Abstract. We prove that the geodesic flow of a Kupka-Smale riemannian metric on a closed surface has homoclinic orbits for all of its hyperbolic closed geodesics.

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2020 Mathematics Subject Classification. 37D40, 53D25, 37C29.
Gonzalo Contreras is partially supported by CONACYT, Mexico, grant A1-S-10145.
1. Introduction.

Let \((M, \rho)\) be a closed (i.e. compact, boundaryless) riemannian surface. Let \(SM = \{(x, v) : \rho(v, v) = 1\}\) be its unit tangent bundle with projection \(\pi : SM \to M, \pi(x, v) = x\). The geodesic flow \(\phi_t : SM \to SM\) of \((M, \rho)\) is defined by \(\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))\), where \(\gamma\) is the unit speed geodesic with \((\gamma(0), \dot{\gamma}(0)) = (x, v)\).

A closed orbit for \(\phi_t\) is hyperbolic if its Floquet multipliers do not have modulus 1. The (strong) stable and unstable manifolds of a point \(z \in SM\) are

\[ W^{s,u}(z) := \{w \in SM : \lim_{t \to \pm \infty} d(\phi_t(w), \phi_t(z)) = 0\}, \]

respectively. For a subset \(A \subset SM\) define

\[ W^{s,u}(A) = \bigcup_{a \in A} W^{s,u}(a). \]

For a hyperbolic closed geodesic \(\gamma\) the sets \(W^s(\dot{\gamma})\), \(W^u(\dot{\gamma})\) are immersed submanifolds of \(SM\) either diffeomorphic to a cylinder with one boundary \(\dot{\gamma}\) or to a Möbius band where \(\dot{\gamma}\) is its equator, according to wether the Floquet multipliers of \(\dot{\gamma}\) are positive or negative respectively. A homoclinic point of a hyperbolic closed geodesic \(\gamma\) is a point in \((W^s(\dot{\gamma}) \cap W^u(\dot{\gamma})) \setminus \dot{\gamma}\). A heteroclinic point is a point in \((W^s(\dot{\gamma}) \setminus \dot{\gamma}) \cap (W^u(\dot{\eta}) \setminus \dot{\eta})\), where \(\dot{\gamma}, \dot{\eta}\) are two hyperbolic closed orbits of \(\phi_t\).

Homoclinic points where first discovered by Henri Poincaré in 1889 (cf. Andersson [1]) and named in Poincaré [34, §395]. It is well known the paragraph of Poincaré [34, vol. III, §397], [1, §5] describing his admiration of the complexity of the dynamics implied by the existence of a transversal homoclinic point.

We say that the riemannian metric \(\rho\) is Kupka-Smale if

(i) The Floquet multipliers of every periodic orbit are not roots of unity.
(ii) The heteroclinic intersections of hyperbolic orbits \(W^s(\dot{\gamma}) \cap W^u(\dot{\eta})\) are transversal.

For any \(r \in \mathbb{N}, r \geq 2\), the set of \(C^r\) riemannian metrics whose geodesic flow is Kupka-Smale is residual in the set of \(C^r\) riemannian metrics in \(M\), see Contreras, Paternain [10, Thm. 2.5]. Clarke [5] proves that Kupka-Smale metrics are also residual in the \(C^\omega\) topology for analytic hypersurfaces of \(\mathbb{R}^n, n \geq 3\). Here we prove

**Theorem A.**

For a Kupka-Smale riemannian metric on a closed surface every hyperbolic closed geodesic \(\gamma\) has homoclinic orbits in all the components of \(W^s(\dot{\gamma}) \setminus \dot{\gamma}\) and of \(W^u(\dot{\gamma}) \setminus \dot{\gamma}\) and satisfy \(\overline{W}^s(\dot{\gamma}) = \overline{W}^u(\dot{\gamma})\).
The importance of finding homoclinic orbits is that in any neighborhood of the homoclinic orbit one finds a horseshoe with complicated dynamics. This dynamics can be coded using symbolic dynamics and implies positive (local) topological entropy, infinitely many periodic orbits shadowing the homoclinic, infinitely many homoclinics and exponential growth of periodic orbits in a neighborhood of the homoclinic. Homoclinics prevent integrability [28, §III.6], and can also be used to obtain Birkhoff sections [8]. They are also the basic skeleton for Mather acceleration theorems in Arnold diffusion [27], [4], [12], [18].

Also theorem A may help to prove that the closed orbits for the geodesic flow of surfaces are generically dense in the phase space. A conjecture by Poincaré [33, vol. I, p.82 §36] stated for the three body problem. By now it is only known that their projection to the surface is generically dense, Irie [22].

It is well known that $C^2$, $r \geq 2$, generic riemannian surfaces of genus $g \geq 2$ have homoclinic orbits, see e.g. [10]. Contreras and Paternain [10] proved that $C^2$ generic metrics on $S^2$ or $\mathbb{R}P^2$ have some orbits with homoclinic orbits. Knieper and Weiss [24] extended this result to the $C^\infty$ topology and Clarke [5] proved it for analytic convex surfaces in $\mathbb{R}^3$ and the $C^\omega$ topology. Xia and Zhang [35] prove that for a $C^\infty$ generic metric of positive curvature in $S^2$, every hyperbolic periodic orbit has homoclinics. Contreras [6] proves that $C^2$ generic metrics on any closed manifold have homoclinics.

The $C^\infty$, $C^\omega$ results [24], [5], [35] in the sphere use the annular Birkhoff section [3, §VI.10, p.180] for the spheres with positive curvature. Then they apply the techniques of Pixton [32] and Oliveira [29] for area preserving maps on surfaces of genus 0 to obtain the homoclinics. These techniques extend to genus 1 but not to higher genus. The problem with riemannian surfaces which are not spheres of positive curvature is that they have Birkhoff sections in the Kupka-Smale case [9], [7] but their genera is not known.

Instead we construct what we call a complete system of surfaces of section of genera $g \leq 1$. With this we complete a program initiated by Birkhoff in [2, §28, p.281] with formal justifications using the curve shortening flow [19], [16]. But now the Poincaré maps to these surfaces of section are not continuous. They are essentially discontinuous\(^1\). And the standard (continuity) arguments of Mather [26] and Oliveira [29] for area preserving homeomorphisms can not be applied. We show how to take advantage of the discontinuities of the Poincaré map to obtain homoclinic orbits for certain closed orbits. For the remaining hyperbolic orbits we develop in [30] and [31] the theories of Mather and Oliveira for partially defined area preserving homeomorphisms so that they can be applied to our situation.

\(^1\)There are arbitrarily small curves whose image under the Poincaré map have infinite length and large diameter.
We also remark that in [31] we show that the usual hypothesis of Moser stability for elliptic periodic points in Mather [26] is not needed. We use instead Theorem 1.2.(4) from [31] which allows to use only condition (i) from our Kupka-Smale definition. Nevertheless, as observed by Xia and Zhang [35], Fayad and Krikorian [14] prove that elliptic periodic points are Moser stable if their Floquet multipliers are diophantine, which is a generic condition for geodesic flows by the Bumpy Metric Theorem.

The Kupka-Smale condition has been chosen in order to have a unified approach using the results from [9], [30], [31]. But the transversality condition (ii) can be relaxed to asking (ii) only for periodic orbits of small period, in order to obtain a Birkhoff section [9]; and a no heteroclinic connections\(^2\) condition instead of the transversality (ii). Moreover, since the theorems that we use from [31] on homoclinic points are about fixed points; in order to get an homoclinic orbit for an orbit \(\dot{\gamma}\) we only need to ask for such generic conditions on periodic orbits of smaller period than \(\dot{\gamma}\). We shall not pursue such refinements here.

For an elliptic periodic orbit with Floquet multipliers \(\sigma\) satisfying \(\sigma^k \neq 1\) for \(1 \leq k \leq 4\), its Poincaré map on a local transversal section can be written in Birkhoff normal form as

\[
P(z) = z e^{i(\omega+\beta|z|^2)} + R(z),
\]

with \(\omega, \beta \in \mathbb{R}\) and \(R(z)\) with zero 4-jet at \(z = 0\). The condition \(\beta \neq 0\) is residual for 4-jets of \(P\). By theorem 2.5 in [10] this condition on all elliptic orbits is residual for \(C^r\) riemannian metrics with the \(C^r\) topology, \(r \geq 5\). The condition \(\beta \neq 0\) implies that the Poincaré map \(P\) is locally a twist map. Kupka-Smale twist maps have hyperbolic minimizing orbits with homoclinics for every rational rotation number in an interval \([\omega, \omega + \epsilon]\) or \([\omega - \epsilon, \omega]\), depending on the sign of \(\beta\), see [25], [36], [17]. These periodic orbits accumulate on the fixed point \(z = 0\). Therefore for \(C^r\), \(r \geq 5\), generic riemannian metrics on closed surfaces every closed geodesic is accumulated by homoclinic orbits, and the closure of the periodic orbits is the same as the closure of the homoclinic orbits.

The proof of theorem A needs results in dynamics of area preserving maps, Reeb flows and geodesic flows.

A contact 3-manifold is a pair \((N, \lambda)\) where \(N\) is a closed 3-manifold and \(\lambda\) is a 1-form in \(N\) such that \(\lambda \wedge d\lambda = 0\). The Reeb vector field \(X\) of \((N, \lambda)\) is defined by \(i_X d\lambda \equiv 0\) and \(\lambda(X) \equiv 1\). The Reeb flow \(\psi_t : N \to N\) of \((N, \lambda)\) is the flow of \(X\). The Liouville form of a riemannian surface \((M, \rho)\), given by

\[
\lambda(v)(\xi) := \rho(v, d\pi(\xi)), \quad v \in TM, \quad \xi \in T_v TM,
\]

is a contact form on \(SM\). The Reeb flow of \((SM, \lambda)\) is the geodesic flow of \((M, \rho)\).

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\(^2\) A heteroclinic connection is the case in which two components of \(W^s(\dot{\gamma}) \setminus \dot{\gamma}\) and \(W^u(\dot{\eta}) \setminus \dot{\eta}\) are equal.
A surface of section for the Reeb flow \( \psi_t \) is a compact immersed surface with boundary \( \Sigma \subset N \), whose interior is embedded and transversal to the Reeb vector field and whose boundary is a cover of a finite union of closed orbits of \( \psi_t \).

A Birkhoff section is a connected immersed surface \( \Sigma \subset N \) whose interior is embedded and transversal to the vector field. Its boundary is a cover of finitely many closed orbits and there is \( \ell > 0 \) such that for all \( z \in N \), \( \psi_{[0,\ell]}(z) \cap \Sigma \neq \emptyset \) and \( \psi_{[-\ell,0]}(z) \cap \Sigma \neq \emptyset \).

Contreras and Mazzucchelli proved in [9, Thm. A] that every Kupka-Smale Reeb flow on a closed contact 3-manifold \((N, \lambda)\) has a Birkhoff section. The first return map of the interior of a Birkhoff section is a diffeomorphism which preserves the area form \( d\lambda \).

In order to use the results in [30], [31] to obtain homoclinic orbits we need to have area preserving maps defined on surfaces of genus 0 or 1. In higher genus, the time one map of an area preserving flow without heteroclinic connections is an example of a Kupka-Smale map without homoclinics.

In general we don’t know the genus of the Birkhoff sections obtained in [9] or [7]. Instead we use a complete system of surfaces of section with genus 0 or 1, (definition 2.4). This is a finite collection of surfaces of section which intersect every orbit and such that the points which do not return to the collection of surfaces are in the stable or unstable manifold of a finite set of hyperbolic closed orbits \( K_{fix} \), called non rotating boundary orbits, which are some of the boundaries of the surfaces of section of the system. The other closed orbits in the boundaries of the sections are called rotating boundary orbits, their union is denoted \( K_{rot} \). They have the property that there is a neighborhood of \( K_{rot} \) where the return times to the system of sections is uniformly bounded.

If \( \gamma \) is a hyperbolic orbit of a Reeb flow in a 3-manifold we call separatrices the connected components of \( W^s(\gamma) \setminus \gamma \) and of \( W^u(\gamma) \setminus \gamma \). Since the contact manifold \((N, \lambda)\) is orientable, they separate any small tubular neighborhood \( U \) of \( \gamma \) into 2 or 4 connected components. The germ of these components obtained by shrinking \( U \) are called sectors of \( \gamma \). We say that a separatrix accumulates on a sector if it intersects such sector for any tubular neighborhood \( U \). A separatrix is adjacent to a sector if both the closure of the sector and the separatrix contain a component of a local invariant manifold \( W^s,u_{\varepsilon}(\gamma) \).

A Kupka-Smale contact manifold means that its Reeb flow satifies (i) and (ii).

**Theorem B.**

Let \((N, \lambda)\) be a Kupka-Smale closed contact 3-manifold.

(1) For any hyperbolic closed orbit \( \gamma \) of \((N, \lambda)\), all the connected components of \( W^s(\gamma) \setminus \gamma \) and \( W^u(\gamma) \setminus \gamma \) have the same closure equal to \( \overline{W^s(\gamma)} = \overline{W^u(\gamma)} \).

Moreover, each separatrix of \( \gamma \) accumulates on both of its adjacent sectors.
(2) If \((N, \lambda)\) has a Birkhoff section \(\Sigma\) of genus 0 or 1, then every hyperbolic orbit intersecting the interior of \(\Sigma\) has homoclinics in all its separatrices.

A hyperbolic boundary orbit in \(\partial \Sigma\) has homoclinics in all its separatrices provided that \(\Sigma\) has genus 0 or if \(\Sigma\) has genus 1 and the union of its local separatrices intersect \(\Sigma\) in at least 4 curves.

(3) Suppose that \((N, \lambda)\) admits a complete system of surfaces of section. Then:

Every non rotating boundary orbit in \(K_{fix}\) has homoclinics in all its separatrices.

If the system contains a component \(S\) of genus 0 or 1, then every periodic orbit which intersects the interior of \(S\) has homoclinics in all its separatrices.

A hyperbolic rotating boundary orbit in \(K_{rot} \cap \partial S\) has homoclinics in all its separatrices provided that \(S\) has genus 0 or if \(S\) has genus 1 and the union of its local separatrices intersect \(S\) in at least 4 curves.

See also proposition 2.10 which has no genus restriction.

Observe that the condition of four intersections is satisfied if the hyperbolic boundary orbit has positive Floquet multipliers. Because in that case the separatrices divide a tubular neighborhood of the orbit into four sectors and the trace of the Birkhoff section must turn around the four sectors.

Recall that Hofer, Wysocki and Zehnder prove in [21] corollary 1.8, that any non degenerate tight contact form on the 3-sphere \(S^3\) admits a finite energy foliation whose leaves have genus 0. The rigid surfaces of the finite energy foliation form a complete system of surfaces of section. We check in §3.8 that the transversality condition in item (iii) of definition 2.4 holds. Therefore we get

1.1. Corollary.

**Any Kupka-Smale tight contact form on \(S^3\) has homoclinic orbits in all branches of all of its hyperbolic closed orbits.**

Since a homoclinic orbit implies the existence of a horseshoe we also obtain

1.2. Corollary.

**If a Kupka-Smale tight contact form on \(S^3\) contains a hyperbolic periodic orbit then it has infinitely many periodic orbits.**

The geodesic flow is the Reeb flow of the Liouville form in the unit tangent bundle. By lifting the geodesic flow to a double covering if necessary, in order to obtain homoclinic orbits for geodesic flows it is enough to consider orientable surfaces. Theorem A follows from theorem B and the following theorem 1.3 once the conditions on the rotating boundary orbits in \(K_{rot}\) in item (3) of theorem B are checked.
1.3. Theorem (Contreras, Knieper, Mazzucchelli, Schulz [7, Thm. E]).

Let \((M, \rho)\) be a closed connected orientable surface all of whose simple contractible closed orbits without conjugate points are non degenerate. Then there is a complete system of surfaces of section for the geodesic flow of \((M, \rho)\) whose components have genus 0 or 1.

The ideas in theorem 1.3 date back to Birkhoff [2] section 28, together with the modern version of the curve shortening lemma by Grayson [19]. We also provide a proof theorem 1.3 in section §4, theorem 4.8 with a different construction. And we check the conditions on the rotating boundary orbits in item (3) of theorem B. In our case the system has two embedded surfaces of section of genus 1 and finitely many Birkhoff annuli of disjoint simple closed geodesics. This proves theorem A.

For area preserving maps the auto accumulation of invariant manifolds as in item (1) of theorem B usually requires the Kupka-Smale condition and also the condition that elliptic periodic orbits are Moser stable. This is a fundamental step to obtain homoclinics. Instead, using our results in [31], we only use the non-degeneracy condition (i) from our Kupka-Smale definition. In our application the first return map to the complete system of sections is not globally defined. Special care has been taken in [30], [31] to deal with this case.

In section 2.1 we prove theorem B using our results in area preserving maps from [30], [31]. In section 3 we show that the return map in a neighborhood of \(K_{rot}\) extends to the boundary and that the extension of hyperbolic rotating boundary orbits give rise to saddle periodic orbits for the return map. In section 4 we give a proof of theorem 1.3 adapted to our application.

2. Proof of Theorem B.

2.1. Auto-accumulation of invariant manifolds. Proof of item (1).

2.1. Theorem (Contreras, Mazzucchelli [9] Thm. A).

Any closed contact 3-manifold satisfying the Kupka-Smale condition admits a Birkhoff section for its Reeb flow.

2.2. Definition. Let \(S\) be a compact orientable surface with boundary. Suppose that \(f : S \to S\) is a orientation preserving homeomorphism. We say that a periodic point \(x \in \text{Fix}(f^n)\) is hyperbolic or of saddle type if there is an open neighborhood \(V\) and a local chart \(h : V \to W\) such that \(W = ]-1,1[^2\), if \(p \in \text{int}(S)\) or \(W = ]-1,1[ \times [0,1]\), if \(p \in \partial S\), \(h(p) = (0,0)\) and \(h \circ f \circ h^{-1} = g\), where \(g(x, y) = (\lambda x, \lambda^{-1} y)\) with \(\lambda \in \mathbb{R}, \lambda \notin \{-1, 0, 1\}\).

In such coordinates the set \(\{(x,y) \in V \mid x \neq 0 \text{ and } y \neq 0\}\) has two or four connected components that contain \(p\) in their closures. We call them sectors of \(p\). If \(\Sigma\) is one of these sectors and \(\Sigma'\) is a sector of \(p\) defined by means of another coordinate neighborhood \(V'\) of
Reeb flow \( \psi \) and no eigenvalue of \( df \) the closure of \( \Sigma \) in \( S \). We say that a branch \( L \) is area preserving homeomorphism of \( S \) which is positive on open sets and \( f \) are going to apply to \( \Sigma \) are defined by

\[
W^s(p, f) = \{ q \in S : \lim_{m \to +\infty} d(f^m(p), f^m(q)) = 0 \},
\]

\[
W^u(p, f) = \{ q \in S : \lim_{m \to -\infty} d(f^m(p), f^m(q)) = 0 \}.
\]

The branches of \( p \) are the connected components of \( W^s(p, f) \setminus \{p\} \) or of \( W^u(p, f) \setminus \{p\} \). A connection between two periodic points \( p, q \in S \) is a branch of \( p \) which is also a branch of \( q \), i.e. a whole branch which is contained in \( W^s(p, f) \cap W^u(q, f) \) or in \( W^u(p, f) \cap W^s(p, f) \).

