Asymptotic enumeration of sparse connected 3-uniform hypergraphs

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Abstract

We derive an asymptotic formula for the number of connected 3-uniform hypergraphs with vertex set $[N]$ and $M$ edges for $M = N/2 + R$ as long as $R$ satisfies $R = o(N)$ and $R = \omega(N^{1/3} \ln^2 N)$. This almost completely fills the gap in the range of $M$ for which the formula is known. We approach the problem using an ‘inside-out’ approach of an earlier paper of Pittel and the second author, for connected graphs. A key part of the method uses structural components of connected hypergraphs called cores and kernels. These are structural components of connected hypergraphs. Our results also give information on the numbers of them with a given number of vertices and edges, and hence their typical size in random connected 3-uniform hypergraphs with $N$ vertices and $M$ edges, for the range of $M$ we consider.

1 Introduction

The problem of counting connected graphs with given number of vertices and edges has been intensively studied throughout the years. One of the best results is an asymptotic formula by Bender, Canfield and McKay [3] that works when the excess $m - n \to \infty$ as $n \to \infty$, where $m = m(n)$ is the number of edges and $n$ is the number of vertices. Pittel and Wormald [9] rederived this formula with improved error bounds for some ranges. Significantly less is known about connected hypergraphs. Karoński and Łuczak [5] derived an asymptotic formula for the number of connected $k$-uniform hypergraphs on $[N]$ with $M$ hyperedges for the range of small excess where $M = N/(k - 1) + o(\ln N/\ln \ln N)$, which is a range with small excess. This was later extended by Andriamampianina and Ravelomanana [1] for $M = N/(k - 1) + o(N^{1/3})$, which still has very small excess. Regarding results for denser hypergraphs, Behrisch, Coja-Oghlan and Kang [2] provided an asymptotic formula for the case $M = N/(k - 1) + \Theta(N)$. Thus, there is a gap between the case $M - N/(k - 1) = o(N^{1/3})$ and the linear case $M - N/(k - 1) = \Omega(N)$ in which no asymptotic formulae were found. The case $M - N/(k - 1) = \omega(N)$ is also open.

In this paper, we obtain an asymptotic formula for the number of connected 3-uniform hypergraphs with vertex set $[N]$ and $M$ edges for $M = N/2 + R$ as long as $R$ satisfies $R = o(N)$ and $R = \omega(N^{1/3} \ln^2 N)$. This leaves only a tiny remaining gap, between $M - N/2 = o(N^{1/3})$ and $M - N/2 = \omega(N^{1/3} \ln^2 N)$. Our technique is based on an approach that Pittel and Wormald [9] used to the enumerate connected graphs. With this technique, we also obtain information on the sizes of a kind of core and kernel in the graphs being counted. We restrict ourselves to the 3-uniform case because the complexity of this approach increases for the more general case. The results in this paper are contained in the PhD thesis [10] of the first author.

Behrisch, Coja-Oghlan and Kang [2] obtained their enumeration result by precisely estimating the joint distribution of the number of vertices and the number of edges in the giant component of the random hypergraph. We remark that the distribution of the number of vertices and edges has recently been

∗This research was supported in part by the Canada Research Chairs Program, and currently by an ARC Australian Laureate Fellowship.
described independently by Bollobás and Riordan [4], but that result does not provide point probabilities, which would allow the enumeration result to be deduced.

From our results, it will be possible to determine the joint distribution of the giant component’s excess, core size and kernel size in the random \( k \)-uniform hypergraph with \( n \) vertices and \( m \) random edges, for the range of density that we have covered in this paper. This should be straightforward along the lines of the analogous argument for the case \( k = 2 \) in [9].

2 Main result

A hypergraph is a pair \((V, \mathcal{E})\), where \( V \) is a finite set and \( \mathcal{E} \) is a subset of nonempty sets in \( 2^V \), which is the set of all subsets of \( V \). The elements in \( V \) are called vertices and the elements in \( \mathcal{E} \) are called hyperedges. For any integer \( k \geq 2 \), a \( k \)-uniform hypergraph is a hypergraph where each hyperedge has size \( k \). For any hypergraph \( G \), a path is a (finite) sequence \( v_1E_1v_2E_2\ldots v_kE_k \), where \( v_1, \ldots, v_k \) are distinct vertices and \( E_1, \ldots, E_{k-1} \) are distinct hyperedges such that \( v_i, v_{i+1} \in E_i \) for all \( i \in [k-1] \). We say that a hypergraph is connected if, for any vertices \( u \) and \( v \), there exists a path from \( u \) to \( v \).

An \((N, M, k)\)-hypergraph is a \( k \)-uniform hypergraph with \( V = [N] \) and \( M \) edges. Let \( C(N, M) \) denote the number of connected \((N, M, 3)\)-hypergraphs. Our main result is an asymptotic formula for \( C(N, M) \) for a sparse range of \( M \). For \( k \geq 0 \), define \( g_k(\lambda) = \exp(\lambda) + k \) and recall that \( f_k(\lambda) = \exp(\lambda) - \sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \).

Theorem 2.1. Let \( M = M(N) = N/2 + R \) be such that \( R = o(N) \) and \( R = \omega(N^{1/3}\ln^2 N) \). Then

\[
C(N, M) \sim \sqrt{\frac{3}{\pi N}} \exp \left( N\phi(\tilde{n}^*) + N \ln N - N \right),
\]

where

\[
\left(1 - x\right) \ln(1 - x) + \frac{1 - x}{2} + \frac{2R}{N} \ln(N) - \left(\ln(2) + 2\right) \frac{R}{N} - \frac{1}{2} \ln(2)x + \frac{R}{N} \ln \left( \frac{g_1(\lambda^{**})}{\lambda^{**}f_1(\lambda^{**})} \right) + \frac{1}{2} x \ln \left( \frac{f_1(\lambda^{**})g_1(\lambda^{**})}{\lambda^{**}} \right),
\]

\[
\tilde{n}^* = \frac{f_2(2\lambda^{**})}{f_1(\lambda^{**})g_1(\lambda^{**})},
\]

and \( \lambda^{**} \) is the unique positive solution of

\[
\lambda \frac{e^{2\lambda} + e^\lambda + 1}{f_1(\lambda)g_1(\lambda)} = \frac{3M}{N}. \tag{1}
\]

Our proof basically follows one of the two approaches that Pittel and Wormald [9] use to the enumerate connected graphs in the sparser range. This involves decomposing a connected graph into two parts: a cyclic structure and an acyclic structure. The cyclic structure is a pre-kernel, which is a 2-core without isolated cycles. The acyclic structure is a rooted forest where the roots are the vertices of the pre-kernel. A rooted forest with roots \( r_1, \ldots, r_t \) (that are vertices in the forest) simply is a forest such that each component contains exactly one of the roots. The graph can then be obtained by ‘gluing’ these two structures together. Pittel and Wormald obtain an asymptotic formula for the number of the cyclic structures and combine it with a known formula for the acyclic parts to obtain an asymptotic formula for the number of connected graphs with given number of vertices and edges.
We will also decompose a connected 3-uniform hypergraph into two parts: a cyclic structure (which we will also call pre-kernel) and an acyclic structure (a forest rooted on the vertices of the pre-kernel). We will also obtain asymptotic formulae for these structures and then combine them to obtain an asymptotic formula for the number of connected \((N, M, k)\)-hypergraphs.

From now on, we will deal with 3-uniform hypergraphs most of the time, so henceforth, for convenience, we will use the word ‘graph’ to denote 3-uniform hypergraphs. When we want to refer to graphs in the usual sense, we will call them ‘2-uniform hypergraphs’. We will also use the word ‘edge’ instead of ‘hyperedge’.

We often give the results of routine algebraic manipulation and expansions for which we have used Maple. The interested reader can see [10, Appendix A] for more details of these computations. Please note that we give a glossary of notation at the end of this article.

3 Relation to a known formula

As we mentioned before, Behrisch, Coja-Oghlan and Kang [2] provided an asymptotic formula for the number of connected \((N, M, k)\)-hypergraphs for the range \(M = N/(k-1) + \Omega(N)\). In this section, we show that, for \(k = 3\), their formula is asymptotic to ours when \(R = M - N = o(N)\). Behrisch, Coja-Oghlan and Kang obtained their result by computing the probability that the random hypergraph \(H_k(N, M)\) with uniform distribution on all \((N, M, k)\)-hypergraphs is connected.

**Theorem 3.1** ([2] Theorem 5). Let \(k \geq 2\) be a fixed integer. For any compact set \(J \subset (k(k-1)^{-1}, \infty)\), and for any \(\delta > 0\) there exists \(N_0 > 0\) such that the following holds. Let \(M = M(N)\) be a sequence of integers such that \(\zeta = \zeta(N) = kM/N \in J\) for all \(N\). Then there exists a unique number \(0 < r = r(N) < 1\) such that

\[
r = \exp \left( -\zeta \frac{(1-r)(1-r^{-k})}{1-r^k} \right).
\]

Let \(\Phi(\zeta) = r^{\gamma/(1-r)}(1-r)^{1-\zeta(1-rk)^{\zeta/k}}\). Furthermore, let

\[
R_2(N, M) = \frac{1 + r - \zeta r}{\sqrt{(1-r)^2 - 2r}} \exp\left( \frac{2\zeta r + \zeta^2 r}{2(1+r)} \right) \Phi(\zeta)^N, \quad \text{and set}
\]

\[
R_k(N, M) = \frac{1 - r^k - (1-r)\zeta(k-1)r^{-k+1}}{\sqrt{(1-r^k + \zeta(k-1)(1-r^{-k-1})(1-r^{-k}) - \zeta kr(1-r^{-k-1})^2}}
\times \exp\left( \frac{\zeta(k-1)(r - 2r^k + r^{-k-1})}{2(1-r^k)} \right) \Phi(\zeta)^N, \quad \text{if } k > 2.
\]

For \(N > N_0\), the probability that \(H_k(N, M)\) is connected is in \(((1-\delta)R_k(N, M), (1+\delta)R_k(N, M))\).

From this theorem, it is immediate that the number of connected \((N, M, k)\)-graphs is asymptotic to

\[
\binom{N}{k} R_k(N, M) =: D(N, M, k)
\]

when \(R = M - N/2 = \Omega(N)\). Next we assume \(R/N = o(1)\) and do some simplifications in \(D(N, M, 3)\). So suppose \(R = M - N/2 = o(N)\). First we compare \(r\) in [2] with \(\lambda^{**}\) in [11]:

\[
r = \exp \left( -\frac{3M}{N} \frac{(1-r)(1-r^2)}{1-r^3} \right) \quad \text{and} \quad \lambda^{**} e^{2\lambda^{**}} + e^{\lambda^{**}} + 1 = \frac{3M}{N}.
\]
By taking the logs in both sides of the definition of $r$, it is obvious that $r = \exp(-\lambda^{**})$. As we will see later, $\lambda^{**} \to 0$ and so $r \to 1$. Then by expanding we find that

$$
\lim_{r \to 1} \frac{1 - r^3 - 2(1 - r)\zeta r^2}{\sqrt{(1 - r^3 + 2\zeta(r - r^2))(1 - r^3) - 3\zeta r(1 - r^2)^2}} = \sqrt{3}
$$

and

$$
\lim_{r \to 1} \frac{\zeta(r - 2r^3 + r^2)}{1 - r^3} = 3/2.
$$

Thus,

$$
D(N, M, k) \sim \left(\frac{\binom{N}{3}}{M}\right) \sqrt{3} \exp(3/2) \Phi(3M/N)^N
$$

and

$$
\sim \sqrt{2\pi M} \left(\frac{M}{e}\right)^M \sqrt{2\pi \left(\binom{N}{3} - M\right) \left(\frac{N}{3} - M\right)} \binom{N}{3} \cdot \sqrt{3} \exp(3/2) \Phi(3M/N)^N,
$$

by Stirling’s approximation. Thus, using $M = N/2 + R$ and $R = o(N)$,

$$
D(N, M, k) \sim \sqrt{\frac{3}{\pi N}} \exp \left(\frac{3}{2} + \left(\frac{N}{3}\right) \ln \left(\frac{N}{3}\right) - M \ln M - \left(\frac{N}{3} - M\right) \ln \left(\frac{N}{3} - M\right) + N \Phi(3M/N)\right)
$$

$$
= \sqrt{\frac{3}{\pi N}} \exp \left(\frac{3}{2} - \left(\frac{N}{3} - M\right) \ln \left(1 - \frac{M}{\binom{N}{3}}\right) - M \ln M + M \ln \left(\frac{N}{3}\right) + N \Phi(3M/N)\right)
$$

$$
= \sqrt{\frac{3}{\pi N}} \exp \left(\frac{3}{2} - \left(\frac{N}{3} - M\right) \left(-\frac{M}{\binom{N}{3}} + O\left(\frac{M^2}{\binom{N}{3}^2}\right)\right) - M \ln M + M \ln \left(\frac{N}{3}\right) + N \Phi(3M/N)\right).
$$

Thus,

$$
D(N, M, k) = \sqrt{\frac{3}{\pi N}} \exp \left(\frac{3}{2} + M - M \ln M + M \ln \left(\frac{N}{3}\right) + N \Phi(3M/N) + o(1)\right)
$$

$$
= \sqrt{\frac{3}{\pi N}} \exp \left(\frac{3}{2} + M - M \ln M + M \ln \frac{N^3}{6} + M \ln \frac{N(N - 1)(N - 2)}{N^3} + N \Phi(3M/N) + o(1)\right)
$$

$$
= \sqrt{\frac{3}{\pi N}} \exp \left(\frac{3}{2} + M - M \ln M + 3M \ln N - M \ln 6 + M \ln \left(1 - \frac{3N - 2}{N^2}\right) + N \Phi(3M/N) + o(1)\right)
$$

$$
= \sqrt{\frac{3}{\pi N}} \exp \left(\frac{3}{2} + M - M \ln M + 3M \ln N - M \ln 6 - M \frac{3N}{N^2} + N \Phi(3M/N) + o(1)\right)
$$

$$
\sim \sqrt{\frac{3}{\pi N}} \exp \left(M - M \ln M + 3M \ln N - M \ln 6 + N \Phi(3M/N)\right),
$$

which is exactly the same as our formula in Theorem 2.1 after a series of routine algebraic simplifications.
4 Basic definitions and results for hypergraphs

In this section, we present some basic definitions for hypergraphs and show how to decompose a hypergraph into a cyclic structure and an acyclic structure.

A cycle in a hypergraph $G = (V, E)$ is a (finite) sequence $(v_0, E_0, \ldots, v_k, E_k)$ such that $v_1, \ldots, v_k \in V$ are distinct vertices, $E_1, \ldots, E_k \in E$ are distinct edges with $v_i \in E_i$ and $v_{i+1} \in E_i$ for every $0 \leq i \leq k$ (operations in the indices are in $\mathbb{Z}_{k+1}$). A tree is an acyclic connected hypergraph and a forest is an acyclic hypergraph. A rooted forest $G = (V, E)$ with set of roots $S \subseteq V$ is a forest such that each component of the forest has exactly one vertex in $S$. See Figure 1 for a rooted forest.

The degree of a vertex $v$ in a hypergraph $G$ is the number of edges in $G$ containing $v$. Recall that we use the word ‘graph’ to denote 3-uniform hypergraphs. The core of a graph is its maximal induced subgraph such that every edge contains at least two distinct vertices of degree at least 2. To see that the core of a hypergraph is unique, it suffices to notice that the union of two cores would also be a core. We remark that the $k$-core of a graph is usually defined as the maximal subgraph such that every vertex has degree at least $k$. The core we defined contains the 2-core of the hypergraph and it allows some vertices of degree 1. We also say that a graph is a core, when its core is the graph itself. For an example of a core see Figure 2.

Every edge in a core has either one vertex of degree 1, or none. We say that an edge is a 2-edge if it has a vertex of degree 1 and that it is a 3-edge if it has no vertex of degree 1. It is easy to see that the core of graph can be obtained by iteratively removing edges that are not 2-edges nor 3-edges until all edges are 2-edges or 3-edges, and then deleting all vertices of degree 0. See Figure 3 for an example of this procedure.

We also define cycles as graphs. We say that a graph $G = (V, E)$ is a cycle if there is an ordering $(v_0, \ldots, v_k)$ of a subset of $V$ and an ordering $(E_0, \ldots, E_k)$ of $E$ such that $(v_0, E_0, \ldots, v_k, E_k)$ is a cycle in $G$ and every $v \in V$ is in some $E \in E$. Note that, if a graph is a cycle, then all edges are actually 2-edges. An isolated cycle in a graph is a component that is a cycle. A pre-kernel is a core without isolated cycles (see Figure 2). So, every connected core that is not just a cycle is also a pre-kernel.

The following proposition explains how to decompose a graph into its core and a rooted forest.
Proposition 4.1. Let $G$ be a connected graph with a nonempty core. The graph obtained from $G$ by deleting the edges of the core of $G$ and by setting all vertices in the core as roots is a rooted forest with $(N - n)/2$ edges, where $N$ is the number of vertices in $G$ and $n$ is the number of vertices in the core. Moreover, the core of $G$ is connected.

Proof. As we already mentioned, the core of $G$ can be obtained by iteratively deleting edges that contain at most one vertex of degree at least 2. More precisely, start with $G' = G$ and while there is an edge in $G'$ containing less than 2 vertices of degree at least 2 in $G'$, redefine $G'$ by deleting one such edge. When this procedure stops, $G'$ is the core of $G$. Let $F$ be the graph with vertex set $[N]$ with the deleted edges as its set of edges. Suppose for a contradiction that $F$ has a cycle. Such a cycle is a cycle in $G$ too. Let $E$ be the first edge of the cycle that was deleted by the procedure described above. All other edges in the cycle were still present in the graph $G'$ when $E$ was deleted. Thus, since $E$ was in the cycle, it had a least 2 vertices of degree at least 2. Hence, $E$ could not have been deleted at this point, which shows that $F$ has no cycles.

Suppose for contradiction that the core of $G$ is not connected. Then it has at least 2 components that are joined by a path in $G$ with all edges in $F$ since $G$ is connected. The union of these 2-components and the path is a 2-core, which is a contradiction. Thus, the core is connected. This argument also shows that each component of $F$ has at most one vertex in the core. Every component of $F$ must have one vertex in the core, otherwise it is disconnected from the core and so $G$ would not be connected.

Now we determine the number of edges in $F$. As we discussed above, each component of $F$ has exactly one vertex in the core. In the deletion procedure, for the initial $G'$ (that is, $G$), every edge has at least one vertex of degree at least 2 since otherwise $G$ would not be connected. We claim that the deletion procedure will only delete edges that contain exactly one vertex of degree 2 in the current $G'$. If not, let $E$ be an edge that contained no vertex of degree at least 2 in $G'$ in the moment it was deleted. Let $v_0$ be the vertex of the core in the same component of $E$ in $F$. Then there is a path $(v_0, E_0, \ldots, E_{k-1} v_k)$ in $F$, where $E_{k-1} = E$. The edge $E_0$ cannot be $E$ since the vertex $v_0$ must have degree at least 2 the moment $E_0$ is deleted. A trivial induction proof then shows that the deletion procedure cannot delete any of the edges $E_0, \ldots, E_{k-2}$ before deleting $E_{k-1}$, which shows that the moment $E$ was deleted the vertex $v_{k-1}$ still was in 2 edges: $E$ and $E_{k-2}$. This is a contradiction. Thus, the moment any edge is deleted is has exactly one vertex of degree at least 2. This means that, for every deleted edge, we also delete exactly 2 vertices that are not in the core. Since there $N - n$ vertices to be deleted, the number of edges in $F$ is $(N - n)/2$. 

For any graph $G$ with $N$ vertices and $M$ edges such that its core has $n$ vertices and $m$ edges, we have that

$$m - n/2 = M - (N - n)/2 - n/2 = M - N/2$$

(3)

since $m = M - (N - n)/2$ by Proposition 4.1. Intuitively speaking, this says that the ‘excess’ of edges $(M - N/2)$ in the graph is transferred to its core.
Let $g_{\text{forest}}(N, n)$ denote the number of forests with vertex set $[N]$ and $[n]$ as its set of roots. Let $g_{\text{pre}}(n, m)$ denote the number of connected pre-kernels with vertex set $[n]$ and $m$ edges. Next, we show how to write $C(N, M)$ using $g_{\text{forest}}$ and $g_{\text{pre}}$.

**Proposition 4.2.** For $M = M(N)$ such that $R := M - N/2 \to \infty$, we have that

$$C(N, M) = \sum_{1 \leq n \leq N \atop (N - n) / 2 \in \mathbb{Z}} g_{\text{forest}}(N, n)g_{\text{pre}}(n, M - (N - n)/2),$$

for $N$ sufficiently large.

**Proof.** In view of Proposition 4.1, it suffices to show that, for any connected graph $G$ with $N$ vertices and $M$ edges, the core of $G$ is a pre-kernel. If it is not, either the core is empty or it is a cycle. If the core is empty, then the graph $G$ is a forest and so $M < N/2$, which is impossible since $M = N/2 + R$ with $R \to \infty$. If the core is a cycle, then $3m = 2(n - m) + m$ since each edge in the core has two vertices of degree 2 and one of degree 1. Thus, in this case, we have that $m = n/2$, which is impossible since $m - n/2 = M - N/2 = R \to \infty$ by (3).

Basically, our approach to compute an asymptotic formula for $C(N, M)$ will be to analyse the summation in (4).

We will work with random graphs. More precisely, we will work with random multihypergraphs and then deduce results for simple graphs. A $k$-uniform multihypergraph is a triple $G = (V, E, \Phi)$, where $V$ and $E$ are finite sets and $\Phi : E \times [k] \to V$. We say that $V$ is the vertex set of $G$ and $E$ is the edge set of $G$. From now on, we will use the word ‘multigraph’ to denote 3-uniform multihypergraphs.

Given a multigraph $G = (V, E, \Phi)$, a loop is an edge $E \in E$ such that there exist distinct $j, j' \in \{1, 2, 3\}$ such that $\Phi(E, j) = \Phi(E, j')$, a pair of double edges is a pair $(E, E')$ of distinct edges in $E$ such that the collection $\{\Phi(E, 1), \Phi(E, 2), \Phi(E, 3)\}$ is the same as the collection $\{\Phi(E', 1), \Phi(E', 2), \Phi(E', 3)\}$. A multigraph $G$ with no loops nor double edges corresponds naturally to a graph because each edge corresponds to a unique subset of $V$ of size 3. In this case we say that the multigraph is simple. Let $S(n, m)$ denote the set of simple multigraphs with vertex set $[n]$ and edge set $[m]$. We have the following relation between simple multigraphs and graphs:

**Lemma 4.3.** For any $G = ([n], [m], \Phi) \in S(n, m)$, let $s(G)$ be the graph with vertex set $[n]$ obtained by including one edge for each $i \in [m]$ incident to the vertices $\Phi(i, 1), \Phi(i, 2)$ and $\Phi(i, 3)$. Let $G'$ be a graph with vertex set $[n]$ with $m$ edges. Then $|s^{-1}(G')| = m!6^m$, that is, each graph corresponds to $m!6^m$ simple multigraphs.

**Proof.** Let $G = ([n], [m], \Phi) \in S(n, m)$ be such that $s(G) = G'$. For any permutation $g$ of $[m]$, the multigraph $G_g := ([n], [m], \Phi')$ satisfies $s(G_g) = G'$, where $\Phi'(i, j) = \Phi(g(i), j)$ for each $i \in [m]$ and $j \in \{1, 2, 3\}$. (That is, any permutation of the label of the edges generates the same graph.) Moreover, for each $i \in [m]$ and permutation $g_i$ of $[3]$, the function $\Phi''(i, j) = \Phi''(i, g(j))$ satisfies $s([n], [m], \Phi'') = G'$. Since there are $m!$ permutations on $[m]$ and $3!$ permutations of $[3]$, the number of graphs $G \in S(n, m)$ with $s(G) = G'$ is $m!3!6^m$.

We extend the definitions of path and connectedness for multihypergraphs. For any multihypergraph $G = (V, E, \Phi)$, a path is a (finite) sequence $v_1E_1v_2E_2\ldots v_k$, where $v_1, \ldots, v_k$ are distinct vertices and $E_1, \ldots, E_{k-1}$ are distinct hyperedges such that $v_i, v_{i+1} \in \text{Im}(\Phi(E_i, \cdot))$ for all $i \in [k - 1]$. We say that a multihypergraph is connected if, for any vertices $u$ and $v$, there exists a path from $u$ to $v$. 
5 Overview of proof

In this section, we give an overview of our proof of the asymptotic formula for \( C(N, M) \) in Theorem 2.1. Recall that \( R = M - N/2 = o(N) \) and \( R = \omega(N^{1/3} \log^2 N) \). Our approach is to analyse \( C(N, M) \) by using (4), which shows how to obtain \( C \) from formulae for the number of rooted forests \( g_{\text{forest}} \) and the number of pre-kernels \( g_{\text{pre}} \). The proof consists of the following steps.

1. We obtain an exact formula \( g_{\text{forest}}(N, n) \) for the number of rooted forests with set of roots \([n]\) and vertex set \([N]\). We show that, for even \( N - n \),

\[
g_{\text{forest}}(N, n) = \frac{n}{N} \cdot \frac{(N - n)!N^{(N-n)/2}}{((N - n)/2)!2^{(N-n)/2}},
\]

and, for odd \( N - n \), \( g_{\text{forest}}(N, n) = 0 \). The proof is in Section 9 and is a simple proof by induction.

2. We show that the number of cores with vertex set \([n]\) and \( m \) edges is at most the following function of \( n \) and \( m \)

\[
g_{\text{core}}(n, m) := \alpha n \sqrt{m} \cdot n! \exp(n f_{\text{core}}(\hat{n}_1^*)) \text{, for } m - n/2 \to \infty,
\]

where \( \alpha \) is a constant, the function \( f_{\text{core}} \) is defined in Section 8 and \( \lambda^* \) is the unique positive solution for \( \lambda f_1(\lambda)g_2(\lambda)/f_2(2\lambda) = 3m/n \) and \( \hat{n}_1^* = 3m/(\pi g_2(\lambda^*) \). The proof is in Section 8.

3. We obtain an asymptotic formula for the number \( g_{\text{pre}}(n, m) \) of simple connected pre-kernels with \( n \to \infty \) vertices and \( m = n/2 + \gamma n \) edges when \( R = o(n) \) and \( R = \omega(n^{1/2} \log^3/2 n) \). We show that

\[
g_{\text{pre}}(n, m) \sim \sqrt[3]{\frac{3}{\pi n}} \cdot n! \exp(n f_{\text{pre}}(\hat{x}^*)),
\]

where \( f_{\text{pre}} \) is defined in Section 9 and \( \hat{x}^* \in \mathbb{R}^4 \) will be determined using \( \lambda^* \) as defined in the previous step.

4. We define a set \( I \subseteq \mathbb{Z} \) such that (5) holds for every \( n \in I \) with \( m = M - (N - n)/2 \). Using (5), we show that, for \( n \in I \) and \( m = M - (N - n)/2 \),

\[
g_{\text{pre}}(n, m) \sim \sqrt[3]{\frac{3}{\pi n}} \cdot n! \exp(n f_{\text{core}}(\hat{n}_1^*)),
\]

where \( \hat{n}_1^* \) is defined in Step 2. Using (8), we then show that

\[
\sum_{n \in I} \binom{N}{n} g_{\text{forest}}(N, n) g_{\text{pre}}(n, m) \sim \sqrt{\frac{3}{\pi N}} \exp \left( N t(\hat{x}^*) + N \ln N - N \right)
\]

where \( t \) is defined in Section 10, \( \lambda^{**} \) is the unique positive solution of the equation \( \lambda(e^{2\lambda} + e^\lambda + 1)/(f_1(\lambda)g_1(\lambda)) = 3M/N \) and \( \hat{x}^* = f_2(2\lambda^{**})/(f_1(\lambda^{**})g_1(\lambda^{**})) \).

5. Since every pre-kernel is a core and \( g_{\text{core}}(n, m) \) is an upper bound for the number of cores with vertex set \([n]\) and \( m \) edges, we have that \( g_{\text{pre}}(n, m) \leq g_{\text{core}}(n, m) \). Using this relation together with (2), we show that

\[
\sum_{n \in [N] \setminus I} \binom{N}{n} g_{\text{forest}}(N, n) g_{\text{pre}}(n, m) \leq \sum_{n \in [N] \setminus I} \binom{N}{n} g_{\text{forest}}(N, n) g_{\text{core}}(n, m)
\]

\[
= o \left( \frac{1}{\sqrt{\pi N}} \exp \left( N t(\hat{x}^*) + N \ln N - N \right) \right).
\]
Hence, together with (7), we have that
\[ C(N, M) \sim \sqrt{\frac{3}{\pi \lambda^*}} N \exp \left( Nt(\bar{n}^*) + N \ln N - N \right). \]

6. The conclusion is then easily obtained by simplifying \( t(\bar{n}^*) \).

6 Counting forests

In this section we prove an exact formula for rooted forests. In this section we consider \( k \)-uniform hypergraphs, for any \( k \geq 2 \). We remark that this formula has also been proved in a note by Lavault [6] around the same time we obtained it. Lavault shows a one-to-one correspondence between rooted forests and a set of tuples whose size can be easily computed.

Recall that \( g_{\text{forest}}(N, n) \) is the number of rooted forests on \([N]\) with set of roots \([n]\). (See Figure 1 for a rooted forest.)

**Theorem 6.1.** For integers \( N \geq n \geq 0 \) and any integer \( k \geq 2 \),

\[
g_{\text{forest}}(N, n) = \begin{cases} 
\frac{n(N-n)!N^{m'-1}}{m'!(k-1)!m''}, & \text{if } m' = \frac{N-n}{k-1} \text{ is a nonnegative integer;} \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** A connected \( k \)-uniform hypergraph is a tree if and only if, by iteratively deleting edges that have at least \( k-1 \) vertices of degree 1, we delete all edges. It then is obvious that \( m' \) is the number of edges in the forest. We remark that the a tree can be seen as a 2-uniform hypergraph where each block is a clique on \([n-1]\) vertices (which is known as a clique tree).

