Twisted Bundle On Quantum Torus and BPS States in Matrix Theory

Pei-Ming Ho

Department of Physics, Jadwin Hall, Princeton University, Princeton, NJ 08544

pmho@feynman.princeton.edu

Abstract

Following the recent work of Connes, Douglas and Schwarz, we study the M(atrix) model compactified on a torus with a background of the three-form field. This model is given by a super Yang-Mills theory on a quantum torus. To consider twisted gauge field configurations, we construct twisted $U(n)$ bundles on the quantum torus as a deformation of its classical counterpart. By properly taking into account membranes winding around the light-cone direction, we derive from the M(atrix) model the BPS spectrum which respects the full $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ U-duality in M theory.

1on leave from Department of Physics, University of Utah, Salt Lake City, Utah 84112.
1 Introduction

In this paper we look into more details of the M(atrix) model first considered by Connes, Douglas and Schwarz [1], which is a super Yang-Mills theory living on a quantum torus. It is conjectured that [1, 2] this describes the discrete light-cone quantization (DLCQ) of the M theory [3] compactified on a torus with a nonvanishing three-form field background. This idea has been further developed in [4, 5, 6, 7].

The purpose for this work is to derive from this M(atrix) model the BPS spectrum expected in M theory [1] which respects the full $SL(2, \mathbb{Z})_C \times SL(2, \mathbb{Z})_N$ U-duality. Here $SL(2, \mathbb{Z})_C$ denotes the group of modular transformations for the classical torus on which the M theory compactifies, and $SL(2, \mathbb{Z})_N$ is the non-classical duality group which acts on the three-form field $C_{123}$. For M theory compactified on $T^3$ with volume $V$, $SL(2, \mathbb{Z})_N$ acts on $\tau \equiv C_{123} + iV$ by the linear fractional transformation $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$ for any integers $a, b, c, d$ satisfying $ac - bd = 1$. In our case we have one side of the $T^3$ on a ligh-cone direction so $V = 0$ and $\tau = C_{-12}$. The symmetry of $SL(2, \mathbb{Z})_C$ is relatively trivial to check in the M(atrix) model. In this paper we will only focus on the $SL(2, \mathbb{Z})_N$ symmetry for the BPS spectrum in M(atrix) model.

However, since we are interested in the BPS configurations with constant field strength on the quantum torus, the first question we meet is how to define twisted (non-trivial) $U(n)$ bundles on a quantum torus. The general definition of a bundle on a quantum space is that it is a projective module of the algebra of functions on the quantum space [8]. In [9] the projective module for twisted $U(n)$ bundles over the quantum torus was found and in fact they were used in [1]. Here we will take a naive, classical approach to construct twisted bundles on the quantum torus from the algebra of functions on the quantum plane. Although the results are the same, in this way we show what is the classical counterpart for this projective module.

The second problem we meet is how to describe in M(atrix) model membranes winding around one transverse direction as well as the light-cone direction, because such states are needed to complete the $SL(2, \mathbb{Z})_N$ multiplet of BPS states [10]. We will argue that in M(atrix) model the winding numbers for such winding states appear as the quantum numbers parametrizing the Wilson lines for the electric field background.

In Sec.2 we review the Matrix compactification on a torus, which generically results in the dual space of a quantum torus. In Sec.3 we review how sections on twisted bundles over a classical torus can be constructed for a constant field strength by imposing twisted boundary conditions. Then we follow the same steps for the quantum

---

1 A projective module is a direct summand of a (finite-dimensional) free module.
case in Sec.4. Finally, in Sec.5 we modify the action of M(atrix) model by inserting appropriate Wilson lines and calculate the BPS spectrum, which is formally the same as the conjectured BPS spectrum in [1] obtained from M theory arguments, but we find more independent quantum numbers.

