Maker–Breaker domination number for $P_2 \Box P_n$

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Abstract

We study the Maker–Breaker domination game played by Dominator and Staller on the vertex set of a given graph. Dominator wins when the vertices he has claimed form a dominating set of the graph. Staller wins if she makes it impossible for Dominator to win, or equivalently, she is able to claim some vertex and all its neighbours. Maker–Breaker domination number $\gamma_{MB}(G)$ ($\gamma'_{MB}(G)$) of a graph $G$ is defined to be the minimum number of moves for Dominator to guarantee his winning when he plays first (second). We investigate these two invariants for the Cartesian product of any two graphs. We obtain upper bounds for the Maker–Breaker domination number of the Cartesian product of two arbitrary graphs. Also, we give upper bounds for the Maker–Breaker domination number of the Cartesian product of the complete graph with two vertices and an arbitrary graph. Most importantly, we prove that $\gamma_{MB}(P_2 \Box P_n) = n - 2$ for $n \geq 13$ and $\gamma'_{MB}(P_2 \Box P_n) = n$ for $n \geq 1$. Also, for the unions of $P_2 \Box P_n$s, we show that $\gamma'_{MB}(\bigcup_{i=1}^k (P_2 \Box P_n)_i) = k \cdot n$ ($n \geq 1$), and that $k \cdot n \geq \gamma_{MB}(\bigcup_{i=1}^k (P_2 \Box P_n)_i) \geq k \cdot n - 2$ ($n \geq 13$).

Keywords: Positional game, Maker–Breaker domination game, Maker–Breaker domination number, domination number, winning strategy on grids.

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1 Introduction

1.1 Background

In this paper we study the Maker–Breaker domination game, first introduced in the literature by Duchêne, Gledel, Parreau and Renault in [5]. This game combines two following

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research directions. In the original domination game, introduced by Brešar, Klavžar, and Rall in [2], two players, Dominator and Staller, alternately take a turn in claiming vertices from the finite graph $G$, which were not yet chosen in the course of the game. Dominator has a goal to dominate the graph in as few moves as possible while Staller tries to prolong the game as much as possible.

The Maker–Breaker games, introduced by Erdős and Selfridge in [6], are played on a finite hypergraph $(X, \mathcal{F})$ with the vertex set $X$ and a set $\mathcal{F} \subseteq 2^X$ of hyperedges. The set $X$ is called the board of the game, and $\mathcal{F}$ the family of winning sets. Two players, Maker and Breaker take turns in claiming previously unclaimed elements of $X$. Maker wins the game if, by the end of the game, he has claimed all elements of some $F \in \mathcal{F}$. Otherwise, Breaker wins. For a deeper and more comprehensive analysis of Maker–Breaker games, see the book of Beck [1], and the recent monograph of Hefetz, Krivelevich, Stojaković and Szabó [9].

The Maker–Breaker domination game (MBD for short) is played on graph $G = (V,E)$ by two players Dominator and Staller. The aim of Dominator is to build a dominating set of the graph, which is a set $T$ such that every vertex not in $T$ has a neighbour in $T$. The aim of Staller is to claim a vertex from the graph $G$ and all its neighbours, so that the Dominator cannot dominate all of the vertices. As introduced in [5], Staller is the Maker and Dominator is the Breaker. Then, the board of the game is the vertex set of a given graph $G$, and the winning sets are closed neighbourhoods of every vertex of graph $G$.

When it is not hard to determine the identity of the winner in some Maker–Breaker game, then a more interesting question to ask is how fast the player with the winning strategy can win. Fast winning strategies for Maker in the Maker–Breaker games have received a lot of attention in recent years (see e.g. [3, 4, 8]). Specifically, for the Maker–Breaker domination game the smallest number of moves for Dominator is studied in [7], where Gledel, Iršič, and Klavžar introduced the Maker–Breaker domination number $\gamma_{MB}(G)$ of a graph $G$, as the minimum number of moves for Dominator to win in the game on $G$ where he is the first player. If Dominator is the second player, then the corresponding invariant is denoted by $\gamma'_{MB}(G)$ in their paper. In [7], the authors proved that $\gamma_{MB}(G) = \gamma(G) = 2$ if and only if $G$ has a vertex which lies in at least two $\gamma$-sets of $G$, where $\gamma(G)$ is the domination number of $G$ — the order of a smallest dominating set of $G$ — and $\gamma$-set is a dominating set of size $\gamma(G)$.

1.2 Main results

In the paper [7], the authors proposed finding the minimum number of moves for Dominator in the MBD game on the Cartesian product of two graphs. Motivated by a given problem, we give the upper bounds for Maker–Breaker domination number for the Cartesian product of $K_2$ and a general graph, and that of two general graphs as well, in Section 2.

**Theorem 1.1.** Let $G$ be a graph on $n$ vertices. Then Dominator can win the game on $G \square K_2$ in at most $n$ moves. If Dominator has a winning strategy as the first and as the second player in the game on $G$, then $\gamma_{MB}(G \square K_2) \leq \min\{\gamma_{MB}(G) + \gamma'_{MB}(G), n\}$ and
\[ \gamma'_M(B \sqcap K_2) \leq \min\{2 \cdot \gamma'_M(G), n\} \].

**Theorem 1.2.** Let \(G\) and \(H\) be two arbitrary graphs on \(n\) and \(m\) vertices, respectively. Suppose that Dominator has a winning strategy in MBD game on at least one of these two graphs as the first and as the second player. Then

\[ \gamma_M(B \sqcap H) \leq \min\{\gamma_M(G) + (m - 1) \cdot \gamma'_M(G), \gamma_M(H) + (n - 1) \cdot \gamma'_M(H)\} \]

and

\[ \gamma'_M(B \sqcap H) \leq \min\{m \cdot \gamma_M(G), n \cdot \gamma'_M(H)\} \].

Most importantly, we focus on determining how fast can Dominator win on the graphs \(P_2 \sqcap P_n\), for \(n \geq 1\). We prove the following two theorems in Section 2.1.

**Theorem 1.3.** \(\gamma'_M(P_2 \sqcap P_n) = n\) for \(n \geq 1\), and Dominator cannot skip any moves, otherwise he cannot win.

**Theorem 1.4.** \(\gamma_M(P_2 \sqcap P_n) = n - 2\), for \(n \geq 13\).

From the above two results, we get two results on the disjoint union of \(P_2 \sqcap P_n\)-s, as follows. We denote by \(\sqcup\) the disjoint union, and by \(\sqcup_{i=1}^{k} (G)\), \(i\)-many copies of disjoint union of graph \(G\).

**Theorem 1.5.** \(\gamma'_M(\sqcup_{i=1}^{k} (P_2 \sqcap P_n)) = k \cdot n\), \(n \geq 1\).

**Theorem 1.6.** \(k \cdot n - 2 \leq \gamma_M(\sqcup_{i=1}^{k} (P_2 \sqcap P_n)) \leq k \cdot n\), \(n \geq 13\).

### 1.3 Preliminaries

Before we continue with the main part of the paper, let us come to some preliminaries. Assume that the MBD game is in progress. We denote by \(D_1, D_2, \ldots\) the sequence of vertices chosen by Dominator and by \(S_1, S_2, \ldots\) the sequence of vertices chosen by Staller. At any given moment of this game, we denote the set of vertices claimed by Dominator by \(D\) and the set of vertices claimed by Staller by \(S\). As in [7], we say that the game is the \(D\)-game if Dominator is the first to play, i.e. one round consists of a move by Dominator followed by a move of Staller. In the \(S\)-game, one round consists of a move by Staller followed by a move of Dominator. We say that the vertex \(v\) is isolated by Staller if \(v\) and all its neighbours are claimed by Staller. For a given graph \(G\), by \(V(G)\) and \(E(G)\) we denote its vertex set and edge set, respectively.

### 2 MBD game on the Cartesian product of two graphs

In this section, we prove the two theorems regarding to Cartesians of general graphs.
Proof of Theorem 1.1. Let $V(G) = \{v_1, v_2, ..., v_n\}$. Let $H$ be a copy of the graph $G$ and let $V(H) = \{v'_1, v'_2, ..., v'_n\}$, where $v'_i$ is the copy of $v_i$ for $1 \leq i \leq n$. Then $V(G \square K_2) = V(G) \cup V(H) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}$ and $E(G \square K_2) = E(G) \cup E(H) \cup \{\{v_1, v'_1\}, \{v_2, v'_2\}, ..., \{v_n, v'_n\}\}$.

Dominator can always use the pairing strategy to guarantee his winning. That is to say, when Staller claims $v_i$ (or $v'_i$), for some $i \in \{1, 2, ..., n\}$, Dominator responds by claiming vertex $v'_i$ (or $v_i$). So, Dominator wins in at most $n$ moves. To see that bound is tight consider $G$ as the disjoint union of $K_1$s.

Suppose that Dominator can win in the game on the graph $G$ as the first and as the second player. Assume that Dominator starts the game. Note that $\gamma_{MB}(G) = \gamma_{MB}(H)$ and $\gamma'_{MB}(G) = \gamma'_{MB}(H)$. By $S_D$ and $S'_D$ denote Dominator’s winning strategy on $G$ (and also on $H$) in the $D$-game and the $S$-game, respectively.

