Non-perturbative aspects of gauge and string theories and their holographic relations

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To my parents
Abstract

In this thesis we discuss non-perturbative phenomena emerging in gauge as well as in string/supergravity theories. We discuss instantons in supersymmetric Yang-Mills theories. An interesting class of theories are obtained by adding adjoint hypermultiplets to pure $\mathcal{N} = 2$ theories. These theories, called $\mathcal{N} = 2^*$, are massive deformations of $\mathcal{N} = 4$ super Yang-Mills (SYM) and can be thought of as a minimal supersymmetric five dimensional theory compactified on a circle. We compute the partition function of 5D minimal supersymmetric $U(1)$ gauge theory with extra adjoint matter in general $\Omega$-background. It is well known that such partition functions encode very rich topological information. We show in particular that unlike the case with no extra matter, the partition function with extra adjoint at some special values of the parameters directly reproduces the generating function for the Poincaré polynomial of the moduli space of instantons.

Instantons play also a very crucial role in string theory, specifically in the context of string dualities. They have also interesting phenomenological implications. We discuss the basic aspects of worldsheet and penta-brane instantons as well as (unoriented) D-brane instantons and threshold corrections to BPS-saturated couplings in superstring theories. Then we consider non-perturbative superpotentials generated by ‘gauge’ and ‘exotic’ instantons living on D3-branes at orientifold singularities. We also discuss the interplay between worldsheet and D-string instantons on $T^4/\mathbb{Z}_2$. We focus on a 4-fermi amplitude, give Heterotic and perturbative Type I descriptions, and offer a multi D-string instanton interpretation.

Furthermore, instantons give a non-trivial check of the $AdS/CFT$ correspondence. $AdS/CFT$ dualities relate Type IIB superstring theory or M-theory compactified on an anti-de Sitter space-time times a compact space to conformally invariant field theories. In particular, in $AdS_5/CFT_4$, Type IIB D-instantons correspond to usual gauge instantons in dual $\mathcal{N} = 4$ SYM theory. Another interesting application of the holography principle is $AdS_4/CFT_3$ correspondence. This allows to investigate the worldvolume theory of M2-branes, the basic objects of M-theory. In this context we consider $\mathcal{N} = 8$ supergravity on $AdS_4 \times S^7$, which is the low energy limit of M-theory compactified on $S^7$. We revisit Kaluza-Klein compactification of 11-d supergravity on $S^7/\mathbb{Z}_k$ using group theory techniques that may find application in other flux vacua with internal coset spaces. Among the
SO(2) neutral states, we identify marginal deformations and fields that couple to the recently discussed world-sheet instanton of Type IIA on $\mathbb{CP}^3$. We also discuss charged states, dual to monopole operators, and the $\mathbb{Z}_k$ projection of the $Osp(8|4)$ singleton and its tensor products. In particular, we show that the doubleton spectrum may account for $\mathcal{N} = 6$ higher spin symmetry enhancement in the limit of vanishing ’t Hooft coupling in the boundary Chern-Simons theory.
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Introduction

We are introducing the interested reader to the fascinating subject of non-perturbative effects generated by unoriented D-brane instantons. After a very short reminder of Yang-Mills (YM) instantons and the ADHM construction, we discuss basic aspects of worldsheet and D-brane instantons and their original applications to threshold corrections.

Then we discuss non-perturbative superpotentials generated by ‘gauge’ and ‘exotic’ instantons living on D3-branes at orientifold singularities. We consider the interplay between worldsheet and D-brane instantons on $T^4/\mathbb{Z}_2$. We focus on a specific 4-hyperin amplitude, give Heterotic and perturbative Type I descriptions, and offer a multi D-string instanton interpretation.

Several good reviews are available on the subject [4], that is also covered in textbooks [5].

We investigate the partition function of a five dimensional supersymmetric $U(1)$ gauge theory with an extra adjoint hypermultiplet. Such partition functions encode very rich topological information. As a manifestation we argue that unlike the case with no extra matter, at some special values of the parameters this partition function directly reproduces the generating function of the Poincarè polynomial for the moduli space of instantons. We check this conclusion explicitly computing the partition function in the case of gauge group $U(1)$.

We revisit Kaluza-Klein (KK) compactification of 11-d supergravity on $S^7$. The spectrum of KK excitations in flux vacua plays an important role both in attempts to embed the Standard Model in String Theory and in the holographic correspondence. In the spirit of holography, the seminal observation of Schwarz’s [74] and the subsequent work of Bagger and Lambert [75–78] and, independently, of Gustavsson [79], motivated Aharony, Bergman, Jafferis and Maldacena (ABJM) [81, 82] to propose
a duality between superconformal Chern-Simons (CS) theories in $d = 3$ dimensions and String / M-theory on $AdS_4$.

The duality has been thoroughly tested and extended to cases with lower supersymmetry [83–88]. In particular the superconformal index has been matched both in the regime $k >> 1$ ($SO(2)$ singlets) [89, 90] and at finite $k$ [91, 92]. A detailed analysis of the (BPS) spectrum and the supermultiplet structure is however still incomplete. We fill in this gap and perform precision spectroscopy of 11-d supergravity on $AdS_4 \times S^7/Z_k$ or, equivalently, Type IIA on $AdS_4 \times \mathbb{CP}^3$. We will also discuss higher spin symmetry enhancement in the limit of vanishing 't Hooft coupling in the boundary $\mathcal{N} = 6$ Chern-Simons theory.

After reviewing the ABJM model, presenting both bulk and boundary vantage points, we revisit KK reduction of 11-d supergravity on $S^7$ [93] and then perform the decomposition of $SO(8)$ into $SO(6) \times SO(2)$ so as to derive the KK excitations of $\mathcal{N} = 6$ gauged supergravity [94], including states charged under $SO(2)$ that are expected to be dual to 'monopole' operators on the boundary [81,82]. Since we rely on group theory techniques which are not easily found in the available literature, we try to make this part of the presentation as pedagogical as possible, also in view of applications to other flux vacua with internal coset manifolds $G/H$. We then compare the resulting bulk spectrum with the spectrum of gauge-invariant operators on the boundary. Finally we compute the partition function of the boundary theory performing an orbifold projection on the parent theory ($k = 1, 2$ cases) and examine the higher spin content of the theory. Various appendices summarize useful $SO(8)$ and $SO(6)$ group theory formulae.

The plan of the thesis is the following.

In Chapter 1 we give a review of instantons in gauge theories, discuss ADHM construction. Then we discuss instantons in string theories, their possible generated effects in diverse string compactifications. We discuss a little bit thresholds in toroidal compactifications, emphasize some phenomenological aspects.

In Chapter 2 we compute the partition function of five dimensional supersymmetric $U(1)$ gauge theory with adjoint matter ($\mathcal{N} = 2^*$) in general $\Omega$-background. Such partition functions encode very rich topological information. We show that the partition function with extra adjoint matter at
some special values of the parameters directly reproduces the generating function for the Poincare polynomial of the moduli space of instantons.

In Chapter 3 we discuss instantons in string theories. In particular, we explain how the rather non-trivial ADHM construction arises very intuitively and naturally in string theory in the context of Dp-branes inside D(p+4) ones. Particularly we discuss D3/D(-1) system. Then we mention the vertex operators for ‘gauge’ and ‘stringy’ instantons. D-branes at orbifolds, unoriented projection, in particular $R^6/\mathbb{Z}_3$ projection, non-perturbative superpotential for $Sp(6) \times U(2)$ and for $U(4)$ are discussed as well. Then we discuss exotic/stringy instantons. One section is devoted to the effect of fluxes.

Chapter 4 is devoted to the relation of worldsheet and D-brane instantons in different string theories. We discuss how perturbatively different string theories may be shown to be equivalent once non-perturbative effects are taken into account. In particular, Heterotic-Type I duality requires that the Heterotic fundamental string and the Type I D-string to be identified. Then we consider compactification on $T^4/\mathbb{Z}_2$ to $D = 6$ of Type I and Heterotic theories. We discuss duality and dynamics in $D = 6$, where $\phi_H$ and $\phi_I$ are independent. To further test the correspondence and gain new insights into multi D-brane instantons, we consider a four-hyperini Fermi type interaction that is generated by instantons and corresponds to a ‘chiral’ (1/2 BPS) coupling in the $\mathcal{N} = (1, 0)$ low energy effective action.

In Chapter 5 we discuss the role of instantons in AdS/CFT correspondence. First we give introductory section on gauge theory/string theory dualities, then we discuss the general aspects of $AdS_5 \times S^5$ and instantons which play an important role in proving the conjectural duality between Type IIB superstring theory compactified on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$. Then we consider other gauge theory/string theory dualities, in particular, we concentrate on $AdS_4 \times S^7$ case. M-theory on $AdS_4 \times S^7$ is dual to three dimensional superconformal field theory. This three dimensional theory was identified by Aharony, Bergman, Jafferis and Maldacena and goes under the name ABJM model. We give a brief description of the ABJM model, bulk and boundary theories. Then, by giving group theoretical methods, we show how one can break the supersymmetry from $\mathcal{N} = 8$ of M-theory on $AdS_4 \times S^7$ to $\mathcal{N} = 6$ of Type IIA theory compactified on $AdS_4 \times \mathbb{CP}^3$. This is not a spontaneous supersymmetry breaking. We give all KK tow-
ers keeping track of $SO(2)$ charge. Among the neutral states we identify marginal deformations and fields that couple to world-sheet instanton of Type IIA on $\mathbb{CP}^3$, which we discuss later in Chapter 8. Charged states are dual to monopole operators.

In Chapter 6 we discuss the higher spin (HS) extension of $\mathcal{N} = 6$ gauged supergravity in $AdS_4$. We perform $\mathbb{Z}_k$ projection of the $Osp(8|4)$ singleton and its tensor products. We show that the spectrum arising from symmetric doubleton is precisely the spectrum of ‘massless’ states of $\mathcal{N} = 8$ gauged supergravity on $AdS_4$.

In Chapter 7 we discuss instantons in $\mathbb{CP}^3$.

Appendices include different formulae.

This thesis is based on the papers [1], [2], [3].
Chapter 1

Instantons from Fields to Strings

1.1 Yang-Mills Instantons: a reminder

Instantons (anti-instantons) are self-dual (anti-self-dual) classical solutions of the equations of motions of pure Yang-Mills theory in Euclidean space-time.

\[ F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \]  

(1.1)

with \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \). In quantum theory they can be thought of as gauge configurations bridging quantum tunnelling among topologically distinct vacua. It is remarkable that self-dual (anti-self-dual) gauge fields automatically satisfy YM equations in vacuo as a result of the Bianchi identities. These solutions are classified by a topological charge:

\[ K = \frac{g^2}{32\pi^2} \int d^4 x \tilde{F}_{\mu\nu}^{a} \tilde{F}_{\mu\nu}^{a} \]  

(1.2)

an integer, which computes how many times an \( SU(2) \) subgroup of the gauge group is wrapped by the classical solution while its space-time location spans the \( S_3 \)-sphere at infinity. The action of a self-dual (or anti-self-dual) instanton configuration turns out to be

\[ S_I = \frac{8\pi^2}{g^2} |K|. \]  

(1.3)

Yang-Mills instantons are interesting from both physical and mathematical point of view. They give non-perturbative contributions to the functional integral in the semi-classical approximation. Instantons are very
interesting both in phenomenological models of QCD and for describing exact non-perturbative phenomena in supersymmetric gauge theories. In mathematics instantons play a central role in Donaldson’s construction of topological invariants of four-manifolds.

In \( \mathcal{N} = 1 \) theories, instantons may generate superpotentials, chiral condensates and lead to the dynamical supersymmetry breaking.

In \( \mathcal{N} = 2 \) theories, instantons correct the analytic prepotential, give exact spectrum of \( \frac{1}{2} \) BPS states. In \( \mathcal{N} = 2 \) SYM theory the instanton calculus produces the correct coefficients of \( \mathcal{N} = 2 \) prepotential derived in the Seiberg-Witten construction.

In \( \mathcal{N} = 4 \) theories, instantons interfere with perturbation theory (no \( R \)-symmetry anomaly), give non-perturbative corrections to correlation functions and anomalous dimensions (\( S \)-duality).

**ADHM construction**

An elegant algebro-geometric construction of YM instantons was elaborated by Atiyah, Drinfeld, Hitchin and Manin and goes under the name of ADHM construction [6].

For \( SU(N) \) groups, the ADHM ansatz for a self-dual gauge field with topological charge \( K \), written as a traceless hermitean \( N \times N \) matrix, reads

\[
(A_\mu)_{uv}(x) = g^{-1} \bar{U}_u^i \partial_\mu U_{i v},
\]

where \( U_{\lambda u}(x) \) with \( u = 1, \ldots, N \) and \( \lambda = 1, \ldots, N + 2K \) are \( (N + 2K) \times N \) complex ‘matrices’ whose columns are the basis ortho-normal vectors for the \( N \) dimensional null-space of a complex \( 2K \times (N + 2K) \) ‘matrix’ \( \Delta(x) \), i.e. satisfy

\[
\Delta_i^{\lambda\dot{\lambda}} U_{\lambda u} = 0 = U_u^\lambda \Delta_{\lambda \dot{\alpha}}
\]

for \( i = 1, \ldots, K, \alpha, \dot{\alpha} = 1, 2 \). Remarkably, \( \Delta_{\lambda \dot{\alpha}}(x) \) turns out to be at most linear in \( x \). In quaternionic notation\(^1\) for \( x \),

\[
\Delta_{\lambda \dot{\alpha}}(x) = a_{\lambda \dot{\alpha}} + b_{\lambda \dot{\alpha}} x_{\alpha \dot{\alpha}}, \quad \bar{\Delta}^{\dot{\alpha}\lambda}(x) = \bar{a}^{\dot{\alpha}\lambda} + \bar{b}^{\dot{\alpha}\lambda} \bar{x}_{\alpha \dot{\alpha}} \equiv (\Delta_{\lambda \dot{\alpha}})^*.
\]

The complex constant ‘matrices’ \( a \) and \( b \) form a redundant set of collective coordinates that include the moduli space \( \mathcal{M}_K \). Decomposing the index \( \lambda \)

\(^1\)Any real 4-vector \( V_\mu \) can be written as a ‘real’ quaternion \( V_{\alpha \dot{\alpha}} = V_\mu \sigma_\mu^{\alpha \dot{\alpha}} \) with \( \sigma_\mu = \{1, -i\sigma^3\} \).
as $\lambda = u + i\alpha$, with no loss of generality, one can choose a simple canonical form for $b$

$$b^\beta_{\lambda j} = b^\beta_{(u+ia)j} = \begin{pmatrix} 0 \\ \delta^\beta_\alpha \delta_{ij} \end{pmatrix}, \quad \bar{b}^\lambda_{\beta j} = \bar{b}^{(u+ia)}_{\beta j} = \begin{pmatrix} 0 \\ \delta^\beta_\alpha \delta_{ij} \end{pmatrix} \quad (1.7)$$

One can also split $a$ in a similar way as:

$$a_{\lambda j\dot{\alpha}} = a_{(u+ia)j\dot{\alpha}} = \begin{pmatrix} w_{uj\dot{\alpha}} \\ (X_{a\dot{\alpha}})_{ij} \end{pmatrix}, \quad \bar{a}^{\dot{\alpha} \lambda j} = \bar{a}^{\dot{\alpha}(u+ia)}_{\dot{\alpha}j} = \begin{pmatrix} \bar{w}^{\dot{\alpha}u} \\ (\bar{X}^{\dot{\alpha}a})_{ji} \end{pmatrix} \quad (1.8)$$

In order to ensure self-duality of the connection, the ‘ADHM data’ $\{w, \bar{w}, X, \bar{X}\}$ with $X_\mu^\dagger = X_\mu$ must satisfy algebraic constraints, known as the ADHM equations, that can be written in the form

$$w_{ui\dot{\alpha}} (\sigma^a)^{\dot{\alpha}}_\beta \bar{w}^{\dot{\beta}u} + \eta_{\mu\nu} [X_\mu, X_\nu]_{ij} = 0 \quad (1.9)$$

for later comparison with the D-brane construction. Note the $U(K)$ invariance of the above $3K \times K$ equations. For a recent review of supersymmetric instanton calculus see [7].

The ADHM construction for unitary groups can be generalized to orthogonal and symplectic groups. It is quite remarkable how the rather abstract ADHM construction can be made very intuitive using D-branes and $\Omega$-planes [8] as we will see later on.
Chapter 2

The $U(1)$ theory with adjoint matter

Recent progress in understanding non-perturbative phenomena in supersymmetric Yang-Mills (SYM) theories due to direct multi-instanton calculations is quite impressive. Two main ideas played essential role in all this developments. First was the realization that the SYM action induced to the moduli space of instantons can be represented in terms of closed, equivariant with respect to the diagonal part of the gauge group, forms [62]. This observation leads to a crucial simplifications reducing SYM path integral to an integral over the stable, with respect to the action of the diagonal part of the gauge group, subset of the moduli space of instantons. The next brilliant idea, which is the corner stone for all further developments, was suggested by Nekrasov in [63]. The idea is to generalize the theory involving into the game in equal setting besides the already mentioned global diagonal gauge transformations also the diagonal part of the (Euclidean) space-time rotations. Why this is so crucial because the subset of the instanton moduli space invariant under this combined group action appears to consist only of finite number of points.

In the case of the gauge group $U(N)$ this fixed point set is in one to one correspondence with the set of array of Young diagrams $\vec{Y} = (Y_1, ..., Y_N)$ with the total number of boxes $|\vec{Y}|$ being equal to the instanton charge $k$. Thus, to calculate path integral for the various ‘protected’ by supersymmetry physical quantities one needs to know only the pattern how the combined group acts in the neighborhoods of the fixed point. All this information can be encoded in the character of the group action in the
tangent space at given fixed points. An elegant formula for this character which played a significant role in both physical and mathematical applications was proposed in [64] (see eq. (2.1)). Let us note at once, that combining space time rotations with gauge transformations, besides giving huge computational advantage due to the finiteness of the fixed point set, has also a major physical significance generalizing the theory to the case with certain nontrivial graviphoton backgrounds [63]. In order to recover the standard flat space quantities (say the Seiberg-Witten prepotential of $\mathcal{N} = 2$ SYM theory) one should take the limit when the space time rotation angles vanish. It is shown by Nekrasov and Okounkov [65] that in this limit the sum over the arrays of Young diagrams is dominated by a single array with specific ‘limiting shape’. This enables one to handle in this limit the entire instanton sum expressing all relevant quantities in terms of emerging Seiberg-Witten curve [66]. This is essential, since only the entire sum and not its truncated part exhibits remarkable modular properties, which allows one to investigate rich phase structure of SYM theories. This is why all the attempts to investigate the instanton sums also in general case seems quite natural. Unfortunately, there was a little progress till now in this direction besides the simplest case of the gauge group $U(1)$. Though the $U(1)$ 4D theory in flat background is trivial, the general 5D $U(1)$ theory compactified on a circle\footnote{Roughly speaking the main technical difference between 4D and 5D cases is that in the former case the above mentioned combined group enters into the game in the infinitesimal level while in the latter case the main role is played by finite group elements.} being rather nontrivial nevertheless in many cases admits full solution. We investigate the partition function of 5D gauge theory with an extra adjoint hypermultiplet. It is not surprising that such partition functions encode very rich topological information. As a manifestation we argue that unlike the case with no extra matter, at some special values of the parameters this partition function directly reproduces the generating function of the Poincare polynomial for the moduli space of instantons. We check this conclusion explicitly computing the partition function in the case of gauge group $U(1)$. We compare our result with that of recently obtained by Iqbal et. al. [67] who used the refined topological vertex method [73] to find the same partition function and present our comments on discrepancies we found.

The weight decomposition of the torus action on the tangent space at
the fixed point $\vec{Y} = (Y_1, \ldots, Y_N)$ is given by [64]

$$\chi = \sum_{\alpha, \beta = 1}^{N} e_\beta e_\alpha^{-1} \left\{ \sum_{s \in Y_\alpha} \left( T_1^{-l_\beta(s)} T_2^{-a_\alpha(s)+1} \right) + \sum_{s \in Y_\beta} \left( T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)} \right) \right\}, \quad (2.1)$$

where $e_1, \ldots, e_N$ are elements of (complexified) maximal torus of the gauge group $U(N)$ and $T_1, T_2$ belong to the maximal torus of the (Euclidean) space-time rotations, $a_\alpha(s)$ ($l_\alpha(s)$) measures the distance from the location of the box $s$ to the edge of the young diagram $Y_\alpha$ in the vertical (horizontal) direction.

The 5D partition function can be read off from the above character

$$Z = \sum_{\vec{Y}} \left\{ \frac{q^{\mid \vec{Y} \mid}}{\prod_{\alpha, \beta = 1}^{N} \prod_{s \in Y_\alpha} \left( 1 - e_\beta e_\alpha^{-1} T_1^{-l_\beta(s)} T_2^{-a_\alpha(s)+1} \right)} \times \frac{1}{\prod_{\alpha, \beta = 1}^{N} \prod_{s \in Y_\beta} \left( 1 - e_\beta e_\alpha^{-1} T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)} \right)} \right\} \quad (2.2)$$

From the mathematical point of view this quantity could be regarded as the character of the torus action on the space of holomorphic functions of the moduli space of instantons. The Nekrasov’s partition function for 4D theory could be obtained tuning the parameters $q \rightarrow \beta^{2N} q$, $T_1 \rightarrow \exp -\beta \epsilon_1$, $T_2 \rightarrow \exp -\beta \epsilon_2$, $e_\alpha \rightarrow -\beta v_\alpha$ and tending $\beta \rightarrow 0$, where $v_1, \ldots, v_N$ are the expectation values of the chiral superfield and $\epsilon_1, \epsilon_2$ characterize the strength of the graviphoton background (sometimes called $\Omega$-background).

Fortunately, instanton counting is powerful enough to handle also the cases when an extra hypermultiplet in adjoint or several fundamental hypermultiplets are present. In the case with adjoint hypermultiplet instead of (2.1) one starts with the (super) character [68]

$$\chi = (1 - T_m) \times \sum_{\alpha, \beta = 1}^{N} e_\beta e_\alpha^{-1} \left\{ \sum_{s \in Y_\alpha} \left( T_1^{-l_\beta(s)} T_2^{-a_\alpha(s)+1} \right) + \sum_{s \in Y_\beta} \left( T_1^{l_\alpha(s)+1} T_2^{-a_\beta(s)} \right) \right\}. \quad (2.3)$$

One way to interpret this character is to imagine that each (complex) 1d eigenspace of the torus action is complemented by a grassmanian eigenspace with exactly the same eigenvalues of the torus action. In addition an extra $U(1)$ action is introduced so that $T_m \in U(1)$ acts trivially
on bosonic directions while acting on each grassmanian coordinate in its fundamental representation. Then (2.3) is the super-trace of the extended torus action on the super-tangent space at given fixed point. The corresponding 5D partition function now reads:

\[ Z_{\text{adj}} = \sum_{Y} q^{|Y|} \prod_{\alpha, \beta = 1}^{N} \prod_{s \in Y_{\alpha}} \left( \frac{1 - T_{m} e_{\beta} e_{\alpha}^{-1} T_{1}^{-l_{\beta}(s)} T_{2}^{a_{\alpha}(s)+1}}{1 - e_{\beta} e_{\alpha}^{-1} T_{2}^{-a_{\beta}(s)+1}} \right) \prod_{s \in Y_{\beta}} \left( \frac{1 - T_{m} e_{\beta} e_{\alpha}^{-1} T_{1}^{l_{\alpha}(s)+1} T_{2}^{-a_{\beta}(s)+1}}{1 - e_{\beta} e_{\alpha}^{-1} T_{2}^{a_{\alpha}(s)+1}} \right) \]

(2.4)

Each term here could be thought of as trace over the space of local holomorphic forms, with parameter \( T_{m} \) counting the degrees of forms. Hence the sum over the fixed points is expected to give the super-trace over the globally defined holomorphic forms. We see that \( Z_{\text{adj}} \) is an extremely rich quantity from both physical and mathematical point of view. It is interesting to note, that at the special values of the parameters, \( Z_{\text{adj}} \) directly reproduces the generating function for the Poincare polynomial of the moduli space of \( U(N) \) instantons. Indeed, following [69] part (3.3) let us assume that \( T_{2} \gg T_{a_{1}} > \cdots > T_{a_{N}} \gg T_{1} > 0 \). It is easy to see, that in the limit when all these parameters go to zero, each fraction under the products in (2.3) tends to \( T_{m} \) or 1 depending whether we have a negative weight direction or not (see the classification of negative directions in [69], proof of corollary 3.10). We will see this explicitly in the simplest case \( N = 1 \) when the moduli space of instantons coincides with the Hilbert scheme of points on \( \mathbb{C}^{2} \).

