A second-order accurate implicit difference scheme for time fractional reaction-diffusion equation with variable coefficients and time drift term

Yong-Liang Zhao\textsuperscript{a,*}, Pei-Yong Zhu\textsuperscript{a}, Xian-Ming Gu\textsuperscript{b,*}, Xi-Le Zhao\textsuperscript{a}

\textsuperscript{a}School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, P.R. China
\textsuperscript{b}School of Economic Mathematics/Institute of Mathematics, Southwestern University of Finance and Economics, Chengdu, Sichuan 611130, P.R. China

Abstract

An implicit finite difference scheme based on the $L_2-\sigma$ formula is presented for a class of one-dimensional time fractional reaction-diffusion equations with variable coefficients and time drift term. The unconditional stability and convergence of this scheme are proved rigorously by the discrete energy method, and the optimal convergence order in the $L_2$-norm is $O(\tau^2 + h^2)$ with time step $\tau$ and mesh size $h$. Then, the same measure is exploited to solve the two-dimensional case of this problem and a rigorous theoretical analysis of the stability and convergence is carried out. Several numerical simulations are provided to show the efficiency and accuracy of our proposed schemes and in the last numerical experiment of this work, three preconditioned iterative methods are employed for solving the linear system of the two-dimensional case.

Keywords: Caputo fractional derivative, $L_2-\sigma$ formula, Finite difference scheme, Time fractional reaction-diffusion equation, Iterative method.

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1. Introduction

In the past decades, fractional calculus has growing interest been paid in modelling applications, including the spread of HIV infection of CD4+ T-cells\footnote{The spread of HIV infection of CD4+ T-cells is a critical issue in global health, influencing both individual and public health strategies.}, entropy\footnote{Entropy is a fundamental concept in thermodynamics and information theory, representing disorder or randomness.}, hydrology\footnote{Hydrology studies the amount, distribution, and movement of water on Earth's surface.}, soft tissues such as mitral valve in the human heart\footnote{The mitral valve is a critical component of the heart, and its dysfunction can lead to serious health issues.}, anomalous diffusion in transport dynamics of complex systems\footnote{Anomalous diffusion refers to processes where the mean squared displacement of particles does not scale linearly with time.}, engineering and physics. Many other examples can be found in Refs.\footnote{References to these examples are not provided here but can be found in the cited literature.}. In these models, the fractional diffusion equation (FDE) has been studied by many researchers, see\footnote{This reference indicates a broader discussion on fractional diffusion equations.} and references therein.

Since the solution of fractional operator at a given point depends on the solution behavior on the entire domain, i.e., the fractional operators are nonlocal, fractional diffusion equations (FDEs) tend to be more
appropriate for the description of various materials and processes with memory and hereditary properties than the normal integer-order counterparts. At the same time, the nonlocal nature of fractional operators has an inherent challenge when facing FDEs, namely the analytical solutions of FDEs are difficult to obtain, except for some special cases \[19\]. For this reason, the proposal and study of numerical methods that are efficient, accurate and easy to implement, are quite essential in obtaining the approximate solutions of FDEs. Without doubt, it is worth noting that there still are a few effective numerous analytical methods, for instance the Laplace transform method, the Fourier transform method and Adomian decomposition method. Up to now abundant numerical methods have been proposed for solving the FDEs, e.g., finite difference method \[11, 14, 20, 21\], finite element method \[22, 24\], collocation method \[23\], meshless method \[25\] and spectral method \[26\]. Among them, the finite difference scheme is one of the most popular numerical schemes employed to solve space and/or time FDEs, and we only mention some works in the next.

For the space FDEs, Meerschaert and Tadjeran \[27\] used the implicit Euler method based on the standard Grünwald-Letnikov formula to discrete space-fractional advection-dispersion equation with first order accuracy, but the obtained implicit difference scheme (IDS) is unstable. To overcome this problem, Meerschaert and Tadjeran \[27\] first proposed the shifted Grünwald-Letnikov formula, which is unconditionally stable. After their study, second-order approximations to space FDEs have been investigated, Sousa and Li \[28\] derived an unconditionally stable weighted average finite difference formula for one-dimensional FDE with convergence \(O(\tau + h^2)\) where \(\tau\) and \(h\) are time step and mesh size, respectively. Tian et al. \[29\] proposed a class of second-order approximations, which are termed as the weighted and shifted Grünwald difference (abbreviated as WSGD) operators, to solve the two-sided one-dimensional space FDE numerically. As expected, the convergence rate of their IDS is \(O(\tau^2 + h^2)\) by combining the Crank-Nicolson method (C-N). Adopting the same idea and utilizing the quasi-compact numerical technique, Zhou et al. \[30\] obtained a numerical approximate scheme with convergence \(O(\tau^2 + h^2)\). Subsequently, Hao et al. \[31\] applied a new fourth-order difference approximation, which was derived by using the weighted average of the shifted Grünwald formulae and combining with the compact numerical technique, to solve the two-sided one-dimensional space FDE. They proved that the proposed quasi-compact difference scheme is unconditionally stable and convergent in \(L_2\)-norm with the optimal order \(O(\tau^2 + h^4)\). On the other hand, for the time FDEs, many early researches \[32, 34\] employed the \(L_1\) formula to obtain their difference schemes. Then, Gao et al. \[33\] applied their new fractional numerical differentiation formula (called the \(L_1\)-2 formula) to solve the time-fractional sub-diffusion equations with accuracy \(O(\tau^{3-\alpha} + h^2)\) (\(0 < \alpha < 1\)). Alikhanov \[36\] proposed a modified scheme, which is of second order accuracy. The stability of his scheme was then proved and numerical evidence has shown that this scheme for the \(\alpha\)-order Caputo fractional derivative is of second order accuracy. Later, based on this modified scheme, Yan et al. \[37\] designed a fast high-order accurate numerical scheme (named \(FL2-1,\omega\)) to speed up the evaluation of the Caputo fractional derivative. This scheme efficiently reduces the computational storage and cost for solving the time FDEs. Although there are many studies on the
space/time FDEs, numerical studies on space-time FDEs are still not extensive, the readers are suggested to see [11, 38–40] and references therein.

In this manuscript, a second-order IDS is concerned for solving the initial-boundary value problem of the one-dimensional (1D) time fractional reaction-diffusion equation (TFRDE) with variable coefficients and time drift term:

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} + D_{0,t}^{\alpha} u(x,t) &= \mathcal{L} u(x,t) + f(x,t), & 0 \leq x \leq L, & 0 \leq t \leq T, \\
\quad u(x,0) &= u_0(x), & 0 \leq x \leq L, \\
\quad u(0,t) &= \phi_1(t), & u(L,t) = \phi_2(t), & 0 \leq t \leq T,
\end{aligned}
$$

where

$$
\mathcal{L} u(x,t) = \frac{\partial}{\partial x} \left( k(x,t) \frac{\partial u(x,t)}{\partial x} \right) - q(x,t) u(x,t),
$$

$k(x,t) \geq C > 0$, $q(x,t) \geq 0$ and $f(x,t)$ are sufficiently smooth functions. Moreover, the time fractional derivative in (1.1) is the Caputo fractional derivative [19] with order $\alpha \in (0,1]$ denoted by

$$
D_{0,t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^{\alpha}}.
$$

The time drift term $\frac{\partial u(x,t)}{\partial t}$ is added to describe the motion time, and this helps to distinguish the status of particles conveniently. In particular, when $k(x,t) \equiv k$ is a constant and $q(x,t) = 0$, Eq. (1.1) reduces to a special time fractional mobile/immobile transport model introduced in [41, 42].

The rest of the paper is organized as follows: For clarity of presentation, in the next section the full discretization of Eq. (1.1) is introduced first, then the stability analysis of the discrete scheme is carried out, and an error estimate shows that the discrete scheme accuracy is of $O(\tau^2 + h^2)$. In Section 3, we extend the TFRDE to two dimension, and the unconditionally stable and convergence are also proved. Numerical examples are presented in Section 4 to illustrate the effectiveness of our proposed methods. At last, some conclusions are drawn in Section 5.

2. An implicit difference scheme for TFRDE

In this section, an IDS is derived to discretize the TFRDE defined in (1.1), and the stability and error estimate of the IDS are analyzed in detail.

2.1. Derivation of the second-order difference scheme

To establish the numerical simulation scheme, we first discrete the solution region and let the mesh

\[ \bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau, \]

where $\bar{\omega}_h = \{x_i = ih, \ i = 0,1,\cdots,N; \ x_0 = 0, \ x_N = L\}$ and $\bar{\omega}_\tau = \{t_j = j\tau, \ j = 0,1,\cdots,M\}$.

The time fractional derivative is approximated by

$$
D_{0,t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^{\alpha}}
$$

The space fractional derivative is approximated by

$$
D_{x}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(x-x_0)^{\alpha}}
$$

The space-time fractional derivative is approximated by

$$
D_{x,t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^1 \int_0^{\xi} \frac{\partial u(x,\eta,\xi)}{\partial \eta} \frac{d\eta}{(x-x_0)^{\alpha}} \frac{d\xi}{(t-\xi)^{\alpha}}
$$

The time drift term is approximated by

$$
\frac{\partial u(x,t)}{\partial t} = \frac{u(x,t+\tau) - u(x,t)}{\tau}
$$

The space drift term is approximated by

$$
\frac{\partial u(x,t)}{\partial x} = \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}
$$

The solution at the next time step is obtained by

$$
u(x,t+\tau) = u(x,t) + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^t \int_0^\xi \frac{\partial u(x,\eta,\xi)}{\partial \eta} \frac{d\eta}{(x-x_0)^{\alpha}} \frac{d\xi}{(t-\xi)^{\alpha}}
$$

The stability and error estimate of the IDS are analyzed in detail.
0, 1, · · · , M; \( t_M = T \), in which \( h = \frac{L}{N} \), \( \tau = \frac{T}{M} \) are the uniform spatial and temporal mesh sizes respectively, and \( N, M \) are two positive integers. Let

\[ S_h = \{ v | v = (v_0, v_1, \cdots, v_N), \quad v_0 = v_N = 0 \} \]

be defined on \( \bar{\omega}_h \). Then about the discretization of Caputo fractional derivative, we utilize the \( L_2-1\sigma \) formula derived by Alikhanov in [36], and some helpful properties for later analysis in next subsection are reviewed therewith.

