Analogue gravitational field from nonlinear fluid dynamics

Satadal Datta\textsuperscript{1} and Uwe R. Fischer\textsuperscript{1}

\textsuperscript{1}Seoul National University, Department of Physics and Astronomy, Center for Theoretical Physics, Seoul 08826, Korea

(Dated: January 20, 2022)

The dynamics of sound in a fluid is intrinsically nonlinear. We derive the consequences of this fact for the analogue gravitational field experienced by sound waves, by first describing generally how the nonlinearity of the equation for phase fluctuations back-reacts onto the definition of the background providing the effective space-time metric. Subsequently, we use the the analytical tool of Riemann invariants in one-dimensional motion to derive source terms of the effective gravitational field stemming from nonlinearity. Finally, we show that the consequences of nonlinearity we derive can be observed with Bose-Einstein condensates in the ultracold gas laboratory.

I. INTRODUCTION

The basic assumption underlying the simulation of curved space-times, coined analogue gravity \cite{1}, is the separation of the underlying classical or quantum field into a perturbation field and a background field. Such a separation is conventionally possible when one has a control parameter (such as the number of particles in mean-field approaches to condensed matter systems), which separates the field into a large background and its perturbations. In addition, one needs to separate the length and frequency scales of background and perturbation to have them well defined and separable. Under rather general conditions for the action of the system under consideration, then a wave equation results which is identical to that of a scalar (in the simplest case), minimally coupled to gravity \cite{2}. This standard paradigm, originally due to Unruh \cite{3} (see also for an early precursor \cite{4}) has yielded analogues of classical and quantum field propagation in flat and curved space-time for a multitude of physical contexts, e.g., analogues of Lorentzian signature space-times and the associated kinematical effects of quantum fields on these, experimentally as well as theoretically. A non-exhaustive list of examples comprises black holes via (shallow water) gravity waves \cite{5–8}, black holes in fluids of light \cite{9, 10}, numerous studies on Hawking radiation, e.g., \cite{11–15}, the inflationary Universe and Hubble dynamics \cite{16–20}, the pair-production of cosmological quasiparticles \cite{21–23} and their associated degree of entanglement \cite{24–26}, the Unruh and Gibbons-Hawking effects as manifestations of the observer dependent content of quantum fields in flat and curved space-time \cite{27–30}, the quantum back-reaction on a classical background \cite{31}, and to probe analogue trans-Planckian effects on low-energy phenomena \cite{17, 32}. Furthermore, this standard paradigm has been harnessed to investigate the black hole lasing phenomenon for black-white hole configurations \cite{33, 34}, black hole superradiance \cite{35–37} and quasinormal black hole modes \cite{38}, as well as analogues of gravitational waves \cite{39, 40}, and to address aspects of the black hole information paradox \cite{41}.

The underlying nonrelativistic medium in the laboratory, in its continuum description, is usually however intrinsically nonlinear. For instance, one encounters, in the case of a fluid, in the course of time unavoidably the fluid-dynamical nonlinearity will enter the dynamics of the fluid velocity and density, and the basic linearization premise on which conventional analogue gravity is based will break down. Even when initially a linear description applies, eventually the nonlinear dynamics of the fundamental variables becomes manifest, and finally a shock wave singularity will develop.

The standard paradigm of quantum field theory, leading to the definition of (quasi-)particles, is in fact precisely this linearization procedure on top of an essentially inert background. It underlies the majority of derivations of phenomena which are described by quantum fields propagating in \textit{fixed} curved space-time \cite{42}. However, for concreteness, in the arena of the nonlinear dynamics of fluids, there are only two variables, density and velocity of the fluid, and the separation into background and perturbations, when one goes beyond simply linearizing the equations governing the perturbations, needs to be readdressed.

Our aim in this paper is to address how the intrinsic nonlinearity of perfect fluid dynamics affects the concept of analogue gravity and the definition of the space-time metric which is attached to the background field. We show, in particular, that the space-time metric which affords a suitable description of the effective analogue gravitational field which furnishes the wave equation in curved space-time changes when one has to take into account that the background solution derives from the solution of the full nonlinear fluid-dynamical equations. As a further consequence of nonlinearity, we then demonstrate dynamical aspects of metric perturbations above a background metric, consisting in the emergence of source terms in the wave equation for the metric perturbations.

To treat the problem of fluid-dynamical nonlinearity in a tractable manner, we then consider the 1+1D of Riemann wave equation and the theory of Riemann invariants, which are furnishing an analytical description of the emergence of shock waves. Using the Riemann approach, we obtain source terms which are constituting sources of the propagating gravitational perturbation field (such as gravitational waves), which sources themselves depend
nonlinearly on the metric perturbations. We are thus supplying a concrete experimentally testable setup, in an analogue gravity setup, for the emergence of a curved space-time metric from a Minkowski metric due to the nonlinearity of the underlying (scalar) field theory [43].

We finally provide concrete estimates for the experimental manifestations of such a nonlinear analogue gravity in Bose-Einstein condensates (BECs). In particular, we show that the time periods $t_{\text{shock}}$ for shock waves to emerge, for current BEC setups, also and in particular those which study analogue gravity phenomena, are much less than the lifetime of typical experimental runs. Hence we argue that, when studying analogue gravity, the nonlinearity of fluid dynamics in the perfect fluid BEC must generally be taken into account.

II. ACTION PRINCIPLE FOR NONLINEAR FLUID DYNAMICS

We assume in the following that we are in the nondisperessive limit of the fluid dynamics, which for a BEC in particular implies the neglect of the quantum pressure term involving density gradients (the so-called Thomas-Fermi limit in BECs). The action for an inviscid irrotational barotropic fluid then is [44],

$$S = -\int d^4x \left\{ \rho \Phi + \frac{1}{2} \rho (\nabla \Phi)^2 + u(\rho) + V_{\text{ext}} \rho \right\}.$$  

(1)

Here, $\rho(x,t), \Phi(x,t), V_{\text{ext}}(x,t)$ are fluid density, velocity potential and a scalar potential corresponding to external conservative force, respectively: $u(\rho)$ is internal energy density. The Inviscid irrotational fluid equations of motion are derivable by varying the above action:

$$\partial_t \rho + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0,$$  

(2)

$$\rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + c_s^2 \nabla \rho + \rho \nabla V_{\text{ext}} = 0,$$  

(3)

where fluid velocity $\mathbf{v}(x,t) = \nabla \Phi$, fluid pressure $p(x,t) = -\left(\rho \Phi + \frac{1}{2} \rho (\nabla \Phi)^2 + u(\rho)\right)$, and $c_s^2 = c_s^2(\rho) = \frac{dp}{d\rho}$, where $c_s$ is the sound speed. The fluid equations Eq. (2)-(3) represent a system of first order quasilinear partial differential equations, a particular type of nonlinear partial differential equation [45]. A well posed boundary value problem gives a unique solution in the domain of $x - t$. If one considers, in particular, two solutions of $\rho$ and $\mathbf{v}$ originating from two boundary value problems, one may select one of them as the background. Therefore, in general the definition of a background is arbitrary. Conventionally, one selects a solution which varies slowly with $x$, and $t$ as a background or “mean” flow. Any solution of flow can be decomposed into a mean flow plus a perturbation terms in density and velocity. For a given solution chosen as the background flow, a variation in the boundary value problem produces perturbation terms ($\delta \rho$ and $\delta \mathbf{v}$). We then expand the action in Eq. (1) as follows,

$$S = S_0 + \delta S.$$  

(4)

Here, $S_0$ is the action corresponding to the background flow. The term linear in perturbations, vanishes because the background by definition satisfies itself the fluid-dynamical equations. Hence, $\delta S$ consists of a term quadratic in perturbations and higher order terms. Thus we have

$$\delta S = -\int d^4x \left\{ \delta \rho \delta \Phi + \delta \rho \mathbf{v}_0 \cdot \nabla \phi + \frac{1}{2} \rho_0 (\nabla \delta \Phi)^2 \right\}$$

$$+ \frac{1}{2} \delta \rho (\nabla \delta \Phi)^2 + \delta \psi \right).$$  

(5)

We denote the background quantities with suffix 0, and $\delta \psi$ is the series of terms in $\delta u$ starting from terms of order $\delta \rho^2$ and higher orders.

$$u(\rho) = u(\rho_0 + \delta \rho) = u(\rho_0) + \delta u$$

$$= u(\rho_0) + \frac{du}{\rho_0} \rho_0 \delta \rho + \frac{1}{2} \frac{d^2 u}{\rho_0^2} \rho_0 \delta \rho^2 + ..$$

$$= u(\rho_0) + \frac{du}{\rho_0} \rho_0 \delta \rho + \delta \psi.$$  

(6)

The specific enthalpy, $h(\rho) = \frac{du}{\rho}$ is $\delta \psi$. $\delta \rho$ can be found from the equation of motion of $\delta \rho$ in the action, $S_{\delta}$ of Eq. (5):

$$\delta \Phi + \mathbf{v}_0 \cdot \nabla \delta \Phi + \frac{1}{2} \nabla (\nabla \delta \Phi)^2 + \delta h = 0,$$  

