A Diagrammer’s Note on Superconducting Fluctuation Transport for Beginners: Supplement. Jonson-Mahan Transmutation

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Abstract

The Ward identity for the heat current vertex is illustrated in terms of Feynman diagrams. The Jonson-Mahan transmutation, the way how the kinetic energy at the heat current vertex for the free propagator is transmuted into the frequency for the full propagator, is the key of the illustration.

1 Introduction

This Note is the Supplement to the series of Notes [I] on the superconducting fluctuation transport where the quasi-particle transport is also discussed. The necessity of this Supplement is noticed in §14 of [I].

I think that the Jonson-Mahan transmutation [II], the way how the kinetic energy at the heat current vertex for the free propagator is transmuted into the frequency for the full propagator, is the key to understand the Ward identity for the heat current vertex but is not recognized by most authors and that the ignorance of it leads to some confusions seen in the literatures.

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1The Notes are quoted as [I] and [II] where [I] ≡ arXiv:1112.1513 and [II] ≡ arXiv:1203.0127

2See the footnote for 33 in this Supplement. For example, in the perturbational calculation in the Appendix-B of Michaeli and Finkel’stein: Phys. Rev. B 80, 115111 (2009), the frequency is erroneously employed for the free vertex. Originally, the definition of the free vertex (2) in this work is incorrect. Including this work the use of $c_{p}^{\sigma}(\varepsilon_{n})$
In the following I illustrate in terms of Feynman diagrams the way how the Ward identity is satisfied for the heat current vertex. The Jonson-Mahan transmutation [I] is the key of the illustration.

The symbols that have appeared in [I] and [II] are used here without explanations.

2 Heat-Current Operator

The charge current of electrons, (59) in [I], is Fourier-transformed into

$$ j^c(k) = e \sum_p \sum_{\sigma} \frac{v_p + v_{p+k}}{2} c_{p\sigma}^{\dagger} c_{p+k\sigma}, \quad (1) $$

which is equivalent to (73) in [I]. Here the operator is transformed as (75) in [I]. The uniform current is

$$ j^c(k=0) = e \sum_p \sum_{\sigma} v_p c_{p\sigma}^{\dagger} c_{p\sigma}. \quad (2) $$

The heat current of electrons, (65) in [I], is Fourier-transformed into

$$ j^Q(k) = \sum_p \sum_{\sigma} \frac{\xi_p + \xi_{p+k}}{2} v_p + v_{p+k} c_{p\sigma}^{\dagger} c_{p+k\sigma}, \quad (3) $$

for free electrons.\footnote{For free electrons $v_p = \frac{p}{m}$, $\xi_p = \frac{p^2}{2m} - \mu$.} The uniform current is

$$ j^Q(k=0) = \sum_p \sum_{\sigma} \xi_p v_p c_{p\sigma}^{\dagger} c_{p\sigma}. \quad (4) $$

Thus each component $c_{p\sigma}^{\dagger} c_{p\sigma}$ carries the charge $e$ in (2) and the kinetic energy $\xi_p$ in (4).
For interacting electrons the heat current also carries the interaction energy as follows. The Fourier transform of (88) in [I] is given as

\[
J^Q(k=0) = \lim_{\tau' \to \tau} \frac{1}{2} \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'} \right) \sum_p V_p \left( c^\dagger_{p\uparrow}(\tau)c_{p\uparrow}(\tau') + c^\dagger_{p\downarrow}(\tau)c_{p\downarrow}(\tau') \right). \tag{5}
\]

The time-dependence is determined by (8) in [I] with

\[
K = K_0 + V, \tag{6}
\]

where

\[
K_0 = \sum_p \xi_p \left( c^\dagger_{p\uparrow}c_{p\uparrow} + c^\dagger_{p\downarrow}c_{p\downarrow} \right), \tag{7}
\]

and

\[
V = \lambda \sum_{p'} \sum_{p''} \sum_q c^\dagger_{p'+q/2\uparrow}c^\dagger_{-p'+q/2\downarrow}c_{-p''-q/2\downarrow}c_{p''+q/2\uparrow}. \tag{8}
\]

