Massive 4D Abelian 2-Form Theory: Nilpotent Symmetries from the (Anti-)Chiral Superfield Approach

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Abstract: The off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetries are obtained by using the (anti-)chiral superfield approach (ACSA) to Becchi-Rouet-Stora-Tyutin (BRST) formalism for the four (3+1)-dimensional (4D) Stuckelberg-modified massive Abelian 2-form gauge theory. We perform exactly similar kind of exercise for the derivation of the off-shell nilpotent (anti-)co-BRST symmetry transformations, too. In the above derivations, the symmetry invariant restrictions on the superfields play very important and decisive roles. To prove the sanctity of the above nilpotent symmetries, we generalize our 4D ordinary theory (defined on the 4D flat Minkowskian spacetime manifold) to its counterparts (4,1)-dimensional (anti-)chiral super sub-manifolds of the (4, 2)-dimensional supermanifold which is parameterized by the superspace coordinates \( Z^M = (x^\mu, \theta, \bar{\theta}) \) where \( x^\mu(\mu = 0, 1, 2, 3) \) are the bosonic coordinates and a pair of Grassmannian variables \((\theta, \bar{\theta})\) are fermionic: \( \theta^2 = \bar{\theta}^2 = 0, \ \theta \bar{\theta} + \bar{\theta} \theta = 0 \) in nature. One of the novel observations of our present endeavor is the derivation of the Curci-Ferrari (CF)-type restrictions from the requirement of the symmetry invariance of the coupled (but equivalent) Lagrangian densities of our theory within the framework of ACSA to BRST formalism. We also exploit the standard techniques of ACSA to capture the off-shell nilpotency and absolute anticommutativity of the conserved (anti-)BRST as well as the conserved (anti-)co-BRST charges. In a subtle manner, the proof of the absolute anticommutativity of the above conserved charges also implies the existence of the appropriate CF-type restrictions on our theory. This proof is also a novel result.

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Keywords: 4D massive Abelian 2-form gauge theory; (anti-)BRST symmetries; (anti-)co-BRST symmetries; coupled Lagrangian densities; Curci–Ferrari type restrictions; conserved charges; ACSA to BRST formalism; symmetry invariant restrictions; nilpotency and absolute anticommutativity properties.
1 Introduction

The superfield approach to Becchi-Rouet-Stora-Tyutin (BRST) formalism [1-8] provides the geometrical origin and interpretation for the nilpotency and absolute anticommutativity properties that are associated with the quantum BRST and anti-BRST symmetries corresponding to a given classical local gauge symmetry transformation for a given p-form \( (p = 1, 2, 3, \ldots) \) gauge theory which is generated by the first-class constraints [9, 10] that characterize such a theory. One of the key features of the usual superfield approach (USFA) to BRST formalism [4-6], proposed by Bonora and Tonin (BT), is the derivation of (i) the exact off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations, and (ii) the (anti-)BRST invariant CF-type restriction(s). The existence of the latter is the signature [11, 12] of the quantum version of a gauge theory that is discussed and described within the framework of BRST formalism. The other characteristic feature of USFA (proposed by BT) is the observation that it leads only to the derivation of the nilpotent (anti-)BRST symmetries for the gauge and associated (anti-)ghost fields of the given p-form quantum gauge theory.

The above USFA has been systemically generalized in our earlier works (see, e.g. [13-16]) which lead to the derivation of proper (anti-)BRST symmetry transformations for the matter fields in addition to the gauge and (anti-)ghost fields of an interacting p-form (non-)Abelian gauge theory. The generalized version of the USFA has been christened as the augmented version of superfield approach (AVSA) to BRST formalism (see, e.g. [13-16]). In a very recent set of works (see, e.g. [17-20]), we have incorporated the diffeomorphism transformation in the BT-superfield formalism which has been called as the modified BT-superfield approach to BRST formalism. One of the common features of the above superfield approaches [1-8, 13-20] is the observation that the fields of an ordinary D-dimensional gauge/diffeomorphism invariant theory have been generalized onto a suitably chosen \((D, 2)\)-dimensional supermanifold which is parameterized by the super space coordinates \( Z^M = (x^\mu, \theta, \bar{\theta}) \) where coordinates \( x^\mu (\mu = 0, 1, 2, \ldots, D - 1) \) define the bosonic \( D \)-dimensional ordinary coordinates of the \( D \)-dimensional flat Minkowskian sub-manifold of the above \((D, 2)\)-dimensional supermanifold and the Grassmannian variables \((\theta, \bar{\theta})\) satisfy: \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} = 0 \) demonstrating that they are fermionic in nature. Furthermore, it has been observed that the full expansions of the superfields have been taken along all the possible Grassmannian directions of the \((D, 2)\)-dimensional supermanifold in all the above superfield approaches to BRST formalism [1-8, 13-20].

In our very recent set of works (see, e.g. [21-23] for details), we have applied the simplified version of the AVSA/USFA as well as the modified BT-superfield approaches where the ordinary fields of a given ordinary \( D \)-dimensional gauge theory have been generalized onto a couple of \((D, 1)\)-dimensional (anti-)chiral super sub-manifolds of the general \((D, 2)\)-dimensional supermanifold that has been considered in [1-8, 13-20]. The purpose of our present investigation is to apply the simplified version (see, e.g. [21-23]) of the superfield approach to BRST formalism where only the (anti-)chiral super expansions are taken into account for the (anti-)chiral superfields [defined on the \((D, 1)\)-dimensional (anti-)chiral super sub-manifolds] and apply the symmetry invariant restrictions on them to obtain the (anti-)BRST as well as the (anti-)co-BRST transformations for our 4D St"uckelberg-modified...
massive Abelian 2-form theory which has been proven by us to be a massive model for the Hodge theory in physical (3+1)-dimensional (4D) spacetime [24].

In our present investigation, we briefly mention the most general forms of the Lagrangian densities [cf. Eqs. (10), (11) below] by linearizing the kinetic and gauge-fixing terms for the fields ($B_{\mu\nu}, \phi_\mu, \bar{\phi}_\mu$) by invoking the Nakanishi-Lautrup type auxiliary fields. These coupled (but equivalent) Lagrangian densities are the generalizations of the ordinary Lagrangian density [cf. Eq. (6) below] where the kinetic term for the $B_{\mu\nu}$ field and gauge-fixing terms for ($B_{\mu\nu}, \phi_\mu, \bar{\phi}_\mu$) fields are not linearized. We focus on the (anti-)BRST and (anti-)co-BRST symmetries of these Lagrangian densities and obtain the Euler-Lagrange equations of motion (EL-EOMs) as well as the CF-type restrictions from them. The main results of our present investigation are the derivations of the (anti-)BRST, (anti-)co-BRST symmetries and CF-type restrictions by using the ACSA to BRST formalism. Furthermore, we express the coupled Lagrangian densities and (anti-)BRST as well as (anti-)co-BRST charges in terms of the (anti-)chiral superfields which are obtained after the applications of the (anti-)BRST and (anti-)co-BRST invariant restrictions on the (anti-)chiral superfields. We prove the existence of the CF-type restrictions on our theory by demanding the (anti-) BRST and (anti-)co-BRST symmetry invariance of the super Lagrangian densities as well as by proving the absolute anticommutativity of the (anti-)BRST as well as (anti-) co-BRST charges that are present in our theory.

The following motivating factors have spurred our curiosity in perusing our present investigation. First of all, we have demonstrated the existence of the proper (anti-)BRST, (anti-)co-BRST symmetries and CF-type restrictions in our earlier work [24] on the 4D massive Abelian 2-form gauge theory. To prove the sanctity of the above continuous symmetries and the CF-type restrictions, it is essential to verify them within the framework of ACSA to BRST formalism. Second, we have taken (anti-)ghost part of the coupled (but equivalent) Lagrangian densities of our 4D theory to be the same [24]. We provide the precise arguments for the above correct choice in our present endeavor starting from the (dual-)gauge symmetry transformations (cf. Sec. 2). Finally, we know, from our earlier works [11, 12], that the existence of the CF-type restriction(s) is a decisive feature of a gauge theory (described within the framework of BRST formalism). We verify their existence, using the ACSA to BRST formalism in our present endeavor.

Our present paper is organized as follows. In Sec. 2, we discuss the bare essentials of the continuous symmetry properties of the St\"{u}ckelberg-modified Lagrangian density for the massive 4D Abelian 2-form theory. Our Sec. 3 is devoted to the discussion of the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries for the coupled (but equivalent) Lagrangian densities. We derive the (anti-)BRST as well as the (anti-)co-BRST symmetries within the framework of ACSA to BRST formalism and comment on their absolute anticommutativity property in Sec. 4. In Sec. 5 of our present endeavor, we show the existence of the CF-type restrictions by proving the invariance of the super Lagrangian densities within the framework of ACSA to BRST formalism. Our Sec. 6 deals with the proof of

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*Our present 4D massive Abelian 2-form theory is interesting because there is some physical relevance of this theory in the context of dark matter, dark energy and cosmological models of the universe (cf. Sec. 7 below for detail). It is precisely because of this reason that we have concentrated seriously on the proof of its quantum symmetries as well as the CF-type restrictions within the framework of ACSA to BRST formalism.
nilpotency and absolute anticommutativity of the (anti-)BRST and (anti-)co-BRST conserved charges within the framework of ACSA. We also demonstrate the existence of the CF-type restrictions, in a subtle manner, in the proof of the absolute anticommutativity property of the conserved and nilpotent charges. Finally, in Sec. 7, we make some concluding remarks and point out a few future directions.

In our Appendices A and B, we perform some explicit computations which complement the theoretical contents of our sub-sections 5.2 and 6.1 in the main body of the text of our present investigation.

Convention and Notations: We adopt the convention of the left derivative w.r.t. all the fermionic fields of our theory. We take the 4D flat Minkowskian metric tensor $\eta_{\mu\nu}$ as: $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ so that the dot product between two non-null 4D vectors $P_{\mu}$ and $Q_{\mu}$ is defined as: $P \cdot Q = \eta_{\mu\nu}P^{\mu}Q^{\nu} \equiv P_{0}Q_{0} - P_{i}Q_{i}$ where the Greek indices $\mu, \nu, \lambda... = 0, 1, 2, 3$ stand for the time and space directions and Latin indices $i, j, k... = 1, 2, 3$ correspond to the 3D space directions only. We denote the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations (of all kinds) by $s_{(a)b}$ and $s_{(a)d}$, respectively. The corresponding conserved charges are represented by $Q_{(a)b}$ and $Q_{(a)d}$. The 4D Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$ is chosen such that $\varepsilon_{0123} = +1 = -\varepsilon_{0123}$ and $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} = -3! \delta_{\rho}^{\sigma} \delta_{\sigma}^{\rho} \delta_{\rho}^{\sigma} = -2! (\delta_{\rho}^{\sigma} \delta_{\sigma}^{\rho} - \delta_{\rho}^{\sigma} \delta_{\sigma}^{\rho})$, etc. We also adopt the convention: $(\delta B_{\mu\nu}/\delta B_{\rho\sigma}) = \frac{1}{2!} (\delta_{\mu}^{\rho} \delta_{\nu}^{\sigma} - \delta_{\mu}^{\sigma} \delta_{\nu}^{\rho})$, etc. for the tensorial differentiation/variation.

2 Preliminaries: Lagrangian Formulation

In this section, we discuss the infinitesimal and continuous (dual-)gauge symmetry transformations of the Stückelberg-modified Lagrangian densities before their generalizations to the off-shell nilpotent and continuous (anti-)BRST and (anti-)co-BRST symmetry invariant coupled (but equivalent) Lagrangian densities for our present 4D massive Abelian 2-form theory. Our present section is divided into two parts as discussed and described below:

2.1 Infinitesimal Gauge Symmetry Transformations

We begin with the four (3 + 1)-dimensional (4D) Kalb-Ramond Lagrangian density $L_{(0)}$ for the free Abelian 2-form massive theory (with the rest mass equal to $m$) as follows (see, e.g. [25] for details)

$$L_{(0)} = \frac{1}{12} H^{\mu\nu\eta} H_{\mu\nu\eta} - \frac{m^2}{4} B^{\mu\nu} B_{\mu\nu},$$

where the antisymmetric $(B_{\mu\nu} = -B_{\nu\mu})$ tensor field $B_{\mu\nu}$ is the 4D massive Abelian 2-form $[B^{(2)} = \frac{1}{2!} (dx^\mu \wedge dx^\nu) B_{\mu\nu}]$ field and the curvature (i.e. field strength) tensor $H_{\mu\nu\eta} = \partial_{\mu}B_{\nu\eta} + \partial_{\nu}B_{\eta\mu} + \partial_{\eta}B_{\mu\nu}$ is derived from the 3-form $[H^{(3)} = dB^{(2)} = \frac{1}{3!} (dx^\mu \wedge dx^\nu \wedge dx^\eta) H_{\mu\nu\eta}]$ where $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative of differential geometry. We note that the mass dimension of $B_{\mu\nu}$ is $[M]$ in the natural units (where we take $\hbar = c = 1$) for our 4D theory. Due to the existence of mass term $(-\frac{m^2}{4} B_{\mu\nu} B^{\mu\nu})$, the gauge invariance...
is lost because the above Lagrangian density is endowed with the second-class constraints [26] in the terminology of Dirac's prescription for the classification scheme of constraints [9, 10]. We can find out the Euler-Lagrange equation of motion (EL-EOM) from \( \mathcal{L}(0) \) as:

\[
\partial_\mu H^{\mu\nu} + m^2 B^{\nu\eta} = 0.
\]

At this stage, it is evident that, for \( m^2 \neq 0 \), we have \( \partial_\mu B^{\mu\nu} = 0 = \partial_\nu B^{\mu\nu} \). The latter conditions (i.e. \( \partial_\mu B^{\mu\nu} = 0, \partial_\nu B^{\mu\nu} = 0 \)) emerge out because of the totally antisymmetric nature of \( H^{\mu\nu} \) (present in the original equation: \( \partial_\mu H^{\mu\nu} + m^2 B^{\nu\eta} = 0 \)). Ultimately, we obtain the usual Klein-Gordon equation [i.e. \((\Box + m^2) B^{\mu\nu} = 0\)] for the massive Abelian 2-form field \( (B^{\mu\nu}) \). This observation, in a subtle way, implies that all the numerical factors in Eq. (1) are correct with their proper signatures.

By the application of St"uckelberg’s technique, it can be checked that, we have the following appropriate transformation [24] of the antisymmetric tensor field \( (B^{\mu\nu}) \):

\[
B^{\mu\nu} \rightarrow B^{\mu\nu} - \frac{1}{m} \Phi^{\mu\nu} - \frac{1}{2m} \varepsilon_{\mu\nu\eta\kappa} \tilde{\Phi}^{\eta\kappa} \equiv B^{\mu\nu} - \frac{1}{m} \left( \partial_\mu \phi_\nu - \partial_\nu \phi_\mu - \varepsilon_{\mu\nu\eta\kappa} \partial^\eta \phi^\kappa \right) \equiv B^{\mu\nu} - \frac{1}{m} \Phi^{\mu\nu} - \frac{1}{m} F^{\mu\nu}.
\] (2)

In the above, the Abelian 2-form \( \Phi^{(2)} = \frac{1}{2} (dx^\mu \wedge dx^\nu) \Phi^{\mu\nu} \equiv d \Phi^{(1)} \) (with vector 1-form \( \Phi^{(1)} = dx^\mu \phi_\mu \rightarrow \Phi^{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \)) is obtained from a vector field \( \phi_\mu \). On the contrary, the dual antisymmetric tensor \( \tilde{\Phi}_{\mu\nu} = \partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu \) is constructed with the help of an axial-vector field \( \tilde{\phi}_\mu \) which is defined through the axial-vector 1-form \( \tilde{\Phi}^{(1)} = dx^\mu \tilde{\phi}_\mu \). In other words, the axial Abelian 2-form: \( \tilde{\Phi}^{(2)} = d \tilde{\Phi}^{(1)} \equiv \left( \frac{d x^\mu \wedge d x^\nu}{2!} \right) \tilde{\Phi}^{\mu\nu} \) leads to the derivation: \( \tilde{\Phi}^{\mu\nu} = \partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu \). We would like to add that we have \( F^{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa} \tilde{\Phi}^{\eta\kappa} \), where \( \tilde{\Phi}^{\mu\nu} = \partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu \) is defined, as argued earlier, from the axial Abelian 2-form \( \tilde{\Phi}^{(2)} \) to maintain the parity invariance in our Abelian 2-form massive gauge theory. We shall comment on the specific structure of the antisymmetric tensor: \( \partial_\mu \phi_\nu - \partial_\nu \phi_\mu + \varepsilon_{\mu\nu\eta\kappa} \partial^\eta \phi^\kappa \) and its connection with the source-free Maxwell’s theory in our Conclusions section (cf. Sec. 7 below). Thus, we observe that, under the above transformations (2), the Lagrangian density \( \mathcal{L}(0) \) transforms (i.e. \( \mathcal{L}(0) \rightarrow \mathcal{L}(1) \)) into the following form\(^1\):

\[
\mathcal{L}(0) \rightarrow \mathcal{L}(1) = \frac{1}{12} H^{\mu\nu\eta} H_{\mu\nu\eta} - \frac{m^2}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} \Phi^{\mu\nu} \Phi_{\mu\nu} + \frac{1}{4} \tilde{\Phi}^{\mu\nu} \tilde{\Phi}_{\mu\nu} + \frac{m}{2} B^{\mu\nu} \Phi_{\mu\nu} + \frac{m}{4} \varepsilon_{\mu\nu\eta\kappa} B_{\mu\nu} \tilde{\Phi}_{\eta\kappa},
\] (3)

The above St"uckelberg’s modified Lagrangian density respects the following continuous and infinitesimal gauge symmetry transformations (\( \delta_g \))

\[
\delta_g B^{\mu\nu} = - \left( \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \right), \quad \delta_g \phi_\mu = \left( \partial_\mu \Lambda - m \Lambda_\mu \right), \quad \delta_g \tilde{\phi}_\mu = 0,
\] (4)

where \( \Lambda_\mu \) and \( \Lambda \) are the vector and scalar gauge transformation parameters. Under the continuous gauge symmetry transformations (4), the Lagrangian density \( \mathcal{L}(1) \) transforms to the following total spacetime derivative, namely;

\[
\delta_g \mathcal{L}(1) = \partial_\mu \left[ - m \varepsilon_{\mu\nu\eta\kappa} \Lambda_\nu \partial_\eta \tilde{\phi}_\kappa \right].
\] (5)

\(^1\)It should be noted that the kinetic term (i.e. \( \frac{1}{4} H^{\mu\nu\eta} H_{\mu\nu\eta} \)) transforms, under (2), to itself plus extra terms [25]. The latter terms, however, turn out to be derivatives of third and fourth order. Hence, they have been ignored keeping in mind the renormalizability of our theory [in four \((3+1)\)-dimensions of spacetime],
As a consequence, the action integral \( S = \int d^4x L_1 \) remains invariant under (4) for the physically well-defined fields which vanish off at infinity.

We would like to end this sub-section with the following remarks. First, we observe that the kinetic term \( \left( \frac{1}{12} H_{\mu\nu\eta} H^{\mu\nu\eta} \right) \) remains invariant under the infinitesimal gauge symmetry transformations (4). To be precise, we note that the curvature (i.e. the field strength) tensor \( H_{\mu\nu\eta} \), owing its origin to the exterior derivative \( d = d x^\mu \partial_\mu \) \((d^2 = 0)\) of the de Rham cohomological operators [27-31], remains invariant (i.e. \( \delta g H_{\mu\nu\eta} = 0 \)) under the infinitesimal gauge symmetry transformations (4). Second, we do note that the axial-vector field \( \tilde{\Phi}_\mu \) and, hence \( \tilde{\Phi}_{\mu\nu} = \partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu \) as well as the kinetic term for this field, remain invariant (i.e. \( \delta g \tilde{\phi}_\mu = 0, \delta g \tilde{\Phi}_{\mu\nu} = 0 \)) under the gauge symmetry transformations (4). Third, we have not invoked any new fields in the theory besides the St"uckelberg fields (e.g. vector field \( \phi_\mu \) and axial-vector field \( \tilde{\phi}_\mu \)) due to the dimensionality of the spacetime and antisymmetric \((B_{\mu\nu} = -B_{\nu\mu})\) nature of the gauge field \((B_{\mu\nu})\). Finally, we note that, so far, our gauge symmetry transformations (4) are classical. However, they can be elevated to their quantum counterparts [i.e. nilpotent and anticommuting (anti-)BRST symmetries] within the framework of BRST formalism (see, Sec. 3 below).

2.2 Infinitesimal Dual-Gauge Symmetry Transformations

To discuss the dual-gauge symmetry transformations, we have to add the gauge-fixing term to the St"uckelberg-modified Lagrangian density \( L_1 \) [cf. Eq. (3)] which respects the classical gauge symmetry transformations quoted in Eq. (4). Furthermore, we have to modify the kinetic term as well as the gauge-fixing term by invoking some new additional fields. This has already been done systematically in our earlier work [24] where physical and mathematical arguments have been provided for the same. The total Lagrangian density with the modified kinetic term, gauge-fixing term and a mass term is as follows (see, e.g. [24] for details)

\[
L_1 \rightarrow L_2 = -\frac{1}{2} \left( \frac{1}{2} \varepsilon_{\mu
u\rho} \partial^{\nu} B^{\rho}_{\eta\kappa} - \frac{1}{2} \partial_\mu \tilde{\phi} + m \tilde{\phi}_\mu \right)^2 - \frac{m^2}{4} B^{\mu\nu} B_{\mu\nu}
- \frac{1}{4} \Phi^{\mu\nu} \Phi_{\mu\nu} + \frac{m}{2} B^{\mu\nu} \Phi_{\mu\nu} + \frac{1}{4} \tilde{\Phi}^{\mu\nu} \tilde{\Phi}_{\mu\nu} + \frac{m}{4} \varepsilon^{\mu\nu\rho\kappa} B_{\mu\nu} \tilde{\Phi}_{\rho\kappa}
+ \frac{1}{2} \left( \partial^{\nu} B_{\nu\mu} - \frac{1}{2} \partial_\mu \phi + m \phi_\mu \right)^2
- \frac{1}{2} \left( \partial_\mu \phi^{\mu} + \frac{1}{2} m \phi \right)^2
+ \frac{1}{2} \left( \partial_\mu \tilde{\phi}^{\mu} + \frac{1}{2} m \tilde{\phi} \right)^2 ,
\]

where the (pseudo-)scalar fields \((\tilde{\phi})\phi\) are the new fields and the gauge-fixing term \((\partial^{\nu} B_{\nu\mu})\) for the antisymmetric tensor field owes its origin to the co-exterior derivative \( \delta = \pm \ast d \ast \) \((\delta^2 = 0)\) of the de Rham cohomological operators of differential geometry [26-30] because \( \delta B^{(2)} = (\partial^{\nu} B_{\nu\mu}) \, dx^{\mu} \). Here * is the Hodge duality operator. It should be noted that the last two terms in Eq. (6) are nothing but the gauge-fixing terms for the vector and axial-vector fields \( \phi_\mu \) and \( \tilde{\phi}_\mu \), respectively. At this stage, it can be noted that the new fields \((\phi, \tilde{\phi}, \phi_\mu, \tilde{\phi}_\mu)\) have mass dimension \([M]\) in the natural units (i.e. \( h = c = 1 \)) for our present theory.
It is obvious that the above modified Lagrangian density (6) would not have the perfect gauge symmetry transformations (4) because of the additional terms. However, it is very interesting to note that under the following infinitesimal and continuous transformations

\[
\delta_{dg} B_{\mu\nu} = -\varepsilon_{\mu\nu\eta\kappa} \partial^{\eta} \Sigma^\kappa, \quad \delta_{dg} \tilde{\phi} = \frac{1}{2} \partial \Sigma + \frac{1}{2} m \Sigma, \quad \delta_{dg} \phi_\mu = 0,
\]

\[
\delta_{dg} \phi = -\sigma, \quad \delta_{dg} \varphi = 0, \quad \delta_{dg} \left[ \partial^\nu B_{\nu\mu} - \frac{1}{2} \partial_\mu \varphi + m \phi_\mu \right] = 0,
\]

\[
\delta_g B_{\mu\nu} = -\left( \partial_\nu \Lambda_\mu - \partial_\mu \Lambda_\nu \right), \quad \delta_g \phi_\mu = \left( \partial_\mu \Lambda - m \Lambda_\mu \right), \quad \delta_g \tilde{\phi}_\mu = 0,
\]

\[
\delta_g \varphi = \tilde{\lambda}, \quad \delta_g \tilde{\phi} = 0, \quad \delta_g \left[ \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa} \partial^\nu B^{\eta\kappa} - \frac{1}{2} \partial_\mu \tilde{\varphi} + m \tilde{\phi}_\mu \right] = 0,
\]

we obtain the following transformations for the Lagrangian density (6):

\[
\delta_{dg} \mathcal{L}_{(2)} = \partial_\mu \left[ \frac{1}{2} \tilde{\varphi} \partial^\mu \left( (\partial \cdot \Sigma) + \frac{1}{2} m \Sigma \right) \right] - \left( \partial \cdot \Sigma \right) \left( \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa} \partial_\nu B_{\eta\kappa} + m \tilde{\phi}_\mu \right) - \frac{1}{2} \tilde{\varphi} \left( \square + m^2 \right) \Sigma + \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa} \partial_\nu \Sigma_\eta \phi_{\kappa} + \left( \partial \cdot \phi \right) \left( \square + m^2 \right) \Sigma - \frac{1}{2} \tilde{\varphi} \left( \square + m^2 \right) \sigma,
\]

\[
\delta_g \mathcal{L}_{(2)} = -\partial_\mu \left[ \frac{1}{2} \varphi \partial^\mu \left( (\partial \cdot \Lambda) - \frac{1}{2} \tilde{\lambda} + m \Lambda \right) \right] - \left( \partial \cdot \Lambda \right) \left( \frac{1}{2} \tilde{\lambda} m \Lambda \right) \left( \partial_\nu B^{\nu\mu} + m \phi_\mu \right) - \frac{1}{2} \varphi \left( \square + m^2 \right) \Lambda + \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa} \Lambda_\nu \phi_{\kappa} - \left( \partial_\nu B^{\nu\mu} - m \phi_\mu \right) \left( \square + m^2 \right) \Lambda - \left( \partial \cdot \phi \right) \left( \square + m^2 \right) \Lambda - \frac{1}{2} \varphi \left( \square + m^2 \right) \tilde{\lambda}.
\]

We christen the above transformations [cf. Eqs. (7), (4)] as the (dual-)gauge symmetry transformations because of the following arguments. First of all, we note that under the gauge symmetry transformations (\(\delta_g\)), the total kinetic term remains invariant. On the other hand, it is the total gauge-fixing term that remains invariant under the dual-gauge symmetry transformations (\(\delta_{dg}\)). Second, as argued earlier, the kinetic term has its origin in the exterior derivative \(d = dx^\mu \partial_\mu\) (with \(d^2 = 0\)) but the gauge-fixing term owes its origin to the dual-exterior (i.e. co-exterior) derivative \(\delta = \pm \ast d \ast\) (with \(\delta^2 = 0\)). Thus, the nomenclature (dual-)gauge symmetry transformation \(\delta_{(dg)}\) sounds appropriate. Finally, even though there are three individual terms in the kinetic and gauge-fixing terms, the origin of the additional terms like \(-\frac{1}{2} \partial_\mu \tilde{\varphi} + m \phi_\mu\) and \(-\frac{1}{2} \partial_\mu \varphi + m \phi_\mu\) is the \(H^{(3)} = d \tilde{B}^{(2)}\) and \(\delta B^{(2)} = (\partial^\nu B_{\nu\mu}) dx^\mu\) which owe their origins to \(d = dx^\mu \partial_\mu\) and \(\delta = \pm \ast d \ast\), respectively. It is evident that the transformation parameters \((\Sigma_\mu, \Sigma, \sigma)\) are the axial-vector and pseudo-scalars (i.e. \(\Sigma, \sigma\)) for the full dual-gauge symmetry transformations \(\delta_{dg}\). On the other
hand, we have already noted that the Lorentz vector ($\Lambda_{\mu}$) and Lorentz scalars ($\Lambda, \tilde{\lambda}$) are the transformation parameters for the full gauge symmetry transformations ($\delta_g$).

At this stage, we now comment on the transformations (8) which have been obtained after the applications of $\delta_{dg}$ and $\delta_\mu$. It is straightforward to note that under the following restrictions on the (dual-)gauge transformation parameters, namely;

$$
(\Box + m^2) \Sigma_\mu = 0, \quad (\Box + m^2) \Sigma = 0, \quad (\Box + m^2) \sigma = 0, \\
(\Box + m^2) \Lambda_\mu = 0, \quad (\Box + m^2) \Lambda = 0, \quad (\Box + m^2) \tilde{\lambda} = 0, 
$$

we achieve the perfect (dual-)gauge symmetry invariance of the Lagrangian density $L_{(2)}$. It is very interesting to pinpoint that the mathematical structure of the restrictions in Eq. (9) is exactly the same on the (dual-)gauge symmetry transformation parameters. Thus, it is very clear that, within the framework of BRST approach, the (anti-)ghost part of the Lagrangian density would be exactly the same for the coupled (but equivalent) Lagrangian densities (cf. Sec. 3 below). The bosonic nature of the transformation parameters ($\Sigma_\mu, \Sigma, \sigma, \Lambda_\mu, \Lambda, \tilde{\lambda}$) implies that, at the quantum level, these parameters would be replaced by the fermionic (anti-)ghost fields within the framework of BRST formalism.

We wrap up this sub-section with the following remarks. We note, as pointed out earlier, that total gauge-fixing term remains invariant under the dual-gauge symmetry transformations ($\delta_{dg}$). Furthermore, the vector and scalar fields ($\phi_\mu$ and $\varphi$) do not transform at all under $\delta_{dg}$. Hence, the kinetic term ($-\frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu}$) for the vector field ($\phi_\mu$) also does not transform under the dual-gauge symmetry transformations. Whereas the gauge symmetry transformations ($\delta_g$) exist at the classical level, we note that the dual-gauge symmetry transformations ($\delta_{dg}$) exist only when the gauge-fixing term is incorporated for the purpose of quantization (or the derivation of the propagator) for the gauge field ($B_{\mu\nu}$) of our massive 4D theory. Finally, we observe that new fields ($\varphi, \tilde{\varphi}$) have been invoked for the discussion of the dual-gauge symmetries transformations ($\delta_{dg}$) in our theory [in contrast to the gauge symmetry transformations ($\delta_g$) where no such fields have been invoked (see, e.g., the discussions after Eq. (5))].

