MULTI-PARTICLE DYNAMICAL LOCALIZATION
IN A CONTINUOUS ANDERSON MODEL
WITH AN ALLOY-TYPE POTENTIAL

VICTOR CHULAEVSKY$^1$, ANNE BOUTET DE MONVEL$^2$, AND YURI SUHOV$^3$

Abstract. This paper is a complement to our earlier work [4]. With the help of the multi-scale analysis, we derive, from estimates obtained in [4], dynamical localization for a multi-particle Anderson model in a Euclidean space $\mathbb{R}^d$, $d \geq 1$, with a short-range interaction, subject to a random alloy-type potential.

1. Introduction

1.1. The model. In this paper we continue our study of a multi-particle Anderson model in $\mathbb{R}^d$ with interaction and in an external random potential of alloy type. The Hamiltonian $H = H(N)(\omega)$ is a random Schrödinger operator of the form

$$H = -\frac{1}{2}\Delta + U(x) + V(\omega; x)$$

acting in $L^2(\mathbb{R}^{Nd})$. This means that we consider a system of $N$ interacting quantum particles in $\mathbb{R}^d$. Here $x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd}$ is for the joint position vector, where each component $x_j \in \mathbb{R}^d$ represents the position of the $j$th particle, $1 \leq j \leq N$. Next, $\Delta$ stands for the Laplacian in $\mathbb{R}^{Nd}$. The interaction energy operator $U(x)$ acts as multiplication by a function $U(x)$. Finally, the term $V(\omega; x)$ represents the operator of multiplication by a function

$$x \mapsto V(x_1; \omega) + \cdots + V(x_N; \omega),$$

where $x \in \mathbb{R}^d \mapsto V(x; \omega)$ is a random external field potential assumed to be of the form

$$V(x; \omega) = \sum_{s \in \mathbb{Z}^d} V_s(\omega) \varphi(x - s).$$

Here and below $V_s$, $s \in \mathbb{Z}^d$, are i.i.d. (independent and identically distributed) real random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ is usually referred to as a “bump” function.

1.2. Basic geometric notations. Throughout this paper, we will fix an integer $N \geq 2$ and work in Euclidean spaces of the form $\mathbb{R}^{ld} \cong \mathbb{R}^d \times \ldots \times \mathbb{R}^d$ ($l$ times) associated with $l$-particle sub-systems where $1 \leq l \leq N$. Correspondingly, the notations $x$, $y$, $\ldots$ will be used for vectors from $\mathbb{R}^{ld}$, depending on the context. Given a vector $x \in \mathbb{R}^{ld}$, we will consider “sub-configurations” $x'$ and $x''$ generated by $x$ for a given partition of an $l$-particle system into disjoint sub-systems with $l'$ and $l''$ particles, where $l' + l'' = l$, $l', l'' \geq 1$; the vectors $x'$ and $x''$ are identified with points from $\mathbb{R}^{l'd}$ and $\mathbb{R}^{l''d}$, respectively, by re-labelling the particles accordingly.
Finally, we say that the interaction is non-negative. We say that this interaction has range $U$. The boundedness condition can be relaxed to include hard-core interactions where

$$
A_L(u) = \{x \in \mathbb{R}^l : |x - u| < L\}. \tag{1.4}
$$

The lattice counterpart for $A_L(u)$ is denoted by $B_L(u)$:

$$
B_L(u) = A_L(u) \cap \mathbb{Z}^l; \quad u \in \mathbb{Z}^l.
$$

Finally, we consider “cells” (cubes of radius 1) centered at lattice points $u \in \mathbb{Z}^l$:

$$
C(u) = A_1(u) \subset \mathbb{R}^l.
$$

The union of all cells $C(u)$, $u \in \mathbb{Z}^l$, covers the entire Euclidean space $\mathbb{R}^l$. For each $i \in \{1, \ldots, l\}$ we introduce the projection $\Pi_i : \mathbb{R}^l \to \mathbb{R}^d$ defined by

$$
\Pi_i : (x_1, \ldots, x_l) \mapsto x_i, \quad 1 \leq i \leq l.
$$

1.3. Interaction potential. The interaction within the system of particles is represented by the term $U(x)$ in the expression \textbf{(1.1)} of the Hamiltonian $H$. As was said, it is the operator of multiplication by a function $x \in \mathbb{R}^l \to U(x) \in \mathbb{R}$, $1 \leq l \leq N$. A usual assumption is that $U(x)$ (considered for $x \in \mathbb{R}^l$ with $1 \leq l \leq N$) is a sum of $k$-body potentials

$$
U(x) = \sum_{k=1}^{l} \sum_{1 \leq i_1 < \ldots < i_k \leq l} U^{(k)}(x_{i_1}, \ldots, x_{i_k}), \quad x = (x_1, \ldots, x_l) \in \mathbb{R}^l.
$$

In this paper we do not assume isotropy, symmetry or translation invariance of this interaction. However, we use the conditions of finite range, nonnegativity and boundedness, as stated below.

