Abstract

In this paper we revisit the instability problem of the relativistic fluid equations, when coupled with the heat-flux constitutive relation arising within the Chapman-Enskog approximation. By considering general perturbations, we report on generic instabilities and ill-posedness of the system when the dynamics is referred to any boosted frame, unlike the comoving frame case previously noticed. Motivated by these results, we discuss the viability of Chapman-Enskog’s constitutive relation to emerge from first order divergence-type fluid theories (DTT’s), which have been shown to be ill-posed in the strongest sense. We find that the most general first-order DTT does not naturally lead to the corresponding constitutive equation, unless some considerations from local-equilibrium configurations are set up.
1 Introduction

The Chapman-Enskog (CE) method consists of a formal expansion procedure on the distribution function under the assumption that the characteristic (macroscopic) length of the gradients of state variables is large when compared to the (microscopic) mean free path of the molecules (see [1, 2, 3] for further references). This method is considered the most successful one in order to rigorously establish the transport coefficients of a dilute (but still not rarefied) gas of classical monoatomic molecules with no internal degrees of freedom. This scheme has been successfully extended to relativistic gases, even though it has been argued that its first solution— which corresponds to the Navier-Stokes regime— leads to unphysical behaviour of the corresponding fluctuations. However, recent numerical studies point to this method as the most appropriate one for particular systems; for example, the ultrarelativistic electron gas (see for example [4]).

One of the issues that arise when considering the CE procedure is the stability of the resulting dissipative relativistic transport equations. In [5], Hiscock and Lindblom carried out a stability analysis for the equilibrium states of a relativistic fluid theory, by considering a heat conduction law such that the hydrodynamic acceleration is considered a heat driving force. Such analysis was carried out over a flat background with respect to $L^2$ perturbations, and the spatial Fourier modes given by plane waves whose frequencies are the roots of the corresponding dispersion relation. They showed that such an equation only leads to growing modes, and thus the solutions are unbounded on successive space-like surfaces, even with very short characteristic growth times; namely, up to $\tau \sim 10^{-34}$ sec. Nevertheless, the authors examined the stability of the corresponding equilibrium solutions with respect to Eckart’s frame, only in the case where a Fourier transform is well-defined by considering plane-wave solutions in the fluid’s co-moving frame. Since unstable modes are found both in the transverse and longitudinal directions, the study of the general case is only carried out afterwards, by considering the Landau’s frame.

However, when the CE procedure is carried out in a complete fashion, the heat flux in a relativistic fluid has been shown to be driven by a density gradient instead of the acceleration term mentioned above. Moreover, it has been recently shown that using such constitutive equation, the generic instability found in [5] is avoided with respect to the fluid’s frame [6, 7]. Although the formalism corresponds to Eckart’s $(3+1)$-decomposition, that
result comes about by establishing the corresponding constitutive equation, using relativistic kinetic theory and choosing $T$, $n$, and $u^\mu$ as state variables (instead of considering the four–acceleration $\dot{u}^\mu$).

These results seem to suggest that CE procedure gives rise to well-behaved and generically stable equations under linear perturbations. In the spirit of providing a concise proof of such a conjecture, in this paper we review the stability problem of the relativistic hydrodynamic equations, coupled with the CE constitutive equation for the heat flux. In particular, both the decoupling and decay of the transverse mode as well as the damping of the longitudinal modes are obtained in a general fashion without any approximation in the fluid’s frame. It is worthwhile to mention at this point that in [6], the curl of the spatial components of the momentum balance equation was calculated in order to decouple the transverse mode. Also, in [7], Mountain’s approximate method is used in order to examine the longitudinal modes.

Although the corresponding perturbations of the CE system of equations are stable from the fluid’s reference frame, we show that this stability disappears when studying linear perturbations with respect to more general (boosted) frames, which are essential for example in the description of rotational fluids. Moreover, we show that the system is ill-posed in this general case, as short wavelength fluctuations can grow arbitrarily fast.

In view of these results, it is interesting to ponder whether the CE closure equation for the heat flux can be framed within Geroch’s first order DTT’s [8,9]. Indeed, it is well known that such theory can be implemented in order to show that any first-order dissipative relativistic fluid theory that can be cast in a particular way as to fit in that prescribed structure (namely, all its dynamical equations can be written as total-divergence differential equations) does not constitute a well-posed initial-value problem, since it fails to satisfy the stability and causality properties that such a physical theory is expected to (for references, see [10,11,12,13]). For instance, the well-know Landau-Lifshitz [14] and Eckart’s [15] fluid theories for relativistic dissipative fluids fall into the DTT group and have been shown to fail Geroch’s criterium [5].

The last ingredient in the current discussion is, thus, the analysis of the CE equation from the point of view of divergence-type fluid theories. We show that, following a similar proposal for a kinetic theory based first-order DTT carried out in [16], it is not possible to recover the corresponding constitutive equation, unless some hypothesis of the CE procedure is introduced within the DTT program. As a result we conclude that first-order Chapman-
Enskog procedure almost fits within Geroch’s formulation and thus, if the assumption mentioned above is considered, can be shown to lead to an ill-posed initial value problem. However, it is key to emphasize that such pathology is only present if no single hypersurface can be chosen for the $(3+1)$–decomposition in the fluid’s frame. Indeed, if the fluid four-velocity forms orthogonal surfaces, the theory is stable and causal, as has been already shown in [6, 7].

In order to address the points described above, the rest of this paper is organized as follows. In Section 2, we review the proof that Eckart’s first order theory for dissipative fluids leads to growing modes. Section 3 is devoted to a general treatment of the transport equations considering the Chapman-Enskog constitutive equation for the heat flux and to showing that no instabilities are present in the co-moving frame but, however, by choosing a general (boosted) frame, exponential growing fluctuations may appear and the problem is shown to be ill-posed. In Section 4, the discussion of obtaining the CE heat flux constitutive equation from first-order DTTs is addressed. Finally, a thorough discussion of our results and final remarks are included in Section 5.

Throughout this work, we will consider the ($-,-,+,+)$ signature for the spacetime background metric. Also, we will adopt a system of units such that $c = m = k_B = 1$, where $c$ is the speed of light in vacuum, $m$ is the molecule’s mass and $k_B$ is the Boltzmann’s constant.

