracles (Simchi-Levi and Xu, 2020) and adapting to low-noise conditions (Foster et al., 2020).

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A Reducing Weighted to Unweighted Regression

**Algorithm 5**: Randomized reduction from weighted to unweighted online regression

**Input**: Online regression oracle $\text{Alg}_{\text{Sq}}$ satisfying Assumption 1.

**Initialize** $w_{\text{max}} \leftarrow 0$

for $t = 1, \ldots, T$ do

1. Receive weight $w_t$ and $x_t$.
2. if $w_t > w_{\text{max}}$ then
   1. Reset $\text{Alg}_{\text{Sq}}$
   2. $w_{\text{max}} \leftarrow 2w_t$.
3. Predict $\hat{y}_t$, where $\hat{y}_t$ is the prediction from $\text{Alg}_{\text{Sq}}$ given $x_t$.
4. Observe $a_t$ and $\ell_t$.
5. if $u_t \sim \text{Ber}(w_t/w_{\text{max}}) = 1$ then
   1. Update $\text{Alg}_{\text{Sq}}$ with $(x_t, a_t, \ell_t)$.

In this section we show how to transform any unweighted online regression oracle $\text{Alg}_{\text{Sq}}$ satisfying Assumption 1 into a weighted regression oracle satisfying Assumption 2. The reduction is given in Algorithm 5, and the performance guarantee is as follows.

**Theorem 5.** If the oracle $\text{Alg}_{\text{Sq}}$ satisfies Assumption 1 with regret bound $\text{Reg}_{\text{Sq}}(T)$, Algorithm 5 satisfies Assumption 2 with the same regret bound.

**Proof.** Let $D_t = (w_t, x_t, a_t, \ell_t)$ and define a filtration $\mathcal{F}_t = \sigma(D(1:t))$, with the convention $E_t[.] = E[.] | \mathcal{F}_t$. Let $\tau_1, \tau_2, \ldots, \tau_I$ denote the timesteps at which the algorithm doubles $w_{\text{max}}$ and resets $\text{Alg}_{\text{Sq}}$, with the convention that for all $n > I$, $\tau_n = T + 1$. Note that these random variables are stopping times with respect to the filtration $\mathcal{F}_{1:T}$, and hence $\mathcal{F}_{\tau_i}$ is well-defined for each $i \in \mathbb{N}$. It will be helpful to note that we have $\tau_{i+1} > \tau_i$ for all $i < I$ by construction, and otherwise $\tau_{i+1} = \tau_i$. We also observe that $\tau_1 = 1$ unless $w_1 = 0$.

For the first step, we show that the conditional regret of Algorithm 5 between any pair of doubling steps is bounded. Let $i \leq I$ and $f \in \mathcal{F}$ be fixed, and observe that $i \leq I$ holds iff $\tau_i \leq T$, which is $\mathcal{F}_{\tau_i}$-measurable. Hence,

\[
E \left[ \sum_{t=\tau_i}^{\tau_{i+1} - 1} w_t \left( \langle a_t, \hat{y}_t \rangle - \ell_t \rangle^2 - \langle a_t, f(x_t) \rangle - \ell_t \rangle^2 \right) | \mathcal{F}_{\tau_i} \right]
\]

\[= E \left[ 2w_{\tau_i} \sum_{t=\tau_i}^{\tau_{i+1} - 1} \frac{w_t}{2w_{\tau_i}} \left( \langle a_t, \hat{y}_t \rangle - \ell_t \rangle^2 - \langle a_t, f(x_t) \rangle - \ell_t \rangle^2 \right) | \mathcal{F}_{\tau_i} \right]
\]

\[= E \left[ 2w_{\tau_i} \sum_{t=\tau_i}^{\tau_{i+1} - 1} \mathbb{E}_t \left[ \langle a_t, \hat{y}_t \rangle - \ell_t \rangle^2 - \langle a_t, f(x_t) \rangle - \ell_t \rangle^2 \right] | \mathcal{F}_{\tau_i} \right]
\]

\[= E \left[ 2w_{\tau_i} \sum_{t=\tau_i}^{\tau_{i+1} - 1} u_t \left( \langle a_t, \hat{y}_t \rangle - \ell_t \rangle^2 - \langle a_t, f(x_t) \rangle - \ell_t \rangle^2 \right) | \mathcal{F}_{\tau_i} \right]
\]

\[\leq E[2w_{\tau_i} | \mathcal{F}_{\tau_i}] \cdot \text{Reg}_{\text{Sq}}(T),
\]

where (a) follows from the conditional independence of $u_t$, (b) is by the tower rule of expectation, and (c) uses Assumption 1 on the set $\{t \in \{\tau_i, \ldots, \tau_{i+1} - 1\} | u_t = 1\}$ (in particular, that regret is bounded by $\text{Reg}_{\text{Sq}}(T)$ on every sequence with probability 1 and $\text{Reg}_{\text{Sq}}(T)$ is non-decreasing in $T$). For $i > I$, the term is 0 since the sum is empty. To complete the proof that Algorithm 5 satisfies Assumption 2, we sum the bound above across all epochs as follows:
\[
E \left[ \sum_{t=1}^{T} w_t \left( (a_t, \tilde{y}_t) - \ell_t \right)^2 - (a_t, f(x_t)) - \ell_t \right)^2 \]
\]
\[
(d) = E \left[ \sum_{t=1}^{\tau + 1} \sum_{t=1}^{\tau + 1} w_t \left( (a_t, \tilde{y}_t) - \ell_t \right)^2 - (a_t, f(x_t)) - \ell_t \right)^2 \]
\]
\[
(e) = E \left[ \sum_{t=1}^{\tau + 1} \sum_{t=1}^{\tau + 1} w_t \left( (a_t, \tilde{y}_t) - \ell_t \right)^2 - (a_t, f(x_t)) - \ell_t \right)^2 \mid \tilde{\mathcal{F}}_{\tau_t} \right] \]
\]
\[
(f) \leq E \left[ \sum_{i=1}^{T} 2w_{\tau_i} \mid \tilde{\mathcal{F}}_{\tau_i} \right] \text{Reg}_{\text{Sq}}(T) \]
\]
\[
(g) = 2E \left[ \sum_{i=1}^{T} w_{\tau_i} \right] \text{Reg}_{\text{Sq}}(T) \]
\]
\[
(h) \leq 2E[2w_{\tau_i}] \text{Reg}_{\text{Sq}}(T) \leq 4E \left[ \max_{t \in [T]} w_t \right] \text{Reg}_{\text{Sq}}(T), \]
\]
where (d) uses that all \( t < \tau \) have \( w_t = 0 \), (e) uses the tower rule of expectation, (f) applies the conditional bound between stopping times above, (g) uses the tower rule of expectation again, (h) holds because the weights at least double between doubling steps, and (i) follows because \( \tau_T \) is a random variable with support over \([T]\).

\[ \square \]

## B Proofs from Section 3

In this section we provide complete proofs for all of the algorithmic results from Section 3.

### B.1 Proofs from Section 3.1

**Proof of Lemma 1.** We begin by showing that the \textit{log-barrier}(\( \bar{\theta}, \gamma \)) distribution takes the form claimed in Eq. (5). The minimization problem of Lemma 1 is strictly convex and the value approaches \( \infty \) at the boundary. Hence the unique solution lies in the interior of the domain. By the K.K.T. conditions, the partial derivatives for each coordinate must coincide for the minimizer \( p^* \). That is, there exists \( \lambda \in \mathbb{R} \) such that

\[
\forall a \in [K] : \frac{\partial}{\partial p_a} \left( \langle p^*, \bar{\theta} \rangle - \frac{1}{\gamma} \sum_{a \in [K]} \log(p_a^*) \right) = \frac{1}{\gamma p_a^*} - \frac{\lambda}{p_a^*} = \bar{\lambda}.
\]

Substituting \( \bar{\lambda} = \min_{a \in [K]} \bar{\theta}_a - \lambda/\gamma \) and rearranging finishes the proof.

We next show that the \textit{log-barrier}(\( \bar{\theta}, \gamma \)) distribution indeed solves the minimax problem Eq. (4), which we rewrite as

\[
\min_{p \in \Delta([K])} \sup_{\epsilon \in [K]} \max_{i \in [K]} \langle \bar{a}_p - \epsilon_i, \theta \rangle - \frac{\epsilon}{4} \| \bar{\theta} - \theta \|^2_{H_p}
\]

\[
= \min_{p \in \Delta([K])} \max_{\epsilon \in [K]} \sup_{i \in [K]} \langle \bar{a}_p - \epsilon_i, \bar{\theta} + \delta \rangle - \frac{\gamma}{4} \| \delta \|^2_{H_p}. \quad (11)
\]

For any fixed \( p \) and \( i^* \), the derivative of the expression in Eq. (11) with respect to \( \delta \) is given by

\[
\frac{\partial}{\partial \delta} \left[ \langle \bar{a}_p - \epsilon_{i^*}, \delta \rangle - \frac{\gamma}{4} \| \delta \|^2_{H_p} \right] = \bar{a}_p - \epsilon_{i^*} - \frac{\gamma}{2} H_p \delta. \quad (12)
\]

For \( p \) on the boundary of \( \Delta([K]) \) (i.e. \( p \) for which there exists \( i \in [K] \) such that \( p_i = 0 \)), the gradient is constant and the supremum has value \( +\infty \). Hence, we only need to consider the case where \( p \) lies in the
interior of $\Delta([K])$, which implies $H_p > 0$. In this case, Eq. (12) is strongly convex in $\delta$ and the unique maximizer is given by $\delta^* = \frac{2}{7} H_p^{-1} (\bar{a}_p - e_i^*)$. Hence, we can rewrite (11) as

$$
\min_{p \in \Delta([K])} \max_{\gamma \in [K]} \max_{\delta \in \mathbb{R}^K} \langle \bar{a}_p - e_i^*, \hat{\theta} + \delta \rangle - \frac{\gamma}{4} \|\delta\|^2_{H_p^{-1}}
$$

$$
= \min_{p \in \Delta([K])} \max_{\gamma \in [K]} \max_{\delta \in \mathbb{R}^K} \langle \bar{a}_p - e_i^*, \hat{\theta} \rangle + \frac{1}{\gamma} \|\bar{a}_p - e_i^*\|_{H_p^{-1}}^2 - \frac{\gamma}{4} \|\delta\|^2_{H_p^{-1}}
$$

$$
\geq \min_{p \in \Delta([K])} \max_{\gamma \in [K]} \mathbb{E}_{\gamma \sim p} \left[ \langle \bar{a}_p - e_i^*, \hat{\theta} \rangle + \frac{1}{\gamma} \|\bar{a}_p - e_i^*\|_{H_p^{-1}}^2 \right]
$$

$$
= \min_{p \in \Delta([K])} \max_{\gamma \in [K]} \mathbb{E}_{\gamma \sim p} \left[ \frac{1}{\gamma} \left( \text{tr}(H_p^{-1} H_p) - \|\bar{a}_p\|_{H_p^{-1}}^2 \right) \right] = \frac{K - 1}{\gamma}.
$$

(13)

Now consider the inequality (13). If we can show that there exists a unique solution $p$ such that this step in fact holds with equality, then we have identified the minimizer over $p \in \Delta([K])$. Consider an arbitrary candidate solution $p$ on the interior of $\Delta([K])$. Then, letting $W_i := \langle \bar{a}_p - e_i^*, \hat{\theta} \rangle + \frac{1}{\gamma} \|\bar{a}_p - e_i^*\|_{H_p^{-1}}^2$, the step (13) lower bounds $\max_{\gamma \in [K]} W_i$ by $\mathbb{E}_{\gamma \sim p} [W_i]$. This step holds with equality if and only if $\mathbb{E}_{\gamma \sim p} [W_i - \max_{\gamma \in [K]} W_i] = 0$. Since all probabilities $p_i$ are strictly positive, this can happen if and only if

$$
\exists \lambda \in \mathbb{R} \text{ such that } \forall i \in [K]: W_i = \langle \bar{a}_p - e_i^*, \hat{\theta} \rangle + \frac{1}{\gamma} \|\bar{a}_p - e_i^*\|_{H_p^{-1}}^2 = \lambda.
$$

Basic algebra shows that

$$
\langle \bar{a}_p - e_i^*, \hat{\theta} \rangle + \frac{1}{\gamma} \|\bar{a}_p - e_i^*\|_{H_p^{-1}}^2 = \sum_{i' \in [K]} p_{i'} \hat{\theta}_{i'} - \hat{\theta} - \frac{1}{\gamma} + \frac{1}{\gamma} p_i = \lambda.
$$

Substituting $\lambda = \sum_{i' \in [K]} p_{i'} \hat{\theta}_{i'} - \min_j \hat{\theta}_j - \frac{\lambda}{\gamma} + \lambda/\gamma$, rearranging, and picking the unique value for $\lambda$ such that the result is a probability distribution leads to the precisely the log-barrier$(\hat{\theta}, \gamma)$ distribution.