From now on we will restrict ourselves to the simplest case of \( U(1) \) gauge group, when the partition function could be computed in a closed way. The partition function of the pure \( \mathcal{N} = 2, U(1) \) theory has the form [70]

\[ Z = \sum_{Y} q^{|Y|} \prod_{s \in Y} \left( 1 - T_{1}^{-l(s)} T_{2}^{a(s)+1} \right) \left( 1 - T_{1}^{l(s)+1} T_{2}^{-a(s)} \right) = \exp \left( \sum_{n=1}^{\infty} \frac{q^{n}}{n(1 - T_{1}^{n})(1 - T_{2}^{n})} \right). \]

(2.5)

This remarkable combinatorial identity in the 4D limit and in ‘self dual’
case $\epsilon_1 = -\epsilon_2$ boils down to the Burnside’s theorem

$$\sum_{|\lambda| = n} (\dim R_\lambda)^2 = n!,$$

(2.6)

where $R_\lambda$ is the irreducible representation of the symmetric group given by the Young diagram $\lambda$.

Now let us turn to the $U(1)$ theory with adjoint matter. Doing low instanton calculations using (2.4) is straightforward and gives

$$\log Z_{adj} = \frac{q(1 + T_m q + T_m^2 q^2 + T_m^3 q^3)(1 - T_m T_1)(1 - T_m T_2)}{(1 - T_1)(1 - T_2)} + \frac{q^2(1 + T_m^2 q^2)(1 - T_m^2 T_1^2)(1 - T_m^2 T_2^2)}{2(1 - T_1^2)(1 - T_2^2)} + \frac{q^3(1 - T_m^3 T_1^3)(1 - T_m^3 T_2^3)}{3(1 - T_1^3)(1 - T_2^3)} + \frac{q^4(1 - T_m^4 T_1^4)(1 - T_m^4 T_2^4)}{4(1 - T_1^4)(1 - T_2^4)} + O(q^4).$$

(2.7)

These drove us to the conjecture that the exact formula is

$$\log Z_{adj} = \sum_{n=1}^{\infty} \frac{q^n(1 - T_m^m T_1^m)(1 - T_m^m T_2^m)}{n(1 - T_1^n)(1 - T_2^n)(1 - T_m^m q^n)},$$

(2.8)

which is equivalent to the following highly nontrivial combinatorial identity

$$Z_{adj} = \sum_Y q^{[Y]} \prod_{s \in Y} \left(1 - T_m T_1^{-l(s)} T_2^{-a(s)+1} \right) \left(1 - T_m T_1^{-l(s)+1} T_2^{-a(s)} \right) \left(1 - T_1^{-l(s)} T_2^{-a(s)+1} \right) \left(1 - T_1^{-l(s)+1} T_2^{-a(s)} \right)$$

$$= \exp \left( \sum_{n=1}^{\infty} \frac{q^n(1 - (T_m T_1)^n)(1 - (T_m T_2)^n)}{n(1 - T_1^n)(1 - T_2^n)(1 - (T_m q)^n)} \right).$$

(2.9)

(2.10)

Indeed calculations with Mathematica code up to 10 instantons further convinced us that this formula is indeed correct. Note that the 4D limit of this identity with a particular choice of graviphoton background $\epsilon_1 = -\epsilon_2$ is mentioned earlier in [65] and was used later in [71] to calculate the expectation value $tr \langle \phi^2 \rangle$.

As a further check let us go to the limit when $T_1 \to 0$, $T_2 \to 0$. As we have explained above one expects to find the generating function of Poincare polynomial for Hilbert scheme of points on $\mathbb{C}^2$. An easy calcula-
CHAPTER 2. THE $U(1)$ THEORY WITH ADJOINT MATTER

tion yields:

$$Z_{adj}|_{T_1, T_2=0} = \exp \sum_{n=1}^{\infty} \frac{q^n}{n(1 - T^n_m q^n)} =$$

$$\exp \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{1+kT^k_m})^n}{n} = \prod_{k=0}^{\infty} \frac{1}{1 - T^k_m q^{k+1}},$$

(2.11)

which indeed after identifying $T_m$ with Poincare parameter $t^2$ reproduces the well known result (see e.g. [72]). Now let us go back to the general case. In various domains of the variables $T_1, T_2$ we can represent (2.8) as infinite product as in (2.11). Let us consider separately the cases:

(a) $|T_1| < 1, |T_2| < 1, |T_m q| < 1$

In this region (2.8) could be rewritten as

$$Z_{adj} = \exp \left\{ \sum_{n=1}^{\infty} \sum_{k,i,j=0}^{\infty} \frac{q^n}{n} T_{1}^{ni} T_{2}^{nj} (T_m q)^{nk} (1 - T^n_m T^n_1) (1 - T^n_m T^n_2) \right\}.$$  

(2.12)

Performing summation over $n$ we get

$$Z_{adj} = \prod_{i,j,k=0}^{\infty} \frac{(1 - q^{k+1} T^{k+1}_m T^{i+1}_1 T^{j+1}_2)}{(1 - q^{k+1} T^{k+1}_m T^{i+1}_1 T^{j+1}_2) (1 - q^{k+1} T^{k+2}_m T^{i+1}_1 T^{j+1}_2)}.$$  

(2.13)

(b) $|T_1| > 1, |T_2| < 1, |T_m q| < 1$

In this region we expand (2.8) over $1/T_1$:

$$Z_{adj} = \exp \left\{ \sum_{n=1}^{\infty} \sum_{k,i,j=0}^{\infty} \frac{-q^n}{n} T_{1}^{-ni} T_{2}^{nj} (T_m q)^{nk} (1 - T^n_m T^n_1) (1 - T^n_m T^n_2) T_1^{-n} \right\},$$

(2.14)

which leads to

$$Z_{adj} = \prod_{i,j,k=0}^{\infty} \frac{(1 - q^{k+1} T^{k+1}_m T^{-i-1}_1 T^{j+1}_2)}{(1 - q^{k+1} T^{k+1}_m T^{-i-1}_1 T^{j+1}_2) (1 - q^{k+1} T^{k+2}_m T^{-i-1}_1 T^{j+1}_2)}.$$  

(2.15)

Recently Iqbal, Kozcaz and Shabir [67] have computed the partition function of these $U(1)$ adjoint theory using the refined topological vertex formalism [73]. And, since the formula (2.8) was known to the present authors for quite a while, we performed a detailed comparison with their results. To make contact with the formulae of Iqbal et. al. we need the following dictionary: $T_m = Q_m(t/q)^{1/2}, T_1 = 1/t, T_2 = q$, $q = Q(q/t)^{1/2}$. In terms of these variables the equations (2.13) and (2.15) take the form:
(a) $|t| > 1, \, |q| < 1, \, |QQ_m| < 1$

$$Z_{\text{adj}} = \prod_{i,j,k=1}^{\infty} \frac{(1 - Q^k Q^k_{m} q_i^{i-1} t^{-j}) (1 - Q^k Q^k_{m} q^{i-1} t^{1-j})}{(1 - Q^k Q^k_{m} q^{i-1} t^{-j} + \frac{1}{2}) (1 - Q^k Q^k_{m} q^{i-1} t^{1 - j} + \frac{1}{2})}, \quad (2.16)$$

and

(b) $|t| < 1, \, |q| < 1, \, |QQ_m| < 1$

$$Z_{\text{adj}} = \prod_{i,j,k=1}^{\infty} \frac{(1 - Q^k Q^k_{m} q^{i-1} t^{-j}) (1 - Q^k Q^k_{m} q^{i-1} t^{1-j})}{(1 - Q^k Q^k_{m} q^{i-1} t^{1-j} + \frac{1}{2}) (1 - Q^k Q^k_{m} q^{i-1} t^{1 - j} + \frac{1}{2})}. \quad (2.17)$$

These equations come rather close, but certainly do not coincide with those given in [67] at the end of the part 3.2. The reason for this discrepancy seems to us as follows. According to [67] the refined topological vertex method for the 5D $U(1)$ theory with adjoint matter leads to (see eq. (4.6) of [67]; below we omit the ‘perturbative part’ $\prod_{i,j=1}^{\infty} (1 - Q^k q^{i-1} t^{-j} + \rho_j)$)

$$Z = \prod_{k=1}^{\infty} (1 - Q^k Q^k_{m} q^{i-1} t^{-j} + \rho_j) \prod_{i,j=1}^{\infty} (1 - Q^k Q^k_{m} q^{i-1} t^{-j} + \rho_j) (1 - Q^k Q^k_{m} q^{i-1} t^{-j} + \rho_j + 1/2) (1 - Q^k Q^k_{m} q^{i-1} t^{-j} + \rho_j - 1/2) (1 - Q^k Q^k_{m} q^{i-1} t^{-j} + \rho_j), \quad (2.18)$$

where $\rho_j = -i + 1/2$. But four factors under the product over $i, j$ have different, excluding each other regions of convergence. Thus this infinite product should be treated very carefully. Unfortunately the authors of [67] do not tell what analytic continuation procedure they have adopted to pass from their eq. (4.6) to those presented at the end of the part 3.2, but we will demonstrate now that one, perhaps the simplest approach directly leads to our conjectural formula (2.8). We simply examine the product over each factor separately within its region of convergence and only after that continue analytically to a common region of the parameters. Thus for the first factor in (2.18) we have

$$\prod_{k=1}^{\infty} (1 - Q^k Q^k_{m})^{n} = \exp \sum_{n,k=1}^{\infty} \frac{(QQ_{m})^{nk}}{n} = \exp \sum_{n=1}^{\infty} \frac{(QQ_{m})^{n}}{n(1 - (QQ_{m})^{n})}. \quad (2.19)$$
For the next factor (assuming $q < 1$, $t < 1$)

$$\prod_{k,i,j=1}^{\infty} (1 - Q^k Q_m^{k-1} q^{-\frac{1}{2} t^{-\frac{1}{2}}}) = \exp \sum_{n,k,i,j=1}^{\infty} \frac{-Q^{kn} Q_m^{(k-1)n} q^{(i-\frac{1}{2})n} t^{(j-\frac{1}{2})n}}{n}$$

$$= \exp \sum_{n=1}^{\infty} \frac{-Q^n q^\frac{n}{2} t^\frac{n}{2}}{(1 - (QQ_m)^n)(1 - q^n)(1 - t^n)} (2.20)$$

Similarly for $q > 1$, $t < 1$

$$\prod_{k,i,j=1}^{\infty} (1 - Q^k Q_m^k q^{-i} t^{-j}) = \exp \sum_{n=1}^{\infty} \frac{-Q^n Q_m^n q^{-n} t^n}{n(1 - (QQ_m)^n)(1 - q^n)(1 - t^n)} (2.21)$$

for $q < 1$, $t > 1$

$$\prod_{k,i,j=1}^{\infty} (1 - Q^k Q_m^k q^{i} t^{j}) = \exp \sum_{n=1}^{\infty} \frac{-Q^n Q_m^n q^n t^n}{n(1 - (QQ_m)^n)(1 - q^n)(1 - t^n)} (2.22)$$

and, finally for $q > 1$, $t > 1$

$$\prod_{k,i,j=1}^{\infty} (1 - Q^k Q_m^{k+1} q^{-i+\frac{1}{2} t^{-\frac{1}{2}}} t^{-j+\frac{1}{2}}) = \exp \sum_{n=1}^{\infty} \frac{-Q^n Q_m^{2n} q^{-\frac{1}{2} t^{-\frac{1}{2}}} t^{-\frac{1}{2}}}{n(1 - (QQ_m)^n)(1 - q^n)(1 - t^n)}. (2.23)$$

Note that the r.h.s.'s of above expressions are defined also outside of their initial convergence region. Combining all these together we get

$$Z = \exp \sum_{n=1}^{\infty} \frac{(QQ_m)^n (q^\frac{n}{2} t^\frac{n}{2} - Q_m^n)(q^\frac{n}{2} t^\frac{n}{2} - Q_m^{-n})}{n(1 - (QQ_m)^n)(1 - q^n)(1 - t^n)}, (2.24)$$

which in terms of the parameters $q$, $T_1$, $T_2$ exactly coincides with our conjectural result (2.8).
Chapter 3

Instantons in String Theory

3.1 Worldsheet instantons

World-sheet instantons in Heterotic and Type II theories correspond to Euclidean fundamental string world-sheets wrapping topologically non-trivial internal cycles of the compactification space and produce effects that scale as $e^{-R^2/\alpha'}$ [9]. Depending on the number of supersymmetries (thus on the number of fermionic zero modes), they can correct the two-derivative effective action or they can contribute to threshold corrections to higher derivative (BPS saturated) couplings [10]. For Type II compactifications on CY three-folds, preserving $\mathcal{N} = 2$ supersymmetry in $D = 4$, holomorphic worldsheet instantons ($\bar{\partial}X = 0$) correct the special Kähler geometry of vector multiplets (Type IIA) or the dual quaternionic geometry of hypermultiplets (Type IIB). For heterotic compactifications with standard embedding of the holonomy group in the gauge group, complex structure deformations are governed by the same special Kähler geometry as in Type IIB on the same CY three-fold, that is not corrected by worldsheet instantons. Complexified Kähler deformations are governed by the same special Kähler geometry as in Type IIA on the same CY three-fold, that is corrected by worldsheet instantons, or equivalently, as a result of mirror symmetry, by the same special Kähler geometry as in Type IIB on the mirror CY three-fold that is tree level exact. For standard embedding, the Kähler metrics of charged supermultiplets in the 27 and $27^*$ representations of the surviving/visible $E_6$ are simply determined by the ones of the neutral moduli of the same kind by a rescaling [11]. For non standard embeddings the situation is much subtler.
**Brane instantons**

Euclidean NS5-branes (EN5-branes) wrapping the 6-dimensional compactification manifold produce non-perturbative effects in $e^{-c/g_s^2}$ (reflecting the NS5-brane tension) that qualitatively correspond to ‘standard’ gauge and gravitational instantons. Euclidean Dp-brane wrapping $(p+1)$-cycles produce instanton effects that scale as $e^{-c_p/g_s}$ (reflecting the EDp-brane tension) [12]. In Type IIB on CY three-fold, ED(-1), ED1-, ED3- and ED5-brane instantons, obtained by wrapping holomorphic submanifolds, correct dual quaternionic geometry in combination with world-sheet (EF1) and EN5-instantons. In Type IIA on CY three-folds, ED2-instantons (D-‘membrane’ instantons) wrapping special Lagrangian submanifolds, correct the dual quaternionic geometry, in combination with EN5-instantons. In both cases, the dilaton belongs to the universal hypermultiplet.

**Unoriented D-brane instantons**

In Type I, the presence of $\Omega 9$-planes severely restricts the possible homologically non trivial instanton configurations. Only ED1- and ED5-branes are homologically stable. Other (Euclidean) branes may be associated to instanton with torsion (K-theory) charges. For other un-oriented strings the situation is similar and can be deduced by means of T-duality: e.g. for intersecting D6-branes one has two different kinds of ED2-branes (ED0- and/or ED4-brane instanton require $b_1,5 \neq 0$), for intersecting D3- and D7-branes one has ED(-1) and ED3-branes. There are two classes of unoriented D-brane instantons depending on the stack of branes under consideration.

- ‘Gauge’ instantons correspond to EDp-branes wrapping the same cycle $\mathcal{C}$ as a stack of background D$(p+4)$-branes. The prototype is the D3, D(-1) system [13], [14] that has 4 N-D directions. The EDp-branes behave as instantons inside D$(p+4)$’s:

$$F = \tilde{F}$$

and produce effects whose strength, given by

$$e^{-W_{p+1}(\mathcal{C})/g_s \ell_s^{p+1}} = e^{-1/g_s^2 M},$$

is precisely the one expected from ‘gauge’ instantons in the effective field-theory.
3.2. ORIGINAL APPLICATIONS AND VARIOUS COMMENTS

• ‘Exotic’ instantons arise from ED$p'$-branes wrap a cycle $C'$ which is not wrapped by any stack of background D($p+4$)-branes. The prototype is the D9, ED1 system with 8 N-D directions and only a chiral fermion at the intersection. In this case

$$F \neq \tilde{F}$$

(3.3)

and the strength is given by

$$e^{-W_{p'+1}(C')/g_s'\theta_s'^{p'+1}} \neq e^{-1/g_{YM}^2}$$

(3.4)

‘Exotic’ instantons may eventually enjoy a field theory description in terms of octonionic instantons or hyper-instantons with $F \wedge F = \ast_8 F \wedge F$.

3.2 Original Applications and Various Comments

Let us now list possible effects generated by (un)oriented D-brane instantons in diverse string compactifications.

• In $\mathcal{N} = 8$ theories (e.g. toroidal compactifications of oriented Type II A/B) D-brane instantons produce threshold corrections to $R^4$ terms and other 1/2 BPS (higher derivative) terms.

• In $\mathcal{N} = 4$ theories (e.g. toroidal compactifications of Type I / Heterotic) D-brane instantons produce threshold corrections to $F^4$ terms and other 1/2 BPS (higher derivative) terms.

• In $\mathcal{N} = 2$ theories (e.g. toroidal orbifolds with $\Gamma \subset SU(2)$) D-brane instantons produce threshold corrections to $F^2$ terms and other 1/2 BPS terms.

• In $\mathcal{N} = 1$ theories (e.g. toroidal orbifolds with $\Gamma \subset SU(3)$) D-brane instantons produce threshold corrections and superpotential terms.

Thresholds in toroidal compactifications

We have not much to add to the vast literature on threshold corrections to $R^4$ terms in $\mathcal{N} = 8$ theories which are induced by oriented D-brane as well as world-sheet instantons\(^1\). We would only like to argue that in unoriented

\(^1\)In $D = 4$ and lower Euclidean NS5-branes can also contribute.
Type I strings and alike these corrections should survive as functions of the unprojected closed string moduli despite some of the corresponding D-brane or worldsheet instantons be not BPS. These and lower derivative ($R^2$) couplings may receive further perturbative corrections from surfaces with boundaries and crosscaps. Viz: $\mathcal{L}_{II} \approx \mathcal{R}^4 f_{II}(\phi, \chi) \rightarrow \mathcal{L}_I \approx \mathcal{R}^4[f_{II}(\phi, \chi = 0) + f_I(\phi)]$.

The original application of unoriented D-brane instanton was in the context of threshold corrections to $F^4$ terms in toroidal compactifications of Type I strings [17]. These are closely related to the threshold corrections to $F^4$ terms for heterotic strings on $T^d$. For later use, let us briefly summarize the structure of the latter. After

- Computing the one-loop correlation function of 4 gauge boson vertex operators $V(0) = A^a_{\mu}(\partial X^\mu + ip^2 \psi^a_\mu)\tilde{J}_a e^{ipx}$
- Taking the limit of zero momentum in the exponential factors i.e. neglecting the factor $\Pi(z_i, p_i) = \prod_{i,j} \exp[-\alpha' p_i \cdot p_j \mathcal{G}(z_{ij})] \rightarrow 1$

or, equivalently,

- computing the character-valued partition function in a constant field-strength background $\nu$
- taking the fourth derivative wrt $\nu$

one arrives at the integral over the one-loop moduli space that receives contribution only from BPS states and schematically reads

$$I_d[\Phi] = \mathcal{V}_d \int_F \frac{d^2 \tau}{T_2^2} \sum_M e^{2\pi i T(M)} e^{-\frac{\pi \alpha' T(M)}{\sqrt{2} \alpha' U(M)}} \tau^{-U(M)^2} \Phi(\tau)$$

(3.5)

where $M = (\vec{n}, \vec{m})$ represent the embedding of the world-sheet torus in the target $T^d$, $\Phi(\tau)$ is some modular form. The induced Kahler $T(M)$ and complex $U(M)$ structures are given by

$$T(M) = \mathcal{B}_{12} + i \sqrt{\det \mathcal{G}} \quad , \quad U(M) = \frac{1}{\mathcal{G}_{11}}(\mathcal{G}_{12} + i \sqrt{\det \mathcal{G}})$$

(3.6)

with $\mathcal{G} = M^t G M$, $\mathcal{B} = M^t B M$ induced metric and $B$-field [19]. The integral can be decomposed into three terms $I_d[\Phi] = I_d^{\text{triv}}[\Phi] + I_d^{\text{deg}}[\Phi] + I_d^{\text{ndeg}}[\Phi]$. The three different orbits are classified as follows: the orbit of $M = 0$ (trivial orbit), degenerate orbits with $\det(M^{i,j}) = 0$ and non-degenerate orbits with some $\det(M^{i,j}) \neq 0$. Let us consider the various contributions.
3.2. ORIGINAL APPLICATIONS AND VARIOUS COMMENTS

- **Trivial orbit**: \( M = 0, \)
  \[
  \mathcal{I}^{\text{triv}}_{d,d}[\Phi] = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \Phi(\tau) \to \mathcal{I}^{\text{triv}}_{d,d}[1] = \frac{\pi^2}{3} V_d
  \] (3.7)

- **Degenerate orbits**: \( M \neq 0, \det(M^i{}^j) = n^i m^j - n^j m^i = 0 \ \forall i, j. \)
  One can choose \( \vec{n} = 0 \) representative and unfold \( \mathcal{F} \) to the strip \( \mathcal{S} = \{|\tau_1| < 1/2, \tau_2 > 0\}, \) then
  \[
  \mathcal{I}^{\text{deg}}_{d,d}[\Phi] = V_d \int_{\mathcal{S}} d^2 \tau \sum_{\vec{m} \neq 0} e^{-\frac{\pi}{\tau_2} \vec{m}^T \vec{m} \Phi(\tau)} \to \mathcal{I}^{\text{deg}}_{d}[1] = V_d \mathcal{E}^{SL(d)}(G).
  \] (3.8)

- **Non degenerate orbits**: at least one \( \det(M^i{}^j) = n^i m^j - n^j m^i \neq 0. \)
  The representative for these orbits may be chosen to be \( \vec{n}^\alpha = 0 \) for \( \alpha = 1,..,k, \ m^\alpha \neq 0, \ n^\alpha > m^\alpha \geq 0 \) and enlarging the region of integration \( \mathcal{F} \) to the full upper half plane \( \mathcal{H}^+ \) one finds:
  \[
  \mathcal{I}^{\text{ndeg}}_{d,d}[\Phi] = V_d \int_{\mathcal{H}^+} d^2 \tau \sum_{(n^\alpha,0;m^\alpha,m^\alpha)} e^{2\pi i T(M)} e^{-\frac{\pi}{\tau_2} |m^\alpha \Phi(\tau)|^2} \to \mathcal{I}^{\text{ndeg}}_{d,d}[1] = V_d \mathcal{E}^{SO(d,d)}(G,B) \quad (\text{generalized Eisenstein series}).
  \] (3.9)

Thanks to Type I / Heterotic duality, heterotic worldsheet instantons are mapped into ED-string instantons. Since \( F^4 \) terms are 1/2 BPS saturated, matching the spectrum of excitations, including their charges, was believed to be sufficient to match the threshold corrections even in the presence of (non)commuting Wilson lines [17,20] or after T-duality [21,22]. More recently, thanks to powerful localization techniques, a perfect match between threshold corrections in Heterotic and Type I (with D7-branes) has been found on \( T^2 \) for the specific choice of commuting Wilson lines breaking \( SO(32) \) to \( SO(8)^4 \) [23]. The somewhat unsatisfactory results of [24] for different breaking patterns with orthogonal or symplectic groups can be either interpreted as a failure of localization or as the need to include higher order terms. Notice that only for \( SO(8) \), ‘exotic’ string instantons should admit a field theory interpretation in terms of ‘octonionic’ instantons. It would be nice to further explore this issue in this or closely related context of \( \mathcal{N} = 1,2 \) theories in D=4 where heterotic worldsheet
instantons correcting the gauge kinetic function should be dual to ED-string (or other ED-brane) instantons [25]. A short review of the strategy to compute similar threshold corrections will be presented later on when we discuss Heterotic / Type I duality on $T^4/\mathbb{Z}_2$.

