UNIQUENESS OF ENTIRE GRAPHS EVOLVING BY
MEAN CURVATURE FLOW

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Abstract. In this paper we study the uniqueness of graphical mean curvature flow with locally Lipschitz initial data. We first prove that rotationally symmetric entire graphs are unique, without any further assumptions. Our methods also give an alternative simple proof of uniqueness in the one dimensional case. In the general case, we establish the uniqueness of entire proper graphs that satisfy a uniform lower bound on the second fundamental form. The latter result extends to initial conditions that are proper graphs over subdomains of $\mathbb{R}^n$. A consequence of our result is the uniqueness of convex entire graphs, which allow us to prove that Hamilton’s Harnack estimate holds for mean curvature flow solutions that are convex entire graphs.

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1. Introduction

The evolution under Mean curvature flow studies a family of immersions $F(\cdot, t): M^n \to \mathbb{R}^{n+1}$, $t \in (0, T)$, of $n$-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ such that

\begin{equation}
\frac{\partial}{\partial t} F(p, t) = H(p, t) \nu(p, t), \quad p \in M^n
\end{equation}

where $H(p, t)$ and $\nu(p, t)$ denote the Mean curvature and inward pointing normal of the surface $M_t := F(M^n, t)$ at the point $F(p, t)$.

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We will assume in this work that $M_t, t \in (0, T]$ is a complete non-compact graph over a domain $\Omega_t \subset \mathbb{R}^n$ (if $\partial \Omega_0 \neq \emptyset$ then $\partial \Omega_t$ will evolve by MCF, that is in general it will change in time). Then, the solution $M_t$ can be written as $M_t = \{(x, u(x, t)) : x \in \Omega_t\}$ for a height function $u(x, t)$. In the case where $\Omega = \mathbb{R}^n$ we will say that $M_t$ is an entire graph.

The height function $u$ satisfies the following quasilinear parabolic initial value problem

$$
\begin{aligned}
  u_t &= \left( \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) D_{ij} u, \\
  u(x, 0) &= u_0(x), \\
  (x, t) &\in \Omega_t \times (0, T], \\
  x &\in \Omega_0,
\end{aligned}
$$

where $M_0 := \{(x, u_0(x)) : x \in \Omega_0\}$. Here we sum over repeated indices. In what follows, we will refer to this equation as graphical mean curvature flow.

Although the Mean curvature flow (MCF) has been extensively studied in the compact case from many points of view (such as existence and regularity, weak solutions, singularities, the extension of the flow through the singularities, flow with surgery) not much has been done in the non-compact case beyond the fundamental works by Ecker-Huisken [9, 10] which deal with graphs over $\mathbb{R}^n$ and the more recent work by the second author and Schnürer [13] which deals with graphs over domains.

The works by Ecker-Huisken [9, 10] establish the existence and local a’priori estimates of the graphical MCF over $\mathbb{R}^n$. Also, in [9] the uniqueness of graphical solutions is addressed in some special cases. The results in [10] show that in some sense the MCF on entire graphs behaves better than the heat equation on $\mathbb{R}^n$, namely an entire graph solution exists for all times, independently from the growth of the initial surface at infinity. The initial entire graph is assumed to be locally Lipschitz. Methods of similar spirit as in [10] are used by the second author and Schnürer in [13] to establish the existence of MCF solutions which are complete non-compact graphs over domains $\Omega_t \subset \mathbb{R}^n$. Note that if $\partial \Omega_0 \neq \emptyset$ then $\partial \Omega_t$ will evolve by MCF, that is in general it will change in time.

While the works [9, 10] and [13] completely address the existence of classical solutions to the graphical MCF with Lipschitz continuous initial data (on $\mathbb{R}^n$ or domains), the uniqueness question in such generality has remained on open question. While the methods in [9, 10] imply that polynomial growth at infinity is preserved by the flow, the question of uniqueness is not addressed in those works. In [5] the authors address uniqueness of graphs in general ambient manifolds and high codimension. However, their result requires a uniform bound on the second fundamental form for all times. Our goal in this work is to address the uniqueness of classical solutions to [13] under minimal assumptions on the behavior of the initial data $u_0(x)$ as $|x| \to +\infty$, and under no assumptions on the behavior of the solutions at infinity.

We will first describe our results in the case of entire graphs, these are Theorems [1, 18] We will then state our result in the case of domains.

For the reader’s convenience let us state the following existence result for graphical MCF over $\mathbb{R}^n$ that follows from the Ecker-Huisken works [9, 10]: Assume that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz
continuous function. Then there exists a solution $u: \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ of the initial value problem

$$
\begin{align*}
(1.3) & \quad \begin{cases}
u_t = \left( \delta^{ij} - \frac{\partial^2 u}{1 + |Du|^2} \right) D_{ij} u, & (x, t) \in \mathbb{R}^n \times (0, T) \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^n
\end{cases}
\end{align*}
$$

with $T = +\infty$ which is continuous up to $t = 0$ and $C^\infty$-smooth for $t > 0$.

The striking feature of the result above is that existence holds for any locally Lipschitz entire graph initial data that is independently from the spatial growth of the initial data $u_0(x)$, as $|x| \to +\infty$. This is in contrast with the heat equation on $\mathbb{R}^n$ where existence is guaranteed only for initial data with at most quadratic exponential growth at infinity. The underlying reason for this difference is that the diffusion coefficient $g^{ij} = \delta^{ij} - \frac{\partial^2 u}{1 + |Du|^2}$ in this non-linear problem becomes small in a maximal direction of the gradient where $|Du| \to +\infty$. This behavior can simply be observed in the one-dimensional case of an entire graph $u: \mathbb{R} \times (0, \infty) \to \mathbb{R}$ evolving by curve shortening flow (CSF), where $u(x, t)$ satisfies the equation

$$
\begin{align*}
(1.4) & \quad u_t = \frac{u_{xx}}{1 + u_x^2},
\end{align*}
$$

or in higher dimensions under rotational symmetry where $x_{n+1} = u(r, t)$, $r = |x|$ evolves by

$$
\begin{align*}
(1.5) & \quad u_t = \frac{u_{rr}}{1 + u_r^2} + \frac{n-1}{r} u_r.
\end{align*}
$$

Note that a similar phenomenon has been observed for quasilinear equations of the form

$$
\begin{align*}
(1.6) & \quad u_t = \Delta u^m, \quad \text{on } \mathbb{R}^n \times (0, \infty)
\end{align*}
$$

in the range of exponents $\frac{(n-2)\cdot 2}{n} < m < 1$ (see in [10] [15] and the references therein). In all cases above the slow diffusion at spatial infinity when $|Du| \to +\infty$ in (1.4) and (1.5), or $u \to +\infty$ (1.6) prevents instant blow up of solutions with large growing initial data as $|x| \to +\infty$.

We will see that in the one-dimensional case of the CSF (equation (1.4)) or the rotationally symmetric case of MCF (equation (1.3)) uniqueness holds for any entire graph solution independently of its growth at infinity. This is in sharp contrast with the heat equation in any dimension. More precisely we will show the following two results. The first shows the uniqueness of entire graph solutions to CSF:

**Theorem 1.1** (Uniqueness of solutions to CSF). Let $u_1, u_2: \mathbb{R} \times (0, T] \to \mathbb{R}$, $T > 0$ be two smooth solutions of equation (1.4) with the same Lipschitz continuous initial data $u_0$, that is $\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0$. Then, $u_1 = u_2$ on $\mathbb{R} \times (0, T]$.

The second, shows the uniqueness of rotationally symmetric entire graph solutions of MCF:

**Theorem 1.2** (Uniqueness of rotationally symmetric MCF solutions). Let $u_1, u_2: \mathbb{R}^n \times (0, T] \to \mathbb{R}$, $T > 0$ be two entire graph rotationally symmetric smooth solutions of (1.3) with the same Lipschitz continuous initial data $u_0(x)$, that is $\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0$. Then, $u_1 = u_2$ on $\mathbb{R}^n \times (0, T]$. 
We remark that Theorem 1.1 is already covered by the results in [3]. However, we provide here a
simpler and more direct proof in the case of entire one-dimensional graphs, in particular pointing
out the similarity with fast-diffusion.

Regarding the general case of proper entire graphs, we establish the uniqueness under a suitable
lower bound on the second fundamental form which prevents large oscillations of the solution in
different directions. We then extend this result to proper graphs over subdomains $\Omega \subset \mathbb{R}^n$. We
begin by recalling the following definition.

**Definition 1.1** (Proper graphs over subdomains $\Omega \subset \mathbb{R}^n$). A graph $M := \{(x, u(x)) : x \in \Omega \}$ over
a subdomain $\Omega \subset \mathbb{R}^n$ defined by the height function $u : \Omega \to \mathbb{R}$ is called proper if $u(x) \to +\infty$ as
$x \to \partial \Omega$ or $|x| \to +\infty$ (the latter is assumed if $\Omega$ is unbounded, in particular when $\Omega = \mathbb{R}^n$).

Let $M_t = \{(x, u(x, t)) : x \in \mathbb{R}^n\}$, $t \in (0, T)$ be a proper entire graph solution to mean curvature
flow (1.1) starting at $M_0$, which is defined by the height function $u : \mathbb{R}^n \times (0, T) \to \mathbb{R}$. We denote
by $v = (e_{n+1}, \nu)^{-1}$ the gradient function of $M_t$, where $\nu$ denotes the inward pointing unit normal
on $M_t$. Since $M_t, t \in (0, T)$ is assumed to be an entire graph, $(e_{n+1}, \nu)$ has always the same sign.
Furthermore, our assumption that $M_t$ is proper, guarantees that

\begin{equation}
(1.7) \quad v = (e_{n+1}, \nu)^{-1} > 0, \quad \text{on } M_t, \ t \in [0, T]
\end{equation}

in which case $v = \sqrt{1 + |Du|^2}$. In our result below we will further assume that $M_t$ satisfies the lower
bound curvature condition

\begin{equation}
(1.8) \quad v h_i^j \geq -c \delta_i^j, \quad \text{on } M_t, \ t \in (0, T]
\end{equation}

for some uniform constant $c > 0$. Here $h_i^j$ is the second fundamental form and in the particular case
of graphs corresponds to $h_i^j = \left(\delta_i^j - \frac{Du_i Du_j}{1 + |Du|^2}\right) \frac{Du_j}{\sqrt{1 + |Du|^2}}$.

Our uniqueness result states as follows:

**Theorem 1.3** (General uniqueness result for entire graphs). Assume that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a locally
Lipschitz function defining a proper entire graph $M_0 = \{(x, u_0(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$.

Let $u_1, u_2 : \mathbb{R}^n \times (0, T] \to \mathbb{R}$ be two smooth solutions of (1.1) defining two entire graph solutions
$M_1^t = \{(x, u_1(x, t)) : x \in \mathbb{R}^n\}$ and $M_2^t = \{(x, u_2(x, t)) : x \in \mathbb{R}^n\}$ of MCF (1.1) which both satisfy
condition (1.8) and have the same initial data $u_0$, that is $\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0$.
Then, $u_1 = u_2$ on $\mathbb{R}^n \times (0, T]$, that is $M_1^t = M_2^t$ for all $t \in (0, T]$.

