A 2-Approximation for the Height of Maximal Outerplanar Graph Drawings

Therese Biedl* Philippe Demontigny†

Abstract

In this paper, we study planar drawings of maximal outerplanar graphs with the objective of achieving small height. A recent paper gave an algorithm for such drawings that is within a factor of 4 of the optimum height. In this paper, we substantially improve the approximation factor to become 2. The main ingredient is to define a new parameter of outerplanar graphs (the so-called umbrella depth, obtained by recursively splitting the graph into graphs called umbrellas). We argue that the height of any poly-line drawing must be at least the umbrella depth, and then devise an algorithm that achieves height at most twice the umbrella depth.

1 Introduction

Graph drawing is the art of creating a picture of a graph that is visually appealing. In this paper, we are interested in drawings of so-called outer-planar graphs, i.e., graphs that can be drawn in the plane such that no two edges have a point in common (except at common endpoints) and all vertices are incident to the outer-face. All drawings are required to be planar, i.e., to have no crossing. The drawing model used is that of flat visibility representations where vertices are horizontal segments and edges are horizontal or vertical segments, but any such drawing can be transformed into a poly-line drawing (or even a straight-line drawings if the width is of no concern) without adding height [6].

Every planar graph has a straight-line drawing in an \( n \times n \)-grid [18, 13]. Minimizing the area is NP-complete [16], even for outer-planar graphs [7]. In this paper, we focus on minimizing just one direction of a drawing (we use the height; minimizing the width is equivalent after rotation). It is not known whether minimizing the height of a planar drawing is NP-hard (the closest related result concerns minimizing the height if edges must connect adjacent rows [15]). Given the height \( H \), testing whether a planar drawing of height \( H \) exists is fixed parameter tractable in \( H \) [10], but the run-time is exceeding large in \( H \). As such, approximation algorithms for the height of planar drawings are of interest.

*David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, Ontario N2L 1A2, Canada. biedl@uwaterloo.ca. Supported by NSERC.
†phdemontigny@gmail.com. Part of this work appeared as the author’s Master’s thesis at UWaterloo.
It is known that any graph $G$ with a planar drawing of height $H$ has $\text{pw}(G) \leq H$ [11], where $\text{pw}(G)$ is the so-called pathwidth of $G$. This makes the pathwidth a useful parameter for approximating the height of a planar graph drawing. For a tree $T$, Suderman gave an algorithm to draw $T$ with height at most $\lceil \frac{3}{2} \text{pw}(T) \rceil$ [19], making this an asymptotic $\frac{3}{2}$-approximation algorithm. It was discovered later that optimum-height drawings can be found efficiently for trees [17]. Approximation-algorithms for the height or width of order-preserving and/or upward tree drawing have also been investigated [1, 2, 8].

For outer-planar graphs, the first author gave two results that will be improved upon in this paper. In particular, every maximal outerplanar graph has a drawing of height at most $3 \log n - 1$ [3] and of height $4 \text{pw}(G) - 3$ [5]. Note that the second result gives a 4-approximation on the height of drawing outerplanar graphs, and improving this “4” is the main objective of the current paper. A number of results for drawing outer-planar graphs have been developed since paper [3]. In particular, any outerplanar graph with maximum degree $\Delta$ admits a planar straight-line drawing with area $O(\Delta n^{1.48})$ [14], or with area $O(\Delta n \log n)$ [12]. The former bound was improved to $O(n^{1.48})$ area [9]. Also, every so-called balanced outer-planar graph can be drawn in an $O(\sqrt{n}) \times O(\sqrt{n})$-grid [9].

In this paper, we present a 2-approximation algorithm for the height of planar drawings of maximal outer-planar graphs. The key ingredient is to define the so-called umbrella depth $\text{ud}(G)$ in Section 3. In Section 4, we show that any outerplanar graph $G$ has a planar drawing of height at most $2 \text{ud}(G) + 1$. (We actually show a height of $2 \text{bd}(G) + 1$, where the bonnet depth $\text{bd}(G)$ is another newly defined graph parameter.) This algorithm is a relatively minor modification of the one in [5], albeit described differently. The bulk of the work for proving a better approximation factor hence lies in proving a better lower bound, which we do in Section 5. Any maximal outerplanar graph $G$ with a planar drawing of height $H$ has $\text{ud}(G) \leq H - 1$. This proves that our result is a 2-approximation for the optimal height, which must fall in the range $[\text{ud}(G) + 1, 2\text{ud}(G) + 1]$.

2 Preliminaries

Throughout this paper, we assume that $G = (V, E)$ is a simple graph with $n \geq 3$ vertices and $m$ edges that is maximal outer-planar. Thus, $G$ has a standard planar embedding in which all vertices are in the outer face (the infinite connected region outside the drawing) and form an $n$-cycle, and all interior faces are triangles. We call an edge $(u, v)$ of $G$ a cutting edge if $G - \{u, v\}$ is disconnected, and a non-cutting edge otherwise. In an outer-planar graph, any cutting edge $(u, v)$ has exactly two cut-components, i.e., there are two maximal outerplanar subgraphs $G_1, G_2$ of $G$ such that $G_1 \cap G_2 = \{u, v\}$ and $G_1 \cup G_2 = G$.

The dual tree $T$ of $G$ is the weak dual graph of $G$ in the standard embedding, i.e., $T$ has a vertex for each interior face of $G$, and an edge between two vertices if their corresponding faces in $G$ share an edge. An outerplanar path $P$ is a maximal outerplanar graph whose dual

\[1\] The cutting edges are exactly those edges for which in the standard embedding both incident faces are interior, but we prefer to phrase this and the following definitions independent of the standard embedding since we do not necessarily draw the graph in the standard embedding.
tree is a path. We say that $P$ connects edges $e$ and $e'$ if $e$ is incident to $f_1$ and $e'$ is incident to $f_k$, where $f_1$ and $f_k$ are the first and last face in the path that is the dual tree of $P$. An outerplanar path $P$ with $n = 3$ is a triangle and connects any pair of its edges. Since any two interior faces are connected by a path in the dual tree, there exists an outerplanar path connecting $e$ and $e'$ for any two edges $e, e'$.

**Graph drawing:** A drawing of a graph consists of a point or an axis-aligned box for every vertex, and a polygonal curve for every edge. We only consider planar drawings where none of the points, boxes, or curves intersect unless the corresponding elements do in the original graph. In this paper, a planar drawing is not required to reflect a graph’s given planar embedding. In a flat visibility representations vertices are represented by horizontal line segments, and edges are vertical or horizontal straight-line segments. (For ease of reading, draw vertices as boxes of small height in our illustrations.) In a poly-line drawing vertices are points and edges are polygonal curves, while in a straight-line drawing vertices are points and edges are line segments. In this paper, we only study planar flat visibility representations, but simply speak of a planar drawing, because it is known that any planar flat visibility representation can be converted into a planar straight-line drawing of the same height and vice versa [6].