We say that a branch \( L \) and a sector \( \Sigma \) are adjacent if a local branch of \( L \) is contained in the closure of \( \Sigma \) in \( S \). Two branches are adjacent if they are adjacent to a single sector.

A periodic point \( p \in \text{Fix}(f^n) \) is irrationally elliptic if \( f \) is \( C^1 \) in a neighborhood of \( p \) and no eigenvalue of \( df^n(p) \) is a root of unity.

Let \((N, \lambda)\) be a Kupka-Smale closed contact 3-manifold and \( \Sigma \) a Birkhoff section for its Reeb flow \( \psi \). The first return times \( \tau_{\pm} : \Sigma \to \mathbb{R}^+ \) and the first return maps \( f^{\pm} : \Sigma \to \Sigma \) to \( \Sigma \) are defined by

\[
\tau_{\pm}(x) := \pm \min\{ t > 0 : \psi_{\pm t}(x) \in \Sigma \}, \quad f^{\pm}(x) := \psi_{\tau_{\pm}(x)}(x).
\]

We have that \( f^{\pm} \) are smooth diffeomorphisms of \( \Sigma \) preserving the area form \( d\lambda \) on \( \Sigma \). We are going to apply to \( f \) the following Theorem:

2.3. **Theorem** (Oliveira, Contreras [31] corollary 4.9).

Let \( S \) be a compact connected orientable surface with boundary provided with a finite measure \( \mu \) which is positive on open sets and \( f : S \to S \) be an orientation preserving and area preserving homeomorphism of \( S \).

1. Suppose that \( L \) is a (periodic) branch of \( f \) and that all periodic points of \( f \) contained in \( \text{cl}_S L \) are of saddle type or irrationally elliptic. Then either \( L \) is a connection or \( L \) accumulates on both adjacent sectors. In the later alternative \( L \subset \omega(L) \).

2. Let \( p \in S - \partial S \) be a periodic point of \( f \) of saddle type and let \( L_1 \) and \( L_2 \) be adjacent branches of \( p \) that are not connections. If all the periodic points of \( f \) contained in \( \text{cl}_S(L_1 \cup L_2) \) are of saddle type or irrationally elliptic, then \( \text{cl}_S L_1 = \text{cl}_S L_2 \).

3. Suppose that \( p \in S - \partial S \) is a periodic point of \( f \) of saddle type. Assume that all the periodic points contained in \( \text{cl}_S(W^u_p \cup W^s_p) \) are of saddle type or irrationally elliptic and \( p \) has no connections. Then the branches of \( p \) have the same closure and each branch of \( p \) accumulates on all the sectors of \( p \).

   If in addition \( S \) has genus 0 or 1, then the four branches of \( p \) have homoclinic points.
(4) Let $C$ be a connected component of $\partial S$ and suppose that all the periodic points $p_1, \ldots, p_{2n}$ of $f$ in $C$ are of saddle type. Let $L_i$ be the branch of $p_i$ contained in $S-\partial S$. Assume that for every $i$ all the periodic points of $f$ contained in $\text{cl}_S L_i$ are of saddle type or irrationally elliptic and that $L_i$ is not a connection. Then for every pair $(i,j)$ the branch $L_i$ accumulates on all the sectors of $p_j$ and $\text{cl}_S L_i = \text{cl}_S L_j$.

If in addition $S$ has genus 0 then any pair $(L_i, L_j)$ of stable and unstable branches intersect. The same happens if the genus of $S$ is 1 provided that there are at least 4 periodic points in $C$.

Item (2) of theorem 2.3 for closed manifolds without boundary and under the further hypothesis that the elliptic periodic points are Moser stable appears in Mather [26] theorem 5.2. It also appears in Franks, Le Calvez [15] theorem 6.2 for $S = S^2$, the 2-sphere, when the elliptic points are Moser stable. The proof of items (3) and (4) on the existence of homoclinic orbits using item (2), appears in the proof of theorem 4.4 of [31] and can be read independently of the rest of the paper.

In section §3 we prove that if $S$ is a Birkhoff section for the Reeb flow $\psi_t$ of $(N, \lambda)$, then there is a continuous extension of the return map $f : S \to S$ to the boundary $\partial S$ which preserves its boundary components as in figure 4. If $\gamma$ is a boundary component of $\partial S$ which is an irrationally elliptic closed orbit then the restriction $f|_{\gamma}$ has no periodic points. If $\gamma$ is a hyperbolic closed orbit then the extension $f|_{\gamma}$ has periodic points which are the limits in $\gamma$ of the intersections $W^s(\gamma) \cap S$ and $W^u(\gamma) \cap S$. The extension $f|_{\gamma}$ corresponds to the action of the derivative of the flow $d\psi_t$ on the projective space of the contact structure $\xi$, transversal to the vector field. Therefore the limits of the intersections $W^s(\gamma) \cap S$ are sources in $f|_{\gamma}$ and the limits of the intersections $W^u(\gamma) \cap S$ are sinks in $f|_{\gamma}$. The other points in $\gamma$ are connections among these sources and sinks, i.e. stable manifolds of sinks which coincide with unstable manifolds of sources inside $\gamma$. These periodic points in $\gamma$ are saddles for $f$ in $\mathbb{S}$. The sinks in $\gamma$ have an unstable manifold in $S$ which is a connected component of $W^u(\gamma) \cap S$. Similarly, the sources for $f|_{\gamma}$ are saddles in $S$ with stable manifold a connected component of $W^s(\gamma) \cap S$. The Kupka-Smale condition for the Reeb flow $\psi_t$ implies that the branches in $S - \partial S$ of periodic points in $\partial S$ are not connections. In fact their intersections with other branches of periodic points of $f$ are transversal.

Therefore we can apply the first part of items (3) and (4) of theorem 2.3 to the return map $f$ of a Birkhoff section for the Reeb flow $\psi_t$. This implies item (1) of theorem B. For periodic points which intersect the interior of $S$ we use item (3) of theorem 2.3 and for hyperbolic periodic orbits in $\partial S$ we use item (4) of theorem 2.3.

### 2.2. Homoclinics for Birkhoff sections.

Proof of item (2).

We saw in the proof of item (1) of theorem B in subsection §2.1 that we can apply theorem 2.3 to the first return map of a Birkhoff section for the Kupka-Smale Reeb flow.
Figure 1. This figure shows in the left how a neighborhood $N$ of a boundary closed orbit $\gamma \in K_{fix}$ arrives to $\gamma$ in a local transversal section to the flow. The figure in the right shows a neighborhood $N$ of a rotating boundary orbit $\gamma \in K_{rot}$ and some of the iterates of $N$ under the Reeb flow.

In the case that the Reeb flow admits a Birkhoff section with genus zero or one, we can also apply the second part of items (3) and (4) of theorem 2.3. This gives homoclinic orbits in every separatrix of all the hyperbolic closed orbits for the Reeb flows $\psi_t$ selected in item (2) of theorem B.

2.3. Complete system of surfaces of section.

Let $(N, \lambda)$ be a compact contact $3$-manifold and $\psi_t$ its Reeb flow. For $Z \subset N$ define the forward trapped set $\text{trap}_+(Z)$ and the backward trapped set $\text{trap}_-(Z)$ as

$$\text{trap}_\pm(Z) = \{ z \in N : \exists \tau \forall t > \tau \phi_{\pm t}(z) \in Z \}. $$

2.4. Definition.

We say that $(\Sigma_1, \ldots, \Sigma_n)$ is a complete system of surfaces of section for $(N, \lambda)$ if

(i) Each $\Sigma_i$ is a connected surface of section for $(N, \lambda)$, i.e. $\Sigma_i \subset N$ is a connected immersed compact surface whose interior $\text{int}(\Sigma_i)$ is embedded and transversal to the Reeb vector field $X$ and its boundary is a cover of a finite collection of closed orbits of $\psi$.

(ii) Separate the boundary orbits in two sets$^3$ $\cup_i \partial \Sigma_i = K_{rot} \cup K_{fix}$. The non rotating periodic orbits in $K_{fix}$ are hyperbolic and have a neighborhood $N$ in $\Sigma_i$ which arrives to the boundary inside a sector as in figure 1. For the rotating boundary orbits$^4$ in $K_{rot}$ there is $\ell > 0$ such that each $\gamma \in K_{rot}$ has a neighborhood $N_\gamma$ in $\Sigma_i$ such that

$$\forall z \in N_\gamma \quad \psi|_{0,\ell}(z) \cap \Sigma_i \neq \emptyset \quad \& \quad \psi|_{-\ell,0}(z) \cap \Sigma_i \neq \emptyset.$$

(iii) At each$^5$ rotating boundary orbit $\gamma \in K_{rot} \cap \Sigma_i$ the extension of $\Sigma_i$ to the unit normal bundle $\mathcal{N}(\gamma)$ of $\gamma$ by blowing up a neighborhood of $\gamma$ using polar coordinates, is an

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$^3$This classification is the same as radial and broken binding orbits for broken book decompositions.

$^4$Rotating boundary orbits can be hyperbolic or elliptic.

$^5$This condition says that the flow rotates more than the surface of section when it approaches its boundary orbit $\gamma$. 

embedded collection of closed curves transversal to the extension of the Reeb vector field to $B_\gamma$.

(iv) Every orbit intersects $\Sigma = \cup_i \Sigma_i$.

(v) $\text{trap}_\pm (N \setminus \Sigma) \subset W^{s,u}(K_{\text{fix}})$.

Recall that a Birkhoff section is a connected embedded surface $\Sigma \subset N$ whose interior is transversal to the vector field. Its boundary is a cover of finitely many closed orbits and there is $\ell > 0$ such that for all $z \in N$, $\psi_{|0,\ell}(z) \cap \Sigma \neq \emptyset$ and $\psi_{[-\ell,0]}(z) \cap \Sigma \neq \emptyset$. We use the same notation $K_{\text{fix}}, K_{\text{rot}}, \partial K_i, \partial \Sigma$ for a collection of periodic orbits or their union. Here $\partial \Sigma := \cup_i \partial \Sigma_i$ and also $\text{int}(\Sigma) := \cup_i \text{int} \Sigma_i$.

2.5. Lemma.

Let $(\Sigma_i)_{i=1}^n$ be a complete system of surfaces of section for $(N, \lambda)$.

Let $\gamma \in K_{\text{fix}}$ and a connected component $L \subset W^{s,u}(\gamma) \setminus \gamma$, then

$$\exists \xi \in K_{\text{fix}} \quad L \cap W^{u,s}(\xi) \neq \emptyset.$$

Proof:

We prove it only for $L \subset W^u(\gamma)$, the other case is similar. Suppose by contradiction that

$$L \cap W^s(K_{\text{fix}}) = \emptyset.$$  

Let $z \in L$. Let $S$ be an essential smooth embedded circle in $L$. By (1) and (v), the first return time $\tau : S \to \mathbb{R}$

$$\tau(x) := \inf \{ t > 0 : \psi_t(x) \in \Sigma \}$$

is well defined and finite on $S$. The return map $f : S \to \text{int}(\Sigma)$, $f(x) = \psi_{\tau(x)}(x)$ is an immersion. Since $S \subset L$ is connected, compact and disjoint from periodic orbits, there is a component $\Sigma_{i_0}$ of $\Sigma$ such that $f : S \to f(S) \subset \Sigma_{i_0}$ is a diffeomorphism. By the intrinsic dynamics of $\psi_t$ on $L$, we have that $f(S)$ is an essential smooth embedded circle in $L$. Repeating this argument there is a component $\Sigma_{i_1}$ of $\Sigma$ and an infinite collection $\{S_k\}_{k \in \mathbb{N}}$ of disjoint essential smooth embedded circles in $L$ given by $S_k = f_k(S)$, where $f_k$ is the $k$-th return of $S$ to $\Sigma_{i_1}$. Observe that for $i < j$, the circles $S_i, S_j$ bound an embedded annulus $A_{i,j}$ in $L$.

The circles $S_k$ are disjoint and embedded in $\Sigma_{i_1}$. By Lemma 3.2 or Theorem 3.3 in [23], there is a free homotopy class in $\Sigma_{i_1}$ which contains infinitely many of them $\{S_{k_n}\}_{n \in \mathbb{N}}$.

If the circles $S_{k_n}$ are contractible in $\Sigma_{i_1}$, they bound disjoint disks $D_{k_n}$ with area

$$\int_{D_{k_n}} d\lambda = \int_{S_{k_n}} \lambda = \int_{A(k_n)} d\lambda + \int_{m \cdot \gamma} \lambda = m \cdot \text{period}(\gamma) > 0,$$
where \( A(k_n) \) is the annulus on \( L \) with boundaries \( \{S_{k_n}, \gamma\} \) and \( m \in \{1, 2\} \) whether \( \partial L \) covers \( \gamma \) \( m \)-times. We have used that \( \lambda(X) \equiv 1 \) and that \( d\lambda|_L \equiv 0 \) because the Reeb vector field is tangent to \( L \). This contradicts the fact that the area of \( \Sigma_{i_1} \) is finite, because

\[
\text{area}(\Sigma_{i_1}) = \int_{\Sigma_{i_1}} d\lambda = \sum_{\gamma \in \partial \Sigma_{i_1}} m_{\gamma} \int_{\gamma} \lambda < +\infty,
\]

where \( \partial \Sigma_{i_1} \) is finite and \( m_{\gamma} \) is the covering number of \( \partial \Sigma_{i_1} \) over \( \gamma \).

If the homotopy class of the \( S_{k_n} \) in \( \Sigma_{i_1} \) is non trivial then \( S_{k_1} \) and \( S_{k_2} \) bound an annulus \( B_{12} \) in \( \Sigma_{i_1} \). The annulus \( B_{12} \) has positive \( d\lambda \)-area because the transversality of \( \Sigma_{i_1} \) to the Reeb vector field implies that \( d\lambda \) is non-degenerate on \( \Sigma_{i_1} \). They also bound the annulus \( A_{k_1k_2} \) in \( L \) with zero \( d\lambda \)-area, because \( d\lambda|_L \equiv 0 \). Therefore

\[
0 = \int_{A_{k_1k_2}} d\lambda = \int_{S_{k_1}} \lambda - \int_{S_{k_2}} \lambda = \int_{B_{12}} d\lambda > 0.
\]

A contradiction.

\( \square \)

2.6. Remark. Using proposition 2.12 instead of proposition 2.1 in [8] it is possible to reproduce the proofs of lemma 5.2 and theorem B in [9] to obtain

\[
\forall \gamma \in K_{fix} \quad W^s(\gamma) = W^u(\gamma),
\]

whenever \((N, \lambda)\) is Kupka-Smale and has a complete system of surfaces of section. Then proposition 2.8 and section 3 in [9] give a Birkhoff section for \((N, \lambda)\) starting from a complete system instead of a broken book decomposition.

Here we will use theorem B.(1), proved in subsection §2.1, to get (2).

2.7. Lemma.

Let \( \alpha, \beta \) be hyperbolic periodic orbits of a Kupka-Smale Reeb flow of a closed contact 3-manifold \((N, \lambda)\). Suppose that \( \beta \) has homoclinic orbits. Let \( Q \) be a separatrix of \( \alpha \). Suppose that

\[
\text{cl}(W^s(\beta) \cup W^u(\beta)) \subset \overline{Q}.
\]

Then

\[
Q \subset W^u(\alpha) \implies Q \cap W^s(\beta) \neq \emptyset,
\]

\[
Q \subset W^s(\alpha) \implies Q \cap W^u(\beta) \neq \emptyset.
\]

Moreover, all the separatrices of \( \alpha \) have homoclinics.

Proof: Let \( D \) be a small disk transversal to the Reeb flow containing a point \( p \in \beta \cap D \) and such that

\[
D \cap \alpha = \emptyset \quad \text{and} \quad \int_D d\lambda < \int_\alpha \lambda.
\]
The Kupka-Smale condition implies that the homoclinic intersections in $W^s(\beta) \cap W^u(\beta)$ are transversal. By the $\lambda$-lemma there are segments of $W^u(\beta) \cap D$ (resp. $W^s(\beta) \cap D$) accumulating in the $C^1$ topology on the whole local component $W^u_\varepsilon(\beta) \cap D$ (resp. $W^s_\varepsilon(\beta) \cap D$). These segments form a grid in $D$ nearby $p$ which contains rectangles of arbitrarily small diameter. Choose small rectangles $A$, $B$ with boundaries in $W^s(\beta) \cup W^u(\beta)$ such that

$$\text{cl}(A) \subset \text{int}(B) \subset \text{cl}(B) \subset \text{int}(D).$$

Since $\text{cl}(W^s(\beta) \cup W^u(\beta)) \subset \overline{Q}$, we have that $Q \cap D$ accumulates on the boundary $\partial A$. Then there is a point $q \in Q \cap \text{int}(B)$. Let $J$ be the connected component of $Q \cap D$ containing $q$. Since by (6) $D \cap \alpha = \emptyset$, $J$ is either a circle or a curve with endpoints in $\partial D$. Suppose first that $J$ is a circle. Since $J$ is transversal to the flow inside $Q$ and there are no periodic orbits in $Q$, by Poincaré-Bendixon theorem, $J$ must be an essential embedded circle in $Q$. Let $A \subset D$ be a disk with $\partial A = J$. Since $Q$ is tangent to the Reeb vector field, $d\lambda|_Q \equiv 0$. By Stokes theorem

$$\int_A d\lambda = \int_J \lambda = k \cdot \int_\alpha \lambda, \quad k \in \{1, 2\},$$

with $k = 2$ if $\alpha$ is negative hyperbolic. But (7) contradicts (6) because $A \subset D$. Therefore $J$ is a curve with endpoints in $\partial D$. Since $q \in J \cap \text{int}(B) \neq \emptyset$ and $B \cap \partial D = \emptyset$, we have that $\exists r \in J \cap \partial B \neq \emptyset$. Then $r \in Q \cap W^u(\beta)$ if $Q \subset W^s(\alpha)$. This proves (4) and (5).

