RATIONALITY OF THE HILBERT SERIES
OF HOPF-INVARIANTS OF FREE ALGEBRAS

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ABSTRACT. It is shown that the subalgebra of invariants of a free associative algebra of finite rank under a linear action of a semisimple Hopf algebra has a rational Hilbert series with respect to the usual degree function whenever the ground field has zero characteristic.

INTRODUCTION

Invariants of free algebras under actions of linear automorphisms have been considered by many authors; e.g. in [La76] and [Kh78] it is proved that if $k\langle X\rangle$ is a free associative algebra and $G$ is a group of linear (therein called homogeneous) automorphisms of $k\langle X\rangle$, then the subalgebra of invariants $k\langle X\rangle^G$ is also free. Moreover, when $X$ is finite, $k\langle X\rangle^G$ is finitely generated as an algebra if, and only if, the elements of $G$ are scalar automorphisms, as shown in [DF82] and [Kh84]. Similar results have been obtained for the subalgebra of constants of Lie algebras of linear derivations of a free algebra (cf. [Jo78], [Kh81], [FM06]).

A natural path of investigation was to consider possible generalizations of the above results to linear Hopf algebra actions on free algebras. The subalgebra of invariants $k\langle X\rangle^H$ of a free algebra $k\langle X\rangle$ under a linear action of a Hopf algebra $H$ was shown to be free (cf. [FMP04]), and Koryukin in [Ko94] produced a criterion for $k\langle X\rangle^H$ to be finitely generated. When $H$ is finite-dimensional and generated by grouplikes and skew-primitives, a simpler criterion, similar to the one in [DF82, Kh84], was presented in [FM07]. Moreover, in [FMP04], a Galois correspondence (inspired by [Kh78]) between right coideal subalgebras of $H$ and free subalgebras of $k\langle X\rangle$ containing $k\langle X\rangle^H$ has been described.

The object under analysis in this note is the Hilbert series of the subalgebra of invariants $k\langle X\rangle^H$, for $H$ a semisimple Hopf algebra. Since a linear action $H$ on $k\langle X\rangle$ is homogeneous, the subalgebra $k\langle X\rangle^H$ inherits the grading of $k\langle X\rangle$ by the usual degree. Therefore, if $X$ is finite, the Hilbert series of $k\langle X\rangle$ with respect to this grading is defined. We show that for a field $k$ with char $k = 0$, this Hilbert series is a rational function. The proof of this fact provides an explicit formula for

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this rational function in terms of characters of the action of $H$ on the subspace generated by $X$. We are eventually able to obtain Dicks-Formanek’s formula for the case of finite groups of automorphisms of $k\langle X \rangle$ in [DF82] as a special case.

Hereafter vectorspaces and algebras will be defined over a field $k$; tensor products are taken over $k$. The symbol $\mathbb{N}$ will denote the set of nonnegative integers and $\mathbb{Q}$ the field of rational numbers.

We refer the reader to [DNR01] or [Mo93] for the usual notation, definitions and general facts on Hopf algebra theory. In particular, we shall denote the comultiplication and counit maps of a Hopf algebra by $\Delta$ and $\varepsilon$, respectively, and we shall adopt Sweedler’s notation $\Delta(h) = \sum (h) h(1) \otimes h(2)$ for an element $h$ in a Hopf algebra.

1. Graded module algebras

Given an algebra $H$ and an $H$-module $W$, we let

$$\rho_W : H \rightarrow \text{End} W$$

denote the algebra map that affords the $H$-module structure on $W$; that is, $\rho_W$ is defined by

$$\rho_W(h)(w) = h \cdot w, \quad \text{for all } h \in H \text{ and } w \in W.$$

If $W$ is finite-dimensional, denote by $\chi_W$ the character of $W$; that is, $\chi_W : H \rightarrow k$ is the linear map given by

$$\chi_W(h) = \text{tr}(\rho_W(h)), \quad \text{for all } h \in H,$$

where $\text{tr}$ stands for the trace of a linear endomorphism.

Now suppose that $H$ is a Hopf algebra. In this case, $W^H = \{ w \in W : h \cdot w = \varepsilon(h)w, \text{ for all } h \in H \}$ is an $H$-submodule of $W$.

