Macroscopic traversable wormhole solutions to Einstein’s field equations in (2+1) and (3+1) dimensions with a cosmological constant are investigated. Ensuring traversability severely constrains the material used to generate the wormhole’s spacetime curvature. Although the presence of a cosmological constant modifies to some extent the type of matter permitted (for example it is possible to have a positive energy density for the material threading the throat of the wormhole in (2+1) dimensions), the material must still be “exotic”, that is matter with a larger radial tension than total mass-energy density multiplied by \( c^2 \). Two specific solutions are applied to the general cases and a partial stability analysis of a (2+1) dimensional solution is explored.
1 Introduction

Wormholes are tunnels in the geometry of space and time that connect two separate and distinct regions of spacetime. These regions may either be part of the same universe or be regions of two different universes. Although such objects were long known to be solutions to Einstein’s equations, early work on macroscopic wormholes (those considered large enough for interstellar travel) led to the conclusion that at best they were either unstable or hidden behind event horizons, so as not to be traversable by living entities. For this reason wormholes have only intermittently been studied, even though they predate black holes as objects of interest in relativity.

Recently, a renaissance in the study of wormholes has taken place, instigated by a closer investigation of conditions necessary to ensure their traversability [1]. This interest is motivated in part by the possibility that quantum gravity might permit the formation of “exotic” material (ie. matter which violates the positive energy condition) necessary to construct traversable wormholes. Furthermore, processes involving quantum gravity are important in both early universe cosmology [2] and the final stages of black hole evaporation [3].

The main focus of this paper is traversable macroscopic wormholes in (2 + 1) and (3 + 1) dimensional spacetimes with a cosmological constant. Previous investigations into wormholes have not considered this case [1, 4], although there has been some investigation of the Wheeler-DeWitt equation for wormholes in spacetimes with a cosmological constant [5]. As the cosmological constant can be interpreted as the energy density of the vacuum one might expect it to modify the form of the exotic matter required. As we shall see, this does in fact occur, although not in such a manner as to avoid violation of the weak-energy condition. A positive constant will produce an expansion term that counteracts gravity, while a negative constant will aid gravitational collapse of the wormhole. We will consider the case of a positive constant (ie. anti-de Sitter spacetime) in this paper unless otherwise stated.

Proceeding as in ref. [4], we consider what constraints are required for the wormhole to be traversable in (2 + 1) and (3 + 1) dimensions. Section III presents the needed tensors and solves Einstein’s field equations. Through the use of embedding, inequality restrictions on the type of matter that is needed to generate the wormhole’s spacetime curvature are presented, modi-
fying the results of ref. [3]. Section 4 discusses the mathematical equivalents of the traversability criteria. These criteria are utilized in Section 5 where the zero-tidal force solutions for both (2 + 1) and (3 + 1) dimensions. Additionally the “Junction Condition” Formalism solution with stability analysis for the simplest case is investigated in (2 + 1) dimensions.

Wormhole solutions that connect regions in the same universe or regions in two different universes are the same, but the topology for each case, which is not constrained by the field equations, is different. The important facet of the wormhole is the throat which is a finite spatial region between the wormhole ‘mouths’, at which the embedding surface (discussed below) will be vertical. This exists at a (possibly large) set of circles (spheres) of minimum radius $r$, where $r$ is the radial coordinate, producing a throat that travellers could use as their bridge.

We employ (in the context of the Einstein field equations) the traversability properties of ref. [1] in (3 + 1) dimensions and of [4] in (2 + 1) dimensions. Briefly, these are

1. The metric should be spherically (radially) symmetric and static (i.e. no time dependence).
2. By definition, the solution must have a throat which connects two regions of spacetime.
3. There should be no horizon so as to permit two way travel.
4. The tidal gravitational forces experienced by a traveller must be reasonably small (e.g. approximately one Earth gravity).
5. The time to traverse the wormhole must be reasonably short (e.g. one year) as measured by both the traveller and any observers who wait on the outside of the wormhole.

Criteria (1) through (3) can be thought of as basic to constructing wormholes, while properties (4) and (5) are usability criteria. Property (1) is not a requirement for the wormhole solution, but it greatly simplifies the calculations. It is still possible that the wormhole might be unstable to radial or non-radial perturbations, a subject which we will investigate to a certain extent in section 5.
We choose the following metrics for the \((2 + 1)\) and \((3 + 1)\) cases respectively:

\[
ds^2 = -e^{2\Phi(r)}c^2 dt^2 + \frac{dr^2}{\Lambda r^2 - M(r)} + r^2 d\phi^2,
\]

\[
ds^2 = -e^{2\Phi(r)}c^2 dt^2 + \frac{dr^2}{\Lambda r^2 - M(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2),
\]

where \(\Lambda\) is the cosmological constant with units of \(\text{cm}^{-2}\) and \(M\) and \(\Phi\) are arbitrary functions of the radial coordinate \(r\) (unless otherwise specified \(M \equiv M(r)\) and \(\Phi \equiv \Phi(r)\)). The \((3 + 1)\) dimensional metric is similar to the DeSitter metric with the constant incorporated into the function \(M\) to simplify calculations and make it easier to identify the physics involved. The shape of the wormhole is specifically chosen by the designer and is modified by the “shape function” \(M\) \(\square\). \(\Phi\) determines the gravitational redshift and will be referred to as the “redshift” function. We see that the metrics are respectively circularly and spherically symmetric, and static. Choosing \(\Phi(r)\) finite ensures that there is no horizon (the time component does not vanish), and note that there is in fact a throat when \(\Lambda r^2\) equals \(M\), and when \(\Lambda r^2/3\) equals \(M/r\) respectively.

