WELL-POSEDNESS
FOR THE CAUCHY PROBLEM OF
THE MODIFIED ZAKHAROV-KUZNETSOV EQUATION

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Abstract. This paper is concerned with the Cauchy problem of the modified Zakharov-Kuznetsov equation on \( \mathbb{R}^d \). If \( d = 2 \), we prove the sharp estimate which implies local in time well-posedness in the Sobolev space \( H^s(\mathbb{R}^2) \) for \( s \geq 1/4 \). If \( d \geq 3 \), by employing \( U^p \) and \( V^p \) spaces, we establish the small data global well-posedness in the scaling critical Sobolev space \( H^{s_c}(\mathbb{R}^d) \) where \( s_c = d/2 - 1 \).

1. Introduction

We consider the Cauchy problem of the generalized Zakharov-Kuznetsov equation

\[
\begin{cases}
\partial_t u + \partial_{x_1} \Delta u = \partial_{x_1} (u^{k+1}), & (t, x_1, \cdots, x_d) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0, \cdot) = u_0 \in H^s(\mathbb{R}^d),
\end{cases}
\]

where \( d \geq 2, k \in \mathbb{N}, u = u(t, x_1, \cdots, x_d) \) is a real valued function and \( \Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 \) is the Laplacian. When \( k = 1 \), (1.1) is called the Zakharov-Kuznetsov equation which was introduced by Zakharov and Kuznetsov in [30] as a model for the propagation of ion-sound waves in magnetic fields for \( d = 3 \). See also [20]. In [21], Lannes, Linares and Saut derived the Zakharov-Kuznetsov equation in dimensions 2 and 3 rigorously as a long-wave limit of the Euler-Poisson system. The generalized Zakharov-Kuznetsov equation can be seen as a multi-dimensional extension of the generalized KdV equation

\[
\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\]

There are lots of works on the generalized Zakharov-Kuznetsov equation (1.1). For \( d = 2 \), we refer to the papers [2], [4], [8], [17], [22], [25] for the case \( k = 1 \), [1], [2], [5], [22], [23], [26] for the case \( k = 2 \), and see [5], [7], [23], [26] for \( k \geq 3 \). For \( d = 3 \), we refer to [24], [27] for the case \( k = 1 \), and [6] for \( k = 2 \), and [7] for \( k \geq 3 \).

The aim of the paper is to establish well-posedness of (1.1) when \( k = 2 \):

\[
\begin{cases}
\partial_t u + \partial_{x_1} \Delta u = \partial_{x_1} (u^3), & (t, x_1, \cdots, x_d) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0, \cdot) = u_0 \in H^s(\mathbb{R}^d),
\end{cases}
\]

which we call the modified Zakharov-Kuznetsov equation (mZK). The paper is divided into two parts. The first part is devoted to the well-posedness of the 2D (mZK) and in the latter part we...
Theorem 1.2. Let 

Theorem 1.1. is no transformation to symmetrize the modified Zakharov-Kuznetsov equation. Convenient that we consider the symmetrized equation (1.3) instead of (1.2). While for $H$ is equivalent to (1.2) when $d = 2$. We will see that, because of the symmetry of $x$ and $y$, it is convenient that we consider the symmetrized equation (1.3) instead of (1.2). While for $d \geq 3$ there is no transformation to symmetrize the modified Zakharov-Kuznetsov equation.

We now state the main results.

Theorem 1.1. Let $d = 2$ and $s \geq 1/4$. Then the Cauchy problem (1.3) is locally well-posed in $H^s(\mathbb{R}^2)$.

Theorem 1.2. Let $d \geq 3$. Then the Cauchy problem (1.2) is small data globally well-posed in $H^s(\mathbb{R}^d)$.

We give a comment on Theorem 1.1. For $d = 2$, in [22], Linares and Pastor proved the local well-posedness of (mZK) for $s > 3/4$. After that, the local well-posedness of the 2D (mZK) for $s > 1/4$, which is the best known result so far, was established by Ribaud and Vento in [26]. The global results of 2D (mZK) can be found in [1] and [23]. When $d = 2$, the scaling critical index $s_\ast$ of (mZK) is 0. In [22], Linares and Pastor proved that (1.3) is ill-posed in $H^s(\mathbb{R}^2)$ if $s \leq 0$ in the sense that the data-to-solution map fails to be uniformly continuous. As far as we know, there are no results for the case $0 < s \leq 1/4$. Theorem 1.1 establishes the well-posedness at $s = 1/4$ which is in fact optimal for the Picard iteration approach, as the following theorem shows.

Theorem 1.3. Let $s < 1/4$. Then for any $T > 0$, the data-to-solution map $u_0 \mapsto u$ of (1.3), as a map from the unit ball in $H^s(\mathbb{R}^2)$ to $C([0, T]; H^s)$ fails to be $C^3$.

Proof. We follow the Bourgain’s argument which was introduced in [3]. See also Section 6 in [14]. It should be noted that the function we choose below is essentially the same as the one which was employed to show the not-$C^3$ result of the modified KdV equation in [3].

It suffices to show that if $s < 1/4$ for any $C > 0$ there exists a real-valued function $\varphi \in \mathcal{S}(\mathbb{R}^2)$ such that $\|\varphi\|_{H^s(\mathbb{R}^2)} \sim 1$ and

$$\left\| \int_0^t e^{-t-t'}(\partial_2^2 + \partial_3^2)(\partial_x + \partial_y) \left( (e^{-t'(\partial_2^2 + \partial_3^2)}\varphi)(e^{-t'(\partial_2^2 + \partial_3^2)}\varphi)(e^{-t'(\partial_2^2 + \partial_3^2)}\varphi) \right) dt' \right\|_{H^s} \geq C. \quad (1.4)$$

Define real-valued even functions $\psi_{N, \xi}, \psi_\eta \in \mathcal{S}(\mathbb{R})$ as

$$\psi_{N, \xi}(\xi) = \begin{cases} 1 & \text{if } N \leq |\xi| \leq N + N^{-\frac{1}{2}} \\ 0 & \text{if } |\xi - N - 2^{-1}N^{-\frac{1}{2}}| \geq N^{-\frac{1}{2}} \end{cases}, \quad \psi_{\eta}(\eta) = \begin{cases} 1 & \text{if } |\eta| \leq 1 \\ 0 & \text{if } |\eta| \geq 2, \end{cases}$$

and $\varphi_N$ by $(F_{x,y}\varphi_N)(\xi, \eta) = N^{-s+1/4}\psi_{N, \xi}(\xi)\psi_{\eta}(\eta)$. Then $\|\varphi_N\|_{H^s(\mathbb{R}^2)} \sim 1$. Let