We require that all defining features (points, endpoints of segments, bends) are placed at points with integer $y$-coordinates. A layer (or row) is a horizontal line with integer $y$-coordinate that intersects elements of the drawing, and the height is the number of layers. We do not enforce integer $x$-coordinates since we do not focus on minimizing the width. We can always achieve $O(n)$ width (without adding height) for visibility representations and for the poly-line drawings obtained from them [6].
3 Umbrellas, bonnets and systems thereof

In this section, we introduce a method of splitting maximal outerplanar graphs into systems of special outerplanar graphs called umbrellas and bonnets.

**Definition 1.** Let $G$ be a maximal outer-planar graph, let $U$ be a subgraph of $G$ with $n \geq 3$, and let $(u,v)$ be a non-cutting edge of $G$. We say that $U$ is an umbrella with cap $(u,v)$ if it can be written as the union of three outerplanar paths $P$, $F_1$, and $F_2$ such that:

1. $P$ (the handle) connects $(u,v)$ to some other non-cutting edge of $G$,
2. $F_1$ (the fan at $u$) contains only $u$ and neighbours of $u$. $F_2$ (the fan at $v$) contains only $v$ and neighbours of $v$.
3. $F_1$ and $F_2$ are edge-disjoint. $F_1$ and $P$ have exactly one edge (incident to $u$) in common; $F_2$ and $P$ have exactly one edge (incident to $v$) in common.
4. All neighbours of $u$ and $v$ belong to $U$.

See also Figure 2(a). We allow the fans to be empty, but $P$ must have at least one interior face (the one incident to $(u,v)$). Any edge $(a,b)$ of $U$ that is a cutting edge of $G$, but not of $U$, is called an anchor-edge of $U$ in $G$. (In the standard embedding, such edges are on the outer-face of $U$ but not on the outer-face of $G$.) The hanging subgraph with respect to anchor-edge $(a,b)$ of $U$ in $G$ is the cut-component $S_{a,b}$ of $G$ with respect to cutting-edge $(a,b)$ that does not contain the cap $(u,v)$ of $U$. We often omit “of $U$ in $G$” when umbrella and super-graph are clear from the context.

**Definition 2.** Let $G$ be a maximal outerplanar graph with $n \geq 3$, and let $(u,v)$ be a non-cutting edge of $G$. An umbrella system $U$ on $G$ with root-edge $(u,v)$ is a collection $U = \{U_0\} \cup U_1 \cup \cdots \cup U_k$ of subgraphs of $G$ for some $k \geq 0$ that satisfy the following:

1. $U_0$ (the root umbrella) is an umbrella with cap $(u,v)$,
2. $U_0$ has $k$ anchor-edges. We denote them by $(u_i,v_i)$ for $i = 1, \ldots, k$, and let $S_i$ be the hanging subgraph with respect to $(u_i,v_i)$.
3. For $i = 1, \ldots, k$, $U_i$ (the hanging umbrella system) is an umbrella system of $S_i$ with root-edge $(u_i,v_i)$.

The depth of such an umbrella system is $d(U) := 1 + \max_i d(U_i)$.

Define $\ud(G;u,v)$ (the (rooted) umbrella depth of $G$) to be the maximum depth over all umbrella systems with root-edge $(u,v)$. Note that the umbrella depth depends on the choice of the root-edge; define the free umbrella depth $\udfree(G)$ to be the minimum umbrella depth over all possible root-edges. (One can show that the free umbrella depth is at most one unit less than the rooted umbrella depth for any choice of root-edge; see the appendix.)

**Bonnets:** A bonnet is a generalization of an umbrella that allows two handles, as long as they go to different sides of the face $(u,v)$. Thus, condition (1) of the definition of an umbrella gets replaced by

4
Figure 2: (a) An umbrella system of depth 3. The root umbrella is shaded, with its handle darker shaded. (b) The same graph has a bonnet system of depth 2, with the root bonnet shaded and its ribbon darker shaded.

\[(1') P \text{ (the ribbon)} \text{ connects two non-cutting edges and contains } u, v \text{ and their common neighbour.}\]

Other than that, bonnets are defined exactly like umbrellas. See also Figure 2(b). We can also define a bonnet system, root bonnet, etc., exactly as for an umbrella system, except that “bonnet” is substituted for “umbrella” everywhere. Let \(bd(G; u, v)\) (the rooted bonnet-depth of \(G\)) be the minimum possible depth of a bonnet system with root-edge \((u, v)\), and let \(bd^{\text{free}}(G)\) be the minimum bonnet-depth over all choices of root-edge. Since any umbrella is a bonnet, clearly \(bd(G; u, v) \leq ud(G; u, v)\) for all root-edges \((u, v)\).

We would like to emphasize that the root bonnet \(U_0\) of a bonnet system must contain all edges incident to the ends \(u, v\) of the root-edge. If follows that no edge incident to \(u\) or \(v\) can be an anchor-edge of \(U_0\), else the hanging subgraph at it would contain further neighbours of \(u\) (resp. \(v\)). We note this trivial but useful fact for future reference:

**Observation 1.** In a bonnet system with root-edge \((u, v)\), no edge incident to \(u\) or \(v\) is an anchor-edge of the root bonnet.

### 4 From Bonnet System to Drawing

In this chapter, we show how to create a flat visibility representation, given a maximal outerplanar graph \(G\) and a bonnet system of \(G\). The drawings we create will not be in the standard embedding of \(G\), as we will place drawings of hanging subgraphs inside an inner face of the root bonnet. For merging purposes, we draw the root-edge \((u, v)\) in a special way: It spans the top layer, which means that \(u\) touches the top left corner of the drawing, and \(v\) touches the top right corner, or vice versa (see for example Figure 3(b)). We first explain how to draw the root bonnet.

**Lemma 1.** Let \(U_0\) be the root bonnet of a bonnet system with root-edge \((u, v)\). Then there exists a flat visibility representation \(\Gamma\) of \(U_0\) on three layers such that...
1. \((u, v)\) spans the top layer of \(\Gamma\).

