Nonlinear dynamics of running: Speed, stability, symmetry and the effects of leg amputations

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Abstract

In this paper, we study dynamic stability during running, focusing on the effects of speed and the use of a leg prosthesis. We compute and compare the maximal Lyapunov exponents of kinematic time-series data from subjects with and without unilateral transtibial amputations running at a wide range of speeds. We find that the dynamics of the affected leg with the running-specific prosthesis are less stable than the dynamics of the unaffected leg, and also less stable than the biological legs of the non-amputee runners. Surprisingly, we find that the center-of-mass dynamics of runners with two intact biological legs are slightly less stable than those of runners with amputations.
Our results suggest that while leg asymmetries may be associated with instability, runners may compensate for this effect by increased control of their center-of-mass dynamics.
Lead Paragraph

In order to understand the combined effects of speed, stability, and the use of leg prostheses, it is important to explore the dynamical details of running. Nonlinear time-series analysis of kinematic gait data can effectively elucidate these details. There have been a number of experimental studies of the dynamics of running (e.g., [18]). To our knowledge, however, no one has explored the stability dynamics of runners with leg amputations, a population to whom dynamical stability seems an especially important issue. Using nonlinear time-series analysis on motion-capture data from treadmill studies, we analyzed the gait dynamics of runners with and without a unilateral transtibial amputation, from a slow run up to each individual’s top speed. We used standard delay-coordinate embedding techniques to reconstruct the dynamics from scalar time-series traces of the positions of various anatomical markers (e.g., the height of the sacrum or the sagittal-plane angle of the right knee), then we calculated the maximal Lyapunov exponent $\lambda_1$ of each resulting trajectory. We found that stability decreased at faster speeds for all runners, with or without amputations. We also found that lower-limb dynamics were less stable (viz., higher $\lambda_1$) for the affected leg of runners with an amputation than for their unaffected leg—and less stable than either leg of the non-amputee runners. The $\lambda_1$ values increased with running speed, but the inter-leg and inter-group relationships remained largely the same. Surprisingly, the results showed that the center-of-mass dynamics of non-amputee runners were slightly less stable than for runners with a unilateral transtibial amputation. This suggests that asymmetries may lead to instability in the leg dynamics that are compensated for by increased control of the center of mass.

1 Introduction

Analysis of the dynamics of locomotion elucidates temporal variations in gait patterns and also leads to a better understanding of stability. Nonlinear time-series analysis techniques have been used to study various aspects of human walking, including differences between normal and pathological walking gait (e.g., [7, 16]), the effects of age and illness [5, 27], synchronization when two people walk side-by-side [26], recognition of an individual from his or her gait [11], and stability of walking in the face of continuous perturbations [23]. The goal of our study was to explore the effects of speed, stability, and leg prosthesis use in the dynamics of a different locomotion pattern: running. At moderate speeds, a runner can be modelled as a bouncing spring-mass system, whereas walking can be represented as a series of inverted-pendulum arcs. A number of interesting models of the dynamics of running have been developed in the biomechanics, robotics, and nonlinear
dynamics communities (e.g., [17]), some of which were specifically constructed to explore stability issues [6]. Only a few studies involved nonlinear analysis of laboratory data from human runners (e.g., [24]), but none have explored the temporal details of the dynamics. Further, the effects of prosthesis use on these dynamics have not, to our knowledge, been studied at all.

To explore these dynamics, we collected data from 17 subjects running on an instrumented treadmill across a wide range of speeds (3-9 m/s). Six of these subjects had a unilateral transtibial amputation and eleven had two intact biological legs. The time-series data included the \(xyz\) positions of reflective markers placed on the body, gathered via motion-capture cameras over a number of gait cycles. We reconstructed the center-of-mass dynamics using delay-coordinate embedding on various scalar projections of these raw data. We reconstructed the limb dynamics by converting the 3D positions to joint angles and then embedded those angle traces. Finally, we calculated the maximal Lyapunov exponent \(\lambda_1\) of each of the embedded trajectories using the algorithm of Kantz [19]. Quantifying the dynamic stability of human locomotion, which is defined as resistance to change under perturbation, is not trivial. Full et al., for instance, suggested a detailed approach that decomposes each trajectory into limit cycles and quantifies the rates of recovery from perturbations in different state-space directions [13]. For the purposes of a time-series analysis study with finite amounts of data, \(\lambda_1\) is a proxy for stability that has been used extensively in human walking studies (e.g., [3, 4, 7, 22]).