Assume a partition of a configuration $x \in \mathbb{Z}^l$ is given, into complementary sub-configurations $x_{\mathcal{J}} = (x_j)_{j \in \mathcal{J}}$ and $x_{\mathcal{J}^c} = (x_j)_{j \in \{1, \ldots, l\} \setminus \mathcal{J}}$, where $\emptyset \neq \mathcal{J} \subset \{1, 2, \ldots, l\}$. The energy of interaction between $x_{\mathcal{J}}$ and $x_{\mathcal{J}^c}$ is defined by

$$
U(x_{\mathcal{J}} | x_{\mathcal{J}^c}) := U(x) - U(x_{\mathcal{J}}) - U(x_{\mathcal{J}^c}). \tag{1.5}
$$

Next, define

$$
\rho(x_{\mathcal{J}}, x_{\mathcal{J}^c}) := \min \left[ |x_i - x_j| : i \in \mathcal{J}, j \in \mathcal{J}^c \right]. \tag{1.6}
$$

We say that this interaction has range $r_0 \in (0, \infty)$ if, for all $l = 1, \ldots, N$ and $x \in \mathbb{R}^ld$,

$$
\rho(x_{\mathcal{J}}, x_{\mathcal{J}^c}) > r_0 \implies U(x_{\mathcal{J}} | x_{\mathcal{J}^c}) = 0. \tag{1.7}
$$

Finally, we say that the interaction is non-negative and bounded if

$$
\inf_{x \in \mathbb{R}^l} U(x) \geq 0 \quad \text{and} \quad \sup_{x \in \mathbb{R}^l} U(x) < +\infty, \quad 1 \leq l \leq N. \tag{1.8}
$$

The boundedness condition can be relaxed to include hard-core interactions where $U(x) = +\infty$ if $|x_i - x_j| \leq a$, for some given $a \in (0, r_0)$. 

All Euclidean spaces will be endowed with the max-norm denoted by $| \cdot |$. We will consider $ld$-dimensional cubes of integer size in $\mathbb{R}^{ld}$ centered at lattice points $u \in \mathbb{Z}^{ld} \subset \mathbb{R}^{ld}$ and with edges parallel to the co-ordinate axes. The cube of edge length $2L$ centered at $u$ is denoted by $A_L(u)$; in the max-norm it represents the ball of radius $L$ centered at $u$: 

$$
A_L(u) = \{x \in \mathbb{R}^l : |x - u| < L\}.
$$
1.4. **Assumptions.** Our assumptions on the interaction potential $U$ are borrowed from [4]:

(E1) $U$ is non-negative, bounded and has a finite range $r_0 \geq 0$.

Similarly, we use assumptions on the i.i.d. random variables $V_s$, $s \in \mathbb{Z}^d$, and the bump function $\varphi$ introduced in [4]:

(E2) There exists a constant $v \in (0, \infty)$ such that
\[ P\{0 \leq V_0 \leq v\} = 1 \quad (1.9) \]
and
\[ \forall \epsilon > 0 \quad P\{V_0 \leq \epsilon\} > 0. \quad (1.10) \]

(E3) *Uniform Hölder continuity.* There exist constants $a, b > 0$ such that for all $\epsilon \in [0, 1]$, the common distribution function $F$ of the random variables $V_s$ satisfies
\[ \sup_{y \in \mathbb{R}} [F(y + \epsilon) - F(y)] \leq a \epsilon^b. \quad (1.11) \]

(E4) The function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is bounded, nonnegative and compactly supported:
\[ \text{diam}(\text{supp } \varphi) \leq r_1 < \infty. \quad (1.12) \]

(E5) For all $L \geq 1$ and $u \in \mathbb{Z}^d$,
\[ \sum_{s \in A_L(u) \cap \mathbb{Z}^d} \varphi(x - s) \geq 1_{A_L(u)}(x). \quad (1.13) \]

Here and below, $1_A$ stands for the indicator function of a set $A$.

Henceforth, we suppose that $d$ and $N$ are fixed, as well as the interaction $U$ and the structure of the external potential (i.e., the distribution function $F$ and the bump function $\varphi$). All constants emerging in various bounds below are introduced under this assumption.

