C-CONFORMAL METRIC TRANSFORMATIONS ON FINSLERIAN HYPERSURFACE

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Abstract. The purpose of the paper is to give some relation between the original Finslerian hypersurface and other C-conformal Finslerian hypersurfaces. In this paper we define three types of hypersurfaces, which were called a hyperplane of the 1st kind, hyperplane of the 2nd kind and hyperplane of the 3rd kind under consideration of C-conformal metric transformation.

Key words: Finsler spaces, Finsler hypersurface, Conformal, C-conformal, Hyperplane of 1st kind, 2nd kind and 3rd kind.

Abstrak. Tujuan dari paper ini adalah untuk memberikan beberapa kaitan antara hypersurface Finsler asal dengan hypersurfaces C-konformal Finsler yang lain. Dalam tulisan ini kami mendefinisikan tiga jenis hypersurfaces, yang disebut hyperplane jenis pertama, hyperplane jenis kedua dan hyperplane jenis ketiga berdasarkan transformasi metrik C-konformal.

Kata kunci: Ruang Finsler, hypersurface Finsler, konformal, C-konformal, hyperplane jenis pertama, jenis kedua dan jenis ketiga.

1. Introduction

The conformal theory and its related concepts of Finsler spaces was initiated by Knebelman in 1929. M. Hashiguchi [1] introduced a special change called C-conformal change which satisfies C-condition. The theory of Special Finsler spaces and their properties were studied by M. Matsumoto [8], C. Shibata [13] et al and authors like H. Izumi [2], S. Kikuchi [4] et al have given the condition for Finsler space to be conformally flat. C. Shibata and H. Azuma [13] have studied C-conformal transformazioni.
invariant tensor of Finsler metric. The author M. Kitayama ([5], [6], [7]) have studied Finsler spaces admitting a parallel vector field and also studied Finslerian hypersurface and metric transformations. The authors H.G. Nagaraja, C.S. Bagewadi and H. Izumi [9] have published a paper on infinitesimal h-conformal motions of Finsler metric.

The authors S.K. Narasimhamurthy and C.S. Bagewadi ([10], [11]) have published a paper on C-conformal Special Finsler spaces admitting a parallel vector field and the same authors have also studied on Infinitesimal C-conformal motions of special Finsler spaces.

Throughout the paper, terminology and notations are referred to [1], [8] and [12].

2. Preliminaries

A Finsler space, we mean a triple $F^n = (M, D, L)$, where $M$ denotes n-dimensional differentiable manifold, $D$ is an open subset of a tangent vector bundle $TM$ and $L: D \rightarrow R$ is a differentiable mapping having the properties

i) $L(x, y) > 0$, for $(x, y) \in D$,

ii) $L(x, \lambda y) = |\lambda|L(x, y)$, for any $(x, y) \in D$ and $\lambda \in R$ such that $(x, \lambda y) \in D$,

iii) $g_{ij}(x, y) = \frac{1}{2} \partial_i \partial_j L^2$, $(x, y) \in D$, is positive definite, where $\partial_i = \frac{\partial}{\partial y^i}$.

The metric tensor $g_{ij}(x, y)$ and Cartan’s C-tensor $C_{ijk}$ are given by [12]:

$$g_{ij}(x, y) = \frac{1}{2} \partial_i \partial_j L^2, \quad g^{ij} = (g_{ij})^{-1},$$

$$C_{ijk} = \frac{1}{2} \partial_k g_{ij}, \quad C_i^j = \frac{1}{2} g^{im}(\partial_k g_{mj}),$$

where $\partial_j = \frac{\partial}{\partial y^j}$ and $\partial_i = \frac{\partial}{\partial x^i}$. We use the following [12]:

a) $l_i = \partial_i L, \quad l^i = y^i/L, \quad h_{ij} = g_{ij} - l_il_j$,

b) $\gamma^i_{jk} = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} - \delta_r g_{jk})$,

c) $G^i = \frac{1}{2} \gamma^i_{jk} y^j y^k, \quad G^i_j = \partial_j G^i, \quad G^i_{jk} = \partial_k G^i_j, \quad G^i_{jkl} = \partial_l G^i_{jk}, \quad (1)$

d) $F^i_{jk} = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \delta_k g_{rj} - \delta_r g_{jk})$,

e) $N^i_j = N^i_j - y_j \sigma^i + \sigma_0 \delta^i_j + \sigma_j y^i$,

where $\delta_j = \partial_j - G^r_j \partial_r$.