The approach outlined in this paper is useful not only for exploring the nonlinear dynamics of running, but also for assessing the sensitivity of those dynamics to perturbations. A better understanding of these effects could inform the design of better prostheses for this activity. A careful assessment of dynamics is also useful for understanding the intertwined roles of symmetry and stability. Seeley et al. [28] and Gundersen et al. [15], for instance, demonstrated that healthy walking gait is bilaterally symmetrical, even though slight asymmetries may develop to accommodate for changing environmental factors. Skinner & Effeney [29] found significant bilateral asymmetries in the lower-limb kinematics of people with leg amputations during walking; Enoka et al. [10] found similar asymmetries in running. During running and sprinting, Grabowski et al. determined that people with a unilateral transtibial amputation applied significantly less force to the ground with their affected leg than their unaffected leg [14]. It is not known, however, if that kind of force asymmetry affects the dynamic stability of gait. Variability and asymmetry are not necessarily detrimental; in the introduction to the 2009 focus issue of CHAOS on “Bipedal Locomotion–From Robots to Humans,” Milton [25] writes, “Thus it is possible that a certain amount of kinematic variability in certain aspects of performance might be indicative of a healthier dynamical system.” A comparison of the gait dynamics of non-amputee runners to those of runners with
a unilateral transtibial amputation may elucidate these subtle effects. The research reported in this paper was driven by three hypotheses:

1. For individuals with or without a transtibial amputation, dynamic stability will decrease at faster running speeds. This hypothesis is based on the work of England & Granata, who found that faster walking speeds lead to larger $\lambda_1$ (viz., less stability) \[9\]. We expected a similar relationship between speed and stability during running.

2. The $\lambda_1$ of the lower-limb dynamics of runners with a unilateral transtibial amputation will be asymmetric, across all speeds. This followed from the geometric asymmetry of the dynamical system, defined as the notable anthropomorphic differences (mass and moment of inertia) between the affected and unaffected legs, as well as the loss of muscular control in the affected leg.

3. The $\lambda_1$ of the center-of-mass dynamics of runners with a unilateral transtibial amputation will be greater than in non-amputee runners. We based this hypothesis on the rationale that symmetry in the lower limbs poses a challenge to maintaining overall stability during locomotion.

Our study confirmed our first two hypotheses. The $\lambda_1$ values increased with running speed, while the inter-leg and inter-subject relationships remained largely the same across speed. We found that lower-limb mechanics were generally less stable (viz., higher $\lambda_1$) for the affected leg of runners with amputations than for their unaffected leg—or than for either leg of the non-amputee runners. Surprisingly, though, our results showed that the center-of-mass dynamics of non-amputee runners were slightly less stable than in runners with a unilateral transtibial amputation.

The following two sections describe how the data for this study were collected and analyzed. The results are presented in Section 4 and discussed in Section 5.

### 2 Data Collection

A total of 17 subjects—6 runners (4 male and 2 female) with a unilateral transtibial amputation and 11 runners (8 male and 3 female) without amputations—participated in the study described in this paper. In the rest of this document, members of these two groups are designated with the acronyms WA and NA, respectively. All of the experiments occurred at the Biomechanics Laboratory of the Orthopedic Specialty Hospital (Murray, Utah). A photograph of the setup is shown in Figure[1] All subjects gave informed written consent according to the Intermountain
Figure 1: Subject with a unilateral transtibial amputation running on a high-speed instrumented treadmill

Healthcare IRB approved protocol. Each WA subject used his or her own sprint-specific passive-elastic prosthesis during the tests. We measured each subject’s height, mass, prosthesis mass, and standing leg lengths. We defined leg length as the vertical distance from the greater trochanter to the floor during standing. We measured the length of the affected leg of each WA subject when it was unloaded, by having the individual stand with a 2” wooden block under the unaffected leg.

