Proper actions on reductive homogeneous spaces

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Let $G$ be a linear semisimple real Lie group and $H$ be a reductive subgroup of $G$. We give a necessary and sufficient condition for the existence of a nonabelian free discrete subgroup $\Gamma$ of $G$ acting properly on $G/H$. For instance, such a group $\Gamma$ does exist for $\text{SL}(2n, \mathbb{R})/\text{SL}(2n-1, \mathbb{R})$ but does not for $\text{SL}(2n+1, \mathbb{R})/\text{SL}(2n, \mathbb{R})$ with $n \geq 1$.

1 Introduction

In this introduction, we state our results for the base field $k = \mathbb{R}$. Later in the text we shall prove analogous results for any local field $k$.

1.1 Proper actions of free groups

Let $G$ be a connected linear semisimple real Lie group, $H$ a reductive connected closed subgroup in $G$ (i.e. such that the adjoint action of $H$ in the Lie algebra of $G$ is semisimple). We know that $G$ contains an infinite discrete subgroup $\Gamma$ acting properly on $G/H$ if and only if $\text{rank}_\mathbb{R}(G) \neq \text{rank}_\mathbb{R}(H)$: this is the Calabi-Markus phenomenon ([Kob1]).

The main goal of this paper is to give a necessary and sufficient condition for existence of a nonabelian free discrete subgroup $\Gamma$ of $G$ that acts properly on $G/H$.

To state this condition, we need a few notations. Let $A_H$ be a Cartan subspace of $H$ (i.e. a connected abelian subgroup composed of hyperbolic elements and maximal for these properties), $A$ a Cartan subspace of $G$ containing $A_H$, $\Sigma := \Sigma(A, G)$ the restricted root system of $A$ in $G$, $W$ the Weyl group of $\Sigma$, $\Sigma^+$ a choice of positive roots, $A^+ := \{a \in A \mid \forall \chi \in \Sigma^+, \chi(a) \geq 1\}$ the corresponding closed Weyl chamber, $\iota$ the opposition involution for $A^+$ (for $a$ in $A^+$, $\iota(a)$ is the unique element of $A^+$ conjugate to $a^{-1}$) and $B^+ := \{a \in A^+ \mid a = \iota(a)\}$.

Note that $B^+$ differs from $A^+$ if and only if one of the connected components of the Dynkin diagram of $\Sigma$ is of type $A_n$ with $n \geq 2$, $D_{2n+1}$ with $n \geq 1$ or $E_6$.

We say that a group $\Gamma$ is virtually abelian if it contains a finite-index abelian subgroup.

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Theorem. With these notations, $G$ contains a non virtually abelian discrete subgroup $\Gamma$ acting properly on $G/H$ if and only if for every $w \in W$, $wA_H$ does not contain $B^+$. In this case, we can choose $\Gamma$ to be free and Zariski-dense in $G$.

Example 1. Here are thus some homogeneous spaces for which such a subgroup $\Gamma$ does not exist:

- $\text{SL}(n, \mathbb{R})/(\text{SL}(p, \mathbb{R}) \times \text{SL}(n-p, \mathbb{R}))$ where $1 \leq p < n$ and $p(n-p)$ is even;
- $\text{SL}(2p, \mathbb{R})/\text{SL}(p, \mathbb{R})$, $\text{SL}(2p, \mathbb{R})/\text{SO}(p, p)$ and $\text{SL}(2p + 1, \mathbb{R})/\text{SO}(p, p + 1)$ where $p \geq 1$;
- $\text{SO}(p + 1, q)/\text{SO}(p, q)$ when $p \geq q$ or when $p = q - 1$ is even;
- $G_C/H_C$ where $G_C$ is a complex simple Lie group and $H_C$ is the set of fixed points of a complex involution of $G_C$, except for $\text{SO}(4n, \mathbb{C})/(\text{SO}(p, \mathbb{C}) \times \text{SO}(4n - p, \mathbb{C}))$ for $n \geq 2$ and $p$ odd.

Example 2. Here are now some homogeneous spaces for which such a subgroup $\Gamma$ does exist:

- $\text{SL}(n, \mathbb{R})/(\text{SL}(p, \mathbb{R}) \times \text{SL}(n-p, \mathbb{R}))$ where $1 \leq p < n$ and $p(n-p)$ is odd;
- $\text{SL}(n, \mathbb{R})/\text{SO}(p, n-p)$ where $1 \leq p < \left\lfloor \frac{n}{2} \right\rfloor$;
- $\text{SO}(p + 1, q)/\text{SO}(p, q)$ when $0 \leq p \leq q - 2$ or when $p = q - 1$ is odd;
- $\text{SO}(4n, \mathbb{C})/(\text{SO}(p, \mathbb{C}) \times \text{SO}(4n - p, \mathbb{C}))$ for $n \geq 2$ and $p$ odd.

1.2 Compact quotients

We say that a homogeneous space $G/H$ admits a compact quotient if there exists a discrete subgroup $\Gamma$ of $G$ acting properly on $G/H$ and such that the quotient $\Gamma \backslash G/H$ is compact.

Determining whether a given homogeneous space admits a compact quotient is a question for which only partial answers are known (see [Ku], [Kob 1], [K-O], [Kob2], [B-L1] and [Zi]).

Corollary 1. Keep the same notations, and assume that $G/H$ is not compact and that for a suitable choice of $\Sigma^+$, $A_H$ contains $B^+$. Then $G/H$ does not have a compact quotient.

In particular, none of the homogeneous spaces from Example 1 have a compact quotient.

Among these examples, the simplest homogeneous space for which the answer was not already known is $\text{SL}(3, \mathbb{R})/\text{SL}(2, \mathbb{R})$ (see for example Question 3 from the introduction to [Zi]). Very recently and independently, G. Margulis also proved that the example $\text{SL}(2n + 1, \mathbb{R})/\text{SL}(2n, \mathbb{R})$ has no compact quotient (personal communication).
Corollary 2. Let $G_C$ be a complex simple Lie group, $\sigma$ a complex involution of $G_C$ and $H_C := \{ g \in G_C \mid \sigma(g) = g \}$. Then the symmetric space $G_C/H_C$ has no compact quotient, except possibly for the case where $G_C/H_C = \text{SO}(4n, \mathbb{C})/\text{SO}(4n-1, \mathbb{C})$ for $n \geq 2$.

For previous results going in this direction, see ([Kob2 1.9]).

Here is a more geometric way to state Corollary 1 for the homogeneous space $S^{p,q} := \text{SO}(p+1,q)/\text{SO}(p,q)$.

Corollary 3. No complete compact pseudoriemannian manifold $V$ of signature $(p,q)$ with constant sectional curvature $+1$ exists for $p = 2n$ and $q = 2n + 1$ with $n \geq 1$.

Indeed such a manifold $V$ would be a compact quotient of $S^{p,q}$.

Furthermore it is known ([C-M], [Wo], [Ku]) that no such manifolds exist when $p \geq q$ (the Calabi-Markus phenomenon) or when $pq$ is odd (because of the Gauss-Bonnet formula).

When $p = 1$ and $q = 2n$ (resp. when $p = 3$ and $q = 4n$) with $n \geq 1$, there is an abundant supply of such manifolds $V$ as the group $U(1,n)$ (resp. $\text{Sp}(1,n)$) acts properly and transitively on $S^{p,q}$.

Let us now describe the four main steps in the proof of the theorem and of its corollaries.

1.3 The Cartan projection of $\Gamma$ (see Chapter 3)

Let $K$ be a maximal compact subgroup of $G$. We have the Cartan decomposition $G = KA^+K$ and the Cartan projection $\mu : G \to A^+$: for $g \in G$, $\mu(g)$ is the unique element of $KgK \cap A^+$ (see [He] Ch. 9). For example if $G = \text{SL}(n, \mathbb{R})$, we may take $K = \text{SO}(n)$ and $A^+ = \{ \text{diag}(\sigma_1, \ldots, \sigma_n) \in G \mid \sigma_1 \geq \cdots \geq \sigma_n > 0 \}$; we then have $\mu(g) = \text{diag}(\sigma_1(g), \ldots, \sigma_n(g))$, where $\sigma_i(g)^2$ is the $i$-th eigenvalue of $t^gg$.

The first step consists in studying the set $\mu(\Gamma)$.

Proposition. Let $\Gamma$ be a discrete subgroup of $G$ that is not virtually abelian. Then there exists a compact subset $M$ of $A$ such that the set $\mu(\Gamma) \cap B^+M$ is infinite. In particular, for every closed subgroup $H'$ of $G$ containing $B^+$, $\Gamma$ does not act properly on $G/H'$.

We explicitly construct infinitely many elements in $\mu(\Gamma) \cap B^+M$: we choose $f$ and $g$ in $\Gamma$ in a suitable way and we take the elements $\mu(g^pfg^{-p})$, for $p \geq 1$. To check that these elements work, we estimate the norms of their images in a sufficient number of representations of $G$.

The “only if” part of Theorem 1.1 is of course a consequence of this proposition.

1.4 Actions of nilpotent groups (see Chapter 4)

Corollary 4 follows from this proposition and from the following proposition which shows that virtually abelian groups $\Gamma$ do not provide compact quotients either.
Proposition. Let $N$ be a nilpotent subgroup of $G$. Then the quotient $N \backslash G/H$ is not quasicompact.

The proof of this proposition is based on reduction to the case of real rank 1.

1.5 Properness criterion (see Chapter 5)

Let $H_1$, $H_2$ be two closed subgroups of $G$. We then prove a criterion for properness of the action of $H_1$ on $G/H_2$. This criterion depends only on the subsets $\mu(H_1)$ and $\mu(H_2)$ and on the commutative group $A$. It generalizes two known criteria, one due to Kobayashi ([Kob1]) when $H_1$ and $H_2$ are reductive Lie subgroups and the other due to Friedland ([Fr]) when $G = \text{SL}(n, \mathbb{R})$ and $H_2 = \text{SL}(p, \mathbb{R}) \times \text{I}_{n-p}$. Here it is:

Proposition. $H_1$ acts properly on $G/H_2$ if and only if for every compact subset $M$ of $A$, the set $\mu(H_1) \cap \mu(H_2)M$ is compact.

Example. A discrete subgroup $\Gamma$ of $\text{SL}(2p, \mathbb{R})$ acts properly on $\text{SL}(2p, \mathbb{R})/\text{Sp}(p, \mathbb{R})$ if and only if for every $R > 0$, the set $\Gamma_R := \{ g \in \Gamma \mid \frac{1}{R} \leq \sigma_i(g)\sigma_{2p+1-i}(g) \leq R, \forall i = 1, \ldots, p \}$ is finite.

The main idea of the proof is to estimate, as in 1.3, the Cartan projection $\mu(g)$ of an element $g \in G$ using the norms of the images of $g$ in a sufficient number of representations of $G$.

1.6 Construction of free groups (see Chapter 7)

In the last step, we construct Zariski-dense free subgroups of $G$ using ideas from [Ti2], [Mar1], [G-M] and [B-L2].

We denote by $\log$ the logarithm map that identifies $A$ with its Lie algebra $\mathfrak{a}$. We say that a subset $\Omega$ of $A^+$ is a convex cone if $\log(\Omega)$ is a convex cone in $\mathfrak{a}$. For example $A^+$ and $B^+$ are convex cones in $A$.

Proposition. Let $\Omega$ be a nonempty open convex cone in $A^+$ that is stable by the opposite involution. Then there exists in $G$ a discrete subgroup $\Gamma$ that is free on two generators, Zariski-dense in $G$ and such that $\mu(\Gamma)$ is contained in $\Omega \cup \{e\}$.

G. Margulis told me that he was also aware of this proposition (personal communication).

The group $\Gamma$ we construct is “$\varepsilon$-Schottky”, which means that its image in a sufficient number of representations $V$ of $G$ is “$\varepsilon$-Schottky on $\mathbb{P}(V)$”. Chapter 6 is dedicated to defining and studying linear groups that are “$\varepsilon$-Schottky on $\mathbb{P}(V)$”.

The “if” part of Theorem 1.1 is a consequence of Propositions 1.5 and 1.6.

1.7

The proof of Theorem 1.1 and of Propositions 1.3, 1.5 and 1.6 remains valid without any significant changes on any local field $k$ (see respectively Theorems 7.5, 3.3, 5.2 and 7.4).
The proof of Proposition 1.4 and consequently also that of Corollary 1 remains valid without any changes on any local field of characteristic 0 (see respectively Corollaries 4.1 and 7.6). Its adaptation to a local field of nonzero characteristic is also possible, but will not be discussed in this paper.

These results were announced in [Be].

2 Preliminaries

2.1 Local fields

Let \( k \) be a local field, i.e. either \( \mathbb{R} \) or \( \mathbb{C} \) or a finite extension of \( \mathbb{Q} \) or \( \mathbb{Q}_p \) or \( \mathbb{F}_p[^T-1, T] \) for some integer prime \( p \). Let \( |.| \) be a continuous absolute value on \( k \).