The Berwald connection and the Cartan connection of $F^n$ are given by $B^\Gamma = (G^i_{jk}, N^i_j, 0)$ and $C^\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ respectively.
A hypersurface $M^{n-1}$ of the underlying smooth manifold $M^n$ may be parametrically represented by the equation

$$x^i = x^i(u^\alpha),$$

where $u^\alpha$ are Gaussian coordinates on $M^{n-1}$ and Greek indices take values 1 to $n-1$. Here we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \partial x^i / \partial u^\alpha$ is of rank $(n-1)$. The following notations are also employed [6]:

$$B^i_\alpha\beta = \partial x^i / \partial u^\alpha \partial u^\beta, \quad B^i_\alpha\beta\ldots = B^i_\alpha B^j_\beta \ldots.$$  

If the supporting element $y^i$ at a point $(u^\alpha)$ of $M^{n-1}$ is assumed to be tangential to $M^{n-1}$, we may then write

$$y^i = B^i_\alpha(u^\alpha),$$

i.e., $v^\alpha$ is thought of as the supporting element of $M^{n-1}$ at a point $(u^\alpha)$. Since the function $L(u, v) = L(x(u), y(u, v))$ gives rise to a Finsler matrix of $M^{n-1}$, we get a $(n-1)$-dimensional Finsler space $F^{n-1} = (M^{n-1}, L(u, v))$.

At each point $(u^\alpha)$ of $F^{n-1}$, the unit normal vector $N^i(u, v)$ is defined by

$$g_{ij} B^i_\alpha N^j = 0, \quad g_{ij} N^i N^j = 1.$$  

(2)

If $(B^\alpha_i, N_i)$ is the inverse matrix of $(B^\alpha_i, N^i)$, we have

$$B^i_\alpha B^i_\beta = \delta^i_\alpha, \quad B^i_\alpha N_i = 0, \quad N^i B^i_\alpha = 0, \quad N^i N_i = 1,$$

and further

$$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$  

Making use of the inverse $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^i_\alpha = g^{\alpha\beta} g_{ij} B^j_\beta, \quad N_i = g_{ij} N^j.$$  

For the induced Cartan connections $IC\Gamma = (F^\alpha_{\beta\gamma}, N^\alpha_i, C^\alpha_{\beta\gamma})$ on $F^{n-1}$, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature tensor $H_\alpha$ are given by

i) \quad $H_{\alpha\beta} = N_i (B^i_\alpha B^i_\beta + F^i_{jk} B^{jk}_\alpha) + M_\alpha H_\beta,$

ii) \quad $H_\alpha = N_i (B^i_0_\alpha + N^i B^i_\alpha),$

respectively, where $M_\alpha = C_{ijk} B^i_\alpha N^j N^k$ and $B^i_0_\alpha = B^i_\alpha v^\beta$. Transvecting $H_{\alpha\beta}$ by $v^\beta$, we get $H_{0\alpha} = H_{\beta\alpha} v^\beta = H_\alpha$.

Further more we have to put

$$M_{\alpha\beta} = C_{ijk} B^{ij}_\alpha N^k.$$  

(4)
3. C-Conformal Finsler Space

We shall consider conformal change of a Finsler metric formed by 
\[ L \rightarrow L = e^\sigma(x) L, \]
where \( \sigma \) is conformal factor depends on the point \( x \) only and under this change we have another Finsler space \( F^n = (M^n, L) \) on the same underlying manifold \( M^n \).