2 Stress-Energy Constraints

Consider metric \((\square)\) above. By introducing a set of orthonormal (hatted) basis vectors (that is the reference frame of a set of observers who remain always at rest in the coordinate system), the mathematics and physical interpretations become greatly simplified. The new basis vectors are,

\[
e_{\hat{t}} = e^{-\Phi}e_t, \quad e_{\hat{r}} = (\Lambda r^2 - M)^{1/2}e_r, \quad e_{\hat{\phi}} = r^{-1}e_{\phi},
\]

where \(e_t = c^{-1}\partial/\partial t, e_r = \partial/\partial r,\) and \(e_{\phi} = \partial/\partial \phi\). This gives \(g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}},\) and we find the following Riemann tensor components:

\[
R_{\hat{t}\hat{r}\hat{r}} = e^{-2\Phi}(\Lambda r^2 - M)R_{trtr},
\]

\[
= (\Lambda r^2 - M) \left[ \Phi'' + \frac{2\Lambda r - M'}{2(\Lambda r^2 - M)}\Phi' + (\Phi')^2 \right],
\]

\[
R_{\hat{t}\hat{\phi}\hat{\phi}} = e^{-2\Phi}r^{-2}R_{t\phi\phi} = (\Lambda r^2 - M)\Phi'/r,
\]
\[ R_{\hat{r}\hat{r}\hat{\phi}\hat{\phi}} = r^{-2}(\Lambda r^2 - M)R_{\hat{r}\hat{r}\hat{\phi}\hat{\phi}} = \frac{M' - 2\Lambda r}{2r}, \]  

(6)

where the prime denotes a derivative with respect to \( r \). The Einstein tensor \( G_{\hat{\alpha}\hat{\beta}} \equiv R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2} g_{\hat{\alpha}\hat{\beta}} R \) has the following non-zero components:

\[ G_{\hat{t}\hat{t}} = \frac{M'}{r} - \Lambda, \]  

(7)

\[ G_{\hat{r}\hat{r}} = \frac{(\Lambda r^2 - M) \Phi'}{r}, \]  

(8)

\[ G_{\hat{\phi}\hat{\phi}} = (\Lambda r^2 - M) \left[ \Phi'' + \frac{2\Lambda r - M'}{2(\Lambda r^2 - M)} \Phi' + (\Phi')^2 \right]. \]  

(9)

Using the same method on (2), we get the following for the (3+1) dimensional case:

\[ G_{\hat{t}\hat{t}} = \frac{1 + M' - \Lambda r^2}{r^2}, \]  

(10)

\[ G_{\hat{r}\hat{r}} = \frac{2\Phi'}{r^2} \left( \frac{\Lambda r^3}{3} - M \right) + \frac{\Lambda r^3 - 3M - 3r}{3r^3}, \]  

(11)

\[ G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \left( \frac{\Lambda r^3}{3} - M \right) \left[ \Phi'' + \Phi' \left( \frac{2\Lambda r^3 - 3r M' + 3M}{2r(\Lambda r^3 - 3M)} + \frac{1}{r} \right) \right. \]

\[ \left. + (\Phi')^2 + \frac{2\Lambda r^3 - 3r M' + 3M}{2r^2(\Lambda r^3 - 3M)} \right]. \]  

(12)

The field equations

\[ G_{\hat{\alpha}\hat{\beta}} - \Lambda g_{\hat{\alpha}\hat{\beta}} = \kappa T_{\hat{\alpha}\hat{\beta}}, \]  

(13)

where \( \kappa \equiv 8\pi G c^{-4} \) imply that the stress-energy tensor \( T_{\hat{\alpha}\hat{\beta}} \) must be of the same form as the Einstein tensor. Hence the only non-zero components are

\[ T_{\hat{t}\hat{t}} = \rho(r)c^2, \ T_{\hat{r}\hat{r}} = -\tau(r), \ T_{\hat{\theta}\hat{\theta}} = p(r) = T_{\hat{\phi}\hat{\phi}}, \]  

(14)

where \( \rho(r) \) is the total density of mass-energy that the static observers measure in units of \( g/cm^2 \) (\( g/cm^3 \) for the \( 3 + 1 \) case); \( \tau(r) \) is the radial tension per unit area that they measure [it is the negative of the radial pressure and has units dyn/cm]; and \( p(r) \) is the lateral pressure that they measure in directions orthogonal to the radial direction.
Solving these equations for $\rho(r)$, $\tau(r)$, and $p(r)$ we get:

$$\rho(r) = \frac{c^2}{8\pi G} \left( \frac{M'}{2r} \right),$$

(15)

$$\tau(r) = \frac{c^4}{8\pi G} \left( \Lambda - \frac{\Phi'}{r}(\Lambda r^2 - M) \right),$$