When $\gamma_{MB}(G) + \gamma'_{MB}(G) \geq n$, Dominator can use the pairing strategy. So, suppose that $\gamma_{MB}(G) + \gamma'_{MB}(G) < n$. For his first move Dominator chooses a vertex from $V(G)$ according to his winning strategy $S_D$. In this way he starts the $D$-game on $G$. In every other round $r \geq 2$, Dominator looks at the $(r - 1)^{th}$ move of Staller. If Staller claims a vertex from $V(G)$, Dominator responds by claiming a vertex from $V(G)$ according to $S_D$ and if Staller claims a vertex from $V(H)$, Dominator also claims a vertex from $V(H)$ according to $S'_D$. In this way, after $\gamma_{MB}(G) + \gamma'_{MB}(G)$ moves, Dominator will be able to dominate all vertices in $V(G) \cup V(H)$. Dominator can win in at most $\gamma_{MB}(G) + \gamma'_{MB}(G)$ moves.

Next, assume that Staller starts the game on $G \square K_2$. If $2 \cdot \gamma'_{MB}(G) \geq n$, Dominator can use the pairing strategy. Suppose that $2 \cdot \gamma'_{MB}(G) < n$. If Staller claims a vertex in $G$, Dominator react according to $S'_D$ on $G$; if Staller claims a vertex in $H$, Dominator react according to $S'_D$ on $H$. So, to win in the game on $G \square K_2$, he needs to play at most $2 \cdot \gamma'_{MB}(G)$ moves. □

Remark 2.1. We observe that the domination number of the $r \times l$ rook’s graph $K_r \square K_l$ is $\min(r, l)$.

Proof of Theorem 1.2. Consider, first, the $D$-game on $G \square H$. Suppose that Dominator has a winning strategy as the second player on $G$. By $G^{(1)}, G^{(2)}, ..., G^{(m)}$ denote copies of the graph $G$. By $S_D$ and $S'_D$ denote Dominator’s winning strategy on $G$ in the $D$-game and the $S$-game, respectively.

Dominator will play his first move on one copy of $G$ according to his winning strategy $S_D$. In every other round $i \geq 2$, he looks on the $(i - 1)^{th}$ move of Staller. If Staller in his $(i - 1)^{th}$ move claims a vertex from $V(G^{(j)})$, Dominator responds by claiming a vertex from the same set $V(G^{(j)})$ according to the corresponding winning strategy $S_D$ or $S'_D$ on graph $G^{(j)}$. Since Staller can be the first player on at most $m - 1$ copies of the graph $G$, we know that Dominator can win within $\gamma_{MB}(G) + (m - 1) \cdot \gamma'_{MB}(G)$ many moves.

When Dominator has a winning strategy on $H$ both as the first and as the second player, we obtain analogously that he can win within $\gamma_{MB}(G) + (m - 1) \cdot \gamma'_{MB}(G)$ many moves. Combining the above analysis, we conclude that the statement holds. □
2.1 MBD game on $P_2 \Box P_n$

In order to prove our main results on MBD games on grids, namely Theorem 1.3 and Theorem 1.4, we need to introduce several “MBD graphs” and also two types of traps that Staller can make so as to win the game, as preparations.

2.2 MBD graphs

An MBD graph is a pair $(G, I)$, where $G = (V, E)$ is a graph and function $I : V \rightarrow \{s, d, n\} \times \{0, 1\}$ assigns to each vertex a pair from $\{s, d, n\} \times \{0, 1\}$, describing the current situation of the vertex.

To say it more intuitively, $s$ means the vertex is already claimed by Staller, $d$ means the vertex is already claimed by Dominator and $n$ refers to “null”, which mean that the vertex is still free — not yet claimed by any player in the current game. And 1 means that the vertex is not yet dominated, i.e., none of its neighbors has been assigned $d$ yet. In the sequel, we define several MBD subgraphs of the cartesian graph $P_2 \Box P_m$. They will be helpful in our later proofs. Let $V_m = \{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ and $E_m = \{\{u_i, u_{i+1}\} \mid i = 1, 2, \ldots, m-1\} \cup \{\{v_i, v_{i+1}\} : i = 1, 2, \ldots, m-1\} \cup \{\{u_i, v_i\} : i = 1, 2, \ldots, m\}$. It is easy to check that $P_2 \Box P_m = (V_m, E_m)$.

Suppose that Maker–Breaker domination game on $P_2 \Box P_m$ is in progress, where $m \geq 5$ is an integer.

1. By $X_m$ ($m \geq 1$) denote the MBD graph $(G, I)$ with $G = P_2 \Box P_m$, $I$ assigns to vertex $u_1$ with $(n, 1)$ and to all other vertices $(n, 0)$. That is to say, vertices $u_1$ is already dominated in the ongoing game. Therefore, in the remaining game, Dominator does not need to consider dominating $u_1$. (see Figure 1(a))

2. By $Y_m$ ($m \geq 3$) denote the MBD graph $(G, I)$ with $G = P_2 \Box P_m$, $I$ assigns to vertex $u_1$ value $(n, 1)$, to vertex $v_2$ value $(s, 0)$, to vertices $u_m, v_m$ the value $(n, 1)$ and to all other vertices $(n, 0)$. When consider the $D$-game on $Y_m$, we set $S_0 = v_2$ — consider the MBD game played on this graph, but Staller has claimed the vertex $v_2$ and vertices $u_m$ and $v_m$ are already dominated (but not claimed by any players) before the game starts. (see Figure 1(b))

3. By $Z_m$ ($m \geq 1$) denote the MBD graph $(G, I)$ with $G = P_2 \Box P_m$, $I$ assigns to vertices $u_1$ and $v_1$ the value $(n, 1)$, to all other vertices value $(n, 0)$. That is to say, vertices $u_1$ and $v_1$ are already dominated in the ongoing game. Therefore, in the remaining game, Dominator does not need to consider dominating $u_1$ or $v_1$. (see Figure 1(c))

4. By $W_m$ ($m \geq 1$) denote the MBD graph $(G, I)$, where $G = (V, E)$ and $V = V_m \cup \{v_0\}$, $E = E_m \cup \{v_0, v_1\}$; $I(v_0) = I(u_1) = (n, 1)$ and $I$ assigns to all other vertices value $(n, 0)$. That is to say, $v_0$ and $u_1$ are already dominated in the current game. (see Figure 1(d))
5. By $\rho_m (m \geq 0)$ denote the MBD graph $(G, I)$ with $G = P_2 \Box P_m$ and $I(u_1) = (n, 1)$, $I(v_2) = (s, 0)$; $I$ assigns to all other vertices value $(n, 0)$. When consider the $D$-game on $\rho_m$, we set $S_0 = v_2$ — consider the MBD game played on this graph, but Staller has claimed the vertex $v_2$ and $u_1$ is already dominated by Dominator before the game starts. Note that here we use $S_0$ to denote the “move” of Staller before the game starts. (Figure 1(e)).

![Subfigures](a) $X_m$ (b) $Y_m$ (c) $Z_m$ (d) $W_m$ (e) $\rho_m$. Two incident edges being dotted indicates that the vertex is already dominated by Dominator but not claimed by any players; the cross indicates that the vertex is claimed by the Staller.

We define the domination number of an MBD graph naturally as follows. Dominator intends to dominate all undominated vertices, namely those vertices with the second coordinate of $I(v)$ being 0; both players can only claim the unclaimed vertices, namely vertices $v$ with the first coordinate of $I(v)$ being $n$. We use the same notations $\gamma_{MB}$ and $\gamma'_{MB}$ for the minimum number of moves for Dominator to guarantee his winning in $D$-game and $S$-game on the MBD graph, respectively. Actually a normal graph can be viewed as a special case of an MBD graph, namely with all vertices assigned by $(n, 0)$ under the function $I$.

Now we define two types of traps that Staller can create in the MBD game on $P_2 \Box P_n$ for $n \geq 3$, so as to prevent the winning of Dominator. These two strategies of Staller will be used a lot later in the proofs.
Figure 2: Illustrations of the two type of traps. (a) Sequence of triangle traps $v_3 - v_7$; (b) Sequence of line traps $u_3 - u_7$.

**Trap 1 - triangle trap.** We say that Staller created a **triangle trap** if after her move Dominator is forced to claim the vertex $v_i$ (or $u_i$) — so as to dominate $v_i$ (or $u_i$). This is because all neighboring vertices of $v_i$ (or $u_i$) are already claimed by Staller and she can isolate $v_i$ (or $u_i$) by claiming it in her next move, if Dominator does not claim it. We say that Staller created a **sequence of triangle traps** $v_i - v_j$ (or $v_i - u_j$), $i < j$, if Dominator is consecutively forced to claim vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, ..., v_j$ (or $v_i, u_{i+1}, u_{i+2}, u_{i+3}, ..., u_j$) because of the triangle traps one after another set up by Staller.

