Multichannel generalization of eigenphase-preserving supersymmetric transformations

Andrey M Pupasov-Maksimov

Departamento de Matemática, Universidade Federal de Juiz de Fora, Juiz de Fora, MG, Brazil
OOO Expert Energo, Moscow, Russia

E-mail: pupasov@phys.tsu.ru

Received 18 January 2013, in final form 7 March 2013
Published 25 April 2013
Online at stacks.iop.org/JPhysA/46/195201

Abstract
We generalize eigenphase-preserving (EPP) supersymmetric (SUSY) transformations to an \( N > 2 \) channel Schrödinger equation with equal thresholds. It is established that EPP SUSY transformations exist only in the case of an even number of channels, \( N = 2M \). A single EPP SUSY transformation provides an \( M(M-1)+2 \) parametric deformation of the matrix Hamiltonian without affecting eigenphase shifts of the scattering matrix.

PACS numbers: 03.65.Nk, 24.10.Eq

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we study an \( N \)-channel radial Schrödinger equation with equal thresholds. Such an equation may describe the scattering of particles with internal structure, for instance, with spin [1–3]. Supersymmetrizations allow for analytical studies of the Schrödinger equation with a wide class of interaction potentials [4–7]. In particular, the inverse scattering problem [8, 9] for a two-channel Schrödinger equation with equal thresholds may be treated by combined usage of single-channel SUSY transformations [10] and eigenphase-preserving (EPP) SUSY transformations [11]. These transformations conserve the eigenvalues of the scattering matrix and modify its eigenvectors (the coupling between channels), in contrast to phase-equivalent SUSY transformations which do not modify the scattering matrix at all [12–15].

In [3] the two-channel neutron–proton potential was reproduced by a chain of SUSY transformations, where the coupled channel inverse scattering problem was decomposed into the fitting of the channel phase shifts [10, 16] and the fitting of the mixing between channels. The fitting of the mixing between channels was provided by the EPP SUSY transformations.

This paper extends the two-channel EPP SUSY transformations to include a higher number of channels.

The paper is organized as follows. We first establish notation and recall the basics of SUSY transformations [6, 17–19]. Given the explicit form of a second-order SUSY transformation
operator we study the physical sector of SUSY transformations between real and symmetric Hamiltonians. We analyze the most general form of a second-order SUSY transformation for the case of mutually conjugated factorization energies. Then we discuss the applications of SUSY transformations to the scattering problems and calculate how the S-matrix transforms.

We start section 3 by re-examining the conservation of its eigenvalues in the two-channel case. There is the following asymptotic condition for EPP SUSY transformations: the term with a first-order derivative in the operator of the EPP SUSY transformation vanishes at large distances.

To generalize EPP supersymmetry for an arbitrary number of channels we study the matrix equation which comes from this asymptotic condition. We find that EPP SUSY transformations may exist for $2^M$ channels only and obtain their general form. A four-channel example explicitly shows how EPP SUSY transformations act. We conclude with a summary of the obtained results and a discussion of the possible applications.

2. Second-order SUSY transformations

2.1. Definition of SUSY transformations

SUSY transformations of the stationary matrix Schrödinger equation are well-known [17, 19]. Therefore, in this subsection we only establish the notation used in this study.

Consider a family of matrix Hamiltonians $\{H_a\}$

$$H_a = -i_N \frac{d^2}{dr^2} + V_a(r),$$

where $i_N$ is the $N \times N$ identity matrix and $V_a(r)$ is the $N \times N$ real symmetric matrix potential. The multi-index $a$ parameterizes a family of potentials $V_a(r)$; the matrix Hamiltonian defines the system of ordinary differential equations

$$H_a \varphi_a(k, r) = k^2 \varphi_a(k, r)$$

on $N \times N$ matrix functions $\varphi_a(k, r)$.

A (polynomial) SUSY transformation of (2) is a map of solutions

$$L_{ba} : \varphi_a(k, r) \rightarrow \varphi_b(k, r) = L_{ba} \varphi_a(k, r),$$

provided by a differential matrix operator

$$L_{ba} = A_n \frac{d^n}{dr^n} + A_{n-1} \frac{d^{n-1}}{dr^{n-1}} + \cdots + A_1 \frac{d}{dr} + A_0,$$

where $A_j, j = 0, \ldots, n$, are some matrix-valued functions. This differential matrix operator obeys the intertwining relation

$$L_{ba} H_a = H_b L_{ba}.$$  

The intertwining relation (5) defines both the operator $L_{ab}$ and the transformed Hamiltonian $H_b$. We will consider a family of Hamiltonians $\{H_b\} = \{H_b | L_{ba} H_a = H_b L_{ba}\}$ related to the given Hamiltonian $H_a$ by the transformation operator (4). In the next subsection we present the explicit form of the second-order transformation operator and the transformed Hamiltonian.

