BAR RECURSION AS PRIMITIVE RECURSION WITH NONSTANDARD NUMBERS

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ABSTRACT. Bar recursion is a form of higher-type recursion, originally introduced by Spector, which allows for the extraction of constructive information from proofs involving the axiom of dependent choice. Alternative approaches with similar aim are: open recursion, update recursion, the so-called BBC-functional, and selection functions. Unfortunately, none of these forms of recursion, including modified bar recursion, are Kleene-S1-S9-computable over the total continuous functionals, and a more direct or effective approach has been called for. To this end, we show in this paper that bar recursion becomes primitive recursive inside Nelson’s syntactic framework of Nonstandard Analysis. In particular, we show that modified bar recursion of type zero, which corresponds to the Gandy-Hyland functional, equals a primitive recursive functional involving nonstandard numbers. Similar results for more general bar recursion are discussed.

1. INTRODUCTION

Bar recursion is a form of higher-type recursion, first introduced by Spector [27], which plays a central role in so-called proof mining [17]. The aim of the latter is to extract constructive information from the proof of a given mathematical theorem. Such information usually takes the form of upper bounds or even witnessing functionals for the existential quantifiers in the theorem at hand. Bar recursion provides such information for proofs involving (versions of) the axiom of choice (See e.g. [7, Theorem 2.5]). An overview of the various kinds of bar recursion may be found in [7].

Alternative forms of recursion with similar aims are: open recursion, update recursion, the so-called BBC-functional, and selection functions (See [23] for an overview). By [23, Theorem 5.1], these forms of recursion are all equivalent to modified bar recursion in a precise technical sense. Together with the results in [10, 12], bar recursion thus permeates various research areas; It is for instance essential for the computation of certain Nash equilibria.

Unfortunately, modified bar recursion is not computable over the total continuous functionals in the sense of Kleene’s schemas S1-S9 (See [19, Def. 1.10] for the latter). Another disadvantage of bar recursion is that its definition exhibits

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non-wellfounded self-reference. This is best understood by considering the Gandy-Hyland functional $\Gamma$, as follows:

\[
(\text{GH}) \quad \Gamma(Y^2, s^0) := Y(s \ast 0 \ast (\lambda n^0) \Gamma(Y, s \ast (n + 1))).
\]

Clearly, in order to compute $\Gamma$ at $s^0$, one needs the values of $\Gamma$ at all child nodes of $s^0$, as is clear from the right-hand side of (GH). In turn, to compute the value of $\Gamma$ at the child nodes of $s$, one needs the value of $\Gamma$ at all grand-child nodes of $s$, and so on. Hence, repeatedly applying the definition of $\Gamma$ seems to result in a non-terminating recursion. By contrast, primitive recursion is well-founded as it reduces the case for $n + 1$ to the case for $n$, and the case for $n = 0$ is given.

The observations in the previous paragraph suggest that it is desirable to have a more effective or direct alternative to bar recursion. This desire is also expressed in [2, §8] and [23, §5]. Indeed, open induction (See [6]) and the BBC-functional were even introduced with this goal in mind, but [23, Theorem 5.1] shows that these forms of recursion are merely ‘bar recursion in disguise’.

In this paper, we show that Nonstandard Analysis (following Nelson’s syntactic approach from [18]) provides a rather effective and direct version of bar recursion. In particular, we show that the following primitive recursive functional

\[
H(Y^2, s^0, M) = \begin{cases} 
Y(\Sigma M \ast 0 \ldots) & |s| \geq M \\
Y(s \ast 0 \ast H(Y, s \ast 1, M) \ast \cdots \ast H(Y, s \ast M, M) \ast 00 \ldots) & \text{otherwise}
\end{cases}
\]

equals the $\Gamma$-functional from (GH) for standard input and any nonstandard number $M^0$. Note that one need only apply the definition of $H$ at most $M$ times to terminate in the first case. In other words, the extra case ‘$|s| \geq M$’ provides a nonstandard stopping condition which ‘unwinds’ the non-terminating recursion in $\Gamma$ to the terminating one in $H$. Or: one can trade in self-reference for nonstandard numbers.

Here, it is worth recalling Nelson’s dictum from [18, p. 1166] that Every specific object of conventional mathematics is standard. Hence, $\Gamma(\cdot)$ equals $H(\cdot, M)$ for infinite $M$ and standard input, which is ‘where it matters’. Furthermore, a general treatment of bar recursion involving a nonstandard stopping condition is explained in Section 3.4 but the technical details are beyond the scope of this paper.

In conclusion, we can view bar recursion as primitive recursion with a nonstandard stopping condition, but independent of the choice of the particular infinite number $M$. In other words, one can trade in bar recursion’s apparent self-reference for nonstandard numbers; The meaning of the latter in the setting of a computer program is of course a matter of debate.

As to the structure of this paper, we introduce some suitable nonstandard systems in Section 2 based on Nelson’s syntactic approach [18]. Our main results are proved in Section 3. In particular, we prove that $H(\cdot, M)$ equals $\Gamma(\cdot)$ in both classical and (semi-)constructive systems, for varying classes of input. We also prove equivalences between continuity principles on one hand, and the statement that

\footnote{Modified bar recursion of type 0 corresponds to (GH) by [7, §4].}

\footnote{The fact that $H(\cdot)$ is primitive recursive follows from the results in [11]. We urge the reader to consult Remark [10] where it is shown that ‘primitive recursion in Nelson’s framework’ is exactly primitive recursion itself.}
\( \Gamma(\cdot) \) equals \( H(\cdot, M) \) on the other hand. We prove similar results for Spector and modified bar recursion. We formulate our conclusion in Section 4.

Proofs of the main theorems may be found in the Appendix at the end of this paper. Finally, we urge the reader to first consult Remark 10 concerning Tennenbaum’s theorem and its relation to Nelson’s framework.

2. A nonstandard version of Gödel’s T

In this section, we introduce the systems \( T_0 \) and \( T_1 \) in which we will prove our main results. In Section 3 it will become clear why we cannot work with Nelson’s IST from [18] directly. Furthermore, we discuss some basic results and introduce some notation in this section. We assume familiarity with the usual notations for concatenation of strings from [7]. We introduce \( T_0 \) and \( T_1 \) in Section 2.1 and discuss these systems’ axioms in more detail in Section 2.2.

2.1. The systems \( T_0 \) and \( T_1 \). In two words, both systems \( T_0 \) and \( T_1 \) are conservative extensions of Gödel’s system \( T \) (of all primitive recursive functionals of higher type) with certain axioms from Nelson’s Internal Set Theory IST [18] based on the approach from [3,4]. We introduce these systems in this section, and discuss their axioms in more detail in Section 2.2.

In Nelson’s syntactic approach to Nonstandard Analysis [18], as opposed to Robinson’s semantic one [24], a new predicate ‘\( \text{st}(x) \)’, read as ‘\( x \) is standard’ is added to the language of ZFC. The notations \((\forall^{\text{st}} x)\) and \((\exists^{\text{st}} y)\) are short for \((\forall x)(\text{st}(x) \to \ldots)\) and \((\exists y)(\text{st}(y) \land \ldots)\). The three new axioms Idealization, Standard Part, and Transfer of IST govern the new predicate ‘\( \text{st} \)’ and give rise to a conservative extension of ZFC set theory. Nelson’s approach has been studied in the context of higher-type arithmetic in e.g. [1,3,4], where the aforementioned new axioms of IST are weakened to be conservative over systems of arithmetic.