**Lemma 2.1.** ([36, Lemma 2]) Suppose \( 0 < \alpha < 1 \), \( \sigma = 1 - \frac{\alpha}{2} \), \( y(t) \in C^3[0, T] \), and \( t_{j+\sigma} = (j + \sigma)\tau \). Then

\[ |D_0^\alpha y(t) - \Delta_0^\alpha y(t)| = \mathcal{O}(\tau^{3-\alpha}), \]

where

\[ \Delta_0^\alpha y(t) = \frac{\sigma^{-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j} \epsilon_{j-s}^{(\alpha,\sigma)} [y(t_{s+1}) - y(t_s)], \quad (2.1) \]

and for \( j = 0 \),

\[ \epsilon_0^{(\alpha,\sigma)} = a_0^{(\alpha,\sigma)}, \]

for \( j \geq 1 \),

\[ \epsilon_{j-s}^{(\alpha,\sigma)} = \begin{cases} a_0^{(\alpha,\sigma)} + b_1^{(\alpha,\sigma)}, & s = 0, \\ a_s^{(\alpha,\sigma)} + b_{s+1}^{(\alpha,\sigma)} - b_s^{(\alpha,\sigma)}, & 1 \leq s \leq j - 1, \\ a_j^{(\alpha,\sigma)} - b_j^{(\alpha,\sigma)}, & s = j. \end{cases} \]

In which

\[ a_0^{(\alpha,\sigma)} = \sigma^{1-\alpha}, \quad a_l^{(\alpha,\sigma)} = (l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha} \quad (l \geq 1), \]

and

\[ b_l^{(\alpha,\sigma)} = \frac{1}{2-\alpha} [(l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha}] = \frac{1}{2} [(l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha}]. \]

Here, the properties of \( \epsilon_j^{(\alpha,\sigma)} \) proved in [36] are revisited as below.

**Lemma 2.2.** ([36, Lemma 4]) For any \( \alpha \) \((0 < \alpha < 1)\) and \( \epsilon_j^{(\alpha,\sigma)} \) defined in Lemma 2.1, it holds

\[ \epsilon_j^{(\alpha,\sigma)} > \frac{1-\alpha}{2} (j + \sigma)^{-\alpha}, \]

\[ \epsilon_0^{(\alpha,\sigma)} > \epsilon_1^{(\alpha,\sigma)} > \epsilon_2^{(\alpha,\sigma)} > \cdots > \epsilon_{j-1}^{(\alpha,\sigma)} > \epsilon_j^{(\alpha,\sigma)}, \]
where \( \sigma = 1 - \frac{\alpha}{2} \).

As for approximation of the time drift term \( \frac{\partial u(x,t)}{\partial t} \), the Taylor expansion of the function \( u(t) \) is employed for \( t = t_{j+1}, t_j \) and \( t_{j-1} \) at the point \( t_{j+\sigma} \), respectively. Thus, the next lemma can be easily obtained after simple calculation.

**Lemma 2.3.** Suppose \( u \in C^3[0,T] \), we have

\[
\delta_t^\alpha u(t_j) = \frac{1}{2\tau} \left[ (2\sigma + 1)u(t_{j+1}) - 4\sigma u(t_j) + (2\sigma - 1)u(t_{j-1}) \right] = \frac{du(t_{j+\sigma})}{dt} + O(\tau^2), \quad j \geq 1.
\]

On the other hand, [36] also proved that

\[
L u(x,t) \big|_{(x_i,t_{j+\sigma})} = \sigma \Lambda u(x_i,t_{j+1}) + (1 - \sigma) \Lambda u(x_i,t_j) + O(\tau^2 + h^2),
\]

where \( \Lambda \) is a difference operator, which approximates the continuous operator \( L \), defined by

\[
\Lambda u(x_i,t_j) = \frac{1}{h^2} \left[ k(x_{i+\frac{1}{2},t_{j+\sigma}})u(x_{i+1},t_j) - k(x_{i-\frac{1}{2},t_{j+\sigma}})u(x_{i-1},t_j) \right] - q(x_i,t_{j+\sigma})u(x_i,t_j).
\]

Assume \( u(x,t) \) is a sufficiently smooth solution of the TFRDE (1.1). For the sake of simplification, some symbols are introduced:

\[
u_{i,j}^{t+\sigma} = \sigma u_{i,j}^{t+1} + (1 - \sigma) u_{i,j}^t, \quad k_{i,j}^{t+\sigma} = k(x_i,t_{j+\sigma}), \quad q_{i,j}^{t+\sigma} = q(x_i,t_{j+\sigma}), \quad f_{i,j}^{t+\sigma} = f(x_i,t_{j+\sigma}).
\]

Using (2.1)-(2.2) and omitting the small term, the solution of (1.1) can be approximated by the following IDS for \( (x_i,t_{j+\sigma}) \in \bar{\omega}_{h\tau}, \quad i = 1,2,\ldots,N-1, \quad j = 0,1,\ldots,M-1 : \)

\[
\delta_t^\alpha u_{i,j}^t + \Delta_0^{\alpha,t_{j+\sigma}} u_i = \Lambda u_{i,j+\sigma}^t + f_{i,j}^{t+\sigma}.
\]

There is a problem that cannot be ignored in the above equation: when \( j = 0 \), then \( u_{i,j}^{-1} = u_{i}^{-1} \) is defined outside of \([0,T] \). In numerical calculation, we handle with this problem mainly by using the neighboring function values to approximate \( u_{i}^{-1} \), that is,

\[
u_{i}^{-1} = u_{i}^0 - \tau \frac{\partial u_{i}^0}{\partial t} + O(\tau^2).
\]

If \( \frac{\partial u_{i}^0}{\partial t} \neq 0 \), our IDS only has first-order temporal accuracy. Thus, in order to obtain the second-order
accuracy in time, we suppose \( \frac{\partial u(x, t)}{\partial t} = 0 \), then set \( u_{i}^{0} = u_{i}^{0} \). The IDS with the accuracy order \( O(\tau^{2} + h^{2}) \) is:

\[
\begin{align*}
\delta_{t}u_{i}^{0} + \Delta_{t}^{\alpha}u_{i} = \Delta_{t}^{\alpha}u_{i}^{+} + f_{i}^{+}, & \quad 1 \leq i \leq N - 1, \quad 0 \leq j \leq M - 1, \\
u_{i}^{0} = u_{0}(x_{i}), & \quad u_{i}^{-} = u_{i}^{0}, \quad 0 \leq i \leq N, \\
u_{i}^{j} = \phi_{1}(t_{j}), & \quad u_{N}^{j} = \phi_{2}(t_{j}), \quad 0 \leq j \leq M.
\end{align*}
\]

(2.3)

It is interesting to note that for \( \alpha \to 1 \), the Crank-Nicolson difference scheme is obtained. In the next subsection, we will proof the unconditional stability and give error estimate about this approximate scheme.

2.2. Stability analysis and optimal error estimates

Before exploring the stability and convergence of Eq. (2.3), an inner product and the corresponding norm are introduced:

\[
(u, v) = h \sum_{i=1}^{N-1} u_{i}v_{i}, \quad \|u\| = \sqrt{(u, u)},
\]

here \( u, v \in S_{h} \) are arbitrary vectors. Meanwhile, we need another two lemmas, which are essential for our proof, see [36, 44].

Lemma 2.4. ([36, Corollary 1]) Let \( V_{\tau} = \{u \mid u = (u_{0}, u_{1}, \ldots, u_{M})\} \). For any \( u \in V_{\tau} \), one has the following inequality

\[
[\sigma u_{i+1}^{j} + (1 - \sigma)u_{i}^{j}] \Delta_{t}^{\alpha}u_{i} \geq \frac{1}{2} \Delta_{t}^{\alpha}(u)^{2}.
\]

Lemma 2.5. ([44, Lemma 3.5]) For any grid functions \( u^{0}, u^{1}, \ldots, u^{N} \in S_{h} \), we have

\[
(\delta_{t}u^{k}, \sigma u^{k+1} + (1 - \sigma)u^{k}) \geq \frac{1}{4r}(E^{k+1} - E^{k}), \quad k \geq 1,
\]

with

\[
E^{k+1} = (2\sigma + 1)\|u^{k+1}\|^{2} - (2\sigma - 1)\|u^{k}\|^{2} + (2\sigma^{2} + \sigma - 1)\|u^{k+1} - u^{k}\|^{2}, \quad k \geq 0.
\]

In addition, it holds

\[
E^{k+1} \geq \frac{1}{\sigma}\|u^{k+1}\|^{2}, \quad k \geq 0.
\]

From Lemma 2.4, we obtain \( E^{0} = 2\|u^{0}\|^{2} \). With this in hand, the next theorem can be established.