**Remark 1.1.**

i) Theorem 1.3 implies that uniqueness holds under convexity with no other growth
conditions on the initial data (see in the last section [5]). As a consequence Hamilton’s differential
Harnack inequality holds for convex graphs evolving under Mean Curvature Flow (see Corollary
5.3 in Section 5). A related result was recently discussed in [1] in the context of translating
solutions.

ii) Theorem 1.3 shows that uniqueness holds for initial data $u_0(x)$ which has arbitrarily large
growth as $|x| \to +\infty$, as long as the lower curvature bound (1.8) holds.
iii) Theorem 1.3 only assumes the lower bound (1.8) in comparison with the results in [5] which assume upper and lower bounds on the second fundamental form.

At last we will discuss the uniqueness for graphs over subdomains of $\mathbb{R}^n$. In that context, the result in [13] guarantees the existence of smooth solutions: Let $\Omega_0 \subset \mathbb{R}^{n+1}$ be a bounded open set and $u_0: \Omega_0 \to \mathbb{R}$ a locally Lipschitz continuous function with $u_0(x) \to \infty$ for $x \to x_0 \in \partial \Omega_0$. Then there exists $(\mathcal{D}, u)$, where $\mathcal{D} \subset \mathbb{R}^{n+1} \times [0, \infty)$ is relatively open, such that $u$ is a solutions of the graphical mean curvature flow

\[
\begin{aligned}
& (1.9) \\
& u_t = \left( \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) D_{ij} u, \quad (x, t) \in \mathcal{D} \setminus (\Omega_0 \times \{0\}) \\
& u(x, 0) = u_0(x), \quad x \in \Omega_0
\end{aligned}
\]

The function $u$ is smooth for $t > 0$ and continuous up to $t = 0$, $u(\cdot, 0) = u_0$ in $\Omega_0$ and $u(x, t) \to \infty$ as $(x, t) \to \partial \mathcal{D}$, where $\partial \mathcal{D}$ is the relative boundary of $\mathcal{D}$ in $\mathbb{R}^{n+1} \times [0, \infty)$.

It is relevant to remark that in this theorem the domain of definition for the function $u$ changes in time and it is given by the mean curvature flow evolution of $\partial \Omega_0$ (see the discussion in [13]). More precisely, $u(x, t)$ is a graph over $\Omega_t$ where $\Omega_t \times \{t\} = \mathcal{D} \cap \left( \mathbb{R}^n \times \{t\} \right)$ and $\partial \Omega_t$ agrees with the evolution by mean curvature flow of $\partial \Omega_0$ at time $t$, provided that this evolution is smooth. In addition, it is possible to see from the proof in [13] that if $(x_k, t_k) \to (\bar{x}, \bar{t}) \in \partial \mathcal{D}$ and $|\bar{x}| \leq R$ for some $R > 0$, then $u(x_k, t_k) \to \infty$.

Our uniqueness result for graphs over subdomains states as follows:

**Theorem 1.4** (General uniqueness result for subdomains). Let $\Omega_0 \subset \mathbb{R}^n$ be an open set such that $\partial \Omega_0$ has a unique smooth evolution by Mean Curvature Flow in $(0, T]$ and let $\Omega_t$ be such that $\partial \Omega_t$ agrees with the evolution of $\partial \Omega_0$ at time $t$. Assume that $u_0: \Omega_0 \to \mathbb{R}$ is a locally Lipschitz function defining a proper graph $M_0 = \{(x, u_0(x)) : x \in \Omega_0\} \subset \mathbb{R}^{n+1}$.

Let $u_1, u_2: \Omega_t \times (0, T] \to \mathbb{R}$ be two smooth solutions of (1.9) defining two proper entire graph solutions $M_1^t = \{(x, u_1(x, t)) : x \in \Omega_t\}$ and $M_2^t = \{(x, u_2(x, t)) : x \in \Omega_t\}$ of MCF (1.1), both satisfying condition (1.8), and having the same initial data $u_0$, that is $\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0$. Assume in addition that if $(x_k, t_k) \to (\bar{x}, \bar{t}) \in \partial \mathcal{D}$ and $|\bar{x}| \leq R$ for some $R > 0$ then $u_i(x_k, t_k) \to \infty$. Then, $u_1 = u_2$ on $\mathcal{D} = \bigcup_{t \in [0, T]} \Omega_t \times \{t\}$, that is $M_1^t = M_2^t$ for all $t \in (0, T]$.

The organization of this paper is as follows: In Sections 2 and 3 we give the proofs of Theorems 1.1 and 1.2 respectively. Section 4 is devoted to the proofs of Theorems 1.3 and 1.4. Finally, Section 5 is devoted to the proof of Hamilton’s differential Harnack inequality.

We conclude this section with the following remarks.

**Remark 1.2.** In Theorem 1.4, if the evolution of $\partial \Omega_0$ is not unique, it follows from the proof of the result that for each evolution $\Omega_t$ there is at most one proper graphical solution satisfying assumption (1.8).
Remark 1.3. Uniqueness for other non-compact flows has been discussed in other works. For instance, uniqueness results for complete Ricci Flow are discussed in [4] and [15]. The uniqueness for complete Yamabe flow in hyperbolic space is discussed in [14].

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2. Curve shortening flow - Theorem 1.1

In this section we will show that entire graph smooth solutions to Curve Shortening Flow (that is \( n = 1 \) and \( \Omega = \mathbb{R} \)) are unique without any growth assumptions at spatial infinity. This result is in contrast with the case of the heat equation where at most quadratic exponential growth at infinity is required for uniqueness. As mentioned in the introduction Theorem 1.1 is already covered by the results in [3]. We provide here a simpler and more direct proof in the case of entire graphs.

The evolution of a curve \( y = u(x,t) \) on the plane is given by \( u_t = \frac{u_{xx}}{1 + u_x^2} \) which can be also written in divergence form as

\[
(2.1) \quad u_t = (\arctan(u_x))_x.
\]

Differentiating in \( x \) we see that \( v := u_x \) satisfies the equation

\[
(2.2) \quad v_t = (\arctan v)_x x.
\]

The proof of Theorem 1.1 will be based on the following simple observation which we prove next.

Lemma 2.1. For any \( \gamma \in (0,1] \), the following holds

\[
(\arctan v_1 - \arctan v_2)_+ \leq 2 (v_1 - v_2)_+, \quad \forall v_1, v_2 \in [0, +\infty).
\]

Proof of Lemma 2.1. Fix a number \( \gamma \in (0,1] \). We may assume that \( v_1 > v_2 \) and write

\[
(\arctan v_1 - \arctan v_2)_+ = \int_{v_2}^{v_1} \frac{1}{1 + s^2} ds.
\]

Assume first that \( v_1 > v_2 \geq 1 \). In this case, for any number \( \gamma \in (0,1] \) we have \( v_1 \geq (v_1 - v_2)^{1-\gamma} \), so that the above gives

\[
(\arctan v_1 - \arctan v_2)_+ \leq \int_{v_2}^{v_1} \frac{1}{s^2} ds = \frac{v_1 - v_2}{v_1 v_2} \leq \frac{v_1 - v_2}{v_1} \leq \frac{v_1 - v_2}{(v_1 - v_2)^{1-\gamma}} \leq (v_1 - v_2)^{\gamma}_+.
\]

In the case that \( 0 < v_2 < 1 < v_1 \) we have

\[
(\arctan v_1 - \arctan v_2)_+ \leq \int_{v_2}^{1} ds + \int_{1}^{v_1} \frac{1}{s^2} ds \leq (1 - v_2) + \frac{v_1 - 1}{v_1} \leq 2 (v_1 - v_2)^{\gamma}_+.
\]

since for any \( \gamma \in (0,1] \) we have \( 1 - v_2 < (1 - v_2)^{\gamma} < (v_1 - v_2)^{\gamma} \) and \( \frac{v_1 - 1}{v_1} < \frac{v_1 - v_2}{v_1 - v_2} < (v_1 - v_2)^{\gamma} \). The last inequality follows from \( v_1 > (v_1 - v_2)^{1-\gamma} \) which holds in this case. Finally, for \( 0 < v_2 < v_1 \leq 1 \), we have

\[
(\arctan v_1 - \arctan v_2)_+ \leq (v_1 - v_2)_+ \leq (v_1 - v_2)^{\gamma}_+.
\]
Proof of Theorem 1.1. The proof follows the method by Herrero and Pierre in [8]. Let \( v_1 = u_{1x} \) and \( v_2 = u_{2x} \). We will first show that \( v_1 \equiv v_2 \) on \( \mathbb{R} \times [0, T) \). To this end, we set \( w = (v_1 - v_2)_+ \). Since \( v_1, v_2 \) satisfy equation (3.2), Kato’s inequality implies that \( w \) satisfies the differential inequality

\[
(2.3) \quad w_t \leq (aw)_{xx}, \quad \text{on } \mathbb{R} \times (0, T)
\]

in the sense of distributions, where

\[
a := \frac{\arctan v_1 - \arctan v_2_+}{(v_1 - v_2)_+}.
\]

Our observation in Lemma 2.1 shows that for any \( \gamma \in (0, 1) \) we have

\[
0 \leq a \leq 2 w^{-1+\gamma}.
\]

We will use that momentarily.