Suppose now that $Q \subset W^u(\alpha)$, the case $Q \subset W^s(\alpha)$ is similar. By (4)

$$Q \cap W^s(\beta) \neq \emptyset.$$  (8)

But by B.(1), $\overline{W^s(\alpha)} = \overline{W^u(\alpha)} = \overline{Q}$. Hence $\text{cl}(W^s(\beta) \cup W^u(\beta)) \subset \overline{Q} = \overline{W^s(\alpha)}$. By (5) applied to a separatrix in $W^s(\alpha)$ we have that

$$W^u(\beta) \cap W^s(\alpha) \neq \emptyset.$$  (9)

Since by the Kupka-Smale condition the heteroclinic intersections are transversal, equations (8), (9) and the $\lambda$-lemma imply that $Q \cap W^s(\alpha) \neq \emptyset$. Thus the separatrix $Q$ has homoclinics. Now observe that by B.(1) the condition (3) is satisfied by all the separatrices of $\alpha$.

\begin{flushright} $\Box$ \end{flushright}

2.8. Proposition.

Let $(N, \lambda)$ be a closed contact 3-manifold satisfying the Kupka-Smale condition. Let $(\Sigma_1, \ldots, \Sigma_m)$ be a complete system of surfaces of section with boundary components $K = K_{rot} \cup K_{fix}$. Then every component of $W^\pm_{\Sigma}(\gamma) \setminus \gamma$ of every non rotating boundary orbit $\gamma \in K_{fix}$ has homoclinics and $\overline{W^s(\gamma)} = \overline{W^u(\gamma)}$. 

Proof:

Write \((\kappa_1, \ldots, \kappa_n) \in \Gamma\) if \(\forall i \kappa_i \in K_{\text{fix}}\) and \(W^u(\kappa_i) \cap W^s(\kappa_{i+1}) \neq \emptyset\) for \(1 \leq i < n\). The definition of \(\Gamma\) implies that

\[(\kappa_1, \kappa_2) \in \Gamma, (\kappa_2, \kappa_3) \in \Gamma \implies (\kappa_1, \kappa_2, \kappa_3) \in \Gamma.\]  

The \(\lambda\)-lemma implies that

\[(\kappa_1, \ldots, \kappa_n) \in \Gamma \implies (\kappa_1, \kappa_n) \in \Gamma.\]  

By lemma 2.5

\[(\forall \beta \in K_{\text{fix}} \exists \alpha, \gamma \in K_{\text{fix}} \{ (\alpha, \beta), (\beta, \gamma) \} \subset \Gamma).\]  

By (12) for any \(\alpha \in K_{\text{fix}}\) there is an infinite sequence \((\alpha, \kappa_1, \kappa_2, \ldots) \in \Gamma\). Since \(K_{\text{fix}}\) is finite, there are \(n \neq m\) such that \(k_n = k_m =: \beta\). By properties (10) and (11), \((\alpha, \beta, \beta) \in \Gamma\).

Thus

\[(\forall \alpha \in K_{\text{fix}} \exists \beta \in K_{\text{fix}} (\alpha, \beta, \beta) \in \Gamma).\]

Let \(\alpha \in K_{\text{fix}}\) and let \(Q\) be a component of \(W^{s,u}(\alpha) \setminus \alpha\). By theorem B.(1), proved in §2.1, \(Q = W^s(\alpha) = W^u(\alpha)\).

Let \(\beta \in K_{\text{fix}}\) be given by (13). Since the intersection \(W^u(\alpha) \cap W^s(\beta)\) is transversal, by the \(\lambda\)-lemma \(W^u(\beta) \subset W^u(\alpha)\). Then by theorem B.(1),

\[cl(W^s(\beta) \cup W^u(\beta)) = W^u(\beta) \subset W^u(\alpha) = Q.\]

Then lemma 2.7 implies that \(Q\) has homoclinics.

\[\square\]

2.4. Homoclinics for complete systems. Proof of item B.(3).

We shall use the following

2.9. Theorem (The accumulation lemma).

Let \(S\) be a connected surface with compact boundary provided with a Borel measure \(\mu\) such that open non-empty subsets have positive measure and compact subsets have finite measure. Let \(S_0 \subset S\) be an open subset with \(\text{fr}_S S_0\) compact.

Let \(f, f^{-1} : S_0 \to S\) be an area preserving homeomorphism of \(S_0\) onto open subsets \(f(S_0), f^{-1}(S_0)\) of \(S\). Let \(K \subset S_0\) be a compact connected invariant subset of \(S_0\).

If \(L \subset S_0\) is a branch of \(f\) and \(L \cap K \neq \emptyset\), then \(L \subset K\).

This version of theorem 2.9 is proved in [30, Thm. 4.3], its proof also applies to branches \(L\) of saddle points in the boundary \(\partial S_0\). Theorem 2.9 was originally proved in Mather [26, corollary 8.3] for surfaces without boundary and global maps \((S_0 = S)\). It is also proved
in Franks, Le Calvez [15, lemma 6.1] for \( S = S_0 = S^2 \), the 2-sphere. This version is needed to prove theorem 2.13 in [31]. In proposition 2.10 we only use its global version \( S_0 = S \), but in corollary 2.15 we use this version for partially defined maps.

2.10. Proposition.

Let \((N, \lambda)\) be a closed contact 3-manifold satisfying the Kupka-Smale condition with a given complete system of surfaces of section. Let \( K_{fix} \) be the set of non rotating boundary orbits let

\[ \mathcal{W} = (W^s(K_{fix}) \cup W^u(K_{fix})) \setminus K_{fix}. \]

Let \( \gamma \) be a hyperbolic closed orbit of the Reeb flow of \((N, \lambda)\). Let \( Q \) be a separatrix of \( \gamma \). If \( Q \cap \mathcal{W} \neq \emptyset \), then all the separatrices of \( \gamma \) have homoclinics.

Proof:

Assume that \( Q \subset W^u(\gamma) \), the case \( Q \subset W^s(\gamma) \) is similar.

By theorem 2.1 there is a Birkhoff section \( \mathcal{B} \) for the Kupka-Smale Reeb flow of \((N, \lambda)\).

Let \( f \) be the first return map of the Reeb flow to \( \mathcal{B} \).

In section 3 we show that in case \( \gamma \subset \partial \mathcal{B} \) is a hyperbolic periodic orbit, then the connected components of \( Q \cap \mathcal{B} \) are interior branches of saddle points in \( \gamma \) for the return map \( f \) to \( \mathcal{B} \). Choose a connected component \( L_p \) of \( Q \cap \mathcal{B} \). Then \( L_p \) is a branch of a periodic point \( p \in \mathcal{B} \) of \( f \), possibly at the boundary \( p \in \partial \mathcal{B} \) if \( \gamma \subset \partial \mathcal{B} \). Let \( n \) be the minimal period of \( L_p \), \( f^n(L_p) = L_p \). In particular \( f^n(p) = p \). Then \( L_p \) is a branch of \( W^u(p, f^n) \). And \( K = L_p \) is a compact \( f^n \)-invariant subset of \( \mathcal{B} \).

Let \( \kappa \in K_{fix} \) be a non rotating boundary orbit\(^6\) and let \( q \in \kappa \cap \mathcal{B} \) be a (saddle) periodic point for \( f \). Choose a multiple \( m \) of \( n \) such that \( f^m(q) = q \), then \( \{p,q\} \subset \text{Fix}(f^m) \) and \( f^m(K) = K \). We will apply the accumulation lemma 2.9 to \( f^m \) and the compact \( f^m \)-invariant set \( K \). Let \( L_q \subset \text{int}(\mathcal{B}) \) be an interior branch of \( q \), i.e. a connected component of \( W^\tau(q, f^m) \setminus \{q\}, \tau \in \{s, u\} \), which is also a connected component of \( W^\tau(\kappa) \cap \mathcal{B} \).

By the accumulation lemma 2.9,

\[ L_q \cap K \neq \emptyset \quad \Rightarrow \quad L_q \subset K. \]

This implies in the Reeb flow that

\[ (W^\tau(\kappa) \setminus \kappa) \cap \overline{\mathcal{Q}} \neq \emptyset \quad \Rightarrow \quad \text{(a component of } W^\tau(\kappa) \setminus \kappa) \subset \overline{\mathcal{Q}}. \]

By item (1) of theorem B we have that \( \overline{W^s(\kappa)} = \overline{W^u(\kappa)} = \overline{W^\tau(\kappa)} \) is also the closure of any component of \( W^\tau(\kappa) \setminus \kappa \), therefore we get

\[ \exists \tau \in \{s, u\} \quad \overline{\mathcal{Q}} \cap (W^\tau(\kappa) \setminus \kappa) \neq \emptyset \quad \Rightarrow \quad \text{cl}(W^s(\kappa) \cup W^u(\kappa)) \subset \overline{\mathcal{Q}}. \]

\(^6\)In lemma 3.3 in [9] an argument of Fried is used to show that, since by proposition 2.8 an orbit \( \kappa \in K_{fix} \) has homoclinics in all its branches, one can obtain a Birkhoff section \( \mathcal{B} \) which intersects \( \kappa \) in its interior.
Suppose that $\overline{Q} \cap \mathbb{W} \neq \emptyset$. Then there is $\kappa \in K_{fix}$ and $\tau \in \{s, u\}$ such that $\overline{Q} \cap (W^\tau(\kappa) \setminus \kappa) \neq \emptyset$. By (14),

$$cl(W^s(\kappa) \cup W^u(\kappa)) \subset \overline{Q}.$$ 

By proposition 2.8 we have that $\kappa$ has transversal homoclinics. Then by lemma 2.7 all the separatrices of $\gamma$ have homoclinics.

Let $S$ be a component of a complete system of surfaces of section for a closed contact 3-manifold $(N, \lambda)$. Define the first return times $\tau_{\pm}$ to $S$ and the first return maps $f^{\pm 1}$ as

$$\tau_{\pm} : \text{int}(S) \rightarrow [0, +\infty] \cup \{+\infty\} \quad \text{and} \quad \tau_{\pm} : \text{int}(S) \rightarrow (-\infty, 0] \cup \{-\infty\} \quad \text{by}$$

$$(15) \quad \tau_{\pm}(z) := \pm \inf\{t > 0 : \phi_{\pm t}(x) \in S\}. $$

$$(16) \quad f(x) := \phi_{\tau_+(x)}(x), \quad f^{\pm 1}(x) := \phi_{\tau_-(x)}(x).$$

By the implicit function theorem $f$ and $f^{\pm 1}$ are defined in the open subsets $[\tau_+ < +\infty]$ and $[\tau_- > -\infty]$ of $\text{int}(S)$ respectively.

In section 4 we show that $f$ and $f^{\pm 1}$ extend to a neighborhood of $K_{rot} \cap \partial S$ as in figure 4. All the periodic points for $f^{\pm 1}$ in any $\gamma \in K_{rot} \cap \partial S$ are of saddle type, their invariant manifolds for $f^{\pm 1}$ are either the intersections $W^{s,u}(\gamma) \cap S$ or heteroclinic connections in $\gamma$. Irrationally elliptic orbits in $\partial S$ are in $K_{rot} \cap \partial S$, but they have no periodic orbits for $f^{\pm 1}$. And $f^{\pm 1}$ are not defined\footnote{In fact the natural extension of $f$ to a point $x \in K_{fix}$ would be the whole circle of a first intersection of a component of $W^u(\gamma) \setminus \gamma$, $\gamma = \psi_R(x)$, with $S$. See figure 1.} on $K_{fix} \cap \partial S$. Let

$$(17) \quad S_0 = ([\tau_+ < +\infty] \cap [\tau_- > -\infty]) \cup (K_{rot} \cap \partial S).$$

By condition 2.4.(ii) the maps $\tau_{\pm}$ are finite in a neighborhood in $\text{int}(S)$ of $K_{rot} \cap \partial S$. Then $S_0$ is an open submanifold of $S$ with compact boundary $\partial S_0 \subset \partial S$ and $f^{\pm 1} : S_0 \rightarrow S$ are differentiable, area preserving and $f(\partial S_0) \subset \partial S_0$.

Observe that condition 2.4.(ii) implies that $K_{fix} \subset \cap_{n \in \mathbb{N}} \{ |\tau_{\pm}| > n \}$. In section 3 we see that the functions $\tau_{\pm}$ can be extended to $K_{rot}$. So we use the notation

$$(18) \quad K_{fix} \subset [\tau_{\pm} = \pm \infty], \quad K_{rot} \subset [|\tau_{\pm}| < \infty],$$

$$(19) \quad S_0 = [\tau_+ < +\infty] \cap [\tau_- > -\infty].$$
2.11. Lemma.

(20) \( \forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \, x \in \text{int}(S) \, \& \, N \leq |\tau_+(x)| < \infty \implies d(x, |\tau_+| = \infty) < \varepsilon. \)

Proof:

We only prove it for \( \tau_+ \). For \( n \in \mathbb{N} \cup \{+\infty\} \) let \( A_n := [\tau_+ < n] \). Then \( A_n \) is an increasing family of open sets in the closure \( \text{cl}(S) \) with \( A_\infty = \bigcup_{n \in \mathbb{N}} A_n \). For \( \delta > 0 \) let \( B(\partial A_\infty, \delta) := \{ x \in S : d(x, \partial A_\infty) < \delta \} \). Observe that \( \partial A_\infty = [\tau_+ = \infty] \) in \( \text{cl}(S) \). Indeed, by Poincaré recurrence theorem, \( A_\infty \) has total measure in \( S \), then \( K_{\text{fix}} \subset \partial A_\infty \). It is enough to prove that

\[
(21) \quad \forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \, A_\infty \subset A_N \cup B(\partial A_\infty, \varepsilon).
\]

Let \( K_\varepsilon := A_\infty \setminus B(\partial A_\infty, \varepsilon) \). Then \( K_\varepsilon \) is compact and \( \{A_n\}_{n \in \mathbb{N}} \) is an open cover of \( K_\varepsilon \). Since the family \( \{A_n\} \) is increasing, there is \( N \in \mathbb{N} \) such that \( K_\varepsilon \subset A_N \). Then \( A_\infty \subset K_\varepsilon \cup B(\partial A_\infty, \varepsilon) \subset A_N \cup B(\partial A_\infty, \varepsilon) \).

\[ \square \]

2.12. Proposition (M. Mazzucchelli).

Let \( N \) be a compact 3-manifold with a flow \( \psi_i \). Let \( \Sigma \) be a finite union of connected surfaces of section and \( K \) a finite collection of hyperbolic periodic orbits in \( \partial \Sigma \). Suppose that

(a) Every orbit of \( \psi \) intersects \( \Sigma \).
(b) \( z \in N \, \& \, \psi_{[0, +\infty[}(z) \cap \Sigma = \emptyset \implies z \in W^s(K) \).

Let \( \Sigma_1 \) be a connected component of \( \text{int}(\Sigma) \). Let \( \tau : \Sigma_1 \to [0, +\infty[ \cup \{+\infty\} \) be the first return time to the component \( \Sigma_1 \), i.e.

\[
\tau(z) := \inf\{ t > 0 \mid \psi_t(z) \in \Sigma_1 \}.
\]

Let \( \alpha : [0, 1[ \to \Sigma_1 \) be continuous and suppose that

(i) \( \forall s \in [0, 1[ \quad \tau(\alpha(s)) < +\infty \).
(ii) \( \exists \{s_n\}_{n \in \mathbb{N}} \subset [0, 1[ \lim \alpha(s_n) = w \in \text{int}(\Sigma_1), \, \, \tau(w) = +\infty \).

Then \( w \in W^s(K) \).

Proof: Let

\[
k(s) = \#\{ t \in [0, \tau(\alpha(s))] : \psi_t(\alpha(s)) \in \Sigma \}.
\]

By the implicit function theorem \( k(s) = k_0 \) is constant in \( s \in [0, 1[ \).

Suppose by contradiction that \( w \notin W^s(K) \). Then \( \psi_t(w) \notin W^s(K) \) for all \( t > 0 \). Hypothesis (b) then implies that \( \psi_{[0, +\infty[}(w) \cap \Sigma \) is infinite. Let \( T > 0 \) be such that \( \psi_{[0,T[}(w) \) intersects \( k_0 + 1 \) times the surface \( \Sigma \). By the implicit function theorem there is
a neighborhood $U$ of $w$ in $\text{int}(\Sigma_1)$ such that for each $x \in U$, the curve $\psi_{[0,T]}(x)$ intersects $(k_0 + 1)$-times $\Sigma$. Therefore $\tau(\alpha(s)) \leq T$ whenever $\alpha(s) \in U$.

Thus $\tau(\alpha(s_n)) \leq T$ for $n$ large enough. There is a subsequence $s_{n_k}$ such that $0 < \tau_1 := \lim_k \tau(\alpha(s_{n_k})) \leq T$ exists. Then

$$\psi_{\tau_1}(w) = \lim_k \psi_{\tau(\alpha(s_{n_k}))}(\alpha(s_{n_k})) \in \Sigma_1.$$ 

Therefore $\tau(w) \leq \tau_1 < +\infty$. A contradiction.

The previous results will allow us to obtain homoclinics for branches of periodic points whose closure is not included in $S_0$. For the remaining case we will use theorem 2.13.

We remark that the proof of existence of homoclinic orbits in [31], once the auto accumulation of invariant manifolds is known, only uses the dynamics of the map in a neighborhood of the invariant manifolds. We will use the following

2.13. Theorem (Oliveira, Contreras [31], corollary 4.10).

Let $S$ be a compact connected orientable surface with boundary. Let $S_0 \subset S$ be a submanifold with compact boundary $\partial S_0 \subset \partial S$ and let $f,f^{-1} : S_0 \to S$ be an orientation preserving and area preserving homeomorphism of $S_0$ onto open subsets $f S_0$, $f^{-1} S_0$ of $S$ with $f(\partial S_0) \subset \partial S_0$.

(1) Let $p \in S_0 - \partial S$ be a periodic point of $f$ of saddle type. Assume that the branches of $p$ have closure included in $S_0$. Assume also that each branch of $p$ accumulates on both of its adjacent sectors and that all the branches of $p$ have the same closure in $S$. If in addition $S$ has genus 0 or 1, then the four branches of $p$ have homoclinic points.

(2) Let $C$ be a connected component of $\partial S_0$ and suppose that all the periodic points $p_1, \ldots, p_{2n}$ of $f$ in $C$ are of saddle type. Let $L_i$ be the branch of $p_i$ contained in $S - \partial S$. Assume that for every $i$, $L_i$ is not a connection and $\text{cl}_S L_i = \text{cl}_S L_j \subset S_0$ for every pair $(i, j)$.

If in addition $S$ has genus 0, then every pair $L_i, L_j$ of stable and unstable branches intersect. The same happens if the genus of $S$ is 1 provided that there are at least 4 periodic points in $C$.

Theorem 2.13 is the version for periodic points of theorem 4.4 in [31]. The proof of theorem 4.4 in [31] can be read independently of the rest of the paper.

Proof of item (3) of theorem B:

By proposition 2.8 every orbit $\gamma \in K_{fix}$ has homoclinics in all its separatrices.
When \( \lim_n \alpha(s_n) \in \eta \subset K_{fix} \), the forward orbits of the \( \alpha(s_n) \) approach \( W^s(\eta) \setminus \eta \).