### 3 Coupled Lagrangian Densities: Off-Shell Nilpotent (Anti-)BRST and (Anti-)co-BRST Symmetry Transformations

In this section, we concisely mention the off-shell nilpotent symmetries and the CF-type restrictions for the most generalized version of the Lagrangian density (6) where the Nakanishi-Lautrup type auxiliary fields are invoked for the linearizations of the kinetic and gauge-fixing terms for the $B_{\mu\nu}$ field and gauge-fixing terms for the fields $\phi_\mu$ and $\tilde{\phi}_\mu$. The central theme and purpose of this section is to mention all the appropriate equations in the 4D ordinary spacetime which are important in the context of superfield approach to BRST formalism (cf. Secs. 4, 5, 6 below). We begin with the following coupled (but
equivalent) (dual-)BRST invariant Lagrangian densities (see, e.g. [2 4] for details)

\[ \mathcal{L}_{(B,B)} = \frac{1}{2} B_\mu B^\mu - B^\mu \left( \frac{1}{2} \epsilon_{\mu \nu \eta \kappa} \partial^\nu B^\eta B^\kappa - \frac{1}{2} \partial_\mu \tilde{\phi} + m \tilde{\phi}_\mu \right) - \frac{m^2}{4} B^{\mu \nu} B_{\mu \nu} \]

\[ - \frac{1}{4} \Phi^{\mu \nu} \Phi_{\mu \nu} + \frac{m}{2} B^{\mu \nu} \Phi_{\mu \nu} + \frac{1}{4} \tilde{\Phi}^{\mu \nu} \tilde{\Phi}_{\mu \nu} + \frac{m}{4} \epsilon^{\mu \nu \rho \sigma} B_{\mu \nu} \Phi_{\rho \sigma} - \frac{1}{2} B^\mu B_\mu \]

\[ + B^\mu \left( \partial^\nu B_{\nu \mu} - \frac{1}{2} \partial_\mu \varphi + m \phi_\mu \right) + \frac{1}{2} B^2 + B \left( \partial_\mu \tilde{\phi}^\mu + \frac{m}{2} \varphi \right) \]

\[ - \frac{1}{2} B^2 - B \left( \partial_\mu \tilde{\phi}^\mu + \frac{m}{2} \varphi \right) + (\partial_\mu \tilde{C} - m \tilde{C}_\mu) (\partial^\mu C - m C^\mu) \]

\[ - \left( \partial_\mu \tilde{\phi}^\mu + m \phi_\mu \right) \rho - \frac{1}{4} \rho \right) \rho, \] \hspace{1cm} (10)

\[ \mathcal{L}_{(\bar{B},\bar{B})} = \frac{1}{2} \bar{B}_\mu \bar{B}^\mu + \bar{B}^\mu \left( \frac{1}{2} \epsilon_{\mu \nu \eta \kappa} \partial^\nu B^\eta B^\kappa - \frac{1}{2} \partial_\mu \tilde{\phi} + m \tilde{\phi}_\mu \right) - \frac{m^2}{4} B^{\mu \nu} B_{\mu \nu} \]

\[ - \frac{1}{4} \Phi^{\mu \nu} \Phi_{\mu \nu} + \frac{m}{2} B^{\mu \nu} \Phi_{\mu \nu} + \frac{1}{4} \tilde{\Phi}^{\mu \nu} \tilde{\Phi}_{\mu \nu} + \frac{m}{4} \epsilon^{\mu \nu \rho \sigma} B_{\mu \nu} \Phi_{\rho \sigma} - \frac{1}{2} \bar{B}^\mu \bar{B}_\mu \]

\[ - \bar{B}^\mu \left( \partial^\nu B_{\nu \mu} + \frac{1}{2} \partial_\mu \varphi + m \phi_\mu \right) + \frac{1}{2} \bar{B}^2 - \bar{B} \left( \partial_\mu \tilde{\phi}^\mu - \frac{m}{2} \varphi \right) \]

\[ - \frac{1}{2} \bar{B}^2 + \bar{B} \left( \partial_\mu \tilde{\phi}^\mu + \frac{m}{2} \varphi \right) + (\partial_\mu \bar{C} - m \bar{C}_\mu) (\partial^\mu C - m C^\mu) \]

\[ - \left( \partial_\mu \tilde{\phi}^\mu + m \phi_\mu \right) \rho - \frac{1}{4} \rho \right) \rho, \] \hspace{1cm} (11)

where the auxiliary fields \((B_\mu, B_\mu, B, \bar{B}_\mu, \bar{B}_\mu, \bar{B}, \bar{B})\) are nothing but the bosonic Nakanishi-Lautrup type auxiliary fields. The fermionic (anti-)ghost fields\(^3\) are: \((\bar{C}_\mu) C_\mu\), \((\bar{C}) C\), \((\rho) \lambda\) and bosonic (anti-)ghost fields are \((\bar{\beta}) \beta\). Because of the stage-one reducibility in our theory, we have the ghost-for-ghost bosonic fields \((\bar{\beta}) \beta\). It should be noted that the ghost-part of the Lagrangian densities (10) and (11) are same. We have provided some arguments regarding it (cf. Sec. 2) in the language of the (dual-)gauge symmetry transformations and the restrictions on the transformation gauge parameters [cf. Eq. (9)] for their invariance. We also note here that the fields \((\rho) \lambda\) are auxiliary fields but they are fermionic in nature and they carry the ghost number \((-1)+1\), respectively. In addition, the ghost numbers for the fermionic (anti-)ghost fields \((\bar{C}_\mu) C_\mu\) and \((\bar{C}) C\) are \((-1)+1\) and that of the bosonic (anti-)ghost fields \((\bar{\beta}) \beta\) are \((-2)+2\), respectively.

\(^3\)These fermionic (anti-)ghost fields are the generalizations of the bosonic (dual-)gauge symmetry transformation parameters \((\Sigma_\mu, \Sigma_\mu, \sigma, \Lambda_\mu, \Lambda, \lambda)\) which have been mentioned at the fag end of Sec. 2 of our present endeavor.
We observe that the following, nilpotent \((s_{(a)b}^2 = 0)\) (anti-)BRST transformations \((s_{(a)b})\)

\[
\begin{align*}
s_{ab}B_{\mu\nu} &= - (\partial_{\mu}C_{\nu} - \partial_{\nu}C_{\mu}), \\
s_{ab}\tilde{C}_{\mu} &= - \partial_{\mu}\tilde{\beta}, \\
s_{ab}C_{\mu} &= \tilde{B}_{\mu}, \\
s_{ab}\beta &= - \lambda, \\
s_{ab}\tilde{\phi} &= \partial_{\mu}\tilde{C} - m \tilde{C}_{\mu}, \\
s_{ab}B &= - m \rho, \\
s_{ab}\varphi &= \rho, \\
s_{ab}B_{\mu} &= - \partial_{\mu}\rho,
\end{align*}
\]

(12)

leave the action integrals \(S_1 = \int d^4x \mathcal{L}_{(B,B)}\) and \(S_2 = \int d^4x \mathcal{L}_{(B,B)}\) invariant because the Lagrangian densities transform to the total spacetime derivatives under the \(s_{(a)b}\) as follows [24]:

\[
\begin{align*}
s_{ab}\mathcal{L}_{(B,B)} &= - \partial_{\mu} \left[ m \varepsilon^{\mu\nu\rho\kappa} \tilde{\phi}_{\nu} (\partial_{\rho}C_{\kappa}) - B_{\nu} (\partial^{\mu}C^{\nu} - \partial^{\nu}C^{\mu}) + \frac{1}{2} B^{\mu} \rho \right] \\
&\quad + \tilde{B} (\partial^{\mu}C - m C^{\mu}) - \frac{1}{2} (\partial^{\mu}\tilde{\beta}) \lambda, \\
s_{b}\mathcal{L}_{(B,B)} &= - \partial_{\mu} \left[ m \varepsilon^{\mu\nu\rho\kappa} \tilde{\phi}_{\nu} (\partial_{\rho}C_{\kappa}) + B_{\nu} (\partial^{\mu}C^{\nu} - \partial^{\nu}C^{\mu}) + \frac{1}{2} B^{\mu} \lambda \right] \\
&\quad - B (\partial^{\mu}C - m C^{\mu}) - \frac{1}{2} (\partial^{\mu}\beta) \rho.
\end{align*}
\]

(14)

(15)

We point out that the above action integrals remain invariant due to Gauss’s divergence theorem (as all the physical fields vanish off at infinity). It is very interesting to note that the above coupled (but equivalent) Lagrangian densities \(\mathcal{L}_{(B,B)}\) and \(\mathcal{L}_{(\tilde{B},\tilde{B})}\) also respect another set of off-shell nilpotent \((s_{(a)d}^2 = 0)\) symmetries which are known as the (anti-)co-BRST [or (anti-)dual BRST] symmetries \([s_{(a)d}]\). This is due to the fact that we observe the following transformations for the coupled (but equivalent) Lagrangian densities:

\[
\begin{align*}
s_{ad}\mathcal{L}_{(B,B)} &= - \partial_{\mu} \left[ m \varepsilon^{\mu\nu\rho\kappa} \tilde{\phi}_{\nu} (\partial_{\rho}C_{\kappa}) + B_{\nu} (\partial^{\mu}C^{\nu} - \partial^{\nu}C^{\mu}) + \frac{1}{2} \tilde{B}^{\mu} \lambda \right] \\
&\quad - \tilde{B} (\partial^{\mu}C - m C^{\mu}) + \frac{1}{2} (\partial^{\mu}\beta) \rho, \\
s_{d}\mathcal{L}_{(B,B)} &= - \partial_{\mu} \left[ m \varepsilon^{\mu\nu\rho\kappa} \tilde{\phi}_{\nu} (\partial_{\rho}C_{\kappa}) - B_{\nu} (\partial^{\mu}C^{\nu} - \partial^{\nu}C^{\mu}) + \frac{1}{2} B^{\mu} \rho \right] \\
&\quad + B (\partial^{\mu}\tilde{C} - m \tilde{C}^{\mu}) - \frac{1}{2} (\partial^{\mu}\beta) \lambda.
\end{align*}
\]

(16)

(17)

Hence, the action integrals \(S_1 = \int d^4x \mathcal{L}_{(B,B)}\) and \(S_2 = \int d^4x \mathcal{L}_{(\tilde{B},\tilde{B})}\) remain invariant under the following infinitesimal and continuous (anti-)co-BRST symmetry transformations \([s_{(a)d}]\):

\[
\begin{align*}
s_{ad}B_{\mu\nu} &= - \varepsilon_{\mu\nu\rho\kappa} \partial^{\rho}C^{\kappa}, \\
s_{ad}\tilde{C}_{\mu} &= \tilde{B}_{\mu}, \\
s_{ad}C_{\mu} &= \partial_{\mu}\beta, \\
s_{ad}\tilde{\phi} &= - \lambda, \\
s_{ad}\phi &= \partial_{\mu}C - m C_{\mu}, \\
s_{ad}B &= m \lambda, \\
s_{ad}B_{\mu} &= \partial_{\mu}\lambda, \\
s_{ad}\tilde{\beta} &= \rho, \\
s_{ad}\tilde{C} &= \tilde{B}, \\
s_{ad}[\partial_{\mu}B_{\nu}, B_{\mu}, \tilde{B}_{\mu}, \partial_{\mu}, \tilde{\beta}, \tilde{B}] &= 0.
\end{align*}
\]

(18)
\[ s_d B_{\mu\nu} = -\varepsilon_{\mu\nu\rho\kappa} \partial^\rho \tilde{C}^\kappa, \quad s_d C_\mu = B_\mu, \quad s_d \tilde{C}_\mu = -\partial_\mu \tilde{\beta}, \quad s_d \beta = -\lambda, \]
\[ s_d \tilde{\phi}_\mu = \partial_\mu \tilde{C} - m \tilde{\beta}_\mu, \quad s_d \tilde{C} = B, \quad s_d \tilde{\beta} = -m \beta, \quad s_d \tilde{\beta} = m \rho, \]
\[ s_d \tilde{\beta}_\mu = \partial_\mu \rho, \quad s_d \tilde{\phi} = -\rho, \quad s_d [\phi^\rho B_{\mu\nu}, B_\mu, B_\mu, \tilde{B}_\mu, B, \tilde{B}, \varphi, \phi_\mu, \rho, \lambda, \tilde{\beta}] = 0. \]  

(19)

It is straightforward to note that the (anti-)BRST and (anti-)co-BRST symmetry transformations are off-shell nilpotent \( (s^2_{(a)b} = 0, s^2_{(a)d} = 0) \) of order two and, hence, are fermionic in nature. We note that the total gauge-fixing term for the massive gauge field \( B_{\mu\nu} \) remains invariant under \( s_{(a)d} \). This observation should be contrasted with the (anti-)BRST symmetry transformations where the total kinetic term for the massive gauge field \( B_{\mu\nu} \) is found to be invariant. Furthermore, it is worth pointing out that the BRST as well as anti-co-BRST transformations increase the ghost number by one and anti-BRST as well as the co-BRST symmetry transformations decrease the ghost number by one when they operate on an arbitrary field.

A few comments are in order as far as the absolute anticommutativity \( \{s_b, s_{ab}\} = 0, \{s_d, s_{ad}\} = 0 \) of the (anti-)BRST \( (s_{(a)b}) \) and (anti-)co-BRST \( (s_{(a)d}) \) symmetry transformations are concerned. It can be checked that the following anticommutators, namely;

\[ \{s_b, s_{ab}\} B_{\mu\nu} = -\partial_\mu (B_\nu + \tilde{B}_\nu) + \partial_\nu (B_\mu + \tilde{B}_\mu), \]
\[ \{s_b, s_{ab}\} \phi_\mu = \partial_\mu (B + \tilde{B}) - m (B_\mu + \tilde{B}_\mu), \]
\[ \{s_d, s_{ad}\} B_{\mu\nu} = -\varepsilon_{\mu\nu\rho\kappa} \partial^\rho (B^\kappa + \tilde{B}^\kappa), \]
\[ \{s_d, s_{ad}\} \phi_\mu = \partial_\mu (B + \tilde{B}) - m (B_\mu + \tilde{B}_\mu), \]

are equal to zero only when the following physically allowed CF-type restrictions [24] are imposed from outside, namely;

\[ B_\mu + \tilde{B}_\mu + \partial_\mu \varphi = 0, \quad B + \tilde{B} + m \varphi = 0, \]
\[ B_\mu + \tilde{B}_\mu + \partial_\mu \tilde{\varphi} = 0, \quad B + \tilde{B} + m \tilde{\varphi} = 0. \]
\[ (21) \]

We can explicitly check that the above CF-type restrictions are (anti-)BRST as well as (anti-)co-BRST invariant. To corroborate the above statement, we point out the following precise observations:

\[ s_b [B_\mu + \tilde{B}_\mu + \partial_\mu \varphi] = 0, \quad s_d [B_\mu + \tilde{B}_\mu + \partial_\mu \tilde{\varphi}] = 0, \]
\[ s_{ab} [B_\mu + \tilde{B}_\mu + \partial_\mu \varphi] = 0, \quad s_{ad} [B_\mu + \tilde{B}_\mu + \partial_\mu \tilde{\varphi}] = 0, \]
\[ s_d [B + \tilde{B} + m \varphi] = 0, \quad s_b [B + \tilde{B} + m \tilde{\varphi}] = 0, \]
\[ s_{ad} [B + \tilde{B} + m \varphi] = 0, \quad s_{ab} [B + \tilde{B} + m \tilde{\varphi}] = 0. \]
\[ (22) \]

Thus, for a model of the Hodge theory (i.e. 4D massive Abelian 2-form gauge theory), the CF-type restrictions (21) are physical constraints on the theory because the restrictions (21) are (anti-)BRST as well as (anti-)co-BRST invariant, together. It would be worthwhile to note that some of the pertinent equations of motion from the coupled Lagrangian densities
We note that the (anti-)BRST and (anti-)co-BRST invariant restrictions (21) can be derived from the above equations (23). Thus, in a subtle manner, we provide the derivation of the CF-type restrictions (21) from the coupled (but equivalent) Lagrangian densities $L_{(B,B)}$ and $L_{(\bar{B},\bar{B})}$ in the sense that the appropriate EL-EOM lead to their existence on our theory.

In the context of the existence of the above CF-type restrictions (21), we note the following transformations for the (anti-)BRST and (anti-)co-BRST invariant Lagrangian densities $L_{(B,B)}$ and $L_{(\bar{B},\bar{B})}$:

\[
\begin{align*}
L_{(B,B)} & = \left( \frac{1}{2} \varepsilon_{\mu \nu \eta} \partial^\nu B^{\eta \alpha} - \frac{1}{2} \partial_\mu \phi + m \partial_\mu \phi \right), \quad B = - \left( \partial_\mu \phi^\mu + \frac{m}{2} \phi \right), \\
B_\mu & = \left( \partial^\nu B_{\nu \mu} - \frac{1}{2} \partial_\mu \phi + m \partial_\mu \phi \right), \quad B = - \left( \partial_\mu \phi^\mu + \frac{m}{2} \phi \right), \\
\bar{B}_\mu & = - \left( \frac{1}{2} \varepsilon_{\mu \nu \eta} \partial_\mu B^{\nu \eta} + \frac{1}{2} \partial_\mu \phi + m \partial_\mu \phi \right), \quad \bar{B} = \left( \partial_\mu \phi^\mu - \frac{m}{2} \phi \right), \\
\bar{B}_\mu & = - \left( \partial^\nu B_{\nu \mu} + \frac{1}{2} \partial_\mu \phi + m \partial_\mu \phi \right), \quad \bar{B} = \left( \partial_\mu \phi^\mu - \frac{m}{2} \phi \right). \tag{23}
\end{align*}
\]

We note that the (anti-)BRST and (anti-)co-BRST invariant restrictions (21) can be derived from the above equations (23). Thus, in a subtle manner, we provide the derivation of the CF-type restrictions (21) from the coupled (but equivalent) Lagrangian densities $L_{(B,B)}$ and $L_{(\bar{B},\bar{B})}$ in the sense that the appropriate EL-EOM lead to their existence on our theory.

In the context of the existence of the above CF-type restrictions (21), we note the following transformations for the (anti-)BRST and (anti-)co-BRST invariant Lagrangian densities $L_{(B,B)}$ and $L_{(\bar{B},\bar{B})}$:

\[
\begin{align*}
s_{ab}L_{(B,B)} & = - \partial_\mu \left[ m \varepsilon_{\mu \nu \eta} \phi_{\nu} (\partial_\eta \bar{C}_\kappa) + \left( \partial_\nu B^{\nu \mu} + \frac{1}{2} \bar{B}^\mu + m \phi^\mu \right) \rho \right] \\
& + B_\nu (\partial^\mu C^\nu - \partial^\nu C^\mu) - B (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{1}{2} (\partial^\mu \bar{B}) \lambda \\
& + \frac{1}{2} \left[ \bar{B}_{\mu} + \bar{B}_\mu + \partial_\mu \phi \right] (\partial^\mu \rho) + \partial_\mu \left[ B_{\nu} + \bar{B}_\nu + \partial_\nu \phi \right] (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \\
& + m \left[ B_{\mu} + \bar{B}_\mu + \partial_\mu \phi \right] (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{m}{2} \left[ B + \bar{B} + m \phi \right] \rho \\
& - \partial_\mu \left[ B + B + m \phi \right] (\partial^\mu C - m C^\mu), \tag{24}
\end{align*}
\]

\[
\begin{align*}
s_bL_{(\bar{B},\bar{B})} & = - \partial_\mu \left[ m \varepsilon_{\mu \nu \eta} \phi_{\nu} (\partial_\eta \bar{C}_\kappa) - \left( \partial_\nu B^{\nu \mu} - \frac{1}{2} B^\mu + m \phi^\mu \right) \lambda \right] \\
& - \bar{B}_\nu (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) + \bar{B} (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{1}{2} (\partial^\mu \bar{B}) \rho \\
& + \frac{1}{2} \left[ B_{\mu} + \bar{B}_\mu + \partial_\mu \phi \right] (\partial^\mu \lambda) - \partial_\mu \left[ B_{\nu} + \bar{B}_\nu + \partial_\nu \phi \right] (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \\
& - m \left[ B_{\mu} + \bar{B}_\mu + \partial_\mu \phi \right] (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{m}{2} \left[ B + \bar{B} + m \phi \right] \lambda \\
& + \partial_\mu \left[ B + B + m \phi \right] (\partial^\mu C - m C^\mu), \tag{25}
\end{align*}
\]
\[ s_d \mathcal{L}_{(\bar{B}, \bar{B})} = -\partial_\mu \left[ m \varepsilon^{\mu \nu \rho \kappa} \phi_{\nu} (\partial_\rho \bar{C}_\kappa) - \left( \frac{1}{2} \varepsilon^{\mu \nu \rho \kappa} \partial_\nu B_{\rho \kappa} - \frac{1}{2} \mathcal{B}^\mu + m \bar{\phi}^\mu \right) \rho \right] \\
+ \mathcal{B}_\nu (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) - \bar{\mathcal{B}} (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{1}{2} (\partial^\mu \bar{\beta}) \lambda \]
\[ + \frac{1}{2} \left[ \mathcal{B}_\mu + \bar{\mathcal{B}}_\mu + \partial_\mu \bar{\phi} \right] (\partial^\mu \rho) + \partial_\mu \left[ \mathcal{B}_\nu + \bar{\mathcal{B}}_\nu + \partial_\nu \bar{\phi} \right] (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \]
\[ + m \left[ \mathcal{B}_\mu + \bar{\mathcal{B}}_\mu + \partial_\mu \bar{\phi} \right] (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{m}{2} [\mathcal{B} + \bar{\mathcal{B}} + m \bar{\phi}] \rho + \partial_\mu [\mathcal{B} + \bar{\mathcal{B}} + m \bar{\phi}] (\partial^\mu \bar{C} - m \bar{C}^\mu), \] (26)

\[ s_{ad} \mathcal{L}_{(B, B)} = -\partial_\mu \left[ m \varepsilon^{\mu \nu \rho \kappa} \phi_{\nu} (\partial_\rho C_\kappa) - \left( \frac{1}{2} \varepsilon^{\mu \nu \rho \kappa} \partial_\nu B_{\rho \kappa} + \frac{1}{2} \mathcal{B}^\mu + m \bar{\phi}^\mu \right) \lambda \right] \\
- B_\nu (\partial^\mu C^\nu - \partial^\nu C^\mu) + B (\partial^\mu C - m C^\mu) + \frac{1}{2} (\partial^\mu \beta) \rho \]
\[ + \frac{1}{2} \left[ B_\mu + \bar{B}_\mu + \partial_\mu \beta \right] (\partial^\mu \lambda) - \partial_\mu \left[ B_\nu + \bar{B}_\nu + \partial_\nu \beta \right] (\partial^\mu C^\nu - \partial^\nu C^\mu) \]
\[ - m \left[ B_\mu + \bar{B}_\mu + \partial_\mu \beta \right] (\partial^\mu C - m C^\mu) - \frac{m}{2} [B + \bar{B} + m \beta] \lambda + \partial_\mu [B + \bar{B} + m \beta] (\partial^\mu C - m C^\mu). \] (27)

It is crystal clear, at this stage, that both the Lagrangian densities \( \mathcal{L}_{(B, B)} \) and \( \mathcal{L}_{(\bar{B}, \bar{B})} \) are equivalent in the sense that both of them respect the (anti-)BRST as well as (anti-)co-BRST symmetry transformations together provided our whole theory is considered on the submanifold of space of fields which is defined by the field equations (21). The latter are nothing but the (anti-) BRST and (anti-) co-BRST invariant [cf. Eq. (22)] CF-type restrictions on our theory. We observe that, besides the perfect symmetry invariance(s) [cf. Eqs. (14), (15), (16), (17)], we have the following, namely;

\[ s_{ab} \mathcal{L}_{(B, B)} = -\partial_\mu \left[ m \varepsilon^{\mu \nu \rho \kappa} \phi_{\nu} (\partial_\rho \bar{C}_\kappa) + \left( \partial_\nu B^{\nu \mu} + \frac{1}{2} \bar{B}^\mu + m \bar{\phi}^\mu \right) \rho \right] \\
+ (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu - B (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{1}{2} (\partial^\mu \bar{\beta}) \lambda, \]

\[ s_b \mathcal{L}_{(\bar{B}, \bar{B})} = -\partial_\mu \left[ m \varepsilon^{\mu \nu \rho \kappa} \phi_{\nu} (\partial_\rho C_\kappa) - \left( \partial_\nu B^{\nu \mu} - \frac{1}{2} B^\mu + m \phi^\mu \right) \lambda \right] \\
- (\partial^\mu C^\nu - \partial^\nu C^\mu) \bar{B}_\nu + \bar{B} (\partial^\mu C - m C^\mu) - \frac{1}{2} (\partial^\mu \beta) \rho, \]

\[ s_d \mathcal{L}_{(B, B)} = -\partial_\mu \left[ m \varepsilon^{\mu \nu \rho \kappa} \phi_{\nu} (\partial_\rho \bar{C}_\kappa) - \left( \frac{1}{2} \varepsilon^{\mu \nu \rho \kappa} \partial_\nu B_{\rho \kappa} - \frac{1}{2} \mathcal{B}^\mu + m \bar{\phi}^\mu \right) \rho \right] \\
+ (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{B}_\nu - B (\partial^\mu \bar{C} - m \bar{C}^\mu) - \frac{1}{2} (\partial^\mu \bar{\beta}) \lambda, \]

\[ s_{ad} \mathcal{L}_{(B, B)} = -\partial_\mu \left[ m \varepsilon^{\mu \nu \rho \kappa} \phi_{\nu} (\partial_\rho C_\kappa) + \left( \frac{1}{2} \varepsilon^{\mu \nu \rho \kappa} \partial_\nu B_{\rho \kappa} + \frac{1}{2} \mathcal{B}^\mu + m \phi^\mu \right) \lambda \right] \\
- (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu + B (\partial^\mu C - m C^\mu) + \frac{1}{2} (\partial^\mu \beta) \rho, \] (28)
which are also perfect symmetry transformations on a submanifold of the space of fields where the (anti-)BRST and (anti-)co-BRST invariant CF-type restrictions are satisfied. Hence, the latter are physical constraints on our theory.

We end this section with the remark that the most appropriate generalized versions of the Lagrangian density (6) (that are nothing but $\mathcal{L}_{(B,B)}$ and $\mathcal{L}_{(B,\bar{B})}$) respect both types of off-shell nilpotent symmetries [i.e. (anti-)BRST and (anti-)co-BRST symmetries] provided the whole theory is restricted to be defined on the submanifold of space of fields where the CF-type restrictions (21) are respected. In fact, on this submanifold, the (anti-)BRST and (anti-)co-BRST symmetries are found to be absolutely anticommutating, too. Hence, the submanifold of the space of fields, defined by the (anti-)BRST and (anti-)co-BRST invariant [cf. Eq. (22)] field equations (21), are physical subspace of the quantum fields where the proper off-shell nilpotent symmetries and corresponding proper (i.e. coupled and equivalent) Lagrangian densities $\mathcal{L}_{(\bar{B},\bar{B})}$ and $\mathcal{L}_{(B,B)}$ are defined.

4 Off-Shell Nilpotent (Anti-)BRST and (Anti-)co-BRST Symmetries: ACSA to BRST Formalism

In this section, we exploit the basic tenets of ACSA to BRST formalism [21-23] to derive the (anti-)BRST and (anti-)co-BRST symmetries for the coupled (but equivalent) Lagrangian densities $\mathcal{L}_{(\bar{B},\bar{B})}$ and $\mathcal{L}_{(B,B)}$ [(cf. Eqs. (10),(11))] of our theory by invoking the symmetry invariant restrictions on the (anti-)chiral superfields. Our present section is divided into two sub-sections as described in 4.1 and 4.2.

4.1 (Anti-)BRST Symmetries: ACSA

First of all, we derive the BRST symmetries [cf. Eq. (13)]. For this purpose, we generalize the ordinary fields of the Lagrangian density $\mathcal{L}_{(B,B)}$ onto their counterpart anti-chiral superfields on the (4, 1)-dimensional (anti-)chiral super sub-manifold as:

\[
B_\mu(x) \rightarrow \tilde{B}_\mu(x, \bar{\theta}) = B_\mu(x) + \bar{\theta} R_\mu(x),
\]
\[
C_\mu(x) \rightarrow \tilde{F}_\mu(x, \bar{\theta}) = C_\mu(x) + \bar{\theta} B^{(1)}_\mu(x),
\]
\[
\bar{C}_\mu(x) \rightarrow \tilde{\bar{F}}_\mu(x, \bar{\theta}) = \bar{C}_\mu(x) + \bar{\theta} B^{(2)}_\mu(x),
\]
\[
\beta(x) \rightarrow \tilde{\beta}(x, \bar{\theta}) = \beta(x) + \bar{\theta} f_1(x),
\]
\[
\bar{\beta}(x) \rightarrow \tilde{\bar{\beta}}(x, \bar{\theta}) = \bar{\beta}(x) + \bar{\theta} f_2(x),
\]
\[
\varphi(x) \rightarrow \tilde{\Phi}(x, \bar{\theta}) = \varphi(x) + \bar{\theta} f_3(x),
\]
\[
\phi_\mu(x) \rightarrow \tilde{\phi}_\mu(x, \bar{\theta}) = \phi_\mu(x) + \bar{\theta} R_\mu(x),
\]
\[
C(x) \rightarrow \tilde{F}(x, \bar{\theta}) = C(x) + \bar{\theta} B_1(x),
\]
\[
\bar{C}(x) \rightarrow \tilde{\bar{F}}(x, \bar{\theta}) = \bar{C}(x) + \bar{\theta} B_2(x),
\]
\[
B_\mu(x) \rightarrow \tilde{B}_\mu(x, \bar{\theta}) = B_\mu(x) + \bar{\theta} f^{(1)}_\mu(x),
\]
\[
\bar{B}_\mu(x) \rightarrow \tilde{\bar{B}}_\mu(x, \bar{\theta}) = \bar{B}_\mu(x) + \bar{\theta} f^{(2)}_\mu(x),
\]

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\begin{align*}
B(x) & \rightarrow \tilde{B}(x, \bar{\theta}) = B(x) + \bar{\theta} f_4(x), \\
B(x) & \rightarrow \tilde{B}(x, \bar{\theta}) = B(x) + \bar{\theta} f_5(x), \\
\mathcal{B}_\mu(x) & \rightarrow \tilde{\mathcal{B}}_\mu(x, \bar{\theta}) = \mathcal{B}_\mu(x) + \bar{\theta} f_\mu^{(3)}(x), \\
\mathcal{B}_\mu(x) & \rightarrow \tilde{\mathcal{B}}_\mu(x, \bar{\theta}) = \mathcal{B}_\mu(x) + \bar{\theta} f_\mu^{(4)}(x), \\
\bar{\varphi}(x) & \rightarrow \tilde{\bar{\varphi}}(x, \bar{\theta}) = \bar{\varphi}(x) + \bar{\theta} f_6(x), \\
\phi_\mu(x) & \rightarrow \tilde{\phi}_\mu(x, \bar{\theta}) = \phi_\mu(x) + \bar{\theta} \beta^{(1)}(x), \\
\lambda(x) & \rightarrow \tilde{\lambda}(x, \bar{\theta}) = \lambda(x) + \bar{\theta} B_3(x), \\
\rho(x) & \rightarrow \tilde{\rho}(x, \bar{\theta}) = \rho(x) + \bar{\theta} B_4(x), \\
\mathcal{B}(x) & \rightarrow \tilde{\mathcal{B}}(x, \bar{\theta}) = \mathcal{B}(x) + \bar{\theta} f_7(x), \\
\tilde{\mathcal{B}}(x, \bar{\theta}) & = \mathcal{B}(x) + \bar{\theta} f_8(x). 
\end{align*}

In the above anti-chiral super expansions, it is worthwhile as well as pertinent to point out that the secondary fields \((R_{\mu\nu}, f_1, f_2, f_3, R_{\mu\nu}, f_\mu^{(1)}, f_\mu^{(2)}, f_4, f_5, f_\mu^{(3)}, f_\mu^{(4)}, f_6, R_{\mu\nu}^{(1)}, f_7, f_8)\) are fermionic and \((B_{\mu}^{(1)}, B_{\mu}^{(2)}, B_1, B_2, B_3, B_4)\) are bosonic in nature due to the fermionic \((\bar{\theta}^2 = 0)\) nature of the Grassmannian variable \(\bar{\theta}\) that characterizes the anti-chiral super sub-manifold [along with the bosonic coordinates \(x^\mu(\mu = 0, 1, 2, 3)\)].

