A UNIFORM MODEL FOR KIRILLOV-RESHETIKHIN CRYSTALS II.  
ALCOVE MODEL, PATH MODEL, AND $P = X$

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Abstract. We establish the equality of the specialization $P_{\lambda}(x; q, 0)$ of the Macdonald polynomial at $t = 0$ with the graded character $X_{\lambda}(x; q)$ of a tensor product of “single-column” Kirillov-Reshetikhin (KR) modules for untwisted affine Lie algebras. This is achieved by constructing two uniform combinatorial models for the crystals associated with the mentioned tensor products: the quantum alcove model (which is naturally associated to Macdonald polynomials), and the quantum Lakshmibai-Seshadri path model. We provide an explicit affine crystal isomorphism between the two models, and realize the energy function in both models. In particular, this gives the first proof of the positivity of the $t = 0$ limit of the symmetric Macdonald polynomial in the untwisted and non-simply-laced cases, when it is expressed as a linear combination of the irreducible characters for a finite-dimensional simple Lie subalgebra, as well as a representation-theoretic meaning of the coefficients in this expression in terms of degree functions.

1. Introduction

We prove the equality of the specialization $P_{\lambda}(x; q, 0)$ of the Macdonald polynomial $P_{\lambda}(x; q, t)$ (see [Ma2] for the definition of Macdonald polynomials) at $t = 0$ with the graded character $X_{\lambda}(x; q)$ of a tensor product of “single-column” Kirillov-Reshetikhin (KR) modules [KR] for all untwisted affine Lie algebras. This result follows from another important result in this paper, namely the construction of the explicit isomorphism between two uniform combinatorial models for the affine crystals associated with the mentioned tensor products.

The first model has its origins in the work of Naito and Sagaki [NS1, NS2, NS3, NS5, NS6], who realized the tensor products of single-column KR crystals in terms of so-called projected level-zero affine Lakshmibai-Seshadri (LS) paths. We provide an explicit description of these paths (as quantum LS paths), in terms of the parabolic quantum Bruhat graph [BFP, Po, LS], which originated from (small) quantum cohomology of partial flag manifolds. This part of our work is based on previous results of ours in [LNSSS1], where we study various properties of the parabolic quantum Bruhat graph, including two lifts of it: to the Bruhat order on the affine Weyl group, and to Littelmann’s level-zero weight poset [Li]; we also provided a precise characterization of the latter.

In the second part of the paper, we construct an explicit affine crystal isomorphism between the quantum LS path model and the quantum alcove model in [LL1]. The latter is intimately related to $P_{\lambda}(x; q, 0)$ by the Ram-Yip formula for Macdonald polynomials [RY]. This leads us to our $P_{\lambda} = X_{\lambda}$ result.

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The context of this project has its origins in Ion’s observation [Ion] that, when the affine simple root \( \alpha_0 \) is short (which includes the duals of untwisted affine root systems), \( P_\lambda(x; q, 0) \) is an affine Demazure character (see [Sa] for type \( A \)). On the other hand, Fourier and Littelmann [FL] showed that, for simply-laced untwisted affine Lie algebras, these Demazure characters are graded characters of tensor products of KR modules, and hence of local Weyl modules for current algebras, by using results in [NS2]. Combining [Ion] and [FL], one deduces the equality \( P_\lambda = X_\lambda \) in the simply-laced untwisted cases. Our results go beyond this. Let us note that, in the non-simply-laced untwisted affine types, \( X_\lambda \) is, in general, a positive (but not explicitly described) sum of affine Demazure characters, as proved in [Na, Theorem A]; this means that the underlying tensor product of KR modules is, in fact, larger than a corresponding Demazure module. Because of this, we cannot directly apply the known results and methods in the simply-laced untwisted cases to prove the equality \( P_\lambda = X_\lambda \) for all untwisted cases; this is why we need the affine crystal isomorphism between the quantum LS path model and the quantum alcove model. Here we should mention that the “folding” procedure (according to an affine Dynkin diagram automorphism) does not work for the equality \( P_\lambda = X_\lambda \) in the non-simply-laced untwisted affine types, since this procedure only gives us Macdonald polynomials associated to twisted affine root systems.

Our work reveals other interesting connections, complementing some recent results in the literature. Braverman and Finkelberg [BF2] have shown that, for simply-laced untwisted affine root systems, the characters of the duals of certain current algebra modules, called global Weyl modules, coincide with the characters \( \Psi_\lambda(x; q) \) of the spaces of global sections of line bundles on quasi-maps spaces, which arise in the study of quantum cohomology and quantum K-theory of the flag manifold; in this case, it is also shown that the function \( \Psi_\lambda(x; q) \) is equal to \( P_\lambda(x; q, 0) \) times an explicit product of geometric series whose ratios are powers of \( q \), and these functions are called \( q \)-Whittaker functions due to their appearance in the quantum group version of the Kostant-Whittaker reduction of Etingof [E] and Sevostyanov [Se] for the \( q \)-Toda integrable system. More precisely, the functions \( \Psi_\lambda(x; q) \) are eigenfunctions of the \( q \)-Toda difference operators, and their generating function yields the \( K \)-theoretic \( J \)-function of Givental and Lee [BF1]. Note, however, that in the non-simply-laced untwisted cases, the situation differs considerably: indeed, the proof in [BF2] of the equality between \( \Psi_\lambda(x; q) \) and \( P_\lambda(x; q, 0) \) times the explicit product, does not carry over; this is mainly because \( X_\lambda(x; q) \) is not a single affine Demazure character. Finally, the quantum alcove model arises in Lenart and Postnikov’s conjectural description of the quantum product by a divisor in quantum K-theory [LP]. We summarize these connections in Figure 1.

Combinatorial models for all nonexceptional KR crystals (not just of column-shape) were given in [FOS]. The quantum LS path model and the quantum alcove model uniformly describe tensor products of column-shape KR crystals, for all untwisted affine types. More precisely, these models realize the root operators on the aforementioned tensor products, and also give efficient formulas for the corresponding energy function; the latter can be viewed as an affine grading on a tensor product of KR crystals [NS6, ST], and is used to express one-dimensional configuration sums in statistical mechanics [HKOTT, HKOTY]. Given that, in large rank, certain configuration sums for classical Lie types were shown to coincide with certain parabolic Lusztig \( q \)-analogues of weight multiplicity [LOS], we can also use our energy formula to compute the latter. Moreover, by results in [LOS], we can calculate the energy on certain tensor products of “single-row” KR crystals. Another application of the quantum alcove model, which was given in [LL2], is a uniform realization of the combinatorial \( R \)-matrix (i.e., the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals).
The quantum LS path model and the quantum alcove model were implemented in the computer algebra system SAGE \cite{Sage, Sage-comb}. Using this implementation, we verified some conjectures related to KR crystals in the exceptional types (except for two Dynkin nodes for type $E_8^{(1)}$); these conjectures, which had been previously proved only in the classical types \cite{FOS1}, are concerned with the perfectness property of KR crystals \cite{HKOTT}, and with their graded classical decompositions \cite{HKOTY}.

There have been several developments related to the work in this paper. Based on our results, an interpretation of the specialization $P_\lambda(x; q, 0)$ above is given in \cite{INS, NS7}, in terms of the crystal bases of level-zero extremal weight modules over quantum affine algebras. On another hand, our work was used in \cite{CSSW} to provide the character of a stable level-one Demazure module associated to type $B_n^{(1)}$ as an explicit combination of suitably specialized Macdonald polynomials. In addition, our results were used in a crucial way by Chari and Ion in \cite{CI, Theorem 4.2} to show that Macdonald polynomials at $t = 0$ are characters of local Weyl modules for current algebras. Based on this, they prove a Bernstein-Gelfand-Gelfand (BGG) reciprocity theorem for the category of representations of a current algebra. In related work, Khoroshkin \cite{Kho} exhibits a categorification of Macdonald polynomials, by realizing them as the Euler characteristic of bigraded characters for certain complexes of modules over a current algebra. This realization simplifies considerably if BGG reciprocity holds (the mentioned complexes become actual modules concentrated in homological degree zero).

The paper is organized as follows. In Sections 2 and 3 we review the affine Lakshmibai-Sahadri (LS) and the quantum Lakshmibai-Sahadri path models, respectively. Theorem 3.3 shows that the set of projected level-zero affine LS paths $\mathbb{E}(\lambda)_{cl}$ is the same as the set of quantum LS paths $QLS(\lambda)$, where $\lambda$ is a (level-zero) dominant integral weight. This fact is also proven in \cite{LNSSS2} in a somewhat roundabout way, by providing an explicit description of the image of a quantum LS path under root operators and showing that the set of quantum LS paths is stable under the action of the root operators. (Quantum) LS paths carry a grading by a degree function (which is closely related to the (“right”) energy function on KR crystals). We provide
an explicit formula for the degree function of quantum LS paths in Theorem 4.6 in terms of the parabolic quantum Bruhat graph. For KR crystals, there exist the right and the left energy functions. In Section 4, we also relate the left energy function with the degree function using the Lusztig involution. In Section 5 the quantum alcove model and its crystal structure are defined. In Section 6, we show that there is a bijection between the quantum alcove model and the quantum LS path model by exhibiting a forgetful map and its inverse. We show that up to Kashiwara operators $f_0$ at the end of their strings, there is an affine crystal isomorphism between the quantum alcove model and a tensor product of KR crystals. Section 7 contains the main application of this work: by showing that the energy/degree function under the affine crystal isomorphism maps to a height function in the quantum alcove model, we show that the graded character of a tensor product of single-column KR crystals is equal to a Macdonald polynomial evaluated at $t = 0$ (see Corollary 7.11). We conclude in Section 8 with the proof of Lemmas from various sections. In Appendix A, we verify the conjectures in [HKOTT, HKOTY] mentioned above, in the exceptional types.

We follow the same conventions and notation as in [LNSSSS1].

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2. Lakshmibai-Seshadri paths

In this section we review Lakshmibai-Seshadri paths and the corresponding affine crystal model. As summarized in Theorem 2.7 and Remark 2.8, the crystal of level-zero projected LS paths is isomorphic to tensor products of KR crystals.

2.1. Basic notation. Let $g_{af}$ be an untwisted affine Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{ij})_{i,j \in I_{af}}$. The index set $I_{af}$ of the Dynkin diagram of $g_{af}$ is numbered as in [Kac, Section 4.8, Table Aff 1]. Take the distinguished vertex $0 \in I_{af}$ as in [Kac], and set $I := I_{af} \setminus \{0\}$. Let $h_{af} = (\bigoplus_{j \in I_{af}} \mathbb{C} \alpha_j^\vee) \oplus \mathbb{C}d$ denote the Cartan subalgebra of $g_{af}$, where $\{\alpha_j\}_{j \in I_{af}} \subset h_{af}$ is the set of simple coroots, and $d \in h_{af}$ is the scaling element (or degree operator). Also, we denote by $\{\alpha_j\}_{j \in I_{af}} \subset h_{af}^* := \text{Hom}_\mathbb{C}(h_{af}, \mathbb{C})$ the set of simple roots, and by $\Lambda_j \in h_{af}^*$, $j \in I_{af}$, the fundamental weights; note that $\langle d, \alpha_j \rangle = \delta_{j,0}$ and $\langle d, \Lambda_j \rangle = 0$ for $j \in I_{af}$. Let $\delta = \sum_{j \in I_{af}} a_j \alpha_j \in h_{af}^*$ and $c = \sum_{j \in I_{af}} a_j \alpha_j^\vee \in h_{af}$ denote the null root and the canonical central element of $g_{af}$, respectively. The dual weight lattice $X_{af}^\vee$ and the weight lattice $X_{af}$ are defined...
as follows:

\[(2.1) \quad X_{af}^\vee = \left( \bigoplus_{j \in I_{af}} \mathbb{Z} \alpha_j^\vee \right) \oplus \mathbb{Z} d \subset \mathfrak{b}_{af}^* \quad \text{and} \quad X_{af} = \left( \bigoplus_{j \in I_{af}} \mathbb{Z} \Lambda_j \right) \oplus \mathbb{Z} \delta \subset \mathfrak{b}_{af}^*.
\]

It is clear that \( X_{af} \) contains \( Q_{af} := \bigoplus_{j \in I_{af}} \mathbb{Z} \alpha_j \), and that \( X_{af} \cong \text{Hom}_\mathbb{Z}(X_{af}^\vee, \mathbb{Z}) \). We set \( Q_{af}^+ := \sum_{j \in I_{af}} \mathbb{Z}_{\geq 0} \alpha_j \). Let \( \mathfrak{g} \) be the classical subalgebra of \( \mathfrak{g}_{af} \) and denote the finite weight lattice by \( X = \bigoplus_{i \in I} \mathbb{Z} \varpi_i \), where the \( \varpi \) are the fundamental weights associated with \( \mathfrak{g} \). We set

\[
Q := \bigoplus_{j \in I} \mathbb{Z} \alpha_j, \quad Q^+ := \sum_{j \in I} \mathbb{Z}_{\geq 0} \alpha_j, \quad Q^\vee := \bigoplus_{j \in I} \mathbb{Z} \alpha_j^\vee, \quad Q^{\vee +} := \sum_{j \in I} \mathbb{Z}_{\geq 0} \alpha_j^\vee.
\]

Let \( W_{af} \) (resp. \( W \)) be the affine (resp. finite) Weyl group with simple reflections \( r_i \) for \( i \in I_{af} \) (resp. \( i \in I \)). \( W_{af} \) acts on \( X_{af} \) and \( X_{af}^\vee \) by

\[
\begin{align*}
    r_i \lambda &= \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \\
    r_i h &= h - \langle h, \alpha_i \rangle \alpha_i^\vee
\end{align*}
\]

for \( i \in I_{af} \), \( \lambda \in X_{af} \), and \( h \in X_{af}^\vee \). We denote by \( \ell \) the length function on \( W_{af} \) (resp. \( W \)), and denote by \( e \) the identity element of \( W_{af} \). Note that \( W_{af} \cong W \ltimes Q^\vee \); denote by \( t_\xi \) the image of \( \xi \in Q^\vee \) in \( W_{af} \).

The set of affine real roots (resp. roots) of \( \mathfrak{g}_{af} \) (resp. \( \mathfrak{g} \)) are defined by \( \Phi_{af}^+ = W_{af} \{ \alpha_i \mid i \in I_{af} \} \) (resp. \( \Phi = W \{ \alpha_i \mid i \in I \} \)). The set of positive affine real (resp. positive) roots are the set \( \Phi_{af}^+ = \Phi_{af} \cap Q_{af}^+ \) (resp. \( \Phi^+ = \Phi \cap Q^+ \)). We have \( \Phi_{af} = \Phi_{af}^+ \cup \Phi_{af}^- \), where \( \Phi_{af}^- = -\Phi_{af}^+ \), and \( \Phi = \Phi^+ \cup \Phi^- \), where \( \Phi^- = -\Phi^+ \). We have \( \delta = \alpha_0 + \theta \), where \( \theta \) is the highest root for \( \mathfrak{g} \), and

\[
\Phi_{af}^+ = \Phi^+ \cup (\Phi + \mathbb{Z}_{>0} \delta).
\]

For \( \beta \in \Phi_{af}^+ \), let \( \beta^\vee \) denote the coroot of \( \beta \), and let \( r_\beta \in W_{af} \) denote the associated reflection. Also, we set \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \).

The level of a weight \( \lambda \in X_{af} \) is defined by \( \text{lev}(\lambda) = \langle c, \lambda \rangle \). Since the action of \( W_{af} \) on \( X_{af} \) is level-preserving, the sublattice \( X_{af}^0 \subset X_{af} \) of level-zero elements is \( W_{af} \)-stable. The natural projection \( \text{cl} : X_{af}^0 \to X \) has kernel \( \mathbb{Z} \delta \) and sends \( \Lambda_i - a_i^\vee \Lambda_0 \) to \( \varpi_i \) for \( i \in I \); also, there is a section \( X \to X_{af}^0 \) given by \( \varpi_i \mapsto \Lambda_i - a_i^\vee \Lambda_0 \) for \( i \in I \).

Let \( J \) be a subset of \( I \). Denote by \( W_J \) the parabolic subgroup of \( W \) generated by \( r_i \) for \( i \in J \). For \( w \in W \), we denote by \( \{ w \} = [w]_J \) the minimum-length coset representative in the coset \( wW_J \), and set \( W^J := \{ \{ w \} \mid w \in W \} \). We set \( Q_J := \bigoplus_{j \in J} \mathbb{Z} \alpha_j \), \( Q_J := \bigoplus_{j \in J} \mathbb{Z} \alpha_j \), and \( \Phi_J := \Phi \cap Q_J \), \( \Phi_J^+ := \Phi^+ \cap Q_J \); note that \( \Phi_J = \Phi_J^+ \cup \Phi_J^- \). Also, we set \( \rho_J = \frac{1}{2} \sum_{\alpha \in \Phi_J^+} \alpha \).

Finally, we briefly review the level-zero weight poset. Fix a dominant weight \( \lambda \) in the finite weight lattice \( X \). We view \( X \) as a sublattice of \( X_{af}^0 \). Let \( X_{af}^0(\lambda) \) be the orbit of \( \lambda \) under the action of the affine Weyl group \( W_{af} \).

**Definition 2.1 (Li Section 4)**. A poset structure is defined on \( X_{af}^0(\lambda) \) as the transitive closure of the relation

\[(2.2) \quad \mu < r_\beta \mu \iff \langle \beta^\vee, \mu \rangle > 0,
\]

where \( \beta \in \Phi_{af}^+ \). This poset is called the level-zero weight poset for \( \lambda \).
2.2. Definition of Lakshmibai-Seshadri paths. In this subsection, we fix a dominant integral weight \( \lambda \in X \). We recall the definition of Lakshmibai-Seshadri (LS) paths of shape \( \lambda \) from [LI] Section 4. Let \( X^0_{af}(\lambda) \) be the level-zero weight poset for \( \lambda \).

**Definition 2.2.** For \( \mu, \nu \in X^0_{af}(\lambda) \) with \( \nu > \mu \) and \( b \in \mathbb{Q} \), a \( b \)-chain for \((\nu, \mu)\) is, by definition, a sequence \( \nu = \nu_0 > \nu_1 > \cdots > \nu_m = \mu \) of covers in \( X^0_{af}(\lambda) \) such that \( b(\beta^\vee_k, \mu_k) \in \mathbb{Z} \) for all \( k = 1, 2, \ldots, m \), where \( \beta_k \in \Phi^{af+} \) is the corresponding positive real root for \( \mu_{k-1} > \mu_k \). Here, for \( \mu, \mu' \in X^0_{af}(\lambda) \), the cover \( \mu > \mu' \) in \( X^0_{af}(\lambda) \) means that \( \mu > \mu' \) in \( X^0_{af}(\lambda) \) and that there exists no \( \nu \in X^0_{af}(\lambda) \) such that \( \mu > \nu > \mu' \) in \( X^0_{af}(\lambda) \).

**Definition 2.3.** An LS path of shape \( \lambda \) is, by definition, a pair \( \pi = (\nu; b) \) of a sequence \( \nu : \nu_1 > \nu_2 > \cdots > \nu_s \) of elements in \( X^0_{af}(\lambda) \) and a sequence \( b : 0 = b_0 < b_1 < \cdots < b_s = 1 \) of rational numbers satisfying the condition that there exists a \( b_k \)-chain for \((\nu_k, \nu_{k+1})\) for each \( k = 1, 2, \ldots, s - 1 \).

Denote by \( \mathbb{B}(\lambda) \) the set of all LS paths of shape \( \lambda \). We identify an element

\[ \pi = (\nu_1, \nu_2, \ldots, \nu_s; b_0, b_1, \ldots, b_s) \in \mathbb{B}(\lambda) \]

with the following piecewise-linear, continuous map \( \pi : [0, 1] \rightarrow \mathbb{R} \otimes \mathbb{Z} X^0_{af} \): \(^{(2.3)}\)

\[ \pi(t) = \sum_{k=1}^{l-1} (b_k - b_{k-1}) \nu_k + (t - b_{l-1}) \nu_l \quad \text{for} \quad b_{l-1} \leq t \leq b_l, \quad 1 \leq l \leq s. \]

**Remark 2.4.** It follows from the definition of an LS path of shape \( \lambda \) that \( \pi_\nu : (\nu; 0, 1) \in \mathbb{B}(\lambda) \) for every \( \nu \in X^0_{af}(\lambda) \), which corresponds to the straight line \( \pi_\nu(t) = t \nu, \ t \in [0, 1] \).

Recall that \( X^0_{af} / \mathbb{Z} \delta \cong X \). Denote by

\[ \text{cl} : \mathbb{R} \otimes \mathbb{Z} X^0_{af} \rightarrow \mathbb{R} \otimes \mathbb{Z} X^0_{af} / \mathbb{R} \delta \cong \mathbb{R} \otimes \mathbb{Z} X \]

the canonical projection; remark that \( \text{cl}(X^0_{af}(\lambda)) = W\lambda \cong W^J \) (see [LNSSS1], Lemma 3.1). For \( \pi \in \mathbb{B}(\lambda) \), we define \( \text{cl}(\pi) \) by: \( (\text{cl}(\pi))(t) = \text{cl}(\pi(t)) \) for \( t \in [0, 1] \); note that \( \text{cl}(\pi) \) is a piecewise linear, continuous map from \([0, 1]\) to \( \mathbb{R} \otimes \mathbb{Z} X \). Then we set

\[ \mathbb{B}(\lambda)_{cl} := \{ \text{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda) \}; \]

an element of this set is called a projected level-zero LS path.

2.3. Crystal structures on \( \mathbb{B}(\lambda) \) and \( \mathbb{B}(\lambda)_{cl} \). As in the previous subsection, let \( \lambda \in X \) be a dominant integral weight. We use the following notation:

\[ \alpha_i := \begin{cases} \alpha_i & \text{if } i \neq 0, \\ -\theta & \text{if } i = 0 \end{cases}, \quad s_i := \begin{cases} r_i & \text{if } i \neq 0, \\ r_\theta & \text{if } i = 0, \end{cases} \]

where \( \theta \) is the highest root for \( g \).

