Lower-Dimensional Volumes and Kastler–Kalau–Walze Type Theorem for Manifolds with Boundary*

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Abstract In this paper, we define lower-dimensional volumes of spin manifolds with boundary. We compute the lower-dimensional volume \( \text{Vol}^{(2,2)} \) for 5-dimensional and 6-dimensional spin manifolds with boundary and we also get the Kastler–Kalau–Walze type theorem in this case.

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1 Introduction

The noncommutative residue plays a prominent role in noncommutative geometry.\(^{[1−2]}\) Connes\(^{[3]}\) used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes\(^{[4]}\) proved that the noncommutative residue on a compact manifold \( M \) coincided with the Dixmier’s trace on pseudodifferential operators of order \(-\dim M\). Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein–Hilbert action, which we call the Polyakov action analogy. Connes\(^{[5]}\) gave a brute-force proof of this theorem. Kalau and Walze\(^{[6]}\) proved this theorem in the normal coordinates system simultaneously. 

On the other hand, Fedosov et al. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a unique continuous trace.\(^{[8]}\) Wang\(^{[9−10]}\) generalized some results\(^{[4,11]}\) to the case of manifolds with boundary. Wang\(^{[12]}\) proved a Kastler–Kalau–Walze type theorem for the Dirac operator and the signature operator for 3, 4-dimensional manifolds with boundary. Recently, Ponge defined lower-dimensional volumes of Riemannian manifolds by the Wodzicki residue.\(^{[13]}\) The motivation of this paper is to find a Kastler–Kalau–Walze type theorem for higher dimensional manifolds with boundary and generalize the definition of lower dimensional volumes to manifolds with boundary.

This paper is organized as follows. In Sec. 2, we define lower dimensional volumes of spin manifolds with boundary. In Sec. 3, for 6-dimensional spin manifolds with boundary and the associated Dirac operator \( D \), we compute the lower dimensional volume \( \text{Vol}^{(2,2)} \) and get a Kastler–Kalau–Walze type theorem in this case. In Sec. 4, when \( \partial M \) is flat, we can define \( \int_{\partial M} \text{res}_{2,2}(D^{-2}, D^{-2}) \) and \( \int_{\partial M} \text{res}_{2,3}(D^{-2}, D^{-2}) \) (see Sec. 3) and get that the gravitational action for \( \partial M \) is proportional to \( \int_{\partial M} \text{res}_{2,2}(D^{-2}, D^{-2}) \) and \( \int_{\partial M} \text{res}_{2,3}(D^{-2}, D^{-2}) \), which gives two kinds of operator theoretic explanations of the gravitational action for boundary. For 5-dimensional spin manifolds with boundary and the associated Dirac operator \( D \), we compute the lower dimensional volume \( \text{Vol}^{(2,2)} \).

2 Lower-Dimensional Volumes of Spin Manifolds with Boundary

In order to define lower dimensional volumes of spin manifolds with boundary, we need some basic facts and formulae about Boutet de Monvel’s calculus and the definition of noncommutative residue for manifolds with boundary. We can find them in Sec. 2.3 of Ref. [9] and Sec. 2.1 of Ref. [11].

Let \( M \) be an \( n \)-dimensional compact oriented spin manifold with boundary \( \partial M \). We assume that the metric \( g^M \) on \( M \) has the following form near the boundary,

\[
g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,
\]

where \( g^{\partial M} \) is the metric on \( \partial M \). Let \( D \) be the Dirac operator associated to \( g \) on the spinors bundle \( S(TM) \).

Let \( p_1, p_2 \) be nonnegative integers and \( p_1 + p_2 \leq n \). The definition of this paper is to find a Kastler–Kalau–Walze type theorem for higher dimensional manifolds with boundary and generalize the definition of lower dimensional volumes to manifolds with boundary.