**Phenomenological considerations**

Despite some success in embedding (MS)SM in vacuum configurations with open and unoriented strings, there are few hampering properties at the perturbative level:

- Forbidden Yukawas in $U(5)$ (susy) GUT’s

\[
\begin{align*}
H_{5_{-1}}^d F_{5_{-1}}^c A_{10_{+2}} & \text{ OK but } H_{5_{+1}}^u A_{10_{+2}} A_{10_{+2}} \text{ KO}
\end{align*}
\]

forbidden by (global, anomalous) $U(1)$ invariance, though compatible with $SU(5)$ (yet no way $\epsilon^{abcde}$ from Chan-Paton)

- R-handed (s)neutrino masses $W_M = M_{RNN}$ forbidden by e.g. $U(1)_{B-L}$ in Pati-Salam like models $SO(6) \times SO(4) \to SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$

- $\mu$-term in MSSM $W_\mu = \mu H_1 H_2$ typically forbidden by extra (anomalous) $U(1)'s$

All the above couplings can be generated by ‘stringy’ instantons after integrating over the ‘non-dynamical’ moduli living on the world-volume of the ED$p'$-branes under consideration. These effects scale as $e^{-T_{EDp'}V_{EDp'}}$ and are non-perturbative in $g_s$, since $T_{EDp'} \approx 1/g_s(\alpha')^{p+1/2}$. Yet a priori they depend on different moduli (through the dependence of $V_{EDp'}$ on various $Z$’s) from the ones appearing in the gauge kinetic function(s) of background D$p'$-brane, so they cannot in general be identified with the standard ‘gauge’ instantons. Relying on the $g_s$ power counting introduced in [13], [14] the relevant are disks with insertions of the non-dynamical vertex operators $V_\Theta$ (connecting ED$p'$-ED$p'$) and $V_\lambda$ (connecting ED$p$-D$p$) with or without insertions of dynamical vertex operators $V_A$ etc, which correspond to the massless excitations of the vacuum configuration of (intersecting/magnetized) unoriented D$p$-branes [26], [27]. Disks without dynamical insertions yield the ‘instanton action’, with one dynamical
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vertex they produce classical profiles for $A$ etc. Disks with more insertions contribute to higher-order corrections. One loop diagrams with no insertions produce running couplings and subtle numerical prefactor that can cancel a given type of non-perturbative F-terms [28, 29].

**Anomalous $U(1)$’s and gauged PQ symmetries**

In general, a ‘naked’ chiral field $Z$ whose pseudoscalar axionic components $\zeta = \text{Im} Z$ shift under some local anomalous $U(1)$ cannot appear in a (super)potential term if not dressed with other chiral fields charged under $U(1)$. $U(1)$ invariance puts tight constraints on the form of the possible superpotential terms. Since the axionic shift is gauged it must be a symmetry of the kinetic term. This is only possible when no non-perturbative (world-sheet or D-brane instanton) corrections spoil the tree level (in fact perturbative) PQ symmetry. This means that the gauging procedure corresponds to turning on fluxes such that the potential instanton corrections in $Z$ are in fact disallowed. In practice, this means the corresponding wrapped brane is either anomalous (à la Freed-Witten) [30] or destabilized due to the flux [31].

Moreover, background fluxes (for both open and closed strings) can lift fermionic zero-modes. Various ‘perturbative’ studies have been carried out [32], [33].

### 3.3 ADHM from branes within branes

As already mentioned, the ADHM construction has a rather intuitive description in open string theory, whereby the gauge theory is realized on a stack of Dp-branes. D(p-4)-branes which are localized within the previous stack of branes behave as instantons [8].

Indeed, the WZ couplings on the Dp-brane worldvolume schematically reads

$$S_{WZ} = \int C_{p+1} + \int C_{p-1} \wedge Tr(F) + \int C_{p-3} \wedge Tr(F \wedge F) + ... \quad (3.10)$$

In particular a localized source of $C_{p-3}$ within a Dp-brane behaves like an instanton density $Tr(F \wedge F)$. Moreover, the ADHM data are nothing but the massless modes of open strings connecting the D(p-4)-branes with one another or with the background Dp-branes.
Let us take \( p = 3 \) for definiteness. The low-energy effective theory on the world-volume of \( N \) parallel D3-branes is \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory with gauge group \( U(N) \). Instanton moduli are described by the massless modes of open strings with at least one end on the D(-1)-brane stack. In this system of D3 and D(-1) branes there are three sectors of the open string spectrum to be considered. \( U(N) \) gauge fields and their superpartners are provided by the strings that start and end on D3-branes. Strings stretching between two D(-1)-branes give rise to \( U(K) \) non-dynamical gauge fields and their superpartners. These ‘fields’ represent part of the (super) ADHM data. The remaining (super) ADHM data are provided by the strings with one end on the D3-branes and the other one on the D(-1)-branes and vice-versa.

In the presence of D3-branes, (Euclidean) Lorentz symmetry is broken \( SO(10) \to SO(4) \times SO(6) \) and it is convenient to split ten ‘gauge bosons’, \( A_M \), into four gauge bosons \( a_\mu \), and six real ‘scalars’, \( \chi_i \). Similarly the \( d = 10 \) gauginos produce four non-dynamical Weyl ‘gauginos’, \( \Theta^A_\alpha \) as well as their antiparticles \( \bar{\Theta}^{\dot{\alpha}}_A \).

Introducing, for later convenience, three auxiliary fields \( D^c \), the D(-1)-D(-1) \( U(K) \) ‘geometric’ supermoduli are given by

\[
\begin{align*}
    a_\mu, \quad \chi_i, \quad \Theta^A_\alpha, \quad \bar{\Theta}^{\dot{\alpha}}_A, \quad D^c
\end{align*}
\]

while the 4\( KN \) D(-1)-D(3) ‘gauge’ supermoduli are

\[
\begin{align*}
    w^{\mu}_{\alpha i}, \quad \tilde{w}^i_{\dot{\alpha} u}, \quad \nu^{A u}_i, \quad \bar{\nu}^{A i}_u
\end{align*}
\]

with \( \mu = 1, \ldots, 4 \), \( \alpha, \dot{\alpha} = 1, 2 \) vector and spinor indices of \( SO(4) \), \( i = 1, \ldots, 6 \), \( A = 1, \ldots, 4 \) are vector and spinor indices of \( SO(6) \) respectively and \( c = 1, 2, 3 \). The matrices \( a_\mu, \chi_i \) describe the position of the instanton along the longitudinal and transverse directions to the D3-brane respectively. \( \Theta^A_\alpha \) and \( \bar{\Theta}^{\dot{\alpha}}_A \) are their superpartners. \( w^{\mu}_{\alpha i}, \tilde{w}^i_{\dot{\alpha} u} \) represent the D3-D(-1) open string in the \( NS \) sector, accounting for instanton sizes and orientations, and \( \nu^{A u}_i, \bar{\nu}^{A i}_u \) are their fermionic superpartners.

### 3.4 The D3-D(-1) action

By computing scattering amplitudes on the disk, one can determine the complete action that governs the dynamics of the light modes (or moduli)
of the system of D(-1) branes in the presence of D3-branes. It schematically reads \cite{16}

\[ S_{K,N} = Tr_k \left[ \frac{1}{g_0^2} S_G + S_K + S_D \right] \] (3.11)

with

\[ S_G = -[\chi_i, \chi_j]^2 + i \Theta_{\dot{a}A}[\chi_i^\dagger, \Theta_{\dot{b}B}] - D^c D^c \] (3.12)

\[ S_K = -[\chi_i, a_\mu]^2 + \chi^i \bar{w}^\dot{a} w_{\dot{a} \chi_i} - i \Theta^{\alpha A}[\chi_{AB}, \Theta^B] + 2i \chi_{AB} \bar{v}^A v^B \] (3.13)

\[ S_D = i \left( -[a_{\dot{a} \dot{a}}, \Theta^{A\dot{a}}] + \bar{v}^A w_{\dot{a}} + \bar{w}_{\dot{a}} v^A \right) \Theta^A_{\dot{a}} + D^c \left( \bar{w}^{c \dot{a}} w - i \bar{\eta}^{c \mu} [a^{\mu}, a^\nu] \right) \] (3.14)

where $\chi_{AB} \equiv \frac{1}{2} \Sigma_{AB}^{i} \chi_i$ and $\Sigma_{AB}^{i} = (\eta_{AB}^{\dot{a} \dot{b}}, i \bar{\eta}_{AB}^{\dot{a} \dot{b}})$ are given in terms of t’Hooft symbols and $g_0^2 = 4\pi(4\pi^2 \alpha’)^{-2} g_s$. Note that the action $S_{K,N}$ arises from the dimensional reduction of the D5-D9 action in six dimensions down to zero dimension. If there are v.e.v. for the six $U(N)$-adjoint scalars $\varphi_a$ belonging to the D3-D3 open string sector one has to add the term

\[ S_{\varphi} = tr_k \left[ \bar{w}^A (\varphi^i \varphi_i + 2 \chi^i \varphi_i) w_{\dot{a}} \right] + 2i \bar{v}^A \varphi_{AB} v^B \] (3.15)

to the action $S_{K,N}$. In the limit $g_0 \sim (\alpha’)^{-1} \rightarrow \infty$ ($g_0$ fixed) gravity decouples from the gauge theory and there are no contributions coming from $S_G$. $\Theta_{\dot{a}A}$ and $D^c$ fields become Lagrange multipliers for the super ADHM constraints:

\[ D^a : \quad [a_{\mu}, a_{\nu}] \eta^{\mu \nu}_{\dot{a}} + w_{\sigma_a} \bar{w} = 0 \quad \text{ADHM Eqs} \] (3.16)

\[ \Theta^A_{\dot{a}} : \quad [a_{\mu}, \Theta^A] \sigma_{\mu} + \bar{v}^A + v^A \bar{w} = 0 \quad \text{super ADHM Eqs} \] (3.17)

In this limit the multi-instanton ‘partition function’ becomes

\[ Z_{k,N} = \int_{\mathcal{M}} e^{-S_{k,N} - S_{\varphi}} = \frac{1}{\text{Vol} U(k)} \int_{\mathcal{M}} d\chi dD da d\theta d\bar{w} dw d\nu e^{-S_{k,N} - S_{\varphi}}. \] (3.18)

### 3.5 Vertex operators

Classical actions, (super)instanton profiles and non-perturbative contributions to scattering amplitudes can be derived by computing disk amplitudes with insertions of vertex operators for non-dynamical moduli $V_a$, $V_\chi$, $V_w$, $V_{w^\dagger}$ (ADHM data) and their superpartners \cite{13}, \cite{14}.
Vertex operators for ‘gauge’ instantons

Let us first start considering the vertex operator for a non dynamical gauge boson $a_\mu$ along the four D-D space-time directions. The vertex operator reads

$$V_a = a_\mu e^{-\varphi} \psi^\mu T_{K \times K}$$  \hfill (3.19)

where $\varphi$ arises from the bosonization of the $\beta, \gamma$ worldsheet super-ghosts, $\psi$ are the worldsheet fermions and $T_{K \times K}$ are $U(K)$ Chan-Paton matrices. For the non dynamical transverse scalars $\chi_i$ along the six internal D-D directions, the vertex operator reads

$$V_\chi = \chi_i e^{-\varphi} \psi^i T_{K \times K}$$  \hfill (3.20)

Similarly

$$V_\Lambda = \Theta_a(p) S_a e^{-\varphi/2} T_{K \times K}$$  \hfill (3.21)

with $a = 1, \ldots, 16$, produces four non-dynamical Weyl ‘gauginos’, $\Theta^A_\alpha$, and their antiparticles, $\bar{\Theta}^\dot{\alpha}_A$.

Bosonic vertex operators for low-lying D(p-4)-Dp strings, with multiplicity $K \times N$ and their conjugates are given by

$$V_w = \sqrt{\frac{g_s}{v_{p-3}}} w_\alpha e^{-\varphi} \prod_\mu \sigma_\mu S^\alpha T_{K \times N}$$  \hfill (3.22)

with $S^\alpha$ an $SO(4)$ spin field of worldsheet scaling dimension $1/4$. $\sigma_\mu$ are $\mathbb{Z}_2$ bosonic twist fields along the 4 relatively transverse N-D directions. $\Pi_\mu \sigma_\mu$ has total dimension $1/4 = 4/16$. $T_{K,N}$ denote the $K \times N$ Chan-Paton ‘matrices’. The super-partners of $w_\alpha$ are represented by vertex operators of the form

$$V_\nu = \sqrt{\frac{g_s}{v_{p-3}}} \nu_A e^{-\varphi/2} \prod_\mu \sigma_\mu S^A T_{K \times N}$$  \hfill (3.23)

where $S^A$ is an $SO(6)$ spin field of dimension $3/8$. Note that the overall normalization $\sqrt{g_s/v_{p-3}}$ is crucial for the correct field theory limit ($\alpha' \to 0$).

Vertex operators for ‘stringy’ instantons

Let us now consider ‘stringy’ instantons. The prototype is the D9, D1 system which has 8 N-D directions. The multi-(instanton) configuration of this system was first analyzed in [17]. The lowest lying modes of an
open string stretched between N D9 and K D1 branes are massless fermions with a given chirality (say Right) along the two common N-N directions. For Type I strings there are 32 such chiral fermions ($\lambda^A$) that precisely reproduce the gauge degrees of freedom of the 'dual' heterotic string [18]. In addition, in the $\mathcal{N} = (8,0)$ theory on the D1 world-sheet with $SO(8)$ R-symmetry group, there are 8 transverse bosons $X^I$ in the $8_v$ and as many Green-Schwarz type fermions $S^a$ of opposite chirality (say Left) in the $8_s$ of $SO(8)$. The 32 massless right-moving $\lambda^A$ are inert under the left-moving susy $Q_{a}$ in the $8_c$.

After compactification to $D = 4$ on a manifold with non-trivial holonomy some of the global supersymmetries are broken and the corresponding D1 world-sheet theory changes accordingly. In particular $SO(8)$ breaks to some subgroup.

### 3.6 D-branes at Orbifolds

A particularly promising class of configurations with nice phenomenological perspectives that also allow explicit non-perturbative computations are unoriented D-branes at singularities. Let us consider a stack of D3-branes at the orbifold singularity $T^d/\Gamma \approx R^d/\Gamma$ (locally), and let us take $\Gamma = Z_n$ for simplicity. At the singularity $N$ D3-branes group into stacks of $N_i$ ‘fractional’ branes, that cannot move away from the singularity, with $i = 0, 1, 2, ...$ labelling the conjugacy classes of $Z_n$. The gauge group $U(N)$ decomposes as $\Pi_i U(N_i)$.

\[
(Z_1, Z_2, Z_3) \approx (\omega^{k_1} Z_1, \omega^{k_2} Z_2, \omega^{k_3} Z_3)
\]

(3.24)

for simplicity we take $k_1 + k_2 + k_3 = 0 \ (mod \ n)$ that generically preserves $\mathcal{N} = 1$ supersymmetry.

The action on Chan-Paton factors is given by

\[
\rho(Z_n) = \rho_0(1_{N_0}, \omega^1 1_{N_1}, \omega^2 1_{N_2}, ..., \omega^{n-1} 1_{N_{n-1}})
\]

(3.25)

For $\alpha' \approx 0$, keeping only invariant components under (3.24), the resulting theory turns out to be an $\mathcal{N} = 1$ quiver gauge theory, in which vector multiplets $V$ are in the $N_i \tilde{N}_i$ representation while chiral multiplets $\Phi_i$ are in the $N_j \tilde{N}_i$ representation with $k_i + j - l = 0 \ (mod \ n)$ [36].

Twisted RR tadpole cancellation in sectors with non vanishing Witten
index can be written as $\text{tr} \rho(Z_n) = 0$ that ensures the cancellation of chiral non-abelian anomalies [37].

**Unoriented projection**

Possible unoriented projections depend on the parity of $n$ and the charge of the $\Omega$-plane. For $n$ odd there is only one possibility

$$N_0 = \tilde{N}_0, \quad N_i = \tilde{N}_{n-i} \quad (3.26)$$

For $n$ even there are two possibilities

$$N_0 = \tilde{N}_0, \quad N_i = \tilde{N}_{n-i}, \quad N_{n/2} = \tilde{N}_{n/2} \quad (3.27)$$

$$N_0 = \tilde{N}_{n/2}, \quad N_i = \tilde{N}_{n/2-i} \quad (3.28)$$

One should also impose the twisted RR tadpole cancellation condition (non vanishing Witten index) $\text{tr} \rho(Z_n) = \pm q^{\Omega}_n$ which from the field theory point of view is just the chiral anomaly cancellation [38].

Let us focus on the very rich and instructive case of $T^6/\mathbb{Z}_3 \approx R^6/\mathbb{Z}_3$.

### 3.7 Unoriented $R^6/\mathbb{Z}_3$ projection

In the remaining part of this Section, for illustrative purposes, we will discuss unoriented D-brane instantons on a stack of D3-branes located at an unoriented $R^6/\mathbb{Z}_3$ orbifold singularity.

Since $n = 3$ is odd, there is only one possible embedding in the Chan-Paton group up to the charge of the $\Omega^3^\pm$ planes. Introduction of $\Omega^3^+$-plane combined with local R-R tadpole cancellation leads to a theory with gauge group $G = SO(N_0) \times U(N_0 + 4) \times H_{\text{reg}}$, where $H_{\text{reg}}$ accounts for the Chan-Paton group of the ‘regular’ branes that can move into the bulk. We will henceforth assume that regular branes are far from the singularity and essentially decoupled from the local quiver theory. For $N_0 = 0$, we have $U(4)$ gauge group with 3 chirals in $6_{-2}$. In the presence of $\Omega^3^+$-plane we get a theory with $G = Sp(2N_0) \times U(2N_0 - 4) \times H_{\text{reg}}$ gauge group, *e.g.* for $2N_0 = 6$, we have $Sp(6) \times U(2)$ gauge group with 3 chirals in $(6,2_{+1}) + (1,3_{-2})$. 
3.7. UNORIENTED $R^6/Z_3$ PROJECTION

In both cases the anomalous $U(1)$ mixes with the twisted RR axion $\zeta$ in a closed string chiral (linear) multiplet $Z$ (gauging of axionic shift)

$$\delta A = d\alpha, \quad \zeta = -M_A \alpha$$

(3.29)

$$\mathcal{L}_{ax} = (d\zeta - M_A A)^2 + \frac{1}{f_\zeta} \zeta F \wedge F$$

(3.30)

Anomaly cancellation $\delta_\alpha [\mathcal{L}_{ax} + \mathcal{L}_{1-loop}] = 0 \leftrightarrow M_A / f_\zeta = t_3 = Tr \mathcal{Q}^3$.  

**Field Theory analysis**

As already mentioned ‘gauge’ instantons are expected to generate VY-ADS-like superpotentials. Neglecting $U(1)$’s for the time being, the two choices of $\Omega$-planes and, thus, of gauge group lead to superpotentials of the form

$$SU(4) : W = \frac{\Lambda^9}{\det_{I,J}(\epsilon_{abcd} A_I^{ab} A_J^{cd})}$$

(3.31)

$$Sp(6) \times SU(2) : W = \frac{\Lambda^9}{\det_{6x6}(\Phi^{ai})}$$

(3.32)

In string theory, $\Lambda^\beta = M_s^3 e^{-\frac{S}{T_s} - \frac{2}{T_s}} (\beta = 9$ here), shift of $Z$ compensates the $U(1)$ charge of the denominator! The ‘thumb rule” is that in each case there are two exact/unlifted fermionic zero-modes $n(\lambda_0) - n(\psi_0) = 2$. The rest is lifted by Yukawa interaction $Y_g = g \phi^\dagger \psi \lambda$. We now pass to describe the explicit computations with unoriented D-instantons

$$U(4)_{D3} \rightarrow U(K)_{D(-1)} \quad , \quad Sp(6)_{D3} \rightarrow O(K)_{D(-1)}$$

(3.33)

In both cases, there are two exact un-lifted fermionic zero-modes for $K = 1$.

**Non-perturbative superpotential for $Sp(6) \times U(2)$**

After the projection, in the D(-1)-D(-1) sector one has geometric supermoduli: $a_\mu$ (instanton position) and $\Theta_\alpha^0$ (Grassman coordinate), which

\[ \mathcal{L}_{GCSC} = E_{(ij)k} A^i \wedge A^j \wedge F^k \]  

are needed in the low-energy effective theory with non-trivial phenomenological consequences [40].
yield the $\mathcal{N} = 1$ superspace measure. There is no room for $D^c$ and $\bar{\Theta}_0^\alpha$ in the present case, since the relevant instanton is an $O(1)$ instanton in the $Sp(6)$ group and as such there are no super ADHM constraints.

In the D(-1)-D3 sector the gauge super-moduli are $w_\alpha^u, \nu^0u, \nu^Ia$ with $u = 1, \ldots 6$ $Sp(6)$, $a = 1, 2$ $U(2)$, and $I = 1, 2, 3$ $SU(3)$ 'pseudo' flavor indices\(^3\). Both $\Theta_{0a}$ and $D^c$ are projected out. Taking into account the interactions with the D3-D3 excitations $\Phi^{Iua} = \phi^{Iua} + \ldots$, the instanton action can be reduced to the form

$$S_{D(-1)-D(3)} = w_\alpha^u \tilde{\phi}_{Iua} \phi^{Iua} w_u \dot{\alpha} + \nu^0u \nu^Ia \tilde{\phi}_{uIa}$$  \hspace{1cm} (3.34)

Integrations over gauge super-moduli are gaussian and the final result can be written as

$$\int d^6w d^3\nu d^3\nu I e^{S_{D(-1)-D(3)}} = \frac{\det(\tilde{\phi}_{uIa})}{\det(\phi^{Iua} \phi_{uIa})}$$  \hspace{1cm} (3.35)

Including D(-1)-D(-1) action and one-loop contribution, up to a non-vanishing numerical constant, we get

$$\int d^4ad^2\Theta \mu^9 e^{2\pi i\tau(\mu)} = \int d^4xd^2\theta \frac{\Lambda^9}{\det(\phi^{Iua})}$$  \hspace{1cm} (3.36)

to a non-zero numerical constant.

**Non-perturbative superpotential for $U(4)$**

As explained in [36] for $U(4)$ gauge theory with three chiral multiplets in the $6$, the D(-1)-D(-1) geometric 'supermoduli' are $\alpha^0(0)$ (instanton position), $\chi_{I(-2)}, \lambda^I_{(2)}$ (internal) and $\Theta^0_{\alpha(0)}, \bar{\Theta}_{0\alpha}$ (Grassman coordinates), which give $\mathcal{N} = 1$ superspace measure. D(-1)-D(3) gauge 'supermoduli' are $w_\alpha^u(\nu), \bar{w}_\alpha^{u(1)}, \nu^0(\nu), \nu^Ia(\nu)$ with $u = 1, \ldots 4$ $U(4)$ and $I = 1, 2, 3$ $SU(3)$ 'pseudo' flavor respectively. Notice that the subscript in parentheses represents the charge under $U(1)_{k_1}$. Taking into account the interactions with D3-D3 excitations $\Phi^{Iuv} = \phi^{Iuv} + \ldots$, the fermionic integration will lead to the determinant

$$\Delta_F = \rho^8 \epsilon^{u_1u_2u_3u_4} \epsilon^{v_1v_2v_3v_4} \epsilon^{w_1w_2w_3w_4} X_{u_1u_2v_1v_2} X_{u_3u_4v_3v_4} Y_{w_1w_2} Y_{w_3w_4}$$  \hspace{1cm} (3.37)

with $X = \epsilon^{IJK} \chi_I \chi_J \chi_K$, $Y_{uv} = U_{uv}$, and $\rho, U$ are defined by $w_{ua} = \rho U_{ua}$, $\bar{w}_{\dot{a}u} = \rho \bar{U}_{\dot{a}u}$, $\bar{U}_{\dot{a}u} \bar{U}_{\dot{b}u} = \delta^\beta_\dot{a}$ . Integration over bosonic 'moduli' is more

\(^3\)In string theory, $SU(3)$ is an accidental symmetry of the two-derivative effective action
involved. For arbitrary choices of the v.e.v’s \( \bar{\phi}_{Iuv} \) and \( \phi_{Iuv} \), even along the flat directions, integration over \( U \) represents a difficult task. Fortunately for the choice \( \phi_{Iuv} = \eta_{Iuv} \), the full \( \phi \)-dependence can be factorized. And after rescaling \( \rho^2 \to \rho^2/(\phi \bar{\phi}) \), \( \chi_I \to \phi \chi_I \), \( \bar{\chi}_I \to \bar{\phi} \bar{\chi}_I \), \( X_{u_1v_1u_2v_2} \to \epsilon^{I_1I_2I_3} \bar{\chi}_{I_1} \bar{\eta}_{I_2u_1u_2} \bar{\eta}_{I_3v_1v_2} \) the \( \phi \)-independent integral \( I_B \) becomes

\[
I_B = \int d\rho d^2\mu d^3\chi d^3\bar{\chi} \Delta_F e^{-\tilde{S}_B} \tag{3.38}
\]

where \( \tilde{S}_B = -\rho^2(1 + \eta_{Iuv}^T Y_{uv} \chi_I + \bar{\eta}_{Iuv} \bar{Y}_{uv} \bar{\chi}_I + \bar{\chi}_I \chi^I) \). Restoring the \( SU(4) \) gauge and \( SU(3) \) ‘flavor’ invariance the superpotential follows after promoting \( \phi^I \to \Phi^I \):

\[
S_W = c \int d^4d^2\theta \frac{\epsilon_{I}}{\Phi^6} \frac{e^{2\pi ir(\mu)}}{\Phi^6} I_B = \int d^4x d^2\theta \frac{\Lambda^9}{\det_{3\times 3} \epsilon [e_{u_1...u_4} \Phi^{J_1u_1u_2} \Phi^{J_3u_3u_4}]} \tag{3.39}
\]

up to a non-zero numerical constant.

