RIGOROUS COMPUTATION OF FUNDAMENTAL UNITS IN
ALGEBRAIC NUMBER FIELDS

FELIX FONTEIN AND MICHAEL J. JACOBSON, JR.

Abstract. We present an algorithm that unconditionally computes a representation of the unit group of a number field of discriminant $\Delta_K$, given a full-rank subgroup as input, in asymptotically fewer bit operations than the baby-step giant-step algorithm. If the input is assumed to represent the full unit group, for example, under the assumption of the Generalized Riemann Hypothesis, then our algorithm can unconditionally certify its correctness in expected time $O(\Delta_K^{n/(4n+4)+\epsilon}) = O(\Delta_K^{1/4-1/(8n+4)+\epsilon})$ where $n$ is the unit rank.

1. Introduction

Let $K$ be an algebraic number field of discriminant $\Delta_K$. One of the main computational problems in algebraic number theory is to compute a representation of the group of units of the corresponding maximal order $O_K$. The units are of interest in a number of contexts. As an example, it is well-known that computing the fundamental unit of a real quadratic field is equivalent to solving the Pell equation $x^2 - Dy^2 = 1$.

In general, the unit group consists of a finite torsion subgroup and an infinite part of rank $n$, where $n$ is called the unit rank. A generating system of the infinite part is called a system of fundamental units. Instead of directly computing the units themselves, many algorithms compute a basis of the corresponding logarithm lattice $\Lambda_K$, a rank $n$ lattice in $\mathbb{R}^n$ derived from the Archimedean absolute values of $K$. The fundamental units can be recovered from a basis of $\Lambda_K$ (see, for example, [Thi95]).

The fastest algorithms for unconditionally computing a system of fundamental units, meaning that they generate the entire unit group without having to rely on any unproved assumptions or heuristics, are of exponential complexity in the bit length of the field discriminant. The current state-of-the-art is due to Buchmann [Buc87c], whose algorithm computes a basis of the logarithm lattice in $O(\Delta_K^{1/4+\epsilon})$ bit operations. However, if one is willing to assume the truth of the Generalized Riemann Hypothesis (GRH), then Buchmann’s index-calculus algorithm [Buc90] can be used. This algorithm has subexponential complexity in $\log \Delta_K$ assuming the GRH, but unfortunately the correctness of the output also depends on the GRH.

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1Throughout this paper, the $O$-constants are assumed to be dependent on the degree $[K : \mathbb{Q}]$ of $K$. Furthermore, to simplify notation, expressions involving $\Delta_K$ should be assumed to operate on $|\Delta_K|$. 

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The motivating question for the work in this paper is whether it is possible to certify that the logarithm lattice of a unit group produced by the index-calculus algorithm is unconditionally correct in asymptotically fewer than \( O(\Delta^{1/4+\epsilon}) \) bit operations. More generally, given a full rank sublattice \( \Lambda' \) of the logarithm lattice corresponding to the unit group of a number field \( K \), is it possible to compute the full logarithm lattice in fewer than \( O((\det \Lambda')^{1/2+\epsilon}\Delta_K) \) bit operations, i.e., faster than using baby-step giant-step?

These questions were answered affirmatively for the case of real quadratic fields in \cite{dHJW07}. The unit group of a real quadratic field of discriminant \( \Delta \) has rank one, generated by a single fundamental unit \( \varepsilon_{\Delta} > 1 \). The corresponding lattice of logarithms is generated by a single real number, the regulator \( R_{\Delta} = \log \varepsilon_{\Delta} \).

In \cite{dHJW07}, it is proved that an unconditionally correct approximation of \( R_{\Delta} \) can be computed in time \( O(S^{1/3}\Delta^\epsilon) \) given an integer multiple \( S \) of \( R_{\Delta} \). Furthermore, if it is assumed that \( S \) is the output of the index-calculus algorithm, then, assuming the GRH, \( S \) is the regulator and hence of size \( O(\Delta^{1/2+\epsilon}) \). The end result is an algorithm that unconditionally computes the regulator in expected time \( O(\Delta^{1/6+\epsilon}) \) assuming the GRH. This algorithm was shown to work very well in practice, as demonstrated by the computation of the regulator of a real quadratic field with 65-decimal digit discriminant, the largest such result to-date.

In this paper, we generalize this result to computing a basis of the logarithm lattice corresponding to the unit group of an algebraic number field \( K \) with arbitrary unit rank, given a full rank sublattice \( \Lambda' \) as input. In particular, we describe an algorithm that solves this problem in \( O((\det \Lambda')^{n/(2n+1)+\epsilon}\Delta_K) \) bit operations. For unit rank one fields we recover the same complexity as \cite{dHJW07}, and the algorithm is asymptotically faster than \( O((\det \Lambda')^{1/2+\epsilon}\Delta_K) \) for all \( n \). When \( \Lambda' \) is computed using the index-calculus algorithm, we have, similar to the quadratic case, that it is in fact the full logarithm lattice under the assumption of the GRH. Thus, we obtain an algorithm for computing the logarithm lattice unconditionally in expected \( O(\Delta_K^{n/(4n+2)+\epsilon}) \) bit operations assuming the GRH. Our algorithm is asymptotically faster than \( O(\Delta_K^{1/4+\epsilon}) \) for all \( n \), but the greatest improvements occur for small \( n \). For example, for fields of unit rank one we obtain \( O(\Delta_K^{1/6+\epsilon}) \), the same complexity as \cite{dHJW07} in the real quadratic case, and for unit rank two we obtain \( O(\Delta_K^{1/5+\epsilon}) \).

The paper is organized as follows. Following a presentation of the required notation and background in Section 2, we give an overview of the algorithm in Section 3. The theory behind the algorithm is described in detail in Section 4 and two important subroutines are described in Section 5. The algorithm itself and a proof of its complexity are given in Section 6 and we finish with some concluding remarks.

2. Notation and Background

All required information on number fields can be found in \cite{Neu99}. References are provided for results not appearing in this source.

Let \( K \) be a number field, i.e. a finite extension of \( \mathbb{Q} \). Denote the integral closure of \( \mathbb{Z} \) in \( K \) by \( \mathcal{O}_K \). This is a Dedekind domain. Let \( \|\cdot\|_1, \ldots, \|\cdot\|_{n+1} \) be all \( n+1 \) Archimedean absolute values of \( K \); these correspond to embeddings \( \sigma_i : K \to \mathbb{C} \) up to complex conjugation by \( |f|_i = |\sigma_i(f)|, f \in K, 1 \leq i \leq n+1 \). Let \( \deg \|\cdot\|_i := 1 \) if
The image of the unit group \(\mathcal{O}_K^*\) is a lattice of rank \(n\), denoted by \(\Lambda_K := \Psi(\mathcal{O}_K^*)\). The kernel of \(\Psi|_{\mathcal{O}_K^*} : \mathcal{O}_K^* \to \Lambda_K\) is the group of roots of unity in \(K\), \(\mu_K\), and we have that \(\mathcal{O}_K^* \cong \mu_K \times \Lambda_K \cong \mu_K \times \mathbb{Z}^n\), where the number \(n\) is called the unit rank of \(K\). Thus, every unit in \(\mathcal{O}_K^*\) can be written as \(\zeta \varepsilon_1 \ldots \varepsilon_n\), where \(\zeta \in \mu_K\) and \(\varepsilon_1, \ldots, \varepsilon_n\) are a system of fundamental units of \(\mathcal{O}_K^*\). The regulator \(R_K\) of \(K\) equals \(\det \Lambda_K \cdot \prod_{i=1}^n \deg \langle \bullet \rangle_i\).