Consider the test function \( \varphi(x) = \psi\left(\frac{x}{R}\right) \) where \( \psi(\rho) \) is a smooth cut-off function supported in \((-2, 2)\) such that \( 0 \leq \varphi \leq 1, \varphi(\rho) = 1 \) for \( x \in [-1, 1] \). Integrating the differential inequality (2.3) against \( \varphi \), we obtain

\[
(2.4) \quad \frac{d}{dt} \int w(\cdot, t) \varphi \, dx \leq \int (aw)(\cdot, t) \varphi'' \, dx, \quad t \in (0, T).
\]

For any number \( \gamma \in (0, 1) \) (to be fixed at the end of our proof) we use inequality \( 0 \leq a \leq 2 w^{-1+\gamma} \) to conclude

\[
\frac{d}{dt} \int w \varphi \, dx \leq 2 \int w^\gamma |\varphi''| \, dx \leq C \left( \int w^\gamma \, dx \right)^\gamma \left( \int |\varphi''|^{\frac{1}{1-\gamma}} \varphi^{-\frac{\gamma}{1-\gamma}} \, dx \right)^{1-\gamma}.
\]

Since \( |\varphi''(x)| \leq CR^{-2} |\varphi''(\rho)|, x = R \rho \), and \( \psi \) is supported in the interval \([-2, 2]\) we have

\[
\int |\varphi''|^{\frac{1}{1-\gamma}} \varphi^{-\frac{\gamma}{1-\gamma}} \, dx \leq CR^1 \int |\varphi''|^{\frac{1}{1-\gamma}} \varphi^{-\frac{\gamma}{1-\gamma}} \, d\rho.
\]

For any \( \gamma \in (0, 1) \) we can choose cutoff \( \psi = \psi_\gamma \) such that \( \int |\varphi''|^{\frac{1}{1-\gamma}} \varphi^{-\frac{\gamma}{1-\gamma}} \, d\rho \leq C_\gamma \). We conclude that \( I(t) := \int w(\cdot, t) \varphi \, dx \) satisfies

\[
I'(t) \leq C_\gamma I(t)^\gamma R^{-(1+\gamma)}.
\]

Integrating the last inequality on \([0, \bar{t}]\) for any \( \bar{t} \in (0, T) \) while using that \( \lim_{t \to 0} I(t) = 0 \) (this follows from the fact that that \( v_1(\cdot, 0) = v_2(\cdot, 0) \) a.e.) we obtain

\[
I(\bar{t})^{1-\gamma} \leq C_\gamma \bar{t} R^{-(1+\gamma)} \quad \Rightarrow \quad I(\bar{t}) \leq C_\gamma \bar{t}^{\frac{1}{1-\gamma}} R^{-\frac{1+\gamma}{1-\gamma}}.
\]

Finally recalling that \( \varphi \equiv 1 \) on \([-R, R] \), we get

\[
\int_{-R}^{R} (v_1 - v_2)_+(x, t) \, dx \leq C_\gamma \bar{t}^{\frac{1}{1-\gamma}} R^{-\frac{1+\gamma}{1-\gamma}}.
\]

Letting \( R \to +\infty \) and using monotone convergence we conclude that \( \int_0^\infty (v_1 - v_2)_+(x, t) \, dx = 0 \), for all \( t \in [0, T) \). Therefore, conclude that \( (v_1 - v_2)_+ \equiv 0 \) on \([0, \infty) \times [0, t_0]\), i.e. \( (u_1)_x(\cdot, t) \leq (u_2)_x(\cdot, t) \).
in \( \mathbb{R} \). Similarly \( (u_2)_x(\cdot, t) \leq (u_1)_x(\cdot, t) \) in \( \mathbb{R} \) implying that for any \( t \in [0, T) \), we have \((u_1)_x(\cdot, t) = (u_2)_x(\cdot, t) \) in \( \mathbb{R} \). This and the fact that \( u_1 = u_2 \) at time \( t = 0 \) easily give us that \( u_1 \equiv u_2 \), finishing our proof.

\[ \square \]

3. Rotationally symmetric solutions - Theorem 1.2

In this section we will consider the uniqueness of rotationally symmetric solutions of the initial value problem (1.3) on \( \mathbb{R}^n \times (0, T) \). On a radial solution \( u(r, t) \) the evolution equation in (1.3) becomes

\[ u_t = \frac{u_{rr}}{1 + u_r^2} + \frac{n-1}{r} u_r. \]

Differentiating (3.1) with respect to \( r \) we find that the derivative \( v := u_r \) of any solution \( u \) of (1.5) satisfies the equation

\[ v_t = (\arctan v)_r + \left( \frac{n-1}{r} v \right)_r. \]

**Proof of Theorem 1.3.** The proof follows the method by Herrero and Pierre in [8] and is a generalization of the one-dimensional case with the necessary adaptations. We simply denote by \( u_1(r, t), u_2(r, t) \) the rotational symmetric profiles we let \( v_1 = u_1 r \) and \( v_2 = u_2 r \). Set \( w = (v_1 - v_2)_+ \). Since, \( v_1, v_2 \) both satisfy (3.2), Kato’s inequality implies that \( w := (v_1 - v_2)_+ \) satisfies

\[ w_t \leq \Delta (aw) - \frac{n-1}{r} (aw)_r + \left( \frac{n-1}{r} w \right)_r \]

in the sense of distributions, where

\[ a := \frac{(\arctan v_1 - \arctan v_2)_+}{(v_1 - v_2)_+}. \]

Similarly with the one-dimensional case, the crucial observation is that for any \( \gamma \in (0, 1) \) we have \( 0 \leq a \leq 2 w^{-1+\gamma} \).

Consider the test function

\[ \varphi_R(r, t) = \psi \left( \frac{r^2 + 2(n-1)t}{R^2} \right) \]

where \( \psi(\rho) \) is a smooth cut-off function defined on \( [0, +\infty) \) such that \( 0 \leq \psi \leq 1, \psi(\rho) = 1 \) for \( 0 \leq \rho \leq 1 \) and \( \psi(\rho) \equiv 0 \) for \( \rho \geq 2 \). Then,

\[ (\varphi_R)_t = \frac{2(n-1)}{R^2} \psi', \quad (\varphi_R)_r = \frac{2r}{R^2} \psi' \quad \Rightarrow \quad (\varphi_R)_t = \frac{n-1}{r} (\varphi_R)_r \]

and

\[ (\varphi_R)_{rr} = \frac{4r^2}{R^4} \psi'' + \frac{2}{R^2} \psi' \quad \Rightarrow \quad \Delta \varphi_R = \frac{4r^2}{R^4} \psi'' + \frac{2n}{R^2} \psi'. \]

Hence, using \( (\varphi_R)_r \), we obtain

\[ \frac{d}{dt} \int w \varphi_R r^{n-1} dr = \int w_t \varphi_R r^{n-1} dr + \int w (\varphi_R)_r r^{n-1} dr \]

\[ \leq \int a \varphi_{Rr} r^{n-1} dr - \int \frac{n-1}{r} (aw)_r \varphi_R r^{n-1} dr + \int (\frac{n-1}{r} w)_r \varphi_R r^{n-1} dr + \int \frac{n-1}{r} w (\varphi_R)_r r^{n-1} dr. \]
Performing integration by parts on the second and third terms, using that
\[
\int \frac{n-1}{r} (aw)_r \varphi_R r^{n-1} dr = - \int \frac{n-1}{r} aw (\varphi_R)_r r^{n-1} dr - \int \frac{(n-2)(n-1)}{r^2} aw \varphi_R r^{n-1} dr
\]
we obtain (after cancellations) that
\[
\frac{d}{dt} \int w \varphi_R d\mu \leq \int aw \Delta \varphi_R r^{n-1} dr + \int \frac{n-1}{r} aw (\varphi_R)_r r^{n-1} dr \\
+ \int \frac{(n-2)(n-1)}{r^2} aw \varphi_R r^{n-1} dr - \int \frac{(n-1)^2}{r^2} w \varphi_R r^{n-1} dr.
\]
(3.4)

Next notice that
\[
a := \frac{(\arctan v_1 - \arctan v_2)_+}{(v_1 - v_2)_+} = \frac{1}{1 + \bar{v}^2}
\]
for some \( \bar{v} \) between \( v_1 \) and \( v_2 \), hence \( a \leq 1 \). It follows that
\[
\int \frac{(n-2)(n-1)}{r^2} aw \varphi_R r^{n-1} dr - \int \frac{(n-1)^2}{r^2} w \varphi_R r^{n-1} dr \leq - \int \frac{n-1}{r^2} w \varphi_R r^{n-1} dr \leq 0.
\]

Let \( \gamma \in (0,1) \) be any number (to be chosen at the end of our proof) and use the inequality
\( 0 \leq a \leq 2 w^{-1+\gamma} \) shown in Lemma 2.1 to bound the first two terms on the right hand side of (3.4).
We conclude that
\[
\frac{d}{dt} \int w \varphi_R r^{n-1} dr \leq C \int w \gamma (|\Delta \varphi_R| + |(n-1) r^{-1} (\varphi_R)_r|) r^{n-1} dr \\
\leq C \left( \int w \varphi_R r^{n-1} dr \right)^{\gamma} \left( \int (|\Delta \varphi_R| + |r^{-1} (\varphi_R)_r|) \frac{1}{\gamma} \varphi_R \frac{r^{-1}}{\gamma} r^{n-1} dr \right)^{1-\gamma}.
\]

Observing that for \( 0 \leq t \leq t_0 \) and \( R \gg 1 \) large we have
\[
|\Delta \varphi_R(r,t)| + |r^{-1} (\varphi_R)_r(r,t)| \leq C_n R^{-2} (|\psi''(\rho)| + |\psi'(\rho)|)
\]
where \( \rho := \frac{r^2 + 2(n-1)t}{R^2} \) we get
\[
\left\{ \int \left( |\Delta \varphi_R(r,t)| + |r^{-1} (\varphi_R)_r(r,t)| \right)^{\frac{1}{\gamma}} \varphi_R(r)^{-\frac{1}{\gamma}} r^{n-1} dr \right\}^{1-\gamma} \\
\leq R^{-2} \left\{ \int (|\psi''(\rho)| + |\psi'(\rho)|)^{\frac{1}{\gamma}} \psi(\rho)^{-\frac{1}{\gamma}} r^{n-1} (\rho) d\rho \right\}^{1-\gamma}
\]
where \( r^2(\rho) = R^2 \rho - 2(n-1)t \), which in particular implies \( r \, dr = \frac{R^2}{2} \, d\rho \). Thus,
\[
\int (|\psi''(\rho)| + |\psi'(\rho)|)^{\frac{1}{\gamma}} \psi(\rho)^{-\frac{1}{\gamma}} r^{n-1} (\rho) d\rho \\
= \frac{R^2}{2} \int (|\psi''(\rho)| + |\psi'(\rho)|)^{\frac{1}{\gamma}} \psi(\rho)^{-\frac{1}{\gamma}} \left( R^2 \rho - 2(n-1)t \right)^{-\frac{n-2}{2}} d\rho \\
\leq C_n R^n \int (|\psi''(\rho)| + |\psi'(\rho)|)^{\frac{1}{\gamma}} \psi(\rho)^{-\frac{1}{\gamma}} d\rho
\]
where we have used that on the support of \( \psi' \), \( \psi'' \) where \( \rho \leq 2 \), and for \( 0 \leq t \leq t_0 \) and \( R \gg \max(1,t_0) \), one has \( (R^2 \rho - 2(n-1)t)^{\frac{n-2}{2}} \leq C_n R^{n-2} \).
For any $\gamma \in (0, 1)$ we can choose cutoff $\psi = \psi_\gamma$ for which the support of $\psi', \psi''$ lies in $[1, 2]$ such that
\[
\int_1^2 \left( |\psi''(\rho)| + |\psi'(\rho)| \right) \frac{1}{\rho} \psi(\rho) \frac{1}{\rho^\gamma} d\rho \leq C(n, \gamma).
\]
We then conclude from the above discussion that $I(t) := \int w\varphi_R \tau^{n-1} dr$ satisfies
\[
I'(t) \leq C(n, \gamma) I(t)^\gamma R^{-2+n(1-\gamma)}.
\]
Since $\gamma \in (0, 1)$ can be any number, we may choose $\gamma = \gamma(n) \in (0, 1]$ so that $n(1-\gamma) < 2$, and integrating the last inequality on $[0, \bar{t})$ for any $\bar{t} \in (0, T)$ while using that $I(0) = 0$, we obtain
\[
I(\bar{t})^{1-\gamma} \leq C_n \bar{t} R^{-2+n(1-\gamma)} \implies I(\bar{t}) \leq C_n \bar{t}^{1/(1-\gamma)} R^{n-2/(1-\gamma)}.
\]
Finally recalling that $\varphi_R \equiv 1$ on $[0, R]$, we get
\[
\int_0^R (v_1 - v_2)_+ (r, t) \tau^{n-1} dr \leq R^{n-2/(1-\gamma)}.
\]
Letting $R \to +\infty$, using that $n - \frac{2}{1-\gamma} < 0$, and monotone convergence yields $\int_0^\infty (v_1 - v_2)_+ (\cdot, t) \tau^{n-1} dr = 0$, for all $t \in [0, T)$. Therefore, we conclude that $(v_1 - v_2)_+ \equiv 0$ on $\mathbb{R}^n \times [0, T)$, i.e. $(u_1)_r \leq (u_2)_r$. Similarly $(u_2)_r \leq (u_1)_r$, a.e. in $\mathbb{R}^n \times [0, T)$ implying that $(u_2)_r \equiv (u_1)_r$. This and the fact that $u_1 \equiv u_2$ at time $t = 0$ easily give us that $u_1 \equiv u_2$ on $\mathbb{R}^n \times [0, t_0]$, for all $t_0 < T$ finishing our proof. \hfill \Box