### 3.8 Exotic/Stringy instantons

EDp-branes on unoccupied nodes of the quiver produce exotic instanton effects. The gauge theory on EDp’ is of the same kind as on EDp (like 8 N-D directions, periodic sector).

Grassmann integration over chiral fermions \( \nu \)'s at intersections produces positive powers of \( \Phi \). The resulting non perturbative superpotential can grow at large VEV’s, which is incompatible with field theory intuition (asymptotic freedom) for standard ‘gauge’ instantons. Yet it is compatible with gauge invariance and ‘exotic’ scaling

\[
e^{-A(C')/\ell_s'^{+1}} \neq e^{-1/g^2_M} \tag{3.40}
\]

For generic \( K \), there are many unlifted fermionic zero-modes and one gets higher derivative F-terms, threshold corrections, ... or dangerous bosonic zero-modes. For specific \( K \), there are only two unlifted zero-modes \( (d^2\theta) \) and one gets superpotential terms. For ED1, the relevant \( \nu \)'s are in the direction of the worldsheet.
CHAPTER 3. INSTANTONS IN STRING THEORY

$U(4)$ model: non-perturbative masses

Let us consider $U(4)$ model, $\Theta^0_\alpha, X_\mu$ plus 4 $\nu^\mu$ that couple to one complex component $\phi_{uv}$ (related to $C$) through

$$S_{D3-ED3} = \phi_{uv}^{\mu\nu} + \ldots \quad (3.41)$$

The superpotential generated by ED-strings wrapping 2-cycles $C$ passing through the singularity schematically reads

$$W(\Phi) = \sum_C M_s e^{-A(C)/g_s \alpha'} \Phi_C^2 \quad (3.42)$$

and thus represents a mass terms for $\Phi \approx A$.

Effect of multi-instanton are hard to evaluate. Heterotic/Type I duality may help clarifying the procedure.

3.9 Effect of fluxes

Phenomenological applications of string theory and realistic model building require compactifications. Four dimensional compactifications of Type II string theories which preserve $\mathcal{N} = 1$ supersymmetry in the presence of intersecting or magnetized D-branes are very interesting. Gauge interactions can be realized with space-filling D-branes that partially or totally wrap the internal six dimensional space. Adjoint gauge fields are given by the massless excitations of open strings that start and end on the same stack of D-branes. Open strings stretched between different stacks provide bi-fundamental matter fields. From the closed string point of view D-branes are sources for Type II supergravity fields, which have a non-trivial profile in the bulk. The effective actions of these models describe interactions of both open string (boundary) and closed string (bulk) degrees of freedom and have the generic structure of $\mathcal{N} = 1$ supergravity in four dimensions coupled to vector and chiral multiplets. Four dimensional $\mathcal{N} = 1$ supergravity theories are specified by the choice of the gauge group $G$, by a Kähler potential $K$ and a superpotential $W$. The Kähler potential is real and the superpotential is holomorphic function of some chiral superfields $\Phi^i$. The expectation values of these chiral multiplets, which parametrize the supergravity vacuum minimizes the scalar potential

$$V = e^K (D_i \bar{W} D^i W - 3|W|^2) + D^a D_a, \quad (3.43)$$
3.9. EFFECT OF FLUXES

where $D_i W \equiv \partial_{\Phi} W + (\partial_{\Phi} K) W$ is the Kähler covariant derivative of the superpotential and the $D^a$ with $a = 1, \ldots, \dim(G)$ are the D-terms. Supersymmetric vacua correspond to the solutions of $\partial_{\Phi} V = 0$ equations satisfying D-flatness and F-flatness conditions $D^a = D_i W = 0$. In the case of Type IIB string theory on a Calabi-Yau three-fold in the presence of D3-branes, [33], the chiral superfields $\Phi^i$ consist of the fields $U_r$ and $T_m$, which parametrize the deformations of the complex and Kähler structures of the three-fold, of the axion-dilaton field and also of some other multiplets coming from the open strings stretching between D-branes. The axion-dilaton field $\tau = C_0 + i e^{i \varphi}$ is given in terms of the R-R scalar $C_0$ and the dilaton $\varphi$. The corresponding low energy $\mathcal{N} = 1$ supergravity theory has a highly degenerated vacuum. One way of lifting this degeneracy, at least partially, is to add the internal 3-form fluxes of the bulk theory which generate a superpotential of the form:

$$W_{\text{flux}} = \int G_3 \wedge \Omega$$

(3.44)

where $G_3$ is the complex 3-form flux given in terms of the R-R and NS-NS fluxes $F$ and $H$ via $G_3 = F - \tau H$ and $\Omega$ is a holomorphic $(3,0)$-form of the Calabi-Yau three-fold. The flux superpotential (3.44) depends on $\tau$ and on the complex structure parameters $U_r$ which specify $\Omega$. An unbroken $\mathcal{N} = 1$ supersymmetry requires the flux $G_3$ to be an imaginary anti-selfdual 3-form of $(2,1)$ type, since the F-terms $D_{U_r} W_{\text{flux}}$, $D_{\tau} W_{\text{flux}}$ and $D_{T_m} W_{\text{flux}}$ are proportional to the $(1,2)$, $(3,0)$ and $(0,3)$ components of the $G$-flux, respectively. These F-terms can also be interpreted as the ‘auxiliary’ $\theta^2$-components of the kinetic functions for the gauge theory defined on the space-filling branes, and thus are soft supersymmetry breaking terms for the brane-world effective action. Such soft terms in flux compactifications give effects like, for instance, induced masses for the gauginos and the gravitino. Non-perturbative contributions to the effective actions may also play an important role in the moduli stabilization. They have phenomenologically relevant implications for string theory compactifications. Non-perturbative effects, coming from wrapped Euclidean branes, may lead to the generation of a non-perturbative superpotential of the following form:

$$W_{\text{n.p.}} = \sum_{\{k_A\}} c_{\{k_A\}} (\Phi^i)^{2 \pi i \sum_A k_A \tau_A}$$

(3.45)

where the index $A$ labels the cycles wrapped by the instantonic branes, $\tau_A$ is the complexified gauge coupling of a D-brane wrapping the cycle $A$. 

and the sum is over the instanton numbers $k_A$. $c_{\{k_A\}}(\Phi^i)$ are holomorphic functions of the chiral superfields. The specific forms of these functions depend on the details of the model under consideration. The coupling $\tau_A$ generally depend on the axion-dilaton modulus $\tau$ and the Kähler parameters $T^m$ that describe the volumes of the cycles wrapped by D-branes. When the cycle $A$ is wrapped by some physical D-branes one has gauge instantons. The case when the cycle is not wrapped by any D-brane in the background corresponds to stringy instantons. In both cases the generated superpotential has the form (3.45). Fluxes and non-perturbative effects contribute to the total superpotential $W = W_{\text{flux}} + W_{\text{n.p.}}$. Thus, new possibilities to get supersymmetric vacua arise. Indeed, the derivatives $D_{U_r}W_{\text{flux}}$, $D_\tau W_{\text{flux}}$ and $D_{T^m}W_{\text{flux}}$ can now be compensated by $D_{U_r}W_{\text{n.p.}}$, $D_\tau W_{\text{n.p.}}$ and $D_{T^m}W_{\text{n.p.}}$ so that also the (1,2), (3,0) and (0,3) components of $G_3$ may become compatible with supersymmetry and help to remove the vacuum degeneracy.

Besides generating perturbative superpotential $W_{\text{flux}}$, fluxes lift some of the zero-modes of the instanton background. As a result, new types of non-perturbative couplings arise. Following [33], let us consider the case of D3-branes at a $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity and engineer an $\mathcal{N} = 1$ $U(N_0) \times U(N_1)$ quiver gauge theory with bi-fundamental matter fields. This quiver theory can be thought of as a local description of the Type IIB Calabi-Yau compactification on the toroidal orbifold $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In this context the numbers $N_0$ and $N_1$ of the D3-branes can be arbitrary and orientifold planes for tadpole cancellation are not needed. Gauge and stringy instantons are realized by means of D-instantons. When one introduces background fluxes of type $G_{(3,0)}$ and $G_{(0,3)}$ and studies the induced non-perturbative interactions in the presence of gauge and stringy instantons, one finds a very rich class of non-perturbative effects from ‘exotic’ superpotential terms to non-supersymmetric multi-fermion couplings.

Stringy instantons in the presence of $G$-fluxes can generate non-perturbative interactions even for $U(N)$ gauge theories. In the case without fluxes an orientifold projection is required to solve the problem of the neutral fermionic zero-modes. Since $G_{(3,0)}$ and $G_{(0,3)}$ components of $G_3$ are related to the gaugino and gravitino masses, the non-perturbative flux-induced interactions can be regarded as the analog of the Affleck-Dine-Seiberg (ADS) superpotentials for gauge/gravity theories with soft supersymmetry breaking terms. In particular, the presence of the $G_{(0,3)}$
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flux has no effect on the gauge theory at a perturbative level, but it generates new instanton-mediated effective interactions.

Besides this supergravity method there is also world-sheet approach. This approach allows to obtain the flux induced couplings in a unified way, it is applicable to more general brane configurations, with or without magnetization, with twisted or untwisted boundary conditions. In the four dimensional compactifications of Type II string theories preserving $\mathcal{N} = 1$ supersymmetry in the presence of intersecting or magnetized D-branes one can add internal antisymmetric fluxes both in the NS-NS and in the R-R sector of the bulk theory. These fluxes are important for moduli stabilization, supersymmetry breaking and may also generate non-perturbative superpotentials. At a perturbative level internal 3-form fluxes are encoded in a bulk superpotential from which F-terms can be obtained using standard supergravity methods. These F-terms can be viewed as the $\theta^2$ auxiliary components of the kinetic functions for the gauge theory living on the space-filling branes. Thus, these are soft supersymmetry breaking terms for the brane-world effective action. The non-perturbative sector of the effective action coming from string theory compactifications is also important to study. The computational tool to study these effects using systems of branes with different boundary conditions has been developed in [13], [14], [15]. These techniques allow to reproduce the known instanton calculus of supersymmetric field theories and also can be generalized to more exotic instanton configurations for which there are no field theory methods available. The study of these exotic instantons has led to interesting results related to moduli stabilization, (partial) supersymmetry breaking, fermion masses and Yukawa couplings [26], [57]. There are neutral anti-chiral fermionic zero modes, which totally decouple from all other exotic instanton moduli. In the case of gauge theory instanton this does not happen. In this case neutral anti-chiral fermionic zero modes act as Lagrange multipliers for the fermionic ADHM constraints [13]. Therefore, to get non-vanishing contributions to the effective action from exotic instantons, one needs to remove these anti-chiral zero modes [36], [58] or lift them by some mechanism [59]. The presence of internal background fluxes may allow for such a lifting and gives an idea of the existence of the interplay among soft supersymmetry breaking, moduli stabilization, instantons and more generally non-perturbative effects in the low energy theory.
In [32], [33], by evaluating disk amplitudes involving two open string vertex operators at a generic intersection and one closed string vertex representing the background fluxes, authors have given the couplings of NS-NS and R-R fluxes to various types of D-branes including instantonic ones using world-sheet approach. This approach, being in full agreement with the derivation of the flux couplings in the brane effective actions based on supergravity methods, is applicable also to more general brane configurations which involve fields with twisted boundary conditions. It allows to study the modification of the action by R-R and NS-NS fluxes which gives the measure of integration on the moduli space of instantons. Considering an orbifold compactification of Type IIB string theory with fractional D-branes preserving $\mathcal{N} = 1$ supersymmetry and studying the flux-induced fermionic mass terms on space- filling and on instantonic branes, it has been shown that there is a relation between the soft supersymmetry breaking and the lifting of some instanton fermionic zero-modes. This may lead to new types of non-perturbative couplings in brane-world models.

Let us consider string amplitudes between two massless open string fermions and the background closed string flux. This is a mixed open/closed string amplitude on a disk with mixed boundary conditions in general. For the two fermionic open string vertices and one closed string R-R vertex one has the following amplitude:

$$A_F = \langle V_\Theta(x) V_F(z, \bar{z}) V_{\Theta'}(y) \rangle = c_F \Theta_{A_1} (F R_0)_{A_2 A_3} \Theta'_{A_4} \times A^{A_1 A_2 A_3 A_4}$$

(3.46)

where $V_\Theta$ is the vertex operator for the lowest fermionic excitation $\Theta_A$ of the open string and $V_F$ is the vertex operator for the closed string field strengths of the antisymmetric tensor fields in the R-R sector of Type IIB theory in the $(-\frac{1}{2}, -\frac{1}{2})$ superghost picture. $c_F = c_{(p+1)} \mathcal{N}_\Theta \mathcal{N}_{\Theta'} \mathcal{N}_F$ stands for normalizations of the vertex operators and $c_{p+1}$ is the topological normalization of any disk amplitude with the boundary conditions of a Dp-brane. $F_{AB}$ is the bi-spinor polarization which consist of all R-R field strengths of the Type IIB theory via

$$F_{AB} = \sum_{n=1,3,5} \frac{1}{n!} F_{M_1...M_n} (\Gamma^{M_1...M_n})_{AB}.$$  

(3.47)

Note that we discuss only untwisted closed string vertices. Thus, in order to get a non vanishing amplitude, open string vertices must have opposite twists. In the presence of D-branes the left and right moving components of the vertex operator $V_F$ must be identified via the reflection rules. As
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a consequence $F_{AB}$ is replaced with $(FR_0)_{AB}$, where $R_0$ is the reflection matrix in the adjoint representation of the rotation group. The 4-point correlator $A^{A_1 A_2 A_3 A_4}$ is given by:

$$A^{A_1 A_2 A_3 A_4} = (\Gamma_M)^{A_1 A_4}(\Gamma^M I_1)^{A_2 A_3} + (\Gamma^M I_2)^{A_1 A_3}(\Gamma^M)^{A_2 A_4}$$  \hspace{1cm} (3.48)

where $I_1$ and $I_2$ are $\vec{\nu}$-dependent diagonal matrices with entries:

$$(I_1)_{A_3}^{A_1} = \frac{1}{2} e^{-i\sigma_3 A} \left( e^{-2\pi i(\alpha' t - \vec{\nu} \cdot \vec{e}_3)} - 1 \right) B(\alpha' s; \alpha' t - \vec{\nu} \cdot \vec{e}_3)$$  \hspace{1cm} (3.49)

$$(I_2)_{A_3}^{A_1} = \frac{1}{2} e^{-i\sigma_3 A} \left( e^{-2\pi i(\alpha' t - \vec{\nu} \cdot \vec{e}_3)} - 1 \right) B(\alpha' s + 1; \alpha' t - \vec{\nu} \cdot \vec{e}_3)$$  \hspace{1cm} (3.50)

where $B(a; b)$ is the Euler $\beta$-function. Plugging 4-point correlator (3.48) into (3.46) one finds:

$$A_F = -8c_F \Theta \Gamma^M [FR_0(2 I_1 - I_2)]_M + \frac{4c_F}{3!} \Theta \Gamma^{MNP} [FR_0 I_2]_{MNP}.$$  \hspace{1cm} (3.51)

This amplitude describes the tree-level bilinear fermionic couplings induced by R-R fluxes on a general brane intersection.

Fermionic couplings induced by the NS-NS 3-form flux arise from the following mixed disk amplitude:

$$A_H = \langle V_{\Theta}(x) V_H(z, \bar{z}) V_{\Theta'}(y) \rangle = c_H \Theta \partial BR_0 [\partial BR_0]_{MNP} \Theta^B \times A^{AB; MNP}$$  \hspace{1cm} (3.52)

where the NS-NS 3-form flux $H$ has the vertex operator $V_H(z, \bar{z})$. $A^{AB; MNP}$ is the 4-point correlator and $c_H = C_{p+1} N_\Theta N_\Theta' N_H$ is the normalization factor. $R_0$ is the reflection matrix in the vector representation of the rotation group. The NS-NS counterpart of the R-R amplitude (3.51) on a generic D-brane intersection is

$$A_H = -4c_H \Theta \Gamma^N \Theta^M [\partial BR_0(2 I_1 - I_2)]_{MN} + 2c_H \Theta \Gamma^{MNP} [\partial BR_0 I_2]_{MNP}$$  \hspace{1cm} (3.53)

and shares with it the same type of fermionic structures. At leading order in $\alpha'$ these amplitudes describe fermionic mass terms induced at linear order in the R-R and NS-NS fluxes for open string modes. $R_\sigma$ and $R_\sigma'$ are boundary reflection matrices in the vector and spinor representations of the rotation group, respectively. The boundary conditions are encoded in the reflection matrices and in the open string twists $\vec{\nu}$. We consider constant background fluxes coupled to untwisted open strings, i.e., strings starting and ending on a single stack of D-branes. This corresponds to set $\vec{\nu} = 0$. Constant background fluxes allow to set the momentum of the
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closed string vertices to zero. As a consequence $2I_1 = I_2 = -i\pi$. Using the amplitudes (3.51) and (3.53) one can see that the fermionic couplings with a single $\Gamma$ matrix vanish and only the terms with three $\Gamma$’s survive. Taking into account the fact that in both amplitudes the untwisted fermions $\Theta$ and $\Theta'$ describe the same field and differ only by opposite momentum, the total amplitude becomes:

$$ A \equiv A_F + A_H = -2\pi i \Theta \Gamma^{MNP} \Theta \left[ \frac{c_F}{3} (F^R_0)_{MNP} + c_H (\partial B R_0)_{MNP} \right] $$

(3.54)

It is clear that once the flux configuration is given, the structure of the fermionic couplings for different types of D-branes depends crucially on the boundary reflection matrices $R_0$ and $R_0$. Let us consider only 3-form fluxes. In general the R-R piece of (3.54) is non-zero for 1-form, 3-form, and 5-form fluxes. For the 3-form flux the bi-spinor is:

$$ F_{AB} = \frac{1}{3!} F_{MNP} (\Gamma^{MNP})_{AB} $$

(3.55)

In this case the normalization factors can be better specified and are related via the string coupling constant:

$$ c_F = g_s c_H $$

(3.56)

The amplitude can be rewritten as:

$$ A = -\frac{2\pi i}{3!} c_F \Theta \Gamma^{MNP} \Theta T_{MNP} $$

(3.57)

where

$$ T_{MNP} = (F^R_0)_{MNP} + \frac{3}{g_s} (\partial B R_0)_{[MNP]} $$

(3.58)

To study the flux induced couplings for gauge theories and instantons in four dimensions, one has to split ten dimensional indices $M,N,... = 0,1,...,9$ into four-dimensional space-time indices $\mu,\nu,... = 0,1,2,3$ and six dimensional indices $m,n,... = 4,5,...,9$ for the internal space. Background fluxes, which carry space-time indices break the four-dimensional Lorentz invariance and generically give rise to deformed gauge theories [14], [34], [35]. Let us concentrate only on the internal 3-form fluxes, like $F_{mnp}$ or $(\partial B)_{mnp}$. They preserve four dimensional Lorentz invariance, and the fermionic amplitudes are of the form:

$$ A = -\frac{2\pi i}{3!} c_F \Theta \Gamma^{mnp} \Theta T_{mnp} $$

(3.59)
Let us discuss unmagnetized space-filling D-branes, \( \vec{\vartheta} = 0 \). To do so one has to go to Minkowskian signature. In this case \( \Theta \) becomes a Majorana-Weyl spinor in ten dimensions. For an unmagnetized Dp-brane which fills four dimensional Minkowski space and possibly extends also in some internal directions, the reflection matrices are very simple:

\[
R_0 = \text{diag}(\pm 1, \pm 1, \ldots)
\]

(3.60)

for the reflection matrix in the vector representation. Plus sign of the matrix element refers to the longitudinal direction and the minus sign to the transverse one. The reflection matrix in the spinor representation has the following form:

\[
\mathcal{R}_0 = \Gamma^{p+1} \cdots \Gamma^9
\]

(3.61)

Using this, one observes from (3.58) that \( T_{mnp} \) is a real tensor, so that the total fermionic amplitude (3.59) is also real in view of equation \( \Theta \Gamma^{mnp} \Theta = - (\Theta \Gamma^{mnp} \Theta)^* \), which Majorana-Weyl spinor satisfies in ten dimensions. The explicit expression of \( T_{mnp} \) is particularly simple in the case of brane configurations which respect the 4+6 structure of the spacetime, i.e. D3 and D9-branes. For space-filling D3-branes all the internal directions are transverse, so that \( R_0|_{\text{int}} = -1 \) and \( \mathcal{R}_0 = \Gamma^4 \cdots \Gamma^9 \). So that one has:

\[
T_{mnp} = (*_6 F)_{mnp} - \frac{1}{g_s} H_{mnp}
\]

(3.62)

where \( *_6 \) denotes the Poincaré dual in the six dimensional internal space and \( H_{mnp} = 3 \partial_m B_{np} = \partial_m B_{np} + \partial_n B_{pm} + \partial_p B_{mn} \). For D9-branes all internal indices are longitudinal. The internal longitudinal indices we will denote by \( \hat{m}, \hat{n}, \ldots \). In the D9-brane case \( R_0 = 1 \) and \( \mathcal{R}_0 = \Gamma^4 \cdots \Gamma^9 \). So that one has:

\[
T_{\hat{m}\hat{n}\hat{p}} = F_{\hat{m}\hat{n}\hat{p}} + \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}
\]

(3.63)

For tadpole cancellation there always should be orientifold 9-planes. The corresponding orientifold projection eliminates NS-NS flux \( H_{\hat{m}\hat{n}\hat{p}} \). Then the coupling tensor for D9-branes reduces to

\[
T_{\hat{m}\hat{n}\hat{p}} = F_{\hat{m}\hat{n}\hat{p}}
\]

(3.64)

For D7-branes the longitudinal internal indices \( \hat{m}, \hat{n}, \ldots \) take four values and the transverse indices \( p, q, \ldots \) take two values. Non vanishing components of the \( T \) tensor for D7-branes are the following:

\[
T_{\hat{m}\hat{n}\hat{p}} = \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}, \quad T_{\hat{m}\hat{n}p} = F_{\hat{m}\hat{n}} g_{q p} + \frac{1}{g_s} H_{\hat{m}\hat{n}p}, \quad T_{\hat{m}np} = -\frac{1}{g_s} H_{\hat{m}np}
\]

(3.65)
O7-planes required for tadpole cancellation remove all $F$ and $H$ components with an even number of transverse indices, so that one is left with

$$ T_{\hat{m}\hat{n}\hat{p}} = F_{\hat{m}\hat{n}}^q \epsilon_{qp} + \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}} \quad (3.66) $$

For D5-branes the longitudinal internal indices run over two values and the transverse indices over four values. Non vanishing components of $T$ are:

$$ T_{\hat{m}\hat{n}\hat{p}} = \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}, \quad T_{\hat{m}\hat{n}\hat{p}} = -\frac{1}{2} F_{\hat{m}}^{qr} \epsilon_{qrnp} - \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}, \quad T_{\hat{m}\hat{n}\hat{p}} = -\frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}} \quad (3.67) $$

O5-planes enforce an orientifold projection $\Omega I_4$ which removes the components of $H$ with even number of transverse indices and of $F$ with odd number of transverse indices. Thus $T_{\hat{m}\hat{n}\hat{p}}$ becomes

$$ T_{\hat{m}\hat{n}\hat{p}} = -\frac{1}{2} F_{\hat{m}}^{qr} \epsilon_{qrnp} \quad (3.68) $$

It is interesting to observe that while for D9 and D5-branes the fermionic couplings depend either on $F$ or on $H$, for D3 and D7-branes they depend on a combination of the R-R and NS-NS 3-forms. This follows from the fact that O3 and O7-planes act on the same way on R-R and NS-NS fluxes. By introducing the complex 3-form

$$ G = F - \frac{i}{g_s} H \quad (3.69) $$

one can rewrite the D3-brane coupling (3.62) as

$$ T_{mnp} = (\ast_6 F)_{mnp} - \frac{1}{g_s} H_{mnp} = Re(\ast_6 G - iG)_{mnp} \quad (3.70) $$

This confirms the fact that an imaginary self-dual (ISD) 3-form flux $G$ does not couple to unmagnetized D3- branes. The fermionic couplings for D7-branes (3.66) can also be rewritten:

$$ T_{\hat{m}\hat{n}\hat{i}} = iG_{\hat{m}\hat{n}\hat{i}} \quad T_{\hat{m}\hat{n}\hat{i}} = -iG^*_{\hat{m}\hat{n}\hat{i}} \quad (3.71) $$

where $i$ and $\bar{i}$ denote the complex directions of the plane transverse to the D7-branes. (3.71) is in agreement with the structure of soft fermionic mass terms found in [39].