2. Any anchor-edge of \(U_0\) is drawn horizontally in the middle or bottom layer.

Proof. As a first step, we draw the ribbon of \(U_0\) on 2 layers in such a way that \((u, v)\) and all anchor-edges are drawn horizontally; see Figure 3(a) for an illustration. (This part is identical to [5].) Consider the standard embedding of \(P\) in which the dual tree is a path, say it consists of faces \(f_1, \ldots, f_k\). We draw \(k + 1\) vertical edges between two layers, with the goal that the region between two consecutive ones belong to \(f_1, \ldots, f_k\) in this order. Place \(u\) and \(v\) as segments in the top layer, and with an \(x\)-range such that they touch all the regions of faces that \(u\) and \(v\) are incident to. Similarly create segments for all other vertices. The placement for the vertices is uniquely determined by the standard planar embedding, except for the vertices incident to \(f_1\) and \(f_k\). We place those vertices such that the leftmost/rightmost vertical edge is not an anchor-edge. To see that this is possible, recall that \(P\) connects two non-cutting edges \(e_1, e_2\) of \(G\) that are incident to \(f_1\) and \(f_k\). If \(e_1 \neq (u, v)\), then choose the layer for the vertices of \(f_1\) such that \(e_1\) is drawn vertical. If \(e_1 = (u, v)\), then one of its ends (say \(u\)) is the degree-2 vertex on \(f_1\) and drawn in the top-left corner. The other edge \(e'\) incident to \(u\) is not an anchor-edge of \(U\) by Observation [1] and we draw \(e'\) vertically. So the leftmost vertical edge is either a non-cutting edge (hence not an anchor-edge) or edge \(e'\) (which is not an anchor-edge). We proceed similarly at \(f_k\) so that the rightmost vertical edge is not an anchor-edge. Finally all other vertical edges are cutting edges of \(U_0\) and hence not anchor-edges.

The drawing of \(P\) obtained in this first step has \((u, v)\) in the top layer. As a second step, we now release \((u, v)\) as in [5]. This operation can be applied to any edge that is drawn horizontally in the top layer of a flat visibility representation. It consists of adding a layer above the drawing, moving \((u, v)\) into it, and re-routing edges by expanding vertical ones at \(u\) and \(v\), and turning horizontal ones into vertical ones. In the result, \((u, v)\) spans the top layer. See Figure 3(b) for an illustration and [5] for details.

We now have a drawing of the ribbon \(P\) on 3 layers where \((u, v)\) spans the top, say \(u\) is in the top left corner. As the third and final step, we add the remaining vertices of the two fans. Consider the fan \(F_1\) at \(u\), and let \((u, a_\ell)\) be the edge that it has in common with the ribbon \(P\). We have two possible cases. If \((u, a_\ell)\) is the leftmost vertical edge, then simply place the vertices of the fan to the left of \(a_\ell\) and extend \(u\) (see Figure 3(c)). Else, \((u, a_\ell)\) was drawn horizontally in the drawing of the first step (because it reflects the planar embedding), and therefore after releasing \((u, v)\) there is space to the right of \(a_\ell\). Into this space we insert the remaining vertices of the fan at \(u\). The fan at \(v\) is added in a symmetric fashion. Figure 3(c) illustrates the second case for the fan at \(v\).

This finishes the construction of the drawing of \(U_0\). It remains to show that all anchor-edges are horizontal and in the bottom two layers. We ensured that this is the case in the first step. Releasing \((u, v)\) adds more vertical edges, but all of them are incident to \(u\) or \(v\) and not anchor-edges by Observation [1]. Likewise, all vertical edges added when inserting the fans are incident to \(u\) or \(v\). The only horizontal edge in the top layer is \((u, v)\), which is not an anchor-edge. This finished the proof of Lemma [1].
Now we explain how to merge hanging subgraphs.

**Theorem 1.** Any maximal outerplanar graph $G$ has a planar flat visibility representation of height at most $2bd_{\text{free}}(G) + 1$.

**Proof.** We show by induction that any graph with a bonnet system $U$ of depth $H$ has a drawing $\Gamma$ of height $2H + 1$ where the root-edge $(u, v)$ spans the top layer. This proves the theorem when applying it to a bonnet system $U$ of depth $bd_{\text{free}}(G)$.

Let $U_0$ be the root bonnet of the bonnet system, and draw $U_0$ on 3 layers using Lemma 1. Thus $(u, v)$ spans the top and any anchor-edge $(a, b)$ of $U_0$ is drawn as a horizontal edge in the bottom two layers of $\Gamma_0$. If $H = 1$ then we are done. Else add $2H - 2$ layers to $\Gamma_0$ between the middle and bottom layers. For each anchor-edge $(a, b)$ of $U_0$, the hanging subgraph $S_{a,b}$ of $U_0$ has a bonnet system of depth at most $H - 1$ with root-edge $(a, b)$. By induction $S_{a,b}$ has a drawing $\Gamma_1$ on at most $2H - 1$ layers with $(a, b)$ spanning the top layer.

If $(a, b)$ is in the bottom layer of $\Gamma_0$, then we can rotate (and reflect, if necessary) $\Gamma_1$ so that $(a, b)$ is in the bottom layer of $\Gamma_1$ and the left-to-right order of $a$ and $b$ in $\Gamma_1$ is the same as their left-to-right order in $\Gamma_0$. This updated drawing of $\Gamma_1$ can then be inserted in the space between $(a, b)$ in $\Gamma_0$. This fits because $\Gamma_1$ has height at most $2H - 1$, and in the insertion process we can re-use the layer spanned by $(a, b)$. If $(a, b)$ is in the middle layer of $U_0$, then we can reflect $\Gamma_1$ (if necessary) so that $(a, b)$ has the same left-to-right order in $\Gamma_1$ as in $\Gamma_0$. This updated drawing of $\Gamma_1$ can then be inserted in the space between $(a, b)$ in $\Gamma_0$. See Figure 3(d). Since we added $2H - 2$ layers to a drawing of height 3, the total height of the final drawing is $2H + 1$ as desired.

Our proof is algorithmic, and finds a drawing, given a bonnet system, in linear time. One
can also show (see the appendix) that the rooted bonnet depth, and an associated bonnet system, can be found in linear time using dynamic programming. Hence the run-time to find this drawing is linear.

Comparison to [5]: The algorithm in [5] has only two small difference. The main one is that it does not do the “third step” when drawing the root umbrella, thus it draws the ribbon but not the fans. Thus in the induction step our algorithm always draws at least as much as the one in [5]. Secondly, [5] uses a special construction if \(pw(G) = 1\) to save a constant number of levels. This could easily be done for our algorithm as well in the case where \(pw(G) = 1\) but \(bd(G) = 2\). As such, our construction never has worse height (and frequently it is better).

Comparison to [3]: One can argue that \(bd(G) \leq \log(n + 1)\) (see the appendix). Since [3] uses \(3 \log n - 1\) levels while ours uses \(2bd(G) + 1 \leq 2 \log(n + 1) + 1\) levels, the upper bound on the height is better for \(n \geq 9\).

5 From Drawing to Umbrella System

We now argue that any flat visibility representation of height \(H\) gives rise to an umbrella system of depth at most \(H - 1\), proving a lower bound. We first briefly sketch the idea. We assume that we have a drawing such that for some non-cutting edge \((u, v)\) we have an “escape path”, i.e., a poly-line to the outerface that does not intersect the drawing. Now find an outerplanar path that connects the leftmost vertical edge \((x, y)\) of the drawing with \((u, v)\). This becomes the handle of an umbrella \(U\) with cap \((u, v)\), and the fans consist of all remaining neighbours of \(u\) and \(v\). One can now argue that any hanging subgraph of \(U\) is drawn with height at most \(H - 1\), and furthermore, has an escape path from its anchor-edge. The claim then holds by induction.

