STRONG MINIMIZERS OF THE CALCULUS OF VARIATIONS ON TIME SCALES AND THE WEIERSTRASS CONDITION*

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Abstract

We introduce the notion of strong local minimizer for the problems of the calculus of variations on time scales. Simple examples show that on a time scale a weak minimum is not necessarily a strong minimum. A time scale form of the Weierstrass necessary optimality condition is proved, which enables to include and generalize in the same result both continuous-time and discrete-time conditions.

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1 Introduction

Dynamic equations on time scales is a recent subject that allows the unification and extension of the study of differential and difference equations in one and same theory [10].

The calculus of variations on time scales was introduced in 2004 with the papers of Martin Bohner [6] and Roman Hilscher and Vera Zeidan [15]. Roughly speaking, in [6] the basic problem of the calculus of variations on time scales with given boundary conditions is introduced, and time scale versions of the classical necessary optimality conditions of Euler-Lagrange and Legendre proved, while in [15] necessary conditions as well as sufficient conditions for variable

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end-points calculus of variations problems on time scales are established. Since the two pioneer works [6, 15] and the understanding that much remains to be done in the area [13], several recent studies have been dedicated to the calculus of variations on time scales: the time scale Euler-Lagrange equation was proved for problems with double delta-integrals [9] and for problems with higher-order delta-derivatives [14]; a correspondence between the existence of variational symmetries and the existence of conserved quantities along the respective Euler-Lagrange delta-extremals was established in [5]; optimality conditions for isoperimetric problems on time scales with multiple constraints and Pareto optimality conditions for multiobjective delta variational problems were studied in [20]; a weak maximum principle for optimal control problems on time scales has been obtained in [16]. Such results may also be formulated via the nabla-calculus on time scales, and seem to have interesting applications in economics [1, 2, 3, 21].

In all the works available in the literature on time scales the variational extrema are regarded in a weak local sense. Differently, here we consider strong solutions of problems of the calculus of variations on time scales. In Section 2 we briefly review the necessary results of the calculus on time scales. The reader interested in the theory of time scales is referred to [10, 11], while for the classical continuous-time calculus of variations we refer to [12, 19], and to [18] for the discrete-time setting. In Section 3 the concept of strong local minimum is introduced (cf. Definition 3.1), and an example of a problem of the calculus of variations on the time scale

\[ T = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{ 0 \} \]

is considered showing that the standard weak minimum used in the literature on time scales is not necessarily a strong minimum (cf. Example 3.2). Our main result is a time scale version of the Weierstrass necessary optimality condition for strong local minimum (cf. Theorem 3.3). We end with Section 4 illustrating our main result with the particular cases of discrete-time and the \( q \)-calculus of variations [4].

2 Time Scales Calculus

In this section we introduce basic definitions and results that will be needed for the rest of the paper. For a more general theory of calculus on time scales, we refer the reader to [10, 11].

A nonempty closed subset of \( \mathbb{R} \) is called a \textit{time scale} and it is denoted by \( T \). Thus, \( \mathbb{R}, \mathbb{Z}, \text{ and } \mathbb{N} \), are trivial examples of times scales. Other examples of times scales are: \([-2, 4] \cup \mathbb{N}, h\mathbb{Z} := \{hz \mid z \in \mathbb{Z} \} \) for some \( h > 0 \), \( q^{\mathbb{N}} := \{q^k \mid k \in \mathbb{N}_0 \} \) for some \( q > 1 \), and the Cantor set. We assume that a time scale \( T \) has the topology that it inherits from the real numbers with the standard topology.

The \textit{forward jump operator} \( \sigma : T \to T \) is defined by

\[ \sigma(t) = \inf \{ s \in T : s > t \}, \text{ for all } t \in T, \]

while the \textit{backward jump operator} \( \rho : T \to T \) is defined by

\[ \rho(t) = \sup \{ s \in T : s < t \}, \text{ for all } t \in T, \]
with \( \inf \emptyset = \sup \emptyset \) (i.e., \( \sigma(M) = M \) if \( T \) has a maximum \( M \)) and \( \sup \emptyset = \inf T \) (i.e., \( \rho(m) = m \) if \( T \) has a minimum \( m \)).