Unmagnetized Euclidean branes that are transverse to the four dimensional space-time and extend partially or totally in the internal directions
are relevant to discuss non-perturbative instanton effects in the branes models. In this case it is necessary to work in a space with Euclidean signature. Then the massless fermions $\Theta$ are not Majorana-Weyl spinors anymore. There is no issue about the reality of a fermionic amplitude and the coupling tensor $T$ is in general complex. For D-instantons (or D(-1)-branes) all ten directions are transverse. In this case the reflection matrices in the vector and spinor representations are:

$$R_0 = -1, \quad \mathcal{R}_0 = \Gamma^0 \Gamma^1 \cdots \Gamma^9 \equiv i \Gamma^E_{11}$$  

(3.72)

where $\Gamma^E_{11}$ is the chirality matrix in ten Euclidean dimensions. For D-instantons the $T$ tensor is:

$$T_{mnp} = -i F_{mnp} - \frac{1}{g_s} H_{mnp} = -i G_{mnp}$$  

(3.73)

In the case of Euclidean instantonic 5-branes, E5-branes, extending in the six internal directions the reflection matrices are:

$$R_{0|\text{int}} = 1, \quad \mathcal{R}_0 = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 = -i \Gamma^4 \cdots \Gamma^9 \Gamma^E_{(11)}$$  

(3.74)

The fermionic coupling $T$ for unmagnetized E5-branes has the following form:

$$T_{\hat{m}\hat{n}\hat{p}} = i (\ast_6 F)_{\hat{m}\hat{n}\hat{p}} + \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}$$  

(3.75)

In an orientifold model with O9-planes this coupling reduces to:

$$T_{\hat{m}\hat{n}\hat{p}} = i (\ast_6 F)_{\hat{m}\hat{n}\hat{p}}$$  

(3.76)

E3-branes extend along four of the six internal directions. Flux induced fermionic couplings $T$ for E3 are of the form:

$$T_{\hat{m}\hat{n}\hat{p}} = \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}, \quad T_{\hat{m}\hat{n}\hat{p}} = -\frac{i}{2} \epsilon_{\hat{m}\hat{n}\hat{p}\hat{q}} F^{\hat{p}\hat{q}}_{\hat{m}\hat{n}\hat{p}} + \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}, \quad T_{\hat{m}\hat{n}\hat{p}} = -\frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}$$  

(3.77)

The appropriate orientifold projections remove $H_{\hat{m}\hat{n}\hat{p}}$ and $H_{\hat{m}\hat{n}\hat{p}}$. Thus one has only

$$T_{\hat{m}\hat{n}\hat{p}} = -\frac{i}{2} \epsilon_{\hat{m}\hat{n}\hat{p}\hat{q}} F^{\hat{p}\hat{q}}_{\hat{m}\hat{n}\hat{p}} + \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}$$  

(3.78)

The fermionic couplings for the E1-branes are:

$$T_{\hat{m}\hat{n}\hat{p}} = \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}, \quad T_{\hat{m}\hat{n}\hat{p}} = -i \epsilon_{\hat{m}\hat{n}\hat{q}} F^{\hat{q}}_{\hat{n}\hat{p}} - \frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}, \quad T_{\hat{m}\hat{n}\hat{p}} = -\frac{1}{g_s} H_{\hat{m}\hat{n}\hat{p}}$$  

(3.79)
$H_{\hat{m}\hat{n}\hat{p}}$ gets removed by the orientifold projection when the E1-branes are considered together with D5/D9-branes and the corresponding orientifold planes. One observes that in the presence of E-branes the space-time filling Dp-branes live in the Euclidean ten-dimensional space. The couplings of such Dp-branes are again given by the same linear combinations of $F$ and $H$ like in the Minkowskian case, since $R_0$ and $R_0$ are trivial along the would be time direction.

One can generalize the discussion to the branes with a non-trivial magnetization on their worldvolume for which the longitudinal coordinates satisfy non-diagonal boundary conditions. In the setup discussed above one can introduce a worldvolume gauge field $A$ that couples to the open string end-points and obtain a magnetization $F_0 = F_\pi = 2\pi\alpha'(dA)$. Above discussed R-R and NS-NS background fluxes can be used and one can study the new couplings induced by the worldvolume magnetization using the following reflection matrices:

$$R_\sigma = (1 - F_\sigma)^{-1}(1 + F_\sigma)$$

$$R_\sigma = \pm \prod_{I=1}^{5} e^{i\pi\theta_I^I\Gamma^I} = \pm \prod_{I=1}^{5} \frac{1 + if_I^I\Gamma^I}{\sqrt{1 + (f_I^I)^2}}$$

To complete the analysis and to make the structure of the flux-induced fermionic masses more clear let us write the fermion bilinear $\Theta \Gamma^{mnp}\Theta$ using a four dimensional spinor notation. The anti-chiral ten dimensional spinor $\Theta_A$ has the following 4+6 splitting:

$$\Theta_A \rightarrow (\Theta^{\alpha A}, \Theta^{\dot{\alpha} A})$$

where $\alpha$ ($\dot{\alpha}$) are chiral (anti-chiral) indices in four dimensions and the lower (upper) indices $A$ are chiral (anti-chiral) spinor indices of the internal six dimensional space. $\Gamma$ matrices decompose as:

$$\Gamma^\mu = \gamma^\mu \otimes 1, \quad \Gamma^m = \gamma^{(5)} \otimes \gamma^m$$

Then the fermion bilinear $\Theta \Gamma^{mnp}\Theta$ can be written as:

$$\Theta \Gamma^{mnp}\Theta = -i\Theta^{\alpha A} \Theta^{B}_{\dot{\alpha} A} (\Sigma^{mnp})_{AB} - i\Theta^{\dot{\alpha} A} \Theta^{\dot{\alpha}}_{B} (\Sigma^{mnp})^{AB}$$

where $\Sigma^{mnp}$ and $\Sigma^{mnp}$ are the chiral and anti-chiral blocks of $\gamma^{mnp}$. $\Sigma^{mnp}$ couples only to an imaginary self-dual tensor (ISD) tensor and $\Sigma^{mnp}$ only to imaginary anti-self-dual (IASD) tensor, since $*_6 \Sigma^{mnp} = -i\Sigma^{mnp}$ and
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\[ *_6 \Sigma^{mnp} = +i \Sigma^{mnp} \]. One has:

\[
\Theta \Gamma^{mnp} \Theta T_{mnp} = -i \Theta^\alpha A \Theta_B^B (\Sigma^{mnp})_{AB} T^{IASD}_{mnp} - i \Theta_{\dot{\alpha} A} \Theta^\dot{B} (\Sigma^{mnp})^{AB} T^{ISD}_{mnp} \\
= -i \Theta^\alpha A \Theta_B^B T_{AB} - i \Theta_{\dot{\alpha} A} \Theta^\dot{B} T^{AB}
\]

where in the second line we used a SU(4) \(\sim \) SO(6) notation and defined

\[
T_{AB} = (\Sigma^{mnp})_{AB} T^{IASD}_{mnp}, \quad T^{AB} = (\Sigma^{mnp})^{AB} T^{ISD}_{mnp}
\]

where

\[
T^{ISD}_{mnp} = \frac{1}{2} (T - i *_6 T)_{mnp}, \quad T^{IASD}_{mnp} = \frac{1}{2} (T + i *_6 T)_{mnp}
\]

The upper indices \(A, B\) run over 4 representation of SU(4) and the lower ones over \(\bar{4}\) representation. Fixing a complex structure, the 3-form tensors \(T^{ISD}, T^{IASD}\) can be decomposed into their (3,0), (2,1), (1,2) and (0,3) parts as:

\[
T^{ISD} \rightarrow T_{(0,3)} \oplus T_{(1,2)NP} \oplus T_{(2,1)P} = \bar{1} \oplus \bar{3} \oplus \bar{6} \quad (3.88)
\]

\[
T^{IASD} \rightarrow T_{(3,0)} \oplus T_{(2,1)NP} \oplus T_{(1,2)P} = 1 \oplus 3 \oplus 6 \quad (3.89)
\]

Let us now focus on D3 and D-instantons on flat space. For D3-branes one can use a Minkowski signature and the fermionic bilinear decomposition is given in (3.84), where the four dimensional chiral and anti-chiral components are related by charge conjugation and assembled into four Majorana spinors. These are the four gauginos living on the worldvolume of the D3-brane. Let us denote their chiral and anti-chiral parts as \(\Lambda^\alpha A\) and \(\bar{\Lambda}_{\dot{\alpha} A}\) instead of \(\Theta^\alpha A\) and \(\Theta_{\dot{\alpha} A}\) for future notational convenience. Then plugging (3.84) and (3.70) into (3.59) one finds the following amplitude for D3-branes in flat space:

\[
\mathcal{A}_{D3} = \frac{2\pi i}{3!} c_F \text{Tr} [\Lambda^\alpha A \Lambda_B^B (\Sigma^{mnp})_{AB} G^{IASD}_{mnp} - \bar{\Lambda}_{\dot{\alpha} A} \bar{\Lambda}_{\dot{B} B} (\Sigma^{mnp})^{AB} (G^{IASD})^*_{mnp}]
\]

which encodes the structure of soft symmetry breaking terms in \(\mathcal{N} = 4\) gauge theory induced by NS-NS and RR fluxes. From (3.90) we see that an imaginary anti-self dual G-flux configuration induces a Majorana mass for
the gauginos leading to the supersymmetry breaking on the gauge theory. The prefactor is:

$$c_F = \frac{4}{g_Y} \left(2\pi\alpha'\right)^{-\frac{1}{2}} N_F$$

(3.91)

Notice that the mass term for the two different chiralities are complex conjugate of each other: $T^{IASD} = -iG^{IASD}$ and $T^{ISD} = i(G^{IASD})^*$, which a consequence of the Majorana condition that the four dimensional spinors inherit from the Majorana-Weyl condition of the fermions in the original ten dimensional theory. If we decompose $G^{IASD}$ as in (3.89), we see that a $G$-flux of type $(1, 2)_P$ gives mass to the three gauginos transforming non-trivially under $SU(3)$ but keeps the $SU(3)$-singlet gaugino massless, thus preserving $\mathcal{N} = 1$ upersymmetry. A $G$-flux of type $(3,0)$ or $(2,1)_P$ gives mass also to the $SU(3)$-singlet gaugino.

Using (3.73) one finds the coupling of fluxes to D-instantons:

$$\mathcal{A}_{D(-1)} = \frac{2\pi i}{3!} c_F(\Theta) [\Theta^\alpha A \Theta^B (\Sigma^{mnp})_{AB} G^{IASD}_{mnp} + \bar{\Theta}^{\dot{\alpha} A} \bar{\Theta}^{\dot{\beta}} B (\Sigma^{mnp})^{AB} G^{ISD}_{mnp}]$$

(3.92)

where

$$c_F(\Theta) = \frac{8\pi^2}{g_Y^2} N_F^2 \mathcal{N}_F$$

(3.93)

From (3.92) one observes that both the IASD and the ISD components of the $G$-flux couple to the D-instanton fermions. The couplings are different and independent for the two chiralities since they are not related by complex conjugation, as always in Euclidean spaces. Comparing $\mathcal{A}_{D3}$ and $\mathcal{A}_{D(-1)}$ shows that ISD $G$-flux does not give a mass to any gauginos but instead induces a ‘mass’ term for the anti-chiral instanton zero-modes which are therefore lifted. This effect may play a crucial role in discussing the non-perturbative contributions of the so-called ‘exotic’ D-instantons for which the neutral anti-chiral zero modes $\Theta^{\dot{\alpha} A}$ must be removed or lifted by some mechanism. Introducing ISD $G$-flux is one of such mechanisms.

Twisted fermions stretching between D3-brane and D-instanton represent the charged (or flavored) fermionic moduli of the $\mathcal{N} = 4$ ADHM construction of instantons and are usually denoted as $\mu$ and $\bar{\mu}$ depending on the orientation. The R-R 3-form flux $F_{mnp}$ couplings with these twisted fermions is given by:

$$\mathcal{A}_F \sim \bar{\mu}^A \mu^B (\Sigma^{mnp})_{AB} F^{IASD}_{mnp}$$

(3.94)

The couplings to NS-NS 3-form flux $H_{mnp}$ is:

$$\mathcal{A}_H \sim \bar{\mu}^A \mu^B (\Sigma^{mnp})_{AB} H^{IASD}_{mnp}$$

(3.95)
Collecting (3.94) and (3.95) and reinstating the appropriate normalizations, one finds
\[ A_{D3/D(-1)} \equiv A_F + A_H = \frac{4\pi i}{3!}c_F(\mu)\bar{\mu}^A \mu^B \left( \sum_{mnp} \right)_{AB} G_{mnp}^{LASD}. \] (3.96)

where \( c_F(\mu) = C_{(0)} N_{\mu} N_{\bar{\mu}} N_{F} \) with \( C_{(0)} = \frac{8\pi^2}{g_s M} \). Notice that \( \mu \) and \( \bar{\mu} \) are distinct and independent quantities. This amplitude together with \( A_{D(-1)} \) accounts for the flux induced fermionic couplings on the D-instanton effective action. (3.92) and (3.96) describe deformations of the instanton moduli space of \( \mathcal{N} = 4 \) gauge theory living on the worldvolume of D3-branes.

An extension to less supersymmetric cases can be done straightforwardly. For pure \( \mathcal{N} = 1 \) SYM the flux couplings for gauge and exotic instantons follow from (3.90), (3.92) and (3.96) by restricting the spinor components to \( A = B = 0 \). Contributions to fermionic mass terms come only from the components \( G_{3,0} \) and \( G_{0,3} \) related to the soft symmetry breaking gaugino and gravitino masses. For \( T_6/(\mathbb{Z}_2 \times \mathbb{Z}_2) \) these masses are:
\[ |m_\Lambda| = \left| 4\frac{e^\varphi}{\nu} G_{(3,0)} \right| \] (3.97)
\[ |m_{3/2}| = \left| 4\frac{e^\varphi}{\nu} G_{(0,3)} \right| \] (3.98)

where \( \varphi \) is the dilaton, \( \nu \) is the volume of the \( T_6/(\mathbb{Z}_2 \times \mathbb{Z}_2) \) orbifold. The fermionic flux couplings
\[ A_{D3} = -\frac{i}{16\pi} m_\Lambda e^{-\varphi} \text{Tr}[A^\alpha A_\alpha] + c.c. \] (3.99)
\[ A_{D(-1)} = -i\pi m_\Lambda e^{-\varphi} \theta^\alpha \theta_\alpha + \frac{i\pi}{8} (2\pi\alpha')^2 m_{3/2} e^{-\varphi} \lambda_\dot{\alpha} \lambda^{\dot{\alpha}} \] (3.100)
\[ A_{D3/D(-1)} = -\frac{i}{8} m_\Lambda \bar{\mu}_u \mu^u \] (3.101)
modify the zero mode structure of the instanton and allow for new low energy coupling in the D3-brane action. In the above formulae \( A^\alpha \) is the gaugino, \( \theta^\alpha \) and \( \lambda^{\dot{\alpha}} \) are two chiral and two anti-chiral zero modes respectively coming from the R sector of D(-1)/D(-1) strings. \( \mu^u, \bar{\mu}_u \) scalars come from the R sector of D3/D(-1) and D(-1)/D3 strings. One observes that the presence of \( \lambda \)-fermionic zero modes prevents the generation of non-perturbative superpotentials via exotic instantons. One can overcome this difficulty by introducing an O-plane. This leads to O(1)-instantons without \( \lambda \)-modes. In the case of oriented gauge theories, the presence of the \( \lambda^2 \)-term in amplitudes suggests that R-R and NS-NS fluxes can give an alternative mechanism.
Chapter 4

Worldsheet vs D-brane instantons

4.1 Heterotic-Type I duality in $D \leq 10$

Perturbatively different string theories may be shown to be equivalent once non-perturbative effects are taken into account. Heterotic and Type I string theories with gauge group $SO(32)$, were conjectured in [18,41], to be equivalent. In fact, up to field redefinitions, they share the same low-energy effective field theory. It was shown in [42] that for the equivalence to work, the strong coupling limit of one should correspond to the weak coupling limit of the other\(^1\). In $D = 10$ the strong - weak coupling duality takes the following form [18,41]

$$g_s^H = 1/g_s^I, \quad \alpha'_H = g_s^I \alpha'_I$$ (4.1)

where $g_s^H$, $\alpha'_H$ and $g_s^I$, $\alpha'_I$ are the heterotic and Type I coupling constants and tensions respectively. The simple strong-weak coupling duality $\phi_I = -\phi_H$ in $D = 10$ changes significantly in lower dimensions. Indeed, since the dilaton belongs to the universal sector of the compactification, the relation between the heterotic and Type I dilatons in $D$ dimensions is determined by dimensional reduction to be [43], [44]

$$\phi_I^{(D)} = \frac{(6 - D)}{4} \phi_H^{(D)} - \frac{(D - 2)}{16} \log \det G_H^{(10-D)}$$ (4.2)

\(^1\)Similar situations in which the strong coupling limit of one string theory is the weak coupling of another ‘dual’ string theory were discussed earlier by Duff [48].
where $G^{(10-D)}_H$ is the internal metric in the heterotic-string frame, and there is a crucial sign change at $D = 6$ where $\phi_H$ and $\phi_I$ are independent [45]. It is well known that Type I models exist with different number of tensor multiplets in $D = 6$ [46, 47]. This does not have an analogue in perturbative heterotic compactifications on $K3$. In $D = 6$, the Type I dilaton belongs to a hypermultiplet to be identified with one of the moduli of the $K3$ compactification on the heterotic side. In four dimensional $\mathcal{N} = 1$ models on both sides the dilaton appears in a linear multiplet, and heterotic-type I duality is related to chiral-linear duality. The presence of anomalous $U(1)$’s under which R-R axions shift suggests that the latter correspond to changed scalars on the heterotic side.

Heterotic-Type I duality requires that the heterotic fundamental string and the Type I D-string be identified. The massless fluctuations of a Type I D-string are eight bosons and eight negative chirality fermions in the D1-D1 sector together with 32 positive chirality fermions in the D1-D9 sector. Thus, the world-sheet of the D-string exactly matches the world-sheet of the Heterotic fundamental string. By the same token, the Type I D5-brane should be identified with the heterotic NS5-brane. The latter is a soliton of the effective low-energy heterotic action and its microscopic description is not fully understood. The tensions agree in the two descriptions since $T_{NS5} = 1/(g^H_s\alpha' H)^3 \equiv 1/g^I_s(\alpha' I)^3 = T_{D5}$.

$SO(32)$ Heterotic / Type I duality has been well tested in $D = 10$ and in toroidal compactifications. In $D = 10$ BPS-saturated terms, like $F^4$, $F^2R^2$ and $R^4$, are anomaly related and match in the two theories as a consequence of supersymmetry and absence of anomaly. In toroidal compactifications, the comparison of BPS-saturated terms becomes more involved. The spectra of BPS states become richer and differ on the two sides at the perturbative level.

Non-perturbative corrections to $F^4$, $F^2R^2$ and $R^4$ terms are due to instantons that preserve half of the supersymmetry. In the heterotic string they get perturbative corrections at one loop only and the NS5-brane is the only relevant non-perturbative configuration in $D \leq 4$. Instanton configurations can be provided by taking the world-volume of the NS5-brane to be Euclidean and to wrap supersymmetrically around a compact manifold, so as to keep finite the classical action. This requires at least six-dimensional compact manifold. Therefore, BPS-saturated terms do not receive non-perturbative corrections for toroidal compactifications with more than four
non-compact directions. Thus, the full heterotic result arises from tree level and one loop for \( D > 4 \). In the Type I string both D1- and D5-branes can provide instanton configurations after Euclideanization. D5-brane will contribute in four or less noncompact dimensions, D1-brane can contribute in eight or fewer noncompact dimensions. Thus, in nine dimensions the two theories can be compared in perturbation theory. In eight dimensions the perturbative heterotic result at one-loop corresponds to perturbative as well as nonperturbative Type I contributions coming from the D1-instanton via duality. The heterotic results can be expanded and the Type I instanton terms can be identified. The classical action can be written straightforwardly and it matches with the heterotic result. The determinants and multi instanton summation can also be performed in the Type I theory. In general, world-sheet instantons in heterotic string duals of Type I models help clarifying the rules for multi-instanton calculus with unoriented D-branes. Two prototypical examples are the \( T^4/\mathbb{Z}_3 \) orbifold to \( D = 4 \), that we have already encountered [43], and the \( T^6/\mathbb{Z}_2 \) orbifold to \( D = 6 \) [46], that we are going to discuss in the following.

4.2 Compactification on \( T^4/\mathbb{Z}_2 \) to \( D = 6 \)

Type I description

The Type I theory is an un-oriented projection of the Type IIB theory. Upon compactification on \( T^4/\mathbb{Z}_2 \) to \( D = 6 \), the Type IIB theory has \( \mathcal{N} = (2,0) \) spacetime supersymmetry with 16 supercharges, i.e. those satisfying \( Q = R\bar{Q} \), where \( R \) denotes the inversion of the four coordinates of \( T^4 \). The \( \Omega \) projection preserves only the sum of left- and right-moving supersymmetries \( Q_\alpha + \bar{Q}_\alpha \). The \( \Omega R \) projection preserves the same linear combination since \( Q_\alpha + R\bar{Q}_\alpha \equiv Q_\alpha + \bar{Q}_\alpha \). The massless little group in six dimensions is \( SO(4) = SU(2) \times SU(2) \). The massless bosonic content of the unoriented closed string spectrum contains in untwisted NS-NS sector \((3,3)+11(1,1)\), in the untwisted R-R sector \((3,1)+(1,3)+6(1,1)\), in the twisted NS-NS sector there are 48 \((1,1)\) and in the twisted R-R sector 16 \((1,1)\). This is exactly the bosonic content of the \( D = 6 \) \( \mathcal{N} = (1,0) \) supergravity coupled to one tensor and 20 hypermultiplets.

Let us now discuss the unoriented open string spectrum. Tadpole cancellation conditions imply that the total Chan-Paton dimensionalities
of twisted and untwisted sectors both equal to 32. The $U(16)_9 \times U(16)_5$ model, which arises at the maximally symmetric point, where all the D5-branes are on top of an $\Omega$-plane and no Wilson lines are turned on the D9-branes, is of particular interest. This model was first discussed by Bianchi and Sagnotti and later by Gimon and Polchinski in [46]. The D9-D9 sector contributes a vector multiplet in the adjoint of $U(16)_9$ and hypermultiplets in the $120_+ + 120^*_2$. The D5-D5 gives a vector multiplet in the adjoint of $U(16)_5$ and the hypermultiplet in the $\tilde{120}_+ + \tilde{120}^*_2$. In the D5-D9 ‘twisted’ spectrum there are half-hypers in the $(16_{+1}, \tilde{16}^{*}_{-1}) + (16_{-1}, \tilde{16}^{*}_{+1})$ of $U(16)_9 \times U(16)_5$.