(16)

and

$$p(r) = \frac{c^4}{8\pi G} \left[ (\Lambda r^2 - M) \left( \Phi'' + \frac{2\Lambda r - M'}{2(\Lambda r^2 - M)} \Phi' + (\Phi')^2 \right) - \Lambda \right],$$

(17)

for the $(2+1)$ case and

$$\rho(r) = \frac{c^2}{8\pi G r^2} (1 + M'),$$

(18)

$$\tau(r) = \frac{c^4}{8\pi G r^2} \left[ \left( \frac{2\Lambda r^2}{3} + \frac{M}{r} + 1 \right) - 2\Phi' r \left( \frac{\Lambda r^2}{3} - \frac{M}{r} \right) \right],$$

(19)

and

$$p(r) = \frac{c^4}{8\pi G} \left[ \left( \frac{\Lambda r^2}{3} - \frac{M}{r} \right) \left( \Phi'' + \left( \frac{2\Lambda r^3 - 3r M' + 3M}{2r(\Lambda r^3 - 3M)} + \frac{1}{r} \right) \Phi' \right. \\
\left. + (\Phi')^2 + \frac{2\Lambda r^3 - 3r M' + 3M}{2r^2(\Lambda r^3 - 3M)} \right) - \Lambda \right],$$

(20)

for the $(3+1)$ case. The typical way to solve these equations would be to assume a particular type of material, and then derive equations of state based on that material. This would then give the three equations above with two equations of state (one for $\tau(\rho)$, and one for $p(\rho)$) for a total of five equations with five unknown functions of $r$. Since we are concerned with traversable wormholes with certain properties, the solutions come by tailoring the choices of $\Phi(r)$ and $M(r)$ for a “nice” wormhole. In this fashion we have the three equations above as functions of $r$. The requirement of this method is that the wormhole builders must in some way obtain the material with the appropriate stress-energy tensor so calculated. The above process is applied in Section 4 to the zero tidal-force solution.
Consider next the embedding of the spatial geometry of the wormhole into a flat space of one higher dimension. First, take a “slice” of the metric at constant time (at \( t = t_0 = \text{constant}, \, dt = 0 \)), and embed this slice into the usual flat Lorentzian space:

\[
ds^2 = dz^2 + dr^2 + r^2 d\phi^2,
\]
(21)
and for the \((3+1)\) case

\[
ds^2 = dz^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]
(22)

The geometry of (21), and (22) implies \( z = z(r) \) due to the circular and spherical symmetry. Hence we can solve for \( z(r) \) by setting the metrics for each case equal to get,

\[
\frac{dz}{dr} = \pm \left( \frac{1 - \Lambda r^2 + M}{\Lambda r^2 - M} \right)^{1/2},
\]
(23)
for the \((2+1)\) case and for the \((3+1)\) case:

\[
\frac{dz}{dr} = \pm \left( \frac{1 - \frac{\Lambda r^2}{3} + \frac{M}{r}}{\frac{\Lambda r^2}{3} - \frac{M}{r}} \right)^{1/2}.
\]
(24)

Solving these equations will yield the functional form \( z(r) \) of the two-dimensional embedding surface.

Note that embedding in flat space is possible only if \( \frac{dz}{dr} \) is real, implying

\[
M_{(2+1)} \leq \Lambda r^2 \leq M_{(2+1)} + 1 \quad \text{and} \quad M_{(3+1)} \leq \Lambda r^3/3 \leq M_{(3+1)} + 1
\]
(25)
for the respective cases. This is equivalent to requiring that \(|dr/dl| \leq 1\), where

\[
dl_{2+1} = \frac{\pm dr}{\sqrt{\Lambda r^2 - M}}, \quad dl_{3+1} = \frac{\pm dr}{\sqrt{\frac{\Lambda r^3}{3} - \frac{M}{r}}}.
\]
(26)

The coordinate \( l \) represents the proper radial distance from the wormhole and ranges from zero to infinity.

Geometrical visualization is not the only use of embedding. Since the throat is a minimum radius from the \( z \)-axis and we know that at some point away from the wormhole the space is essentially flat, (this is known since
a positive $\Lambda$ acts as a repulsive force at large distances, and must at some radius balance gravity) we know that the embedding surface flares outward. For this to be true the matter distribution must be small relative to $\Lambda^{-1/2}$ so that the spacetime far from the wormhole is not greatly affected by the matter. Additionally, the wormhole itself must also be small compared to the characteristic length so that it is not locally deSitter and hence indistinguishable from the surrounding Lorentzian space. Assuming these properties we take $d^2r/dz^2 > 0$ and $r(z)$ to be a minimum at the throat. Inverting (23) and (24) and taking the derivative respectively gives the following useful conditions:

$$M' < 2\Lambda r, \quad \text{and} \quad \frac{M'}{r} < \frac{2\Lambda}{3} + \frac{M}{r^2}. \quad (27)$$

By applying the first inequality of (25) and (27) we get

$$\rho(r) < \frac{c^2\Lambda}{8\pi G}, \quad (28)$$

in $(2+1)$ dimensions. Note that when $\Lambda = 0$ [4] $\rho < 0$; we see that the addition of a positive $\Lambda$ increases the energy density for that case. In $(3+1)$ dimensions we get

$$\rho(r) < \frac{c^2}{8\pi Gr^2} \left(1 + \frac{4\Lambda r^2}{3}\right). \quad (29)$$

in agreement the $\Lambda = 0$ case in that both allow for the possibility of a positive mass-energy for a positive $\Lambda$ [4].

