Abnormal diffusion of a single vortex in the two dimensional XY model

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We study thermal diffusion dynamics of a single vortex in two dimensional XY model. By numerical simulations we find an abnormal diffusion such that the mobility decreases with time $t$ as $1/\ln t$. In addition we construct a one dimensional diffusion-like equation to model the dynamics and confirm that it conserves quantitative property of the abnormal diffusion. By analyzing the reduced model, we find that the radius of the collectively moving region with the vortex core grows as $R(t) \propto t^{1/2}$. This suggests that the mobility of the vortex is described by dynamical correlation length as $1/\ln R(t)$.

KEYWORDS: vortex, XY model, abnormal diffusion, Josephson junction array

1. Introduction

Vortices, which are topological defects of U(1) symmetry fields, plays an important role in low dimensional systems such as thin film super fluids, liquid crystals, layer superconductors and Josephson junction arrays (JJAs). A well known example is that two dimensional XY (2dXY) model exhibits a vortex driven KT phase transition while the elastic theory, which ignores the vortices, predicts a unique phase with quasi-long-range order. In this case, i.e., a vortex behaves as two dimensional Coulomb gas, which make dipole pairs in the ordered state.

Vortex is also an important keyword in dynamical property of the system. In the out-of-equilibrium dynamics, relaxation process from certain initial state to the equilibrium at fixed temperature environment, it is pointed out that the critical relaxation of the 2dXY model seems not to be universal; the dynamical exponent $z$, which connects dynamical correlation length $L(t)$ and time $t$ as $L(t) \propto t^{-1/z}$, depends on the initial state. When the initial state is an ordered ground state, $z$ equals 2, which agrees with the result of the elastic description, i.e., the Gaussian model. On the other hand $z \approx 2.35^2$ for process quenching from highly disordered initial state at high temperature. The latter value of $z$ is, however, considered to be a consequence of short time correction caused by topological defects. In the disordered initial state the system is filled with vortices and vortex-antivortex pair annihilation is a main

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process of initial relaxation. This yields logarithmic correction as \( L(t) \sim (t/\ln t)^{-1/2} \), \(^2\text{-}^4\) which resembles \( t^{-1/2} \) when the observing time is not large enough. The long time asymptotic behavior is expressed by \( z = 2 \) as well as in the case of the ordered initial state.

In the phenomena mentioned above, interaction among vortices is important. On the other hand, it is also reported that even a single vortex causes abnormal behavior in transport property of a JJA system. Under very low magnetic field, which yields very diluted vortices, the frequency dependence of the vortex mobility behaves as \( 1/\omega \).\(^5\) This result conflicts with the free Coulomb gas picture.\(^6\text{-}^7\) Korshunov explained this experimental observation (deriving corresponding facts that the mobility of the vortex decreases as \( 1/\ln t \)) by analyzing the 2dXY model assuming an effective action which involves a memory kernel in the interaction term.\(^8\) In this article, we study the diffusion dynamics of a single vortex based on two models and discuss about the origin of the memory effect. At first we show numerical study of the bare 2dXY model. Then we derive a one dimensional model as an approximation of the 2dXY model, which enables us to understand the phenomena more clearly. Analysis of both models reveals that the dynamics of a vortex shows abnormal diffusion, where mean square displacement grows as \( t/\ln t \).

2. Numerical analysis on the two dimensional XY model

We study the XY spin model on a square lattice, whose energy is written as

\[
E = J \sum_{(i,j)} [1 - \cos(\theta_i - \theta_j)].
\]

(1)

Here \( \theta_i \) indicates the angle of the XY spin at the \( i \)-the site and the summation is taken over all nearest neighbor pairs. The dynamics of this model is investigated by the overdamped Langevin equation,

\[
\eta \frac{d}{dt} \theta_i(t) = -J \sum_{j \in \text{n.n.}} \sin(\theta_i(t) - \theta_j(t)) + \zeta_i(t),
\]

(2)

where \( \zeta_i(t) \) is a Gaussian white noise satisfying

\[
\langle \zeta_i \rangle = 0 \quad \text{and} \quad \langle \zeta_i(t)\zeta_j(t') \rangle = 2\eta k_B T \delta_{ij} \delta(t - t').
\]

(3)

Here \( \langle \cdots \rangle \) means the average over independent noise realizations. In the following, we set the coupling constant \( J \), friction coefficient \( \eta \) and the Boltzmann constant \( k_B \) to unity.