Subjects performed a series of constant-speed running trials on a custom high-speed treadmill (Treadmetrix, Park City UT). Each trial consisted of at least ten strides except for top-speed trials, which consisted of 8 strides. After a brief warm up, subjects started the series of running trials at 3 m/s. Each subsequent trial speed was incremented by 1 m/s until subjects reported that they were approaching their top speed. Smaller speed increments were then employed until subjects reached their top speed, defined as the speed at which they could not maintain their position on the treadmill for more than eight strides [31]. Subjects were allowed as much time as desired between trials for full recovery. The pelvis position was defined by reflective markers attached to the anterior and posterior iliac spines and iliac crests of the right and left sides. We used reflective marker clusters to define the thigh and shank segments. In order to define the hip and knee joint centers, we also placed reflective markers on the greater trochanters and the medial and lateral femoral
condyles of the right and left legs. We used a marker placed over the sacrum as a proxy for the center of mass. We used motion-capture cameras (Motion Analysis Corporation, Santa Rosa, CA) to measure the 3D positions of those markers at a rate of 300 frames per second, then calculated the joint angles from those data using Visual3D software (C-Motion Inc., Germantown, MD). We did not normalize the timebase of each data set to the average stride period, as is done in some gait studies, because that operation would obscure the speed effects in which we were interested.

We used the spatial position of the sacrum marker at the base of the spine to study the center-of-mass dynamics. This is, of course, an approximation. The sacrum location is close to the overall body center of mass (COM) when a person stands in the standard anatomical position. However, when a person runs, their COM position moves within the body. It is possible to estimate the COM location using a segmental approach, but this methodology relies on many assumptions and estimates about human body segment parameters. Furthermore, there are no established methodologies for estimating how an amputation and/or the use of a running-specific prosthesis affects the position or movement of the COM. Thus, we chose to use the sacrum marker as a proxy for estimating the COM location and studying its dynamics during running.

3 Time-Series Analysis

The time-series data described in the previous section comprised time-series traces of dozens of joint positions in 3D space. To reconstruct the locomotion dynamics from these data, we used delay-coordinate embedding. Provided that the underlying dynamics and the observation function \( h \) that produces the measurement \( x(t) \) from the underlying state variables \( X \) of the dynamical system are both smooth and generic, the delay-coordinate map

\[
F(\tau, m)(x) = ([x(t) \ x(t+\tau) \ldots \ x(t+(m-1)\tau)])
\]

with delay \( \tau \) from a \( d \)-dimensional smooth compact manifold \( M \) to \( \mathbb{R}^m \) is a diffeomorphism on \( M \) if the embedding dimension \( m \) is greater than \( 2d \) \[^{[30]}\). Here, \( M \) is the dynamics of the human body; \( h \) is the measurement executed by the motion-capture system, plus the post-processing involved in the conversion from marker positions to joint angles.

Since the body is a coupled dynamical system, one should theoretically be able to use delay-coordinate embedding to reconstruct its \( d \)-dimensional dynamics from any single joint position (or angle). Here, though, we wished to focus on smaller units of the body. To this end, we used the medio-lateral \( (x) \), anterior-posterior \( (y) \)
and vertical (z) position coordinates of the sacrum marker to assess the center-of-mass dynamics. To explore the lower-limb dynamics, we used the sagittal plane knee- and hip-joint angles. Examples of these data can be seen in Figure 2 which shows traces of the knee-angle data from two of the runners in this study, one NA and one WA subject. The temporal patterns in the left- and right-knee angles of the NA runner are very similar, though they are of course roughly 180 degrees out of phase. There is an obvious difference, however, between the knee angles of the affected and unaffected legs of the WA subject. All four traces—both knees of both runners—demonstrate largely, but not completely, periodic motion.

To reconstruct the state-space dynamics from these time-series data, we followed standard procedures regarding the choice of appropriate values for the embedding parameters: the minimum of the mutual-information curve as an estimate of the delay $\tau$ and the false-near neighbors technique of with a threshold of 10%, to estimate the embedding dimension $m$. Figure 3 shows the mutual information and false-near neighbor curves for the time-series data of Figure 2. To perform these calculations, we used TISEAN’s mutual and false nearest tools. The corresponding embeddings are shown in Figure 4. For both legs of both subjects, $m = 3$ was sufficient to unfold the dynamics and the first minima of the mutual information curves occurred between $52\Delta t$ and $56\Delta t$, where the sampling rate $\Delta t = 1/300$th of a second. The embedded trajectories have a characteristic figure-eight shape that reflects the general pattern of running gait, but with visible stride-to-stride variations.