B.2 Proofs from Section 3.2

Recall that $\dim(A)$ is the dimension of the smallest affine linear subspace containing $A$. In other words, $\dim(A) = \dim(\text{span}(A - a))$ for all $a \in A$. Our main result in this section is the following slightly stronger version of Lemma 2.

Lemma 3. Any solution $p \in \Delta(A)$ to the problem logdet-barrier$(\hat{\theta}, \gamma; A)$ in Eq. (6) satisfies

$$
\max_{a^* \in A} \sup_{\delta \in A^d} \langle \bar{a}_p - a^*, \theta \rangle - \frac{\gamma}{4} \|\bar{a}_p - a^*\|_{H_p^{-1}}^2 \leq \gamma^{-1} \dim(A).
$$

Since $-\|\hat{\theta} - \theta\|_{H_p^{-1}}^2 \bar{a}_p = -\|\hat{\theta} - \theta\|^2_{H_p^{-1}} + (\hat{\theta} - \theta, \bar{a}_p)^2 \geq -\|\hat{\theta} - \theta\|_{H_p^{-1}}^2$, Lemma 2 is a direct corollary of Lemma 3.

Proof of Lemma 3. We begin by handling the generate case in which $\dim(A) < d$.

Case: $\dim(A) < d$. We first show that if $\dim(A) < d$, there exists a bijection from $A$ to a set $\tilde{A} \subset \mathbb{R}^{\dim(A)}$ and a projection $P$ taking the loss estimator $\hat{\theta}$ into $\mathbb{R}^{\dim(A)}$, such that logdet-barrier$(\hat{\theta}, \gamma; \tilde{A})$ and logdet-barrier$(P(\hat{\theta}), \gamma; \tilde{A})$ are (up to the bijection) identical, and such that the objective in Lemma 3 coincides for $(\hat{\theta}, \gamma, \tilde{A})$ and $(P(\hat{\theta}), \gamma, \tilde{A})$. This implies for all subsequent arguments, we can assume without loss of generality that $\dim(A) = d$, since if this does not hold we can work in the subspace outlined in this section.

Pick an arbitrary anchor $a \in A$, and let $P$ be the projection onto $\text{span}(A - a)$, represented with an arbitrary fixed orthonormal basis for $\text{span}(A - a)$. Let $\tilde{A} = P(A - a)$, and for each $p \in \Delta(A)$, let $\tilde{p} \in \Delta(\tilde{A})$ be such
that \( \tilde{p}_{P(a-a)} = p_a \) (recall that we define \( \Delta(A) \) to have countable support). Observe that for all \( \hat{\theta} \in \mathbb{R}^d \), we have

\[
\langle \tilde{a}_p, \hat{\theta} \rangle = E_{a \sim P} \left[ (P(a - a), P(\hat{\theta})) \right] + \langle a, \hat{\theta} \rangle = \langle \tilde{a}_p, P(\hat{\theta}) \rangle + \langle a, \hat{\theta} \rangle.
\]

Recall that we define the determinant function \( \det(\cdot) \) in logdet-barrier as the product over the first \( \text{dim}(A) \) eigenvalues of \( H_p - \tilde{a}_p \tilde{a}_p^\top \). Let \( (\nu_i)_{i=1}^{\text{dim}(A)} \) denote the corresponding eigenvectors (note that this requires \( \nu_i \in \text{span}(A - a) \)). We have

\[
\log \det(H_p - \tilde{a}_p \tilde{a}_p^\top) = \sum_{i=1}^{\text{dim}(A)} \log(\|\nu_i\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2) = \sum_{i=1}^{\text{dim}(A)} \log(E_{a \sim P}[\nu_i, a - \tilde{a}_p])
\]

\[
= \sum_{i=1}^{\text{dim}(A)} \log(E_{a \sim P}(\nu_i, a - a - \tilde{a}_p)) = \sum_{i=1}^{\text{dim}(A)} \log(E_{a \sim P}(P(\nu_i), P(a - a - \tilde{a}_p))
\]

\[
= \sum_{i=1}^{\text{dim}(A)} \log(\|P(\nu_i)\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2) = \log \det(H_p - \tilde{a}_p \tilde{a}_p^\top),
\]

where we have used the fact that \( P \) only changes the representation on \( \text{span}(A - a) \) and does not change the identity of the eigenvalues. Combining these two results immediately shows that for any \( p \in \logdet-barrier(\hat{\theta}, \gamma; A) \), we have \( \tilde{p} \in \logdet-barrier(P(\hat{\theta}), \gamma; A) \) and vice versa.

For the objective in Lemma 3, we note that

\[
\langle \tilde{a}_p - a^*, \theta \rangle = E_{a \sim P}[P(a - a)] - P(a^* - a), P(\theta)] = \langle \tilde{a}_p - (P(a^* - a)), P(\theta) \rangle.
\]

For the quadratic term, following the same steps as above for \( \nu_i \), we have

\[
\|\tilde{\theta} - \theta\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2 = \|P(\tilde{\theta}) - P(\theta)\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2.
\]

and

\[
\langle \tilde{a}_p - a^*, \theta \rangle - \frac{\gamma}{4} \|\tilde{\theta} - \theta\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2 = \langle \tilde{a}_p - P(a^* - a), P(\theta) \rangle - \frac{\gamma}{4} \|P(\tilde{\theta}) - P(\theta)\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2.
\]

Hence, we have

\[
\max_{a^* \in \mathcal{A}, \theta \in \mathbb{R}^d} \langle \tilde{a}_p - a^*, \theta \rangle - \frac{\gamma}{4} \|\tilde{\theta} - \theta\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2 = \max_{\tilde{a}_p, \theta \in \mathbb{R}^d} \langle \tilde{a}_p - a^*, \theta \rangle - \frac{\gamma}{4} \|P(\tilde{\theta}) - P(\theta)\|_{H_p - \tilde{a}_p \tilde{a}_p^\top}^2.
\]

Case: \( \text{dim}(A) = d \). We now handle the full-dimensional case. Our technical result here is as follows.

**Lemma 4.** When \( \text{dim}(A) = d \), any solution \( p \in \Delta(A) \) to the problem \( \logdet-barrier(\theta, \gamma; A) \) in Eq. (6) satisfies

\[
\forall a \in \mathcal{A} : \langle \tilde{a}_p - a, \theta \rangle + \frac{1}{\gamma} \|\tilde{a}_p - a\|_{H_{p^{-1}} - \tilde{a}_p \tilde{a}_p^\top}^2 \leq \frac{\text{dim}(A)}{\gamma}.
\]

**Proof.** We first observe that any solution \( p \in \Delta(A) \) to the problem \( \logdet-barrier(\hat{\theta}, \gamma; A) \) must be positive definite in the sense that \( H_p - \tilde{a}_p \tilde{a}_p^\top > 0 \), since otherwise the objective has value \( \infty \); note that \( \text{dim}(A) = d \) implies that a distribution \( p \) with \( H_p - \tilde{a}_p \tilde{a}_p^\top > 0 \) indeed exists. Hence, going forward, we only consider \( p \) for which \( H_p - \tilde{a}_p \tilde{a}_p^\top > 0 \).

Recall \( p = \logdet-barrier(\hat{\theta}, \gamma; A) \) is any solution to

\[
\argmin_{p \in \Delta(A)} \left\{ \langle \tilde{a}_p, \hat{\theta} \rangle - \gamma^{-1} \log \det(H_p - \tilde{a}_p \tilde{a}_p^\top) \right\},
\]

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where $\Delta(A)$ is the set of distributions over countable subsets of $A$. Hence we can write

$$
\Delta(A) = \left\{ \sum_{i=1}^{\infty} w_i e_{A_i} \mid w \in \mathbb{R}_+^N, A \in A^N, \sum_{i=1}^{\infty} w_i = 1 \right\},
$$

where $e_a$ denotes the distribution that selects $a$ with probability 1. By first-order optimality, $p$ is a solution to Eq. (6) if and only if

$$
\forall p' \in \Delta(A) : \sum_{a \in \text{supp}(p) \cap \text{supp}(p')} (p'_a - p_a) \frac{\partial}{\partial p_a} \left[ \langle \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \log \det(H_p - \bar{a}_p a_p^\top) \right] \geq 0.
$$

By the K.K.T. conditions, this holds if and only if there exists some $\bar{\lambda} \in \mathbb{R}$ such that

$$
\forall a \in \text{supp}(p) : \frac{\partial}{\partial p_a} \left[ \langle \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \log \det(H_p - \bar{a}_p a_p^\top) \right] = \bar{\lambda}
$$

and

$$
\forall a \in A : \frac{\partial}{\partial p_a} \left[ \langle \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \log \det(H_p - \bar{a}_p a_p^\top) \right] \geq \bar{\lambda}.
$$

To find $\bar{\lambda}$, we calculate the partial derivative for each action $a$ using the chain rule:

$$
\frac{\partial}{\partial p_a} \left[ \langle \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \log \det(H_p - \bar{a}_p a_p^\top) \right] = \langle a, \hat{\theta} \rangle - \frac{\text{det}(H_p - \bar{a}_p a_p^\top) \text{tr}((H_p - \bar{a}_p a_p^\top)^{-1}(aa^\top - \bar{a}_p a_p^\top - \bar{a}_p \bar{a}_p^\top))}{\gamma \det(H_p - \bar{a}_p a_p^\top)}
$$

$$
= \langle a - \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \|a - \bar{a}_p\|^2 \|H_p - \bar{a}_p a_p^\top\|^{-1} + \frac{1}{\gamma} \|\bar{a}_p\|^2 \|H_p - \bar{a}_p a_p^\top\|^{-1} + \langle \bar{a}_p, \hat{\theta} \rangle.
$$

Using Eq. (14) and taking the expectation over $p$ yields

$$
\bar{\lambda} = \mathbb{E}_{a \sim p} \left[ \langle \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \log \det(H_p - \bar{a}_p a_p^\top) \right] = -\frac{d}{\gamma} + \frac{1}{\gamma} \|\bar{a}_p\|^2 \|H_p - \bar{a}_p a_p^\top\|^{-1} + \langle \bar{a}_p, \hat{\theta} \rangle.
$$

Plugging this expression into Eq. (15), we deduce that

$$
\forall a \in A : \langle a - \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \|a - \bar{a}_p\|^2 \|H_p - \bar{a}_p a_p^\top\|^{-1} \geq -\frac{d}{\gamma}.
$$