Next let us introduce the quantities to observe. The mean square displacement (MSD) is calculated as

\[
D(t, t_0) \equiv \langle |X(t) - X(t_0)|^2 \rangle
\]

(4)

where \( X(t) = (X(t), Y(t)) \) is a position of a vortex at time \( t \) and \( t_0 \) is a waiting time. The velocity auto correlation function is derived from \( D(t, t_0) \) from the relation

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial t_0} D(t, t_0) = -2 \langle \dot{X}(t) \dot{X}(t_0) \rangle \equiv -2C(t, t_0).
\]

(5)
When the system has time translational symmetry in stationary state, the MSD is a function of only \( t - t_0 \) and the velocity correlation function is rewritten as

\[
C(t - t_0) = \frac{1}{2} \frac{\partial^2}{\partial t^2} D(t - t_0).
\]

### 2.1 Simulation settings

We numerically integrated eq. (2) by the second-order Runge-Kutta method. The samples are square-shaped including \( L^2 \) spins and open boundary condition is imposed. In the initial state, the phase \( \theta_i(t = 0) \) is given as an angle between the \( x \)-axis and the vector \( \mathbf{r}_i - \mathbf{X}(0) \), where \( \mathbf{r}_i \) is a position vector of the \( i \)-th site and \( \mathbf{X}(0) \) is the initial position of the vortex core set to the center of the sample \( ((L - 1)/2, (L - 1)/2) \).

Next let us explain how to detect the position of the vortex core. We calculate vorticity \( n_i \) from the snapshot at time \( t \) by summing the phase difference along the closed path of each plaquette as

\[
2\pi n_i = [\theta_{i+x} - \theta_i] + [\theta_{i+x+y} - \theta_{i+x}] + [\theta_{i+y} - \theta_{i+x+y}] + [\theta_i - \theta_{i+y}],
\]

where \( [x] = x - 2\pi \text{int}(x/2\pi) \) and then \( [x] \) returns a value between \(-\pi\) and \( \pi \). Thus \( n_i \) can take 0 or \( \pm 1 \). A vortex core exists in the plaquette labeled by index \( i \), where \( n_i = 1 \). The velocity of the vortex is calculated as \( \mathbf{V}(t) = (\mathbf{X}(t + \tau) - \mathbf{X}(t))/\tau \), where \( \tau = m\Delta t \) is time interval between sequential observations and \( \Delta t \) is an incremental time step of the simulation. We set to \( \Delta t = 0.056 \) and \( m = 64 \). When \( \tau \) is set to sufficiently small value, \( \mathbf{V}(t) \) equals zero in the most time steps. This is because the motion of the vortex is discrete and intermittent. In such situation we have to note that this velocity averaged for finite time span barely depends on the choice of \( \tau \).

There are two difficulties in this simulation. One is that pairs of vortex and anti-vortex can be created by thermal fluctuation, which makes it very difficult to find the trajectory of the vortex initially prepared. Such a thermal excitation, however, is observed with extremely small probability at low temperature. We set \( T = 0.250 \approx 0.28T_{KT} \) where \( T_{KT} \) is the critical temperature of the Kosterlitz-Thouless transition. The other problem is that the vortex feels attractive force from the sample edge so that it gets out of the sample in a certain long time. In order to observe the long time steady behavior we make an operation to keep the vortex around the center of the sample as follows. Let us consider an example case that the vortex core moves along the \( x \)-right-direction by \( a (\geq 2) \) steps. At first we delete \( a \) columns of spins from left edge and then move the all remaining spins to the left by \( a \) lattice units. Finally we add spins to every empty columns on the right edge by copying the \((L - a)\)-th column in the same way. We do similar operation when the vortex core moves to the left, up and down. Such operation affects the motion of the vortex core to some extent but this effect will decrease with the system size \( L \) in a systematic way.
2.2 Numerical Result

If the dynamics of the vortex is the so-called “normal diffusion”, the MSD would be proportional to $Tt$ and its coefficient means the mobility. In fact $D(t)$ grows slower than $t$-linear behavior as shown in Fig. 1. In order to show the deviation from normal diffusion apparently we plot $Tt/D(t)$, which can be regarded as a time-dependent effective friction coefficient (or inverse of the effective mobility) of the vortex. The results for different waiting times $t_0$ are plotted together. Since the deviation among them is very small as far as $t - t_0 \ll t_0$, the system is considered to be in a stationary state. The coefficient is found to be proportional to $\ln t$ in long time regime, i.e.,

$$D(t) \propto Tt/\ln t.$$  \hspace{1cm} (9)

Finite size effect is observed in the long time limit.