M. Hashiguchi [1] introduced the special change named C-conformal change which is by definition, a non-homothetic conformal change satisfying
\[ C_{ijk} \sigma_i = 0, \tag{5} \]
where \( C_{ijk} = g^{im} (\partial_j g_{km}) / 2, \quad \sigma_i = g^{im} \sigma_m, \quad \sigma_m = \partial \sigma / \partial x^m, \quad \sigma^j = g^{ij} \sigma_j. \) From (1) and by symmetry of lower indices of \( C_{ijk} \), we have
\[ C_{ijk} \sigma_i = C_{jik} \sigma_i = C_{jki} \sigma_i = 0, \]
also we have
\[ C_{ij} \sigma^i = C_{ij} \sigma^j = C_{ik} \sigma^k = 0. \]

In the following the quantity with bar will be defined in C-conformal Finsler space \( F^n \), and the quantity without bar will be defined in Finsler space \( F^n \).

Under the C-conformal change, we have the following [2], [13]:
\begin{align*}
a) \quad \bar{g}_{ij} &= (L/L)^2 g_{ij}, \quad \bar{g}^{ij} = (L/L)^2 g^{ij}, \\
b) \quad \bar{y}_i &= \bar{g}^{ij} y_j, \\
c) \quad \bar{C}_{ijk} &= C_{ijk}, \quad \bar{C}^{ij} = e^{2a} C^{ij}, \quad \bar{C}_i = e^{-2a} C_i, \\
d) \quad \bar{\gamma}^{ij}_k &= \gamma^{ij}_k + (\sigma_j \delta^i_k + \sigma_k \delta^i_j - g_{jk} \sigma^i), \\
e) \quad \bar{\gamma}^i &= G^i - \frac{1}{2} L^2 \sigma^i + \sigma_0 y^i, \tag{6} \\
f) \quad \bar{C}^{ij}_k &= C^{ij}_k - g_{jk} \sigma^i + \sigma_k \delta^i_j + \sigma_j \delta^i_k, \\
g) \quad \bar{N}^i_j &= N^i_j - y_j \sigma^i + \sigma_0 \delta^i_j + \sigma_j y^i, \\
h) \quad \bar{F}^{ij}_k &= F^{ij}_k - g_{jk} \sigma^i + \sigma_k \delta^i_j + \sigma_j \delta^i_k + \sigma_0 C^{ij}_k. 
\end{align*}

4. Hypersurface Given by a C-Conformal Change

We now consider a Finsler hypersurface \( F^{n-1} = (M^{n-1}, L(u, v)) \) of the Finsler space \( F^n \) and another Finsler hypersurface \( \bar{F}^{n-1} = (M^{n-1}, \bar{L}(u, v)) \) of the Finsler space \( F^n \) given by the C-conformal change.

Let \( N^i(u, v) \) be a unit normal vector at each point of the \( F^{n-1} \), and as component of \( n-1 \) linearly independent tangent vectors of \( F^{n-1} \) and they are invariant under the C-conformal change. Thus we shall show that a unit normal vector \( \bar{N}^i(u, v) \) of \( \bar{F}^{n-1} \) is uniquely determined by
\[ g_{ij} \bar{B}^i_\alpha \bar{N}^j = 0, \quad g_{ij} \bar{N}^i \bar{N}^j = 1. \tag{7} \]
By means of (2) and (6), we get

\[ g_{ij}(\pm e^{-\sigma} N^i)(\pm e^{-\sigma} N^j) = 1. \]

Therefore we can put

\[ N^i = e^{-\sigma} N^i, \]

where we have chosen the sign ‘+’ in order to fix an orientation. It is obvious that \( N_i(u, v) \) satisfies (2), hence we obtain:

**Lemma 4.1.** For a field of linear frame \((B^i_1, B^i_2, ..., B^i_{n-1}, N^i)\) of \( F^n \), there exists a field of linear frame \((B^i_1, B^i_2, ..., B^i_{n-1}, \overline{N}^i = e^{-\sigma} N^i)\) of the \( F^n \) given by the \( C\)-conformal change such that (7) satisfied along \( F^{n-1} \).

