Antimagic orientations of disconnected even regular graphs

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A labeling of a digraph $D$ with $m$ arcs is a bijection from the set of arcs of $D$ to \{1, 2, \ldots, m\}. A labeling of $D$ is antimagic if no two vertices in $D$ have the same vertex-sum, where the vertex-sum of a vertex $u \in V(D)$ for a labeling is the sum of labels of all arcs entering $u$ minus the sum of labels of all arcs leaving $u$. An antimagic orientation $D$ of a graph $G$ is antimagic if $D$ has an antimagic labeling. Hefetz, Mütze and Schwartz in [J. Graph Theory 64(2010)219-232] raised the question: Does every graph admit an antimagic orientation? It had been proved that for any integer $d$, every $2d$-regular graph with at most two odd components has an antimagic orientation. In this paper, we consider the $2d$-regular graph with many odd components. We show that every $2d$-regular graph with any odd components has an antimagic orientation provided each odd component has enough order.

Keywords: Regular graph; Antimagic labeling; Antimagic orientation.

1 Introduction

All graphs in this paper are finite and simple. For a graph $G$, let $|G|$ denote the number of vertices of $G$. For a path $P$, let $|P|$ denote the length of $P$. For an orientation $D$ of graph $G$, $D$ is a digraph, we use $A(D)$ and $V(D)$ to denote the set of arcs and vertices of $D$, respectively. We define $[i, j] := \{i, i+1, \ldots, j\}$, for any two positive integers $i$ and $j$. A labeling of $D$ with $m$ arcs is a bijection from $A(D)$ to $[1, m]$. A labeling of $D$ is antimagic if no two vertices in $D$ have the same vertex-sum, where the vertex-sum of a vertex $u \in V(D)$ for a labeling is the sum of labels of all arcs entering $u$ minus the sum of labels of all arcs leaving $u$. An antimagic orientation $D$ of $G$ is antimagic if $D$ has an antimagic labeling. A graph $G$ has an antimagic orientation if an antimagic orientation of $G$ is antimagic. Let $D$ be an orientation of a graph $G$ with $m$ edges. For any labeling $c : A(D) \rightarrow [1, m]$ of $D$ and any vertex $u \in V(D)$, we use $s_D(u)$ to denote the vertex-sum of $u$ for the labeling $c$ of orientation $D$.

Hefetz, Mütze and Schwartz [5] raised the question: Does every graph admits an antimagic orientation? For this question and any integer $d \geq 1$, they proved the following solutions: (a) every $(2d - 1)$-regular graph admits an antimagic orientation; (b) every connected $2d$-regular graph $G$ admits an antimagic orientation if $G$ has a matching covers all but at most one vertex of $G$. Alon et al [1] obtained that the dense graphs are antimagic. Cranston [3] proved that regular bipartite graphs are antimagic. Chang et al. [2] discussed
the antimagic labeling of regular graphs. Cranston et al. [1] proved that regular graphs of odd degree are antimagic. Recently, Shan et al. [2] support this conjecture by proving that every biregular bipartite graph admits an antimagic orientation. Li et al. [3] proved that every connected 2d-regular graph admits an antimagic orientation. Let $G$ be a 2d-regular graph, where $d \geq 2$ is an integer. The result that $G$ admits an antimagic orientation if $G$ has at most two odd components is proved in [6].

It remained unknown whether every disconnected 2d-regular graph admits an antimagic orientation. In this paper, first we find an orientation of the disconnected 2d-regular graph, then we find a labeling based on this orientation and finally we show that this labeling is antimagic provide each component with enough order. The main results of this paper are Theorem 1 and Theorem 2.

**Theorem 1.** For any integer $d \geq 2$, let $G$ be a 2d-regular graph with components $G_1, G_2, \ldots, G_q$, where $G_1, G_2, \ldots, G_k$ are odd components such that $|G_1| \leq |G_2| \leq \ldots \leq |G_k|$. Then the following results hold.

