THE LOGBOOK OF POINTED HOPF ALGEBRAS OVER THE SPORADIC SIMPLE GROUPS

N. ANDRUSKIEWITSCH, F. FANTINO, M. GRAÑA AND L. VENDRAMIN

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Introduction

In these notes, we give details on the proofs performed with GAP of the theorems of our paper [AFGV2]. Here we discuss the algorithms implemented for studying

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Nichols algebras over non-abelian groups. For each group we refer to files where the results of our computations are shown. These logs files can be downloaded from our webpages

http://www.mate.uncor.edu/~fantino/afgv-sporadic
http://mate.dm.uba.ar/~lvendram/afgv-sporadic

We believe, and hope, that the details in this work are enough to guide the reader to repeat and corroborate our calculations.

Throughout the paper, we follow the notations and conventions of [AFGV2]. We write ATLAS for any of the references [CC+, WPN+, WWT+].

1. Algorithms

In this section, we explain our algorithms to implement the techniques presented in [AFGV2] Subsect. 2.2).

1.1. Algorithms for type D. Algorithm 1 checks if a given conjugacy class $O_r^G$ of a finite group $G$ is of type D.

**Algorithm 1: Type D**

```plaintext
for s ∈ $O_r^G$ do
    if $(rs)^2 \neq (sr)^2$ then
        Compute the group $H = \langle r, s \rangle$
        if $O_r^H \cap O_s^H = \emptyset$ then
            return true /* the class is of type D */
        end
    end
end
return false /* the class is not of type D */
```

Notice that in Algorithm 1 we need to run over all the conjugacy class $O_r^G$ to look for the element $s$ such that the conditions of [AFGV1, Prop. 3.4] are satisfied. This is not always an easy task. To avoid this problem, we have the random variation of Algorithm 1 that is our Algorithm 2. The key is to pick randomly an element $x$ in the group $G$ and to check if $r$ and $s = xrx^{-1}$ satisfy the conditions of [AFGV1].

**Algorithm 2: Random variation of Algorithm 1**

```plaintext
forall i : 1 ≤ i ≤ N do /* the number of iterations */
    /* randomly chosen */
    x ∈ G;
    s = xrx^{-1}
    if $(rs)^2 \neq (sr)^2$ then
        Compute the group $H = \langle r, s \rangle$
        if $O_r^H \cap O_s^H = \emptyset$ then
            return true /* the class is of type D */
        end
    end
end
return false /* the class is not of type D */
```

Notice that in Algorithm 1 we need to run over all the conjugacy class $O_r^G$ to look for the element $s$ such that the conditions of [AFGV1, Prop. 3.4] are satisfied. This is not always an easy task. To avoid this problem, we have the random variation of Algorithm 1 that is our Algorithm 2. The key is to pick randomly an element $x$ in the group $G$ and to check if $r$ and $s = xrx^{-1}$ satisfy the conditions of [AFGV1].
Prop. 3.4]; if not we repeat the process \( N \) times, where \( N \) is fixed. This naive variation of the Algorithm 1 turns out to be very powerful and allows us to study big sporadic groups such as the Janko group \( J_4 \) or the Fischer group \( F_{i'24} \).

For large groups, it is more economical to implement Algorithms 1 and 2 in a recursive way. Let \( G \) be a finite group represented faithfully, for example, as a permutation group or inside a matrix group over a finite field. We compute \( O_{g_1}^G, \ldots, O_{g_n}^G \), the set of conjugacy classes of \( G \). To decide if these conjugacy classes are of type D, we restrict the computations to be done inside a nice subgroup of \( G \).

Assume that the list of all maximal subgroups of \( G \), up to conjugacy, is known, namely \( M_1, M_2, \ldots, M_k \), with non-decreasing order. Also assume that it is possible to restrict our good representation of \( G \) to every maximal subgroup \( M_i \). Here we say that a representation is good if it allows us to perform our computations in a reasonable time.

Fix \( i \in \{1, \ldots, k\} \). Let \( h \in M_i \) and let \( O_{M_i}^h \) be the conjugacy class of \( h \) in \( M_i \). Since \( M_i \) is a subgroup of \( G \), the element \( h \) belongs to a conjugacy class of \( G \), say \( O_{g}^G \). So, if the class \( O_{M_i}^h \) is of type D, then the class \( O_{g}^G \) is of type D too.

**Algorithm 3: Type D: Using maximal subgroups**

\( O_{g_1}^G, \ldots, O_{g_n}^G \) is the set of conjugacy classes of \( G \)
\( S = \{1, 2, \ldots, n\} \)

```
foreach maximal subgroup \( M \) do
    Compute \( O_{h_1}^M, \ldots, O_{h_m}^M \), the set of conjugacy classes of \( M \)
    foreach \( i : 1 \leq i \leq m \) do
        Identify \( O_{h_i}^M \) with a conjugacy class in \( G \): \( h_i \in O_{g_\sigma(i)}^G \)
        if \( \sigma(i) \in S \) then
            if \( O_{h_i}^M \) is of type D then
                Remove \( \sigma(i) \) from \( S \)
                if \( S = \emptyset \) then
                    return true /* the group is of type D */
            end
        end
    end
    foreach \( j \in S \) do
        if \( O_{g_j}^G \) is of type D then
            Remove \( s \) from \( S \)
            if \( S = \emptyset \) then
                return true /* the group is of type D */
        end
    end
end
return \( S \) /* conjugacy classes not of type D */
```

Notice that to implement Algorithm 3 we need to have not only a good representation for the group \( G \), but we need to know how to restrict the good representation of \( G \) to all its maximal subgroups. This information appears in the ATLAS for many of the sporadic simple groups. So, using the GAP interface to the ATLAS we could implement Algorithm 3 for the sporadic simple groups.
1.2. Structure constants. Some conjugacy classes of involutions are studied with [AFGV2] Prop. 1.7, Equation (1.4)]. For example, in the proof of Theorem 2.2 we claim that the conjugacy class 2A of $L_5(2)$ gives only infinite-dimensional Nichols algebras. This follows from [AFGV2] Prop. 1.8] because $S(2A, 3A, 3A) = 42.$

$$\text{gap> ct := CharacterTable("L5(2)");}$$
$$\text{gap> ClassNames(ct);}$$
$$\text{gap> ClassMultiplicationCoefficient(ct, ct.2a, ct.3a, ct.3a); 42}$$

1.3. Bases for permutation groups. Let $G$ be a group acting on a set $X$. A subset $B$ of $X$ is called a base for $G$ if the identity is the only element of $G$ which fixes every element in $B$, see [DM] Subsection 3.3. In other words,

$$\{g \in G \mid g \cdot b = b, \text{ for all } b \in B \} = 1.$$  

**Lemma 1.1.** Let $G$ be a group acting on a set $X$. Let $B$ be a subset of $X$. The following are equivalent:

1. $B$ is a base for $G$.
2. For all $g, h \in G$ we have: $g \cdot b = h \cdot b$ for all $b \in B$ implies $g = h$.

**Proof.** If $B$ is a base, then $g \cdot b = h \cdot b \Rightarrow (h^{-1} g) \cdot b = b \Rightarrow h^{-1} g = 1 \Rightarrow h = g$. The converse is trivial. $\square$

Let $G$ be a permutation group. With the GAP function `BaseOfGroup` we compute a base for $G$. We use `OnTuples` to encode a permutation and `RepresentativeAction` to decode the information. This enables us to reduce the size of our log files.

1.4. An algorithm for involutions. We describe here an algorithm used to discard some conjugacy classes of involutions. In this work, we use this algorithm for the classes called $2A$ in $O_7(3)$, $S_6(2)$, $S_8(2)$, $Co_2$, $Fi_{22}$, $Fi_{23}$ and $B$.

Let $O$ be one of the classes $2A$ in $O_7(3)$, $S_6(2)$, $S_8(2)$ or $Co_2$, and $g \in O$. It is enough to consider the irreducible representations $\rho$ of the corresponding centralizer such that $\rho(g) = -1$, see [AFGV2] (1.3)]. For these conjugacy classes, the remaining representations $\rho$ satisfy $\deg \rho > 4$, as can be seen from the character tables. By [AFGV2] Lemma 1.3, we are reduced to find an involution $x$ such that $gh = h g$, for $h = x g x^{-1}$, and to compute the multiplicities of the eigenvalues of $\rho(h)$. For this last task, we use [AFGV2] Remark 1.4. See the proofs of Theorems [2.1] and Lemma[2.11] for the classes $2A$ of $O_7(3)$, $2A$ of $S_6(2)$ and $2A$ of $S_8(2)$, respectively, and the file `Co2/2A.log` for $2A$ of $Co_2$.

Let $O$ be the class $2A$ of $Fi_{22}$; it has 3510 elements. Let $\rho$ be an irreducible representation of the corresponding centralizer. The group $O_7(3)$ is a maximal subgroup of $Fi_{22}$ and the class $2A$ of $O_7(3)$ is contained in the class $2A$ of $Fi_{22}$ – see the file `Fi22/2A.log`. By [AFGV2] Lemma 1.5, $\dim \mathcal{B}(O, \rho) = \infty$.