To study these patterns and variations, we employed the algorithm of Kantz, as instantiated in TISEAN’s lyap_k tool, to estimate the maximal Lyapunov exponent $\lambda_1$ of the embedded data. First, we plotted the log of the expansion rate $S(\Delta n)$ versus time $\Delta n$ and verified that the curves were of the appropriate shape (i.e., a scaling region followed by a horizontal asymptote, which should occur when the time horizon of the algorithm is large enough to allow neighboring trajectories to stretch across the diameter of the attractor). We then fit a line to that scaling region and determined its slope. All $\lambda_1$ values in this paper are scaled to the inverse of the sampling interval, $\Delta t = 1/300$th second; to convert these $\lambda_1$ values to inverse seconds, one multiplies these values by the sampling frequency (300). Figure 5 shows the log $S(\Delta n)$ versus $\Delta n$ curves for the time-series data in Figure 2 embedded using the $\tau$ and $m$ values suggested by the curves in Figure 3. These results indicated that the dynamics of both knees of each of these two runners was sensitively dependent on initial conditions, with $\lambda_1$ ranging from 0.0084–0.0154 per $\Delta t$, which translates to 2.52–4.62 in units of inverse seconds. In both subjects, the $\lambda_1$ values differed between the two legs, but the difference was more pronounced for the WA subject. This pattern is discussed at more length in the following section.
Figure 2: Sagittal-plane knee angles for (top) a non-amputee runner and (bottom) a runner with a unilateral transtibial amputation, both running at 4 m/s. 0° is full extension; negative angles correspond to flexion of the joint.
Figure 3: Estimating embedding parameters for the data of Figure 2: mutual information as a function of the delay $\tau$, plotted in units of the sampling rate $\Delta t = 1/300^{th}$ sec, and % false near neighbors as a function of the dimension $m$. The minima of the mutual information curves occur near $\tau = 52\Delta t$ (i.e., 173 milliseconds) for both knees of the non-amputee (“NA”) runner and $\tau = 55\Delta t$ (i.e., 183 milliseconds) for both knees of the runner with a unilateral transtibial amputation (the “WA” subject). All four false near neighbor curves drop to 10% at $m = 3$. Color code as in the previous figure: blue and red correspond to the left and right leg, respectively, of the NA runner, and to the affected and unaffected leg, respectively, of the WA subject.
Figure 4: Delay-coordinate embeddings of the traces in Figure 2 with the $\tau$ and $m$ values suggested by the curves in Figure 3. In these plots, time ($t$) is in units of $\Delta t$, the inverse of the 300 Hz sampling rate of the time series.
Figure 5: Lyapunov exponent calculations for the embedded data of Figure [3]. The slopes of the scaling regions of these curves—fit by the superimposed lines in the plots—represent estimates of the maximal Lyapunov exponent $\lambda_1$ of the corresponding trajectories. $\Delta n$ is plotted in units of $\Delta t$, the inverse of the 300Hz sampling rate of the time series.
All of the nonlinear time-series analysis algorithms mentioned in this section are known to be sensitive to data and parameter effects [20]. These systematic uncertainties preclude the use of traditional statistics to assess or compare their results, but there are other ways to do “due diligence.” We validated all \( \lambda_1 \) calculations by repeating them for a range of values of the dimension \( m \) and the critical scale parameter (\( \epsilon \)) in the \( \lambda y a p \_k \) algorithm\(^1\) and discarding any that produced inconsistent results (i.e., large variation with \( m \) and/or \( \epsilon \)). We also discarded all results from \( S(\Delta n) \) versus \( \Delta n \) curves that did not have a clear scaling region. We repeated all of these calculations on seven traces (right and left knee- and hip-joint angles, plus the \( x \), \( y \), and \( z \) positions of the sacrum) for each of the 17 subjects. The following section summarizes the results and examines the differences and similarities between and across subjects, groups, speeds, and legs.

