Internal Exact Controllability of the Heat Equation in Finite Dimension by Strategic Actuators Zone

Mame Libasse Laye Ane\textsuperscript{1}, Cheikh Seck\textsuperscript{2,3*}, Abdoulaye Sène\textsuperscript{1} and Mary Teuw Niane\textsuperscript{3}

\textsuperscript{1}Département de Mathématiques et d’Informatique de la Faculté des Sciences et Techniques de l’Université Cheikh Anta Diop de Dakar, Sénégal.

\textsuperscript{2}Département de Mathématiques de la FASTEF ex ENS de l’Université Cheikh Anta Diop de Dakar, Sénégal.

\textsuperscript{3}Laboratoire d’Analyse Numérique et d’Informatique (LANI) de l’UFR SAT, BP 234, Université Gaston Berger, Saint-Louis, Sénégal.

Authors’ contributions
This work was carried out in collaboration among all the authors. Author MTN designed the study and wrote the first draft of the manuscript. The mathematical analysis of the problem was performed by authors AS and CS. Furthermore authors CS and MLLA managed the structural consistency and the english language. While authors CS and MLLA controlled the calculations and the formulas analysis. The three authors managed the literature review, read and approved the final manuscript.

Article Information
DOI: 10.9734/JAMCS/2021/v36i130332

Editor(s):
(1) Dr. Leo Willyanto Santoso, Petra Christian University, Indonesia.

Reviewers:
(1) A’Qilah Ahmad Dahalan, National Defence University of Malaysia, Malaysia.
(2) Hassan Kamil Jassim, University of Thi-Qar., Iraq.

Complete Peer review History: http://www.sdiarticle4.com/review-history/65548

Received: 05 December 2020
Accepted: 10 February 2021
Published: 11 March 2021

Abstract
In this paper, we show, by a new approach called SCD (Surjectivity by compactness and density), the internal exact controllability of the 1D heat equation by the use of strategic zone actuators. For the achievement of this objective, an operator (linear, continuous and surjective) built directly from the system allowed us to establish the exact result of controllability.

Keywords: Control; controllability; strategic actuators; estimations.

*Corresponding author: E-mail: cheikh5.seck@ucad.edu.sn;
1 Motivations and Statement of Problem

The main mathematical models, used in the most diverse applications, are written in the form of partial differential equations. An equation with partial derivatives (PDE) is therefore a relation between a function of several variables and its derivatives. Note that physical models can lead to solutions that are not derivable in the classical sense. So the theory of control and the exact controllability of distributed systems has in recent years a net renewed interest; thanks to the development by J.L Lions [1] of the HUM method (Hilbert Uniqueness Methods). It is essentially based on the uniqueness of properties suitable for the homogeneous equation by a particular choice of controls, building a Hilbert space and a continuous linear mapping of this space into its dual hilbert which is, in fact, an isomorphism which establishes the exact controllability.

For hyperbolic problems this method has yielded important results Lions [1, 2], Seck [3], Lebeau [4], Fursikov [5],...; Although when the checks are supporting small Niane [6], Jai [7], it seems little operative, as well as for technical reasons, the multiplier method does not lead to.

So for parabolic equations any result (to our knowledge) has been prepared by this method. Also, the harmonic method is ineffective for such equations.

This work is the logical sequence of the publications (see Seck [8]) where a new method has been developped to solves some of these problems types. This method is based on criteria of surjectivity of a linear continuous operator a Hilbert space in another builds directly from the exact controllability problem.

The criteria are of two types:

1. A density criterion which is a consequence of the uniqueness properties (J. L. Lions [1, 2]);
2. A compactness criterion which follows from the parabolic nature of the operator or the regularity of control;

In both cases these criteria are easier to verify than HUM method of Lions.

This method which we call exact controllability by surjective, compactness and density opens up broad prospects for the theory of the exact controllability. Also allows to for the equations parabolic, of Schrödinger, plates, Navier Stokes linearized to solve many issues and so opening up many perspectives.

