Gorenstein injective dimension, Bass formula and Gorenstein rings

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Abstract

Let \((R, m, k)\) be a noetherian local ring. It is well-known that \(R\) is regular if and only if the injective dimension of \(k\) is finite. In this paper it is shown that \(R\) is Gorenstein if and only if the Gorenstein injective dimension of \(k\) is finite. On the other hand a generalized version of the so-called Bass formula is proved for finitely generated modules of finite Gorenstein injective dimension. It also improves the results by Enochs and Jenda \cite{10} and Christensen \cite{5}.

\textit{MSC:} 13D05; 13H10

1 Introduction

The classical homological dimensions are no debt the central homological notions in commutative algebra. The notion of Gorenstein injective dimension of a module has been defined by E. E. Enochs and O. M. G. Jenda \cite{11} in mid nineties. It is a refinement of the classical injective dimension and shares some of its nice properties. One can also consider the Gorenstein injective dimension as the dual notion to the...
Gorenstein dimension introduced by M. Auslander [1]. In this note we try to generalize some of the classical results on injective dimensions to Gorenstein injective dimension.

Recall that the following statement is well-known.

**Theorem** If \((R, \mathfrak{m}, k)\) is a commutative local noetherian ring then \(R\) is regular if and only if the injective dimension of \(k\) is finite.

Enochs and Jenda proved that over a Gorenstein local ring, the Gorenstein injective dimension of every module is finite [8]. Using the so-called Foxby duality, it has been proved that if the local ring \(R\) admits a dualizing complex (i.e. it is a homomorphic image of a Gorenstein local ring), then \(R\) is Gorenstein if and only if its residue field has finite Gorenstein injective dimension (cf. [5] and [6]). In section 2, we prove the same statement over an arbitrary noetherian local ring (Theorem 2.7).

The main theorem of section 3, generalizes the so-called Bass formula. Recall that

**Theorem** If \((R, \mathfrak{m}, k)\) be a commutative noetherian local ring and \(M\) is a finitely generated \(R\)-module of finite injective dimension, then

\[
\text{id}_R M = \text{depth}_R M.
\]

In ([5]; 6.2.15) Christensen has proved the same formula for finitely generated modules of finite Gorenstein injective dimension over a Cohen-Macaulay local ring which admits a dualizing module. More recently, the result has been proved over local rings which admit a dualizing module (cf. [6]).

Theorem 3.1 gives another generalization of the Bass formula. Namely,

**Theorem** Let \(S\) be a commutative noetherian ring. If \(M\) is a finitely generated \(S\)-module of finite Gorenstein injective dimension, then

\[
\text{Gid}_S M = \sup \{ \text{depth}_S p | p \in \text{Supp}(M) \}.
\]

Consequently, the equation \(\text{Gid}_R M = \text{depth}_R M\) holds, for a finitely generated module \(M\) of finite Gorenstein injective dimension over an almost Cohen-Macaulay local
ring $R$ $(3.2)$.

**Convention.** Throughout this note, the rings are assumed to be commutative and noetherian. Furthermore, $(R, \mathfrak{m}, k)$, always denotes a commutative noetherian local ring with the maximal ideal $\mathfrak{m}$ and the residue field $k$.

## 2 Characterization of Gorenstein local rings.

In this section we prove that finiteness of the Gorenstein injective dimension characterizes the Gorenstein local rings. First recall basic definitions and facts. For details and proofs cf. [13] or [5].

**Definition 2.1** An exact complex of injective $R$-modules,

$$I = \ldots \rightarrow I_2 \rightarrow I_1 \rightarrow I_0 \rightarrow I_{-1} \rightarrow I_{-2} \rightarrow \ldots$$

is a complete injective resolution if and only if the complex $\text{Hom}_R(J, I)$ is exact for every injective $R$-module $J$. A module $M$ is said to be Gorenstein injective if and only if it is the 0-th kernel of a complete injective resolution.

It is clear that every injective module is Gorenstein injective. Then one can construct a Gorenstein injective resolution for any module.