Another physical aspect of the wormhole can be investigated by forming the dimensionless ratio [4]

$$\zeta \equiv \frac{\tau - \rho c^2}{|\rho c^2|}. \quad (30)$$

For regular matter (that respects the weak energy condition) this ratio is negative; using conditions (27) and (24) at the throat (where the restrictions are the most severe) along with the fact that $\Phi'$ is finite, we find for both cases that

$$\zeta_0 \equiv \frac{\tau_0 - \rho_0 c^2}{|\rho_0 c^2|} > 0. \quad (31)$$

Here the subscript 0 signifies that the quantity is evaluated at the minimum radius. That is $r = r_0 \neq 0 \Rightarrow \Lambda r_0^2 - M_0 = 0$ for (23), and $\Lambda r_0^3 = 3M_0$ for equation (24), where $M_0 \equiv M(r_0)$. As in ref. [4] we shall call material
with the property $\tau > \rho c^2 > 0$, “exotic”. This terminology arises because an observer moving through the throat with sufficiently large velocity will necessarily see a negative mass-energy density.

A negative density of mass-energy implies a violation of the “weak energy condition” $[1]$. This in itself gives no basis for immediately discarding solutions as the weak energy condition has been experimentally shown to be false in $(3+1)$ dimensions, the Casimir effect being the perhaps the best known. Whether or not exotic matter can be formed in macroscopic quantities is still an open question.

### 3 Traversability Criteria

As discussed in Section 1, in order for the wormhole to be useful for theoretical $(2+1)$ dimensional beings it must take a reasonable time to traverse the wormhole as seen by both the traveller and those waiting outside the wormhole. Additionally, the tidal forces experienced by the traveller must not be too great (say the equivalent to one Earth gravity).

These criteria are analogous to those in references $[1]$ and $[4]$. Consider a wormhole that joins two distinct universes. Assume that the traveller starts at location $l = -l_1$ in a lower universe (pictorially speaking) at rest and ends at $l = l_2$ in an upper universe at rest. As in special relativity, denote $\gamma \equiv (1 - (v/c)^2)^{-1/2}$. Noting that $dl$ is the distance travelled, $dr$ is the radius travelled, $dt$ is the coordinate time lapse, and $d\tau_T$ is the proper time lapse as seen by the observer, we have for the $(2+1)$ case

$$v = \frac{dl}{e^\Phi dt} = \mp \frac{dr}{\sqrt{\Lambda r^2 - M e^\Phi dt}}, \text{ and } v\gamma = \frac{dl}{d\tau_T} = \mp \frac{dr}{\sqrt{\Lambda r^2 - M d\tau_T}}, \quad (32)$$

and for the $(3+1)$ case

$$v = \mp \frac{dr}{\sqrt{\Lambda r^2 - M e^\Phi dt}}, \text{ and } v\gamma = \mp \frac{dr}{\sqrt{\Lambda r^2 - M d\tau_T}}, \quad (33)$$

where the $(-)$ sign refers to the first half of the trip from station $l_1$ to the wormhole and the $(+)$ sign refers to the second half trip from the wormhole to station $l_2$.

It is desirable that the effects of the wormhole are small at the stations. This can be effectively obtained by specifying: i) the positions of the stations
such that the geometry is essentially (to less than one percent) the same as
space which is not affected by the wormhole \((M_{(2+1)} \ll \Lambda r^2 \text{ or } M_{(3+1)}/r \ll \Lambda r^2/3)\); ii) the gravitational red-shift of signals sent off to infinity from the
stations is small \(|\Phi| \ll 1\); iii) the “acceleration of gravity” as measured at
the stations is no more than 1 Earth gravity \(\equiv g_\oplus (= 980 \text{cm/s}^2)\) \[\|\];

\[
|g| \approx |- \Phi' c^2| \leq g_\oplus \tag{34}
\]

Now, the time criteria can be satisfied by the equations

\[
\Delta \tau_T = \int_{-l_1}^{l_2} \frac{dl}{v\gamma} \leq 1 \text{ yr.}, \text{ for the traveler and,} \tag{35}
\]

\[
\Delta t = \int_{-l_1}^{l_2} \frac{dl}{v e} \Phi \leq 1 \text{ yr.}, \text{ for outside observers.} \tag{36}
\]

It is also reasonable to assume that the maximum acceleration the traveller
can experience is about 1 Earth gravity. This gives the restriction

\[
\left| e^{-\Phi} d(\gamma e^\Phi) \right| d\ell \leq \frac{g_\oplus}{c^2} \simeq \frac{1}{0.97} \frac{1}{1 \text{ yr.}} \tag{37}
\]