By proposition 2.10 the same happens for a periodic orbit \( \gamma \) if \( W^s(\gamma) \cup W^u(\gamma) \) intersects

\[
\mathbb{W} := W^s(K_{fix}) \cup W^u(K_{fix}) \setminus K_{fix}.
\]

So assume that \( \gamma \) is a hyperbolic periodic orbit with \( \gamma \cap \overline{S} \neq \emptyset \) and

\[
(\overline{W^s(\gamma)} \cup \overline{W^u(\gamma)}) \cap \mathbb{W} = \emptyset.
\]

Let \( S \) be a component of the complete system. Let \( \tau_{\pm} : S \to \mathbb{R} \cup \{-\infty, +\infty\} \) be the first return times to \( S \), defined in (15), (18), let \( S_0 = [\tau_+ < +\infty] \cap [\tau_- > -\infty] \) be as in (17), (19), and let \( f, f^{-1} : S_0 \to S \) be the extensions of the first return maps as in (16) and §3.

Since \( \gamma \) is a periodic orbit, \( |\tau_{\pm}| \) are finite on \( \gamma \cap S \), bounded by the period of \( \gamma \). Thus \( \gamma \cap S \subset S_0 \). Let \( p \in S_0 = \text{int}(S_0) \cup \partial S_0, p \in \gamma \cap \overline{S} \), be a saddle point for \( f \) and let \( L \subset \text{int}(S) \) be an interior branch of \( p \). Let \( Q \) be the separatrix of \( \gamma \) which contains \( L \).

Suppose that \( \tau_+ \) is unbounded on \( L \). Let \( L_1 \subset L \) be the connected component of \( L \cap [\tau_+ < \infty] \) with \( p \in L_1 \). Then \( \tau_+ \) is unbounded on \( L_1 \). Let \( \alpha : [0,1] \to L_1 \) be a parametrization of \( L_1 \). Then there is a sequence \( s_n \in [0,1] \) with \( \lim_n \tau_+(\alpha(s_n)) = +\infty \). Extracting a subsequence we can assume that \( z_0 = \lim_n \alpha(s_n) \) exists. Since \( [\tau_+ = \infty] \) is compact, lemma 2.11 implies that \( \tau_+(z_0) = \infty \). But condition 2.4.(ii) implies that \( \tau_+ \) is bounded in a neighborhood of \( K_{rot} \cap \partial S \). Thus \( z_0 \in \text{int}(S) \cup K_{fix} \). If \( z_0 \in \text{int}(S) \) then proposition 2.12 implies that \( z_0 \in W^s(K_{fix}) \). Therefore \( z_0 \in L \cap (W^s(K_{fix}) \setminus K_{fix}) \neq \emptyset \). This contradicts (22).

Then \( z_0 = \lim_n \alpha(s_n) \in K_{fix} \cap \partial S \). Let \( \eta = \psi_R(z_0) \in K_{fix} \). The surface \( S \) approaches the non rotating boundary orbit \( \eta \subset \partial S \cap K_{fix} \) through a quadrant of \( \eta \) as in figure 2. There are \( t_n \geq 0 \) such that \( \lim_n (\psi_{t_n}(\alpha(s_n))) \subset W^u(\eta) \setminus \eta \). Since \( W^u(\eta) \setminus \eta \) does not contain periodic orbits, this limit is in \( W^u(\eta) \setminus K_{fix} \). Since \( L_1 \subset Q \) and \( Q \) is invariant, \( \psi_{t_n}(\alpha(s_n)) \in Q \). Therefore \( \overline{Q} \cap (W^u(\eta) \setminus K_{fix}) \neq \emptyset \). This contradicts (22).
This proves that $\tau_+$ is bounded on $L$. A similar\textsuperscript{8} proof shows that $\tau_-$ is bounded on $L$.

Now assume that $\tau_\pm$ are bounded on $L$ and $\text{genus}(S) \leq 1$. Then there is $N > 0$ such that

$$T \subset [\tau_+ \leq N] \cap [\tau_- \leq N] \subset S_0,$$

as required in theorem 2.13. In order to apply theorem 2.13 we need to show that

(a) If $p \in \text{int}(S)$ then each branch of $p$ accumulates on both of its adjacent sectors and all branches of $p$ in $\text{int}(S)$ have the same closure.

(b) If $p \in \partial S$ (and hence $\gamma = \psi_{\mathbb{R}}(p) \subset K_{rot}\cap\partial S$), then all the components of $(W^s(\gamma) \setminus \gamma) \cap S$ and of $(W^u(\gamma) \setminus \gamma) \cap S$ have the same closure.

Then corollary 2.15 finishes the proof of item 3 of theorem B.

\textbf{2.14. Lemma.}

Let $(N, \lambda)$ be a Kupka-Smale closed contact 3-manifold. Let $S$ be a component of a complete systems of surfaces of section for $(N, \lambda)$. Let $S_0$ be as in (19) and let $f : S_0 \cup \partial S_0 \to S$ be the extension of the first return map to $S$ made in section 3. Let $\gamma$ be a hyperbolic closed orbit for $(N, \lambda)$, $\gamma \notin K_{fix}$, such that

$$S \cap (W^s(\gamma) \cup W^u(\gamma)) \subset S_0.$$ \hfill (23)

Let $p \in \gamma \cap (S \cup \partial S)$ be a periodic point for $f$ and let $L_p \subset S \setminus \partial S$ be an interior branch of $p$.

Then $L_p$ accumulates on both of its adjacent sectors.

\textbf{Proof:}

Let $\tau_+$ and $f$ be from (15) and (16). The branches in $S \setminus \partial S$ of $p$ for $f$ are the connected components of the intersection of the separatrices of $\gamma$ with $S$ that contain $p$ as an endpoint. Let $Q$ be the separatrix of $\gamma$ containing the branch $L_p$. By item (1) of theorem B we know that $Q$ accumulates on both of its adjacent sectors in $(N, \lambda)$.

By hypothesis (23), all the branches of the $f$-orbit of $p$ in $S$ have closure included in $S_0$. The map $f : S_0 \to S$ is well defined in $S_0$ and is an injective immersion. In particular $f$ is continuous in a neighborhood in $S$ of the branches of the orbit of $p$. And every connected component of $S \cap (W^s(\gamma) \cup W^u(\gamma))$ has an endpoint in an element of the $f$-orbit of $p$.

Let $A$ be a sector for $(f, p)$ in $S$ adjacent to $L_p$. Let $n$ be the minimal period of $p$, $f^n(p) = p$. There are at most $2n$ connected components of $Q \cap S$ and they are branches of the iterates $f^i(p)$. At least one of these components accumulates on the sector $A$.

\textsuperscript{8}For the boundedness of $\tau_-$ we apply proposition 2.12 to the inverse flow $\psi_{-t}$.
Suppose first that $\gamma$ is a positive hyperbolic orbit. Then for every $i \in \mathbb{Z}$, $f^i(L_p)$ is the unique connected component of $Q \cap S$ with endpoint $f^i(p)$. Let $L_k$ be a connected component of $Q \cap S$ which accumulates on the sector $A$. Then $L_k$ is a branch of an $f$-periodic point $p_k \in \gamma \cap S$. There is $0 \leq k < n$ such that $f^k(p) = p_k$. If $k = 0$ then the lemma holds. Assume $k \geq 1$. Since $L_k$ accumulates on the sector $A$ adjacent to $L_p$, we have that $L_p \cap \overline{L_k} \neq \emptyset$. The compact set $\overline{L_k}$ is invariant under $f^n$ and $\overline{L_k} \subset S_0$. By the accumulation lemma 2.9 applied to $f^n$, $L_p \subset \overline{L_k}$.

Observe that $f^k(L_k) = L_{2k}$ is a connected component of $Q \cap S$ with endpoint $p_{2k} = f^k(p_k)$ and it accumulates on the sector $f^k(A)$ of $p_k$. Similarly $L_k \subset \overline{L_{2k}}$. And then $L_p \subset \overline{L_k} \subset \overline{L_{2k}}$. Inductively $L_{nk} = f^{nk}(L_p)$ is a component of $Q \cap S$ with endpoint $p_{nk} = f^{nk}(p) = p$. Thus $L_{nk} = L_p$. Moreover $\overline{L_k} \subset \overline{L_{2k}} \subset \cdots \subset \overline{L_{nk}} = \overline{L_p}$. Therefore $L_p$ accumulates on the sector $A$.

Suppose now that $p \in \gamma \subset \partial S$ is a boundary $f$-periodic point. The return map $f$ preserves the area form of $S$ and hence it preserves orientation. This implies that $p$ is a positive hyperbolic orbit for $f$. The orbit $\gamma$ for the flow may be negative hyperbolic but the return map $f$ permutes the interior components of $Q \cap S$. The previous proof of the positive hyperbolic case applies here.

Now suppose that $p \in S \setminus \partial S$ is a negative hyperbolic periodic point for $f$. Let $n > 0$ be its minimal period, $f^n(p) = p$. Let $J_k$ be a connected component of $Q \cap S$ which accumulates on the sector $A$. Let $p_k$ be the endpoint of $J_k$ and let $k \in \mathbb{N}$ be such that $f^k(p) = p_k$. In the case $L_k := f^k(L_p) = J_k$ the same proof as in the positive hyperbolic case follows, with $L_j := f^j(L_p)$ and $\overline{L_k} \subset \overline{L_{2k}} \subset \cdots \subset \overline{L_{nk}} = \overline{L_p}$. The accumulation lemma 2.9 is applied to $f^{2n}$ which leaves the branches $L_{nk}$ invariant. We iterate $2n$ times $f^k$, because the map $f^{2nk}$ fixes the branch $L_p$.

Suppose then that $f^k(L_p) \neq J_k$. For $j \in \mathbb{Z}$, let $L_j := f^j(L_p)$ and $A_j := f^j(A)$. Let $K_j$ be the other component of $Q \cap S$ with endpoint $f^j(p)$. In local coordinates $K_j$ is the branch $-L_j$ adjacent to the sector $-A_j$. Then $J_k = K_k$ and $K_j = f^j(K_0)$. The branch $K_k$ accumulates on the sector $A$. Then the branch $L_k = f^n(K_k)$ accumulates on the sector $f^n(A) = -A$, adjacent to the branch $K_0$. By the accumulation lemma 2.9 applied to $f^{2n}$, for which the branches are invariant, $L_p \subset \overline{K_p}$ and $K_p := K_0 \subset \overline{L_k}$.

Since the branch $K_k$ accumulates on the sector $A$, we have that the branch $K_{2k} = f^k(K_k)$ accumulates on the sector $A_k = f^k(A)$, adjacent to $K_k$. And using $f^n$, the branch $L_{2k} := f^{2k}(L_p) = f^n(K_{2k})$ accumulates on the sector $-A_k = f^n(A_k)$, adjacent to $K_k$. Using the accumulation lemma 2.9 we get that

$$L_k \subset \overline{K_{2k}} \quad \text{and} \quad K_k \subset \overline{L_{2k}}.$$
Applying $f^j$ to the inclusions in (24) and using that $L_j = f^j(L_0)$, $K_j = f^j(K_0)$ we have that

$$L_p \subset \overline{K}_k \subset \overline{L}_{2k} \subset \overline{K}_{3k} \subset \overline{L}_{4k} \subset \cdots$$

In the iterate $2nk$ we have that

$$K_k \subset \overline{L}_{2nk} = f^{2nk}(\overline{L}_p) = \overline{L}_p,$$

then $L_p$ accumulates on the sector $A$.

\[\square\]

The hypothesis in theorem 2.13 asks for more than lemma 2.14, namely

2.15. Corollary.

Let $p \in S_0 \cup \partial S_0$ be a saddle point for the extension $f : S_0 \to S$ of the return map. Assume that

(25) \( \gamma := \psi_R(p), \quad \overline{S \cap W^s(\gamma)} \subset S_0 \quad \text{and} \quad \overline{S \cap W^u(\gamma)} \subset S_0. \)

Then

(1) If $p \in S \setminus \partial S$, all the branches of $p$ have the same closure and accumulate on all the sectors of $p$.

(2) If $p \in \partial S$, $\gamma = \psi_R(p)$, $p_1, \ldots, p_{2m}$ are the periodic points of $f$ in $\gamma \subset \partial S_0$ and $L_i \subset S \setminus \partial S$ is the interior branch of $p_i$, then $\overline{L_i} = \overline{L_j}$ for every $(i, j)$.

Proof:

(1). By lemma 2.14, it is enough to prove that the branches of $p$ in $S \setminus \partial S$ have the same closure. Let $n$ be such that $f^n(p) = p$. Let $L_0$, $L_1$ be two branches of $p$ adjacent to the same sector $A$. Since by lemma 2.14, $L_0$ accumulates on the sector $A$; we have that $L_1 \cap L_0 \neq \emptyset$. By hypothesis $\overline{L_0} \subset S_0$. Also $f^{2n}(L_0) = L_0$. By the accumulation lemma 2.9 applied to $f^{2n}$, we have that $L_1 \subset \overline{L_0}$. Iterating this argument we have that all branches of $p$ have the same closure.
Let \( p = p_0, p_1, \ldots, p_{2m} = p_0 \) be the ordered periodic points of \( f \) in \( \gamma \subset \partial S \), as in figure 3. Let \( L_p = L_0 = L_2m \) and let \( L_i \) be the branch of \( p_i \) in \( S \setminus \partial S \). Then the \( L_i \)’s are connected components of \( S \cap W^{s,u}(\gamma) \). By (25), \( \overline{L_i} \subset S_0 \) and the \( L_i \)’s are all the components of \( S \cap W^{s,u}(\gamma) \). Let \( A_{2i}, A_{2i+1} \) be the sectors adjacent to \( p_i \) chosen so that \( A_{2i-1} \) and \( A_{2i} \) are adjacent to a connection between \( p_{i-1} \) and \( p_i \).

By lemma 2.14, \( L_i \) accumulates in its adjacent sector \( A_{2i+1} \). Due to the connection, \( L_i \) also accumulates on the sector \( A_{2i+2} \), adjacent to \( L_{i+1} \). Then \( \overline{L_i} \cap \overline{L_{i+1}} \neq \emptyset \). By the accumulation lemma 2.9 applied to \( f^N \) where \( N \) is a multiple of the periods of \( L_i \) and \( L_{i+1} \), we have that \( L_{i+1} \subset \overline{L_i} \). Thus

\[
\overline{L_0} \supset \overline{L_1} \supset \overline{L_2} \supset \cdots \supset \overline{L_{2m}} = \overline{L_0}.
\]

Therefore \( \overline{L_i} = \overline{L_j} \) for every \((i, j)\).

\[\square\]

3. The extension to the boundary of the return map.

In this section we study the extension to the boundary at a rotating boundary orbit of the return map to a component of a complete system of surfaces of section a Reeb flow. On non rotating boundary orbits the return time is infinite and the return map is not defined.

We first consider the case in which rotating boundary orbit \( \Gamma \) at the boundary of a component \( \Sigma \) is irrationally elliptic. In that case we prove that if the Floquet multipliers of the elliptic orbit are not roots of unity then the extension to the boundary has no periodic point.

Afterwards we deal with the case in which the periodic orbit \( \Gamma \) at the boundary of the component \( \Sigma \) is hyperbolic. We show that there exist a continuous extension of the Poincaré map to the boundary \( \Gamma \) of \( \Sigma \). This extension is a Morse-Smale map in \( \Gamma \) with periodic points on \( \Gamma \) that, seen in \( \Sigma \), are saddle points. These periodic points do not correspond to other periodic orbits of the Reeb flow but their interior invariant manifolds \( W^s, W^u \) are the intersections of the invariant manifolds \( W^u(\Gamma), W^s(\Gamma) \) of the closed orbit \( \Gamma \) with the component \( \Sigma \) and hence their intersections with other invariant manifolds of the return map to \( \Sigma \) are transversal if the Reeb flow is Kupka-Smale.

3.1. The elliptic case.

3.1. Proposition.

If the binding periodic orbit \( \Gamma \) is elliptic and its Floquet multipliers are not roots of unity, then the extension of the return map to the boundary \( \Gamma = \partial \Sigma \) has no periodic points.
Figure 4. The figure shows the dynamics of the extension of the return map $P$ to the surface of section $\Sigma$ with hyperbolic rotating boundary orbits.

**Proof:** As we shall see in §3.6 the component $\Sigma$ has an asymptotic direction $e(t) \in \xi(\Gamma(t))$, where $\xi$ is the contact structure. The direction of $e(t)$ turns more slowly than its movement $s \mapsto d\varphi_s(e(t_0))$ under the linearized Reeb flow. Then the extension $P$ of the return map to $\Gamma = \partial \Sigma$ is given by $\Gamma(t) \mapsto \Gamma(b(t))$, where

$$b(t) = t + \min \{ s > 0 \mid \exists \lambda > 0 : d\varphi_s(e(t)) = \lambda e(t+s) \}.$$

If the extension $P$ has a periodic point at $\Gamma(t)$ then the subspace generated by $e(t)$ is invariant under $d\varphi_T$, where $T$ is a multiple of the period of $\Gamma$. Then some iterate of the Poincaré map of the periodic orbit $\Gamma$ has an invariant 1-dimensional subspace. Since $\Gamma$ is elliptic, this implies that the Poincaré map of $\Gamma$ has an eigenvalue which is a root of unity. This is a contradiction. $\square$

### 3.2. The hyperbolic case. Sketch of the proof.

Let $\Gamma$ be a periodic orbit for the Reeb flow of $(N, \lambda)$. The contact structure $\xi = \ker \lambda$ is a subspace transversal to the Reeb vector field which is non integrable. The image of the exponential map of a small ball $B_\delta(0) \cap \xi(\Gamma(t))$, $\Xi(\Gamma(t)) = \exp_{\Gamma(t)}(\xi(\Gamma(t)) \cap B_\delta(0))$ is a system of transversal sections to the Reeb flow in a neighborhood of $\Gamma$ which is tangent to the contact structure at $\Gamma$.

The picture of the return map to the surface of section nearby the boundary orbit and its extension to the boundary is clear when the flow is linear in a neighborhood of the biding orbit $\Gamma$, the contact structure $\xi$ is orthogonal to the vector field $X(\Gamma) = \Gamma$ and the intersections, near the boundary $\partial \Sigma$, of the surface of section $\Sigma$ with the transversals $\Xi$, are images under the exponential map of straight lines. In this case the set of images of straight lines in $\xi$ passing through $\Gamma$ is a singular foliation invariant$^9$ by the flow.

---

$^9$This does not happen on a contact flow but may happen for a reparametrization of the flow.
In the following paragraphs we show that the return map to $\Sigma$ is conjugate to the situation described above. We choose a coordinate system in which the strong invariant manifolds $W^s(\psi_t z), W^u(\psi_t z)$ coincide with coordinate axes, and the flow is linear. In these coordinates the surface of section turns around the axis at least an angle $\pi$. We construct an open book decomposition $F$ of a neighborhood of $\Gamma$ with spine $\Gamma$, which is invariant under the Reeb flow, whose intersections with the transversals $\Xi$ are straight lines in these coordinates.

