Dirichlet-Neumann and Neumann-Neumann Waveform Relaxation for the Wave Equation

Martin J. Gander¹, Felix Kwok¹ and Bankim C. Mandal¹

1 Introduction

We present two new types of Waveform Relaxation (WR) methods for hyperbolic problems based on the Dirichlet-Neumann and Neumann-Neumann algorithms, and present convergence results for these methods. The Dirichlet-Neumann algorithm for elliptic problems was first considered by Bjørstad & Widlund [3] and further studied in [5], [20] and [19]; the Neumann-Neumann algorithm was introduced by Bourgat et al. [4], see also [6] and [22]. The performance of these algorithms for elliptic problems is now well understood, see for example the book [23].

To solve time-dependent problems in parallel, one can either discretize in time to obtain a sequence of steady problems, and then apply domain decomposition algorithms to solve the steady problems at each time step in parallel, or one can first discretize in space and then apply WR to the large system of ordinary differential equations (ODEs) obtained from the spatial discretization. WR has its roots in the work of Picard [21] and Lindelöf [17], who studied existence and uniqueness of solutions of ODEs in the late 19th century. Lelarasmee, Ruehli and Sangiovanni-Vincentelli [16] rediscovered WR as a parallel method for the solution of ODEs. The main computational advantage of WR is parallelization, and the possible use of different discretizations in different space-time subdomains.

Domain decomposition methods for elliptic PDEs can be extended to time-dependent problems by using the same decomposition in space. This leads to WR type methods, see [2] and [14]. The systematic extension of the classical Schwarz method to time-dependent parabolic problems was started independently in [12] [13]. Like WR algorithms in general, the so-called Schwarz Waveform Relaxation algorithms (SWR) converge relatively slowly, except if the time window size is short. A remedy is to use optimized transmission conditions, which leads to much faster algorithms, see [8] [1] for parabolic problems, and [9] [7] for hyperbolic problems. More recently, we studied the WR extension of the Dirichlet-Neumann and Neumann-Neumann methods for parabolic problems [10] [18] [15]. We proved for the heat equation that on finite time intervals, the Dirichlet-Neumann Waveform Relaxation (DNWR) and the Neumann-Neumann Waveform Relaxation (NNWR) methods converge superlinearly for an optimal choice of the relaxation parameter. DNWR and NNWR also converge faster than classical and optimized SWR in this case.

¹Department of Mathematics, University of Geneva, Switzerland. Martin.Gander@unige.ch, Felix.Kwok@unige.ch, Bankim.Mandal@unige.ch
In this paper, we define and study the convergence of DNWR and NNWR for the second order wave equation
\[
\begin{align*}
\partial_t u - c^2 \Delta u &= f(x, t), & x \in \Omega, 0 < t < T, \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = v_0(x), & x \in \Omega, \\
u(x, t) &= g(x, t), & x \in \partial \Omega, 0 < t < T,
\end{align*}
\]
where \(\Omega \subset \mathbb{R}^d\), \(d = 1, 2, 3\), is a bounded domain with a smooth boundary, and \(c\) denotes the wave speed.

2 Domain decomposition and algorithms

To explain the new algorithms, we assume for simplicity that the spatial domain \(\Omega\) is partitioned into two non-overlapping subdomains \(\Omega_1\) and \(\Omega_2\). We denote by \(u_i\) the restriction of the solution \(u\) of (1) to \(\Omega_i\), \(i = 1, 2\), and by \(n_i\) the unit outward normal for \(\Omega_i\) on the interface \(\Gamma := \partial \Omega_1 \cap \partial \Omega_2\).

The Dirichlet-Neumann Waveform Relaxation algorithm (DNWR) consists of the following steps: given an initial guess \(u^0_i(x, t), i \in (0, T)\) along the interface \(\Gamma \times (0, T)\), compute for \(k = 1, 2, \ldots\)
\[
\begin{align*}
\partial_t u^k_1 - c^2 \Delta u^k_1 &= f, & \text{in } \Omega_1, \\
u^k_1(x, 0) &= u^0_1(x), & \text{in } \Omega_1, \\
\partial_t u^k_2 - c^2 \Delta u^k_2 &= f, & \text{in } \Omega_2, \\
u^k_2(x, 0) &= u^0_2(x), & \text{in } \Omega_2, \\
u^k_1 = n_1 \cdot \nabla u^k_1 &= -n_1 \cdot \nabla u_1, & \text{on } \Gamma, \\
u^k_1 = \theta u^k_2 |_{\Gamma \times (0, T)} + (1 - \theta) u^{k-1}_1(x, t), & \text{on } \partial \Omega_1 \setminus \Gamma,
\end{align*}
\]
where \(\theta \in (0, 1]\) is a relaxation parameter.