1. For $k \in [0, 5d + 4]$, $G$ admits an antimagic orientation.
2. For $k \geq 5d + 5$, if $|G_1| \geq 2x_0 + 5$, then $G$ admits an antimagic orientation. Where $x_0$ is the unique positive integer solution for one of equations: $k = (2d - 2)(x + 2) + 0$, $k = (2d - 2)(x + 2) + 1, \ldots$, $k = (2d - 2)(x + 2) + (d + 8), k = (2d - 2)(x + 1) + (d + 9)$, $k = (2d - 2)(x + 1) + (d + 10), \ldots, k = (2d - 2)(x + 1) + (2d - 3)$.

From Theorem 1 the following result is derived directly.

**Theorem 2.** For any integer $d \geq 2$, let $G$ be a 2d-regular graph with $q$ components. If each odd component of $G$ has enough order, then $G$ admits an antimagic orientation.

## 2 Proof of Theorem 1

A closed walk in a graph is an Euler tour if it traverses every edge of the graph exactly once. There is a Euler theorem that a connected graph admits an Euler tour if and only if every vertex has even degree.

In this section, we study the antimagic orientations of disconnected 2d-regular graphs for any integer $d \geq 2$. We will prove Theorem 1 by the following process: Firstly, we find an orientation $D^*$ of the given graph $G$. Secondly, we label the edges of $G$ by three algorithms. Thirdly, we show that the orientation $D^*$ is antimagic by proving the labeling of $D^*$ which is given by the algorithm is antimagic.

Since $G_i$ is an 2d-regular connected graph with $d \geq 2$ and $i \in [1, q]$, $G_i$ is an Euler graph. Let $C_i^*$ be an Euler tour of $G_i$. For each vertex $u \in V(G_i)$, $C_i^*$ should pass through each vertex $d$ times. Let $u \in V(G_i)$. Pick a fixed one of the $d$ copies of $u$ on $C_i^*$ as a real vertex and the remaining $d - 1$ copies of $u$ as imaginary vertices. Then regarding $C_i^*$ as a circuit, say $C_i$, with $|G_i|$ real vertices. For any $i \in [1, q]$, we may assume that $|G_i| = t_i$. Let $n_i = \sum_{j=1}^{t_i} |E(C_i)|$. Then $|E(C_i)| = n_i - n_{i-1}$. Let $V_R = \{v_{i,1}, v_{i,2}, \ldots, v_{i,t_i}\}$ for $i \in [1, q]$. Then $V_R = \bigcup_{i=1}^{q} V_R^i$ is the set of real vertices of $\bigcup_{i=1}^{q} C_i$, where $v_{i,j}$ denote the $j$th real vertex of $C_i$. Let $V_I = V(\bigcup_{i=1}^{q} C_i) \setminus V_R$ be the set of imaginary vertices of $\bigcup_{i=1}^{q} C_i$. By renaming the vertices in $V_R$ if necessary, we label the vertices of $V_R$ on $C_i$ with $v_{i,1}, v_{i,2}, v_{i,4}, \ldots, v_{i,t_i-1}, v_{i,t_i}, v_{i,t_i-2}, \ldots, v_{i,5}, v_{i,3}$ in clockwise, if $i \in [1, k]$ as depicted in Figure 1 and $v_{i,1}, v_{i,2}, v_{i,4}, \ldots, v_{i,t_i-1}, v_{i,t_i-1}, v_{i,t_i-3}, \ldots, v_{i,5}, v_{i,3}$ in clockwise if $i \in [k+1, q]$ as depicted in Figure 2.
Let $P_{j,k}^i(j < k)$ be the path between $v_{i,j}$ and $v_{i,k}$ on $C_i$ such that all internal vertices of $P_{j,k}^i$ are not real vertices. Next, we will find an orientation $D$ of $\bigcup_{i=1}^q C_i$.