## 2 Generic instabilities in Eckart’s framework

The heat flux constitutive equation that arises in Eckart’s first order theory is established from phenomenological grounds, by imposing linear relations for the dissipative fluxes and ensuring the second law is satisfied. The term “first-order” arises from the fact that the entropy production is written as a positive semi-definite quantity, by imposing first order corrections to equilibrium fluxes. In such framework, the heat flux law reads

$$ q^\mu = -\kappa h^{\mu\nu} \left( \frac{\nabla_\nu T}{T} + u^\alpha \nabla_\alpha u_\nu \right), $$

where $h_{\mu\nu} := \delta_{\mu\nu} + u^\mu u_\nu$ is the orthogonal projector to the space perpendicular to $u^\mu$ at each event. At this order, the entropy production through
shear and heat dissipation is shown to be quadratic in them, and since in equilibrium entropy must vanish, then all dissipative fluxes must also vanish.

By perturbing the fluid equations around an arbitrary equilibrium state, we get the following set of equations:

$$\nabla \mu \delta N^\mu = 0 \quad (2)$$

$$\nabla \mu \delta T^{\mu \nu} = 0 \quad (3)$$

$$\delta q^\mu = -\kappa h^{\mu \nu} \left( \frac{\nabla \nu \delta T}{T} + \delta u^\alpha \nabla_\alpha u_\nu + u^\alpha \nabla_\alpha \delta u_\nu \right) \quad (4)$$

where

$$\delta T^{\mu \nu} = \delta (n \varepsilon) u^\mu u^\nu + 2(n \varepsilon + p) u^{(\mu} \delta u^{\nu)} + \delta p h^{\mu \nu} + 2 u^{(\mu} \delta q^{\nu)}, \quad (5)$$

and

$$\delta N^\mu = \delta nu^\mu + n \delta u^\mu. \quad (6)$$

For a classical relativistic ideal gas (see [2] for example), the thermodynamic identities

$$p = nT,$$

and

$$n \varepsilon = nm \left( 3z + \frac{K_1 (\frac{1}{z})}{K_2 (\frac{1}{z})} \right),$$

lead to the relation

$$n \varepsilon + p = nm \mathcal{G} (z), \quad (7)$$

were $z = T$ (in this units) and

$$\mathcal{G} (z) = \frac{K_3 (\frac{1}{z})}{K_2 (\frac{1}{z})}, \quad (8)$$

where $K_n (x)$ stands for the $n$-th modified Bessel function (see also Kremer, [2]). Introducing the heat capacity

$$c_n := \left( \frac{\partial \varepsilon}{\partial T} \right)_n, \quad (9)$$
the perturbed energy-momentum tensor reads

$$\delta T^{\mu\nu} = (nc_n\delta T + \varepsilon\delta n)u^\mu u^\nu + 2nmG(z)u^{(\mu}\delta u^{\nu)} + (n\delta T + T\delta n)h^{\mu\nu} + 2u^{(\mu}\delta q^{\nu)}. \tag{10}$$

Thus, the set of dynamical equations for the perturbations can be written as

$$M^{AB}\delta Y^B = 0, \tag{11}$$

with

$$\delta Y^B = \{\delta T, \delta n, \delta u_x, \delta q_x, \delta u_y, \delta q_y, \delta u_z, \delta q_z\} \tag{12}$$

By assuming plane-wave solutions in the co-moving frame given by

$$\delta Q = \delta Q_0 \exp(ikx + st), \tag{13}$$

the matrix $M^{AB}$ can be written in a diagonal form (see Appendix A for the details):

$$M^{AB} = \begin{pmatrix} Q & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}, \tag{14}$$

where the subsidiary matrices $Q$ and $R$ are given by

$$Q = \begin{pmatrix} 0 & s & ink & 0 \\ nc_n s & 0 & nTik & ik \\ nik & Tik & nmG(z)s & s \\ \frac{ik}{T} & 0 & s & \frac{1}{\kappa} \end{pmatrix} \tag{15}$$

and

$$R = \begin{pmatrix} nG(z)s & s \\ s & \frac{1}{\kappa} \end{pmatrix}. \tag{16}$$

The matrix $Q$ is associated with the “longitudinal” modes (parallel to the direction of the perturbation), while the matrix $R$ is associated to the “transverse” ones. The dynamics of the fluctuations depend on the nature of the roots of the dispersion equation

$$\det(M^{AB}) = 0, \tag{17}$$

where

$$\det(M^{AB}) = (\det Q)(\det R)^2.$$
The condition \( \det(R) = 0 \) yields
\[
s^2 - \frac{nG(z)}{\kappa} s = 0, \quad (18)
\]
whose positive real root leads to a growing mode. On the other hand,
\[
\frac{\det(Q)}{c_n} = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4, \quad (19)
\]
where
\[
\begin{align*}
a_1 &= -\frac{nG(z)}{\kappa}, \\
a_2 &= \frac{n k^2}{c_n}, \\
a_3 &= -\frac{n z k^2}{\kappa} \left( 1 + \frac{G(z)}{zc_n} \right), \\
a_4 &= -\frac{k^4}{c_n}.
\end{align*}
\]
Clearly, since \( a_4 < 0 \), the dispersion relation has at least one positive real root, which implies that at least one longitudinal mode is unstable.

3 Fluctuations in Chapman-Enskog solution

3.1 Dynamics in the co-moving frame

By following Chapman-Enskog’s procedure, it is possible to arrive at a constitutive equation which is written in terms of gradients of dynamical variables as follows [17]:
\[
q^{\mu} = -h^{\mu\nu} \left( \frac{\kappa \nabla_{\nu} T}{T} - \lambda \frac{\nabla_{\nu} n}{n} \right) \quad (20)
\]
while the corresponding fluctuations are written as
\[
\delta q^{\mu} = -h^{\mu\nu} \left( \frac{\kappa \nabla_{\nu} \delta T}{T} - \lambda \frac{\nabla_{\nu} \delta n}{n} \right) \quad (21)
\]
Using the plane-wave Ansatz for the perturbations used before, we straightforwardly get

\[
\begin{align*}
\delta q^1 &= -ik \left( \frac{\kappa}{T} \delta T - \frac{\lambda}{n} \delta n \right), \\
\delta q^2 &= \delta q^3 = 0.
\end{align*}
\]

Plugging them into the perturbed equation \( \nabla_\mu \delta T^{\mu\nu} = 0 \), we get the conditions

\[
\delta u^2 = \delta u^3 = 0. \tag{22}
\]

The subsidiary system of equations can thus be written as \( \tilde{M}^{AB} \delta \tilde{Y}^B = 0 \), where

\[
\delta \tilde{Y}^B = \{ \delta T, \delta n, \delta u_x, \delta q_x \}; \tag{23}
\]

\[
\tilde{M}^{AB} = \begin{pmatrix} 0 & s & ink & 0 \\
nc_n s & 0 & nTik & ik \\
nik & Tik & nG(z) & s \\
\frac{ik\kappa}{T} - \frac{ik\lambda}{n} & 0 & 1 \end{pmatrix}. \tag{24}
\]