In general, the tidal acceleration is given by

\[
\Delta a^\alpha = -c^2 R^\alpha_{\beta \gamma \delta} u^\beta \xi^\gamma u^\delta. \tag{38}
\]

where \(\xi^\mu\) is a spacelike vector which denotes the separation of two parts
of the traveller’s body and \(u^\mu\) is the four-velocity. Note that \(u^\alpha = \delta^\alpha_0\) and
\(\xi^\alpha = 0\) in the travellers frame. Additionally since the Riemann tensor is anti-
symmetric in its first two indices, the tidal accelerations are purely spatial
with components,

\[
\Delta a^\alpha = -c^2 R^\alpha_{\beta \gamma \delta} u^\beta \xi^\gamma u^\delta. \tag{39}
\]

Taking the size of a human to be \(|\xi| \sim 2\text{m and } |\Delta a| \leq g_\oplus\) for \(\xi\) oriented
along any spatial direction in the travelers frame, then the Riemann tensor
components are constrained to obey,

\[
|R^\alpha_{\beta \gamma \delta}| = \left| (\Lambda r^2 - M) \left( \Phi'' + \frac{2\Lambda r - M'}{2(\Lambda r^2 - M)} \Phi' + (\Phi')^2 \right) \right| \leq \frac{g_\oplus}{c^2 \cdot 2 \text{m}} \simeq \frac{1}{(10^{10}\text{cm})^2}. \tag{40}
\]
and

\[ |R_{2\psi'2\phi'}| = \left| \frac{\gamma^2}{2r} \left[ \left( \frac{v}{c} \right)^2 (M' - 2\Lambda r) - 2\Phi'(\Lambda r^2 - M) \right] \right| \leq \frac{g_{\oplus}}{c^2 \cdot 2m} \approx \frac{1}{(10^{10}\text{cm})^2}, \quad (41) \]

for the (2 + 1) case and for the (3 + 1) case

\[ |R_{1\psi'1\phi'}| = \left| \left( \frac{\Lambda r^2}{3} - \frac{M}{r} \right) \left( \Phi'' + \frac{2\Lambda r^3 - 3rM' + 3M}{2r(\Lambda r^3 - 3M)} \Phi' + (\Phi')^2 \right) \right| \leq \frac{g_{\oplus}}{c^2 \cdot 2m} \approx \frac{1}{(10^{10}\text{cm})^2}, \quad (42) \]

and

\[ |R_{2\psi'2\phi'}| = \left| \frac{\gamma^2}{2r} \left[ \left( \frac{v}{c} \right)^2 \left( \frac{2\Lambda r}{3} - \frac{M' + M}{r^2} \right) + 2\Phi' \left( \frac{\Lambda r^2}{3} - \frac{M}{r} \right) \right] \right| \leq \frac{g_{\oplus}}{c^2 \cdot 2m} \approx \frac{1}{(10^{10}\text{cm})^2}. \quad (43) \]

Equations (40) and (42) represent the radial tidal force constraint and can be regarded as constraining the function \( \Phi(r) \) while equations (41) and (43) represent the lateral tidal force and restrict the speed \( v \) of the traveller while crossing the wormhole.

4 Wormhole Solutions

The first solution presented results when \( \Phi = 0 \) everywhere. This solution is called the zero tidal force because a stationary observer \( (v = 0) \) will not experience any tidal forces [c.f. Eqs. (40), (41) and (42), (43)]. Consider the following choices which satisfy the conditions of Section 1, equations (25), (27), and (28), and allow for fairly simple integral equations;

\[ \Phi = 0, \quad M_{(2+1)} = -k\Lambda r^2 + k/100, \quad M_{(3+1)} = -k\Lambda r^3/3 + kr/100 \quad (44) \]

where \( k \) is some constant greater than zero so that the signature of \( dr \) is maintained and we have a throat. Substitution of (44a,b) into (49), (10),
and (17), and (44a,c) into (18), (19), and (20) gives:

\[
\frac{c^2 \rho(r)}{k} = -\tau(r) = p(r) = \frac{-c^4 \Lambda}{8\pi G}, \tag{45}
\]

for the (2 + 1) case and for the (3 + 1) case:

\[
\rho(r) = \frac{c^2}{8\pi Gr^2} (1 + k/100 - k\Lambda r^2), \tag{46}
\]
\[
\tau(r) = \frac{c^4}{8\pi Gr^2} \left( \frac{\Lambda r^2}{3} (2 - k) + \frac{k}{100} + 1 \right), \tag{47}
\]
\[
p(r) = \frac{\Lambda c^4}{24\pi G} (k - 2). \tag{48}
\]

Equation (45) shows that the energy-density of exotic matter depends
on the value of \(\Lambda\) and is constant throughout the universe. This makes the
solution unrealistic. A possible way around this problem would be to create
the wormhole within a relativistic vacuum bubble as described in [4]; the
interior of the bubble would have a non-zero \(\Lambda\), but the exterior could have
\(\Lambda = 0\). Such a bubble would contain the exotic matter in some finite volume
of spacetime. Travellers wishing to go through the wormhole would have to
pass through the discontinuity at the surface of the bubble. Equation (46)
shows that the exotic matter extends out to infinity and \(\rho(r)\) approaches a
constant as \(r \to \infty\). A vacuum bubble can only partially help in this case.
We would still need a very large amount of exotic material, extending out to
infinity, to create such a wormhole. The only other way around this is too
have a radial cutoff of the stress-energy (see [4]).

