EXPLICIT SERRE DUALITY ON COMPLEX SPACES

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Abstract. In this paper we use recently developed calculus of residue currents together with integral formulas to give a new explicit analytic realization, as well as a new analytic proof of Serre duality on any reduced pure n-dimensional paracompact complex space \(X\). At the core of the paper is the introduction of concrete fine sheaves \(A^{n,q}_X\) of certain currents on \(X\) of bidegree \((n,q)\), such that the Dolbeault complex \((A^{n,\bullet}_X, \bar{\partial})\) becomes, in a certain sense, a dualizing complex. In particular, if \(X\) is Cohen-Macaulay (e.g., Gorenstein or a complete intersection) then \((A^{n,\bullet}_X, \bar{\partial})\) is an explicit fine resolution of the Grothendieck dualizing sheaf.

1. Introduction

Let \(X\) be a complex \(n\)-dimensional manifold and let \(F \to X\) be a complex vector bundle. Let \(E^{0,q}(X,F)\) denote the space of smooth \(F\)-valued \((0,q)\)-forms on \(X\) and let \(E^{n,q}_c(X,F^*)\) denote the space of smooth compactly supported \((n,q)\)-forms on \(X\) with values in the dual vector bundle \(F^*\). Serre duality, [28], can be formulated analytically as follows: There is a non-degenerate pairing

\[
H^q(E^{0,\bullet}(X,F), \bar{\partial}) \times H^{n-q}(E^{n,\bullet}_c(X,F^*), \bar{\partial}) \to \mathbb{C},
\]

\[
([\varphi]_\bar{\partial}, [\psi]_\bar{\partial}) \mapsto \int_X \varphi \wedge \psi,
\]

provided that \(H^q(E^{0,\bullet}(X,F), \bar{\partial})\) and \(H^{q+1}(E^{0,\bullet}(X,F), \bar{\partial})\) are Hausdorff considered as topological vector spaces. If we set \(\mathcal{F} := \mathcal{O}(F)\) and \(\mathcal{F}^* := \mathcal{O}(F^*)\) and let \(\Omega^n_X\) denote the sheaf of holomorphic \(n\)-forms on \(X\), then one can, via the Dolbeault isomorphism, rephrase Serre duality more algebraically: There is a non-degenerate pairing

\[
H^q(X, \mathcal{F}) \times H^{n-q}_c(X, \mathcal{F}^* \otimes \Omega^n_X) \to \mathbb{C},
\]

realized by the cup product, provided that \(H^q(X, \mathcal{F})\) and \(H^{q+1}(X, \mathcal{F})\) are Hausdorff. In this formulation Serre duality has been generalized to complex spaces, see, e.g., Hartshorne [19], [20], and Conrad [15] for the algebraic setting and Ramis-Ruget [26] and Andreotti-Kas [11] for the analytic. In fact, if \(X\) is a pure \(n\)-dimensional paracompact complex space that in addition is Cohen-Macaulay, then again there is a perfect pairing (1.2) if we construe \(\Omega^n_X\) as the Grothendieck dualizing sheaf that we will get back to shortly. If \(X\) is not Cohen-Macaulay things get more involved and \(H^{n-q}_c(X, \mathcal{F}^* \otimes \Omega^n_X)\) is replaced by \(\text{Ext}^{-q}_c(X; \mathcal{F}, K^*)\), where \(K^*\) is the dualizing complex in the sense of [26]; a certain complex of \(\mathcal{O}_X\)-modules with coherent cohomology.

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To our knowledge there is no such explicit analytic realization of Serre duality as (1.1) in the case of singular spaces. In fact, *verbatim* the pairing (1.1) cannot realize Serre duality in general since the Dolbeault complex $(\mathcal{E}^*_X, \bar{\partial})$ in general does not provide a resolution of $\mathcal{O}_X$. In this paper we replace the sheaves of smooth forms by concrete fine sheaves of certain currents $\mathcal{A}_{p,q}$, $p = 0$ or $p = n$, that are smooth on $X_{\text{reg}}$ and such that (1.1) with $\mathcal{E}$ replaced by $\mathcal{A}$ indeed realizes Serre duality.

We will say that a complex $(\mathcal{D}_i^+, \delta)$ of fine sheaves is a *dualizing Dolbeault complex* for a coherent sheaf $\mathcal{F}$ if $(\mathcal{D}_i^+, \delta)$ has coherent cohomology and if there is a non-degenerate pairing $H^q(X, \mathcal{F}) \times H^n-q(\mathcal{D}_i^+(X), \delta) \to \mathbb{C}$. In this terminology, $(\mathcal{A}^*_X, \bar{\partial})$ thus is a dualizing Dolbeault complex for $\mathcal{O}_X$.

At this point it is appropriate to mention that Ruget in [27] shows, using Coleff-Herrera residue theory, that there is an injective morphism $K^*_X \to \mathcal{C}^{n,\bullet}_X$, where $\mathcal{C}^{n,\bullet}_X$ is the sheaf of germs of currents on $X$ of bidegree $(n, \bullet)$.

Let $X$ be a reduced complex space of pure dimension $n$. Recall that every point in $X$ has a neighborhood $V$ that can be embedded into some pseudoconvex domain $D \subset \mathbb{C}^n$, $i: V \to D$, and that $\mathcal{O}_V \cong \mathcal{O}_D / \mathcal{J}_V$, where $\mathcal{J}_V$ is the radical ideal sheaf in $D$ defining $i(V)$. Similarly, a $(p, q)$-form $\varphi$ on $V_{\text{reg}}$ is said to be smooth on $V$ if there is a smooth $(p, q)$-form $\hat{\varphi}$ in $D$ such that $\varphi = i^* \hat{\varphi}$ on $V_{\text{reg}}$. It is well known that the so defined smooth forms on $V$ define an intrinsic sheaf $\mathcal{E}^q_{\text{reg}}$ on $X$. The currents of bidegree $(p, q)$ on $X$ are defined as the dual of the space of compactly supported smooth $(n-p, n-q)$-forms on $X$ with a certain topology. More concretely, given a local embedding $i: V \to D$, for any $(p, q)$-current $\mu$ on $V$, $\tilde{\mu} := i_* \mu$ is a current of bidegree $(p+N-n, q+N-n)$ in $D$ with the property that $\tilde{\mu} | \xi = 0$ for every test form $\xi$ in $D$ such that $i^* \xi | V_{\text{reg}} = 0$. Conversely, if $\tilde{\mu}$ is a current in $D$ with this property, then it defines a current on $V$ (with a shift in bidegrees). We will often suggestively write $\int \mu \wedge \xi$ for the action of the current $\mu$ on the test form $\xi$.

A current $\mu$ on $X$ is said to have the *standard extension property* (SEP) with respect to a subvariety $Z \subset X$ if $\chi([h]/\epsilon) \mu \to \mu$ as $\epsilon \to 0$, where $\chi$ is a smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$ and $h$ is a holomorphic tuple such that $\{h = 0\}$ has positive codimension and intersects $Z$ properly; if $Z = X$ we simply say that $\mu$ has the SEP on $X$. In particular, two currents with the SEP on $X$ are equal on $X$ if and only if they are equal on $X_{\text{reg}}$.

We will say that a current $\mu$ on $X$ has *principal value-type singularities* if $\mu$ is locally integrable outside a hypersurface and has the SEP on $X$. Notice that if $\mu$ has principal value-type singularities and $h$ is a generically non-vanishing holomorphic tuple such that $\mu$ is locally integrable outside $\{h = 0\}$, then the action of $\mu$ on a test form $\xi$ can be computed as

$$\lim_{\epsilon \to 0} \int_X \chi([h]/\epsilon) \mu \wedge \xi,$$

where the integral now is an honest integral of an integrable form on the manifold $X_{\text{reg}}$.

By using integral formulas and residue theory, Andersson and the second author introduced in [7] fine sheaves $\mathcal{A}^{0,q}_{X_{\text{reg}}}$ (i.e., modules over $\mathcal{E}^{0,0}_{X_{\text{reg}}}$) of $(0, q)$-currents with the SEP on $X$, containing $\mathcal{E}^{0,q}_X$, and coinciding with $\mathcal{E}^{0,q}_{X_{\text{reg}}}$ on $X_{\text{reg}}$, such that the

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1See below for the definition of $\mathcal{E}^{p,q}_X$; the sheaf of smooth $(p, q)$-forms on $X$.
associated Dolbeault complex yields a resolution of $\mathcal{O}_X$. We introduce our sheaves $\mathcal{A}^{n,q}_X$ of $(n,q)$-currents in a similar way and we show that such currents have the SEP on $X$, that $\mathcal{E}^{n,q}_X \subset \mathcal{A}^{n,q}_X$, and that $\mathcal{A}^{n,q}_X$ coincides with $\mathcal{E}^{n,q}_X$ on $X_{\text{reg}}$; cf. Proposition 4.3. Moreover, by Theorem 4.4, $\bar{\partial}: \mathcal{A}^{n,q}_X \to \mathcal{A}^{n,q+1}_X$, where of course $\bar{\partial}$ is defined by duality: $\int \partial \mu \wedge \xi := \pm \int \mu \wedge \partial \xi$ for currents $\mu$ and test forms $\xi$ on $X$. By adapting the constructions in [7] to the setting of $(n,q)$-forms we get the following semi-global homotopy formula for $\bar{\partial}$.

**Theorem 1.1.** Let $V$ be a pure $n$-dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$, and put $V' = V \cap D'$. There are integral operators

$$\bar{\partial} : \mathcal{A}^{n,q}(V) \to \mathcal{A}^{n,q-1}(V'), \quad \bar{\partial} : \mathcal{A}^{n,q}(V) \to \mathcal{A}^{n,q}(V'),$$

such that if $\psi \in \mathcal{A}^{n,q}(V)$, then the homotopy formula

$$\psi = \bar{\partial} \mathcal{H} \psi + \mathcal{H}(\bar{\partial} \psi) + \mathcal{P} \psi$$

holds on $V'$.

The integral operators $\mathcal{H}$ and $\tilde{\mathcal{P}}$ are given by kernels $k(z, \zeta)$ and $p(z, \zeta)$ that are respectively integrable and smooth on $\text{Reg}(V_z) \times \text{Reg}(V'_z)$ and that have principal value-type singularities at the singular locus of $V \times V'$. In particular, one can compute $\mathcal{H} \psi$ and $\tilde{\mathcal{P}} \psi$ as

$$\mathcal{H} \psi(\zeta) = \lim_{\epsilon \to 0} \int_{V_z} \chi(|h(z)|/\epsilon) k(z, \zeta) \psi(z), \quad \tilde{\mathcal{P}} \psi(\zeta) = \lim_{\epsilon \to 0} \int_{V_z} \chi(|h(z)|/\epsilon) p(z, \zeta) \psi(z),$$

where $\chi$ is a smooth approximation of the characteristic function of $[1, \infty) \subset \mathbb{R}$, $h$ is a holomorphic tuple cutting out $V_{\text{sing}}$, and where the limit is understood in the sense of currents. We use our integral operators to prove the following result.

**Theorem 1.2.** Let $X$ be a reduced complex space of pure dimension $n$. The cohomology sheaves $\omega^{n,q}_X := \mathcal{H}^q(\mathcal{A}_X^{n,*}, \bar{\partial})$ of the sheaf complex

$$(1.3) \quad 0 \to \mathcal{A}^{n,0}_X \xrightarrow{\partial} \mathcal{A}^{n,1}_X \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{A}^{n,n}_X \to 0$$

are coherent. If $X$ is Cohen-Macaulay, then

$$(1.4) \quad 0 \to \omega^{n,0}_X \to \mathcal{A}^{n,0}_X \xrightarrow{\bar{\partial}} \mathcal{A}^{n,1}_X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{n,n}_X \to 0$$

is exact.

In fact, our proof of Theorem 1.2 shows that if $V \subset X$ is identified with an analytic codimension $p$ subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, then $\omega^{n,q}_V \cong \mathcal{E}xt^{p+q}(\mathcal{O}_D/\mathcal{J}_V, \mathcal{O}_D^N)$, where $\mathcal{O}_D^N$ is the canonical sheaf on $D$. Hence, we get a concrete analytic realization of these $\mathcal{E}xt$-sheaves.

The sheaf $\omega^{n,0}_V$ of $\bar{\partial}$-closed currents in $\mathcal{A}^{n,0}_V$ is in fact equal to the sheaf of $\bar{\partial}$-closed meromorphic currents on $V$ in the sense of Henkin-Passare [21, Definition 2], cf. [7, Example 2.8]. This sheaf was introduced earlier by Barlet in a different way in [12]; cf. also [21, Remark 5]. In case $X$ is Cohen-Macaulay $\mathcal{E}xt^p(\mathcal{O}_D/\mathcal{J}_V, \mathcal{O}_D^N)$ is by definition the Grothendieck dualizing sheaf. Thus, (1.4) can be viewed as a concrete analytic fine resolution of the Grothendieck dualizing sheaf in the Cohen-Macaulay case.
Let \( \varphi \) and \( \psi \) be sections of \( \mathcal{A}_X^{0,q} \) and \( \mathcal{A}_X^{n,q'} \) respectively. Since \( \varphi \) and \( \psi \) then are smooth on the regular part of \( X \), the exterior product \( \varphi|_{X_{reg}} \wedge \psi|_{X_{reg}} \) is a smooth \((n, q + q')\)-form on \( X_{reg} \). In Theorem 5.1 we show that \( \varphi|_{X_{reg}} \wedge \psi|_{X_{reg}} \) has a natural extension across \( X_{sing} \) as a current with principal value-type singularities; we denote this current by \( \varphi \wedge \psi \). Moreover, it turns out that the Leibniz rule \( \bar{\partial}(\varphi \wedge \psi) = \bar{\partial}\varphi \wedge \psi + (-1)^q \varphi \wedge \bar{\partial}\psi \) holds. Now, if \( q' = n - q \) and either \( \varphi \) or \( \psi \) has compact support, then \( \int \varphi \wedge \psi \) (i.e., the action of \( \varphi \wedge \psi \) on 1) gives us a complex number.

Since the Leibniz rule holds we thus get a pairing, a trace map, on cohomology level:

\[
Tr : H^q(\mathcal{A}_X^{0,\bullet}(X), \bar{\partial}) \times H^{n-q}(\mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \to \mathbb{C},
\]

\[
Tr([\varphi], [\psi]) = \int_X \varphi \wedge \psi,
\]

where \( \mathcal{A}_X^{0,q}(X) \) denotes the global sections of \( \mathcal{A}_X^{0,q} \) and \( \mathcal{A}_X^{n,q}(X) \) denotes the global sections of \( \mathcal{A}_X^{n,q} \) with compact support. It causes no problems to insert a locally free sheaf: If \( \mathcal{F} \to X \) is a vector bundle, \( \mathcal{F} = \mathcal{O}(F) \) the associated locally free sheaf, and \( \mathcal{F}^* = \mathcal{O}(F^*) \) the dual sheaf, then the trace map gives a pairing \( \mathcal{F} \otimes \mathcal{A}_X^{0,q}(X) \times \mathcal{F}^* \otimes \mathcal{A}_c^{n,0}(X) \to \mathbb{C} \).

**Theorem 1.3.** Let \( X \) be a paracompact reduced complex space of pure dimension \( n \) and \( \mathcal{F} \) a locally free sheaf on \( X \). If \( H^q(X, \mathcal{F}) \) and \( H^{q+1}(X, \mathcal{F}) \), considered as topological vector spaces, are Hausdorff (e.g., finite dimensional) then the pairing

\[
H^q(\mathcal{F} \otimes \mathcal{A}_X^{0,\bullet}(X), \bar{\partial}) \times H^{n-q}(\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \to \mathbb{C},
\]

\[
([\varphi], [\psi]) \mapsto \int_X \varphi \wedge \psi
\]

is non-degenerate.

By [7, Corollary 1.3], the complex \( (\mathcal{F} \otimes \mathcal{A}_X^{0,\bullet}, \bar{\partial}) \) is a fine resolution of \( \mathcal{F} \) and so, via the Dolbeault isomorphism, Theorem 1.3 gives us a non-degenerate pairing

\[
H^q(X, \mathcal{F}) \times H^{n-q}(\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \to \mathbb{C}.
\]

The complex \( (\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}, \bar{\partial}) \) is thus a concrete analytic dualizing Dolbeault complex for \( \mathcal{F} \). If \( X \) is Cohen-Macaulay, then \( (\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}, \bar{\partial}) \) is, by Theorem 1.2, a fine resolution of the sheaf \( \mathcal{F}^* \otimes \omega_X^{n,0} \) and so Theorem 1.3 yields in this case a non-degenerate pairing

\[
H^q(X, \mathcal{F}) \times H^{n-q}(X, \mathcal{F}^* \otimes \omega_X^{n,0}) \to \mathbb{C}.
\]

In Section 7 we show that this pairing also can be realized as the cup product in Čech cohomology.