First of all, we define the system \( T_0 \) as

\[
(1) \quad \text{E-PA}_{\text{st}}^{\omega} + \text{I} + \text{HAC}_{\text{int}}
\]

from [3, §7]; Here, E-PA\( ^{\omega} \) defines all extensional primitive recursive functionals of higher type and E-PA\( ^{\omega}_{\text{st}} \) is a conservative extension with only trivial axioms of Nonstandard Analysis. Furthermore, the system \( T_1 \) is defined as:

\[
(2) \quad \text{E-PA}_{\text{st}}^{\omega} + \text{I} + \text{HAC}_{\text{int}} + \text{PF-TP}_\forall
\]

discussed in [4, §3.2-3.3]. We discuss the axioms of \( T_0 \) and \( T_1 \) in more detail in Section 2.2.

Intuitively speaking, to guarantee that our systems are still conservative extensions of Peano arithmetic (PA), Nelson’s axiom Standard part must be limited to \( \Omega-\text{CA} \) defined below (which derives from \( \text{HAC}_{\text{int}} \) as in (1) above), while Nelson’s axiom Transfer has to be limited to universal formulas without parameters, as in \( \text{PF-TP}_\forall \) in (2).

**Theorem 1.** The systems \( T_0 \) and \( T_1 \) are conservative extensions of E-PA\( ^{\omega} \).

**Proof.** See [3] Corollary 7.7] and [4] Theorem 5. \( \square \)
2.2. The Transfer principle of $T_1$. In this section, we discuss the Transfer principle included in $T_1$, which is as follows.

**Principle 2 (PF-TP$\forall$).** For any internal formula $\varphi(x^\tau)$ with all parameters shown, we have $(\forall x^\tau)\varphi(x) \rightarrow (\forall x)\varphi(x)$.

A special case of this axiom is found in Avigad’s system NPRA$^\omega$ from [1]. The omission of parameters in PF-TP$\forall$ is essential, as is clear from the following theorem, relating to:

- $(\Pi_1^{\text{-TRANS}})$ $(\forall x^\tau f^1)(\forall n)[(\forall x f^1) = 0 \rightarrow (\forall n)f(n) = 0]$,
- $(\exists^2)$ $(\exists^2\varphi)(\exists^2 g^1)(\exists^2 x^0)[(\exists x^0)g(x) = 0 \leftrightarrow \varphi(g) = 0]$.

Note that standard parameters are allowed in $f$, and that $(\exists^2)$ computes the halting problem. Furthermore, $T_3$ is the system from [4, Cor. 12], a conservative extension of $I\Delta_0 + \text{EXP}$.

**Theorem 3.** The system $T_3$ proves $\Pi_1^{\text{-TRANS}} \leftrightarrow (\exists^2)$

**Proof.** By [4, Cor. 12]. □

Besides being essential for the proof of the previous theorem, PF-TP$\forall$ is also convenient for other reasons. Indeed, as discussed in the next remark, we may assume all functionals defined without parameters are standard, thanks to PF-TP$\forall$.

**Remark 4 (Standard functionals).** First of all, given the existence of a functional, like e.g. the existence of the fan functional (See [7, §2.5]) as follows:

$$(\exists^2)^3(\forall x^2)(\exists^2 f^1, g^1)(\exists^2)(\exists^2 x^0)[fg = x^0 \leftrightarrow \varphi(f) = \varphi(g)]$$

we immediately obtain, via the contraposition of PF-TP$\forall$, that

$$(\exists^2)^3(\forall x^2)(\exists^2 f, g)(\exists^2 x^0)[fg = x^0 \leftrightarrow \varphi(f) = \varphi(g)]$$

In other words, we may assume that the fan functional (if it exists) is standard. The same holds for any functional of which the definition does not involve additional parameters, like the $\Gamma$-functional given as in [GH].

We again recall Nelson’s dictum from [18, p. 1166] that Every specific object of conventional mathematics is standard. By the previous, this statement remains true in $T_1$.

2.3. The Standard part principle of $T_0$ and $T_1$. In this section, we discuss the Standard Part principle called $\Omega$-CA included in our systems (in the guise of HAC$_{\text{int}}$). Intuitively speaking, a Standard Part principle allows us to convert nonstandard into standard objects. By way of example, the following type 1-version of the Standard part principle is essentially weak König’s lemma (See [15]).

$$(\forall X^1)(\exists^\omega x^0)(\exists^\omega x^0)(\exists^\omega x^0)(x \in X \leftrightarrow x \in Y).$$

Here, we have used set notation to increase readability: We assume that sets $X^1$ are given by their characteristic functions $f^1_X$, i.e. $(\forall x^0)(x \in X \leftrightarrow f_X(x) = 1]$. The set $Y$ from is also called the standard part of $X$.

We now discuss the Standard Part principle $\Omega$-CA, a very practical consequence of the axiom HAC$_{\text{int}}$. Intuitively speaking, $\Omega$-CA expresses that we can obtain the standard part (in case $G$) of $\Omega$-invariant nonstandard objects (in case $F(\cdot, M)$).

Note that we write ‘$M \in \Omega$’ as short for $\neg \text{st}(M^0)$.
Definition 5. [Ω-invariance] Let $F^{(\sigma \times 0) \rightarrow 0}$ be standard and fix $M^0 \in \Omega$. Then $F(\cdot, M)$ is Ω-invariant if

$$\forall^{st} N^0 \in \Omega \left[ F(x, M) =_0 F(x, N) \right].$$

Principle 6 (Ω-CA). Let $F^{(\sigma \times 0) \rightarrow 0}$ be standard and fix $M \in \Omega$. For Ω-invariant $F(\cdot, M)$, there is standard $G^{\sigma \rightarrow 0}$ such that

$$\forall^{st} x^\sigma (\forall N^0 \in \Omega) \left[ G(x) =_0 F(x, N) \right].$$

The axiom Ω-CA provides the standard part of a nonstandard object, if the latter is independent of the choice of infinite number used in its definition.

Theorem 7. The system $T_0$ proves the axiom Ω-CA.

Proof. We sketch the derivation of Ω-CA from HAC_{int}. The latter takes the form

$$\forall^{st} x^\sigma (\exists y)(\forall^{st} x^\sigma)(\forall N^0 \in \Omega) \left[ \varphi(x, y) \rightarrow (\exists \Phi)(\forall^{st} x^\sigma)(\exists y \in \Phi(x)) \varphi(x, y), \right]$$

where $\varphi(x, y)$ is internal, i.e. not involving the standardness predicate ‘st’, and where $\Phi(x)$ is a finite sequence of objects of the type of $y$. Thus, HAC_{int} does not provide a witness $y$, but only a sequence of possible witnesses.

Assuming $F(\cdot, M^0)$ is Ω-invariant, i.e. we have

$$\forall^{st} x^\sigma (\forall N^0, M^0 \in \Omega) \left[ F(x, M) =_0 F(x, N) \right],$$

it is easy to obtain (e.g. via minimisation present in PA) that

$$\forall^{st} x^\sigma (\exists k^0)(\forall N^0, M^0 \geq k) \left[ F(x, M) =_0 F(x, N) \right].$$

Indeed, note that (5) trivially implies (take any $m \in \Omega$) that

$$\forall^{st} x^\sigma (\exists m)(\forall N^0, M^0 \geq m) \left[ F(x, M) =_0 F(x, N) \right],$$

and obtain the least such $m$ using the minimisation axioms present in PA. By (5), this least number must be standard, and we obtain (6).

