Pleasant extensions retaining algebraic structure, I

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Abstract

In the recent papers [4] and [3] we introduced some new techniques for constructing an extension of a probability-preserving system $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ that enjoys certain desirable properties in connexion with the asymptotic behaviour of some related nonconventional ergodic averages.

The present paper is the first of two that will explore various refinements and extensions of these ideas. This first part is dedicated to some much more general machinery for the construction of extensions that can be used to recover the results of [4, 3]. It also contains two relatively simple new applications of this machinery to the study of certain families of nonconventional averages, one in discrete and one in continuous time (convergence being a new result for the latter).

In the forthcoming second part [2] we will introduce the problem of describing the characteristic factors and the limit of the linear nonconventional averages

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{nP_i}$$

when the directions $p_1, p_2, \ldots, p_k \in \mathbb{Z}^d$ are not assumed to be linearly independent, and provide a fairly detailed solution in the case when $k = 3$, $d = 2$ and any pair of directions is linearly independent. This will then be used to prove the convergence in $L^2(\mu)$ of the quadratic nonconventional averages

$$\frac{1}{N} \sum_{n=1}^{N} (f_1 \circ T_1^{n^2})(f_2 \circ T_2^{n^2}T_2^n).$$

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1 Introduction

Suppose that $T_1, T_2, \ldots, T_k \curvearrowright (X, \mu)$ is a system of commuting invertible transformations on a standard Borel probability space. To such a system we can associate various ‘nonconventional’ ergodic averages, such as the averages

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T_{n}^{m_i}$$

or their more complicated relatives of the form

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{p_i(n)}$$

for an action $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ and polynomial mappings $p_i : \mathbb{Z} \rightarrow \mathbb{Z}^d$ for $i = 1, 2, \ldots, k$ (we sometimes refer to these as ‘linear’ and ‘polynomial’ averages respectively).

The linear averages have been the subject of considerable study since they first emerged from Furstenberg’s ergodic theoretic approach to Szemerédi’s Theorem and generalizations [14, 15], and more recently their polynomial relatives have received similar attention since the extension of Furstenberg’s work to an appropriate polynomial setting [5, 6]. These applications to Arithmetic Ramsey Theory typically require that certain related scalar averages ‘stay large’ as $N \rightarrow \infty$, but it
quickly became clear that the more fundamental question of their norm convergence in $L^2(\mu)$ posed an interesting challenge in its own right. After several important partial results [8, 9, 10, 28, 30, 16, 19, 21, 31], the convergence of the linear averages in general was settled by Tao in [29]. By contrast, our understanding of the polynomial case remains poor.

In [4] we gave a new proof of convergence in the linear case, more classically ergodic theoretic than Tao’s (which relies on a conversion of the problem into an equivalent quantitative assertion about the shift transformations on $(\mathbb{Z}/N\mathbb{Z})^d$). In this paper and its sequel [2] we shall further develop the methods of [4] to provide some more versatile machinery, and illustrate its use with a proof of norm convergence for the new polynomial instance

$$\frac{1}{N} \sum_{n=1}^{N} (f_1 \circ T_1^{n^2})(f_2 \circ T_1^{n^2} T_2^{m}).$$

Most analyses of such questions rest on the fundamental notion of ‘characteristic factors’ for a system of averages, first made explicit by Furstenberg and Weiss in their work [16] on a case of polynomial averages involving only a single underlying transformation. Here a tuple of factors $\xi_i : (X, \mu, T) \to (Y_i, \nu_i, S_i)$, $i = 1, 2, \ldots, k$, of a $\mathbb{Z}^d$-system will be termed characteristic for some tuple of polynomial mappings $p_i : \mathbb{Z} \to \mathbb{Z}^d$ if

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{d} f_i \circ T_i^{p_i(n)} \sim \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{d} E_\mu(f_i | \xi_i) \circ T_i^{p_i(n)},$$

for any $f_1, f_2, \ldots, f_d \in L^\infty(\mu)$, where we write $f_N \sim g_N$ to denote that $\|f_N - g_N\|_2 \to 0$ as $N \to \infty$.

In the case $d = 1$ (so we deal only with powers of a single transformation) the possible structures of such characteristic factors have been completely understood as those of pro-nilsystems, first in the case of linear averages [21, 32] and then also for their higher-degree polynomial relatives [20, 24].

However, for general $d$, even when each $p_i$ is a linear mapping the possible characteristic factors seem much more complicated; although it can be shown abstractly that minimal characteristic factors exist, they have so far largely resisted useful description. The chief innovations of the papers [4, 3] were methods for constructing an extension a system $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ to a larger system $\tilde{T} : \mathbb{Z}^2 \curvearrowright (\tilde{X}, \tilde{\mu})$ (which depends on the linear mappings $p_i$) in which a characteristic tuple of factors for the convergence of these linear averages could be found with an especially
simple structure. Knowing this structure then enabled (together with an appeal to various existing machinery) new ergodic-theoretic proofs of first the convergence of such averages in $L^2(\mu)$ and then the Multidimensional Multiple Recurrence Theorem.

Effectively, this approach shifts our viewpoint from the setting of an individual system of commuting transformations (and factors thereof) to the category of all such systems. While a given system can fail to exhibit among its own factors all of the structural features that can be exploited to examine nonconventional averages, these emerge upon passing to a sufficiently enriched extension of that system (or, equivalently, upon considering more general joinings of it to other systems of various special kinds). We should note that it seems likely that the extensions with improved behaviour for our purposes seem to be highly arbitrary and will often be obtained after several steps that both involve some arbitrary choices and could be performed in different orders. In this sequence of papers we will generally refer to extensions that admit some simple characteristic factors for a given system of averages as pleasant extensions of the original system, although the particular class of ‘simple’ characteristic factors that we use will vary from one context to the next.

In fact, the strategy of passing to an extension where the behaviour of nonconventional averages is more easily described also has a precedent in various earlier papers, notably the work [16] of Furstenberg and Weiss (we will return to the relationship between their work and ours several times later). However, while in their work it can later be proved that the characteristic factors of the original system must have taken the same form as those obtained in the extension, in the multidimensional setting the passage to an extension leads to a genuine reduction in complexity of description of the characteristic factors, and recent works in this area have begun to exploit this idea much more extensively. (For example, Bernard Host [18] has given a new and rather more efficient construction of an extension with much the same desirable properties as those constructed in [4].)

Here we develop further this approach to nonconventional averages. An undesirable feature of the constructions of extensions in [4, 3, 18] is that the lifted commuting transformations $\tilde{T}_i$ generally lose any algebraic structure that might have been known to hold a priori among the $T_i$. Two particular kinds of structure that can be of interest for applications are:

- the existence of roots, such as some $S_i$ such that $T_i = S_i^2$;
- linear relations that may hold among the original transformations $T_i$, such as $T_3 = T_1T_2$. 
In this paper and its sequels we will begin to see what pleasant extensions can be found that retain such additional features. Concerning the existence of roots, in Section 4.2 we will find that one can recover essentially the same result as in [4]:

**Theorem 1.1** (Pleasant extensions of linearly independent linear averages). Any $\mathbb{Z}^d$-system $(X, \mu, T)$ has an extension $\pi : (\tilde{X}, \tilde{\mu}, \tilde{T}) \to (X, \mu, T)$ such that for any linearly independent $p_1, p_2, \ldots, p_k \in \mathbb{Z}^d$ the averages

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{d} f_i \circ \tilde{T}^{np_i}$$

admit a characteristic tuple of factors of the form

$$\xi_i := \zeta_0^{\tilde{T}^{p_i}} \lor \bigvee_{j \neq i} \zeta_0^{\tilde{T}^{p_i-p_j}},$$

where for a transformation $S$ we write $\zeta_0^S$ for some factor map generating, up to negligible sets, the $\sigma$-algebra of sets left invariant by $S$.

The point here is that, setting $\Lambda := p_1 + \cdots + p_d \leq \mathbb{Z}^d$, the older constructions give only an extension of the subaction $T|\Lambda$. Notice that we obtain a single extended system $(\tilde{X}, \tilde{\mu}, \tilde{T})$ that enjoys the above simplified characteristic factors for every tuple of linearly independent directions. We will refer to such an extension as a **pleasant extension for linearly independent linear averages**.

More importantly than this result, this first paper will introduce various general ideas needed in preparation for the more sophisticated results of the sequel [2], especially the notion of ‘satedness’ for probability-preserving systems (Subsection 3.1 below) which will be relied on repeatedly in proving all the main results of that paper.

In [2] we shall begin to address the second kind of algebraic structure listed above. We will examine one simple case in detail, but conjecture that our methods could eventually be extended to a much more general result. We will consider the case of three directions $T^{p_1}, T^{p_2}, T^{p_3}$ in a $\mathbb{Z}^2$-system that are in general position with the origin $0$: that is, such that no three of the points $p_1, p_2, p_3$ and $0$ lie on a line. For the associated linear averages we will show how to construct an extended $\mathbb{Z}^2$-system in which the characteristic factors take a special form, which will give us a notion of pleasant extensions for triple linear averages subject to this kind of linear dependence.

Of course, we do pay a price for insisting that the $\mathbb{Z}^2$-structure of the action be preserved, in that the characteristic factors we eventually obtain are not as simple.
as the pure joins of isotropy factors that emerge in the linearly independent case. The additional ingredients we need are $\mathbb{Z}^2$-actions given by pairs of commuting rotations on two-step pro-nilmanifolds (or, to be precise, direct integrals of such actions).

**Theorem 1.2** (Pleasant extensions for linearly dependant triple linear averages). Any system $T : \mathbb{Z}^2 \curvearrowright (X, \mu)$ has an extension $\pi : (\tilde{X}, \tilde{\mu}, \tilde{T}) \to (X, \mu, T)$ such that for any $p_1, p_2, p_3 \in \mathbb{Z}^2$ that are in general position with the origin the averages

$$\frac{1}{N} \sum_{n=1}^{N} (f_1 \circ \tilde{T}^{n p_1})(f_2 \circ \tilde{T}^{n p_2})(f_3 \circ \tilde{T}^{n p_3}), \quad f_1, f_2, f_3 \in L^\infty(\tilde{\mu}),$$

admit characteristic factors $\xi_i, \ i = 1, 2, 3$, of the form

$$\xi_i = \zeta_{T}^{p_1} \vee \zeta_{T}^{p_2} \vee \zeta_{T}^{p_3} \vee \zeta_{\text{nil},2},$$

where $\zeta_{\text{nil},2}$ is the maximal factor of $(X, \mu, T)$ generated by direct integrals of two-step nilsystems (the precise meaning of this will be elaborated in [2]).

From this we will also be able to deduce a pleasant-extensions result for certain double quadratic averages, following an application of the well-known van der Corput estimate.

**Theorem 1.3** (Pleasant extensions for some double quadratic averages). Any system of two commuting transformations $T_1, T_2 : \mathbb{Z} \curvearrowright (X, \mu)$ has an extension $\pi : (\tilde{X}, \tilde{\mu}, \tilde{T}_1, \tilde{T}_2) \to (X, \mu, T_1, T_2)$ in which the averages

$$\frac{1}{N} \sum_{n=1}^{N} (f_1 \circ \tilde{T}_1^{n^2})(f_2 \circ \tilde{T}_2^{n^2}), \quad f_1, f_2 \in L^\infty(\tilde{\mu}),$$

admit characteristic factors of the form

$$\xi_1 = \xi_2 := \bigvee_{m \geq 1} \zeta_{T}^{n^2} \vee \zeta_{T}^{m} \vee \zeta_{\text{nil},2}.$$
Although this new convergence result is modest in itself — it is only one special case of the much more general conjecture of norm convergence for all multidimensional polynomial averages, which remains out of reach for the time being — we suspect that the methods developed in this paper and its sequel will ultimately have more far-reaching relevance to this question, and potentially to other questions on the structure of joinings between different classes of system in the ergodic theory of $\mathbb{Z}^d$-actions.

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2 Background definitions and general results

2.1 Measurable functions and probability kernels

We will work exclusively in the category of standard Borel probability spaces $(X, \Sigma_X, \mu)$, and so will often suppress mention of their $\sigma$-algebras. Given standard Borel spaces $X$, $Y$, and $Z$, a completed Borel probability measure $\mu$ on $X$, a $\sigma$-subalgebra $\Xi \subseteq \Sigma_X$ and measurable functions $\phi : X \to Y$, $\psi : X \to Z$, we will write that $\psi$ is $\mu$-virtually $\Xi$-measurable if there is some $\Xi$-measurable map $\psi_1 : X \to Z$ such that $\psi = \psi_1$ $\mu$-almost everywhere, or similarly that $\psi$ is $\mu$-virtually a function of $\phi$ if there is some measurable function $\theta : Y \to Z$ such that $\psi = \theta \circ \phi$; these two definitions are related by the usual correspondence (up to negligible sets) between $\sigma$-subalgebras and maps that generate them for standard Borel spaces.

Factor maps from one probability space to another comprise the simplest class of morphisms between such spaces, but we will sometimes find ourselves handling also a weaker class of morphisms. Suppose that $Y$ and $X$ are standard Borel spaces. Then by a probability kernel from $Y$ to $X$ we understand a function $P : Y \times \Sigma_X \to [0, 1]$ such that

- the map $y \mapsto P(y, A)$ is $\Sigma_Y$-measurable for every $A \in \Sigma_X$;
- the map $A \mapsto P(y, A)$ is a probability measure on $\Sigma_X$ for every $y \in Y$. 