When \( k = \mathbb{R} \) or \( \mathbb{C} \), we set \( k^0 := (0, \infty) \), \( k^+ := [1, \infty) \) and \( k^{++} := (1, \infty) \).

When \( k \) is non-Archimedean, we call \( \mathcal{O} \) the ring of integers of \( k \), \( \mathcal{M} \) the maximal ideal of \( \mathcal{O} \) and we choose a uniformizer, i.e. an element \( \pi \) of \( \mathcal{M} - 1 \) which is not in \( \mathcal{O} \). We then set \( k^0 := \{ \pi^n \mid n \in \mathbb{Z} \} \), \( k^+ := \{ \pi^n \mid n \geq 0 \} \) and \( k^{++} := \{ \pi^n \mid n \geq 1 \} \).

Let \( V \) be a finite-dimensional vector space over \( k \). To every basis \( v_1, \ldots, v_n \) of \( V \), we associate norms on \( V \) and on \( \text{End}(V) \) defined, for every \( v = \sum_{1 \leq i \leq n} x_i v_i \) in \( V \) and for every \( g \) in \( \text{End}(V) \), by

\[
\|v\| := \sup_{1 \leq i \leq n} |x_i| \quad \text{and} \quad \|g\| := \sup_{v \in V, \|v\|=1} \|g \cdot v\|.
\]

Of course two different bases of \( V \) give rise to equivalent norms.

2.2 Cartan decomposition

For every \( k \)-group \( G \), we denote by \( G \) or by \( G_k \) the set of its \( k \)-points and by \( g \) its Lie algebra.

Let \( G \) be a semisimple \( k \)-group. For example \( G = \text{SL}(n, k) \), \( \text{Sp}(n, k) \) or the Spin group of a nondegenerate quadratic form (if \( \text{char} \ k \neq 2 \)).

Let \( A \) be a maximal \( k \)-split torus of \( G \) and (in accordance with our conventions) \( A = A_k \). The dimension \( r \) of \( A \) is by definition the \( k \)-rank of \( G \): \( r = \text{rank}_k(G) \). We denote by \( X^*(A) \) the set of characters of \( A \) (this is a free \( \mathbb{Z} \)-module of rank \( r \)) and we set \( E := X^*(A) \otimes_{\mathbb{Z}} \mathbb{R} \). We denote by \( \Sigma = \Sigma(A, G) \) the set of roots of \( A \) in \( G \), i.e. the nontrivial weights of \( A \) in the adjoint representation of the group \( G \), also called \( k \)-roots or restricted roots. Then \( \Sigma \) is a root system in \( E \) ([B-T1] § 5). We choose a system of positive roots \( \Sigma^+ \) and we set

\[
A^0 := \{ a \in A \mid \forall \chi \in X^*(A), \chi(a) \in k^0 \}; \\
A^+ := \{ a \in A^0 \mid \forall \chi \in \Sigma^+, \chi(a) \in k^+ \}; \\
A^{++} := \{ a \in A^0 \mid \forall \chi \in \Sigma^+, \chi(a) \in k^{++} \}.
\]
Let $N$ be the normalizer of $A$ in $G$, $L$ be the centralizer of $A$ in $G$ and $W := N/L$ the small Weyl group of $G$: it can be identified with the Weyl group of the root system $\Sigma$. The subset $A^+$ is called the positive Weyl chamber. We have the equality $A^0 = \bigcup_{w \in W} wA^+$.

We now assume $G$ to be simply connected; this assumption is innocuous, as we can reduce the problem to this case by standard methods (see [Mar2] I.1.5.5 and I.2.3.1).

There exists a maximal compact subgroup $K$ of $G$ such that $N = (N \cap K) \cdot A$. We then have the equality $G = KA^+K$, called the Cartan decomposition of $G$. It follows that for every $g$ in $G$, there exists an element $\mu(g)$ in $A^+$ such that $g$ is in $K\mu(g)K$. This element $\mu(g)$ is unique. We call $\mu$ the Cartan projection this map $\mu : G \to A^+$. It is a continuous and proper map. For all of this, we refer to ([He] 9.1.1) in the Archimedean case and to ([Mac] 2.6.11) in the non-Archimedean case.

Let $w_0$ be the “longest” element of the Weyl group relative to $A^+$: it is the unique element of $W$ such that for every $a$ in $A^+$, we have $w_0(a^{-1})$ in $A^+$. The map $\iota : A^+ \to A^+$ given by

$$\iota(a) = w_0(a^{-1})$$

is called the opposition involution. We then have the formula, for every $g$ in $G$:

$$\mu(g^{-1}) = \iota(\mu(g)).$$

Finally let $B^+ := \{a \in A^+ \mid \iota(a) = a\}$ be the set of fixed points of $\iota$.

**Example.**

- If $G = \text{SL}(n, k)$ and $k = \mathbb{R}$ or $\mathbb{C}$, we may take $K = \{g \in G \mid g^g = 1\}$ and $A^+ = \left\{ \begin{pmatrix} \sigma_1 & 0 \\ \vdots & \ddots \\ 0 & \sigma_n \end{pmatrix} \in G \mid \forall i, \sigma_i \in \mathbb{R} \text{ and } \sigma_1 \geq \cdots \geq \sigma_n > 0 \right\}$.

- If $G = \text{SL}(n, k)$ and $k$ is non-Archimedean, we may take $K = \text{SL}(n, \mathcal{O})$ and $A^+ = \left\{ \begin{pmatrix} \pi^{q_1} & 0 \\ \vdots & \ddots \\ 0 & \pi^{q_n} \end{pmatrix} \in G \mid \forall i, q_i \in \mathbb{Z} \text{ and } q_1 \geq \cdots \geq q_n \right\}$.

- In both cases, for $g$ in $G$, we denote by $\sigma_j(g)$ the diagonal coefficients of $\mu(g)$: these are the singular values of $g$. When $k = \mathbb{R}$ or $\mathbb{C}$, these are the eigenvalues of $(g^g)^{1/2}$.

### 2.3 Representations of $G$

Though the reminders given in this paragraph are valid on any infinite field $k$, we keep the notations from 2.2. Let $\rho$ be a representation of $G$ on a finite-dimensional $k$-vector space $V$. More precisely, $\rho$ is a $k$-morphism of $k$-groups $\rho : G \to \text{GL}(V)$. For $\chi \in X^*(A)$, we denote by $V_\chi := \{v \in V \mid \forall a \in A, \rho(a)v = \chi(a)v\}$ the corresponding eigenspace. We denote by $\Sigma(\rho) := \{\chi \in X^*(A) \mid V_\chi \neq 0\}$ the set of $k$-weights of $V$. This set is invariant by the action of the Weyl group $W$ and we have

$$V = \bigoplus_{\chi \in \Sigma(\rho)} V_\chi.$$
We endow $X^*(\mathfrak{a})$ with the order given by
\[
\chi_1 \leq \chi_2 : \iff \chi_2 - \chi_1 \in \sum_{\chi \in \Sigma^+} \mathbb{N}\chi.
\]

We assume that $\rho$ is irreducible. The set $\Sigma(\rho)$ then has a unique element $\lambda$ that is maximal for this order, called the highest $k$-weight of $V$. When $G$ is $k$-split, we have $\dim V_\lambda = 1$.

We will need the following well-known lemma.

**Lemma.** There exist $r$ irreducible representations $\rho_i$ of the group $G$ on $k$-vector spaces $V_i$ whose highest $k$-weights $(\omega_i)_{1 \leq i \leq r}$ form a basis of the $\mathbb{R}$-vector space $E$ and such that $\dim(V_i)_{\omega_i} = 1$.

For complete results concerning classification of representations of $G$, we refer to [B-T1, B-T2] as well as to [Ti1].

**Proof.** We choose irreducible representations $\sigma_i$ of $G$ on $k$-vector spaces $W_i$ whose highest $k$-weights $(\lambda_i)_{1 \leq i \leq r}$ form a basis of the $\mathbb{R}$-vector space $E$ ([Ti1] 7.2). We set $d_i := \dim(W_i)_{\lambda_i}$, $\omega_i := d_i\lambda_i$ and we take for $V_i$ the simple subquotient of $\Lambda^{d_i}(W_i)$ having $\omega_i$ as a $k$-weight.

**Example.** When $G = \text{SL}(n,k)$, we have $r = n - 1$ and $V_i = \Lambda^i(k^n)$, for $1 \leq i \leq n - 1$.

### 2.4 Cartan projection and representations of $G$

The following lemma is easy and fundamental. In the light of Lemma 2.3, it says that, up to a bounded multiplicative constant, computing $\mu(g)$ is equivalent to computing the norms $\|\rho_i(g)\|$ for $i = 1, \ldots, r$.

**Lemma.** For every irreducible representation $(V, \rho)$ of $G$ with highest $k$-weight $\chi$ and for every norm on $V$, there exists a constant $C_\chi > 0$ such that, for every $g \in G$, we have
\[
C_\chi^{-1} \leq \frac{|\chi(\mu(g))|}{\|\rho(g)\|} \leq C_\chi.
\]

**Proof.** We may assume that the chosen norm corresponds to a basis formed by eigenvectors for the action of $A$ (see 2.1) so that, for $a$ in $A^+$, we have
\[
|\chi(a)| = \|\rho(a)\|.
\]

Take $a = \mu(g)$ so that $g = k_1ak_2$ with $k_1$, $k_2$ in $K$. We then have
\[
\frac{|\chi(\mu(g))|}{\|\rho(g)\|} = \frac{\|\rho(a)\|}{\|\rho(k_1ak_2)\|} \in [C_\chi^{-1}, C_\chi]
\]
where $C_\chi = \sup_{k \in K} \|\rho(k)\|^2$. This is what we wanted.
3 The Cartan projection of $\Gamma$

The goal of this section is to prove Theorem 3.3 which generalizes Proposition 1.3.

3.1 $H$-proper pairs

Let us start with a few easy observations whose power will become apparent later.

**Definition.** Let $H$ be a locally compact group and $H_1$, $H_2$ two closed subsets of $H$. We shall say that $(H_1, H_2)$ is $H$-proper if for every compact subset $L$ of $H$, the intersection $H_1 \cap LH_2L$ is compact.

**Remarks.**

1. If $(H_1, H_2)$ is $H$-proper, then $(H_2, H_1)$ is $H$-proper and moreover for any $h_1$, $h_2$ in $H$, $(h_1H_1h_1^{-1}, h_2H_2h_2^{-1})$ is $H$-proper.

2. This definition may be generalized as follows: let $E$ and $X$ be locally compact topological spaces and $a : E \times X \to X$ a continuous map, in other terms $a$ is a continuous family $e \mapsto a_e$ of continuous transformations of $X$. We shall say that this family is proper if for every compact subset $L$ of $X$, the set $E_L := \{ e \in E \mid eL \cap L \neq \emptyset \}$ is compact. When $E$ is a semigroup and $a$ is an action, we get the usual definition of a proper action.

Saying that $(H_1, H_2)$ is $H$-proper is equivalent to saying that the family of transformations of $H$

$$(H_1 \times H_2) \times H \to H$$

$$((h_1, h_2), h) \mapsto a_{h_1, h_2}(h) := h_1hh_2^{-1}$$

is proper.

The following lemma is a direct application of the definitions. Its verification is left to the reader.

**Lemma 3.1.1.** Let $H$ be a locally compact group, $H_1$ a closed subsemigroup of $H$ and $H_2$ a closed subgroup of $H$. We have the equivalence:

$(H_1, H_2)$ is $H$-proper $\iff$ $H_1$ acts properly on $G/H_2$.

**Definition.** Let $H$ be a locally compact group and $H_1$, $H'_1$ two closed subsets of $H$. We shall say that $H'_1$ is contained in $H_1$ modulo the compacts of $H$ if there exists a compact subset $L$ of $H$ such that $H'_1 \subset LH_1L$.

**Lemma 3.1.2.** Let $H$ be a locally compact group, and let $H_1$, $H'_1$, $H_2$, $H'_2$ be four closed subsets of $H$ such that $(H_1, H_2)$ is $H$-proper and $H'_j$ is contained in $H_j$ modulo the compacts of $H$ for $j = 1, 2$. Then the pair $(H'_1, H'_2)$ is $H$-proper.

This lemma is also an immediate consequence of the definitions. It is the conjunction of this lemma with the Cartan decomposition that explains the Calabi-Markus phenomenon.
3.2 Growth of $g_1^p f g_2^p$

Let us use the notations of 2.2 again. Let $G$ be a simply-connected semisimple $k$-group, $G := G_k$ and $\mu : G \to A^+$ a Cartan projection.