In this case, the dispersion relation is a cubic polynomial, given by

\[
s^3 + \lambda k^2 b_1 s^2 + k^2 b_2 s + \lambda k^4 b_3 = 0, \tag{25}
\]

with

\[
\begin{align*}
b_1 &= \frac{1}{nG(z)} \left[ \frac{1}{c_n} \left( \frac{G(z)}{z} - 1 \right) + 1 \right], \\
b_2 &= \frac{z}{G(z)} \left( 1 + \frac{1}{c_n} \right), \\
b_3 &= \frac{1}{nG(z)c_n} \left( 1 + \frac{\kappa}{\lambda} \right).
\end{align*}
\]

Now, from the Boltzmann equation, it is possible to deduce the following identity:

\[
\frac{\kappa}{\lambda} = \frac{G(z)}{z} - 1, \tag{26}
\]

which directly implies that

\[
\begin{align*}
b_1 &= \frac{1}{nG(z)} \left[ c_n \left( \frac{G(z)}{z} - 1 \right)^2 + 1 \right], \\
b_3 &= \frac{1}{nzc_n}.
\end{align*}
\]
Since all coefficients \( b_n \) are positive, the Routh-Hurwitz criterion requires only one additional condition for stability, namely \( b_1 b_2 > b_3 \). In this case, we get
\[
\frac{b_1 b_2}{b_3} = \left( \frac{z}{\mathcal{G}(z)} \right)^2 (1 + c_n) (1 + c_n \chi^2) > 1 ,
\] (27)
where
\[
\chi = \frac{1}{c_n} \left( \frac{\mathcal{G}(z)}{z} - 1 \right).
\]
Using the equilibrium properties (for references, see [2])
\[
\frac{\mathcal{G}(z)}{z} > 4, \quad \frac{1}{3} < \frac{1}{c_n} < \frac{2}{3}
\] (28)
the last stability condition follows, and all the roots of the dispersion relation fall on the left quadrants of the complex plane. This finally shows that the Chapman-Enskog constitutive equation in the comoving frame does not lead to the generic instability reported in the previous section. It is worthwhile to notice at this point that the procedure here carried out does not require any approximations and is thus a definite proof of the stability of the theory when the constitutive equation [20] is considered and the hydrodynamic velocity forms surfaces.

### 3.2 Dynamics in a boosted frame: ill-posedness

The procedure carried out above is not completely general since the fluid’s motion may include rotation, in which case one cannot build space-like hyper-surfaces orthogonal to \( u^\mu \) everywhere. In order to address the general case, one considers a Fourier-Laplace transform in a more general frame, with a wave-like Ansatz for perturbations
\[
\delta Q = \delta Q_0 \exp (iK \bar{x} + S \bar{t}) ,
\]
where the corresponding Lorentz transformation to the new coordinates reads
\[
k = \gamma (K + ivS), \quad s = \gamma (S - ivK) .
\] (29)
Introducing Eq. (29) in Eq. (25) leads to a quartic complex polynomial, which can be written as
\[
\sum_{r=0}^{4} (iS)^r K^{3-r} (K \tau \mu(z) \alpha_r + \beta_r i) = 0 ,
\] (30)
where the coefficients are given by

\[
\begin{align*}
\alpha_0 &= -n \left( b_3 - b_1 v^2 \right), \\
\alpha_1 &= -n \left( 4b_3 - 2b_1 \left( 1 + v^2 \right) \right) v, \\
\alpha_2 &= n \left[ b_1 \left( 1 + v^2 \right)^2 + 2v^2 \left( b_1 - 3b_3 \right) \right], \\
\alpha_3 &= -n \left( 4b_3 v^2 - 2b_1 \left( 1 + v^2 \right) \right) v, \\
\alpha_4 &= -n \left( b_3 v^2 - b_1 \right) v^2, \\
\beta_0 &= \left( v^2 - b_2 \right) v, \\
\beta_1 &= 3v^2 - b_2 \left( 1 + 2v^2 \right), \\
\beta_2 &= \left( 3 - b_2 \left( 2 + v^2 \right) \right) v, \\
\beta_3 &= \left( 1 - b_2 v^2 \right), \\
\beta_4 &= 0.
\end{align*}
\]

Also, the coefficient \( \lambda \) has been expressed in terms of a characteristic microscopic time \( \tau \) as \( \lambda = -n \tau \mu(z) \), and the function \( \mu(z) \) includes all the temperature dependence of this transport coefficient (for further references, see [2,17]). By defining the dimensionless parameter \( \zeta = \tau/L \) (where \( L \) is the characteristic scale of the system), and the dimensionless variables \( \hat{K} := LK \) and \( \hat{S} := \tau S \), it is possible to rewrite Eq. (30) as

\[
\sum_{r=0}^{4} \left( i \hat{S} \right)^r \hat{K}^{4-r} \left( \hat{K} \zeta \gamma \mu(z) \alpha_r + \beta_r i \right) = 0. \tag{31}
\]

In this way, Eq. (31) can be solved numerically for specific values of the parameters \( \zeta, v \) and \( z \). Figures 1 and 2 show the positive real part of the roots of the dispersion relation as a function of \( K \) for some values of the corresponding parameters.

This dispersion relation in Eq. (31) can be easily analyzed by considering the case \( K = 0 \); that is, homogeneous fluctuations in the general frame. In such a case, one obtains

\[
S^3 \left[ \lambda \gamma v^2 \left( b_3 v^2 - b_1 \right) S + 1 - b_2 v^2 \right] = 0, \tag{32}
\]

which has a positive real root that can be written as

\[
S = -\frac{nz c_n}{\gamma \lambda v^2} \left[ \frac{\frac{g(z)}{z} - v^2 \left( 1 + \frac{1}{c_n} \right)}{v^2 \frac{g(z)}{z} - c_n - \left( \frac{g(z)}{z} - 1 \right)^2} \right]. \tag{33}
\]
Figure 1: Solutions of the dispersion relation in equation (30) with positive real part, for $\zeta = v = z = 0.1$.

Figure 2: Solutions of the dispersion relation in equation (30) with positive real part, for $\zeta = v = 0.5$ and $z = 0.7$. 
The term in parenthesis can be shown to be negative by using the properties given in Eq. (28), together with $v < 1$. The implication of this result is that, in any case of rotational fluids, the Chapman-Enskog closure relation leads to exponentially growing solutions.