The proof is by induction on \( N \). We have that \( g_{\text{forest}}(1, 1) = 1 = 1(1-1)!0^{-1}/(0!(1-1)!0) = 1 \) and the formula also works for \( g_{\text{forest}}(N, 0) = 0 \). So assume that \( N > 1 \) and \( n \geq 1 \). We will show how to obtain a recurrence relation for \( g_{\text{forest}}(N, n) \). Suppose that the vertex 1 is in \( j \) edges, where \( 0 \leq j \leq m' \). We choose \((k-1)j\) other vertices to be in these \( j \) edges. There are \( \binom{N-n}{(k-1)j} \) ways to choose these vertices. The number of ways we can split the vertices into the edges is

\[
\binom{(k-1)j}{k-1} \binom{(k-1)j-(k-1)}{k-1} \cdots \binom{k-1}{k-1} \frac{1}{j!} = \frac{(k-1)j!}{(k-1)!j!}.
\]

We can build the rooted forest by first choosing the edges containing 1 and then deleting 1 and considering the other \((k-1)j\) vertices in these edges as new roots. This gives us the following recurrence:

\[
g_{\text{forest}}(N, n) = \sum_{j=0}^{m'} \binom{N-n}{(k-1)j} \frac{(k-1)j!}{(k-1)!j!} g_{\text{forest}}(N-1, n-1 + (k-1)j).
\]

Note that \( 0 \leq n-1 + (k-1)j \leq N-1 \) since \( j \in [0, m'] \). The new number of edges is \( m'' = \frac{1}{k-1}((N-1) -
(n - 1 + (k - 1)j)) = m' - j. So, by induction hypothesis,

\[
g_{\text{forest}}(N, n) = \sum_{j=0}^{m'} \binom{N - n}{(k-1)j} \frac{(n - 1 + (k - 1)j)(N - n - (k - 1)j)!}{(k - 1)!j!} \frac{(n - 1 + (k - 1)j)(N - n - (k - 1)j)!}{(m' - j)!j!(m' - j - 1)!}
\]

\[
= \frac{(N - n)!}{(N - 1)(k - 1)!m'} \sum_{j=0}^{m'} \frac{(n - 1 + (k - 1)j)(N - 1)^{m' - j}}{j!(m' - j)!}
\]

\[
= \frac{(N - n)!}{m'!(N - 1)(k - 1)!m'} \sum_{j=0}^{m'} \binom{m'}{j} (n - 1 + (k - 1)j)(N - 1)^{m' - j}
\]

\[
= \frac{(N - n)!}{m'!(N - 1)(k - 1)!m'} \left( (n - 1) \sum_{j=0}^{m'} \binom{m'}{j} (N - 1)^{m' - j} + (k - 1) \sum_{j=0}^{m'} \binom{m'}{j} j(N - 1)^{m' - j} \right).
\]

Using the Binomial Theorem,

\[
\sum_{j=0}^{m'} \binom{m'}{j} (N - 1)^{m' - j} = N^{m'}
\]

and by differentiating both sides with respect to \(N\),

\[
\sum_{j=0}^{m'} \binom{m'}{j} (m' - j)(N - 1)^{m' - j - 1} = m'N^{m' - 1},
\]

and so

\[
\sum_{j=0}^{m'} \binom{m'}{j} j(N - 1)^{m' - j} = m' \sum_{j=0}^{m'} \binom{m'}{j} (N - 1)^{m' - j} - m'N^{m' - 1}(N - 1)
\]

\[
= m'N^{m'} - m'N^{m' - 1}(N - 1) = m'N^{m' - 1}.
\]

Hence,

\[
g_{\text{forest}}(N, n) = \frac{(N - n)!}{m'!(N - 1)(k - 1)!m'} \left( (n - 1)N^{m'} + (k - 1)m'N^{m' - 1} \right)
\]

\[
= \frac{(N - n)!N^{m' - 1}}{m'!(N - 1)(k - 1)!m'} (N(n - 1) + N - n)
\]

\[
= \frac{n(N - n)!N^{m' - 1}}{m'!(k - 1)!m'},
\]

and we are done. \(\square\)

7 Tools

In this section, we include some definitions and computations that will be used a number of times throughout the proofs.

Given a nonnegative real number \(\lambda\) and a nonnegative integer \(k\), we say that a random variable \(Y\) is a truncated Poisson with parameters \((k, \lambda)\) if, for every \(j \in \mathbb{N}\),

\[
P(Y = j) = \begin{cases} 
\frac{\lambda^j}{j!f_k(\lambda)}, & \text{if } j \geq k; \\
0, & \text{otherwise.}
\end{cases}
\]
where
\[ f_k(\lambda) := e^\lambda - \sum_{i=0}^{k-1} \frac{\lambda^i}{i!}. \] (9)

We use Po\((k, \lambda)\) to denote the distribution of a truncated Poisson random variable with parameters \((k, \lambda)\). Throughout this paper, we often use properties of truncated Poisson random variables proved by Pittel and Wormald [8].

Pittel and Wormald [8, Lemma 1] showed that, for every \(c > k\), there exists a unique positive real \(\lambda\) such that
\[ \frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = c. \] (10)

Note that this implies that, for any \(c > k\), there exists \(\lambda > 0\) such that the expectation of a random variable with distribution Po\((k, \lambda)\) is \(c\).

The first derivative of \(\lambda f_{k-1}(\lambda)/f_k(\lambda)\) is obviously a continuous function and it is positive for \(\lambda > 0\) (as compute in the proof of [8, Lemma 1]). From this, one obtains the following lemma:

**Lemma 7.1.** Let \(\gamma\) and \(k\) be positive integer constants with \(\gamma > k\). Let \(\alpha(n), \beta(n)\) be function such that \(k < \alpha(n) < \beta(n) < \gamma\) and \(|\alpha(n) - \beta(n)| = o(\phi)\) where \(\phi = o(1)\). Then \(|\lambda(k, \alpha) - \lambda(k, \beta)| = o(\phi)\).

Let \(k\) be a positive integer. Let \(c : \mathbb{R} \to \mathbb{R}\) so that \(c(y) > k\) for all \(y \in \mathbb{R}\). Let \(\lambda(y)\) be defined by
\[ \frac{\lambda(y) f_{k-1}(\lambda(y))}{f_k(\lambda(y))} = c(y). \]

The existence and uniqueness of \(\lambda(y)\) follow from [8, Lemma 1]. We compute the derivative \(\lambda'\) of \(\lambda(y)\) by implicit differentiation. Assuming that \(c\) is differentiable with derivative \(c'\):
\[ \lambda' \frac{f_{k-1}(\lambda(y))}{f_k(\lambda(y))} \left(1 + \frac{\lambda(y) f_{k-2}(\lambda(y))}{f_{k-1}(\lambda(y))} - \frac{\lambda(y) f_{k-1}(\lambda(y))}{f_k(\lambda(y))}\right) = c'. \] (11)

Let \(T, t : \mathbb{R} \to \mathbb{R}\) be differentiable functions be such that \(T(y)/t(y) > k\) for all \(y \in \mathbb{R}\). Let \(t'\) and \(T'\) denote the derivatives of \(t\) and \(T\), resp. We will compute the derivative of \(t(y) \log f_k(\lambda(y)) - T(y) \log(\lambda(y))\). For \(c(y) = T(y)/t(y) = \lambda(y) f_{k-1}(\lambda(y))/f_k(\lambda(y))\) and \(\eta(y) = \lambda(y) f_{k-2}(\lambda(y))/f_{k-1}(\lambda(y))\), and using (11),
\[ \frac{d}{dy} \left(t(y) \log f_k(\lambda(y)) - T(y) \log(\lambda(y))\right) = \]
\[ = t' \log f_k(\lambda(y)) + \lambda' \frac{t(y) f_{k-1}(\lambda(y))}{f_k(\lambda(y))} - T' \log(\lambda(y)) - \lambda' \frac{T(y)}{\lambda(y)} \]
\[ = t' \log f_k(\lambda(y)) + \lambda' \frac{t(y) f_{k-1}(\lambda(y))}{f_k(\lambda(y))} - \lambda' \frac{\lambda(y) c'}{c'(y)(1 + \eta(y) - c(y))} \]
\[ - T' \log(\lambda(y)) - \lambda' \frac{T(y)}{\lambda(y) c(y)(1 + \eta(y) - c(y))} \]
\[ = t' \log f_k(\lambda(y)) + \frac{t(y) c'}{1 + \eta(y) - c(y)} - T' \log(\lambda(y)) - \frac{t(y) c'}{1 + \eta(y) - c(y)} \]
\[ = t' \log f_k(\lambda(y)) - T' \log(\lambda(y)). \] (12)

The following lemma is an application of standard results concerning Gaussian functions and the definition of Riemann integral.
Lemma 7.2. Let $\phi(n) \to 0$, $\psi(n) \to 0$, $T_n \to \infty$ and $s_n \to \infty$. Let $f_n = \exp(-\alpha x^2 + \beta x + \phi x^2 + \psi x)$ with constants $\alpha > 0$ and $\beta$. Let $P_n = z + Z$, where $z \in \mathbb{R}$. Then
\[
\frac{1}{s_n} \sum_{x \in P_n/s_n} f_n(x) \sim \exp \left( \frac{\beta^2}{4\alpha} \right) \sqrt{\frac{\pi}{\alpha}}.
\]

Proof. Let $\varepsilon \in (0, \min(\alpha, \beta))$ and let $f^+(x) = \exp(-\alpha x^2 + \beta x + \varepsilon x^2 + \varepsilon x)$ and $f^-(x) = \exp(-\alpha x^2 + \beta x - \varepsilon x^2 - \varepsilon x)$. Since $\phi = o(1)$ and $\psi = o(1)$, we may assume $f^-(x) \leq f_n(x) \leq f^+(x)$. We will show that
\[
\frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) \sim \exp \left( \frac{(\beta + \varepsilon)^2}{4(\alpha + \varepsilon)} \right) \sqrt{\frac{\pi}{\alpha + \varepsilon}}.
\]
and
\[
\frac{1}{s_n} \sum_{x \in P_n/s_n} f^-(x) \sim \exp \left( \frac{(\beta - \varepsilon)^2}{4(\alpha - \varepsilon)} \right) \sqrt{\frac{\pi}{\alpha - \varepsilon}}.
\]
Since we can choose $\varepsilon$ arbitrarily close to zero, this proves the lemma. We will only show the proof for (13) since the proof for (14) is very similar. We have that
\[
\int_{-\infty}^{\infty} f^+(x) dx = \lim_{C \to \infty} \int_{-C}^{C} f^+(x) dx
\]
and
\[
\lim_{-\infty}^{\infty} f^+(x) dx = e^{\frac{(\beta + \varepsilon)^2}{4(\alpha + \varepsilon)}} \sqrt{\frac{\pi}{\alpha + \varepsilon}}.
\]
So it suffices to show that,
\[
\left| \lim_{n \to \infty} \frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) - \lim_{C \to \infty} \int_{-C}^{C} f^+(x) dx \right| = 0.
\]
We have that
\[
\left| \lim_{n \to \infty} \frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) - \lim_{C \to \infty} \int_{-C}^{C} f^+(x) dx \right| \leq
\]
\[
\left| \lim_{n \to \infty} \frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) - \lim_{C \to \infty} \frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) \right|
\]
\[
+ \left| \lim_{C \to \infty} \frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) - \lim_{C \to \infty} \int_{-C}^{C} f^+(x) dx \right|
\]
the last term goes to is zero by the definition of Riemann integral. It is known that the tail for Gaussian functions is very small. More precisely, for each $\varepsilon' > 0$ there exists $n_0$ such that, for each $n \geq n_0$,
\[
\left| \frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) - \lim_{n \to \infty} \frac{1}{s_n} \sum_{x \in P_n/s_n} f^+(x) \right| \leq \varepsilon'.
\]
Since $C \to \infty$, $C$ is eventually bigger than $T_{n_0}$. And we are done since we can choose $\varepsilon' > 0$ arbitrarily small. 

\section{Counting cores}

In this section we obtain an upper bound for the number of cores with vertex set $[n]$ and $m = n/2 + R$ edges, when $R \to \infty$. We remark that the asymptotics in this section are for $n \to \infty$. We will always use $r$ to denote $R/n$.

For $n_1 \in \mathbb{R}$, define

\[ n_2(n_1) = n - n_1, \]
\[ m_3(n_1) = m - n_1, \]
\[ Q_2(n_1) = 3m - n_1, \]
\[ c_2(n_1) = Q_2(n_1)/n_2(n_1) = (3m - n_1)/(n - n_1). \]

For any symbol $y$ in this section, we use $\hat{y}$ to denote $y/n$.

We will use $n_1$ as the number of vertices of degree 1 in the core. Then $n_2(n_1)$ is the number of vertices of degree at least 2, $m_3(n_1)$ is the number of 3-edges, $Q_2(n_1)$ is the sum of degrees of vertices of degree at least 2, and $c_2(n_1)$ is the average degree of the vertices of degree at least 2. We omit the argument $n_1$ when it is obvious from the context.

Let $J_m$ denote the set of reals $n_1$ such that $\max\{0, 2n - 3m\} \leq n_1 \leq \min\{n, m\}$. The lower bound $2n - 3m$ is used to ensure that $c_2(n_1) \geq 2$ for $n_1 \in J_m$. Let $\hat{J}_m = \{x/n : x \in J_m\}$, that is, $\hat{J}_m$ is a scaled version of $J_m$. For $n_1 \in \hat{J}_m \setminus \{2n - 3m\}$, let $\lambda_{n_1}$ be the unique positive solution of

\[ \frac{\lambda f_1(\lambda)}{f_2(\lambda)} = c_2(n_1). \tag{15} \]

Such a solution exists and is unique since $c_2(n_1) = (3m - n_1)/(n - n_1)$ and $n_1 > 2n - 3m$ ensures that $3m - n_1 > 2(n - n_1)$ (see \[8, Lemma 1\]). By continuity reasons, we define $\lambda_{2n-3m} = 0$.

Let

\[ \eta_2(n_1) = \frac{\lambda_{n_1} \exp(\lambda_{n_1})}{f_1(\lambda_{n_1})}. \tag{16} \]

Let $h_n(x) = x \ln(xn) - x$ and define, for $\hat{n}_1$ in the interior of $\hat{J}_m$,

\[ f_{\text{core}}(\hat{n}_1) = h_n(\hat{Q}_2(n_1)) - h_n(\hat{n}_2(n_1)) - h_n(\hat{n}_1) - h_n(m_3(n_1)) - \hat{n}_1 \ln(2) - \hat{m}_3(n_1) \ln(6) + \hat{n}_2(n_1) \ln(f_2(\lambda_{n_1})) - \hat{Q}_2(n_1) \ln(\lambda_{n_1}), \tag{17} \]

We extend the definition $f_{\text{core}}$ to $\hat{J}_m$ by setting the $f_{\text{core}}(\hat{n}_1)$ to be the limit of $f_{\text{core}}(x)$ as $x \to \hat{n}_1$, for the points $\hat{n}_1 \in \hat{J}_m \cap \{0, 1, \hat{n}, 2 - 3\hat{n}\}$. For all points in $\hat{J}_m \cap \{0, 1, \hat{n}, 2 - 3\hat{n}\}$ except $2 - 3\hat{n}$, this only means that $0 \log 0$ should be interpreted as 1. For $\hat{n}_1 = 2 - 3\hat{n}$, as we already mentioned, $\lambda_{2n-3m} = 0$ by continuity reasons. But then $\hat{n}_2(n_1) \ln(f_2(\lambda_{n_1})) - \hat{Q}_2(n_1) \ln(\lambda_{n_1})$ is not defined (and note that $\hat{n}_2(n_1) \ln(f_2(\lambda_{n_1}))$ and $\hat{Q}_2(n_1) \ln(\lambda_{n_1})$ appear in the definition of $f_{\text{core}}$). For $\hat{n}_1 = 2 - 3\hat{n}$,

\[ \lim_{\lambda \to 0} \left( \hat{n}_2(n_1) \ln(f_2(\lambda)) - \hat{Q}_2(n_1) \ln(\lambda) \right) = \hat{n}_2(n_1) \lim_{\lambda \to 0} \left( \ln(f_2(\lambda)) - 2 \ln(\lambda) \right) = \hat{n}_2(n_1) \ln \left( \frac{1}{2} \right). \]
Thus, 
\[ f_{\text{core}}(2 - 3\hat{m}) = h_n(Q_2) - h_n(\hat{n}_2) - h_n(\hat{n}_1) - h_n(\hat{m}_3) - \hat{n}_1 \ln(2) - \hat{m}_3 \ln(6) - \hat{n}_2 \ln 2. \] (18)

We will show that the \( n! \exp(n f_{\text{core}}(\hat{n}_1)) \) approximates the exponential part of the number of cores with \( n_1 \) vertices of degree 1. We obtain an upper bound for the number of cores with vertex set \([n]\) and \(m\) edges.

**Theorem 8.1.** Let \( m(n) = n/2 + R \) with \( R \to \infty \). There exists a constant \( \alpha \) such that, for \( n \geq 1 \), the number of cores with vertex set \([n]\) and \(m\) edges is at most

\[ g_{\text{core}}(n, m) := \alpha n \sqrt{m} \cdot n! \exp\left(n f_{\text{core}}(\hat{n}_1^*)\right). \] (19)

where \( \hat{n}_1^* = 3m/(ng_2(\lambda^*)) \) and \( \lambda^* \) is the unique positive solution of

\[ \frac{\lambda f_1(\lambda)g_2(\lambda)}{f_2(2\lambda)} = \frac{3m}{n}. \] (20)

We will show that the point \( \hat{n}_1^* \) maximizes \( f_{\text{core}} \) in \( J_m \).

For all subsections of this section, let \( \Sigma_{n_1} \) denote the event that a random vector \( Y = (Y_1, \ldots, Y_{n-n_1}) \) satisfies \( \sum_i Y_i = 3m - n_1 \), where the \( Y_i \)'s are independent random variables with truncated Poisson distribution \( \text{Po}(2, \lambda_{n_1}) \). Also, whenever symbols \( y \) and \( \hat{y} \) appear in the same computation, \( \hat{y} \) denotes \( y/n \).

### 8.1 Random cores

Recall that our aim in Section 8 is to obtain the upper bound \( g_{\text{core}}(n, m) \) for the number of cores with vertex set \([n]\) and \(m\) edges. Note that up to this point there is no random graph involved in the problem. However, we show how to reduce the asymptotic enumeration problem to approximating the expectation, in a probability space of random sequences \( Y \), of the probability that a certain type of random multigraph with given degree sequence \( Y \) is simple.

For integer \( n_1 \in J_m \), let \( \mathbb{D}_{n_1} \) be the set of all \( d \in (\mathbb{N} \setminus \{0, 1\})^{n-n_1} \) with \( \sum_{i=1}^{n-n_1} d_i = 3m - n_1 \). For \( n_1 \in J_m \cap \mathbb{Z} \) and \( d \in \mathbb{D}_{n_1} \), let \( \mathcal{G}(n_1, d) = \mathcal{G}_{n,m}(n_1, d) \) be the multigraph obtained by the following procedure. We will start by creating for each edge one bin/set with 3 points inside it. These bins are called **edge-bins**.

We also create one bin for each vertex with the number of points inside it equal to the degree of the vertex. These bins are called **vertex-bins**. Each point in a vertex-bin will be matched to a point in an edge-bin with \( d_i \) points inside it. The multigraph can then be obtained by creating one edge for each edge-bin \( i \) such that the vertices incident to the edge are the vertices with points matched to the edge-bin of \( i \). We describe the procedure in detail now. In the following, in each step, every choice is made u.a.r. among all possible choices satisfying the stated constraints:

1. **(Edge-bins)** For each \( i \in [m] \), create an edge-bin \( i \) with 3 points labelled 1, 2 and 3.

2. **(Vertex-bins)** Choose a set \( V_1 \) of \( n_1 \) vertices in \([n]\) to be the vertices of degree 1. For each \( v \in V_1 \), create one vertex-bin \( v \) with one point inside each. Let \( v_1 < \cdots < v_{n-n_1} \) be an enumeration of the vertices in \([n] \setminus V_1 \). For each \( i \in [n-n_1] \) create a vertex-bin \( v_i \) with \( d_i \) points.

3. **(Matching)** Match the points from the vertex-bins to the points in edge-bins so that each edge-bin has at most one point being matched to a point in a vertex-bin of size 1. This matching is called a configuration.

4. **(Multigraph)** \( \mathcal{G}(n_1, d) = ([n], [m], \Phi) \), where \( \Phi(i, j) = v \), where \( v \) is the vertex-bin containing the point matched to \( j \).
Proof. First we compute the total number of configurations that can be generated. There are \( \binom{n}{n_1} \) ways of choosing the vertices of degree 1 in Step 2. We can split Step 3 by first choosing the vertices in the set of vertices of degree at least 2, although \( d \in \mathbb{N}^{n-n_1} \) is such that, given the set \( V_1 \) of vertices of degree 1 and an enumeration \( v_1 < \ldots < v_{n-n_1} \) of the vertices in \([n] \setminus V_1\), the degree of \( v_i \) is \( d_i \). We say that \( d \) is the degree sequence for the vertices of degree at least 2, although \( d \) is not indexed by the set of vertices of degree at least 2. The following proposition relates \( g_{\text{core}}(n,m,n_1) \) and \( g_{\text{core}}(n,m,n_1,d) \) to \( g_{n,m}(n_1,d) \) and \( Y \). Recall that \( S(n,m) \) is defined in Section 4 as the set of multigraphs with vertex set \([n]\) and \( m \) edges corresponding to simple graphs. Let \( U(n_1,d) \) denote the probability that \( g_{n,m}(n_1,d) \in S(n,m) \).

**Proposition 8.2.** We have that, for any integer \( n_1 \in J_m \)

\[
g_{\text{core}}(n,m,n_1,d) = n! \frac{Q_2(n_1)!}{n_2(n_1)!n_1!m_3(n_1)!2^{n_1}6^{m_3(n_1)}} \frac{1}{\prod_i d_i!} \mathbb{P}(U(n_1,d)), \tag{21}
\]

and, for any integer \( n_1 \in J_m \setminus \{2n-3m\} \),

\[
g_{\text{core}}(n,m,n_1) = \frac{n!}{n_2(n_1)!n_1!m_3(n_1)!2^{n_1}6^{m_3(n_1)}} \frac{Q_2(n_1)!f_2(\lambda_{n_1})^{n_2(n_1)}}{\lambda_{n_1}^n} \mathbb{E}(U(n_1,Y)|\Sigma_{n_1}) \mathbb{P}(\Sigma_{n_1}), \tag{22}
\]

where \( \Sigma_{n_1} \) is the event that a random vector \( Y = (Y_1, \ldots, Y_{n-n_1}) \) satisfies \( \sum_i Y_i = 3m - n_1 \) and the \( Y_i \)'s are independent random variables with truncated Poisson distribution \( \text{Po}(2, \lambda_{n_1}) \).

**Proof.** First we compute the total number of configurations that can be generated. There are \( \binom{n}{n_1} \) ways of choosing the vertices of degree 1 in Step 2. We can split Step 3 by first choosing the \( n_1 \) edge-bins and one point in each of these edge-bins to be matched to the points inside vertex-bins of size 1. There are \( \binom{m}{n_1}3^{n_1} \) possible choices for these edge-bins and the points inside them. There are \( n_1! \) ways of matching these points to the points in vertex-bins of size 1 and there are \( Q_2(n_1)! \) ways of matching the remaining points in the edge-bins to the vertex-bins of size at least 2. Thus, the total number of configurations is

\[
\binom{n}{n_1} \binom{m}{n_1} 3^{n_1} n_1! Q_2(n_1)! =: \beta. \tag{23}
\]

It is straightforward to see that every multigraph with degree sequence \( d \) for the vertices of degree at least 2 is generated by \( \prod_{i=1}^{n-n_1} d_i! \) configurations. Together with Lemma 4.3 this implies that every graph...
with degree sequence $d$ for the vertices of degree at least 2 is generated by

$$\alpha = m!6^n \prod_{i=1}^{n-n_1} d_i!$$

(24)

configurations. Thus, since each configuration is generated with the same probability,

$$g_{\text{core}}(n, m, n_1, d) = \frac{\beta}{\alpha} U(n_1, d).$$

(25)

Together with (23) and (24), and trivial simplifications, this implies (21).

We now prove (22). The proof is very similar to [8, Equation (13)]. Recall that $D_{n_1}$ be the set of all $d \in (N \setminus \{0,1\})^{n_3(n_1)}$ with $\sum_i d_i = Q_2(n_1)$. We have that

$$g_{\text{core}}(n, m, n_1) := \sum_{d \in D_{n_1}} g_{\text{core}}(n, m, n_1, d)$$

$$= n! \sum_{d \in D_{n_1}} \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} m_3(n_1)!} \frac{1}{\prod_{i=1}^{n_3(n_1)} d_i!} U(n_1, d)$$

$$= n! \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} m_3(n_1)!} \sum_{d \in D_{n_1}} U(n_1, d) \prod_{i=1}^{n_3(n_1)} \frac{\lambda_{d_i}^{d_i}}{d_i! f_2(\lambda_{d_i})}$$

$$= n! \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} m_3(n_1)!} \sum_{d \in D_{n_1}} U(n_1, d) \mathbb{P}(Y = d)$$

$$= n! \frac{Q_2(n_1)!}{n_2(n_1)! n_1! m_3(n_1)! 2^{n_1} m_3(n_1)!} \mathbb{E}(U(n_1, Y)|\Sigma_{n_1}) \mathbb{P}(\Sigma_{n_1}),$$

which proves (22). We remark that the only reason why the above proof does not work for $n_1 = 2n - 3m$ (and so for the whole $J_m \cap \mathbb{Z}$) is that $\lambda_{2n-3m} = 0$ (by continuity), which would cause a division by zero in (22).

8.2 Proof of Theorem 8.1

In this section we present the proof that the number of cores with vertex set $[n]$ and $m$ edges is at most $g_{\text{core}}(n, m)$, which is defined in Theorem 8.1 as $\alpha n \sqrt{m} \cdot n! \exp\left(n f_{\text{core}}(\tilde{n}_1)\right)$, where $\alpha$ is a constant, $\tilde{n}_1 = 3m/(ng_2(\lambda^*))$, and $\lambda^*$ is the unique positive solution of

$$\frac{\lambda f_1(\lambda) g_2(\lambda)}{f_2(2\lambda)} = \frac{3m}{n}. $$

(26)

First we show that $\lambda^*$ is well-defined.

**Lemma 8.3.** The equation $\lambda f_1(\lambda) g_2(\lambda)/f_2(2\lambda) = \alpha$ has a unique positive solution $\lambda^*_\alpha$ for any $\alpha > 3/2$. Moreover, for any positive constant $\varepsilon$, there exists a positive constant $\varepsilon'$, such that, if $\alpha, \beta \in (0, \varepsilon)$, then $|\lambda^*_\alpha - \lambda^*_\beta| \leq \varepsilon' |\alpha - \beta|$.

**Proof.** It suffices to show that $f(\lambda) := \lambda f_1(\lambda) g_2(\lambda)/f_2(2\lambda)$ is a strictly increasing function of $\lambda$ with $\lambda > 0$ and $\lim_{\lambda \to 0^+} f(\lambda) = 3/2$. By computing the series of $f(\lambda)$ with $\lambda \to 0$, we obtain

$$f(\lambda) = \frac{3}{2} + \frac{\lambda}{4} + O(\lambda^2).$$

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The derivative of \( f \) is
\[
\frac{df(\lambda)}{d\lambda} = \frac{2 + 2e^{2\lambda} - e^\lambda - 4e^{2\lambda} - e^{3\lambda} - 2e^\lambda + e^{4\lambda} + e^{3\lambda} - 3e^{2\lambda} - e^\lambda}{f(2\lambda)^2},
\]
which we want to show that is positive for any \( \lambda > 0 \). Let \( F(\lambda) \) denote the numerator in the above. It suffices to show that \( F(\lambda) \) is positive for \( \lambda > 0 \). Let \( F^{(0)} = F \). We will use the following strategy: starting with \( i = 1 \), we check that \( F^{(i-1)}(0) \geq 0 \) and compute the derivative \( F^{(i)} \) of \( F^{(i-1)} \). If for some \( i \) we can show that \( F^{(i)}(\lambda) > 0 \) for any \( \lambda > 0 \), then we obtain \( F(\lambda) > 0 \) for \( \lambda > 0 \). Otherwise, we try to simplify the derivative. If \( \exp(\lambda) \) appears in every term of \( F^{(i)} \), we redefine \( F^{(i)} \) by dividing it by \( \exp(\lambda) \). Eventually, we obtain
\[
216e^{2\lambda} - 24\lambda e^\lambda - 44e^\lambda - 16\lambda - 52,
\]
which is trivially positive since \( \exp(2x) \geq \exp(x) \geq 1 + x \) for \( x \geq 0 \) and the sum of the coefficients of the negative terms is less than 216.

The proof of the second statement in the lemma follows trivially from the fact that the first derivative is always positive and, with \( \lambda \to 0 \),
\[
\frac{df(\lambda)}{d\lambda} = \frac{\lambda^4 + O(\lambda^5)}{4\lambda^4 + O(\lambda^5)} \to \frac{1}{4} > 0.
\]

\( \square \)

Since \( 3\hat{m} = 3/2 + 3r \), for \( r = o(1) \) we have that \( \lambda^* \) is well-defined and \( \lambda^* \to 0 \), by Lemma 8.3. Let

\[
w_{core}(n_1) = \begin{cases} 
n! \frac{Q_2(n_1)!f_2(\lambda_{n_1})^{n_2(n_1)}}{n_2(n_1)!n_1!m_3(n_1)!!2^n!6^{m_3(n_1)}\lambda_{n_1}^{Q_2(n_1)-n_2(n_1)}}, & \text{if } n_1 \notin J_m \setminus \{2n - 3m\}; \\
n! \frac{Q_2(n_1)!}{n_2(n_1)!n_1!m_3(n_1)!!2^n!6^{m_3(n_1)}}, & \text{if } n_1 = 2n - 3m \in J_m. 
\end{cases}
\]  

Then, Proposition 8.2 implies that

\[
g_{core}(n, m) = \sum_{n_1 \in J_m \setminus \{2n - 3m\}} w_{core}(n_1)\mathbb{E}(G(n, m, n_1, Y) \text{ simple} | \Sigma_{n_1}) \mathbb{P}(\Sigma_{n_1}) \\
+ 1_{2n - 3m \in J_m} w_{core}(2n - 3m) \mathbb{P}(G(n, m, n_1, 2) \text{ simple}),
\]

where the last term comes from \( n_1 = 2n - 3m \) and \( J_{2n-3m} = \{2\} \).

Recall that \( h_n(x) = x \ln(xn) - x \) and

\[
f_{core}(\hat{n}_1) = h_n(\hat{Q}_2) - h_n(\hat{n}_2) - h_n(\hat{n}_1) - h_n(\hat{m}_3) \\
- \hat{n}_1 \ln(2) - \hat{m}_1 \ln(6) \\
+ \hat{n}_2 \ln(f_2(\lambda_{n_1})) - \hat{Q}_2 \ln(\lambda_{n_1}).
\]

The function \( n! \exp(nf_{core}(\hat{n}_1)) \) is an approximation for the exponential part of \( w_{core}(n_1) \). We will analyse \( f_{core} \) and use it to draw conclusions about \( w_{core} \). It will be useful to know the asymptotic values of \( \hat{n}_1^* \) and some functions of it. In Equation (26), the RHS is \( 3m/n = 3/2 + 3r \) and so we can write \( r \) in terms of \( \lambda^* \). Since \( \hat{n}_1^* \) is defined as \( 3\hat{m}/g_2(\lambda^*) \), we can also write it in terms of \( \lambda^* \) and so we can write \( Q_2(n_1^*), n_2(n_1^*) \) and \( m_3(n_1^*) \) in terms of \( \lambda^* \) (and \( n \)). As we have mentioned before, by Lemma 8.3 we have that \( \lambda^* \to 0 \).
By computing the series with $\lambda^* \to 0$, we have that
\[
\lambda^* = 12r + O(r^2);
\]
\[
\hat{n}_1^* = 1/2 - r + O(r^2);
\]
\[
Q_2(n^*_1) = 3m - n^*_1 = n + 4R + o(R);
\]
\[
n_2(n^*_1) = n - n^*_1 = n/2 + R + o(R);
\]
\[
m_2(n^*_1) = m - n^*_1 = 2R + o(R).
\]

Next, we state the main lemmas for the proof of Theorem 8.1. First we show that $\hat{n}_1^*$ achieves the maximum value for $f_{\text{core}}$ in $\hat{J}_m$.

**Lemma 8.4.** The point $\hat{n}_1^*$ is the unique maximum of the function $f_{\text{core}}(\hat{n}_1)$ for $\hat{n}_1 \in \hat{J}_m$. Moreover, we have that $f'_{\text{core}}(\hat{n}_1^*) = 0$, and $f'_{\text{core}}(\hat{n}_1) > 0$ for $\hat{n}_1 < \hat{n}_1^*$ and $f'_{\text{core}}(\hat{n}_1) < 0$ for $\hat{n}_1 > \hat{n}_1^*$.

**Proof of Lemma 8.4.** Using (12) with $T = \hat{Q}_2$ and $t = \hat{n}_2$, the derivative of $f_{\text{core}}(\hat{n}_1)$ is
\[
- \ln(\hat{Q}_2) + \ln(\hat{n}_2) - \ln(\hat{n}_1) + \ln(\hat{n}_3) + \ln(3) - \ln f_2(\lambda) + \ln \lambda.
\]
The second derivative is
\[
\frac{1}{\hat{Q}_2} - \frac{1}{\hat{n}_2} - \frac{1}{\hat{n}_1} - \frac{m_3}{\hat{Q}_2(1 + \eta_2 - c_2)} < 0,
\]
because $1/\hat{Q}_2 < 1/\hat{n}_3$.