If \( \sigma(t) > t \), we say that \( t \) is right-scattered, while if \( \rho(t) < t \) we say that \( t \) is left-scattered. Also, if \( t < \sup T \) and \( \sigma(t) = t \), then \( t \) is called right-dense, and if \( t > \inf T \) and \( \rho(t) = t \), then \( t \) is called left-dense. The set \( T^\rho \) is defined as \( T \) without the left-scattered maximum of \( T \) (in case it exists).

The graininess function \( \mu : T \to [0, \infty) \) is defined by

\[
\mu(t) = \sigma(t) - t, \text{ for all } t \in T.
\]

**Example 2.1.** If \( T = \mathbb{R} \), then \( \sigma(t) = \rho(t) = t \) and \( \mu(t) = 0 \). If \( T = \mathbb{Z} \), then \( \sigma(t) = t + 1, \rho(t) = t - 1 \), and \( \mu(t) = 1 \). On the other hand, if \( T = \mathbb{Q}^{\infty} \), where \( q > 1 \) is a fixed real number, then we have \( \sigma(t) = qt, \rho(t) = q^{-1}t \), and \( \mu(t) = (q - 1)t \).

A function \( f : T \to \mathbb{R} \) is regulated if the right-hand limit \( f(t+) \) exists (finite) at all right-dense points \( t \in T \) and the left-hand limit \( f(t-) \) exists at all left-dense points \( t \in T \). A function \( f \) is \( rd \)-continuous (we write \( f \in C_{rd} \)) if it is regulated and if it is continuous at all right-dense points \( t \in T \). Following [15], a function \( f \) is piecewise \( rd \)-continuous (we write \( f \in C_{prd} \)) if it is regulated and if it is \( rd \)-continuous at all, except possibly at finitely many, right-dense points \( t \in T \).

We say that a function \( f : T \to \mathbb{R} \) is delta differentiable at \( t \in T^\rho \) if there exists a number \( f^\Delta(t) \) such that for all \( \varepsilon > 0 \) there is a neighborhood \( U \) of \( t \) (i.e., \( U = (t - \delta, t + \delta) \cap T \) for some \( \delta > 0 \)) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \text{ for all } s \in U.
\]

We call \( f^\Delta(t) \) the delta derivative of \( f \) at \( t \) and say that \( f \) is delta differentiable on \( T^\rho \) provided \( f^\Delta(t) \) exists for all \( t \in T^\rho \). Note that in right-dense points \( f^\Delta(t) = \lim_{s \to t^-} \frac{f(\sigma(s)) - f(\sigma(t))}{\sigma(s) - \sigma(t)} \), provided this limit exists, and in right-scattered points \( f^\Delta(t) = \frac{f(\sigma(t)) - f(\sigma(s))}{\sigma(t) - \sigma(s)} \) provided \( f \) is continuous at \( t \).

**Example 2.2.** If \( T = \mathbb{R} \), then \( f^\Delta(t) = f'(t) \), i.e., the delta derivative coincides with the usual one. If \( T = \mathbb{Z} \), then \( f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t) \). If \( T = \mathbb{Q}^{\infty} \), \( q > 1 \), then \( f^\Delta(t) = \frac{f(\sigma(t)) - f(\sigma(s))}{\sigma(t) - \sigma(s)} \), i.e., we get the usual derivative of Quantum calculus [17].