These are equal to (2), (3) and (5) in [I] when we put \(\lambda = -g\). The time-derivative is calculated as

\[
\frac{\partial}{\partial \tau} c^\dagger_{p\uparrow}(\tau) = [K, c^\dagger_{p\uparrow}] = \xi_{p\uparrow} c^\dagger_{p\uparrow} + \lambda \sum_{p'} \sum_{p''} \sum_q c^\dagger_{p'+q/2\uparrow}c^\dagger_{-p'+q/2\downarrow}c_{-p''-q/2\downarrow}c_{p''+q/2\uparrow} \delta_{p,p'+q/2}, \tag{9}
\]

\[
\frac{\partial}{\partial \tau'} c_{p\uparrow}(\tau') = [K, c_{p\uparrow}] = -\xi_{p\uparrow} c_{p\uparrow} - \lambda \sum_{p'} \sum_{p''} \sum_q c^\dagger_{-p'+q/2\downarrow}c_{-p''-q/2\downarrow}c_{p''+q/2\uparrow} \delta_{p,p'+q/2}, \tag{10}
\]

and so on. Thus \(\text{(5)}\) leads to

\[
J^Q(k=0) = \sum_p \sum_\sigma \sum_\sigma \xi_p V_p c^\dagger_{p\sigma} c_{p\sigma} \]

\[
+ \frac{\lambda}{2} \sum_p \sum_{p'} \sum_q \left( V_{p'\uparrow+q/2} + V_{p'\downarrow+q/2} + V_{-p'\uparrow+q/2} + V_{-p'\downarrow+q/2} \right) \]

\[
\times c^\dagger_{p'+q/2\uparrow}c^\dagger_{-p'+q/2\downarrow}c_{-p''-q/2\downarrow}c_{p''+q/2\uparrow}. \tag{11}
\]

This quartic interaction term in the heat current, which does not exist in the quadratic charge current, makes the following discussions complicated.
Figure 1: Diagrams for un-renormalized current vertex. In the case of heat current the right-hand side of (11) is represented by these two diagrams. In the case of charge current (2) is represented only by the left diagram.

3 Diagrammatics for Charge Vertex

Let us consider the Ward identity for the charge current vertex in terms of electron-boson interaction

\[ V_{boson} = \lambda \sum_{p'} \sum_q \sum_\sigma \chi_q c_{p'+q,\sigma} c_{p',\sigma}, \]  

(12)

\[ V_{phonon} = \sum_{p'} \sum_q \sum_\sigma \gamma_q \varphi_{p'+q,\sigma} c_{p',\sigma}, \]

with

\[ \varphi_q = a_q + a_{-q}^\dagger, \]

which is the operator for the absorption/emission of the phonon.

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4 This interaction has the same structure as the electron-phonon interaction. See, for example, (4-26) and (4-27) in [2]. Neglecting the polarization of the phonon the interaction is expressed as

\[ V_{phonon} = \sum_{p'} \sum_q \sum_\sigma \gamma_q \varphi_{p'+q,\sigma} c_{p',\sigma}, \]
shown in Fig. 2 (left). The interaction between two electrons is represented by the exchange of the boson. The self-energy $\Sigma$ of electrons is given as

$$\Sigma(p) = -\lambda \sum_q D(q)G(p-q)\Lambda(p-q, p). \quad (13)$$

Figure 2: Diagrams for electron-boson interaction (left) and self-energy (right). The line with an arrow represents the free propagator $G_0$ (thin) or the full propagator $G$ (thick) for electrons. The wavy line represents the free propagator $D_0$ (thin) or the full propagator $D$ (thick) for bosons. The shaded circle is the full interaction vertex $\Lambda$.

The Ward identity for electron-boson system is thoroughly clarified in the study of QED.

The four-divergence of the free vertex satisfies

$$\sum_{\mu=0}^{3} k_{\mu}\gamma_{\mu}^e(p + k, p) = eG_0(p)^{-1} - eG_0(p + k)^{-1}. \quad (14)$$

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5 Using the faithful representation mentioned in the footnote 13 of [II]

$$-\Sigma(p) = [-\lambda] \sum_q [-D(q)][-G(p-q)][-\Lambda(p-q, p)].$$

6 We use the thermal Green functions

$$G(p, \tau) = -\langle T_{\tau}\{c_{p\sigma}(\tau)c_{p\sigma}^\dagger\} \rangle,$$

for electrons and

$$D(q, \tau) = -\langle T_{\tau}\{\chi_{q}(\tau)\chi_{-q}\} \rangle,$$

for bosons. The zeroth component of the four-momentum is given by $p_0 = -i\varepsilon_n$ or $q_0 = -i\omega_m$ as in [I]. The summation $\sum_q$ means $\sum_m \sum_q$.