We shall make use of characters of modules over a semisimple Hopf algebra. For that, we recall some facts on semisimple Hopf algebras. (See [DNR01] for proofs.) A Hopf algebra $H$ is said to be semisimple if it is semisimple as an algebra. An element $t$ of a Hopf algebra $H$ is called a left integral in $H$ if $ht = \varepsilon(h)t$, for all $h \in H$. Semisimple Hopf algebras can be characterized in terms of integrals via the following version of Maschke’s theorem: a Hopf algebra $H$ is semisimple if and only if there exists a left integral $t$ in $H$ with $\varepsilon(t) = 1$. Moreover, semisimple Hopf algebras are finite-dimensional. Because left integrals with counit 1 in a semisimple Hopf algebra are central idempotent elements, we have the following (easily proved) well-known facts.

**Proposition 1.** Let $H$ be a semisimple Hopf algebra, let $t \in H$ be a left integral with $\varepsilon(t) = 1$ and let $W$ be a finite-dimensional $H$-module. Then $\rho_W(t)$ is an $H$-module endomorphism of $W$ satisfying $\rho_W(t)^2 = \rho_W(t)$ and $\text{im} \rho_W(t) = W^H$. In particular, if $\text{char } k = 0$, then $\dim W^H = \chi_W(t)$. □

We now specialize to Hopf algebra actions on graded algebras.

An algebra $A$ is said to be graded if there exist subspaces $A_n$, for $n \in \mathbb{N}$, satisfying

$$A = \bigoplus_{n \in \mathbb{N}} A_n \quad \text{and} \quad A_m A_n \subseteq A_{m+n}, \text{ for all } m, n \in \mathbb{N}.$$
When \( \dim A_n \) is finite, for all \( n \in \mathbb{N} \), we associate to the grading of \( A \) a formal power series with integer coefficients, called the Hilbert (or Poincaré) series of \( A \), defined by

\[
P(A, z) = \sum_{n \in \mathbb{N}} (\dim A_n) z^n \in \mathbb{Z}[[z]].
\]

Now let \( H \) be a Hopf algebra and suppose that the graded algebra \( A \) is an \( H \)-module algebra. We say that the action of \( H \) on \( A \) is homogeneous with respect to the grading if \( H \cdot A_n \subseteq A_n \), for all \( n \in \mathbb{N} \), or, equivalently, if \( A_n \) is an \( H \)-submodule of \( A \), for all \( n \in \mathbb{N} \). In this case, \( A^H \) is a subalgebra of \( A \) which inherits the grading from \( A \):

\[
A^H = \bigoplus_{n \in \mathbb{N}} (A_n \cap A^H).
\]

Hence, it makes sense to speak of the Hilbert series \( P(A^H, z) \) of \( A^H \); since \( A_n \cap A^H = A^H_n \), it follows that \( P(A^H, z) = \sum_{n \in \mathbb{N}} (\dim A^H_n) z^n \). We shall make use of Proposition \[ \square \] in order to determine the coefficients of this series. In the above context we shall denote the character of the \( H \)-module \( A_n \) by, simply, \( \chi_n \). So, applying Proposition \[ \square \] to the \( H \)-modules \( A_n \), we obtain the following.

Corollary 2. Let \( H \) be a semisimple Hopf algebra and let \( t \) be a left integral in \( H \) with \( \varepsilon(t) = 1 \). Let \( A = \bigoplus_{n \in \mathbb{N}} A_n \) be a graded \( H \)-module algebra with a homogeneous action of \( H \) and suppose that \( \dim A_n \) is finite for all \( n \in \mathbb{N} \). Then \( P(A^H, z) = \sum_{n \in \mathbb{N}} \chi_n(t) z^n \). \[ \square \]

2. Linear actions

In this section, we investigate the Hilbert series of invariants of tensor algebras under homogeneous actions of a semisimple Hopf algebra.

Let \( H \) be Hopf algebra and let \( V \) be an \( H \)-module. Let \( T(V)_0 = k \), viewed as the trivial \( H \)-module, and for \( n \in \mathbb{N}, n \geq 1 \), let \( T(V)_n = V \otimes T(V)_{n-1} \). The \( H \)-module structure on \( V \) induces an \( H \)-module structure on \( T(V)_n \) such that

\[
h \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_{(h)} (h_1) \cdot v_1 \otimes \cdots \otimes (h_n) \cdot v_n
\]

for \( h \in H \) and \( v_i \in V \) for \( i = 1, \ldots, n \).