**Remark 1.4.** By [26, Théorème 2] there is another non-degenerate pairing

\[
H^q_c(X, \mathcal{F}) \times \text{Ext}^{-q}(X; \mathcal{F}, K_X^\bullet) \to \mathbb{C}
\]

if \( H^q_c(X, \mathcal{F}) \) and \( H^{q+1}_c(X, \mathcal{F}) \) are Hausdorff. In view of this we believe that one can show that, under the same assumption, the pairing

\[
H^q(\mathcal{F} \otimes \mathcal{A}_c^{0,\bullet}(X), \bar{\partial}) \times H^{n-q}(\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \to \mathbb{C},
\]

\[
([\varphi], [\psi]) \mapsto \int_X \varphi \wedge \psi
\]

is non-degenerate but we do not pursue this question in this paper.

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Our considerations here are local or semi-global so let \( V \) be a pure \( n \)-dimensional analytic subset of a pseudoconvex domain \( D \subseteq \mathbb{C}^N \).

2. Preliminaries

2.1. Pseudomeromorphic currents on a complex space. In \( \mathbb{C}_z \) the principal value current \( 1/z^m \) can be defined, e.g., as the limit as \( \epsilon \to 0 \) in the sense of currents of \( \chi((h(z))/\epsilon)/z^m \), where \( \chi \) is a smooth regularization of the characteristic function of \( [1, \infty) \subseteq \mathbb{R} \) and \( h \) is a holomorphic function vanishing at \( z = 0 \), or as the value at \( \lambda = 0 \) of the analytic continuation of the current-valued function \( \lambda \mapsto |h(z)|^{2\lambda}/z^m \). Regularizations of the form \( \chi((h)/\epsilon)\mu \) of a current \( \mu \) occur frequently in this paper and throughout \( \chi \) will denote a smooth regularization of the characteristic function of \( [1, \infty) \subseteq \mathbb{R} \). The residue current \( \bar{\partial}(1/z^m) \) can be computed as the limit of \( \bar{\partial}\chi((h(z))/\epsilon)/z^m \) or as the value at \( \lambda = 0 \) of \( \lambda \mapsto \bar{\partial}|h(z)|^{2\lambda}/z^m \). Since tensor products of currents are well-defined we can form the current

\[
\tau = \bar{\partial}\frac{1}{z_1^{m_1}} \wedge \cdots \wedge \bar{\partial}\frac{1}{z_r^{m_r}} \wedge \frac{\gamma(z)}{z_{r+1}^{m_{r+1}} \cdots z_n^{m_n}}
\]

in \( \mathbb{C}^n_z \), where \( m_1, \ldots, m_r \) are positive integers, \( m_{r+1}, \ldots, m_n \) are nonnegative integers, and \( \gamma \) is a smooth compactly supported form. Notice that \( \tau \) is anti-commuting in the residue factors \( \bar{\partial}(1/z_j^{m_j}) \) and commuting in the principal value factors \( 1/z_k^{m_k} \). We say that a current of the form (2.1) is called an elementary pseudomeromorphic current and we say that a current \( \mu \) on \( V \) is pseudomeromorphic, \( \mu \in \mathcal{PM}(V) \), if it is a locally finite sum of pushforwards \( \pi_\ast \tau = \pi_1^\ast \cdots \pi_r^\ast \tau \) under maps

\[
V^\ell \xrightarrow{\pi^\ell} \cdots \xrightarrow{\pi^2} V^1 \xrightarrow{\pi^1} V^0 = V,
\]

where each \( \pi^j \) is either a modification, a simple projection \( V^j = V^{j-1} \times Z \to V^{j-1} \), or an open inclusion, and \( \tau \) is an elementary pseudomeromorphic current on \( V^\ell \). The sheaf of pseudomeromorphic currents on \( V \) is denoted \( \mathcal{PM}_V \). Pseudomeromorphic currents were originally introduced in [10] but with a more restrictive definition; simple projections were not allowed. In this paper we adopt the definition of pseudomeromorphic currents in [7].

**Example 2.1.** Let \( f \in \mathcal{O}(V) \) be generically non-vanishing and let \( \alpha \) be a smooth form on \( V \). Then \( \alpha/f \) is a semi-meromorphic form on \( V \) and it defines a semi-meromorphic current, also denoted \( \alpha/f \), on \( V \) by

\[
(2.2) \quad \xi \mapsto \lim_{\epsilon \to 0} \int_V \chi((|h|)/\epsilon) \frac{\alpha}{f} \wedge \xi,
\]

where \( \xi \) is a test form on \( V \) and \( h \in \mathcal{O}(V) \) is generically non-vanishing and vanishes on \( \{ f = 0 \} \). That (2.2) indeed gives a well-defined current is proved in [22]; the existence of the limit in (2.2) relies on Hironaka’s theorem on resolution of singularities. Let \( \pi: \tilde{V} \to V \) be a smooth modification such that \( \{ \pi^\ast f = 0 \} \) is a normal crossings divisor. Locally on \( \tilde{V} \) one can thus choose coordinates so that \( \pi^\ast f \) is a monomial. One can then show that the semi-meromorphic current \( \alpha/f \) is the push forward under \( \pi \) of elementary pseudomeromorphic currents (2.1) with \( r = 0 \); hence, \( \alpha/f \in \mathcal{PM}(V) \).

The \((0, 1)\)-current \( \bar{\partial}(1/f) \) is the residue current of \( f \). Since the class of elementary pseudomeromorphic currents is closed under \( \bar{\partial} \) it follows that also \( \bar{\partial}(1/f) \in \mathcal{PM}(V) \). Moreover, since the action of \( 1/f \) on test forms is given by (2.2) with \( \alpha = 1 \) it follows
from Stokes’ theorem that
\[ \bar{\partial} \frac{1}{f} \xi = \lim_{\epsilon \to 0} \int_{V} \bar{\partial} \chi(\frac{|h|}{\epsilon}) \wedge \xi. \]

One crucial property of pseudomeromorphic currents is the following, see, e.g., [7, Proposition 2.3].

**Dimension principle.** Let \( \mu \in \mathcal{PM}(V) \) and assume that \( \mu \) has support on the subvariety \( Z \subset V \). If \( \dim V - \dim Z > q \) and \( \mu \) has bidegree \((\ast, q)\), then \( \mu = 0 \).

The subsheaf of \( \mathcal{PM}_V \) of currents with the SEP is denoted \( \mathcal{W}_V \). It is closed under multiplication by smooth forms and if \( \pi: V \to V \) is either a modification or a simple projection then \( \pi_*: \mathcal{W}(V) \to \mathcal{W}(V) \). A natural subclass of \( \mathcal{W}(V) \) is the class of *almost semi-meromorphic currents* on \( V \); a current \( \mu \) on \( V \) is said to be almost semi-meromorphic if there is a smooth modification \( \pi: V \to V \) and a semi-meromorphic current \( \hat{\mu} \) on \( V \) such that \( \pi_* \hat{\mu} = \mu \), see [7]. Notice that almost semi-meromorphic currents are generically smooth and have principal value-type singularities.

**Proposition 2.2** (Proposition 2.7 in [7]). Let \( \alpha \) be an almost semi-meromorphic current on \( V \) and let \( \mu \in \mathcal{W}(V) \). Then the current \( \alpha \wedge \mu \), a priori defined where \( \alpha \) is smooth, has a unique extension to a current in \( \mathcal{W}(V) \).

We will also have use for the following slight variation of [6, Theorem 1.1 (ii)].

**Proposition 2.3.** Let \( Z \subset V \) be a pure dimensional analytic subset and let \( J \subset \mathcal{O}_V \) be the ideal sheaf of holomorphic functions vanishing on \( Z \). Assume that \( \tau \in \mathcal{PM}(V) \) has the SEP with respect to \( Z \) and that \( h\tau = dh \wedge \tau = 0 \) for all \( h \in J \). Then there is a current \( \mu \in \mathcal{PM}(Z) \) with the SEP such that \( i_* \mu = \tau \), where \( i: Z \hookrightarrow V \) is the inclusion.

**Proof.** Let \( i: V \hookrightarrow D \) be the inclusion. By [6, Theorem 1.1 (i)] we have that \( i_* \tau \in \mathcal{PM}(D) \). It is straightforward to verify that \( i_* \tau \) has the SEP with respect to \( Z \) considered now as a subset of \( D \) and that \( h i_* \tau = dh \wedge i_* \tau = 0 \) for all \( h \in J \), where we now consider \( J \) as the ideal sheaf of \( Z \) in \( D \). Hence, it is sufficient to show the proposition when \( V \) is smooth. To this end, we will see that there is a current \( \mu \) on \( Z \) such that \( i_* \mu = \tau \); then the proposition follows from [6, Theorem 1.1 (ii)].

The existence of such a \( \mu \) is equivalent to that \( \tau \xi = 0 \) for all test forms \( \xi \) such that \( \iota^* \xi = 0 \) on \( Z_{\text{reg}} \). By, e.g., [7, Proposition 2.3] and the assumption on \( \tau \) it follows that \( \tilde{h} \tau = dh \wedge \tau = h \tau = dh \wedge \tau = 0 \) for every \( h \in J \). Using this it is straightforward to check that if \( p \in Z_{\text{reg}} \) and \( \xi \) is a smooth form such that \( \iota^* \xi = 0 \) in a neighborhood of \( p \), then \( \xi \wedge \tau = 0 \) in a neighborhood of \( p \). Thus, if \( g \) is a holomorphic tuple in \( V \) cutting out \( Z_{\text{sing}} \), then \( \chi(|g|/\epsilon) \tau \xi = 0 \) for any test form \( \xi \) such that \( \iota^* \xi = 0 \) on \( Z_{\text{reg}} \). Since \( \tau \) has the SEP with respect to \( Z \) it follows that \( \tau \xi = 0 \) for all test forms \( \xi \) such that \( \iota^* \xi = 0 \) on \( Z_{\text{reg}} \). \( \square \)

### 2.2. Residue currents

We briefly recall the the construction in [9] of a residue current associated to a generically exact complex of Hermitian vector bundles.

Let \( J \) be the radical ideal sheaf in \( D \) associated with \( V \subset D \). Possibly after shrinking \( D \) somewhat there is a free resolution
\[
(2.3) \quad 0 \to \mathcal{O}(E_m) \xrightarrow{f_m} \cdots \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0),
\]
Hence, Corollary 20.14 in [17] that somehow measure the singularities of $V \overset{\text{sing}}{\to}$ and these sets are in fact independent of the choice of resolution (2.3) and of the embedding $V \to D$, i.e., they are invariants of the sheaf $\mathcal{O}_V = \mathcal{O}_D/\mathcal{J}$, and they somehow measure the singularities of $V$. Since $V$ has pure dimension it follows from Corollary 20.14 in [17] that

$$\dim V^r < n - r, \quad r \geq 0.$$  

Hence, $V^n = \emptyset$ and so $f_N$ has optimal rank everywhere; we may thus assume that $m \leq N - 1$ in (2.3). Notice also that $V^r = \emptyset$ for $r \geq 1$ if and only if there is a resolution (2.3) with $m = p$ of $\mathcal{O}_V$, i.e., if and only if $V$ is Cohen-Macaulay.

Given Hermitian metrics on the $E_j$, following [9], one can construct a smooth form $u = \sum_{k \geq 1} u_k$ in $D \setminus V$, where $u_k$ is a $(0, k - 1)$-form taking values in $E_k$, such that

$$f_1 u_1 = 1, \quad f_{k+1} u_{k+1} = \bar{\partial} u_k, \quad k = 1, \ldots, m - 1, \quad \bar{\partial} u_m = 0 \quad \text{in } D \setminus V. \tag{2.5}$$

Moreover, if $F$ is a holomorphic tuple in $D$ vanishing on $V$, then it is proved that

$$\lambda \mapsto |F|^{2\lambda} u, \tag{2.6}$$

a priori defined for $\Re \lambda \gg 0$, has an analytic continuation as a current-valued function to a neighborhood of the origin. The value at $\lambda = 0$ is a pseudomeromorphic current $U = \sum_{k \geq 1} U_k$, where $U_k$ is a $(0, k - 1)$-current taking values in $E_k$, that one should think of as a generalization of the meromorphic current $1/f$ in $D$ when $V = f^{-1}(0)$ is a hypersurface. The residue current $R = \sum_{k \geq 0} R_k$ associated with $V$ is then defined by

$$R_0 = 1 - f_1 U_1, \quad R_k = \bar{\partial} U_k - f_{k+1} U_{k+1}, \quad k = 1, \ldots, m - 1, \quad R_m = \bar{\partial} U_m. \tag{2.7}$$

Hence, $R_k$ is a pseudomeromorphic $(0, k)$-current in $D$ with values in $E_k$, and from (2.5) it follows that $R_k$ has support on $V$. By the dimension principle, thus $R = R_p + \cdots + R_m$. Notice that if $V$ is Cohen-Macaulay then $R = R_p$ and $\bar{\partial} R = 0$. By [9, Theorem 1.1] we have that if $h \in \mathcal{O}_D$ then

$$h R = 0 \quad \text{if and only if} \quad h \in \mathcal{J}. \tag{2.7}$$

For future reference we note that

$$\lambda \mapsto \bar{\partial} |F|^{2\lambda} \wedge u, \tag{2.8}$$

a priori defined for $\Re \lambda \gg 0$, has an analytic continuation as a current-valued function to a neighborhood of the origin and the value at $\lambda = 0$ is $R$; cf. (2.6).

\(^2\)For $j \leq p$, the set where $f_j$ does not have optimal rank is $V$. 

of $\mathcal{O}_D/\mathcal{J}$, where $E_k$ are trivial vector bundles, $E_0$ is the trivial line bundle, $f_k$ are holomorphic mappings, and $m \leq N$. The resolution (2.3) induces a complex of vector bundles

$$0 \to E_m \overset{f_m}{\to} \cdots \overset{f_2}{\to} E_1 \overset{f_1}{\to} E_0$$

that is pointwise exact outside $V$. Let $p$ be the codimension of $V$ in $D$, let for $r \geq 1$ $V^r$ be the set where $f_{p+r}: E_{p+r} \to E_{p+r-1}$ does not have optimal rank\(^2\), and let $V^0 := V_{\text{sing}}$. Then

$$\cdots \subset V^{k+1} \subset V^k \subset \cdots \subset V^1 \subset V^0 \subset V \tag{2.4}$$

and these sets are in fact independent of the choice of resolution (2.3) and of the embedding $V \to D$, i.e., they are invariants of the sheaf $\mathcal{O}_V = \mathcal{O}_D/\mathcal{J}$, and they somehow measure the singularities of $V$. Since $V$ has pure dimension it follows from Corollary 20.14 in [17] that

$$\dim V^r < n - r, \quad r \geq 0.$$  

Hence, $V^n = \emptyset$ and so $f_N$ has optimal rank everywhere; we may thus assume that $m \leq N - 1$ in (2.3). Notice also that $V^r = \emptyset$ for $r \geq 1$ if and only if there is a resolution (2.3) with $m = p$ of $\mathcal{O}_V$, i.e., if and only if $V$ is Cohen-Macaulay.