We note that the following non-trivial quantities are BRST invariant in view of the symmetry transformations \((13)\), namely:

\begin{align*}
 s_b(\lambda \varphi) = 0, & \quad s_b(\bar{\beta} \lambda - \varphi \rho) = 0, & \quad s_b(C_\mu \partial^\mu \rho - \partial_\mu \bar{\beta} \partial^\mu \beta) = 0, \\
 s_b(C_\mu \partial^\mu \varphi + \bar{\phi}_\mu \partial^\mu \beta) = 0, & \quad s_b(\bar{B}_\mu + \partial_\mu \varphi) = 0, & \quad s_b(\bar{B} + m \varphi) = 0, \\
 s_b[C_\mu (m C^\mu - \partial^\mu C) + B_\mu \phi_\mu] = 0, & \quad s_b[\bar{C}(m C_\mu - \partial_\mu C) + B \phi_\mu] = 0, \\
 s_b[m B_{\mu\nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)] = 0, & \quad s_b[m \bar{\beta} \beta + \rho C] = 0.
\end{align*}

In addition, we have some trivial BRST invariant quantities as: \(s_b[B, B_\mu, \rho, \lambda, \beta, \mathcal{B}_\mu, \bar{\mathcal{B}}, \varphi, \phi_\mu, \mathcal{B}, \bar{\mathcal{B}}, H_{\mu\nu\lambda}] = 0\) [cf. Eq. \((13)\)]. It is because of the latter observation that we have the following trivial super-expansions and equalities for the appropriate anti-chiral superfields

\begin{align*}
\tilde{\mathcal{B}}^{(b)}(x, \bar{\theta}) & = B(x), \quad \tilde{\rho}^{(b)}(x, \bar{\theta}) = \rho(x), \quad \tilde{\lambda}^{(b)}(x, \bar{\theta}) = \lambda(x), \\
\tilde{\mathcal{B}}^{(b)}_\mu(x, \bar{\theta}) & = \tilde{\mathcal{B}}_\mu(x), \quad \tilde{\varphi}^{(b)}(x, \bar{\theta}) = \varphi(x), \quad \tilde{\phi}^{(b)}_\mu(x, \bar{\theta}) = \phi_\mu(x), \\
\tilde{\mathcal{B}}^{(b)}_\mu(x, \bar{\theta}) & = \tilde{\mathcal{B}}_\mu(x), \quad \tilde{\mathcal{B}}^{(b)}_\lambda(x, \bar{\theta}) = \tilde{\mathcal{B}}(x), \quad \tilde{\phi}^{(b)}_\mu(x, \bar{\theta}) = B_\mu(x), \\
\tilde{\beta}^{(b)}(x, \bar{\theta}) & = \beta(x), \quad \tilde{\mathcal{B}}^{(b)}(x, \bar{\theta}) = \mathcal{B}(x), \quad \tilde{H}^{(b)}_{\mu\nu\lambda}(x, \bar{\theta}) = H_{\mu\nu\lambda}(x).
\end{align*}

where the superscript \((b)\) denotes the superfields that have been obtained after the applications of the BRST invariant trivial restrictions. The above equation \((31)\) also implies that we have some of the secondary fields trivially equal to zero: \(B_3 = B_4 = 0, f_1 = f_4 = f_6 = f_7 = f_8 = f_\mu^{(1)} = f_\mu^{(3)} = f_\mu^{(4)} = R_{\mu\nu}^{(1)} = 0\). The above expansions \((31)\) can now be utilized in
the non-trivial BRST-invariant equalities as follows:

\[
\begin{align*}
\tilde{\lambda}^{(b)}(x, \theta) \tilde{\Phi}(x, \bar{\theta}) &= \lambda(x) \varphi(x), \quad \tilde{B}_\mu(x, \bar{\theta}) + \partial_\mu \tilde{\Phi}(x, \bar{\theta}) = \tilde{B}_\mu(x) + \partial_\mu \varphi(x), \\
\tilde{\beta}(x, \theta) \tilde{\lambda}^{(b)}(x, \bar{\theta}) - \tilde{\Phi}(x, \theta) \tilde{\rho}^{(b)}(x, \bar{\theta}) &= \tilde{\beta}(x) \lambda(x) - \varphi(x) \rho(x), \\
m \tilde{\beta}(x, \bar{\theta}) \tilde{\beta}^{(b)}(x, \bar{\theta}) + \tilde{\rho}^{(b)}(x, \bar{\theta}) \tilde{F}_\mu(x, \bar{\theta}) &= m \tilde{\beta}(x) \beta(x) + \rho(x) C(x), \\
\tilde{\mu}_\mu(x, \bar{\theta}) \partial^\nu \tilde{F}_\mu(x, \bar{\theta}) + \tilde{\Phi}_\mu(x, \theta) \partial^\nu \tilde{\beta}^{(b)}(x, \bar{\theta}) &= C_\mu(x) \partial^\nu C(x) \\
+ \phi_\mu(x) \partial^\nu \beta(x), \quad \tilde{B}(x, \theta) + m \tilde{\Phi}(x, \theta) &= \tilde{B}(x) + m \varphi(x), \\
\tilde{\mu}_\mu(x, \bar{\theta}) [m \tilde{\Phi}_\mu(x, \theta) - \partial^\nu \tilde{F}_\mu(x, \bar{\theta})] + \tilde{B}^{(b)}(x, \bar{\theta}) \tilde{\Phi}_\mu(x, \theta) &= C(x) [m C_\mu(x) \\
- \partial_\mu C(x)] + B(x) \phi_\mu(x), \quad m \tilde{B}_\mu(x, \bar{\theta}) - [\partial_\mu \tilde{\Phi}_\nu(x, \theta) - \partial_\nu \tilde{\Phi}_\mu(x, \theta)] \\
= m B_\mu(x) - [\partial_\mu \phi_\nu(x) - \partial_\nu \phi_\mu(x)], \quad \tilde{F}_\mu(x, \bar{\theta}) \partial^\nu \tilde{\beta}^{(b)}(x, \bar{\theta}) \\
- \partial_\mu \tilde{\beta}(x, \bar{\theta}) \partial^\nu \tilde{\beta}^{(b)}(x, \bar{\theta}) &= C_\mu(x) \partial^\nu \rho(x) - \partial_\mu \tilde{\beta}(x) \partial^\nu \beta(x).
\end{align*}
\]

(32)

Here we have utilized the basic tenets of ACSA to BRST formalism and imposed the condition that the BRST invariant (physical) quantities should be independent of the fermionic \((\bar{\theta}^2 = 0)\) Grassmannian variable \(\bar{\theta}\). The above restrictions on the superfields lead to the derivation of all the rest of the secondary fields of the super expansion (29) in terms of the basic and auxiliary fields of the Lagrangian density \(\mathcal{L}_{(B,S)}\) as:

\[
\begin{align*}
R_\mu(x) &= - (\partial_\mu C_\nu(x) - \partial_\nu C_\mu(x)), \quad B^{(1)}_\mu(x) = - \partial_\mu \beta(x), \quad B^{(2)}_\mu(x) = B_\mu(x), \\
f_2(x) &= - \rho(x), \quad R_\mu(x) = \partial_\mu C(x) - m C_\mu(x), \quad B_1(x) = - m \beta(x), \\
f_3(x) &= - m \lambda(x), \quad f^{(2)}_\mu(x) = - \partial_\mu \lambda(x), \quad f_3(x) = \lambda(x), \quad B_2(x) = B(x).
\end{align*}
\]

(33)

Plugging in these inputs into the anti-chiral super expansion (29), we obtain the coefficients of \(\bar{\theta}\) as the non-trivial BRST symmetry transformations (13) as illustrated below:

\[
\begin{align*}
B_\mu(x) &\rightarrow \tilde{B}^{(b)}_\mu(x, \bar{\theta}) \quad = \quad B_\mu(x) + \bar{\theta} \left[ - (\partial_\mu C_\nu(x) - \partial_\nu C_\mu(x)) \right] \\
&\equiv \tilde{B}_\mu(x) + \bar{\theta} \left[ s_b B_\mu(x) \right], \\
C_\mu(x) &\rightarrow \tilde{\mu}_\mu(x, \bar{\theta}) \quad = \quad C_\mu(x) + \bar{\theta} \left( - \partial_\mu \beta(x) \right) \equiv C_\mu(x) + \bar{\theta} \left[ s_b C_\mu(x) \right], \\
\tilde{C}_\mu(x) &\rightarrow \tilde{\mu}^{(b)}_\mu(x, \bar{\theta}) \quad = \quad \tilde{C}_\mu(x) + \bar{\theta} \left( B_\mu(x) \right) \equiv \tilde{C}_\mu(x) + \bar{\theta} \left[ s_b \tilde{C}_\mu(x) \right], \\
\beta(x) &\rightarrow \tilde{\beta}^{(b)}(x, \bar{\theta}) \quad = \quad \beta(x) + \bar{\theta} \left( 0 \right) \equiv \beta(x) + \bar{\theta} \left[ s_b \tilde{\beta}(x) \right], \\
\tilde{\beta}(x) &\rightarrow \tilde{\beta}^{(b)}(x, \bar{\theta}) \quad = \quad \tilde{\beta}(x) + \bar{\theta} \left( - \rho(x) \right) \equiv \tilde{\beta}(x) + \bar{\theta} \left[ s_b \tilde{\beta}(x) \right], \\
\varphi(x) &\rightarrow \tilde{\phi}^{(b)}(x, \theta) \quad = \quad \varphi(x) + \bar{\theta} \left( \lambda(x) \right) \equiv \varphi(x) + \bar{\theta} \left[ s_b \varphi(x) \right], \\
\phi_\mu(x) &\rightarrow \tilde{\phi}^{(b)}_\mu(x, \theta) \quad = \quad \phi_\mu(x) + \bar{\theta} \left[ \partial_\mu C(x) - m C_\mu(x) \right] \\
&\equiv \phi_\mu(x) + \bar{\theta} \left[ s_b \phi_\mu(x) \right], \\
C(x) &\rightarrow \tilde{C}^{(b)}(x, \bar{\theta}) \quad = \quad C(x) + \bar{\theta} \left( - m \beta(x) \right) \equiv C(x) + \bar{\theta} \left[ s_b C(x) \right], \\
\tilde{C}(x) &\rightarrow \tilde{\phi}^{(b)}(x, \bar{\theta}) \quad = \quad \tilde{C}(x) + \bar{\theta} \left( B(x) \right) \equiv \tilde{C}(x) + \bar{\theta} \left[ s_b \tilde{C}(x) \right],
\end{align*}
\]

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According to the basic tenets of ACSA to BRST formalism, the above quantities should be independent of the Grassmannian variable \( \theta \) of the \((4, 1)\)-dimensional chiral super sub-manifold of the \((4, 2)\)-dimensional general supermanifold (on which our 4D ordinary massive Abelian 2-form theory is generalized).

We now concentrate on the derivation of the off-shell nilpotent anti-BRST symmetry transformations (12) for \( \mathcal{L}_{(B, \bar{B})} \) within the framework of ACSA to BRST formalism. It may be mentioned here that the anti-BRST transformations (12) are perfect symmetry transformations for \( \mathcal{L}_{(B, \bar{B})} \) [cf. Eq. (14)]. Keeping this objective in our mind, first of all, we note that the following interesting and useful quantities are anti-BRST invariant in view of the quantum nilpotent, infinitesimal and continuous anti-BRST symmetry transformations (12), namely:

\[
\begin{align*}
B_\mu(x) \rightarrow \bar{B}_\mu^{(b)}(x, \bar{\theta}) &= B_\mu(x) + \bar{\theta}(0) \equiv B_\mu(x) + \bar{\theta}[s_b B_\mu(x)], \\
\bar{B}_\mu(x) \rightarrow \bar{\bar{B}}_\mu^{(b)}(x, \bar{\theta}) &= \bar{B}_\mu(x) + \bar{\theta}(-\partial_\mu \lambda(x)) \equiv \bar{B}_\mu(x) + \bar{\theta}[s_b \bar{B}_\mu(x)], \\
B(x) \rightarrow \bar{B}^{(b)}(x, \bar{\theta}) &= B(x) + \bar{\theta}(0) \equiv B(x) + \bar{\theta}[s_b B(x)], \\
\bar{B}(x) \rightarrow \bar{\bar{B}}^{(b)}(x, \bar{\theta}) &= \bar{B}(x) + \bar{\theta}(-m \lambda(x)) \equiv \bar{B}(x) + \bar{\theta}[s_b \bar{B}(x)], \\
B_\mu(x) \rightarrow \bar{B}_\mu^{(b)}(x, \bar{\theta}) &= B_\mu(x) + \bar{\theta}(0) \equiv B_\mu(x) + \bar{\theta}[s_b B_\mu(x)], \\
\bar{B}_\mu(x) \rightarrow \bar{\bar{B}}_\mu^{(b)}(x, \bar{\theta}) &= \bar{B}_\mu(x) + \bar{\theta}(0) \equiv \bar{B}_\mu(x) + \bar{\theta}[s_b \bar{B}_\mu(x)] ,
\end{align*}
\]

(34)

In the above, the superscript \((b)\) denotes the fact that the super expansions have been obtained after exploiting the basic tenets of ACSA and coefficients of \( \bar{\theta} \) lead to the determination of the BRST symmetry transformations (13). It is elementary to note that \( \partial_{\bar{\theta}} \Omega^{(b)}(x, \bar{\theta}) = s_b \omega(x) \) where \( \Omega^{(b)}(x, \bar{\theta}) \) is the generic superfield obtained after the application of the BRST invariant restrictions (31) and (32) and \( \omega(x) \) is the ordinary \((4D)\) generic field of the Lagrangian density \( \mathcal{L}_{(B, \bar{B})} \). Thus, we note that the nilpotency \((s_b^2 = 0)\) property of \( s_b \) is deeply connected with the nilpotency \((\partial_{\bar{\theta}}^2 = 0)\) of the translational generators \((\partial_{\bar{\theta}})\) along the \( \theta \)-direction of the \((4, 1)\)-dimensional anti-chiral super sub-manifold of the \((4, 2)\)-dimensional general supermanifold (on which our 4D ordinary massive Abelian 2-form theory is generalized).
manifold on which the ordinary fields of $\mathcal{L}_{(\mathcal{B}, \mathcal{B})}$ are generalized in the following manner

\[
\begin{align*}
B_{\mu\nu}(x) & \rightarrow \tilde{B}_{\mu\nu}(x, \theta) = B_{\mu\nu}(x) + \theta \tilde{R}_{\mu\nu}(x), \\
C_{\mu}(x) & \rightarrow \tilde{F}_{\mu}(x, \theta) = C_{\mu}(x) + \theta \tilde{B}_{\mu}^{(1)}(x), \\
\tilde{C}_{\mu}(x) & \rightarrow \tilde{F}_{\mu}(x, \theta) = \tilde{C}_{\mu}(x) + \theta \tilde{B}_{\mu}^{(2)}(x), \\
\beta(x) & \rightarrow \tilde{\beta}(x, \theta) = \beta(x) + \theta \tilde{f}_{1}(x), \\
\tilde{\beta}(x) & \rightarrow \tilde{\beta}(x, \theta) = \tilde{\beta}(x) + \theta \tilde{f}_{2}(x), \\
\varphi(x) & \rightarrow \tilde{\Phi}(x, \theta) = \varphi(x) + \theta \tilde{f}_{3}(x), \\
\phi_{\mu}(x) & \rightarrow \tilde{\Phi}_{\mu}(x, \theta) = \phi_{\mu}(x) + \theta \tilde{R}_{\mu}(x), \\
C(x) & \rightarrow \tilde{C}(x, \theta) = C(x) + \theta B_{1}(x), \\
\tilde{C}(x) & \rightarrow \tilde{F}(x, \theta) = \tilde{C}(x) + \theta B_{2}(x), \\
B_{\mu}(x) & \rightarrow \tilde{B}_{\mu}(x, \theta) = B_{\mu}(x) + \theta \tilde{f}_{4}(x), \\
\tilde{B}_{\mu}(x) & \rightarrow \tilde{B}_{\mu}(x, \theta) = \tilde{B}_{\mu}(x) + \theta \tilde{f}_{5}(x), \\
\phi_{\mu}(x) & \rightarrow \tilde{\varphi}_{\mu}(x, \theta) = \phi_{\mu}(x) + \theta \tilde{R}_{\mu}(x), \\
\tilde{\varphi}_{\mu}(x) & \rightarrow \tilde{\Phi}_{\mu}(x, \theta) = \tilde{\varphi}_{\mu}(x) + \theta \tilde{f}_{6}(x), \\
\lambda(x) & \rightarrow \tilde{\lambda}(x, \theta) = \lambda(x) + \theta B_{3}(x), \\
\rho(x) & \rightarrow \tilde{\rho}(x, \theta) = \rho(x) + \theta B_{4}(x), \\
B(x) & \rightarrow \tilde{B}(x, \theta) = B(x) + \theta \tilde{f}_{7}(x), \\
\tilde{B}(x) & \rightarrow \tilde{B}(x, \theta) = \tilde{B}(x) + \theta \tilde{f}_{8}(x),
\end{align*}
\]  

(36)

where the fermionic ($\theta^{2} = 0$) Grassmannian variable $\theta$ [along with the bosonic coordinates $x^{\mu}(\mu = 0, 1, 2, 3)$] characterize the chiral supermanifold. We note that the secondary fields ($\tilde{R}_{\mu\nu}, \tilde{f}_{1}, \tilde{f}_{2}, \tilde{f}_{3}, \tilde{R}_{\mu}, \tilde{f}_{4}^{(1)}, \tilde{f}_{5}^{(1)}, \tilde{f}_{6}, \tilde{R}_{\mu}^{(2)}, \tilde{f}_{7}, \tilde{f}_{8}$) are fermionic in nature and the other secondary fields ($B_{\mu}^{(1)}, \tilde{B}_{\mu}^{(2)}, \tilde{B}_{1}, \tilde{B}_{2}, \tilde{B}_{3}, \tilde{B}_{4}$) are bosonic. These secondary fields are to be determined in terms of the basic and auxiliary fields of the Lagrangian density $\mathcal{L}_{(\mathcal{B}, \mathcal{B})}$. At this stage, as argued earlier, the following restrictions on the specific combination of the chiral superfields, namely:

\[
\begin{align*}
\tilde{\rho}^{(ab)}(x, \theta) \tilde{\Phi}(x, \theta) = \rho(x) \varphi(x), \\
\tilde{B}_{\mu}(x, \theta) + \partial_{\mu} \tilde{\Phi}(x, \theta) = B_{\mu}(x) + \partial_{\mu} \varphi(x), \\
\tilde{\beta}(x, \theta) \tilde{\varphi}_{\mu}^{(ab)}(x, \theta) - \tilde{\Phi}(x, \theta) \lambda^{(ab)}(x, \theta) = \beta(x) \rho(x) - \varphi(x) \lambda(x), \\
m \beta(x, \theta) \tilde{\beta}^{(ab)}(x, \theta) + \lambda^{(ab)}(x, \theta) \tilde{F}(x, \theta) = m \beta(x) \beta(x) + \lambda(x) \tilde{C}(x), \\
\tilde{F}_{\mu}(x, \theta) \partial^{\mu} \tilde{F}(x, \theta) + \tilde{\Phi}_{\mu}(x, \theta) \partial^{\mu} \tilde{\beta}_{\mu}(x, \theta) = \tilde{C}_{\mu}(x) \partial^{\mu} \tilde{C}(x), \\
+ \phi_{\mu}(x) \partial^{\mu} \tilde{\beta}(x), \\
\tilde{F}_{\mu}(x, \theta) \partial^{\mu} \tilde{F}(x, \theta) + \tilde{\Phi}_{\mu}(x, \theta) \partial^{\mu} \tilde{\beta}_{\mu}(x, \theta) + \tilde{B}^{(ab)}(x, \theta) \tilde{\Phi}_{\mu}(x, \theta),
\end{align*}
\]
are to be satisfied due to the basic tenets of ACSA to BRST formalism. The above equalities [cf. Eq. (37)] are nothing but the generalizations of the useful and interesting anti-BRST invariant quantities (35) onto the chiral super sub-manifold [of the general (4, 2)-dimensional supermanifold] with the following inputs [due to the trivial anti-BRST invariant quantities: $s_{ab}[\bar{B}, \rho, \lambda, \beta, B_{\mu}, B_{\mu}, \phi_{\mu}, \varphi, B, \bar{B}, H_{\mu\nu\kappa}] = 0$ that are useful and important for our purpose], namely:

$$
\begin{align*}
\tilde{B}^{(ab)}(x, \theta) &= \bar{B}(x), & \tilde{\phi}^{(ab)}(x, \theta) &= \rho(x), & \tilde{\lambda}^{(ab)}(x, \theta) &= \lambda(x), \\
\tilde{B}^{(ab)}_{\mu}(x, \theta) &= \bar{B}_{\mu}(x), & \tilde{\phi}^{(ab)}_{\mu}(x, \theta) &= \varphi(x), & \tilde{\phi}^{(ab)}_{\mu}(x, \theta) &= \phi_{\mu}(x), \\
\tilde{B}^{(ab)}_{\mu}(x, \theta) &= B_{\mu}(x), & \tilde{B}^{(ab)}(x, \theta) &= \bar{B}(x), & B^{(ab)}(x, \theta) &= B(x), \\
\tilde{H}^{(ab)}_{\mu\nu}(x, \theta) &= H_{\mu\nu\eta}(x), & \tilde{B}^{(ab)}_{\mu}(x, \theta) &= \bar{B}_{\mu}(x), & \tilde{\beta}^{(ab)}_{\mu}(x, \theta) &= \beta(x).
\end{align*}
$$

The above equalities/restrictions [i.e. Eq. (38)] also imply that the secondary fields: $\tilde{f}_2 = \tilde{f}_3 = \tilde{f}_5 = \tilde{f}_6 = \tilde{f}_7 = \tilde{f}_8 = B_{\bar{B}} = \bar{B}_{\bar{B}} = \tilde{F}^{(2)}_{\mu} = \tilde{F}^{(3)}_{\mu} = \tilde{F}^{(4)}_{\mu} = \tilde{R}^{(1)}_{\mu} = 0$ in the chiral super expansions of the chiral superfields in (36). In other words, the coefficients of $\theta$ in the chiral super expansions (36) (that correspond to the anti-BRST symmetry transformations ($s_{ab}$) are trivially zero for all the ordinary fields that are present on the r.h.s. of (38).

The anti-BRST invariant restrictions (37) lead to the following precise expressions for the secondary fields in terms of the basic and auxiliary fields of Lagrangian density $L_{(B, \bar{B})}$:

$$
\begin{align*}
\tilde{R}_{\mu\nu}(x) &= -[\partial_{\mu} C_{\nu}(x) - \partial_{\nu} C_{\mu}(x)], & \tilde{B}^{(2)}_{\mu}(x) &= - \partial_{\mu} \beta(x), & \tilde{B}^{(1)}_{\mu}(x) &= \bar{B}_{\mu}(x), \\
\tilde{f}_{1} &= - \lambda(x), & \tilde{R}_{\mu}(x) &= \partial_{\mu} C(x) - m C_{\mu}(x), & \tilde{B}_{2}(x) &= - m \beta(x), \\
\tilde{B}_{1}(x) &= \bar{B}(x), & \tilde{f}_{4}(x) &= - m \rho(x), & \tilde{f}_{3}(x) &= \rho(x).
\end{align*}
$$

The above derivations are straightforward as there are no complicated tricks involved in their deductions. Substitutions of the above precise values of the secondary fields into the chiral super expansions (36) lead to the determination of anti-BRST symmetries ($s_{ab}$) [cf. Eq. (12)] as the coefficients of $\theta$ as illustrated in the following super expansions:
\[ \bar{\beta}(x) \rightarrow \tilde{\bar{\beta}}^{(ab)}(x, \theta) = \bar{\beta}(x) + \theta(0) \equiv \bar{\beta}(x) + \theta[s_{ab} \bar{\beta}(x)], \]

\[ \varphi(x) \rightarrow \tilde{\varphi}^{(ab)}(x, \theta) = \varphi(x) + \theta(\rho(x)) \equiv \varphi(x) + \theta[s_{ab} \varphi(x)], \]

\[ \phi_\mu(x) \rightarrow \tilde{\phi}_\mu^{(ab)}(x, \theta) = \phi_\mu(x) + \theta[\partial_\mu \bar{C}(x) - m \bar{\phi}_\mu(x)] \equiv \phi_\mu(x) + \theta[s_{ab} \phi_\mu(x)], \]

\[ C(x) \rightarrow \tilde{C}^{(ab)}(x, \theta) = C(x) + \theta(\bar{B}(x)) \equiv C(x) + \theta[s_{ab} C(x)], \]

\[ \bar{C}(x) \rightarrow \tilde{\bar{C}}^{(ab)}(x, \theta) = \bar{C}(x) + \theta(-m \bar{\beta}(x)) \equiv \bar{C}(x) + \theta[s_{ab} \bar{C}(x)], \]

\[ B_\mu(x) \rightarrow \tilde{B}_\mu^{(ab)}(x, \theta) = B_\mu(x) + \theta(-\partial_\mu \rho) \equiv B_\mu(x) + \theta[s_{ab} B_\mu(x)], \]

\[ \bar{B}_\mu(x) \rightarrow \tilde{\bar{B}}_\mu^{(ab)}(x, \theta) = \bar{B}_\mu(x) + \theta(0) \equiv \bar{B}_\mu(x) + \theta[s_{ab} \bar{B}_\mu(x)], \]

\[ B(x) \rightarrow \tilde{B}^{(ab)}(x, \theta) = B(x) + \theta(-m \rho(x)) \equiv B(x) + \theta[s_{ab} B(x)], \]

\[ B_\mu(x) \rightarrow \tilde{B}_\mu^{(ab)}(x, \theta) = B_\mu(x) + \theta(0) \equiv B_\mu(x) + \theta[s_{ab} B_\mu(x)], \]

\[ B(x) \rightarrow \tilde{B}^{(ab)}(x, \theta) = B(x) + \theta(0) \equiv B(x) + \theta[s_{ab} B(x)]. \]

It is self-evident that we have an interesting relationship: \( \partial_\theta \Omega^{(ab)}(x, \theta) = s_{ab} \omega(x) \) where the generic superfield \( \Omega^{(ab)}(x, \theta) \) represents nothing but the chiral superfields present on the l.h.s. of Eqs. (38) as well as (40) and \( \omega(x) \) denotes nothing but the generic 4D field which stands for the ordinary basic and auxiliary fields of the 4D ordinary Lagrangian density \( L_{(B, B)} \) (that respects anti-BRST symmetry transformations (12) in a perfect manner [cf. Eq. (14)])

In other words, the translation of the superfields (obtained after the application of anti-BRST invariant restrictions) along the chiral \( \theta \)-direction of the chiral (4, 1)-dimensional super sub-manifold generates the anti-BRST symmetry transformations \( s_{ab} \) in the ordinary 4D space [cf. Eq. (12)]. We also observe that the nilpotency \( s^2_{ab} = 0 \) of \( s_{ab} \) and the nilpotency \( \partial_\theta^2 = 0 \) of the translational generator \( \partial_\theta \) (along the \( \theta \)-direction of the chiral super sub-manifold) are deeply related to each other.

