Quantifying Coherence and Entanglement via Simple Measurements

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Coherence and entanglement are fundamental properties of quantum systems, promising to power the near future quantum computers, sensors and simulators. Yet, their experimental detection is challenging, usually requiring full reconstruction of the system state. We show that one can extract quantitative bounds to the relative entropy of coherence and the coherent information, coherence and entanglement quantifiers respectively, by a limited number of purity measurements. The scheme is readily implementable with current technology to verify quantum computations in large scale registers, without carrying out expensive state tomography.

The superposition principle is one of the pillars of quantum mechanics. Coherent superpositions of multipartite states can yield entanglement, a property of such states that are nonfactorizable. Entanglement has been extensively investigated [1], having being identified as a key property since the pioneering studies in quantum communication and cryptography protocols, and for quantum computational speed-up [2–5]. Entanglement is also crucial in quantum condensed matter theory, because it underpins fundamental properties of many-body systems such as critical behaviors [6] as well as improved metrology beyond the standard quantum limit [7]. Once confined to suggestive thought experiments, highly coherent quantum systems are nowadays observed and manipulated in the laboratory.

The quantification of entanglement is thus of great interest. Except for the case of bipartite pure states [8] quantifying entanglement is complicated. For mixed states, there are many different measures (see [9] for a review). Depending on one’s purpose, one of more of these may be appropriate. They include the distillable entanglement [10], the entanglement of formation [10], the entanglement cost [11], the concurrence [12], and the log negativity [13]. The most informative entanglement measures have some operational meaning. For example, the distillable entanglement is how much pure entanglement can be extracted from the quantum state in question, while the entanglement cost is how much pure entanglement is needed to create a state asymptotically. Because of their asymptotic nature, these can be extremely difficult to calculate. Indeed, there seems to be a tradeoff between operational meaning and computational accessibility. The log negativity, for example, is simple to calculate given a full description of a quantum state, its density matrix, but lacks operational interpretation.

In general, it is easier to detect, or certify, entanglement than to quantify it. A heavily employed solution is to verify quantum non-locality by violation of Bell inequalities, which is observed in a subset of the entangled states [14]. Another possibility is to design and measure witnesses [15]. These are observables which verify the presence of entanglement whenever their value is above (or below) a threshold. For example, entanglement-enhanced precision in phase estimation protocols is certified by super-linear scaling of the quantum Fisher information [16–18]. These customary detection methods usually certify nonseparability but offer no quantitative information about the usefulness of the entanglement present, and often fail to detect the entanglement at all. In this work we seek to quantify entanglement rather than merely detect it, without requiring extraordinarily difficult experiments.

This most direct way to estimate the entanglement of a state is by full tomographic reconstruction. This task is computationally and experimentally challenging, scaling exponentially with the number of qubits. It is therefore desirable to have a way to quantify the entanglement present in a system without doing full tomography. In Ref. [19], it was proposed how to directly confirm the presence of entanglement by interfering two identical copies of a state and extracting a quantity \( \gamma = \text{Tr} \rho^2 \) which is known as the purity of a quantum state. In Ref. [20], such an experiment was carried out on ultracold bosonic atoms in an optical lattice. Using the methods developed below, it is possible to not only to verify that the systems were entangled, but quantify the amount of entanglement present.

By measuring the purity of a bipartite system \( \gamma_{AB} \), as well as the purity of (at least) one of the subsystems, one certifies that the systems \( A, B \) are entangled whenever the global purity is lower than the subsystem purity. This can be expressed in terms of the Renyi coherent information

\[
S_R = S_2(\rho_B) - S_2(\rho_{AB}),
\]

as \( S_R > 0 \). Here \( \rho_{AB} \) is the density matrix of the entire
system and $\rho_B$ is the reduced density matrix on system $B$, while

$$S_2(\rho) = - \log \text{Tr}\rho^2 = - \log \gamma .$$

(2)

Positivity of $S_2$ implies that the system is entangled [1]. Unfortunately, this Renyi quantity, while able to certify entanglement (it is an entanglement witness [15]), does not quantify it.