One can recover a unit \(\varepsilon\) from its image \(\Psi(\varepsilon)\) up to a root of unity. If one sets \(t_i := \log |\varepsilon|_i\), \(1 \leq i \leq n\) and \(t_{n+1} := -\frac{1}{\deg \langle \bullet \rangle_{n+1}} \sum_{i=1}^n t_i \deg \langle \bullet \rangle_i\), one has that \(\mu_K \varepsilon \cup \{0\} = \{f \in \mathcal{O}_K \mid \log |f|_i \leq t_i\ \text{for}\ 1 \leq i \leq n+1\}\). Thus, computing a basis of \(\Lambda_K\) allows us to recover a system of fundamental units, thereby completely determining the unit group of \(\mathcal{O}_K\).

Another important invariant of \(K\) is the discriminant \(\Delta_K\); it is defined as follows. The ring \(\mathcal{O}_K\) is a free \(\mathbb{Z}\)-module of rank \(d = [K : \mathbb{Q}]\); let \(v_1, \ldots, v_d \in \mathcal{O}_K\) be a \(\mathbb{Z}\)-basis of \(\mathcal{O}_K\). Moreover, as \(K/\mathbb{Q}\) is separable, one has \(d\) distinct embeddings \(\sigma_1, \ldots, \sigma_{n+1}, \sigma_{n+2}, \ldots, \sigma_d : K \to \mathbb{C}\). The discriminant \(\Delta_K\) is defined as \(\det(A)^2\), where \(A := (\sigma_i(v_j))_{1 \leq i, j \leq d} \in \mathbb{C}^{n \times n}\); it can be shown that \(\Delta_K \in \mathbb{Z} \setminus \{0\}\), with \(\Delta_K \neq \pm 1\) for \(K \neq \mathbb{Q}\). In order to simplify the notation, \(\Delta_K\) should be understood to be in absolute value when required in arithmetic expressions and complexity statements.

Let \(g : \mathbb{R}^{l+1} \to \mathbb{R}_{\geq 0}\) be a function and \(x_1, \ldots, x_l\) be parameters which depend on the number field \(K\); examples are \(\Delta_K\), \(R_K\) and \(n\). We say that a quantity \(f(x_1, \ldots, x_l)\) is in \(O(g(x_1, \ldots, x_n, \epsilon))\), if there exist a family of constants \(C_{[K : \mathbb{Q}], \epsilon, \sigma} > 0\), only depending on \([K : \mathbb{Q}]\) and \(\epsilon\), such that for all \(\epsilon > 0\) and all number fields \(K\), \(f(x_1(K), \ldots, x_n(K)) \leq C_{[K : \mathbb{Q}], \epsilon} \cdot g(x_1(K), \ldots, x_n(K), \epsilon)\) for sufficiently large \(x_1(K), \ldots, x_n(K)\). In that case, we write \(f = O(g(x_1, \ldots, x_n, \epsilon))\). This simply means that the \(O\)-constant depends only on the extension degree \([K : \mathbb{Q}]\), and not on any other information of \(K\) or any other parameter.

In the following, we will use that \(R_K = O(\Delta_K^{1/2+\epsilon})\) by a result of Sands [San91], as well as that \(\det \Lambda_K = R_K / \prod_{i=1}^n \deg \langle \bullet \rangle_i\) can be bounded from below only in terms of \([K : \mathbb{Q}]\) by a result of Remak [Rem2]. The latter means that for any sublattice \(\Lambda' \subseteq \Lambda_K\), we have \([\Lambda_K : \Lambda'] = \det \Lambda' / \det \Lambda_K = O(\det \Lambda')\). Moreover, we will use that arithmetic in \(K\) can be done in \(O(\Delta_K^{1/2+\epsilon})\) bit operations; see, for example, [Bue77a, Bue77c].

Finally, for \(v \in \mathbb{R}^n\) and \(M \subseteq \mathbb{R}\), we set \(Mv := \{vm \mid m \in M\}\), and for subsets \(M', M'' \subseteq \mathbb{R}^n\), we set \(M' + M'' := \{m' + m'' \mid m', m'' \in M'\}\). We equip \(\mathbb{R}^n\) with the Euclidean norm, denoted by \(\| \bullet \|\), as well as with the Lebesgue measure, denoted by \(\text{vol}\).

3. Overview of the Algorithm

Our algorithm will, given a sublattice \(\Lambda' \subseteq \Lambda_K\) of full rank \(n\), compute \(\Lambda_K\) in \(O((\det \Lambda')^{2+\epsilon} \Delta_K^{1/2})\) bit operations, using \(O((\det \Lambda')^{2+\epsilon} \Delta_K)\) bits of storage.

The idea can be sketched as follows. Since \(\Lambda'\) is of full rank, the quotient group \(\Lambda_K / \Lambda'\) is finite. Denote its order by \(i_{\Lambda'}\). Now we do not know \(\Lambda_K\) or \(i_{\Lambda'}\), but there is an effective test whether a prime \(p\) divides the index \(i_{\Lambda'}\) based on the following proposition, which we will prove in Section 4.
Proposition 1. Assume that $\Lambda' = \sum_{i=1}^{n} \mathbb{Z}v_i$ for a basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$. A prime $p$ divides $i_{\Lambda'}$ if, and only if, there is an element of $\Lambda_K$ in

$$
\bigcup_{k=1}^{n} \left\{ \sum_{i=1}^{k-1} \frac{a_i}{p} v_i + \frac{1}{p} v_k \mid a_1, \ldots, a_{k-1} \in \{0, \ldots, p-1\} \right\}.
$$

If such an element $v$ exists, set $\Lambda'' = \Lambda' + \mathbb{Z}v$. This is a sublattice of $\Lambda_K$ with $i_{\Lambda''} = [\Lambda_K : \Lambda''] = \frac{i_{\Lambda'}}{p}$.

The search set in the proposition is shown in Figure 1a. If we would have a finite set of candidates for prime divisors of $i_{\Lambda'}$, we could iterate through the set of candidates and use the proposition to determine the prime divisors of $i_{\Lambda'}$, their multiplicities and, most importantly, $\Lambda_K$ itself. Unfortunately, as $i_{\Lambda'} = O(\det \Lambda')$, this method would in general be slower than baby-step giant-step.

Alternatively, one could simply search a fundamental parallelepiped of $\Lambda'$, such as $\sum_{i=1}^{n} [0, 1]v_i$, for elements of $\Lambda_K$. Using Buchmann’s baby-step giant-step method for number fields as presented in [Buc87c], this can be done in $O((\det \Lambda')^{1/2} \Delta_K)$ bit operations. But instead, one could also directly apply Buchmann’s method to compute a basis for $\Lambda_K$ and compare it to $\Lambda'$; if $i_{\Lambda'} > 1$, this would actually be faster.