The sequence of triangle traps $v_3 - v_7$ is illustrated on Figure 2(a). The first trap shows up when Staller has claimed $v_2, v_3$ and $v_4$, which forces Dominator to claim $v_3$; immediately after, Staller claims $u_5$ to force Dominator to claim $u_4$, which creates the second trap. Notice that after the Dominator claims $v_8$, Dominator has to claim $v_7$; but Staller can then claim $u_8$, which will isolate $u_8$ and further leads to the winning of the game. This strategy will be applied a lot later on in our proofs. The sequences of triangle traps $u_i - v_j$ and $u_i - u_j$ are defined analogously.

**Trap 2 - line trap.** We say that Staller created a **line trap** if after her move Dominator is forced to claim the vertex $v_i$ (or $u_i$) — so as to dominate $u_i$ (or $v_i$). This is because all other neighbors of $u_i$ (or $v_i$) and $u_i$ (or $v_i$) itself are already claimed by Staller. Staller can isolate $u_i$ (or $v_i$) by claiming $v_i$ (or $u_i$) in her next move. We say that Staller creates a **sequence of line traps** $v_i - v_j$ (or $u_i - u_j$), $i < j$, if Dominator is consecutively forced to claim the vertices $v_i, v_{i+1}, v_{i+2}, v_{i+3}, ..., v_j$ (or $u_i, u_{i+1}, u_{i+2}, u_{i+3}, ..., u_j$) because of the line traps one after another set up by Staller.

The sequence of line traps $u_3 - u_7$ is illustrated on Figure 2(b). The first trap shows up when Staller has claimed $v_2, v_3$ and $v_4$, which forces Dominator to claim $v_3$ (in order to dominate $v_3$); directly after, Staller claims $v_5$, which forces Dominator to claim $u_4$, which creates the second trap. Notice that after Dominator claims $u_7$, Staller can then claim $u_8$, which makes it impossible for Dominator to dominate $v_8$ anymore, then win the game. This strategy will be employed a lot later on in our proofs.

With this we finished introducing the basic definitions which we will need later on in the proofs of our main results.
2.3 Domination numbers of MBD graphs

In this section, we introduce some results on the domination numbers on MBD graphs, which then will lead to the completion of the proofs of Theorem 1.3 and Theorem 1.4.

**Proposition 2.2.** \( \gamma_{MB}(\rho_m) = m \) for \( m \geq 0 \); when \( m \geq 2 \), Dominator cannot skip any move, otherwise he would lose the game.

**Proof.** It is not hard to verify that \( \gamma_{MB}(\rho_0) = 0 \), \( \gamma_{MB}(\rho_1) = 1 \), \( \gamma_{MB}(\rho_2) = 2 \) and Dominator cannot skip any move in a D-game on graph \( \rho_2 \). On \( \rho_1 \), the Dominator cannot skip any move if the Staller starts first (on any vertex).

Assume that \( \gamma_{MB}(\rho_k) \leq k \) holds when \( 2 \leq k \leq m - 1 \), then when \( k = m \), let Dominator adopt the strategy for \( \rho_{m-1} \) on the (left) subgraph \( \rho_{m-1} \), let him adopt the pairing strategy for the \( K_2 \) (complete graph with two vertices) on the right. We then obtain that \( \gamma_{MB}(\rho_m) \leq \gamma_{MB}(\rho_{m-1}) + 1 \leq m \). Hence by induction we obtain that \( \gamma_{MB}(\rho_m) \leq m \) for any \( m \geq 0 \). We only need to show that \( \gamma_{MB}(\rho_m) \geq m \) and Dominator cannot skip any move in the game. Assume \( \gamma_{MB}(\rho_k) \geq k \) and Dominator cannot skip any move in D-game on graph \( \rho_k \) for any \( k < m \). Now we consider the situation when \( k = m \); notice that here we can assume \( m \geq 3 \).

If Dominator skips the first move, denote it as \( D_1 = \emptyset \). We propose the following strategy for Staller: Let \( S_1 = u_1 \), which forces \( D_2 \) to be \( u_1 \). Then the Staller starts to make line trap from \( v_3 \) to \( v_n \), which forces Dominator to respond on \( u_2 \) until \( u_{m-1} \). Then let Staller claim \( u_m \), the Dominator cannot dominate \( v_m \) anymore. Hence Dominator cannot skip first move.

So now we consider all possibilities for \( D_1 \). And we will propose correspondingly the Staller strategy.

- **Case 1:** \( D_1 = u_i \), \( i \geq 2 \). Let \( S_i = v_1 \) which forces \( D_1 = u_1 \). Then if \( i \neq 2 \), let Staller make line trap from \( v_3 \) until \( v_i \), which forces Dominator to claim from \( u_2 \) until \( u_{i-1} \). Then let Staller claim \( u_{i+2} \), on the right we obtain graph \( \rho_{m-i} \). If \( m - i = 1 \), just let the Staller claim any remaining vertex on the rightmost of the graph; this trick for \( \rho_1 \) on the right we will apply more times later, we will not specifically mention. By induction hypothesis we know that Dominator cannot skip any move and he needs \( m \) moves to win.

- **Case 2:** \( D_1 = v_i \), \( i \geq 3 \). Suppose \( i > 5 \). Let \( S_i = u_2 \), which forces \( D_2 = v_1 \) or \( u_1 \). But no matter which choice \( D_1 \) take, Staller will claim \( v_3 \), \( u_3 \) in the next two rounds so that either Dominator cannot dominate \( v_4 \) or he cannot dominate \( u_4 \). Hence \( i = 3 \) or \( 4 \).

  Case 2.1: \( i = 3 \). Staller will then claim \( u_1 \) and \( u_3 \), which forces Dominator to respond on \( v_1 \) and \( u_2 \), then Staller will claim \( v_5 \), on the right we can use induction hypothesis.

  Case 2.2: \( i = 4 \). Let Staller claim \( u_2 \), then Dominator needs to claim \( u_1 \) or \( v_1 \). When \( D_2 = u_1 \), then let Staller claim \( v_3 \) which forces \( D_3 = v_1 \). then let \( S_3 = u_4 \),
which forces $D_4 = u_3$. Then $S_4 = u_6$, we can use induction hypothesis for the graph on the right.

- **Case 3:** $D_1 = u_1$, let $S_1 = v_3$. In this case if $D_2 = \emptyset$, Staller can make a square trap with claiming $u_2, v_2, u_3, v_3$, then Dominator cannot win. So he cannot skip $D_2$. For the same reason, we know that $D_2 \in \{v_1, u_2, u_3, v_4, v_1\}$. However if $D_2 \not\in \{v_1, u_2\}$, then the Staller can make line traps by claiming $v_4$ until $v_n$, then at the last step claiming $u_n$, which makes Dominator lose the game. Hence $D_2 \in \{u_3, u_4, v_1\}$.

  Case 3.1: $D_2 = u_3$. Let $S_2 = u_2$, forcing $D_3 = v_1$. Then $S_3 = v_5$, we can use induction for the graph on the right.

  Case 3.2: $D_2 = u_4$. Let $S_2 = u_2$, forcing $D_3 = v_1$. Let $S_3 = v_4$, forcing $D_4 = u_3$. Then $S_4 = v_6$, we can use induction for the graph on the right.

  Case 3.3: $D_2 = v_4$. Let $S_2 = u_2$, forcing $D_3 = v_1$. Let $S_3 = u_4$, forcing $D_4 = u_3$. Then $S_4 = u_6$, we can use induction for the graph on the right.

- **Case 4:** $D_1 = v_1$, let $S_1 = u_3$. In this case if $D_2 = \emptyset$, Staller can make a claim $u_2, v_2, u_3, v_3$ in the next rounds, then Dominator cannot win. So he cannot skip $D_2$. For the same reason, we know that $D_2 \in \{u_1, u_2, v_3, u_4, v_1\}$. However if $D_2 \not\in \{u_1, u_2\}$, then the Staller can make triangle traps by claiming $v_4$ until $v_n$, then at the last step claiming $u_m/v_m$ (depending on $m$ is odd or even), then at the last step claiming $u_m$, which makes Dominator lose the game. Hence $D_2 \in \{v_3, u_4, v_1\}$.

  Case 4.1: $D_2 = v_3$. Let $S_2 = u_2$, forcing $D_3 = u_1$. Then $S_3 = u_5$, we can use induction for the graph on the right.

  Case 4.2: $D_2 = u_4$. Let $S_2 = u_2$, forcing $D_3 = u_1$. Let $S_3 = v_4$, forcing $D_4 = v_3$. Then $S_4 = v_6$, we can use induction for the graph on the right.

  Case 4.3: $D_2 = v_4$. Let $S_2 = u_2$, forcing $D_3 = u_1$. Let $S_3 = u_4$, forcing $D_4 = v_3$. Then $S_4 = u_6$, we can use induction for the graph on the right.

Notice that in some case it may happen that the graph is not big enough for the strategy of Staller, but it is not hard to verify that in those cases we simply have a $\rho_0$ or $\rho_1$ on the right, still we can use induction hypothesis.