$$\Phi(\xi_1, \xi_2, \xi_3) := (\xi_1 + \xi_2 + \xi_3)^3 - \xi_1^3 - \xi_2^3 - \xi_3^3.$$
We easily observe that if
\[
|\xi_1 - N - 2^{-1}N^{-\frac{1}{2}}| < N^{-\frac{1}{2}}, \; |\xi_2 - N - 2^{-1}N^{-\frac{1}{2}}| < N^{-\frac{1}{2}}, \; |\xi_3 + N + 2^{-1}N^{-\frac{1}{2}}| < N^{-\frac{1}{2}},
\]
we have \(|\Phi(\xi_1, \xi_2, \xi_3)| \lesssim 1\). Let \(t\) be sufficiently small. By Plancherel’s theorem, we get
\[
\left\| \int_0^t e^{-t(t')\left(\partial_x^2 + \partial_y^2\right)}(\partial_x + \partial_y) \left(e^{-t'(\partial_x^2 + \partial_y^2)}\varphi_N\right)\right\|_{H^s} \gtrsim N^{-2s+7/4} \int_0^t e^{-it'(\Phi(\xi_1, \xi_2 - \xi_1, \xi_2 - 2\xi_2)) + \Phi(\eta_1, \eta_2)} d\eta_1 d\eta_2 dt,
\]
\[
\int \psi_N,\xi(\xi_1) \psi_N,\xi(\xi_2 - \xi_1) d\xi_1 d\xi_2 \int \psi_\eta(\eta_1) \psi_\eta(\eta_2 - \eta_1) d\eta_1 d\eta_2 dt \gtrsim N^{-2s+1/2}.
\]
This completes the proof of (1.4).

Next we comment on Theorem 1.2. For the 3D (mZK), in [6], Grünrock established the local well-posedness in the full subcritical regime \(s > 1/2\). Theorem 1.2 is an extension of the result by Grünrock. To be specific, Theorem 1.2 establishes the small data global well-posedness in the scaling critical regularity Sobolev space for \(d \geq 3\). The key ingredient in the proof of Theorem 1.2 is that we employ \(U^p\), \(V^p\) spaces which were introduced by Koch and Tataru in [18] and [19]. See also [11] and [12].

The paper is organized as follows. In Section 2, we introduce notations, \(X^{s,b}\) space and estimates for the proof of Theorem 1.1. Section 3 is devoted to the proof of the key estimate which establishes Theorem 1.1 immediately. In Sections 4 and 5, we consider Theorem 1.2. In the former section, we introduce \(U^p\) and \(V^p\) spaces and fundamental estimates. Lastly, in Section 4, we will prove the key estimate which immediately provedes Theorem 1.2.

Throughout the paper, we use the following notations. \(A \lesssim B\) means that there exists \(C > 0\) such that \(A \leq CB\). Also, \(A \sim B\) means \(A \lesssim B\) and \(B \lesssim A\). Let \(N, L \geq 1\) be dyadic numbers, i.e. there exist \(n_1, n_2 \in \mathbb{N}_0\) such that \(N = 2^{n_1}\) and \(L = 2^{n_2}\), and \(\psi \in C_0^\infty((-2, 2))\) be an even, non-negative function which satisfies \(\psi(t) = 1\) for \(|t| \leq 1\) and letting \(\psi_N(t) := \psi(t N^{-1}) - \psi(2t N^{-1})\), \(\psi_1(t) := \psi(t)\), the equality \(\sum_N \psi_N(t) = 1\) holds. Here we used \(\sum = \sum_{N \in 2^\mathbb{N}_0}\) for simplicity. We also use the notations \(\sum_L = \sum_{L \in 2^\mathbb{N}_0}\) and \(\sum_{N,L} = \sum_{N,L \in 2^\mathbb{N}_0}\) throughout the paper.

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2. Preliminaries for Theorem 1.1

In this section, we introduce notations and estimates which will be utilized to establish the key bilinear estimate for Theorem 1.1. Let \(u = u(t, x, y)\) with \((t, x, y) \in \mathbb{R} \times \mathbb{R}^2\). \(\mathcal{F}_t u, \mathcal{F}_{x,y} u\) denote
the Fourier transform of $u$ in time, space, respectively. $\mathcal{F}_{t,x,y} u = \hat{u}$ denotes the Fourier transform of $u$ in space and time. We define frequency and modulation projections $P_N$, $Q_L$ as

$$(\mathcal{F}^{-1}_{x,y} P_N u)(\xi, \eta) := \psi_N(|(\xi, \eta)|) (\mathcal{F}_{x,y} u)(\xi, \eta),$$

$$(\mathcal{F}^{-1}_{x,y} Q_L u)(\tau, \xi, \eta) := \psi_L(\tau - \xi^3 - \eta^3) \hat{u}(\tau, \xi, \eta).$$

Let $s, b \in \mathbb{R}$. We define $X^{s,b}(\mathbb{R}^3)$ spaces.

$$X^{s,b}(\mathbb{R}^3) := \{ f \in S'((\mathbb{R}^3) \mid \| f \|_{X^{s,b}} < \infty \},$$

$$\| f \|_{X^{s,b}} := \left( \sum_{N, L} N^{2s} L^{2b} \| P_N Q_L f \|_{L^2_{x,y,t}}^2 \right)^{1/2}.$$

For convenience, we define the set in frequency as

$$G_{N,L} := \{ (\tau, \xi, \eta) \in \mathbb{R}^3 \mid \psi_L(\tau - \xi^3 - \eta^3) \psi_N(|(\xi, \eta)|) \neq 0. \}$$

Next we observe the fundamental properties of $X^{s,b}$. A simple calculation gives the following.

(i) $X^{s,b} = X^{-s,-b}$, (ii) $(X^{s,b})^* = X^{-s,-b},$

for $s, b \in \mathbb{R}$.

Recall the Strichartz estimates for the unitary group $\{ e^{-t(\partial_x^3 + \partial_y^3)} \}$.

**Lemma 2.1** (Theorem 3.1. [15]). Let $\varphi \in L^2(\mathbb{R}^2)$. Then we have

$$\| \nabla_x \partial_x^2 \partial_y^2 \varphi \|_{L^p_x L_y^{2q}} \lesssim \| \varphi \|_{L^2_x L_y^{2q}}, \quad \text{if } \frac{2}{p} + \frac{2}{q} = 1, p > 2, \quad (2.1)$$

$$\| e^{-t(\partial_x^3 + \partial_y^3)} \varphi \|_{L^p_x L_y^{2q}} \lesssim \| \varphi \|_{L^2_x L_y^{2q}}, \quad \text{if } \frac{3}{p} + \frac{2}{q} = 1, p > 3, \quad (2.2)$$

where $|\nabla|^s := \mathcal{F}^{-1}_x|\xi |^s \mathcal{F}_x$ and $|\nabla|^s := \mathcal{F}^{-1}_y|\eta |^s \mathcal{F}_y$ denote the Riesz potential operators with respect to $x$ and $y$, respectively.

The Strichartz estimates above provide the following estimates. See [10].

$$\| \nabla_x \partial_x^2 \partial_y^2 Q_L u \|_{L^p_x L_y^{2q}} \lesssim L^{\frac{1}{2}} \| Q_L u \|_{L^2_x L_y^{2q}}, \quad \text{if } \frac{2}{p} + \frac{2}{q} = 1, p > 2, \quad (2.3)$$

$$\| Q_L u \|_{L^p_x L_y^{2q}} \lesssim L^{\frac{1}{2}} \| Q_L u \|_{L^2_x L_y^{2q}}, \quad \text{if } \frac{3}{p} + \frac{2}{q} = 1, p > 3. \quad (2.4)$$

**Remark 2.1.** Since the estimates (2.3) and (2.4) are almost equivalent to (2.1) and (2.2), respectively, we frequently call (2.3) and (2.4) Strichartz estimates in the paper.