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The first of the above conditions is then the natural sense in which the assignment $y \mapsto P(y, \cdot)$ of a probability distribution is measurable in $y$; indeed, a popular alternative definition of probability kernel is as a measurable function from $Y$ to the set $\Pr X$ of Borel probability measures on $X$. In ergodic theory this notion is also often referred to as a ‘quasifactor’: see, for example, Chapter 8 of Glasner [17], where this alternative convention and notation are used. We will write $P : Y \xrightarrow{\mathbb{P}} X$ when $P$ is a probability kernel from $Y$ to $X$.

Given a kernel $P : Y \xrightarrow{\mathbb{P}} X$ and a probability measure $\nu$ on $Y$, we define the pushforward measure $P_\# \nu$ on $X$ by

$$P_\# \nu(A) := \int_Y P(y, A) \nu(dy);$$

this measure on $X$ can be interpreted as the law of a member of $X$ selected randomly by first selecting a member of $Y$ with law $\nu$ and then selecting a member of $X$ with law $P(y, \cdot)$. This extends standard deterministic notation: given a measurable function $\phi : Y \to X$, we may associate to it the deterministic probability kernel given by $P(y, \cdot) = \delta_{\phi(y)}$ (the point mass at the image of $y$ under $\phi$), and now $P_\# \nu$ is the usual push-forward measure $\phi_\# \nu$.

Certain special probability kernels naturally serve as adjoints to probability-preserving maps, in the sense of the following theorem.

**Theorem 2.1.** Suppose that $Y$ and $X$ are standard Borel spaces, that $\mu$ is a probability measure on $X$ and that $\phi : X \to Y$ is a measurable map. Then, denoting the pushforward $\phi_\# \mu$ by $\nu$, there is a $\nu$-almost surely unique probability kernel $P : Y \xrightarrow{\mathbb{P}} X$ such that $\mu = P_\# \nu$ and which represents the conditional expectation with respect to $\phi$: for any $f \in L^1(\mu)$, the function

$$x_1 \mapsto \int_X f(x) P(\phi(x_1), dx)$$

is a version of the $\mu$-conditional expectation of $f$ with respect to $\phi^{-1}(\Sigma_Y)$.

We also write that this $P$ represents the disintegration of $\mu$ over $\phi$. A general probability kernel $P : Y \xrightarrow{\mathbb{P}} X$ represents the disintegration over $\phi$ of some measure that pushes forward onto $\nu$ if and only if $\int_A P(x, \cdot) \nu(dx)$ and $\int_B P(y, \cdot) \nu(dy)$ are mutually singular whenever $A \cap B = \emptyset$.

**Proof** See Theorem 6.3 in Kallenberg [22].
2.2 Systems, subactions and factors

In this paper we shall spend a great deal of time passing up and down from systems to extensions or factors. Moreover, sometimes one system will appear as a factor of a ‘larger’ system in several different ways (most obviously, when we work with a system recovered under the different coordinate projections from some self-joining). For this reason the notational abuse of referring to one system as a factor of another but leaving the relevant factor map to the understanding of the reader, although popular and useful in modern ergodic theory, seems dangerous here, and we shall carefully avoid it. In its place we substitute the alternative abuse, slightly safer in our circumstances, of often referring only to the factor maps we use, and leaving either their domain or target systems to the reader’s understanding. Let us first set up some notation to support this practice.

If \( \Gamma \) is a l.c.s.c. group, by a \( \Gamma \)-system (or, if \( \Gamma \) is clear, just a system) we understand a coarsely-continuous probability-preserving action \( T : \Gamma \curvearrowright (X, \mu) \) on a standard Borel probability space. We will often alternatively denote this space and action by \( (X, \mu, T) \), or by a single boldface letter such as \( X \). If \( \Lambda \leq \Gamma \) is a closed subgroup we denote by \( T|\Lambda : \Lambda \curvearrowright (X, \mu) \) the action defined by \( (T|\Lambda)\gamma := T\gamma \) for \( \gamma \in \Lambda \), and refer to this as a subaction, and if \( X = (X, \mu, T) \) is a \( \Gamma \)-system we write similarly \( X|\Lambda \) for the system \( (X, \mu, T|\Lambda) \) and refer to it as a subaction system.

A factor from one system \( (X, \mu, T) \) to another \( (Y, \nu, S) \) is a Borel map \( \pi : X \to Y \) such that \( \pi \# \mu = \nu \) and \( \pi \circ T\gamma = S\gamma \circ \pi \) for all \( \gamma \in \Gamma \). Given such a factor, we sometimes write \( T|\pi \) to denote the action \( S \) with which \( T \) is intertwined by \( \pi \).

Any factor \( \pi : X \to Y \) specifies a globally \( T \)-invariant \( \sigma \)-subalgebra of \( \Sigma_X \) in the form of \( \pi^{-1}(\Sigma_Y) \). Two factors \( \pi \) and \( \psi \) are equivalent if these \( \sigma \)-subalgebras of \( \Sigma_X \) that they generate are equal up to \( \mu \)-negligible sets, in which case we shall write \( \pi \simeq \psi \); this clearly defines an equivalence relation among factors.

It is a standard fact that in the category of standard Borel spaces equivalence classes of factors are in bijective correspondence with equivalence classes of globally invariant \( \sigma \)-subalgebras under the relation of equality modulo negligible sets. A treatment of these classical issues may be found, for example, in Chapter 2 of Glasner [17]. Given a globally invariant \( \sigma \)-subalgebra in \( X \), a choice of factor \( \pi : X \to Y \) generating that \( \sigma \)-subalgebra will sometimes be referred to as a coordinatization of the \( \sigma \)-subalgebra. Importantly for us, some choices of coordinatizing factor \( \pi \) may reveal some additional structure of the factors more clearly than others. For this reason, given one coordinatization \( \pi : X \to Y \) and an isomorphism \( \psi : Y \to Y' \), we shall sometimes refer to the composition \( \psi \circ \pi : X \to Y' \)
as a **recoordinatization** of \( \pi \).

More generally, the factor \( \pi : (X, \mu, S) \to (Y, \nu, S) \) **contains** \( \psi : (X, \mu, T) \to (Z, \theta, R) \) if \( \pi^{-1}(\Sigma_Y) \supseteq \psi^{-1}(\Sigma_Z) \) up to \( \mu \)-negligible sets. It is a classical fact that in the category of standard Borel spaces this inclusion is equivalent to the existence of a **factorizing** factor map \( \phi : (Y, \nu, S) \to (Z, \theta, R) \) with \( \psi = \phi \circ \pi \) \( \mu \)-a.s., and that a measurable analog of the Schroeder-Bernstein Theorem holds: \( \pi \simeq \psi \) if and only if a single such \( \phi \) may be chosen that is invertible away from some negligible subsets of the domain and target. If \( \pi \) contains \( \psi \) we shall write \( \pi \trianglerighteq \psi \) or \( \psi \triangleleft \pi \). It is clear that (up to set-theoretic niceties) this defines a partial order on the class of \( \simeq \)-equivalence classes of factors of a given system. We will extend the above terminology to that of **coordinatizations** and **recoordinatizations** of families of factors of a system in terms of the appropriate commutative diagram of isomorphisms.

Given a \( \Gamma \)-system \( X = (X, \mu, T) \), the \( \sigma \)-algebra \( \Sigma_X^T \) of sets \( A \in \Sigma_X \) for which \( \mu(A \triangle T^\gamma(A)) = 0 \) for all \( \gamma \in \Gamma \) is \( T \)-invariant, so defines a factor of \( X \). More generally, if \( \Gamma \) is Abelian and \( \Lambda \leq \Gamma \) is closed then we can consider the \( \sigma \)-algebra \( \Sigma_X^{T|\Lambda} \) generated by all \( T|\Lambda \)-invariant sets: we refer to this as the \( \Lambda \)-**isotropy factor** and write \( Z^{T|\Lambda}_0 \) for some new system that we adopt as the target for a factor map \( \zeta^{T|\Lambda} : X \to Z^{T|\Lambda}_0 \) that generates \( \Sigma_X^{T|\Lambda} \), and similarly. Note that in this case the Abelianness condition (or, more generally, the condition that \( \Lambda \leq \Gamma \)) is needed for this to be a globally \( T \)-invariant factor. If \( T_1 \) and \( T_2 \) are two commuting actions of the same group \( \Gamma \) on \( (X, \mu) \) then we can define a third action \( T_1 T_2^{-1} \) by setting \( (T_1 T_2^{-1})^\gamma := T_1^\gamma T_2^{-\gamma} \), and in this case we sometimes write \( \zeta^{T_1 T_2^{-1}} : X \to Z^{T_1 T_2^{-1}}_0 \) in place of \( \zeta^{T_1 T_2^{-1}} : X \to Z^{T_1 T_2^{-1}}_0 \). Similarly, if \( S \subseteq \Gamma \) and \( \Lambda \) is the group generated by \( \Lambda \), we will sometimes write \( Z^{T|\sigma}_{0} \) for \( Z^{T|\Lambda}_0 \), and similarly.

An important construction of new systems from old is that of **relatively independent products**. If \( Y = (Y, \nu, S) \) is some fixed system and \( \pi_i : X_i = (X_i, \mu_i, T_i) \to Y \) is an extension of it for \( i = 1, 2, \ldots, k \) then we define the relatively independent product of the systems \( X_i \) over their factor maps \( \pi_i \) to be the system

\[
\prod_{\{\pi_1 = \pi_2 = \ldots = \pi_k\}} X_i = \left( \prod_{\{\pi_1 = \pi_2 = \ldots = \pi_k\}} X_i, \bigotimes_{\{\pi_1 = \pi_2 = \ldots = \pi_k\}} \mu_i, T_1 \times T_2 \times \cdots \times T_k \right)
\]
where

\[
\prod_{\{\pi_1=\pi_2=\ldots=\pi_k\}} X_i := \{(x_1, x_2, \ldots, x_k) \in X_1 \times X_2 \times \cdots \times X_k : \\
\pi_1(x_1) = \pi_2(x_2) = \ldots = \pi_k(x_k)\},
\]

\[
\bigotimes_{\{\pi_1=\pi_2=\ldots=\pi_k\}} \mu_i = \int_Y \bigotimes_{i=1}^k P_i(y, \cdot) \nu(dy)
\]

and \( P_i : Y \overset{P_i}{\to} X_i \) is a probability kernel representing the disintegration of \( \mu_i \) over \( \pi_i \). In case \( k = 2 \) we will write this instead as \( X_1 \times_{\{\pi_1=\pi_2\}} X_2 \), and in addition if \( X_1 = X_2 = X \) and \( \pi_1 = \pi_2 = \pi \) then we will abbreviate this further to \( X \times_{\pi} X \), and similarly for the individual spaces and measures.

### 2.3 Measurable selectors

At several points in this paper we need to appeal to some basic results on the existence of measurable selectors, often as a means of making rigorous a selection of representatives of one or another kind of data above the ergodic components of a non-ergodic system.

**Theorem 2.2.** Suppose that \((X, \Sigma_X)\) and \((Y, \Sigma_Y)\) are standard Borel spaces, that \(A \subseteq X\) is Borel and that \(\pi : X \to Y\) is a Borel surjection. Then the image \(\pi(A)\) lies in the \(\nu^c\)-completion of \(\Sigma_Y\) for every Borel probability measure \(\nu\) on \((Y, \Sigma_Y)\) with completion \(\nu^c\), and for any such \(\nu\) there is a map \(f : B \to A\) with domain \(B \in \Sigma_Y\) such that \(B \subseteq \pi(A)\), \(\nu^c(\pi(A) \setminus B) = 0\) and \(\pi \circ f = \text{id}_B\).

**Proof** See, for example, 423O and its consequence 424X(h) in Fremlin [13].

**Definition 2.3** (Measurable selectors). We refer to a map \(f\) as given by the above theorem as a **measurable selector** for the set \(A\).

**Remark** We should stress that this is only one of several versions of the ‘measurable selector theorem’, due variously to von Neumann, Jankow, Lusin and others. Note in particular that in some other versions a map \(f\) is sought that select points of \(A\) for strictly all points of \(\pi(A)\). In the above generality we cannot guarantee that a strictly-everywhere selector \(f\) is Borel, but only that it is Souslin-analytic and hence universally measurable (of course, from this the above version follows at once). On the other hand, if the map \(\pi|_A\) is countable-to-one, then a version of
the result due to Lusin does guarantee a strictly-everywhere Borel selector \( f \). This version has already played a significant rôle in our corner of ergodic theory in the manipulation of the Conze-Lesigne equations (see, for example, [8, 16, 7]), and so we should be careful to distinguish it from the above. A thorough account of all these different results and their proofs can be found in Sections 423, 424 and 433 of Fremlin [13].

In the right circumstances it is possible to strengthen Theorem 2.2 to obtain a Borel selector that is invariant under a group of transformations, by making use of a coordinatization of the invariant factor.

**Proposition 2.4.** Suppose that \((X, \Sigma_X)\) and \((Y, \Sigma_Y)\) are standard Borel spaces, \(A \subseteq X\) is Borel and \(\pi : X \to Y\) is a surjective Borel map, and in addition that \(T : \Gamma \curvearrowright (X, \Sigma_X)\) is a jointly measurable action of a l.c.s.c. group such that \(\pi\) is a factor map, so \(\pi \circ T^\gamma = S^\gamma \circ \pi\) for some jointly measurable action \(S : \Gamma \curvearrowright (Y, \Sigma_Y)\), and that \(A\) is \(T\)-invariant. Then for any \(S\)-invariant probability measure \(\nu\) on \((Y, \Sigma_Y)\) with completion \(\nu^c\) there are an \(S\)-invariant set \(B \subseteq \Sigma_Y\) such that \(B \subseteq \pi(A)\) and \(\nu^c(\pi(A) \setminus B) = 0\) and an \(S\)-invariant map \(f : B \to A\) such that \(\pi \circ f = \text{id}_B\).