(7)

along with $h = \int \frac{dp}{\rho}$ and $\delta h = \frac{\partial h}{\partial \rho} \delta \rho$. Therefore, the inverse function of $h$ (assuming its existence) reads

$$h^{-1} \left( \frac{h(\rho)}{h(\rho_0)} \right)$$

$$= h^{-1} \left( 1 - \frac{1}{h(\rho_0)} \left( \delta \Phi + \mathbf{v}_0 \cdot \nabla \delta \Phi + \frac{1}{2} \nabla (\nabla \delta \Phi)^2 \right) \right).$$  

(8)

We consider the simplest possible case, such that $\frac{h(\rho)}{h(\rho_0)} = h(\rho/\rho_0)$. Therefore $h(\rho) \propto \rho^q$, $q$ is some real number. This is the case for an ideal gas with polytropic equation of state: $p = K \rho^q$, where $K$ is a constant, and $q$ is the ratio of the specific heat capacities; then $q = \gamma - 1$. We may then functionally express the density variations as

$$\delta \rho = \rho_0 \left( \mathcal{F} \right)^{\frac{1}{\gamma - 1}} - 1,$$  

(9)

where the functional $\mathcal{F}$ is given by

$$\mathcal{F} \left( \delta \Phi, \nabla \delta \Phi \right) = 1 - \frac{\gamma - 1}{c_s^2} \left( \delta \Phi + \mathbf{v}_0 \cdot \nabla \delta \Phi + \frac{1}{2} \nabla (\nabla \delta \Phi)^2 \right).$$  

(10)

Thus $\delta \rho = \rho \left( \delta \Phi, \nabla \delta \Phi, x, t \right)$. The perturbation Lagrangian density $\mathcal{L}$ corresponding to the action $\delta S$ is:

$$\delta \mathcal{L} \left( \delta \Phi, \nabla \delta \Phi, x, t \right)$$

$$= -\rho_0 \left( \mathcal{F} \right)^{\frac{1}{\gamma - 1}} \left( \delta \Phi + \mathbf{v}_0 \cdot \nabla \delta \Phi + \frac{1}{2} \nabla (\nabla \delta \Phi)^2 \right)$$

$$+ \frac{1}{2} \rho_0 (\nabla \delta \Phi)^2 + \delta \psi.$$  

(11)
Therefore, using the expression (10),
\[
\delta \mathcal{L} \left( \delta \Phi, \nabla \delta \Phi, x, t \right) = -\frac{1}{2} f^{\mu \nu}_0 \partial_{\mu} \delta \Phi \partial_{\nu} \delta \Phi + \delta \mathcal{L}_I, \tag{12}
\]
where we define the coefficient matrix \( (\mu, \nu = t, x, y, z) \)
\[
f^{\mu \nu}_0 := \frac{\rho_0}{c_{s_0}^2} \begin{bmatrix}
-1 & \cdots & -v_0 \\
\cdots & \cdots & \cdots \\
-v_0^T & \cdots & -v_0 \otimes v_0.
\end{bmatrix}. \tag{13}
\]
Note that the self-interaction part \( \delta \mathcal{L}_I \) of Eq. (12) is an infinite series in \( \delta \Phi \) and \( \nabla \delta \Phi \), except for the particular (BEC) case of \( \gamma = 2 \), where the series exactly truncates at cubic order,
\[
\delta \mathcal{L}_I = \frac{1}{2} \frac{\omega_0}{c_{s_0}^2} \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right) (\nabla \delta \Phi)^2 \quad \text{(BEC)}. \tag{14}
\]

### III. EFFECTIVE SPACE-TIME METRIC

#### A. General equation of motion for phase perturbations

The equation of motion for \( \delta \Phi \) can be found from (12) as follows

\[
\begin{align*}
\partial_{\mu} \left( f^{\mu \nu}_0 \partial_{\nu} \delta \Phi \right) + \partial_t & \left( -\frac{\rho_0}{c_{s_0}^2} \frac{1}{2} (\nabla \delta \Phi)^2 \right) + \nabla \cdot \left( -\frac{\rho_0 v_0}{c_{s_0}^2} \frac{1}{2} (\nabla \delta \Phi)^2 \right) + (2 - \gamma) \frac{1}{c_{s_0}^2} (\delta \Phi + v_0 \cdot \nabla \delta \Phi) \nabla \cdot (\rho_0 \nabla \delta \Phi) \\
+ \nabla \cdot \left( -\frac{\rho_0}{c_{s_0}^2} (\delta \Phi + v_0 \cdot \nabla \delta \Phi) \nabla \delta \Phi \right) + \frac{(2 - \gamma)}{2 \rho_0^2} (\nabla \delta \Phi)^2 \nabla \cdot (\rho_0 \nabla \delta \Phi) + \nabla \cdot \left( -\frac{\rho_0}{2 \rho_0^2} (\nabla \delta \Phi)^2 \nabla \delta \Phi \right) &= 0. \tag{15}
\end{align*}
\]

The above equation is the basic underlying nonlinear wave equation for \( \delta \Phi \) on which the following considerations will be based. We note there that even though the Lagrangian of Eq. (12) is an infinite series (for \( \gamma \neq 2 \)) in the interaction term, the equation of motion for \( \delta \Phi \) terminates at cubic order. Furthermore we observe that when we take nonlinearity into account, evidently the wave equation for \( \delta \Phi \) cannot be brought into the form of a (massless) Klein-Gordon (KG) wave equation. Difference between any two solutions of the fluid equations, Eq. (2)-Eq. (3) originating from two different boundary conditions can be expressible in terms \( \delta \Phi \) satisfying Eq. (15).

#### B. On the choice of background

Using the decomposition \( \rho = \rho_0 + \delta \rho \), \( \mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v} \) of the solution \( \rho \), \( \mathbf{v} \) of the fundamental fluid-dynamical equations, it is not always possible to physically distinguish a background \( (\rho_0, \mathbf{v}_0) \), and identify it uniquely. For concreteness, to illustrate this, say we consider an ideal gas in a box, for which \( \rho_0 \) is constant everywhere. One of the walls of the box can be moved by a piston to introduce perturbations. We then increase the pressure in the box adiabatically by pushing the piston. After equilibrium has been reached, a pressure change obtains everywhere (as a result of a polytropic equation of state), and the change in velocity is zero, so that \( \nabla \delta \Phi = 0 \), where \( \delta \Phi = Ct \), \( C \) a constant, can be found from Eq. (9). In Eq. (15), all the nonlinear parts involve \( \nabla \delta \Phi \), therefore \( \delta \Phi = Ct \), which satisfies \( \partial_{\mu} (f^{\mu \nu}_0 \partial_{\nu} \delta \Phi) = 0 \), which is the usual wave equation in a static uniform background. Such a wave equation has a general solution of the form \( f(x - c_{s_0} t) + g(x + c_{s_0} t) \), where \( f \) and \( g \) are two well behaved functions, and the phase perturbation is \( \delta \Phi = Ct = -A (x - c_{s_0} t) + A (x + c_{s_0} t) \) where \( A \) is another constant, satisfying \( Ac_{s_0} = C/2 \). However, this perturbation \( \delta \Phi \) is achievable from an infinite number of backgrounds which we would start from.

We see from this simple example that, formally, it is certainly correct to write any solution of the fluid equations as perturbations on top of a background, but it is not always physically possible and meaningful to uniquely identify perturbations and background. Conversely, if we imagine the piston executes a small-amplitude oscillatory motion, the separation into perturbation and background is meaningful. Thus an arbitrary solution of fluid equations has a physical background and perturbations on top of it only when the solution can be separated into two parts; one having slow variations in space and time, i.e., the background, and the other having relatively fast variations in space and time, i.e., the perturbation (the sound wave).

#### C. Linearized regime

Only when we are operating in the linearized in \( \delta \Phi \) regime of (15), the equation of motion for \( \delta \Phi \) can be put into the form of the massless KG equation. Then one may readily define the effective metric from \( f^{\mu \nu}_0 \). In
3+1D, it reads
\[
g_{\mu\nu} \equiv \frac{\rho_0}{c_{s0}} \begin{bmatrix} -c_{s0}^2 - v_0^2 & -v_0^T \\ \vdots & \vdots & \ddots \\ -v_0 & \vdots & \cdots & \| \end{bmatrix}
\] (16)
which corresponds to the conventional acoustic metric of the standard paradigm of analogue gravity.