Let \( f, g : T \to \mathbb{R} \) be delta differentiable at \( t \in T^\rho \). Then (see, e.g., [10]),

(i) the product \( fg \) is delta differentiable at \( t \) with

\[
(fg)^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t);
\]

(ii) if \( g(t)g^\sigma(t) \neq 0 \), then \( \frac{f}{g} \) is delta differentiable at \( t \) with

\[
\left( \frac{f}{g} \right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)},
\]

where we abbreviate here and throughout the text \( f \circ \sigma \) by \( f^\sigma \).
A function \( f \) is \( r d \)-continuously delta differentiable (we write \( f \in C^1_{rd} \)) if \( f^\triangle \) exists for all \( t \in T^c \) and \( f^\triangle \in C_{rd} \). A continuous function \( f \) is piecewise \( rd \)-continuously delta differentiable (we write \( f \in C^1_{prd} \)) if \( f \) is continuous and \( f^\triangle \) exists for all, except possibly at finitely many, \( t \in T^c \) and \( f^\triangle \in C_{rd} \). It is known that piecewise rd-continuous functions possess an antiderivative, i.e., there exists a function \( F \) with \( F^\triangle = f \), and in this case the delta integral is defined by \( \int_c^d f(t) \Delta t = F(d) - F(c) \) for all \( c, d \in T \).

**Example 2.3.** Let \( a, b \in T \) with \( a < b \). If \( T = \mathbb{R} \), then \( \int_a^b f(t) \Delta t = \int_a^b f(t) dt \), where the integral on the right-hand side is the classical Riemann integral. If \( T = \mathbb{Z} \), then \( \int_a^b f(t) \Delta t = \sum_{k=a}^{b-1} f(k) \). If \( T = q^{\mathbb{N}_0}, q > 1 \), then \( \int_a^b f(t) \Delta t = (1 - q) \sum_{t \in [a,b]} tf(t) \).

The delta integral has the following properties (see, e.g., [10]):

(i) if \( f \in C_{prd} \) and \( t \in T^c \), then
\[
\int_t^{\sigma(t)} f(\tau) \Delta \tau = \mu(t) f(t);
\]

(ii) if \( c, d \in T \) and \( f, g \in C_{prd} \), then
\[
\int_c^d f(\sigma(t)) g^\triangle(t) \Delta t = [(fg)(t)]_t=c^d - \int_c^d f^\triangle(t) g(t) \Delta t;
\]
\[
\int_c^d f(t) g^\triangle(t) \Delta t = [(fg)(t)]_t=c^d - \int_c^d f^\triangle(t) g(\sigma(t)) \Delta t.
\]

### 3 The Weierstrass Necessary Condition

Let \( T \) be a bounded time scale. Throughout we let \( t_0, t_1 \in T \) with \( t_0 < t_1 \). For an interval \([t_0, t_1] \cap T \) we simply write \([t_0, t_1] \). The problem of the calculus of variations on time scales under consideration has the form

\[
\text{minimize } \mathcal{L}[x] = \int_{t_0}^{t_1} f(t, x(\sigma(t)), x^\Delta(t)) \Delta t, \tag{3.1}
\]

over all \( x \in C^1_{prd} \) satisfying the boundary conditions

\[
x(t_0) = \alpha, \quad x(t_1) = \beta, \quad \alpha, \beta \in \mathbb{R}, \tag{3.2}
\]

where \( f : [t_0, t_1]^c \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \).

A function \( x \in C^1_{prd} \) is said to be admissible if it satisfies conditions (3.2).

Let us consider two norms in \( C^1_{prd} \):

\[
\|x\|_1 = \sup_{t \in [t_0, t_1]} |x(\sigma(t))| + \sup_{t \in [t_0, t_1]} |x^\Delta(t)|,
\]
where here and subsequently $T$ denotes the set of points of $[t_0, t_1]$ where $x^\wedge(t)$ does not exist, and

$$||x||_0 = \sup_{t \in [t_0, t_1]} |x^\sigma(t)|.$$  

The norms $|| \cdot ||_0$ and $|| \cdot ||_1$ are called the strong and the weak norm, respectively. The strong and weak norms lead to the following definitions for local minimum:

**Definition 3.1.** An admissible function $\tilde{x}$ is said to be a strong local minimum for (3.1–3.2) if there exists $\delta > 0$ such that $L[\tilde{x}] \leq L[x]$ for all admissible $x$ with $||x - \tilde{x}||_0 < \delta$. Likewise, an admissible function $\tilde{x}$ is said to be a weak local minimum for (3.1–3.2) if there exists $\delta > 0$ such that $L[\tilde{x}] \leq L[x]$ for all admissible $x$ with $||x - \tilde{x}||_1 < \delta$.