Next we introduce the bilinear transversal inequality. For $f : \mathbb{R}^3 \rightarrow \mathbb{C}$, we use the following notation hereafter.

$$\text{supp}_{\xi, \eta} f := \{ (\xi, \eta) \in \mathbb{R}^2 \mid \text{There exists } \tau \in \mathbb{R} \text{ such that } (\tau, \xi, \eta) \in \text{supp } f. \}$$

**Proposition 2.2.** Let $N_2 \leq N_1$, $\varphi(\xi, \eta) = \xi^3 + \eta^3$. Suppose that

$$\text{supp } \hat{u}_{N_1 L_1} \subset G_{N_1, L_1}, \quad \text{supp } \hat{u}_{N_2 L_2} \subset G_{N_2, L_2},$$

and

$$|\nabla \varphi(\xi_1, \eta_1) - \nabla \varphi(\xi_2, \eta_2)| \gtrsim N_1^2,$$
Thus, it suffices to show

This implies that, without loss of generality, we may assume that

\[ N \leq (L_1 L_2) \frac{1}{2} \parallel u_{N_1, L_1} \parallel_{L_t^2 L_x^2_y} \parallel u_{N_2, L_2} \parallel_{L_t^2 L_x^2_y}. \]

(2.5)

In particular, if \( N_2 \leq 2^{-3} N_1 \) and

\[ \text{supp} \hat{u}_{N_1, L_1} \subset G_{N_1, L_1}, \quad \text{supp} \hat{v}_{N_2, L_2} \subset G_{N_2, L_2}, \]

we have

\[ \parallel u_{N_1, L_1} v_{N_2, L_2} \parallel_{L_t^2 L_x^2_y} \lesssim N_1^{-1} N_2^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \parallel u_{N_1, L_1} \parallel_{L_t^2 L_x^2_y} \parallel v_{N_2, L_2} \parallel_{L_t^2 L_x^2_y}. \]

(2.6)

Proof. First we consider (2.5). By Plancherel’s theorem, it suffices to show

\[ \left\| \int \hat{u}_{N_1, L_1} (\tau_1, \xi_1, \eta_1) \hat{v}_{N_2, L_2} (\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) d\tau_1 d\xi_1 d\eta_1 \right\|_{L^2} \]

\[ \lesssim N_1^{-1} N_2^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \left\| \hat{u}_{N_1, L_1} \right\|_{L^2} \left\| \hat{v}_{N_2, L_2} \right\|_{L^2}. \]

(2.7)

By the almost orthogonality, we may assume that \( \text{supp}_{\xi, \eta} \hat{u}_{N_1, L_1} \) and \( \text{supp}_{\xi, \eta} \hat{v}_{N_2, L_2} \) are confined to balls whose radius \( r \) such that \( r \ll N_2 \), respectively. Since \( \varphi \) is a cubic polynomial, we deduce from \( N_2 \leq N_1 \) that

\[ \left\| \parallel \partial_\xi \partial_\eta \varphi (\xi_1, \eta_1) \right\| + \left\| \partial_\xi \partial_\eta \varphi (\xi - \xi_1, \eta - \eta_1) \right\| \ll N_1. \]

Therefore, we easily observe

\[ \left| \nabla \varphi (\xi, \eta) - \nabla \varphi (\xi', \eta') \right| \ll N_1 N_2 \quad \text{if} \quad \left| (\xi, \eta) - (\xi', \eta') \right| \ll N_2. \]

This implies that, without loss of generality, we may assume that

\[ \left\| \partial_\xi \varphi (\xi_1, \eta_1) - \partial_\xi \varphi (\xi_2, \eta_2) \right\| \ll N_1, \]

(2.8)

for all \( (\xi_1, \eta_1) \in \text{supp}_{\xi, \eta} \hat{u}_{N_1, L_1}, \ (\xi_2, \eta_2) \in \text{supp}_{\xi, \eta} \hat{v}_{N_2, L_2} \). Now we turn to (2.7). By the Cauchy-Schwarz inequality, we get

\[ \left\| \int \hat{u}_{N_1, L_1} (\tau_1, \xi_1, \eta_1) \hat{v}_{N_2, L_2} (\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) d\tau_1 d\xi_1 d\eta_1 \right\|_{L^2} \]

\[ \leq \left\| \left[ \left\| \hat{u}_{N_1, L_1} \right\|_{L^2} \right]^2 \left\| \hat{v}_{N_2, L_2} \right\|_{L^2} \right\|_{L^2} \]

\[ \leq \sup_{\tau, \xi, \eta} \left\| E(\tau, \xi, \eta) \right\|_{L^2} \left\| \hat{u}_{N_1, L_1} \right\|_{L^2} \left\| \hat{v}_{N_2, L_2} \right\|_{L^2}, \]

where \( E(\tau, \xi, \eta) \subset \mathbb{R}^3 \) is defined by

\[ E(\tau, \xi, \eta) := \{ (\tau_1, \xi_1, \eta_1) \in \text{supp} \hat{u}_{N_1, L_1} \mid (\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) \in \hat{v}_{N_2, L_2} \}. \]

Thus, it suffices to show

\[ \left| E(\tau, \xi, \eta) \right| \lesssim N_1^{-2} N_2 L_1 L_2. \]

(2.9)

If we fix \( (\xi_1, \eta_1) \), it is easily observed that

\[ \left| \{ \tau_1 \mid (\tau_1, \xi_1, \eta_1) \in E(\tau, \xi, \eta) \} \right| \lesssim \min (L_1, L_2). \]

(2.10)
Next, if we fix $\eta_1$, since $\max(L_1, L_2) \gtrsim |\varphi(\xi_1, \eta_1) + \varphi(\xi - \xi_1, \eta - \eta_1) - \tau|$, the inequality (2.8) implies that $\xi_1$ is confined to an interval whose length is comparable to $\max(L_1, L_2)/N_4^2$. This, combined with (2.10) and $(\tau_1, \xi_1, \eta_1) \in \supp \hat{u}_{N_1, L_1}$ which implies $O(\eta_1) \leq N_2$, yields (2.9).

To see (2.6), it suffices to show

$$
|\xi_1, \eta_1| \geq 2|\xi_2, \eta_2| \implies |\nabla \varphi(\xi_1, \eta_1) - \nabla \varphi(\xi_2, \eta_2)| \gtrsim |\xi_1, \eta_1|^2,
$$

which is verified by a simple calculation. \hfill \Box

3. **Proof of the Key estimate for Theorem 1.1**

In this section, we establish the key estimate which gives Theorem 1.1 by a standard iteration argument, see [10], [16], and [29]. In this paper, we omit the details of the proof of Theorem 1.1 and focus on showing the following key estimate.

