MONOIDAL CATEGORIES OF CORINGS

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Abstract. We introduce a monoidal category of corings using two different notions of corings morphisms. The first one is the (right) coring extensions recently introduced by T. Brzeziński in [2], and the other is the usual notion of morphisms defined in [5] by J. Gómez-Torrecillas.

Introduction

The word coring appeared for the first time in the literature in the famous paper of M. Sweedler [8], where he showed that this notion can be used to give a simplest proof of the first Galois-correspondence theorem for division rings. It turns out that corings and their comodules unify many kind of relative modules, like as graded modules, Doi-Hopf modules, and more generally entwined modules. This was shown in another famous paper due to T. Brzeziński [1].

Corings are, in some sense, a generalization of coalgebras to the case of non-commutative scalars base rings. They have bimodule structure rather than module one. Thus a tensor product in the category of bimodules hampers any attempt to give a compatibility with multiplication and comultiplication. This is a well known problem to define bialgebras using bimodules. The first approach to generalize the bialgebras to the case of bimodules was given by M. Sweedler in [9]. If we see that those bialgebras should be defined as monad or comonad in an appropriate monoidal category, then we should look first to the possible monoidal categories. In this note we prove that there is a more than one monoidal category whose objects are all corings.

We work over a unital commutative ring $k$. All our algebras $A, A', B, B'...$, are an unital associative $k$–algebras. For any algebra we denote by $\mathcal{M}_A$ its category of all unital right $A$–modules; we use the notation $\mathcal{M}_A$ to denote the category of unital left $A$–modules. Bimodules are assumed to be central $k$–bimodules, and their category is denoted by $\mathcal{M}_B$. If $A\mathcal{M}_B$ and $B\mathcal{N}_C$ are, respectively, an $(A, B)$–bimodule and $(B, C)$–bimodule, then their tensor product $M \otimes_B N$ will be considered, in a canonical way, as an $(A, C)$–bimodule.

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An $A$–coring is a three-tuple $(\mathcal{C}, \Delta_\mathcal{C}, \varepsilon_\mathcal{C})$ consisting of an $A$–bimodule $\mathcal{C}$ and the two $A$–bilinear maps
\[ \mathcal{C} \xrightarrow{\Delta_\mathcal{C}} \mathcal{C} \otimes_A \mathcal{C}, \quad \mathcal{C} \xrightarrow{\varepsilon_\mathcal{C}} A \]
such that $(\Delta_\mathcal{C} \otimes_A \mathcal{C}) \circ \Delta_\mathcal{C} = (\mathcal{C} \otimes_A \Delta_\mathcal{C}) \circ \Delta_\mathcal{C}$ and $(\varepsilon_\mathcal{C} \otimes_A \mathcal{C}) \circ \Delta_\mathcal{C} = (\mathcal{C} \otimes_A \varepsilon_\mathcal{C}) \circ \Delta_\mathcal{C} = \mathcal{C}$. A right $\mathcal{C}$–comodule is a pair $(M, \rho_M)$ consisting of right $A$–module and a right $A$–linear map $\rho_M : M \rightarrow M \otimes_A \mathcal{C}$, called right $\mathcal{C}$–coaction, such that $(M \otimes_A \Delta_\mathcal{C}) \circ \rho_M = (\rho_M \otimes_A \mathcal{C}) \circ \rho_M$ and $(M \otimes_A \varepsilon_\mathcal{C}) \circ \rho_M = M$. Left $\mathcal{C}$–comodules are symmetrically defined. For instance, $(\mathcal{C}, \Delta_\mathcal{C})$ is a left and right $\mathcal{C}$–comodule. We use Sweedler’s notation for comultiplications, that is $\Delta_\mathcal{C}(c) = c^{(1)} \otimes_A c^{(2)}$, for every $c \in \mathcal{C}$ (the finite sums are understood). We also use lower indices Sweedler’s notation for coactions: $\rho_M(m) = m^{(0)} \otimes_A m^{(1)}$, for all $m \in M$. But, if two different coactions are managed, its convenient to use upper indices: $\rho^N(n) = n^{[0]} \otimes_B n^{[1]}$, for a right $D$–comodule $N$. A source of basic notions on corings, categories of comodules, bicomodules, and cotensor product is [3].