We first must clarify some definitions. Let \(\Gamma\) be a flat visibility representation, and let \(B_\Gamma\) be a minimum-height bounding box of \(\Gamma\). A vertex \(v \in G\) has a left escape path in \(\Gamma\) if there exists a polyline inside \(B_\Gamma\) from \(v\) to a point on the left side of \(B_\Gamma\) that is vertex-disjoint from \(\Gamma\) except at \(v\), and for which all bends are on layers. We say that \((\ell_1, \ell_2)\) is a left-free edge of \(\Gamma\) if it is vertical, and any layer intersected by \((\ell_1, \ell_2)\) is empty to the left of \((\ell_1, \ell_2)\). In particular, both \(\ell_1\) and \(\ell_2\) have a left escape path by going leftwards in their respective layers. Define right escape paths and right-free edges symmetrically; we use escape path for either a left escape path or a right escape path. See Figure 4(a).

It is easy to see that any flat visibility representation has a left-free edge, presuming the graph has minimum degree at least 2. Let \((v, w)\) be the leftmost vertical edge (breaking ties arbitrarily); there must be such an edge because the leftmost vertex in each layer has at most one horizontal incident edge, and therefore at least one vertical one. In any layer spanned by \((v, w)\), no vertical edge is farther left by assumption. No vertex can be farther left either, else the incident vertical edge of the leftmost of them would be farther left. So \((v, w)\) is left-free.

For the proof of the lower bound, we use as handle an outerplanar path connecting to a left-free edge. Recall that the definition of handle requires that it connects to a non-cutting
edge, so we need a left-free edge that is not a cutting edge. This does not exist in all drawings (see e.g. Figure 4(a)), but as we show now, we can modify the drawing without increasing height such that such an edge exists. To be able to apply it later, we must also show that this modification does not destroy a given escape path.

**Lemma 2.** Let $\Gamma$ be a flat visibility representation of a maximal outerplanar graph $G$.

1. Let $(r_1, r_2)$ be a right-free edge of $\Gamma$, and let $w$ be a vertex that has a right escape path. Then there exists a drawing $\Gamma'$ in which $w$ has a right escape path, $(r_1, r_2)$ is a right-free edge, and there exists a left-free edge that is not a cutting edge of $G$.

2. Let $(\ell_1, \ell_2)$ be a left-free edge of $\Gamma$, and let $w$ be a vertex that has a left escape path. Then there exists a drawing $\Gamma'$ in which $w$ has a left escape path, $(\ell_1, \ell_2)$ is a left-free edge, and there exists a right-free edge that is not a cutting edge of $G$.

In either case, the $y$-coordinates of all vertices in $\Gamma$ are unchanged in $\Gamma'$, and in particular both drawings have the same height.

**Proof.** We prove the claim by induction on $n$ and show only the first claim (the other is symmetric). Let $(\ell_1, \ell_2)$ be the leftmost vertical edge of $\Gamma$; this is left-free as argued above. If $(\ell_1, \ell_2)$ is not a cutting edge of $G$, then we are done with $\Gamma' = \Gamma$. In particular, this is the case if $n = 3$ when $G$ is a triangle and has no cutting edge.

So assume $n \geq 4$ and $(\ell_1, \ell_2)$ is a cutting edge of $G$. Let $A$ and $B$ be the cut-components of $(\ell_1, \ell_2)$, named such that $w \in A$. Let $\Gamma_A$ [resp. $\Gamma_B$] be the drawing of $A$ [resp. $B$] induced by $\Gamma$. Edge $(\ell_1, \ell_2)$ is left-free for both $\Gamma_A$ and $\Gamma_B$. Reflect $\Gamma_B$ horizontally (this makes $(\ell_1, \ell_2)$ right-free) to obtain $\Gamma'_B$. By induction, we can create a drawing $\Gamma''_B$ from $\Gamma'_B$ in which $(\ell_1, \ell_2)$ is right-free and there is a left-free edge $(\ell'_1, \ell'_2)$ that is not a cutting edge of $B$. We have $(\ell'_1, \ell'_2) \neq (\ell_1, \ell_2)$, because the common neighbour of $\ell_1, \ell_2$ in $B$ forces a vertex or edge to reside to the left of the right-free edge $(\ell_1, \ell_2)$. So $(\ell'_1, \ell'_2)$ is not a cutting edge of $G$ either.

Create a new drawing that places $\Gamma''_B$ to the left of $\Gamma_A$ and extends $\ell_1$ and $\ell_2$ to join the two copies; this is possible since $(\ell_1, \ell_2)$ has the same $y$-coordinates in $\Gamma_A, \Gamma_B$ and $\Gamma''_B$, and it is left-free in $\Gamma_A$ and right-free in $\Gamma''_B$. Also delete one copy of $(\ell_1, \ell_2)$. See Figure 4(b).
The drawing \( \Gamma_A \) is unchanged, so \( w \) will have the same right escape path in \( \Gamma' \) as in \( \Gamma \), and \( \Gamma' \) will have right-free edge \( (r_1, r_2) \) and left-free non-cutting edge \( (\ell'_1, \ell'_2) \), as desired.

We are now ready to prove the lower bound if there is an escape path.

**Lemma 3.** Let \( \Gamma \) be a flat visibility representation of a maximal outerplanar graph \( G \) with height \( H \), and let \((u, v)\) be a non-cutting edge of \( G \). If there exists an escape path from \( u \) or \( v \) in \( \Gamma \), then \( G \) has an umbrella system with root-edge \((u, v)\) and depth at most \( H - 1 \).

**Proof.** We proceed by induction on \( H \). Assume without loss of generality that there exists a right escape path from \( v \) (all other cases are symmetric). Using Lemma 2, we can modify \( \Gamma \) without increasing the height so that \( v \) has a right escape path, and there is a left-free edge \((\ell_1, \ell_2)\) in \( \Gamma \) that is not a cutting edge of \( G \). Let \( P \) be the outerplanar path that connects edge \((\ell_1, \ell_2)\) and \((u, v)\). Let \( U_0 \) be the union of \( P \), the neighbors of \( u \), and the neighbors of \( v \); we use \( U_0 \) as the root umbrella of an umbrella system.

We now must argue that all hanging subgraphs of \( U_0 \) are drawn with height at most \( H - 1 \) and have escape paths from their anchor-edges; we can then find umbrella systems for them by induction and combining them with \( U_0 \) gives the umbrella system for \( G \) as desired.