1.5. **Dynamical localization.** The main result of this paper, Theorem 1.1, establishes the so-called “strong dynamical localization” for the operator $H(\omega)$ defined in (1.1) near the lower edge $E^0$ of its spectrum. More precisely, let $E^0$ be the lower edge of the spectrum $\text{spec}(H^0)$ of the $N$-particle operator without interaction,
\[ H^0 = -\frac{1}{2}\Delta + \sum_{j=1}^N V(x_j; \omega). \quad (1.14) \]

Actually, it follows from our conditions (1.9) and (1.10) that $E^0 = 0$. Owing to the non-negativity of the interaction potential $U$, the lower edge of the spectrum of $H$ is bounded from below by $E^0$. Moreover, $H$ has a non-empty spectrum in the interval $[E^0, E^0 + \epsilon]$, for any $\epsilon > 0$. This follows, e.g., from a result by Klopp and Zenk [8] which says that the integrated density of states for a multi-particle system with a decaying interaction is the same as for the system without interaction.

Denote by $X$ the operator of multiplication by the norm of $x$, i.e.,
\[ X f(x) = |x| f(x), \quad x \in \mathbb{R}^N. \quad (1.15) \]

The main result of this paper is the following

\[ 1^\text{The Hölder continuity can be relaxed to the log-Hölder continuity.} \]
Theorem 1.1. Consider the operator $H$ from (1.1) and assume that conditions (E1)–(E5) are fulfilled. Then for any $Q > 0$ there exists a nonrandom number $\eta = \eta(Q) > 0$ such that for any compact subset $K \subset \mathbb{R}^d$ the following bound holds:

$$\mathbb{E} \left[ \sup_{t \in \mathbb{R}} \left\| X^{Q} e^{-i t H} P_{I(\eta)}(H(\omega)) 1_{K} \right\|_{L^{2}(\mathbb{R}^d)} \right] < \infty, \quad (1.16)$$

where $P_{I(\eta)}(H)$ is the spectral projection of the Hamiltonian $H$ on the interval $I(\eta) = [E^0, E^0 + \eta]$.

Remark 1.2. The interval $I(\eta)$ is a sub-interval of the interval of energies $[E^0, E^0 + \eta^*]$ for which the spectrum of $H$ was proven to be pure point (and the eigenfunctions to be decaying exponentially); see [4].

2. Results of the multi-particle MSA

The MSA works with the finite-volume approximations $H_{\Lambda_L(u)}$ of $H$, relative to the cubes $\Lambda_L(u)$. More precisely, $H_{\Lambda_L(u)}$ is an operator in $L^2(\Lambda_L(u))$, given by the same expression as in (1.1) (for $x \in \Lambda_L(u)$), with Dirichlet’s boundary conditions on $\partial \Lambda_L(u)$; see [4]. Specifically, the Green operator $G_{\Lambda_L(u)}(E)$ is of particular interest:

$$G_{\Lambda_L(u)}(E) = (H_{\Lambda_L(u)} - E)^{-1}, \quad (2.1)$$

defined for $E \in \mathbb{R} \setminus \text{spec}(H_{\Lambda_L(u)})$.

Let $\lfloor \cdot \rfloor$ denote the integer part. For a cube $\Lambda_L(u)$ we denote

$$\Lambda_L^{\text{int}}(u) = \Lambda_{\lfloor L/3 \rfloor}(u), \quad \Lambda_L^{\text{out}}(u) = \Lambda_L(u) \setminus \Lambda_{L-2}(u). \quad (2.2)$$

Next, given two points $v, w \in B_L(u)$ such that $C(v), C(w) \subset \Lambda_L(u)$, set

$$G_{v,w}^{\Lambda_L(u)}(E) := 1_{C(v)} G_{\Lambda_L(u)}(E) 1_{C(w)}. \quad (2.3)$$

Following a long-standing tradition, we use a parameter $\alpha \in (1, 2)$ in the definition of a sequence of scales $L_k$ (cf. Equ (2.5)); For our purposes, it suffices to set $\alpha = 3/2$; this will be always assumed below.

Definition 2.1. A cube $\Lambda_L(u)$ is called $(E, m)$-non-singular ($(E, m)$-NS, in short) if for any $v \in B_{\lfloor L/\alpha \rfloor}(u)$ and $y \in \Lambda_L^{\text{out}}(u) \cap \mathbb{Z}^d$ the norm of the operator $G_{v,y}^{\Lambda_L(u)}(E)$ satisfies

$$\left\| G_{v,y}^{\Lambda_L(u)}(E) \right\|_{L^2(\Lambda_L(u))} \leq e^{-mL}. \quad (2.4)$$

Otherwise, it is called $(E, m)$-singular ($(E, m)$-S).