**Proof** Let \(f_0 : B_0 \to A\) be an ordinary measurable selector as given by Theorem 2.2 and let \(\nu\) be any \(S\)-invariant probability measure on \((Y, \Sigma_Y)\). This \(B_0\) must be \(\nu\)-almost \(S\)-invariant, simply because \(\pi(A)\) is \(T\)-invariant and \(\nu^c(B \Delta \pi(A)) = 0\). Using local compactness and second countability, let \((F_i)_{i \geq 1}\) be a countable compact cover of \(\Gamma\), and also let \(m_\Gamma\) be a left-invariant Haar measure on \(\Gamma\). From the joint measurability of \(T\) it follows that the set

\[
B := \{y \in Y : m_\Gamma \{y \in Y : S^\gamma(y) \in Y \setminus B_0\} = 0\} = \bigcap_{i \geq 1} \{y \in Y : m_\Gamma \{y \in F_i : S^\gamma(y) \in B_0\} = m_\Gamma(F_i)\}
\]

is Borel, \(T\)-invariant and satisfies \(\nu(B_0 \Delta B) = 0\).

We now let \(\zeta : (Y, \Sigma_Y, \nu) \to (Z, \Sigma_Z, \theta)\) be any coordinatization of the invariant factor \(\Sigma^\Gamma_Y\); it is easy to see that this may be chosen so that there exists some \(C \in \Sigma_Z\) such that \(B = \zeta^{-1}(C)\). We can now use \(B\) and \(\zeta\) to ‘tidy up’ our original selector \(f_0\). Indeed, by the \(S\)-invariance of \(\zeta\) and the fact that for every \(y \in B\) we have \(S^\gamma(y) \in B_0\) for some (indeed, almost all) \(\gamma \in \Gamma\), we must have \(\zeta(B_0) \supseteq C\). Therefore by applying the ordinary Measurable Selector Theorem a second time we can find a Borel subset \(D \subseteq \Sigma_Z\) with \(D \subseteq C\) and \(\theta(C \setminus D) = 0\) and a Borel section \(\eta : D \to B_0\) such that \(\zeta \circ \eta = \text{id}_D\); and so now replacing \(B_0\) with \(B\) and the map \(f_0\) with \(f : y \mapsto f_0(\eta(\zeta(y)))\) completes the proof.
Definition 2.5 (Invariant measurable selectors). We refer to a map \( f \) as given by the above proposition as a \( T \)-invariant measurable selector for the set \( A \).

3 Idempotent classes of systems

3.1 Basic properties of idempotent classes

Central to this paper will be a systematic exploitation of a property of certain systems according to which they can be joined to certain other classes of system only in simple ways. This key definition, although very abstract and very simple, will repeatedly prove surprisingly powerful. We will introduce it after some other preliminaries about classes of \( \Gamma \)-systems.

Lemma 3.1. Suppose that \( C \) is a class of \( \Gamma \)-systems (formally, \( C \) is a subcategory of \( \Gamma \text{-Sys} \)) that contains the trivial system and is closed under isomorphisms, arbitrary joinings and inverse limits. Then any \( \Gamma \)-system \( X \) has an essentially unique maximal factor in the class \( C \).

Proof. It is clear that under the above assumption the class

\[ \{ \Xi \subseteq \Sigma_X : \Xi \text{ is a } T\text{-invariant } \sigma\text{-subalgebra such that the associated factor is in } C \} \]

is nonempty (it contains \( \{ \emptyset, X \} \)), upwards directed (because \( C \) is closed under joinings) and closed under taking \( \sigma \)-algebra completions of increasing unions (because \( C \) is closed under inverse limits). There is therefore a maximal \( \sigma \)-subalgebra in this set.

Definition 3.2 (Idempotence). A class of systems \( C \) is idempotent if it contains the trivial system and is closed under isomorphisms, joinings and inverse limits. In this case, we will write that \( X \) is a \( C \)-system if \( X \) is a system in the class \( C \), and for arbitrary \( X \) we write \( \zeta_X^C : X \to CX \) for an arbitrarily-chosen coordinatization of its maximal \( C \)-factor given by the above lemma.

It is clear that if \( \pi : X \to Y \) then \( \zeta_X^C \supseteq \zeta_Y^C \circ \pi \), and so there is an essentially unique factorizing map, which we denote by \( C\pi \), that makes the following diagram commute:
In addition, we shall abbreviate \( X \times \zeta \times X \) to \( X \times \zeta X \), and similarly for the individual spaces and measures defining these systems.

The reason for this terminology lies in the observation that the assignment \( X \mapsto \zeta X \times \zeta X \) defines an autofunctor of the category \( \Gamma\text{-Sys} \), and the assignment \( X \mapsto \zeta X \) defines a natural transformation from the identity functor to this autofunctor. We can work with such functors quite generally, and a simple definition-chase shows that a functor \( F : \Gamma\text{-Sys} \to \Gamma\text{-Sys} \) together with a natural transformation \( \text{id}_{\Gamma\text{-Sys}} \to F \) correspond to a class of systems \( C \) as above if and only if \( F \) is idempotent (that is, \( F(FX) = FX \) for every \( X \)) and the natural transformation is the identity on the subcategory of systems of the form \( FX \). Indeed, it would be possible to develop the theory of the coming sections by working entirely with autofunctors rather than classes, but I do not know of any examples of autofunctors that are useful as such but do not arise from idempotent classes as above.

The name we give for our next definition is also motivated by this relationship with functors.

**Definition 3.3 (Order continuity).** A class of \( \Gamma \)-systems \( C \) is **order continuous** if whenever \( (X_m)_{m \geq 0}, (\psi_{(k)})_{m \geq k \geq 0} \) is an inverse sequence of \( \Gamma \)-systems with inverse limit \( X \), \( (\psi_{(m)})_{m \geq 0} \) we have

\[
\zeta X = \bigvee_{m \geq 0} \zeta X_{(m)} \circ \psi_{(m)} :\]

that is, the maximal \( C \)-factor of the inverse limit is simply given by the (increasing) join of the maximal \( C \)-factors of the contributing systems.

**Examples** The following idempotent classes will be of particular importance (in all cases idempotence is routine to check):

1. Given a fixed normal subgroup \( \Lambda \trianglelefteq \Gamma \), \( Z_0^\Lambda \) denotes the class of systems for which the \( \Lambda \)-subaction is trivial.
2. More generally, for $\Lambda$ as above and any $n \in \mathbb{N}$ we let $\mathcal{Z}_n^\Lambda$ denote the class of systems on which the $\Lambda$-subaction is a distal tower of height at most $n$, in the sense of direct integrals of compact homogeneous space data introduced in [1] to allow for the case of non-ergodic systems.

3. We can modify the previous example by placing some additional restrictions on the permissible distal towers: for example, $\mathcal{Z}_{\Lambda,Ab}^n$ comprises the systems with $\Lambda$-subaction a distal tower of height at most $n$ and in which each isometric extension is Abelian. We will meet other, even more restricted idempotent classes contained within this one later.

(We note in passing that an example of a natural class that is not closed under joinings when $\Gamma = \mathbb{Z}$ is the class $\mathcal{W}M^\perp$ of systems that are system-disjoint from all weakly-mixing systems, as has been proved by Lemańczyk and Parreau in [25].)

**Example**  Although all the idempotent classes that will matter to us in this paper can be shown to be order continuous, it may be instructive to exhibit one that is not. Let us say that a system $X$ has a finite-dimensional Kronecker factor if its Kronecker factor $\zeta^X_1 : X \to \mathbb{Z}_1^X$ can be coordinatized as a direct integral (see Section 3 of [1]) of rotations on some measurably-varying compact Abelian groups all of which can be isomorphically embedded into a fibre repository $T^D$ for some fixed $D \in \mathbb{N}$ (this includes the possibility that the Kronecker factor is finite or trivial). It is now easy to check that the class of $\mathbb{Z}$-systems comprising all those that are either themselves finite-dimensional Kronecker systems, or have a Kronecker factor that is not finite-dimensional (so we exclude just those systems that have a finite-dimensional Kronecker factor but properly contain it), is idempotent but not order continuous, since any infinite-dimensional separable group rotation can be identified with an inverse limit of finite-dimensional group rotations.

**Definition 3.4** (Hereditariness). An idempotent class $\mathcal{C}$ is hereditary if it is also closed under taking factors.

**Example**  Examples 1, 2 and 3 in the first list above are hereditary. In the case of $\mathcal{Z}_0^\Lambda$ this is obvious; for the higher distal classes $\mathcal{Z}_n^\Lambda$ or their Abelian subclasses it is an easy consequence of the Relative Factor Structure Theorem 6.4 of [1] (applied to the relatively independent self-joining of the $n$-step distal system in question over the factor we wish to analyze). On the other hand, the separate example above involving the dimensionality of the Kronecker factors is clearly not hereditary.  

**Definition 3.5** (Join). If $\mathcal{C}_1$, $\mathcal{C}_2$ are idempotent classes, then the class $\mathcal{C}_1 \vee \mathcal{C}_2$ of all joinings of members of $\mathcal{C}_1$ and $\mathcal{C}_2$ is clearly also idempotent. We call $\mathcal{C}_1 \vee \mathcal{C}_2$ the join of $\mathcal{C}_1$ and $\mathcal{C}_2$. 

[15]
Lemma 3.6 (Join preserves order continuity). If $C_1$ and $C_2$ are both order continuous then so is $C_1 \lor C_2$.

Proof. Let $(X_{(m)})_{m \geq 0}$, $(\psi^{(m)}_{(k)})_{m \geq k \geq 0}$ be an inverse sequence with inverse limit $X$, $(\psi^{(m)}_{(k)})_{m \geq k \geq 0}$. Then $\zeta^X_{C_1 \lor C_2}$ is the maximal factor of $X$ that is a joining of a $C_1$-factor and a $C_2$-factor (so, in particular, it must be generated by its own $C_1$- and $C_2$-factors), and hence it is equivalent to $\zeta^X_{C_1} \lor \zeta^X_{C_2}$. Therefore any $f \in L^\infty(\mu)$ that is $\zeta^X_{C_1} \lor \zeta^X_{C_2}$-measurable can be approximated by some function of the finite-sum form $\sum_p g_{p,1} : g_{p,2}$ with each $g_{p,i} \in L^\infty(\mu)$ being $C_i$-measurable, and now since each $C_i$ is order continuous we may further approximate each $g_{p,i}$ by some $h_{p,i} \circ \psi_{(m)}$ for a large integer $m$ and some $C_i$-measurable $h_{p,i} \in L^\infty(\mu_{(m)})$. Combining these approximations completes the proof. ☐

Examples. Of course, we can form the joins of any of our earlier examples of idempotent classes: for example, given $\Gamma = \mathbb{Z}^2$ and $p_1, p_2, p_3 \in \mathbb{Z}^2$ we can form $\mathbb{Z}_{p_1} \lor \mathbb{Z}_{p_1-p_2} \lor \mathbb{Z}_{p_1-p_3}$. This particular example and several others like it will appear frequently throughout the rest of this paper. We will remark shortly that joins of hereditary idempotent classes need not be hereditary. ☐

The following terminology will also prove useful.

Definition 3.7 (Joining to an idempotent class; adjoining). If $X$ is a system and $C$ is an idempotent class then a joining of $X$ to $C$ or a $C$-adjoining of $X$ is a joining of $X$ and $Y$ for some $Y \in C$.

Definition 3.8 (Subjoining). Given idempotent classes $C_1, C_2, \ldots, C_k$, a system $X$ is a subjoining of $C_1, C_2, \ldots, C_k$ if it is a factor of a member of $C_1 \lor C_2 \lor \cdots \lor C_k$.

Finally we have reached the key definition that will drive much of the rest of this paper.

Definition 3.9 (Sated system). Given an idempotent class $C$, a system $X$ is $C$-sated if whenever $\pi : \tilde{X} = (\tilde{X}, \tilde{\mu}, \tilde{T}) \rightarrow X$ is an extension, the factor maps $\pi$ and $\zeta_{C}^\tilde{X}$ on $\tilde{X}$ are relatively independent over $\zeta_{C}^X \circ \pi = C \pi \circ \zeta_{C}^\tilde{X}$ under $\tilde{\mu}$.

An inverse sequence is $C$-sated if it has a cofinal subsequence all of whose systems are $C$-sated.

Remark. This definition has an important precedent in Furstenberg and Weiss’ notion of a ‘pair homomorphism’ between extensions elaborated in Section 8 of [16]. Here we shall make much more extensive use of this basic idea. ☐

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Lemma 3.10. If $C$ is an idempotent class, $(X, \mu, T)$ is $C$-sated, $(Y, \nu, S) \in C$ and $\lambda$ is a $(T \times S)$-invariant $(\mu, \nu)$-joining then $\zeta_C^{(X \times Y, \lambda, T \times S)} \simeq \zeta_C^{(X, \mu, T)} \times \text{id}_Y$.

Proof Let $\pi_1, \pi_2 : (X \times Y, \lambda) \to (X, \mu), (Y, \nu)$ be the first and second coordinate projections respectively. The relation $\zeta_C^{(X \times Y, \lambda, T \times S)} \simeq \zeta_C^{(X, \mu, T)} \times \text{id}_Y$ is clear. On the other hand, the $C$-satedness of $(X, \mu, T)$ applied to the factor map $\pi_1$ implies that $\pi_1$ is relatively independent from $\zeta_C^{(X \times Y, \lambda, T \times S)} \circ \pi_1$ under $\lambda$, and this implies the reverse containment. This completes the proof.

The crucial technical fact that turns satedness into a useful tool is the ability to construct sated extensions of arbitrary systems. This can be seen as a natural abstraction from Proposition 4.6 of [4] and Corollary [3].

