Abstract

We study a contextual bandit setting where the learning agent has access to sampled bandit instances from an unknown prior distribution \( \mathcal{P} \). The goal of the agent is to achieve high reward on average over the instances drawn from \( \mathcal{P} \). This setting is of a particular importance because it formalizes the offline optimization of bandit policies, to perform well on average over anticipated bandit instances. The main idea in our work is to optimize differentiable bandit policies by policy gradients. We derive reward gradients that reflect the structure of our problem, and propose contextual policies that are parameterized in a differentiable way and have low regret. Our algorithmic and theoretical contributions are supported by extensive experiments that show the importance of baseline subtraction, learned biases, and the practicality of our approach on a range of classification tasks.

1 Introduction

A stochastic multi-armed bandit [32, 9, 34] is an online learning problem where a learning agent takes a sequence of actions, with the goal of learning to act optimally in an unknown environment. Actions (treatments in a clinical trial or ads on a website) are called arms, which when pulled, provide the agent with a stochastic reward. A contextual bandit [35, 5] generalizes this setting: the reward depends on context in each round (patient medical history or targeted user demographics), which is observed by the agent. In this case, the mean reward of an arm is an unknown function of the context. This function is often parametric and its parameters are learned.

We focus on a contextual form of Bayesian bandits [13], where the learning agent has access to bandit instances sampled from an unknown prior distribution \( \mathcal{P} \), with the goal of achieving high reward on average over the instances drawn from \( \mathcal{P} \). Early works on Bayesian bandits were concerned with understanding conditions on \( \mathcal{P} \) under which Bayes optimal policies take a simple form in the context-free case [25, 26]. Unfortunately, these results are not helpful when \( \mathcal{P} \) is unknown, or when the context is present. We address the problem from a computational point of view, where the prior \( \mathcal{P} \) is treated as a simulator and bandit policies are directly optimized against it. This can be viewed as a model-based approach to bandit policy learning, which received very little attention in the prior work. It is also an application of meta-learning [49, 50, 10, 11] to bandit policies.

Our prime motivation is that bandit policies are rarely put into practice using theoretically analyzed parameters. It is well known that careful tuning may reduce their regret by an order of magnitude [53, 37, 29, 28]. However, practical tuning tends to be ad-hoc. Our objective is to develop a general, systematic, and data-dependent method for learning bandit policies. In this work, we focus on policy-gradient optimization [54, 47, 12]. The data dependence emerges because we directly optimize the quantity of interest, the actual regret, against a prior distribution over problem instances, which can be fit to any historical data. To make our approach practical, we incorporate a novel form of baseline subtraction to reduce variance. In addition, we propose special policy classes. Both the baselines
We propose a general and data-dependent method for learning bandit policies. The key idea is to project the context into a relevant subspace to increase the statistical efficiency of learning. This projection is optimized to minimize the Bayes regret. We apply this idea to linear Thompson sampling (LinTS) \[5\] and a novel softmax policy \text{CoSoftElim}. Unlike other randomized exploration schemes \[4, 5, 43, 31, 30\], \text{CoSoftElim} does not add noise to the maximum likelihood estimate to explore, and thus is algorithmically novel. To justify that \text{CoSoftElim} gives rise to a reasonable policy class, we prove that it has a sublinear \( n \)-round regret with appropriate parameter choices.

The paper is organized as follows. In Section 2, we introduce our setting. In Section 3, we propose gradient-based optimization of contextual bandit policies, derive the reward gradient, and suggest baseline subtraction to make the optimization practical. In Section 4, we present our differentiable policies. The \( n \)-round regret of \text{CoSoftElim} is bounded in Section 5. In Section 6, we evaluate our proposed policies and their optimization, on both synthetic and multi-class classification problems. We discuss related work in Section 7 and conclude in Section 8.

2 Setting

We use the following notation. We define \([n] = \{1, \ldots, n\}\). We denote by \(x \oplus y\) the concatenation of vectors \(x\) and \(y\). For any positive semi-definite (PSD) matrix \(M\), we define \(\|x\|_M = \sqrt{x^\top M x}\). A \(d \times d\) identity matrix is \(I_d\). We use \(O\) for the big-O notation up to logarithmic factors.

A Bayesian multi-armed bandit \[25, 13\] is an online learning problem where the learning agent interacts with problem instances that are drawn i.i.d. from a known prior distribution. We study a contextual variant of this problem, with \(K\) arms (actions) and \(n\) rounds of interaction. The context in round \(t\) is a \(d\)-dimensional vector \(X_t \in \mathbb{R}^d\) and \(X = (X_t)_{t=1}^n \in \mathbb{R}^{d \times n}\) are realized contexts in all rounds. A problem instance is a tuple \(P = (\theta, X)\), where \(\theta \in \mathbb{R}^d\) are model parameters. The mean reward of arm \(i \in [K]\) in context \(x \in \mathbb{R}^d\) is \(f_i(x, \theta_i)\), where \(f_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\) is a known function. One may think of \(f_i(x, \theta_i)\) as linear in \(x\) and \(\theta_i\), though we do not need this assumption until the analysis in Section 5. We assume that \(Y_{i,t}\), the realized reward of arm \(i\) in round \(t\), is drawn i.i.d. from a \(\sigma^2\)-sub-Gaussian distribution with mean \(\mathbb{E}[Y_{i,t} | P] = f_i(X_t, \theta_i)\). The realized rewards of all arms in round \(t\) are \(Y_{t} = (Y_{i,t})_{i=1}^K\) and all realized rewards are \(Y = (Y_{t})_{t=1}^n \in \mathbb{R}^{K \times n}\).

Let \(P\) be a prior distribution over problem instances. Before the agent starts interacting with the environment, we sample \(P = (\theta, X) \sim P\) and all rewards \(Y \sim P\), as described above. We do not make any assumption on how contexts \(X_t\) are generated. Then, in round \(t\), the agent observes context \(X_t\), pulls arm \(I_t \in [K]\), and gains reward \(Y_{I_t,t}\). We define the history of the agent at the beginning of round \(t\) as \(H_t = (I_1, \ldots, I_{t-1}, X_1, \ldots, X_t, Y_{1,t}, \ldots, Y_{I_{t-1},t-1})\), which is its past observations and the context in round \(t\). The agent is a randomized policy or controller. The probability that the agent pulls arm \(i\) in round \(t\), given history \(H_t\) and policy parameters \(\pi \in \Pi\), is

\[
p(i \mid H_t; \pi),
\]

where \(\Pi\) is the space of feasible parameters. Thus \(I_t \sim p(\cdot \mid H_t; \pi)\). We refer to \(\pi\) as a policy when the controller is fixed. Let \(\mathbb{P}_t(\cdot) = \mathbb{P}(\cdot \mid H_t)\) and \(\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot \mid H_t]\) be the conditional probability and expectation associated with history \(H_t\). We also define \(I_{i,j} = (I_t)_{t=j}^i\) and \(I = I_{1:n}\).

The \(n\)-round Bayes reward of policy \(\pi\) is \(r(n; \pi) = \mathbb{E} \left[ \sum_{t=1}^n Y_{I_{t,t}} \right]\). We stress that the expectation is over problem instances \(P\), realized rewards \(Y\), and pulled arms \(I\). Our goal is to maximize \(r(n; \pi)\) over \(\pi\). This is equivalent to minimizing the \(n\)-round Bayes regret

\[
R(n; \pi) = \sum_{t=1}^n \mathbb{E} \left[ Y_{I_{*,t},t} - Y_{I_{t,t}} \right],
\]

where \(I_{*,t} = \arg \max_{i \in [K]} f_i(X_t, \theta_i)\) is the best arm in round \(t\) in problem instance \(P = (\theta, X)\).

3 Bandit Policy Optimization

We propose a general and data-dependent method for learning bandit policies. The key idea is to maximize the Bayes reward \(r(n; \pi)\) by gradient ascent, on sampled problem instances from \(P\).
We introduce CoGradBand. Its pseudocode is in Algorithm 1. CoGradBand is initialized with policy \( \pi_0 \in \Pi\). In iteration \( \ell \), the last policy \( \pi_{\ell-1} \) is updated by gradient ascent using \( \hat{g}(n; \pi_{\ell-1}) \), an empirical estimate of the reward gradient at the last policy, \( \nabla r(n; \pi_{\ell-1}) \). To obtain \( \hat{g}(n; \pi_{\ell-1}) \), we apply \( \pi_{\ell-1} \) to \( m \) sampled problem instances from \( \mathcal{P} \). The \( j \)-th sampled instance is \( P^j \). The corresponding contexts are \( X^j \in \mathbb{R}^{d \times n} \), realized rewards are \( Y^j \in \mathbb{R}^{K \times n} \), and pulled arms are \( I^j \in [K]^n \). Note that \( \mathcal{P} \) is not needed to compute the gradient. Only \( (X^j)_{j=1}^m \) and \( (Y^j)_{j=1}^m \) are needed.

CoGradBand is general, data-dependent, and directly optimizes the quantity of interest, the Bayes reward \( r(n; \pi) \). However, since \( r(n; \pi) \) is a complex function of the adaptive bandit policy \( \pi \) and environment, it is hard to provide meaningful guarantees on learned policies. This is one reason why bandit papers analyze theoretically manageable regret upper bounds and not the regret itself. Limited guarantees exist when \( r(n; \pi) \) has a closed form. For instance, Boutilier et al. [14] showed that \( r(n; h) \) of an explore-then-commit policy [33] is concave in its exploration horizon \( h \) in a 2-armed Gaussian bandit. So the optimal \( h \) can be found by gradient ascent. We generalize this result to the contextual setting in Appendix D. Results like these, and other empirical evidence [28], justify gradient-based optimization of bandit policies. In this work, we focus on designing contextual controllers in (1) that can be optimized in a stable manner. This means that the convergence of CoGradBand to good policies must be validated empirically (Section 6).