To prove the height-bound, we define “dividing paths” as follows. The outer-face of \( U_0 \) in the standard embedding contains \((\ell_1, \ell_2)\) (since it is not a cutting edge) as well as \( v \). Let \( P_1 \) and \( P_2 \) be the two paths from \( \ell_1 \) and \( \ell_2 \) to \( v \) along this outer-face in the standard embedding. Define the dividing path \( \Pi_i \) (for \( i = 1, 2 \)) to be the poly-line in \( \Gamma \) that consist of the left escape path from \( \ell_i \), then the drawing of the path \( P_i \) (i.e., the vertical segments of its edges and parts of the horizontal segments of its vertices), and then the right escape path from \( v \). See Figure 5.

![Figure 5: Extracting dividing paths from a flat visibility representation. \( P_1/\Pi_1 \) is dotted while \( P_2/\Pi_2 \) is dashed.](image)

Now consider any hanging subgraph \( S_{a,b} \) of \( U_0 \) with anchor-edge \((a, b)\). No edge incident to \( v \) is an anchor-edge, and neither is \((\ell_1, \ell_2)\), since it is not a cutting edge. So \((a, b)\) is an edge of \( P_1 \) or \( P_2 \) (say \( P_1 \)) that is not incident to \( v \). Therefore \((a, b)\) (and with it \( S_{a,b} \)) is vertex-disjoint from \( P_2 \). In consequence, the drawing \( \Gamma_S \) of \( S_{a,b} \) induced by \( \Gamma \) is disjoint from the dividing path \( \Pi_2 \). Since \( \Pi_2 \) connects a point on the left boundary with a point on the right boundary, therefore \( \Gamma_S \) must be entirely above or entirely below \( \Pi_2 \), say it is above. Since \( \Pi_2 \) has all bends at points with integral \( y \)-coordinate, therefore the bottom layer of \( \Gamma \) is not available for \( \Gamma_S \). In consequence \( \Gamma_S \) has height at most \( H - 1 \) as desired.
Recall that \((a, b)\) belongs to \(P_1\) and is not incident to \(v\). After possible renaming of \(a\) and \(b\), we may assume that \(b\) is closer to \(\ell_1\) along \(P_1\) than \(a\). Then the sub-path of \(P_1\) from \(b\) to \(\ell_1\) is interior-disjoint from \(S_{a,b}\). This path, together with the left escape path from \(\ell_1\), is a left escape path from \(b\) that resides within the top \(H − 1\) layers, because it does not contain \(v\) and hence is disjoint from \(\Pi_2\). We can hence apply induction to \(S_{a,b}\) to obtain an umbrella system of depth at most \(H − 2\) with root-edge \((a, b)\). Repeating this for all hanging subgraphs, and combining the resulting umbrella systems with \(U_0\), gives the result.

\[\text{Theorem 2. Let } G \text{ be a maximal outerplanar graph. If } G \text{ has a flat visibility representation } \Gamma \text{ of height } H, \text{ then } ud^{free}(G) \leq H − 1.\]

\[\text{Proof. Using Lemma 2, we can convert } \Gamma \text{ into a drawing } \Gamma' \text{ of the same height in which some edge } (u, v) \text{ is a right-free non-cutting edge. This implies that there is a right escape path from } v, \text{ and by Lemma 3 we can find an umbrella system of } G \text{ with root-edge } (u, v) \text{ and depth } H − 1. \text{ So } ud^{free} \leq ud(G; u, r) \leq H − 1.\]

6 Conclusions and Future Work

We presented an algorithm for drawing maximal outerplanar graphs that is a 2-approximation for the optimal height. To this end, we introduced the umbrella depth as a new graph parameter for maximal outerplanar graphs, and used as key result that any drawing of height \(H \) implies an umbrella-depth of at least \(H − 1\). Our result significantly improves the previous best result, which was based on the pathwidth and gave a 4-approximation. We close with some open problems:

- Our result only holds for maximal outerplanar graphs. Can the algorithm be modified so that it works for all outerplanar graphs? Specifically, can we make an outerplanar graph maximal in such a way that the umbrella depth does not increase (much)?

- The algorithm from Section 4 creates a drawing that does not place all vertices on the outer face. Can we create an algorithm that minimizes or approximates the optimal height when the standard planar embedding must be respected?

- What is the width achieved by the algorithm from Section 4? Any visibility representation can be modified without changing the height so that the width is at most \(m + n\), where \(m\) is the number of edges and \(n\) is the number of vertices. Thus the width is \(O(n)\), but what is the constant?

- Is it possible to determine the optimal height for maximal outerplanar graphs in polynomial time?

Finally, are there approximation algorithms for the height or the area of drawings for other, more general planar graph classes?
References

[1] Md. J. Alam, Md. A.H. Samee, M. Rabbi, and Md. S. Rahman. Minimum-layer upward drawings of trees. *J. Graph Algorithms Appl.*, 14(2):245–267, 2010.

[2] J. Batzill and T. Biedl. Order-preserving drawings of trees with approximately optimal height (and small width), 2016. CoRR 1606.02233 [cs.CG]. In submission.

[3] T. Biedl. Drawing outer-planar graphs in $O(n \log n)$ area. In *Graph Drawing (GD’01)*, volume 2528 of LNCS, pages 54–65. Springer-Verlag, 2002. Full version appeared in [4].

[4] T. Biedl. Small drawings of outerplanar graphs, series-parallel graphs, and other planar graphs. *Discrete and Computational Geometry*, 45(1):141–160, 2011.

[5] T. Biedl. A 4-approximation algorithm for the height of drawing 2-connected outerplanar graph. In *Workshop on Approximation and Online Algorithms (WAOA ’12)*, volume 7846 of LNCS, pages 272–285. Springer-Verlag, 2013.

[6] T. Biedl. Height-preserving transformations of planar graph drawings. In *Graph Drawing (GD’14)*, volume 8871 of LNCS, pages 380–391. Springer, 2014.

[7] T. Biedl. On area-optimal planar grid-drawings. In *International Colloquium on Automata, Languages and Programming (ICALP ’14)*, volume 8572 of LNCS, pages 198–210. Springer-Verlag, 2014.

[8] T. Biedl. Ideal tree-drawings of approximately optimal width (and small height), 2015. CoRR 1502.02753 [cs.CG]. Conditionally accepted in JGAA.

[9] G. Di Battista and F. Frati. Small area drawings of outerplanar graphs. *Algorithmica*, 54(1):25–53, 2009.

[10] V. Dujmovic, M. Fellows, M. Kitching, G. Liotta, C. McCartin, N. Nishimura, P. Ragde, F. Rosamond, S. Whitesides, and D. Wood. On the parameterized complexity of layered graph drawing. *Algorithmica*, 52:267–292, 2008.

[11] S. Felsner, G. Liotta, and S. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. *J. Graph Alg. Appl.*, 7(4):335–362, 2003.

[12] F. Frati. Straight-line drawings of outerplanar graphs in $O(dn \log n)$ area. *Comput. Geom.*, 45(9):524–533, 2012.

[13] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10:41–51, 1990.

