LEVI DECOMPOSITION OF NILPOTENT CENTRALISERS IN CLASSICAL GROUPS

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Abstract. We check that the connected centralisers of nilpotent elements in the orthogonal and symplectic groups have Levi decompositions in even characteristic. This provides a justification for the identification of the isomorphism classes of the reductive quotients as stated in [Liebeck, Seitz; Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras].

1. Introduction

Let $G$ be a linear algebraic group over an arbitrary field $k$ with unipotent radical $U := \mathcal{R}_u(G)$. Then $U$ is by definition a subgroup of $G_{\bar{k}}$, where $G_{\bar{k}}$ is the base change of $G$ to the algebraic closure $\bar{k}$ of $k$. In fact, the subgroup $U$ is defined to be the largest smooth, connected, unipotent normal subgroup of $G_{\bar{k}}$. We say $G$ has a Levi subgroup $L$ if $G_{\bar{k}} = L_{\bar{k}} U$ and $L_{\bar{k}} \cap U = \{1\}$, scheme-theoretically; that is to say, that the following conditions hold:

(1) $L_{\bar{k}}(\bar{k}) \cap U(\bar{k}) = \{1\}$;

(2) $\text{Lie}(L_{\bar{k}}) \cap \text{Lie}(U) = 0$.

The existence (or otherwise) of Levi subgroups is a central issue to address in understanding the subgroup structure of linear algebraic groups. When $k$ is a field of characteristic 0, it is an old theorem of G. D. Mostow [Mos56] that all linear algebraic groups have Levi subgroups. Essentially, the proof relies on Lie’s theorem and exponentiation, both of which fail over fields of characteristic $p > 0$. Indeed, algebraic groups need not have Levi subgroups over such fields. The points $G(W_2(k))$ of a reductive $k$-group $G$ over the length 2 Witt vectors $W_2(k)$ furnish an example of such an algebraic group; see [CGP10, §A.6] for a full account. (Also note that a minimal dimensional faithful representation for $G = \text{SL}_2(W_2(k))$ is constructed in [McN03].) In this case one has a short exact sequence $1 \to g^{[1]} \to G(W_2(k)) \to G \to 1$, where $g = \text{Lie}(G)$ and $g^{[1]}$ is its first Frobenius twist as a $G$-module. Then the (unipotent) vector subgroup $g^{[1]} \subseteq G(W_2(k))$ coincides with the unipotent radical of the latter. One can see that this sequence corresponds to an element of the rational (Hochschild) cohomology group $H^2(G, g^{[1]})$ and indeed one has a suite of examples of $G$-modules $V$ where $H^2(G, V) \neq 0$ each giving rise to a non-split extension of $V$ by $G$ such that $V$ is the unipotent radical of the extension $E$ with no Levi factor. By contrast, if $G$ is a connected linear algebraic group over $k$ with unipotent radical $U$ which is defined over $k$ then [McN14, Thm. B] (see also [Ste13, Thm. 3.3.5]) shows that one can find a filtration of $U$ such that the sections have the structure of modules for $G/U$, and [McN10] points out that the vanishing of the second Hochschild cohomology of these modules is enough to guarantee a Levi subgroup.

Certain interesting situations arise over an imperfect field $k$ since it is possible that the unipotent radical $U$ may fail to be defined over $k$. This can happen in particular when one considers the

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case that $G$ is a pseudo-reductive group. The main result of the monograph [CGP10] asserts that most pseudo-reductive groups arise from Weil restriction of a reductive group across an inseparable extension of $k$. Moreover, if $G'$ is a reductive group that happens to be defined over $k$ and $k'/k$ is an inseparable extension, then the Weil restriction $R_{k'/k}(G')$ is a non-reductive linear algebraic group $G$ whose unipotent radical $U$ is not defined over $k$ but which contains a canonical copy of $G'$ as a Levi subgroup. For a general result on the existence of Levi subgroups in pseudo-reductive groups, see [CGP10, Thm. 3.4.6].

In [Jan04, Prop. 5.10], Jantzen shows, using arguments from [Ric67] that when the characteristic $p$ of $k$ is good for $G$ the (smooth) centraliser $C_G(e)$ of a nilpotent elements $e \in \text{Lie}(G)$ for $G$, a reductive group always has a Levi subgroup. In bad characteristic, this can apparently fail in the exceptional groups (see [LS12, p283]), though we have not found explicit examples. However, in this short note we wish to make the observation:

**Theorem.** Let $G$ be a simple algebraic group of classical type over $k = \bar{k}$ of characteristic 2 and $e \in \text{Lie}(G)$ a nilpotent element. Then $C_G(e)^\circ_{\text{red}}$ has a Levi decomposition.

