Pion Beta Decay and CKM Unitarity

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Pion beta decay, $\pi^+ \rightarrow \pi^0 e^+ \nu(\gamma)$, provides a theoretically clean $\pm 0.3\%$ determination of the CKM matrix element $V_{ud}$. That value falls short of super-allowed nuclear beta decays where an order of magnitude better precision already exists. We advocate a new strategy for utilizing pion beta decay, based on its role in determining $V_{us}/V_{ud}$ via the ratio $R_V = \Gamma(K \rightarrow \pi\nu(\gamma))/\Gamma(\pi^+ \rightarrow \pi^0 e^+ \nu(\gamma))$. $R_V$ measures $f_K^S(0)|V_{us}|/f_\pi(0)|V_{ud}|$ and is insensitive to the Fermi constant and some radiative corrections. Its dependence on the ratio of form factors may also prove useful for lattice gauge theory calculations. Employing a lattice based value $f_K^S(0)/f_\pi(0) = 0.970(2)$, we find $V_{us}/V_{ud} = 0.22918(88)$ compared to $V_{us}/V_{ud} = 0.23131(45)$ obtained from $R_A = \Gamma(K \rightarrow \mu\nu(\gamma))/\Gamma(\pi \rightarrow \mu\nu(\gamma))$. Those vector and axial-vector based $V_{us}/V_{ud}$ values exhibit a $2.2\sigma$ discrepancy. That tension may be relieved by a shift in the lattice $f_K^S(0)/f_\pi(0)$ towards the consistency range $0.961(4)$, changes in experimental input or new physics. Other implications of $R_V$ and $R_A$ are also discussed.

I. INTRODUCTION

The Standard Model (SM) of particle physics includes a three-generation Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix, $V_{ij}$, $i = u, c, t$, $j = d, s, b$, which satisfies unitarity, $V^{\dagger} = V^{-1}$. That condition gives rise to orthonormal relationships among its rows and columns. Of special interest is the first row constraint

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1. \tag{1}$$

Neglecting the tiny $|V_{ub}|^2 \simeq 2 \times 10^{-5}$ contribution [3], it simplifies to approximately the original Cabibbo [1] two-generation relationship

$$|V_{ud}|^2 + |V_{us}|^2 = 1. \tag{2}$$

A significant experimental deviation from eq. (2) would signal the presence of “new physics” beyond SM expectations. Examples of such “new physics” include charged Higgs scalars or leptoquarks, $Z'$ loop effects, exotic muon decay modes that modify the Fermi constant value etc.

Up until recently, eq. (2) appeared to be well satisfied [3, 4]. However, a novel dispersion relation (DR) approach [5] to super-allowed nuclear beta decay loop effects found an increase in the electroweak radiative corrections that reduced $V_{ud}$ from 0.97420(21) to

$$|V_{ud}| = 0.97370(10)_{NP}(10)_{RC} \quad \text{DR Result} \tag{3},$$

where NP (nuclear physics) and RC (radiative corrections) label the uncertainties. A more recent calculation [6] employing a different approach found

$$|V_{ud}| = 0.97389(10)_{NP}(14)_{RC} \quad \text{CMS} \tag{4}.$$

When used with the $V_{us}$ average [3] from $K_{l3}$ and $K_{l2}$ decays

$$|V_{us}| = 0.2243(9) \quad \text{Error scale factor 1.8}, \tag{5}$$

the two approaches lead to roughly 3 and 2 $\sigma$ deviations from unitarity respectively. That discrepancy could be a hint of “new physics” starting to show up as an effective deviation from CKM unitarity. Alternatively, nuclear physics (NP) effects and theoretical uncertainties might resolve the problem [7].

The $V_{us}$ situation also calls for some explanation. $K_{l3}$ $(K \rightarrow \pi l\nu)$ decays considered alone give a relatively small

$$|V_{us}| = 0.2231(8) \quad K_{l3} \text{ decays}, \tag{6}$$
while the ratio 

\[ R_A = \frac{\Gamma(K \rightarrow \mu \nu(\gamma))}{\Gamma(\pi \rightarrow \mu \nu(\gamma))}, \]  

is generally considered a more dependable constraint, since common uncertainties tend to cancel in the ratio. That is particularly important for lattice gauge theory calculations of \( \frac{f_{K^+}}{f_{\pi^+}} \). From the experimental constraint \( R_A = 1.3367(28) \), one finds

\[ \frac{|V_{us}| f_{K^+}}{|V_{ud}| f_{\pi^+}} = 0.2760(3). \]  

Then using the lattice value \[ f_{K^+} = 1.1932(19), \]  

one obtains

\[ \frac{|V_{us}|}{|V_{ud}|} = 0.2313(45), \]  

which assuming agreement with three-generation unitarity, implies

\[ |V_{ud}| = 0.97428(10) \text{ and } |V_{us}| = 0.2253(4). \]  

Those SM expectations are 2 or more \( \sigma \) different than some of the current values shown above. Since \( R_A \) is rather free of various theoretical and experimental uncertainties, it probably represents our best first row CKM constraint and must be taken seriously.

