Wrinkling transition in quenched disordered membranes at two loops

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One investigates the flat phase of quenched disordered polymerized membranes by means of a two-loop, weak-coupling, computation performed near their upper critical dimension $D_{uc} = 4$, generalizing the one-loop computation of Morse, Lubensky and Grest [Phys. Rev. A 45, R2151 (1992), Phys. Rev. A 46, 1751 (1992)]. Our work confirms the existence of the finite-temperature, finite-disorder, wrinkling transition, which has been recently identified by Coquand et al. [Phys. Rev E 97, 030102 (2018)] using a nonperturbative renormalization group approach. One also points out ambiguities in the two-loop computation that prevent the exact identification of the properties of the novel fixed point associated with the wrinkling transition, which very likely requires a three-loop order approach.

Introduction

The critical and, more generally, the long-distance equilibrium statistical physics of pure, homogeneous, systems is now widely understood. By contrast, quenched, random heterogeneities, like defects or impurities, inevitably present in most of real physical systems, are known to give rise to a wide spectrum of new phenomena. Quenched disordered membranes occupy a special place, see e.g. [1], as their most famous physical realizations seem to bring out the physical effects of both random bonds and random fields, see [2–8] for reviews. For instance, in a series of experiments beginning in the early 90’s, Mutz et al. [9] then Chaieb et al. [10, 11] have shown that, upon cooling below the chain melting temperature, photo-induced partially polymerized vesicles made of diacetylenic phospholipid undergo a transition from a smooth structure, at high polymerization, to a wrinkled structure, at low polymerization, with randomly frozen normals that could characterize a glassy phase. This transition and the resulting wrinkled phase have been conjectured to result from the joint effects of random heterogeneities on both the internal metric and the curvature of the membrane [12] that bear formal similarities with, respectively, random mass/random bond and random field in magnetic systems, see below. More recently, in the context of the rapidly growing defect engineering [13–16] of graphene [17, 18], it has been shown that by thoroughly damaging a clean sheet of this material with a laser beam, it was possible to induce a crystal-to-glass transition giving rise to a vacancy-amorphized graphene structure [19–21]. Here also, the inclusion of lattice defects – foreign adatoms or substitutional impurities – is expected to lead, in addition to metric alterations, to a rearrangement of sp²-hybridized carbon atoms into nonhexagonal structures and, thus, to the generation of nonvanishing curvature structures showing again that the underlying physics could rely on the coexistence of the two kinds of disorder.

Disordered membranes stand also out from the crowd by the theoretical investigations to which they have been subjected. Stimulated by the work of Mutz et al. [9] on partially polymerized vesicles, the first attempt to probe the effects of disorder in membranes has been realized by Nelson and Radzihovsky [12, 22], who have focused their study on the role of disorder in the preferred metric. Performing a weak-coupling expansion near the upper critical dimension $D_{uc} = 4$ they have shown that, for $D < 4$, while the disorder is irrelevant and the renormalization group (RG) flow driven toward the disorder-free fixed point $– P_4 –$ at any finite temperature, an instability close to $T = 0$ could be accompanied by a diverging Edwards-Anderson correlation length, leading to a glassy phase. Then Radzihovsky and Le Doussal [23], employing a large embedding dimension $d$-expansion, have confirmed such a possibility, finding an instability of the flat phase toward a spin-glass-like phase. However no quantitative or qualitative empirical confirmation of this scenario have been provided. Morse et al. [24, 25] have then reconsidered the weak-coupling analysis of Ref.[12, 22] within an approach including both metric and curvature disorders. They have confirmed the irrelevance of the disorder in $D < 4$ but shown that the presence of curvature disorder gives rise to a new and vanishing temperature fixed point $– P_5 –$ stable with respect to randomness but unstable with respect to the temperature. Somewhat disappointing from the point of view of the search of a new exotic phase, these works have been followed mainly by mean-field approximations involving either short-range [23, 26–31] or long-range disorder [32, 33] leading to conjecture the appearance of a glassy phase at any temperature and for large enough disorder; see [34] for a review. Again no confirmation of this conjecture has been provided. However very recently, an approach based on the nonperturbative renormalization group (NRPG), following those performed on disorder-free membranes [35–40] has been realized by Coquand et al. [41] on the model considered by Morse and Lubensky displaying both metric and curvature disorders. Their main result has been to identify a novel finite temperature, finite disorder fixed-point $– P_c –$ once unstable, thus associated with a second-order

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phase transition and making the $T = 0$ fixed point fully attractive provided $T < T_c$. This study has allowed to identify three distinct universal scaling behaviors\footnote{Corresponding both qualitatively and quantitatively to those observed in the experiments of Chaieb et al\cite{10, 11}.}

