The relation between the Kochen-Specker theorem and bivalence

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Abstract
In the paper it is argued that the Kochen-Specker theorem necessitates a conclusion that for a quantum system it is possible to find a set of projection operators which is not truth-value bivalent; that is, a bivalent truth-value assignment function imposed on such a set cannot be total. This means that at least one proposition associated with the said set must be neither true nor false.

Keywords: Kochen-Specker theorem; Truth-value assignment; Bivalence; Many-valued logics; Partial semantics.

1 Introduction
The main implication of the result of the Kochen-Specker (KS) theorem [1, 2, 3] is that quantum theory fails to allow a non-contextual hidden variable model. But what is more, this theorem proves that quantum theory fails to allow a logic that obeys the principle of bivalence.

To state this in more detail, let us recall that each projection operator leaves invariant any vector lying in its range, ran(·), and annihilates any vector lying in its null space, ker(·). Next, suppose that a quantum system is prepared in a pure state $|\Psi\rangle$ that lies in the range of the projection operator $\hat{P}_A$ associated with the proposition $A$, whose valuation is denoted by $[A]_v$. Being in the state $|\Psi\rangle$ is subject to the assumption that the truth-value assignment function $v$ must assign the truth value 1 (denoting the truth) to the proposition $A$ and, thus, the operator $\hat{P}_A$, namely, $v(\hat{P}_A) = [A]_v = 1$, since $\hat{P}_A|\Psi\rangle = 1 \cdot |\Psi\rangle$. In an analogous manner, if the system is prepared in a pure state $|\Psi\rangle$ lying in the null space of the projection operator $\hat{P}_A$, then the function $v$ must assign the truth value 0 (denoting the falsity) to $\hat{P}_A$, namely, $v(\hat{P}_A) = [A]_v = 0$, since $\hat{P}_A|\Psi\rangle = 0 \cdot |\Psi\rangle$.

Because ran($\hat{P}_A$) $\neq$ ker($\hat{P}_A$), for any nontrivial vector $|\Psi\rangle$ it must be that $|\Psi\rangle \notin$ ker($\hat{P}_A$) if $|\Psi\rangle \in$ ran($\hat{P}_A$) as well as $|\Psi\rangle \notin$ ran($\hat{P}_A$) if $|\Psi\rangle \in$ ker($\hat{P}_A$). Thus, the definiteness of the proposition $A$ in the prepared pure state $|\Psi\rangle$ can be written down as its bivaluation, explicitly, $[A]_v \neq 0$.

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if $[A]_v = 1$ and $[A]_v \neq 1$ if $[A]_v = 0$.\footnote{Due to the fundamental tenet of quantum theory, to represents a realizable pure state, the vector $|\Psi\rangle$ must be different from the null vector $|0\rangle$.}

Let us assume that the valutational axiom

$$v(\hat{P}_\diamond) = [\diamond]_v ,$$  \hspace{1cm} (1)

where the symbol $\diamond$ can be replaced by any proposition (compound or simple), holds as a general principle.

Such an assumption brings the following question: Can the bivalent truth-values be assigned to all propositions related to the quantum system?

To be more specific, suppose that the sum

$$\sum_{i=1}^{n} \hat{P}_{\diamond i} = \hat{1} ,$$  \hspace{1cm} (2)

where $\hat{1}$ stands for the operator of the identity, is the resolution of the identity associated with the projection operators $\hat{P}_{\diamond i}$. Then the question is, does there exist a truth-value assignment function $v$ such that if $v(\hat{1}) = 1$ then exactly one of $\hat{P}_{\diamond i}$ has the truth value 1? In other words, is it possible to find a bivaluation

$$v(\hat{P}_{\diamond i}) = \left[ \diamond_{i} \right]_v \in \{1, 0\}$$  \hspace{1cm} (3)

such that the following entailment

$$v \left( \sum_{i=1}^{n} \hat{P}_{\diamond i} \right) = 1 \implies \sum_{i=1}^{n} v(\hat{P}_{\diamond i}) = 1$$  \hspace{1cm} (4)

hold even before the measurement of $\hat{P}_{\diamond i}$ (i.e., the verification of $\diamond_{i}$)?

The KS theorem shows that the answer is no for a system whose Hilbert space $\mathcal{H}$ has dimension greater than two. This answer can be interpreted as showing that prior to their verification the propositions related to the quantum system do not obey a bivalent logic, which means that at least one of them must be neither true nor false.