Integrating (26) for these solutions yields the proper radial distances as
functions of \(r\):

\[
l(r) = \pm \frac{1}{\sqrt{b}} \ln \left[ \frac{r + \sqrt{r^2 - \frac{k}{100}}}{\sqrt{\frac{k}{100}}} \right], \tag{49}
\]

where \(b_{(2+1)} = \Lambda(1 + k)\), and \(b_{(3+1)} = \Lambda(1 + k)/3\). The two stations
are located at a distance \(\Lambda r^2 - M \approx 1\) for the (2 + 1) case and \(\Lambda r^2/3 - M/r \approx 1\)
for the (3 + 1) case. Taking this to be true to 1% gives for both cases
\(r \approx \sqrt{(100 + k)/(100b)}\). Note that the stations are located at the maximum
value of \(r\) that still allows embedding. In order to use these solutions we must
therefore have a method of cutting off larger $r$ values (see above paragraph). These values of $r_0$ and $r$ increasingly conform to the requirement that they be smaller than the characteristic length as $k$ increases.

Assuming that $(v/c) \ll 1$ gives $\gamma \approx 1$, with (41) and (43) we get:

$$v \leq \frac{1}{\sqrt{b}} \text{ ms}^{-1}. \quad (50)$$

where $b$ is defined for each case as before. Hence (43) and (46) show us that for $\gamma \approx 1$:

$$\Delta \tau_T \approx \Delta t \approx \int_{-l_1}^{l_2} \frac{dl}{v} \approx 2\ln \left[ \frac{10 + \sqrt{100 + k}}{\sqrt{k}} \right] \text{ s}. \quad (51)$$

These times depend only on the value of the constant $k$. Large $k$ values are better as they keep the wormhole smaller than the characteristic length, but also reduce the proper time ($t \to 0$ as $k \to \infty$). The advantages of such a wormhole for human use is evident.

The next solution limits the exotic material to a circle of radius $a$ around the wormhole. Here we use the “Junction Condition” or “Boundary Layer” formalism [7, 8, 9, 10]. This solution will only be attempted for the (2 + 1) dimensional case. The (3 + 1) dimensional case can be done in an identical manner if needed. The model is constructed by surgically grafting the usual spacetime of (1) between two identical (by oppositely directed) spacetimes of the form

$$ds^2 = -(\Lambda r^2 - M_+)c^2 dt^2 + \frac{dr^2}{\Lambda r^2 - M_+} + r^2 d\phi^2, \quad (52)$$

and,

$$ds^2 = -(\Lambda r^2 - M_-)c^2 dt^2 + \frac{dr^2}{\Lambda r^2 - M_-} + r^2 d\phi^2. \quad (53)$$

Note that $M_+$, and $M_-$ are constants, not functions. By symmetry, and the matching of the metrics for a continuous solution, it should be evident that $M_+ = M_- = M$. Although the value of $M$ will be equivalent to that of $M(r = a)$ in equation (1), we will not let $M(a) = M$ as the derivatives will most likely not be equal. This solution is a little different than previous solutions as we now have a three part sandwich instead of typical solutions which have simply two oppositely directed identical spacetimes.
The two outer solutions have a zero stress-energy, while the two boundary layers will both have a non-zero stress-energy. The magnitude of this stress-energy can be calculated in terms of the second fundamental forms (equivalent to the extrinsic curvature tensor components) at the boundaries. For the following general formulas see ref. [7]. Adopting Riemann normal coordinates at the junctions: \( \eta \) denotes the coordinate normal to the junction with \( \eta \) positive in the manifold described by (52) and negative in the manifold described by (1); and \( x^\mu = (\tau, \phi, \eta) \). The second fundamental forms are given by:

\[
K_{ij}^{\pm} = \frac{1}{2} g^{ik} \frac{\partial g_{kj}}{\partial \eta} \bigg|_{\eta=\pm 0} = \frac{1}{2} \frac{\partial r}{\partial \eta} \bigg|_{r=a} g^{ik} \frac{\partial g_{kj}}{\partial r} \bigg|_{r=a}.
\]  

(54)

The discontinuity in the second forms is then given by,

\[
K_{ij} \equiv K_{ij}^+ - K_{ij}^-.
\]

(55)

Conservation of stress-energy constrains the line stress-energy so that the only non-zero components are \( S_{ij}^\eta \) with \( S^{\eta \eta} = 0 = S^{\eta i} \). Additionally, the Einstein field equations lead to

\[
S_{ij} = -\frac{c^4}{8\pi G} (K_{ij}^\eta - \delta^\eta_j K_k^\eta).
\]

(56)

Circular symmetry allows us to write

\[
K_{ij} = \begin{pmatrix} K_{\tau \tau} & 0 \\ 0 & K_{\theta \theta} \end{pmatrix},
\]