CoGradBand is a meta-algorithm. To use it, we must compute the empirical gradient \( \hat{g}(n; \pi_{\ell-1}) \). We derive it below and show how to reduce its variance. This generalizes results for GradBand [14] to incorporate context. We must also design controllers in (1), which we propose in Section 4.

**Reward gradient:** In Lemma 3 in Appendix A, we show that the gradient of the \( n \)-round Bayes reward of any policy \( \pi \) is \( \nabla r(n; \pi) = \sum_{t=1}^n \mathbb{E} \left[ \nabla \log p(I_t | H_t; \pi) \sum_{s=1}^n Y_{t,s} \right] \). The gradient has \( O(n^2) \) terms, because the change in the policy in any round \( t \) affects \( n \) \( t + 1 \) future rewards. So the empirical estimate of \( \nabla r(n; \pi) \) is expected to have large variance. To reduce the variance, we incorporate baseline subtraction [54, 47]. Specifically, let \( b_t : [K]^{t-1} \times \mathbb{R}^{d \times n} \times \mathbb{R}^{K \times n} \rightarrow \mathbb{R} \) be any function of previous \( t - 1 \) pulled arms, problem instance, and all realized rewards. A baseline is a collection of functions \( b = (b_t)_{t=1}^n \). Then we have the following result, which extends the analogous result [14] to the contextual case.

**Lemma 1.** For any baseline \( b = (b_t)_{t=1}^n \),

\[
\nabla \pi r(n; \pi) = \sum_{t=1}^n \mathbb{E} \left[ \nabla \log p(I_t | H_t; \pi) \left( \sum_{s=1}^n Y_{t,s} - b_t(I_{1:t-1}, P, Y) \right) \right].
\]

The proof of Lemma 1 (Appendix A) uses the fact that \( b \) is independent of the future actions of \( \pi \). The induced empirical gradient from \( m \) samples is

\[
\hat{g}(n; \pi) = \frac{1}{m} \sum_{j=1}^m \sum_{t=1}^n \nabla \log p(I_t^j | H_t^j; \pi) \left( \sum_{s=1}^n Y_{t,s}^j - b_t(I_{1:t-1}^j, P^j, Y^j) \right),
\]

(3)

where \( j \) indexes the \( j \)-th random experiment in CoGradBand.

Our baselines are motivated by the structure of our problem. Most bandit policies are designed to have sublinear regret in any problem instance [34]. When the number of arms is finite, this means...
We propose a contextual softmax policy, \( \hat{b}^{\text{opt}}(I_{1:t-1}, P, Y) = \sum_{s} Y_{I_{1:s}, s} \). The optimal arm in round \( s \), \( I_{1:s} \), is defined as in (2). Note that \( \hat{b}^{\text{opt}} \) can be only constructed because of the specific form of our gradient (Lemma 1), where \( b_i \) is a function of the same \( X \) and \( Y \) as \( \pi \).

Unfortunately, \( \hat{b}^{\text{opt}} \) reduces variance ineffectively if \( \pi \) has high regret. In this case, a better idea is to subtract the sum of rewards of an independent run of \( \pi \). We call this baseline “self” and define it as \( \hat{b}^{\text{self}}(I_{1:t-1}, P, Y) = \sum_{s} Y_{J_{s}, s} \), where \( \{J_s\}_{s=1} \) are arms pulled in the independent run. Critically, \( \hat{b}^{\text{self}} \) has the same expected reward as \( \pi \), and thus performs well even if \( \pi \) has high regret. When \( \pi \) has low regret, \( \hat{b}^{\text{self}} \) is by definition comparable to pulling optimal arms, and also reduces variance.

4 Differentiable Bandit Algorithms

The above discussion suggests that regret-minimizing policies may have low-variance gradients when paired with our baselines. This motivates our choice of controllers in (1). In particular, we focus on bandit algorithms that have provably sublinear regret. The controllers are based on linear models, since we understand well how to minimize uncertainty in these models [19, 1].

A state-of-the-art approach in meta-learning of linear models is projecting features into a relevant subspace to speed up learning [15, 52]. This motivates a specific parameterization of our controllers: we project context \( X_t \) to \( AX_t \), where \( A \in \mathbb{R}^{d \times d} \) is a projection. The projection \( A \) is optimized to minimize the Bayes regret of \textit{learning to act in the subspace}. For instance, the columns of \( A \) that correspond to irrelevant features are zeroed out. Once projected, we assume that the mean arm reward is linear in \( AX_t \). Then, using \( AX_t \) as features, the maximum likelihood estimate (MLE) of model parameters of arm \( i \) after \( t \) observations is

\[
\hat{\theta}_{i,t} = G_{t,i}^{-1} \sum_{\ell=1}^{t} \mathbb{1}\{I_{\ell} = i\} AX_{\ell} Y_{\ell,i,t}, \quad G_{i,t} = \sum_{\ell=1}^{t} \mathbb{1}\{I_{\ell} = i\} AX_{\ell} X_{\ell}^\top + \lambda I_d, \tag{4}
\]

where \( G_{i,t} \) is the corresponding sample covariance matrix and \( \lambda > 0 \) is a regularization parameter. In round \( t \), the learning agent has access to past \( t - 1 \) observations. So the MLE of the mean reward of arm \( i \) in context \( x \in \mathbb{R}^d \) is \( (Ax)_{i}^\top \hat{\theta}_{i,t-1} \) and its variance is \( \|Ax\|^2_{G_{i,t-1}^{-1}} = (Ax)^\top G_{i,t-1}^{-1} Ax \).

In the rest of this section, we propose three bandit controllers and derive their gradients. The gradients can be directly used in (3). Two of the controllers are known to have sublinear regret [5, 34], and we provide guarantees for the third. To simplify notation, we use \( p_{i,t} = p(i | H_{i}; \pi) \).

4.1 Contextual Soft Elimination

We propose a contextual softmax policy, CoSoftElim, which stands for \textit{contextual soft elimination}. The policy generalizes SoftElim [14] to contextual bandits, and is reminiscent of arm elimination algorithms [8]. CoSoftElim pulls arm \( i \) in round \( t \) with probability

\[
p_{i,t} = \exp[-S_{i,t}]/\sum_{j=1}^{K} \exp[-S_{j,t}], \tag{5}
\]

where \( S_{i,t} \geq 0 \) is the \textit{score} of arm \( i \) in round \( t \), which depends on history \( H_{i} \) including context \( X_{t} \), and projection \( A \). We define a specific score \( S_{i,t} \) below. Since \( \exp[-S_{i,t}] \in [0, 1] \), it can be viewed as a soft indicator of whether the arm can be pulled, and hence the name of CoSoftElim.

One advantage of \( p_{i,t} \) above is that it is easy to differentiate with respect to \( A \). In particular,

\[
\nabla_A \log p_{i,t} = -\nabla_A S_{i,t} - \nabla_A \log \sum_{j=1}^{K} \exp[-S_{j,t}] = \frac{\sum_{j=1}^{K} \exp[-S_{j,t}] \nabla_A S_{j,t}}{\sum_{j=1}^{K} \exp[-S_{j,t}]} - \nabla_A S_{i,t}. \tag{6}
\]

The score of arm \( i \) in round \( t \) is defined as

\[
S_{i,t} = \gamma (\mu_{\text{max},t} - \hat{\mu}_{i,t})^2 / \|AX_t\|_{G_{i,t-1}^{-1}}^2, \tag{7}
\]

where \( \hat{\mu}_{i,t} = (AX_t)^\top \hat{\theta}_{i,t-1} \) is the estimated reward of arm \( i \), \( \mu_{\text{max},t} = \arg \max_{i \in [K]} \hat{\mu}_{i,t} \) is the arm with the highest reward, and \( \hat{\mu}_{i,t} \) is its reward. The parameter \( \gamma > 0 \) is tunable. We experiment
Thompson sampling (TS) is a state-of-the-art randomized bandit algorithm and we derive a sublinear regret bound for CoSoftElim \( \epsilon \). This policy depends only on a single parameter \( \epsilon \) and we use it as a baseline in our experiments, to observe empirically (Section 6).

\subsection{Contextual Thompson Sampling}

Contextual Thompson sampling (TS) [48, 18, 4] is a state-of-the-art randomized bandit algorithm and LinTS [5] is its contextual variant. We adapt LinTS to our MLE in (4) as follows. In round \( t \), the estimated reward of arm \( i \) is sampled as \( \hat{\mu}_{i,t} \sim N((AX_i)^\top \hat{\theta}_{i,t-1}, \sigma^2 \|AX_i\|^2_{G_{i,t-1}}) \). Then we pull the arm with the highest estimated reward \( I_t = \arg \max_{i \in [K]} \hat{\mu}_{i,t} \).

It is difficult to compute \( \nabla A \log p_{i,t} \) directly, because the log probability is a result of an argmax over posterior-sampled means \( \hat{\mu}_{i,t} \). Therefore, in Lemma 4 (Appendix A), we rederive a complete reward gradient with \( \hat{\mu}_{i,t} \), where we work with probabilities \( p_{i,t} \) instead. Interestingly, the gradient has the same form as that in Lemma 1, except that \( \nabla \pi \log p(I_t \mid H_t; \pi) \) becomes
\[
\nabla_A \log p_{i,t} = \sum_{i=1}^K \nabla_A N(\hat{\mu}_{i,t}; (AX_i)^\top \hat{\theta}_{i,t-1}, \sigma^2 \|AX_i\|^2_{G_{i,t-1}}),
\]
where \( N(\cdot; \mu, \sigma^2) \) is a normal density with mean \( \mu \) and variance \( \sigma^2 \). Note that this is the first derived reward gradient for TS ever. Given the practical importance of TS, we believe that this is a major result. The additional randomness due to \( \hat{\mu}_{i,t} \), in comparison to (6), increases variance, which we observe empirically (Section 6).