7 The free vertex is given as $\gamma_0^e(p + k, p) = e$ and $\gamma_\mu^e(p + k, p) = \frac{e}{m}(p_\mu + k_\mu)$ for
The four-divergence of the full vertex satisfies the same relation as the free vertex
\[ \sum_{\mu=0}^{3} k_{\mu} \Gamma_{\mu}^e(p + k, p) = eG(p)^{-1} - eG(p + k)^{-1}. \]  
This relation for the full vertex is the Ward identity. If we decompose the full vertex as
\[ \Gamma_{\mu}^e(p + k, p) = \gamma_{\mu}^e(p + k, p) + \tilde{\Gamma}_{\mu}^e(p + k, p), \]
the interaction part satisfies
\[ \sum_{\mu=0}^{3} k_{\mu} \tilde{\Gamma}_{\mu}^e(p + k, p) = e\Sigma(p + k) - e\Sigma(p). \]
Almost all of the contributions to the four-divergence of the full vertex cancel out so that only end-diagrams in Fig. 4 contain non-canceling contributions. The cancelation results from the four-divergence of the free vertex multiplied by \(G_0(p)G_0(p + k)\)
\[ G_0(p)\left[ \sum_{\mu=0}^{3} k_{\mu} \gamma_{\mu}^e(p + k, p) \right] G_0(p + k) = eG_0(p + k) - eG_0(p), \]
\(\mu = 1, 2, 3\) so that
\[ \sum_{\mu=0}^{3} k_{\mu} \gamma_{\mu}^e(p + k, p) = e\left[ k_0 + \frac{1}{m} k \cdot (p + \frac{k}{2}) \right]. \]
One the other hand,
\[ G_0(p)^{-1} - G_0(p + k)^{-1} = k_0 + \frac{1}{m} k \cdot (p + \frac{k}{2}), \]
since
\[ G_0(p)^{-1} = -p_0 - \xi_p. \]
where the left-hand side is depicted as the left diagram in Fig. 5.

![Diagrams](image)

**Figure 4:** End-diagrams which contain the surviving contributions against the cancellation. The left end of each diagram links to the external electron propagator $G(p)$ and the right end to $G(p + k)$.

Only the contribution whose end electron propagator, which directly links to the external electron propagator (not shown in the figures) couples to the external field, has no canceling partner. The left end-propagator which couples to the external field shown in Fig. 5 is replaced by $eG(p + k - q)$ shown in Fig. 6 after taking the four-divergence.

Thus the non-canceling contribution of the four-divergence of the left end-diagram in Fig. 4 becomes $e\Sigma(p + k)$. By the same way the right end-diagram in Fig. 4 gives $-e\Sigma(p)$. Consequently we obtain (17).

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8 See, for example, Peskin and Schroeder: *An Introduction to Quantum Field Theory* (Westview, Boulder, 1995). All the internal electron propagators in $\Gamma^e$ are classified into two types. One is the line and the other is the loop. Elementary canceling pair for the line is shown in Fig. 4. Such a pair for the loop is shown in Fig. 8. While the end-contributions remain from the line process, the similar contributions from the loop process cancel out after integrating over the internal variable of the loop. Thus we consider only end-contributions from the line process.
Figure 5: Left end-propagator. It is a part of the propagator coupling to external field used in the left diagram in Fig. 4. Only the end free-propagator $G_0(p-q)$, which directly links to the external propagator $G_0(p)$, couples to external field in the left end-propagator. The gray circle is the electron self-energy $\Sigma$. Taking the four-divergence and using (18) we obtain two types of contributions. One which has $-G_0(p-q)$ at its left end is canceled if it is used in Fig. 4. An example of the cancelation is shown in Fig. 7. The other which has $G_0(p+k-q)$ at its left end remains, because it has no canceling partner.

Figure 6: Remaining contribution of the left end-propagator in Fig. 5 after taking the four-divergence. Every propagator carries the four-momentum $p+k-q$. Via Dyson equation it is identical to $G(p+k-q)$.