**Proposition.** Let $g_1, g_2$ be two elements of $G$ and $F$ a nonempty subset of $G$. Then there exists a nonempty subset $F'$ of $F$ which is Zariski-open in $F$ and such that for any $f, f'$ in $F'$, there exists a compact subset $M_{f,f'}$ of $A$ such that for every $p \geq 1$, we have

$$\mu(g_1^p f g_2^p) \cdot \mu(g_1^p f' g_2^p)^{-1} \in M_{f,f'}.$$

**Remark.** In this statement, the phrase "Zariski-open in $F$" means open with respect to the topology induced on $F$ by the Zariski topology of $G$. In other terms, $F'$ is a nonempty subset of $F$ whose complement is defined by polynomial equations.

**Proof.** We may assume that $F$ is Zariski-connected so that the intersection of two nonempty Zariski-open subsets of $F$ is still a nonempty Zariski-open subset of $F$.

The proposition is then a consequence of Lemma 2.4 and of the following elementary lemma applied to the subsets $\rho_i(F)$ where the $\rho_i$ are the representations of $G$ introduced in Lemma 2.3.

Let $V$ be a $k$-vector space of dimension $d$. Let us take the notations of 2.1.

**Lemma.** Let $g_1, g_2$ be elements of $\text{GL}(V)$ and let $F$ be a nonempty subset of $\text{End}(V) \setminus \{0\}$. Then there exists a nonempty subset $F'$ of $F$, Zariski-open in $F$ and such that for any $f, f'$ in $F'$, there exists a constant $C_{f,f'} > 1$ such that for every $p \geq 1$, we have

$$C_{f,f'}^{-1} \leq \|g_1^p f g_2^p\| \|g_1^p f' g_2^p\|^{-1} \leq C_{f,f'}.$$

We suggest to the reader to prove this lemma in the particular case where $g_1$ and $g_2$ are diagonal matrices before reading the complete proof.

**Proof.** We denote by $\phi$ the endomorphism of $\text{End}(V)$ given by $\phi(f) := g_1 f g_2$. Replacing if necessary $k$ by a finite extension, we may assume that the eigenvalues of $\phi$ are all in $k$.

We endow the set $(0, \infty) \times \mathbb{N}$ (where $\mathbb{N}$ stands for the set of nonnegative integers) with the lexicographic order:

$$(\lambda, r) \leq (\lambda', r') :\iff \lambda < \lambda' \text{ or } (\lambda = \lambda' \text{ and } r \leq r').$$

We introduce, for $\lambda$ in $(0, \infty)$ and $r$ in $\mathbb{N}$, the following vector subspace of $\text{End}(V)$:

$$W^{\lambda,r} := \sum_{(z,s) \in k \times \mathbb{N}} \text{Ker}((\phi - z)^s).$$

Let $(\lambda, r)$ be the greatest element of $(0, \infty) \times \mathbb{N}$ such that $F \not\subset W^{\lambda,r}$ and let $F' := F \setminus (F \cap W^{\lambda,r})$. It is clear that for every $f$ in $F'$, there exists a constant $A_f > 0$ such that the sequence $p \mapsto \|\phi^p(f)\| = \|g_1^p f g_2^p\|$ is equivalent to $A_f p^\lambda$. The conclusion follows.
3.3 Fixed points of the opposition involution

**Theorem.** Let $k$ be a local field, $G$ a semisimple $k$-group, $A$ a maximal $k$-split torus of $G$, $A^+$ a positive Weyl chamber, $\iota$ the opposition involution, $B^+ := \{ a \in A^+ \mid \iota(a) = a \}$ and $\Gamma$ a discrete subgroup of $G := G_k$. We assume $\text{char } k = 0$ (resp. char $k \neq 0$).

If the pair $(\Gamma, B^+)$ is $G$-proper then $\Gamma$ is virtually abelian (resp. nilpotent).

In particular, if $H$ is a subgroup of $G$ containing $B^+$ and if $\Gamma$ acts properly on $G/H$ then $\Gamma$ is virtually abelian (resp. nilpotent).

**Proof.** We may assume $G$ to be simply connected and $\Gamma$ to be Zariski-connected. We denote by $\mu : G \to A^+$ a Cartan projection. Let $g$ be an element of $\Gamma$. By the previous proposition, there exists a nonempty subset $\Gamma'$ of $\Gamma$ which is Zariski-open in $\Gamma$ and such that for every $f$ in $\Gamma'$, there exists a compact subset $M_f$ of $A$ such that, for every $p \geq 1$,

$$\mu(g^pfg^{-p}) \cdot \mu(g^pfg^{-1}g^{-p})^{-1} \in M_f.$$ 

Hence by the lemma below, there exists a compact subset $M'_f$ in $A$ such that, for every $p \geq 1$,

$$\mu(g^pfg^{-p}) \in M'_fB^+.$$ 

By assumption, the set $\mu(\Gamma) \cap M'_fB^+$ is compact. Since $\Gamma$ is discrete and $\mu$ is proper, the set $\{g^pfg^{-p} \mid p \geq 0\}$ is finite. Let $Z_f$ be the centralizer of $f$ in $\Gamma$ and $Z_{\Gamma}$ be the center of $\Gamma$. Hence there exists $p_0 \geq 1$ such that $g^{p_0}$ is in $Z_f$.

By Noetherianness, there exists a finite subset $\Gamma_0$ of $\Gamma$ such that

$$Z_{\Gamma} = \bigcap_{\gamma \in \Gamma_0} Z_{\gamma}.$$ 

Since $\Gamma'$ generates $\Gamma$, we may assume that $\Gamma_0$ is contained in $\Gamma'$. Hence there exists $p \geq 1$ such that $g^p$ is in $Z_{\Gamma}$.

The group $\Gamma/Z_{\Gamma}$ is a linear torsion group. The claim below shows that $\Gamma/Z_{\Gamma}$ contains a finite-index nilpotent subgroup. Since $\Gamma$ is Zariski-connected, $\Gamma$ is nilpotent.

Now if $k$ has zero characteristic, since $\Gamma$ is nilpotent and discrete, $\Gamma$ is finitely generated. But then $\Gamma/Z_{\Gamma}$ is a finite group. This is what we wanted. \hfill \qed

In this proof, we used the following lemma and claim.

**Lemma.** Let $M$ be a compact subset of $A$. Then there exists a compact subset $M'$ of $A$ such that, for every $a$ in $A^+$, we have the implication

$$a \cdot \iota(a)^{-1} \in M \implies a \in B^+M'.$$

**Proof.** When $k = \mathbb{R}$, the logarithm map identifies the connected component $A_e$ of $A$ to an $\mathbb{R}$-vector space, the involution $\iota$ to a linear symmetry and $A^+$ to a convex cone invariant by $\iota$. We may then take $M' := \{ m \in A \mid m^2 \in M \}$.

The general case is no harder. There exists a finite subset $L$ of $A^+$ such that for every $a$ in $A^+$, there exists $l$ in $L$ and $c$ in $A^+$ such that $a = c^2l$. We may then write $a = bm$ where $b := c \cdot \iota(c)$ is in $B^+$ and $m := l \cdot c \cdot \iota(c)^{-1}$ is in the compact set $M' := \{ m \in A \mid m^2 \in ML\iota(L)^{-1} \}$. \hfill \qed
Claim. Let $k$ be a field, $V$ a finite-dimensional $k$-vector space and $\Gamma$ a torsion subgroup of $\text{GL}(V)$.

a) (Schur, see [C-R] p. 258) If $\text{char}(k) = 0$, $\Gamma$ contains a finite-index abelian subgroup whose elements are all semisimple.

b) ([Ti2] 2.8) If $\text{char}(k) \neq 0$, we denote by $k_a$ the algebraic closure in $k$ of the prime subfield of $k$. Then every simple subquotient $V'$ of $V$ has a basis in which the coefficients of the elements of $\Gamma$ lie in $k_a$. In particular, if $k_a$ is finite (which is the case when $k$ is local), $\Gamma$ has a finite-index nilpotent subgroup whose elements are all unipotent.

c) (Burnside) In both cases, if $\Gamma$ is finitely generated, then $\Gamma$ is finite.

4 Action of nilpotent groups

In this section $k$ is a local field of characteristic zero. The goal of this section is to prove Theorem 4.1 and its corollary which generalizes Proposition 1.4.

4.1 Non-quasicom pactness of $N \backslash G/H$

Theorem. Let $k$ be a local field of characteristic zero, $G$ a reductive $k$-group, $H$ a $k$-subgroup of $G$, $G := G_k$, $H := H_k$, and $N$ a nilpotent subgroup of $G$.

If $N \backslash G/H$ is quasicompact, then $H$ contains a maximal unipotent $k$-subgroup of $G$.

The proof of this proposition is based on a reduction to the case of $k$-rank one. It is done in sections 4.2 to 4.5. First of all let us state a corollary of this theorem.

Corollary. Same notations. We assume that $G/H$ is not compact.

a) If $H$ is reductive then $N \backslash G/H$ is not quasicompact.

b) $(k = \mathbb{R})$ If $N$ acts properly on $G/H$ then $N \backslash G/H$ is not compact.

Proof.

a) We may assume that $G$ is simply connected. We decompose $G$ into a product $G = G_{\text{an}} \times G_{\text{is}}$ where $G_{\text{an}}$ is the largest anisotropic connected normal $k$-subgroup of $G$ and $G_{\text{is}}$ is the largest connected normal $k$-subgroup of $G$ that has no anisotropic factor.

Otherwise $H$ contains a maximal unipotent $k$-subgroup of $G$. Since $H$ is reductive, $H$ contains $G_{\text{is}}$ hence $G/H$ is a quotient of the group of $k$-points $G_{\text{an}}$ which is compact ([B-T1] 9.4). Contradiction.

b) Otherwise $H$ contains a maximal unipotent $k$-subgroup $U$ of $G$. There exists a maximal compact subgroup $K$ of $G$ such that $G = KUK$ ([Kos] 5.1). It follows that $G = KHK$ and $G$ is contained in $H$ modulo the compacts of $G$. Lemma 3.1.1 then shows that the pair $(N, G)$ is $G$-proper. Hence $N$ is a compact group. Contradiction. \qed
4.2 Parabolic $k$-subgroups

Let us introduce a few classical notations (see [Bor] §21) that will be useful in this section only.

Let $A$ be a maximal $k$-split torus of $G$, $\Sigma = \Sigma_G$ the system of the $k$-roots of $A$ in $G$, $\Sigma^+$ a choice of positive roots, $\Pi$ the simple $k$-roots of $\Sigma^+$, $L$ the centralizer of $A$ in $G$ and $g$ the Lie algebra of $G$. As usual, we denote by the corresponding Roman letter $A$, $L$ etc. the group of $k$-points.

For $\alpha$ in $\Sigma$, we denote by $g_\alpha := \{X \in g \mid \forall a \in A, \text{Ad} a(X) = \alpha(a)X\}$ the corresponding root space, $g_{(\alpha)} := g_\alpha \oplus g_{2\alpha}$ and $U_\alpha$ the unique unipotent $k$-subgroup (normalized by $L$) with Lie algebra $g_{(\alpha)}$. We say that a subset $\Theta$ of $\Sigma$ is closed if $\alpha, \beta \in \Theta$, $\alpha + \beta \in \Sigma \implies \alpha + \beta \in \Theta$. We denote by $(\Theta)$ the smallest closed subset of $\Sigma$ containing $\Theta$. For every closed subset $\Theta$ of $\Sigma^+$, we call $U_\Theta$ the unique unipotent $k$-subgroup (normalized by $L$) with Lie algebra $g_\Theta := \bigoplus_{\alpha \in \Theta} g_\alpha$. We set $U := U_{\Sigma^+}$, this is a maximal unipotent $k$-subgroup of $G$. The $k$-group $P := LU$ is a minimal parabolic $k$-subgroup of $G$. For every subset $\Theta$ of $\Pi$, we denote by $A^\Theta$ the Zariski-connected component of $\bigcap_{\alpha \in \Theta} \text{Ker}(\alpha)$, $L_\Theta$ the centralizer of $A^\Theta$, $U^\Theta := U_{\Sigma^+ \setminus \Theta}$ and $P_\Theta := L_\Theta U^\Theta$ the standard parabolic $k$-subgroup associated to $\Theta$. We also have the equality for the $k$-points $P_\Theta = L_\Theta U^\Theta$ ([B-T1] 3.14).

In the following well-known lemma, we do not need to assume that $k$ is a local field.

**Lemma 4.2.1.** $(\text{char}(k) = 0)$

a) Let $U'$ be a $k$-subgroup of $U$ and $[U, U]$ the derived subgroup of $U$. If we have $U = U' \cdot [U, U]$, then $U' = U$.

b) We have $[U, U] = U_{\Sigma^+ \setminus \Pi}$.

**Proof.**

a) This holds for any unipotent group: if $U' \neq U$, by Engel’s theorem, we may assume that $U'$ has codimension 1; $U'$ is then normal in $U$ and $U/U'$ is abelian. Hence $U \neq U' \cdot [U, U]$. Contradiction.

b) This follows from the equality $[g_\alpha, g_\beta] = g_{\alpha + \beta}$ for any $\alpha, \beta$ in $\Sigma^+$. \hfill $\Box$

The following lemma will allow us to assume that $H$ is solvable $k$-split.