4. The general case

Our goal in this section is to give the proof of our general uniqueness results, Theorem 1.3 and Theorem 1.4. We will see that the proof of the latter theorem is almost identical to the proof of the former. Hence, we will omit most of the proof of Theorem 1.4 pointing out only the minor differences.

For the sake of completeness we show next that for entire graphs the condition $u_0 \geq C$ is preserved under the flow, which implies that if the initial condition is a proper entire graph, then the solution is proper as well, uniformly in time. Both facts will be used our proofs. Because we are dealing with non-compact solutions, we will use the localization techniques developed in [10].

Lemma 4.1. Let $u$ be a solution to \eqref{1.3} on $\mathbb{R}^n \times (0, T)$ and assume that $u_0(x) \geq C$ on $|x - x_0| \leq R$, $x = (x, u_0(x))$, for some fixed point $x_0 \in \mathbb{R}^{n+1}$ and some number $R > 1$. Then, we have
\[
u(x, t) \geq C - \frac{10}{R} t
\]
on the parabolic ball $|x - x_0|^2 + 2nt \leq \frac{R^2}{2}$, $x = (x, u(x, t))$ (provided it is non-empty).

In particular, if $u_0 \geq C$ on $\mathbb{R}^n$, then for every $t \in (0, T)$ we have $u(\cdot, t) \geq C$ on $\mathbb{R}^n$.

Proof. We will do all calculations in geometric coordinates, that is we assume that our solutions are given by the embedding $x = F(p, t)$ as in \eqref{1.1} and we define
\[
U_R(p, t) := (u - C) \left( 1 - \frac{|x - x_0|^2 + 2nt}{R^2} \right)_+ + \frac{5}{R} t
\]
where \( u := (F, e_{n+1}) \) and \( x = F(p, t) \). Our assumption \( u_0 \geq C \) in \( B_R(x_0) \), gives \( U_R \geq 0 \) at \( t = 0 \). Furthermore,

\[
(U_R)_t - \Delta U_R = -2\nabla u \cdot \frac{(x - x_0)^T}{R^2} + \frac{5}{R} \geq -\frac{4}{R} + \frac{5}{R} > 0.
\]

The maximum principle implies that \( U_R \) does not have any interior minima and \( U_R \geq 0 \). In particular, if \(|x - x_0|^2 + 2nt \leq \frac{R^2}{2}\) then

\[
0 \leq \frac{u - C}{2} + \frac{5}{R} t,
\]

and the first result follows.

In the case where \( u_0 \geq C \) globally on \( \mathbb{R}^n \), then for any \( x_0 \in \mathbb{R}^n \), \( t \in (0, T) \), we apply the above result taking \( x_0 = (x_0, u_0(x)) \) and choosing \( R \gg 1 \) so that \(|x - x_0|^2 + 2nt \leq \frac{R^2}{2}\) if \( x = (x_0, u(x, t)) \).

We readily conclude that \( u(x_0, t) \geq C - \frac{10}{R^2}t \) and by taking \( R \to \infty \) we obtain that \( u(x_0, t) \geq C \). Since \( x_0 \in \mathbb{R}^n \) and \( t \in (0, T) \) are arbitrary, the second result follows.

**Corollary 4.2.** Let \( u \) be a solution to \( (1.3) \) on \( \mathbb{R}^n \times (0, T) \) and assume that \( \lim_{|x| \to +\infty} u_0(x) = +\infty \). Then, we have

\[
\lim_{|x| \to +\infty} u(x, t) = +\infty, \quad \text{uniformly in } t \in (0, T).
\]

**Proof.** We begin by observing that our assumption that \( \lim_{|x| \to +\infty} u_0(x) = +\infty \) implies that \( u_0 \geq C \) for some \( C \in \mathbb{R} \) and hence by the previous lemma, \( u \geq C \) as well.

Now, for every \( k \gg 1 \) let \( R_k > k \) be a sufficiently large number so that \( u_0(x) \geq k \) for \(|x| \geq R_k \). For any \( x_0 \in \mathbb{R}^n \) such that \(|x_0| > 4R_k \), let \( x_0 = (x_0, 0) \). Then,

\[
u_0(x) \geq k, \quad \text{on } |x - x_0| \leq 2R_k, \quad x = (x, u_0(x))
\]

and hence, by the previous lemma, for any \( t \in (0, T) \), we have

\[
|u(x, t)| \geq k - \frac{5}{R_k}t, \quad \text{on } |x - x_0|^2 + 2nt \leq 4R_k^2, \quad x = (x, u(x, t)).
\]

We may choose \( k, R_k \gg 1 \) so that \( 2nt < R_k^2 \) and \( \frac{5}{R_k}T < 1 \). Evaluating the above estimate at \( x = (x_0, u_0(x_0)) \), for any \( t \in (0, T) \), it gives us that

\[
u(x_0, t) \geq k - 1, \quad \text{provided } |x - x_0| = |u(x_0, t)| \leq R_k.
\]

We conclude that for any \(|x_0| \geq 4R_k \) and \( t \in (0, T) \) we either have \( u(x_0, t) \geq k - 1 \) or \(|u(x_0, t)| \geq R_k \). Since, \( u \geq C \) (be our initial observation) and \( R_k \geq k \), we conclude that in either case \( u(x_0, t) \geq k - 1 \), for all \( t \in (0, T) \) and all \(|x_0| \geq 4R_k \). Since, \( R_k \) is independent of \( t \), the result readily follows. \( \square \)

One may ask whether condition \( (1.3) \) is preserved in time, namely if \( vh_t^i \geq -c \) at time \( t = 0 \) implies that \( vh_t^i \geq c \) for \( t > 0 \). Although this is easy to verify for the evolution of compact manifolds, in the non-compact setting it becomes challenging. Actually, even the case where \( c = 0 \) is not known to hold in the general graphical non-compact setting. In the lemma below we show that the condition is preserved under a suitable polynomial growth condition on the solution (which is expected to be preserved by the flow from the results in \([9]\)).
Lemma 4.3. Assume that \( vh^j_t \geq -c \) at time \( t = 0 \), for some constant \( c > 0 \) and that for all times we have \( |h^j_t v| \leq C|x|^q \) and that \( |\nabla v| \leq C|x| \). Then, Condition (1.8) holds for every \( t \geq 0 \).

Proof. Let \( f^j = h^j v + c \). Then, following [9] we have in geometric coordinates that

\[
\left( \frac{d}{dt} - \Delta_M \right) f^j = -\frac{2}{v} (\nabla \left( f^j \right), \nabla v).
\]

Let \( \gamma = |x|^2 + 2nt + 1 \) and \( p > q \) (for instance \( p = q + 1 \)) and define \( F = e^{-Kt} \gamma^{-p} f^j v \). From our assumption \( F \to 0 \) as \(|x| \to \infty \). Assume that there is an interior minimum that is negative. Then

\[
0 \geq \left( \frac{d}{dt} - \Delta_M \right) F = -\frac{2e^{-Kt} \gamma^{-p}}{v} (\nabla \left( f^j \right), \nabla v) - p(p + 1)e^{-Kt} \gamma^{-p-2} f^j v |x|^2 \tag{1.8}
\]

Observe that \(|x|^2 \leq \gamma\). Then at the interior critical point we have \( \gamma^{-p} \nabla \left( f^j \right) = p \gamma^{-p} f^j v x_T \) and

\[
0 \geq \left( \frac{d}{dt} - \Delta_M \right) F \geq F(C - p(p + 1)\gamma^{-1} + 2p^2 \gamma^{-1} - K).
\]

Since \( \gamma \geq 1 \), by choosing \( K \) large enough (depending on \( C \) and \( p \)) we have that the right hand side is positive when \( F < 0 \) which is a contradiction. \( \square \)

Remark 4.1. Note that \( |\nabla v| \leq |A|v \). Then, the results in [9] imply that if \(|A|v \leq |x| \) holds at \( t = 0 \), then this is preserved in time and the condition of our lemma is met with \( q = 1 \).

4.1. Proof of Theorem (1.3)

Proof. To simplify the notation in this proof we denote \( u = u_1 \) and \( \bar{u} = u_2 \), that is we assume that \( u, \bar{u} : \mathbb{R}^n \times (0, T] \to \mathbb{R} \) are the two smooth solutions to (1.3) with initial data \( u_0 \) as in the statement of Theorem (1.3). Since \( u_0 \) is proper we have \( u_0 \geq -C \) for some constant \( C > 0 \). Hence, by adding \( u_0 \) the constant \( C + 1 \) we may assume without loss of generality that \( u_0 \geq 1 \). Lemma 4.1 implies that

\[
u, \bar{u} \geq 1, \quad \text{on } \mathbb{R}^n \times (0, T].
\]

To show that \( \bar{u} = u \), it is sufficient to prove that \( u \leq \bar{u} \), since the same argument will also imply that \( u \leq \bar{u} \), thus showing that \( u = \bar{u} \).