Figure 7: Elementary cancelation process between the canceling pair. The incoming electron, outgoing electron and boson carry $p$, $p+k-q$ and $q$ respectively. After taking the four-divergence the contribution of the left diagram reduces to $-\lambda e \left[ G_0(p+k)-G_0(p) \right] D_0(q)G_0(p+k-q)$ and the right to $-\lambda e G_0(p)D_0(q) \left[ G_0(p+k-q)-G_0(p-q) \right]$ by (18). The terms proportional to $G_0(p)D_0(q)G_0(p+k-q)$ cancel out each other. The other terms remain as the end-contribution.
4 NCA for Charge Vertex

Since the interaction vertex $\Lambda$ in Fig. 2 plays no crucial role to establish (15) as seen in the previous section, the Ward identity holds within the non-crossing approximation (NCA)\(^9\) which neglects the vertex correction to the electron-boson coupling ($\Lambda \rightarrow \lambda$).

\[
\Gamma_{\mu}^e(p+k,p) = \gamma_{\mu}^e(p+k,p) - \lambda^2 \sum_{p'} G(p'+k)G(p')D(p-p')\Gamma_{\mu}^e(p'+k,p'). \tag{19}
\]

The iterative solution of (19) reduces to the ladder series.

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\(^9\) The discussion in this section is based on [2].
\(^{10}\) The summation $\sum_{p'}$ means $T\sum_{\nu} \sum_{p'}$ as the footnote for (13).
To see the consistency between the Ward identity (15) and the ladder approximation (19), we use
\[ \sum_{\mu=0}^{3} k_\mu \Gamma_\mu^e(p' + k, p') = eG(p')^{-1} - eG(p' + k)^{-1}, \] (20)
in the four-divergence of (19) and obtain
\[ \sum_{\mu=0}^{3} k_\mu \Gamma_\mu^e(p + k, p) = \sum_{\mu=0}^{3} k_\mu \gamma_\mu^e(p + k, p) - \lambda^2 e \sum_{p'} G(p' + k)D(p - p') + \lambda^2 e \sum_{p'} G(p')D(p - p'). \] (21)

Using (14) and the self-energy in NCA
\[ \Sigma(p) = -\lambda^2 \sum_{p'} G(p')D(p - p'), \] (22)
(21) is written as
\[ \sum_{\mu=0}^{3} k_\mu \Gamma_\mu^e(p + k, p) = eG_0(p)^{-1} - eG_0(p + k)^{-1} + e\Sigma(p + k) - e\Sigma(p). \] (23)

Using the Dyson equation
\[ G(p)^{-1} = G_0(p)^{-1} - \Sigma(p), \] (24)
it is proven that the ladder approximation for the charge current vertex (19) satisfies the Ward identity (15). Thus the skeleton behavior of the Ward identity is seen in NCA.

5 NCA for Heat Vertex

Since we can learn the skeleton behavior of the Ward identity from NCA as seen in the previous section, we investigate the integral equation for the heat current vertex within NCA\footnote{The discussion in this section is based on \cite{3}.}.

Un-renormalized current vertexes depicted in Fig. 1 are replaced by Fig. 10 in the case of electron-boson interaction.
Figure 10: Diagrams for un-renormalized current vertex $\gamma^Q_{\mu}$ (left) and $\alpha^Q_{\mu}$ (right) in the case of electron-boson interaction. In the left diagram the incoming electron propagator is $G(p)$ and the outgoing one is $G(p + k)$. In the right diagram the pair of incoming and outgoing electron propagators is $G(p)G(p + k - q)$ or $G(p - q)G(p + k)$.

Since

$$j^Q(-k) = \frac{1}{m} \sum_p \sum_\sigma \left( p + \frac{k}{2} \right) \left[ \frac{\xi_p + \xi_{p+k}}{2} c^\dagger_{p+k\sigma} c_{p\sigma} + \frac{\lambda}{2} \sum_q (\chi_q c^\dagger_{p+k-q\sigma} c_{p\sigma} + \chi_q c^\dagger_{p+k\sigma} c_{p-q\sigma}) \right],$$

(25)

the vertexes are given as

$$\gamma^Q_{\mu}(p + k, p) = \frac{1}{m} \left( p_\mu + \frac{k_\mu}{2} \right) \left[ \frac{\xi_p + \xi_{p+k}}{2} \right] c^\dagger_{p+k\sigma} c_{p\sigma}, \quad \alpha^Q_{\mu} = \frac{\lambda}{2m} \left( p_\mu + \frac{k_\mu}{2} \right),$$

(26)

for $\mu = 1, 2, 3$. For the zeroth component in the limit of $k \to 0$

$$\gamma^Q_0(p + k, p) = \xi_p, \quad \alpha^Q_0 = \lambda,$$

(27)

The Fourier transform of (88) in [I] leads to

$$j^Q(-k) = \lim_{\tau' \to \tau} \frac{1}{2} \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'} \right) \frac{1}{m} \sum_p \left( p + \frac{k}{2} \right) c^\dagger_{p+k\sigma}(\tau) c_{p\sigma}(\tau').$$

In the case of $K = K_0 + V_{\text{boson}}$ the time-derivatives are calculated as

$$\left[ K, c^\dagger_{p+k\sigma} \right](\tau) = \xi_{p+k} c^\dagger_{p+k\sigma} + \lambda \sum_q \chi_q c^\dagger_{p+k+q\sigma},$$

$$\left[ K, c_{p\sigma} \right](\tau') = -\xi_p c_{p\sigma} - \lambda \sum_q \chi_q c_{p-q\sigma}.$$
from (67) in [I].