On the other hand, in order to assess the well or ill-posedness of Eq. (31), notice that $\zeta$ is a small parameter, being the ratio between micro and macroscopic quantities, since one requires the characteristic wavelength of fluctuations to be large enough for collisions to take place within them. Because of this, when considering growing wave numbers $\hat{K}$, the product $\hat{K}\zeta$ still needs to be bounded. Taking this into account, one may look into the behaviour of the solutions $\hat{S}(\hat{K})$ of Eq. (31). Proposing $\hat{S} = S_0 + S_1 K + S_2 K^2$ and considering only higher order terms (with fixed $\hat{K}\zeta$), it is easy to see that as long as $\alpha_4 \neq 0$, one has $S_2 = 0$. This implies that solutions grow linearly with $\hat{K}$ as $\hat{S} = \epsilon \hat{K}$, where $\epsilon$ is given by the solution of

$$\sum_{r=0}^{4} (i\epsilon)^r \left( \hat{K}\zeta \gamma \mu(z) \alpha_r + \beta_r i \right) = 0. \quad (34)$$

We conclude, thus, that the system is not only unstable but also ill-posed, since in the high frequency limit ($\hat{K} \to \infty$), there are solutions which grow arbitrarily fast.

## 4 Linking Chapman-Enskog equation with first order DTTs

Motivated by the previous result, we now explore whether or not Eq. (20) emerges within some first-order DTT. In order to address this issue, we use some tools of kinetic theory with the aim of obtaining the most general constitutive tensor field for a first order divergence-type theory. With this, we show that the set of equations that arises when using Chapman-Enskog’s procedure can be cast as a first order DTT under a key assumption, and in this sense leads to an ill-posed theory for relativistic fluids.

First, we give a brief review about divergence-type theories, and then we compute the most general constitutive tensor for first-order DTT’s. In order to accomplish this, we consider the Jüttner equilibrium distribution
function in the mass-shell $p^\mu p_\mu = -1$. Finally, we recover the heat flux of the theory by considering the corresponding projection for the divergence of the constitutive tensor. We address the problem in both Eckart and Landau frames.

Roughly speaking, a set of dynamic equations is considered a divergence-type theory if it can be written as a set of equations on the divergence of the corresponding dynamical variables. For instance, any fluid theory which is governed by a set of conservation laws (particle number density, energy and momentum densities, etc) constitutes indeed a divergence-type theory. However, constitutive equations, which play a major role in the assessment of causality and hyperbolicity through Geroch’s criterion need to be also expressed in divergence form. In other words, an additional tensor containing the thermodynamic forces is needed, and the equation for its divergence should lead to the corresponding constitutive equations.

Formally speaking, DTT’s are the type of theories which satisfy the following three conditions:

(i) The dynamical variables are given by the energy-momentum tensor $T^{\mu\nu}$ of the fluid and the particle number current, $N^\mu$;

(ii) The dynamical equations are given by

\begin{align}
\nabla_\mu N^\mu &= 0 \quad (35) \\
\nabla_\mu T^{\mu\nu} &= 0 \quad (36) \\
\nabla_\mu A^{\mu\nu\sigma} &= I^{\nu\sigma} \quad (37)
\end{align}

Here, both the constitutive tensor $A^{\mu\nu\sigma}$ and the source tensor $I^{\nu\sigma}$ are algebraic functions of the dynamical variables $N^\mu$ and $T^{\mu\nu}$, and $I^{\nu\sigma}$ is symmetric and traceless.

(iii) There exists a four-current $S^\alpha$, which is also a local algebraic function of $T^{\mu\nu}$ and $N^\mu$ that satisfies, as a consequence of the dynamical equations,

$$\nabla_\alpha S^\alpha = \sigma,$$

with $\sigma \geq 0$ an algebraic function of $T^{\mu\nu}$ and $N^\mu$ and $\sigma = 0$ if and only if $I^{\nu\sigma} \equiv 0$.

\footnote{As in the rest of the paper, we have normalized the mass of each fluid component as $m \equiv 1$.}
These conditions are, by nature, phenomenological. However, the quantities $S$, $N^\mu$ and $T^{\mu\nu}$ can be easily identified as the entropy, particle flux and energy momentum tensor respectively, such that conditions (iii) and part of (ii) are satisfied. This serves as a link to the microscopic approach from which one can obtain such quantities as

$$
N^\mu = \int p^\mu f \, d\Omega; 
$$

$$
T^{\mu\nu} = \int p^\mu p^\nu f \, d\Omega; 
$$

$$
S^\alpha = \int p^\alpha \ln(f) \, d\Omega; 
$$

and the third rank constitutive tensor, whose divergence equation includes the constitutive relations, may be related to the third order moment by the following relation:

$$
A^{\mu\nu\sigma} = \int \left( p^\mu p^\nu p^\sigma + \frac{1}{4} p^\mu \eta^{\nu\sigma} \right) f \, d\Omega, 
$$

where the second term in parenthesis is included in order to enforce the trace-free condition

$$
\eta_{\nu\sigma} A^{\mu\nu\sigma} = 0.
$$

As an interesting consequence of conditions (i), (ii) and (iii), there exists a scalar generating function $\chi$ that depends on a new set of variables $(\xi, \xi^\mu, \xi^\mu_{\nu})$ such that

$$
N^\mu = \frac{\partial^2 \chi}{\partial \xi \partial \xi^\mu}, \quad T^{\mu\nu} = \frac{\partial^2 \chi}{\partial \xi^\mu \partial \xi^\nu}, 
$$

and

$$
A^{\mu\nu\sigma} = \frac{\partial^2 \chi}{\partial \xi_{\mu} \partial \xi_{\nu\sigma}} - \frac{\eta_{\alpha\beta}}{4} \frac{\partial^2 \chi}{\partial \xi^\mu \partial \xi^{\alpha\beta}} \eta^{\nu\sigma}. 
$$

The variable $\xi^\mu$ is associated with the degrees of freedom of the four-velocity of the fluid, $u^\mu$, while dissipative degrees of freedom are associated with those that come from variable $\xi_{\mu\nu}$, that is symmetric and trace-free. Moreover, the entropy creation through dissipation is given by $\sigma = -T^{\mu\nu} \xi_{\mu\nu}$.