Let $O$ be the class $2A$ of $Fi_{23}$. Let $\rho$ be an irreducible representation of the corresponding centralizer. The group $S_8(2)$ is a maximal subgroup of $Fi_{23}$ and the class $2A$ of $S_8(2)$ is contained in the class $2A$ of $Fi_{23}$. By [AFGV2] Lemma 1.5, $\dim \mathcal{B}(O, \rho) = \infty$.

Let $O$ be the class $2A$ of $B$. Let $\rho$ be an irreducible representation of the corresponding centralizer. The group $Fi_{23}$ is a maximal subgroup of $B$ and the class $2A$ of $Fi_{23}$ is contained in the class $2A$ of $B$ – see the file `B/step2.log`. By [AFGV2] Lemma 1.5, $\dim \mathcal{B}(O, \rho) = \infty$.

2. Some auxiliary groups

In this section we study some groups that appear as subquotients of some sporadic simple groups.
2.1. The groups $A_9$, $A_{11}$, $A_{12}$, $S_{12}$. In [AFGV1], we classify the conjugacy classes of type D in alternating and symmetric groups. In Table 1 we list all non-trivial permutations such that their conjugacy classes are not of type D for the groups $A_9$, $A_{11}$, $A_{12}$, $S_{12}$.

| Group | Not of type D | Log file |
|-------|---------------|----------|
| $A_9$ | $(1 2 3)$     | $A9/A9.log$ |
| $A_{11}$ | $(1 2 3)$, $(1234567891011)$, $(1234567891110)$ | $A11/A11.log$ |
| $A_{12}$ | $(1 2 3)$, $(1234567891011)$, $(1234567891012)$ | $A12/A12.log$ |
| $S_{12}$ | $(1 2)$, $(1 2 3)$ | $S12/S12.log$ |

2.2. The group $L_5(2)$. This group has order 9999360. It has 27 conjugacy classes. To study Nichols algebras over this group we use the representation inside $S_{31}$ given in the ATLAS.

**Lemma 2.1.** Every non-trivial conjugacy class of $L_5(2)$, except $2A$ and those with representatives of order $31$, is of type D.

**Proof.** We perform Algorithm I see the file $L5(2)/L5(2).1og$ for details. □

**Theorem 2.2.** The group $L_5(2)$ collapses.

**Proof.** Let $O$ be the class $2A$: then $\dim \mathcal{B}(O, \rho) = \infty$ for any irreducible representation $\rho$ of the corresponding centralizer since $S(2A, 3A, 3A) = 42$ and [AFGV2 Prop. 1.8] applies. Now the result follows from Lemma 2.1 and from [AFGV2 Lemma 1.2] for the conjugacy classes with representatives of order 31 since these are quasi-real of type $j = 2$ and $g_j^2 \neq g$. □

2.3. The group $O_7(3)$. This group has order 4585351680. It has 58 conjugacy classes. For computations we use a representation inside $S_{351}$ given in the ATLAS.

**Lemma 2.3.** Every non-trivial conjugacy class of $O_7(3)$, except $2A$, is of type D.

**Proof.** We perform Algorithm II see the file $O7(3)/O7(3).1og$ for details. □

**Theorem 2.4.** The orthogonal group $O_7(3)$ collapses.

**Proof.** By Lemma 2.3 it remains to study the conjugacy class $2A$. For this conjugacy class we use [AFGV2 Lemma 1.3]. See the file $O7(3)/2A.1og$ for details. □

2.4. The group $O_8^+(2)$. This group has order 174182400. It has 53 conjugacy classes. For the computations we construct a permutation representation.

**Lemma 2.5.** Every non-trivial conjugacy class with representative of order distinct from 2, 3 is of type D.

**Proof.** We perform Algorithm II see the file $O8+(2)/O8+2(2).1og$ for details. □

**Theorem 2.6.** The group $O_8^+(2)$ collapses.
Table 2. Involutions in $O_8^+(2)$.

| Class | Size  | $S(2A, 3E, 3E) = 81$ |
|-------|-------|---------------------|
| 2A    | 1575  |                     |
| 2B    | 3780  | $S(2B, 3E, 3E) = 108$ |
| 2C    | 3780  | $S(2C, 3E, 3E) = 108$ |
| 2D    | 3780  | $S(2D, 3E, 3E) = 108$ |
| 2E    | 56700 | $S(2E, 3E, 3E) = 486$ |

Proof. By Lemma 2.5 it remains to consider the conjugacy classes with representative of order 2 or 3. For the five conjugacy classes of involutions in $O_8^+(2)$ we use [AFGV2, Prop. 1.8]. See Table 2 for details.

On the other hand, the conjugacy classes $3A$, $3B$, $3C$, $3D$, $3E$ of $O_8^+(2)$ are real, so [AZ, Lemma 2.2], cf. [AFGV2, Lemma 1.2], applies, and the result follows. □

2.5. The group $O_{10}^+(2)$. This group has order 25 015 379 558 400. It has 115 conjugacy classes. For the computations we use the representation inside $S_{495}$ given in the ATLAS.

Lemma 2.7. Every non-trivial conjugacy class, except the conjugacy classes $2A$, $3A$, $11A$, $11B$, $33A$, $33B$, $33C$, $33D$, is of type D.

Proof. We perform Algorithm 2 see the file $010-(2)/010-(2).log$ for details. □

Theorem 2.8. The group $O_{10}^+(2)$ collapses.

Proof. By Lemma 2.7 it remains to study the conjugacy classes $2A$, $3A$, $11A$, $11B$, $33A$, $33B$, $33C$, $33D$. Let $O$ be the class $2A$; then $\dim \mathfrak{B}(O, \rho) = \infty$ for any irreducible representation $\rho$ of the corresponding centralizer since $S(2A, 3F, 3F) = 243$ and [AFGV2, Prop. 1.8] applies. For the class $3A$ use [AZ, Lemma 2.2], since it is a real conjugacy class. And for the classes $11A$, $11B$, $33A$, $33B$, $33C$, $33D$ use [AFGV2, Lemma 1.2], since the classes $11A$, $11B$ (resp. $33A$, $33B$, $33C$, $33D$) are quasi-real of type $j = 3$ (resp. $j = 4$) with $g^j \neq g$. □

2.6. The exceptional group $G_2(4)$. In this section we prove that the group $G_2(4)$ collapses. This group has order 251 596 800. It has 32 conjugacy classes. In particular, the conjugacy classes with representatives of order 2 or 3 are the following:

| Name | Centralizer size |
|------|------------------|
| 2A   | 61440            |
| 2B   | 3840             |
| 3A   | 60480            |
| 3B   | 180              |

Lemma 2.9. The conjugacy classes of $G_2(4)$ with representatives of order distinct from 2, 3 are of type D.

Proof. We perform Algorithm 3 see the file $G2(4)/G2(4).log$ for details. □

Theorem 2.10. The group $G_2(4)$ collapses.

Proof. By Lemma 2.9 it remains to study the conjugacy classes with representatives of order 2 or 3. For the two conjugacy classes of involutions use [AFGV2, Prop. 1.8], because $S(2A, 3B, 3B) = 171$ and $S(2B, 3A, 3A) = 126$. For the conjugacy classes with representatives of order 3 use [AFGV2, Lemma 1.2], because these conjugacy classes are real. □
2.7. The exceptional groups $G_2(3)$ and $G_2(5)$. For the orders and number of conjugacy classes of the groups $G_2(3)$ and $G_2(5)$ see Table 3. We have the following result.

**Theorem 2.11.** The groups $G_2(3)$ and $G_2(5)$ are of type D. Hence, they collapse.

**Proof.** We perform Algorithm 3, see Table 3 for the log files.

**Table 3.** Some Chevalley groups of type D.

| Group   | Order       | Conjugacy classes | Log file            |
|---------|-------------|-------------------|---------------------|
| $G_2(3)$ | 4 245 696   | 23                | $G2(3)/G2(3).log$  |
| $G_2(5)$ | 5 859 000 000 | 44             | $G2(5)/G2(5).log$  |

2.8. The symplectic group $S_6(2)$. This group has order 1 451 520. It has 30 conjugacy classes. For the computations we use the representation of $S_6(2)$ inside $S_{28}$ given in the ATLAS.

**Lemma 2.12.** Every non-trivial conjugacy class of $S_6(2)$, with the possible exception of 2A, 2B, 3A, is of type D.

**Proof.** We perform Algorithm 2, see the file $S6(2)/S6(2).log$ for details.

**Theorem 2.13.** The group $S_6(2)$ collapses.

**Proof.** By Lemma 2.12 it remains to study the classes 2A, 2B, 3A. For the conjugacy class 2A use [AFGV2, Lemma 1.3], see the file $S6(2)/2A.log$ for details. For the conjugacy class 2B use [AFGV2, Prop. 1.8], since $S(2B,3C,3C) = 27$. For the conjugacy class 3A use [AZ, Lemma 2.2], since this conjugacy class is real.

