On the integrability of a lattice equation
with two continuum limits

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Abstract

We study a new example of lattice equation being one of the key equations
of a recent generalized symmetry classification of five-point differential-difference
equations. This equation has two different continuum limits which are the well-
known fifth order partial-differential equations, namely, the Sawada-Kotera and
Kaup-Kupershmidt equations. We justify its integrability by constructing an \( L-A \)
pair and a hierarchy of conservation laws.

1 Introduction

We consider the differential-difference equation

\[
\begin{align*}
  u_{n,t} = (u_n + 1) \left( \frac{u_{n+2}u_{n+1} + 1}{u_{n+1}} - \frac{u_{n-2}u_{n-1} + 1}{u_{n-1}} + (2u_n + 1)(u_{n+1} - u_{n-1}) \right),
\end{align*}
\]

where \( n \in \mathbb{Z} \), while \( u_n(t) \) is an unknown function of one discrete variable \( n \) and one
continuous variable \( t \), and the index \( t \) in \( u_{n,t} \) denotes the time derivative. Equation (1)
is obtained as a result of the generalized symmetry classification of five-point differential-
difference equations

\[
\begin{align*}
  u_{n,t} = F(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}),
\end{align*}
\]

carried out in [8–10]. Equation (1) coincides with the equation [9, (E16)]. Even earlier
this equation has been obtained in [2]. Equations of the form (2) play an important role
in the study of four-point discrete equations on the square lattice, which are very relevant
for today, see e.g. [1, 5, 6, 17].

At the present time there is very little information on equation (1). It has been shown
in [9] that equation (1) possesses a nine-point generalized symmetry of the form

\[
\begin{align*}
  u_{n,\theta} = G(u_{n+4}, u_{n+3}, \ldots, u_{n-4}).
\end{align*}
\]

As for relations to the other known integrable equations of the form (2), nothing useful
from the viewpoint of constructing solutions is known, see details in the next Section.
However, this equation occupies a special place in the classification \[8–10\]. In particular, it possesses a remarkable property discovered in \[7\]. This equation has two different continuum limits being the well-known Kaup-Kupershmidt and Sawada-Kotera equations, details will be given below. For this reason, equation (1) deserves a more detailed study.

In Section 2 we discuss the known properties of equation (1). In order to justify the integrability of (1), we construct an \(L − A\) pair in Section 3 and show that it provides an infinity hierarchy of conservation laws in Section 4.

## 2 Special place of equation (1) in the classification \[8–10\]

In two lists of integrable equations of the form (2) presented in \[9, 10\], the following four equations occupy a special place: those are (1) and

\[
u_{n,t} = (u_{n}^{2} - 1)(u_{n+2}^{2} - u_{n+1}^{2} - 1) - u_{n-2}^{2} - 1), \quad (3)
\]

\[
u_{n,t} = u_{n}^{2}(u_{n+2}u_{n+1} - u_{n-1}u_{n-2}) - u_{n}(u_{n+1} - u_{n-1}), \quad (4)
\]

\[
u_{n,t} = u_{n+1}u_{n+1}^{2}u_{n+1} - u_{n+1}u_{n-1} - u_{n+1}u_{n-1}^{2} - u_{n}(u_{n+1} - u_{n-1}). \quad (5)
\]

Equations (3-5) correspond to equations (E17), (E15) of \[9\] and (E14) of \[10\], respectively. Equation (4) is known for a long time \[19\].

All other equations of \[9, 10\] go over in the continuum limit into the third order equations of the form

\[U_{\tau} = U_{xxx} + F(U_{xx}, U_{x}, U), \quad (6)\]

where the indices \(\tau\) and \(x\) denote \(\tau\) and \(x\) partial derivatives, and mainly into the Korteweg-de Vries equation. These four equations correspond in the continuum limit to the fifth order equations of the form:

\[U_{\tau} = U_{xxxxx} + F(U_{xxxx}, U_{xxx}, U_{xx}, U_{x}, U). \quad (7)\]

For all the four equations (1-5) we get in the continuum limit one of the two well-known equations. One of them is the Kaup-Kupershmid equation \[4, 12\]:

\[U_{\tau} = U_{xxxxx} + 5UU_{xxx} + \frac{25}{2}U_{x}U_{xx} + 5U^{2}U_{x}, \quad (8)\]

and the second one is the Sawada-Kotera equation \[18\]:

\[U_{\tau} = U_{xxxxx} + 5UU_{xxx} + 5U_{x}U_{xx} + 5U^{2}U_{x}. \quad (9)\]