Figure 1 shows that the coefficients saturate to a finite values which are roughly proportional to the logarithm of the system size $L$.

Next we calculate the velocity auto-correlation function by using the Fourier series $V_k$ of the $V(t)$

$$C(n\tau) = \sum_{k=1}^{n^* - 1} |V_k|^2 e^{-2\pi ink/N} \hspace{1cm} (10)$$

$$V_k = \frac{1}{\sqrt{n^*}} \sum_{n=0}^{n^* - 1} V(t_0 + n\tau)e^{2\pi ink/N}, \hspace{1cm} (11)$$

where $n^*$ is the total number of observations of the velocity with constant interval $\tau$. Although the above $C(t)$ barely depends on $\tau$, a normalized function $C(t)/C(0)$ does not for $t \gg \tau$. Note that finite time observation results non-vanishing constant in $C(t)$ for large $t$ (we set the observation time equivalent to the waiting time $t_0$). Ignoring this finite time effect the observation of Fig. 2 supports that

$$C(t) \propto \frac{d^2}{dT^2} Tt = -\frac{1}{t\ln t^2} \left(1 - \frac{2}{\ln t}\right). \hspace{1cm} (12)$$

The correlation function is negative for $t > 0$. This is natural because the local motion of the vortex core driven by random force usually raise the interaction energy with surrounding region. Thus restoring force works on the vortex core. (Of course the system has an energy invariance against the global translation of spins if boundary effect can be ignored.) The response time should correlate with the range of dragged region. For a single vortex, there is no characteristic length scale except the lattice unit and therefore the system has infinitely long-time memory.

3. Reduced model

In the previous section, we saw that a single vortex does not behave as a Brownian particle with normal diffusion but the mobility decreases as $1/\ln t$. For the aim to study the origin
Fig. 1. Mean square displacement is plotted with $t - t_0$. The result for three waiting times, five system sizes ($L=16, 32, 64, 128, 256$) are shown together. Average is taken over 8192 samples.

Fig. 2. Vortex velocity auto-correlation function multiplied by $t - t_0$. The divergent behavior for large $t$ is due to the finite observation time, which is set to be same length with $t_0$.

of the abnormal diffusion the 2dXY model is hard to analyze and numerical simulation is rather heavy (note that the observation time of the present simulations is not long enough to eliminate the possibility that $D(t) \propto t^{0.8}$, which is difficult to distinguish from $t/\ln t$ for small $t$).

For this reason we propose a reduced model of this system. The fundamental idea is as follows. Considering circles with various radii centered on the vortex core, all XY spins in the energy minimal state with single vortex are directed the radial direction (see the left figure in Fig. 3). We assume that the excited state can be described only the motion of these circles and spins on each circle are always along the radial direction (see the right figure in Fig. 3 and note that spins do not change there position). By taking the center positions as a degrees
Fig. 3. Schematic diagram of the restricted deformation in the reduced model. We consider virtual circles with radius \( r \) and center \( \mathbf{X}(r) \). In fact there are infinite number of circles, whose radius varies continuously. The spins direct to the radial direction of the circle on which they are put on.

The deformation is described only with the positions of the circles.

of freedom, the resultant equation of motion in continuum limit is

\[
\frac{d}{dt} \mathbf{X}(r, t) = 2 \left( \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \right) \mathbf{X}(r, t) + \mathbf{Z}(r, t).
\]

\[
\langle \mathbf{Z}(r, t) \mathbf{Z}(r', t') \rangle = 2 \frac{r}{\pi} T \delta(r - r') \delta(t - t').
\]  

(13)

The detail of the derivation is shown in the appendix. Here \( \mathbf{X}(r, t) \) is the \( x \)-component of the center of the circle with radius \( r \). The \( y \)-component obeys the same equation and decoupled with \( \mathbf{X}(r, t) \). The position of the vortex core is identified with \( \lim_{r \to 0} \mathbf{X}(r, t) \).