Hence notice that in all above cases Dominator in total needs $m$ moves to win and cannot skip any move during the game. By induction, we conclude that for any $m \geq 2$, in the D-game on graph $\rho_m$, Dominator needs at least $m$ steps to win and he cannot skip any move in the game. \(\square\)

**Remark 2.3.** Note that the D-game on graph $\rho_m$ can be considered as the S-game on $X_m$ where $S_1 = v_2$.

**Proposition 2.4.** $\gamma_{MB}(Y_m) = m - 1$, $m \geq 3$.

**Proof.** Vertex $v_2$ is pre-claimed by Staller: we denote this by $S_0 = v_2$. The proof and case analysis can be done analogously with Proposition \(2.2\). \(\square\)
Proposition 2.5. \( \gamma'_{MB}(Z_m) = m - 1, m \geq 1 \).

Proof. It is not hard to check that \( \gamma'_{MB}(Z_m) = m - 1 \) holds for \( m = 1, 2, 3 \). Assume that \( \gamma'_{MB}(Z_k) \leq k - 1 \) for all \( 3 \leq k < m \). When \( k = m \), we view graph \( Z_m \) as two parts --- \( Z_{m-1} \) and \( u_m, v_m \) with the edge connecting them; of course the two parts are connected via two edges in \( Z_m \), namely \( \{u_{m-1}, u_m\} \) and \( \{v_{m-1}, v_m\} \) --- but we do not consider them for now, since the extra edges will just make it easier for Dominator to win. We propose the following strategy for Dominator. If Staller play on the \( Z_{m-1} \) part of the graph, let Dominator respond on \( Z_{m-1} \) with his strategy on \( Z_{m-1} \); if Staller play on the other part, Dominator will use the pairing strategy, namely claim the other vertex in this part. In this way, by induction hypothesis, Dominator can win within \( \gamma'_{MB}(Z_{m-1}) + 1 \leq (m - 2) + 1 = m - 1 \) steps. Hence, \( \gamma'_{MB}(Z_m) \leq m - 1, m \geq 1 \).

For the other direction of the proof, we need to show that \( \gamma'_{MB}(Z_m) \geq m - 1, m \geq 4 \). We prove it by proposing a strategy for Staller. Let \( S_1 = u_m \), then we have \( D_1 \in \{u_{m-1}, v_{m-1}, v_m\} \); otherwise \( S_2 = v_m \), and at least one of \( v_m \) and \( u_m \) can get isolated after \( S_3 \) (Staller’s third step) --- after \( D_2 \) (Dominator’s second step) at least one of \( u_{m-1} \) and \( v_{m-1} \) will still be free, so Staller can choose this free vertex at her third step. Now we make case distinctions for this three choices of \( D_1 \).

1. \( D_1 = v_{m-1} \). Let \( S_2 = u_{m-1} \): this forces \( D_2 = v_m \) (in order to dominate \( u_m \)). Let \( S_3 = v_{m-3} \). The remaining MBD graph in this game is an “upside-down” \( Y_{m-2} \) with vertex set \( V(Y_{m-2}) = \{u_1, u_2, ..., u_{m-2}, v_1, v_2, ..., v_{m-2}\} \). It is Dominator’s turn now. Therefore he needs \( \gamma_{MB}(Y_{m-2}) = m - 3 \) (by Proposition 2.4) more moves to win. So he needs in total \( m - 1 \) steps to win in this case.

2. \( D_1 = u_{m-1} \). Let \( S_2 = v_{m-1} \): this forces \( D_2 = v_m \) (in order to dominate \( v_m \)). Let \( S_3 = v_{m-3} \). The remaining part of this case is the same as the last case. Dominator needs at total \( m - 1 \) steps to win.

3. \( D_1 = v_m \). Let \( S_2 = u_{m-2} \): the remaining MBD graph in this game is an “upside-down” \( Y_{m-1} \). Dominator needs \( \gamma_{MB}(Y_{m-1}) = m - 2 \) many moves to win on this part. He needs in total \( (m - 2) + 1 = m - 1 \) many moves to win in this case.

Hence, Staller has a strategy such that Dominator needs at least \( m - 1 \) steps in order to win in an S-game on the MBD graph \( Z_m \). We obtain that \( \gamma'_{MB}(Z_m) \geq m - 1, m \geq 4 \).

Proposition 2.6. \( \gamma'_{MB}(W_m) = m \) when \( 1 \leq m \leq 3 \) and \( \gamma'_{MB}(W_m) = m - 1 \) when \( m \geq 4 \).

Proof. One can check that \( \gamma'_{MB}(W_m) = m, 1 \leq m \leq 3 \). Let \( m \geq 4 \). Since \( W_m \) has one more undominated vertex than \( Z_m \) (namely \( v_1 \)), Dominator needs to play at least as many moves on \( W_m \) as on \( Z_m \). Therefore we have \( \gamma'_{MB}(W_m) \geq \gamma'_{MB}(Z_m) = m - 1 \), which shows the lower bound. For the upper bound, we start our consideration from \( m = 4 \). When \( m = 4 \), we propose strategies for Dominator accordingly to the steps of Staller.

- Case 1: \( S_1 = v_2 \). Let \( D_1 = u_3 \).
If \( S_2 = v_1 \), then let \( D_2 = v_3 \). Dominator only needs one step to dominate \( v_1 \) before his winning; so as to do this, he only needs to claim \( v_0 \) or \( u_1 \) in the next move. So he needs in total 3 moves to win.

If \( S_2 \neq v_1 \), then let \( D_2 = v_1 \). Then Dominator just needs one more step to dominate \( v_4 \). In total he needs 3 steps to win.

- **Case 2: \( S_1 \neq v_2 \).**

  If \( S_1 = u_4 \) (or \( v_4 \)). Let \( D_1 = u_3 \). If \( S_2 = v_3 \), then let \( D_2 = v_1 \) (or \( u_4 \)) and let \( D_3 \in \{v_1, v_2\} \). If \( S_2 = v_4 \) (or \( u_4 \)), let \( D_2 = v_3 \) and \( D_3 \in \{v_1, v_2\} \). So Dominator needs in total 3 steps to win.

  If \( S_1 \notin \{u_4, v_4\} \). Let \( D_1 = v_2 \). Dominator needs at most two more moves to dominate the remaining vertices.

So far, we obtain that \( \gamma'_MB(W_4) \leq 3 \). Suppose that \( \gamma'_MB(W_{m-1}) \leq m - 2 \) for \( m \geq 5 \). Then consider the MBD graph \( W_m \) \((m \geq 5)\) as two parts (of course plus edges connecting the two parts): one is the rightmost \( K_2 \), namely \( u_m, v_m \) and the edge connecting them; the other is on the left an MBD subgraph \( W_{m-1} \). We let Dominator react on the left \( W_{m-1} \) according to his strategy on \( W_{m-1} \) whenever Staller claims a vertex on this subgraph; otherwise we let Dominator claim the remaining vertex among \( u_m \) and \( v_m \). By induction hypothesis, Dominator needs no more than \( (m - 2) + 1 = m - 1 \) moves to win. Hence \( \gamma'_MB(W_m) \leq m - 1 \). \( \square \)

The following result is the essence for proving Theorem 1.4. The case analyses in the proof can be lengthy. We provide a strategy tree (see Figure 3) showing the idea of the case studies, which, although not covering all cases, but can at least express the idea behind. We leave it as an exercise for readers, to complete the whole strategy tree taking into consideration of all cases.

**Theorem 2.7.** \( \gamma_{MB}(X_1) = 1 \), \( \gamma_{MB}(X_m) = m - 1 \) when \( 2 \leq m \leq 5 \) and \( \gamma_{MB}(X_m) = m - 2 \) for \( m \geq 6 \).

**Proof.** For \( m \in \{1, 2, 3\} \) the situation is not hard to directly see. For \( m = 4 \) and \( m = 5 \) simple cases analysis gives the result. Let \( m \geq 6 \). First, we consider the \( D \)-game on \( X_6 \). Let \( D_1 = v_2 \), then we see an MBD subgraph \( W_4 \) on the rightmost which already contains all the undominated vertices. By Proposition 2.6, we know that \( \gamma'_MB(W_4) = 3 \). Hence, \( \gamma_{MB}(X_6) = 4 \).

Suppose that \( \gamma_{MB}(X_k) \leq k - 2 \) for \( k < m \). When \( k = m \), consider the graph as two parts (plus edges connecting the two parts): one is the \( K_2 \) on the rightmost, the other is the MBD subgraph \( X_{m-1} \) on the left. Let Dominator adopt the strategy for \( X_{m-1} \) on the left subgraph whenever Staller plays on that subgraph and let him claim the remaining vertex among \( u_m \) and \( v_m \) when Staller claims one of them. In this way, by induction hypothesis we know that Dominator needs at most \( (m - 3) + 1 = m - 2 \) steps to win. Hence \( \gamma_{MB}(X_m) \leq m - 2 \) for \( m \geq 6 \).
Figure 3: This is a strategy tree showing the strategies for Staller under the situation when $D_1 = u_i (i \geq 3), D_2 = u_j (j \geq 1)$. At the end of each branch, we explain at least how many steps Dominator would need in order to win the game. We see that in all these ten cases, Dominator needs at least $m - 2$ steps to win the game.
We prove the lower bound by induction and proposing strategies for Staller. Suppose that $\gamma_{MB}(X_k) \geq k - 2$ for $4 \leq k < m$, $m \geq 6$. Now consider the case when $k = m$, $m \geq 6$. We want to show that $\gamma_{MB}(X_m) \geq m - 2$, $m \geq 6$. According to the first step of Dominator, there are three cases: $D_1 \in \{u_i, v_1, u_2, v_2\}$, $D_1 = u_i$ ($i \geq 3$), or $D_1 = v_i$ ($i \geq 3$).