Following [LI], we give \( \mathbb{B}(\lambda) \) and \( \mathbb{B}(\lambda)_{cl} \) crystal structures with the weight lattices \( X^0_{af} \) and \( \text{cl}(X^0_{af}) \cong X \), respectively. Here we focus on the crystal structure on \( \mathbb{B}(\lambda)_{cl} \) for the crystal structure on \( \mathbb{B}(\lambda) \), in the argument below, replace \( \eta \in \mathbb{B}(\lambda)_{cl} \) with \( \pi \in \mathbb{B}(\lambda) \), and then replace \( \alpha_j \in \Phi \) and \( s_j \in W \) with \( \alpha_j \in \Phi^{af} \) and \( r_j \in W_{af} \).

Let \( \eta \in \mathbb{B}(\lambda)_{cl} \). We see from [LI Lemma 4.5 a)] that \( \eta(1) \in \text{cl}(X^0_{af}) \cong X \). So we set

\[ \text{wt}(\eta) := \eta(1) \in X. \]
Next we define root operators $e_j$ and $f_j$ for $j \in I_{af} = I \sqcup \{0\}$ as follows (see [Li, Section 1]). We set
\[
H(t) = H^0_j(t) := \langle \tilde{\alpha}_j^\vee, \eta(t) \rangle \quad \text{for} \quad t \in [0, 1],
\]
\[
m = m^0_j := \min \{ H^0_j(t) \mid t \in [0, 1] \}.
\]
It follows from [Li, Lemma 4.5 d)] that all local minima of $H(t)$ are integers; in particular, $m \in \mathbb{Z}_{\leq 0}$. If $m = 0$, then $e_j \eta := 0$, where $0$ is an extra element not contained in $\mathfrak{B}(\lambda)_{cl}$. If $m \leq -1$, then set
\[
t_1 := \min \{ t \in [0, 1] \mid H(t) = m \},
\]
\[
t_0 := \max \{ t \in [0, t_1] \mid H(t) = m + 1 \}.
\]

**Remark 2.5.**

(1) Recall that all local minima of $H(t)$ are integers by [Li, Lemma 4.5 d)]. Hence we deduce that $H(t)$ is strictly decreasing on $[t_0, t_1]$.

(2) Because $H(t)$ attains the minimum $m$ at $t = t_1$, it follows immediately that $H(t_1 + \epsilon) \geq H(t_1)$ for sufficiently small $\epsilon > 0$.

(3) We deduce that $H(t_0 - \epsilon) \geq H(t_0)$ for sufficiently small $\epsilon > 0$. Indeed, suppose that $H(t_0 - \epsilon) < H(t_0)$. Then the minimum $m'$ of $H(t)$ on $[0, t_0]$ is less than $H(t_0) = m + 1$. Since all local minima of $H(t)$ are integers, we obtain $m' = m$. However, this contradicts the definition of $t_1$; recall that $t_0 < t_1$.

Define $e_j \eta$ for $j \in I_{af}$ by:
\[
(e_j \eta)(t) = \begin{cases} 
\eta(t) & \text{if } 0 \leq t \leq t_0, \\
\eta(t_0) + s_j(\eta(t) - \eta(t_0)) & \text{if } t_0 \leq t \leq t_1, \\
\eta(t) + \tilde{\alpha}_j & \text{if } t_1 \leq t \leq 1,
\end{cases}
\]
where $s_j \in W$ is the reflection with respect to $\tilde{\alpha}_j \in \Phi$. We see from [Li, Corollary 2 a)] that $e_j \eta \in \mathfrak{B}(\lambda)_{cl}$. The definition of $f_j \eta$ is similar (see also [NS6, Section 2.2]). In addition, for $\eta \in \mathfrak{B}(\lambda)_{cl}$ and $j \in I_{af}$, we set
\[
\varepsilon_j(\eta) := \max \{ n \geq 0 \mid e_j^n \eta \neq 0 \}, \quad \varphi_j(\eta) := \max \{ n \geq 0 \mid f_j^n \eta \neq 0 \}.
\]
We see from [Li, Section 2] that the set $\mathfrak{B}(\lambda)_{cl}$ together with the map $\text{wt} : \mathfrak{B}(\lambda)_{cl} \to X$, the root operators $e_j, f_j, j \in I_{af}$, and the maps $\varepsilon_j, \varphi_j, j \in I_{af}$, becomes a crystal with $\text{cl}(X_{af}^0(\lambda)) \cong X$ the weight lattice.

**Remark 2.6.** It is easily verified that
\[
\text{wt}(\text{cl}(\pi)) = \text{cl}(\text{wt}(\pi)) \quad \text{for} \quad \pi \in \mathfrak{B}(\lambda),
\]
\[
\text{cl}(e_j \pi) = e_j \text{cl}(\pi) \quad \text{and} \quad \text{cl}(f_j \pi) = f_j \text{cl}(\pi) \quad \text{for} \quad \pi \in \mathfrak{B}(\lambda) \text{ and } j \in I_{af},
\]
\[
\varepsilon_j(\text{cl}(\pi)) = \varepsilon_j(\pi) \quad \text{and} \quad \varphi_j(\text{cl}(\pi)) = \varphi_j(\pi) \quad \text{for} \quad \pi \in \mathfrak{B}(\lambda) \text{ and } j \in I_{af}.
\]

We know the following theorem from [NS1, NS2, NS3].

**Theorem 2.7.**

(1) For each $i \in I$, the crystal $\mathfrak{B}(\varpi_i)_{cl}$ is isomorphic to the crystal basis of $W(\varpi_i)$, the level-zero fundamental representation of the quantum affine algebra $U_q'(\mathfrak{g}_{af})$ (without the degree operator), introduced by Kashiwara [Kas].
(2) The crystal graph of $\mathcal{B}(\lambda)_{\text{cl}}$ is connected.

(3) Let $i = (i_1, i_2, \ldots, i_p)$ be an arbitrary sequence of elements of $I$ (with repetitions allowed), and set $\lambda_i := w_{i_1} + w_{i_2} + \cdots + w_{i_p}$. Then, there exists an isomorphism $\Psi_i : \mathcal{B}(\lambda_i)_{\text{cl}} \rightarrow \mathcal{B}(w_{i_1})_{\text{cl}} \otimes \cdots \otimes \mathcal{B}(w_{i_p})_{\text{cl}}$ of crystals.

**Remark 2.8.** It is known that the fundamental representation $W(\varpi_i)$ of level-zero is isomorphic to the Kirillov-Reshetikhin (KR) module $W_i^{(i)}$ in the sense of [HKOTT] Section 2.3 (for the Drinfeld polynomials of $W(\varpi_i)$, see [N] Remark 3.3). Also we can prove that the crystal basis of $W(\varpi_i) \cong W_i^{(i)}$ is unique, up to a nonzero constant multiple (see also [NS4, Lemma 1.5.3]); we call this crystal basis a (one-column) KR crystal, and denote by $B_i^{\pm 1}$. By the theorem above, the crystal $\mathcal{B}(\lambda)_{\text{cl}}$ of projected level-zero LS paths of shape $\lambda$ is a model for the corresponding tensor product of KR crystals.

In this paper we use the Kashiwara convention for the tensor product. More precisely, for two (normal) crystals $B_1$ and $B_2$, the tensor product $B_1 \otimes B_2$ as a set is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\text{wt}(b) = \text{wt}(b_1) + \text{wt}(b_2)$. In the Kashiwara convention the crystal operators are given by

$$f_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes f_i(b_2) & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ f_i(b_1) \otimes b_2 & \text{otherwise,} \end{cases}$$

and similarly for $e_i(b)$, where $\varepsilon_j$ and $\varphi_j$ are defined as in (2.7).

### 3. Quantum Lakshmibai-Seshadri paths

In this section, we introduce quantum Lakshmibai-Seshadri paths, which are defined in terms of the parabolic quantum Bruhat graph. The main result of this section is Theorem 3.3, which shows that projected level-zero LS paths are QLS paths. Although this result is proved in [LNSSS2], the proof given there is somewhat roundabout, and heavily depends on the connectedness of the affine crystal $\mathcal{B}(\lambda)_{\text{cl}}$, which itself is a deep result; in contrast, the proof given here is much more direct, and we need not use root operators.

#### 3.1. The parabolic quantum Bruhat graph

The quantum Bruhat graph was first introduced in a paper by Brenti, Fomin and Postnikov [BFP] motivated by work of Fomin, Gelfand and Postnikov [FGP] in type $A$. It later appeared in connection with the quantum cohomology of flag varieties in a paper by Fulton and Woodward [FW].

Let $J$ be a subset of $I$. We denote by $\text{QB}(W^J)$ the parabolic quantum Bruhat graph. Its vertex set is $W^J$. There are two kinds of directed edges. Both are labeled by some $\alpha \in \Phi^+ \setminus \Phi^+_J$. For $w \in W^J$, there is a directed edge $w \xrightarrow{\alpha} [wr_{\alpha}]$ (recall that $[wr_{\alpha}]$ denotes the minimum-length coset representative in the coset $wr_{\alpha}W_J$) if $\alpha \in \Phi^+ \setminus \Phi^+_J$ and one of the following holds:

1. (Bruhat edge) $w \triangleleft wr_{\alpha}$ is a covering relation in Bruhat order, that is, $\ell(wr_{\alpha}) = \ell(w) + 1$. (One may deduce that $wr_{\alpha} \in W^J$.)
2. (Quantum edge) $\ell([wr_{\alpha}]) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho - 2\rho_J \rangle$.

(3.1) We define the weight of an edge $w \xrightarrow{\alpha} [wr_{\alpha}]$ in the parabolic quantum Bruhat graph to be either $\alpha^\vee$ or 0, depending on whether it is a quantum edge or not, respectively. Then the weight of a directed path $p$, denoted by $\text{wt}(p) \in Q_{\geq 0}$, is defined as the sum of the weights of its edges.
3.2. Definition of quantum Lakshmibai-Seshadri paths. In this subsection, we fix a dominant integral weight \( \lambda \in X \). Set
\[
J = \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \},
\]
so that \( W_J \) is the stabilizer of \( \lambda \). Given a rational number \( b \), we define \( QB_{b\lambda}(W^J) \) to be the subgraph of the parabolic quantum Bruhat graph \( QB(W^J) \) with the same vertex set but having only the edges:
\[
x \xrightarrow{\alpha} y \quad \text{with} \quad \langle \alpha^\vee, b\lambda \rangle = b(\alpha^\vee, \lambda) \in \mathbb{Z};
\]

\( \text{note that } QB_{b\lambda}(W^J) = QB(W^J) \text{ if } b \in \mathbb{Z}. \)

**Definition 3.1.** A quantum Lakshmibai-Seshadri (QLS) path of shape \( \lambda \) is a pair \( \eta = (x; b) \) of a sequence \( x = (x_1, x_2, \ldots, x_s) \) of elements of \( W^J \) with \( x_k \neq x_{k+1} \) for \( 1 \leq k \leq s-1 \) and a sequence \( b = (b_0 < b_1 < \cdots < b_s = 1) \) of rational numbers satisfying the condition that there exists a directed path from \( x_{k+1} \) to \( x_k \) in \( QB_{b\lambda}(W^J) \) for each \( 1 \leq k \leq s-1 \).

Denote by \( QLS(\lambda) \) the set of QLS paths of shape \( \lambda \). We use the notation \( x \xrightarrow{b\lambda} y \) to indicate that there exists a directed path from \( x \) to \( y \) in \( QB_{b\lambda}(W^J) \), where \( J \) is as in (3.2); so we can write an element
\[
\eta = (x_1, x_2, \ldots, x_s; b_0, b_1, \ldots, b_s)
\]
in \( QLS(\lambda) \) as follows:
\[
x_1 \xrightarrow{b_1\lambda} x_2 \xleftarrow{b_2\lambda} \cdots \xleftarrow{b_{s-1}\lambda} x_s. \tag{3.4}
\]

Since \( W_J \) can be identified with \( W\lambda \) under the canonical bijection \( w \mapsto w\lambda \), we will sometimes think of the elements \( x_i \) as weights. Moreover, we identify \( \eta \) with the following piecewise-linear, continuous map \( \eta : [0,1] \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} X \):
\[
\eta(t) = \sum_{k=1}^{l-1} (b_k - b_{k-1}) x_k \lambda + (t - b_{l-1}) x_l \lambda \quad \text{for} \quad b_{l-1} \leq t \leq b_l, \ 1 \leq l \leq s. \tag{3.5}
\]

**Remark 3.2.** It follows from the definition of a QLS path of shape \( \lambda \) that \( \eta_x := (x; 0, 1) \in QLS(\lambda) \) for every \( x \in W^J \), with \( J \) as in (3.2), which corresponds to the straight line \( \eta_x(t) = tx\lambda \), \( t \in [0,1] \). We can easily see that \( \text{cl}(\pi_\nu) = \eta_{\text{cl}(\nu)} \) for \( \nu \in X^0_{\text{al}}(\lambda) \); recall that \( \text{cl}(X^0_{\text{al}}(\lambda)) = W\lambda \cong W^J \).

3.3. Relation between LS paths and QLS paths. We now establish the correspondence between projected level-zero LS paths and QLS paths. As before, \( \lambda \in X \) is a fixed dominant integral weight, and \( J = \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \).

**Theorem 3.3.** \( B(\lambda)_{\text{cl}} = QLS(\lambda) \) as sets of piecewise-linear, continuous maps from \([0,1]\) to \( \mathbb{R} \otimes_{\mathbb{Z}} X \) (see Section 2.2 and (3.5)).

In order to prove this theorem, we need the following lemma.

**Lemma 3.4.** Let \( b \in \mathbb{Q} \).

1. Let \( \mu, \nu \in X^0_{\text{al}}(\lambda) \). If there exists a \( b \)-chain for \( (\nu, \mu) \), then there exists a directed path from \( \text{cl}(\mu) \) to \( \text{cl}(\nu) \) in \( QB_{b\lambda}(W^J) \).

2. Let \( w, w' \in W^J \). If there exists a directed path from \( w \) to \( w' \) in \( QB_{b\lambda}(W^J) \), then for each \( \mu \in X^0_{\text{al}}(\lambda) \) with \( \text{cl}(\mu) = w \), there exists a \( b \)-chain for \( (\nu, \mu) \) for some \( \nu \in X^0_{\text{al}}(\lambda) \) with \( \text{cl}(\nu) = w' \).
Proof. (1) It suffices to show the assertion in the case that \( \nu \) is a cover of \( \mu \), i.e., \( \mu \preceq \nu \) in \( X^0(\lambda) \). Let \( \beta \in \Phi^{af} \) be such that \( r_\beta \mu = \nu \) and \( b(\beta', \mu) \in \mathbb{Z} \). Then, \( \beta \in \Phi^+ \) or \( \beta \in \delta - \Phi^+ \) (see \cite[Lemma 6.5 (1)]{LNSSS1}). Set \( w := \text{cl}(\mu) \in \mathcal{W} \lambda \cong \mathcal{W}^J \). If \( \beta \in \Phi^+ \), then it follows from \cite[Theorem 6.5]{LNSSS1} that \( \gamma := w^{-1} \beta \in \Phi^+ \setminus \Phi^+_J \) and \( \text{cl}(\mu) = w \stackrel{\gamma}{\rightarrow} \text{cl}(\nu) \) in \( \mathcal{Q}^B(W^J) \). In addition, we see that \( b(\gamma^\vee, \lambda) = b(\beta', \mu) \in \mathbb{Z} \), which implies that \( \text{cl}(\mu) = w \stackrel{\gamma}{\rightarrow} \text{cl}(\nu) \) in \( \mathcal{Q}^B_{b\lambda}(W^J) \). Similarly, if \( \beta \in \delta - \Phi^+ \), then it follows from \cite[Theorem 6.5]{LNSSS1} that \( \gamma := w^{-1}(\beta - \delta) \in \Phi^+ \setminus \Phi^+_J \) and \( \text{cl}(\mu) = w \stackrel{\gamma}{\rightarrow} \text{cl}(\nu) \) in \( \mathcal{Q}^B(W^J) \). We see that \( b(\gamma^\vee, \lambda) = b(\beta' - c, \mu) = b(\beta', \mu) \in \mathbb{Z} \), which implies that \( \text{cl}(\mu) = w \stackrel{\gamma}{\rightarrow} \text{cl}(\nu) \) in \( \mathcal{Q}^B_{b\lambda}(W^J) \). Thus we have proved part (1).

(2) Fix \( \mu \in X^0_{al}(\lambda) \) such that \( \text{cl}(\mu) = w \). Assume that
\[
 w = x_0 \gamma_1 \rightarrow x_1 \gamma_2 \rightarrow \ldots \rightarrow x_m = w'
\]
is a directed path from \( w \) to \( w' \) in \( \mathcal{Q}^B_{b\lambda}(W^J) \). We show the assertion by induction on the length \( m \) of the directed path above. Assume first that \( m = 1 \); for simplicity of notation, we set \( \gamma := \gamma_1 \). Set
\[
 \beta := \begin{cases} 
 w\gamma & \text{if } w \stackrel{\gamma}{\rightarrow} w' \text{ is a Bruhat edge}, \\
 \delta + w\gamma & \text{if } w \stackrel{\gamma}{\rightarrow} w' \text{ is a quantum edge}.
\end{cases}
\]
It follows from \cite[Theorem 6.5]{LNSSS1} that \( \beta \in \Phi^{af} \) and \( \mu \not\preceq r_\beta \mu =: \nu \). Also, we see that \( \text{cl}(\nu) = w' \). In addition, \( b(\beta', \mu) = b(\nu^\vee, \lambda) \in \mathbb{Z} \). Thus, \( \mu \preceq \nu \) is a \( b \)-chain for \((\nu, \mu)\). Assume that \( m \geq 2 \). By our induction hypothesis, there exists a \( b \)-chain for \((\mu', \mu)\) for some \( \mu' \in X^0_{al}(\lambda) \) with \( \text{cl}(\mu') = x_{m-1} \). Also, by our induction hypothesis, there exists a \( b \)-chain for \((\nu, \mu')\) for some \( \nu \in X^0_{al}(\lambda) \) with \( \text{cl}(\nu) = x_{m} = w' \). Concatenating these \( b \)-chains, we obtain a \( b \)-chain for \((\nu, \mu)\). Thus we have proved the lemma.

Proof of Theorem 3.3 First, let us show that \( \mathbb{B}(\lambda)_{cl} \subseteq \mathbb{QLS}(\lambda) \). Let
\[
(3.6) \quad \pi = (\nu_1, \nu_2, \ldots, \nu_{s-1}, \nu_s; b_0, b_1, b_2, \ldots, b_{s-1}, b_s) \in \mathbb{B}(\lambda).
\]
We show \( \text{cl}(\pi) \in \mathbb{QLS}(\lambda) \) by induction on \( s \). If \( s = 1 \), then the assertion is obvious by Remark 3.2. Assume that \( s > 1 \). Set
\[
(3.7) \quad \pi' := (\nu_2, \ldots, \nu_{s-1}, \nu_s; b_0, b_2, \ldots, b_{s-1}, b_s).
\]
Then we see that \( \pi' \in \mathbb{B}(\lambda) \), and hence \( \text{cl}(\pi') \in \mathbb{QLS}(\lambda) \) by our induction hypothesis. Write \( \text{cl}(\pi') \) as:
\[
(3.8) \quad \text{cl}(\pi') := (y_1, y_2, \ldots, y_u; c_0, c_1, \ldots, c_{u-1}, c_u)
\]
for some \( y_1, y_2, \ldots, y_u \in W^J \) and \( 0 = c_0 < c_1 < \cdots < c_{u-1} < c_u = 1 \). Here we claim that \( 0 < b_1 < b_2 \leq c_1 \) and \( y_1 = \text{cl}(\nu_2) \); notice that the inequalities \( 0 < b_1 < b_2 \) are obvious by the definition of LS paths. We show that \( b_2 \leq c_1 \) and \( y_1 = \text{cl}(\nu_2) \). By (2.3) and (3.7), we have
\[
(3.9) \quad \text{cl}(\pi')(t) = \begin{cases} 
 t\nu_2 & \text{for } t \in [0, b_2], \\
 b_2\nu_2 + (t - b_2)\nu_3 & \text{for } t \in [b_2, b_3].
\end{cases}
\]
Hence we have
\[
(3.9) \quad \text{cl}(\pi')(t) = \begin{cases} 
 t\text{cl}(\nu_2) & \text{for } t \in [0, b_2], \\
 b_2\text{cl}(\nu_2) + (t - b_2)\text{cl}(\nu_3) & \text{for } t \in [b_2, b_3];
\end{cases}
\]
note that \( \text{cl}(\nu_2) \) and \( \text{cl}(\nu_3) \) are direction vectors of \( \text{cl}(\pi') \) for the intervals \([0, b_2]\) and \([b_2, b_3]\), respectively. Therefore,
(a) if $\text{cl}(\nu_2) \neq \text{cl}(\nu_3)$, then the first turning point of $\text{cl}(\pi')$ is equal to $b_2$,
(b) if $\text{cl}(\nu_2) = \text{cl}(\nu_3)$, then the first turning point of $\text{cl}(\pi')$ is greater than or equal to $b_2$;

in particular, the first turning point of $\text{cl}(\pi')$ is greater than or equal to $b_2$. Next, by (3.8), we have

$$
\text{(3.10)} \quad (\text{cl}(\pi'))(t) = \begin{cases} 
    t(y_1\lambda) & \text{for } t \in [0, c_1], \\
    c_1(y_1\lambda) + (t - c_1)(y_2\lambda) & \text{for } t \in [c_1, c_2].
\end{cases}
$$

Since $y_1 \in W^J$ is not equal to $y_2 \in W^J$ in (3.8), we have $y_1\lambda \neq y_2\lambda$, which implies that $c_1$ is nothing but the first turning point of $\text{cl}(\pi')$. Because the first turning point of $\text{cl}(\pi')$ is greater than or equal to $b_2$ as seen above, we obtain $b_2 \leq c_2$. Moreover, by comparing the first direction vector of $\text{cl}(\pi')$ in (3.9) and that in (3.10), we obtain $y_1 = \text{cl}(\nu_2)$, as desired.