Then we show that nearby $\Gamma$ there is an isotopy $\Sigma_s$ between $\Sigma$ and a surface $\Sigma_1$, with the properties that for all $s$, $\Sigma_s$ is transversal to the Reeb vector field $X$ and that, for the final surface, $\Sigma_1 \cap \Xi(\Gamma(t))$ is a leaf of $F \cap \Xi(\Gamma(t))$ for all $t$. Then the return map is conjugated to the map that arises in the simpler situation described above.

For this, in §3.4 we compute the condition for a small cylinder, with $\Gamma$ as one boundary component, to be tangent to the Reeb vector field $X$. In §3.5 we give sufficient conditions for such isotopy to give surfaces transversal to $X$. Finally, in §3.6, we use the transversal approach of $\Sigma$ to $\Gamma$ to prove that the isotopy is made by surfaces transverse to $X$.

3.3. Coordinates and preliminary equations.

Let $(N, \lambda)$ be a 3-dimensional contact manifold, let $X$ be its Reeb vector field, $\psi_t$ its Reeb flow and $\xi = \ker \lambda$ its contact structure. Let $\Sigma$ be a surface of section for $X$ and $\Gamma \subset \partial \Sigma$ a rotating boundary periodic orbit. This means that there is a neighborhood $U$ of $\Gamma$ in $\Sigma$ such that the first arrival times of $\psi_t$ and $\psi_{-t}$ from $U$ to $\Sigma$ are bounded.

From now on we assume that the boundary periodic orbit $\Gamma$ is hyperbolic. For simplicity assume that the periodic orbit $\Gamma$ has period 1. To simplify the notation we shall also assume that $\Gamma$ has negative eigenvalues and that each local invariant manifold $W^s_\varepsilon(\Gamma), W^u_\varepsilon(\Gamma)$ intersects $\Sigma$ in a neighborhood of $\Gamma$ in 3 connected components. The other cases are similar.

Since $\Gamma$ has negative eigenvalues we can choose a smooth coordinate system $(x, y, z)$ near the periodic orbit $\Gamma$ such that

$$\Gamma(t) = (0, 0, t + Z) \in \mathbb{R}^2 \times S^1 / \equiv, \quad S^1 = \mathbb{R} / \mathbb{Z}, \quad (x, y, t + 1) \equiv (-x, -y, t),$$

$$W^u_\varepsilon(\Gamma(t)) = \{(x, 0, t \mod 1) : |x| < \varepsilon \},$$

$$W^s_\varepsilon(\Gamma(t)) = \{(0, y, t \mod 1) : |y| < \varepsilon \},$$

and along the periodic orbit one has

$$e_u := \frac{\partial}{\partial x} \in \mathbb{E}^u, \quad e_s := \frac{\partial}{\partial y} \in \mathbb{E}^s, \quad d\eta(e_u, e_s) = 1 \quad \text{and} \quad \frac{\partial}{\partial z} = X,$$

where $\mathbb{E}^s, \mathbb{E}^u$ are the stable and unstable subspaces for $\Gamma$:

$$d\psi_s(\mathbb{E}^s, \Gamma(t))) = \mathbb{E}^s, \Gamma(t + s)), \quad ||d\psi_1|\mathbb{E}^s|| < 1, \quad ||d\psi_1|\mathbb{E}^u|| > 1.$$
Consider the derivative $DX(0,0,t)$ of the vector field along the orbit $\Gamma(t)$. Since the subspaces $E^s, E^u$ are invariant under the linearized flow, we have that

$$
(27) \quad DX(0,0,t) = \begin{bmatrix} \lambda_t & 0 & 0 \\ 0 & \mu_t & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

The derivative of the flow at the periodic orbit $F(t) := d\varphi_t(0,0,0)$ satisfies the differential equation $\dot{F} = DX(0,0,t) F$. Its solution is

$$
d\varphi_t \left[ x_0 e_u(\tau) + y_0 e_s(\tau) + z_0 X(0,0,\tau) \right] = x_0 e^{\int_0^t \lambda_s \, ds} e_u(t+\tau) + y_0 e^{\int_0^t \mu_s \, ds} e_s(t+\tau) + z_0 X(0,0,t+\tau).
$$

Since $\varphi_t$ preserves $d\eta$, $d\eta [d\varphi_t(e_u), d\varphi_t(e_s)] \equiv 1$. This implies that $\mu_t = -\lambda_t$ for all $t$.

Since $d\varphi_t$ is 1-periodic in $t$, the unique invariant (i.e. 1-periodic) subspace transversal to $X$ is given by $E_\xi = \text{span} \{e_u, e_s\}$, which necessarily coincides with the contact structure $\xi = \ker \eta$ along $\Gamma$. Indeed, if in our coordinates the invariant transversal subspace $E_t$ at $(0,0,t)$ is given by $E_t: a_t x + b_t y + c_t z = 0$, with 1-periodic functions $a_t, b_t, c_t$. Then for any $(x,y,z)$ such that

$$
a_\tau x + b_\tau y + c_\tau z = 0,
$$

we must have that $D(0,0,\tau)\varphi_n(x, y, z) \in E_{\tau+n}$, i.e.

$$
a_{\tau+n} x e^{\int_0^\tau \lambda_s \, ds} + b_{\tau+n} y e^{\int_0^\tau \mu_s \, ds} + c_{\tau+n} z = 0,
$$

for all $n \in \mathbb{Z}$. Since $a_t, b_t, c_t$ are 1-periodic, this implies that $a_\tau = b_\tau = 0$ for all $\tau$.

Thus the equation for the derivative of the flow $d\psi_{\xi} |_{\xi}$ restricted to the contact structure $\xi$ over $\Gamma$ is

$$
(28) \quad \dot{w} = A(t) \cdot w, \quad A(t) = \begin{bmatrix} \lambda_t & 0 \\ 0 & -\lambda_t \end{bmatrix}.
$$

We write now this differential equation in polar coordinates.

Let $w(t) = (x(t), y(t)) = \zeta(t) u_\beta(t)$ be a solution of (28), where $u_\beta = (\cos \beta, \sin \beta)$. Then $w(t) = (x_0 e^{\Lambda_t}, y_0 e^{-\Lambda_t})$, $\Lambda_t = \int_0^t \lambda_s \, ds$, $w(0) = (x_0, y_0)$. Hence

$$
\tan \beta(t) = e^{-2\Lambda_t} \tan \beta(0).
$$

Differentiating this equation we obtain

$$
(\sec^2 \beta) \dot{\beta} = -2\lambda_t \tan \beta,
$$

$$
\dot{\beta} = -2\lambda_t \cos^2 \beta \tan \beta,
$$

$$
(29) \quad \dot{\beta} = -\lambda_t \sin(2\beta).
$$
Also,
\[
\begin{align*}
\zeta(t)^2 &= e^{2\Lambda_1} x_0^2 + e^{-2\Lambda_1} y_0^2 \\
2 \zeta \dot{\zeta} &= 2\lambda_t \left(e^{2\Lambda_1} x_0^2 - e^{-2\Lambda_1} y_0^2\right) \\
\dot{\zeta} &= \lambda_t \zeta^2 (\cos^2 \beta - \sin^2 \beta)
\end{align*}
\]
(30)
\[
\dot{\zeta} = \lambda_t \zeta \cos(2\beta).
\]

The vector field \( X \) satisfies \( X(x, y, t) = (\lambda_t x, -\lambda_t y, 1) + \mathcal{O}(x^2 + y^2) \). Let
\[
\gamma(t) = (\rho(t) u_{\alpha(t)}, z(t))
\]
be a solution of \( \dot{\gamma} = X(\gamma) \). Then \( \dot{\gamma} = (\dot{\rho} u_{\alpha} + \rho \dot{\alpha} u_{\alpha}^1, \dot{z}) \), \( u_{\alpha}^1 := (-\sin \alpha, \cos \alpha) \), and
\[
\begin{align*}
\dot{\rho} &= \langle \gamma, u_{\alpha} \rangle = \langle X(\gamma), u_{\alpha} \rangle = \langle A(t) \gamma, u_{\alpha} \rangle + \mathcal{O}(\rho^2), \\
\rho \dot{\alpha} &= \langle \dot{\gamma}, u_{\alpha}^1 \rangle = \langle X(\gamma), u_{\alpha}^1 \rangle = \langle A(t) \gamma, u_{\alpha}^1 \rangle + \mathcal{O}(\rho^2).
\end{align*}
\]
Therefore, using (30) and (29),
\[
\begin{align*}
\dot{\rho} &= \lambda_t \rho \cos(2\alpha) + \mathcal{O}(\rho^2), \\
\rho \dot{\alpha} &= -\lambda_t \rho \sin(2\alpha) + \mathcal{O}(\rho^2), \\
\dot{\alpha} &= -\lambda_t \sin(2\alpha) + \mathcal{O}(\rho).
\end{align*}
\]

Writing \( X = (f_1, f_2, f_3) \) let \( Y = \frac{1}{f_3} X \) and let \( \phi_t \) be the flow of \( Y \). Then \( \phi_t \) is the reparametrization of \( X \) which preserves the solution \( \Gamma(t) \) and for which the foliation “\( z = \text{constant} \)” is invariant. Observe that from (27), \( \frac{\partial f_2}{\partial z}|_{(0,0,z)} \equiv 0 \). The vector field \( Y \) is not a Reeb vector field of a contact form but it is smooth and along \( \Gamma \), \( Y = X \) and \( DY = DX \). In particular, the arguments above remain valid for \( Y \).

Consider the \( \phi \)-invariant foliation \( \mathcal{F} \) of \( U_\varepsilon \setminus \{0\} \times S^1 \), where
\[
U_\varepsilon := D_\varepsilon \times S^1, \quad D_\varepsilon = \{z \in \mathbb{R}^2 : |z| < \varepsilon\}, \quad S^1 = \mathbb{R}/\mathbb{Z},
\]
whose leaves are
\[
\mathcal{F}_t(u_{\alpha_i}) := \{\phi_r(r u_{\alpha_i}, s) : r \in [0, \delta[, s \in S^1 \} \cap U_\varepsilon, \quad \alpha_i = \frac{\pi}{4} i, \quad i = 0, 1, \ldots, 7,
\]
with \( 0 < \varepsilon \ll \delta \ll 1 \). This is a “radial” foliation which satisfies \( \phi_s(\mathcal{F}_t(\alpha_i)) = \mathcal{F}_{s+t}(\alpha_i) \). Observe that \( \mathcal{F}_s(u_0) = \mathcal{F}_t(u_0) \subset W^u(\Gamma) \) for all \( s, t \) and also \( \mathcal{F}_t(u_{\alpha_2}), \mathcal{F}_t(u_{\alpha_3}) \subset W^s(\Gamma) \) and \( \mathcal{F}_t(u_{\alpha_7}) \subset W^s(\Gamma) \).

Let \( \Sigma \) be a surface of section having \( \Gamma \) as a rotating boundary orbit. We will construct an isotopy of \( \Sigma \cap U_\varepsilon \) along surfaces \( \Sigma_\sigma \), \( \sigma \in [0, 1] \) such that \( \partial \Sigma_\sigma \cap \text{int}(U_\varepsilon) = \Gamma \), \( \Sigma_0 = \Sigma \), \( \Sigma_\sigma \) is transversal to \( Y \) and such that for all \( \tau \in S^1 \), \( \Sigma_1 \cap [z = \tau] \) is included in one leaf of \( \mathcal{F} \).
3.4. The tangency condition.

Consider an annular smooth surface $S$ in $U_\varepsilon$ with boundary $\partial S \cap \text{int}(U_\varepsilon) = \Gamma$ which has a well defined limit tangent space at the points in $\partial S = \Gamma$. Then, for $\varepsilon > 0$ small enough, $S \cap U_\varepsilon$ is the image of a map $F : [0, \varepsilon] \times S^1 \to U_\varepsilon$, $F(r, t) = (ru_{\theta(r,t)}, t)$, where $\theta$ is a $C^1$ map, with continuous derivatives at $r = 0$. We obtain now the conditions for $S$ to be tangent to the vector field $Y$ on $r > 0$:

3.2. Lemma. If the surface $F(r, t) = (ru_{\theta(r,t)}, t)$ is tangent to the vector field $Y$ at a point $(r, t)$ then $\theta(r, t)$ satisfies

\[
\theta_t = -\lambda_t \left[ \sin(2\theta) + r \theta_r \cos(2\theta) \right] + O(r) \quad \text{at} \quad (r, t).
\]

Proof: The tangent plane to $S$ is generated by

\[
\begin{align*}
    r \cdot \frac{\partial F}{\partial r} &= (ru_{\theta} + r^2 \theta_r u_{\theta}^\perp, 0), \\
    \frac{\partial F}{\partial t} &= (r \theta_t u_{\theta}^\perp, 1).
\end{align*}
\]

Let $\gamma(t) = (\rho(t) u_{\alpha(t)}, t)$ be an orbit of the flow $\phi$ of $Y$. Then $\rho(t)$ and $\alpha(t)$ also satisfy equations (31) and (32). The surface $S$ is tangent to the vector field $Y$ at $ru_{\theta} = \rho u_{\alpha}$ if and only if there exists $a, b \in \mathbb{R}$ such that

\[
\begin{align*}
    (\dot{\rho} u_{\alpha} + \rho \dot{\alpha} u_{\alpha}^\perp, 1) &= a \left( r u_{\theta} + r^2 \theta_r u_{\theta}^\perp, 0 \right) + b \left( r \theta_t u_{\theta}^\perp, 1 \right) \\
    &= a \left( r u_{\theta} + r^2 \theta_r u_{\theta}^\perp, 0 \right) + b \left( r \theta_t u_{\theta}^\perp, 1 \right).
\end{align*}
\]

In this case $b = 1$ and, using (31) and (32),

\[
\begin{align*}
    \dot{\rho} &= a r = \lambda_t \rho \cos(2\alpha) + O(\rho^2), \\
    \rho \dot{\alpha} &= a r^2 \theta_r + r \theta_t = -\rho \lambda_t \sin(2\alpha) + O(\rho^2).
\end{align*}
\]

Since $\rho = r$ and $\alpha = \theta$, from (35) we get that $a = \lambda_t \cos(2\theta) + O(r)$. Substituting $a$ in (36) we get (34).

\[
\square
\]

3.5. The isotopy.

Let $F : [0, \varepsilon] \times S^1 \to U_\varepsilon$, $F(r, t) = (ru_{\theta(r,t)}, t)$ be a local parametrization of the surface of section $\Sigma =: \Sigma_0$. Let $\overline{\theta}(t) := \theta(0, t)$. Let $G : [0, \varepsilon] \times S^1 \to U_\varepsilon$, $G(r, t) = (ru_{\omega(r,t)}, t)$ be defined by $G([0, \varepsilon] \times t) \in \mathbb{F} \cap (\mathbb{R}^2 \times \{t\})$, where $\mathbb{F}$ is the leaf of the foliation $\mathcal{F}$ in (33) such that its tangent space at $(0, 0, t)$ is $E = \text{span}(\theta, 1, (u_{\theta(t)}, 0))$. 

3.3. Lemma.

Let $\theta, \omega : [0, \varepsilon] \times S^1 \to \mathbb{R}$ be of class $C^1$. Write $\overline{\theta}(t) := \theta(0, t)$ and $\overline{\omega}(t) := \omega(0, t)$. For $\mu \in [0, 1]$ write $\varphi^\mu(r, t) := \mu \omega(r, t) + (1 - \mu) \theta(r, t)$. Suppose that

\begin{equation}
\overline{\theta}_t < -\lambda_t \sin(2\overline{\theta}) \quad \text{and} \quad \overline{\omega}(t) = \overline{\theta}(t) \quad \text{for all } t \in S^1.
\end{equation}

Then there is $\rho_0 > 0$ such that for all $\mu \in [0, 1]$, the surface $H^\mu(r, t) = (r, u_{\varphi^\mu(r, t)}, t)$ is transversal to the vector field $Y$ at all points $(r, t)$ with $0 < r < \rho_0$, $t \in S^1$.

**Proof:** Since $S^1$ is compact, there exists $\varepsilon > 0$ such that

$$\overline{\theta}_t < -\lambda_t \sin(2\overline{\theta}) - 3\varepsilon \quad \text{for all } t \in S^1.$$ 

Choose $\rho_1 > 0$ such that for all $0 \leq r < \rho_1$ and $t \in S^1$,

$$|\lambda_t| r |\theta_r| < \frac{\varepsilon}{4}, \quad |\lambda_t| r |\omega_r| < \frac{\varepsilon}{4},$$

$$|\lambda_t| |\overline{\theta}(t) - \theta(r, t)| < \frac{\varepsilon}{16}, \quad |\lambda_t| |\overline{\theta}(t) - \omega(r, t)| < \frac{\varepsilon}{16} \quad \text{and}$$

$$\theta_t < -\lambda_t \left[\sin(2\theta) + r \theta_r \cos(2\theta)\right] - 2\varepsilon,$$

$$\omega_t < -\lambda_t \left[\sin(2\omega) + r \omega_r \cos(2\omega)\right] - 2\varepsilon.$$

Then

$$|\lambda_t| |\varphi^\mu - \overline{\theta}| < \frac{\varepsilon}{16}, \quad |\lambda_t| |\varphi^\mu - \theta| < \frac{\varepsilon}{8}, \quad |\lambda_t| |\varphi^\mu - \omega| < \frac{\varepsilon}{8},$$

$$|\lambda_t| |\sin(2\varphi^\mu) - \sin(2\overline{\theta})| < \frac{\varepsilon}{8} \quad \text{and}$$

$$|\lambda_t| |\sin(2\varphi^\mu) - \sin(2\theta)| < \frac{\varepsilon}{4}, \quad |\lambda_t| |\sin(2\varphi^\mu) - \sin(2\omega)| < \frac{\varepsilon}{4}.$$

Hence

$$|\lambda_t| \left|\sin(2\varphi^\mu) - [\mu \sin(2\omega) + (1 - \mu) \sin(2\theta)]\right| < \frac{\varepsilon}{4}.$$ 

Also, for all $(r, t) \in [0, \rho_1] \times S^1$ and $\mu \in [0, 1]$, since $\varphi^\mu_t = \mu \omega_r + (1 - \mu) \theta_r$, we have that

$$|\lambda_t r \theta_r \cos(2\theta)| < \frac{\varepsilon}{4}, \quad |\lambda_t r \omega_r \cos(2\omega)| < \frac{\varepsilon}{4} \quad \text{and} \quad |\lambda_t r \varphi^\mu r \cos(2\varphi^\mu)| < \frac{\varepsilon}{4}.$$

Thus, we have that

$$\varphi^\mu_t = \mu \omega_t + (1 - \mu) \theta_t$$

$$< -2\varepsilon - \lambda_t \left[\mu \sin(2\omega) + (1 - \mu) \sin(2\theta)\right] - \mu \lambda_t r \omega_r \cos(2\omega) - (1 - \mu) \lambda_t r \theta_r \cos(2\theta)$$

$$< -2\varepsilon + \frac{\varepsilon}{4} - \lambda_t \sin(2\varphi^\mu) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$$

$$< -2\varepsilon + \frac{\varepsilon}{4} - \lambda_t \sin(2\varphi^\mu) + \frac{\varepsilon}{2} - \lambda_t r \varphi^\mu r \cos(2\varphi^\mu) + \frac{\varepsilon}{4}$$

$$< -\lambda_t \left[\sin(2\varphi^\mu) + r \varphi^\mu r \cos(2\varphi^\mu)\right] - \varepsilon.$$

Then there is $0 < \rho_0 < \rho_1$ such that if $0 < r < \rho_0$,

$$\varphi^\mu_t < -\lambda_t \left[\sin(2\varphi^\mu) + r \varphi^\mu r \cos(2\varphi^\mu)\right] + O(r),$$

where $O(r)$ is from lemma 3.2. Then lemma 3.2 finishes the proof. \qed
3.6. The transversality condition.