The solutions \( u, \bar{u} \) satisfy equations

\[
u_t = \left( \delta_{ij} - \frac{D_i u D_j \bar{u}}{1 + |D \bar{u}|^2} \right) D_{ij} u, \quad \bar{u}_t = \left( \delta_{ij} - \frac{D_i \bar{u} D_j \bar{u}}{1 + |D \bar{u}|^2} \right) D_{ij} \bar{u}.
\]

Set \( a_{ij} = \delta_{ij} - \frac{D_i u D_j \bar{u}}{1 + |D \bar{u}|^2}, \quad \bar{a}_{ij} = \delta_{ij} - \frac{D_i \bar{u} D_j \bar{u}}{1 + |D \bar{u}|^2} \) and define

\[
w := u - \bar{u}.
\]

Then, subtracting the above equations, we find that the function \( w \) satisfies the equation

\[
w_t - a_{ij} D_{ij} w = (a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u} \tag{4.1}
\]
The main idea in the proof is to introduce the supersolution
\[ \zeta(x, t) := \epsilon(t + \epsilon) u^2(x, t) \]
for any given \( \epsilon > 0 \) small. At the end we will let \( \epsilon \to 0 \). First, we use \( u_t - a_{ij} D_{ij} u = 0 \) and find that \( \zeta \) satisfies
\[ \zeta_t - a_{ij} D_{ij} \zeta = -2\epsilon(t + \epsilon) a_{ij} D_j u D_i u + \epsilon u^2, \]
where
\[ a_{ij} D_j u D_i u = \left( \delta^{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) D_i u D_j u = \delta^{ij} D_i u D_j u - \frac{(D_i u)^2 (D_j u)^2}{1 + |Du|^2} = |Du|^2 (1 - \frac{|Du|^2}{1 + |Du|^2}) = \frac{|Du|^2}{1 + |Du|^2}. \]
Combining the above gives
\[ \zeta_t - a_{ij} D_{ij} \zeta = -2\epsilon(t + \epsilon) \frac{|Du|^2}{1 + |Du|^2} + \epsilon u^2 \geq \epsilon(u^2 - 2(t + \epsilon)). \]
Since \( u \geq 1 \), we conclude that for \( t \leq 1/4 \) and \( \epsilon < 1/10 \), we have
\[ \zeta_t - a_{ij} D_{ij} \zeta > \frac{\epsilon}{2} u^2. \]
Set next
\[ W := w - \zeta = u - \bar{u} - \epsilon(t + \epsilon) u^2. \]
By (4.2) we find that \( W \) satisfies
\[ W_t - a_{ij} D_{ij} W < (a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u} - \frac{\epsilon}{2} u^2. \]
Furthermore, our assumption that \( u = \bar{u} \) at \( t = 0 \) (in the sense that \( \lim_{t \to 0} [u(\cdot, t) - \bar{u}(\cdot, t)] = 0 \)) we have
\[ \lim_{t \to 0} W(x, t) = -\epsilon^2 u(x, 0) \leq -\epsilon^2 < 0, \quad \text{uniformly on any } K \subset \mathbb{R}^n \text{ compact}. \]
The uniform convergence on compact sets follows from the bounds in [10] which give us local bounds on the second fundamental form \( |A| \leq C/\sqrt{T} \) for both solutions \( u, \bar{u} \) where \( C \) depends on the initial data.

Let
\[ T^* = \min \left( T, \frac{1}{4}, \frac{1}{10\epsilon} \right) \]
where \( c \) is the constant in (1.8). We will use (4.3) - (4.4) and the maximum principle to conclude that \( W \leq 0 \) for all \( t \in [0, T^*] \). To this end, observe first that \( u, \bar{u} \geq 1 \) implies that for every fixed \( \epsilon > 0 \) and for all \( t \in (0, T) \),
\[ m^* := \sup_{(x, t) \in \mathbb{R}^n \times (0, T]} W(x, t) \leq \frac{1}{\epsilon^2}. \]
Indeed, notice that if there is a point \( (x, t) \in \mathbb{R}^n \times (0, T^*) \) where \( W(x, t) \geq 0 \), then since \( \bar{u} \geq 1 \), at such point we have \( u \geq \bar{u} + \epsilon(t + \epsilon) u^2 \geq \epsilon^2 u^2 \), that is \( u(x, t) \leq \epsilon^{-2} \). Hence, \( W(x, t) \leq u(x, t) \leq \epsilon^{-2} \) and the same holds for the supremum \( m^* \).
Claim 4.1. We have

\[ m^* := \sup_{(x,t) \in \mathbb{R}^n \times (0,T^*)} W(x,t) \leq 0 \]

provided that \( \epsilon \) is sufficiently small.

Once this claim is shown, the theorem will follow by simply letting \( \epsilon \to 0 \) to show that \( u \leq \bar{u} \) and then switching the roles of \( u \) and \( \bar{u} \).

Proof of Claim 4.1. To prove the claim, we assume by contradiction, that

\[ m^* > 0. \]

Since \( \lim_{|x| \to +\infty} u(x,t) = +\infty \) uniformly in \([0,T]\) and \( \bar{u} \geq 1 \), the supremum \( m^* \) cannot be attained at infinity. Hence, we have

\[ m^* = W(x_{\text{max}}(t_0), t_0) \]

for some point \( t_0 \in (0, T^*) \) and \( x_{\text{max}}(t_0) \in \mathbb{R}^n \). Then at such point

\[ (4.6) \quad (1 - \epsilon(t_0 + \epsilon)u) u = \bar{u} + m^* \quad \text{and} \quad (1 - 2\epsilon(t_0 + \epsilon)u) D_t u = D_t \bar{u} \]

Note that the first equality, \( m^* > 0 \) and \( \bar{u} \geq 1 \) imply that \( 1 - \epsilon(t_0 + \epsilon)u > 0 \) at the maximum point, which will be used below. We will now use the second equality in (4.6) to evaluate the right hand side of (4.3) at the maximum point. First, we have

\[ a_{ij} - \bar{a}_{ij} = \frac{D_i \bar{u} D_j \bar{u}}{1 + |D \bar{u}|^2} - \frac{D_i u D_j u}{1 + |Du|^2} = (1 - 2\epsilon(t_0 + \epsilon)u)^2 \frac{D_i u D_j u}{1 + |Du|^2} - \frac{D_i \bar{u} D_j \bar{u}}{1 + |D \bar{u}|^2} \]

(4.7)

\[ = \frac{D_i u D_j u}{1 + |Du|^2} \left[ (1 - 2\epsilon(t_0 + \epsilon)u)^2 (1 + |Du|^2) - (1 + |D \bar{u}|^2) \right] \]

\[ = -4\epsilon(t_0 + \epsilon)u(1 - \epsilon(t_0 + \epsilon)u) \frac{D_i u D_j u}{1 + |Du|^2} \]

To derive the last equality we used \( (1 - 2\epsilon(t_0 + \epsilon)u)^2 |Du|^2 = |D \bar{u}|^2 \) which gave us

\[ (1 - 2\epsilon(t_0 + \epsilon)u)^2 (1 + |Du|^2) - (1 + |D \bar{u}|^2) = (1 - 2\epsilon(t_0 + \epsilon)u)^2 - 1 = -4\epsilon(t_0 + \epsilon)u(1 - \epsilon(t_0 + \epsilon)u). \]

Combining the above with (4.3) we find that at the point \( (x_{\text{max}}(t_0), t_0) \) we have

\[ (4.8) \quad 0 \leq W_t - a_{ij} D_i J W < -4\epsilon(t_0 + \epsilon)u(1 - \epsilon(t_0 + \epsilon)u) \frac{D_i \bar{u} D_j \bar{u}}{1 + |Du|^2} \]

\[ \frac{1 + |D \bar{u}|^2}{1 + |Du|^2} - \epsilon u^2. \]

We next use the lower bound on the second fundamental form in (1.8) which implies that

\[ \tilde{v} h_j D_i u D_j u \geq -c |Du|^2. \]
On the other hand, since \( \widetilde{h}_j = \frac{D_{ij} \tilde{u}}{\sqrt{1 + |Du|^2}} D_i u D_j u \), it follows that at the maximum point \((x_{\text{max}}(t_0), t_0)\) we have
\[
\tilde{v} \tilde{h}_j D_i u D_j u = \left( D_{ij} \tilde{u} - \frac{D_{ij} \tilde{u} D_i \tilde{u} D_j \tilde{u}}{1 + |D\tilde{u}|^2} \right) D_i u D_j u
\begin{align*}
&= D_{ij} \tilde{u} D_i u D_j u - \langle D\tilde{u}, Du \rangle \frac{D_{ij} \tilde{u}}{1 + |D\tilde{u}|^2} D_i \tilde{u} D_j u \\
&= (1 + |D\tilde{u}|^2 - (1 - 2\epsilon (t_0 + \epsilon) u)^2|Du|^2) \frac{D_{ij} \tilde{u}}{1 + |D\tilde{u}|^2} D_i u D_j u \\
&= \frac{D_{ij} \tilde{u} D_i u D_j u}{1 + |D\tilde{u}|^2}.
\end{align*}
\] Combining the last two formula gives
\begin{equation}
\frac{D_{ij} \tilde{u} D_i u D_j u}{1 + |D\tilde{u}|^2} = \tilde{h}_j \tilde{v} D_i u D_j u \geq -c|Du|^2.
\end{equation}
Inserting this bound in (4.8) implies that at the point \((x_{\text{max}}(t_0), t_0)\) we have
\begin{equation}
0 \leq W_i - a_{ij} D_{ij} W < 4c \epsilon (t_0 + \epsilon) u (1 - \epsilon(t_0 + \epsilon) u) \frac{|Du|^2}{1 + |Du|^2} - \frac{\epsilon}{2} u^2 \\
\leq 4c \epsilon (t_0 + \epsilon) u (1 - \epsilon(t_0 + \epsilon) u) - \frac{\epsilon}{2} u^2.
\end{equation}
We conclude from (4.10) that at the maximum point \((x_{\text{max}}(t_0), t_0)\) we have
\[
4c \epsilon (t_0 + \epsilon) u (1 - \epsilon(t_0 + \epsilon) u) - \frac{\epsilon}{2} u^2 > 0
\] holds at the maximum point \((x_{\text{max}}(t_0), t_0)\), that is
\[
u < 8c t_0 (1 - \epsilon(t_0 + \epsilon) u) < 8c (t_0 + \epsilon)
\] holds, since \(1 - \epsilon(t_0 + \epsilon) u > 0\). Then \(u \geq 1\) yields that \(t_0 + \epsilon > \frac{1}{8c}\), where \(c\) is the constant from (4.8). Since we have assumed that \(t_0 \in (0, T^*]\) and \(T^* \leq \frac{1}{10c}\) we derive a contradiction by choosing \(\epsilon\) sufficiently small. This shows, that contrary to our assumption, \(W^*(t_0) < 0\), finishing the proof of the claim.

We have just seen that \(W := u - \tilde{u} - \epsilon(t + \epsilon) u^2 \leq 0\) on \(\mathbb{R}^n \times (0, T^*]\). Let \(\epsilon \to 0\) to obtain that \(u \leq \tilde{u}\) on \(\mathbb{R}^n \times (0, T^*]\). Similarly, \(\tilde{u} \leq u\) on the same interval, which means that \(u = \tilde{u}\). By repeating the same proof starting at \(t = T^*\) we conclude after finite many steps that \(u \equiv \tilde{u}\) on \(\mathbb{R}^n \times (0, T)\), finishing the proof of the theorem.