[14] A. Garg and A. Rusu. Area-efficient planar straight-line drawings of outerplanar graphs. *Discrete Applied Mathematics*, 155(9):1116–1140, 2007.
[15] L.S. Heath and A.L. Rosenberg. Laying out graphs using queues. *SIAM Journal on Computing*, 21(5):927–958, 1992.

[16] M. Krug and D. Wagner. Minimizing the area for planar straight-line grid drawings. In *Graph Drawing (GD’07)*, volume 4875 of *LNCS*, pages 207–212. Springer-Verlag, 2007.

[17] D. Mondal, Md. J. Alam, and Md. S. Rahman. Minimum-layer drawings of trees. In *Algorithms and Computations (WALCOM 2011)*, volume 6552 of *LNCS*, pages 221–232. Springer, 2011.

[18] W. Schnyder. Embedding planar graphs on the grid. In *ACM-SIAM Symposium on Discrete Algorithms (SODA ’90)*, pages 138–148, 1990.

[19] M. Suderman. Pathwidth and layered drawings of trees. *Intl. J. Comp. Geom. Appl*, 14(3):203–225, 2004.
A Computing the Depth

We now introduce a dynamic programming algorithm for finding the rooted bonnet depth of a maximal outerplanar graph $G$, relative to a given fixed root-edge $(u, v)$. (A very similar algorithm finds the umbrella depth; we leave those details to the reader.)

As before, for any cutting edge $(a, b)$ of $G$ let $S_{a,b}$ be the cut-component of $G$ that does not contain $(u, v)$. It will also be convenient to define $S_{a,b} := \{(a, b)\}$ if $(a, b)$ is a non-cutting edge with $(a, b) \neq (u, v)$, and to define $S_{u,v} := G$. We also use the notation $ud(a, b) := ud(S_{a,b}; a, b)$ and define $ud(a, b) = 0$ if $S_{a,b}$ is a single edge.

We first sketch the overall idea. Consider one subgraph $S_{a,b}$ and the root bonnet $U_0$ used in the bonnet system of minimum depth. The ribbon of $U_0$ is an outerplanar path connecting edges $e_1$ and $e_2$, but we can view it as two outerplanar paths, connecting the root-edge $(a, b)$ to one of $e_1$ and $e_2$. If $c$ is the common neighbour of $a$ and $b$, then we can hence split the bonnet into two parts by removing the face $\{a, b, c\}$. Each part looks much like an umbrella, except that they are rooted at $(a, c)$ and $(c, b)$, respectively, and each of them has no fan at $c$. See also Figure 6(a). Therefore the minimum bonnet depth can be found by finding the minimum-depth in the two subgraphs $S_{a,c}$ and $S_{b,c}$, under the restriction that the root bonnets in the corresponding systems are such partial umbrellas. We must repeat the argument for the partial umbrellas, breaking them into a fan and a handle, and hence end up computing 6 different types of depths for each anchor edge, but 6 types are enough and hence the overall run-time is linear. We now define these variants of bonnet-depth for $S_{a,b}$:

- A **handle** is an outerplanar path that connects $(a, b)$ to some non-cutting edge. See Figure 6(b). Define
  \[ bd^h(a, b) := 1 + \min_{U^h} \left\{ \max_{(x,y)} bd(x, y) \right\} \]
  where $U^h$ is a handle and $(x, y)$ is a non-cutting edge of $U^h$.

- The **full fan at $a$** [resp. full fan at $b$] consists of $a$ [resp. $b$] and all neighbours of $a$ [resp. $b$]. See Figure 6(c). Define
  \[ bd^{f_a}(a, b) := 1 + \max_{(x,y)} bd(x, y) \]
  where $(x, y)$ is a non-cutting edge of the full fan at $a$. Symmetrically define $bd^{f_b}(a, b)$ using the full fan at $b$.

- A **partial-$a$ umbrella** [resp. partial-$b$ umbrella] is an umbrella with cap $(a, b)$ that contains all neighbours of $a$ [resp. $b$] and in which the fan at $b$ [resp. $a$] is empty. See Figure 6(d). Define
  \[ bd^{p_a}(a, b) := 1 + \min_{U^{p_a}} \left\{ \max_{(x,y)} bd(x, y) \right\} \]
  where $U^{p_a}$ is a partial-$a$ umbrella and $(x, y)$ is a non-cutting edge of $U^{p_a}$. Symmetrically define $bd^{p_b}(a, b)$ using a partial-$b$ umbrella.
(a) Bonnet.

(b) Handle.

(c) Full fan.

(d) Partial umbrella.

Figure 6: For the recursive formulas. Edges that may be anchor-edges are dashed.

Lemma 4. We have $ud^t(a, b) = 0$ if $S_{a,b}$ is a single edge, where $t \in \{\emptyset, h, f_a, f_b, p_a, p_b\}$. Else, letting $c$ be the common neighbour of $a$ and $b$, we have

1. $bd(a, b) = \max \{bd^{p_a}(a, c), bd^{p_b}(c, b)\}$
2. $bd^h(a, b) = \min \left[ \max \{bd^h(a, c), 1 + bd(c, b)\}, \max \{1 + bd(a, c), bd^h(c, b)\} \right]$
3. $bd^{f_a}(a, b) = \max \{bd^{f_a}(a, c), 1 + bd(c, b)\}$
4. $bd^{f_b}(a, b) = \max \{1 + bd(a, c), bd^{f_b}(c, b)\}$
5. $bd^{p_a}(a, b) = \min \left[ \max \{bd^{p_a}(a, c), 1 + bd(c, b)\}, \max \{bd^{f_a}(a, c), bd^h(c, b)\} \right]$ 
6. $bd^{p_b}(a, b) = \min \left[ \max \{1 + bd(a, c), bd^{p_b}(c, b)\}, \max \{bd^h(a, c), bd^{f_b}(c, b)\} \right]$

Proof. All these formulas are proved in a similar way: Consider the root bonnet $U_0$ of the bonnet system that achieves the depth on the left side. When splitting this bonnet into two by removing the face $\{a, b, c\}$, we obtain two bonnets for the two subgraphs $S_{a,c}$ and $S_{c,b}$ and can argue what type they are. (Sometimes there may be two possibilities, depending on which direction the handle of $U_0$ went, in case of which the one that yields the smaller depth is used.) This proves “$\geq$”, and equality is easily shown by putting together bonnet systems of $S_{a,c}$ and $S_{c,b}$ of the appropriate type.

We demonstrate this in detail for (5); see also Figure 6(d). So assume we have an umbrella system $\mathcal{U}$ with root-edge $(a, b)$ that has depth $bd^{p_a}(a, b)$ and where the root-bonnet $U_0$ is a partial-$a$ umbrella. So $U_0$ consists of a handle $P$ and a fan at $a$ that includes all neighbours of $a$ not in $P$. Since $P$ is a handle, it connects to $(a, b)$, therefore not both edges $(a, c)$ and $(c, b)$ can be cutting edges of $P$. 