Here we point out and examine a weak vector current analog of \( R_A \) for which short-distance as well as some long-distance radiative corrections and muon lifetime normalization dependence (introduced for all beta decays including \( K_{l3} \) decays) also cancel in the ratio. A vector current analog for which the lattice gauge theory calculations may also be made more dependable The specific ratio we first consider is

\[ R_V = \frac{\Gamma(K_L \rightarrow \pi^0 e^+ \nu(\gamma))}{\Gamma(\pi^+ \rightarrow \pi^0 e^+ \nu(\gamma))}, \]

which compares radiative inclusive \( K_{l3} \) for the \( K_L \) and \( \tau_{e3} \) decay rates. We begin by recalling properties of the denominator which is generally viewed as a theoretically pristine quantity for measuring \( V_{ud} \); although not competitive with more precise determinations of that quantity. Our analysis will show that \( R_V \) is competitive as a normalization for \( K_{e3} \) and determination of \( V_{us} \) in much the same way and with nearly the same precision as \( R_A \).

**II. PION BETA DECAY**

The branching ratio \[ \Gamma(\pi^+ \rightarrow \pi^0 e^+ \nu(\gamma)) = 0.3988(23) \text{ s}^{-1}, \]

which can be compared with the rather precise SM theoretical prediction \[ \Gamma(\pi^+ \rightarrow \pi^0 e^+ \nu(\gamma)) = \frac{G^2_\mu |V_{us}|^2 m^2_{\pi^+} |f_{\pi^+}(0)|^2}{64 \pi^3} (1 + R_C) I_\pi, \]  

where

\[ R_C = 0.0334(10), \]

\[ I_\pi = \frac{32}{15} \left( 1 - \frac{\Delta}{2 m_{\pi^+}} \right)^3 \left( \frac{\Delta}{m_{\pi^+}} \right)^5 f(\epsilon, \Delta) = 7.376(1) \times 10^{-8}, \]

\[ f(\epsilon, \Delta) = \sqrt{1 - \epsilon} \left( 1 - \frac{9 \epsilon}{2} - 4 \epsilon^2 \right) + \frac{15}{2} \epsilon^2 \ln \frac{1 + \sqrt{1 - \epsilon}}{\sqrt{\epsilon}} - \frac{3}{7} \frac{\Delta^2}{(m_{\pi^+} + m_{\pi^0})^2}, \]

\[ \Delta = m_{\pi^+} - m_{\pi^0}, \quad \epsilon = m^2/\Delta^2, \]
where the $+0.0334(10)$ includes electroweak and quantum electrodynamics (QED) radiative corrections. Solving for $V_{ud}$ leads to

$$|V_{ud}| = 0.9739(29),$$  \hspace{2cm} (19)

That value is in accord with expectations from CKM unitarity; but not competitive with super-allowed nuclear beta decays which are more precise than the $|V_{us}|$ value. Even improvement by a factor of 2 or 3, which appears to be difficult but possible, would not make pion beta decay directly competitive for determining $V_{ud}$. However, currently, the $\pm 0.6\%$ fractional uncertainty in the pion beta decay rate is similar to individual $K_{l3}$ rates used in the determinations of $V_{us}$. So, it can be used to normalize $K_{l3}$ decay widths without a significant increase in the overall uncertainty. That will be an important feature that supports our following discussion.