2.2. Second-order SUSY algebra of the matrix Schrödinger equation

Given the initial matrix Hamiltonian $H_0$, we choose two $N \times N$ matrix solutions

$$H_0 \varphi_j = E_j \varphi_j, \quad j = 1, 2,$$

2
where \( E_1, E_2 \) are factorization constants. These functions determine a second-order operator \( L_{20}[u_1, u_2] \):

\[
L_{20} f(r) = \left[ i_{20} \partial^2 - V_0 + E_1 + (E_2 - E_1)(w_1 - w_2)^{-1}(w_1 - \partial_r) \right] f(r),
\]

where

\[
w_j(r) = u_j^\prime (r) u_j^{-1} (r), \quad w_j^2 + w_j' + E_j = V_0, \quad j = 1, 2.
\]

Functions \( w_j(r) \) are called superpotentials. We also introduce a second-order superpotential

\[
W_2(r) = (E_2 - E_1)[w_1(r) - w_2(r)]^{-1}.
\]

The more symmetric and compact form of formula (7) reads

\[
L_{20} f(r) = \left[ -H_0 + \frac{E_2 + E_1}{2} + W_2 \left( \frac{w_1 + w_2}{2} - \partial_r \right) \right] f(r).
\]

Operator \( L_{20} \) and Hamiltonian \( H_0 \) obey the following algebra

\[
L_{20} H_0 = H_2 L_{20}, \quad H_2 = H_0 - 2W_2.
\]

(11)

The new (transformed) potential is expressed in terms of a second-order superpotential

\[
W_2 \text{ as follows}
\]

\[
V_2 = V_0 - 2W_2.
\]

(13)

The transformation operator has a global symmetry

\[
L_{20}[u_1, u_2] = L_{20}[u_1(r) U_1, u_2 U_2], \quad \det U_{1,2} \neq 0.
\]

(14)

The second-order SUSY transformations of an initial Hamiltonian \( H_0 \) form the family \( HH_0 = [H_2 L_{20}, H_0 = H_2 L_{20}] \). We will work only with Hamiltonians from \( HH_0 \) and we omit subscripts in the notation of the transformation operator \( L_{20} \rightarrow L \).

2.3. Restrictions to the SUSY transformations

We restrict our consideration only to second-order SUSY transformations with mutually
congruent factorization energies \( E_1 = E^*_2 = \mathcal{E} \). Potentials \( V_0 \) and \( V_2 \) are supposed to be real and symmetric. Hence the transformation functions \( u_1 \) and \( u_2 \) have to be mutually
congruent, \( u_1 = u^*_2 = u \). The symmetry of \( V_2 \) demands the symmetry of superpotentials

\[
(8), \quad w_2^1 = w_1 = w, \quad w_2^2 = w_2 = u^*,
\]

Defining the Wronskian of two matrix functions as

\[
W[u_1, u_2](r) = u_1^T(r) u_2'(r) - u_1'(r) u_2(r)
\]

(15)

\[
= u_1^T(r) [w_2(r) - w_2^T(r)] u_2(r),
\]

(16)

we see that the symmetry of superpotential \( w \) implies a self-vanishing Wronskian \( W[u, u] = 0 \) of transformation functions \([2]\).

We present the second-order superpotential \( W_2 \) in terms of the matrix Wronskian for

\[
W_2(r) = (E_1 - E_2) u_2(r) W[u_1, u_2]^{-1}(r) u_1^T(r).
\]

(17)

To specify an acceptable choice of transformation solutions explicitly, we choose the basis
in the solution space. The natural basis for the radial problem, \( r \in (0, \infty) \), is formed by Jost solutions \( f(\pm k, r) \) with the exponential asymptotic behavior

\[
f(k, r \to \infty) \to I_N e^{i\delta r}.
\]

(18)
Let us expand the transformation functions in the Jost basis
\[ u(r) = f_0(-K, r)C_j + f_0(K, r)D, \]
where \( K = k_c + ik_l, K^2 = \mathcal{E}, k_l > 0. \) The complex constant matrices \( C \) and \( D \) should provide the self-vanishing Wronskian \( W[u, u] = 0. \) The Wronskian of two solutions with the same \( k \) is a constant. For instance, \( W[f(-k, r), f(k, r)] = 2ik\mathcal{N}. \) Then, calculating \( W[u, u] \) we get a constraint on the possible choice of matrices \( C \) and \( D, \)
\[ D^T C = C^T D. \]  

Matrices \( C \) and \( D \) have an ambiguity due to symmetry (14). The rank of matrix \( C = M \leq N, \) determines the structure of the transformation operator. The sum of ranks \( \text{rank} C + \text{rank} D \geq N, \) otherwise operator \( L \) is undefined. Using (14) we may transform \( C \) to the form where only the first \( M \) columns are non-zero and linearly independent. Reordering the channels (by permutations of rows in the system of equations (2)) we can put nontrivial \( M \times M \) minor of \( C \) into the upper left corner. Then, \( C \) and \( D \) obey the following canonical form,
\[ C = \begin{pmatrix} I_M & 0 \\ Q & 0 \end{pmatrix}, \quad D = \begin{pmatrix} X & -Q^T \\ 0 & I_{N-M} \end{pmatrix}, \]
where \( X = X^T \) is a symmetric \( M \times M \) complex matrix and \( Q \) is an \( (N-M) \times M \) complex matrix. This canonical form is a gauge which fixes the ambiguity (14) of the transformation solutions.

2.4. Application to scattering theory

In concrete physical applications of SUSY transformations we may further restrict the class of Hamiltonians. In particular, in scattering theory [1] we work with the radial problem, \( r \in (0, \infty). \) The interaction potentials decrease sufficiently fast at large distances and may contain a centrifugal term
\[ \lim_{r \to \infty} r^2V(r) = l(l + \mathcal{I}_N), \quad l = \text{diag}(l_1, \ldots, l_N), \quad e^{i\pi} = \pm \mathcal{I}_N. \]  
The physical solution has the following asymptotic behavior
\[ \psi(k, r \to \infty) \propto k^{-1/2}[e^{-i\mathcal{I}_N}e^{i\mathcal{I}_N}/e^{-i\mathcal{I}_N}e^{i\mathcal{I}_N}S(k)], \]
where the matrix coefficient \( S(k) \) is the scattering matrix.