### 4.2 (Anti-)co-BRST Symmetries: ACSA

We focus now on the derivation of the (anti-)co-BRST symmetry transformations \( s_{(a)d} \) by applying the ACSA to BRST formalism. For this purpose, we use the (anti-)chiral super expansions (29) and (36) for the sake of brevity. First of all, we derive the co-BRST symmetry transformations \( s_{d} \) by taking into account the chiral super expansions

\[ s^{(a)d}_{(a)d} = (\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2) \text{ with the fermionic properties: } \theta_1^2 = \theta_2^2 = 0, \bar{\theta}_1^2 = \bar{\theta}_2^2 = 0, \theta_1 \bar{\theta}_1 + \bar{\theta}_1 \theta_1 = 0, \theta_2 \bar{\theta}_2 + \bar{\theta}_2 \theta_2 = 0, \text{ etc.} \]
\(s_d[\rho \ddot{\varphi}] = 0, \quad s_d[\beta \rho + \ddot{\varphi} \lambda] = 0, \quad s_d[C_\mu \partial^\mu \lambda - \partial_\mu \beta \partial^\mu \bar{C}] = 0,
\)
\(s_d[C_\mu \partial^\mu \bar{C} + \tilde{\phi}_\mu \partial^\mu \beta] = 0, \quad s_d[B_\mu + \partial_\mu \ddot{\varphi}] = 0, \quad s_d[\bar{B} + m \ddot{\varphi}] = 0,
\)
\(s_d[C_\mu (m \bar{C}^\mu - \partial^\mu \bar{C}) + B_\mu \tilde{\phi}_\mu] = 0, \quad s_d[C (m \bar{C}^\mu - \partial^\mu \bar{C}) + B \tilde{\phi}_\mu] = 0,
\)
\(s_d[m B_{\mu \nu} - \varepsilon_{\mu \nu \rho \beta} \partial^\beta \bar{C}] = 0, \quad s_d[m \bar{\beta} \beta + \lambda \bar{C}] = 0, \quad (41)\)

should be independent of the Grassmannian variables \(\theta\) when they are generalized onto the \((4, 1)\)-dimensional \textit{chiral} super sub-manifold [of the \textit{general} \((4, 2)\)-dimensional supermanifold]. In other words, we demand the following conditions on the \textit{chiral} superfields

\[
\begin{align*}
\hat{\rho}^{(d)}(x, \theta) &\Phi(x, \theta) = \rho(x) \dot{\varphi}(x), \quad \hat{B}_\mu(x, \theta) + \partial_\mu \hat{\Phi}(x, \theta) = \bar{B}_\mu(x) + \partial_\mu \ddot{\varphi}(x), \\
\hat{\beta}(x, \theta) &\tilde{\rho}^{(d)}(x, \theta) + \tilde{\Phi}(x, \theta) \tilde{\lambda}^{(d)}(x, \theta) = \beta(x) \rho(x) + \ddot{\varphi}(x) \lambda(x), \\
\Phi^{(d)}(x, \theta) \bar{B}_\mu(x, \theta) - \partial_\mu \tilde{\Phi}(x, \theta) &+ \bar{B}^{(d)}(x, \theta) \tilde{\Phi}(x, \theta) = C_\mu(x) - \partial_\mu C(x) + B(x) \tilde{\phi}_\mu(x), \\
\hat{\beta}(x, \theta) &\tilde{\beta}^{(d)}(x, \theta) = \beta(x) \beta(x) + \lambda(x) C(x), \quad \tilde{\Phi}(x, \theta) \tilde{\Phi}(x, \theta) + \tilde{\Phi}^{(d)}(x, \theta) \tilde{\Phi}(x, \theta) = m B_{\mu \nu}(x, \theta) - \varepsilon_{\mu \nu \rho \beta} \partial^\beta \bar{C}(x, \theta), \\
\tilde{B}^{(d)}(x, \theta) &\tilde{B}^{(d)}(x, \theta) = C_\mu(x) [m \bar{C}^\mu(x) - \partial^\mu \bar{C}(x)], \quad (42)
\end{align*}
\]

where the superfields with the superscript \((d)\) have been derived from our earlier observation: \(s_d[\partial^n B_{\mu \nu}, B_\mu, B_\mu, B, B, B_\mu, B_\mu, \Phi_\mu, \beta, \varphi, \rho, \lambda] = 0\) [cf. Eq. \((19)\)] which imply the following trivial restrictions on the \textit{chiral} superfields:

\[
\begin{align*}
\partial^n \hat{B}^{(d)}_{\nu \mu}(x, \theta) &\hat{B}^{(d)}_{\nu \mu}(x, \theta) = \partial^n B_{\nu \mu}(x), \quad \hat{\rho}^{(d)}(x, \theta) = \rho(x), \quad \tilde{\lambda}^{(d)}(x, \theta) = \lambda(x), \\
\hat{B}^{(d)}_{\mu}(x, \theta) &\hat{B}^{(d)}_{\mu}(x, \theta) = B_{\mu}(x), \quad \hat{\Phi}^{(d)}_{\mu}(x, \theta) = \phi_\mu(x), \quad \tilde{B}^{(d)}(x, \theta) = B(x), \\
\tilde{B}^{(d)}(x, \theta) &\tilde{B}^{(d)}(x, \theta) = B(x), \quad \tilde{B}^{(d)}_{\mu}(x, \theta) = B_{\mu}(x), \\
\hat{\beta}^{(d)}(x, \theta) &\hat{\beta}^{(d)}(x, \theta) = \beta(x), \quad \hat{\Phi}^{(d)}(x, \theta) = \varphi(x), \quad \tilde{B}^{(d)}_{\mu}(x, \theta) = B_{\mu}(x). \quad (43)
\end{align*}
\]

\((\partial_{\theta_1}, \partial_{\tilde{\theta}_1})\) can be associated with the BRST and anti-BRST symmetries [keeping the pair \((\theta_2, \tilde{\theta}_2)\) intact]. On the other hand, the pair \((\theta_2, \tilde{\theta}_2)\) and corresponding derivatives \((\partial_{\theta_2}, \partial_{\tilde{\theta}_2})\) could be associated with the co-BRST and anti-co-BRST symmetries [keeping the pair \((\theta_1, \tilde{\theta}_1)\) intact]. This is required because the (anti-)BRST and (anti-)co-BRST symmetries are independent of each-other as are the exterior and co-exterior derivatives of differential geometry. However, for the sake of brevity, we have considered only a single pair [i.e. \((\theta, \tilde{\theta})\)] of Grassmannian variables \(\theta\) and \(\tilde{\theta}\) for the discussions of the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries \textit{together} in our present endeavor.
The restrictions (42) on the chiral superfields, along with the inputs from (43), lead to the determination of the secondary fields [in terms of the basic and auxiliary fields of $\mathcal{L}_{(B,S)}$] of the chiral expansions in (36) as:

$$\tilde{R}_{\mu \nu} = -\varepsilon_{\mu \nu \rho} \partial^\rho \bar{C}^\kappa, \quad \tilde{B}^{(1)}_\mu = B_\mu, \quad \tilde{B}^{(2)}_\mu = -\partial_\mu \tilde{\beta}, \quad \tilde{f}_1 = -\lambda, \quad \tilde{f}^{(4)}_\mu = \partial_\mu \rho,$$

$$\tilde{R}^{(1)}_\mu = \partial_\mu \bar{C} - m \bar{C}_\mu, \quad \tilde{B}_1 = B, \quad \tilde{B}_2 = -m \tilde{\beta}, \quad \tilde{f}_8 = m \rho, \quad \tilde{f}_6 = -\rho. \quad (44)$$

Substitutions of these precise values into the chiral expansions (36) lead to the following in terms of the super expansions along $\theta$, namely:

$$B_{\mu \nu}(x) \rightarrow \tilde{B}^{(d)}_{\mu \nu}(x, \theta) = B_{\mu \nu}(x) + \theta [-\varepsilon_{\mu \nu \rho} \partial^\rho \bar{C}^\kappa(x)]$$

$$= B_{\mu \nu}(x) + \theta [s_d B_{\mu \nu}(x)],$$

$$C_\mu(x) \rightarrow \tilde{F}^{(d)}_\mu(x, \theta) = C_\mu(x) + \theta (B_\mu(x)) \equiv C_\mu(x) + \theta [s_d C_\mu(x)],$$

$$\bar{C}_\mu(x) \rightarrow \tilde{F}^{(d)}_{\mu \nu}(x, \theta) = C_\mu(x) + \theta (-\partial_\mu \tilde{\beta}(x)) \equiv C_\mu(x) + \theta [s_d \bar{C}_\mu(x)],$$

$$\beta(x) \rightarrow \beta^{(d)}(x, \theta) = \beta(x) + \theta (-\lambda(x)) \equiv \beta(x) + \theta [s_d \beta(x)],$$

$$\bar{\beta}(x) \rightarrow \bar{\beta}^{(d)}(x, \theta) = \bar{\beta}(x) + \theta (0) \equiv \bar{\beta}(x) + \theta [s_d \bar{\beta}(x)],$$

$$\phi(x) \rightarrow \tilde{\phi}^{(d)}(x, \theta) = \varphi(x) + \theta ((0)) \equiv \varphi(x) + \theta [s_d \varphi(x)],$$

$$\bar{\phi}(x) \rightarrow \tilde{\phi}^{(d)}_{\mu}(x, \theta) = \phi_\mu(x) + \theta (0) \equiv \phi_\mu(x) + \theta [s_d \phi_\mu(x)],$$

$$C(x) \rightarrow \tilde{C}^{(d)}(x, \theta) = C(x) + \theta (B(x)) \equiv C(x) + \theta [s_d C(x)],$$

$$\bar{C}(x) \rightarrow \tilde{C}^{(d)}(x, \theta) = \bar{C}(x) + \theta (-m \tilde{\beta}(x)) \equiv \bar{C}(x) + \theta [s_d \bar{C}(x)],$$

$$B_\mu(x) \rightarrow \tilde{B}^{(d)}_\mu(x, \theta) = B_\mu(x) + \theta (0) \equiv B_\mu(x) + \theta [s_d B_\mu(x)],$$

$$\bar{B}_\mu(x) \rightarrow \tilde{B}^{(d)}_{\mu \nu}(x, \theta) = \bar{B}_\mu(x) + \theta (0) \equiv \bar{B}_\mu(x) + \theta [s_d \bar{B}_\mu(x)],$$

$$B(x) \rightarrow \tilde{B}^{(d)}(x, \theta) = B(x) + \theta (0) \equiv B(x) + \theta [s_d B(x)],$$

$$\tilde{B}(x) \rightarrow \tilde{B}^{(d)}(x, \theta) = \tilde{B}(x) + \theta (0) \equiv \tilde{B}(x) + \theta [s_d \tilde{B}(x)],$$

$$B_\mu(x) \rightarrow \tilde{B}^{(d)}_{\mu \nu}(x, \theta) = B_\mu(x) + \theta (0) \equiv B_\mu(x) + \theta [s_d B_\mu(x)],$$

$$\bar{B}_\mu(x) \rightarrow \tilde{B}^{(d)}_{\mu \nu}(x, \theta) = \bar{B}_\mu(x) + \theta (0) \equiv \bar{B}_\mu(x) + \theta [s_d \bar{B}_\mu(x)],$$

$$\tilde{B}(x) \rightarrow \tilde{B}^{(d)}(x, \theta) = \tilde{B}(x) + \theta (0) \equiv \tilde{B}(x) + \theta [s_d \tilde{B}(x)], \quad (45)$$

where the superscript $(d)$ denotes the chiral super expansions of the superfields that lead to the derivation of the co-BRST transformations (19) as the coefficients of $\theta$. It is crystal clear that $\partial_\theta \Omega^{(d)}(x, \theta) = s_d \omega(x)$ where the generic superfield $\Omega^{(d)}(x, \theta)$ stands for all the chiral superfields that are present on the l.h.s. of Eqs. (43) as well as (45) and the ordinary
generic field $\omega(x)$ corresponds to all the basic and auxiliary fields of the Lagrangian density $\mathcal{L}_{(B,B)}$. We also note that the off-shell nilpotency $(s_d^2 = 0)$ of $s_d$ is deeply connected with the nilpotency $(\partial_0^2 = 0)$ of the translational generator $(\partial_0)$ along the $\theta$-direction of the $(4,1)$-dimensional chiral super sub-manifold [of the general $(4,2)$-dimensional supermanifold].

We devote time on the derivation of the anti-co-BRST symmetry transformations $s_{ad}$ by applying the basic tenets of ACSA to BRST formalism. In this connection, first of all, we observe that the following anti-co-BRST invariant quantities of interest, namely:

$$s_{ad}[\rho \beta] = 0, \quad s_{ad}[\beta \lambda + \bar{\varphi} \rho] = 0, \quad s_{ad}[C_\mu \partial^\mu \rho - \partial_\mu \bar{\beta} \partial^\mu \beta] = 0,$$

$$s_{ad}[C_\mu \partial^\mu C + \bar{\phi}_\mu \partial^\mu \beta] = 0, \quad s_{ad}[\mathcal{B}_\mu + \partial_\mu \bar{\varphi}] = 0, \quad s_{ad}[\mathcal{B} + m \bar{\varphi}] = 0,$$

$$s_{ad}[C (m C_\mu - \partial_\mu C) + \mathcal{B} \bar{\phi}_\mu] = 0,$$

$$s_{ad}[m B_{\mu \nu} - \varepsilon_{\mu \nu \rho \sigma} \partial^\rho \bar{\phi}^\sigma] = 0, \quad s_{ad}[m \bar{\beta} \beta + \rho C] = 0,$$

(46)

can be generalized onto the suitably chosen $(4,1)$-dimensional anti-chiral super sub-manifold where we have to take into account the anti-chiral superfield expansions (29). However, before we perform that, we note that (due to the trivial anti-co-BRST symmetry invariance $s_{ad}[\partial^\nu B_{\nu \mu}, B_\mu, \bar{B}_\mu, B, \bar{B}, \bar{B}, \bar{B}, \bar{B}, \bar{B}, \varphi, \rho, \lambda] = 0$), we have the following

$$\partial^\nu \bar{B}_{(ad)}^\mu(x, \bar{\theta}) = \partial^\nu B_{\nu \mu}(x), \quad \bar{\rho}_{(ad)}^\mu(x, \bar{\theta}) = \rho(x), \quad \bar{\lambda}_{(ad)}(x, \bar{\theta}) = \lambda(x),$$

$$\bar{B}_{(ad)}^\mu(x, \bar{\theta}) = \bar{B}_\mu(x), \quad \bar{\phi}_{(ad)}^\mu(x, \bar{\theta}) = \phi_\mu(x), \quad \bar{B}^\mu_{(ad)}(x, \bar{\theta}) = B(x),$$

$$\bar{B}(x, \bar{\theta}) = B(x), \quad \bar{B}_{(ad)}^\mu(x, \bar{\theta}) = B_{(ad)}^\mu(x, \bar{\theta}) = B_\mu(x),$$

$$\bar{\beta}_{(ad)}(x, \bar{\theta}) = \beta(x), \quad \bar{\phi}_{(ad)}^\mu(x, \bar{\theta}) = \varphi(x), \quad \bar{B}^\mu_{(ad)}(x, \bar{\theta}) = B^\mu_{(ad)}(x, \bar{\theta}),$$

(47)

where the superscript $(ad)$ denotes the anti-chiral superfields which have been obtained due to the trivial anti-co-BRST invariance [cf. Eq. (18)]. Another way of saying the fact that the secondary fields $(R_\mu, B_3, B_4, f_1, f_2, f_3, f_4, f_5, f_6, f^{(1)}_\mu, f^{(2)}_\mu, f^{(3)}_\mu)$ are trivially equal to zero so that we can have $s_{ad}[\partial^\nu B_{\nu \mu}, B_\mu, \bar{B}_\mu, B, \bar{B}, \bar{B}, \bar{B}, \bar{B}, \bar{B}, \varphi, \rho, \lambda] = 0$ as the coefficients of $\bar{\theta}$ in the super expansions that have been listed in Eq. (29) and represented in the super expansions (47).

Using the trivial equalities (47), we have the following generalizations of the anti-co-BRST invariant quantities (46) [located inside the square brackets] in terms of the anti-chiral superfields, namely;

$$\bar{\rho}_{(ad)}^\mu(x, \bar{\theta}) \bar{\beta}(x, \bar{\theta}) = \rho(x) \bar{\beta}(x), \quad \bar{B}_{\mu}(x, \bar{\theta}) + \partial_\mu \bar{\phi}(x, \bar{\theta}) = B_\mu(x) + \partial_\mu \bar{\varphi}(x),$$

$$\bar{\beta}(x, \bar{\theta}) \bar{\lambda}_{(ad)}(x, \bar{\theta}) + \bar{\phi}(x, \bar{\theta}) \bar{\rho}_{(ad)}^\mu(x, \bar{\theta}) = \bar{\beta}(x) \lambda(x) + \bar{\varphi}(x) \rho(x),$$

$$m \bar{\beta}(x, \bar{\theta}) \bar{\beta}_{(ad)}^\mu(x, \bar{\theta}) + \bar{\rho}_{(ad)}^\mu(x, \bar{\theta}) \bar{F}(x, \bar{\theta}) = m \bar{\beta}(x) \beta(x) + \rho(x) C(x),$$

$$\bar{F}_{\mu}(x, \bar{\theta}) \partial^\nu \bar{F}(x, \bar{\theta}) + \bar{\phi}_{\mu}(x, \bar{\theta}) \partial^\nu \bar{\beta}_{(ad)}^\mu(x, \bar{\theta}) = C_\mu(x) \partial^\nu C(x) + \bar{\phi}_{\mu}(x) \partial^\nu \beta(x),$$

$$\bar{B}(x, \bar{\theta}) + m \bar{\phi}(x, \bar{\theta}) = B(x) + m \bar{\varphi}(x), \quad \bar{F}_{\mu}(x, \bar{\theta}) \left[ m \bar{F}^\mu(x, \bar{\theta}) - \partial^\mu \bar{F}(x, \bar{\theta}) \right]$$

$$+ \bar{B}^\mu_{(ad)}(x, \bar{\theta}) \bar{\phi}_{\mu}(x, \bar{\theta}) = C_\mu(x) \left[ m C^\mu(x) - \partial^\mu C(x) \right] + \bar{B}_\mu(x) \bar{\phi}_{(ad)}^\mu(x),$$

23
\[
\begin{align*}
\dot F(x, \bar \theta) &= [m \dot F_\mu(x, \bar \theta) - \partial_\mu \dot F(x, \bar \theta)] + \dot B^{(ad)}(x, \bar \theta) \dot \Phi_\mu(x, \bar \theta) = C(x) [m C_\mu(x) \\
&= -\partial_\mu C(x)] + \dot B(x) \dot \Phi_\mu(x, \bar \theta), \\
&= m \dot B_\mu(x) \bar \theta - \varepsilon_{\mu \nu \eta \kappa} \partial^\eta \dot \Phi_\kappa(x, \bar \theta) = m B_\mu(x) \\
-\varepsilon_{\mu \nu \eta \kappa} \partial^\eta \dot \Phi_\kappa(x, \bar \theta), \\
\dot F_\mu(x, \bar \theta) &= \partial^\mu \rho^{(ad)}(x, \bar \theta) - \partial_\mu \beta(x, \bar \theta) \partial^\mu \bar \beta^{(ad)}(x, \bar \theta) \\
&= C_\mu(x) \partial^\mu \rho(x) - \partial_\mu \bar \beta(x) \partial^\mu \beta(x). \quad (48)
\end{align*}
\]

At this stage, we substitute the anti-chiral super expansions (29) into the above equalities which lead to the determination of secondary fields in terms of the basic and auxiliary fields of the Lagrangian density \( \mathcal{L}_{(B, \bar B)} \) as:

\[
\begin{align*}
R_{\mu \nu} &= -\varepsilon_{\mu \nu \eta \kappa} \partial^\eta C^\kappa, \quad B^{(2)}_\mu = B_\mu, \quad B^{(1)}_\mu = \partial_\mu \beta, \quad f_6 = -\lambda, \quad \tilde f^{(3)}_\mu = \partial_\mu \rho, \\
R^{(1)}_{\mu \nu} &= \partial_\mu C - m C_\mu, \quad B_2 = \bar B, \quad B_1 = m \beta, \quad f_7 = m \lambda, \quad f_2 = \rho. \quad (49)
\end{align*}
\]

The substitutions of (49) and observations in (47) enable us to write the anti-chiral super expansions (29), in terms of the anti-co-BRST symmetry transformations (18), as:

\[
\begin{align*}
B_\mu(x) &\longrightarrow \tilde B^{(ad)}_\mu(x, \bar \theta) = B_\mu(x) + \bar \theta [-\varepsilon_{\mu \nu \eta \kappa} \partial^\eta \tilde C^\kappa(x)] \\
&\equiv B_\mu(x) + \bar \theta [s_{ad} B_\mu(x)], \\
C_\mu(x) &\longrightarrow \tilde F^{(ad)}_\mu(x, \bar \theta) = C_\mu(x) + \bar \theta (\partial_\mu \beta(x)) \equiv C_\mu(x) + \bar \theta [s_{ad} C_\mu(x)], \\
\beta(x) &\longrightarrow \tilde \beta^{(ad)}(x, \bar \theta) = \beta(x) + \bar \theta (0) \equiv \beta(x) + \bar \theta [s_{ad} \beta(x)], \\
\bar \beta(x) &\longrightarrow \tilde \bar \beta^{(ad)}(x, \bar \theta) = \bar \beta(x) + \bar \theta (\rho(x)) \equiv \bar \beta(x) + \bar \theta [s_{ad} \bar \beta(x)], \\
\varphi(x) &\longrightarrow \tilde \varphi^{(ad)}(x, \bar \theta) = \varphi(x) + \bar \theta (0) \equiv \varphi(x) + \bar \theta [s_{ad} \varphi(x)], \\
\phi_\mu(x) &\longrightarrow \tilde \phi^{(ad)}_\mu(x, \bar \theta) = \phi_\mu(x) + \bar \theta (0) \equiv \phi_\mu(x) + \bar \theta [s_{ad} \phi_\mu(x)], \\
C(x) &\longrightarrow \tilde C^{(ad)}(x, \bar \theta) = C(x) + \bar \theta (m \beta(x)) \equiv C(x) + \bar \theta [s_{ad} C(x)], \\
B_\mu(x) &\longrightarrow \tilde B^{(ad)}_\mu(x, \bar \theta) = B_\mu(x) + \bar \theta (0) \equiv B_\mu(x) + \bar \theta [s_{ad} B_\mu(x)], \\
\bar B_\mu(x) &\longrightarrow \tilde \bar B^{(ad)}_\mu(x, \bar \theta) = \bar B_\mu(x) + \bar \theta (0) \equiv \bar B_\mu(x) + \bar \theta [s_{ad} \bar B_\mu(x)], \\
B(x) &\longrightarrow \tilde B^{(ad)}(x, \bar \theta) = B(x) + \bar \theta (0) \equiv B(x) + \bar \theta [s_{ad} B(x)], \\
\bar B(x) &\longrightarrow \tilde \bar B^{(ad)}(x, \bar \theta) = \bar B(x) + \bar \theta (0) \equiv \bar B(x) + \bar \theta [s_{ad} \bar B(x)], \\
B_\mu(x) &\longrightarrow \tilde B^{(ad)}_\mu(x, \bar \theta) = B_\mu(x) + \bar \theta (\partial_\mu \lambda(x)) \equiv B_\mu(x) + \bar \theta [s_{ad} B_\mu(x)], \\
\bar B_\mu(x) &\longrightarrow \tilde \bar B^{(ad)}_\mu(x, \bar \theta) = \bar B_\mu(x) + \bar \theta (0) \equiv \bar B_\mu(x) + \bar \theta [s_{ad} \bar B_\mu(x)], \\
\tilde \varphi(x) &\longrightarrow \tilde \tilde \varphi^{(ad)}(x, \bar \theta) = \tilde \varphi(x) + \bar \theta (-\lambda) \equiv \tilde \varphi(x) + \bar \theta [s_{ad} \tilde \varphi(x)], \\
\tilde \phi_\mu(x) &\longrightarrow \tilde \tilde \phi^{(ad)}_\mu(x, \bar \theta) = \tilde \phi_\mu(x) + \bar \theta [\partial_\mu \bar C(x) - m \bar C_\mu(x)] \\
&\equiv \tilde \phi_\mu(x) + \bar \theta [s_{ad} \tilde \phi_\mu(x)],
\end{align*}
\]
\[ \lambda(x) \rightarrow \tilde{\lambda}^{(ad)}(x, \bar{\theta}) = \lambda(x) + \bar{\theta}(0) \equiv \lambda(x) + \bar{\theta}[s_{ad} \lambda(x)], \]
\[ \rho(x) \rightarrow \tilde{\rho}^{(ad)}(x, \bar{\theta}) = \rho(x) + \bar{\theta}(0) \equiv \rho(x) + \bar{\theta}[s_{ad} \rho(x)], \]
\[ B(x) \rightarrow \tilde{B}^{(ad)}(x, \bar{\theta}) = B(x) + \bar{\theta}(m \lambda) \equiv B(x) + \bar{\theta}[s_{ad} B(x)], \]
\[ \bar{B}(x) \rightarrow \tilde{\bar{B}}^{(ad)}(x, \bar{\theta}) = \bar{B}(x) + \bar{\theta}(0) \equiv \bar{B}(x) + \bar{\theta}[s_{ad} \bar{B}(x)]. \quad (50) \]

The above final expansions explicitly show that we have already derived the anti-co-BRST symmetry transformations \( s_{ad} \) as the coefficients of \( \bar{\theta} \). In other words, we note that \( \partial_\theta \Omega^{(ad)}(x, \bar{\theta}) = s_{ad} \omega(x) \) where \( \Omega^{(ad)}(x, \bar{\theta}) \) is the generic superfield that stands for all the \( \textit{anti-chiral} \) superfields which are present on the l.h.s. of Eqs. (47) as well as (50) and \( \omega(x) \) is the generic field which corresponds to the \textit{ordinary} basic and \textit{auxiliary} fields of \( \mathcal{L}(\tilde{B}, \tilde{\bar{B}}) \) that are the \textit{first} terms on the r.h.s. of Eqs. (47) as well as (50). This observation also implies that we have interconnection between the nilpotency \( s_{ad}^2 = 0 \) of \( s_{ad} \) and nilpotency \( (\partial^{(ad)}_\theta = 0) \) of the translational generator \( \partial_\theta \) along \( \bar{\theta} \)-direction of the anti-chiral super sub-manifold as: \( s_{ad}^2 = 0 \leftrightarrow \partial^{(ad)}_\theta = 0 \).

5 Invariance of the Lagrangian Densities: ACSA to BRST Formalism

In this section, we establish the existence of the CF-type restrictions [cf. Eq. (21)] within the framework ACSA by capturing the symmetry invariance of the Lagrangian densities \( \mathcal{L}(\tilde{B}, \tilde{\bar{B}}) \) and \( \mathcal{L}(\tilde{B}, \tilde{\bar{B}}) \). This exercise also proves the equivalence of the coupled Lagrangian densities \( \mathcal{L}(\tilde{B}, \tilde{\bar{B}}) \) and \( \mathcal{L}(\tilde{B}, \tilde{\bar{B}}) \) w.r.t. the (anti-)BRST and (anti-)co-BRST symmetry transformations in the space of fields where the CF-type restrictions [cf. Eq. (21)] are satisfied. Our present section is divided into two parts. In sub-section 5.1, we discuss the (anti-) BRST invariance and the CF-type restrictions (associated with the nilpotent (anti-)BRST symmetries). Our sub-section 5.2 is devoted to the discussion of the (anti-)co-BRST invariance and derivation of the CF-type restrictions (associated with these nilpotent symmetries).

5.1 (Anti-)BRST Invariance and CF-Type Restrictions

In this sub-section, we discuss the (anti-)BRST invariance and derivation of the proper CF-type restrictions within the framework of ACSA to BRST formalism. Toward this objective in mind, we perform the following generalization of the \textit{ordinary} Lagrangian density: \( \mathcal{L}(\tilde{B}, \tilde{\bar{B}}) \rightarrow \tilde{\mathcal{L}}^{(ac)}(\tilde{B}, \tilde{\bar{B}})(x, \bar{\theta}) \), namely;
\[
\tilde{\mathcal{L}}^{(ac)}(\tilde{B}, \tilde{\bar{B}})(x, \bar{\theta}) = \frac{1}{2} \bar{\mathcal{B}}_{\mu}(x) \mathcal{B}^{\mu}(x) - \mathcal{B}^{\mu}(x) \left( \frac{1}{2} \varepsilon_{\mu\nu\eta\kappa} \partial^{\nu} \tilde{\mathcal{B}}^{\eta\kappa}(b)(x, \bar{\theta}) \right) - \frac{1}{2} \partial_{\mu} \tilde{\varphi}(x) + m \tilde{\varphi}_{\mu}(x) \right) - \frac{m^2}{4} \tilde{\mathcal{B}}^{\mu\nu}(b) \tilde{\bar{B}}_{\mu}(b)(x, \bar{\theta}) \tilde{B}^{\nu}(b)(x, \bar{\theta}) \]
\[ - \frac{1}{2} \partial^{\mu} \tilde{\Phi}^{\nu}(b)(x, \bar{\theta}) \left( \partial_{\mu} \tilde{\Phi}^{\nu}(b)(x, \bar{\theta}) - \partial_{\nu} \tilde{\Phi}_{\mu}^{(b)}(x, \bar{\theta}) \right) + m \tilde{\mathcal{B}}^{\mu\nu}(b)(x, \bar{\theta}) \partial_{\mu} \tilde{\Phi}_{\nu}^{(b)}(x, \bar{\theta}) + \frac{1}{4} \tilde{\Phi}^{\mu\nu}(x) \tilde{\Phi}_{\mu\nu}(x) \]

25
\[
\begin{align*}
&+ \frac{m}{2} \varepsilon^{\mu \nu \kappa} \tilde{B}^{(b)}_{\mu \nu}(x, \tilde{\theta}) \partial_\kappa \phi(x) - \frac{1}{2} B^\mu(x) B_\mu(x) \\
&+ B^\mu(x) \left( \partial^\nu \tilde{B}^{(b)}_{\nu \mu}(x, \tilde{\theta}) - \frac{1}{2} \partial_\nu \tilde{\phi}^{(b)}(x, \tilde{\theta}) + m \tilde{\phi}^{(b)}(x, \tilde{\theta}) \right) \\
&+ \frac{1}{2} B(x) B(x) + B(x) \left( \partial_\mu \tilde{\phi}^{(b)}(x, \tilde{\theta}) + \frac{m}{2} \tilde{\phi}^{(b)}(x, \tilde{\theta}) \right) \\
&- \frac{1}{2} B(x) B(x) - B(x) \left( \partial_\mu \tilde{\phi}^{(b)}(x, \tilde{\theta}) + \frac{m}{2} \tilde{\phi}^{(b)}(x, \tilde{\theta}) \right) \\
&+ \left( \partial_\mu \tilde{\mathcal{F}}^{(b)}(x, \tilde{\theta}) - m \tilde{\mathcal{F}}^{(b)}_{\mu}(x, \tilde{\theta}) \right) \\
&\left( \partial^\mu \tilde{\mathcal{F}}^{(b)}(x, \tilde{\theta}) - m \tilde{\mathcal{F}}^{\mu(b)}(x, \tilde{\theta}) \right) \\
&- \left( \partial_\mu \tilde{\mathcal{F}}^{(b)}_{\nu}(x, \tilde{\theta}) - \partial_\nu \tilde{\mathcal{F}}^{(b)}_{\mu}(x, \tilde{\theta}) \right) \left( \partial^\mu \tilde{\mathcal{F}}^{\nu(b)}(x, \tilde{\theta}) \right) \\
&- \frac{1}{2} \partial_\mu \tilde{\phi}^{(b)}(x, \tilde{\theta}) \partial^\mu \beta(x) + \frac{1}{2} m^2 \tilde{\phi}^{(b)}(x, \tilde{\theta}) \beta(x) \\
&- \frac{1}{2} \left( \partial_\mu \tilde{\mathcal{F}}^{\mu(b)}(x, \tilde{\theta}) + m \tilde{\mathcal{F}}^{(b)}(x, \tilde{\theta}) + \frac{1}{4} \rho(x) \right) \lambda(x) \\
&- \frac{1}{2} \left( \partial_\mu \tilde{\mathcal{F}}^{\mu(b)}(x, \tilde{\theta}) + m \tilde{\mathcal{F}}^{(b)}(x, \tilde{\theta}) - \frac{1}{4} \lambda(x) \right) \rho(x),
\end{align*}
\]

where the superscript \((ac)\) on the super Lagrangian density denotes that we have taken into account the anti-chiral superfields in the anti-chiral super Lagrangian density \(\mathcal{L}^{(ac)}_{(B, B)}(x, \tilde{\theta})\) which incorporates a combination of the ordinary fields and anti-chiral superfields with superscript \((b)\) that have been derived earlier in Eq. (34). We would like to point out that the ordinary fields are those which are trivially BRST invariant [cf. Eq. (13)]. Hence, they are independent of the Grassmannian variable. It is now straightforward to check that we have the following expression when we apply \(\partial_\theta\) on (51), namely;

\[
\frac{\partial}{\partial \theta} \left[ \mathcal{L}^{(ac)}_{(B, B)} \right] = -\partial_\mu \left[ m \varepsilon^{\mu \nu \kappa} \tilde{\phi}_\nu (\partial_\kappa C_\mu) + B_\nu (\partial^\mu C^\nu - \partial^\nu C^\mu) \right] \\
+ \frac{1}{2} B^\mu \lambda - B \left( \partial^\mu C - m C^\mu \right) - \frac{1}{2} (\partial^\mu \beta) \rho \equiv s_b \mathcal{L}_{(B, B)},
\]

which demonstrates that the r.h.s. is a total spacetime derivative that has been derived earlier in Eq. (15) due to the BRST transformation of the Lagrangian density \(\mathcal{L}_{(B, B)}\) in the ordinary space.