4. Numerical integration of reduced model

To confirm the validity of the above one dimensional model we numerically integrate the equation of motion. We write the elastic energy

\[
E_{el} = \frac{1}{2} \sum_j \frac{|X_{j+1} - X_j|^2}{r_j} = \frac{1}{2} \sum_j \frac{\Delta X_j^2}{r_j}
\]

(14)

as a discrete version of eq. (13) where \( X_i(t) \) \( (i = 0, 1, 2, \cdots, L - 1) \) is a degree of freedom on the lattice points and \( \Delta X_i = X_{i+1} - X_i \). We use a system with reflective symmetry, i.e., \( r_j = j + 1/2 \) for \( j \leq L - 1 \) and \( r_j = L - j - 1/2 \) for \( j \geq L \). Therefore both of \( X_0 \) and \( X_{L-1} \) represents the position of the vortex core. The Langevin equation is written as

\[
\frac{d}{dt} X_i = -i \frac{\partial E_{el}}{\partial X_i} + Z_i(t)
\]

\[
= \frac{1}{1 - 1/4i^2} \left[ (\Delta X_i - \Delta X_{i-1}) - \frac{\Delta X_i + \Delta X_{i-1}}{2i} \right] + Z_i(t).
\]

(15)

where

\[
\langle Z_i(t) Z_j(t') \rangle = 2Ti \delta_{ij} \delta(t - t')
\]

(16)

In this equation temperature can be absorbed by scaling \( X \) with \( \sqrt{T} \) and therefore we set \( T = 1 \). We set \( \Delta X_{-1} = \Delta X_{L-1} = 0 \) as an open boundary condition. Actually we approximate the denominator in eq. (15), \( 1 - 1/4i^2 \), with unity.
Fig. 4. Mean square displacement of the vortex center $X_0(t)$ for the reduced one dimensional model.

Fig. 5. Vortex velocity auto-correlation function multiplied by $t - t_0$ for the reduced one dimensional model.

4.1 Abnormal diffusion

In order to compare the present reduced model with the original two dimensional XY model, we calculate the MSD and velocity auto correlation function of $X_0(t)$ and $X_{L-1}(t)$. Figure 4 and 5 is a result of numerical calculation. The behaviors agree with those of the two dimensional XY model; logarithmic correction to the normal diffusion is observed. It can be said that the present model holds the essential property of the abnormal diffusion of the original model. Furthermore we can perform much longer time simulation on this model than on the 2dXY model and observe logarithmic property more clearly.

4.2 Dynamical correlation length

On the reduced model we can easily investigate the behavior of off-core region. The MSD of $i$-th site

$$D_i(t - t_0) = \langle |X_i(t) - X_i(t_0)|^2 \rangle$$ (17)
is shown in Fig. 6. For small $t$, $D_i(t)$ is proportional to $t^{1/2}$. This is a standard behavior of an stochastic diffusion equation,

$$\frac{\partial}{\partial t} X(x, t) = \frac{\partial^2}{\partial x^2} X(x, t) + Z(x, t),$$  \hspace{1cm} (18)

which lacks the gradient term in eq. (13). Since local temperature increases with $i$, the initial coefficient of $t^{1/2}$ term does as well. The core region, however, shows different behavior. Seeing in logarithmic scale the growth rate is larger than that for off-core region. This ease to move is due to the weak confinement in the vicinity of the free edge on the one side. After $D_0(t) = \langle X_0(t)^2 \rangle$ catches up with $D_i(t)$, $D_i(t)$ coincides with $D_0(t)$. As time goes by, more and more regions moves with the core. Therewith the growth $D_0(t)$ becomes slower with factor $1/\ln t$ (seems to become faster in double-logarithmic scale). This suggests that the dynamics of vortex is a collective one and the mobility becomes smaller as its effective radius of the collective motion becomes large.

The range of collective motion can be estimated by the correlation function;

$$C(i, t) = \frac{\langle X_0(t)X_i(t)\rangle}{\sqrt{D_0(t)D_i(t)}}.$$  \hspace{1cm} (19)

A universal scaling function is found so that

$$C(i, t) = F(i/R(t)) \approx e^{-i/R(t)}$$  \hspace{1cm} (20)

where

$$R(t) \propto t^{1/2}.$$  \hspace{1cm} (21)

In Fig. 7 $C(i, t)$ is plotted as a function of $i$ scaled by $t^{1/2}$.

5. Discussions

We have shown that a single vortex exhibits abnormal diffusion even though there is no inter-vortex interaction. What is important is that vortex is not a mere point particle but
Fig. 7. The strain correlation function is plotted as a function of site $i$. The data at different time is plotted together, which collapse to a universal curve by scaling horizontal axis by $\sqrt{t}$.

its motion drags the phase field of the surrounding region. As a natural result velocity autocorrelation function has memory with negative correlation. For an isolated vortex, the influence of its core motion spreads infinitely large range and then correlation time also diverges.