\subsection{\( \epsilon \)-Greedy Policy}

The \( \epsilon \)-greedy policy [46, 9] is popular in practice because it can be combined with any generalization model. The policy pulls arm \( i \) in round \( t \) with probability \( p_{i,t} = (1-\epsilon)1\{I_{\max,t} = i\} + \epsilon K^{-1} \), where \( \epsilon \in [0, 1] \) is the exploration rate. Thus \( p_{i,t} \) can be easily differentiated with respect to \( \epsilon \),
\[
\nabla_\epsilon \log p_{i,t} = p_{i,t}^{-1} \nabla \epsilon p_{i,t} = p_{i,t}^{-1} \left(K^{-1} - 1\{I_{\max,t} = i\}\right).
\]
This policy depends only on a single parameter \( \epsilon \) and we use it as a baseline in our experiments, to show the benefits of learning the whole projection \( A \). The generalization model is the same as in (4), except that we set \( A = I_d \).

\section{Analysis of CoSoftElim}

We derive a sublinear regret bound for CoSoftElim in linear models. That is, we assume that for any arm \( i \) and context \( x \), \( f_i(x, \theta_s) = x^\top \theta_{i,s} \) for some fixed unknown \( \theta_{i,s} \in \mathbb{R}^d \). The joint parameter vector is \( \theta_s = \theta_{1,s} \oplus \cdots \oplus \theta_{K,s} \). Our result holds for any \( \theta_s \) and contexts \( X = (X_i)_{i=1}^n \). Since \( X_i \) are not random anymore, we write \( x_i \) instead of \( X_i \).

To simplify notation, we assume that \( A = I_d \). This corresponds to dropping all matrices \( A \) in (4). Let \( L = \max_{x \in [n]} \|x_i\|_2 \), \( L_\star = \|\theta_\star\|_2 \), and \( \Delta_{\max} \) be the maximum gap. Now we are ready to present our regret bound, which is proved in Appendix B.

\begin{theorem}
Let the expected \( n \)-round regret of CoSoftElim in problem instance \( P \) be \( R(n, P) = \mathbb{E} \left[ \sum_{t=1}^n Y_{i_s,t} - Y_{i_s,t} \mid P \right] \), where \( I_{s,t} \) is defined as in (2). Then for any problem instance \( P \), \( \lambda \geq L^2 \) in (4), and \( \gamma = c_1^{-2} \) in (7), we have
\[
R(n, P) \leq (12K \epsilon + 3)c_1 \sqrt{cn \log n} + (K + 1)\Delta_{\max},
\]
where \( c_1 = \tilde{O}(\sqrt{Kd}) \) (Lemma 5 in Appendix B) and \( c_2 = \tilde{O}(Kd) \) (Lemma 6 in Appendix B).
\end{theorem}

Our regret bound is \( \tilde{O}(K_2d^{1/2}\sqrt{n}) \) and has an optimal dependence on the number of rounds \( n \). As the number of features is \( Kd \), it has an extra factor of \( K \) when compared to LinUCB [1]. LinTS [5] has
an extra factor of $\sqrt{Kd}$ in our setting. So our bound is tighter whenever $K < d$, which is expected in our setting. This is because Thompson sampling adds noise in $Kd$ directions. On the other hand, CoSoftElim only samples from a softmax over $K$ arms.

**Key steps in the proof of Theorem 2:** The proof relies on equivalence of our problem with a linear bandit with $Kd$ features, which is clear from the definition of the MLE in (4). Thus we can reuse two results from the analysis of LinUCB [1], the concentration of the MLE (Lemma 5 in Appendix B) and that the sum of squared confidence widths of pulled arms is $\tilde{O}(Kd)$ (Lemma 6 in Appendix B).

The last, and most novel, part of the analysis is an upper bound on the expected regret in round $t$ by the expected confidence widths of pulled arms. This upper bound is conditioned on the history $H_t$ and relies heavily on the properties of softmax, in (5) and (7). The argument proceeds as follows. Let $\Delta_{i,t} = \max_{j \in [K]} x_i^T \theta_{j,t} - x_i^T \hat{\theta}_{i,t}$ be the gap of arm $i$ in round $t$. First, we show for any arm $i$ and “undersampled” arm $j$, an arm with a lot of uncertainty in direction $x_i$,

$$\Delta_{i,t} p_{i,t} \leq 3c_1 \sqrt{\log n} |x_i|_{G_{i,t}}^{-1} p_{i,t} + 12c_1 \sqrt{\log n} |x_i|_{G_{i,t}}^{-1} p_{i,t} + \Delta_{\max} n^{-1}. \quad (8)$$

Roughly speaking, the bound is proved as follows. If arm $i$ is “undersampled”, its gap is bounded by its confidence width; and thus term 1. If arm $i$ is “oversampled” and $\hat{\mu}_{j,t}$ is sufficiently high, arm $i$ is unlikely to be pulled; and thus term 3. In all other cases, the gap of arm $i$ can be bounded by the confidence width of arm $j$; and thus term 2. This is proved in Lemma 7 (Appendix B).

Second, we choose an appropriate “undersampled” arm $j$ to get an upper bound on the last $p_{i,t}$ in (8), in the form of $p_{j,t}$. Finally, we sum up the upper bounds over all arms $i$ and get $\mathbb{E}_t [\Delta_{i,t}] \leq (12Kc + 3)c_1 \sqrt{\log n} \mathbb{E}_t |x_i|_{G_{i,t}}^{-1} + K \Delta_{\max} \delta$. This is proved in Lemma 8 (Appendix B).

### 6 Experiments

We conduct three experiments. In Section 6.1, we demonstrate the importance of baseline subtraction. In Section 6.2, we show that CoGradBand can effectively learn subspaces with parameter vectors. In Section 6.3, we evaluate our approach on 6 multi-class classification bandit problems. The shaded areas in plots show standard errors.
6.1 Baseline Subtraction

We first show the benefits of baseline subtraction. When no subtraction is used, we view this as a special baseline $b_{\text{NONE}}(I_{1:t-1}, P, Y) = 0$. We experiment with Problem 1, a contextual Bayesian bandit with $K = 4$ arms, $d = 8$ features, and horizon $n = 200$. The context in round $t$ is sampled as $X_t \sim N(\mu_x, \Sigma_x)$, where $\mu_x = 1$ and $\Sigma_x = I_d$. The parameter vector of arm $i$ is $\theta_i, \sim N(\mu_\theta, \Sigma_\theta)$, where $\mu_\theta = 0$ and $\Sigma_\theta$ is defined in Figure 3. The reward of arm $i$ in round $t$ is $Y_{i,t} = X_t^T \theta_i, + Z_{i,t}$, where $Z_{i,t} \sim N(0, \sigma^2)$ and $\sigma = 0.5$. In this problem, $\theta_i, lie in 4 dimensions out of 8; and we expect CoSoftElim to learn policies that ignore the other 4 dimensions.

CoSoftElim (Section 4.1) and TS (Section 4.2) are parameterized by $A \in \mathbb{R}^{d \times d}$, and initialized at $A_0 = I_d$. The $\varepsilon$-greedy policy (Section 4.3) is parameterized by $\varepsilon$, initialized at $\varepsilon_0 = 0.2$. The policies are optimized by CoGradBand for $L = 100$ iterations with learning rate $\alpha = c^{-1}L^{-\frac{1}{2}}$ and batch size $m = 500$. We set $c$ automatically so that $||\hat{g}(n; \pi_0)|| \leq c$ holds with a high probability. This obviates the need for learning rate tuning in each problem. CoGradBand is implemented in TensorFlow on 112 cores and with 392 MB RAM. One iteration takes less than a second.

Our results are reported in Figure 1. We observe five trends. First, optimization of CoSoftElim and TS fails without a baseline, showing the importance of baselines. Second, the best optimized policy is CoSoftElim with $b_{\text{SELF}}$. Its regret decreases by 28%, from 55.74 to 40.01. Third, optimization of TS is noisier due to the Monte Carlo approximation in Section 4.2. This affects the quality of learned policies. Fourth, the $\varepsilon$-greedy policy performs poorly because it optimizes only a single parameter. Finally, $b_{\text{SELF}}$ is the best baseline overall. We observed the same trends in all remaining experiments and do not report them due to space constraints. We use baseline $b_{\text{SELF}}$ in the rest of the section.

6.2 Subspace Recovery

In the second experiment, we demonstrate that CoGradBand can learn effective subspace representations, those containing $\theta_i, \sim N(0, \sigma^2)$ and $\sigma = 0.5$. In Problem 4, $\theta_i, lie in 4 dimensions out of 8; and we expect CoSoftElim to learn policies that ignore the other 4 dimensions.

Our results are reported in Figure 2. The best optimized policy is CoSoftElim. Its regret always decreases by about 50%. The $\varepsilon$-greedy policy performs poorly, as its optimization is limited to a single parameter. Figure 2 also shows the learned projections $A$ in CoSoftElim at $\ell = 100$. The projections resemble $\Sigma_\theta$ in Figure 3, which indicates that we learn the correct subspace.