Theorem 3.11 (Idempotent classes admit multiply sated extensions). If $(C_i)_{i \in I}$ is a countable family of idempotent classes then any system $X_0$ admits an extension $\pi : X \to X_0$ such that

- $X$ is $C_i$-sated for every $i \in I$;
- the factors $\pi$ and $\bigsqcup_{i \in I} \zeta^X_{C_i}$ generate the whole of $X$.

We shall prove this result after a preliminary lemma.

Lemma 3.12. If $C$ is an idempotent class then the inverse limit of any $C$-sated inverse sequence is $C$-sated.

Proof By passing to a subsequence if necessary, it suffices to suppose that $(X_{(m)})_{m \geq 0}, (\psi_{(k)})_{m \geq k \geq 0}$ is an inverse sequence of $C$-sated systems with inverse limit $X_{(\infty)}$, $(\psi_{(m)})_{m \geq 1}$, and let $\pi : \tilde{X} \to X_{(\infty)}$ be any further extension and $f \in L^\infty(\mu_{(\infty)})$. We will commit the abuse of identifying such a function with its lift to any given extension when the extension in question is obvious. With this in mind, we need to show that

$$E(f \mid \zeta_{\tilde{X}}) = E(f \mid \zeta_{X_{(\infty)}}).$$

However, by the $C$-satedness of each $X_{(m)}$, we certainly have

$$E(E(f \mid \psi_{(m)}) \mid \zeta_{\tilde{X}}) = E(f \mid \zeta_{X_{(m)}}),$$

and now as $m \to \infty$ this equation converges in $L^2(\mu)$ to

$$E(f \mid \zeta_{\tilde{X}}) = E(f \mid \lim_{m \to \infty} (\zeta_{C_i}^{X_{(m)}} \circ \psi_{(m)})).$$
By monotonicity we must have
\[ \zeta_C \overset{\bowtie}{\preceq} \zeta_C \overset{\bowtie}{\preceq} \lim_{m \to \infty} (\zeta_C \circ \psi(m)), \]
and so by sandwiching we must also have the equality of conditional expectations desired. \[ \square \]

**Proof of Theorem 3.11** We first prove this for \( I \) a singleton, and then in the general case.

**Step 1** Suppose that \( I = \{i\} \) and \( C_i = C \). This case will follow from a simple ‘energy increment’ argument.

Let \( (f_r)_r \geq 1 \) be a countable subset of the \( L^\infty \)-unit ball \( \{ f \in L^\infty(\mu) : \|f\|_\infty \leq 1 \} \) that is dense in this ball for the \( L^2 \)-norm, and let \( (r_i)_i \geq 1 \) be a member of \( \mathbb{N}^\infty \) in which every non-negative integer appears infinitely often.

We will now construct an inverse sequence \( (\mathbf{X}(m))_{m \geq 0}, (\psi(m))_{m \geq k \geq 0} \) starting from \( \mathbf{X}(0) := \mathbf{X}_0 \) such that each \( \mathbf{X}(m+1) \) is a \( C \)-adjoining of \( \mathbf{X}(m) \). Suppose that for some \( m_1 \geq 0 \) we have already obtained \( (\mathbf{X}(m))_{m = 0}^{m_1}, (\psi(m))_{m \geq m_1 \geq 0} \) such that \( \text{id}_{\mathbf{X}(m_1)} \simeq \zeta_C(\mathbf{X}(m_1)) \vee \psi(m_1) \). We consider two separate cases:

- **If** there is some further extension \( \pi : \mathbf{X} \to \mathbf{X}(m_1) \) such that
  \[ \|E_{\mu}(f_{r_m} \circ \psi(0) \circ \pi | \zeta_C)\|_2^2 > \|E_{\mu}(f_{r_m} \circ \psi(0) | \zeta_C)\|_2^2 + 2^{m_1}, \]
  then choose a particular \( \pi : \mathbf{X} \to \mathbf{X}(m_1) \) such that the increase
  \[ \|E_{\mu}(f_{r_m} \circ \psi(0) \circ \pi | \zeta_C)\|_2^2 - \|E_{\mu}(f_{r_m} \circ \psi(0) | \zeta_C)\|_2^2 \]
is at least half its supremal possible value over all extensions. By restricting to the possibly smaller subextension of \( \mathbf{X} \to \mathbf{X}(m_1) \) generated by \( \pi \) and \( \zeta_C \) we may assume that \( \mathbf{X} \) is itself a \( C \)-adjoining of \( \mathbf{X}(m_1) \) and hence of \( \mathbf{X}_0 \), and now we let \( \mathbf{X}(m_1+1) := \mathbf{X} \) and \( \psi(m_1+1) := \pi \) (the other connecting factor maps being now determined by this one).

- **If**, on the other hand, for every further extension \( \pi : \mathbf{X} \to \mathbf{X}(m_1) \) we have
  \[ \|E_{\mu}(f_{r_m} \circ \psi(0) \circ \pi | \zeta_C)\|_2^2 \leq \|E_{\mu}(f_{r_m} \circ \psi(0) | \zeta_C)\|_2^2 + 2^{m_1} \]
  then we simply set \( \mathbf{X}(m_1+1) := \mathbf{X}(m_1) \) and \( \psi(m_1+1) := \text{id}_{\mathbf{X}(m_1)} \).
Finally, let \( X_{(\infty)} \) be the inverse limit of this sequence. We have

\[
\text{id}_{X_{(\infty)}} \simeq \bigvee_{m \geq 0} \psi(m) \simeq \bigvee_{m \geq 0} (\zeta_C^{X(m)} \lor \psi'_0(m)) \circ \psi(m)
\]

\[
\simeq \bigvee_{m \geq 0} (\zeta_C^{X(m)} \circ \psi(m)) \lor \bigvee_{m \geq 0} (\psi'(0) \circ \psi(m)) \simeq \zeta_C^{X(\infty)} \lor \psi(0),
\]

so \( X_{(\infty)} \) is still a C-adjoining of \( X_0 \). To show that it is C-sated, let \( \pi : \tilde{X} \to X_{(\infty)} \) be any further extension, and suppose that \( f \in L^\infty(\mu_{(\infty)}) \). We will complete the proof for Step 1 by showing that

\[
\mathbb{E}_{\tilde{\mu}}(f \circ \pi | \zeta_C^{X}) = \mathbb{E}_{\mu_{(\infty)}}(f | \zeta_C^{X(\infty)}) \circ \pi.
\]

Since \( X_{(\infty)} \) is a C-adjoining of \( X \), this \( f \) may be approximated arbitrarily well in \( L^2(\mu_{(\infty)}) \) by finite sums of the form \( \sum_p g_p \cdot h_p \) with \( g_p \) being bounded and \( \zeta_C^{X(\infty)} \)-measurable and \( h_p \) being bounded and \( \psi(0) \)-measurable, and now by density we may also restrict to using \( h_p \) that are each a scalar multiple of some \( f_{r_p} \circ \psi(0) \), so by continuity and multilinearity it suffices to prove the above equality for one such product \( g \cdot (f \circ \psi(0)) \). Since \( g \) is \( \zeta_C^{X(\infty)} \)-measurable, this requirement now reduces to

\[
\mathbb{E}_{\tilde{\mu}}(f \circ \psi(0) \circ \pi | \zeta_C^{X}) = \mathbb{E}_{\mu_{(\infty)}}(f \circ \psi(0) | \zeta_C^{X(\infty)}) \circ \pi.
\]

Since \( \zeta_C^{X} \simeq \zeta_C^{X(\infty)} \circ \pi, \) this will follow if we only show that

\[
\| \mathbb{E}_{\tilde{\mu}}(f \circ \psi(0) \circ \pi | \zeta_C^{X}) \|_2^2 = \| \mathbb{E}_{\mu_{(\infty)}}(f \circ \psi(0) | \zeta_C^{X(\infty)}) \|_2^2.
\]

Now, by the martingale convergence theorem we have

\[
\| \mathbb{E}_{\mu_{(m)}}(f \circ \psi'(m) | \zeta_C^{X(m)}) \|_2^2 \uparrow \| \mathbb{E}_{\mu_{(\infty)}}(f \circ \psi(0) | \zeta_C^{X(\infty)}) \|_2^2
\]

as \( m \to \infty \). It follows that if

\[
\| \mathbb{E}_{\tilde{\mu}}(f \circ \psi(0) \circ \pi | \zeta_C^{X}) \|_2^2 > \| \mathbb{E}_{\mu_{(\infty)}}(f \circ \psi(0) | \zeta_C^{X(\infty)}) \|_2^2
\]

then for some sufficiently large \( m \) we would have \( r_m = r \) (since each integer appears infinitely often as some \( r_m \)) but also

\[
\| \mathbb{E}_{\mu_{(m+1)}}(f \circ \psi(m+1) | \zeta_C^{X(m+1)}) \|_2^2 - \| \mathbb{E}_{\mu_{(m)}}(f \circ \psi(m) | \zeta_C^{X(m)}) \|_2^2
\]

\[
\leq \| \mathbb{E}_{\mu_{(\infty)}}(f \circ \psi(0) | \zeta_C^{X(\infty)}) \|_2^2 - \| \mathbb{E}_{\mu_{(m)}}(f \circ \psi(m) | \zeta_C^{X(m)}) \|_2^2
\]

\[
< \frac{1}{2}
\]
and

\[ \|E_\mu(f \circ \psi^{(m)}_0 \circ \pi | \zeta_C^X)\|^2 \geq \|E_{\mu(m)}(f \circ \psi^{(m)}_0 | \zeta_C^{X(m)})\|^2 + 2^{-m}, \]

so contradicting our choice of \( X_{(m+1)} \to X_{(m)} \) in the first alternative in our construction above. This contradiction shows that we must actually have the equality of \( L^2 \)-norms asserted above, as required.

**Step 2** The general case follows easily from Step 1 and a second inverse limit construction: choose a sequence \((i_m)_{m \geq 1} \in I^N\) in which each member of \( I \) appears infinitely often, and form an inverse sequence \((X_{(m)})_{m \geq 0}, (\psi^{(m)}_{(k)})_{m \geq k \geq 0}\) starting from \( X_{(0)} := X_0 \) such that each \( X_{(m)} \) is \( C_{i_m} \)-sated for \( m \geq 1 \). The inverse limit \( X \) is now sated for every \( C_i \), by Lemma 3.12.

**Remark** Thierry de la Rue has shown me another proof of Theorem 3.11 that follows very quickly from ideas contained in his paper [27] with Lesigne and Rittaud, and which has now received a nice separate writeup in [11]. The key observation is that

An idempotent class \( C \) if hereditary if and only if every system is \( C \)-sated.

This in turn follows from a striking result of Lemańczyk, Parreau and Thouvenot [26] that if two systems \( X \) and \( Y \) are not disjoint then \( X \) shares a nontrivial factor with the infinite Cartesian power \( Y^{\infty} \). Given now an idempotent class \( C \) and a system \( X \), let \( C^* \) be the hereditary idempotent class of all factors of members of \( C \), and let \( Y \) be any \( C \)-system admitting a factor map \( \pi : Y \to C^*X \) (such exists because by definition \( C^*X \) if a factor of some \( C \)-system). Now forming \( \bar{X} := X \times_{\{\zeta_C^X = \pi\}} Y \), a quick check using the above fact shows that \( CX = C^*\bar{X} \), and that this is equivalent to the \( C \)-satedness of \( \bar{X} \).

As remarked previously, a routine argument shows that our basic examples \( Z_0^\Lambda, Z_n^\Lambda \) and \( Z_{\text{Ab},n}^\Lambda \) are all hereditary, and hence that any system is sated with respect to any of them. However, joins of several hereditary idempotent classes need not be hereditary. For example, let \( X = (T^2, \text{Haar}, R_{(\alpha,0)}, R_{(0,\alpha)}) \), where \( R_\beta \) denotes a rotation by \( \beta \in \mathbb{T}^2 \) and \( \alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z} \), and let \( e_1, e_2 \) be the standard basis vectors of \( \mathbb{Z}^2 \). Then clearly \( \zeta^{Te_i}_0 = \pi_{3-i} \), the projection onto the \((3-i)\)th coordinate, and so \( \zeta^{Te_1}_0 \vee \zeta^{Te_2}_0 \simeq \text{id}_X \) and therefore \( X \) is itself a system in the class \( Z_0^{e_1} \vee Z_0^{e_2} \). However, \( Z_0^{e_1} + e_2 \) is the factor generated by the SW-NE diagonal circles in \( T^2 \), so \( Z_0^{e_1} + e_2 \) is a factor of a system of class \( Z_0^{e_1} \vee Z_0^{e_2} \) but \( Z_0^{e_1} + Z_0^{e_2} (Z_0^{e_1} + e_2 X) \) is the trivial system, so \( Z_0^{e_1} + e_2 X \) is not \( (Z_0^{e_1} \vee Z_0^{e_2}) \)-sated. (Nevertheless, some preliminary results in Section 7 of [11] indicate that such counterexamples must be
rather special, and we suspect that the machinery of that paper and the present one may have more to say on this question in the future.)

It will serve us well to adopt a special name for a particular case of the above multiple satedness that will recur frequently.

**Definition 3.13** (Full isotropy-satedness). A system $T : \mathbb{Z}^d \curvearrowright (X, \mu)$ is **fully isotropy-sated** (FIS) if whenever $p_i : \mathbb{Z}^d \hookrightarrow \mathbb{Z}^d$, $i = 1, 2, \ldots, k$, are isomorphic embeddings then the system $(X, \mu, T)$ is $(\mathbb{Z}^{p_1}_0 \vee \mathbb{Z}^{p_2}_0 \vee \cdots \vee \mathbb{Z}^{p_k}_0)$-sated.

