Abstract: The topological index of graph has a wide range of applications in theoretical chemistry, network design, data transmission, etc. In fuzzy graph settings, these topological indices have completely different definitions and connotations. In this work, we define new Wiener index and connectivity index for bipolar fuzzy incidence graphs, and obtain the characteristics of these indices by means of the definition of fuzzy membership functions. Furthermore, the interrelationship between Wiener index and connectivity index is considered.

Keywords: fuzzy graph, topology index, bipolar fuzzy incidence graph

1 Introduction

Compared with regular data, such as images, other tools are often needed to describe irregular data. Among them, the graph model is an effective tool for describing irregular data, such as human skeleton, molecular structure, meteorological data, traffic flow data, etc., all of which can be described and analyzed using graphic models. Therefore, topological indices are defined on the graph to describe the characteristics of the graph. For example, the topological indices defined on molecular structure are often used to describe the physical and chemical characteristics of the compound, such as PI index, Wiener index, Szeged index, and so on (Ghorbani et al. [1] and Jafarzadeh and Iranmanesh [2]).

Fuzzy theory is used to describe things and data that have fuzzy relationships and it is introduced by Zhang [3] into bipolar fuzzy setting, to record the bipolar behavior of the object. Since the topological index is widely studied in various settings [4–11], it is natural to study the general characteristics of important topological indices in special bipolar fuzzy graph (BFG) setting.

In fact, BFGs have been widely used in the field of chemistry. The following example illustrates the application of fuzzy graph theory in compound storage management.

Example 1. Chemical management is a very meticulous task, since many chemicals are incompatible with each other and they will react when mixed. In this case, the recyclables can be dangerous, and must be taken care when trying to mix or store these chemicals. This example uses a fuzzy graph to describe the fuzzy characteristics of the compounds that are compatible with each other. Each vertex represents a type of compound. If there is an edge between two vertices, it means that the corresponding kinds of compounds are relatively safe to mix. If there is no edge between adjacent vertices, it means that the corresponding types of compounds are relatively dangerous to mix. Figure 1 shows a fuzzy graph of some common kinds of compounds mixed with each other. The membership function of each vertex indicates how the same type of compounds react positively or negatively when mixed.

For example, the degree of membership (0.95, −0.05) at the vertex of “Inorganic acids” means that these chemicals are approximately 95% compatible, and approximately 5% have a chance to explode. Here the negative degree of membership describes the incompatibility of the group of chemicals. In Figure 1, due to space constraint, we did not give the values of membership function of the edges. It is just an example to illustrate the application of BFGs in the field of chemistry.

In chemical graph theory, we use vertices to represent atoms, and the edges between vertices are used to denote the chemical bonds between the atoms. In the
standard model, the positions of all the vertices and edges are the same, and only the topological structure of the molecular graph can distinguish the topological characteristics of each vertex. Obviously, this model cannot represent chemical properties, for example, the kind of atom, the kind of chemical bond the edge represents, the kind of bonding force between the chemical bond and the atom, and so on. All of these need to introduce a certain measure in the molecular graph, and some developments have already been made in the literature [12]. In this direction, we now introduce fuzzy graph theory to solve the above-mentioned problems to a certain extent. In this work, we mainly study the Wiener index and connectivity index of particular BFGs, hence new definitions are introduced and several characteristics are determined.

2 Definitions in BFG setting

Let \( V \) be a set with at least one element. The set \( A = \{(v, \mu^P_A(v), \mu^N_A(v)) : v \in V\} \) is a bipolar fuzzy set in \( V \), if two maps satisfy \( \mu^P_A : V \rightarrow [0, 1] \) and \( \mu^N_A : V \rightarrow [-1, 0] \).

A mapping \( B = (\mu^P_B, \mu^N_B) : V \times V \rightarrow [0, 1] \times [-1, 0] \) is a bipolar fuzzy relation on \( V \), if \( \mu^P_B(v, v') \in [0, 1] \), \( \mu^N_B(v, v') \in [-1, 0] \), \( \mu^P_B(v, v') \leq \min(\mu^P_A(v), \mu^P_A(v')) \), and \( \mu^N_B(v, v') \geq \max(\mu^N_A(v), \mu^N_A(v')) \) for any \( (v, v') \in V \times V \).

Fixed the set \( V, \sim \) is an equivalence relation on \( V \times V \), which is defined as \((v_1, v_2) \sim (v'_1, v'_2)\), if and only if \((v_1, v_2) = (v'_1, v'_2)\) or \((v_1, v_2) = (v'_2, v'_1)\). The quotient set yielded here is denoted by \( \mathcal{V}/\sim \), and we denote \( vv' \) or \( v' v \) by the equivalent class which contains the element \((v, v')\).
Yang et al. [13] proposed the generalization of BFGs as follows. If 
\( A = \{(v, \mu^A_k(v), \mu^N_k(v)) : v \in V\} \) is a bipolar fuzzy set on an underlying set \( V \) and \( B = (\mu^B_k, \mu^N_k) \) is a bipolar fuzzy set in \( \mathbb{V}^2 \), where \( \mu^B_0(v, v') \leq \min(\mu^B_0(v), \mu^B_0(v')) \) and \( \mu^B_0(v, v') \geq \max(\mu^N_0(v), \mu^N_0(v')) \) for any \( (v, v') \in \mathbb{V}^2 \) and \( \mu^B_0(v, v') = \mu^N_0(v, v') = 0 \) for any \( (v, v') \in \mathbb{V}^2 - E \), then \( G = (V, A, B) \) is a BFG of the graph \( G = (V, E) \).

Matthew et al. [14] further gave the following concepts: a BFG \( G' = (V, A', B') \) is a partial bipolar fuzzy subgraph of \( G \), if \( \mu^B_0(v) \leq \mu^B_0(v) \) and \( \mu^N_0(v) \geq \mu^N_0(v) \) for any \( v \in V \) and \( \mu^B_0(v, v') \leq \mu^B_0(v, v') \) and \( \mu^B_0(v, v') \geq \mu^B_0(v, v') \) for any \( (v, v') \in E(G) \); a BFG \( G' = (V', A', B') \) is a bipolar fuzzy subgraph of \( G \), if \( \mu^B_0(v) = \mu^B_0(v) \) and \( \mu^N_0(v) = \mu^N_0(v) \) for any \( v \in V' \) and \( \mu^B_0(v, v') = \mu^B_0(v, v') \) and \( \mu^B_0(v, v') = \mu^B_0(v, v') \) for any \( (v, v') \in E(G') \).