2 Matrix Compactification On Torus

In [4] and [5] we gave the general formulation of Matrix compactification on the quotient space $\mathbb{R}^d/\Gamma$, where $\Gamma$ is a discrete subgroup of Euclidean motions in $\mathbb{R}^d$. The resulting theory is generically a super Yang-Mills theory (SYM) on a dual quantum space. The data needed to specify a Matrix compactification include the action $\Phi$ of $\Gamma$ on $\mathbb{R}^d$ and a function $\alpha(g, h)$ with $g, h \in \Gamma$. We use $\alpha$ to define a projective representation of $\Gamma$ as

$$U_g U_h = e^{i\alpha(g, h)}U_{gh},$$

and then impose the quotient conditions

$$U_g^\dagger X^\mu U_g = \Phi^\mu_g(X).$$

Physically inequivalent choices of $\alpha(g, h)$ are one-to-one corresponding to elements in the second Hochschild cohomology $H^2(\Gamma, U(1))$ defined as follows [5]. We call an angular function (defined up to $2\pi$) with $k$ arguments in $\Gamma$ a $k$-cochain. The coboundary $\delta$ is an operation which maps a $k$-cochain $\omega(g_1, \cdots, g_k)$ to a $k+1$-cochain given by

$$(\delta \omega)(g_0, \cdots, g_k) = \omega(g_1, \cdots, g_k) + \sum_{l=1}^{k} (-1)^l \omega(g_0, \cdots, g_{l-1}g_l, \cdots, g_k) + (-1)^{k+1} \epsilon(g_0, \cdots, g_{k-1})\epsilon(g_k),$$

where $\epsilon(g) = \pm 1$ is the $\mathbb{Z}_2$-grading on $\Gamma$ corresponding to the orientifolding. We have $\epsilon(g_1g_2) = \epsilon(g_1)\epsilon(g_2)$. It follows that $\delta^2 = 0$. For orbifolds $\epsilon(g) = 1$ for all $g$. A cochain annihilated by $\delta$ is called a cocycle and $H^2(\Gamma, U(1))$ is the group of equivalence classes of 2-cocycles defined up to the coboundary of 1-cochains. It is very important to find out the corresponding physical degrees of freedom for $H^2(\Gamma, U(1))$.

Now we focus on the case of the compactification on a 2-torus. In this case $\Gamma = \mathbb{Z}^2$ and for a 2-torus with radii $R_1, R_2$ the quotient conditions are

$$U_i^\dagger X_j U_i = X_j + 2\pi R_j \delta_{ij}, \quad i, j = 1, 2.$$
For simplicity we will consider a straight torus. It is straightforward to generalize it to a slanted torus. The algebra of $U_1$ and $U_2$ should satisfy

$$U_1 U_2 = e^{i2\pi \theta} U_2 U_1,$$

which happens to define the algebra of functions $\mathcal{A}(T^2_\theta)$ on a quantum torus $T^2_\theta$. This $\theta$ is the unique parameter for this compactification since $H^2(\Gamma, U(1)) = U(1)$. It is proposed that [1, 2] it corresponds to the three-form field background $C_{-12}$ for the DLCQ M theory on a torus in directions $X_1$ and $X_2$:

$$\theta = RC_{-12},$$

where $R$ is the radius for light-cone quantization.

For the case $\theta = 0$ the solutions of $X$ to the quotient conditions are the covariant derivatives on the dual torus [1, 2]:

$$X_i = -i2\pi R_i \partial_i + A_i.$$  

In general, the solutions of $X$ can be viewed as the covariant derivatives on the dual quantum torus $T^2_{(-\theta)}$ [1, 4, 5]. The algebra of functions on the dual quantum torus $\mathcal{A}(T^2_{(-\theta)})$ is simply the opposite algebra of $\mathcal{A}(T^2_\theta)$. We denote the generators of functions on the dual torus by $\hat{U}_1$ and $\hat{U}_2$, with $\hat{U}_1 \hat{U}_2 = e^{-i2\pi \theta} \hat{U}_2 \hat{U}_1$ and $[\hat{U}_i, U_j] = 0$. Then

$$X_i = -i2\pi R_i \partial_i + A_i(\hat{U}_1, \hat{U}_2), \quad i = 1, 2,$$

where $\partial_i$ are the derivatives on $T^2_{(-\theta)}$:

$$\partial_i \hat{U}_j = \hat{U}_j (\partial_i + i\delta_{ij}), \quad \partial_i U_j = U_j (\partial_i + i\delta_{ij}),$$

and

$$\partial_1 \partial_2 = \partial_2 \partial_1.$$  

The Matrix compactification specified by [1] and [5] results in the $U(n)$ SYM on the dual quantum torus $T^2_{(-\theta)}$. When the $U(n)$ bundle on the dual torus is untwisted, the gauge fields $A_i$ are $n \times n$ matrices of functions on $T^2_{(-\theta)}$. A state in the Hilbert space is then an $n$-vector of functions on $T^2_{(-\theta)}$. When the $U(n)$ bundle is twisted, the Hilbert space is a projective module of $\mathcal{A}(T^2_{(-\theta)})$. We will construct the twisted bundle on the quantum torus by following the naive classical treatment of $U(n)$ bundles on a torus and view the quantum case as a deformation.