The motivation of this paper is to find a Kastler–Kalau–Walze type theorem for higher dimensional manifolds with boundary and the associated Dirac operator \( D \), we compute the lower dimensional volume \( \text{Vol}^{(2,2)} \) and get a Kastler–Kalau–Walze type theorem in this case. In Sec. 4, when \( \partial M \) is flat, we can define \( \int_{\partial M} \text{res}_{2,2}(D^{-2}, D^{-2}) \) and \( \int_{\partial M} \text{res}_{2,3}(D^{-2}, D^{-2}) \) (see Sec. 3) and get that the gravitational action for \( \partial M \) is proportional to \( \int_{\partial M} \text{res}_{2,2}(D^{-2}, D^{-2}) \) and \( \int_{\partial M} \text{res}_{2,3}(D^{-2}, D^{-2}) \), which gives two kinds of operator theoretic explanations of the gravitational action for boundary. For 5-dimensional spin manifolds with boundary and the associated Dirac operator \( D \), we compute the lower dimensional volume \( \text{Vol}^{(2,2)} \).

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where
\[
\Phi = \int_{|\xi'|=1}^{+\infty} \sum_{k=0}^{\infty} \sum_{j,k=0} (-i)^{[a+j+k+1]} \sigma_{j+k}^{(a)}(D^{-p_1}) \left( x', 0, \xi', \xi_n \right)
\]
\[
\times \text{trace}_{S(TM)} \left[ \partial_{\xi_n} \partial^{\phi_{\xi_n}} \hat{g}_{\xi_n} \sigma_{+}^{(a)} \sigma_{-}^{(a)} \sigma_{\text{vol}}^{(a)} \sigma_{0}(D^{-p_1}) \right] (x', 0, \xi', \xi_n)
\]
\times \partial_{\xi_n} \hat{\xi}_{\xi_n}^{(a)} \sigma_{0}(D^{-p_2}) (\xi', 0, \xi_n, \xi_n) \text{d}c_{\xi_n}(\xi') dx',
\]
where the sum is taken over $r - k - |\alpha| + l - j = -n, r \leq -p_1, l \leq -p_2$. Since $[\sigma_{-n}(D^{-p_1-p_2})]_M$ has the same expression as $\sigma_{-n}(D^{-p_1-p_2})$ in the case of manifolds without boundary, so locally we can use the computations\cite{5-6,13} to compute the first term. The following proposition is the motivation of the definition of lower dimensional volumes of spin manifolds with boundary.

**Proposition**
(i) When $p_1 + p_2 = n$, then $\text{Vol}_{p_1,p_2} M = c_0 \text{Vol}_M$.
(ii) When $p_1 + p_2 = n$ mod 1, $\text{Vol}_{p_1,p_2} M = f_{\partial M} \Phi$.
(iii) $\text{Vol}_{p_1,0}^{(1,1)} = -\Omega^4_3 / 3 \int_M s \text{dvol}_M$, $\text{Vol}_{p_1,0}^{(1,1)} = c_1 \text{Vol}_{\partial M}$.

**Proof**
(i) Comes from (2.3) and (2.2) of Ref.~[13]. (ii) Comes from the proposition 2.3 and 3.2 of Ref.~[13]. (iii) Comes from Theorems 2.5 and 5.1 of Ref.~[12].

### 3 A Kastler–Kalau–Walze Type Theorem for 6-Dimensional Spin Manifolds with Boundary

In this section, we compute the lower dimensional volume $\text{Vol}_{p_1}^{(2,2)}$ for 6-dimensional spin manifolds with boundary and get a Kastler–Kalau–Walze type theorem in this case.

Firstly, we recall the symbol expansion of $D^{-2.5}$. Recall the definition of the Dirac operator $D$\cite{14-15}. Let $\nabla^L$ denotes the Levi-Civita connection about $g^L$. In the local coordinates $\{ x_i ; 1 \leq i \leq n \}$ and the fixed orthonormal frame $\{ \hat{e}_1, \ldots, \hat{e}_n \}$, the connection matrix $(\omega_{s,t})$ is defined by
\[
\nabla^L(\hat{e}_1, \ldots, \hat{e}_n) = (\hat{e}_1, \ldots, \hat{e}_n)(\omega_{s,t}).
\]
$c(\hat{e}_i)$ denotes the Clifford action. The Dirac operator
\[
D = \sum_{i=1} \{ c(\partial_i) (\partial_i + \delta_i) \}
\]
\[
\delta_i = -\frac{1}{4} \sum_{s,t} \omega_{s,t} (\partial_i c(\hat{e}_s) c(\hat{e}_t)).
\]