Now apply HAC_{int} to (6) to obtain standard $\Phi^{st \rightarrow 0}$ such that

$$\forall^{st} x^\sigma (\exists k^0 \in \Phi(x))(\forall N^0, M^0 \geq k) \left[ F(x, M) =_0 F(x, N) \right].$$

Define $\Psi(x) := \max_{i \in \Phi(x)} \Phi(x)(i)$ and note that

$$\forall^{st} x^\sigma (\forall N^0, M^0 \geq \Psi(x)) \left[ F(x, M) =_0 F(x, N) \right].$$

Finally, define $G(x) := F(x, \Psi(x))$ and note that the latter is as in Ω-CA.

The axiom Ω-CA can be generalised to functionals $F^{(\sigma \times 0) \rightarrow \tau}$ using the approximate equality ‘$\approx \tau$’ defined in the next section. However, for modified bar recursion, the above version suffices, as the former has output type zero.

Finally, we note that Ω-invariance can be viewed as the nonstandard version of computability. Indeed, by Post’s Theorem (See [26], p. 64), a set is computable if and only if it has a $\Delta^0_1$-definition. It is not difficult to show that sets with a $\Delta^0_1$-definition (relative to ‘st’) are also Ω-invariant.
2.4. Notations regarding $T_0$ and $T_1$. We finish this section with some remarks regarding notation; in general, we follow the notations from [4].

Remark 8 (Standardness). As suggested above, we write $(\forall x^\tau)\Phi(x^\tau)$ and $(\exists x^\tau)\Psi(x^\tau)$ as short for $(\forall x^\tau)[\text{st}(x^\tau) \to \Phi(x^\tau)]$ and $(\exists x^\tau)[\text{st}(x^\tau) \land \Psi(x^\tau)]$. We also write $(\forall x^0 \in \Omega)\Phi(x^0)$ and $(\exists x^0 \in \Omega)\Psi(x^0)$ as short for $(\forall x^0)[\neg \text{st}(x^0) \to \Phi(x^0)]$ and $(\exists x^0)[\neg \text{st}(x^0) \land \Psi(x^0)]$. Furthermore, if $\neg \text{st}(x^0)$ (resp. $\text{st}(x^0)$), we also say that $x^0$ is ‘infinite’ (resp. finite) and write ‘$x^0 \in \Omega$’. Finally, a formula $A$ is ‘internal’ if it does not involve st, and $A^\text{st}$ is defined from $A$ by appending ‘st’ to all quantifiers (except bounded number quantifiers).

As we will see below, the notion of equality in our systems $T_0$ and $T_1$ is quite important.

Remark 9 (Equality). Our systems only include equality between natural numbers ‘$=^0$’ as a primitive. Equality ‘$=^\tau$’ for type $\tau$-objects $x, y$ is then defined as follows:

$$[x =^\tau y] \equiv (\forall z^1 \ldots z^k)[xz_1 \ldots z_k =^0 yz_1 \ldots z_k]$$

if the type $\tau$ is composed as $\tau = (\tau_1 \to \ldots \to \tau_k \to 0)$. In the spirit of Nonstandard Analysis, we define ‘approximate equality $\approx^\tau$’ as follows:

$$[x \approx^\tau y] \equiv (\forall z^1 \ldots z^k)[xz_1 \ldots z_k =^0 yz_1 \ldots z_k]$$

with the type $\tau$ as above. Furthermore, our systems include the axiom of extensionality as follows:

$$(\forall \varphi^\rho \cdot \forall \rho \in \sigma \cdot x =^\tau y \to \varphi(x) =^\tau \varphi(y)).$$

By [3, p. 1973] however, the axiom of standard extensionality $\text{[E]}^\text{st}$ is problematic and cannot be included in our systems.

We finish this section with a discussion of an often encountered fallacy pertaining to Tennenbaum’s famous result.

Remark 10. Tennenbaum’s theorem [14, §11.3] ‘literally’ states that any nonstandard model of PA is not computable. What is meant is that for a nonstandard model $M$ of PA, the operations $+_M$ and $\times_M$ cannot be computably defined in terms of the operations $+_N$ and $\times_N$ of the standard model $N$.

Tennenbaum’s theorem is clearly of interest to the semantic approach to Nonstandard Analysis involving nonstandard models, but our systems are based on Nelson’s syntactic framework. Therefore Tennenbaum’s theorem does not apply; in fact, any attempt at defining the function ‘$+^\text{st}$ limited to the standard numbers’ is an instance of illegal set formation, forbidden in Nelson’s internal framework (See [18, p. 1165]).

To be absolutely clear, lest we be misunderstood, Nelson’s framework internal set theory forbids the formation of external sets such as $\{x \in A : \text{st}(x)\}$ or external functions ‘$f(x)$ limited to standard $x$’. Therefore, any appeal to Tennenbaum’s theorem to claim the ‘non-computable’ nature of ‘$+$ and $\times$ from our systems is blocked, for the simple reason that the (external) functions ‘$+$ and/or $\times$ limited to the standard numbers’ do not exist.

Finally, in light of its importance, we repeat once more Nelson’s dictum from [18, p. 1166], as follows:
Every specific object of conventional mathematics is a standard set.

It remains unchanged in the new theory [IST].

In other words, the operations ‘+’ and ‘×’, but equally so primitive recursion, in (subsystems of) IST, are exactly the same familiar operations we know from (subsystems of) ZFC.

3. Bar recursion as primitive recursion with nonstandard numbers

3.1. The Gandy-Hyland functional. In this section, we list our main results concerning the Γ-functional and its so-called canonical approximation $H$, both introduced in the first section. As to its provenance, we recall that the Γ-functional was introduced in [13] as an example of a functional not Kleene-S1-S9-computable over the total continuous functionals, even with the fan functional as an oracle (See [19, §4] for these results).

We shall also study the following functional $G$, a ‘less finitary’ version of $H$, but which is still primitive recursive by the results in [11].

$$G(Y, s, N) = \begin{cases} Y(s * 00 \ldots) & |s| \geq N \\ Y(s * 0 * (\lambda n)G(Y, s * (n + 1), N)) & \text{otherwise} \end{cases}$$

As noted in the first section, the Γ-functional corresponds to modified bar recursion of type 0 (See [7, §4]). Since bar recursion holds in the model of all total continuous functionals (See [5, 9]), the easiest way of obtaining Γ from $H$ or $G$ seems to be adding the following continuity axioms to $T_0$ and $T_1$ (Recall Remark 9):

(NPC) \hspace{1cm} (\forall^t Y^2, f^1)(\forall g^1)(f \approx_1 g \rightarrow Y(f) =_0 Y(g)).

(MPC) \hspace{1cm} (\exists \Psi^3)(\forall Y^2, f^1, g^1)(\exists \Psi(Y, f) = \exists \Psi(Y, f) \rightarrow Y(f) = Y(g)).

Note that these axioms contradict classical mathematics\(^\text{3}\) which is the reason we cannot use Nelson’s IST directly. Also note that the first axiom expresses the nonstandard (pointwise) continuity of all type two functionals, while (MPC) states the existence of a modulus-of-(pointwise)-continuity functional.

Furthermore, according to [7, p. 167], the role of the continuity principle and bar induction in [7, Theorem 2.5] is to verify the correctness of the [bar recursive] witnessing functional. As will be proved in Section A.1, the principle (STP) from Section 2.3 implies a version of bar induction. Hence, we obtain the following theorem.