The scattering matrix is related to the Jost matrix
\[ S(k) = e^{i\mathcal{I}_N}F(-k)F^{-1}(k) e^{i\mathcal{I}_N}, \]
where the Jost matrix reads
\[ F(k) = \lim_{r \to 0} [f(k, r)v^r][(2v-1)]^{-1}. \]  
The diagonal matrix \( v \) indicates the strength of the singularity in the potential near the origin
\[ V(r \to 0) = v(v + \mathcal{I}_N)r^{-2} + O(1). \]  
Knowledge of the Jost solutions allows one to define the scattering matrix. SUSY transformations of the Hamiltonian and solutions induce the transformation of the scattering matrix. A formal approach to the calculations of the \( S \)-matrices was developed in the work of Amado [20].

Let us consider how the Jost solution transforms asymptotically,
\[ (Lf_0)(k, r \to \infty) = \left[ -k^2 + \frac{E_2 + E_1}{2} + \left( \sum_{j=1}^{N} \frac{w_j + w_2}{2} \right) \right] (r \to \infty) ~ -i k \mathcal{W}_2 (r \to \infty) \exp(i kr). \]
Assume that there exists the following limit

\[ U_\infty(k) = \lim_{r \to \infty} \left[ -k^2 + \frac{E_2 + E_1}{2} + W_2 \frac{w_1 + w_2}{2} - iW_2 \right]. \] (28)

Then the transformed Jost solution reads

\[ f_2(k, r) = (L f_0)(k, r) U^{-1}_\infty(k). \] (29)

By similarly manipulating the physical solution (23) we establish the form of the transformed \( S \)-matrix

\[ S_\infty(k) = e^{i\frac{\pi}{2}} U_\infty(k) e^{-i\frac{\pi}{2}} S_0(k) e^{i\frac{\pi}{2}} U^{-1}_\infty(k) e^{-i\frac{\pi}{2}}. \] (30)

In the case of our second-order SUSY transformation, the transformed \( S \)-matrix depends on the factorization energy \( E \), and the parameters \( Q \) and \( X \) through the matrix multipliers \( U_\infty(k) \) and \( U^{-1}_\infty(k) \). In general, this dependence is very complicated. Moreover, the scattering matrix \( S_2 \) may have unphysical low- and high-energy behavior.

SUSY transformations that deform the scattering matrix in a simple way are useful tools to solve the inverse scattering problem. In the two-cannel case there is a special kind of deformation, when \( U_\infty(k) \) becomes an orthogonal matrix [11]. We call such deformations EPP transformations.

3. Eigenphase-preserving SUSY transformations

3.1. The two-channel case

Let us analyze the conditions that make a two-channel SUSY transformation be an EPP one [11]. In this case the parameters of the transformation, \( Q = q \), \( X = x \), are just some numbers. Matrix \( U_\infty(k) \) depends on \( q \) only and becomes orthogonal when \( q = \pm i \). The determinant of \( u \) vanishes at large distances, \( \det u(r \to \infty) \to 0 \) with such a choice of \( q \). Let \( \det u(r \to \infty) \simeq \epsilon \), then the superpotential \( w \) diverges as \( w(r \to \infty) \simeq \epsilon^{-1} \) and the two-fold superpotential \( W_2 \) vanishes as \( w_2(r \to \infty) \simeq \epsilon \). As a result, the limit (28) contains only even powers of \( k \)

\[ U_\infty(k) = \lim_{r \to \infty} \left[ -k^2 + \frac{E_2 + E_1}{2} + W_2 \frac{w_1 + w_2}{2} \right]. \] (31)

The cancellation of the odd powers of \( k \) is a necessary condition to provide EPP SUSY transformations. In the next subsection we establish the most general form of the matrix \( Q \) which leads to the vanishing limit

\[ \lim_{r \to \infty} W_2 = 0, \] (32)

for the case \( N > 2 \).

Parameter \( x \) should also be fixed to provide \( \det W[u, u^*] \neq 0 \) for all \( r > 0 \) which leads to a finite \( V_2 \).

3.2. Asymptotic SUSY transformation at large distances for arbitrary \( N \)

The transformation function \( u \) (19) has the following asymptotic behavior at large distances

\[ u(r \to \infty) \to u_\infty(I_N + \Lambda r^{-1} + o(r^{-1})), \quad u_\infty = A e^{-iKr \Sigma}, \] (33)

where \( \Lambda \) is a constant matrix

\[ A = \begin{pmatrix} I_M & -Q^T \\ Q & I_{N-M} \end{pmatrix}, \quad \Sigma_{M,N-M} = \begin{pmatrix} I_M & 0 \\ 0 & -I_{N-M} \end{pmatrix}. \] (34)
For each concrete EPP transformation $N$ and $N - M$ are fixed, therefore we will use notation $\Sigma$ instead of $\Sigma_{M,N-M}$. The matrix $\Lambda$ is determined by the centrifugal terms entering the potential.

The leading term of the asymptotic behavior of the two-fold superpotential at large distances

$$
\lim_{r \to \infty} W_2 = W_{2,\infty} = (\mathcal{E} - \mathcal{E}^*) u_{\infty}^* W[u_{\infty}, u_{\infty}^*]^{-1} u_{\infty}^T
$$

is a constant matrix

$$
W_{2,\infty} = 2i \epsilon^{\text{int}} A^* e^{i k r / \Sigma} W[u_{\infty}, u_{\infty}^*]^{-1} e^{-i k r / \Sigma} A^T
$$

Equation (39) may be satisfied if and only if matrix

$$
W_{\infty} = K^* A^T \Lambda^* \Sigma + K \Sigma A^T A^* = 2 \left( k_i (I_M + Q^T Q^* - Q^* Q) \right),
$$

Limit (32) leads to the following matrix equation

$$
W_{2,\infty} = 0 \Rightarrow A^* W_{\infty}^{-1} A^T = 0, \quad \det W_{\infty} \neq 0,
$$

which provides asymptotic cancellation of $k$ in (28). In the two-channel case these equations fix $Q$ uniquely. When $N > 2$, these equations determine a set of $Q$ values.