While Renyi entropies often play a similar role, the most useful entanglement quantities are usually given in terms of von Neumann entropy $S(\rho) = - \text{Tr}\rho \log \rho$. Then, a more operational quantity is the coherent information, which is defined like $S_B$, except in terms of von Neumann entropy instead of Renyi:

$$I(A)B = S(\rho_B) - S(\rho_{AB}) .$$

(3)

Coherent information characterizes the degree to which error-correction can maintain coherence in the system. As can be seen in Fig. 1, there are many states with high coherent information but very low Renyi coherent information, and vice versa.

The coherent information is harder to measure than the Renyi version, because to calculate von Neumann entropies one needs to know the eigenvalues of a system, not just its purity. However, one can obtain quantitative upper and lower bounds on the von Neumann entropy in terms of the global and marginal purities using the method of Lagrange multipliers (see the Appendix for details).

Given the spectral decomposition $\rho = \sum_{i=1}^d x_{i,\rho} |\psi_i\rangle \langle \psi_i|$, $\sum_{i=1}^d x_{i,\rho} = 1$, $\langle \psi_i | \psi_j \rangle = \delta_{ij}$, $x_{1,\rho} \geq x_{2,\rho} \geq \ldots \geq x_{d,\rho}$, we determine the extreme values of the state entropy $S(\rho) = - \sum_{i=1}^d x_{i,\rho} \log x_{i,\rho}$ at fixed purity $\gamma_{\rho} := \sum_{i=1}^d x_{i,\rho}^2$, where the logarithm is written in base $2$ [21]. The spectrum $\{x_{i,\rho}^M\}$ that maximizes $S(\rho)$ is the following:

$$x_{1,\rho}^M = \frac{1}{d} + \sqrt{\frac{d-1}{d} \left( \gamma_{\rho} - \frac{1}{d} \right)},$$

$$x_{2,\rho}^M = x_{3,\rho}^M = \ldots = x_{d,\rho}^M = \frac{1 - x_{1,\rho}^M}{d-1}.$$
The spectrum \( \{x_{1,\rho}^m\} \) that minimizes \( S(\rho) \) is given by
\[
x_{1,\rho}^m = x_{2,\rho}^m = \cdots = x_{k_{\rho}-1,\rho}^m = \frac{1 - \alpha_{\rho}}{k_{\rho} - 1}, \quad x_{k_{\rho},\rho}^m = \alpha_{\rho},
\]
where \( \alpha_{\rho} = 1/k_{\rho} - \sqrt{(1 - 1/k_{\rho})(\gamma_{\rho} - 1/k_{\rho})} \) and \( k_{\rho} \) is the integer such that \( \frac{1}{k_{\rho}} \leq \gamma_{\rho} \leq \frac{1}{k_{\rho}-1} \). We can immediately use these results to bound the coherent information as follows:

**Result 1.**—Given a quantum state \( \rho_{AB} \in \mathcal{H}_{d_A} \otimes \mathcal{H}_{d_B} \), its coherent information \( I(A|B) \) is bounded as follows:
\[
l_c(\rho_{AB}) \leq I(A|B) \leq u_c(\rho_{AB}),
\]
where \( \rho_{B} = \text{Tr}_{A}(\rho_{AB}) \). These bounds are very good, see Figure 1 and Figure 2.

We pause here to note that coherent information is not a full-fledged entanglement measure [22], since it can be zero or even negative (even for states that do have entanglement). Yet, it characterizes many uses of bipartite entanglement in quantum computation and communication protocols. The coherent information measures the capacity for noiseless quantum communication of a quantum channel between \( A \) and \( B \), when maximized over the sets of possible inputs, in the asymptotic limit of an infinite number of trials [23–26]. It also quantifies the sets of possible inputs, in the asymptotic limit of an infinite number of trials [23–26]. It also quantifies the use these results to bound the coherent information as follows:

\[
l_c(\rho_{AB}) = (\log d) \log \rho_{AB} - \sum_{i,j} \rho_{ij} \log \rho_{ij} - \sum_{i,j} \rho_{ij} \log \rho_{ij}.
\]

where \( \rho_{B} = \text{Tr}_{A}(\rho_{AB}) \). These bounds are very good, see Figure 1 and Figure 2.