A weak minimum may not necessarily be a strong minimum:

**Example 3.2.** Consider the variational problem

$$L[x] = \int_0^1 [x^\wedge(t)^2 - x^\wedge(t)^4] \Delta t, \quad x(0) = 0, \quad x(1) = 0,$$

on the time scale $T = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ (note that we need to add zero in order to have a closed set). Let us show that $\tilde{x}(t) = 0$, $0 \leq t \leq 1$ is a weak local minimum for (3.3). In the topology induced by $|| \cdot ||_1$ consider the open ball of radius 1 centered at $\tilde{x}$, i.e.,

$$B^1_1(\tilde{x}) = \{x \in C^{1}_{prd} : ||x - \tilde{x}||_1 < 1\}.$$

We use the notation $B^k_1$ for the ball of radius $r$ in norm $|| \cdot ||_k$, $k = 1, 2$. For every $x \in B^1_1(\tilde{x})$ we have

$$|x^\wedge(t)| \leq 1, \quad \forall t \in [0, 1],$$

hence $L[x] \geq 0$. This proves that $\tilde{x}$ is a weak local minimum for (3.3) since $L[\tilde{x}] = 0$. Now let us consider the function defined by

$$x_d(t) = \begin{cases} d & \text{if } t = \sigma(t_0) \\ 0 & \text{otherwise} \end{cases}, \quad t_0 \in (0, 1) \cap T, \quad \sigma(t_0) \neq 1, \quad d \in \mathbb{R} \setminus \{0\}.$$

Function $x_d$ is admissible and $||x_d||_0 = \sup_{t \in [0, 1]} |x_d^\sigma(t)| = |d|$. Therefore, for every $\delta > 0$ there is a $d$ such that

$$x_d \in B^0_\delta(\tilde{x}) = \{x \in C^{1}_{prd} : ||x - \tilde{x}||_0 < \delta\}.$$

We have

$$x_d^\wedge(t_0) = -\frac{d}{\mu(t_0)}, \quad x_d^\wedge(\sigma(t_0)) = \frac{-d}{\mu(\sigma(t_0))},$$
and $x_d^\triangle(t) = 0$ for all $t \neq t_0, \sigma(t_0)$. Hence, $|x_d^\triangle(t)|, 0 \leq t \leq 1$, can take arbitrary large values since $\mu(t) = \frac{1}{t^2} \to 0$ as $t \to 0$. Note that for every $\delta > 0$ we can choose $d$ and $t_0$ such that $x_d \in B_\delta(\bar{x})$ and $\frac{d}{\mu(\sigma(t_0))} > 1$. Finally,

$$
\mathcal{L}[x_d] = \int_0^1 [x_d^\triangle(t)^2 - x_d^\triangle(t)^4] \Delta t
= \mu(t_0) \left[ \left( \frac{d}{\mu(t_0)} \right)^2 - \left( \frac{d}{\mu(t_0)} \right)^4 \right] + \mu(\sigma(t_0)) \left[ \left( \frac{d}{\mu(\sigma(t_0))} \right)^2 - \left( \frac{d}{\mu(\sigma(t_0))} \right)^4 \right]
= \frac{d^2}{\mu(t_0)} \left[ 1 - \frac{d^2}{\mu^2(t_0)} \right] + \frac{d^2}{\mu(\sigma(t_0))} \left[ 1 - \frac{d^2}{\mu^2(\sigma(t_0))} \right] < 0.
$$

Therefore, the trajectory $\dot{x}$ cannot be a strong minimum for (3.3).

