Ground state energy of a homogeneous Bose-Einstein condensate beyond Bogoliubov

Christoph Weiss and André Eckardt
Institut für Physik, Carl von Ossietzky Universität, D-26111 Oldenburg, Germany

(Dated: November 10, 2018)

The standard calculations of the ground-state energy of a homogeneous Bose gas rely on approximations which are physically reasonable but difficult to control. Lieb and Yngvason [Phys. Rev. Lett. 80, 2504 (1998)] have proved rigorously that the commonly accepted leading order term of the ground state energy is correct in the zero-density-limit. Here, strong indications are given that also the next to leading term is correct. It is shown that the first terms obtained in a perturbative treatment provide contributions which are lost in the Bogoliubov approach.

PACS numbers: 03.75.Hh 67.40.Db 05.30.Jp

As dilute weakly interacting Bose gases are experimentally realisable since 1995 [2, 3], there is a renewed interest in principal results for the ground state energy derived some 50 years ago [4, 5, 6, 7, 8, 9, 10]. In recent years, several subtle issues have been clarified [11, 12, 13, 14]. Consider a weakly interacting translationally invariant Bose gas in the thermodynamic limit with non-negative two-particle interaction potentials excluding bound states of two or more particles. In momentum representation, the $N$-particle Hamiltonian in second quantisation reads [15]

$$\hat{H} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \frac{1}{2} \sum_{\{\vec{k}_i\}} \langle \vec{k}_1 \vec{k}_2 | U | \vec{k}_3 \vec{k}_4 \rangle \hat{a}_{\vec{k}_1}^\dagger \hat{a}_{\vec{k}_2}^\dagger \hat{a}_{\vec{k}_3} \hat{a}_{\vec{k}_4},$$  \hspace{1cm} (1)

where the matrix elements

$$\langle \vec{k}_1 \vec{k}_2 | U | \vec{k}_3 \vec{k}_4 \rangle = \delta_{\vec{k}_1+\vec{k}_2,\vec{k}_3+\vec{k}_4} \hat{U}(\vec{k}_2 - \vec{k}_4)$$  \hspace{1cm} (2)

are given by the Fourier transform of the potential

$$\hat{U}(\vec{k}) \equiv \frac{1}{V} \int_V U(\vec{r}) e^{i\vec{k} \cdot \vec{r}} d^3r.$$  \hspace{1cm} (3)

This approach is valid both for soft potentials (non-pseudo-potentials for which the Fourier transform exists) and the pseudo-potential usually used to model hard sphere interaction

$$U_0(\vec{r}) = \frac{4\pi a^2}{m} \delta(\vec{r}) \partial_r,$$  \hspace{1cm} (4)

where $a$ is the hard sphere diameter.

For the pseudo-potential [4], neglecting terms of the Hamiltonian which are believed to be small leads to the Lee-Yang formula [4, 5] for the ground-state energy per particle $e_0 \equiv E_0/N$

$$e_0 = \frac{2\pi a^2 \hbar^2}{m} n \left[ 1 + \frac{128}{15\sqrt{\pi}} \sqrt{n a^3} + o \left( \sqrt{n a^3} \right) \right],$$  \hspace{1cm} (5)

where $n \equiv N/V$ is the density. The $o \left( \sqrt{n a^3} \right)$ includes a term proportional to $n a^3 \ln (n a^3)$ as well as higher order terms (see ref. [3] and references therein) which are very small for realistic experimental values of $n a^3$ [17]. According to the so-called “Landau postulate”, eq. (5) is the ground state energy for any non-negative interaction potential if the hard sphere diameter $a$ is replaced by the $s$-wave scattering length [16, 17]. This is supported by the fact that known potential dependent corrections to $e_0$ are given by $C n a^3 2\pi a^2 \hbar^2 / m$ [3]. The constant $C$ has only recently been calculated explicitly by Braaten et al. in terms of a quantity defined by the $3 \to 3$ scattering amplitude [12, 13].