Let us present such an interpretation of the KS theorem in this paper.
2 Preliminaries

Consider the Hilbert space $\mathcal{H} = \mathbb{C}^{4 \times 4}$ formed by complex $4 \times 4$ matrices related to the states for the spin of the composite system containing two spin-$\frac{1}{2}$ particles, namely,

$$\ket{\Psi_{j\alpha k\beta}} = \ket{\Psi^{(1)}_{j\alpha}} \otimes \ket{\Psi^{(2)}_{k\beta}},$$

(5)

where $j$ and $k$ are elements of the set $\{x,y,z\}$, $\alpha$ and $\beta$ are elements of the set $\{+,−\}$, $\ket{\Psi^{(i)}}$ represent the normalized eigenvectors of the Pauli matrices for each particle.

Let $O$ be a set of 12 projection operators $\hat{P}_{j\alpha\beta}$ on $\mathbb{C}^{4 \times 4}$ which are defined by

$$\hat{P}_{j\alpha\beta} = |\Psi^{(1)}_{j\alpha}\rangle \langle \Psi^{(1)}_{j\beta}| \otimes |\Psi^{(2)}_{j\alpha}\rangle \langle \Psi^{(2)}_{j\beta}|$$

(6)

and associated with the propositions $J_{\alpha\beta}$ in a way that

$$\nu(\hat{P}_{j\alpha\beta}) = [J_{\alpha\beta}]_e,$$

(7)

where $J \in \{X,Y,Z\}$.

Let the set $O$ be separated into three subsets: $C_z = \{\hat{P}_{z\alpha\beta}\}$, $C_x = \{\hat{P}_{x\alpha\beta}\}$ and $C_y = \{\hat{P}_{y\alpha\beta}\}$, each called a context, explicitly,

$$C_z = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\},$$

(8)

$$C_x = \left\{ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right\},$$

(9)

$$C_y = \left\{ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right\}. $$

(10)

It is not difficult to see that within each context $C_j$ the projection operators $\hat{P}_{j\alpha\beta}$ give the resolution of the identity

$$\sum_{\alpha\beta} \hat{P}_{j\alpha\beta} = \hat{1}_4,$$

(11)
and, additionally, their product is a projection operator as well, namely,
\[ \hat{P}_{j\alpha\beta} \cdot \hat{P}_{j\gamma\delta} = \hat{P}_{j\gamma\delta} \cdot \hat{P}_{j\alpha\beta} = \hat{0}_4 \quad , \] (12)

where \( \hat{1}_4 \) and \( \hat{0}_4 \) are the identity and zero matrices, respectively, \( \gamma \) and \( \delta \) are elements of the set \( \{+,-\} \) different from \( \alpha \) and \( \beta \), respectively.

In contrast, the projection operators \( \hat{P}_{j\alpha\beta} \) and \( \hat{P}_{k\epsilon\zeta} \) taken from different contexts \( C_j \) and \( C_k \), where \( \epsilon \) and \( \zeta \) are elements of the set \( \{+,-\} \) (not necessarily different from \( \alpha \) and \( \beta \)), do not commute, that is,
\[ \hat{P}_{j\alpha\beta} \cdot \hat{P}_{k\epsilon\zeta} \neq \hat{P}_{k\epsilon\zeta} \cdot \hat{P}_{j\alpha\beta} \quad , \] (13)

therefore, neither \( \hat{P}_{j\alpha\beta} \cdot \hat{P}_{k\epsilon\zeta} \) nor \( \hat{P}_{k\epsilon\zeta} \cdot \hat{P}_{j\alpha\beta} \) is a projection operator on \( \mathbb{C}^{4 \times 4} \).

Let us introduce a lattice \( L(\mathbb{C}^{4 \times 4}) \) of the subspaces of \( \mathbb{C}^{4 \times 4} \), specifically, \( \text{ran}(\hat{P}_\circ) \), where the partial order \( \leq \) is the inclusion relation \( \subseteq \) on a set of \( \text{ran}(\hat{P}_\circ) \), the meet \( \cap \) is their intersection \( \cap \) and the join \( \sqcup \) is the span of their union \( \cup \). The lattice \( L(\mathbb{C}^{4 \times 4}) \) is bounded, with the trivial space \( \{0\} \) equal to the range of the zero matrix, \( \text{ran}(\hat{0}_4) = \{0\} \), as the bottom and the space \( \mathbb{C}^{4 \times 4} \) equal to the range of the identity matrix, \( \text{ran}(\hat{1}_4) = \mathbb{C}^{4 \times 4} \), as the top.