1. Tensor product of corings

In this section we recall the tensor product of two corings over different scalars base rings.

Let $\mathcal{C}$ and $\mathcal{C}'$ are, respectively, an $A$–bimodule and $A'$–bimodule. We consider the tensor product $\mathcal{C} \otimes_k \mathcal{C}'$ as an $A \otimes_k A'$–bimodule using the canonical bi-action
\[
(a \otimes_k a')(c \otimes_k c')(b \otimes_k b') = (acb) \otimes_k (a'c'b'),
\]
for all $(a, b) \in A \times A$, $(a', b') \in A' \times A'$, and $(c, c') \in \mathcal{C} \times \mathcal{C}'$. The following well known lemma will be used frequently; for completeness we include the proof.

**Lemma 1.1.** For every pair of modules $(M_A, N_{A'}) \in \mathcal{M}_A \times \mathcal{M}_{A'}$, there exists a right $A \otimes_k A'$–linear map
\[
\eta_{(M_A, N_{A'})} : M \otimes_A \mathcal{C} \otimes_k N \otimes_A \mathcal{C}' \xrightarrow{\sim} (M \otimes_k N) \otimes_{A \otimes_k A'} (\mathcal{C} \otimes_k \mathcal{C}')
\]
\[ m \otimes c \otimes n \otimes c' \xrightarrow{\sim} (m \otimes_k n) \otimes_{A \otimes_k A'} (c \otimes_k c'), \]
which becomes an $(A, A \otimes_k A')$–bilinear map if $M \in \mathcal{A}_M$. Furthermore,
\[
\eta_{(-, -)} : - \otimes_A \mathcal{C} \otimes_k - \otimes_{A'} \mathcal{C}' \longrightarrow (- \otimes_k -) \otimes_{A \otimes_k A'} (\mathcal{C} \otimes_k \mathcal{C}')
\]
is a natural isomorphism.
Proof. It is clear that \( A \otimes_A C \otimes_k A' \otimes_A C' \cong (A \otimes_k A') \otimes_{A \otimes_k A'} (C \otimes_k C') \), via the map sending \( a \otimes_A c \otimes a' \otimes_A c' \mapsto (a \otimes_k a')(c \otimes_k c') = ac \otimes_k a'c' \). If \((f, g) : (A, A') \to (A, A')\) is any arrow in the product category \( \mathcal{M}_A \times \mathcal{M}_{A'} \), then its also clear that

\[
\begin{array}{ccc}
A \otimes_A C \otimes_A A' \otimes_A C' & \xrightarrow{f \otimes g \otimes C'} & A \otimes_A C \otimes_A A' \otimes_A C' \\
\cong & & \cong \\
(A \otimes_k A') \otimes_{A \otimes_k A'} (C \otimes_k C') & \xrightarrow{(f \otimes_k g) \otimes_{A \otimes_k A'} (C \otimes_k C')} & (A \otimes_k A') \otimes_{A \otimes_k A'} (C \otimes_k C')
\end{array}
\]

is a commutative diagram. Since \((A, A')\) is a projective generator in \( \mathcal{M}_A \times \mathcal{M}_{A'} \) and the tensor product commute with direct limits, Mitchell’s Theorem \([7, \text{Theorem } 5.4, \text{p. } 109]\), implies that there exists a unique natural isomorphism

\[
\eta(-, -) : - \otimes_A C \otimes_k - \otimes_A C' \to (- \otimes_k -) \otimes_{A \otimes_k A'} (C \otimes_k C').
\]

Let \( m \in M, n \in N, c \in C \) and \( c' \in C' \), the form of the image \( \eta_{(M, N)}(m \otimes_A c \otimes_k n \otimes_A c') \), is computed by using the morphisms \((A, A') \to (mA, nA')\) and the naturality of \( \eta \), where \( mA \) and \( nA \) denote the cyclic submodules.

Let \((C, \Delta_C, \varepsilon_C)\) and \((C', \Delta_{C'}, \varepsilon_{C'})\) are, respectively, an \( A \)-coring and \( A' \)-coring, and consider \( C \otimes_k C' \) canonically as an \( A \otimes_k A' \)-bimodule.