The following concepts are introduced by Akram [15] and Akram and Karunambigai [16]. Let \( G = (V, A, B) \) be a BFG, \( P = v_0v_1 \ldots v_k \) is a path with length \( k \) from \( v = v_0 \) to \( y = v_k \) in \( G \) and is a sequence of different vertices, where \( \mu^B_0(v_{i-1}, v_i) > 0, \mu^N_0(v_{i-1}, v_i) < 0 \) for \( i \in (1, \ldots, k) \). Then (throughout the article, \( \wedge \) means minimum and \( \vee \) implies maximum),

\[
(\mu^B_0(x, y))^k = \sup(\mu^B_0(x, v_1) \wedge \mu^B_0(v_1, v_2) \wedge \cdots \wedge \mu^B_0(v_{k-1}, y))
\]

and

\[
(\mu^N_0(x, y))^k = \inf(\mu^N_0(x, v_1) \vee \mu^N_0(v_1, v_2) \vee \cdots \vee \mu^N_0(v_{k-1}, y)).
\]

Set

\[
\text{CONN}^B_0(x, y) = \sup_{k \in \mathbb{N}}((\mu^B_0(x, y))^k),
\]

\[
\text{CONN}^N_0(x, y) = \inf_{k \in \mathbb{N}}((\mu^N_0(x, y))^k).
\]

Then, the strength of connectedness between arbitrary \( x, y \in V(G) \) in BFG \( G \) is denoted by

\[
\text{CONN}^B_0(x, y) = (\text{CONN}^B_0(x, y), \text{CONN}^N_0(x, y)) = ((\mu^B_0(x, y))^\infty, (\mu^N_0(x, y))^\infty).
\]

If \( \mu^B_0(v, v') > 0 \) and \( \mu^N_0(v, v') < 0 \) for any \( (v, v') \in G \), then the BFG \( G \) is connected. If \( \mu^B_0(v, v') = \min(\mu^B_0(v), \mu^B_0(v')) \) and \( \mu^B_0(v, v') = \max(\mu^N_0(v), \mu^N_0(v')) \) for any \( v, v' \in V \), then the BFG \( G \) is called a complete BFG.

The following concepts were introduced by Karunambigai et al. [17], Akram and Farooq [18], Singh and Kumar [19], and Yang et al. [20].

- For a BFG, if \( \mu^N_0(v, v') \geq (\mu^N_0(v, v'))^\infty \) and \( \mu^N_0(v, v') \geq (\mu^N_0(v, v'))^\infty \), then arc \((v, v')\) is called a strong arc of \( G \).

A path \( v - v' \) is a strong path, if all arcs on the path are strong. The \((\alpha, \beta)\)-cut of a BFG \( G \) is \( G^{(\alpha, \beta)} = (V', A', B') \) with \( \mu^B_0(v) \geq \alpha \) and \( \mu^N_0(v) \leq \beta \) for all \( v \in V(G^{(\alpha, \beta)}) \) and \( \mu^B_0(v, v') \geq \alpha \) and \( \mu^N_0(v, v') \leq \beta \) for all edges \( vv' \in E(G^{(\alpha, \beta)}) \), where \( \alpha, \beta \in [0, 1] \).

- An edge \((v, v')\) is a bipolar fuzzy bridge, if \( (\mu^B_0(v, v'))^\infty < (\mu^N_0(v, v'))^\infty \) and \( (\mu^N_0(v, v'))^\infty > (\mu^B_0(v, v'))^\infty \) (i.e., an edge \( vv' \) is a bipolar fuzzy bridge if the deletion of \( vv' \) reduces the positive connectedness and increases the negative connectedness between certain pairs of vertices in \( G \)).

- Let \( G = (V, A, B) \) and \( G' = (V', A', B') \) be two BFGs of the graph \( G = (V, E) \) and \( G' = (V', E') \). If there is a bijective mapping \( f : V \to V' \) with \( \mu^B_0(v) = \mu^B_0(f(v)) \) and \( \mu^N_0(v) = \mu^N_0(f(v)) \) for all \( v \in V(G) \) and \( \mu^B_0(v, v') = \mu^B_0(f(v), f(v')) \) and \( \mu^N_0(v, v') = \mu^N_0(f(v), f(v')) \) for all \((v, v') \in \mathbb{V}^2 \), then the mapping \( f : V \to V' \) is called an isomorphism. In this case, \( G \cong G' \).

The connectivity index of BFG is defined by Poulik and Ghorai [21], which is formulated by

\[
\text{Cl}^B_0(G) = (\text{Cl}^B_0(G), \text{Cl}^N_0(G))
\]

\[
= \left\{ \sum_{x,y \in V} \mu^B_0(x) \mu^B_0(y) \text{CONN}^B_0(x, y), \right. \times \sum_{x,y \in V} \mu^N_0(x) \mu^N_0(y) \text{CONN}^N_0(x, y) \right\},
\]

where \( \text{Cl}^B_0(G) \) and \( \text{Cl}^N_0(G) \) are positive connectivity index and negative connectivity index of \( G \), respectively. Poulik and Ghorai [22, 23] defined the geodesic distance and Wiener index as follows: the bipolar fuzzy geodesic distance from \( v \) to \( v' \) is the any smallest strong path between them; the Wiener index of a bipolar fuzzy graph is denoted by

\[
\text{W}^B_0(G) = (\text{W}^B_0(G), \text{W}^N_0(G))
\]

\[
= \left\{ \sum_{x,y \in V} \mu^B_0(x) \mu^B_0(y) d^B_0(x, y), \times \sum_{x,y \in V} \mu^N_0(x) \mu^N_0(y) d^N_0(x, y) \right\},
\]

where \( \text{W}^B_0(G) \) and \( \text{W}^N_0(G) \) are the positive and negative Wiener indices of \( G \) and \( d^B_0(x, y) \) and \( d^N_0(x, y) \) are the minimum and maximum values of the sums of all positive and negative membership values of all edges on the strong geodesics from \( x \) to \( y \).
3 Bipolar fuzzy incidence graph

The aim of this Section is to introduce the concepts of bipolar fuzzy incidence graph, and the main notations are followed from Fang et al. [24], but extended to bipolar fuzzy setting. An incidence graph of graph \( G = (V, E) \) is denoted by \( G = (V, E, I) \) with \( I \subseteq V \times E \). If \( (v, v') \) in the incidence graph, then \( (v, v') \) is an incidence pair or shortly pair. A sequence \( v_0, (v_0, v_1), v_1, . . . , v_{n-1}, (v_{n-1}, v_n), v_n, . . . , v_0 \), \( v_0 \) is a walk in the incidence graph. If \( v_0 = v_n \), then it is said to be closed. A walk is a path, if all the vertices are different, and an incidence graph is connected, if all the pairs of vertices are connected by a path.