2.9. The symplectic group $S_8(2)$. This group has order 47 377 612 800. It has 81 conjugacy classes. For the computations we use the representation of $S_8(2)$ inside $S_{120}$ given in the ATLAS.

**Lemma 2.14.** Every non-trivial conjugacy class, except 2A, 2B, 3A, is of type D.

**Proof.** We perform Algorithm 2, see the file $S8(2)/S8(2).log$ for details.

**Lemma 2.15.** Let $O$ be one of the classes 2A, 2B, 3A. Then $\dim \mathfrak{b}(O,\rho) = \infty$ for any irreducible representation $\rho$ of the corresponding centralizer.

**Proof.** For the real conjugacy class 3A use [AFGV2, Lemma 1.2]. For the conjugacy class 2A use [AFGV2, Lemma 1.3], see the file $S8(2)/2A.log$ for details. For the conjugacy class 2B use [AFGV2, Prop. 1.8], since $S(2B,3C,3C) = 135$.

2.10. The automorphism group of the Tits group. In this subsection we study Nichols algebras over the group Aut($^2F_4(2)'$) $\simeq$ Aut($^2F_4(2)$) $\simeq$ $^2F_4(2)$, the automorphism group of the Tits group, see [GL]. This group has order 35 942 400. It has 29 conjugacy classes.

**Lemma 2.16.** Every non-trivial conjugacy class of $^2F_4(2)$, with the exception of the class 2A, of size 1755, is of type D.

**Proof.** We perform Algorithm 1. See the file $2F4(2)/2F4(2).log$ for details.

**Theorem 2.17.** The group $^2F_4(2)$ collapses.

**Proof.** By Lemma 2.16 it remains to study the conjugacy class 2A. For this conjugacy class use [AFGV2, Prop. 1.8], because $S(2A,3A,3A) = 27$. 

- Theorem 2.11.
- Table 3.
- Lemma 2.12.
- Lemma 2.13.
- Lemma 2.14.
- Lemma 2.15.
- Theorem 2.16.
- Theorem 2.17.
2.11. **Direct products.** In this subsection we study some direct products of groups that appear as subgroups or subquotients of the sporadic simple groups.

The group $A_4 \times G_2(4)$. In this group every non-trivial conjugacy class with representative of order distinct from $2, 3, 6$ is of type D. This follows from Lemma 2.9 and [AFGV2, Lemma 2.8].

The group $3 \times G_2(3)$. In this group every non-trivial conjugacy class with representative of order distinct from $3$ is of type D since every non-trivial conjugacy class of $G_2(3)$ is of type D. This follows from Theorem 2.11 and [AFGV2, Lemma 2.8].

The group $A_9 \times S_3$. In this group every non-trivial conjugacy class with representative of order distinct from $2, 3, 6$ is of type D. See the file $A9xS3/A9xS3.log$ for the computations.

The group $S_5 \times L_3(2)$. In this group every non-trivial conjugacy class with representative of order distinct from $2, 3, 4, 6, 7$ is of type D. This was proved with Algorithm 1 – see the file $S5xL3(2)/S5xL3(2).log$ for details.

3. **Proof of Theorem II**

In Table 4 we list the conjugacy classes of the sporadic simple groups that are not necessarily of type D. We use the phrase “all collapse” to indicate those groups where all non-trivial conjugacy classes are of type D. Also we give a reference about either the algorithms or the subsections where these groups are treated.

In the next subsections, we deal with some large sporadic groups. In order to study these groups we study their maximal subgroups with Algorithms 1 or 2. The fusion of the conjugacy classes of the maximal subgroups of a sporadic group is stored in the ATLAS, up to the Monster group $M$. For this group the fusion of the conjugacy classes is known only for some of its maximal subgroups; furthermore, the list of all maximal subgroups of $M$ is not known. In the case of the Baby Monster group $B$ all the maximal subgroups and the fusions of conjugacy classes are known except the fusion of the conjugacy classes of the sixth maximal subgroup.

In each case we split the proof into several steps according to the corresponding maximal subgroup. We collect in tables and logs the relevant information.

3.1. **The Lyons group $Ly$.**

**Step 1.** The maximal subgroup $M_1 \simeq G_2(5)$.

In $G_2(5)$ every non-trivial conjugacy class is of type D, see Theorem 2.11. Therefore, the conjugacy classes $2A, 3A, 3B, 4A, 5A, 5B, 6A, 6B, 6C, 7A, 8A, 8B, 10A, 10B, 12A, 12B, 15A, 15B, 15C, 20A, 21A, 21B, 24A, 24B, 24C, 25A, 30A, 30B, 31A, 31B, 31C, 31D, 31E of $Ly$ are of type D.

**Step 2.** The maximal subgroup $M_4 \simeq 2.A_{11}$. 

### Table 4. Conjugacy classes not known of type D.

| $G$     | Conjugacy classes not necessarily of type D | Reference |
|---------|---------------------------------------------|-----------|
| $M_{11}$ | 8A, 8B, 11A, 11B                             | Algorithm 3 |
| $M_{12}$ | 11A, 11B                                     | Algorithm 3 |
| $M_{22}$ | 11A, 11B                                     | Algorithm 3 |
| $M_{23}$ | 23A, 23B                                     | Algorithm 3 |
| $M_{24}$ | 23A, 23B                                     | Algorithm 3 |
| $J_1$   | 15A, 15B, 19A, 19B, 19C                      | Algorithm 3 |
| $J_2$   | 2A, 3A                                       | Algorithm 3 |
| $J_3$   | 5A, 5B, 19A, 19B                             | Algorithm 3 |
| $Suz$   | 3A                                           | Algorithm 3 |
| $Ru$    | 29A, 29B                                     | Algorithm 3 |
| $HS$    | 11A, 11B                                     | Algorithm 3 |
| $He$    | all collapse                                 | Algorithm 3 |
| $McL$   | 11A, 11B                                     | Algorithm 3 |
| $Co_3$  | 23A, 23B                                     | Algorithm 3 |
| $Co_2$  | 2A, 23A, 23B                                 | Algorithm 3 |
| $O'N$   | 31A, 31B                                     | Algorithm 3 |
| $Fi_{22}$ | 2A, 22A, 22B                                 | Algorithm 3 |
| $T = 2F_4(2)'$ | 2A | Algorithm 3 |
| $Co_1$  | 3A, 23A, 23B                                 | §3.6 |
| $Fi_{23}$ | 2A, 23A, 23B                                 | §3.4 |
| $HN$    | all collapse                                 | §3.6 |
| $Th$    | all collapse                                 | §3.2 |
| $Ly$    | 33A, 33B, 37A, 37B, 67A, 67B, 67C             | §3.1 |
| $J_4$   | 29A, 37A, 37B, 37C, 43A, 43B, 43C             | §3.9 |
| $Fi_{24}$ | 23A, 23B, 27B, 27C                           | §3.7 |
| $B$     | 2A, 16C, 16D, 32A                            | §3.8 |
|         | 32B, 32C, 32D, 34A                           | §3.8 |
|         | 46A, 46B, 47A, 47B                           | §3.8 |
| $M$     | 32A, 32B, 41A, 46A, 46B                       | §3.9 |
|         | 47A, 47B, 59A                                | §3.9 |
|         | 59B, 69A, 69B, 71A, 71B                      | §3.9 |
|         | 87A, 87B, 92A, 92B, 94A, 94B                 | §3.9 |

Consider the short exact sequence $1 \rightarrow 2 \rightarrow 2.A_{11} \rightarrow A_{11} \rightarrow 1$. From Table II every non-trivial conjugacy class of $A_{11}$ is of type D except the class of the 3-cycles and the classes of the 11-cycles. Thus, every non-trivial conjugacy class in $2.A_{11}$ with representative of order distinct from 2, 3, 6, 11, 22 is of type D, by [AFGV2, Lemma 2.7]. Hence the conjugacy classes 9A, 14A, 18A, 28A, 40A, 40B, 42A, 42B of $Ly$ are of type D.

**Step 3.** The maximal subgroup $M_6 \simeq 3^5 : (2 \times M_{11})$.

We construct a permutation representation of $M_6$ and apply Algorithm II in this maximal subgroup. We check that all conjugacy classes of $M_6$ with representative of order 11, 22 are of type D – see the file $Ly/step3.log$. By the fusion of the conjugacy classes, the conjugacy classes 11A, 11B, 22A, 22B of $Ly$ are of type D.