Rearranging finishes the proof. \qed

We now conclude the proof of Lemma 3. Recall that for any solution $p \in \Delta(A)$ to the problem $\log \text{det-barrier}(\theta, \gamma; A)$, the matrix $H_p - \bar{a}_p a_p^\top$ is positive definite. In this case, for any fixed $a^* \in A$, the function

$$
\theta \mapsto \langle \bar{a}_p - a^*, \theta \rangle - \frac{\gamma}{4} \|\theta - \bar{\theta}\|^2_{H_p - \bar{a}_p a_p^\top}
$$

is strictly concave in $\theta$, and the maximizer $\theta^*$ may be found by setting the derivative with respect to $\theta$ to 0. In particular,

$$
\frac{\partial}{\partial \theta} \left[ \langle \bar{a}_p - a^*, \theta \rangle - \frac{\gamma}{4} \|\theta - \bar{\theta}\|^2_{H_p - \bar{a}_p a_p^\top} \right] = \bar{a}_p - a^* + \frac{\gamma}{2} (H_p - \bar{a}_p a_p^\top)(\bar{\theta} - \theta),
$$

so that the maximizer is given by

$$
\theta^* = \bar{\theta} + \frac{\gamma}{2} (H_p - \bar{a}_p a_p^\top)^{-1}(a_p - a^*).
$$

Substituting in this choice, we have that

$$
\max_{a^* \in A} \sup_{\theta \in \mathbb{R}^d} \langle \bar{a}_p - a^*, \theta \rangle - \frac{\gamma}{4} \|\theta - \bar{\theta}\|^2_{H_p - \bar{a}_p a_p^\top} = \max_{a \in A} \langle \bar{a}_p - a, \hat{\theta} \rangle + \frac{1}{\gamma} \|\bar{a}_p - a\|^2 \|H_p - \bar{a}_p a_p^\top\|^{-1}.
$$

The result is obtained by applying Lemma 4 to the right-hand side above. \qed
B.3 Proofs from Section 3.3

Proof of Theorem 3. Let \( m \) be fixed. To keep notation compact, we abbreviate \( q_t \equiv q_{t,h}, \rho_t \equiv \rho_{p,m}, \gamma_t \equiv \gamma_{t,m}, Z_t \equiv Z_{t,m}, \) and so forth. Consider a fixed sequence \( S, \) and let \( f^* \) be any predictor achieving the value of \( \varepsilon_T(S) \) (if the infimum is not achieved, we can consider a limit sequence; we omit the details). Recall that since we assume an oblivious adversary, \( f^* \) is fully determined before the interaction protocol begins. Finally, let us abbreviate \( \theta_t^* \equiv f^*(x_t), a_t^* \equiv \pi_{f^*}(x_t), \) and \( \pi_t^*(x_t) = \arg\min_{a \in A_t} \mu(a, x_t), \) where ties are broken arbitrarily. Then we can bound

\[
\text{Reg}_{\text{Imp}}(T) = E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} (\mu(a_t, x_t) - \mu(\pi_t^*(x_t), x_t)) \right] \\
\leq E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} (\langle a_t, \pi_t^*(x_t), \theta_t^* \rangle + 2 \max_{a \in A_t} |\mu(a, x_t) - \langle a, \theta_t^* \rangle|) \right] \\
\leq (a) E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} (\langle a_t, \pi_t^*(x_t), \theta_t^* \rangle) \right] + 2\varepsilon_T T \\
\leq (b) E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} (a_t - a_t^*, \theta_t^*) \right] + 2\varepsilon_T T \\
\leq (c) E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} (\langle \tilde{a}_t, a_t^*, \theta_t^* \rangle - \frac{\gamma_t}{4}\|\tilde{\theta}_t - \theta^*\|_{H_p}^2 + \frac{\gamma_t}{4}\|\tilde{\theta}_t - \theta^*\|_{H_p}^2) \right] + 2\varepsilon_T T \\
\leq (d) E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} (\frac{\dim(A_t)}{\gamma_t} + \frac{\gamma_t}{4}\|\tilde{\theta}_t - \theta^*\|_{H_p}^2) \right] + 2\varepsilon_T T \\
\leq (e) E \left[ \max_{t \in [T]} \gamma_t^{-1} \sum_{t=1}^{T} \dim(A_t) + \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t (\langle a_t, \hat{\theta}_t \rangle - \langle a_t, \theta_t^* \rangle)^2 \right] + 2\varepsilon_T T.
\]

Here (a) follows from the fact that \( E[Z_t] = q_t \) and the Cauchy-Schwarz inequality, together with the definition of \( \varepsilon_T; \) (b) follows from the definition of the policy \( \pi_{f^*}; \) (c) is due to the fact that, conditioned on \( Z_t = 1, \) we sample \( a_t \sim p_t \) with \( E_{a_t \sim p_t}[a_t] = \hat{a}_p; \) (d) uses Lemma 2; (e) uses \( E_{a_t \sim p_t}[a_t a_t^\top] = H_p. \) Continuing with squared error term above, we have

\[
E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t (\langle a_t, \hat{\theta}_t \rangle - \langle a_t, \theta_t^* \rangle)^2 \right] \\
= E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t \left( (\langle a_t, \hat{\theta}_t \rangle - \ell_t)^2 - (\langle a_t, \theta_t^* \rangle - \ell_t)^2 + 2(\ell_t - \langle a_t, \theta_t^* \rangle)(\langle a_t, \hat{\theta}_t \rangle - \langle a_t, \theta_t^* \rangle) \right) \right] \\
\leq (a) E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t \left( (\langle a_t, \hat{\theta}_t \rangle - \ell_t)^2 - (\langle a_t, \theta_t^* \rangle - \ell_t)^2 + 2(\mu(a_t, x_t) - \langle a_t, \theta_t^* \rangle)(\langle a_t, \hat{\theta}_t \rangle - \langle a_t, \theta_t^* \rangle) \right) \right],
\]

where (a) uses that \( \ell_t \) is conditionally independent of \( Z_t \) and \( a_t. \) We bound the term involving the difference of squares as

\[
E \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t ((\langle a_t, \hat{\theta}_t \rangle - \ell_t)^2 - (\langle a_t, \theta_t^* \rangle - \ell_t)^2) \right] \leq E \left[ \max_{t \in [T]} \frac{\gamma_t}{q_t} \right] \text{Reg}_{\text{Sq}}(T),
\]
by Assumption 2.⁴ For the linear term, we apply the sequence of inequalities

\[ 2\mathbb{E} \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t (\mu(a_t, x_t) - \langle a_t, \theta_t^* \rangle)(a_t, \hat{\theta}_t - \theta_t^*) \right] \]

\[ \leq 2\mathbb{E} \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t ((\mu(a_t, x_t) - \langle a_t, \theta_t^* \rangle)^2 + \frac{1}{4}(a_t, \hat{\theta}_t - \theta_t^*)^2 \right] \]

\[ \leq 2\mathbb{E} \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t \max_{a \in A_t} ((\mu(a, x_t) - \langle a, \theta_t^* \rangle)^2 + \frac{1}{4}(a_t, \hat{\theta}_t - \theta_t^*)^2 \right] \]

\[ \leq 2\mathbb{E} \left[ \max_{t \in [T]} \gamma_t \right] \varepsilon_T^2 T + \frac{1}{2} \mathbb{E} \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t (\langle a_t, \hat{\theta}_t \rangle - \langle a_t, \theta_t^* \rangle)^2 \right], \]

where (a) is by the AM-GM inequality: \(2ab \leq a^2 + \frac{1}{2}b^2\); (b) follows from the fact that \(Z_t\) is conditionally independent of \(\gamma_t\), and the definition of \(\varepsilon_T\).

Altogether, we have

\[ \mathbb{E} \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t (\langle a_t, \hat{\theta}_t \rangle - \langle a_t, \theta_t^* \rangle)^2 \right] \leq 2\mathbb{E} \left[ \max_{t \in [T]} \gamma_t \right] \text{Reg}_{\text{Seq}}(T) + 4\mathbb{E} \left[ \max_{t \in [T]} \gamma_t \right] \varepsilon_T^2 T. \]

Rearranging yields

\[ \mathbb{E} \left[ \sum_{t=1}^{T} \frac{Z_t}{q_t} \gamma_t (\langle a_t, \hat{\theta}_t \rangle - \langle a_t, \theta_t^* \rangle)^2 \right] \leq 2\mathbb{E} \left[ \max_{t \in [T]} \gamma_t \right] \text{Reg}_{\text{Seq}}(T) + 2\mathbb{E} \left[ \max_{t \in [T]} \gamma_t \right] \varepsilon_T^2 T. \]

Combining all of the developments so far, we have

\[ \text{Reg}_{\text{Imp}}(T) \leq \sum_{t=1}^{T} \mathbb{E} \left[ \gamma_t^{-1} \right] \text{dim}(A_t) + \frac{1}{2} \mathbb{E} \left[ \max_{t \in [T]} \gamma_t \right] \text{Reg}_{\text{Seq}}(T) + \mathbb{E} \left[ \max_{t \in [T]} \gamma_t \right] \varepsilon_T^2 T + 2\varepsilon_T T. \] (16)

The proof is completed by noting that the learning rate \(\gamma_t = \min \left\{ \frac{\sqrt{d}}{\varepsilon_T}, \sqrt{dT/\rho_t \text{Reg}_{\text{Seq}}(T)} \right\} \) is non-increasing, but \(\gamma_t \rho_t \geq \frac{\sqrt{d}}{\varepsilon_T}\) is non-decreasing. Hence, we can upper bound the expression above by

\[ \text{Reg}_{\text{Imp}}(T) \leq \mathbb{E} \left[ \gamma_t^{-1} \right] dT + \frac{1}{2} \mathbb{E} \left[ \gamma_t \rho_t \right] \text{Reg}_{\text{Seq}}(T) + \mathbb{E} [\gamma_t] \varepsilon_T^2 T + 2\varepsilon_T T \]

\[ \leq \left( \frac{\varepsilon_T^2}{\sqrt{d}} + \mathbb{E} [\sqrt{\rho t}] \sqrt{\frac{\text{Reg}_{\text{Seq}}(T)}{dT}} \right) dT + \frac{1}{2} \mathbb{E} [\sqrt{\rho t}] \sqrt{dT \text{Reg}_{\text{Seq}}(T)} + \frac{\sqrt{d}}{\varepsilon_T} \varepsilon_T^2 T + 2\varepsilon_T T. \]

**Proof of Theorem 1.** Let \(m^* := \arg\min_{m \in [M]} \frac{\hat{\ell}_{t,A_t} - \ell_{t,m^*}}{\varepsilon_m} \) if \(\varepsilon_T \geq T^{-1}\) and \(m^* := M\) otherwise. We begin by formally verifying the claim

\[ \text{Reg}(T) = \mathbb{E} \left[ \sum_{t=1}^{T} \hat{\ell}_{t,A_t} - \hat{\ell}_{t,m^*} \right] + \text{Reg}_{\text{Imp}}^m(T). \] (17)

⁴Note that Assumption 2 holds with bound \(\text{Reg}_{\text{Seq}}(T)\) even if Alg_{Seq} is run for less than \(T\) timesteps, since we could extend the sequence with 0 weight until time \(T\).
By the definition \( \tilde{\ell}_{t,A_i} := \ell_t + 1 \), we have
\[
\mathbb{E} \left[ \tilde{\ell}_{t,A_i} - \tilde{\ell}_{t,m^*} \right] = \mathbb{E} \left[ \ell_t + 1 - \frac{Z_{t,m^*}}{p_{t,m^*}}(\ell_t + 1) \right] = \mathbb{E} \left[ \mu(a_t, x_t) - \frac{Z_{t,m^*}}{p_{t,m^*}} \mu(a_t, x_t) \right].
\]

On the other hand, the second term on the right-hand side of Eq. (17) is
\[
\text{Reg}_{\text{Imp}}^m(T) = \mathbb{E} \left[ \sum_{t=1}^T \frac{Z_{t,m^*}}{p_{t,m^*}}(\mu(a_t, x_t) - \mu(\pi^*_t(x_t), x_t)) \right] = \mathbb{E} \left[ \sum_{t=1}^T \frac{Z_{t,m^*}}{p_{t,m^*}}(\mu(a_t, x_t) - \mu(\pi^*_t(x_t), x_t)) \right].
\]
Combining both lines leads to the identity in Eq. (17).