**D. Redefining the space-time metric taking nonlinearity into account**

If the solution of Eq. (15) is known, the density follows from Eq. (9), and the velocity is \( v = v_0 + \nabla \delta \Phi \). The difference between two solutions of the fluid-dynamical equations originating from two different boundary value problems imposed on the system can always be expressed by a single scalar which is \( \delta \Phi \). We can then choose to define a background solution of the fluid equations. We may define a new **background (ii)** (see table 1) by \( \rho(0) = \rho_0 + \delta \rho, v(0) = v_0 + \nabla \delta \Phi \), where \( \delta \Phi \) is a (sufficiently slowly varying) solution of the full nonlinear (15). The fluid equations linearized over this new background give a wave equation for the first-order perturbation \( \Phi(1) \) which again takes the form of the massless KG equation. The new perturbation \( \Phi(1) \) is not aware of the original background (i), it couples to the new background (ii) in a similar manner to what is postulated in the scalar theory of gravity proposed in [46], and as in other field theories of gravity over curved or Minkowski background space-times [47–50]. Therefore by linearizing with respect to the **background (ii)**, we have
\[
\partial_{\mu}(f_{\mu\nu}^{(0)} \partial_{\nu} \Phi(1)) = 0,
\] (17)
where \( f_{\mu\nu}^{(0)} \) is given by
\[
f_{\mu\nu}^{(0)} := \frac{\rho(0)}{c_{s0}^2} \begin{bmatrix} -(c_{s0}^2 - v_0^2) & -v_0^T \\ \vdots & \vdots & \ddots \\ -v_0 & \vdots & \cdots & \| \end{bmatrix}.
\] (18)

The acoustic metric then is again of the form of (16), replacing \( 0 \to (0) \),
\[
g_{\mu\nu} = \rho(0) \frac{c_{s0}}{c_s} \begin{bmatrix} -(c_{s0}^2 - v_0^2) & -v_0^T \\ \vdots & \vdots & \ddots \\ -v_0 & \vdots & \cdots & \| \end{bmatrix}.
\] (19)

The spacet-time metrics associated to the two backgrounds are generally related by
\[
g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu},
\] (20)
where the \( h_{\mu\nu} \) represent the difference in acoustic metrics between **background (ii)** and background (i), and can be expressed as functions of \( \partial_{\mu} \delta \Phi \). Therefore, here gravity can be generated from a single self-interacting scalar over an arbitrary background, cf. [48, 49]. Expanding up to second order in \( \delta \Phi \), we obtain
\[
\begin{align*}
h_{tt} &= \frac{\rho_0}{c_{s0}} \left[ \left( \frac{\gamma + 1}{2} - \frac{(3 - \gamma)}{2} \frac{v_0^2}{c_{s0}^2} \right) \delta \Phi + v_0 \cdot \nabla \delta \Phi \left( \frac{5}{2} - \frac{(3 - \gamma)}{2} \frac{v_0^2}{c_{s0}^2} \right) \right] \\
&\quad - \frac{\rho_0}{c_{s0}} \frac{\delta \Phi^2}{2c_{s0}^2} \left( (\gamma - 1) + \frac{1}{2} \frac{v_0^2}{c_{s0}^2} \left( \frac{5 - 3\gamma}{4} \right) \right) + \left( \delta \Phi v_0 \cdot \nabla \delta \Phi \right) \left( \frac{3 - \gamma}{2c_{s0}^2} \right) \left( 2\gamma + \frac{1}{2} \frac{v_0^2}{c_{s0}^2} \left( \frac{5 - 3\gamma}{2} \right) \right) \\
&\quad + \frac{\rho_0}{c_{s0}} \left( \nabla \delta \Phi \right)^2 \left( \frac{\gamma}{2} + 1 - \frac{v_0^2}{c_{s0}^2} \left( \frac{3 - \gamma}{4} \right) \right) - \left( v_0 \cdot \nabla \delta \Phi \right)^2 \left( \frac{3 - \gamma}{2c_{s0}^2} \right) \left( \gamma + 1 + 1 - \frac{v_0^2}{c_{s0}^2} \left( \frac{5 - 3\gamma}{4} \right) \right),
\end{align*}
\] (21)
\[
\begin{align*}
h_{ii} &= \frac{\rho_0}{c_{s0}} \left[ \left( \frac{3 - \gamma}{2c_{s0}^2} \right) \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right) - \partial_i \delta \Phi \right] + \left( \frac{3 - \gamma}{2c_{s0}^2} \right) \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right) \partial_i \delta \Phi \\
&\quad + \frac{\rho_0}{c_{s0}} \frac{v_0^2}{2c_{s0}^2} \left( \frac{5 - 3\gamma}{4c_{s0}^2} \right) \left( \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right)^2 \right),
\end{align*}
\] (22)
\[
\begin{align*}
h_{ij} &= \frac{\rho_0}{c_{s0}} \delta_{i,j} \left[ \left( \frac{3 - \gamma}{2c_{s0}^2} \right) \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right) + \left( \frac{3 - \gamma}{2c_{s0}^2} \right) \left( \left( \frac{5 - 3\gamma}{4c_{s0}^2} \right) \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right)^2 \right) \right],
\end{align*}
\] (23)
where \( i, j = x, y, z \).

We shall see in Sec. IV that nonlinearity comes into play over time for the simplest possible nonlinear wave in fluid. With progressing time, starting from a linear
approximation, $\delta \Phi$ reveals the nonlinearity of (15), as a consequence changing the proper definition of background, and produces the new metric $g_{\mu \nu}$.

| Background (i) | Background (ii) |
|----------------|-----------------|
| $\delta \Phi$: linearized Eq. (15) | $\delta \Phi$: full nonlinear Eq. (15) |
| $\rho_0, v_0$ | $\rho_0, v_0$ |
| $g_{\mu \nu}$ | $g_{\mu \nu}$ |

**TABLE I.** Definition of background variables and their associated metrics. In Background (i), the velocity potential perturbation $\delta \Phi$ represents a massless scalar field, whereas in Background (ii) it satisfies the full nonlinear wave equation (15).

E. Wave equation for time independent backgrounds

If the background (i) is time independent, the equations satisfied by $\delta \Phi$ have time translation symmetry; $t \rightarrow t' = t - \epsilon$, $\delta \Phi \rightarrow \delta \Phi - \epsilon \delta \zeta$, $\partial_t \delta \Phi = -\delta \zeta$, where $\delta \zeta$ is Bernoulli’s function. Substituting the new time-translated $\delta \Phi$ in Eq. (7) yields

$$\frac{\rho}{c_s^2} \partial_t \delta \zeta - \frac{\rho}{c_s^2} \mathbf{v} \cdot \nabla \delta \zeta + \partial_t \partial_t \rho = 0.$$  

Taking another partial time derivative of Eq. (24), and using the continuity equation, we find

$$\partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \delta \zeta) = 0,$$  

Equation (25), and as a consequence, Eq. (25) are valid without the requirement of enthalpy $h$ being in the specific form (the case of polytropic equation of state) discussed after Eq. (8). Nevertheless, Eq. (25) can also be found from the equation of motion of $\delta \Phi$ if we consider the background (i) as a uniform stationary medium. Thus here the time derivative $\delta \Phi = -\delta \zeta$, instead of $\delta \Phi$ itself, behaves like a massless scalar field over a curved space-time, where the nonlinear self-interaction of $\delta \Phi$ is responsible for generating $h_{\mu \nu}$ in addition to the original Minkowski background.

IV. RIEMANN WAVE EQUATION AND RIEMANN INVARIANTS

We now treat a case where analytical techniques to study nonlinear sound are well established. We consider a one-dimensional sound wave propagating in a uniform static medium, i.e., a background of type (i) is our starting point. Therefore, from Eq. (15), we have

$$-\partial_t^2 + c_{s0}^2 \partial_x^2 \delta \Phi = 2(\partial_x \partial_x \delta \Phi) \partial_t \delta \Phi + (\gamma - 1) (\partial_x^2 \delta \Phi) \partial_t \delta \Phi + \frac{\gamma - 1}{2} (\partial_x \delta \Phi)^2 \partial_x^2 \delta \Phi.$$  

FIG. 1. Simple wave solution of the Riemann wave equation (31), displayed in the comoving frame $x'$ frame, $x' = x - c_{s0} t$, for wave profile with initial configuration: $v = A \cos k x$ at $t = 0$; $A = 0.05$ and $A = 0.025$; and $k = 1$, $h = 1$ in units of $c_{s0}, c_{s0}$ is set to unity. For a BEC, $\gamma = 2$. According to the linearized solution from Eq. (35), the wave does not change shape over time, the red dotted lines are the solutions for all $t$. Due to nonlinearity, points of larger velocity $v$ move with higher speed, resulting in a change in shape of the profile as displayed, where we consider times $t < t_{\text{shock}}$, where $t_{\text{shock}} = 40/3$ for the cosine wave with amplitude $A = 0.05$, and $t_{\text{shock}} = 80/3$ for the cosine wave with $A = 0.025$. The degree of nonlinearity is different for the two waves at the same instant; e.g., at $t = 13$, the profile with $A = 0.05$ is closer to approach the shock than that with $A = 0.025$.

Instead of directly starting from this equation for $\delta \Phi$, we reinstate the problem in terms of Riemann invariants, the powerful method being due to the seminal paper of Riemann in 1860 [51], solving the problem of 1D shock waves analytically. The fluid equations in one spatial dimension lead to the Riemann invariants being given by the partial differential equations [52]:

$$\left[ \frac{\partial}{\partial t} + \left( v + c_s \right) \frac{\partial}{\partial x} \right] J_+ = 0,$$  

$$\left[ \frac{\partial}{\partial t} + \left( v - c_s \right) \frac{\partial}{\partial x} \right] J_- = 0.$$  

The total sound speed is $c_s = c_{s0} + \delta c_s$, and the invariants can be expanded $J_\pm = J_{0\pm} + \delta J_\pm = v \pm 1$. In a polytropic medium, we have [52]

$$J_{0\pm} = \pm \frac{2 c_{s0}}{\gamma - 1},$$  

$$v = \frac{\delta J_+ + \delta J_-}{2} = J_0^+, J_0^-, \quad \rho = \rho_0 \left\{ \left( \frac{\gamma - 1}{\gamma - 1} \right) (J_+ - J_-) \right\}^{\frac{\gamma - 1}{\gamma - 1}}.$$  

A. Simple wave solution

Now, we consider the simplest possible case, i.e., a wave traveling in a particular direction, called a simple wave [52]. Then, $J_-$ is constant everywhere ($\delta J_- = 0 \rightarrow$...
\( J_- = J_0^- \), and the variation of \( J_+ \) represents the propagation of the Riemann wave.