### 3 Twisted Bundles On Classical Torus

Just like we can only talk about functions on a quantum space rather than points on the quantum space, we can only talk about sections on a quantum bundle rather than
points on the bundle. Hence we first review the classical construction for sections on a twisted bundle. Since we are considering $U(n)$ bundles, we can treat sections of the twisted bundle of fundamental representation as states in a Hilbert space so that sections of the bundle of adjoint representation as well as covariant derivatives act as operators on the Hilbert space.

Consider the classical case of a constant curvature with

$$[D_1, D_2] = -\frac{i}{2\pi} \frac{m}{n}.$$  \hspace{1cm} (10)

By fixing a gauge, let

$$D_1 = \frac{\partial}{\partial \sigma_1}, \quad D_2 = \frac{\partial}{\partial \sigma_2} - i \frac{m \sigma_1}{n \cdot 2\pi}.$$  \hspace{1cm} (11)

The gauge fields are not well-defined functions on $T^2$ but are functions on $\mathbb{R}^2$ satisfying the twisted boundary conditions

$$D_i(\sigma_1 + 2\pi, \sigma_2) = \Omega_1(\sigma_2) D_i(\sigma_1, \sigma_2) \Omega_1^\dagger(\sigma_2),$$  \hspace{1cm} (12)

$$D_i(\sigma_1, \sigma_2 + 2\pi) = \Omega_2(\sigma_1) D_i(\sigma_1, \sigma_2) \Omega_2^\dagger(\sigma_1),$$  \hspace{1cm} (13)

where

$$\Omega_1(\sigma_2) = e^{i m \sigma_2 / n} U, \quad \Omega_2(\sigma_1) = V,$$  \hspace{1cm} (14)

and $U, V$ are $n \times n$ matrices satisfying

$$UV = e^{-i 2\pi m / n} VU.$$  \hspace{1cm} (15)

The reason for the constant curvature to be quantized by $m/n$ is that there is a consistency condition on $\Omega_i$:

$$\Omega_1(\sigma_2 + 2\pi) \Omega_2(\sigma_1) = \Omega_2(\sigma_1 + 2\pi) \Omega_1(\sigma_2).$$  \hspace{1cm} (16)

Sections on the twisted bundle of the fundamental representation of $U(n)$ on a classical 2-torus are $n$-vectors of functions on $\mathbb{R}^2$ satisfying the twisted boundary conditions

$$\phi(\sigma_1 + 2\pi, \sigma_2) = \Omega_1(\sigma_2) \phi(\sigma_1, \sigma_2),$$  \hspace{1cm} (17)

$$\phi(\sigma_1, \sigma_2 + 2\pi) = \Omega_2(\sigma_1) \phi(\sigma_1, \sigma_2).$$  \hspace{1cm} (18)

In the basis where

$$U_{kl} = e^{i 2\pi k m / n} \delta_{kl}, \quad V_{kl} = \delta_{(k+1)l}, \quad k, l = 1, \cdots, n,$$  \hspace{1cm} (19)
it is not difficult to find the general solution to these boundary conditions as \[12\]

\[
\phi_k(\sigma_1, \sigma_2) = \sum_{s \in \mathbb{Z}} \sum_{j=1}^{m} \exp \left\{ i \left( \frac{m}{n} (\sigma_2/2\pi + k + ns) + j \right) \sigma_1 \right\} \tilde{\phi}_j(\sigma_2/2\pi + k + ns).
\] (20)

So a section of the fundamental bundle is specified by \( m \) functions \( \tilde{\phi}_j(x) \) with \( x \in \mathbb{R} \).

Similarly, sections on the twisted bundle of the adjoint representation are matrices of functions on \( \mathbb{R}^2 \) satisfying the twisted boundary conditions of the same form as \((12,13)\). Without loss of generality, we assume that \((n, m)\) are coprime. Then one can check that two particular sections are given by

\[
Z_1 = e^{i\sigma_1/n}V^b, \quad Z_2 = e^{i\sigma_2/n}U^{-b},
\] (21)

for any two integers \( a, b \) satisfying \( an - bm = 1 \). In fact, for any fixed pair of \( a \) and \( b \) they generate the algebra of sections on the adjoint bundle. The actions of \( D_i \) and \( Z_i \) on the fundamental sections \( \phi_k(\sigma_1, \sigma_2) \) induces their actions on \( \tilde{\phi}_j(x) \). These can be read off from the quantum case in the next section by setting \( \theta = 0 \).