Whereas the NPRG results have been challenged within a recent SCSA approach\footnote{They have been confirmed within a large $d$ approach performed at next-to-leading order in $1/d$\cite{43} although in a model involving only curvature disorder. In this controversial context, it was compelling to further investigate the model of Morse, Lubensky and Grest\cite{24, 25}. In that respect an essential feature of the novel fixed point $P_c$ found in\cite{41} is that its coordinates near $D_{uc} = 4$ differ only from those of the vanishing temperature fixed point $P_5$ by terms of order $\epsilon^2$, with $\epsilon = 4 - D$, strongly suggesting that $P_c$ could be also identified within a perturbative $\epsilon$-expansion up to this order. This is the reason why one investigates, in this paper, quenched disordered membranes at two-loops in the vicinity of the upper critical dimension, extending both the one-loop computation of Morse, Lubensky and Grest performed thirty years ago\cite{24, 25}, at the next order and the recent two-loop computation of Coquand, Mouhanna and Teber\cite{51} on disorder-free membranes, to the disordered case. One derives the RG equations, analyze them and provide the critical quantities, notably the anomalous dimension $\eta$, at order $\epsilon^4$. Our approach confirms unambiguously the existence of the once-unstable fixed point $P_4$ characterizing a phase transition between a high-temperature phase controlled by the disorder-free fixed point $P_4$ and a low-temperature phase controlled by the vanishing-temperature, finite-disorder, fixed point $P_5$\cite{52}. It nevertheless reveals also a drawback of the perturbative approach at two-loop order which manifests through the impossibility to identify exactly the location and properties of the fixed points $P_4$ and $P_5$ at this order; one argues that this should very likely be raised by a three-loop order computation.

The action

A membrane is modeled by a $D$-dimensional surface embedded in a $d$-dimensional Euclidean space. A point on the membrane is thus identified by $D$-dimensional vector $x$ and a configuration of the membrane in the Euclidean space is described through the embedding $x \rightarrow R(x)$ with $R \in \mathbb{R}^d$. In the flat phase one defines the average position of a point $x$:

$$R^0(x) = [R(x)] = \zeta e_i$$

(1)

where the $e_i$ form an orthonormal set of $D$ vectors and $\zeta$ is the stretching factor taken to be one in what follows. In Eq. (1) $\ldots$ and $\langle \ldots \rangle$ denote averages over disorder and thermal fluctuations respectively. The fluctuations around the configuration (1) are parametrized by writing $R(x) = R^0(x) + u(x) + h(x)$ with $h.e_i = 0$. The fields $u$ and $h$ represent $D$ longitudinal – phonon – and $d - D$ transverse – flexural – modes, respectively. The long-distance, effective, action is given by\cite{24, 25}:

$$S = \int d^Dx \left\{ \frac{\kappa}{2} (\nabla u_i(x))^2 + \frac{\lambda}{2} u_{ii}(x)^2 + \mu u_{ij}(x)^2 - c(x).\Delta h(x) - \sigma_{ij}(x)u_{ij}(x) \right\}$$

(2)

In Eq. (2) the first term represents the curvature energy with bending rigidity $\kappa$ while the second and third terms represent the elastic energies with $u_{ij}$ being the strain tensor which, truncated to its most relevant part reads $u_{ij} \approx \frac{1}{2} [\partial_i u_j + \partial_j u_i + \partial_i h_i \partial_j h_j]$; $\lambda$ and $\mu$ are the Lamé coefficients; The fourth and fifth terms in Eq. (2) represent disorder fields $c$ and $\sigma_{ij}$ that couple respectively to the curvature $\Delta h$ – thus linearly to $h$ as a random field\cite{53} – and to strain tensor $u_{ij}$ – thus quadratically to $h$ as a random mass. These fields are chosen to be short-ranged quenched Gaussian ones with zero-mean value and variances given by\cite{24, 25}:

$$[c_i(x) c_j(x')] = \Delta_\kappa \delta_{ij} (x - x')$$

$$[\sigma_{ij}(x) \sigma_{kl}(x')] = (\Delta_\lambda \delta_{ij} \delta_{kl} + 2 \Delta_\mu I_{ijkl}) \delta(D)(x - x')$$

(3)

where $I_{ijkl} = \frac{1}{4} (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl})$, with $i,j,k,l \equiv 1 \ldots D$. Stability considerations require that the coupling constants $\kappa$, $\mu$, and $\lambda + (2/D)\mu$ as well as $\Delta_\kappa$, $\Delta_\mu$ and $\Delta_\lambda + (2/D)\Delta_\mu$ are positive.

Disorder averages are performed through the replica trick which leads to the effective action\cite{24, 25}:

$$S = \int d^Dx \left\{ \frac{Z_{\alpha\beta}}{2} \Delta h^\alpha(x) \Delta h^\beta(x) + \frac{\lambda_{\alpha\beta}}{2} u_{ii}(x) u_{jj}^\alpha(x) + \mu_{\alpha\beta} u_{ij}(x) u_{ij}^\alpha(x) \right\}$$

(4)