**Definition 2.2** Let $M$ be an $R$-module. A Gorenstein injective resolution of $M$ is an exact sequence

$$0 \rightarrow M \rightarrow G_0 \rightarrow G_1 \rightarrow \cdots$$

such that $G_i$ is Gorenstein injective for all $i \geq 0$. We say that $M$ has Gorenstein injective dimension less than or equal to $n$, $\text{Gid}_{R}M \leq n$, if $M$ has a Gorenstein injective resolution

$$0 \rightarrow M \rightarrow G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_n \rightarrow 0.$$

It is well-known that one always has

$$\text{Gid}_{R}M \leq \text{id}_{R}M.$$

The equality holds if $\text{id}_{R}M < \infty$. The Gorenstein injective dimension can be computed using Ext functors.
Theorem 2.3 Let $M$ be an $R$-module of finite Gorenstein injective dimension. Then

$$\text{Gid}_R^{} M = \sup\{i \mid \text{Ext}^i_R(J, M) \neq 0 \text{ for an } R-\text{module } J \text{ with } \text{id}_R^{} J < \infty\}.$$  

We start with proving some preliminary lemmas. Recall that over a local noetherian ring, $(R, \mathfrak{m}, k)$, injective dimension of a finite $R$-module $M$ is the supremum of integers $i$ such that $\text{Ext}^i_R(k, M)$ is non-zero (cf. [3]).

The following lemma shows that the residue field can be replaced with its injective envelope when the module has finite injective dimension.

Lemma 2.4 Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local ring homomorphism of noetherian local rings and let $M$ be a finitely generated $S$-module with finite injective dimension over $R$. Then

$$\text{id}_R^{} M = \sup\{i \mid \text{Ext}^i_R(E(k), M) \neq 0\}.$$  

Proof. Let $\text{id}_R^{} M = t$. The exact sequence

$$0 \to k \to E(k) \to C \to 0$$

induces the long exact sequence

$$\cdots \to \text{Ext}^t_R(E(k), M) \to \text{Ext}^t_R(k, M) \to \text{Ext}^{t+1}_R(C, M) \to \cdots.$$  

Since $\text{Ext}^{t+1}_R(C, M) = 0$ and $\text{Ext}^t_R(k, M) \neq 0$ (cf. [2]; 5.5) one has

$$\text{Ext}^t_R(E(k), M) \neq 0$$

and this proves the assertion. □

Note that in 2.4 the finitely generated condition for $M$ is necessary.

Example. Let $\phi$ be the identity homomorphism over $R$. If $M = E(R/p)$, then $\text{id}_R^{} M = 0$. Let $\psi : E(k) \to E(R/p)$ be an $R$ homomorphism. If $x \in E(k)$ then module $Rx$ is of finite length. Thus $R\psi(x)$ is a submodule of $E(R/p)$ which has finite length. Since $E(R/p)$ is an essential extension of $R/p$ non of its non-trivial submodules have finite length. Therefore $\psi(x) = 0$. That is $\text{Hom}_R^{}(E(k), E(R/p)) = 0$.  

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Lemma 2.5 Let \((R, \mathfrak{m}, k)\) be a noetherian local ring and let \(M\) be an \(R\)-module. Then for any \(p \neq \mathfrak{m}\) and any \(i \geq 0\), \(\Ext^i_R(E(R/p), M) = 0\) if one of the following conditions hold.

(a) \(M\) has finite length.

(b) \(R\) is complete and \(M\) is finitely generated.

Proof. (a) Since \(p \neq \mathfrak{m}\) there exists \(x \in R - p\) such that \(xM = 0\). But multiplication by \(x\) is an automorphism on \(E(R/p)\). So the assertion holds.

(b) See the proof of ([14]; 2.2). □

The following corollary, which generalizes 2.4, shows that the Gorenstein injective dimension of a finith-length module, if it is finite, can be computed in terms of vanishing of the \(\Ext^i_R(E(k))\) functors.

Corollary 2.6 Let \((R, \mathfrak{m}, k)\) be a noetherian local ring and let \(M\) be an \(R\)-module of finite length which has finite Gorenstein injective dimension. Then

\[
\Gid_R M = \sup \{ i \mid \Ext^i_R(E(k), M) \neq 0 \}.
\]

Proof. It is proved ([15]; 2.29) that for any \(R\)-module \(M\) with finite Gorenstein injective dimension we have

\[
\Gid_R M = \sup \{ i \mid \exists p \in \text{Spec}R : \Ext^i_R(E(R/p), M) \neq 0 \}.
\]

Now the assertion follows from 2.5. □

Now we are ready to prove the main theorem of this section. It gives a characterization of Gorenstein local rings in terms of the finiteness of Gorenstein dimension of modules. In [6], using hyperhomological techniques, a similar result has been proved, provided that the ring admits a dualizing complex (equivalently, is a homomorphic image of a Gorenstein local ring). Theorem 2.7 generalizes that theorem with a rather simpler proof.