Equation (39) may be satisfied if and only if matrix $A$ is singular. Matrix $W_{\infty}$ is invertible, rank $W_{\infty} = N$. Let rank $A = n$, then rank $A^* = \text{rank} A^T = n$ and rank $(W_{\infty}^{-1} A^T) = \dim \text{Im} (W_{\infty}^{-1}) = n$. The dimension of kernels $\dim \text{Ker} A = \dim \text{Ker} A^* = \dim \text{Ker} A^T = N - n$. Equation (39) implies that $\text{Im} (W_{\infty}^{-1} A^T) \subset \text{Ker} A^*$, hence $n \leq N - n$. Therefore equation (39) has solutions only if $n \leq \frac{1}{2} N$. On the other hand, from the explicit form of matrix $A$, (34), its rank $n \geq \text{max}(M, N - M)$. That is, (39) has solutions if and only if

$$
\text{rank} A = \frac{N}{2}, \quad N = 2M.
$$

From here it follows that for an odd number of channels equation (39) has no solutions.

Consider $2M \times 2M$ matrix $A$

$$
A = \begin{pmatrix} I_M & -Q^T \\ Q & I_M \end{pmatrix},
$$

with rank $A = M$. Two of its rectangular sub matrices have the same rank

$$
\text{rank} \begin{pmatrix} I_M \\ Q \end{pmatrix} = \text{rank} \begin{pmatrix} -Q^T \\ I_M \end{pmatrix} = M.
$$

We can take first the $M$ columns of $A$ as linearly independent, then from (40), (41) and (42) it follows that there exists an $M \times M$ matrix $Z$, such that

$$
\begin{pmatrix} I_M \\ Q \end{pmatrix} Z = \begin{pmatrix} -Q^T \\ I_M \end{pmatrix} = M.
$$

Solving this equation we obtain $Z = -Q^T$ and $QQ^T = -I_M$.

Let us extract $i$ from $Q$,

$$
Q = \pm i B, \quad B^T B = BB^T = I_M
$$

$$
(44)
$$
and substitute $Q$ in this form into (39). First of all we invert matrix $W_\infty$. This matrix can be factorized in two ways

$$\frac{1}{2} W_\infty = \begin{pmatrix} k_i B^T (B + B^*) & -k_i (B^T + B^*) \\ -k_i (B + B^*) & -k_i B (B^T + B^*) \end{pmatrix} \begin{pmatrix} k_i (B^T + B^*)B^* & -k_i (B^T + B^*) \\ -k_i (B + B^*) & -k_i (B + B^*)B^* \end{pmatrix}$$

$$= \begin{pmatrix} k_i B^T & -k_i I_M \\ -k_i I_M & -k_i B \end{pmatrix} \begin{pmatrix} (B + B^*)^{-1} & 0 \\ 0 & (B^T + B^*)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (B^T + B^T) \ 0 \\ 0 \ (B + B^*) \end{pmatrix} \begin{pmatrix} k_i B^* & -k_i I_M \\ -k_i I_M & -k_i B^* \end{pmatrix}.$$  

We note that

$$W_\infty W_\infty^{-1} = 4 (k_i^2 + k_i^2) \begin{pmatrix} (B^T + B^T) & 0 \\ 0 & (B^T + B^T) \end{pmatrix} \begin{pmatrix} (B + B^*) & 0 \\ 0 & (B^T + B^T) \end{pmatrix}.$$ (45)

Therefore the inverse matrix reads

$$W_\infty^{-1} = \frac{1}{2 |K|^2} k_i B^T \begin{pmatrix} k_i B^* & -k_i I_M \\ -k_i I_M & -k_i B^* \end{pmatrix} \begin{pmatrix} (B^T + B^*)^{-1} & 0 \\ 0 & (B^T + B^*)^{-1} \end{pmatrix}$$

$$= \frac{1}{2 |K|^2} k_i B^T \begin{pmatrix} (B^T + B^T)^{-1} & 0 \\ -k_i (B + B^*)^{-1} & -k_i (B + B^*)^{-1} \end{pmatrix}.$$ (46)

Let us introduce notations for the auxiliary matrices

$$\tilde{B} = \begin{pmatrix} (B^T + B^T) & 0 \\ 0 & (B + B^*) \end{pmatrix},$$

$$B_k = \begin{pmatrix} k_i B^T & -k_i I_M \\ -k_i I_M & -k_i B^* \end{pmatrix}.$$ Then $W_{2,\infty} = E_m A_k^* B_k \tilde{B}^{-1} A_k^T / (k_i^2 + k_i^2)$.