4 Results

4.1 Knee Dynamics

Our analysis of the embedded knee-joint dynamics supports our first hypothesis: stability decreases (viz., increasing \( \lambda_1 \)) at faster speeds for all runners, with and without amputations. The average \( \lambda_1 \) of the right and left knee angles of the NA runners increased from 0.095 and 0.101 at 3 m/s to 0.137 and 0.136 at 9 m/s, respectively. The overall speed-stability trends—higher \( \lambda_1 \) at faster running speeds—were similar for the WA subjects, further supporting our first hypothesis. The average \( \lambda_1 \) of WA subjects were 0.103 and 0.090 for affected and unaffected legs, respectively, at 3 m/s; at 9 m/s, the corresponding values were 0.138 and 0.124. See Table 1 for \( \lambda_1 \) values for both groups across all running speeds.

The symmetry of the system—the subject of the second hypothesis—is reflected in the similarities and differences between the numbers in Table 1. As one would expect, the left and right knee dynamics were quite similar in the NA subjects, who have two intact biological legs. Not surprisingly, there were differences between legs in the WA subjects. This result is consistent with our second hypothesis regarding inter-leg asymmetry in this group.

4.2 Hip Dynamics

The \( \lambda_1 \) values for the reconstructed hip-joint dynamics were also consistent with our first two hypotheses. The average \( \lambda_1 \) of NA runners were 0.098 and 0.103 at

\(^1\)This parameter specifies the size of the neighborhood whose points are tracked for the calculation of the spreading factor \( S \). Too-small values of \( \epsilon \) cause numerical problems because the neighborhood contains only a few points; too-large values cause the calculation to sample the dynamics too broadly.
NA subjects

| Speed (m/s) | Sample Size | Left $\lambda_1$ | Right $\lambda_1$ |
|-------------|-------------|------------------|-------------------|
| 3           | 9           | 0.101 $\pm$ 0.016 | 0.095 $\pm$ 0.017 |
| 4           | 11          | 0.096 $\pm$ 0.017 | 0.096 $\pm$ 0.016 |
| 5           | 10          | 0.100 $\pm$ 0.009 | 0.104 $\pm$ 0.009 |
| 6           | 10          | 0.110 $\pm$ 0.020 | 0.113 $\pm$ 0.020 |
| 7           | 10          | 0.113 $\pm$ 0.020 | 0.115 $\pm$ 0.016 |
| 8           | 8           | 0.117 $\pm$ 0.012 | 0.121 $\pm$ 0.013 |
| 9           | 8           | 0.136 $\pm$ 0.016 | 0.137 $\pm$ 0.007 |

WA subjects

| Speed (m/s) | Sample Size | Unaffected $\lambda_1$ | Affected $\lambda_1$ |
|-------------|-------------|-------------------------|-----------------------|
| 3           | 4           | 0.098 $\pm$ 0.018       | 0.103 $\pm$ 0.009     |
| 4           | 5           | 0.103 $\pm$ 0.012       | 0.107 $\pm$ 0.009     |
| 5           | 5           | 0.102 $\pm$ 0.102       | 0.124 $\pm$ 0.019     |
| 6           | 6           | 0.104 $\pm$ 0.104       | 0.118 $\pm$ 0.017     |
| 7           | 5           | 0.113 $\pm$ 0.113       | 0.133 $\pm$ 0.017     |
| 8           | 4           | 0.119 $\pm$ 0.119       | 0.137 $\pm$ 0.024     |
| 9           | 4           | 0.124 $\pm$ 0.124       | 0.138 $\pm$ 0.037     |

Table 1: $\lambda_1$ values for the embedded knee-joint dynamics of non-amputees and subjects with amputations. The values reported are averages across all traces in the corresponding class at that speed (e.g., the average of the right-knee $\lambda_1$ values of all NA runners at 3 m/s was $0.095$ in units of inverse $\Delta t$, the 300Hz sampling interval of the data, with a standard deviation of $0.017$).
Table 2: $\lambda_1$ values for the embedded hip-joint dynamics of non-amputees and subjects with amputations. The values reported are averages across all traces in the corresponding class at that speed (e.g., the average of the right-hip $\lambda_1$ values of all NA runners at 3 m/s was 0.098 in units of inverse $\Delta t$, the 300Hz sampling interval of the data, with a standard deviation of 0.013).