### 4 Twisted Bundle On Quantum Torus

Twisted bundles on the quantum torus can be viewed as a deformation of its classical counterpart. As in the classical case, sections on a twisted bundle are not functions well defined on the quantum torus but they are functions on the quantum plane with twisted boundary conditions. So we need to extend the algebra of functions on the torus, which is generated by the two operators \( \tilde{U}_1 \) and \( \tilde{U}_2 \), to the algebra \( \mathcal{A}(\mathbb{C}_{(-\theta)}) \) of functions on the quantum plane, which is generated by \( \sigma_1 \) and \( \sigma_2 \) satisfying

\[
[\sigma_1, \sigma_2] = i2\pi \theta.
\] (22)

The algebra of functions on \( \mathcal{T}_{(-\theta)}^2 \) can be realized as a subalgebra of \( \mathcal{A}(\mathbb{C}_{(-\theta)}) \) by

\[
\tilde{U}_i = e^{i\sigma_i}.
\] (23)

The derivatives \( \partial_i \) act on \( \sigma_i \) by

\[
[\partial_i, \sigma_j] = \delta_{ij}.
\] (24)

Now we can repeat what we just did for the classical case. By fixing a gauge, let

\[
D_1 = \frac{\partial}{\partial \sigma_1}, \quad D_2 = \frac{\partial}{\partial \sigma_2} - if\sigma_1.
\] (25)
for the field strength of
\[ [D_1, D_2] = -if. \]  
(26)
The boundary conditions for \( D_i \) are \([12][13]\) with
\[ \Omega_1 = e^{i\alpha_2}U, \quad \Omega_2 = V, \]  
(27)
where \( \alpha = \frac{2\pi f}{1 + hf} \). The consistency condition
\[ \Omega_1(\sigma_2 + 2\pi)\Omega_2(\sigma_1) = \Omega_2(\sigma_1 + 2\pi)\Omega_1(\sigma_2) \]  
(28)
implies that \( \alpha = m/n \). Therefore
\[ 2\pi f = \frac{m}{n - \theta_m}. \]  
(29)

The solutions for the boundary conditions for sections on a fundamental bundle are given by
\[ \phi_k(\sigma_1, \sigma_2) = \sum_{s \in \mathbb{Z}} \sum_{j=1}^{m} E\left(\frac{m}{n}(\sigma_2/2\pi + k + ns) + j, i\sigma_1\right) \tilde{\phi}_j(\sigma_2/2\pi + k + ns), \]  
(30)
where the function \( E \) is an analogue of the exponential function but is “normal ordered”:
\[ E(A, B) \equiv \frac{1}{1 - [A, B]} \sum_{l=0}^{\infty} \frac{1}{l!} A^l B^l, \]  
(31)
where we assumed that \([A, B]\) is proportional to the unity. It has the following useful properties
\[ E(-B, A)E(A, B) = 1, \]  
(32)
\[ E(A + c, B) = E(A, B)e^{cB}, \]  
(33)
\[ E(A, B + c) = e^{cA}E^{A,B} \]  
(34)
for \( c \in \mathbb{C} \). As in the classical case, the sections on the fundamental bundle are given by \( m \) functions \( \tilde{\phi}_j \) on \( \mathbb{R} \).

Similarly, the sections on adjoint bundles are generated by
\[ Z_1 = e^{i\sigma_1/(n - \theta_m)}V^b, \quad Z_2 = e^{i\sigma_2/n}U^{-b}, \]  
(35)
where \( b \) is an integer satisfying \( an - bm = 1 \) for some integer \( a \). Its algebra is
\[ Z_1Z_2 = e^{i2\pi \theta'} Z_2Z_1, \]  
(36)
where
\[
\theta' = \frac{b - \theta a}{n - \theta m}.
\] (37)

As the \(U(n)\) SYM on a classical manifold can also be viewed as the \(U(1)\) SYM on a noncommutative space involving a factor of \(\mathbb{Z}_n\) [13], the twisted bundle with twisting number \(m\) on \(T^2_{(-\theta)}\) can be viewed as the \(U(1)\) SYM on a dual quantum torus \(T^2_{\theta'}\) [1]. Note that generically \(\theta' \neq 0\) even when \(\theta = 0\).