We also use this experiment to compare to prior approaches. First, note that TS at $\ell = 1$ (Figure 2) is standard LinTS (Section 4.2) with $A = I_d$. Both optimized TS and CoSoftElim outperform it by a large margin. This shows the importance of learning a suitable projection $A$. However, the learning objective is also critical. To show this, we use the method-of-moments (MOM) estimator for meta-learning in linear models [52] to learn projection $A$ (Appendix C) and then use it in LinTS. CoSoftElim outperforms this approach in all problems (Figure 2) because efficient exploration is not only about subspace recovery, but also about scaling $A$ relative to the rest of $A$. Finally, Cella et al. [17] recently proposed a LinUCB-like approach to meta-learning in linear bandits, where the learned parameter vector is biased towards the mean task vector. In our problems, the mean task vector is $\mu_\theta = 0$. In this case, an idealized variant of their approach reduces to LinUCB, which has regret of $116.02 \pm 0.78$, $99.01 \pm 0.69$, $134.57 \pm 0.98$, and $156.98 \pm 1.15$ in Problems 1-4. This is inferior to our approaches.

6.3 Multi-Class Classification Experiments

To show the generality of our approach, we experiment with multi-class classification bandit problems [2, 42] where arms represent labels. In round $t$, the agent observes feature vector $X_t$ and pulls an arm. The reward is one if the pulled arm is the correct label, and zero otherwise. We experiment with
| Dataset                | K  | d  | Examples |
|-----------------------|----|----|----------|
| Australian Statlog    | 2  | 14 | 690      |
| Breast Cancer         | 2  | 10 | 683      |
| Iris                  | 3  | 4  | 150      |
| Image Segmentation    | 7  | 19 | 2310     |
| Vehicle Statlog       | 4  | 18 | 846      |
| Wine                  | 3  | 13 | 178      |

Figure 4: Left. Multi-class classification bandit problems in Section 6.3. Center. The Bayes regret of CoSoftElim, TS, and ε-greedy policies on Wine problem. The regret is averaged over 20 runs. We also show the learned projection A in CoSoftElim. Right. The same for Iris problem.

6 datasets from the UCI ML Repository [7], with up to 7 classes and 19 features. The datasets are described in Figure 4. The horizon is n = 500.

The results for all problems are reported in Appendix C. In summary, we observe the same trends as in Section 6.2. Figure 4 shows the best and worst of our results. In Wine problem, the regret of CoSoftElim is reduced more than 5 fold, from 45.16 to 8.18. In Iris problem, on the other hand, the regret of CoSoftElim is reduced by mere 8%, from 72.21 to 66.27.

7 Related Work

The closest related works are Duan et al. [22] and Boutilier et al. [14], who optimized bandit policies by policy gradients. These works do not consider context. Our work generalizes Boutilier et al. [14] to include context. It is known that offline tuning of bandit algorithms reduces regret [53, 37, 29, 28]. None of these works used policy gradients or even the sequential character of n-round rewards.

Our problem is an instance of reinforcement learning (RL) [45] where the state is the history H_t of the learning agent. The main challenge is that the number of dimensions in H_t grows linearly with the number of rounds t. Therefore, in the absence of any additional structure, RL methods must deal with the curse of dimensionality. Our approach is a policy-gradient method [54, 47] with Monte-Carlo returns [12]. Baseline substraction in policy gradients [27, 40, 55, 21, 36] is used to reduce the variance of estimated gradients, which tends to blow up with the horizon. Our baselines (Section 3) differ from those in RL. For instance, b^{OPT} relies on the best arm in hindsight; and both b^{OPT} and b^{RL} use the fact that we can simulate all rewards in any problem instance P. Until recently, there were no guarantees for policy-gradient methods. Agarwal et al. [3] and Mei et al. [38] provided asymptotic and finite, respectively, guarantees for softmax policies. These results are either for small state spaces, discounted rewards, or noise-free gradients; or all of these. Therefore, none of them can be used to analyze CoGradBand.

Our approach is a form of meta-learning [49, 50], where we learn from a sample of tasks to perform well across tasks from the same distribution [10, 11]. Meta-learning showed a lot of promise in deep RL [23, 24, 39]. Sequential multitask learning [16] was studied in contextual bandits by Deshmukh et al. [20]. In contrast, our setting is offline. A general template for sequential meta-learning was presented in Ortega et al. [41]. Meta-learning in linear models was recently analyzed [15, 52]. Our algorithm design is motivated by these works and we compare to them in Section 6.2. Cella et al. [17] proposed a UCB-like algorithm for meta-learning in linear bandits. This work only considers a simple bias, the mean task vector, and does not seem competitive with learning subspaces (Section 6.2). We consider a more general setting, but do not provide guarantees on meta-learned policies.

8 Conclusions

In this work, we meta-learn contextual bandit policies by policy gradients, to perform well on average over bandit instances from some distribution P. The key ideas in our solution are bandit-specific baselines in policy gradients, which significantly reduce variance; and contextual bandit policies that are parameterized in a differentiable way and have a provably low regret. Our approach is general and works well in practice, as validated by extensive experiments.
Our work can extended in several directions. For instance, CoGradBand could be combined with more advanced optimizers, such as TRPO [44]. We focused on gradient ascent to simplify exposition. Also, based on recent analyses of policy gradients in RL [3, 38], we believe that the optimization of complex adaptive softmax policies, such as CoSoftElim, could be analyzed in the future.

References

[1] Yasin Abbasi-Yadkori, David Pal, and Csaba Szepesvari. Improved algorithms for linear stochastic bandits. In Advances in Neural Information Processing Systems 24, pages 2312–2320, 2011.

[2] Alekh Agarwal, Daniel Hsu, Satyen Kale, John Langford, Lihong Li, and Robert Schapire. Taming the monster: A fast and simple algorithm for contextual bandits. In Proceedings of the 31st International Conference on Machine Learning, pages 1638–1646, 2014.

[3] Alekh Agarwal, Sham Kakade, Jason Lee, and Gaurav Mahajan. Optimality and approximation with policy gradient methods in Markov decision processes. CoRR, abs/1908.00261, 2019. URL http://arxiv.org/abs/1908.00261.

[4] Shipra Agrawal and Navin Goyal. Analysis of Thompson sampling for the multi-armed bandit problem. In Proceeding of the 25th Annual Conference on Learning Theory, pages 39.1–39.26, 2012.

[5] Shipra Agrawal and Navin Goyal. Thompson sampling for contextual bandits with linear payoffs. In Proceedings of the 30th International Conference on Machine Learning, pages 127–135, 2013.

[6] V. M. Aleksandrov, V. I. Sysoyev, and V. V. Shemeneva. Stochastic optimization. Engineering Cybernetics, 5:11–16, 1968.

[7] A. Asuncion and D. J. Newman. UCI machine learning repository, 2007. URL http://www.ics.uci.edu/$\sim$mlearn/{MLR}epository.html.

[8] Peter Auer and Ronald Ortner. UCB revisited: Improved regret bounds for the stochastic multi-armed bandit problem. Periodica Mathematica Hungarica, 61(1-2):55–65, 2010.

[9] Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. Machine Learning, 47:235–256, 2002.

[10] Jonathan Baxter. Theoretical models of learning to learn. In Learning to Learn, pages 71–94. 1998.

[11] Jonathan Baxter. A model of inductive bias learning. Journal of Artificial Intelligence Research, 12:149–198, 2000.

[12] Jonathan Baxter and Peter Bartlett. Infinite-horizon policy-gradient estimation. Journal of Artificial Intelligence Research, 15:319–350, 2001.

[13] Donald Berry and Bert Fristedt. Bandit Problems: Sequential Allocation of Experiments. 1985.

[14] Craig Boutilier, Chih-Wei Hsu, Branislav Kveton, Martin Mladenov, Csaba Szepesvari, and Manzil Zaheer. Differentiable bandit exploration. CoRR, abs/2002.06772, 2020. URL http://arxiv.org/abs/2002.06772.

[15] Brian Bullins, Elad Hazan, Adam Kalai, and Roi Livni. Generalize across tasks: Efficient algorithms for linear representation learning. In Proceedings of the 30th International Conference on Algorithmic Learning Theory, pages 235–246, 2019.

[16] Rich Caruana. Multitask learning. Machine Learning, 28:41–75, 1997.

[17] Leonardo Cella, Alessandro Lazaric, and Massimiliano Pontil. Meta-learning with stochastic linear bandits. CoRR, abs/2005.08531, 2020. URL http://arxiv.org/abs/2005.08531.
[18] Olivier Chapelle and Lihong Li. An empirical evaluation of Thompson sampling. In Advances in Neural Information Processing Systems 24, pages 2249–2257, 2012.

[19] Varsha Dani, Thomas Hayes, and Sham Kakade. Stochastic linear optimization under bandit feedback. In Proceedings of the 21st Annual Conference on Learning Theory, pages 355–366, 2008.

[20] Aniket Anand Deshmukh, Ururun Dogan, and Clayton Scott. Multi-task learning for contextual bandits. In NIPS, pages 4848–4856, 2017.

[21] Travis Dick. Policy gradient reinforcement learning without regret. Master’s thesis, University of Alberta, 2015.

[22] Yan Duan, John Schulman, Xi Chen, Peter Bartlett, Ilya Sutskever, and Pieter Abbeel. RL²: Fast reinforcement learning via slow reinforcement learning. CoRR, abs/1611.02779, 2016. URL http://arxiv.org/abs/1611.02779.

[23] Chelsea Finn, Pieter Abbeel, and Sergey Levine. Model-agnostic meta-learning for fast adaptation of deep networks. In ICML, pages 1126–1135, 2017.