(57)

while the line stress-energy tensor in terms of the line energy density \( \sigma \) and line tension \( \vartheta \) is

\[
S_{ij} = \begin{pmatrix} -\sigma & 0 \\ 0 & -\vartheta \end{pmatrix}.
\]

(58)

We see from (56) that the field equations become

\[
\sigma = -\frac{c^4}{8\pi G} K_{\theta \theta} \quad \text{and} \quad \vartheta = -\frac{c^4}{8\pi G} K_{\tau \tau}.
\]

(59)

Considering the positive side of the top boundary [c.f. (52)] we note that \( \partial r/\partial \eta = \sqrt{\Lambda r^2 - M} \) and \( \partial r/\partial \eta = -\sqrt{\Lambda r^2 - M(r)} \) being assigned to the
negative side of the top boundary. Armed with (54), (55), and (59), and letting \( r = a \) at the boundary, we readily get

\[
\sigma = -\frac{c^4}{8\pi Ga} \left( \sqrt{\Lambda a^2 - M} + \sqrt{\Lambda a^2 - M(a)} \right),
\]

(60)

\[
\vartheta = -\frac{c^4}{8\pi G} \left( \frac{\Lambda a}{\sqrt{\Lambda a^2 - M}} + \Phi'(a)\sqrt{\Lambda a^2 - M(a)} \right).
\]

(61)

As expected, the energy density is negative at the throat of the wormhole. This simply implies that we have exotic matter, as previously discussed. The line tension is also negative, which implies that there is a line pressure as opposed to a line tension. This was also expected as a pressure would be needed to prevent the collapse of the wormhole throat.

A dynamic analysis of this solution can be obtained by letting the radius be a function of time \( a \mapsto a(t) \). This method closely parallels that of [7]. Let the throat by described by \( x^\mu(t, \phi) = (t, a(t), \phi) \). For the top boundary we get the three-velocity of a piece of stress energy at the throat given by:

\[
U^\mu = \left( \frac{dt}{d\tau}, \frac{da}{d\tau}, 0 \right) = \left( \frac{\sqrt{\Lambda a^2 - M + \dot{a}^2}}{\Lambda a^2 - M}, \dot{a}, 0 \right).
\]

(62)

The unit normal to the boundary can be calculated from the conditions \( U^\mu \xi_\mu = 0 \) and \( \xi^\mu \xi_\mu = 1 \). The result is

\[
\xi^\mu = \left( \frac{\dot{a}}{\Lambda a^2 - M}, \frac{\sqrt{\Lambda a^2 - M + \dot{a}^2}}{\Lambda a^2 - M}, 0 \right).
\]

(63)

The \( \phi\phi \) component is easily calculated from:

\[
K^\phi_\phi = \frac{1}{r} \frac{\partial r}{\partial \eta} \bigg|_{r=a},
\]

(64)

giving

\[
K^\phi_\phi = \frac{\sqrt{\Lambda a^2 - M + \dot{a}^2}}{a}.
\]

(65)

The \( \tau\tau \) component is more difficult using brute force. Another method is as follows. First note that (see [7])

\[
K^\tau_\tau = \xi_\mu (U^\mu, U^\nu) = \xi_\mu A^\mu,
\]

(66)
where $A^\mu$ is the three-acceleration of the throat. Now $A^\mu \equiv A_\xi^\mu$ by circular symmetry so that, $K^\tau_\tau = A \equiv$ magnitude of the three-acceleration. $A$ is determined by using the Killing vector $k^\mu \equiv (\partial / \partial t)^\mu = (1, 0, 0)$ for the underlying geometry. In addition $k^\mu = -(\Lambda a^2 - M, 0, 0)$, and

$$k_\mu \xi^\mu = -\dot{a},$$

$$k_\mu U^\mu = -\sqrt{\Lambda a^2 - M + \dot{a}^2}. \quad (67)$$

Comparing

$$\frac{D}{D\tau} (k_\mu U^\mu) = k_{\mu;\nu} U^\nu U^\mu + k_\mu \frac{DU^\mu}{D\tau} = -A\dot{a}, \quad (69)$$

to the actual derivative of (68),

$$\frac{D}{D\tau} (k_\mu U^\mu) = -\dot{a} \frac{a + \ddot{a}}{\sqrt{\Lambda a^2 - M + \dot{a}^2}}, \quad (70)$$

we finally get

$$K^\tau_\tau = \frac{\Lambda a + \ddot{a}}{\sqrt{\Lambda a^2 - M + \dot{a}^2}}. \quad (71)$$

Similar calculations for the middle section (1) give:

$$K^{\phi}_\phi = -\sqrt{\Lambda a^2 - M(a) + \dot{a}^2} \frac{a + \ddot{a}}{a}, \quad (72)$$

$$K^{\tau}_\tau = \Phi' \sqrt{\Lambda a^2 - M(a) + \dot{a}^2} + \frac{\ddot{a}}{\sqrt{\Lambda a^2 - M(a) + \dot{a}^2}}$$

$$-\frac{\dot{a}^2 (2\Lambda a - M'(a))}{2\sqrt{\Lambda a^2 - M(a) + \dot{a}^2 (\Lambda a^2 - M(a))}}. \quad (73)$$