2 Notations and Some Functional Analysis Reminders

2.1 Functional analysis reminders

Let $\Omega$ be a regular nonempty bounded domain of $\mathbb{R}^N$ of border $\Gamma$, of outer unit normal $\nu(\sigma)$ at point $\sigma \in \Gamma$.

We denote $A$ the unbounded operator of $L^2(\Omega)$ defined by

$$D(A) = \{ u \in H^1_0(\Omega) / - \triangle u \in L^2(\Omega) \}$$

$$Au = -\triangle u, \forall u \in D(A)$$

The operator $A$ has a Hilbert base of eigenfunctions $(w_k)_{k \geq 1}$ such that the sequence of associated eigenvalues $(\lambda_k)_{k \geq 1}$ is decreasing (see Jai [9]).

Let $T > 0$, we consider the following Hilberts spaces and their respective dual.
Let $\alpha \in \mathbb{R}, \delta \in \{1, 0, -1\}$.

$$F_T^{\alpha, \delta} = \left\{ u = \sum_{k=1}^{+\infty} u_k w_k / \sum_{k=1}^{+\infty} \lambda_k u_k^2 2\lambda_k T^\delta < +\infty \right\} \quad (2.3)$$

We equip $F_T^{\alpha, \delta}$ with the natural scalar product

$$(x, y)_{F_T^{\alpha, \delta}} = \sum_{k=1}^{+\infty} x_k y_k \quad (2.4)$$

and the associated norm is $||\cdot||_{F_T^{\alpha, \delta}}$.

The dual of $F_T^{\alpha, \delta}$ is $F_T^{-\alpha,-\delta}$. If $x \in F_T^{\alpha, \delta}$ and $y \in F_T^{-\alpha,-\delta}$ we have:

$$(y, x)_{F_T^{\alpha, \delta} \times F_T^{-\alpha,-\delta}} = \sum_{k=1}^{+\infty} x_k y_k \quad (2.5)$$

**Remark 2.1.** We can notice that:

i. $F_T^{0,0} = L^2(\Omega)$,

ii. $F_T^{1,0} = H^1_0(\Omega)$,

iii. $F_T^{-1,0} = H^{-1}(\Omega)$

### 2.2 Case of heat equation 1-D

Let $I = \Omega = [0, \pi]$ an open interval of $\mathbb{R}$ and the 1D heat equation defined as follows:

$$\begin{cases}
  y'(t, x) - \Delta y(t, x) = \beta(t)\mu(t, x) \text{ in } [0, T] \times I \\
  \gamma y = 0 \text{ in } [0, T] \times \Gamma \\
  y(0, x) = y_0(x) \text{ in } I
\end{cases} \quad (2.6)$$

We set the operator $A$ defined by

$$D(A) = \left\{ u \in H^1_0(I) / \frac{d^2 u}{dx^2} \in L^2(I) \right\}; \forall u \in D(A), \quad A u = \frac{d^2 u}{dx^2}$$

According to the spectral theory Brezis [10], Hörmander [11], $A$ admits a Hilbert base of $L^2(I)$ of eigenfunctions $(w_k)_{k \geq 1}$ whose associated eigenvalues are $(\lambda_k)_{k \geq 1}$ rows in the decreasing direction where

$$\begin{cases}
  w_k(x) = \sqrt{\frac{2}{k^2}} \sin(kx) \\
  \lambda_k = \frac{k^2}{2}
\end{cases} \quad \text{and}$$

See also Lebeau [7] and Fursikov [6].

**Definition 2.1.** An integrable square function $\mu : I \subset \Omega \rightarrow \mathbb{R}$ is called strategic (see El. Jai[9]) if it satisfies,

for all $\phi_0 \in L^2(I)$, the solution $\phi^+$ of heat equation

$$\begin{cases}
  \phi^+(t, x) - \Delta \phi^+(t, x) = 0 \text{ in } Q_T = [0, +\infty[ \times I \\
  \gamma \phi^+(t, x) = 0 \text{ in } \Sigma_T = [0, +\infty[ \times \partial I \\
  \phi^+(0) = \phi_0 \text{ in } I
\end{cases} \quad (2.7)$$
satisfies:
\[
\forall \ t > 0, \int_I \mu(t, x) \phi^+(t, x) dx = 0 \quad \text{then} \quad \phi_0 = 0. \tag{2.8}
\]

**Remark 2.2.** 1. In 1 dimension (see Seck [8] and Ane [12]), it has been proved that there are internal strategic actuators, this is the case of certain particular geometries.