Using the following matrix identities

$$B^T (B^T + B^T)^{-1} = (B^* + B)^{-1} B^*, \quad B^T (B^T + B^T)^{-1} = (B^* + B)^{-1} B^T,$$ (47)

$$B^T (B^T + B^T)^{-1} B_k = B^* (B^* + B)^{-1}, \quad (B^T + B^T)^{-1} B_k = B^* (B^* + B)^{-1},$$ (48)

$$B^{-1} A_k^T = \begin{pmatrix} (B^T + B^T)^{-1} & 0 \\ 0 & (B + B^*)^{-1} \end{pmatrix} \begin{pmatrix} I_M & iB^T \\ -iB^* & I_M \end{pmatrix}$$

$$= \begin{pmatrix} I_M & iB^T \\ -iB^* & I_M \end{pmatrix} \begin{pmatrix} (B^T + B^T)^{-1} & 0 \\ 0 & (B + B^*)^{-1} \end{pmatrix},$$ (49)

$$A_k^* B_k = K^* \begin{pmatrix} B^+ & -iI_M \\ -iI_M & -B^* \end{pmatrix},$$ (50)

we see that

$$A_k^* B_k \tilde{B}^{-1} A_k = K^* \begin{pmatrix} B^+ & -iI_M \\ -iI_M & -B^* \end{pmatrix} \begin{pmatrix} B^+ & -iI_M \\ -iI_M & -B^* \end{pmatrix} \begin{pmatrix} (B^T + B^*)^{-1} & 0 \\ 0 & (B + B^*)^{-1} \end{pmatrix} = 0.$$ Thus (44) gives solutions of (39).

Now we can calculate the asymptotic form of a transformation operator explicitly. To calculate the asymptotic of $W_2 w$ we use the symmetry of superpotential $w = u' u^{-1} = (u^T)^{-1} (u^T)'$. The asymptotic form reads

$$\lim_{t \to \infty} W_2 w_{2,\infty} w_{\infty} = 2 i E_{\text{Im}} u_{\infty} u_{\infty}^* W_{1,\infty} u_{1,\infty}^* u_{\infty}^* (u_{\infty}^*)^{-1} (u_{\infty}^*)^{-1} \left( \begin{array} {cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$= -2i K E_{\text{Im}} A_k^* W_{\infty}^{-1} \Sigma A_k^T = -2 E_{\text{Im}} \left( \begin{array} {cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) .$$
Matrix $W_{2,\infty}$ is real, therefore

$$
\Omega = W_{2,\infty} \frac{w_\infty' + \omega_\infty'}{2} = \text{Re}(W_{2,\infty} w_\infty')
= 2k_1k_2 \left( \begin{array}{cc} i(B^T - B^*) & 2IM \\ i(B - B^*) & -2IM \end{array} \right) \left( \begin{array}{cc} (B^T + B^*)^{-1} & 0 \\ 0 & (B + B^*)^{-1} \end{array} \right).
$$

The matrix $U_\infty$ defined in (28) reads

$$
U_\infty(k^2) = (-k^2 + k_1^2 - k_2^2)I_N + \Omega. \quad (51)
$$

Matrix $\Omega$ is real, orthogonal (up to a normalization), $\Omega^T \Omega = 4k_1^2k_2^2I_N$ and antisymmetric $\Omega = -\Omega^T$. To establish its orthogonality and antisymmetry one should use relations (47) and (48). With these two properties of $\Omega$ the matrix $U_\infty(k^2)$ becomes proportional to the orthogonal matrix

$$
U_\infty(k^2)U_\infty^T(k^2) = \left( (-k^2 + k_1^2 - k_2^2)^2 + 4k_1^2k_2^2 \right)I_N.
$$

That is the Jost solutions at large distances are rotated by the orthogonal matrix $U_\infty$

$$(L_f)(k, r \to \infty) \to U_\infty \exp(ikr). \quad (53)$$

In this case the $S$-matrix transformation (30) is just an energy-dependent orthogonal transformation

$$
S_2(k) = R_5(k^2)S_0(k)R_5^T(k^2),
$$

with the orthogonal matrix $R_5^T R_5 = I_N$,

$$
R_5 = e^{i\frac{\pi}{2}}U_\infty e^{-i\frac{\pi}{2}}\left( (-k^2 + k_1^2 - k_2^2)^2 + 4k_1^2k_2^2 \right)^{-1/2}. \quad (55)
$$

That is we have obtained the desired generalization of two-channel EPP SUSY transformations.

The above analysis is valid for an arbitrary $M \times M$ symmetric matrix $X$. A transformed $S$-matrix $S_2$ depends on matrix $Q$ only. Therefore $X$ might provide an additional $M(M + 1)/2$ parametric deformation of potential $V_2$ without affecting the $S$-matrix. On the other hand, the possibility of such deformations contradicts the uniqueness of the inversion of the complete set of scattering data. Therefore, there may exist only one matrix $X$ corresponding to one physical potential $V_2$. The EPP SUSY transformation should be uniquely determined by the factorization energy, $M \times M$ complex orthogonal matrix $B$ and a sign factor. In the next subsection we show how to fix matrix $X$ and prove that the corresponding potential $V_2$ is regular for all $r > 0$.

3.3. Eigenphase-preserving SUSY transformation near the origin

To analyze the properties of EPP SUSY transformations in the vicinity of $r = 0$ we will use the solution

$$
\varphi_0(k, r) = \frac{i}{2k} [f_0(-k, r)f_0(k) - f_0(k, r)f_0(-k)], \quad (56)
$$

vanishing at the origin

$$
\varphi_0(k, r \to 0) \to \text{diag} \left( \frac{r^{\nu+1}}{(2\nu_l + 1)!!} \cdots \frac{r^{\nu+1}}{(2\nu_N + 1)!!} \right), \quad (57)
$$

where $f_0(k)$ is the Jost matrix (25). We rewrite the transformation solution in the basis $(\varphi_0(K, r), f_0(K, r))$ expressing $f_0(-K, r)$ from (56)

$$
f_0(-K, r) = \frac{2K}{i} \varphi_0(K, r) f_0^{-1}(K) + f_0(K, r) f_0(-K) f_0^{-1}(K). \quad (58)
$$
and substituting in (19)

\[ u(r) = \frac{2K}{i} \varphi_0(K, r) F_0^{-1}(K) C + f_0(K, r) (D + s_0C), \]

where

\[ s_0 = F_0(-K) F_0^{-1}(K) = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad e^{i\tilde{\gamma}} s_0 e^{i\tilde{\gamma}} = S_0. \]