**Theorem 11.** In $T_0 + (\text{NPC}) + (\text{STP})$, we have

\[ (9) \quad (\forall^t Y^2, s^0)(\forall M, N \in \Omega)(H(Y, s, N) =_0 H(Y, s, M)), \]

\[ (10) \quad (\forall^t Y^2, s^0)(\forall M, N \in \Omega)(G(Y, s, N) =_0 G(Y, s, M)), \]

i.e. the canonical approximations of Γ are Ω-invariant.

\(^3\)By the results in [10], the existence of a discontinuous function is equivalent to (\exists^2) from Section 2.3 and the weaker principle weak König’s lemma is consistent with the statement that all functions are continuous.
Proof. We sketch the proof of the theorem and provide a detailed version in the Appendix.

First of all, one derives a version of bar induction from (STP). Secondly, one uses this bar induction to prove that $G(Y, s, M)$ is standard for standard $Y^2$, $s^0$ and infinite $M^0$. Thirdly, one applies this bar induction again to prove that (9) holds for fixed inputs.

The result for $H(\cdot , M)$ is now straightforward. Intuitively speaking, $H$ and $G$ only really differ in the second case of their definition for elements in the sequence with infinite index, which does not matter due to nonstandard continuity. □

The following corollary expresses that the standard Gandy-Hyland functional equals its canonical approximation.

Corollary 12. In $T_0 + \text{(NPC)} + \text{(STP)}$, the standard Gandy-Hyland functional exists and equals both canonical approximations for standard inputs, i.e. we have $(\text{GH})^{st}$ and

$$(\forall^s Y^2, s^0)(\forall N \in \Omega)(G(Y, s, N) = H(Y, s, N) = \Gamma(Y, s)).$$

Proof. By (10), the functional $G(Y, s, M)$ is $\Omega$-invariant and $\Omega$-CA yields the standard part of $G(Y, s, M)$, denoted $\Gamma_0(Y, s)$. Thus, for standard $Y^2$, $s^0$ and $M \in \Omega$, we have

$$\Gamma_0(Y, s) = G(Y, s, M) = Y(s \ast 0 \ast G(Y, s \ast 1, M) \ast G(Y, s \ast 2, M) \ast \ldots)$$
$$= Y(s \ast 0 \ast \Gamma_0(Y, s \ast 1) \ast \Gamma_0(Y, s \ast 2) \ast \ldots),$$

where we used (NPC) in the final step. Hence, the standard part $\Gamma_0(\cdot)$ of $G(\cdot, M)$ as provided by $\Omega$-CA is indeed the Gandy-Hyland functional and $(\text{GH})^{st}$ follows. □

This corollary sports a ‘more standard’ base theory.

Corollary 13. In $T_1 + \text{(MPC)} + \text{(STP)}$, the Gandy-Hyland functional exists and equals its canonical approximations for standard inputs, i.e. we have $(\text{GH})^{st}$ and

$$(\forall^s Y^2, s^0)(\forall N \in \Omega)(G(Y, s, N) = H(Y, s, N) = \Gamma(Y, s)).$$

Proof. As noted in Remark 4, the functional $\Psi$ from (MPC) may be assumed to be standard. But then (NPC) is immediate as $\Psi(\varphi, f)$ is standard for standard input. To show that $(\text{GH})^{st}$ implies $(\text{GH})$, proceed in the same way as in [4, §3.3 and Cor. 12] for Feferman’s non-constructive $\mu$-operator ($\mu^2$).

In more detail, the (languages of the) systems in [4] include a symbol $\mu_0$ representing the standard $\mu$-operator if the latter exists. Due to the uniqueness of the $\mu$-operator, one may replace the latter by $\mu_0$ in the defining formula $(\mu^2)^{st}$ of the standard $\mu$-operator. The resulting sentence has no more parameters, and applying PF-TP$_\nu$ yields the full $\mu$-operator ($\mu^2$). Similarly, the systems in [4] and $T_1$ contain a symbol $\Lambda_0$ in their language for the (unique by the proof of Corollary 12) Gandy-Hyland functional, if the latter exists. Thus, replace $\Gamma$ in $(\text{GH})^{st}$ by $\Lambda_0$ and the resulting formula has no more parameters; Applying PF-TP$_\nu$ now yields $(\text{GH})$. □

While the combination of (MPC) and (STP) suffices for our purposes, the proof of Theorem 11 in Section A.2 goes through for other principles appearing in the context of bar recursion. Hence, consider the the ‘full’ fan functional (FFF):

$$(\text{FFF}) \quad (\exists \Omega^3)(\forall Y^2, h^1)(\forall f^1, g^1 \leq_1 h) Y(\Omega(Y, h) = 0 \Omega(Y, h) \rightarrow Y(f) = 0 Y(g),$$
and the weak continuity functional (PWC) as follows:

**Principle 14 (PWC).** There is a functional \( \Upsilon^3 \) such that

\[
(\forall Y^2, f^1, g^1)[\Upsilon(Y, f) = \Upsilon(Y, f) \rightarrow Y(g) \leq \Upsilon(Y, f)],
\]

and \( \Upsilon(Y, f) \) is the least number with this property.

We refer to [5, §2.5 and Lemma 5.4] for more details on these functionals and their relation to bar recursion.

**Theorem 15.** The proof of Theorem 11 in \( T_1 + \text{(MPC)} + \text{(STP)} \) goes through for \( \text{(MPC)} \) replaced by \( \text{(FFF)} \) and (PWC).

**Proof.** It is a tedious verification that the proof of Theorem 11 indeed goes through for (NPC) (which follows from (MPC)) replaced by the fan and weak continuity functional. We provide a sketch in Section A.3 in the Appendix. □

Other replacements giving rise to a proof of Theorem 11:

**Remark 16.**

1. Replace (STP) by (EBI) from Section A.1 and:

\[
(\forall^{st} Y^2)(\forall f^1)[(\forall^{st} n)(\exists^{st} m)(f(n) = 0 \rightarrow st(Y(f))].
\]

2. Limit the previous formula to functionals \( \mu_{sp}(Y, \cdot) \) where \( \mu_{sp} \) is Spector’s search functional from Section 3.3.

The attentive reader has noted that Corollary 12 also holds for nonstandard \( s^0 \) such that \( s(n) \) is standard for standard \( n^0 \), i.e. \( s^0 \) has a standard part. We now prove that this generalisation gives rise to nice equivalences.

**Principle 17 (GHS).** The standard Gandy-Hyland functional \( \Gamma(Y, s) \) exists and equals its canonical approximation \( H(Y, s, M) \) for standard \( Y^2 \), \( M \in \Omega \), and any \( s^0 \) with a standard part.

**Theorem 18.** In \( T_0 + \text{(STP)} + \text{(FFF)} \), we have \( \text{(NPC)} \leftrightarrow \text{(GHS)} \).

**Proof.** The forward implication is immediate by Corollary 12. To obtain (NPC) from (GHS), one computes \( Y(f) \) and \( Y(g) \) using the latter and notes that they are identical if \( f \approx_1 g \) for standard \( f \). A more detailed proof may be found in Section B of the Appendix. □

Let (PWC)\(^-\) be (PWC) without the minimality condition. It is easy to verify that the proof in Section A.3 of the Appendix still goes through with this weakening. We also have the following theorem.