(The centralisers of nilpotent elements in bad characteristic need not be smooth; the group $C_G(e)^\circ_{\text{red}}$ is the unique smooth group whose $k$-points are the same as that of $C_G(e)^\circ$, hence is the centraliser in the sense of [Spr98].) Most of our work is done by [LS12], which finds a subgroup $L$ of $C_G(e)^\circ$ satisfying (1) above. It remains to show that (2) holds. Chasing through the proof of [LS12, Prop. 5.11] and applying a result of Vasiu we show this is the case.

Having established the existence of a subgroup $L$ satisfying (1), the authors of [LS12] do not appear to have made an attempt to justify their statement in [LS12, Thm. 5.6] that there is an isomorphism $C_G(e)^\circ_{\text{red}}/\mathcal{R}_u(C_G(e)^\circ_{\text{red}}) \cong L$ as algebraic groups and indeed this map can fail to be an isomorphism of algebraic groups, precisely when (2) does not hold. Hence our theorem provides the missing justification.

2. **Proof of the theorem**

In this section $k$ will denote an algebraically closed field of characteristic 2.

The following is a brief version of [LS12, Thm. 5.6]. As explained in the introduction, the proof in op. cit. only establishes the isomorphisms at the level of the abstract groups of points.

**Theorem 2.1.** Let $e$ be a nilpotent element of $\text{Lie}(G)$ where $G = \text{Sp}(V)$ or $\text{O}(V)$ and $V$ is a vector space over $k$. Then there are integers $m_i$ and $a_i$ such that:

(i) If $G = \text{Sp}(V)$, then $C_G(e)^\circ_{\text{red}}/\mathcal{R}_u(C_G(e)) \cong \prod_i \text{Sp}_{2a_i}$.
(ii) If $G = \text{O}(V)$ then $C_G(e)^\circ_{\text{red}}/\mathcal{R}_u(C_G(e)) \cong \prod_{m_i} \text{Sp}_{2a_i} \times \prod_{m_i} I_{a_i}$, where $I_{a_i}$ is either $\text{SO}_{2a_i}$ or $\text{SO}_{2a_i+1}$.

A technical condition related to the action of $e$ on $V$ determines the integers $a_i$ and $m_i$ and the condition by which one decides the isomorphism class of $I_{a_i}$. Then [LS12, Prop. 5.11] finds subgroups $C$ such that $C_G(e)_{\text{red}} = C\mathcal{R}_u(C_G(e))$.

To prove our theorem, we use [Vas05, Thm. 1.2]. Recall that for a field $k$ of characteristic $p$, $\alpha_p$ denotes the height 1 group scheme whose representing Hopf algebra is $k[X]/(X^p)$, the comultiplication being determined by $\Delta(X) = 1 \otimes X + X \otimes 1$. (It is also the first Frobenius kernel of the smooth additive group $\mathbb{G}_a$.) For us, loc. cit. takes the form:
Theorem 2.2 (Vasiu). Let $G$ be a reductive group over $k$. If $G$ has a non-trivial normal unipotent subgroup scheme then $\text{char } k = 2$ and $G$ has a direct factor isomorphic to $\text{SO}_{2n+1}$. Furthermore, if $G = \text{SO}_{2n+1}$ then $U \cong \alpha_2^{2n}$ is the unique such; and $\text{Lie}(U)$ is a $2n$-dimensional module for $\text{SO}_{2n+1}$ of high weight $\varpi_1$.

**Remark 2.3.** In the theorem above, the $2n$-dimensional module $L(\varpi_1)$ is obtained as a quotient of the ‘natural’ Weyl module $V(\varpi_1)$ by the radical of its form; see [Jan03, II.8.21] for a brief discussion.

As is rather well-known (see [Vas05, 2.1]) we have that $\text{SO}_{2n+1}/U \cong \text{Sp}_{2n}$, where $U \cong \alpha_2^{2n}$ is its infinitesimal unipotent normal subgroup. The following is now immediate from the theorem and the fact that $L(\varpi_1)$ is irreducible.

**Corollary 2.4.** Let $G$ be a linear algebraic group over $k$ admitting a reductive subgroup $C$ such that $G = C\mathcal{R}_u(G)$. Then either the quotient map $q : G \to G/\mathcal{R}_u(G)$ restricts to an isomorphism on $C$ or $C$ contains a direct factor $H$ isomorphic to $\text{SO}_{2n+1}$ and the image of $H$ under $q$ is isomorphic to $\text{Sp}_{2n}$.