We will work with a sequence of “scales” $L_k$ (positive integers) defined recursively by

$$L_k := \lfloor L_{k-1}^\alpha \rfloor + 1, \quad \text{where} \quad \alpha = \frac{3}{2}. \quad (2.5)$$

The sequence $L_k$ is determined by an initial scale $L_0 \geq 2$. Most of arguments in Sect. 3 require $L_0$ to be large enough, to fulfill some specific numerical inequalities. In addition, we also assume that $L_0 \geq r_1$ (defined in (1.12)) in order to simplify some cumbersome technicalities.

We will use a well-known property of generalized eigenfunctions of the operator $H$ which can be found, e.g., in [9, Lemma 3.3.2];
Lemma 2.2. For every bounded set $I_0 \subset \mathbb{R}$ there exists a constant $C^{(0)} = C^{(0)}(I_0)$ such that, for any cube $A_L(u)$ with $L > 7$, any point $w \in B_L(u)$ with $C(w) \subseteq A_L^{int}(u)$ and every generalized eigenfunction $\Psi$ of $H$ with eigenvalue $E \in I_0$, the norm of the vector $1_{C(w)}\Psi$ satisfies

$$\|1_{C(w)}\Psi\| \leq C^{(0)}(1)_{A_L^{int}(u)} G_{A_L(u)}(E) 1_{C(w)} \| \cdot 1_{A_L^{int}(u)}\Psi\|. \quad (2.6)$$

(From now on we omit the subscript indicating the $L^2$-space where a given norm is considered, as this will be clear in the context of the argument.)

The following geometric notion is used in the forthcoming analysis.

Definition 2.3. (see [4]). Let $J$ be a non-empty subset of $\{1, \ldots, N\}$.

We say that the cube $A_L(y)$ is $J$-separable from the cube $A_L(x)$ if

$$\left(\bigcup_{j \in J} \Pi_j A_{L+1}(y)\right) \cap \left(\bigcup_{i \notin J} \Pi_i A_{L+1}(y) \bigcup \Pi A_{L+1}(x)\right) = \emptyset \quad (2.7)$$

where $\Pi A_{L+1}(x) = \bigcup_{j=1}^N \Pi_j A_{L+1}(x)$.

A pair of cubes $A_L(x), A_L(y)$ is separable if, for some $J \subseteq \{1, \ldots, N\}$, either $A_L(y)$ is $J$-separable from $A_L(x)$, or $A_L(x)$ is $J$-separable from $A_L(y)$.

We will use the following easy assertion (see [4]):

Lemma 2.4. For any $L > 1$ and $x \in \mathbb{R}^{Nd}$, there exists a collection of $N$-particle cubes $A_{2N(L+1)}(x^{(i)})$, $i = 1, \ldots, K(x,N)$, with $K(x,N) \leq N^N$, such that if a vector $y \in \mathbb{Z}^{Nd}$ satisfies

$$y \notin \bigcup_{l=1}^{K(x,N)} A_{2N(L+1)}(x^{(i)}), \quad (2.8)$$

then two cubes $A_L(x)$ and $A_L(y)$ with $\operatorname{dist}(A_L(x), A_L(y)) > 2N(L+1)$ are separable. In particular, assuming $L \geq 1$, a pair of cubes $A_L(x)$, $A_L(y)$ is separable if

$$|y| > |x| + (4N + 2)L. \quad (2.9)$$

Since $N \geq 2$, one can replace the condition (2.9) by

$$|y| > |x| + 5NL. \quad (2.10)$$

In particular, two cubes of the form $A_L(0)$, $A_L(y)$ with $|y| > 5NL$ are always separable.

The main outcome of [4] is summarized in the following Theorem 2.5.

Theorem 2.5 (see [4]). For any large enough $p > 0$ there exist $m^*(p) > 0$, $\eta^*(p) > 0$, and $L^*_0(p) > 0$ such that

(i) if $L_0 \geq L^*_0(p)$ then for all $k \geq 0$ and for any pair of separable cubes $A_{L_k}(x)$, $A_{L_k}(y)$ with $x, y \in \mathbb{Z}^{Nd}$,

$$\mathbb{P}\left(\exists \ E \in [E^0, E^0 + \eta^*] : A_{L_k}(x) \text{ and } A_{L_k}(y) \text{ are } (E, m) \text{-S}\right) \leq L_k^{-2p}, \quad (2.11)$$

(ii) with probability one, the spectrum of $H$ in the interval $I = [E^0, E^0 + \eta^*(p)]$ is pure point, and the eigenfunctions $\Phi_n$ of $H$ with eigenvalues $E_n \in I$ satisfy

$$\|\Phi_n 1_{C(w)}\| \leq C_n(\omega) e^{-m^*(p)|w|}, \quad w \in \mathbb{Z}^{Nd}, \quad C_n(\omega) < \infty. \quad (2.12)$$