**Remark 3.1.** Not necessarily of type D: 33A, 33B, 37A, 37B, 67A, 67B, 67C.
Table 5. Log files for the sporadic groups studied with Algorithm 3

| $G$ | Log file | $G$ | Log file |
|-----|----------|-----|----------|
| $M_{11}$ | M11/M11.log | $Ru$ | Ru/Ru.log |
| $M_{12}$ | M12/M12.log | $HS$ | HS/HS.log |
| $M_{22}$ | M22/M22.log | $He$ | He/He.log |
| $M_{23}$ | M23/M23.log | $McL$ | McL/McL.log |
| $M_{24}$ | M24/M24.log | $Co_3$ | Co3/Co3.log |
| $J_1$ | J1/J1.log | $Co_2$ | Co2/Co2.log |
| $J_2$ | J2/J2.log | $ON$ | ON/ON.log |
| $J_3$ | J3/J3.log | $Fi_{22}$ | Fi22/Fi22.log |
| $Suz$ | Suz/Suz.log | $T$ | T/T.log |

3.2. The Thompson group $Th$. See the file Th/fusions.log for the fusion of conjugacy classes.

**Step 1.** The maximal subgroup $M_3 \simeq 2^{1+8}.A_9$.

Consider the short exact sequence $1 \to 2^{1+8} \to 2^{1+8}.A_9 \to A_9 \to 1$. From Table 1, every non-trivial conjugacy class of $A_9$ is of type D except the class of the 3-cycles. By [AFGV2, Lemma 2.7], every conjugacy class in $2^{1+8}.A_9$ with representative of order 5, 7, 9, 10, 14, 15, 18, 20, 28, 30, 36 is of type D. Hence the conjugacy classes $5A, 7A, 10A, 14A, 15A, 15B, 18A, 18B, 20A, 28A, 30A, 30B, 36A, 36B, 36C$ of $Th$ are of type D.

**Step 2.** The maximal subgroup $M_2 \simeq 2^5.L_5(2)$.

Consider the short exact sequence $1 \to 2^5 \to 2^5.L_5(2) \to L_5(2) \to 1$. By Lemma 2.7 in $L_5(2)$ every non-trivial conjugacy class with representative of order distinct from 2 and 31 is of type D. Therefore, by [AFGV2, Lemma 2.7], every conjugacy class in $M_2$ with representative of order 3, 6, 12, 21, 24 is of type D. Hence the classes $3A, 3C, 12D, 21A, 24A, 24B$ of $Th$ are of type D.

On the other hand, each group $L_5(2)$ and $2^5.L_5(2)$ have six classes of elements of order 31; let $\{O_1, \ldots, O_6\}$ be these classes in $L_5(2)$. We can check that for every $i, j$, with $1 \leq i \neq j \leq 6$, there exist $r \in O_i$ and $s \in O_j$ such that $(rs)^2 \neq (sr)^2$. Then the same occurs in the corresponding classes of $2^5.L_5(2)$. Since the fusion of the conjugacy classes from $2^5.L_5(2)$ to $Th$ establish that $31a, 31c, 31d$ go to $31A$, and $31b, 31e, 31f$ go to $31B$, then the conjugacy classes $31A$ and $31B$ of $Th$ are of type D. We note that here the classes of this maximal subgroup are named in lower case letter because they are not necessarily named as in the ATLAS.

**Step 3.** The maximal subgroup $M_{12} \simeq L_2(19).2$.

In this maximal subgroup, the conjugacy classes with representatives of order $2, 3, 6, 19$ are of type D – see the file Th/step3.log. Then, by the fusion of the conjugacy classes, the conjugacy classes $2A, 3B, 19A$ of $Th$ are of type D.

**Step 4.** The maximal subgroup $M_5 \simeq (3 \times G_2(3)) : 2$.

From Theorem 2.11 the group $G_2(3)$ is of type D. Thus, the conjugacy class $13A$ of $Th$ is of type D. On the other hand, by the Subsubsection 2.11 and [AFGV2, Lemma 2.8], the conjugacy classes $39A, 39B$ of $Th$ are of type D. See the file Th/step4.log for the fusion of the conjugacy classes $3 \times G_2(3) \to (3 \times G_2(3)) : 2$.

**Step 5.** The maximal subgroup $M_6 \simeq 3.3^2.3.(3 \times 3^2).3^2 : 2S_4$.

By the fusion of the conjugacy classes and Algorithm 3 we see that the classes $4A, 8A, 12A, 12B, 12C$ of $Th$ are of type D. See the file Th/step5.log for details.
Step 6. The maximal subgroup $M_7 \simeq 3^2.3^3.3^2.3^2 : 2S_4$.

By the fusion of the conjugacy classes and Algorithm 1 we see that the conjugacy classes 4B, 6A, 6B, 6C, 8B, 9A, 9B, 9C, 24C, 24D, 27A, 27B, 27C of $Th$ are of type D. See the file Th/step6.log for details.

3.3. The Janko group $J_4$. See the file J4/fusions.log for the fusion of conjugacy classes.

Step 1. The maximal subgroup $M_1 \simeq 2^{11} : M_{24}$.

We use the maximal subgroup $M_1 \simeq 2^{11} : M_{24}$. Consider the short exact sequence $1 \to 2^{11} \to 2^{11} : M_{24} \to M_{24} \to 1$. We know that every non-trivial conjugacy class of $M_{24}$ with representative of order distinct from 23 is of type D. By [AFGV2, Lemma 2.7], every non-trivial conjugacy class in $2^{11} : M_{24}$ with representative of order distinct from 2, 4, 8, 16, 23 is of type D. Hence the conjugacy classes 3A, 5A, 6A, 6B, 6C, 7A, 7B, 10A, 10B, 12A, 12B, 12C, 14A, 14B, 14C, 14D, 15A, 20A, 20B, 21A, 21B, 24A, 24B, 28A, 28B, 30A of $J_4$ are of type D. Also, this maximal subgroup has a primitive permutation representation on $2^{11}$ points. We construct this primitive group and use Algorithm 2 to determine that the only conjugacy class with representative of order 16 in $M_1$ is of type D – see the file J4/step1.log. Hence, the conjugacy class 16A of $J_4$ is of type D.

Step 2. The maximal subgroup $M_2 \simeq 2^{3+12}.(S_5 \times L_3(2))$.

Consider the short exact sequence $1 \to 2^{3+12} \to 2^{3+12}.(S_5 \times L_3(2)) \to (S_5 \times L_3(2)) \to 1$. Every non-trivial conjugacy class of $S_5 \times L_3(2)$ with representative of order distinct from 2, 3, 4, 6, 7 is of type D, see Subsubsection 2.11. By [AFGV2, Lemma 2.7], every conjugacy class in $2^{3+12}.(S_5 \times L_3(2))$ with representative of order 35, 42 is of type D. Hence the conjugacy classes 35A, 35B, 42A, 42B of $J_4$ are of type D.

Step 3. The maximal subgroup $M_3 \simeq U_3(11).2$.

We perform Algorithm 2 – see the file J4/step3.log. Hence the conjugacy classes 8A, 8B, 11A, 11B of $J_4$ are of type D.

Step 4. The maximal subgroup $M_4 \simeq M_{22}.2$.

We perform Algorithm 1 – see the file M22/M22.2.log. Hence the conjugacy classes 4B, 8C of $J_4$ are of type D.

Step 5. The maximal subgroup $M_5 \simeq 11^{1+2} : (5 \times 2S_4)$.

We perform Algorithm 1 – see the file J4/step5.log. Hence the conjugacy classes 4A, 22A, 22B, 40A, 40B, 44A, 66A, 66B of $J_4$ are of type D.

Step 6. The maximal subgroup $M_6 \simeq L_2(32).5$.

We perform Algorithm 1 – see the file J4/step6.log. Hence the conjugacy classes 31A, 31B, 31C, 33A, 33B of $J_4$ are of type D.

Step 7. The maximal subgroup $M_7 \simeq L_3(23).2$.

We perform Algorithm 1 – see the file J4/step7.log. Hence the conjugacy classes 2A, 2B, 23A of $J_4$ are of type D.

Step 8. The maximal subgroup $M_{13} \simeq 37 : 12$.

We construct a permutation representation of this maximal subgroup. By the fusion of the conjugacy classes, the classes 4a and 4b of $M_{13}$ go to the conjugacy class 4C of $J_4$. We find $r$ in 4A, $s$ in 4B of $M_{13}$ such that $(rs)^2 \neq (sr)^2$ – see
the file J4/step8.log. Hence the conjugacy class 4C of $J_4$ is of type D. We note that here the classes of this maximal subgroup are named with lower case letters because they are not necessarily named as in the ATLAS.

**Remark 3.2.** Not necessarily of type D: 29A, 37A, 37B, 37C, 43A, 43B, 43C.

3.4. **The Fischer group $Fi_{23}$.** See the file F123/fusions.log for the fusion of conjugacy classes.