[24] Chelsea Finn, Kelvin Xu, and Sergey Levine. Probabilistic model-agnostic meta-learning. In NIPS, pages 9537–9548, 2018.

[25] John Gittins. Bandit processes and dynamic allocation indices. Journal of the Royal Statistical Society. Series B (Methodological), 41:148–177, 1979.

[26] John Gittins, Kevin Glazebrook, and Richard Weber. Multi-Armed Bandit Allocation Indices. John Wiley & Sons, 2011.

[27] Evan Greensmith, Peter Bartlett, and Jonathan Baxter. Variance reduction techniques for gradient estimates in reinforcement learning. Journal of Machine Learning Research, 5:1471–1530, 2004.

[28] Chih-Wei Hsu, Branislav Kveton, Ofer Meshi, Martin Mladenov, and Csaba Szepesvari. Empirical Bayes regret minimization. CoRR, abs/1904.02664, 2019. URL http://arxiv.org/abs/1904.02664.

[29] Volodymyr Kuleshov and Doina Precup. Algorithms for multi-armed bandit problems. CoRR, abs/1402.6028, 2014. URL http://arxiv.org/abs/1402.6028.

[30] Branislav Kveton, Csaba Szepesvari, Mohammad Ghavamzadeh, and Craig Boutilier. Perturbed-history exploration in stochastic linear bandits. In Proceedings of the 35th Conference on Uncertainty in Artificial Intelligence, 2019.

[31] Branislav Kveton, Csaba Szepesvari, Sharan Vaswani, Zheng Wen, Mohammad Ghavamzadeh, and Tor Lattimore. Garbage in, reward out: Bootstrapping exploration in multi-armed bandits. In Proceedings of the 36th International Conference on Machine Learning, pages 3601–3610, 2019.

[32] T. L. Lai and Herbert Robbins. Asymptotically efficient adaptive allocation rules. Advances in Applied Mathematics, 6(1):4–22, 1985.

[33] John Langford and Tong Zhang. The epoch-greedy algorithm for multi-armed bandits with side information. In Advances in Neural Information Processing Systems 20, pages 817–824, 2008.

[34] Tor Lattimore and Csaba Szepesvari. Bandit Algorithms. Cambridge University Press, 2019.

[35] Lihong Li, Wei Chu, John Langford, and Robert Schapire. A contextual-bandit approach to personalized news article recommendation. In Proceedings of the 19th International Conference on World Wide Web, 2010.

[36] Hao Liu, Yihao Feng, Yi Mao, Dengyong Zhou, Jian Peng, and Qiang Liu. Action-dependent control variates for policy optimization via Stein’s identity. In Proceedings of the 6th International Conference on Learning Representations, 2018.
[37] Francis Maes, Louis Wehenkel, and Damien Ernst. Meta-learning of exploration/exploitation strategies: The multi-armed bandit case. In Proceedings of the 4th International Conference on Agents and Artificial Intelligence, pages 100–115, 2012.

[38] Jincheng Mei, Chenjun Xiao, Csaba Szepesvari, and Dale Schuurmans. On the global convergence rates of softmax policy gradient methods. CoRR, abs/2005.06392, 2020. URL http://arxiv.org/abs/2005.06392.

[39] Nikhil Mishra, Mostafa Rohaninejad, Xi Chen, and Pieter Abbeel. A simple neural attentive meta-learner. In ICLR, 2018.

[40] Remi Munos. Geometric variance reduction in Markov chains: Application to value function and gradient estimation. Journal of Machine Learning Research, 7:413–427, 2006.

[41] Pedro Ortega, Jane Wang, Mark Rowland, Tim Genewein, Zeb Kurth-Nelson, Razvan Pascanu, Nicolas Heess, Joel Veness, Alexander Pritzel, Pablo Sprechmann, Siddhant Jayakumar, Tom McGrath, Kevin Miller, Mohammad Gheshlaghi Azar, Ian Osband, Neil Rabinowitz, Andras Gyorgy, Silvia Chiappa, Simon Osindero, Yee Whye Teh, Hado van Hasselt, Nando de Freitas, Matthew Botvinick, and Shane Legg. Meta-learning of sequential strategies. CoRR, abs/1905.03030, 2019. URL http://arxiv.org/abs/1905.03030.

[42] Carlos Riquelme, George Tucker, and Jasper Snoek. Deep Bayesian bandits showdown: An empirical comparison of Bayesian deep networks for Thompson sampling. In Proceedings of the 6th International Conference on Learning Representations, 2018.

[43] Daniel Russo, Benjamin Van Roy, Abbas Kazerouni, Ian Osband, and Zheng Wen. A tutorial on Thompson sampling. Foundations and Trends in Machine Learning, 11(1):1–96, 2018.

[44] John Schulman, Sergey Levine, Pieter Abbeel, Michael Jordan, and Philipp Moritz. Trust region policy optimization. In Proceedings of the 32nd International Conference on Machine Learning, pages 1889–1897, 2015.

[45] Richard Sutton. Learning to predict by the methods of temporal differences. Machine Learning, 3:9–44, 1988.

[46] Richard Sutton and Andrew Barto. Reinforcement Learning: An Introduction. MIT Press, Cambridge, MA, 1998.

[47] Richard Sutton, David McAllester, Satinder Singh, and Yishay Mansour. Policy gradient methods for reinforcement learning with function approximation. In Advances in Neural Information Processing Systems 12, pages 1057–1063, 2000.

[48] William R. Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. Biometrika, 25(3-4):285–294, 1933.

[49] Sebastian Thrun. Explanation-Based Neural Network Learning - A Lifelong Learning Approach. PhD thesis, University of Bonn, 1996.

[50] Sebastian Thrun. Lifelong learning algorithms. In Learning to Learn, pages 181–209. 1998.

[51] Y.L. Tong. The Multivariate Normal Distribution. Springer Series in Statistics. Springer New York, 2012. ISBN 9781461396550. URL https://books.google.com/books?id=FtHgBWAAQBAJ.

[52] Nilesh Tripuraneni, Chi Jin, and Michael Jordan. Provable meta-learning of linear representations. CoRR, abs/2002.11684, 2020. URL http://arxiv.org/abs/2002.11684.

[53] Joannes Vermorel and Mehryar Mohri. Multi-armed bandit algorithms and empirical evaluation. In ECML, pages 437–448, 2005.

[54] Ronald Williams. Simple statistical gradient-following algorithms for connectionist reinforcement learning. Machine Learning, 8(3-4):229–256, 1992.

[55] Tingting Zhao, Hirotaka Hachiya, Gang Niu, and Masashi Sugiyama. Analysis and improvement of policy gradient estimation. In Advances in Neural Information Processing Systems 24, pages 262–270, 2011.
A Technical Lemmas

Lemma 3. The gradient of the $n$-round Bayes reward with respect to $\pi$ is

$$\nabla_\pi r(n; \pi) = \sum_{t=1}^{n} \mathbb{E} \left[ \nabla_\pi \log p(I_t \mid H_t; \pi) \sum_{s=t}^{n} Y_{t,s} \right].$$

Proof. The $n$-round Bayes reward can be written as $r(n; \pi) = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{t=1}^{n} Y_{t,t} \mid P, Y \right] \right]$, where the outer expectation is over problem instances $P = (\theta_*, X)$ and their realized rewards $Y$, which do not depend on $\pi$. Thus

$$\nabla_\pi r(n; \pi) = \mathbb{E} \left[ \sum_{t=1}^{n} \nabla_\pi \mathbb{E} \left[ Y_{t,t} \mid P, Y \right] \right].$$

Only the pulled arm is random in the inner expectation. Therefore, for any $t \in [n]$, we have

$$\mathbb{E} \left[ Y_{t,t} \mid P, Y \right] = \sum_{i_{1:t}} \mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y) Y_{i,t}.$$

Now note that $\mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y)$ can be decomposed by the chain rule of probabilities as

$$\mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y) = \prod_{s=1}^{t} \mathbb{P} (I_s = i_s \mid I_{1:s-1} = i_{1:s-1}, P, Y). \quad (9)$$

Since the policy does use $\theta_*$, future contexts, and future rewards, we have for any $s \in [n]$ that

$$\mathbb{P} (I_s = i_s \mid I_{1:s-1} = i_{1:s-1}, P, Y) = p(i_s \mid i_{1:s-1}, X_1, \ldots, X_s, Y_1, \ldots, Y_{i-1}, \ldots, Y_{i-1,s-1}; \pi). \quad (10)$$

Finally, note that $\nabla_\pi f(\pi) = f(\pi) \nabla_\pi \log f(\pi)$ holds for any non-negative differentiable function $f$. This is known as the score-function identity \cite{6} and is the basis of policy-gradient methods. We apply it to $\mathbb{E} [Y_{t,t} \mid P, Y]$ and obtain

$$\nabla_\pi \mathbb{E} \left[ Y_{t,t} \mid P, Y \right] = \sum_{i_{1:t}} Y_{i,t} \nabla_\pi \mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y)$$

$$= \sum_{i_{1:t}} Y_{i,t} \mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y) \nabla_\pi \log \mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y)$$

$$= \sum_{t=1}^{n} \mathbb{E} \left[ Y_{t,t} \nabla_\pi \log p(I_s \mid H_s; \pi) \mid P, Y \right],$$

where the last equality is by (9) and (10). Now we chain all equalities and rearrange the result as

$$\nabla_\pi r(n; \pi) = \sum_{t=1}^{n} \sum_{s=1}^{t} \mathbb{E} \left[ Y_{t,t} \nabla_\pi \log p(I_s \mid H_s; \pi) \right] = \sum_{t=1}^{n} \mathbb{E} \left[ \nabla_\pi \log p(I_t \mid H_t; \pi) \sum_{s=t}^{n} Y_{t,s} \right].$$