The constant $r_1$ is defined in [11.12].
3. Derivation of dynamical localization from MSA estimates

In this section we prove a statement that is slightly more general than Theorem 1.1. Namely, given $Q > 0$, the interval $I = I(\eta) = [E^0_0, E^0_0 + \eta]$ with $\eta = \eta(Q)$, and a compact subset $K \subset \mathbb{R}^d$, there exists a constant $C(Q, K) \in (0, \infty)$ such that for any bounded measurable function $\xi: \mathbb{R} \to \mathbb{C}$ with $\text{supp} \xi \subset I(\eta)$,

$$\mathbb{E} \left[ \| X^Q_0 (\xi(H(\omega))) 1_K \| \right] < C(Q, K) \| \xi \|_\infty < \infty. \quad (3.1)$$

Moreover, $Q > 0$ can be made arbitrarily large, by choosing $\eta = \eta(Q) > 0$ sufficiently small. Theorem 1.1 follows from (3.1) applied to the functions $\xi(s) = e^{-it^1_1} 1_{I(\eta)}(s)$, parametrised by $t \in \mathbb{R}$.

Throughout the section, we assume that the parameter $p$ from (2.11) satisfies

$$2p > 3Nd\alpha + \alpha Q. \quad (3.2)$$

More precisely, given $Q > 0$ and $p$ satisfying (3.2), we work with $\eta = \eta(Q) \in (0, \eta^*(p))$ and $m = m^*(p) > 0$, (3.3)

where $\eta^*(p)$ and $m^*(p)$ are specified in Theorem 2.5. Further, for $p$ satisfying (3.2) we introduce the event $\Omega_1 = \Omega_{1(p)} \subseteq \Omega$ of probability $\mathbb{P}(\Omega_1) = 1$, defined by

$$\Omega_1 = \{ \omega \in \Omega : \text{the spectrum of } H(\omega) \text{ in } [E^0_0, E^0_0 + \eta^*(p)] \text{ is pure point} \}. \quad (3.4)$$

3.1. Probability of “bad samples”. Given $j \geq 1$, consider the event

$$S_j = \{ \omega : \text{there exists } E \in I \text{ and } y, z \in B_{5NL_j+1}(0) \text{ such that } \Lambda_{L_j}(y), \Lambda_{L_j}(z) \text{ are separable and } (m, E)\text{-S} \}. \quad (3.5)$$

Further, for $k \geq 1$ we denote

$$\Omega^\text{bad}_k = \bigcup_{j \geq k} S_j. \quad (3.6)$$

Lemma 3.1. There exists a constant $c_1 \in (0, \infty)$ such that for all $k \geq 1$,

$$\mathbb{P}\{ \Omega^\text{bad}_k \} \leq c_1 L_k^{-(2p-2Nd\alpha)}. \quad (3.7)$$

Proof. The number of separable pairs $\Lambda_{L_j}(x)$, $\Lambda_{L_j}(y)$ with $x, y \in B_{5NL_{j+1}}(0)$ is bounded by $(10NL_{j+1} + 1)^2 < C(N)L_{j+1}^2$. We can apply the bound (2.11) and write

$$\mathbb{P}\{ S_j \} \leq C(N)L_{j+1}^{2Nd}L_{j+1}^{-2p} \leq L_{j+1}^{-2p+2Nd\alpha}. \quad (3.8)$$

Therefore,

$$\Omega^\text{bad}_k \leq L_k^{-2p+2Nd\alpha} \sum_{i \geq 0} \left( \frac{L_{k+i}}{L_k} \right)^{-2p+2Nd\alpha}. \quad (3.9)$$

For $2p > 2Nd\alpha$ and $L_0 \geq 2$ the claim follows from the inequality

$$\frac{L_{k+i}}{L_k} \geq \left[ L_k^{\alpha^i-1} \right].$$

□
3.2. Centers of localization. Denote by \( \Phi_n = \Phi_n(\omega) \) the normalized eigenfunctions of \( H(\omega), \omega \in \Omega_1 \), with corresponding eigenvalues \( E_n = E_n(\omega) \in I \). For each \( n \) we define a center of localization for \( \Phi_n \) as a point \( \hat{x} \in \mathbb{Z}^d \) such that

\[
\| 1_{C(\hat{x})} \Phi_n \| = \max_{y \in \mathbb{Z}^d} \| 1_{C(y)} \Phi_n \| .
\]  

(3.6)

Since \( \| \Phi_n \| = 1 \), for any given \( n \) such centers exist and their number is finite. We will assume that, for any eigenfunction \( \Phi_n \), the centers of localization \( \hat{x}_{n,a}, a = 1, \ldots, \hat{C}(n) \), are enumerated in such a way that \( |\hat{x}_{n,1}| = \min_a |\hat{x}_{n,a}| \).