The idea of our algorithm is to combine both approaches. First, we test all primes $p$ below a bound $B$ using an algorithm based on Proposition 1. After that, we use Buchmann’s algorithm to search a small subset of the fundamental parallelepiped for elements of $\Lambda_K$. Note that the set of elements we have to search for lies in a small subset of the fundamental parallelepiped, as illustrated in Figure 1a. More precisely, if $\Lambda' = \sum_{i=1}^{n} \mathbb{Z}v_i$ as in the proposition, the search set for a prime $p$ lies in

$$
V_p := \sum_{i=1}^{n-1} [0, 1]v_i + [0, \frac{1}{p}]v_n.
$$

![Figure 1. Overview of the Algorithm](image-url)
Moreover, if $q \leq p$, then $V_q \subseteq V_p$. Therefore, if we use the method from Proposition 1 for all primes $\leq B$, then it suffices to search the set $V_B$ using Buchmann’s method, as illustrated in Figure 1b. Finding an optimal value of $B$ that minimizes the total running time of the two parts of the algorithm gives us the results.

Note that we ignore all approximation issues in this algorithm, and refer to the discussion in Sections 13 and 16 of [Buc87c].

4. Lattice Maximization

Lattice maximization refers to the process described in the previous section, proving that $\Lambda' = \Lambda_K$ or finding a sublattice $\Lambda''$ with $\Lambda' \subseteq \Lambda'' \subseteq \Lambda_K$. In this section, we describe in more detail the lattice maximization algorithm outlined in the previous section, and prove the results required to establish its correctness and complexity.

We begin with a lemma which allows us to determine whether an integer is coprime to the index $i_{\Lambda'}$.

**Lemma 1.** An integer $t > 0$ has a non-trivial common divisor with $i_{\Lambda'}$ if, and only if, $\Lambda' \nsubseteq \frac{1}{t} \Lambda' \cap \Lambda_K$. Moreover, any element $v \in (\frac{1}{t} \Lambda' \cap \Lambda_K) \setminus \Lambda'$ gives rise to a sublattice $\Lambda'' := \Lambda' + Zv \nsubseteq \Lambda'$ with $\Lambda'' \subseteq \Lambda_K$, and $[\Lambda'' : \Lambda']$ is a divisor of $t$.

**Proof.** First, assume that $d = \gcd(t, i_{\Lambda'}) > 1$. Let $p$ be a prime dividing $d$. Then there exists an element $v \in \Lambda_K \setminus \Lambda'$ with $pv \in \Lambda'$. But then, $v \in (\Lambda_K \cap \frac{1}{t} \Lambda') \setminus \Lambda'$.

On the contrary, assume that there exists some $v \in (\Lambda_K \cap \frac{1}{t} \Lambda') \setminus \Lambda'$. Then $tv \in \Lambda'$, whence the order of $v$ in $\Lambda'/\frac{1}{t} \Lambda'$ is a non-trivial divisor of $t$. But since the order divides $|\Lambda'/\frac{1}{t} \Lambda'| = i_{\Lambda'}$, we see that $\gcd(t, i_{\Lambda'}) > 1$.

For any $v \in (\frac{1}{t} \Lambda' \cap \Lambda_K) \setminus \Lambda'$, we have $\Lambda'' := \Lambda' + Zv \subseteq \Lambda_K$ and $\Lambda' \nsubseteq \Lambda''$, and since $tv \in \Lambda'$ we see that $\Lambda''/\Lambda' = (v + \Lambda')Z$ is cyclic of order dividing $t$. $\square$ $\square$

Note that we have a tower of subgroups $\Lambda' \subseteq \frac{1}{t} \Lambda' \cap \Lambda_K \subseteq \frac{1}{t} \Lambda'$. The lemma says that $\gcd(t, i_{\Lambda'}) > 1$ if, and only if, $(\frac{1}{t} \Lambda' \cap \Lambda_K)/\Lambda'$ is not the trivial subgroup of $\frac{1}{t} \Lambda'/\Lambda'$. Hence, if we let $p$ be a prime divisor of $i_{\Lambda'}$, we can replace $\Lambda'$ by a sublattice $\Lambda''$ with $i_{\Lambda''} = \frac{1}{t}$ by searching a set of representatives of $\Lambda'/\frac{1}{t} \Lambda'$. But this can be done more efficiently, as hinted in Proposition 1. This is provided by the following result; note that $\frac{1}{t} \Lambda'/\Lambda' \cong (\mathbb{Z}/t\mathbb{Z})^n$.

**Proposition 2.** Let $G$ be a finite group, and let $H \subseteq G$ be a subgroup. Let $S$ be the set of all cyclic subgroups of prime order of $G$, and let $\tilde{S} \subseteq G$ such that for every $U \in S$, there exists a unique element $g \in \tilde{S}$ with $U = \langle g \rangle$.

(a) The subgroup $H$ of $G$ is trivial if, and only if, $\tilde{S} \cap H = \emptyset$.

(b) If $G = (\mathbb{Z}/p\mathbb{Z})^m$ for a prime $p$ and $m \in \mathbb{N}$, we can choose the set $\tilde{S}$ to be a subset of

$$\{(a_1, \ldots, a_m) + p\mathbb{Z}^m \mid (a_1, \ldots, a_m) \in \{0, \ldots, p-1\}^m, a_m \leq 1\}.$$ 

**Proof.**

(a) The neutral element $e$ generates the trivial subgroup of $G$. Hence, if $H = \{e\}$, then $H \cap \tilde{S} = \emptyset$. Conversely, assume that $|H| > 1$. Then there exists an element $g \in H$ of prime order, and $\langle g \rangle$ is a non-trivial cyclic subgroup of prime
order of $G$. Hence, $(g) \in S$, and there exists some $\tilde{g} \in \tilde{S}$ with $(g) = (\tilde{g})$. In particular, $\tilde{g} \in (g) \subseteq H$, whence $\tilde{S} \cap H \neq \emptyset$.

(b) Let $v = (v_1, \ldots, v_m) + p\mathbb{Z}^m \in G$. If $p \mid v_m$, set $\lambda := 1$. Otherwise, let $\lambda \in \mathbb{N}$ such that $\lambda v_m \equiv 1 \pmod{p}$. Set $\tilde{v}_i := \lambda v_i \mod p$; then $\tilde{v}_i \in \{0, \ldots, p-1\}$ and $\tilde{v}_m \in \{0, 1\}$, and we have $(\tilde{v}_1, \ldots, \tilde{v}_m) + p\mathbb{Z}^m = \lambda v$ and $\lambda + p\mathbb{Z} \in (\mathbb{Z}/p\mathbb{Z})^\times$. Since $pG = \{(0, \ldots, 0) + \mathbb{Z}^m\}$, we see that every non-trivial cyclic subgroup of $G$ is of order $p$, and the previous discussion shows that every such subgroup is generated by at least one element in the set from the statement of the lemma. □

In fact, we can also write down a minimal such set $\tilde{S}$ for $(\mathbb{Z}/p\mathbb{Z})^m$ directly as

$$\tilde{S} = \left\{ (v_1, \ldots, v_i, 1, 0, \ldots, 0) + p\mathbb{Z}^m \mid i \in \{0, \ldots, m-1\}, \quad v_1, \ldots, v_i \in \{0, \ldots, p-1\} \right\}.$$ 

This shows that $|\tilde{S}| = 1 + p + p^2 + \cdots + p^{m-1} = \frac{p^m-1}{p-1}$. For our algorithm, we can restrict to a subset of $p^{m-1}$ elements, since we also search the volume

$$V := \sum_{i=1}^{n-1} [0, 1]v_i + [0, \frac{1}{B}]v_n,$$

where the $v_i$ are a basis of $\mathbb{R}^n$. Then we only need the elements of the form $(v_1, \ldots, v_{m-1}, 1) + p\mathbb{Z}^m$.