**Definition 1.** Let \( G = (V, E) \) be a graph, \( \eta^p : V \rightarrow [0, 1], \eta^N : V \rightarrow [-1, 0], \theta^P : E \rightarrow [0, 1], \theta^N : E \rightarrow [-1, 0], \psi^P : V \times E \rightarrow [0, 1], \psi^N : V \times E \rightarrow [-1, 0] \). Let \( \Psi^P(v, e) \leq \eta^P(v) + \theta^P(e) \) and \( \Psi^N(v, e) \geq \eta^N(v) + \theta^N(e) \) for any \( v \in V \) and \( e \in E \), then \( (\Psi^P, \Psi^N) \) is called a bipolar fuzzy incidence of \( G \).

In the fuzzy molecular graph, the relationship between atoms and chemical bonds can be expressed to describe the stability of the molecular structure. The bipolar fuzzy membership function can describe the relative properties of atoms and edges in molecular structures from two aspects. In addition, because in each edge of the incidence molecular graph, the chemical bond needs to be associated with the vertices on the two sides to perform the relationship calculations and then the relationship between the two atoms and the chemical bond can be described separately.

**Definition 2.** Let \( G = (V, E) \) be a graph, and \((\eta^P, \eta^N, \theta^P, \theta^N)\) be a bipolar fuzzy subgraph of \( G \). If \((\Psi^P, \Psi^N)\) is a bipolar fuzzy incidence of \( G \), then \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \psi^P, \psi^N) \) is called a bipolar fuzzy incidence graph of \( G \).

**Remark 1.** Here we explain that the bipolar fuzzy incidence graph is different from directed graph with double weighted values [25–27]. First, the meaning of the expression is completely different. The membership function expresses a kind of uncertainty, which has a fundamentally different motivation from the edge weight function. Second, in the bipolar fuzzy incidence graph, the fuzzy value of the edge is determined by \( \theta^P \) and \( \theta^N \), while \( \psi^P \) and \( \psi^N \) denote the fuzzy relationship between the vertex and the edge from two different directions. Finally, fuzzy graphs have strict restrictions on the values of the membership functions of vertices and edges.

**Definition 3.** Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \psi^P, \psi^N) \) be a bipolar fuzzy incidence graph. Then, \( H = (\kappa^P, \kappa^N, \phi^P, \phi^N, \Omega^P, \Omega^N) \) is a bipolar fuzzy incidence subgraph of \( G \) if \( \kappa^P(v) \leq \eta^P(v) \) and \( \kappa^N(v) \geq \eta^N(v) \) for any \( v \in V \), \( \phi^P(e) \leq \theta^P(e) \) and \( \phi^N(e) \geq \theta^N(e) \) for any \( e \in E \), and \( \Omega^P(v, e) \leq \psi^P(v, e) \) and \( \Omega^N(v, e) \geq \psi^N(v, e) \) for any \( (v, e) \in E \).

**Definition 4.** Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \psi^P, \psi^N) \) be a bipolar fuzzy incidence graph. Set \( \eta^*, \theta^*, \text{ and } \Psi^* \) be the vertex set, edge set, and incidence set (set of \( V \times E \)) of \( G \) (these three sets represent the specific component elements of bipolar fuzzy incidence graph \( G \)), respectively. A fuzzy incidence path \( \lambda \) from \( v \) to \( v' \) (\( v' \in \eta^* \cup \theta^* \)) is a sequence of elements \( \eta^*, \theta^*, \text{ and } \Psi^* \). The minimum value of \( \psi^P(v, v') \) is called the positive incidence strength and the maximum value of \( \psi^N(v, v') \) is called the negative incidence strength, where \( (v, v') \in \lambda \).

Note that in bipolar fuzzy incidence graph, the incidence paths can take different forms.

**Definition 5.** Let \( G \) be a bipolar fuzzy incidence graph. An incidence pair \((v, v')\) is strong if \( \psi^P(v, v') \geq \text{ICONN}_{G}(v, v') \) and \( \psi^N(v, v') \leq \text{ICONN}_{G}(v, v') \), where \( \text{ICONN}_{G}(v, v') \) is the maximum positive incidence strength of \( a - ab \) and \( \text{ICONN}_{G}(v, v') \) is the minimum negative incidence strength of \( a - ab \). If \( \psi^P(v, v') > \text{ICONN}_{G}(v, v') \) and \( \psi^N(v, v') < \text{ICONN}_{G}(v, v') \), then the pair \((v, v')\) is called \( \alpha \)-strong. If \( \psi^P(v, v') = \text{ICONN}_{G}(v, v') \) and \( \psi^N(v, v') = \text{ICONN}_{G}(v, v') \), then the pair \((v, v')\) is called \( \beta \)-strong.

**Remark 2.** Here ICONN is different from CONN. The former is defined in bipolar incidence fuzzy graph and latter is defined in BFG.

**Definition 6.** Let \( G \) be a bipolar fuzzy incidence graph. If all pairs of \( \lambda \) are strong, then an incidence path \( \lambda \) in \( G \) is called the strong incidence path.