For first-order theories, one should consider a generating function that is linear in $\xi_{\mu\nu}$. This implies that, by definition, the constitutive tensor $A^{\mu\nu\sigma}$
cannot depend on $\xi_{\mu\nu}$. In other words, it may only depend on $u^\mu$ and the metric tensor; namely,

$$A^{\mu\nu\sigma} = A_0 u^\mu u^\nu u^\sigma + A_1 u^\mu \eta^{\nu\sigma} + \left( \frac{A_0}{2} - 2A_1 \right) (\eta^{\mu\nu} u^\sigma + \eta^{\mu\sigma} u^\nu).$$

(44)

In order to compute the coefficients $A_i$, we consider the Jüttner distribution function for local equilibrium,

$$f^{(0)} = \frac{n}{4\pi^2 K_2 (\frac{1}{z})} \exp \left( \frac{p_\alpha u^\alpha}{z} \right),$$

(45)

from which we get (see Appendix B for a detailed calculation)

$$A_0 = n (1 + 6zG(z)),$$

(46)

and

$$A_1 = zG(z) + \frac{1}{4}.$$  

(47)

We recall that in Eq. (41), the local equilibrium distribution function is considered as a weight function for the third-order moment. This is due to the fact that, in the context of DTTs, the source $I^{\mu\nu}$ on the right hand side contains all dissipative fluxes, while the divergence on the left hand side has the corresponding driving forces. Since the latter are given in terms of the gradients of the state variables, in this case to first order, only local equilibrium quantities may appear in $A^{\mu\nu\sigma}$.

Now, we use the most general source tensor field that can be constructed as an algebraic function of the dynamical variables for first-order DTTs. In the Eckart’s frame, where

$$N^\mu = n u^\mu,$$

(48)

and

$$T^{\mu\nu} = (n\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} + 2u^\mu q^\nu + \pi^{\mu\nu},$$

(49)

we get

$$I^{\alpha\beta} = I_0 u^\alpha u^\beta + I_1 u^{(\alpha} q^{\beta)} + I_2 I^{\alpha\beta},$$

(50)

for some functions $I_0$, $I_1$ and $I_2$ in such a way that $\eta_{\mu\nu} I^{\mu\nu} = 0$ and $-\xi_{\mu\nu} I^{\mu\nu} \geq 0$ are guaranteed. Using the constitutive equations (37), the heat flux can be obtained by the following projection:

$$q^\gamma = -\frac{2}{I_1} u_\sigma h^\gamma_{\nu\mu} \nabla_\mu A^{\mu\nu\sigma}.$$  

(51)
From the previous expression for $A^{\mu\nu\sigma}$, a direct but careful calculation (see Appendix B for details) yields

$$q^\gamma \propto \left[(zG'(z) + G(z))\nabla_\alpha T + zG(z)\frac{\nabla_\alpha n}{n}\right]h^{\alpha\gamma} + (1 + 5zG(z))\dot{u}^\gamma,$$

(52)

where $\dot{u}^\gamma = u^\alpha\nabla_\alpha u^\gamma$ is the fluid’s 4–acceleration at the leading order. The conservation of the energy–momentum tensor given by Eq. (49), leads to the following expression for $\dot{u}^\gamma$:

$$\dot{u}^\gamma = \dot{u}_{PF}^\gamma + \dot{u}_{\text{dis}}^\gamma,$$

(53)

where

$$\dot{u}_{PF}^\gamma = -\frac{z}{G(z)}h^{\gamma\alpha}\left(\frac{\nabla_\alpha T}{T} + \frac{\nabla_\alpha n}{n}\right)$$

(54)

is the 4–acceleration at zeroth order; i.e., the one that corresponds to the perfect fluid solution; and

$$\dot{u}_{\text{dis}}^\gamma = -\frac{\dot{q}^\gamma + \theta q^\gamma + h^{\gamma\beta}(q^\alpha\nabla_\alpha u^\beta + \nabla_\alpha \pi_{\alpha\beta})}{n\bar{G}(z)}$$

(55)

is the corresponding dissipative contribution, with $\theta = \nabla_\alpha u^\alpha$. Then, introducing Eqs. (53), (54) and (55) in the expression (52) for the heat flux, one is led to

$$q^\gamma = q_{\text{CE}}^\gamma + \delta q^\gamma,$$

(56)

where

$$q_{\text{CE}}^\gamma = -z\bar{G}(z)h^{\gamma\mu}\left(\frac{\kappa}{\lambda}\frac{\nabla_\mu T}{T} - \frac{\nabla_\mu n}{n}\right),$$

(57)

$$\delta q^\gamma = G(z)[G(z) - \bar{G}(z)]\dot{u}_{\text{dis}}^\gamma,$$

(58)

and the function $\bar{G}(z)$ is given by

$$\bar{G}(z) := G(z) - \frac{1 + 5zG(z)}{G(z)}.$$

Thus, one can conclude that the most general first-order DTT does not lead directly to Chapman–Enskog’s constitutive relation, since there appears a new term coming up from the dissipative contribution of the 4–acceleration
of the fluid at the leading order. However, it must be noticed that Chapman-Enskog’s equation (Eq. (20)) is fully recovered when considering the 4–acceleration coming from the equilibrium configuration (the perfect fluid solution). This fact is quite relevant, since this is one of the hypotheses of Champan-Enskog’s procedure and moreover, classical irreversible thermodynamics predicts—both from phenomenological as well as microscopic grounds—the coupling of dissipative fluxes with the so-called thermodynamic forces, namely the gradients of the state variables.

Indeed, in the Chapman-Enskog method, a key step in order to recover well-tested laws, such as the Navier-Newton and Fourier ones, is to substitute the time derivatives of $n$, $T$ and $u^\mu$ into the linearized Boltzmann equation by their corresponding expressions obtained from the lower order (that is, from the local equilibrium) solution. If this step is not carried out, one would be led to erroneous predictions, as well as values for the transport coefficients inconsistent with experimental data. Thus, this consideration must be understood at this point as a necessary step in order to recover the constitutive equation given by the Chapman-Enskog formalism.

Furthermore, one may justify the substitution of solely $\dot{u}_{PF}^\mu$ in Eq. (53) by invoking the fundamental hypotheses of linear irreversible thermodynamics [18] and the experimental (in the non-relativistic framework) and numerical (in the relativistic framework) evidences that the coupling of dissipative fluxes with the gradients of state variables is correct (see [4] for details). Indeed, by doing so, the driving force for the heat flux is obtained as shown in Eq. (20) with the ratio of transport coefficients given by equation (26), which arises directly from the manipulation of the linearized Boltzmann equation alone. It is worthwhile to mention that such relation is obtained both for the relaxation time approximation as with the complete collision kernel (see for example references [2, 17]).

