All-electrically tunable networks of Majorana bound states

Song-Bo Zhang,1,* Alessio Calzona,1,* and Björn Trauzettel1,2

1 Institute for Theoretical Physics and Astrophysics, University of Würzburg, D-97074 Würzburg, Germany
2 Würzburg-Dresden Cluster of Excellence ct.qmat, Germany

Second-order topological superconductors (SOTSs) host localized Majorana fermions and provide a new platform for topological quantum computation. We propose a remarkable and feasible way to realize networks based on SOTSs which allow to nucleate and braid Majorana bound states (MBSs) in an all-electrical manner without fine-tuning. The proposed setups are scalable in a straightforward way and can accommodate any even number of MBSs. Moreover, the MBSs in the networks allow defining qubits whose states can be initialized and read out by measuring Josephson currents flowing between SOTS islands. Our proposal can be implemented in monolayers of FeTe$_{1-x}$Se$_x$ and inverted Hg(Cd)Te quantum wells in proximity to conventional superconductors.

Introduction.— Second-order topological superconductors (SOTSs) are characterized by topologically protected midgap bound states with zero excitation energy and codimension two [1–12]. These midgap states behave like Majorana fermions which constitute their own anyonic particles [13]. They obey non-Abelian exchange statistics and could find promising applications in topological quantum computation [14–20]. Recently, SOTSs have been predicted in certain candidate systems [6–10, 21–37]. Hence, they provide a feasible platform for implementing topological quantum gates. A few theoretical proposals have been made to explore the exchange of Majorana bound states (MBSs) in SOTSs [38–41]. However, they are restricted to only a single pair of MBSs or require to locally tune magnetic fields. To define a multidimensional computational ground-state manifold suitable for implementing non-Abelian quantum gates, four or more MBSs are required [16, 42]. Moreover, simpler manipulation schemes based on electrical controls are advantageous in experimental implementation and runtime for quantum gates.

In this Letter, we propose a novel way to realize electrically tunable networks of MBSs based on SOTSs. We take full advantage of the special role played by the sample geometry in SOTSs and conceive setups whose building blocks consist of isosceles right triangle islands (IR-TIs) of SOTSs. By modulating local gate voltages on the islands, it is possible to nucleate an arbitrary even number of MBSs and control their positions on the networks, allowing for non-Abelian braiding. The magnetic order in our setups is uniform. It can be in-plane ferromagnetism (FM), antiferromagnetism (AFM), Zeeman fields, or a mixture of them. Moreover, the qubit states defined by the MBSs in the network can be initialized and readout, for instance, by measuring Josephson currents flowing between the SOTS islands. Importantly, our proposal can be implemented in a variety of candidate systems, including inverted Hg(Cd)Te quantum wells with proximity-induced superconductivity and FeTe$_{1-x}$Se$_x$ monolayers with intrinsic superconductivity.

MBSs on open boundaries of SOTSs.— To begin with, we consider two-dimensional SOTSs which are realized by introducing $s$-wave pairing potential in combination with in-plane FM or AFM to quantum spin Hall insulators. The SOTSs can be described by

\[ H(k) = n(k) \tau_z \sigma_z + A \sin k_x s_x \sigma_x + A \sin k_y \tau_z \sigma_y - \mu \tau_x + \Delta_0 \tau_y s_y + H_M \]  

(1)

in the basis \( \{ c_a\uparrow, c_{b\downarrow}, c_{a\downarrow}, c_{b\uparrow} \} \), where \( c_{a\uparrow} \) is the fermion operator with orbitial (or sublattice) index \( \sigma \in \{a,b\} \) and spin index \( s \in \{\uparrow,\downarrow\} \); \( n(k) = 2m \cos k_x + 2m \cos k_y + m_0 - 4m \) with \( m_0m > 0 \); \( \mu \) denotes the chemical potential and is controllable by external gates. The Pauli matrices \( s, \sigma \) and \( \tau \) act on spin, orbital and Nambu spaces, respectively. The term \( H_M \) describes the magnetic order. It can be induced by close proximity to ferromagnets or antiferromagnets or by applying in-plane magnetic fields. For concreteness, we focus on the case of FM with strength \( M_0 \) in \( x \) direction, \( H_M = M_0 \tau_x s_x \), in the following. We note, however, that our main results discussed below also apply to other cases, for instance, AFM with \( H_M \) given by \( M_0 \tau_z s_y \).