**Step 1.** The maximal subgroup $M_1 \simeq 2.Fi_{22}$.

Consider the short exact sequence $1 \to 2 \to 2.Fi_{22} \to Fi_{22} \to 1$. By Table 1, every non-trivial conjugacy class of $Fi_{22}$ with representative of order distinct from 2, 22 is of type D. By [AFGV2, Lemma 2.7], every conjugacy class in $2.Fi_{22}$ with representative of order 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 20, 21, 26, 42 is of type D. Then the conjugacy classes 3A, 3B, 3C, 3D, 5A, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 6K, 6L, 6M, 6N, 6O, 7A, 7B, 7C, 7D, 7E, 8A, 8B, 8C, 8D, 8E, 8F, 8G, 8H, 8I, 8J, 8K, 8L, 8M, 8N, 8O, 9A, 9B, 9C, 9D, 9E, 9F, 10A, 10B, 10C, 10D, 10E, 10F, 10G, 10H, 10I, 10J, 11A, 11B, 11C, 11D, 11E, 11F, 11G, 11H, 11I, 11J, 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12I, 12J, 12K, 12L, 12M, 12N, 13A, 13B, 13C, 13D, 14A, 14B, 14C, 14D, 14E, 14F, 14G, 14H, 14I, 14J, 15A, 15B, 15C, 15D, 15E, 15F, 15G, 15H, 15I, 15J, 16A, 16B, 16C, 16D, 16E, 16F, 16G, 16H, 16I, 16J, 17A, 17B, 17C, 17D, 17E, 17F, 17G, 17H, 17I, 17J, 18A, 18B, 18C, 18D, 18E, 18F, 18G, 18H, 18I, 18J, 19A, 19B, 19C, 19D, 19E, 19F, 19G, 19H, 19I, 19J, 20A, 20B, 20C, 20D, 21A, 21B, 21C, 21D, 21E, 21F, 21G, 21H, 21I, 21J, 22A, 22B, 22C, 22D, 22E, 22F, 22G, 22H, 22I, 22J, 23A, 23B are of type D.

**Step 2.** The maximal subgroup $M_4 \simeq S_8(2)$. 

By Lemma 2.14 and the fusion of the conjugacy classes, the conjugacy classes 2B, 2C, 4A, 4B, 4C, 4D, 8C, 15A, 15B, 17A of $Fi_{23}$ are of type D.

**Step 3.** The maximal subgroup $M_5 \simeq O_7(3) : S_3$.

By Lemma 2.3 and [AFGV2, Lemma 2.8], every non-trivial conjugacy class of $O_7(3) : S_3$ with representative of order distinct from 2, 3, 6 is of type D. Thus by the fusion of the conjugacy classes, the conjugacy classes 9D, 12D, 18A, 18B, 18C, 18D, 18E, 18F, 18G, 18H, 18I, 18J, 20A, 20B, 20C, 20D, 20E, 20F, 20G, 20H, 20I, 20J, 30A, 30B, 30C, 30D, 30E, 30F, 30G, 30H, 30I, 30J, 35A, 35B, 40A, 40B, 40C, 40D, 40E, 40F, 40G, 40H, 40I, 40J, 50A, 50B, 50C, 50D, 50E, 50F, 50G, 50H, 50I, 50J, 60A, 60B, 60C, 60D, 60E, 60F, 60G, 60H, 60I, 60J of $Fi_{23}$ are of type D.

**Step 4.** The maximal subgroup $M_3 \simeq 2^2.U_6(2).2$.

We perform Algorithm 2 in this maximal subgroup to see that the conjugacy classes 22A, 22B, 22C of $Fi_{23}$ are of type D. See the file F123/step4.log for details.

**Step 5.** The maximal subgroup $M_2 \simeq O_7(3) : S_3$.

We perform Algorithm 2 to see that the conjugacy class 27A of $Fi_{23}$ is of type D. See the file F123/step5.log for details.

**Step 6.** The maximal subgroup $M_10 \simeq (2^2 \times 2^{1+8}).(3 \times U_4(2)).2$.

We perform Algorithm 1 in this maximal subgroup to see that the classes 9A, 18D, 18G, 24A, 24B, 24C, 36A of $Fi_{23}$ are of type D. See F123/step6.log for details.

**Step 7.** The maximal subgroup $M_12 \simeq S_4 \times S_6(2)$.

We perform Algorithm 1 in this maximal subgroup to see that the conjugacy class 36B of $Fi_{23}$ is of type D. See the file F123/step7.log for details.

**Step 8.** The maximal subgroup $M_9 \simeq S_{12}$.

From Table 1, every non-trivial conjugacy class of $S_{12}$ with representative of order distinct of 2, 3, 11 is of type D. Thus, from the fusion of the conjugacy classes, the conjugacy classes 8A, 8B, 28A, 30A, 30B, 30C, 35A, 42A, 60A of $Fi_{23}$ are of type D.

**Remark 3.3.** Not necessarily of type D: 2A, 23A, 23B.
3.5. The Conway group $Co_1$. See the file $Co1/fusions.log$ for the fusion of conjugacy classes.

**Step 1.** The maximal subgroups $M_1 \simeq Co_2$ and $M_4 \simeq Co_3$.

From Table 4 and the fusion of the conjugacy classes, the conjugacy classes 3B, 3C, 4A, 4B, 4C, 4D, 4F, 5B, 5C, 6C, 6D, 6E, 6F, 6G, 6I, 7B, 7E, 8B, 8C, 8D, 8E, 9B, 9C, 10D, 10E, 10F, 11A, 14B, 15D, 15E, 16A, 16B, 20B, 20C, 21C, 22A, 28A, 30D, 30E of $Co_1$ are of type D.

**Step 2.** The maximal subgroup $M_{16} \simeq A_9 \times S_3$.

We know that every non-trivial conjugacy class of $A_9 \times S_3$ with representative of order distinct from 2, 3 is of type D, see Subsubsection 2.11. Then, by the fusion of conjugacy classes, the classes 4E, 5A, 6A, 6B, 6H, 9A, 9B, 10A, 10B, 12L, 12M, 15A, 15C, 30A, 30C of $Co_1$ are of type D.

**Step 3.** The maximal subgroup $M_3 \simeq 2^{11}:M_{24}$.

Consider the short exact sequence $1 \to 2^{11} \to 2^{11}:M_{24} \to M_{24} \to 1$. By Table 4 and [AFGV2, Lemma 2.7], every non-trivial conjugacy class of $2^{11}:M_{24}$ with representative of order distinct from 23 is of type D. By the fusion of the conjugacy classes, the conjugacy class 3D of $Co_1$ is of type D.

**Step 4.** The maximal subgroup $M_7 \simeq (A_4 \times G_2(4)) : 2$.

Note that $A_4 \times G_2(4)$ is a subgroup of $M_7$. Now by Subsubsection 2.11 and the fusion of conjugacy classes, the conjugacy classes 14A, 26A of $Co_1$ are of type D.

**Step 5.** The maximal subgroup $M_{14} \simeq (A_6 \times U_3(3)) : 2$.

By Lemma 2.5 and [AFGV2, Lemma 2.7] the conjugacy classes of $2^{1+8}.O_6^+(2)$ with representative of order 5, 7, 9, 10, 14, 15, 18, 20, 28, 30, 36, 40, 60 are of type D. Thus, by the fusion of the conjugacy classes, the conjugacy classes 20A, 36A, 40A, 60A of $Co_1$ are of type D.

On the other hand, to study other conjugacy classes of $Co_1$ we use a script that performs an algorithm similar to Algorithm 2 that we explain briefly here – see the file $Co1/step7.log$ for the computations.

**Step 6.** The maximal subgroup $M_2 \simeq 3.Suz.2$.

We perform Algorithm 2 in this maximal subgroup to see that the conjugacy classes 7A, 8A, 8F, 10C, 13A, 15B, 21A, 21B, 30B, 33A, 39A, 39B, 42A of $Co_1$ are of type D. See the file $Suz/3.Suz.2.1.log$ for the computations.

**Step 7.** The maximal subgroup $M_5 \simeq 2^{1+8}.O_6^+(2)$.

By Lemma 2.5 and [AFGV2, Lemma 2.7] the conjugacy classes of $2^{1+8}.O_6^+(2)$ with representative of order 5, 7, 9, 10, 14, 15, 18, 20, 28, 30, 36, 40, 60 are of type D. Thus, by the fusion of the conjugacy classes, the conjugacy classes 20A, 36A, 40A, 60A of $Co_1$ are of type D.

On the other hand, to study other conjugacy classes of $Co_1$ we use a script that performs an algorithm similar to Algorithm 2 that we explain briefly here – see the file $Co1/step7.log$ for the computations. First we compute all conjugacy classes of $Co_1$ and $M_5$. Then we study the conjugacy classes of $M_5$ with representative of order 12, 18, 24 and discard the corresponding conjugacy classes in $Co_1$ when those are of type D. At the end of the log file we see that only two conjugacy classes of $Co_1$, both with representatives of order 12, were not discarded. One of these classes has centralizer of order 48, the other 72. These are the conjugacy classes 12L and 12M of $Co_1$. 
and these classes were considered in the Step 2. Therefore, besides the conjugacy classes of the previous paragraph, the conjugacy classes 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12I, 12J, 12K, 18A, 18B, 18C, 24A, 24B, 24C, 24D, 24E, 24F of $C_{12}$ are of type D.

**Remark 3.4.** Not necessarily of type D: 3A, 23A, 23B.

3.6. **The Harada-Norton group** $H.N$. See the file $HN/fusions.log$ for the fusion of conjugacy classes.