Lemma 3.2. There exists \( k_0 \) large enough such that, for all \( u \in \mathbb{Z}^d, \omega \in \Omega_1 \) and \( k \geq k_0 \), if \( \hat{x}_{n,a} \in B_{L_k}(u) \) then the box \( A_{L_k}(u) \) is \( (m, E_n) \)-S.

Proof. Assume otherwise. Then from (2.6) it follows that

\[
\| 1_{C(\hat{x}_{n,a})} \Phi_n \| \leq C'e^{-mL_k} \| 1_{A_{L_k}^{\text{out}}(u)} \Phi_n \| .
\]

Since the number of cells in \( A_{L_k}^{\text{out}}(u) \) is bounded by \( L_k^{Nd} \), we conclude that

\[
\| 1_{C(\hat{x}_{n,a})} \Phi_n \| \leq C'e^{-mL_k} L_k^{Nd} \max_{y \in B_{L_k}(u)} \| 1_{C(y)} \Phi_n \| .
\]

If \( k_0 \) is large enough so that \( C'e^{-mL_k} L_k^{Nd} < 1 \) for \( k \geq k_0 \), the above inequality contradicts the definition of \( \hat{x}_{n,a} \) as center of localization.

3.3. Annular regions. From now on we work with the integer \( k_0 \) from Lemma 3.2. Given \( k > k_0 \), set:

\[
\Omega_k^{\text{good}} = \Omega_1 \setminus \Omega_k^{\text{bad}} .
\]

(3.7)

Assume that \( \omega \in \Omega_k^{\text{good}} \). Let \( \hat{x}_{n,a}, \hat{x}_{n,b} \) be two centers of localization for the same eigenfunction \( \Phi_n \). It follows from the definition of the event \( \Omega_k^{\text{good}} \) that the cubes \( A_{L_k}(\hat{x}_{n,a}) \) and \( A_{L_k}(\hat{x}_{n,b}) \) with \( i \geq k - 1 \) cannot be separable, since they must be \( (m, E) \)-S. Further, by Lemma 2.4 if \( L_0 \geq r_1 \) then any cube of the form \( A_{L_k}(y) \) with \( |y| > |\hat{x}_{n,1}| + 5NL_k \) is separable from \( A_{L_k}(\hat{x}_{n,1}) \); this also applies, of course, to any localization center \( y = \hat{x}_{n,a} \) with \( a > 1 \), provided that such centers exist for a given \( n \). Since \( \omega \in \Omega_k^{\text{good}} \), for any eigenfunction \( \Phi_n \) there is no center of localization \( \hat{x}_{n,a} \) either outside the cube \( A_{|\hat{x}_{n,1}| + 5NL_k}(0) \) or inside \( A_{|\hat{x}_{n,1}|}(0) \) (since \( |\hat{x}_{n,1}| = \min_a |\hat{x}_{n,a}| \)). In other words, within the event \( \Omega_k^{\text{good}} \), all centers of localization \( \hat{x}_{n,a} \) with a fixed value of \( n \) are located in the annulus

\[
A_{|\hat{x}_{n,1}| + 5NL_k}(0) \setminus A_{|\hat{x}_{n,1}|}(0)
\]

of width \( 5NL_k \) and of inner radius \( |\hat{x}_{n,1}| \). This explains why, for our purposes, an eigenfunction \( \Phi_n \) can be effectively “labeled” by a single localization center.

In other words, although in this paper we cannot rule out the possibility of existence of multiple centers of localization at arbitrarily large distances (depending on \( \Phi_n \) through \( |\hat{x}_{n,1}| \)), such centers do not contribute to a “radial” quantum transport – away from the origin \( 0 \) – which might have lead to dynamical delocalization.

Lemma 3.3. Given \( k > k_0 \), there exists \( j_0 \) large enough such that if \( j \geq j_0 \), \( \omega \in \Omega_k^{\text{good}} \) and \( \hat{x}_{n,1} \in B_{L_j}(0) \) then

\[
\left\| \left( 1 - 1_{A_{L_{j+2}}(0)} \right) \Phi_n \right\| \leq \frac{1}{4} .
\]
Proof. By Lemma 2.2 (see also (2.10)),
\[ \forall i \geq j + 1, \forall w \in \mathbb{Z}^{Nd} \setminus B_{5NL_i}(0), \text{ the cubes } A_{L_i}(w) \text{ and } A_{L_j}(0) \text{ are separable.} \]
In addition, we take \( j \geq k \), as suggested in the lemma.