Given Hermitian metrics on the $E_j$, following [9], one can construct a smooth form $u = \sum_{k \geq 1} u_k$ in $D \setminus V$, where $u_k$ is a $(0, k - 1)$-form taking values in $E_k$, such that

$$f_1 u_1 = 1, \quad f_{k+1} u_{k+1} = \bar{\partial} u_k, \quad k = 1, \ldots, m - 1, \quad \bar{\partial} u_m = 0 \quad \text{in } D \setminus V. \tag{2.5}$$

Moreover, if $F$ is a holomorphic tuple in $D$ vanishing on $V$, then it is proved that

$$\lambda \mapsto |F|^{2\lambda} u, \tag{2.6}$$

a priori defined for $\Re \lambda \gg 0$, has an analytic continuation as a current-valued function to a neighborhood of the origin. The value at $\lambda = 0$ is a pseudomeromorphic current $U = \sum_{k \geq 1} U_k$, where $U_k$ is a $(0, k - 1)$-current taking values in $E_k$, that one should think of as a generalization of the meromorphic current $1/f$ in $D$ when $V = f^{-1}(0)$ is a hypersurface. The residue current $R = \sum_{k \geq 0} R_k$ associated with $V$ is then defined by

$$R_0 = 1 - f_1 U_1, \quad R_k = \bar{\partial} U_k - f_{k+1} U_{k+1}, \quad k = 1, \ldots, m - 1, \quad R_m = \bar{\partial} U_m. \tag{2.7}$$

Hence, $R_k$ is a pseudomeromorphic $(0, k)$-current in $D$ with values in $E_k$, and from (2.5) it follows that $R_k$ has support on $V$. By the dimension principle, thus $R = R_p + \cdots + R_m$. Notice that if $V$ is Cohen-Macaulay then $R = R_p$ and $\bar{\partial} R = 0$. By [9, Theorem 1.1] we have that if $h \in \mathcal{O}_D$ then

$$h R = 0 \quad \text{if and only if} \quad h \in \mathcal{J}. \tag{2.7}$$

For future reference we note that

$$\lambda \mapsto \bar{\partial} |F|^{2\lambda} \wedge u, \tag{2.8}$$

a priori defined for $\Re \lambda \gg 0$, has an analytic continuation as a current-valued function to a neighborhood of the origin and the value at $\lambda = 0$ is $R$; cf. (2.6).
Example 2.4. Let \( V = f^{-1}(0) \) be a hypersurface in \( D \). Then \( 0 \to \mathcal{O}(E_1) \xrightarrow{f} \mathcal{O}(E_0) \) is a resolution of \( \mathcal{O}/(f) \), where \( E_1 \) and \( E_0 \) are auxiliary trivial line bundles. The associated current \( U \) then becomes \((1/f) \otimes \epsilon_1\), where \( \epsilon_1 \) is a holomorphic frame for \( E_1 \), and the associated residue current \( R \) is \( \bar{\partial}(1/f) \otimes \epsilon_1 \).

Let \( g_1, \ldots, g_p \in \mathcal{O}(D) \) be a regular sequence. Then the Koszul complex associated to the \( g_j \) is a free resolution of \( \mathcal{O}_D/(g_1, \ldots, g_p) \). The associated residue current \( R \) then becomes the Coleff-Herrera product \([14]\)

\[
\prod_{i=1}^p \bar{\partial}_{g_i} = \prod_{i=1}^p \bar{\partial}\frac{1}{g_i}
\]
times an auxiliary frame element, see [2, Theorem 1.7].

2.3. Structure forms of a complex space. Assume first that \( V \) is a reduced hypersurface, i.e., \( V = f^{-1}(0) \subset D \subset \mathbb{C}^N \), \( N = n + 1 \), where \( f \in \mathcal{O}(D) \) and \( df \neq 0 \) on \( V_{\text{reg}} \). Let \( \omega' \) be a meromorphic \((n,0)\)-form in \( D \subset \mathbb{C}^{n+1} \) such that

\[
df \wedge \omega' = 2\pi i dz_1 \wedge \cdots \wedge dz_{n+1} / f(z).
\]

Then \( \omega := i^* \omega' \), where \( i: V \hookrightarrow D \) is the inclusion, is a meromorphic form on \( V \) that is uniquely determined by \( f \); \( \omega \) is the Poincaré residue of the meromorphic form \( 2\pi idz_1 \wedge \cdots \wedge dz_{n+1} / f(z) \). For brevity we will sometimes write \( dz \) for \( dz_1 \wedge \cdots \wedge dz_N \). Leray’s residue formula can be formulated as

\[
\int_V \bar{\partial}_{f} \frac{1}{f} \wedge dz \wedge \xi = \lim_{\epsilon \to 0} \int_V \chi(||h||/\epsilon) \omega \wedge i^* \xi,
\]

where \( \xi \) is a \((0,n)\)-test form in \( D \), the left hand side is the action of \( \bar{\partial}(1/f) \) on \( dz \wedge \xi \) and \( h \) is a holomorphic tuple cutting out \( V_{\text{sing}} \). If we consider \( \omega \) as a meromorphic current on \( V \) we can rephrase this as

\[
\bar{\partial}_{f} \frac{1}{f} \wedge dz = i_* \omega.
\]

Assume now that \( V \hookrightarrow D \subset \mathbb{C}^N \) is an arbitrary pure \( n \)-dimensional analytic subset. From Section 2.2 we have, given a free resolution (2.3) of \( \mathcal{O}_D/J_V \) and a choice of Hermitian metrics on the involved bundles \( E_j \), an associated residue current \( R \) that plays the role of \( \bar{\partial}(1/f) \). By the following result, which is an abbreviated version of [7, Proposition 3.3], there is an almost semi-meromorphic current \( \omega \) on \( V \) such that \( R \wedge dz = i_* \omega \); such a current will be called a structure form of \( V \).

Proposition 2.5. Let (2.3) be a Hermitian free resolution of \( \mathcal{O}_D/J_V \) in \( D \) and let \( R \) be the associated residue current. Then there is a unique almost semi-meromorphic current

\[
\omega = \omega_0 + \omega_1 + \cdots + \omega_{n-1}
\]
on \( V \), where \( \omega_r \) is smooth on \( V_{\text{reg}} \), has bidegree \((n,r)\), and takes values in \( E_{p+r}|V \), such that

\[
R \wedge dz_1 \wedge \cdots \wedge dz_N = i_* \omega.
\]

Moreover,

\[
f_p|V \omega_0 = 0, \quad f_{p+r}|V \omega_r = \bar{\partial} \omega_{r-1}, \quad r \geq 1,
\]
in the sense of currents on \( V \), and there are \((0,1)\)-forms \( \alpha_k \), \( k \geq 1 \), that are smooth outside \( V^k \) and that take values in \( \text{Hom}(E_{p+k-1}|V, E_{p+k}|V) \), such that

\[
\omega_k = \alpha_k \omega_{k-1}, \quad k \geq 1.
\]
It is sometimes useful to reformulate (2.10) suggestively as
\begin{equation}
R \wedge dz_1 \wedge \cdots \wedge dz_N = \omega \wedge [V],
\end{equation}
where $[V]$ is the current of integration along $V$.

The following result will be useful for us when defining our dualizing complex.

**Proposition 2.6** (Lemma 3.5 in [7]). If $\psi$ is a smooth $(n, q)$-form on $V$, then there is a smooth $(0, q)$-form $\psi'$ on $V$ with values in $E^n_q|_V$ such that $\psi = \omega_0 \wedge \psi'$.

### 2.4. Koppelman formulas in $\mathbb{C}^N$

We recall some basic constructions from [1] and [4].

Let $D \subset \mathbb{C}^N$ be a domain (not necessarily pseudoconvex at this point), let $k(z, \zeta)$ be an integrable $(N, N-1)$-form in $D \times D$, and let $p(z, \zeta)$ be a smooth $(N, N)$-form in $D \times D$. Assume that $k$ and $p$ satisfy the equation of currents
\begin{equation}
\bar{\partial}k(z, \zeta) = [\Delta^D] - p(z, \zeta)
\end{equation}
in $D \times D$, where $[\Delta^D]$ is the current of integration along the diagonal. Applying this current equation to test forms $\psi(z) \wedge \varphi(\zeta)$ it is straightforward to verify that for any compactly supported $(p, q)$-form $\varphi$ in $D$ one has the following Koppelman formula
\[
\varphi(z) = \bar{\partial}_z \int_{D_\zeta} k(z, \zeta) \wedge \varphi(\zeta) + \int_{D_\zeta} k(z, \zeta) \wedge \bar{\partial}\varphi(\zeta) + \int_{D_\zeta} p(z, \zeta) \wedge \varphi(\zeta).
\]

In [1] Andersson introduced a very flexible method of producing solutions to (2.12). Let $\eta = (\eta_1, \ldots, \eta_N)$ be a holomorphic tuple in $D \times D$ that defines the diagonal and let $\Lambda_\eta$ be the exterior algebra spanned by $T^*_0(1, 0) -forms on (D \times D)$ and the $(1, 0)$-forms $d\eta_1, \ldots, d\eta_N$. On forms with values in $\Lambda_\eta$ interior multiplication with $2\pi i \sum \eta_j \partial/\partial \eta_j$, denoted $\delta_\eta$, is defined; put $\nabla_\eta = \delta_\eta - \bar{\partial}$.

Let $s$ be a smooth $(1, 0)$-form in $\Lambda_\eta$ such that $|s| \lesssim |\eta|$ and $|\eta|^2 \lesssim |\delta_\eta s|$ and let $B = \sum_{k=1}^N s \wedge (\bar{\partial}s)^{k-1}/(\delta_\eta s)^k$. It is proved in [1] that then $\nabla_\eta B = 1 - [\Delta^D]$. Identifying terms of top degree we see that $\bar{\partial}B_{N, N-1} = [\Delta^D]$ and we have found a solution to (2.12). For instance, if we take $s = \bar{\partial}(-z) \wedge \eta$ and $\eta = \zeta - z$, then the resulting $B$ is sometimes called the full Bochner-Martinelli form and the term of top degree is the classical Bochner-Martinelli kernel.

A smooth section $g(z, \zeta) = g_{0,0} + \cdots + g_{N, N}$ of $\Lambda_\eta$, defined for $z \in D' \subset D$ and $\zeta \in D$, such that $\nabla_\eta g = 0$ and $g_{0,0}|_{\Delta^D} = 1$ is called a weight with respect to $z \in D'$. It follows that $\nabla_\eta (g \wedge B) = g - [\Delta^D]$ and, identifying terms of bidegree $(N, N-1)$, we get that
\begin{equation}
\bar{\partial}(g \wedge B)_{N, N-1} = [\Delta^D] - g_{N, N}
\end{equation}
in $D' \times D$ and hence another solution to (2.12). If $D$ is pseudoconvex and $K$ is a holomorphically convex compact subset, then one can find a weight $g$ with respect to $z$ in some neighborhood $D' \subset D$ of $K$ such that $z \mapsto g(z, \zeta)$ is holomorphic in $D'$ and $\zeta \mapsto g(z, \zeta)$ has compact support in $D$; see, e.g., Example 2 in [4].

### 2.5. Koppelman formulas for $(0, q)$-forms on a complex space

We briefly recall from [7] the construction of Koppelman formulas for $(0, q)$-forms on $V \subset D$. The basic idea is to use the currents $U$ and $R$ discussed in Section 2.2 to construct a weight that will yield an integral formula of division/interpolation type in the same spirit as in [13].
Let (2.3) be a resolution of $\partial_D/J$, where as before $J$ is the sheaf in $D$ associated to $V \rightarrow D$. One can find, see [4, Proposition 5.3], holomorphic $(k-\ell, 0)$-form-valued Hefer morphisms, i.e., matrices $H^k_{k}: E_k \rightarrow E_\ell$ depending holomorphically on $z$ and $\zeta$ such that $H^k_{k} = I_{E_k}$ and

$$\delta_\eta H^k_{k} = H^k_{k-1}f_k(\zeta) - f_{k+1}(z)H^k_{k+1}, \quad k > 1.$$  

Let $U^\lambda = |F(\zeta)|^{2\lambda}u(\zeta)$ and let $R^\lambda = \partial|F(\zeta)|^{2\lambda} \wedge u(\zeta)$, cf. (2.6) and (2.8). Then

$$(2.14) \quad \gamma^\lambda := \sum_{k=0}^N H^0_k R^\lambda_k + f_1(z) \sum_{k=1}^N H^1_k U^\lambda_k,$$

is a weight if $\Re \lambda > 0$. Let also $g$ be an arbitrary weight. Then $\gamma^\lambda \wedge g$ is again a weight and we get

$$(2.15) \quad \bar{\partial}(\gamma^\lambda \wedge g \wedge B)_{N, N-1} = |\Delta^D| - (\gamma^\lambda \wedge g)_{N, N}$$

in the current sense in $D \times D$, cf. (2.13). Let us proceed formally and, also, let us temporarily assume that $R$ is Cohen-Macaulay so that $\partial$ is $\bar{\partial}$-closed. Then, multiplying (2.15) with $R(z) \wedge dz$ and using (2.7) so that $f_1(z)R(z) = 0$, we get that

$$(2.16) \quad \bar{\partial}\left(R(z) \wedge dz \wedge (HR^\lambda \wedge g \wedge B)_{N, N-1}\right) = R(z) \wedge dz \wedge |\Delta^D| - R(z) \wedge dz \wedge (HR^\lambda \wedge g)_{N, N},$$

where $HR^\lambda = \sum_{k=0}^N H^k_{k} R^\lambda_k$, cf. (2.14). In view of (2.11) we have $R(z) \wedge dz |\Delta^D| = \omega \wedge [\Delta^V]$, where $[\Delta^V]$ is the integration current along the diagonal $\Delta^V \subset V \times V \subset D \times D$, and formally letting $\lambda = 0$ in (2.16) we thus get

$$(2.17) \quad \bar{\partial}\left(\omega(z) \wedge [V_z] \wedge (HR \wedge g \wedge B)_{N, N-1}\right) = \omega \wedge [\Delta^V] - \omega(z) \wedge [V_z] \wedge (HR \wedge g)_{N, N}.$$  

To see what this means we will use (2.11). Notice first that one can factor out $d\eta = d\eta_1 \wedge \cdots \wedge d\eta_N$ from $(HR \wedge g \wedge B)_{N, N-1}$ and $(HR \wedge g)_{N, N}$. After making these factorization in (2.17) we may replace $d\eta$ by $C_\eta(z, \zeta)d\zeta$, where $C_\eta(z, \zeta) = N! \det(\partial_{ij}/\zeta_k)$, since $\omega(z) \wedge [V_z]$ has full degree in $dz$. More precisely, let $\epsilon_1, \ldots, \epsilon_N$ be a basis for an auxiliary trivial complex vector bundle over $D \times D$ and replace all occurrences of $d\eta_j$ in $H$, $g$, and $B$ by $\epsilon_j$. Denote the resulting forms by $\hat{H}$, $\hat{g}$, and $\hat{B}$ respectively and let

$$(2.18) \quad k(z, \zeta) = C_{\eta}(z, \zeta) \epsilon^*_{\zeta} \wedge \cdots \wedge \epsilon^*_{\zeta} \sum_{k=0}^n \hat{H}^0_{p+k} \omega_k(\zeta) \wedge (\hat{g} \wedge \hat{B})_{n-k, n-k-1},$$

$$(2.19) \quad p(z, \zeta) = C_{\eta}(z, \zeta) \epsilon^*_{\zeta} \wedge \cdots \wedge \epsilon^*_{\zeta} \sum_{k=0}^n \hat{H}^0_{p+k} \omega_k(\zeta) \wedge \hat{g}_{n-k, n-k}.$$  

Notice that $k$ and $p$ have bidegrees $(n, n-1)$ and $(n, n)$ respectively. In view of (2.11) we can replace $(HR \wedge g \wedge B)_{N, N-1}$ and $(HR \wedge g)_{N, N}$ with $[V_z] \wedge k(z, \zeta)$ and $[V_z] \wedge p(z, \zeta)$ respectively in (2.17). It follows that

$$\bar{\partial}(\omega(z) \wedge k(z, \zeta)) = \omega \wedge [\Delta^V] - \omega(z) \wedge p(z, \zeta)$$

holds in the current sense at least on $V_{reg} \times V_{reg}$. The formal computations above can be made rigorous, see [7, Section 5], and combined with Proposition 2.6 we get Proposition 2.7 below; notice that $\omega = \omega_0$ and $\bar{\partial}\omega = 0$ since we are assuming that $V$ is Cohen-Macaulay.
The following result will be the starting point of the next section and it holds without any assumption about Cohen-Macaulay.

**Proposition 2.7** (Lemma 5.3 in [7]). With $k(z, \zeta)$ and $p(z, \zeta)$ defined by (2.18) and (2.19) respectively we have

$$\bar{\partial} k(z, \zeta) = [\Delta^V] - p(z, \zeta)$$

in the sense of currents on $V_{reg} \times V_{reg}$.

**Remark 2.8.** In [7] it is assumed that the weight $g$ in $k$ and $p$ has compact support in $D_\zeta$ but the proof goes through for any weight.