**Corollary 3.14.** Any $\mathbb{Z}^d$-system admits an FIS extension.

3.2 Subactions and insensitivity of idempotent classes

Given an l.c.s.c. group and closed subgroup $\Lambda \leq \Gamma$ there is an obvious forgetful functor $\Gamma\text{-Sys} \to \Lambda\text{-Sys}$, $X \mapsto X^{\mid \Lambda}$. Some idempotent classes $C$ in $\Gamma\text{-Sys}$ actually make sense in both categories, in the sense that $X \in C$ if and only if $X^{\mid \Lambda} \in C_0$ for some idempotent class $C_0$ of $\Lambda$-systems. Loosely, these are the classes that are defined in terms of properties depending only on the subaction $X^{\mid \Lambda}$, the most obvious example being $\mathbb{Z}^{\mid \Lambda}_0$. Importantly, in this case forming the maximal $\mathbb{Z}_0^{\Lambda}$-factor of a $\Gamma$-system $X$ and of its subaction system $X^{\mid \Lambda}$ give measure-theoretically the same factor space. This is a simple but important phenomenon that we will need to appeal to later (although we shall sometimes suppress the distinction between a class $C$ of $\Gamma$-systems and the corresponding class $C^{\mid \Lambda}$ of $\Lambda$-systems).

**Definition 3.15** (Insensitivity of idempotent classes). An idempotent class $C$ of $\Lambda$-systems is **insensitive to the forgetful functor to $\Lambda$-subactions** if whenever $X$ is a $\Gamma$-system, the factor map $\zeta_X^{\mid \Lambda} : X \to CX$ actually intertwines the whole $\Gamma$-action $T$ with some $\Gamma$-action on $CX$ (equivalently, if the $\sigma$-subalgebra $(\zeta_X^{\mid \Lambda})^{-1}(\Sigma_{CX})$ is globally $\Gamma$-invariant).

In this case we can naturally extend the definition of the class $C$ to the category $\Gamma\text{-Sys}$ by letting $CX$ be the $\Gamma$-action with which $T$ is intertwined by $\zeta_X^{\mid \Lambda} = : \zeta_X^{\Lambda}$, and we will generally use the same letter $C$ for the idempotent class in either category.

In general, an idempotent class $C$ of $\Gamma$-systems is **insensitive to the forgetful functor to $\Lambda$-subactions** if it is the extension to $\Gamma\text{-Sys}$ of an insensitive idempotent class of $\Lambda$-systems.

This notion of insensitivity will be most important to us in view of its consequences for satedness. In order to understand these, however, we must first introduce an
important method for building system extensions with desirable properties. For this we restrict to the setting of countable discrete Abelian groups.

Suppose that $\Gamma$ is a countable discrete Abelian group and $\Lambda \leq \Gamma$ a subgroup, that $X$ is a $\Gamma$-system and that $\xi : X' = (X', \mu', S') \to X^\Lambda$ is an extension of $\Lambda$-subaction system. It can easily happen that there does not exist an action $T' : \Gamma \curvearrowright (X', \mu')$ such that $(T')^\Lambda = S'$. However, if we permit ourselves to pass to further system extensions we can retrieve this situation, and this will be crucial in cases where our analysis of characteristic factors in the first place gives information only about a sublattice of $\mathbb{Z}^d$, rather than the whole group.

Here we will introduce a particular construction of such a further extension (although in general we will use only the abstract fact of the existence of such an extension). We first need a basic result and definition from elementary group theory.

**Definition 3.16** (Remainder map). If $\Lambda \leq \Gamma$ are as above and $\Omega \subseteq \Gamma$ is a fundamental region for the subgroup $\Lambda$ (that is, $\Omega$ is a subset containing exactly one member of every coset in $\Gamma/\Lambda$), then there is a remainder map $\Gamma \to \Omega : \gamma \mapsto R(\gamma)$ with the property that $\gamma - R(\gamma) \in \Lambda$ for all $\gamma \in \Gamma$.

Now suppose that $\xi : X' \to X^\Lambda$ is as above. We will give a construction of a further extension of $X'$ based on a similar idea to that underlying the construction of induced group representations.

Let $\Omega \subseteq \Gamma$ be a fundamental region as above with associated remainder map $R$, chosen so that $e \in \Omega$ where $e$ is the identity of $G$. This defines uniquely an ‘integer part’ map $\Gamma \to \Lambda : \gamma \mapsto \lfloor \gamma \rfloor := \gamma - R(\gamma)$, and it follows at once that $R(\gamma + \gamma') = R(\gamma + (R(\gamma') = R(R(\gamma) + R(\gamma')).$

By Rokhlin’s Skew-Product Representation (see, for example, Section 3.3 of Glasner [17]) the extension $\xi$ of $\Lambda$-systems can be described by a decomposition of $X$ as $\bigcup_{1 \leq n < \infty} A_n$ into $T^\Lambda$-invariant sets and, letting $(Y_n, \nu_n)$ be $\{1, 2, \ldots, n\}$ with uniform measure for $1 \leq n < \infty$ and $(Y_{\infty}, \nu_{\infty})$ be $[0, 1)$ with Lebesgue measure, for each $n$ a Borel cocycle $\Phi_n : \Lambda \times A_n \to \text{Aut}(Y_n, \nu_n)$, so that

$$(S')^\gamma(x, y) = (T^n x, \Phi_n(\gamma, x)(y))$$

for $(x, y) \in A_n \times Y_n$ for all $\gamma \in \Lambda$. We adopt the simplified notations $\Phi(\gamma, x) := \Phi_n(\gamma, x)$ and $(Y_x, \nu_x) := (Y_n, \nu_n)$ when $x \in A_n$. For brevity we will simply write $X' = X \times Y_*$.

We base our construction of $\tilde{X}$ on a specification of new, enlarged fibres above each point $x \in X$. For $x \in X$ let $\bar{Y}_x := \prod_{\omega \in \Omega} Y_{T^{-\omega}x}$, and define $\tilde{X} := X \times \bar{Y}_*$. 

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Because \( \Omega \) is countable this product can be given the structure of a standard Borel space in the obvious way.

We define the \( \Gamma \)-action \( \tilde{T} \) on \( \tilde{X} \) by its Rokhlin representation. Writing a typical element of \( Y^\Omega_\gamma \) as \( (y_\omega)_\omega \), we set

\[
\tilde{T}^\gamma(x, (y_\omega)_\omega) := (T^\gamma x, (\Phi([R(\omega - \gamma) + \gamma], T^{-R(\omega - \gamma)}x)(y_{R(\omega - \gamma)}))_\omega)
\]

(noting that this is well-defined: if \( (y_\omega)_\omega \in \prod_{\omega \in \Omega} Y_{T^{-\omega}x} \), then \( y_\omega \in Y_{T^{-\omega}x} \) for all \( \omega \in \Omega \), and so \( y_{R(\omega - \gamma)} \in Y_{T^{-R(\omega - \gamma)}x} = Y_{T^{-\omega}(T^\gamma x)} \), because the assignment \( x \mapsto Y_x \) is \( \Lambda \)-invariant).

Using our simple identities for \( R \) we can compute that

\[
\tilde{T}^{\gamma_1}(\tilde{T}^{\gamma_2}(x, (y_\omega)_\omega)) = \tilde{T}^{\gamma_1}(T^{\gamma_2}x, (\Phi([R(\omega - \gamma_2) + \gamma_2], T^{-R(\omega - \gamma_2)}x)(y_{R(\omega - \gamma_2)}))_\omega)
\]

\[
= \left(T^{\gamma_1+\gamma_2}x, \left(\Phi([R(\omega - \gamma_1) + \gamma_1], T^{-R(\omega - \gamma_1)}T^{\gamma_2}x) \left(\Phi([R(R(\omega - \gamma_1) - \gamma_2) + \gamma_2], T^{-R(R(\omega - \gamma_1) - \gamma_2)}x(y_{R(R(\omega - \gamma_1) - \gamma_2)}))_\omega\right)\right)\right.
\]

\[
= \left(T^{\gamma_1+\gamma_2}x, \left(\Phi([R(\omega - \gamma_1) + \gamma_1], T^{-R(\omega - \gamma_1)}T^{\gamma_2}x) \left(\Phi([R(\omega - (\gamma_1 + \gamma_2)) + \gamma_2], T^{-R(\omega - (\gamma_1 + \gamma_2))x}(y_{R(\omega - (\gamma_1 + \gamma_2)}))_\omega\right)\right)\right.
\]

and now we note that

\[
T^{[R(\omega - (\gamma_1 + \gamma_2)) + \gamma_2]}T^{-R(\omega - (\gamma_1 + \gamma_2))x} = T^{[\omega - (\gamma_1 + \gamma_2)] + [-\omega - (\gamma_1 + \gamma_2)] - R(\omega - (\gamma_1 + \gamma_2))x}
\]

\[
= T^{[\omega - \gamma_1] + [-\omega - \gamma_1] - (\omega - (\gamma_1 + \gamma_2))x}
\]

\[
= T^{(\omega - \gamma_1) - R(\omega - \gamma_1) - (\omega - (\gamma_1 + \gamma_2))x}
\]

\[
= T^{-R(\omega - \gamma_1)}T^{\gamma_2}x
\]

and so the cocycle equation for \( \Phi \) gives

\[
\Phi([R(\omega - \gamma_1) + \gamma_1], T^{-R(\omega - \gamma_1)}T^{\gamma_2}x) \circ \Phi([R(\omega - (\gamma_1 + \gamma_2)) + \gamma_2], T^{-R(\omega - (\gamma_1 + \gamma_2))x})
\]

\[
= \Phi([R(\omega - \gamma_1) + \gamma_1] + [R(\omega - (\gamma_1 + \gamma_2)) + \gamma_2], T^{-R(\omega - (\gamma_1 + \gamma_2))x})
\]

\[
= \Phi(-[\omega - \gamma_1] + [\omega - \gamma_1] - [\omega - (\gamma_1 + \gamma_2)], T^{-R(\omega - (\gamma_1 + \gamma_2))x}).
\]

Inserting this into the above formula for \( \tilde{T}^{\gamma_1}(\tilde{T}^{\gamma_2}(x, (y_\omega)_\omega)) \) shows that it is equal to \( \tilde{T}^{\gamma_1+\gamma_2}(x, (y_\omega)_\omega) \), and hence that \( \tilde{T} \) is a \( \Gamma \)-action.
Finally, if we let
\[ \pi : (x, (y_\omega)_\omega) \mapsto x \quad \text{and} \quad \alpha : (x, (y_\omega)_\omega) \mapsto (x, y_e) \]
(recalling that \( e \in \Omega \)) then it is routine to check that
\[ \pi(T^\gamma(x, (y_\omega)_\omega)) = T^\gamma x = T^\gamma \pi(x, (y_\omega)_\omega) \quad \forall \gamma \in \Gamma \]
and
\[ \alpha(T^\gamma(x, (y_\omega)_\omega)) = \alpha(T^\gamma x, (\Phi([R(\omega - \gamma)] T^{-R(\omega - \gamma)} x)(y_{R(\omega - \gamma)})))_\omega \]
\[ = (T^\gamma, \Phi([R(-\gamma)] T^{-R(-\gamma)} x)(y_{R(-\gamma)})) = (T^\gamma, \Phi(\gamma, x)(y_e)) = (S')^\gamma(x, y_e) \]
when \( \gamma \in \Lambda \), so we have the required commutative diagram

\[
\begin{array}{ccc}
\tilde{X}^{\Lambda} & \xrightarrow{\pi} & X^{\Lambda} \\
\downarrow{\alpha} & & \downarrow{\xi} \\
X' & & \\
\end{array}
\]

**Definition 3.17** (Fibrewise power extension). *We will refer to the particular extension \( \pi : \tilde{X} \to X \) constructed above as the fibrewise power extension (or FP extension) of \( X \) corresponding to the subgroup \( \Lambda \leq \Gamma \) and system extension \( \xi \).*

**Remark** Interestingly, the appeal made to the discreteness of \( \Gamma/\Lambda \) in the above proof seems to be quite important. While other instances of this theorem are certainly available, it seems to be difficult to prove a comparably general statement for an inclusion \( \Lambda \leq \Gamma \) of arbitrary locally compact second countable Abelian groups; it would be interesting to know whether some alternative construction could be found to handle that setting. It is also worth remarking that there are certainly pairs of locally compact non-Abelian groups for which the conclusion fails: for example, by the Howe-Moore Theorem (see, for instance, Section 3.3 of [12]) any ergodic action of a non-compact connected simple Lie group \( G \) with finite centre is mixing, and so any non-mixing ergodic action of \( \mathbb{R} \) cannot be extended to a larger \( \mathbb{R} \)-system in which the action can be enlarged to the whole group \( G \) for any embedding \( \mathbb{R} \hookrightarrow G \) as a one-parameter subgroup.

We can now quickly derive some consequences for satedness.

**Lemma 3.18.** If \( \Lambda \leq \Gamma \) are as above and \( \mathcal{C} \) is an idempotent class that is insensitive to the forgetful functor to \( \Lambda \)-actions then a \( \Gamma \)-system \( X \) is \( \mathcal{C} \)-sated if and only if \( X^{\Lambda} \) is \( \mathcal{C} \)-sated.
Proof. It is clear that if X admits a C-adjoining that is not relatively independent over CX, then simply applying the forgetful functor gives the same phenomenon among the Λ-subactions: thus the C-satedness of X↾Λ implies that of C. The reverse direction follows similarly, except that a given C-adjoining of X↾Λ may need to be extended further (for example, to an FP extension) to recover an action of the whole of Γ, and this extension of X will then witness that it is not sated.