When $i \in [1, k]$, set $d_D^+(v_{i,1}) = 1$, $d_D^+(v_{i,j}) \in \{0, 2\}$ for any $j \in \{2, 3, \ldots, t_i\}$, and $d_D^+(u_j) = 1$ for each $u_j \in V_I \cap V(C_i)$ by orienting the path $P_{1,2}^i$ from $v_{i,1}$ to $v_{i,2}$, the path $P_{2,4}^i$ from $v_{i,4}$ to $v_{i,2}$, ..., $P_{3,5}^i$ from $v_{i,5}$ to $v_{i,3}$ and $P_{1,3}^i$ from $v_{i,1}$ to $v_{i,3}$ (see Figure 1). When $i \in [k + 1, q]$, set $d_D^+(v_{i,j}) \in \{0, 2\}$ for any $j \in [1, t_i]$, and $d_D^+(u_j) = 1$ for each $u_j \in V_I \cap V(C_i)$ by orienting the path $P_{1,2}^i$ from $v_{i,1}$ to $v_{i,2}$, the path $P_{2,4}^i$ from $v_{i,4}$ to $v_{i,2}$, ..., $P_{3,5}^i$ from $v_{i,5}$ to $v_{i,3}$ and $P_{1,3}^i$ from $v_{i,1}$ to $v_{i,3}$ (see Figure 2).

![Figure 1: The orientation of an odd cycle](image1)

![Figure 2: The orientation of an even cycle](image2)
Algorithm 1 Label the edges of $C_{k+1}, \ldots, C_q$

Data: Even cycles $C_i$ for $i \in [k+1, q]$ with the given orientation $D$

Result: A bijection $c_e : A(C_{k+1} \cup \ldots \cup C_q) \rightarrow [n_k + 1, n_q]$

for $i = k + 1$ to $q$ do

Assign the numbers in $[n_{i-1} + 1, n_{i-1} + |P_{1,2}^i|]$ to the edges of $P_{1,2}^i$ in the increasing order along the orientation of $P_{1,2}^i$;

Assign the numbers in $[n_{i-1} + |P_{1,2}^i| + 1, n_{i-1} + |P_{1,3}^i|]$ to the edges of $P_{1,3}^i$ in the increasing order along the orientation of $P_{1,3}^i$;

Set $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$;

while $V_i \neq \{v_{i,1}, v_{i,2}, \ldots, v_{i,t_i}\}$ do

for $j = 2$ to $t_i - 2$ do

Assign the numbers in $[n_{i-1} + |P_{1,2}^i| + |P_{1,3}^i| + \ldots + |P_{j-1, t_i}^i| + 1, n_{i-1} + |P_{1,2}^i| + |P_{1,3}^i| + \ldots + |P_{j-1, t_i}^i|]$ to the edges of $P_{j,j+2}^i$ in the increasing order along the orientation of $P_{j,j+2}^i$;

Set $V_i$ to be $V_i \cup \{v_{i,j+2}\}$;

end for

Assign the numbers in $[n_{i-1} + |P_{1,2}^i| + |P_{1,3}^i| + \ldots + |P_{t_i-2, t_i}^i| + 1, n_{i-1} + |P_{1,2}^i| + |P_{1,3}^i| + \ldots + |P_{t_i-2, t_i}^i|]$ to the edges of $P_{t_i-1, t_i}^i$ in the increasing order along the orientation of $P_{t_i-1, t_i}^i$;

end while

end for

where $c = c_1 \cup c_2 \cup \ldots \cup c_k \cup c_e$ if $k \leq 9$ and $c = c_1 \cup c_2 \cup \ldots \cup c_9 \cup c' \cup c_e$ for $k \geq 10$.

If $k = 0$, we have $c = c_e$. If $k = 1$, we define the bijection $c_1 : A(C_1) \rightarrow [1, n_1]$ as stated in Algorithm 2.