**Definition 7.** Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \psi^P, \psi^N) \) be a bipolar fuzzy incidence graph. Then, \( H = (\kappa^P, \kappa^N, \phi^P, \phi^N, \Omega^P, \Omega^N) \) is a subgraph of \( G \) if \( \kappa^P(v) = \eta^P(v) \) and \( \kappa^N(v) = \eta^N(v) \) for any \( v \in \kappa^* \), \( \phi^P(e) = \theta^P(e) \), \( \phi^N(e) = \theta^N(e) \) for any \( e \in \phi^* \), and
\[\Omega^P(v, e) = \Psi^P(v, e) \quad \text{and} \quad \Omega^N(v, e) = \Psi^N(v, e) \quad \text{for any} \quad (v, e) \in \Omega^*.\]

**Proposition 1.** Let \(G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N)\) be a bipolar fuzzy incidence graph and \(H = (\kappa^P, \kappa^N, \phi^P, \phi^N, \Omega^P, \Omega^N)\) be a bipolar fuzzy incidence subgraph of \(G\). Then, we have
\[
\text{CONN}^P_{H}(v, v') \leq \text{CONN}^P_{G}(v, v'),
\]
and
\[
\text{CONN}^N_{H}(v, v') \geq \text{CONN}^N_{G}(v, v').
\]

**Definition 8.** A bipolar fuzzy incidence graph \(G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N)\) is said to be complete if \(\Psi^P(v, v') = \eta^P(v) \land \theta^P(v')\) and \(\Psi^N(v, v') = \eta^N(v) \lor \theta^N(v')\) for any \((v, v') \in \Psi^*\).

**Definition 9.** Let \(G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N)\) be a bipolar fuzzy incidence graph. The connectivity index of \(G\) is formulated by
\[
\text{Cl}_{\text{BFI}}(G) = (\text{Cl}^P_{\text{BFI}}(G), \text{Cl}^N_{\text{BFI}}(G)) = \left(\sum_{x,y \in \eta^*} \eta^P(x) \eta^P(y) \text{CONN}^P_G(x, y), \sum_{x,y \in \eta^*} \eta^N(x) \eta^N(y) \text{CONN}^N_G(x, y)\right),
\]
where \(\text{Cl}^P_{\text{BFI}}(G)\) and \(\text{Cl}^N_{\text{BFI}}(G)\) are positive and negative connectivity index of \(G\). \(\text{CONN}^P_G(x, y)\) and \(\text{CONN}^N_G(x, y)\) are the maximum value of positive incidence strength for all the possible incidence paths between \(x\) and \(y\), and the minimum value of negative incidence strength for all the possible incidence paths between \(x\) and \(y\).

For convenience, in the calculations below, we take \(\eta^P(v) = 1\) and \(\eta^N(v) = -1\) for all \(v \in \eta^*\).

**Example 2.** Considering a bipolar fuzzy incidence graph as depicted in Figure 2, we confirm that \(\eta^P \equiv \{v_1, v_2, v_3, v_4\}\), \(\theta^P(v_1v_2) = 0.8\), \(\theta^N(v_1v_2) = -0.6\), \(\theta^P(v_1v_3) = 0.5\), \(\theta^N(v_1v_3) = -0.7\), \(\theta^P(v_2v_3) = 0.7\), \(\theta^N(v_2v_3) = -0.6\), \(\theta^P(v_3v_4) = 0.9\), \(\theta^N(v_3v_4) = -0.8\), \(\Psi^P(v_1, v_2v_3) = 0.8\), \(\Psi^N(v_1, v_2v_3) = -0.5\), \(\Psi^P(v_2, v_2v_4) = 0.4\), \(\Psi^N(v_2, v_2v_4) = -0.3\), \(\Psi^P(v_3, v_3v_4) = 0.3\), \(\Psi^N(v_3, v_3v_4) = -0.2\), \(\Psi^P(v_3, v_3v_4) = 0.6\), \(\Psi^N(v_3, v_3v_4) = -0.5\), \(\Psi^P(v_4, v_4v_3) = 0.5\), \(\Psi^N(v_4, v_4v_3) = -0.3\), \(\Psi^P(v_4, v_4v_3) = 0.3\), \(\Psi^N(v_4, v_4v_3) = -0.7\), \(\Psi^P(v_4, v_4v_3) = 0.4\), \(\Psi^N(v_4, v_4v_3) = -0.5\), \(\Psi^P(v_4, v_4v_3) = 0.2\), \(\Psi^N(v_4, v_4v_3) = -0.7\), and \(\text{Cl}^P_{\text{BFI}}(G) = (1.6, -2)\).

The following proposition shows the relationship of connectivity index between bipolar fuzzy incidence graph and its subgraph.

**Proposition 2.** If \(H = (\kappa^P, \kappa^N, \phi^P, \phi^N, \Omega^P, \Omega^N)\) is a bipolar fuzzy incidence subgraph of \(G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N)\) as defined in Definition 3, then \(\text{Cl}^P_{\text{BFI}}(H) \leq \text{Cl}^P_{\text{BFI}}(G), \text{Cl}^N_{\text{BFI}}(H) \geq \text{Cl}^N_{\text{BFI}}(G)\).

**Proof.** For any \(v, v' \in \kappa^*\), we get \(\kappa^P(v) \leq \eta^P(v), \kappa^N(v) \geq \eta^N(v), \kappa^P(v') \leq \eta^P(v'), \kappa^N(v') \geq \eta^N(v')\) by means of Definition 3. Moreover, we obtain \(\text{Cl}^P_{\text{BFI}}(v, v') \leq \text{Cl}^P_{\text{BFI}}(v', v')\) and \(\text{Cl}^N_{\text{BFI}}(v, v') \geq \text{Cl}^N_{\text{BFI}}(v', v')\). Therefore, we infer
\[
\sum_{x,y \in \eta^*} \eta^P(x) \eta^P(y) \text{CONN}_G^P(x, y) \leq \sum_{x,y \in \eta^*} \eta^P(x) \eta^P(y) \text{CONN}_H^P(x, y),
\]
\[
\sum_{x,y \in \eta^*} \eta^N(x) \eta^N(y) \text{CONN}_G^N(x, y) \geq \sum_{x,y \in \eta^*} \eta^N(x) \eta^N(y) \text{CONN}_H^N(x, y).
\]

It implies the conclusion. \(\square\)

**Proposition 3.** If \(H = (\kappa^P, \kappa^N, \phi^P, \phi^N, \Omega^P, \Omega^N)\) is a subgraph of \(G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N)\) as defined in Definition 7, then \(\text{Cl}^P_{\text{BFI}}(H) \leq \text{Cl}^P_{\text{BFI}}(G)\) and \(\text{Cl}^N_{\text{BFI}}(H) \geq \text{Cl}^N_{\text{BFI}}(G)\).