**Theorem 3.1.** For any $s \geq 1/4$, there exist $b \in (1/2, 1)$, $\varepsilon > 0$ and $C > 0$ such that

$$
(\partial_x + \partial_y)(u_1 u_2 u_3) \leq C \prod_{i=1}^3 \|u_i\|_{X^{s,b-\varepsilon}}.
$$

By a duality argument and dyadic decompositions, we observe that

$$
(3.1) \iff \left| \int u_4(\partial_x + \partial_y)(u_1 u_2 u_3) dt dx dy \right| \lesssim \prod_{i=1}^3 \|u_i\|_{X^{s,b-\varepsilon}}.
$$

$$
\iff \sum_{N_i, L_i (i=1, 2, 3, 4)} N_i \left| \int \prod_{i=1}^4 (Q_{L_i} P_{N_i} u_i) dt dx dy \right| \lesssim \prod_{i=1}^3 \|u_i\|_{X^{s,b-\varepsilon}}.
$$

(3.2)

For simplicity, we use the following notations.

$$
N_{\min} = \min(N_1, N_2, N_3, N_4), \quad N_{\max} = \max(N_1, N_2, N_3, N_4),
$$

$$
L_{\max} = \max(L_1, L_2, L_3, L_4), \quad u_{N_i, L_i} = Q_{L_i} P_{N_i} u_i.
$$

Clearly, (3.2) is verified by showing

$$
\left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dt dx dy \right| \lesssim N_{\min}^2 N_{\max}^2 (L_1 L_2 L_3 L_4)^{\frac{1}{2} - \varepsilon} \prod_{i=1}^4 \|u_{N_i, L_i}\|_{L^2}.
$$

(3.3)

By symmetry, we assume $N_1 \gtrsim N_2 \gtrsim N_3 \gtrsim N_4$. We first note that if $N_1 \sim 1$ we easily obtain (3.3) by using the Strichartz estimates. Further, by the Strichartz estimates, we can see that $L_{\max} \gtrsim N_4^3$ yields (3.3). For simplicity, here we only treat the case $L_4 = L_{\max}$. The other cases can be treated in the same way. By the Hölder’s inequality, the Strichartz estimates (2.4) with $(p, q) = (6, 4)$ and the Sobolev inequality, we get

$$
\left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dt dx dy \right| \lesssim \|u_{N_1, L_1}\|_{L^p_{t,x}} \|u_{N_2, L_2}\|_{L^p_{t,x}} \|u_{N_3, L_3}\|_{L^p_{t,x}} \|u_{N_4, L_4}\|_{L^p_{t,x}}
$$

$$
\lesssim (L_1 L_2 L_3)^{\frac{1}{2}} \|u_{N_1, L_1}\|_{L^2} \|u_{N_2, L_2}\|_{L^2} \|u_{N_3, L_3}\|_{L^2} \|u_{N_4, L_4}\|_{L^2}
$$

$$
\lesssim N_4^2 N_1^{-\frac{2}{3}} (L_1 L_2 L_3)^{\frac{1}{2}} \prod_{i=1}^4 \|u_{N_i, L_i}\|_{L^2}.
$$

This completes the proof of (3.3).
Hereafter, we assume $1 \ll N_1$ and $L_{\text{max}} \ll N_1^3$. We divide the proof into the following three cases.

Case 1: $N_1 \sim N_2 \gg N_3 \gtrsim N_4$.
Case 2: $N_1 \sim N_2 \sim N_3 \gg N_4$.
Case 3: $N_1 \sim N_2 \sim N_3 \sim N_4$.

Case 1: $N_1 \sim N_2 \gg N_3 \gtrsim N_4$. Since $N_1 \gg N_3$ and $N_2 \gg N_4$, this case is easily handled by the bilinear transversal estimate (2.6) as follows.

\[ \left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dt dxdy \right| \lesssim \left\| u_{N_1, L_1} u_{N_2, L_2} \right\|_{L_x^2 L_y^2} \left\| u_{N_3, L_3} u_{N_4, L_4} \right\|_{L_x^2 L_y^2} \lesssim N_1^{-1/2} N_3^{1/2} \left( N_2 N_4 \right)^{1/2} \left( \prod_{i=1}^4 \| u_{N_i, L_i} \|_{L^2} \right), \]

which completes the proof of (3.3).

Case 2: $N_1 \sim N_2 \sim N_3 \gg N_4$. By harmless decompositions, we may assume that $\text{supp}_{\xi, \eta} \hat{u}_{N_j, L_j}$ ($j = 1, 2, 3$) is contained in a ball such that its radius $r$ satisfies $r \ll N_1$. We divide the proof into two cases. First we assume

\[ |\nabla \varphi(\xi_1, \eta_1) - \nabla \varphi(\xi_2, \eta_2)| \gtrsim N_1^2, \]

for all $(\xi_1, \eta_1) \in \text{supp}_{\xi, \eta} \hat{u}_{N_1, L_1}$, $(\xi_2, \eta_2) \in \text{supp}_{\xi, \eta} \hat{u}_{N_2, L_2}$. In this case, (2.5) in Proposition 2.2 gives

\[ \| u_{N_1, L_1} u_{N_2, L_2} \|_{L_x^2 L_y^2} \lesssim N_1^{-1/2} (N_2 L_2)^{1/2} \| u_{N_1, L_1} \|_{L_x^2 L_y^2} \| u_{N_2, L_2} \|_{L_x^2 L_y^2}. \]  

(3.4)

On the other hand, since $N_1 \sim N_3 \gg N_4$, we get

\[ \| u_{N_1, L_1} u_{N_3, L_4} \|_{L_x^2 L_y^2} \lesssim N_1^{-2} N_3^{1/2} (L_3 L_4)^{1/2} \| u_{N_1, L_1} \|_{L_x^2 L_y^2} \| u_{N_3, L_4} \|_{L_x^2 L_y^2}. \]  

(3.5)

Consequently, by the H"older’s inequality and (3.4), (3.5), we have

\[ \left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dt dxdy \right| \lesssim \left\| u_{N_1, L_1} u_{N_2, L_2} \right\|_{L_x^2 L_y^2} \left\| u_{N_3, L_3} u_{N_4, L_4} \right\|_{L_x^2 L_y^2} \lesssim N_1^{-3/2} N_3^{1/2} (L_3 L_4)^{1/2} \left( \prod_{i=1}^4 \| u_{N_i, L_i} \|_{L^2} \right), \]

which completes the proof of (3.3). Next suppose that there exist $(\xi_1, \eta_1) \in \text{supp}_{\xi, \eta} \hat{u}_{N_1, L_1}$, $(\xi_2, \eta_2) \in \text{supp}_{\xi, \eta} \hat{u}_{N_2, L_2}$ such that

\[ |\nabla \varphi(\xi_1, \eta_1) - \nabla \varphi(\xi_2, \eta_2)| \ll N_1^2. \]  

(3.6)