15
We distinguish cases. In the first case, \((c, b)\) is not a cutting edge of \(P\). Since the fan at \(b\) is empty in \(U_0\), \((c, b)\) is not a cutting edge of \(U_0\) either. Thus the formula for \(bd^{u_0}(a, b)\) includes the term \(1 + ud(c, b)\) in the maximum. Furthermore, \(U_0 - \{b\}\) is a partial-\(a\) umbrella for \(S_{a,c}\), and this, together with the subsystems of \(U\) for its hanging subgraphs, gives a bonnet system with depth at least \(bd^{u_0}(a, c)\). Therefore \(bd^{u_0}(a, b) \geq \max\{bd^{u_0}(a, c), 1 + ud(c, b)\}\) in the first case.

In the second case, \((a, c)\) is not a cutting edge of \(P\). Therefore the fan of \(U_0\) at \(a\) consists of \(a\) and all neighbours of \(a\) except \(b\), and thus is the full fan \(F\) at \(a\) in the subgraph \(S_{a,c}\). The subsystems of \(U\) for hanging subgraphs of \(F\) give a bonnet system for \(S_{a,c}\) where the root bonnet is the full fan at \(a\), hence it has depth at least \(bd^{f_0}(a, c)\). Furthermore, \(P - \{a\}\) is a handle for \(S_{c,b}\). So \(U\) implies a bonnet system for \(S_{c,b}\) where the root is a handle, hence it has depth at least \(bd^h(c, b)\). Therefore \(bd^{p_0}(a, b) \geq \max\{bd^{f_0}(a, c), ud^h(c, b)\}\) in the second case. One of the two cases must apply, and so \(bd^{p_0}(a, b)\) is at least as big as the smaller of the two bounds and \(\geq\) holds.

To show \(\leq\), let us assume that the minimum is achieved at \(\max\{bd^{f_0}(a, c), ud^h(c, b)\}\) (the other case is similar). Find a bonnet system \(U^f\) of \(S_{a,c}\) of depth \(bd^{f_0}(a, c)\) where the root bonnet \(F\) is the full fan at \(a\), and a bonnet system \(U^h\) of \(S_{c,b}\) of depth \(ud^h(c, b)\) where the root bonnet \(U^h\) is a handle. Then \(F \cup U^h \cup \{(a, b)\}\) is a partial-\(a\) umbrella \(U_0\), and combining it with the subsystems of \(U^f\) and \(U^h\) gives a bonnet system of \(S_{a,c}\) whose depth is \(\max\{bd^{f_0}(a, c), ud^h(c, b)\}\).

We can convert these formulas into a dynamic programming algorithm to compute the bonnet depth by using the standard bottom-up traversal in a tree. Given a maximal outerplanar graph \(G\), initialize \(ud^t(a, b) = 0\) for all types \(t\) and for all non-cutting-edges \((a, b)\) with the exception of \((u, v)\). Root the dual tree \(T\) at the face incident to root-edge \((u, v)\). Any node of \(T\) is associated with a cutting-edge of \(G\) by taking the dual of the arc that connects the node with its parent in \(T\). Traversing \(T\) bottom up, when we encounter a node \(f\) of \(T\) (hence a face of \(G\)) we have obtained the bonnet depth values for two out of the three edges incident to \(f\) already, and can compute the bonnet depth values for the third using the above formulas. This takes \(O(1)\) time since there are 6 values and each formula can be evaluated in constant time. Finally we evaluate at the root of \(T\), which gives \(bd(G; u, v)\). Since \(T\) has \(n - 3\) nodes, the total run-time is \(O(n)\).

**Theorem 3.** Given a non-cutting edge \((u, v)\), there exists an \(O(n)\) algorithm to find the rooted umbrella depth \(ud(G; u, v)\) of a maximal outerplanar graph \(G\) with \(n\) vertices.

### A.1 Free vs. Rooted Umbrella/Bonnet Depth

Note that our algorithm computes the rooted bonnet depth for \(G\), since the root-edge \((u, v)\) must be given. One way to instead find the free bonnet depth is to repeat the process described above for every choice of root-edge in \(G\). This would give an \(O(n^2)\) algorithm for finding the free bonnet depth. One could likely compute the free bonnet depth in \(O(n)\) time by initializing \(ud^t(a, b) = 0\) for all non-cutting edges, then updating at the face where the
resulting bonnet depth is minimized, and using as root-edge one near where we update last. However, as we will show now, the free bonnet depth is at most one less than the rooted bonnet depth, and therefore it does not seem worth the minor improvement to work out the details of this approach.

**Lemma 5.** Given a maximal outerplanar graph $G$, we have

$$\text{bd}_{\text{free}}(G) = \min_{(u,v)} \{\text{bd}(G, u, v)\} \leq \max_{(u,v)} \{\text{bd}(G, u, v)\} \leq \text{bd}_{\text{free}}(G) + 1$$

where the minimum and maximum are taken over all non-cutting edges $(u,v)$ of $G$.

**Proof.** The first equality holds per definition, and the second inequality is obvious, so we focus on the third inequality. Let $U^*$ be a bonnet system on $G$ with depth $H := \text{bd}_{\text{free}}(G)$ and let $(u^*, v^*)$ be its root-edge, which by definition is not a cutting edge. Let $(u,v)$ be an arbitrary non-cutting edge; it suffices to show $\text{bd}(G; u, v) \leq \text{bd}(G; u^*, v^*) + 1 = H + 1$. Let $P$ be the outer-planar path that connects $(u,v)$ and $(u^*, v^*)$ and define a bonnet $U$ with cap $(u,v)$ to consist of $P$ and the fans at $u$ and $v$ that include all neighbours of $u,v$ not in $P$. We claim that any hanging subgraph $S_{a,b}$ of $U$ with anchor-edge $(a,b)$ has rooted bonnet depth $\text{bd}(S_{a,b}; (a,b)) \leq H$. Observe that $(u^*, v^*)$ is not an edge of $S_{a,b}$, because $(u*,v*)$ is not a cutting edge, and $(u^*, v^*) \in U$ while $S_{a,b}$ is disjoint from $U$ except at cutting edge $(a,b)$. Therefore the cutting edge $(a,b)$ has the root-edge $(u^*, v^*)$ of $U^*$ in one component and $S_{a,b}$ in the other. One easily argues that therefore $\text{bd}(S_{a,b}; (a,b)) \leq d(U^*)$, because the bonnets of $U^*$ can be used to build a bonnet system $U_S$ of $S_{a,b}$ after trimming parts in $G - S_{a,b}$ and expanding each bonnet as to include all neighbours of the ends of its cap. Thus $S_{a,b}$ has an umbrella system $U_S$ with depth at most $d(U^*) = H$. Combining the umbrella systems of these hanging subgraphs with $U$ gives an umbrella system of $G$ with root-edge $(u,v)$ and depth at most $H + 1$ as desired.