Now let

\[
T(V) = \bigoplus_{n \in \mathbb{N}} T(V)_n
\]

be the tensor algebra of \( V \). It is an \( H \)-module; in fact, \( T(V) \) is an \( H \)-module algebra. When the tensor algebra of a vector space \( V \) has an \( H \)-module algebra structure which is induced by an \( H \)-module structure on \( V \), we shall call the action of \( H \) on \( T(V) \) linear.

The tensor algebra \( T(V) \) is, by definition, graded by the tensor powers of \( V \), and a linear action of a Hopf algebra \( H \) on \( T(V) \) is homogeneous. Therefore, we can apply the ideas of the previous section to this case and look into the Hilbert series of the subalgebra of invariants \( T(V)^H = \bigoplus_{n \in \mathbb{N}} T(V)_n^H \) of \( T(V) \) under the action of \( H \).

Remark. Once a basis \( X \) of \( V \) is fixed, we have a natural isomorphism between \( T(V) \) and the free associative algebra \( k(X) \) on \( X \) over \( k \). So if \( \dim V = d \), then \( T(V) \) is a free algebra of rank \( d \) over \( k \). Under this viewpoint a linear action of a
Hopf algebra $H$ on $T(V)$ is just an $H$-module algebra structure on $k\langle X \rangle$ such that for each free generator $x \in X$ and each $h \in H$, there exist scalars $\lambda_y$, $y \in X$, a finite number of which nonzero, such that $h \cdot x = \sum_{y \in X} \lambda_y y$.

We shall present conditions under which the Hilbert series of $T(V)^H$ is a rational function.

Let $H$ be a Hopf algebra, let $V$ be a finite-dimensional vector space and suppose that $T(V)$ is an $H$-module algebra with a linear action of $H$. For each $h \in H$, we shall denote by $\alpha_h(z) \in k[[z]]$ the power series defined by

$$\alpha_h(z) = \sum_{n \in \mathbb{N}} \chi_n(h)z^n,$$

where $\chi_n$ denotes the character of the $H$-module $T(V)_n$.

**Lemma 3.** Let $H$ be a finite-dimensional Hopf algebra with basis $\{h_1, \ldots, h_r\}$. For each $i = 1, \ldots, r$, denote $\alpha_i(z) = \alpha_{h_i}(z)$. Then $\alpha_i(z) \in k(z)$, for every $i = 1, \ldots, r$.

**Proof.** For each $i = 1, \ldots, r$, there exist uniquely determined $a_{ij} \in H$, $j = 1, \ldots, n$, such that

$$\Delta(h_i) = \sum_{j=1}^r a_{ij} \otimes h_j.$$

Let $\lambda_{ij} = \chi_1(a_{ij}) \in k$. So $\chi_0(h_i) = \varepsilon(h_i)$, and for $n \in \mathbb{N}$, $n \geq 1$, we have

$$\chi_n(h_i) = \sum_{j=1}^r \chi_1(a_{ij}) \chi_{n-1}(h_j) = \sum_{j=1}^r \lambda_{ij} \chi_{n-1}(h_j).$$

This is because, being characters, the $\chi_n$ are multiplicative, in the sense that, for all $h \in H$, $\chi_n(h) = \sum_{h_1, h_2} \chi_1(h_1) \chi_{n-1}(h_2)$, for, by definition, $T(V)_n = T(V)_1 \otimes T(V)_{n-1}$ and the action of $H$ on $T(V)_n$ is induced by its action on $T(V)_1$ and on $T(V)_{n-1}$ through $\Delta$.