With this, we see that the theory obtained following Chapman-Enskog method can be given a similar structure to a first-order divergence-type fluid theory, which have been proven to represent in general ill-posed initial-value formulations.

Finally, we recall that the result obtained above is the same independently of the chosen frame, since the distribution function $f^{(0)}$ is Lorentz invariant. In particular, choosing the Landau frame, in which $N^\mu = n u^\mu + \pi^\mu$ and $T^{\mu\nu} = \rho u^\mu u^\nu + \Pi^{\mu\nu}$, the most general source tensor field at first order in dissipative variables one can construct is still given by formula (50), but changing $q^\mu$ by $\pi^\mu$. Thus, the projection needed to recover heat flux constitutive relation is
5 Discussion and final remarks

In the present work, we revisited the stability of the transport equations for a relativistic dilute gas when closed with the heat flux constitutive equation arising from the Chapman-Enskog formalism. Firstly, we revised the generic instability pointed out in [5] using Eckart’s formalism in a general way by applying an analytic criterion to show that not only the transverse modes grow exponentially but also at least one longitudinal mode leads to unstable behaviour.

Following a similar procedure, we could see that if the CE constitutive equation is introduced in the first order relativistic hydrodynamic equations, the system is stable which means that the instability in this scheme resides not necessarily in the equations per se, but in the flux-force relation introduced for the heat flux. However, this result is not completely general, since the equilibrium state is assumed to be such that the hydrodynamic velocity has a unique orthogonal direction. That is, since the 3+1 decomposition is based on $u^\nu$, the temporal direction is fixed by it and in order to be well defined, no rotation in the system must be present. In order to address the stability within the CE formalism in a general scenario, a boosted frame was considered, where a positive real root was found for the dispersion relation in the homogeneus perturbation case. Also, in the general case we numerically obtained exponentially growing modes. These results imply that in this general case, the system is unstable. Furthermore, the system was shown to be ill-posed in this case by analyzing the high frequency growing modes.

In order to explore a step further the nature of the relativistic Navier-Stokes-Fourier system when the heat law is written solely in terms of gradients (in contrast with the acceleration dependent one proposed by Eckart) the possibility of fitting such theory in a DTT structure was examined. The purpose of such analysis lies on the fact that first order DTTs have been formally shown to be unstable.

In this direction, we could see that the CE constitutive equation can be obtained from Eq. (37), provided the Euler equations are used in order to eliminate the acceleration in favor of space gradients of density and temperature. This substitution is consistent with Hilbert’s method for the solution of the integral-differential Boltzmann equation within the CE procedure. How-
ever, in order to justify such step in a macroscopic framework one would need to find some relation between the Knudsen parameter ordering in kinetic theory and the dissipative expansion in DTTs. This is a topic of interest and part of current and future work. If such justification is achieved, one could conclude that in a general framework the CE constitutive equation leads to unstable behavior. However, it is important to emphasize that the theory is still physically sound as long as the hydrodynamic velocity of the system can be assumed to form orthogonal surfaces.

The underlying mathematical mechanism for the inhibition of instabilities in this particular case can be traced down to the first order system being parabolic in nature with either real or imaginary frequencies. Even though from a physical standpoint one would expect a hyperbolic system, the parabolic-damped one generated by using the CE procedure is still physically acceptable and could explain the success of such theory in numerical simulations.

A Details of calculations for Section 2

In this appendix, the steps leading to Eqs. (14)-(16) are shown to some detail. Starting with the particle four-flux conservation equation, Eq. (6), by considering plane wave solutions as specified in Eq. (13), one can write

\[ \nabla_\mu \delta N^\mu = u^\mu \nabla_\mu \delta n + n \nabla_\mu \delta u^\mu \Rightarrow s \delta n + nik \delta u_x \]  

which translates to a row given by

\[ A_B^1 = \{0, s, ink, 0, 0, 0, 0, 0\} \]

in which the arrangement of the variables is the one specified in Eq. (11).

Notice that \(A_B^1\) corresponds to \(Q_1 = M_B^1\) in Ref. [5].

For the energy-momentum balances, one calculates the divergence of \(T^{\mu\nu}\) from Eq. (10), which leads to

\[ \nabla_\mu \delta T^{\mu\nu} = u^\mu u^\nu (nc_n \nabla_\mu \delta T + \varepsilon \nabla_\mu \delta n) + n \mathcal{G}(z) (u^\nu \nabla_\mu \delta u^\mu + u^\mu \nabla_\mu \delta u^\nu) + h^{\mu\nu} (n \nabla_\mu \delta T + T \nabla_\mu \delta n) + (u^\nu \nabla_\mu \delta q^\mu + u^\mu \nabla_\mu \delta q^\nu). \]  

(60)

Considering once again the hypothesis given in Eq. (13) one obtains

\[ \nabla_\mu \delta T^{\mu\nu} = \left[ c_n s u^\nu + (\eta^{1\nu} ik + u^\nu s) n \right] \delta T + n \mathcal{G}(z) (u^\nu ik \delta u^1 + s \delta u^\nu) + (\eta^{1\nu} ik T + u^\nu n \mathcal{G}(z) s) \delta n + u^\nu ik q^1 + s \delta q^\nu \]  

(61)
which, in the direction parallel to the perturbation yields

$$\nabla_\mu \delta T^{\mu 1} = i n k \kappa_B \delta T + i k T \delta n + n \tilde{G}(z) s \delta u^1 + s \delta q^1$$  \hspace{1cm} (62)$$

and is written as the following row:

$$A^2_B = \left\{ n i k, T i k, n \tilde{G} (z) s, \frac{s}{c^2}, 0, 0, 0, 0 \right\}$$