**Lemma 4.2.2.** Let $G$ be a reductive $k$-group, $H$ a $k$-subgroup. Then there exists a maximal $k$-split torus $A$ of $G$ and a maximal unipotent $k$-subgroup $U$ normalized by $A$ such that $H/(H \cap AU)$ is compact.

In other terms, $H$ meets the maximal $k$-split solvable $k$-subgroup $AU$ along a subgroup that is cocompact.

**Proof.** Let $A$ be a maximal $k$-split subtorus of $G$ and $U$ be a maximal unipotent $k$-subgroup normalized by $A$. Then by the Iwasawa decomposition (see ([Hc] IX.1.3)
for the Archimedean case and [Mac] for the non-Archimedean case, the homogeneous space \( G/AU \) is compact.

By \([\text{B-T2}]\) 3.18, the orbits of \( H \) in \( G/AU \) are locally closed. Hence \( H \) has a closed orbit in \( G/AU \). Without loss of generality, it is the orbit of the basepoint. The quotient \( H/(H \cap AU) \) is then homeomorphic to this orbit (loc. cit.) and in particular \( H/(H \cap AU) \) is compact.

\[\square\]

4.3 Reduction to the case of \( k \)-rank one

Let us show that Theorem 4.1 holds in all generality if it holds for the \( k \)-groups \( G \) whose \( k \)-rank is equal to 1.

By 4.2.2, we may assume \( H \subset AU \). By \([\text{Bor}]\) 10.6 and 19.2, we may assume that \( H = A'U' \) where \( A' \) is a subtorus of \( A \) and \( U' \) is a unipotent \( k \)-subgroup of \( U \). We then also have equality for the \( k \)-points: \( H = A'U' \) \([\text{Bor}]\) 15.8.

Similarly, we may assume that \( N \) is the group of \( k \)-points of its Zariski-closure \( N \), that \( N \) is a maximal Zariski-connected nilpotent \( k \)-subgroup of \( AU \) and that \( N = A''U'' \) where \( A'' \) is a subtorus of \( A \) and \( U'' \) is a unipotent \( k \)-subgroup of \( U \). Hence there exists a subset \( \Theta \) of \( \Pi \) such that \( A'' = A^\Theta \) and \( U'' = U_{(\Theta)} \). We still have equality for the \( k \)-points: \( N = A''U'' \).

We want to show that \( U' = U \). If it is not the case, we may assume, thanks to 4.2.1, that \( U' \) contains \( [U, U] \) and that \( U' \) has codimension 1 in \( U \). Let us write \( U_\alpha := U' \cap U_\alpha \) and \( \Xi := \{ \alpha \in \Pi \mid U_\alpha \neq U_\alpha \} \). The set \( \Xi \) is nonempty.

Since \( U' \) has codimension 1 in \( U \) and is normalized by \( A' \), we have \( A' \subset \text{Ker}(\alpha - \alpha') \) for any \( \alpha, \alpha' \in \Xi \). We may suppose that \( A' \) is the connected component of \( \bigcap_{\alpha, \alpha' \in \Xi} \text{Ker}(\alpha - \alpha') \).

The subgroup \( AU \) is closed in \( G \) and contains \( N \) and \( H \). Hence \( N\backslash AU/A'U' \) is quasicompact and so is \( A''\backslash A/A' \). We deduce that \( A''A' \) has finite index in \( A \) \([\text{Bor}]\) 8.5. Hence \( \#(\Xi \cap \Theta) \leq 1 \). For more clarity, let us distinguish two cases:

1st case: \( \Xi \cap \Theta = \emptyset \). Let us fix an element \( \alpha \in \Xi \). Let \( P_\alpha \) denote the standard parabolic \( k \)-subgroup associated to \( \{ \alpha \} \), and let \( P_\alpha = L_\alpha U^\alpha \) be its Levi decomposition. The group of \( k \)-points \( P_\alpha \) is closed in \( G \) and contains \( N \) and \( H \), hence \( N\backslash P_\alpha/H \) is quasicompact.

We have the equality \( U = U_\alpha \cdot U' \) since \( U/U' \) identifies with a one-dimensional \( k \)-vector space and the image of \( U_\alpha \) is a nontrivial \( k \)-vector subspace of that space. Hence we have the equality \( P_\alpha = L_\alpha U = L_\alpha U' \) and the identification

\[ P_\alpha/H \simeq L_\alpha/A'U'_{(\alpha)} \]

Let us study the action of \( N \) on that quotient. On the one hand, the group \( U^\Xi \) is normalized by \( P_\alpha \) and is contained in \( U' \), hence it acts trivially on that quotient. So does its subgroup \( U'' \). On the other hand, the group \( A'' \) is contained in \( L_\alpha \). Hence

\[ N\backslash P_\alpha/H \simeq A''\backslash L_\alpha/A'U'_{(\alpha)} \]

is not quasicompact since the semisimple \( k \)-rank of \( L_\alpha \) is equal to 1. Contradiction.
2nd case: \( \Xi \cap \Theta = \{ \alpha \} \). The quotient \( N\setminus P_\alpha /H \) is still quasicompact and we have the equality \( P_\alpha = L_\alpha U' \) and hence the same identification

\[
P_\alpha /H \simeq L_\alpha /A'U'_\alpha .
\]

This time we have the equality \( N = A''U'' = (A''U_{(\alpha)}) \cdot U''^\alpha \) where \( U''^\alpha := U'' \cap U^\alpha \). Once again \( U''^\alpha \) is contained in \( U^\Xi \) since \( \langle \Theta \rangle \cap \langle \Xi \rangle = \langle \alpha \rangle \) and \( U''^\alpha \) acts trivially on this quotient. On the other hand, \( A'' \) is still included in \( L_\alpha \), it is now even included in the center of \( L_\alpha \). Hence

\[
N\setminus P_\alpha /H \simeq A''U_{(\alpha)}/A'U'_\alpha (\alpha)
\]

is not quasicompact for the same reason. Contradiction.

4.4 The case of \( k \)-rank one

To finish the proof of Theorem 4.1, we may thus assume that the semisimple \( k \)-rank of \( G \) is equal to 1. Let \( Z \) be the center of \( G \); the quotient \( N\setminus G/(ZH)_k \) is still quasicompact. We may thus assume that \( G \) is semisimple and adjoint.

Let \( \alpha \) be the unique element of \( \Pi \) and \( V := U_{(-\alpha)} \). The following lemma is a crucial ingredient of our proof.

**Lemma.** (\( \text{char}(k) = 0, \text{rank}_k(G) = 1 \))

a) Let \( v_n, u_n \) and \( a_n \) be sequences respectively in \( V, U \) and \( A \) such that the sequence \( g_n := v_n u_n a_n \) converges in \( G \). Then the sequence \( u_n \) is bounded in \( U \).

b) There exists an element \( v_0 \) of \( V \) such that if \( b_n, u_n \) and \( a_n \) are sequences respectively in \( A, U \) and \( A \) such that the sequence \( h_n := b_n v_0 u_n a_n \) converges in \( G \), then the sequence \( u_n \) is bounded in \( U \).

**Remarks.**

- The condition “\( u_n \) bounded in \( U \)” means that the sequence \( u_n \) remains in a compact subset of \( U \). It is likely that in both cases, the sequence \( u_n \) converges. We shall not need this fact.

- This lemma is false for \( G = \text{SL}(3, k) \), which shows the necessity of the reduction to the case of \( k \)-rank one. Indeed, take (with \( t = t_n \to 0 \)):

\[
g_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t^{-1} & -t^{-2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & t^{-1} & 0 \\ 0 & 1 & t^2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} = \begin{pmatrix} t & 1 & 0 \\ 0 & t & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Let us start by showing why this lemma implies Theorem 4.1. For the same reasons as in 4.3, we may assume that \( H = AU' = U'A \) where \( U' \) is a unipotent \( k \)-subgroup of codimension 1 in \( U \) which contains \([U, U]\) and that \( N \) is the group of \( k \)-points of a
maximal connected nilpotent $k$-subgroup $N$ of $G$. Up to conjugating by an element of $G$, there are only two possibilities for $N$: $N = V$ or $N = A$ (remember that $V$ and $U$ are conjugate over $k$). The quotient $U/U'$ is homeomorphic to $k$, hence we may choose a sequence $u_n$ in $U$ such that the image of this sequence in $U/U'$ tends to infinity.

**1st case: $N = V$.** Suppose by contradiction that $V \setminus G/U' A$ is quasicompact. Extracting if necessary a subsequence, the image of the sequence $u_n$ in this quotient converges (to a limit that might not be unique!). We may then find sequences $v_n$ in $V$, $u'_n$ in $U'$ and $a_n$ in $A$ such that the sequence $g_n := v_n u_n u'_n a_n$ converges in $G$. The previous lemma proves that the sequence $u_n u'_n$ is bounded in $U$. Contradiction.

**2nd case: $N = A$.** We proceed in the same fashion. Suppose by contradiction that $A \setminus G/U' A$ is quasicompact. Extracting if necessary a subsequence, the image of the sequence $v_0 u_n$ in this quotient converges. We may then find sequences $b_n$ in $A$, $u'_n$ in $U'$ and $a_n$ in $A$ such that the sequence $h_n := b_n v_0 u_n u'_n a_n$ converges in $G$. The previous lemma proves that the sequence $u_n u'_n$ is bounded in $U$. Contradiction.

**4.5 The sequence $u_n$**

It remains to prove Lemma 1.4. The Lie algebra $\mathfrak{g}$ has a decomposition defined on $k$

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}.$$  

We denote by $\mathfrak{g}_k$ the Lie algebra of $G$. We say that $\Sigma$ is reduced if $\mathfrak{g}_{\pm 2\alpha} = 0$. For $\beta$ in $\Sigma \cap \{0\}$, we call $p_\beta$ the projection onto $\mathfrak{g}_\beta$ parallel to the other subspaces $\mathfrak{g}_\gamma$; we call $(\mathfrak{g}_\beta)_k$ the intersection of $\mathfrak{g}_\beta$ with $\mathfrak{g}_k$. We call $H_\alpha$ the element of $\mathfrak{g} := \text{Lie}(A)$ such that $d\alpha(H_\alpha) = 2$. For $p$ in $\mathbb{Z}$, $\mathfrak{g}_{p\alpha}$ is also the eigenspace of $\text{ad} H_\alpha$ for the eigenvalue $2p$.

The following lemma is a variant of the Jacobson-Morozov theorem. It holds for any graded semisimple Lie algebra on a field of characteristic zero.

**Lemma.** (char($k$) = 0) Let $X$ be a nonzero element of $(\mathfrak{g}_{q\alpha})_k$ with $q \neq 0$. Then there exists an SL(2)-triple $(Y, H, X)$ with $Y$ in $(\mathfrak{g}_{-q\alpha})_k$ and $H$ in $(\mathfrak{g}_0)_k$.

**Proof.** We initially follow the proof of the Jacobson-Morozov theorem (see for example [Bou] VIII.11.2). Since $X$ is nilpotent, we may find $H$ in $\text{ad} X(\mathfrak{g}_k)$ such that $[H, X] = 2X$. Replacing if necessary $H$ by $p_0(H)$, we may assume that $H$ is in $(\mathfrak{g}_0)_k$. We may then complete $(H, X)$ to an SL(2)-triple $(Y, H, X)$ (loc. cit.). Replacing if necessary $Y$ by $p_{-q\alpha}(Y)$, we may assume that $Y$ is in $(\mathfrak{g}_{-q\alpha})_k$. \[\square\]

**Remark.** In our situation $G$ has $k$-rank 1. It follows since $H$ generates the Lie algebra of a $k$-split torus that $H$ is in $\mathfrak{a}$ and $d\alpha(H) = 2/q$. Hence $H = q^{-1} H_\alpha$. When $q > 0$, the theory of SL(2)-modules ([Bou] VIII.1.3) proves that the restriction of $\text{ad} X$ to $\mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha}$ is injective and that $\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{2\alpha}$ is contained in the image of $\text{ad} X$; when $q < 0$, the same statement holds with $\alpha$ and $-\alpha$ exchanged.
Proof of Lemma 4.4 when $\Sigma$ is reduced. We choose $v_0$ of the form $e^Y$ with $Y$ a nonzero element of $(\mathfrak{g}_{-\alpha})_k$. Note that the exponential is well-defined because $Y$ is nilpotent. The previous discussion proves that the restriction of $\text{ad} \ Y$ to $\mathfrak{g}_\alpha$ is injective. We write $u_n = e^{X_n}$ with $X_n$ in $(\mathfrak{g}_\alpha)_k$.

a) We have the equality

$$p_\alpha(\text{Ad} \ g_n(H_o)) = p_\alpha(\text{Ad}(v_nu_n a_n)(H_o)) = \text{ad} \ X_n(H_o) = -2X_n.$$ 