Using the substitution

\[u_{n}(t) = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2} \varepsilon^{2}U \left( \tau - \frac{18}{5} \varepsilon \tau, x + \frac{4}{3} \varepsilon t \right), \quad x = \varepsilon n, \quad (10)\]
in equation (5), we get at $\varepsilon \to 0$ the Sawada-Kotera equation (9). All other continuum limits are known, see [1] for (4) and [7] for (1) and (3). Here we explicitly replicate substitutions for equation (11) under study which has two different continuum limits. The substitution

$$u_n(t) = -\frac{4}{3} - \varepsilon^2 U \left( \tau - \frac{18}{5} \varepsilon^5 t, x + \frac{4}{3} \varepsilon t \right), \quad x = \varepsilon n,$$

in (11) leads to equation (8), while the substitution

$$u_n(t) = -\frac{2}{3} + \varepsilon^2 U \left( \tau - \frac{18}{5} \varepsilon^5 t, x + \frac{4}{3} \varepsilon t \right), \quad x = \varepsilon n,$$

leads to equation (9). The link between these discrete and continuous equations is shown in the following diagram:

We see that equation (8) has two different integrable approximations, while equation (9) has three approximations.

As far as we know, there are no relations between (1,3-5) and other known equations of the form (2) presented in [9, 10]. More precisely, we mean relations in the form of the transformations

$$\hat{u}_n = \varphi(u_{n+k}, u_{n+k-1}, \ldots, u_{n+m}), \quad k > m,$$

and their compositions, see a detailed discussion of such transformations in [8]. As for relations among (1,3-5), equation (5) is transformed into (4) by $\hat{u}_n = u_{n+1}u_n$, i.e. (5) is a simple modification of (1). There is a complicated relation between equations (1) and (4) found in [2]. As it is shown in [9], it is a composition of two Miura type transformations. It is very difficult to use that relation for the construction of solutions because the problem is reduced to solving the discrete Riccati type equations [9].

There is a complete list of integrable equations of the form (7), see [3,13,16]. Equations (8) and (9) play the key role in that list, since all the other are transformed into these two by transformations of the form:

$$\dot{U} = \Phi(U, U_x, U_{xx}, \ldots, U_{x^m}).$$

3 \quad L − A pair

As the continuum limit shows, equation (11) should be close to equations (8,9) in its integrability properties, and these equations (8,9) have the $L − A$ pairs defined by $3 \times 3$ matrices [11,7]. Here we construct an $L − A$ pair for equation (11) following [7].

We look for an $L − A$ pair of the form

$$L_n\psi_n = 0, \quad \psi_{n,t} = A_n\psi_n$$

(14)
with the operator $L_n$ of the form

$$L_n = T^2 + l_n^{(1)}T + l_n^{(0)} + l_n^{(-1)}T^{-1}, \quad (15)$$

where $l_n^{(k)}$ with $k = -1, 0, 1$ depend on a finite number of the functions $u_{n+j}$. Here $T$ is the shift operator: $Th_n = h_{n+1}$. In the case of $(15)$ the operator $A_n$ can be chosen as:

$$A_n = a_n^{(1)}T + a_n^{(0)} + a_n^{(-1)}T^{-1}.$$ 

The compatibility condition for system (14) has the form

$$\frac{d(L_n \psi_n)}{dt} = (L_{n,t} + L_n A_n) \psi_n = 0 \quad (16)$$

and it must be satisfied in virtue of the equations (1) and $L_n \psi_n = 0$.

If we suppose that the coefficients $l_n^{(k)}$ depend on $u_n$ only, then we can check that $a_n^{(k)}$ have to depend on $u_{n-1}, u_n$ only. However, in this case the problem has no solution for equation (1). Therefore we proceed to the case when the functions $l_n^{(k)}$ depend on $u_n, u_{n+1}$. Then the coefficients $a_n^{(k)}$ must depend on $u_{n-1}, u_n, u_{n+1}$. In this case we have managed to find operators $L_n$ and $A_n$ with one irremovable arbitrary constant $\lambda$ playing the role of the spectral parameter here:

$$L_n = T^2 - \frac{U_{n+1}}{u_{n+1}}T + \frac{U_{n+1}}{u_n} \left(1 - \frac{u_n}{U_n} T^{-1}\right), \quad (17)$$

$$A_n = \frac{u_n}{U_n} (\lambda T^{-1} - \lambda^{-1} T) + \frac{u_n}{U_n^2} (u_{n-1} + u_{n+1} T^{-1})(T - 1), \quad (18)$$