Proposition 1.2 \([\mathbb{H}]\). The tensor product \( C \otimes_k C' \) is an \( A \otimes_k A' \)-coring with comultiplication given by the composition map

\[
C \otimes_k C' \xrightarrow{\Delta_C \otimes C'} (C \otimes_A C) \otimes_A (C' \otimes_A C') \xrightarrow{\eta_{C, C'}} (C \otimes_k C') \otimes_{A \otimes_k A'} (C \otimes_k C'),
\]

and counit by

\[
C \otimes_k C' \xrightarrow{\varepsilon_C \otimes C'} A \otimes_k A'.
\]

2. Tensor product of right corings extensions

Let \( A \) and \( B \) be \( k \)-algebras, \( C \) an \( A \)-coring and \( D \) a \( B \)-coring. Recall from \([2, \text{Definition } 2.1]\), that \( D \) is called right extension of \( C \) provided \( C \) is a \((C, D)\)-bicomodule with the regular left coaction \( \Delta_C \). This means that \( C \) should be an \((A, B)\)-bimodule and \( \Delta_C \) a right \( B \)-linear map.

Proposition 2.1. Let \( D \) and \( D' \) are, respectively, a \( B \)-coring and \( B' \)-coring. Assume that \( D \) (resp. \( D' \)) is a right extension of \( C \) (resp. of \( C' \)). Then \( D \otimes_k D' \) is a right extension of \( C \otimes_k C' \).
Proof. Denote by $\rho^\mathcal{C} : \mathcal{C} \to \mathcal{C} \otimes_B \mathcal{D}$ ($c \mapsto c(0) \otimes_B c_{(1)}$) and $\rho^{\mathcal{C}'} : \mathcal{C}' \to \mathcal{C}' \otimes_{B'} \mathcal{D}'$ ($c' \mapsto c'_{(0)} \otimes_{B'} c'_{(1)}$) (sums are understood) the attached right coaction to the stated extensions. Define $\rho^{\mathcal{C} \otimes_k \mathcal{C}'}$ as the composition map

$$\begin{align*}
\mathcal{C} \otimes_k \mathcal{C}' &\xrightarrow{\rho^\mathcal{C} \otimes_k \rho^{\mathcal{C}'}} (\mathcal{C} \otimes_B \mathcal{D}) \otimes_k (\mathcal{C}' \otimes_{B'} \mathcal{D}') \\
&\xrightarrow{\eta \otimes \eta'} (\mathcal{C} \otimes_k \mathcal{C}') \otimes_{B \otimes_k B'} (\mathcal{D} \otimes_k \mathcal{D}').
\end{align*}$$

It is easily checked, using the $B$–linearity of $\Delta_\mathcal{C}$ and the $B'$–linearity of $\Delta_{\mathcal{C}'}$, that this composition is $(A \otimes_k A') - (B \otimes_k B')$–bilinear map. Since $\rho^\mathcal{C}$ and $\rho^{\mathcal{C}'}$ are, respectively, right $\mathcal{D}$–coaction and right $\mathcal{D}'$–coaction and $\eta_{-,-}$ is a natural transformation, $\rho^{\mathcal{C} \otimes_k \mathcal{C}'}$ is also a right $\mathcal{D} \otimes_k \mathcal{D}'$–coaction. Furthermore, $\rho^\mathcal{C}$, $\rho^{\mathcal{C}'}$, and the comultiplications of $\mathcal{C}$ and $\mathcal{C}'$, enjoy the following four commutative diagrams:

1. \[
\begin{align*}
\mathcal{C} \otimes_k \mathcal{C}' &\xrightarrow{\rho^\mathcal{C} \otimes_k \rho^{\mathcal{C}'}} (\mathcal{C} \otimes_B \mathcal{D}) \otimes_k (\mathcal{C}' \otimes_{B'} \mathcal{D}') \\
&\xrightarrow{\eta \otimes \eta'} (\mathcal{C} \otimes_k \mathcal{C}') \otimes_{B \otimes_k B'} (\mathcal{D} \otimes_k \mathcal{D}').
\end{align*}
\]