2. Similarly, the Exact controllability border of the heat equation in 1D by strategic actuators area by the SCD Method (surjectivity by compactness and density) was done see Seck [8] and Jai [7].

### 3 Internal Temporally Strategic Actuator

**Definition 3.1.** An integrable square function

\[
\mu: \left( \mathbb{R}_+^* \times I \to \bar{\mathbb{R}} \right), \quad (t, x) \mapsto \mu(t, x)
\]

is called temporally strategic if for all \( \phi_0 \in L^2(I) \) such that

\[
\forall t \in [0, +\infty], \quad \int_I \mu(t, x) \phi^+(t, x) dx = 0 \tag{3.1}
\]

then \( \phi_0 = 0. \)

**Remark 3.1.** If the application

\[
t \mapsto \int_I \mu(t, x) \phi^+(t, x) dx \tag{3.2}
\]

is analytic, it is enough that the relation is true for all \( t \in ]0, T[ \) so that it is true for all \( t \in \mathbb{R}_+^* \).

An example of a temporally strategic actuator is given:

**Theorem 3.1.** If \( \phi_0 \in L^2(I) \) and verify: \( \forall k \in \mathbb{N}^*, \ \phi_{0k} \neq 0 \) for all open non-empty \( O \) content in \( I \) then \( \mu(t, x) = \chi_{O} \phi^+(t, x) \) is a temporally strategic actuator.

**Proof.** We have:

\[
\mu_k(t, x) = \int_I \chi_{O} \phi^+(t, x) w_k(x) dx = \int_O \phi^+(t, x) w_k(x) dx = \sum_{h=1}^{+\infty} \phi_{0h} e^{-\lambda_h t} \int_O w_h w_k dx
\]

Assume \( \alpha_{kh} = \int_O w_h w_k dx \) we obtain

\[
\mu_k(t, x) = \sum_{h=1}^{+\infty} \phi_{0h} \alpha_{kh} e^{-\lambda_h t}
\]

Let \( \zeta_0 \in L^2(I) \) such that:

\[
\forall t > 0, \quad \int_I \mu(t, x) \zeta^+(t, x) dx = 0 \quad \text{then} \quad \sum_{h=1}^{+\infty} \zeta_{0h} \mu_k(t, x) e^{-\lambda_h t} = 0 \tag{3.3}
\]
let
\[ \forall t > 0, \quad \sum_{k=1}^{\infty} \sum_{h=1}^{\infty} \zeta_{0k} \phi_{0h}(t) \alpha_{kh} e^{-(\lambda_k + \lambda_h)t} = 0 \] (3.4)
so \( \zeta_{0k} \phi_{0h}(t) \alpha_{kh} = 0 \) \( \forall h, k \).

As \( \phi_{0h} \neq 0 \) then \( \forall h, \forall k \) \( \zeta_{0k} \alpha_{kh} = 0 \).

Suppose it exists \( k_0 \in \mathbb{N}^* \) such that \( \forall h \in \mathbb{N}^*, \alpha_{kh_0} = 0 \).

Also
\[ \forall h \in \mathbb{N}^*, \quad \int_I \chi_{\omega_{k_0}}(x) w_h(x) dx = 0 \] (3.5)
so \( \chi_{\omega_{k_0}} \) is identically null.

Or \( w_h \) is a non-zero eigen function, this is impossible \( \Box \)

**Corollary 3.2.** For any open non-empty contained in \( I \), there is a temporarily strategic \( \mu \) actuator whose support contained in \( \mathbb{R}_+ \times \mathcal{O} \).