The SOTSs feature zero-energy MBSs when open boundary conditions are enforced. To better understand this, it is instructive to derive a low-energy effective Hamiltonian on boundaries. We start with the low-energy limit of \( H(k) \) and consider the SOTSs in a disk geometry of radius \( R \). In the absence of \( M_0 \) and \( \Delta_0 \), we find helical states (\( \Psi_{e,\uparrow}, \Psi_{e,\downarrow}, \Psi_{h,\uparrow}, \Psi_{h,\downarrow} \)) on the disk boundary. Using these helical states as a basis and projecting the full Hamiltonian \( H(k) \) on these states, the boundary Hamiltonian is constructed as

\[ \tilde{H}(\varphi) = -Ap_\varphi s_z + \Delta_0 \tau_y s_y - \tilde{M} e^{-i\tau_z \varphi} s_y - \mu \tau_z, \]  

(2)

where \( \varphi \) is the azimuthal coordinate and \( p_\varphi \equiv -i\partial_\varphi / R \) the corresponding momentum defined along the boundary. The boundary states possess effective pairing potential \( \Delta_0 \) and magnetization \( \tilde{M} = M_0 \sin \varphi \), as induced from the bulk. When \( M_0 > \tilde{\Delta} \equiv (\Delta_0^2 + \mu^2)^{1/2} \), we find that the energy bands of Eq. (2) change their order at

\[ \varphi_{1/4} = \pm \arcsin(\tilde{\Delta} / M_0), \quad \varphi_{2/3} = \varphi_{4/1} + \pi \]  

(3)
along the boundary, due to the oscillatory behavior of $M$. The changes of band order indicate the appearance of four MBSs $\gamma_i$ with $i \in \{1, 2, 3, 4\}$, exponentially localized at $\varphi_i$. When they are well separated from each other, the four MBSs are at zero energy and it is possible to analytically derive their wavefunctions $\Psi_i$. Details about these derivations are provided in the Supplemental Material (SM) [43]. Importantly, the chemical potential $\mu$ controls the angles $\varphi_i$, according to Eq. (3). This enables us to manipulate the positions of the MBSs, and eventually their fusion and braiding in an all-electrical manner, as discussed below.

**Fusion properties of MBSs.**—When two MBSs are brought close together, their wavefunctions start to overlap and their energies become finite. This process, known as fusion, is mediated by the electron hopping in the SOTs. According to Eq. (1), the hopping corresponds to the operator $T = iA(s_1\sigma_x + \tau_3\sigma_y)/2 + 2n\tau_z\sigma_z$. Thus, the fusion strength between two MBSs, say $\gamma_i$ and $\gamma_j$, can be estimated as $F_{\gamma_i,\gamma_j} = |\langle \Psi_i | T | \Psi_j \rangle|$. On a single island, we find that the fusion strengths $F_{\gamma_1,\gamma_2}$ and $F_{\gamma_3,\gamma_4}$ are proportional to $\cos \vartheta$, while $F_{\gamma_1,\gamma_4}$ and $F_{\gamma_2,\gamma_3}$ are linear in $\sin \vartheta$, where $\vartheta = \arctan(\mu/\Delta_0)$. By contrast, the fusion between $\gamma_1$ and $\gamma_3$ (or $\gamma_2$ and $\gamma_4$) is strictly forbidden, due to inversion symmetry of the SOTs [43].

The fusion properties become richer when we consider two sets of MBSs $\{\gamma_i\}$ and $\{\gamma'_i\}$ (with $i \in \{1, 2, 3, 4\}$) belonging to two different islands, featuring a finite pairing phase difference. In this case, when two MBSs from different islands are brought close together, they can always fuse in general. The mutual fusion strengths $F_{\gamma_i,\gamma'_j}$ are summarized in Table I and depend sinusoidally on the pairing phase difference $2\phi$ and the chemical potentials $\mu$ and $\mu'$ of the two islands.