Let $g^{ij} = g(dx_i, dx_j)$ and
\[
\nabla^L \partial_j = \sum_k \Gamma^k_{ij} \partial_k, \quad \Gamma^k = g^{ij} \Gamma^k, \quad \delta_j = g^{ij} \delta_i.
\]

Let the cotangent vector $\xi = \sum_j \xi_j dx_j$ and $\xi^i = g^{ij} \xi_j$. Then we have

**Lemma**\cite{5}
\[
\sigma_{-2}(D^{-2}) = |\xi|^2,
\]
\[
\sigma_{-3}(D^{-2}) = -\sqrt{-1} |\xi|^4 \xi_k (\Gamma^k - 2\delta^k) - \sqrt{-1} |\xi|^6 2\xi^{ij} \xi_\beta \partial_j g^{\alpha \beta}.
\]

Since $\Phi$ is a global form on $\partial M$, so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$) and compute $\Phi(x_0)$ in the coordinates $\hat{U} = U \times [0, 1) \subset M$ and the metric $(1/h(x_n)) g^{\partial M} + dx_n^2$. For details, see Section 2.2.2\cite{12}.

Now we can compute $\Phi$ (see formula (4) for the definition of $\Phi$), since the sum is taken over $-r - l + k + j + |\alpha| = -5, r, l \leq -2$, then we have the following five cases:

**Case 1**
(i) $r = -2, l = -2, k = j = 0, |\alpha| = 1$.

By Eq. (4), we get
\[
\text{Case 1 (i)} = -\int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-2})]
\times \partial_{\xi_n} \partial_{\xi_n} \sigma_{-2}(D^{-2}) (\xi) dx'.
\]

By Lemma 2.2\cite{12} for $i < n$, then
\[
\partial_{\xi_n} \sigma_{-2}(D^{-2})(\xi) = \partial_{\xi_n}(\xi (-2)(\xi)) = -\partial_{\xi_n}(\frac{\xi}{|\xi|^2})(\xi) = 0.
\]

so Case 1 (i) vanishes.

(ii) $r = -2, l = -2, k = |\alpha| = 0, j = 1$.

By Eq. (4), we get
\[
\text{Case 1 (ii)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-2})]
\times \partial_{\xi_n} \sigma_{-2}(D^{-2})(\xi) dx'.
\]

By Lemma 2.2\cite{12} we have
\[
\partial_{\xi_n} \sigma_{-2}(D^{-2})(\xi)|_{|\xi'|=1} = -\frac{h'(0)}{(1 + \xi^2)^2}.
\]

By Eq. (12) and the Cauchy integral formula and (2.1.1)\cite{12} then
\[
\pi_{\xi_n}^+ \partial_{\xi_n} \sigma_{-2}(D^{-2})(\xi)|_{|\xi'|=1}
= -h'(0) \frac{1}{2\pi i}
\int_{|\xi|=1} \frac{1}{(\xi_n + i\eta)(\xi_n - i\eta)^2} d\eta
= h'(0) \frac{i\xi_n + 2}{4(\xi_n - i)^2},
\]
\[
\partial_{\xi_n}^2(\xi (\xi)^{-2})(\xi) = -2 + 6\xi_n^2
= \frac{(1 + \xi^2)^4}{1 - 10\xi_n^2 + 5\xi_n^4},
\]

We note that
\[
\int_{|\xi|=1} \frac{i\xi_n + 2}{4(\xi_n - i)^2} d\xi_n
= \frac{3\xi_n^4 + 6\xi_n^2 - \xi_n - 2}{4} d\xi_n
= \frac{2\pi i}{16} \left[ \frac{3\xi_n^4 + 6\xi_n^2 - \xi_n - 2}{(\xi_n + i)^4} \right] \bigg|_{|\xi_n|=1}
= \frac{\pi}{16}.
\]

Since $n = 6$, $\text{tr}_{S(TM)}[id] = \dim (\wedge^*_0(3)) = 8$. So by Eqs. (11)–(15), we get Case 1 (ii) $= -(5/8 \pi h'(0)) \Omega_4 dx'$, where $\Omega_4$ is the canonical volume of $S^4$.