Lieb et al. [1, 18] have calculated rigorous bounds to prove that in the limit of vanishing density $n$, the leading order term of the ground state of the complete Hamiltonian is indeed given by the leading term of the Lee-Yang formula:

$$1 - C(n a^3)^{1/17} \leq \frac{e_0}{2\pi a^2 \hbar^2 / m n} \leq 1 + 11.3(n a^3)^{1/3},$$  \hspace{1cm} (6)

*Electronic address: weiss@theorie.physik.uni-oldenburg.de*
However, despite many efforts, the next to leading term could not be proved rigorously; neither its sign nor the exponent of $na^3$ has been established beyond doubt.

A standard textbook derivation of equation \( \text{(4)} \) uses the Bogoliubov approximation to replace the quartic Hamiltonian \( \hat{H} \) by a quadratic Hamiltonian which can be diagonalised exactly with the help of the Bogoliubov transformation. Repeating the same analysis for soft potentials leads to the formula of Brueckner and Sawada \( \text{(6)} \text{, 11, 19} \) extended to non-regular potentials like the pseudo-potential \( \text{(4)} \) will be given in a subsequent paper.

As we also improve the upper bound \( \text{(6)} \) for the ground state energy. Lieb et al. used a product-ansatz for the \( N \)-particle wave function to calculate their upper bound according to the variational theorem. Here, we use the Bogoliubov ground state which is the exact quantum-mechanical ground state of an approximate Hamiltonian to calculate an upper bound on the ground state energy of the complete Hamiltonian.

Although we do not use the Born approximation for the scattering length at any point in our calculations, several expressions can be identified with terms of the Born series which are derived in a first step. Before we can calculate a standard textbook derivation of equation \( \text{(5)} \) uses the Bogoliubov approximation to replace the quartic Hamiltonian with a rapidly converging Born series. However, our approach is not restricted to such potentials; an extension to non-regular potentials like the pseudo-potential \( \text{(4)} \) will be given in a subsequent paper.

In this letter we use quantum mechanical perturbation theory to show how the missing terms of the Born series can be recovered. Furthermore, for very soft potentials, already first order perturbation theory indicates that the next to leading term of the ground state energy of the full Hamiltonian is also given by the term of the Lee-Yang formula. If one accepts the Landau postulate, any experimentally realistic interaction potential can be replaced by a soft potential with a rapidly converging Born series. However, our approach is not restricted to such potentials; an extension to non-regular potentials like the pseudo-potential \( \text{(4)} \) will be given in a subsequent paper.

We also improve the upper bound \( \text{(6)} \) for the ground state energy. Lieb et al. used a product-ansatz for the \( N \)-particle wave function to calculate their upper bound according to the variational theorem. Here, we use the Bogoliubov ground state which is the exact quantum-mechanical ground state of an approximate Hamiltonian to calculate an upper bound on the ground state energy of the complete Hamiltonian.

Although we do not use the Born approximation for the scattering length at any point in our calculations, several expressions can be identified with terms of the Born series which are derived in a first step. Before we can calculate the upper bound, we describe the properties of the Bogoliubov ground state for soft potentials.

### I. BORN SERIES

Often, the scattering length is approximated by the first term of the Born series

$$
a_0 = \frac{m}{4\pi\hbar^2} \int U(\vec{r}) \, d^3r .
$$

The next two terms of the Born series are given by

$$
a_1 = -\left( \frac{m}{4\pi\hbar^2} \right)^2 \int U(\vec{r}) \int \frac{U(\vec{r}_1)}{|\vec{r} - \vec{r}_1|} \, d^3r_1 \, d^3r \tag{9}
$$

$$
a_2 = \left( \frac{m}{4\pi\hbar^2} \right)^3 \int U(\vec{r}) \left[ \int \frac{U(\vec{r}_1)}{|\vec{r} - \vec{r}_1|} \left( \int \frac{U(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} \, d^3r_2 \right) \, d^3r_1 \right] \, d^3r \tag{10}
$$