If $\text{cl}(\nu_1) = \text{cl}(\nu_2)$, then it follows immediately that $\text{cl}(\pi) = \text{cl}(\pi')$, and hence $\text{cl}(\pi) \in \text{QLS}(\lambda)$. Assume that $\text{cl}(\nu_1) \neq \text{cl}(\nu_2) = y_1$; set $x_1 := \text{cl}(\nu_1) \in W\lambda \cong W^J$. Because there exists a $b_1$-chain for $(\nu_1, \nu_2)$ by the definition of an LS path, we deduce from Lemma 3.4(1) that there exists a directed path from $y_1 = \text{cl}(\nu_2)$ to $x_1 = \text{cl}(\nu_1)$ in $\text{QB}_{b_1\lambda}(W^J)$. Therefore, we see that

$$(x_1, y_1, y_2, \ldots, y_u; c_0, b_1, c_1, \ldots, c_u)$$

is a QLS path of shape $\lambda$, which is identical to $\text{cl}(\pi)$. Thus we obtain $\text{cl}(\pi) \in \text{QLS}(\lambda)$, as desired.

Next, let us show the opposite inclusion, i.e., $\mathcal{B}(\lambda)_{cl} \supset \text{QLS}(\lambda)$. Let

$$\eta = (x_1, x_2, \ldots, x_s, y_s; b_0, b_1, b_2, \ldots, b_{s-1}, b_s) \in \text{QLS}(\lambda).$$

We show by induction on $s$ that there exists $\pi \in \mathcal{B}(\lambda)$ such that $\text{cl}(\pi) = \eta$. If $s = 1$, then the assertion is obvious by Remark 3.2. Assume that $s > 1$. We see that

$$\eta' := (x_2, \ldots, x_s, x_s; b_0, b_2, \ldots, b_{s-1}, b_s)$$

is contained in $\text{QLS}(\lambda)$. Hence, by our induction hypothesis, there exists $\pi' \in \mathcal{B}(\lambda)$ such that $\text{cl}(\pi') = \eta'$. Write $\pi'$ as:

$$\pi' = (\mu_1, \mu_2, \ldots, \mu_u; c_0, c_1, \ldots, c_{u-1}, c_u)$$

for some $\mu_1, \mu_2, \ldots, \mu_u \in X^0_{af}(\lambda)$ and $0 = c_0 < c_1 < \ldots < c_{u-1} < c_u = 1$; we remark that $0 < b_1 < b_2 \leq c_1$ and $\text{cl}(\mu_1) = x_2$. Because there exists a directed path from $x_2 = \text{cl}(\mu_1)$ to $x_1$ in $\text{QB}_{b_1\lambda}(W^J)$, it follows from Lemma 3.4(2) that there exists a $b_1$-chain for $(\nu_1, \mu_1)$ for some $\nu_1 \in X^0_{af}(\lambda)$ with $\text{cl}(\nu_1) = x_1$. Therefore, we have

$$\pi := (\nu_1, \mu_1, \mu_2, \ldots, \mu_u; c_0, b_1, c_1, \ldots, c_{u-1}, c_u) \in \mathcal{B}(\lambda).$$

Also, it is easily seen that $\text{cl}(\pi) = \eta$. Thus we have proved the opposite inclusion, thereby completing the proof of the theorem.

4. Formula for the degree function

Throughout this section, we fix a dominant integral weight $\lambda \in X$, and set $J := \{i \in I \mid \langle \alpha_i^y, \lambda \rangle = 0\}$. We define the degree function on projected level-zero LS paths in Section 4.2 and recall the relation with the energy function on KR crystals in Theorem 4.5. Theorem 4.6 is the main result of this section and provides an explicit expression for the degree function as sums of weights of shortest paths in the parabolic quantum Bruhat graph.
4.1. **Weights of directed paths.** We know the following proposition from [LNSSSI Propo-
sition 8.1].

**Proposition 4.1.** Let \( x, y \in W^J \), where \( J = \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). Let \( p \) and \( q \) be a shortest
and an arbitrary directed path from \( x \) to \( y \) in \( QB(W^J) \), respectively. Then there exists \( h \in Q^\vee_+ \)
such that

\[
wt(q) - wt(p) \equiv h \mod Q^\vee_J.
\]

In addition, if \( q \) is also shortest, then \( wt(q) \equiv wt(p) \mod Q^\vee_J \).

Let \( x, y \in W^J \), where \( J = \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). By Proposition 4.1, the value \( \langle wt(p), \lambda \rangle \)
for a shortest directed path \( p \) from \( x \) to \( y \) in \( QB(W^J) \) does not depend on the choice of such a
shortest directed path; we denote this value by \( wt_\lambda(x \Rightarrow y) \).

The following is a corollary to [LNSSSI Lemma 7.7]; for \( \tilde{\alpha}_j \) and \( s_j \), see [2.4].

**Corollary 4.2.** Let \( w_1, w_2 \in W^J \), and \( j \in I_{af} \).

1. If \( \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle > 0 \) and \( \langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle \leq 0 \), then
   \[
   wt_\lambda([s_jw_1] \Rightarrow w_2) = wt_\lambda(w_1 \Rightarrow w_2) - \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle.
   \]

2. If \( \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle < 0 \) and \( \langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle < 0 \), then
   \[
   wt_\lambda([s_jw_1] \Rightarrow [s_jw_2]) = wt_\lambda(w_1 \Rightarrow w_2) - \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle + \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle.
   \]

3. If \( \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle \geq 0 \) and \( \langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle < 0 \), then
   \[
   wt_\lambda(w_1 \Rightarrow [s_jw_2]) = wt_\lambda(w_1 \Rightarrow w_2) + \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_2 \lambda \rangle.
   \]

**Proof.** We give a proof only for part (1); the proofs for parts (2) and (3) are similar. Let \( p \) be
a shortest directed path from \( w_1 \) to \( w_2 \). Then it follows from [LNSSSI Lemma 7.7 (3) and (5)]
that there exists a shortest directed path \( p' \) from \( [s_jw_1] \) to \( w_2 \) such that

\[
wt(p') = wt(p) - \delta_{j,0} w_1^{-1} \tilde{\alpha}_j^\vee.
\]

Hence,

\[
wt_\lambda([s_jw_1] \Rightarrow w_2) = \langle wt(p'), \lambda \rangle = \langle wt(p), \lambda \rangle - \delta_{j,0} \langle w_1^{-1} \tilde{\alpha}_j^\vee, \lambda \rangle
\]

\[
= wt_\lambda(w_1 \Rightarrow w_2) - \delta_{j,0} \langle \tilde{\alpha}_j^\vee, w_1 \lambda \rangle.
\]

Thus we have proved the corollary. \( \square \)

4.2. **Definition of the degree function.** Let us recall from [NS6 Section 3.1] the definition
of the degree function

\[
\text{Deg} = \text{Deg}_\lambda : \mathbb{B}(\lambda)_{cl} \rightarrow \mathbb{Z}_{\leq 0}.
\]

Denote by \( \mathbb{B}_0(\lambda) \) the connected component of \( \mathbb{B}(\lambda) \) containing the straight line \( \pi_\lambda = (\lambda; 0, 1) \).
Also, for \( \pi = (\nu_1, \ldots, \nu_s; b_0, \ldots, b_s) \in \mathbb{B}(\lambda) \), we set \( \iota(\pi) := \nu_1 \), and call it the initial direction
of \( \pi \); note that \( \iota(\pi) = \pi(\varepsilon)/\varepsilon \) for sufficiently small \( \varepsilon > 0 \). We know from [NS6 Proposition
3.1.3] that for each \( \eta \in \mathbb{B}(\lambda)_{cl} \), there exists a unique \( \pi_\eta \in \mathbb{B}_0(\lambda) \) satisfying the conditions that
\( \text{cl}(\pi_\eta) = \eta \) and \( \iota(\pi_\eta) \in \lambda - Q^+ \); recall that \( Q^+ = \sum_{j \in I} \mathbb{Z}_{\geq 0} \alpha_j \). Then it follows from [NS6
Lemma 3.1.1] that \( \pi_\eta(1) \in X^0_{af} \) is of the form:

\[
\pi_\eta(1) = \lambda - \beta + K\delta
\]
for some $\beta \in Q^+$ and $K \in \mathbb{Z}_{\geq 0}$. We define the degree $\text{Deg}(\eta) \in \mathbb{Z}_{\leq 0}$ of $\eta \in \mathcal{B}(\lambda)_{\text{cl}}$ by:

$$
\text{Deg}(\eta) = -K \in \mathbb{Z}_{\leq 0}.
$$

**Example 4.3.** Assume that $\mathfrak{g}$ is of type $A_2^{(1)}$, and let $\lambda = \varpi_1 + \varpi_2$. Note that $J = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$ is the empty set, and hence $W^J = W$; for the quantum Bruhat graph $QB(W)$, see [LNSS1, Fig. 1].

(1) Set $\eta_1 := (e, w_0; 0, 1/2, 1)$. Since $w_0 = r_0 \theta \rightarrow e$ is a quantum edge in $QB(W)$, which is also an edge in $QB(1/2)_{\lambda}(W)$, we see that $\eta_1 \in \text{QLS}(\lambda) = \mathbb{B}(\lambda)_{\text{cl}}$. We claim that $\pi_{\eta_1} = (\lambda, r_0\lambda; 0, 1/2, 1)$. First we see that $e_0\pi_{\lambda} = (\lambda, r_0\lambda; 0, 1/2, 1)$, which implies that $(\lambda, r_0\lambda; 0, 1/2, 1)$ is contained in $\mathbb{B}_0(\lambda)$. Since $\text{cl}(\lambda) = e$ and $\text{cl}(r_0\lambda) = w_0$, the image of $(\lambda, r_0\lambda; 0, 1/2, 1)$ under the map $\text{cl}$ is identical to $\eta$. Also, the initial direction of this element is equal to $\lambda \in \lambda - Q^+$. Therefore, we deduce that $\pi_{\eta_1} = (\lambda, r_0\lambda; 0, 1/2, 1)$, as desired. Since $\pi_{\eta_1}(1) = \lambda + \alpha_0 = \lambda - \theta + \delta$, we have $\text{Deg}(\eta_1) = -1$.

(2) Set $\eta_2 := (r_1r_2, r_2; 0, 1/2, 1)$. Since $r_2 \theta \rightarrow r_1r_2$ is a Bruhat edge in $QB(W)$, which is also an edge in $QB(1/2)_{\lambda}(W)$, we see that $\eta_2 \in \text{QLS}(\lambda) = \mathbb{B}(\lambda)_{\text{cl}}$. As above, we deduce that $\pi_{\eta_2} = (r_1r_2, r_2; 0, 1/2, 1)$. Since $\pi_{\eta_2}(1) = \lambda - (\alpha_1 + \alpha_2)$, we have $\text{Deg}(\eta_2) = 0$.

**Remark 4.4.** It is known (see, e.g., [NS6, Proposition 4.3.1]) that for each $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$, there exist $j_1, j_2, \ldots, j_s \in I_{af}$ such that $e_{j_1}e_{j_2} \ldots e_{j_s} \eta = \eta$; recall from Remark 3.2 that $\eta_0 = \eta_1 = \text{cl}(\pi_\lambda)$. Therefore, we deduce from [NS6, Lemma 3.2.1] that $\text{Deg} = \text{Deg}_\lambda : \mathbb{B}(\lambda)_{\text{cl}} \rightarrow \mathbb{Z}_{\leq 0}$ is a unique function satisfying the following conditions:

(i) $\text{Deg}(\eta_0) = 0$;

(ii) for $\eta \in \mathbb{B}(\lambda)_{\text{cl}}$ and $j \in I_{af}$ with $e_j \eta \neq 0$,

$$
\text{Deg}(e_j \eta) = \begin{cases} 
\text{Deg}(\eta) - 1 & \text{if } j = 0 \text{ and } \iota(e_0\eta) = \iota(\eta), \\
\text{Deg}(\eta) - \langle \alpha_i^\vee, \iota(\eta) \rangle - 1 & \text{if } j = 0 \text{ and } \iota(e_0\eta) = s_0(\iota(\eta)), \\
\text{Deg}(\eta) & \text{if } j \neq 0,
\end{cases}
$$

where $\iota(\eta) := \eta(\varepsilon)/\varepsilon$ for sufficiently small $\varepsilon > 0$.

### 4.3. Relation between the degree function and the energy function.

Write $\lambda$ as $\lambda = \varpi_{i_1} + \varpi_{i_2} + \cdots + \varpi_{i_p}$, with $i_1, i_2, \ldots, i_p \in I$. By Theorem 2.7 (3), there exists an isomorphism

$$
\Psi : \mathbb{B}(\lambda)_{\text{cl}} \cong \mathbb{B}(\varpi_{i_1})_{\text{cl}} \otimes \mathbb{B}(\varpi_{i_2})_{\text{cl}} \otimes \cdots \otimes \mathbb{B}(\varpi_{i_p})_{\text{cl}} =: \mathcal{B}
$$

of crystals. Here we should recall from Remark 2.8 that $\mathbb{B}(\varpi_{i_l})_{\text{cl}}$ is isomorphic to the one-column KR crystal $B^{1,1}$. Also, recall from Section 2.3 that we are using the Kashiwara convention for tensor products in this paper. So, following [HKOTY, Section 3] and [HKOTT, Section 3.3] (see also [SS] and [NS6, Section 4.1]), we define the energy function $D = D_\lambda : \mathcal{B} \rightarrow \mathbb{Z}_{\leq 0}$ on $\mathcal{B}$ as follows. First, for each $1 \leq k, l \leq p$, there exists a unique isomorphism (called a combinatorial $R$-matrix)

$$
R_{k,l} : \mathbb{B}(\varpi_{i_k})_{\text{cl}} \otimes \mathbb{B}(\varpi_{i_l})_{\text{cl}} \cong \mathbb{B}(\varpi_{i_l})_{\text{cl}} \otimes \mathbb{B}(\varpi_{i_k})_{\text{cl}}
$$

of crystals. Also, there exists a unique $\mathbb{Z}$-valued function (called a local energy function) $H_{k,l} : \mathbb{B}(\varpi_{i_k})_{\text{cl}} \otimes \mathbb{B}(\varpi_{i_l})_{\text{cl}} \rightarrow \mathbb{Z}$ satisfying the following conditions (H1) and (H2):
(H1) For $\eta_k \otimes \eta_l \in B(\omega_{i_k})_{cl} \otimes B(\omega_{i_l})_{cl}$ and $j \in \mathbb{I}_d$ such that $e_j(\eta_k \otimes \eta_l) \neq 0$,

$$H_{k,l}(e_j(\eta_k \otimes \eta_l)) =
\begin{cases}
H_{k,l}(\eta_k \otimes \eta_l) + 1 & \text{if } j = 0, \text{ and if } e_0(\eta_k \otimes \eta_l) = e_0\eta_k \otimes \eta_l, e_0(\tilde{\eta}_l \otimes \tilde{\eta}_k) = e_0\tilde{\eta}_l \otimes \tilde{\eta}_k, \\
H_{k,l}(\eta_k \otimes \eta_l) - 1 & \text{if } j = 0, \text{ and if } e_0(\eta_k \otimes \eta_l) = \eta_k \otimes e_0\eta_l, e_0(\tilde{\eta}_l \otimes \tilde{\eta}_k) = \tilde{\eta}_l \otimes e_0\tilde{\eta}_k,
\end{cases}$$

where we set $\tilde{\eta}_l \otimes \tilde{\eta}_k := R_{k,l}(\eta_k \otimes \eta_l) \in B(\omega_{i_k})_{cl} \otimes B(\omega_{i_l})_{cl}$.

(H2) $H_{k,l}(\eta_{\omega_{i_k}} \otimes \eta_{\omega_{i_l}}) = 0$.

Now, for each $1 \leq k < l \leq p$, there exists a unique isomorphism

$$B(\omega_{i_k})_{cl} \otimes B(\omega_{i_{k+1}})_{cl} \otimes \cdots \otimes B(\omega_{i_{l-1}})_{cl} \otimes B(\omega_{i_l})_{cl} \xrightarrow{\sim} B(\omega_{i_k})_{cl} \otimes B(\omega_{i_{k+1}})_{cl} \otimes \cdots \otimes B(\omega_{i_{l-2}})_{cl} \otimes B(\omega_{i_{l-1}})_{cl}$$

of crystals, which is given by composition of combinatorial $R$-matrices. Given $\eta_k \otimes \eta_{k+1} \otimes \cdots \otimes \eta_l \in B(\omega_{i_k})_{cl} \otimes B(\omega_{i_{k+1}})_{cl} \otimes \cdots \otimes B(\omega_{i_l})_{cl}$, we define $\eta^{(k)}_l \in B(\omega_{i_l})_{cl}$ to be the first factor of the image of $\eta_k \otimes \eta_{k+1} \otimes \cdots \otimes \eta_l$ under the above isomorphism of crystals. For convenience, we set $\eta^{(l)}_l := \eta_l$ for $\eta_l \in B(\omega_{i_l})_{cl}, 1 \leq l \leq p$. In addition, for each $1 \leq k \leq p$, take (and fix) an arbitrary element $\eta^{(k)}_k \in B(\omega_{i_k})_{cl}$ such that $f_j \eta^{(k)}_k = 0$ for all $j \in \mathbb{I}$. Then we define the energy function $D = D_B : B = B(\omega_{i_1})_{cl} \otimes B(\omega_{i_2})_{cl} \otimes \cdots \otimes B(\omega_{i_p})_{cl} \to \mathbb{Z}$ by:

$$D(\eta_1 \otimes \eta_2 \cdots \otimes \eta_p) =
\sum_{1 \leq k < l \leq p} H_{k,l}(\eta_k \otimes \eta_l^{(k+1)}) + \sum_{k=1}^{p} H_{k,k}(\eta^{(k)}_k \otimes \eta^{(1)}_k).$$

(4.4)

We know the following theorem from [NS6 Theorem 4.1.1].

**Theorem 4.5.** Using the same notation as above, we have

$$\text{Deg}(\eta) = D(\Psi(\eta)) - D^\text{ext} \quad \text{for every } \eta \in B(\lambda)_{cl},$$

where $D^\text{ext} = D^\text{ext}_B \in \mathbb{Z}$ is a constant defined by

$$D^\text{ext} = D^\text{ext}_B := \sum_{k=1}^{p} H_{k,k}(\eta^{(k)}_k \otimes \eta_{\omega_{i_k}});$$

here, $\Psi : B(\lambda)_{cl} \xrightarrow{\sim} B$ is the isomorphism of crystals given in (4.3).

### 4.4. Formula for the degree function.

Let $\eta \in B(\lambda)_{cl}$. Because $B(\lambda)_{cl} = \text{QLS}(\lambda)$ by Theorem 3.3, we can write $\eta$ as:

$$\eta = (x_1, x_2, \ldots, x_s; b_0, b_1, \ldots, b_s) \in \text{QLS}(\lambda)$$

for some $x_1, x_2, \ldots, x_s \in W^J$ and $0 = b_0 < b_1 < \cdots < b_s = 1$; note that $\varepsilon(\eta) = x_1 \lambda$. 


Theorem 4.6. With the same notation as above, we have

\begin{equation}
\text{Deg}(\eta) = - \sum_{k=1}^{s-1} (1 - b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k).
\end{equation}

Proof. For \( \eta \in \text{QLS}(\lambda) = \mathbb{B}(\lambda)_{cl} \), we define \( F(\eta) \) to be the right-hand side of (4.7). It suffices to show that \( F \) satisfies conditions (i) and (ii) in Remark 4.4, i.e.,

(i) \( F(\eta_\lambda) = 0 \);

(ii) for \( \eta \in \mathbb{B}(\lambda)_{cl} \) and \( j \in I_{af} \) with \( e_j \eta \neq 0 \),

\begin{equation}
F(e_j \eta) = \begin{cases} 
F(\eta) - 1 & \text{if } j = 0 \text{ and } \iota(\eta_0 \eta) = \iota(\eta), \\
F(\eta) - \langle \alpha_0^\vee, \iota(\eta) \rangle - 1 & \text{if } j = 0 \text{ and } \iota(\eta_0 \eta) = s_0(\iota(\eta)), \\
F(\eta) & \text{if } j \neq 0.
\end{cases}
\end{equation}

It is obvious that \( F \) satisfies condition (i). Let us show that \( F \) satisfies condition (ii). Let \( \eta \in \mathbb{B}(\lambda)_{cl} \) and \( j \in I_{af} \) be such that \( e_j \eta \neq 0 \). We see that the point \( t_1 = \min \{ t \in [0,1] \mid H_j^\eta(t) = m_j^\eta \} \) is equal to \( b_p \) for some \( 0 < p \leq s \). Let \( 0 < q \leq p \) be such that \( b_{q-1} \leq t_0 < b_q \); recall that \( t_0 = \max \{ t \in [0,t_1] \mid H_j^\eta(t) = m_j^\eta + 1 \} \). It follows from the definition of the root operator \( e_j \) that \( e_j \eta \in \text{QLS}(\lambda) \) can be written as follows:

\[
\begin{array}{c}
x_1 \xleftarrow{b_1} \cdots \xleftarrow{b_{p-2}} x_{q-1} \xleftarrow{b_{q-1}} x_q \xleftarrow{t_0} s_j x_q \xleftarrow{b_j} \cdots \xleftarrow{b_{p+1}} x_{p+1} \xleftarrow{b_{p+1}} x_p \xleftarrow{b_{p+1}} \cdots \xleftarrow{b_{s-1}} x_s. \\
\end{array}
\]

Here, if \( b_{q-1} = t_0 \), then we drop (a) from the above; note that in this case, \( x_{q-1} \neq [s_j x_q] \) since \( \langle \alpha_j^\vee, x_{q-1} \rangle \leq 0 \) and \( \langle \alpha_j^\vee, s_j x_q \lambda \rangle = -\langle \alpha_j^\vee, x_{q} \lambda \rangle > 0 \) by Remark 2.5(1), (3). Also, if \( [s_j x_p] = x_{p+1} \), then we replace (b) by \( x_{p+1} \) (or \( [s_j x_p] \)) in the above path. We see from the definition (2.6) of the root operator \( e_j \) that \( \iota(e_j \eta) = s_j \iota(\eta) \) if and only if \( t_0 = b_0 = 0 \); in this case, \( m_j^\eta = -1 \) (see (2.5)) since \( H_j^\eta(t_0) = H_j^\eta(0) = 0 \).