From now on we assume that $f : [t_0, t_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has partial continuous derivatives $f_x$ and $f_r$, respectively with respect to the second and third variables, for all $t \in [t_0, t_1]$, and $f(t, x, v), f_x(\cdot, x, v)$ and $f_r(\cdot, x, v)$ are continuous.

Let $E : [t_0, t_1] \times \mathbb{R}^3 \to \mathbb{R}$ be the function with the values

$$
E(t, x, r, q) = f(t, x, q) - f(t, x, r) - (q - r)f_r(t, x, r).
$$

This function, called the Weierstrass excess function, is utilized in the following theorem:

**Theorem 3.3** (Weierstrass necessary optimality condition on time scales). Let $T$ be a time scale, $t_0, t_1 \in T$, $t_0 < t_1$. Assume that the function $f(t, x, r)$ in problem (3.1), (3.2) satisfies the following condition:

$$
\mu(t)f(t, x, \gamma r_1 + (1 - \gamma)r_2) \leq \mu(t)\gamma f(t, x, r_1) + \mu(t)(1 - \gamma)f(t, x, r_2)
$$

for each $(t, x) \in [t_0, t_1] \times \mathbb{R}$, all $r_1, r_2 \in \mathbb{R}$ and $\gamma \in [0, 1]$. Let $\bar{x}$ be a piecewise continuous function. If $\bar{x}$ is a strong local minimum for (3.1), (3.2), then

$$
E[t, \bar{x}^\gamma(t), \bar{x}^\triangle(t), q] \geq 0
$$

for all $t \in [t_0, t_1]$ and all $q \in \mathbb{R}$, where we replace $\bar{x}^\triangle(t)$ by $\bar{x}^\triangle(t^-)$ and $\bar{x}^\triangle(t^+)$ at finitely many points $t$ where $\bar{x}^\triangle(t)$ does not exist.

**Proof.** Assume that $\bar{x}$ is a strong local minimum for (3.1), (3.2). We consider two cases. First, suppose that $a \in [t_0, t_1]$ is a right-scattered point. If $\bar{x}$ is a strong minimizer for the problem (3.1), (3.2), then it the restriction of $\bar{x}$ to $[a, \sigma(a)] \cap T$ is a strong minimizer for the problem (see [22])

$$
\int_a^{\sigma(a)} f(t, x^\circ(t), x^\triangle(t)) \Delta t \longrightarrow \min
\quad x(a) = \bar{x}(a), \quad x(\sigma(a)) = \bar{x}(\sigma(a)).
$$
We define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(q) = \int_a^{\sigma(a)} f(t, x^\sigma(t), q) \Delta t$. Hence, $h(q) = \mu(a)f(a, x^\sigma(a), q)$. By assumption (3.1), we have immediately that

$$h(q) - h(\bar{x}^\Delta(a)) - (q - \bar{x}^\Delta(a))h'(\bar{x}^\Delta(a)) \geq 0.$$ 

This gives

$$E[a, x^\sigma(a), \bar{x}^\Delta(a), q] \geq 0.$$ 

Second, we suppose that $a \in [t_0, t_1]$, $a < t_1$, is a right-dense point and $[a, b] \cap T$ is an interval between two successive points where $\bar{x}^\Delta(t)$ does not exist. Then, there exists a sequence $\{\varepsilon_k : k \in \mathbb{N}\} \subset [t_0, t_1]$ with $\lim_{k \to \infty} \varepsilon_k = a$. Let $\tau$ be any number such that $\sigma(\tau) \in [a, b)$ and $q \in \mathbb{R}$. We define the function $x : [t_0, t_1] \cap T \to \mathbb{R}$ as follows:

$$x(t) = \begin{cases} \bar{x}(t) & \text{if } t \in [t_0, a] \cup [b, t_1] \\ X(t) & \text{if } t \in [a, \tau], \\ \phi(t, \tau) & \text{if } t \in [\tau, b], \end{cases}$$

where

$$X(t) = \bar{x}(a) + q(t - a), \quad q \in \mathbb{R},$$

$$\phi(t, \tau) = \bar{x}(t) + \frac{X(\tau) - \bar{x}(\tau)}{b - \tau}(b - t).$$