Theorem 2.1. Denote \( u^{j+1} = (u_{1}^{j+1}, u_{2}^{j+1}, \ldots, u_{N-1}^{j+1})^{T} \) and \( \|f^{j+\sigma}\|^{2} = h \sum_{i=1}^{N-1} f_{i}^{2}(x_{i}, t_{j+\sigma}) \). Then the IDS
is unconditionally stable, and the following two priori estimates hold:

\[ \|u^1\|^2 \leq \left( \frac{4\sigma T^{1-\alpha}}{\Gamma(2-\alpha)} + 2\sigma \right) \|u^0\|^2 + 4\sigma T^{1+\alpha} \Gamma(2-\alpha) f^\sigma \|^2, \tag{2.4} \]

\[ \|u^k\|^2 \leq C_1 \|u^1\|^2 + \left( \frac{2\sigma T^{1-\alpha}}{\Gamma(2-\alpha)} + 4\sigma^2 \right) \|u^0\|^2 + 8\sigma T^{1+\alpha} \Gamma(1-\alpha) \sum_{j=1}^{k-1} \|f^{j+\sigma}\|^2, \quad k \geq 2, \tag{2.5} \]

where \( C_1 = \frac{T^{1-\alpha}}{\Gamma(1-\alpha)} + \frac{2\sigma(4-3\alpha)T^{1-\alpha}}{\Gamma(3-\alpha)} + 4\sigma^2 - 2\sigma. \)

**Proof.** Taking the inner product of (2.3) with \( u^{j+\sigma} \), it has

\[ (\hat{\delta}_t u^j, u^{j+\sigma}) + (\Delta_0^{\alpha} u^{j+\sigma}, u^{j+\sigma}) = (\Lambda u^{j+\sigma}, u^{j+\sigma}) + (f^{j+\sigma}, u^{j+\sigma}). \]

Using Lemmas 2.4 and noticing \((\Lambda u^{j+\sigma}, u^{j+\sigma}) \leq 0\), it can be obtained that

\[ \frac{1}{4\tau} (E^{j+1} - E^j) + \frac{1}{2} \Delta_0^{\alpha} \|u\|^2 \leq (f^{j+\sigma}, u^{j+\sigma}). \tag{2.6} \]

**Step 1.** When \( j = 0 \), from the inequality (2.6), we have

\[ \frac{1}{4\tau} (E^1 - E^0) + \frac{1}{2T^{\alpha} \Gamma(2-\alpha)} a_0^{\alpha} (||u^1||^2 - ||u^0||^2) \leq (f^\sigma, u^\sigma). \]

With the help of virtue Cauchy-Schwarz inequality, we arrive at

\[ \|u^1\|^2 + \frac{2\tau^\alpha}{T^{\alpha} \Gamma(2-\alpha)} a_0^{\alpha} \|u^1\|^2 \leq \frac{2\tau^\alpha}{T^{\alpha} \Gamma(2-\alpha)} a_0^{\alpha} \|u^0\|^2 + 2\tau \|u^0\|^2 + \frac{\tau^\alpha}{\epsilon_1} \|f^\sigma\|^2 + 8\tau \sigma \epsilon_1 (||u^1||^2 + ||u^0||^2), \quad \epsilon_1 > 0. \]

Let \( \epsilon_1 = \frac{1}{2T^{\alpha} \Gamma(2-\alpha)} a_0^{\alpha} \), it gives immediately the estimate for \( u^j \), that is

\[ \|u^j\|^2 \leq \left( \frac{4\tau^\alpha}{T^{\alpha} \Gamma(2-\alpha)} a_0^{\alpha} + 2\sigma \right) \|u^0\|^2 + \frac{4\tau^\alpha T^{1+\alpha} \Gamma(2-\alpha)}{a_0^{\alpha}} \|f^\sigma\|^2 \leq \left( 4T^{1-\alpha} \sigma \right) \|u^0\|^2 + 4\sigma T^{1+\alpha} \Gamma(2-\alpha) \|f^\sigma\|^2. \]

**Step 2.** When \( j \geq 1 \), summing up for \( j \) in (2.6) from 1 to \( k - 1 \) and doing some simple manipulations, it
obtains

\[
\frac{1}{4\tau}(E^k - E^1) + \frac{1}{2\tau^\alpha \Gamma(2 - \alpha)} \left[ c_0^{(\alpha, \sigma)} \sum_{j=1}^{k-1} \|u^{j+1}\|^2 - \sum_{j=1}^{k-1} \sum_{s=2}^{j} (c_s^{(\alpha, \sigma)} - c_{j-s+1}^{(\alpha, \sigma)}) \|u^s\|^2 \right] \\
\leq \frac{1}{2\tau^\alpha \Gamma(2 - \alpha)} \|u^0\|^2 \sum_{j=1}^{k-1} c_j^{(\alpha, \sigma)} + \frac{1}{2\tau^\alpha \Gamma(2 - \alpha)} \|u^1\|^2 \sum_{j=1}^{k-1} (c_j^{(\alpha, \sigma)} - c_{j-1}^{(\alpha, \sigma)}) \\
+ \sum_{j=1}^{k-1} \|f^{j+\sigma}\| \cdot (\sigma \|u^{j+1}\| + (1 - \sigma)\|u^j\|). 
\]

(2.7)

To estimate the second term on the left hand side of inequality (2.7), Lemma 2.2 is applied. Then

\[
\frac{1}{2\tau^\alpha \Gamma(2 - \alpha)} \left[ c_0^{(\alpha, \sigma)} \sum_{j=1}^{k-1} \|u^{j+1}\|^2 - \sum_{j=1}^{k-1} \sum_{s=2}^{j} (c_s^{(\alpha, \sigma)} - c_{j-s+1}^{(\alpha, \sigma)}) \|u^s\|^2 \right] \\
= \frac{1}{2\tau^\alpha \Gamma(2 - \alpha)} \sum_{j=2}^{k} c_{j-k}^{(\alpha, \sigma)} \|u^j\|^2 \geq \frac{1}{2\tau^\alpha \Gamma(2 - \alpha)} \frac{1 - \alpha}{2} (j - 1 + \sigma)^{-\alpha} \sum_{j=2}^{k} \|u^j\|^2 \\
\geq \frac{1}{4T^\alpha \Gamma(1 - \alpha)} \sum_{j=2}^{k} \|u^j\|^2.
\]

Bringing above estimate to inequality (2.7) gives

\[
\|u^k\|^2 + \frac{\tau \sigma}{T^\alpha \Gamma(1 - \alpha)} \sum_{j=2}^{k} \|u^j\|^2 \\
\leq \sigma E^1 + 2\tau^{1-\alpha} \sigma (4 - 3\alpha)(k - 1 + \sigma)^{1-\alpha} \|u^1\|^2 + \frac{2\tau^{1-\alpha} \sigma (k - 1 + \sigma)^{1-\alpha}}{\Gamma(2 - \alpha)} \|u^0\|^2 \\
+ 4\tau \sigma \varepsilon_2 \sum_{j=1}^{k-1} \left( \sigma \|u^{j+1}\| + (1 - \sigma)\|u^j\| \right)^2 + \frac{\tau \sigma}{\varepsilon_2} \sum_{j=1}^{k-1} \|f^{j+\sigma}\|^2 \\
\leq \sigma \left[ 4\sigma^2 + 4\sigma - 1 \right] \|u^1\|^2 + 4\sigma^2 \|u^0\|^2 + \frac{2\sigma (4 - 3\alpha) T^{1-\alpha}}{\Gamma(3 - \alpha)} \|u^1\|^2 \\
+ \frac{2\sigma T^{1-\alpha}}{\Gamma(2 - \alpha)} \|u^0\|^2 + 8\tau \sigma \varepsilon_2 \sum_{j=1}^{k} \|u^j\|^2 + \frac{\tau \sigma}{\varepsilon_2} \sum_{j=1}^{k-1} \|f^{j+\sigma}\|^2, \ \varepsilon_2 > 0,
\]

(2.8)
where $E^1 \leq (4\sigma^2 + 4\sigma - 1)\|u^1\|^2 + 4\sigma^2\|u^0\|^2$. Taking $\varepsilon_2 = \frac{1}{8\tau\sigma\Gamma(1 - \alpha)}$, inequality (2.8) leads to

$$
\|u^k\|^2 \leq \left[ \frac{\tau\sigma}{\Gamma(1 - \alpha)} + \frac{2\sigma(4 - 3\alpha)\Gamma^{1 - \alpha}}{\Gamma(3 - \alpha)} + 4\sigma^3 + 4\sigma^2 - \sigma \right] \|u^1\|^2
$$

where

$$
\sum_{j=1}^{k-1} \|f^{j+\sigma}\|^2
$$

and

$$
\|u^0\|^2 + 8\tau\sigma\Gamma(1 - \alpha)\sum_{j=1}^{k-1} \|f^{j+\sigma}\|^2.
$$

The proof of Theorem 2.1 is completed.

With the above proof, the convergence of the difference scheme (2.8) is easy to obtain.

**Theorem 2.2.** Let $u(x,t)$ be the sufficiently smooth exact solution of (1.1), $\{u^j_i \mid x_i \in \mathcal{W}_h, 0 \leq j \leq M\}$ be the solution of the problem (2.3). Let $e^j_i = u(x_i,t_j) - u^j_i$ $(0 \leq i \leq N, 0 \leq j \leq M)$ and $e^j = [e^j_1, e^j_2, \cdots, e^j_N]^{T}$ $(0 \leq j \leq M)$. Then, for $j = 0, 1, 2, \cdots, M$, we have

$$
\|e^j\| \leq C_2(\tau^2 + h^2), 0 \leq j \leq M,
$$

where $C_2$ is a positive constant, which may depend on $\alpha$ and $T$.

**Proof.** Subtracting (2.3) from (1.1), the error equations are represented as:

$$
\begin{align*}
\delta_t e^j_i + \Delta_{x,\tau,x} e^i_i = \Lambda e^{i+\sigma} + R^j_i, & \quad 1 \leq i \leq N - 1, \quad 0 \leq j \leq M - 1, \\
e^{i-1}_i = e^0_i = 0, & \quad 0 \leq i \leq N, \\
e^0_i = e^{N-1}_i = 0, & \quad 0 \leq j \leq M,
\end{align*}
$$

with $R^j_i = O(\tau^2 + h^2)$. Then the following procedure is similar to Theorem 2.1, the error $e^j$ yields

$$
\|e^j\| \leq C_2(\tau^2 + h^2), 0 \leq j \leq M,
$$

where $C_2$ is a positive constant, which may depend on $\alpha$ and $T$.