We can transform matrix \((D + s_0C)\) to the form of matrix \(D\) (without affecting \(C\)) multiplying \(u\) from the right

\[ (D + s_0C) \begin{pmatrix} I_M \\ -(s_2 \pm is_3B) \end{pmatrix} I_M = \begin{pmatrix} \tilde{X} \\ 0 \end{pmatrix} \mp iB^T \left( \begin{array}{c} 0 \\ I_M \end{array} \right), \]

where

\[ \tilde{X} = X + s_1 \pm i(s_2^T B + B^T s_2) - B^T s_3B. \]

Then the transformation solution reads

\[ u(r) = \frac{2K}{i} \varphi_0(K, r) F_0^{-1}(K) \begin{pmatrix} I_M \\ \pm iB \end{pmatrix} + f_0(K, r) \begin{pmatrix} \tilde{X} \\ 0 \end{pmatrix} \mp iB^T \left( \begin{array}{c} 0 \\ I_M \end{array} \right). \]

Consider the case \(\tilde{X} = 0\). In this case the potential \(V_2\) is regular for all \(r > 0\). Let us prove this. According to the Wronskian representation of the second-order superpotential \(W_2(16)\), the potential \(V_2\) will be regular if and only if \(\det W[u, u^*](r) \neq 0\).

The derivative of the Wronskian \(W[u, u^*]\) reads

\[ W[u, u^*]'(r) = (\mathcal{E} - \mathcal{E}^*) u^T(r) u^*(r). \]

By construction \(W[u, u^*]\) is an anti-Hermitian matrix, i.e. \(W[u, u^*] = -W^*[u, u^*]\). We represent the transformation solution in block-diagonal form

\[ u(r) = \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix}, \]

with \(M \times M\) matrix blocks. When \(\tilde{X} = 0\) these blocks obey the following boundary conditions:

\[ u_{11}(0) = 0, \quad u_{21}(0) = 0, \quad u_{12}(\infty) = 0, \quad u_{22}(\infty) = 0. \]

As a result the Wronskian

\[ W[u, u^*] = \begin{pmatrix} u_{11}^T & u_{12}^T \\ u_{21}^T & u_{22}^T \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11}^T u_{11} + u_{12}^T u_{21} \\ u_{21}^T u_{12} + u_{22}^T u_{22} \end{pmatrix}. \]

has vanishing blocks at the origin, \(W_{11}(0) = 0\), and at infinity \(W_{22}(\infty) = 0\). The boundary behavior of \(W_{12}\) is not determined.

Now we can calculate the diagonal blocks of the Wronskian by integrating its derivative

\[ \frac{W[u, u^*]'}{(\mathcal{E} - \mathcal{E}^*)} = \begin{pmatrix} W_{11}' & W_{12}' \\ W_{21}' & W_{22}' \end{pmatrix} = \begin{pmatrix} u_{11}' u_{11} + u_{12}' u_{21} \\ u_{21}' u_{12} + u_{22}' u_{22} \end{pmatrix}. \]

Integration of (67) with the established boundary conditions yields

\[ W[u, u^*]'(r) = \left( \mathcal{E} - \mathcal{E}^* \right) \left( \int_0^r \left( u_{11}' u_{11} + u_{12}' u_{21} \right) dt \right) \frac{W_{12}}{W_{12}'}, \]

Assume that there is a point \(r_0\) where \(\det W[u, u^*](r_0) = 0\). Hence, the matrix \(W[u, u^*](r_0)\) has at least one zero eigenvalue

\[ W[u, u^*](r_0) \tilde{v} = 0. \]

Let us represent the \(N\)-dimensional eigenvector \(\tilde{v}\) as two \(M\)-dimensional vectors \(\tilde{v}_u\) and \(\tilde{v}_d\) and rewrite (69) as a system of equations

\[ W_{11} \tilde{v}_u + W_{12} \tilde{v}_d = 0, \]

\[ W_{21} \tilde{v}_u + W_{22} \tilde{v}_d = 0. \]
Its scattering matrix is diagonal and reads asymptotic uniqueness of

\[ \lim_{r \to \infty} \psi_{\text{out}, j} = \exp(\text{i}kr) \psi_{\text{in}, j} \] (78)

Therefore \( (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = n_u < 0 \) is real and negative. Now calculating the scalar product

\[ (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = 0 \] (71)

The first term of the scalar product \( (\vec{v}_d, \vec{W}_{11} \vec{v}_d) + (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = 0 \), \((a, b) = a^* b\) is positive,

\[ (\vec{v}_d, \vec{W}_{11} \vec{v}_d) = \left( \vec{v}_d, \int_0^r \exp(iu_1 u_1' + iu_2 u_2') \vec{v}_d \right) = \int_0^r ((u_1 u_1' + u_2 u_2')) \vec{v}_d \ dt > 0, \]

therefore \( (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = n_u < 0 \) is real and negative. Now calculating the scalar product

\[ (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = 0 \] (71)