2. \[
\begin{align*}
(\mathcal{C} \otimes \mathcal{D}) \otimes_k (\mathcal{C}' \otimes \mathcal{D}') &\xrightarrow{\cong} (\mathcal{C} \otimes_k \mathcal{C}') \otimes_{B \otimes_k B'} (\mathcal{D} \otimes_k \mathcal{D}').
\end{align*}
\]

3. \[
\begin{align*}
(\mathcal{C} \otimes_k \mathcal{C}') \otimes (\mathcal{C} \otimes_k \mathcal{C}') &\xrightarrow{\cong} (\mathcal{C} \otimes_k \mathcal{C}') \otimes_k (\mathcal{C}' \otimes \mathcal{C}')
\end{align*}
\]

4. \[
\begin{align*}
\mathcal{C} \otimes_k \mathcal{C}' &\xrightarrow{\cong} (\mathcal{C} \otimes \mathcal{C}') \otimes_k (\mathcal{C}' \otimes \mathcal{C}')
\end{align*}
\]
If we put those diagrams in the following form

\[(\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{D}) \otimes_{\mathcal{A} \otimes \mathcal{A}'} (\mathcal{C}' \otimes \mathcal{C}' \otimes \mathcal{D}') \cong (\mathcal{C} \otimes \mathcal{C}) \otimes_{\mathcal{A} \otimes \mathcal{A}'} (\mathcal{C}' \otimes \mathcal{C}' \otimes \mathcal{D}') \otimes_{\mathcal{B} \otimes \mathcal{B}'} (\mathcal{D} \otimes \mathcal{D}')\]

where the isomorphisms maps are defined by the natural isomorphism of Lemma 1.1.

If we put those diagrams in the following form

\[
\begin{array}{c|c}
(1) & (2) \\
\hline
(3) & (4)
\end{array}
\]

we then get another commutative diagram which shows that \(\rho^{\mathcal{C} \otimes \mathcal{C}'}\) is left \(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}'\)-colinear with respect to regular left coaction \(\Delta_{\mathcal{C} \otimes \mathcal{C}'}\), and this finishes the proof. \(\square\)

3. A MONOIDAL CATEGORY

Let us recall from [2] the category of corings \(\text{CrgExt}_k^\mathcal{C}\). The object of this category are corings understood as pair \((\mathcal{C} : A)\) (that is \(\mathcal{C}\) is an \(A\)-coring), and morphisms \((\mathcal{C} : A) \to (\mathcal{D} : B)\) are pairs \((\rho_\mathcal{C}, \rho_\mathcal{D})\) where \(\rho_\mathcal{C} : \mathcal{C} \otimes B \to \mathcal{C}\) is left \(\mathcal{C}\)-colinear right \(B\)-action, and \(\rho_\mathcal{D} : \mathcal{C} \to \mathcal{C} \otimes B \mathcal{D}\) is a left \(\mathcal{C}\)-colinear right \(\mathcal{D}\)-coaction (that is \(\mathcal{D}\) is right extension of \(\mathcal{C}\)). The identity arrow of an object \((\mathcal{C} : A)\) is given by the pair \(id_{(\mathcal{C}, A)} = (\rho_\mathcal{C}, \rho_\mathcal{D})\) where \(\rho_\mathcal{C} = \iota_\mathcal{C} : \mathcal{C} \otimes_k A \to \mathcal{C}\) is the initial right \(A\)-action and \(\rho_\mathcal{D} = \Delta_\mathcal{C}\) is the comultiplication of \(\mathcal{C}\). The composition law is given as follows. If \((\rho_\mathcal{C}, \rho_\mathcal{D}) : (\mathcal{C} : C) \to (\mathcal{C} : A)\) and \((\rho_\mathcal{E}, \rho_\mathcal{F}) : (\mathcal{C} : A) \to (\mathcal{D} : B)\), then

\[\rho_\mathcal{E} \circ (\rho_\mathcal{C}, \rho_\mathcal{D}) = (\rho_\mathcal{E} \circ \rho_\mathcal{C}, \rho_\mathcal{E} \circ \rho_\mathcal{D})\]