**Case 1:** $D_1 \in \{u_1, v_1, u_2, v_2\}$.

If $D_1 \in \{u_1, v_1\}$, then we see that the remaining game is the S-game on the MBD subgraph $W_{m-1}$. By Proposition 2.6, $\gamma_{MB}(W_m) = m - 1$ when $m \geq 4$. Hence Dominator needs $m - 2$ more steps to win; he needs in total $m - 1$ steps to win.

If $D_1 = v_2$, then we see that on the right is an MBD subgraph $W_{m-2}$. By Proposition 2.6, Dominator needs in total $(m - 3) + 1 = m - 2$ moves to win.

If $D_1 = u_2$, then on the left is the MBD subgraph $W_{m-2}$, while on the right Dominator needs at most one more step, so as to dominate $v_1$. Hence Dominator needs at least $(m - 3) + 1 = m - 2$ steps to win.

**Case 2:** $D_1 = u_i$, $i \geq 3$. Let $S_1 = v_2$, then according to the second move of Dominator, we carry out case distinctions.

(2.1) $D_2 = u_1$. Then we further consider two subcases: $i = 3$ or $i \geq 4$.

(2.1.1) If $D_1 = u_3$, then we see that the remaining game is the S-game on the MBD subgraph $W_{m-3}$ on the left of $X_m$; where on the right Dominator needs at most one more step, so as to dominate $v_2$. Hence he needs in total at least $(m - 4) + 2 = m - 2$ steps to win.

(2.1.2) If $D_1 = u_i$, $i \geq 4$, we let $S_2 = v_3$. We need to further distinct cases according to the choice of $D_3$.

(2.1.2.1) $D_3 \in \{v_1, u_2\}$. Let Staller take the strategy of creating a sequence of line traps $u_3 - u_{i-1}$ which ends with $S_{i-1} = v_i$ and forces $D_i = u_{i-1}$. Then, if $m - i \geq 2$, let $S_i = v_{i+2}$. We see that the rightmost is the MBD graph $\rho_{m-i}$. Dominator wins after the D-game on the rightmost $\rho_{m-i}$. By Proposition 2.2, $\gamma_{MB}(\rho_{m-i}) = m - i$. Hence Dominator needs in total $m$ moves to win. If $m - i = 1$, then let $S_i \in \{u_m, v_m\}$, Dominator needs only one more step to win. If $m - i = 0$, the game finished already. In either case, Dominator wins with $m$ steps.

(2.1.2.2) $D_3 = u_j$, $j \geq 3$ or $D_3 = v_j$, $j \geq 4$. Suppose that $\min\{i, j\} \notin \{3, 4\}$. Let $S_3 = u_2$, which forces $D_4 = v_1$. Then let $S_4 = u_3$, Dominator cannot dominate both $u_4$ and $v_4$, hence he cannot win. Therefore, $\min\{i, j\} \in \{3, 4\}$.
(2.1.2.2.a) When $D_3 = u_j$, where $j \geq 3$ and $j < i$, we know that $j \in \{3, 4\}$. When $j = 3$, we have $D_3 = u_3$. Recall that $D_1 = u_i$, $i \geq 4$, $S_1 = v_2$, $D_2 = u_1$, $S_2 = v_3$, $D_3 = u_3$. Now let $S_3 = u_2$, which forces $D_4 = v_1$. We see on the rightmost the MBD graph $X_{m-3}$, but, with $u_i$ being claimed by Dominator already. Now it is Staller’s turn. By induction hypothesis, we know that Dominator needs at least $\gamma_{MB}(X_{m-3}) - 1 \geq m - 6$ more steps to win the game. In total, he needs $(m - 6) + 4 = m - 2$ steps.

(2.1.2.2.b) When $j = 4$, we have $D_3 = u_4$. Let $S_3 = u_2$, which forces $D_4 = v_1$. Then let $S_4 = v_4$, which forces $D_5 = u_3$. Similarly as in the last case, the remaining graph is the MBD graph $X_{m-4}$ on the right, but with $u_i$ being claimed by Dominator already. Now it is Staller’s turn. By induction hypothesis, we know that Dominator needs at least $\gamma_{MB}(X_{m-4}) - 1 \geq m - 7$ more steps to win the game. In total, he needs $(m - 7) + 5 = m - 2$ steps.

(2.1.2.2.c) When $D_3 = u_j$, where $j > i$, we know that $i \in \{3, 4\}$. Since $i \geq 4$, we know that $D_1 = u_i = u_4$. Let Staller take the same strategy as in the last case, we get that Dominator needs at least $m - 2$ steps to win.

(2.1.2.2.d) When $D_3 = u_j$, where $j < i$, since $S_2 = v_3$, we know that $j = 4$, i.e., $D_3 = v_4$. Let $S_3 = u_2$, which forces $D_4 = v_1$. Then let $S_4 = u_4$, which forces $D_5 = u_3$. The remaining graph on the right is the MBD graph $X_{m-4}$, but with $u_i$ claimed by Dominator already, and now it is Staller’s turn. By induction hypothesis, we know that Dominator needs at least $\gamma_{MB}(X_{m-4}) - 1 \geq m - 7$ more steps to win the game. In total, he needs $(m - 7) + 5 = m - 2$ steps.

(2.1.2.2.e) When $D_3 = v_j$, where $j = i$, since $S_2 = v_3$, we know that $j = i = 4$, that is, $D_1 = u_4$ and $D_3 = v_4$. Let $S_3 = v_2$, which forces $D_4 = v_1$. The remaining graph on the right is the MBD graph $Z_{m-4}$, and now it is Staller’s turn. By Proposition 2.5, Dominator needs in $\gamma_{MB}(Z_{m-4}) = m - 5$ more steps to win. In total, he needs $(m - 5) + 4 = m - 1$ steps to win.

(2.1.2.2.f) When $D_3 = v_j$, where $j > i$, then $i \in \{3, 4\}$, that is, $D_1 \in \{u_3, u_4\}$. The analyses are analogous to items (a) and (b). Dominator needs in total at least $m - 2$ steps to win.

(2.2) $D_2 = u_2$. Recall that we are currently under the case where $D_1 = u_i$, $i \geq 3$, $S_1 = v_2$. Let $S_2 = u_1$, which forces $D_3 = v_1$. The remaining graph is the MBD graph $X_{m-2}$ on the right, but with $u_i$ claimed already by Dominator, and now it is Staller’s
turn. By induction hypothesis, Dominator needs at least $m - 5$ more steps to win. In total, he needs at least $(m - 5) + 3 = m - 2$ steps to win.

(2.3) $D_2 = u_j$, where $j \geq 3$. Recall that we are currently under the case where $D_1 = u_i$, $i \geq 3$, $S_1 = v_2$. Now let $S_2 = v_3$, which forces $D_3 = u_1$. Let $l := \min \{i, j\}$ and $h := \max \{i, j\}$. Let Staller create a sequence of line traps $u_2 - u_{l-1}$. The remaining graph is the MBD graph $X_{m-l}$ on the right, but with $u_h$ claimed already by Dominator, and now it is Staller’s turn. By induction, Dominator needs at least $(m - 5) + 3 = m - 2$ steps to win.

(2.4) $D_2 = v_1$.

(2.4.1) $D_1 = u_3$. Recall that $S_1 = v_2$, $D_2 = v_1$. The remaining graph is the MBD graph $W_{m-3}$ on the right, and now it is Staller’s turn. By Proposition 2.6, we know that Dominator needs $\gamma'_{MB}(W_{m-3}) \geq m - 4$ more steps to win. In total, he needs at least $(m - 4) + 2 = m - 2$ steps to win.

(2.4.2) $D_1 = u_i$, where $i \geq 4$. Let $S_2 = u_3$. We need to further distinct cases according to the choice of $D_3$.

(2.4.2.1) $D_3 \in \{u_1, u_2\}$.

(2.4.2.1.a) When $i$ is even, let Staller create a sequence of triangle traps $v_3 - v_{i-1}$, which ends with $S_{i-1} = v_{i-1}, D_{i} = v_{i-1}$. Then, let $S_i = v_{i+2}$ if $m - i \geq 2$. The remaining graph is the MBD graph $\rho_{m-i}$ and now it is Dominator’s turn. By Proposition 2.2, we know that Dominator needs $\gamma_{MB}(\rho_{m-i}) = m - i$ more steps to win. In total, he needs $(m - i) + i = m$ steps to win.

(2.4.2.1.b) When $i$ is odd, let Staller create a sequence of triangle traps $v_3 - v_{i-2}$, which ends with $S_{i-2} = v_{i-1}, D_{i-1} = v_{i-2}$. The remaining graph is the MBD graph $W_{m-i}$ on the right and now it is Staller’s turn. By Proposition 2.6 Dominator needs at least $\gamma'_{MB}(W_{m-i}) \geq m - i - 1$ more steps to win. In total, he needs at least $(m - i - 1) + i - 1 = m - 2$ steps to win.