The first term of the scalar product \( (\vec{v}_d, \vec{W}_{11} \vec{v}_d) + (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = 0 \) with a negative second term

\[ (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = - \left( \vec{v}_d, \int_0^\infty \exp(iu_1 u_1' + iu_2 u_2') \vec{v}_d \right) \]

\[ = - \int_0^\infty ((u_1 u_1' + u_2 u_2') \vec{v}_d) \ dt < 0, \]

we obtain a contradiction, \( (\vec{v}_d, \vec{W}_{12} \vec{v}_d) = n_u > 0 \). This contradiction proves that the Wronskian \( W[u, u^*] \) has only non-zero eigenvalues for all \( r > 0 \). As a result \( W[u, u^*] \) is invertible and hence both \( V_2 \) and \( V_3 \) are regular (finite) for all \( r > 0 \). Any non-zero \( \vec{X} \) will lead to the potential \( V_2 \), which is singular at some point \( r_0 \) (it was checked numerically in the four-channel case). We do not have the general mathematical proof that any non-zero \( \vec{X} \) will give non-physical (singular) potential. There is the following argument supporting the uniqueness of \( \vec{X} \). Both the scattering matrix \( S_2 \) and asymptotic normalization constants of (possible) bound states of \( H_2 \) depend on \( \vec{Q} \) only. However these data determine the physical \( V_2 \) uniquely (by the Marchenko transformation). Therefore \( \vec{X} = 0 \) is the only acceptable choice.

Thus we complete our construction of multi-channel EPP SUSY transformations leading to physical potentials. In the next subsection we present an illustrative example.

### 3.4. A four-channel coupled potential

We construct our initial four-channel potential with \( l = 0 \),

\[ V_0(r) = \text{diag} [v_0(r, a_1), v_0(r, a_2), v_0(r, a_3), v_0(r, a_4)], \] (72)

from four copies of the single-channel potential

\[ v_0(r, a) = \frac{2a^2}{\sinh^2 (ar)}. \] (73)

Its scattering matrix is diagonal and reads

\[ S_0(k) = \text{diag} [s_0(k, a_1), s_0(k, a_2), s_0(k, a_3), s_0(k, a_4)], \quad s_0(k, a) = \frac{a - ik}{a + ik}. \] (74)

Consider an incoming wave in the \( j \)th channel

\[ \psi_{\text{in}, j} = \exp(-\text{i}kr)(\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j})^T, \] (75)

where \( \delta_{ij} \) is the Kronecker delta symbol. The incoming wave is just a first term of the long-range asymptotic

\[ [\exp(-\text{i}kr) - \exp(\text{i}kr) S_0(k)](\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j})^T. \] (76)

The scattering of such a wave on the potential (72) results only in a phase shift of the outgoing wave

\[ \psi_{\text{out}, j} = \exp(\text{i}kr + 2\text{i}(\delta_0(k, a_j))) (\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j})^T, \] (77)

by the eigenphase

\[ \delta_0(k, a_j) = - \arctan \frac{k}{a_j} \] (78)

without mixing between channels.
The strength of coupling increases with \( \arg \mathcal{E} \). This results in the mixing of different channels in the outgoing wave, completely defining the EPP SUSY transformation.

Provided by EPP SUSY transformations for three values of \( J \). Vectors \( \psi \) depend on the energy of the incoming wave, with \( \arg \mathcal{E} \) decreasing from 0.78\( \pi \) to \( \pi /2 \).

Using an EPP SUSY transformation we deform the potential to introduce coupling between channels. The diagonal components of the basis \( (\varphi_0, f_0) \) explicitly read

\[
\varphi_0(k, r; a) = \frac{1}{k^2 + a^2} (k \cos(kr) - a \coth(ar) \sin(kr)),
\]

\[
f_0(k, r; a) = \exp(ikr) \frac{k + ia \coth(ar)}{k + ia}.
\]

These solutions, together with matrix \( B \) depending on a single complex number \( b = b_r + ib_i \),

\[
B = \begin{pmatrix} b & 0 \\ -\sqrt{1 - b^2} & b \end{pmatrix},
\]

completely define the EPP SUSY transformation.

Let us fix all parameters of the model \( a_j, b_r, b_i \) except the factorization energy \( \mathcal{E} \) (\( a_1 = 1.1, a_2 = 1.5, a_3 = 2.1, a_4 = 2.5, b_r = 2.5, b_i = 1.3 \)). In figures 1 and 2 we show the potential \( V_2 \) provided by EPP SUSY transformations for three values of \( \mathcal{E} = -2 + 1.5i; -1.25 + 3i; 4.5i \). The strength of coupling increases with \( \arg \mathcal{E} \) decreasing from 0.78\( \pi \) to \( \pi /2 \). One can check that \( \arg \mathcal{E} = 0 \) corresponds to zero-coupling, \( V_2 = V_0 \). For our choice of matrix \( B \) and parameters, the matrix \( \Omega \) reads

\[
\Omega = \mathcal{E}_{\text{em}} \begin{pmatrix} 0 & -0.936848 & 0.305791 & -0.16973 \\ 0.936848 & 0 & 0.16973 & 0.305791 \\ -0.305791 & -0.16973 & 0 & 0.936848 \\ 0.16973 & -0.305791 & -0.936848 & 0 \end{pmatrix}.
\]

The matrix \( S_2 \) (54) has the same eigenvalues, but the non-diagonal character of the potential results in the mixing of different channels in the outgoing wave,

\[
\begin{pmatrix} e^{-ikr \omega^2 R_3} - e^{-ikr \omega^2 R_0} \\ R_3 \end{pmatrix} (\delta_{1j}, \delta_{3j}, \delta_{1j}, \delta_{4j})^T.
\]