| Speed (m/s) | Sample Size | Unaffected $\lambda_1$ | Affected $\lambda_1$ |
|------------|-------------|-------------------------|---------------------|
| 3          | 9           | 0.075 ± 0.020           | 0.113 ± 0.017       |
| 4          | 5           | 0.097 ± 0.014           | 0.116 ± 0.017       |
| 5          | 5           | 0.095 ± 0.010           | 0.109 ± 0.017       |
| 6          | 6           | 0.098 ± 0.009           | 0.117 ± 0.021       |
| 7          | 5           | 0.104 ± 0.012           | 0.115 ± 0.014       |
| 8          | 4           | 0.111 ± 0.011           | 0.119 ± 0.014       |
| 9          | 4           | 0.122 ± 0.004           | 0.131 ± 0.005       |

3 m/s for the right and left legs, respectively; these values increased to 0.119 and 0.116 at 9 m/s (Table 2). Again, the values for the right and left legs were similar for the NA subjects, reflecting the symmetry in their running gait. As in the case of the knee-angle results in the previous section, the embedded hip-joint data provide some indications of asymmetry in the dynamics between unaffected and affected legs of the WA subjects, again supporting our second hypothesis. The average $\lambda_1$ of WA subjects were 0.075 and 0.113 at 3 m/s for the affected and unaffected legs, respectively; these values increased to 0.122 and 0.131 at 9 m/s. This convergence of $\lambda_1$ values at higher running speeds—a reduction in the asymmetry in the dynamics—might indicate that while there could be many different mechanical choices to run slowly (i.e., fewer constraints), there may only be one effective way to run at faster speeds. The average $\lambda_1$ increased at faster running speeds for
both legs in both groups, again supporting our first hypothesis. Since the knee and hip angles are measurements of the same dynamical system—essentially, different measurement functions $h$ applied to the same underlying dynamics $M$—these corroborations are not surprising. We discuss this at more length in Section 5.

4.3 Center-of-Mass Dynamics

As discussed on page 7, we used the sacrum marker at the base of the spine as a proxy for the center-of-mass position. See Table 3 for $\lambda_1$ values for the dynamics reconstructed from the medio-lateral and vertical positions of this marker. (The anterior-posterior position of the sacrum during treadmill running reflects more about the subjects’ ability to match treadmill speed than anything else, and hence was not included in these analyses.) The $\lambda_1$ of the medio-lateral dynamics of the sacrum marker generally increased with running speed, which is in accordance with our first hypothesis. However, the dynamics reconstructed from time-series data of the vertical position of the sacrum exhibited a different pattern. In NA runners, the $\lambda_1$ of these embedded dynamics did not show any clear pattern with increasing speed; in WA runners, $\lambda_1$ decreased with speed. These patterns are significantly different from those in the hip- and knee-joint dynamics. Since all of these data are simultaneous measurements of different macroscopic variables in the same dynamical system, this discrepancy between joint dynamics and center-of-mass dynamics is a puzzling finding from a dynamical-systems standpoint; see the following section for more discussion of this issue.

The sacrum position data also had interesting implications regarding our third hypothesis (that the center-of-mass dynamics of WA runners will be less stable than in NA runners). The answer appears not to be so simple. Across all speeds, $\lambda_1$ was smaller in the vertical dynamics for WA runners—i.e., those dynamics were more stable. In the medio-lateral direction, WA runners were more stable than NA runners at slower speeds, but at faster speeds, NA runners were more stable.

5 Discussion

A nonlinear time-series analysis of knee, hip, and sacrum dynamics of runners with and without a unilateral transtibial amputation confirmed our hypothesis that $\lambda_1$ generally increases with running speed, with one exception: the vertical position of the sacrum, where the $\lambda_1$ of the embedded data decreased with running speed.