Constancy of \( J_- \) throughout the whole \( x, t \) domain implies, from Eqs. (29)

\[
\rho = \rho_0 \left[ 1 + \left( \frac{\gamma - 1}{2} \right) \frac{v}{c_{s0}} \right]^{\frac{2}{\gamma - 1}}.
\] (30)

Hence, a simple wave can be described by a single variable \( v \) or \( \rho \). The equation (27) for \( J_+ \) gives the Riemann wave equation [52]:

\[
\frac{\partial v}{\partial t} + \left[ c_{s0} + \left( \frac{\gamma + 1}{2} \right) v \right] \frac{\partial v}{\partial x} = 0.
\] (31)

This equation has an analytic solution [52]

\[
v = f(\xi),
\]

\[x = [c_{s0} + \left( \frac{\gamma + 1}{2} \right) v] t + \xi.
\] (32)

The above relation between \( \xi \) and \( x, t \) represents a simple wave solution traveling along positive \( x \) axis. In the linearized limit, the Riemann wave equation reduces to

\[
\frac{\partial v}{\partial t} + c_{s0} \frac{\partial v}{\partial x} = 0.
\] (34)

Using again the method of characteristics gives the solution in the form

\[
v = f(\xi),
\]

\[x = c_{s0} t + \xi.
\] (35)

The linearized solution and the nonlinear solution are depicted in the Fig. 1, which depicts how the solution deviates from the linearized solution over time. Initially, the linearized description affords a sufficiently accurate description and the wave corresponds in the analogue gravity context to a massless scalar field over flat Minkowski space-time. However, with advancing time, that is as \( t \) approaches \( t_{\text{shock}} \), this does not hold anymore, requiring the background to be redefined as a background of type (ii), which is represented by the Riemann wave itself, with the metric \( g_{\mu\nu} \). Thus the field \( \Phi \), satisfying Eq. (26), changes the background and effectively creates gravity, that is a curved space-time, from a Minkowski space-time.

The shock time \( t_{\text{shock}} \) is defined as the instant when at the shock location \( x = x_{\text{shock}} \), \( \frac{\partial v}{\partial x} \) goes to infinity. This can be analytically shown from the method of characteristics to solve the Riemann wave equation [52]. For a simple wave with initial profile (at \( t=0 \)) \( v = A \cos kx \),

\[t_{\text{shock}} = \frac{1}{2(v_{\text{shock}})}.
\] [53]

Inspecting Fig. 1, one may choose an instant \( t_{\text{lin}} \) which is setting an upper limit in time until which the solution of the Riemann wave equation can be approximately regarded as residing in a linearized regime. The choice of \( t_{\text{lin}} \) depends on the required precision of reproducing the exact solution while still staying in that linearized regime. If the observation time in any experiment fulfills \( t_{\text{obs}} < t_{\text{lin}} \), then the solution of the Riemann wave equation can be considered as a massless field over the background (i), with metric (absorbing a constant conformal factor \( \rho_0/c_{s0} \)). Writing the originally 3+1D metric in a quasi-1D system in 1+1D form, we have [54],

\[
g_{\mu\nu} := \begin{bmatrix} -c_{s0}^2 & 0 \\ 0 & 1 \end{bmatrix}.
\] (37)

On the other hand, if \( t_{\text{obs}} \gtrsim t_{\text{lin}} \), there are two possible procedures. As a first option (1), for \( t < t_{\text{lin}} \), the Riemann wave is considered as a massless scalar field; for \( t \gtrsim t_{\text{lin}} \), background (i) is reverted to background (ii), and one defines a new perturbation \( \Phi_{(1)} \), which again varies faster (in space and time) than the Riemann wave. The background is redefined, the Riemann wave is itself the background, leading to the new acoustic 1+1D metric of type (ii), again taking over the conformal factors from the 3+1D embedding space of a quasi-1D system [53],

\[
g_{\mu\nu}(r, t) := \frac{c_{s0} \rho_0}{\rho_0} \begin{bmatrix} -c_{s0}^2 & 0 \\ 0 & -v_0 \end{bmatrix} \begin{bmatrix} -v_0 & 1 \\ 0 & -v_0 \end{bmatrix}.
\] (38)

As a second option (2) one can instead consider the Riemann wave as the background from the very beginning \( (t=0) \), and linearize the fluid equations to find the corresponding massless scalar field, \( \Phi_{(1)} \). That linear perturbation \( \Phi_{(1)} \) again can behave nonlinearly in some domain \( x - t \), therefore one needs to redefine the background again.

\section*{B. Metric components due to nonlinearity}

The new acoustic 1+1D metric can be decomposed as follows

\[
g_{\mu\nu}(x, t) = g_{\mu\nu} + h_{\mu\nu}
\]

\[
= \begin{bmatrix} -c_{s0}^2 & 0 \\ 0 & 1 \end{bmatrix} + h_{\mu\nu}(x, t), \quad (\mu, \nu = t, x). \] (39)

The metric perturbations \( h_{\mu\nu}(x, t) \) are to second order given by [53]

\[
h_{tt} = \frac{\gamma + 1}{2} \frac{\delta \rho}{\rho_0} + \frac{(\gamma^2 - 1)}{8} \left( \frac{\delta \rho}{\rho_0} \right)^2 + v^2,
\] (40)

\[
h_{tx} = h_{xt} = -v \left( 1 + \frac{(3 - \gamma)}{2} \frac{\delta \rho}{\rho_0} \right),
\] (41)

\[
h_{xx} = \frac{(3 - \gamma)}{2} \frac{\delta \rho}{\rho_0} - \frac{(3 - \gamma)(\gamma - 1)}{8} \left( \frac{\delta \rho}{\rho_0} \right)^2.
\] (42)

The \( h_{\mu\nu} \) are expressible in terms of a single variable \( v \) from Eq. (30).

The perturbations on top of the background back-react therefore on the definition of the background and the resulting acoustic metric. We limit ourselves to the nondispersive (Thomas-Fermi) limit of negligible density variations. As the wave slopes in Fig. 1 become steeper and steeper with time, the Thomas-Fermi assumption breaks down. We may take this limitation into account by imposing \( t \lesssim t_{\text{shock}} \).
V. SOURCE TENSOR OF THE EFFECTIVE GRAVITATIONAL FIELD

From Eqs. (39)-(42), and using Eq. (30), we find the metric components \( h_{\mu\nu} = h_{\mu\nu}(v) \). We observe the Riemann Eq. (31) remains true for any analytical function of \( v = v(x, t) \): Given any such function \( \mathcal{F}(v) \), for simple waves along positive \( x \) axis, we thus have \( \frac{\partial \mathcal{F}(v)}{\partial t} + (c_0 + \frac{1}{2} v) \frac{\partial \mathcal{F}(v)}{\partial x} = 0 \). Hence, from Eq. (31), we have the following relation: \( \frac{\partial}{\partial t} + (c_0 + \frac{1}{2} v) \frac{\partial}{\partial x} = 0 \). We introduce \( \Box_+ = \left( \frac{1}{c_0} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right), \Box_- = - \frac{1}{c_0} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \) and thus have \( \Box = \Box_- \Box_+ = \Box_+ \Box_- = - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \). Therefore, we obtain the wave equations

\[
\Box h_{\mu\nu} = S_{\mu\nu},
\]

where the nonlinear source term is given by,

\[
S_{\mu\nu} = \frac{\gamma + 1}{2c_0^2}(\partial_\mu h_{\nu\tau})(\partial_\tau h_{\mu\nu} - c_0 \partial_\nu \partial_\mu h_{\tau\tau} + 2c_0 \partial_\tau h_{\mu\tau}) + \frac{\gamma + 1}{2c_0^2}((2 - \gamma) h_{\mu\nu} + 2c_0 h_{\mu\tau} h_{\tau\nu} - c_0 \partial_\nu \partial_\mu h_{\tau\tau} + \partial_\tau \partial_\mu h_{\tau\mu}).
\]

Note that this source term appears if and only if the non-linearity in the Riemann wave equation (31) is present, and hence vanishes in the limit of linearized acoustics. In Appendix A, we derive the source term of the wave equation (43) for the more general case of a non-simple wave.