| $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ |
|-----------|-----------|-----------|-----------|
| $\sin \vartheta \sin \delta \phi$ | $\cos \vartheta \cos \delta \phi$ | $\cos \vartheta \sin \delta \phi$ | $\sin \vartheta \cos \delta \phi$ |
| $\cos \vartheta \cos \delta \phi$ | $\sin \vartheta \sin \delta \phi$ | $\sin \vartheta \cos \delta \phi$ | $\cos \vartheta \sin \delta \phi$ |
| $\cos \vartheta \sin \delta \phi$ | $\sin \vartheta \cos \delta \phi$ | $\sin \vartheta \sin \delta \phi$ | $\cos \vartheta \cos \delta \phi$ |
| $\sin \vartheta \cos \delta \phi$ | $\cos \vartheta \sin \delta \phi$ | $\cos \vartheta \cos \delta \phi$ | $\sin \vartheta \sin \delta \phi$ |

Table I. Fusion strength $F_{\gamma_i,\gamma'_j}$ of MBSs $\{\gamma_i\}$ and $\{\gamma'_j\}$ belonging to two SOTS islands. The table displays the dependence of $F_{\gamma_i,\gamma'_j}$ on $\mu$ and $\mu'$ and on $\delta \phi$. We define $\vartheta = (\vartheta + \vartheta')/2$, $\vartheta = \arctan(\mu/\Delta_0)$, and $\vartheta' = \arctan(\mu'/\Delta_0)$. Results for the fusion of MBSs belonging to the same island can be obtained by taking $\gamma'_i = \gamma_i$, $\delta \phi = 0$ and $\mu' = \mu$.

**Basic manipulation of MBSs in IRTIs.**—In order to obtain a scalable platform hosting any even number of MBSs which can be manipulated by purely electrical means, it is essential to go beyond the simple disk geometry presented so far. Particularly, we focus on IRTIs, the short sides of which are orientated in $x$ and $y$ directions, as depicted in Fig. 1. To develop some intuition about the appearance of MBSs in the IRTIs, one can relate the latter to the disk geometry in the following way: the dotted lines normal to the triangle sides define three arcs of the disk boundary (dashed curves); all the points belonging to the same arc reduce to the corresponding vertex of the triangle (colored arrows); conversely, each side of the triangle reduces to a single point on the disk. Out of the four MBSs hosted by the disk (gray dots), two of them must locate on the same arc meaning that, in the triangle, they fuse on the same vertex. By contrast, the two remaining MBSs locate on different arcs and thus stay robustly as zero-energy corner states (blue dots) in the IRTI. Which vertices host the zero-energy MBSs crucially depends on the angles $\varphi_i$ (angle $\varphi_1$ is depicted in red) and, therefore, on the value of chemical potential $\mu$.

In particular, for $|\mu| < \mu_c \equiv |M_0^2 \sin^2(\pi/4) - \Delta_0^2|^{1/2}$, the four MBSs on the disk are sketched in Fig. 1(a). For $|\mu| > \mu_c$, the MBSs are located as shown in Fig. 1(b). By tuning $\mu$ across $\mu_c$ in the IRTI, say from $\mu_4(< \mu_c)$ to $\mu_4(> \mu_c)$, we can thus adiabatically move one MBS along the diagonal while the other one stays fixed at the right-angle vertex. We observe that a finite $\mu_c$ requires $M_0 > \sqrt{2}\Delta_0$. When $\mu$ is close to $\mu_c$, the localization length of the movable MBS along the diagonal is approximately proportional to $A\Delta_0/(|\mu_c|\mu - |\mu_c|)$. Therefore, larger islands pose weaker constraints on the tunability range of the chemical potential $\mu_4 - \mu_4$. The possibility to move MBSs along the diagonal of the triangle is confirmed numerically [43]. These results apply to any IRTIs with the short sides in $x$ and $y$ directions.

**Building networks of MBSs.**—By properly connecting several IRTIs, networks of diagonals can be defined, for example, as sketched in Fig. 2 (more network examples can be found in the SM [43]). When two or more vertices get in contact, there is a finite overlap between the wavefunctions of different MBSs, which fuse according to the inter- and intra-island fusion strengths summarized in Table I. The latter clearly depends on the chemical potential and the superconducting phases $\Phi_j$ of the ad-
to $\mu \cdot \mu$ (whose diagonals are highlighted in red) have been tuned MBSs. In Fig. 2(b), the chemical potentials of two IRTIs are illustrated in Fig. 2. In Fig. 2(a), all chemical potentials of our networks, we now show how to braid a couple of MBSs, thus implementing a phase gate on a Majorana qubit. The latter consists of four MBSs, which can be hosted by the six-island structure depicted in Fig. 3. We label the IRTIs by $T_j \in \{1, \cdots, 6\}$ and the corresponding chemical potentials and superconducting phases by $\mu_j$ and $\Phi_j$, respectively. For the numerical simulation illustrated in Fig. 3, we considered $\Phi_5 = \Phi_6 = \pi/2$ and $\Phi_j = 0$ otherwise.