Even within NCA we have to consider the contributions, besides the ladder series in Fig. 9 depicted in Fig. 11 in the integral equation for the vertex.

![Diagram](image)

**Figure 11:** Non-crossing contributions to the heat current vertex besides the ladder series. The process at the left vertex is $\chi_{q}c_{p+k}^{\dagger}c_{p-\sigma}$. The process at the right vertex is $\chi_{q}c_{p+k}^{\dagger}c_{p-\sigma}$.

Thus the integral equation becomes

$$
\Gamma_{\mu}^{Q}(p + k, p) = \gamma_{\mu}^{Q}(p + k, p) - \lambda^{2} \sum_{p'} G(p' + k)G(p')D(p - p')\Gamma_{\mu}^{Q}(p' + k, p')
$$

$$
- \alpha_{\mu}^{Q} \lambda \sum_{q} [G(p + k - q)D(q) + G(p - q)D(q)].
$$

Taking the four-divergence and using the Ward identity

$$
\sum_{\mu=0}^{3} k_{\mu} \Gamma_{\mu}^{Q}(p' + k, p') = p_{0}'G(p' + k)^{-1} - (p_{0}' + k_{0})G(p')^{-1},
$$

(28) and (27) we obtain

$$
\sum_{\mu=0}^{3} k_{\mu} \Gamma_{\mu}^{Q}(p + k, p) = k_{0}\xi_{p} + \frac{1}{2m}k \cdot \left( p + k \right) \left[ \xi_{p} + \xi_{p+k} + \Sigma(p) + \Sigma(p + k) \right]
$$

$$
+ k_{0}\left[ \Sigma(p) + \Sigma(p + k) \right] - \lambda^{2} \sum_{p'} D(p - p') \left[ p_{0}'G(p') - (p_{0}' + k_{0})G(p' + k) \right],
$$

(30)
where the self-energy is given by (22).

Using the Jonson-Mahan transmutation \[1\]

\[ \xi_p + \Sigma(p) = -p_0 - G(p)^{-1}, \]  

(31)

the first line of the right-hand side of (30) is written as

\[ k_0 \xi_p - \frac{1}{2m} \mathbf{k} \cdot \left( \mathbf{p} + \frac{\mathbf{k}}{2} \right) [2p_0 + k_0 + G(p)^{-1} + G(p + k)^{-1}], \]  

(32)

and \( k_0 \) in [ ] is negligible in the limit of \( k_0 \to 0 \). We expect that this line is equal to

\[ p_0 G_0(p + k)^{-1} - (p_0 + k_0) G_0(p)^{-1} = k_0 \xi_p - \frac{1}{m} \mathbf{k} \cdot \left( \mathbf{p} + \frac{\mathbf{k}}{2} \right) p_0. \]  

(33)

But there is no counter term to cancel out the term proportional to \( G(p)^{-1} + G(p + k)^{-1} \) in (32) within NCA. Since it does not contribute to the conductivity, this unnecessary term was neglected in \[1\]. However, we should find the counter term to establish the Ward identity. This task is accomplished by the exact diagrammatics in the next section. In the following discussion within this section we neglect this unnecessary term.

It should be noted that the relation

\[ \sum_{\mu=0}^{3} k_\mu \gamma_\mu^Q(p + k, p) = p_0 G_0(p + k)^{-1} - (p_0 + k_0) G_0(p)^{-1}, \]  

(34)

that is the same relation as satisfied by the full quantities (29), does not hold\(^{13}\) in contrast to \[14\]. Actually

\[ \sum_{\mu=0}^{3} k_\mu \gamma_\mu^Q(p + k, p) = k_0 \xi_p + \frac{1}{2m} \mathbf{k} \cdot \left( \mathbf{p} + \frac{\mathbf{k}}{2} \right) (\xi_p + \xi_{p+k}). \]  

(35)