The integral operators $\mathcal{K}$ and $\mathcal{P}$ for forms in $W^{0,q}$ introduced in [7] are defined as follows. Let $\tilde{\pi} : V_z \times V_\zeta \to V_z$ be the natural projection onto $V_z$, let $g$ in (2.18) and (2.19) be a weight with respect to $z$ in some $D' \subseteq D$ and with compact support in $D_\zeta$, and let $\mu \in W^{0,q}(D)$. Since $\omega$ and $B$ are almost semi-meromorphic $k(z, \zeta)$ and $p(z, \zeta)$ are also almost semi-meromorphic and it follows from Proposition 2.2 that $k(z, \zeta) \wedge \mu(\zeta)$ and $p(z, \zeta) \wedge \mu(\zeta)$ are in $W(V' \times V)$, where $V' = D' \cap V$. It follows that

$$\mathcal{K} \mu(z) := \tilde{\pi}_*(k(z, \zeta) \wedge \mu(\zeta)), \quad \mathcal{P} \mu(z) := \tilde{\pi}_*(p(z, \zeta) \wedge \mu(\zeta)),$$

are in $W(V'_z)$. The sheaves $\mathcal{A}^{0,q}_V$ are then morally defined to be the smallest sheaves that contain $\mathcal{E}^{0,q}_V$ and are closed under operators $\mathcal{K}$ and under multiplication with $\mathcal{E}^{0,q}_V$. More precisely, the stalk $\mathcal{A}^{0,q}_{x}$ consists of those germs of currents which can be written as a finite sum of of terms

$$\xi_m \wedge \mathcal{K}_m (\cdots \xi_1 \wedge \mathcal{K}_1(\xi_0) \cdots),$$

where $\xi_j$ are smooth $(0,*)$-forms and $\mathcal{K}_j$ are integral operators at $x$ of the above form; cf. [7, Definition 7.1].

**Theorem 2.9** (Theorem 1.2 [7]). Let $X$ be a reduced complex space of pure dimension $n$. The sheaves $\mathcal{A}^{0,q}_X$ are fine sheaves of $(0,q)$-currents on $X$, they contain $\mathcal{E}^{0,q}_X$, and moreover

1. $\oplus_q \mathcal{A}^{0,q}_X$ is a module over $\mathcal{E}^{0,q}_X$,
2. $\mathcal{A}^{0,q}_X|_{X_{reg}} = \mathcal{E}^{0,q}_X|_{X_{reg}}$,
3. the complex $(\mathcal{A}^{0,q}_X, \bar{\partial})$ is a resolution of $\mathcal{O}_X$.

3. **Koppelman formulas for $(n,q)$-forms**

Let $V$ be a pure $n$-dimensional analytic subset of a pseudoconvex domain $D \subseteq \mathbb{C}^N$ and let $\omega$ be a structure form on $V$. Let $k(z, \zeta)$ and $p(z, \zeta)$ be the kernels defined respectively in (2.18) and (2.19). Since $k$ and $p$ are almost semi-meromorphic it follows from Proposition 2.2 that if $\mu = \mu(z) \in W^{m,q}(V)$, then $k(z, \zeta) \wedge \mu(z)$ and $p(z, \zeta) \wedge \mu(z)$ are well-defined currents in $W(V \times V)$. Assume that the weight $g$ in (2.18) and (2.19) has compact support in $V_z$ or that $\mu$ has compact support in $V_z$. Let $\pi : V_z \times V_\zeta \to V_\zeta$ be the natural projection and define

$$\mathcal{K} \mu(\zeta) := \pi_*(k(z, \zeta) \wedge \mu(z)), \quad \mathcal{P} \mu(\zeta) := \pi_*(p(z, \zeta) \wedge \mu(z)).$$
It follows that \( \mathcal{H} \mu \) and \( \mathcal{P} \mu \) are well-defined currents in \( \mathcal{W} \). Notice that \( \mathcal{P} \mu \) is of the form \( \sum_{r} \omega_{r} \wedge \xi_{r} \), where \( \xi_{r} \) is a smooth \((0,*)\)-form (with values in an appropriate bundle) in general, and holomorphic if the weight \( g(z, \zeta) \) is chosen holomorphic in \( \zeta \); cf. (2.19). It is natural to write

\[
\mathcal{H} \mu(\zeta) = \int_{V_{z}} k(z, \zeta) \wedge \mu(z), \quad \mathcal{P} \mu(\zeta) = \int_{V_{z}} p(z, \zeta) \wedge \mu(z).
\]

We have the following analogue of Proposition 6.3 in [7].

**Proposition 3.1.** Let \( \mu(z) \in \mathcal{W}^{m,q}(V) \) and assume that \( \partial \mu \in \mathcal{W}^{m,q+1}(V) \). Let \( g \) in (2.18) and (2.19) be a weight with respect to \( \zeta \) in some \( D' \subset D \). If either \( \mu \) has compact support in \( V \) or \( g \) has compact support in \( D_{z} \), then

\[
(3.3) \quad \mu = \partial \mathcal{H} \mu + \mathcal{H}(\partial \mu) + \mathcal{P} \mu
\]

in the sense of currents on \( V'_{\text{reg}} = D' \cap V_{\text{reg}} \).

**Proof.** If \( \varphi = \varphi(\zeta) \) is a \((0, n-q)\)-test form on \( V'_{\text{reg}} \) it follows, cf. the beginning of Section 2.4, from Proposition 2.7 that

\[
\varphi(z) = \partial_{\zeta} \int_{\mathcal{V}_{\zeta}} k(z, \zeta) \wedge \varphi(\zeta) + \int_{\mathcal{V}_{\zeta}} k(z, \zeta) \wedge \partial \varphi(\zeta) + \int_{\mathcal{V}_{\zeta}} p(z, \zeta) \wedge \varphi(\zeta)
\]

for \( z \in V_{\text{reg}} \). By [7, Lemma 6.1] and since \( p(z, \zeta) \) is smooth in \( z \) each term on the right hand side is smooth on \( V \). Moreover, since \( k \) and \( p \) have compact support in \( z \) each term is in fact a test form on \( V \) so that \( \mu \) acts on each term. Thus (3.3) follows provided that \( \mu \) has compact support in \( V_{\text{reg}} \).

For the general case, let \( h = h(z) \) be a holomorphic tuple cutting out \( V_{\text{sing}} \) and let \( \chi_{\epsilon} = \chi(|h|/\epsilon) \). Then the proposition holds for \( \chi_{\epsilon} \mu \) (since \( k \) and \( p \) have compact support in \( z \)). Since \( k(z, \zeta) \wedge \mu(z) \) and \( p(z, \zeta) \wedge \mu(z) \) are in \( \mathcal{W}(V \times V') \) it follows that \( \mathcal{H}(\chi_{\epsilon} \mu) \rightarrow \mathcal{H} \mu \) and that \( \mathcal{P}(\chi_{\epsilon} \mu) \rightarrow \mathcal{P} \mu \) in the sense of currents, and consequently \( \partial \mathcal{H}(\chi_{\epsilon} \mu) \rightarrow \partial \mathcal{H} \mu \) in the current sense. It remains to see that \( \lim_{\epsilon \rightarrow 0} \mathcal{H}(\partial(\chi_{\epsilon} \mu)) = \mathcal{H}(\partial \mu) \). In fact, since by assumption \( \partial \mu \in \mathcal{W}(V) \) it follows that \( \mathcal{H}(\chi_{\epsilon} \partial \mu) \rightarrow \mathcal{H}(\partial \mu) \) and so

\[
(3.4) \quad \lim_{\epsilon \rightarrow 0} \mathcal{H}(\partial(\chi_{\epsilon} \mu)) = \mathcal{H}(\partial \mu) + \lim_{\epsilon \rightarrow 0} \mathcal{H}(\partial \chi_{\epsilon} \wedge \mu);
\]

it also follows that

\[
(3.5) \quad \partial \chi_{\epsilon} \wedge \mu = \partial(\chi_{\epsilon} \mu) - \chi_{\epsilon} \partial \mu \rightarrow \partial \mu - \partial \mu = 0.
\]

Now, if \( \zeta \) is in a compact subset of \( V'_{\text{reg}} \) and \( \epsilon \) is sufficiently small, then \( k(z, \zeta) \wedge \chi_{\epsilon} \) is a smooth form times \( \omega = \omega(\zeta) \). Since \( \mu(z) \wedge \omega(\zeta) \) is just a tensor product it follows from (3.5) that \( \partial \chi_{\epsilon} \wedge \mu(z) \wedge \omega(\zeta) \rightarrow 0 \). Hence, \( \mathcal{H}(\partial \chi_{\epsilon} \wedge \mu) \rightarrow 0 \) as a current on \( V'_{\text{reg}} \) and so by (3.4) we have \( \lim_{\epsilon \rightarrow 0} \mathcal{H}(\partial(\chi_{\epsilon} \mu)) = \mathcal{H}(\partial \mu) \).

\[\square\]

4. The dualizing Dolbeault complex of \( \mathcal{A}_{X}^{n,q} \)-currents

Let \( X \) be a reduced complex space of pure dimension \( n \). We define our sheaves \( \mathcal{A}_{X}^{n,*} \) in a way similar to the definition of \( \mathcal{A}_{X}^{n,*} \); see the end of Section 2.5. In a moral sense \( \mathcal{A}_{X}^{n,q} \) then becomes the smallest sheaf that contains \( \mathcal{A}_{X}^{n,q} \) and that is closed under integral operators \( \mathcal{H} \) and exterior products with elements of \( \mathcal{A}_{X}^{n,q} \).

\[\text{The proof goes through also in our setting, i.e., when } g \text{ not necessarily has compact support in } D_{\zeta} \text{ but } \varphi(\zeta) \text{ has.}\]
**Definition 4.1.** We say that an \((n, q)\)-current \(\psi\) on an open set \(V \subset X\) is a section of \(\mathcal{A}^{n,q}_X\), \(\psi \in \mathcal{A}^{n,q}(V)\), if, for every \(x \in V\), the germ \(\psi_x\) can be written as a finite sum of terms
\[
(4.1) \quad \xi_m \wedge \mathcal{K}_m \left( \cdots \xi_1 \wedge \mathcal{K}_1(\omega \wedge \xi_0) \cdots \right),
\]
where \(\xi_j\) are smooth \((0, *)\)-forms, \(\mathcal{K}_j\) are integral operators at \(x\) given by (3.1) with kernels of the form (2.18), and \(\omega\) is a structure form at \(x\).

Notice that \(\omega\) takes values in some bundle \(\oplus_j E_j\) so we let \(\xi_0\) take values in \(\oplus_j E_j^*\) to make \(\omega \wedge \xi_0\) scalar valued.

It is clear that \(\mathcal{K}\) preserves \(\oplus_q \mathcal{A}^{n,q}_X\). Notice that we allow \(m = 0\) in the definition above so that \(\mathcal{A}^{n,q}_X\) contains all currents of the form \(\omega \wedge \xi_0\), where \(\xi_0\) is smooth with values in \(\oplus_j E_j^*\). Since \(\partial \mu\) is of the form \(\omega \wedge \xi\) for a smooth \(\xi\), also \(\partial\) preserves \(\oplus_q \mathcal{A}^{n,q}_X\).

Recall that if \(\mu \in \mathcal{W}^{m,*}(V)\), then \(\mathcal{K} \mu \in \mathcal{W}^{m,*}(V')\), where \(V'\) is a relatively compact subset of \(V\). Since \(\omega \wedge \xi_0 \in \mathcal{W}^{n,q}_X\) it follows that \(\mathcal{K}^{n,q}_X\) is a subsheaf of \(\mathcal{W}^{n,q}_X\). In fact, by Proposition 4.3 below we can say more.

**Definition 4.2.** A current \(\mu \in \oplus_q \mathcal{W}^{n,q}_X\) is said to be in the domain of \(\partial\), \(\mu \in \text{Dom } \partial\), if \(\partial \mu \in \oplus_q \mathcal{W}^{n,q}_X\).

Assume that \(\mu \in \mathcal{W}^{n,q}_X\) is smooth on \(X_{\text{reg}}\), let \(h\) be a holomorphic tuple cutting out \(X_{\text{sing}}\), and let \(\chi_\epsilon = \chi(|h|/\epsilon)\). Then \(\partial \chi_\epsilon \mu \to \partial \mu\) since \(\mu\) has the SEP. In view of the first equality in (3.5) it follows that \(\partial \mu\) has the SEP if and only if \(\partial \chi_\epsilon \wedge \mu \to 0\) as \(\epsilon \to 0\); this last condition can be interpreted as a “boundary condition” on \(\mu\) at \(X_{\text{sing}}\).

**Proposition 4.3.** Let \(X\) be a reduced complex space of pure dimension \(n\). Then
\[
\begin{align*}
(i) \quad & \mathcal{A}^{n,q}_X|_{X_{\text{reg}}} = \mathcal{E}^{n,q}_X|_{X_{\text{reg}}}; \\
(ii) \quad & \mathcal{E}^{n,q}_X \subset \mathcal{A}^{n,q}_X \subset \text{Dom } \partial.
\end{align*}
\]

**Proof.** Part (i) is proved in the same way as part (i) of Lemma 6.1 in [7].

Let \(\psi\) be a smooth \((n, q)\)-form on \(X\) and let \(\omega = \sum r \omega_r\) be a structure form. Then, by Proposition 2.6, there is smooth \((0, q)\)-form \(\xi\) (with values in the appropriate bundle) such that \(\psi = \omega_0 \wedge \xi\) and so \(\mathcal{E}^{n,q}_X \subset \mathcal{A}^{n,q}_X\).

To prove the second inclusion of (ii) we may assume that \(\mu\) is of the form (4.1). Let \(k_j(w^{j-1}, w^j), j = 1, \ldots, m\), be the integral kernel corresponding to \(\mathcal{K}_j\); \(w^j\) are coordinates on \(V\) for each \(j\). We define an almost semi-meromorphic current \(T\) on \(V^{m+1}\) (the \(m + 1\)-fold Cartesian product) by
\[
(4.2) \quad T := \bigwedge_{j=1}^m k_j(w^{j-1}, w^j) \wedge \omega(w^0),
\]
and we let \(T_r\) be the term of \(T\) corresponding to \(\omega_r\). Notice that \(\pi_*(\xi \wedge T) = \mu\) for a suitable smooth \((0, *)\)-form \(\xi\) on \(V^{m+1}\), where \(\pi: V^{m+1} \to V_w^m\) is the natural projection. We will prove that
\[
(4.3) \quad \lim_{\epsilon \to 0} \partial \chi(|h(w^m)|/\epsilon) \wedge T_r = 0,
\]
where \(h\) is a holomorphic tuple cutting out \(V_{\text{sing}}\), by double induction over \(m\) and \(r\); cf. the discussion after Definition 4.2.
If $m = 0$ then $T = \omega(w^0)$ and, since $\bar{\partial}\omega_r = f_{r+1}|V\omega_{r+1}$ by (2.5), it follows that $\bar{\partial}T$ has the SEP, i.e., $\lim_{r \to 0} \partial_X (|h|/r) \wedge T = 0$.

Assume that (4.3) holds for $m \leq k - 1$ and all $r$. The left hand side of (4.3), with $m = k$, defines a pseudomeromorphic current $\tau_r$ of bidegree $(*, kn - k + r + 1)$ since each $k_j$ has bidegree $(*, n - 1)$ and clearly $\mathrm{supp} \tau_r \subset \operatorname{Sing}(V_{w^m}) \times V^m$. If $w^j \neq w^{j-1}$, then $k_j(w^{j-1}, w^j)$ is a smooth form times some structure form $\tilde{\omega}(w^j)$. Thus $T$, with $m = k$, is a smooth form times the tensor product of two currents, each of which is of the form (4.2) with $m < k$. By the induction hypothesis, it follows that (4.3), with $m = k$, holds outside $\{w^j = w^{j-1}\}$ for all $j$. Hence, $\tau_r$ has support in $\{w^1 = \cdots = w^k\} \cap (\operatorname{Sing}(V_{w^m}) \times V^m)$, which has codimension at least $kn + 1$ in $V^{k+1}$. Since $\tau_0$ has bidegree $(*, kn - k + 1)$, $k \geq 1$, it follows from the dimension principle that $\tau_0 = 0$.

By Proposition 2.5, there is a $(0, 1)$-form $\alpha_1$ such that $\omega_1 = \alpha_1 \omega_0$ and $\alpha_1$ is smooth outside $V^1$ (cf. (2.4)) which has codimension at least 2 in $V$. Since $\tau_1 = \alpha_1(w^0)\tau_0$ outside $V^1_{w^0}$ and $\tau_0 = 0$ it follows that $\tau_1$ has support in $\{w^1 = \cdots = w^k\} \cap (V_{w^1} \times V^m)$. This set has codimension at least $kn + 2$ in $V^{m+1}$ and $\tau_1$ has bidegree $(*, kn - k + 2)$ so the dimension principle shows that $\tau_1 = 0$. Continuing in this way we get that $\tau_r = 0$ for all $r$ and hence, (4.3) holds with $m = k$.

**Theorem 4.4.** Let $X$ be a reduced complex space of pure dimension $n$. Then $\bar{\partial}_p \mathcal{A}^{n,q} = \mathcal{A}^{n,q+1}$.