By setting the derivative of $f_{\text{core}}$ in (29) to 0 and using the definition of $\lambda_{n_1}$ in (15), we obtain the Equation (20), which has a unique positive solution $\lambda^*$ by Lemma 8.3. The second derivative computation in (30) implies that $f_{\text{core}}$ is strictly concave and so $\lambda^*$ is the unique maximum.

We are now ready to prove Theorem 8.1. We discuss the relation of $w_{\text{core}}$ and $f_{\text{core}}$ more precisely here. The function $n! \exp(n f_{\text{core}}(\hat{n}_1))$ can be obtained from the definition of $w_{\text{core}}(n_1)$ in (27) as follows: replace $Q_2(n_1)!$ by $\exp(h_n(\hat{Q}_2))$, and do the same for $n_1!, n_2(n_1)!$, and $m_3(n_1)!$. That is, $n! \exp(n f_{\text{core}}(\hat{n}_1))$ can be obtained from $w_{\text{core}}(n_1)$ by replacing each factorial involving $n_1$ by its Stirling approximation (but ignoring the polynomials terms). By Stirling’s approximation, there exists constants $\alpha_1$ and $\alpha_2$ such that, for every $x \in \mathbb{N}$,
\[
\alpha_1 \sqrt{x} \left(\frac{x}{e}\right)^x \leq x! \leq \alpha_2 \sqrt{x} \left(\frac{x}{e}\right)^x,
\]
and so, there exists a constant $\alpha$ such that
\[
w_{\text{core}}(n_1) \leq \alpha \sqrt{m} \exp(n f_{\text{core}}(\hat{n}_1)).
\]
Together with Lemma 8.3, this immediately implies (19).

### 9 Counting pre-kernels

In this section we obtain an asymptotic formula for the number of pre-kernels with vertex set $[n]$ with $m = n/2 + R$ edges, when $R = \omega(n^{1/2} \log^{3/2} n)$ and $R = o(n)$. We remark that the asymptotics in this section are for $n \to \infty$. We will always use $r$ to denote $R/n$. 

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For \( x = (n_1, k_0, k_1, k_2) \in \mathbb{R}^4 \), let
\[
\begin{align*}
n_2(x) &= k_0 + k_1 + k_2, \\
n_3(x) &= n - n_1 - n_2(x) = n - n_1 - k_0 - k_1 - k_2, \\
m_2(x) &= n_1, \\
m_2^+(x) &= n_1 - k_0, \\
P_2(x) &= 2m_2^+(x) = 2n_1 - 2k_0, \\
m_3(x) &= m - n_1, \\
P_3(x) &= 3m_3(x) = 3m - 3n_1, \\
Q_3(x) &= 3m - n_1 - 2n_2(x) = 3m - n_1 - 2k_0 - 2k_1 - 2k_2, \\
T_3(x) &= P_3(x) - k_1 - 2k_2 = 3m - 3n_1 - k_1 - 2k_2, \\
T_2(x) &= P_2(x) - k_1 = 2n_1 - 2k_0 - k_1, \\
\end{align*}
\] (32)

For any symbol \( y \) in this section (and following subsections), we use \( \hat{y} \) to denote \( y/n \).

We will have \( n_1 \) as the number of vertices of degree 1, \( k_0 \) as the number of vertices of degree 2 such that the two edges incident to it are 2-edges, \( k_2 \) as the number of vertices of degree 2 such that the two edges incident to it are 3-edges, and \( k_1 \) as the remaining vertices of degree 2. Then it is clear that \( n_2 \) is the number of vertices of degree 2, \( n_3 \) is the number of vertices of degree at least 3, \( Q_3 \) is the sum of degrees of vertices of degree at least 3, \( m_3 \) is the number of 3-edges, \( m_2 \) is the number of 2-edges, and \( m_2^+ \) is the number of 2-edges that contain exactly two vertices of degree 2. We omit the argument \( x \) when it is obvious from the context.

For \( x \in \mathbb{R}^4 \), let
\[
c_3(x) = \frac{Q_3(x)}{n_3(x)} = \frac{3m - n_1 - 2n_2(x)}{n - n_1 - n_2(x)},
\]
that is, \( c_3 \) is the average degree of the vertices of degree at least 3. Note that \( c_3(x) = \hat{Q}_3(x)/\hat{n}_3(x) = \hat{c}_3(x) \).

For \( x \in \mathbb{R}^4 \) such that \( Q_3(x) > 3\hat{n}_3(x) > 0 \), let \( \lambda = \lambda(x) \) be the unique positive solution of
\[
\frac{\lambda f_3(\lambda)}{f_3'(\lambda)} = c_3(x).
\] (33)

Such \( \lambda(x) \) always exists and is unique by Lemma [8, Lemma 1]. By continuity reasons, we define \( \lambda(x) = 0 \) when \( c_3(x) = 3 \).

Let \( S_m \) be the region of \( \mathbb{R}^4 \) such that \( x = (n_1, k_0, k_1, k_2) \in S_m \) if all of the following conditions hold:

- \( n_1, k_0, k_1, k_2 \in [0, n] \);
- \( Q_3(x) \geq 3n_3(x) \geq 0 \), and \( Q_3(x) = 0 \) whenever \( n_3(x) = 0 \);
- \( m_3(x), m_2(x), m_2^+(x), T_3(x), T_2(x) \geq 0 \).

We will work with pre-kernels with \( n_1 \) vertices of degree 1 and \( k_i \) vertices of degree 2 incident to exactly \( i \) 3-edges, for \( i = 0, 1, 2 \). We say that such pre-kernels have parameters \( (n_1, k_0, k_1, k_2) \). The region \( S_m \) is defined so that all tuples \( (n_1, k_0, k_1, k_2) \) for which there exists a pre-kernel with such parameters are included. Let \( \hat{S}_m = \{ x/n : x \in S_m \} \) denote the scaled version of \( S_m \). The set \( S_m \) is not closed because \( Q_3(x) = 0 \) whenever \( n_3(x) = 0 \). This constraint is added because \( Q_3(x) \) should be the sum of the degrees of vertices of degree at least 3 and \( n_3(x) \) should be the number of vertices of degree at least 3.
For \( \hat{x} = (\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2) \) in the interior of \( \hat{S}_m \), define
\[
\lim_{\lambda \to 0} \left( \frac{3\hat{m}}{g_2(\lambda^*)} \right)
\]
\[
= h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2)
- h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3)
- h_n(\hat{P}_3 - \hat{k}_1 - 2\hat{k}_2) - h_n(\hat{P}_2 - \hat{k}_1) - 2h_n(\hat{m}_2)
\hat{k}_2 \ln 2 - \hat{m}_2 \ln 2 - \hat{m}_3 \ln 6
\hat{n}_3 \ln f_3(\lambda) - \hat{Q}_3 \ln \lambda,
\]
where \( \lambda = \lambda(x) \). As we will see later, for \( x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4 \), we have that \( n! \exp(nf_{\text{pre}}(\hat{x})) \) approximates the exponential part of the number of pre-kernels with parameters \( (n_1, k_0, k_1, k_2) \).

We extend the definition of \( f_{\text{pre}} \) for points \( \hat{x} \in \hat{S}_m \) that are in the boundary of \( \hat{S}_m \) as the limit of \( f_{\text{pre}}(x^{(i)}) \) on any sequence of points \( (x^{(i)})_{i \in \mathbb{N}} \) in the interior of \( \hat{S}_m \) with \( x^{(i)} \to \hat{x} \). One of the reasons the points \( x \) with \( Q_3(x) > c_3(x) = 0 \) are not allowed is that \( f_{\text{pre}}(x_i) \) does not necessarily converge on a sequence of points \( (x^{(i)})_{i \in \mathbb{N}} \) converging to \( x \). For the points in the boundary where \( Q_3(x) > c_3(x) \), this only means that \( 0 \log 0 \) should be interpreted as 1. For \( \hat{x} \in \hat{S}_m \) such that \( \hat{Q}_3(\hat{x}) = 3\hat{n}_3(\hat{x}) \), we have that \( \lambda(\hat{x}) = 0 \). This means that \( \hat{n}_3(x) \ln f_3(\lambda(\hat{x})) - \hat{Q}_3(\hat{x}) \ln \lambda(x) \) is not defined (and note that \( \hat{n}_3(x) \ln f_3(\lambda(x)) \) and \( -\hat{Q}_3(\hat{x}) \ln \lambda(x) \) are the last two terms in the definition of \( f_{\text{pre}}(\hat{x}) \)). We compute \( \lim_{\lambda \to 0}(\hat{n}_3(x) \ln f_3(\lambda) - \hat{Q}_3(\hat{x}) \ln \lambda) \). We have that \( \lim_{\lambda \to 0}(\ln f_3(\lambda) - 3 \ln \lambda) = -6 \). Thus, \( \lim_{\lambda \to 0}(\ln f_3(\lambda) - 3 \ln \lambda) = -6 \). Thus, \( \lim_{\lambda \to 0}(\ln f_3(\lambda) - 3 \ln \lambda) = -6 \).

We obtain the following asymptotic formula for the number of pre-kernels with \( n \) vertices and \( m = m(n) \) edges.

**Theorem 9.1.** Let \( m = m(n) = n/2 + R \) such that \( R = o(n) \) and \( R = \omega(n^{1/2} \log^{3/2} n) \). Then
\[
g_{\text{pre}}(n, m) \sim \frac{\sqrt{3}}{\pi n} n! \exp(nf_{\text{pre}}(\hat{x}^*))
\]
where \( \hat{x}^* \) is defined as \( (\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*) \) with
\[
\hat{n}_1^* = \frac{3\hat{m}}{g_2(\lambda^*)},
\hat{k}_0^* = \frac{3\hat{m}}{g_2(\lambda^*)} \frac{2\lambda^*}{f_1(\lambda^*)g_1(\lambda^*)},
\hat{k}_1^* = \frac{3\hat{m}}{g_2(\lambda^*)} \frac{2\lambda^*}{g_1(\lambda^*)},
\hat{k}_2^* = \frac{3\hat{m}}{g_2(\lambda^*)} \frac{\lambda^* f_1(\lambda^*)}{2g_1(\lambda^*)},
\]
and \( \lambda^* = \lambda^*(n) \) is the unique nonnegative solution for the equation
\[
\frac{\lambda f_1(\lambda)g_2(\lambda)}{f_2(2\lambda)} = 3\hat{m}.
\]
We discussed the existence and uniqueness of \( \lambda^* \) in Section 8. Also, note that (37), implies

\[
1 \frac{\lambda^* f_1(\lambda^*) g_2(\lambda^*)}{f_2(2\lambda^*)} = \frac{1}{2}.
\]

(38)

We will show that the point \( \hat{x}^* \) maximizes \( f_{\text{pre}} \) in a region that contains all points \((n_1, k_0, k_1, k_2)\) for which there exists a pre-kernel with parameters \((n_1, k_0, k_1, k_2)\). The result is then obtained basically by expanding the summation around \( \hat{x}^* \) in a region such that each term in (32) are nonnegative and \( c_3 \geq 3 \). This approach is similar to the one in Section 8 in which we analyse cores, but it will require much more work since we are now dealing with a 4-dimensional space. We remark that \( \lambda^* = \lambda(x^*) \), that is, \( \lambda^* f_2(\lambda^*)/f_3(\lambda^*) = c_3(x^*) \).

Similarly to Section 8 that deals with cores, it will be useful to know approximations for some parameters at the point \( \hat{x}^* = (\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*) \) that achieves the maximum. For \( r = o(1) \), we proved in Lemma 8.4 that \( \lambda^* = o(1) \). From (37), we can write \( r \) in terms of \( \lambda^* \) and so we can write \( \hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^* \) and \( \hat{k}_2^* \) in terms of \( \lambda^* \).

Thus, using (36), and computing the series of each function in (32) as \( \lambda^* \to 0 \), we have

\[
\begin{align*}
\hat{n}_1^* &= \frac{1}{12} \lambda^* + \frac{1}{360}(\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{k}_0^* &= \frac{1}{2} - \frac{7}{12} \lambda^* - \frac{2}{360}(\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{k}_1^* &= \frac{1}{2} \lambda^* - \frac{7}{12} (\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{k}_2^* &= \frac{1}{3}(\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{n}_3^* &= \frac{1}{6} \lambda^* + \frac{1}{12} (\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{Q}_3 &= \frac{1}{2} \lambda^* + \frac{1}{12} (\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{m}_3^* &= \frac{1}{2} \lambda^* + \frac{1}{12} (\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{m}_2^* &= \frac{1}{2} \lambda^* - \frac{1}{4} (\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{T}_2^* &= \frac{1}{2} \lambda^* - \frac{1}{4} (\lambda^*)^2 + O((\lambda^*)^3) \\
\hat{T}_3^* &= \frac{1}{4} (\lambda^*)^2 + O((\lambda^*)^3)
\end{align*}
\]

(39)

In the following subsections, we will use \( Y = (Y_1, \ldots, Y_{n_3}) \) to denote a vector of independent random variables \( Y_1, \ldots, Y_{n_3} \) such that each \( Y_i \) has truncated Poisson distribution with parameters \((3, \lambda(x))\) and \( \Sigma(x) \) to denote the event \( \sum_i Y_i = Q_3 \).

9.1 Kernels

In this section, we define the notion of kernels of pre-kernels, which will be useful to study properties of pre-kernels and to generate random pre-kernels.

Recall that the pre-kernel is a core with no isolated cycles. Let the kernel of a pre-kernel \( G \) be the multihypergraph obtained as follows. Start by obtaining \( G' \) from \( G \) by deleting all vertices of degree 1 and replacing each edge containing a vertex of degree 1 by a new edge of size 2 incident to the other two vertices (and note that the multihypergraph is not necessarily uniform anymore). While there is a vertex \( v \) of degree 2 in \( G' \) such that the two edges incident to \( v \) have size 2, update \( G' \) by deleting both edges, and adding a new edge of size 2 containing the vertices other than \( v \) that were in the deleted edges. When this procedure is finished, delete all vertices of degree less than 2. The final multihypergraph is the kernel of \( G \). This procedure obviously produces a unique multihypergraph (disregarding edge labels). See Figure 5 for an example of the procedure above.

This procedure is similar to the one for obtaining kernels of 2-uniform hypergraphs described by Pittel and Wormald [9]: the kernel of a pre-kernel is obtained by repeatedly replacing edges \( uv \) and \( vw \), where \( v \) is a vertex of degree 2, by a new edge \( uw \) until no vertex of degree 2 remains, and then deleting all isolated vertices. Note that in our procedure there may be vertices of degree 2 in the kernel while there is no vertex of degree 2 in the kernel of a 2-uniform hypergraph.

In the kernel all edges have size 2 or 3. We call these edges 2-edges and 3-edges in the kernel, resp. It is trivial from the description above that in the kernel every degree 2 vertex is contained in at least
one 3-edge. We say that any multihypergraph in which all edges have size 2 or 3, there are no vertices of degree 1, and every vertex of degree 2 is in at least one edge of size 3, is a kernel. The reason for this is that given such a multihypergraph, one can create a pre-kernel following the procedure we discuss next.

Consider the following operation: we split one 2-edge with vertices $u$ and $v$ by deleting the edge and adding a new vertex $w$ and two new 2-edges, one containing $u$ and $w$ and the other containing $v$ and $w$. Given the kernel of a pre-kernel, one can split edges from the kernel in a way that it reverses the steps of the procedure for finding the kernel. After including the vertices of degree 1 in the 2-edges, the resulting graph is the pre-kernel. Note that, by replacing 2-edges in the kernel by splitting 2-edges and adding new vertices (of degree 1) to all 2-edges, the resulting multigraph does not have any isolated cycle. Thus, whenever the resulting multigraph is simple, it is a pre-kernel.

**9.2 Random kernels and pre-kernels**

Recall that our aim in Section 9 is to find an asymptotic formula for $\text{g}_{\text{pre}}(n, m)$, the number of connected pre-kernels with vertex set $[n]$ and $m$ edges. Similarly to Section 8.1 about random cores, we show how to reduce the enumeration problem for pre-kernels to approximating the expectation, in a probability space of random degree sequences, of the probability that a random graph with given degree sequence is connected and simple.

We will describe a procedure to generate pre-kernels. For $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$, let $\mathcal{D}(x) \subseteq \mathbb{N}^{n_3(x)}$ be such that $d \in \mathcal{D}(x)$ if $d_i \geq 3$ for all $i$ and $\sum_{i=1}^{n_3(x)} d_i = Q_3(x)$. Our strategy to generate a random pre-kernel is the following. We start by choosing the vertices and 3-edges that will be in the kernel. We then generate a random kernel with degree sequence $d$ for the vertices of degree at least 3, $k_1 + k_2$ vertices of degree 2, $m_2$ 2-edges and $m_3$ 3-edges so that $k_i$ vertices of degree 2 are contained in exactly $i$ 3-edges for $i = 1, 2$. The pre-kernel is then obtained by splitting 2-edges $k_0$ times and assigning the vertices of degree 1.

The kernel is generated in a way similar to the random cores in Section 8 but with different restrictions. For each vertex, we create a bin/set with the number of points inside it equal to the degree of the vertex. These bins are called *vertex-bins*. For each edge, we create one bin/set with 2 or 3 points inside it, depending on whether it is a 2-edge or a 3-edge. These bins are called *edge-bins*. Each point in a vertex-bin will be matched to a point in an edge-bin with some constraints. The kernel can then be obtained by creating one edge for each edge-bin $i$ such that the vertices incident to it are the vertices with points matched to point in the edge-bin $i$. We describe how to generate a random kernel $\mathcal{K}(V, M_3, k_1, k_2, d)$ where $V \subseteq [n]$ is a set of size $k_1 + k_2 + n_3$ and $M_3 \subseteq [m]$ is a set of size $m_3$. In each step, every choice is made u.a.r. among all
possible choices satisfying the stated conditions:

1. (Vertex-bins) Choose a set $V_3$ of $n_3$ vertices in $V$ to be the vertices of degree at least 3. Let $v_1 < \cdots < v_{n_3}$ be an enumeration of $V_3$. For each $i \in [n_3]$, create a vertex-bin $v_i$ with points labelled $1, \ldots, d_i$ inside it. For each $v \in V \setminus V_3$, create a vertex-bin $v$ with points labelled 1 and 2 inside it.

2. (Edge-bins) For each $i \in M_3$, create one edge-bin with points labelled 1, 2, and 3 inside it. Let $M_2 = \{(i, 0) : i \in [m_2^-]\}$. For each $i \in M_2$, create one edge-bin with points labelled 1 and 2 inside it.

3. (Matching) Match the points from the vertex-bins to the points in edge-bins so that, for $i = 1, 2, k_i$ vertex-bins with two points have exactly $i$ points being matched to an edge-bin of size 3. This matching is called a kernel-configuration with parameters $(V, M_3, k_1, k_2, d)$.

4. (Kernel) The kernel $K(V, M_2, k_1, k_2) = (V, M_2 \cup M_3, \Phi)$ is the multihypergraph such that for each $E \in M_2 \cup M_3$, we have that $\Phi(E, i) = v$, where $v$ is the vertex corresponding to the vertex-bin containing $j$ and $j$ is the point matched to point $i$ in the edge-bin $E$ in the previous step.

See Figure 6 for an example of this procedure. The constraints in Step 3 ensures that each vertex of degree 2 is contained by at least one 3-edge and so the procedure above always generates a kernel. It is also trivial that all kernels (with edges $M_3 \cup M_2$) can be generated by this procedure.

We now describe the pre-kernel model precisely. For $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$ and $d \in D(x)$, let $\mathcal{P}(x, d) = \mathcal{P}_{n,m}(x, d)$ be the random graph generated as follows. In each step, every choice is made u.a.r. among all possible choices satisfying the stated conditions:

1. (Kernel) Let $V$ be a subset of $[n]$ of size $n - n_1 - k_0$ and $M_3$ be a subset of $[m]$ of size $m_3(x)$. Let $K = (V, M_K, \Phi_K)$ be the random kernel $K(V, M_3, k_1, k_2, d)$.

2. (Splitting edges) Let $V_{k_0}$ be a subset of $[n]\setminus V$ of size $k_0$. This set will be the set of vertices of degree at 2 contained by two 2-edges. Let $v_1, \ldots, v_{k_0}$ be an enumeration of the vertices in $V_{k_0}$. Let $P = K$. For $i = 1$ to $k_0$, do the following operation: split a 2-edge of $P$ with new vertex $v_i$ and update $P$.

3. (Assigning 2-edges and vertices of degree 1) Let $V_1$ be a subset of size $n_1$ in $[n]\setminus V$. These will be the vertices of degree 1 in the multigraph. Assign for each 2-edge $E$ of $P$ a (unique) edge $E'$ from $[m]\setminus M_3$ and a (unique) vertex $u$ in $V_1$. Place a perfect matching $M_{E'}$ between the collection $\{\Phi_K(E, 1), \Phi_K(E, 2), u\}$ and $\{1, 2, 3\}$. We call this matching together with the sequence of splittings in the previous step a splitting-configuration.

![Figure 6: A kernel generated with vertex and edge-bins](image-url)
4. *(Pre-kernel)* Let $\mathcal{P}(n_1, k_0, k_1, k_2, d) = ([n], [m], \Phi)$, where $\Phi(E, i) = \Phi_k(E, i)$ if $E \in M_3$ and, otherwise, $\Phi(E, i) = v$, where $v$ is the vertex matched to $i$ in $M_E$.

When the procedure above results in a (simple) graph, it is a pre-kernel since it is obtained by splitting the 2-edges of a kernel and assigning vertices of degree 1 to the 2-edges. It is trivial all pre-kernels are generated since all kernels and the ways of splitting the edges are considered.

For $(n_1, k_0, k_1, k_2) \in S_m$, let $g_{pre}(n, m, n_1, k_0, k_1, k_2)$ denote the number of connected (simple) pre-kernels with vertex set $[n]$ and $m$ edges such that $n_1$ vertices have degree 1, and $k_0 + k_1 + k_2$ vertices have degree 2 so that $k_i$ of the degree 2 vertices are incident to exactly $i$ 3-edges for $i = 0, 1, 2$. For $d \in D(n_1, k_0, k_1, k_2)$, let $g_{pre}(n, m, n_1, k_0, k_1, k_2, d)$ denote the number of such pre-kernels with the additional constraint that $d$ is the degree sequence of the vertices of degree at least 3.

In order to analyse $g_{pre}(n, m, n_1, k_0, k_1, k_2, d)$ it will be useful to know the number of kernel-configurations.

**Lemma 9.2.** Let $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$ and $d \in D(x)$. The number of kernel-configurations with parameters $(V, M_3, k_1, k_2, d)$, where $V$ is a set of size $k_1 + k_2 + n_3$ and $M_3$ is a set of size $m_3$, is

$$\binom{k_1 + k_2 + n_3}{n_3} \binom{k_1 + k_2}{k_1} 2^{k_1} P_3(k_1 + 2k_2) \binom{P_2}{k_1} k_1! Q_3! = \frac{(k_1 + k_2 + n_3)! P_3! P_2! Q_3! 2^{k_1}}{n_3! k_1! k_2! T_3! T_2!}.$$  

Moreover, each kernel with parameters $(V, M_3, k_1, k_2, d)$ is generated by exactly $2^{k_1 + k_2} \prod_{i=1}^{n_3} d_i!$ kernel-configurations.

**Proof.** There are $(k_1 + k_2 + n_3)$ ways of choosing the vertices of degree at least 3 in the first step. The step where the kernel-configuration is created can be described in the following more detailed way:

1. Choose $k_1$ vertex-bins of size 2. Let $U$ be a set containing exactly one point of each of these vertex-bins and let $D$ be the set consisting of all points in vertex-bins of size 2 that are not in $U$.
2. Choose $k_1 + 2k_2$ points inside edges-bins of size 3 and match them to points in $D$.
3. Choose $k_1$ points inside edges-bins of size 2 and match them to points in $U$.
4. Match the remaining unmatched $Q_3$ points from the vertex-bins to the unmatched points in the edge-bins.

In Step 1, there are $\binom{k_1 + k_2}{k_1}$ choices for the vertex-bins of size 2 and $2^{k_1}$ choices for $U$. There are $\binom{P_3}{k_1 + 2k_2} (k_1 + 2k_2)!$ choices for Step 2, $(P_2)^k_1$ for Step 3 and $Q_3!$ choices for Step 4. The first part of the lemma then follows trivially.

Each kernel with parameters $(V, M_3, k_1, k_2, d)$ is generated by $2^{k_1 + k_2} \prod_{i=1}^{n_3} d_i!$ distinct kernel-configurations, because any permutation of the points inside vertex-bins can be done without changing the resulting kernel.

The following proposition relates $g_{pre}(n, m, n_1, k_0, k_1, k_2, d)$ and $g_{pre}(n, m, n_1, k_0, k_1, k_2)$ to the random pre-kernels $\mathcal{P}(x, d)$ and random degree sequences. The proof is similar to the proof of Proposition 8.2. We include it here for completeness.

**Proposition 9.3.** For $x = (n_1, k_0, k_1, k_2) \in S_m \cap \mathbb{Z}^4$ and $d \in D(x)$,

$$g_{pre}(n, m, n_1, k_0, k_1, k_2, d) = \frac{n! P_3! P_2! Q_3!(m_2 - 1)! \mathbb{P}(\mathcal{P}(n_1, k_0, k_1, k_2, d) \text{ simple and connected})}{k_0! k_1! k_2! n_3! m_3! T_3! T_2! (m_2 - 1)! m_2! 2^{k_1 + 2k_2} 6^{n_3} \prod_{i=1}^d d_i!}.$$
and, if $Q_3(x) > n_3(x)$, then

$$g_{pre}(n, m, n_1, k_0, k_1, k_2) = n! \frac{P_3! P_2! Q_3!(m_2 - 1)!}{k_0! k_1! k_2! n_3! m_2! (P_3 - k_1 - 2k_2)! (P_2 - k_1)! (m_2 - 1)! m_2! 2^{k_2} 2^{m_2} 6^{m_3}}$$

where

$$Q_3(x) = \left( \frac{f_3(\lambda)n_3}{\lambda Q_3} \right) \mathbb{E} \left[ \prod_{i=1}^n P\left( \mathcal{P}(n_i, k_0, k_1, k_2, Y) \text{ simple and connected} \right) \left( \sum x \right) \mathbb{P}\left( \Sigma(x) \right) \right],$$

and $Y = (Y_1, \ldots, Y_{n_3})$ is a vector of independent random variables $Y_1, \ldots, Y_{n_3}$ such that each $Y_i$ has a truncated Poisson distribution with parameters $(3, \lambda(x))$ and $\Sigma(x)$ denotes the event $\sum_i Y_i = Q_3$.

**Proof.** Any multigraph obtained by the process for $\mathcal{P}(x, d)$ is generated by $2^{k_1+k_2} (\prod_{i=1}^{n_3} d_i)! m_2^{-1} 2^{m_2} m! 6^m$ combinations of kernel-configurations and splitting-configurations. This is because each kernel is generated by $2^{k_1+k_2} (\prod_{i=1}^{n_3} d_i)!$ kernel-configurations by Lemma 9.2 and permuting the labels and points inside each of the 2-edges in the kernel do not change the resulting multigraph. Thus, by Lemma 4.3, each pre-kernel with parameters $(x, d)$ is generated by

$$\alpha := 2^{k_1+k_2} \left( \prod_{i=1}^{n_3} d_i! \right) m_2^{-1} 2^{m_2} m! 6^m$$

combinations of kernel-configurations and splitting-configurations. Next we compute the total number of such combinations. In Step 1 in which we generate the kernel, there are $(\begin{smallmatrix} n \\ k_0 + k_1 + k_2 + n_3 \end{smallmatrix})$ ways of choosing $V$ and $(\begin{smallmatrix} m_3 \\ m \end{smallmatrix})$ ways of choosing $M_3$. The number of ways of generating the kernel is

$$\frac{(k_1 + k_2 + n_3)! P_3! P_2! Q_3! 2^{k_1}}{n_3! k_1! k_2! T_3! T_2!}$$

by Lemma 9.2. In Step 2, there are $(\begin{smallmatrix} n_1 + k_0 \\ k_0 \end{smallmatrix})$ ways of choosing $V_0$ and $m_2^{-1} (m_2^{-1} + 1) \cdots (m_2^{-1} + k_0 - 1) = (m_2 - 1)!/(m_2^{-1} - 1)!$ ways of splitting the edges. In Step 3, that are $(m_2)!^2$ ways of assigning the 2-edges and vertices of degree 1 and $6^{m_2}$ ways of placing the matchings. Thus, the total number of combinations of kernel-configurations and splitting-configurations is

$$\left( \begin{smallmatrix} n \\ k_0 + k_1 + k_2 + n_3 \end{smallmatrix} \right) \left( \begin{smallmatrix} m_3 \\ m \end{smallmatrix} \right) \frac{(k_1 + k_2 + n_3)! P_3! P_2! Q_3! 2^{k_1}}{n_3! k_1! k_2! T_3! T_2!} \left( \begin{smallmatrix} n_1 + k_0 \\ k_0 \end{smallmatrix} \right) \frac{(m_2 - 1)!}{(m_2^{-1} - 1)!} (m_2)!^2 6^{m_2} =: \beta.$$

Hence, since each combination is generated with the same probability, we have that

$$g_{pre}(n_1, k_0, k_1, k_2, d) = \frac{\beta}{\alpha} \mathbb{P}(\mathcal{P}(n_1, k_0, k_1, k_2, d) \text{ simple and connected}),$$

where $\beta$ is the total number of configurations. which together with (42) and trivial simplifications implies (9.3).

We now prove (10). Again, the proof is very similar to $\mathcal{S}$ Equation (13). For $x = (n_1, k_0, k_1, k_2)$, let
U(x, d) denote the probability that P_{n,m}(n_1, k_0, k_1, k_2, d) is simple and connected. For x = (n_1, k_0, k_1, k_2),

\[
g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2) := \sum_{d \in D(x)} g_{\text{pre}}(n, m, n_1, k_0, k_1, k_2, d)
\]

\[
= n! \sum_{d \in D(x)} \frac{P_3!P_2!Q_3!(m_2 - 1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2 - 1)!m_2^22k_2^2m_2^26m_3} \prod_i d_i! U(n_1, k_0, k_1, k_2, d)
\]

\[
= n! \sum_{d \in D(x)} \frac{P_3!P_2!Q_3!(m_2 - 1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2 - 1)!m_2^22k_2^2m_2^26m_3} \frac{f_3(\lambda(x))^{n_3}}{\lambda(\lambda(x))^{n_3}} \sum_{d \in D(x)} \prod_i d_i! f_3(\lambda(x)) U(x, d)
\]

\[
= n! \sum_{d \in D(x)} \frac{P_3!P_2!Q_3!(m_2 - 1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2 - 1)!m_2^22k_2^2m_2^26m_3} \frac{f_3(\lambda(x))^{n_3}}{\lambda(\lambda(x))^{n_3}} \sum_{d \in D(x)} U(x, d) \mathbb{P}(Y = d)
\]

which proves (10).