By Algorithm 2, observe that the edges are labeled in $C_1$ in the order of $P_{1,2}^1, P_{1,3}^1, P_{2,4}^1, P_{3,5}^1, P_{4,6}^1, \ldots, P_{t_1-2, t_1}^1, P_{t_1-1, t_1}^1$ by using the numbers in $[1, n_1]$ with the increasingly order along the orientation of each path. If $k \in [2, 9]$, we modify the label order of some fixed paths based on the Algorithm 2 to define the bijections $c_2, \ldots, c_9$. That is, when $k = 2$, we label the edges in $C_2$ in the order of $P_{2,3}^2, P_{2,4}^2, P_{3,5}^2, P_{4,6}^2, \ldots, P_{t_2-2, t_2}^2, P_{t_2-1, t_2}^2$ by using the numbers in $[n_1 + 1, n_2]$ with the increasingly order along the orientation of each path; when $3 \leq k \leq 8$, for every $i \in [3, k]$, we label the edges in $C_i$ in the order of $P_{3,5}^i, P_{2,4}^i, P_{1,3}^i, P_{3,5}^i, P_{4,6}^i, \ldots, P_{t_i-2, t_i}^i, P_{t_i-1, t_i}^i$ by using the numbers in $[n_{i-1} + 1, n_i]$ with the increasingly order along the orientation of each path; when $k = 9$, we label the edges in $C_9$ in the order of $P_{9,10}^9, P_{9,10}^9, P_{9,10}^9, P_{9,10}^9, P_{9,10}^9, \ldots, P_{t_9-2, t_9}^9, P_{t_9-1, t_9}^9$ by using the numbers in $[n_8 + 1, n_9]$ with the increasing order along the orientation of each path. If $k \geq 10$, we define the bijection $c'$ such that $c' : A(C_{10} \cup \ldots \cup C_k) \rightarrow [n_9 + 1, n_k]$ is the same as stated in Algorithm 3.

It remains to verify that the bijection $c$ is an antimagic labeling of $D^*$, where $c = c_1 \cup c_2 \cup \ldots \cup c_k \cup c_e$ if $k \leq 9$ and $c = c_1 \cup c_2 \cup \ldots \cup c_9 \cup c' \cup c_e$ for $k \geq 10$.

Since $C_i$ corresponds to $C_i'$, and $C_i'$ can be reordered if necessary, so $C_i$ can satisfy the following conditions according to the different values of $k$, respectively.

1. For $k \in [0, 5d + 4]$, if $k \in [3, 6]$, let $|P_{2,4}^i| = i - 2$ for $i \in [3, k]$; If $k = 7$ or $k = 8$, let $|P_{2,4}^i| = i - 2$ for $i \in [3, k]$ and $|P_{2,4}^i| \geq 3$; If $k = 9$, let $|P_{2,4}^i| = i - 2$ for $i \in [3, k]$, $|P_{3,5}^i| = 1$ and $|P_{2,4}^i| \geq 4$; If $k \in [10, 5d + 4]$, let $|P_{1,2}^i| = |P_{1,3}^i| = |P_{3,5}^i| = |P_{1,3}^i| = 1$ for $i \in [10, k]$, respectively.
Algorithm 2 Label the edges of $C_1$