Since $|\xi_1, \eta_1| \geq N_1/2$, without loss of generality, we can assume $|\xi_1| \geq N_1/4$. This and (3.6) imply

\[ |\partial_1 \varphi(\xi_1, \eta_1) - \partial_1 \varphi(\xi_2, \eta_2)| \ll N_1^2 \]

\[ \iff 3|\xi_1 - \xi_2|(|\xi_1 + \xi_2| \ll N_1^2 \]

\[ \implies |\xi_2(2\xi_1 + \xi_2)| \gtrsim N_1^2 \]

\[ \implies |\partial_1 \varphi(\xi_1, \eta_1) - \partial_1 \varphi(\xi_1 + \xi_2 + \eta_2)| \gtrsim N_1^2. \]
Thus, because $N_1 \gg N_4$, we may assume

$$|\nabla \varphi(\xi_1, \eta_1) - \nabla \varphi(\xi_3, \eta_3)| \gtrsim N_1^2,$$

for all $(\xi_1, \eta_1) \in \text{supp} \xi, \eta \sim \hat{u}_{N_1, L_1}$, $(\xi_3, \eta_3) \in \text{supp} \xi, \eta \sim \hat{u}_{N_3, L_3}$. Therefore, in the same manner as for the former case, Proposition 2.2 provides

$$\left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dtdxdy \right| \lesssim \|u_{N_1, L_1} u_{N_2, L_2} \|_{L^2_x L^2_y} \|u_{N_2, L_2} u_{N_3, L_3} \|_{L^2_x L^2_y} \lesssim N_1^{-\frac{3}{2}} N_2^\frac{1}{2} (L_1 L_2 L_3 L_4)^\frac{1}{2} \prod_{i=1}^4 \|u_{N_i, L_i}\|_{L^2}.$$

**Case 3: $N_1 \sim N_2 \sim N_3 \sim N_4$.** Similarly to the previous case, by performing harmless decompositions, we assume that \(\text{supp} \xi, \eta \sim \hat{u}_{N_i, L_i} \) \((i = 1, 2, 3, 4)\) is contained in a ball whose radius \(r\) satisfies \(r \ll N_1\). First we deal with the simple case $|\xi_1| \sim |\eta_1| \sim N_1 \) \((i = 1, 2, 3, 4)\). By employing the Strichartz estimate (2.3) with $p = q = 4$, we have

$$\|u_{N_i, L_i}\|_{L_{t,x}^4} \lesssim N_1^{-\frac{3}{2}} L_1^\frac{1}{2} \|u_{N_i, L_i}\|_{L^2}, \quad (3.7)$$

which immediately yields (3.3) as follows.

$$\left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dtdxdy \right| \lesssim \prod_{i=1}^4 \|u_{N_i, L_i}\|_{L^2} \lesssim N_1^{-1} (L_1 L_2 L_3 L_4)^{\frac{1}{2}} \prod_{i=1}^4 \|u_{N_i, L_i}\|_{L^2}.$$

Thus, without loss of generality, we may assume that $|\eta_4| \ll N_1$. We divide the proof into three cases.

1. $\min(|\xi_1|, |\xi_2|, |\xi_3|) \ll N_1$,
2. $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim N_1$, $L_{\max}^8 \leq \max(|\eta_1|, |\eta_2|, |\eta_2|, |\eta_4|)$,
3. $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim N_1$, $\max(|\eta_1|, |\eta_2|, |\eta_2|, |\eta_4|) \leq L_{\max}^8$.

We consider the first case. Without loss of generality, we can assume $|\xi_3| \ll N_1$. Note that, since $N_1 \sim N_3 \sim N_4$, it holds that $|\eta_1| \sim |\xi_4| \sim N_1$, and then it is easily obtained $|\nabla \varphi(\xi_3, \eta_3)| \gtrsim N_1^2$ which, by utilizing Proposition 2.2, yields

$$\|u_{N_3, L_3} u_{N_4, L_4} \|_{L^2_x L^2_y} \lesssim N_1^{-\frac{3}{2}} (L_3 L_4)^{\frac{1}{2}} \|u_{N_3, L_3}\|_{L^2_x L^2_y} \|u_{N_4, L_4}\|_{L^2_x L^2_y}. \quad (3.8)$$

We consider two cases. First assume that $|\xi_1| \sim |\eta_1| \sim |\xi_2| \sim |\eta_2| \sim N_1$. Recall that this condition provides the $L^4$ Strichartz estimate (3.7) for $i = 1, 2$. Then by (3.8), we have

$$\left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dtdxdy \right| \lesssim \|u_{N_1, L_1}\|_{L^4} \|u_{N_2, L_2}\|_{L^4} \|u_{N_3, L_3}\|_{L^4} \|u_{N_4, L_4}\|_{L^2} \lesssim N_1^{-1} (L_1 L_2 L_3 L_4)^{\frac{1}{2}} \prod_{i=1}^4 \|u_{N_i, L_i}\|_{L^2}.$$

Next we treat the case $\min(|\xi_1|, |\eta_1|, |\xi_2|, |\eta_2|) \ll N_1$. Without loss of generality, assume $|\xi_2| \ll N_1$. Clearly, this implies $|\xi_1| \sim N_1$ since $|\xi_3| \ll N_1$ and $|\xi_4| \sim N_1$. Therefore we get $|\partial_1 \varphi(\xi_1, \eta_1) -$
\( \partial_t \varphi(\xi_2, \eta_2) \gtrsim N_1^2 \) which yields
\[
\| u_{N_1, L_1} u_{N_2, L_2} \|_{L^2_t L^2_y} \lesssim N_1^{-\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| u_{N_1, L_1} \|_{L^2_t L^2_y} \| u_{N_2, L_2} \|_{L^2_t L^2_y}.
\]
This and (3.8) verify the desired estimate.

To deal with the second case, we introduce the following bilinear estimate.

**Proposition 3.2.** Let \( A \) be dyadic such that \( 1 \leq A \leq N_1 L_{\max}^{-\frac{8e}{c}} \). Suppose that
\[
supp \tilde{u}_{N_1, L_1} \subset G_{N_1, L_1} \cap \{(\tau, \xi, \eta) \mid |\xi| \sim N_1, |\eta| \sim A^{-1} N_1\},
\]
\[
supp \tilde{v}_{N_2, L_2} \subset G_{N_2, L_2} \cap \{(\tau, \xi, \eta) \mid |\xi| \sim N_1, |\eta| \ll A^{-1} N_1\},
\]
Then we have
\[
\| u_{N_1, L_1} v_{N_2, L_2} \|_{L^2_t L^2_y} \lesssim A^\# N_1^{-\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| u_{N_1, L_1} \|_{L^2_t L^2_y} \| v_{N_2, L_2} \|_{L^2_t L^2_y}. \tag{3.9}
\]