These two results imply Proposition 1. Moreover, we combine them as sketched in Section 3 to obtain our algorithm. The following corollary presents the preceding material in a way which leads directly to the algorithm and its correctness. It is also helpful to compare it with the sketch in Figure 1b.

**Corollary 1.** Assume that $\Lambda = \sum_{i=1}^{n} \mathbb{Z}v_i$, and let $B > 0$ be arbitrary. Let $p_1, \ldots, p_t$ be all primes $\leq B$. For $i \in \{1, \ldots, t\}$, set

$$\tilde{S}_i := \left\{ \frac{1}{p_i} (a_1v_1 + \cdots + a_{n-1}v_{n-1} + v_n) \mid a_1, \ldots, a_{n-1} \in \{0, \ldots, p_i-1\} \right\}.$$ 

Moreover, define the volume

$$V_B := \sum_{i=1}^{n-1} [0, 1]v_i + [0, \frac{1}{B}]v_n.$$ 

Then $\Lambda_K = \Lambda'$, if and only if, $\Lambda_K \cap (V_B \cup \bigcup_{i=1}^{t} \tilde{S}_i) = \{0\}$. Otherwise, any non-trivial element $v$ of $\Lambda_K \cap (V_B \cup \bigcup_{i=1}^{t} \tilde{S}_i)$ gives a lattice $\Lambda'' := \Lambda' + \mathbb{Z}v$ with $\Lambda' \not\subseteq \Lambda'' \subseteq \Lambda_K$.

Moreover, $\text{vol}(V_B) = \frac{1}{B} \det \Lambda'$, $t = O\left(\frac{B}{\log B}\right)$ and $\sum_{i=1}^{t} |\tilde{S}_i| = O(B^n/\log B)$.

**Proof.** Clearly

$$\left( V_B \cup \bigcup_{i=1}^{t} \tilde{S}_i \right) \cap \Lambda_K = \{0\}.$$ 

Now assume that $\Lambda' \not\subseteq \Lambda$. Let $p$ be a prime dividing $i_{\Lambda'}$. We define $\tilde{S} := \left\{ \frac{1}{p} (a_1v_1 + \cdots + a_{n-1}v_{n-1} + v_{1+i}) \mid a_1, \ldots, a_{n-1} \in \{0, \ldots, p-1\} \right\}$.

by Proposition 2. $\tilde{S}$ must contain a non-trivial element of $\Lambda_K$. In case $p > B$, we have $\tilde{S} \subseteq V_B$; and if $p \leq B$, say $p = p_i$, we have $\tilde{S} \subseteq \tilde{S}_i \cup V_B$. 


Now vol($V_B$) = $\frac{1}{\mu} \log |B|$ vol($\sum_{i=1}^{n} |0, 1| v_i$) = $\frac{1}{\mu} \log |\det(A')$. Moreover, by the Prime Number Theorem, $t = O(\frac{B}{\log B})$. Finally, $|S'_i| = p_i^{n-1} \leq B^{n-1}$, whence $\sum_{i=1}^{t} |S'_i| \leq tB^{n-1} = O(\frac{B^n}{\log B})$.

We have seen how the idea sketched in the last section can be made rigorous. It translates in a straightforward manner into an algorithm, as we will see in Section 6. The only missing pieces are how to search for elements in $V \cap \Lambda_K$, and how to test whether some $v \in \mathbb{R}^n$ lies in $\Lambda_K$. We will investigate this in the next section.

5. Baby-Step Giant-Step Search and Existence Testing

We will now investigate how to search for elements of $\Lambda_K$ in the set $V = \sum_{i=1}^{n} |0, 1| v_i$, where $v_1, \ldots, v_n$ is a basis of $\mathbb{R}^n$. We assume that this basis is mostly orthogonal, i.e. $|\det(v_1, \ldots, v_n)|^{-1} \prod_{i=1}^{n} \|v_i\| = O(1)$. This means that vol($V$) = $\mathcal{O}(\prod_{i=1}^{n} \|v_i\|)$. The algorithm requires $\mathcal{O}(\text{vol}(V)^{1/2} \Delta_K)$ bit operations and was first described by Buchmann in [Buc87c]. We will also describe how to test whether an element $v \in \mathbb{R}^n$ lies in $\Lambda_K$.

For describing these algorithms, we need fractional ideals and the notion of minima of these. A fractional ideal is a finitely generated $\mathcal{O}_K$-submodule of $K$; it is always of the form $\frac{1}{a}$, where $a$ is an (integral) ideal of $\mathcal{O}_K$ in the usual sense and $f \in \mathcal{O}_K \setminus \{0\}$. As $\mathcal{O}_K$ is a Dedekind domain, the nonzero fractional ideals form a free abelian group $\text{Id}(K)$ generated by the prime ideals of $\mathcal{O}_K$.

To define a minimum of an ideal, we use methods from Minkowski’s geometry of numbers. Set $W_i := \mathbb{R}$ if $\deg |v_i| = 1$ and $W_i := \mathbb{C}$ otherwise. Then

$$\Phi : K \to W_K := \prod_{i=1}^{n+1} W_i, \quad f \mapsto (\sigma_1(f), \ldots, \sigma_{n+1}(f))$$

is injective and maps every fractional ideal $a \in \text{Id}(K)$ onto a lattice in the $[K : \mathbb{Q}]$-dimensional real vector space $W_K \cong K \otimes_{\mathbb{Q}} \mathbb{R}$. For $a \in \text{Id}(K)$ and $t_1, \ldots, t_{n+1} \in \mathbb{R}_{>0}$, define

$$B(a, t_1, \ldots, t_{n+1}) := \{ f \in a \mid \|f\| \leq t_i \}.$$  

Then $\Phi$ identifies $B(a, t_1, \ldots, t_{n+1})$ with the finite set of elements in $\Phi(a)$ which lie in the bounded area $\{(v_1, \ldots, v_{n+1}) \in W_K \mid v_i \leq t_i \}$. For convenience, define

$$B(a, f, f') := B(a, \max\{|f|_1, |f'|_1\}, \ldots, \max\{|f|_{n+1}, |f'|_{n+1}\})$$

and

$$B(a, f, f') := B(a, f, f) = B(a, |f|_1, \ldots, |f|_{n+1})$$

for $f, f' \in K^*$. Using this notation, we have that $B(\mathcal{O}_K, \varepsilon) = \{0\} \cup \mu_K \varepsilon$ if $\varepsilon \in \mathcal{O}_K^*$. 

Let $a \in \text{Id}(K)$. We say that $\mu \in a$ is a minimum of $a$ if $f \in B(a, \mu) \setminus \{0\}$ implies $|f|_i = |\mu|_i$ for some $i$. Denote the set of all minima of $a$ by $\mathcal{E}(a)$. We say that $a$ is reduced if $1 \in \mathcal{E}(a)$. Note that $\mathcal{O}_K$ itself is reduced. 