We next study another feature of quantum systems, their coherence. In a way similar to how non-factorizable superpositions of multipartite states, e.g. \( \sum_i c_i |ii\ldots i\rangle \), yield entanglement, the quantumness of a system can be identified with the degree of coherence of its state \( |\psi\rangle = \sum_i c_i |ii\rangle \), \( \sum_i |c_i|^2 = 1 \), in a given basis \( \{|i\rangle\} \). Surprisingly, an information-theoretic characterization of coherence has been developed only in recent works [29], in contrast to entanglement which has been extensively investigated [11].

A natural way to quantify the coherence of a state in a reference basis \( \{|1\rangle, \{2\rangle, \ldots, |d\rangle\} \) of a \( d \)-dimensional Hilbert space \( \mathcal{H}_d \) is by measuring how far it is to the set of incoherent states \( \mathcal{I} \) [30, 31]. The choice of distance function is in principle arbitrary. Yet, an important operational interpretation is enjoyed by the relative entropy of coherence
\[
C_{RE}(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho||\sigma) = S(\rho_d) - S(\rho),
\]
where \( \rho_d = \sum_i |i\rangle \langle i| \rho |i\rangle \langle i| \) is the state after dephasing in the reference basis. In other words, coherence is evaluated by how much mixedness a dephasing channel adds to the system state. The relative entropy of coherence is the distillable coherence of a state [32]. That is, in the asymptotic limit of infinite system preparations, the maximal rate of extraction of maximally coherent qubit states \( 1/2 \sum_{i,j=0,1} |i\rangle \langle j| \) by incoherent operations. This quantity is again easily bounded by purity measurements.

**Result 2** — The relative entropy of coherence \( C_{RE}(\rho) \) is bounded as follows:
\[
l_c(\rho) \leq C_{RE}(\rho) \leq u_c(\rho),
\]
where \( \rho_{B} = \text{Tr}_{A}(\rho_{AB}) \). These bounds are very good, see Figure 1 and Figure 2.

To summarize, we provided quantitative bounds to coherent information and relative entropy of coherence in terms of global and marginal purities. We now describe the experimental setting required for measuring state purity. The purity of a state \( \rho \) can be measured on just two copies, \( \rho \otimes \rho \)—using precisely the same data as used in [20].

This can be done in two ways. The first method, illustrated in Fig. 1, is to measure the expectation value of the swap operator \( V \) on \( \rho \otimes \rho \), taking advantage of the identity \( \text{Tr}(\rho^2) = \text{Tr}(V\rho \otimes \rho) \). This can be accomplished using an ancillary qubit and a controlled swap [33–35]. For multiple qubit systems, implementing a full controlled swap appears difficult. However, observing that the swap is factorizable, one can perform controlled swaps sequentially on the individual corresponding pairs of qubits from each copy of \( \rho \). We note that the purity of the dephased state is also measurable by applying dephasing before the interaction gate to just one copy of the state, as \( \text{Tr}(\rho_d^2) = \text{Tr}(\rho_{pd}) = \text{Tr}(V(\rho_d \otimes \rho)) \).

The measurement can also be accomplished without ancilla by measuring in the Bell basis. This is because
\[
\text{Tr}(V \rho \otimes \rho) = \text{Tr} \left( (I \otimes 2)(\Psi^-)(\Psi^-) \right) \rho = 1 - 2(\Psi^- \rho \otimes \rho)(\Psi^-)
\]
where \( \Psi^\pm = \sqrt{1/2} \left( |01\rangle \pm |10\rangle \right) \) is the antisymmetric singlet state. This second method, ideal for bosonic states, can be achieved by interfering two copies of a state on a beamsplitter [19, 20, 36]. When a photon is detected
at both output ports of the beamsplitter, the state is projected into the singlet. From repeated experiments the probability that the output state is the singlet can be determined. Again, if the state \( \rho \) is of many qubits, the beamsplitter can be performed on individual qubits. Here there is the drawback that the probability of measuring the output as all singlets goes down exponentially in the number of qubits in the state, so many measurements will be needed to evaluate this probability. This scheme is shown in Fig. 4.