**Data:** The odd cycle $C_1$ with the given orientation $D$

**Result:** A bijection $c_1 : A(C_1) \rightarrow [1, n_1]$.

1. Assign the numbers in $[1, \lvert P_{1,2}^1 \rvert]$ to the edges of $P_{1,2}^1$ in the increasing order along the orientation of $P_{1,2}^1$;
2. Assign the numbers in $[\lvert P_{1,2}^1 \rvert + 1, \lvert P_{1,2}^1 \rvert + \lvert P_{1,3}^1 \rvert]$ to the edges of $P_{1,3}^1$ in the increasing order along the orientation of $P_{1,3}^1$;
3. Set $V_1 = \{v_{1,1}, v_{1,2}, v_{1,3}\}$;
4. While $V_1 \neq \{v_{1,1}, v_{1,2}, \ldots, v_{1,t_1}\}$ do
   a. For $j = 2$ to $t_1 - 2$ do
      i. Assign the numbers in $[\lvert P_{1,2}^1 \rvert + \lvert P_{1,3}^1 \rvert + \ldots + \lvert P_{j-1,j+1}^1 \rvert + 1, \lvert P_{1,2}^1 \rvert + \lvert P_{1,3}^1 \rvert + \ldots + \lvert P_{j,j+2}^1 \rvert]$ to the edges of $P_{j,j+2}^1$ in the increasing order along the orientation of $P_{j,j+2}^1$;
      ii. Set $V_1$ to be $V_1 \cup \{v_{1,j+2}\}$;
   b. Assign the numbers in $[\lvert P_{1,2}^1 \rvert + \lvert P_{1,3}^1 \rvert + \ldots + \lvert P_{t_1-2,t_1}^1 \rvert + 1, \lvert P_{1,2}^1 \rvert + \lvert P_{1,3}^1 \rvert + \ldots + \lvert P_{t_1-1,t_1}^1 \rvert]$ to the edges of $P_{t_1-1,t_1}^1$ in the increasing order along the orientation of $P_{t_1-1,t_1}^1$;
5. End while

Algorithm 3 For $k \geq 10$, label the edges of $C_{10}, \ldots, C_k$

**Data:** Odd cycles $C_i$ for $i \in [10, k]$ with the given orientation $D$

**Result:** A bijection $c' : A(C_{10} \cup \ldots \cup C_k) \rightarrow [n_9 + 1, n_k]$.

1. For $i = 10$ to $k$ do
   a. Assign the numbers in $[n_{i-1} + 1, n_{i-1} + \lvert P_{1,2}^i \rvert]$ to the edges of $P_{1,2}^i$ in the increasing order along the orientation of $P_{1,2}^i$;
   b. Assign the numbers in $[n_{i-1} + \lvert P_{1,2}^i \rvert + 1, n_{i-1} + \lvert P_{1,2}^i \rvert + \lvert P_{1,3}^i \rvert]$ to the edges of $P_{1,3}^i$ in the increasing order along the orientation of $P_{1,3}^i$;
   c. Set $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$;
   d. While $V_i \neq \{v_{i,1}, v_{i,2}, \ldots, v_{i,t_i}\}$ do
      i. For $j = 2$ to $t_i - 2$ do
         a. Assign the numbers in $[n_{i-1} + \lvert P_{1,2}^i \rvert + \lvert P_{1,3}^i \rvert + \ldots + \lvert P_{j-1,j+1}^i \rvert + 1, n_{i-1} + \lvert P_{1,2}^i \rvert + \lvert P_{1,3}^i \rvert + \ldots + \lvert P_{j,j+2}^i \rvert]$ to the edges of $P_{j,j+2}^i$ in the increasing order along the orientation of $P_{j,j+2}^i$;
         b. Set $V_i$ to be $V_i \cup \{v_{i,j+2}\}$;
   e. Assign the numbers in $[n_{i-1} + \lvert P_{1,2}^i \rvert + \lvert P_{1,3}^i \rvert + \ldots + \lvert P_{t_i-2,t_i}^i \rvert + 1, n_{i-1} + \lvert P_{1,2}^i \rvert + \lvert P_{1,3}^i \rvert + \ldots + \lvert P_{t_i-1,t_i}^i \rvert]$ to the edges of $P_{t_i-1,t_i}^i$ in the increasing order along the orientation of $P_{t_i-1,t_i}^i$;
   f. End while
2. End for
$|P_{2,4}^i| = i - 2$ for $i \in [3,9]$, $|P_{1,2}^i| = i - 8$ for $i \in [10,k]$ and $|P_{2,4}^1| \geq 5d - 6$.

(2) For $k \geq 5d + 5$, based on the orientation and $|G_1| \geq 2x_0 + 5$, we can let $|P_{1,2}^i| = |P_{0,2}^i| = |P_{1,3}^i| = 1$ for $i \in [10,k]$, $|P_{2,4}^i| = i - 2$ for $i \in [3,9]$, $|P_{1,2}^i| = i - 8$ for $i \in [10,k]$ and $|P_{2,4}^1| \geq (2d - 2)x_0 + 5d - 6$.