Now we view the fundamental sections as the Hilbert space and let
\[
\hat{\phi}_j(x) = \tilde{\phi}_j(x - \frac{n}{m}j).
\] (38)

This Hilbert space is equivalent to the projective module used in [1]. The action of \(D_i\) and \(Z_i\) on \(\phi_k(\sigma_1, \sigma_2)\) induces their action on \(\hat{\phi}_j(x)\), thus \(D_i\) and \(Z_i\) can be realized as operators [1]:
\[
\begin{align*}
(D_1 \hat{\phi})_j(x) &= i 2\pi f x \hat{\phi}_j(x), \\
(D_2 \hat{\phi})_j(x) &= \frac{1}{2\pi} \frac{\partial}{\partial x} \hat{\phi}_j(x), \\
(Z_1 \hat{\phi})_j(x) &= \hat{\phi}_{j-a}(x - \frac{1}{m}), \\
(Z_2 \hat{\phi})_j(x) &= e^{-i2\pi j/m} e^{i\frac{2\pi j}{m}x} \hat{\phi}_j(x).
\end{align*}
\] (39) (40)

It is then easy to compute, e.g.
\[
[D_i, Z_j] = \frac{i}{n - \theta m} \delta_{ij} Z_j.
\] (41)

The trace \(Tr\) on the adjoint bundle should be a composition of the trace \(tr\) over \(n \times n\) matrices and the integration over \(\sigma_1\) and \(\sigma_2\), i.e., up to normalization,
\[
Tr(f(Z_1, Z_2)) = \int d\sigma_1 d\sigma_2 tr(f(Z_1, Z_2))
\] (42)

for a section \(f\) of the adjoint bundle. From [13] we see that
\[
Tr(Z_1^{u_1} Z_2^{u_2}) = (n - \theta m) \delta_{u_1}^0 \delta_{u_2}^0,
\] (43)

where the normalization is chosen such that \(Tr(i[D_1, D_2]) = m \in \mathbb{Z}\). While this is the natural normalization for the trace from the mathematical point of view [1], it also turns out to be the right choice for the M(atrix) model to respect the \(SL(2, \mathbb{Z})_N\) duality.

5 BPS Spectrum

As a test of the conjecture [1, 2] that the SYM on a quantum torus describes the DLCQ of M theory on a torus with a three-form field background, we calculate the
BPS spectrum for the M(atrix) model and show that it respects the $SL(2,\mathbb{Z})_N$ duality with the quantum torus parameter $\theta$ interpreted as the three-form field background.

The bosonic part of the Lagrangian for the Matrix model in uncompactified space-time is

$$L = Tr \left( \frac{1}{2R} (D_0 X_{\mu})^2 + (2\pi T_2^M)^2 \frac{R}{4} [X_{\mu}, X_{\nu}]^2 \right),$$

(44)

where $\mu, \nu = 1, 2, \cdots, 9$ and $T_2^M$ is the membrane tension in M theory. We will set $2\pi T_2^M = 1$. Since we will only consider BPS states without fermions, we will ignore the fermionic part for simplicity.

We need to modify this action because it misses the description of the BPS states of membranes winding $m_{i-}$ times around $X_i$ ($i = 1, 2$) and the light-cone direction $X_-$. To see how to do this modification, we note that we can view this action as the action of Matrix strings [14] obtained from M theory compactified on $X_1$. On the other hand, the world-sheet action for a string with a $B$ field background is known to be

$$L = \frac{1}{2} \partial X^\mu \partial X^\nu (g_{\mu\nu} + B_{\mu\nu}),$$

(45)