**Corollary 19.** In \( T_1 + \text{(STP)} + \text{(FFF)} \), we have:

\[
(11) \quad \text{(MPC)} \leftrightarrow \text{(GHS)} \leftrightarrow \text{(PWC)}^{-}\.
\]

**Proof.** In the proof of Corollary 13, it is proved that (MPC) implies (NPC). In turn, the latter implies (MPC)\(^st\) thanks to the idealisation axiom I present in \( T_0 \) and \( T_1 \). As in the proofs of Corollaries 13 and 23, the systems in 4 and \( T_1 \) contain a symbol \( \Lambda_0 \) in their language representing a standard modulus-of-continuity functional as in (MPC)\(^st\) if the latter exists. Replacing \( \Psi \) in the latter by \( \Lambda_0 \), the resulting formula is actually a sentence, and PF-TP\(_\Psi^\Lambda_0 \) yields (MPC). Finally, (MPC) clearly implies (PWC)\(^-\) and it is easy to verify that the proof in Section A.3 of the Appendix still goes through for this weakening. □
The previous theorem is interesting in its own right, but its proof also gives rise to the following ‘practical’ corollary.

**Corollary 20.** In $T_0 + \{\text{MPC}\} + \{\text{STP}\}$, we have
\[
(\forall^s Y^2, \alpha^1)(\exists^s K)(\forall M \geq K)(\forall s^0)[s + 0 \ldots K \leq \pi K \rightarrow H(Y, s, M) = \Gamma(Y, s)].
\]

**Proof.** Immediate from the proof in Section 13. □

Intuitively speaking, the previous corollary expresses that one can uniformly compute $\Gamma(Y, s)$ via the number $H(Y, s, k)$ for some $k$ independent of $s^0$ in a standard compact space. It should be noted that proof mining is known for extracting highly uniform bounds from proofs dealing with compact spaces (See [17, Introduction]).

### 3.2. Classical results.

In this section, we show that results similar to the above ones can be obtained in systems of classical mathematics. To this end, let $\{\text{NPC}\}_0$, $\{\text{MPC}\}_0$, and $\{\text{GH}\}_0$ be the associated principles limited to $Y^2 \in C$, where the latter is:

\[
(\forall f^1)(\exists N^0)(\forall g^1)[f^N = 0 \rightarrow g^N = Y(f) = Y(g)],
\]
i.e. the usual definition of pointwise continuity. We have the following theorem.

**Theorem 21.** In $T_0 + \{\text{NPC}\}_0 + \{\text{STP}\}$, we have
\[
(\forall^s Y^2 \in C, s^0)(\forall M, N \in \Omega)[H(Y, s, N) = 0 \rightarrow H(Y, s, M)]
\]
\[
(\forall^s Y^2 \in C, s^0)(\forall M, N \in \Omega)[G(Y, s, N) = 0 \rightarrow G(Y, s, M)]
\]

**Proof.** Limit the proof of Theorem 11 to $Y^2 \in C$. □

Despite the presence of ‘$\in C$’, the corollaries of Theorem 11 also go through for classical systems.

**Corollary 22.** In $T_0 + \{\text{NPC}\}_0 + \{\text{STP}\}$, we have $\{\text{GH}\}_0^s$ and
\[
(\forall^s Y \in C, s^0)(\forall N \in \Omega)[G(Y, s, N) = H(Y, s, N) = \Gamma(Y, s)].
\]

**Proof.** Following the proof of Theorem 7, we obtain a standard part for $G(\cdot, M)$. By Theorem 21, $G(\cdot, M)$ is $\Omega$-invariant for $Y \in C$, implying
\[
(\forall^s Y \in C, s^0)(\exists^s k)(\forall M, N \geq k)[G(Y, s, N) = G(Y, s, M)],
\]
in the same way as for (6). Using classical logic, this yields:
\[
(\forall^s Y^2, s^0)(\exists^s k)(\forall M, N \geq k)(Y \in C \rightarrow G(Y, s, N) = G(Y, s, M)).
\]
As $\{\text{GH}\}_0^s$ is internal, $\text{HAC}_{\text{int}}$ applies to the previous formula. In the same way as in the proof of Theorem 7, we obtain the standard part of $G(\cdot, M)$, say $\Gamma_1$. Finally, repeat (??) for the latter, but with additionally $Y^2 \in C$. □

**Corollary 23.** In $T_1 + \{\text{MPC}\}_0 + \{\text{STP}\}$, we have $\{\text{GH}\}_0$ and
\[
(\forall^s Y \in C, s^0)(\forall N \in \Omega)[G(Y, s, N) = H(Y, s, N) = \Gamma(Y, s)].
\]

**Proof.** Similar to the proof of Corollary 13. In particular, to show that $\{\text{GH}\}_0^s$ implies $\{\text{GH}\}_0$, one can still proceed in the same way as in [21, §3.3 and Cor. 12] for Feferman’s non-constructive $\mu$-operator, as ‘$Y \in C$’ is an internal formula. □
In conclusion, we observe that to obtain the Gandy-Hyland functional, we have a choice between a) working in semi-intuitionistic systems involving (NPC) or (MPC) and obtain the ‘full version’ (GH), or b) working in a classical system involving (NPC)₀ or (MPC)₀ and obtain the ‘partial version’ (GH)₀. By the previous, Nelson’s IST can replace the system in Corollary 23.

Remark 24. Normann provides a characterisation of the continuous functionals in [20] using Nonstandard Analysis, while a more general result is obtained in [19]. Our results are much ‘finer’ as Normann seems to use the non-constructive Transfer and Standard part principles arbitrarily in [20], while the model-theoretic techniques from [21] are also claimed to be non-constructive by the authors.

3.3. Specter bar recursion. In this section, we treat special cases of Spector’s bar recursion using the techniques from the previous section.

3.3.1. Specter search functional. In this section, we show that the Specter search functional \( \mu_{sp} \) (See [22] §1.2 or [7] §6, the latter also for notations) equals its canonical approximation, defined as:

\[
\mu_{sp}(Y^2, f^1) := \text{the least } n^0 \text{ such that } Y(f \ast 00 \ldots) < n,
\]

\[
\nu(Y^2, f^1, M) := (\mu n \leq M)(Y(f \ast 00) < n).
\]

Let (SSF) be the first formula with (\forall Y^2, f^1) prepended. We have the following theorem.

Theorem 25. In \( T_0 + (\text{NPC}) \), the Specter search functional exists as in (SSF) and equals its canonical approximation for standard input.

Proof. Let \( Y^2 \) and \( f \) be standard; Since we have (NPC), \( \overline{f} M \approx_1 f \), implies \( Y(f) = Y(f \ast 00) \) for \( M \in \Omega \). Hence, \( Y(f \ast 00 \ldots) < M \) for any \( M \in \Omega \), as \( Y(f) \) is standard. But then \( \nu(Y, f, M) \) is standard and the choice of \( M \in \Omega \) does not matter. Thus, \( \nu(\cdot, M) \) is \( \Omega \)-invariant, and applying \( \Omega \)-CA yields the functional \( \mu_{sp} \). □

Corollary 26. In \( T_1 + (\text{MPC}) \), the Specter search functional exists as in (SSF), and equals its canonical approximation for standard input.