Theorem 2.7 Let \((R, \mathfrak{m}, k)\) be a noetherian local ring. The following are equivalent.
(i) $R$ is Gorenstein.

(ii) $\text{Gid}_R k$ is finite.

(iii) $\text{Gid}_R M$ is finite for any finitely generated $R$–module $M$.

(iv) $\text{Gid}_R M$ is finite for any $R$–module $M$.

Proof. (iv)⇒(iii) and (iii)⇒(ii) are obvious. For (ii)⇒(i), let $i \geq 0$. One has

$$\text{Ext}^i_R(E(k), k) = \text{Ext}^i_R(E(k), \text{Hom}_R(k, E(k)))$$

$$\cong \text{Ext}^i_R(k, \text{Hom}_R(E(k), E(k)))$$

$$\cong \text{Ext}^i_R(k, \hat{R})$$

$$\cong \text{Ext}^i_{\hat{R}}(k, \hat{R})$$

where $\hat{R}$ is the completion of $R$ in $\mathfrak{m}$-adic topology. Now using 2.6, we have

$$\text{id}_{\hat{R}} \hat{R} = \text{Gid}_R k < \infty.$$ 

Thus $\hat{R}$ and hence $R$ are Gorenstein.

(i)⇒(iv). cf. [8] or [5].

Let $R$ be a local ring. It is well-known that $R$ is regular (respectively, Gorenstein, Cohen-Macaulay) if and only if there exists a simple (respectively, cyclic, finitely generated) $R$–module with finite injective dimension. (For the first and last statement cf. [3] and for the second one cf. [17].)

Theorem 2.7 shows that if there exists a simple $R$-module of finite Gorenstein injective dimension then $R$ is Gorenstein. Now it is natural to ask

**Question.** What one can say about a ring $R$ which admits a cyclic (respectively, finitely generated) module of finite Gorenstein injective dimension?

### 3 Bass formula.

Recall that if a finitely generated module over a noetherian local ring has finite injective dimension then its injective dimension is equal to the depth of the base ring. This is known as Bass formula. In ([5]; 6.2.5) Christensen has proved that
over a Cohen-Macaulay local ring with a dualizing module, one can replace injective dimension with Gorenstein injective dimension. In this section we try to generalize this result. The main result of this section is the following theorem.

**Theorem 3.1** Let $S$ be a commutative noetherian ring. If $M$ is a finitely generated $S$-module of finite Gorenstein injective dimension, then

$$
\text{Gid}_SM = \sup \{ \text{depth } S_p \mid p \in \text{Supp}(M) \}.
$$

**Proof.** Let $\text{Gid}_SM = 0$ and suppose that $\text{depth } S_p > 0$ for some $p \in \text{Supp}(M)$. Then there exists an $S_p$-regular element $x \in pS_p$. The exact sequence

$$
0 \to S_p \xrightarrow{x} S_p \to S_p/xS_p \to 0
$$

induces the following exact sequence.

$$
M_p \xrightarrow{x} M_p \to \text{Ext}^1_{S_p}(S_p/xS_p, M_p) \to 0
$$

Since $M$ is finitely generated, using Nakayama’s lemma, multiplication by $x$ is not surjective over $M_p$ and so $\text{Ext}^1_{S_p}(S_p/xS_p, M_p) \neq 0$.

On the other hand, $M$ is a Gorenstein injective $R$-module and then an exact sequence $0 \to N \to I \to M \to 0$, with $I$ an injective $S$-module, exists. Hence $\text{Ext}^1_{S_p}(S_p/xS_p, M_p) \cong \text{Ext}^2_{S_p}(S_p/xS_p, N_p) = 0$, which is a contradiction. Then depth $S_p$ should be zero and the claim is proved in this case.

Now let $\text{Gid}_SM = n > 0$. We prove that $M$ can be written as a homomorphic image of an $S$-module with injective dimension $n$. Furthermore, we will show that localizations of this module to every prime ideal outside $\text{Supp}(M)$ is injective.