Now, by the definition of \( F \), we have

\begin{equation}
F(e_j \eta) = - \sum_{k=1}^{q-2} (1 - b_k) \text{wt}_\lambda(x_{k+1} \Rightarrow x_k) + R + \sum_{k=q}^{p-1} (1 - b_k) \text{wt}_\lambda([s_j x_{k+1}] \Rightarrow [s_j x_k])
\end{equation}
where

\[
R := \begin{cases} 
(1 - b_{q-1}) w_{t_\lambda}(x_q \Rightarrow x_{q-1}) + (1 - t_0) w_{t_\lambda}([s_j x_q] \Rightarrow x_q) & \text{if } t_0 \neq b_{q-1}, \\
(1 - b_{q-1}) w_{t_\lambda}([s_j x_q] \Rightarrow x_{q-1}) & \text{if } q > 1 \text{ and } t_0 = b_{q-1}, \\
0 & \text{if } q = 1 \text{ and } t_0 = b_0 = 0;
\end{cases}
\]

(4.10)

If \( p = s \) (resp., \( q = 1 \)), then \( w_{t_\lambda}(x_{p+1} \Rightarrow [s_j x_p]) \) in \( U_3 \) (resp., \( w_{t_\lambda}(x_q \Rightarrow x_{q-1}) \) in \( R \)) is understood to be 0; notice that the equality (4.9) is valid even when \([s_j x_p] = x_{p+1}\). Also, observe that in (4.10),

\[
w_{t_\lambda}([s_j x_q] \Rightarrow x_q) = -\delta_{j,0}(\tilde{\alpha}_j', x_q\lambda).
\]

(4.11)

Note that if \( q > 1 \) and \( t_0 = b_{q-1} \) (in the second case of (4.10)), then

\[
R = (1 - b_{q-1}) w_{t_\lambda}(x_q \Rightarrow x_{q-1}) - (1 - t_0) \delta_{j,0}(\tilde{\alpha}_j', x_q\lambda).
\]

Indeed, note that \( \langle \tilde{\alpha}_j', x_{q-1}\lambda \rangle \leq 0 \) and \( \langle \tilde{\alpha}_j', s_j x_q\lambda \rangle = -\langle \tilde{\alpha}_j', x_q\lambda \rangle \geq 0 \) by Remark 2.5 (1), (3). Therefore, applying Corollary 4.2 (1) to \( w_1 = [s_j x_q] \) and \( w_2 = x_{q-1} \), we see that

\[
R = (1 - b_{q-1}) w_{t_\lambda}([s_j x_q] \Rightarrow x_{q-1}) - (1 - t_0) \delta_{j,0}(\tilde{\alpha}_j', x_q\lambda),
\]

as desired. By combining (4.9) and (4.12), we obtain

\[
U_1 = \begin{cases} 
\sum_{k=1}^{q-1} (1 - b_k) w_{t_\lambda}(x_{k+1} \Rightarrow x_k) - (1 - t_0) \delta_{j,0}(\tilde{\alpha}_j', x_q\lambda) & \text{if } t_0 \neq 0, \\
0 & \text{if } t_0 = 0.
\end{cases}
\]

(4.13)

Recall that the function \( H_j^\circ(t) \) is strictly decreasing on \([t_0, t_1]\) (see Remark 2.5 (1)), which implies that \( \langle \tilde{\alpha}_j', x_q\lambda \rangle < 0 \) for all \( q \leq k \leq p \). Hence, by Corollary 4.2 (2), we have

\[
w_{t_\lambda}([s_j x_{k+1}] \Rightarrow [s_j x_k]) = w_{t_\lambda}(x_{k+1} \Rightarrow x_k) - \delta_{j,0}(\tilde{\alpha}_j', x_{k+1}\lambda) + \delta_{j,0}(\tilde{\alpha}_j', x_k\lambda)
\]

for each \( q \leq k \leq p - 1 \). From this, we see that

\[
U_2 = \sum_{k=q}^{p-1} (1 - b_k) w_{t_\lambda}(x_{k+1} \Rightarrow x_k) - \delta_{j,0} \sum_{k=q}^{p-1} (1 - b_k) \langle \tilde{\alpha}_j', x_{k+1}\lambda \rangle + \delta_{j,0} \sum_{k=q}^{p-1} (1 - b_k) \langle \tilde{\alpha}_j', x_k\lambda \rangle
\]

\[= \sum_{k=q}^{p-1} (1 - b_k) w_{t_\lambda}(x_{k+1} \Rightarrow x_k) + \delta_{j,0} (1 - b_q) \langle \tilde{\alpha}_j', x_q\lambda \rangle - \delta_{j,0} (1 - b_{p-1}) \langle \tilde{\alpha}_j', x_{p-1}\lambda \rangle.
\]

(4.14)
Finally, let us show that

\[(4.15) \quad U_3 = (1 - b_p) wt_\lambda(x_{p+1} \Rightarrow x_p) + \delta_{j,0}(1 - b_p)\langle \widehat{\alpha}_j^\vee, x_p \lambda \rangle,\]

where if \(p = s\), then \(wt_\lambda(x_{p+1} \Rightarrow x_p)\) is understood to be 0. If \(p = s\), then the equality obviously holds. Assume that \(p < s\). Then, since \(\langle \widehat{\alpha}_j^\vee, x_p \lambda \rangle < 0\) and \(\langle \widehat{\alpha}_j^\vee, x_{p+1} \lambda \rangle \geq 0\) by Remark 2.5(2), the equality \[(4.15)\]

follows immediately from Corollary 4.2(3) (applied to \(w_1 = x_{p+1}\) and \(w_2 = x_p\)).

Substituting \[(4.13), (4.14), (4.15)\] into \[(4.9)\], we conclude that

\[
F(e_j \eta) = - \sum_{k=1}^{s-1} (1 - b_k) wt_\lambda(x_{k+1} \Rightarrow x_k) - T + \delta_{j,0} \left\{ \langle b_q - t_0 \rangle \langle \widehat{\alpha}_j^\vee, x_q \lambda \rangle + \sum_{k=q+1}^{p} (b_k - b_{k-1}) \langle \widehat{\alpha}_j^\vee, x_k \lambda \rangle \right\},
\]

where

\[
T := \begin{cases} 0 & \text{if } t_0 \neq 0, \\ \delta_{j,0} \langle \widehat{\alpha}_j^\vee, x_1 \lambda \rangle = \delta_{j,0} \langle \widehat{\alpha}_j^\vee, \iota(\eta) \rangle & \text{if } t_0 = 0. \end{cases}
\]

Here, observe that

\[
V = H_j^\eta(b_p) - H_j^\eta(t_0) = H_j^\eta(t_1) - H_j^\eta(t_0) = m_j - (m_j^\eta + 1) = -1.
\]

Thus we have shown that \(F\) satisfies \((4.8)\), thereby completing the proof of Theorem 4.6. \(\square\)

4.5. **Lusztig involution** \(S\) on \(\mathbb{B}(\lambda)_{cl} = \text{QLS}(\lambda)\). The results in this subsection will be used in Section 6 to define the bijection between the quantum alcove model \(A(\lambda)\) and the set \(\text{QLS}(\lambda)\) (not \(\text{QLS}(-w_\alpha\lambda)\)). Let \(w_\circ \in W\) be the longest element in \(W\), and let \(\omega : I \rightarrow I\) be the Dynkin diagram automorphism for \(\mathfrak{g}\) induced by \(w_\circ\), i.e., \(w_\circ \alpha_j = -\alpha_{\omega(j)}\) for \(j \in I\). Note that \(\omega\) acts as \(-w_\circ\) on the integral weight lattice \(X\) and also on the Cartan subalgebra \(\mathfrak{h}\) of \(\mathfrak{g}\). Then, \(\omega(\theta) = \theta\), and \(\omega \gamma_j = r_{\omega(j)} \gamma_j\) on \(X\), and on \(\mathfrak{h}\) for \(j \in I\). There exists a group automorphism, denoted also by \(\omega\), of the Weyl group \(W\) such that \(\omega(r_j) = r_{\omega(j)}\) for all \(j \in I\); notice that \(\ell(\omega(v)) = \ell(v)\) for \(v \in W\), and \(\omega(r_\alpha) = r_{\omega(\alpha)}\) for \(\alpha \in \Phi^+\).

**Lemma 4.7.**

1. If \(v \in W^J\), then \(\omega(v) \in W^{\omega(J)}\).
2. Let \(b\) be a rational number. For \(x_1, x_2 \in W^J\) and \(\beta \in \Phi^+ \setminus \Phi^J\),

\[
x_1 \xrightarrow{\beta} x_2 \quad \text{in } \text{QB}_b(\lambda)(W^J) \iff [w_\circ x_1]^J \xleftarrow{v_\circ \beta} [w_\circ x_2]^J \in \text{QB}_b(\lambda)(W^J)
\]

\[
\iff \omega(x_1) \xrightarrow{\omega(\beta)} \omega(x_2) \quad \text{in } \text{QB}_b(\lambda)(W^{\omega(J)})
\]

\[
\iff [x_1 w_\circ]^J \xleftarrow{v_\circ \omega(\beta)} [x_2 w_\circ]^J \quad \text{in } \text{QB}_b(\lambda)(W^{\omega(J)}),
\]

where \(u_\circ\) and \(v_\circ\) are the longest elements in \(W_J\) and \(W_{\omega(J)}\), respectively. In addition, the types (i.e., Bruhat or quantum) of these four edges coincide.
Lemma 4.7 that \( x \in \{ 1 \} \). Proof. Part (1) is obvious. Part (2) follows from part (1) and [LNSSSS1], Proposition 4.3, together with the fact that \( vw = w_0 \omega(v) \) for all \( v \in W \). Part (3) follows from part (2) since \( w_0 \lambda = \lambda \) and \( v_0 \omega(\lambda) = \omega(\lambda) \). \( \square \)

Now, let

\[
(4.16) \quad \eta = (x_1, \ldots, x_s; b_0, b_1, \ldots, b_s) \in \text{QLS}(\lambda) = B(\lambda)_{cl},
\]

with \( x_1, \ldots, x_s \in W^J \) and rational numbers \( 0 = b_0 < \cdots < b_s = 1 \). Then we see from Lemma 4.7 that

\[
(4.17) \quad \eta^* := ([x_s w_0]^\omega(J), \ldots, [x_1 w_0]^\omega(J); 1 - b_s, 1 - b_{s-1}, \ldots, 1 - b_0)
\]

is a QLS path of shape \( \omega(\lambda) = -w_0 \lambda \); note that the stabilizer of the dominant integral weight \( \omega(\lambda) = -w_0 \lambda \) in \( W \) is identical to the parabolic subgroup \( W_\omega(J) \). Because \( \eta^*(t) = \eta(1 - t) - \eta(1) \) for \( t \in [0, 1] \), it follows from [Li, Lemma 2.1(e)] that

\[
(4.18) \quad \text{wt}(\eta^*) = -\text{wt}(\eta), \quad (e_j \eta)^* = f_j \eta^*, \quad (f_j \eta)^* = e_j \eta^*
\]

for \( \eta \in B(\lambda)_{cl} \) and \( j \in I_{af} \).

Next, for \( \eta \in \text{QLS}(\lambda) = B(\lambda)_{cl} \) of the form (4.16), we define \( \omega(\eta) \) by:

\[
(4.19) \quad \omega(\eta) = (\omega(x_1), \ldots, \omega(x_s); b_0, b_1, \ldots, b_s).
\]

We can easily check by using Lemma 4.7 that \( \omega(\eta) \) is a QLS path of shape \( \omega(\lambda) = -w_0 \lambda \). Since \((\omega(\eta))(t) = \omega(\eta(t)) \) for \( t \in [0, 1] \), we see that

\[
\text{wt}(\omega(\eta)) = \omega(\text{wt}(\eta)), \quad \omega(e_j \eta) = e_{w_j} \omega(\eta), \quad \omega(f_j \eta) = f_{w_j} \omega(\eta)
\]

for \( \eta \in B(\lambda)_{cl} \) and \( j \in I_{af} \).

Finally, we set \( S(\eta) := \omega(\eta)^* = (\omega(\eta))^* \) for each \( \eta \in \text{QLS}(\lambda) = B(\lambda)_{cl} \); by the argument above, we see that \( S(\eta) \in B(\lambda)_{cl} \). Moreover, it is easily checked that \( S \) is an involution on \( B(\lambda)_{cl} \), which we call the Lusztig involution (see also [LeS] for the affine version of the Lusztig involution in type C), such that

\[
(4.20) \quad \text{wt}(S(\eta)) = -\omega(\text{wt}(\eta)) = w_0(\text{wt}(\eta)), \quad S(e_j \eta) = f_{w_j}(S(\eta)), \quad S(f_j \eta) = e_{w_j}(S(\eta))
\]

for \( \eta \in B(\lambda)_{cl} \) and \( j \in I_{af} \). We remark that if \( \eta \) is of the form (4.16), then

\[
(4.21) \quad S(\eta) = ([\omega(x_s w_0]^{\omega(J)}, \ldots, [\omega(x_1 w_0]^{\omega(J)}; 1 - b_s, 1 - b_{s-1}, \ldots, 1 - b_0)
\]

Corollary 4.8. Let \( \eta = (x_1, x_2, \ldots, x_s; b_0, b_1, \ldots, b_s) \in \text{QLS}(\lambda) = B(\lambda)_{cl} \). Then,

\[
\text{Deg}_\lambda(S(\eta)) = -\sum_{k=1}^{s-1} b_k \text{wt}_\lambda(x_{k+1} \Rightarrow x_k)
\]
Proof. It follows from Theorem 4.6 and (4.21) that
\[
\text{Deg}_\lambda(S(\eta)) = -\sum_{k=1}^{s-1} \left\{ 1 - (1 - b_{s-k}) \right\} \text{wt}_\lambda([w_0 x_{s-k}]^J \Rightarrow [w_0 x_{s-k+1}]^J) = -\sum_{k=1}^{s-1} b_k \text{wt}_\lambda([w_0 x_k]^J \Rightarrow [w_0 x_{k+1}]^J).
\]
Also, by Lemma 4.7(3), we have \(\text{wt}_\lambda([w_0 x_k]^J \Rightarrow [w_0 x_{k+1}]^J) = \text{wt}_\lambda(x_{k+1} \Rightarrow x_k)\) for all \(1 \leq k \leq s-1\). This proves the corollary.

As in Section 4.3, we write \(\lambda = \varpi_{i_1} + \varpi_{i_2} + \cdots + \varpi_{i_p}\), with \(i_1, i_2, \ldots, i_p \in I\), and let
\[
\Psi : \mathcal{B}(\lambda)_{\text{cl}} \xrightarrow{\sim} \mathcal{B}(\varpi_{i_1})_{\text{cl}} \otimes \mathcal{B}(\varpi_{i_2})_{\text{cl}} \otimes \cdots \otimes \mathcal{B}(\varpi_{i_p})_{\text{cl}} =: \mathcal{B}
\]
be the isomorphism of crystals; recall again that \(\mathcal{B}(\varpi_i)_{\text{cl}} \cong B^{i,1}\) as crystals. Also, let
\[
\Psi^{\text{rev}} : \mathcal{B}(\lambda)_{\text{cl}} \xrightarrow{\sim} \mathcal{B}(\varpi_{i_p})_{\text{cl}} \otimes \mathcal{B}(\varpi_{i_{p-1}})_{\text{cl}} \otimes \cdots \otimes \mathcal{B}(\varpi_{i_1})_{\text{cl}} =: \mathcal{B}^{\text{rev}}
\]
be the isomorphism of crystals in Theorem 2.7. Furthermore, we define \(S : \mathcal{B} \to \mathcal{B}^{\text{rev}}\) by:
\[
(4.22) \quad S(\eta_1 \otimes \cdots \otimes \eta_p) = S(\eta_p) \otimes \cdots \otimes S(\eta_1)
\]
for \(\eta_1 \otimes \cdots \otimes \eta_p \in \mathcal{B} = \mathcal{B}(\varpi_{i_1})_{\text{cl}} \otimes \cdots \otimes \mathcal{B}(\varpi_{i_p})_{\text{cl}}\). Then we deduce that
\[
(4.23) \quad \text{wt}(S(\eta)) = -\omega(\text{wt}(\eta)) = w_0(\text{wt}(\eta)),
\]
\[
S(e_j \eta) = f_{\omega(j)} S(\eta), \quad S(f_j \eta) = e_{\omega(j)} S(\eta)
\]
for \(\eta \in \mathcal{B}\) and \(j \in I_{\text{af}}\). By the connectedness of the crystals (see Theorem 2.7(2), (3)) and (4.20), (4.23), we have the following commutative diagram:
\[
(4.24) \quad \begin{array}{ccc}
\mathcal{B}(\lambda)_{\text{cl}} & \xrightarrow{\Psi} & \mathcal{B} \\
\Downarrow S & & \Downarrow S \\
\mathcal{B}(\lambda)_{\text{cl}} & \xrightarrow{\Psi^{\text{rev}}} & \mathcal{B}^{\text{rev}}
\end{array}
\]
The next corollary follows immediately from Theorem 4.5 (applied to \(\mathcal{B}^{\text{rev}}\)) and the commutative diagram (4.24) above.

Corollary 4.9. For each \(\eta \in \mathcal{B}(\lambda)_{\text{cl}}\),
\[
\text{Deg}_\lambda(S(\eta)) = D_{{\mathcal{B}^{\text{rev}}}}(\Psi^{\text{rev}}(S(\eta))) - D_{{\mathcal{B}^{\text{rev}}}}^\text{ext} = D_{{\mathcal{B}^{\text{rev}}}}(S(\Psi(\eta))) - D_{{\mathcal{B}^{\text{rev}}}}^\text{ext}.
\]

Remark 4.10. The energy function \(D = D_\mathcal{B}\) in Section 4.3 corresponds to the “right” energy function \(D^R\) in [LeS, Section 2.4]; we should remark that the order of tensor products of crystals in [LeS] is “opposite” to that in this paper. Hence the composite \(D_{{\mathcal{B}^{\text{rev}}}} \circ S : \mathcal{B} \to \mathbb{Z}_{\leq 0}\) corresponds to the “left” energy function \(D^L\) in [LeS, Section 2.4].

5. The Quantum Alcove Model

Now let us recall the quantum alcove model [LL1]. Throughout this section, we refer to roots and weights in the corresponding finite lattices. Fix a dominant integral weight \(\lambda \in X\).
5.1. The objects of the model. We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves $A$ and $B$, we write $A \overset{\beta}{\rightarrow} B$ for $\beta \in \Phi$ if the common wall is orthogonal to $\beta$ and $\beta$ points in the direction from $A$ to $B$. Recall that alcoves are separated by hyperplanes of the form

$$H_{\beta,l} = \{ \mu \in h^*_R | \langle \beta^\vee, \mu \rangle = l \},$$

where $h^*_R = \mathbb{R} \otimes X$. We denote by $r_{\beta,l}$ the affine reflection in this hyperplane.

**Definition 5.1** ([LP]). An alcove path is a sequence of alcoves $(A_0, A_1, \ldots, A_m)$ such that $A_{j-1}$ and $A_j$ are adjacent, for $j = 1, \ldots, m$. We say that $(A_0, A_1, \ldots, A_m)$ is reduced if it has minimal length among all alcove paths from $A_0$ to $A_m$.

Let $A_\lambda = A_0 + \lambda$ be the translation of the fundamental alcove $A_0$ by the weight $\lambda$. The fundamental alcove is defined as

$$A_\lambda = \{ \mu \in h^*_R | 0 < \langle \alpha^\vee, \mu \rangle < 1 \quad \text{for all } \alpha \in \Phi^+ \}.$$

**Definition 5.2** ([LP]). The sequence of roots $(\beta_1, \beta_2, \ldots, \beta_m)$ is called a $\lambda$-chain if

$$A_0 = A_\lambda \overset{-\beta_1}{\rightarrow} A_1 \overset{-\beta_2}{\rightarrow} \cdots \overset{-\beta_m}{\rightarrow} A_m = A_{-\lambda}$$

is a reduced alcove path.

A reduced alcove path $(A_0 = A_\lambda, A_1, \ldots, A_m = A_{-\lambda})$ can be identified with the corresponding total order on the hyperplanes $H_{\beta,-l}$, to be called $\lambda$-hyperplanes, which separate $A_\lambda$ from $A_{-\lambda}$ (i.e., are subject to $\beta \in \Phi^+$ and $0 \leq l < \langle \beta^\vee, \lambda \rangle$); we refer here to the sequence $H_{\beta_i,-l_i}$ for $i = 1, \ldots, m$, where $H_{\beta_i,-l_i}$ contains the common wall of $A_{i-1}$ and $A_i$. Note also that a $\lambda$-chain $(\beta_1, \ldots, \beta_m)$ determines the corresponding reduced alcove path. Indeed, we can recover the corresponding sequence $(l_1, \ldots, l_m)$, to be called the height sequence, by setting $l_i := |\{ j < i | \beta_j = \beta_i \}|$. Therefore, we will sometimes refer to the sequence of $\lambda$-hyperplanes considered above as a $\lambda$-chain.

**Remark 5.3.** An alcove path corresponds to the choice of a reduced word for the affine Weyl group element sending $A_\lambda$ to $A_{-\lambda}$ [LP Lemma 5.3]. Another equivalent definition of an alcove path/$\lambda$-chain, based on a root interlacing condition which generalizes a similar condition characterizing reflection orderings, can be found in [LP1] Definition 4.1 and Proposition 10.2].

We will work with a special choice of a $\lambda$-chain in [LP1] Section 4], which we now recall.

**Proposition 5.4** ([LP1]). Given a total order $I = \{1 < 2 < \cdots < r\}$ on the set of Dynkin nodes, one may express a coroot $\beta^\vee = \sum_{i=1}^r c_i \alpha_i^\vee$ in the $\mathbb{Z}$-basis of simple coroots. Consider the total order on the set of $\lambda$-hyperplanes defined by the lexicographic order on their images in $\mathbb{Q}^{r+1}$ under the map

$$H_{\beta_i,-l_i} \mapsto \frac{1}{\langle \beta_i^\vee, \lambda \rangle} (l_i, c_1, \ldots, c_r).$$

This map is injective, thereby endowing the set of $\lambda$-hyperplanes with a total order, which is a $\lambda$-chain. We call it the lexicographic (lex) $\lambda$-chain.