The array $A^2_B$ corresponds to $Q_3 = M^3_B$ in Ref. [5]. In the perpendicular direction, one has

$$\nabla_\mu \delta T^{\mu 2,3} = n \tilde{G}(z)s \delta u^{2,3} + \frac{s}{c^2} \delta q^{2,3}$$  \hspace{1cm} (63)$$

for which we write

$$A^3_B = \{ 0, 0, 0, 0, n \tilde{G}(z)s, s, 0, 0 \}$$

and

$$A^4_B = \{ 0, 0, 0, 0, 0, 0, n \tilde{G}(z)s, s \}$$

corresponding to $(R_1)_1 = M^8_B$ and $(R_1)_2 = M^{11}_B$ in Ref. [5], respectively. In order to obtain the internal energy balance, one starts by projecting the energy-momentum tensor in the direction $u_\nu$:

$$u_\nu \nabla_\mu \delta T^{\mu \nu} = - n c_n s \delta T - \varepsilon s \delta n - n \tilde{G}(z)i k \delta u^1 - i k \delta q^1$$  \hspace{1cm} (64)$$

which, using Eq. (59) and the identity $n \tilde{G}(z) = n \varepsilon + p$, can also be written as

$$u_\nu \nabla_\mu \delta T^{\mu \nu} = - n c_n s \delta T - p i k \delta u^1 - i k \delta q^1$$  \hspace{1cm} (65)$$

and thus

$$A^5_B = \{ c_n s, 0, n T i k, i k, 0, 0, 0, 0 \}$$

which corresponds to $Q_2 = M^2_B$ in Ref. [5]. Finally, for the heat flux constitutive equation, Eq. (4), one has

$$\delta q^\mu = - \frac{\kappa}{T} (\eta^\mu i k + u^\mu s) \delta T - \kappa s \delta u^\mu$$  \hspace{1cm} (66)$$

which in the parallel direction yields

$$\delta q^1 + \frac{\kappa}{T} i k \delta T + \kappa s \delta u^1 = 0$$  \hspace{1cm} (67)$$
and corresponds to $Q_5 = M_B^5$ in Ref. [5]. In the perpendicular direction one has
\[ \delta q^{2,3} + \kappa s \delta u^{2,3} = 0 \]  
which is written as the following rows:
\[
A_B^7 = \left\{ 0, 0, 0, 0, s, \frac{1}{\kappa}, 0, 0 \right\} \\
A_B^8 = \left\{ 0, 0, 0, 0, 0, s, \frac{1}{\kappa} \right\}
\]
, corresponding to $(R_2)_1 = M_B^9$ and $(R_2)_2 = M_B^{12}$ in Ref. [5], respectively. Rearranging the rows as follows
\[
A_B^1 = M_B^1, \quad A_B^5 = M_B^2 \\
A_B^2 = M_B^3, \quad A_B^6 = M_B^4 \\
A_B^3 = M_B^5, \quad A_B^7 = M_B^6 \\
A_B^4 = M_B^8, \quad A_B^8 = M_B^7
\]
the matrix can thus be written as the following block diagonal array:
\[
M_B^A = \begin{pmatrix}
  0 & s & i n k & 0 & 0 & 0 & 0 & 0 \\
  c_n s & 0 & n T i k & i k & 0 & 0 & 0 & 0 \\
  n i k & T i k & n G(z)s & s & 0 & 0 & 0 & 0 \\
  i k/T & 0 & s & \frac{1}{\kappa} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & n G(z)s & s & 0 & 0 \\
  0 & 0 & 0 & 0 & s & \frac{1}{\kappa} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & n G(z)s & s \\
  0 & 0 & 0 & 0 & 0 & 0 & s & \frac{1}{\kappa}
\end{pmatrix}
\]
which is written in the main text as Eqs. (14)-(16).
B Details of calculations for Section 4

In this appendix we compute the most general constitutive tensor field that can be constructed from first-order divergence-type fluid theories, using a Jüttner distribution function in local equilibrium, $f^{(0)}$. As was pointed out in Section 4, such a constitutive tensor is made up by means of the third moment of $f^{(0)}$, as shown in Eq. (45).

From the macroscopic point of view, the constitutive tensor field must be an algebraic function of the dynamical variables $N^\mu$ and $T^{\mu\nu}$. On the other hand, since we are considering first-order theories and $A^{\mu\alpha\beta}$ includes, by definition, first derivatives with respect to the dissipative tensor, we conclude that it must be of zeroth-order (for that reason we are considering just the Jüttner distribution function). Up to this order, both $N^\mu$ and $T^{\mu\nu}$ are made up in terms of the fluid four-velocity $u^\mu$ and the background metric $\eta_{\mu\nu}$. It is rather straightforward to see that the most general tensor field that satisfies these requirements, and has the symmetries imposed by the theory, is the one given in Eq. (44), where the coefficients $A_o$ and $A_1$ can be directly obtained from the expressions (46) and (47). In fact, recalling that the possible $p^\mu$ are those restricted to the mass-shell $p^\mu p_\mu = -1$ (where the mass of each fluid component is normalized to $m = 1$), we get

$$A_o = -\frac{2}{3} A^{\mu\alpha\beta} (2u_\mu u_\alpha + h_{\mu\alpha}) u_\beta$$

$$= 2 \int f^{(0)} (-u_\alpha p^\alpha) \left[ (-u_\alpha p^\alpha)^2 - \frac{1}{2} \right] d\Omega. \quad (69)$$

By the change of variables

$$\epsilon = -u_\alpha p^\alpha,$$

we get

$$f^{(0)} = \frac{n}{4\pi z K_2 (1/z)} e^{-\epsilon/z},$$

and $d\Omega = \sqrt{\epsilon^2 - 1} d\epsilon dp_\theta dp_\varphi$. Thus, integral (69) reduces to

$$A_o = \frac{2n}{z K_2 (1/z)} \int_1^\infty e^{-\epsilon/z} \epsilon \left( \epsilon^2 - \frac{1}{2} \right) \sqrt{\epsilon^2 - 1} d\epsilon$$

$$= \frac{2n}{z K_2 (1/z)} \left( \frac{1}{2} I_1(z) + I_2(z) \right),$$

The details of the integrals $I_1(z)$ and $I_2(z)$ are given in the Appendix C.
where, for $\ell \in \mathbb{N}$,

$$I_\ell(z) := \int_1^\infty e^{-\varepsilon/z} (\varepsilon^2 - 1)^{\ell-1/2} d\varepsilon. \quad (70)$$

The integral in Eq. (70) can be easily computed by means of some properties of modified Bessel functions, as shown in the following

**Proposition B.1.** Let $K_\ell$ be the $\ell$-th modified Bessel function. Then, the following identities hold for any $\ell \in \mathbb{N}$:

(i) $\int_1^\infty e^{-\varepsilon/z} (\varepsilon^2 - 1)^{\ell-1/2} d\varepsilon = \frac{\Gamma \left( \ell + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} (2z)^\ell K_{\ell+1} \left( \frac{1}{z} \right). \quad (71)$

(ii) $\frac{dK_\ell(x)}{dx} = \frac{\ell K_\ell(x)}{x} - K_{\ell+1}(x). \quad (72)$

**Proof.** Identity (ii) is a direct consequence of the derivative formula

$$\frac{d}{dx} \left[ \frac{K_\ell(x)}{x^\ell} \right] = -\frac{K_{\ell+1}(x)}{x^\ell}.$$ 