\(^{13}\) This fact is not widely recognized. Even in \[3\] (34) is assumed so that the definition of \( \gamma_\mu^Q \) (4-13a) is incorrect. Since we consider the limit of \( k \to 0 \), neglecting the difference between \( \xi_{p+k} \) and \( \xi_p \) (35) gives

\[ \sum_{\mu=0}^{3} k_\mu \gamma_\mu^Q(p + k, p) = \xi_p \left[ k_0 + \frac{1}{m} \mathbf{k} \cdot \left( \mathbf{p} + \frac{\mathbf{k}}{2} \right) \right] = \xi_p [G_0(p)^{-1} - G_0(p + k)^{-1}]. \]

This relation is parallel to

\[ \sum_{\mu=0}^{3} k_\mu \gamma_\mu^Q(p + k, p) = \epsilon [G_0(p)^{-1} - G_0(p + k)^{-1}]. \]
The right-hand side of (33) results from (35) plus the contribution depicted in Fig. 11.

The last term in the right-hand side of (30) is written as

\[ p_0 \Sigma(p) - (p_0 + k_0) \Sigma(p + k) + \lambda^2 \sum_q \left[ q_0 D(q) - (q_0 + k_0) D(q + k) \right] G(p - q), \]  

(36)

where the self-energy is given by (22). Since we need only \( p_0 \Sigma(p) - (p_0 + k_0) \Sigma(p + k) \) in (36), we should have included

\[ -\lambda^2 \sum_q D(q) D(q + k) G(p - q) \Delta^Q_\mu(q + k, q), \]  

(37)

in (28). The vertex for bosons \( \Delta^Q_\mu \) in NCA is depicted in Fig. 12-(left) and (37) is depicted in Fig. 12-(right). Even if the direct coupling of bosons to external field is absent, the intermediate electron-hole pair state in the boson propagator couples to external field as depicted in Fig. 8 so that non-zero \( \Delta^Q_\mu \) arises.

By putting above considerations together we manage to reach the expected form of the Ward identity

\[ \sum_{\mu=0}^{3} k_\mu \Gamma^Q_\mu(p + k, p) = p_0 G_0(p + k)^{-1} - (p_0 + k_0) G_0(p)^{-1} - p_0 \Sigma(p + k) + (p_0 + k_0) \Sigma(p) \]

\[ = p_0 G(p + k)^{-1} - (p_0 + k_0) G(p)^{-1}. \]  

(39)

14 In [3] a coupled equation for \( \Gamma^Q_\mu \) and \( \Delta^Q_\mu \) is discussed in the case of electron-phonon system. See Fig. 2 there.

15 Assuming that \( \Delta^Q_\mu \) in NCA satisfies the Ward identity, (144) in [I],

\[ \sum_{\mu=0}^{3} k_\mu \Delta^Q_\mu(q + k, q) = q_0 D(q + k)^{-1} - (q_0 + k_0) D(q)^{-1}, \]

the four-divergence of (37) reduces to

\[ -\lambda^2 \sum_q \left[ q_0 D(q) - (q_0 + k_0) D(q + k) \right] G(p - q), \]  

(38)

and cancels out the last term in (36).

16 In the case of charge current vertex the coupling of bosons to external field vanishes, because the boson is charge neutral and does not carry charge. On the other hand, bosons carry heat so that \( \Delta^Q_\mu \) is non-zero. See arXiv:1109.1404 for detail.
Figure 12: Heat current vertex for bosons (left) and vertex correction for electrons (right). Both processes vanish in the case of charge current vertex, since the boson is charge neutral and does not carry charge.

However, this discussion is incomplete, since I have shut my eyes to the unnecessary term.  

6 Diagrammatics for Heat Vertex

Now the skeleton behavior of the heat current vertex has been clarified in NCA. However, the unnecessary term remains so that the Ward identity is not completely satisfied within NCA. Thus we shall show the complete explanation of the Ward identity by exact diagrammatics in this section.

In the case of the line-diagram we should add the processes in Fig. 13 to the canceling pair in Fig. 7 to form the canceling quartet.