The quantities \( B^i_\alpha \) are uniquely defined along \( F^{n-1} \) by

\[ B^i_\alpha = \overline{g}^{\alpha\beta} g_{ij} B^j_\beta, \]

where \((\overline{g}^{\alpha\beta})\) is the inverse metric of \((g_{\alpha\beta})\). If \((B^i_\alpha, N^i)\) is the inverse vector of \((B^i_\alpha, N^i)\), then we have

\[ B^i_\alpha N^i = 0, \quad \overline{N}^i B^j_\alpha = 0, \quad \overline{N}^i \overline{N}^j = 1, \]

and also

\[ B^i_\alpha B^j_\alpha + \overline{N}^i \overline{N}^j = \delta^i_j. \]

Also we get \( \overline{N}^i = \overline{g}_{ij} \overline{N}^j \), that is

\[ \overline{N}^i = e^{\sigma} N^i. \]  \hspace{1cm} (8)

We have from (6(e)),

\[ D^i = \overline{G}^i - G^i = \sigma_0 y^i - \frac{L^2}{2} \sigma^i, \quad \text{where} \quad \sigma_0 = \sigma_0 y^r. \]  \hspace{1cm} (9)

Differentiating (9) by \( y^j \) and from (6(f)), we obtain

\[ D^i_j = \overline{D}^i_{(j)}, \]

\[ = \overline{G}^i_j - G^i_j, \]

\[ = \overline{N}^i_j - N^i_j, \]

\[ = -y_j \sigma^i + \sigma_0 \delta^i_j + \sigma_j y^i, \]

where \( D^i_{(j)} = \partial_j D^i \). From (9), we have

\[ N_i D^i = \sigma_0 N_i y^i - \frac{L^2}{2} N_i \sigma^i. \]

We assume that \( N_i \sigma^i = 0 \). i.e., \( \sigma^i(x) \) is tangential to \( F^{n-1} \) and using the condition \( N_i y^i = 0 \), then we have

\[ N_i D^i = 0. \]  \hspace{1cm} (10)
Differentiating (10) by $y^j$, we have
\[ N_i D^i_{(j)} + D^i (N^i)_{(j)} = 0, \]
\[ N_i D^i_j + D^i_j (\hat{\partial}_j N_i) = 0. \]
Transvecting above equation by $B^j_\alpha$, we get
\[ N_i D^i_j B^j_\alpha + D^i_j (\hat{\partial}_j N_i) B^\alpha_\beta = 0, \]
\[ N_i D^i_j B^\alpha_\beta = 0, \quad (11) \]
where we used
\[ B^j_\alpha (\hat{\partial}_j N_i) = M_\alpha N_i = C_{ij\alpha} B^i_j N^i N^k N_i = 0. \]

**Definition 4.1.** If each path of the hypersurface $F^{n-1}$ with respect to the induced connection is also a path of the ambient space $F^n$, then $F^{n-1}$ is called a ‘hyperplane of the 1st kind’.

A hyperplane of the 1st kind is characterized by $H_\alpha = 0$.

From (3(ii)) and using (8), we have
\[ H_\alpha = N_i (B^i_\alpha + \bar{N}^i_j B^j_\alpha). \]
Thus
\[ H_\alpha - e^\sigma H_\alpha = N_i (B^i_\alpha + \bar{N}^i_j B^j_\alpha) - e^\sigma N_i (B^i_\alpha + \bar{N}^i_j B^j_\alpha), \]
\[ = e^\sigma (N_i B^i_\alpha + N_i \bar{N}^i_j B^j_\alpha) - e^\sigma (N_i B^i_\alpha + N_i N^i_j B^j_\alpha), \]
\[ = e^\sigma N_i (\bar{N}^i_j - N^i_j) B^j_\alpha, \]
\[ = e^\sigma N_i D^i_j B^j_\alpha. \]
Thus we have
\[ H_\alpha = e^\sigma (H_\alpha + N_i D^i_j B^j_\alpha). \]

Hence we state the following:

**Theorem 4.1.** A Finsler hypersurface $F^{n-1}$ is a hyperplane of 1st kind if and only if C-conformal Finsler hypersurface $F^{n-1}$ is a hyperplane of 1st kind, provided $N_i \sigma^i = 0$, i.e., $\sigma^i(x)$ is tangential to $F^{n-1}$.