---

**B Comparison with Other Graph Parameters**

In this section, we compare the bonnet depth and the umbrella depth to the pathwidth (the other graph parameter previously used for graph drawing purposes), as well as the so-called rooted pathwidth. We define these parameters first. Let $T$ be a tree. The **pathwidth** $\text{pw}(T)$ of $T$ is 0 if $T$ is a single node, and $1 + \min_P \max_{T' \subset T - P} \text{pw}(T')$ otherwise. Here $P$ is an arbitrary path in $T$, and $T'$ is any subtree that remains after deleting the vertices of $P$. Any path where the minimum is achieved is called a **main path** of $T$. See [19] for more details. The rooted pathwidth is quite similar, but forces the path to end at the root. Thus, let $T$ be a rooted tree. The **rooted pathwidth** $\text{rpw}(T)$ is 0 if $T$ is empty and $1 + \min_{P_r} \max_{T' \subset T - P_r} \text{rpw}(T')$ otherwise. Here $P_r$ is a path in $T$ that ends at the root of $T$ and $T'$ is any subtree that remains after deleting the vertices of $P_r$, where $T'$ is rooted as

---

\(^2U\) is actually an umbrella, and indeed the same chain of inequalities holds if we replace ‘bd’ by ‘ud’ everywhere.
induced by the root of $T$. Any path where the minimum is achieved is called an rpw-main path. See [8] for more details. We write $\text{rpw}(T, r)$ if the root $r$ is not clear from the context, and define $\text{rpw}^\text{free}(T) := \min_r \text{rpw}(T, r)$. It is not hard to see that $\min_r \text{rpw}(T, r)$ is attained at a leaf $r$, because for any interior node $r$ we could have used an even longer rpw-main path to reach a leaf without making any subtree bigger.

**Lemma 6.** Let $G$ be a maximal outerplanar graph $G$ with dual tree $T$. Then

$$\frac{1}{2} \text{pw}(T) \leq \text{bd}^\text{free}(G) \leq \text{ud}^\text{free}(G) \leq \text{rpw}^\text{free}(T) \leq \max\{1, 2\text{pw}(T)\}.$$  

**Proof.** We first show that $\text{pw}(T) \leq 2\text{bd}(G; u, v)$ for any choice of root-edge $(u, v)$. Fix any bonnet system that has depth $H := \text{bd}(G; u, v)$ and root-edge $(u, v)$, and let $U_0$ be its root-umbrella. Recall that $U_0$ is split into the ribbon $P$, which is an outer-planar path, and the two fans $F_1, F_2$, which are also outer-planar paths. Use the dual tree $P^*$ of $P$ as main path for $T$, and for $i = 1, 2$, use the dual tree $F_i^*$ of $F_i$ in the subtree $T_i$ of $T - P^*$ that contains $F_i^*$. Any subtree $T'$ of $T - P^* - F_1^* - F_2^*$ corresponds to a hanging subgraph of $U$ that has bonnet depth at most $H - 1$. By induction $\text{pw}(T') \leq 2H - 2$, and $\text{pw}(T) \leq 2H$ as desired.

Since an umbrella is a bonnet, we have $\text{bd}(G; u, v) \leq \text{ud}(G; u, v)$ for all non-cutting edges $(u, v)$, and the second inequality holds.

For the third inequality, assume that $H := \text{rpw}^\text{free}(T)$ is attained when rooting $T$ at leaf $r$, and let $(u, v)$ be a non-cutting edge of $G$ incident to $r$. We claim that $\text{ud}(G; u, v) \leq \text{rpw}(T, r) = H$. Let $P^*$ be an rpw-main path of $T$; without loss of generality we may assume that $P^*$ connects from $r$ to a leaf of $T$ (else a longer path could be used). Let $P$ be the outerplanar path whose dual tree is $P^*$; it connects $(u, v)$ to some non-cutting edge since $P^*$ connects $r$ to a leaf of $T$. Let $U_0$ be the umbrella obtained by adding all other neighbours of $u$ and $v$ to $P$. For any hanging subgraph $S_{a, b}$ of $P$, the dual tree $T_S$ is a subtree of $T - P^*$, and therefore $\text{rpw}(T_S) \leq H - 1$. By induction, $\text{ud}(S_{a, b}; a, b) \leq H - 1$, and combining the umbrella systems of the hanging subgraphs with $U_0$ hence shows $\text{ud}(G; u, v) \leq H$ as desired.

For the last inequality, we already know that $\text{rpw}(T, r) \leq 2\text{pw}(T) + 1$ for all choices of the root $r$ [8]. To prove the slightly tighter bound, assume that $P$ is a main path of $T$ and let $r$ be its end. Root $T$ at $r$, and use $P$ as rpw-main path. If $T = P$, then $\text{rpw}(T) = 1$ and we are done. Else $\text{pw}(T) \geq 1$ and any subtree $T'$ of $T - P$ has $\text{pw}(T') \leq \text{pw}(T) - 1$. The bound in [8] gives $\text{rpw}(T') \leq 2\text{pw}(T') - 1$. Thus $\text{rpw}(T, r) \leq 1 + \max_{T'} \text{rpw}(T') \leq 2\text{pw}(T)$.

It is known that $\text{rpw}(T) \leq \log(n + 1)$ [8], and therefore $\text{bd}^\text{free}(G) \leq \log(n + 1)$ as claimed earlier. All bounds in Lemma 6 are tight, except for a ‘+1’ term. Both graphs to show this are constructed as follows. Define a small graph $G_1$ that has a marked root-edge $(u, v)$ and some marked anchor-edges. Obtain graph $G_i$ by starting with graph $G_1$ and attaching copies of $G_{i-1}$ such that the root-edge of each $G_{i-1}$ is one of the anchor-edges of $G_1$. Let $T_i$ be the dual tree of graph $G_i$. We leave to the reader to verify the following claims:

- For the construction using the graph in Figure 7(a), we have $\text{pw}(T_i) \geq 2i$ while $\text{ud}(G_i; u, v) \leq i$. Therefore

$$i \leq \frac{1}{2} \text{pw}(T_i) \leq \text{bd}^\text{free}(G_i) \leq \text{ud}^\text{free}(G_i) \leq \text{ud}(G_i; u, v) \leq i,$$
and the left two inequalities are tight.

- For the construction using the graph in Figure 7(b), we have $pw(T_i) \leq i$ and $ud(G_i; u, v) \geq 2i$. Therefore

$$2i \leq ud(G_i; u, v) \leq ud^{free}(G_i) + 1 \leq rpw^{free}(G_i) + 1 \leq 2pw(T_i) + 1 \leq 2i + 1,$$

and the third and fourth inequality are tight up to a '+1' term.

![Figure 7: Constructions to prove tightness.](image)

Figure 7: Constructions to prove tightness.