In general relativity, the right-hand side of (43) contains the gravitational Landau-Lifshitz (LL) energy-momentum pseudo-tensor \( t_{\mu\nu} \), in the form \( t_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu} T^\lambda \), in traceless-transverse gauge, with \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) [55]. The proper gravitational source term and its analogue model counterpart share some common properties, but there are also notable differences. Both in gravity proper and within our analogue model they are quadratic in \( h_{\mu\nu} \) and contain its first and second order space-time coordinate derivatives, and the components of the GW act as a source themselves [55]. The presence of the lab frame, with absolute Newtonian time \( t \), however engenders that the (coordinate reparametrization) general covariance property of general relativity is not reflected in (so far existing) analogue models [56]. The energy-momentum conservation law from the Bianchi identities thus does not hold in the analogue model.

One may contrast this with the tensor describing conserved energy and momentum canonically derived from the Lagrangian density \( \delta \mathcal{L} \) as \( T^\mu_\nu = \frac{\partial \delta \mathcal{L}}{\partial (\partial_\mu \delta \Phi)} \partial_\nu \delta \Phi - \frac{\partial {\delta \mathcal{L}}}{\partial \delta \Phi} \delta_\mu \delta \mathcal{L}, \) which can be defined for a uniform and stationary background. Considering up to cubic terms of \( \delta \Phi \) in \( \delta \mathcal{L} \), one has for the interaction part of the Lagrangian density, from (11),

\[
\delta \mathcal{L} = \frac{\rho_0}{2c_0^2} \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right) \left( \nabla \delta \Phi \right)^2 - \frac{(2 - \gamma)}{3c_0^4} \left( \delta \Phi + v_0 \cdot \nabla \delta \Phi \right)^2
\]

\[
+ \mathcal{O}(\delta \Phi)^4.
\]

Then one obtains for the canonical energy-momentum tensor, to \( \mathcal{O}(\delta \Phi)^4 \)

\[
T^t_t = \frac{\rho_0}{2c_0^2} \left( \nabla \delta \Phi \right)^2 - \frac{(2 - \gamma)}{3c_0^4} \delta \Phi^3,
\]

\[
T^i_i = \frac{\rho_0}{c_0^2} \delta \Phi \partial_i \delta \Phi + \frac{\rho_0}{c_0^2} \left( \nabla \delta \Phi \right)^2 - \frac{(2 - \gamma)}{c_0^2} \delta \Phi^2 \partial_i \delta \Phi,
\]

\[
T^i_j = \left( \frac{\rho_0}{c_0^2} \delta \Phi \partial_i \delta \Phi - \rho_0 \partial_j \delta \Phi \right) \partial_j \delta \Phi - \delta^i_j \delta \mathcal{L}.
\]

One readily verifies that there is no one-to-one correspondence between \( T_{\mu\nu} \) and \( S_{\mu\nu} \). The latter involves derivatives of density and velocity perturbations (gradients of the \( h_{\mu\nu} \)), while \( T_{\mu\nu} \) contains only algebraic functions of the \( h_{\mu\nu} \).

VI. EXPERIMENTAL CONSIDERATIONS

A. Shock times \( t_{\text{shock}} \)

Here we show that the nonlinearity after imprinting a wave profile becomes manifest for typical BEC parameters, e.g., in \(^7\text{Rb}\) on time scales much less than their lifetime, also and in particular for realized analogue gravity setups [15].

For a BEC, the time after which the shock singularity is reached after initially imprinting a cosine profile for the velocity is given by

\[
t_{\text{shock}} = \frac{2}{3\Lambda k} \quad \text{(BEC)}.
\]

We specify, setting \( h = m = 1 \), the wave vector \( k \) in units of \( 1/\xi_c := c_s \) and \( A \) also in units of \( c_s \) or \( 1/\xi_c \). Furthermore, \( \mu = 1/\xi_c^2 = c_s^2 \), which means that \( t_{\text{shock}} \) can be expressed in units of \( 1/\mu \).

We should have as a minimal requirement for non-linearity to be observable that \( t_{\text{shock}} \ll \tau \), the lifetime of the BEC (which is mainly limited by three-body recombination). We assume that the laser wavelength for phase imprinting (also see below subsection) is scaled in units of \( 2\pi\xi_c \), and \( k \) expressed in units of \( 1/\xi_c = c_s \), so...
that $t_{\text{shock}}[1/\mu] \sim \frac{2}{\mu A_0}$. In Ref. [57], the chemical potential is $\mu \sim 2\text{kHz} \ [100\text{nK}]$. With $A = 0.05$ used in Fig. 1, for $k = 1$, then, returning to dimensionful units via $t_{\text{shock}}[1/\mu] = t_{\text{shock}}[1/2\pi\mu \text{[Hz]}]$, we have $t_{\text{shock}} \simeq 1.1 \text{msec} \ (\simeq 2.2 \text{msec} \text{ for } A = 0.025)$, which is much less than the lifetime of order seconds which was observed in Ref. [57]. In the $^{87}\text{Rb}$ analogue black hole experiment [15], which provided an observation of analogue Hawking radiation, the chemical potential as defined from $\xi_c$ is much less, of order $\mu \sim 30 \text{Hz} \ [1.4 \text{nK}]$. To derive this $\mu$ from $1/\xi_c^2$, we use as the coherence length the geometrically averaged quantity defined in [15] (averaged between upstream and downstream regions relative to the horizon) which is $\xi_c \sim 1.8 \mu\text{m}$. The Hawking temperature in [15] is $T_H \sim \mu/4$. Setting $k = 1/4$ ($\lambda = 8\pi$), which is of order the wave vector of the dominant Hawking modes (in the infrared), we then have $t_{\text{shock}} \sim 0.57 \text{sec}$ for $A = 0.025$, choosing here a value for the velocity perturbation amplitude which is order-of-magnitude consistent with the density-density Hawking correlations measured in [15]. This value for $t_{\text{shock}}$ is still much less than the lifetime of the experiment, which strongly increases due to the much lower densities used in the $^{87}\text{Rb}$ experiment of [15] when compared to that of [57]. We also note that shock waves in a BEC have in fact been observed, e.g., in [58], with the theory developed in [59]. Finally, similar considerations can be performed for fluids of light, in which shock wave dynamics has been observed and analyzed as well [60].

While the above estimates for $t_{\text{shock}}$ were derived for a strictly one-dimensional flow which obeys the Riemann wave equation, using parameters from previously conducted experiments on $^{87}\text{Rb}$ which have not been conducted in quasi-one-dimensional setups (Ref. [15] operates in the transition region to quasi-1D), we conclude that to consider the nonlinearity of the BEC fluid is in general unavoidable, also and in particular in typical analogue gravity setups.

B. Mass flux as a signature of nonlinearity

To derive a simple experimental measure of nonlinearity, we compute in this subsection the time-averaged mass flux through $x = \xi_i$ (we use here background ($i$), see table I, left column). We consider a homogeneous cloud of ultracold $^{87}\text{Rb}$ in a cylindrical box trap, generated e.g. by the methods used in [61], with radius $R (\ll \xi_c)$ and length $L (\gg \xi_c)$, choosing a region of length $n\lambda$ at the left end of the cylinder, denoted $S$ for source region henceforth; the observation region on the right is abbreviated RO, see Fig. 2; here, $n \in \mathbb{N}$. We suggest to employ the phase imprinting technique, readily available in the quantum optical setup of ultracold gases [62, 63]. We create a spatial variation in the phase of the initial condensate wave function, within $S$, by red-detuned laser light turned on for a short duration $T$ (as short as to stay within the Raman-Nath regime of simple diffraction). The super-

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{Phase in the expression (51) produces a monochromatic $v = A\cos kx$. Mass then flows from source ($S$) to observation region (RO). For the final equilibration stage, when the full wave train has entered the RO, we separate RO and $S$ by a repulsive laser sheet barrier, at time $n\lambda/c_{\text{rel}}$, and absorption imaging determines the total mass displacement $\Delta M$ as a signature of nonlinearity.}
\end{figure}

fluid phase pattern $\Phi(x) \propto I(x)/\delta$, where $I(x)$ is laser intensity and $\delta$ detuning from resonance, is then imprinted in $S$, where

$$\Phi(x) = (A/k) \sin(kx) + C. \quad (51)$$

Here, we put $A, C \geq 0$, and the constant $C$ is chosen such that $\Phi(x) > 0$ within $S$ (red-detuning $\forall x \in S$). The thus created bipartite 1D configuration, cf. Fig. 2, produces simple waves [52].

We calculate the current averaged over time to be, see for details Appendix B,

$$\langle j \rangle = \rho_0 \frac{3 - \gamma}{c_{\text{rel}}} \frac{A^2}{8}. \quad (52)$$

For a BEC ($\gamma = 2$), $\langle j \rangle > 0$, indicating the mass amount $\Delta M = \pi R^2 \langle j \rangle \Delta t$ with $\Delta t = 2\pi n/\omega c_{\text{rel}}$ is flowing to RO from $S$. The total mass displacement in (52) is proportional to $A^2\Delta t$ which is the hallmark of nonlinearity we aimed to derive.

VII. CONCLUSION AND DISCUSSION

At the very core of the analogue gravity concept is the separation of the underlying field(s) into a background and small perturbations propagating on top of that background, where the acoustic metric components are functions of the background flow solution (density and velocity of the fluid medium). Here, we have tested this assumption of linearizing perturbations over a background flow. We find that generally, and worked out in detail for the simplest possible case of a one-dimensional wave over a uniform static background (corresponding to a Minkowski space-time for sound), that the assumption of linearity breaks down over the course of time. Beginning with a nonlinear perturbation over a background
flow, we have demonstrated how the presence of nonlinearity in the perturbation equations of motion back-reacts on and thus changes the background flow, so that the acoustic metric is modified.