The equation for the dynamics under $d\psi_t$ of subspaces along the periodic orbit $\Gamma(t)$ is (29). Then equation (37) just says that the limit direction $\mathbf{\ell}$ of the surface of section at $\Gamma$ turns more slowly than its iteration under the linearized Reeb flow. Condition 2.4.(iii) in the definition of complete systems implies (37).

We check in §3.8 equation (37) for surfaces of section which are projections of pseudo holomorphic curves in a symplectization, and in §3.7 for Birkhoff annular surfaces of section. The surfaces of section that we use in theorem 1.3 are obtained by topological surgery from Birkhoff annuli. These surgeries maintain inequality (37).

Since a surface of section is transversal to the Reeb flow in its interior, a weak inequality equation (37) must hold at a rotating boundary orbit. If needed one can modify the surface nearby its boundary binding orbit so that the asymptotic rotation of the surface in the boundary is uniform with respect to the rotation of the flow, satisfying (37).

3.7. The transversality condition for a Birkhoff annulus.

Let $M$ be a closed oriented riemannian surface, $SM$ its unit tangent bundle and let $\phi_t : SM \leftrightarrow$ be the geodesic flow of $M$. Let $\lambda$ be the Liouville form on $TM$:

$$\lambda_{(x,v)}(\xi) = \langle v, d\pi(\xi) \rangle_x, \quad (x,v) \in TM, \quad \xi \in T_{(x,v)}TM.$$  

Let $V = \ker d\pi$, and $H = \ker K$ be the vertical and horizontal subspaces, where $K : TTM \to TM$ is the connection. The subbundle $N = \ker \lambda$ of $T(SM)$,

$$(38) \quad N = \ker \lambda = \{ (h, w) \in T_0SM \subset H \oplus V \mid \langle h, \theta \rangle_{\pi(\theta)} = 0 \},$$

is invariant under the linearized geodesic flow $d\phi_t$, which is given by

$$d\phi_t(J(0), \dot{J}(0)) = (J(t), \dot{J}(t)) \in N \subset H \oplus V,$$

where $t \mapsto J(t)$ is a Jacobi field along a geodesic $c(t) = \pi J(t)$ which is orthogonal to $\dot{c}(t)$. The tangent space to the unit tangent bundle $SM$ is

$$T_0SM = \langle X(\theta) \rangle \oplus N(\theta),$$

where $X$ is the geodesic vector field.

Let $\mathcal{J} : T_\pm M \leftrightarrow$ be the rotation of angle $\pm \frac{\pi}{2}$. Given a simple closed geodesic $\gamma(t)$ parametrized with unit speed, define its Birkhoff annulus by

$$A(\dot{\gamma}) := \{ (x,v) : \exists t, \ x = \gamma(t), \ \langle v, \mathcal{J}\dot{\gamma}(t) \rangle_{\gamma(t)} \geq 0 \}$$

The interior of $A(\dot{\gamma})$ is transversal to the geodesic flow. The tangent space of the Birkhoff annulus at a boundary point $\pm \dot{\gamma}(t)$, is generated by the geodesic vector field $X(\dot{\gamma}(t))$ and the vertical direction $V \cap N$. 

In order to obtain the transversality condition \( \overline{\theta}_t < -\lambda t \sin(2\overline{\theta}) \) it is enough to show that the (vertical) limit tangent space of the Birkhoff annulus moves slower than the movement of the vertical subspace under the derivative of the flow. This is done as follows:

Let \( J(t) \in T_{\gamma(t)} M \) be an orthogonal Jacobi field. Since both \( J \) and \( \dot{J} \) are multiples of the orthogonal vector \( \dot{\gamma}(t) \perp \) they can be regarded as scalar quantities. When \( (J, \dot{J}) \) is not horizontal, i.e. when \( \dot{J}(t) \neq 0 \), define \( W(t) = J(t)/\dot{J}(t) \). From the Jacobi equation

\[ \ddot{J} + KJ = 0 \quad \text{and} \quad J = W \dot{J}, \]

we get

\[ \dot{J} = \dot{W} \dot{J} - WKJ. \]

Replacing \( J = W \dot{J} \) when \( \dot{J} \neq 0 \) one obtains the Riccati equation

(39) \[ \dot{W} = KW^2 + 1. \]

A solution \( W(t) \) is the slope of the iteration under \( d\phi_t \) of a linear subspace, i.e. if

\[ W_0 = \text{graph } W(0) = \{(W(0)v, v) \in \mathbb{H} \oplus \mathbb{V} \mid v \in \mathbb{H}\} \]

then \( d\phi_t(W_0) = \text{graph}(W(t)) \). The subspace \( W_0 \) is the vertical subspace \( \mathbb{V} \) precisely when \( W(0) = 0 \). In this case, from (39) we have that \( \dot{W}(0) = 1 \). If \( V(t) \) is the slope of the vertical subspace \( \mathcal{N} \cap \mathbb{V} \), then \( V(t) \equiv 0 \) and \( \dot{V}(0) = 0 \). This means that the iteration \( W(t) \) of the vertical subspace under the linearized geodesic flow moves faster than the vertical subspace \( V(t) \) (tangent to the Birkhoff annulus).

### 3.8. The transversality condition for finite energy surfaces.

In this section we prove that condition (37) holds for projections on \( S^3 \) of pseudo holomorphic curves in the simplectization of a tight contact form on \( S^3 \). Then we can apply item (3) of theorem B to the complete system of surfaces of section of genus 0 obtained by Hofer, Wysocki, Zehnder in [21, Cor. 1.8], in order to obtain Corollary 1.1.

In this case the complete system is given by the rigid surfaces of the finite energy foliation. Let \( \Sigma \) be a rigid surface and let \( \Gamma \) be a boundary periodic orbit of \( \Sigma \) where the foliation is radial. The equation for the dynamics under \( d\psi_t \) of subspaces along a periodic orbit \( \Gamma(t) \) is (29). So we want to prove that the limit direction \( \overline{\theta} \) of the surface of section \( \Sigma \) at \( \Gamma \) turns slower than its iteration under the linearized Reeb flow.

Recall that the contact structure \( \xi \) is invariant under the Reeb flow \( \psi_t \). The linearized Reeb flow on \( \xi \) satisfies \( v(t) = d\psi_t(v(0)) \) where \( v(0) \in \xi \) and

(40) \[ \dot{v} = DX(\psi_t(\Gamma(0))) \cdot v = S(t) v. \]

Here the matrix \( S(t) = DX(\Gamma(t)) \) is symmetric on symplectic linear coordinates in \( \xi(\Gamma(t)) \) and \( v(t) = \zeta(t) u_{\beta(t)} \) satisfies (29) and (30).
From theorem 1.4 in [20] (where \( S_\infty = -J_0 S(t) \) and \( J_0 = J|_{\xi} \)), there is a periodic vector \( e(t) = \varepsilon(t) u_{\overline{\mu}(t)} \in \xi(\Gamma(t)) \) in the asymptotic direction of the rigid surface \( \Sigma \) which satisfies the (eigenvalue) equation

\[
(41) \quad \dot{e} = S(t) \dot{e} + \mu J e
\]

with \( \mu < 0 \) and \( J: \xi \leftrightarrow \) an almost complex structure on \( \xi \).

Our choice of coordinates (26) about \( \Gamma(t) \) is symplectic and the almost complex structure can be taken \( J(x, y) = (-y, x) \) in these coordinates. Comparing equations (40) and (41) at an initial condition for \( v(t) \) such that \( v(t_0) = e(t_0) \) we get that the rigid surface \( \Sigma \) turns slower than the linearized flow.

Indeed, in polar coordinates \( ru_\beta = v(t_0) = e(t_0) = \varepsilon u_\theta, \) from (40), at \( t = t_0 \) we have that

\[
S(t_0) e = \dot{v} = \dot{r} u_\beta + r \dot{\beta} u_\beta^\perp.
\]

\[
\dot{e} = \dot{\varepsilon} u_\theta + \varepsilon \dot{\theta} u_\theta^\perp,
\]

\[
= S(t_0) e + \mu J e,
\]

where \( J e = \varepsilon u_\theta^\perp \). In the component \( u_\theta^\perp \) these equations are

\[
\dot{\varepsilon} = \dot{\varepsilon} + \varepsilon \dot{\theta} + \mu \varepsilon.
\]

From (29), \( \dot{\beta} = -\lambda_t \sin 2\beta = -\lambda_t \sin 2\theta \). Therefore

\[
\dot{\theta} = -\lambda_t \sin 2\theta + \mu,
\]

with the (constant) eigenvalue \( \mu < 0 \). This implies (37).

### 3.9. The return map.

Lemma 3.3 gives an isotopy of the local surface of section \( \Sigma_0 \) by surfaces \( \Sigma_\mu, \mu \in [0, 1] \), which are transversal to the vector field. Then the return map of the Reeb flow to the final surface \( \Sigma_1 \) is topologically conjugate to the return map to the surface \( \Sigma_0 \) in a neighbourhood of its boundary. Moreover, the intersections \( \Sigma_1 \cap (\mathbb{R}^2 \times \{t\}) \) are included in a leaf of the radial invariant foliation \( \mathcal{F} \). This implies that the return map to \( \Sigma_1 \) near the boundary \( \partial \Sigma_1 \) preserves the foliation of \( \Sigma_1 \) given by the sections \( \langle \Sigma_1 \cap (\mathbb{R}^2 \times \{t\}) \rangle_{t \in \mathbb{S}^1} \).

The surface \( \Sigma_1 \) is parametrized by

\[
G(r, t) = (r u_{\omega(r, t)}, t).
\]

At its boundary points \((0, t)\), the surface \( \Sigma_1 \) has a well defined tangent plane generated by \((u_{\omega(0, t)}, 0)\) and the Reeb vector field \( X = (0, 0, 1) \). Here \( \omega(0, t) = \overline{\omega}(t) = \overline{\theta}(t) = \theta(0, t) \) is the same angular approach of the surface \( \Sigma_0 \), which by §3.6 satisfies

\[
\overline{\theta}_t < -\lambda_t \sin 2\theta.
\]
Let \( P : \Sigma_1 \to \Sigma_1 \) be the first return map to \( \Sigma_1 \). Then in coordinates \((r, t)\) given by the parametrization \( G(r, t) \) we have that
\[
P(r, t) = (a(r, t), b(t)).
\]
Here \( b(t) \) is given by the time in which the leaf \( G([0, \varepsilon[, t) \) returns to \( \Sigma_1 \). This is the same as the time in which the derivative of the flow sends the tangent subspace \( T_{(0,t)}\Sigma_1 \) to \( T_{(0,b(t))}\Sigma_1 \). The equations for the derivative of the flow in polar coordinates are (29) and (30). Then \( b(t) \) is determined by the minimal \( b(t) > t \) satisfying
\[
\beta(t) = \bar{\theta}(t), \quad \beta(b(t)) = \bar{\theta}(b(t)), \quad \dot{\beta} = -\lambda_t \sin(2\beta).
\]
This is the first return map \( Q \) of the flow \( t \mapsto (t, \beta(t)) \) of the differential equation (29) to the graph of \( \bar{\theta} \) (see figures 5, 6).

The graph of \( \bar{\theta} \), \( \mathcal{G}(\bar{\theta}) = \{(t, \bar{\theta}(t)) \mid t \in \mathbb{S}^1 \} \), is transversal to the flow lines of (29). We are assuming that the angle \( \bar{\theta}(t) \) of \( \Sigma \) turns \( 3\pi \) in one period \( t \in [0, 1] \). The return map \( Q : \mathcal{G}(\bar{\theta}) \to \mathcal{G}(\bar{\theta}) \) under the flow of the differential equation (43) for \( \beta \) is the continuous extension of the return map of \( \psi_t \) to \( \Sigma_1 \) to the boundary \([r = 0] \approx \mathcal{G}(\bar{\theta}) \approx \mathbb{R}/3\pi \mathbb{Z} \subset \partial \Sigma_1 \).

The periodic orbit at \( \bar{\theta} = 0, \pi, 2\pi \) is a repellor and the periodic orbit at \( \bar{\theta} = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \) is an attractor. Lemma 3.4 shows that there are no other periodic points for the return map to \( \mathcal{G}(\bar{\theta}) \).

3.4. Lemma. There are no periodic points \((\tau, \bar{\theta}(\tau))\) for the return map \( Q : \mathcal{G}(\bar{\theta}) \to \mathcal{G}(\bar{\theta}) \)
\[
Q(t, \bar{\theta}(t)) = (b(t), \bar{\theta}(b(t))), \quad t \in \mathbb{R}/\mathbb{Z}, \quad \bar{\theta} \in \mathbb{R}/3\pi \mathbb{Z}
\]
with \( \bar{\theta}(\tau) \notin \frac{\pi}{2} \mathbb{Z} \). The periodic orbit \( \bar{\theta} = 0, \pi, 2\pi \) is a repellor and the periodic orbit \( \bar{\theta} = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \) is an attractor.

Proof: Observe that since \( t \mapsto \lambda_t \) is a 1-periodic function, the equation (29) defines a 1-periodic flow \( \phi_t(s, \beta(s)) = (s + t, \beta(s + t)) \) on \( \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z} \).

Since the line \( \bar{\theta} = \frac{\pi}{2} \) corresponds to the unstable subspace \( \mathbb{E}^u \) of \( \Gamma \) and \( t \mapsto \beta(t) \) describes the dynamics of the linearized flow on 1-dimensional subspaces along \( \Gamma \), we have that
\[
0 < \beta(0) < \pi \quad \Rightarrow \quad \lim_{t \to +\infty} \beta(t) = \frac{\pi}{2},
\]
\[
\pi < \beta(0) < 2\pi \quad \Rightarrow \quad \lim_{t \to +\infty} \beta(t) = \frac{3\pi}{2}.
\]
This implies that the periodic orbit \( \bar{\theta} = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \) for \( Q \) is an attractor. Similarly, the periodic orbit \( \bar{\theta} = 0, \pi, 2\pi \) is a repellor because it corresponds to the stable subspace \( \mathbb{E}^s \).
Figure 5. This figure shows the flow of the differential equation (29) for $\beta(t)$ describing the action of the derivative of the Reeb flow on 2-planes tangent to the periodic orbit $\gamma$. It also shows the curve $(t, \overline{\theta}(t))$, which corresponds to the movement of the limit tangent plane of the surface of section $\Sigma$ along its rotating boundary periodic orbit $\gamma$. The graph of $\overline{\theta}$ is transversal to the flow of $\beta$ and $\overline{\theta}(t+1) = \overline{\theta}(t) + 3\pi$, but $\overline{\theta}(t)$ may not be monotonous. Observe that the first return map of the flow of (29) to the graph of $\overline{\theta}$ has repelling periodic points at $\overline{\theta} = 0, \pi, 2\pi$ and attracting periodic points at $\overline{\theta} = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.

Suppose that there is a periodic point $(\tau, \overline{\theta}(\tau))$ of the return map $Q$ with $\overline{\theta}(\tau) \notin -\frac{\pi}{2}\mathbb{Z}$. Then there is $n \in \mathbb{Z}^+$ such that $Q(\tau, \overline{\theta}(\tau)) = (\tau + n, \overline{\theta}(\tau))$. The solution $\beta$ of (29) with $\beta(\tau) = \overline{\theta}(\tau)$ satisfies $\beta(\tau + n) = \overline{\theta}(\tau)$ and hence it is $n$-periodic. This contradicts (44). □
The figure shows the dynamics of the extension to $\partial \Sigma$ of the return map $P$ to the surface of section $\Sigma$ in a neighborhood of a rotating boundary component $\Gamma \subset \partial \Sigma$, when the periodic orbit $\Gamma$ is hyperbolic. Here the map $P$ has two periodic points in $\Gamma = \partial \Sigma$ of period 3 which are saddles on $\Sigma$. The periodic points correspond to the times in which the stable and unstable subspaces intersect the tangent space of the section $\Sigma$ at its boundary.

The vertical axis is the time parameter and the periodic orbit $\Gamma$ which is supposed to have period 1. The three shadowed rectangles are copies of the 2-torus formed by the periodic orbit (the time circle) and the one dimensional subspaces orthogonal to the periodic orbit, parametrized by their angle with one branch of the stable subspace. The movement of the stable subspace $E^s$ is represented by the angles $0$ and $\pi$ and the unstable subspace $E^u$ by the angles $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The periodic orbit has negative eigenvalues, then after one period the normal subspaces are identified by a rotation of angle $\pi$. For example, the 0 branch of the stable subspace $E^s$ is identified with the $\pi$ branch of $E^s$. This can be seen in the picture as a shift of length $\pi$ in the second shadowed square. The black lines are the dynamics of linear subspaces orthogonal to $\Gamma$ under the derivative of the flow, we will call it the projective flow. The subspaces converge to the unstable subspace $E^u$ in the future and to $E^s$ in the past.

The transversal lines are the graph of the asymptotic limit $\overline{\theta}(t)$ of the surface of section $\Sigma$. We have assumed that this graph intersects three times $E^s$, $E^u$ in one period. The dynamics of the extension to the boundary in this figure is given by the return map of the projective flow in the figure, to the graph of the asymptotic direction $\overline{\theta}(t)$. The periodic orbit corresponding to the unstable subspace $E^u$ is shown in the figure with the numbers 1, 2, 3, in the order of the orbit. It is an attracting periodic orbit, and the stable subspace in $\partial \Sigma$ gives a repelling periodic orbit. The extension of the return map to $\partial \Sigma$ is Morse Smale.
3.5. Proposition.