Hence the sequence $X_n$ converges and so does $u_n$.

b) We have the equality

$$p_0(\text{Ad} \ h_n(H_o)) = p_0(\text{Ad}(b_n v_0 u_n a_n)(H_o)) = H_o + \text{ad} \ Y(\text{ad} \ X_n(H_o)) = H_o - 2\text{ad} \ Y(X_n).$$

Hence the sequence $\text{ad} \ Y(X_n)$ converges, so does $X_n$ and so does $u_n$. \(\square\)

Proof of Lemma 4.4 when $\Sigma$ is not reduced. We choose $v_0$ of the form $e^Y$ with $Y$ some nonzero element of $(\mathfrak{g}_{-2\alpha})_k$. As previously, the restriction of $\text{ad} \ Y$ to $\mathfrak{g}_{2\alpha}$ is injective. We write $u_n = e^{X_n + Z_n}$ with $X_n$ in $(\mathfrak{g}_\alpha)_k$ and $Z_n$ in $(\mathfrak{g}_{2\alpha})_k$. We set $\psi(X_n) := (\text{ad} \ X_n)^2 \circ p_0$.

a) We have the equality

$$p_{2\alpha}(\text{Ad} \ g_n(H_o)) = p_{2\alpha}(\text{Ad}(v_n u_n a_n)(H_o)) = \frac{1}{2}(\text{ad} \ X_n)^2(H_o) + \text{ad} \ Z_n(H_o) = -4Z_n.$$ 

Hence the sequence $Z_n$ converges. More generally, we have the equality

$$p_{2\alpha} \circ \text{Ad} \ g_n \circ p_0 = \frac{1}{2} \psi(X_n) + \text{ad} \ Z_n \circ p_0.$$ 

Hence the sequence $\psi(X_n)$ converges. The lemma below proves that the sequence $X_n$ is bounded, hence so is $u_n$.

b) We have the equality

$$p_0(\text{Ad} \ h_n(H_o)) = p_0(\text{Ad}(b_n v_0 u_n a_n)(H_o)) = H_o + \text{ad} \ Y(\text{ad} \ Z_n(H_o)) = H_o - 4\text{ad} \ Y(Z_n).$$

Hence the sequence $Z_n$ converges. More generally, we have the equality

$$p_0 \circ \text{Ad} \ h_n \circ p_0 = \frac{1}{2} \text{ad} \ Y \circ \psi(X_n) + \text{ad} \ Y \circ \text{ad} \ Z_n \circ p_0.$$ 

Hence the sequence $\psi(X_n)$ converges, $X_n$ is bounded, and so is $u_n$. \(\square\)

We used the following lemma.

**Lemma.** (char$(k) = 0$, rank$_k(G) = 1$ and $\Sigma$ not reduced) The map

$$\psi : (\mathfrak{g}_\alpha)_k \rightarrow \text{Hom}_k(\mathfrak{g}_0, \mathfrak{g}_{2\alpha})$$

$$X \mapsto \psi(X) = (\text{ad} \ X)^2 \circ p_0$$

is proper.
Proof. The previous lemma and its remark prove that if $X$ is nonzero in $(g_\alpha)_k$, the space $g_{2\alpha}$ is contained in the image of $(\text{ad } X)^2$ hence $\psi(X)$ is nonzero. Our lemma is then a consequence of the following elementary claim whose proof we omit. \hfill $\square$

Claim. Let $E$, $F$ be two finite-dimensional $k$-vector spaces and $\psi : E \to F$ a continuous map that is homogeneous of degree 2 (i.e. $\psi(\lambda x) = \lambda^2 \psi(x)$ for every $\lambda$ in $k$ and $x$ in $E$) and such that $\psi^{-1}(0) = \{0\}$. Then $\psi$ is proper.

5 Properness criterion

The goal of this section is to prove Theorem 5.2 which generalizes Proposition 1.5. We reuse the notations from 2.2.

5.1 Inclusion modulo compacts

Proposition. Let $G$ be a simply-connected semisimple $k$-group, $G := G_k$ and $\mu : G \to A^+$ a Cartan projection. Then for every compact subset $L$ of $G$, there exists a compact subset $M$ of $A$ such that for every $g$ in $G$, we have $\mu(LgL) \subset \mu(g)M$.

Proof. Let us fix a compact subset $L$ of $G$ such that $L = L^{-1}$. Let $(V, \rho)$ be an irreducible representation of $G$ with highest $k$-weight $\chi$. By Lemma 2.3, it suffices to show that there exists a constant $C > 1$ such that for every $g$ in $G$ and $l_1, l_2$ in $L$, we have

$$C^{-1} \leq |\chi(\mu(l_1gl_2))| \cdot |\chi(\mu(g))|^{-1} \leq C.$$

According to Lemma 2.4, it suffices to find a constant $C' > 1$ such that for every $g$ in $G$ and $l_1, l_2$ in $L$, we have

$$C'^{-1} \leq \|\rho(l_1gl_2)\| \cdot \|\rho(g)\|^{-1} \leq C'.$$

We may take $C' = \sup_{l \in L} \|\rho(l)\|^2$. \hfill $\square$

5.2 Properness criterion

Theorem. Let $k$ be a local field, $G$ a simply-connected semisimple $k$-group, $G := G_k$ and $\mu : G \to A^+$ a Cartan projection.

Let $H_1$, $H_2$ be two closed subsets of $G$. The pair $(H_1, H_2)$ is $G$-proper if and only if the pair $(\mu(H_1), \mu(H_2))$ is $A$-proper.

The following corollary is a reformulation of this theorem for subgroups, made possible by Lemma 3.1.1.

Corollary. We keep the notations of the theorem. Let $H_1$, $H_2$ be two closed subgroups of $G$. The group $H_1$ acts properly on $G/H_2$ if and only if for every compact subset $M$ of $A$, the set $\mu(H_1) \cap \mu(H_2)M$ is compact.
Remark. Let us give a geometric interpretation of this criterion when $k = \mathbb{R}$. Endow $A$ with an $A$-invariant Riemannian metric and denote by $d$ the corresponding distance. The criterion is that $(H_1, H_2)$ is $G$-proper if and only if for every $R > 0$, the set of pairs of points $(a_1, a_2)$ in $\mu(H_1) \times \mu(H_2)$ such that $d(a_1, a_2) \leq R$ is compact. In other terms, $\mu(H_1)$ and $\mu(H_2)$ get infinitely far apart from each other when approaching infinity.

Proof. Suppose first that $(H_1, H_2)$ is $G$-proper. By definition, for $j = 1, 2$, $\mu(H_j)$ is contained in $H_j$ modulo the compacts of $G$ (see 3.1); we also have the opposite inclusion... but we will not need it. Lemma 3.1.2 proves that the pair $(\mu(H_1), \mu(H_2))$ is $G$-proper, hence it is $A$-proper.

Conversely, suppose that $(\mu(H_1), \mu(H_2))$ is $A$-proper. Let $L$ be a compact subset of $G$ such that $L = KLK$ and $M$ be a compact subset of $A$ as given by Proposition 5.1. We have the inclusion

$$\mu(H_1 \cap LH_2L) \subset \mu(H_1) \cap \mu(H_2)M.$$ 

Since $\mu$ is proper, $H_1 \cap LH_2L$ is compact and $(H_1, H_2)$ is $G$-proper. 

5.3 Examples

Here are a few particular cases of this theorem.

The first one is due to Kobayashi ([Kob1] 4.1) when $k = \mathbb{R}$.

Corollary 5.3.1. With the notations of 2.2. Let $G$ be a semisimple $k$-group, $A$ a maximal $k$-split torus of $G$, $H_1$ and $H_2$ two reductive $k$-subgroups of $G$. For $j = 1, 2$, we denote by $A_j$ a maximal $k$-split torus of $H_j$. We suppose that the $A_j$ are contained in $A$ (we easily reduce the problem to this case, by conjugating the $H_j$ by some element of $G$).

We then have the equivalence:

$$H_1 \text{ acts properly on } G/H_2 \iff \forall w \in W, A_1 \cap wA_2 \text{ is finite.}$$

Proof. We may assume $G$ to be simply connected. By the Cartan decomposition for $H_j$, the group $H_i$ is contained in $A_j$ modulo the compacts of $G$. We consequently have the equivalences:

$$H_1 \text{ acts properly on } G/H_2 \iff A_1 \text{ acts properly on } G/A_2$$

(by Lemma 3.1.2)

$$\iff (\mu(A_1), \mu(A_2)) \text{ is } A\text{-proper}$$

(by the theorem)

$$\iff \forall w \in W, A_0 \cap (A_1 \cap wA_2) \text{ is compact}$$

(since $\mu(A_j) = \bigcup_{w \in W} (wA_j \cap A_+)$)

$$\iff \forall w \in W, A_1 \cap wA_2 \text{ is finite.}$$

Remark. When $k = \mathbb{R}$, the corollary and its proof still hold under the weaker assumption that $H_i$ is reductive in $G$ (i.e. $H_i$ is a connected closed subgroup of $G$ such that the adjoint action of $H_i$ on the Lie algebra of $G$ is semisimple). This latter formulation is the one used in [Kob1].
The second corollary is due to Friedland ([F1]) when $k = \mathbb{R}$.

**Corollary 5.3.2.** Let $G = \text{SL}(n, k)$, $H = \text{SL}(m, k) \times I_{n-m}$. A closed subgroup $\Gamma$ of $G$ acts properly on $G/H$ if and only if for every compact subset $C$ of $k$, the set

$$\Gamma^H_C := \{ g \in \Gamma \mid g \text{ has } m \text{ singular values in } C \}$$

is compact.

**Proof.** Let us take the notations of Example 2.2 and let us write, for $a \in A^+$,

$$a = \begin{pmatrix} \sigma_1 & 0 \\ \vdots & \ddots \\ 0 & \sigma_n \end{pmatrix}.$$ 

This corollary is a consequence of Corollary 5.2 because the Weyl group is the group of permutations of the coordinates and because we have

$$\mu(H) = \{ a \in A^+ \mid \exists i \in \{1, \ldots, n-m+1\}, \sigma_i = \sigma_{i+1} = \cdots = \sigma_{i+m-1} = 1 \}.$$

Let us give just three more examples... but we could easily continue the list.

**Corollary 5.3.3.** The statement of Corollary 5.3.2 also holds for $G = \text{SL}(n, k)$ and

a) $H = \text{SL}(m, k) \times \text{SL}(n-m, k)$, if we take

$$\Gamma^H_C := \{ g \in \Gamma \mid \text{there exist } m \text{ singular values of } g \text{ whose product is in } C \};$$

b) $n = 2m$ and $H = \text{Sp}(m, k)$, if we take

$$\Gamma^H_C := \{ g \in \Gamma \mid \forall i = 1, \ldots, m, \sigma_i(g)\sigma_{2m+1-i}(g) \in C \};$$

c) $\text{char}(k) \neq 2$ and $H = \text{SO}(b)$ is the stabilizer of a nondegenerate symmetric bilinear form $b$ of index $d$, if we take

$$\Gamma^H_C := \left\{ g \in \Gamma \left| \begin{array}{c} \forall i = 1, \ldots, d, \sigma_i(g)\sigma_{n+1-i}(g) \in C; \\
\forall j = d+1, \ldots, n-d, \sigma_j(g) \in C. \end{array} \right. \right\}.$$ 

**Proof.** This is a consequence of Corollary 5.3.2 since we have, respectively in each of the three cases:

$$\mu(H) = \{ a \in A^+ \mid \exists i_1 < \cdots < i_m, \sigma_{i_1} \cdots \sigma_{i_m} = 1 \};$$

$$\mu(H) = \{ a \in A^+ \mid \forall i = 1, \ldots, m, \sigma_i\sigma_{2m+1-i} = 1 \};$$

$$\mu(H) = \{ a \in A^+ \mid \forall i = 1, \ldots, d, \sigma_i\sigma_{n+1-i} = 1; \\
\forall j = d+1, \ldots, n-d, \sigma_j = 1. \}.$$ 

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6 Proximality

This chapter mostly consists of preliminaries about proximal maps which will play a central role in defining and studying the properties of \(\varepsilon\)-Schottky groups in the next chapter.

6.1 Notations

Let \(k\) be a local field, \(V\) a finite-dimensional \(k\)-vector space, \(X := \mathbb{P}(V)\) the projective space of \(V\): it is the set of vector lines in \(V\). We endow \(V\) with a norm \(\|\cdot\|\), and we define on \(X\) a distance \(d\) by

\[
d(x_1, x_2) := \inf \{\|v_1 - v_2\| \mid v_i \in x_i \text{ and } \|v_i\| = 1, \forall i = 1, 2\}.
\]

If \(X_1\) and \(X_2\) are two closed subsets of \(X\), we set

\[
\delta(X_1, X_2) = \inf \{d(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\},
\]

and we denote by

\[
d(X_1, X_2) = \sup \{\delta(x_i, X_{3-i}) \mid x_i \in X_i \text{ and } i = 1, 2\}
\]

the Hausdorff distance between \(X_1\) and \(X_2\).