where Greek indices are associated with the $n$ replica. In Eq. (4) one has rescaled the fields $h \rightarrow T^{1/2} Z^{1/2} \kappa^{-1/2} h$, $u \rightarrow T Z \kappa^{-1} u$ where $Z$ is a field renormalization and introduced the running coupling constants: $\tilde{\lambda} = \lambda T Z^2 \kappa^{-2}$, $\tilde{\mu}_k = \mu T Z \kappa^{-2}$, $\tilde{\Delta}_\lambda = \Delta_\lambda Z^2 \kappa^{-2}$, $\tilde{\Delta}_\mu = \Delta_\mu Z^2 \kappa^{-2}$, $\tilde{\Delta}_\kappa = \Delta_\kappa T^{-1} Z \kappa^{-1}$ and $\tilde{Z}_{\alpha\beta} = Z^{\alpha\beta} - \Delta_\alpha Z^{\beta\alpha} - \Delta_\beta Z^{\alpha\beta}$, $\tilde{\mu}_{\alpha\beta} = \mu \delta^{\alpha\beta} - \Delta_\mu J^{\alpha\beta}$ and $\tilde{\lambda}_{\alpha\beta} = \tilde{\lambda} \delta^{\alpha\beta} - \Delta_\lambda J^{\alpha\beta}$ where $J^{\alpha\beta} \equiv 1$ $\forall \alpha, \beta$. Note that $\tilde{\mu}$ and $\tilde{\lambda}$ can be used as a measure of the temperature $T$ while $\tilde{\Delta}_\lambda$ diverges at vanishing temperatures. Finally, as usual, on defines the correlation functions $G_{h_i h_j}(q) = \langle \delta h_i(q)\delta h_j(-q) \rangle$ and $G_{u_{ij}(q)} = \langle u_{ij}(q) u_{ij}(-q) \rangle$ as well the thermal $-\chi(q)$ – and disorder $-C(q)$ – ones through\cite{24, 25}:

$$G_{h_i h_j}(q) = \langle \delta h_i(q)\delta h_j(-q) \rangle = T \chi_{h_i h_j}(q) + C_{h_i h_j}(q)$$

(5)

and

$$G_{u_{ij}(q)} = \langle \delta u_i(q)\delta u_j(-q) \rangle = T \chi_{u_i u_j}(q) + C_{u_i u_j}(q)$$

(6)
with \( \delta h_i(q) = h_i(q) - \langle h_i(q) \rangle \), \( \delta u_i(q) = u_i(q) - \langle u_i(q) \rangle \). At low momenta one expects the scaling behaviors [24, 25]:

\[
\begin{align*}
\chi_{h,h_i}(q) & \sim q^{-(4-n)} , \quad C_{h,h_i}(q) \sim q^{-(4-n')} \\
\chi_{u,u_i}(q) & \sim q^{-(4-n_u)} , \quad C_{u,u_i}(q) \sim q^{-(4-n'_u)} .
\end{align*}
\] (7)

Ward identities relate these quantities [24, 25] through \( \eta_n + 2\eta' = 4 - D \) and \( \eta'_u + 2\eta' = 4 - D \). Finally one defines [24, 25, 34], from \( \eta \) and \( \eta' \), the exponent \( \phi = \eta' - \eta \) that determines which kind of – thermal or disorder – fluctuations dominates at a given fixed point: (i) if \( \phi > 0 \), the fixed point behavior is dominated by thermal fluctuations (ii) if \( \phi < 0 \) the fixed point behavior is dominated by disorder fluctuations (iii) if \( \phi = 0 \) both fluctuations coexist; the fixed point is said to be marginal.

Renormalization group equations and fixed points

As in the disorder-free [45, 47, 51] case Ward identities associated with a partial rotation invariance ensure the renormalizability of the theory. Also only the renormalizations of phonon and flexural mode propagators are required. As in [51] one has treated the massless theory using the modified minimal subtraction scheme and used standard techniques for computing massless Feynman diagram calculations; see, e.g., [49]. As usual one defines dimensionless coupling constants \( \bar{\pi} = Z^{-2} k^{D-4} \bar{\mu} \), \( \bar{\chi} = Z^{-2} k^{D-4} \bar{\chi} \), \( \bar{\lambda} = Z^{-2} k^{D-4} \bar{\lambda} \), \( \bar{\Delta}_\chi = Z^{-1} \bar{\Delta}_\chi \) and \( \bar{\Delta}_\lambda = Z^{-1} \bar{\Delta}_\lambda \). The running anomalous dimension is given by \( \eta_\mu = \partial \ln Z + \omega_\mu = \eta'_\mu - \eta_\mu = \partial \ln \bar{\Delta}_\chi \) [54] where \( t = \ln \bar{k} \), \( \bar{k} \) being a renormalization momentum scale [55]. The RG equations are given in Appendix A and computational details will be given in a forthcoming publication [48]. Note finally that our computations have been checked using the effective-field theory required. As in [51] one has treated the massless theory.

Let us first recall the one-loop results [24, 25]. At this order one finds, in \( D < 4 \), two nontrivial physical fixed points, located on the hypersurfaces \( \bar{\chi} = \bar{\lambda} = \bar{\Delta}_\chi = \bar{\Delta}_\lambda = 0 \). First, the disorder-free fixed point, \( P_4 \), for which \( \bar{\mu} = \phi = 0 \), is left attractive and thus controls the long distance behaviour of both disordered and disorder-free membranes. This fixed point is – obviously – dominated by thermal fluctuations. There is another fixed point, \( P_5 \), located at vanishing temperature, i.e. \( \bar{\mu} = 0 \). To get this fixed point from the RG equations one has to consider the coupling constant \( \bar{g}_\mu = \bar{\mu} \bar{\Delta}_\mu \) that stays finite at vanishing temperature while \( \bar{\Delta}_\mu \) is diverging. \( P_5 \) is characterized by \( \bar{\Delta}_\mu = 24 \pi^2 \bar{\mu} \), \( \bar{g}_\mu = 48 \pi^2 \bar{\mu} \) and \( \bar{\mu} = \eta' = 3 \bar{\mu} \). At this fixed point one has \( \phi = 0 \); it is thus marginal. A further analysis taking account of nonlinear contributions shows that \( P_5 \) is marginally relevant [24, 25].