where

\[
\rho_\mathcal{E} \circ \rho_\mathcal{C} : \mathcal{C} \otimes B \xrightarrow{\rho_\mathcal{C} \otimes B} \mathcal{C} \otimes A \mathcal{C} \otimes B \xrightarrow{\rho_\mathcal{E} \otimes A} \mathcal{C} \otimes A \mathcal{C} \xrightarrow{\mathcal{C} \otimes \rho_\mathcal{E} \otimes \mathcal{C}} \mathcal{C} \otimes A \mathcal{C} \cong \mathcal{C}
\]

and

\[
\rho_\mathcal{E} \circ \rho_\mathcal{D} : \mathcal{C} \xrightarrow{\rho_\mathcal{D}} \mathcal{C} \otimes_B \mathcal{D} \xrightarrow{\mathcal{C} \otimes \rho_\mathcal{D}} \mathcal{C} \otimes_B \mathcal{C} \mathcal{D} \xrightarrow{\rho_\mathcal{E} \otimes \mathcal{D}} \mathcal{C} \otimes_B \mathcal{D} \cong \mathcal{C} \otimes_B \mathcal{D}.
\]

Explicitly, the bullet compositions are given as follows: for \(e \in \mathcal{C}\) and \(b \in B\)

\[
\rho_\mathcal{E} \circ \rho_\mathcal{C}(e \otimes_k b) = e_0 \varphi_A(e_1 b),
\]

\[
\rho_\mathcal{E} \circ \rho_\mathcal{E}(e) = e_0 \varphi_A(e_1^{[0]} \otimes_B e_1^{[1]}).
\]

where \(\rho_\mathcal{E}(e) = e_0 \otimes_A e_1\), and \(\rho_\mathcal{E}(e_1) = e_1^{[0]} \otimes_B e_1^{[1]}\), for all \(e_1\).
The tensor product of two morphisms \((\rho_C, \rho^C) : (C : A) \rightarrow (D : B)\) and \((\rho_C', \rho^{C'}) : (C' : A') \rightarrow (D' : B')\) in \(\text{CrgExt}^r_k\), is defined as in the proof of Proposition 2.1 that is by the following morphism:

\[
(3.2) \quad (\rho_{\otimes k}^C, \rho_{\otimes k}^{C'}) : (C \otimes_k C' \otimes_k (B \otimes_k B')) \rightarrow (C \otimes_k C' \otimes_k (D \otimes_k D'))
\]

where \(\rho_{\otimes k}^C = cb\), \(\rho_{\otimes k}^{C'} = c'b'\), and

\[
(\rho_{\otimes k}^{C,C'}) : (c \otimes_k c') \otimes_k (b \otimes_k b') \rightarrow (cb) \otimes_k (c'b')
\]

Proposition 3.1. Let \(k\) be an unital commutative ring. Consider the category \(\text{CrgExt}^r_k\) of corings with morphisms right coring extensions, and denote by \(k := (k : k)\) the trivial \(k\)-coring \(k\). There exists a covariant bi-functor

\[
- \otimes_k - : \text{CrgExt}^r_k \times \text{CrgExt}^r_k \rightarrow \text{CrgExt}^r_k
\]

where \((\rho_{\otimes k}^C, \rho_{\otimes k}^{C'})\) is the morphism defined in equation (3.2). Moreover,

\[
k \otimes_k (C : A) \cong (C : A) \text{ and } (C : A) \otimes_k k \cong (C : A)
\]

natural isomorphisms in \(\text{CrgExt}^r_k\). In particular, \(\text{CrgExt}^r_k\) is a monoidal category with unit \(k\).

Proof. After all, the Propositions 2.1 and 2.1 imply that the stated functor is well defined. Now, by definition of the comultiplication of the tensor product of two corings, the identity arrow of any pair of corings is mapped by \(- \otimes_k -\) to the identity arrow of their tensor product; that is in the above notation, we have

\[
id_{(C : A)} \otimes_k \text{id}_{(C' : A')} = (t_C, \Delta_C) \otimes_k (t_{C'}, \Delta_{C'}) = (t_{\otimes_k C, C'}, \Delta_{\otimes_k C, C'}) = \text{id}_{(C \otimes_k C' : A \otimes_k A')}
\]