(iii) $r = -2, l = -2, j = |\alpha| = 0, k = 1$.

By Eq. (4) and an integration by parts, we get

\[
\int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-2}(D^{-2})]
\times \partial_{\xi_n} \sigma_{-2}(D^{-2})(\xi) dx'.
\]
Case 1 (iii) = \( \frac{1}{2} \int_{|\xi|=1}^{+\infty} \frac{1}{1-|\xi|^2} \frac{1}{(1+\xi_n)^3} dx \) 
\[ = \frac{1}{2} \int_{|\xi|=1}^{+\infty} \frac{1}{(1+\xi_n)^3} dx \] 
By Lemma 2.2, we have 
\[ \frac{\partial^{2} \pi_{\xi_{n}}^{+}}{\partial x_{n}} \sigma_{-2}(D^{-2})(x_0)|_{\xi_{n}}=1 = \frac{-i}{(\xi_{n}-i)} \] 
By Eqs. (12) and (17), we have 
\[ \text{Case 1 (iii)} = 4ih'(0) \int_{|\xi|=1}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \sigma(\xi')dx'. \] 
Thus the sum of Case 1 (ii) and Case 1 (iii) is zero.

Case 2 \( r = -2, \ l = -3, \ k = j = |\alpha| = 0. \)

By Eq. (4) and an integration by parts, we get 
\[ \text{Case (ii)} = -i \int_{|\xi|=1}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \sigma(\xi')dx'. \] 
By Lemma 2.2, we have 
\[ \frac{\partial \pi_{\xi_{n}}^{+}}{\partial x_{n}} \sigma_{-2}(D^{-2})(x_0)|_{\xi_{n}}=1 = \frac{i}{2(\xi_{n}-i)^{2}} \] 
In the normal coordinate, \( g^{(s)}(x_0) = \delta_{k}^{i} \) and \( \partial_{x_{j}}(g^{a\beta})(x_0) = 0, \) if \( j < n; \) \( \partial_{x_{j}}(g^{a\beta})(x_0) = h'(0)\delta_{j}^{i}, \) if \( j = n. \) So by Lemma A.2, we have \( \Gamma^{(s)}(x_0) = (5/2)h'(0) \) and \( \Gamma^{(s)}(x_0) = 0 \) for \( k < n. \) By the definition of \( \delta_{k}^{i} \) and Lemma 2.3, we have 
\[ \sigma_{-3}(D^{-2})(x_0)|_{\xi_{n}}=1 = -\sqrt{-1|\xi|^{-4}} \xi_{k}(\Gamma^{(k)} - 2\delta^{k})(x_0)|_{\xi_{n}}=1 - \sqrt{-1|\xi|^{-6}} 2i \xi_{n} \xi_{k} \xi_{j} \partial_{x_{j}} g^{a\beta}(x_0)|_{\xi_{n}}=1 \] 
\[ = -\frac{i}{(1+\xi_{n}^{2})^{2}} \left( \frac{1}{2(\xi_{n}-i)^{2}} \right) + \frac{2i}{(1+\xi_{n}^{2})^{2}} \right] d\xi_{n} \sigma(\xi')dx'. \]
We note that \( \int_{|\xi|=1} \xi_{1} \cdot \xi_{2} \cdot \xi_{3} \cdot \sigma(\xi') = 0, \) so the first term in Eq. (20) has no contribution for computing Case 2.