**Example 5.5.** As in Example 4.3, assume that $\mathfrak{g}$ is of type $A_2^{(1)}$, and let $\lambda = \varpi_1 + \varpi_2$. Then the $\lambda$-hyperplanes are $H_{\alpha_1,0}$, $H_{\alpha_2,0}$, $H_{\theta,0}$, and $H_{\theta,-1}$, where $\theta = \alpha_1 + \alpha_2$. Since

$$H_{\alpha_1,0} \mapsto (0, 1, 0), \quad H_{\alpha_2,0} \mapsto (0, 0, 1), \quad H_{\theta,0} \mapsto (0, 1/2, 1/2), \quad H_{\theta,-1} \mapsto (1/2, 1/2, 1/2)$$
under the map \([5.1]\), the corresponding total order \(<\) on the set of \(\lambda\)-hyperplanes is \(H_{\theta, 0} < H_{\alpha_2, 0} < H_{\alpha_1, 0} < H_{\theta, -1}\).

The objects of the quantum alcove model are defined next.

**Definition 5.6** ([LL1]). Given a \(\lambda\)-chain \(\Gamma = (\beta_1, \ldots, \beta_m)\), a subset \(A = \{j_1 < j_2 < \cdots < j_s\}\) of \([m] := \{1, \ldots, m\}\) (possibly empty) is an admissible subset if we have the following path in the quantum Bruhat graph \(QB(W)\) on \(W\):

\[
e^{-\beta_1} r_{\beta_1}^{-1} r_{\beta_1} r_{\beta_{j_2}}^{-1} r_{\beta_{j_2}} \cdots r_{\beta_{j_s}}^{-1} r_{\beta_{j_s}}.
\]

The weight of \(A\) (not necessarily admissible) is defined by

\[
\text{wt}(A) := -r_{\beta_{j_1}}^{-1} \cdots r_{\beta_{j_s}}^{-1} \lambda.
\]

We let \(A(\Gamma)\) be the collection of all admissible subsets of \([m]\).

**Example 5.7.** Keep the notation and setting of Example \([5.5]\) recall that \(\Gamma = (\beta_1 = \alpha_2, \beta_2 = \theta, \beta_3 = \alpha_1, \beta_4 = \theta)\) is the lex \(\lambda\)-chain. We see that the admissible subsets of \(\{1, 2, 3, 4\}\) are

\[
\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\},
\]

\[
\{1, 2, 3\}, \{1, 2, 3, 4\}.
\]

**Remark 5.8.** If we restrict to admissible subsets for which the path \([5.2]\) has no quantum edges, we recover the classical alcove model in \([LP, LP1]\).

5.2. **Root operators in the quantum alcove model.** We continue to use the notation in Subsection \([5.1]\). Fix a \(\lambda\)-chain \(\Gamma = (\beta_1, \ldots, \beta_m)\) and the corresponding reduced alcove path \((A_0, A_1, \ldots, A_m)\). In this subsection, we recall from \([LL1]\) the construction of (combinatorial) root operators in the quantum alcove model, namely on the collection \(A(\Gamma)\) of admissible subsets of \([m]\).

Let \(A = \{j_1 < j_2 < \cdots < j_s\}\) be an arbitrary subset of \([m]\). The elements of \(A\) are called folding positions. We “fold” the reduced alcove path \((A_0, A_1, \ldots, A_m)\) in the hyperplanes corresponding to these positions and obtain a “folded” alcove path; this can be recorded by a sequence of roots, namely \(\Gamma(A) = (\gamma_1, \gamma_2, \ldots, \gamma_m)\); here

\[
\gamma_i := r_{\beta_{j_1}} r_{\beta_{j_2}} \cdots r_{\beta_{j_s}}(\beta_i),
\]

with \(j_k\) the largest folding position less than \(i\). We define \(\gamma_\infty := r_{\beta_{j_1}} r_{\beta_{j_2}} \cdots r_{\beta_{j_s}}(\rho)\), where \(\rho = (1/2)\sum_{\alpha \in \Phi^+} \alpha\). Upon folding, the hyperplane separating the alcoves \(A_{i-1}\) and \(A_i\) is mapped to

\[
H_{|\gamma_i|, -l^A} = r_{\beta_{j_1}}^{-1} l_1 r_{\beta_{j_2}}^{-1} l_2 \cdots r_{\beta_{j_k}}^{-1} l_k (H_{\beta_i, -l_i}),
\]

for some \(l_i^A\), which is defined by this relation; here we write \(|\alpha| := \text{sgn}(\alpha)\alpha\), where \(\text{sgn}(\alpha)\) is the sign of the root \(\alpha\).

Given \(A \subseteq [m]\) and \(\alpha \in \Phi\), we will use the following notation:

\[
I_\alpha = I_\alpha(A) := \{i \in [m] \mid \gamma_i = \pm \alpha\}, \quad I_\alpha^* = I_\alpha(A) := I_\alpha \cup \{\infty\},
\]

and \(l_\infty^A := \langle \text{sgn}(\alpha)\alpha^\vee, \text{wt}(A) \rangle\).

Let \(A\) now be an admissible subset, so \(A \in A(\Gamma)\). Fix \(p \in I_\alpha^*, \) so \(\check{\alpha}_p\) is a simple root if \(p \neq 0\), or \(-\theta\) if \(p = 0\), see \([2.7]\). We set

\[
M := \max \{\langle \text{sgn}(\check{\alpha}_p)l_i^A \mid i \in I_{\check{\alpha}_p} \rangle \cup \{\text{sgn}(\check{\alpha}_p)l_\infty^A\} \}.
\]
Let $\ell$ be the minimum index $i$ in $\hat{T}_A$, for which we have $\text{sgn}(\hat{\alpha}_p)l_i = M$. It was proved in [LL1] that, if $M \geq \delta_{p,0}$, then either $\ell \in A$ or $\ell = \infty$; furthermore, if $M > \delta_{p,0}$, then $\ell$ has a predecessor $k$ in $\hat{T}_A$, and we have $k \not\in A$. We define

$$f_p(A) := \begin{cases} (A \setminus \{\ell\}) \cup \{k\} & \text{if } M > \delta_{p,0} \\ 0 & \text{otherwise.} \end{cases}$$

(5.7)

Now we define $e_p$. Let $M$ be as in (5.6). Assuming that $M > \langle \hat{\alpha}_p \rangle, \text{wt}(A)$, let $k$ be the maximum index $i$ in $\hat{T}_A$ for which we have $\text{sgn}(\hat{\alpha}_p)l_i = M$, and let $\ell$ be the successor of $k$ in $\hat{T}_A$. Assuming also that $M \geq \delta_{p,0}$, it was proved in [LL1] that $k \in A$, and either $\ell \not\in A$ or $\ell = \infty$. Define

$$e_p(A) := \begin{cases} (A \setminus \{k\}) \cup \{\ell\} & \text{if } M > \langle \hat{\alpha}_p \rangle, \text{wt}(A) \text{ and } M \geq \delta_{p,0} \\ 0 & \text{otherwise.} \end{cases}$$

(5.8)

In the above definitions, we use the convention that $A \setminus \{\infty\} = A \cup \{\infty\} = A$.

**Example 5.9.** Keep the notation and setting of Examples 5.5 and 5.7. We can verify that $f_0(\{1, 2, 3\}) = \{1, 2, 3, 4\}$. We set $A = \{1, 2, 3\}$. Then,

$$\text{wt}(A) = -r_{\beta_1, -1} r_{\beta_2, -2} r_{\beta_3, -3} (-\lambda) = -r_{a_0, 0} r_{\theta, 0} r_{a_1, 0} (-\lambda)$$

$$= -r_2 r_{\theta, 1} (-\lambda) = w_0 \lambda.$$

If we write $\Gamma(A) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, then

$$\gamma_1 = \beta_1 = a_2, \quad \gamma_2 = r_{\beta_1}(\beta_2) = r_2(\theta) = a_1, \quad \gamma_3 = r_{\beta_1} r_{\beta_2}(\beta_3) = r_2 r_{\theta}(a_1) = a_2, \quad \gamma_4 = r_{\beta_1} r_{\beta_2} r_{\beta_3}(\beta_4) = r_2 r_{\theta, 1} = -\theta,$$

and

$$l_1^A = 0 \quad \text{since } H_{\beta_1, -1} = H_{a_2, 0},$$

$$l_2^A = 0 \quad \text{since } r_{\beta_1, -1} H_{\beta_2, -2} = r_{a_2, 0} H_{\theta, 0} = H_{a_1, 0},$$

$$l_3^A = 0 \quad \text{since } r_{\beta_1, -1} r_{\beta_2, -2} H_{\beta_3, -3} = r_{a_2, 0} r_{\theta, 0} H_{a_1, 0} = H_{a_2, 0},$$

$$l_4^A = -1 \quad \text{since } r_{\beta_1, -1} r_{\beta_2, -2} r_{\beta_3, -3} H_{\beta_4, -4} = r_{a_2, 0} r_{\theta, 0} r_{a_1, 0} H_{\theta, -1} = H_{\theta, 1}.$$

Also, since $\alpha_0 = -\theta$, we have $I_{\alpha_0}(A) = \{4\}$. Because

$$\text{sgn}(\alpha_0) l_0^\infty \alpha_0 = (-1) \times \langle (-1) \times (-\theta) \rangle, \text{wt}(A) = 2 > 1 = \text{sgn}(\alpha_0) l_4^A,$$

we obtain $M = 2$ (which implies that $f_0 A \neq 0$), and hence $\ell = \infty, k = 4$. Therefore, we conclude that

$$f_0 A = (A \setminus \{\infty\}) \cup \{4\} = \{1, 2, 3, 4\}.$$

Next, let us check that $f_0 B = 0$, with $B = \{1, 2, 3, 4\}$. We have

$$\text{wt}(B) = -r_{\beta_1, -1} r_{\beta_2, -2} r_{\beta_3, -3} r_{\beta_4, -4} (-\lambda) = -r_{a_2, 0} r_{\theta, 0} r_{a_1, 0} r_{\theta, -1} (-\lambda) = 0,$$

and hence $\text{sgn}(\alpha_0) l_0^\infty = 0$. Furthermore, by a computation similar to the one above for $A$, we deduce that $\Gamma(B) = (a_2, a_1, a_2, -\theta)$, and that $l_i^B = l_i^A$ for all $1 \leq i \leq 4$. Also, since $\alpha_0 = -\theta$, we have $I_{\alpha_0}(B) = \{4\}$. Since $\text{sgn}(\alpha_0) l_4^B = 1$, it follows that $M = 1$, which implies that $f_0 B = 0$ by the definition.
The following theorem about root operators on \( A(\Gamma) \) was proved in \cite{LL1}.

**Theorem 5.10** (\cite{LL1} Theorem 3.8).

1. If \( A \) is an admissible subset and if \( f_p(A) \neq 0 \), then \( f_p(A) \) is also an admissible subset. Similarly for \( e_p(A) \). Moreover, \( f_p(A) = A' \) if and only if \( e_p(A') = A \).
2. We have \( \varphi_p(A) = M - \delta_{p,0} \) if \( M \geq \delta_{p,0} \), then \( \varphi_p(A) = M - \langle \alpha_p^\vee, wt(A) \rangle \), while otherwise \( \varphi_p(A) = \varepsilon_p(A) = 0 \).

**Remark 5.11.** Let \( A = \{ j_1 < \cdots < j_s \} \) be an admissible subset, and \( w_i := r_{\beta_{j_1}} r_{\beta_{j_2}} \cdots r_{\beta_{j_i}} \). Let \( M, \ell, k \) be as in the above definition of \( f_p(A) \), assuming \( M > \delta_{p,0} \). Now assume that \( \ell \neq \infty \), and let \( a < b \) be such that

\[
j_a < k < j_{a+1} < \cdots < j_b = \ell < j_{b+1} ;
\]

if \( a = 0 \) or \( b + 1 > s \), then the corresponding indices \( j_a \), respectively \( j_{b+1} \), are missing. In the proof of Theorem 5.10 in \cite{LL1}, it was shown that \( f_p \) has the effect of changing the path in the quantum Bruhat graph

\[
ed = w_0 \to \cdots \to w_a \to w_{a+1} \to \cdots \to w_{b-1} \to w_b \to \cdots \to w_s
\]

corresponding to \( A \) into the following path corresponding to \( f_p(A) \):

\[
ed = w_0 \to \cdots \to w_a \to s_p w_a \to s_p w_{a+1} \to \cdots \to s_p w_{b-1} = w_b \to \cdots \to w_s,
\]

where \( s_p \) is the simple reflection \( r_p \) if \( p \neq 0 \), or \( r_0 \) if \( p = 0 \), see (2.4). The case \( \ell = \infty \) is similar.

6. **The bijection between the quantum LS path model and the quantum alcove model**

The main result of this section is the crystal isomorphism between the QLS paths of Section 3 and the quantum alcove model of Section 5 as stated in Theorem 6.11.

6.1. **The forgetful map.** Fix a dominant integral weight \( \lambda \), and set \( J = \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). Recall from Sections 3.2 and 5.1 the notation related to QLS paths and the quantum alcove model, respectively.

We will now define a forgetful map from the quantum alcove model based on a lex \( \lambda \)-chain \( \Gamma_{\text{lex}} = (\beta_1, \ldots, \beta_m) \) (see Proposition 5.4), namely from \( \mathcal{A}(\lambda) := \mathcal{A}(\Gamma_{\text{lex}}) \), to the set \( QLS(-w_0 \lambda) \) of QLS paths of shape \(-w_0 \lambda\). Given an index \( i \in [m] \), we let \( t_i := l_i / \langle \beta_i^\vee, \lambda \rangle \), where \( l_i \) is the height defined in Section 5. Note that \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_m \), by the definition of \( \Gamma_{\text{lex}} \). Consider an admissible subset \( A = \{ j_1 < j_2 < \cdots < j_s \} \), and let

\[
0 = b_0 < b_1 < \cdots < b_p := \{ t_{j_1} \leq t_{j_2} \leq \cdots \leq t_{j_s} \} \cup \{0\}.
\]

Let \( 0 = n_0 \leq n_1 < \cdots < n_{p+1} = s \) be such that \( t_{j_h} = b_k \) if and only if \( n_k < h \leq n_{k+1} \), for \( k = 0, \ldots, p \). Define Weyl group elements \( u_h \) for \( h = 0, \ldots, s \) and \( w_k \) for \( k = 0, \ldots, p \) by \( u_0 := e, u_h := r_{\beta_{j_1}} \cdots r_{\beta_{j_h}}, \) and \( w_k := u_{n_{k+1}} \). For any \( k = 1, \ldots, p \), we have the following directed path in the quantum Bruhat graph \( QB(W) \):

\[
w_{k-1} = u_{n_k} \xrightarrow{\beta_{n_k+1}} u_{n_k+1} \xrightarrow{\beta_{n_k+2}} \cdots \xrightarrow{\beta_{n_{k+1}}} u_{n_{k+1}} = w_k.
\]
We claim that this is a directed path in $QB_{b_k}(W)$. Indeed, for $n_k < h \leq n_{k+1}$, we have
$$b_k \langle \beta_{j_h}^\vee, \lambda \rangle = l_{j_h} \in \mathbb{Z}_{\geq 0},$$
by the definition of $b_k = t_{j_h}$. By Lemma 6.2 below, for each edge $u_i \to u_{i+1}$ in the path (6.2), there is a directed path from $[u_i]^J$ to $[u_{i+1}]^J$ in $QB_{b_k}(W^J)$. Concatenating these directed paths in $QB_{b_k}(W^J)$, we obtain a directed path from $[w_k]^J$ to $[w_{k-1}]^J$ in $QB_{b_k}(W^J)$. Hence it follows from Lemma 4.7 that there is a directed path from $[w_k w_0]^\omega(J)$ to $[w_{k-1} w_0]^\omega(J)$ in $QB_{b_k \omega(\lambda)}(W^\omega(J))$; note that $\omega(\lambda) = -w_0 \lambda$. Thus we conclude that

$$[w_0 w_0]^\omega(J) \xleftarrow{-b_1 w_0 \lambda} [w_1 w_0]^\omega(J) \xleftarrow{-b_2 w_0 \lambda} \ldots \xleftarrow{-b_p w_0 \lambda} [w_p w_0]^\omega(J)$$

is a QLS path of shape $-w_0 \lambda$. We denote this QLS path of shape $-w_0 \lambda$ by $\Pi(A)$, and the dual QLS path of shape $\lambda$ defined in Section 4.5 by $\Pi^*(A)$; that is, $\Pi^*(A) \in \text{QLS}(\lambda)$ is of the form:

$$[w_p]^J \xleftarrow{(1-b_p \lambda)} [w_{p-1}]^J \xleftarrow{(1-b_{p-1} \lambda)} \ldots \xleftarrow{(1-b_1 \lambda)} [w_1]^J \xleftarrow{(1-b_1 \lambda)} [w_0]^J,$$

and we have the following commutative diagram:

$$\begin{array}{c}
\mathcal{A}(\lambda) \\
\Pi \quad QLS(-w_0 \lambda) \\
\Pi^* \\
\downarrow \\
\text{QLS}(\lambda).
\end{array}$$

**Example 6.1.** Keep the notation and setting of Example 5.7.

1. Let us compute the image of $A_1 = \{1, 2, 3, 4\} \in \mathcal{A}(\lambda)$ under the map $\Pi^*: \mathcal{A}(\lambda) \to \text{QLS}(\lambda)$. Recall that $t_i = l_i/\langle \beta_i^\vee, \lambda \rangle$ for $1 \leq i \leq 4$. Since $l_1 = l_2 = l_3 = 0$ and $l_4 = 1$, and since $\langle \beta_1^\vee, \lambda \rangle = \langle \beta_2^\vee, \lambda \rangle = 1$ and $\langle \beta_3^\vee, \lambda \rangle = \langle \beta_4^\vee, \lambda \rangle = 2$, we see that $t_1 = t_2 = t_3 = 0 < 1/2 = t_4$. Hence, by (6.1), we have $b_0 = 0$ and $b_1 = 1/2$ (with $p = 1$). Also, we see that $n_1 = 3$ and $n_2 = 4$, and hence that

$$w_0 = u_{n_1} = r_2 r_\vartheta r_1 = r_\theta = w_0, \quad w_1 = u_{n_2} = r_2 r_\vartheta r_1 r_\theta = e.$$ 

Therefore, by definition (6.4), we obtain

$$\Pi^*(A_1) = (e, w_0; 0, 1/2, 1),$$

which is equal to $\eta_1 \in \text{QLS}(\lambda)$ in Example 4.3.

2. Let us compute the image of $A_2 = \{1, 4\} \in \mathcal{A}(\lambda)$ under the map $\Pi^*: \mathcal{A}(\lambda) \to \text{QLS}(\lambda)$. As seen above, $t_1 = 0 < 1/2 = t_4$. Hence, by (6.1), we have $b_0 = 0$ and $b_1 = 1/2$ (with $p = 1$). Also, we see that $n_1 = 1$ and $n_2 = 2$, and hence that

$$w_0 = u_{n_1} = r_2, \quad w_1 = u_{n_2} = r_2 r_\vartheta = r_1 r_2.$$ 

Therefore, by definition (6.4), we obtain

$$\Pi^*(A_2) = (r_1 r_2, r_2; 0, 1/2, 1),$$

which is equal to $\eta_2 \in \text{QLS}(\lambda)$ in Example 4.3.

3. By computations similar to the ones above, we can verify that

$$\emptyset \mapsto (e; 0, 1), \quad \{1\} \mapsto (r_2; 0, 1),$$

$$\{1, 2\} \mapsto (r_1 r_2; 0, 1), \quad \{1, 2, 3\} \mapsto (w_0; 0, 1)$$

under the map $\Pi^*: \mathcal{A}(\lambda) \to \text{QLS}(\lambda)$. 
Lemma 6.6. Consider result.

Lemma 6.2. Let $w \twoheadrightarrow wr$ be an edge in $QB_{b\lambda}(W)$ for some rational number $b$, which is viewed as a path $q$. Then there exists a path $p$ from $[w]$ to $[wr]$ in $QB_{b\lambda}(W^J)$ (possibly of length 0), such that $wt(p) \equiv wt(q) \pmod{Q^J}$.

Remarks 6.3.

(1) The special case of the lemma corresponding to the $b$-Bruhat order on $W$ and $W^J$ (i.e., the subgraphs of $QB_{b\lambda}(W)$ and $QB_{b\lambda}(W^J)$ with no quantum edges) was proved in [LeSh] Lemma 4.16. For $b = 0$, i.e., the usual Bruhat order, the latter result is well-known; see, e.g., [BB] Proposition 2.5.1.

(2) Based on the strong connectivity of the quantum Bruhat graph (cf. Theorem 6.4 below), the lemma implies the same property for the parabolic quantum Bruhat graph. Note that this was proved by different methods (and, in fact, in a slightly stronger form) in [LNSSS1] Lemma 6.12.

6.2. The inverse map. Next we prove that the forgetful map in Section 6.1, from the quantum alcove model to quantum LS paths, is a bijection, by exhibiting the inverse map. We will use the shellability of the quantum Bruhat graph $QB(W)$ with respect to a reflection ordering on the positive roots $[Dy]$, which we now recall.

Theorem 6.4 ([BFP]). Fix a reflection ordering on $\Phi^+$. Fix a reflection ordering on $\Phi^+$.

(1) For any pair of elements $v, w \in W$, there is a unique path from $v$ to $w$ in the quantum Bruhat graph $QB(W)$ such that its sequence of edge labels is strictly increasing (resp., decreasing) with respect to the reflection ordering.

(2) The path in (1) has the smallest possible length $\ell(v \rightarrow w)$ and is lexicographically minimal (resp., maximal) among all shortest paths from $v$ to $w$.

In [LeSh] Section 4.3], we constructed a reflection ordering $<_\lambda$ on $\Phi^+$ which depends on $\lambda$. The bottom of the order $<_\lambda$ consists of the roots in $\Phi^+ \setminus \Phi^+_J$. For two such roots $\alpha$ and $\beta$, define $\alpha < \beta$ whenever the hyperplane $H_{(\alpha, 0)}$ precedes $H_{(\beta, 0)}$ in the lex $\lambda$-chain (see Proposition 5.4). This forms an initial section $[Dx]$ of $<_\lambda$. The top of the order $<_\lambda$ consists of the positive roots for the Weyl group $W_J$, and we fix any reflection ordering for them. We refer to the reflection ordering $<_\lambda$ throughout this section.