Proceeding with the proof, recall that the losses \( \ell \) satisfy \( \tilde{\ell}_{t,m} \in [0,2] \) for all \( m \), since \( \ell_t \in [-1,1] \) and we shift the loss by 1. Hence, we can apply Corollary 2 with \( \alpha = \frac{1}{2} \) and \( R = \frac{3}{2} \sqrt{dT \text{Reg}_{\text{sq}}(T)} \) to obtain
\[
\mathbb{E} \left[ \sum_{t=1}^T \tilde{\ell}_{t,A_i} - \tilde{\ell}_{t,m^*} \right] \leq 4\sqrt{2MT} + 3\sqrt{dT \text{Reg}_{\text{sq}}(T)} M - \frac{3}{2} \mathbb{E} \left[ \sqrt{\rho T,m^*} \right] \sqrt{dT \text{Reg}_{\text{sq}}(T)},
\]
and by Theorem 3,
\[
\text{Reg}_{\text{Imp}}^m(T) \leq \left( \left( \frac{\varepsilon_{m^*}}{\varepsilon_T} + \frac{\varepsilon_T}{\varepsilon_{m^*}} \right) \sqrt{d} + 2 \right) \varepsilon_T T + \frac{3}{2} \mathbb{E} \left[ \sqrt{\rho T,m^*} \right] \sqrt{dT \text{Reg}_{\text{sq}}(T)}.\]

We now consider two cases. First, if \( \varepsilon_T > T^{-1} \), we can pick \( m^* \) such that \( \varepsilon_{m^*} \in [\varepsilon_T, c\varepsilon_T] \), which ensures that \( \left( \frac{\varepsilon_{m^*}}{\varepsilon_T} + \frac{\varepsilon_T}{\varepsilon_{m^*}} \right) \leq e + e^{-1} \). Otherwise, we pick \( \varepsilon_{m^*} = T^{-1} \) so that the misspecification term is bounded by
\[
\left( \left( \frac{\varepsilon_{m^*}}{\varepsilon_T} + \frac{\varepsilon_T}{\varepsilon_{m^*}} \right) \sqrt{d} + 2 \right) \varepsilon_T T = \left( \left( \frac{\varepsilon_{m^*}}{\varepsilon_T} + \frac{\varepsilon_T}{\varepsilon_{m^*}} \right) \sqrt{d} + 2 \varepsilon_T \right) T \leq 2\sqrt{d} + 2.
\]
Summing the regret bounds for the base and master algorithms completes the proof.

\[ \Box \]

### B.4 Proofs from Section 3.5

**Algorithm description** We begin by outlining the algorithm that achieves the bound in Theorem 4. The algorithm proceeds in episodes. At the begin of episode 1, the algorithm defines \( D_1 := \sum_{t=1}^T \dim(A_t) \leq 2T \) and \( \tau_1 = 1 \) and plays the algorithm from Theorem 1 with the learning rate tuned for \( d = D_1 \). Within each episode \( i \geq 1 \), if the agent observes at time \( t \) that \( \sum_{t=t_i}^{t_i+1} \dim(A_s) > D_1 \), it restarts the algorithm from Theorem 1 with \( D_{i+1} := 2D_i \); we denote this time by \( \tau_{i+1} \). Note that we can assume \( \dim(A) \leq d < T \) without loss of generality (otherwise the result is trivial), and hence we never need to double more than once at each time step.

To analyze this algorithm, we first show that the bound from Theorem 1 continues to hold even if the learner plays only on a subset of time steps.

**Proposition 2.** Let \( T \subset [T] \) be an obliviously chosen subset of timesteps. Then the upper bound from Theorem 1 on a sequence \( S \) continues to hold if the algorithm is run on a sub-sequence \( S_T \).

**Proof.** We extend the sequence \( S \) by adding an “end” sequence \( E = (\{0\}, x)^T \), where \( x \in X \) is picked such that \( \mu(0,x) = 0 \) (If there is no such context, we add a context with that property to \( X \)). Let the extended sequence be \( S' = S + E \) and consider the sequence \( \hat{S} = S_{T \cup \{T+1, \ldots, 2T-|T|\}} \), which has length \( T \). The contribution to regret from regreting on the \( E \) section on the sequence is always 0, since there is only one action. Furthermore \( \varepsilon_T(\hat{S}) \leq \varepsilon_T(S) \). Hence, by Theorem 1, we have
\[
\mathbb{E} \left[ \sum_{t \in T} \mu(a_t, x_t) - \min_{a \in A_t} \mu(a, x_t) \right] = \mathbb{E} \left[ \sum_{t \in T \cup \{T+1, \ldots, 2T-|T|\}} \mu(a_t, x_t) - \min_{a \in A_t} \mu(a, x_t) \right] \leq O \left( \sqrt{d \varepsilon_T(S) T} + \sqrt{d \text{Reg}_{\text{sq}}(T) \log(T)} \right).
\]

\[ \Box \]
Note that since \( d_{\text{avg}}(\tilde{S}) \leq d_{\text{avg}}(S) \), this argument also applies to the refined version of the bound where \( d \) is replaced by \( d_{\text{avg}} \), as long as the algorithm’s parameters are tuned accordingly.

**Proof of Theorem 4.** Let \( \tau_1, \ldots, \tau_L \) denote the times where the algorithm is restarted, with \( \tau_1 = 1 \) and \( \tau_{L+1} = T+1 \) by convention. Since the adversary fixes the action sets in advance, these doubling times are deterministic. The regret is given by

\[
\text{Reg}(T) = \mathbb{E} \left[ \sum_{t=1}^{T} \mu(a_t, x_t) - \min_{a \in \mathcal{A}_t} \mu(a, x_t) \right] \\
\leq \sum_{i=1}^{L} \mathbb{E} \left[ \sum_{t=\tau_{i-1}+1}^{\tau_i} \mu(a_t, x_t) - \min_{a \in \mathcal{A}_t} \mu(a, x_t) \right].
\]

By applying Proposition 2 to each episode, we have

\[
\mathbb{E} \left[ \sum_{t=\tau_{i-1}+1}^{\tau_i} \mu(a_t, x_t) - \min_{a \in \mathcal{A}_t} \mu(a, x_t) \right] = \sqrt{2^i} \cdot O\left( \varepsilon_T(S)T + \sqrt{\text{Reg}_{\text{Sq}}(T) \log(T)} \right).
\]

Summing over these terms and observing that

\[
\sum_{i=1}^{L} 2^{i/2} = O(2^{L/2}) = O(1) \cdot \sqrt{\frac{1}{T} \sum_{t=1}^{T} \dim(\mathcal{A}_t)} = O\left( d_{\text{avg}}^{1/2} \right)
\]

completes the proof. \( \square \)

**C Improved Master Algorithms for Bandit Aggregation**

In this section we present a new family of algorithms that can be used for the master algorithm within the framework of Algorithm 3. For the remainder of this section, we work in a generic adversarial multi-armed bandit setting, at each time step, the agent selects an action \( A_t \in [M] \), then observes a loss \( \ell_{t,A_t} \in [0,L] \) for the action they selected. Compared to the log-barrier-based master algorithm used within the original CORRAL algorithm of Agarwal et al. (2017), the algorithms we describe here are simpler to analyze, more flexible, and have improved logarithmic factors.

**C.1 Background and Motivation**

The CORRAL algorithm is a special case of Algorithm 3 that uses a bandit variant of the Online Mirror Descent (OMD) algorithm with log-barrier regularization as the master.\(^5\) The bandit variant of the OMD algorithm used within CORRAL is parameterized by a Legendre potential \( F(x) = \sum_{i=1}^{d} \eta_i^{-1} f(x_i) \) where \( \eta_1, \ldots, \eta_d \) are per-coordinate learning rates. It is initialized using the distribution \( p_1 = \arg\min_{p \in \Delta([M])} F(p) \). Then, at each time \( t \), the bandit OMD algorithm samples an arm \( A_t \sim p_t \), observes the loss \( \ell_{t,A_t} \), and constructs an unbiased importance-weighted loss estimator \( \hat{\ell}_t = \frac{\ell_{t,A_t}}{p_t,A_t} e_{A_t} \). It then updates the action distribution via

\[
p_{t+1} = \arg\min_{p \in \Delta([M])} \left\{ \langle p, \hat{\ell} \rangle + D_F(p, p_t) \right\},
\]

where \( D_F(x, y) := F(x) - F(y) - \langle x-y, \nabla F(y) \rangle \) is the Bregman divergence associated with \( F \). An important feature which leads to the guarantee for the CORRAL master is a time-dependent learning rate schedule for each of the per-arm learning rates, which increases the learning rate for each arm whenever the probability for that arm falls below a certain threshold.\(^6\)

\(^5\)Note that the use of the log-barrier in CORRAL is not related to our use of the log-barrier within the contextual bandit framework.

\(^6\)For time-dependent learning rates, we replace \( \eta \) by \( \eta_t \) in the update rule of Eq. (18).
Online Mirror Descent is closely related to the Follow-the-Regularized-Leader (FTRL) algorithm. In particular, for any sequence of loss vector estimates \((\hat{\ell}_i)_{i=1}^T\), there exists a sequence of (vector) biases \(b_i\) such that FTRL running on the loss sequence \((\hat{\ell}_i - b_i)_{i=1}^T\) using the same learning rate as its OMD counterpart has an identical trajectory of plays \(p_t\). We can view the CORRAL master through the lens of FTRL. In particular, the FTRL variant of the algorithm performs two steps whenever it increases the learning rate of arm \(i\). First it subtracts a bias \(b_{t,i} > 0\) from the loss estimates for arm \(i\). Then it increases the learning rate for that arm. We show that only the former step is actually required, while the latter is unnecessary. This motivates the \((\alpha, R)\)-hedged FTRL algorithm, which achieves a slightly improved guarantee by removing the per-coordinate learning rates.

C.2 The Hedged FTRL Algorithm

Following the intuition in the prequel, we present \((\alpha, R)\)-hedged FTRL, a modified variant of the FTRL algorithm with strong guarantees for aggregating bandit algorithms. To do so, we first describe a basic bandit variant of FTRL algorithm.