Based on the reduced one-dimensional model, clearer picture of the phenomena is obtained. In addition, we can perform numerical simulations for large scale both in time and space. We found that the correlation length $R(t)$ grows as $t^{1/2}$ not as $(t/\ln t)^{1/2}$. This means that the logarithmic correlation does not come from the growth law of the correlation length $R(t)$ itself but originates with the size dependence of the mobility as $1/\ln R(t)$. This is consistent with the finite size effect of the mobility in the 2dXY model.

The reduced model, eq. (13), has two points that ordinary stochastic diffusion equation does not have. Those are the gradient term and the position dependence of temperature. We found that the $i$-linear dependence of the local temperature is not essential for the logarithmic correction but the gradient term in eq. (13) is important (not shown here). This term has a function to make the deformation amplitude $X(r)$ propagate to the positive direction of $r$. This makes the diffusion of the end of chain slower than that of the ordinary elastic chain. This is in contrast with the Bessel equation with $n = 0$ which has positive sign on the gradient term. Since the derivative kernel of the eq. (A·20) is that of the 1st Bessel equation (see the appendix), expansion of the solution with the Bessel functions will be useful and the exact analysis of the present reduced model may be possible. This is challenging open question.

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Appendix: Derivation of the reduced model

A.1 Elastic energy of single vortex

At first we consider the interaction energy of the two dimensional XY model, eq. (1). In the elastic continuum approximation, which is justified for the region away from vortex core, the energy is written as

\[ E = \int dr^2 \frac{1}{2} |\nabla \theta(r)|^2, \]  

(A-1)

where \( r = (x, y) \). A metastable state having an isolated vortex is written as

\[ \theta_0(r) = \arctan(y/x). \]  

(A-2)

(Strictly speaking we have to add or subtract \( \pi \) for \( x < 0 \) since the arctangent function returns the value between \( -\pi/2 \) and \( \pi/2 \).) The elastic energy of this state is

\[ E_0 = \frac{1}{2} \int_a^R dr^2 \pi r \left( \frac{1}{r} \right)^2 = \pi \ln \frac{R}{a}, \]  

(A-3)

where \( a \) is an ultraviolet cut-off length.

We evaluate the integral in eq. (A-1) supposing \( \theta(r) = \theta_0(r') \) and \( r' \) is related to \( r \) as

\[ r = r' + X(r'), \quad r' = |r'|. \]  

(A-5)

To transform the integral variable from \( r \) to \( r' \) we first evaluate the Jacobian \( J \) to scale \( d^2r = J d^2r' \):

\[ J = \frac{\partial(x, y)}{\partial(x', y')} = \det \begin{pmatrix} 1 + \hat{x}'X' & \hat{y}'X' \\ \hat{x}'Y' & 1 + \hat{y}'Y' \end{pmatrix} = 1 + \hat{r}' \cdot X', \]  

(A-6)

where \( X' = \frac{dX}{dr} \), and \( \hat{r}' = (\hat{x}' = x'/r', \hat{y}' = y'/r') \) is the unit vector parallel to \( r' \). To evaluate the gradient

\[ \nabla \theta(r) = \left( \frac{\partial \theta'}{\partial x} \frac{\partial \theta'}{\partial y} \right) \nabla' \theta_0(r'), \]  

(A-7)

we need the transform matrix (the Hessian)

\[ \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} 1 + \hat{y}'Y' & -\hat{x}'Y' \\ -\hat{y}'X' & 1 + \hat{x}'X' \end{pmatrix}. \]  

(A-8)

Knowing the gradient with respect to \( r' \)

\[ \nabla' \theta_0(r') = \frac{1}{r'} \begin{pmatrix} -\hat{y}' \\ \hat{x}' \end{pmatrix}, \]  

(A-9)
we obtain
\[ \nabla \theta(r) = \frac{1}{Jr'} \left( -\hat{y}' - Y' \right) \]
and the integrand of \( E \) reads
\[
d^2r |\nabla \theta(r)|^2 = d^2r \frac{1 + 2\hat{r}' \cdot X' + |X'|^2}{Jr'^2} \]
\[
= d^2r \frac{1 + \hat{r}' \cdot X' + |X'|^2 - (\hat{r}' \cdot X')^2}{r'^2} + O(|X'|^3). \tag{A-11}
\]
The integration of the first term corresponds to the undistorted vertex energy \( E_0 \) while that of the second term vanishes due to the rotational symmetry with respect to \( r' \). Thus we finally find
\[
E_{el} = E - E_0 = \frac{1}{2} \int_a^R dr' \pi \frac{|X'|^2}{r'^2} + O(|X'|^3). \tag{A-12}
\]
Note that two degrees of freedom \( X(r) \) and \( Y(r) \) are decoupled. Therefore all we have to treat is one component field \( X(r) \) with one dimensional parameter \( r \).