Now from (6(h)), the so called difference tensor $D^i_{jk}$ has the following form
\[ D^i_{jk} = F^i_{jk} - F^i_{jk}, \]
\[ = -g_{ij} \sigma^i + \sigma_k \delta^i_j + \sigma_j \delta^i_k + \sigma_0 C^i_{jk}. \]
Contracting above equation by $N_i$ and $B^i_\alpha$, we get
\[ N_i D^i_{jk} B^j_\alpha = -N_i g_{ij} \sigma^i B^j_\alpha + \sigma_k N_i \delta^i_j B^j_\alpha + \sigma_j N_i \delta^i_k B^j_\alpha + \sigma_0 C^i_{jk} N_i B^j_\alpha, \]
\[ = 0. \]
Where we use $\sigma_0 = \sigma_i y^i$ and equation (5). Thus we state the following:
Lemma 4.2. Assuming that $\sigma_i(x)$ is tangential to $F^{n-1}$, then the tensor $N_i D_{jk}^i B_{\alpha}^j$ is vanishes if and only if it satisfies (5).

Definition 4.2. If each h-path of a hypersurface $F^{n-1}$ with respect to the induced connection is also h-path of the ambient space $F^n$, then $F^{n-1}$ is called a ‘hyperplane of the 2nd kind’.

A hyperplane of the 2nd kind is characterized by $H_{\alpha \beta} = 0$.

From (3(i)), we have

$$H_{\alpha \beta} = N_i (B_{\alpha \beta}^i + F_{jk}^i B_{\alpha \beta}^{jk}) + M_{\alpha} H_{\beta}.$$  

(12)

Under the C-conformal change, (12) can be written as

$$\overline{H}_{\alpha \beta} = N_i (B_{\alpha \beta}^i + \overline{F}_{jk}^i B_{\alpha \beta}^{jk}) + \overline{M}_{\alpha} \overline{H}_{\beta}.$$  

(13)

Using equations (12) and (13), we get

$$\overline{H}_{\alpha \beta} - e^\sigma H_{\alpha \beta} = [N_i (B_{\alpha \beta}^i + \overline{F}_{jk}^i B_{\alpha \beta}^{jk}) + \overline{M}_{\alpha} \overline{H}_{\beta}] - e^\sigma N_i (B_{\alpha \beta}^i + F_{jk}^i B_{\alpha \beta}^{jk}) - e^\sigma M_{\alpha} H_{\beta},$$

(14)

that implies

$$\overline{H}_{\alpha \beta} - e^\sigma H_{\alpha \beta} = e^\sigma N_i (\overline{F}_{jk}^i - F_{jk}^i) B_{\alpha \beta}^{jk},$$

(15)

Thus by virtue of lemma (4.1), therefore we state the following:

Theorem 4.2. A Finsler hypersurface $F^{n-1}$ is a hyperplane of the 2nd kind if and only if the C-conformal Finsler hypersurface $F^{n-1}$ is a hyperplane of the 2nd kind, provided $\sigma_i(x)$ is tangential to $F^{n-1}$.

Definition 4.3. If the unit normal vector of $F^{n-1}$ is parallel along each curve of $F^{n-1}$, then $F^{n-1}$ is called a ‘hyperplane of the 3rd kind’.

A hyperplane of the 3rd kind is characterized by $H_{\alpha \beta} = M_{\alpha \beta} = 0$.

From (4), under C-conformal change the tensor $M_{\alpha \beta}$ can be written as

$$M_{\alpha \beta} = C_{ijk} B_{\alpha \beta}^{ij} N_k,$$

(16)

$$= e^{-\sigma} C_{ijk} B_{\alpha \beta}^{ij} N_k,$$

$$= e^{-\sigma} M_{\alpha \beta}.$$

By characterization of hyperplane of the 3rd kind and (15), we have $\overline{H}_{\alpha \beta} = \overline{M}_{\alpha \beta} = 0$.

Thus by virtue of lemma (4.1), we state the following:

Theorem 4.3. A Finsler hypersurface $F^{n-1}$ is a hyperplane of the 3rd kind if and only if C-conformal Finsler hypersurface $F^{n-1}$ is a hyperplane of the 3rd kind, provided $\sigma_i(x)$ is tangential to $F^{n-1}$.
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