Our results Eqs. \((17)\) and \((16)\) rely on bounding the von Neumann entropy by quadratic polynomials, i.e. purity. This represents the leading order term of the von Neumann entropy Taylor expansion about pure states. We anticipate that tightened bounds can be extracted by evaluating the higher order terms \( \text{Tr}(\rho^2), \text{Tr}(\rho^4), \ldots, \text{Tr}(\rho^k) \). Each \( k \)-th degree polynomial \( \text{Tr}(\rho^k) \) can be estimated by upgrading the scheme in Fig. 4 to interfere \( k \) copies of the state, and evaluating the shift (generalized swap) operator \( V_k(\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_k) = \phi_k \otimes \phi_1 \otimes \ldots \otimes \phi_{k-1} \), \( \text{Tr}(V_k(\bigotimes_{i=1}^{k} \rho_i)) = \text{Tr}(\Pi_{i=1}^{k} \rho_i), \forall \rho_1, \rho_2, \ldots, \rho_k \). The protocol would still exponentially outperform full state reconstruction.

One can extend the method proposed here to determine directly measurable bounds to the total correlations in multipartite systems \( \{ A_i \} \). Let us consider a geometric measure of correlations, the multi-information given by the relative entropy between the state under study and the closest product state,

\[
\mathcal{I}(\rho_{A_1, \ldots, A_n}) = \min_{\rho_{A_i}} S(\rho_{A_1, \ldots, A_n} || \bigotimes_i \rho_{A_i}). \tag{9}
\]

The quantity is the quantum analogue of the classical multi-information between random variables [37]. It is easy to verify that the product of the state marginals \( \bigotimes_i \rho_{A_i} \) solves the minimization, \( \mathcal{I}(\rho_{A_1, \ldots, A_n}) = \sum_i S(\rho_{A_i}) - S(\rho_{A_1, \ldots, A_n}) \) [38]. Thus, quantitative bounds to the total system correlations in terms of purities are given by a straightforward generalization of Eq. \((17)\).

In this letter, we have provided a strategy to evaluate coherence and entanglement with limited laboratory resources. We have derived bounds to the relative entropy of coherence and the coherent information, which can be experimentally extracted by purity measurements. Although controlled swaps of large-dimensional systems are hard to implement, in the case where the systems factorize into qubits we can do the controlled swaps piece by piece [19] [39]. We verified the accuracy of our method by evaluating how tight our approximations are to the actual entanglement/purity measures are.

The scheme is readily implementable in standard quantum information testbeds, as optical lattices, ion traps and NMR (Nuclear Magnetic Resonance) systems. The scalability of the measurement network makes purity detection employable in testing the successful preparation of quantum superpositions in large computational registers, certifying that a complex device has run a truly quantum computation. The proposal could simplify the study of key properties and structure of many-body complex systems, e.g. by investigating phase transition of condensed matter through coherence and entanglement detection.

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APPENDIX: DERIVATION OF THE BOUNDS TO RELATIVE ENTROPY OF COHERENCE AND COHERENT INFORMATION, Eqs. 4,6 OF THE MAIN TEXT

Given a quantum state $\rho$ in a $d$-dimensional Hilbert space, our task is to bound the Von Neumann entropy of $\rho$ with a function of the state purity $\gamma(\rho) := \text{Tr}(\rho^2)$. The spectral decomposition of the quantum state is $\rho = \sum_{i=1}^{d} x_i |\psi_i\rangle \langle \psi_i|$, where $\{|\psi_i\rangle\}$ forms an orthonormal basis of the $d$-dimensional Hilbert space. The variational problem is then formulated as

$$\max \ / \min S(\rho) = -\sum_{i=1}^{d} x_i \log(x_i)$$

s.t. $\sum_{i=1}^{d} x_i^2 = \gamma$

$\sum_{i=1}^{d} x_i = 1$

$0 \leq x_i \leq 1, \forall i,$

(10)

where $\gamma = \text{Tr}\rho^2$ is the purity of $\rho$.

Intuitively, the the vector $x$ that maximize $S$ is the one that spread as uniformly as possible; while the vector $x$ that minimize $S$ is the one that has the minimal number of nonzero large values. In the following, we will analytically solve this problem and confirm this intuition.