### III. THE RATIO $R_V$

We begin by considering the $K_L(3\pi)$ partial decay width, traditionally normalized in terms of the muon lifetime derived Fermi constant, $G_\mu$. It has a form similar to eq. (14),

$$\Gamma(K_L \rightarrow \pi^+ e^+ \nu(\gamma)) = \frac{G_\mu^2 |V_{us}|^2 m_{K_L}^5 |f_K^Z(0)|^2}{192\pi^3} (1 + RC_K) I_K,$$  \hspace{2cm} (20)

with

$$RC_K = 0.0334(20),$$

$$I_K = 2 \left( \frac{m_{\pi}^+}{m_{K_L}} \right)^4 \left[ 1 + 2\lambda_+ \frac{m_{K_L}^2 + m_{\pi}^2}{m_{\pi}^+} \right] \left[ \frac{\beta_m (5\beta_m^2 - 3) E_{\pi}^4}{m_{\pi}^2} + 3 \ln \frac{E_m (1 + \beta_m)}{m_{\pi}^+} \right] - \frac{64\lambda_+ \beta_m^5 E_{\pi}^5}{5m_{\pi}^2 m_{K_L}^2}$$

$$= 0.15455(15)$$

and

$$\beta_m = \sqrt{1 - \left( \frac{m_{\pi}^+/E_m} \right)^2},$$

where $E_m = 0.26838$ GeV is the maximum pion energy in the $K_L$ rest frame and $\lambda_+ = 0.0282(4)$ parametrizes the energy dependence of the form factor $f_K^Z$. The radiative corrections (RC$_{\pi,K}$) in eq. (14) and eq. (20) are to a good approximation equal in magnitude and will cancel (up to the uncertainties) when we take the ratio. Cancellation also occurs for common uncertainties in short distance effects as well as some examples of “new physics.” SM electromagnetic effects for other kaon modes will differ from RC$_K = 0.0334(20)$ and do not fully cancel. Their uncertainties are included as part of the kaon contribution.

The usual method for extracting $V_{us}$ is to compare the $K_L(3\pi)$ partial width theory with experiment to obtain the constraint $f_K^Z(0)V_{us} = 0.2165(6)$. Employing a lattice gauge theory value for the form factor is then generally used to determine $V_{us}$. That number depends on the electroweak short and QED long-distance radiative corrections and the Fermi constant. As an alternative, we now normalize relative to pion beta decay which provides a different perspective on testing CKM unitarity and a means to search for the presence of “new physics”. Of course, a similar exercise can be carried out for any of the $K(3l)$ neutral and charged kaon decay modes. Then the results can be averaged to reduce the uncertainties.

The ratio $R_V$, defined in eq. (12), has the current experimental value

$$R_V^{exp} = \frac{\tau_\pi \times BR(K_L \rightarrow \pi^+ e^+ \nu(\gamma))}{\tau_{K_L} \times BR(\pi^+ \rightarrow \pi^0 e^+ \nu(\gamma))} = \frac{26.033(5)\times 0.4056(9)}{51.16(21)\times 1.038(6) \times 10^{-8}} = 1.9884(115)(93) \times 10^7,$$  \hspace{2cm} (23)

where the first uncertainty stems from the pion partial width and the second from the $K_L$ lifetime and branching ratio. The latter can be reduced by roughly a factor 1/2 by averaging over all $K_{l3}$ partial widths after accounting for differences in phase space, QED corrections, particle masses, a second form factor for muon modes and in the case of charged kaons strong isospin breaking. In that way, the average kaon partial width uncertainty is reduced to 0.3%, about half that due to the pion partial width.

The experimental value of eq. (23) should be compared with the SM prediction

$$R_V^{theory} = \frac{1}{3} \left( \frac{m_{K^+}}{m_{\pi^+}} \right)^5 \left( \frac{f_K^Z(0)}{f_\pi^Z(0)} \right)^2 V_{us}^2 I_K \frac{\lambda_{\pi}}{\lambda_{\pi}},$$

$$\hspace{2cm} \times \frac{V_{ud}}{V_{us}}$$

Equating theory and experiment leads to

$$\frac{f_K^Z(0)}{f_\pi^Z(0)} V_{us} = 0.2230(64)(54),$$

$$\hspace{2cm} (24)$$
with the first and second error coming from the pion and kaon widths respectively. The second error in eq. (25) includes small uncertainties from $I_K$ and $RC_K$. The radiative corrections (RC$_\pi$,$K$) and Fermi constant have cancelled, $I_K$ and $I_\pi$ are phase space integrals and $f^K_+(0)$ is the $K\pi$ vector transition form factor, which is 1 up to second order in SU(3) flavor breaking for the neutral kaon. (The charged kaon form factor requires a 2.8(3)% increase to fully account for its difference with the neutral kaon case. That shift is consistent with expectations due to strong isospin breaking resulting from pion-eta mixing.) The pion form factor, $f^\pi_+(0)$, is essentially 1 in the SU(2) flavor limit. The deviation from 1 is $O(10^{-5})$ and can be usually ignored. However, we retain the form factor ratio $f^K_+(0)/f^\pi_+(0)$ in our discussion, since for some lattice calculations the ratio may provide a means for extraneous lattice artifacts to cancel.