There is another set of incoming waves

\[
\psi_{\text{in}, j} = \exp(-ikr) R_j(k^2),
\]

\[
R_3 = (R_1, R_2, R_3, R_4),
\]

given by columns \( R_j(k^2) \) of matrix \( R_3 \), which scatter just with a phase shift (78),

\[
\psi_{\text{out}, j} = \exp(ikr + \alpha b_0(k, a_j)) R_j(k^2).
\]

Vectors \( \vec{R}_j \) depend on the energy of the incoming wave. In figure 3 we show this dependence for a particular example \( \mathcal{E} = 4.5i \). By changing two complex parameters \( b \) and \( E \) we can manipulate...
Figure 2. Diagonal entries of the exactly solvable potential matrix $V_2$ obtained from the uncoupled potential (72) with parameters $a_1 = 1.1, a_2 = 1.5, a_3 = 2.1, a_4 = 2.5, b_r = 2.5, b_i = 1.3$, for three choices of factorization energy $\mathcal{E} = -2 + 1.5i; -1.25 + 3i; 4.5i$.

Figure 3. Eigenvectors of the scattering matrix $S_2$. 
the transitions between channels which may open a way for broad physical application of EPP SUSY transformations.

4. Conclusion

In the present paper we have generalized two-channel eigenphase-preserving (EEP) supersymmetric (SUSY) transformations to the multichannel case, $N = 2M > 2$. It was surprising that such generalization exists for even numbers of channels only. A single EPP SUSY transformation depends on a complex factorization energy $E$ and an $M \times M$ complex matrix $B$, such that $B^T B = I_N$. Therefore a single EPP SUSY transformation provides an $M(M-1) + 2$ parametric deformation of the scattering matrix without affecting eigenphase shifts.

There are several possible applications of the presented results. One can use EPP SUSY transformations to solve inverse scattering problems by deforming a diagonal $S$-matrix as in [3]. We also may consider the $S$-matrix eigenvalues which are conserved under $M(M-1) + 2$ parametric deformation as integrals of motion for dynamical systems associated with the matrix Schrödinger equation [4]. In this context it is interesting to establish how this dynamical system may look. We expect that in this way new, exactly solvable, nonlinear equations may be discovered.

Acknowledgments

AMP thanks the Brazilian foundation CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) for their financial support. This work is supported by RFBR (Russian Foundation for Basic Research) grant number 12-02-31552.

References

[1] Taylor J R 1972 Scattering Theory: The Quantum Theory on Nonrelativistic Collisions (New York: Wiley)
[2] Sparenberg J-M, Pupasov A M, Samsonov B F and Baye D 2008 Exactly-solvable coupled-channel models from supersymmetric quantum mechanics Mod. Phys. Lett. B 22 2277–86
[3] Pupasov A, Samsonov B F, Sparenberg J M and Baye D 2011 Phys. Rev. Lett. 106 152301
[4] Matveev V and Salle M 1991 Darboux Transformations and Solitons (New York: Springer)
[5] Cannata F and Ioffe M V 1993 Coupled channel scattering and separation of coupled differential equations by generalized Darboux transformations J. Phys. A: Math. Gen. 26 L89–92
[6] Samsonov B F, Sparenberg J-M and Baye D 2007 Supersymmetric transformations for coupled channels with threshold differences J. Phys. A: Math. Theor. 40 4225–40
[7] Pupasov A M, Samsonov B F and Sparenberg J-M 2008 Exactly-solvable coupled-channel potential models of atom–atom magnetic Feshbach resonances from supersymmetric quantum mechanics Phys. Rev. A 77 012724 (arXiv:quant-ph/0709.0343)
[8] Levitan B M 1984 Inverse Sturm–Liouville Problems (Moscow: Nauka)
[9] Chadan K and Sabatier P C 1989 Inverse Problems in Quantum Scattering Theory 2nd edn (New York: Springer)
[10] Baye D and Sparenberg J-M 2004 Inverse scattering with supersymmetric quantum mechanics J. Phys. A: Math. Gen. 37 10223–49
[11] Pupasov A M, Samsonov B F, Sparenberg J M and Baye D 2010 Eigenphase preserving two-channel SUSY transformations J. Phys. A: Math. Theor. 43 155201
[12] Baye D 1987 Supersymmetry between deep and shallow nucleus–nucleus potentials Phys. Rev. Lett. 58 2738–41
[13] Sparenberg J-M and Baye D 1996 Supersymmetry between deep and shallow optical potentials for $^{16}$O + $^{16}$O scattering Phys. Rev. C 54 1309–21
[14] Sparenberg J-M and Baye D 1997 Supersymmetry between phase-equivalent coupled-channel potentials Phys. Rev. Lett. 79 3802–5
[15] Samsonov B F and Stancu F 2002 Phase equivalent chains of Darboux transformations in scattering theory Phys. Rev. D 66 034001
[16] Samsonov B F and Stancu F 2003 Phase shifts effective range expansion from supersymmetric quantum mechanics Phys. Rev. C 67 054005
[17] Amado R D, Cannata F and Dedonder J-P 1988 Coupled-channel supersymmetric quantum mechanics Phys. Rev. A 38 3797–800
[18] Amado R D, Cannata F and Dedonder J-P 1990 Supersymmetric quantum mechanics coupled channels scattering relations Int. J. Mod. Phys. A 5 3401–15
[19] Leeb H, Sofianos S A, Sparenberg J-M and Baye D 2000 Supersymmetric transformations in coupled-channel systems Phys. Rev. C 62 064603
[20] Amado R D, Cannata F and Dedonder J-P 1988 Formal scattering theory approach to S-matrix relations in supersymmetric quantum mechanics Phys. Rev. Lett. 61 2901–4