Corollary 3.19. For any subgroups Λ1, Λ2, . . . , Λk ≤ Zd a system X is ( ⋁i≤k Z0 Ai)-sated if and only if X↾(Λ1+Λ2+...+Λk) is ( ⋁i≤k Z0 Ai)-sated.

4 Some applications to characteristic factors

4.1 The Furstenberg self-joining

Consider a Zd-system X = (X, µ, T). As have many previous works in this area, our analysis of characteristic factors associated to different sequences of linear averages will make heavy use of a particular class of self-joinings of X. Given k directions p1, p2, . . . , pk ∈ Zd, let us here write

\[ S_N(f_1, f_2, \ldots, f_k) := \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{n p_i} \]

for the associated k-fold linear nonconventional averages. Now for A1, A2, . . . , Ak ∈ ΣX we can define

\[ \mu^F_{T^{p_1}, T^{p_2}, \ldots, T^{p_k}}(A_1 \times A_2 \times \cdots \times A_k) := \lim_{N \to \infty} \int_X S_N(1_{A_1}, 1_{A_2}, \ldots, 1_{A_k}) \, d\mu, \]

and now it is routine to show (recalling that X is standard Borel) that this extends by linearity and continuity to a k-fold self-joining of µ on Xk, which is invariant under not only the Zd-action T×k but also the ‘diagonal transformation’ \( \vec{T} := T^{p_1} \times T^{p_2} \times \cdots \times T^{p_k} \). This is the Furstenberg self-joining of µ associated to the transformations Tp1, Tp2, . . . , Tpk, and will be denoted by \( \mu^F_{T^{p_1}, T^{p_2}, \ldots, T^{p_k}} \) or \( \mu^F_T \), or sometimes abbreviated to \( \mu^F \). Clearly whenever \( f_i \in L^\infty(\mu) \) for i = 1, 2, . . . , k we have also

\[ \int_X f_1 \otimes f_2 \otimes \cdots \otimes f_k \, d\mu^F_{T^{p_1}, T^{p_2}, \ldots, T^{p_k}} = \lim_{N \to \infty} \int_X S_N(f_1, f_2, \ldots, f_k) \, d\mu. \]

Of course, the existence of the above limit follows from the known convergence of linear nonconventional averages. (Once convergence of the relevant polynomial
nonconventional averages has been established, a similar definition can be made corresponding to such polynomial averages, but in the nonlinear case these have yet to prove similarly useful.)

In fact, in [4] the convergence of the above values is proved alongside the convergence of the nonconventional averages as part of a zigzag induction (one claim for a given \(k\) implies the other for that \(k\), which then implies the first for \(k + 1\), and so on). This is possible in view of a reduction of the above limits to the study of linear averages involving only \(k - 1\) transformations, which will also be important for us here: simply because \(\mu\) is \(T\)-invariant, and working now in terms of bounded functions rather than sets, we can re-write the above limit as

\[
\int_X f_1 \otimes f_2 \otimes \cdots \otimes f_k \, d\mu_{TP_1, TP_2, \ldots, TP_k}^F
= \int_X f_1 \cdot \left( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=2}^{k} (f_i \circ T^n(p_i - p_1)) \right) \, d\mu.
\]

Knowing that the Furstenberg self-joining \(\mu_F^{T_{p_1, TP_2, \ldots, TP_k}}\) exists, a basic application of the Hilbert space version of the classical van der Corput estimate gives us a way to use it to control the asymptotic behaviour of \(S_N(f_1, f_2, \ldots, f_k)\), in the sense of the following estimate taken from Lemma 4.7 of [4]. (We have modified the statement to give explicit bounds, but the proof is unchanged.)

**Lemma 4.1.** If \(f_1, f_2, \ldots, f_k \in L^\infty(\mu)\) are 1-bounded and \(\varepsilon > 0\) is such that

\[
\lim_{N \to \infty} \|S_N(f_1, f_2, \ldots, f_d)\|_2 \geq \varepsilon,
\]

then there is a 1-bounded \(\vec{T}\)-invariant function \(g \in L^\infty(\mu_F^\vec{T})\) such that

\[
\left| \int_X \prod_{i=1}^{k} (f_i \circ \pi_i) \cdot g \, d\mu_F^\vec{T} \right| \geq \varepsilon^2.
\]

This now implies a useful sufficient condition for characteristicity of a tuple of factors.

**Corollary 4.2.** A tuple of factors \(\xi_i : X \to Y_i, i = 1, 2, \ldots, k\), is characteristic for the averages \(S_N\) if for any choice of \(f_i \in L^\infty(\mu), i = 1, 2, \ldots, k\), and \(\vec{T}\)-invariant \(g \in L^\infty(\mu_F^\vec{T})\) we have

\[
\int_X \prod_{i=1}^{k} (f_i \circ \pi_i) \cdot g \, d\mu_F^\vec{T} = \int_X \prod_{i=1}^{k} (E_\mu(f_i | \xi_i) \circ \pi_i) \cdot g \, d\mu_F^\vec{T}.
\]
Proof  Suppose that \( f_i \in L^\infty(\mu) \) for each \( i = 1, 2, \ldots, k \). We need to show that

\[
S_N(f_1, f_2, \ldots, f_k) \sim S_N(E_\mu(f_1 | \xi_1), E_\mu(f_2 | \xi_2), \ldots, E_\mu(f_k | \xi_k))
\]
as \( N \to \infty \), but by replacing each function with its conditional expectation in turn it clearly suffices to show that

\[
S_N(f_1, f_2, \ldots, f_k) \sim S_N(E_\mu(f_1 | \xi_1), f_2, \ldots, f_k).
\]

This, in turn, is equivalent to

\[
S_N(f_1 - E_\mu(f_1 | \xi_1), f_2, \ldots, f_k) \to 0,
\]

and this now follows from the assumption and Lemma 4.1 because for any \( \overline{T} \)-invariant \( g \in L^\infty(\mu^E_{\overline{T}}) \) we have

\[
\int_{X^k} \prod_{i=1}^k (f_i \circ \pi_i) \cdot g \, d\mu^E_{\overline{T}} = \int_{X^k} \prod_{i=1}^k (E_\mu(f_i | \xi_i) \circ \pi_i) \cdot g \, d\mu^E_{\overline{T}} = \int_{X^k} E_\mu(f_1 | \xi_i) \cdot \prod_{i=2}^k (f_i \circ \pi_i) \cdot g \, d\mu^E_{\overline{T}}
\]

and so

\[
\int_{X^d} (f_1 - E_\mu(f_1 | \xi_1)) \cdot \prod_{i=2}^d (f_i \circ \pi_i) \cdot g \, d\mu^E_{\overline{T}}.
\]

Remark  An alternative to the Furstenberg self-joining that can sometimes be put to similar uses has been constructed by Host and Kra, first in the case of powers of a single transformation in [21] and then for several commuting transformations by Host in [18]. This is defined in terms of a tower of iterated relatively independent self-products, and so has the advantage over the Furstenberg self-joining that it does not require an appeal to a previously-known nonconventional convergence result for its definition. Although we shall focus on the Furstenberg self-joining here for consistency, I suspect that the present paper and its sequel could be reworked to use a Host-Kra self-joining throughout, and that neither presentation would be substantially easier. (In early drafts of these papers, the preference for the Furstenberg self-joining was dictated by a particular appeal to it in the last stages of proving Theorem 1.4 but subsequent improvements to that proof have made these unnecessary.)
It is easy to see that the tuple of factors \((\xi_1, \xi_2, \ldots, \xi_d)\) is characteristic for the averages \(S_N\) if and only if each of the \(d\) tuples
\[(\xi_1, \text{id}_X, \text{id}_X, \ldots, \text{id}_X),\]
\[(\text{id}_X, \xi_2, \text{id}_X, \ldots, \text{id}_X),\]
\[\vdots\]
\[(\text{id}_X, \text{id}_X, \text{id}_X, \ldots, \xi_d)\]
is characteristic for them. A slightly more subtle property that we will find useful later is the following.

**Lemma 4.3.** For any factor \(\xi : X \to Y\) the tuple \((\xi, \text{id}_X, \ldots, \text{id}_X)\) is characteristic for the nonconventional averages
\[
\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{n p_i} f_1, f_2, \ldots, f_k \in L^\infty(\mu)
\]
if and only if the tuple \((\text{id}_X, \xi, \text{id}_X, \ldots, \text{id}_X)\) is characteristic for the nonconventional averages
\[
\frac{1}{N} \sum_{n=1}^{N} (f_0 \circ T^{-n p_j}) \prod_{i \leq k, i \neq j} f_i \circ T^{n(p_i - p_j)} f_0, f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_k \in L^\infty(\mu)
\]
for every \(j = 2, 3, \ldots, k\).

**Proof** This follows from a similar re-arrangement to those we have already seen above. By symmetry it suffices to treat only one of the needed implications, so let us suppose that \((\xi, \text{id}_X, \ldots, \text{id}_X)\) is characteristic for
\[
\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{n p_i}
\]
and show that \((\text{id}_X, \xi, \text{id}_X, \ldots, \text{id}_X)\) is characteristic for
\[
\frac{1}{N} \sum_{n=1}^{N} (f_0 \circ T^{-n p_2}) \prod_{i \leq k, i \neq 2} f_i \circ T^{n(p_i - p_2)} f_0, f_1, f_3, \ldots, f_k \in L^\infty(\mu).
\]
Replacing \(f_1\) by \(f_1 - E_\mu(f_1 | \xi_1)\), it will suffice to show that if the latter averages do not tend to zero in \(L^2(\mu)\) for some choice of \(f_0, f_3, \ldots, f_k\) then also the former do

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not tend to zero for some choice of \( f_2, f_3, \ldots, f_k \). Thus, suppose that \( f_0, f_3, \ldots, f_k \) are such that the latter averages do not tend to zero, and now let

\[
f_2 := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (f_0 \circ T^{-np_2}) \cdot \prod_{i \leq k, i \neq 2} f_i \circ T^{n(p_i-p_2)}.
\]

The condition that \( f_2 \neq 0 \) and a change of variables now give

\[
0 \neq \lim_{N \to \infty} \int_X f_2 \cdot \left( \frac{1}{N} \sum_{n=1}^{N} (f_0 \circ T^{-np_2}) \cdot \prod_{i \leq k, i \neq 2} f_i \circ T^{n(p_i-p_2)} \right) d\mu
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f_2 \cdot (f_0 \circ T^{-np_2}) \cdot \prod_{i \leq k, i \neq 2} f_i \circ T^{n(p_i-p_2)} d\mu
\]

\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f_0 \cdot (f_2 \circ T^{np_2}) \cdot \prod_{i \leq k, i \neq 2} f_i \circ T^{np_i} d\mu
\]

\[
= \lim_{N \to \infty} \int_X f_0 \cdot \left( \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{np_i} \right) d\mu,
\]

and so we must also have

\[
\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{np_i} \not\to 0,
\]

as required.

**Example** An easily-generalized argument of Ziegler [32] in the case \( d = 1 \) and \( p_i = a_i \in \mathbb{Z} \) shows that there is always a unique minimal characteristic factor tuple: a characteristic tuple of factors \( \xi_i : X \to Y_i, i = 1, 2, \ldots, k \), such that any other characteristic tuple of factors \( \xi'_i : X \to Y'_i \) must satisfy \( \xi'_i \preceq \xi_i \) for all \( i \leq k \). However, it is worth noting that the members of this tuple can depend on the whole system \( X \), in that if we restrict to averages involving functions \( f_j \) that are all lifted from \( Y_i \) for some fixed \( i \), then \( Y_i \) may in turn admit a characteristic tuple of smaller factors.

For example, when \( d = 2 \) and \( p_i = e_i \) for \( i = 1, 2 \), consider three irrational and rationally independent points on the circle \( r, s, t \in \mathbb{T} \), and let \( X \) be the \( \mathbb{Z}^2 \) system on \((X, \mu) = (\mathbb{T}^2, m_{\mathbb{T}^2})\) generated by \( T_1 := R_x \times R_t \) and \( T_2 := R_r \times R_s \). In this simple setting we can use Fourier analysis to obtain that the minimal characteristic factors \( \xi_1, \xi_2 \) are equivalent to the first and second coordinate projections \( \mathbb{T}^2 \to \mathbb{T} \).
respectively. However, after passing down through the first coordinate projection, it is equally easy to compute that the minimal characteristic factors of the resulting system on \((\mathbb{T}, m_\mathbb{T})\) are both trivial.

It follows that there is in general no tuple of idempotent classes of systems \(C_1, C_2, \ldots, C_k\) such that \(Y_i = C_i X\) for every \(X\). This contrasts interestingly with the case \(d = 1\), where the main technical result of Host and Kra \([21]\) and Ziegler \([32]\) can be phrased as asserting that there is such a class, and for \(k\) distinct integers \(p_1, p_2, \ldots, p_k \in \mathbb{Z}\) we have \(C_1 = C_2 = \ldots = C_k\) and it is the class of all ‘direct integrals’ (suitably defined) of inverse limits of \(k\)-step nilsystems. \(\triangleright\)

Let us finish by recording the following useful property of minimal characteristic factors.

**Lemma 4.4.** If \((X_{(m)})_{m \geq 0}, (\psi^{(m)}_{(k)})_{m \geq 0}\) is an inverse system with inverse limit \(X_{(\infty)}\), \((\psi_{(m)})_{m \geq 0}\) and the factors of the minimal characteristic tuples of these systems for some averaging scheme are \(\xi_{(m),i}, i = 1, 2, \ldots, k\) and \(\xi_{(\infty),i}, i = 1, 2, \ldots, k\) respectively then

\[
\xi_{(\infty),i} = \bigvee_{m \geq 0} \xi_{(m),i} \circ \psi_{(m)}.
\]

**Proof** The direction \(\supseteq\) is obvious (since any particular nonconventional averages on system \(X_{(m)}\) can be lifted to \(X_{(\infty)}\), so we need only show the reverse containment.