This is an obvious conclusion. There is no need to prove it here.
Proposition 4. If H is a subgraph of connected bipolar fuzzy incidence graph \( G = (\eta^p, \eta^N, \theta^p, \theta^N, \Psi^p, \Psi^N) \), where \( V(H) = V(G) - \{v\} \), then \( C_{\text{BFI}}^p(H) < C_{\text{BFI}}^p(G) \) and \( C_{\text{BFI}}^N(H) > C_{\text{BFI}}^N(G) \).

Example 3. Considering a bipolar fuzzy incidence graph as depicted in Figure 3, we confirm that \( \eta^* = \{v_1, v_2, v_3\} \), \( \theta^p(v_1v_2) = 0.8 \), \( \theta^N(v_1v_2) = -0.6 \), \( \theta^p(v_1v_3) = 0.5 \), \( \theta^N(v_1v_3) = -0.7 \), \( \theta^p(v_2v_3) = 0.7 \), \( \theta^N(v_2v_3) = -0.6 \), \( \Psi^p(v_1, v_2v_3) = 0.8 \), \( \Psi^N(v_1, v_2v_3) = -0.5 \), \( \Psi^p(v_2, v_2v_3) = 0.4 \), \( \Psi^N(v_2, v_2v_3) = -0.3 \), \( \Psi^p(v_3, v_2v_3) = 0.3 \), \( \Psi^N(v_3, v_2v_3) = -0.7 \), and \( C_{\text{BFI}}(G) = (1, -0.9) \).

Definition 10. Let \( G_1 = (\eta^p_1, \eta^N_1, \theta^p_1, \theta^N_1, \Psi^p_1, \Psi^N_1) \) and \( G_2 = (\eta^p_2, \eta^N_2, \theta^p_2, \theta^N_2, \Psi^p_2, \Psi^N_2) \) be two bipolar fuzzy incidence graphs. If there is a bijective mapping \( f : V(G_1) \to V(G_2) \) with \( \eta^p_1(v) = \eta^p_2(f(v)) \) and \( \eta^N_1(v) = \eta^N_2(f(v)) \) for all \( v \in V(G_1) \), then the mapping \( f \) is called an isomorphism and we denote \( G_1 \cong G_2 \).

Definition 11. The Wiener index of a bipolar fuzzy incidence graph \( G = (\eta^p, \eta^N, \theta^p, \theta^N, \Psi^p, \Psi^N) \) is denoted by

\[
W_{\text{BFI}}(G) = (W_{\text{BFI}}^p(G), W_{\text{BFI}}^N(G))
\]

\[
= \left( \sum_{x, y \in \eta^*} \eta^p(x)\eta^p(y)d^p(x, y), \sum_{x, y \in \eta^*} \eta^N(x)\eta^N(y)d^N(x, y) \right).
\]

4 Main results and proofs

Our first main conclusion reveals the bounds for connectivity index of bipolar fuzzy incidence graph which is manifested as follows.

Theorem 1. Let \( G = (\eta^p, \eta^N, \theta^p, \theta^N, \Psi^p, \Psi^N) \) be a bipolar fuzzy incidence graph with \( |\eta^p| = n \) and \( G_1 = (\eta^p_1, \eta^N_1, \theta^p_1, \theta^N_1, \Psi^p_1, \Psi^N_1) \) be the completion of the bipolar fuzzy incidence graph \( G \). Then, we have \( 0 \leq C_{\text{BFI}}^p(G_1) \leq C_{\text{BFI}}^p(G) \) and \( C_{\text{BFI}}^N(G_1) \leq C_{\text{BFI}}^N(G) \).

Proof. If \( n = 1 \), then \( 0 = C_{\text{BFI}}^p(G_1) = C_{\text{BFI}}^p(G) \) and \( C_{\text{BFI}}^N(G_1) = C_{\text{BFI}}^N(G) = 0 \).

Considering \( n \geq 2 \), we confirm that \( \eta^p_1(v) = \eta^p(v) \) and \( \eta^N_1(v) = \eta^N(v) \) for every \( v \in \eta^* \) and \( \theta^p_1(e) \geq \theta^p(e), \theta^N_1(e) \leq \theta^N(e) \) for every \( e \in \theta^* \). Also, we have \( \text{ICONN}^0(x, y) \leq \text{ICONN}^N_0(x, y) \) and \( \text{ICONN}^N_0(x, y) \leq \text{ICONN}^0(x, y) \) for any \( x, y \in \eta^* \). Therefore, we have \( 0 \leq C_{\text{BFI}}^p(G_1) \leq C_{\text{BFI}}^p(G) \) and \( C_{\text{BFI}}^N(G_1) \leq C_{\text{BFI}}^N(G) \).

Theorem 1 reveals that the complete bipolar fuzzy incidence graph has the largest positive connectivity index and the lowest negative connectivity index.

Example 4. The bipolar fuzzy incidence graph in Figure 4 is a complete bipolar fuzzy incidence graph. We deduce \( \eta^* = \{v_1, v_2, v_3\} \), \( \theta^p(v_1v_2) = 0.8 \), \( \theta^N(v_1v_2) = -0.6 \), \( \theta^p(v_1v_3) = 0.5 \), \( \theta^N(v_1v_3) = -0.7 \), \( \theta^p(v_2v_3) = 0.7 \), \( \theta^N(v_2v_3) = -0.6 \), \( \Psi^p(v_1, v_2v_3) = 0.8 \), \( \Psi^N(v_1, v_2v_3) = -0.5 \), \( \Psi^p(v_2, v_2v_3) = 0.4 \), \( \Psi^N(v_2, v_2v_3) = -0.3 \), \( \Psi^p(v_3, v_2v_3) = 0.3 \), \( \Psi^N(v_3, v_2v_3) = -0.7 \), \( \Psi^p(v_3, v_2v_3) = -0.6 \), \( \Psi^N(v_3, v_2v_3) = -0.7 \), \( \Psi^p(v_3, v_2v_3) = 0.5 \), \( \Psi^N(v_3, v_2v_3) = -0.7 \), and \( C_{\text{BFI}}(G) = (2.2, -1.9) \).