The FTRL family of algorithms is parameterized by a potential \(F\) and learning rate \(\eta > 0\). At each round \(t\), the algorithm selects

\[
p_t = \arg\min_{p \in \Delta([M])} \left\{ \langle p, \hat{L}_{t-1} \rangle + \eta^{-1} F(p) \right\}, \quad \text{where } \hat{L}_t = \sum_{s=1}^t \hat{\ell}_s.
\]

Two relevant properties of \(F\) that arise in our analysis are stability and diameter. Define

\[
\hat{F}_\eta^*(-L) = \max_{p \in \Delta([M])} \left\{ \langle p, -L \rangle - \eta^{-1} F(p) \right\}.
\]

The stability \(\text{stab}(F)\) and diameter \(\text{diam}(F)\) of \(F\) for loss range \([0, L]\) are defined as follows:

\[
\text{stab}(F) = \sup_{\eta > 0} \sup_{x \in \Delta([M])} \sup_{\ell \in [0, L]^M} \eta^{-1} \mathbb{E}_{A \sim x} \left[ D_{\hat{F}_\eta^*} \left( \eta^{-1} \nabla F(x) - \hat{\ell}_A e_A, \eta^{-1} \nabla F(x) \right) \right],
\]

\[
\text{diam}(F) = \max_{p \in \Delta([M])} \left( F(p) - \min_{p \in \Delta([M])} F(p) \right).
\]

Given a potential with bounded \(\text{stab}(F)\) and \(\text{diam}(F)\), setting the learning rate as \(\eta = \sqrt{\text{diam}(F)/(\text{stab}(F)T)}\) leads to regret at most \(2\sqrt{\text{stab}(F)\text{diam}(F)T}\) (Abernethy et al., 2015).\(^7\) Well-known algorithms that arise as special cases of this result include:

- EXP3 (Auer et al., 2002) is an instantiation of bandit FTRL with \(F(x) = \sum_{i=1}^M x_i \log(x_i)\), \(\text{diam}(F) = \log(M)\) and \(\text{stab}(F) \leq \frac{L^2M}{\eta}\).
- Tsallis-INF (Audibert and Bubeck, 2009; Abernethy et al., 2015; Zimmert and Seldin, 2019) is an instantiation of bandit FTRL that gives the best known regret bound for multi-armed bands. It is given by \(F(x) = -2 \sum_{i=1}^M \sqrt{x_i}\), which has \(\text{diam}(F) \leq 2\sqrt{M}\) and \(\text{stab}(F) \leq L^2\sqrt{M}\).

We now present the \((\alpha, R)\)-hedged FTRL algorithm. The algorithm augments the basic bandit FTRL strategy using an additional pair of parameters \(\alpha \in (0, 1)\), \(R \in \mathbb{R}\). The algorithm begins by initializing a collection of parameters \((B_{0,i})_{i=1}^M\) with \(B_{0,i} = \rho_{t,0} R\). At each step \(t\), it plays \(A_t \sim p_t\), then computes

\[
\hat{p}_{t+1} = \arg\min_{p \in \Delta([M])} \left\{ \langle p, \hat{L}_t - \langle B_{t-1} - B_0 \rangle \rangle + \eta^{-1} F(p) \right\}, \quad \text{where } \hat{L}_t = \sum_{s=1}^t \hat{\ell}_s.
\]

If \(\hat{p}_{t+1,A_t} R \leq B_{t-1,A_t}\), the algorithm sets \(B_t = B_{t-1}\) and \(p_{t+1} = \hat{p}_{t+1}\). Otherwise it chooses the unique \(b_t > 0\), such that for \(B_t = B_{t-1} + b_t e_{A_t}\), the following properties hold simultaneously:

\[
p_{t+1} = \arg\min_{p \in \Delta([M])} \left\{ \langle p, \hat{L}_t - \langle B_t - B_0 \rangle \rangle + \eta^{-1} F(p) \right\} \quad \text{and} \quad \hat{p}_{t+1,A_t} R = B_{t,A_t}.
\]

This algorithm is always well defined when the potential \(F\) is symmetric; see Appendix C.3 for details. Letting \(p_{t,i} = \max_{s \leq t} p_{s,i}\), the main regret guarantee is as follows.

\(^7\)Abernethy et al. (2015) present this result slightly differently. See our proof of Theorem 6 with \(R = 0\) for an alternative.
Theorem 6. For any potential $F$ with $\text{stab}(F), \text{diam}(F) < \infty$, the pseudo-regret $\text{Reg}_\alpha(T) = \mathbb{E} \left[ \sum_{t=1}^{T} \ell_{t,a_t} - \ell_{t,a^*} \right]$ for $(\alpha, R)$-hedged FTRL with learning rate $\eta = \sqrt{\text{diam}(F)/\text{stab}(F)T}$ is bounded by

$$\text{Reg}_\alpha(T) \leq 2\sqrt{\text{stab}(F) \text{diam}(F)T} + \left[ \frac{\alpha}{1 - \alpha} \sum_{i=1}^{M} \left( \rho_{\alpha,i} - \mathbb{E}[\rho_{\alpha,i}] \right) \right] \cdot R$$

for all arms $a^* \in [M]$.

This algorithm may be viewed “hedging” against the event that the arm $a^*$ experiences a very small probability, as this leads to a negative regret contribution proportional to $\rho_{\alpha,i}R$.

C.3 Proofs

Before proving the main result, we first establish that the $(\alpha, R)$-hedged FTRL strategy as described is in fact well-defined. Recall that the algorithm initializes with $B_0$ such that $\nabla \tilde{F}^*_\eta(B_0) - \rho_{\alpha}^i R = B_{0,i}$. For symmetric potentials $F(x) = \sum_{i=1}^{M} f(x_i)$, $\nabla \tilde{F}^*_\eta(c1_M) = \frac{1}{M} \mathbf{1}_M$ for all $c \in \mathbb{R}$. Hence $B_0 = M^{-\alpha} R 1_M$ satisfies the initialization condition. Otherwise a solution exists by the observation that $\nabla \tilde{F}^*_\eta(B_0)^{-\alpha}R$ is a continuous, decreasing function in $B_0$ that has positive values at $B_0 = 0$. Hence a solution to the equation must exist.

The same argument holds during the update at subsequent rounds $t$. Only the arm that was played can decrease in probability, which means we only need to ensure that $\rho_{\alpha,i+1} R = B_{t,A_t}$. The left-hand side is continuously decreasing with increasing $b_t$, while the right-hand side is increasing. Hence, the optimal value must exist, is unique, and lies in $[0, \hat{\ell}_{t,A_t}]$.

Proof of Theorem 6. We follow the standard FTRL analysis. Let $\hat{B}_i = B_i - B_0$ and note that $p_t = \nabla \tilde{F}^*_\eta(-\hat{L}_{t-1} + \hat{B}_{t-1})$, so $\langle p_t, \hat{L}_t \rangle = \langle \nabla \tilde{F}^*_\eta(-\hat{L}_{t-1} + \hat{B}_{t-1}), \hat{L}_t - \hat{L}_{t-1} \rangle$. Hence, we can write

$$\mathbb{E} \left[ \sum_{t=1}^{T} \ell_{t,A_t} - \ell_{t,a^*} \right] = \mathbb{E} \left[ \sum_{t=1}^{T} \langle p_t, \hat{L}_t \rangle - \hat{\ell}_{t,a^*} \right]$$

$$= \mathbb{E} \left[ \sum_{t=1}^{T} D_{\tilde{F}_\eta}(-\hat{L}_t + \hat{B}_{t-1} - \hat{L}_{t-1} + \hat{B}_{t-1}) \right]$$

$$+ \mathbb{E} \left[ \sum_{t=1}^{T} \left( \tilde{F}_\eta(-\hat{L}_t + \hat{B}_{t-1}) - \tilde{F}_\eta(-\hat{L}_{t-1} + \hat{B}_{t-1}) - \hat{\ell}_{t,a^*} \right) \right].$$

Note that there exists $\lambda$ such that $-\hat{L}_{t-1} + \hat{B}_{t-1} = \lambda 1_M + \eta^{-1} \nabla F(p_t)$. Furthermore, adding or subtracting the same $\lambda 1_M$ term to both arguments does not change the value of the Bregman divergence, because $\tilde{F}_\eta(-L + \lambda 1_M) = F_\eta(-L) + \lambda$. Thus,

$$\mathbb{E} \left[ \sum_{t=1}^{T} D_{\tilde{F}_\eta}(-\hat{L}_t + \hat{B}_{t-1} - \hat{L}_{t-1} + \hat{B}_{t-1}) \right]$$

$$= \eta \mathbb{E} \left[ \sum_{t=1}^{T} \eta^{-1} D_{\tilde{F}_\eta}(\eta^{-1} \nabla F(p_t) - \hat{L}_t, \eta^{-1} \nabla F(p_t)) \right] \leq \eta \text{stab}(F)T.$$

Rearranging the second term gives

$$\sum_{t=1}^{T} \left( \tilde{F}_\eta(-\hat{L}_t + \hat{B}_{t-1}) - \tilde{F}_\eta(-\hat{L}_{t-1} + \hat{B}_{t-1}) - \hat{\ell}_{t,a^*} \right)$$

$$= \tilde{F}_\eta(0) - \tilde{F}_\eta(-\hat{L}_T + \hat{B}_{T-1}) - \hat{\ell}_{T,a^*} + \sum_{t=1}^{T-1} \tilde{F}_\eta(-\hat{L}_t + \hat{B}_t) - \tilde{F}_\eta(-\hat{L}_t + \hat{B}_{t-1}).$$
Note that \( \tilde{F}_\eta^*(\hat{L}_t + \hat{B}_t) = \langle p_{t+1}, -\hat{L}_t + \hat{B}_t \rangle + \eta^{-1}F(p_{t+1}) \). Furthermore we have the bounds
\[
- \tilde{F}_\eta^*(-\hat{L}_T + \hat{B}_{T-1}) \leq - \left( \langle e^*, -\hat{L}_T + \hat{B}_{T-1} \rangle - \eta^{-1}F(e^*) \right),
\]
and
\[
- \tilde{F}_\eta^*(-\hat{L}_t + \hat{B}_{t-1}) \leq - \left( \langle p_{t+1}, -\hat{L}_t + \hat{B}_{t-1} \rangle - \eta^{-1}F(p_{t+1}) \right).
\]

Plugging these inequalities in above leads to
\[
\tilde{F}_\eta^*(0) - \tilde{F}_\eta^*(-\hat{L}_T + \hat{B}_{T-1}) - \hat{L}_{T,a^*} + \sum_{t=1}^{T-1} \tilde{F}_\eta^*(-\hat{L}_t + \hat{B}_t) - \tilde{F}_\eta^*(-\hat{L}_t + \hat{B}_{t-1}) \\
\leq \frac{F(e^*) - F(p_1)}{\eta} - \tilde{B}_{T-1,a^*} + \sum_{t=1}^{T-1} \langle p_{t+1}, \hat{B}_t - \hat{B}_{t-1} \rangle \\
\leq (\rho^o_{1,a^*} - \rho^o_{T,a^*})R + \frac{\text{diam}(F)}{\eta} + \sum_{t=1}^{T-1} \langle p_{t+1}, B_t - B_{t-1} \rangle.
\]
To bound the final sum, note that for each coordinate \( i \), the difference \( B_{t,i} - B_{t-1,i} \) can be non-zero only if \( p_{t+1,i} \) satisfies \( p_{t+1,i} = \tilde{r}_{t+1,i} \). It follows that
\[
p_{t+1,i}(B_{t,i} - B_{t-1,i}) = R\tilde{r}_{t+1,i}^{-1} (\rho^o_{t+1,i} - \rho^o_{t,i}) \\
= \alpha R \int_{\rho_{t+1,i}}^{\rho^o_{t+1,i}} x^{\alpha-1} \rho_{t+1,i}^{-1} dx \\
\leq \alpha R \int_{\rho_{t+1,i}}^{\rho^o_{t+1,i}} x^{\alpha-2} dx \\
= \frac{\alpha R}{1 - \alpha} (\rho^o_{t+1,i} - \rho^o_{t,i}).
\]
Applying this bound to each coordinate, we have
\[
\sum_{t=1}^{T-1} \langle p_{t+1}, B_t - B_{t-1} \rangle = \sum_{i=1}^{M} \alpha R \left( \frac{\rho^o_{1,i} - \rho^o_{T,i}}{1 - \alpha} \right) = \sum_{i=1}^{M} \alpha R \left( \frac{\rho^o_{1,i} - \rho^o_{T,i}}{1 - \alpha} \right).
\]
Combining all of the bounds above concludes the proof.