Remark 6.5. Given a $\lambda$-hyperplane $H_{\beta, -t}$, we call the first component of the vector associated with it in (5.1), namely $t/(\beta^\vee, \lambda)$, the relative height of $H_{\beta, -t}$. It is not hard to see that, in the lex $\lambda$-chain, the order on the $\lambda$-hyperplanes $H_{\beta, -t}$ with the same relative height is given by the order $<_\lambda$ on the corresponding roots $\beta$. We will use this fact implicitly below.

Recall from [LNSSS1] Proposition 7.2] that there exists a unique element $x \in wW_J$ such that $\ell(v \rightarrow x)$ attains its minimum value as a function of $x \in wW_J$, for fixed $v, w \in W$. We refer also to [LNSSS1] Theorem 7.1], stating that the mentioned minimum is, in fact, attained by the minimum of the cost $wW_J$ with respect to the $v$-tilted Bruhat order $\preceq_v$ on $W$ [BFP]; therefore, it makes sense to denote it by $\min(wW_J, \preceq_v)$, although we will not use this stronger result.

Lemma 6.6. Consider $\sigma, \tau \in W^J$ and $wJ \in W_J$. Write $\min(\tau wJ, \preceq_{wJ}) \in \tau wJ$ as: $\tau w'_J = \min(\tau wJ, \preceq_{wJ})$, with $w'_J \in W_J$. 

(1) There is a unique path in $QB(W)$ from $\sigma w_J$ to some $x \in \tau W_J$ whose edge labels are increasing and lie in $\Phi^+ \setminus \Phi^+_J$. This path ends at $\tau w'_J$.

(2) Assume that there is a path from $\sigma$ to $\tau$ in $QB_{b\lambda}(W')$ for some $b \in \mathbb{Q}$. Then the path in (1) from $\sigma w_J$ to $\tau w'_J$ is in $QB_{b\lambda}(W)$.

Proof. The first part is just the content of [LNSSS1] Lemmas 7.4 and 7.5, based on the results recalled just above the lemma. For the second part, we start by considering a second path in $QB(W)$ from $\sigma w_J$ to $\tau w'_J$, beside the one given by (1). This path is formed by concatenating the following:

- a path from $\sigma w_J$ to $\sigma$ with only quantum edges and all edge labels in $\Phi^+_J$ (for instance simple roots in $\Phi_J$);
- a path from $\sigma$ to $\tau$ constructed from the path in $QB(W')$ between the same two elements by replacing each edge $u \xrightarrow{\alpha} [ur_\alpha]$ with $u \xrightarrow{\alpha} ur_\alpha$ (cf. [LNSSS1] Condition (2') in Section 4.2)] followed by a path from $ur_\alpha$ to $[ur_\alpha]$ with only quantum edges and all edge labels in $\Phi^+_J$ (for instance simple roots in $\Phi_J$);
- a path from $\tau$ to $\tau w'_J$ with only Bruhat edges and all edge labels in $\Phi^+_J$.

By Theorem 6.4 (2), we know that the first of the two paths above is a shortest one (from $\sigma w_J$ to $\tau w'_J$). Furthermore, by the hypothesis, the second path is in $QB_{b\lambda}(W)$ (any edge in $QB(W)$ labeled by a root in $\Phi^+_J$ is by default in $QB_{b\lambda}(W)$). So we can apply Lemma 6.7 below and deduce that the first path is also in $QB_{b\lambda}(W)$. \hfill \Box

Let us now state Lemma 6.7 which will be proved in Section 8.2 below.

**Lemma 6.7.** Consider two paths in $QB(W)$ between some $v$ and $w$. Assume that the first one is a shortest path, while the second one is in $QB_{b\lambda}(W)$, for some rational number $b$. Then the first path is in $QB_{b\lambda}(W)$ as well.

We now construct the inverse of the forgetful map in Section 6.1. We begin with a QLS path in $QLS(-w_0\lambda)$, which is written in the form

$$\sigma'_0 \leftrightarrow -b_0 w_0 \lambda \leftrightarrow \sigma'_1 \leftrightarrow -b_2 w_0 \lambda \cdots \leftrightarrow -b_p w_0 \lambda \leftrightarrow \sigma'_p,$$

where $\sigma'_j \in W^{w(J)}$, and $0 = b_0 < b_1 < \cdots < b_p < 1$. Set $\sigma_i := [\sigma'_j w_0] J \in W^J$; we see from Lemma 4.7 that

$$\sigma_0 b_1 \lambda \leftrightarrow \sigma_1 b_2 \lambda \cdots b_p \lambda \leftrightarrow \sigma_p.$$

We will now associate with it an admissible subset (see Definition 5.6), i.e., a lex increasing sequence of $\lambda$-hyperplanes, and the corresponding path in $QB(W)$ defined in (5.2).

We start by defining the sequence $w_{-1}, w_0, \ldots, w_p$ in $W$ recursively by $w_{-1} = e$, and by $w_i = \min(\sigma_i W_J, \preceq_{w_{i-1}})$ for $i = 0, \ldots, p$. Note that $w_0 = \sigma_0$. For each $i = 0, \ldots, p$, consider the unique path in $QB(W)$ with increasing edge labels (with respect to $\prec_{\lambda}$, cf. Theorem 6.4) from $w_{i-1}$ to $w_i$. Note that the path corresponding to $i = 0$ is, in fact, a saturated chain in the Bruhat order on $W^J$. By (6.7) and Lemma 6.6, for any $i$, the edges of the corresponding path are in $QB_{b\lambda}(W)$ and no edge label is in $\Phi^+_J$. We define the path in the quantum alcove model corresponding to the QLS path (6.7) by concatenating the paths constructed above, for $i = 0, \ldots, p$. The corresponding sequence of $\lambda$-hyperplanes is defined by associating with a label $\beta$ in the path from $w_{i-1}$ to $w_i$ the $\lambda$-hyperplane $\langle \beta, -b_i(\beta^\vee, \lambda) \rangle$; this is indeed a $\lambda$-hyperplane: $b_i(\beta^\vee, \lambda)$ is an integer, and we have $0 \leq b_i(\beta^\vee, \lambda) < \langle \beta^\vee, \lambda \rangle$, as $0 \leq b_i < 1$ and $\beta \in \Phi^+ \setminus \Phi^+_J$, so
$\langle \beta^y, \lambda \rangle > 0$. The constructed sequence of $\lambda$-hyperplanes is lex increasing because the sequence $(b_i)$ is increasing and the edge labels in the path from $w_{j-1}$ to $w_j$ increase (with respect to $<_\lambda$). So we constructed an admissible sequence, which is now associated with the QLS path in (6.7).

**Proposition 6.8.** The forgetful map $A \mapsto \Pi^*(A)$ is a weight-preserving bijection from $\mathcal{A}(\lambda)$ to QLS($\lambda$).

**Proof.** We need to show that the maps in Sections 6.1 and 6.2 are mutually inverse. The crucial fact to check is that the map $\Pi$ followed by the backward one is the identity. This follows from QLS($\lambda$).

Moreover, noting that in the notation of Section 6.1, with $Q \vdash \beta, \lambda$, and set $A^1 := \{j_1 < \cdots < j_s \}$. Then, by (6.4), the QLS path $\Pi^*(A^1) \in$ QLS($\lambda$) is of the form (where we dropped the superscript $J$):

$$[w_p] \left< \frac{1}{1-b_p} \right| [w_{p-1}] \left< \frac{1}{1-b_{p-1}} \right| \cdots \left< \frac{1}{1-b_1} \right| [w_1] \left< \frac{1}{1-b_1} \right| [w_0]$$

in the notation of Section 6.1, with $s-1$ instead of $s$; note that $w_p = r_{\beta_{j_1}} \cdots r_{\beta_{j_{s-1}}}$ and $b_p = t_{j_{s-1}}$. Moreover, noting that $A = A^1 \cup \{j_s \}$, we deduce that the QLS path $\Pi^*(A) \in$ QLS($\lambda$) is of the form:

$$\left\{ \begin{array}{ll}
[w_p r_{\beta_{j_s}}] & \left< \frac{1}{1-b_p} \right| [w_{p-1}] \left< \frac{1}{1-b_{p-1}} \right| \cdots \left< \frac{1}{1-b_1} \right| [w_0] & \text{if } t_{j_s} = b_p, \\
[w_p r_{\beta_{j_s}}] & \left< \frac{1}{1-t_{j_s}} \right| [w_p] \left< \frac{1}{1-b_p} \right| [w_{p-1}] \left< \frac{1}{1-b_{p-1}} \right| \cdots \left< \frac{1}{1-b_1} \right| [w_0] & \text{if } t_{j_s} > b_p.
\end{array} \right.$$ 

From this, by direct calculation using (3.5), we can show that in both cases above,

$$\text{wt}(\Pi^*(A)) = (\Pi^*(A))(1) = (\Pi^*(A'))(1) - \langle \beta^y_{j_s}, \lambda \rangle (1-t_{j_s}) w_p \beta_{j_s}$$

$$= \text{wt}(\Pi^*(A')) - \langle \beta^y_{j_s}, \lambda \rangle (1-t_{j_s}) r_{\beta_{j_1}} \cdots r_{\beta_{j_{s-1}}},$$

thus obtaining a relation between $\text{wt}(\Pi^*(A))$ and $\text{wt}(\Pi^*(A'))$.

If we set

$$z_{A'} := r_{\beta_{j_1}} \cdots r_{\beta_{j_{s-1}}} \cdot l_{j_{s-1}} \in W_{af},$$

then we have $z_{A'} \mu = \gamma_{A'} + w_{A'} \mu$ for all $\mu \in X$, where $w_{A'} := r_{\beta_{j_1}} \cdots r_{\beta_{j_{s-1}}} \in W$ and $\gamma_{A'}$ is an element of $Q$. It follows that by (5.3),

$$\text{wt}(A') = -r_{\beta_{j_1}} \cdots r_{\beta_{j_{s-1}}} \cdot l_{j_{s-1}} (-\lambda) = -z_{A'} (-\lambda)$$

$$= -\gamma_{A'} - w_{A'} (-\lambda).$$

Also, we have

$$r_{\beta_{j_s}} \cdot l_{j_s} (-\lambda) = -\lambda + (\langle \beta^y_{j_s}, \lambda \rangle - l_{j_s}) \beta_{j_s}.$$ 

Therefore, again by (5.3), we see that

$$\text{wt}(A) = -r_{\beta_{j_1}} \cdots r_{\beta_{j_{s-1}}} \cdot l_{j_{s-1}} \cdot r_{\beta_{j_s}} \cdot l_{j_s} (-\lambda)$$

$$= -z_{A'} r_{\beta_{j_s}} \cdot l_{j_s} (-\lambda) = -z_{A'} (-\lambda + (\langle \beta^y_{j_s}, \lambda \rangle - l_{j_s}) \beta_{j_s})$$

$$= -\gamma_{A'} - w_{A'} (-\lambda) - (\langle \beta^y_{j_s}, \lambda \rangle - l_{j_s}) w_{A'} \beta_{j_s}$$

$$= \text{wt}(A') - (\langle \beta^y_{j_s}, \lambda \rangle - l_{j_s}) r_{\beta_{j_1}} \cdots r_{\beta_{j_{s-1}}} \beta_{j_s}.$$
Since \( t_{j_s} = \langle \beta'_{j_s}, \lambda \rangle t_{j_s} \) by the definition of \( t_{j_s} \), we conclude that
\[
\text{wt}(A) = \text{wt}(A') - \langle \beta'_{j_s}, \lambda \rangle (1 - t_{j_s}) r_{\beta_{j_s}} \cdots r_{\beta_{j_{s-1}}} \beta_{j_s},
\]
thus obtaining the same relation between \( \text{wt}(A) \) and \( \text{wt}(A') \) as the one between \( \text{wt}(\Pi^{*}(A)) \) and \( \text{wt}(\Pi^{*}(A')) \). This proves that \( \text{wt}(\Pi^{*}(A)) = \text{wt}(A) \) by induction on \( s \).

\[\Box\]

6.3. **The crystal isomorphism between** \( \mathcal{A}(\lambda) \) **and** \( \mathcal{B} \). **We will now prove that, up to the** \( f_0 \) **arrows at the end of a string, we can view** \( \mathcal{A}(\lambda) \) **as a model for the tensor product of KR crystals** \( \mathcal{B} \) **via the bijection** \( \tilde{\Psi} := \Psi \circ \Pi^{*} \), see (4.3);

\[
\mathcal{A}(\lambda) \xrightarrow{\Pi} \text{QLS}(\omega_0 \lambda)
\]

\[
\Pi^{*} \downarrow
\]

\[
\text{QLS}(\lambda) \xrightarrow{\Psi} \mathcal{B}.
\]

**Definition 6.9.** Let \( b \to f_i(b) \) be an arrow in \( \mathcal{B} \). It is called a **Demazure arrow** if \( i \neq 0 \), or \( i = 0 \) and \( \varepsilon_0(b) \geq 1 \). It is called a **dual Demazure arrow** if \( i \neq 0 \), or \( i = 0 \) and \( \varphi_0(b) \geq 2 \).

**Remark 6.10.** In the case when all of the tensor factors of \( \mathcal{B} \) are perfect crystals (see Definition A.1), the subgraph of \( \mathcal{B} \) consisting of the dual Demazure arrows is connected. See the discussion in Section A below about which column shape KR crystals are perfect.

We now state the main result of this section, relating the crystal structures in the QLS path model and the quantum alcove model.

**Theorem 6.11.** Consider the root operator \( e_p \) (and the corresponding map \( \varepsilon_p \)) for QLS paths, as defined in Section 2.3, and the root operator \( f_p \) in the quantum alcove model defined in Section 5.2. Given \( A \) in \( \mathcal{A}(\lambda) \), we have \( f_p(A) \neq 0 \) if and only if \( \varepsilon_p(\Pi(A)) > \delta_{p,0} \); in this case, we have
\[
eq e_p(\Pi(A)) = \Pi(f_p(A)).
\]

**Proof.** The proof of the similar result for the classical alcove model, namely [LP1, Theorem 9.4], carries through (cf. Remark 5.8). The main fact underlying this proof is the similarity between the definition (2.6) of \( e_p \) for QLS paths, and the change under \( f_p \) of the relevant path in the quantum Bruhat graph, which is explained in Remark 5.11. Note that in both cases the reflection \( s_p \) is applied to a segment of the corresponding path.

To be more precise, the proof is based on deforming the path \( \Pi(A) \) to a path \( \Pi_\varepsilon(A) \) between the same endpoints (where \( \varepsilon \) is a sufficiently small positive real number), such that the latter does not pass through the intersection of two or more \( \lambda \)-hyperplanes (here we exclude the endpoints). The path \( \Pi_\varepsilon(A) \) encodes the same information as the “folded” alcove path corresponding to \( A \) or, equivalently, the sequence of roots \( \Gamma(A) \); see Section 5.2. Therefore, the actions of \( e_p \) on \( \Pi_\varepsilon(A) \) and of \( f_p \) on \( A \) (where the latter is based on \( \Gamma(A) \)) are equivalent. The proof concludes by taking the limit \( \varepsilon \to 0 \), under which \( \Pi_\varepsilon(A) \) goes to \( \Pi(A) \).

We will now point out the additional elements in the proof. First, some results invoked in the proof of [LP1, Theorem 9.4] need to be replaced, as follows: [LP1, Corollary 6.11] with [LL1, Propositions 3.15 and 3.18], [LP1, Corollary 6.12] with [LL1, Propositions 3.16 and 3.19], and [LP1, Proposition 7.3] with Remark 5.11. Other than this, there is just one notable addition...
A UNIFORM MODEL FOR KR CRYSTALS II. PATH MODELS AND $P = X$

to the proof, which has to do with the case $p = 0$. Consider the number $M$ in the definition of $f_0(A)$, and assume for the moment that $M \geq 1$. By the same reasoning as in \cite{LP1}, we can see that the minimum of the function $t \mapsto \langle \tilde{\alpha}^+_0, \tilde{\Pi}_\varepsilon(A)(t) \rangle$ is $-M$. Therefore, as discussed in \cite[Section 2.2]{LNSSS2}, cf. also \cite[Lemma 2.1 (c)]{Li}, the maximum number of times $e_0$ can be applied to $\tilde{\Pi}_\varepsilon(A)$ is $M$. Meanwhile, the maximum number of times $f_0$ can be applied to $A$ is $M - 1$, by Theorem 5.10 (2). In the remaining case, namely $M < 1$, we have $f_0(A) = 0$, and the minimum of the function mentioned above is 0, so $e_0$ is not defined on $\tilde{\Pi}_\varepsilon(A)$. We conclude that $f_0(A) \neq 0$ if and only if $\varepsilon_0(\tilde{\Pi}_\varepsilon(A)) \geq 2$. The rest of the argument is identical to the one in \cite{LP1}.

\section*{Remarks 6.12.}

(1) The forgetful map $\Pi$ from the quantum alcove model to the QLS path model is a very natural map. Therefore, we think of the former model as a mirror image of the latter, via this bijection. If we use the mentioned identification to construct the non-dual Demazure arrows in the quantum alcove model, we quickly realize that, in general, the constructions are considerably more involved than (5.7) and (5.8), see \cite[Example 4.9]{LL1}.

(2) Although the quantum alcove model so far misses the non-dual Demazure arrows, it has the advantage of being a discrete model. Therefore, combinatorial methods are applicable, for instance in proving the independence of the model from the choice of an initial alcove path (or $\lambda$-chain of roots), see below, including the application in Remark 6.15 (2). This should be compared with the subtle continuous arguments used for the similar purpose in the Littelmann path model \cite{Li}.

Based on (4.18), we immediately obtain the following corollary of Theorems 2.7, 3.3, 6.11, and Proposition 6.8.

\section*{Corollary 6.13.}

The bijection $\tilde{\Psi} = \Psi \circ \Pi^*$ is a weight-preserving affine crystal isomorphism from $A(\lambda)$ to the subgraph of $B$ consisting of the dual Demazure arrows.

Recall that the set $A(\lambda) = A(\Gamma_{\text{lex}})$ in Corollary 6.13 is based on a lex $\lambda$-chain $\Gamma_{\text{lex}}$. In fact, the following stronger version of Corollary 6.13 is proved in \cite{LL2}.

\section*{Theorem 6.14. \cite{LL2}}

Given any $\lambda$-chain $\Gamma$, there is a weight-preserving affine crystal isomorphism between $A(\Gamma)$ and the subgraph of $B$ consisting of the dual Demazure arrows.

The proof uses Corollary 6.13 as the starting point. Then, given two $\lambda$-chains $\Gamma$ and $\Gamma'$, we construct a bijection between $A(\Gamma)$ and $A(\Gamma')$ preserving the dual Demazure arrows, as well as the weights and heights of the vertices (see Definition \ref{definition} below); this means that the quantum alcove model does not depend on the choice of a $\lambda$-chain. The mentioned construction is based on generalizing to the quantum alcove model the so-called Yang-Baxter moves in \cite{Le1}. As a result, we obtain a collection of a priori different bijections between $B$ and $A(\Gamma)$.

\section*{Remarks 6.15.}

(1) We believe that the bijections mentioned above are identical. In fact, this is clearly the case if all the tensor factors of $B$ are perfect crystals. Indeed, since the subgraph of $B$ consisting of the dual Demazure arrows is connected, there is no more than one isomorphism between it and $A(\Gamma)$.

(2) In the case when all the tensor factors of $B$ are perfect crystals, a corollary of the work in \cite{LL2} is the following application of the quantum alcove model, cf. Remark 6.15 (1). By making
specific choices for the \( \lambda \)-chains \( \Gamma \) and \( \Gamma' \), the bijection between \( \mathcal{A}(\Gamma) \) and \( \mathcal{A}(\Gamma') \) mentioned above gives a uniform realization of the combinatorial \( R \)-matrix (i.e., the unique affine crystal isomorphism commuting factors in a tensor product of KR crystals). In fact, we believe that this statement would hold in full generality, rather than just the perfect case.

(3) In \cite{LL1} we proved Theorem 6.14 in types \( A \) and \( C \), for certain \( \lambda \)-chains different from the lex ones, via certain bijections constructed in \cite{Le2}. Here we used the realization of the corresponding crystal \( \mathcal{B} \) in terms of Kashiwara-Nakashima (KN) columns \cite{KN}.

7. The energy function in the quantum alcove model and \( P = X \)

We use the notation in Section 5. Given the lex \( \lambda \)-chain \( \Gamma = (\beta_1, \ldots, \beta_m) \) with height sequence \( (l_1, \ldots, l_m) \), we define the complementary height sequence \( (\overline{l_1}, \ldots, \overline{l_m}) \) by \( \overline{l_i} := (\beta_i^\vee, \lambda - l_i) \). In other words, \( \overline{l_i} = \{|j \geq i \mid \beta_j = \beta_i\} \).

**Definition 7.1.** Given \( A = \{j_1 < \cdots < j_s\} \in \mathcal{A}(\Gamma) \), we let

\[
A^- := \{i \in A \mid r_{\beta_{j_1}} \cdots r_{\beta_{j_{i-1}}} > r_{\beta_{j_1}} \cdots r_{\beta_{j_{i-1}}} r_{\beta_{j_i}}\}
\]

(in other words, we record the quantum steps in the path \( (5.2) \)). We also define

\[
(7.1) \quad \text{height}(A) := \sum_{j \in A^-} \overline{l_j}.
\]

For examples, we refer to \cite{Le2}.

Our goal is to show that the bijection in Section 6.1 translates the height statistic in the quantum alcove model to the energy statistic in the quantum LS path model given in Section 4.3. For this, we need the following lemma, whose proof is given in Section 8.3.

**Lemma 7.2.** Let \( \sigma, \tau \in W^J \), and \( v \in \sigma W_1, w \in \tau W_1 \). Consider a shortest path \( p \) from \( \sigma \) to \( \tau \) in \( \text{QB}(W^J) \), as well as a shortest path \( q \) from \( v \) to \( w \) in \( \text{QB}(W) \). Then, \( \langle \text{wt}(p), \lambda \rangle \equiv \langle \text{wt}(q), \lambda \rangle \).

We can now state one of our main results.