The initial configuration, Fig. 3(a), features $\mu_j = \mu_u$ for $j \in \{1, 5, 6\}$ and $\mu_j = \mu_d$ otherwise. In this state, we can observe four MBSs which are indicated by the black localized densities and labeled by $\gamma_\alpha, \gamma_b, \gamma_c$ and $\gamma_d$. In order to braid $\gamma_a$ and $\gamma_b$, the chemical potentials $\mu_4, \mu_5$ and $\mu_6$ must be adiabatically tuned in time, according to Fig. 3(i). This results in the motion of $\gamma_a$ and $\gamma_b$ along the diagonals of $T_4, T_5$ and $T_6$, as shown in Fig. 3 (a-g). At the end of the protocol, while the system has the same parameters as in the initial state, the positions of $\gamma_a$ and $\gamma_b$ are exchanged. Importantly, during the whole process, the four MBSs stay robustly at zero energy [red bands in Fig. 3(h)]. They are always separated from excited states (blue bands) by a finite energy gap. Similar procedures can be applied to exchange other MBS pairs [43].

Because of the non-Abelian nature of MBSs, the braiding of $\gamma_a$ and $\gamma_b$ results in a nontrivial unitary operation $U_{ab} = \exp(i\pi\gamma_a\gamma_b/4)$ on the Majorana qubit [14]. It corresponds to a quantum gate that implements a $\pi/2$ rotation of $\gamma_a$ and $\gamma_b$. In (a-g), seven subsequent snapshots show the positions of the four MBSs (black localized densities). During the protocol, $\mu_4, \mu_5$ and $\mu_6$ are varied in time, according to (i), while $\mu_1 = \mu_u$ and $\mu_2 = \mu_3 = \mu_d$ are fixed. (h) The energy spectrum of the system during the process. It is symmetric with respect to zero energy. The parameters are $\mu_u = 0.15m_0, \mu_d = 0.05m_0, \Phi_5 = \Phi_6 = \pi/2$ and $\Phi_j = 0$ otherwise, $\Delta_0 = 0.25m_0, M_0 = 0.4m_0, A = m = 0.5m_0 = 1$, the short-side length of the IRTIs is $L = 35$ and the lattice constant $a = 1$. 

![Fig. 2. Networks of connected IRTIs. The cyan and yellow colors distinguish between two pairing phases on the islands. In (a), all the IRTIs have the same chemical potential $\mu_a$ and the network hosts four MBSs indicated by the blue dots. In (b), the chemical potentials of triangles whose diagonals are marked in dashed red have been tuned to $\mu_u$, resulting in the moving of the top right MBS and in the nucleation of an additional couple of MBSs.](image)

![Fig. 3. Numerical simulation of braiding $\gamma_a$ and $\gamma_b$. In (a-g), seven subsequent snapshots show the positions of the four MBSs (black localized densities). During the protocol, $\mu_4, \mu_5$ and $\mu_6$ are varied in time, according to (i), while $\mu_1 = \mu_u$ and $\mu_2 = \mu_3 = \mu_d$ are fixed. (h) The energy spectrum of the system during the process. It is symmetric with respect to zero energy. The parameters are $\mu_u = 0.15m_0, \mu_d = 0.05m_0, \Phi_5 = \Phi_6 = \pi/2$ and $\Phi_j = 0$ otherwise, $\Delta_0 = 0.25m_0, M_0 = 0.4m_0, A = m = 0.5m_0 = 1$, the short-side length of the IRTIs is $L = 35$ and the lattice constant $a = 1$.](image)
tion on the Bloch sphere, around an axis whose direction depends on the choice of the computational basis. This can be experimentally confirmed by measuring the parity of two different couples of MBSs, $P_{bc} = i \gamma_b \gamma_c$ and $P_{ac} = i \gamma_a \gamma_c$. The former one can be used to initialize the qubit, say in the eigenstate of $P_{bc} | 0 \rangle = | 0 \rangle$. Then, the braiding rotates the initial state to $U_{ab} | 0 \rangle$ which is an eigenstate of $P_{ac} U_{ab} | 0 \rangle = U_{ab} | 0 \rangle$. The validity of this result can be straightforwardly verified by measuring $P_{ac}$.