Consider the following four morphisms in \(\text{CrgExt}^r_k\)

\[
(C : C) \xrightarrow{(\rho_C, \rho^C)} (C : A) \xrightarrow{(\rho_C, \rho^C)} (D : B)
\]

\[
(C' : C') \xrightarrow{(\rho_{C'}, \rho^{C'})} (C' : A') \xrightarrow{(\rho_{C'}, \rho^{C'})} (D' : B')
\]
and put \((\rho_C \bullet \rho_C, \rho_C \bullet \rho_C) \otimes_k (\rho_C \bullet \rho_C, \rho_C \bullet \rho_C) = (\overline{\rho}_{\mathcal{E} \otimes \mathcal{E}}, \overline{\rho}_{\mathcal{E} \otimes \mathcal{E}})\). By definition \(\overline{\rho}_{\mathcal{E} \otimes \mathcal{E}}: (\mathcal{E} \otimes_k \mathcal{E}) \otimes_k (B \otimes_k B') \to \mathcal{E} \otimes_k \mathcal{E}\) sends \((e \otimes_k e') \otimes_k (b \otimes_k b') \mapsto (eb) \otimes_k (e'b')\), where \(eb = \rho_C \bullet \rho_C(e \otimes_k b) = e(0) \varepsilon e(e(1)b)\) and \(e'b' = \rho_C \bullet \rho_C(e' \otimes_k b') = e'(0) \varepsilon e(e'(1)b')\). That is
\[
\overline{\rho}_{\mathcal{E} \otimes \mathcal{E}}((e \otimes_k e') \otimes_k (b \otimes_k b')) = e(0) \varepsilon e(e(1)b) \otimes_k e'(0) \varepsilon e(e'(1)b')
\]
for every \(e \in \mathcal{E}, e' \in \mathcal{E}, b \in B, \) and \(b' \in B'\). Thus \(\overline{\rho}_{\mathcal{E} \otimes \mathcal{E}} = \rho_{\mathcal{E} \otimes \mathcal{E}} \cdot \rho_{\mathcal{E} \otimes \mathcal{E}}\).

From another hand the map \(\overline{\rho}_{\mathcal{E} \otimes \mathcal{E}}\) is defined by the composition
\[
\mathcal{E} \otimes_k \mathcal{E} \xrightarrow{(\rho_C \bullet \rho_C) \otimes (\rho_C \bullet \rho_C)} (\mathcal{E} \otimes B \mathcal{D}) \otimes_k (\mathcal{E}' \otimes B' \mathcal{D'}) \xrightarrow{\eta_{\mathcal{E}, \mathcal{E}'}} (\mathcal{E} \otimes_k \mathcal{E}') \otimes_{B \otimes_k B'} (\mathcal{D} \otimes_k \mathcal{D}')
\]
sending
\[
e \otimes_k e' \longmapsto \left((- \otimes_k (e(0) \varepsilon e(e(1))) \otimes_k (e'(0) \varepsilon e(e'(1))))\right) \otimes_{B \otimes_k B'} (e(1) \otimes_k e'(1))
\]
that is \(\overline{\rho}_{\mathcal{E} \otimes \mathcal{E}}(e \otimes_k e') = \rho_{\mathcal{E} \otimes \mathcal{E}} \cdot \rho_{\mathcal{E} \otimes \mathcal{E}}(e \otimes_k e')\), for every \((e, e') \in \mathcal{E} \times \mathcal{E}'\). Hence \(\overline{\rho}_{\mathcal{E} \otimes \mathcal{E}} = \rho_{\mathcal{E} \otimes \mathcal{E}} \cdot \rho_{\mathcal{E} \otimes \mathcal{E}}\). Therefore,
\[
(\rho_C \bullet \rho_C, \rho_C \bullet \rho_C) \otimes_k (\rho_C \bullet \rho_C, \rho_C \bullet \rho_C) = (\rho_{\mathcal{E} \otimes \mathcal{E}}, \rho_{\mathcal{E} \otimes \mathcal{E}}) \cdot (\rho_{\mathcal{E} \otimes \mathcal{E}}, \rho_{\mathcal{E} \otimes \mathcal{E}})
\]
which shows that \(- \otimes_k -\) is a covariant functor. The last assertion is obvious, and the particular one is clear, since the associativity of the bi-functor \(- \otimes_k -\) is up to natural isomorphisms. \(\Box\)

Remark 3.2. Of course, we have a similar results on the left hand. That is, the category \(\text{CrgExt}\) whose object are corings and morphisms are left coring extensions is also a monoidal category with the same unit \(k = (k : k)\). If we denote by \(\text{CrgExt}\) the category whose object are all corings and morphisms are the left and right (at the same times) coring extensions. Then the study of \(\text{CrgExt}\) can be also posed, viewing it as a subcategory of both \(\text{CrgExt}^r\) and \(\text{CrgExt}^l\). Examples of morphisms in this subcategory are morphisms between corings with the same scalars base ring.