To this end, suppose that \(f_i \in L^\infty(\mu_{(\infty)})\) for \(i = 1, 2, \ldots, k\). Then by the definition of the inverse limit, we know that we can approximate these functions arbitrarily well in \(L^2(\mu)\) by functions of the form \(g_i \circ \psi_{(m)}\) for \(g_i \in L^\infty(\mu_{(m)})\) with \(\|g_i\|_{\infty} \leq \|f_i\|_{\infty}\) and \(m\) sufficiently large. This approximation now clearly gives

\[
S_{(\infty),N}(f_1, f_2, \ldots, f_k) \approx S_{(m),N}(g_1, g_2, \ldots, g_k) \circ \psi_{(m)} \quad \text{in } L^2(\mu)
\]

uniformly in \(N\), and this latter behaves asymptotically as

\[
S_{(m),N}(E_{\mu_{(m)}}(g_1 | \xi_{(m),1} \circ \psi_{(m)}), E_{\mu_{(m)}}(g_2 | \xi_{(m),2} \circ \psi_{(m)}), \ldots, E_{\mu_{(m)}}(g_k | \xi_{(m),k} \circ \psi_{(m)})) \circ \psi_{(m)} = S_{(\infty),N}(E_{\mu_{(\infty)}}(g_1 \circ \psi_{(m)} | \xi_{(m),1} \circ \psi_{(m)}), E_{\mu_{(\infty)}}(g_2 \circ \psi_{(m)} | \xi_{(m),2} \circ \psi_{(m)}), \ldots, E_{\mu_{(\infty)}}(g_k \circ \psi_{(m)} | \xi_{(m),k} \circ \psi_{(m)}))
\]

in \(L^2(\mu)\) as \(N \to \infty\), by the defining property of \(\xi_{(m),1}, \xi_{(m),2}, \ldots, \xi_{(m),k}\).

This, in turn, is approximately equal to

\[
S_{(\infty),N}(E_{\mu_{(\infty)}}(f_1 | \xi_{(m),1} \circ \psi_{(m)}), E_{\mu_{(\infty)}}(f_2 | \xi_{(m),2} \circ \psi_{(m)}), \ldots, E_{\mu_{(\infty)}}(f_k | \xi_{(m),k} \circ \psi_{(m)})),
\]

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and so letting \( m \to \infty \) and observing that each \( (E_{\mu(\infty)}(f_i | \xi(m),i \circ \psi(m)))_{m \geq 1} \) for \( i = 1, 2, \ldots, d \) is a uniformly bounded martingale, we obtain

\[
S(\infty),N(f_1, f_2, \ldots, f_k) \\
\sim S(\infty),N(E_{\mu(\infty)}(f_1 | \xi_1^0), E_{\mu(\infty)}(f_2 | \xi_2^0), \ldots, E_{\mu(\infty)}(f_k | \xi_k^0))
\]

as \( N \to \infty \) with

\[
\xi_i^0 := \bigvee_{m \geq 0} \xi(m),i \circ \psi(m),
\]

and hence \( \xi(\infty),i \simeq \xi_i^0 \), as required.

\[\square\]

### 4.2 Linearly independent directions in discrete time

In this section we will address the easier of the questions posed in the introduction: whether we can construct pleasant extensions while retaining the existence of roots for our transformations. In fact it will follow quite easily from the machinery developed above that FIS extensions achieve this goal.

**Proposition 4.5** (FIS extensions are pleasant). If \((X, \mu, T)\) is an FIS system and \(p_1, p_2, \ldots, p_k \in \mathbb{Z}^d\) are linearly independent then the tuple of factors

\[
\xi_i := \zeta_0^{Tp_i} \lor \bigvee_{j \in \{1, 2, \ldots, k\} \setminus \{i\}} \zeta_0^{Tp_i = Tp_j} \quad i = 1, 2, \ldots, k
\]

is characteristic for the associated linear nonconventional averages

\[
\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i \circ T^{np_i}.
\]

The case in which in addition \( \mathbb{Z}p_1 + \mathbb{Z}p_2 + \cdots + \mathbb{Z}p_k = \mathbb{Z}^d \) (so necessarily \( k = d \)) is implicitly contained in [4]; the point here is to handle the case when the \( p_j \) generate a proper sublattice.

Before turning to Proposition 4.5 we illustrate our basic method by proving the following useful lemma (which is, in turn, implicitly contained in [3]). For our purposes a tuple of isomorphic embeddings \( p_j : \mathbb{Z}^{r_j} \hookrightarrow \mathbb{Z}^d \), \( j = 1, 2, \ldots, k \) is **totally linearly independent** if

\[
p_1(n_1) + p_2(n_2) + \ldots + p_k(n_k) = 0 \in \mathbb{Z}^d \quad \Rightarrow \quad n_j = 0 \in \mathbb{Z}^{r_j} \forall j \leq k.
\]
Lemma 4.6. If \((X, \mu, T)\) is an FIS system then whenever \(p_j : \mathbb{Z}^{r_j} \rightarrow \mathbb{Z}^d\) are totally linearly independent isomorphic embeddings for \(j = 1, 2, \ldots, k\) we have

\[
\zeta_0^{Tp_1} \land \left( \bigvee_{j=2}^k \zeta_0^{Tp_j} \right) \simeq \bigvee_{j=2}^k \zeta_0^{Tp_1 \oplus p_j},
\]

where \(T^{p_j}\) is the \(\mathbb{Z}^{r_j}\)-action \(n \mapsto T^{p_j}(n)\).

Proof Let \(\Lambda := p_1(\mathbb{Z}^{r_1}) + p_2(\mathbb{Z}^{r_2}) + \cdots + p_k(\mathbb{Z}^{r_k}) \leq \mathbb{Z}^d\). We first suppose \(\Lambda = \mathbb{Z}^d\), and then use this to prove the general case. Note that since our \(p_j\) are injective and linearly independent, in this special case they together define an isomorphism \(\mathbb{Z}^d \cong \mathbb{Z}^{r_1} \oplus \mathbb{Z}^{r_2} \oplus \cdots \oplus \mathbb{Z}^{r_k}\).

It is clear that for any system \((X, \mu, T)\) we have

\[
\zeta_0^{Tp_1} \land \left( \bigvee_{j=2}^k \zeta_0^{Tp_j} \right) \succ \bigvee_{j=2}^k \zeta_0^{Tp_1 \oplus p_j},
\]

so we need only prove the reverse containment. Let

\[(\tilde{X}, \tilde{\mu}) := (X \times \zeta_0^{Tp_1} X, \mu \otimes \zeta_0^{Tp_1} \mu)\]

and \(\pi_1\) and \(\pi_2\) be respectively the first and second coordinate projections \(\tilde{X} \rightarrow X\), and define a \(\mathbb{Z}^d\)-action \(\tilde{T}\) on \((\tilde{X}, \tilde{\mu})\) by setting

\[
\tilde{T}^{p_i}(\cdot) := \begin{cases} T^{p_1}(\cdot) \times \text{id}_X & \text{if } i = 1 \\ (T \times 2)^{p_i}(\cdot) & \text{if } i = 2, 3, \ldots, k. \end{cases}
\]

and extending additively. This is easily to seen to be a well-defined probability-preserving \(\mathbb{Z}^d\)-system and an extension of \((X, \mu, T)\) through \(\pi_1\). Now note that the whole second coordinate in \(\tilde{X}\) is \(\tilde{T}^{p_1}\)-invariant, and hence that

\[
\bigvee_{j=2}^k \zeta_0^{T^{p_1 \oplus p_j}} \succ \left( \bigvee_{j=2}^k \zeta_0^{T^{p_j}} \right) \circ \pi_2
\]

\[
\succ \left( \zeta_0^{T^{p_1}} \land \bigvee_{j=2}^k \zeta_0^{T^{p_j}} \right) \circ \pi_2 \simeq \left( \zeta_0^{T^{p_1}} \land \bigvee_{j=2}^k \zeta_0^{T^{p_j}} \right) \circ \pi_1,
\]

where the last equivalence holds because \(\zeta_0^{T^{p_1}} \circ \pi_1 \simeq \zeta_0^{T^{p_1}} \circ \pi_2\) by construction. On the other hand since \((X, \mu, T)\) is FIS the factors \(\bigvee_{j=2}^k \zeta_0^{T^{p_1 \oplus p_j}}\) and \(\pi_1\) must be
relatively independent over
\[ \pi_1 \wedge \left( \bigvee_{j=2}^k \tilde{T}_0^p \right) \cong \left( \bigvee_{j=2}^k \zeta_0^p \right) \circ \pi_1, \]
so in fact we have
\[ \left( \bigvee_{j=2}^k \zeta_0^p \right) \circ \pi_1 \succ \left( \bigvee_{j=2}^k \zeta_0^p \right) \circ \pi_1 \]
and hence
\[ \bigvee_{j=2}^k \zeta_0^p \succ \zeta_0^p \wedge \bigvee_{j=2}^k \zeta_0^p, \]
as required.

Finally, for a general \( \Lambda \) Corollary 3.19 tells us that the subaction system \( X^\Lambda \) is still FIS, and so since all joins of the idempotent classes \( Z_{p_0}^p \) are insensitive to the forgetful functor to \( \Lambda \)-subactions the special case treated above completes the proof. \( \square \)

**Proof of Proposition 4.5** Let \( \Lambda := \mathbb{Z}p_1 + \mathbb{Z}p_2 + \cdots + \mathbb{Z}p_k \).

Once again we first treat the case \( \Lambda = \mathbb{Z}^d \); this is already covered in [4] in slightly different terms, and the underlying idea here is as in that paper. Write \( T_i := T_{p_i} \) and fix some \( j \leq k \). Consider the extension \( \pi : \tilde{X} \to X \) built from the Furstenberg self-joining by

- letting \( (\tilde{X}, \tilde{\mu}) := (X^k, \mu^F) \),
- defining the lifted transformations \( \tilde{T}_i \) by
  \[ \tilde{T}_i = \begin{cases} T_1 \times T_2 \times \cdots \times T_k & \text{for } i = j \\ T_{i \times k} & \text{for } i \in \{1, 2, 3, \ldots, k\} \setminus \{j\}, \end{cases} \]
- writing \( \pi_i : X^k \to X, i = 1, 2, \ldots, k \), for the coordinate projections,
- and taking \( \pi := \pi_j \).

Now let \( f_i \in L^\infty(\mu) \) for \( i = 1, 2, \ldots, k \) and let \( g \in L^\infty(\mu^F) \) be \( \tilde{T}_j \)-invariant. Observe from the above choice of the lifted transformations that \( f_i \circ \pi_i \) is a \( \tilde{T}_i \tilde{T}_j^{-1} \)-invariant function on \( \tilde{X} \) for each \( i \neq j \), and so the function \( \prod_{i \leq k, i \neq j} (f_i \circ \pi_i) \cdot g \) on
\(\tilde{X}\) is \((\zeta_{T_j}^0 \lor \bigvee_{i \leq k, i \neq j} \zeta_{T_i}^0 = T_j)\)-measurable. Since \(X\) is FIS, under \(\mu^F\) this function is relatively independent from \(f_j \circ \tau_j^{T_j} \circ \pi_j\) that is, writing \(\xi_j := \zeta_{T_j}^0 \lor \bigvee_{i \leq k, i \neq j} \zeta_{T_i}^0 = T_j\), we have

\[
\int_{X^k} \prod_{i=1}^k (f_i \circ \pi_i) \cdot g \, d\mu^F = \int_{X^k} (E_{\mu}(f_j | \xi_j) \circ \pi_j) \cdot \prod_{i \leq k, i \neq j} (f_i \circ \pi_i) \cdot g \, d\mu^F.
\]

Using this argument to replace \(f_j\) with \(E_{\mu}(f_j | \xi_j)\) for each \(j\) in turn, Lemma 4.2 tells us that \((\xi_1, \xi_2, \ldots, \xi_k)\) is characteristic, as required.

Now if \(\Lambda\) is a general sublattice, we observe that for a given tuple of factors of our FIS system \(X\), their characteristicity depends only on the subaction system \(X|_\Lambda\), which is also FIS by Corollary 3.19, and so as for the preceding lemma the above special case completes the proof. \(\square\)

### 4.3 An example in continuous time

In addition to the above description of pleasant extensions for certain linear averages (by itself only a very modest generalization of technical results from [4]), we will now offer an application of sated extensions to a different convergence problem for nonconventional averages. This problem is ‘quadratic’ and ‘two-dimensional’, which features introduce new difficulties, but it is also in ‘continuous time’, and we will find that this allows us to recover a fairly short proof.

Thus, we now switch to the setting of a jointly measurable action \(\mathbb{R}^2 \curvearrowright (X, \mu)\), which we denote by \(\mathbb{R}^2 \rightarrow \text{Aut}_0(X, \mu) : \nu \mapsto \tau^\nu\). We also let \(e_1, e_2\) be the standard basis of \(\mathbb{R}^2\).

**Theorem 4.7.** The averages

\[
S_T(f_1, f_2) := \frac{1}{T} \int_0^T (f_1 \circ \tau^{t^2 e_1})(f_2 \circ \tau^{t^2 e_1 + t e_2}) \, dt
\]

converge in \(L^2(\mu)\) as \(T \rightarrow \infty\) for any \(f_1, f_2 \in L^\infty(\mu)\).

As in [4] this will follow once we ascend to a suitable pleasant extension.