By taking variation of the energy function in eq. (A-12) we obtain the energy minimal condition
\[
\frac{\delta E_{el}}{\delta X(r)} = -2\pi \partial_r \frac{1}{r} \partial_r X(r) = 0. \tag{A-13}
\]
On the boundary condition; \( X(0) = X_0 \) and \( X(R) = 0 \), the solution is obtained as
\[
X(r) = X_0 \left[ 1 - \left( \frac{r}{R} \right)^2 \right]. \tag{A-14}
\]

A.2 Energy dissipation

On the next step, we derive effective friction force for \( X \)'s. By using phase variable the energy dissipation rate of the whole system can be written as
\[
d\frac{E_{el}}{dt} = -\int dr^2 |\dot{\theta}(r, t)|^2. \tag{A-15}
\]
Remembering \( \tan \theta(r) = (y - Y) / (x - X) \), we obtain
\[
\frac{\partial \theta(r)}{\partial X} = \frac{\sin \theta(r)}{r'}, \quad \frac{\partial \theta(r)}{\partial Y} = \frac{\cos \theta(r)}{r'} \tag{A-16}
\]
By using this,
\[
\frac{dE}{dt} = -\int dx dy \left| \frac{dX}{dt} \frac{\partial \theta}{\partial X} + \frac{dY}{dt} \frac{\partial \theta}{\partial Y} \right|^2 \\
= -\int r' dr' \int_0^{2\pi} d\varphi' \frac{1}{r'} \left| \frac{dX}{dt} \right|^2 \sin^2 \varphi' + \cdots + O(X^3) \\
= -\int dr' \pi \frac{1}{r'} \left( \left| \frac{dX}{dt} \right|^2 + \left| \frac{dY}{dt} \right|^2 \right) + O(X^3). \tag{A-17}
\]
Again \( X \) and \( Y \) are decoupled. Therefore friction force acting on the region \((r', r' + dr')\) is
\[
-\frac{dr'}{r'} \left( \frac{dX}{dt}, \frac{dY}{dt} \right) \tag{A-18}
\]
A.3 Equation of motion

We can construct overdamped equation of motion for $X(r, t)$ by considering local balance between the elastic and friction forces,

$$\frac{\pi}{r} \frac{d}{dt} X(r, t) = 2\pi \frac{1}{r} \partial_r X(r, t).$$

$$\frac{d}{dt} X(r, t) = 2 \left( \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \frac{\partial}{\partial r} \right) X(r, t). \quad (A.19)$$

By putting $u(r, t) = X(r, t)/r$, we obtain

$$\frac{d}{dt} u(r, t) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) u(r, t). \quad (A.20)$$

Assuming separable solution $u(r, t) = \Upsilon(kr) \exp(-k^2 t)$, we obtain the Bessel equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} + 1 - \frac{n^2}{x^2} \right) \Upsilon(x) = 0 \quad (A.21)$$

with $n = 1$. It is natural that deformation is not isotropic ($n = 0$) but anisotropic ($n = 1$) since the translational motion of vortex breaks the circular symmetry.

A.4 Thermal noise

Finally we consider the property of thermal noise. We introduce Gaussian white noise $Z(r, t)$,

$$\frac{d}{dt} X(r, t) = -\frac{r}{\pi} \delta E \langle X \rangle + Z(r, t). \quad (A.22)$$

This force $Z(r, t)$ have to satisfy the fluctuation-dissipation relation

$$\langle Z(r, t) Z(r', t') \rangle = 2\frac{r}{\pi} T \delta(r - r') \delta(t - t') \quad (A.23)$$

to realize the canonical distribution at temperature $T$ in equilibrium. The deviation of this random force is not uniform in space but proportional to $r$. It can be said that effective temperature becomes higher with $r$. 


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