where

$$U_n = \frac{u_n}{1 + u_n}. \quad (19)$$

The $L - A$ pair $(17,18)$ can be rewritten in the standard matrix form with $3 \times 3$ matrices $\tilde{L}_n, \tilde{A}_n$:

$$\Psi_{n+1} = \tilde{L}_n \Psi_n, \quad \Psi_{n,t} = \tilde{A}_n \Psi_n,$$

where $\Psi_n$ is a spectral vector-function, whose standard form is

$$\Psi_n = \begin{pmatrix} \psi_{n+1} \\ \psi_n \\ \psi_{n-1} \end{pmatrix}. $$

Here we slightly change $\Psi$ by gauge transformation to simplify the matrices $\tilde{L}_n, \tilde{A}_n$:

$$\Psi_n = \begin{pmatrix} U_n (\lambda \psi_{n+1} + \frac{1}{u_n} \psi_n) \\ \psi_n \\ \psi_{n-1} \end{pmatrix}. $$
and now $\tilde{L}_n, \tilde{A}_n$ read:

$$
\tilde{L}_n = \begin{pmatrix}
0 & -1 & \frac{1}{U_n} \\
\lambda U_n & \frac{u_n}{u_{n-1}} & 0 \\
0 & 1 & 0
\end{pmatrix},
$$

(20)

$$
\tilde{A}_n = \begin{pmatrix}
\frac{u_{n-1} u_{n-2}}{U_{n-1}^2} & -\frac{u_{n+1} u_n}{U_n^2} & -u_n + u_{n-1} \\
\lambda (1 + u_n) u_{n-1} - u_n & \frac{u_{n-1} u_{n-2}}{U_{n-1}^2} & \frac{u_{n-1} u_{n-2}}{U_{n-1}} - \lambda\frac{u_n}{U_n} \\
\lambda u_{n-1} - (1 + u_{n-1}) u_n & \frac{u_{n-1} u_{n-2}}{U_{n-1}^2} - \frac{u_{n-1} u_{n-2}}{U_{n-1}} - \lambda\frac{u_n}{U_n} & \lambda - \frac{u_{n-1} u_{n-2}}{U_{n-1}^2} + \frac{u_{n-1} u_{n-2}}{U_{n-1}}
\end{pmatrix}.
$$

(21)

In this case, unlike (10), the compatibility condition can be represented in a form which does not use the spectral vector-function $\Psi_n$. It will be the following matrix form in terms of $3 \times 3$ matrices:

$$
\tilde{L}_{n,t} - \tilde{A}_{n+1} \tilde{L}_n - \tilde{L}_n \tilde{A}_n.
$$

(22)

## 4 Conservation laws

As far as we know, there exist two methods to construct the conservation laws by using the matrix $L - A$ pair (22), see [3][11][14]. However, we do not see how to apply those methods in the case of matrices (20) and (21). Here we will use a different scheme, presented in [7], for deriving conservation laws from the $L - A$ pair (14). In [7] that scheme was applied to one equation (3) only. Here we check it again by example of one more equation (1).

The structure of operators (17,18) allows us to rewrite the $L - A$ pair (14) in form of the Lax pair. The operator $L_n$ has a linear dependence on $\lambda$:

$$
L_n = P_n - \lambda Q_n,
$$

(23)

where

$$
P_n = T^2 - \frac{U_{n+1}}{u_{n+1}} T, \quad Q_n = -\frac{U_{n+1}}{u_n} \left(1 - \frac{u_n}{U_n} T^{-1}\right),
$$

and $U_n$ is defined by (19). Introducing $\hat{L}_n = Q_n^{-1} P_n$ we get an equation of the form:

$$
\hat{L}_n \psi_n = \lambda \psi_n.
$$

(24)

The functions $\lambda \psi_n$ and $\lambda^{-1} \psi_n$ in the second equation of (14) can be expressed in terms of $\hat{L}_n$ and $\psi_n$, using (24) and its consequence $\lambda^{-1} \psi_n = \hat{L}_n^{-1} \psi_n$. As a result we have:

$$
\psi_{n,t} = \hat{A}_n \psi_n,
$$

(25)

where

$$
\hat{A}_n = \frac{u_n}{U_n} (T^{-1}Q_n P_n - TP_n^{-1} Q_n) + \frac{u_n}{U_n^2} (u_{n-1} + u_{n+1} T^{-1})(T - 1).
$$

It is important that the new operators $\hat{L}_n$ and $\hat{A}_n$ in the $L - A$ pair (24,25) do not depend on the spectral parameter $\lambda$. For this reason, the compatibility condition can be written in the operator form, without using the $\psi$-function:

$$
\hat{L}_{n,t} - \hat{A}_{n+1} \hat{L}_n - \hat{L}_n \hat{A}_n = [\hat{A}_n, \hat{L}_n],
$$

(26)
and this is nothing but the Lax equation. The difference between this \( L - A \) pair and the well-known Lax pairs for the Toda and Volterra equations is that the operators \( \hat{L}_n \) and \( \hat{A}_n \) are nonlocal. Nevertheless, using the definition of inverse operators:

\[
P_n P_n^{-1} = P_n^{-1} P_n = 1, \quad Q_n Q_n^{-1} = Q_n^{-1} Q_n = 1
\]

and the fact that they are linear, we can check that (26) is true by direct calculation.