By ([15], 2.45), there exists a Gorenstein injective $S$-module $G$ and an $R$-module $C$ with $\text{id}_SC = \text{Gid}_SC = n - 1$, such that the following sequence is exact.

$$
0 \to M \to G \to C \to 0
$$

Since $G$ is a Gorenstein injective $S$-module, there exists an injective $S$-module $E$ and an exact sequence

$$
0 \to K \to E \to G \to 0
$$

with $K$ a Gorenstein injective $S$-module, too. So the isomorphisms $C \cong G/M$ and $G \cong E/K$ of $S$-modules hold and then there exists a submodule $L$ of $E$ such that $K \subseteq L$ and $M \cong L/K$ and therefore $C \cong E/L$. 

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Considering the following exact sequence, we get $\text{id}_S L \leq \text{id}_S C + 1 = n$.

$$0 \to L \to E \to C \to 0$$

On the other hand, since $\text{Gid}_S M = n$, there exists an injective $S$-module $J$ with $\text{Ext}_S^n(J, M) \neq 0$. The exact sequence

$$0 \to K \to L \to M \to 0$$

induces the following exact sequence.

$$\text{Ext}_S^n(J, L) \to \text{Ext}_S^n(J, M) \to \text{Ext}_S^{n+1}(J, K) = 0$$

Therefore, we have $\text{Ext}_S^n(J, L) \neq 0$ and then $\text{id}_S L \geq n$. Hence $\text{id}_S L = n = \text{Gid}_S M$.

By Chouinard’s equality ([4]), we have

$$\text{id}_S L = \sup \{ \text{depth} S_p - \text{width} S_p L_p \mid p \in \text{Supp}(L) \}.$$

For every $p \in \text{Supp}(L)$, we have the following exact sequence.

$$0 \to K_p \to L_p \to M_p \to 0$$

If $p \in \text{Supp}(M)$, then from the induced exact sequence $L_p/pL_p \to M_p/pM_p \to 0$ and the fact that $M_p/pM_p \neq 0$, we get $\text{width} S_p L_p = 0$.

If $p \not\in \text{Supp}(M)$, then $L_p \cong K_p$. Therefore $L_p$ is an injective $S_p$-module and then using Chouinard’s equality, we have

$$\text{depth} S_p - \text{width} S_p L_p \leq \text{id}_S L_p = 0.$$

Therefore

$$\text{Gid}_S M = \text{id}_S L$$

$$= \sup \{ \text{depth} S_p - \text{width} S_p L_p \mid p \in \text{Supp}(L) \}$$

$$= \sup \{ \text{depth} S_p \mid p \in \text{Supp}(M) \}$$

This finishes the proof. \qed

**Corollary 3.2** Let $(R, m, k)$ be an almost Cohen-Macaulay local ring (i.e. $\dim R - \text{depth} R \leq 1$) and let $M$ be a finite $R$-module. If $\text{Gid}_R M < \infty$ then

$$\text{Gid}_R M = \text{depth} R.$$
Proof. Use \ref{3.1} and the fact that over an almost Cohen-Macaulay ring, for every two prime ideals \( p \) and \( q \) with \( p \in q \), the inequality \( \text{depth } R_p \leq \text{depth } R_q \) holds. \( \square \)

In \cite{18}, Salarian, Sather-Wagstaff and Yassemi prove that over local rings, Gorenstein injective dimension "behaves well with respect to killing a regular element". Namely, if \( M \) is a finitely generated module over a local ring \((R, \mathfrak{m}, k)\) then \( \text{Gid}_R M < \infty \) implies \( \text{Gid}_{R/xR} M/xM < \infty \), when \( x \in \mathfrak{m} \) is \( R \)- and \( M \)-regular. The similar result was proved in \cite{6} when \( R \) is assumed to admit a dualizing complex. The next corollary of \ref{3.1} expresses how the values of the Gorenstein injective dimensions relate.

**Corollary 3.3** Let \((R, \mathfrak{m}, k)\) be a noetherian local ring and \( M \) a finitely generated \( R \)-module. If \( x \in \mathfrak{m} \) is an \( R \)- and \( M \)-regular element, then

\[
\text{Gid}_{R/xR} M/xM \leq \text{Gid}_R M - 1.
\]

Furthermore, the equality holds when \( R \) is almost Cohen-Macaulay and \( \text{Gid}_R M \) is finite.