The periodic points at a hyperbolic rotating boundary of a surface of section are saddles for the return map.

**Proof:** In a neighbourhood of the periodic orbit $\Gamma$, the foliation of the surface $\Sigma_1$ whose leaves are $G([0, \varepsilon[, t), t \in [0, 1]$ is invariant under the return map $P: \Sigma_1 \to \Sigma_1$. Let $u, s \in [0, 1]$ be such that $\bar{\theta}(u) \in \{0, \pi\}$ and $\bar{\theta}(s) \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. Then $G([0, \varepsilon[, u)$ and $G([0, \varepsilon[, s)$ are components of $W^u(\gamma) \cap \Sigma_1$ and $W^s(\gamma) \cap \Sigma_1$ respectively. For such $u$'s, using formula (42), the third iterate $P^3$ of return map $r \mapsto a(r, u) \in [0, \varepsilon[, which is the dynamics in $W^u(\gamma) \cap \Sigma_1$, is expanding with fixed point $r = 0$ and on the components of $W^s(\gamma) \cap \Sigma_1$ it is a contraction with fixed point at the boundary of $\Sigma_1$.
Figure 10. The minimizing geodesics in the homotopy classes of $\gamma_i$ separate $M$ into four simply connected regions $R_j$ and the smoothing of their Birkhoff annuli

$$T = \partial R_1 \cup \partial R_3, \quad -T = \partial R_2 \cup \partial R_4,$$

are two embedded surfaces of section of genus 1. The orbits $\dot{\gamma}_i$, $-\dot{\gamma}_i$ are simply covered rotating boundary orbits for $T$ and $-T$.

Consider a small curve $\alpha$ transversal to $W^u(\gamma) \cap \Sigma_1$ as in figure 9. The inverse image $P^{-3}(\alpha)$ intersects a larger set of leaves of the $P$-invariant foliation $\mathcal{F} = \mathcal{F} \cap \Sigma_1$, this depends only on the dynamics of $b(t)$. Extend $\{\alpha, P^{-3}(\alpha)\}$ to a 1-dimensional foliation $A$ on $\Sigma_1$ between $\alpha$ and $P^{-3}(\alpha)$. By the $\lambda$-lemma, the backward flow $\phi_{-T}(\phi_{[0,\varepsilon]}(\alpha))$ of $\phi_{[0,\varepsilon]}(\alpha)$ approaches in the $C^1$ topology to the stable manifold $W^s(\gamma)$. The intersection of $\phi_{-T}(\phi_{[0,\varepsilon]}(\alpha))$ with $\Sigma_1$ are leaves of $P^{-3n}(A)$, which approach the boundary of $\Sigma$. Extend the foliation by iteration to a neighbourhood $\bigcup_{n \in \mathbb{N}} P^{-3n}(A)$ of the fixed point at the boundary $r = 0$, $t = u$. Use the foliations $\mathcal{F}$ and $A$ as in figure 8, to construct a coordinate system in a neighbourhood of the fixed point $r = 0$, $t = u$ which conjugates the dynamics to two sectors of a saddle fixed point. A similar construction can be made in a neighbourhood of the periodic points $r = 0$, $t = s$.

□

4. The complete system for geodesic flows.

The set of ideas in this section descent from G. Birkhoff, notably [2] section §28. By using an orientable double cover of $M$ if necessary, for theorem A it is enough to assume that the surface $M$ is orientable.

We denote $SM = \{(x,v) \in TM : \rho(v,v) = 1\}$ the unit tangent bundle, $\pi : SM \to M$ the projection, $\phi_t$ the geodesic flow on $SM$ and $SA = \pi^{-1}(A) \cap SM$ for every $A \subset M$.

4.1. Two surfaces of section of genus 1.

Let $\gamma_1, \ldots, \gamma_{2g+2}$ be minimizing geodesics in the homotopy classes of the curves shown in figure 10. We show now that they divide the surface $M$ into four regions $R_1, \ldots, R_4$ which are simply connected.
Figure 11. Fried surgery for a double crossing. The new surface in the center does not self intersect and is transversal to the flow. The right figure shows that the surgery is obtained by cutting the surfaces along two segments and gluing them. The gluing is uniquely determined by the conditions of transversality to the flow and non intersection.

Figure 12. Fried surgery for an intersection of a boundary orbit. The surgery is obtained by cutting along two segments and gluing. The gluing is uniquely determined by the flow. The resulting surface can be realized in an arbitrarily small neighborhood of the original surfaces. At interior points it is the same surgery as in figure 11.

A bigon is a simply connected open subset of $M$ whose boundary is two geodesic segments. Two minimizing geodesics in their homotopy classes can not form a bigon. Then they must have minimal intersection number in their homotopy classes c.f. [13, Prop. 1.7]. Therefore

$$|\#(\gamma_i \cap \gamma_j)| = \delta_{i,j-1} + \delta_{i,j+1} \quad \text{if} \quad i \neq j.$$ 

Now $M \setminus (\gamma_1 \cup \gamma_3 \cup \cdots \cup \gamma_{2g+1})$ is the union of two surfaces $N_1$, $N_2$ of genus zero with $2g + 1$ boundary components. The segments $\gamma_{2i} \cap N_j$ are curves connecting the boundary components $\gamma_{2i-1}$ and $\gamma_{2i+1}$. They form two simple closed curves bounding two simply connected regions $R_{2j-1}$, $R_{2j}$.

Let $\mathcal{J} : TM \to TM$ be a linear map such that $(v, \mathcal{J}v)$ is an oriented orthonormal basis for every unit vector $v$. Given an oriented simple closed geodesic $\gamma$, define the Birkhoff annulus of $\dot{\gamma}$ as

$$A(\dot{\gamma}) := \{(x,v) \in SM \mid \exists t, \ x = \gamma(t), \ \langle v, \mathcal{J} \dot{\gamma}(t) \rangle \geq 0 \}.$$ 

Then $A(\dot{\gamma})$ is an annulus in $SM$ with boundaries $\dot{\gamma}$, $-\dot{\gamma}$ whose interior is transversal to the geodesic flow. Because other geodesics intersecting $\gamma$ must be transversal to $\gamma$.

We perform the Fried surgeries described in figures 11, 13 to the collection of Birkhoff annuli $A(\dot{\gamma}_1), \ldots, A(\dot{\gamma}_{2g+2}), A(-\dot{\gamma}_1), \ldots, A(-\dot{\gamma}_{2g+2})$. Observe that there are not triple intersections of the interior of these annuli because there are no triple intersections of their projected geodesics. We need to use the surgery in figure 13 instead of figure 12 because the annuli $A(\dot{\gamma}_i), A(-\dot{\gamma}_i)$ meet at their boundaries.
Figure 13. The Birkhoff annuli $A(\dot{\gamma}_i)$ and $A(-\dot{\gamma}_i)$ intersect at $\dot{\gamma}_i$ and $-\dot{\gamma}_i$, and the orbit $\dot{\gamma}_i$ intersects transversely the annulus $A(\dot{\gamma}_{i+1})$ so it is necessary to perform the surgery in figure 12 twice.

We prove that the result are two surfaces of section $S_1$, $S_2$, of genus 1, each of them with the $4g+4$ boundary components $\{\dot{\gamma}_1, \ldots, \dot{\gamma}_{2g+2}, -\dot{\gamma}_1, \ldots, -\dot{\gamma}_{2g+2}\}$. Observe that any orbit $\Gamma$ with

$$\forall i \pi \Gamma \neq \dot{\gamma}_i \text{ and } \pi \Gamma \cap \bigcup_{i=1}^{2g+2} \dot{\gamma}_i \neq \emptyset$$

intersects $S_1$ or $S_2$ transversely.

Let $A(\partial R_i)$, $1 \leq i \leq 4$ be the closure of the set of unit vectors based at $\partial R_i$ pointing outside of $R_i$. Then each $A(\partial R_i)$ is a cylinder whose boundary projects to $\partial R_i$. Figure 14 shows the cylinders $A(\partial R_1)$, $A(\partial R_3)$ and how they are glued after performing the surgeries in figures 11, 13. Figure 16 shows how the surgeries of figure 13 glue the segments $a$ and $b$ in figure 14. Then figure 15 is the same as a figure 14 with the boundaries curved and rotated in order to show how the two cylinders $A(\partial R_1)$, $A(\partial R_3)$ glue after the surgery to form a torus $S_1$ with $4g+4$ holes. Similarly $S_2$ is obtained from $A(\partial R_2)$ and $A(\partial R_4)$.

Now we prove that the boundary components $\dot{\gamma}_i$, $-\dot{\gamma}_i$ are hyperbolic and that their local invariant manifolds $W_{loc}^s \cup W_{loc}^u$ intersect four times each section $S_1$, $S_2$.

The geodesics $\gamma_i$ are minimizers in their homotopy class. Since $M$ is a surface their multiples $\gamma_i^n(t) := \gamma_i(nt)$ are local minimizers, because a curve $\eta$ homotopic to $\gamma_i^n$ contained in a small tubular neighborhood of $\gamma_i$ can be separated into $n$ closed curves homotopic to $\gamma_i$. Then the length $L(\eta) \geq n \cdot L(\gamma_i) = L(\gamma_i^n)$. This implies that the whole geodesic $\gamma_i(t)$, $t \in \mathbb{R}$ has no conjugate points. Since $\gamma_i$ is non-degenerate, then it must be hyperbolic. Since $M$ is orientable $\gamma_i$, is positive hyperbolic.

By section 3.7 the vertical subspace is not invariant. Then its forward iterates $d\phi_t(V(\dot{\gamma}_i))$ must converge to the unstable subspace $E^u(\dot{\gamma}_i)$. But $d\phi_t(V(\dot{\gamma}_i))$ can not approach the vertical $V(\dot{\gamma}_i)$ because by section 3.7 it would intersect the vertical non trivially, producing
The sets \( A(\partial R_1) \) and \( A(\partial R_3) \) are a collage of half of the Birkhoff cylinders \( A(\dot{\gamma}_i) \) and \( A(-\dot{\gamma}_i) \) respectively. Both \( A(\partial R_1) \) and \( A(\partial R_3) \) are cylinders.

This is the same as figure 14 where the segments of \( \dot{\gamma}_i, -\dot{\gamma}_i \) has been drawn as curves and their adjacent vertical dashed segments have been drawn horizontal. Each of the figures is a cylinder, glued at its sides. The two figures are glued at the horizontal dashed lines. They form a torus \( T = A(\partial R_1) \cup A(\partial R_3) \) with \( 4g + 4 \) holes. The complete system contains another torus \( -T = A(\partial R_2) \cup A(\partial R_4) \) which corresponds to a similar construction using the regions \( R_2 \) and \( R_4 \) with the vectors opposite to those of \( T \). Both tori intersect pairwise at their boundaries.
Figure 16. This figure shows in more detail how the annuli $A(\dot{\gamma}_i)$, $A(\dot{\gamma}_{i+1})$, $A(-\dot{\gamma}_i)$ are glued in figure 14 after the surgery in figure 13.

Figure 17. This figure shows how the local invariant manifolds $W^s(\dot{\gamma}_i)$, $W^u(\dot{\gamma}_i)$ intersect the surface $S_j$ over the point $\gamma_i \cap \gamma_{i+1}$. The surface $S_j$ stays near the Birkhoff annuli $A(\dot{\gamma}_i)$, $A(\dot{\gamma}_{i+1})$, $A(-\dot{\gamma}_i)$. The annulus $A(\dot{\gamma}_i)$ is included in the vertical fibre $S_{\dot{\gamma}_i}$. The local manifolds do not intersect $S_{\dot{\gamma}_i} \setminus \dot{\gamma}_i$ because $\gamma_i$ has no conjugate points. There are other intersections over the point $\gamma_{i-1} \cap \gamma_i$.

where $X$ is the geodesic vector field. Then the invariant subspaces $E^s(\dot{\gamma}_i)$, $E^u(\dot{\gamma}_i)$ are bounded away from $T_{\dot{\gamma}_i}A(\dot{\gamma}_i)$. This implies that the local invariant manifolds $W^s(\dot{\gamma}_i)$, $W^u(\dot{\gamma}_i)$ do not intersect the interior of $A(\dot{\gamma}_i)$. Figure 17 shows how each of the local invariant manifolds intersect once the surface $S_j$ over the intersection $\gamma_{i-1} \cap \gamma_i$ and once more over $\gamma_i \cap \gamma_{i+1}$. By section 3 this gives four saddle periodic points for the return map at each boundary component $\dot{\gamma}_i$ or $-\dot{\gamma}_i$ of each surface of section $S_1$, $S_2$.

Observe that there is $\ell > 0$ and a neighborhood $N$ of $\partial S_j = \cup_i (-\dot{\gamma}_i \cup \dot{\gamma}_i)$ such that

$$\forall z \in N \quad \phi_{[0,\ell]}(z) \cap S_j \neq \emptyset \quad \& \quad \phi_{[-\ell,0]}(z) \cap S_j \neq \emptyset.$$
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Therefore the orbits $\pm \dot{\gamma}_i$ will be rotating boundary orbits for the sections $S_j$.

4.2. Applications of the curve shortening flow.

Here we follow section §2 of [7]. Let $(M, \rho)$ be an oriented riemannian surface. Let $S^1 = \mathbb{R}/\mathbb{Z}$. For an embedding $\gamma : S^1 \hookrightarrow M$, let $\nu_\gamma$ be its positively oriented normal vector field and let $k_\gamma$ be the curvature of $\gamma$. Let $\text{Emb}(S^1, M)$ be the space of smooth embedded circles in $M$ endowed with the $C^\infty$ topology. Let

$$L(\gamma) = \int_{S^1} \|\dot{\gamma}\|_\rho$$

be the length functional. The curve shortening flow is a continuous map

$$\mathcal{U} \longrightarrow \text{Emb}(S^1, M), \quad (s, \gamma_0) \mapsto \Psi_s(\gamma_0) =: \gamma_s,$$

defined on a maximal open neighborhood $\mathcal{U} \subset [0, \infty[ \times \text{Emb}(S^1, M)$ of $\{0\} \times \text{Emb}(S^1, M)$ by the following PDE:

$$\partial_s \gamma_s = k_{\gamma_s} \nu_{\gamma_s}.$$

The following properties are proved in [19], [11]:

(i) $\Psi_0 = \text{id}$ and $\Psi_s \circ \Psi_t = \Psi_{s+t}$ for all $s, t \geq 0$.

(ii) $\Psi_s(\gamma \circ \theta) = \Psi_s(\gamma) \circ \theta$ for all $\gamma \in \text{Emb}(S^1, M)$ and $\theta \in \text{Diff}(S^1)$.

(iii) $\frac{d}{ds} L(\psi_s(\gamma)) \leq 0$ for all $\gamma \in \text{Emb}(S^1, M)$, with equality if and only if the image of $\gamma$ is a geodesic.

(iv) Given $\gamma \in \text{Emb}(S^1, M)$ let $s_\gamma = \sup \{ s \geq 0 \mid (s, \gamma) \in \mathcal{U} \}$. Then $s_\gamma$ is finite if and only if $\Psi_s(\gamma)$ converges to a constant when $s \to s_\gamma$.

A path-connected subset $U \subset M$ is weakly convex if for any pair $x, y \in U$ that can be joined by an absolutely continuous curve in $U$ of length smaller than the injectivity radius $\text{inj}(M, g)$, the shortest geodesic joining $x$ and $y$ is contained in $U$. Another useful property of $\Psi_s$ is that it preserves weakly convex sets, namely

(v) If $U \subset M$ is weakly convex then

$$\gamma \in \text{Emb}(S^1, U) \quad \Rightarrow \quad \forall s \in [0, s_\gamma[ \quad \psi_s(\gamma) \in \text{Emb}(S^1, U).$$

This flow is used in [7] to prove the following lemmata.

4.1. Lemma ([7] lemma 2.1).

Let $U \subseteq M$ be a weakly convex subset that is not simply connected. Let $C \subset \text{Emb}(S^1, U)$ be a connected component containing loops that are non-contractible in $U$. Then, there exists a sequence $\gamma_n \in C$ converging in the $C^2$-topology to a simple closed geodesic $\gamma$ contained in $\overline{U}$ of length

$$L(\gamma) = \inf_{\zeta \in C} L(\zeta) > 0.$$
4.2. Lemma ([7] lemma 2.2).

If \( U \subset M \) is weakly convex and \( K \subset SM \) is invariant by the geodesic flow (i.e. \( \forall t \in \mathbb{R} \ \phi_t(K) = K \)) and such that \( \pi(K) \subset U \), then any path-connected component of \( U \setminus \pi(K) \) is weakly convex.

A closed geodesic \( \gamma : S^1 \to M \) is called a waist when any absolutely continuous curve \( \zeta \) which is sufficiently \( C^0 \)-close to \( \gamma \) satisfies \( L(\zeta) \geq L(\gamma) \). By the argument before (45), non degenerate waists are positive hyperbolic and have no conjugate points.

4.3. Lemma.

A simple nondegenerate closed geodesic \( \gamma \) is a waist if and only if it has no conjugate points.

Proof:

Suppose that \( \gamma \) is nondegenerate and has no conjugate points, we prove that it is a waist. The converse is standard. Consider the geodesic lagrangian \( L : TM \to \mathbb{R} \) and hamiltonian \( H : T^*M \to \mathbb{R} \)

\[
L(x,v) = \frac{1}{2} |v|^2_x, \quad H(x,p) = \sup_{v \in T_x M} \{ p(v) - L(x,v) \}, \quad H(x,p) = \frac{1}{2} |p|^2_x.
\]

The Legendre transform \( L(x,v) = \langle v, \cdot \rangle_x \) conjugates the geodesic flow to the hamiltonian flow of \( H \) on the energy level \( H \equiv \frac{1}{2} \). Also \( L(\pi^{-1}\{x\}) = \pi^{-1}\{x\} \) identifies the vertical fibers. Observe that \( \gamma \) must be positive hyperbolic. Since (45) holds in the hamiltonian flow there is a neighborhood \( U \) of \( \gamma \) where \( W^s(\gamma) \subset T^*M \) is a graph:

\[
T^*U \cap W^s(\gamma) = \{ (x, \omega(x)) \in T^*_x M : x \in U \}.
\]

Then \( \omega \in \Lambda^1(U) \) is a 1-form on \( U \) which is closed because \( W^s(\gamma) \) is a lagrangian submanifold. And \( (dx \wedge dp)|_{W^s(\gamma)} \equiv 0 \) because \( W^s(\gamma) \) is tangent to the Reeb vector field of \( \langle H^{-1}\{\frac{1}{2}\}, p \ dx \rangle \).

Since \( H(x, \omega(x)) \equiv \frac{1}{2} \), equation (46) implies that

\[
\forall(x,v) \in TM \quad \omega(x)(v) \leq L(x,v) + \frac{1}{2}.
\]

For \( x \in \gamma \), we have that \( \omega(x) = L(x, \dot{x}) = \langle \dot{x}, \cdot \rangle_x \). Therefore \( \omega(\gamma) \cdot \dot{\gamma} \equiv 1 \).