4.2. Proof of Theorem 1.4.

Proof: The proof of Theorem 1.4 is very similar to that of Theorem 1.3. We briefly outline it in what follows. As before, let \(u, \tilde{u} : \mathcal{D} := \bigcup_{t \in (0, T)} (\Omega_t \times \{t\}) \to \mathbb{R}\) be the two smooth solutions to (I.9) with initial data \(u_0\) as in the statement of Theorem 1.3 (as above, we simplify the notation by calling \(u = u_1\) and \(\tilde{u} = u_2\)). Our assumption that \(u_0\) is proper implies that \(u_0 \geq -C\) for some constant \(C > 0\) and hence Lemma 4.1 implies that \(u, \tilde{u} \geq -C\), for \(t > 0\) (possibly for a different
constant $C > 0$ which is uniform in $t$ for $t < \min(1, T)$, where $T$ is the maximal existence time. By adding on both solutions the constant $C + 1$ we may assume that $u, \bar{u} \geq 1$. As in the proof of Theorem 1.3 we take

$$W := w - \zeta - \epsilon = u - \bar{u} - \epsilon (t + \epsilon) u^2.$$ 

Let $m^* := \sup_{(x,t) \in D} W(x,t)$ and assume that $m^* > 0$.

We first remark that Lemma 4.1 and Corollary 1.2 can be directly extended to estimate the infimum of $u$ in $D \cap B_R(x_0)$ (instead of $\mathbb{R}^n \cap B_R(x_0)$). Hence we have that if $u_0$ is proper then $u(x,t) \to \infty$ uniformly in $t$ as $|x| \to \infty$.

Let $(x_k, t_k)$ be a sequence of points in $D$ such that $W(x_k, t_k) \to m^*$. Note that from our definition and the previous remark we have that if $t_k \to \bar{t}$, and either $x_k \to \partial \Omega_t$ or $|x_k| \to +\infty$, then $u(x_k, t_k) \to \infty$ and $W \to -\infty$. Hence, we may assume that that supremum of $W$ is attained in the interior of $\Omega$. Now we conclude the desired result by following the the proof of Theorem 1.3. \hfill \Box

4.3. Extension of uniqueness for entire graphs (not necessarily proper). In this section we provide extensions to our result in Theorem 1.3. We will consider graphical solutions that are not necessarily proper, but their initial height function $u_0$ and its gradient function $v_0$ satisfy the following assumption

$$\tag{4.11} \text{for every } M \text{ there is a constant } c(M) \text{ such that } \sup_{\{x : u_0(x) < M\}} u_0 \leq c(M).$$

This condition can be understood as excluding oscillatory behavior in the set where the height function $u_0$ is bounded at the initial time. Then our result states as follows:

**Theorem 4.4.** Assume that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function (not necessarily proper) defining an entire graph hypersurface $M_0 = \{(x, u_0(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ whose height function $u_0$ is bounded from below and also satisfies condition (4.11).

Let $u_1, u_2 : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ be two smooth solutions of (1.3) defining two entire graph solutions $M_1 = \{(x, u_1(x,t)) : x \in \mathbb{R}^n\}$ and $M_2 = \{(x, u_2(x,t)) : x \in \mathbb{R}^n\}$ of MCF (1.1) satisfying condition (1.8) and having the same initial data $u_0$, that is $\lim_{t \to 0} u_1(\cdot, t) = \lim_{t \to 0} u_2(\cdot, t) = u_0$. Then, $u_1 = u_2$ on $\mathbb{R}^n \times (0,T)$, that is $M_1 = M_2$ for all $t \in (0,T)$.

We will first show that condition (4.11) is preserved in time and that implies uniform local bounds for the second fundamental form on the set where $\{u \leq M\}$ (these bounds depend only on $M$).

**Proposition 4.5.** Assume that $u \geq 0$ is a smooth solution of (1.3) with initial data $u_0$ and that (4.11) holds. Then,

i) $(M - u)_{+}^2 \leq M^2 c(M)$ holds for all $t \in (0,T]$.

ii) If, we further assume that $|A|^2(x,0) \leq c(M)$ in the set $\{x : u_0(x) \leq M\}$ (without loss of generality we can take $c(M)$ to be the same as in (4.11)), then

$$\tag{4.12} |A|^2 (M - u)_{+}^2 \leq \max\{c(M) M^2, k^{-1} (3 + k^{-1}) M\}$$
when \( u(x, t) \leq M \) and \( k = \frac{1}{2M^2 c(M)} \).

**iii)** Without any assumption on the second fundamental form at the initial time, we have instead

\[
|A|^2 (M - u)^2(x, t) \leq 2k^{-1}(3 + k^{-1})M + M^2
\]

if \( u(x, t) \leq M \) and \( k = \frac{1}{2M^2 c(M)} \).

**Proof.**

i) Consider the cut-off function (in terms of both \( u \) and \( x \)) given by

\[
\eta_R(x, t) = \left( (M - u) + \frac{1}{R^2}(1 - |x|^2 + 2nt) - \frac{4}{R} \right)_+
\]

A direct calculation shows that

\[
(\eta_R)_t - \Delta \eta_R = \frac{2}{R^2} \langle \nabla u, \nabla |x|^2 \rangle - \frac{4}{R} \leq 0.
\]

In the last line we used that \( |\nabla |x|^2| = 2|x^T| \leq 2R \) in the set that \( 1 - \frac{|x|^2 + 2nt}{R^2} \geq 0 \) and that \( |\nabla u| \leq 1 \). Recalling also that

\[
v_t - \Delta v = -|A|^2 v - 2|\nabla v|^2 v
\]

and defining \( V_R = v \eta_R^2 \) we have

\[
(V_R)_t - \Delta V_R = \eta_R^2 \left( -|A|^2 v - 2|\nabla v|^2 v \right) - 2v|\nabla \eta_R|^2 v - 4\eta \langle \nabla v, \nabla \eta \rangle
\]

\[
\leq \eta_R^2 \left( -|A|^2 v - 2|\nabla v|^2 v \right) - 2v|\nabla \eta_R|^2 v + 2\eta_R^2 \frac{|\nabla v|^2 v}{v} + 2\eta_R |\nabla \eta_R|^2 v
\]

\[
= -\eta_R^2 |A|^2 v < 0.
\]

A standard application of the maximum principle shows that \( V_R \) does not have any interior maximum and hence

\[
V_R \leq \max V_R(\cdot, 0) \leq M^2 c(M).
\]

The result follows by taking \( R \to \infty \).

ii) We follow the proof in [10] replacing the localization function in that paper by \( \eta_R^2 \) (where \( \eta_R \) is defined by (4.14)). The proof is analogous and we only point out the main steps and differences.

Following [10] we define \( k \) such that \( kv^2 \leq \frac{1}{2} \) in the set that \( \eta_R \neq 0 \) and define the function

\[
g = \frac{v^2 |A|^2}{1 - kv^2}.
\]

Then

\[
g_t - \Delta g \leq -2kg^2 - \frac{2k}{(1 - kv^2)^2} |\nabla v|^2 g - 2\frac{v^{-1}}{1 - kv^2} \langle \nabla v, \nabla g \rangle.
\]

A similar calculation as in [10] where we use (4.14) gives that

\[
(\eta_R^2 g)_t - \Delta (\eta_R^2 g) \leq -2k\eta_R^2 g^2 - \frac{2k}{(1 - kv^2)^2} |\nabla v|^2 \eta_R^2 g - 2\eta_R^2 \frac{v^{-1}}{1 - kv^2} \langle \nabla v, \nabla g \rangle
\]

\[
- 2g |\nabla \eta_R|^2 - 4\eta_R \langle \nabla \eta_R, \nabla g \rangle.
\]
Following again [10] we can find a vector function \( b \) (that can be explicitly computed, but it is not important) such that

\[
(\eta^2 R g)_t - \Delta(\eta^2 R g) \leq -2k\eta^2 R g^2 + (6 + 2k^{-1}v^{-2})g|\nabla \eta R|^2 + \langle \nabla (g\eta^2 R), b \rangle.
\]

Then, observing that \(|\nabla \eta R|^2 \leq M\) we conclude that if \( \eta^2 R g \) has an interior maximum then

\[
0 \leq -2k\eta^2 R g^2 + (6 + 2k^{-1}v^{-2})g\eta R^2 \leq -2k\eta^2 R g^2 + (6 + 2k^{-1}v^{-2})gM
\]

or equivalently,

\[
\eta^2 R g \leq k^{-1}(3 + k^{-1}v^{-2})M.
\]

Taking \( R \to \infty \) (4.12) follows since \( v \geq 1 \).

iii) Finally, consider \( t\eta^2 R g \). Then, we have

\[
(t\eta^2 R g)_t - \Delta(t\eta^2 R g) \leq -2k\eta^2 R g^2 + (6 + 2k^{-1}v^{-2})g|\nabla \eta R|^2 + \langle \nabla (g\eta^2 R), b \rangle + \eta^2 R g.
\]

At a maximum holds

\[
t\eta^2 R g \leq k^{-1}(3 + k^{-1}v^{-2})M + M^2,
\]

and we conclude (4.13) by taking \( R \to \infty \).

We will now prove Theorem 4.4:

**Proof of Theorem 4.4.** As in the proof of Theorem 1.3 we set \( u = u_1, \bar{u} = u_2 \) and assume without loss of generality that \( u_0 \geq 1 \) in which case \( u, \bar{u} \geq 1 \) (this follows from \( u_0 \geq 1 \) and Lemma 4.1). We define as before

\[
W := w - \zeta = u - \bar{u} - \epsilon (t + \epsilon)u^2
\]

and set

\[
T^* = \min \left( T, \frac{1}{4}, \frac{1}{10\epsilon} \right)
\]

where \( \tilde{c} \) is a uniform constant (to be determined later) and depends on the constant \( c \) in (1.8).

We proceed as in the proof of Theorem 1.3 but we need to consider an additional case: *the supremum \( m^* \) is attained at infinity.* This means, there exists a sequence of points \( y_k \in \mathbb{R}^n \) with \( |y_k| \to +\infty \) and a sequence of times \( s_k \in (0, T^*), s_k \to t_0 \) such that

\[
W(y_k, s_k) > \frac{m^*}{2} > 0.
\]

Applying the maximum principle we will deduce that \( t_0 > 1/8c \) deriving a contradiction to the definition of \( T^* \). Notice that since our initial data is complete non-compact and the convergence of our solutions to the initial data is assumed to be uniform only on compact subsets of \( \mathbb{R}^n \), it is not a’priori guaranteed that \( t_0 > 0 \), that is at this point we assume that \( s_k \to t_0 \in [0, T^*] \).