Theorem 2.2 implies that our numerical scheme converges to the optimal order $O(\tau^2 + h^2)$ in the $L_2$-norm, when the solution of Eq. (1.1) is sufficiently smooth. If the solution of Eq. (1.1) is non-smooth, several interesting alternative approaches [45, 46] have been introduced to address this problem.
For convenience, Eq. (2.9) can be rewritten into the equivalent matrix form:

\[
\mathcal{M}^{j+1} u^{j+1} = B^j u^j - \frac{2}{\Gamma(2-\alpha)} h^2 \sum_{s=0}^{j-1} c_{j-s}^{(\alpha, \sigma)} (u^{s+1} - u^s) + 2\tau h^2 f^{j+\sigma} + \eta^{j+\sigma}, \quad 0 \leq j \leq M - 1,
\]

where \( f^{j+\sigma} = [f_1^{j+\sigma}, f_2^{j+\sigma}, \ldots, f_{N-1}^{j+\sigma}]^T, u_0 = [u_0(x_1), u_0(x_2), \ldots, u_0(x_{N-1})]^T, \)

\[
\eta^{j+\sigma} = 2\tau \sigma k_1^{j+\sigma} u_0^j + (1 - \sigma) k_1^{j+\sigma} u_0^j, 0, \ldots, 0, \sigma k_N^{j+\sigma} u_0^j + (1 - \sigma) k_{N-1}^{j+\sigma} u_0^j\]

and

\[
\mathcal{M}^{j+1} = \left[ h^2 (2\sigma + 1) + \frac{2\tau^{-\alpha} h^2}{\Gamma(2-\alpha)} \right] I - 2\sigma \tau (A^{j+\sigma} - h^2 Q^{j+\sigma}),
\]

\[
B^j = \left( 4h^2 \sigma + \frac{2\tau^{-\alpha} h^2}{\Gamma(2-\alpha)} \right) I + 2(1 - \sigma) \tau (A^{j+\sigma} - h^2 Q^{j+\sigma}).
\]

Whereas

\[
A^{j+\sigma} = -\text{diag} \left( \left[ k_1^{j+\sigma}, k_1^{j+\sigma}, k_2^{j+\sigma}, k_2^{j+\sigma}, \ldots, k_N^{j+\sigma}, k_{N-1}^{j+\sigma} \right] \right) + \text{diag} \left( \left[ k_1^{j+\sigma}, k_2^{j+\sigma}, k_3^{j+\sigma}, \ldots, k_{N-1}^{j+\sigma}, 1 \right] \right) + \text{diag} \left( \left[ k_1^{j+\sigma}, k_2^{j+\sigma}, k_3^{j+\sigma}, \ldots, k_{N-2}^{j+\sigma} \right] \right),
\]

\( Q^{j+\sigma} = \text{diag}(q_1^{j+\sigma}, q_2^{j+\sigma}, \ldots, q_{N-2}^{j+\sigma}) \) and \( I \) is the identity matrix with an appropriate size. Upon above definitions, it is obvious that the coefficient matrix \( \mathcal{M}^{j+1} \) is a symmetric tridiagonal matrix.

3. The two-dimensional problem of TFRDE

In practical applications, one-dimensional problems are rare, therefore in this section, the two-dimensional (2D) TFRDE is studied:

\[
\frac{\partial u(x, y, t)}{\partial t} + D_0^\alpha_t u(x, y, t) = \mathcal{N} u(x, y, t) + f(x, y, t), \quad (x, y) \in [0, L_x] \times [0, L_y], \quad 0 \leq t \leq T,
\]

(3.1)

with initial condition

\[
u(x, y, 0) = u_0(x, y), \quad (x, y) \in [0, L_x] \times [0, L_y],
\]

(3.2)

and boundary value conditions

\[
u(0, y, t) = \psi_1(y, t), \quad u(L_x, y, t) = \psi_2(y, t), \quad 0 \leq t \leq T,
\]

(3.3)

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and presented in Eq. (2.2), we have

\[ u(x, 0, t) = g_1(x, t), \quad u(x, L_y, t) = g_2(x, t), \quad 0 \leq t \leq T, \]  

(3.4)

where

\[ Nu(x, y, t) = \frac{\partial}{\partial x} \left( d(x, y, t) \frac{\partial u(x, y, t)}{\partial x} \right) + \frac{\partial}{\partial y} \left( k(x, y, t) \frac{\partial u(x, y, t)}{\partial y} \right) - q(x, y, t)u(x, y, t), \]

\[ d(x, y, t) \geq C_3 > 0, \quad k(x, y, t) \geq C_4 > 0, \quad q(x, y, t) \geq 0 \]  

and \( f(x, y, t) \) are sufficiently smooth functions. In the rest of this section, we will deduce a second-order difference scheme and investigate its stability and convergence.

### 3.1. Difference scheme for the 2D TFRDE

Taking two positive integers \( N_x \) and \( N_y \), then \( h_x = \frac{L_x}{N_x}, \quad h_y = \frac{L_y}{N_y} \). Denote

\[ \tilde{\omega} = \{ x_i = ih_x, \quad y_l = lh_y, \quad 0 \leq i \leq N_x, \quad 0 \leq l \leq N_y; \quad x_0 = x_{N_x} = 0, \quad y_0 = y_{N_y} = 0 \}, \]

and

\[ \tilde{S} = \{ v \mid v = (v_{il})_{0 \leq i \leq N_x, \quad 0 \leq l \leq N_y}; \quad v_{0l} = v_{N_xl} = 0, \quad v_{il} = v_{iN_y} = 0 \}. \]

Now the fully discrete scheme is derived. Let

\[ \tilde{\Lambda}u(x_i, y_l, t_j) = \frac{1}{h_x^2} \left[ d(x_i - \frac{1}{2}, y_l, t_{j+\sigma})u(x_{i-1}, y_l, t_j) - d(x_i + \frac{1}{2}, y_l, t_{j+\sigma})u(x_{i+1}, y_l, t_j) \right] \]

\[ \times u(x_i, y_l, t_j) + \frac{1}{h_y^2} \left[ k(x_i, y_l - \frac{1}{2}, t_{j+\sigma})u(x_i, y_{l-1}, t_j) - k(x_i, y_l + \frac{1}{2}, t_{j+\sigma})u(x_i, y_{l+1}, t_j) \right] \]

\[ + k(x_i, y_l, t_{j+\sigma})u(x_i, y_l, t_j) - q(x_i, y_l, t_{j+\sigma})u(x_i, y_l, t_j) \]

be a difference operator approximates the continuous operator \( \mathcal{N} \). Afterwards, similar implementation as presented in Eq. (2.2), we have

\[ \mathcal{N}u(x, y, t) \mid_{(x_i, y_l, t_{j+\sigma})} = \sigma \tilde{\Lambda}u(x_i, y_l, t_{j+\sigma}) + (1 - \sigma) \tilde{\Lambda}u(x_i, y_l, t_j) + \mathcal{O} \left( \tau^2 + h_x^2 + h_y^2 \right), \]

and some other new notations are given based on Section 2

\[ u_{il}^{j+\sigma} = \sigma u_{il}^{j+1} + (1 - \sigma)u_{il}^{j}, \quad d_{il}^{j+\sigma} = d(x_i, y_l, t_{j+\sigma}), \quad k_{il}^{j+\sigma} = k(x_i, y_l, t_{j+\sigma}), \]

\[ q_{il}^{j+\sigma} = q(x_i, y_l, t_{j+\sigma}), \quad f_{il}^{j+\sigma} = f(x_i, y_l, t_{j+\sigma}). \]
Similar to the process of dealing with 1D case in Section 2 obtains

\[ \delta_t u^j_i + \Delta_{ij}^a \sigma = \bar{\Lambda} u^{j+\sigma}_i + f^{j+\sigma}_i. \]

When \( j = 0 \), \( u^{j-1}_i = u^{-1}_i \) is defined outside of \([0,T]\), in the same way as Section 2,

\[ u^{-1}_i = u^0_0 - \tau \frac{\partial u^0_0}{\partial t} + O(\tau^2). \]

In order to obtain the second-order accuracy in time, we assume \( \frac{\partial u(x,y,0)}{\partial t} = 0 \), thus set \( u^{-1}_i = u^0_0 \). Adding the discrete initial-boundary conditions, our approximate scheme for the problem \( \text{(3.1)-(3.4)} \) is

\[
\begin{cases}
\delta_t u^j_i + \Delta_{ij}^a \sigma = \bar{\Lambda} u^{j+\sigma}_i + f^{j+\sigma}_i, & 1 \leq i \leq N_x, 1 \leq l \leq N_y, 1 \leq j \leq M, \\
u^0_0 = u_0(x_i, y_l), & 0 \leq i \leq N_x, 0 \leq l \leq N_y, \\
u^0_j = \psi_1(y_l, t_j), & 0 \leq l \leq N_y, 0 \leq j \leq M, \\
u^j_0 = \psi_2(y_l, t_j), & 0 \leq i \leq N_x, 0 \leq j \leq M.
\end{cases}
\]

(3.5)

3.2. Stability and convergence analysis

In order to probe into the scheme \( \text{(3.5)} \), an inner product and the corresponding norm are defined to facilitate our subsequent analysis

\[ (u, v) = h_x h_y \sum_{i=1}^{N_x-1} \sum_{l=1}^{N_y-1} u_{il} v_{il}, \quad \|u\| = \sqrt{(u, u)}, \quad \forall u, v \in \tilde{\mathcal{S}}. \]

The priori estimate of \( \text{(3.5)} \) is given.