where $\mu, \nu = 0, 2, \cdots, 9, 11$. Let $X_{\pm} = \frac{1}{\sqrt{2}}(X_0 \pm X_{11})$. $X_-$ and $X_2$ are compactified on circles with radius $R$ and $R_2$ and the string lives in a background with $B_{-2} = C_{-12} = \theta/R$. For strings winding $m_{1-}$ times around $X_-$ we have $X_- = m_{1-}R\sigma$. These are understood as membranes winding $m_{1-}$ times around $X_1$ and $X_-$ in M theory. Thus in the light-cone gauge ($X_+ = P_+\tau$), the Lagrangian contains a term linear in $\dot{X}_2$ with the coefficient of $m_{1-}R\beta_{-2}$. It follows that in the Hamiltonian the kinetic term $\frac{1}{2}P_2^2$ is modified to be $\frac{1}{2}(P_2 + m_{1-}R\beta_{-2})^2$ with $P_2 = n_2/R_2$. Dividing this Hamiltonian by $P_+$ and treating it as an Hamiltonian for $X_2, \cdots, X_9$, with no reference to $X_{11}$, one finds the corresponding Lagrangian to be $L = \frac{P_2}{2} \dot{X}_2^2 - \dot{X}_2 m_{1-}R\beta_{-2}$, where $\mu = 2, \cdots, 9$. (If $B_{-2} \neq 0$ there would be an additional term of $\frac{m_{1-}BB_{-2}}{P_+}$ in $L$.) This suggests that in the Matrix string Lagrangian we need to add the term $\frac{m_{1-}BB_{-2}}{Tr(\gamma)}Tr(\dot{X}_2)$. Similarly, we can repeat the same argument with the roles of $X_1, X_2$ interchanged.

Inserting such appropriate Wilson lines, in the temporal gauge the action for the M(atrix) model becomes

$$L = Tr \left( \frac{1}{2R} \dot{X}_i^2 + F_{0i} \dot{X}_i + \frac{1}{2R} \dot{X}_a^2 + \frac{R}{2} ([X_1, X_2] + F_{12})^2 + \frac{R}{2} [X_i, X_a]^2 + \frac{R}{4} [X_a, X_0]^2 \right),$$

(46)

with

$$F_{0i} = \frac{\theta m_i}{(n - \theta m)R_i}, \quad F_{12} = i2\pi R_1 2\pi R_2 \gamma,$$

(47)
where $m_i = \epsilon_{ij}m_{j-}$, $i = 1, 2$ and $a, b = 3, \ldots, 9$. The new term linear in $\dot{X}_i$ appears as a constant electric field background which was also considered in [15, 16]. The other modification is that we allowed the background of a constant magnetic field $\gamma$ on the dual torus, which may also be interpreted as the background of a gauge field coupled to membranes.

At first it may come as a surprise that the quantum numbers $m_i$ appear in the M(atrix) model not as the eigenvalues of dynamical variables but as the coefficients of new terms in the action. However note that these numbers are associated with states with zero longitudinal momentum, which are supposed to be integrated out to obtain the light cone theory [17, 18]. It is thus natural to expect that $m_i$ appear as a background or the coefficients of new terms in the action. Perhaps $\gamma$ can also have a similar interpretation.

From the point of view of SYM theory, the term linear in $\dot{X}_i$ is a topological term which does not affect the equations of motion for $X_i$. Imposing periodic boundary conditions in time on $X_i$, i.e., $X_i$(final) − $X_i$(initial) = $2\pi k_iR_i$ for some integers $k_i$, one finds that the coefficient of $\dot{X}_i$ is only defined up to an integer over $R_i$ [15, 16]. Since $\theta$ is defined for the quantum torus only up to an integer, $F_{0i}$ is quantized by an integer $m_i$ as above.

It is straightforward to calculate the spectrum for BPS states with

$$X_i = -i2\pi R_iD_i + p_it, \quad X_a = \alpha_aZ_1^{w_1}Z_2^{w_2}e^{i\omega t} + \alpha_a^\dagger Z_2^{-w_2}Z_1^{-w_1}e^{-i\omega t},$$  \hspace{1cm} (48)

where $\omega = \frac{2\pi R}{n-\theta m}\sqrt{(R_1w_1)^2 + (R_2w_2)^2}$. We have

$$[X_i, Z_1^{w_1}Z_2^{w_2}] = 2\pi R_i \frac{w_i}{n-\theta m}Z_1^{w_1}Z_2^{w_2}$$  \hspace{1cm} (49)

and the conjugate momentum for $X_i$ is quantized to be $n_i/R_i$ for the K-K modes. The BPS spectrum is thus

$$H = R \left( \frac{1}{2(n-\theta m)} \left( \frac{n_i - \theta m}{R_i} \right)^2 + \frac{1}{2} V^2 \left( \frac{m + (n-\theta m)\gamma}{n-\theta m} \right)^2 \right)$$

$$+ \frac{2\pi R}{(n-\theta m)}\sqrt{(R_1w_1)^2 + (R_2w_2)^2},$$  \hspace{1cm} (50)

where $V = 2\pi R_12\pi R_2$.