The set $\Psi(\mathcal{E}(a))$ is distributed rather uniformly in $\mathbb{R}^n$; here, $\Psi$ is as defined in Section 2. More precisely, Buchmann showed the following.

Proposition 3 ([Buc87b] [Buc87c]). Let $V = \sum_{i=1}^{n} [a_i, b_i] v_i$, where $(v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$ and $a_i < b_i$.

(a) Assuming that the $v_i$’s are mostly orthogonal, the set $\Psi(\mathcal{E}(a)) \cap V$ contains $O(\text{vol}(V))$ elements.
(b) If \( V \) contains a sphere of radius \( \frac{1}{4}\sqrt{n}\log\Delta_K \), \( V \cap \Psi(\mathcal{E}(a)) \neq \emptyset \).

Note that \( f \in \mathcal{E}(f(a)) \) for \( f \in K^* \) and \( \mu \in \mathcal{E}(a) \), and moreover that \( 1 \in \mathcal{E}(\mathcal{O}_K) \). This implies that \( \mathcal{O}_K^\ast \) operates on \( \mathcal{E}(a) \) and that \( \mathcal{O}_K^\ast \subseteq \mathcal{E}(\mathcal{O}_K) \). It turns out that \( \mathcal{E}(a)/\mathcal{O}_K^\ast \) is finite and contains \( O(R_K) \) elements \([Buc87b] \) Theorem 2.1. Moreover, note that the map \( \mathcal{E}(a)/\mathcal{O}_K^\ast \to \text{Id}(K) \), \( \mu \mathcal{O}_K^\ast \mapsto \frac{1}{\mu}a \) is a bijection between \( \mathcal{E}(a)/\mathcal{O}_K^\ast \) and the set of reduced ideals equivalent to \( a \). Denote this set of ideals by \( \text{Red}(a) \). This allows one to represent an element \( \mu \) of \( \mathcal{E}(a) \) up to a root of unity by the pair \((\frac{1}{\mu}a, \Psi(\mu))\). In practice, one stores \( \frac{1}{\mu}a \) together with an approximation of \( \Psi(\mu) \).

The set of minima of an ideal modulo units is known as the infrastructure of that ideal. More precisely, consider the map \( \Psi \), together with the lattice \( \Lambda_K = \Psi(\mathcal{O}_K^\ast) \); the map \( d^n : \mathcal{E}(a)/\mathcal{O}_K^\ast \to \mathbb{R}^n/\Lambda_K \), \( \mu \mapsto \Psi(\mu) + \Lambda_K \), respectively \( d^n : \text{Red}(a) \to \mathbb{R}^n/\Lambda_K \), \( \frac{1}{\mu}a \mapsto \Psi(\mu) + \Lambda_K \), is called the distance map.

We now discuss how to search for all minima \( \mu \in \mathcal{E}(a) \) with \( \Psi(\mu) \in V \). For that, we need the notion of neighboring minima as described in \([Buc87a] \). Two minima \( \mu, \mu' \in \mathcal{E}(a) \) are said to be neighbors if \( f \in B(a, \mu, \mu') \setminus \{0\} \) implies \( |f|_i = \max(|\mu|_i, |\mu'|_i) \) for some \( i \). This relation defines a graph structure on \( \mathcal{E}(a) \) and \( \mathcal{E}(a)/\mathcal{O}_K^\ast \), and Buchmann showed that this graph is connected \([Buc87a] \). Moreover, Buchmann showed that if \( a \) is a reduced ideal, one can compute the set of all neighbors of \( 1 \in \mathcal{E}(a) \) in \( O(\Delta_K^2) \) bit operations; in fact, the number of neighbors is in \( O((\log \Delta_K)^n) \).

Using this, one can compute the set of all minima of \( a \) in \( V \cap \Lambda_K \) bit operations. Moreover, one can test whether \( \mu \mathcal{O}_K^\ast = \mu' \mathcal{O}_K^\ast \) by computing \( \frac{1}{\mu}a \) and \( \frac{1}{\mu'}a \) and comparing these. In fact, if one works with \( \frac{1}{\mu}a, \Psi(\mu) \) instead of \( \mu \) directly, one can do this easily by comparing the ideals in the representations. Another reason to use this representation of \( \mathcal{E}(a)/\mu_K \) is that this representation is small: Thiel showed that one can represent a reduced ideal with at most \( (\lceil K : \mathbb{Q} \rceil^2 + 1) \log_2 \sqrt{\Delta_K} \) bits \([Th95] \) Corollary 3.7]. Hence, the storage required to store all minima \( \mu \in \mathcal{E}(a) \) with \( \Psi(\mu) \in V \) is \( O(\log(\Delta_K)^n) \) bits.

We can use this to employ a baby-step giant-step strategy similar to the one in \([Buc87c] \) to search for elements in \( V \cap \Lambda_K \), where \( V = \sum_{i=1}^n [0, 1]|v_i| \). Select integers \( a_1, \ldots, a_n > 0 \) and set

\[
B := -\sum_{i=1}^n v_i[0, \frac{1}{a_i}] + S \quad \text{and} \quad G := \left\{ \sum_{i=1}^n \frac{b_i}{a_i} v_i \mid b_i \in \mathbb{N}, 0 \leq b_i < a_i \right\},
\]

where

\[
S := \{ v \in \mathbb{R}^n \mid \|v\| \leq \frac{1}{4}\sqrt{n}\log\Delta_K \}.
\]

The sets \( B \) and \( G \) are depicted in Figure 2a.

Let \( \mathcal{E}_B := \{ (\frac{1}{\mu}a, \Psi(\mu)) \mid \mu \in \mathcal{E}(a), \ \Psi(\mu) \in B \} \); this set is called the baby stock. For every \( v \in G \), one can find at least one \( \mu \in \mathcal{E}(a) \) with \( \Psi(\mu) \in v + S \) by Proposition \( \text{K} \) (b); choose an arbitrary such \( \mu \) as \( \mu_v \) and set \( \mathcal{E}_G := \{ (\frac{1}{\mu_v}a, \Psi(\mu_v)) \mid v \in G \} \). Finding \( \mu_v \) from \( v \) is called a giant step. Using the strategy in Section 11 of \([Buc87c] \), \( (\frac{1}{\mu}\mathcal{O}_K, \Psi(\mu_v)) \) can be computed in \( O(\log \|v\| \cdot \Delta_K^2) \) bit operations. The elements of \( \mathcal{E}_B \) and \( \mathcal{E}_G \) are depicted in Figure 2b.

**Proposition 4.** For every \( \lambda \in V \cap \Lambda_K \), there exists an ideal \( a \in \text{Red}(\mathcal{O}_K) \) such that \( (a, v) \in \mathcal{E}_B, (a, w) \in \mathcal{E}_G \) for some \( v, w \in \mathbb{R}^n \) such that \( \lambda = w - v \).