Proof. As in the last part of the proof of Corollary 13. □

In general, the Specter search functional has type \( ((\rho + 1) \times \rho^0) \rightarrow 0 \) and is defined as:

\[
\mu_{sp}(\varphi, \alpha) := \text{the least } n^0 \text{ such that } \varphi(\alpha, n) < n,
\]

where \( \overline{\alpha, k} = \overline{\pi k \ast 000 \ldots} \) and \( k^0 \) is the type \( \rho \rightarrow 0 \) functional which is constant \( k \). Clearly, the generalisation to higher-type functionals of Theorem 25 is immediate, assuming a modulus-of-continuity functional or nonstandard continuity.

3.3.2. Specter bar recursion of low type. We investigate Spector’s bar recursion limited to functionals of low type, inspired by our results on the Gandy-Hyland functional. We sketch what kind of difficulties appear when attempting to generalise our results to higher types.
We refer to [7] for notations and the following definition of Spector’s bar recursion:

\[
\text{SBR}_{\rho,\tau}(Y, G, H, s) := \left\{ \begin{array}{ll}
G(s) & \text{if } Y(\hat{s}) < |s| \\
H(s, (\lambda x)\text{SBR}_{\rho,\tau}(Y, G, H, s * x)) & \text{otherwise}
\end{array} \right.
\]

Let us denote by \((\text{SBR}_{\rho,\tau})\) the previous formula with the quantifier \((\forall Y, G, H, s)\) of the right type in front. The canonical approximation of Spector bar recursion is:

\[
S_{\rho,\tau}(Y, G, H, s, M) := \left\{ \begin{array}{ll}
G(s) & \text{if } Y(\hat{s}) < |s| \lor |s| \geq M \\
H(s, (\lambda x)S_{\rho,\tau}(Y, G, H, s * x, M)) & \text{otherwise}
\end{array} \right.
\]

We have the following theorems.

**Theorem 27.** In \(T_0 + (\text{NPC}) + (\text{STP})\), we have

\[(\forall^* Y^2, G^1, H^2, s^0)(\forall M, N \in \Omega)( S_{0,0}(Y, G, H, s, N) =_0 S_{0,0}(Y, G, H, s, M)),\]

i.e. the canonical approximation of \(\text{SBR}_{0,0}\) is \(\Omega\)-invariant.

*Proof.* Similar to the proof of Theorem 11. \(\square\)

**Corollary 28.** In \(T_0 + (\text{NPC}) + (\text{STP})\), the standard Spector bar recursion functional \(\text{SBR}_{0,0}\) exists and equals its canonical approximation, i.e. we have \((\text{SBR}_{0,0})^*\) and

\[(\forall^* Y^2, G^1, H^2, s^0)(\forall N \in \Omega)( S_{0,0}(Y, G, H, s, N) =_0 \text{SBR}_{0,0}(Y, G, H, s)).\]

*Proof.* The proof is similar to the proof of Corollary 12. By Theorem 27, we may take the standard part, say \(T(\cdot)\) of the canonical approximation \(S_{0,0}(\cdot, M)\). For standard \(s\) and \(Y\), if \(Y(s \ast 0 \ldots) < |s|\), then \(T(Y, G, H, s)\) equals \(G(s)\); If not:

\[
T(Y, G, H, s) = H(s, (\lambda x)S_{0,0}(Y, G, H, s * t * x, N))
\]

\[
= H(s, (\lambda x)T(Y, G, H, s * t * x)),
\]

where the final equality follows by continuity. Hence \(T\) is indeed the functional \(\text{SBR}_{0,0}\) for standard inputs. \(\square\)

**Corollary 29.** In \(T_1 + (\text{MPC}) + (\text{STP})\), the Spector bar recursion functional \(\text{SBR}_{0,0}\) exists and equals its canonical approximation, i.e. we have \((\text{SBR}_{0,0})^*\) and

\[(\forall^* Y^2, G^1, H^2, s^0)(\forall N \in \Omega)( S_{0,0}(Y, G, H, s, N) =_0 \text{SBR}_{0,0}(Y, G, H, s)).\]

*Proof.* Similar to the proof of Corollary 13. \(\square\)

Clearly, we could limit the previous results to functionals \(Y^2 \in C\) and hence classical systems.

We now sketch a similar result for \(\text{SBR}_{0,1}\), and provide a hint of what kind of problems await if one attempts to generalise the above results to higher types.

To obtain \(\text{SBR}_{0,1}\) in the above way, one requires a stronger version of bar induction, \((\text{MPC})\) and \((\text{NPC})\) generalised to \(Y^{(0 \rightarrow 1)^{1}}\), and \((\text{STP})\) generalised to \(X^{0 \rightarrow 1}\), i.e. sequences of sets. All proofs proceed along the lines of Theorem 11 and the aforementioned generalised principles are of strength similar to the original ones. Indeed, (13) below can be obtained in \(T_1\) plus the aforementioned generalisations of bar induction, continuity and \((\text{STP})\); Note the approximate equality.

\[(\forall^* Y^2, G, H, s^0)(\forall N \in \Omega)S_{0,1}(Y, G, H, s, N) \approx_1 \text{SBR}_{0,1}(Y, G, H, s).\]
However, SBR\(_{0,1}\) seems to be as far as one can push the above approach: The first two of the aforementioned generalisations (involving bar induction and continuity) are almost generic in light of [7, Theorem 2.5], but the third one is not: The generalisation of \((\text{STP})\) to type two objects implies \((\exists^2)\), which implies the existence of discontinuous functions by the results in [16]. Hence, we need a more constructive alternative to generalisations of \((\text{STP})\).

Furthermore, for the general treatment of SBR\(_{\rho,\tau}\), the associated version of (13) will involve the approximate equality ’\(\approx\)’ as follows:

\[
(\forall s Y,G,H,s)(\forall N \in \Omega) S_{\rho,\tau}(Y,G,H,s,N) \approx_{\tau} SBR_{\sigma,\tau}(Y,G,H,s).
\]

The presence of approximate equality in (14) complicates matters even further.

In light of the previous observations, to treat more general versions of Spector bar recursion in a semi-intuitionistic setting, a novel approach is needed, unfortunately beyond the scope of this paper. We next briefly discuss modified bar recursion, which has some advantages over Spector bar recursion.

### 3.4. Modified bar recursion

In this section, we briefly sketch the canonical approximation of modified bar recursion. The latter schema is defined as follows in [7, Def. 2.1], where notations are also given.

\[
MBR_{\rho}(Y,H,s) := Y(s@H(s, (\lambda x^\rho)MBR_{\rho}(Y,H,s \circledast x))).
\]

Here, ‘\(s@\alpha\)’ is short for ‘\(\alpha\) with its first \(|s| - 1\) entries overwritten with \(s\)’. By [7, Thm. 3.1], modified bar recursion may be bootstrapped from a weak version for \(\rho \equiv \tau^\omega\):

\[
wMBR_{\rho}(Y,H,s) := Y(s@((\lambda k^0)H(s, (\lambda x^\rho)wMBR_{\rho}(Y,H,s \circledast x))).
\]

We note two advantages of wMBR over other versions of bar recursion: The former has output type zero and it has input types based on sequences. While the treatment of full modified bar recursion is beyond the scope of this paper, the former is -intuitively speaking- easier due to the aforementioned features. Finally, the canonical approximation of wMBR is:

\[
W_{\rho}(Y,H,s,N^0) := \begin{cases} Y(\pi N \ast 0^\rho \ldots) & |s| \geq N \\ Y(s@((\lambda k^0)H(s, (\lambda x^\rho)W_{\rho}(Y,H,s \circledast x,N)))) & \text{otherwise} \end{cases}
\]

In the right system, we can obtain for \(\rho \equiv \tau^\omega\)

\[
(\forall s Y,H,s)(\forall N \in \Omega)(W_{\rho}(Y,H,s,N) =_0 wMBR_{\rho}(Y,H,s)),
\]

and from this full modified bar recursion by [7, Thm. 3.1].