Let \( \eta \) be an absolutely continuous curve \( C^0 \) close to \( \gamma \) in \( U \) parametrized by arc length. Then \( L(\eta, \dot{\eta}) \equiv \frac{1}{2} \) and

\[
L(\gamma) = \int_\gamma \omega = \int_\eta \omega \leq \int_\eta L + \frac{1}{2} = L(\eta),
\]

where the second inequality holds because \( \eta \) is homotopic to \( \gamma \) inside \( U \).
We need the following min-max lemma. These geodesics have conjugate points because minimax critical points cannot be local minima.

4.4. Lemma ([7] lemma 2.4).

Let \((M, \rho)\) be an orientable riemannian surface.

(i) If \(A \subset M\) is an annulus bordered by two waists, then \(\text{int}(A)\) contains a non contractible simple closed geodesic with conjugate points.

(ii) If \(D \subset M\) is a compact disk bounded by a waist, then \(\text{int}(D)\) contains a simple closed geodesic with conjugate points.

4.5. Lemma ([11] lemma 5.9).

Let \((M, \rho)\) be a riemannian surface, and \(\gamma: [-T, T] \to M\) a geodesic arc parametrized with unit speed whose interior \(\gamma|_{[-T,T]}\) contains a pair of conjugate points. Then there exists an open neighborhood \(U \subset SM\) of \((\gamma(0), \dot{\gamma}(0))\) such that, for each \((x, v) \in SU\), the geodesic \(\zeta(t) = \exp_x(tv)\) intersects \(\gamma\) for some \(t \in [-T, T]\).

Lemma 4.5 implies the following corollary:

4.6. Corollary.

Let \((M, \rho)\) be an orientable riemannian surface and \(\gamma\) a simple closed geodesic with conjugate points.

(i) There exists \(T > 0\) and an open neighborhood \(V \subset SM\) of the lift \(\dot{\gamma}\) such that, for each \(z \in V\), the geodesic \(\zeta(t) := \pi \circ \phi_t(z)\) intersects \(\gamma\) on some positive time \(t_1 \in [0, T]\) and some negative time \(t_2 \in [-T, 0]\).

(ii) There exists \(T > 0\) and an open neighborhood \(U \subset M\) of \(\gamma\) such that, for each \(z \in SU\), the geodesic arc \(\zeta|_{[-T, T]}\) intersects \(\gamma\) on some \(t \in [-T, T]\).

A geodesic polygon in a riemannian surface is a simple closed curve which is a union of finitely many distinct geodesic arcs that is not one closed geodesic. Observe that necessarily the geodesic arcs are transversal. Therefore we have

4.7. Remark.

(i) If \(P\) is a geodesic polygon then there exist a neighborhood \(V \subset SM\) of the lift \(\dot{P}\) and \(\ell > 0\) such that for every \(z \in V\) and \(\zeta(t) := \pi \phi_t(z)\), both geodesic arcs \(\zeta|_{[0, \ell]}, \zeta|_{[-\ell, 0]}\) intersect \(P\).

(ii) If \(P\) is a geodesic polygon there exists a neighborhood \(U \subset M\) of \(P\) and \(\ell > 0\) such that for every \(z \in SU\) and \(\zeta(t) := \pi \phi_t(z)\), the geodesic arc \(\zeta|_{[-\ell, \ell]}\) intersects \(P\).
4.3. Complementary Birkhoff annuli.

In this section we obtain a complete system of surfaces of sections for \((M, \rho)\) provided that all waists are nondegenerate (i.e. hyperbolic). This is done by adding disjoint Birkhoff annuli to the surfaces obtained in section 4.1. The Birkhoff annuli have genus 0, so for them we don’t need to check the condition in theorem B.(3) on the number of intersections of the separatrices.

In fact some are Birkhoff annuli of waists which are in \(K_{fix}\) and other are Birkhoff annuli of minimax orbits which have index 1 and are in \(K_{rot}\). If these minimax orbits are hyperbolic, then their Floquet multipliers are negative and their invariant subspaces \(E^s, E^u\) intersect the vertical bundle \(V = \ker d\pi, \pi : TM \to M\), twice along one period. So each local invariant manifold \(W^s(\gamma), W^u(\gamma)\) intersects each Birkhoff annuli \(A(\dot{\gamma}), A(-\dot{\gamma})\) only once.

We prove the following.

4.8. Theorem.

Let \((M, \rho)\) be an orientable riemannian surface of genus \(g\) with all its waists non-degenerate. There are a finite number of surfaces of section \(\Sigma_1, \ldots, \Sigma_{2n}\) such that

(a) If \(g = 0\) then \(\Sigma_1, \Sigma_2\) are the Birkhoff annuli of a minimax simple closed geodesic.
(b) If \(g > 0\), \(\Sigma_1, \Sigma_2\) are the surfaces of genus 1 and \(4G + 4\) boundary components described in subsection 4.1.
(c) \(\Sigma_3, \ldots, \Sigma_{2n}\) are Birkhoff annuli of \(n - 1\) mutually disjoint simple closed geodesics.
(d) \(\Sigma_3, \ldots, \Sigma_{2n}\) are disjoint from \(\Sigma_1, \Sigma_2\).
(e) Every geodesic orbit intersects \(\Sigma_1 \cup \cdots \cup \Sigma_{2n}\).
(f) Let \(K_{fix}\) be the union of the set of closed orbits without conjugate points in \(\cup_{i=3}^{2n} \partial \Sigma_i\) and let \(K_{rot} = \cup_{i=1}^{2n} \partial \Sigma_i \setminus K_{fix}\). There are \(0 < \ell < \infty\) and a neighborhood \(U\) of \(K_{rot}\) in \(SM\) such that

\[
\forall z \in U \quad \phi_{[0, \ell]}(z) \cap \Sigma \neq \emptyset \quad \& \quad \phi_{[-\ell, 0]}(z) \cap \Sigma \neq \emptyset, \quad \Sigma := \cup_{i=1}^{2n} \Sigma_i.
\]

(g) If \(\gamma\) is a geodesic with \(\hat{\gamma}([0, +\infty]) \cap \Sigma = \emptyset\) then \(\hat{\gamma}(t) \in W^s(z_t)\) for some \(z_t \in K_{fix}\).
(h) If \(\gamma\) is a geodesic with \(\hat{\gamma}([-\infty, 0]) \cap \Sigma = \emptyset\) then \(\hat{\gamma}(t) \in W^u(z_t)\) for some \(z_t \in K_{fix}\).

The following proposition is proved in lemmas 3.8 and 3.7 in [7], using examples 3.2 and 3.3 in [7].

4.9. Proposition ([7] lemmas 3.8, 3.7).

Let \((M, \rho)\) be a riemannian surface and let \(D \subset M\) be a simply connected open set whose boundary \(\partial D = P\) is a geodesic polygon or a simple closed geodesic with conjugate points. Suppose that every simple closed geodesic without conjugate points contained in \(D\) is non-degenerate. Then every collection of mutually disjoint simple closed geodesics contained in \(D\) is finite.
These are examples of a decomposition of a bowl and a corset in lemma 4.10 when there is a new invariant subset projecting in its interior. The star \( \star \) marks a point in the projection \( \pi \Lambda \) of the new invariant subset and determines the homotopy class of the new waist.

A corset \((A, w)\) in \((M, \rho)\) is an annulus \(A \subset M\) such that \(\text{int}(A)\) contains a simple closed geodesic \(w\) which is a waist and that the boundary components of \(\partial A\) are either a polygon or a simple closed geodesic with conjugate points. A bowl is a disk \(D \subset M\) whose boundary \(\partial D\) is either a geodesic polygon or a simple closed geodesic with conjugate points. We further require that corsets and bowls are connected components of the complement of finitely many geodesics. Observe that by lemma 4.2, corsets and bowls are weakly convex.

4.10. Lemma.

(1) If \((A, w_1)\) is a corset, \(U = \text{int}(A \setminus w_1)\) and \(\cap_{t \in \mathbb{R}} \phi_{-t}(SU) \neq \emptyset\); then there are two corsets \((A_1, w_1), (A_2, w_2)\) with \(A = A_1 \cup A_2\) and \(A_1 \cap A_2 = \partial A_1 \setminus \partial A\), \(i = 1, 2\).

(2) If \(D\) is a bowl, \(V = D \setminus \partial D\) and \(\cap_{t \in \mathbb{R}} \phi_{-t}(SV) \neq \emptyset\); then there is a corset \((A, w)\) and a bowl \(B\) such that \(D = A \cup B\) and \(A \cap B = \partial B = \partial A \setminus \partial D\).

Proof:

(1). Write \(\Lambda := \cap_{t \in \mathbb{R}} \phi_{-t}(SU)\). By corollary 4.6.(ii) and remark 4.7.(ii) there is a neighborhood \(N\) of \(\partial A\) such that \(\Lambda \cap SN = \emptyset\). Since \(U \cap w_1 = \emptyset\) we have that \(\pi \Lambda \subset w_1\). By lemma 4.2 any path-connected component of \(A \setminus (w_1 \cup \pi \Lambda)\) is weakly convex. Let \(a_2\) be the connected component of \(\partial A\) which is included in a connected component of \(A \setminus w_1\) which intersects \(\pi \Lambda\). Let \(a_1\) be the other component of \(\partial A\). Let \(W\) be the connected component of \(A \setminus (w_1 \cup \pi \Lambda)\) which contains \(a_2\). Observe that \(a_2\) is not homotopic to \(w_1\) in \(W\).

Let \(x \in \pi \Lambda \setminus w_1\), and \(\varepsilon > 0\) with \(d(x, w_1) > \varepsilon\). We claim that if a closed curve \(\gamma \subset W\) is homotopic to \(a_2\) inside \(W\) then there is \(y_\gamma \in \gamma\) such that \(d(y_\gamma, w_1) \geq \varepsilon\). For if

\[
\gamma \subset B(w_1, \varepsilon) := \{ z \in A : d(z, w_1) < \varepsilon \},
\]

then \(\gamma\) is homotopic to \(w_1\) inside \(A \setminus \{x\}\). Thus \(\gamma\) is non homotopic to \(a_2\) inside \(A \setminus \{x\}\).

Then \(\gamma\) is non homotopic to \(a_2\) inside \(W \subset A \setminus \{x\}\). A contradiction. Consequently, if \(\eta\) is a \(C^0\) limit of curves \(\gamma_n \subset W\) homotopic to \(a_2\) inside \(W\), then

\[
\eta \neq w_1.
\]
Let \( C \) be the connected component of \( \text{Emb}(S^1, W) \) containing a curve homotopic to \( a_2 \). By lemma 4.1 there is a sequence \( \gamma_n \in C \) converging in the \( C^2 \) topology to a simple closed geodesic \( w_2 \) in \( \overline{W} \) of length \( L(w_1) = \inf_{\zeta \in C} L(\zeta) > 0 \). Then \( w_2 \) is a waist and by (47), \( w_2 \neq w_1 \). Since \( w_1, w_2 \) are waists in the same homotopy class in \( A \), we have that \( w_1 \cap w_2 = \emptyset \) (c.f. [13, Prop. 1.7]).

Then \( w_2 \subset \overline{W} \subset A \), if \( w_2 \cap a_2 \neq \emptyset \) then \( w_2 \) and \( a_2 \) would be tangent geodesics (segments) and hence the same geodesic. But this is not possible because \( a_2 \) has conjugate points or is a polygon and \( w_1 \) is a waist.

In the annulus \( A \) the curves \( a_1, w_1, w_2, a_2 \) are all disjoint and homotopic. There is an annulus \( A(w_1, w_2) \) in \( A \) with boundaries \( w_1 \) and \( w_2 \). By lemma 4.4.(i) there is a non contractible simple closed geodesic \( h \) with conjugate points in \( \text{int}(A(w_1, w_2)) \). In particular \( h \) is disjoint and homotopic to \( w_i, a_i, i = 1, 2 \). Denote the annuli \( A_1 := A(a_1, h), A_2 := A(h, a_2) \) with boundaries \( (a_1, h), (h, a_2) \) respectively. Then \( (A_1, w_1), (A_2, w_2) \) are the desired corsets.

(2). Write \( \Lambda := \cap_{t \in \mathbb{R}} \phi_t(SV) \). By corollary 4.6.(ii) and remark 4.7.(ii) there is a neighborhood \( N \) of \( \partial D \) such that \( \Lambda \cap SN = \emptyset \). Let \( W \) be the connected component of \( D \setminus \pi \Lambda \) which contains \( \text{int}(N) \). Observe that \( \partial D \) is non-contractible in \( \overline{W} \). Let \( C \) be a connected component in \( \text{Emb}(S^1, W) \) containing curves homotopic to \( \partial D \). By lemma 4.1 applied to \( C \), there is a waist \( w_1 \) in \( \overline{W} \). The waist \( w_1 \) bounds a disk \( D_1 \) in \( D \). By lemma 4.4.(i) there is a simple closed geodesic \( h_1 \) in \( \text{int}(D_1) \) with conjugate points. Let \( B_1 \) be the disk in \( D \) with \( \partial B_1 = h_1 \). Let \( A_1 := A(\partial D, h_1) \) be the annulus with boundary \( \partial D \cup h_1 \). Then \( B_1 \) is a bowl and \( (A_1, w_1) \) is a corset with disjoint interiors and \( D = A_1 \cup B_1 \) as required.

\[ \square \]

**Proof of theorem 4.8:**

If \( M = S^2 \) let \( \Sigma_1, \Sigma_2 \) be the Birkhoff annuli of a simple closed minimax geodesic \( m \) and let \( R_1, R_2 \) be the two disks bounded by \( m \). Otherwise let \( R_1, \ldots, R_4 \) be the disks in \( M \) and \( \Sigma_1, \Sigma_2 \) be the surfaces of section of genus 1 obtained in subsection 4.1.

Given and open subset \( V \subset SM \) define the **forward trapped set** \( \text{trap}_+(V) \) and the **backward trapped set** \( \text{trap}_-(V) \) as

\[ \text{trap}_\pm(V) = \{ z \in SM : \exists \tau \quad \forall t > \tau \quad \phi_{\pm t}(z) \in V \}. \]
4.11. CLAIM.

For each $i = 1, \ldots, \{2, 4\}$ there are finitely many corsets $(A_1^i, w_1^i), \ldots, (A_m^i, w_m^i)$ and a bowl $B_m^i$ with disjoint interiors such that $R_i = A_1^i \cup \cdots \cup A_m^i \cup B_m^i$ and letting

$$K_i := \partial R_i \cup \bigcup_{j=1}^{m_i} (\partial A_j^i \cup w_j^i) \cup \partial B_m^i,$$

$$\gamma \text{ geodesic } \gamma(0) \in R_i \implies \gamma(\mathbb{R}) \cap K_i \neq \emptyset,$$

$$\text{trap}_\pm S U_i \subset \bigcup_{j=1}^{m_i} W^{s,u}(\dot{w}_j^i) \cup W^{s,u}(-\dot{w}_j^i), \quad U_i := R_i \setminus K_i.$$

Assume claim 4.11 holds. Let $\Sigma_3, \ldots, \Sigma_{2n}$ be the collection of the two Birkhoff annuli of the geodesics in $K_i \cap \text{int } R_i$ whenever $\text{trap}_\pm S(R_i \setminus \partial R_i) \neq \emptyset$. Then 4.8.(a)-(d) hold. Also (49) implies 4.8.(e). By (50) we have that

$$\text{trap}_\pm (SM \setminus \bigcup_{j=1}^{2n} \Sigma_j) \subset \bigcup_{i=1}^{\{2, 4\}} \bigcup_{j=1}^{m_i} W^{s,u}(\dot{w}_j^i) \cup W^{s,u}(-\dot{w}_j^i).$$

This implies 4.8.(g) and 4.8.(h).

We have that $K_{fix} = \bigcup_{i,j} \{\dot{w}_j^i, -\dot{w}_j^i\}$ and $K_{rot} = \bigcup_{i=1}^{2n} \partial \Sigma_i \setminus K_{fix}$. Since the orbits in $K_{rot}$ are either in a polygon or have conjugate points, corollary 4.6.(i) and remark 4.7.(i) imply 4.8.(f).

We now prove claim 4.11. Observe that the disks $R_1, \ldots, R_{\{2, 4\}}$ are bowls. Recall that the surfaces $\Sigma_1, \Sigma_2$ can be constructed inside an arbitrarily small neighborhood of $S(\bigcup_{i=1}^{\{2, 4\}} \partial R_i)$.

If $\cap_{t \in \mathbb{R}} \phi_t(S(\text{int } R_i)) \neq \emptyset$, by lemma 4.10.(2) we can add a corset $(A_1, w_1)$ and a bowl $B_1$ with $R_i = A_1 \cup B_1$, $A_1 \cap B_1 = \partial B_1 = \partial A_1 \setminus \partial R_i =: b_1$. Observe that $\{\dot{w}_1, b_1\}$ is a set of pairwise disjoint simple closed geodesics in $R_i$. Inductively, suppose we have corsets
Here the closed simple geodesics $R \cap (51)$ continue as long as
where $(A_j, w_j)$ are corsets and $B_{m+1}$ is a bowl, all with disjoint interiors. The process can continue as long as
\begin{equation}
\cap_{t \in \mathbb{R}} \phi_t(SU) \neq \emptyset, \quad U = \cup_{j=1}^{m+1} (\text{int } A_i - w_j) \cup \text{int } B_{m+1}.
\end{equation}

Here the closed simple geodesics $\{w_j\}_{j=1}^{m+1}$ are mutually disjoint waists and contained in $R_i$. By proposition 4.9 this process must stop. Then for each $R_i, i = 1, \ldots, \{2, 4\}$ there is $m =: m_i - 1$ for which condition (51) does not hold. This implies (49).

Let $U_i = R_i \setminus K_i$ be from (50). Suppose that $z \in \text{trap}_+(SU_i)$ then its $\omega$-limit
\[
\omega(z) := \bigcap_{t > T} \bar{\phi}_{[T, +\infty]}(z)
\]
is an invariant set with projection $\pi(\omega(z)) \subset \overline{U_i}$. Since condition (51) does not hold for $U = U_i$ we have that $\pi(\omega(z)) \cap U_i = \emptyset$. Therefore $\pi(\omega(z))$ is a connected component of $K_i$ in (48). By corollary 4.6(i) and remark 4.7(i) the forward orbit of $z$ can not approach the boundary orbits in $S(\partial A_j)$ or $SB_m$ without intersecting $SK_i$. Thus $\pi(\omega(z)) = w_j^i$ for some $1 \leq j \leq m_i$. Since $\omega(z)$ is invariant, this implies that $\omega(z) = \pm \tilde{w}_j^i$. This proves (50).

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