The Neumann-Neumann Waveform Relaxation algorithm (NNWR) starts with an initial guess \(u^0_1(x, t), i \in (0, T)\) along the interface \(\Gamma \times (0, T)\) and then computes simultaneously for \(i = 1, 2\) with \(k = 1, 2, \ldots\)
\[
\begin{align*}
\partial_t u^k_1 - c^2 \Delta u^k_1 &= f, & \text{in } \Omega_i, \\
u^k_1(x, 0) &= u^0_1(x), & \text{in } \Omega_i, \\
\partial_t u^k_2 - c^2 \Delta u^k_2 &= f, & \text{in } \Omega_i, \\
u^k_2(x, 0) &= u^0_2(x), & \text{in } \Omega_i, \\
u^k_1 = n_1 \cdot \nabla u^k_1 &= -n_1 \cdot \nabla u_1, & \text{on } \Gamma, \\
u^k_2 = \theta u^k_2 |_{\Gamma \times (0, T)} + (1 - \theta) u^{k-1}_2(x, t), & \text{on } \partial \Omega_2 \setminus \Gamma,
\end{align*}
\]
where \(\theta \in (0, 1]\) is also a relaxation parameter.
3 Kernel estimates and convergence analysis

We present here the case $d = 1$, with $\Omega = (-a, b)$, $\Omega_1 = (-a, 0)$ and $\Omega_2 = (0, b)$. By linearity, it suffices to study the error equations, $f(x, t) = 0$, $g(x, t) = 0$, $u_0(x) = v_0(x) = 0$ in (2) and (3), and to examine convergence to zero.

Our convergence analysis is based on Laplace transforms. The Laplace transform of a function $u(x, t)$ with respect to time $t$ is defined by $\hat{u}(x, s) = \mathcal{L}\{u(x, t)\} := \int_0^\infty e^{-st}u(x, t)\,dt$, $s \in \mathbb{C}$. Applying a Laplace transform to the DNWR algorithm in (2) in 1d, we obtain for the transformed error equations

\[
(s^2 - c^2 \partial_x^2)\hat{u}_k^1 = 0 \quad \text{in} \ (-a, 0), \quad (s^2 - c^2 \partial_x^2)\hat{u}_k^2 = 0 \quad \text{in} \ (0, b), \\
\hat{u}_k^1(-a, s) = 0, \quad \partial_x\hat{u}_k^2(0, s) = \partial_x\hat{u}_k^1(0, s), \\
\hat{h}_k^1(s) = \hat{h}_k^{k-1}(s), \quad \hat{u}_k^2(b, s) = 0, \quad \partial_x\hat{h}_k(s) = \theta \hat{u}_k^2(0, s) + (1 - \theta)\hat{h}_k^{k-1}(s). \tag{4}
\]

Solving the two-point boundary value problems in (4), we get

\[
\hat{u}_k^1(x, s) = \frac{\hat{h}_k^{k-1}(s)}{\sinh(as/c)} \sinh \left( (x + a)\frac{s}{c} \right), \quad \hat{u}_k^2(x, s) = \frac{\hat{h}_k^{k-1}(s)}{\cosh(bs/c)} \sinh \left( (x - b)\frac{s}{c} \right),
\]

and inserting them into the updating condition (last line in (4)), we get by induction

\[
\hat{h}_k(s) = [1 - \theta - \theta \coth(as/c) \tanh(bs/c)]^k \hat{h}_0(s), \quad k = 1, 2, \ldots \tag{5}
\]