**Step 1. The maximal subgroup** $M_1 \simeq A_{12}$.

By Table 1 in this maximal subgroup every non-trivial conjugacy class with representative of order distinct from 3, 11 is of type D. Therefore, the conjugacy classes 2A, 2B, 5A, 5E, 6A, 6B, 6C, 7A, 9A, 15A, 20C, 21A, 30A, 35A, 35B of $H.N$ are of type D. It remains to prove that the conjugacy class 11A of $H.N$ is of type D. For that purpose, let $r = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11)$ and $s = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 11 \ 10)$ be elements in $A_{12}$. It is easy to see that $(rs)^2 \neq (sr)^2$ and that $r$ and $s$ belong to different conjugacy classes in the group $\langle r, s \rangle \simeq A_{11}$. Then, the conjugacy class 11A of $H.N$ is of type D.

**Step 2. The maximal subgroup** $M_{11} \simeq M_{12.2}$.

We perform Algorithm 1 and obtain that every non-trivial conjugacy class is of type D – see the file $M_{12}/M_{12.2}.log$. By the fusion of the conjugacy classes, the conjugacy classes 3A, 3B, 4A, 4B, 4C, 12C of $H.N$ are of type D.

**Step 3. The maximal subgroup** $M_2 \simeq 2.HS.2$.

We perform Algorithm 2 and obtain that every conjugacy class with representative of order 5, 8, 10, 12, 14, 20, 22, 40 is of type D – see the file $HS/2.HS.2.log$. Therefore, the conjugacy classes 5B, 8A, 8B, 10A, 10B, 10C, 10F, 10G, 10H, 12A, 12B, 14A, 20A, 20B, 22A, 40A, 40B of $H.N$ are of type D.

**Step 4. The maximal subgroup** $M_{14} \simeq 3^{(1+4)} : 4.A_5$.

We use Algorithm 1 and obtain that every conjugacy class with representative of order 5, 10, 15, 20, 30 is of type D – see the file $HN/step4.log$. Therefore, the conjugacy classes 5C, 5D, 10D, 10E, 15B, 15C, 20D, 20E, 30B, 30C of $H.N$ are of type D.

**Step 5. The maximal subgroup** $M_3 \simeq U_3(8).3_1$.

We perform Algorithm 2 and obtain that every conjugacy class with representative of order 19 is of type D – see the file $HN/step5.log$. Therefore, the conjugacy classes 19A, 19B of $H.N$ are of type D.

**Step 6. The maximal subgroup** $M_{10} \simeq 5^{2+1+2}.4.A_5$.

We perform Algorithm 1 and obtain that every conjugacy class with representative of order 25 is of type D – see the file $HN/step6.log$. Therefore, the conjugacy classes 25A, 25B of $H.N$ are of type D.

3.7. **The Fischer group** $Fi_{24}$. The log files concerning this group are stored in the folder $F3+$. See the file $F3+/fusions.log$ for the fusion of the conjugacy classes. The list of (representatives of conjugacy classes of) maximal subgroups of $Fi_{24}$ can be found in [LW].

**Step 1. The maximal subgroup** $M_1 \simeq Fi_{23}$. 
By Section 3.4, we know that every non-trivial conjugacy class of $Fi_{23}$ with representative of order distinct from 2, 23 is of type D. Hence, the conjugacy classes $3A, 3B, 3C, 3D, 4A, 4B, 4C, 5A, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 7A, 8A, 8B, 9A, 9B, 9C, 9E, 9F, 10A, 10B, 11A, 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12K, 12L, 12M, 13A, 14A, 15A, 15C, 16A, 17A, 18A, 18B, 18C, 18D, 18E, 18F, 20A, 21A, 22A, 24A, 24B, 24E, 26A, 27A, 28A, 30A, 30B, 35A, 36C, 36D, 39A, 39B, 42A, 60A of $Fi'_{24}$ are of type D.

**Step 2.** The maximal subgroups $M_{13} \simeq He.2$ and $M_{14} \simeq He.2$.

We perform Algorithm 3 and obtain that every non-trivial conjugacy class is of type D – see the file $He/He.2.log$. Then the conjugacy classes $3A, 3B, 3C, 3D, 4A, 4B, 4C, 5A, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 7A, 8A, 8B, 9A, 9B, 9C, 9E, 9F, 10A, 10B, 11A, 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12K, 12L, 12M, 13A, 14A, 15A, 15C, 16A, 17A, 18A, 18B, 18C, 18D, 18E, 18F, 20A, 21A, 22A, 24A, 24B, 24E, 26A, 27A, 28A, 30A, 30B, 35A, 36C, 36D, 39A, 39B, 42A, 60A of $Fi'_{24}$ are of type D.

**Step 3.** The maximal subgroup $M_4 \simeq O^-_{10}(2)$.

By Lemma 2.7 and the fusion of the conjugacy classes, the conjugacy classes $8C, 15B, 18G, 18H, 20B$ of $Fi'_{24}$ are of type D.

**Step 4.** The maximal subgroup $M_5 \simeq 3^7.O_7(3)$.

We consider the short exact sequence $1 \rightarrow 3^7 \rightarrow 3^7.O_7(3) \rightarrow O_7(3) \rightarrow 1$. By Lemma 2.7, every non-trivial conjugacy class of $O_7(3)$ with representative of order distinct from 2 is of type D. Therefore, the conjugacy classes $24F, 24G, 36A, 45A, 45B$ of $Fi'_{24}$ are of type D.

**Step 5.** The maximal subgroup $M_{20} \simeq h_6 \times L_2(8) : 3$.

We perform Algorithm 1 and obtain that every conjugacy class with representative of order 9 is of type D – see the file $F3+/step5.log$. Therefore, the conjugacy class $9D$ of $HN$ is of type D.

**Remark 3.5.** Not necessarily of type D: $23A, 23B, 27B, 27C, 29A, 29B, 33A, 33B, 39C, 39D$.

3.8. The Baby Monster group $B$. For this group we compute the fusion of the conjugacy classes in each step.

**Step 1.** The maximal subgroup $M_2 \simeq 2^{1+22}.Co_2$.

We consider the short exact sequence $1 \rightarrow 2^{1+22} \rightarrow 2^{1+22}.Co_2 \rightarrow Co_2 \rightarrow 1$. By Table 2, every non-trivial conjugacy class of $Co_2$ with representative of order distinct from 2, 23 is of type D. Therefore, by Lemma 2.7, the conjugacy classes $5A, 5B, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 7A, 9A, 9B, 10A, 10B, 10C, 10D, 10E, 10F, 11A, 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12I, 12J, 12K, 12L, 12M, 12N, 12O, 12P, 12Q, 12R, 12S, 14A, 14B, 14C, 14D, 14E, 15A, 15B, 15C, 15D, 15E, 15F, 15G, 15H, 15I, 15J, 15K, 15L, 15M, 15N, 15O, 15P, 15Q, 15R, 15S, 15T, 15U, 15V, 15W, 15X, 15Y, 15Z$ of $B$ are of type D. See the file $B/step1.log$ for the fusion of the conjugacy classes $2^{1+22}.Co_2 \rightarrow B$.

**Step 2.** The maximal subgroup $M_3 \simeq Fi_{23}$.

By Subsection 3.3, every non-trivial conjugacy class of $Fi_{23}$ distinct of $2A, 23A, 23B$ is of type D. Therefore, the conjugacy classes $2B, 2D, 3A, 3B, 4D, 4E, 4G, 4H, 7A, 7B, 7C, 7D, 7E, 7F, 7G, 7H, 7I, 7J, 7K, 7L, 7M, 7N, 7O, 7P, 7Q, 7R, 7S, 7T, 7U, 7V, 7W, 7X, 7Y, 7Z$ of $B$ are of type D. See the file $B/step2.log$ for the fusion of conjugacy classes $Fi_{23} \rightarrow B$. 
Step 3. The maximal subgroup \( M_{18} \cong S_5 \times M_{22} : 2 \).

We perform Algorithm 1 and obtain that every conjugacy class of \( M_{18} \) with representative of order 4, 8, 33, 42, 55, 66, 70 is of type D. Hence the conjugacy classes 4A, 4B, 4C, 4F, 8B, 8C, 8E, 8I, 33A, 42C, 55A, 66A, 70A of \( B \) are of type D. See the file \texttt{B/step3.log} for the fusion of conjugacy classes \( S_5 \times M_{22} : 2 \rightarrow B \) and the computations.

Step 4. The maximal subgroup \( M_4 \cong 2^{9+16}.S_8(2) \).

By Lemma 2.14, every non-trivial conjugacy class, except 2A, 2B, 3A is of type D. By [AFGV2, Lemma 2.7] and the fusion of conjugacy classes – see the file \texttt{B/step4.log} – the conjugacy classes 34B, 34C, 40E, 42A, 60A, 60B, 60C of \( B \) are of type D.

Step 5. The maximal subgroup \( M_{16} \cong S_4 \times 2.F_4(2) \).

We perform Algorithm 1 and obtain that every conjugacy class of \( M_{16} \) with representative of order 4 is of type D. Hence the conjugacy classes 4A, 4B, 4C, 4F, 8A, 8D, 8F, 8G, 8H, 8L, 8N, 12T, 16A, 16B, 16E, 16F, 26A, 52A of \( B \) are of type D.