(2.4.2.2) $D_3 = u_j$, $j \geq 4$ or $D_3 = v_j$, $j \geq 3$. A similar reasoning as in case (2.1.2.2) shows that $\min \{i, j\} \in \{3, 4\}$ must hold.
(2.4.2.2.a) When $D_3 = u_j$, where $j < i$; since $S_2 = u_3$, we know that $j = 4$, that is, $D_3 = u_4$. Let $S_3 = u_2$, which forces $D_4 = u_1$; then let $S_4 = v_4$, which forces $D_3 = v_3$. The remaining graph is the MBD graph $X_{m-4}$, but with $u_j$ already claimed by Dominator, and now it is Staller’s turn. By induction hypothesis, Dominator needs at least $m - 6 - 1 = m - 7$ more steps to win. In total, he needs $(m - 7) + 5 = m - 2$ steps to win.

(2.4.2.2.b) When $D_3 = u_j$, where $j > i$; since $S_2 = u_3$, we know that $i = 4$, i.e., $D_1 = u_4$. With the same strategy as in item (a), we obtain that Dominator needs at least $m - 2$ steps to win.

(2.4.2.2.c) When $D_3 = v_j$, where $j > i$; since $S_2 = u_3$, we know that $i = 4$, i.e., $D_1 = u_4$. With the same strategy as in item (a), we obtain that Dominator needs at least $m - 2$ steps to win.

(2.4.2.2.d) When $D_3 = v_j$, where $j = i$; since $S_2 = u_3$, we know that $i = j = 4$, i.e., $D_1 = u_4$, $D_3 = v_4$. Let $S_3 = u_2$, which forces $D_4 = u_1$. The remaining graph is the MBD graph $Z_{m-4}$ on the right and it is Staller’s turn. By Proposition 2.5, Dominator needs at least $\gamma'_MB(Z_{m-4}) = m - 5$ more steps to win.

(2.4.2.2.e) When $D_3 = v_j$, where $j < i$, then $j \in \{3, 4\}$. When $j = 3$, $D_3 = v_3$. Let $S_3 = u_2$, which forces $D_4 = u_1$. Consider the remaining graph as the MBD graph $X_{m-3}$, but with $u_j$ already claimed by Dominator, and now it is Staller’s turn. By induction hypothesis, Dominator needs at least $k - 6$ more steps to win. In total, he needs at least $k - 2$ steps to win. When $j = 4$, $D_3 = v_4$. Let $S_3 = u_2$, which forces $D_4 = u_1$. Then let $S_4 = u_4$, which forces $D_5 = v_3$. We get the MBD graph $X_{m-4}$ with $u_i$ already claimed by Dominator on the right. By induction hypothesis, Dominator needs at least $k - 7$ steps to win. In total, he needs at least $k - 2$ steps to win.

(2.5) $D_2 = v_j$, where $j > i$.

Recall that we are currently under the case where $D_1 = u_i$, $i \geq 3$, $S_1 = v_2$. Let $S_2 = v_1$, which forces $D_3 = u_1$. Then let Staller create a sequence of line traps $u_2 - u_{i-1}$. Then the remaining graph is the MBD graph $X_{m-i}$ on the right, but with $v_j$ already claimed by Dominator, and now it is Staller’s turn. By induction hypothesis, Dominator needs at least $m - i - 3$ more steps to win. In total, he needs $(m - i - 3) + i + 1 = m - 2$ steps to win.
Case 3: $D_1 = u_i$, $i \geq 3$. Let $S_1 = v_2$, then according to the second move of Dominator, we carry out case distinctions.

(2.6) $D_2 = v_j$, where $j = i \geq 3$.
Recall that $D_1 = u_i$, $i \geq 3$, $S_1 = v_2$, $D_2 = v_i$. Let $S_2 = v_1$, which forces $D_3 = u_1$. Then let Staller create a sequence of line traps $u_2 - u_{i-2}$. If $i = m$, then Dominator already won with $m$ steps; otherwise, the remaining graph is the MBD graph $Z_{m-i}$ on the right. By Proposition 2.5, Dominator needs $\gamma'_{MB}(Z_{m-i}) = m - i - 1$ more steps to win. In total, he needs $(m - i - 1) + i = m - 1$ steps to win. In either case, he needs more than $m - 2$ steps to win.

(2.7) $D_2 = v_j$, where $i > j \geq 2$ and $j$ is even.
Recall that $D_1 = u_i$, $i \geq 3$, $S_1 = v_2$, so actually $j \geq 4$. Let $S_2 = u_2$, then we claim that $D_3 \in \{u_1, v_1\}$. Suppose not. After $D_3$ either $v_3$ or $u_3$ should be free. If $v_3$ is free after $D_3$, let $S_3 = v_1$ and $S_4 \in \{u_1, v_3\}$. We see that Dominator cannot dominate both $u_1$ and $v_2$, hence he cannot win. If $u_3$ is free after $D_3$, let $S_3 = u_1$ and $S_4 \in \{v_1, u_3\}$. We see that Dominator cannot dominate both $v_1$ and $u_2$, hence he cannot win.

(2.7.a) $D_3 = u_1$. Recall that $D_1 = u_i$, $i \geq 3$, $S_1 = v_2$, $D_2 = v_j$, $i > j \geq 4$ is even, $S_2 = u_2$. Let $S_3 = v_3$, which forces $D_4 = v_1$. Then let Staller create a sequence of triangle traps $u_3 - u_{j-1}$. The remaining graph is the MBD graph $X_{m-j}$ on the right, but with $u_i$ claimed already by Dominator, and now it is Staller’s turn. By induction hypothesis, Dominator needs at least $m - j - 2$ more steps to win. In total, he needs $(m - j - 2) + j = m - 2$ steps to win.

(2.7.b) $D_3 = v_1$. Recall that $D_1 = u_i$, $i \geq 3$, $S_1 = v_2$, $D_2 = v_j$, $i > j \geq 4$ is even, $S_2 = u_2$. Let $S_3 = u_3$, which forces $D_4 = u_1$. Actually $j = 4$ must hold, otherwise in the next move Staller can either isolate $u_4$ or $v_4$. Let $S_4 = u_4$, which forces $D_5 = v_3$. The remaining graph is the MBD graph $X_{m-4}$ on the right, but with $u_i$ already claimed by Dominator, and now it is Staller’s turn. By induction hypothesis, Dominator needs at least $m - 7$ more steps to win. In total, he needs at least $(m - 7) + 5 = m - 2$ steps to win.

(2.8) $D_2 = v_j$, where $i > j \geq 2$ and $j$ is odd. Let $S_2 = u_1$, which forces $D_3 = v_1$. Then let Staller create a sequence of triangle traps $u_2 - u_{j-1}$. The remaining graph is the MBD graph $X_{m-j}$ on the right, but with $u_i$ already claimed by Dominator, and now it is Staller’s turn. By induction hypothesis, Dominator needs at least $m - j - 3$ more steps to win. In total, he needs at least $(m - j - 3) + j + 1 = m - 2$ steps to win.
(3.1) \( D_2 = u_1 \).

(3.1.1) If \( i = 3 \), i.e., \( D_1 = v_3 \), then the remaining graph is the MBD graph \( W_{m-3} \) on the right, and it is Staller’s turn now. By Proposition 2.6, Dominator needs at least \((m - 3) - 1 = m - 4\) more steps to win. In total, he needs \((m - 4) + 2 = m - 2\) steps to win.

(3.1.2) If \( i \geq 3 \), let \( S_2 = v_3 \). We carry out case distinctions according to the third move of Dominator in the sequel.

(3.1.2.1) \( D_3 = v_1 \). If \( i = 4 \), i.e., \( D_1 = v_4 \), on the right we see the MBD graph \( W_{m-4} \), and it is Staller’s turn now. Dominator needs at least \((m - 4) - 1 = m - 5\) more steps to dominate those undominated vertices in this graph \( W_{m-4} \). Hence he needs in total at least \((m - 5) + 3 = m - 2\) steps to win. Otherwise, if \( i > 4 \), let \( S_3 = v_4 \) starting a sequence of line traps \( u_3 - u_{i-2} \). The remaining graph is the MBD graph \( W_{m-i} \) on the right, and it is Staller’s turn now. By Proposition 2.6, Dominator needs at least \((m - i) - 1\) more steps to win. In total, he needs at least \((m - i - 1) + (i - 1) = m - 2\) steps to win.

(3.1.2.2) \( D_3 = v_j, j \geq 4 \) or \( D_3 = u_j, j \geq 3 \). A similar reasoning as in case (2.1.2.2) shows that \( \min\{i, j\} \in \{3, 4\} \) must hold.

(3.1.2.2.1) \( D_3 = v_j, j \geq 4 \). Denote by \( l := \min\{i, j\}, h := \max\{i, j\} \), then we know that \( l = 4 \) since \( v_3 \) is claimed by the second move of Staller. Let \( S_3 = u_2 \), which forces \( D_4 = v_1 \). Then let \( S_4 = u_4 \), which forces \( D_5 = u_3 \) by making a triangle trap. The remaining graph is the MBD graph \( X_{m-4} \) on the right, but with \( u_h \) claimed by Dominator, and it is Staller’s turn now. By induction hypothesis, Dominator needs at least \((m - 6) - 4 = m - 2\) steps to win. In total, he needs at least \((m - 5) + 4 = m - 2\) steps to win.