Next, we divide the complement \( \mathbb{R}^{Nd} \setminus A_{5NL_{j+2}}(0) \) into annular regions
\[ M_i(0) := A_{5NL_i+1}(0) \setminus A_{5NL_i}(0), \quad i \geq j + 2, \quad (3.8) \]
and write
\[ \left\| \left( 1 - 1_{A_{L_{j+2}}(0)} \right) \Phi_n \right\|^2 = \sum_{i \geq j+2} \left\| 1_{M_i(0)} \Phi_n \right\|^2 \leq \sum_{i \geq j+2} \sum_{w \in M_i(0)} \left\| 1_{C(w)} \Phi_n \right\|^2. \]
Furthermore, \( \hat{x}_{n,1} \in B_{L_j}(0) \subset B_{L_{j-1}}(0) \), so that by Lemma 3.2 the cube \( A_{L_j}(0) \) must be \( (m, E_n) \)-S. Therefore, by the definition of the event \( \Omega_k^{\text{good}} \), the cube \( A_{L_j}(w) \) is \( (m, E_n) \)-NS. Applying Lemma 2.2 to the cube \( A_{L_j}(w) \) and to the cell \( C(w) \), we obtain
\[ \left\| 1_{C(w)} \Phi_n \right\|^2 \leq e^{-2mL_j}. \]
Since the volume \( [M_i(0)] \) of the annular region \( M_i(0) \) grows polynomially in \( L_i \), the assertion of Lemma 3.3 follows. □

3.4. Bounds on concentration of localization centers.

Lemma 3.4. There exists a constant \( c_2 \in (0, \infty) \) such that for \( \omega \in \Omega_k^{\text{bad}}, j \geq k, \)
\[ \text{card} \{ n : \hat{x}_{n,1} \in B_{L_{j+1}}(0) \} \leq c_2 L_j^{Nd}. \quad (3.9) \]
Proof. The left-hand-side of (3.9) is nondecreasing in \( j \), so we can restrict ourselves to the case \( j \geq j_0 \). First, observe that, with \( A_{L_{j+2}} = A_{L_{j+2}}(0) \)
\[ \sum_{n: \hat{x}_{n,1} \in B_{L_{j+1}}(0)} \left( 1_{A_{L_{j+2}}} P_l 1_{A_{L_{j+2}}} \Phi_n, \Phi_n \right) \leq \text{tr} \left( 1_{A_{L_{j+2}}} P_l \right). \quad (3.10) \]
Each term in the above sum is not less than \( 1/2 \). Indeed,
\[ \left( 1_{A_{L_{j+2}}} P_l 1_{A_{L_{j+2}}} \Phi_n, \Phi_n \right) \]
\[ = \left( 1_{A_{L_{j+2}}} P_l \Phi_n, \Phi_n \right) - \left( 1_{A_{L_{j+2}}} P_l 1 - 1_{A_{L_{j+2}}} \right) \Phi_n, \Phi_n \]
\[ \geq \left( 1_{A_{L_{j+2}}} \Phi_n, \Phi_n \right) - \frac{1}{4} \quad \text{(using Lemma 3.3)} \]
\[ \geq \frac{1}{2}. \]
(3.11) (3.12)
Substituting the lower bounds from (3.11) – (3.12) under the trace in Eqn (3.10), we get the desired upper bound on the LHS of Eqn (3.9). □

3.5. Bounds for “good” samples of potential.

Lemma 3.5. There exists an integer \( k_1 = k_1(L_0) \) such that \( \forall k \geq k_1, \omega \in \Omega_k^{\text{good}} \) and \( x \) from the annular region \( M_{k+1} \) defined in (3.3),
\[ \left\| 1_{A_{L_k}(x)} P_l \xi(H) 1_{A_{L_k}(0)} \right\| \leq e^{-mL_k/2} \| \xi \|_{\infty}. \quad (3.13) \]
Proof. It suffices to prove (3.13) in the particular case where \( \| \xi \|_\infty \leq 1 \), which we assume below. First, we bound the LHS of (3.13) as follows:

\[
\| 1_{A_{L_k}}(x) \mathcal{P}_I \xi(H) 1_{A_{L_k}}(0) \| \leq \sum_{E_n \in I} |\xi(E_n)| \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \|
\]

\[
\leq \sum_{E_n \in I} \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \|
\]

since \( \| \eta \|_\infty \leq 1 \). Now divide the sum according to where \( \tilde{x}_{n,1} \) are located and write

\[
\sum_{E_n \in I} \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \| = \sum_{E_n \in I} \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \|
\]

\[
+ \sum_{j=k+1}^\infty \sum_{\tilde{x}_{n,1} \in M_j(0)} \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \|
\]

with \( M_j(0) \) defined in (3.9). Since \( x \in M_{k+1}(0) \), we have \( B_{L_k}(x) \cap B_{L_k}(0) = \emptyset \). Then, by Lemma 2.3, the two cubes \( B_{L_k}(x) \) and \( B_{L_k}(0) \) are separable. In turn, this implies that one of these cubes is \( (m, E_n) \)-NS. Therefore, by Lemma 3.4

\[
\sum_{E_n \in I} \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \| \leq c_2 C' L_k^{N_d} e^{-mL_k}.
\]