**Compactification on $T^4/Z_2$ to $D = 6$: Heterotic description**

The Type I model corresponds to a compactification without vector structure [49], [50]:

$$\tilde{\omega}_{2,YM}^{SW} \neq 0 \approx B_2^{NS-NS} = 1/2 (mod 1) \quad (4.3)$$

where $\tilde{\omega}_{2,YM}^{SW}$ is modified second Stieffel-Whitney class (obstruction to vector structure).

The $Z_2$ orbifold (besides its geometrical action) acts on the 32 heterotic fermions as $\lambda^A_{us} \rightarrow (i)\lambda^u_{us}, (-i)\lambda^\bar{u}_{us}$, which breaks the gauge group $SO(32)$ to $U(16)$. The resulting massless spectrum is as follows. In the untwisted sector we have four neutral hypers, charged hypers in $120_+ + 120^*_2$, vector in the adjoint, one tensor and the $N = (1, 0)$ supergravity multiplet.

The twisted sector (16 fixed points) does not contain neutral hypermultiplets, it has charged half hypermultiplets in the $16_{-3} + 16^{*}_{+3}$.

**Matching the spectrum**

In order to match the massless spectrums of the two descriptions one has to distribute one ‘fractional’ D5-brane per each fixed point, thus breaking the D5-brane gauge group $U(16)_5 \rightarrow U(1)^{16}_5$ [50].

In six dimensions the full gauge plus gravitational anomaly can be written as [51]

---

2 In the sense that the 4 N-D directions have half-integer bosonic modes.
\[ I_8 = \sum_i \left( X_i^2 \wedge X_i^6 + X_i^4 \wedge \bar{X}_i^4 \right) \]  

(4.4)

The GSS counterterm reads as \( L_{GSS} = C_2^{RR} X_4 + \sum_f C_0^{RR} X^f_6 \), so that Type I photons become massive by eating twisted RR axions: \( \partial C_0^{RR} \rightarrow DC_0^{RR} = \partial C_0^{RR} + 4A^{(9)} + A^{(5)}_f \). The Type I combination \( A^I = A^{(9)} - 4 \sum_f A^{(5)}_f \) decouples from twisted closed string scalars and matches with the heterotic photon \( A^H \). The vector multiplets get massive by eating neutral closed string hypers. Thus we have a supersymmetric Higgs-like mechanism: full hypers are eaten.

### 4.3 Duality and dynamics in \( D = 6 \)

In order to further test the correspondence and gain new insights into multi D-brane instantons, we are going to consider a four-hyperini Fermi type interaction that is generated by instantons and corresponds to a ‘chiral’ \((1/2 \text{ BPS})\) coupling in the \( \mathcal{N} = (1, 0) \) low energy effective action. If the four hyperini are localized at four different fixed points, this coupling is absent to any order in perturbation theory. This is so, because twisted fields at different fixed points do not interact perturbatively. ED1-brane or worldsheet instantons which connect the four fixed points can generate such a term. The contributions will be exponentially suppressed with the area of the cycle wrapped by the instanton.

Let us mention what kind of corrections one expects in the two descriptions before describing the computation. In \( D = 6 \) Heterotic / Type I duality implies

\[ \phi_H = \omega_I, \quad \phi_I = \omega_H \]  

(4.5)

where \( \phi \) is the dilaton and \( \omega \) is the volume modulus. Supersymmetry implies that there are no neutral couplings between vectors and hypers. The gauge couplings can only depend (linearly) on the scalar \( \phi_H = \omega_I \) in the unique tensor multiplet, while \( \phi_I = \omega_H \) belongs to a neutral hyper.

For these reasons in the heterotic description the hypermultiplet geometry is tree-level exact, but may get worldsheet instanton corrections \( e^{-h(C)/\alpha'} \), where neutral hypers \( h \) determine the size of 2-cycles \( C \) in \( T^4/\mathbb{Z}_2 \).

\[ ^3 \text{This is an efficient, not fully exploited mechanism for moduli stabilization even in } D = 4. \]

The remnant of the \( D = 6 \) anomaly in \( D = 4 \) is massive ‘non-anomalous’ \( U(1) \)’s [52].
the Type I description, hypers receive both perturbative (string loops) and non-perturbative corrections from BPS Euclidean D-string instantons wrapping susy 2-cycles \( C \) in \( T^4/\mathbb{Z}_2 \). The Type I gauge couplings are completely determined by disk amplitudes. In the heterotic string, they receive (only) a one-loop correction.

## 4.4 Four-hyperini amplitude

### Computational strategy

Let us summarize our strategy:

- Focus on a specific 4-hyperini amplitude

\[
\mathcal{A}_{4\text{hyper}}^{f_1 f_2 f_3 f_4} = \langle V_{16}^\zeta f_1 V_{16}^\zeta f_2 V_{16}^\zeta f_3 V_{16}^\zeta f_4 \rangle
\]

(absent at tree level for particular choices of fixed points)

- Compute \( \mathcal{A}_{4\text{hyper}}^{f_1 f_2 f_3 f_4} \) in the limit of vanishing momenta

- Start with heterotic string, where it is tree level exact and extract worldsheet instanton corrections

- Translate into Type I language and interpret the result in terms of perturbative and non-perturbative contributions

- Learn new rules for unoriented multi D-brane instantons

### Heterotic description

To compute the four-hyperini Fermi interaction in the heterotic description we need the hyperini vertex operators

\[
V_{16/16^*}^\zeta = \zeta_{f,a}^u z^{-\varphi/2}(z) \tilde{\Sigma}^u_{\bar{u}}(\bar{z}) \sigma_f e^{ibX}(z,\bar{z})
\]

(4.7)

where \( \sigma_f \) is the bosonic \( \mathbb{Z}_2 \)-twist field \( (h = 1/4) \), \( \tilde{\Sigma}^u_{\bar{u}} =: e^{\pm i\tilde{\phi}_u} \prod_v e^{\mp i\tilde{\phi}_v/4} \) are twisted ground-states \( (h = 3/4) \) for heterotic fermions \( \tilde{\lambda}_{u/\bar{u}} \), \( S^a \) are \( SO(5,1) \) spin fields, \( \varphi \) and \( \tilde{\phi}_u \) are the bosonizations of the superghost and \( SO(32) \) gauge fermions respectively. One can use \( SL(2,\mathbb{C}) \) invariance on the sphere to set \( z_1 \to \infty, z_2 \to 1, z_3 \to z, z_4 \to 0 \) with cross ratio \( z = z_1 z_3 z_4 / z_2 z_4 \). Then the string amplitude will depend on the \( SL(2,\mathbb{C}) \) invariant cross ratio \( z \).
The $\mathbb{Z}_2$-twist field correlator is given by [53]

$$
\langle \prod_{i=1}^{4} \sigma_{f_i}(z_i, \bar{z}_i) \rangle \rightarrow |z_\infty|^{-1} \Psi_{qu}(z, \bar{z}) \Lambda_{cl} \left[ \frac{f_{12}}{f_{13}} \right] (z, \bar{z}) \quad (4.8)
$$

The quantum part $\Psi_{qu}$ is independent of the twist-fields locations i.e. of the choice of 4 out of 16 fixed points $\bar{f}_i = 1/2(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4)$ with $\epsilon^i = 0, 1$ and in order to get a non-trivial coupling the $\bar{f}_i$ should satisfy $\sum_i \bar{f}_i = \bar{0}$ mod $\Lambda(T^4)$. $\Lambda_{cl} = \sum e^{-S_{inst}}$ is the classical part accounting for worldsheet instantons depending on the relative positions $\bar{f}_{ij} = \bar{f}_i - \bar{f}_j$. The $\mathbb{Z}_2$-twist field correlator can be mapped into the torus doubly covering the sphere with two $\mathbb{Z}_2$ branch cuts using the relation between the cross-ratio $z$ and the Teichmüller parameter of the torus $\tau(z)$

$$
z = \theta^4_3(\tau)/\theta^4_4(\tau). \quad (4.9)
$$

The quantum and classical parts of the 4-twist correlator read

$$
\Psi_{qu}(z, \bar{z}) = 2^{-8/3} |z(1-z)|^{-1/3} \tau_2^{-2} |\eta(\tau)|^{-8} \quad (4.10)
$$

$$
\Lambda_{cl} \left[ \frac{f_{12}}{f_{13}} \right] (z, \bar{z}) = \sum_{\bar{m}, \bar{n}} e^{-\frac{\tau_2}{\bar{m}} \tau (\bar{m} + \bar{n} \bar{\tau} + \bar{f}_{12} \bar{\tau}) - (G + B)(\bar{m} + \bar{n} \bar{\tau} + \bar{f}_{13} \bar{\tau})} \quad (4.11)
$$

where $G_{ij}$ is the metric and $B_{ij}$ is the antisymmetric tensor of $T^4/\mathbb{Z}_2$ (neutral hypers). Writing the $z$-integral as integral over the torus modulus $\tau$ (for $s, t \rightarrow 0$) one finds

$$
\mathcal{A}_{u_1u_2u_3u_4} = \mathcal{V}(T^4) \int_{\mathcal{F}_2} \frac{d^2 \tau}{\tau_2} \left( \frac{\partial^4}{\partial^4_3} \delta_{u_1u_2} \delta_{u_3u_4} - \frac{\partial^4}{\partial^4_2} \delta_{u_1u_4} \delta_{u_2u_3} \right) \Lambda_{cl} \left[ \frac{f_{12}}{f_{13}} \right]. \quad (4.12)
$$

The integral goes over the fundamental domain $\mathcal{F}_2$ of the index 6 subgroup $\Gamma_2$ of $SL(2, \mathbb{Z})$, leaving invariant $\vartheta_{even}$ [54]. The region $\mathcal{F}_2$ can be decomposed into 6 domains each of which is an image of the fundamental domain $\mathcal{F}$ of $SL(2, \mathbb{Z})$ under the action of the 6 elements of $SL(2, \mathbb{Z})/\Gamma_2$

$$
\int_{\mathcal{F}_2} \frac{d^2 \tau}{\tau_2} \Phi(\tau, \bar{\tau}) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \sum_{s=1}^{6} \Phi(\tau_s, \bar{\tau}_s) \quad (4.13)
$$

where $\tau_s = \gamma_s(\tau)$, $\gamma_s = \{1, S, T, TS, ST, TST\}$. For the 4-hyperini amplitude one gets

$$
\Phi(\tau, \bar{\tau}) = \left( \frac{\partial^4}{\partial^4_3} \delta_{u_1u_2} \delta_{u_3u_4} - \frac{\partial^4}{\partial^4_2} \delta_{u_1u_4} \delta_{u_2u_3} \right) \Lambda_{cl} \left[ \frac{f_{12}}{f_{13}} \right]. \quad (4.14)
$$
In the special case when all 4-hyperini are located at the same fixed point \( \vec{f}_{12} = \vec{f}_{13} = (\vec{0}) \), the amplitudes receive contribution only from BPS-like modes as in Type I (see later). The instanton sum \( \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{0} \right] \) is modular invariant. Sums over 6 images produce

\[
\sum_{s=1}^{6} \frac{\partial \mathcal{A}}{\partial \vec{f}} (\vec{r}_s) = 3 , \quad \sum_{s=1}^{6} \frac{\partial \mathcal{A}}{\partial \vec{f}} (\vec{r}_s) = -3 \tag{4.15}
\]

and the final expression for the amplitude with \( \vec{f}_1 = \vec{f}_2 = \vec{f}_3 = \vec{f}_4 \) is given by

\[
\mathcal{A}_{u_1 u_2 u_3 u_4}^{f_1,f_2,f_3,f_4} = 3 (\delta_{u_1 u_2} \delta_{u_3 u_4} + \delta_{u_1 u_3} \delta_{u_2 u_4}) \mathcal{V} (T^4) \int \frac{d^2 \tau}{T^2} \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{0} \right] . \tag{4.16}
\]

Next consider the case when hyperini are located in pairs at two different fixed point:

- \( \vec{f}_{12} = \vec{f}_{13} = \vec{f} \) for \( \delta_{u_1 u_2} \delta_{u_3 u_4} \) structure in 4-hyperini amplitude

\[
\mathcal{A}_{u_1 u_2 u_3 u_4}^{f_1,f_2,f_3,f_4} = \mathcal{V} (T^4) \int \frac{d^2 \tau}{T^2} \left( \Lambda_{\text{cl}} \left[ \vec{f} \big/ \vec{0} \right] + \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{f} \right] + \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{0} \right] \right) . \tag{4.17}
\]

- \( \vec{f}_{12} = \vec{0}, \vec{f}_{13} = \vec{h} \) for \( \delta_{u_1 u_4} \delta_{u_2 u_3} \) structure in 4-hyperini amplitude

\[
\mathcal{A}_{u_1 u_2 u_3 u_4}^{f_1,f_2,f_3,f_4} = \mathcal{V} (T^4) \int \frac{d^2 \tau}{T^2} \left( \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{h} \right] + \Lambda_{\text{cl}} \left[ \vec{h} \big/ \vec{0} \right] + \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{0} \right] \right) . \tag{4.18}
\]

These are the same integrals as for BPS saturated thresholds to \( F^4 \) in \( T^4 \) compactifications (with shifts) \([54]\). Since the pieces proportional to \( \delta_{u_1 u_2} \delta_{u_3 u_4} \) and \( \delta_{u_1 u_4} \delta_{u_2 u_3} \) are related by a simple relabeling of the fixed points \( f_i \)’s, we have restricted our attention onto the amplitude with color structure \( \delta_{u_1 u_2} \delta_{u_3 u_4} \) for the first and the amplitude with color structure \( \delta_{u_1 u_4} \delta_{u_2 u_3} \) for the second case. Performing Poissson resummation over \( \vec{m} \) in \( \Lambda_{\text{cl}} \) one finds

\[
\Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{f} \right] = \frac{T^2}{\mathcal{V} (T^4)} \sum_{\vec{k},\vec{\alpha}} (-)^{2f \cdot \vec{k}} q^{\vec{k}^2/2} q^{\vec{\alpha}^2/2} \tag{4.19}
\]

where \( \vec{p}_{L/R} = \frac{1}{\sqrt{2}}(E^{-1} \vec{k} + E^t \vec{m}) \), \( EE^t = G \) and we have set \( B = 0 \) for simplicity. One can recognize the shifted orbifold partition function

\[
\Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{f} \right] (G) + \Lambda_{\text{cl}} \left[ \vec{f} \big/ \vec{0} \right] (G) + \Lambda_{\text{cl}} \left[ \vec{f} \big/ \vec{f} \right] (G) = 2 \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{0} \right] (G) (G_{\vec{f}}) - \Lambda_{\text{cl}} \left[ \vec{0} \big/ \vec{0} \right] (G) . \tag{4.20}
\]

The toroidal metric \( G_{\vec{f}} \) is ‘halved’ along the direction \( \vec{v} = 2\vec{f} \) by \( SO(d,d) \) transformation. This is similar to what one gets for the threshold corrections to \( F^4 \) terms in toroidal compactifications.
4.4. FOUR-HYPERINI AMPLITUDE

Type I description

As we already noticed, in the Type I description hypers can receive both perturbative and non-perturbative corrections since the dilaton belongs to a hypermultiplet. Some scattering amplitudes may vanish in perturbation theory and receive only contributions from non-perturbative effects. The four-hyperini Fermi interaction term does not get perturbative contributions for $f_i$ all different from one another or for $f_1 = f_3$ (same charge). When all fixed points $f_i$ are equal or are equal in pairs $f_1 = f_2$ or $f_2 = f_3$ (opposite charge) there is a perturbative correction which matches with the contribution of the degenerate orbit in the heterotic description.

The open string vertex operators are given by

$$V_{16}^\xi = \xi_\alpha(p) S^\alpha e^{-\varphi/2} \sigma_f e^{i p \cdot X} \Lambda^\alpha_\beta \quad V_{16}^{\xi*} = \xi_\alpha(p) S^\alpha e^{-\varphi/2} \sigma_f e^{i p \cdot X} \bar{\Lambda}_\beta^\alpha,$$

They involve Chan-Paton matrices $\Lambda_\beta^\alpha$ in the bifundamental of $U(16) \times U(1)_5$ rather than heterotic fermions $\lambda$ yielding

$$Tr(\Lambda_{f_1}^{\bar{u}_1} \Lambda_{f_2}^{\bar{u}_2} \Lambda_{f_3}^{\bar{u}_3} \Lambda_{f_4}^{\bar{u}_4}) = \delta_{f_1 f_2} \delta_{f_3 f_4} \delta^{u_1 \bar{u}_4} \delta^{u_2 \bar{u}_3} + \delta_{f_1 f_4} \delta_{f_2 f_3} \delta^{u_1 \bar{u}_2} \delta^{u_3 \bar{u}_4}. \quad (4.22)$$

Consider $Z_2$-twist field correlator for open strings with 4 N-D boundary conditions [55]. The quantum part of the 4-twist correlator, which is independent of location $f_i$ of twist fields, is given by

$$\Psi_{qu} = [x(1 - x)]^{-1/3} t(x)^{-2} \eta(it)^{-4} \quad (4.23)$$

where $x = \psi_3^4(it)/\psi_4^4(it)$ is $SL(2, R)$ invariant ratio with $t$ the modular parameter of the annulus doubly covering the disk. The classical part from exchange of (massive) open string modes stretched between (different) fixed points is

$$\Lambda_{\ell f}[f_{12}] = \delta_{f_2}^{f_1} \delta_{f_4}^{f_3} \sum_{\vec{n}} e^{-\pi t(\bar{n} + f_{12})^T G(\bar{n} + f_{12})} + \delta_{f_4}^{f_1} \delta_{f_2}^{f_3} \sum_{\vec{n}} e^{-\pi t(\bar{n} + f_{12})^T G(\bar{n} + f_{12})}.$$

Plugging into the open string amplitude and taking the limit $s,t \to 0$ one finds a perfect agreement with heterotic degenerate orbits, which is independent of $B_2^H \approx C_2^{-R}$ but only on $G$ and $\phi$ (recall $\omega_H = \phi_I$, $\phi_H = \omega_I$). Terms involving $B_2^H$ have no disk counterpart in the Type I description since the dual $C_2^{R\ell}$ couples to (E)D-strings.
ED-string corrections

We then consider non-perturbative corrections to the four-hyperini coupling in the Type I description. When the four hyperini are located at the different fixed points $f_i$, we have only non-perturbative contribution from (regular) ED-strings wrapping supersymmetric (untwisted) two cycles $C \approx T^2/\mathbb{Z}_2 = S^2$, passing through the four fixed points. Fractional ED-strings wrapping the 16 collapsed ‘rigid’ 2-cycles (since corresponding moduli are eaten by anomalous $U(1)^{16}_5$) may contribute to amplitudes having also perturbative contributions.

Using by now the well established Heterotic / Type I duality we will deduce the ‘exact’ 4-hyperini amplitude and determine ED-string corrections. Then we interpret these corrections in terms of symmetric orbifold CFT [56].

Let us describe the spectrum of ED-strings. The instanton dynamics is governed by a gauge theory describing the excitations of unoriented strings connecting E1, D5 and D9 branes. Three sectors of open string excitations are

- E1-E1 strings (2 N-N, 8 D-D): $X^I$, $S^a$, $\tilde{S}^{\dot{a}}$ with $I = 1, \ldots, 8$, $a, \dot{a} = 1, \ldots, 4$

- E1-D9 strings (2 N-N, 8 N-D): $\lambda^u$, $\lambda^{\bar{u}}$ with $u, \bar{u} = 1, \ldots, 16$

- E1-D5 strings (2 D-D, 8 N-D): $\mu^f$, $\mu_{\bar{f}}$ with $f = f_1, f_2, f_3, f_4$

Alternatively, after T-duality along the wrapped 2-cycle we will have $E1 \rightarrow E(-1)$, $D9 \rightarrow D7_9$, $D5 \rightarrow D7_5$. The residual (super)symmetry of the spectrum is

$$\mathcal{N} = (8, 0) \rightarrow \mathcal{N} = (4, 0).$$ (4.25)

And the spacetime symmetry breaks according to

$$SO(9, 1) \rightarrow SO(5, 1) \times SU(2) \times SU(2) \rightarrow SO(5, 1) \times SO(2)_E \times SO(2).$$ (4.26)

$$\mathcal{N} = (4, 0)$$ gauge theory in IR flows to symmetric product CFT

$$(R^6 \times T^4/\mathbb{Z}_2)^k/S_k.$$ (4.27)
4.4. FOUR-HYPERINI AMPLITUDE

ED-string wraps two-cycle $C$ inside $T^4/\mathbb{Z}_2$ which is specified by the two vectors $M_k = (\vec{k}_1, \vec{k}_2)$ each made out of four integers with greatest common divisor 1. $\vec{k}_{1,2}$ show how many times the two 1-cycles of $C$ wrap around 1-cycle of $T^4/\mathbb{Z}_2$.

Heterotic vertex operator can be derived from interaction term with hyperino:

$$\mathcal{L}_{4F} = (\zeta_a)^u \mu^f_{\mu} \lambda^i_{\nu} S^a = V^H_\zeta$$

(4.28)

Only $(\ell)^m$-twisted sectors (with $m\ell = k$ and $Z^s_{\ell}$ projection) with exactly four fermionic zero modes of $S^a$ contribute. So, we can fold the $k$ copies of fields and form a single field on a worldsheet with the following Kahler and complex structures:

$$\mathcal{T}(M) = k \mathcal{T}(M_k) \quad , \quad \mathcal{U}(M) = \frac{m \mathcal{U}(M_k) + s}{\ell}$$

(4.29)

where $M = M_k \begin{pmatrix} l & s \\ 0 & m \end{pmatrix}$. This is in perfect agreement with the heterotic result $\mathcal{T}^{\text{deg}}_{d,d}$ for the four-hyperino coupling on $T^4/\mathbb{Z}_2$.

This is the generalized Hecke transform as in the Heterotic computation!
Chapter 5

AdS/CFT correspondence and instantons

5.1 Gauge theory/string theory dualities

Gauge theory/string theory duality states that string theory or M-theory in the near-horizon geometry of a collection of coincident D-branes or M-branes is equivalent to the low-energy world-volume theory of the corresponding branes. AdS/CFT dualities relate Type IIB superstring theory or M-theory compactified on an anti-de Sitter space-time times a compact space to conformally invariant field theories. Anti-de Sitter space is a maximally symmetric space-time with a negative cosmological constant. The space-time manifold of the conformal field theory is associated with the conformal boundary of the AdS. This boundary lies at the infinity of the AdS space-time. Usually AdS/CFT dualities are such that when one description is weakly coupled, the dual description is strongly coupled. Using an information in the weakly coupled theory allows to learn non trivial facts about strongly coupled dual theory. There are three basic examples of AdS/CFT duality. They all have maximal supersymmetry (32 supercharges). Superconformal field theory on the world-volume of $N$ parallel D3-branes corresponds to the type IIB theory on $AdS_5 \times S^5$. M-theory on $AdS_7 \times S^4$ is dual to superconformal field theory on $N$ M5-branes. M-theory on $AdS_4 \times S^7$ corresponds to superconformal field theory which lives on the world-volume of $N$ parallel M2-branes. In each case the sphere surrounds the branes. Each of these branes has one unit of the appropriate type of charge. Thus, the background has nonvanishing an-
tisymmetric tensor gauge field with \( N \) units of flux. Gauss’s law requires that these fluxes thread the sphere.

## 5.2 General aspects of \( AdS_5 \times S^5 \)

Let us consider how the symmetries of bulk and boundary theories are related in this case. The isometry group \( SO(6) \) of \( S^5 \) and the isometry group \( SO(4, 2) \) of \( AdS_5 \) correspond respectively to R-symmetry group and conformal group of the boundary theory. The radii of \( AdS_5 \) and \( S^5 \) are equal and are related to the ’t Hooft parameter \( \lambda = g_{YM} N \) of the gauge theory by \( L = \lambda^2 l_s \) with \( l_s = \sqrt{\alpha'} \). The dual theories have the same symmetry. In each case the supergroup is \( PSU(2, 2|4) \) which has a bosonic subgroup \( SU(2, 2) \times SU(4) \) and 32 fermionic generators transforming as \((4, 4) + (\bar{4}, \bar{4}) \) under \( SU(2, 2) \times SU(4) \). First let us consider the string theory side. The covering groups of \( SO(4, 2) \) and of \( SO(6) \) are \( SU(2, 2) \) and \( SU(4) \) respectively. Since there are fermions in the theory belonging to the spinor representations one needs to consider these covering groups. Thus, the bosonic subgroup is realized by the geometry. The 32 supersymmetries of type IIB superstring theory are realized as vacuum symmetries. The conserved charges transform as \((4, 4) + (\bar{4}, \bar{4}) \) under \( SU(2, 2) \times SU(4) \) and together with spacetime isometries give \( PSU(2, 2|4) \). In the dual \( \mathcal{N} = 4 \) SYM theory \( SU(4) \) symmetry is a global R-symmetry, which does not commute with supersymmetries. The four fermions of one chirality are in the \( 4 \) representation and the others with opposite chirality are in the \( \bar{4} \) of \( SU(4) \). The six scalar fields transform as \( 6 \) of \( SU(4) \). The 32 supersymmetries are also realized. So one gets \( PSU(2, 2|4) \) superconformal algebra.