(3.1.2.2.2) \( D_3 = u_j, j > i \). Since \( v_3 \) is already claimed, we know that \( i = 4 \), that is, \( D_1 = v_4 \). The reasoning for this case is analogous to the last case (case (3.1.2.2.1)).

(3.1.2.2.3) \( D_3 = u_j, j = i \). Since \( v_3 \) is already claimed, we know that \( i = j = 4 \). The MBD graph on the right is \( Z_{m-4} \), and it is Staller’s turn now. By Proposition 2.5, Dominator needs at least \((m - 4) - 1 = m - 5\) more steps to dominate those undominated vertices in \( Z_{m-4} \), and he needs one more step to dominate \( v_2 \). Hence he needs at least \((m - 5) + 4 = m - 1\) steps to win.
(3.1.2.2.4) \[ D_3 = u_j, \ j < i, \text{ hence } j \in \{3, 4\}. \]

(3.1.2.2.4.a) If \( j = 3 \), i.e., \( D_3 = u_3 \). Let \( S_3 = v_1 \), which forces \( D_4 = u_2 \) by making a line trap. The remaining graph on the right is \( X_{m-3} \), but with \( v_i \) claimed by Dominator already, and it is Staller’s turn now. By induction hypothesis, Dominator needs at least \((m - 3) - 2 = m - 5\) more steps to win. In total, he needs at least \((m - 5) + 3 = m - 2\) steps to win.

(3.1.2.2.4.b) If \( j = 4 \), i.e., \( D_3 = u_4 \). Let \( S_3 = u_2 \), which forces \( D_4 = v_1 \). Then let \( S_4 = v_4 \), which forces \( D_5 = u_3 \) by making a line trap. The remaining graph on the right is \( X_{m-4} \), but with \( v_i \) claimed by Dominator already, and it is Staller’s turn now. By induction hypothesis, Dominator needs at least \((m - 4) - 2 = m - 6\) more steps to win. In total, he needs at least \((m - 6) + 4 = m - 2\) steps to win.

(3.2) \[ D_2 = v_1. \] Let \( S_2 = u_3 \), we carry out case distinctions according to the third move of Dominator in the sequel.

(3.2.1) \[ D_3 \in \{u_1, u_2\}. \]

(3.2.1.1) \( i \) is even. Let Staller create a sequence of triangle traps \( v_3 - u_{i-2} \). The remaining graph is the MBD graph \( W_{m-i} \) on the right, and it is Staller’s turn now. By Proposition \ref{prop:MBD_properties}, Dominator needs at least \((m - i) - 1\) more steps to win. In total, he needs at least \((m - i - 1) + (i - 1) = m - 2\) steps to win.

(3.2.1.2) \( i \) is odd. Let Staller create a sequence of triangle traps \( v_3 - u_{i-1} \), which ends with \( S_{i-1} = u_i, D_i = u_{i-1} \). Let \( S_i = u_{i+2} \), which creates the MBD graph \( \rho_{m-i} \) on the right, and it is Dominator’s turn now. By Proposition \ref{prop:MBD_properties}, \( \gamma_{\text{MB}}(\rho_{m-i}) = m - i \), hence Dominator needs at least \( m - i \) steps to win. In total, he needs at least \((m - i) + i = m \) moves to win.

(3.2.2) \[ D_3 = u_j, \ j \geq 4. \] A similar reasoning as in case (2.1.2.2) shows that \( \min\{i, j\} \in \{3, 4\} \) must hold.

(3.2.2.a) \( j < i \). Since \( u_3 \) is claimed, \( j = 4 \), i.e., \( D_3 = u_4 \). The analysis is analogous to case (2.4.2.2.a).
(3.2.2.b) \( j = i \). Since \( u_3 \) is claimed, \( i = j = 4 \), that is, \( D_1 = v_4 \), \( D_3 = u_4 \). Let \( S_3 = u_2 \), which forces \( D_4 = u_1 \). Then the remaining graph is the MBD graph \( Z_{m-4} \) on the right, and it is Staller’s turn now. By Proposition 2.5, Dominator needs at least \((m - 4) - 1 = m - 5\) more steps to win. In total, he needs at least \((m - 5) + 4 = m - 1\) steps to win.

(3.2.2.c) \( j > i \), hence \( i \in \{3, 4\} \). If \( i = 3 \), that is, \( D_1 = v_3 \), then let \( S_3 = u_2 \), which forces \( D_4 = u_1 \). We get the MBD graph \( X_{m-3} \) on the right, but with \( v_j \) claimed already, and it is Staller’s turn now. By induction hypothesis, Dominator needs at least \((m - 3) - 3 = m - 6\) more steps to win. In total, he needs at least \((m - 6) + 4 = m - 2\) steps to win. If \( i = 4 \), that is, \( D_1 = v_4 \). Let \( S_3 = u_2 \), which forces \( D_4 = u_1 \). Then let \( S_4 = u_4 \), which forces \( D_5 = v_3 \) by creating a line trap. Then consider the MBD graph \( X_{m-4} \) on the right. Again, Dominator needs in total at least \( m - 2 \) steps to win.

(3.2.3) \( D_3 = v_j \), \( j \geq 3 \). A similar reasoning as in case (2.1.2.2) shows that \( \min\{i, j\} \in \{3, 4\} \) must hold.

(3.2.3.a) \( j < i \). The reasoning is analogous to cases (3.2.2.c).

(3.2.3.b) \( j > i \). The reasoning is analogous to cases (3.2.2.c).

(3.3) \( D_2 = u_j \), where \( j \geq 2 \), \( j > i \) and \( i \) is even. Let \( S_2 = u_2 \), the remaining reasoning is analogous to case (2.7).

(3.4) \( D_2 = u_j \), where \( j \geq 2 \), \( j > i \) and \( i \) is odd. Let \( S_2 = u_1 \), the remaining reasoning is analogous to case (2.8).

(3.5) \( D_2 = u_j \), where \( j = i \geq 2 \). Let \( S_2 = v_1 \), the remaining reasoning is analogous to case (2.6).

(3.6) \( D_2 = u_j \), where \( 2 \leq j < i \). Let \( S_2 = v_1 \), the remaining reasoning is analogous to case (2.5).

(3.7) \( D_2 = v_j \), \( j \geq 2 \), such that \( \min\{i, j\} \) is odd. Let \( S_2 = u_1 \), the remaining reasoning is analogous to case (2.8).
(3.8) \( D_2 = v_j, j \geq 2, \) such that \( \min \{i, j\} \) is even. Let \( S_2 = u_2, \) the remaining reasoning is analogous to case (2.7).

Since we have gone through all cases, it turns out that Dominator anyway needs at least \( m - 2 \) steps to win the game. By induction, \( \gamma_{MB}(X_m) \geq m - 2, \) when \( m \geq 6. \)

So to sum up, \( \gamma_{MB}(X_m) = m - 2 \) for \( m \geq 6. \) \( \square \)

2.4 Domination number of \( P_2 \square P_n \)

Proof of Theorem 2.3. To prove the upper bound \( \gamma'_{MB}(P_2 \square P_n) \leq n, \) let Dominator adopt the pairing strategy: whenever Staller claims \( u_i (v_i), \) let Dominator claim \( v_i (u_i) \) in the next round. It is not hard to see that Dominator will win within \( n \) moves.

To prove the lower bound \( \gamma'_{MB}(P_2 \square P_n) \geq n, \) we propose the following strategy for Staller. Let Staller claim \( v_2 \) as her first move if \( n \geq 2, \) let Staller claim \( u_1 \) when \( n = 1. \)

The remaining game is harder for Dominator than the D-game on \( \rho_n, \) since \( \rho_n \) has one less undominated vertex than \( P_2 \square P_n \) with \( v_2 \) claimed by Staller. By Proposition 2.2, we know that Dominator needs at least \( n \) more steps to win, and he cannot skip any moves. When \( n = 1, \) it is easy to see that he also cannot skip any moves. Hence \( \gamma'_{MB}(\rho_n) \geq n, \) and Dominator cannot skip any moves, otherwise he cannot win. \( \square \)

Next, we would like to address a concrete example of \( P_2 \square P_n, \) namely when \( n = 13. \)

Before which, we need two more auxiliary results.

Claim 2.8. Consider the S-game on the MBD graph \( W_4. \) Let Dominator skip his first move, namely \( D_1 = \emptyset. \) In addition, assume that \( S_1 \notin \{u_3, v_3, u_4, v_4\}. \) Then Dominator can win within 4 moves.

Proof. The proof can be done by case distinctions, we do not go into details here. \( \square \)

Claim 2.9. Consider the S-game on the MBD graph \( W_6. \) Let Dominator skip his first move, namely \( D_1 = \emptyset. \) In addition, assume that \( S_1 = v_2. \) Then Dominator can win within 6 moves.