Conversely, given an ideal \( a \) such that \( (a, v) \in \mathcal{E}_B, (a, w) \in \mathcal{E}_G \) for some \( v, w \in \mathbb{R}^n \), then \( w - v \in \Lambda_K \) with \( w - v \in V + 2S \).
Corollary 2. \[ \text{in } [Buc87c], \text{ this yields the following. } \]

Proof. First, note that \( \frac{1}{\mu a} = \frac{1}{\mu'} a \) if, and only if, \( \mu^{-1}\mu' \in \mathcal{O}_K^* \); therefore, \( \frac{1}{\mu a} = \frac{1}{\mu'} a \) if, and only if, \( \Psi(\mu') - \Psi(\mu) \in \Psi(\mathcal{O}_K^*) = \Lambda_K \).

Now if \( \lambda \in V \cap \Lambda_K \), we can write \( \lambda = \sum_{i=1}^n \lambda_i v_i \) with \( \lambda_i \in [0, 1] \). Write \( \lambda_i = \mu_i + \frac{b_i}{n} \) with \( b_i \in \mathbb{N}, \mu_i \in [0, \frac{1}{n}] \). Set \( w := \sum_{i=1}^n \frac{b_i}{n} v_i \); then \( (\frac{1}{\mu w} a, \Psi(\mu w)) \in \mathcal{E}_G \) and \( \lambda := \Psi(\mu w) - w \in S \). Now \( v := -\sum_{i=1}^n \mu_i v_i + \tilde{w} \in B \); we have to show that \( (\frac{1}{\mu w} a, v) \in \mathcal{E}_B \), as \( \Psi(\mu w) - v = w + \tilde{w} - v = \sum_{i=1}^n \frac{b_i}{n} v_i + \tilde{w} + \sum_{i=1}^n \mu_i v_i - \tilde{w} = \lambda \).

For that, let \( \varepsilon \in \mathcal{O}_K^* \) with \( \Psi(\varepsilon) = \lambda \). Now \( \Psi(\varepsilon) = \lambda = \Psi(\mu w) - v \), whence \( v = \Psi(\mu w \varepsilon^{-1}) \). But \( \frac{1}{\mu w} a = \frac{1}{\mu w} \varepsilon a \), whence \( (\frac{1}{\mu w} a, v) = (\frac{1}{\mu w} \varepsilon a, \Psi(\mu w \varepsilon^{-1})) \in \mathcal{E}_B \).

Hence, to find all elements in \( V \cap \Lambda_K \), one can enumerate and store \( \mathcal{E}_B \), enumerate all elements \( v \in G \), compute a corresponding \( \mu_v \), and see if \( (\frac{1}{\mu w} a, v) \in \mathcal{E}_B \) for some \( v \in \mathbb{R}^n \). If that is the case, one obtains an element of \( \Lambda_K \cap (V + S) \), and the proposition shows that every element of \( \Lambda_K \cap V \) can be obtained in this way. As in \([Buc87c] \), this yields the following.

**Corollary 2.** Let \( R = \text{vol}(V) \). The strategy sketched above computes all elements in \( V \cap \Lambda_K \) in \( O((R \prod_{i=1}^n n_i^{-1} + \prod_{i=1}^n n_i \cdot \log \max_{i=1,\ldots,n} \|v_i\|) \Delta_K') \) bit operations and requires \( O(R \prod_{i=1}^n n_i^{-1} \cdot \Delta_K') \) bits of storage.

Note that the running time is minimized if \( \prod_{i=1}^n n_i \approx \sqrt{R} \).

Proof. The storage requirements follow from Proposition 3.7 (a) and \([Thi95, \text{ Corollary 3.7}] \). Using the enumeration technique by Buchmann \([Buc87a, Buc87c] \), one can compute \( \mathcal{E}_B \) in \( O(R \prod_{i=1}^n n_i^{-1} \cdot \Delta_K') \) bit operations since \( \text{vol}(B) = O(R \cdot \prod_{i=1}^n n_i^{-1} \cdot \Delta_K') \). Finally, one can compute the elements in \( \mathcal{E}_G \) in \( O(|\mathcal{E}_G| \cdot \Delta_K' \cdot \log \max_{i=1,\ldots,n} \|v_i\|) \) bit operations.

Finally, we discuss how to test whether \( v \in \Lambda_K \) for some \( v \in \mathbb{R}^n \). We use the giant step strategy mentioned above to compute some \( \mu \in \mathcal{E}(\mathcal{O}_K) \) with \( \Psi(\mu) \in V \cap \Lambda_K \).
v + S. Then, one uses the above strategy to enumerate all minima \( \mu' \in \mathcal{E}(\frac{1}{\mu}O_K) \) with \( \Psi(\mu') \in S \) to check whether a minimum \( \mu' \) with \( \Psi(\mu') + \Psi(\mu) = v \) and \( \frac{1}{\mu'}(\frac{1}{\mu}O_K) = O_K \) exists.

**Lemma 2.** Let \( \mu \in \mathcal{E}(O_K) \) and \( v \in \mathbb{R}^n \). Then there exists a minimum \( \mu' \in \mathcal{E}(\frac{1}{\mu}O_K) \) with \( \Psi(\mu') + \Psi(\mu) = v \) such that \( \frac{1}{\mu'}(\frac{1}{\mu}O_K) = O_K \) if, and only if, \( v \in \Lambda_K \).

**Proof.** First, assume that \( \Psi(\mu') + \Psi(\mu) = v \) and \( \frac{1}{\mu'}(\frac{1}{\mu}O_K) = O_K \). Then \( \mu \mu' \in O_K \) and \( v = \Psi(\mu' \mu) \in \Lambda_K \). Conversely, assume that \( v \in \Lambda_K \), say \( v = \Psi(\varepsilon) \) with \( \varepsilon \in O_K^* \).

During the course of the algorithm, we try to keep the basis vectors \( v_1, \ldots, v_n \) as orthogonal as possible; in that case, we have \( |\det(v_1, \ldots, v_n)| \approx \prod_{i=1}^{n} ||v_i|| \). Such a basis can be computed as in Algorithm 16.10 of [vzGG03] and is called a reduced basis.

We now analyze the asymptotic running time and memory consumption of Algorithm 1. Recall that \([K : \mathbb{Q}] = O(1)\); note that the \(O\)-constants are assumed to be exponentially dependent on \( n \) (compare [Bue87] p. 5).

**Theorem 3.** Algorithm 1 requires

\[
O((\frac{1}{B} \det \Lambda')^\delta + (\frac{1}{B} \det \Lambda')^{1-\delta} + B^n (\log B)^{-1})(\Delta_K \det \Lambda')^\varepsilon
\]

bit operations and \(O((\frac{1}{B} \det \Lambda')^\delta \Delta_K^\varepsilon)\) bits of storage.

**Proof.** First, assume that \( \Lambda' = \Lambda_K \), i.e. no element in \( \Lambda_K \setminus \Lambda' \) is found \( \Lambda' \) is not replaced by a larger sublattice of \( \Lambda_K \).

During the course of the algorithm, we try to keep the basis vectors \( v_1, \ldots, v_n \) as orthogonal as possible; in that case, we have \( |\det(v_1, \ldots, v_n)| \approx \prod_{i=1}^{n} ||v_i|| \). Such a basis can be computed as in Algorithm 16.10 of [vzGG03] and is called a reduced basis.

We now analyze the asymptotic running time and memory consumption of Algorithm 1. Recall that \([K : \mathbb{Q}] = O(1)\); note that the \(O\)-constants are assumed to be exponentially dependent on \( n \) (compare [Bue87] p. 5).
Algorithm 1 Find $\Lambda_K \subseteq \mathbb{R}^n$, given a sublattice $\Lambda'$ of full rank.