The goal of the next lemmas is to show that the expectation in (10) goes to 1 for points x close to x*. For x \in S_m \cap \mathbb{Z}^4 and \phi = \phi(n) > 0, let

\[
\mathcal{D}_\phi(x) = \{d \in D(x) : |\eta(d) - \mathbb{E}(\eta(Y))| \leq R\phi\}
\]

where \eta(d) := \sum_{i=1}^{n_3} d_i(d_i - 1)/(2m_i), and recall that R = m - n/2. We will show that, for some function \phi = o(1), conditioned upon \Sigma(x), the probability that Y is in \mathcal{D}_\phi(x) goes to 1. Intuitively, this means that the set \mathcal{D}_\phi(x) contains all ‘typical’ degree sequences for points x \in S that are close to x*. For \psi = \psi(n) = o(1), let

\[
S_\psi^* = \left\{ x = (n_1, k_0, k_1, k_2) \in S : \left| \tilde{n}_1 - \frac{1}{2} \right| \leq \psi r; \left| \tilde{k}_0 - \frac{1}{2} \right| \leq \psi r; \left| \tilde{k}_1 - 6r \right| \leq \psi r; \left| \tilde{k}_2 - 18r^2 \right| \leq \psi r^2; \left| \tilde{n}_3 - 2r \right| \leq \psi r; \left| \tilde{Q}_3 - 6r \right| \leq \psi r; \left| \tilde{m}_3 - 2r \right| \leq \psi r; \left| \tilde{\hat{m}} - 6r \right| \leq \psi r; \left| \tilde{T}_3 - 6r \right| \leq \psi r^2 \right\}
\]

We define S_\psi^* this way so that all points in it are close to x*, where we are using (39) to find around which values each of the functions in the definition of S_\psi^* should be concentrated. The idea is to define \psi later in a way that it is small enough so that we can approximate the summation of n!exp(nf_{pre}(\hat{x})) in the integer points x in S_\psi^*, but large enough so that what is not included do not significant effect in the summation \sum_{x \in S_m \cap \mathbb{Z}^4} n!exp(nf_{pre}(\hat{x})).

Next we show that for points in x \in S_\psi^* with \psi = o(1) the set \mathcal{D}_\phi(x) (for some \phi = o(1)) is a set of ‘typical’ degree sequences.

**Lemma 9.4.** Let \psi = o(1). There exists \phi = o(1) such that, for every integer point x \in S_\psi^*, we have that \mathbb{P}(Y \in \mathcal{D}_\phi(x) | \Sigma(x)) = 1 - o(1).
We then show that for \( x = x(n) \in S^*_\psi \cap \mathbb{Z}^4 \) and \( d \in \tilde{D}_\phi(x) \), the random pre-kernel \( P(x, d) \) is connected and simple a.a.s.

**Lemma 9.5.** Assume \( R = o(n) \). Let \( \psi, \phi = o(1) \). Let \( x = x(n) \in S^*_\psi \) be an integer point and \( d = d(n) \in \tilde{D}_\phi(x) \). Then \( P(x, d) \) is simple a.a.s.

**Lemma 9.6.** Assume \( R = o(n) \). Let \( \psi, \phi = o(1) \). Let \( x \in S^*_\psi \) be an integer point and \( d = d(n) \in \tilde{D}_\phi(x) \). Then \( P(x, d) \) is connected a.a.s.

The proofs for Lemmas 9.4, 9.5, and 9.6 are presented in Sections 9.3, 9.4, and 9.5, respectively. We now show how to prove that the expectation in (40) goes to 1 assuming Lemmas 9.4, 9.5, and 9.6.

**Corollary 9.7.** Let \( \psi = o(1) \) and let \( x = (n_1, k_0, k_1, k_2) \in S^*_\psi \cap \mathbb{Z}^4 \). Then

\[
\mathbb{E} \left( \frac{P((n_1, k_0, k_1, k_2, Y) \text{ simple and connected})}{\Sigma(x)} \right) \sim 1.
\]

**Proof.** Let \( U(Y) \) denote the probability that \( P(x, Y) \) is connected and simple. Let \( \phi = o(1) \) be given by Lemma 9.4. We have that

\[
\mathbb{E} \left( U(Y) \right) \geq \sum_{d \in \tilde{D}_\phi(x)} P(U(d)) P(Y = d | \Sigma(x)).
\]

By Lemmas 9.5 and 9.6, we have that \( P(U(d)) = 1 - o(1) \) for every \( d = d(n) \in \tilde{D}_\phi(x) \). Since \( \tilde{D}_\phi(x) \) is a finite set for each \( n \), this implies that there exists a function \( q(n) = o(1) \) such that \( P(U(d)) \geq 1 - q(n) \) for every \( d \in \tilde{D}_\phi(x) \). Thus,

\[
\mathbb{E} \left( U(Y) \right) \geq (1 - q(n)) P(Y \in \tilde{D}_\phi(x)) = 1 - o(1).
\]

by Lemma 9.4.

### 9.3 Typical degree sequences

In this section, given an integer point \( x \in S_m \) ‘close’ to the point \( x^* \) (more precisely \( x \in S^*_\psi \) with \( \psi = o(1) \)), we show that, for a random vector of \( Y = (Y_1, \ldots, Y_{n_3(x)}) \) of independent truncated Poisson random variables with parameters \( (3, \lambda(x)) \) conditioned upon the event \( \Sigma(x) \) that \( \sum_{i=1}^{n_3(x)} Y_i = Q_3(x) \), the value of \( \sum_{i=1}^{n_3(x)} Y_i \) is concentrated around its expected value. More specifically, we present the proof for Lemma 9.4.

Recall that \( \tilde{D}_\phi(x) = \{ d \in D(x) : |\eta(d) - \mathbb{E}(\eta(Y))| \leq R\phi \} \)

where \( \eta(d) = \sum_{i=1}^{n_3} d_i (d_i - 1)/(2m) \). We want to show that, given \( x \in S^*_\psi \cap \mathbb{Z}^4 \) with \( \psi = o(1) \), there exists \( \phi = o(1) \) such that \( P(Y \in \tilde{D}_\phi(x) \mid \Sigma(x)) > 1 - \phi \), where \( Y = (Y_1, \ldots, Y_{n_3}) \) is a vector of independent random variables with distribution \( \text{Po}(3, \lambda(x)) \).

Recall that \( n_3 \sim 2rn = 2R \to \infty \), and \( Q_3/n_3 \sim 6r/(2r) = 3 \) for \( x \in S^*_\psi \). Thus, by the definition of \( \lambda(x) \) (in (33)) and [3] Lemma 1], we must have \( \lambda(x) = o(1) \). Then by [3] Lemma 2], \( \text{Var}(Y_i | Y_i - 1) = \Theta(\lambda) \).

Thus, by Chebyshev’s inequality,

\[
\mathbb{P} \left( |\mathbb{E}(\eta(Y)) - \mathbb{E}(\eta(Y))| \geq R\phi \right) \leq \frac{\text{Var}(\eta(Y))}{R^2 \phi^2} = \frac{n_3 \Theta(\lambda)}{R^2 \phi^2} = o \left( \frac{n_3}{R^2 \phi^2} \right).
\]

If \( R_3 := Q_3 - 3n_3 \leq \log n_3 \), by [3] Theorem 4] and Stirling’s approximation

\[
\mathbb{P}(\Sigma(x)) = (1 + o(1)) e^{-R_3} \frac{R_3^{R_3}}{R_3!} = \Omega \left( \frac{1}{\sqrt{R_3}} \right) = \Omega \left( \frac{1}{\sqrt{\log n_3}} \right).
\]
If $Q_3 - 3n_3 \geq \log n_3$, by [8] Theorem 4,
\[
\mathbb{P}(\Sigma(x)) \sim \frac{1}{\sqrt{2\pi n_3 c_3(1 + n_3 - c_3)}} = \Omega\left(\frac{1}{\sqrt{n_3}}\right),
\]
where $c_3 = Q_3/n_3$ and $\eta_3 = \lambda(x)f_1(\lambda(x))/f_2(\lambda(x))$, and we used [8] Lemma 2. Thus,
\[
\mathbb{P}\left(|\eta(Y) - \mathbb{E}(\eta(Y))| \geq R\phi|\Sigma|\right) = O\left(\frac{n_3}{R^2\phi^2 \sqrt{n_3}}\right) = O\left(\frac{1}{R^1/2\phi^2}\right)
\]
since $n_3 \sim 2R$ and so it is suffices to choose $\phi^2 = \omega(\sqrt{1/R})$. This finishes the proof of Lemma 9.4.

### 9.4 Simple pre-kernels

In this section, given an integer point $x \in S_m$ ‘close’ to the point $x^*$ and $d \in \mathbb{N}^{n_3}$ with some constraints (more precisely $x \in S_\psi^*$ and $d \in \tilde{D}_\phi(x)$ with $\psi, \phi = o(1)$), we show that the random multigraph $\mathcal{P}(x, d)$ defined in Section 9.2 is simple a.a.s., thus proving Lemma 9.5. Recall that a multigraph is simple if it has no loops and no double edges (as defined in Section 4). Any loop (or double edge) involving only 3-edges in the kernel remains a loop (or double edge) in the pre-kernel. Any double edge involving 2-edges in the kernel will not be a double edge in the pre-kernel, because each 2-edge will be assigned a unique vertex of degree 1 in the procedure that creates the pre-kernel from the kernel. A loop in the kernel that is an 2-edge will cease to be a loop in the pre-kernel if it is split at least once. Note that, if a 2-edge that is a loop in the kernel is split exactly once, the two 2-edges created will not form a double edge in the final multigraph since the assignment of vertices of degree 1 to the 2-edges eliminates all double edges involving 2-edges. It is clear that no other loops or double edges can be created. We rewrite these conditions for the kernel-configuration: the pre-kernel $\mathcal{P} = \mathcal{P}(x, d)$ is simple if and only if

(A) **(No loops in 3-edges)** No edge-bin of size 3 has at least 2 points matched to points from the same vertex-bin.

(B) **(No double 3-edges)** Assuming no loops in 3-edges, no pair of edges-bins of size 3 has their points matched to points in the same 3 vertices.

(C) **(No loops in 2-edges)** For every edge-bin of size 2, its points are matched to points from distinct vertex-bins or the 2-edge corresponding to this edge-bin is split at least once in the process that obtains the pre-kernel from the kernel.

We will show that, for $x \in S_\psi^*$ and $d \in \tilde{D}_\phi(x)$ with $\psi, \phi = o(1)$, the random multigraph $\mathcal{P}(x, d)$ is simple a.a.s., which proves Lemma 9.5. We need to show that each of the conditions (A), (B) and (C) holds a.a.s. We will use the detailed procedure for obtaining kernel-configurations described in the proof of Lemma 9.2. We work in the probability space conditioned upon the vertices of degree 3 and the points in $U$ being already chosen, since the particular choices of these vertices and points do not affect the probability of loops or double edges in the kernel.

First we prove (A) holds a.a.s. Consider the case that the loop is on a vertex of degree 2. There are $k_2$ possible choices for the vertex-bin. There are $m_3$ choices for the edge-bin of size 3 and $3 \cdot 2$ choices for the points inside of the edge-bin to be matched to the points in the vertex-bin of size 2. Thus, we have $6k_2m_3$ choices. Following the proof of Lemma 9.2 after the vertices of degree 3 and $U$ are chosen, there are

\[
\binom{P_3}{k_1 + 2k_2}(k_1 + 2k_2)! \binom{P_2}{k_1} k_1! Q_3!
\]

(44)
ways of completing the kernel-configuration. The number of completions of kernel-configurations containing a given matching that matches 2 points in a vertex-bin of size 2 to 2 points in an edge-bin of size 3 is then

\[
\binom{P_3 - 2}{k_1 + 2k_2 - 2} (k_1 + 2k_2 - 2)! \binom{P_2}{k_1} k_1! Q_3!
\]

Thus, using the definition of \( S_\phi^* \), the probability that there is a loop on a vertex of degree 2 in a 3-edge is at most

\[
6k_2m_3 \frac{1}{P_3(P_3 - 1)} = O \left( \frac{k_2m_3}{P^2_3} \right) = O \left( \frac{(r^2n)(rn)}{(rn)^2} \right) = O(r) = o(1).
\]

Now consider the case that the loop is on a vertex of degree at least 3. There are \( \sum_{i=1}^{n_3} \binom{d_i}{2} \) possible choices for the vertex-bin and 2 points inside it. Since \( d \in \mathcal{D}(x) \) and \( \mathbb{E}(\eta(Y)) = n_3 \mathbb{E}(Y_1(Y_1 - 1)) \sim 6n_3 = \Theta(R) \),

\[
\eta(d) = \Theta(n_3).
\]

There are \( m_3 \) choices for the edge-bin of size 3 and \( 3 \cdot 2 \) choices for the points inside of the edge-bin to be matched to the chosen points in the vertex-bin. Thus, we have \( O(n_3m_3) \) choices. The number of completions of kernel-configurations containing one given matching that matches 2 points in a vertex-bin of size at least 3 and 2 points in a edge-bin of size 3 is

\[
\binom{P_3 - 2}{k_1 + 2k_2} (k_1 + 2k_2)! \binom{P_2}{k_1} k_1! (Q_3 - 2)!
\]

Thus, using (44), the probability that there is a loop on a vertex-bin of size at least 3 in edge-bin of size 3 is

\[
O \left( n_3m_3 \cdot \frac{(P_3 - 2)!}{P_3!} \frac{(Q_3 - 2)!}{Q_3!} \frac{(T_3)!}{(T_3 - 2)!} \right) = O \left( \frac{n_3m_3 T^2_3}{P^2_3 Q_3^2} \right) = O \left( \frac{(rn)(rn)(r^2n)^2}{(rn)^2(rn)^2} \right) = O(r^2) = o(1).
\]

This finishes the proof that Condition (A) holds a.a.s. Now we prove that Condition (B) holds a.a.s. We consider 4 cases:

(B1) The edge-bins corresponding to the double edge have their points matched to points in 3 vertex-bins all of size 2.

(B2) The edge-bins corresponding to the double edge have their points matched to points in 2 vertex-bins of size 2 and 1 vertex-bin of size at least 3.

(B3) The edge-bins corresponding to the double edge have their points matched to points in 1 vertex-bin of size 2 and 2 vertex-bins of size at least 3.

(B4) The edge-bins corresponding to the double edge have none of their points matched to points in vertex-bins of size 2.

Let us start with (B1). We have \( O(k_2^2m_3^2) \) choices for the 3 vertex-bins and 2 edge-bins involved. There are \( O(1) \) matchings between the points 6 in these vertex-bins and the 6 points in these edge-bins that creates a double edge. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

\[
\binom{P_3 - 6}{k_1 + 2k_2 - 6} (k_1 + 2k_2 - 6)! \binom{P_2}{k_1} k_1! Q_3!,
\]
where we are following the proof of Lemma 9.2, after the vertices of degree 3 and $U$ are chosen. Thus, using (44), the expected number of double edges as in (B1) is at most

$$O \left( \frac{k_2^3 m_3^2}{P_3} \right) \left( \frac{P_3 - 6}{P_3!} \right) \frac{(P_3 - 6)!}{P_3!} = O \left( \frac{k_2^3 m_3^2}{P_3^m} \right) \frac{P_3^m}{P_3^m} \frac{(P_3 - 6)!(P_3 - 6)}{P_3^m P_3^m} = O \left( \frac{(P_3 - 6)!}{(P_3 - 6)!} \right) = O \left( \frac{r^2}{n} \right).$$

Now let us consider (B2). We have $O(k_2^2 \eta(d)m_3^2)$ choices for the 2 vertex-bins of size 2 and the points inside them, the vertex-bin of size at least 3 and the points inside them, and the 2 edge-bins of size 3 involved in the double edge. We match 4 points from the 2 vertex-bins of size 2 to the 4 points in the vertex-bins creating a double edge. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

$$O \left( \frac{k_2^2 m_3^2}{P_3^m} \right) \left( \frac{P_3 - 6}{P_3} \right) \left( \frac{P_3 - 6}{P_3} \right) \left( \frac{P_3 - 6}{P_3} \right) = O \left( \frac{k_2^2 m_3^2 T_3^4}{P_3^m Q_3^4} \right) = O \left( \frac{(r^2 n)^2 (r n)^2}{(r n)^6 (r n)^6} \right) = O \left( \frac{r^3}{n} \right).$$

We analyse (B3) now. There are 2 vertex-bins of size at least 3 involved. We have $O(k_2 \eta(d)^2 m_3^2)$ choices for the vertex-bin of size 2, the 2 vertex-bins of size at least 3 and the points inside them, and the 2 edge-bins involved. There are $O(1)$ matchings between the 6 points in the vertex-bins (2 in the vertex-bin of size 2 and 4 in the other vertex-bins) and the 6 points in the edge-bins creating a double edge. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

$$O \left( \frac{k_2 n_3^2 m_3^2}{P_3!} \right) \left( \frac{P_3 - 6}{P_3!} \right) \left( \frac{P_3 - 6}{P_3!} \right) = O \left( \frac{k_2 n_3^2 m_3^2 T_3^4}{P_3^m Q_3^4} \right) = O \left( \frac{(r^2 n)^2 (r n)^2}{(r n)^6 (r n)^6} \right) = O \left( \frac{r^4}{n} \right).$$

We analyse (B4) now. We have $O(\eta(d)^3 m_3^2)$ choices for the 3 vertex-bins of size at least 3 and the points inside them and the 2 edge-bins involved. There are $O(1)$ matchings between the 6 points in the vertex-bins and the 6 points in the edge-bins creating a double edge. The number of completions for the kernel-configurations containing a giving matching creating such a double edge is

$$O \left( \frac{n_3^3 m_3^2}{P_3!} \right) \left( \frac{P_3 - 6}{P_3!} \right) \left( \frac{P_3 - 6}{P_3!} \right) = O \left( \frac{n_3^3 m_3^2 T_3^6}{P_3^m Q_3^4} \right) = O \left( \frac{(r n)^3 (r n)^2}{(r n)^6 (r n)^6} \right) = O \left( \frac{r^5}{n} \right).$$
Thus, using (44) and the definition of $S_{\psi^*}$, the expected number of loops as in (C) is at most

$$\frac{(P_2 - 2)!}{P_2!} \cdot \frac{(Q_3 - 2)!}{Q_3!} \cdot \frac{Q_2!}{(T_2 - 2)!} \cdot O \left( n_3 m_2 \right) = O \left( \frac{n_3 m_2 T_2^2}{P_2^2 Q_3^2} \right) = O \left( \frac{(r n)(r n)(r n)^2}{(r n)^2 (r n)^2} \right) = O(1).$$

So let $\alpha(n) \to \infty$ such that $\alpha r \to 0$. Then the number of edge-bins corresponding to 2-edges that are loops in the kernel is less than $\alpha$ a.a.s. For any 2-edge in the kernel, let $A_i$ be the event that it is not split by the $i$-th splitting operation performed when creating the pre-kernel from the kernel. Then

$$\Pr \left( \bigcap_{i=1}^{k_0} A_i \right) = \prod_{i=1}^{k_0} \Pr \left( A_i \right) \bigcap_{j=1}^{j-1} A_j = \frac{m_2 - 1}{m_2} \cdot \frac{m_2}{m_2 + 1} \cdot \cdots \cdot \frac{m_2 + k_0 - 2}{m_2 + k_0 - 1} \sim \frac{6 rn}{(1/2)n} \sim 12r.$$

This together with the fact the expected number of 2-edges that are loops in the kernel is less than $\alpha$ a.a.s. implies that the probability there is a 2-edge that is a loop in the pre-kernel is $O(\alpha r) + o(1) = o(1)$. This finishes the proof of Lemma 9.5.

### 9.5 Connected pre-kernels

In this section, we analyse the probability that the random multigraph $P(x, d)$ is connected for $x$ ‘close’ to $x^*$ and $d \in \mathbb{N}^{n^3}$ with some constraints (more precisely $x \in S_{\psi^*}$ and $d \in \hat{D}_\psi(x)$ with $\psi, \phi = o(1)$). We will show that $P(x, d)$ is connected a.a.s., proving Lemma 9.6. Our strategy has some similarities with the proof by Luczak[7] for connected random 2-uniform hypergraphs with given degree sequence and minimum degree at least 3. The main difference is that, in our case, we have some vertices of degree 2 and the matching on the set of points in the bins has some constraints because of these vertices. This makes it more difficult to compute the probability of connectedness.

A pre-kernel is connected if and only of its kernel is connected, since the pre-kernel is obtained by splitting 2-edges of the kernel and assigning vertices of degree 1. Thus, we only need to analyse the connectivity of the kernel. Let $d$ denote the degree sequence of the vertices of degree at least 3, $k_i$ the number of vertices of degree 2 that are in exactly $i$ 3-edges (for $i = 1, 2$), $m_2$ the number of 2-edges and $m_3$ the number of 3-edges.

We say that a kernel-configuration is connected if the 2-uniform multigraph described as follows is connected: contract each vertex-bin and each edge-bin into a single vertex and add one edge $uv$ for each edge $ij$ of the matching in the kernel-configuration such that $i$ is in the bin corresponding to $u$ and $j$
is in the bin corresponding to \( v \). Given a kernel-configuration with matching \( M \), perform the following operations:

1. For each vertex-bin \( v \) with more than 6 points, partition the points of \( v \) into new vertex-bins so that each of the new vertex-bins has 3, 4 or 5 points. Delete \( v \) and keep \( M \) unchanged. See Figure 7.

2. For each edge-bin \( e \) of size 2 such that exactly one of its points, say \( p_v \), in a vertex-bin \( v \) of size 2, do the following. Let \( p'_e \) be the point in \( e \) other than \( p_v \) and let \( p'_v \) be the point in \( v \) other than \( p_v \). Let \( i \) be the point matched to \( p'_e \) in \( M \) and let \( j \) be the point matched to \( p'_v \) in \( M \). Delete \( v \) and \( e \) from the kernel-configuration. Add a new edge to \( M \) connecting \( i \) and \( j \). See Figure 8.

3. For each edge-bin \( e \) of size 2 such that both of its points \( p_v \) and \( p'_e \) are matched to points \( p_v \) and \( p_w \) in vertex-bins \( v \) and \( w \) of size 2, do the following. Let \( p'_v \) be the point in \( v \) other than \( p_v \) and let \( p'_w \) be the point in \( w \) other than \( p_w \). Let \( i \) be the point matched to \( p'_v \) in \( M \) and let \( j \) be the point matched to \( p'_w \) in \( M \). Delete \( v \), \( w \) and \( e \) from the kernel-configuration. Create a new vertex-bin of size 2 with points \( p'_v \) and \( p'_w \) and add the edges \( p'_v i \) and \( p'_w j \) to \( M \). See Figure 9.

See Figure 7 for an example of the procedure. If the kernel-configuration created in Step 1 is connected, the original kernel-configuration was also connected, since splitting vertex-bins cannot turn a disconnected vertex-bin into smaller pieces.

Figure 7: Breaking a vertex-bin into smaller pieces.

Figure 8: Transforming an edge-bin of size 2 matched to a vertex-bin of size 2 into an edge of the matching is in the bin corresponding to \( v \). Given a kernel-configuration with matching \( M \), perform the following operations:

1. For each vertex-bin \( v \) with more than 6 points, partition the points of \( v \) into new vertex-bins so that each of the new vertex-bins has 3, 4 or 5 points. Delete \( v \) and keep \( M \) unchanged. See Figure 7.

2. For each edge-bin \( e \) of size 2 such that exactly one of its points, say \( p_v \), in a vertex-bin \( v \) of size 2, do the following. Let \( p'_e \) be the point in \( e \) other than \( p_v \) and let \( p'_v \) be the point in \( v \) other than \( p_v \). Let \( i \) be the point matched to \( p'_e \) in \( M \) and let \( j \) be the point matched to \( p'_v \) in \( M \). Delete \( v \) and \( e \) from the kernel-configuration. Add a new edge to \( M \) connecting \( i \) and \( j \). See Figure 8.

3. For each edge-bin \( e \) of size 2 such that both of its points \( p_v \) and \( p'_e \) are matched to points \( p_v \) and \( p_w \) in vertex-bins \( v \) and \( w \) of size 2, do the following. Let \( p'_v \) be the point in \( v \) other than \( p_v \) and let \( p'_w \) be the point in \( w \) other than \( p_w \). Let \( i \) be the point matched to \( p'_v \) in \( M \) and let \( j \) be the point matched to \( p'_w \) in \( M \). Delete \( v \), \( w \) and \( e \) from the kernel-configuration. Create a new vertex-bin of size 2 with points \( p'_v \) and \( p'_w \) and add the edges \( p'_v i \) and \( p'_w j \) to \( M \). See Figure 9.

See Figure 7 for an example of the procedure. If the kernel-configuration created in Step 1 is connected, the original kernel-configuration was also connected, since splitting vertex-bins cannot turn a disconnected vertex-bin into smaller pieces.

Figure 8: Transforming an edge-bin of size 2 matched to a vertex-bin of size 2 into an edge of the matching is in the bin corresponding to \( v \). Given a kernel-configuration with matching \( M \), perform the following operations:

1. For each vertex-bin \( v \) with more than 6 points, partition the points of \( v \) into new vertex-bins so that each of the new vertex-bins has 3, 4 or 5 points. Delete \( v \) and keep \( M \) unchanged. See Figure 7.

2. For each edge-bin \( e \) of size 2 such that exactly one of its points, say \( p_v \), in a vertex-bin \( v \) of size 2, do the following. Let \( p'_e \) be the point in \( e \) other than \( p_v \) and let \( p'_v \) be the point in \( v \) other than \( p_v \). Let \( i \) be the point matched to \( p'_e \) in \( M \) and let \( j \) be the point matched to \( p'_v \) in \( M \). Delete \( v \) and \( e \) from the kernel-configuration. Add a new edge to \( M \) connecting \( i \) and \( j \). See Figure 8.

3. For each edge-bin \( e \) of size 2 such that both of its points \( p_v \) and \( p'_e \) are matched to points \( p_v \) and \( p_w \) in vertex-bins \( v \) and \( w \) of size 2, do the following. Let \( p'_v \) be the point in \( v \) other than \( p_v \) and let \( p'_w \) be the point in \( w \) other than \( p_w \). Let \( i \) be the point matched to \( p'_v \) in \( M \) and let \( j \) be the point matched to \( p'_w \) in \( M \). Delete \( v \), \( w \) and \( e \) from the kernel-configuration. Create a new vertex-bin of size 2 with points \( p'_v \) and \( p'_w \) and add the edges \( p'_v i \) and \( p'_w j \) to \( M \). See Figure 9.

See Figure 7 for an example of the procedure. If the kernel-configuration created in Step 1 is connected, the original kernel-configuration was also connected, since splitting vertex-bins cannot turn a disconnected vertex-bin into smaller pieces.
Figure 10: Modifying a kernel-configuration
kernel-configuration into a connected one. We say that the structures in Step 2 and Step 3 are connected if the 2-uniform hypergraph obtained by contracting each bin into a single vertex is connected. It is trivial that, if the structure obtained is connected, then the original kernel-configuration was connected.

Recall that $M$ is chosen u.a.r. from all possible matchings when generating a random kernel as described in Section 9.2. This implies that, in the structure obtained after Step 3, the resulting matching has uniform distribution among the perfect matchings on the set of points in the bins such that each point in an edge-bin of size 2 is matched to a point in a vertex-bin of size at least 3, each point in a vertex-bin of size 2 is matched to a point in an edge-bin of size 3, each point in a vertex-bin of size at least 3 is matched to a point in an edge-bin, and each point in an edge-bin of size 3 is matched to a point in a vertex-bin.

Here we describe a new model to generate structures as the one obtained by the process above. Let and $t \in \{3, 4, 5\}^N$ and let $t' \in \{3, 4, 5\}^{N'}$. Let $L \leq \sum_i t_i/2$ and $L' \leq \sum_i t'_i/2$ be such that $\sum_i (t_i - 2L) = \sum_i (t'_i - 2L') =: K$. Let $B(t, t', L, L')$ be generated as follows. In each step, every choice is made u.a.r.:

1. **(Left-bins)** For each $i \in [N]$, create one bin/set with $t_i$ points in it. We call these bins left-bins.
2. **(Right-bins)** For each $i \in [N']$, create one bin/set with $t'_i$ points in it. We call these bins right-bins.
3. **(Left-connectors)** Create $L$ bins with 2 points inside each. We call these bins left-connectors.
4. **(Right-connectors)** Create $L'$ bins with 2 points inside each. We call these bins right-connectors.
5. **(Matching)** Choose a perfect matching such that each point in a left-connector is matched to a point in a left-bin, each point in a right-connector is matched to a point in a right-bin, each point in a left-bin is either matched to a point in a left-connector or in a right-bin, and each point in a right-bin is either matched to a point in a right-connector or in a left-bin. The edges in the matching from points in right-bins to points in left-bins are called across-edges.

In the structure we obtained from the kernel-configuration, vertex-bins of size at least 3 have the same role as the left-bins, edge-bins of size 3 have the same role as the right-bins, vertex-bins of size 2 have the same role as the right-connectors, and edge-bins of size 2 have the same role as the left-connectors. See Figure 10.

We will prove that $B(t, t', L, L')$ with $K \to \infty$ is connected a.a.s.

**Lemma 9.8.** Let and $t \in \{3, 4, 5\}^N$ and let $t' \in \{3, 4, 5\}^{N'}$. Let $L \leq \sum_i t_i/2$ and $L' \leq \sum_i t'_i/2$ be such that $\sum_i (t_i - 2L) = \sum_i (t'_i - 2L') =: K$. If $K \to \infty$, then $B(t, t', L, L')$ is connected a.a.s.

Before presenting the proof for this lemma, we explain how to prove Lemma 9.6 assuming Lemma 9.8 holds. In the structure obtained from the kernel-configuration, the number of points from vertex-bins of size at least 3 (which corresponds to left-bins) that are matched to points in edge-bins of size 3 (which corresponds to right-bins) is $T_3 + m_2(1)$, where $m_2(1)$ is the number of edge-bins as described in Step 2 of the procedure. In order to use Lemma 9.8 to conclude that the kernel-configuration is connected a.a.s. (and thus proving Lemma 9.6), it suffices to show that $m_2(1) \to \infty$ a.a.s. (which ensures that the condition $K \to \infty$ is satisfied).

Let $U$ be the set of points in vertex-bins of size 2 that will be matched to points in edge-bins of size 2. (See Step 3 in the proof of Lemma 9.2.) There are $inom{2m_2}{k_1}k_1!$ ways of matching the points in $U$ to points in edge-bins of size 2. For every edge-bin $i$ of size 2, let $X_i$ be the indicator random for the event that $i$ has both of its points matched to points in $U$. For $x \in S_\psi^*$, we
have that $m_2 \sim k_1$ and so

$$
P(X_i = 1) = \frac{\binom{k_1}{2} 2! \binom{2m_2 - 2}{k_1 - 2} (k_1 - 2)!}{\binom{2m_2}{k_1} k_1!} \sim \frac{1}{4},$$

$$
P(X_i = 1, X_j = 1) = \frac{\binom{k_1}{4} 4! \binom{2m_2 - 4}{k_1 - 4} (k_1 - 4)!}{\binom{2m_2}{k_1} k_1!} \sim \frac{1}{16},$$

for $i \neq j$.

and so $\mathbb{E}(\sum_i X_i) \sim m_2/4$ and $\text{Var}(\sum_i X_i) = o(\mathbb{E}(\sum_i X_i)^2)$. Thus, by Chebyshev’s inequality,

$$
P\left( |\sum_i X_i - \mathbb{E}(\sum_i X_i)| \geq t \mathbb{E}(\sum_i X_i) \right) = o\left(\frac{1}{t^2}\right)
$$

and so we can choose $t$ going to 0 sufficiently slowly so that $m_2(2) = \sum_i X_i \sim m_2/4$ a.a.s. Similarly, $m_2(0) = \sum_i X_i \sim m_2/4$ a.a.s. Thus,

$$
m_2(1) \geq (1 + o(1)) \frac{m_2}{2} \to \infty
$$

since $x \in S^*_\psi$.

We finish this section by presenting the proof for Lemma 9.8.