At two-loop order one recovers the disorder-free fixed point \( P_4 \) whose coordinates and anomalous dimension have been given in [51]. Using the variables relevant to study the vanishing temperature one also identifies a fixed point with \( \bar{\mu} = 0 \) that coincides with the fixed point \( P_5 \) found at one-loop order. Note however that one is not able to fully characterize this fixed point – see below. Finally the search for a new fixed point is inspired by the NPRG results [41] where one recalls that the coordinates of \( P_5 \) in the vicinity of \( D_{uc} = 4 \) are given at leading nontrivial order in \( \epsilon \) by [41, 42]:

\[
\bar{\mu} = 4 \pi^2 \epsilon^2 (5 d_{ec} + 27)/15 d_{ec}^2 + O(\epsilon^3), \quad \bar{\lambda} = -1/3 \bar{\mu} + O(\epsilon^3), \quad \bar{\Delta}_\mu = 24 \pi^2 \epsilon/d_{ec} + O(\epsilon^3), \quad \bar{\Delta}_\chi = -1/3 \bar{\Delta}_\mu + O(\epsilon^3)
\]

while the anomalous dimension is given by:

\[
\frac{\eta_\mu^{\text{prg}}}{\bar{\mu}} = \frac{3 \epsilon}{d_{ec}} - \frac{d_{ec} (425 d_{ec} + 2556)}{240 d_{ec}^3} \epsilon^2
\] (8)

with \( \eta_\mu^{\text{prg}} = \eta_\mu^{\text{prg}} \). Within the perturbative context one thus considers for the various coupling constants the ansatz:

\[
\bar{\chi} = C_\chi^{(1)} \epsilon + C_\chi^{(2)} \epsilon^2 \quad \text{for} \quad \bar{\chi} = \{ \bar{\chi}, \bar{\mu}, \bar{\lambda}, \bar{\Delta}_\mu \}
\] (9)

where the \( C_\chi^{(1)} \) are given by the coordinates of the vanishing temperature fixed point \( P_3 \) at one-loop order [56], and the usual – singular – behaviour for \( \bar{\Delta}_\mu \):

\[
\bar{\Delta}_\mu = C_{\Delta_\mu}^{(-1)} / \epsilon + C_{\Delta_\mu}^{(0)}.
\] (10)

Canceling the RG equations at (next-to-leading) order \( \epsilon^3 \) for the \( \bar{\chi} \)'s and at order \( \epsilon \) for \( \bar{\Delta}_\mu \) one finds a new fixed point \( P^* \) with parameters:

\[
\begin{align*}
C_\chi^{(2)} &= -\frac{C_\chi^{(2)}}{3} \\
C_{\Delta_\mu}^{(2)} &= -\frac{C_{\Delta_\mu}^{(2)}}{6} - \frac{2 (6 d_{ec} + 83) \pi^2}{5 d_{ec}^3}, \\
C_{\Delta_\mu}^{(2)} &= -\frac{C_{\Delta_\mu}^{(2)}}{2} - \frac{6 (14 d_{ec} + 37) \pi^2}{5 d_{ec}^3}, \\
C_{\Delta_\mu}^{(0)} &= -\frac{2 (d_{ec} + 3)}{d_{ec}} - \frac{4 (28 d_{ec} + 27) \pi^2}{5 C_{\Delta_\mu}^{(2)} d_{ec}^3}.
\end{align*}
\] (11)

As seen on these expressions one of the parameters entering in (9)-(11), here \( C_\mu^{(2)} \), is left undetermined. An analysis of the NPRG approach [41] shows that by using the ansatz (9)-(10) and by canceling the corresponding NPRG equations at the same order in \( \epsilon \) leads to the same difficulty, i.e. the same indetermination of \( C_\chi^{(2)} \), which is thus a feature of the \( \epsilon \)-expansion and not of the loop expansion. It is thus judicious to go beyond the former expansion. One can first analyze the RG equations numerically. Doing this one clearly identifies a once-unstable
fixed point in the vicinity of \( D = 4 \) with coordinates of
the type (9)-(11). Thereafter, in order to identify analytically this
fixed point one can push the solution of the RG equations beyond
next-to-leading order, notably by canceling the equation for \( \Delta \) order \( \epsilon^2 \). This raises the
indetermination on \( C^{(2)} \) which is found to be equal to:

\[
C^{(2)} = \frac{4 \pi^2 (3075 d_c^2 + 16850 d_c - 576)}{15 d_c^2 (166 + 169 d_c + 20 d_c^2)}
\]

(12)

where the index 2 refers to the two-field (phonon-flexuron) theory. Note that this value is approximate
as one expects three-loop contribution to (12). However
with the expressions (11) and (12) one reproduces very
satisfactorily the numerical results in the extreme vicinity
of \( D = 4 \), e.g. for \( \epsilon \) of order \( 10^{-3} \) the errors for the
coordinates are at worst of order \( 10^{-8} \). One now gives the
eigenvalues around \( P^* \) at leading non-vanishing order:

\[
\left\{ \frac{3d_c C^{(2)}}{8 d_c^2} \epsilon^2 ; - \frac{d_c}{d_c} \epsilon ; \frac{d_c}{d_c} \epsilon ; - \epsilon ; - \epsilon \right\}
\]

with \( C^{(2)} \) given by (12) which is positive for any physical
value of \( d_c \). Having one repulsive direction the fixed point
\( P^* \) is associated with a second order phase transition. It is
caracterized by the anomalous dimensions:

\[
\eta_c^{(2)} = \frac{3 \epsilon}{d_c} - \frac{d_c}{d_c} (5 C^{(2)} d_c^2 + 2(407 + 60 d_c) \epsilon^2}{80 \pi^2 \epsilon^2}
\]