The next theorem shows the characteristics of vertex deleting and edge removing bipolar fuzzy incidence subgraphs with connectivity index.

Theorem 2. Let \( G = (\eta^p, \eta^N, \theta^p, \theta^N, \Psi^p, \Psi^N) \) be a bipolar fuzzy incidence graph and \( G_1 = (\eta^p_1, \eta^N_1, \theta^p_1, \theta^N_1, \Psi^p_1, \Psi^N_1) \) be the bipolar fuzzy incidence subgraph of \( G \) by removing an incidence pair \( (v, v') \in \Psi^* \). Then, \( C_{\text{BFI}}^p(G_1) < C_{\text{BFI}}^p(G) \) and \( C_{\text{BFI}}^N(G_1) > C_{\text{BFI}}^N(G) \), if and only if \( (v, v') \) is a fuzzy bridge.

Proof. Set pair \( (v, v') \) is a fuzzy bridge of \( G \). Then, in view of its definition, we yield \( \square \).
\[ \text{ICONN}^N_{(v,v')}(v,v') > \text{ICONN}^N_{(v,v')}(v,v'), \]

which implies \( \text{Cl}_{\text{BFI}}^N(G_1) < \text{Cl}_{\text{BFI}}^N(G) \) and \( \text{Cl}_{\text{BFI}}^N(G) < \text{Cl}_{\text{BFI}}^N(G_2) \).

On the contrary, we assume that \( \text{Cl}_{\text{BFI}}^N(G_1) < \text{Cl}_{\text{BFI}}^N(G) \) and \( \text{Cl}_{\text{BFI}}^N(G_1) > \text{Cl}_{\text{BFI}}^N(G_2) \). We further suppose that \((v,v')\) is not a fuzzy bridge. Then, for any pair of \((x, y)\) in \(G\), we have \( \text{ICONN}^N_{(x,y)}(v,v') \) or \( \text{ICONN}^N_{(x,y)}(v,v') \geq \text{ICONN}^N_{(x,y)}(v,v') \Rightarrow \sum_{x,y \in \eta^*} \eta^N(x)\eta^N(y) \text{ICONN}^N_{(x,y)}(x,y) \geq \sum_{x,y \in \eta^*} \eta^N(x)\eta^N(y) \text{ICONN}^N_{(x,y)}(x,y) \) or both \( \Rightarrow \text{Cl}_{\text{BFI}}^N(G) \leq \text{Cl}_{\text{BFI}}^N(G_1) \) or \( \text{Cl}_{\text{BFI}}^N(G) \leq \text{Cl}_{\text{BFI}}^N(G_2) \) or both, which lead to a contradiction. Therefore, \((v,v')\) is a fuzzy bridge.

**Theorem 3.** Let \( G_1 = (\eta^P_1, \eta^N_1, \theta^P_1, \theta^N_1, \Psi^P_1, \Psi^N_1) \) and \( G_2 = (\eta^P_2, \eta^N_2, \theta^P_2, \theta^N_2, \Psi^P_2, \Psi^N_2) \) be two isomorphic bipolar fuzzy incidence graphs. Then, \( \text{Cl}_{\text{BFI}}^N(G_1) = \text{Cl}_{\text{BFI}}^N(G_2) \), i.e., \( \text{Cl}_{\text{BFI}}^N(G_1) = \text{Cl}_{\text{BFI}}^N(G_2) \) and \( \text{Cl}_{\text{BFI}}^N(G_1) = \text{Cl}_{\text{BFI}}^N(G_2) \).

**Proof.** Let \( f \) be the bijection between two bipolar fuzzy incidence graphs as described in Definition 10. We have \( \text{ICONN}^N_{(x,y)}(f(x), f(y)) = \text{ICONN}^N_{(x,y)}(f(x), f(y)) \) for any \( x, y \in \eta^* \). Hence,

\[
\sum_{x,y \in \eta^*} \eta^N(x)\eta^N(y) \text{ICONN}^N_{(x,y)}(f(x), f(y)) = \sum_{x,y \in \eta^*} \eta^N(x)\eta^N(y) \text{ICONN}^N_{(x,y)}(f(x), f(y))
\]

Thus, we have \( \text{Cl}_{\text{BFI}}^N(G_1) = \text{Cl}_{\text{BFI}}^N(G_2) \) and \( \text{Cl}_{\text{BFI}}^N(G_1) = \text{Cl}_{\text{BFI}}^N(G_2) \).

**Definition 12.** Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N) \) be a bipolar fuzzy incidence graph with \( n \) vertices. The average connectivity index of \( G \) is given by

\[
\text{ACI}_{\text{BFI}}(G) = \left( \frac{1}{n} \right) \left( \sum_{x,y \in \eta^*} \eta^N(x)\eta^N(y) \text{ICONN}^N_{(x,y)}(f(x), f(y)) \right).
\]

**Example 5.** Considering the bipolar fuzzy incidence graph in Figure 2, we have \( \text{ACI}_{\text{BFI}}(G) = \left( \frac{4}{15}, -\frac{1}{3} \right) \). For the bipolar fuzzy incidence graph in Figure 3, we have \( \text{ACI}_{\text{BFI}}(G) = \left( \frac{1}{3}, -0.3 \right) \).

**Definition 13.** Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N) \) be a bipolar fuzzy incidence graph and \( v \in \eta^* \).

- If \( \text{ACI}_{\text{BFI}}(G - \{v\}) < \text{ACI}_{\text{BFI}}(G) \) and \( \text{ACI}_{\text{BFI}}(G - \{v\}) > \text{ACI}_{\text{BFI}}(G) \), then \( v \) is called a bipolar fuzzy incidence connectivity-reducing vertex;
- If \( \text{ACI}_{\text{BFI}}(G - \{v\}) > \text{ACI}_{\text{BFI}}(G) \) and \( \text{ACI}_{\text{BFI}}(G - \{v\}) < \text{ACI}_{\text{BFI}}(G) \), then \( v \) is called a bipolar fuzzy incidence connectivity-enhancing vertex;
- If \( \text{ACI}_{\text{BFI}}(G - \{v\}) = \text{ACI}_{\text{BFI}}(G) \) and \( \text{ACI}_{\text{BFI}}(G - \{v\}) = \text{ACI}_{\text{BFI}}(G) \), then \( v \) is called a bipolar fuzzy incidence connectivity-neutral vertex.