Finally, we will consider the category of corings \(\text{Corings}\) where the objects are corings understood also as pairs \((\mathcal{C} : A)\) and morphisms are in the sense of [5]; that is a pair of maps \((\phi, \varphi) : (\mathcal{C} : A) \to (\mathcal{D} : B)\), where \(\varphi : A \to B\) is an algebra map and
$\phi : \mathcal{C} \rightarrow \mathcal{D}$ is an $A$–bilinear map ($\mathcal{D}$ is an $A$–bimodule by scalars restriction) such that the following diagrams commute

$\begin{array}{c}
\mathcal{C} \\
\downarrow \phi \\
\mathcal{D}
\end{array} \xrightarrow{\Delta_C} \begin{array}{c}
\mathcal{C} \otimes_A \mathcal{C} \\
\downarrow \phi \otimes A \phi \\
\mathcal{D} \otimes_A \mathcal{D},
\end{array} \quad \begin{array}{c}
\mathcal{C} \\
\downarrow \varepsilon_C \\
A
\end{array}$

where $\omega_{A,B} : \mathcal{D} \otimes_A \mathcal{D} \rightarrow \mathcal{D} \otimes_B \mathcal{D}$ is the obvious map associated to $\varphi$. The identity arrow of an object ($\mathcal{C}: A$) is $id_{(\mathcal{C}, A)} = (id_{\mathcal{C}}, id_A)$, and the composition law is componentwise composition.

**Lemma 3.3.** Let $(\phi, \varphi) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ and $(\phi', \varphi') : (\mathcal{C}' : A') \rightarrow (\mathcal{D}' : B')$ are a corings morphisms. Then

$$(\phi \otimes_k \phi', \varphi \otimes_k \varphi') : (\mathcal{C} \otimes_k \mathcal{C}' : A \otimes_k A') \rightarrow (\mathcal{D} \otimes_k \mathcal{D}', B \otimes_k B')$$

is also a corings morphism.

**Proof.** Analogue to that of [4, Proposición 1.1.20]. Obviously $\varphi \otimes_k \varphi'$ is an algebra map. Since $\phi$ and $\phi'$ are, respectively, $A$–bilinear and $A'$–bilinear, it is easily checked that $\phi \otimes_k \phi'$ is $A \otimes_k A'$–bilinear. The counit property of $\phi \otimes_k \phi'$ is given by the following commutative diagram

$\begin{array}{c}
\mathcal{C} \otimes_k \mathcal{C}' \\
\downarrow \phi \otimes_k \phi' \\
\mathcal{D} \otimes_k \mathcal{D}'
\end{array} \xrightarrow{\omega_{\mathcal{C} \otimes_k \mathcal{C}', \mathcal{D} \otimes_k \mathcal{D}'}} \begin{array}{c}
A \otimes_k A' \\
\downarrow \varphi \otimes_k \varphi' \\
B \otimes_k B'.
\end{array}$

For the coassociativity, we denote by $\eta_{A,A'}$ and $\eta_{B,B'}$ the natural isomorphism of Lemma [4] to distinguish between the tensor product algebra $A \otimes_k A'$ and $B \otimes_k B'$. With this notation, we have