(13)

and \( \eta_c' = \eta_c \) implying \( \phi = 0 \) so that \( P^* \) is marginal.
The result (13) is also found within the effective (pure
flexuron) approach of the theory, see [48] and Appendix
B, which is a strong confirmation of the validity of our
result. Note however that in the latter case the approxi-
mate expression of \( C^{(2)} \) slightly differs and is given by:

\[
C^{(2)} = \frac{4 \pi^2 (3450 d_c^2 + 19100 d_c - 576)}{15 d_c^2 (166 + 169 d_c + 20 d_c^2)}
\]

(14)

However this change affects extremely weakly the physical
results – see below.

All the qualitative properties of \( P^* \) – one marginally
relevant direction of order \( \epsilon^2 \), one coupling constant \( \mu
\) of order \( \epsilon^2 \) – are shared with those of the fixed point
\( P^* \) found in [41, 42] using a NPRG approach. Moreover the
agreement between the anomalous dimension obtained
within the present work – (13) with \( C^{(2)} \) given by (12) or
(14) – and that computed with the NPRG approach (8)
is remarkable, see Fig.(1) where one has represented the
two-loop corrections \( \Delta_\mu \) defined as \( \Delta_c = \eta_c + \eta_c^{(2)} \epsilon^2 \).

In the physical situation \( d_c \) is \( 1 \) – they are given by:

\[
\eta_c^{(2)} = 0.0362 \quad \eta_c^{(2)} = 0.0366 \quad \eta_c^{(2)} = 0.0370
\]

One thus identifies \( P^* \) with \( P_c \) and fully confirms the existence of a – wrinkling – phase transition at finite
temperature.

Concerning the fixed point \( P_3 \) one finds – numerically
– that it is, in fact marginally irrelevant – in agreement
with the unstable character of \( P_c \) and with the NPRG
approach. However as said above there are difficulties to
characterize \( P_3 \) as well as \( P_c \) – at low-temperatures. Indeed
this implies to use the low-temperature variables \( g_\lambda = \lambda \Delta_\lambda \) and \( g_\mu = \mu \Delta_\mu \) at order \( \epsilon^2 \). But, in the vicinity
of \( P_3 \) or \( P_c \) – one has at next-to-leading order in \( \epsilon \):

\[
\eta_\mu = C^{(2)} \eta_\mu^{(2)} (1 - \epsilon^2) + C^{(3)} \eta_\mu^{(3)} \epsilon^2 + C^{(2)} \eta_\mu^{(0)} \epsilon^2 + O(\epsilon^3)
\]

where it appears that, due to the specific scaling of \( \Delta_\lambda
\) with \( \epsilon \) that involves negative powers of this parameter,
the subsubleading contribution in \( \epsilon \) to \( \mu - C^{(3)}_\mu \) is needed but is obviously lacking within the present – two-loop
order – computation.

**Conclusion**

One has investigated quenched disordered membranes
by means of a two-loop order perturbative approach. As
a main result our approach clearly confirms the finding
obtained with the NPRG approach [41], i.e. the exis-
tence of a richer phase diagram than that expected from
previous investigations: the existence of a novel fixed
point \( P_3 \) characterizing a wrinkling phase transition
occurring at a temperature \( T_c \) separating a disorder-free
phase at \( T > T_c \) controlled by the vanishing-disorder at-
ttractive fixed point \( P_3 \) and a low-temperature \( T < T_c
\) “glassy-phase” controlled by the vanishing-temperature,
finite-disorder, attractive fixed point \( P_3 \). One thus has
reached a consistent picture of disordered membranes at
finite temperatures and in particular of the occurrence of a
wrinkling transition. Our work reinforces the interest
to investigate experimentally or numerically this tran-
sition in several systems involving both curvature and
stretching disorder. This includes (i) a further study of
partially polymerized fluid vesicles that have been al-
ready investigated by Chaieb et al. and have shown

![FIG. 1: The correction of order O(\epsilon^2) to \eta_c, \eta_c^{(2)}, as function of d_c at the fixed point P_c. In full line the prediction from the NPRG approach [41]; in dashed line, the two-loop/two-field result (present work); in dotted line, the two-loop/effective result (present work).](image-url)
to be qualitatively and quantitatively well explained by the scenario proposed in [41, 42] and (ii) a careful investigation of graphene and graphene-like materials with quenched lattice defects. Moreover our work, by confirming the attractive character of the vanishing temperature fixed point \( P_1 \), opens the possibility of a low-temperature phase controlled by a complex energy landscape and a genuine “glassy phase” that have been intensively looked for theoretically. It is thus pressing to probe this phase experimentally and numerically, notably in the context of the physics of graphene where it would be of prime interest to study the effects induced by disorder on the electronic and transport properties of graphene and graphene-like materials in this phase. Finally, from a more formal point of view our work strongly suggests to investigate deeply the nature of the perturbative series in the vicinity of the fixed points \( P_3 \) and \( P_c \). In particular it would be of interest, even if it would be represent a very substantial amount of work, to see whether the three-loop contributions indeed raise the ambiguities encountered within the two-loop order computation when studying the wrinkling transition.