Step 6. The maximal subgroup \( M_{28} \cong L_2(17).2 \).

We use this maximal subgroup to show that the classes 23A, 23B of \( B \) are of type D. Indeed, the classes 23a and 23d of \( M_{30} \) fuse into the class 23A of \( B \) and there exist elements \( r \) and \( s \) in the classes 23a and 23d of \( M_{30} \), respectively, such that \((rs)^2 \neq (sr)^2 \). Then the class 23A of \( B \) is of type D. On the other hand, the classes 23u and 23v of \( M_{30} \) fuse into the class 23B of \( B \) and there exist elements \( r \) and \( s \) in the classes 23u and 23v of \( M_{30} \), respectively, such that \((rs)^2 \neq (sr)^2 \). Then the class 23B of \( B \) is of type D. See the file \texttt{B/step6.log} for the details. We note that here the classes of this maximal subgroup are named in lower case letter because they are not necessarily named as in the ATLAS.

Step 7. The maximal subgroup \( M_5 \cong Th \).

By Subsection 3.2 every non-trivial conjugacy class in \( Th \) is of type D. Therefore, by the fusion of the conjugacy classes – see the file \texttt{B/step7.log} – the conjugacy classes 4J, 19A, 24N, 31A, 31B of \( B \) are of type D.

Step 8. The maximal subgroup \( M_{30} \cong 47 : 23 \).

Step 9. The maximal subgroup \( M_1 \cong 2.(2E_6(2)) : 2 \).

We know that \( 2 \times U_3(8) \) is a subgroup of \( B \), because \( 2 \times U_3(8) \) is a subgroup of \( M_1 \), see [W]. In \( U_3(8) \) there exist non-conjugate elements \( r \) and \( s \), both of order 19, such that \((rs)^2 \neq (sr)^2 \) – see the file \texttt{B/step9.log}. Since \( B \) has only one conjugacy class of elements of order 38, the conjugacy class 38A of \( B \) is of type D.

Remark 3.6. Not necessarily of type D: 2A, 16C, 16D, 32A, 32B, 32C, 32D, 34A, 46A, 46B, 47A, 47B
3.9. **The Monster group** $M$. For the known maximal subgroups of $M$, we use the order in which they appear listed in the ATLAS.

**Step 1. The maximal subgroup $M_1 \simeq 2.B$.**

By Subsection 3.8 and [AFGV2, Lemma 2.7], the conjugacy classes 3A, 3B, 5A, 5B, 6A, 6B, 6C, 6D, 7A, 8A, 8B, 8C, 8D, 8E, 8F, 9A, 9B, 10A, 10B, 10C, 10D, 10E, 11A, 12A, 12B, 12C, 12E, 12F, 12G, 12H, 12I, 13A, 14A, 14B, 17A, 18A, 18B, 18C, 18D, 18E, 19A, 21A, 22A, 22B, 23A, 23B, 24A, 24B, 24C, 24D, 24F, 24G, 24H, 24I, 25A, 26A, 27A, 28A, 28B, 28C, 33B, 35A, 36A, 36B, 36C, 36D, 38A, 39A, 40A, 40B, 40C, 40D, 42A, 44A, 44B, 48A, 50A, 52A, 54A, 55A, 66A, 70A, 78A, 84A, 104A, 104B, 110A of $M$ are of type D. See the file $M/step1.log$ for the fusion of conjugacy classes $2.B \to M$.

**Step 2. The maximal subgroup $M_2 \simeq 2^{1+24}.Co_1$.**

By Subsection 3.5 and [AFGV2, Lemma 2.7], the conjugacy classes 7B, 13B, 14C, 15A, 15B, 15C, 15D, 20A, 20B, 20C, 20D, 20E, 21B, 21D, 26B, 28D, 30A, 30B, 30C, 30D, 30E, 30F, 30G, 33A, 35B, 39C, 39D, 42B, 42C, 42D, 52B, 56A, 56B, 56C, 60A, 60B, 60C, 60D, 60E, 60F, 66B, 70B, 78B, 78C, 84B, 88A, 88B of $M$ are of type D. See the file $M/step2.log$ for the fusion of conjugacy classes $2^{1+24}.Co_1 \to M$.

**Step 3. The maximal subgroup $M_3 \simeq S_3 \times Th$.**

By Subsection 3.2 and [AFGV2, Lemma 2.8], every non-trivial conjugacy class in this maximal subgroup, with representative of order distinct from 2, 3 is of type D. Moreover, in this maximal subgroup, there is only one conjugacy class with representative of order 3 that is not of type D: that corresponding to the 3-cycles in $S_3$. Hence, the conjugacy classes 3C, 4A, 4D, 6F, 12D, 12J, 21C, 24E, 24J, 27A, 28A, 28B, 28C, 33B, 35A, 36A, 36B, 36C, 36D, 38A, 39A, 40A, 40B, 40C, 40D, 42A, 44A, 44B, 48A, 50A, 52A, 54A, 55A, 66A, 70A, 78A, 84A, 104A, 104B, 110A of $M$ are of type D. See the file $M/step3.log$ for the fusion of conjugacy classes $S_3 \times Th \to M$.

**Step 4. The maximal subgroup $M_4 \simeq L_2(29).2 \simeq PGL(2,29)$.**

We use a representation of this maximal subgroup inside $S_{30}$ given in the ATLAS, see [BW]. This maximal subgroup has only one conjugacy class of elements of order 29 which is of type D. Therefore, the conjugacy class 29A of $M$ is of type D. See the file $M/step4.log$ for details.

**Step 5. The maximal subgroup $M_{23} \simeq (A_5 \times U_3(8):3_1):2$.**

We use a representation of this maximal subgroup inside $S_{184}$ given in the ATLAS, see [BW]. In this maximal subgroup the conjugacy classes with representatives of order 16, 34, 51, 68, 119 are of type D. Therefore, the conjugacy classes 16A, 16B, 16C, 34A, 51A, 68A, 119A, 119B of $M$ are of type D. See the file $M/step5.log$ for details.

**Step 6. The maximal subgroup $M_{21} \simeq (A_5 \times U_3(8):3_1):2$.**

We use a representation of this maximal subgroup inside $S_{518}$ given in the ATLAS, see [BW]. In this maximal subgroup the conjugacy classes with representatives of order 95 are of type D. Therefore, the conjugacy classes 95A, 95B of $M$ are of type D. See the file $M/step6.log$ for details.

**Step 7. The maximal subgroup $M_3 \simeq 3.Fi_{24}$.**

Let $H = Fi_{24}$. From the ATLAS we know that the group $K = S_5 \times S_9$ is a maximal subgroup of $H$. From [2.11] every conjugacy class of $K$ with representative of order 4 is of type D. Also, from the fusion of conjugacy classes $K \to H$, every
conjugacy class of $F_{i24}$ with representative of order 4 is of type D. Therefore, by [AFGV2, Lemma 2.7], the conjugacy classes 4A, 4B, 4C, 4D of $M$ are of type D. Also, since $F'_{i24}$ is a maximal subgroup of $F_{i24}$, the conjugacy classes of $F_{i24}$ with representatives of order 15, 45 are of type D – see the Subsection 3.7. Therefore, by [AFGV2, Lemma 2.7] and the fusion of conjugacy classes $F'_{i24} \rightarrow F_{i24}$, the conjugacy class 4A of $M$ is of type D. For details about these observations and the fusion of conjugacy classes $3.F_{i24} \rightarrow M$ see the file M/step7.log.

Step 8. The subgroup $HN$.

Since $(D_{10} \times HN).2$ is a maximal subgroup of $M$, $HN$ is a subgroup of $M$. Therefore, by Subsection 3.6 and the fusion of the conjugacy classes $HN \rightarrow M$, the conjugacy classes 2A, 2B of $M$ are of type D. For the fusion of the conjugacy classes see the file M/step8.log.

Step 9. The subgroup $2 \times 47 : 23$

We use this subgroup to show that the classes 46C, 46D of $M$ are of type D. Indeed, the classes 46A and 46B of $2 \times 47 : 23$ fuse into the class 46C of $M$ and there exist elements $r$ and $s$ in the classes 46A and 46B of $2 \times 47 : 23$, respectively, such that $(rs)^2 \neq (sr)^2$. Then the class 46C of $M$ is of type D. The conjugacy class 46D is treated analogously. See the file M/step9.log for the details.

Remark 3.7. Not necessarily of type D: 32A, 32B, 41A, 46A, 46B, 47A, 47B, 59A, 59B, 69A, 69B, 71A, 71B, 87A, 87B, 92A, 92B, 94A, 94B.

Appendix. Real and quasi-real conjugacy classes

In this appendix we list all real and quasi-real conjugacy classes of the groups studied in [AFGV2]. The information about real conjugacy classes of a given group $G$ is easy to obtain from the character table of $G$ using the GAP function RealClasses. Similarly, with the GAP function PowerMaps it is easy to determine the quasi-real conjugacy classes of a given group.