Given that there is a one-to-one correspondence between \( \text{ran}(\hat{P}_\circ) \) and the corresponding projection operators \( P_\circ \), one can take \( \hat{P}_\circ \) to be the elements of \( L(\mathbb{C}^{4 \times 4}) \).

Specifically, as
\[ \text{ran}(\hat{P}_{j\alpha\beta}) \subseteq \ker(\hat{P}_{j\gamma\delta}) = \text{ran}(\hat{1}_4 - \hat{P}_{j\gamma\delta}) \quad , \] (14)

one can define the partial order \( \leq \) within each context \( C_j \subset O \) by setting
\[ \hat{P}_{j\alpha\beta} \leq (\hat{1}_4 - \hat{P}_{j\gamma\delta}) \quad \text{iff} \quad \hat{P}_{j\alpha\beta} \cap (\hat{1}_4 - \hat{P}_{j\gamma\delta}) = \hat{P}_{j\alpha\beta} \quad , \] (15)

which means that the meet of \( \hat{P}_{j\alpha\beta} \) and \( \hat{P}_{j\gamma\delta} \) in \( L(\mathbb{C}^{4 \times 4}) \) can be defined as
\[ \hat{P}_{j\alpha\beta} \cap \hat{P}_{j\gamma\delta} = \hat{P}_{j\alpha\beta} \cdot \hat{P}_{j\gamma\delta} = \hat{0}_4 \quad . \] (16)

Since the subspaces \( \text{ran}(\hat{P}_{j\alpha\beta}) \) and \( \text{ran}(\hat{P}_{j\gamma\delta}) \) satisfy the following property
\[ \text{ran}(\hat{P}_{j\alpha\beta}) \cap \text{ran}(\hat{P}_{j\gamma\delta}) = \text{ran}(\hat{P}_{j\alpha\beta} \cdot \hat{P}_{j\gamma\delta}) = \text{ran}(\hat{0}_4) = \{0\} \quad , \] (17)

the join of the projection operators taken from the same context can be defined as their sum, i.e.,
\[ \bigcup_{\alpha \beta} \hat{P}_{j\alpha \beta} = \sum_{\alpha \beta} \hat{P}_{j\alpha \beta} = \hat{1}_4 \].  

As the identity matrix \( \hat{1}_4 \) leaves invariant any column vector \(|\Psi\rangle\) lying in the space \( \mathbb{C}^{4 \times 4} \), the range of \( \hat{1}_4 \), a proposition represented by \( \hat{1}_4 \) must be true in any state of the system, i.e., such a proposition must be a tautology \( \top \). Also, as the zero matrix \( \hat{0}_4 \) annihilates any column vector in \( \mathbb{C}^{4 \times 4} \), the null space of \( \hat{0}_4 \), a proposition represented by \( \hat{0}_4 \) must be false in any state of the system, in other words, this proposition must be a contradiction \( \bot \).

This can be written as
\[
|\Psi\rangle \in \text{ran}(\hat{1}_4) = \mathbb{C}^{4 \times 4} \implies v(\hat{1}_4) = [\top]_v = 1, \quad (19)
\]
\[
|\Psi\rangle \in \ker(\hat{0}_4) = \mathbb{C}^{4 \times 4} \implies v(\hat{0}_4) = [\bot]_v = 0. \quad (20)
\]

In keeping with the valuational axiom (1), let us assume that conjunction and disjunction on the propositions \( J_{\alpha \beta} \) represented by the projection operators taken from the same context \( C_j \) are defined respectively as
\[
v\left( \hat{P}_{j\alpha \beta} \cap \hat{P}_{j\gamma \delta} \right) = v(\hat{0}_4) = [J_{\alpha \beta} \land J_{\gamma \delta}]_v = 0, \quad (21)
\]
\[
v\left( \bigcup_{\alpha \beta} \hat{P}_{j\alpha \beta} \right) = v(\hat{1}_4) = [\lor J_{\alpha \beta}]_v = 1. \quad (22)
\]