Maximization

First, we focus on the maximization problem with $d = 3$. Note that when $d = 2$, the solution to the constraints of Eq. (10) is unique and the optimization problem will be trivial. Without loss of generality, we assume $x_1 \geq x_2 \geq x_3$. Then the problem can be stated as

$$\max S(\rho) = -x_1 \log(x_1) - x_2 \log(x_2) - x_3 \log(x_3)$$

s.t. $x_1^2 + x_2^2 + x_3^2 = \gamma$

$x_1 + x_2 + x_3 = 1$

$1 \geq x_1 \geq x_2 \geq x_3 \geq 0.$

(11)

We prove that the maximum is reached with the following Lemma.

**Lemma 1.** The solution to the maximization problem in Eq. (11) is given by

$$x_1 = \frac{1}{3} + \frac{1}{\sqrt{3}} \left( \frac{2}{\gamma - 1} \right),$$

$$x_2 = x_3 = \frac{1 - x_1}{2}.$$

**Proof.** The differential of the entropy function $S(\rho)$ and the constraints are given by

$$dS = -(1 + \log x_1)dx_1 - (1 + \log x_2)dx_2 - (1 + \log x_3)dx_3$$

and

$$x_1dx_1 + x_2dx_2 + x_3dx_3 = 0$$

$$dx_1 + dx_2 + dx_3 = 0,$$

respectively. We rewrite Eq. (12) to

$$dx_1 = -\frac{(x_3 - x_2)}{x_1 - x_2} dx_3,$$

$$dx_2 = -\frac{(x_1 - x_3)}{x_1 - x_2} dx_3.$$
Thus, the differential of the entropy function becomes
\[
dS(\rho) = \frac{dx_3}{x_1-x_2} \left[ (x_3 - x_2) \log x_1 + (x_1 - x_3) \log x_2 + (x_2 - x_1) \log x_3 \right]
\]
\[
= (x_2 - x_3) \left[ -\frac{\log x_1 - \log x_2}{x_1-x_2} + \frac{\log x_3 - \log x_2}{x_3-x_2} \right] dx_3
\]
Since the function \(\log x\) is convex for \(x \in [0,1]\), for \(x_1 \geq x_2 \geq x_3\),
\[
\frac{\log x_1 - \log x_2}{x_1-x_2} \leq \frac{\log x_3 - \log x_2}{x_3-x_2}.
\]
Thus, \(dS(\rho)/dx_3 \geq 0\). To reach the maximum of \(S(\rho)\), we thus only need to set \(x_3\) to be its maximum, which happens when \(x_2 = x_3\). Together with the constraints, then we can solve the equations and show that the solution to the maximization problem is given in Eq. (12).

Now, we can solve the maximization problem of Eq. (10) for a general case of \(d\).

**Theorem 1.** Suppose \(x_1 \geq x_2 \geq \ldots \geq x_d\), the solution to the maximization problem in Eq. (10) is
\[
x_1 = \frac{1}{d} + \sqrt{\frac{d-1}{d} \left( \frac{\gamma}{d} - \frac{1}{d} \right)},
\]
\[
x_2 = x_3 = \cdots = x_d = \frac{1 - x_1}{d - 1}.
\]