$$\begin{align*}
\omega_{A \otimes_k A', B \otimes_k B'} & \circ ((\phi \otimes_k \phi') \otimes_{A \otimes_k A'} (\phi \otimes_k \phi')) \circ \Delta_{\mathcal{C} \otimes_k \mathcal{C}'} \\
= & \omega_{A \otimes_k A', B \otimes_k B'} \circ ((\phi \otimes_k \phi') \otimes_{A \otimes_k A'} (\phi \otimes_k \phi')) \circ \eta_{\mathcal{C} \otimes_k \mathcal{C}', \mathcal{D} \otimes_k \mathcal{D}'} \\
= & \omega_{A \otimes_k A', B \otimes_k B'} \circ \eta_{\mathcal{C} \otimes_k \mathcal{C}', \mathcal{D} \otimes_k \mathcal{D}'} \circ ((\phi \otimes_A \phi) \otimes_k (\phi' \otimes_{A'} \phi')) \circ (\Delta_{\mathcal{C} \otimes_k \mathcal{C}'} \circ \Delta_{\mathcal{C} \otimes_k \mathcal{C}'}) \\
= & \eta_{B,B'} \circ (\omega_{A,B} \otimes_k \omega_{A',B'}) \circ ((\phi \otimes_A \phi) \otimes_k (\phi' \otimes_{A'} \phi')) \circ (\Delta_{\mathcal{C} \otimes_k \mathcal{C}'} \circ \Delta_{\mathcal{C} \otimes_k \mathcal{C}'}) \\
= & \eta_{B,B'} \circ (\Delta_{\mathcal{D} \otimes_k \mathcal{D}'} \circ (\phi \otimes_k \phi') = \Delta_{\mathcal{D} \otimes_k \mathcal{D}'} \circ (\phi \otimes_k \phi'),
\end{align*}$$
where we have used the naturality of \( \eta_{\cdot \cdot} \) and the coassociativity of \( \phi \) and \( \phi' \); this means the coassociative property of the proposed map.

\[\square\]

**Proposition 3.4.** Consider the category \( \text{Corings} \) of all corings, and denote by \( k = (k : k) \) the trivial \( k \)-coring. The following

\[\begin{array}{c}
\text{Corings} \times \text{Corings} \\
\downarrow \\
\text{Corings}
\end{array}\]

\[\begin{array}{c}
((C : A), (C' : A')) \\
\downarrow \\
(C \otimes_k C' : A \otimes_k A')
\end{array}\]

\[\begin{array}{c}
((\phi, \varphi), (\phi', \varphi')) \\
\downarrow \\
(\phi \otimes_k \phi', \varphi \otimes_k \varphi')
\end{array}\]

establishes a covariant bi-functor. Moreover,

\[k \otimes_k (C : A) \cong (C : A) \quad \text{and} \quad (C : A) \otimes_k k \cong (C : A)\]

natural isomorphisms in \( \text{Corings} \). In particular, \( \text{Corings} \) is a monoidal category with unit \( k \).

**Proof.** Consequence of Lemma 3.3. \[\square\]

**Remark 3.5.** The relationship between the morphisms of \( \text{Corings} \) and those of \( \text{CrgExt}^+_k \) can be given as follows. Let \( (\phi, \varphi) : (C : A) \to (D : B) \) be a morphism in \( \text{Corings} \), and consider the \( B \)-coring \( B \otimes_A C \otimes_A B \) called the base ring extension of \( C \), see [3, 17.2]. Denote by \( \rho_{B \otimes_A C \otimes_A B} : B \otimes_A C \otimes_A B \otimes_k B \to B \otimes_A C \otimes_A B \) the right scalar multiplication map, and define

\[\rho^{B \otimes_A C \otimes_A B} : B \otimes_A C \otimes_A B \to B \otimes_A C \otimes_A B \otimes_B D \cong B \otimes_A C \otimes_A D \]

\[b \otimes_A c \otimes_A b' \mapsto b \otimes_A c(1) \otimes_A \phi(c(2))b' \]

where \( \Delta_C(c) = c(1) \otimes_A c(2) \). It is easily checked that the pair

\[\rho_{B \otimes_A C \otimes_A B}^{-1} \otimes_A \rho^{B \otimes_A C \otimes_A B} : (B \otimes_A C \otimes_A B : B) \to (D : B)\]

is now a morphism in the category \( \text{CrgExt}^+_k \).

In fact \( B \otimes_A C \otimes_A B \) is a right and left coring extensions of \( D \). That is to any morphism \( (\phi, \varphi) \) in \( \text{Corings} \), one can associate a morphism in the category \( \text{CrgExt}^+_k \) described in Remark 3.2.

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