Remarkably, our all-in-one setup allows for initialization, braiding, and readout. Indeed, because of the possibility to move and fuse arbitrary couples of MBSs on the network, we can measure a generic parity operator $P_{ac}$. For concreteness, we now focus on the measurement of $P_{bc}$ in the six-island architecture described before. In this case, one must fuse $\gamma_b$ and $\gamma_c$ by moving them in the region which defines the Josephson junction between islands with different pairing phases [Fig. 4(a)]. The effective Hamiltonian which describes the coupling between the two MBSs reads $H_{bc} = \Gamma \cos(\delta \Phi) P_{bc}$, where $2\delta \Phi$ is the pairing phase difference and $\Gamma$ is the coupling strength factor that depends on the chemical potentials and wavefunction overlap. The two eigenenergies are therefore $E_g = \pm \Gamma \cos(\delta \Phi)$ [orange curves in Fig. 4(b)]. At zero temperature, the Josephson current across the junction can be evaluated as $I = I_0 e^{\Gamma \sin(\delta \Phi)/2h} = I_0 (1 + I_{mbbs})$, where $I_{mbbs}$ and $I_0$ are the contributions from the MBSs and ordinary fermions, respectively [44]. As long as $2\delta \Phi \neq 0$, by probing $I$ flowing between the islands one can therefore measure $P_{bc}$ [Fig. 4(b)]. In principle, other measurement schemes based on quantum dots are also possible [43, 45].

Feasibility and summary.—We now discuss the experimental feasibility of our proposal. Remarkably, FeTe$_{1-x}$Se$_x$ monolayers have been shown to possess a band inversion at the Gamma point [46–48] and intrinsic high-temperature superconductivity [49, 50]. The magnetic order may be induced by putting (anti)ferromagnets, e.g., FeSe or FeTe layers [51–53], on top of FeTe$_{1-x}$Se$_x$ monolayers or by applying in-plane magnetic fields. We note that the sustenance of superconductivity under strong in-plane magnetic fields in this material has been reported experimentally [54]. Interestingly, FeSe monolayers coupling to substrates may have all the desired ingredients for realizing SOTs (namely, band inversions at the $M$ points, superconductivity [55–57] and AFM order [58]) intrinsically within one material. Quantum spin Hall insulators, such as inverted Hg(Cd)Te and InAs/GaSb quantum wells [59–65], in proximity to conventional superconductors could offer another candidate system.

In general, the control of local chemical potentials on the islands might be a challenging task. However, it is important to stress that, because of the topological nature of our setup, it is by no means necessary to fine tune the chemical potentials to specific values of $\mu_a$ and $\mu_b$. The only requirements are (i) the possibility to tune $\mu$ across its critical value, i.e., $\mu_a < \mu_c < \mu_b$ and (ii) that, at $\mu = \mu_a$ and $\mu_b$, the MBSs are well localized at the vertices of IRTIs. Importantly, we numerically prove that inhomogeneities of chemical potential within each IRTI are not detrimental to our proposal [43]. Finally, we remark that field effects on (superconducting) thin films have proven to be a valid alternative to conventional chemical doping in order to tune the carrier density [66–68], suggesting the feasibility of controlling local chemical potentials with external gates.

An important issue, when it comes to Majorana-based quantum computation, is represented by quasiparticle poisoning (QP) [69–71], causing detrimental flips in the total fermion parity of individual qubits. In this respect, the large superconducting gap of FeTe$_{1-x}$Se$_x$ monolayers (up to 16.5 meV [49]) represents a prime advantage: on the one hand, it is likely to decrease the QP rate; on the other hand, it allows for faster adiabatic qubit operations. Moreover, it might be possible to implement quasiparticle filters which have proven, at least for quantum wires, to increase the characteristic QP time up to $(1/200)$s [72].

In summary, we have proposed a feasible way to realize networks of SOTs which can accommodate any (even) number of topologically protected MBSs. The MBSs can be generated, moved and fused by full electrical tuning of local gates. As a simple application, our proposal allows to define a qubit, braid the MBSs of which it consists, and measure the nontrivial outcome of this operation.

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![Fig. 4.](image-url)
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