The definition $V_R$ and $V_I$ and the labeling method of even cycles are the same as the method in [4].

**Claim 1.** If $D$ is antimagic, then $D^*$ is antimagic.

**Proof of Claim 1.** By the three algorithms, $s_D(u_i) = -1$ for all $u_i \in V_I$. We may assume that $V(G) = V_R$. For each $v \in V(G)$, $s_{D^*}(v) = s_D(v) + (d - 1)s_D(u^*) = s_D(v) - (d - 1)$, where $u^*$ is one of the $d - 1$ imaginary vertices of $v$. Therefore, if $D$ is antimagic, for any $u,v \in V_R$ with $u \neq v$, $s_D(u) \neq s_D(v)$. Then, for any $u,v \in V(G)$ with $u \neq v$, one has that $s_{D^*}(u) \neq s_{D^*}(v)$. That is, $D^*$ is antimagic.

By Claim 1, it suffices to show that for any $u,v \in V_R$ with $u \neq v$, $s_D(u) \neq s_D(v)$.

**Claim 2.** For any $u,v \in V(C_{k+1} \cup \ldots \cup C_q) \cap V_R$ with $u \neq v$, one has that $s_D(u) \neq s_D(v)$.

**Proof of Claim 2.** By the orientation of even cycle, observe that the two edges incident with each real vertex be either both entering the vertex or both leaving the vertex. Choose two different real vertices $u$ and $v$ from all even cycles. Clearly, if $d_D^-(u) = 0$ and $d_D^+(v) = 2$ or $d_D^+(u) = 0$ and $d_D^-(v) = 2$, we have $s_D(u) \cdot s_D(v) < 0$. If $d_D^-(u) = d_D^+(v) = 0$ or $d_D^+(u) = d_D^-(v) = 2$, by Algorithm 1, the labels of two edges incident with one vertex must be strictly less than the labels of two edges incident with the other vertex, respectively. One has that $|s_D(u)| < |s_D(v)|$ or $|s_D(v)| < |s_D(u)|$. Thus, $s_D(u) \neq s_D(v)$.

**Claim 3.** For any $u \in V(C_1 \cup \ldots \cup C_k) \cap V_R$ and $v \in V(C_{k+1} \cup \ldots \cup C_q) \cap V_R$, one has that $s_D(u) \neq s_D(v)$.

**Proof of Claim 3.** By the definition of the labeling $c$, $A(C_1 \cup \ldots \cup C_k) \rightarrow [1,n_k]$ and $A(C_{k+1} \cup \ldots \cup C_q) \rightarrow [n_k + 1,n_q]$. The labels of two edges incident with $u$ must be strictly less than the labels of two edges incident with $v$, respectively. If $u$ is in an odd cycle $C_i$ but $u$ is not the first vertex $v_{i,1}$ of $C_i$, it is clearly that $s_D(u) \neq s_D(v)$. If $u = v_{i,1}$ in $C_i$ for $i \in [1,k]$, assume that the labels of the edges entering $u$ is $a$ and leaving $u$ is $b$. Assume that the labels of two edges incident with $v$ are $c$ and $d$ respectively. If $s_D(u) \cdot s_D(v) < 0$, then $s_D(u) \neq s_D(v)$. If $s_D(u) = a - b < 0$ and $s_D(v) = c + d > 0$, then $a - b < a + b < c + d$, so $s_D(u) \neq s_D(v)$. Otherwise, $s_D(u) = a - b < 0$ and $s_D(v) = -(c + d) < 0$, one has that $-(c + d) < -(a + b) < a - b$; it also implies that $s_D(u) \neq s_D(v)$.

By Claims 2 and 3, it suffices to prove Claim 4.

**Claim 4.** For any $u$ and $v \in V(C_1 \cup \ldots \cup C_k) \cap V_R$ with $u \neq v$, one has that $s_D(u) \neq s_D(v)$.