Figure 13: Processes necessary to form the canceling quartet.

\[ ^{17} \text{Although is written under the assumption that the Ward identity for the heat current holds within NCA, it is not the case.} \]
The renormalized vertex $\tilde{\gamma}^Q_\mu$ defined as the sum of the free vertex $\gamma^Q_\mu$ and the processes in Fig. 11 with $\mu = 1, 2, 3$, satisfies

$$\sum_{\mu=0}^{3} k_\mu \tilde{\gamma}^Q_\mu (p + k, p) = p_0 G_0(p + k)^{-1} - (p_0 + k_0) G_0(p)^{-1}, \quad (40)$$

as the discussion in the previous section. Since the disposal of the unnecessary term is possible in the exact diagrammatics, (40) holds exactly. Corresponding to (18) the four-divergence of the coupling to external field in Fig. 7-(left) satisfies

$$G_0(p) \left[ \sum_{\mu=0}^{3} k_\mu \tilde{\gamma}^Q_\mu (p + k, p) \right] G_0(p + k) = p_0 G_0(p) - (p_0 + k_0) G_0(p + k), \quad (44)$$

when we use the renormalized vertex. By the same manner the renormalized coupling in Fig. 13-(left) satisfies

$$D_0(q) \left[ \sum_{\mu=0}^{3} k_\mu \tilde{\beta}^Q_\mu (q + k, q) \right] D_0(q + k) = q_0 D_0(q) - (q_0 + k_0) D_0(q + k). \quad (45)$$

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18 The self-energy in the skeleton analysis of NCA is (22) but the fully renormalized self-energy used in this section is (13).

19 By the renormalization procedure $\gamma^Q \to \tilde{\gamma}^Q$ an unnecessary term

$$\frac{1}{2m} k \cdot \left( p + \frac{k}{2} \right) G(p) \lambda D(q) G(p + k - q), \quad (41)$$

arises from Fig. 7-(left) via

$$- \frac{1}{2m} k \cdot \left( p + \frac{k}{2} \right) \left[ G(p)^{-1} + G(p + k)^{-1} \right] \frac{1}{2} \left( -1 \right)^4 G(p + k) \lambda D(q) G(p + k - q). \quad (42)$$

It is canceled out by the three-divergence of Fig. 13-(right)

$$\frac{\lambda}{2m} k \cdot \left( p + \frac{k}{2} \right) \left( -1 \right)^4 G(p) D(q) G(p + k - q). \quad (43)$$

Here I have used the full propagators $G$ and $D$. 

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After taking the four-divergence the process in Fig. 7-(left) gives
\[ -\lambda [p_0 G_0(p) - (p_0 + k_0)G_0(p + k)] D_0(q) G_0(p + k - q), \tag{46} \]
the process in Fig. 7-(right) gives
\[ -\lambda G_0(p) D_0(q) \left[ (p_0 - q_0) G_0(p - q) - (p_0 + k_0 - q_0) G_0(p + k - q) \right], \tag{47} \]
the process in Fig. 13-(left) gives
\[ -\lambda G_0(p) \left[ (q_0 - k_0) D_0(q - k) - q_0 D_0(q) \right] G_0(p + k - q), \tag{48} \]
and the process in Fig. 13-(right) gives
\[ -\lambda k_0 G_0(p) D_0(q) G_0(p + k - q). \tag{49} \]
Then the terms proportional to \( G_0(p) D_0(q) G_0(p + k - q) \) in these four equations cancel out.

Figure 14: Coupling of boson to external field.

\[ \text{Before taking the four-divergence the process in Fig. 7-(left) gives} \]
\[ \left[ -\lambda \right] \left[ -G_0(p) \right] \tilde{\gamma}_Q \left[ -G_0(p + k) \right] \left[ -D_0(q) \right] \left[ -G(p + k - q) \right], \]
and the process in Fig. 13-(right) gives
\[ \lambda \left[ -G_0(p) \right] \left[ -D_0(q) \right] \left[ -G_0(p + k - q) \right], \]
which results from the coupling \( \alpha_Q^2 \).

\[ \text{See Fig. 14 for a simple example. The term proportional to } q_0 D(q) \text{ forms the canceling} \]
\[ \text{quartet concerning the lower electron line. On the other hand, the term proportional to} \]
\[ (q_0 - k_0) D(q - k) \text{ forms the canceling quartet concerning the upper electron line.} \]
\[ \text{As discussed in the footnote for the disposal, the three-divergence is renormalized} \]
\[ \text{into } \tilde{\gamma}_Q \text{ so that the remaining is the zeroth component only.} \]
In the case of the loop-diagram the cancelation is not complete\textsuperscript{23} so that the external field couples to bosons via indirect processes as shown in Fig. 8.