As already mentioned for D3-branes AdS/CFT conjecture says that type IIB superstring theory on \( AdS_5 \times S^5 \) is dual to \( \mathcal{N} = 4, d = 4 \) super Yang-Mills theory with \( SU(N) \) gauge group. It is known that in type II superstring theories the world-volume theory of \( N \) coincident BPS Dp-branes is a maximally supersymmetric \( U(N) \) gauge theory at the lowest order in \( \alpha' \) and in the absence of background fields. The low energy effective action on the world-volume of \( N \) coincident Dp-branes is given by dimensional reduction of supersymmetric \( U(N) \) gauge theory in ten dimensions down to \( p + 1 \) dimensions. A \( U(N) \) gauge theory is equivalent to a free \( U(1) \) vector multiplet times an \( SU(N) \) gauge theory, up to some
5.2. GENERAL ASPECTS OF $\text{ADS}_5 \times S^5$

$\mathbb{Z}_N$ identifications, which affect only global issues. In the dual string theory all modes interact with gravity, so there are no decoupled modes. Hence, the bulk AdS theory describes the $SU(N)$ part of the gauge theory. There are also some zero modes living in the region connecting the near horizon region (the ‘throat’) with the bulk. These zero modes correspond to the above mentioned $U(1)$ degrees of freedom. The $U(1)$ vector supermultiplet includes six scalars related to the center of mass motion of all the branes. These zero modes live at the boundary from the AdS point of view. It seems one might or might not decide to include them in the AdS theory. Depending on this choice one could have a correspondence to an $SU(N)$ or $U(N)$ theory. The $U(1)$ center of mass degree of freedom is related to the topological theory of $B$-fields on AdS. If one imposes local boundary conditions for these $B$-fields at the boundary of AdS one finds a $U(1)$ gauge field living at the boundary, as in Chern-Simons theories. These modes living at the boundary are sometimes called singletons or doubletons. There is a distinction between these two theories with $SU(N)$ and $U(N)$, which is actually a subleading effect in the large $N$-limit.

Gauge fields are the massless modes of open strings. Hence the super Yang-Mills and the open string coupling constants are the same. For D$p$-branes with $p \neq 0$ the relation is the following:

$$g_{YM}^2 = \frac{4\pi g_s}{V_{p-3}/(\alpha')^{\frac{p-3}{2}}}$$

where $V_{p-3}$ is the volume of the cycle wrapped by the branes and where one has taken into account the relation between open and closed string coupling constants. The ’t Hooft coupling constant is:

$$\lambda = g_{YM}^2 N$$

which is held constant in the large $N$ expansion of the gauge theory. The D3-branes are sources for non-vanishing Ramond-Ramond five form field strength $F_5 = \star F_5$. The five-sphere surrounds D3-branes and according to the Gauss’s law there is a flux through $S^5$. There is a very nice fact of the duality saying that the rank of the gauge group (the number of branes with unit charge) corresponds to the five-form flux through the five-sphere:

$$\int_{S^5} F_5 = N$$

Constant Ramond-Ramond axionic background $\tilde{C}^{(0)}$ is proportional to the
Yang-Mills vacuum angle via

\[ 2\pi \tilde{C}^{(0)} = \theta_{YM} \quad (5.4) \]

Therefore, the complex Yang-Mills coupling is identified with the constant boundary value of the complex scalar field of the type IIB superstring

\[ \tau \equiv \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g_{YM}^2} = \tilde{C}^{(0)} + \frac{i}{g_s} \quad (5.5) \]

Boundary values of the bulk superstring theory fields are in correspondence with gauge invariant operators of four-dimensional boundary \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory. The lowest Kaluza-Klein modes of the graviton supermultiplet couple to the superconformal multiplet of Yang-Mills currents. To every Kaluza-Klein excitation one can associate a gauge-singlet composite Yang-Mills operator. The type IIB supergravity effective action evaluated on a solution of the equations of motion with boundary conditions is equal to the generating functional of connected gauge-invariant correlation functions in the Yang-Mills theory. The connection between bulk and boundary theory can be schematically given as

\[ \exp(-S_{IIB}[\Phi_m(J)]) = \int DA \exp(-S_{YM} [A] + \mathcal{O}_\Delta [A] J) \quad (5.6) \]

where \( S_{IIB} \) is the effective action of the IIB superstring or its low energy supergravity limit, \( \Phi_m(J) \) denote boundary values of the bulk ‘massless’ supergravity fields and their Kaluza-Klein descendents, \( A \) denotes fluctuating boundary \( \mathcal{N} = 4 \) supersymmetric Yang-Mills fields and \( \mathcal{O}(A) \) is the set of gauge-invariant composite operators to which \( J \) couples. Each field propagating on \( AdS \) space has its correspondent operator in the field theory. Each field propagating on AdS space is in one to one correspondence with an operator in the field theory. There is a relation between the mass of the field and the conformal dimension of the corresponding operator in the conformal field theory. The mass-dimension formula changes for particles with spin. These formulae in \( AdS_{d+1} \) with unit radius are the
5.2. GENERAL ASPECTS OF $ADS_5 \times S^5$

following:

\[
\begin{align*}
\Delta_+ &= \frac{1}{2}(d \pm \sqrt{d^2 + 4m^2}) \quad \text{scalars} \\
\Delta &= \frac{1}{2}(d \pm 2|m|) \quad \text{spinors} \\
\Delta_\pm &= \frac{1}{2}(d \pm \sqrt{(d-2)^2 + 4m^2}) \quad \text{vectors} \\
\Delta &= \frac{1}{2}(d \pm \sqrt{(d-2)^2 + 4m^2}) \quad \text{p-forms} \\
\Delta &= \frac{1}{2}(d + 2|m|) \quad \text{first order } d/2 \text{ forms (}d\text{ even}) \\
\Delta &= \frac{1}{2}(d + 2|m|) \quad \text{spin-3/2} \\
\Delta &= d \quad \text{massless spin-2}
\end{align*}
\]

The choice $\Delta = \Delta_+$ is clear from the unitary bound. Only the positive branch $\Delta = \Delta_+$ is relevant for the lowest ‘mass’ supergravity multiplet.

To compute correlation functions one needs bulk-to-boundary Green functions. These are specific normalized limits of bulk-to-bulk Green functions when one point is taken to the $AdS$ boundary. The precise forms of these propagators depend on the spin and mass of the field. For scalar field with conformal dimension $\Delta$ the normalized bulk-to-boundary Green function has the following form:

\[
G_\Delta(x, \rho, \omega; x', 0, \omega') = c_\Delta K_\Delta(x^\mu, \rho; x'^\mu, 0) \tag{5.8}
\]

where $\omega$ are the coordinates on $S^5$ and $z^M \equiv (x^\mu, \rho)$ are the $AdS_5$ coordinates, $M = 0, 1, 2, 3, 5$ and $\mu = 0, 1, 2, 3$, $\rho \equiv z_5$ is the coordinate transverse to the boundary.

\[
c_\Delta = \frac{\Gamma(\Delta)}{\pi^2 \Gamma(\Delta - 2)} \tag{5.9}
\]

and

\[
K_\Delta(x^\mu, \rho; x'^\mu, 0) = \frac{\rho^\Delta}{(\rho^2 + (x - x')^2)\Delta} \tag{5.10}
\]

Note that due to (5.8) this Green function is independent of $\omega$. The bulk field is given by:

\[
\Phi_m(z; J) = c_\Delta \int d^4x' K_\Delta(x^\mu, \rho; x'^\mu, 0)J_\Delta(x'^\mu) \tag{5.11}
\]

In the limit $\rho \to 0$, $\rho^{\Delta-4}K_\Delta$ reduces to the $\delta$-function on the boundary and one has

\[
\Phi_m(x, \rho; J) \approx \rho^{4-\Delta}J_\Delta(x) \tag{5.12}
\]

In particular, for the massless scalar ($\Delta_+ = 4$) in the limit $\rho \to 0$ the propagator reduces to $\delta^{(4)}(x^\mu - x'^\mu)$. It is crucial to observe that in the case
\[ \Delta_+ = 4 \] the propagator (5.10) is exactly equal to \( \text{Tr}(F_{\mu\nu})^2 \), where \( F_{\mu\nu} \) is the field strength corresponding to the Yang-Mills instanton with \( \rho \) identified with instanton size. On the other hand (5.10) has precisely the same form as the five-dimensional profile of a D-instanton centered on the point \( z^M \) and evaluated at the boundary point \( (x^\mu, 0) \). This allows to identify D-instanton effects of the bulk theory with those of YM instanton of the boundary theory. The correspondence between YM and D-instantons is nicely clarified in [61], where the classical D-instanton solution of the Type IIB supergravity equations in the \( AdS_5 \times S^5 \) is presented. It is shown that though D-instanton solution does not affect the \( AdS_5 \times S^5 \) geometry but it creates wormhole (in string frame) which leads to interesting modification of the geometry in the large instanton number limit.

5.3 General aspects of \( AdS_7 \times S^4 \)

There is a six dimensional superconformal field theory living on the world-volume of M5-branes and it is dual to M-theory compactified on \( AdS_7 \times S^4 \). The isometry group of \( AdS_7 \times S^4 \) metric is \( SO(6, 2) \times SO(5) \approx Spin(6, 2) \times USp(4) \). After including supersymmetries one gets \( OSp(6, 2|4) \) as a complete isometry superalgebra which contains 32 fermionic generators transforming as \( (8, 4) \) under \( Spin(6, 2) \times USp(4) \). The ‘mysterious’ \( \mathcal{N} = (2, 0) \) superconformal theory is related to SCFT in \( d = 4 \) à la AGT [80] after compactification on Riemann surfaces. We will not discuss it anymore here.

5.4 General aspects of \( AdS_4 \times S^7 \)

A stack of M2-branes has \( AdS_4 \times S^7 \) near-horizon geometry. M-theory on \( AdS_4 \times S^7 \) is dual to three dimensional superconformal field theory. M-theory does not contain a dilaton field, which means that there is no weak-coupling limit. Hence the dual field theory is strongly coupled and as a result does not need to have a classical Lagrangian description. One can think about this three dimensional conformal field theory in the following way. Remember that the low energy effective theory on the world-volume of \( N \) coincident D2-branes of type IIA superstring theory is a maximally supersymmetric three dimensional Yang- Mills theory with gauge group \( U(N) \). Yang-Mills coupling in three dimensions is dimensionful and intro-
produces a scale. This means this theory is not conformal. From the other side type IIA coupling constant is proportional to the radius of a circle on which eleventh dimension is compactified. When this coupling constant becomes large the gauge theory coupling constant also increases. This corresponds to going to the infrared in the gauge theory. Also the circular eleventh dimension is increasing. In the limit where the coupling constant becomes infinite one reaches the conformally invariant fixed-point theory describing a stack of coincident M2-branes in 11 dimensions. This theory has $SO(8)$ R-symmetry which corresponds to the rotations in the eight dimensions transverse to the M2-branes in 11 dimensions. The isometry group of $AdS_4 \times S^7$ metric is $SO(3,2) \times SO(8) \approx Sp(4) \times Spin(8)$. $SO(3,2)$ is the symmetry of $AdS_4$ and in the dual theory it corresponds to the conformal symmetry group. The isometry group $SO(8)$ of $S^7$ corresponds to the R-symmetry of the dual gauge theory. There are 32 conserved supercharges (maximal supersymmetry). In the dual gauge theory 16 of these supersymmetries are realized linearly and the other 16 are conformal supersymmetries. The isometry superalgebra becomes $OSp(8|4)$ once these supersymmetries are included. The 32 fermionic generators transform as $(8,4)$ under $Spin(8) \times Sp(4)$. It was mysterious how to get the dual $d = 3$ SCFT. ABJM have conjectured that 11-d supergravity on $AdS_4 \times S^7 / \mathbb{Z}_k$, corresponding to the near horizon geometry of $N$ M2-branes at a $\mathbb{C}^4 / \mathbb{Z}_k$ singularity, be dual to $\mathcal{N} = 6$ CS theory in $d = 3$ with gauge group $U(N)_k \times U(N)_{-k}$ and opposite CS couplings $k_1 = k = -k_2$ [81,82]. Remind that the near-horizon geometry of a stack of $N$ M2-branes is $AdS_4 \times S^7$ with $N$ units of $F_4$ flux along $AdS_4$ and as many units of its dual $F_7$ along $S^7$ [95]. The metric reads

$$ds^2_{11} = \frac{1}{4} L^2 ds^2_{AdS} + L^2 ds^2_{S^7}$$  \hspace{1cm} (5.13)$$

for later use, note that $L_{AdS} = L/2$ with $L$ the radius of $S^7$ and henceforth the metrics of the subspaces are for unit curvature radii.

5.5 Supergravity description

The Type IIA solution corresponding to the ABJM model reads

$$ds^2_{IIA} = 4 \frac{L^2}{L^2} dx \cdot dx + 4 \frac{L^2}{4\rho^2} d\rho^2 + L^2 ds^2_{\mathbb{C}P^3} = \frac{1}{4} L^2 ds^2_{AdS} + L^2 ds^2_{\mathbb{C}P^3}$$  \hspace{1cm} (5.14)$$
where

\[ L = \left( \frac{32\pi^2 N}{k} \right)^{1/4} \]  

(5.15)

is the curvature radius in string units. The string coupling, related to the VEV of the dilaton, is given by

\[ g_s = L/k = \left( \frac{32\pi^2 N}{k} \right)^{1/4} \]  

(5.16)

Thus the perturbative Type IIA description should be valid for \( L \gg 1 \) and \( g_s \ll 1 \), i.e., for \( N^{1/5} \ll k \ll N \) while \( \lambda = N/k \) is the ’t Hooft coupling of the boundary CS theory.

In the 11-d uplift, \( \mathbb{CP}^3 \) becomes the base of a Hopf fibration \( S^7 = \mathbb{CP}^3 \rtimes S^1 \) whose metric reads

\[ ds_{S^7}^2 = ds_{\mathbb{CP}^3}^2 + (d\tau + \mathcal{A})^2 \]  

(5.17)

with \( d\mathcal{A} = 2\mathcal{J}_{\mathbb{CP}^3} \), the Kähler form on \( \mathbb{CP}^3 \) normalized so that \( dV(\mathbb{CP}^3) = \mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J}/6 \) and \( V(\mathbb{CP}^3) = \pi^3/6 \). The solution is supported by R-R fluxes

\[ g_s F_2 = 2L\mathcal{J} \quad , \quad g_s F_4 = 6L^3dV(AdS_4) \quad , \quad g_s F_6 = 6L^5dV(\mathbb{CP}^3) \]  

(5.18)

In the ABJM model, corresponding to \( \mathcal{N} = 6 \) \( U(N)_k \times U(N)_{-k} \) CS theory on the boundary, \( B_2 = 0 \). For fractional M2-branes, one has the ABJ model corresponding to \( \mathcal{N} = 6 \) CS theory with \( U(N)_k \times U(N+k-l)_{-k} \) [96] on the boundary, \( B_2 = \mathcal{J}l/k \), with \( l = 1, \ldots, k-1 \). Boundary CS theories with \( \sum_i k_i \neq 0 \) and lower susy should be dual to turning on a non-zero Romans mass \( (F_0 \neq 0) \) in the bulk Type IIA description [97–99].

The 11-d supergravity approximation should be valid in the double-scaling limit \( k \to \infty, N \to \infty \) with \( \lambda = N/k \) fixed and large. The CFT description, to which we momentarily turn our attention, should instead be valid when \( \lambda \ll 1 \), i.e., \( k \gg N \). As \( \lambda \to 0 \) higher spin symmetry enhancement takes place as we will eventually see.

### 5.6 Boundary CFT description

\( \mathcal{N} = 6 \) CS theories are conveniently constructed from \( \mathcal{N} = 3 \) CS theories. The case \( \mathcal{N} = 3 \) arises in turn from the \( \mathcal{N} = 4 \) case obtained after dimensional reduction of \( \mathcal{N}' = 2 \) in \( d = 4 \). In this way, each vector multiplet
5.7. A QUICK LOOK AT THE SPECTRUM

includes an \( \mathcal{N} = 2 \) (i.e. \( \mathcal{N} = 1' \) in \( d = 4 \)) chiral multiplet in the adjoint \( \Phi = \Phi_at^a \) and couples to various hypers \( Q \) and \( \tilde{Q} \) in real (reducible) representations. Adding to the ‘standard’ \( \mathcal{N} = 4 \) superpotential

\[
W = \tilde{Q}\Phi Q
\]  

(5.19)

the CS term, giving a mass \( m = g^2YM/\pi \) to the vectors, and a CS superpotential

\[
W = \frac{k}{8\pi} Tr\Phi^2
\]  

(5.20)

breaks \( \mathcal{N} = 4 \) to \( \mathcal{N} = 3 \). Integrating out \( \Phi \) yields

\[
W = \frac{4\pi}{k} (\tilde{Q}t^aQ)(\tilde{Q}t^aQ)
\]  

(5.21)

The resulting \( \mathcal{N} = 3 \) theory has no marginal susy preserving deformations [97–99]. In the process R-symmetry is reduced to \( SO(3) \approx SU(2) \) for \( \mathcal{N} = 3 \) from the original \( SO(4) \) of \( \mathcal{N} = 4 \).

The case \( \mathcal{N} = 6 \) is special. Starting with the \( \mathcal{N} = 3 \) theory with \( G = U(N)_k \times U(N)_{-k} \) and two pairs of hypers, \( A_r \in (N, N^*) \) and \( B_m \in (N^*, N) \) and integrating out \( \Phi_1 \) and \( \Phi_2 \) one gets

\[
W = \frac{2\pi}{k} e^{rs}e^{\tilde{m}\tilde{n}}Tr(A_rB_mA_sB_{\tilde{n}})
\]  

(5.22)

Since the manifest ‘flavour’ symmetry of \( W \) under \( SU(2) \times SU(2) \times U(1)_B \) does not commute with R-symmetry \( SO(3) \approx SU(2) \) under which \( A \) and \( B \) form doublets, the full theory has a larger \( SU(4) \approx SO(6) \) symmetry which is the R-symmetry of \( \mathcal{N} = 6 \). To expose the symmetry it is convenient to define \( X^i = (A_1, A_2, B^*_1, B^*_2) \) and their conjugate \( X^*_i \) that together transform as \( 4_{+1} + 4^*_{-1} \) of \( SO(6) \times SO(2) \). As we will momentarily see, \( SO(2) \approx U(1) \) acts as a baryonic symmetry. Further (super)symmetry enhancement to \( \mathcal{N} = 8 \) with \( SO(8) \) R-symmetry takes place for \( k = 1 \) and \( k = 2 \). The former corresponds to compactification on \( S^7 \) the latter to \( S^7/\mathbb{Z}_2 \) (only ‘even’ spherical harmonics).

5.7 A quick look at the spectrum

The (ungauged) \( \mathcal{N} = 6 \) supergravity multiplet consists of the graviton \( g_{\mu\nu} \), 6 gravitini \( \psi_{\mu}^{i} \), 16 graviphotons \( A_{\mu}^{[ij]} \) and \( A_{\mu}^{0} \), 26 dilatini \( \lambda^{ijkl} \) and \( \lambda_i \), and 30 scalars \( \phi^{ijkl} \) and \( \phi_{ij} \). The latter parameterize the moduli space \( \mathcal{M} = SO^*(12)/U(6) \). After ‘gauging’ \( SO(6) \times SO(2) \) a scalar potential
is generated and the two sets of 150 scalars become ‘massive’ or rather ‘tachyonic’ i.e. \((ML_{AdS})^2 = -2\), safely above the B-F bound \((ML_{AdS})^2 = -9/4\).

Compactification of Type IIA supergravity on \(\mathbb{CP}^3\) was studied in [100]. KK excitations with \(Q = 0\), i.e. neutral wrt \(SO(2)\), were identified there. The non-perturbative spectrum, contains various wrapped branes, including D0-branes that are charged wrt \(SO(2)\). The latter correspond to 11-d KK modes along the compact circle that can be obtained by a \(\mathbb{Z}_k\) projection of the M-theory compactification on \(S^7\). The dual to \(SO(2)\) charged states are monopole operators on the boundary [82, 101, 102]. Although the fundamental fields \((A_r, B_s)\) are neutral wrt the diagonal \(U(1)\) that couples to \(A_\mu^+ = A_\mu^1 + A_\mu^2\), the orthogonal combination \(A_\mu^- = A_\mu^1 - A_\mu^2\) acts as a baryonic symmetry. The corresponding current, \(J_B = *F^+\), is conserved thanks to Bianchi identities. Due to the CS coupling \(k \int A^- \wedge F^+\), configurations with \(A^+\) magnetic charge are electrically charged wrt \(A^-\).

Alternatively one can introduce a Lagrange multiplier \(\tau\) for \(dF^+ = 0\) (on-shell \(kA^- = d\tau\)) and form combinations \(e^{in\tau}\) that can screen the baryonic charge of matter field composites. In general one can consider magnetic monopoles charged under \(U(1)^N \subset U(N)\) with \(H = (Q_1, ..., Q_N)\). Without loss of generality one can take \(Q_1 \geq Q_2 \geq ... \geq Q_N\). Since elementary fields have unit charges and transform in the fundamental of \(SU(N)\), these monopole operators correspond to Young Tableaux with \(kQ_i\) boxes in the \(i^{th}\) row. For \(k = 1, 2\) dressing composite vector currents in the \(6_{\pm 2}\) and scalar operators in the \(10_{\pm 2}\) and \(10^*_{\pm 2}\) (with \(\Delta_\pm = 1, 2\)) with charge 2 monopole operators is crucial to the enhancement of supersymmetry to \(\mathcal{N} = 8\) with full \(SO(8)\) R-symmetry [82]. Monopole and anti-monopole operators however appear in the spectrum even when \(k \geq 3\) and no (super)symmetry enhancement takes place [101, 102].

Before concluding this preliminary look, let us note that out of the two \(U(1)\) in the boundary CS theory only the Baryonic \(U(1)_B = U(1)_-\) is visible as a global symmetry, whose \(\mathbb{Z}_k\) subgroup is gauged, in the bulk description. The fate of the other \(U(1)\) is a sort of Higgs mechanism, under which \(A_M \rightarrow A_\mu\) and \(C_{MNP} \rightarrow C_\mu J_{ab}\) mix. Only the combination \(kA_\mu + NC_\mu\) remains massless and couples to \(U(1)_B\) while the orthogonal combination \(NA_\mu - kC_\mu\) becomes massive by ‘eating’ the (pseudo)scalar \(\beta\) from \(B_2 = \beta J\). A 5-brane instanton is thus expected to mediate processes in which \(k\) D0-branes transform into \(N\) D4-branes wrapped around \(\mathbb{CP}^2 \subset\)
5.8 Compactification on $S^7$ revisited

For the later use let us briefly review the mass spectrum of the Freund-Rubin solution of $d = 11$ supergravity on $S^7$ [93, 103, 104]. The gravitino field as well as all the fermions are set to zero, the $AdS_4$ Riemann tensor and the three-form field strength are given by:

\[
R_{\mu\nu\rho\sigma} = -4(g_{\mu\rho}(x)g_{\nu\sigma}(x) - g_{\mu\sigma}(x)g_{\nu\rho}(x)) \quad (5.23)
\]

\[
F_{\mu\nu\rho\sigma} = 3\sqrt{2}\sqrt{-\det g_{\mu\nu}(x)}\varepsilon_{\mu\nu\rho\sigma} \quad (5.24)
\]

where $\varepsilon_{0123} = -1$. The metric and the three form field with mixed indices vanish:

\[
g_{\mu\alpha} = F_{\mu\nu\rho\alpha} = F_{\mu\nu\alpha\beta} = F_{\mu\alpha\beta\gamma} = 0 \quad (5.25)
\]

and also

\[
F_{\alpha\beta\gamma\delta}(y) = 0 \quad (5.26)
\]

\[
R_{\alpha\beta} = -6g_{\alpha\beta}(y) \quad (5.27)
\]

$\mu, \nu, \rho = 0, ..., 3$ are $d = 4$ indices, $\alpha, \beta, \gamma = 1, ..., 7$ are internal indices.