The phenomenon of emergence of a new metric due to nonlinearity of the underlying field(s), which arises naturally in analogue gravity, can be mapped to field theoretical formulations of gravity. From a historical perspective, the background field method was introduced by Feynman, Deser, and Gupta [47–50] to quantize gravity. A spacetime (classical background) is supposed to exist, and then the equations of motion for the metric perturbations are studied. In our case of analogue gravity in fluids, the metric perturbations (which change the initial concept of background) are shown to be functions of nonlinear perturbations in the velocity scalar; hence gravity is a scalar field in the analogue context. The idea of a scalar theory of gravity can be traced back to Newtonian gravity, where the gravitational potential satisfies Poisson’s equation. A generalization of the Newtonian gravitational potential within special relativity, initially proposed by Einstein and Grossmann [64], lacked general covariance (diffeomorphism invariance). More recently, a modern geometric scalar theory of gravity, respecting diffeomorphism invariance, has been proposed [46], in which a nonlinear self-interacting field produces metric perturbations over Minkowski space-time. Matter fields do not perceive the latter space-time, they couple to the modified metric by the scalar field. We have demonstrated that in essentially the same way the self-interacting nonlinear terms in the perturbation of the scalar velocity potential are responsible for generating metric perturbations, and thus the acoustic metric is changed. Introducing linear perturbations (on top of the new background type (ii)), they do not interact with the Minkowskian background, and instead minimally couple to the modified acoustic metric $\bar{g}_{\mu\nu}$, due to the nonlinear perturbation generated by the velocity potential scalar. The idea of an emergence of a curved space-time metric from a Minkowski metric due to the nonlinearity of an underlying scalar field theory [43] can thus be tested with the established tools of fluid dynamics in the context of analogue gravity.

Finally, we assumed in this work the nondispersive limit of fluid dynamics (which in BECs amounts to the Thomas-Fermi limit). Within the context of analogue gravity, the dispersive limit (that is, e.g., including the quantum pressure term in a BEC [65]), has been studied under the banner of rainbow gravity [66]. Going beyond the nondispersive limit and hence approaching very closely the instant of the shock (which represents an analogue spacetime singularity) will be the subject of a future study.

VIII. ACKNOWLEDGMENTS

This work has been supported by the National Research Foundation of Korea under Grants No. 2017R1A2A2A05001422 and No. 2020R1A2C2008103.

Appendix A: Non-simple waves in one spatial dimension

1. Inhomogeneous wave equations for Riemann invariants

We find from Eq. (27),

$$\square \delta J_{\pm} = \left( \mp \frac{(\delta J_+ + \delta J_-)}{2c_{s0}} - \frac{(\gamma - 1)}{4c_{s0}} (\delta J_+ - \delta J_-) \right) \frac{\partial \delta J_{\pm}}{\partial x},$$

(A1)

where $\square_{\pm} = \pm \frac{1}{c_{s0}^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$. Therefore, the perturbation terms of Riemann invariants, $\delta J_{\pm}$ satisfy the following inhomogeneous wave equation

$$\square \delta J_{\pm} = S_{\pm},$$

(A2)

where the source terms are

$$S_{\pm} = \frac{(\gamma + 1)}{4c_{s0}^2} \left( \delta J_{\pm} \frac{\partial^2 \delta J_{\pm}}{\partial x^2} \mp c_{s0} \delta J_{\pm} \frac{\partial^2 \delta J_{\pm}}{\partial x^2} \right)$$

$$+ \frac{(\gamma + 1)}{4c_{s0}^2} \left( \frac{\partial \delta J_{\pm}}{\partial t} \frac{\partial \delta J_{\pm}}{\partial x} \mp c_{s0} \left( \frac{\partial \delta J_{\pm}}{\partial t} \frac{\partial \delta J_{\pm}}{\partial x} \right)^2 \right)$$

$$- \frac{(\gamma - 3)}{4c_{s0}^2} \left( \frac{\partial \delta J_{\pm}}{\partial t} \frac{\partial \delta J_{\pm}}{\partial x} \mp c_{s0} \delta J_{\pm} \frac{\partial^2 \delta J_{\pm}}{\partial x^2} \right)$$

$$- \frac{(\gamma - 3)}{4c_{s0}^2} \left( \frac{\partial \delta J_{\pm}}{\partial t} \frac{\partial \delta J_{\pm}}{\partial x} \mp c_{s0} \left( \frac{\partial \delta J_{\pm}}{\partial t} \frac{\partial \delta J_{\pm}}{\partial x} \right)^2 \right).$$

(A3)

with $\square = \square_{-} \square_{+} = \left( - \frac{1}{c_{s0}^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)$.

2. Source terms

The expression (A3) represents an exact expression with source terms quadratic in $\delta J_{\pm}$. The acoustic metric of background type (ii) is $g_{\mu\nu}(x,t) = g_{\mu\nu}(\rho, v) = g_{\mu\nu}(J_+, J_-) = g_{\mu\nu}(J_0 + \delta J_+, J_0 + \delta J_-) = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = g_{\mu\nu}(J_0, J_0)$, is the Minkowski metric with light speed replaced by the sound speed $c_{s0}$, and $h_{\mu\nu}$ consists of all the other terms in the Taylor series of $g_{\mu\nu}$. We then have $h_{\mu\nu} = h_{\mu\nu}(\delta J_+, \delta J_-)$. Let us consider a general function $f = f(\delta J_+, \delta J_-)$.

$$\square f = g_{+} \square \delta J_{+} + g_{-} \square \delta J_{-} + g_{++}(\square \delta J_{+})(\square \delta J_{+})$$

$$+ g_{-+}(\square \delta J_{-})(\square \delta J_{+}) + g_{+-}(\square \delta J_{-})(\square \delta J_{-}) + g_{-\cdot}(\square \cdot \delta J_{-})(\square \cdot \delta J_{+}),$$

(A4)

where $g_{+} = \frac{\partial f}{\partial \delta J_{+}}$, $g_{-} = \frac{\partial f}{\partial \delta J_{-}}$, $g_{++} = \frac{\partial^2 f}{\partial \delta J_{+} \delta J_{+}}$, $g_{+-} = \frac{\partial^2 f}{\partial \delta J_{-} \delta J_{+}}$, $g_{-\cdot} = \frac{\partial^2 f}{\partial \delta J_{-} \partial \delta J_{-}}$. The first two terms in the right-hand side of Eq. (A4) are at least quadratic in $\delta J_{\pm}$, whereas the second, third and fifth terms are at
least cubic in $\delta J_\pm$ according to Eq. (A1); and the fourth term is at least quadratic in $\delta J_\pm$. We have

$$h_{xx} = \left(1 + \frac{\delta J_+ - \delta J_-}{\Delta J_0}\right)^{\frac{3}{2}} - 1, \quad (A5)$$

$$h_{xt} = -\frac{\delta J_+ + \delta J_-}{\Delta J_0}(h_{xx} + 1), \quad (A6)$$

$$h_{tt} = (1 + h_{xx})(c_0^2(1 + h_{xx})^{\frac{2(\gamma - 1)}{\gamma - 1}} + \frac{h_{xt}^2}{(1 + h_{xx})^2}) + c_0^2, \quad (A7)$$

where $\Delta J_0 = \frac{\gamma c_0^2}{r}$. The number of independent components is two because $\delta J_+$ and $\delta J_-$ are independent, and $\delta J_\pm$ are derivable from the two Riemann-invariant equations. The perturbation terms of $J_\pm$ can be written in terms of $h_{xt}$ and $h_{xx}$,

$$\delta J_\pm = \pm \frac{\Delta J_0}{2} \left( (1 + h_{xx})^{\frac{2}{\gamma - 1}} - 1 \right) - \left( \frac{h_{xt}}{1 + h_{xx}} \right). \quad (A8)$$

Therefore, using Eq. (A4), we can compute the source terms as a functions of the $h_{\mu \nu}$. Source terms $S_{\mu \nu}$ are functions of $h_{xx}$, $h_{xt}$ and all first order and second order partial derivatives of $h_{xt}$ and $h_{xx}$. As an example, we first consider the case of simple waves [52]. For a simple wave propagating along the positive $x$ axis, $\delta J_-$ is zero. The dynamics has only one degree of freedom corresponding to $\delta J_+$. This yields from the expression (A8),

$$h_{xt} = -\frac{\Delta J_0}{2} \left( 1 + h_{xx} \right) \left( 1 + \frac{h_{xt}}{1 + h_{xx}} \right). \quad (A9)$$

From Eq. (A3), we also have

$$S_+ = \frac{(\gamma + 1)}{2c_0^2} \left\{ \delta J_+ \frac{\partial \delta J_+}{\partial \partial_t} - c_0 \delta J_+ \frac{\partial^2 \delta J_+}{\partial x^2} \right\}$$

$$+ \frac{(\gamma + 1)}{2c_0^2} \left\{ \left( \frac{\partial \delta J_+}{\partial x} \right) \left( \frac{\partial \delta J_+}{\partial \partial_t} \right) - c_0 \left( \frac{\partial \delta J_+}{\partial x} \right)^2 \right\}, \quad (A10)$$

$$S_- = 0. \quad (A11)$$

Limiting ourselves to second order perturbations,

$$S_{\mu \nu} = \frac{(\gamma + 1)}{2c_0^2} \left\{ (2 - \gamma)h_{tx} + 2c_0h_{xx} \right\} \left( -c_0 \frac{\partial^2 h_{\mu \nu}}{\partial x^2} + \partial_{\mu} \partial_{\nu} h_{\mu \nu} \right)$$

$$+ \frac{(\gamma + 1)}{2c_0^2} \left( \partial_{\mu} h_{\mu \nu} \right) \left( \partial_{\nu} c_0 - \partial_{\nu} \partial_{\nu} \right) \left( (2 - \gamma)h_{tx} + 2c_0h_{xx} \right). \quad (A12)$$

However, the above expression can also be written in terms of $h_{xx}$ or $h_{xt}$, or a different linear combination of $h_{xx}$ and $h_{xt}$: In the case of simple waves, $h_{xx}$ and $h_{xt}$ are not independent due to relation (A9).