3 Logical account of the KS theorem

Imagine that the pair of spin–\( \frac{1}{2} \) particles is prepared in a correlated spin state \(|\Psi_{j\alpha k\beta}\rangle\) where \( j = k \), say, such one that is represented by the column vector \(|\Psi_{z+}\rangle\)

\[
|\Psi_{z+}\rangle = |\Psi_{z+}^{(1)}\rangle \otimes |\Psi_{z+}^{(2)}\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

sitting in the range of the projection operator \( \hat{P}_{z+} \)

\[
\text{ran}(\hat{P}_{z+}) = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}
\]

and the null spaces of the projection operators \( \hat{P}_{z-}, \hat{P}_{z=} \) and \( \hat{P}_{z=} \).
and null space are as follows:

\[
\ker(\hat{P}_{z+}) = \left\{ \begin{bmatrix} a \\ 0 \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} ,
\]

\[
\ker(\hat{P}_{z-}) = \left\{ \begin{bmatrix} a \\ b \\ 0 \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} ,
\]

\[
\ker(\hat{P}_{z-+}) = \left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\} .
\]

Consider a projection operator on \( C_z \) among all the propositions \( J_{\alpha\beta} \) associated with the context \( C_z \) whose range and null space are as follows:

\[
\ker(\hat{P}_{z++}) = \left\{ \begin{bmatrix} a \\ 0 \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} ,
\]

\[
\ker(\hat{P}_{z+-}) = \left\{ \begin{bmatrix} a \\ b \\ 0 \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} ,
\]

\[
\ker(\hat{P}_{z-+}) = \left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\} .
\]

Clearly, being in the state \( |\Psi_{z++}\rangle \) causes all the propositions \( Z_{\alpha\beta} \) associated with the context \( C_z \) become bivalent, that is,

\[
|\Psi_{z++}\rangle \in \begin{cases} \text{ran}(\hat{P}_{z++}) \\ \ker(\hat{P}_{z+-}) \\ \ker(\hat{P}_{z-+}) \\ \ker(\hat{P}_{z-}) \end{cases} \implies \begin{cases} v(\hat{P}_{z++}) = \|Z_{++}\|_v = 1 \\ v(\hat{P}_{z+-}) = \|Z_{+-}\|_v = 0 \\ v(\hat{P}_{z-+}) = \|Z_{-+}\|_v = 0 \\ v(\hat{P}_{z-}) = \|Z_{-}\|_v = 0 \end{cases} .
\]

One can infer from here that in any correlated spin state \( |\Psi_{j\alpha\beta}\rangle \) there is a context \( C_j \) such that among all the propositions \( J_{\alpha\beta} \) associated with \( C_j \) one is true while the others are false, and, hence, the entailment \([4] \) is valid:

\[
|\Psi_{j\alpha\beta}\rangle \in \begin{cases} \text{ran}(\hat{P}_{j\alpha\beta}) \\ \ker(\hat{P}_{j\gamma\delta}) \end{cases} \implies \begin{cases} v(\hat{P}_{j\alpha\beta}) = \|J_{\alpha\beta}\|_v = 1 \\ v(\hat{P}_{j\gamma\delta}) = \|J_{\gamma\delta}\|_v = 0 \end{cases} .
\]

As follows, in the said context \( C_j \), the truth values of conjunctions and disjunctions can be expressed with the basic operations of arithmetic or by the min and max functions, namely,

\[
\|J_{\alpha\beta} \land J_{\gamma\delta}\|_v = \|J_{\alpha\beta}\|_v \cdot \|J_{\gamma\delta}\|_v = \min \{ \|J_{\alpha\beta}\|_v, \|J_{\gamma\delta}\|_v \} = 0 ,
\]

\[
\|\bigvee_{\alpha\beta} J_{\alpha\beta}\|_v = \sum_{\alpha\beta} \|J_{\alpha\beta}\|_v = \max_{\alpha\beta} \{ \|J_{\alpha\beta}\|_v \} = 1 .
\]

Consider a projection operator on \( \mathbb{C}^4 \) that is not an element of the preselected (by the preparation of the composite system’s state) context \( C_z \): Take, for example, the operator \( \hat{P}_{y++} \) whose range and null space are as follows:

\[
\text{ran}(\hat{P}_{y++}) = \left\{ \begin{bmatrix} a \\ ia \\ ia \\ -a \end{bmatrix} : a \in \mathbb{C} \right\} ,
\]

\[
\|J_{\alpha\beta} \land J_{\gamma\delta}\|_v = \|J_{\alpha\beta}\|_v \cdot \|J_{\gamma\delta}\|_v = \min \{ \|J_{\alpha\beta}\|_v, \|J_{\gamma\delta}\|_v \} = 0 ,
\]

\[
\|\bigvee_{\alpha\beta} J_{\alpha\beta}\|_v = \sum_{\alpha\beta} \|J_{\alpha\beta}\|_v = \max_{\alpha\beta} \{ \|J_{\alpha\beta}\|_v \} = 1 .
\]
\[ \ker(\hat{P}_{y^+}) = \left\{ \begin{bmatrix} ia + ib + c \\ a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{C} \right\} . \]  