Clearly, given $\delta > 0$, for any $q$ one can choose $\tau$ such that $\|x - \bar{x}\|_0 < \delta$. Let us now consider the function $K$ defined for all $\tau \in [a, b) \cap T$ such that $\sigma(\tau) \in [a, b) \cap T$ with the values $K(\tau) = \mathcal{L}[x] - \mathcal{L}[\bar{x}]$. Since $\mathcal{L}[x] \geq \mathcal{L}[\bar{x}]$, by hypothesis, $K(\tau) \geq 0$ and $K(a) = 0$, it follows by Theorem 1.12 in [11] that $K^\Delta(a) \geq 0$. By the definition of $x$, we have

$$K(\tau) = \int_a^{\tau} \{f[t, X^\sigma(t), X^\Delta(t)] - f[t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)]\} \Delta t + \int_\tau^b \{f[t, \phi(\sigma(t), \tau), \phi^\Delta(t, \tau)] - f[t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)]\} \Delta t$$

so that, by Theorem 5.37 in [7] and Theorem 7.1 in [8], we obtain

$$K^\Delta(\tau) = f[\tau, X^\sigma(\tau), X^\Delta(\tau)] - f[\tau, \phi(\sigma(\tau), \sigma(\tau)), \phi^\Delta(\tau, \sigma(\tau)) ] + \int_\tau^b \{f_x[t, \phi(\sigma(t), \tau), \phi^\Delta(t, \tau)] \phi^\Delta_1(\sigma(t), \tau) + f_\tau[t, \phi(\sigma(t), \tau), \phi^\Delta(t, \tau)] \phi^\Delta_1(\tau, \tau) \} \Delta t.$$ 

Invoking the relation $\phi^\Delta_1^\Delta_2 = \phi^\Delta_2^\Delta_1$ (see Theorem 6.1 in [8]), integration by parts gives

$$\int_\tau^b f_x \phi^\Delta_2^\Delta_1(t, \tau) \Delta t = f_x \phi^\Delta_2(t, \tau)|_\tau^b - \int_\tau^b f_{\phi^\Delta_1}(t, \tau) \phi^\Delta_2(\sigma(t), \tau) \Delta t.$$ 

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Thus, \((3.7)\) becomes
\[
\int_{\tau}^{b} \left[ f_x - f_{r_1}^\triangle \right] \phi^\triangle_2(\sigma(t), \tau) \Delta t
+ f_r \phi^\triangle_2(t, \tau) \bigg|_{\tau}^{b}.
\] (3.8)

From the definition of \(\phi(t, \tau)\) we have
\[
\phi^\triangle_2(t, \tau) = \frac{(X^\triangle(\tau) - \bar{x}^\triangle(\tau))(b - \tau) + X(\tau) - \bar{x}(\tau)(b - t)}{(b - \tau)(b - \sigma(\tau))}
\]
so that \(\phi^\triangle_2(b, a) = 0, \phi^\triangle_2(a, a) = X^\triangle(a) - \bar{x}^\triangle(a)\). Also, \(\phi(\sigma(t), a) = \bar{x}(\sigma(t))\), \(\phi^\triangle_1(t, a) = \bar{x}^\triangle(t)\). Thus, letting \(\tau = a\) in \((3.8)\) we obtain
\[
-f_x[a, \bar{x}(\sigma(a)), \bar{x}^\triangle(a)]|X^\triangle(a) - \bar{x}^\triangle(a)|.
\]

Since \(\bar{x}\) verifies the Euler-Lagrange equation (see \([6]\)), we get
\[
\int_{\tau}^{b} \left\{ f_x(t, \bar{x}(\sigma(t)), \bar{x}^\triangle(t)) - f_r^\triangle(t, \bar{x}(\sigma(t)), \bar{x}^\triangle(t)) \right\} \phi^\triangle_2(\sigma(t), \tau) \Delta t = 0.
\]