Identity (i) is also consequence of the above formula and the following important one:

$$K_\ell(x) = \left( \frac{x}{2} \right)^\ell \frac{\Gamma(1/2)}{\Gamma(\ell + 1/2)} \int_1^\infty e^{-xy} (y^2 - 1)^{\ell-1/2} dy.$$  

\[\Box\]

By applying the proposition above we get, then, $I_1 = zK_2(1/z)$, $I_2 = 3z^2K_3(1/z)$ and

$$A_o = n \left( 1 + 6zG(z) \right),$$

where $G(z)$ is given by Eq. (8). Analogously, for $A_1$ we get

$$A_1 = \frac{1}{3} A^{\mu\alpha\beta} u_\mu u_\alpha u_\beta$$

$$= \frac{n}{3zK_2(1/z)} \int_1^\infty e^{-\varepsilon/z} \left( \varepsilon^2 - \frac{1}{4} \right)^{1/2} \sqrt{\varepsilon^2 - 1} d\varepsilon$$

$$= \frac{n}{3zK_2(1/z)} \left( \frac{3}{4} I_1(z) + I_2(z) \right)$$

$$= n \left( zG(z) + \frac{1}{4} \right). \quad (73)$$
Now, in order to obtain the Chapman-Enskog constitutive relation, we proceed to project the constitutive tensor in the space perpendicular to $u^\mu$. In order to do so, we find it useful to express the constitutive tensor as a sum of three contributions, namely

$$ A^{\mu\alpha\beta} = A_{1}^{\mu\alpha\beta} + A_{2}^{\mu\alpha\beta} + A_{3}^{\mu\alpha\beta}, \quad (74) $$

where

$$ A_{1}^{\mu\alpha\beta} = A_{o} u^\mu u^\alpha u^\beta, \quad A_{2}^{\mu\alpha\beta} = A_{1} u^\mu \eta^{\alpha\beta}, \quad A_{3}^{\mu\alpha\beta} = (A_{o} - 4A_{1}) \eta^{\mu(\alpha} u^{\beta)}. $$

Then,

$$ h_{\alpha}^{\gamma} \nabla_{\mu} A_{1}^{\mu\alpha\beta} = h_{\alpha}^{\gamma} \left[ u^{\alpha} u^{\beta} \dot{A}_{o} + A_{o} \nabla_{\mu} (u^{\mu} u^{\alpha}u^{\beta}) \right] = A_{o} u^{\beta} \dot{u}^{\gamma}, \quad (75) $$

yielding

$$ u_{\beta} h_{\alpha}^{\gamma} \nabla_{\mu} A_{1}^{\mu\alpha\beta} = -A_{o} \dot{u}^{\gamma}. $$

By similar calculations, we get

$$ h_{\alpha}^{\gamma} \nabla_{\mu} A_{2}^{\mu\alpha\beta} = \left( \dot{A}_{1} + A_{1} \nabla_{\mu} u^{\mu} \right) h^{\gamma\beta}, $$

which implies $u_{\beta} h_{\alpha}^{\gamma} \nabla_{\mu} A_{2}^{\mu\alpha\beta} = 0$. Finally, defining

$$ A_{3} := \frac{A_{o}}{2} - 2A_{1}, $$

we have

$$ u_{\beta} h_{\alpha}^{\gamma} \nabla_{\mu} A_{3}^{\mu\alpha\beta} = 2u_{\beta} h_{\alpha}^{\gamma} \left[ \eta^{\mu(\alpha} u^{\beta)} \nabla_{\mu} A_{3} + A_{3} \nabla^{\alpha} u^{\beta} \right] = -h^{\gamma\mu} (\nabla_{\mu} A_{3}) + A_{3} \dot{u}^{\gamma} $$

Plugging all together, we get

$$ q^{\gamma} \propto \begin{align*}
& u_{\beta} h_{\alpha}^{\gamma} \nabla_{\mu} A^{\mu\alpha\beta} \\
\propto & h^{\gamma\mu} \left( \nabla_{\mu} A_{3} + A_{3} \frac{\nabla_{\mu} n}{n} \right) + \left( \frac{A_{o}}{2} + 2A_{1} \right) \dot{u}^{\gamma} \\
= & zh^{\gamma\mu} \left( (zG(z))' \frac{\nabla_{\mu} T}{T} + G(z) \frac{\nabla_{\mu} n}{n} \right) + (1 + 5zG(z)) \dot{u}^{\gamma}. \quad (76)
\end{align*} $$

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Then, we use the local expression (53) for the acceleration at the leading order, and the following formula for the derivative of $G(z)$:

$$G'(z) = -\frac{1}{z^2} \left[ G^2(z) - 5zG(z) - 1 \right], \quad (77)$$

which implies that

$$(zG(z))' = -\frac{1}{z} \left[ G^2(z) - 6zG(z) - 1 \right]$$

$$= G(z) \left[ 1 - \frac{\bar{G}(z)}{z} \right], \quad (78)$$

where

$$\bar{G}(z) := G(z) - \frac{1 + 5zG(z)}{G(z)}.$$

Finally, recalling the contribution to the heat flux $\delta q^\gamma$ given in Eq. (58) by taking into account first-order corrections in the four-acceleration, we get

$$q^\gamma \propto h^{\gamma\mu} \bar{G}(z) \left[ \left( 1 - \frac{\bar{G}(z)}{z} \right) \frac{\nabla_\mu T}{T} + \frac{\nabla_\mu n}{n} \right] - \frac{1 + 5z\bar{G}(z)}{\bar{G}(z)} h^{\gamma\mu} \left( \frac{\nabla_\mu T}{T} + \frac{\nabla_\mu n}{n} \right) + \delta q^\gamma$$

$$= h^{\gamma\mu} G(z) \left[ \left( 1 - \frac{G(z)}{z} \right) \frac{\nabla_\mu T}{T} + \frac{\nabla_\mu n}{n} \right] + h^{\gamma\mu} \left[ \bar{G}(z) - G(z) \right] \left( \frac{\nabla_\mu T}{T} + \frac{\nabla_\mu n}{n} \right) + \delta q^\gamma$$

$$= h^{\gamma\mu} \bar{G}(z) \left[ \left( 1 - \frac{G(z)}{z} \right) \frac{\nabla_\mu T}{T} + \frac{\nabla_\mu n}{n} \right] + \delta q^\gamma$$

$$= -h^{\gamma\mu} \bar{G}(z) \left( \frac{\kappa \nabla_\mu T}{\lambda T} - \frac{\nabla_\mu n}{n} \right) + \delta q^\gamma. \quad (79)$$

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