(33)

If \([Z_{++}]_v = 1\), then the proposition \(Y_{++}\) cannot be bivalent under the truth-value assignment function \(v\), otherwise one would get a contradiction \(1 = 0\), to be exact,

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \left\{ \begin{bmatrix} a \\ ia \\ ia \\ -a \end{bmatrix} : a \in \mathbb{C} \right\} ,
\]

(34)

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \left\{ \begin{bmatrix} ia + ib + c \\ a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{C} \right\} .
\]

(35)

The same contradiction would obviously appear for any other projection operator on \(\mathbb{C}^{4 \times 4}\) not belonging to \(C_z\). Thus, in any correlated spin state \(|\Psi_{j\alpha\beta}\rangle\) there is only one context \(C_j\) in the set \(O\) where the sole projection operator can be assigned the truth-value 1 at the same time as all the rest are assigned the truth-value 0.

Next, imagine that the pair of spin–\(\frac{1}{2}\) particles is prepared in an uncorrelated spin state \(|\Psi_{j\alpha k\beta}\rangle\) where \(j \neq k\). Due to the one-to-one correspondence between column spaces \(\text{ran}(\cdot)\) and projection operators, the column vector \(|\Psi_{j\alpha k\beta}\rangle\) does not lie in the column or null space of either projection operator \(\hat{P}_{j\alpha\beta}\), which implies that in this case no projection operator in the set \(O\) can be bivalent under \(v\), specifically,

\[
j \neq k : |\Psi_{j\alpha k\beta}\rangle \notin \text{ran}(\hat{P}_{j\alpha\beta}) \implies \begin{cases} v(\hat{P}_{j\alpha\beta}) = [J_{\alpha\beta}]_v \neq 1 \\ v(\hat{P}_{j\alpha\beta}) = [J_{\alpha\beta}]_v \neq 0 \end{cases} .
\]

(36)

This means that independently of the state \(|\Psi_{j\alpha k\beta}\rangle\) in which the pair of spin–\(\frac{1}{2}\) particles can be prepared, the set \(O\) cannot be truth-value bivalent under \(v\); otherwise stated, \(v\) cannot be a total two-valued function, namely,

\[
v : O \mapsto \{1, 0\} .
\]

(37)

4 Concluding remarks

The fact that the truth-value assignment function \(v\) imposed on the set \(O\) can be only partial indicates that unless they are associated with the preselected context, prior to their verification the propositions \(J_{\alpha\beta}\) have either a truth-value \(v\) different from 1 and 0, explicitly,
\[ v(\hat{P}_{j\alpha\beta}) = [J_{\alpha\beta}]_v \in \{0 < v < 1 \mid v \in \mathbb{R}\} \quad \text{(38)} \]

or absolutely no truth-value, that is,

\[ \left\{ v(\hat{P}_{j\alpha\beta}) \right\} = \{[J_{\alpha\beta}]_v \} = \emptyset \quad \text{(39)} \]

In the first case, prior to the verification the propositions \( J_{\alpha\beta} \) obey many-valued semantics, for example, the Lukasiewicz-Pykacz model of infinite-valued logic \[5, 6\]. Within this model, the entailment (38) fails because \( v(\sum_{\alpha\beta} \hat{P}_{j\alpha\beta}) = 1 \) means that \( \sum_{\alpha\beta} v(\hat{P}_{j\alpha\beta}) \geq 1 \), i.e., more than one \( \hat{P}_{j\alpha\beta} \) can have non-zero truth-value.

In contrast, in the second case, before the verification the propositions \( J_{\alpha\beta} \) comply with partial semantics having truth-value gaps, such as supervaluationism \[7, 8\]. According to supervaluationism, the entailment (38) fails because \( \bigvee_{\alpha\beta} J_{\alpha\beta} \) should be true regardless of whether or not its disjuncts \( J_{\alpha\beta} \) have a truth value (supervaluationism describes \( \bigvee_{\alpha\beta} J_{\alpha\beta} \) as “supertrue”).

Mathematically though, partial semantics are not very different from many-valued semantics. Moreover, for any partial (“gappy”) semantics, one can construct a gapless many-valued semantics which will define the same logic \[9\].

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