To apply the maximum principle, we employ a parabolic version of the Omori-Yau maximum principle (see for example in [12]). We define the functions

\[
W_k(x, t) = W(x, t) - t \frac{|x|^2}{C_k^2}, \quad \text{for } C_k = \max\{|y_k|^2, k\}
\]
and we look at the supremum of $W_k$ in $\mathbb{R}^n \times (0, s_k]$. If this supremum is less than $m^*/4$, then $W(y_k, s_k) \leq \frac{m^*}{4} + t \frac{|y_k|^2}{2}$ and from our choice of $C_k$ we have $W(y_k, s_k) \leq \frac{3m^*}{8} < \frac{m^*}{8}$ for $k \gg 1$, contradicting our assumption.

We deduce that $m_k := \sup_{\mathbb{R}^n \times (0, s_k]} W_k > \frac{m^*}{4} > 0$. Since $W$ is uniformly bounded (see (4.5)) this supremum is attained in the interior at a point $(x_k, t_k) \in \mathbb{R}^n \times (0, s_k]$. At this point necessarily we have

\[
W(x_k, t_k) \geq t_k \frac{|x_k|^2}{C_k^2} > 0, \quad W_t(x_k, t_k) = (W_k)_t(x_k, t_k) + \frac{|x_k|^2}{C_k^2} \geq 0
\]

(4.16)

\[
DW(x_k, t_k) = \frac{2t_k |x_k|^2}{C_k^2}, \quad D_{ij} W(x_k, t_k) \leq \frac{2t_k \delta_{ij}}{C_k^2} \leq \frac{2t_k \delta_{ij}}{k^2}
\]

where the last inequality is understood in the sense of quadratic forms, that is for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $D_{ij} W(x_k, t) \xi_i \xi_j < \frac{2}{k^2} |\xi|^2$ holds. Furthermore, notice that since $(x_k, t_k)$ is the maximum for $W_k$ on $\mathbb{R}^n \times (0, s_k]$, we have $W(x_k, t_k) - t_k \frac{|x_k|^2}{C_k^2} \geq W(0, 0)$, and because $W \leq \epsilon^2$, we have

\[
\frac{t_k |x_k|^2}{C_k^2} \leq W(x_k, t_k) - W(0, 0) \leq \epsilon^2 - W(0, 0) = \epsilon^2 + \epsilon^2 u^2(0, 0) =: M_\epsilon.
\]

Then

\[
|DW(x_k, t_k)| = \frac{2t_k |x_k|^2}{C_k^2} \leq \frac{2 \sqrt{t_k} M_\epsilon}{C_k} \leq \frac{2 \sqrt{t_k} M_\epsilon}{k} = O(\sqrt{t_k / k}).
\]

Moreover, since $W_k(x_k, t_k) = m_k^* > \frac{m^*}{4} > 0$ we have $W(x_k, t_k) = W_k(x_k, t_k) + t_k \frac{|x_k|^2}{C_k^2} > \frac{m^*}{4} > 0$. Combining these with (4.16) we conclude the following:

(4.18) $W(x_k, t_k) > \frac{m^*}{4} > 0$, $W_t(x_k, t_k) \geq 0$, $|DW(x_k, t_k)| \leq O(\sqrt{t_k / k})$, $D_{ij} W(x_k, t_k) \leq \frac{2 \delta_{ij}}{k^2}$.

Hence, we deduce from (4.11), (4.2), (4.18) and the uniform ellipticity of the matrix $a_{ij}$, that

(4.19) $-\frac{C}{k} \leq W_t - a_{ij} D_{ij} W < (a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u} - \frac{\epsilon}{2} u^2$

holds at each point $(x_k, t_k)$. Furthermore from $W(x_k, t_k) > 0$ we have

(4.20) $(1 - \epsilon(t_k + \epsilon) u) u(x_k, t_k) + \epsilon > \bar{u}(x_k, t_k)$.

Next, observe that the fact that $W(x_k, t_k) > 0$ implies that $u(x_k, t_k)$ is bounded (otherwise if $u(x_k, t_k) \to +\infty$ for some subsequence, then $\lim_{t \to +\infty} W(x_k, t_k) \to -\infty$). Furthermore, $u(x_k, t_k)$ bounded and $u, \bar{u} \geq 1$ imply that $\bar{u}(x_k, t_k)$ is bounded as well. Hence, we may assume without loss of generality that

(4.21) $u(x_k, t_k) \to u^*$, $\bar{u}(x_k, t_k) \to \bar{u}^*$ and $1 \leq u(x_k, t_k), \bar{u}(x_k, t_k) \leq u^* + 1$.

Therefore, our assumption that $u, \bar{u}$ satisfy condition (4.11) and the first assertion in Proposition 4.5 applied to $M = u^* + 2$ yield

(4.22) $|Du(x_k, t_k)| \leq C(u^*)$ and $|D\bar{u}(x_k, t_k)| \leq C(u^*)$.

Furthermore, by the third assertion in Proposition 4.5 we have

\[
t_k |A|^2(x_k, t_k) \leq C(u^*) \quad \text{and} \quad t_k |\bar{A}|^2(x_k, t_k) \leq C(u^*).
\]
It follows that at the points \((x_k, t_k)\) we have

\[
\sqrt{t_k} v|\tilde{h}_t^j|(x_k, t_k) \leq C(u^*) \quad \text{and} \quad \sqrt{t_k} v|\tilde{h}_t^i|(x_k, t_k) \leq C(u^*)
\]

and also

\[
\sqrt{t_k} \frac{|D_{ij} u|}{\sqrt{1 + |Du|^2}} \leq C(u^*) \quad \text{and} \quad \sqrt{t_k} \frac{|D_{ij} \bar{u}|}{\sqrt{1 + |\bar{u}|^2}} \leq C(u^*).
\]

These bounds will be used momentarily.

We will next analyze the main term on right hand side of (4.19). From the definition of \(W\) we have that \(D\bar{u}(x_k, t_k) = (1 - 2\epsilon (t_k + \epsilon) u)Du - DW\). Then, similarly to (4.7) (the computation here has more terms since \(DW \neq 0\)) we get

\[
a_{ij} - \bar{a}_{ij} = \frac{D_i \bar{u} D_j \bar{u}}{1 + |\bar{u}|^2} - \frac{D_i u D_j u}{1 + |u|^2} = (1 - 2\epsilon (t_0 + \epsilon) u)^2 \frac{D_i u D_j u}{1 + |u|^2} - \frac{D_i u D_j u}{1 + |u|^2} + \frac{D_i W D_j W - (1 - 2\epsilon (t_k + \epsilon) u)(D_i u D_j W + D_i W D_j u)}{1 + |u|^2}
\]

where \(b = 2(1 - 2\epsilon (t_k + \epsilon) u)Du - DW\). Denoting

\[
B_{ij} = (DW, b) \frac{D_i u D_j u}{(1 + |u|^2)(1 + |\bar{u}|^2)} + \frac{D_i W D_j W - (1 - 2\epsilon (t_k + \epsilon) u)(D_i u D_j W + D_i W D_j u)}{1 + |u|^2}
\]

we can then express the main term \((a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u}\) on right hand side of (4.19) as

\[
(4.25) \quad (a_{ij} - \bar{a}_{ij}) D_{ij} \bar{u} = -4\epsilon (t_0 + \epsilon) u (1 - \epsilon (t_0 + \epsilon) u) \frac{D_i \bar{u} D_i u D_j u}{(1 + |u|^2)(1 + |\bar{u}|^2)} + B_{ij} D_{ij} \bar{u}.
\]

Next observe that from (4.17) at \((x_k, t_k)\) we have that

\[
|B_{ij}| \leq C(u^*) \frac{\sqrt{t_k}}{k}
\]

which combined with (4.24) yields

\[
(4.26) \quad |B_{ij} D_{ij} \bar{u}| \leq C \frac{\sqrt{t_k}}{k}(\sqrt{t_k})^{-\frac{1}{2}} = O\left(\frac{1}{k}\right).
\]

To bound the first term on the right-hand side of (4.25) we use (4.8) which in particular implies that

\[
(4.27) \quad \bar{v} h_j^i D_i u D_j u \geq -c |Du|^2.
\]
On the other hand, \( \tilde{h}_j^i \) is given by

\[
\tilde{h}_j^i = \frac{D_{ij} \bar{u}}{\sqrt{1 + |D\bar{u}|^2}} - \frac{D_{ij} \bar{D}_i \bar{u} \bar{D}_j \bar{u}}{(1 + |D\bar{u}|^2)^{3/2}}
\]

implies

\[
\tilde{h}_j^i D_i u D_j u = \left( D_{ij} \bar{u} - \frac{D_{ij} \bar{D}_i \bar{u} \bar{D}_j \bar{u}}{1 + |D\bar{u}|^2} \right) D_i u D_j u
\]

\[
= D_{ij} \bar{D}_i u D_j u - (\bar{D}_i u, D_j u) \frac{D_{ij} \bar{u}}{1 + |D\bar{u}|^2} D_i u D_j u
\]

\[
= (1 + |D\bar{u}|^2 - (1 - 2 \varepsilon (t_k + \varepsilon) u)^2 |Du|^2) \frac{D_{ij} \bar{u}}{1 + |D\bar{u}|^2} D_i u D_j u
\]

\[- (1 - 2 \varepsilon (t_k + \varepsilon) u) \frac{|Du|^2}{1 + |D\bar{u}|^2} D_{ij} u D_j u D_i W - \frac{(D W, D u)}{1 + |D\bar{u}|^2} D_{ij} u D_i \bar{u} D_j u
\]

\[
= (1 + |D\bar{u}|^2 - (1 - 2 \varepsilon (t_k + \varepsilon) u)^2 |Du|^2) \frac{D_{ij} \bar{u}}{1 + |D\bar{u}|^2} D_i u D_j u + O(\frac{1}{k})
\]

where to derive the last line we combined (4.17) and (4.24) (following a similar estimate as the one we did for \( B_{ij} D_i \bar{u} \)).

To further estimate the last line above, we use

\[
|D\bar{u}|^2 - (1 - 2 \varepsilon (t_k + \varepsilon) u)^2 |Du|^2 = (D W, D \bar{u} + (1 - 2 \varepsilon (t_k + \varepsilon) u) D u) = O(\frac{\sqrt{t_k}}{k})
\]

concluding that

\[
\tilde{h}_j^i D_i u D_j u = \frac{D_{ij} \bar{u}}{1 + |D\bar{u}|^2} D_i u D_j u + O(\frac{1}{k})
\]

which in turn, combined with (4.27) yields

\[
(4.28) \quad \frac{D_{ij} \bar{u}}{1 + |D\bar{u}|^2} D_i u D_j u \geq -c |Du|^2 + O(\frac{1}{k}).
\]

Finally, (4.19), (4.26), (4.20) and (4.28) together imply that as \( k \to \infty \)

\[
0 \leq 4 \varepsilon c (t_0 + \varepsilon) u^* (1 - \varepsilon (t_0 + \varepsilon) u^*) - \frac{\varepsilon}{2} (u^*)^2.
\]

We now use the same argument as in the proof of Theorem 1.3 to conclude that this is not possible provided that \( t_0 + \varepsilon > \frac{1}{8c} \), where \( c \) is the constant from (1.8). Since we have assumed that \( t_0 \in (0, T^*) \) and \( T^* \leq \frac{1}{10c} \) we derive a contradiction by choosing \( \varepsilon \) sufficiently small. This shows, that contrary to our assumption, \( W^*(t_0) < 0 \), finishing the proof of the claim.

\[ \square \]

5. The convex case and Harnack inequality

In this final section we will state and prove an existence and uniqueness result for convex, proper, non-compact entire graphs Mean curvature flow solutions and show that Hamilton’s Harnack inequality holds.