**Theorem 7.3.** Consider an admissible subset \( A \) in \( \mathcal{A}(\lambda) \), and the corresponding QLS path \( \Pi(A) \in \text{QLS}(-w_0\lambda) \). Write \( \Pi(A) \) as:

\[
\sigma_0 \xleftarrow{b_1w_0\lambda} \sigma_1 \xleftarrow{b_2w_0\lambda} \cdots \xleftarrow{b_{p}w_0\lambda} \sigma_p,
\]

with \( \sigma'_i \in W^{w(w)} \) and \( 0 = b_0 < b_1 < \cdots < b_p < 1 \). Set \( \sigma_i := [\sigma'_i w_0]^J \in W^J \); note that \( \sigma_0 \xrightarrow{b_1} \sigma_1 \xrightarrow{b_2} \cdots \xrightarrow{b_p} \sigma_p \) (cf. (6.6) and (6.7)). Then, we have

\[
(7.2) \quad \text{height}(A) = \sum_{i=1}^{p}(1 - b_i) \text{wt}_\lambda(\sigma_{i-1} \Rightarrow \sigma_i).
\]

**Proof.** Recall from Section 6.2 that \( A \) (in fact, the corresponding path \( (5.2) \) in \( \text{QB}(W) \)) can be reconstructed from the quantum LS path by first defining recursively a sequence \( w_i \in \sigma_i W_1, i = 0, \ldots, p \) (and \( w_{-1} = 1 \)), and then by concatenating the unique paths \( q_i \) with increasing edge labels (with respect to \( <_\lambda \), cf. Section 6.2) between \( w_{i-1} \) and \( w_{i} \), for \( i = 0, \ldots, p \). By Theorem 6.4 (2), the paths \( q_i \) are shortest ones. Therefore, by Lemma 7.2, we have

\[
(7.2) \quad \text{wt}_\lambda(\sigma_{i-1} \Rightarrow \sigma_i) = \langle \text{wt}(q_i), \lambda \rangle \quad \text{for} \quad i = 1, \ldots, p.
\]
we also have \( \text{wt}(q_0) = 0 \).

Consider a quantum edge in some path \( q_i \), and let \( \beta_j \) be the root in the lex \( \lambda \)-chain labeling it (so \( j \in A^- \)). As discussed in Section 6.2 we have

\[
(7.3) \quad b_i \langle \beta_j', \lambda \rangle = l_j, \quad \text{so} \quad (1 - b_i) \langle \beta_j', \lambda \rangle = \tilde{l}_j.
\]

By noting that \( \text{wt}(q_i) \) is the sum of \( \beta_j' \) for all such \( \beta_j \), and by combining (7.1), (7.2), and (7.3), the statement of the theorem follows.

**Corollary 7.4.** 
*Keep the notation of Sections 4.5 and 6.3. For each \( A \in A(\lambda) \), we have* \( -\text{height}(A) = \text{Deg}_{-w_0 \lambda}(\Pi(A)) = \text{Deg}_{\lambda}(\omega(\Pi(A))) \)

\[
= \text{Deg}_{\lambda}(S(\Pi^*(A))) = D_{\text{rev}}(S(\tilde{\Psi}(A))) - D_{\text{ext}}^\lambda.
\]

*Namely, the following diagram commutes:*

\[
\begin{array}{cccccc}
\mathbb{Z}_{\leq 0} & \xleftarrow{\text{height} \mathcal{T} \mathcal{L} \mathcal{S}(4.1)} & A(\lambda) & \xrightarrow{\Pi(4.1)} & \Pi^* & \xrightarrow{\omega(4.5)} & \mathbb{Z}_{\leq 0} \\
& \downarrow{\Psi} & \downarrow{\Psi^*} & \downarrow{\Psi^*} & \downarrow{\Psi^*} & \downarrow{\Psi^*} & \downarrow{\Psi^*} \\
& \mathbb{Z}_{\leq 0} & \xrightarrow{\text{id}} & \mathbb{Z}_{\leq 0} & \xrightarrow{\text{id}} & \mathbb{Z}_{\leq 0} & \xrightarrow{\text{id}} \mathbb{Z}_{\leq 0} \\
\end{array}
\]

**Proof.** As in Theorem 7.3 write \( \Pi(A) \in \mathcal{QLS}(-w_0 \lambda) \) as:

\[
\sigma'_0 \overset{b_1 w_0 \lambda}{\leftrightarrow} \cdots \overset{-b_p w_0 \lambda}{\leftrightarrow} \sigma'_p,
\]

with \( \sigma'_i \in W^{\omega(A)} \) and \( 0 = b_0 < b_1 < \cdots < b_p < 1 \), and set \( \sigma_i := [\sigma'_i w_0]J \in W^J \). It follows from Theorem 4.6 that

\[
\text{Deg}_{-w_0 \lambda}(\Pi(A)) = -\sum_{i=1}^p (1 - b_i) \text{wt}_{-w_0 \lambda}(\sigma'_i \Rightarrow \sigma'_{i-1}).
\]

Also, we see from Lemma 4.7(3) that

\[
\text{wt}_{-w_0 \lambda}(\sigma'_i \Rightarrow \sigma'_{i-1}) = \text{wt}_{\lambda}(\sigma_{i-1} \Rightarrow \sigma_i) \quad \text{for all} \ 1 \leq i \leq p.
\]

Therefore, we obtain

\[
\text{Deg}_{-w_0 \lambda}(\Pi(A)) = -\sum_{i=1}^p (1 - b_i) \text{wt}_{\lambda}(\sigma_{i-1} \Rightarrow \sigma_i) = -\text{height}(A) \quad \text{by Theorem 7.3}
\]

which proves the first equality. For the second equality, observe

\[
\text{wt}_{-w_0 \lambda}(\sigma'_i \Rightarrow \sigma'_{i-1}) = \text{wt}_{\lambda}(\omega(\sigma'_i) \Rightarrow \omega(\sigma'_{i-1})) \quad \text{for all} \ 1 \leq i \leq p \quad \text{by Lemma 4.7(3)}.
\]
Using this, we deduce that
\[
\text{Deg}_{-w_0 \lambda}(\Pi(A)) = - \sum_{i=1}^{p} (1 - b_i) \text{wt}_\lambda(\omega(\sigma'_i) \Rightarrow \omega(\sigma'_{i-1}))
\]
\[
= \text{Deg}_\lambda(\omega(\Pi(A))) \quad \text{by Theorem 4.6 and (4.19)},
\]
as desired. Since \(S \circ \Pi^* = \omega \circ \Pi\) by the definitions of these maps, the third equality follows. The last equality follows from Corollary 4.9 since \(\tilde{\Psi} = \Psi \circ \Pi^*\).

Based on Theorem 6.14 cf. also the discussion following it, we have the strengthening of Corollary 7.4 stated below.

**Theorem 7.5.** Corollary 7.4 holds for \(A(\Gamma)\), where \(\Gamma\) is an arbitrary \(\lambda\)-chain, with \(\tilde{\Psi}\) replaced with one of the isomorphisms in Theorem 6.14.

**Remark 7.6.** In [LeS] the energy function in types \(A\) and \(C\) was realized in terms of a statistic in the model based on KN columns, which is known as charge. Furthermore, in [Le2] it was shown that this statistic is the translation of the height statistic via the bijections constructed there (also mentioned in Remark 6.15 (3)), between the corresponding quantum alcove model and models based on KN columns. This should be compared with Corollary 7.4 and Theorem 7.5 where the constant \(D^\text{ext}\) is 0 in these cases.

The following is due to Ion [Ion, Theorem 4.2] for the dual of an untwisted affine root system.

**Lemma 7.7.** For \(\lambda\) dominant,
\[
(7.5) \quad P_\lambda(x; q, 0) = E_{w_0 \lambda}(x; q, 0)
\]
where \(E_\mu\) is the nonsymmetric Macdonald polynomial [Ma2].

**Proof.** Applying [Ma2] (5.7.8)] and its notation, at \(t = 0\) we have \(\xi_{\mu} \to 0\) if \(\mu\) is not the unique antidominant element \(w_0 \lambda\) in the finite Weyl group orbit of \(\lambda\). Indeed, letting \(v(\mu) = r_{i_1} r_{i_2} \cdots r_{i_p}\) be a reduced expression of the shortest element \(v(\mu)\) in the finite Weyl group such that \(v(\mu)\mu\) is antidominant, we obtain
\[
\xi_{\mu} = \prod_{k=1}^{p} \frac{t q^{-\langle \beta'_k, \mu \rangle} - t^{-\langle v(\mu) \beta'_k, \rho \rangle}}{q^{-\langle \beta'_k, \mu \rangle} - t^{-\langle v(\mu) \beta'_k, \rho \rangle}},
\]
where \(\beta_k := r_{i_k} \cdots r_{i_{k+1}} \alpha_{i_k}\) for \(1 \leq k \leq p\); here we note that \(\langle \beta'_k, \mu \rangle > 0\) and \(\langle v(\mu) \beta'_k, \rho \rangle < 0\) for all \(1 \leq k \leq p\) since the elements \(\beta_k, 1 \leq k \leq p\), comprise the inversion set for \(v(\mu)\). \(\square\)

We now recall the specialization of the Ram-Yip formula [RY] for the nonsymmetric Macdonald polynomial \(E_{w_0 \lambda}(x; q, t)\) at \(t = 0\), which was worked out by Orr-Shimozono [OS]. Let us consider a reduced alcove path
\[
\Gamma := (A_0 = A_0 \to -\beta_1 \to A_1 \to -\beta_2 \to \cdots \to -\beta_m \to A_m = A_0 + w_0 \lambda),
\]
where \((\beta_1, \beta_2, \ldots, \beta_m)\) is the corresponding \((-w_0 \lambda)\)-chain of roots, and let \(H_{\beta_i, -l_i}\) denote the hyperplane separating \(A_{i-1}\) and \(A_i\) for \(1 \leq i \leq m\). Then, an admissible subset \(A = \{j_1 < \cdots < j_s\} \in A(\Gamma)\) can be interpreted as a “folding” of the alcove path \(\Gamma\) along the hyperplanes \(H_{\beta_i, -l_i}\), where \(i\) ranges over \(A\). With this notation, the Orr-Shimozono formula for the specialization \(E_{w_0 \lambda}(x; q, 0)\) can be stated as follows.
Proposition 7.8 ([OS Corollary 4.4]). For \( \lambda \) dominant,
\[
E_{w_0 \lambda}(x; q, 0) = \sum_{A \in A(\Gamma)} q^{\text{height}(A)} x^{\text{wt}(A)};
\]
here, for an admissible subset \( A = \{ j_1 < \cdots < j_s \} \in A(\Gamma) \),
\[
\text{height}(A) = \sum_{j \in A^{-}} ( -I_{j} + \langle \beta_{j}', -w_0 \lambda \rangle ),
\]
\[
-\text{wt}(A) = r_{\beta_{j_1}, -I_{j_1}} \cdots r_{\beta_{j_s}, -I_{j_s}} (w_0 \lambda).
\]

Sketch of proof. Let \( t_{w_0 \lambda} = r_{i_1} \cdots r_{i_m} \pi \) be the reduced expression (in the extended affine Weyl group) corresponding to \( \Gamma \), where \( \pi \) is an element of length zero. Rewriting \( t_{w_0 \lambda} \) as \( \pi r_{i_1} \cdots r_{i_m} \), we set
\[
\hat{\beta}_{j} := r_{i_m} \cdots r_{i_{j+1}} \alpha_{i_j}, \quad 1 \leq j \leq m,
\]
and define a sequence of alcoves
\[
\Gamma' := (A'_0 \to A'_1 \to \cdots \to A'_{m} = A_0)
\]
in such a way that the \( \hat{\beta}_{j} \) corresponds to the hyperplane separating \( A'_{j-1} \) and \( A'_{j} \) for \( 1 \leq j \leq m \). Then, by applying the identity
\[
 r_{i_1} \cdots r_{i_j} = t_{w_0 \lambda} \pi^{-1} r_{i_m} \cdots r_{i_{j+1}} = t_{w_0 \lambda} r_{i_m} \cdots r_{i_{j+1}} \pi^{-1}
\]
to \( A_0 \), we obtain
\[
A_j = A'_j + w_0 \lambda \quad \text{for all } 1 \leq j \leq m;
\]
this explains why the exponent of \( q \) in \( E_{w_0 \lambda}(x; q, 0) \), as given in [OS Corollary 4.4], can be written as desired. Also, it is not hard to see that the exponent of \( x \) in \( E_{w_0 \lambda}(x; q, 0) \) can be written as \( r_{\beta_{j_1}, -I_{j_1}} \cdots r_{\beta_{j_s}, -I_{j_s}} (w_0 \lambda) \).

Theorem 7.9. For \( \lambda \) dominant,
\[
P_{\lambda}(x; q, 0) = \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}(\eta)} x^{\text{wt}(\eta)} = \sum_{A \in A(\lambda)} q^{\text{height}(A)} x^{\text{wt}(A)}.
\]

Proof. For simplicity of notation, we set \( \mu := -w_0 \lambda = \omega(\lambda) \), where \( \omega \) is the Dynkin diagram automorphism given by \( -w_0 \alpha_j = \alpha_{\omega(j)} \) for \( j \in I \). By Lemma 7.7 and Proposition 7.8 we obtain
\[
P_{\lambda}(x; q, 0) = \sum_{A \in A(\mu)} q^{\text{height}(A)} x^{-\text{wt}(A)}.
\]
Moreover, by Proposition 6.8 applied to \( \mu = -w_0 \lambda \) and the first equality of Corollary 7.4, we deduce that
\[
\sum_{A \in A(\mu)} q^{\text{height}(A)} x^{-\text{wt}(A)} = \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}(\eta)} x^{\text{wt}(\eta)},
\]
which proves the first equality of [7.4].

Now, we have
\[
P_{\lambda}(x; q, 0) = \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}(\eta)} x^{\text{wt}(\eta)} = \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}(\lambda(\eta))} x^{\text{wt}(\eta)}
\]
\[
= \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}(\lambda(\eta))} x^{\text{wt}(\eta)} = \sum_{\eta \in \text{QLS}(\lambda)} q^{-\text{Deg}(\lambda(\eta))} x^{\text{wt}(\eta)},
\]
the last equality follows from the fact that \( P_\lambda(x; q, t) \) is symmetric, i.e., invariant under the action of the Weyl group \( W \), in the variable \( x \). Because \( \Pi^*: A(\lambda) \rightarrow QLS(\lambda) \) is a weight-preserving bijection, it follows that

\[
\sum_{\eta \in QLS(\lambda)} q^{-\text{Deg}_\lambda(S(\eta))} x^{\text{wt}(\eta)} = \sum_{A \in A(\lambda)} q^{-\text{Deg}_\lambda(S(\Pi^*(A)))} x^{\text{wt}(A)}.
\]

Also, by Corollary 7.4 we get

\[
\sum_{A \in A(\lambda)} q^{-\text{Deg}_\lambda(S(\Pi^*(A)))} x^{\text{wt}(A)} = \sum_{A \in A(\lambda)} q^{\text{height}(A)} x^{\text{wt}(A)},
\]

which proves the second equality of (7.7).

\[\square\]

Remark 7.10. The formula (7.7) holds for any \( \lambda \)-chain.

Now define the graded character corresponding to the KR crystal \( B \) (see for example [HKOTT, HKOTY]) by

(7.8)

\[
X_\lambda(x; q) := \sum_{b \in B} q^{D_b - D_b^{\text{ext}}} x^{\text{wt}(b)},
\]

where \( \text{wt}(b) \) is the weight of the crystal element \( b \). From Theorems 4.5 and 7.9, we immediately derive our main result.

Corollary 7.11. We have

\[
P_\lambda(x; q^{-1}, 0) = X_\lambda(x; q).
\]

Remark 7.12. In type \( A \), the Macdonald polynomial at \( t = 0 \) can be expanded in terms of Schur functions with Kostka-Foulkes polynomials as the transition matrix [Ma, Chapter III.6]. These in turn can be expressed as one-dimensional configuration sums \( X \) [NY], which implies the \( P = X \) result in type \( A \). In all simply-laced types it was known by combining the results in [Ion] and [FL], which equate a certain affine Demazure character with \( P \) and \( X \), respectively. It was also known in type \( C \) by [Le2, LeS].

8. Proofs of the lemmas in Sections 6.1, 6.2, and 7

8.1. Proof of Lemma 6.2. In the proof of this lemma, a dotted (resp., plain) edge represents a quantum (resp., Bruhat) edge in QB(\( W \)) or QB(\( W_J \)), while a dashed edge can be of both types. Define \( \beta \in \Phi^{af+} \) by

(8.1)

\[
\beta := \begin{cases} 
  w\gamma & \text{if } w\gamma \in \Phi^+ \\
  \delta + w\gamma & \text{if } w\gamma \in \Phi^-.
\end{cases}
\]

As in the proof of one of the main results in [LNSSS1], namely Theorem 6.5 (more precisely, the converse statement), we proceed by induction on the height of \( \beta \) (i.e., the sum of the coefficients in its expansion in the basis of affine simple roots). The base case, when \( \beta \) is an affine simple root, is treated in the following lemma.

Lemma 8.1. In QB\( b_{\lambda}(W) \) we have an edge \( \xrightarrow{w^{-1} \alpha} r_\alpha w \) for a finite simple root \( \alpha \) with \( w^{-1} \alpha \not\in \Phi_J \) (resp. \( \xrightarrow{-w^{-1} \theta} r_\theta w \), where \( w^{-1} \theta \not\in \Phi_J \)) if and only if in QB\( b_{\lambda}(W_J) \) we have \( \xrightarrow{|w|^{-1} \alpha} r_\alpha |w| \) (resp. \( \xrightarrow{|w|^{-1} \theta} r_\theta |w| \)).
Lemma 8.2. Let us first ignore the parameter $b$ (or just assume $b = 0$). By the trichotomy of cosets in [LNSSSS1], Propositions 5.10 and 5.11, there is a simple way to test whether we have the mentioned edges in $\text{QB}(W)$, namely $w^{-1}\tilde{\alpha} \in \Phi^\pm \setminus \Phi_J^\pm$, where $\tilde{\alpha}$ is the simple root $\alpha$ or $-\theta$, respectively; similarly for the mentioned edges in $\text{QB}(W^J)$, with $w$ replaced by $|w|$. The proof is completed by noting that

$$w^{-1}\tilde{\alpha} \in \Phi^\pm \setminus \Phi_J^\pm \iff |w|^{-1}\tilde{\alpha} \in \Phi^\pm \setminus \Phi_J^\pm,$$

where $\tilde{\alpha}$ can be any root, in fact; indeed, writing $w = |w|w_J$, we have $|w|^{-1} = w_Jw^{-1}$, and we know that the elements of $W_J$ permute $\Phi^\pm \setminus \Phi_J^\pm$. For an arbitrary $b$ (and $\tilde{\alpha}$), we observe that

$$b(w^{-1}\tilde{\alpha}^\vee, \lambda) = b(\tilde{\alpha}^\vee, w\lambda) = b(\tilde{\alpha}^\vee, |w|\lambda) = b(|w|^{-1}\tilde{\alpha}^\vee, \lambda).$$

We need the following result from [LNSSSS1], which we recall.

**Lemma 8.2** ([LNSSSS1] Lemma 6.10). Let $w \in W$, and let $\gamma \in \Phi^\pm \setminus \Phi_J^\pm$. Define $\beta \in \Phi_{af}^+$ as in (8.1). There exists an affine simple root $\alpha$ (in fact, $\alpha \neq \alpha_0$ if $w\gamma \in \Phi^+$) such that $\langle \alpha^\vee, \beta \rangle > 0$, and we have the edge in $\text{QB}(W^J)$ indicated either in case (1) or (2) below, where $z$ is defined by $r_\theta[w\gamma] = [r_\theta[w\gamma]]z = [r_\theta w\gamma]z$:

(1) \[
\begin{align*}
[w] & \xrightarrow{w^{-1}\alpha} r_\alpha[w] & \text{if } \alpha \neq \alpha_0 \\
[w] & \xrightarrow{w^{-1}\theta} r_\theta[w] & \text{if } \alpha = \alpha_0,
\end{align*}
\]

(2) \[
\begin{align*}
r_\alpha[w\gamma] & \xrightarrow{w^{-1}\alpha} r_\alpha[w\gamma] & \text{if } \alpha \neq \alpha_0 \\
r_\theta[w\gamma] & \xrightarrow{w^{-1}\theta} r_\theta[w\gamma] & \text{if } \alpha = \alpha_0.
\end{align*}
\]

We also need the following lemma.

**Lemma 8.3.** Consider any one of the diamonds in the parabolic quantum Bruhat graph $\text{QB}(W^J)$ listed in [LNSSSS1] Lemma 5.14. If one of the two paths (of length 2) is in $\text{QB}_b(\lambda)(W^J)$, for some fixed $b$, then the other one is too.