$$f^{K}_+(0) = 0.22230(64)(32)$$

Equating $R_V$ experiment and theory followed by averaging over all $K_{l3}$ modes, as shown in Table I, leads to

$$f^{K}_+(0)V_{ud} = 0.22230(64)(32),$$

where the central value has remained essentially unchanged; but, the kaon dependent uncertainty has been reduced by about a factor of $3/s$. We note that the $\chi^2$/degree of freedom for the five $K_{l3}$ modes was found to be an acceptable 0.984; so, it is not necessary to scale up the error. The goodness of the fit also helps validate the sign and relative size of the radiative corrections applied to the different $K$ decay modes.

Requiring $R_V$ and $R_A$ to have the same $V_{us}/V_{ud} = 0.23131(45)$ within errors implies $f^{K}_+(0)/f^{\pi}_+(0) = 0.9610(36)$. That value is in tension with the prevailing lattice result of roughly 0.970(2) for 2+1+1 quark flavors by about 2.2σ. In that comparison, we are ignoring cross correlations between $R_A$ and $R_V$ which would somewhat increase the significance. The discrepancy can also be illustrated by inserting $f^{K}_+(0)/f^{\pi}_+(0) = 0.970(2)$ in eq. (26) which leads to $V_{us}/V_{ud} = 0.22918(88)$. It differs from eq. (10) by a related 2.2σ.

The roughly 2σ discrepancy was observed already some time ago in global $K$ decay fits [19]. Our use of $R_V$ suggests that it is not likely to be an artifact of short-distance electroweak radiative corrections, muon lifetime $G_u$ normalization or “new physics” contributions common to numerator and denominator which cancel in the ratio. We note that a similar discrepancy occurs in 2+1 flavor lattice gauge theories where comparison shows preference for $f^{K}_+(0)/f^{\pi}_+(0) = 0.961$ as compared with the 0.968 lattice value. Indeed, it suggests that a reduction to $f^{K}_+(0)/f^{\pi}_+(0) = 0.9614$ may be a prerequisite for CKM unitarity. To resolve this problem additional lattice studies are needed.

Employing $V_{us}/V_{ud} = 0.22918(88)$ and three-generation unitarity implies $V_{ud} = 0.97473(19)$ which exceeds eq. (11). That makes it more difficult to reconcile with super-allowed nuclear beta decay, further arguing against the 0.970 value for $f^{K}_+(0)/f^{\pi}_+(0)$.

We averaged over all five $K_{l3}$ decay modes to reduce the second error in eq. (25) by roughly a factor of $3/s$. Future improvements in kaon measurements are expected to further reduce that part of the uncertainty by another $3/s$, leaving the partial width of pion beta decay as the dominant uncertainty in the error budget by about a factor of 3. Improving the pion partial beta decay width by a factor of 2 to 3 would bring the overall experimental error budget for $R_V$ down by roughly a factor of 2. Such a reduction in the $V_{us}/V_{ud}$ uncertainty derived from $R_V$ along with a similar improvement in $R_A$ will together strongly restrict or provide evidence for the existence of “new physics” at rather high significance. The latter scenario would be more likely if the $V_{ud}$ from super-allowed beta decays continues to show a deviation from CKM unitarity. Neutron lifetime and decay asymmetry precision measurements should also help resolve the $V_{ud}$ problem. Indeed, the larger radiative corrections found in [5, 6] combined with a unitarity favored 0.97428 for $|V_{ud}|$ and 1.2762 for $g_A$ predict a neutron lifetime of about 878 s with a small uncertainty.

More precise experimental measurements are clearly needed to reconcile CKM unitarity or unveil evidence for “new physics.” Our study of pion beta decay and the utility of $R_V$ will hopefully reinvigorate interest in that experimental effort. More specifically, on the basis of its complementary role, we strongly advocate a new experiment on pion beta
decay designed to improve measurement of that rare branching ratio by an overall factor of 2 to 3 [10]. In addition, we encourage the lattice gauge theory community to examine the possibility of a reduced uncertainty to 0.001 and check for a possible shift in the $f_K^+(0)$ central value when the ratio $f_K^+(0)/f_K^-(0)$ is computed in coincidence.

Note Added: While this work was being written up, an interesting paper [30] relevant for our discussion was posted.

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