To capture the anti-BRST invariance of \(\mathcal{L}_{(B, B)}\) within the framework of ACSA, we perform the generalization: \(\mathcal{L}_{(B, B)} \rightarrow \mathcal{L}^{(c)}_{(B, B)}(x, \theta)\) as follows

\[
\begin{align*}
\mathcal{L}^{(c)}_{(B, B)}(x, \theta) &= \frac{1}{2} \tilde{B}_\mu(x) B^\mu(x) + B^\mu(x) \left( \frac{1}{2} \varepsilon^{\mu \nu \kappa} \partial^\nu \tilde{B}^{\nu \kappa}(x, \tilde{\theta}) \right) \\
&+ \frac{1}{2} \partial_\mu \tilde{\phi}(x) + m \tilde{\phi}(x) \right) - \frac{m^2}{4} \tilde{B}^{\nu \mu(ab)}(x, \theta) \tilde{B}^{\mu(ab)}(x, \theta) \\
&- \frac{1}{2} \partial^\mu \tilde{\phi}^{\mu(ab)}(x, \theta) \left( \partial_\mu \tilde{\phi}^{(ab)}(x, \theta) - \partial_\nu \tilde{\phi}^{(ab)}(x, \theta) \right).
\end{align*}
\]
density is made up of the ordinary Lagrangian densities generalizing the Lagrangian density and capturing the perfect anti-BRST invariance of the Lagrangian density $L$. To establish the existence of the CF-type restrictions [cf. Eq. (21)] and $(\bar{B}, c \bar{B}) \equiv \partial \bar{\theta}$ (that is characterized by a superscript $(c)$), we have: $\partial \bar{\theta} L^{(c)}_{(B, B)} \equiv s_{ab} L_{(B, B)}$. 

Keeping in mind the mapping $\partial \theta \leftrightarrow s_{ab}$ [4-6], it is straightforward to note that we have captured the perfect anti-BRST invariance of the Lagrangian density $L_{(B, B)}$ [as we have: $\partial \theta L^{(c)}_{(B, B)} \equiv s_{ab} L_{(B, B)}$].

To establish the existence of the CF-type restrictions [cf. Eq. (21)] and equivalence of the Lagrangian densities $L_{(B, B)}$ and $L_{(B, B)}$ w.r.t. the nilpotent symmetries $s_{(ab)}$, we generalize the Lagrangian density $L_{(B, B)}$ to its chiral counterpart: $L_{(B, B)} \rightarrow \tilde{L}^{(c)}_{(B, B)}$ as

$$
\tilde{L}^{(c)}_{(B, B)}(x, \theta) = \frac{1}{2} B^\mu(x) \, B^\mu(x) - B^\mu(x) \left( \frac{1}{2} \varepsilon_{\mu
u\kappa} \partial^\nu \tilde{B}^\nu(x, \theta) \right)
- \frac{1}{2} \partial^\mu \tilde{\varphi}(x) + m \tilde{\varphi}(x) \right) - \frac{m^2}{4} \tilde{B}^{\nu(ab)}(x, \theta) \tilde{B}^{\nu(ab)}(x, \theta)
$$

where we have obtained a chiral super Lagrangian density from the ordinary Lagrangian density $L_{(B, B)}$ [that is characterized by a superscript $(c)$]. This chiral super Lagrangian density is made up of the ordinary fields as well as the chiral superfields with superscript $(ab)$ that have been obtained in Eq. (40). Now we are in the position to apply a derivative $(\partial_\theta)$ w.r.t. the Grassmannian variable $\theta$ on (53). This operation leads to the following:

$$
\frac{\partial}{\partial \theta} \left[ \tilde{L}^{(c)}_{(B, B)} \right] = -\partial_\mu \left[ m \varepsilon^{\mu\nu\kappa} \tilde{\phi}_\nu \left( \partial_\eta \bar{C}_\kappa \right) - \bar{B}_\nu \left( \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu \right) \right]
+ \frac{1}{2} \tilde{B}^\mu \rho + \bar{B} \left( \partial^\mu \bar{C} - m \bar{C}^\mu \right) - \frac{1}{2} \left( \partial_\mu \tilde{\varphi} \right) \lambda \right] \equiv s_{ab} L_{(B, B)}. 
$$

Keeping in mind the mapping $\partial_\theta \leftrightarrow s_{ab}$ [4-6], it is straightforward to note that we have captured the perfect anti-BRST invariance of the Lagrangian density $L_{(B, B)}$ [as we have: $\partial_\theta L^{(c)}_{(B, B)} \equiv s_{ab} L_{(B, B)}$].

To establish the existence of the CF-type restrictions [cf. Eq. (21)] and equivalence of the Lagrangian densities $L_{(B, B)}$ and $L_{(B, B)}$ w.r.t. the nilpotent symmetries $s_{(ab)}$, we generalize the Lagrangian density $L_{(B, B)}$ to its chiral counterpart: $L_{(B, B)} \rightarrow \tilde{L}^{(c)}_{(B, B)}$ as
\[ - \frac{1}{2} \partial_\mu \tilde{\Phi}^{(ab)}(x, \theta) \left( \partial_\mu \tilde{\Phi}^{(ab)}(x, \theta) - \partial_\nu \tilde{\Phi}_{\mu}(x, \theta) \right) + m \tilde{B}^{(ab)}(x, \theta) \partial_\mu \tilde{\Phi}_{\nu}(x, \theta) + \frac{1}{4} \tilde{\Phi}_{\mu}(x) \tilde{\Phi}_{\mu}(x) + \frac{m}{2} \tilde{\Phi}(x) \tilde{\Phi}(x) + B^{(ab)}(x, \theta) \partial_\mu \tilde{\Phi}_{\nu}(x, \theta) + m \tilde{\Phi}_{\mu}(x, \theta) + \frac{1}{2} B^{(ab)}(x, \theta) B^{(ab)}(x, \theta) - \frac{1}{2} \tilde{\Phi}_{\mu}(x, \theta) \tilde{\Phi}_{\mu}(x, \theta) + \frac{1}{2} B(x, \theta) B(x, \theta) \left( \partial_\mu \tilde{\Phi}(x) + \frac{m}{2} \tilde{\Phi}(x) \right) + \left( \partial_\mu \tilde{F}^{(ab)}(x, \theta) - m \tilde{F}^{(ab)}(x, \theta) \right) \left( \partial_\mu \tilde{F}^{(ab)}(x, \theta) - m \tilde{F}^{(ab)}(x, \theta) \right) - \left( \partial_\mu \tilde{F}^{(ab)}(x, \theta) - \partial_\nu \tilde{F}^{(ab)}(x, \theta) \right) \left( \partial_\mu \tilde{F}^{(ab)}(x, \theta) \right) - \frac{1}{2} \partial_\mu \tilde{\beta}(x) \partial_\nu \tilde{\beta}^{(ab)}(x, \theta) + \frac{1}{2} m^2 \tilde{\beta}(x) \tilde{\beta}^{(ab)}(x, \theta) - \frac{1}{2} \left( \partial_\mu \tilde{F}^{(ab)}(x, \theta) + m \tilde{F}^{(ab)}(x, \theta) + \frac{1}{4} \rho(x) \right) \lambda(x) - \frac{1}{2} \left( \partial_\mu \tilde{F}^{(ab)}(x, \theta) + \frac{1}{4} \lambda(x) \right) \rho(x), \right) \tag{55} \]

where the superfields with the superscript \((ab)\) have been obtained earlier [cf. Eq. (40)] after the applications of anti-BRST invariant restrictions and superscript \((c)\) on \(\mathcal{L}_{(B,B)}\) denotes that we have taken the chiral generalization of the Lagrangian density \(\mathcal{L}_{(B,B)}\). It will be noted that there are superfields in Eq. (55) as the ordinary fields because they are anti-BRST invariant fields. It is now elementary to check that we have the following when \(\partial_\theta\) operates on \(\tilde{\mathcal{L}}^{(c)}_{(B,B)}\), namely;

\[
\frac{\partial}{\partial \theta} \left[ \tilde{\mathcal{L}}^{(c)}_{(B,B)} \right] = -\partial_\mu \left[ m \varepsilon^{\mu\nu\kappa} \tilde{\phi}_\nu (\partial_\nu \tilde{C}_\kappa) + \left( \partial_\nu B^{(ab)} + \frac{1}{2} \tilde{B}^{(ab)} + m \tilde{\phi}^{(ab)}(x) \right) \rho \right. + B_\nu \left( \partial_\mu \tilde{C}^{(ab)} - \partial^\mu \tilde{C}^{(ab)} \right) - B \left( \partial_\mu \tilde{C} - m \tilde{C}^{(ab)} \right) - \frac{1}{2} \left( \partial_\mu \tilde{\beta}^{(ab)}(x, \theta) \right) \lambda(x) \right] + \frac{1}{2} \left[ B_\mu + B_\mu + \partial_\mu \varphi \right] \left( \partial^\mu \rho \right) + \partial_\mu \left[ B_\nu + B_\nu + \partial_\nu \varphi \right] \left( \partial^\mu \tilde{C}^{(ab)} - \partial^\mu \tilde{C}^{(ab)} \right) + m \left[ B_\mu + \tilde{B}_\mu + \partial_\mu \varphi \right] \left( \partial^\mu \tilde{C} - m \tilde{C}^{(ab)} \right) - \frac{m}{2} \left[ B + \tilde{B} + m \varphi \right] \rho - \partial_\mu \left[ B + \tilde{B} + m \varphi \right] \left( \partial^\mu \tilde{C} - m \tilde{C}^{(ab)} \right) \equiv s_{ab} \mathcal{L}_{(B,B)}. \tag{56} \]

The above equation establishes (keeping in our mind \(\partial_\theta \leftrightarrow s_{ab}\)) that when the anti-BRST symmetry operates on the Lagrangian density \(\mathcal{L}_{(B,B)}\) in the ordinary spacetime, we obtain
the variation of $\mathcal{L}_{(B,B)}$ such that it transforms to a total spacetime derivative plus terms that vanish off in the space of fields where the CF-type restrictions [cf. Eq. (21)] are satisfied. Hence, the Lagrangian density $\mathcal{L}_{(B,B)}$ respects both the BRST and anti-BRST symmetry transformations together provided we consider the whole theory on the sub-space of fields (defined on the flat 4D Minkowskian spacetime manifold) on which the CF-type restrictions [cf. Eq. (21)] are fully satisfied together.

Now we capture the BRST invariance of the Lagrangian density $\mathcal{L}_{(B,B)}$ within the framework of ACSA. Toward this goal in mind, we generalize the ordinary work of ACSA. Keeping in our mind the mapping $\partial_\theta \to \theta$ and the variation of $\mathcal{L}$ satisfied. Hence, the Lagrangian density

$$\mathcal{L}_{(B,B)}(\bar{B}, \bar{B}, B, \bar{B}, \Phi, \bar{\Phi}, \theta, \bar{\theta})$$

onto the (4, 1)-dimensional anti-chiral supermanifold as the anti-chiral super Lagrangian density $\tilde{\mathcal{L}}^{(ac)}_{(B,B)}(x, \bar{\theta})$. The explicit and lucid form of $\tilde{\mathcal{L}}^{(ac)}_{(B,B)}(x, \bar{\theta})$ is

$$\tilde{\mathcal{L}}^{(ac)}_{(B,B)}(x, \bar{\theta}) = \frac{1}{2} \bar{B}_\mu(x) \bar{B}^\mu(x) + B^\mu(x) \left( \frac{1}{2} \varepsilon_{\mu
u\rho\kappa} \partial^\nu \bar{B}^{\rho\kappa}(x, \bar{\theta}) \right)$$

$$+ \frac{1}{2} \partial_\mu \bar{\varphi}(x) + m \bar{\varphi}_\mu(x) \right) - \frac{m^2}{4} \bar{B}^{\mu\nu}(x, \bar{\theta}) \bar{B}_{\mu\nu}(x, \bar{\theta})$$

$$- \frac{1}{2} \partial_{\mu} \bar{\Phi}_{\nu}^{(b)}(x, \bar{\theta}) \left( \partial_{\nu} \bar{\Phi}_{\mu}^{(b)}(x, \bar{\theta}) - \partial_{\nu} \tilde{\Phi}_{\mu}^{(b)}(x, \bar{\theta}) \right)$$

$$+ m \bar{B}^{\mu\nu}(x, \bar{\theta}) \partial_{\mu} \bar{\Phi}_{\nu}^{(b)}(x, \bar{\theta}) + \frac{1}{4} \tilde{\Phi}_{\mu\nu}(x) \tilde{\Phi}_{\mu\nu}(x)$$

$$+ \frac{m}{2} \varepsilon^{\mu\nu\rho\kappa} \bar{B}_{\mu\nu}(x, \bar{\theta}) \partial_{\rho} \bar{\phi}_{\kappa}(x) - \frac{1}{2} \tilde{B}_{\mu}^{(b)}(x, \bar{\theta}) \tilde{B}_{\mu}^{(b)}(x, \bar{\theta})$$

$$- \tilde{B}_{\mu}^{(b)}(x, \bar{\theta}) \left( \partial_{\mu} \tilde{\Phi}_{\nu}^{(b)}(x, \bar{\theta}) + \frac{1}{2} \partial_{\mu} \bar{\Phi}_{\nu}^{(b)}(x, \bar{\theta}) \right)$$

$$- \frac{1}{2} \bar{B}(x) \bar{B}(x) + B(x) \left( \partial_{\mu} \bar{\phi}_{\mu}(x) - \frac{m}{2} \bar{\varphi}(x) \right)$$

$$+ \left( \partial_{\mu} \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) - m \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) \right) \left( \partial_{\mu} \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) - m \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) \right)$$

$$- \left( \partial_{\mu} \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) - \partial_{\nu} \bar{F}_{\mu}^{(b)}(x, \bar{\theta}) \right) \left( \partial_{\mu} \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) \right)$$

$$- \frac{1}{2} \partial_{\mu} \bar{\beta}_{\nu}^{(b)}(x, \bar{\theta}) \partial_{\nu} \beta(x) + \frac{1}{2} m^2 \bar{\beta}_{\nu}^{(b)}(x, \bar{\theta}) \beta(x)$$

$$- \frac{1}{2} \left( \partial_{\mu} \bar{\beta}_{\nu}^{(b)}(x, \bar{\theta}) + \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) + \frac{1}{4} \rho(x) \right) \lambda(x)$$

$$- \frac{1}{2} \left( \partial_{\mu} \bar{\beta}_{\nu}^{(b)}(x, \bar{\theta}) + m \bar{F}_{\nu}^{(b)}(x, \bar{\theta}) \right) \lambda(x)$$

where the superscript $(b)$ on the superfields denotes that these anti-chiral superfields have been obtained after the applications of BRST-invariant restrictions [cf. Eq. (34)]. In the above super Lagrangian density $\tilde{\mathcal{L}}^{(ac)}_{(B,B)}(x, \bar{\theta})$, we have also 4D ordinary fields due to the fact that these fields are BRST-invariant. Keeping in our mind the mapping $\partial_{\bar{\theta}} \leftrightarrow s_b$ [4-6],

29
we operate \( \partial_b \) on the above anti-chiral super Lagrangian density that leads to:

\[
\frac{\partial}{\partial \theta} \left[ \tilde{L}^{(ac)}_{(B,B)} \right] = -\partial_{\mu} \left[ m \varepsilon^{\mu \nu \rho \kappa} \tilde{\phi}_\nu (\partial_\eta C_\kappa) - \left( \partial_\nu B^{\mu} - \frac{1}{2} B^\mu + m \phi^\mu \right) \lambda \right. \\
- B_\nu \left( \partial^\mu C^\nu - \partial^\nu C^\mu \right) + B \left( \partial^\mu C - m C^\mu \right) - \frac{1}{2} \left( \partial^\mu \beta \right) \rho \\
+ \frac{1}{2} \left[ B_\mu + \bar{B}_\mu + \partial_\mu \varphi \right] (\partial^\mu \lambda) \\
- \partial_\mu \left[ B_\nu + \bar{B}_\nu + \partial_\nu \varphi \right] (\partial^\mu C^\nu - \partial^\nu C^\mu) \\
- m \left[ B_\mu + \bar{B}_\mu + \partial_\mu \varphi \right] (\partial^\mu C - m C^\mu) - \frac{m}{2} \left[ B + \bar{B} + m \varphi \right] \lambda \\
+ \partial_\mu \left[ B + \bar{B} + m \varphi \right] (\partial^\mu C - m C^\mu) \equiv s_b \tilde{L}^{(B,B)}. \tag{58}
\]

Thus, we have captured the BRST symmetry transformation \( s_b \tilde{L}^{(B,B)} \) [cf. Eq. (25)] that has been derived [see, the r.h.s. of Eq. (25)] in the ordinary space (in the terminology of ACSA to BRST formalism).

We end this sub-section with the remarks that we have expressed the (anti-) BRST invariance [cf. Eqs. (56), (58)] within the framework of ACSA. Further, we have derived the CF-type restrictions: \( B_\mu + \bar{B}_\mu + \partial_\mu \tilde{\phi} = 0 \) and \( B + \bar{B} + m \tilde{\phi} = 0 \). First of all, we concentrate on the co-BRST and anti-co-BRST symmetry invariance of our 4D ordinary Lagrangian densities \( \tilde{L}^{(B,B)} \) and \( \tilde{L}^{(B,B)} \) w.r.t. the nilpotent (anti-)BRST symmetry transformations while expressing the transformations \( \left[ (s_b \tilde{L}^{(B,B)} \right) \) and \( \left( s_{ab} \tilde{L}^{(B,B)} \right) \) in the terminology of ACSA. In other words, we note that the r.h.s. of equations (56) and (58) would become total spacetime derivatives [cf. Eqs. (24),(25)] if and only if we impose the (anti-)BRST invariant CF-type restrictions (21) from outside.

### 5.2 (Anti-)co-BRST Invariance and CF-Type Restrictions: ACSA

In this sub-section, we prove the (anti-)co-BRST invariance of the Lagrangian densities \( \tilde{L}^{(B,B)} \) and \( \tilde{L}^{(B,B)} \) and derive the corresponding CF-type restrictions: \( B_\mu + \bar{B}_\mu + \partial_\mu \tilde{\phi} = 0 \) and \( B + \bar{B} + m \tilde{\phi} = 0 \). First of all, we concentrate on the co-BRST and anti-co-BRST symmetry invariance of our 4D ordinary Lagrangian densities \( \tilde{L}^{(B,B)} \) and \( \tilde{L}^{(B,B)} \), respectively. In this connection, we generalize the Lagrangian density \( \tilde{L}^{(B,B)} \) to its counterpart chiral super Lagrangian density: \( \tilde{L}^{(c,d)}_{(B,B)} \rightarrow \tilde{L}^{(c,d)}_{(B,B)}(x, \theta) \) as follows

\[
\tilde{L}^{(c,d)}_{(B,B)}(x, \theta) = \frac{1}{2} B_\mu(x) B^\mu(x) \\
- B_\mu(x) \left( \frac{1}{2} \varepsilon^{\mu \nu \rho \kappa} \partial^\nu \tilde{B}^{\rho \kappa}(x, \theta) - \frac{1}{2} \partial_\mu \tilde{\Phi}^{(d)}(x, \theta) + m \tilde{\Phi}^{(d)}_\mu(x, \theta) \right) \\
- \frac{m^2}{4} \tilde{B}^{\mu \nu}(x, \theta) \tilde{B}^{\mu \nu}(x, \theta) - \frac{1}{2} \partial^\mu \phi^\nu(x) \left( \partial_\mu \phi(x) - \partial_\nu \phi(x) \right) \\
+ m \tilde{B}^{\mu \nu}(x, \theta) \partial_\mu \phi(x) + \frac{1}{2} \partial^\mu \tilde{\Phi}^{(d)}(x, \theta) \left( \partial_\mu \tilde{\Phi}^{(d)}(x, \theta) - \partial_\nu \tilde{\Phi}^{(d)}(x, \theta) \right) \\
+ \frac{m}{2} \varepsilon^{\mu \nu \rho \kappa} \tilde{B}^{\mu \nu}(x, \theta) \partial_\eta \tilde{\Phi}^{(d)}_\kappa(x, \theta) - \frac{1}{2} B^\mu(x) B_\mu(x)
\]
\[ + B^\mu(x) \left( \partial^\nu \tilde{B}^{(d)}_{\mu\nu}(x, \theta) - \frac{1}{2} \partial_\mu \varphi(x) + m \phi_\mu(x) \right) + \frac{1}{2} B(x) B(x) \]
\[ + B(x) \left( \partial_\mu \phi^\mu(x) + \frac{m}{2} \varphi(x) \right) - \frac{1}{2} B(x) B(x) \]
\[ - B(x) \left( \partial^\mu \tilde{\Phi}^{(d)}_{\mu}(x, \theta) + \frac{m}{2} \tilde{\phi}^{(d)}(x, \theta) \right) \]
\[ + \left( \partial_\mu \tilde{F}^{(d)}_{\mu}(x, \theta) - m \tilde{F}^{(d)}_{\mu}(x, \theta) \right) \left( \partial^\mu \tilde{F}^{(d)}_{\nu}(x, \theta) - m \tilde{F}^{(d)}_{\nu}(x, \theta) \right) \]
\[ - \frac{1}{2} \partial_\mu \tilde{\beta}(x) \partial^\mu \tilde{\beta}^{(d)}(x, \theta) + \frac{1}{2} m^2 \tilde{\beta}(x) \tilde{\beta}^{(d)}(x, \theta) \]
\[ - \frac{1}{2} \left( \partial_\mu \tilde{F}^{(d)}_{\mu}(x, \theta) + m \tilde{F}^{(d)}_{\mu}(x, \theta) + \frac{1}{4} \rho(x) \right) \lambda(x) \]
\[ - \frac{1}{2} \left( \partial_\mu \tilde{F}^{(d)}_{\mu}(x, \theta) + m \tilde{F}^{(d)}_{\mu}(x, \theta) - \frac{1}{4} \lambda(x) \right) \tilde{\rho}(x), \quad (59) \]

where the superscript \((c, d)\) on the super Lagrangian density denotes that we have taken the \textit{chiral} superfields that have been derived after the applications of the co-BRST invariant restrictions [cf. Eq. (41)]. At this stage, keeping in our mind the mapping \( s_d \leftrightarrow \partial_\theta \) [4-6], we observe the operation of \( \partial_\theta \) on the super Lagrangian density \( \tilde{L}^{(c,d)} \) as:

\[
\frac{\partial}{\partial \theta} \left[ \tilde{L}^{(c,d)}(B,B) \right] = - \partial_\mu \left[ m \varepsilon^{\mu\nu\kappa} \phi_\nu (\partial_\eta \tilde{C}_\kappa) - B_\nu (\partial^\mu \tilde{C}^{\nu} - \partial^\nu \tilde{C}^\mu) + \frac{1}{2} B^\mu \rho \right]
\]
\[ + B \left( \partial^\mu \tilde{C} - m \tilde{C}^\mu \right) - \frac{1}{2} \left( \partial^\mu \tilde{\beta} \right) \lambda \equiv s_d \tilde{L}^{(B,B)}. \quad (60) \]

The above equation captures the co-BRST invariance of the Lagrangian density \( \tilde{L}^{(B,B)} \) in the \textit{ordinary} space because the total spacetime derivative is \textit{exactly} the same as the one we have derived in the \textit{ordinary} space [cf. Eq. (17)]. Hence, the action integral \( S = \int d^4x \tilde{L}^{(B,B)} \) would remain invariant for the physically well-defined fields which vanish off at infinity due to the sanctity of Gauss’s divergence theorem.

As far as the anti-co-BRST invariance of \( \tilde{L}^{(B,B)} \) is concerned [cf. Eq. (16)], we generalize this Lagrangian density to its counterpart \textit{anti-chiral} super Lagrangian density \( \tilde{L}^{(ac,ad)} \rightarrow \tilde{L}^{(ac,ad)}(x, \bar{\theta}) \) on the \((4, 1)\)-dimensional \textit{anti-chiral} super sub-manifold as follows

\[
\tilde{L}^{(ac,ad)}(x, \bar{\theta}) = \frac{1}{2} \tilde{B}_\mu(x) \tilde{B}^\mu(x) 
\]
\[ + \tilde{B}^{\mu}(x) \left( \frac{1}{2} \varepsilon^{\mu\nu\kappa} \partial^\nu \tilde{B}^{\eta\nu\kappa}(x, \bar{\theta}) \right) 
\]
\[ + \frac{1}{2} \partial_\mu \tilde{\phi}(ad)(x, \bar{\theta}) + m \tilde{\phi}(ad)(x, \bar{\theta}) \]
\[ - \frac{m^2}{4} \tilde{B}^{\mu\nu}(x, \bar{\theta}) \tilde{B}^{\mu\nu}(x, \bar{\theta}) - \frac{1}{2} \partial^\mu \phi^\nu(x) 
\]
\[ \left( \partial_\mu \phi^\nu(x) - \partial_\nu \phi^\mu(x) \right) + m \tilde{B}^{\mu\nu}(x, \bar{\theta}) \partial_\mu \phi^\nu(x) \]
follows chiral to its counterpart (anti-)co-BRST symmetry transformations [cf. Eqs. (18), (19)] within the framework of dimensional super sub-manifold leads to a total spacetime ordinary and super Lagrangian density \( \tilde{L} \) the superfields that have been incorporated into the Lagrangian density \( \tilde{L} \) where the superscript \((ac, ad)\) on the super anti-chiral Lagrangian density denotes that all the superfields that have been incorporated into the Lagrangian density \( \tilde{L}_{(B,B)} \) are the ones which have been derived after the applications of the anti-co-BRST invariant restrictions [cf. Eq. (50)]. It is crystal clear to note that we have

\[
\begin{align*}
\partial \left[ \tilde{L}_{(B,B)}^{(ac,ad)} \right] &= -\partial \left[ m \varepsilon^{ \mu \nu \kappa \lambda } \phi_\nu (\partial_\kappa C_\lambda ) + \bar{B}_\nu (\partial^\mu C^\nu - \partial^\nu C^\mu ) + \frac{1}{2} \bar{B}^\mu \lambda \\
&- \bar{B} (\partial^\mu C - m C^\mu ) + \frac{1}{2} (\partial^\mu \beta ) \rho \right] = s_{ad} \tilde{L}_{(B,B)},
\end{align*}
\]

where we take into account the mapping: \( \partial_\theta \leftrightarrow s_{ad} \). Thus, we conclude that the anti-chiral super Lagrangian density \( \tilde{L}_{(B,B)}^{(ac,ad)} \) is the sum of a unique combination of the anti-chiral superfields (obtained after the applications of the anti-co-BRST invariant restrictions) and ordinary 4D fields such that its translation along \( \theta \)-direction of the anti-chiral \((4, 1)\)-dimensional super sub-manifold leads to a total spacetime derivative in the ordinary space.