**Proof of Lemma 9.8** Let $Q = \sum_i t_i$ and let $Q' = \sum_i t'_i$. The number of choices for the matching in Step 5 is

$$
\binom{Q}{2L}(2L)! \binom{Q'}{2L'}(2L')! K! = \frac{Q!Q'!}{K!}.
$$

Let $A$ be a set of left-bins with $P$ points of which $S$ points are matched to a set of left-connectors (covering all points in these left-connectors). Similarly, let $A'$ be a set of right-bins with $P'$ points of which $S'$ points are matched to a set of right-connectors. Note that $S$ and $S'$ must be even numbers. We compute the number of configurations such that $A, A'$ form a connected component with $r := P - S = P' - S'$ across-edges:

$$
\times r!(K - r)! \\
\times \binom{L'}{S'/2} \binom{P'}{S'} S'! \binom{Q' - P'}{2L' - S'} (2L' - S')!
$$

Thus, the probability that $A, A'$ form a connected component (with parameters $S, S'$) is exactly

$$
\frac{\binom{L}{S/2} \binom{L'}{S'/2} \binom{K}{r} \binom{Q}{P} \binom{Q'}{P'}}{Q!Q'!}.
$$
So we want to bound the summation:

\[
\sum_{(P,S,n)} \sum_{(A,A')} \frac{L}{(S/2)} \frac{L'}{(S'/2)} \frac{K}{r} \frac{N}{n} \frac{N'}{n'} = O \left( \frac{L^{S/2}(L')^{S'/2}K^{rN(n'(N'))^{n'}}}{Q^P(Q')^{P'}} \right)
\]

where the second summation is over the pairs \((A,A')\) where \(A\) is a set of \(n\) left-bins with \(P\) points and \(S\) points matched to left-connectors and \(A'\) is a set of \(n'\) right-bins with \(P'\) points and \(S'\) points matched to right-connectors; and \(r = P - S = P' - S'\). Let \(C\) be an integer constant to be determined later.

First consider the case where

\[P \leq C\] and \[P' \leq C\],

or

\[Q - P \leq C\] and \[Q' - P' \leq C\].

We only need to check one of the options above because if \(A \cup A'\) is disconnected from the rest of the graph the same is true for the \(\overline{A} \cup \overline{A'}\) where \(\overline{A}\) is the complement of \(A\) in the set of left-bins and \(\overline{A'}\) is the complement of \(A'\) in the set of right-bins. So let us assume \(P \leq C\) and \(P' \leq C\). Then the number of choices for \((P,S,n)\) and \((P',S',n')\) is \(O(1)\). Moreover, there are at most \(\binom{N}{n}\) choices for \(A\) and \(\binom{N'}{n'}\) choices for \(A'\), where \(N\) is the number of left-bins and \(N'\) is the number of right-bins. Then the summation in (45) for this case is at most

\[
= O \left( \frac{Q^P(Q')^{P'}}{Q^{P/6}(Q')^{P'/6}} \right) = o(1),
\]

since \(P - S/2 - r/2 - n \geq P - S/2 - (P - S)/2 - P/3 = P/6\) (and similarly for \(P' - S'/2 - r/2 - n'\)) and \(P\) or \(P'\) is at least 1.

Now consider the case where

\[P \leq C\] and \[Q' - P' \leq C\],

or

\[Q - P \leq C\] and \[P' \leq C\].

If \(P \leq C\) and \(Q' - P' \leq C\). Then \(r = P - S \leq C\) and \(r = P' - S' \geq P' - 2L' \geq Q - C - 2L' = K - C\), which is impossible since \(K \to \infty\) and \(C = O(1)\).

Finally consider the case

\[P \geq C\] and \[P' \geq C\],

or

\[Q - P \geq C\] and \[Q' - P' \geq C\].
Using Stirling’s approximation, there is a positive constant $\alpha$ such that
\[
\frac{\binom{K}{r}}{\left(\frac{K}{2}\right)^{\frac{K}{2}} \left(\frac{r}{2}\right)^{\frac{r}{2}}} \leq \alpha \sqrt{K}.
\]
Thus, for $P$ and $P'$ in this range,
\[
\sum_{(P',S',n')}(A,A') \binom{L}{S/2} \binom{L'}{S'/2} \binom{K}{r} \frac{Q}{P} \frac{Q'}{P'} \leq \alpha \sum_{(P,S,n)}(A,A') \binom{L}{S/2} \binom{L'}{S'/2} \sqrt{K} \frac{[K/2]}{[r/2]} \frac{[K/2]}{[r/2]}
\]
\[
\leq \alpha \sum_{(P,S,n)}(P',S',n') \binom{N}{n} \binom{N'}{n'} \binom{L}{S/2} \binom{L'}{S'/2} \sqrt{K} \frac{[K/2]}{[r/2]} \frac{[K/2]}{[r/2]}
\]
\[
\leq \alpha \binom{N}{n} \binom{N'}{n'} \sqrt{K} \frac{Q - L - [K/2]}{P - S/2 - [r/2]} \frac{Q' - L' - [K/2]}{P' - S'/2 - [r/2]}
\]
\[
= \alpha \frac{\binom{N}{n} \binom{N'}{n'}}{P' - d(r)} \frac{Q/2 - u(K)}{P' - d(r)} \frac{Q'/2 - d(K)}{P'/2 - d(r)},
\]
where $u(x) := \lfloor x/2 \rfloor - x/2$ and $d(x) := x/2 - \lfloor x/2 \rfloor$. Note that, for $P' \leq Q'/2$,
\[
\frac{\binom{N'}{n'}}{Q'/2 - d(K)} \frac{P'/2 - d(r)}{P' - d(r)} \leq \frac{Q'/3}{P' - d(K)} \frac{P'/2 - d(r)}{P'/2 - d(r)} \leq \frac{1}{Q'/6 - d(K)} \frac{P'/6 - d(r)}{P'/2 - d(r)} \leq 1,
\]
and for $P' \geq Q'/2$
\[
\frac{\binom{N'}{n'}}{Q'/2 - d(K)} \frac{P'/2 - d(r)}{P' - d(r)} = \frac{\binom{N'}{N' - n'}}{Q'/2 - d(K)} \frac{Q'/2 - P'/2 - d(K) + d(r)}{P' - d(K)} \leq \frac{Q'/3}{Q'/2 - P'/2 - d(K) + d(r)} \frac{Q'/2 - d(K)}{Q'/6 - P'/6 - d(K) + d(r)} \leq \frac{1}{Q'/6 - P'/6 - d(K) + d(r)} \leq 1,
\]
Thus, for $P' \leq Q'$,
\[
\frac{\binom{N'}{n'}}{Q'/2 - d(K)} \frac{P'/2 - d(r)}{P' - d(r)} \leq 1.
\]

(46)
For $C \leq P \leq \beta \log Q$, 
\[
\frac{\binom{N}{n}}{(Q/2 - u(K)) (P/2 - d(r))} \leq \frac{(Q/3)^{P/3}}{(Q/2 - u(K)) (P/2 - u(r))} \leq \left(\frac{Q/6 - u(K)}{P/6 - u(r)}\right)^{-1} = O\left(\frac{Q}{\beta \log Q}\right)^{-P/6 + u(r)}
\]
and so by choosing $C$ big enough and using (46),
\[
\sum_{\binom{P,S,n}{(P',S',n')}}_{C \leq P \leq \beta \log Q} \frac{\binom{N}{n}}{(Q/2 - u(K)) (Q'/2 - d(K))} \leq Q^{11/2} \log Q \cdot O\left(\frac{\beta \log Q}{Q}\right)^6 = o(1).
\]
The range $Q - \beta \log Q \leq P \leq Q - C$ can be treated similarly.

There exists a constant $\gamma > 0$ such that, for $\beta \log Q \leq P \leq Q/2$, 
\[
\frac{\binom{N}{n}}{(Q/2 - u(K)) (P/2 - u(r))} \leq \frac{(Q/3)^{P/3}}{(Q/2 - u(K)) (P/2 - u(r))} \leq \left(\frac{Q/6 - u(K)}{P/6 - u(r)}\right)^{-1} = O(\gamma^{P/6 - u(r)}),
\]
and so, by (46),
\[
\sum_{\binom{P,S,n}{(P',S',n')}}_{\beta \log Q \leq P \leq Q/2} \frac{\binom{N}{n}}{(Q/2 - u(K)) (Q'/2 - d(K))} \leq Q^{13/2} \cdot O\left(\gamma^\beta \log N\right) = o(1),
\]
for sufficiently large constant $\beta$. The range $Q/2 \leq P \leq Q - \beta \log Q$ can be treated similarly. The same argument works for $(P',S',n')$ and $Q'$. We are done because $P \leq Q - C$ or $P' \leq Q' - C$ (otherwise, it falls in a case that has already been treated). \(\square\)

### 9.6 Proof of Theorem 9.1

In this section we obtain an asymptotic formula for the number of connected pre-kernels with vertex set $[n]$ and $m = n/2 + R$ edges, when $R = \omega(n^{1/2} \log^{3/2} n)$ and $R = o(n)$. The complete proof is contained in this section together with Sections 9.7, 9.8, and 9.9 in which we prove some lemmas we state in this section. This proves Theorem 9.1.

We rewrite the conditions defining $S_m \subseteq \mathbb{R}^4$. We have that $(n_1, k_0, k_1, k_2) \in S_m$ if all of the following conditions are satisfied:

- (C1) $n_1, k_0, k_1, k_2 \geq 0$;
- (C2) $T_2 \geq 0$ (equivalently, $2n_1 - 2k_0 - k_1 \geq 0$);
- (C3) $T_3 \geq 0$ (equivalently, $3n_1 + k_1 + 2k_2 \leq 3m$);
- (C4) $Q_3 \geq 3n_3 \geq 0$ (equivalently, $k_0 - k_1 - k_2 \leq 3m - n$ and $n_1 - k_0 - k_1 - k_2 \leq n$);
- (C5) $Q_3 = 0$ whenever $n_3 = 0$. 

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For $x = (n_1, k_0, k_1, k_2) \in S_m$, let

$$w_{pre}(x) = \begin{cases} \frac{P_3!P_3!Q_3!(m_2 - 1)!}{k_0!k_1!k_2!n_3!m_3!T_3!T_2!(m_2 - 1)!m_2^{-1}2^{k_2}2^{m_2}6^{m_3}} f_3(\lambda)^{n_3}, & \text{if } Q_3 > 3n_3; \\
\frac{f_3(\lambda)^{n_3}}{1}, & \text{otherwise.}
\end{cases} \quad (47)$$

Recall that $\hat{x}^* = (\hat{n}_i^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*)$ is defined as

$$\hat{n}_i^* = \frac{3\hat{m}}{g_2(\lambda^*)}, \quad \hat{k}_0^* = \frac{3\hat{m}}{g_2(\lambda^*)} 2\lambda^*, \quad \hat{k}_1^* = \frac{3\hat{m}}{g_2(\lambda^*)} g_1(\lambda^*), \quad \hat{k}_2^* = \frac{3\hat{m}}{g_2(\lambda^*)} \lambda^* f_1(\lambda^*)$$

where $\lambda^* = \lambda^*(n)$ is the unique nonnegative solution of the equation

$$\frac{\lambda f_1(\lambda) g_2(\lambda)}{f_2(2\lambda)} = 3\hat{m}.$$

The existence and uniqueness of $\lambda^*$ was discussed in Lemma 8.3.

We will show that $\hat{x}^*$ is the unique point achieving the maximum for $f_{pre}$ in the set $\hat{S}_m$ and then we will expand the summation around $\hat{x}^*$. To determine the region where the summation will be expanded we will analyse the Hessian of $f_{pre}$. Let

$$H_0 = \frac{1}{36} \begin{pmatrix} 33 & 12 & 15 & 18 \\ 12 & 6 & 6 & 6 \\ 15 & 6 & 7 & 8 \\ 18 & 6 & 8 & 12 \end{pmatrix}$$

and

$$T = \frac{1}{30} \begin{pmatrix} -47 & -16 & -11 & -6 \\ -16 & 22 & 12 & 2 \\ -11 & 12 & 31/3 & -4/3 \\ -6 & 2 & -4/3 & -4/3 \end{pmatrix} \quad (48)$$

Later we will see that the Hessian of $f_{pre}$ at $\hat{x}^*$ is $(-1/r^2)H_0 - (1/r)T + O(J)$, where $J$ denotes the $4 \times 4$ matrix of all 1’s. For two matrices $A, B$ of same dimensions, we say that a matrix $A = O(B)$ if $A_{ij} = O(B_{ij})$ for all $i, j$.

Let $z_1 = (1, 1, -3, 0)$. Then $z_1$ is an eigenvector of $H_0$ with eigenvalue 0. Let $e_i \in \mathbb{R}^4$ be the vector such that the $i$-th coordinate is 1 and all the others are 0. Let

$$B := \left\{ x \in \mathbb{R}^4 : x = \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4, \ |\gamma_1| \leq \hat{\delta} n \text{ and } |\gamma_i| \leq \hat{\delta} n \text{ for } i = 2, 3, 4 \right\},$$

and let $\hat{B} = \{(n_1/n, k_0/n, k_1/n, k_2/n) : (n_1, k_0, k_1, k_2) \in B\}$, that is, $\hat{B}$ is a scaled version of $B$. We will choose $\delta$ and $\hat{\delta}$ later. The set $x^* + B$ (this is the Minkowski sum of $\{x^*\}$ and $B$) is the region where we will approximate $\sum_x n! \exp(n f_{pre}(\hat{x}))$ by using Taylor’s approximation. For this, we show that, for an appropriate choice for $\delta_1$ and $\delta$, the set $x^* + B$ is contained in $S_m$.

Lemma 9.9. Suppose that $\delta_1 = o(r)$ and that $\delta = o(r^2)$. Let $x \in B$. For any function $F$ among $n_1(x+x^*)$, $k_i(x^*+x)$ for $i = 0, 1, 2, Q_3(x^* + x) - 3n_3(x^* + x)$, and the linear functions defined in (32), we have that $F(x^* + x) \sim F(x^*)$. Moreover, $\lambda(x) \sim \lambda(x^*)$.

Proof. Write $x$ as $x = \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4$ with $|\gamma_1| \leq \delta_1$ and $|\gamma_i| \leq \delta$ for $i = 2, 3, 4$. We will show that $F(\gamma_1 z_1) = o(F(x^*))$ and $F(\gamma_i e_i) = o(F(x^*))$ for $i = 2, 3, 4$. Since $F$ is a linear function, this implies that $F(x^* + x) = F(x^*) + F(x) = F(x^*) + o(F(x^*))$, proving the first statement in the lemma.

Using (39), we have that $F(x^*) = O(r^2 n)$ for all the functions $F$ under consideration and so, for $i = 2, 3, 4$, we have that $F(\gamma_i e_i) = o(r^2 n) = o(F(x^*))$ since $|\gamma_i| \leq \delta n = o(r^2 n)$.
Using (39), we have that \(F(x^*) = \Omega(rn)\) for all \(F\) under consideration except \(k_2, T_3\) and \(Q_3 - 3n_3\). Since \(|\gamma_1| < \delta_1 n = o(rn)\), we have that \(F(\gamma_1 z_1) = o(rn) = o(F(x^*))\) for all \(F\) under consideration, except \(k_2, T_3\) and \(Q_3 - 3n_3\). So let \(F\) be one of the functions \(k_2, T_3\) or \(Q_3 - 3n_3\). Then, using \(z_1 = (1, 1, -3, 0)\), we have that \(F(z_1) = 0\) and so \(F(x^* + x) = F(x^*)\), finishing the proof of the first statement in the lemma.

Since \(Q_3(x^* + x) \sim Q_3(x^*)\) and \(n_3(x^* + x) \sim n_3(x^*)\), we have that \(c_3(x + x^*) = Q_3(x^* + x)/n_3(x^* + x) \sim c_3(x^*)\). Thus, since \(\lambda(y)\) is defined as the unique solution of \(\lambda f_2(\lambda)/f_3(\lambda) = c_3(y)\), we have that \(\lambda(x) \sim \lambda(x^*)\) by Lemma 7.1.

**Corollary 9.10.** Suppose that \(\delta_1 = o(r)\) and that \(\delta = o(r^2)\). Let \(x \in B\). Then there exists \(\psi = o(1)\) such that \(x^* + x \in S^*_\psi\) and \(x^* + x\) is in the interior of \(S_m\).

**Proof.** Recall that \(S^*_\psi\) is defined in (43). Lemma 9.9 and the definition of \(S^*_\psi\) immediately imply the first part of the conclusion.

We check whether \(x^* + x\) satisfies the conditions (C1)–(C5) strictly. We have that \(x^*\) satisfies the constraints (C1)–(C4) with slack \(\Omega(r^2n)\) by (39) and recall that \(r^2n \to \infty\). By Lemma 9.9 we have that \(x^* + x\) also satisfies all the constraints (C1)–(C4) with slack \(\Omega(r^2n)\).

It remains to check (C5). We have that \(n_3(x^* + x) \sim n_3(x^*) = \Omega(rn) = \omega(1)\) and so (C5) is satisfied strictly. We conclude that \(x^* + x\) is in the interior of \(S_m\).

The following lemmas are the main steps in the proof of Theorem 9.1. We show that \(\hat{x}^*\) is the unique maximum for \(f_{\text{pre}}\) in \(\hat{S}\) and compute a bound for any other local maximum.

**Lemma 9.11.** The point \(\hat{x}^* = (\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*)\) is the unique maximum for \(f_{\text{pre}}\) in \(\hat{S}_m\) and

\[
\begin{align*}
\hat{f}_{\text{pre}}(\hat{x}^*) = 2r \ln n - 4r \ln r + \left(-\frac{2}{3} \ln(2) - \frac{1}{3} \ln(3) + \frac{1}{3}\right) \lambda^* + \\
\left(-\frac{2}{9} \ln(2) - \frac{1}{9} \ln(3) + \frac{7}{36}\right) (\lambda^*)^2 + O((\lambda^*)^3).
\end{align*}
\]

Moreover, there exists a constant \(\beta < -(2/9) \ln(2) - (1/9) \ln(3) + (7/36)\) such that any other local maximum in \(\hat{S}_m\) has value at most

\[
2r \ln n - 4r \ln r + \left(-\frac{2}{3} \ln(2) - \frac{1}{3} \ln(3) + \frac{1}{3}\right) \lambda^* + \beta(\lambda^*)^2.
\]

We then estimate the summation of \(\exp(n f_{\text{pre}}(\hat{x} + \hat{x}^*))\) over points \(x \in B\) such that \(x + x^*\) is integer.

**Lemma 9.12.** Suppose that \(\delta_1^1 = o(r/n)\) and \(\delta_2^1 = \omega(r/n)\), and \(\delta^3 = o(r^4/n)\) and \(\delta^2 = \omega(r^2/n)\). Then

\[
\sum_{x \in B \atop x + x^* \in \mathbb{Z}^d} \exp \left(n f_{\text{pre}}(\hat{x} + \hat{x}^*)\right) \sim 144 \sqrt{3\pi^2} n^{2r^2/7} \exp(n f_{\text{pre}}(\hat{x}^*)).
\]

Finally, we bound the contribution from points far from the maximum.

**Lemma 9.13.** Suppose that \(\delta_1^1 = o(r/n)\) and \(\delta_2^1 = \omega(r \ln n/n)\), and \(\delta_3 = o(r^4/n)\) and \(\delta^2 = \omega(r^2 \ln n/n)\). We have that

\[
\sum_{x \in S \setminus (x^* + B) \atop x \in \mathbb{Z}^d} w_{\text{pre}}(x) = o(n! \exp(n f_{\text{pre}}(\hat{x}^*))).
\]
The proof of Lemma 9.14 is deferred to Section 9.8. The proofs of Lemmas 9.12 and 9.13 are presented in Section 9.9. We are now ready to prove Theorem 9.1.

In order to use Lemmas 9.12 and 9.13, we need to check if there exists $\delta_1$ such that $\delta_1^3 = o(r/n)$ and $\delta_2^2 = \omega(r \ln n/n)$, and $\delta$ such that $\delta^3 = o(r^4/n)$ and $\delta^2 = \omega(r^2 \ln n/n)$. There exists such $\delta_1$ if and only if $(r/n)^2 = \omega((r \ln n/n)^3)$, which is true if and only if $n/r = \omega(n^3/n)$, which is true since $r = o(1)$. There exists such $\delta$ if and only if $(r^4/n)^2 = \omega((r^2 \ln n/n)^3)$, which is true if and only if $r^2 = \omega(n^3/n)$, which is one of the hypotheses of the theorem.

By Proposition 9.3 and Lemma 9.9, we have that, for $x \in (x^* + B)$,

$$g_{\text{pre}}(x) = w_{\text{pre}}(x) \mathbb{E} \left( \mathbb{P} \left( \mathcal{P}(x, Y) \text{ simple and connected} \right) \bigg| \Sigma(x) \right) \mathbb{P} (\Sigma(x)),$$

where $\Sigma(x)$ is the event that a random vector $Y = (Y_1, \ldots, Y_{n_3(x)})$ of independent truncated Poisson random variables with parameters $(3, \lambda(x))$ satisfy $\sum_{i=1}^{n_3(x)} = Q_3(x)$. By Corollary 9.7 and Lemma 9.9

$$\mathbb{E} \left( \mathbb{P} \left( \mathcal{P}(x, Y) \text{ simple and connected} \right) \bigg| \Sigma(x) \right) \sim 1. \quad (49)$$

By Stirling’s approximation, the definition of $f_{\text{pre}}$ (in (34) and (35)), and definition of $w_{\text{pre}}$ (in (17), we have that

$$w_{\text{pre}}(x) \sim n! \frac{1}{(2\pi n)^{5/2}} \left( \frac{\hat{P}_2 \hat{P}_3 Q_3}{k_0 k_1 k_2 \bar{n}_3 \bar{m}_3 \bar{T}_2 \bar{m}_2} \right)^{1/2} \exp(n f_{\text{pre}}(\hat{x})).$$

Since $x \in (x^* + B)$, by Lemma 9.9, we have that

$$\frac{\hat{P}_3 \hat{P}_2 Q_3}{k_0 k_1 k_2 \bar{n}_3 \bar{m}_3 \bar{T}_2 \bar{m}_2} \sim \frac{\hat{P}_3^* \hat{P}_2^* Q_3^*}{k_0^* k_1^* k_2^* \bar{n}_3^* \bar{m}_3^* \bar{T}_2^* \bar{m}_2^*} \sim \frac{1}{r^{5/2} \sqrt{6}}.$$

Next we estimate $\mathbb{P}(\Sigma(x))$. We will use [8, Theorem 4], applied with $n_3$ as the parameter $n$ in [8, Theorem 4], and $c_3 = Q_3/n_3$ as $c$ in [8, Theorem 4]. By Lemma 9.9 and (39), we have that $Q_3(x) - 3n_3(x) \sim (Q_3(x^*) - n_3(x^*)) \sim 12R^2/n = \omega(\ln(n))$. Thus, by [8, Theorem 4],

$$\mathbb{P}(\Sigma(x)) \sim \frac{1}{\sqrt{2\pi Q_3(x)(1 + \eta_3(x) - c_3(x))}},$$

where $\eta_3(x) = \lambda(x)f_1(\lambda(x))/f_2(\lambda(x))$ and $c_3(x) = Q_3(x)/n_3(x) = \lambda(x)f_2(\lambda(x))/f_3(\lambda(x))$. Since $Q_3(x)/n_3(x) \sim Q_3(\hat{x}^*)/n_3(\hat{x}^*)$, Lemma 7.1 implies that $\lambda(x) \sim x^* \to 0$ and so (omitting the (x) in the following)

$$1 + \eta_3 - c_3 = \frac{f_2(\lambda)f_3(\lambda) + \lambda f_1(\lambda)f_3(\lambda) - \lambda f_2(\lambda)^2}{f_2(\lambda)f_3(\lambda)} = \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \frac{\lambda^4}{24} + \lambda \left( \frac{\lambda^2}{2} + \frac{\lambda^3}{6} \right) \left( \frac{\lambda^2}{6} + \frac{\lambda^3}{24} \right) + \lambda \left( \frac{\lambda^2}{2} + \frac{\lambda^3}{6} \right)^2 + O(\lambda^7)$$

$$= \lambda^6/144 \frac{1}{\lambda^5/12} \left( 1 + O(\lambda) \right) \sim \frac{\lambda}{12} \sim \frac{\lambda^*}{12} \sim r;$$

by Lemma 9.9 and (39). Moreover, $Q_3 \sim 6R$ by (39). Hence,

$$\mathbb{P}(\Sigma) \sim \frac{1}{\sqrt{2\pi(6R)(1 + \eta_3 - c_3)}} \sim \frac{1}{r \sqrt{12\pi n}}.$$
Thus,

\[ g_{\text{pre}}(x) = n! \frac{1}{144(\pi n)^{3r/2}} \sum_{x \in B} \exp(nf_{\text{pre}}(x))(1 + o(1)), \]

for all \( x \in (x^* + B) \). Since \((x^* + B) \cap \mathbb{Z}^4 \) is a finite set for each \( n \), we have that there is a function \( q(n) = o(1) \) such that the error in (50) is bounded by \( q(n) \) uniformly for all \( x \in (x^* + B) \cap \mathbb{Z}^4 \). Thus,

\[
\sum_{x \in (x^* + B) \cap \mathbb{Z}^4} g_{\text{pre}}(x) \sim n! \frac{1}{144(\pi n)^{3r/2}} \sum_{x \in (x^* + B)} \exp(nf_{\text{pre}}(x)) \\
\sim n! \frac{1}{144(\pi n)^{3r/2}} \cdot 144\sqrt{3}\pi^2 n^2 r^{7/2} \exp(nf_{\text{pre}}(x^*)),
\]

by Lemma 9.12. Thus,

\[
\sum_{x \in (x^* + B) \cap \mathbb{Z}^4} g_{\text{pre}}(x) \sim n! \sqrt{3} \frac{1}{\pi n} \exp(nf_{\text{pre}}(x^*)).
\]

Together with Lemma 9.13, this finishes the proof of Theorem 9.1.

### 9.7 Partial derivatives

In this section, we will analyse the first, second, and third partial derivatives of \( f_{\text{pre}} \). This will be used in the proof that \( \hat{x}^* \) achieves the maximum for \( f_{\text{pre}} \) (Lemma 9.11) and also to approximate the summation around \( \hat{x}^* \) (Lemma 9.12).

Recall that \( h_n(y) = y \ln(yn) - y \) and, for \( \hat{x} = (\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2) \),

\[ f_{\text{pre}}(\hat{x}) = h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2) \]

\[ - h_n(k_0) - h_n(k_1) - h_n(k_2) - h_n(\hat{n}_3) - h_n(m_3) \]

\[ - h_n(T_3) - h_n(T_2) - 2h_n(m_2) \]

\[ - \hat{k}_2 \ln 2 - \hat{n}_2 \ln 2 - \hat{m}_3 \ln 6 \]

\[ + \hat{n}_3 \ln f_3(\lambda(x)) - \hat{Q}_3 \ln \lambda(x), \]

where \( \lambda(x) \) is the unique positive solution to \( \lambda f_2(\lambda)/f_3(\lambda) = c_3 \), where \( c_3 = \hat{Q}_3/\hat{n}_3 \).

Using (12) to compute the partial derivatives of \( \hat{n}_3 \ln f_3(\lambda(x)) - \hat{Q}_3 \ln \lambda(x) \) (w.r.t. \( \hat{n}_1, \hat{k}_0, \hat{k}_1 \) and \( \hat{k}_2 \)), we obtain

\[
\frac{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{n}_1} \right)}{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{k}_0} \right)} = \frac{4\hat{T}_3^2\hat{n}_3\hat{n}_1\lambda}{9\hat{m}_3^2\hat{Q}_3\hat{T}_2^2f_3^2\lambda};
\]

\[
\frac{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{k}_0} \right)}{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{k}_1} \right)} = \frac{\hat{n}_3\hat{T}_2^2\lambda^2}{2\hat{Q}_3^2\hat{k}_0f_3(\lambda)};
\]

\[
\frac{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{k}_1} \right)}{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{k}_2} \right)} = \frac{\hat{T}_3^2\hat{n}_3\lambda^2}{\hat{k}_1\hat{Q}_3^2f_3(\lambda)};
\]

\[
\frac{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{k}_2} \right)}{\exp \left( \frac{df_{\text{pre}}(x)}{d\hat{k}_3} \right)} = \frac{\hat{T}_2^2\hat{n}_3\lambda^2}{2\hat{k}_2\hat{Q}_3^2f_3(\lambda)};
\]

For the second partial derivatives, we need to compute

\[
\frac{\partial^2 (\hat{n}_3 \ln f_3(\lambda(x)) - \hat{Q}_3 \ln \lambda(x))}{\partial a \partial b},
\]

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for any \(a, b \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}\). Using (12), this is

\[
\frac{\partial}{\partial a} \left( \frac{\partial \hat{n}_3}{\partial b} \ln f_3(\lambda) - \frac{\partial \hat{Q}_3}{\partial b} \ln \lambda \right) = \frac{\partial}{\partial a} \left( - \ln f_3(\lambda) - \frac{\partial \hat{Q}_3}{\partial b} \ln \lambda \right) = \frac{\partial \lambda}{\partial a} \left( - f_2(\lambda) - \frac{\partial \hat{Q}_3}{\partial b} \lambda \right) \\
= \frac{\partial c_3}{\partial a} \frac{\lambda}{c_3(1 + \eta_3 - c_3)} \left( - f_2(\lambda) - \frac{\partial \hat{Q}_3}{\partial b} \lambda \right),
\]

\[
= \left( \frac{\partial \hat{Q}_3}{\partial a} \frac{1}{\hat{n}_3} - \frac{\partial \hat{n}_3}{\partial a} \frac{3}{\hat{n}_3^2} \right) \frac{1}{c_3(1 + \eta_3 - c_3)} \left( - c_3 - \frac{\partial \hat{Q}_3}{\partial b} \lambda \right),
\]

\[
= - \left( c_3 + \frac{\partial \hat{Q}_3}{\partial a} \right) \left( c_3 + \frac{\partial \hat{Q}_3}{\partial b} \right) \frac{1}{Q_3(1 + \eta_3 - c_3)}. \tag{55}
\]

The second partial derivatives now are

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{n}_1} = \frac{9}{P_3} + \frac{4}{P_2} - \frac{9}{T_3} + \frac{1}{Q_3} - \frac{1}{n_3} - \frac{1}{m_3} - \frac{4}{T_2} - \frac{2}{m_2} + \frac{1}{n_1} + D_1
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{k}_0} = \frac{4}{P_2} + \frac{2}{Q_3} - \frac{1}{n_3} + \frac{4}{T_2} + \frac{2}{m_2} + D_k
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{k}_1} = \frac{3}{T_3} + \frac{2}{Q_3} - \frac{1}{n_3} + \frac{2}{T_2} + D_k
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{n}_1 \partial \hat{k}_2} = - \frac{6}{T_3} + \frac{2}{Q_3} - \frac{1}{n_3} + D_k
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_0 \partial \hat{k}_0} = \frac{4}{P_2} + \frac{4}{Q_3} - \frac{1}{n_3} - \frac{4}{T_2} - \frac{2}{m_2} - \frac{1}{k_0} + D_{kk}
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_0 \partial \hat{k}_1} = \frac{4}{Q_3} - \frac{1}{n_3} - \frac{2}{T_2} + D_{kk}
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_0 \partial \hat{k}_2} = \frac{4}{Q_3} - \frac{1}{n_3} + D_{kk}
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_1 \partial \hat{k}_1} = - \frac{1}{k_1} - \frac{1}{T_3} + \frac{4}{Q_3} - \frac{1}{n_3} - \frac{1}{T_2} + D_{kk}
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_1 \partial \hat{k}_2} = - \frac{2}{T_3} + \frac{4}{Q_3} - \frac{1}{n_3} + D_{kk}
\]

\[
\frac{\partial^2 f_{\text{pre}}(\hat{x})}{\partial \hat{k}_2 \partial \hat{k}_2} = - \frac{1}{k_2} - \frac{4}{T_3} + \frac{4}{Q_3} - \frac{1}{n_3} + D_{kk},
\]

where

\[
D_1 = -\frac{(c_3 - 1)^2}{(1 + \eta_3 - c_3)Q_3};
\]

\[
D_k = -\frac{(c_3 - 1)(c_3 - 2)}{(1 + \eta_3 - c_3)Q_3};
\]

\[
D_{kk} = -\frac{(c_3 - 2)^2}{(1 + \eta_3 - c_3)Q_3}.
\]
In the next lemma, we find an approximation for the Hessian $f_{\text{pre}}$ at $\hat{x}^*$. It follows immediately by computing the series of each partial second derivative with $\lambda \to 0$.