**Proof.** The solution in Eq. (12) is exactly determined when setting \(x_2 = x_3 = \cdots = x_d\). Suppose the maximization problem solution is not this one, then we must have that \(x_2 > x_d\). In the following, we prove the contradiction by showing that changing the values of \(x_1, x_2, x_d\) would make the entropy \(S(\rho)\) larger, while fixing all other values \((x_3, x_4, \ldots, x_{d-1})\) and the constraints. Now the constraints for \(x_1, x_2,\) and \(x_d\) becomes
\[
x_1^2 + x_2^2 + x_d^2 = a
\]
\[
x_1 + x_2 + x_d = b.
\]
By defining \(x_1' = x_1/b, x_2' = x_2/b, x_d' = x_d/b\), the relations become
\[
x_1'^2 + x_2'^2 + x_d'^2 = a/b^2
\]
\[
x_1' + x_2' + x_d' = 1.
\]
The entropy function is
\[
S(\rho) = -\sum_{i=1}^{d} x_i \log(x_i),
\]
\[
= S_{1,2,d}(\rho) + S_r(\rho),
\]
where \(S_{1,2,d}(\rho) = -x_1 \log(x_1) - x_2 \log(x_2) - x_d \log(x_d)\) and \(S_r(\rho) = -\sum_{i=3}^{d-1} x_i \log(x_i)\). Since \(S_r(\rho)\) is fixed, we need to maximize \(S_{1,2,d}(\rho)\), which can also be represented as
\[
S_{1,2,d}(\rho) = -bx_1' \log(bx_1') - bx_2' \log(bx_2') - bx_d' \log(bx_d')
\]
\[
= b \left[ -x_1' \log(x_1') - x_2' \log(x_2') - x_d' \log(x_d') \right] - b \log b
\]
Denoting \(S'_{1,2,d}(\rho) = -x_1' \log(x_1') - x_2' \log(x_2') - x_d' \log(x_d')\), this optimization problem has the same form of Eq. (11). Then Lemma 1 indicates that the maximum of \(S'_{1,2,d}(\rho)\) given the constraints in Eq. (13) is reached when \(x_2' = x_d'\). In other words, the maximum of \(S_{1,2,d}(\rho)\) given the constraints of Eq. (13) is saturated with \(x_2 = x_d\), which contradicts with \(x_2 > x_d\). Therefore, the solution to the maximization problem is given by Eq. (12). \[\square\]
Now, we consider the solution to the minimization of Eq. (10). Similarly, we first consider the minimization with \( d = 3 \) and \( x_1 \geq x_2 \geq x_3 \),

\[
\min S(\rho) = -x_1 \log(x_1) - x_2 \log(x_2) - x_3 \log(x_3)
\text{s.t. } x_1^3 + x_2^3 + x_3^3 = \gamma
\quad x_1 + x_2 + x_3 = 1
\quad 1 \geq x_1 \geq x_2 \geq x_3 \geq 0.
\] (13)

**Lemma 2.** The solution to the minimization problem in Eq. (13) is reached either when \( x_1 = x_2 \) or \( x_3 = 0 \).

**Proof.** From the proof of Lemma 1, we already showed that \( \frac{dS(\rho)}{dx_3} \geq 0 \). Therefore, the lower bound of \( S(\rho) \) is reached when \( x_3 \) takes its minimum. As \( 2(x_1^2 + x_2^2) \geq (x_1 + x_2)^2 \), according to Eq. (13), we have \( 2(\gamma - x_3^2) \geq (1 - x_3)^2 \).

The lower bound for \( x_3 \) is

\[
x_3 \geq \max \left\{ 0, \frac{1 - \sqrt{6\gamma - 2}}{3} \right\}
\]

Thus, when \( \gamma \geq 1/2 \), the minimal possible value for \( x_3 \) is 0. When \( 1/3 \leq \gamma < 1/2 \), the minimal possible value for \( x_3 \) is \( \frac{1 - \sqrt{6\gamma - 2}}{3} \) and \( x_1 = x_2 = (1 - x_3)/2 \). Note that \( \gamma \geq 1/3 \) for \( d = 3 \).

Now, we can show the general solution to the minimization of Eq. (10).

**Theorem 2.** Suppose \( x_1 \geq x_2 \geq \ldots \geq x_k \), the solution to the minimization problem in Eq. (10) is

\[
x_1 = x_2 = \cdots = x_{k-1} = \frac{1 - \alpha}{k - 1},
\]
\[
x_k = \alpha,
\]
\[
x_{k+1} = \cdots = x_d = 0.
\] (14)

Here,

\[
\alpha = \frac{1}{k} - \sqrt{(1 - 1/k)(\gamma - 1/k)}
\] (15)

and \( k \) is the integer such that \( \frac{1}{k} \leq \gamma \leq \frac{1}{k-1} \).

**Proof.** Suppose we always have the solution in the form as

\[
x_1 = x_2 = \cdots = x_{k-1}, x_k, x_{k+1} = \ldots x_d = 0.
\]

Otherwise, there must exist three \( x_i, x_j, x_k \) such that \( x_i \geq x_j \geq x_k \) and \( x_k \neq 0 \). Following a similar argument in the proof of Theorem 1, we can show that this contradicts Lemma 2.