This concludes the proof. \hfill $\Box$

Lemma 1. For any baseline $b = (b_t)_{t=1}^{n}$,

$$\nabla_\pi r(n; \pi) = \sum_{t=1}^{n} \mathbb{E} \left[ \nabla_\pi \log p(I_t \mid H_t; \pi) \left( \sum_{s=t}^{n} Y_{t,s} - b_t(I_{1:t-1}, P, Y) \right) \right].$$

Proof. Fix round $t$. We want to show that $b_t$ does not change the expectation in Lemma 3. That is,

$$\mathbb{E} [b_t(I_{1:t-1}, P, Y) \nabla_\pi \log p(I_t \mid H_t; \pi)] = 0.$$

We proceed as follows. Since $b_t$ does not depend on $I_t$,

$$\mathbb{E} [b_t(I_{1:t-1}, P, Y) \nabla_\pi \log p(I_t \mid H_t; \pi)] = \mathbb{E} [b_t(I_{1:t-1}, P, Y) \mathbb{E} \left[ \nabla_\pi \log p(I_t \mid H_t; \pi) \mid I_{1:t-1}, P, Y \right]].$$
Thompson sampling (Section 4.2) is

\[ \mathbb{E} [\nabla_\pi \log p(I_t \mid H_t; \pi) \mid I_{1:t-1}, P, Y] = \sum_{i=1}^{K} \mathbb{P} (I_t = i \mid I_{1:t-1}, P, Y) \nabla_\pi \log p(i \mid H_t; \pi) \]

\[ = \sum_{i=1}^{K} p(i \mid H_t; \pi) \nabla_\pi \log p(i \mid H_t; \pi) \]

\[ = \nabla_\pi \sum_{i=1}^{K} p(i \mid H_t; \pi). \]

Since \( \sum_{i=1}^{K} p(i \mid H_t; \pi) = 1 \), we have \( \nabla_\pi \sum_{i=1}^{K} p(i \mid H_t; \pi) = 0 \). This concludes the proof. \( \square \)

Lemma 4. For any baseline \( b = (b_t)_{t=1}^{n} \), the gradient of the \( n \)-round Bayes reward of contextual Thompson sampling (Section 4.2) is

\[ \nabla_\pi r(n; \pi) = \sum_{t=1}^{n} \mathbb{E} \left[ \left( \sum_{i=1}^{K} \nabla_\pi \log p_i(\tilde{\mu}_{i,t} \mid H_t; \pi) \right) \left( \sum_{s=1}^{n} Y_{I_s,s} - b_t(I_{1:t-1}, P, Y) \right) \right], \]

where \( p_i(\cdot \mid H_t; \pi) \) is the posterior distribution of arm \( i \) in round \( t \). That is, \( \tilde{\mu}_{i,t} \sim p_i(\cdot \mid H_t; \pi) \) and \( I_t = \arg \max_{i \in [K]} \tilde{\mu}_{i,t}. \)

Proof. The key idea is to rederive \( \nabla_\pi \mathbb{E} [Y_{I_t,t} \mid P, Y] \) in Lemma 3, with sampled posterior means in Thompson sampling. The remaining steps are the same as in Lemmas 1 and 3.

Let \( \tilde{\mu}_{t} = (\tilde{\mu}_{1,t}, \ldots, \tilde{\mu}_{K,t}) \) be all posterior-sampled means in round \( t \) and \( \tilde{\mu}_{1,t} \) be all posterior-sampled means in the first \( t \) rounds. Then, analogously to the chain rule in (9), we have

\[ \mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y) = \prod_{s=1}^{t} \mathbb{P} (I_s = i_s \mid I_{1:s-1} = i_{1:s-1}, P, Y) \]

\[ = \prod_{s=1}^{t} \int_{\tilde{\mu}_s} \mathbb{P} (I_s = i_s, \tilde{\mu}_s \mid I_{1:s-1} = i_{1:s-1}, P, Y) \, d\tilde{\mu}_s \]

\[ = \prod_{s=1}^{t} \int_{\tilde{\mu}_s} \mathbb{P} (I_s = i_s) \mathbb{P} (\tilde{\mu}_s \mid I_{1:s-1} = i_{1:s-1}, P, Y) \, d\tilde{\mu}_s \]

\[ = \int_{\tilde{\mu}_{1:t}} \left( \prod_{s=1}^{t} \mathbb{P} (I_s = i_s | \tilde{\mu}_s) \right) \left( \prod_{s=1}^{t} \mathbb{P} (\tilde{\mu}_s | I_{1:s-1} = i_{1:s-1}, P, Y) \right) \, d\tilde{\mu}_{1:t}. \]

The third equality holds because \( I_s \) depends only on \( \tilde{\mu}_s \).

Since \( \mathcal{I}(\tilde{\mu}_{1:t}) \) is independent of \( \pi \), we have

\[ \nabla_\pi \mathbb{P} (I_{1:t} = i_{1:t} \mid P, Y) = \nabla_\pi \int_{\tilde{\mu}_{1:t}} \mathcal{I}(\tilde{\mu}_{1:t}) \, d\tilde{\mu}_{1:t} \]

\[ = \int_{\tilde{\mu}_{1:t}} \nabla_\pi \mathcal{I}(\tilde{\mu}_{1:t}) \, d\tilde{\mu}_{1:t} \]

\[ = \int_{\tilde{\mu}_{1:t}} \mathcal{I}(\tilde{\mu}_{1:t}) \nabla_\pi \log \mathcal{M}(\tilde{\mu}_{1:t}) \, d\tilde{\mu}_{1:t} \]

\[ = \int_{\tilde{\mu}_{1:t}} \mathbb{P} (I_{1:t} = i_{1:t}, \tilde{\mu}_{1:t} \mid P, Y) \nabla_\pi \log \mathcal{M}(\tilde{\mu}_{1:t}) \, d\tilde{\mu}_{1:t}. \]

The third equality is by the score-function identity [6].
Finally, as in Lemma 3, we have

\[
\nabla_\pi \mathbb{E}[Y_{t_1:t} | P, Y] = \sum_{i_{1:t}} Y_{i_{1:t}} \nabla_\pi \mathbb{P}(I_{1:t} = i_{1:t} | P, Y)
\]

\[
= \sum_{i_{1:t}} Y_{i_{1:t}} \int \mathbb{P}(I_{1:t} = i_{1:t}, \hat{\mu}_{1:t} | P, Y) \nabla_\pi \log \mathcal{M}(\hat{\mu}_{1:t}) \, d\hat{\mu}_{1:t}
\]

\[
= \mathbb{E}[Y_{1:t}, \nabla_\pi \log \mathcal{M}(\hat{\mu}_{1:t}) | P, Y]
\]

\[
= \sum_{s=1}^{t} \mathbb{E}[Y_{1:t}, \nabla_\pi \log \mathbb{P}(\hat{\mu}_s | I_{1:s-1}, P, Y) | P, Y]
\]

\[
= \sum_{s=1}^{t} \mathbb{E}[Y_{1:t}, \sum_{i=1}^{K} \nabla_\pi \log p_i(\hat{\mu}_{i,s} | H_s; \pi) | P, Y].
\]

The last equality follows from the fact that the posterior mean of each arm in round \( s \) is sampled independently and depends only on \( H_s \). Thus

\[
\mathbb{P}(\hat{\mu}_s | I_{1:s-1}, P, Y) = \prod_{i=1}^{K} p_i(\hat{\mu}_{i,s} | H_s; \pi).
\]

This concludes the proof. \(\square\)
B Proof of Theorem 2

Our proof relies on equivalence between our problem and a linear bandit with $Kd$ features, which we discuss next. Let $u_{i,t} \in \mathbb{R}^{Kd}$ be a context vector where $x_i$ is at entries $d(i-1)+1, \ldots, di$ and all remaining entries are zeros. Let the joint parameter vector be $\theta_* = \theta_{1,*} \oplus \cdots \oplus \theta_{K,*} \in \mathbb{R}^{Kd}$ and the joint estimated vector be $\hat{\theta}_t = \hat{\theta}_{1,t} \oplus \cdots \oplus \hat{\theta}_{K,t} \in \mathbb{R}^{Kd}$. Let $M_t \in \mathbb{R}^{Kd \times Kd}$ be a block-diagonal matrix with blocks $G_{1,t}, \ldots, G_{K,t}$. Then, for any arm $i$ in round $t$,

$$u_{i,t}^\top \theta_* = x_i^\top \theta_{i,*}, \quad u_{i,t}^\top \theta_{t-1} = x_i^\top \theta_{i,t-1}, \quad \|u_{i,t}\|_{M_t^{-1}} = \|x_t\|_{G_{t-1}}^{-1}.$$ 

The equivalence is useful because it allows us to reuse two existing results from the analysis of LinUCB [1], the concentration of the MLE (Lemma 5 in Appendix B) and that the sum of squared confidence widths of pulled arms is $\tilde{O}(Kd)$ (Lemma 6 in Appendix B).

The concentration part is solved as follows. Let $E_{1,t} = \{\varnothing \in [K] : |x_i^\top \hat{\theta}_{i,t-1} - x_i^\top \theta_{i,*}| \leq c_1 \|x_t\|_{G_{t-1}}^{-1}\}$ (11) be the event that all estimated arm means in round $t$ are “close” to their actual means. Let $E_1 = \cap_{t=1}^n E_{1,t}$ and $\bar{E}_1$ be its complement. The next lemma shows how to choose $c_1$ in (11) such that event $E_2$ is unlikely.