**Theorem 3.1.** Suppose \( \{u^{j+1}_i \mid 0 \leq i \leq N_x, 0 \leq l \leq N_y, 0 \leq j \leq M \} \) be the solution of \( \text{(3.5)} \) and denote \( \|f^{j+\sigma}\|^2 = h_x h_y \sum_{i=1}^{N_x-1} \sum_{l=1}^{N_y-1} f^2(x_i, y_l, t_{j+\sigma}) \). Then the IDS \( \text{(3.5)} \) is unconditionally stable, and the following two priori estimates hold:

\[ \|U^1\|^2 \leq \left( \frac{4T^{1-\alpha}\sigma}{\Gamma(2-\alpha)} + 2\sigma \right) \|U^0\|^2 + 4\sigma T^{1+\alpha}\Gamma(2-\alpha) \|f^\sigma\|^2, \quad (3.6) \]

\[ \|U^k\|^2 \leq C_1 \|U^1\|^2 + \left( \frac{2\sigma T^{1-\alpha}}{\Gamma(2-\alpha)} + 4\sigma^3 \right) \|U^0\|^2 + 8\sigma T^{1+\alpha}\Gamma(1-\alpha) \sum_{j=1}^{k-1} \|f^{j+\sigma}\|^2, \quad k \geq 2, \quad (3.7) \]

where \( C_1 \) is given in Theorem 2.7.
Proof. In this proof, we take advantage of the method in Theorem 2.1 again. Taking the inner product of \( U^{j+\sigma} = \sigma U^{j+1} + (1 - \sigma) U^j \), it results

\[
(\delta_j U^j, U^{j+\sigma}) + (\Delta^\alpha_{0,t_j+\sigma} U_j U^{j+\sigma}) = (\bar{M} U^{j+\sigma}, U^{j+\sigma}) + (f^{j+\sigma}, U^{j+\sigma}).
\]

Using Lemmas 2.4-2.5 and noticing \( (\bar{M} U^{j+\sigma}, U^{j+\sigma}) \leq 0 \), one obtains

\[
\frac{1}{4\tau} (E^{j+1} - E^j) + \frac{1}{2\tau} \Delta^\alpha_{0,t_j+\sigma} \| U \|^2 \leq (f^{j+\sigma}, U^{j+\sigma}). \tag{3.8}
\]

**Step 1.** When \( j = 0 \). From the inequality (3.8), it has

\[
\frac{1}{4\tau} (E^1 - E^0) + \frac{1}{2\tau} \Delta^\alpha_{0,t_j+\sigma} \| U \|^2 \leq (f^\sigma, U^\sigma).
\]

With the aid of virtue Cauchy-Schwarz inequality, we arrive at

\[
\| U^1 \|^2 + \frac{2\tau\sigma}{\Gamma(2 - \alpha)} a^{(\alpha,\sigma)}_0 (\| U^1 \|^2 - \| U^0 \|^2) \leq \frac{2\tau\sigma}{\Gamma(2 - \alpha)} a^{(\alpha,\sigma)}_0 \| U^0 \|^2 + 2\sigma \| U^0 \|^2 + \frac{\tau\sigma}{\varepsilon_3} \| f^\sigma \|^2 + 8\sigma \varepsilon_3 (\| U^1 \|^2 + \| U^0 \|^2), \quad \varepsilon_3 > 0,
\]

Let \( \varepsilon_3 = \frac{1}{4T^\alpha \Gamma(2 - \alpha)} a^{(\alpha,\sigma)}_0 \), it gives immediately the estimate for \( U^1 \), that is

\[
\| U^1 \|^2 \leq \left( \frac{4\tau\sigma}{\Gamma(2 - \alpha)} a^{(\alpha,\sigma)}_0 + 2\sigma \right) \| U^0 \|^2 + \frac{4\tau\sigma T^\alpha \Gamma(2 - \alpha)}{a^{(\alpha,\sigma)}_0} \| f^\sigma \|^2 \leq \frac{4T^{1-\alpha}\sigma}{\Gamma(2 - \alpha)} + 2\sigma \right) \| U^0 \|^2 + 4\sigma T^{1-\alpha} \Gamma(2 - \alpha) \| f^\sigma \|^2.
\]

**Step 2.** When \( j \geq 1 \), summing up for \( j \) in (3.8) from 1 to \( k - 1 \) and doing some simple manipulations, it results

\[
\frac{1}{4\tau} (E^k - E^1) + \frac{1}{2\tau \Gamma(2 - \alpha)} \left[ c^{(\alpha,\sigma)}_0 \sum_{j=1}^{k-1} \| U^{j+1} \|^2 - \sum_{j=1}^{k-1} \sum_{s=2}^{j} (c^{(\alpha,\sigma)}_j - c^{(\alpha,\sigma)}_{j-1}) \| U^s \|^2 \right]
\leq \frac{1}{2\tau \Gamma(2 - \alpha)} \| U^0 \|^2 \sum_{j=1}^{k-1} c^{(\alpha,\sigma)}_j + \frac{1}{2\tau \Gamma(2 - \alpha)} \| U^1 \|^2 \sum_{j=1}^{k-1} (c^{(\alpha,\sigma)}_j - c^{(\alpha,\sigma)}_{j-1}) \| U^s \|^2 + \sum_{j=1}^{k-1} \| f^{j+\sigma} \| \cdot \| U^{j+1} \|$$. \tag{3.9}
To estimate the second term on the left hand side of inequality (3.9), Lemma 2.2 is applied. Then

\[
\frac{1}{2\tau^\alpha \Gamma(2-\alpha)} \left[ \sum_{j=1}^{k-1} \left( \| U_{j+1} \|^2 - \sum_{j=1}^{k-1} \sum_{s=2}^{j} (\epsilon_{j-s}^{(a,\sigma)} - \epsilon_{j-s+1}^{(a,\sigma)}) \| U_s \|^2 \right) \right] = \frac{1}{2\tau^\alpha \Gamma(2-\alpha)} \sum_{j=0}^{k} \epsilon_{k-j}^{(a,\sigma)} \| U_j \|^2 \\
\geq \frac{1}{2\tau^\alpha \Gamma(2-\alpha)} \frac{1}{2} (j - 1 + \sigma)^{-\alpha} \sum_{j=2}^{k} \| U_j \|^2 \\
\geq \frac{1}{4\tau^\alpha \Gamma(1-\alpha)} \sum_{j=2}^{k} \| U_j \|^2.
\]

Bringing above estimate to inequality (3.9) gets

\[
\| U^k \|^2 + \frac{\tau \sigma}{T^\alpha \Gamma(1-\alpha)} \sum_{j=2}^{k} \| U_j \|^2 \\
\leq \sigma E^1 + \frac{2\tau^{1-\alpha} \sigma (4\sigma + 3\lambda)(k - 1 + \sigma)^{1-\alpha}}{\Gamma(3-\alpha)} \| U^1 \|^2 + \frac{2\tau^{1-\alpha} \sigma (k - 1 + \sigma)^{1-\alpha}}{\Gamma(2-\alpha)} \| U^0 \|^2 \\
+ 4\sigma \varepsilon_4 \sum_{j=1}^{k-1} (\sigma \| U_{j+1} \| + (1 - \sigma) \| U_j \|)^2 + \frac{\tau \sigma}{\varepsilon_4} \sum_{j=1}^{k-1} \| f^{j+\sigma} \|^2 \\
\leq \sigma \left[ (4\sigma^2 + 4\sigma - 1) \| U^1 \|^2 + 4\sigma^2 \| U^0 \|^2 \right] + \frac{2\sigma (4\sigma + 3\lambda)^{1-\alpha}}{\Gamma(3-\alpha)} \| U^1 \|^2 \\
+ \frac{2\sigma T^{1-\alpha}}{\Gamma(2-\alpha)} \| U^0 \|^2 + 8\sigma \varepsilon_4 \sum_{j=1}^{k} \| U_j \|^2 + \frac{\tau \sigma}{\varepsilon_4} \sum_{j=1}^{k-1} \| f^{j+\sigma} \|^2, \varepsilon_4 > 0, \quad (3.10)
\]

where \( E^1 \leq (4\sigma^2 + 4\sigma - 1) \| U^1 \|^2 + 4\sigma^2 \| U^0 \|^2 \). Taking \( \varepsilon_4 = \frac{1}{8\sigma \Gamma(1-\alpha)} \), inequality (3.10) leads to

\[
\| U^k \|^2 \leq \left[ \frac{\tau \sigma}{T^\alpha \Gamma(1-\alpha)} + \frac{2\sigma (4\sigma + 3\lambda) T^{1-\alpha}}{\Gamma(3-\alpha)} + 4\sigma^3 + 4\sigma^2 - \sigma \right] \| U^1 \|^2 \\
+ \left( \frac{2\tau T^{1-\alpha}}{\Gamma(2-\alpha)} + 4\lambda^3 \right) \| U^0 \|^2 + 8\sigma T^\alpha \Gamma(1-\alpha) \sum_{j=1}^{k-1} \| f^{j+\sigma} \|^2 \\
\leq \left[ \frac{T^{1-\alpha} \sigma}{\Gamma(1-\alpha)} + \frac{2\sigma (4\sigma + 3\lambda) T^{1-\alpha}}{\Gamma(3-\alpha)} + 4\sigma^3 + 4\sigma^2 - \sigma \right] \| U^1 \|^2 \\
+ \left( \frac{2\tau T^{1-\alpha}}{\Gamma(2-\alpha)} + 4\lambda^3 \right) \| U^0 \|^2 + 8\sigma T^{1+\alpha} \Gamma(1-\alpha) \sum_{j=1}^{k-1} \| f^{j+\sigma} \|^2.
\]

Hence, the targeted results are immediately completed. \( \square \)

Next, the convergence of (3.5) is discussed.

**Theorem 3.2.** Assume \( u(x, y, t) \) be the sufficiently smooth exact solution of (3.1) - (3.5), \( u_i(t) \mid x_i \in \tilde{\omega}, 0 \leq x_i \leq R \).
Let $\xi^j_{il} = u(x_i, y_j, t_l) - u^j_{il}$ \((0 \leq i \leq N_x, 0 \leq l \leq N_y, 0 \leq j \leq M)\). Then, for $j = 0, 1, 2, \cdots, M$, we have

$$
\|\xi^j\| \leq C_3 (r^2 + h_x^2 + h_y^2), \quad 0 \leq j \leq M,
$$

where $C_3$ is a positive constant, which may depend on $\alpha$ and $T$.