One need only substitute into equations (53) and (53) to get the time dependent field equations

$$\sigma = -\frac{c^4}{8\pi Ga} \left[ \sqrt{\Lambda a^2 - M + \dot{a}^2} + \sqrt{\Lambda a^2 - M(a) + \dot{a}^2} \right], \quad (74)$$
\[
\vartheta = -\frac{c^4}{8\pi Ga} \left[ \frac{\Lambda a + \ddot{a}}{\sqrt{\Lambda a^2 - M + \dot{a}^2}} - \Phi' \sqrt{\Lambda a^2 - M(a) + \dot{a}^2} - \frac{\ddot{a}}{\sqrt{\Lambda a^2 - M(a) + \dot{a}^2}} \right]
\]
\[
+ \frac{\dot{a}^2 (2\Lambda a - M' (a))}{2\sqrt{\Lambda a^2 - M(a) + \dot{a}^2 (\Lambda a^2 - M(a))}} \right].
\] (75)

For a constant \(\Phi(a)\) such that \(\Phi'(a) = 0\), the conservation of stress-energy implies [4],
\[
\dot{\sigma} = -\frac{\dot{a}}{a} (\sigma - \vartheta).
\] (76)

Introducing the length of the stress-energy ring \(\ell = 2\pi a\), it is easily seen that (76) may be rewritten in the more recognizable form,
\[
\frac{D}{D\tau} (\ell \sigma) = \vartheta \frac{D}{D\tau} (\ell).
\] (77)

Assuming the equation of state \(\sigma = \vartheta\) holds for time dependent wormholes, we immediately have \(\sigma = \text{constant}\) from (76). Rearranging (74) and using geometrodynamic units \(c \equiv 1 \equiv G\) we get the unsimplified differential equation,
\[
\sqrt{\Lambda a^2 - M + \dot{a}^2} + \sqrt{\Lambda a^2 - M(a) + \dot{a}^2} = -8\pi \sigma a.
\] (78)

This equation is very difficult to solve for most functions \(M(a)\). The simplest case where \(M(a) = M\) is the only one considered here. Equation (78) then becomes after some manipulation,
\[
\dot{a}^2 - a^2 (16\pi^2 \sigma^2 - \Lambda) - M = 0.
\] (79)

There are two cases to consider; the first one is where \(\Lambda > 16\pi^2 \sigma^2\), and the second case is the opposite (\(\Lambda < 16\pi^2 \sigma^2\)). For case 1, equation (79) has a cosine solution as follows:
\[
a(\tau) = \frac{\sqrt{M}}{\sqrt{\Lambda - 16\pi^2 \sigma^2}} \cos \sqrt{\Lambda - (16\pi^2 \sigma^2)} \tau.
\] (80)

This is a stable oscillatory solution. In addition, the idea of a large \(\Lambda\) has already been introduced and conceivably possible using relativistic bubbles.

For the second case we set \(a = r_0\) at \(\tau = 0\), where \(r_0\) is the initial radius of the wormhole to get the solution,
\[
a(\tau) = \frac{1}{-2\sqrt{16\pi^2 \sigma^2 - \Lambda}} \left[ Q_+ e^{\pm \sqrt{16\pi^2 \sigma^2 - \Lambda} \tau} + Q_- e^{\mp \sqrt{16\pi^2 \sigma^2 - \Lambda} \tau} \right],
\] (81)
with \( Q_{\pm} \equiv (\sqrt{16\pi^2\sigma^2 - \Lambda})r_0 \pm \sqrt{(16\pi^2\sigma^2 - \Lambda)r_0^2 - 4M} \). We see that as \( \tau \to \infty \), \( a \to \infty \). That is, the solution is unstable to explosion. However, for no \( \tau \) does \( a \) go to zero. Hence the solution is stable to collapse. This is in agreement to the similar solution for the \( (2 + 1) \) case without a cosmological constant \([4]\). A similar approach can be used to analyse the \((3 + 1)\) dimensional case.

5 Discussion

Inclusion of a cosmological constant modifies to some extent the structure of a wormhole, permitting, for example, positive \( \rho \) in \((2 + 1)\) dimensions. The zero-tidal force solution gave limits on the velocity and showed a possibly very quick transit time. However, these results were only valid for relatively small velocities. Additionally, this type of solution needed an extremely large amount of exotic material. Using a bubble to modify the constant could solve this problem for the \((2 + 1)\) dimensional case but would create the additional concern of the discontinuous boundary at the bubble surface. The wormhole constructed by surgically grafting two solutions devoid of matter around the general wormhole manifold gave negative line energy density and line tension. The addition of the cosmological constant to the field equations allowed for a stable solution not found in previous papers.

The possibility for all of the above mentioned wormhole universes hinge on the existence of exotic matter. Although no macroscopic exotic matter is known to exist, quantum theory shows tantalizing hints that it may be possible to manufacture exotic matter if it does not exist normally.

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Note added: Upon completion of this manuscript we became aware of a preprint by S.W. Kim, H. Lee, S. Kim and J. Yang (“(2 + 1) Dimensional Schwarzschild de Sitter Wormhole” SNUTP-93-51) in which \((2 + 1)\) dimensional wormholes in de-Sitter space were analyzed.
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