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\[\partial_t \mu = -\mu \eta_{\text{int}} + \frac{d_c\mu^2}{6(16\pi^2)}(1 + 2\Delta_\kappa) + \frac{d_c\mu^2}{216(16\pi^2)^2}(\lambda + \mu)(\lambda + 2\mu) - 227\Delta_\lambda \mu^2(4\Delta_\kappa + 1) - 227\Delta_\mu(4\Delta_\kappa + 1)(\lambda^2 + 2\lambda\mu + 2\mu^2),\]

\[\partial_t \lambda = -\lambda \eta_{\text{int}} + \frac{d_c(6\lambda^2 + 6\lambda\mu + \mu^2)}{6(16\pi^2)}(1 + 2\Delta_\kappa) + \frac{d_c}{216(16\pi^2)^2}(\lambda + 2\mu)^2 \left[\mu(\lambda + 2\mu) - 2\Delta_\kappa^2(108(d_c + 9)\lambda^3 + 18(16d_c + 63)\lambda^2\mu + (156d_c + 239)\lambda\mu^2 + (24d_c + 77)\mu^3) - 4\Delta_\kappa(54(d_c + 7)\lambda^3 + 72(2d_c + 3)\lambda^2\mu + (78d_c - 179)\lambda\mu^2 + (12d_c - 17)\mu^3)\right]
\]

\[\partial_t \Delta_\mu = -\Delta_\mu \eta_{\text{int}} + \frac{d_c(2\Delta_\mu(2\Delta_\mu + 1) - \Delta_\mu^2)}{6(16\pi^2)} + \frac{d_c}{108(16\pi^2)^2}(\lambda + 2\mu)^2 \left[\Delta_\lambda(343\Delta_\mu^2(\lambda^2 + 8\mu + 6\mu^2) + 908\Delta_\kappa(\lambda + \mu)(\lambda + 2\mu) + 227(\lambda + \mu)(\lambda + 2\mu)) - \Delta_\mu^2(32\Delta_\kappa + 227)(\lambda + \mu)(\lambda + 2\mu) - 227\Delta_\mu(4\Delta_\kappa + 1)(\lambda^2 + 2\lambda\mu + 2\mu^2)\right],\]

\[\partial_t \Delta_\lambda = -\Delta_\lambda \eta_{\text{int}} - \frac{d_c(6\lambda^2 + 6\lambda\mu + \mu^2)}{6(16\pi^2)} + \frac{2d_c(3\Delta_\lambda(2\Delta_\lambda + 1) + \Delta_\lambda(3\Delta_\lambda + \mu))}{6(16\pi^2)}(1 + 2\Delta_\kappa) - \frac{d_c}{108(16\pi^2)^2}(\lambda + 2\mu)^2 \left[\Delta_\kappa^2(\Delta_\lambda \mu(216(d_c + 9)\lambda^3 + 18(52d_c + 387)\lambda^2\mu + 72(16d_c + 63)\lambda\mu^2 + (288d_c + 401)\mu^3) + \Delta_\mu(108(d_c + 9)\lambda^4 + 36(16d_c + 63)\lambda^3\mu + 3(348d_c + 995)\lambda^2\mu^2 + 16(45d_c + 79)\lambda\mu^3 + 6(24d_c + 77)\mu^4)\right]
\]

\[+ (4\Delta_\kappa + 1)(\lambda + 2\mu)(9d_c(3\lambda + \mu)(\lambda + 2\mu)(2\Delta_\lambda + \Delta_\mu(\lambda + \mu)) + \mu(\lambda + \mu)(\Delta_\lambda(378\lambda - 81\mu) - \Delta_\mu(81\lambda + 17\mu))\right]
\]

\[+ \Delta_\kappa^2(\lambda + 2\mu)(4\Delta_\kappa + 1)(\lambda + 2\mu)(27(\lambda + 11)\lambda^4 + 9(8d_c + 51)\lambda^3\mu + (39d_c + 209)\lambda^2\mu^2 + (6d_c + 47)\mu^3)
\]

\[-(54(d_c + 7)\lambda^3 + 72(2d_c + 3)\lambda^2\mu + (78d_c - 179)\lambda\mu^2 + (12d_c - 17)\mu^3)
\]