**Proof.** Let $\psi$ be a germ of a current in $\mathcal{A}^{n,q}_X$ at some point $x$; we may assume that
\[
\psi = \zeta_m \wedge \mathcal{H}_m(\cdots \zeta_1 \wedge \mathcal{H}_1(\omega \wedge \zeta_0) \cdots),
\]
see Definition 4.1.

We will prove the theorem by induction over $m$. Assume first that $m = 0$ so that $\psi = \omega \wedge \zeta_0$; recall that $\zeta_0$ takes values in $\oplus_j E_j$ so that $\psi$ is scalar valued. Then, by Proposition 2.5, we have that
\[
\bar{\partial}_p \psi = \bar{\partial} \zeta \wedge \xi_0 \pm \omega \wedge \bar{\partial} \xi_0 = f \omega \wedge \xi_0 \pm \omega \wedge \bar{\partial} \xi_0 = \omega \wedge f^* \xi_0 \pm \omega \wedge \bar{\partial} \xi_0,
\]
where $f = \oplus_{j=0}^r f_{j+1}|V$ and $f^*$ is the transpose of $f$. Hence, $\bar{\partial}_p \psi$ is in $\mathcal{A}^{n,q+1}_X$. Assume now that $\bar{\partial}_p \psi' \in \oplus_q \mathcal{A}^{n,q}_X$, where
\[
\psi' = \zeta_{m-1} \wedge \mathcal{H}_{m-1}(\cdots \zeta_1 \wedge \mathcal{H}_1(\omega \wedge \zeta_0) \cdots).
\]
Then $\psi' \in \operatorname{Dom} \bar{\partial}_p \subset \mathcal{W}_X$ and by Proposition 4.3 $\psi'$ is smooth on $X_{\text{reg}}$. Thus, from Proposition 3.1 it follows that
\[
\psi' = \bar{\partial} \mathcal{H}_m \psi' + \mathcal{H}_m(\bar{\partial} \psi') + \mathcal{D}_m \psi',
\]
in the current sense on $V_{\text{reg}}$, where $V$ is some neighborhood of $x$. By the induction hypothesis, $\partial \psi' \in \oplus_q \mathcal{A}^{n,q}_X$ and since $\mathcal{H}_m$ and $\mathcal{D}_m$ preserve $\oplus_q \mathcal{A}^{n,q}_X$ and furthermore $\oplus_q \mathcal{A}^{n,q}_X \subset \operatorname{Dom} \bar{\partial}_p$ it follows that every term of (4.4) has the SEP. Thus, (4.4) holds in fact on $V$. Finally, notice that $\psi = \zeta_m \wedge \mathcal{H}_m \psi'$ and so, since $\psi'$, $\mathcal{H}_m(\bar{\partial} \psi')$, and $\mathcal{D}_m \psi'$ all are in $\oplus_q \mathcal{A}^{n,q}_X$, it follows that $\bar{\partial} \psi \in \mathcal{A}^{n,q+1}_X$. 

**Proof of Theorem 1.1.** Choose a weight $g$ with respect to $\zeta \in D'$ and with compact support in $D_\zeta$ in the kernels $k(z, \zeta)$ and $p(z, \zeta)$, cf. (2.18) and (2.19), and let $\mathcal{H}$ and $\mathcal{D}$ be the associated integral operators.
Let $\psi \in \mathcal{A}^{n,q}(V)$. By Proposition 3.1,

$$
(4.5) \quad \psi = \partial \mathcal{H} \psi + \mathcal{H}(\overline{\partial} \psi) + \mathcal{H} \psi
$$

holds on $V'_{reg}$. Since $\mathcal{H}$ and $\mathcal{H}'$ map $\oplus_q \mathcal{A}^{n,q}(V)$ to $\oplus_q \mathcal{A}^{n,q}(V')$ it follows from Theorem 4.4 that every term of (4.5) has the SEP. Hence, (4.5) holds on $V'$ and the theorem follows.

\[\square\]

**Proof of Theorem 1.2.** Let $V$ be a pure $n$-dimensional analytic subset of a pseudo-convex domain $D \subset \mathbb{C}^N$, let $\mathcal{J}$ be the sheaf in $D$ defined by $V$, let $i: V \hookrightarrow D$ be the inclusion, and let $p = N - n$ be the codimension of $V$ in $D$. Let (2.3) be a resolution of $\mathcal{O}_D/\mathcal{J}$ in (possibly a slightly smaller domain still denoted) $D$ and let $\omega = \sum_r \omega_r$ be an associated structure form.

Taking $\mathcal{H}$om's from the complex (2.3) into $\mathcal{O}_D$ and tensoring with the invertible sheaf $\Omega^N_D$ gives the complex

$$
(4.6) \quad 0 \rightarrow \mathcal{O}(E_0^*) \otimes \mathcal{O}_D \mathcal{N}^D \xrightarrow{f_1} \cdots \xrightarrow{f_m} \mathcal{O}(E_m^*) \otimes \mathcal{O}_D \mathcal{N}^D \rightarrow 0.
$$

It is well-known that the cohomology sheaves of (4.6) are isomorphic to $\mathcal{O}(\mathcal{J}/\mathcal{J}, \mathcal{N}_D^N)$ and that $\mathcal{O}(\mathcal{J}/\mathcal{J}, \mathcal{N}_D^N) = 0$ for $k \neq p$. Notice that if $V$ is Cohen-Macaulay, i.e., if we can take $m = p = \text{codim} V$ in (2.3), then $\mathcal{O}(\mathcal{J}/\mathcal{J}, \mathcal{N}_D^N) = 0$ for $k \neq p$.

We define mappings $\varrho_k: \mathcal{O}(E_{p+k}^*) \otimes \mathcal{N}_D^N \rightarrow \mathcal{A}_V^{n,k}$ by letting $\varrho_k(hdz) = 0$ for $k < 0$ and $\varrho_k(hdz) = \omega_k \cdot h$ for $k \geq 0$; here we let $\mathcal{A}_V^{n,k} := 0$ for $k < 0$ and $\mathcal{O}(E_k^*) \otimes \mathcal{N}_D^N := 0$ for $k > m$. We get a map

$$
(4.7) \quad \varrho_k: (\mathcal{O}(E_{p+k}^*) \otimes \mathcal{N}_D^N, f_{p+k}^*) \rightarrow (\mathcal{A}_V^{n,k}, \partial)
$$

which is a morphism of complexes since if $h \in \mathcal{O}(E_{p+k}^*)$, then, by Proposition 2.5,

$$
\partial \varrho_k(hdz) = \Delta \omega_k \cdot h = f_{p+k+1}^* \omega_{k+1} \cdot h = \omega_{k+1} \cdot f_{p+k+1}^* h = \varrho_{k+1}(f_{p+k+1}^* h).
$$

Hence, (4.7) induces a map on cohomology. We claim that $\varrho_k$ in fact is a quasi-isomorphism, i.e., that $\varrho_k$ induces an isomorphism on cohomology level. Given the claim it follows that $\mathcal{H}^k(\mathcal{A}_V^{n,k})$ is coherent since the corresponding cohomology sheaf of $(\mathcal{O}(E_{p+k}^*) \otimes \mathcal{N}_D^N, f_{p+k}^*)$ is $\mathcal{O}(\mathcal{J}/\mathcal{J}, \mathcal{N}_D^N)$, which is coherent.

To prove the claim, recall first that $i_* \omega_k = R_k \wedge dz$. Thus, by [5, Theorem 7.1] the mapping on cohomology is injective. For the surjectivity, choose integral operators $\mathcal{H}$ and $\mathcal{H}'$ corresponding to integral kernels (2.18) and (2.19) respectively, where $g$ is a weight with respect to $\zeta \in D'$ that is holomorphic in $\zeta$ and has compact support in $D_z$. Let $\psi \in \mathcal{A}^{n,k}(V)$ be $\partial$-closed. By Theorem 1.1 we get

$$
\psi(\zeta) = \partial \int_{V_z} k(z, \zeta) \wedge \psi(z) + \int_{V_z} p(z, \zeta) \wedge \psi(z)
$$

in $V \cap D'$. Hence, the $\partial$-cohomology class of $\psi$ is represented by the last integral. For degree reasons it follows from (2.19) that this integral is of the form

$$
\omega_k(\zeta) \wedge \int_{V_z} G(z, \zeta) \wedge \psi(z),
$$

where $G$ takes values in $E_k^*$ and $G(z, \zeta)$ is holomorphic in $\zeta$ since we have chosen the weight $g$ to be. Thus, the class of $\psi$ is in the image of $\varrho_k$. 

\[\square\]
If $V$ is Cohen-Macaulay, then (4.6) is exact except for at level $p$ and so $(\mathcal{A}^{n,q}_X, \bar{\partial})$ is exact except for at level 0 where the cohomology is $\omega^{n,0}_V = \ker (\bar{\partial} : \mathcal{A}^{n,0}_V \to \mathcal{A}^{n,1}_V)$. Thus, (1.4) is exact.

5. The trace map

The basic result of this section is the following theorem. It is the key to define our trace map.

**Theorem 5.1.** Let $X$ be a reduced complex space of pure dimension $n$. There is a unique map

$$\wedge : \mathcal{A}^{n,q}_X \times \mathcal{A}^{0,q'}_X \to \mathcal{W}^{n,q+q'}_X \cap \text{Dom} \bar{\partial}$$

extending the exterior product on $X_{\text{reg}}$.

It follows that one can compute this product in the following way: Let $\psi \in \mathcal{A}^{n,q}_X$, let $\varphi \in \mathcal{A}^{0,q'}_X$ and let $V \subset X$ be an open set where both $\psi$ and $\varphi$ are defined. Then, if $h$ is a generically non-vanishing holomorphic tuple such that $V_{\text{sing}} \subset \{ h = 0 \}$, we have that

$$\psi \wedge \varphi = \lim_{\epsilon \to 0} \chi(|h|/\epsilon) \psi \wedge \varphi,$$

where the limit is understood in the sense of currents.

Let $\psi \in \mathcal{A}^{n,q}(X)$ and $\varphi \in \mathcal{A}^{0,n-\eta}(X)$ and assume that at least one of $\psi$ and $\varphi$ has compact support. Then, by Theorem 5.1, we can define our trace map on the level of currents by mapping $(\psi, \varphi)$ to the action of $\psi \wedge \varphi$ on the function 1; explicitly, if $h$ is a generically non-vanishing holomorphic tuple such that $X_{\text{reg}} \subset \{ h = 0 \}$, then

$$\lim_{\epsilon \to 0} \int_X \chi(|h|/\epsilon) \psi \wedge \varphi.$$

This map induces a trace map on cohomology. Indeed, assume that $\psi$ and $\varphi$ are $\bar{\partial}$-closed and that one of them, say $\varphi$, is $\bar{\partial}$-exact so that there is a $\tilde{\varphi} \in \mathcal{A}^{0,n-\eta-1}(X)$, which has compact support if $\varphi$ has, such that $\varphi = \bar{\partial} \tilde{\varphi}$. By Theorem 5.1, $\psi \wedge \tilde{\varphi}$ is in the domain of $\bar{\partial}$, which implies that $\bar{\partial} \chi_{\epsilon} \wedge \psi \wedge \tilde{\varphi} \to 0$ as $\epsilon \to 0$, where $\chi_{\epsilon} = \chi(|h|/\epsilon)$; cf. the first equality in (3.5). Hence

$$\lim_{\epsilon \to 0} \int_X \chi_{\epsilon} \psi \wedge \varphi = \lim_{\epsilon \to 0} \int_X \chi_{\epsilon} \psi \wedge \bar{\partial} \tilde{\varphi} = (-1)^{n+\eta} \lim_{\epsilon \to 0} \left( \int_X \bar{\partial} (\chi_{\epsilon} \psi \wedge \tilde{\varphi}) - \int_X \bar{\partial} \chi_{\epsilon} \wedge \psi \wedge \tilde{\varphi} \right) = 0.$$

**Proof of Theorem 5.1.** Notice first that if there is a current $\mu \in \mathcal{W}_X$ coinciding with $\psi|_{X_{\text{reg}}} \wedge \varphi|_{X_{\text{reg}}}$ on $X_{\text{reg}}$, then it must be unique. Moreover, then $\mu = \lim_{\epsilon \to 0} \chi_{\epsilon} \mu = \lim_{\epsilon \to 0} \chi_{\epsilon} \psi \wedge \varphi$. We now prove that such a $\mu$ exists.

Let $V$ be a relatively compact open subset of a pure $n$-dimensional analytic subset of some pseudoconvex domain in some $\mathbb{C}^N$. Let $\psi \in \mathcal{A}^{n,q}(V)$ and $\varphi \in \mathcal{A}^{0,q}(V)$. The tensor product $\psi(w) \wedge \varphi(z)$ is of course well defined on $V \times V$. Let $\phi = (\phi_1, \ldots, \phi_s)$ be generators for the radical ideal sheaf over $V \times V$ associated to the diagonal $\Delta^V \subset V \times V$. Let

$$M^\lambda = \bar{\partial} |\phi|^{2\lambda} \wedge \frac{\partial \log |\phi|^2}{2\pi i} \wedge (dd^c \log |\phi|^2)^{n-1},$$

where $\lambda \in \mathbb{C}$, $\text{Re} \lambda \gg 0$, and $dd^c = i\partial\bar{\partial}/2\pi$. It is proved in [3] that $\lambda \mapsto M^\lambda$ has an analytic continuation as a current-valued function to a neighborhood of $\lambda = 0$. 

Moreover, by [8, Theorem 1.2], \( M := M^\lambda|_{\lambda=0} = \beta[\Delta^V] \), where \( \beta \) is the generic multiplicity of \( \Delta^V \) in \( V \times V \). Hence, \( M = [\Delta^V] \).

**Claim:**

\[ \lambda \mapsto M^\lambda \wedge \psi(w) \wedge \varphi(z) \]

has an analytic continuation to a neighborhood of \( \lambda = 0 \) and \( M^\lambda \wedge \psi(w) \wedge \varphi(z)|_{\lambda=0} \) defines an intrinsic \( \mathcal{PM} \)-current on \( \Delta^V \simeq V \) with the SEP, i.e., there is a current \( \mu \in \mathcal{W}(\Delta^V) \) such that \( i_*\mu = M^\lambda \wedge \psi(w) \wedge \varphi(z)|_{\lambda=0} \), where \( i : \Delta^V \to V \times V \) is the inclusion.

Assume the claim for the moment. Since \( M^\lambda|_{\lambda=0} = [\Delta^V] \) and \( \psi \) and \( \varphi \) are smooth on \( V_{reg} \times V_{reg} \) it follows, after making the identification \( \Delta^V \simeq V \), that \( \mu = \psi \wedge \varphi \) on \( V_{reg} \). Thus \( \psi \wedge \varphi|_{V_{reg}} \) has a \( \mathcal{W}(V) \)-extension to \( V \) (namely \( \mu \)); we denote the extension by \( \psi \wedge \varphi \) as well.