**Lemma 9.14.** The Hessian of $f_{\text{pre}}$ at $\hat{x}^*$ is $-(1/r^2)H_0 - (1/r)T + O(J)$, where $H_0$ and $T$ are defined in (18) and $J$ is a $4 \times 4$ matrix with all entries equal to 1.

We will bound the third partial derivatives for points close to $x^*$.

**Lemma 9.15.** Suppose that $\delta_1^3 = o(r/n)$ and $\delta_4^3 = o(r^4/n)$. Then for any $x \in B$ we have that

$$n\lambda \partial f_{\text{pre}}(\hat{x}^* + \hat{x}) t_1(\hat{x})t_2(\hat{x})t_3(\hat{x}) = o(1),$$

for any $t_1, t_2, t_3 \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$.

**Proof.** Let $x \in B$. Then $x = \alpha z_1 + b$, where $|\alpha| \leq \delta_1$ and $b = (0, b_2, b_3, b_4)$ and $|b_i| \leq \delta$ and $b^T z_1 = 0$. Recall that $z_1 = (1, 1, -3, 0)$ and so $x = (\alpha, \alpha + b_2, -3\alpha + b_3, b_4)$. Then, by using (50), we may compute each partial derivative $\partial f_{\text{pre}}(\hat{x})/\partial t_1(\hat{x})t_2(\hat{x})t_3(\hat{x})$ exactly. We omit the lengthy computations here. The third derivative is the sum of the part involving $\lambda$ and the part that does not involve $\lambda$. The part not involving $\lambda$ can be written as

$$\sum_{a=(a_1,a_2,a_3,a_4)\in\mathbb{N}^4, a_1+a_2+a_3+a_4=3} T(a)\alpha^{a_1}(\alpha + b_2)^{a_2}(-3\alpha + b_3)^{a_3}b_4^{a_4},$$

where each $T(a)$ is a sum of terms in the form $1/z^2$, where

$$z \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2, \hat{n}_3, \hat{P}_2, \hat{P}_3, \hat{Q}_3, \hat{T}_2, \hat{T}_3\}.$$

This can be expanded so that it is

$$\sum_{f=(f_1,f_2,f_3,f_4)\in\{0,1,2,3\}^4 \times \{0,1\}^3, f_1+f_2+f_3+f_4=3} T_2(f)\alpha^{f_1}b_2^{f_2}b_3^{f_3}b_4^{f_4},$$

where each $T_2(f)$ is also a sum of terms in the form $1/z^2$, where

$$z \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2, \hat{n}_3, \hat{P}_2, \hat{P}_3, \hat{Q}_3, \hat{T}_2, \hat{T}_3\}.$$

Since $\delta_1^3 = o(r/n)$ and $\delta_4^3 = o(r^4/n)$ and $R^3 = \omega(N)$, we have that $\delta_1 = o(r)$ and $\delta = o(r^2)$. Thus, by Lemma 9.3, we have that $z \sim z(x^*)$. Using this fact and computing the series of each term with $r \to 0$, we obtain $T_2(f) = O(1/r^{4-f_1})$, and so $\delta = o(r^4/n)$ and $\delta_1 = o(1/\sqrt{n})$ ensure $|\alpha^{f_1}b_2^{f_2}b_3^{f_3}b_4^{f_4}T_2(f)| \leq \delta_1^{f_1}\delta_2^{f_2}\delta_3^{f_3}\delta_4^{f_4}|T_2(f)| = o(1)$.

Similarly the part involving $\lambda$ can be written as

$$\sum_{f=(f_1,f_2,f_3,f_4)\in\{0,1,2,3\}^4 \times \{0,1\}^3, f_1+f_2+f_3+f_4=3} U(f)\alpha^{f_1}b_2^{f_2}b_3^{f_3}b_4^{f_4},$$

where each $U(f)$ is a sum of terms in the following format

$$\frac{1}{Q_3(1+\eta-c_3)} \left( - \frac{(c_3-e_1)(2c_3-e_2-e_3)}{n_3} + (c_3-e_3)(c_3-e_3) \left( \frac{e_1}{Q_3} + \frac{c_3-e_1}{1+\eta-c_3} \left( \frac{\eta(1+\lambda e^{4}/f_1(\lambda)-\eta)}{Q_3(1+\eta-c_3)} - \frac{1}{n_3} \right) \right) \right),$$

where $e_1, e_2, e_3 \in \{1,2\}$. Since $\delta_1 = o(r)$ and $\delta = o(r^2)$, by Lemma 9.3, we have that $\lambda(x^*+x) \sim \lambda(x^*)$.

Using this fact and computing the series of $U(f)$ with $r \to 0$, we have that $U(f) = O(1/r^{4-f_1})$, and so $\delta = o(r^4/n)$ and $\delta_1^3 = o(r/n)$ ensure $|\alpha^{f_1}b_2^{f_2}b_3^{f_3}b_4^{f_4}U(f)| \leq \delta_1^3\delta_2^3\delta_3^3\delta_4^3|U(f)| = o(1)$. \qed

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9.8 Establishing the maximum

In this section, we prove Lemma 9.11 which establishes the maximum of $f_{\text{pre}}$ in $\hat{S}$. Recall that the region $\hat{S}$ where we want to optimise $f_{\text{pre}}(\hat{x})$ over is defined by conditions (C1)–(C4). We rewrite these conditions as follows:

(D1) $\hat{Q}_3 \geq 3\hat{n}_3 \geq 0$ and, if $\hat{n}_3 = 0$, then $\hat{Q}_3 = 0$.

(D2) $\hat{P}_2 \geq 0$;

(D3) $\hat{P}_3 \geq 0$;

(D4) $\hat{k}_0, \hat{k}_1, \hat{k}_2 \geq 0$ and $\hat{T}_2 \geq 0$ and $\hat{T}_3 \geq 0$;

These conditions are obviously a subset of the conditions (C1)–(C5), with the $\hat{n}_1 \geq 0$ being the only constraint missing, which is implied by $\hat{T}_2 \geq 0$. First we will show that $\hat{x}^*$ is the only local maximum in the interior of $\hat{S}$:

**Lemma 9.16.** The point $\hat{x}^* = (\hat{n}_1^*, \hat{k}_0^*, \hat{k}_1^*, \hat{k}_2^*)$ is the unique local maximum for $f_{\text{pre}}$ in the interior of $\hat{S}$ and its value is

$$2r \ln n - 4r \ln r + \left( \frac{2}{3} \ln(2) - \frac{1}{3} \ln(3) + \frac{1}{3} \right) \lambda^* + \left( -\frac{2}{9} \ln(2) - \frac{1}{9} \ln(3) + \frac{7}{36} \right) (\lambda^*)^2 + O((\lambda^*)^3).$$

We will then analyse local maximums when some condition in (D1)–(D4) is tight. The following lemma will be useful to reduce the number of cases to be analysed by giving sufficient conditions for a point not being a local maximum.

**Lemma 9.17.** Let $k$ be a fixed positive integer and $S \subseteq \mathbb{R}$ be a bounded set. Let $f : S \rightarrow \mathbb{R}$ be a continuous function such that $f(x) = -\sum_{i=1}^{\ell} \ell_i(x) \ln \ell_i(x) + g(x)$, where $\ell_i(x) = \sum_{j=1}^{k} \alpha_{i,j} x_j \geq 0$ for all $x \in S$. Suppose $x^{(0)} \in S$ is such that $\ell_i(x^{(0)}) = 0$ for some $i$. Suppose there is $v \in \mathbb{R}^k$ such that $x^{(0)} + tv$ is in the interior of $S$ for small enough $t$ and

$$\frac{d g(x^{(0)} + tv)}{dt} \bigg|_{t=0} > C,$$

for some (possibly negative) constant $C$. Then $x^{(0)}$ is not a local maximum for $f$ in $S$.

The following lemma gives a bound for the value of $f_{\text{pre}}(\hat{x})$ for any local maximum other than $\hat{x}^*$:

**Lemma 9.18.** Let $\hat{S}_1$ be the points in $\hat{S}_m$ such that any of the constraints in (D1)–(D4) is tight. There exists a constant $\beta < -(2/9) \ln(2) - (1/9) \ln(3) + (7/36)$ such that any local maximum of $\hat{S}_m$ in $\hat{S}_1$ for $f_{\text{pre}}$ has value at most

$$2r \ln n - 4r \ln r + \left( \frac{2}{3} \ln(2) - \frac{1}{3} \ln(3) + \frac{1}{3} \right) \lambda^* + \beta (\lambda^*)^2.$$

Note that the constraint $\hat{Q}_3 = 0$ whenever $\hat{n}_3 = 0$ makes $\hat{S}_m$ not closed. We analyse the value of any sequence of points converging to a point with $\hat{Q}_3 > 0$ and $\hat{n}_3 = 0$:

**Lemma 9.19.** Let $(\hat{x}(i))_{i \in \mathbb{N}}$ be a sequence of points in $\hat{S}_m$ converging to a point $z$ with $\hat{Q}_3(z) > 0$ and $\hat{n}_3(z) = 0$. Then $\lim_{i \to \infty} f_{\text{pre}}(x_i) = -\infty$.

Lemma 9.11 is trivially implied by Lemmas 9.16, 9.18, and 9.19. In the rest of this section, we prove these lemmas.
Proof of Lemma 9.16. The computations in this proof are elementary (such as computing resultants) but very lengthy.

Since any local maximum must have value $\exp\left(\frac{d f_{loc}}{dt}\right) = 1$ for any $t \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$, by (61)

$$
4\hat{T}_3^3\hat{n}_3\hat{n}_1\lambda - 9\hat{n}_3^3Q_3\hat{T}_2^2f_3(\lambda) = 0
$$

(57)

$$
\hat{n}_3\hat{T}_2^2\lambda^2 - 2\hat{Q}_3^2\hat{k}_0f_3(\lambda) = 0
$$

(58)

$$
\hat{T}_3\hat{n}_3\hat{T}_2\lambda^2 - \hat{k}_1\hat{Q}_3^2f_3(\lambda) = 0
$$

(59)

$$
\hat{T}_3^2\hat{n}_3\lambda^2 - 2\hat{k}_2\hat{Q}_3^2f_3(\lambda) = 0.
$$

(60)

Next we proceed to take resultants between the LHS of these equations to show that there is only one solution in the interior of $\hat{S}_m$ satisfying all of them. In these computations, we consider $f_3(\lambda)$ and $\lambda$ as independent variables. The resultant of the RHS of (58) and (59) by eliminating $f_3(\lambda)$ is

$$
\lambda^2\hat{T}_2\hat{n}_3\hat{Q}_3^2(6\hat{n}_1\hat{k}_0 + 4\hat{k}_2\hat{k}_0 + 2\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 6\hat{n}\hat{k}_0) = 0
$$

and, since only the last term may possibly be zero in the interior of $\hat{S}_m$, this implies that any local maximum in the interior of $\hat{S}_m$ must satisfy

$$
6\hat{n}_1\hat{k}_0 + 4\hat{k}_2\hat{k}_0 + 2\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 6\hat{n}\hat{k}_0 = 0,
$$

(61)

and note that this determines $\hat{k}_2$ in terms of $\hat{n}_1$, $\hat{k}_1$ and $\hat{k}_0$ for any local maximum in the interior of $\hat{S}_m$. Similarly, the resultant of the RHS of (59) and (60) by eliminating $\lambda$ is

$$
f_3(\lambda)^2\hat{T}_2\hat{n}_3\hat{Q}_3^2(4\hat{k}_2\hat{k}_0 + 3\hat{m}\hat{k}_1 - 3\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 4\hat{n}_1\hat{k}_2) = 0
$$

and it implies that any local maximum in the interior of $\hat{S}_m$ must satisfy

$$
4\hat{k}_2\hat{k}_0 + 3\hat{m}\hat{k}_1 - 3\hat{n}_1\hat{k}_1 - \hat{k}_1^2 - 4\hat{n}_1\hat{k}_2 = 0.
$$

(62)

The resultant of the RHS of (61) and (62) by eliminating $\hat{k}_2$ is

$$
4\hat{T}_2(3\hat{m}\hat{k}_0 - 3\hat{n}_1\hat{k}_0 - \hat{n}_1\hat{k}_1) = 0,
$$

and it implies that any local maximum in the interior of $\hat{S}_m$ must satisfy

$$
3\hat{m}\hat{k}_0 - 3\hat{n}_1\hat{k}_0 - \hat{n}_1\hat{k}_1 = 0,
$$

(63)

which gives determines $\hat{k}_1$ in terms of $\hat{k}_0$ and $\hat{n}_1$.

Taking the resultant of the RHS of (57) and (61) by eliminating $\hat{k}_2$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$
- 4\lambda\hat{k}_1^3\hat{n}_1\hat{k}_0 - 2\lambda\hat{k}_1^3\hat{n}_1^2 + \lambda\hat{k}_1^3\hat{n}_1 + 2\lambda\hat{k}_1^3\hat{n}_1^2\hat{k}_0 + 6\lambda\hat{k}_1^3\hat{n}_1\hat{k}_0 + 4\lambda\hat{k}_1^3\hat{n}_1\hat{k}_0 + 4\lambda\hat{k}_1^3\hat{n}_1\hat{k}_0 + 36\hat{k}_0^3f_3(\lambda)\hat{k}_1\hat{n}_1\lambda
$$

(64)

$$
- 72\hat{k}_0^3f_3(\lambda)\hat{k}_1\hat{n}\hat{n}_1 + 36\hat{k}_0^3f_3(\lambda)\hat{k}_1\hat{m}^2 + 72f_3(\lambda)\hat{k}_0^3\hat{n}_1\hat{n}\hat{m} + 72f_3(\lambda)\hat{n}_1\hat{k}_0\hat{m}^2 + 72f_3(\lambda)\hat{n}_1\hat{k}_0\hat{m}^2 = 0
$$

and then we take the resultant of the RHS of (63) and (64) by eliminating $\hat{k}_1$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$
27\hat{k}_0\lambda\hat{n}_1^3 - 45\hat{k}_0\lambda\hat{n}_1^2 + 21\hat{k}_0\lambda\hat{n}_1\hat{n}_0^2 - 3\lambda\hat{n}_1^3\hat{k}_0 + 12\hat{n}_1\hat{n}_0^3f_3(\lambda) + 12\hat{n}_1^3\lambda\hat{n}_0
$$

(65)

$$
- 12\lambda\hat{n}_1\hat{n}_0^2 - 12\hat{n}_1^3\lambda - 4f_3(\lambda)\hat{n}_1^3 + 12\lambda\hat{n}_1^3 = 0.
$$
Taking the resultant of the RHS of (58) and (61) by eliminating $\hat{k}_2$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$8\hat{k}_0^2f_3(\lambda) + 4\lambda^2\hat{k}_0^2 + 6\lambda^2\hat{m}\hat{k}_0 + 4\lambda^2\hat{k}_1\hat{k}_0 - 2\lambda^2\hat{n}_1\hat{k}_0 + 8\hat{k}_1f_3(\lambda)\hat{k}_0 + \lambda^2\hat{k}_1^2 + 2f_3(\lambda)\hat{k}_1^2 - 2\lambda^2\hat{n}_1\hat{k}_1 = 0 \quad (66)$$

and then we take the resultant of the RHS of (63) and (66) by eliminating $\hat{k}_1$ and ignoring the factors that cannot be zero in $\hat{S}_m$ gives us

$$2\hat{k}_0f_3(\lambda)\hat{n}_1^2 + \lambda^2\hat{n}_1^2\hat{k}_0 - 6\lambda^2\hat{m}\hat{n}_1\hat{k}_0 - 12\hat{k}_0f_3(\lambda)\hat{m}\hat{n}_1 + 9\lambda^2\hat{m}^2\hat{k}_0 + 18\hat{k}_0f_3(\lambda)\hat{m}^2 - 4\lambda^2\hat{n}_1^2 - 4\lambda^2\hat{n}_1^3 = 0, \quad (67)$$

and note that this determines $\hat{k}_0$ in terms of $\hat{n}_1$ and $\lambda$.

Finally we take the resultant of the RHS of (65) and (67) by eliminating $\hat{k}_0$ and ignoring the factors that cannot be zero in $\hat{S}_m$, we get

$$6\lambda\hat{m}\hat{n}_1 + 6f_3(\lambda)\hat{m}\hat{n}_1 + 3\lambda^2\hat{m}\hat{n}_1 - 6\hat{m}\lambda - 6\lambda\hat{n}_1^2 - 2f_3(\lambda)\hat{n}_1^2 - \lambda^2\hat{n}_1^2 + 6\hat{n}_1\lambda = 0. \quad (68)$$

We can then use the equation determining $\lambda$ (that is, $\lambda f_2(\lambda)/f_3(\lambda) = \hat{Q}_3/\hat{n}_3$) by replacing $\hat{k}_0$, $\hat{k}_1$ and $\hat{k}_2$ by the values determined by $\hat{n}_1$, $\lambda$ and $m$ and taking the resultant with (68) by eliminating $\hat{k}_0$ and ignoring the factors that cannot be zero in $\hat{S}_m$:

$$3\hat{m}e^{2\lambda} - 9\hat{m}e^{2\lambda} + 3\hat{m}\lambda e^{2\lambda} + 3\hat{m}\lambda - \lambda e^{2\lambda} + 3\hat{m}\lambda - e^{2\lambda} + 2\lambda - 12\hat{m}\lambda + 9\hat{m}^2 - 3\hat{m} + 18\lambda\hat{m}^2 = 0$$

which has two solutions for $\hat{m}$: $\hat{m} = 1/3$ (which is false) or

$$\hat{m} = \frac{1}{3} \frac{\lambda f_1(\lambda)g_2(\lambda)}{f_2(2\lambda)},$$

which has a unique positive solution $\lambda^*$ by Lemma 8.3, which defines $\hat{x}^*$. Thus, $\hat{x}^*$ is the only point in the interior of $\hat{S}_m$ such that all partial derivatives at it are zero. We now show that $\hat{x}^*$ is a local maximum. Using the second partial derivatives computed in (59) and the series of the determinants of each leading principal submatrix with $\lambda \to 0$, we have that the Hessian at $\hat{x}^*$ is negative definite, which implies that $\hat{x}^*$ is a local maximum.

By writing $f_{pre}(x^*)$ in terms of $\lambda^*$ and computing its series with $\lambda \to 0$, we obtain

$$2r\ln r - 4r\ln r + \left( -\frac{2}{3}\ln(2) - \frac{1}{3}\ln(3) + \frac{1}{3} \right) \lambda^* + \left( -\frac{2}{9}\ln(2) - \frac{1}{9}\ln(3) + \frac{7}{36} \right) (\lambda^*)^2 + O((\lambda^*)^3).$$
Proof of Lemma 9.17. Let $I \in [q]$ be the set of indices such that $\ell_i(x(0)) = 0$. We compute the derivative of $f(x(0) + tv)$ at $t = 0$, using the fact that $\ell_i$ is a linear function,

$$
\frac{d f(x(0) + tv)}{d t} \bigg|_{t=0} \geq C + \sum_{i=1}^{q} \lim_{t \to 0^+} \frac{(-\ell_i(x(0) + tv) \ln \ell_i(x(0) + tv) + \ell_i(x(0)) \ln \ell_i(x(0)))}{t} 
\begin{align*}
&= C + \sum_{i=1}^{q} \lim_{t \to 0^+} \left( -\ell_i(tv) \ln \ell_i(x(0) + tv) + \ell_i(x(0)) \ln \ell_i(x(0)) - \ln(\ell_i(x(0) + tv)) \right) \\
&= C + \sum_{i=1}^{q} \lim_{t \to 0^+} \left( -\ell_i(v) \ln \ell_i(x(0) + tv) \right) - \sum_{i \in [q] \setminus I} \lim_{t \to 0^+} \frac{\ell_i(x(0))}{t} \ln \left( 1 + t \frac{\ell_i(v)}{\ell_i(x(0))} \right) \\
&= C + \sum_{i=1}^{q} \lim_{t \to 0^+} \left( -\ell_i(v) \ln \ell_i(x(0) + tv) \right) - \sum_{i \in [q] \setminus I} \ell_i(v). 
\end{align*}
$$

Since $x_0 + tv$ is in the interior of $S$ for small enough but positive $t$, we have that $\ell_i(v) > 0$ for all $i \in I$. For $i \in [q] \setminus I$, we have that $\ell_i(v) \ln \ell_i(x(0) + tv) + \ell_i(v)$ is bounded. For $i \in I$, using the fact that $\ell_i(v) > 0$, we have that $\ell_i(v) \lim_{t \to 0^+} \ln \ell_i(x(0) + tv) = -\infty$. Thus, we conclude that

$$
\frac{d f(x(0) + tv)}{d t} \bigg|_{t=0} > 0,
$$

which shows that $x(0)$ is not a local maximum. \qed

Proof of Lemma 9.18. We want to find the local maxima in $\hat{S}_1$, which is the set of points in $\hat{S}_m$ such that any of the constraints in (D1)–(D4) is tight. Recall that the constraints (D1)–(D4) are the following:

(D1) $\hat{Q}_3 \geq 3\hat{n}_3 \geq 0$ and, if $\hat{n}_3 = 0$, then $\hat{Q}_3 = 0$.

(D2) $\hat{P}_2 \geq 0$;

(D3) $\hat{P}_3 \geq 0$;

(D4) $\hat{k}_0, \hat{k}_1, \hat{k}_2 \geq 0$ and $\hat{T}_2 \geq 0$ and $\hat{T}_3 \geq 0$;

We split the analysis in the following cases:

Case 1: $\hat{Q}_3 = \hat{n}_3 = 0$;

Case 2: $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 = 0$;

Case 3: $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_2 = 0$;

Case 4: $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 \neq 0$ and $\hat{P}_2 \neq 0$;

Case 5: $\hat{Q}_3 > 3\hat{n}_3 > 0$ and $\hat{P}_3 = 0$;

Case 6: $\hat{Q}_3 > 3\hat{n}_3 > 0$ and $\hat{P}_2 = 0$.

We will use the definitions in (32) many times in the analysis. Maple was used for several computations in the following.

Case 1: Assume that $\hat{Q}_3 = \hat{n}_3 = 0$. Recall that, by definition, we have that $\hat{Q}_3 = 3\hat{n} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2$, $\hat{T}_2 = 2\hat{n}_1 - 2\hat{k}_0 - \hat{k}_1$, and $\hat{T}_3 = 3\hat{n} - 3\hat{n}_1 - \hat{k}_1 - 2\hat{k}_2$. Thus, $\hat{Q}_3 = \hat{T}_2 + \hat{T}_3$. (69)
Moreover, $\hat{T}_2 \geq 0$ and $\hat{T}_3 \geq 0$ are constraints in the definition of $\hat{S}_n$. Thus, since $\hat{Q}_3 = 0$, we have that $\hat{T}_2 = \hat{T}_3 = 0$. Recall that $\hat{n}_3 = 1 - \hat{n}_1 - \hat{k}_0 - \hat{k}_1 - \hat{k}_2$. Hence, we obtain the following equations:

\[
\begin{align*}
1 - \hat{n}_1 - \hat{k}_0 - \hat{k}_1 - \hat{k}_2 &= 0, \\
2\hat{n}_1 - 2\hat{k}_0 - \hat{k}_1 &= 0, \\
3\hat{m} - 3\hat{n}_1 - \hat{k}_1 - 2\hat{k}_2 &= 0.
\end{align*}
\]

By solving this system of equations, we obtain the following values for $\hat{n}_1$, $\hat{k}_1$, and $\hat{k}_2$ in terms of $\hat{k}_0$ and $\hat{m}$:

\[
\begin{align*}
\hat{n}_1 &= 2 - 3\hat{m}; \\
\hat{k}_1 &= 4 - 6\hat{m} - 2\hat{k}_0; \\
\hat{k}_2 &= -5 + 9\hat{m} + \hat{k}_0.
\end{align*}
\]

Moreover, $\hat{P}_3 = 3(m - \hat{n}_1) = -6 + 12\hat{m}$, $\hat{P}_2 = 2(\hat{n}_1 - \hat{k}_0) = 4 - 6\hat{m} - 2\hat{k}_0$. Thus, $f_{\text{pre}}(x)$ depends only on $\hat{k}_0$ and we get

\[
f_{\text{pre}}(x) = f(\hat{k}_0) := h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{m}_2) - h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{m}_3) - 2h_n(\hat{m}_2) - \hat{k}_2 \ln 2 - \hat{m}_2 \ln 2 - \hat{m}_3 \ln 6,
\]

where $\hat{k}_0 \in [5 - 9\hat{m}, 2 - 3\hat{m}]$. We have that

\[
\exp\left(\frac{df}{dk_0}\right) = \frac{(3\hat{m} - 2 + \hat{k}_0)^2}{(-5 + 9\hat{m} + \hat{k}_0)\hat{k}_0}
\]

and

\[
\frac{d^2f}{dk_0^2} = \frac{3\hat{m}\hat{k}_0 - \hat{k}_0 + 33\hat{m} - 10 - 27\hat{m}^2}{\hat{k}_0(-5 + 9\hat{m} + \hat{k}_0)(3\hat{m} - 2 + \hat{k}_0)}
\]

For $\hat{k}_0 \in [5 - 9\hat{m}, 2 - 3\hat{m}]$, the denominator of the second derivative is always nonnegative and its numerator is always negative for sufficiently small $r$ (that is, sufficiently large $n$). Hence, $f$ is strictly concave. Thus, there is a unique maximum and it satisfies:

\[
\frac{(3\hat{m} - 2 + \hat{k}_0)^2}{(-5 + 9\hat{m} + \hat{k}_0)\hat{k}_0} = 1,
\]

that is,

\[
\hat{k}_0 = \frac{(3\hat{m} - 2)^2}{3\hat{m} - 1}.
\]

We then compute the series for $f(\hat{k}_0)$ at this point with $\lambda^*$ going to zero (by using (38)):

\[
f(\hat{k}_0) = 2r \ln n - 4r \ln r + \left(-\ln(2) - \frac{1}{3} \ln(3) + \frac{1}{3}\right) \lambda^* + (\lambda^*)^2 \ln(\lambda^*) + O((\lambda^*)^2).
\]

**Case 2:** Assume that $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 = 0$. Since $\hat{P}_3 = 0$ and $\hat{P}_3 = 3(m - \hat{n}_1)$ by definition (see (32)), we have that $\hat{n}_1 = \hat{m}$. Moreover, since $\hat{T}_3 = \hat{P}_3 - \hat{k}_1 - 2\hat{k}_2$ and $\hat{T}_3, \hat{k}_1, \hat{k}_2 \geq 0$ are constraints in the definition of $\hat{S}_1$, we have that $\hat{k}_1 = 0$ and $\hat{k}_2 = 0$. Using $\hat{Q}_3 = 3\hat{n}_3$ and their definitions in (32), we have that $3\hat{m} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2 = 3(1 - \hat{n}_1 - \hat{k}_0 - \hat{k}_1 - \hat{k}_2)$ and so $\hat{k}_0 = 3 - 3\hat{m} - 2\hat{n}_1 = 3 - 5\hat{m}$. Thus, we only have to compute the value of $f_{\text{pre}}$ in the point $(\hat{m}, 3 - 5\hat{m}, 0, 0)$. By computing the series of $f_{\text{pre}}$ in this point with $\lambda^*$ going to zero (by using (38)), we get

\[
2r \ln(n) - 4r \ln r + \left(\frac{1}{3} - \ln(2) - \frac{1}{3} \ln(3)\right) \lambda^* + O((\lambda^*)^2).
\]

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Case 3: Assume that $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_2 = 0$. Since $\hat{P}_2 = 0$ and $\hat{P}_2 = 2(\hat{n}_1 - \hat{k}_0)$ by definition (see (32)), we have that $\hat{k}_0 = \hat{n}_1$. Moreover, since $\hat{T}_2 = \hat{P}_2 - \hat{k}_1$ and $\hat{T}_2, \hat{k}_1 > 0$ are constraints in the definition of $\hat{S}_m$, we have that $\hat{k}_1 = 0$. Using $\hat{Q}_3 = 3\hat{n}_3$ and their definition in (32), we have that $3\hat{m} - \hat{n}_1 - 2\hat{k}_0 - 2\hat{k}_1 - 2\hat{k}_2 = 3(1 - \hat{n}_1 - \hat{k}_0 - \hat{k}_1 - \hat{k}_2)$ and so $\hat{k}_2 = 3 - 3\hat{m} - 3\hat{n}_1$. So let $f(\hat{n}_1) := f_{\text{pre}}(\hat{n}_1, \hat{n}_1, 0, 3 - 3\hat{m} - 3\hat{n}_1)$ and $\hat{n}_1 \in [2 - 3\hat{m}, 1 - \hat{m}]$. We have that

$$\exp\left(\frac{d f}{d \hat{n}_1}\right) = \frac{8(1 - \hat{n}_1 - \hat{m})^3}{(\hat{m} - \hat{n}_1)^2(-2\hat{n}_1 + 3\hat{m})} \quad \text{and} \quad \frac{d^2 f}{d^2 \hat{n}_1} = \frac{(2\hat{m} - 1)(4 - 7\hat{m} - \hat{n}_1)}{(-2 + \hat{n}_1 + 3\hat{m})(1 - \hat{n}_1 - m)(\hat{m} - \hat{n}_1)}$$

For $\hat{n}_1 \in [2 - 3\hat{m}, 1 - \hat{m}]$, the denominator of the second derivative is always nonnegative and its numerator is always negative for sufficiently small $r$. Hence, $f$ is strictly concave. Thus, there is unique maximum satisfying

$$8(1 - \hat{n}_1 - \hat{m})^3 - (\hat{m} - \hat{n}_1)^2(-2\hat{n}_1 + 3\hat{m}) = 0,$$

which has a unique real solution at $1/2 + \alpha r$, where $\alpha \approx -2.03566$, which is the real solution for

$$9\alpha^3 + 25\alpha^2 + 19\alpha + 11 = 0.$$

We then compute the value of the function $f$ at $1/2 + \alpha r$:

$$2r \ln(n) + 2r \ln r + \beta,$$

with $\beta \approx 1.9389$.

Case 4: Now suppose that $\hat{Q}_3 = 3\hat{n}_3 > 0$ and $\hat{P}_3 > 0$ and $\hat{P}_2 > 0$. By Lemma 9.17, we do not need to consider the cases $\hat{k}_0 = 0$, $\hat{k}_1 = 0$, $\hat{k}_2 = 0$, $\hat{T}_2 = 0$, $\hat{T}_1 = 0$ and $\hat{m}_3 = 0$.