**Theorem 4.** Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N) \) be a bipolar fuzzy incidence graph, \( n = |\eta^*| \geq 3, v \in \eta^* \), \( a = \frac{\text{Cl}_{\text{BFI}}^P(G)}{\text{Cl}_{\text{BFI}}^P(G - \{v\})} \), and \( b = \frac{\text{Cl}_{\text{BFI}}^N(G)}{\text{Cl}_{\text{BFI}}^N(G - \{v\})} \). Then,

- \( v \) is a bipolar fuzzy incidence connectivity-enhancing vertex, if and only if \( a < \frac{n}{n - 2} \) and \( b > \frac{n}{n - 2} \);
- \( v \) is a bipolar fuzzy incidence connectivity-reducing vertex, if and only if \( a > \frac{n}{n - 2} \) and \( b < \frac{n}{n - 2} \);
- \( v \) is a bipolar fuzzy incidence connectivity-neutral vertex, if and only if \( a = \frac{n}{n - 2} \) and \( b = \frac{n}{n - 2} \).
Proof. We only prove the second part of the theorem, and the first and the third parts can be done by means of similar tricks.

Let \( v \) be a bipolar fuzzy incidence connectivity-reducing vertex. Using its definition, we infer
\[
ACI_{BFI}(G - \{v\}) < ACI_{BFI}(G) \quad \text{and} \quad ACI_{BFI}(G - \{v\}) > ACI_{BFI}(G) \\
\Rightarrow \frac{\text{cl}_{BFI}(G - \{v\})}{\binom{n-1}{2}} < \frac{\text{cl}_{BFI}(G)}{\binom{n}{2}} \quad \text{and} \quad \frac{\text{cl}_{BFI}(G - \{v\})}{\binom{n-1}{2}} > \frac{\text{cl}_{BFI}(G)}{\binom{n}{2}} \\
\Rightarrow \frac{\frac{1}{2}}{\binom{n-1}{2}} < \frac{\frac{1}{2}}{\binom{n}{2}} \quad \text{and} \quad \frac{\frac{1}{2}}{\binom{n-1}{2}} > \frac{1}{\binom{n}{2}} \\
\Rightarrow a > \frac{n}{n-2} \quad \text{and} \quad b < \frac{n}{n-2}. 
\]

On the contrary, let \( a > \frac{n}{n-2} \) and \( b < \frac{n}{n-2} \).
\[
\Rightarrow \frac{\text{cl}_{BFI}(G - \{v\})}{\binom{n-1}{2}} > \frac{\text{cl}_{BFI}(G)}{\binom{n}{2}} \quad \text{and} \quad \frac{\text{cl}_{BFI}(G - \{v\})}{\binom{n-1}{2}} < \frac{\text{cl}_{BFI}(G)}{\binom{n}{2}} \\
\Rightarrow \frac{\frac{1}{2}}{\binom{n-1}{2}} < \frac{\frac{1}{2}}{\binom{n}{2}} \quad \text{and} \quad \frac{\frac{1}{2}}{\binom{n-1}{2}} > \frac{1}{\binom{n}{2}} \\
\Rightarrow ACI_{BFI}(G - \{v\}) < ACI_{BFI}(G) \quad \text{and} \quad ACI_{BFI}(G - \{v\}) > ACI_{BFI}(G) \\
\Rightarrow v \) is a bipolar fuzzy incidence connectivity-reducing vertex.

From Theorem 4, we directly get the following corollary.

Corollary 1. Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N) \) be a bipolar fuzzy incidence graph, \( n = |\eta^*| \geq 3 \), \( v \in \eta^* \), \( a = \frac{\text{cl}_{BFI}(G)}{\text{cl}_{BFI}(G - \{v\})} \).

- If \( v \) is a bipolar fuzzy incidence-connectivity-enhancing vertex, then \( a < b \);
- If \( v \) is a bipolar fuzzy incidence-connectivity-reducing vertex, then \( a > b \);
- If \( v \) is a bipolar fuzzy incidence-connectivity-neutral vertex, then \( a = b \).

Definition 14. Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N) \) be a connected bipolar fuzzy incidence graph with \( n = |\eta^*| \geq 3 \).

- If \( G \) has at least one bipolar fuzzy incidence connectivity-enhancing vertex, then \( G \) is called bipolar fuzzy incidence connectivity enhancing graph;
- If \( G \) has one bipolar fuzzy incidence connectivity-reducing vertex, then \( G \) is called bipolar fuzzy incidence connectivity reducing graph;
- If all the vertices in \( G \) are bipolar fuzzy incidence connectivity-neutral vertices, then \( G \) is called bipolar fuzzy incidence connectivity neutral graph.

We now consider the Wiener index of a bipolar fuzzy incidence graph. Considering \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N) \) is a bipolar fuzzy incidence graph and \( H \) is a subgraph of \( G \), note that \( WI_{BFI}^P(H) \leq WI_{BFI}^P(G) \) and \( WI_{BFI}^N(H) \geq WI_{BFI}^N(G) \) may not hold, and we give the following example to explain this point.

Example 6. Considering a bipolar fuzzy incidence graph is depicted in Figure 5, we have \( \eta^* = \{v_1, v_2, v_3\} \), \( \theta^P(v_1v_2) = 0.7, \quad \theta^P(v_1v_3) = 0.7, \quad \theta^N(v_2v_3) = -0.7, \quad \theta^P(v_1v_2) = 0.8, \quad \theta^P(v_1v_3) = -0.8, \quad \Psi^P(v_1, v_2v_3) = 0.6, \quad \Psi^P(v_2, v_1v_3) = -0.6, \quad \Psi^P(v_2, v_1v_3) = 0.6, \quad \Psi^P(v_2, v_1v_3) = 0.6, \quad \Psi^P(v_1, v_2v_3) = -0.6, \quad \Psi^P(v_1, v_2v_3) = 0.8, \quad \Psi^P(v_1, v_2v_3) = -0.8, \quad \Psi^P(v_1, v_2v_3) = -0.8, \quad \Psi^P(v_1, v_2v_3) = -0.8 \). Clearly, each incidence pair is a strong pair and by simple computation we get \( WI_{BFI}^P(G) = 4 \) and \( WI_{BFI}^N(G) = -4 \).