**Proof.** By Plancherel’s theorem, it suffices to show
\[
\left\| \int \tilde{u}_{N_1, L_1}(\tau_1, \xi_1, \eta_1) \tilde{v}_{N_2, L_2}(\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) d\sigma_1 \right\|_{L^2_t L^2_y} \lesssim A^\# N_1^{-\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| u_{N_1, L_1} \|_{L^2_t L^2_y} \| v_{N_2, L_2} \|_{L^2_t L^2_y}, \tag{3.10}
\]
where \( \sigma_1 = d\tau_1 d\xi_1 d\eta_1 \). The proof is divided into two cases. \(|\xi_1^2 - (\xi - \xi_1)^2| \gg A^{-3/2} N_1^2\) and \(|\xi_1^2 - (\xi - \xi_1)^2| \ll A^{-3/2} N_1^2\). Note that the latter condition means that either \(|\xi_1 + (\xi - \xi_1)| \ll A^{-3/2} N_1\) or \(|\xi_1 - (\xi - \xi_1)| \ll A^{-3/2} N_1\) holds. Thus, by the almost orthogonality, we can assume that \( \xi_1 \) is confined to an interval whose length is \( A^{-3/2} N_1 \). By following a standard argument, we observe that
\[
\left\| \int \tilde{u}_{N_1, L_1}(\tau_1, \xi_1, \eta_1) \tilde{v}_{N_2, L_2}(\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) d\sigma_1 \right\|_{L^2_t L^2_y} \leq \sup_{\tau, \xi, \eta} |E(\tau, \xi, \eta)|^{1/2} \left\| \tilde{u}_{N_1, L_1} \right\|_{L^2_t L^2_y} \left\| \tilde{v}_{N_2, L_2} \right\|_{L^2_t L^2_y},
\]
where \( E(\tau, \xi, \eta) \subset \mathbb{R}^3 \) is defined by
\[
E(\tau, \xi, \eta) := \{(\tau_1, \xi_1, \eta_1) \in \text{supp} \tilde{u}_{N_1, L_1} \mid (\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) \in \text{supp} \tilde{v}_{N_2, L_2}\}.
\]
Then it suffices to show \(|E(\tau, \xi, \eta)| \lesssim A^{1/2} N_1^{-1} L_1 L_2\). The condition of the former case \(|\xi_1^2 - (\xi - \xi_1)^2| \gg A^{-3/2} N_1\) implies \(|\partial_1 \varphi(\xi_1, \eta_1) - \partial_1 \varphi(\xi - \xi_1, \eta - \eta_1)| \gtrsim A^{-3/2} N_1^2\). This and \(|\eta_1| \sim A^{-1} N_1\) give \(|E(\tau, \xi, \eta)| \lesssim A^{1/2} N_1^{-1} L_1 L_2\). Similarly, as above, the condition of the latter case allows us to assume that \( \xi_1 \) is confined to an interval whose length is \( A^{-3/2} N_1 \). This and \(|\partial_2 \varphi(\xi_1, \eta_1) - \partial_2 \varphi(\xi - \xi_1, \eta - \eta_1)| \gtrsim A^{-2} N_1^2\) yield \(|E(\tau, \xi, \eta)| \lesssim A^{1/2} N_1^{-1} L_1 L_2\).

**Proof of (3.3) under the conditions (2).** For \( 1 \leq A \leq N_1 L_{\max}^{-\frac{8e}{c}} \), assume
\[
A^{-1} N_1 \leq \max(|\eta_1|, |\eta_2|, |\eta_2|, |\eta_4|) \leq 2A^{-1} N_1.
\]
Our goal is to establish
\[ \left\| \int u_{N_1,L_1}u_{N_2,L_2}u_{N_3,L_3}u_{N_4,L_4}dtdxdy \right\| \lesssim A_{1}^\frac{1}{2}N_1^{-\frac{1}{2}}(L_1L_2L_3L_4)^{\frac{1}{4}} \prod_{i=1}^{4} \| u_{N_i,L_i} \|_{L^2}. \] (3.11)

It is clear that (3.11) gives (3.3) under the conditions (2). We divide the proof of (3.11) into the following three cases.

(2a) |\eta_1| \sim |\eta_2| \sim |\eta_3| \sim |\eta_4|,
(2b) |\eta_1| \sim |\eta_2| \sim |\eta_3| \gg |\eta_4|,
(2c) |\eta_1| \sim |\eta_2| \gg |\eta_3| \gg |\eta_4|.

The proofs are quite simple. The case (2a) can be treated by the following $L^4$ Strichartz estimate which is given by (2.3) with $p = q = 4$.

\[ \| u_{N_i,L_i} \|_{L^4_t L^\infty_y} \lesssim A_{1}^\frac{1}{2}N_1^{-\frac{1}{2}}L_1^\frac{3}{2} \| u_{N_i,L_i} \|_{L^2}. \] (3.12)

The second case (2b) is handled by (3.12) and Proposition 3.2. To be precise, we use
\[
\begin{align*}
\| u_{N_1,L_1} \|_{L^4_t L^\infty_y} & \lesssim A_{1}^\frac{1}{2}N_1^{-\frac{1}{2}}L_1^\frac{3}{2} \| u_{N_1,L_1} \|_{L^2} \| u_{N_2,L_2} \|_{L_t^4 L_y^\infty} \lesssim A_{1}^\frac{1}{2}N_1^{-\frac{1}{2}}L_1^\frac{3}{2} \| u_{N_2,L_2} \|_{L^2}, \\
\| u_{N_3,L_3}u_{N_4,L_4} \|_{L^2} & \lesssim A_{1}^\frac{1}{2}N_1^{-\frac{1}{2}}(L_3L_4)^{\frac{1}{2}} \| u_{N_3,L_3} \|_{L^2} \| u_{N_4,L_4} \|_{L^2}.
\end{align*}
\]

For the last case, we employ Proposition 3.2 which provides
\[
\begin{align*}
\| u_{N_1,L_1} \|_{L^4_t L^\infty_y} & \lesssim A_{1}^\frac{1}{2}N_1^{-\frac{1}{2}}(L_1L_3)^{\frac{1}{2}} \| u_{N_1,L_1} \|_{L^2} \| u_{N_3,L_3} \|_{L^2}, \\
\| u_{N_2,L_2}u_{N_4,L_4} \|_{L^2} & \lesssim A_{1}^\frac{1}{2}N_1^{-\frac{1}{2}}(L_2L_4)^{\frac{1}{2}} \| u_{N_2,L_2} \|_{L^2} \| u_{N_4,L_4} \|_{L^2}.
\end{align*}
\]

These immediately establish (3.11).

We lastly consider the case (3). The following proposition plays a key role.

**Proposition 3.3.** Suppose that
\[
\begin{align*}
supp \hat{u}_{N_1,L_1} & \subset G_{N_1,L_1} \cap \{ (\tau, \xi, \eta) \mid |\xi| \sim N_1, \ |\eta| \leq L_{\max}^{8e} \}, \\
supp \hat{v}_{N_2,L_2} & \subset G_{N_2,L_2} \cap \{ (\tau, \xi, \eta) \mid |\xi| \sim N_1, \ |\eta| \leq L_{\max}^{8e} \}.
\end{align*}
\]

Then we have
\[ \| u_{N_1,L_1} v_{N_2,L_2} \|_{L_t^4 L_y^{\infty}} \lesssim N_1^{-\frac{1}{4}}(L_{\max}^{2e}L_1L_2)^{\frac{1}{2}} + L_{\max}^{10e} \min(L_1,L_2)^{\frac{1}{2}} \| u_{N_1,L_1} \|_{L_t^4 L_y^{\infty}} \| v_{N_2,L_2} \|_{L_t^4 L_y^{\infty}}. \] (3.13)