On account of the above, from \((3.6) - (3.7)\) we have
\[
K^\triangle(a) = f[a, X^\sigma(a), X^\triangle(a)] - f[a, \phi(\sigma(a), \sigma(a)), \phi^\triangle_1(a, \sigma(a))]
- f_x[a, \bar{x}(\sigma(a)), \bar{x}^\triangle(a)]|X^\triangle(a) - \bar{x}^\triangle(a)|.
\]

However, \(X^\sigma(a) = \bar{x}^\sigma(a), X^\triangle(a) = q, \phi(\sigma(a), \sigma(a)) = \bar{x}^\sigma(a), \phi^\triangle_1(a, \sigma(a)) = \bar{x}^\triangle(a)\). Therefore,
\[
K^\triangle(a) = f[a, \bar{x}^\sigma(a), q] - f[a, \bar{x}^\sigma(a), \bar{x}^\triangle(a)] - f_x[a, \bar{x}^\sigma(a), \bar{x}^\triangle(a)]|q - \bar{x}^\triangle(a)|,
\]
and from this
\[
K^\triangle(a) = E[a, \bar{x}^\sigma(a), \bar{x}^\triangle(a), q] \geq 0.
\]

To establish the condition \((3.5)\) for all \(t \in [t_0, t_1]^\kappa\), we consider the limit \(t \to t_1\) from left when \(t_1\) is left-dense, and the limit \(t \to t_p\) from left and from right when \(t_p \in T\).

**Remark 3.4.** For \(T = \mathbb{R}\) problem \((3.1) - (3.2)\) coincides with the classical problem of the calculus of variations. Condition \((3.4)\) is then trivially satisfied and Theorem \((3.3)\) is known as the Weierstrass necessary condition.

**Remark 3.5.** Let \(T\) be a time scale with \(\mu(t)\) depending on \(t\) and such that the time scale interval \([t_0, t_1]\) may be written as follows: \([t_0, t_1] = L \cup U\) with \(\mu(t) \neq 0\) for all \(t \in L\) and \(\mu(t) = 0\) for all \(t \in U\). An example of such time scale is the Cantor set \([10]\). Then, for \(t \in U\) the condition \((3.4)\) is trivially satisfied, while for \(t \in L\) \((3.4)\) is nothing more than convexity of \(f\) with respect to \(r\).
4 Special Cases

Let \( T = \mathbb{Z} \). If \( \bar{x} \) is a local minimum of the problem

\[
\text{minimize } L[x] = \sum_{t=t_0}^{t_1-1} f(t, x(t+1), \Delta x(t)),
\]

\[
x(t_0) = \alpha, \quad x(t_1) = \beta, \quad \alpha, \beta \in \mathbb{R},
\]

and the function \( f(t, x, r) \) is convex with respect to \( r \in \mathbb{R} \) for each \((t, x) \in [t_0, t_1 - 1] \times \mathbb{R}\), then \( E[t, \bar{x}(t+1), \Delta \bar{x}(t), q] \geq 0 \) for all \( t \in [t_0, t_1 - 1] \) and all \( q \in \mathbb{R} \).

Let now \( T = q^N \), \( q \geq 1 \). If \( \bar{x} \) is a local minimum of the problem

\[
\text{minimize } L[x] = \sum_{t \in [t_0, t_1)} (q-1)f \left( t, x(qt), \frac{x(qt) - x(t)}{qt - t} \right),
\]

\[
x(t_0) = \alpha, \quad x(t_1) = \beta, \quad \alpha, \beta \in \mathbb{R},
\]

and the function \( f(t, x, r) \) is convex with respect to \( r \in \mathbb{R} \) for each \((t, x) \in [t_0, t_1) \times \mathbb{R} \), then

\[
E \left[ t, \bar{x}(qt), \frac{\bar{x}(qt) - \bar{x}(t)}{qt - t}, p \right] \geq 0
\]

for all \( t \in [t_0, t_1) \) and all \( p \in \mathbb{R} \).

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