higher order terms can be calculated analogously. The identity

$$
\frac{1}{|\vec{r} - \vec{r}_1|} = \frac{4\pi}{V} \sum_{\vec{k} \neq \vec{0}} \frac{\hat{\vec{k}} \cdot (\vec{r} - \vec{r}_1)}{k^2} + \mathcal{O}(V^{-1/3})
$$

is useful to rewrite the terms of the Born series to expressions in momentum space. Using eq. \( \text{(8)} \), we obtain both the well known formula (see e.g. ref. \( \text{17} \))

$$
a_1 = -\frac{V}{4\pi} \left( \frac{m}{\hbar^2} \right)^2 \sum_{\vec{k} \neq \vec{0}} \frac{(\hat{U}(\vec{k}))^2}{k^2} \tag{12}
$$

and

$$
a_2 = \frac{V}{4\pi} \left( \frac{m}{\hbar^2} \right)^3 \sum_{\vec{k}_1, \vec{k}_2 \neq \vec{0}} \frac{\hat{U}(\vec{k}_1)\hat{U}(\vec{k}_2)\hat{U}(\vec{k}_1 - \vec{k}_2)}{k_1^2k_2^2} \tag{13}
$$
II. BOGOLIUBOV GROUND STATE

For any (normalised) $N$-particle wave-function $|\psi_0\rangle$, the variational theorem yields an upper bound on the ground state energy per particle $e_0$ of the complete Hamiltonian (1)

$$e_0 \leq \frac{1}{N} \langle \psi_0 | \hat{H} | \psi_0 \rangle.$$  \hspace{1cm} (14)

Here, we use the Bogoliubov ground state as our $|\psi_0\rangle$. The idea of the Bogoliubov transformation is to express the operators $\hat{a}_\vec{k}\dagger$ and $\hat{a}_\vec{k}$ which create or annihilate particles with momentum $\vec{k}$ by quasi-particle operators $\hat{b}_\vec{k}\dagger$ and $\hat{b}_\vec{k}$:

$$\hat{a}_\vec{k} = u_\vec{k} \hat{b}_\vec{k} - v_\vec{k} \hat{b}_\vec{k}\dagger, \quad \vec{k} \neq \vec{0},$$  \hspace{1cm} (15)

$$\hat{a}_\vec{0} = \hat{b}_\vec{0}$$  \hspace{1cm} (16)

with

$$[\hat{b}_\vec{k}, \hat{b}_\vec{k}\dagger] = \delta_{\vec{k_1}, \vec{k_2}}, \quad [\hat{b}_\vec{k}\dagger, \hat{b}_\vec{k}^{(t)}] = 0.$$  \hspace{1cm} (17)

Without the unnecessary approximation $\hat{U}(\vec{k}) \approx \hat{U}(\vec{0})$ (which leads to unphysical divergences in both the Born series and in the calculations for the ground state energy [17, 19]), the (real) transformation parameters for isotropic interaction potentials (for which $\hat{U}(\vec{k}) = \hat{U}(\vec{-k})$) are defined via:

$$u_\vec{k} = \frac{1}{\sqrt{1 - L_\vec{k}}}, \quad v_\vec{k} = \frac{L_\vec{k}}{\sqrt{1 - L_\vec{k}}},$$  \hspace{1cm} (18)

$$L_\vec{k} = 1 + \frac{\hbar^2 \vec{k}^2}{2m} - \varepsilon(\vec{k}),$$  \hspace{1cm} (19)

$$\varepsilon(\vec{k}) = \left[ \frac{\hbar^2 \vec{k}^2}{2m} + N\hat{U}(\vec{k}) \right]^{1/2}.$$  \hspace{1cm} (20)