**Proof of Claim 4.** The proof will be given by the induction on the number of odd cycles $k \geq 2$.

If $k = 2$, for any $u$ and $v \in V(C_1 \cup C_2) \cap V_R$, one has that $s_D(u) \neq s_D(v)$, which has been proved in [4], as the labeling method of $V(C_1 \cup C_2)$ is the same as the method in [4].

Assume that the number of odd cycles is not more than $k - 1$, the result is true. Now suppose that there are $k$ odd cycles with $k \geq 3$. 


If \( u \) and \( v \in V(C_1 \cup \ldots \cup C_{k-1}) \cap V_R \), since \( D \setminus C_k \) contains \( k-1 \) odd cycles, so \( s_{D \setminus C_k}(u) \neq s_{D \setminus C_k}(v) \) by the inductive hypothesis. Since \( s_{D \setminus C_k}(u) = s_D(u) \) and \( s_{D \setminus C_k}(v) = s_D(v) \), so \( s_D(u) \neq s_D(v) \). Thus we need to consider only the following two cases.

Case 1. \( u, v \in V(C_k) \cap V_R \). Without loss of generality, let \( u = v_{k,\ell} \) and \( v = v_{k,t} \) and \( t < \ell \).

If \( s_D(u) \cdot s_D(v) < 0 \), one has that \( s_D(u) \neq s_D(v) \). So we only need to consider \( s_D(u) \cdot s_D(v) > 0 \).

By the labeling of \( C_k \), the labels of two edges incident with \( u \) must be strictly less than the labels of two edges incident with \( v \), respectively. If \( s_D(u) \) and \( s_D(v) \) are both positive, then \( s_D(u) < s_D(v) \); if \( s_D(u) \) and \( s_D(v) \) are both negative, then \( s_D(u) > s_D(v) \). Thus, \( s_D(u) \neq s_D(v) \).

Case 2. \( u \in V(C_1 \cup \ldots \cup C_{k-1}) \cap V_R \) and \( v \in V(C_k) \cap V_R \).

Let \( X = \{v_{1,1}, v_{2,1}, \ldots, v_{k-1,1}\} \). There are the following three subcases.

Subcase 2.1. \( u \in X \) and \( v = v_{k,1} \).

Since \( s_D(v_{1,1}) = 1 \), \( s_D(v_{j,1}) = -(j-1) \) for \( j \in [2,9] \), and \( s_D(v_{j,1}) = j-8 \) for \( j \in [10,k] \), we have \( s_D(u) \neq s_D(v) \).

Subcase 2.2. \( u \in V(C_1 \cup \ldots \cup C_{k-1}) \cap V_R \) and \( v \in V(C_k) \cap V_R \setminus \{v_{k,1}\} \).

By the definition of \( c \), \( A(C_1 \cup \ldots \cup C_{k-1}) \rightarrow [1,n_{k-1}] \) and \( A(C_k) \rightarrow [n_{k-1}+1,n_k] \). Using the similar arguments in the proof of Claim 3, we can easily show that \( s_D(u) \neq s_D(v) \).

Subcase 2.3. \( u \in V(C_1 \cup \ldots \cup C_{k-1}) \cap V_R \setminus X \) and \( v = v_{k,1} \).

If \( k \in [3,9] \), then \( s_D(v_{k,1}) = -(k-1) < 0 \). If \( s_D(u) > 0 \), we have \( s_D(u) \neq s_D(v) \).

Let us consider the set, say \( W_1 \), of all vertices \( w \in V(C_1 \cup \ldots \cup C_{k-1}) \cap V_R \setminus X \) such that \( s_D(w) < 0 \). Note that among all the vertices of \( W_1 \), the labels of two edges incident with \( v_{1,3} \) are minimum. By the conditions, one has that \( |s_D(v_{1,3})| = (|P_{1,2}^1|+1)+(|P_{1,2}^1|+|P_{1,3}^1|)+|P_{2,4}^1|+1) = |P_{2,4}^1|+5 \geq k-1 = s_D(v_{k,1}) \). That is, \( s_D(w) > s_D(v_{1,3}) > s_D(v_{k,1}) \) for all \( w \in W_1 \). Thus, if \( s_D(u) < 0 \), one still has that \( s_D(u) \neq s_D(v) \).