Input: A basis $(v_1, \ldots, v_n)$ of $\Lambda' \subseteq \Lambda_K$, a parameter $B \geq 1$, a parameter $\delta \in (0, 1)$.
Output: A basis of $\Lambda_K$.

1. Reduce the basis $(v_1, \ldots, v_n)$, i.e., make it mostly orthogonal.
2. for all primes $p$ with $2 \leq p \leq B$ do
3.   for all $(a_1, \ldots, a_{n-1}) \in \{0, \ldots, p - 1\}^{n-1}$ do
4.       Set $v = \frac{1}{p}(a_1 v_1 + \cdots + a_{n-1} v_{n-1} + v_n)$.
5.       if $v \in \Lambda_K$ then
6.         Compute a reduced basis $(\hat{v}_1, \ldots, \hat{v}_n)$ of $(v_1, \ldots, v_n, v)_{\mathbb{Z}}$.
7.         Replace $(v_1, \ldots, v_n)$ by $(\hat{v}_1, \ldots, \hat{v}_n)$ and restart the loop in line 8.
8.         Determine $a_1, \ldots, a_n \in \mathbb{N}_{>0}$ such that $\prod_{i=1}^{n} a_i \approx (\frac{1}{p^2} \det \Lambda')^{3-\delta}$.
9.       if all $\mu \in \mathcal{E}(O_K)$ with $\Psi(\mu) \in \sum_{i=1}^{n} v_i [-\frac{1}{v_i}, 0] + S$ do /* $S$ as in Section 5 */
10.      Store $(\frac{1}{p} O_K, \Psi(\mu))$ in the set $\mathcal{E}_B$.
11.     if some $(O_K, v) \in \mathcal{E}_B$ with $v \notin (v_1, \ldots, v_n)_{\mathbb{Z}}$ is found then
12.       Compute a reduced basis $(\hat{v}_1, \ldots, \hat{v}_n)$ of $(v_1, \ldots, v_n, v)_{\mathbb{Z}}$.
13.       Replace $(v_1, \ldots, v_n)$ by $(\hat{v}_1, \ldots, \hat{v}_n)$ and go back to line 8.
14.       for all $w \in \{\sum_{i=1}^{n} a_i v_1 \mid a_i \in \mathbb{N}, 0 \leq a_i < b_i\}$ do
15.          Compute some $(\frac{1}{p} O_K, \Psi(\mu))$ with $\mu \in \mathcal{E}(O_K)$ and $\Psi(\mu) \in w + S$.
16.       if $(\frac{1}{p} O_K, v)$ is found in $\mathcal{E}_B$ for some $v \in \mathbb{R}^n$ with $\Psi(\mu) - v \notin (v_1, \ldots, v_n)_{\mathbb{Z}}$ then
17.          Compute a reduced basis $(\hat{v}_1, \ldots, \hat{v}_n)$ of $(v_1, \ldots, v_n, \Psi(\mu) - v)_{\mathbb{Z}}$.
18.          Replace $(v_1, \ldots, v_n)$ by $(\hat{v}_1, \ldots, \hat{v}_n)$ and go back to line 8.
19. return $(v_1, \ldots, v_n)$.

Now, every time one finds an element in $\Lambda_K \setminus \Lambda'$, the index $[\Lambda' : \Lambda_K]$ and $\det \Lambda'$ are divided by at least two. Hence, $\Lambda'$ is replaced at most $\log_2 [\Lambda' : \Lambda_K]$ times. Now $[\Lambda' : \Lambda_K] = O(\det \Lambda')$; therefore, the above bounds for the number of bit operations needs to be multiplied by $\log_2 \det \Lambda' = O((\det \Lambda')^\epsilon)$.

Note that we can ignore the running time for the orthogonalization process. By Theorem 16.11 in [vzGG03], the running time of the basis reduction algorithm is bounded by $O(n^4 \log A)$ arithmetic operations on integers of length $O(n \log A)$, where $A = \max\{\|v_1\|, \ldots, \|v_n\|\}$. Since $n = O(1)$ in our notation, the running time is bounded by $O((\det \Lambda')^\epsilon)$ bit operations. \hfill $\Box$ \hfill $\Box$

We now optimize the running time for two situations. For our optimizations, we simplify the upper bound from Theorem 3 by omitting the $\log B$ factor; then the running time is bounded by

$$O\left((\frac{1}{B^2} \det \Lambda')^\delta + (\frac{1}{B} \det \Lambda')^{1-\delta} + B^n)(\det \Lambda')^\epsilon\right)$$

bit operations. Moreover, we ignore the $(\det \Lambda')^\epsilon$ part, i.e., we assume that all three operations (existence testing, baby stock computation, giant steps) are equally fast. Hence, we need to minimize the term $(\frac{1}{B^2} \det \Lambda')^\delta + (\frac{1}{B} \det \Lambda')^{1-\delta} + B^n$.

Note that these two simplifications are justified. If we minimize the original formula, the difference to our minimal running time can be bounded by $O((\det \Lambda')^\epsilon)$, i.e., can be ignored since we have the factor $O((\det \Lambda')^\epsilon)$ anyway.

First, we optimize without any restrictions on the amount of available memory.
Corollary 4. If $B$ and $\delta$ can be chosen freely, optimal performance of Algorithm 7 is obtained for $\delta = \frac{1}{2}$ and $B = (\det\Lambda)^{1/N}n^{-\frac{1}{2}}$. In that case, one needs $O((\det\Lambda')^{\frac{1}{2}+\epsilon}\Delta_K)$ bit operations and $O((\det\Lambda')^{\frac{1}{2}+\epsilon}\Delta_K')$ bits of storage.

Proof. For fixed $B$, the expression $\frac{n}{B} \det\Lambda'^\delta + (\frac{1}{B} \det\Lambda')^{1-\delta} + B^n$ is minimal for $\delta = \frac{1}{2}$; in that case, it attains the value $2B^{-1/2}\sqrt{\det\Lambda'} + B^n$.

Differentiating this by $B$, we obtain $-\sqrt[4]{\det\Lambda'}B^{-3/2} + nB^{n-1}$. This is zero if, and only if, $B = (\det\Lambda')^{1/N}n^{-1/2}$. In that case, it attains the value $2(\det\Lambda')^{1/4}\sqrt{n\det\Lambda'} + (\det\Lambda')^{n/2}n^{-\frac{1}{2}}$. Plugging these choices for $\delta$ and $B$ in gives the result. □ □

Next, we investigate the situation in which the available memory is insufficient to store the optimal number of baby steps.

Corollary 5. Assume that storage is limited to $T$ baby steps, and that one has less memory than required for the optimal running time of Algorithm 7 as in Corollary 4. Under this assumption, optimal performance of Algorithm 7 is obtained for $\delta = \frac{1}{2} \frac{(1+n)\log T}{\log T + n \log \det\Lambda'}$ and $B = (\det\Lambda')^{\frac{1}{N}+\epsilon}$. In that case, one needs $O\left(\left(T + (\det\Lambda')^{\frac{1}{N}+\epsilon}\right)\Delta_K\right) = O((\det\Lambda')^{\frac{1}{2}+\epsilon}\Delta_K)$ bit operations.