The following lemma will be useful to us.

**Lemma.** For every \(\varepsilon > 0\), there exists a constant \(r_\varepsilon > 0\) such that, for every hyperplane \(V'\) of \(V\) and for every pair of points \(v_1, v_2\) of \(V\) of norm 1 satisfying \(\delta(kv_i, \mathbb{P}(V')) \geq \varepsilon\) for \(i = 1, 2\), the number \(\alpha \in k\) defined by \(v_1 - \alpha v_2 \in V'\) satisfies \(r_\varepsilon^{-1} \leq |\alpha| \leq r_\varepsilon\).

**Proof.** This follows from compactness of the set of such triples \((V', v_1, v_2)\) and from continuity of the map \((V', v_1, v_2) \mapsto |\alpha|\). \(\square\)

6.2 \(\varepsilon\)-proximality

For \(g\) in \(\text{End}(V) \setminus \{0\}\), we write \(\lambda_1(g) := \sup \{\|\alpha\| \mid \alpha\ \text{eigenvalue of } g\}\). Of course an eigenvalue of \(g\) is generally in a finite extension \(k'\) of \(k\). We have implicitly introduced on this extension the unique absolute value that extends the absolute value of \(k\). Note that \(\lambda_1(g) \leq \|g\|\).

**Definition.** The element \(g\) is said to be proximal if it has a unique eigenvalue \(\alpha\) such that \(|\alpha| = \lambda_1(g)\) and if this eigenvalue has multiplicity 1. This eigenvalue \(\alpha\) is then in \(k\) and we call \(x_\alpha^+ \in X\) the corresponding eigenline. We call \(v_\alpha^+\) a vector of \(x_\alpha^+\) of norm 1, \(V_\alpha^+\) the \(g\)-invariant hyperplane transverse to \(x_\alpha^+\) and \(X_\alpha^+ := \mathbb{P}(V_\alpha^+)\).

The set of proximal maps is an open subset of \(\text{End}(V)\), for the norm topology. On this open set, the maps \(g \mapsto x_\alpha^+\) and \(g \mapsto X_\alpha^+\) are continuous.
In the following definition, we impose a uniform control on proximality. This definition is very close to that of [A-M-S]. We fix $\varepsilon > 0$ and we call
\[
\begin{align*}
\mathcal{B}_g^\varepsilon &:= \{x \in X \mid d(x, x_g^+) \leq \varepsilon\}; \\
\mathcal{B}_g^{\varepsilon} &:= \{x \in X \mid \delta(x, X_g^\varepsilon) \geq \varepsilon\}.
\end{align*}
\]
Note that $\mathcal{B}_g^\varepsilon$ is contained in $\mathcal{B}_g^{\varepsilon}$ as soon as $\delta(x_g^+, X_g^\varepsilon) \geq 2\varepsilon$.

**Definition.** A proximal element $g$ is said to be $\varepsilon$-proximal if $\delta(x_g^+, X_g^\varepsilon) \geq 2\varepsilon$, $g(\mathcal{B}_g^\varepsilon) \subset \mathcal{B}_g^\varepsilon$ and $g|_{\mathcal{B}_g^\varepsilon}$ is $\varepsilon$-Lipschitz.

**Remarks.**
- The “$\varepsilon$-Lipschitz” condition means that for every $x, y$ in $\mathcal{B}_g^\varepsilon$, $d(gx, gy) \leq \varepsilon d(x, y)$.
- If $g$ is $\varepsilon$-proximal then $g^n$ is $\varepsilon$-proximal, for every $n \geq 1$.
- For every proximal element $g$ and for every $\varepsilon > 0$ such that $2\varepsilon \leq \delta(x_g^+, X_g^\varepsilon)$, there exists $n_0 \geq 1$ such that for every $n \geq n_0$, $g^n$ is $\varepsilon$-proximal.
- However, there exist proximal elements which are not $\varepsilon$-proximal for any value of $\varepsilon$.

Here is a sufficient condition for proximality.

**Lemma.** Let $g$ be in $\text{End}(V) \setminus \{0\}$, $x^+$ in $\mathbb{P}(V)$, $W$ a hyperplane of $V$ and $\varepsilon > 0$. We write $Y := \mathbb{P}(W)$, $\mathcal{B}^\varepsilon := \{x \in X \mid d(x, x^+) \leq \varepsilon\}$ and $\mathcal{B}_g^{\varepsilon} := \{x \in X \mid \delta(x, Y) \geq \varepsilon\}$.

Suppose that $\delta(x^+, Y) \geq 6\varepsilon$, $g(\mathcal{B}^\varepsilon) \subset \mathcal{B}^\varepsilon$ and $g|_{\mathcal{B}^\varepsilon}$ is $\varepsilon$-Lipschitz.

Then $g$ is $2\varepsilon$-proximal, $d(x_g^+, x^+) \leq \varepsilon$ and $d(X_g^\varepsilon, Y) \leq \varepsilon$.

**Proof.** The restriction of $g$ to $\mathcal{B}^\varepsilon$ is an $\varepsilon$-Lipschitz contraction. It consequently has an attracting fixed point $x_g^+$. Hence $g$ is proximal. Since $g(\mathcal{B}^\varepsilon) \subset \mathcal{B}^\varepsilon$, we have $d(x_g^+, x^+) \leq \varepsilon$.

Since $\mathcal{B}_g^{\varepsilon}$ is contained in the basin of attraction of $x_g^+$, we have $X_g^{\varepsilon} \cap \mathcal{B}_g^{\varepsilon} = \emptyset$, or in other terms $d(X_g^{\varepsilon}, Y) \geq \varepsilon$.

We deduce that $\delta(x_g^+, X_g^{\varepsilon}) \geq 4\varepsilon$, then that $g(\mathcal{B}_g^{2\varepsilon}) \subset g(\mathcal{B}^\varepsilon) \subset \mathcal{B}^\varepsilon \subset \mathcal{B}_g^{2\varepsilon}$ and finally that $g|_{\mathcal{B}_g^{2\varepsilon}}$ is $\varepsilon$-Lipschitz. Hence $g$ is $2\varepsilon$-proximal.

\vspace{10pt}

**6.3 Norm and largest eigenvalue**

**Lemma.** Let $\varepsilon > 0$. The set of $\varepsilon$-proximal elements of $\text{End}(V)$ is a closed subset of $\text{End}(V) \setminus \{0\}$.

**Proof.** Let $(g_n)$ be a sequence of $\varepsilon$-proximal maps that converges to some nonzero element $g$. Let us first show that $g$ is proximal. Extracting if necessary a subsequence, we lose no generality in assuming that $\lim_{n \to \infty} x_g^{n^+} = x^+$ and $\lim_{n \to \infty} X_g^{n^\varepsilon} = Y$, with the latter limit taken with respect to Hausdorff distance. We introduce the notations $\mathcal{B}^\varepsilon := \{x \in X \mid d(x, x^+) \leq \varepsilon\}$ and $\mathcal{B}_g^{\varepsilon} := \{x \in X \mid \delta(x, Y) \geq \varepsilon\}$. We then have $\lim_{n \to \infty} \mathcal{B}_g^{n^\varepsilon} = \mathcal{B}^\varepsilon$ and $\lim_{n \to \infty} \mathcal{B}_g^{n^\varepsilon} = \mathcal{B}_g^{\varepsilon}$. It follows that $\delta(x^+, Y) \geq 2\varepsilon$, $g(\mathcal{B}^\varepsilon) \subset \mathcal{B}^\varepsilon$ and
Proof. Let $n$ and $g$ convention $j = 1$ then $x^j$ ensures that $x = x^j$ and $Y = X^<_g$, hence that $g$ is $\varepsilon$-proximal. □

The following corollary says that for an $\varepsilon$-proximal element $g$, $\lambda_1(g)$ is a good approximation for the norm of $g$.

**Corollary.** Let $V$ be a finite-dimensional $k$-vector space and $\varepsilon > 0$. There exists a constant $c_\varepsilon \in (0, 1)$ such that for every $\varepsilon$-proximal linear transformation $g$ of $V$, we have

$$c_\varepsilon \|g\| \leq \lambda_1(g) \leq \|g\|.$$  

Proof. This follows from compactness of the set of $\varepsilon$-proximal linear transformations of $V$ having norm 1 as well as from continuity of the map $g \mapsto \lambda_1(g)$. □

### 6.4 Product of $\varepsilon$-proximal elements

The following proposition gives an approximation of $\|g\|$ when $g$ is a word whose letters are $\varepsilon$-proximal elements.

**Proposition.** For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ with the following property. Let $g_1, \ldots, g_l$ be any tuple of $\varepsilon$-proximal linear transformations of $V$ satisfying (using the convention $g_0 := g_l$):

$$\delta(x^+_{y_{j-1}}, X^<_y) \geq 6\varepsilon \quad \text{for } j = 1, \ldots, l.$$  

Then for any $n_1, \ldots, n_l \geq 1$, the product $g = g_{l}^{n_l} \cdots g_i^{n_i}$ is $2\varepsilon$-proximal and we have

$$C_\varepsilon^{-1} \leq \lambda_1(g) \cdot \prod_{1 \leq j \leq l} \lambda_1(g_j)^{-n_j} \leq C_\varepsilon^l$$  

and

$$C_\varepsilon^{-l-1} \leq \|g\| \cdot \prod_{1 \leq j \leq l} \lambda_1(g_j)^{-n_j} \leq C_\varepsilon^{l+1}.$$  

Proof. Let $x^+_j, v^+_j$, $X^<_j$, $V^<_j$, $B^<_j$, $b^<_j$ denote respectively $x^+_g$, $v^+_g$ etc. Given that $g^+_j$ is $\varepsilon$-proximal, that $x^+_j = x^+_g$ and that $X^<_j = X^<_g$, we may assume that $n_j = 1$ for every $j = 1, \ldots, l$.

We have the inclusion $g_1(B^<_1) \subset b^<_1 \subset B^<_2$ since $\delta(x^+_1, X^<_2) \geq 2\varepsilon$. Similarly we have $g_2 g_1(B^<_1) \subset b^<_2$; by iterating, we get that $g(B^<_1) \subset b^<_l$ and that $g|_{B^<_1}$ is $\varepsilon$-Lipschitz. We may apply Lemma 6.2 since $\delta(x^+_1, X^<_1) \geq 6\varepsilon$. We get that $g$ is $2\varepsilon$-proximal and that $x^+_g \in b^<_l$.

Let $w_0 := v^+_g$, $y_0 := x^+_g$ and, for $j = 1, \ldots, l$,

$$w_j := g_j w_{j-1} \text{ and } y_j := g_j y_{j-1}.$$
By construction, we have

\[
\begin{cases}
  y_j \in b_j^\varepsilon \quad \text{for } j = 0, \ldots, l; \\
  w_l = \lambda_1(g)w_0.
\end{cases}
\]

For \( j = 1, \ldots, l \), let \( \alpha_j \in k \) be the number defined by the equality

\[ w_{j-1} = \alpha_j v^+_j \pmod{V_j^\varepsilon}. \]

Since \( \delta(y_{j-1}, X_j^\varepsilon) > 5\varepsilon \) and \( \delta(x_j^+, X_j^\varepsilon) \geq 2\varepsilon \), Lemma 6.1 shows that

\[ r^{-1}_\varepsilon \leq \frac{|\alpha_j|}{\|w_{j-1}\|} \leq r_\varepsilon. \]

We also have

\[ w_j = \alpha_j\lambda_1(g_j)v^+_j \pmod{V_j^\varepsilon}. \]

Since \( \delta(y_j, X_j^\varepsilon) \geq \varepsilon \), the same Lemma 6.1 shows that

\[ r^{-1}_\varepsilon \leq \frac{|\alpha_j|\lambda_1(g_j)}{\|w_j\|} \leq r_\varepsilon. \]

These two inequalities yield

\[ r^{-2}_\varepsilon \leq \frac{\|w_j\|}{\|w_{j-1}\|} \lambda_1(g_j)^{-1} \leq r^2_\varepsilon. \]

Multiplying these \( l \) inequalities together and remarking that \( \frac{\|w_l\|}{\|w_0\|} = \lambda_1(g) \), we get

\[ r^{-2l}_\varepsilon \leq \lambda_1(g) \cdot \prod_{1 \leq j \leq l} \lambda_1(g_j)^{-1} \leq r^{2l}_\varepsilon, \]

and then using Corollary 6.3

\[ r^{-2l}_\varepsilon \leq \|g\| \cdot \prod_{1 \leq j \leq l} \lambda_1(g_j)^{-1} \leq r^{2l}_\varepsilon c^{-1}_\varepsilon. \]

This proves our proposition if we set \( C_\varepsilon := \max(r^2_\varepsilon, c^{-1}_\varepsilon) \).