**Proof.** By [LNSSSS1] Lemma 5.14, we know that, up to sign and left multiplication by elements of $W_J$, the pairs of labels on the two paths are $\{w^{-1}\tilde{\alpha}, \gamma\}$ and $\{\gamma, r_\gamma w^{-1}\tilde{\alpha}\}$, for some $\gamma \in \Phi^\pm \setminus \Phi_J^\pm$ and $w \in W_J$, while $\tilde{\alpha}$ is a finite simple root or $-\theta$. The equivalence of the integrality conditions with respect to $b$ for the two pairs follows from the simple calculation

$$b(r_\gamma w^{-1}\tilde{\alpha}^\vee, \lambda) = b(w^{-1}\tilde{\alpha}^\vee + l\gamma^\vee, \lambda) = b(w^{-1}\tilde{\alpha}^\vee, \lambda) + l \left(b(\gamma^\vee, \lambda)\right),$$

where $l$ is an integer. On another hand, it is clear that mapping roots via elements of $W_J$ preserves the integrality condition. □

**Proof of Lemma 6.2.** We can assume that $\gamma \notin \Phi_J$, as otherwise the statement is obvious. As stated above, we proceed by induction on the height of the affine root $\beta$. If $\beta$ is an affine simple root, the conclusion follows directly from Lemma 8.1. Otherwise, we apply Lemma 8.2 for $\text{QB}(W^J)$; this gives an affine simple root $\alpha$ satisfying

(8.2) \[
\alpha \neq \beta, \quad \langle \alpha^\vee, \beta \rangle > 0,
\]

and either condition (1) or (2) in the mentioned lemma. Assume that condition (1) holds, as the reasoning is completely similar if condition (2) holds. By Lemma 8.1 we have

(8.3) \[
\begin{align*}
[w] & \xrightarrow{w^{-1}\alpha} r_\alpha[w] & \text{if } \alpha \neq \alpha_0, \text{ where } w^{-1}\alpha \notin \Phi_J, \\
[w] & \xrightarrow{w^{-1}\theta} r_\theta[w] & \text{if } \alpha = \alpha_0, \text{ where } w^{-1}\theta \notin \Phi_J.
\end{align*}
\]
By Lemma 8.2, we have one of the following three cases:

\[(8.4) \quad (\beta \in \Phi^+, \alpha \neq \alpha_0), \quad (\beta \in \delta - \Phi^+, \alpha \neq \alpha_0), \quad (\beta \in \delta - \Phi^+, \alpha = \alpha_0).\]

By LNSSS1, Lemma 5.14, known as the “diamond lemma”, we have the left diamonds in LNSSS1 Eqs. (5.3), (5.4), and (5.7), respectively. Note that all the necessary conditions for applying the diamond lemma are checked as in the proof of the converse statement of LNSSS1 Theorem 6.5. We can represent the diamonds in the three cases (8.4) using the single diagram below, where \(\tilde{\alpha} := \alpha\) if \(\alpha \neq \alpha_0\), and \(\tilde{\alpha} := -\theta\), otherwise.

\[
\begin{array}{c}
r_{\tilde{\alpha}}w - \gamma \leftarrow r_{\tilde{\alpha}}wr_{\gamma} \\
\downarrow \quad \downarrow \\
|w^{-1}\tilde{\alpha}| \quad |r_{\gamma}w^{-1}\tilde{\alpha}| \\
\downarrow \quad \downarrow \\
w - \gamma \quad \rightarrow wr_{\gamma},
\end{array}
\]

where \(|\xi| := \xi\) (resp., \(|\xi| := -\xi\)) for \(\xi \in \Phi^+\) (resp., \(\xi \in \Phi^-\)). Recall that the bottom edge is viewed as a path \(q\); similarly, we view the top edge as a path \(q'\), and we clearly have \(\text{wt}(q) = \text{wt}(q')\) if \(\tilde{\alpha} \neq -\theta\).

Define \(\beta'\) for the top edge of the diamond (8.5) in the same way as \(\beta\) was defined for the bottom one in (8.1). As in the proof of the converse statement of LNSSS1 Theorem 6.5, we can check in all three cases (8.4) that \(\beta' = r_{\alpha}\beta\). Since \(\langle \alpha', \beta \rangle > 0\), this implies that the height of \(\beta'\) is strictly smaller than that of \(\beta\). Therefore, by applying the induction hypothesis to the top edge of the diamond (8.5), which is clearly in QB_{b\lambda}(W), we obtain a path in QB_{b\lambda}(W^J):

\[
(8.6) \quad p' : \quad [r_{\tilde{\alpha}}w] = y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_n = [r_{\tilde{\alpha}}wr_{\gamma}].
\]

By induction, we have \(\text{wt}(p') \equiv \text{wt}(q') \mod Q_{\gamma}^J\).

**Case 1.** Assume for the moment that \(r_{\gamma}w^{-1}\tilde{\alpha} \notin \Phi_J\). By Lemma 8.1, the right edge in (8.5) implies that in QB(W^J) we have an edge

\[
(8.7) \quad [wr_{\gamma}] \rightarrow [r_{\tilde{\alpha}}wr_{\gamma}].
\]

Assuming that the diamond lemma can be successively applied based on (8.6) and (8.7), we exhibit the diamonds as in (8.8) below in QB(W^J), from right to left, where \(y_i = [r_{\tilde{\alpha}}x_i]\).

\[
\begin{array}{c}
[r_{\tilde{\alpha}}w] = y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_n = [r_{\tilde{\alpha}}wr_{\gamma}] \\
\downarrow \quad \downarrow \\
|w| = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = [wr_{\gamma}]
\end{array}
\]

Note that the labels on the top edges are the same as those on the corresponding edges on the bottom, or at most differ from those by elements of \(W_J\); so all the bottom edges are in QB_{b\lambda}(W^J) too, and we can define \(p\) to be the path formed by them.

Now let us prove that \(\text{wt}(p) \equiv \text{wt}(q) \mod Q_{\gamma}^J\). By LNSSS1 Lemma 5.14, the weights of all paths from \([w]\) to \([r_{\tilde{\alpha}}wr_{\gamma}]\) in (8.8) are congruent mod \(Q_{\gamma}^J\). If \(\tilde{\alpha} \neq -\theta\), then all the vertical edges in (8.8) are Bruhat edges, so \(\text{wt}(p) \equiv \text{wt}(p') \mod Q_{\gamma}^J\). Applying the induction hypothesis and the fact that \(\text{wt}(q) = \text{wt}(q')\) concludes the induction step in this case. If \(\tilde{\alpha} = -\theta\), then we are in the third case in (8.4), and so diagram (8.5) is the left one in LNSSS1 Eq. (5.7); thus its top edge is a Bruhat edge, which implies \(\text{wt}(p') = \text{wt}(q') = 0\,\text{and its bottom edge is a}...
quantum one, in particular $w\gamma \in \Phi^-$. Moreover, all the vertical edges in (8.8) are quantum ones; in particular, the leftmost and the rightmost ones have weights

$$-[w]^{-1}\theta^\vee \equiv -w^{-1}\theta^\vee \mod Q'_{J}, \text{ and } -[wr_{\gamma}]^{-1}\theta^\vee \equiv -r_{\gamma}w^{-1}\theta^\vee \mod Q'_{J},$$

respectively. Then, by the above observation about the paths from $[w]$ to $[r_{\alpha}wr_{\gamma}]$ in (8.8), we have mod $Q'_{J}$:

$$\text{wt}(p) \equiv \text{wt}(p') - w^{-1}\theta^\vee + r_{\gamma}w^{-1}\theta^\vee = -\langle \theta^\vee, -w_{\gamma} \rangle \gamma^\vee = \gamma^\vee = \text{wt}(q).$$

Here we see that $\langle \theta^\vee, -w_{\gamma} \rangle = 1$ by using the well-known fact that if $\phi \neq \theta$ is a positive root, then $\langle \theta^\vee, \phi \rangle$ is 0 or 1; indeed, in our case we saw that $-w_{\gamma} \in \Phi^+$, while we have $-w_{\gamma} \neq \theta$ and $\langle \theta^\vee, -w_{\gamma} \rangle \neq 0$ by (8.2).

**Case 2.** The reasoning in Case 1 fails, i.e., we cannot apply the diamond lemma at some point, if we have the following situation for some $i \leq n$.

$$y_{i-1} \rightarrow \cdots \rightarrow y_i \rightarrow \cdots \rightarrow y_{i+1} \rightarrow \cdots \rightarrow y_n = [r_{\alpha}wr_{\gamma}]$$

$$x_{i-1} \rightarrow \cdots \rightarrow x_i \rightarrow \cdots \rightarrow x_{i+1} \rightarrow \cdots \rightarrow x_n = [wr_{\gamma}]$$

In this case, the edge $x_{i-1} \rightarrow \cdots \rightarrow y_i$ is in QB$\lambda(W)$, since it coincides with the edge $y_{i-1} \rightarrow \cdots \rightarrow y_i$, which has this property by the induction hypothesis, cf. (8.6). By Lemma 8.3, all vertical edges are also in QB$\lambda(W)$, in particular the rightmost one. By Lemma 8.1, the edge $wr_{\gamma} \rightarrow \cdots \rightarrow r_{\alpha}wr_{\gamma}$ in (8.5) is in QB$\lambda(W)$. By applying Lemma 8.3 to (8.5) this time, we conclude that the edge $w \rightarrow \cdots \rightarrow r_{\alpha}w$ is in QB$\lambda(W)$. But we showed in (8.3) that $w^{-1}\alpha \notin \Phi_{J}$, so by Lemma 8.1 the edge $[w] \rightarrow [r_{\alpha}w]$ is in QB$\lambda(W)$. We now define $p$ to be the following path in QB$\lambda(W)$:

$$p: \quad x_0 = [w] \rightarrow [r_{\alpha}w] = y_0 \rightarrow \cdots \rightarrow y_{i-1} = x_i \rightarrow \cdots \rightarrow y_n = [wr_{\gamma}].$$

We then prove that $\text{wt}(p) \equiv \text{wt}(q) \mod Q'_{J}$ in a way completely similar to Case 1, which concludes the induction step.

**Case 3.** The last case to consider is the one when $r_{\gamma}w^{-1}\alpha \notin \Phi_{J}$. We still have the edge $wr_{\gamma} \rightarrow \cdots \rightarrow r_{\alpha}wr_{\gamma}$ in QB$\lambda(W)$, because $\langle r_{\gamma}w^{-1}\alpha, \lambda \rangle = 0$. So we can reason as in the previous paragraph in order to prove that the edge $[w] \rightarrow [r_{\alpha}w]$ is in QB$\lambda(W)$. We now define $p$ to be the following path in QB$\lambda(W)$:

$$p: \quad [w] \rightarrow [r_{\alpha}w] = y_0 \rightarrow \cdots \rightarrow y_{i-1} = y_i \rightarrow \cdots \rightarrow y_n = x_n = [wr_{\gamma}].$$

Note that this is the only case when the induction step produces a path of a different length (more precisely, longer by 1) based on the path in the induction hypothesis.

Now let us prove that $\text{wt}(p) \equiv \text{wt}(q) \mod Q'_{J}$. If $\alpha \neq -\theta$, then the first edge of $p$ is a Bruhat edge, so $\text{wt}(p) = \text{wt}(p')$. Applying the induction hypothesis and the fact that $\text{wt}(q) = \text{wt}(q')$
concludes the induction step in this case. If $\tilde{\alpha} = -\theta$, then by the same reasoning as in Case 1, we deduce
\[ \text{wt}(p') = \text{wt}(q') = 0, \quad \text{wt}(q) = \gamma^\vee, \quad w\gamma \in \Phi^+. \]

We conclude that $\text{wt}(p) \equiv -w^{-1}\theta^\vee \mod Q^\vee_f$ (cf. Case 1), so we need to prove that $-w^{-1}\theta^\vee \equiv \gamma^\vee \mod Q^\vee_f$. This follows from
\[ \Phi^\vee_f \ni r\gamma - w^{-1}\theta^\vee = w^{-1}\theta^\vee - \langle w^{-1}\theta^\vee, \gamma \rangle \gamma^\vee = w^{-1}\theta^\vee + \gamma^\vee. \]

Here we have used the fact that $\langle w^{-1}\theta^\vee, \gamma \rangle = -1$, as in Case 1. \qed

8.2. Proof of Lemma 6.7. We require some notation and results from [LS]. Let $W_{af}^\ominus$ denote the set of minimum coset representatives in $W_{af}/W$. Write $y < x$ for the covering relation in the (strong) Bruhat order on $W_{af}$. For $M \in \mathbb{Z}_{>0}$, say that $\xi \in Q^\vee$ is $M$-superantidominant if $\langle \xi, \alpha_i \rangle \leq -M$ for every positive root $\alpha_i \in \Phi^+$. We fix once and for all a sufficiently large $M \in \mathbb{Z}_{>0}$ ($M = 2|W| + 2$ is sufficient).

**Lemma 8.4 ([LS] Lemma 3.3).** Let $w \in W$ and $\xi \in Q^\vee$. Then $\text{wt}_\xi \in W_{af}^\ominus$ if and only if $\xi$ is antidominant (that is, $\langle \xi, \alpha_i \rangle \leq 0$ for all $i \in I$) and $w \in W^L$, where $L := \{ i \in I \mid \langle \xi, \alpha_i \rangle = 0 \}$.

**Proposition 8.5 ([LS] Proposition 4.4).** Let $\xi \in Q^\vee$ be $M$-superantidominant and let $x = \text{wt}_\xi$ with $v, w \in W$. Then $y = xx_{v\alpha+\rho/a} < x$ if and only if one of the following conditions holds:

1. $\ell(vw) = \ell(wv\alpha) - 1$ and $n = \langle \xi, \alpha \rangle$, giving $y = wv\alpha t_v\xi$;
2. $\ell(vw) = \ell(wv\alpha) + \langle \alpha^\vee, 2\rho \rangle - 1$ and $n = \langle \xi, \alpha \rangle + 1$, giving $y = wv\alpha t_v(\xi + \alpha^\vee)$;
3. $\ell(v) = \ell(v\alpha) + 1$ and $n = 0$, giving $y = wv\alpha t_v\alpha \xi$;
4. $\ell(v) = \ell(v\alpha) - \langle \alpha^\vee, 2\rho \rangle + 1$ and $n = -1$, giving $y = wv\alpha t_v(\xi + \alpha^\vee)$.

We start with the following lemma. We need the $b$-Bruhat order on $W_{af}$, denoted $<_b$, which is defined by a condition completely similar to $[3.3]$ applied to the covers in $W_{af}$.

**Lemma 8.6.** Assume that in $W_{af}^\ominus$ we have
\[ \text{vt}_\xi > \text{wt}_h, \quad \text{vt}_\xi > \text{wt}_h', \quad \text{where} \quad h' - h \in Q^\vee+, \]
and $\xi, h, h' \in Q^\vee$ are $M$-superantidominant. Then $\text{vt}_\xi > \text{wt}_h$, and in fact any saturated chain between these elements is a chain in $b$-Bruhat order.

**Proof.** We claim that $\text{wt}_h > \text{wt}_h'$ using a downward chain in $W_{af}^\ominus$. It suffices to prove this when $h' - h = \alpha_i^\vee$ for some $i \in I$. Suppose this is the case. Suppose first that $w\alpha_i < w$. By Proposition 8.5 we have $\text{wt}_h > w\alpha_i t_h + \alpha_i^\vee > \text{wt}_h + \alpha_i^\vee$ as required. Otherwise we have $w\alpha_i > w$. Then by Proposition 8.5 we have $\text{wt}_h > w\alpha_i t_h > \text{wt}_h + \alpha_i^\vee$ as required.

Knowing this, using Proposition 8.5 we pick a downward saturated chain from $\text{vt}_\xi$ to $\text{wt}_h$, followed by one from $\text{wt}_h$ to $\text{wt}_h'$, all in $W_{af}^\ominus$. By the hypothesis, there is a downward saturated chain in $b$-Bruhat order from $\text{vt}_\xi$ to $\text{wt}_h$. By [LeSh] Lemma 4.15, we know that the first chain is in $b$-Bruhat order too, which concludes the proof. \qed
Proof of Lemma 6.7. By Proposition 8.5 we can lift both paths to downward saturated chains in $W_{af}$ starting at $vt_ξ$, where $ξ$ is a fixed $M$-superantidominant element in $Q^\vee$. Denote the endpoints of the two chains by $wt_ξ + h$ and $wt_ξ + h'$, respectively. Recall that $h$ and $h'$ are the sums of the coroots corresponding to (the labels of) the quantum edges in the paths in $QB(W)$ which are lifted. Since the first path in $QB(W)$ is a shortest one, by [Po, Lemma 1], we have $h' - h \in Q^\vee +$. Furthermore, by the hypothesis, the second chain in $W_{af}$ is in $b$-Bruhat order. Thus the hypotheses of Lemma 8.6 are all satisfied, so we conclude that the first chain in $W_{af}$ is also in $b$-Bruhat order, and therefore the first path in $QB(W)$ is in $QB_{b\lambda}(W)$. □

8.3. Proof of Lemma 7.2. Let us first recall Proposition 4.1 which is the parabolic generalization of a lemma due to Postnikov [Po].

Proof of Lemma 7.2. By Lemma 6.2, we can construct a path from $p'$ from $σ$ to $τ$ in $QB(W^J)$ with

$$\text{wt}(p') \equiv \text{wt}(q) \mod Q^\vee_J;$$

(8.9) namely, we simply concatenate the paths in $QB(W^J)$ that correspond, by the mentioned lemma, to each edge of $q$, cf. the construction of the forgetful map in Section 6.1. By Proposition 4.1, we have

$$\langle \text{wt}(p'), \lambda \rangle \geq \langle \text{wt}(p), \lambda \rangle.$$  

(8.10)

We then exhibit a path $q'$ from $v$ to $w$ as in the proof of Lemma 6.6 (on which the construction of the inverse map in Section 6.2 is based); we refer to this proof for the details. Namely, we concatenate the following:

- a path from $v$ to $σ$ with only quantum edges and all edge labels in $Φ^+_J$;
- a path from $σ$ to $τ$ constructed based on $p$;
- a path from $τ$ to $w$ with only Bruhat edges and all edge labels in $Φ^+_J$.

Note that

$$\langle \text{wt}(q'), \lambda \rangle = \langle \text{wt}(p), \lambda \rangle,$$

(8.11) since all the edges in the first segment of $q'$, as well as the extra edges introduced in the second segment, have labels orthogonal to $λ$. On another hand, by Proposition 4.1 (in fact, we only need here the original version [Po, Lemma 1 (3)]), we have

$$\langle \text{wt}(q'), \lambda \rangle \geq \langle \text{wt}(q), \lambda \rangle.$$  

(8.12)

The proof is concluded by combining (8.9), (8.10), (8.11), and (8.12). □

Appendix A. Perfectness and classical decomposition

The notion of perfectness plays an important role for level-zero crystals. It ensures for example that the Kyoto path model is applicable, which gives a model for highest weight affine crystals as a semi-infinite tensor product of Kirillov–Reshetikhin crystals. Let us define perfect crystals, see for example [HK]. Given a crystal $B$ and $b \in B$, we need the definition

$$ε(b) = \sum_{i ∈ I_{af}} ε_i(b)Λ_i \quad \text{and} \quad φ(b) = \sum_{i ∈ I_{af}} φ_i(b)Λ_i.$$
with \( \varepsilon_i(b) \) and \( \varphi_i(b) \) as defined in (2.7). Furthermore, denote by \( \mathcal{X}_{af}^+ = \{ \lambda \in \mathcal{X}_{af}^+ | \text{lev}(\lambda) = \ell \} \) the set of dominant weights of level \( \ell \), where \( \mathcal{X}_{af}^+ := \bigoplus_{i \in I_{af}} Z_{\geq 0} \Lambda_i \).

**Definition A.1.** For a positive integer \( \ell > 0 \), a crystal \( \mathcal{B} \) is called a perfect crystal of level \( \ell \), if the following conditions are satisfied:

1. \( \mathcal{B} \) is isomorphic to the crystal graph of a finite-dimensional \( U'_q(\mathfrak{g}_{af}) \)-module.
2. \( \mathcal{B} \otimes \mathcal{B} \) is connected.
3. There exists a \( \lambda \in \bigoplus_{i \in I_{af}} Z_{\geq 0} \Lambda_i \), such that wt(\( \mathcal{B} \)) \( \subset \lambda + \sum_{i \in I_{af}} Z_{\leq 0} \alpha_i \) and there is a unique element in \( \mathcal{B} \) of classical weight \( \lambda \).
4. \( \forall b \in \mathcal{B}, \ \text{lev}(\varepsilon(b)) \geq \ell. \)
5. \( \forall \Lambda \in \mathcal{X}_{af}^{+\ell}, \) there exist unique elements \( b_\Lambda, b^{\Lambda} \in \mathcal{B} \), such that \( \varepsilon(b_\Lambda) = \Lambda = \varphi(b^{\Lambda}). \)

We denote by \( \mathcal{B}_{\text{min}} \) the set of minimal elements in \( \mathcal{B} \), namely
\[
\mathcal{B}_{\text{min}} = \{ b \in \mathcal{B} | \text{lev}(\varepsilon(b)) = \ell \}.
\]
Note that condition (5) of Definition A.1 ensures that \( \varepsilon, \varphi : \mathcal{B}_{\text{min}} \to \mathcal{X}_{af}^{+\ell} \) are bijections.

Recall from Section 2.1 that \( \delta = \sum_{j \in I_{af}} a_j \alpha_j \in \mathfrak{h}_{af}^* \) and \( c = \sum_{j \in I_{af}} a_j^\vee \alpha_j^\vee \in \mathfrak{h}_{af} \). Define \( c_r = \max\{ \frac{a_r}{a_r^\vee}, a_0^\vee \} \).

**Conjecture A.2.** [HKOTT, Conjecture 2.1] The Kirillov-Reshetikhin crystal \( B^{r,s} \) is perfect if and only if \( \frac{s}{c_r} \) is an integer. If \( B^{r,s} \) is perfect, its level is \( \frac{s}{c_r} \).

For all nonexceptional types this conjecture was proven in [FOS1]. Given the explicit models for \( B^{r,1} \) for all untwisted types in this paper and their implementation into SAGE [Sage, Sage-comb], we have verified Conjecture A.2 also for untwisted exceptional types when \( s = 1 \). For type \( G_2^{(1)} \), perfectness was also treated in [Y].

**Theorem A.3.** Conjecture A.2 holds for \( B^{r,1} \) for types \( G_2^{(1)}, F_4^{(1)}, E_6^{(1)}, E_7^{(1)} \) for all Dynkin nodes, and type \( E_8^{(1)} \) for all nodes (except possibly 5, 8 in the labeling of [HKOTT]). In addition, the graded classical decompositions of [HKOTY, Appendix A] were verified (except for type \( E_8^{(1)} \)).

For the other nodes in type \( E_8^{(1)} \) the program is currently too slow to test it.

**Proof.** Point (1) of Definition A.1 follows from Remark 2.8. Point (2) can be deduced from [Kas]. Points (3)-(5) were checked explicitly on the computer using the implementation of level-zero LS paths in SAGE [Sage, Sage-comb] (version sage-7.1 or higher), see for example

```
sage: C = CartanType(['E',6,1])
sage: R = RootSystem(C)
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(La[1])
sage: LS.is_perfect()
True
```
This showed that \( B^{r,1} \) is perfect.
• for all nodes of type $E_{6,7}^{(1)}$ and the nodes specified in the theorem for type $E_{8}^{(1)}$;
• the first 2 nodes of $F_{4}^{(1)}$ (long roots);
• the second node of $G_{2}^{(1)}$ (long root).

This confirms the perfectness claim of the theorem. The graded classical decompositions of [HKOTY] Appendix A were also confirmed by computer. □

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