**Proof.** The proof is almost the same as that for Proposition 3.2. We will establish
\[
\left\| \int \hat{u}_{N_1,L_1}(\tau_1,\xi_1,\eta_1)\hat{v}_{N_2,L_2}(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)d\sigma_1 \right\|_{L_t^4 L_y^{\infty}} \lesssim N_1^{-\frac{1}{4}}(L_{\max}^{2e}L_1L_2)^{\frac{1}{2}} + L_{\max}^{10e} \min(L_1,L_2)^{\frac{1}{2}} \| u_{N_1,L_1} \|_{L_t^4 L_y^{\infty}} \| v_{N_2,L_2} \|_{L_t^4 L_y^{\infty}}, \]
(3.14)

where $\sigma_1 = d\tau_1 d\xi_1 d\eta_1$. We consider two cases. $|\xi_1^2 - (\xi - \xi_1)^2| \gg N_1^{1/2} L_{\max}^{12e}$ and $|\xi_1^2 - (\xi - \xi_1)^2| \lesssim N_1^{1/2} L_{\max}^{12e}$. As we saw in the proof of Proposition 3.2, the latter condition means that we can assume that $\xi_1$ is confined to an interval whose length is $N_1^{-1/2} L_{\max}^{12e}$. In the same manner as in the proof of Proposition 3.2, we will show
\[ |E(\tau,\xi,\eta)| \lesssim N_1^{-\frac{1}{4}}(L_{\max}^{4e}L_1L_2 + L_{\max}^{20e} \min(L_1,L_2)) \]
(3.15)
where \( E(\tau, \xi, \eta) \subset \mathbb{R}^3 \) is defined by 
\[
E(\tau, \xi, \eta) := \{(\tau_1, \xi_1, \eta_1) \in \text{supp} \, \hat{u}_{N_1, L_1} \mid (\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) \in \text{supp} \, \hat{u}_{N_2, L_2}\}.
\]

For the former case, we have \( |\partial_1 \varphi(\xi_1, \eta_1) - \partial_1 \varphi(\xi - \xi_1, \eta - \eta_1)| \gtrsim N_1^{1/2} L_{\max}^{12e} \) which, combined with \( |\eta_1| \leq L_{\max}^{8e} \), gives \( |E(\tau, \xi, \eta, \eta_1)| \lesssim N_1^{1/2} L_{\max}^{-4e} L_1 L_2 \). The latter term can be handled by the inequalities \( O(\xi_1) \lesssim N_1^{1/2} L_{\max}^{12e} \) and \( |\eta_1| \leq L_{\max}^{8e} \), which yield \( |E(\tau, \xi, \eta)| \lesssim N_1^{1/2} L_{\max}^{-2e} \min(L_1, L_2) \). 

**Proof of (3.3) under the conditions (3).** Suppose that \( \varepsilon > 0 \) is sufficient small. By using Proposition 3.3, we easily obtain 
\[
\left| \int u_{N_1, L_1} u_{N_2, L_2} u_{N_3, L_3} u_{N_4, L_4} dt \, dx \, dy \right| \lesssim \|u_{N_1, L_1} u_{N_2, L_2} \|_{L^2_t L^2_y} \|u_{N_3, L_3} u_{N_4, L_4} \|_{L^2_t L^2_y} \lesssim N_1^{-\frac{1}{2}} (L_1 L_2 L_3 L_4)^{\frac{1}{4}} \prod_{i=1}^4 \|u_{N_i, L_i}\|_{L^2}.
\]

This completes the proof of (3.3).

## 4. Preliminaries for Theorem 1.2

In this section, we collect notations and estimates that we utilize in the proof of the key estimate for Theorem 1.2. We begin with the definitions of \( U^p \) and \( V^p \) spaces which were exploited in [18]. The definitions and notations of \( U^p \) and \( V^p \) correspond to [11] and [12]. Let \( u = u(t, x) \) with \( (t, x) = (t, x_1, \ldots, x_d) \in \mathbb{R} \times \mathbb{R}^d \). \( \mathcal{F}_t u \) and \( \mathcal{F}_x u \) denote the Fourier transform of \( u \) in time and space, respectively. \( \mathcal{F}_{t,x} u = \hat{u} \) denotes the Fourier transform of \( u \) in space and time. We define frequency and modulation projections \( P_N, Q_L \) as 
\[
(P_N u)(\xi) := \psi_N(\|\xi\|)(\mathcal{F}_x u)(\xi),
\]
\[
(Q_L u)(\tau, \xi) := \psi_L(\tau - \xi \|\xi\|) \hat{u}(\tau, \xi),
\]
where \( (\tau, \xi) = (\tau_1, \xi_1, \ldots, \xi_d) \in \mathbb{R} \times \mathbb{R}^d \) are time and space frequencies. Let \( Z \) be the set of finite partitions \(-\infty = t_0 < t_1 \cdots < t_K = \infty \) and let \( Z_0 \) be the set of finite partitions \(-\infty < t_0 < t_1 \cdots < t_K \leq \infty \) and let \( Z_0 \). We first define \( U^p \) space.

**Definition 1** (Definition 2.1. [11]). Let \( 1 \leq p < \infty \). For \( \{t_k\}_{k=0}^K \in Z \) and \( \{\phi_k\}_{k=0}^{K-1} \subset L^2 \) with 
\( \sum_{k=0}^{K-1} \|\phi_k\|_{L^2} = 1 \) and \( \phi_0 = 0 \) we call the function \( a : \mathbb{R} \to L^2 \) given by 
\[
a = \sum_{k=1}^K \chi_{[t_{k-1}, t_k)} \phi_{k-1}
\]
a \( U^p \)-atom. Furthermore, we define the atomic space 
\[
U^p := \left\{ u = \sum_{j=1}^\infty \lambda_j a_j \, | \, a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^\infty |\lambda_j| < \infty \right\}
\]
with norm 
\[
\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| \mid u = \sum_{j=1}^\infty \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j : U^p\text{-atom} \right\}.
\]

Next we define \( V^p \) space.
Definition 2 (Definition 2.3. [11] and (iii). [12]). Let \( 1 \leq p < \infty \). \( V^p \) space is defined as the normed space of all functions \( v : \mathbb{R} \rightarrow L^2 \) such that \( \lim_{t \to \pm} v(t) \) exist and for which the norm

\[
\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^N} \left( \sum_{k=1}^N \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p}
\]

is finite, where we use the convention that \( v(-\infty) = \lim_{t \to -\infty} v(t) \) and \( v(\infty) = 0 \). Likewise, let \( U^p \) denote the closed subspace of all \( v \in V^p \) with \( \lim_{t \to -\infty} v(t) = 0 \).

For the properties of \( U^p \) and \( V^p \) spaces, see Propositions 2.2 and 2.4 in [11], respectively. See also [12].

We next introduce the important connection between \( U^p \) and \( V^p \).

**Proposition 4.1** (Theorem 2.8 and Proposition 2.10. [11]). Let \( 1 < p < \infty \), \( u \in V^1 \) be absolutely continuous on compact intervals and \( v \in V^p \). Then,

\[
\|u\|_{U^p} = \sup_{\|v\|_{V^p}=1} \left| \int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle_{L^2} dt \right| .
\]

The following definitions correspond to Definition 2.15 in [11].