**6.5 \( \varepsilon \)-Schottky subgroup on \( \mathbb{P}(V) \)**

The following definition is motivated by Proposition 6.4.

**Definition.** Let \( \varepsilon > 0 \) and \( t \geq 2 \). We say that a subsemigroup (resp. subgroup) \( \Gamma \) of \( \text{GL}(V) \) with generators \( \gamma_1, \ldots, \gamma_t \) is \( \varepsilon \)-Schottky on \( \mathbb{P}(V) \) if it satisfies the following properties. We set \( E := \{\gamma_1, \ldots, \gamma_t\} \) (resp. \( E := \{\gamma_1, \ldots, \gamma_t, \gamma_1^{-1}, \ldots, \gamma_t^{-1}\} \)).

i) For any \( h \in E \), \( h \) is \( \varepsilon \)-proximal.

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ii) For any \( h, h' \) in \( E \) (resp. \( h, h' \) in \( E \) with \( h' \neq h^{-1} \), \( \delta(x_h^+, X_{h'}^-) \geq 6\varepsilon \).

Remarks.

- Of course, this definition depends on the choice of the generators \( \gamma_j \) and of the norm on \( V \).
- If the semigroup (resp. group) \( \Gamma \) with generators \( \gamma_1, \ldots, \gamma_t \) is \( \varepsilon \)-Schottky on \( P(V) \), then so is the semigroup (resp. group) \( \Gamma_m \) with generators \( \gamma_1^m, \ldots, \gamma_t^m \), for every \( m \geq 1 \).
- A subgroup \( \Gamma \) with generators \( \gamma_1, \ldots, \gamma_t \) that is \( \varepsilon \)-Schottky on \( P(V) \) is discrete in \( GL(V) \) and is a free group on these generators \( \gamma_1, \ldots, \gamma_t \). This follows from the ping-pong lemma (see \[ Ti2 \] 1.1).

7 Schottky groups

The goal of this section is to prove Theorem 7.4 which generalizes Proposition 1.5. We then deduce Theorem 7.5 which generalizes Theorem 1.1. We finish this section by proving the corollaries from the introduction.

7.1 Jordan decomposition

Let \( G \) be a semisimple \( k \)-group with \( k \)-rank \( r \geq 1 \) (in other terms \( G \) is isotropic) and \( G := G_k \). In order to simplify some formulations, we shall embed the semigroup \( A^+ \) into a salient convex cone with nonempty interior \( A^\ast \), contained in some \( r \)-dimensional \( \mathbb{R} \)-vector space \( A^\ast \).

When \( k = \mathbb{R} \) or \( \mathbb{C} \), we set \( A^\ast := A^0 \) and \( A^\ast := A^\ast \) (see 2.2). The identification of \( A^\ast \) with its Lie algebra makes it an \( \mathbb{R} \)-vector space.

When \( k \) is non-Archimedean, \( A^0 \) is a free \( \mathbb{Z} \)-module of rank \( r \). We set \( A^\ast := A^0 \otimes \mathbb{Z} \mathbb{R} \) and we define \( A^\ast \) to be the convex hull of \( A^+ \) in \( A^\ast \).

The goal of this subsection is to define a map \( \lambda : G \rightarrow A^\ast \) that we shall call the Lyapunov projection. Though it is not strictly necessary, it makes things clearer to treat the Archimedean and non-Archimedean cases separately.

Suppose first that \( k = \mathbb{R} \) or \( \mathbb{C} \). This is the easier case. Every element \( g \) of \( G \) has a unique decomposition \( g = g_\ell g_h g_u \) into a product of three pairwise commuting elements of \( G \), with \( g_\ell \) elliptic, \( g_h \) hyperbolic and \( g_u \) unipotent (see for example \[ Kos \] 2.1). We set \( \lambda(g) \) to be the unique element of \( A^+ \) that is conjugate to \( g_h \).

Suppose now that \( k \) is non-Archimedean. Then such a decomposition of \( g \) does not always exist. Example: \( g = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \in G = SL(4, \mathbb{Q}_p) \) where \( h = \begin{pmatrix} 1 & p-1 \\ 1 & -1 \end{pmatrix} \) has characteristic polynomial \( X^2 - p \). However, we have the following lemma which is probably well-known.

**Lemma.** There exists \( n \geq 1 \) such that for every element \( g \) of \( G \), the element \( g' := g^n \) has a unique decomposition \( g' = g'_\ell g'_h g'_u \) into a product of three pairwise commuting
elements of $G$, with $g^\prime_u$ elliptic (i.e. $\text{Ad}(g^\prime_u)$ is semisimple with eigenvalues of modulus 1), $g^\prime_u$ unipotent and $g^\prime_h$ conjugate to an element $a'$ of $A^\ast$. This element $a'$ is unique.

**Definition.** We set $\lambda(g) := \frac{1}{n} a' \in A^\ast$.

It is clear that $\lambda(g)$ does not depend on the choice of $n$.

**Proof.** Let us realize $G$ as a $k$-subgroup of $\text{SL}(V)$ where $V$ is a $k$-vector space of dimension $d$ whose $k$-weights generate $X^\ast(A)$.

Let $g$ be an element of $G$. It has a unique so-called Jordan decomposition $g = g_u g_s$ into a product of two commuting elements of $G$, with $g_u$ semisimple and $g_s$ unipotent. We may assume that $g_s$ and $g_u$ are in $G$: this always holds when $\text{char}(k) = 0$; if $\text{char}(k) = p > 0$, it suffices to replace $g$ by $g^p$ since $g^p_u = 1$.

Let $n := d!$. The moduli of the eigenvalues of $g' := g^n$ are in $|\pi|^2$. Let $g' = g'_s g'_u$ be the Jordan decomposition of $g'$. Then there exists a unique decomposition of $g'_u$ into a product $g'_s = g'_s g'_h$ of two commuting semisimple elements of $\text{SL}(V)$ such that $g'_s$ (resp. $g'_h$) has eigenvalues of modulus 1 (resp. in $k^o$). By construction, every vector subspace of the algebra $k[V]$ of polynomial functions on $V$ that is $g'_s$-invariant is also $g'_u$- and $g'_h$-invariant. It follows that $g'_s$ and $g'_h$ are in $G$ and commute with $g'_u$. The element $g'_h$ is in a $k$-split one-dimensional torus. Hence it is conjugate to an element $a'$ of $A$. Since the eigenvalues of $a'$ are in $k^o$, $a'$ is in $A^o$. Replacing if necessary $a'$ by some $w \cdot a'$ where $w$ is in the Weyl group, the element $a'$ is in $A^\ast$.

Uniqueness of $a'$ is clear since, with the notations of 2.3 for every $i = 1, \ldots, r$, $|\omega_i(a')|$ is the largest among the moduli of the eigenvalues of $\rho_i(g')$.

The opposition involution $\iota : A^\ast \to A^\ast$ uniquely extends to an $\mathbb{R}$-linear map from $A^\ast$ to itself that preserves $A^\ast$, which we shall also denote by $\iota$. We have the equality, for every $g$ in $G$:

$$\lambda(g^{-1}) = \iota(\lambda(g)).$$

**Definition.** We say that a subset $\Omega$ of $A^\ast$ is a convex open cone if $\Omega$ is the intersection of $A^\ast$ with some convex open cone $\Omega^\ast$ in the $\mathbb{R}$-vector space $A^\ast$.

For $g$ in $G$ such that $\lambda(g) \neq 1$, we call $A_g$ the half-line of $A^\ast$ containing $\lambda(g)$.

### 7.2 $\varepsilon$-Schottky groups

We choose $r$ irreducible representations $(V_i, \rho_i)$ of $G$ whose highest $k$-weights $\omega_i$ have multiplicity one and are $r$ independent characters of $A$ (Lemma 2.3). We fix some norm on each of the $k$-vector spaces $V_i$.

**Definition.** Let $\varepsilon > 0$ and $t \geq 2$. We say that a subsemigroup (resp. subgroup) $\Gamma$ of $G$ with generators $\gamma_1, \ldots, \gamma_t$ is $\varepsilon$-Schottky if, for $i = 1, \ldots, r$, the subsemigroup (resp. subgroup) $\rho_i(\Gamma)$ of $\text{GL}(V_i)$ with generators $\rho_i(\gamma_1), \ldots, \rho_i(\gamma_t)$ is $\varepsilon$-Schottky on $\mathbb{P}(V_i)$.

We then set

$$E_\Gamma := \{\gamma_1, \ldots, \gamma_t\} \quad (\text{resp. } E_\Gamma := \{\gamma_1, \ldots, \gamma_t, \gamma_1^{-1}, \ldots, \gamma_t^{-1}\}).$$
Remarks.

- Of course this definition depends on the choice of the representations \( \rho_i \), of the norms on \( V_i \) and of the generators \( \gamma_j \).

- The remarks made in 6.5 for \( \varepsilon \)-Schottky groups on \( \mathbb{P}(V) \) are also valid for \( \varepsilon \)-Schottky groups.

Definition. A word \( w = g_l \cdots g_1 \) with \( g_j \) in \( E\Gamma \) is said to be reduced if \( g_{j-1}^{-1} \neq g_j^{-1} \) for \( j = 2, \ldots, l \), and very reduced if additionally we have \( g_1 \neq g_l^{-1} \).

When \( \Gamma \) is a Schottky subsemigroup, every word is very reduced since, for any \( h, h' \) in \( E\Gamma \), we have \( h' \neq h^{-1} \).

The following lemma allows us to construct \( \varepsilon \)-Schottky semigroups (resp. groups).

Lemma. (\( G \) is an isotropic semisimple \( k \)-group) Let \( a_1, \ldots, a_j, \ldots \) be elements of \( A^{++} \).

a) We may choose elements \( \gamma_1, \ldots, \gamma_t \) of \( G \) such that, setting \( E := \{ \gamma_1, \ldots, \gamma_t, \gamma_1^{-1}, \ldots, \gamma_t^{-1} \} \), we have:

i) \( \lambda(\gamma_j) = a_j \) for every \( j = 1, \ldots, t \). In particular, for every \( h \) in \( E \) and \( i = 1, \ldots, r \), \( \rho_i(h) \) is proximal.

ii) \( x^+_{\rho_i(h)} \notin X^<_{\rho_i(h')} \) for any \( h, h' \) in \( E \) (resp. for any \( h, h' \) in \( E \) with \( h' \neq h^{-1} \)) and \( i = 1, \ldots, r \).

iii) The semigroup generated by \( \gamma_j \) is Zariski-connected, for every \( j = 1, \ldots, t \).

iv) The semigroup \( \Gamma \) generated by \( E \) is Zariski-dense in \( G \).

b) For every such choice, there exists \( m_a \geq 1 \) and \( \varepsilon > 0 \) such that for every \( m \geq m_a \), the subsemigroup (resp. subgroup) \( \Gamma_m \) with generators \( \gamma_1^m, \ldots, \gamma_t^m \) is \( \varepsilon \)-Schottky and Zariski-dense.

Proof.

a) For every element \( a \) of \( A^{++} \) and \( i = 1, \ldots, r \), the elements \( \rho_i(a) \) and \( \rho_i(a^{-1}) \) are proximal. The points \( x^+_i := x^+_{\rho_i(a)} \) and \( x^-_i := x^+_{\rho_i(a^{-1})} \), as well as the sets \( X^<_i := X^<_i(\rho_i(\gamma_j)) \) and \( X^>_i := X^<_i(\rho_i(\gamma_j)^{-1}) \), do not depend on the choice of \( a \) in \( A^{++} \). Also the semigroup generated by \( a \) is Zariski-connected.

We may assume that \( G \) is simply connected. We then decompose \( G \) into a product of a \( k \)-group \( G_{an} \) that is anisotropic and of a \( k \)-group \( G_{an} \) that has no anisotropic factor. The group \( A \) is contained in \( G_{as} \) and we have \( \rho_i(G_{an}) = 1 \) for every \( i = 1, \ldots, r \).

We shall construct by induction on \( j \) elements \( \gamma_j \) satisfying i), ii) and iii). Let \( b \) be an element of \( G_{an} \) that generates a Zariski-connected subsemigroup and that
is not contained in any proper normal $k$-subgroup of $G_{an}$. It suffices to take
$\gamma_j = h_j ba_j h_j^{-1}$ where $h_j$ is in the Zariski-open set

$$U_j := \left\{ h \in G \left| \begin{array}{l}
\rho_i(h) \cdot x_i^\alpha \notin \rho_i(h_j')(X_i^< \cup X_i^>) \\
\rho_i(h_j') \cdot x_i^\alpha \notin \rho_i(h)(X_i^< \cup X_i^>)
\end{array} \right. \quad \forall \alpha = \pm, \forall i = 1, \ldots, r
\right\}.$$

This Zariski-open set is nonempty since $G$ is Zariski-connected and $\rho_i$ is an irreducible representation.