**Theorem 5.1** (Uniqueness of convex entire graph solutions). Assume that \( u_0 : \mathbb{R}^n \to \mathbb{R} \) is a convex function defining a proper entire graph convex hypersurface \( M_0 = \{ (x, u_0(x)) : x \in \mathbb{R}^n \} \subset \mathbb{R}^{n+1} \).

Let \( u_1, u_2 : \mathbb{R}^n \times (0, T) \to \mathbb{R} \) be two solutions of (1.3) defining two proper smooth convex entire graph solutions \( M_1 = \{ (x, u_1(x, t)) : x \in \mathbb{R}^n \} \) and \( M_2 = \{ (x, u_2(x, t)) : x \in \mathbb{R}^n \} \) of MCF (1.1) with the
same initial data $u_0$, that is $\lim_{t \to 0} u_1(., t) = \lim_{t \to 0} u_2(., t) = u_0$. Then, $u_1 = u_2$ on $\mathbb{R}^n \times (0, T)$, that is $M_1^t = M_2^t$ for all $t \in (0, T)$.

Proof. Now, since our initial data is a convex proper entire graph over $\mathbb{R}^n$, we may assume that it lies above the $e_n+1 = 0$ plane, that is $u_0(x) \geq 0$ for all $x \in \mathbb{R}^n$. Furthermore, we have $\lim_{x \to +\infty} u_0(x) = +\infty$ and the same holds for both solutions $u_i(x, t)$, $i = 1, 2$, namely $u_i(., t) \geq 0$ and $\lim_{x \to +\infty} u_i(x, t) = +\infty$, for all $t > 0$.

One then can then apply the maximum principle argument in Theorem 1.3 (actually in the convex case the computation is simpler) to show that for any small number $\epsilon > 0$, one has $u_1 - u_2 \leq \epsilon t u_1^2 + \epsilon$ and, similarly, $u_2 - u_1 \leq \epsilon t u_2^2 + \epsilon$, for all $t \in (0, T)$. Taking $\epsilon \to 0$ readily gives that $u_1 = u_2$ for all $t \in (0, T)$. \hfill $\square$

An immediate consequence of the previous result is that convex graphical MCF solutions can be smoothly approximated by compact ones. For any two compact convex hypersurfaces $\Sigma_1, \Sigma_2$ we write that $\Sigma_1 \prec \Sigma_2$ if $\Sigma_2$ encloses $\Sigma_1$ (allowing $\Sigma_1 \cap \Sigma_2 \neq \emptyset$).

**Corollary 5.2.** Let $M_t = \{(x, u(x, t)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$, $t \in (0, +\infty)$, be a smooth entire graph Mean Curvature Flow solution with initial data $M_0 = \{(x, u_0(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ which is a proper convex entire graph, normalized in such a way that $u(0) = \min_{x \in \mathbb{R}^n} u_0(x) = 0$.

Then, $M_t$ can be approximated by a sequence $M^t_i$ of compact convex Mean curvature flow solutions. More precisely, the surfaces $\Sigma^i_t$ are reflection symmetric with respect to the hyperplane $\{x_{n+1} = i\}$ and their lower parts $\tilde{\Sigma}^i_t := \Sigma^i_t \cap \{x_{n+1} < i\}$ converge, as $i \to +\infty$, to $M_t$, smoothly on compact subsets of $\mathbb{R}^{n+1} \times (0, +\infty)$.

Proof. From our assumptions we have $M_t = \{(x, u(x, t)) : x \in \mathbb{R}^n\}$, for all $t \in (0, +\infty)$ and that $u(., t) \geq 0$ for all $t \geq 0$, since we have normalized our initial data so that $u(0) = \min_{x \in \mathbb{R}^n} u_0(x) = 0$. Furthermore, since $u_0(x)$ is assumed to be proper we have $\lim_{x \to +\infty} u(x, t) = +\infty$ for all $t \geq 0$.

For each integer $i \geq 1$, we define the Lipschitz domains

$$D_0^i = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : u_0(x) \leq x_{n+1} \leq 2i - u_0(x)\}$$

and we let $\Sigma_0^i = \partial D_0^i$. Our assumption that $u(0) = 0$ guarantees that $D_0^i \neq \emptyset$ for all $i \geq 1$. Note that $\Sigma_0^i \subset \mathbb{R}^{n+1}$ is just the closed hypersurface that consists by $M_0 \cap \{x_{n+1} \leq i\}$ and its reflection with respect to the hyperplane $x_{n+1} = i$. Furthermore, each $\Sigma_0^i$ is convex and Lipschitz continuous.

Standard MCF theory shows that for any $i \geq 1$, there exists a unique smooth mean curvature flow $\Sigma^i_t$ starting at $\Sigma_0^i$. The solutions $\Sigma^i_t$ exists up to times $T^i$, they satisfy $\Sigma^i_t \prec \Sigma^{i+1}_t$ ($\Sigma^{i+1}_t$ encloses $\Sigma^i_t$), and $\lim_{t \to +\infty} T^i = +\infty$. The strong maximum principle guarantees that each $\Sigma^i_t$, $0 < t < T^i$ is strictly convex. Furthermore, $\Sigma^i_t$ is reflection symmetric with respect to the hyperplane $\{x_{n+1} = i\}$, since $\Sigma_0^i$ is by construction.

Denote by $\tilde{\Sigma}^i_t$ to be the lower half of $\Sigma^i_t$, that is

$$\tilde{\Sigma}^i_t := \Sigma^i_t \cap \{x_{n+1} < i\}.$$ 

Also, for any point $x_0 \in \mathbb{R}^{n+1}$ let us denote by $B^{n+1}_R(x_0)$ the ball in $\mathbb{R}^{n+1}$ of radius $R$ centered at $x_0$. 


Claim 5.1. Fix $T > 0$. For any $R > 1$, there exists an integer $i_R$ such as long as $i \geq i_R$, the lower part of $\hat{\Sigma}^i \cap B^{n+1}_{3R}(0)$, $t \in [0, T]$ can be written as a graph $\{(x, u^i(x, t)) : |x| \leq R\}$ and satisfies a uniform in $i$ gradient bound which is independent of $i$ and depends only on $R$ and $M_0$.

Proof. Fix $T > 0$ and assume that $i$ is chosen sufficiently large so that $T^i > T$. Furthermore, given any $R > 1$, we may choose $i_R$ sufficiently large so that $T \ll R$ and if $x_0^i = (0, i) \in \mathbb{R}^{n+1}$, then $B^{n+1}_{4R}(x_0^i) \subset \hat{\Sigma}^i$, for all $i \geq i_R$ and all $t \in [0, T]$.

The convexity and symmetry of the solutions $\hat{\Sigma}^i$ then imply that for any $i \geq i_R$, $\hat{\Sigma}^i \cap B^{n+1}_R(0)$, $t \in [0, T]$ can be written as a graph $\{(x, u^i(x, t)) : |x| \leq 3R\}$. Hence, it remains to show the uniform gradient bound of $\hat{\Sigma}^i \cap B^{n+1}_R(0)$, $t \in [0, T]$ for all $i \geq i_R$. This readily follows from the local gradient bound in [10] and the fact that $u^i(x, 0) = u_0(x)$ for all $i \geq i_R$, which implies that $\hat{\Sigma}^i \cap B^{n+1}_R(0)$, $i \geq i_R$ satisfy a uniform gradient bound.

The results in [10] then imply that $\hat{\Sigma}^i \cap B^{n+1}_R(0)$, $t \in [0, T], i \geq i_R$ have uniformly bounded second fundamental forms. More precisely, there exists a constant $C_{R,T}$ that is independent of $i$ such that the second fundamental form $|A^i|$ of $\Sigma^i$ satisfies the bound

$$\sup_{\hat{\Sigma}^i \cap B^{n+1}_R(0)} |A^i| \leq C_{R,T} t^{-1/2}, \quad t \in (0, T]$$

provided that $i \geq i_R$.

One can then pass to the limit (over a subsequence $i_k \to +\infty$) and obtain a smooth entire graph mean curvature flow solution $\hat{M}_t$, $t \in (0, T)$ whose second fundamental form satisfies the bound

$$\sup_{\hat{M}_t \cap B^{n+1}_R(0)} |A| \leq C_{R,T} t^{-1/2}, \quad t \in (0, T].$$

Standard arguments then imply that if $\hat{M}_t = \{(x, \hat{u}(x, t)) : x \in \mathbb{R}^n\}$, then $\lim_{t \to 0} \hat{u}(x, t) = u_0(x)$. Since $x_{n+1} = u_0(x)$ is proper, $x_{n+1} = \hat{u}(x, t)$ is proper as well. Hence, Theorem 5.1 guarantees that $u = \hat{u}$ on $\mathbb{R}^n \times (0, T)$. Since $T > 0$ was arbitrary, we conclude that $u = \hat{u}$ on $\mathbb{R}^n \times (0, +\infty)$ finishing the proof of the corollary.

Remark 5.1. Our methods can be applied to study the uniqueness of the (convex) solutions that are analyzed by X.-J. Wang in [16]. More precisely, in that paper, the author studies convex translating solutions to Mean Curvature flow via a level set method. In the non-compact case, those solutions are obtained via taking limits and our techniques can be used as an alternative proof of the uniqueness of such limits. We leave the details to the interested reader.

An immediate consequence of Corollary 5.2 is that Hamilton’s Harnack inequality holds for entire convex graphs.

Corollary 5.3 (Hamilton’s Harnack estimate). Any smooth convex proper entire graph solution $M_t$, $t \in (0, +\infty)$ of Mean curvature flow satisfies Hamilton’s Harnack differential inequality, namely for
any tangent vector field \( V \),

\[
\frac{\partial H}{\partial t} + 2\langle \nabla H, V \rangle + h(V, V) + \frac{H}{2t} \geq 0.
\]

**Proof.** Let \( \Sigma_t^1 \) be approximating sequence of compact convex solutions which were constructed in Corollary [5.2]. Each of them satisfy the Harnack differential inequality (5.3). Passing to the smooth limit on compact sets, we conclude that (5.3) also holds for our complete non-compact solution \( M_t \), for all \( t \in (0, +\infty) \).

\[\qed\]

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