We finish this section with a canonical approximation of the fan functional using bar recursion.

**Remark 30.** The fan functional is defined as \(\Psi(.,(),\Phi)\) in [5 §4], where \(\Psi\) and \(\Phi\) are defined via (respectively Kohlenbach and modified) bar recursion. Now define \(\psi(.,M)\) and \(\phi(.,M)\) as \(\Psi(\cdot)\) and \(\Phi(\cdot)\) with the extra stopping condition \(|s| \geq M\) as in [GH]. It is not difficult to show that \(\psi(.,(),\phi(.,M),M)\) is \(\Omega\)-invariant and equals the fan functional for standard input, assuming the right uniform continuity axioms on Cantor space.
4. Conclusion

We have shown that bar recursion (modified or Spector) of low type can be expressed as primitive recursion with a nonstandard stopping condition ‘|s| ≥ M’ for any nonstandard number M. In other words, one can trade in the apparent self-reference in bar recursion for nonstandard number parameters.

The proof of these facts takes place in a nonstandard version of Gödel’s system T, extended with continuity axioms and bar induction, which is the usual setup for bar recursion. An equivalence between the continuity axioms and the approximation of \( \Gamma(\cdot) \) by \( H(\cdot, M) \) was also obtained. We noted that similar results for all finite types require a novel approach, which is unfortunately beyond the scope of this paper.

Finally, it is not inconceivable that apparent self-reference can be removed from other phenomena using nonstandard numbers. The meaning of nonstandard numbers in the setting of a computer program is of course a matter of debate.

In this technical appendix, the proofs of the above theorems may be found.

**Appendix A. Proof of Theorem 11**

We first derive a version of bar induction from \([STP]\). We then prove Theorem 11 in Section A.2.

A.1. External bar induction. First of all, we derive a version of bar induction from \([STP]\). The former is a generalisation of the well-known principle of (mathematical) induction of arithmetic. While induction takes place ‘along the natural numbers’ (or any well-order), bar induction takes place ‘down a tree’. As an example, we consider the following principle.

**Principle 31** \((BI_0)\). For internal quantifier-free \(Q(x^0)\), if

\[
(\forall \alpha^1)(\exists n)Q(\alpha^n) \land (\forall t^0)[(\forall x^0)Q(t * (x)) \to Q(t)]
\]

then we have \(Q(\langle \rangle)\).

Intuitively speaking, bar induction \(BI_0\) expresses that we may conclude \(Q(x)\) for \(x = \langle \rangle\) from the fact that \(Q\) is implied ‘downwards’ from child nodes to parent nodes (second conjunct of (15)) and that \(Q\) holds eventually along any path (first conjunct of (15)). On a technical note, \(BI_0\) is essentially \(BI_{qf}\) from [28, p. 78] for \(P(n) \equiv Q(n)\) quantifier-free.

Nonstandard Analysis also has a distinct kind of induction, called external induction. We consider the following example.

**Principle 32** \((ExInd)\). For standard \(F^{(0 \times 0) \to 0}\) and \(M \in \Omega\), if

\[
(\forall \alpha^1)(\forall t^0)(\forall n)[st(F(n, M)) \to st(F(n + 1, M))],
\]

then \(\forall n^0(st(F(n, M)))\).

Intuitively speaking, \((ExInd)\) tells us that we may use induction on the new standardness predicate along the standard numbers (and obviously not along the numbers). Although seemingly more general than normal induction, we now prove external induction from \([STP]\).

**Theorem 33.** In \(T_0 + [STP]\), we have \((ExInd)\).
Proof. Fix standard $F^{(0 \times 0)}$ and $M \in \Omega$, and suppose the antecedent of $\text{(16)}$ holds. Use $\text{(STP)}$ to obtain $X$, the standard part of the set consisting of the tuples $(n, F(n, M))$. Define $Y$ as the projection of $X$ to the first coordinate. By our assumption $\text{(16)}$, we have $0 \in Y$ since $F(0, M)$ is standard. Also, by the second conjunct of $\text{(16)}$, we have $n \in Y \rightarrow (n + 1) \in Y$ for standard $n$. By quantifier-free induction, we obtain $n \in Y$ for all standard $n$, implying the consequent of $\text{(16)}$. □

Note that the same proof goes through for variations of $(\text{ExInd})$, e.g. if the induction hypothesis involves $(\forall k \leq n)(\text{st}(F(k, M)))$ instead of $\text{st}(F(n, M))$.

We now formulate external bar induction, which is nothing more than bar induction on the external standardness predicate.

**Principle 34 (EBI).** For standard $F^{(0 \times 0)}$ and $M \in \Omega$, if

\[
\begin{align*}
(17) & \forall \alpha^1 \exists\beta^0 \left[ \text{st}(F(\langle \beta \rangle, M)) \right] \\
(18) & \forall \alpha^0 \left[ \text{st}(F(t \ast \langle x \rangle, M)) \rightarrow \text{st}(F(t, M)) \right]
\end{align*}
\]

then $\text{st}(F(\langle \rangle, M))$.

As it happens, external bar induction (EBI) is not much stronger than normal bar induction, as we now prove (EBI) from $\text{(STP)}$ and $(\text{Bl}_0)^{st}$.

**Theorem 35.** In $\mathcal{T}_0 + \text{(STP)} + (\text{Bl}_0)^{st}$, we have (EBI).

Proof. Fix standard $F^{(0 \times 0)}$ and $M \in \Omega$, and suppose $\text{(17)}$ and $\text{(18)}$. Use $\text{(STP)}$ to obtain $X$, the standard part of the set consisting of the tuples $(x, F(x, M))$. Define $Y$ as the projection of $X$ to the first coordinate. By our assumption $\text{(17)}$, we have $\forall \alpha^1 \exists\beta^0 (\forall x \in Y)$. Also, by $\text{(18)}$:

\[
(\forall \alpha^0 \left[ \text{st}(F(t \ast \langle x \rangle, M)) \rightarrow (t \in Y) \right]
\]

By $(\text{Bl}_0)^{st}$, we obtain $\langle \rangle \in Y$, implying (EBI). □

Finally, we prove $(\text{Bl}_0)^{st}$ form $\text{(STP)}$.

**Theorem 36.** In $\mathcal{T}_0 + \text{(STP)}$, we have $(\text{Bl}_0)^{st}$.

Proof. Assume $(\text{15})^{st}$ and suppose we have $\neg Q(\langle \rangle)$. Now define $F(x, M) := (\mu n \leq M)\neg Q(x \ast \langle m \rangle)$ and put $G(0) := F(\langle \rangle, M)$ and $G(n + 1) := F(G(0) \ast \cdots \ast G(n), M)$. By $(\text{15})^{st}$ and the assumption $\neg Q(\langle \rangle)$, $G(0)$ is standard. Furthermore, we also have that $G(n + 1)$ is standard if $G(k)$ is standard, for standard $n$ and $k \leq n$, by $(\text{15})^{st}$, implying that $G(n)$ is standard for all standard $n$ by (a trivial extension of) external induction. This in turn implies that $\neg Q(G(0) \ast \cdots \ast G(k))$ for standard $k$, by quantifier-free induction and $(\text{15})^{st}$. Now consider the sequence $\beta^1 := G(0) \ast G(1) \ast G(2) \ast \cdots$ and take its standard part $\alpha$. Finally, apply the first conjunct of $(\text{15})^{st}$ and obtain a contradiction. □

The above results are not that surprising: $\text{(STP)}$ is the nonstandard version of weak Kőnig’s lemma (See $\text{(15)}$), the latter is equivalent to a version of dependent choice (See $\text{[25, VIII.2.5]}$), and bar induction is a version of the latter.
A.2. **Proof of Theorem 11 using external bar induction.** We now prove Theorem 11.