It remains to check iv). Let $G_j$ denote the Zariski-closure in $G$ of the semigroup generated by $\gamma_1, \ldots, \gamma_j$. If $G_j - 1 \neq G$, we can choose $h_j$ in the Zariski-open set $U_j \cap V_j$ where $V_j := \{ h \in G \left| hbah^{-1} \notin G_{j-1} \right. \}$. This Zariski-open set is nonempty since $ba$ is not contained in any proper normal $k$-subgroup of $G$. The increasing sequence of Zariski-connected and Zariski-closed subgroups $G_j$ is necessarily stationary. Hence we have $G_j = G$ for sufficiently large $j$.

b) It suffices to take $\varepsilon$ such that, for every $h, h' \in E$ (resp. $h, h' \in E$ with $h' \neq h^{-1}$) and $i = 1, \ldots, r$, we have

$$6\varepsilon \leq \frac{\lambda(\omega(h) \cdot X_i^< \cup X_i^>)}{\lambda(h) \cdot X_i^< \cup X_i^>}.$$

We then use Remark 6.2 to deduce that the group $\Gamma_m$ is $\varepsilon$-Schottky with generators $\gamma^m_1, \ldots, \gamma^m_t$. The Zariski-closure of $\Gamma_m$ contains each $\gamma_j$ by iii), hence it is equal to $G$ by iv).

7.3 Cartan projection of an $\varepsilon$-Schottky group

Now that we know how to construct $\varepsilon$-Schottky groups, we need to calculate their Cartan projection. So let us assume $G$ is simply connected and let $\mu : G \to A^+$ be some Cartan projection.

**Proposition.** For every $\varepsilon > 0$, there exists a compact subset $M_\varepsilon$ of $A^*$ that has the following property.

For every $\varepsilon$-Schottky subsemigroup (resp. subgroup) $\Gamma$ of $G$ with generators $\gamma_1, \ldots, \gamma_t$ and for every very reduced word $w = g^n_t \cdots g^n_1$ with $g_j$ in $E_{\Gamma}$ and $n_j \geq 1$, we have

$$\lambda(w) - \sum_{1 \leq j \leq l} n_j \lambda(g_j) \in l \cdot M_\varepsilon \quad \text{and} \quad \mu(w) - \sum_{1 \leq j \leq l} n_j \lambda(g_j) \in (l + 1) \cdot M_\varepsilon.$$

**Remark.** We have used additive notation for addition in $A^*$, even though it extends multiplication in $A^n$ for which we had used multiplicative notation!

**Proof.** The morphisms $a \mapsto |\omega_i(a)|$ from $A^+$ to $(0, \infty)$ can be uniquely extended to continuous morphisms from the group $A^*$ to the multiplicative group $(0, \infty)$, that we shall denote by $\theta_i$. 27
We have for every \( g \) in \( G \)
\[
\theta_i(\lambda(g)) = \lambda_1(\rho_i(g)).
\]

Let us denote by \( C_{\omega_i} \) the constants introduced in (2.4) and let \( C := \sup_{1 \leq i \leq r} C_{\omega_i} \). We then have, for every \( g \) in \( G \) and \( i = 1, \ldots, r \),
\[
C^{-1} \| \rho_i(g) \| \leq \theta_i(\mu(g)) \leq C \| \rho_i(g) \|.
\]

Let \( C_\varepsilon \) be the constant introduced in (6.4) for a \( k \)-vector space with larger dimension than all of the \( V_i \), let \( C'_\varepsilon := C C_\varepsilon \) and let us introduce the compact subset of \( A^* \)
\[
M_\varepsilon := \{ a \in A^* \mid C'^{-1}_\varepsilon \leq \theta_i(a) \leq C'_\varepsilon \quad \forall i = 1, \ldots, r \}.
\]

Our statement is then a consequence of the following upper bounds given by Lemma 2.4 and Proposition 6.4

\[
\theta_i \left( \lambda(w) - \sum_{1 \leq j \leq l} n_j \lambda(g_j) \right) = \frac{\lambda_1(\rho_i(w))}{\prod_{1 \leq j \leq l} \lambda_1(\rho_i(g_j))^{n_j}} \in [C^{-1}_\varepsilon, C'_i] \quad \text{and}
\]
\[
\theta_i \left( \mu(w) - \sum_{1 \leq j \leq l} n_j \lambda(g_j) \right) = \frac{\theta_i(\mu(w))}{\| \rho_i(w) \|} \frac{\| \rho_i(w) \|}{\prod_{1 \leq j \leq l} \lambda_1(\rho_i(g_j))^{n_j}} \in [C^{-1}_\varepsilon, C'^{-1}_i, C'^{l+1}_i].
\]

**Corollary.** Let \( \Gamma \) be an \( \varepsilon \)-Schottky subsemigroup (resp. subgroup) of \( G \) with generators \( \gamma_1, \ldots, \gamma_l \), and let \( \Omega^* \) be an open convex cone of \( A^* \) containing the half-lines generated by \( \lambda(\gamma_1), \ldots, \lambda(\gamma_l) \) (resp. \( \lambda(\gamma_1), \ldots, \lambda(\gamma_l), \lambda(\gamma_l^{-1}), \ldots, \lambda(\gamma_1^{-1}) \)).

Then there exists \( m_0 \geq 1 \) such that, for all \( m \geq m_0 \), the \( \varepsilon \)-Schottky subsemigroup (resp. subgroup) \( \Gamma_m \) of \( G \) with generators \( \gamma_1^m, \ldots, \gamma_l^m \) satisfies \( \mu(\Gamma_m) \subset \Omega \cup \{ 1 \} \).

**Proof.** Let us deal with the case where \( \Gamma \) is a group (the case of a semigroup is easier). Let us first of all introduce, using Proposition 6.4 a compact subset \( M \) of \( A^* \) such that, for every \( w \) in \( G \),
\[
\mu \left( \{ w, w \gamma_1^{-1}, w \gamma_2^{-1} \} \right) \subset \mu(w) + M.
\]

Let \( g \) be an element of \( \Gamma_m \). We express it as a reduced word \( g = g_1^m \cdots g_l^m \). One of the three words \( w = g, g \gamma_1 \) or \( g \gamma_2 \) is very reduced. Hence we can apply the previous proposition to that word. Let \( M' := \{ 0, \lambda(\gamma_1), \lambda(\gamma_2) \} \). We then have
\[
\mu(g) \in \mu(w) + M \subset \left( \sum_{1 \leq j \leq l} m \lambda(g_j) \right) + M + M' + (l + 2) M_\varepsilon \subset \sum_{1 \leq j \leq l} (m \lambda(g_j) + M''),
\]
where \( M'' \) is some convex compact subset of \( A^* \) containing both 0 and \( M + M' + 3M_\varepsilon \). Hence it suffices to take \( m \) large enough for \( \lambda(g_j) + \frac{1}{m} M'' \) to be contained in \( \Omega \). □
Remark. Let \( M_\Gamma \) and \( \Lambda_\Gamma \) denote the smallest closed convex cones in \( A^\bullet \) containing respectively \( \mu(\Gamma) \) and \( \lambda(\Gamma) \). This calculation also proves that as \( m \) tends to infinity, \( M_{\Gamma_m} \) and \( \Lambda_{\Gamma_m} \) converge, in the sense of the Hausdorff distance in the projective space corresponding to \( A^\bullet \), to the convex hull of the half-lines \( \Lambda_{\gamma_1}, \ldots, \Lambda_{\gamma_t} \) (resp. \( \Lambda_{\gamma_1}, \ldots, \iota(\Lambda_{\gamma_1}), \ldots, \iota(\Lambda_{\gamma_t}) \)).

7.4 Construction of the group \( \Gamma \)

**Theorem.** Let \( k \) be a local field, \( G \) a simply-connected isotropic semisimple \( k \)-group, \( G := G_k, \mu : G \to A^+ \) some Cartan projection and \( \iota : A^+ \to A^+ \) the opposition involution.

a) Let \( \Omega \) be a nonempty convex open cone in \( A^+ \) (see 7.1). Then there exists a discrete subsemigroup \( \Gamma \), Zariski-dense in \( G \), such that \( \mu(\Gamma) \subset \Omega \cup \{1\} \).

b) If additionally \( \iota(\Omega) = \Omega \), then there exists a discrete free subgroup \( \Gamma \), Zariski-dense in \( G \), such that \( \mu(\Gamma) \subset \Omega \cup \{1\} \).

**Proof.** Let us choose some points \( a_j \) in \( \Omega \cap A^{++} \). We start by constructing elements \( \gamma_1, \ldots, \gamma_t \) of \( G \) as in Lemma 7.2. We then have \( \lambda(\gamma_j) = a_j \in \Omega \) and, since \( \iota(\Omega) = \Omega \), we have \( \lambda(\gamma_j^{-1}) = \iota(a_j) \in \Omega \). Then Lemma 7.2 and Corollary 7.3 imply that there exists \( m \geq 1 \) such that the semigroup (resp. group) \( \Gamma_m \) with generators \( \gamma_1^m, \ldots, \gamma_t^m \) is \( \varepsilon \)-Schottky hence in particular discrete and free, that it is Zariski-dense and that \( \mu(\Gamma) \subset \Omega \cup \{1\} \). \( \square \)

7.5 Criterion for existence of a free subgroup acting properly on \( G/H \)

**Theorem.** Let \( k \) be a local field, \( G \) a semisimple \( k \)-group, \( H \) a reductive \( k \)-subgroup of \( G \), \( A_H \) a maximal \( k \)-split torus of \( H \), \( A \) a maximal \( k \)-split torus of \( G \) containing \( A_H \), \( G, H, A, A_H, W \) the Weyl group of \( G \) in \( A \), \( A^+ \) a positive Weyl chamber and \( B^+ \) the subset of \( A^+ \) formed by the fixed points of the opposition involution (see 2.2).

There exists a discrete, non virtually unipotent subgroup \( \Gamma \) acting properly on \( G/H \) if and only if for every \( w \) in \( W \), \( wA_H \) does not contain \( B^+ \).

In this case, we can always choose \( \Gamma \) to be free and Zariski-dense in \( G \).

**Remark.** When \( \text{char}(k) = 0 \), we can replace the “non virtually nilpotent” condition by the “non virtually abelian” condition.

**Proof.** By (Mar2) 1.1.5.5 and I.2.3.1) we may assume that \( G \) is simply connected. We have the equality

\[
\mu(H) = \mu(A_H) = \bigcup_{w \in W} (wA_H \cap A^+).
\]

If there exists \( w \) in \( W \) such that \( wA_H \) contains \( B^+ \), then \( \mu(H) \) also contains \( B^+ \) and the conclusion follows from Theorem 3.3 and Corollary 4.1.

Otherwise, we can find a convex open cone \( \Omega^* \) in \( A^* \) that is invariant by \( \iota \) and whose closure has trivial intersection with each of the \( \mathbb{R} \)-vector subspaces of \( A^* \) generated by
some $\omega^* \cap A^+$. We set $\Omega := \Omega^* \cap A^+$. We then choose a discrete free subgroup $\Gamma$ Zariski-dense in $G$ such that $\mu(\Gamma) \subset \Omega \cap \{1\}$ (Theorem 7.4). Corollary 5.2 then proves that this group $\Gamma$ acts properly on $G/H$. \qed

For semigroups, the same proof furnishes the following result.

**Proposition.** Let $G$ be a semisimple $k$-group, $H$ a reductive $k$-subgroup of $G$, and $H$ the $k$-points. Suppose that $\text{rank}_k(H) \neq \text{rank}_k(G)$.

Then there exists a Zariski-dense discrete free subsemigroup of $G$ that acts properly on $G/H$.

**Remark.** ([Kob1]) When $\text{rank}_k(H) = \text{rank}_k(G)$, the only discrete subsemigroups acting properly on $G/H$ are finite.

### 7.6 Proof of the corollaries in the introduction

Corollary 1 is a particular case of the following corollary.

**Corollary.** (char $k = 0$) We keep the notations of Theorem 7.5.

If there exists $w$ in $W$ such that $\omega A_H$ contains $B^+$, then $G/H$ has no compact quotient.

**Proof.** This follows from Theorem 7.5 and Corollary 1.1 \qed

To prove Corollary 2, it suffices to verify, by ([Kob2] 1.9), that the following examples have no compact quotients:

$$
G_C/H_C = \text{SO}(4n + 2, \mathbb{C})/\text{SO}(4n + 1, \mathbb{C}) \quad (n \geq 1),
$$

$$
\text{SL}(2n, \mathbb{C})/\text{Sp}(n, \mathbb{C}) \quad (n \geq 2)
$$

and

$$
E_{6, C}/F_{4, C}.
$$

To prove Corollary 3, we must verify that

$$
G/H = \text{SO}(2n + 1, 2n + 1)/\text{SO}(2n, 2n + 1) \quad (n \geq 1)
$$

has no compact quotient.

In each of these cases, we verify that for a suitable choice of $A^+$, the set $B^+$ is contained in $H$; and we apply the previous corollary.

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