**Proof.** Subtracting (3.5) from (3.1)-(3.4), the error equations are

$$
\begin{aligned}
\delta_t \xi^j_{il} + \Delta_y \xi^j_{il} = \tilde{A} \xi^{j+\sigma}_{il} + \tilde{R} \xi^{j+\sigma}_{il}, \quad &1 \leq i \leq N_x - 1, 1 \leq l \leq N_y - 1, 0 \leq j \leq M - 1, \\
\xi^{-1}_{il} = \xi^0_{il} = 0, \quad &0 \leq i \leq N_x, 0 \leq l \leq N_y, \\
\xi^j_{il} = \xi^j_{N_x,l} = 0, \quad &1 \leq l \leq N_y, 0 \leq j \leq M, \\
\xi^j_{il} = \xi^j_{l,N_y} = 0, \quad &0 \leq i \leq N_x, 0 \leq j \leq M,
\end{aligned}
$$

with $\tilde{R} = O(r^2 + h_x^2 + h_y^2)$. After that, the following procedure is similar to Theorem 3.1 and the error $\xi^j$ yields

$$
\|\xi^j\|_{2D} \leq C_3 (r^2 + h_x^2 + h_y^2), \quad 0 \leq j \leq M,
$$

where $C_3$ is a positive constant, which may depend on $\alpha$ and $T$. \qed

Similar to the 1D case, if the solution of Eq. (3.1) is non-smooth, several alternative approaches can be used to address this problem. Some additional symbols are needed for presentation of the equivalent matrix form of the IDS (3.5).
and

\[ v_1^{j+\sigma} = \begin{bmatrix} k_{1,1}^{j+\sigma} & \sigma u_1^{j+1} + (1 - \sigma) u_1^j & \ldots & k_{N_{\sigma}-1,1}^{j+\sigma} & \sigma u_{N_{\sigma}-1}^{j+1} + (1 - \sigma) u_{N_{\sigma}-1}^j \end{bmatrix}, \]

\[ v_2^{j+\sigma} = \begin{bmatrix} k_{1,1}^{j+\sigma} & \sigma u_{1,1}^{j+1} + (1 - \sigma) u_{1,1}^j & \ldots & k_{N_{\sigma}-1,1}^{j+\sigma} & \sigma u_{N_{\sigma}-1,1}^{j+1} + (1 - \sigma) u_{N_{\sigma}-1,1}^j \end{bmatrix}. \]

Finally, the equivalent matrix form of (3.5) below is derived to complete this section.

\[
\begin{aligned}
S^{j+1} \tilde{U}^{j+1} &= \mathcal{P}^{j} \tilde{U}^{j} - (2\sigma - 1)h_x^2 h_y^2 \tilde{U}^{j-1} - \frac{2\tau^{1-\alpha} h_x^2 h_y^2}{\Gamma(2 - \alpha)} \sum_{s=0}^{j-1} \epsilon_s^{(\alpha,\sigma)} \left( \tilde{U}^{s+1} - \tilde{U}^{s} \right) \\
&\quad + 2\tau h_x^2 h_y^2 \tilde{f}^{j+\sigma} + 2\tau h_x^2 h_y^2 \tilde{c}(\xi_1^{j+\sigma} + \xi_2^{j+\sigma}), \quad 0 \leq j \leq M - 1,
\end{aligned}
\]

(3.11)

in which

\[
S^{j+1} = \left( 2\sigma + 1 \right) h_x^2 h_y^2 + \frac{2\tau^{1-\alpha} h_x^2 h_y^2}{\Gamma(2 - \alpha)} \epsilon_0^{(\alpha,\sigma)} I - 2\tau \sigma \left( h_x^2 \tilde{A}^{j+\sigma} + h_y^2 \tilde{B}^{j+\sigma} - h_x^2 h_y^2 \tilde{Q}^{j+\sigma} \right),
\]

\[
P^{j} = \left[ 4\sigma h_x^2 h_y^2 + \frac{2\tau^{1-\alpha} h_x^2 h_y^2}{\Gamma(2 - \alpha)} \epsilon_0^{(\alpha,\sigma)} \right] I + 2\tau (1 - \sigma) \left( h_x^2 \tilde{A}^{j+\sigma} + h_y^2 \tilde{B}^{j+\sigma} - h_x^2 h_y^2 \tilde{Q}^{j+\sigma} \right)
\]

and

\[
\tilde{u}_0 = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_{N_{\sigma}-2} \\ \tilde{u}_{N_{\sigma}-1} \end{bmatrix}, \quad \tilde{\xi}_1^{j+\sigma} = \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \vdots \\ \tilde{\xi}_{N_{\sigma}-2} \\ \tilde{\xi}_{N_{\sigma}-1} \end{bmatrix}, \quad \tilde{\xi}_2^{j+\sigma} = \begin{bmatrix} v_1^{j+\sigma} \\ 0 \\ \vdots \\ 0 \\ v_2^{j+\sigma} \end{bmatrix}, \quad (N_{\sigma}-1) \times (N_{\sigma}-1)
\]

whereas

\[
\tilde{A}^{j+\sigma} = \text{diag} \left( \tilde{A}_1^{j+\sigma}, \tilde{A}_2^{j+\sigma}, \ldots, \tilde{A}_N_{\sigma-1}^{j+\sigma} \right), \quad \tilde{Q}^{j+\sigma} = \text{diag} \left( \tilde{Q}_1^{j+\sigma}, \tilde{Q}_2^{j+\sigma}, \ldots, \tilde{Q}_{N_{\sigma}-1}^{j+\sigma} \right)
\]
\[ \hat{B}^{j+\sigma} = \text{diag} \left( \left[ B_1^{j+\sigma}, B_2^{j+\sigma}, \ldots, B_{N_y+1}^{j+\sigma} \right] \right) + \text{diag} \left( \left[ B_2^{j+\sigma}, B_3^{j+\sigma}, \ldots, B_{N_y+1}^{j+\sigma} \right], -1 \right) \\
\quad + \text{diag} \left( \left[ B_2^{j+\sigma}, B_3^{j+\sigma}, \ldots, B_{N_y+1}^{j+\sigma} \right], 1 \right). \]

Investigation on the expression of \( S^{j+1} \), it can be found that the coefficient matrix \( S^{j+1} \) is a large sparse banded symmetric matrix. For the sake of clarity, Fig. 1 is an example of \( S^{j+1} \) corresponding to \( h_x = h_y = \frac{1}{8} \) and \( \tau = \frac{1}{5} \).

![Fig. 1: Sparsity pattern of \( S^{j+1} \) with \( h_x = h_y = \frac{1}{8} \) and \( \tau = \frac{1}{5} \). Left: \( j = 0 \); Right: \( j = 1 \).](image)

4. Numerical results

Numerical results are provided to validate the error estimates obtained in Theorems 2.2 and 3.2 for the proposed difference schemes (2.3) and (3.5), respectively, which are given in Examples 1-2. Moreover, the method proposed in [53] (denote as Gao’s method) is employed to solve (1.1), and the corresponding errors also reported in Examples 1-2. In Example 3, three preconditioned iterative methods are employed for solving the linear system of the two-dimensional case. For simplicity, when test Examples 2, we take \( h_x = h_y = h \) in this manuscript, and let

\[
\begin{align*}
\text{Error}_1(h, \tau) &= \max_{0 \leq j \leq M} \| e_j \|_\infty, & \text{Error}_2(h, \tau) &= \max_{0 \leq j \leq M} \| e_j \|, \\
\text{Error}_3(h, \tau) &= \max_{0 \leq j \leq M} \| \xi_j \|_\infty, & \text{Error}_4(h, \tau) &= \max_{0 \leq j \leq M} \| \xi_j \|, \\
\text{rate}_{1\tau} &= \frac{\log_{h_x/\tau_2} \text{Error}_1(h_1, \tau_1)}{\text{Error}_1(h_1, \tau_2)}, & \text{rate}_{1h} &= \frac{\log_{h_1/\tau} \text{Error}_1(h_1, \tau)}{\text{Error}_1(h_2, \tau)}, \\
\text{rate}_{2\tau} &= \frac{\log_{h_x/\tau_2} \text{Error}_2(h_1, \tau_1)}{\text{Error}_2(h_1, \tau_2)}, & \text{rate}_{2h} &= \frac{\log_{h_1/\tau} \text{Error}_2(h_1, \tau)}{\text{Error}_2(h_2, \tau)}.
\end{align*}
\]
\[ \text{rate3}_t = \log_{\tau_1/\tau_2} \frac{\text{Error}_3(h, \tau_1)}{\text{Error}_3(h, \tau_2)}, \quad \text{rate3}_h = \log_{h_1/h_2} \frac{\text{Error}_3(h_1, \tau)}{\text{Error}_3(h_2, \tau)} \]

\[ \text{rate4}_t = \log_{\tau_1/\tau_2} \frac{\text{Error}_4(h, \tau_1)}{\text{Error}_4(h, \tau_2)}, \quad \text{rate4}_h = \log_{h_1/h_2} \frac{\text{Error}_4(h_1, \tau)}{\text{Error}_4(h_2, \tau)} \]

All experiments were performed on a Windows 10 (64 bit) desktop-Intel(R) Xeon(R) E5504 CPU 2.00GHz (two processors), 48GB of RAM using MATLAB R2015b.