**Lemma 5.** For any $\sigma, \lambda, \delta > 0$, and

$$c_1 = \sigma \sqrt{Kd \log \left(\frac{1 + nL^2/(Kd\lambda)}{\delta}\right)} + \lambda^{\frac{3}{2}}L_*,$$

event $E_1$ occurs with probability at least $1 - \delta$.

**Proof.** Fix arm $i$ and round $t$. By the Cauchy-Schwarz inequality,

$$u_{i,t}^\top \theta_{t-1} - u_{i,t}^\top \theta_* = u_{i,t}^\top M_t^{-\frac{1}{2}} (\hat{\theta}_{t-1} - \theta_*) \leq \|\hat{\theta}_{t-1} - \theta_*\|_{M_t^{-1}} \|u_{i,t}\|_{M_t^{-1}}.$$ 

By Theorem 2 of Abbasi-Yadkori et al. [1], $\|\hat{\theta}_{t-1} - \theta_*\|_{M_t^{-1}} \leq c_1$ holds jointly in all rounds $t \in [n]$ with probability of at least $1 - \delta$. This concludes the proof.

We also use Lemma 11 of Abbasi-Yadkori et al. [1], which bounds the sum of squared confidence widths of pulled arms.

**Lemma 6.** For any $\lambda \geq L^2$, we have

$$\sum_{i=1}^n \|x_i\|_{C_{i,t-1}}^2 \leq c_2 = 2Kd \log(1 + nL^2/(Kd\lambda)).$$

Let $\Delta_{i,t} = \max_{j \in [K]} x_i^\top \theta_{j,*} - x_i^\top \theta_{i,*}$ be the gap of arm $i$ in round $t$. Now we are ready to prove our main result.

**Theorem 2.** Let the expected $n$-round regret of CoSoftElim in problem instance $P$ be $R(n,P) = \mathbb{E} \left[\sum_{t=1}^n Y_{i,t} - Y_{i,t} \mid P\right]$, where $I_{i,t}$ is defined as in (2). Then for any problem instance $P, \lambda \geq L^2$ in (4), and $\gamma = c_1^{-2}$ in (7), we have

$$R(n,P) \leq (12Ke + 3)c_1 \sqrt{c_2 n \log n} + (K + 1)\Delta_{\max},$$

where $c_1 = \tilde{O}(\sqrt{Kd})$ (Lemma 5 in Appendix B) and $c_2 = \tilde{O}(Kd)$ (Lemma 6 in Appendix B).

**Proof.** First, we split the $n$-round regret by event $E_1$ and apply Lemma 5 with $\delta = 1/n$,

$$R(n,P) = \sum_{t=1}^n \mathbb{E}\left[\Delta_{i,t}\right] \leq \sum_{t=1}^n \mathbb{E}\left[\Delta_{i,t} I\{E_1,t\}\right] + n\Delta_{\max} \mathbb{P}(E_1) \leq \sum_{t=1}^n \mathbb{E}\left[\mathbb{E}_t\left[\Delta_{i,t}\right] I\{E_1,t\}\right] + \Delta_{\max}.$$

15
Second, we bound \( E_t [\Delta_{i,t}] I \{ E_{1,t} \} \) from above using Lemma 8 with \( \delta = 1/n \) and get

\[
R(n, P) \leq (12K e + 3)c_1 \sqrt{\log n} \mathbb{E} \left[ \sum_{t=1}^n \| x_t \|_{G_{G_i,t-1}}^{-1} \right] + (K + 1) \Delta_{\text{max}}.
\]

By the Cauchy-Schwarz inequality and Lemma 6,

\[
\sum_{t=1}^n \| x_t \|_{G_{G_i,t-1}}^{-1} \leq \sqrt{\sum_{t=1}^n \| x_t \|_{G_{G_i,t-1}}^{-2}} \leq \sqrt{c_2 n}.
\]

This concludes the proof. \( \square \)

Our key lemmas are stated and proved below. We denote the mean reward of arm \( i \) in round \( t \) by \( \mu_{i,t} = x_t^\top \theta_{i,t} \). Let \( i_{\ast,t} = \arg \max_{i \in [K]} \mu_{i,t} \) be the optimal arm in round \( t \) and \( \mu_{\ast,t} \) be its mean reward. The key concepts in our analysis are undersampled and oversampled arms. We say that arm \( i \) is undersampled in round \( t \) when \( c_1 \| x_t \|_{G_{i,t-1}}^{-1} \geq \frac{\Delta_{i,t}}{3 \sqrt{\log(1/\delta)}} \). Otherwise the arm is oversampled.

When \( \log(1/\delta) \geq 1 \), an oversampled arm \( i \) satisfies \( |\hat{\mu}_{i,t} - \mu_{i,t}| < \Delta_{i,t}/3 \) on event \( E_{1,t} \). To simplify notation, we drop subindexing by \( t \) in the proofs of the lemmas.

**Lemma 7.** Fix history \( H_t \) and assume that event \( E_{1,t} \) occurs. Let \( \gamma = c_1^{-2} \) and \( \delta \in (0, 1] \) be chosen such that \( \log(1/\delta) \geq 1 \). Then for any arm \( i \) and undersampled arm \( j \),

\[
\Delta_{i,t} \leq 3c_1 \sqrt{\log(1/\delta)} \| x_t \|_{G_{i,t-1}}^{-1} \Delta_{\text{max}} + 12c_1 \sqrt{\log(1/\delta)} \| x_t \|_{G_{j,t-1}}^{-1} \Delta_{\text{max}} \Delta_j.
\]

**Proof.** We consider four cases. Case 1 is that arm \( i \) is undersampled. Then trivially

\[
\Delta_i \leq 3c_1 \sqrt{\log(1/\delta)} \| x \|_{G_{i,t-1}}^{-1}.
\]

Case 2 is that arm \( i \) is oversampled and the gap of arm \( j \) is “large”, \( \Delta_j \geq \Delta_i/4 \). Then

\[
\Delta_i \leq 4 \Delta_j \leq 12c_1 \sqrt{\log(1/\delta)} \| x \|_{G_{j,t-1}}^{-1}.
\]

Case 3 is that arm \( i \) is oversampled; the gap of arm \( j \) is “small”, \( \Delta_j < \Delta_i/4 \); and \( \hat{\mu}_j \geq \mu_{\ast} - \Delta_i/3 \). In this case, arm \( i \) is unlikely to be pulled for \( \gamma = c_1^{-2} \).

\[
p_t \leq \exp \left[ -\gamma \frac{(\mu_{\ast,t} - \hat{\mu}_j)^2}{\| x \|_{G_{j,t-1}}^{-2}} \right] \leq \exp \left[ -\gamma \frac{(\hat{\mu}_j - \hat{\mu}_i)^2}{\| x \|_{G_{i,t-1}}^{-2}} \right] \leq \exp \left[ -\gamma \frac{\Delta_j^2}{9} \frac{9c_1^2 \log(1/\delta)}{\Delta_i^2} \right] = \delta.
\]

The first inequality holds because the denominator in \( p_t \) is at least 1. The second inequality follows from \( \hat{\mu}_{\ast,t} - \hat{\mu}_j \geq \hat{\mu}_j - \hat{\mu}_i \), which holds from our assumption on \( \hat{\mu}_j \) and that arm \( i \) is oversampled. The last inequality follows from \( \hat{\mu}_j - \hat{\mu}_i \geq \Delta_j/3 \) and that arm \( i \) is oversampled. Since \( \Delta_i \leq \Delta_{\text{max}} \), we have that \( \Delta_i p_t \leq \Delta_{\text{max}} \Delta_j \).

Case 4 is that arm \( i \) is oversampled, \( \Delta_j < \Delta_i/4 \), and \( \hat{\mu}_j < \mu_{\ast} - \Delta_i/3 \). Then

\[
\Delta_i \leq 3(\mu_{\ast} - \hat{\mu}_j) = 3(\mu_{\ast} - \mu_j + \mu_j - \hat{\mu}_j) \leq 3(\Delta_j + c_1 \| x \|_{G_{j,t-1}}^{-1}) \leq 12c_1 \sqrt{\log(1/\delta)} \| x \|_{G_{j,t-1}}^{-1}.
\]

Now we combine all four cases and get our claim. \( \square \)

The above lemma is critical to prove Lemma 8 below, which bounds the expected confidence widths of pulled arms.

**Lemma 8.** Fix history \( H_t \) and assume that event \( E_{1,t} \) occurs. Let \( \gamma = c_1^{-2} \) and \( \delta \in (0, 1] \) be chosen such that \( \log(1/\delta) \geq 1 \). Then

\[
E_t [\Delta_{i,t}] \leq (12K e + 3)c_1 \sqrt{\log(1/\delta)} \mathbb{E}_t \left[ \| x_t \|_{G_{G_i,t-1}}^{-1} \right] + K \Delta_{\text{max}} \Delta_i.
\]

16
Proof. The proof has two parts. First, we bound $p_i$ from above using pulled undersampled arms. We consider two cases. Case 1 is that the best empirical arm $i_{\text{max}} = \arg\max_{i \in [K]} \hat{\mu}_i$ is undersampled. Since $i_{\text{max}}$ has the highest empirical mean, $p_{i_{\text{max}}} \geq 1/K$ and we have $p_i \leq K p_i p_j$ for $j = i_{\text{max}}$.