\[-(4\Delta_\kappa + 1)(\Delta_\lambda(378\lambda - 81\mu) - \Delta_\mu(81\lambda + 17\mu))\right] + \Delta_\kappa^2(\Delta_\lambda(\lambda^2 + 2\lambda\mu + 2\mu^2))\right].\]
\[ \partial_t \Delta = \Delta \eta_t - \frac{5 \Delta (\Delta_\lambda \mu^2 + \Delta \mu (\lambda^2 + 2 \lambda \mu + \mu^2))}{16 \pi^2 (\lambda + 2 \mu)^2} \]
\[ + \frac{\Delta_\lambda}{36 (16 \pi^2)^2 (\lambda + 2 \mu)^2} \left[ - \Delta_\lambda^2 \mu^2 (\lambda + 2 \mu)^2 (39 d_c + 10) \lambda^2 + 2 (39 d_c + 110) \lambda \mu + (81 d_c + 130) \mu^2 \right] \]
\[ + \mu (\lambda + 2 \mu) \left( 5 \Delta_\mu \mu^2 (3 (15 d_c + 47) \Delta_\mu + (15 d_c + 32) \mu + 411 \Delta_\lambda \lambda + 122 \lambda) \right) \]
\[ + \mu^2 (\lambda^2 (39 d_c + 415) \Delta_\mu + (39 d_c + 340)) + 3 \lambda^2 \mu (234 d_c \Delta_\mu + 78 d_c + 435 \Delta_\lambda + 70) \]
\[ + 2 \lambda^2 \mu^2 (477 d_c + 600) \Delta_\lambda + (159 d_c + 50)) + 2 \mu^2 (3 (81 d_c + 55) \Delta_\mu + (81 d_c - 20)) \right) \]
\[ + 30 \mu^2 (-53 \Delta_\lambda^2 \mu^2 - \Delta_\lambda \Delta_\mu (61 \lambda^2 + 32 \lambda \mu + 32 \mu^2)) - 30 \Delta_\mu^2 (17 \lambda^4 + 14 \lambda^3 \mu + 10 \lambda^2 \mu^2 - 8 \lambda \mu^3 - 4 \mu^4) \right], \]

with \[ \eta_t = \frac{5 \mu (\lambda + \mu)}{16 \pi^2 (\lambda + 2 \mu)^2} - \frac{5 \mu (2 \Delta_\mu (\lambda^2 + 2 \lambda \mu + \mu^2) - \mu (\Delta_\lambda (\lambda^2 + 3 \lambda \mu + 2 \mu^2) - 2 \Delta_\mu))}{16 \pi^2 (\lambda + 2 \mu)^2} \]
\[ + \mu^2 \left[ (-39 d_c - 340) \lambda^2 - 4 (39 d_c + 35) \lambda \mu + (20 - 81 d_c) \mu^2 \right] \]
\[ - \frac{1}{72 (16 \pi^2)^2 (\lambda + 2 \mu)^2} \left[ 2 (\lambda + 2 \mu) \left( \mu (2 \Delta_\lambda + 1) \left( \Delta_\mu ((-39 d_c - 340) \lambda^3 - 6 (39 d_c + 35) \lambda \mu - 2 (159 d_c + 50) \lambda \mu^2 \right) \right) \right. \]
\[ + 2 (20 - 81 d_c) \mu^3) - 5 \Delta_\lambda \mu^2 ((15 d_c + 32) \mu + 122 \lambda) \right] - 2880 \pi^2 (\lambda + 2 \mu) (\Delta_\lambda \mu^2 + \Delta_\mu (\lambda^2 + 2 \lambda \mu + 2 \mu^2)) \right) \]
\[ + 3 \Delta_\lambda \mu^2 (\lambda + 2 \mu)^2 \left( \lambda^2 (39 d_c + 10) \Delta_\mu + 39 d_c + 340) + 4 \lambda \mu (39 d_c + 60) \Delta_\lambda + 39 d_c + 35) \right. \]
\[ + \mu^2 ((81 d_c + 30) \Delta_\lambda + 81 d_c - 20)) + 20 \left( 53 \Delta_\lambda \mu^2 + \Delta_\lambda \mu^2 (61 \lambda^2 + 32 \lambda \mu + 32 \mu^2) \right) \]
\[ + \Delta_\mu^2 (17 \lambda^4 + 14 \lambda^3 \mu + 10 \lambda^2 \mu^2 - 8 \lambda \mu^3 - 4 \mu^4) \right] \]

and \[ \eta_{ut} = \epsilon - 2 \eta_t . \]

Appendix B: Renormalization group equations: effective field theory

The effective field theory is obtained after an integration over the phonon field \( u \) in Eq.(4), see [51]:
\[ S_{\text{eff}} = \int \frac{Z_{a\beta}}{2} k^4 h^a(k)h^\beta(-k) \]
\[ + \frac{1}{4} \int_{k_1,k_2,k_3,k_4} h^a(k_1)h^a(k_2) \tilde{R}^{a,b,c,d}_{\alpha\beta} (q)k^b_{1}k^c_{2}k^d_{3}k^e_{4}h^e(k_3)h^e(k_4) \]

where \( \int_k = \int d^D k/(2\pi)^D \), \( q = k_1 + k_2 = -k_3 - k_4 \), and the interaction tensor \( \tilde{R}^{a,b,c,d}_{\alpha\beta} \) is defined as follows:
\[ \tilde{R}^{a,b,c,d}_{\alpha\beta} (q) = \tilde{\tilde{b}}^{a,b} N_{a,b,c,d}(q) + \tilde{\mu}^{\alpha\beta} M_{a,b,c,d}(q) . \] (B1)

In this expression the transverse tensors \( N \) and \( M \) are defined as a function of the projector transverse to \( q \), \( P_{a,b}^{T} = \delta_{a,b} - q_a q_b/q^2 \), by:
\[ \tilde{N}_{ab,c,d}(q) = \frac{1}{D - 1} P_{a,b}^{T}(q) P_{c,d}^{T}(q) \]
\[ \tilde{M}_{ab,c,d}(q) = \frac{1}{2} \left[ P_{a,c}^{T}(q) P_{b,d}^{T}(q) + P_{a,d}^{T}(q) P_{b,c}^{T}(q) \right] - N_{ab,c,d}(q) . \]