In the Bogoliubov ground state $|\psi_0\rangle$ on average $N_0$ particles are in the ground state characterised by $\vec{k} = \vec{0}$. The macroscopically occupied ground state allows a grand-canonical description of the excited states. One has

$$\langle \psi_0 | \hat{b}_\vec{0}\dagger \hat{b}_\vec{0} | \psi_0 \rangle = N_0,$$  \hspace{1cm} (21)

$$\langle \psi_0 | \hat{b}_\vec{0}\dagger \hat{b}_\vec{0} | \psi_0 \rangle = N_0,$$  \hspace{1cm} (22)

$$\hat{b}_\vec{k} | \psi_0 \rangle = 0 \quad \text{for} \quad \vec{k} \neq \vec{0},$$  \hspace{1cm} (23)

where the first equation defines $N_0$ whereas the second is only true in the thermodynamic limit when $N_0 \rightarrow \infty$ as $N \rightarrow \infty$ (such that the error in the approximation $\sqrt{N_0 + 1} \approx \sqrt{N_0}$ is negligible). It relies on the fact that the number of particles in the ground state fluctuates for a weakly interacting Bose gas even at zero temperatures. The contribution to the total wave function of wave functions with exactly $N_0 + \nu$ or exactly $N_0 + \nu + 1$ particles are practically the same. However, we would like to stress at this point that the calculation of an upper bound does not use special assumptions about the ground state — it is of course possible to construct a wave function which satisfies eqs. (21) to (23). A wave function with a phase factor in eq. (22) or even with $\langle \psi_0 | \hat{b}_\vec{0}\dagger \hat{b}_\vec{0} | \psi_0 \rangle = 0$ would lead to a higher upper bound and thus can be discarded here.

III. CALCULATING THE UPPER BOUND

Let us now write the full Hamiltonian (1) in the form

$$\hat{H} = \hat{H}_I + \hat{H}_{II} + \hat{H}_{III} + \hat{H}_{IV},$$  \hspace{1cm} (24)
where the first part includes both the kinetic energy and those terms with all four \( k_i = \vec{0} \), the second those contributions with two momenta equal to zero. In the third part, \( \hat{H}_{III} \), only one \( k_i \) vanishes whereas in the last all are different from zero:

\[
\begin{align*}
\hat{H}_I &= \frac{1}{2} \hat{U}(\vec{0}) \hat{a}^\dagger_0 \hat{a}_0 + \sum_k \frac{\hbar^2 k^2}{2m} \hat{a}^\dagger_k \hat{a}_k \\
\hat{H}_{II} &= \frac{1}{2} \sum_{k \neq \vec{0}} \left[ \hat{U}(\vec{k}) \hat{a}^\dagger_{k \vec{0}} \hat{a}_{-k \vec{0}} + \hat{U}(\vec{k}) \hat{a}^\dagger_{-k \vec{0}} \hat{a}_{k \vec{0}} + \hat{U}(\vec{k}) \hat{a}^\dagger_{k \vec{0}} \hat{a}_{-k \vec{0}} \right] \\
\hat{H}_{IV} &= \frac{1}{2} \sum_{\{k_i \neq \vec{0}\}} \langle k_1 k_2 | U | \hat{k}_3 \hat{k}_4 \rangle \hat{a}^\dagger_{k_1} \hat{a}_{k_2} \hat{a}_{k_3} \hat{a}_{k_4}.
\end{align*}
\]

We have

\[
\langle \psi_0 | \hat{H}_{III} | \psi_0 \rangle = 0
\]

as the total number of creation and annihilation operators for \( \vec{k} \neq \vec{0} \) is odd. Thus, the precise equation for \( \hat{H}_{III} \) is not relevant here.