Case 2 is that arm $i_{\text{max}}$ is oversampled. Because of that, $\hat{\mu}_{i_{\text{max}}} \leq \mu^*$. Since the optimal arm $i^*$ is undersampled by definition, we have for $j = i^*$ that

$$p_i = \frac{1}{p_j} p_i p_j \leq K \exp \left[ \frac{\gamma (\hat{\mu}_{i_{\text{max}}} - \hat{\mu}_j)^2}{\|x\|^{2}_{G_j^{-1}}} \right] p_i p_j \leq K \exp \left[ \frac{\gamma (\mu_j - \hat{\mu}_j)^2}{\|x\|^{2}_{G_j^{-1}}} \right] p_i p_j \leq K e p_i p_j.$$ 

By Lemma 7 and from above, there exists an undersampled arm $j$ such that for any arm $i$,

$$\Delta_i p_i \leq 3 c_1 \sqrt{\log(1/\delta)} \|x\|^{-1}_{G_i} p_i + 12 K c_1 \sqrt{\log(1/\delta)} \|x\|^{-1}_{G_j} p_i p_j + \Delta_{i_{\text{max}}}/\delta.$$ 

Finally, we sum over all arms $i$ and get

$$\mathbb{E}_t [\Delta_{i_t}] = \sum_{i=1}^{K} \Delta_i p_i \leq 3 c_1 \sqrt{\log(1/\delta)} \left( \sum_{i=1}^{K} \|x\|^{-1}_{G_i} p_i \right) + 12 K e c_1 \sqrt{\log(1/\delta)} \|x\|^{-1}_{G_j} \sum_{i=1}^{K} p_i p_j + K \Delta_{i_{\text{max}}}/\delta$$

$$\leq 3 c_1 \sqrt{\log(1/\delta)} \left( \sum_{i=1}^{K} \|x\|^{-1}_{G_i} p_i \right) + 12 K e c_1 \sqrt{\log(1/\delta)} \|x\|^{-1}_{G_j} p_j + K \Delta_{i_{\text{max}}}/\delta$$

$$\leq (12 K e + 3) c_1 \sqrt{\log(1/\delta)} \left( \sum_{i=1}^{K} \|x\|^{-1}_{G_i} p_i \right) + K \Delta_{i_{\text{max}}}/\delta$$

$$= (12 K e + 3) c_1 \sqrt{\log(1/\delta)} \mathbb{E}_t \left[ \|x\|^{-1}_{G_{i_t}} \right] + K \Delta_{i_{\text{max}}}/\delta.$$ 

This concludes the proof. \qed
Figure 5: Projections $A$ estimated by MOM.

Figure 6: The Bayes regret of CoSoftElim, TS, and $\varepsilon$-greedy policies on all UCI ML repository problems in Section 6.3. The regret is averaged over 20 runs. We also show the learned projection $A$ in CoSoftElim.

C Supplementary Experiments

C.1 Subspace Recovery

We use the method-of-moments (MOM) estimator for meta learning in linear models (Algorithm 1 of Tripuraneni et al. [52]) to learn projection $A$. In particular, we generate $n = 100 000$ i.i.d. pairs $(X_t, Y_t)$ as

$$X_t \sim \mathcal{N} (\mu_x, \Sigma_x), \quad \theta_* \sim \mathcal{N} (\mu_\theta, \Sigma_\theta), \quad Z_t \sim \mathcal{N} (0, \sigma^2), \quad Y_t = X_t^\top \theta_* + Z_t;$$

and then estimate the subspace by applying PCA to $\sum_{t=1}^n Y_t^2 X_t X_t^\top$. The learned projections $A$, together with the dimensionality of subspace $r$, are reported in Figure 5. We hand-tuned $r$ to get good empirical performance on our bandit problems in Section 6.2.

C.2 Real-World Experiments

Results for all UCI ML Repository datasets in Section 6.3 are reported in Figure 6. We observe significant improvements due to optimizing CoSoftElim. In particular, the regret decreases as

- Australian Statlog: From $82.58 \pm 0.11$ to $69.71 \pm 0.42$, by 16%.
- Breast Cancer: From $29.08 \pm 0.05$ to $15.10 \pm 0.57$, by 48%.
- Iris: From $72.21 \pm 0.09$ to $66.27 \pm 0.16$, by 8%.
- Image Segmentation: From $197.82 \pm 0.19$ to $150.53 \pm 1.81$, by 24%.
- Vehicle Statlog: From $218.16 \pm 0.22$ to $149.75 \pm 0.38$, by 31%.
- Wine: From $45.16 \pm 0.05$ to $8.18 \pm 0.24$, by 82%.
Algorithm 2 Randomized contextual explore-then-commit policy.

1: **Inputs:** Continuous exploration horizon \( h \)

2: \( \hat{h} \leftarrow \lfloor h \rfloor + Z \), where \( Z \sim \text{Ber}(h - \lfloor h \rfloor) \) \>
\begin{align*}
\triangleright \text{Randomized horizon rounding}
\end{align*}

3: \( \forall i \in [2], j \in [L] : \mu_{i,j} \leftarrow 0 \) \>
\begin{align*}
\triangleright \text{Initialize estimated mean rewards of all arms}
\end{align*}

4: for \( t = 1, \ldots, n \) do

5: \[ j \leftarrow X_t \]

6: \[ s \leftarrow \sum_{t=1}^{l-1} \mathbb{1}\{X_t = j\} \]
\begin{align*}
\triangleright \text{Number of past observations in context } j
\end{align*}

7: if \( s \leq 2h \) then \>
\begin{align*}
\triangleright \text{Explore}
\end{align*}

8: if \( s \) is even then

9: Pull arm 1 and observe its reward \( Y_{1,t} \)

10: \[ \hat{\mu}_{1,j} \leftarrow (\hat{\mu}_{1,j}N + Y_{1,t})/(N + 1) \text{ where } N \leftarrow s/2 \]

11: else

12: Pull arm 2 and observe its reward \( Y_{2,t} \)

13: \[ \hat{\mu}_{2,j} \leftarrow (\hat{\mu}_{2,j}N + Y_{2,t})/(N + 1) \text{ where } N \leftarrow (s-1)/2 \]

14: else \>
\begin{align*}
\triangleright \text{Exploit}
\end{align*}

15: if \( \hat{\mu}_{1,j} > \hat{\mu}_{2,j} \) then Pull arm 1 else Pull arm 2

---

D On Concave Bayes Reward in Contextual Bandits

In this section, we derive a policy and prior distribution \( \mathcal{P} \) pair such that \( r(n; h) \) is concave in the parameter of that policy \( h \). The result is that CoGradBand can find the optimal value of \( h \), with the same convergence guarantees as gradient descent for convex functions. To the best of our knowledge, this is the first such result for optimizing a contextual bandit policy.

The number of arms is \( K = 2 \), the number of contexts is \( L \), and the number of rounds \( n \) is a multiple of \( L \). The last assumption is to simplify exposition only. The key step in our argument is to show that the expected \( n \)-round reward is concave in \( h \) for carefully-chosen problem instances \( P \). Then \( r(n; h) \) is concave for any distribution over \( P \).

We define the problem instance \( P = (\theta_* , X) \) next. The model parameters are

\[
\theta_* = (\mu_{1,1}, \mu_{1,2}) \oplus \cdots \oplus (\mu_{1,L}, \mu_{2,L}),
\]

where \( \mu_{i,j} \) is the expected reward of arm \( i \) in context \( j \). Without loss of generality, we assume that \( \mu_{1, j} \geq \mu_{2, j} \) for any \( j \in [L] \). The context in round \( t \) is \( X_t \in [L] \) and \( (X_t)_{t=1}^{n} \) is fixed. To simplify exposition, we assume that contexts are equally frequent, that is \( \sum_{t=1}^{n} \mathbb{1}\{X_t = j\} = n/L \) for any \( j \in [L] \). Note that we do not make any assumption on the order of \( X_t \). The realized reward of arm \( i \) in round \( t \) is \( Y_{i,t} \sim \mathcal{N}(\mu_{i,X_t}, \sigma^2) \) for some \( \sigma > 0 \).

The policy is a contextual variant of a randomized explore-then-commit policy [14] and we show it in Algorithm 2. It is parameterized by a real-valued exploration horizon \( h \), which is randomly rounded to the nearest integer \( \hat{h} \). In each context, each arm is explored \( h \) times. After that, the policy commits to the arm with the highest empirical mean in that context.

Let \( r(n; P; h) \) be the expected \( n \)-round reward of Algorithm 2 in instance \( P \) and \( r_j(n; P; h) \) be its portion in context \( j \). The problem in context \( j \) is a non-contextual bandit with horizon \( n/L \), which was studied by Boutiller et al. [14] in Appendix A. For any integer horizon \( h \in [\lfloor n/(2L) \rfloor] \), they showed that

\[
r_j(n; P; h) = \mu_{1,j}n - \Delta_j [h + \mathbb{P}(\hat{\mu}_{1,j} < \hat{\mu}_{2,j}) (n/L - 2h)]
\]

where \( \Delta_j = \mu_{1,j} - \mu_{2,j} \) and \( \hat{\mu}_{i,j} \) is the estimated mean reward of arm \( i \) in context \( j \) after exploring context \( j \). The above function is concave in \( h \). Since \( r(n, P; h) = \sum_{j=1}^{L} r_j(n, P; h) \), \( r(n, P; h) \) is concave in \( h \); and so is \( r(n; h) = \mathbb{E}[r(n, P; h)] \), for any prior \( \mathcal{P} \) over our problem instances \( P \).

Finally, note that the randomized horizon rounding in Algorithm 2 does not break anything. For any continuous \( h \in [1, \lfloor n/(2L) \rfloor] \), the Bayes reward is

\[
([h] - h) r(n; [h]) + (h - [h]) r(n; [h]),
\]

and is concave in \( h \).