**Proof of Corollary 2.** Recall that the Tsallis regularizer is given by
\[
F(x) = - \sum_{i=1}^{M} 2\sqrt{x_i}.
\]
For loss range \([0, L]\), the regularizer has stability at most \( L^2 \sqrt{M} \) and diameter at most \( \sqrt{M} \) (Zimmert and Seldin, 2019).\(^8\) Furthermore, since the potential is symmetric, we have \( \forall i : p_{1,i} = 1/M \). Using Theorem 6 with the loss range \([0, 2]\) leads to
\[
\text{Reg}_M(T) \leq 4\sqrt{2MT} + \left[ \frac{\alpha}{1 - \alpha} \sum_{i=1}^{M} (M^{\alpha-1} - \mathbb{E}[\rho^o_{T,i}]) + M^{\alpha} - \mathbb{E}[\rho^o_{T,m^*}] \right] R \\
\leq 4\sqrt{2MT} + \left[ \frac{\alpha}{1 - \alpha} M^{\alpha} \left( 1 - M^{1-\alpha} \min_{j \in [M]} \mathbb{E}[\rho^o_{T,j}] \right) + M^{\alpha} - \mathbb{E}[\rho^o_{T,m^*}] \right] R.
\]
\(^8\)Zimmert and Seldin (2019) show this for \( L = 1 \), but the extension to general \( L \) is trivial.
Dropping the negative $-M^{1-\alpha} \min_{j \in [M]} \mathbb{E}[\rho_{T,j}^{\alpha-1}]$ term above leads to the first part of the $\min \{ \cdot \}$ expression in Eq. (10). For the other term in the $\min \{ \cdot \}$, note that the function

$$\alpha \mapsto \frac{\alpha}{1 - \alpha} (1 - z^{\alpha-1})$$

is monotonically increasing in $\alpha$, with

$$\lim_{\alpha \to 1} \frac{\alpha}{1 - \alpha} (1 - z^{\alpha-1}) = \log(z).$$

Bounding $\log(\max_{j \in [M]} \mathbb{E}[\rho_{T,j}]/M) + 1$ by $2 \log(\max_{j \in [M]} \mathbb{E}[\rho_{T,j}])$ using that $\rho_{1,i} = M$ completes the proof.

\[ \square \]

D Algorithms for the Log-Determinant Barrier Problem

Recall that at each step, SquareCBLin (Algorithm 2) samples from the logdet-barrier $(\hat{\theta}, \gamma; A)$ distribution, which we define as any (not necessarily unique) distribution in the set

$$p^* \in \arg\min_{p \in \Delta(A)} \gamma \langle \bar{a}_p, \hat{\theta} \rangle - \frac{1}{\gamma} \log \det \left( H_p - \bar{a}_p \bar{a}_p^\top \right),$$

where $\bar{a}_p = \mathbb{E}_{a \sim p}[a]$ and $H_p = \mathbb{E}_{a \sim p}[aa^\top]$. In this section, we develop optimization algorithms to efficiently find approximate solutions to the problem Eq. (19). Our main result here is to prove Proposition 1 as a consequence of a more general result, Theorem 7.

While Eq. (19) is a convex optimization problem, developing efficient algorithms presents a number of technical difficulties. First, the optimization problem is non-smooth due to the presence of the log-determinant function, which prevents us from applying standard first-order methods such as gradient descent out of the box. Second, representing distributions in $\Delta(A)$ naively requires $\Omega(|A|)$ memory. To get the result in Proposition 1, we employ a specialized Frank-Wolfe-type method, which maintains a sparse distribution and requires only $O(\log |A|)$ memory.

As a first step toward solving Eq. (19) numerically, we move to an equivalent but slightly more convenient formulation which lifts the actions to $d + 1$ dimensions. Define the lifting operator, which adds a new coordinate with 1 to each vector, by

$$\tilde{a} := \begin{pmatrix} a \\ 1 \end{pmatrix},$$

and define

$$\tilde{a}_p := \mathbb{E}_{a \sim p}[\tilde{a}], \quad \tilde{H}_p := \mathbb{E}_{a \sim p}[\tilde{a} \tilde{a}^\top], \quad \tilde{\theta} := \begin{pmatrix} \hat{\theta} \\ 0 \end{pmatrix}, \quad \text{and} \quad \tilde{d} := d + 1.$$

Finally, define

$$G(p) = \langle \tilde{a}_p, \tilde{\theta} \rangle - \frac{1}{\gamma} \log \det(\tilde{H}_p).$$

**Proposition 3.** The set of solutions for the lifted problem

$$\arg\min_{p \in \Delta(A)} G(p) = \arg\min_{p \in \Delta(A)} \langle \tilde{a}_p, \tilde{\theta} \rangle - \frac{1}{\gamma} \log \det(\tilde{H}_p),$$

is identical to the set of solutions for Eq. (19), and vice-versa.

**Proof.** By Lemma 4, any solution $p^*$ to Eq. (19) must satisfy the optimality condition

$$\forall a \in A: \langle \tilde{a}_{p^*} - a, \tilde{\theta} \rangle + \frac{1}{\gamma} \| \tilde{a}_{p^*} - a \|^2 (\tilde{H}_{p^*} - \tilde{a}_{p^*} \tilde{a}_{p^*}^\top)^{-1} \leq \frac{d}{\gamma}.$$
Now, let \( p^* \) be a minimizer for the optimization problem in (21). By first order optimality, we have

\[
\forall p' \in \Delta(A) : \sum_{a \in \text{supp}(p^*) \cup \text{supp}(p')} (p'_a - p^*_a) \left( \langle \hat{a}, \hat{\theta} \rangle - \frac{1}{\gamma} \| \hat{a} \|_{\hat{H}^{-1}_{p^*}}^2 \right) \geq 0.
\]

By the K.K.T. conditions, this condition holds if and only if there exists \( \lambda \in \mathbb{R} \) such that

\[
\forall a \in \text{supp}(p^*) : \langle \hat{a}, \hat{\theta} \rangle - \frac{1}{\gamma} \| \hat{a} \|_{\hat{H}^{-1}_{p^*}}^2 = \lambda \quad (22)
\]

and

\[
\forall a \in A : \langle \hat{a}, \hat{\theta} \rangle - \frac{1}{\gamma} \| \hat{a} \|_{\hat{H}^{-1}_{p^*}}^2 \geq \lambda. \quad (23)
\]

Note that Eq. (22) implies that

\[
\mathbb{E}_{a \sim p^*} \left[ \langle \hat{a}, \hat{\theta} \rangle - \frac{1}{\gamma} \| \hat{a} \|_{\hat{H}^{-1}_{p^*}}^2 \right] = \langle \hat{a}_{p^*}, \hat{\theta} \rangle - \frac{\bar{d}}{\gamma} = \lambda.
\]

Combining this identity with Eq. (23) and rearranging, we conclude that

\[
\forall a \in A : \langle \hat{a}_{p^*} - a, \hat{\theta} \rangle + \frac{1}{\gamma} \| \hat{a} \|_{\hat{H}^{-1}_{p^*}}^2 \leq \frac{\bar{d}}{\gamma}. \quad (24)
\]

Finally, observe that for any \( p \in \Delta(A) \)

\[
\hat{H}_p = \begin{pmatrix} H_p & a_p \\ \bar{a}_p & 1 \end{pmatrix}, \quad \text{and} \quad \hat{H}_p^{-1} = \begin{pmatrix} \left( H_p - a_p \bar{a}_p^T \right)^{-1} & - \left( H_p - a_p \bar{a}_p^T \right)^{-1} a_p \\ - \bar{a}_p^T \left( H_p - a_p \bar{a}_p^T \right)^{-1} & 1 + \| \bar{a}_p \|_{\left( H_p - a_p \bar{a}_p^T \right)^{-1}}^2 \end{pmatrix},
\]

where the second expression uses the identity for the Schur complement. Using the latter expression, we have that

\[
\| \hat{a} \|_{\hat{H}^{-1}_{p^*}}^2 = \| a \|_{\left( H_p - a_p \bar{a}_p^T \right)^{-1}}^2 - 2 \bar{a}^T \left( H_p - a_p \bar{a}_p^T \right)^{-1} a_p + \| \bar{a}_p \|_{\left( H_p - a_p \bar{a}_p^T \right)^{-1}}^2 + 1
\]

\[
= \| a - \bar{a}_p \|_{\left( H_p - a_p \bar{a}_p^T \right)^{-1}}^2 + 1. \quad (25)
\]

By plugging this expression into Eq. (24), it follows that the optimality conditions for the problems (21) and (19) are identical. Any solution \( p^* \) to the problem (21) yields a solution to the problem (19), and vice-versa. \( \square \)

In light of Proposition 3, we work exclusively with the lifted problem going forward. Before describing our algorithm, it will be useful to introduce the following approximate version of the optimality condition in Eq. (4), which quantifies the quality of a candidate solution \( p \in \Delta(A) \).

**Definition 2.** For any action set \( A \), parameter \( \hat{\theta} \in \mathbb{R}^d \), and learning rate \( \gamma > 0 \), a distribution \( p \in \Delta(A) \) is called an \( \eta \)-rounding for \( \logdet\text{-barrier}(A, \theta, \gamma) \) if it satisfies

\[
\forall a \in A : \frac{1}{\gamma} \| \hat{a} \|_{\hat{H}_p}^2 \leq (1 + \eta) \left( \frac{\bar{d}}{\gamma} + \langle \hat{a} - \bar{a}_p, \hat{\theta} \rangle \right). \quad (26)
\]

The following lemma quantifies the loss in regret incurred by sampling from an \( \eta \)-rounding for the \( \logdet\text{-barrier} \) objective rather than an exact solution.

**Lemma 5.** Suppose that for all steps \( t \), we sample from an \( \eta \)-rounding for \( \logdet\text{-barrier}(A_t, \hat{\theta}_t, \gamma/(1 + \eta)) \) within Algorithm 2. Then the regret bound from Lemma 4 increases by at most a factor of \( 1 + 2\eta \).
Lemma 5 implies that to achieve the regret bound from Theorem 2 up to a factor of 2, it suffices to find a $1/2$-rounding.

**Proof.** We first prove an analogue of the inequality in Lemma 4. Let $\eta$ be fixed and abbreviate $\tilde{\eta} := \hat{\eta}$. Assume without loss of generality that $d = \dim(A)$. For an $\eta$-rounding $p$ that satisfies Eq. (26) with learning rate $\gamma' := \gamma/(1 + \eta)$, by the identity (25) the following inequalities are equivalent:

$$\frac{1}{\gamma'} \|\tilde{a}\|_{\tilde{H}_p}^2 \leq (1 + \eta) \left( \frac{d}{\gamma'} + \langle a - \tilde{a}, \tilde{\eta} \rangle \right)$$

$$\iff \frac{1 + \eta}{\gamma} \|\tilde{a}\|_{\tilde{P}_p}^2 \leq (1 + \eta) \left( \frac{d(1 + \eta)}{\gamma} + \langle a - \tilde{a}, \tilde{\eta} \rangle \right)$$

$$\iff \frac{1}{\gamma} \left( \|a - \tilde{a}_p\|_{(H_p - \tilde{a}_p \hat{a}_p)\tilde{P}_p} - 1 \right) \leq \frac{d(1 + \eta)}{\gamma} + \langle a - \tilde{a}, \hat{\eta} \rangle$$

$$\iff \langle \tilde{a} - a, \hat{\eta} \rangle + \frac{1}{\gamma} \|a - \tilde{a}_p\|_{(H_p - \tilde{a}_p \hat{a}_p)\tilde{P}_p}^2 - 1 \leq \frac{d}{\gamma} \left( 1 + \eta + \frac{\eta}{d} \right).$$

It follows that the bound from Lemma 4 increases by at most a factor of $(1 + \eta + \frac{\eta}{d}) < 1 + 2\eta$ if we use an $\eta$-rounding rather than an exact solution. 