For a non-simple wave, we have to consider both $\delta J_+$ and $\delta J_-$, which are related to $h_{xx}$ and $h_{xt}$ by Eq. (A8). Therefore, we only look at the source terms for the $h_{tx}$ and $h_{xx}$ components. Note $h_{tt}$ is not an independent quantity, but is related to $h_{xt}$ and $h_{xx}$ via Eq. (A7). Using Eq. (A4), the source terms evaluated for $h_{xx}$ and $h_{tx}$ are, up to second power in $h_{\mu \nu},$

$$S_{xx} = \frac{r}{\Delta J_0} \left[ \frac{(5 - 3\gamma)\Delta J_0}{4c_0^2 r} \frac{\partial^2}{\partial \partial_t h_{tx} h_{xx}} - \frac{1}{c_0^2} \frac{\partial^2}{\partial x^2} h_{tx}^2 \right] + \frac{r}{\Delta J_0} \left[ -\frac{(\gamma - 1)}{2c_0^2} \frac{\Delta J_0}{2r} \frac{\partial^2}{\partial x^2} h_{xx} \right]$$

$$+ \frac{r(r - 1)}{\Delta J_0^2} \left[ \frac{1}{c_0^2} \frac{\partial h_{tx}}{\partial t} + \frac{\Delta J_0}{2r} \frac{\partial h_{xx}}{\partial x} \right]^2 - \frac{1}{c_0^2} \frac{\partial h_{tx}}{\partial x} + \frac{\Delta J_0}{2r c_0^2} \frac{\partial h_{xx}}{\partial t} \right]^2, \quad (A13)$$

$$S_{tx} = S_{xt} = -\frac{1}{2} \left[ \frac{1}{c_0^2} \frac{\partial^2}{\partial t \partial x} h_{tx}^2 + \left( \frac{\Delta J_0}{2r} \right) \frac{\partial^2}{\partial x^2} h_{tx}^2 \right] - \left[ \frac{1}{r} \frac{\partial^2}{\partial x^2} (h_{tx} h_{xx}) + \frac{\partial}{\partial x} (h_{tx} \partial_x h_{xx}) \right], \quad (A14)$$

where we defined $r = \frac{3 - \gamma}{\gamma - 1}$.

Appendix B: Calculating the mass flux

We here compute the time-averaged mass flux through $x = \xi_i$, at $\xi_i$, the boundary between source region and region of observation, $v = 0$ initially. The mass flux in a one-dimensional flow is $j$ is given by

$$j = \rho v$$

$$\Rightarrow \langle j \rangle = \rho_0 v + \langle \delta \rho v \rangle. \quad (B1)$$

Taking into account up to second order in $v$ terms, and using Eq. (30),

$$\langle j \rangle \approx \rho_0 \left( c_0 \langle v \rangle + \langle v^2 \rangle \right). \quad (B2)$$
Here, \( \langle v \rangle \) at \( x = \xi_i \) is given by

\[
\langle v \rangle = \frac{k c_0}{2\pi} \int_0^{k c_0} v dt = \frac{k c_0}{2\pi} \int_{\xi_i}^{\xi_i - \lambda} f(\xi) \frac{1}{\partial_\xi \xi} d\xi ,
\]

where \( \partial_\xi \xi \), the partial derivative of \( \xi \) with respect to \( t \) is evaluated at \( x = \xi_i \).

Using Eq. (33), \( \partial_\xi \xi \) at \( x = \xi_i \) is given by

\[
\partial_\xi \xi = -\left( \frac{c_0 + \frac{\gamma + 1}{2} f(\xi)}{1 + \frac{\gamma + 1}{2} f'(\xi) \frac{\xi_i - \xi}{c_0 + \frac{\gamma + 1}{2} f(\xi)}} \right) .
\]

We have the initial cosine profile, \( f(\xi) = A \cos k \xi \). Considering again terms up to second order, now in \( f(\xi) \), we evaluate the integral in Eq. (B3), and after some further manipulations, we find from Eq. (B2) the final result for the averaged mass flux

\[
\langle j \rangle = \rho_0 \frac{3 - \gamma A^2}{8 c_0} ,
\]

which is Eq. (52) in the main text.

[1] Carlos Barceló, Stefano Liberati, and Matt Visser, “Analogue Gravity,” *Living Reviews in Relativity* **14**, 3 (2011).

[2] Carlos Barceló, Stefano Liberati, and Matt Visser, “Analogue gravity from field theory normal modes?” *Classical and Quantum Gravity* **18**, 3595–3610 (2001).

[3] W. G. Unruh, “Experimental Black-Hole Evaporation?” *Phys. Rev. Lett.* **66**, 1351–1353 (1981).

[4] Trautman, Andrzej, “Comparison of Newtonian and Relativistic Theories of Space-time,” in *Perspectives in Geometry and Relativity*, Essays in Honor of Václav Hlavatý (Indiana University Press, 1966).

[5] Ralf Schützhold and William G. Unruh, “Gravity wave analogues of black holes,” *Phys. Rev. D* **66**, 044019 (2002).

[6] Silke Weinfurtner, Edmund W. Tedford, Matthew C. J. Pennrice, William G. Unruh, and Gregory A. Lawrence, “Measurement of Stimulated Hawking Emission in an Analogue System,” *Phys. Rev. Lett.* **106**, 021302 (2011).

[7] L.-P. Euvé, F. Michel, R. Parentani, T. G. Philbin, and M. Rousseaux, “Observation of Noise Correlated by the Hawking Effect in a Water Tank,” *Phys. Rev. Lett.* **117**, 121301 (2016).

[8] Léo-Paul Euvé, Scott Robertson, Nicolas James, Alessandro Fabbrì, and Germain Rousseaux, “Scattering of Co-Current Surface Waves on an Analogue Black Hole,” *Phys. Rev. Lett.* **124**, 141101 (2020).

[9] Francesco Marino, “Acoustic black holes in a two-dimensional “Photon fluid”,” *Phys. Rev. A* **78**, 063804 (2008).

[10] H. S. Nguyen, D. Gerace, I. Carusotto, D. Sanvitto, E. Galopin, A. Lemaître, I. Sagnes, J. Bloch, and A. Amo, “Acoustic Black Hole in a Stationary Hydrodynamic Flow of Microcavity Polaritons,” *Phys. Rev. Lett.* **114**, 036402 (2015).

[11] Iacopo Carusotto, Serena Fagnocchi, Alessio Recati, Roberto Balbinot, and Alessandro Fabbri, “Numerical observation of Hawking radiation from acoustic black holes in atomic Bose–Einstein condensates,” *New Journal of Physics* **10**, 103001 (2008).

[12] Jean Macher and Renaud Parentani, “Black-hole radiation in Bose-Einstein condensates,” *Phys. Rev. A* **80**, 043601 (2009).

[13] Dario Gerace and Iacopo Carusotto, “Analog Hawking radiation from an acoustic black hole in a flowing polaron superfluid,” *Phys. Rev. B* **86**, 144505 (2012).

[14] J. Steinhauser, “Observation of quantum Hawking radiation and its entanglement in an analogue black hole,” *Nat. Phys.* **12**, 959–965 (2016).

[15] Juan Ramón Muñoz de Nova, Karatrice Golubkova, Victor I. Kohlov, and Jeff Steinhauser, “Observation of thermal Hawking radiation and its temperature in an analogue black hole,” *Nature* **569**, 688–691 (2019).

[16] U. R. Fischer and R. Schützhold, “Quantum simulation of cosmic inflation in two-component Bose-Einstein condensates,” *Phys. Rev. A* **70**, 063615 (2004).

[17] Seok-Yeong Cha and Uwe R. Fischer, “Probing the Scale Invariance of the Inflationary Power Spectrum in Expanding Quasi-Two-Dimensional Dipolar Condensates,” *Phys. Rev. Lett.* **118**, 130404 (2017).