The conservation laws of equation (11), which are expressions of the form

\[
\rho_n^{(k)} = (T - 1)\sigma_n^{(k)}, \quad k \geq 0,
\]

can be derived from the Lax equation (26), notwithstanding the nonlocal structure of \( \hat{L}_n, \hat{A}_n \), see [20]. For this, first of all, we have to represent the operators \( \hat{L}_n, \hat{A}_n \) as formal series in powers of \( T^{-1} \):

\[
H_n = \sum_{k \geq 0} h_n^{(k)} T^k.
\]

Formal series of this kind can be multiplied according the rule: \( (a_n T^k)(b_n T^j) = a_n b_{n+k} T^{k+j} \).

The inverse series of the form (28) can be easily obtained by definition (27), for instance:

\[
Q_n^{-1} = - (1 + q_n T^{-1} + (q_n T^{-1})^2 + \ldots + (q_n T^{-1})^k + \ldots) \frac{u_n}{U_{n+1}}, \quad q_n = \frac{u_n}{U_n}.
\]

The series \( \hat{L}_n \) has the second order:

\[
\hat{L}_n = \sum_{k \leq 2} l_n^{(k)} T^k = - \left( \frac{u_n}{U_{n+1}} T^2 + u_n \left( \frac{u_{n-1}}{U_n^2} - \frac{1}{u_{n+1}} \right) T + \frac{u_{n-1}}{U_n} \left( \frac{u_n u_{n-2}}{U_n^2 - 1} \right) T^0 + \ldots \right).
\]

The conserved densities \( \rho_n^{(k)} \) of equation (11) can be found as:

\[
\rho_n^{(0)} = \log l_n^{(2)}, \quad \rho_n^{(k)} = \text{res} \hat{L}_n^k, \quad k \geq 1,
\]

where the residue of formal series (28) is defined by the rule: \( \text{res} H_n = h_n^{(0)} \), see [20]. Corresponding functions \( \sigma_n^{(k)} \) can easily be found by direct calculation.

The conserved densities \( \bar{\rho}_n^{(k)} \) below have been found in this way and then simplified in accordance with the rule:

\[
\bar{\rho}_n^{(k)} = c_k \rho_n^{(k)} + (T - 1) g_n^{(k)},
\]

where \( c_k \) is a constant and \( g_n^{(k)} \) is a function. The first three densities of equation (11) read:

\[
\bar{\rho}_n^{(0)} = \log (u_n + 1),
\]

\[
\bar{\rho}_n^{(1)} = \frac{V_{n+1} u_{n-1}}{U_n},
\]

\[
\bar{\rho}_n^{(2)} = \frac{u_{n+2} u_n + u_{n+1} u^2_n - u_{n-1} u_{n-2}}{U^2_{n+1} U_n U^2_{n-1}} + u_{n+1} u_{n-1} \left( \frac{V^2_n - u_n u_{n-1}}{U_n U_{n-1}} \right) + \frac{u_n^2 u_{n-1}^2}{2 U^3_{n+1} U^3_n} - \frac{u_{n+1} u_{n-1}}{U_{n+1} U_n} + \frac{u_{n+1} u_{n-1} (V_{n+1} - 1) V_n}{2 U_{n+1} U^2_n} + \frac{u_{n-1}^2}{2 U^2_n},
\]

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where
\[ V_n = u_n u_{n-1} + u_n + u_{n-1}. \]

We can easily check that
\[ \frac{\partial^2 \rho_n^{(1)}}{\partial u_{n+1} \partial u_{n-1}} \neq 0, \quad \frac{\partial^2 \rho_n^{(2)}}{\partial u_{n+2} \partial u_{n-2}} \neq 0. \]

Therefore, in accordance with a theory of the review [20], the conserved densities \( \rho_n^{(0)} \), \( \rho_n^{(1)} \), \( \rho_n^{(2)} \) have the orders 0, 2, 4 respectively. This means that we have got three conserved densities, which are nontrivial and essentially different.

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