To prove the claim we may assume, cf. Definition 4.1 and the end of Section 2.5, that

\[ \psi = \xi_m \wedge \mathcal{H}_m \left( \cdots \xi_1 \wedge \mathcal{H}_1 (\omega \wedge \xi_0) \cdots \right), \quad \varphi = \xi_\ell \wedge \mathcal{H}_\ell \left( \cdots \xi_1 \wedge \mathcal{H}_1 (\xi_0) \cdots \right), \]

where \( \xi_i \) and \( \xi_\ell \) are smooth \((0,*)\)-forms, \( \omega = \sum_k \omega_k \) is a structure form, and \( \mathcal{H}_i \) and \( \mathcal{H}_j \) are integral operators for \((n,*)\)-forms and \((0,*)\)-forms respectively. Let \( k_j(w^{j-1}, w^j) \) be the integral kernel corresponding to \( \mathcal{H}_j \) and let \( k_j(z^{j-1}, z^j) \) be the integral kernel corresponding to \( \mathcal{H}_j \); \( w^j \) and \( z^j \) are coordinates on \( V \). We will assume that for each \( j \), \( z^j \mapsto k_{j+1}(z^j, z^{j+1}) \) has compact support where \( z^j \mapsto k_j(z^{j-1}, z^j) \) is defined and similarly for \( k_j \); possibly we will have to multiply by a smooth cut-off function that we however will suppress. The kernels \( \bar{k}_j \) and \( k_j \) are almost semi-meromorphic and \( M^\lambda \) is as smooth as we want if \( \Re \lambda \) is sufficiently large and hence, cf. Proposition 2.2,

\[ T^\lambda := M^\lambda(z^\ell, w^m) \wedge \bigwedge_{j=1}^m \bar{k}_j(w^{j-1}, w^j) \wedge \omega(w^0) \wedge \bigwedge_{j=1}^\ell k_j(z^{j-1}, z^j) \]

is an almost semi-meromorphic current on \( V^{\ell+m+2} \). \(^4\) We will consider \( \phi = \phi(z^\ell, w^m), \bar{k}_j(w^{j-1}, w^j) \), etc. as functions (or forms) on \( V^{\ell+m+2} \). By resolution of singularities, there is a modification \( \Pi : Y \to V^{\ell+m+2} \), with \( Y \) smooth, such that (locally on \( Y \)) we have \( \Pi^* \phi = \phi^0 \phi' \), where \( \phi^0 \) is a holomorphic function and \( \phi' \) is a non-vanishing holomorphic tuple, and

\[ \Pi^* \left( \bigwedge_{j=1}^m \bar{k}_j(w^{j-1}, w^j) \wedge \omega(w^0) \wedge \bigwedge_{j=1}^\ell k_j(z^{j-1}, z^j) \right) = s_0/a, \]

where \( s_0 \) is smooth and \( a \) is a holomorphic function. A straightforward computation then shows that (locally on \( Y \)) we have

\[ \Pi^* M^\lambda = \frac{\overline{\partial}\phi^0 \phi'^{2\lambda}}{\phi^0} \wedge s_1 + \overline{\partial}(\phi^0 \phi'^{2\lambda} \wedge s_2), \quad \Re \lambda \gg 0, \]

where the \( s_1 \) and \( s_2 \) are smooth. From, e.g., [24, Lemma 6] it follows that \( \lambda \mapsto \overline{\partial}(\phi^0 \phi'^{2\lambda} / (\phi^0 a)) \) has an analytic continuation as a current-valued function to a neighborhood of \( \lambda = 0 \) and that the value at \( \lambda = 0 \) is a \( \mathcal{PM} \)-current on \( Y \). Hence,

\(^4\)In this proof \( V^j \) will mean either the Cartesian product of \( j \) copies of \( V \) or the \( j^\text{th} \) set in (2.4). We hope that it will be clear from the context what we are aiming at.
\( \lambda \mapsto \Pi^*(M^\lambda) \wedge s_0/a = \Pi^*T^\lambda \) has an analytic continuation to a neighborhood of \( \lambda = 0 \) and so \( \lambda \mapsto T^\lambda \) has an analytic continuation to a neighborhood of \( \lambda = 0 \) and

\[
T := T^\lambda|_{\lambda=0}
\]

is a \( PM \)-current on \( V^{\ell+m+2} \). Moreover, it is clear that the support of \( T \) must be contained in \( \{ z^\ell = w^m \} \).

Let \( \pi : V^{\ell+m+2} \to V_{w^m} \times V_{z^\ell} \) be the natural projection. Since \( M^\lambda \wedge \psi(w) \wedge \varphi(z) \) is \( \pi_* (T^\lambda) \) times a smooth form it is sufficient to prove our claim with \( M^\lambda \wedge \psi(w) \wedge \varphi(z) \) replaced by \( \pi_* (T^\lambda) \). We know already that \( \lambda \mapsto \pi_* (T^\lambda) \) has an analytic continuation to a neighborhood of \( \lambda = 0 \) and that \( \tau := \pi_* (T^\lambda)|_{\lambda=0} \) is a \( PM \)-current on \( V_{w^m} \times V_{z^\ell} \) with support in \( \Delta^V \). We will now use the following lemma.

**Lemma 5.2.** Let \( h = h(z^\ell, w^m) \) be a holomorphic tuple such that \( H = \{ h = 0 \} \subset V^{\ell+m+2} \) intersects \( \{ z^\ell = w^m \} \) properly and let \( \chi_\ell = \chi((h)/\epsilon) \). Let also \( g = g(z^\ell, w^m) \) be a holomorphic function vanishing on \( \{ z^\ell = w^m \} \). Then

\[
\begin{align*}
(i) & \quad 1_H T := T - \lim_{\epsilon \to 0} \chi_\ell T = 0 \\
(ii) & \quad \lim_{\epsilon \to 0} \bar{\partial} \chi_\ell \wedge T = 0 \\
(iii) & \quad g T = 0.
\end{align*}
\]

From part (i) of the lemma it follows that \( \tau \) has the SEP with respect to \( \Delta^V \) and from part (iii) it follows that \( h \tau = 0 \) for any holomorphic function vanishing on \( \Delta^V \). Moreover, in \( \text{Reg}(V_{w^m}) \times \text{Reg}(V_{z^\ell}) \) we have that \( \tau \) equals \( |\Delta^V| \) times a smooth form and since \( dh \wedge |\Delta^V| = d(h|\Delta^V|) = 0 \) for any holomorphic function vanishing on \( \Delta^V \) it follows that \( dh \wedge \tau = 0 \) in \( \text{Reg}(V_{w^m}) \times \text{Reg}(V_{z^\ell}) \) for any holomorphic function vanishing on \( \Delta^V \). Since \( \tau \) has the SEP with respect to \( \Delta^V \) this holds in fact on \( V_{w^m} \times V_{z^\ell} \). Thus, by Proposition 2.3, there is a \( \mu \in \mathcal{W}(\Delta^V) \) such that \( i_* \mu = \tau \) and the claim follows.

It remains to show that our current \( \psi \wedge \varphi \in \mathcal{W}(V) \) is in the domain of \( \bar{\partial} \). Let \( \bar{h} \) and \( \chi_\ell \) be as in Lemma 5.2. From part (ii) of Lemma 5.2 it follows that \( \lim_{\epsilon \to 0} \bar{\partial} \chi_\ell \wedge \tau = 0 \). Since \( \tau = i_* \mu \) and \( \psi \wedge \varphi = \mu \) (after identifying \( V \simeq \Delta^V \)) we see that \( \lim_{\epsilon \to 0} \bar{\partial} \chi((h)/\epsilon) \wedge \psi \wedge \varphi = 0 \) for any generically non-vanishing holomorphic tuple \( \bar{h} \) on \( V \) such that \( \{ h = 0 \} \supset V_{\text{sing}} \). From the discussion after Definition 4.2 it follows that \( \psi \wedge \varphi \) indeed is in the domain of \( \bar{\partial} \).

**Proof of Lemma 5.2.** Let \( T_k \) be the component of \( T \) corresponding to \( \omega_k(w^0) \). We will show the lemma by double induction over \( \ell \) and \( \ell + m \) by using the dimension principle, cf. the proof of Proposition 4.3. Notice first that \( 1_H T \) and \( \lim_{\epsilon \to 0} \bar{\partial} \chi_\ell \wedge T \) have support contained in \( \{ z^\ell = w^m \} \cap H \).

Consider first the case \( \ell + m = 0 \); then \( T = M^\lambda(z^0, w^0) \wedge \omega(w^0)|_{\lambda=0} \) and part (i) means precisely that \( T \) has the SEP with respect to the diagonal \( \Delta^V \subset V_{z^0} \times V_{w^0} \). The \( PM \)-current \( 1_H T_0 \) has bidegree \( (2n, n) \) and support contained \( \{ z^0 = w^0 \} \cap H \), which has codimension at least \( n + 1 \) in \( V_{z^0} \times V_{w^0} \), and hence \( 1_H T_0 = 0 \) by the dimension principle. By Proposition 2.5 there are \( \alpha_k \) that are smooth outside \( V^k \) such that \( \omega_k = \alpha_k \omega_{k-1} \). Thus, \( 1_H T_0 = 0 \) it follows that \( 1_H T_1 = 0 \) outside \( V_{z^0} \times V_{w^0} \). Moreover, since \( 1_H T_1 \) is also 0 outside \( \{ z^0 = w^0 \} \), the support of \( 1_H T_1 \) is contained in \( V_{z^0} \times V_{w^0}^1 \cap \{ z^0 = w^0 \} \), which has codimension \( \geq n + 2 \) in \( V_{z^0} \times V_{w^0} \). Noticing that \( 1_H T_1 \) has bidegree \( (2n, n+1) \) the dimension principle again shows that \( 1_H T_1 = 0 \). Continuing in this way we see that \( 1_H T_k = 0 \) for all \( k \), and so (i) holds in the case \( \ell + m = 0 \).
To see that (iii) holds in the case $\ell + m = 0$, notice that if $w^0 \in V_{\text{reg}}$ then $T_0$ is the integration current over $\{z \equiv w^0\}$ times a smooth form. Since $g = 0$ on $\{z \equiv w^0\}$ it follows that $gT_0 = 0$ at least outside of $\{z \equiv w^0, \quad w^0 \in V_{\text{sing}}\}$. But then, as above, $gT_0 = 0$ and $gT_k = 0$ inductively by the dimension principle.

To see that also (ii) holds in the case $\ell + m = 0$ we proceed as follows: Let $\tilde{h}$ be a generically non-vanishing holomorphic function on $V$ such that $\{\tilde{h} = 0\} \supset V_{\text{sing}}$ and let $\tilde{\chi}_\delta = \chi(\tilde{h}(w^0)/\delta)$. We have proved that $T_k$ has the SEP with respect to $\Delta V \subset V_{z^0} \times V_{w^0}$ and so

$$T_k = \lim_{\delta \to 0} \tilde{\chi}_\delta M(z^0, w^0) \wedge \omega_k(w^0).$$

Let $i: \Delta V \hookrightarrow V_{z^0} \times V_{w^0}$ be the inclusion of the diagonal. Since $\omega$ is smooth outside $V_{\text{sing}}$ and $M$ is the integration current over $\{z^0 = w^0\}$ we have

$$(5.4) \quad \bar{\partial} \chi \wedge \tilde{\chi}_\delta M(z^0, w^0) \wedge \omega_k(w^0) = i_*(-\bar{\partial} \chi \wedge \tilde{\chi}_\delta \omega_k).$$

As $\delta \to 0$, the left hand side of (5.4) goes to $\bar{\partial} \chi \wedge T_k$. Since $\omega$ has the SEP with respect to $V \simeq \Delta V$ it follows that the right hand side goes to $i_*(-\bar{\partial} \chi \wedge \omega_k)$ as $\delta \to 0$. A straightforward computation and Proposition 2.5 show that

$$\bar{\partial} \chi \wedge \omega_k = \bar{\partial} (\chi \wedge \omega_k) - \chi \wedge f_{k+1} \omega_{k+1} \to \bar{\partial} \omega_k - f_{k+1} \omega_{k+1} = 0, \quad \epsilon \to 0.$$ 

Hence, $i_*(\bar{\partial} \chi \wedge \omega_k) \to 0$ as $\epsilon \to 0$ and (ii) follows for $\ell + m = 0$.

Assume now that the lemma holds for $\ell + m \leq s - 1$, where $s \geq 1$, and let $T$ be given by (5.2) and (5.3) with $\ell + m = s$. Let $1 \leq j \leq \ell$; if $z^{j-1} \neq z^j$ then $k_j(z^{j-1}, z^j)$ is a smooth form times some structure form $\tilde{\omega}(z^j)$. Hence, outside $\{z^j = z^{j-1}\}$, $T$ is (ignoring smooth factors) the tensor product of

$$\tilde{\omega}(z^j) \bigwedge_{i=1}^{j-1} k_i(z^{i-1}, z^i),$$

and some current $\tilde{T}$, where $\tilde{T}$ is of the form (5.2) and (5.3) but with $\ell + m = s - j$. From the induction hypothesis it thus follows that $1_H T$, $\lim_{\epsilon \to 0} \bar{\partial} \chi \wedge T$, and $gT$ have supports contained in $\{z^0 = \ldots = z^s\}$. Similarly, let $1 \leq j \leq m$. If $w^{j-1} \neq w^j$ then $k_j(w^{j-1}, w^j)$ is a smooth form times some structure form $\tilde{\omega}(w^j)$ and so, outside $\{w^j = w^{j+1}\}$, $T$ is (again ignoring smooth factors) the tensor product of

$$\bigwedge_{i=1}^{j-1} k_i(w^{i-1}, w^i) \wedge \omega(w^0)$$

and a current of the form (5.2) and (5.3) with $\ell + m = s - j$. Thus, again from the induction hypothesis, it follows that $1_H T$, $\lim_{\epsilon \to 0} \bar{\partial} \chi \wedge T$, and $gT$ have supports contained in $\{w^0 = \ldots = w^m\}$. In addition, since $T$ vanishes outside $\{z^\ell = w^m\}$, we have that the supports of $1_H T$, $\lim_{\epsilon \to 0} \bar{\partial} \chi \wedge T$, and $gT$ must be contained in the diagonal $\Delta V \subset V^{\ell + m + 2}$.

The currents $1_H T_0$ and $gT_0$ both have bidegree $(*, n(\ell + m + 1) - (\ell + m))$ and since $\Delta V \simeq V$ has codimension $n(\ell + m + 1)$, the dimension principle shows that $1_H T_0 = gT_0 = 0$. As in the beginning of the proof, one inductively shows that $1_H T_k = gT_k = 0$ using that $\omega_k(w^0) = \alpha_k(w^0) \omega_{k-1}(w^0)$ and the dimension principle. Hence (i) and (iii) hold.

To prove (ii) we notice that $\lim_{\epsilon \to 0} \bar{\partial} \chi \wedge T$ vanishes outside $H$. Its support is thus contained in $H \cap \Delta V$, which has codimension at least $n(\ell + m + 1) + 1$. Since
lim_{c \to 0} \partial \chi_c \wedge T_0 has bidegree (*, n(\ell + m + 1) - (\ell + m) + 1) the dimension principle shows that lim_{c \to 0} \partial \chi_c \wedge T_0 = 0. As above, it follows inductively that lim_{c \to 0} \partial \chi_c \wedge T_k = 0.

\[ \square \]

6. Serre duality

6.1. Local duality. Let V be a pure n-dimensional analytic subset of a pseudoconvex domain D \subset \mathbb{C}^N, let D' \subset D be a strictly pseudoconvex subdomain, and let V' = V \cap D'. Consider the complexes

\begin{align}
(6.1) & \quad 0 \to \mathcal{A}^{0,0}(V') \xrightarrow{\partial} \mathcal{A}^{0,1}(V') \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{A}^{0,n}(V') \to 0 \\
(6.2) & \quad 0 \to \mathcal{A}^{n,0}(V') \xrightarrow{\partial} \mathcal{A}^{n,1}(V') \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{A}^{n,n}(V') \to 0.
\end{align}

From Section 5 we know that we have a well-defined trace map on the level of currents and that it induces a trace map on the level of cohomology

\[ (6.3) \quad Tr: H^0(\mathcal{A}^{0,\bullet}(V')) \times H^n(\mathcal{A}^{n,\bullet}(V')) \to \mathbb{C}, \quad Tr([\varphi], [\psi]) = \int_{V'} \varphi \psi. \]

By Theorem 2.9 (iii) the complex (6.1) is exact except for at the level 0 where the cohomology is \( \mathcal{O}(V') \).

**Theorem 6.1.** The complex (6.2) is exact except for at the top level and the pairing (6.3) makes \( H^n(\mathcal{A}^{n,\bullet}(V')) \) the topological dual of the Fréchet space \( H^0(\mathcal{A}^{0,\bullet}(V')) = \mathcal{O}(V') \); in particular (6.3) is non-degenerate.

**Proof.** Let \( \psi \in \mathcal{A}^{n,q}(V') \) be \( \partial \)-closed. We choose integral kernels \( k(z, \zeta) \) and \( p(z, \zeta) \) corresponding to a weight \( g \), with respect to \( z \) in some neighborhood of \( \text{supp} \psi \) in \( D' \), such that \( g \) depends holomorphically on \( z \) and has compact support in \( D_c' \). Since \( \psi \) has compact support in \( V' \), Theorem 1.1 shows that

\[ (6.4) \quad \psi(\zeta) = \partial_c \int_{V'_c} k(z, \zeta) \wedge \psi(z) + \int_{V'_c} k(z, \zeta) \wedge \partial \psi(z) + \int_{V'_c} p(z, \zeta) \wedge \psi(z), \]

holds on \( V' \). The second term on the right hand side vanishes since \( \partial \psi = 0 \). Since \( g \) is holomorphic in \( z \) the kernel \( p \) has degree 0 in \( d\bar{z}_j \) and hence, also the last term vanishes if \( q \neq n \). The first integral on the right hand side is in \( \mathcal{A}^{n,q-1}(V') \) since \( g \) has compact support in \( D_c' \) and so (6.2) is exact except for at level \( n \).