Finally the coupling constant \( \tilde{b}_{\alpha\beta} = \tilde{\tilde{b}}_{\alpha\beta} - \tilde{\Delta}_b J_{\alpha\beta} \) is related to the bare couplings \( \tilde{\tilde{b}}, \tilde{\lambda}, \tilde{\Delta}_\mu \) and \( \tilde{\Delta}_\lambda \) by:

\[ \tilde{\Delta}_b = \frac{D \tilde{\lambda}^2 + 2 \tilde{\Delta}_\lambda \tilde{\mu}^2 - 2 \tilde{\mu}(\tilde{\Delta}_\mu \tilde{\lambda} - 2 \tilde{\Delta}_\mu \tilde{\lambda} \tilde{\mu} + \tilde{\lambda} \tilde{\mu})}{(\lambda + 2 \mu)^2} . \]
The two-loop RG equations are then given by:

\[
\partial_t \mu = -\mu \eta_{\mu t} + \frac{d_c \mu^2}{6(16\pi^2)^2} (1 + 2\Delta_\kappa) + \frac{d_c \mu^2}{1296(16\pi^2)^2} \left[ \mu (574 + 2296\Delta_\kappa + 1732\Delta_\kappa^2) + b(107 + 428\Delta_\kappa + 326\Delta_\kappa^2) - (574\Delta_\mu + 107\Delta_\kappa)(1 + 4\Delta_\kappa) \right],
\]

\[
\partial_t b = -b \eta_{bt} + \frac{5d_c b^2}{12(16\pi^2)^2} (1 + 2\Delta_\kappa) - \frac{5d_c b^2}{2592(16\pi^2)^2} \left[ (178\Delta_\mu - 91\Delta_\kappa)(1 + 4\Delta_\kappa) + b(91 + 364\Delta_\kappa + 394\Delta_\kappa^2) - 2\mu(89 + 356\Delta_\kappa + 146\Delta_\kappa^2) \right],
\]

\[
\partial_t \Delta_\mu = -\Delta_\mu \eta_{\mu t} - \frac{d_c \mu (\mu \Delta_\kappa^2 - 2\Delta_\mu(1 + 2\Delta_\kappa))}{6(16\pi^2)}
- \frac{d_c \mu}{648(16\pi^2)^2} \left[ 107\Delta_\mu \Delta_\mu (1 + 4\Delta_\kappa) - b\Delta_\mu(107 + 428\Delta_\kappa + 326\Delta_\kappa^2) + 574\Delta_\mu^2(1 + 4\Delta_\kappa) - 163\mu \Delta_\kappa \Delta_\kappa^2
+ b\mu \Delta_\kappa^2(107 + 112\Delta_\kappa) - 2\mu(287 + 1148\Delta_\kappa + 1299\Delta_\kappa^2) + 574\Delta_\kappa^2\mu^2 + 584\Delta_\kappa^3\mu^2 \right],
\]

\[
\partial_t \Delta_\kappa = \Delta_\kappa \eta_{\kappa t} - \frac{5(\Delta_\kappa + 2\Delta_\mu)}{6(16\pi^2)} \Delta_\kappa
+ \frac{\Delta_\kappa}{1296(16\pi^2)^2} \left[ 5\mu^2(15d_c - 242)\Delta_\kappa^2 - 2(795\Delta_\kappa^2 - 870\Delta_\kappa\Delta_\mu - 60\Delta_\mu^2 + 5\Delta_\mu(58 + 129\Delta_\kappa)
- 2\mu(111d_c(1 + 3\Delta_\kappa) + 5(33\Delta_\kappa - 4)) + 2\mu^2 \Delta_\kappa^2(130 + 111d_c) \right)
- 5\mu \Delta_\mu(15d_c(1 + 3\Delta_\kappa) - 212 - 681\Delta_\kappa) + 2(\Delta_\mu(58 + 129\Delta_\kappa) - 56\mu\Delta_\kappa^2) \right],
\]

with

\[
\eta_t = \frac{5(b + 2\mu)}{6(16\pi^2)^2} (1 + \Delta_\kappa) - \frac{5(\Delta_\kappa + 2\Delta_\mu)}{6(16\pi^2)}
+ \frac{1}{2592(16\pi^2)^2} \left[ -1060\Delta_\kappa^2 + 5b(15d_c(1 + 3\Delta_\kappa + 3\Delta_\kappa^2) - 2(106 + 318\Delta_\kappa + 333\Delta_\kappa^2)) + 80\Delta_\mu^2
+ 8\mu\Delta_\mu(1 + 2\Delta_\kappa)(111d_c - 20) - 4\mu^2(111d_c(1 + 3\Delta_\kappa + 3\Delta_\kappa^2) + 10(9\Delta_\kappa^2 - 6\Delta_\kappa - 2))
+ 1160\Delta_\mu(\Delta_\mu - (1 + 2\Delta_\kappa)\mu) - 10b(\Delta_\mu(15d_c - 212)(1 + 2\Delta_\kappa) + 116\Delta_\mu(1 + 2\Delta_\kappa) - 4\mu(29 + 87\Delta_\kappa + 72\Delta_\kappa^2)) \right],
\]

and \( \eta_{\mu t} = \epsilon - 2\eta_t. \)