The stage is set now to derive the CF-type restrictions connected with the nilpotent (anti-)co-BRST symmetry transformations [cf. Eqs. (18), (19)] within the framework of ACSA to BRST formalism. In this connection, we generalize the Lagrangian density \( \tilde{L}_{(B,B)} \) to its counterpart chiral super Lagrangian density \( \tilde{L}_{(B,B)}^{(c,d)} \) [i.e. \( \tilde{L}_{(B,B)} \rightarrow \tilde{L}_{(B,B)}^{(c,d)}(x, \theta) \)] as follows

\[
\tilde{L}_{(B,B)}^{(c,d)}(x, \theta) = \frac{1}{2} \tilde{B}_\mu^\mu (x, \theta) \tilde{B}^\mu (x, \theta) + \bar{B}_\mu (x, \theta)
\]
\[
\begin{align*}
& \left( \frac{1}{2} \varepsilon_{\mu \nu \kappa} \partial^\nu \tilde{B}^{\nu \kappa}(x, \theta) + \frac{1}{2} \partial_\mu \tilde{\phi}^{(d)}(x, \theta) \right) + m \tilde{\phi}^{(d)}(x, \theta) - \frac{m^2}{4} \tilde{B}^{\mu \nu}(x, \theta) \tilde{B}^{\nu \mu}(x, \theta) \\
& - \frac{1}{2} \partial^\mu \phi^\nu(x) \left( \partial_\mu \phi_\nu(x) - \partial_\nu \phi_\mu(x) \right) + m \tilde{B}^{\mu \nu}(x, \theta) \partial_\theta \tilde{\phi}^{(d)}(x, \theta) \\
& + \frac{1}{2} \partial^\mu \tilde{\phi}^{(d)}(x, \theta) \left( \partial_\mu \tilde{\phi}^{(d)}(x, \theta) - \partial_\nu \tilde{\phi}^{(d)}(x, \theta) \right) \\
& + \frac{m}{2} \varepsilon^{\mu \nu \kappa} \tilde{B}^{(d)}_{\mu \nu}(x, \theta) \partial_\theta \tilde{\phi}^{(d)}(x, \theta) - \frac{1}{2} \tilde{B}^\mu(x) \tilde{B}^\mu(x) \\
& - B^\mu(x) \left( \partial^\nu \tilde{B}^{\nu \mu}(x, \theta) + \frac{1}{2} \partial_\mu \varphi(x) + m \phi_\mu(x) \right) \\
& + \frac{1}{2} \tilde{B}(x) \tilde{B}(x) - \tilde{B}(x) \left( \partial_\mu \phi^\mu(x) - \frac{m}{2} \varphi(x) \right) \\
& - \frac{1}{2} \tilde{B}^{(d)}(x, \theta) \tilde{B}^{(d)}(x, \theta) + \tilde{B}^{(d)}(x, \theta) \left( \partial_\mu \tilde{\phi}^{(d)}(x, \theta) \right) \\
& - \frac{m}{2} \tilde{\phi}^{(d)}(x, \theta) \left( \partial_\mu \tilde{F}^{(d)}(x, \theta) - m \tilde{F}^{(d)}_\mu(x, \theta) \right) \\
& + \left( \partial_\nu \tilde{F}^{\nu \mu}(x, \theta) - \partial_\mu \tilde{F}^{\nu \mu}(x, \theta) \right) \left( \partial^\nu \tilde{F}^{\nu \mu}(x, \theta) \right) \\
& - \frac{1}{2} \partial_\mu \tilde{B}^{(d)}(x, \theta) \partial_\nu \tilde{B}^{(d)}(x, \theta) + \frac{1}{2} m^2 \tilde{B}^{(d)}(x, \theta) \\
& - \frac{1}{2} \left( \partial_\mu \tilde{F}^{\mu \nu}(x, \theta) + m \tilde{F}^{(d)}(x, \theta) + \frac{1}{4} \varphi(x) \right) \lambda(x) \\
& - \frac{1}{2} \left( \partial_\mu \tilde{F}^{\mu \nu}(x, \theta) + m \tilde{F}^{(d)}(x, \theta) - \frac{1}{4} \lambda(x) \right) \rho(x), \\
& \text{where the superscript } (c, d) \text{ denotes that the chiral super Lagrangian density } \tilde{\mathcal{L}}^{(c,d)}_{(B, \tilde{B})} \text{ incorporates the chiral superfields [cf. Eq. (45)] derived after the imposition(s) of the co-BRST invariant restriction(s) [cf. Eq. (42)] and the ordinary 4D fields. These ordinary 4D fields are nothing but the trivially co-BRST invariant fields [cf. Eq. (19)]. Keeping in our mind the mapping } \partial_\theta \leftarrow s_d, \text{ we observe the following relationship is true, namely;}
\end{align*}
\]

\[
\begin{align*}
& \frac{\partial}{\partial \theta} \left[ \tilde{\mathcal{L}}^{(c,d)}_{(B, \tilde{B})} \right] = -\partial_\mu \left[ m \varepsilon^{\mu \nu \kappa} \phi_\nu \left( \partial_\kappa \tilde{C} + \sigma_\rho \right) - \left( \frac{1}{2} \varepsilon^{\mu \nu \kappa} \partial_\nu B_{\eta \kappa} - \frac{1}{2} B^\mu + m \tilde{\phi}^\mu \right) \rho \right] \\
& + \tilde{B}_\nu \left( \partial^\nu \tilde{C} - \partial^\nu \tilde{C} \right) - \tilde{B} \left( \partial^\nu \tilde{C} - m \tilde{C} \right) - \frac{1}{2} \left( \partial^\nu \tilde{B}^\nu \right) \lambda \\
& + \frac{1}{2} \left[ B_{\mu} + \tilde{B}_\mu + \partial_\mu \tilde{\phi} \right] \left( \partial^\nu \phi^{\nu} \right) + \partial_\mu \left[ B_{\nu} + \tilde{B}_\nu + \partial_\nu \tilde{\phi} \right] \left( \partial^\nu \tilde{C} - \partial^\nu \tilde{C} \right) \\
& + m \left[ B_{\mu} + \tilde{B}_\mu + \partial_\mu \tilde{\phi} \right] \left( \partial^\nu \tilde{C} - m \tilde{C} \right) - \frac{m}{2} \left[ B + \tilde{B} + m \tilde{\phi} \right] \rho \\
& - \partial_\mu \left[ B + \tilde{B} + m \tilde{\phi} \right] \left( \partial^\nu \tilde{C} - m \tilde{C} \right) \equiv s_d \mathcal{L}_{(\tilde{B}, \tilde{B})}. \\
& \text{(64)}
\end{align*}
\]
Thus, we note that we have captured the variation $s_d \mathcal{L}_{(\bar{B}, B)}$ [cf. Eq. (28)] in the terminology of the ACSA to BRST formalism. It is crystal clear that if we impose the CF-type restrictions: $B_\mu + \bar{B}_\mu + \partial_\mu \bar{\phi} = 0, B + \bar{B} + m \bar{\phi} = 0$ from outside, we obtain $s_d \mathcal{L}_{(\bar{B}, B)}$ as a total spacetime derivative [cf. Eq. (25)]. It is straightforward to check the invariance $s_{ad} \mathcal{L}_{(B, B)}$ on exactly similar lines as given in Eqs. (63) and (64). We perform this exercise concisely in our Appendix A to complement the write-up in our present sub-section.

We wrap up this sub-section with the remarks that we have captured the (anti-)co-BRST invariance [cf. Eqs. (18), (19)] as well as we have established the existence of the CF-type restrictions: $B_\mu + \bar{B}_\mu + \partial_\mu \bar{\phi} = 0$ and $B + \bar{B} + m \bar{\phi} = 0$ on our theory. In fact, it is straightforward to note that if our whole theory is considered on the space of fields (in the 4D Minkowskian flat spacetime manifold) where the CF-type restrictions: $B_\mu + \bar{B}_\mu + \partial_\mu \bar{\phi} = 0$ and $B + \bar{B} + m \bar{\phi} = 0$ are satisfied, then, both the Lagrangian densities $\mathcal{L}_{(B, B)}$ and $\mathcal{L}_{(\bar{B}, B)}$ would respect both the nilpotent symmetries (i.e. co-BRST and anti-co-BRST) together. In other words, we shall have $s_d \mathcal{L}_{(B, B)}$ and $s_{ad} \mathcal{L}_{(B, B)}$ as the total spacetime derivatives [cf. Eqs. (16), (17)] as well as the transformations $s_d \mathcal{L}_{(B, B)}$ and $s_{ad} \mathcal{L}_{(B, B)}$ would also turn out to be the total spacetime derivatives [cf. Eqs. (26), (27)]. In a subtle manner, the observations in Eqs. (56) and (64) establish the existence of the CF-type restrictions which are the hallmarks of a BRST quantized gauge theory.

6 Conserved Charges and Their Nilpotency and Absolute Anticommutativity Properties: ACSA

In this section, first of all, we derive the conserved Noether currents, corresponding conserved charges and prove their off-shell nilpotency as well as absolute anticommutativity properties (in the ordinary space) using the BRST formalism. We corroborate the above properties and provide their proof within the framework of ACSA to BRST formalism, too. The proof of the absolute anticommutativity property of the (anti-)BRST as well as (anti-)co-BRST conserved charges is a novel as well as surprising result in the sense that we have taken into account only the (anti-)chiral super expansions of all the appropriate superfields. Our present section is divided into three sub-sections as illustrated below:

6.1 Conserved Currents and Charges: Ordinary Space

According to Noether’s theorem, the invariance of the action integrals, corresponding to the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries, leads to the derivation of the conserved currents. We note, in this connection, that the action integral, corresponding to the Lagrangian density $\mathcal{L}_{(B, B)}$, remains perfectly invariant [cf. Eqs. (15), (17)] under the BRST and co-BRST symmetry transformations (without any use of the CF-type restrictions and/or EL-EOMs). Hence, the conserved BRST and co-BRST Noether currents (corresponding to the Lagrangian density $\mathcal{L}_{(B, B)}$) are:

$$J^\mu_b = \varepsilon^{\mu
u\kappa\lambda} \left( m \bar{\phi}_\nu - B_\nu \right) \left( \partial_\kappa C_\lambda \right) + \left( m B^{\mu\nu} - \Phi^{\mu\nu} \right) \left( \partial_\nu C - m C_\nu \right)$$

$$+ B \left( \partial^\mu C - m C^\mu \right) + m \beta \left( \partial^\mu \bar{C} - m \bar{C}^\mu \right) - \left( \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu \right) \left( \partial_\nu \beta \right)$$
The conservation law \( (\partial_{\mu} J^\mu_{(r)}) = 0, r = b, d \) can be proven by taking into account the EL-EOM corresponding to the Lagrangian density \( L_{(B,B)} \) [cf. Eq. (67) below]. These conserved currents lead to the derivation of the conserved and nilpotent charges \( (Q_b = \int d^3 x \; J^0_b, \; Q_d = \int d^3 x \; J^0_d) \) as:

\[
Q_b = \int d^3 x \left[ \epsilon^{ijk} (m \phi_i - B_i) \partial_j \bar{C}_k + \left( m B^{0i} - \Phi^{0i} \right) (\partial_i C - m C_i) \right. \\
- B_i (\partial^i C^j - \partial^j C^i) + B (\partial^0 C - m C^0) + m \beta (\partial^0 \bar{C} - m \bar{C}^0) \\
- \left( \partial_i \beta \right) (\partial^i \bar{C}^j - \partial^j \bar{C}^i) + \frac{1}{2} \left( \partial^0 \beta \right) \rho - \frac{1}{2} B^0 \lambda, \\
Q_d = \int d^3 x \left[ \epsilon^{ijk} (m \phi_i - B_i) \partial_j \bar{C}_k + \frac{m}{2} \epsilon^{ijk} B_{jk} + \tilde{\Phi}^{0i} \right) (\partial_i C - m C_i) \\
+ B_i (\partial^i C^j - \partial^j C^i) - B (\partial^0 C - m C^0) - m \bar{\beta} (\partial^0 \bar{C} - m \bar{C}^0) \\
+ \left( \partial_i \bar{\beta} \right) (\partial^i \bar{C}^j - \partial^j \bar{C}^i) + \frac{1}{2} \left( \partial^0 \bar{\beta} \right) \lambda - \frac{1}{2} B^0 \rho. \\
\]

The above conserved charges can be expressed in different (but equivalent) forms by using the following EL-EOMs that are derived from \( L_{(B,B)} \), namely:

\[
\epsilon^{\eta \kappa \mu} \partial_{\mu} B_{\eta} + m^2 \left( B^{\eta \kappa} - \frac{1}{m} \Phi^{\eta \kappa} \right) - \frac{1}{2m} \epsilon^{\mu \nu \kappa} \Phi_{\mu \nu} \right) + \left( \partial^\eta B^\kappa - \partial^\kappa B^\eta \right) = 0, \\
\epsilon^{\eta \kappa \mu} \partial_{\mu} B_{\nu} + \frac{m^2}{2} \epsilon^{\mu \kappa \nu} \left( B_{\mu \nu} - \frac{1}{m} \Phi_{\mu \nu} - \frac{1}{2m} \epsilon_{\mu \nu \zeta \sigma} \Phi_{\zeta \sigma} \right) \\
- \left( \partial^\eta B^\kappa - \partial^\kappa B^\eta \right) = 0, \quad \partial_{\mu} \Phi_{\mu \nu} - m \left( \partial_{\mu} B_{\nu} - B^\nu \right) - \partial^\nu B = 0, \\
\partial_{\mu} B^\mu + m B = 0, \quad \partial_{\mu} B^\mu + m B = 0, \quad \square \bar{C} - m \left( \partial_{\mu} \bar{C}^\mu + \frac{B^0}{2} \right) = 0, \\
\partial_{\mu} \bar{\Phi}^\mu + m \left( - \frac{1}{2} \epsilon^{\mu \nu \kappa} \partial_{\mu} B_{\nu \kappa} + B^\nu \right) - \partial^\nu B = 0, \quad \left( \square + m^2 \right) \beta = 0, \\
\lambda = 2 \left( \partial_{\mu} C^\mu + m C \right), \quad \rho = -2 \left( \partial_{\mu} \bar{C}^\mu + m \bar{C} \right), \\
\square \bar{C} - m \left( \partial_{\mu} \bar{C}^\mu - \frac{\lambda}{2} \right) = 0, \quad \left( \square + m^2 \right) C^\mu - \partial_{\mu} \left( \partial_{\nu} C^\nu + m C - \frac{\lambda}{2} \right) = 0, \\
\left( \square + m^2 \right) \bar{\beta} = 0, \quad \left( \square + m^2 \right) \bar{C}^\mu - \partial_{\mu} \left( \partial_{\nu} \bar{C}^\nu + m \bar{C} + \frac{\rho}{2} \right) = 0.
\]

As an additional remark, we mention here that (with \( \epsilon^{0ijk} = \epsilon^{ijk} \) as 3D Levi-Civita tensor) the above equations (i.e. EL-EOMs) are useful in the proof of the conservation \( (\partial_{\mu} J^\mu_{(r)}) = 0, r = b, d \) law, too.

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In exactly similar fashion, we note that the Lagrangian density $\mathcal{L}_{(B,\bar{B})}$ respects perfect [cf. Eqs. (14), (16)] anti-BRST and anti-co-BRST symmetries in the sense that the corresponding action integral remains invariant (without any use of CF-type restrictions). As a consequence, we have the following:

$$J_{ab}^{\mu} = \varepsilon^{\mu\nu\rho} \left( m \dot{\phi}_v + \bar{B}_v \right) \left( \partial_\nu \bar{C}_\rho \right) + \left( m \bar{B}^{\mu\nu} - \Phi^{\mu\nu} \right) \left( \partial_\nu \bar{C} - m \bar{C}_\nu \right)$$

$$- \bar{B} \left( \partial^\mu \bar{C} - m \bar{C}^\mu \right) - m \beta \left( \partial^\mu \bar{C} - m \bar{C}^\mu \right) + \left( \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu \right) \left( \partial_\nu \beta \right)$$

$$+ \bar{B}_v \left( \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu \right) + \frac{1}{2} \left( \partial^\mu \beta \right) \lambda - \frac{1}{2} \bar{B}^\mu \rho,$$

$$J_{ad}^{\mu} = \varepsilon^{\mu\nu\rho} \left( m \phi_v + \bar{B}_v \right) \left( \partial_\nu C_\rho \right) + \left( \frac{m}{2} \varepsilon^{\mu\nu\rho} B_{\eta\kappa} + \bar{\Phi}^{\mu\nu} \right) \left( \partial_\nu C - m \bar{C}_\nu \right)$$

$$+ \bar{B} \left( \partial^\mu C - m \bar{C}^\mu \right) - m \beta \left( \partial^\mu \bar{C} - m \bar{C}^\mu \right) + \left( \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu \right) \left( \partial_\nu \beta \right)$$

$$- \bar{B}_v \left( \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu \right) - \frac{1}{2} \lambda \left( \partial^\mu \beta \right) \rho - \frac{1}{2} \bar{B}^\mu \lambda. \quad (68)$$

The conservation law $(\partial_\mu J_{(r)}^{\mu} = 0, r = ab, ad)$ can be proven by using the following EL-EOMs [besides the ones quoted in Eq. (23)], namely:

$$\varepsilon^{\eta\mu\nu} \partial_\mu B_\nu - \frac{m^2}{2} \varepsilon^{\eta\mu\nu} \left( B_{\nu\mu} - \frac{1}{m} \Phi_{\nu\mu} - \frac{1}{2m} \varepsilon^{\eta\mu\xi\sigma} \Phi_{\xi\sigma} \right) + \frac{1}{2} \left( \partial^\eta \bar{B}^\kappa - \partial^\kappa \bar{B}^\eta \right) = 0,$$

$$\varepsilon^{\eta\mu\nu} \partial_\mu \bar{B}_\nu - \frac{m^2}{2} \varepsilon^{\eta\mu\nu} \left( B_{\nu\mu} - \frac{1}{m} \Phi_{\nu\mu} - \frac{1}{2m} \varepsilon^{\eta\mu\xi\sigma} \Phi_{\xi\sigma} \right)$$

$$- \left( \partial^\eta B^\kappa - \partial^\kappa B^\eta \right) = 0, \quad \partial_\mu \Phi_{\nu\mu} - m \left( \partial_\mu B^{\nu\mu} + \bar{B}^{\nu} \right) + \partial^\nu \bar{B} = 0, \quad \partial_\mu \bar{B}^{\mu} - \frac{1}{2} \varepsilon^{\mu\nu\rho} \partial_\mu B_{\nu\rho} + \bar{B}^{\nu} \right) + \partial^\nu \bar{B} = 0, \quad \partial_\mu \bar{B}^{\mu} + m \bar{B} = 0. \quad (69)$$

The conserved $(Q_{ad} = \dot{Q}_{ab} = 0)$ and nilpotent $(Q_{ad}^2 = Q_{ab}^2 = 0)$ charges $(Q_{ab}, Q_{ad})$ that emerge out from the above conserved currents are:

$$Q_{ab} = \int d^3 x \left[ \varepsilon^{ijk} \left( m \dot{\phi}_i + \bar{B}_i \right) \partial_j \bar{C}_k + \left( m B^{0i} - \Phi^{0i} \right) \left( \partial_j \bar{C} - m \bar{C}_j \right) \right]$$

$$+ \bar{B}_i \left( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 \right) - \bar{B} \left( \partial^0 \bar{C} - m \bar{C}^0 \right) - m \beta \left( \partial^0 \bar{C} - m \bar{C}^0 \right)$$

$$+ \left( \partial_i \beta \right) \left( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 \right) + \frac{1}{2} \left( \partial^0 \beta \right) \lambda - \frac{1}{2} \bar{B}^0 \rho \right],$$

$$Q_{ad} = \int d^3 x \left[ \varepsilon^{ijk} \left( m \phi_i + \bar{B}_i \right) \partial_j C_k + \left( \frac{m}{2} \varepsilon^{ijk} B_{jk} + \bar{\Phi}^{0i} \right) \left( \partial_j C - m C_j \right) \right]$$

$$- \bar{B}_i \left( \partial^0 C^i - \partial^i C^0 \right) + \bar{B} \left( \partial^0 C - m C^0 \right) - m \beta \left( \partial^0 C - m C^0 \right)$$

$$+ \left( \partial_i \beta \right) \left( \partial^0 C^i - \partial^i C^0 \right) - \frac{1}{2} \left( \partial^0 \beta \right) \rho - \frac{1}{2} \bar{B}^0 \lambda \right]. \quad (70)$$

Thus, we have derived the (anti-)BRST $(Q_{(a)b})$ and (anti-)co-BRST $(Q_{(a)d})$ conserved charges [cf. Eqs. (66), (70)] from the perfect invariance of the action integrals corresponding to the Lagrangian densities $\mathcal{L}_{(B,\bar{B})}$ and $\mathcal{L}_{(\bar{B},\bar{B})}$. In the above, we have denoted the totally antisymmetric 3D Levi-Civita tensor as $\varepsilon^{ijk} \equiv \varepsilon^{0ijk}$ (with only space indices $i, j, k = 1, 2, 3$).
It is interesting that the above conserved charges $Q_{(a)b}$ and $Q_{(a)d}$ can be expressed in their equivalent and useful forms for our further discussions. For instance, using the EOMs (23) and (67), the BRST charge $(Q_b)$ can be re-written as:

$$Q_b^{(1)} = \int d^3x \left[ 2 \epsilon^{ijk} (m \dot{\phi}_i - B_i) \partial_j C_k + 2 (m B^{0i} - \Phi^{0i}) (\partial_i C - m C_i) \right. $$

$$- B_i (\partial^0 C^i - \partial^0 C^0) + B (\partial^0 C - m C^0) + m \beta (\partial^0 \dot{C} - m \dot{C}^0) $$

$$+ C (\partial^0 B - m B^0) - (\partial_i \beta) (\partial^0 \dot{C}^i - \partial^i \dot{C}^0) - C_i (\partial^0 B^i - \partial^i B^0) $$

$$+ \frac{1}{2} (\partial^0 \beta) \rho - \frac{1}{2} B^0 \lambda, $$

$$Q_b^{(2)} = \int d^3x \left[ B \dot{C} - B C + B_i C^i - B_i \dot{C}^i - m (\beta \dot{C} - \dot{\beta} C) + (\partial_i \beta) \dot{C}^i $$

$$- (\partial_i \dot{\beta}) \dot{C}^i + \dot{\beta} \rho - \beta \dot{\rho} \right].$$

The above expressions are very interesting for us because they can be expressed in a BRST-exact form as follows:

$$Q_b^{(1)} = \int d^3x \left[ - \epsilon^{ijk} (m \dot{\phi}_i - B_i) B_{jk} + 2 (m B^{0i} - \Phi^{0i}) \phi_i + B_i B^{0i} \right. $$

$$+ B \phi_0 + (\partial^0 C - m C^0) C - (\partial_0 \dot{C}_i - \partial_i \dot{C}_0) C^i - \frac{1}{2} \dot{\beta} \beta - \frac{1}{2} C_0 \lambda, $$

$$Q_b^{(2)} = \int d^3x \left[ C \dot{C} - \dot{C} C + \dot{C}_i C^i - \dot{C}_i \dot{C}^i + \beta \dot{\beta} - \dot{\beta} \beta \right].$$

As a consequence of the above observations, we prove the off-shell nilpotency of the above charges in a straightforward fashion because we note that:

$$s_b Q_b^{(1)} = -i \{Q_b^{(1)}, Q_b^{(1)}\} = 0 \iff s_b^2 = 0 \iff [Q_b^{(1)}]^2 = 0,$$

$$s_b Q_b^{(2)} = -i \{Q_b^{(2)}, Q_b^{(2)}\} = 0 \iff s_b^2 = 0 \iff [Q_b^{(2)}]^2 = 0.$$

Hence, we have proven the off-shell nilpotency property of the BRST charges. We note here that the off-shell nilpotency ($s_b^2 = 0$) of the BRST symmetry transformations ($s_b$) and off-shell nilpotency ($[Q_b^{(1,2)}]^2 = 0$) of the conserved BRST charges ($Q_b^{(1,2)}$) are deeply related with each-other [cf. Eq. (73)].

We go a step further and, once again, using the EOMs (23) and (67), we obtain another equivalent form of the BRST charge as:

$$Q_b^{(3)} = \int d^3x \left[ B\dot{C} - \dot{B}C + B_i \dot{C}^i - B_i \dot{C}^i + \frac{1}{2}(\dot{\beta} \rho - \beta \dot{\rho}) \right].$$

Using the appropriate CF-type restrictions: $B_i + \dot{B}_i + \partial_i \varphi = 0$, $B + \dot{B} + m \varphi = 0$ as well as some appropriate EL-EOM (67), we can recast the above BRST charge $Q_b^{(3)}$ in a very interesting form (see, e.g. Appendix B below) as:

$$Q_b^{(3)} = \int d^3x \left[ \dot{B} C - B \dot{C} + B_i \dot{C}^i - B_i \dot{C}^i + \frac{1}{2}(\dot{\beta} \rho - \beta \dot{\rho}) + \frac{1}{2}(\dot{\varphi} \lambda - \varphi \dot{\lambda}) \right].$$
To be precise, we have used the EL-EOMs: $\partial_\mu C^\mu + m C = \frac{1}{2}, (\Box + m^2) \varphi = (\Box + m^2) C_\mu = 0$ which emerge out from $L_{(B, \bar{B})}$ as well as $L_{(\bar{B}, B)}$. This happens because of the fact that the ghost part of the coupled (but equivalent) Lagrangian densities [cf. Eqs. (10), (11)] is the same. The above expression can be written as an anti-BRST exact expression because we have the following explicit form of the BRST charge, namely:

$$Q_b^{(3)} = \int d^3 x \ s_{ab} \left[ \hat{C} C + C_\mu \hat{C}^\mu + \frac{1}{2} (\hat{\beta} \varphi - \beta \varphi) \right], \quad (76)$$

where $s_{ab}$ stands for the anti-BRST symmetry transformations (12) under which the Lagrangian density $L_{(\bar{B}, B)}$ has perfect invariance [cf. Eq. (14)] without any use of the CF-type restrictions and/or EL-EOMs.

We now concentrate on the anti-BRST charge $Q_{ab}$ [cf. Eq. (70)] which has been derived from the perfect anti-BRST symmetry of $L_{(\bar{B}, B)}$. Using the EL-EOMs (23) and (69), we observe that $Q_{ab}$ can be re-expressed as follows:

$$Q_{ab}^{(1)} = \int d^3 x \left[ 2 \varepsilon^{ijk} (m \tilde{\phi}_i + \bar{B}_i) \partial_j \tilde{C}_k + 2 (m B^{0i} - \Phi^{0i}) (\partial_i \bar{C} - m \tilde{C}_i) + \bar{B}_i (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) - \tilde{B} \left( \partial^0 \tilde{C} - m \tilde{C}^0 \right) \right] + \frac{1}{2} (\partial^0 \bar{\beta}) \lambda - \frac{1}{2} B^0 \rho]
$$

$$Q_{ab}^{(2)} = \int d^3 x \left[ \tilde{B} \bar{C} - \tilde{B} \tilde{C} + B_\mu \tilde{C}^\mu - \tilde{B}_\mu \bar{C}^\mu + m (\tilde{\beta} \dot{C} - \hat{\beta} C) \right]$$

The above expressions are very interesting because they can be re-written in the following exact form w.r.t. the anti-BRST symmetry $s_{ab}$ as:

$$Q_{ab}^{(1)} = \int d^3 x \ s_{ab} \left[ - \varepsilon^{ijk} (m \tilde{\phi}_i + \bar{B}_i) B_{jk} + 2 (m B^{0i} - \Phi^{0i}) \phi_i - \bar{B}_i B^{0i} - \tilde{B} \phi_0 + \tilde{C} \left( \partial^0 \tilde{C} - m \tilde{C}^0 \right) - \tilde{C}_i \left( \partial^0 C^i - \partial^i C^0 \right) - \frac{1}{2} \tilde{\beta} \beta - \frac{1}{2} C_0 \rho \right]
$$

$$Q_{ab}^{(2)} = \int d^3 x \ s_{ab} \left[ \hat{C} C - C \tilde{C} + C_\mu \tilde{C}^\mu - \hat{C}_\mu C^\mu + \tilde{\beta} \hat{\beta} - \beta \bar{\beta} \right]. \quad (78)$$

Now it is straightforward to note that we have

$$s_{ab} Q_{ab}^{(1)} = - i \{ Q_{ab}^{(1)}, Q_{ab}^{(1)} \} = 0 \quad \iff \quad s_{ab}^2 = 0 \quad \iff \quad [Q_{ab}^{(1)}]^2 = 0,$n$$

$$s_{ab} Q_{ab}^{(2)} = - i \{ Q_{ab}^{(2)}, Q_{ab}^{(2)} \} = 0 \quad \iff \quad s_{ab}^2 = 0 \quad \iff \quad [Q_{ab}^{(2)}]^2 = 0, \quad (79)$$

where the anti-BRST symmetry transformations ($s_{ab}$) have been quoted in their full blaze of glory in Eq. (12). Thus, we observe that the off-shell nilpotency $s_{ab}^2 = 0$ of the anti-BRST symmetry transformations as well as the anti-BRST charge $Q_{ab}^{(1,2)}$ are deeply inter connected. We further note that, using the CF-type restrictions $B_\mu + \bar{B}_\mu + \partial_\mu \phi = 0, B + \bar{B} + m \phi = 0$, we can also recast the anti-BRST charge in an exact form w.r.t. the BRST.
symmetry transformations \((s_b)\). For this purpose, first of all, we have an equivalent form of the anti-BRST charge as follows:

\[
Q_{ab}^{(3)} = \int d^3x \left[ \hat{B} \hat{C} - \hat{B} \hat{C} + B_\mu \hat{C}^\mu - \hat{B}_\mu \hat{C}^\mu + \frac{1}{2} (\hat{\beta} \lambda - \hat{\beta} \hat{\lambda}) \right].
\]  

(80)

As argued earlier, the above expression can be re-written, using the appropriate CF-type restrictions [cf. Eq. (21)] as well as some appropriate EL-EOM derived from \(L_{(B, B)}\) and/or \(L_{(B, \bar{B})}\) [cf. Eqs. (67), (69)] as a BRST-exact quantity, namely;

\[
Q_{ab}^{(3)} = \int d^3x \left[ B \hat{C} - \hat{B} \hat{C} + B_\mu \hat{C}^\mu - \hat{B}_\mu \hat{C}^\mu + m(\varphi \hat{C} - \varphi \hat{C}) \right.
\]

\[
+ \left. (\partial_\mu \varphi) \hat{C}^\mu - (\partial_\mu \varphi) \hat{C}^\mu + \frac{1}{2} (\hat{\beta} \lambda - \hat{\beta} \hat{\lambda}) \right].
\]

\[
\equiv \int d^3x \, s_b \left[ \hat{C} \hat{C} + \hat{C} \hat{C} + \frac{1}{2} (\hat{\beta} \varphi - \hat{\beta} \hat{\varphi}) \right],
\]

(81)

where the transformations \((s_b)\) are the off-shell nilpotent \((s_b^2 = 0)\) BRST symmetry transformations quoted in Eq. (13) of our Sec. 3.