Case 2 \( = ih'(0) \int_{|\xi|=1}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \sigma(\xi')dx'. \] 
\[ = 2ih'(0) \Omega_{4} \int_{|\xi|=1}^{+\infty} \frac{5\xi_{n}^{2} + 9\xi_{n}}{(\xi_{n}-i)^{2}(\xi_{n}+i)^{2}} d\xi_{n} dx' = 2ih'(0) \Omega_{4} \frac{2\pi i}{4!} \left[ \frac{5\xi_{n}^{4} + 9\xi_{n}}{(\xi_{n}+i)^{2}} \right]^{(1)} \bigg|_{\xi_{n}=i} dx' = \frac{15}{8} \pi h'(0) \Omega_{4} dx'. \]

Case 3 \( r = -3, \ l = -2, \ k = j = |\alpha| = 0. \)

By Eq. (4), we get 
\[ \text{Case 3} = -i \int_{|\xi|=1}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \sigma(\xi')dx'. \]

By the Leibniz rule, trace property and “+++” and “−” vanishing after the integration over \( \xi_{n}, \) then 
\[ \int_{-\infty}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \]
\[ = \int_{-\infty}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} - \int_{-\infty}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \]
\[ = \int_{-\infty}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} - \int_{-\infty}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \]
\[ = \int_{-\infty}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} - \int_{-\infty}^{+\infty} \frac{1}{(\xi_{n}-i)^{5}(\xi_{n}+i)^{2}} d\xi_{n} \]
We note that we cannot get the sum of Cases 2 and 3 is zero by computations in Eq. (23). In order to compute Case 2, by Eq. (23), we have

\[ \int_{\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-2}(D^{-2})\sigma_{-3}(D^{-2})]d\xi_n + \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-2}(D^{-2})\sigma_{2}(D^{-2})]d\xi_n = \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-2}(D^{-2})\sigma_{-3}(D^{-2})]d\xi_n. \] (23)

By Eq. (23), we have

Case 3 = Case 2 - \( i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-2}(D^{-2})\sigma_{-3}(D^{-2})]d\xi_n \sigma(\xi')dx' \) \( \times (D^{-2}) \sigma_{-3}(D^{-2})]d\xi_n \sigma(\xi')dx' \). (24)

We note that we cannot get the sum of Cases 2 and 3 is zero by computations in Eq. (23). In order to compute Case 3, we only need compute the last term in Eq. (24). By Eq. (20) and

\[ \partial_{\xi_n} \sigma_{-2}(D^{-2})(x_0)|_{|\xi'|=1} = -\frac{2\xi_n}{(\xi_n^2 + 1)^2}, \] (25)

we have

\[ -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-2}(D^{-2})\sigma_{-3}(D^{-2})]d\xi_n \sigma(\xi')dx' \]

\[ = 8h'(0) \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{5\xi_n^4 + 9\xi_n^2}{(\xi_n+i)^2(\xi_n-i)^2}d\xi_n \sigma(\xi')dx' = \frac{15}{4}\pi h'(0)\Omega_4 dx'. \] (26)

By Eqs. (21), (24), and (26), we have the sum of Cases 2 and 3 is zero. Now \( \Phi \) is the sum of the Cases 1, 2, and 3, so is zero. Then we get

**Theorem 1** Let \( M \) be a 6-dimensional compact spin manifold with the boundary \( \partial M \) and the metric \( g^M \) as above and \( D \) be the Dirac operator on \( M \), then

\[ \text{Vol}_6^{(2,2)} = \text{WRes}[(\pi^+ D^{-2})^2] = -\frac{5\Omega_6}{3} \int_M \text{sdvol}_M. \] (27)

### 4 Gravitational Action for 6-Dimensional Manifolds with Boundary

Firstly, we recall the Einstein–Hilbert action for manifolds with boundary,[15–16]

\[ I_{Gr} = \frac{1}{16\pi} \int_M \text{sdvol}_M + 2 \int_{\partial M} K \text{dvol} \partial M : = I_{Gr,i} + I_{Gr,b}, \] (28)

where

\[ K = \sum_{1 \leq i,j \leq n-1} K_{i,j} g_{\partial M}^{i,j}, \quad K_{i,j} = -\Gamma_{ij}^n, \] (29)

and \( K_{i,j} \) is the second fundamental form, or extrinsic curvature. Taking the metric in Sec. 2, then by Lemma A.2,[12] for \( n = 6 \), then

\[ K(x_0) = -\frac{5}{2} h'(0), \quad I_{Gr,b} = -5h'(0)\text{Vol}_\partial M. \] (30)