According to Eq. (16), we have

\[
(k - 1)x_1^2 + x_k^2 = \gamma,
\]
\[
(k - 1)x_1 + x_k = 1,
\]
\[
k \leq d
\]

We can show that the possible integer value for \( k \) is unique. That is,

\[
k[(k - 1)x_1^2 + x_k^2] \geq [(k - 1)x_1 + x_k]^2
\]
\[
\geq (k - 1)[(k - 1)x_1^2 + x_k^2]
\]
Equivalently, we have
\[ k\gamma \geq 1 \geq (k - 1)\gamma, \]
hence
\[ \frac{1}{k} \leq \gamma \leq \frac{1}{k - 1}. \]

Upper and lower bounds to coherence and entanglement

We now call \( \{x^M_{i,\rho}\}, \{x^m_{i,\rho}\} \) the vectors solving the maximization and the minimization, respectively. By minimizing (maximizing) the coherence of the dephased state \( \rho_d = \sum_i |i\rangle \langle i| \rho |i\rangle \langle i| \), and maximizing (minimizing) the coherence of the state under study, we obtain lower (upper) bounds to the relative entropy of coherence:

**Result 1** — The relative entropy of coherence \( C_{RE}(\rho) \) is bounded as follows:

\[ \underline{l_c}(\rho) \leq C_{RE}(\rho) \leq \overline{u_c}(\rho), \quad (16) \]

\[ \underline{l_c}(\rho) = -(1 - x^m_{k,\rho_d,\rho_d}) \log x^m_{k,\rho_d,\rho_d} - x^m_{k,\rho_d,\rho_d} \log x^m_{k,\rho_d,\rho_d} + \frac{(d - 1)}{d} \log \left( \frac{1 - x^M_{1,\rho}}{d} \right) + \frac{x^M_{1,\rho}}{d} \log x^M_{1,\rho}, \]

\[ \overline{u_c}(\gamma_{\rho_d,\rho_d}) = (1 - x^m_{k,\rho_d,\rho_d}) \log x^m_{k,\rho_d,\rho_d} + x^m_{k,\rho_d,\rho_d} \log x^m_{k,\rho_d,\rho_d} - \frac{(d - 1)}{d} \log \left( \frac{1 - x^M_{1,\rho_d,\rho_d}}{d} \right) - \frac{x^M_{1,\rho_d,\rho_d}}{d} \log x^M_{1,\rho_d,\rho_d}. \]

On the same hand, given a bipartite state \( \rho_{AB} \), by minimizing (maximizing) the marginal purity on \( B \) subsystem and maximizing (minimizing) the global purity, one has

**Result 2.** — Given a quantum state \( \rho_{AB} \in \mathcal{H}_{d_A} \otimes \mathcal{H}_{d_B} \), and defining \( \rho_B = \text{Tr}_A \rho_{AB} \), its coherent information \( I(A)B \) is bounded as follows:

\[ \underline{l_c}(\rho) \leq I(A)B \leq \overline{u_c}(\rho), \quad (17) \]

\[ \underline{l_c}(\rho) = -(1 - x^m_{k,\rho_B,\rho_B}) \log x^m_{k,\rho_B,\rho_B} - x^m_{k,\rho_B,\rho_B} \log x^m_{k,\rho_B,\rho_B} + \frac{(d - 1)}{d} \log \left( \frac{1 - x^M_{1,\rho_{AB}}}{d} \right) + \frac{x^M_{1,\rho_{AB}}}{d} \log x^M_{1,\rho_{AB}}, \]

\[ \overline{u_c}(\gamma_{\rho_{AB},\rho_B}) = (1 - x^m_{k,\rho_{AB},\rho_{AB}}) \log x^m_{k,\rho_{AB},\rho_{AB}} + x^m_{k,\rho_{AB},\rho_{AB}} \log x^m_{k,\rho_{AB},\rho_{AB}} - \frac{(d - 1)}{d} \log \left( \frac{1 - x^M_{1,\rho_B}}{d} \right) - \frac{x^M_{1,\rho_B}}{d} \log x^M_{1,\rho_B}. \]