**Proof.** Just take \( \omega \subseteq \mathcal{O}, \phi_0 \in L^2(I) \) such that \( \forall k \in \mathbb{N}, \phi_{0k} \neq 0 \) and, to pose \( \mu(t, x) = \chi_{\omega_{k_0}}(x) \phi^+(t, x) \)
\( \Box \)

4 Main Result: Exact Internal Controllability of the Heat Equation

4.1 Some functional analysis notes

It is recalled that the exact controllability is reflected here by this definition:

**Definition 4.1.** (reminder) The \( L \) operator defined under (4.3). The system defined by (4.6) will be say **exactly controllable on time** \( T \) if and only if \( L \) is surjective (see Ramdani [13, 14]).

We define the operator \( \phi_{\tau} \), which to a command \( v \in L^2(0, \tau; \mathcal{O}) \subset L^1(0, \tau; \mathcal{O}) \) associates the solution \( z \) solution of the system (4.6) at the moment \( \tau \):
\[ \phi_{\tau} : L^2(0, \tau; \mathcal{O}) \rightarrow X \]
\[ v \mapsto z(\tau) = \int_0^\tau e^{(\tau-s)A} \beta v(s) ds \] (4.1)

it’s clear that \( \phi_{\tau} \in \mathcal{L}(L^2(0, \tau; \mathcal{O}), X) \), since there is a constant \( k_{\tau} > 0 \) dependent only of \( \tau \), of \( A \) and of \( \beta \) such that
\[ ||\phi_{\tau}(v)|| \leq k_{\tau} ||v||_{L^2(0, \tau; \mathcal{O})}, \quad \forall v \in L^2(0, \tau; \mathcal{O}) \] (4.2)

Moreover, the adjunct linear operator \( \phi^*_{\tau} \) of \( \phi_{\tau} \) is given by the following proposition

**Proposition 4.1.** Let \( \phi_{\tau} \in \mathcal{L}(L^2(0, \tau; \mathcal{O})) \) the operator defined by system (4.6). His adjunct \( \phi^*_{\tau} \in \mathcal{L}(X, L^2(0, \tau; X)) \) is the operator
\[ \phi^*_{\tau} : X \rightarrow L^2(0, \tau; \mathcal{O}) \]
\[ z_0 \mapsto v \] (4.3)
where \( v(t) = \beta^* e^{-(\tau-t)A} z_0, \forall t \in (0, \tau) \)
Proof. Since \( \phi \) is a bounded operator, it suffices to notice that for all \( v \in L^2(0, \tau; X) \) and for all \( z_0 \in X \), we have (since \( A^* = -A \))

\[
(\phi^*, z_0) = \int_0^\tau (e^{(\tau-s)A} \beta \psi(s), z_0) ds = \int_0^\tau (\psi(s), \beta^* e^{-sA} z_0) ds = (v, w)_{(L^2(0, \tau; X))}
\]

so that we have \( w = \phi^* z_0 \).

Let the following two fundamental lemmas

**Lemma 4.1.** Let \( Y, Z \) be two Hilbert spaces and \( G \in \mathcal{L}(Y, Z) \). So, \( G \) is surjective if and only its adjunct \( G^* \) is lower bounded, i.e., there exist \( C > 0 \) such that

\[
||G^* z||_Y \geq C ||z||_Z, \quad \forall z \in Z
\]

**Proof.** It suffices to apply proposition 4.1 below by taking \( X = Z \) and as operator \( F \) the identity over \( Z \).

**Lemma 4.2.** Given three Hilbert spaces \( X, Y, Z \), let \( F \in \mathcal{L}(X, Z) \) and \( G \in \mathcal{L}(Y, Z) \). So, the following assertions are equivalent:

(i) \( \text{Im} F \subset \text{Im} G \).

(ii) There exist a constant \( C > 0 \) such that

\[
||G^* z||_Y \geq C ||F^* z||_X, \quad \forall z \in Z.
\]

(iii) There exist \( H \in \mathcal{L}(X, Y) \) such that \( F = GH \).