**D.1 Algorithm**

**Preliminaries.** To keep notation compact, throughout this section we drop the learning rate parameter and work with the objective

$$G(p) := \langle \tilde{a}, \hat{\eta} \rangle - \log \det(H_p), \quad \text{and} \quad p^* \in \arg\min_{p \in \Delta(A)} G(p).$$

Note that this suffices to capture the case where $\gamma \neq 1$ (Eq. (20)), since we can multiply both terms by $\gamma^{-1}$ and absorb a gamma factor into $\hat{\eta}$. Consequently, for the remainder of the section we work under the assumption that $\|\theta\| \leq \gamma$ rather than $\|\theta\| \leq 1$. The definition of an $\eta$-rounding remains unaffected, since we can multiply both sides in Eq. (26) by $\gamma$.

**Additional notation.** For each $a \in A$, let $e_a \in \Delta(A)$ be the distribution that selects $a$ with probability 1. For distributions $p_1, p_2 \in \Delta(A)$, let $\text{conv}\{p_1, p_2\} = \{\lambda p_1 + (1 - \lambda)p_2 | \lambda \in [0, 1]\}$ be their convex hull. To improve readability, we abbreviate $\| \cdot \|_{\tilde{P}_p}$ to $\| \cdot \|_p$ in this section.

**Algorithm.** Our main algorithm is stated in Algorithm 6. The algorithm is a generalization of Khachiyan’s algorithm for optimal experimental design (Khachiyan and Todd, 1990). It maintains a finitely supported distribution over arms in $A$ and adds a single arm to the support at each step.

In more detail, the algorithm proceeds as follows. At step $k$, the algorithm checks whether the current iterate $p_{k-1}$ is an $\eta$-rounding. If this is the case, the algorithm simply terminates, as we are done. Otherwise, with $a^* := \arg\min_{a \in A} \langle a, \theta \rangle$, the algorithm first checks whether the current distribution satisfies $d + \langle a^* - \tilde{a}_{p_{k-1}}, \theta \rangle \geq 1$. If that condition is violated, we define a new distribution $p'_{k-1}$ by choosing the distribution in $\text{conv}\{p_{k-1}, e_{a^*}\}$ that minimizes $G(p)$. This ensures that $\frac{\partial}{\partial x}|_{x=0} G(p'_{k-1} + x(e_{a^*} - p_{k-1})) = 0$, i.e.

$$\langle a^*, \theta \rangle - \|a^*\|_{p_{k-1}}^2 = E_{a \sim p_{k-1}} \left[ \langle a, \theta \rangle - \|a\|_{p_{k-1}}^2 \right] = \langle \tilde{a}_{p_{k-1}}, \theta \rangle - d,$$

and hence $\min_{a \in A} d + \langle a - \tilde{a}_{p_{k-1}}, \theta \rangle = \|a^*\|_{p_{k-1}}^2 \geq 1$. In particular, this implies that

$$\eta_k := \max_{a \in A} \|\tilde{a}\|_{p_{k-1}}^2 / (d + \langle a - \tilde{a}_{p_{k-1}}, \theta \rangle).$$

(28)
is well defined. To conclude the iteration, the algorithm selects an action $a_k$ that attains the maximum in Eq. (28) and adds it to the support of $p_{k-1}'$, yielding $p_k$.

**Algorithm 6:** Frank-Wolfe for minimizing the logdet-barrier objective

| Function: p₀ ∈ Δ(A), A, θ, η |
|---|
| Input: $a^* = \arg\min_{a \in A}\langle a, \theta \rangle$, $k = 1$. |
| while $p_{k-1}$ is not an $η$-rounding (Eq. (26)) do |
| if $\tilde{d} + \langle a^* - \tilde{a}_{p_{k-1}}, \theta \rangle < 1$ then |
| Solve $p_{k-1}' = \arg\min_{p \in \text{conv}[p_{k-1}, e_{a^*}]} G(p)$. |
| else |
| $p_{k-1}' = p_{k-1}$. |
| Pick any $a_k \in \arg\max ||\tilde{a}||^2_{p_{k-1}'} / (\tilde{d} + \langle a - \tilde{a}_{p_{k-1}}, \theta \rangle)$ (ties broken arbitrarily). |
| Solve $p_k = \arg\min_{p \in \text{conv}[p_{k-1}', e_{a_k}]} G(p)$. |
| Increment $k$. |

**D.2 Analysis**

In this section, we prove a number of intermediate results used to bound the iteration complexity of Algorithm 6, culminating in our main convergence guarantee, Theorem 7. The total computational complexity is summarized at the end of the section in Appendix D.2.1.

We begin by relating the $η$-rounding property to the suboptimality gap for the objective $G(p)$.

**Lemma 6.** If $p \in Δ(A)$ is an $η$-rounding, then

$$G(p) - G(p^*) \leq \log(1 + η)\tilde{d}.$$  

**Proof of Lemma 6.** By the optimality conditions in Eqs. (22) to (24), we are guaranteed that

$$\forall a \in \text{supp}(p^*) : \tilde{d} + \langle a, \theta \rangle = ||\tilde{a}||^2_{\tilde{H}^{p^*}} + \langle \tilde{a}^*, \theta \rangle.$$  

Hence, combining this statement with the $η$-rounding condition for $p$, we have that

$$\forall a \in \text{supp}(p^*) : ||\tilde{a}||^2_{\tilde{H}^p} \leq (1 + η) \left( ||\tilde{a}||^2_{\tilde{H}^{p^*}} + \langle \tilde{a}^*, \tilde{a}_p \rangle \right).$$  

Taking the expectation over $a \sim p^*$ on both sides above and rearranging leads to

$$\langle \tilde{a}_p - \tilde{a}^*_p, \theta \rangle \leq \tilde{d} - \frac{\text{tr}(\tilde{H}_{p^*}^\dagger \tilde{H}^{-1}_{p^*})}{1 + η} = \tilde{d} - \frac{\text{tr}(\tilde{H}_{p^*}^\dagger \tilde{H}^{-1}_{p^*} \tilde{H}_{p^*}^\dagger)}{1 + η}.$$  

From the definition of $G(p)$, this implies that

$$G(p) - G(p^*) \leq \tilde{d} - \frac{\text{tr}(\tilde{H}_{p^*}^\dagger \tilde{H}^{-1}_{p^*} \tilde{H}_{p^*}^\dagger)}{1 + η} + \log \det(\tilde{H}_{p^*}^\dagger \tilde{H}^{-1}_{p^*} \tilde{H}_{p^*}^\dagger),$$  

where we recall that $\det(\tilde{H}_{p^*}^\dagger \tilde{H}^{-1}_{p^*} \tilde{H}_{p^*}^\dagger) = \det(\tilde{H}_{p^*} \tilde{H}^{-1}_{p^*}) > 0$, since $\tilde{H}_{p^*}, \tilde{H}_{p^*}$ are positive definite. Now, let $(\lambda_i)_{i=1,...,\tilde{d}}$ be the eigenvalues of $\tilde{H}_{p^*}^\dagger \tilde{H}^{-1}_{p^*} \tilde{H}_{p^*}^\dagger$. Then we have

$$G(p) - G(p^*) = \sum_{i=1}^{\tilde{d}} 1 - \frac{\lambda_i}{1 + η} + \log(\lambda_i) \leq \tilde{d} \max_{\lambda > 0} \left\{ 1 - \frac{\lambda}{1 + η} + \log(\lambda) \right\} = \tilde{d} \log(1 + η).$$  

Our next lemma lower bounds the rate at which the suboptimality gap improves at each iteration.
Lemma 7. In each iteration of Algorithm 6, the suboptimality gap improves by at least

$$G(p_{k-1}) - G(p_k) \geq \Omega \left( \min\{\eta_k, 1\}^2 / d \right),$$

(29)

where we recall that $$\eta_k := \|a_k\|^2_{p_{k-1}} / (\tilde{d} + (a_k - \tilde{a}_{p_{k-1}}))$$. Furthermore, if $$\eta_k \geq 2\tilde{d}$$, then it also holds that

$$G(p_k) - G(p^*) \leq \left( 1 - \frac{1}{2d} \right) (G(p_{k-1}) - G(p^*)).$$

(30)

Proof. We first prove that Eq. (29) holds. Let $$k$$ be fixed, and let $$\alpha \in [0, 1]$$ such that $$p_k = (1-\alpha)p_{k-1} + \alpha e_{a_k}$$. Then we have

$$G(p_k) = \langle \tilde{a}_{p_k}, \theta \rangle - \log \det \left( \tilde{H}_{p_k} \right)$$

$$= (1 - \alpha)\langle \tilde{a}_{p_{k-1}}, \theta \rangle + \alpha \langle \tilde{a}_k, \theta \rangle - \log \det \left( (1 - \alpha)\tilde{H}_{p_{k-1}} + \alpha \tilde{a}_k \tilde{a}_k^T \right)$$

$$= \langle \tilde{a}_{p_{k-1}}, \theta \rangle + \alpha \langle \tilde{a}_k - \tilde{a}_{p_{k-1}}, \theta \rangle - \log \left( \det \left( (1 - \alpha)\tilde{H}_{p_{k-1}} \right) \cdot \left( 1 + \frac{\alpha}{1 - \alpha} \|\tilde{a}_k\|^2_{p_{k-1}} \right) \right)$$

$$= G(p_{k-1}) + \alpha \langle \tilde{a}_k - \tilde{a}_{p_{k-1}}, \theta \rangle - (\tilde{d} - 1) \log(1 - \alpha) - \log \left( 1 + \alpha \|\tilde{a}_k\|^2_{p_{k-1}} \right),$$

where the third equality uses the matrix determinant lemma. Now, recall that by the definition of $$a_k$$, we have $$\|\tilde{a}_k\|^2_{p_{k-1}} = (1 + \eta_k)(\tilde{d} + (\tilde{a}_k - \tilde{a}_{p_{k-1}}, \theta))$$. Let us abbreviate $$Z_k := \|\tilde{a}_k\|^2_{p_{k-1}} \geq 1 + \eta_k$$. We proceed as

$$G(p_{k-1}) - G(p_k) \geq G(p_{k-1}) - G(p_k)$$

$$= \alpha \langle \tilde{a}_{p_{k-1}} - \tilde{a}_k, \theta \rangle + (\tilde{d} - 1) \log(1 - \alpha) + \log \left( 1 + \alpha \|\tilde{a}_k\|^2_{p_{k-1}} \right)$$

$$= \alpha \left( \tilde{d} - \frac{Z_k}{1 + \eta_k} \right) + (\tilde{d} - 1) \log(1 - \alpha) + \log \left( 1 + \alpha (Z_k - 1) \right)$$

$$= \max_{\alpha' \in [0, 1]} \left\{ \alpha' \left( \tilde{d} - \frac{Z_k}{1 + \eta_k} \right) + (\tilde{d} - 1) \log(1 - \alpha') + \log (1 + \alpha' (Z_k - 1)) \right\}. \quad (31)$$

where the last equality uses that $$\alpha$$ is chosen such that $$G(p_k)$$ is minimized. Next, recalling the elementary fact that for all $$x \geq -\frac{1}{2}$$, $$\log(1 + x) \geq x - x^2$$, we have in particular that

$$G(p_{k-1}) - G(p_k) \geq \max_{\alpha' \geq \frac{1}{2}} \left\{ \alpha' \left( \tilde{d} - \frac{Z_k}{1 + \eta_k} \right) + (\tilde{d} - 1)(-\alpha' - \alpha'^2) + \alpha' (Z_k - 1) - \alpha'^2 (Z_k - 1)^2 \right\}$$

$$= \max_{\alpha' \geq \frac{1}{2}} \left\{ \alpha' \frac{\eta_k Z_k}{1 + \eta_k} - \alpha'^2 (\tilde{d} - 1 + (Z_k - 1)^2) \right\}.$$ 

Note that $$\tilde{d} \geq 3$$ and $$\max_{x > 0} \frac{\eta_k Z_k}{1 + \eta_k} x (x - 1)^2 \leq 1$$, so if we choose

$$\alpha' = \frac{\eta_k Z_k}{2(1 + \eta_k) \left( \tilde{d} - 1 + (Z_k - 1)^2 \right)} \leq \frac{1}{2},$$

we get the lower bound

$$G(p_{k-1}) - G(p_k) \geq \frac{\eta_k^2 Z_k^4}{4(1 + \eta_k)^2 \left( \tilde{d} - 1 + (Z_k - 1)^2 \right)}.$$ 

The proof of Eq. (29) now follows by noting that $$\frac{x^2}{x + (x - 1)^2} \geq \frac{1}{4}$$ for all $$x \geq 1$$.