This spectrum (50) is invariant under the transformation

$$\theta \to -1/\theta, \quad n \to -m, \quad m \to n,$$  \hspace{1cm} (51) \hspace{1cm} (52)
\[ n_i \rightarrow -m_i, \quad m_i \rightarrow n_i, \quad (53) \]
\[ w_i \rightarrow w_i, \quad (54) \]
\[ \gamma \rightarrow \theta(\theta \gamma - 1), \quad (55) \]
\[ R_i \rightarrow \theta^{-2/3} R_i, \quad R \rightarrow \theta^{-1/3} R. \quad (56) \]

Together with another symmetry
\[ \theta \rightarrow \theta + 1, \quad (57) \]
\[ n \rightarrow n + m, \quad m \rightarrow m, \quad (58) \]
\[ n_i \rightarrow n_i + m_{i-}, \quad m_{i-} \rightarrow m_{i-}, \quad (59) \]

with everything else unchanged, they generate the \( SL(2, \mathbb{Z})_N \) symmetry. The \( SL(2, \mathbb{Z})_N \) transformation on \( \theta \) is given by
\[ \theta \rightarrow \frac{A\theta + B}{C\theta + D} \quad (60) \]
with any integers \( A, B, C, D \) satisfying \( AD - BC = 1 \). Simultaneously,
\[ \begin{pmatrix} n \\ m \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}, \quad \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad (61) \]
so that \( \theta' \) \((37)\) is invariant. This implies that the algebra \((36)\) of sections on the bundle of adjoint representation is \( SL(2, \mathbb{Z})_N \)-invariant, although the gauge group is not invariant. This can be viewed as a hint that the \( SL(2, \mathbb{Z})_N \) symmetry can be extended to non-BPS states as well.

The interpretation of the quantum numbers is as follows. We have \( n \) as the number of D0-branes, \( m \) as the winding number of a membrane (or the D2-brane number) on the torus of \( X_1, X_2 \). The momenta of the K-K modes are \( n_i/R_i \) and \( m_{i-} \) is the winding number around \( X_{i-} \) and \( X_i \).

In M(atrix) theory the total momentum \( P_i = w_i + m\epsilon_{ij}n_j \) on the dual torus has the same interpretation as the winding number \( nm_{i-} \) \([13, 14]\). Thus physically we expect to have the constraint on the winding number \( w_i \) given by \( w_i = nm_{i-} - m\epsilon_{ij}n_j \) as in \([4]\). In string theory this is understood as the level matching condition \( L_0 - \bar{L}_0 = 0 \). For \( \theta = \gamma = m_{i-} = 0 \) this condition enables one to write the BPS energy as a complete square
\[ H = \frac{R}{2n} \left( \sqrt{(n_1/R_1)^2 + (n_2/R_2)^2 + mV} \right)^2, \quad (62) \]
which is necessary to interpret the M(atrix) theory on \( T^2 \) as type IIB string theory on \( S^1 \) \([20]\). However we do not find such a constraint in the M(atrix) model at finite \( N \).
The quantum numbers $m_i$ are independent of the others. This problem of missing the level matching condition already appears for the M(atrix) model compactified on $S^1$. In order to obtain type IIA string theory from the M(atrix) model on $S^1$ one has to consider the special large $N$ limit and the level matching condition is achieved only in this limit [14]. It is unclear how to obtain the level matching condition in M(atrix) theory for a generic background.

It is pointed out in [6] that to describe the D0-brane in a B field background in the M(atrix) model limit of [21] the SYM on a noncommutative torus is not sufficient, rather the Dirac-Born-Infeld action has to be used. Hence it would be interesting to see how the SYM results should be modified.

**Acknowledgement**

The author thanks Shyamoli Chaudhuri, Edna Cheung, Michael Douglas, Eli Hawkins, Danial Kabat, Morton Krogh, Sanjaye Ramgoolam, Marc Rieffel, Yi-Yen Wu and Yong-Shi Wu for discussions and comments. In particular he thanks Michael Douglas for valuable help. This work was supported in part by NSF grant No. PHY-9601277.