Since $\hat{Q}_3 = 3\hat{n}_3$, we have that $\hat{k}_0 = 3 - 3\hat{m} - 2\hat{n}_1 - \hat{k}_1 - \hat{k}_2$. Thus we analyse the function

$$f(\hat{n}_1, \hat{k}_1, \hat{k}_2) := f_{\text{pre}}(\hat{n}_1, 3 - 3\hat{m} - 2\hat{n}_1 - \hat{k}_1 - \hat{k}_2, \hat{k}_1, \hat{k}_2).$$

We have that, for any local maximum in this case,

$$\exp\left(\frac{d f}{d \hat{n}_1}\right) = \frac{8\hat{P}_3\hat{k}_0\hat{n}_3^2\hat{n}_1}{\hat{m}_3^2\hat{T}_2^6} = 1;$$

$$\exp\left(\frac{d f}{d \hat{k}_1}\right) = \frac{2\hat{P}_3\hat{k}_0}{\hat{T}_2^2\hat{k}_1} = 1;$$

$$\exp\left(\frac{d f}{d \hat{k}_2}\right) = \frac{\hat{P}_3^2\hat{k}_0}{\hat{T}_2^2\hat{k}_2} = 1;$$

and so

$$8\hat{P}_3^2\hat{k}_0^2\hat{n}_3^2\hat{n}_1 - \hat{n}_3^2\hat{T}_2^6 = 0; \quad (70)$$

$$2\hat{P}_3\hat{k}_0 - \hat{T}_2\hat{k}_1 = 0; \quad (71)$$

$$\hat{P}_3^2\hat{k}_0 - \hat{T}_2^2\hat{k}_2 = 0. \quad (72)$$

By taking the resultant of the RHS of (71) and (72), by eliminating $\hat{k}_1$, we get

$$9\hat{n}_3^2(\hat{k}_2 - 3 + 3\hat{m} + 4\hat{n}_1)$$

$$+(9\hat{m}^2\hat{k}_2 - 6\hat{m}\hat{n}_1\hat{k}_2 + n_1^2\hat{k}_2 + 27\hat{m}^3 - 27\hat{m}^2 - 36\hat{m}^2\hat{n}_1 - 9\hat{m}\hat{n}_1^2 + 54\hat{m}\hat{n}_1 - 27\hat{n}_1^2 + 18\hat{n}_1^3) = 0.$$
Using $\dot{Q}_3 = 3\dot{n}_3$ and their definition in (32), we have that $3\dot{m} - \dot{n}_1 - 2\dot{k}_0 - 2\dot{k}_1 - 2\dot{k}_2 = 3(1 - \dot{n}_1 - \dot{k}_0 - \dot{k}_1 - \dot{k}_2)$ and so $3\dot{m} - 3 = \dot{k}_0 + \dot{k}_1 + \dot{k}_2 - 2\dot{n}_1$. Thus, $\dot{k}_2 + 3 + 3\dot{m} + 4\dot{n}_1 = \dot{k}_0 + \dot{k}_1 + 2\dot{k}_2 + 2\dot{n}_1 > 0$ since we already excluded the case $\dot{k}_0 = 0$. Recall that in this case we have $\dot{n}_3 > 0$. Thus, for any local maximum in this case,

$$9\dot{m}^2\dot{k}_2 - 6\dot{m}\dot{n}_1\dot{k}_2 + n^1\dot{k}_2 + 27\dot{m}^3 - 27\dot{m}^2 - 36\dot{m}^2\dot{n}_1 - 9\dot{m}\dot{n}_1^2 + 54\dot{m}\dot{n}_1 - 27\dot{n}_1^2 + 18\dot{n}_1^3 = 0.$$  

(73)

This implies that $\dot{k}_2$ can be determined in terms of $\dot{n}_1$:

$$\dot{k}_2 = \frac{9(-3\dot{m} + 3 - 2\dot{n}_1)(\dot{m} - \dot{n}_1)^2}{(3\dot{m} - \dot{n}_1)^2}.$$  

By taking the resultant of the RHS of (71) and (72), by eliminating $\dot{k}_2$, we get

$$44\dot{n}_3^2(\dot{k}_1 - 2\dot{n}_1)(9\dot{m}^2\dot{k}_1 - 6\dot{m}\dot{n}_1\dot{k}_1 + n^1\dot{k}_1 + 36\dot{m}^2\dot{n}_1 - 36\dot{m}\dot{n}_1 + 36\dot{n}_1^2 - 24\dot{n}_1^3 - 12\dot{m}\dot{n}_1^2) = 0.$$  

In this case $\dot{n}_3 > 0$. Moreover, $2\dot{n}_1 - \dot{k}_1 = 0$ implies, by the definitions in (32), that $\dot{T}_2 = \dot{P}_2 - \dot{k}_1 = 2\dot{n}_1 - 2\dot{k}_0 - \dot{k}_1 \leq 0$ since $\dot{k}_0 \geq 0$ in $\dot{S}_m$. But we have already excluded the case $\dot{T}_2 = 0$. Thus,

$$9\dot{m}^2\dot{k}_1 - 6\dot{m}\dot{n}_1\dot{k}_1 + n^1\dot{k}_1 + 36\dot{m}^2\dot{n}_1 - 36\dot{m}\dot{n}_1 + 36\dot{n}_1^2 - 24\dot{n}_1^3 - 12\dot{m}\dot{n}_1^2 = 0.$$  

(74)

This implies that $\dot{k}_1$ can be determined in terms of $\dot{n}_1$:

$$\dot{k}_1 = \frac{12\dot{n}_1(-3\dot{m} + 3 - 2\dot{n}_1)(\dot{m} - \dot{n}_1)}{(3\dot{m} - \dot{n}_1)^2}.$$  

We take the resultant of the RHS of (70) and (74), by eliminating $\dot{k}_1$ and then the resultant of the polynomial obtained with the RHS of (73) by eliminating $k_2$ and ignoring the factors that cannot be zero in $\dot{S}_m$ and we obtain:

$$18\dot{m} - 36\dot{m}^2 + 18\dot{m}^3 - 18\dot{n}_1 + 18\dot{m}\dot{n}_1 - 3\dot{m}^2\dot{n}_1 + 22\dot{n}_1^2 - 16\dot{m}\dot{n}_1^2 - 7\dot{n}_1^3 = 0.$$  

This cubic equation has one real solution for $\dot{n}_1$ and two complex solutions because the discriminant $\Delta$ of the polynomial above is $-63/4 + O(r)$, which is negative for sufficiently large $n$. For we have that the real solution is $1/2 - r - 6r^2 - O(r^3)$ and so the value of the function $f_{pre}$ at this point is, by using (38),

$$2r \ln n - 4r \ln r + \left(\frac{1}{3} - \frac{1}{3} \ln(3) - \frac{2}{3} \ln(2)\right) \lambda^* + \left(\frac{11}{72} - \frac{2}{9} \ln(2) - \frac{1}{9} \ln(3)\right) (\lambda^*)^2 + O((\lambda^*)^3).$$

**Case 5:** Now suppose that $\dot{P}_3 = 0$ and $\dot{Q}_3 > \dot{n}_3 > 0$. Since $\dot{P}_3 = 0$ and $\dot{P}_3 = 3(\dot{m} - \dot{n}_1)$ by definition (see (32)), we have that $\dot{n}_1 = \dot{m}$. Moreover, since $\dot{T}_3, \dot{k}_1, \dot{k}_2 \geq 0$ in $\dot{S}_m$ and $\dot{T}_3 = \dot{P}_3 - \dot{k}_1 - 2\dot{k}_2$ by definition, we have that $\dot{k}_1 = 0$ and $\dot{k}_2 = 0$. Thus, for any local maximum with $\dot{P}_3 = 0$, it suffices to analyse

$$f(\dot{n}_3) := f_{pre}(\dot{m}, 1 - \dot{m} - \dot{n}_3, 0, 0),$$

where $\dot{n}_3 \in (0, 1 - \dot{m})$, since by definition $\dot{k}_0 = 1 - \dot{n}_1 - \dot{k}_1 - \dot{k}_2 - \dot{n}_3 = 1 - \dot{m} - \dot{n}_3 \geq 0$ and $\dot{Q}_3 = 3\dot{m} - \dot{n}_1 - 2\dot{k}_0 - 2\dot{k}_1 - 2\dot{k}_2 = 4\dot{m} - 2 - 2\dot{n}_3 \geq 0$. We do not have to analyse the value at the endpoints of the interval for $\dot{n}_3$ as they were already considered in cases before. Also, in this case $\dot{Q}_3 = \dot{P}_2$, thus we do not have to check the case $\dot{P}_2 = 0$. Thus, it suffices to consider points satisfying

$$\exp \left( \frac{df}{d\dot{n}_3} \right) = \frac{2(1 - \dot{m} - \dot{n}_3)f_3(\lambda)}{\dot{n}_3 \lambda^2} = 1.$$
where \( \lambda f_2(\lambda)/f_3(\lambda) = \hat{Q}_3/\hat{n}_3 \). The equation below is equivalent to

\[
\hat{n}_3 = \frac{(1 - m)f_3(\lambda)}{f_2(\lambda)}.
\]

Combining this with the equation defining \( \lambda \) implies:

\[
r = 1 - 2e^\lambda + 2 + \lambda + \lambda e^\lambda,
\]

and since \( r \) goes to zero so does \( \lambda \). We have that

\[
r = \frac{1}{24} \lambda + O(\lambda^2),
\]

which implies

\[
\lambda = 2\lambda^* + O(\lambda^*)^2.
\]

We then compute the series of \( f(\hat{n}_3) \) with \( \lambda \) going to zero:

\[
2r \ln n - 4r \ln r + \left( -\frac{1}{2} \ln(2) - \frac{1}{6} \ln(3) + \frac{1}{6} \right) \lambda + O(\lambda^2)
\]

\[
= 2r \ln n - 4r \ln r + \left( -\ln(2) - \frac{1}{3} \ln(3) + \frac{1}{3} \right) \lambda^* + O((\lambda^*)^2).
\]

**Case 6:** Now suppose that \( \hat{P}_2 = 0 \) and \( \hat{Q}_3 > \hat{n}_3 > 0 \). We have that \( \hat{P}_2 = 2(\hat{n}_1 - \hat{k}_0) \). Thus, we have \( \hat{n}_1 = \hat{k}_0 \) since \( \hat{P}_2 = 0 \). Moreover, since \( \hat{T}_2, \hat{k}_1 \geq 0 \) in \( \hat{S}_m \) and \( \hat{T}_2 = \hat{P}_2 - \hat{k}_1 \) by definition, we have that \( \hat{k}_1 = 0 \). Thus, we only need to analyse

\[
f(\hat{n}_1, \hat{k}_2) := f_{\text{pre}}(\hat{n}_1, \hat{n}_1, 0, \hat{k}_2),
\]

where \( \hat{Q}_3 > 3\hat{n}_3 > 0 \) and \( \hat{P}_3 > 0 \). Thus, it suffices to consider points satisfying

\[
\exp \left( \frac{d f}{d \hat{n}_1} \right) = \frac{2}{9 \hat{n}_3^2 f_3(\lambda)^2} = 1 \quad \text{and} \quad \exp \left( \frac{d f}{d \hat{k}_2} \right) = \frac{1}{2} \frac{\hat{n}_3 \lambda^2}{k_2 f_3(\lambda)} = 1,
\]

where \( \lambda f_2(\lambda)/f_3(\lambda) = \hat{Q}_3/\hat{n}_3 \). The second equation implies that for any local maximum

\[
\hat{k}_2 = \frac{1}{2} \frac{(1 - 2\hat{n}_1)\lambda^2}{f_2(\lambda)}.
\]

By using this with the derivative w.r.t. \( \hat{n}_1 \), we get

\[
\hat{n}_1 = \frac{\lambda^{3/2} \sqrt{2} - 3 f_2(\lambda) \hat{n}}{2 \lambda^{3/2} \sqrt{2} - 3 f_2(\lambda)}.
\]

By putting this together with the equation defining \( \lambda \), we have that

\[
\frac{(-e^\lambda + 1 + \sqrt{2} \lambda)\lambda}{f_3(\lambda)} = 0,
\]

which has a unique solution \( \ell^* \approx 0.8267 \). For \( \lambda = \ell^* \), we have \( \hat{n}_1 = \frac{1}{2} + \alpha r \), with \( \alpha \approx 1.4887 \) and \( \hat{k}_2 = \beta(1/2 - \hat{n}_1) \) with \( \beta \approx 0.1173 \). By using this values of \( \hat{n}_1 \) and \( \hat{k}_2 \), we evaluate the function \( f(\hat{n}_1, \hat{k}_2) \) as

\[
2r \ln n - 4r \ln r + 6 \ln r + O(r),
\]

since \( \alpha < 0 \), \( 0 < \beta < 2 \) and \( \lambda > 0 \). \( \square \)
Proof of Lemma 9.19 Let $\hat{x}(i) = (\hat{n}_1(i), \hat{k}_0(i), \hat{k}_1(i), \hat{k}_2(i))$ and similarly for $\hat{Q}_3(i), \hat{n}_3(i)$, etc. Let $\lambda(i)$ be such that $\lambda(i)f_2(\lambda(i))/f_1(\lambda(i)) = \hat{Q}_3(i)/\hat{n}_3(i)$. Recall that
\[
\begin{align*}
f_{\text{pre}}(\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2) &= h_n(\bar{P}_3) + h_n(\bar{P}_2) + h_n(\bar{Q}_3) + h_n(\bar{m}_2) \\
&\quad - h_n(\bar{k}_0) - h_n(\bar{k}_1) - h_n(\bar{k}_2) - h_n(\bar{n}_3) - h_n(\bar{m}_3) \\
&\quad - h_n(\bar{T}_3) - h_n(\bar{T}_2) - 2h_n(\bar{m}_2) \\
&\quad - \hat{k}_2 \ln 2 - \hat{m}_2 \ln 2 - \hat{m}_3 \ln 6 \\
&\quad + \hat{n}_3 \ln f_3(\lambda) - \hat{Q}_3 \ln \lambda.
\end{align*}
\]
Since $S \subseteq [0, 1]^4$ and the fact that $|y \ln y| \leq 1/e$ for $y \in [0, 1]$, we have that $f_{\text{pre}}(x) \leq C + \hat{n}_3 \ln f_3(\lambda) - \hat{Q}_3 \ln \lambda$ for some constant $C$. Thus, it suffices to show that $\hat{n}_3(i)\ln f_3(\lambda(i)) - \hat{Q}_3(i)\ln \lambda(i) \to -\infty$ as $i \to \infty$.

Since $\hat{Q}_3(i)$ converges to a positive number and $\hat{n}_3(i)$ converges to 0, we have that $\hat{Q}_3(i)/\hat{n}_3(i) \to \infty$. This implies that $\lambda(i) \to \infty$. Thus,
\[
\begin{align*}
\hat{n}_3(i)\ln f_3(\lambda(i)) - \hat{Q}_3(i)\ln \lambda(i) \\
&\leq \hat{n}_3(i)\lambda(i) - \hat{Q}_3(i)\ln \lambda(i), \quad \text{since } f_3(\lambda) \leq \exp(\lambda) \\
&\leq \hat{n}_3(i)\frac{\hat{Q}_3(i)}{\hat{n}_3(i)} - \hat{Q}_3(i)\ln \lambda(i), \quad \text{since } \lambda(i) \leq \hat{Q}_3(i)/\hat{n}_3(i) \\
&= \hat{Q}_3(i)(1 - \ln(\lambda(i))) \to -\infty, \quad \text{since } \lambda(i) \to \infty \text{ and } \liminf_{i \to \infty} \hat{Q}_3(i) > 0.
\end{align*}
\]

\[\square\]

9.9 Approximation around the maximum and bounding the tail

In this section, we approximate the sum of $\exp(n f_{\text{pre}}(x))$ over a set of points ‘close’ to $x^*$ and bound the sum for the points ‘far’ from $x^*$. More specifically, we prove Lemmas 9.12 and 9.13.

Proof of Lemma 9.12 We use Lemma 9.15 and Lemma 9.16 which were proved in Section 9.7 and Section 9.8 respectively. Let $x \in B$. By Lemma 9.15, since $\delta^3 = o(r/n)$ and $\delta^3 = o(r^4/n)$, we have that
\[
n\frac{\partial f_{\text{pre}}(\hat{x}^* + \hat{x})}{\partial t_1 \partial t_2 \partial t_3} t_1(\hat{x})t_2(\hat{x})t_3(\hat{x}) = o(1),
\]
for any $t_1, t_2, t_3 \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$. By Lemma 9.11, we have that
\[
\frac{\partial f_{\text{pre}}(\hat{x}^*)}{\partial t} = 0,
\]
for any $t \in \{\hat{n}_1, \hat{k}_0, \hat{k}_1, \hat{k}_2\}$. Thus, by Taylor’s approximation,
\[
\exp(n f_{\text{pre}}(\hat{x}^* + \hat{x})) = \exp(n f_{\text{pre}}(\hat{x}^*) + \frac{n\hat{x}^TH\hat{x}}{2} + o(1)),
\]
where $H$ is the Hessian of $f_{\text{pre}}$ at $x^*$. Using the fact that $\hat{B} \cap ((\mathbb{Z}^4 - x^*)/n)$ is a finite set for each $n$, this implies that
\[
\sum_{\hat{x} \in \hat{B} \cap ((\mathbb{Z}^4 - x^*)/n)} \exp(n f_{\text{pre}}(\hat{x}^* + \hat{x})) \sim \sum_{\hat{x} \in \hat{B} \cap ((\mathbb{Z}^4 - x^*)/n)} \exp(n f_{\text{pre}}(\hat{x}^*) + \frac{n\hat{x}^TH\hat{x}}{2}).
\]
So we need to show that
\[
\sum_{\hat{x} \in \hat{B}} \exp\left(\frac{n\hat{x}^T H \hat{x}}{2}\right) \sim 144\sqrt{3} \pi^{r^2/2} n^2.
\] (77)

Let
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-3 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We rewrite the summation in the LHS of (77) over \(\hat{C} := \{y : Ay \in \hat{B}\}\) as
\[
\sum_{y \in \hat{C}} \exp\left(\frac{ny^T (A^T HA)y}{2}\right)
\]
\[
= \sum_{y \in \hat{C}} \exp\left(\frac{-ny^T (A^T H_0 A) y}{2r^2} - \frac{ny^T (A^T TA)y}{2r} + O(ny^T Jy)\right),
\]
\[
(78)
\]
by Lemma 9.14 (for the definitions of \(H_0\) and \(T\), see (48)). Note that the condition “\(\hat{x} \in (Z^4 - x^*)/n\)” became “\(y \in (Z^4 - A^{-1} x^*)/n\)” because \(A\) is an integer invertible matrix and
\[
A^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
is also an integer matrix. Using the definition of \(H_0\) and \(T\) in (48), we have that
\[
\sum_{y \in \hat{C}} \exp\left(\frac{-ny^T (A^T H_0 A)y}{2r^2} - \frac{ny^T (A^T TA)y}{2r} + nO(y^T Jy)\right) =
\]
\[
= \sum_{y \in \hat{C}} \exp\left(-\frac{n}{12r^2} y_2^2 - \frac{n}{6r^2} y_2 y_3 - \frac{n}{6r^2} y_2 y_4 - \frac{7n}{12r^2} y_3^2 + \frac{4n}{9r^2} y_3 y_4 - \frac{n}{6r^2} y_4^2 - \frac{n}{r} y_2^2 + \frac{n}{r} y_1 y_2 + \frac{n}{r} y_1 y_3
\]
\[
- \frac{11n}{30r^2} y_2^2 - \frac{2n}{5r} y_2 y_3 - \frac{n}{15r} y_2 y_4 - \frac{31n}{180r^2} y_3^2 + \frac{2n}{45r} y_3 y_4 + \frac{n}{45r} y_4^2 + nO(y^T Jy)\right)
\]
\[
(79)
\]
The set \(\hat{C} = \{y : Ay \in \hat{B}\}\) can be described as
\[
\hat{C} = \{y \in \mathbb{R}^4 : |y_1| \leq \delta_1, |y_i| \leq \delta \text{ for } i = 2, 3, 4\},
\]
since \(\hat{B}\) was defined as
\[
\hat{B} = \{\hat{x} \in \mathbb{R}^4 : \hat{x} = \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4, |\gamma_1| \leq \delta_1 \text{ and } |\gamma_i| \leq \delta \text{ for } i = 2, 3, 4\}.
\]
Thus, the ranges of the summation for different variables $y_i$’s are independent. We have that

$$\sum_{|y_i| \leq \delta_1 \frac{n}{r}} \exp \left( -\frac{n}{2r} y_i^2 + \frac{n}{r} y_1 y_2 + \frac{n}{r} y_1 y_3 + \sum_{j=1}^{4} O(ny_1 y_j) \right)$$

$$= \sum_{|y_i| \leq \delta_1 \frac{n}{r}} \sum_{\tilde{y}_i \in (\mathbb{Z} - (A^{-1} x^*)_1) / \sqrt{n}} \exp \left( -\frac{n}{2r} y_i^2 + \tilde{y}_1 \tilde{y}_2 + \tilde{y}_1 \tilde{y}_3 + \sum_{j=1}^{4} O(r\tilde{y}_1 \tilde{y}_j) \right),$$

where $\tilde{y}_i = \sqrt{n} y_i / \sqrt{r}$ for $i = 2, 3$. We apply Lemma 7.2 with $\alpha = 1/2$, $\beta = \tilde{y}_2 + \tilde{y}_3$, $\phi = O(r) = o(1)$, $\psi = O(r\tilde{y}_2 + r\tilde{y}_3 + r\tilde{y}_4) = O(r) = o(1)$, $s_n = \sqrt{rn} \to \infty$ and $T_n = \delta_1 \sqrt{n/r} \to \infty$:

$$\sum_{|y_i| \leq \delta_1 \frac{n}{r}} \exp \left( -\frac{n}{2r} y_i^2 + \tilde{y}_1 \tilde{y}_2 + \tilde{y}_1 \tilde{y}_3 + \sum_{j=1}^{4} O(r\tilde{y}_1 \tilde{y}_j) \right) \sim \sqrt{2rn} \exp((\tilde{y}_2 + \tilde{y}_3)^2 / 2).$$

We then proceed similarly for $y_2$, $y_3$ and $y_4$. Fix $y_3$ and $y_4$. Set $\tilde{y}_i = \sqrt{n} y_i / r$ for $i = 3, 4$. We apply Lemma 7.2 with $\alpha = 1/12$, $\beta = -(1/6)\tilde{y}_3 - (1/6)\tilde{y}_4$, $\phi = -(2r/15) + O(r^2) = o(1)$, $\psi = (3r/5)\tilde{y}_3 - r/15 + \sum_{j=3}^{4} O(r^2 \tilde{y}_j) = o(1)$, $s_n = \sqrt{rn} \to \infty$ and $T_n = \delta \sqrt{n/r} \to \infty$:

$$\sum_{|y_2| \leq \delta \frac{n}{r}} \exp \left( -\frac{n}{12r^2} y_2^2 - \frac{n}{6r^2} y_2 y_3 - \frac{n}{6r^2} y_2 y_4 - \frac{2n}{15r} y_2^2 + \frac{3n}{5r} y_2 y_3 - \frac{n}{15r} y_2 y_4 + \sum_{j=2}^{4} O(ny_2 y_j) \right)$$

$$= \sum_{|y_2| \leq \delta \frac{n}{r}} \exp \left( -\frac{1}{12} \tilde{y}_2^2 - \frac{1}{6} \tilde{y}_2 \tilde{y}_3 - \frac{1}{6} \tilde{y}_2 \tilde{y}_4 - \frac{2r}{15} \tilde{y}_2^2 + \frac{3r}{5} \tilde{y}_2 \tilde{y}_3 - \frac{r}{15} \tilde{y}_2 \tilde{y}_4 + \sum_{j=2}^{4} O(r^2 \tilde{y}_2 \tilde{y}_j) \right)$$

$$\sim 2\sqrt{\frac{3}{\pi r n}} \exp((\tilde{y}_2 + \tilde{y}_4)^2 / 12).$$

Fix $y_4$ and set $\tilde{y}_4 = \sqrt{n} y_4 / r$. We apply Lemma 7.2 with $\alpha = 1/72$, $\beta = -(1/8)\tilde{y}_4$, $\phi = O(r) = o(1)$, $\psi = O(r\tilde{y}_4) = O(r) = o(1)$, $s_n = \sqrt{rn} \to \infty$ and $T_n = \delta \sqrt{n/r} \to \infty$:

$$\sum_{|y_3| \leq \delta \frac{n}{r}} \exp \left( -\frac{n}{72r^2} y_3^2 - \frac{n}{18r^2} y_3 y_4 + \sum_{j=3}^{4} O(ny_3 y_j / r) \right)$$

$$= \sum_{|y_3| \leq \delta \frac{n}{r}} \exp \left( -\frac{1}{72} \tilde{y}_3^2 - \frac{1}{18} \tilde{y}_3 \tilde{y}_4 + \sum_{j=3}^{4} O(r\tilde{y}_3 \tilde{y}_j) \right)$$

$$\sim 6\sqrt{\frac{2}{\pi r n}} \exp(\tilde{y}_4^2 / 18)).$$

Finally, for $y_4$, we apply Lemma 7.2 with $\alpha = 1/36$, $\beta = 0$, $\phi = O(r) = o(1)$, $\psi = 0$, $s_n = \sqrt{rn} \to \infty$ and $T_n = \delta \sqrt{n/r} \to \infty$:

$$\sum_{|y_4| \leq \delta \frac{n}{r}} \exp \left( -\frac{n}{36r^2} y_4^2 + O(ny_4 y_4 / r) \right) = \sum_{|y_4| \leq \delta \frac{n}{r}} \exp \left( -\frac{1}{36} y_4^2 + O(r\tilde{y}_4 y_4) \right)$$

$$\sim 6\sqrt{\frac{2}{\pi n}}.$$

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Hence,
\[
\sum_{\hat{x} \in B} \exp \left( \frac{\hat{x}^T H \hat{x}}{2} \right) \sim \sqrt{2\pi n} \cdot 2^{3/2} \pi \sqrt{n} \cdot 6^{3/2} \pi \sqrt{n} = 144 \sqrt{3} \pi^2 n^{3/2},
\]
completing the proof.

Proof of Lemma 9.13. Recall that \( h_n(y) = y \ln(y) - y \),
\[
w_{pre}(x) = \begin{cases} 
\frac{P_3! P_2! Q_3! (m_2 - 1)!}{k_0! k_1! k_2! n_3! m_3! T_3! T_2! (m_2 - 1)!} f_3(\lambda)^{n_3} \lambda Q_3, & \text{if } Q_3 > 3n_3; \\
\frac{P_3! P_2! Q_3! (m_2 - 1)!}{k_0! k_1! k_2! n_3! m_3! T_3! T_2! (m_2 - 1)!} 1, & \text{otherwise.}
\end{cases}
\]
and, for \( x \in S \) such that \( Q_3 > 3n_3 \)
\[
f_{pre}(\hat{x}) = h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2)
- h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3)
- h_n(\hat{P}_3 - \hat{k}_1 - 2\hat{k}_2) - h_n(\hat{P}_2 - \hat{k}_1) - 2h_n(\hat{m}_2)
- \hat{k}_2 \ln 2 - \hat{m}_2 \ln 2 - \hat{n}_3 \ln 6
+ \hat{n}_3 \ln f_3(\lambda) - \hat{Q}_3 \ln \lambda,
\]
and, if \( Q_3 = 3n_3 \),
\[
f_{pre}(\hat{x}) = h_n(\hat{P}_3) + h_n(\hat{P}_2) + h_n(\hat{Q}_3) + h_n(m_2)
- h_n(\hat{k}_0) - h_n(\hat{k}_1) - h_n(\hat{k}_2) - h_n(\hat{n}_3) - h_n(\hat{m}_3)
- h_n(\hat{P}_3 - \hat{k}_1 - 2\hat{k}_2) - h_n(\hat{P}_2 - \hat{k}_1) - 2h_n(\hat{m}_2)
- \hat{k}_2 \ln 2 - \hat{m}_2 \ln 2 - \hat{n}_3 \ln 6
- \hat{n}_3 \ln 6.
\]
Thus, by Stirling’s approximation, there is a polynomial \( Q(n) \) such that for \( \hat{x} \in \hat{S}_m \)
\[
w_{pre}(x) \leq Q(n)n! \exp(nf_{pre}(\hat{x})).
\]
Hence, if we obtain an upper bound for the tail \( \sum_{x \in (S \setminus \{x^* + B\}) \cap \mathbb{Z}^4} n! \exp(nf_{pre}(\hat{x})) \), we also get an upper bound for the tail \( \sum_{x \in (S \setminus \{x^* + B\}) \cap \mathbb{Z}^4} w_{pre}(x) \) although it is a weaker bound because of the polynomial factor \( Q(n) \).

Let \( x \in (S \setminus \{x^* + B\}) \cap \mathbb{Z}^4 \). Let \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) be such that \( x = x^* + \gamma_1 z_1 + \gamma_2 e_2 + \gamma_3 e_3 + \gamma_4 e_4 \). Let \( \delta'_1 = \omega(\delta_1) \) be such that \( \delta'_1 / \delta_1 \) goes to infinity arbitrarily slowly and let \( \delta' \) be such that \( \delta' / \delta \) goes to infinity arbitrarily slowly. If \( \delta_1 \leq |\gamma_i| \leq \delta'_1 \) and \( \delta \leq |\gamma_i| \leq \delta' \) for \( i = 2, 3, 4 \), by (75),
\[
\exp(nf_{pre}(\hat{x})) \sim \exp \left( \frac{n(\hat{x} - \hat{x}^*)^T H(\hat{x} - \hat{x}^*)}{2} \right),
\]
where \( H \) is the Hessian of \( f_{pre} \) at \( x^* \). Recall that, by Lemma 9.14 \( H = (-1/r^2)H_0 - (1/r)T + J \), where \( H_0 \) and \( T \) are defined in (48) and \( J = J(n) \) is a matrix with bounded entries. Thus, there exists a positive
constant \( \alpha \) such that
\[
\exp \left( \frac{n(\hat{x} - \hat{x}^*)^T H (\hat{x} - \hat{x}^*)}{2} \right) = \exp \left( \frac{n(\hat{x} - \hat{x}^*)^T ((-1/r^2)H_0 - (1/r)T + J) (\hat{x} - \hat{x}^*)}{2} \right)
\leq \exp \left( -\frac{\alpha \gamma^2 n}{r} - \sum_{i=2}^{4} \frac{\alpha \gamma^2 n}{r^2} \right)
\leq \max \left( \exp \left( -\frac{\alpha \delta^2 n}{r} \right), \exp \left( -\frac{\alpha \delta^2 n}{r^2} \right) \right)
= \frac{1}{\exp(n^{o(1)})},
\]
where the last relation follows from \( \delta^2 n/r = \omega(\ln n) \) and \( \delta^2 n/r^2 = \omega(\ln n) \).

By Lemma 9.11 for any local maximum \( x \) in \( S \) other than \( x^* \),
\[
\frac{\exp(n f_{\text{pre}}(x))}{\exp(n f_{\text{pre}}(x^*))} = \frac{1}{\exp(\Omega(R^2 n))} = \frac{1}{\exp(\Omega(\ln^{3/2} n))}.
\]
since \( R = \omega(n^{1/2} \ln^{3/2}(n)) \). Hence, for any \( x \in S \setminus (x^* + B) \),
\[
\frac{\exp(n f_{\text{pre}}(x))}{\exp(n f_{\text{pre}}(x^*))} = \frac{1}{\exp(\Omega(\ln^{3/2} n))}.
\]

Thus,
\[
\sum_{x \in (S \setminus (x^* + B)) \cap \mathbb{Z}^4} w_{\text{pre}}(x) \leq Q(n)n! \sum_{x \in (S \setminus (x^* + B)) \cap \mathbb{Z}^4} \exp(n f_{\text{pre}}(x)) \leq Q(n)n! n^4 \frac{\exp(n f_{\text{pre}}(\hat{x}^*))}{\exp(\Omega(\ln^{3/2} n))} = o \left( n! \exp(n f_{\text{pre}}(\hat{x}^*)) \right).
\]

\[\square\]

## 10 Combining pre-kernels and forests

In this section, we will obtain a formula for the number of connected graphs with vertex set \([n]\) and \(m\) edges, proving Theorem 2.1. We defer the proof of some lemmas to Section 10.1. We will perform Steps 4 and 5 as described in the overview of the proof in Section 5.