To see that \( H^n(\mathcal{A}^{n,\bullet}(V')) \) is the topological dual of \( \mathcal{O}(V') \), recall that the topology on \( \mathcal{O}(V') \cong \mathcal{O}(D')/\mathcal{J}(D') \) is the quotient topology, where \( \mathcal{J} \) is the sheaf in \( D \) associated with \( V \subset D \). It is clear that each \( [\psi] \in H^n(\mathcal{A}^{n,\bullet}(V')) \) yields a continuous linear functional on \( \mathcal{O}(V') \) via (6.3). Moreover, if \( q = n \) and \( \int_{V'} \varphi \psi = 0 \) for all \( \varphi \in \mathcal{O}(V') \) then, since \( p(z, \zeta) \) is holomorphic in \( z \) by the choice of \( g \), the last integral on the right hand side of (6.4) vanishes and thus \( [\psi] = 0 \). Hence, \( H^n(\mathcal{A}^{n,\bullet}(V')) \) is a subset of the topological dual of \( \mathcal{O}(V') \).

To see that there is equality, let \( \lambda \) be a continuous linear functional on \( \mathcal{O}(V') \). By composing with the projection \( \mathcal{O}(D') \to \mathcal{O}(D')/\mathcal{J}(D') \) we get a continuous functional \( \tilde{\lambda} \) on \( \mathcal{O}(D') \). By definition of the topology on \( \mathcal{O}(D') \), \( \tilde{\lambda} \) is carried by some compact subset \( K \subset D' \). By the Hahn-Banach theorem, \( \tilde{\lambda} \) can be extended to a continuous linear functional on \( C^0(D') \) and so it is given as integration against some measure \( \mu \) on \( D' \) that has support in a neighborhood \( U(K) \subset D' \) of \( K \). Let \( \tilde{p}(z, \zeta) \) be an integral kernel, as in (2.19), corresponding to a weight \( \tilde{g} \) with respect to \( z \in U(K) \).
such that \( \hat{g} \) has compact support in \( D'_{\zeta} \) and depends holomorphically on \( z \in U(K) \). Let \( f \in \mathcal{O}(V') \) and define the sequence \( f_\epsilon(z) \in \mathcal{O}(K) \) by

\[
f_\epsilon(z) = \int_{V'_{\zeta}} \chi(|h|/\epsilon) \tilde{p}(z, \zeta) f(\zeta),
\]

where \( h = h(\zeta) \) is a holomorphic tuple cutting out \( V_{\text{sing}} \). For each \( z \) in a neighborhood in \( V' \) of \( K \cap V' \) we have that \( \lim_{\epsilon \to 0} f_\epsilon(z) = \mathcal{P} f(z) = f(z) \) by [7, Theorem 1.4]. We claim that \( f_\epsilon \) in fact converges uniformly in a neighborhood of \( K \) in \( D' \) to some \( f \in \mathcal{O}(K) \), which then is an extension of \( f \) to a neighborhood in \( D' \) of \( K \). To see this, first notice by (2.19) that \( \tilde{p}(z, \zeta) \) is a sum of terms \( \omega_k(\zeta) \wedge p_k(z, \zeta) \) where \( p_k(z, \zeta) \) is smooth in both variables and holomorphic for \( z \in U(K) \). By Proposition 2.5, the \( \omega_k \) are almost semi-meromorphic. The claim then follows from a simple instance of [18, Theorem 1]\(^5\). We now get

\[
\lambda(f) = \lim_{\epsilon \to 0} \int_{V'} f_\epsilon(z) d\mu(z) = \lim_{\epsilon \to 0} \int_{V'_{\zeta}} \chi(|h|/\epsilon) \tilde{p}(z, \zeta) f(\zeta) d\mu(z)
\]

\[
= \lim_{\epsilon \to 0} \int_{V'_{\zeta}} f(\zeta) \chi(|h|/\epsilon) \int_{V'_{\zeta}} \tilde{p}(z, \zeta) d\mu(z)
\]

\[
= \lim_{\epsilon \to 0} \int_{V'_{\zeta}} f(\zeta) \chi(|h|/\epsilon) \sum_{k} \omega_k(\zeta) \wedge \int_{V'_{\zeta}} p_k(z, \zeta) d\mu(z)
\]

\[
= \int_{V'_{\zeta}} f(\zeta) \sum_{k} \omega_k(\zeta) \wedge \int_{V'_{\zeta}} p_k(z, \zeta) d\mu(z).
\]

But \( \zeta \mapsto \int_{V'_{\zeta}} p_k(z, \zeta) d\mu(z) \) is smooth and compactly supported in \( D' \) and so \( \lambda \) is given as integration against some element \( \psi \in \mathcal{A}_c^{n,n}(V') \); hence \( \lambda \) is realized by the cohomology class \([\psi]\) and the theorem follows.

\[ \square \]

**Corollary 6.2.** Let \( F \to V \) be a vector bundle, \( \mathcal{F} = \mathcal{O}(F) \) the associated locally free \( \mathcal{O} \)-module, and \( \mathcal{F}^* = \mathcal{O}(F^*) \). Then the following pairing is non-degenerate

\[
\text{Tr}: H^0(V', \mathcal{F}) \times H^n(\mathcal{F}^* \otimes \mathcal{A}_c^{n,*}(V')) \to \mathbb{C}, \quad ([\varphi], [\psi]) \mapsto \int_{V'} \varphi \psi.
\]

By Theorem 1.2, if \( X \) is Cohen-Macaulay, then the complex \((\mathcal{F}^* \otimes \mathcal{A}_c^{n,*}, \bar{\partial})\) is a resolution of \( \mathcal{F}^* \otimes \omega_V^{n,0} \) and so we get a non-degenerate pairing

\[
H^0(V', \mathcal{F}) \times H^n_c(V', \mathcal{F}^* \otimes \omega_V^{n,0}) \to \mathbb{C}.
\]

**6.2. Global duality.** The global duality follows from the local one by an abstract patching argument, see [26], cf. also [11, Theorem (I)]. We will need to make this argument explicit and for this we will use the following perhaps non-standard formalism for Čech cohomology: cf. [23, Section 7.3]

Let \( \mathcal{F} \) be a sheaf on \( X \) and let \( \mathcal{V} = \{ V_j \} \) be a locally finite covering of \( X \). We let \( C^k(V, \mathcal{F}) \) be the group of formal sums

\[
\sum_{i_0 \cdots i_k} f_{i_0 \cdots i_k} V_{i_0} \wedge \cdots \wedge V_{i_k}, \quad f_{i_0 \cdots i_k} \in \mathcal{F}(V_{i_0} \cap \cdots \cap V_{i_k})
\]

\[ \text{Take } p = 0, q = 1, \text{ and } \mu = 1 \text{ in this theorem.} \]
with the suggestive computation rules, e.g., \( f_{12}V_1 \land V_2 + f_{21}V_2 \land V_1 = (f_{12} - f_{21})V_1 \land V_2 \).

Each element of \( C^k(\mathcal{V}, \mathcal{F}) \) thus has a unique representation of the form
\[
\sum_{i_0 < \cdots < i_k} f_{i_0 \cdots i_k} V_{i_0} \land \cdots \land V_{i_k}
\]
that we will abbreviate as \( \sum_{|I|=k+1} f_I V_I \). The coboundary operator \( \delta : C^k(\mathcal{V}, \mathcal{F}) \to C^{k+1}(\mathcal{V}, \mathcal{F}) \) can in this formalism be taken to be the formalism
\[
\delta(\sum_{|I|=k+1} f_I V_I) = (\sum_{|I|=k+1} f_I V_I) \land (\sum_{j} V_j).
\]

If \( \mathcal{V} \) is a Leray covering for \( \mathcal{F} \), then \( H^k(C^*(\mathcal{V}, \mathcal{F}), \delta) \cong H^k(X, \mathcal{F}) \). Indeed, let \((\mathcal{F}^*, d)\) be a flabby resolution of \( \mathcal{F} \). Then \( H^k(X, \mathcal{F}) = H^k(\mathcal{F}^*(X), d) \) and applying standard homological algebra to the double complex \( C^*(\mathcal{V}, \mathcal{F}^*) \) one shows that \( H^k(C^*(\mathcal{V}, \mathcal{F}), \delta) \cong H^k(\mathcal{F}^*(X), d) \). If \( \mathcal{F} \) is fine, i.e., a \( \mathcal{E}^{0,0} \)-module, then the complex \( (\mathcal{F}^*, \delta) \) is exact except for at level 0 where \( H^0(\mathcal{F}^*, \delta) \cong H^0(X, \mathcal{F}) \).

Let \( \mathcal{G}' \) be a precosheaf on \( X \). Recall, see, e.g., [11, Section 3], that a precosheaf of abelian groups is an assignment that to each open set \( V \) associates an abelian group \( \mathcal{G}'(V) \), together with inclusion maps \( i^V_W : \mathcal{G}'(V) \to \mathcal{G}'(W) \) for \( V \subset W \) such that \( i^V_{W'} = i^V_W i^W_{V'} \) if \( V' \subset V \subset W \). We define \( C^{-k}(\mathcal{V}, \mathcal{G}') \) to be the group of formal sums
\[
\sum_{i_0 \cdots i_k} g_{i_0 \cdots i_k} V^*_{i_0} \land \cdots \land V^*_{i_k},
\]
where \( g_{i_0 \cdots i_k} \in \mathcal{G}'(V_{i_0} \cap \cdots \cap V_{i_k}) \) and only finitely many \( g_{i_0 \cdots i_k} \) are non-zero; for \( k < 0 \) we let \( C^{-k}(\mathcal{V}, \mathcal{G}') = 0 \). We define a coboundary operator \( \delta^*: C^{-k}(\mathcal{V}, \mathcal{G}') \to C^{-k+1}(\mathcal{V}, \mathcal{G}') \) by formal contraction
\[
\delta^*(\sum_{|I|=k+1} g_I V_I^*) = \sum_{j} V_j \sum_{|I|=k+1} g_I V_I^*,
\]
see (6.5) and (6.6) below. If \( \mathcal{G} \) is a sheaf (of abelian groups), then \( V \to \mathcal{G}(V) \) is a precosheaf \( \mathcal{G}' \) by extending sections by 0. We will write \( C^{-k}(\mathcal{V}, \mathcal{G}) \) in place of \( C^{-k}(\mathcal{V}, \mathcal{G}') \).

Assume now that there, for every open \( V \subset X \), is a map \( \mathcal{F}(V) \otimes \mathcal{G}'(V) \to \mathcal{F}'(V) \) where \( \mathcal{F}' \) and \( \mathcal{G}' \) are precosheaves on \( X \). We then define a contraction map \( \cdot : C^k(\mathcal{V}, \mathcal{F}) \times C^{-k}(\mathcal{V}, \mathcal{G}') \to C^{k-\ell}(\mathcal{V}, \mathcal{F}') \) by using the following computation rules.

(6.5) \[
V_{i,j} V_{j^*} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}
\]

(6.6) \[
V_{i,j} (V_{0} \land \cdots \land V_{j^*}) = \sum_{m=0}^{\ell} (-1)^m V_{j_0} \land \cdots \land (V_{i,j} V_{j^*} \cdots \land V_{j^*}),
\]

\[
(V_{0} \land \cdots \land V_{k}) \cdot V_{j}^* = \begin{cases} 0, & k > |J| \\ ((V_{0} \land \cdots \land V_{k-1})) \cdot (V_{k} \land V_{j}^*), & k \leq |J| \end{cases}
\]

If \( \mathcal{F}' \) and \( \mathcal{G}' \) are sheaves we define in a similar way also the contraction \( \cdot : C^{-k}(\mathcal{V}, \mathcal{G}') \times C^{\ell}(\mathcal{V}, \mathcal{F}) \to C^{-k-\ell}(\mathcal{V}, \mathcal{F}') \). If \( g = g_V V^*_V \) and \( f = f_J V_J \), then \( g \cdot f = g_{1f} f_{1} V^*_V \cdot V^*_J \), where \( g_{1f} \) is the extension to \( \cap_{i \in J} V_i \) by 0; this is well-defined since \( g_{1f}f \) is 0 in a neighborhood of the boundary of \( \cap_{j \in J} V_j \) in \( \cap_{i \in J \setminus I} V_i \).
Lemma 6.3. If $\mathcal{G}$ is a fine sheaf, then
\[
H^{-k}(C^*_c(V, \mathcal{G}), \delta^*) = \begin{cases} 
0, & k \neq 0 \\
H^0_c(X, \mathcal{G}), & k = 0
\end{cases}.
\]

Proof. Let \( \{\chi_j\} \) be a smooth partition of unity subordinate to \( V \) and let \( \chi = \sum_j \chi_j V_j^* \). Since \( \delta^* \chi = \sum \chi_j = 1 \) we have
\[
\delta^*(\chi \wedge g) = \delta^*(\chi) \cdot g - \chi \wedge \delta^*(g) = g - \chi \wedge \delta^*(g)
\]
for \( g \in C^{c^{-k}}(V, \mathcal{G}) \). Hence, if \( g \) is \( \delta^* \)-closed, then \( g \) is \( \delta^* \)-exact. It follows that the complex
\[
\cdots \xrightarrow{\delta^*} C^{c^{-1}}_c(V, \mathcal{F}^*) \xrightarrow{\delta^*} C^0_c(V, \mathcal{G}) \xrightarrow{\delta^*} H^0_c(X, \mathcal{G}) \to 0
\]
is exact and so the lemma follows. \( \square \)

Let \( X \) be a paracompact reduced complex space of pure dimension \( n \). Let \( \mathcal{N} \) be the precosheaf on \( X \) defined by
\[
\mathcal{N}(V) = H^n(\mathcal{O}_c^n(V), \bar{\partial}),
\]
\[
i^V_W: \mathcal{N}(V) \to \mathcal{N}(W), \quad i^V_W([\psi]) = [\bar{\psi}],
\]
where \( \psi \in \mathcal{O}_c^n(V) \) and \( \bar{\psi} \) is the extension of \( \psi \) by 0.\(^6\) Let \( \mathcal{V} = \{V_j\} \) be a suitable locally finite Leray covering of \( X \) and consider the complexes
\[
0 \to C^0(V, \mathcal{O}_X) \xrightarrow{\delta} C^1(V, \mathcal{O}_X) \xrightarrow{\delta} \cdots (6.7)
\]
\[
\cdots \xrightarrow{\delta^*} C^{c^{-1}}_c(V, \mathcal{N}) \xrightarrow{\delta^*} C^0_c(V, \mathcal{N}) \to 0. (6.8)
\]
By Theorem 6.1 we have non-degenerate pairings
\[
\text{Tr}: C^k(V, \mathcal{O}_X) \times C^{c^{-k}}_c(V, \mathcal{N}) \to \mathbb{C}, \quad \text{Tr}(f, g) = \int_X f \cdot g,
\]
induced by the trace map (6.3); in fact, Theorem 6.1 shows that these pairings make the complex (6.8) the topological dual of the complex of Frechét spaces (6.7). Moreover, if \( f \in C^{c^{-k}}(V, \mathcal{O}_X) \) and \( g \in C^c(V, \mathcal{N}) \) we have
\[
\text{Tr}(\delta f, g) = \int_X (\delta f) \cdot g = \int_X (f \wedge \sum_j V_j) \cdot g = \int_X f \cdot (\sum_j V_j) \cdot g)
\]
\[
= \int_X f \cdot (\delta^* g) = \text{Tr}(f, \delta^* g).
\]
Hence, we get a well-defined pairing on cohomology level
\[
\text{Tr}: H^k(C^*(V, \mathcal{O}_X)) \times H^{-k}(C^*_c(V, \mathcal{N})) \to \mathbb{C}, \quad \text{Tr}([f], [g]) = \int_X f \cdot g.
\]
Since \( V \) is a Leray covering we have
\[
H^k(C^*(V, \mathcal{O}_X)) \cong H^k(X, \mathcal{O}_X) \cong H^k(\mathcal{O}^{0, \cdot}(X)) \text{ ,}
\]
and these isomorphisms induce canonical topologies on \( H^k(X, \mathcal{O}_X) \) and \( H^k(\mathcal{O}^{0, \cdot}(X)) \); cf. [26, Lemma 1]. To understand \( H^{-k}(C^*_c(V, \mathcal{N})) \), consider the double complex
\[
K^{-i,j} := C^{-i}_c(V, \mathcal{F}^{n,j}_X),
\]
\(^6\)In view of Theorem 6.1 and [11, Proposition 8 (a)], \( \mathcal{N} \) is in fact a cosheaf.
where the map $K^{-i,j} \to K^{-i+1,j}$ is the coboundary operator $\delta^*$ and the map $K^{-i,j} \to K^{-i-j+1}$ is $\partial$. We have that $K^{-i,j} = 0$ if $i < 0$ or $j < 0$ or $j > n$. Moreover, the “rows” $K^{-i,*}$ are, by Theorem 6.1, exact except for at the $n$th level where the cohomology is $C_c^{-i}(\mathcal{V}, \mathcal{N})$; the “columns” $K^{-*,j}$ are exact except for at level 0 where the cohomology is $\mathcal{A}_c^{-n,j}(X)$ by Lemma 6.3 since the sheaf $\mathcal{A}_c^{-n,j}$ is fine. By standard homological algebra (e.g., a spectral sequence argument) it follows that

\[(6.12) \quad H^{-k}(C_c^*(\mathcal{V}, \mathcal{N})) \cong H^{n-k}(\mathcal{A}_c^{n-*}(X), \bar{\partial}), \]

cf. also the proof of Theorem 1.3 below. The vector space $C_c^{-k}(\mathcal{V}, \mathcal{N})$ has a natural topology since it is the topological dual of the Frechét space $C^k(\mathcal{V}, \mathcal{O}_X)$; therefore (6.12) gives a natural topology on $H^{n-k}(\mathcal{A}_c^{n-*}(X))$.