[18] S. Eckel, A. Kumar, T. Jacobson, I. B. Spielman, and G. K. Campbell, “A Rapidly Expanding Bose-Einstein Condensate: An Expanding Universe in the Lab,” *Phys. Rev. X* **8**, 021021 (2018).

[19] Stephen Eckel and Ted Jacobson, “Phonon redshift and Hubble friction in an expanding BEC,” *SciPost Phys.* **10**, 64 (2021).

[20] S. Banik, M. Gutierrez Galan, H. Sosa-Martinez, M. Anderson, S. Eckel, I. B. Spielman, and G. K. Campbell, “Hubble Attenuation and Amplification in Expanding and Contracting Cold-Atom Universes,” *SciPost Phys.* **10**, 063615 (2021).

[21] C. Barceló, S. Liberati, and M. Visser, “Probing semiclassical analog gravity in Bose-Einstein condensates with widely tunable interactions,” *Phys. Rev. A* **68**, 053613 (2003).

[22] P. O. Fedichev and U. R. Fischer, “Cosmological” quasiparticle production in harmonically trapped superfluid gases,” *Phys. Rev. A* **69**, 033602 (2004).

[23] Jeff Steinhauser, Murad Abuzarli, Tanguy Aladjidi, Tom Bienaimé, Clara Piekarski, Wei Liu, Elisabeth Giacobino, Alberto Bramati, and Quentin Glorieux, “Analogue cosmological particle creation in an ultracold quantum fluid of light,” (2021), arXiv:2102.08279 [cond-mat.quant-gas].
[24] Xavier Busch, Iacopo Carusotto, and Renaud Parentani, “Spectrum and entanglement of phonons in quantum fluids of light,” Phys. Rev. A 89, 043819 (2014).

[25] Scott Robertson, Florent Michel, and Renaud Parentani, “Assessing degrees of entanglement of phonon states in atomic Bose gases through the measurement of commutating observables,” Phys. Rev. D 96, 045012 (2017).

[26] Zehua Tian, Seok-Young Ch¯a, and Uwe R. Fischer, “Rotation entanglement in quenched dipolar Bose-Einstein condensates,” Phys. Rev. A 97, 063611 (2018).

[27] P. O. Fedichev and U. R. Fischer, “Gibbons-Hawking Effect in the Sonic de Sitter Space-Time of an Expanding Bose-Einstein-Condensed Gas,” Phys. Rev. Lett. 91, 240407 (2003).

[28] A. Retzker, J. I. Cirac, M. B. Plenio, and B. Reznik, “Methods for Detecting Acceleration Radiation in a Bose-Einstein Condensate,” Phys. Rev. Lett. 101, 110402 (2008).

[29] Arkadiusz Kosior, Maciej Lewenstein, and Alessio Celli, “Unruh effect for interacting particles with ultracold atoms,” SciPost Phys. 5, 61 (2018).

[30] Cisco Gooding, Steffen Biermann, Sebastian Erne, Jorma Louko, William G. Unruh, Jörg Schmiedmayer, and Silke Weinfurtner, “Interferometric Unruh Detectors for Bose-Einstein Condensates,” Phys. Rev. Lett. 125, 213603 (2020).

[31] Ralf Schützhold, Michael Uhlmann, Yan Xu, and Uwe R. Fischer, “Quantum backreaction in dilute Bose-Einstein condensates,” Phys. Rev. D 72, 105005 (2005).

[32] Zehua Tian and Jiangfeng Du, “Probing low-energy Lorentz violation from high-energy modified dispersion in dipolar Bose-Einstein condensates,” Phys. Rev. D 103, 085014 (2021).

[33] Steven Corley and Ted Jacobson, “Black hole lasers,” Phys. Rev. D 59, 124011 (1999).

[34] S. Finazzi and R. Parentani, “Black hole lasers in Bose–Einstein condensates,” New Journal of Physics 12, 095015 (2010).

[35] Soumen Basak and Parthasarathi Majumdar, “Super-resonance from a rotating acoustic black hole,” Classical and Quantum Gravity 20, 3907–3913 (2003).

[36] Theo Torres, Sam Patrick, Antonin Coutant, Maurício Richartz, Edmund W. Tedford, and Silke Weinfurtner, “Rotational superradiant scattering in a vortex flow,” Nature Physics 13, 833–836 (2017).

[37] Angus Prain, Calum Maitland, Daniele Faccio, and Francesco Marino, “Superradiant scattering in fluids of light,” Phys. Rev. D 100, 024037 (2019).

[38] Theo Torres, Sam Patrick, Maurício Richartz, and Silke Weinfurtner, “Quasinormal mode oscillations in an analogue black hole experiment,” Phys. Rev. Lett. 125, 011301 (2020).

[39] Daniel Hartley, Tupac Bravo, Dennis Rätzle, Richard Howl, and Ivette Fuentes, “Analogue simulation of gravitational waves in a 3 + 1-dimensional Bose-Einstein condensate,” Phys. Rev. D 98, 025011 (2018).

[40] Satadal Datta, “Acoustic analog of gravitational wave,” Phys. Rev. D 98, 064049 (2018).

[41] Stefano Liberati, Giovanni Tricella, and Andrea Trombettoni, “The Information Loss Problem: An Analogue Gravity Perspective,” Entropy 21, 940 (2019).

[42] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1982).

[43] M. Novello and E. Goulart, “Beyond analog gravity: the case of exceptional dynamics,” Classical and Quantum Gravity 28, 145022 (2011).

[44] Michael Stone, “Acoustic energy and momentum in a moving medium,” Phys. Rev. E 62, 1341–1350 (2000).

[45] In a quasilinear partial differential equation, the highest order derivatives of dependent variables occur linearly, with their coefficients functions of only lower order derivatives, whereas any term with lower order derivatives can occur nonlinearly.

[46] L. P. Grishchuk, A. N. Petrov, and A. D. Popova, “Exact theory of the (Einstein) gravitational field in an arbitrary background space-time,” Communications in Mathematical Physics 94, 379 – 396 (1984).

[47] S. Deser, “Gravity from self-interaction in a curved background,” Classical and Quantum Gravity 4, L99-L105 (1987).

[48] Suraj N. Gupta, “Gravitation and electromagnetism,” Phys. Rev. 96, 1683–1685 (1954).

[49] Bernhard Riemann, “Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsgewichte,” Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen 8, 43–66 (1860).

[50] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Second Edition: Volume 6 (Course of Theoretical Physics), 2nd ed. (Butterworth-Heinemann, 1987).

[51] Satadal Datta and Uwe R. Fischer, “Fluid-dynamical analogue of nonlinear gravitational wave memory,” arXiv:2011.05837 [gr-qc].

[52] See for the derivation of the 1+1D metric from the embedding 3+1D metric Ref. [53].

[53] Steven Weinberg, Gravitation and cosmology: principles and applications of the general theory of relativity (Wiley, 1972).

[54] See, e.g., the discussion of diffeomorphism invariance contained in the review [1].

[55] J. Söding, D. Guéry-Odelin, P. Desholees, F. Chevy, H. Inamori, and J. Dalibard, “Three-body decay of a rubidium Bose–Einstein condensate,” Applied Physics B 69, 257–261 (1999).

[56] R. Meppelink, S. B. Koller, J. M. Vogels, P. van der Straten, E. D. van Ooijen, N. R. Heckenberg, H. Rubinsztein-Dunlop, S. A. Haine, and M. J. Davis, “Observation of shock waves in a large Bose-Einstein condensate,” Phys. Rev. A 80, 043606 (2009).

[57] Bogdan Damski, “Formation of shock waves in a Bose-Einstein condensate,” Phys. Rev. A 69, 043610 (2004).

[58] T. Bienaimé, M. Isoard, O. Fontaine, A. Bramati, A. M. Kamchatnov, Q. Glorieux, and N. Pavloff, “Quantitative Analysis of Shock Wave Dynamics in a Fluid of Light,” Phys. Rev. Lett. 126, 183901 (2021).

[59] Alexander L. Gaunt, Tobias F. Schmidt, Igor Gottliebovych, Robert P. Smith, and Zoran Hadzibabic, “Bose-Einstein Condensation of Atoms in a Uniform Potential,” Phys. Rev. Lett. 110, 200406 (2013).
[62] J. Denschlag, J. E. Simsarian, D. L. Feder, Charles W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, “Generating Solitons by Phase Engineering of a Bose-Einstein Condensate,” Science 287, 97–101 (2000).

[63] A. E. Leanhardt, A. G"orlitz, A. P. Chikkatur, D. Kielpinski, Y. Shin, D. E. Pritchard, and W. Ketterle, “Imprinting Vortices in a Bose-Einstein Condensate using Topological Phases,” Phys. Rev. Lett. 89, 190403 (2002).

[64] Albert Einstein, Collected papers (Princeton University Press Princeton, 1987).

[65] Carlos Barceló, S Liberati, and Matt Visser, “Analogue gravity from Bose-Einstein condensates,” Classical and Quantum Gravity 18, 1137–1156 (2001).

[66] Silke Weinfurtner, Piyush Jain, Matt Visser, and C. W. Gardiner, “Cosmological particle production in emergent rainbow spacetimes,” Classical and Quantum Gravity 26, 065012 (2009).