Similarly, the Laplace transform of the NNWR algorithm in (3) for the error equations yields for the subdomain solutions

\[
\hat{u}_k^1(x, s) = \frac{\hat{w}_k^{k-1}(s)}{\sinh(as/c)} \sinh \left( (x + a)\frac{s}{c} \right), \quad \hat{u}_k^2(x, s) = \frac{\hat{w}_k^{k-1}(s)}{\sinh(bs/c)} \sinh \left( (x - b)\frac{s}{c} \right), \\
\hat{\psi}_k^1(x, s) = \frac{\hat{w}_k^{k-1}(s)}{\cosh(as/c)} \sinh \left( (x + a)\frac{s}{c} \right), \quad \hat{\psi}_k^2(x, s) = \frac{\hat{w}_k^{k-1}(s)}{\cosh(bs/c)} \sinh \left( (x - b)\frac{s}{c} \right),
\]

where $\Psi(s) = [\coth(as/c) + \coth(bs/c)]$. Therefore, in Laplace space the updating condition in (3) becomes

\[
\hat{w}_k(s) = \left[ 1 - \theta \left( 2 + \frac{\coth(as/c)}{\coth(bs/c)} + \frac{\coth(bs/c)}{\coth(as/c)} \right) \right]^k \hat{w}_0(s), \quad k = 1, 2, \ldots \tag{6}
\]

**Theorem 1 (Convergence, symmetric decomposition).** For a symmetric decomposition, $a = b$, convergence is linear for the DNWR (2) with $\theta \in (0, 1)$, $\theta \neq \frac{1}{2}$, and for the NNWR (3) with $\theta \in (0, 1)$, $\theta \neq \frac{1}{4}$. If $\theta = \frac{1}{2}$ for DNWR, or $\theta = \frac{1}{2}$ for NNWR, convergence is achieved in two iterations.

**Proof.** For $a = b$, equation (5) reduces to $\hat{h}_k(s) = (1 - 2\theta)^k \hat{h}_0(s)$, which has the simple back transform $h_k(t) = (1 - 2\theta)^k h_0(t)$. Thus for the DNWR method, the
Theorem 2 (Convergence of DNWR, asymmetric decomposition). Let $\theta = \frac{4}{5}$. Then the DNWR algorithm converges in at most $k + 1$ iterations for two subdomains of lengths $a \neq b$, if the time window length $T$ satisfies $T/k \leq 2 \min \{a/c, b/c\}$, where $c$ is the wave speed.

Proof. With $\theta = \frac{4}{5}$ we obtain from (7) for $k = 1, 2, \ldots$

$$\hat{h}^k(s) = \left(1 - 2\theta\right) - \theta G^0_b(s) \right)^k \hat{h}^0(s).$$

Now if $\theta = \frac{4}{5}$, we see that the linear factor in (7) vanishes, and convergence will be governed by convolutions of $G^0_b(s)$. We show next that this choice also gives finite step convergence, but the number of steps depends on the length of the time window $T$.

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Similarly, we get \(G^0_b(s) = 1 + \frac{2}{2\pi} \sum e^{-\frac{2m\pi}{c}}\), and multiplying the two and subtracting 1, we obtain the expression for \(G^0_b(s)\) in the Lemma.

Using \(G^0_b(s)\) from Lemma 1 we obtain for (5)

$$\hat{h}^k(s) = \{\left(1 - 2\theta\right) - \theta G^0_b(s)\}^k \hat{h}^0(s).$$

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Using G transform and obtain therefore one more iteration produces the desired solution on the entire domain.

Then the NNWR algorithm converges in at most \( k \) iterations for two subdomains of lengths \( a \neq b \), if the time window length \( T \) satisfies \( T \leq 2k \min \{a/c, b/c\} \), then \( h^k(t) = 0 \), and therefore one more iteration produces the desired solution on the entire domain. \( \square \)

Using \( G^d_{\theta}(s) \) from Lemma 11 we can also rewrite (6) in the form

\[
\hat{w}^k(s) = \left\{ (1 - 4\theta) - \theta \left( G^d_{\theta}(s) + G^b_{\theta}(s) \right) \right\}^k \hat{w}^0(s), \quad k = 1, 2, \ldots, \quad (10)
\]

and we see that for NNWR, the choice \( \theta = \frac{1}{4} \) removes the linear factor.

**Theorem 3 (Convergence of NNWR, asymmetric decomposition).** Let \( \theta = \frac{1}{4} \). Then the NNWR algorithm converges in at most \( k+1 \) iterations for two subdomains of lengths \( a \neq b \), if the time window length \( T \) satisfies \( T/k \leq 4 \min \{a/c, b/c\} \), \( c \) being again the wave speed.