Proof. By Theorem 36, we may freely use external bar induction. We first prove that \( G(m, M) \) is standard for standard input and infinite \( M \). To this end, fix standard \( Y^2, s^0 \) and \( M \in \Omega \), and define \( F(x^0, M) := G(Y, s \ast x, M) \).

To prove (17), fix standard \( \gamma \) and \( N \in \Omega \). We have

\[
F(\gamma N, M) = G(Y, s \ast \gamma N, M)
\]

\[
= Y(s \ast \gamma N \ast 0 \ast (\lambda n)G(Y, s \ast \gamma N \ast (n + 1), M))
\]

(19)

\[
= Y(s \ast \gamma),
\]

(20)

where the final step follows by nonstandard continuity as \( s \ast \gamma \approx \zeta \), where the latter is the sequence in (19). We have proved that \( \forall K \in \Omega \), \( F(\gamma K) = Y(s \ast \gamma) \) and minimisation provides the least \( k_0 \) such that \( \forall K \geq k_0 \), \( F(\gamma K) = Y(s \ast \gamma) \) in the same way as for (6). Clearly, \( k_0 \) is standard and since the number in (20) is standard by definition, we have \( (\exists n)(st(F(\gamma n, M))) \) and (17).

To prove (18), assume the antecedent of the latter for standard \( t \), and consider

\[
F(t, M) = G(Y, s \ast t, M)
\]

\[
= Y(s \ast t \ast 0 \ast (\lambda n)G(Y, s \ast t \ast (n + 1), M))
\]

(21)

which follows by the definitions of \( F \) and \( G \). However, the antecedent of (18) tells us that \( F(t \ast \langle m \rangle, M) \) are standard for any standard \( m \). Hence, the sequence

\[
F(t, M) = G(Y, s \ast t, M)
\]

\[
= Y(s \ast t \ast 0 \ast (\lambda n)F(t \ast (n + 1), M))
\]

has a standard part by (STP), say \( \gamma \), and (NPC) yields \( F(t, M) = Y(\gamma) \), which is standard. Hence, we obtain (18) and (EBI) implies that \( F(\langle \rangle, M) = G(Y, s, M) \) is standard.

Now define the function \( F(x, M) \) as follows:

\[
F(x, M) := \begin{cases} 
0 & G(Y, s \ast x, M) = G(Y, s \ast x, M + 1) \\
M & \text{otherwise}
\end{cases}
\]

where \( Y \) and \( s \) are standard again. Repeating the steps from the previous paragraph of the proof, we note that \( F(\langle \rangle, M) \) satisfies (17) and (18) for any \( M \in \Omega \). Hence, (EBI) yields that \( F(\langle \rangle, M) \) is standard for any infinite \( M \); As a consequence, we have \( G(Y, s, M) = G(Y, s, M + 1) \) by definition, for any infinite \( M \). Hence, (10), the second half of Theorem 11 is proved. The proof of the first part is completely analogous.

A.3. **Proof of Theorem 11 using alternative principles.** We prove Theorem 11 in \( T_1 + \text{(STP)} + \text{(FFF)} + \text{(PWC)} \).

Proof. In the proof of Theorem 11 in the previous section, every instance of nonstandard continuity can be replaced by combining the fan and weak continuity functional.

For instance, to obtain that (19) is standard, apply the weak continuity functional to \( Y \) and \( s \ast \gamma \). Similarly, the sequence in (21) has standard part, say \( \gamma \) by
The weak continuity functional and (STP). Applying the weak continuity functional again, (21) is also a standard number.

The second part of the proof is similar: One uses the weak continuity functional and (STP) to obtain a standard upper bound, and the fan functional takes the latter as input and provides the required continuity. □

Appendix B. Proof of Theorem 18

Proof. The forward implication follows from Corollary 12. Indeed, it is clear that this corollary generalises to sequences $s^0$ having a standard part. Now assume (GHS) and consider

$$ (\forall^\ast Y^2, \alpha^1)(\forall M \in \Omega)(\forall s^0)[ s^0 \leq M \rightarrow H(Y, s, M) = \Gamma(Y, s)], $$

which holds by (GHS). Indeed, if the antecedent in the square brackets of (22) holds, then $s^0$ is either standard or has a standard part. Then (22) yields:

$$ (\forall^\ast Y^2, \alpha^1)(\exists K)(\forall M \geq K)(\forall s^0)[ s^0 \leq K \rightarrow H(Y, s, M) = \Gamma(Y, s)], $$

as any infinite $K$ will do. By minimisation, there is a least such $K$ (for standard $Y, \alpha$) and this number must be standard by (22) and (23). Hence, we obtain

$$ (\forall^\ast Y^2, \alpha^1)(\exists^\ast K)(\forall M \geq K)(\forall s^0)[ s^0 \leq K \rightarrow H(Y, s, M) = \Gamma(Y, s)], $$

where the innermost universal formula is internal. Applying HAC$_{int}$ to the previous formula yields a standard functional $\Xi^3$ such that for all standard $Y^2, \alpha^1$, we have

$$ (\forall M \geq \Xi(Y, \alpha))(\forall s^0)[ s^0 \leq \Xi(Y, \alpha) \rightarrow H(Y, s, M) = \Gamma(Y, s)]. $$

Now fix standard $\alpha^1$ and any $\beta^1$ such that $\alpha \approx_1 \beta$, and $M \in \Omega$. By the existence of the fan functional, there are $N_0, N_1 \geq M$ such that $Y(\alpha) = Y(\overline{\alpha N_0} * 0 \ldots)$ and $Y(\beta) = Y(\overline{\beta N_1} * 0 \ldots)$. The following equalities now follow from the definitions of $H, N_0,$ and $N_1$, the formula (24), and extensionality. Note that it is essential that $\Xi(Y, \alpha)$ is standard.

$$ Y(\alpha) = Y(\overline{\alpha N_0} * 0 \ldots) = H(Y, \overline{\alpha N_0}, N_0) = \Gamma(Y, \overline{\alpha N_0}) = H(Y, \overline{\alpha N_0}, \Xi(Y, \alpha)) = Y(\overline{\alpha \Xi(Y, \alpha)} * 0 \ldots) = Y(\overline{\beta \Xi(Y, \alpha)} * 0 \ldots) = H(Y, \overline{\beta \Xi(N_1)}, \Xi(Y, \alpha)) = \Gamma(Y, \overline{\beta \Xi(N_1)}) = H(Y, \overline{\beta \Xi(N_1)}, N_1) = Y(\overline{\beta \Xi(N_1)} * 0 \ldots) = Y(\beta). $$

Hence, we obtain (NPC) and the first equivalence in (11) is done. For the final equivalence in the latter, (NPC) is equivalent to (MPC), which implies (PWC). □

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