At this juncture, we dwell a bit on the equivalent forms of the co-BRST charge \(Q_d\) by exploiting the potential and power of EL-EOMs (23) and (67) that have been derived from the Lagrangian density \(L_{(B, B)}\). It is straightforward to note that we have the following equivalent forms of \(Q_d\) due to the EL-EOM given in (67), namely;

\[
Q_{d1}^{(1)} = \int d^3x \left[ 2e^{ijk} (m \phi_i - B_i) \partial_j \hat{C}_k + 2 \left( \frac{m}{2} e^{ijk} B_{jk} + \tilde{\phi}_0 \right) (\partial_i \hat{C} - m \hat{C}_i) \right.
\]

\[
- \left. B (\partial^0 \hat{C} - m C^0) - m \beta (\partial^0 C - m C^0) - \hat{C} (\partial^0 B - m B^0) \right]
\]

\[
+ \left( \partial_\mu \beta \right) (\partial^0 C^i - \partial^i C^0) + \hat{C}_i (\partial^0 B^i - \partial^i B^0) + \frac{1}{2} (\partial^0 \hat{\beta} \lambda
\]

\[
- \frac{1}{2} B^0 \rho + \hat{B}_i \left( \partial^0 \hat{C}^i - \partial^i \hat{C}^0 \right). \right]
\]

\[
Q_{d2}^{(2)} = \int d^3x \left[ B \hat{C} - \hat{B} \hat{C} + B_\mu \hat{C}^\mu - \hat{B}_\mu \hat{C}^\mu + m (\beta \hat{C} - \beta \hat{C}) \right.
\]

\[
+ \left. (\partial_\mu \beta) \hat{C}^\mu - (\partial_\mu \beta) \hat{C}^\mu - \beta \lambda - \beta \hat{\lambda} \right].
\]

(82)

It is very interesting to point out that both the above forms of charges can be precisely re-written in the exact forms w.r.t. the off-shell nilpotent co-BRST symmetry transformations \((s_d)\), namely;

\[
Q_{d1}^{(1)} = \int d^3x \, s_d \left[ - 2 (m \phi_i - B_i) B^0_i + 2 \left( \frac{m}{2} e^{ijk} B_{jk} + \tilde{\phi}_0 \right) \hat{\phi}_i \right.
\]

\[
- \left. \frac{1}{2} \beta \beta + \frac{1}{2} e^{ijk} B_i B_{jk} - B \tilde{\phi}_0 + \tilde{C} (\partial^0 C - m C^0) \right]
\]

\[
- \hat{C}_i (\partial^0 C^i - \partial^i C^0) - \frac{1}{2} C_0 \rho \right],
\]

\[
Q_{d2}^{(2)} = \int d^3x \, s_d \left[ \hat{C} \hat{C} - \hat{C} \hat{C} + \hat{C}_\mu \hat{C}^\mu - \hat{C}_\mu \hat{C}^\mu + \beta \hat{\beta} - \beta \hat{\beta} \right].
\]

(83)

\footnote{To be precise, we have used the equations of motion: \(\partial_\mu \hat{C}^\mu + m \hat{C} = -\frac{\phi}{2} (\Box + m^2) \hat{C}_\mu = (\Box + m^2) \phi = 0\) which emerge out from \(L_{(B, B)}\) and/or \(L_{(\bar{B}, \bar{B})}\) as the EL-EOMs.}
In other words, we have been able to express \( Q_d^{(1,2)} \) in the co-exact form w.r.t. the co-BRST symmetry transformation \( (s_d) \). As a consequence, we have:

\[
\begin{align*}
\text{s}_d Q_d^{(1)} &= -i \{ Q_d^{(1)}, Q_d^{(1)} \} = 0 \quad \Leftrightarrow \quad s_d^2 = 0 \quad \Leftrightarrow \quad [Q_d^{(1)}]^2 = 0, \\
\text{s}_d Q_d^{(2)} &= -i \{ Q_d^{(2)}, Q_d^{(2)} \} = 0 \quad \Leftrightarrow \quad s_d^2 = 0 \quad \Leftrightarrow \quad [Q_d^{(2)}]^2 = 0. \quad (84)
\end{align*}
\]

Thus, the off-shell nilpotency \( (s_d^2 = 0) \) of the co-BRST symmetry \( (s_d) \) and corresponding charges \( Q_d^{(1,2)} \) are deeply interconnected. Using the EL-EOMs (23) and (67), it can be checked that the co-BRST charge \( Q_d \) can be further re-expressed in an equivalent form as:

\[
Q_d^{(3)} = \int d^3x \bigg[ \mathcal{B} \ddot{C} - \mathcal{B} \dot{C} + \mathcal{B}_\mu \dot{C}^\mu - \mathcal{B}_\mu \dot{C}^\mu + \frac{1}{2} (\beta \lambda - \bar{\beta} \bar{\lambda}) \bigg]. \quad (85)
\]

The above expression, using the CF-type restrictions \( \mathcal{B}_\mu + \bar{\mathcal{B}}_\mu + \partial_\mu \bar{\phi} = 0 \) and \( \mathcal{B} + \bar{\mathcal{B}} + m \bar{\phi} = 0 \) as well as some appropriate\(^1\) and useful EL-EOMs from (67) and/or (69), can be re-written in an exact form w.r.t. the anti-co-BRST symmetry transformations \( (s_{ad}) \) as (see Appendix B for details)

\[
Q_d^{(3)} = \int d^3x \text{s}_{ad} \left[ \ddot{C} + \dot{C} + \dot{C}_\mu \dot{C}^\mu + \frac{1}{2} (\beta \dot{\phi} - \bar{\beta} \dot{\bar{\phi}}) \right], \quad (86)
\]

where the off-shell nilpotent anti-co-BRST symmetry transformations \( s_{ad} \) have been explicitly quoted in Eq. (18) [cf. Sec. 3].

At our present stage, we now express the anti-co-BRST charge in an appropriate and interesting form by using the EL-EOM (23) and (69) which have been derived from \( \mathcal{L}_{(B,B)} \). It turns out that the following couple of equivalent forms

\[
\begin{align*}
Q_{ad}^{(1)} &= \int d^3x \left[ 2 \epsilon^{ijk} (m \phi_i + \bar{B}_i) \partial_j C_k + 2 \left( \frac{m}{2} \epsilon^{ijk} B_{jk} + \bar{\phi} \dot{B} \right) \left( \partial_i C - m C_i \right) \\
&\quad + \mathcal{B} (\partial^0 C - m C^0) - m \beta (\partial^0 \bar{C} - m \bar{C}^0) + C (\partial^0 \mathcal{B} - m \mathcal{B}^0) \\
&\quad + (\partial \beta) \left( \partial^0 C^i - \partial^i C^0 \right) - C_\mu (\partial^0 \mathcal{B}^i - \partial^i \mathcal{B}^0) - \frac{1}{2} (\partial^0 \beta) \rho - \frac{1}{2} \mathcal{B}^0 \lambda \\
&\quad - \bar{B}_i \left( \partial^0 C^i - \partial^i C^0 \right), \\
Q_{ad}^{(2)} &= \int d^3x \left[ \mathcal{B} \ddot{C} - \mathcal{B} \dot{C} + \dot{\mathcal{B}}_\mu \dot{C}^\mu - \ddot{\mathcal{B}}_\mu \dot{C}^\mu + m(\beta \dot{C} - \bar{\beta} \ddot{C}) \\
&\quad + (\partial \beta) \dot{C}^\mu - (\partial \beta) \dot{\bar{C}}^\mu + \beta \dot{\rho} - \bar{\beta} \ddot{\bar{\rho}}, \quad (87)
\end{align*}
\]

can be expressed in the exact forms w.r.t. the off-shell nilpotent anti-co-BRST symmetry

---

\(^1\)The EL-EOMs that have been exploited for our purpose are: \( \partial_\mu C^\mu + m \bar{C} = -\frac{\bar{\phi}}{2} \), \( (\Box + m^2) \bar{C}_\mu = (\Box + m^2) \bar{\phi} = 0 \) and these emerge out from \( \mathcal{L}_{(B,B)} \) and/or \( \mathcal{L}_{(B,B)} \).
transformations (19) as [cf. Sec. 3 for details]:

\[ Q_{ad}^{(1)} = \int d^3x \ s_{ad} \left[ -2 \left( m \phi_i + \tilde{B}_i \right) B^{0i} \right.
\begin{align*}
&+ 2 \left( \frac{m}{2} \epsilon^{ijk} B_{jk} + \tilde{\Phi}^{0i} \right) \tilde{\phi}_i + \tilde{B} \tilde{\phi}_0 \\
&- \frac{1}{2} \epsilon^{ijk} \tilde{B}_i B_{jk} + (\partial^0 \tilde{C} - m \tilde{C}^0) \tilde{C} - (\partial_0 \tilde{C}_i - \partial_i \tilde{C}_0) \tilde{C}^i - \frac{1}{2} \beta \beta - \frac{1}{2} \tilde{C}_0 \lambda \right],
\end{align*}

\[ Q_{ad}^{(2)} = \int d^3x \ s_{ad} \left[ \dot{C} \dot{C} - \dot{\tilde{C}} \dot{C} + \dot{\tilde{C}} \mu \dot{C}^\mu - \dot{C}_\mu \dot{C}^\mu + \beta \dot{\beta} - \beta \dot{\beta} \right]. \tag{88} \]

Thus, it is straightforward to point out that the off-shell nilpotency of the equivalent forms of the charges \( Q_{ad}^{(1)} \) and \( Q_{ad}^{(2)} \) can be proven as

\[ s_{ad} Q_{ad}^{(1)} = -i \left\{ Q_{ad}^{(1)}, Q_{ad}^{(1)} \right\} = 0 \iff s_{ad}^2 = 0 \iff [Q_{ad}^{(1)}]^2 = 0, \]

\[ s_{ad} Q_{ad}^{(2)} = -i \left\{ Q_{ad}^{(2)}, Q_{ad}^{(2)} \right\} = 0 \iff s_{ad}^2 = 0 \iff [Q_{ad}^{(2)}]^2 = 0, \tag{89} \]

where we have used the deep relationship between the continuous symmetry transformations (\( s_{ad} \)) and their generators (\( Q_{ad}^{(1,2)} \)). We note that, using the EL-EOMs (23) and (69), we have yet another interesting and equivalent form of the conserved and off-shell nilpotent anti-co-BRST charge, namely:

\[ Q_{ad}^{(3)} = \int d^3x \left[ \mathcal{B} \dot{C} - \dot{\mathcal{B}} C + \mathcal{B}_\mu C^\mu - \dot{\mathcal{B}}_\mu \dot{C}^\mu + \frac{1}{2} (\beta \dot{\rho} - \dot{\beta} \rho) \right], \tag{90} \]

which can be recast in a different form by using the CF-type restrictions: \( \mathcal{B}_\mu + \dot{\mathcal{B}}_\mu + \partial_\mu \tilde{\phi} = 0 \) and \( \mathcal{B} + \dot{\mathcal{B}} + m \tilde{\phi} = 0 \) and some appropriate EL-EOMs. The ensuing interesting form is:

\[ Q_{ad}^{(3)} = \int d^3x \left[ \mathcal{B} \dot{C} - \dot{\mathcal{B}} C + \mathcal{B}_\mu \dot{C}^\mu - \dot{\mathcal{B}}_\mu C^\mu + \frac{1}{2} (\beta \dot{\rho} - \dot{\beta} \rho) + \frac{1}{2} (\dot{\tilde{\phi}} \lambda - \dot{\tilde{\phi}} \dot{\lambda}) \right], \]

\[ \equiv \int d^3x \ s_d \left[ \dot{C} C + C_\mu \dot{C}^\mu + \frac{1}{2} (\dot{\tilde{\phi}} \lambda - \dot{\tilde{\phi}} \dot{\lambda}) \right]. \tag{91} \]

where the co-BRST symmetry transformations (\( s_d \)) have been quoted in our Eq. (18) [cf. Sec. 3]. Thus, it is crystal clear that (using the appropriate CF-type restrictions and EL-EOMs), the anti-co-BRST charge can be written in a co-exact form w.r.t. the co-BRST symmetry transformations (\( s_d \)).

At this crucial juncture, we now comment on the absolute anticommutativity of the (anti-)BRST and (anti-)co-BRST charges (i.e \( \{ Q_b, Q_{ab} \} = 0 \) and \( \{ Q_d, Q_{ad} \} = 0 \)) which is one of the decisive features of the conserved and off-shell nilpotent (anti-) BRST and (anti-)co-BRST conserved charges within the framework of the BRST formalism. In this context, we recall that, using the appropriate CF-type restrictions (21), we have been able to express (i) the BRST charge (\( Q_b^{(3)} \)) as an exact form w.r.t. the anti-BRST transformations (\( s_{ab} \)) [cf. Eq. (76)], (ii) the anti-BRST charge (\( Q_{ab}^{(3)} \)) as a BRST-exact expression [cf. Eq. (81)], (iii) the co-BRST charge (\( Q_{ad}^{(3)} \)) as an exact form w.r.t. the anti-co-BRST transformation \( s_d \) [cf. Eq. (86)], and (iv) the anti-co-BRST charges (\( Q_{ad}^{(3)} \)) as the co-exact form w.r.t.

\[ **The EL-EOMs derived from \mathcal{L}_{(B, B)} \text{ and/or } \mathcal{L}_{(\tilde{B}, \tilde{B})} \text{ that have come in handy are: } \partial_\mu \tilde{C}^\mu + m \tilde{C} = \dot{\tilde{\phi}}, \ (\square + m^2) \tilde{C}_\mu = (\square + m^2) \tilde{\phi} = 0. \]
the co-BRST transformations $s_d$ [cf. Eq. (91)]. These observations, ultimately, lead to the following proof of the absolute anticommutativity:

$$
\begin{align*}
    s_b Q_{ab}^{(3)} &= -i \{ Q_{ab}^{(3)}, Q_b^{(3)} \} = 0 \iff s_b^2 = 0, \\
s_{ab} Q_b^{(3)} &= -i \{ Q_b^{(3)}, Q_{ab}^{(3)} \} = 0 \iff s_{ab}^2 = 0, \\
s_d Q_{ad}^{(3)} &= -i \{ Q_{ad}^{(3)}, Q_d^{(3)} \} = 0 \iff s_d^2 = 0, \\
s_{ad} Q_d^{(3)} &= -i \{ Q_d^{(3)}, Q_{ad}^{(3)} \} = 0 \iff s_{ad}^2 = 0.
\end{align*}
$$

(92)

We note that the absolute anticommutativity of the BRST charge with the anti-BRST charge is deeply connected with the nilpotency ($s_{ab}^2 = 0$) of the anti-BRST symmetries ($s_{ab}$). On the other hand, the absolute anticommutativity of the anti-BRST charge with that of the BRST charge is intimately connected with the nilpotency ($s_b^2 = 0$) of the BRST symmetry ($s_b$). In exactly similar fashion, the absolute anticommutativity of the co-BRST charge with the anti-co-BRST charge is related with the nilpotency ($s_{ad}^2 = 0$) of the anti-co-BRST symmetry ($s_{ad}$). On the other hand, the absolute anticommutativity of the anti-co-BRST charge with the co-BRST charge is deeply related with the off-shell nilpotency ($s_d^2 = 0$) of the co-BRST (dual-BRST) transformations ($s_d$). These observations should be contrasted with the off-shell nilpotency properties where one finds that the nilpotency ($s_r^2 = 0, r = b, ab, d, ad$) of all the fermionic symmetries and corresponding fermionic ($Q_r^2 = 0, r = b, ab, d, ad$) charges $Q_r$ (with $r = b, ab, d, ad$) are individually deeply connected [cf. Eqs. (73), (79), (84), (89)] with each-other.

### 6.2 Nilpotency and Absolute Anticommutativity Properties of (Anti-)BRST Charges: ACSA

In this subsection, we capture the off-shell nilpotency and absolute anticommutativity of the (anti-)BRST charges within the framework of ACSA to BRST formalism. First of all, we focus on the conserved BRST charges ($Q_b^{(1,2)}$) that have been expressed in (72) as the BRST-exact forms. It is straightforward to note that the expression for $Q_b^{(2)}$ is simpler than the expression for $Q_b^{(1)}$. Thus, keeping in mind the mapping: $\partial_\theta \leftrightarrow s_b$ [4-6], it can be seen that we can express $Q_b^{(2)}$ (within the framework of ACSA to BRST formalism) as

$$
\begin{align*}
    Q_b^{(2)} &= \frac{\partial}{\partial \theta} \int d^2x \left[ \tilde{F}^{(b)}(x, \bar{\theta}) \dot{F}^{(b)}(x, \bar{\theta}) - \dot{\tilde{F}}^{(b)}(x, \bar{\theta}) \tilde{F}^{(b)}(x, \bar{\theta}) \\
    &\quad + \tilde{F}_\mu^{(b)}(x, \bar{\theta}) \tilde{F}_\mu^{(b)}(x, \bar{\theta}) - \dot{\tilde{F}}_\mu^{(b)}(x, \bar{\theta}) \dot{\tilde{F}}_\mu^{(b)}(x, \bar{\theta}) + \tilde{\beta}^{(b)}(x, \bar{\theta}) \dot{\tilde{\beta}}^{(b)}(x, \bar{\theta}) \\
    &\quad - \tilde{\beta}^{(b)}(x, \bar{\theta}) \dot{\tilde{\beta}}^{(b)}(x, \bar{\theta}) \right] \\
    &\equiv \int d\bar{\theta} \int d^2x \left[ \tilde{F}^{(b)}(x, \bar{\theta}) \dot{F}^{(b)}(x, \bar{\theta}) - \dot{\tilde{F}}^{(b)}(x, \bar{\theta}) \tilde{F}^{(b)}(x, \bar{\theta}) \\
    &\quad + \tilde{F}_\mu^{(b)}(x, \bar{\theta}) \dot{F}_\mu^{(b)}(x, \bar{\theta}) - \dot{\tilde{F}}_\mu^{(b)}(x, \bar{\theta}) \dot{\tilde{F}}_\mu^{(b)}(x, \bar{\theta}) + \dot{\tilde{\beta}}^{(b)}(x, \bar{\theta}) \tilde{\beta}^{(b)}(x, \bar{\theta}) \\
    &\quad - \dot{\tilde{\beta}}^{(b)}(x, \bar{\theta}) \tilde{\beta}^{(b)}(x, \bar{\theta}) \right],
\end{align*}
$$

(93)
where the superscript \((b)\) on the anti-chiral superfields denotes the fact that these superfields have been derived after the applications of BRST-invariant restrictions. In other words, we have used the super expansions [that have been already written in (34)] which lead to the derivation of the BRST symmetry transformations (13) as the coefficients of the Grassmannian variable \(\bar{\theta}\) [cf. Eq. (34) for details]. It is now crystal clear that:

\[
\partial_b Q_b^{(2)} = 0 \iff \partial_b^2 = 0 \iff s_{ab}^2 = 0. \tag{94}
\]

In ordinary space, the above equation captures the nilpotency property of Eq. (73). Hence, we observe that it is the nilpotency \((\partial_b^2 = 0)\) of the translational generator \((\partial_b)\) along the \(\bar{\theta}\)-direction of the \((4, 1)\)-dimensional anti-chiral super sub-manifold that is responsible for the off-shell nilpotency of the BRST charge \(Q_b^{(2)}\). We would like to emphasize that both \(Q_b^{(1)}\) and/or \(Q_b^{(2)}\) can be expressed in terms of the derivative \(\partial_b\) and anti-chiral superfields [cf. Eq. (34)]. However, for the sake of brevity, we have chosen \(Q_b^{(2)}\) for our purpose. The same type of exercise can be performed for \(Q_b^{(1)}\), too.

At this stage, we concentrate on capturing the off-shell nilpotency of the anti-BRST charges \(Q_{ab}^{(1,2)}\) that have been written in (78) as an exact form w.r.t. the anti-BRST symmetry transformations \(s_{ab}\) [cf. Eq. (12)]. We capture the expression for \(Q_{ab}^{(2)}\), for the shake of brevity, within the framework of ACSA as

\[
Q_{(ab)}^{(2)} = \frac{\partial}{\partial \bar{\theta}} \int d^3 x \left[ \bar{F}^{(ab)}(x, \theta) \bar{F}^{(ab)}(x, \theta) - \bar{F}^{(ab)}(x, \theta) \bar{F}^{(ab)}(x, \theta) + \bar{F}_{\mu}^{(ab)}(x, \theta) \bar{F}_{\mu}^{(ab)}(x, \theta) - \bar{F}_{\mu}^{(ab)}(x, \theta) \bar{F}_{\mu}^{(ab)}(x, \theta) + \bar{\beta}^{(ab)}(x, \theta) \bar{\beta}^{(ab)}(x, \theta) - \bar{\beta}^{(ab)}(x, \theta) \bar{\beta}^{(ab)}(x, \theta) \right] \equiv \int d\theta \int d^3 x \left[ \bar{F}^{(ab)}(x, \theta) \bar{F}^{(ab)}(x, \theta) - \bar{F}^{(ab)}(x, \theta) \bar{F}^{(ab)}(x, \theta) + \bar{F}_{\mu}^{(ab)}(x, \theta) \bar{F}_{\mu}^{(ab)}(x, \theta) - \bar{F}_{\mu}^{(ab)}(x, \theta) \bar{F}_{\mu}^{(ab)}(x, \theta) + \bar{\beta}^{(ab)}(x, \theta) \bar{\beta}^{(ab)}(x, \theta) - \bar{\beta}^{(ab)}(x, \theta) \bar{\beta}^{(ab)}(x, \theta) \right], \tag{95}
\]

where superscript \((ab)\) denotes that the chiral superfields have been derived after the applications of the anti-BRST invariant restrictions. In other words, we have taken into account the super expansion that have been listed in Eq. (40). It is straightforward to note that we have the following:

\[
\partial_b Q_{ab}^{(2)} = 0 \iff \partial_b^2 = 0 \iff s_{ab}^2 = 0. \tag{96}
\]

In the ordinary space, the above equation is equivalent to the off-shell nilpotency property \(((Q_{(ab)}^{(2)})^2 = 0)\) of the anti-BRST charge \(Q_{ab}^{(2)}\) [that has been quoted in Eq. (79)] in view of the mapping: \(s_{ab} \leftrightarrow \partial_b [4-6]\).

At this crucial juncture, we discuss the absolute anticommutativity of the BRST charge with anti-BRST charge in the terminology of ACSA to BRST formalism. We note that one of the equivalent forms of the BRST charge is Eq. (76) where the BRST charge has been
expressed in the exact form w.r.t. the anti-BRST symmetry transformations \(s_{ab}\) of Eq. (12). Keeping in mind the mapping: \(\partial_\theta \leftrightarrow s_{ab}\), it is straightforward to express (76) as

\[
Q_{(3)}^{(b)} = \frac{\partial}{\partial \theta} \int d^3x \left[ \hat{F}^{(b)}(x, \theta) \hat{F}^{(b)}(x, \theta) + \hat{F}_\mu^{(b)}(x, \theta) \hat{F}^{(b)}(x, \theta) \right] + \frac{1}{2} \left( \hat{\beta}^{(b)}(x, \theta) \hat{P}^{(b)}(x, \theta) - \hat{\beta}^{(b)}(x, \theta) \hat{P}^{(b)}(x, \theta) \right),
\]

where the superscript \((ab)\) stands for the chiral superfields that have been expanded in Eq. (40). In exactly similar fashion, to capture the absolute anticommutativity of the \((\text{anti-})\)BRST charge with \(\text{BRST charge}\), we focus on the expression for one of the equivalent forms of the anti-BRST charge \(Q_{ab}^{(3)}\) that has been quoted in Eq. (81) as a BRST-exact quantity. This expression can be expressed, keeping in mind the mapping: \(\partial_\theta \leftrightarrow s_b\) (within the framework of ACSA to BRST formalism) as follows

\[
Q_{(ab)}^{(3)} = \frac{\partial}{\partial \theta} \int d^3x \left[ \hat{F}^{(b)}(x, \theta) \hat{F}^{(b)}(x, \theta) + \hat{F}_\mu^{(b)}(x, \theta) \hat{F}^{(b)}(x, \theta) \right] + \frac{1}{2} \left( \hat{\beta}^{(b)}(x, \theta) \hat{P}^{(b)}(x, \theta) - \hat{\beta}^{(b)}(x, \theta) \hat{P}^{(b)}(x, \theta) \right),
\]

where the superscript \((b)\) denotes the \((\text{anti-})\)chiral superfields that have been quoted in Eq. (34). It is elementary to note now that we have the following:

\[
\partial_\theta Q_{(ab)}^{(3)} = \partial_\theta Q_{(b)}^{(3)} = 0 \quad \iff \quad \partial_\theta^2 = 0, \quad \partial_\theta = 0.
\]

In the ordinary space, the above relationships are nothing but the absolute anticommutativity of the \((\text{anti-})\)BRST charges in Eq. (92). It is now very interesting to pinpoint the distinct differences between \(\{Q_{b}^{(3)} , Q_{ab}^{(3)} \} = 0\) and \(\{Q_{ab}^{(3)} , Q_{b}^{(3)} \} = 0\) within the framework of ACSA to BRST formalism. It turns out that the absolute anticommutativity of the BRST charge with \(\text{anti-BRST charge}\) is closely connected with the nilpotency \((\partial_\theta^2 = 0)\) of the translational generator \((\partial_\theta)\) along the chiral (i.e. \(\theta\)) direction of the \((4, 1)\)-dimensional super sub-manifold. On the other hand, the absolute anticommutativity of the \(\text{anti-BRST charge}\) with \(\text{BRST charge}\) is deeply related with the nilpotency \((\partial_\theta^2 = 0)\) of the translational generator along \(\theta\)-direction of the \((4, 1)\)-dimensional \((\text{anti-})\)chiral super-submanifold.

### 6.3 Nilpotency and Absolute Anticommutativity of the \((\text{Anti-})\)co-BRST Charges: ACSA

We dwell, in this subsection, on the proof of the off-shell nilpotency as well as absolute anticommutativity properties of the \((\text{anti-})\)co-BRST charges within the framework of ACSA
to BRST formalism. In this context, we note that we have expressed the conserved (anti-)co-BRST charges (i.e. $Q^{(1,2)}_d, Q^{(1,2)}_{ad}$) in the (co-)exact forms in Eqs. (83) and (87) w.r.t. the (anti-)co-BRST symmetries $s_{(a)d}$. Keeping in our mind the mappings: $\partial_\theta \leftrightarrow s_d$ and $\partial_{\bar{\theta}} \leftrightarrow s_{ad}$, we express the simpler versions of the co-BRST and anti-co-BRST charges $Q_d^{(2)}$ and $Q_{ad}^{(2)}$ [cf. Eqs. (83), (87)] as

\begin{align}
Q_d^{(2)} &= \frac{\partial}{\partial \theta} \int d^3x \left[ \tilde{\mathcal{F}}^{(d)}(x, \theta) \tilde{\mathcal{F}}^{(d)}(x, \theta) - \tilde{\mathcal{F}}^{(d)}(x, \theta) \tilde{\mathcal{F}}^{(d)}(x, \theta) \right] \\
&+ \frac{\tilde{F}_\mu^{(d)}}{\mu}(x, \theta) \tilde{\mathcal{F}}^{(d)}(x, \theta) - \frac{\tilde{F}_\mu^{(d)}}{\mu}(x, \theta) \tilde{\mathcal{F}}^{(d)}(x, \theta) + \frac{\tilde{\beta}}{\beta}(x, \theta) \\
&\quad \quad + \frac{\tilde{\beta}}{\beta}(x, \theta) - \frac{\tilde{\beta}}{\beta}(x, \theta) \frac{\tilde{\beta}}{\beta}(x, \theta),
\end{align}

where the superscript $(d)$ and $(ad)$ denote the chiral and anti-chiral super expansions that have been quoted in Eqs. (45) and (50). It is now elementary exercise to check the following:

\begin{align}
\partial_\theta Q_d^{(2)} = 0 &\iff \partial_\theta^2 = 0 \iff s_d^2 = 0, \\
\partial_{\bar{\theta}} Q_{ad}^{(2)} = 0 &\iff \partial_{\bar{\theta}}^2 = 0 \iff s_{ad}^2 = 0.
\end{align}

Thus, it is crystal clear that the nilpotency property of the co-BRST charge $Q_d^{(2)}$ is deeply connected with the nilpotency ($\partial_\theta^2 = 0$) of the translational generator along $\theta$-direction of the chiral super-submanifold. On the other hand, we observe that the nilpotency ($\partial_{\bar{\theta}}^2 = 0$) of the translational generator along $\bar{\theta}$-direction of the anti-chiral super submanifold is responsible for the off-shell nilpotency of the anti-co-BRST charge ($Q_{ad}^{(2)}$).

Within the framework of ACSA to BRST approach, we are now in the position to capture the absolute anticommutativity (i.e. $\{Q_d, Q_{ad}\} = 0$) of the (anti-)co-BRST charges. In this context, we note that the (anti-)co-BRST charges $[Q_{ad}^{(2)} Q_d^{(2)}]$ have been written in the exact forms w.r.t. the co-BRST and anti-co-BRST symmetry transformations [cf. Eqs. (91), (86)]. Keeping in our mind the mappings: $s_d \leftrightarrow \partial_\theta, s_{ad} \leftrightarrow \partial_{\bar{\theta}}$, we can express the charges in (86) and (91), within the framework of the ACSA, as follows

\begin{align}
Q_d^{(3)} &= \frac{\partial}{\partial \theta} \int d^3x \left[ \tilde{\mathcal{F}}^{(ad)}(x, \theta) \tilde{\mathcal{F}}^{(ad)}(x, \theta) + \tilde{\mathcal{F}}^{(ad)}(x, \theta) \tilde{\mathcal{F}}^{(ad)}(x, \theta) \right] \\
&+ \frac{1}{2} \left[ \tilde{\beta}^{(ad)}(x, \theta) \tilde{\Phi}^{(ad)}(x, \theta) - \tilde{\beta}^{(ad)}(x, \theta) \tilde{\Phi}^{(ad)}(x, \theta) \right],
\end{align}

\begin{align}
Q_{ad}^{(3)} &= \frac{\partial}{\partial \bar{\theta}} \int d^3x \left[ \tilde{\mathcal{F}}^{(d)}(x, \theta) \tilde{\mathcal{F}}^{(d)}(x, \theta) - \tilde{\mathcal{F}}^{(d)}(x, \theta) \tilde{\mathcal{F}}^{(d)}(x, \theta) \right] \\
&+ \frac{1}{2} \left[ \tilde{\beta}^{(d)}(x, \theta) \tilde{\Phi}^{(d)}(x, \theta) - \tilde{\beta}^{(d)}(x, \theta) \tilde{\Phi}^{(d)}(x, \theta) \right],
\end{align}

(102)
where the superscripts \((d)\) and \((ad)\) on the superfields stand for the superfield expansions \((45)\) and \((50)\). It is now straightforward to note that:

\[
\begin{align*}
\partial_\theta Q^{(3)}_{d} &= 0 \quad \iff \quad \{Q^{(3)}_{d}, Q^{(3)}_{ad}\} = 0 \quad \iff \quad \partial^2_{\theta} = 0, \\
\partial_\theta Q^{(3)}_{ad} &= 0 \quad \iff \quad \{Q^{(3)}_{ad}, Q^{(3)}_{d}\} = 0 \quad \iff \quad \partial^2_{\theta} = 0.
\end{align*}
\]

(103)

In the above, the anticommutators emerge from the realizations of the \(\partial_\theta Q^{(3)}_{d} = 0\) and \(\partial_\theta Q^{(3)}_{ad} = 0\) in the terminology of the symmetry transformations where \(\partial_\theta \leftrightarrow s_{ad}\) and \(\partial_\theta \leftrightarrow s_{d}\). In other words, we have the following

\[
\begin{align*}
\partial_\theta Q^{(3)}_{d} &= 0 \quad \iff \quad s_{ad}Q^{(3)}_{d} = -i \{Q^{(3)}_{d}, Q^{(3)}_{ad}\} = 0, \\
\partial_\theta Q^{(3)}_{ad} &= 0 \quad \iff \quad s_{d}Q^{(3)}_{ad} = -i \{Q^{(3)}_{ad}, Q^{(3)}_{d}\} = 0.
\end{align*}
\]

(104)

in the ordinary 4D Minkowskian flat space. Thus, we have been able to differentiate and discern between the anticommutators \(\{Q^{(3)}_{d}, Q^{(3)}_{ad}\} = 0\) and \(\{Q^{(3)}_{ad}, Q^{(3)}_{d}\} = 0\) within the framework of ACSA to BRST formalism because of the observations \(\partial_\theta Q^{(3)}_{d} = 0 = \partial_\theta Q^{(3)}_{ad}\).