**Proof.** With \( \theta = \frac{1}{4} \) we obtain from (10) with a similar calculation as in (5)

\[
\hat{w}^k(s) = \left( -\frac{1}{4} \right)^k \left[ G^d_{\theta}(s) + G^b_{\theta}(s) \right]^k \hat{w}^0(s) = \left[ -\sum_{m=1}^{\infty} \left( e^{-\frac{4am}{c}} + e^{-\frac{4bm}{c}} \right) \right]^k \hat{w}^0(s) = (-1)^k e^{-\frac{4bk}{c}} \hat{w}^0(s)
\]

\[+ (-1)^k e^{-\frac{4bk}{c}} \hat{w}^0(s) + \left( \sum_{l>k}^{\infty} d_{l}^{(k)} e^{-\frac{4al}{c}} + \sum_{l>k}^{\infty} q_{l}^{(k)} e^{-\frac{4bl}{c}} + \sum_{m+n \geq k}^{\infty} j_{m,n}^{(k)} e^{-\frac{2(2am+bn)}{c}} \right) \hat{w}^0(s), \]

where \( d_{l}^{(k)}, q_{l}^{(k)}, j_{m,n}^{(k)} \) are the corresponding coefficients. Now we use (9) to back transform and obtain
\[ w^k(t) = (-1)^k w^0(t - 4ak/c) H(t - 4ak/c) + (-1)^k w^0(t - 4bk/c) H(t - 4bk/c) + \sum_{l \geq k} d^{(k)}_l w^0(t - 4al/c) H(t - 4al/c) + \sum_{l \geq k} z^{(k)}_l w^0(t - 4bl/c) H(t - 4bl/c) + \sum_{m+n \geq 2k} j^{(k)}_{mn} w^0(t - 2(am + bn)/c) H(t - 2(4m + 4n)/c). \]

So for \( T \leq 4k \min \{ \frac{a}{c}, \frac{b}{c} \} \), we get \( w^k(t) = 0 \), and the conclusion follows. \( \square \)

4 Numerical Experiments

Having obtained convergence bounds at the continuous level in the previous section, we perform now numerical experiments to measure the actual convergence rate of the discretized DNWR and NNWR algorithms for the model problem

\[ \partial_t u - \partial_{xx} u = 0, \quad x \in (-3, 2), t > 0, \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = 2xe^{-x}, \quad -3 < x < 2, \]
\[ u(-3, t) = t, \quad u(2, t) = te^{-t}, \quad t > 0, \] (11)

with \( \Omega_1 = (-3, 0) \) and \( \Omega_2 = (0, 2) \), so that \( a = 3 \) and \( b = 2 \) in (4, 5, 6). We discretize the wave equation using the Leapfrog scheme on a grid with \( \Delta x = \Delta t = 2 \times 10^{-2} \). The error is calculated by \( \| u - u^k \|_{L^\infty(0,T;L^2(\Omega))} \), where \( u \) and \( u^k \) are respectively the exact solution and the solution in \( k \)-th iteration.

We test the DNWR algorithm by choosing \( h^0(t) = t^2, t \in (0, T] \) as an initial guess. In Figure 1 we show the convergence behavior for different values of the parameter \( \theta \) for \( T = 16 \) on the left, and on the right for the best parameter \( \theta = \frac{1}{2} \) for different time window length \( T \).

For the NNWR method, using the same initial guess, we show in Figure 2 on the left the convergence curves for different values of the parameter \( \theta \) for \( T = 16 \),
we use for the overlapping Schwarz variant an overlap of length $24\Delta x$, where $\Delta x = 1/50$. We observe that the DNWR and NNWR algorithms converge as fast as the Schwarz WR algorithms for smaller time windows $T$.

5 Conclusions

We introduced the Dirichlet-Neumann and Neuman-Neumann Waveform Relaxation algorithms for the second order wave equation, and analyzed their convergence properties for the one dimensional case and a two subdomain decomposition. We showed that for a particular choice of the relaxation parameter, convergence can be achieved in a finite number of steps. Choosing the time window length carefully, these algorithms can be used to solve such problems in two iterations only. For a detailed analysis for the case of multiple subdomains, see [11].
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