**References**

[1] A. Connes, M. R. Douglas, A. Schwarz, “Noncommutative Geometry and Matrix Theory: Compactification on Tori”, hep-th/9711162.

[2] M. R. Douglas, C. Hull: “D-branes and Noncommutative Torus”, hep-th/9711165.

[3] L. Susskind, ‘Another Conjecture About M(atrix) Theory”, hep-th/9704080.

[4] P.-M. Ho, Y.-Y. Wu, Y.-S. Wu, “Towards a Noncommutative Geometric Approach To Matrix Compactification”, hep-th/9712201.

[5] P.-M. Ho, Y.-S. Wu, “Noncommutative Gauge Theories in Matrix Theory”, hep-th/9801147.

[6] M. Li: “Comments on Supersymmetric Yang-Mills Theory on a Noncommutative Torus”, hep-th/9802052.
[7] N. Nekrasov, A. Schwarz: “Instantons on Noncommutative $R^4$ and (2, 0) Superconformal Six-Dimensional Theory”, hep-th/9802068; M. Berkooz: “Nonlocal Field Theories and the Noncommutative Torus”, hep-th/9802069; Y.-K. E. Cheung, M. Krogh: “Noncommutative Geometry From 0-Branes in a Background $B$ Field”, hep-th/9803031; F. Ardalan, H. Arfaei, M. M. Sheikh-Jabbari: “Mixed Branes and M(atrix) Theory on Noncommutative Torus”, hep-th/9803067.

[8] A. Connes, “Noncommutative Geometry”, Academic Press, 1994.

[9] A. Connes, C.R. Acad. Sci. Paris S’er. A-B 290 (1980) A599; M. Pimsner and D. Voiculescu, J. Operator Theory 4 93 (1980); A. Connes and M. A. Rieffel, “Yang-Mills for noncommutative two-tori”, in Operator Algebras and Mathematical Physics (Iowa City, Iowa, 1985), pp.237-266, Contemp. Math. Oper. Algebra. Math. Phys. 62, AMS 1987; M. A. Rieffel, “Projective Modules Over Higher-dimensional Non-commutative Tori”, Can. J. Math. 40 257 (1988); M. A. Rieffel, “Non-Commutative Tori-A Case Study of Non-Commutative Differentiable Manifolds”, Contemp. Math. 105, 191-211 (1990).

[10] F. Hacquebord, H. Verlinde, “Duality Symmetry of $\mathcal{N} = 4$ Yang-Mills Theory on $T^3$”, hep-th/9707179, Nucl. Phys. B508, 609-622 (1997).

[11] W. Taylor, IV, “D-Brane Field Theory On Compact Spaces”, hep-th/9611042, Phys. Lett. B394, 283 (1997).

[12] O.J. Ganor, S. Ramgoolam and W. Taylor IV, “Branes, Fluxes and Duality in M(atrix) Theory”, hep-th/9611202, Nucl. Phys. B492 (1997) 191.

[13] P.-M. Ho, Y.-S. Wu, “D-branes and Noncommutative Geometry”, hep-th/9611233, Phys. Lett. B398, 251 (1997).

[14] R. Dijkgraaf, E. Verlinde, H. Verlinde, “Matrix String Theory”, hep-th/9703030, Nucl. Phys. B500, 43 (1997).

[15] E. Witten, “Bound States of Strings and $p$-Branes”, hep-th/9510135, Nucl. Phys. B460, 335 (1996).
[16] P.-M. Ho, Y.-S. Wu, “II B/ M Duality and Longitudinal Membranes in M(atrix) Theory”, hep-th/9703016, Phys. Rev. D57, 2571-2579 (1998).

[17] M. R. Douglas, H. Ooguri: “Why Matrix Theory Is Hard”, hep-th/9710178.

[18] S. Hellerman, J. Polchinski: “Compactification in the Lightlike Limit”, hep-th/9711037.

[19] T. Banks, N. Seiberg, S. Shenker: “Branes From Matrices”, hept-th/9612157, Nucl. Phys. B490, 91-106 (1997).

[20] J. H. Schwarz: “An SL(2, Z) Multiplet of Type II B Superstrings”, hep-th/9508143, Phys. Lett. B360, 13-18 (1995), Erratum-ibid. B364, 252 (1995).

[21] N. Seiberg: “Why Is The Matrix Model Correct?”, hep-th/9710009, Phys. Rev. Lett. 79, 3577-3580 (1997).