**Proof.** See Karim Ramdani [13].

### 4.2 Main result

**Theorem 4.3.** If \( \mu \) is a temporally strategic actuator, for all \( y_0 \in F_1^{0, -1} \), there exists \( \beta \in L^2(0, T) \) such that the solution \( y \) of the equation

\[
\begin{aligned}
\frac{dy}{dt} - \Delta y(t, x) &= \beta(t) \mu(t, x) \quad \text{in } ]0, T[ \times I \\
\gamma y &= 0 \quad \text{in } ]0, T[ \times \Gamma \\
y(0, x) &= y_0(x) \quad \text{in } I
\end{aligned}
\]

verify \( y(T) = 0 \).

**Proof. First step:**

Formal construction of the \( L \) operator?

We first give a notation: If \( f \in L^2(0, +\infty \times I) \) and for \( t \) fixed, we write \( f^+(s, t, x) \) the solution of the direct homogeneous heat equation of the initial data \( f(t, x) \). So we have:

\[
f^+(s, t, x) = \sum_{k=1}^{+\infty} f_k(t)e^{-\lambda_k s} w_k(x)
\]

where \( (f_k)_{k \geq 1} \) is given by \( f(t, x) = \sum_{k=1}^{+\infty} f_k(t)w_k \).
In the following, we will also use the functional notation \( f^+ (s, t) \).

Let \( \psi_0 \in F_0^{0,1} \), multiply the equation (4.6) by \( \psi^+ \) and integrating by parts:

\[
\int_I y(T)\psi(T)dx - \langle y_0, \psi_0 \rangle_{F_0^{0,1} \times F_0^{0,1}} = \int_0^T \int_I \mu(t, x)\psi(t, x)dxdt
\]

let

\[
\langle y_0, \psi_0 \rangle_{F_0^{0,1} \times F_0^{0,1}} = -\int_0^T \beta(t) < \mu(t, x), \psi_0 >_{F_0^{0,1} \times F_0^{0,1}} dt
\]

from where

\[
y_0 = -\int_0^T \beta(t)\mu(t, x)dt
\]

so

\[
L_1 (\beta) = -\int_0^T \beta(t)\mu(t, x)dt
\]

We are posing now: \( L(\beta) = -(L_1 (\beta))^+ (T) \) so

\[
L(\beta) = -\int_0^T \beta(t)\mu^+(T - t, x)dt \tag{4.8}
\]

Now apply the following theory to the \( F_T \) and \( F_T^\ast \) spaces by taking:

\[
F_T^\ast \xrightarrow{I_d} F_T^\ast
\]

and

\[
L^2 ([0, T]) \xrightarrow{L^1} F_T^\ast
\]

then

\[
F_T^\ast \subset \text{Im} (L) \iff \forall z^\ast \in F_T, \text{there exist } C > 0 \text{ such that } |z| \leq C.||L^\ast (z)||
\]

Thus the operator \( L^\ast \) is bounded below.

Second step.

\( L \in L (L^2 ([0, T]), L^2 (I)) \)?

Let \( \beta \in L^2 ([0, T]) \), we have

\[
\sum_{k=1}^{\infty} \left( \int_0^T \beta(t)\mu_k(t, x)e^{-\lambda_k(T-t)}dt \right)^2 \leq \sum_{k=1}^{\infty} ||\beta||_{L^2 ([0, T])}^2 \int_0^T \mu_k^2(t)e^{-2\lambda_k(T-t)}dt
\]

\[
\leq \sum_{k=1}^{\infty} ||\beta||_{L^2 ([0, T])}^2 ||\mu_k||_{L^2 ([0, T])}^2
\]

\[
\leq ||\beta||_{L^2 ([0, T])}^2 ||\mu||_{L^2 ([0, T] \times I)}^2
\]

\( L(\beta) \in L^2 (I) \) and

\[
||L(\beta)|| \leq ||\beta||_{L^2 ([0, T])} ||\mu||_{L^2 ([0, T] \times I)}^2 \tag{4.9}
\]