2The second hypothesis is not at issue in this section, since the sacrum position data do not effectively isolate the dynamics of the individual lower limbs.
| NA subjects | Speed (m/s) | Sample Size | Medio-Lateral $\lambda_1$ | Vertical $\lambda_1$ |
|-------------|-------------|-------------|--------------------------|-------------------|
|             | 3           | 9           | 0.016 ± 0.006            | 0.130 ± 0.016     |
|             | 4           | 11          | 0.031 ± 0.010            | 0.121 ± 0.021     |
|             | 5           | 10          | 0.034 ± 0.018            | 0.130 ± 0.016     |
|             | 6           | 10          | 0.039 ± 0.015            | 0.142 ± 0.035     |
|             | 7           | 10          | 0.095 ± 0.040            | 0.127 ± 0.062     |
|             | 8           | 8           | 0.100 ± 0.041            | 0.104 ± 0.066     |
|             | 9           | 8           | 0.128 ± 0.015            | 0.128 ± 0.044     |
| WA subjects | Speed (m/s) | Sample Size | Medio-Lateral $\lambda_1$ | Vertical $\lambda_1$ |
|-------------|-------------|-------------|--------------------------|-------------------|
|             | 3           | 4           | 0.015 ± 0.011            | 0.100 ± 0.031     |
|             | 4           | 5           | 0.046 ± 0.040            | 0.123 ± 0.033     |
|             | 5           | 5           | 0.051 ± 0.039            | 0.114 ± 0.038     |
|             | 6           | 6           | 0.073 ± 0.041            | 0.119 ± 0.056     |
|             | 7           | 5           | 0.074 ± 0.043            | 0.087 ± 0.053     |
|             | 8           | 4           | 0.093 ± 0.056            | 0.084 ± 0.051     |
|             | 9           | 4           | 0.084 ± 0.062            | 0.052 ± 0.012     |

Table 3: $\lambda_1$ values for the embedded sacrum-position dynamics of non-amputees and subjects with amputations. The $\lambda_1$ values reported in the two right-hand columns are averages across all traces in the corresponding class at that speed (e.g., the average of the medio-lateral $\lambda_1$ values of all NA runners at 3 m/s was 0.016 in units of inverse $\Delta t$, the 300Hz sampling interval of the data, with a standard deviation of 0.006).
for WA subjects and remained roughly the same for NA subjects. We have two conjectures about this result:

- It may be the case that runners exert increased control of the core to balance decreased stability elsewhere, and this effect may be more pronounced in WA runners.

- Because running involves very little side-to-side motion, the associated vertical movement of the sacrum may dominate the dynamics. That is, there could be a strongly stable limit cycle in the vertical sacrum dynamics due to the mechanical energy storage/return mechanism that governs vertical interactions with the ground.

Our second hypothesis concerned symmetry in the embedded lower-limb dynamics for WA subjects. Our analysis indicated that the $\lambda_1$ of the embedded time-series data from the affected leg was indeed higher than for the unaffected leg—except for knee data at low speeds.

Our third hypothesis—that the $\lambda_1$ of the center-of-mass dynamics of WA runners would be higher than for NA runners—was not verified by this analysis, except for a few midrange speeds in the medio-lateral sacrum position data. This may be due to the effects discussed in the first paragraph of this section. It is also important to note that we observed small amounts of nonstationarity in the medio-lateral data due to subtle changes of the subjects position on the treadmill. Although others have minimized nonstationarities in the signal using divided difference methods prior to computing Lyapunov exponents [8], we avoided that approach because those methods amplify any inherent noise in the signal. In addition, we believe that these slight nonstationarities represents behavior that is dynamically meaningful, as opposed to the kind of unavoidable drift that occurs in a measurement sensor. In our view, slight changes in the subject’s position from step to step may represent responses to local disturbances during running, thus providing additional insight into dynamic stability.

Readers from the biomechanics community will have noted that we did not do any of the traditional statistics analyses on these results—e.g., fitting a regression line to the data in the tables and giving an $R^2$ value to quantify our certainty about whether or not those data validate a particular hypothesis. Numerical algorithms that extract important properties of complicated nonlinear dynamical systems are based on approximations of the associated theory. They involve a number of parameters that strongly affect the results, and they are notoriously sensitive to noise, data length, and other sampling effects. A systematic exploration of these effects is mandatory if one is to believe the results: minimally, a comparison of the results of different algorithms and a careful exploration of the parameter space of each one.
(We performed all of these kinds of checks on our results.) And since algorithms like \texttt{lyap.k} inject systematic biases in the results, the underlying assumptions of traditional statistics—e.g., normally distributed errors—do not hold.\(^3\)