Proof. The proof can be done by case distinctions, we do not go into details here. \( \square \)

Now we can prove the following result about \( P_2 \square P_{13}. \)

Theorem 2.10. \( \gamma_{MB}(P_2 \square P_{13}) = 11. \)

Proof. Since \( P_2 \square P_{13} \) has one more undominated vertex than \( X_{13}, \) \( \gamma_{MB}(P_2 \square P_{13}) \geq X_{13}. \)

By Theorem 2.7, \( \gamma_{MB}(X_{13}) = 11. \) Hence \( \gamma_{MB}(P_2 \square P_{13}) \geq 11. \) Denote by \( L = (V_L, E_L) \) the subgraph of \( P_2 \square P_{13} \) induced by vertex set \( \{u_1, v_1, u_2, v_2, \ldots, u_6, v_6\}, \) and by \( R = (V_R, E_R) \) the subgraph of \( P_2 \square P_{13} \) induced by vertex set \( \{u_7, v_7, u_8, v_8, \ldots, u_{13}, v_{13}\}. \) For the upper bound, we propose strategy for Dominator.

Let \( D_1 = v_7. \) It suffices to consider Staller’s response on \( V_L \cup \{u_7\}, \) by the symmetric property of \( P_2 \square P_{13}. \) We carry out case distinctions according to the choice of \( S_1 \) as follows:
• Case 1: $S_1 = u_7$. In this case, we see an MBD graph $X_6$ on both subgraphs $L$ and $R$. Let Dominator carry out the next move on the right MBD graph $X_6$. Then, whenever Staller claims some vertex on $R$, let Dominator adopt his strategy on the right MBD graph $X_6$; whenever Staller claims some vertex on $L$, let Dominator respond on the left MBD graph $X_6$, using the pairing strategy. By Theorem 2.7, Dominator needs 4 steps to dominate all vertices on the right. He needs 6 steps to dominate all vertices on the left. In total, he can win within $6 + 1 + 4 = 11$ steps.

• Case 2: $S_1 = u_5$, then let $D_2 = u_9$. Then let Dominator respond on $L$ if Staller claims any vertex of $L$, and let him respond on $R$ if Staller claims any vertex of $R$. On the right is the MBD graph $W_4$, while on the left it is the situation described in Claim 2.9. By Proposition 2.6, Dominator needs $\gamma(MB(W_4)) = 3$ steps to dominate all vertices of $R$. Claim 2.9 tells us that Dominator needs within 6 steps to dominate all vertices of $L$. Hence, he needs in total within $6 + 3 + 2 = 11$ steps to win the game.

• Case 3: $S_1 \in \{u_3, v_3, u_4, v_4, u_5, v_6\}$, then let $D_2 = u_5$. Then we have the MBD graph $W_6$ on the right. Let Dominator respond on $V_R \cup \{u_7\}$ whenever Staller claims any vertex in $V_R \cup \{u_7\}$, and let him respond on $V_L$ otherwise. If $S_1 \in \{u_6, v_6\}$, on the left would be $W_4$. In this case, Dominator needs at most $\gamma(MB(W_4)) + \gamma(MB(W_6)) + 2 = 3 + 5 + 2 = 10$ steps to win. Otherwise, $S_1 \in \{u_3, v_3, u_4, v_4, v_5\}$, then the graph on the left fulfills the description in Claim 2.8, where he needs within 4 steps to win. In total, he needs at most $4 + 5 + 2 = 11$ steps to win.

• Case 4: $S_1 \in \{u_2, v_2\}$, then let $D_2 = u_3$. Then let Dominator respond on $V_R \cup \{u_7\}$ whenever Staller claims any vertex of this set. Let Dominator adopt the pairing strategy on the left, where the pairing sets of vertices are $\{u_1, v_1\}$, $\{v_4, v_5\}$, $\{u_5, u_6\}$. In addition, he needs at most one more step to dominate $v_2$. By Proposition 2.6, $\gamma(MB(W_6)) = 5$, hence Dominator needs in total at most $3 + 1 + 5 + 2 = 11$ steps to win the game.

• Case 5: $S_1 \in \{u_1, v_1\}$, then let $D_2 = v_2$. Let Dominator adopt strategy for the S-game on $W_6$ on the right, and the pairing strategy with pairing sets $\{u_3, u_4\}$, $\{v_4, v_5\}$, $\{u_5, u_6\}$. In addition, he needs at most one more step to dominate $u_1$. In total he needs within $3 + 1 + 5 + 2 = 11$ steps to win the game.

To sum up, $\gamma(MB(P_2 \square P_{13})) \leq 11$. Combine with the lower bound, we obtain that $\gamma(MB(P_2 \square P_{13})) = 11$.

**Proof of Theorem 1.4** By Theorem 2.10, $\gamma(MB(P_2 \square P_{13})) = 11$. When $n > 13$, consider the graph as the two subgraphs $A := P_2 \square P_{13}$ and $B \cong P_2 \square P_{n-13}$ connected by edges $\{u_{13}, u_{14}\}$, $\{v_{13}, v_{14}\}$, up to isomorphism. Let Dominator resond on $A$ whenever Staller claims a vertex of $A$, and let Dominator respond on $B$ with the pairing strategy whenever Staller claims a vertex of $B$. In this way, we see that he needs within $11 + (n - 13) = n - 2$ steps in total, in order to win. Therefore, $\gamma(MB(P_2 \square P_{13})) \leq n - 2$, $n \geq 13$. 

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For the lower bound, $\gamma_{MB}(P_2 \Box P_{13}) \geq \gamma_{MB}(X_n)$ since $P_2 \Box P_{13}$ has one more undominated vertex than the MBD graph $X_n$. By Theorem 2.7, $\gamma_{MB}(X_n) \geq n - 2, n \geq 6$. Hence $\gamma_{MB}(P_2 \Box P_{13}) \geq \gamma_{MB}(X_n) \geq n - 2, n \geq 13$. To conclude, $\gamma_{MB}(P_2 \Box P_{13}) = n - 2, n \geq 13$. 

2.5 Union of grids

In this subsection, we prove the two results on the disjoint union of $P_2 \Box P_n$s, using Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.5. By the pairing strategy where the pairing sets are $\{u_i, v_i\}$s in each copy, Dominator can win within $k \cdot n$ steps. However, by Theorem 1.3, we know that no matter Staller claim a vertex in which copy, Dominator has to respond on the same copy, otherwise he would lose the game. Hence he needs at least $k \cdot \gamma'_{MB}(P_2 \Box P_n) = k \cdot n$ steps to win. So we get $\gamma'_{MB}(\bigcup_{i=1}^{k}(P_2 \Box P_n)_i) = k \cdot n, n \geq 1$.

Proof of Theorem 1.6. The upper bound can be argued by letting Dominator adopt the pairing strategy. The lower bound can be argued by letting Staller start her move on any copy where no vertex is not claimed by Dominator and continue on it until Dominator has won on this copy. By Theorem 1.3, we know that Dominator cannot skip any move during this process and needs $n$ steps to dominate all vertices of this copy. Then, let Staller repeat this strategy on all the remaining copies except for the one where the first vertex that is claimed by Dominator belongs to. After this process, let Staller continue by responding on that remaining copy, then it should be considered as a D-game on $P_2 \Box P_n$, where by Theorem 1.4 Dominator needs $n - 2$ steps to win, when $n \geq 13$.

3 Concluding remarks

We proved that Dominator needs exactly $n$ moves to win in the $S$-game on $P_2 \Box P_n$ for every $n \geq 1$, while in the $D$-game he needs exactly $n - 2$ moves for $n \geq 13$. We showed that for $k$ copies of $P_2 \Box P_n$, Dominator needs exactly $k \cdot n$ steps to win in the $S$-game, and we gave upper and lower bounds for the steps Dominator needs in order to win in $D$-games. The exact result for general cartesians $P_m \Box P_n$ does not seem easy, so it would be interesting to consider the situation for $P_3 \Box P_n$, as a starting point.

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References

[1] J. Beck, Combinatorial Games: Tic-Tac-Toe Theory, Encyclopedia of Mathematics and Its Applications 114, Cambridge University Press, (2008).

[2] B. Brešar, S. Klavžar and D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math., 24 (2010), pp. 979–991.

[3] D. Clemens, A. Ferber, M. Krivelevich, A. Liebenau, Fast strategies in Maker–Breaker games played on random boards, Combin. Probab. Comput. 21 (2012) 897–915.

[4] D. Clemens, A. Ferber, R. Glebov, D. Hefetz and A. Liebenau, Building spanning trees quickly in Maker–Breaker games, SIAM Journal on Discrete Mathematics, 29(3) (2015), pp. 1683–1705.

[5] E. Duchêne, V. Gledel, A. Parreau and G. Renault, Maker–Breaker domination game, Discrete Mathematics, 342(9) (2020).

[6] P. Erdős and J. L. Selfridge, On a combinatorial game, J. Combinatorial Theory Ser. A 14 (1973) pp. 298–301.

[7] V. Gledel, V. Iršič and S. Klavžar, Maker–Breaker domination number, Bulletin of the Malaysian Mathematical Sciences Society, 42(4) (2019), pp. 1773–1789.

[8] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Fast winning strategies in Maker–Breaker games, Journal of Combinatorial Theory Series B 19 (2009), 39–47.

[9] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Positional Games, Oberwolfach Seminars 44, Birkhäuser/Springer Basel, 2014.