**Proof of Theorem.** In [LS12, Prop. 5.11] a subgroup $C \subseteq C_G(e)$ is constructed such that $C_G(e)\cap u = C\mathcal{R}_u(C_G(e))$. One finds that $C$ contains direct factors of type $\text{SO}_{2n+1}$ only if $G$ is $O(V)$ for some $V$, hence Corollary 2.4 implies $\text{Lie}(C) \cap \text{Lie}(\mathcal{R}_u(G))$ is trivial when $G = \text{Sp}_{2n}$.

Hence we assume $G$ is $O(V)$ and $C$ contains a direct factor isomorphic to $\text{SO}_{2r+1}$. The proof of [LS12, Prop. 5.11] proceeds by finding an orthonormal basis for $V$ and describing explicitly the action of $e$ on $V$. One finds that the action of $e$ on $V$ is constructed as a direct sum of non-isomorphic indecomposable $ke$-modules which are labelled $W(m_i)$ and $W_i(n)$; a basis of these modules and explicit action of $e$ is given in [LS12, §5.1]. The multiplicity of the module $W(m_i)$ is labelled $a_i$, thus $W(m_i)^{a_i}$ appears as a direct $ke$-summand of $V$. Furthermore, a certain 1-dimensional torus $T \subseteq G$ associated to $e$ is constructed which stabilises each of the indecomposable $ke$-modules above. Then $C$ is constructed as a subgroup of $C_G(T,e) = C_G(T) \cap C_G(e)$. It turns out that the non-zero weight spaces of $T$ on $C_G(e)$ are all of positive weight; denoting the corresponding subgroup by $C_G(e)_{>0}$ we have $C_G(e)_{>0} \subseteq \mathcal{R}_u(C_G(e))$. Thus it suffices to show that $\mathcal{R}_u(C_G(T,e)) \cap C = \{1\}$, scheme-theoretically.

We proceed by identifying, for each direct factor $H$ of type $\text{SO}_{2r+1}$ in $C_G(e)$, a $C_G(T,e)$-submodule of $V$ on which $H$ acts faithfully and on which $\mathcal{R}_u(C_G(T,e))$ acts trivially. This is enough to prove the theorem.

Since $C$ contains a direct factor isomorphic to $\text{SO}_{2r+1}$ we have from [LS12, Lem. 5.10] that $V$ contains a summand of the form $W_l(n)$ with $2(n-l) \leq m_i \leq 2l - 1$. Then following the proof of *loc. cit.* we obtain an action of $\text{SO}_{2n+1}$ on the zero weight space $Z_0$ of the module $Z := W(m_i)^{(a_i)} \perp W_l(n)$. Given the explicit description of the modules $W(m_i)$ and $W_l(n)$ from [LS12, §5.1], we have that $Z_0$ is non-degenerate of dimension $2a_i + 2$. Then the proof of [LS12, Lem. 5.10] describes $\text{SO}_{2n+1}$ as acting on $Z_0$ as the indecomposable module with successive factors being the trivial module $k$, $L(\varpi_1)$ and $k$ again (or $k$, $L(2\varpi_1)$, $k$ if $Y \cong \text{SO}_3 = \text{PGL}_2$). Since the natural module for $\text{SO}_{2n+1}$ is isomorphic to the unique codimension 1-submodule of $Z_0$, we have that $\text{SO}_{2n+1}$ acts faithfully on this module. As is well-known, $\text{SO}_{2n+1}$ is contained in no parabolic subgroup of $\text{O}_{2n+2}$. Hence by the Borel–Tits theorem, the image of $C_G(T,e)$ in $\text{O}_{2n+2}$ must be reductive. Thus its unipotent radical $\mathcal{R}_u(C_G(T,e))$ acts trivially on the faithful $\text{SO}_{2n+1}$-module $Z_0$ as required. □
3. A Question

It is possible for a reductive subgroup $L$ of an algebraic group $G = LU$ to satisfy (1) but not (2). This occurs specifically when $L = \text{SO}_{2n+1} \subset G := \text{Sp}_{2n} \rtimes V$ where $V$ is the natural module for $\text{Sp}_{2n}$. Nevertheless, $G$ evidently does have a Levi subgroup. In light of this, we raise the following question.

**Question 3.1.** Suppose $G$ is an algebraic group over $k = \bar{k}$ with unipotent radical $U$, and $L$ is a subgroup which satisfies $G(k) = L(k)U(k)$. Must $G$ have a Levi factor $L'$ such that $G = L' \rtimes U$?

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