All sums are expanded in powers of \( n \) with the following procedure: if \( \left( \frac{d}{dn} \right)^r \sum \ldots \) is finite and the next derivative with respect to \( n \) diverges, the expression \( \left( \frac{d}{dn} \right)^r \sum \ldots \) is evaluated. In the thermodynamic limit, the resulting integrals can be evaluated for low densities after the substitution \( \hbar^2 k^2 = 2mN \hat{U}(\vec{0}) y^2 \). To calculate the depletion \( N - N_0 \), we also apply the fact that a complete set of eigenfunctions was used to obtain the second quantised Hamiltonian in the form \( \hat{H} \). Thus, the conservation of the total number of particles \( N \) can be expressed as

\[
N = \langle \psi_0 | \hat{a}^\dagger_{\vec{0}} \hat{a}_{\vec{0}} | \psi_0 \rangle + \sum_{\vec{k} \neq \vec{0}} \langle \psi_0 | \hat{a}^\dagger_k \hat{a}_k | \psi_0 \rangle.
\]

Because of \( \hat{a}_{\vec{0}} = \hat{b}_{\vec{0}} \) and \( \langle \psi_0 | \hat{b}^\dagger_{\vec{0}} \hat{b}_{\vec{0}} | \psi_0 \rangle = N_0 \), this leads to

\[
N - N_0 = \sum_{\vec{k} \neq \vec{0}} n^2_k = N \frac{8}{3\sqrt{\pi}} \sqrt{n a_0^3}.
\]

The fact that in this formula for the depletion, \( a_0 \) rather than \( a \), appears is due to the approximate ground state wave-function used at this point.

Even for the anomalous condensate fluctuations of a homogeneous Bose gas we can replace \( \langle \psi_0 | \hat{a}^\dagger_0 \hat{a}_0 | \psi_0 \rangle / \sqrt{N} \) by \( (\psi_0 | \hat{a}_{\vec{0}}^2 | \psi_0 \rangle)^2 / N \) for large \( N \). We get including the order \( \mathcal{O} \left( n \sqrt{n a_0^3} \right) \)

\[
\frac{\langle \psi_0 | \hat{H}_I | \psi_0 \rangle}{N} = \frac{2\pi \hbar^2 a_0}{m} - \frac{2\pi \hbar^2 a_1}{m} - \frac{2\pi \hbar^2 a_0}{m} n \frac{16}{3\sqrt{\pi}} \sqrt{n a_0^3} + \frac{2\pi \hbar^2 a_0}{m} n \frac{64}{5\sqrt{\pi}} \sqrt{n a_0^3} n \]

\[
\frac{\langle \psi_0 | \hat{H}_{II} | \psi_0 \rangle}{N} = \frac{2\pi \hbar^2 a_0}{m} (\frac{16}{\sqrt{\pi}} - \frac{2\pi \hbar^2 a_1}{m} n \frac{16}{3\sqrt{\pi}} + \frac{2\pi \hbar^2 a_0}{m} n \frac{32}{3\sqrt{\pi}}) \sqrt{n a_0^3} n.
\]

The only non-zero contributions to \( \frac{1}{N} \langle \psi_0 | \hat{H}_{IV} | \psi_0 \rangle \) come from expressions where at least two \( k_i \) have the same modulus. We find:

\[
\frac{\langle \psi_0 | \hat{H}_{IV} | \psi_0 \rangle}{N} = \frac{1}{N} \sum_{\vec{k}, \vec{k} \neq \vec{0}} \frac{1}{2} v_k v_{\vec{k}} v_{\vec{k}} v_{\vec{k}} \hat{U}(\vec{k} - \vec{k}') = \frac{2\pi \hbar^2}{m} a_{2n} + \frac{16}{\sqrt{\pi}} \frac{2\pi \hbar^2}{m} a_{2n} n \frac{128}{15\sqrt{\pi}} \sqrt{n a_0^3} + \frac{2\pi \hbar^2 a_1}{m} n \frac{32}{3\sqrt{\pi}} \sqrt{n a_0^3},
\]

as the sums \( \frac{1}{N} \sum_{\vec{k}, \vec{k} \neq \vec{0}} \frac{1}{2} v_k v_{\vec{k}} v_{\vec{k}} v_{\vec{k}} \hat{U}(\vec{k}) \) and \( \frac{1}{N} \sum_{\vec{k}, \vec{k} \neq \vec{0}} \frac{1}{2} v_k v_{\vec{k}} v_{\vec{k}} v_{\vec{k}} \hat{U}(\vec{k} - \vec{k}') \) are of order \( n^2 a_0^3 \). Thus, including orders of \( \mathcal{O} \left( n \sqrt{n a_0^3} \right) \) we get the upper bound \( \langle \psi_0 | \hat{H} | \psi_0 \rangle / N / \sqrt{N} \):