Therefore,

$$\alpha_i(z) = \sum_{n \in \mathbb{N}} \chi_n(h_i)z^n = \varepsilon(h_i) + \sum_{n \geq 1} \chi_n(h_i)z^n$$

$$= \varepsilon(h_i) + \sum_{n \geq 1} \sum_{j=1}^r \lambda_{ij} \chi_{n-1}(h_j)z^n$$

$$= \varepsilon(h_i) + \sum_{j=1}^r \lambda_{ij} z \sum_{n \geq 1} \chi_{n-1}(h_j)z^{n-1}$$

$$= \varepsilon(h_i) + \sum_{j=1}^r \lambda_{ij} z \alpha_j(z).$$

It follows that, for each $i = 1, \ldots, r$, we have

$$(\lambda_{ii}z - 1)\alpha_i(z) + \sum_{j \neq i} \lambda_{ij} z \alpha_j(x) = -\varepsilon(h_i).$$
That is, \((\alpha_1(z), \ldots, \alpha_r(z))\) is a solution of the linear system \(M(z)X = \eta\) over \(\mathbb{k}(z)\), where
\[
M(z) = \begin{bmatrix}
\lambda_{11}z - 1 & \lambda_{12}z & \cdots & \lambda_{1r}z \\
\lambda_{21}z & \lambda_{22}z - 1 & \cdots & \lambda_{2r}z \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{r1} & \lambda_{r2}z & \cdots & \lambda_{rr}z - 1
\end{bmatrix}
\]
and \(\eta = -\begin{bmatrix}
\varepsilon(h_1) \\
\varepsilon(h_2) \\
\vdots \\
\varepsilon(h_r)
\end{bmatrix}\).

Now \(\det M(0) = (-1)^r\) and, hence, \(M(z)\) is an invertible matrix over \(\mathbb{k}(z)\). It follows, by Cramer’s rule, that \(\alpha_i(z) \in \mathbb{k}(z)\), for all \(i = 1, \ldots, r\).

We point out that the proof of Lemma 3 not only does give the rationality of the series \(\alpha_i(z)\), but also provides explicit formulas for them as quotients of polynomials with coefficients in \(k\). We shall explore this fact below.

**Theorem 4.** Let \(H\) be a semisimple Hopf algebra, let \(V\) be a finite-dimensional vector space and suppose that the tensor algebra \(T(V)\) of \(V\) is an \(H\)-module algebra with a linear action of \(H\). If \(\text{char} \, \mathbb{k} = 0\), then \(P(T(V)^H, z) \in \mathbb{Q}(z)\).

**Proof.** Let \(\{h_1, \ldots, h_r\}\) be the basis for \(H\), let \(t\) be a left integral in \(H\) with \(\varepsilon(t) = 1\) and let \(\mu_i \in \mathbb{k}\), \(i = 1, \ldots, r\), be scalars such that \(t = \sum_{i=1}^r \mu_i h_i\). Then, by Corollary 2, we get
\[
P(T(V)^H, z) = \sum_{n \in \mathbb{N}} \chi_n(t) z^n = \sum_{i=1}^r \mu_i \alpha_i(z),
\]
where \(\alpha_i(z) = \sum_{n \in \mathbb{N}} \chi_n(h_i) z^n\). It follows from Lemma 3 that \(P(T(V)^H, z) \in \mathbb{k}(z)\). Now, since \(P(T(V)^H, z)\) has rational coefficients and is a rational function over \(\mathbb{k} \supset \mathbb{Q}\), it follows from Kronecker’s theorem on Hankel operators (see, e.g., [Pa88, Th. 3.11] or [Co06 Cor. 1.5.7]) that \(P(T(V)^H, z) \in \mathbb{Q}(z)\).

If \(G\) is a finite group and we let \(H = \mathbb{k}G\) be the group algebra of \(G\) over \(\mathbb{k}\) with its usual Hopf algebra structure, then the proofs of Theorem 4 and Lemma 3 using \(G\) as a \(k\)-basis for \(H\), allow us to reconstruct Dicks and Formanek’s formula for the subalgebra of invariants of a free algebra under the action of a finite group of linear automorphisms when \(\text{char} \, \mathbb{k} = 0\) (see [DF82]):
\[
P(T(V)^H, z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - \chi_1(g) z}.
\]

In [FM10], the authors have also explored the particular case of \(H = (\mathbb{k}G)^*\), the dual Hopf algebra of the group algebra \(\mathbb{k}G\) of a finite group \(G\). In this case, a linear \(H\)-module algebra structure on \(T(V)\) amounts to a “linear” grading on \(T(V)\) by the group \(G\). It has been possible to obtain a sufficiently explicit formula for the Hilbert series of the homogeneous component associated to the identity of \(G\) in order to produce a criterion for finite generation of this subalgebra.

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