Let \( M \) be 6-dimensional manifolds with boundary and \( P, P' \) be two pseudodifferential operators with transmission property[9,17] on \( M \). Motivated by Eq. (4), we define locally

\[ \text{res}_{2,2}(P, P') := -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \text{tr}[\partial_{\xi_n} \pi^+ \sigma_{-2}(P) \partial_{\xi_n} \sigma_{-2}(P')]d\xi_n \sigma(\xi')dx', \] (31)

\[ \text{res}_{2,3}(P, P') := -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\pi^+ \sigma_{-2}(P) \partial_{\xi_n} \sigma_{-3}(P')]d\xi_n \sigma(\xi')dx'. \] (32)

By Eqs. (31) and (32), so

- Case 1 (ii) = \( \text{res}_{2,2}(D^{-2}, D^{-2}) \),
- Case 2 = \( \text{res}_{2,3}(D^{-2}, D^{-2}) \).

Now, we assume \( \partial M \) is flat, then \( \{dx_i = e_i\}, g_{\partial M}^{i,j} = \delta_{i,j}, \partial_{\xi_n} g_{\partial M}^{i,j} = 0 \). So \( \text{res}_{2,2}(D^{-2}, D^{-2}) \) and \( \text{res}_{2,3}(D^{-2}, D^{-2}) \) are two global forms locally defined by the above oriented orthonormal basis \( \{dx_i\} \). By Case 1 (ii) and Case 2, then we have:

**Theorem 2** Let \( M \) be a 6-dimensional compact spin manifold with the boundary \( \partial M \) and the metric \( g^M \) as above and \( D \) be the Dirac operator on \( M \). Assume \( \partial M \) is flat, then

\[ \int_{\partial M} \text{res}_{2,2}(D^{-2}, D^{-2}) = \frac{\pi}{8} \Omega_4 I_{Gr,b}, \] (34)

\[ \int_{\partial M} \text{res}_{2,3}(D^{-2}, D^{-2}) = \frac{3\pi}{8} \Omega_4 I_{Gr,b}. \] (35)
Nextly, for 5-dimensional spin manifolds with boundary, we compute \( \text{Vol}^{(2,2)}_5 \). By Proposition (ii), we have
\[
\tilde{\text{Wres}}[(\pi^+ D^{-2})^2] = \int_{\partial M} \Phi.
\]
When \( n = 5 \), then in Eq. (4), \( r - k - |\alpha| + l - j + 1 = -5 \), \( r, l \leq -2 \), so we get \( r = l = -2, k = |\alpha| = j = 0 \), then
\[
\Phi = \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}_{S(TM)}[\sigma^+_\xi \sigma^-_{-2}(D^{-2})(x', 0, \xi', \xi_n)]d\xi_5 \sigma(\xi')dx'.
\]
By Eq. (25) and
\[
\pi^+_{\xi_n} \sigma^-_{-2}(x_0)|_{|\xi'| = 1} = \frac{1}{2i(\xi_n - 1)},
\]
\[\text{tr}(id) = \dim(S(TM)) = 4,\]
we can get \( \text{Vol}^{(2,2)}_5 = (\pi i/2)\Omega^3 \text{Vol}_{\partial M} \). By \( I_{\text{Gr}, b} = -4h'(0)\text{Vol}_{\partial M} \), we have

**Theorem 3** Let \( M \) be a 5-dimensional compact spin manifold with the boundary \( \partial M \) and the metric \( g^{\nu} \) as in Sec. 2 and \( D \) be the Dirac operator on \( \hat{M} \), then
\[
\text{Vol}^{(2,2)}_5 = \tilde{\text{Wres}}[(\pi^+ D^{-2})^2] = \frac{\pi i}{2} \Omega^3 \text{Vol}_{\partial M},
\]
\[
I_{\text{Gr}, b} = \frac{8ih'(0)}{\pi \Omega^3} \tilde{\text{Wres}}[(\pi^+ D^{-2})^2],
\]
where \( \text{Vol}_{\partial M} \) denotes the canonical volume of \( \partial M \).

**Remark** By Theorem 3, we know that \( \tilde{\text{Wres}}[(\pi^+ D^{-1})^2] \) is proportional to the gravitational action for boundary for 5-dimensional manifolds with boundary. But the constant depends on \( h'(0) \).

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