We now prove that the second part of the lemma, Eq. (30), holds. Suppose $$\eta_k > 2\tilde{d}$$. We return to Eq. (31) and this time select

$$\alpha' = \frac{\sqrt{\eta_k}}{Z_k - 1} \leq \frac{1}{\sqrt{\eta_k}} \leq \frac{1}{2}.$$ 

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Using the approximation $\log(1+x) \geq x - x^2$ only for the first term in \((31)\), we get

\[
G(p_{k-1}) - G(p_k) \geq \alpha' \left( \frac{d - Z_k}{1 + \eta_k} \right) - (d - 1)(\alpha' + \alpha'^2) + \log(1 + \alpha'(Z_k - 1))
\]

\[
\geq -\frac{\sqrt{\eta_k}}{1 + \eta_k} - \frac{d - 1}{\eta_k} + \log(1 + \sqrt{\eta_k})
\]

\[
= -\frac{\sqrt{\eta_k}}{1 + \eta_k} - \frac{d - 1}{\eta_k} + \log(1 + \sqrt{\eta_k}) - \frac{1}{4} \log(1 + \eta_k) + \frac{1}{4} \log(1 + \eta_k)
\]

\[
\geq -\frac{\sqrt{\eta_k}}{1 + \eta_k} - \frac{1}{2} + \frac{1}{\eta_k} + \log(1 + \sqrt{\eta_k}) - \frac{1}{4} \log(1 + \eta_k) + \frac{1}{4} \log(1 + \eta_k),
\]

where the last line uses that $\eta_k \geq 2d$. Now observe that for $x \geq 6$

\[
\frac{\partial}{\partial x} \left( -\frac{\sqrt{x}}{1 + x} + \frac{1}{x} + \log(1 + \sqrt{x}) - \frac{1}{4} \log(1 + x) \right)
\]

\[
= \frac{1}{2\sqrt{x}(1+x)^2} - \frac{1}{x^2} + \frac{1}{2(\sqrt{x} + x)} - \frac{1}{4(1+x)}
\]

\[
= \frac{x^2 + 5x^2 + 7x^2 - 12x^2 - 8x - 4x^2 - 4}{4x^2(1 + \sqrt{x})(1 + x)^2}
\]

\[
\geq \frac{7x^2 + 60x^2 - 12x^2 - 8x - 4x^2 - 4}{4x^2(1 + \sqrt{x})(1 + x)^2} \geq 0.
\]

Hence

\[
-\frac{\sqrt{\eta_k}}{1 + \eta_k} - \frac{1}{2} + \frac{1}{\eta_k} + \log(1 + \sqrt{\eta_k}) - \frac{1}{4} \log(1 + \eta_k)
\]

\[
\geq -\frac{\sqrt{6}}{1 + 6} - \frac{1}{2} + \frac{1}{6} + \log(1 + \sqrt{6}) - \frac{1}{4} \log(1 + 6) > 0.
\]

It follows that

\[
G(p_{k-1}) - G(p_k) \geq \frac{1}{4} \log(1 + \eta_k).
\]

\[\square\]

The next lemma ensures we can efficiently find a good initial distribution $p_0$.

**Lemma 8 (Kumar and Yildirim (2005), Lemma 3.1).** There exists an algorithm that terminates in $\mathcal{O}(|\mathcal{A}|d^2)$ time and finds a distribution $p_0 \in \Delta(\mathcal{A})$ with $|\text{supp}(p_0)| \leq 2d$ such that

\[-\log \det(\mathcal{H}_{p_0}) + \min_{p \in \Delta(\mathcal{A})} \log \det(\mathcal{H}_p) = \mathcal{O}(d \log(d)).\]

The memory required by the algorithm is at most $\mathcal{O}(d^2 + \log(|\mathcal{A}|d))$.

**Corollary 3.** The distribution $p_0$ described in Lemma 8 has initial suboptimality gap at most

\[G(p_0) - G(p^*) = \mathcal{O}(d \log(d) + \gamma).\]

**Proof.** Recall that

\[G(p_0) - G(p^*) = \langle \tilde{a}_{p_0} - \bar{a}_{p^*}, \theta \rangle - \log \det(\mathcal{H}_{p_0}) + \log \det(\mathcal{H}_{p^*}).\]

The difference between the log-det terms is bounded by $\mathcal{O}(d \log(d))$ using Lemma 8, while the difference between the linear terms is bounded by

\[\langle \tilde{a}_{p_0} - \bar{a}_{p^*}, \theta \rangle \leq \|\tilde{a}_{p_0} - \bar{a}_{p^*}\| \cdot \|\theta\| \leq 2\gamma.\]

\[\square\]
Theorem 7. If Algorithm 6 is initialized using the distribution from Lemma 8, then it requires $O(d \log(d) + \log(\gamma))$ iterations to reach a $2^d$-rounding. Moreover,

- After reaching the $2d$-rounding above, the algorithm requires $O((\log(d) d^2)$ additional iterations to reach a 1-rounding.
- After reaching a 1-rounding, the algorithm requires $O(d^2/\eta)$ additional iterations to reach an $\eta$-rounding for any $\eta < 1$.

Altogether, for any $\eta > 0$, Algorithm 6—when initialized using Lemma 8—requires

$$O(d \log(\gamma) + d^2(\log(d) + 1/\eta))$$

total steps to reach an $\eta$-rounding.

Proof. By Corollary 3, the initial distribution $p_0$ satisfies

$$G_0 := G(p_0) - G(p^*) = O(d \log(d) + \gamma).$$

We first bound the number of steps required to reach a $2d$-rounding. Let $k_0$ denote the first step $k$ in which $p_k$ is a $2d$-rounding. Then every $k < k_0$ has $\eta_k > 2d$, so in light of Lemma 7, all such $k$ have

$$G(p_k) - G(p_0) \leq \left(1 - \frac{1}{2d}\right)(G(p_{k-1}) - G(p_0))$$

and

$$G(p_k) \leq G(p_{k-1}) - \Omega(1/d).$$

It follows that as long as $\eta_k > 2d$, the suboptimality gap will reach 1 in most $O(d \log(G_0)) = O(d \log(d) + \log(\gamma))$ iterations. Moreover, since the absolute decrease in function value is at least $\Omega(1/d)$, the gap would reach zero after another $O(d)$ iterations of this type. We conclude that after $O(d(\log(d) + \log(\gamma)))$ iterations, the algorithm must find a $2d$-rounding.

We now bound the number of steps to reach a 1-rounding from the first step where we have a $2d$-rounding. By Lemma 6, the suboptimality gap of any $2d$-rounding is at most $O(d \log(d))$. Moreover, as long as we haven’t reached a 1-rounding, Lemma 7 guarantees that the suboptimality gap will decrease by $\Omega(1/d)$ per step. Hence, we must reach a 1-rounding within $O(d^2 \log(d))$ iterations.

Finally we bound the number of steps required to reach an $\eta$-rounding for any $\eta < 1$, starting from the first iteration where we reach a 1-rounding. We adapt an argument of Kumar and Yildirim (2005). Given an $\eta_k$-rounding for $\eta_k \leq 1$, we need $O(d^2/\eta_k)$ iterations to reach an $(\eta_k/2)$-rounding. This follows from the same argument as above: the suboptimality gap is at most $O(d \eta_k)$ by Lemma 6 (using that $\log(1 + \eta_k) \leq \eta_k$) and we reduce it by $\Omega(\eta_k^2/d)$ as long as we have not found an $(\eta_k/2)$-rounding (by Lemma 7). Summing up the required number of iterations to get from precision $1$ to $1/2$ to $1/4$ to $\ldots$ to $1/2^{\lceil \log(1/\eta) \rceil}$ shows that $O(d^2/\eta)$ total iterations suffice.

D.2.1 Total Computational Complexity

The computational complexity per iteration for our method is comparable to similar algorithms for the D-optimal design problem, which we recall is the case where $\theta = 0$ (Khachiyan and Todd, 1990; Kumar and Yildirim, 2005; Todd and Yildirim, 2007). We walk calculate the complexity step-by-step for completeness, and to handle differences arising from our generalization to the $\theta \neq 0$ case. The first difference is that our method solves an intermediate optimization problem over the line $\text{conv}(p_{k-1}, e_{a^*})$. This step increases the computational complexity by a factor of 2. At each iteration, Algorithm 6 computes

$$\arg\max_{a \in A} \frac{||\hat{a}||^2_{p_{k-1}}}{d + \langle a - \hat{a}_{p_{k-1}}', \theta \rangle}.$$
For generic action sets, this can be computed in time $O(|A|d^2)$, given that $\tilde{H}^{-1}_{\mathcal{P}_{k-1}}$ has already been computed. In the next step, the algorithm solves the one dimensional optimization problem

$$\max_{\alpha' \in [0, 1]} \left( \alpha' \left( \tilde{d} - \frac{Z_k}{1 + \eta_k} \right) + (\tilde{d} - 1) \log(1 - \alpha') + \log (1 + \alpha' (Z_k - 1)) \right),$$

where $Z_k = ||\tilde{a}_k||^2_{\mathcal{P}_{k-1}}$. This can be done in time $O(1)$, since it is equivalent to solving the quadratic problem

$$\left( \tilde{d} - \frac{Z_k}{1 + \eta_k} \right) - \frac{\tilde{d} - 1}{1 - x} + \frac{Z_k - 1}{1 + x (Z_k - 1)} = 0.$$

Finally we need to update $\bar{a}_p$, which costs $O(d)$, and update $\tilde{H}^{-1}_{\mathcal{P}}$, which can be done in time $O(d^2)$ using a rank-one update.

Across all iterations, we require a total of $\tilde{O}(d^4|A|)$ arithmetic operations, with $p_k$ never exceeding support size $O(d^2 \log(d) + d \log(\gamma))$, since we add at most one arm to the support in any iteration. We can store $p_k$ as a sparse vector of key and value pairs, where each entry uses memory complexity of $O(\log(|A|))$ to represent the key.