**Proposition 4.8** (Pleasant extensions for continuous-time quadratic averages). If the \(\mathbb{R}^2\)-system \((X, \mu, \tau)\) is sated for the idempotent class \(\zeta_{0}^{e_1} \lor \zeta_{0}^{e_2}\) then the factors

\[\xi_1 = \xi_2 := \zeta_{0}^{e_1} \lor \zeta_{0}^{e_2}\]

are characteristic for the above averages.
Proof of Theorem 4.7 from Proposition 4.8: Given Proposition 4.8 it suffices to consider the averages $S_T(f_1, f_2)$ with each $f_i$ measurable with respect to $\zeta_0^{\Re e_1} \cup \zeta_0^{\Re e_2}$. By a simple approximation in $L^2(\mu)$ and multilinearity, the convergence of these follows in turn if we know it when $f_i = g_i \cdot h_i$ for some $g_1, g_2$ that are $\tau^{\Re e_1}$-invariant and $h_1, h_2$ that are $\tau^{\Re e_2}$-invariant.

Substituting this form into the definition of $S_T$, and using first the invariance of $g_1$ and then that of $h_1$, we are left with the averages

$$S_T(f_1, f_2) = g_1 \cdot \frac{1}{T} \int_0^T (h_1 \circ \tau^{t e_1})((g_2 \cdot h_2) \circ \tau^{t e_1 + t e_2}) \, dt$$

and these latter are now conventional polynomial ergodic averages, which converge in $L^2(\mu)$ simply by spectral theory and the corresponding result for the scalar averages $\frac{1}{T} \int_0^T \exp(2\pi (at^2 + bt)i) \, dt$, whose convergence follows from the classical scalar-valued van der Corput estimate (or, indeed, in the only nontrivial case $a \neq 0$, from a change of variables and the classical evaluation of Fresnel integrals).

Remark: Note that Proposition 4.8 is formulated in terms of satedness with respect to a single idempotent class, rather than by appeal to a continuous analog of the blanket notion of full isotropy satedness (Definition 3.13). This is because the continuous group $\mathbb{R}^2$ has uncountably many subgroups, and so we should need in turn an analog of Theorem 3.11 that allows for uncountably many idempotent classes. This is impossible in general without leaving the class of standard Borel spaces (although presumably the class of ‘perfect’ measure spaces, in the sense of Section 451 of [13], is still large enough), and so would entail a barrage of new technical measure-theoretic details that we prefer to avoid.

Proof of Proposition 4.8: This proof that starts with an important initial twist. We first note that we may change variables in the integral

$$\frac{1}{T} \int_0^T (f_1 \circ \tau^{t e_1})(f_2 \circ \tau^{t e_1 + t e_2}) \, dt$$

to $u := t^2$, and so obtain

$$S_{\sqrt{U}}(f_1, f_2) = \frac{1}{\sqrt{U}} \int_0^U (f_1 \circ \tau^{u e_1})(f_2 \circ \tau^{u e_1 + \sqrt{u} e_2}) \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{2} S'_{\sqrt{U}}(f_1, f_2) + \frac{1}{\sqrt{U}} \int_0^U \frac{1}{4} V^{-1/2} \cdot S'_V(f_1, f_2) \, dV$$

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where
\[ S'_U(f_1, f_2) := \frac{1}{U} \int_0^U (f_1 \circ \tau^{ue_1})(f_2 \circ \tau^{ue_1+\sqrt{ue_2}}) \, du. \]

Thus, this change of variables has revealed that the averages \( S_T(f_1, f_2) \) are actually ‘smoother’ than the averages \( S'_U(f_1, f_2) \), which involve only linear and sub-linear exponents. In spite of of the non-integer power \( \sqrt{u} \) that has now appeared, we will now see that these are quite simple for our purposes. (This crucial trick was pointed out to me by Vitaly Bergelson.)

To complete the proof we show that
\[ S'_U(f_1, f_2) \not\to 0 \quad \Rightarrow \quad E_{\mu}(f_1 | \zeta^{\tau|Re_1} \lor \zeta^{\tau|Re_2}) \neq 0. \]

As usual, this begins with the van der Corput estimate (in its version for continuous families of vectors, which is exactly analogous to the discrete setting: see, for example, Section 1.9 of Kuipers and Niederreiter [23]), which after a little rearrangement gives that
\[ \frac{1}{H} \int_0^h \frac{1}{U} \int_0^U \int_X \left((f_1 \circ \tau^{he_1} \cdot \bar{f}_1) \circ \tau^{ue_1}\right) \cdot \left((f_2 \circ \tau^{he_1+(\sqrt{u+h}-\sqrt{u})e_2} \cdot \bar{f}_2) \circ \tau^{ue_1+\sqrt{ue_2}}\right) \, d\mu \, du \, dh \]
\[ = \frac{1}{H} \int_0^h \frac{1}{U} \int_0^U \int_X \left(f_1 \circ \tau^{he_1} \cdot \bar{f}_1\right) \cdot \left((f_2 \circ \tau^{he_1+(\sqrt{u+h}-\sqrt{u})e_2} \cdot \bar{f}_2) \circ \tau^{\sqrt{ue_2}}\right) \, d\mu \, du \, dh \not\to 0 \quad \text{as } U \to \infty \text{ and then } H \to \infty. \]

The important feature here is that for each fixed \( h \) we have
\[ \sqrt{u+h} - \sqrt{u} \to 0 \quad \text{as } u \to \infty, \]
and hence by the strong continuity of \( \tau \) it follows that
\[ \| f_2 \circ \tau^{he_1+(\sqrt{u+h}-\sqrt{u})e_2} - f_2 \circ \tau^{he_1} \|_2 \to 0 \]
as \( u \to \infty \). From this it follows that for any fixed \( h \) we have

\[
\frac{1}{U} \int_0^U \int_X (f_1 \circ \tau^{0 \cdot h} \cdot \tilde{f}_1)((f_2 \circ \tau^{0 \cdot h}) + (\sqrt{u} + \sqrt{h - \sqrt{u}}) \cdot \tilde{f}_2) \circ \tau^{0 \cdot h} \cdot \tilde{f}_2) \, d\mu \, du
\]

\[
\sim \frac{1}{U} \int_0^U \int_X (f_1 \circ \tau^{0 \cdot h} \cdot \tilde{f}_1)((f_2 \circ \tau^{0 \cdot h}) \cdot \tilde{f}_2) \circ \tau^{0 \cdot h} \cdot \tilde{f}_2) \, d\mu \, du
\]

\[
\to \int_X (f_1 \circ \tau^{0 \cdot h} \cdot \tilde{f}_1) E_\mu(f_2 \circ \tau^{0 \cdot h} \cdot \tilde{f}_2) \, d\mu \quad \text{as } U \to \infty
\]

\[
= \int_{X^2} (f_1 \circ \tau^{0 \cdot h} \cdot \tilde{f}_1) \otimes (f_2 \circ \tau^{0 \cdot h} \cdot \tilde{f}_2) \, d(\mu \otimes \zeta_{\tau^{0 \cdot h} \cdot \tilde{f}_2} \cdot \mu)
\]

(this crucial simplification resulting from our change-of-variables was pointed out to me by Vitaly Bergelson). Now letting \( h \to \infty \) this simply tends to

\[
\int_{X^2} (f_1 \otimes f_2) \cdot g \, d(\mu \otimes \zeta_{\tau^{0 \cdot h} \cdot \tilde{f}_2} \cdot \mu)
\]

for the \((\tau^{0 \cdot h})^{\otimes 2}_{\Re e_1}\)-invariant function

\[
g := E_{\mu \otimes \zeta_{\tau^{0 \cdot h} \cdot \tilde{f}_2} \cdot \mu}(f_1 \otimes f_2) \mid \zeta_{\tau^{0 \cdot h} \cdot \tilde{f}_2}^{(\otimes 2)} \cdot \mu
\]

Hence, letting \((\tilde{X}, \tilde{\mu}) = (X^2, \mu \otimes \zeta_{\tau^{0 \cdot h} \cdot \tilde{f}_2} \cdot \mu)\), letting \( \pi : \tilde{X} \to X \) and lifting \( \tau \) to the action \( \tilde{\tau} \) defined by

\[
\tilde{\tau}^{s e_1 + t e_2} := \tau^{s e_1 + t e_2} \times \tau^{s e_1}
\]

(notating that \( \mu \otimes \zeta_{\tau^{0 \cdot h} \cdot \tilde{f}_2} \cdot \mu \) is also invariant under the flow \( t \mapsto \text{id}_X \times \tau^{t e_2} \)), we see that we have found an extension of \((X, \mu, \tau)\) in which

\[
E_{\tilde{\mu}}(f_1 \circ \pi) \mid \zeta_{\tau^{0 \cdot h} \cdot \tilde{f}_2}^{(\otimes 2) \cdot \mu} \neq 0,
\]

and hence by satedness the analogous non-vanishing must have held inside the original system \((X, \mu, \tau)\), as required.

\[\square\]

**Remark** Although rather simple, it is worth noting that the use of Satedness still played a crucial rôle in the above proof. Without the initial assumption of Satedness, the above appeal to the van der Corput estimate combined with considerations of the structure of the Furstenberg self-joining tell us that for the averages \( S_U(f_1, f_2) \) the pair of factors \( \xi_1, \xi_2 \) is characteristic, where \( \xi_i \) coordinatizes the maximal \( \tau^{\otimes e_i} \)-isometric extension of the isotropy factor \( \zeta_{0 \cdot e_i}^{(\otimes 2)} \).
This much can be argued following the same lines as Conze and Lesigne’s initial analysis in [8, 9, 10] of double linear averages in discrete time for some system $T : \mathbb{Z}^2 \curvearrowright (X, \mu)$. Thus, allowing ourselves to assume that each $f_i$ is $\xi_i$-measurable, it now follows that each $f_i$ may be approximated by a function residing in a finite-rank $\tau^{[\mathbb{R}e_1]}$-invariant module over the factor $\mathcal{C}_0^{[\mathbb{R}(e_1-e_2)]}$. In Conze and Lesigne’s setting (with $T$ and $\mathbb{Z}$ in place of $\tau$ and $\mathbb{R}$) this leads directly to a proof of convergence, because when written in terms of unitary cocycles describing these finite-rank modules the double linear averages become simply averages for some new ‘combined’ finite rank module over the single system $T|_{\mathcal{C}_0^{[\mathbb{Z}(e_1-e_2)]}}$, to which the usual mean ergodic theorem can be applied. However, in our setting matters are not so simple, since even after approximating and then using a representation in terms of unitary cocycles in this way, the expression that results still involves two different polynomials in the exponents, and so it is not clear how to realize it as some kind of more classic ergodic average. Once we assumed satedness, this problem vanished because the structure of a finite-rank $\tau^{[\mathbb{R}e_1]}$-invariant module over $\mathcal{C}_0^{[\mathbb{R}(e_1-e_2)]}$ is replaced by that of the factor $\mathcal{C}_0^{[\mathbb{R}e_1]} \vee \mathcal{C}_0^{[\mathbb{R}(e_1-e_2)]}$, for which more explicit simplifications to our averages are possible, as exhibited above.

It would be interesting to know how far into the world of polynomial nonconventional averages in continuous time one could penetrate using the method of sated extensions together with the important change-of-variables trick shown above. I strongly suspect that in this setting it would at least be possible to prove convergence for a much wider range of averages involving quadratics in the exponents. However, this seems to require the organization of a new induction scheme that allows for working with fractional-power polynomials (as a result of the changing of variables), and this would take us too far from the setting of discrete averages that the present sequence of papers aims to examine, so we will not pursue this further here.

5 Next steps

This paper has begun to showcase the far-reaching consequences of satedness in the study of nonconventional ergodic averages, but its larger purpose is to prepare the ground for its sequel [2]. There we will turn to nonconventional averages which require rather more elaborate new arguments. Indeed, only after several more technical steps will we be able to address even one new case of convergence.
for polynomial averages in discrete time: that of
\[ \frac{1}{N} \sum_{n=1}^{N} (f_1 \circ T_1^n)(f_2 \circ T_2^n) \] for \((T_1, T_2) : \mathbb{Z}^2 \curvearrowright (X, \mu), f_1, f_2 \in L^\infty(\mu)\) (as in Theorem 1.4 above).

The analysis of these will rely heavily on some auxiliary results concerning the triple linear averages
\[ \frac{1}{N} \sum_{n=1}^{N} (f_1 \circ T_1^{np_1})(f_2 \circ T_2^{np_2})(f_3 \circ T_2^{np_3}) \]
for some action \(T : \mathbb{Z}^2 \curvearrowright (X, \mu)\) and three directions \(p_1, p_2, p_3 \in \mathbb{Z}^2\) enjoying some linear dependence. Such linear averages arise naturally from the above quadratic averages upon a single application of the van der Corput estimate, and so we will construct a useful notion of pleasant extension for the quadratic averages by first developing such a notion for these triple linear averages and then showing how the resulting characteristic factors can be simplified further.

The point is that although convergence is known for triple linear averages such as the above, the use of extensions to prove this in [4] forgets the linear dependence of the \(p_i\), effectively replacing the \(T^{np_i}\) with three independent commuting transformations on the extended system. We cannot afford this freedom in the study of the quadratic averages, because after passing to such a \(\mathbb{Z}^3\)-system it is not clear how the quadratic averages of interest can even be sensibly interpreted. Our main task, therefore, will be to see how simple a triple of characteristic factors can be obtained for the above linear nonconventional averages while preserving the algebraic relations of the original \(\mathbb{Z}^2\)-action. It will turn out that we can do quite well, except that in addition to the factors that contribute to each \(\xi_i\) in Theorem 1.1 we must now involve some systems on which our \(\mathbb{Z}^2\)-action is by commuting rotations on a two-step nilmanifold, thus re-establishing contact with earlier works such as [21, 32] and their forerunners in which the relevance of these was made clear in the setting of \(\mathbb{Z}\)-actions.

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