Proof. In this case, the number of operations required for the “baby steps” in the loop in lines 6-13 of the algorithm is $O(T\Delta_K)$. As optimal performance as in Corollary 4 can not be obtained, one needs to balance the number of operations for the loop in lines 2-7 and the one in lines 14-18. i.e. one needs to choose $\delta$ and $B$ such that $(\frac{1}{B} \det\Lambda')^\delta \approx T$ and $(\frac{1}{B} \det\Lambda')^{1-\delta} \approx B^n(\log B)^{-1}$. For simplicity, we ignore the factor of $\frac{1}{\log B}$ as in Corollary 4 and replace “$\approx$” by “$=.$”.

The first equality gives $B = T^{-1/\delta} \det\Lambda'$, whence the second translates to $T^{\frac{1}{\delta}} = T^{-n/\delta} (\det\Lambda')^n$. But this gives $(T^{1+n})^\delta = T(\det\Lambda')^n$, i.e. $\delta = \frac{(1+n)\log T}{\log T + n \log \det\Lambda'}$ and, hence, $B = (\det\Lambda')^{\frac{1}{N}+\epsilon}$. Plugging this in, we obtain the given bound. □ □

7. Conclusions

We have seen that our algorithm computes $\Lambda_K$ in

$$O((\det\Lambda')^{\frac{1}{2}+\epsilon}\Delta_K') = O((\det\Lambda')^{1/2} \sqrt{n\det\Lambda'})$$

bit operations, using $O((\det\Lambda')^{1/4} \sqrt{n\det\Lambda'})$ bits of storage. In particular, our algorithm generalizes the algorithm in [117] to number fields of arbitrary unit rank, with the same complexity as [117] being obtained in our algorithm for unit rank 1. In the case that memory is too limited for the optimal method, we determined for the value of $B$ for which optimal performance is obtained when using a restricted amount of memory.

If $\det\Lambda' = O(\Delta_K^{1/2+\epsilon})$, for example when $\Lambda'$ is computed using Buchmann’s index-calculus algorithm and is correct assuming the GRH, we obtain a complexity of $O(\Delta_K^{1/4} \sqrt{n\det\Lambda'})$ bit operations. Thus, computing $\Lambda'$ with Buchmann’s algorithm followed by our’s to verify that $\Lambda' = \Lambda_K$ yields an algorithm that computes $\Lambda_K$ unconditionally with expected complexity $O(\Delta_K^{1/4} \sqrt{n\det\Lambda'})$ bit operations. Only the complexity is dependent on the GRH, for both the running time and correctness (required to bound the size of $\det\Lambda'$) of Buchmann’s algorithm. This is always
asymptotically better than Buchmann’s baby-step giant-step method for computing \( \Lambda_K \), whose running time is \( O(\Delta_K^{1/4+\epsilon}) \) bit operations. For unit rank one, i.e. for \( n = 1 \), we obtain \( O(\Delta_K^{1/6+\epsilon}) \) bit operations; this is the same complexity as in [dHJW07]. For unit rank two, we obtain \( O(\Delta_K^{1/5+\epsilon}) \) bit operations; this is faster than any other known algorithm for computing the units of a number field of unit rank two whose correctness of the output does not depend on the GRH.

Even though the baby stock computation, giant step computation and existence testing of lattice elements roughly need \( O(\Delta_K^{\epsilon}) \) bit operations, with some factor polynomial in the logarithms of the dimensions of the involved objects, the running times of these three operations vary a lot in practice. In particular, computing all neighbors of a minimum is very slow compared to reducing an ideal, which is the main operation when computing giant steps. Therefore, in practice, it makes sense to first sample the running times of these three operations, and to find optimal values of \( \delta \) and \( B \) that take this into account in a manner similar to the algorithm in [dHJW07]. Moreover, it is also possible re-adjust \( \delta \) and \( B \) after an element in \( \Lambda_K \setminus \Lambda' \) is found, as this changes \( \det \Lambda' \). One can also optimize the running time by reusing the already computed part of \( E_B \) when updating \( \Lambda' \) in line 13.

Another possible practical improvement is to parallelize parts of the algorithm. In particular, the loops in lines 3–7 and 2–7 can easily be parallelized. The loops in lines 9–13 and 14–18 can be parallelized in a similar manner to all baby-step giant-step type algorithms. As in [dHJW07], it is possible to re-optimize the running time to find optimal values of \( \delta \) and \( B \) that take into account parallelization and the number of processors used.

Note that these optimizations do not affect the asymptotic complexity of our algorithm. However, as in the case of real quadratic fields [dHJW07], we expect that they will have a significant impact on its practical performance.

So far, we do not have an implementation of our algorithm. The main problem is that the methods in Section 5, or more precisely computing all neighbors of 1 in a reduced ideal, are not implemented in any number theory library to our knowledge. All libraries and computer algebra systems which provide methods for computing units of number fields use Buchmann’s subexponential algorithm [Buc90]. An implementation is not yet available, but is currently work in progress. It will be interesting to see how our algorithm performs in practice.

References

[Buc87a] J. A. Buchmann. On the computation of units and class numbers by a generalization of Lagrange’s algorithm. *J. Number Theory*, 26(1):8–30, 1987.

[Buc87b] J. A. Buchmann. On the period length of the generalized Lagrange algorithm. *J. Number Theory*, 26(1):31–37, 1987.

[Buc87c] J. A. Buchmann. Zur Komplexität der Berechnung von Einheiten und Klassenzahl algebraischer Zahlkörper. Habilitationsschrift, October 1987.

[Buc90] J. A. Buchmann. A subexponential algorithm for the determination of class groups and regulators of algebraic number fields. In C. Goldstein, editor, *Séminaire de Théorie des Nombres, Paris 1988–1989*, volume 91 of *Progr. Math.*, pages 27–41, Boston, MA, 1990. Birkhäuser Boston.

[dHJW07] R. de Haan, M. J. Jacobson, Jr., and H. C. Williams. A fast, rigorous technique for computing the regulator of a real quadratic field. *Math. Comp.*, 76(260):2139–2160 (electronic), 2007.

[Neu99] J. Neukirch. *Algebraic number theory*. Springer-Verlag, Berlin, 1999.
[Rem32] R. Remak. Über die Abschätzung des absoluten Betrages des Regulators eines algebraischen Zahlkörpers nach unten. *J. Reine Angew. Math.*, 167:360–378, 1932.

[San91] J. W. Sands. Generalization of a theorem of Siegel. *Acta Arith.*, 58(1):47–57, 1991.

[Thi95] C. Thiel. Short proofs using compact representations of algebraic integers. *J. Complexity*, 11(3):310–329, 1995.

[vzGG03] J. von zur Gathen and J. Gerhard. *Modern computer algebra*. Cambridge University Press, Cambridge, second edition, 2003.

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF CALGARY, 2500 UNIVERSITY DRIVE NW, CALGARY, ALBERTA, CANADA T2N 1N4

E-mail address: fwfontei@ucalgary.ca

DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF CALGARY, 2500 UNIVERSITY DRIVE NW, CALGARY, ALBERTA, CANADA T2N 1N4

E-mail address: jacobs@ucalgary.ca