Table 1: $L_2$-norm and maximum norm errors and convergence orders for Example 1 where $h = 1/2000$.

| $\alpha$ | $\tau$ | Our method | Gao's method |
|----------|--------|-------------|--------------|
|          |        | $\text{Error}_1(h, \tau)$ | rate1 | $\text{Error}_2(h, \tau)$ | rate2 | $\text{Error}_1(h, \tau)$ | rate1 | $\text{Error}_2(h, \tau)$ | rate2 |
| 0.10     | 1/8    | 6.9433e-05  | -     | 5.9972e-05  | 2.0127 | 7.0252e-04  | -     | 4.4325e-04  | -     |
| 1/16     | 1.8594e-05 | 1.9007      | 1.351e-05 | 1.9152      | 1.7601e-04 | 1.9969      | 1.110e-04 | 1.9969      | 1.110e-04 |
| 1/32     | 4.7487e-06 | 1.9693      | 3.4269e-06 | 1.9795      | 4.3992e-05 | 2.0003      | 2.775e-05 | 2.0005      | 2.775e-05 |
| 1/64     | 1.1958e-06 | 1.9896      | 8.6184e-07 | 1.9914      | 1.0966e-05 | 2.0042      | 6.9152e-06 | 2.0047      | 6.9152e-06 |
| 1/128    | 3.0034e-07 | 1.9933      | 2.1639e-07 | 1.9938      | 2.7094e-06 | 2.0170      | 1.7057e-06 | 2.0194      | 1.7057e-06 |
| 0.50     | 1/8    | 5.5452e-04  | -     | 3.4264e-04  | 2.0127 | 9.0968e-04  | -     | 5.7256e-04  | -     |
| 1/16     | 1.4046e-04 | 1.9810      | 8.6759e-05 | 1.9816      | 2.2859e-04 | 1.9926      | 1.4385e-04 | 1.9929      | 1.4385e-04 |
| 1/32     | 3.5268e-05 | 1.9938      | 2.1774e-05 | 1.9944      | 5.7192e-05 | 1.9989      | 3.5981e-05 | 1.9993      | 3.5981e-05 |
| 1/64     | 8.8017e-06 | 2.0025      | 5.4299e-06 | 2.0036      | 1.4267e-05 | 2.0031      | 8.9715e-06 | 2.0038      | 8.9715e-06 |
| 1/128    | 2.1711e-06 | 2.0194      | 1.3362e-06 | 2.0227      | 3.5351e-06 | 2.0129      | 2.2206e-06 | 2.0147      | 2.2206e-06 |
| 0.90     | 1/8    | 1.1057e-03  | -     | 6.9253e-04  | 2.0127 | 1.1521e-03  | -     | 7.2351e-04  | -     |
| 1/16     | 2.7756e-04 | 1.9941      | 1.7381e-04 | 1.9944      | 2.8877e-04 | 1.9963      | 1.8131e-04 | 1.9966      | 1.8131e-04 |
| 1/32     | 6.9391e-05 | 2.0000      | 4.3440e-05 | 2.0004      | 2.7214e-05 | 2.0010      | 4.5288e-05 | 2.0013      | 4.5288e-05 |
| 1/64     | 1.7304e-05 | 2.0036      | 1.0928e-05 | 2.0043      | 1.7987e-05 | 2.0039      | 1.1296e-05 | 2.0046      | 1.1296e-05 |
| 1/128    | 4.2930e-06 | 2.0110      | 2.6833e-06 | 2.0127      | 4.4632e-06 | 2.0108      | 2.7976e-06 | 2.0123      | 2.7976e-06 |
| 0.99     | 1/8    | 1.2116e-03  | -     | 7.6041e-04  | 2.0127 | 1.2156e-03  | -     | 7.6310e-04  | -     |
| 1/16     | 3.0380e-04 | 1.9958      | 1.9066e-04 | 1.9958      | 3.0475e-04 | 1.9960      | 1.9131e-04 | 1.9960      | 1.9131e-04 |
| 1/32     | 7.5972e-05 | 1.9996      | 4.7675e-05 | 1.9997      | 7.6204e-05 | 1.9997      | 4.7834e-05 | 1.9998      | 4.7834e-05 |
| 1/64     | 1.8967e-05 | 2.0020      | 1.1900e-05 | 2.0023      | 1.9024e-05 | 2.0020      | 1.1938e-05 | 2.0024      | 1.1938e-05 |
| 1/128    | 4.7154e-06 | 2.0081      | 2.9550e-06 | 2.0093      | 4.7296e-06 | 2.0080      | 2.9656e-06 | 2.0093      | 2.9656e-06 |

4.1. The 1D case

At first, the 1D TFRDE with zero boundary condition is considered.

Example 1. In this example, we consider the Eq. (1.1) on space interval $[0, L] = [0, 1]$ and time interval $[0, T] = [0, 1]$ with the coefficients $k(x, t) = x \exp(-t) + 1, q(x, t) = t^2 \cos(x)$, and the source term

\[ f(x, t) = x^2(1-x)^2 \left( (3+\alpha) t^{1+\alpha} + \frac{\Gamma(4+\alpha)}{\Gamma(4)} t \right) - t^{1+\alpha} \left\{ \exp(-t) \left( 16x^3 - 18x^2 + 4x \right) - (12x^2 - 12x + 2) \right\} - t^2 \cos(x)x^2(1-x)^2. \]

For the above values, the exact solution is $u(x, t) = t^{1+\alpha}x^2(1-x)^2$.

Firstly, fixing the spatial step $h = 1/2000$ and taking different temporal steps. Table 1 displays the maximum norm errors, $L_2$-norm errors and temporal convergence orders of the IDS for $\alpha = 0.1, 0.5, 0.9, 0.99$. It shows that the convergence order of the scheme in temporal direction is $O(\tau^2)$. It is in accord with the theoretical result in Section 2.2. Although the temporal convergence orders of the proposed method are smaller than the Gao’s method, the errors of the proposed method are slightly better than the Gao’s method. Afterwards, we investigate the spatial convergence rate for a fixed temporal step size $\tau = h$. Table 2 lists the
maximum norm errors, $L_2$-norm errors and spatial convergence rates of the scheme \([2,3]\). From Table 2, the errors of the Gao’s method are smaller than our method. However, the spatial convergence orders of our method are slightly better than the Gao’s method. As predicted by the theoretical estimates, the temporal and spatial approximation orders of our proposed scheme \([2,3]\) are close to 2, i.e., the slopes of the error curves in Fig. 2 is 2, for $\alpha = 0.1, 0.5, 0.9, 0.99$.

4.2. The 2D case

In this subsection, we think about the 2D TFRDE with zero boundary condition.

Example 2. In (3.1)-(3.4), take $L_x = L_y = 1$, $T = 1$ and coefficients $d(x, y, t) = 2 - \sin(xyt)$, $k(x, y, t) = 19$.
exact solutions can be clearly seen. Further illustrate the efficiency of the proposed difference scheme (3.5), Fig. 3 shows surface solutions at $h = 1/1000$. Table 4 also displays that the errors and convergence orders of the two methods are almost the same. To see from Tables 3-4, the numerical solution provided by the difference approximation (3.5) is in good agreement with our theoretical analysis. In Table 3, fix $h = 1/1000$, the errors in maximum norm and $L_2$-norm decrease steadily with the shortening of time step, and the convergence order of time is the expected $O(\tau^2)$. Furthermore, from Table 3 although the temporal convergence orders of the proposed method are slightly bigger than Gao’s method, the errors of our method are smaller than Gao’s method. While in Table 4, the mesh size $\tau = 1/1000$ is chosen and the spatial convergence rates of the scheme (3.5) are also near to two, for $\alpha = 0.1, 0.5, 0.9, 0.99$, which is consistent with the theoretical result in Section 3.2. Table 4 also displays that the errors and convergence orders of the two methods are almost the same. To further illustrate the efficiency of the proposed difference scheme (3.5), Fig. 8 shows surface solutions at $t = 1$ with the mesh sizes $\tau = 1/1000$, $h_x = h_y = 1/32$. The good agreement of simulate solutions with the exact solutions can be clearly seen.

$$f(x, y, t) = x^2(1-x)^2y^2(1-y)^2 \left\{ (4 + \alpha)\tau^{3+\alpha} + \frac{\Gamma(5 + \alpha)}{\Gamma(5)} t^4 \right\} - \left\{ y^2(1-y)^2 \times \right.$$ 

$$\left[ -y\tau \cos(xyt) \left( 4x^3 - 6x^2 + 2x \right) + (2 - \sin(xyt)) \left( 12x^2 - 12x + 2 \right) \right] +$$

$$x^2(1-x)^2 \left[ x\tau \exp(-t) \left( 4y^3 - 6y^2 + 2y \right) + (1 + x\tau \exp(-t)) \left( 12y^2 - 12y + 2 \right) \right]$$

$$- x^2(1-x)^2y^2(1-y)^2(x+y)\tau \right\} (t^{4+\alpha} + 1).$$

Hence the causal solution is $u(x, y, t) = (t^{4+\alpha} + 1)x^2(1-x)^2y^2(1-y)^2$.

As one can see from Tables 3-4, the numerical solution provided by the difference approximation (3.5) is in good agreement with our theoretical analysis. In Table 3, fix $h = 1/1000$, the errors in maximum norm and $L_2$-norm decrease steadily with the shortening of time step, and the convergence order of time is the expected $O(\tau^2)$. Furthermore, from Table 3 although the temporal convergence orders of the proposed method are slightly bigger than Gao’s method, the errors of our method are smaller than Gao’s method. While in Table 4, the mesh size $\tau = 1/1000$ is chosen and the spatial convergence rates of the scheme (3.5) are also near to two, for $\alpha = 0.1, 0.5, 0.9, 0.99$, which is consistent with the theoretical result in Section 3.2. Table 4 also displays that the errors and convergence orders of the two methods are almost the same. To further illustrate the efficiency of the proposed difference scheme (3.5), Fig. 8 shows surface solutions at $t = 1$ with the mesh sizes $\tau = 1/1000$, $h_x = h_y = 1/32$. The good agreement of simulate solutions with the exact solutions can be clearly seen.

| $\alpha$ | $\tau$ | $h = 1/1000$ | $h = 1/2000$ | $h = 1/4000$ | $h = 1/8000$ |
|----------|--------|-------------|-------------|-------------|-------------|
| 0.10     | 1/5    | 1.1189e-05  | 1.5537e-05  | 1.9885e-05  | 2.4233e-05  |
| 0.10     | 1/10   | 2.7763e-06  | 3.2273e-06  | 3.6783e-06  | 4.1293e-06  |
| 0.20     | 1/20   | 6.7644e-07  | 7.8257e-07  | 8.8870e-07  | 9.9483e-07  |
| 0.40     | 1/40   | 1.5089e-07  | 1.8823e-07  | 2.2557e-07  | 2.6291e-07  |
| 0.80     | 1/80   | 3.1178e-08  | 3.7444e-08  | 4.3710e-08  | 5.0976e-08  |

1 + xy \exp(-t), q(x, y, t) = (x + y)t and the source term

$$f(x, y, t) = x^2(1-x)^2y^2(1-y)^2 \left\{ (4 + \alpha)\tau^{3+\alpha} + \frac{\Gamma(5 + \alpha)}{\Gamma(5)} t^4 \right\} - \left\{ y^2(1-y)^2 \times$$

$$\left[ -y\tau \cos(xyt) \left( 4x^3 - 6x^2 + 2x \right) + (2 - \sin(xyt)) \left( 12x^2 - 12x + 2 \right) \right] +$$

$$x^2(1-x)^2 \left[ x\tau \exp(-t) \left( 4y^3 - 6y^2 + 2y \right) + (1 + x\tau \exp(-t)) \left( 12y^2 - 12y + 2 \right) \right]$$

$$- x^2(1-x)^2y^2(1-y)^2(x+y)\tau \right\} (t^{4+\alpha} + 1).$$

Hence the causal solution is $u(x, y, t) = (t^{4+\alpha} + 1)x^2(1-x)^2y^2(1-y)^2$.