We end this sub-section with the following remarks. We observe that the off-shell nilpotency \((Q^2_{(ad)} = 0)\) of the (anti-)co-BRST charge \((Q_{(ad)})\) is connected with the nilpotent \((\partial^2_{\theta} = \bar{\partial}^2_{\theta} = 0)\) translational generators along \((\bar{\theta})\theta\)-directions of the (anti-) chiral super submanifolds. This result is very much expected within the framework of ACSA to BRST formalism. However, the interesting and intriguing observations are (i) the absolute anticommutativity of the co-BRST charge with the anti-co-BRST charge is intimately related with the nilpotency \((\partial^2_{\theta} = 0)\) of the translational generator \((\partial_\theta)\) along the anti-chiral \(\theta\)-direction of the anti-chiral super submanifold, and (ii) the absolute anticommutativity of the anti-co-BRST charge with the co-BRST charge, however, is intimately related with the nilpotency \((\bar{\partial}^2_{\theta} = 0)\) of the translational generator \((\bar{\partial}_\theta)\) along the chiral \(\theta\)-direction of the \((4, 1)\)-dimensional chiral super sub-manifold [of the general \((4, 2)\)-dimensional supermanifold on which our present ordinary 4D massive Abelian 2-form theory is generalized within the framework of ACSA to BRST formalism].

7 Conclusions

In our earlier work [24], we have already established that the 4D massive Abelian 2-form gauge theory (without any interaction with matter fields) is a massive model of Hodge theory in exactly same manner as the 2D Proca (i.e. 2D massive Abelian 1-form) theory is (see, e.g. [31-33] for details). In our present endeavor, we have corroborated the correctness of the nilpotent (fermionic) symmetries of the former theory by exploiting the basic tenets and techniques of ACSA to BRST formalism. We would like to lay emphasis on the fact that the existence of the (anti-)BRST and (anti-)co-BRST symmetries for the massive 4D Abelian 2-form gauge theory is very fundamental as they provide the physical realizations of

\[\text{[The fundamental off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations “gauge away” all the spurious degrees of freedom that are connected with the modified form of St"uckelberg’s formalism where the polar-vector field (\(\varphi_{\mu}\)), axial-vector field (\(\tilde{\varphi}_{\mu}\)), scalar field (\(\varphi\)) and pseudo-scalar field (\(\tilde{\varphi}\)) appear. At the end, we have only three physical degrees of freedom (of the BRST-quantized theory) that are connected with the massive 4D free Abelian 2-form theory.} \]
of the nilpotent (co-)exterior derivatives of differential geometry at the algebraic level. The bosonic symmetry transformations (i.e. the analogue of the Laplacian operator) are derived from the above fundamental off-shell nilpotent symmetries. Thus, our present work essentially corroborates the correctness of the off-shell nilpotent symmetries that have been discussed in our earlier work [24] for the massive 4D Abelian 2-form theory that has been turned out to be physically interesting, too. Hence, our present endeavor is important in its own right as far as the sanctity of the nilpotent symmetries and CF-type restrictions of our 4D massive theory are concerned.

We would like to comment on the combination of fields that appear in a specific manner in the transformation (2) where $B_{\mu\nu} \rightarrow B_{\mu\nu} - \frac{1}{m}(\partial_{\mu}\phi_{\nu} - \partial_{\nu}\phi_{\mu} + \varepsilon_{\mu\nu\eta\kappa}\partial^{\eta}\tilde{\phi}^{\kappa})$. In our earlier work on the local duality invariance [34] of the source-free Maxwell’s equations, we have taken the field strength tensor $F_{\mu\nu}$ for the $U(1)$ Abelian 1-form theory as: $F_{\mu\nu} = (\partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu} + \varepsilon_{\mu\nu\eta\kappa}\partial^{\eta}A^{\kappa})$ where $V_{\mu}$ and $A_{\mu}$ are the vector and axial-vector potentials. It is interesting to note that, in the context of our present 4D massive Abelian 2-form gauge theory, the above specific structure appears very naturally. We would like to point out that the two potential approach to electrodynamics has been considered by other(s), too (see, e.g. [35] and references therein). It is essential to pinpoint that the nature of vector and axial-vector ($\phi_{\mu}, \tilde{\phi}_{\mu}$) fields in our theory is quite different from the properties of the vector and axial-vector (i.e. $V_{\mu}, A_{\mu}$) fields that are present in our earlier work [34].

One of the novel results of our present investigation is the observation that the off-shell nilpotent (anti-)BRST and (anti-)co-BRST charges are found to be absolutely anticommuting despite the fact that we have taken only the (anti-)chiral superfields within the framework of ACSA to BRST formalism. In fact, right in the beginning, we have utilized the CF-type restrictions to recast the off-shell nilpotent charges in a specific form so that the proof of absolute anticommutativity property could become straightforward. Thus, in a subtle manner, this proof establishes the existence of the CF-type restrictions on our theory. We have also established their existence by proving the invariances of the Lagrangian densities $L_{(B,B)}$ and $L_{(\bar{B},\bar{B})}$ within the framework of ACSA to BRST formalism in Sec. 5. These observations should be contrasted with the ACSA approach to the $\mathcal{N} = 2$ SUSY quantum mechanical models where we do not obtain the absolute anticommutativity of the conserved and nilpotent $\mathcal{N} = 2$ SUSY charges (see, e.g. [36-38] for details). Thus, it is self-evident that the observation of the absolute anticommutativity property between the (anti-)BRST and (anti-)co-BRST charges is indeed a novel observation in our present investigation.

Another interesting observation in the context of ACSA to BRST formalism is the result that it distinguishes (cf. Sec. 6 for details) between the absolute anticommutativity property of (i) the BRST charge with the anti-BRST charge, (ii) the anti-BRST charge with the BRST charge, (iii) the co-BRST charge with the anti-co-BRST charge, and (iv) the anti-co-BRST charge with the co-BRST charge. We observe that the proof of the off-shell nilpotency of all the conserved charges is as expected within the framework of ACSA to BRST formalism. However, the proof of the absolute anticommutativity property, within the framework of ACSA to BRST formalism, yields some non-trivial and novel results. We have discussed these issues elaborately in the terminology of the translational generators $(\partial_{\theta}, \partial_{\bar{\theta}})$ along the Grassmannian directions $(\theta, \bar{\theta})$ of the $(4, 1)$-dimensional chiral and anti-chiral super sub-manifolds of the general $(4, 2)$-dimensional supermanifold (cf. Sec. 6) on
which our *ordinary* 4D massive Abelian 2-form theory has been generalized.

The model under consideration (i.e. 4D massive Abelian 2-form theory) is *physically* interesting because it has led to the existence of fields with *negative* kinetic terms [24]. These fields have turned out to be pseudo-scalar and axial-vector fields which are invoked in the theory on symmetry grounds. They possess well-defined *mass* but their kinetic terms are forced to be *negative* if we wish to have our theory to be a 4D massive field-theoretic model of Hodge theory [24]. Such fields have been found to be one of the possible candidates of dark matter and dark energy [39, 40]. Furthermore, in the context of cosmological models of Universe, these kinds of fields have been found to be useful for explaining cyclic, self-acceleration and bouncing phenomena of the cohomological models (see, e.g. [41-43]).

We would like to pinpoint the fact that the application of ACSA to BRST formalism enables us to derive the proper (anti-)BRST and (anti-)co-BRST symmetry transformations *without* any application of the *formal* mathematical techniques like: horizontality condition, differential geometry, dual-horizontality condition, etc. However, it would be nice to corroborate our observations by the formal mathematical techniques as well. Some of us, at present, are trying to do that [44]. We have proven the 6D Abelian 3-form gauge theory (without any interaction with matter fields) to be a tractable field-theoretic example for the Hodge theory in our earlier work [see, e.g. [45] for a brief review] where we have discussed the (anti-)BRST and (anti-)co-BRST symmetry transformations. It would be very nice future endeavor to apply the theoretical tricks of ACSA to BRST formalism and derive the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries for the 6D Abelian 3-form *massive* gauge theory by applying the St"uckelberg formalism. We plan to accomplish this goal in our forthcoming future publications [46].

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**Appendix A: On the Anti-co-BRST Invariance of the Lagrangian Density**

\[ \mathcal{L}_{(B,B)} \]

Using ACSA

To complement the contents of our sub-section 5.2, we capture the anti-co-BRST symmetry invariance of \( \mathcal{L}_{(B,B)} \) within the framework of ACSA to BRST formalism. Keeping this goal in mind, we generalize the *ordinary* 4D Lagrangian density \( \mathcal{L}_{(B,B)} \) to its counterpart *anti-chiral* super Lagrangian density \( \tilde{\mathcal{L}}^{(ac,ad)}_{(B,B)}(x, \bar{\theta}) \) as:

\[
\tilde{\mathcal{L}}^{(ac,ad)}_{(B,B)}(x, \bar{\theta}) = \frac{1}{2} \tilde{B}_{\mu}^{(ad)}(x, \bar{\theta}) \tilde{B}^{(ad)}_{\mu}(x, \bar{\theta})
\]
\[ -\mathcal{B}^{(ad)}(x, \bar{\theta}) \left( -\frac{1}{2} \varepsilon_{\mu\nu\kappa} \partial^\nu \mathcal{B}^{(ad)}_{\mu\kappa}(x, \bar{\theta}) - \frac{1}{2} \partial_{\mu} \tilde{\Phi}^{(ad)}(x, \bar{\theta}) + m \tilde{\phi}^{(ad)}_{\mu}(x, \bar{\theta}) \right) \]
\[-\frac{m^2}{4} \mathcal{B}^{\mu;(ad)}(x, \bar{\theta}) \mathcal{B}^{(ad)}_{\mu}(x, \bar{\theta}) - \frac{1}{2} \partial^\mu \phi^\nu(x) \left( \partial_{\mu} \phi_{\nu}(x) - \partial_{\nu} \phi_{\mu}(x) \right) \]
\[ + m \mathcal{B}^{\mu;(ad)}(x, \bar{\theta}) \partial_{\mu} \phi_{\nu}(x) + \frac{1}{2} \partial^\mu \tilde{\Phi}^{(ad)}_{\nu}(x, \bar{\theta}) \left( \partial_{\mu} \tilde{\Phi}^{(ad)}_{\nu}(x, \bar{\theta}) - \partial_{\nu} \tilde{\Phi}^{(ad)}_{\mu}(x, \bar{\theta}) \right) \]
\[ + \mathcal{B}^{\mu}(x) \left( \partial^\mu \tilde{\Phi}^{(ad)}_{\nu}(x, \bar{\theta}) - \frac{1}{2} \partial_{\mu} \phi(x) + m \phi_{\mu}(x) \right) \]
\[ + \frac{1}{2} \mathcal{B}(x, \bar{\theta}) \mathcal{B}(x) + \mathcal{B}(x) \left( \partial_{\mu} \phi_{\mu}(x) + \frac{m}{2} \phi(x) \right) \]
\[ - \frac{1}{2} \mathcal{B}(x, \bar{\theta}) \mathcal{B}^{(ad)}(x, \bar{\theta}) - \mathcal{B}^{(ad)}(x, \bar{\theta}) \left( \partial_{\mu} \tilde{\Phi}^{(ad)}_{\nu}(x, \bar{\theta}) + \frac{m}{2} \tilde{\phi}^{(ad)}(x, \bar{\theta}) \right) \]
\[ + \left( \partial_{\mu} \tilde{\mathcal{F}}^{(ad)}_{\nu}(x, \bar{\theta}) - \frac{m}{2} \tilde{\mathcal{F}}^{(ad)}_{\mu}(x, \bar{\theta}) \right) \left( \partial^\mu \tilde{\mathcal{F}}^{(ad)}_{\nu}(x, \bar{\theta}) - \frac{m}{2} \tilde{\mathcal{F}}^{(ad)}_{\mu}(x, \bar{\theta}) \right) \]
\[ - \left( \partial_{\mu} \tilde{\mathcal{F}}^{(ad)}_{\nu}(x, \bar{\theta}) - \partial_{\nu} \tilde{\mathcal{F}}^{(ad)}_{\mu}(x, \bar{\theta}) \right) \left( \partial^\mu \tilde{\mathcal{F}}^{(ad)}_{\nu}(x, \bar{\theta}) - \partial^\nu \tilde{\mathcal{F}}^{(ad)}_{\mu}(x, \bar{\theta}) \right) \]
\[ - \frac{1}{2} \partial_{\mu} \tilde{\beta}^{(ad)}(x, \bar{\theta}) \partial^\mu \beta(x) + \frac{1}{2} m^2 \tilde{\beta}^{(ad)}(x, \bar{\theta}) \beta(x) \]
\[ - \frac{1}{2} \left( \partial_{\mu} \tilde{\mathcal{F}}^{(ad)}_{\nu}(x, \bar{\theta}) + \frac{m}{2} \tilde{\mathcal{F}}^{(ad)}(x, \bar{\theta}) + \frac{1}{4} \lambda(x) \right) \lambda(x) \]
\[ - \frac{1}{2} \left( \partial_{\mu} \tilde{\mathcal{F}}^{(ad)}_{\nu}(x, \bar{\theta}) + \frac{m}{2} \tilde{\mathcal{F}}^{(ad)}(x, \bar{\theta}) - \frac{1}{4} \lambda(x) \right) \lambda(x), \quad (A.1) \]

where the superscript \((ad)\) on the \textit{anti-chiral} superfields denotes the super expansions in Eq. (50). We note that the above super Lagrangian density is a combination of anti-chiral superfields [cf. Eq. (50)] and some \textit{ordinary} fields that remain invariant under the anti-co-BRST symmetry transformations \((s_{ad})\). It is straightforward to note that:

\[
\frac{\partial}{\partial \bar{\theta}} \left[ \tilde{L}^{(ac,ad)}_{(B,B)} \right] = -\partial_{\mu} \left[ m \varepsilon_{\mu\nu\kappa} \phi_{\nu}(\partial_{\kappa} C) + \left( \frac{1}{2} \varepsilon_{\mu\nu\kappa} \partial_{\nu} B_{\kappa} + \frac{1}{2} \mathcal{B}_{\mu} + m \tilde{\phi} \right) \lambda \right] \]
\[ - \mathcal{B}_{\nu} \left( \partial^\nu C - \partial_{\nu} C \right) + \mathcal{B} \left( \partial^\mu C - m C_{\mu} \right) + \frac{1}{2} \left( \partial^\mu \beta \right) \rho \]
\[ + \frac{1}{2} \left[ \mathcal{B}_{\mu} + \mathcal{B}_{\mu} + \partial_{\mu} \tilde{\phi} \right] \left( \partial^\mu \lambda \right) - \partial_{\mu} \left[ \mathcal{B}_{\nu} + \mathcal{B}_{\nu} + \partial_{\nu} \tilde{\phi} \right] \left( \partial^\mu C - \partial^\nu C \right) \]
\[ - m \left[ \mathcal{B}_{\mu} + \mathcal{B}_{\mu} + \partial_{\mu} \tilde{\phi} \right] \left( \partial^\mu C - m C_{\mu} \right) - \frac{m}{2} \left[ \mathcal{B} + \mathcal{B} + m \tilde{\phi} \right] \lambda \]
\[ + \partial_{\mu} \left[ \mathcal{B} + \mathcal{B} + m \tilde{\phi} \right] \left( \partial^\mu C - m C_{\mu} \right) \equiv s_{ad} \mathcal{L}_{(B,B)}. \quad (A.2) \]

It is now self-evident that we can have \textit{perfect} anti-co-BRST invariance [cf. Eq. (28)] of the Lagrangian density \( \mathcal{L}_{(B,B)} \) only when the (anti-)co-BRST invariant CF-type restrictions:
\(B_\mu + \bar{B}_\mu + \partial_\mu \bar{\varphi} = 0\) and \(B + \bar{B} + m \varphi = 0\) would be imposed from outside. In a subtle manners, in other words, we have derived the (anti-)co-BRST invariant CF-type restrictions from the symmetry invariance of the Lagrangian densities.

We wrap up this Appendix with the remark that the requirement of the anti-co-BRST invariance of the Lagrangian density \(L_{(B \bar{B})}\) [cf. Eq. (28)] leads to the derivation of the CF-type restrictions: \(B_\mu + \bar{B}_\mu + \partial_\mu \bar{\varphi} = 0\), \(B + \bar{B} + m \varphi = 0\) which are responsible for (i) the equivalence of the Lagrangian densities \(L_{(B\bar{B})}\) and \(L_{(B\bar{B})}\) w.r.t. the (anti-)co-BRST symmetry transformations \(s_{(a)d}\), and (ii) the absolute anticommutativity (i.e. \(\{s_d, s_{ad}\} = 0\) of the (anti-)co-BRST symmetry transformations \((s_{(a)d})\) [and corresponding (anti-)co-BRST charges \((Q_{(a)d})\)] which can be captured within the framework of ACSA to BRST formalism. The derivation of the CF-type restriction is novel result in our present investigation.

**Appendix B: On the Derivation of the Conserved Charges \(Q_{(b)d}^{(3)}\) from \(Q_{(b)d}^{(2)}\)**

To supplement the contents of Subsecs. 6.1 and 6.2, we perform here the explicit algebraic computations to show that, with the helps of EL-EOMs and CF-type restrictions, we can derive the expressions for the conserved and nilpotent \(Q_{(b)d}^{(3)}\) (i.e. the exact forms w.r.t. the anti-BRST and anti-co-BRST symmetry transformations \(s_{ad}\) and \(s_{ad}\), respectively) from the conserved and nilpotent charges \(Q_{(b)d}^{(2)}\) (i.e. the exact forms w.r.t. the BRST and co-BRST symmetry transformations \(s_b\) and \(s_d\), respectively). In this connection, first of all, we begin with \(Q_{(b)d}^{(2)}\) [cf. Eq. (71)] which can be re-written as

\[
Q_{(b)d}^{(2)} = \int d^3x \left[(B \dot{C} - \dot{B} C) + (\dot{B}_\mu C^\mu - B_\mu \dot{C}^\mu) + \partial_\mu (\beta \dot{C}_\mu - \dot{B} \bar{C}_\mu)
\right.
\]

\[
+ \beta (\partial_\mu \bar{C}^\mu + m \bar{C}) - \beta (\partial_\mu \dot{C}^\mu + m \dot{C}) + (\dot{\beta} \rho - \dot{\beta} \dot{\rho}) \right],
\]

where we have taken the total derivatives and re-arranged the rest of the terms. Using the EL-EOM: \(\partial_\mu C^\mu + m \bar{C} = -\frac{\rho}{2}\) and throwing away the total space derivative terms, we obtain:

\[
Q_{(b)d}^{(2)} = \int d^3x \left[(B \dot{C} - \dot{B} C) + (\dot{B}_\mu C^\mu - B_\mu \dot{C}^\mu) + (\beta \ddot{C}^0 - \dot{\beta} \bar{C}^0) + \frac{1}{2}(\dot{\beta} \rho - \dot{\beta} \dot{\rho}) \right].
\]

Now taking the helps of the following EL-EOMs, namely;

\[
(\Box + m^2) \beta = 0 \implies \ddot{\beta} = -\partial_i \partial^i \beta - m^2 \beta,
\]

\[
(\Box + m^2) \bar{C}^\mu = 0 \implies \ddot{\bar{C}}^\mu = -\partial_i \partial^i \bar{C}^\mu - m^2 \bar{C}^\mu,
\]

the above charge, with the application of Gauss’s divergence theorem, can be re-expressed as:

\[
Q_{(b)d}^{(2)} = \int d^3x \left[(B \dot{C} - \dot{B} C) + (\dot{B}_\mu C^\mu - B_\mu \dot{C}^\mu) + \frac{1}{2}(\dot{\beta} \rho - \dot{\beta} \dot{\rho}) \right].
\]
The stage is set to apply the CF-type restrictions: $B_\mu + \dot{B}_\mu + \partial_\mu \phi = 0$ and $B + \dot{B} + m \phi = 0$ on the above expression to obtain

$$Q^{(3)}_{(b)} = \int d^3 x \left[ (\dot{B} C - \dot{B} \dot{C}) + (B_\mu \dot{C}^\mu - \dot{B}_\mu C^\mu) + \frac{1}{2} (\dot{\beta} \rho - \beta \dot{\rho}) \\
+ (\partial_\mu \phi) \dot{C}^\mu - (\partial_\mu \phi) C^\mu \right],$$  \hspace{1cm} (B.5)

where we have denoted the BRST charge by the symbol $Q^{(3)}_{b}$ because we have already used the CF-type restrictions. We take the helps of the total derivatives and re-arrange the terms of the above expression to get:

$$Q^{(3)}_{(b)} = \int d^3 x \left[ (\dot{B} C - \dot{B} \dot{C}) + (B_\mu \dot{C}^\mu - \dot{B}_\mu C^\mu) + \frac{1}{2} (\lambda \dot{\phi} - \dot{\lambda} \phi) \\
+ \frac{1}{2} (\dot{\beta} \rho - \beta \dot{\rho}) + \partial_\mu (\phi \dot{C}^\mu - \dot{\phi} C^\mu) \right].$$  \hspace{1cm} (B.6)

At this juncture, we substitutes the EL-EOM: $\partial_\mu C^\mu + m C = \frac{\lambda}{2}$ and $\partial_\mu \dot{C}^\mu + m \dot{C} = \frac{\dot{\lambda}}{2}$ to recast the above charge into the following form:

$$Q^{(3)}_{(b)} = \int d^3 x \left[ (\dot{B} C - \dot{B} \dot{C}) + (B_\mu \dot{C}^\mu - \dot{B}_\mu C^\mu) + \frac{1}{2} (\lambda \dot{\phi} - \dot{\lambda} \phi) \\
+ \frac{1}{2} (\dot{\beta} \rho - \beta \dot{\rho}) + \partial_\mu (\phi \dot{C}^\mu - \dot{\phi} C^\mu) \right].$$  \hspace{1cm} (B.7)

Applying the Gauss divergence theorem, we drop the total space derivative terms which leads to:

$$Q^{(3)}_{(b)} = \int d^3 x \left[ (\dot{B} C - \dot{B} \dot{C}) + (B_\mu \dot{C}^\mu - \dot{B}_\mu C^\mu) + \frac{1}{2} (\lambda \dot{\phi} - \dot{\lambda} \phi) \\
+ \frac{1}{2} (\dot{\beta} \rho - \beta \dot{\rho}) + (\phi \ddot{C}^0 - \dot{\phi} C^0) \right].$$  \hspace{1cm} (B.8)

As mentioned earlier, we have $(\Box + m^2) C^\mu = 0$ that implies that $\ddot{C}^0 = -\partial_i \partial^i C^0 - m^2 C^0$ and the EL-EOM $\partial_\mu B^\mu + m B = 0$ leads to $(\Box + m^2) \phi = 0$ [provided we use the expressions for $B$ and $B_\mu$ in terms of $\phi$ and $\partial_\mu \phi$ as given in Eq. (23)]. This implies that $\ddot{\phi} = -\partial_i \partial^i \phi - m^2 \phi$.

Once again, we use the Gauss divergence theorem to obtain the following

$$Q^{(3)}_{(b)} = \int d^3 x \left[ (\dot{B} C - \dot{B} \dot{C}) + (B_\mu \dot{C}^\mu - \dot{B}_\mu C^\mu) + \frac{1}{2} (\dot{\phi} \lambda - \dot{\lambda} \phi) + \frac{1}{2} (\dot{\beta} \rho - \beta \dot{\rho}) \right],$$  \hspace{1cm} (B.9)

which is quoted in Eq. (75) and re-expressed as an exact form w.r.t. the anti-BRST symmetry transformations $(s_{ab})$ that are quoted in Eq. (12).

We now concentrate on the crucial steps that are useful in the derivation of the conserved co-BRST charge $Q^{(3)}_d$ [cf. Eq. (86)] from the earlier expression $Q^{(2)}_d$ [cf. Eq. (82)]. In this context, first of all, we note that $Q^{(2)}_d$ of Eq. (82) can be expressed in terms of the total spacetime derivatives as:

$$Q^{(2)}_d = \int d^3 x \left[ (\dot{B} C - B \dot{C}) + (B_\mu \dot{C}^\mu - B_\mu C^\mu) + \partial_\mu (\dot{\beta} C^\mu - \beta \dot{C}^\mu) \right]$$
where

$$B \at \juncture, we ta ke the helps of EL-EOMs: (\partial_{\mu} C^\mu + m C) = \frac{\lambda}{2} \text{ and } \partial_{\mu} \dot{C}^\mu + m \dot{C} = \frac{\lambda}{2},$$
we obtain the following form of the above charge, namely:

$$Q^{(2)}_{(d)} = \int d^3 x \left[ (\dot{B} \dot{C} - \dot{B} \dot{C}) + (\partial_{\mu} \dot{C}^\mu - \partial_{\mu} \dot{C}^\mu) + \frac{1}{2} (\dot{\beta} \lambda - \dot{\beta} \lambda) \right], \quad (B.11)$$

where we have re-arranged the terms and have applied the Gauss divergence theorem to drop the total *space* derivative terms. At this stage, we use the EL-EOMs: $\Box + m^2 \beta = 0$ and $(\Box + m^2) C^\mu = 0$ to re-express (C.11) as

$$Q^{(2)}_{(d)} = \int d^3 x \left[ (\dot{B} \dot{C} - \dot{B} \dot{C}) + (\partial_{\mu} \dot{C}^\mu - \partial_{\mu} \dot{C}^\mu) + \frac{1}{2} (\dot{\beta} \lambda - \dot{\beta} \lambda) \right], \quad (B.12)$$

where we have used $\dot{\beta} = - \partial_t \beta - m^2 \beta$ and $\ddot{C}^0 = - \partial_t \dot{C}^0 - m^2 C^0$ and performed the partial integration in the evaluation of the term $(\dot{\beta} C^0 - \dot{\beta} \dot{C}^0)$ which turns out to contribute *zero* because of Gauss’s divergence theorem. At this stage, we apply the CF-type restriction: $\mathcal{B} + \dot{\mathcal{B}} + m \ddot{\varphi} = 0$ and $\partial_{\mu} \ddot{C}^\mu + m \ddot{C} = - \frac{\lambda}{2}$ and Gaussian’s divergence theorem to express the above form of the conserved charge as:

$$Q^{(3)}_{(d)} = \int d^3 x \left[ (\dot{B} \dot{C} - \dot{B} \dot{C}) + \partial_{\mu} (\dot{\varphi} \dot{C}^\mu - \dot{\varphi} \dot{C}^\mu) + \varphi (\partial_{\mu} \dot{C}^\mu + m \dot{C}) \right.
\left. - \dot{\varphi} (\partial_{\mu} \ddot{C}^\mu + m \ddot{C}) + (\dot{\varphi} \dot{C}^\mu - \dot{\varphi} \dot{C}^\mu) + \frac{1}{2} (\dot{\beta} \lambda - \dot{\beta} \lambda) \right], \quad (B.13)$$

where we have denoted the charge by the symbol $Q^{(3)}_{(d)}$ after the applications of the CF-type restrictions. We exploit the usefulness of EL-EOMs: $\partial_{\mu} \ddot{C}^\mu + m \ddot{C} = - \frac{\lambda}{2}$ and $\partial_{\mu} \ddot{C}^\mu + m \ddot{C} = - \frac{\lambda}{2}$ and Gaussian’s divergence theorem to express the above form of the conserved charge as:

$$Q^{(3)}_{(d)} = \int d^3 x \left[ (\dot{B} \dot{C} - \dot{B} \dot{C}) + (\dot{B} \dot{C} - \dot{B} \dot{C}) + \frac{1}{2} (\dot{\beta} \lambda - \dot{\beta} \lambda) \right]
\left. + \frac{1}{2} (\dot{\varphi} \rho - \dot{\varphi} \rho) + (\varphi \ddot{C}^0 - \varphi \ddot{C}^0) \right], \quad (B.14)$$

At this juncture, we take the helps of EL-EOMs: $(\Box + m^2) \ddot{C} = 0$ and $(\Box + m^2) \varphi = 0$ where the latter EOM has been derived from $\partial_{\mu} \mathcal{B}^\mu + m \dot{\mathcal{B}} = 0$ with the inputs from Eq. (23) where $\mathcal{B}^\mu$ and $\mathcal{B}$ have been expressed in terms of $\dot{\varphi}$ and $\ddot{\varphi}$. The substitutions of the following

$$\dot{\varphi} = - \partial_t \dot{\varphi} - m^2 \varphi, \quad \ddot{C}^0 = - \partial_t \dot{C}^0 - m^2 \ddot{C}^0, \quad (B.15)$$

and the use of Gaussian’s divergence theorem demonstrate that the actual contribution of the *last* term in (C.14) is zero. Thus, finally, we obtain the following form of the conserved co-BRST charge

$$Q^{(3)}_{(d)} = \int d^3 x \left[ (\dot{B} \dot{C} - \dot{B} \dot{C}) + (\dot{B} \dot{C} - \dot{B} \dot{C}) + \frac{1}{2} (\dot{\beta} \lambda - \dot{\beta} \lambda) + \frac{1}{2} (\dot{\varphi} \rho - \dot{\varphi} \rho), \quad (B.16)$$

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which has been mentioned in the main body of our text as Eq. (86) and has been expressed as an exact form w.r.t. the anti-co-BRST symmetry \( \delta_{ad} \).

We end this Appendix with the closing remarks that exactly similar types of exercises have been performed to obtain \( Q^{(3)}_{ab} \) and \( Q^{(3)}_{ad} \) [cf. Eqs. (80), (91)] from the expressions for the same charges as \( Q^{(2)}_{ab} \) and \( Q^{(2)}_{ad} \) [cf. Eqs. (78), (88)] which are very interesting and illuminating as far as the exact forms of the conserved charges are concerned. Furthermore, we observe that our present discussion in this Appendix is relevant to the Subsecs. 6.1 as well as 6.2 where we have discussed the off-shell nilpotency and absolute anticommutativity properties of the conserved charges in the ordinary space and superspace (by exploiting the ACSA to BRST formalism), respectively.

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