Third step:

\( L(L^2 ([0, T])) \) is dense in \( L^2 (I) \)?
Let $\psi_0 \in F_T$ such that $\psi(T)$ is orthogonal to $\mathcal{L}(\mathcal{L}^2([0,T]))$; let $\beta \in L^2([0,T])$, we have

$$\langle \mathcal{L}(\beta), \psi(T) \rangle_{L^2([0,T])} = 0$$

so

$$\int_0^T \beta(t) \mu^+(T-t,x) \psi(T)dxdt = 0 \implies \int_0^T \beta(t) \mu^+(T-t,x) \psi(T)dxdt = 0 \implies \int_0^T \beta(t) \mu(t,x) \psi(t)dxdt = 0$$

from where

$$\forall t \in [0,T], \quad \int_t^T \mu(t,x) \psi(t)dt = 0$$

so $\psi_0 \equiv 0$ from where $\psi_0(T) = 0$.

**Fourth step:**

$L^*L$ is compact of $L^2([0,T])$ in $L^2([0,T])$?

$$\langle L(\beta), L(\gamma) \rangle_{L^2([0,T])} = \int_0^T \int_0^T \beta(t) \mu^+(T-t,x) \gamma(s) \mu^+(T-s,x)dsdt$$

$$= \int_0^T \int_0^T \beta(s) \gamma(t) \mu^+(T-t,x)dsdt$$

$$= \int_0^T \gamma(t) \int_0^T \beta(s) \left( \int_t^T \mu^+(T-s,x)dx \right) dsdt$$

so

$$L^*L(\beta)(t) = \int_0^T \beta(s) \int_t^T \mu^+(T-s,x) \mu^+(T-t,x)dxds$$

$$= \int_0^T \beta(s) \sum_{k=1}^{+\infty} \mu_k(t,x) \mu_k(s,x) e^{-\lambda_k(T-t)} e^{-\lambda_k(T-s)}ds$$

$$= \int_0^T \beta(s) K(t,s)ds$$

where

$$K(t,s) = \sum_{k=1}^{+\infty} \mu_k(t,x) \mu_k(s,x) e^{-\lambda_k(T-t)} e^{-\lambda_k(T-s)}$$

$K$ is continuous on $[0,T] \times [0,T]$, just take $\mu(t,x)$ regular and the choice of $\phi_0$ regulates the problem.

**Step five:** Conclusion

Let $y_0 \in F_{T_0}^{0,-1}$ then it exists $\beta \in L^2([0,T])$ such that

$$y^+(T) = -\int_0^T \beta(t) \mu^+(T-t,x)dt$$
Let $\psi_0 \in L^0_T$, we have
\[
\langle y^+(T), \psi(T) \rangle_{L^2(I)} = \langle y_0, \psi_0 \rangle_{F_T^0, F_T^{0,1}}
\]
\[
= - \int_I \left( \int_0^T \mu^+(T-t, x)dt \right) \psi(T)dx
\]
\[
= - \int_0^T \int_I \mu(t, x)\psi(t)dxdt
\]

On the other hand, there is the solution of
\[
\begin{cases}
y' - \Delta y = \beta(t)\mu(t, x) \text{ in } ]0, T[ \times I \\
\gamma y = 0 \text{ in } ]0, T[ \times \Gamma} \\
y(0) = y_0 \text{ in } I
\end{cases}
(4.10)
\]
then
\[
\int_I y(T)\psi(T)dx - \langle y_0, \psi_0 \rangle_{F_T^0, F_T^{0,1}} = \int_0^T \int_I \mu(t, x)\beta^-(t)dxdt
\]
so
\[
\int_I y(T)\psi(T)dx = 0
\]
from where $y(T) = 0$.  \qed

Remark 4.1. We can choose the control as regular as we want.