**Proof.** If \( \text{Gid}_R M \) is not finite then the inequality is clear. Now assume that \( M \) has finite Gorenstein injective dimension, then to prove the inequality, it is sufficient to use \ref{3.1} and the following facts.

- \( \text{Supp}(M/xM) = \{ p/xR \mid p \in \text{Supp}(M) \text{ and } x \in p \} \).
- If \( x \in p \) then \( \text{depth } (R/xR)_{p/xR} = \text{depth } R_p - 1 \).

The last part of the corollary is clear from \ref{3.2}. \( \square \)

The following immediate corollary of \ref{3.1} shows that finite Gorenstein injective dimension does not exceed after localization.

**Corollary 3.4** Let \( S \) be a commutative noetherian ring and \( M \) a finitely generated \( S \)-module such that \( \text{Gid}_{S_p} M_p < \infty \) for all prime ideals \( p \). Then if \( p \) and \( q \) are two prime ideals with \( p \subset q \), we have

\[
\text{Gid}_{S_p} M_p \leq \text{Gid}_{S_q} M_q.
\]
**Remark.** Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let $M$ and $N$ be finitely generated $R$-modules. If $\text{id}_R N < \infty$ then in [16] Ischebeck showed that  

$$\text{depth } R - \text{depth}_R M = \sup \{ i \mid \text{Ext}^i_R(M, N) \neq 0 \}.$$ 

It is natural to ask whether one could replace $N$ by a finite module of finite Gorenstein dimension. The answer is negative.

**Example.** If $M$ has finite projective dimension then the above equality holds for every finitely generated $R$-module $M$. Now let $R$ be a Gorenstein local ring which is not regular. Let $k$ be the residue field of $R$. One has $\text{pd}_R k = \infty$ and then $\sup \{ i \mid \text{Ext}^i_R(k, k) \neq 0 \} = \infty$. But it is clear (cf. 2.7) that $\text{Gid}_R k < \infty$.

The following statement is a partial generalization of the Ischebeck’s result in another direction. It proves the equality for an $S$-module $M$ (not necessarily finitely generated) with $\text{depth}_R M = 0$.

**Proposition 3.5** Let $\phi : (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a local ring homomorphism of noetherian local rings and let $M$ be an $S$-module. Then for any finitely generated $S$-module $N$ with finite injective dimension over $R$ the following equality holds.

$$\text{depth } R - \text{depth}_R M = \sup \{ i \mid \text{Ext}^i_R(M, N) \neq 0 \}$$

Provided that $\text{depth}_R M = 0$ or $M$ is finitely generated $R$-module.

**Proof.** Set $\text{id}_R N = t$. If $\text{depth}_R M = 0$ then there exists a short exact sequence

$$0 \rightarrow k \rightarrow M \rightarrow C \rightarrow 0$$

which induces a long exact sequence

$$\cdots \rightarrow \text{Ext}^t_R(M, N) \rightarrow \text{Ext}^t_R(k, N) \rightarrow \text{Ext}^{t+1}_R(C, N) \rightarrow \cdots .$$

Since $\text{Ext}^{t+1}_R(C, N) = 0$ and $\text{Ext}^t_R(k, N) \neq 0$, we have $\text{Ext}^t_R(M, N) \neq 0$ and then

$$\sup \{ i \mid \text{Ext}^i_R(M, N) \neq 0 \} \geq t.$$ 

The inverse inequality holds clearly. If $M$ is finitely generated we use induction on $\text{depth}_R M$ to prove the desired equality. If $\text{depth}_R M > 0$ then there exist an
$M$-regular element $x \in \mathfrak{m}$. Using the long exact sequence induced by the following exact sequence, the equality can be proved from induction’s hypothesis.

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

The following corollary of 3.5 is a generalization of Bass formula. The statement has been appeared in [19], too.

**Corollary 3.6** Let $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local ring homomorphism of noetherian local rings. For any finitely generated $S$-module $N$ with finite injective dimension over $R$, the following equality holds.

$$\text{depth } R = \text{id}_{R} N$$

**Proof.** In 3.5, set $M = R/\mathfrak{m}$ and use ([2]; 5.5). □

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