**Definition 3.** Let \( S = -\partial_x \Delta \). We define

(i) \( U^p_S = e^{-S}U^p \) with norm \( \|u\|_{U^p_S} = \|e^{-S}u\|_{U^p} \),

(ii) \( V^p_S = e^{-S}V^p \) with norm \( \|u\|_{V^p_S} = \|e^{-S}u\|_{V^p} \).

Now we define the solution space \( Y^* \) as the closure of all \( u \in C(\mathbb{R}; H^s(\mathbb{R}^d) \cap (\nabla_x)^{-s}U^2_S \) such that

\[
\|u\|_{Y^*} := \left( \sum_{\lambda \in \Lambda} N^{2s} \|P_N u \|^2_{L^2} \right)^{1/2} < \infty
\]

We collect the fundamental estimates of the Zakharov-Kuznetsov equation. The first estimate is included in Proposition 1.3 in [28] which was obtained in the same way as for the Strichartz estimate of a higher dimensional version of the Benjamin-Ono equation. See the proof of Theorem 1.1 in [13]. It should be noted that we can get the \( L^4 \) Strichartz estimate below by following the proof of Theorem 2 in [9]. See also [6].

**Proposition 4.2.** Let \( d \geq 3 \). Then we have

\[
\|e^{iS} \varphi\|_{L^4_x L^2_t} \lesssim \|\nabla \frac{d-3}{2} \varphi\|_{L^2_t}.
\]

**Remark 4.1.** By using Proposition 2.19 in [11], the above \( L^4 \) Strichartz estimate yields

\[
\|u\|_{L^4_t L^4_x} \lesssim \|\nabla \frac{d-3}{2} u\|_{U^3}.
\]

The following bilinear transversal estimate can be found in [28].

**Proposition 4.3** (Proposition 1.2 in [28] with \( a = 2 \)). Let \( d \geq 2 \), \( N_2 \ll N_1 \) and define \( u_{N_1} = P_{N_1} u \), \( v_{N_2} = P_{N_2} v \). Then we have

\[
\|e^{iS} u_{N_1} e^{iS} v_{N_2}\|_{L^4_x L^2_t} \lesssim N_1^{-1} N_2^{\frac{d-1}{2}} \|u_{N_1}\|_{L^4_t L^2_x} \|v_{N_2}\|_{L^4_t L^2_x}.
\]
Remark 4.2. By the same argument as of the proof of Corollary 2.21 in [11], we can see that (4.2) implies
\[ \|u_{N_1}v_{N_2}\|_{L_t^2 L_x^2} \lesssim N_1^{-1}N_2^{d-1} \|u_{N_1}\|_{U^1_3} \|v_{N_2}\|_{U^1_3}. \] (4.3)

If \( d \geq 3 \), by interpolating the above bilinear estimate and
\[ \|u_{N_1}v_{N_2}\|_{L_t^2 L_x^2} \lesssim (N_1N_2)^{\frac{d-1}{2}} \|u_{N_1}\|_{U^1_3} \|v_{N_2}\|_{U^1_3}, \]
which follows from (4.1), for any \( \varepsilon > 0 \), we get
\[ \|u_{N_1}v_{N_2}\|_{L_t^2 L_x^2} \lesssim N_1^{-1+\varepsilon}N_2^{\frac{d-1}{2}-\varepsilon} \|u_{N_1}\|_{V^1_{\frac{3}{2}}} \|v_{N_2}\|_{V^1_{\frac{3}{2}}}. \] (4.4)

5. PROOF OF KEY ESTIMATE FOR THEOREM 1.2

We show the following key estimate which immediately yields Theorem 1.2. We omit the proof of Theorem 1.2 here and focus on the key estimate. To complete the proof of Theorem 1.2, see the proof of Theorem 1.1 in [11].

**Theorem 5.1.** Let \( T \in (0, \infty) \). We define the Duhamel term as
\[ I_T(u_1, u_2, u_3)(t) := \int_0^t \chi_{[0,T]}(t-t') \partial_x(u_1u_2u_3)(t')dt'. \]

Then there exists \( C > 0 \) such that
\[ \|I_T(u_1, u_2, u_3)\|_{Y^{\infty}} \leq C \|u_1\|_{Y^{\infty}} \|u_2\|_{Y^{\infty}} \|u_3\|_{Y^{\infty}}. \] (5.1)

By using Proposition 4.1 above and Proposition 2.4 in [11], it suffices to show the following proposition.

**Proposition 5.2.** Let \( N_1 \geq N_2 \geq N_3 \geq N_4 \) and \( u_{N_i} = P_{N_{N_i}}u_i \) where \( i = 1, 2, 3, 4 \). Then we have
\[ \left\| \int u_{N_1}u_{N_2}u_{N_3}u_{N_4}dt \right\| \leq N_1^{-\frac{1}{2}+\varepsilon}N_3^{\frac{d-2}{2}}N_4^{\frac{d-1}{2}-\varepsilon} \prod_{i=1}^4 \|u_{N_i}\|_{V^1_{\frac{3}{2}}}. \]

**Proof.** We divide the proof into three cases.

Case 1: \( N_1 \sim N_2 \gg N_3 \gg N_4 \),
Case 2: \( N_1 \sim N_2 \sim N_3 \gg N_4 \),
Case 3: \( N_1 \sim N_2 \sim N_3 \sim N_4 \).

Case 1. The first case can be handled by (4.4) as follows.
\[ \left\| \int u_{N_1}u_{N_2}u_{N_3}u_{N_4}dt \right\| \lesssim \|u_{N_1}\|_{L_t^2 L_x^2} \|u_{N_2}\|_{L_t^2 L_x^2} \|u_{N_3}\|_{L_t^2 L_x^2} \|u_{N_4}\|_{L_t^2 L_x^2} \lesssim N_1^{-2+2\varepsilon}N_3^{\frac{d-1}{2}-\varepsilon}N_4^{\frac{d-1}{2}-\varepsilon} \prod_{i=1}^4 \|u_{N_i}\|_{V^1_{\frac{3}{2}}}. \]

Case 2. It follows from (4.1) and (4.4) that
\[ \left\| \int u_{N_1}u_{N_2}u_{N_3}u_{N_4}dt \right\| \lesssim \|u_{N_1}\|_{L_t^2 L_x^2} \|u_{N_2}\|_{L_t^2 L_x^2} \|u_{N_3}\|_{L_t^2 L_x^2} \|u_{N_4}\|_{L_t^2 L_x^2} \lesssim N_1^{\frac{d-1}{2}+\varepsilon}N_4^{\frac{d-1}{2}-\varepsilon} \prod_{i=1}^4 \|u_{N_i}\|_{V^1_{\frac{3}{2}}}. \]
Lastly, (4.1) gives

\[ \int \int \int u_{N_1} u_{N_2} u_{N_3} u_{N_4} dt \, dx \lesssim \left\| u_{N_1} \right\|_{L^4} \left\| u_{N_2} \right\|_{L^4} \left\| u_{N_3} \right\|_{L^4} \left\| u_{N_4} \right\|_{L^4} \]

\[ \lesssim N_1^{d-3} \prod_{i=1}^{4} \left\| u_{N_i} \right\|_{V^2} , \]

which completes the proof of Theorem 5.2. \( \square \)

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