**Theorem 4.4.** Let $\mathcal{O}$ an open part of $I$ not empty, for all $y_0 \in L^0_T$, it exists $\mu \in L^2(]0, T[ \times I)$
to support contained in $]0, T[ \times \mathcal{O}$ such that if $y$ is solution
\[
\begin{cases}
y' - \Delta y = \beta(t)\mu(t, x) \text{ in } ]0, T[ \times I \\
\gamma y = 0 \text{ in } ]0, T[ \times \Gamma} \\
y(0) = y_0 \text{ in } I
\end{cases}
(4.11)
\]
then $y(T) = 0$.

**Proof.** Take $\eta$ functions to support contained in $\mathcal{O}$, belonging to $D(I)$ non-zero on an open content in $\mathcal{O}$.

Let $\psi_0 \in D(A)$ such that $\forall k \in \mathbb{N}^*, \phi_{\psi_0} \neq 0$ then $\mu(t, x) = \eta\phi^+(t)$ is temporally strategic and we apply the theorem 4.3  \qed

5 Conclusion and Perspective

In this work, we established the exact controllability of the heat equation in any finite dimension by the use of strategic zone actuators.

a Thus, by a new approach called (Surjectivity by compactness and Density), the internal exact controllability of the 1D heat equation by the use of strategic zone actuators was done.
Indeed, by the construction of a compact surjective linear operator resulting from the system we have succeeded in achieving the exact internal controllability of the heat equation in any dimension by using the notion of strategic actuators.

In the near future, we would want to extend these results in non-convex domains, with cracks and with corners. Also, we would to investigate others kind linear EDP. For example explore the case of equations in evolutions of wave type, Schrödinger, plates and Navier Stokes linearized.

Acknowledgement

The authors thank the referees in advance for their comments and suggestions.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Lions JL. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2, Recherches en Mathématiques Appliquées, Research in Applied Mathematics. Perturbations, Masson, Paris. 1988;9.

[2] Lions JL. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, Recherches en Mathématiques Appliquées. Research in Applied Mathematics. Paris. 1988;8.

[3] Seck C, Sène A, Niane MT. Estimates related to the extended Spectral Control of the Wave Equation. Journal of Mathematical Research. Published by Canadian Centre of Science and Education. 2018;10(4).

[4] Lebeau G, Robbiano L. Contrôle exact de l’équation de la Chaleur. Prépublications Univ. Paris-Sud;1994.

[5] Fursikov AV. Imanuvilov O. Yu. Controllability of Evolution Equations.-Séoul National University; 1996.

[6] Niane MT. Contrôlabilité spectrale élargie des systèmes distribués par une action sur une petite partie analytique arbitraire de la frontière. C.R. Acad. Sci. Paris, t. 1989;309(1):335-340.

[7] Jai A. El. Analyse régionale des systèmes distribués. ESAIM: Control, Optimisation and Calculus of Variation. 2002;8:663-692.

[8] Seck C. Exact controllability of the heat equation by temporarily strategic actuators borders. Journal of Mathematical Research, Published by Canadian Centre of Science and Education; 2019.

[9] Jai A. El. Quelques problèmes de contrôle propres aux systèmes distribués. Annals of University of Craiova. Math. Comp. Sci. Ser. 2003;30:137-153.

[10] Brezis Ham. Analyse fonctionnelle, Théorie et applications, Masson; 1983.
[11] Hörmander Lars. Linear partial differential operators. Springer Verlag, Berlin; 1976;285:35-XX (46FXX),0404822(53#8622).

[12] Ane L, Seck C, Sène Abdoulaye. Boundary exact controllability of the heat equation in 1D by strategic actuators and a linear surjective compact operator. Applied Mathematics, Scientific Research Publishing. Applied Mathematics. 2020;11:639-649. available:https://www.scirp.org/journal/am

[13] Ramdani K, Takahashi T, Tenenbaum G, Tucsnak M. A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator. J. Funct. Anal. 2005;226:193-229.

[14] Ramdani K. Approches fréquentielle et spectrale pour le contrôle des systèmes linéaires conservatifs en dimension infinie. Journée TIPE; 2012 -2013.

© 2021 Ane et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/65548