If \( k \in [10,5d+4] \), then \( s_D(v_{k,1}) = k-8 > 0 \). If \( s_D(u) < 0 \), one has that \( s_D(u) \neq s_D(v) \). Let us consider the set, say \( W_2 \), of all vertices \( w \in V(C_1 \cup \ldots \cup C_{k-1}) \cap V_R \setminus X \) such that \( s_D(w) > 0 \). Note that among all the vertices of \( W_2 \), the labels of two edges incident with \( v_{1,2} \) are minimum. By the conditions that \( |P_{2,4}^1| \geq 5d-6 \), one has that \( s_D(v_{1,2}) = |P_{1,2}^1|+|P_{1,2}^1|+|P_{1,3}^1|+|P_{1,4}^1| = |P_{2,4}^1|+3 \geq 5d-3 > k-8 = s_D(v_{k,1}) \). That is, \( s_D(v_{k,1}) < s_D(v_{1,2}) < s_D(w) \) for all \( w \in W_2 \). Thus, if \( s_D(u) > 0 \), we still have \( s_D(u) \neq s_D(v) \).

If \( k \geq 5d+5 \), then \( s_D(v_{k,1}) = k-8 > 0 \). If \( s_D(u) < 0 \), we have \( s_D(u) \neq s_D(v) \). Let us consider the set, say \( W_3 \), of all vertices \( w \in V(C_1 \cup \ldots \cup C_{k-1}) \cap V_R \setminus X \) such that \( s_D(w) > 0 \). Note that among all the vertices of \( W_3 \), the labels of two edges incident with \( v_{1,2} \) are minimum. If \( k \geq 5d+5 \), by the condition of Theorem \( \square \), \( |G_1| \geq 2x_0+5 \), where \( x_0 \) is a unique positive integer solution for one of equations: \( k = (2d-2)(x+2)+0 \), \( k = (2d-2)(x+2)+1, \ldots, k = (2d-2)(x+2)+(d+8), k = (2d-2)(x+1)+(d+9), k = (2d-2)(x+1)+(d+10), \ldots, k = (2d-2)(x+1)+(2d-3). \) It implies that, \( x_0 \) is a positive integer solution for one of equations: \( k = (2d-2)x+3d+7, k = (2d-2)x+3d+8, \ldots, k = (2d-2)x+5d+4. \) So, \( k \leq (2d-2)x_0+5d+4. \) (Otherwise, \( k > (2d-2)x_0+5d+4. \) Assume \( k = (2d-2)x_0+5d+t \) for \( t \geq 5 \). It implies that \( k = (2d-2)(x_0+1)+3d+t+2 \) with \( 3d+t+2 \geq 3d+7 \). It implies that \( x_0+1 \) or \( x_0+i \) for \( i > 1 \) is the positive integer solution, which contradicts with \( x_0 \) is the unique positive integer solution.) Since \( s_D(v_{1,2}) = |P_{1,2}^1|+|P_{1,2}^1|+|P_{1,3}^1|+|P_{2,4}^1| \geq |P_{2,4}^1|+3 \) and \( |P_{2,4}^1| \geq (2d-2)x_0+5d-6 \) given in the definition
of $c$, $s_D(v_{1,2}) \geq (2d-2)x_0+5d-3 > k-8 = s_D(v_{k,1})$. That is, $s_D(v_{k,1}) < s_D(v_{1,2}) < s_D(w)$ for all $w \in \mathcal{W}_3$. Thus, if $s_D(u) > 0$, we still have $s_D(u) \neq s_D(v)$.

Therefore, the result holds if the number of odd cycles is $k$ with $k \geq 3$.

This completes the proof of Theorem 1. 

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