Now remove the edge \( v_1v_3 \) from \( G \), we construct a subgraph \( H \) of \( G \) which is presented in Figure 6. The strong geodesic from \( v_1 \) to \( v_3 \) is \( v_1v_2v_3 \), \( d^P(v_1, v_2) = \Psi^P(v_1, v_2v_3) + \Psi^P(v_2, v_1v_3) = 1.2 \) and \( d^N(v_1, v_2) = \Psi^P(v_1, v_2v_3) + \Psi^P(v_2, v_1v_3) = -1.2 \). In terms of the same trick, we confirm \( d^P(v_1, v_3) = 2.4 \), \( d^N(v_1, v_3) = 2.4 \), \( d^P(v_2, v_3) = 1.2 \), and \( d^N(v_2, v_3) = -1.2 \). Thus, \( WI_{BFI}^P(H) = 4.8 \) and \( WI_{BFI}^N(H) = -4.8 \), which imply that \( WI_{BFI}^P(G) < WI_{BFI}^P(H) \) and \( WI_{BFI}^N(G) > WI_{BFI}^N(H) \).

The following result reveals the inner connection between the Wiener index and connectivity index of bipolar fuzzy incidence graph.

Theorem 5. Let \( G = (\eta^P, \eta^N, \theta^P, \theta^N, \Psi^P, \Psi^N) \) be a bipolar fuzzy incidence graph with \( |\eta^*| \geq 3 \). Then, \( WI_{BFI}^P(G) > CI_{BFI}^P(G) \) and \( WI_{BFI}^N(G) < CI_{BFI}^N(G) \).

Proof. Consider a bipolar fuzzy incidence graph with \( |\eta^*| \geq 3 \). For any \( x, y \in \eta^* \), the sum of positive membership values and negative membership values of each incidence pairs connecting \( x \) and \( y \) are \( d^P(x, y) \) and \( d^N(x, y) \), respectively. While the minimum positive membership value and the maximum negative membership value of all strong incidence pairs are \( \text{ICONN}^P(x, y) \) and \( \text{ICONN}^N(x, y) \), respectively. Hence, we have \( \text{ICONN}^P(x, y) < d^P(x, y) \) and \( \text{ICONN}^N(x, y) > d^N(x, y) \), which imply
\[
\sum_{x, y \in \eta^*} \eta^P(x)\eta^P(y)\text{ICONN}^P(x, y) < \sum_{x, y \in \eta^*} \eta^P(x)\eta^P(y)d^P(x, y), \\
\sum_{x, y \in \eta^*} \eta^N(x)\eta^N(y)\text{ICONN}^N(x, y) > \sum_{x, y \in \eta^*} \eta^N(x)\eta^N(y)d^N(x, y). 
\]
Therefore, we get $\text{WIF}^P(G) > \text{ClIF}^P(G)$ and $\text{WIF}^N(G) < \text{ClIF}^N(G)$.

**Theorem 6.** Let $G$ be a complete bipolar fuzzy incidence graph with two vertices $v_1$ and $v_2$. Assume $\eta^P(v_1) = \eta^P(v_2) = 1$ and $\eta^N(v_1) = \eta^N(v_2) = -1$. Then, $\text{WIF}^P(G) = 2\text{ClIF}^P(G)$ and $\text{WIF}^N(G) = 2\text{ClIF}^N(G)$.

**Proof.** According to the assumption of the above theorem, the sum of positive membership values and negative membership values of each incidence pairs connecting $v_1$ and $v_2$ are $d^P_1(v_1, v_2)$ and $d^N_1(v_1, v_2)$, respectively. While the minimum positive membership value and the maximum negative membership value of all strong incidence pairs are $\text{ICONN}_0(P)(v_1, v_2)$ and $\text{ICONN}_0(N)(v_1, v_2)$, respectively. This reveals that $2\text{ICONN}_0(P)(v_1, v_2) = d^P_1(v_1, v_2)$ and $2\text{ICONN}_0(N)(v_1, v_2) = d^N_1(v_1, v_2)$. Hence,$$
2\eta^P(v_1)\eta^P(v_2)\text{ICONN}_0^P(v_1, v_2) = \eta^P(v_1)\eta^P(v_2)d^P_1(v_1, v_2),
$$
$$
2\eta^N(v_1)\eta^N(v_2)\text{ICONN}_0^N(v_1, v_2) = \eta^N(v_1)\eta^N(v_2)d^N_1(v_1, v_2).
$$

Therefore, the desired conclusion holds. $\square$

**Example 7.** Considering the following complete bipolar fuzzy incidence graph with two vertices denoted by $v_1$ and $v_2$ which is depicted in Figure 7, we check that $\theta^P(v_1v_2) = 0.4$, $\theta^N(v_1v_2) = -0.4$, $\psi^P(v_1, v_1v_2) = 0.4$, $\psi^N(v_1, v_1v_2) = -0.4$, $\text{ClIF}^P(G) = 0.4$, $\text{ClIF}^N(G) = -0.4$, $\text{WIF}^P(G) = 0.8$, and $\text{WIF}^N(G) = -0.8$.

When it comes to the molecular graph setting, we need to define the incidence membership function on the molecular fuzzy graph, which reflects to a certain extent the closeness between the atoms and the chemical bonds which connect the forces between the molecules, the respective differences between the atoms and the chemical bonds, and the related attributes. Hence, such definition of the membership function requires the participation of a chemist. On the other hand, not all graphs are molecular graphs [28]. Only when the graph structure can represent a certain chemical substance, the graph can be called a molecular graph. Due to the interaction between atoms and chemical bonds, spins and other physical effects are produced and many chemical structures are unstable. Such unstable chemical structures can be characterized by membership functions of atoms, chemical bonds, and their mutual relations. This part of the content requires chemical data or the assistance of a chemist and can be considered for future work.

**5 Conclusion**

In this article, we first introduce the BFG and two kinds of topological index, then define concepts of bipolar fuzzy incidence graphs, and finally we deduce the characteristics of Wiener index and connectivity index on bipolar fuzzy incidence graphs. The results obtained in this study have potential applications in many fields such as fuzzy reasoning and intelligent decision systems.
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