\[
eq c_0 \leq \frac{2\pi \hbar^2 (a_0 + a_1 + a_2)}{m} n + \frac{2\pi \hbar^2 a_0}{m} n \frac{128}{15\sqrt{\pi}} \sqrt{n a_0^3} + \frac{2\pi \hbar^2 a_1}{m} n \frac{32}{3\sqrt{\pi}} \sqrt{n a_0^3},
\]
which lies above the Lee-Yang formula (for non-negative potentials, the Born series is an alternating series with \( a \leq a_0 + a_1 + a_2 \)). Had we chosen a wave-function with \( \langle \psi_0 | \hat{b} \hat{b}^\dagger | \psi_0 \rangle = 0 \), the first three terms in eq. \( \text{(33)} \) would then read \( 2 \pi \hbar^2 (a_0 + a_1 + a_2) n/m \) which because of \( a_1 < 0 \) lies above our bound. For physically reasonable densities of up to \( 128(na_1^3)^{1/3} / (15 \sqrt{\pi}) \approx 0.03 \) [10] and a potential such that the Born series converges fast enough, we have \( (a_0 + a_1 + a_2 - a) / a \ll 11.3(na_1^3)^{1/3} \approx 0.4 \). Thus, with respect to the exponent of the next to leading term, our bound constitutes a significant improvement of the upper bound [10].

IV. GROUND STATE ENERGY BEYOND THE BOGOLIUBOV APPROXIMATION

We note that the Bogoliubov transformation exactly diagonalises the Hamiltonian \( \hat{H}_0 \simeq \hat{H}_1 + \hat{H}_2 \) (see e.g. ref. [19]):

\[
\hat{H}_0 = \frac{N}{2} \hat{U}(0) N + \sum \frac{\hbar^2 k^2}{2m} \hat{a}_k \hat{a}_k + \frac{N}{2} \sum_{k \neq \bar{0}} \left\{ \hat{U}(\bar{k}) (\hat{a}_{\bar{k}}^\dagger \hat{a}_{-\bar{k}} + \hat{a}_{\bar{k}} \hat{a}_{-\bar{k}}) + 2 \hat{U}(\bar{k}) \hat{a}_{\bar{k}}^\dagger \hat{a}_{\bar{k}} \right\}
\]

\[
= Ne_0^{(0)} + \sum_{k \neq \bar{0}} \varepsilon(\bar{k}) \hat{b}_k^\dagger \hat{b}_k\tag{34}
\]

with quasi-particle energies \( \varepsilon(\bar{k}) \) given by eq. (20) and a ground state energy per particle \( e_0^{(0)} \) given by the Brueckner-Sawada-formula [1]. We now employ \( \hat{H}_0 \) as the starting point for perturbation theory,

\[
\hat{H} = \hat{H}_0 + (\hat{H} - \hat{H}_0) \tag{35}
\]

and again assume an inter-particle interaction potential such that the Born series rapidly converges. Then already the first order correction to the ground state energy (cf. eq. \( \text{(33)} \))

\[
e_0^{(1)} = \frac{1}{N} \langle \psi_0 | \hat{H} - \hat{H}_0 | \psi_0 \rangle = \frac{2 \pi \hbar^2 a_2}{m} n + \frac{2 \pi \hbar^2 a_1}{m} \frac{32}{3 \sqrt{\pi}} n \sqrt{na_0^3}
\]

is small.