Furthermore, for \( k > k_0 \) large enough,

\[
\sum_{\tilde{x}_{n,1} \in \Lambda_{k+1}} \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \| \leq \frac{1}{2} e^{-mL_k/2}.
\]

For any \( j \geq k + 2 \) and \( \tilde{x}_{n,1} \in M_j(0) \), by Lemma 3.2, the cube \( B_{L_j}(\tilde{x}_{n,1}) \) must be \( (m, E_n) \)-S, so that \( B_{L_j}(0) \) has to be \( (m, E_n) \)-NS:

\[
\| 1_{A_{L_k}}(x) \varphi_n \| \leq \| 1_{A_{L_j}}(0) \varphi_n \| \leq C' e^{-mL_j}.
\]

Applying again Lemma 3.4 we see that, if \( k \geq k_1 \), then

\[
\sum_{j=k+2}^\infty \sum_{\tilde{x}_{n,1} \in M_j(0)} \| 1_{A_{L_k}}(x) \varphi_n \| \| 1_{A_{L_k}}(0) \varphi_n \| \leq C \sum_{j=k+2}^\infty e^{-mL_j} L_j^{N_d} \leq \frac{1}{2} e^{-mL_k/2}.
\]

Combining this estimate with (3.14) and (3.15), the assertion of Lemma 3.5 follows. \( \square \)

3.6. Bounds for “bad” samples of potential.

Lemma 3.6. Let \( k_1 \) be as in Lemma 3.5 and assume that \( k \geq k_1 \) and \( x \in M_{k+1}(0) \). We have:

\[
\mathbb{E} \left[ \| 1_{A_{L_k}}(x) \mathcal{P}_I \xi(H) 1_{A_{L_k}}(0) \| \right] \leq \| \xi \|_\infty \left( C L_k^{-2p+2N_d} + e^{-mL_k/2} \right).
\]
Proof. We again assume $\|\xi\|_{\infty} \leq 1$. For $\omega \in \Omega_k^{\text{bad}}$ we can use Sect. 3.1 while for $\omega \in \Omega_k^{\text{good}}$ we can use Sect. 3.5. More precisely, the above expectation is bounded by
\[
P\{\Omega_k^{\text{bad}}\} + e^{-mL_k/2} P\{\Omega_k^{\text{good}}\} \leq CL_k^{-2p+2Nd\alpha} + e^{-mL_k/2}.
\]
3.7. Conclusion. For a compact subset $K \subset \mathbb{R}^{Nd}$ we find an integer $k \geq k_1$ such that
\[
K \subset \Lambda_{L_k}(0).
\]
Then, with the annular regions $M_j(0)$,
\[
\mathbb{E}\left[\|X^Q P_I \xi(H(\omega)) 1_K\|\right] \leq L_k^Q + \sum_{j \geq k+1} \mathbb{E}\left[\|X^Q 1_{M_j(0)} P_I \xi(H) 1_K\|\right]
\]
\[
\leq L_k^Q + \sum_{j \geq k+1} L_j^{Q+1} \mathbb{E}\left[\|1_{\Lambda_{L_k}(w)} P_I \xi(H) 1_{\Lambda_{L_k}(0)}\|\right]
\]
\[
\leq L_k^Q + \sum_{j \geq k+1} L_j^{Q+1} L_j^{Nd\alpha} \left(L_j^{-2p+2Nd\alpha} + e^{-mL_j/2}\right) < \infty,
\]
since $2p > 3Nd\alpha + \alpha Q$ by assumption (3.2), and $L_j \sim \left[L_0^Q\right]$ grow fast enough.

This completes the proof of dynamical localization. \qed

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1DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE REIMS, MOULIN DE LA HOUSSE, B.P. 1039,, 51687 REIMS CEDEX 2, FRANCE, E-MAIL: victor.tchoulaevski@univ-reims.fr
2INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UNIVERSITÉ PARIS DIDEROT, 75156 PARIS, FRANCE, E-MAIL: aboutet@math.jussieu.fr
3STATISTICAL LABORATORY, DPMMS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK, E-MAIL: Y.M.Suhov@statslab.cam.ac.uk