**Lemma 6.4.** Assume that $H^k(X, \mathcal{O}_X)$ and $H^{k+1}(X, \mathcal{O}_X)$, considered as topological vector spaces, are Hausdorff. Then the pairing (6.10) is non-degenerate.

**Proof.** Since (6.8) is the topological dual of (6.7) it follows (see, e.g., [26, Lemma 2]) that the topological dual of

\[(6.13) \quad \text{Ker}(\delta: C^k(\mathcal{V}, \mathcal{O}_X) \to C^{k+1}(\mathcal{V}, \mathcal{O}_X))/\text{Im}(\delta: C^{k-1}(\mathcal{V}, \mathcal{O}_X) \to C^k(\mathcal{V}, \mathcal{O}_X))\]

equals

\[(6.14) \quad \text{Ker}(\delta^*: C_c^{-k}(\mathcal{V}, \omega^n_X) \to C_c^{-k+1}(\mathcal{V}, \omega^n_X))/\text{Im}(\delta^*: C_c^{-k-1}(\mathcal{V}, \omega^n_X) \to C_c^{-k}(\mathcal{V}, \omega^n_X)).\]

Since $H^k(X, \mathcal{O}_X)$ and $H^{k+1}(X, \mathcal{O}_X)$ are Hausdorff it follows that the images of $\delta: C^{k-1} \to C^k$ and $\delta: C^k \to C^{k+1}$ are closed. Since the image of the latter map is closed it follows from the open mapping theorem and the Hahn-Banach theorem that also the image of $\delta^*: C_c^{-k-1} \to C_c^{-k}$ is closed. The images of $\delta$ and $\delta^*$ in (6.13) and (6.14) are thus closed and so the closure signs may be removed. Hence, (6.10) makes $H^{-k}(C_c^*(\mathcal{V}, \omega^n_X))$ the topological dual of $H^k(X, \mathcal{O}_X)$. \hfill $\Box$

**Remark 6.5.** If $X$ is compact the Cartan-Serre theorem says that the cohomology of coherent sheaves on $X$ is finite dimensional, in particular Hausdorff. In the compact case the pairing (6.10) is thus always non-degenerate. The pairing (6.10) is also always non-degenerate if $X$ is holomorphically convex since then, by [25, Lemma II.1], $H^k(X, \mathcal{I})$ is Hausdorff for any coherent sheaf $\mathcal{I}$.

If $X$ is $q$-convex it follows from the Andreotti-Grauert theorem that for any coherent sheaf $\mathcal{I}$, $H^k(X, \mathcal{I})$ is Hausdorff for $k \geq q$. Hence, in this case, (6.10) is non-degenerate for $k \geq q$.

**Proof of Theorem 1.3.** For notational convenience we assume that $\mathcal{I} = \mathcal{O}_X$. By Lemma 6.4 we know that (6.10) is non-degenerate. In view of the Dolbeault isomorphisms (6.11) and (6.12) we get an induced non-degenerate pairing

\[\text{Tr}: H^k(\mathcal{A}_c^{0-*}(X)) \times H^{n-k}(\mathcal{A}_c^{n-*}(X)) \to \mathbb{C}.\]

It remains to see that this induced trace map is realized by $([\varphi], [\psi]) \mapsto \int_X \varphi \wedge \psi$; for this we will make (6.11) and (6.12) explicit.

Let $\{\chi_j\}$ be a partition of unity subordinate to $\mathcal{V}$, and let $\chi = \sum_j \chi_j V_j^*$. We will use the convention that forms *commute* with all $V_i^*$ and $V_j$, i.e., if $\xi$ is a differential form then $\xi V_i^* = V_i^* \xi, \quad V_j^* \xi(V_j) = \xi V_i^* \cdot V_j$. 
Moreover, we let \( \partial(\xi V^*_f) = \bar{\partial}\xi V^*_f \). We now let

\[ T_{k,j} : C^k(\mathcal{V}, \mathcal{O}_X) \to C^{k-j-1}(\mathcal{V}, \mathcal{O}_X^{0,j}), \quad T_{k,j}(f) = (\chi \land (\bar{\partial}\chi)^j)_j f, \]

where we put \( C^{-1}(\mathcal{V}, \mathcal{O}_X^{0,k}) = \mathcal{O}_X^{0,k}(X) \) and \( C^\ell(\mathcal{V}, \mathcal{O}_X^{0,k}) = 0 \) for \( \ell < -1 \).\(^7\) Using that \( \chi_\mathcal{V} = 1 \) it is straightforward to verify that

\[ T_{k,j}(\delta f) = \delta T_{k-1,j-1}(f) + (-1)^{k-j} \bar{\partial} T_{k-1,j-1}(f), \quad \bar{f} \in C^{k-1}(\mathcal{V}, \mathcal{O}_X). \]

It follows that if \( f \in C^k(\mathcal{V}, \mathcal{O}_X) \) is \( \delta \)-closed then \( T_{k,k}(f) \) is \( \bar{\delta} \)-closed and if \( f \) is \( \delta \)-exact then \( T_{k,k}(f) \) is \( \bar{\delta} \)-exact. Thus \( T_{k,k} \) induces a map

\[ \text{Dol} : H^k(\mathcal{O}^*(\mathcal{V}, \mathcal{O}_X)) \to H^k(\mathcal{O}_X^{0,\bullet}(X)), \quad \text{Dol}([f]) = [T_{k,k}(f)]. \]

this is a realization of the composed isomorphism (6.11).

To make (6.12) explicit, let \([g] \in C^{-k}(\mathcal{V}, \mathcal{O}_X)\), where \( g \in C^{-k}(\mathcal{V}, \mathcal{O}_X^{0,n}) \), be \( \delta \)-closed. This means that there is a \( \tau^{n-1} \in C_{c_k}^{-k+1}(\mathcal{V}, \mathcal{O}_X^{0,n-1}) \) such that \( \delta^* g = \bar{\partial}\tau^{n-1} \).

Hence, \( \bar{\partial}\delta^* \tau^{n-1} = \delta^* \bar{\partial}\tau^{n-1} = \delta^* \delta^* g = 0 \) and so by Theorem 6.1 there is a \( \tau^{n-2} \in C_{c_k}^{-k+2}(\mathcal{V}, \mathcal{O}_X^{0,n-2}) \) such that \( \delta \tau^{n-1} = \bar{\partial}\tau^{n-2} \). Continuing in this way we obtain, for all \( j, \tau^{-j} \in C_{c_k}^{-k+j}(\mathcal{V}, \mathcal{O}_X^{0,n-j}) \) such that \( \delta^* \tau^{-j} = \bar{\partial}\tau^{-1} \).

It follows that \( \delta^* \tau^{-j} \in \mathcal{O}_X^{0,n-j}(X) \), cf. the proof of Lemma 6.3, and that it is \( \bar{\delta} \)-closed. One can verify that if \([g] \in C_{c_k}^{-k}(\mathcal{V}, \mathcal{O}_X)\) is \( \delta \)-exact then \( \delta^* \tau^{-j} \) is \( \delta \)-exact and so we get a well-defined map

\[ \text{Dol}^* : H^{-k}(\mathcal{O}^*(\mathcal{V}, \mathcal{O}_X)) \to H^{-n-k}(\mathcal{O}_X^{0,\bullet}(X)), \quad \text{Dol}^*([g]) = [\delta^* \tau^{-j}] \]

this is a realization of the isomorphism (6.12).

Let now \( f \in C^k(\mathcal{V}, \mathcal{O}_X) \) be \( \delta \)-closed and let \([g] \in C_{c_k}^{-k}(\mathcal{V}, \mathcal{O}_X)\) be \( \delta \)-closed. One checks that \( \delta T_{k,0}(f) = (-1)^k f + \text{and thus, by (6.15), we have} \)

\[ \delta T_{k,j}(f) = \begin{cases} (-1)^{k-j} \bar{\partial} T_{k-1,j-1}(f), & 1 \leq j \leq k, \\ (-1)^k f, & j = 0. \end{cases} \]

Using this and the computation in (6.9) we get

\[ \int_X f \cdot g = (-1)^k \int_X \delta T_{k,0}(f) \cdot g = (-1)^k \int_X T_{k,0}(f) \cdot \delta^* g = (-1)^k \int_X T_{k,0}(f) \cdot \bar{\partial}\tau^{n-1} = (-1)^{k+1} \int_X \bar{\partial} T_{k,0}(f) \cdot \tau^{n-1} = (-1)^{2k} \int_X T_{k,1}(f) \cdot \delta^* \tau^{n-1} = \cdots = (-1)^{k(k+1)} \int_X T_{k,k}(f) \cdot \delta^* \tau^{n-k} = \int_X \text{Dol}([f]) \land \text{Dol}^*([g]). \]

}\[\square\]

\[ \text{7. Compatibility with the Cup Product} \]

Assume that \( X \) is compact and Cohen-Macaulay. In view of Theorem 2.9 and Theorem 1.2 we have that

\[ H^k(X, \mathcal{O}_X) \cong H^k(\mathcal{O}_X^{0,\bullet}(X), \bar{\partial}) \quad \text{and} \quad H^k(X, \omega^0_X) \cong H^k(\mathcal{O}_X^{0,\bullet}(X), \bar{\partial}). \]

\(^7\)In fact, the image of \( T_{k,j} \) is contained in \( C^{k-j-1}(\mathcal{V}, \mathcal{O}_X^{0,j}) \).
Now we make these Dolbeault isomorphisms explicit in a slightly different way than
in the previous section: We adopt in this section the standard definition of Čech
cochain groups so that now
\[
C^p(\mathcal{V}, \mathcal{F}) := \prod_{a_0 \neq a_1 \neq \cdots \neq a_p} \mathcal{F}(V_{a_0} \cap \cdots \cap V_{a_p})
\]
for a sheaf \( \mathcal{F} \) on \( X \) and a locally finite open cover \( \mathcal{V} = \{ V_{a} \} \).

Let \( \mathcal{V} \) be a Leray covering and let \( \{ \chi_{\alpha} \} \) be a smooth partition of unity subordinate
to \( \mathcal{V} \). Following [16, Chapter IV, §6], given Čech cocycles \( c \in C^p(\mathcal{V}, \mathcal{A}_X^p) \) and \( c' \in C^q(\mathcal{V}, \omega_X^{n,0}) \) we define Čech cochains \( f \in C^0(\mathcal{V}, \mathcal{A}_X^{0,p}) \) and \( f' \in C^0(\mathcal{V}, \mathcal{A}_X^{n,q}) \) by
\[
f_{\alpha} = \sum_{\nu_0, \ldots, \nu_{p-1}} \tilde{\partial} \chi_{\nu_0} \wedge \cdots \wedge \tilde{\partial} \chi_{\nu_{p-1}} \cdot c_{\nu_0 \cdots \nu_{p-1} \alpha} \quad \text{in} \quad V_{\alpha},
\]
\[
f'_{\alpha} = \sum_{\nu_0, \ldots, \nu_{q-1}} \tilde{\partial} \chi_{\nu_0} \wedge \cdots \wedge \tilde{\partial} \chi_{\nu_{q-1}} \wedge c'_{\nu_0 \cdots \nu_{q-1} \alpha} \quad \text{in} \quad V_{\alpha}.
\]

In fact, \( f \) and \( f' \) are cocycles and define \( \tilde{\partial} \)-closed global sections
\[
\varphi = \sum_{\nu_p} \chi_{\nu_p} f_{\nu_p} = \sum_{\nu_0, \ldots, \nu_p} \chi_{\nu_0} \tilde{\partial} \chi_{\nu_0} \wedge \cdots \wedge \tilde{\partial} \chi_{\nu_{p-1}} \cdot c_{\nu_0 \cdots \nu_p} \in \mathcal{A}^{0,p}(X),
\]
\[
\varphi' = \sum_{\nu_q} \chi_{\nu_q} f'_{\nu_q} = \sum_{\nu_0, \ldots, \nu_q} \chi_{\nu_0} \tilde{\partial} \chi_{\nu_0} \wedge \cdots \wedge \tilde{\partial} \chi_{\nu_{q-1}} \wedge c'_{\nu_0 \cdots \nu_q} \in \mathcal{A}^{n,q}(X).
\]
The Dolbeault isomorphisms (7.1) are then realized by
\[
H^p(X, \mathcal{O}_X) \xrightarrow{\sim} H^p(\mathcal{A}^{0,*}(X)), \quad [c] \mapsto [\varphi], \quad \text{and}
\]
\[
H^q(X, \omega_X^{n,0}) \xrightarrow{\sim} H^q(\mathcal{A}^{n,*}(X)), \quad [c'] \mapsto [\varphi'],
\]
respectively.

We can now show that the cup product is compatible with our trace map on the
level of cohomology.

**Proposition 7.1.** The following diagram commutes.
\[
\begin{array}{ccc}
H^p(X, \mathcal{O}_X) \times H^q(X, \omega_X^{n,0}) & \xrightarrow{\cup} & H^{p+q}(X, \omega_X^{n,0}) \\
\downarrow & & \downarrow \\
H^p(\mathcal{A}^{0,*}(X)) \times H^q(\mathcal{A}^{n,*}(X)) & \xrightarrow{\Lambda} & H^{p+q}(\mathcal{A}^{n,*}(X)),
\end{array}
\]
where the vertical mappings are the Dolbeault isomorphisms.

**Proof.** Let \( \mathcal{V} = \{ V_{a} \} \) be a Leray covering of \( X \). Let \( [c] \in H^p(X, \mathcal{O}_X) \) and \( [c'] \in H^q(X, \omega_X^{n,0}) \), where \( c \in C^p(\mathcal{V}, \mathcal{O}_X) \) and \( c' \in C^q(\mathcal{V}, \omega_X^{n,0}) \) are cocycles. Then \( c \cup c' \in C^{p+q}(\mathcal{V}, \omega_X^{n,0}) \), defined by
\[
(c \cup c')_{a_0 \cdots a_{p+q}} = c_{a_0 \cdots a_p} \cdot c'_{a_p \cdots a_{p+q}} \quad \text{in} \quad V_{a_0} \cap \cdots \cap V_{a_{p+q}},
\]
is a cocycle representing \( [c] \cup [c'] \in H^{p+q}(X, \omega_X^{n,0}) \). The image of \( [c] \cup [c'] \) in \( H^{p+q}(\mathcal{A}^{n,*}(X)) \) is the cohomology class defined by the \( \tilde{\partial} \)-closed current
\[
\sum_{\nu_0, \ldots, \nu_{p+q}} \chi_{\nu_{p+q}} \tilde{\partial} \chi_{\nu_0} \wedge \cdots \wedge \tilde{\partial} \chi_{\nu_{p+1}} \wedge c_{\nu_0 \cdots \nu_p} \cdot c'_{\nu_{p+1} \cdots \nu_{p+q}} \in \mathcal{A}^{n,p+q}(X).
\]
The images of $[c]$ and $[c']$ in Dolbeault cohomology are, respectively, the cohomology classes of the $\bar{\partial}$-closed currents $\varphi$ and $\varphi'$ defined by (7.2) and (7.3). Notice that

$$\varphi|_{V_{\nu}} = \sum_{\nu_0, \ldots, \nu_{p-1}} \bar{\partial}^{\nu_0} \wedge \cdots \wedge \bar{\partial}^{\nu_{p-1}} \cdot c_{\nu_0 \cdots \nu_{p-1}} \nu_p.$$ 

Therefore, $\varphi \land \varphi'$ is given by (7.4) as well. □

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