Analogously, it can be shown that in second order perturbation theory we have

\[
e_0^{(2)} = \frac{2 \pi \hbar^2 a_3}{m} n + O \left( n \sqrt{na_0^3} \right) \tag{36}
\]

as the leading order term. Thus, the terms of the Born series which are missing in eq. 4 are recovered, for low densities, by Rayleigh-Schrödinger perturbation theory, i.e. they stem from contributions to the Hamiltonian which are neglected in the usual Bogoliubov approximation. Analogously, the ground state wave function can be modified to obtain an improved upper bound.

V. CONCLUSION

Using a quantum-mechanical variational ansatz, we have derived an upper bound \( \text{(33)} \) on the ground state energy which lies below the bound [10] for physically reasonable densities if the Born series converges fast enough. We have shown how our approach could in principle be used to further improve this bound. However, more important than an ever improved upper bound is the fact that there are strong indications that a perturbative treatment of the ground state energy starting with the decomposition \( \text{(35)} \), actually converges towards the Lee-Yang formula [5] (if both the Born series and \( a^{3/2} = a_0^{3/2} (1 + a_1/a_0 + a_2/a_0 + \ldots)^{3/2} = a_0^{3/2} (1 + \frac{3}{2} a_1/a_0 + \ldots) \) converge).

Acknowledgments

We would like to thank M. Holthaus for his continuous support.

[1] Lieb E. H. and Yngvason J., Phys. Rev. Lett. 80, 2504 (1998).
[2] Anderson M. H., Ensher J. R., Matthews M. R., Wiemann C. E. and Cornell E. A., Science 269, 198 (1995).
[3] Davis K. B., Mewes M. O., Andrews M. R., van Druten N. J., Durfee D. S., Kurn D. M. and Ketterle W., Phys. Rev. Lett. 75, 3969 (1995).
[4] Lee T. D. and Yang C. N., Phys. Rev. 105, 1119 (1957).
[5] Lee T. D., Huang K. and Yang C. N., Phys. Rev. 106, 1135 (1957).
[6] Brueckner K. A. and Sawada K., Phys. Rev. 106, 1117 (1957).
[7] Dyson F. J., Phys. Rev. 106, 20 (1957).
[8] Hugenholtz N. M. and Pines D., Phys. Rev. 116, 489 (1959).
[9] Lieb E. H., Phys. Rev. 130, 2518 (1963).
[10] Bogoliubov N. N., J. Phys. (USSR) 11, 23 (1947).
[11] Cherny A. Yu. and Shanenko A. A., Phys. Rev. E 62, 1646 (2000).
[12] Braaten E. and Nieto A., Eur. Phys. J. B 11, 143 (1999).
[13] Braaten E., Hammer H. W. and Hermans S., Phys. Rev. A 63, 063609 (2001).
[14] Leggett A. J., New J. Phys. 5, 103 (2003).
[15] Landau L. D. and Lifshitz E. M., Course of theoretical Physics, Vol. 3, Butterworth-Heinemann, Oxford, (2000).
[16] Leggett A. J., Rev. Mod. Phys. 73, 307 (2001).
[17] Lifshitz E. M. and Pitaevskii L. P., Landau and Lifshitz — Course of Theoretical Physics, Vol. 9: Statistical Physics, Part 2, Butterworth-Heinemann, Oxford, (2002).
[18] Lieb E. H., Seiringer R., Solovej J. P. and Yngvason J., Current Developments in Mathematics, 2001, International Press, Cambridge, 131 (2002).
[19] Weiss C., Block M., Boers D., Eckardt A. and Holthaus M., Z. Naturforsch. 59a, 1 (2004).
[20] Cohen-Tannoudji C., Diu B. and Lalöé F., Quantum Mechanics Volume two, Wiley, New York, (1977).
[21] Huang K. and Yang C. N., Phys. Rev. 105, 767 (1957).
[22] Weiss C. and Wilkens M., Opt. Express 1, 272 (1997).
[23] Giorgini S., Pitaevskii, L. P. and Stringari, S., Phys. Rev. Lett. 80, 5040 (1998).
[24] Meier F. and Zwerger W., Phys. Rev. A 60, 5133 (1999).