The function QuasiRealConjugacyClasses returns the list of quasi-real conjugacy classes and its type.

```gap
gap> QuasiRealClasses := function( ct )
  > local nc, oc, a, b, p, c, j, rc;
  > nc := NrConjugacyClasses(ct);
  > oc := OrdersClassRepresentatives(ct);
  > rc := RealClasses(ct);
  > a := [];
  > b := [];
  > for c in [1..nc] do
    > if not c in rc then
      > for j in [2..oc[c]-2] do
        > p := PowerMap(ct, j);
        > if p[c] = c then
          > if j-1 mod oc[c] <> 0 then
            > if not c in b then
              > Add(a, [c, j]);
              > Add(b, c);
            > fi;
          > fi;
        > fi;
      > fi;
    > fi;
  > fi;
```
The group $L_5(2)$. The conjugacy classes $7A$, $7B$, $15A$, $15B$, $21A$, $21B$, $31A$, $31B$, $31C$, $31D$, $31E$, $31F$ are quasi-real of type $j = 2$, and the classes $14A$, $14B$ are quasi-real of type $j = 9$. The remaining conjugacy classes are real.

The groups $O^+_8(2)$, $S_6(2)$ and $S_8(2)$. In these groups every conjugacy class is real.

The group $O^-_{10}(2)$. The conjugacy classes $3B$, $3C$, $6B$, $6C$, $6F$, $6G$, $6H$, $6I$, $6L$, $6M$, $6T$, $6U$ are neither real nor quasi-real. The conjugacy classes $11A$, $11B$, $35A$, $35B$ are quasi-real of type $j = 3$, the classes $9B$, $9C$, $15B$, $15C$, $15F$, $15G$, $33A$, $33B$, $33C$, $33D$ are quasi-real of type $j = 4$, the classes $12B$, $12C$, $12E$, $12F$, $12I$, $12J$, $12N$, $12O$, $12R$, $12S$, $18A$, $18B$, $18C$, $18D$, $24C$, $24D$, $30B$, $30C$ are quasi-real of type $j = 7$. The remaining conjugacy classes are real.

The group $G_2(4)$. The conjugacy classes $12B$, $12C$ are quasi-real of type $j = 7$. The remaining conjugacy classes are real.

The Tits group. The conjugacy classes $8A$, $8B$ are quasi-real of type $j = 5$, the conjugacy classes $16A$, $16B$, $16C$, $16D$ are quasi-real of type $j = 9$. The remaining conjugacy classes are real.

The Mathieu groups. In any of the Mathieu simple groups, every conjugacy class is real or quasi-real. See Table 6 for the details concerning not real but quasi-real conjugacy classes.

Table 6. Mathieu groups: quasi-real classes.

| Classes   | Type |
|-----------|------|
| $M_{11}$  | $8A$, $8B$, $11A$, $11B$ | $j = 3$ |
| $M_{12}$  | $11A$, $11B$ | $j = 3$ |
| $M_{22}$  | $7A$, $7B$ | $j = 2$ |
| $M_{23}$  | $11A$, $11B$ | $j = 3$ |
| $M_{23}$  | $7A$, $7B$, $15A$, $15B$, $23A$, $23B$ | $j = 2$ |
| $M_{24}$  | $11A$, $11B$ | $j = 3$ |
| $M_{24}$  | $14A$, $14B$ | $j = 9$ |
| $M_{24}$  | $7A$, $7B$, $15A$, $15B$, $21A$, $21B$, $23A$, $23B$ | $j = 2$ |
| $M_{24}$  | $14A$, $14B$ | $j = 9$ |

The Conway groups. In the Conway groups $Co_1$, $Co_2$ and $Co_3$ every conjugacy class is real or quasi-real. The quasi-real not real conjugacy classes are listed in Table 6.

The Janko groups. In the Janko groups $J_1$ and $J_2$ every conjugacy class is real. In the Janko group $J_3$ the conjugacy classes $19A$, $19B$ are quasi-real of type $j = 4$ and the remaining conjugacy classes are real. In the group $J_4$ every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

1. $7A$, $7B$, $21A$, $21B$, $35A$, $35B$ (of type $j = 2$);
2. $14A$, $14B$, $14C$, $14D$, $28A$, $28B$ (of type $j = 9$);
3. $42A$, $42B$ (of type $j = 11$).
Table 7. Conway groups: quasi-real classes.

| Classes     | Type |
|-------------|------|
| Co$_1$      |      |
| 23A, 23B, 39A, 39B | $j = 2$ |
| Co$_2$      |      |
| 15B, 15C, 23A, 23B | $j = 2$ |
| 14B, 14C    | $j = 9$ |
| 30B, 30C    | $j = 17$ |
| Co$_3$      |      |
| 23A, 23B    | $j = 2$ |
| 11A, 11B, 20A, 20B, 22A, 22B | $j = 3$ |

The Fischer groups. In the Fischer groups $Fi_{22}$, $Fi_{23}$ and $Fi'_{24}$ every conjugacy class is real or quasi-real. The quasi-real not real conjugacy classes are listed in Table 8.

Table 8. Fischer groups: quasi-real classes.

| Classes     | Type |
|-------------|------|
| $Fi_{22}$   |      |
| 11A, 11B, 16A, 16B, 22A, 22B | $j = 3$ |
| 18A, 18B    | $j = 7$ |
| $Fi_{23}$   |      |
| 16A, 16B, 22B, 22C | $j = 3$ |
| 23A, 23B    | $j = 2$ |
| $Fi'_{24}$  |      |
| 23A, 23B    | $j = 2$ |
| 18G, 18H    | $j = 7$ |

The Highman-Sims group. The conjugacy classes 11A, 11B, 20A, 20B are quasi-real of type $j = 3$. The remaining conjugacy classes are real.

The Lyons group. The conjugacy classes 11A, 11B, 22A, 22B are quasi-real of type $j = 3$, the conjugacy classes 33A, 33B are quasi-real of type $j = 4$. The remaining conjugacy classes are real.

The Harada-Norton group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

1. 19A, 19B (of type $j = 4$);
2. 35A, 35B (of type $j = 3$);
3. 40A, 40B (of type $j = 7$).

The Held group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

1. 7A, 7B, 7D, 7E, 21C, 21D (of type $j = 2$);
2. 14A, 14B, 14C, 14D, 28A, 28B (of type $j = 9$).

The MacLaughlin group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

1. 7A, 7B, 15A, 15B (of type $j = 2$);
2. 11A, 11B (of type $j = 3$);
3. 9A, 9B (of type $j = 4$);
4. 14A, 14B (of type $j = 9$);
5. 30A, 30B (of type $j = 17$).
The O'Nan group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

1. 31A, 31B (of type $j = 2$);
2. 20A, 20B (of type $j = 3$).

The Rudvalis group $Ru$. The conjugacy classes 16A, 16B are quasi-real of type $j = 5$. The remaining conjugacy classes are real.

The Suzuki group $Suz$. The conjugacy classes 6B, 6C, with centralizers of size 1296 are neither real nor quasi-real. The classes 9A, 9B are quasi-real of type $j = 4$, and the classes 18A, 18B are quasi-real of type $j = 7$. The remaining conjugacy classes are real.

The Thompson group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

1. 15A, 15B, 31A, 31B, 39A, 39B (of type $j = 2$);
2. 27B, 27C (of type $j = 4$);
3. 24C, 24D (of type $j = 5$);
4. 12A, 12B, 24A, 24B, 36B, 36C (of type $j = 7$);
5. 30A, 30B (of type $j = 17$).

The Baby Monster group. The conjugacy classes 23A, 23B, 31A, 31B, 47A, 47B are quasi-real of type $j = 2$, the classes 30G, 30H are quasi-real of type $j = 17$, the classes 32C, 32D, 46A, 46B are quasi-real of type $j = 3$. The remaining conjugacy classes are real.

The Monster group. The conjugacy classes 23A, 23B, 31A, 31B, 39C, 39D, 47A, 47B, 69A, 69B, 71A, 71B, 87A, 87B, 93A, 93B, 95A, 95B, 119A, 119B are quasi-real of type $j = 2$. The classes 40C, 40D, 44A, 44B, 46A, 46B, 46C, 46D, 56B, 56C, 59A, 59B, 88A, 88B, 92A, 92B, 94A, 94B, 104A, 104B are quasi-real of type $j = 3$. The classes 62A, 62B, 78B, 78C are quasi-real of type $j = 5$. The remaining conjugacy classes are real.

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N. A., F. F.: Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba. CIEM – CONICET. Medina Allende s/n (5000) Ciudad Universitaria, Córdoba, Argentina

M. G., L. V.: Departamento de Matemática – FCEyN, Universidad de Buenos Aires, Pab. I – Ciudad Universitaria (1428) Buenos Aires – Argentina

L. V.: Instituto de Ciencias, Universidad de Gral. Sarmiento, J.M. Gutierrez 1150, Los Polvorines (1653), Buenos Aires – Argentina

E-mail address: (andrus, fantino)@famaf.unc.edu.ar
E-mail address: (matiasg, lvendramin)@dm.uba.ar