At this point, we are unaware of any other studies that have quantified the nonlinear dynamics of time-series data for individuals with unilateral amputations running across a range of speeds, as described here. Our findings on this unique population of runners, then, are difficult to compare directly to other work. Enoka et al. \cite{enoka10} were the first to provide important insights into the asymmetries that exist between the biological and prosthetic leg in individuals with unilateral amputations. As was normal in that era, the runners with amputations used inelastic prostheses designed for walking, not running. Yet, they were able to run at speeds ranging from 2.7 m/s–8.2 m/s and exhibited notable kinematic intra-limb asymmetries, e.g. significant reductions in the joint angle range of motion of the prosthetic leg compared to the biological leg. The leg prostheses used by runners in our study were designed to mimic the spring-like mechanical behavior of biological legs more closely. Even so, we observed slight asymmetries in the stability of the hip and knee dynamics. We also observed slight asymmetries in the stability of the hip and knee dynamics, indicating that the use of running-specific prostheses do not yet exactly replicate the biomechanical function of biological legs.

Readers from the nonlinear dynamics community will have noted the differences between the \(\lambda_1\) values. This bears some explanation since the different time-series data sets studied here are simultaneous samples of the same nonlinear dynamical system. Theoretically, the \(\lambda_1\) values of the dynamics reconstructed from these different time-series datasets should be the same. This holds if the sensors that measure those different angles effect smooth, generic functions of at least one state variable of that system, and as long as the dynamics themselves are smooth. In practice, the length of the datasets plays a role as well. If the dynamical coupling between parts of the body is weak, that coupling will not manifest during a short time series and thus the “invariants” of the reconstructed dynamics will not be the same from joint to joint. Since we were interested in the dynamics of gaits that could not be sustained indefinitely (viz., running at top speed), gathering longer time series was not an option. The \(\lambda_1\) values reported here, then, are really more like local \(\lambda_s\) \cite{ap}, also known as finite-time Lyapunov exponents, and they should not be expected to be identical across the entire body. There may be other effects at work here: the sharp ground-contact forces of running may disrupt the smoothness of the dynamics, and the movement of the residual limb within the prosthetic socket (“pistoning”) may add dynamics.

\(^3\)Indeed, Kantz & Schreiber \cite{kantz} quote Salman Rushdie to make this point.
This study raises a variety of interesting questions regarding stability, symmetry, and the effects of a running-specific prosthesis. Here, we follow the practice of defining dynamic stability as the resistance to a perturbation [23]. It is not clear, however, whether the body reacts differently to endogenous vs. exogenous perturbations. The analysis presented here studies the stability of running by analyzing the rate of divergence of nearby trajectories in the reconstructed state space. This provides some indication of how the system responds to local perturbations, but it does not distinguish between internal and external perturbations. If the system is autonomous, this distinction is irrelevant. However, if the dynamics are nonstationary—if the forward evolution from a given point in state space depends on how (or when) one got to that point—this distinction may be very important. One could explore this by delivering controlled perturbations to the subject on the treadmill and studying the resulting dynamics. These experiments would be challenging. There are a number of technical issues surrounding sampling movement trajectories following a perturbation, including the short time scales over which the body’s internal controller reacts and the potential hystereses and nonstationarities in that controller: e.g., a shift from feedback to a feedforward control strategy. For instance, the body could learn, over time, to prepare an appropriate response at the expected time of a perturbation. Experiments that could elucidate these effects, while challenging, have the potential to reveal general strategies of how the body’s internal controller deals with external perturbations and whether these responses can be captured by nonlinear time-series analysis.

With regards to dynamic symmetry, the anthropomorphic differences between the affected and unaffected legs of runners with a unilateral transtibial amputation are accompanied by slight asymmetries in stepping kinematics of running and sprinting [14]. Interestingly, adding mass (∼300 g) to the running-specific prosthesis helps to improve kinematic symmetry [14]. Similarly, anthropomorphic and mass differences between the unaffected and affected leg may create stability asymmetries in the dynamics of runners with a unilateral amputation. Our analysis suggests that the ability to respond to small perturbations during running may be compromised in the affected leg as compared to the unaffected leg. It is important to note that the running-specific prosthesis plus the socket together weigh ∼2.3 kg while the biological leg (foot and shank) weighs ∼3.6 kg. The question remains as to whether adding mass to the running-specific prosthesis, as explored in [14], would improve the dynamic symmetry between the unaffected and affected leg in runners with a unilateral amputation.
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