For \( \tilde{n} \in [0,1] \), let
\[
t(\tilde{n}) = -\frac{(1 - \tilde{n})}{2} \ln(1 - \tilde{n}) + \frac{1 - \tilde{n}}{2} + \tilde{n} f_{\text{core}}(\tilde{n}_1^*), \quad (80)
\]
where \( \tilde{n}_1^* = \tilde{n}_1^*(n) = 3\tilde{n}/g_2(\lambda^*) \) and \( \lambda^* = \lambda^*(n) \) is the unique positive solution of the equation \( \lambda f_1(\lambda)g_2(\lambda)/f_2(2\lambda) = 3\tilde{m}/n \). We have already discussed the existence and uniqueness of \( \lambda^* \) in Section 8.

Elementary but lengthy computations show that
\[
t(\tilde{n}) = -\frac{(1 - \tilde{n})}{2} \ln(1 - \tilde{n}) + \frac{1 - \tilde{n}}{2} + 2R \ln(\tilde{n}) + (2 \ln(3) - \ln(2) - 2)R + 2R \ln(\tilde{n}) + \left( \ln 3 - \frac{1}{2} \ln(2) \right) \tilde{n}
+ \ln \left( \frac{f_2(2\lambda^*)}{g_1(\lambda^*)} \right) \tilde{n} + \left( \frac{1}{2} \tilde{n} + \tilde{R} \right) \ln \left( \frac{\tilde{n}^2 g_1(\lambda^*)^3}{g_2(\lambda^*) f_1(\lambda^*)(\lambda^*)^3} \right), \quad (81)
\]
where \( \tilde{R} = R/N \). In this section, we use \( \tilde{y} \) to denote \( y/N \). We obtain the following asymptotic formulae.
Theorem 10.1. We have that
\[
\sum_{\substack{n \in [N] \\
N-n \text{ even}}} \binom{N}{n} g_{\text{forest}}(N,n)g_{\text{pre}}(n,m) \sim \frac{\sqrt{3}}{\sqrt{\pi N}} \exp(\sqrt{t(n^*)} + N \ln N - N)
\]  
(82)
where
\[
\tilde{n}^* = \frac{f_2(2\lambda^{**})}{f_1(\lambda^{**})g_1(\lambda^{**})}.
\]  
(83)
and \(\lambda^{**}\) is the unique positive solution to
\[
\frac{2\lambda f_1(\lambda)g_2(\lambda) - 3f_2(2\lambda)}{f_1(\lambda)g_1(\lambda)} = \frac{6R}{N}.
\]  
(84)

Theorem 2.1 follows immediately from Theorem 10.1 by simplifying \(t(\tilde{n}^*)\) by using (81) with (83) and (20). The rest of this section is dedicated to prove Theorem 10.1.

The following lemma shows that \(\lambda^{**}\) is well-defined.

Lemma 10.2. The equation
\[
\frac{2\lambda f_1(\lambda)g_2(\lambda) - 3f_2(2\lambda)}{f_1(\lambda)g_1(\lambda)} = \alpha_n
\]  
has a unique solution for \(\alpha_n > 0\) and it goes to 0 if \(\alpha_n \to 0\).

Proof. For the first part, it suffices to show that the function
\[
f(\lambda) = \frac{2\lambda f_1(\lambda)g_2(\lambda) - 3f_2(2\lambda)}{f_1(\lambda)g_1(\lambda)}
\]  
is strictly increasing and it goes to zero as \(\lambda \to 0\). By computing the series of \(f(\lambda)\) with \(\lambda \to 0\), we obtain \(f(\lambda) = \lambda^2/2 + O(\lambda^3) \to 0\) as \(\lambda \to 0\). To show \(f(\lambda)\) is strictly increasing, we compute its derivative:
\[
\frac{df(\lambda)}{d\lambda} = \frac{2(e^{4\lambda} + e^{3\lambda} - e^{2\lambda} - 1 - \lambda e^{3\lambda} - 4\lambda e^{2\lambda} - \lambda e^{\lambda})}{f_1(\lambda)^2g_1(\lambda)^2}
\]  
while it is obvious that the denominator is positive for \(\lambda > 0\), it is not immediate that so is the numerator.

Let \(g(\lambda) = e^{4\lambda} + e^{3\lambda} - e^{2\lambda} - 1 - \lambda e^{3\lambda} - 4\lambda e^{2\lambda} - \lambda e^{\lambda}\). We will use the following strategy: starting with \(i = 1\), we check that \(\frac{d^{i-1}g(\lambda)}{d\lambda^{i-1}}|_{\lambda=0} = 0\) and compute \(\frac{d^ig(\lambda)}{d\lambda^i}\). If for some \(i\) we can show that \(\frac{d^ig(\lambda)}{d\lambda^i} > 0\) for any \(\lambda\), then we obtain \(g(\lambda) > 0\) for \(\lambda > 0\). We omit the computations here. We have that
\[
\frac{d^5g(x)}{d^5x} = 2048e^{4x} - 12e^x - 324e^{3x} - 486\lambda e^{3x} - 640e^{2x} - 256\lambda e^{2x} - 2\lambda e^x,
\]  
which is trivially positive since \(\exp(x) > 1 + x\) for all \(x \in \mathbb{R}\) and the sum of the coefficients of the negative terms is less than 2048.

It will be useful to know how \(\lambda^{**}\) compares to \(R\) and \(\tilde{n}^*\). By Lemma 10.2 \(\lambda^{**} = o(1)\) since \(R = o(N)\). We can write \(\tilde{R}\) and \(\tilde{n}^*\) in terms of \(\lambda^{**}\) by using (83) and (84). By expanding the LHS of (84) and the RHS (83) as functions of \(\lambda^{**}\) about 0, we have that
\[
\tilde{R} = \left(\frac{\lambda^{**}}{12}\right)^2 + O((\lambda^{**})^4),
\]  
(85)
\[
\tilde{n}^* = \lambda^{**} - \left(\frac{(\lambda^{**})^2}{3}\right) + O((\lambda^{**})^4).
\]  

Next, we state the main lemmas for the proof of Theorem 10.1. We defer their proofs to Section 10.1.

The next lemma follows from Theorem 9.1 and a series of simplifications that show that \(f_{\text{core}}(n^*_1) = f_{\text{pre}}(x^*)\).
Lemma 10.3. Let $\alpha_1 < \alpha_2$ be positive constants. If $\alpha_1 \sqrt{RN} \leq n \leq \alpha_2 \sqrt{RN}$, then
\[
g_{\text{pre}}(n, m) \sim \frac{\sqrt{3}}{\pi n} \cdot n! \exp(n f_{\text{core}}(\hat{n}_1^*)) \tag{86}
\]
where $\hat{n}_1^* = \hat{n}_1^*(n) = 3m/g_2(\lambda^*)$ and $\lambda^*(n)$ is the unique positive solution of the equation $\lambda f_1(\lambda)g_2(\lambda)/f_2(2\lambda) = 3m/n$.

This will allow us to obtain the formula for connected hypergraphs from the formula for simple hypergraphs. We compute the point of maximum for $t(\tilde{n})$:

Lemma 10.4. The point $\tilde{n}^*$ is the unique maximum of the function $t(\tilde{n})$ in the interval $[0, 1]$. Moreover, $\tilde{n}^*$ is the unique point such that the derivative of $t(\tilde{n})$ is 0 in $(0, 1)$, and $t'(\tilde{n}) > 0$ for $\tilde{n} < \tilde{n}^*$ and $t'(\tilde{n}) < 0$ for $\tilde{n} < \tilde{n}^*$.

We then expand the summation around this maximum and approximate it by an integral that can be easily computed.

Lemma 10.5. Suppose $\delta^3 = o(\lambda^{**}/N)$ and $\delta = \omega(1/N^{1/2})$. Then
\[
\sum_{n \in [n^* - \delta N, n^* + \delta N]} \exp(Nt(\tilde{n})) \sim \sqrt{\frac{\pi N}{2}} \exp(Nt(\tilde{n}^*)).
\]

Finally, we show that the terms far from the maximum do not contribute significantly to the summation:

Lemma 10.6. Suppose that $\delta^3 = o(\lambda^{**}/N)$ and $\delta^2 = \omega((\ln N)/N)$. Then
\[
\sum_{n \in [0, N]\setminus[n^* - \delta N, n^* + \delta N]} \binom{N}{n} g_{\text{forest}}(N, n)g_{\text{pre}}(n, m) = \frac{N! \exp(Nt(\tilde{n}^*))}{N^{\omega(1)}}.
\]

We are now ready to prove Theorem 10.1.

Proof of Theorem 10.1. In order to use Lemma 10.5 and Lemma 10.6 we need to check if there exists $\delta$ such that $\delta^3 = o(\lambda^{**}/N)$ and $\delta^2 = \omega((\ln N)/N)$. This is true if and only if
\[
(\lambda^{**})^2 = \omega\left(\frac{\log^3 N}{N}\right),
\]
which, by (85), is true if and only if
\[
R = \omega(\log^3 N),
\]
which is true by assumption. Thus, assume that $\delta$ satisfies $\delta^3 = o(\lambda^{**}/N)$ and $\delta^2 = \omega((\ln N)/N)$.

Let $J(\delta) = [n^* - \delta N, n^* + \delta N] \cap (2Z - N)$. By (85), we have that
\[
n^* = \Theta(\lambda^{**} N) = \Theta(\sqrt{RN}).
\]

Moreover, since $\delta^3 = o(\lambda^{**}/N)$ and $R \to \infty$,
\[
\delta N = o(\sqrt[6]{RN^3}) = o\left(\frac{\sqrt{RN}}{R^{1/3}}\right) = o(n^*).
\]
Thus, there are constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that any $n \in J(\delta)$ satisfies $\alpha_1 \sqrt{RN} < n < \alpha_2 \sqrt{RN}$. By Lemma 10.3

$$g_{\text{pre}}(n, m) \sim \frac{\sqrt{3}}{\pi n} \cdot n! \exp(n f_{\text{core}}(\hat{n}_1^*))$$

for any $n \in J(\delta)$ and $m = n/2 + R$. By Theorem 6.1 for $n \in J(\delta)$

$$g_{\text{forest}}(n, N) = \frac{n}{N} \cdot \frac{(N - n)! \cdot N^{(N-n)/2}}{(N-2)! \cdot 2^{(N-n)/2}}.$$

Thus, for $n \in J(\delta)$, with $m = n/2 + R$, by Stirling’s approximation and using the fact that $n = o(N)$ by (85),

$$\binom{N}{n} g_{\text{forest}}(N, n) g_{\text{pre}}(n, m) \sim \binom{N}{n} \cdot \frac{n}{N} \cdot \frac{(N - n)! \cdot N^{(N-n)/2}}{(N-2)! \cdot 2^{(N-n)/2}} \cdot \frac{\sqrt{3}}{\pi n} \cdot n! \exp\left(n f_{\text{core}}(\hat{n}_1^*)\right)
= \frac{\sqrt{3}}{\pi N} \cdot \frac{N! \cdot N^{(N-n)/2}}{(N-2)! \cdot 2^{(N-n)/2}} \exp\left(n f_{\text{core}}(\hat{n}_1^*)\right)
\sim \frac{\sqrt{6}}{\pi N} \cdot \exp\left(N t(\hat{n}) + N \ln N - N\right),$$

by (87).

Since $J(\delta)$ is a finite set for each $n$, we have that there exists a function $q(n) = o(1)$ such that the $o(1)$ in (87) is bounded by $q(n)$ for any $n \in J(\delta)$. Thus,

$$\sum_{n \in J(\delta)} \binom{N}{n} g_{\text{forest}}(N, n) g_{\text{core}}(n, m) \sim \sum_{n \in J(\delta)} \frac{\sqrt{6}}{\pi N} \cdot \exp\left(N t(\hat{n}) + N \ln N - N\right)
\sim \frac{\sqrt{6}}{\pi N} \sqrt{\frac{\pi N}{2}} \exp(N t(\hat{n}^*) + N \ln N - N),$$

by Lemma 10.5. Together with Lemma 10.6, this proves Theorem 10.1.

10.1 Proof of the lemmas in Section 10

In this section, we prove Lemmas 10.4, 10.5, and 10.6. We start by computing the derivatives of $t$. For that, we need to compute $\frac{d\lambda^*(\hat{n})}{dn}$. This can be done by implicit differentiation using Equation (20) that defines $\lambda^*$ and recalling $m = n/2 + R$. We obtain

$$\frac{d\lambda^*}{d\hat{n}} = -\frac{\hat{R}}{\hat{n}^2 \hat{m} \alpha(\lambda^*)},$$

where

$$a(\lambda) = \frac{1}{\lambda} + \frac{\exp(\lambda)}{f_1(\lambda)} + \frac{\exp(\lambda)}{g_2(\lambda)} - \frac{2 \exp(2\lambda)}{f_2(2\lambda)} + \frac{2}{f_2(2\lambda)}.$$

Thus, the first derivative of $t(\hat{n})$, which is defined in (81), is

$$\frac{\ln(1 - \hat{n})}{2} + \ln(3) - \frac{\ln 2}{2} + \ln \left(\frac{f_2(2\lambda^*)}{g_1(\lambda^*)}\right) - \frac{1}{2} \ln \left(\frac{\hat{m}^2 g_1(\lambda^*)^3}{g_2(\lambda^*) f_1(\lambda^*) (\lambda^*)^3}\right).$$

The second derivative is

$$-\frac{1}{2(1 - \hat{n})} - \frac{2\hat{R}}{(\hat{n} + 2\hat{R})\hat{n}} - \frac{4\hat{R}^2}{\hat{n}(\hat{n} + 2\hat{R})^2} \frac{b(\lambda^*)}{a(\lambda^*) f_2(2\lambda^*)}.$$
where
\[ b(\lambda) = 2F_1(\lambda) - \frac{f_2(2\lambda) \exp(\lambda)}{g_1(\lambda)}. \]

The third derivative is
\[
-\frac{1}{2(1-\tilde{n})^2} + \frac{4\tilde{R}(\tilde{n} + \tilde{R})}{\tilde{n}^2(\tilde{n} + 2\tilde{R})^2} + \frac{d}{d\tilde{n}} \left( -\frac{4\tilde{R}^2}{\tilde{n}(\tilde{n} + 2\tilde{R})^2} \right) \frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} - \frac{4\tilde{R}^2}{\tilde{n}(\tilde{n} + 2\tilde{R})^2} \frac{d}{d\lambda^*} \left( \frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} \right) \frac{d\lambda^*}{d\tilde{n}}. \tag{93}
\]

**Lemma 10.7.** For \( \delta = o(\tilde{n}^*) \) and \( n \in [n^* - \delta N, n^* + \delta N] \), we have that \( |\lambda^*(n) - \lambda^{**}| = o(\tilde{n}^*) \).

**Proof.** Given a connected \((N, M)\)-graph such that its core has \( n \) vertices and \( m \) edges, we have that \( m = M - (N - n)/2 \). Recall that \( \tilde{n}^* = f_2(2\lambda^*)/f_1(\lambda^*)g_1(\lambda^*) \) by \( \boxed{83} \) and
\[
\frac{6\tilde{R}}{N} = \frac{2\lambda^{**}f_1(\lambda^{**})g_2(\lambda^{**})}{f_1(\lambda^{**})g_1(\lambda^{**})} - 3f_2(2\lambda^{**}),
\]
by \( \boxed{84} \). Thus,
\[
3M = \frac{\lambda^{**}(1 + \exp(2\lambda^{**}) + \exp(\lambda^{**}))}{\exp(2\lambda^{**}) - 1}.
\]
Hence,
\[
\frac{\lambda^*(n^*)f_1(\lambda^*(n^*))g_2(\lambda^*(n^*))}{f_2(2\lambda^*(n^*))} = \frac{3m}{n^*} = \frac{3M}{n^*} - \frac{3}{2n^*} + 3 = \frac{\lambda^{**}f_1(\lambda^{**})g_2(\lambda^{**})}{f_2(2\lambda^{**})}
\]
and so \( \lambda^* = \lambda^*(n^*) \). The lemma then follows directly from the fact that \( \lambda^{**} = \lambda^*(\tilde{n}^*) \) and Lemma \( \boxed{8.3} \) \( \square \).

Now we bound the third derivative for points close to \( \tilde{n}^* \):

**Lemma 10.8.** The third derivative of \( t(\tilde{n}) \) is \( O(1/\lambda^{**}) \) for \( |\tilde{n} - n^*| = o(\tilde{n}^*) \).

**Proof.** We analyse the terms in \( \boxed{93} \). By Lemma \( \boxed{8.3} \) since \( n = \tilde{n}(1 + o(1)) \),
\[
\frac{d}{d\tilde{n}} \left( -\frac{1}{2(1-\tilde{n})^2} + \frac{4\tilde{R}(\tilde{n} + \tilde{R})}{\tilde{n}^2(\tilde{n} + 2\tilde{R})^2} \right) = -\frac{1}{2(1-\tilde{n})^2} + \frac{4\tilde{R}(\tilde{n} + \tilde{R})}{\tilde{n}^2(\tilde{n} + 2\tilde{R})^2}
\]
\[
= \left( -\frac{1}{2(1-\tilde{n})^2} + \frac{4\tilde{R}(\tilde{n}^* + \tilde{R})}{(\tilde{n}^*)^2(\tilde{n}^* + 2\tilde{R})^2} \right) (1 + o(\tilde{n}^*))
\]
\[
= \frac{1}{3\lambda^{**}} + O(1),
\]
where the last equality is obtained by computing the series of the expression in the previous equation using \( \boxed{85} \). For \( \lambda \to 0 \),
\[
a(\lambda) = \frac{1}{6} + \frac{\lambda}{12} + O(\lambda^2) \tag{94}
\]
\[
b(\lambda) = 4\lambda + O(\lambda^2); \tag{95}
\]
Thus, by Lemma \( \boxed{8.3} \) and \( \boxed{85} \),
\[
\frac{d}{d\tilde{n}} \left( -\frac{4\tilde{R}^2}{\tilde{n}(\tilde{n} + 2\tilde{R})^2} \right) \frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} = \frac{4\tilde{R}^2(3\tilde{n} - 2\tilde{R})}{\tilde{n}^2(\tilde{n} + 2\tilde{R})^3} \frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)}
\]
\[
\sim \frac{4\tilde{R}^2(3\tilde{n}^* - 2\tilde{R})}{(\tilde{n}^*)^2(\tilde{n}^* + 2\tilde{R})^3} \frac{b(\lambda^{**})}{a(\lambda^{**})f_2(2\lambda^{**})} = \frac{1}{\lambda^{**}} + O(1).
\]

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We have that
\[
\frac{db(\lambda)}{d\lambda} = 4\exp(2\lambda) - \frac{\exp(\lambda)(3\exp(2\lambda) - 3 - 2\lambda)}{g_1(\lambda)} + \frac{f_2(2\lambda)\exp(2\lambda)}{g_1(\lambda)^2}
\]
and
\[
\frac{da(\lambda)}{d\lambda} = -\frac{1}{\lambda^2} + \frac{\exp(\lambda)}{f_1(\lambda)} - \frac{\exp(2\lambda)}{f_1(\lambda)^2} + \frac{\exp(\lambda) - \exp(2\lambda)}{g_2(\lambda)} - \frac{4\exp(2\lambda)}{g_2(\lambda)^2} - \frac{4f_2(2\lambda)}{f_2(2\lambda)^2}.
\]
Thus, by (89)
\[
\frac{4R^2}{\tilde{n}(\tilde{n} + 2R)^2} \frac{d}{d\lambda} \left( \frac{b(\lambda^*)}{a(\lambda^*)f_2(2\lambda^*)} \right) \frac{d\lambda^*}{d\tilde{n}}
\]
\[
= -\frac{4R^2}{\tilde{n}(\tilde{n} + 2R)^2} \left( \frac{db(\lambda)}{d\lambda} \bigg|_{\lambda=\lambda^*} a(\lambda^*)f_2(2\lambda^*) \frac{1}{a(\lambda^*)^2f_2(2\lambda^*)^2} \right) \left( \frac{da(\lambda)}{d\lambda} \bigg|_{\lambda=\lambda^*} f_2(2\lambda^*) + 2f_1(\lambda^*)a(\lambda^*) \right) \left( -\frac{R}{\tilde{n}^2ma(\lambda^*)} \right).
\]
By Lemma 8.3 the above is the value applied at \( \lambda^* \) with an error of \( o(\lambda^*) \) and the series for it with \( \lambda^* \to 0 \) is
\[
\frac{2}{3\lambda^*} + O(1).
\]

We now present the proofs of Lemmas 10.4, 10.5 and 10.6.

**Proof of Lemma 10.4.** By setting \( 91 \) to zero and using \( \hat{m} = \lambda^*f_1(\lambda^*)g_2(\lambda^*)/f_2(2\lambda^*) \), we get following value for \( \tilde{n} \)
\[
\tilde{n}^* = \frac{f_2(2\lambda^*)}{f_1(\lambda^*)g_1(\lambda^*)}.
\]
We also know that, by (20),
\[
\frac{\lambda^*f_1(\lambda^*)g_2(\lambda^*)}{f_2(2\lambda^*)} = 3\hat{m} = \frac{3}{2} + \frac{3R}{\tilde{n}}.
\]
Thus, by combining (96) and (97), we get the following equation:
\[
\hat{R} = \frac{1}{6} - \frac{3f_2(2\lambda^*) + 2\lambda^*f_1(\lambda^*)g_2(\lambda^*)}{f_1(\lambda^*)g_1(\lambda^*)},
\]
which has a unique solution \( \lambda^* \) for \( \hat{R} > 0 \) by Lemma 10.2. By computing the series of the second derivative as \( \lambda^* \to 0 \), we get that the second derivative at \( \tilde{n}^* \) is
\[
-1 + O(\lambda^*),
\]
which is negative for big enough \( n \) and so \( \tilde{n}^* \) is a local maximum.

**Proof of Lemma 10.5.** Let \( J(\delta) = [n^*-\delta N, n^*+\delta N] \cap (2\mathbb{Z} - N) \). Using Taylor’s approximation, Lemma 10.4 and Lemma 10.8 for \( n \in J(\delta) \),
\[
\exp(Nt(\tilde{n})) = \exp \left( Nt(\tilde{n}^*) + \frac{Nt''(\tilde{n}^*)|\tilde{n} - \tilde{n}^*|^2}{2} + O \left( \frac{\delta^3N}{\lambda^{**}} \right) \right)
\approx \exp \left( Nt(\tilde{n}^*) + \frac{Nt''(\tilde{n}^*)|\tilde{n} - \tilde{n}^*|^2}{2} \right)
\]

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since $\delta^3 = o(\lambda^* / N)$, and so

$$\sum_{\tilde{n} \in J(\delta)} \exp (Nt(\tilde{n})) \sim \sum_{\tilde{n} \in J(\delta)} \exp \left( Nt(\tilde{n}^*) + \frac{Nt''(\tilde{n}^*)|\tilde{n} - \tilde{n}^*|^2}{2} \right)$$

$$= \exp (Nt(\tilde{n}^*)) \sum_{x \in [-\delta N, \delta N]} \exp \left( \frac{t''(\tilde{n}^*)x^2}{2N} \right).$$

(99)

We change variables from $x$ to $y = \ell x / 2$ with $\ell = \sqrt{|t''(\tilde{n}^*)|/2} \sim 1/2$. Using $\delta = \omega(1/\sqrt{N})$ and Lemma 7.2

$$\sum_{x \in [-\delta N, \delta N]} \exp \left( \frac{t''(\tilde{n}^*)x^2}{2N} \right) = \sum_{y \in [-\delta \ell \sqrt{N}/2, \delta \ell \sqrt{N}/2]} \exp (-4y^2) \sim \sqrt{\pi N}/2.

\(\square\)

**Proof of Lemma 10.6.** We have that $g_{\text{pre}}(n, m) \leq g_{\text{core}}(n, m)$ since every pre-kernel is a core and $g_{\text{core}}(n, m)$ is an upper bound for the number of cores with vertex-set $[n]$ and $m$ edges by Theorem 8.1. Using Theorem 6.1 and the definition of $t$, we have that there is a polynomial $Q(N)$ such that

$$\sum_{n \notin [(n^* - \delta)N, (n^* + \delta)N]} \binom{N}{n} g_{\text{forest}}(N, n)g_{\text{pre}}(n, m) \leq \sum_{n \notin [(n^* - \delta)N, (n^* + \delta)N]} \binom{N}{n} g_{\text{forest}}(N, n)g_{\text{core}}(n, m)$$

$$\leq Q(N)N! \sum_{n \in [0, N]} \exp (Nt(n)).$$

Using Lemma 10.4 and (99), we have that

$$\sum_{n \in [0, N]} \sum_{n \notin [(n^* - \delta)N, (n^* + \delta)N]} \exp (Nt(n)) \leq N \exp (Nt(\tilde{n}^*) - \Omega(N\delta^2)),$$

and $N\delta^2 = \omega(\ln N)$ for $\delta^2 = \omega(\ln N)/N$.

\(\square\)

**Acknowledgment**

The authors would like to thank Huseyin Acan for pointing out a mistake in [10] which led to an incorrect claim on the number of cores, not necessarily connected.
Glossary

\( C(N, M) \) number of connected 3-uniform hypergraphs on \([N]\) with \(M\) edges

- \( N \) used for the number of vertices in the graph
- \( M \) used for the number of edges in the graph
- \( R \) \( M - N/2 \) used as an excess function in the graph
- \( n \) used for the number of edges in the core
- \( r \) \( R/n \), scaled \( R \)
- \( f_k(\lambda) = e^\lambda - \sum_{i=0}^{k-1} \lambda^i/i! \)
- \( g_k(\lambda) = e^\lambda + k \)
- \( \lambda(k, c) \) the unique positive solution to \( \lambda f_{k-1}(\lambda)/f_k(\lambda) = c \)
- \( g_{\text{core}}(n, m) \) number of (simple) cores with vertex set \([n]\) and \(m\) edges
- \( g_{\text{forest}}(N, n) \) number of forest with vertex set \([N]\) and \([n]\) as its roots
- \( g_{\text{pre}}(n, m) \) number of (simple) pre-kernels with vertex set \([n]\) and \(m\) edges that are connected
- \( \lambda^* \) unique positive solution to \( \lambda e^{2\lambda} + e^\lambda + 1/(f_1(\lambda)g_1(\lambda)) = 3M/N \). This is used to define a point achieving the maximum when combining cores and pre-kernels, p. 2
- \( \hat{n}^* = f_2(2\lambda^*)/(f_1(\lambda^*)g_1(\lambda^*)) \). This is the point achieving the maximum when combining cores and pre-kernels, p. 2
- 2-edge an edge that contains exactly one vertex of degree 1
- 3-edge an edge that contains no vertices of degree 1

For the core:

For any symbol \( y \), \( \hat{y} = y/n \) denotes the scaled version of \( y \)

- \( h_n(y) = y \ln(yn) - y \)
- \( f_{\text{core}} \) a function used to approximate the exponential part of \( w_{\text{core}} \), p. 13
- \( w_{\text{core}} \) a function used to count cores, p. 17
- \( n_1 \) used as the number of vertices of degree 1
- \( D_{n_1} \) set of all \( d \in (N \setminus \{0, 1\})^{n-n_1} \) with \( \sum_i d_i = 3m - n_1 \)
- \( \lambda_{n_1} \) unique positive solution to \( \lambda f_1(\lambda)/f_2(\lambda) = c_2(n_1) \)
- \( n_2(n_1) \) \( n - n_1 \), the number of vertices of degree at least 2.
- \( m_3(n_1) \) \( m - n_1 \), the number of 3-edges
- \( Q_2(n_1) \) \( 3m - n_1 \), the sum of degrees of vertices of degree at least 2
- \( c_2(n_1) = Q_2(n_1)/n_2(n_1) \), the average degree of the vertices of degree at least 2.
- \( n_2(n_1) \) \( \lambda_{n_1} \exp(\lambda_{n_1}/f_1(\lambda_{n_1})) \)
- \( \mathcal{G}(n_1, d) \) random core with \( n_1 \) vertices of degree 1 and degree sequence \( d \) for the vertices of degree at least 2, p. 14
- \( \lambda^* \) unique positive solution to \( \lambda f_1(\lambda)g_2(\lambda)/f_2(2\lambda) = 3m/n \). This is used to define a point achieving the maximum for \( f_{\text{core}} \), p. 16
$n_1^*$: $3m/n g_2(\lambda^*)$. This the point achieving the maximum for $f_{\text{core}}$, p. 8.

$Y$: $(Y_1, \ldots, Y_{n_2})$, where the $Y_i$'s are independent random variables with truncated Poisson distribution Po$(2, \lambda n_1)$

$\Sigma_{n_1}$: event that a random variable $Y$ satisfies $\sum_i Y_i = 3m - n_1$

**For the pre-kernel:**

For any symbol $y$, $\hat{y} = y/n$ denotes the scaled version of $y$

$h_n(y) = y \ln(yn) - y$.

$f_{\text{pre}}$: a function used to approximate the exponential part of $w_{\text{pre}}$, p. 20

$w_{\text{pre}}$: a function used to count pre-kernels, p. 39

$n_1$: used as the number of vertices of degree 1

$k_0$: used as the number of vertices of degree 2 that are in two 2-edges

$k_1$: used as the number of vertices of degree 2 that are in one 2-edge and in one 3-edge

$k_2$: used as the number of vertices of degree 2 that are in two 3-edges

$x$: used as $(n_1, k_0, k_1, k_2)$

$D(x)$: subset of $\mathbb{N}^{n_3(x)}$ such that $d \in D(x)$ if $d_i \geq 3$ for all $i$ and $\sum_{i=1}^{n_3(x)} d_i = Q_3(x)$, p. 22

$n_2(x) = k_0 + k_1 + k_2$, the number of vertices of degree 2

$n_3(x) = n - n_1 - n_2(x)$, the number of vertices of degree at least 3

$m_2(x) = n_1$, the number of 2-edges in the pre-kernel

$m_2^-(x) = n_1 - k_0$, the number of 2-edges in the kernel

$P_2(x) = 2m_2^-(x)$, the number of points in 2-edges in the kernel

$m_3(x) = m - n_1$, the number of 3-edges in the pre-kernel

$P_3(x) = 3m_3(x)$, the number of points in 3-edges in the pre-kernel

$Q_3(x) = 3m - n_1 - 2n_2(x)$, the sum of the degrees of the vertices of degree at least 3

$c_3(x) = Q_3(x)/n_3(x)$, the average degree of the vertices of degree at least 3

$T_3(x) = P_3(x) - k_1(x) - 2k_2(x)$, the number of points in 3-edges that will be matched to points in vertices of degree at least 3

$T_2(x) = P_2(x) - k_1(x)$, the number of points in 2-edges that will be matched to points in vertices of degree at least 3

$\lambda(x)$: unique positive solution to $\lambda f_2(\lambda)/f_3(\lambda) = c_3(x)$

$\eta_3(x) = Q_3(x)/n_3(x)$

$\lambda^*$: unique positive solution to $\lambda f_1(\lambda) g_2(\lambda)/f_2(2\lambda) = 3m/n$. This is used to define a point achieving the maximum for $f_{\text{pre}}$, p. 16

$x^*$: This the point achieving the maximum for $f_{\text{pre}}$, p. 20

$K$: random kernel (it receives parameters $(V, M_3, k_1, k_2, d)$), p. 22

$\mathcal{P}(x, d)$: random pre-kernel with parameters $x = (n_1, k_0, k_1, k_2)$ and degree sequence $d$ for the vertices of degree at least 3, p. 23
\( \mathbf{Y} = (Y_1, \ldots, Y_n) \), where the \( Y_i \)'s are independent random variables with truncated Poisson distribution \( \text{Po}(3, \lambda(x)) \).

\( \Sigma(x) \) event that a random variable \( \mathbf{Y} \) satisfies \( \sum_i Y_i = 3m - n_1 - 2n_2 \)

\( S^*_\psi \) a set of points ‘close’ to \( x^* \), p. 26

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