Since the elementary cancelation mechanism has become clear by the above explanation, we can discuss the Ward identity by the same procedure as in the case of the charge current vertex. Namely, we only have to consider the end-contributions\textsuperscript{24} to establish the relation corresponding to (17). The end-contribution from the left diagram in Fig. 4 is

$$-(p_0 + k_0)\Sigma(p + k) + \lambda \sum_q q_0 D(q) G(p + k - q) \Lambda(p + k - q, p + k), \quad (50)$$

and the one from the right diagram is

$$p_0 \Sigma(p) - \lambda \sum_q q_0 D(q) G(p - q) \Lambda(p, p - q). \quad (51)$$

The terms proportional to $q_0$ in these contributions are canceled out\textsuperscript{25} by the end-contributions in terms of boson propagator;

$$- \lambda \sum_q (q_0 + k_0) D(q + k) G(p - q) \Lambda(p - q, p + k), \quad (52)$$

from the process in Fig. 15 (left) and

$$\lambda \sum_q q_0 D(q) G(p - q) \Lambda(p, p - q), \quad (53)$$

from the process in Fig. 15 (right).

\textsuperscript{23} For example, when we discuss the cancelation between two processes in Fig. 8 both left and right bosons, which couple to electron loop, can form the canceling quartet and the excess contributions survive against the cancelation. See also the footnote for non-zero $\Delta Q$.

\textsuperscript{24} The contribution via the violation of the loop cancelation is taken into account by the coupling of bosons to external field. Such a coupling is absent in the case of charge current because of the loop cancelation.

\textsuperscript{25} To see the cancelation the second term in (50) is written as

$$\lambda \sum_q (q'_0 + k_0) D(q' + k) G(p - q') \Lambda(p - q', p + k),$$

by the variable change $q - k \rightarrow q'$. 

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Thus the sum of (50), (51), (52) and (53) results in
\[ p_0 \Sigma(p) - (p_0 + k_0) \Sigma(p + k). \] (54)
The end-contributions in terms of \( \alpha^Q_\mu \) depicted in Fig. 16 are\(^{26}\)
\[ k_0 \left[ \Sigma(p) + \Sigma(p + k) \right]. \] (55)

From (54) plus (55) we obtain
\[ (p_0 + k_0) \Sigma(p) - p_0 \Sigma(p + k). \] (56)
From (10) and (56) we obtain the Ward identity
\[ \sum_{\mu=0}^{3} k_\mu \Gamma^Q_\mu(p + k, p) = p_0 \left[ G_0(p + k)^{-1} - \Sigma(p + k) \right] - (p_0 + k_0) \left[ G_0(p)^{-1} - \Sigma(p) \right], \] (57)
for heat current vertex. It should be noted that the interacting process in Fig. 11 plays a dual role. One is to establish (10) and the other (56). Thus (10) does not hold for free electrons.

\(^{26}\) Since the three-divergence is already renormalized into \( \gamma^Q \), we only consider the zeroth component.
7 Remarks

The sections of Exercise and Acknowledgements are common to [I] so that I do not repeat here. Some typographic errors in [I] have been listed in the section of Remarks in [II]. The errors in [II] should be corrected as follows.

- The first line of §4 should be “The linearized GL transport theory in 2D · · · ”.
- The first line of §5 should be “The calculations for electrons in §3 are translated into those [2] for Cooper pairs straightforwardly.”

Exploiting this opportunity I add the following explanation to (147) and (149) in [I]. When we put $k_0 = 0$

$$\sum_{\mu=1}^3 k_\mu \Gamma_\mu^\omega \equiv e \left\{ [\xi_{p+k} + \Sigma'(p + k)] - [\xi_p + \Sigma'(p)] \right\} \equiv e \frac{e}{m^*} \mathbf{k} \cdot \mathbf{p},$$

where $\Sigma'(p)$ is the real part of $\Sigma(p)$ and $\xi_p + \Sigma'(p) \equiv \frac{\mathbf{p}^2}{2m^*} - \mu^*$. Thus we obtain (147) in [I]. Similarly

$$\sum_{\mu=1}^3 k_\mu \Gamma_\mu^Q \equiv -p_0 \left\{ [\xi_{p+k} + \Sigma'(p + k)] - [\xi_p + \Sigma'(p)] \right\} \equiv -\frac{p_0}{m^*} \mathbf{k} \cdot \mathbf{p},$$

leads to (149) in [I] with $k_0 = 0$.

References

[1] Jonson and Mahan: Phys. Rev. B 21, 4223 (1980).
[2] Schrieffer: Theory of Superconductivity (Benjamin-Cummings, Massachusetts, 1964).
[3] Ono: Prog. Theor. Phys. 46, 757 (1971).