As one can see from Tables 3-4, the numerical solution provided by the difference approximation (3.5) is in good agreement with our theoretical analysis. In Table 3, fix $h = 1/1000$, the errors in maximum norm and $L_2$-norm decrease steadily with the shortening of time step, and the convergence order of time is the expected $O(\tau^2)$. Furthermore, from Table 3 although the temporal convergence orders of the proposed method are slightly bigger than Gao’s method, the errors of our method are smaller than Gao’s method. While in Table 4, the mesh size $\tau = 1/1000$ is chosen and the spatial convergence rates of the scheme (3.5) are also near to two, for $\alpha = 0.1, 0.5, 0.9, 0.99$, which is consistent with the theoretical result in Section 3.2. Table 4 also displays that the errors and convergence orders of the two methods are almost the same. To further illustrate the efficiency of the proposed difference scheme (3.5), Fig. 8 shows surface solutions at $t = 1$ with the mesh sizes $\tau = 1/1000$, $h_x = h_y = 1/32$. The good agreement of simulate solutions with the exact solutions can be clearly seen.

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4.3. Preconditioned iterative methods for solving (3.3)

According to the property of operator $\tilde{\Lambda}$ in the proof of Theorem 3.1, the matrix $-2\tau\sigma(h_2^2\tilde{A}^{+\sigma} + h_2^2\tilde{B}^{+\sigma} - h_2^2h_g^2\tilde{Q}^{+\sigma})$ is a symmetric positive definite matrix. Based on this, we can further indicate that the coefficient matrix $S^{i+1}$ in Eq. (3.5) is a sparse symmetric positive definite matrix. Meanwhile, considering that the linear system (3.5) may ill-conditioned, hence, in this work, the aggregation-based multigrid iterative method (AGMG) [48, 49, 51] and the conjugate gradient method (CG) [52] with two preconditioners \footnote{It remarks that such preconditioners are obtained by MATLAB codes: `ichol(S, struct('type', 'ict', 'droptol', 1e-2))` and `ichol(S, struct('type', 'nofill', 'michol', 'on'))`;} are adopted to solve (3.5). For convenience, the two preconditioned CG methods are
Fig. 4: Comparison of the average numbers of iterations for ict(1e-2), michol(0) and AGMG with $\alpha = 0.4, 0.7, 0.99$.

abbreviated as “ict(1e-2)” and “michol(0)” in this subsection, respectively. Two functions $\text{cs\_ltsolve}$ and $\text{cs\_lsolve}$, which are built-in functions of the MATLAB software package CSparse (download from [http://faculty.cse.tamu.edu/davis/SuiteSparse/](http://faculty.cse.tamu.edu/davis/SuiteSparse/)) are used to fast implement $P^{-1}x$, where $P$ represents a preconditioner. See also [50].

\[\text{Py} = @(x) \text{cs\_ltsolve}(L, \text{cs\_lsolve}(L,x))\], where $L$ is a matrix received from $\text{ichol}(S, \text{struct}('type', 'ict', 'droptol', 1e-2))$ or $\text{ichol}(S, \text{struct}('type', 'nofill', 'michol', 'on'))$.

\[\text{2}\text{The MATLAB code is given as } \text{Py} = @(x) \text{cs\_ltsolve}(L, \text{cs\_lsolve}(L,x))\], where $L$ is a matrix received from $\text{ichol}(S, \text{struct}('type', 'ict', 'droptol', 1e-2))$ or $\text{ichol}(S, \text{struct}('type', 'nofill', 'michol', 'on'))$.
Fig. 5: Comparison of CPU time for ict(1e-2), michol(0) and AGMG with $\alpha = 0.4, 0.7, 0.99$.

Table 5: Performance of the three proposed preconditioned iterative methods with $\alpha = 0.40, 0.70, 0.99$.

| $\alpha$ | ict(1e-2) | michol(0) | AGMG |
|----------|-----------|-----------|------|
|          | $\tau = h_x = h_y$ | Time | Iter | Time | Iter | Time | Iter |
| $0.40$   | $2^{-6}$  | 1.87 | 28.5 | 1.76 | 26.1 | 1.82 | 14.1 |
|          | $2^{-7}$  | 17.11 | 45.3 | 13.73 | 30.7 | 13.12 | 15.0 |
|          | $2^{-8}$  | 187.22 | 69.4 | 114.32 | 35.6 | 106.50 | 15.0 |
|          | $2^{-9}$  | 2382.33 | 100.4 | 1059.58 | 40.9 | 914.46 | 16.0 |
|          | $2^{-10}$ | 209402.48 | 142.9 | 9688.17 | 46.4 | 8362.73 | 16.0 |
| $0.70$   | $2^{-6}$  | 1.83 | 26.5 | 1.69 | 24.3 | 1.82 | 14.0 |
|          | $2^{-7}$  | 15.67 | 41.6 | 12.57 | 28.6 | 13.20 | 15.0 |
|          | $2^{-8}$  | 171.83 | 62.8 | 112.13 | 33.3 | 107.05 | 15.0 |
|          | $2^{-9}$  | 1960.39 | 91.6 | 1022.63 | 38.3 | 893.47 | 15.1 |
|          | $2^{-10}$ | 21248.69 | 129.7 | 9436.39 | 43.7 | 8385.46 | 16.0 |
| $0.99$   | $2^{-6}$  | 1.67 | 22.0 | 1.60 | 20.5 | 1.77 | 13.0 |
|          | $2^{-7}$  | 14.02 | 32.4 | 11.40 | 23.5 | 12.86 | 14.0 |
|          | $2^{-8}$  | 143.15 | 46.5 | 101.78 | 26.9 | 107.51 | 15.0 |
|          | $2^{-9}$  | 1528.68 | 64.8 | 909.97 | 30.5 | 891.66 | 15.0 |
|          | $2^{-10}$ | 16215.25 | 91.4 | 8397.78 | 34.6 | 8190.02 | 15.0 |

**Example 3.** Above mentioned preconditioned iterative methods are adopted in this example, the coefficients
\[d(x, y, t), k(x, y, t), q(x, y, t), \text{source term } f(x, y, t) \text{ and the exact solution } u(x, y, t) \] are given in Example 2. In the rest of this work, “Time” denotes CPU time for solving (3.5) with a preconditioned iterative method, and “Iter” represents the average number of iterations required to solve this linear system, i.e.,

\[
\text{Iter} = \frac{1}{M} \sum_{m=1}^{M} \text{Iter}(m),
\]

in which \(\text{Iter}(m)\) is the number of iterations required for solving (3.5). Those preconditioned iterative methods terminate if the relative residual error satisfies \(\|r_k\| \leq 10^{-10}\) or the iteration number is more than 1000, where \(r^k\) is the residual vector of the linear system after \(k\) iteration, and the initial guess at each time step is chosen as the zero vector.

In Table 5, the AGMG method is the cheapest one among these three methods, in aspect of average iteration number. Moreover, it shows that the average iteration numbers of AGMG are not strongly depend on the mesh size, see the blue curves in Fig. 4. On the other hand, Fig. 4 implies that the average numbers of iterations of ict(1e-2) grow more rapidly than michol(0) in a same problem. When \(M = N_x = N_y = 2^8, 2^9\) and \(2^{10}\), the calculation time of AGMG method also is the least one among them. Although the calculation times of ict(1e-2) and michol(0) for small test problems \((M = N_x = N_y = 2^6, 2^7)\) are cheaper than AGMG, the average iteration numbers of them are bigger than AGMG. From another point of view, the log-log curves in Fig. 5 are plotted to further display their performances in CPU time. In addition, the CPU time and average iteration number of all these proposed preconditioned iterative methods are decreasing along with the increase of \(\alpha\). As a conclusion, these results are not very satisfactory. So our further work is to seek more economical preconditioners to solve fast problem (3.5).

5. Conclusion

Two implicit finite difference schemes combined with Alikhanov’s \(L2-1_\sigma\) formula are considered for solving both 1D and 2D time fractional reaction-diffusion equations with variable coefficients and time drift term. The unconditional stability and convergence of the schemes in \(L_2\)-norm are derived by the discreted energy method, and the convergence orders of our obtained schemes are two both in time and space, even under maximum norm. Two numerical experiments are reported to verify the theoretical results, which reflect that the schemes indeed have second order accuracy in both time and space. Considering that sometimes the linear system (3.5) may be ill-conditioned, two preconditioned CG methods and AGMG are adopted for solving (3.5), and numerical results are displayed in Example 3. In the future work, the higher-order interpolation approximation to a nonlinear time and space fractional reaction-diffusion equation with variable coefficients will be taken into account.
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