Let us then consider fluctuations around the Freund-Rubin solution. The linearized field equations are obtained by replacing the background fields in the $d = 11$ field equations by background fields plus arbitrary fluctuations. An elegant and quite general method to determine the complete mass spectrum on any coset manifold relies on generalized harmonic expansion. In our case, one expands the fluctuations in a complete set of spherical harmonics of $S^7 = SO(8)/SO(7)$. The coefficient functions of the spherical harmonics correspond to the physical fields in $d = 4$. In order to diagonalize the linearized equations it turns out to be convenient to parameterize the fluctuations as follows:

\[
g_{\mu\nu}(x, y) = g_{\mu\nu}(x) + h_{\mu\nu}(x, y) \quad (5.28)
\]

\[
h_{\mu\nu}(x, y) = h'_{\mu\nu}(x, y) - \frac{1}{2}g_{\mu\nu}(x)h_{\alpha}(x, y) \quad (5.29)
\]

\[
g_{\alpha\beta}(x, y) = g_{\alpha\beta}(x) + h_{\alpha\beta}(x, y) \quad (5.30)
\]

\[
g_{\mu\alpha}(x, y) = h_{\mu\alpha}(x, y) \quad (5.31)
\]

\[
A_{\mu\nu\rho}(x, y) = A_{\mu\nu\rho}(x) + a_{\mu\nu\rho}(x, y) \quad (5.32)
\]
In particular the Weyl rescaled spacetime metric appears in (5.29) so as to put the \( d = 4 \) Einstein action in canonical form. The spherical harmonic expansions of the fluctuations of the metric and of the antisymmetric tensor fields are given by:

\[
\begin{align*}
 h'_{(\mu\nu)}(x, y) &= \sum H^{N_1}_{\mu\nu}(x) Y^{N_1}(y) \\
 h_{\mu\alpha}(x, y) &= \sum B^{N_7}_{\mu}(x) Y^{N_7}(y) + B^{N_1}_{\mu}(x) D_\alpha Y^{N_1}(y) \\
 h_{(\alpha\beta)}(x, y) &= \sum \phi^{N_7}(x) Y^{N_7}_{(\alpha\beta)}(y) + \phi^{N_1}(x) D_{(\alpha} Y^{N_1}_{\beta)}(y) \\
 h^\alpha(x, y) &= \sum \pi^{N_1}(x) Y^{N_1}(y) \\
 A_{\mu\nu\rho}(x, y) &= \sum a^{N_1}_{\mu\nu\rho}(x) Y^{N_1}(y) \\
 A_{\mu\nu\alpha}(x, y) &= \sum a^{N_7}_{\mu\nu}(x) Y^{N_7}_{\alpha}(y) + a^{N_1}_{\mu\nu}(x) D_\alpha Y^{N_1}(y) \\
 A_{\mu\alpha\beta}(x, y) &= \sum a^{N_7}_{\mu}(x) Y^{N_7}_{\alpha\beta}(y) + a^{N_1}_{\mu}(x) D_{[\alpha} Y^{N_1}_{\beta]}(y) \\
 A_{\alpha\beta\gamma}(x, y) &= \sum a^{N_35}_{\alpha\beta}(x) Y^{N_35}_{\alpha\beta\gamma}(y) + a^{N_21}_{\alpha}(x) D_{[\alpha} Y^{N_21}_{\beta\gamma]}(y) \\
&\text{(5.33)}
\end{align*}
\]

All superscripts \( N_r \) (\( r = 1, 7, 21, 27, 35 \)) have infinite range, since they should provide a basis for arbitrary fields on the 7-sphere. The index \( r \) specifies the \( SO(7) \) representation of the corresponding spherical harmonic. For example, \( Y^{N_35}_{\alpha\beta\gamma} \) is in the third rank totally antisymmetric representation of \( SO(7) \) with dimension 35, while \( Y^{N_7}_{(\alpha\beta)} \) is in the symmetric traceless 27-dimensional representation. Derivatives of \( Y \)'s appear in the expansions since any tensor can be decomposed into its transverse and longitudinal parts. After fixing all local symmetries which do not correspond to gauge invariances of the final \( d = 4 \) theory and by choosing de Donder type, \( D^\alpha h_{(\alpha\beta)}(x, y) = 0 \), and Lorentz type, \( D^\alpha h_{\mu\alpha}(x, y) = 0 \), conditions the last term in \( h_{\mu\alpha} \) and the last two terms in \( h_{(\alpha\beta)} \) drop out. To fix the local symmetries of the antisymmetric tensor fields we choose the Lorentz conditions \( D^\alpha A_{\alpha\beta\gamma}(x, y) = D^\alpha A_{\alpha\beta\mu}(x, y) = D^\alpha A_{\mu\nu\rho}(x, y) = 0 \). As a consequence, also these fields have only transverse harmonics \( a^{N_1}_{\mu\nu}(x) = a^{N_7}_{\mu}(x) = a^{N_21}_{\alpha}(x) = 0 \). Substituting the resulting expansions into the \( d = 11 \) field equations, the coefficients of each independent spherical harmonic yield the \( d = 4 \) field equations.

In the Einstein equation for \( R_{\mu\nu} \) only \( Y^{N_1} \) spherical harmonics appear without derivatives. Thus there is only one field equation, \( i.e. \) one KK tower, for traceless symmetric tensors in \( AdS_4 \).

Examining the Einstein equation for \( R_{\alpha\beta} \) one can see that the vector fields \( B^{N_7}_{\mu} \) are massive and transversal, except for the lowest lying state
5.8. COMPACTIFICATION ON $S^7$ REVISITED

The spin-0 fields $\phi^{N_7}$ have a mass matrix $\Delta_y + 12$ ($\Delta_y$ is the Hodge-de Rham operator). By a judicious gauge choice one can eliminate $H^{N_1}_{\mu\nu}$ in favour of $\pi^{N_1}$ namely $H^{N_1}_{\mu\nu} = \frac{9}{7} \pi^{N_1}$.

Collecting the coefficients of the spherical harmonics $Y^N_{\alpha}$ and $D^{\alpha}Y^N_1$ in the Einstein equation for $R_{\mu\alpha}$, one finds that the spin-1 spectrum consists of linear combinations of $B^{N_7}_\mu$ and $C^{N_7}_\mu$ (from $a^{N_7}_{\rho\sigma}$) and that one can eliminate the divergence $D^\mu H^{N_1}_{\mu\nu}$ in favour of $\pi^{N_1}$, $a^{N_1}_{\rho\sigma\tau}$ except when $Y^{N_1}$ is a constant.

Similarly, inspecting the equations for $p$-form field strengths ($p = 1, 2, 3, 4$), one concludes that field expansions in spherical harmonics can be chosen such that only the first terms in the expansions survive with $Y$s being transversal and traceless.

In particular, from the three-form field strength equation one finds that $a^{N_1}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\lambda} D^\lambda \sigma^{N_1}$. This implies that the divergence of $H^{N_1}_{\mu\nu}$ is proportional to a gradient.

From the four-form field strength equation one gets an equation for $\Box_x \sigma^{N_1}$. Taking the trace of the equations for $R_{\mu\nu}$ and $R_{\alpha\beta}$, an equation involving $\Box_x \sigma^{N_1}$ and $\Box_x H^{N_1}_{\mu\nu}$ arises. Resolving the mixing between $a^{N_1}_{\mu\rho}$ and $H^{N_1}_{\mu\nu}$ produces to independent combinations and as many KK towers of scalars.

From the two-form field strength equation one finds $D^\mu a^{N_7}_{\mu\nu} = 0$, which implies $a^{N_7}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} D_\rho C^{N_7}_\sigma$. Using one of the three-form field strength equations one finds that $C^{N_7}_\mu$ and $B^{N_7}_\mu$ mix. Resolving the mixing one finds two KK towers, one of which starts with a massless vector corresponding to the internal Killing vectors of $S^7$.

After diagonalizing the bosonic field equations one obtains the mass spectrum summarized in Table 5.8. The resulting bosonic spectrum includes the massless graviton, 28 massless vectors of $SO(8)$, corresponding to a combination of $B_\mu$ (in $h_{\mu\alpha}$) and $C_\mu$ (in $A_{\mu\alpha}$), 35 scalars ($\Delta = 1$) and 35s ($\Delta = 2$) pseudoscalars with $(ML_{AdS})^2 = -2$. In the supergravity literature [93, 103, 104] masses of scalars are often shifted by $-R/6$ so that $(ML_{AdS})^2 \rightarrow (\tilde{M}L_{AdS})^2 = (ML_{AdS})^2 + 2$. The 70 (pseudo)scalars in the $\mathcal{N} = 8$ supergravity multiplet are ‘massless’ in the sense that $(\tilde{M}L_{AdS})^2 = 0$. Moreover, there are three families of scalars and two families of pseudoscalar excitations. Three of them ($0^+_2$, $0^+_3$ and $0^-_2$) contain
only states with positive mass square and correspond to irrelevant operators in the dual CFT. The remaining families $0_1^+$ and $0_1^-$ contain states with positive, zero and negative mass squared corresponding to irrelevant, marginal and relevant operators, respectively.

A similar analysis can be performed for fermionic fluctuations. In Table 5.8 we summarize the fermionic mass spectrum.

The KK spectrum does not include the states with * for $\ell = -1$, since they do not propagate in the bulk but live on the conformal boundary of $AdS_4$. They correspond to the singleton representation of $Osp(8|4)$ that consists of $8_v$ bosons $X^i$ with $\Delta = \frac{1}{2}$, $(ML)^2 = -\frac{5}{4}$ and $8_c$ fermions $\psi^a$ with $\Delta = 1$, $ML = \frac{1}{2}$, both at the unitary bound.

The KK excitations on $S^7$ can be put in one-to-one correspondence with ‘gauge-invariant’ composite operators on the boundary. The dictio-

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**Table 5.1: Bosonic KK towers after compactification on $S^7$**

| Spin | Field | $SO(7)$ | $SO(8)$ | $4(ML)^2$ | $\Delta$ | $\ell$ |
|------|-------|---------|---------|------------|---------|-------|
| $2^+$ | $h'_{(\mu \nu \nu)}$ | $N_1$ | $(\ell, 0, 0, 0)$ | $\ell(\ell + 6)$ | $\Delta = \frac{\ell}{2} + 3$ | $\ell \geq 0$ |
| $1^-_1$ | $h_{\mu \alpha}$ | $N_7$ | $(\ell, 1, 0, 0)$ | $\ell(\ell + 2)$ | $\Delta = \frac{\ell}{2} + 2$ | $\ell \geq 0$ |
| $0_1^+$ | $A_{\mu \nu \alpha}$ | $N_7$ | $(\ell - 2, 1, 0, 0)$ | $(\ell + 6)(\ell + 4)$ | $\Delta = \frac{\ell}{2} + 4$ | $\ell \geq 2$ |
| $0_2^+$ | $A_{\mu \alpha \beta}$ | $N_{21}$ | $(\ell - 1, 0, 1, 1)$ | $(\ell + 2)(\ell + 4)$ | $\Delta = \frac{\ell}{2} + 3$ | $\ell \geq 1$ |
| $0_1^0$ | $A_{\mu \nu \rho}$ | $N_1$ | $(\ell + 2, 0, 0, 0)^*$ | $(\ell + 2)(\ell - 4)$ | $\Delta = \frac{\ell}{2} + 1$ | $\ell \geq 0$ |
| $0_2^0$ | $h_{\alpha \alpha}, h'_{\alpha \alpha}$ | $N_1$ | $(\ell - 2, 0, 0, 0)$ | $(\ell + 10)(\ell + 4)$ | $\Delta = \frac{\ell}{2} + 5$ | $\ell \geq 2$ |
| $0_3^0$ | $h_{(\alpha \beta)}$ | $N_{27}$ | $(\ell - 2, 2, 0, 0)$ | $\ell(\ell + 6)$ | $\Delta = \frac{\ell}{2} + 3$ | $\ell \geq 2$ |
| $0_1^- A_{\alpha \beta \gamma}$ | $N_{35}$ | $(\ell, 0, 2, 0)$ | $(\ell - 2)(\ell + 4)$ | $\Delta = \frac{\ell}{2} + 2$ | $\ell \geq 0$ |
| $0_2^- A_{\alpha \beta \gamma}$ | $N_{35}$ | $(\ell - 2, 0, 0, 2)$ | $(\ell + 8)(\ell + 2)$ | $\Delta = \frac{\ell}{2} + 4$ | $\ell \geq 2$ |

**Table 5.2: Fermionic KK towers after compactification on $S^7$**

| Spin | $SO(8)$ | $4(ML)^2$ | $\Delta$ | $\ell$ |
|------|---------|------------|---------|-------|
| $(\frac{3}{2})_1$ | $(\ell, 0, 0, 1)$ | $(\ell + 2)^2$ | $\Delta = \frac{\ell}{2} + \frac{3}{2}$ | $\ell \geq 0$ |
| $(\frac{3}{2})_2$ | $(\ell - 1, 0, 1, 0)$ | $(\ell + 4)^2$ | $\Delta = \frac{\ell}{2} + \frac{3}{2}$ | $\ell \geq 1$ |
| $(\frac{1}{2})_1$ | $(\ell + 1, 0, 1, 0)^*$ | $\ell^2$ | $\Delta = \frac{\ell}{2} + \frac{3}{2}$ | $\ell \geq 0$ |
| $(\frac{1}{2})_2$ | $(\ell - 1, 1, 1, 0)$ | $(\ell + 2)^2$ | $\Delta = \frac{\ell}{2} + \frac{3}{2}$ | $\ell \geq 1$ |
| $(\frac{1}{2})_3$ | $(\ell - 2, 1, 0, 1)$ | $(\ell + 4)^2$ | $\Delta = \frac{\ell}{2} + \frac{3}{2}$ | $\ell \geq 2$ |
| $(\frac{1}{2})_4$ | $(\ell - 2, 0, 0, 1)$ | $(\ell + 6)^2$ | $\Delta = \frac{\ell}{2} + \frac{9}{2}$ | $\ell \geq 2$ |
nary for bosonic operators schematically reads:

\[
\begin{align*}
S & = 2^+ \quad T^{i_1 \ldots i_T}_{\mu_1, \Delta = \frac{7}{2} + 3} = (\partial_{\mu} X_i \partial_{\nu} X^i + \bar{\psi} \gamma_{\mu} \partial_{\nu} \psi) X^{i_1} \ldots X^{i_T} \\
S & = 1^- \quad J^{[i_1 \ldots i_T]}_{\mu, \Delta = \frac{7}{2} + 2} = (X^{[i} \partial_{\mu} X^{j]} + \bar{\psi} \Gamma^{ij} \gamma_{\mu} \psi) X^{i_1} \ldots X^{i_T} \\
S & = 1^- \quad J^{[i_1 \ldots i_{T-2}]}_{\mu, \Delta = \frac{7}{2} + 4} = \partial_{\mu} X_i \tilde{\partial}_{\nu} X^i \bar{\psi} \gamma^\nu \Gamma^{ij} \bar{\psi} X^{i_1} \ldots X^{i_{T-2}} \\
S & = 1^+ \quad J^{ab[1 \ldots i_{T-1}]}_{\mu, \Delta = \frac{7}{2} + 3} = \bar{\psi} \Gamma_{jk} \partial_{\mu} \psi (X_i \Gamma^{ijk})^{ab} X^{i_1} \ldots X^{i_{T-1}} \\
S & = 0^+ \quad \Phi^{i_1 \ldots i_T}_{\Delta = \frac{7}{2} + 1} = X^i X^j X^{i_1} \ldots X^{i_T} \\
S & = 0^+ \quad \Phi^{i_1 \ldots i_{T-2}}_{\Delta = \frac{7}{2} + 5} = \partial_{\mu} X^i \tilde{\partial}_{\nu} X^i \bar{\psi} \gamma^\nu \partial^\rho \psi X^{i_1} \ldots X^{i_{T-2}} \\
S & = 0^+ \quad \Phi^{ij[kl]}_{\Delta = \frac{7}{2} + 3} = (\bar{\psi} \Gamma^{ij} \gamma_{\mu} \psi X^{[k} \partial_{\mu} X^{l]}) X^{i_1} \ldots X^{i_{T-2}} \\
S & = 0^- \quad \Phi^{(ab)[i_1 \ldots i_{T-2}]}_{\Delta = \frac{7}{2} + 2} = \bar{\psi} \gamma^b \psi X^{i_1} \ldots X^{i_{T-2}} \\
S & = 0^- \quad \Phi^{(ab)i_1 \ldots i_{T-2}}_{\Delta = \frac{7}{2} + 4} = (\Gamma^{ijkl})^{ab} X_i \partial_{\mu} X_j \bar{\psi} \Gamma_{kl} \partial_{\nu} \psi X^{i_1} \ldots X^{i_{T-2}}
\end{align*}
\]

A similar dictionary can be compiled for fermions.

### 5.9 Polynomial representations for \( SO(8) \) and \( U(4) \)

In order to decompose KK harmonics on \( S^7 = SO(8)/SO(7) \) into KK harmonics on \( \mathbb{CP}^3 = U(4)/U(3) \times U(1) \), we will present the construction of arbitrary representations of \( SO(8) \) in the space of polynomials of 12 variables. The latter are the coordinates of the subgroup \( Z_{SO(8)}^{SU(3)} \) generated by the raising operators of \( SO(8) \). We will then describe a technique which allows to identify which of the above polynomials correspond to highest weight states of representations of \( U(4) \subset SO(8) \). The method we use is quite standard in representation theory of Lie groups (see e.g. Chapter 16 of [105]).

It is convenient to start with \( SO(8, \mathbb{C}) \) defined as the group of \( 8 \times 8 \) complex matrices which leave invariant the quadratic form \( X^T C^{(8)} X \), where \( X \) is a complex (column) vector whose components will be enumerated as \( X^1, X^2, X^3, X^4, X^5, X^6, X^7, X^8 \) and \( C^{(8)} \) is an \( 8 \times 8 \) matrix with 1’s on SW-NE (anti)diagonal:

\[
C^{(8)}_{ij} = C^{(8)}_{ij} = 0, \quad C^{(8)}_{ij} = C^{(8)}_{ji} = \delta_{ij}, \quad i, j = 1, 2, 3, 4
\]

By definition all matrices \( g \in SO(8) \) satisfy the condition \( g^T C^{(8)} g = C^{(8)} \).
Eventually, in order to select the compact real form $SO(8)$ of our interest, one should identify the coordinates $X^i$ with $\bar{X}^i$ (bar means complex conjugate). A generic $SO(8)$ matrix $g$ can be (uniquely) decomposed as (Gauss decomposition):

$$g = \zeta \lambda z,$$

(5.44)

where $\zeta \in Z_-$, $z \in Z_+$, $\lambda \in \Lambda$ with $Z_+$ ($Z_-$) being the subgroup of lower (upper) triangular matrices with 1's on the diagonal and $\Lambda$ is the subgroup of diagonal matrices (Cartan subgroup). Let’s set

$$\lambda = \text{Diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_4^{-1}, \lambda_3^{-1}, \lambda_2^{-1}, \lambda_1^{-1}).$$

We will realize the irreducible representations of the group $SO(8)$ on some spaces of functions defined on it. In particular, the role of the highest weight vector will be played by the function:

$$\alpha(g) = \lambda_1^{m_1} \lambda_2^{m_2} \lambda_3^{m_3} \lambda_4^{m_4}$$

(5.45)

where $m_1 \geq m_2 \geq m_3 \geq |m_4|$ ($m_i$ are either all integers or all half-integers) uniquely characterize the irrep. The eigenvalues $\lambda_i$ can be expressed in terms of the matrix elements of $g$ explicitly:

$$\lambda_p = \frac{\Delta_p}{\Delta_{p-1}}, \quad p = 1, 2, 3, 4$$

(5.46)

where $\Delta_0 = 1$ and $\Delta_p, p = 1, 2, 3, 4$ are the diagonal minors

$$\Delta_p = \begin{vmatrix}
g_{11} & \cdots & g_{1p} \\
\vdots & \cdots & \vdots \\
g_{p1} & \cdots & g_{pp}
\end{vmatrix}.$$  

(5.47)

Introducing the notation $S_- = \frac{\Delta_1}{\sqrt{\Delta_4}}$, $S_+ = \sqrt{\Delta_4}$ (it is easy to see that $S_{+, -}$ polynomially depend on the matrix elements of $g$) we can rewrite eq. (5.45) as

$$\alpha(g) = \Delta_1^{\ell_1} \Delta_2^{\ell_2} S_+^{\ell_3} S_-^{\ell_4}$$

(5.48)

where $\ell_1 = m_1 - m_2$, $\ell_2 = m_2 - m_3$, $\ell_3 = m_3 - m_4$ and $\ell_4 = m_3 + m_4$ are non-negative integers commonly referred as the Dynkin labels of the irrep. Consider the space $\mathcal{R}_\alpha$ of all linear combinations of the functions $\alpha(\zeta g g_0)$, $g_0 \in SO(8)$. $SO(8)$ is represented in $\mathcal{R}_\alpha$ simply by the right multiplication of the argument. As already mentioned the function $\alpha(g)$ plays the role of the highest weight state. For any function $f(g) \in \mathcal{R}_\alpha$ we have $f(\zeta \lambda z) = \alpha(\lambda) f(z)$ which shows that to restore its full $g$-dependence it is sufficient to only know the values the function assumes on the subgroup $Z_+$. This is why actually we get representation on a space of functions of $z$, in fact
polynomials due to the polynomial dependence on $g$ of $\alpha(g)$ mentioned earlier.

There is an elegant way to characterize this space of polynomials. Consider the four raising generators corresponding to the simple roots

$$
e_1 = E_{12} - E_{21}; \quad e_2 = E_{23} - E_{32};$$

$$e_- = E_{34} - E_{34}; \quad e_+ = E_{34} - E_{43} \quad (5.49)$$

where $E_{pq}$ denotes the $8 \times 8$ matrix whose only non-zero entry 1 is at the position $(p, q)$. Denote their left action on $R_\alpha$ by $D_1, D_2, D_-, D_+$. It is not difficult to prove that

$$D_1^{l_1} \alpha(g) = 0$$

$$D_2^{l_2} \alpha(g) = 0$$

$$D_-^{l_3} \alpha(g) = 0$$

$$D_+^{l_4} \alpha(g) = 0. \quad (5.50)$$

The key observation is that the same equations are valid also for arbitrary functions $f \in R_\alpha$, since they are all generated by $\alpha(g)$ through right multiplications which commute with left multiplications. Below we will use a convenient explicit parametrization of $Z_+ \subset SO(8)$ in terms of two $4 \times 4$ matrices $\eta$ and $a$

$$\eta = \begin{pmatrix} 1 & \eta_{12} & \eta_{13} & \eta_{14} \\ 0 & 1 & \eta_{23} & \eta_{24} \\ 0 & 0 & 1 & \eta_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad a = \begin{pmatrix} a_{14} & a_{13} & a_{12} & 0 \\ a_{24} & a_{23} & 0 & -a_{12} \\ a_{34} & 0 & -a_{23} & -a_{13} \\ 0 & -a_{34} & -a_{24} & -a_{14} \end{pmatrix}. \quad (5.51)$$

Let us further introduce the $8 \times 8$ matrices which in $2 \times 2$ block form read

$$z_0 = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix}; \quad z' = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad (5.52)$$

where

$$\bar{\eta} = \begin{pmatrix} 1 & -\eta_{34} & -\eta_{24} + \eta_{23}\eta_{34} & -\eta_{14} + \eta_{12}\eta_{24} + \eta_{13}\eta_{34} - \eta_{12}\eta_{23}\eta_{34} \\ 0 & 1 & -\eta_{23} & -\eta_{13} + \eta_{12}\eta_{23} \\ 0 & 0 & 1 & -\eta_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.53)$$

An arbitrary $z \in Z_+$ can be (uniquely) represented as

$$z = z' z_0. \quad (5.54)$$
Left multiplication by raising generators (5.49) induces infinitesimal motion on the parameters \( a, \eta \). A straightforward algebra shows that e.g.

\[
(1 + \epsilon e_1)z(a, \eta) = z(a + \delta a, \eta + \delta \eta) + O(\epsilon^2),
\]

where the non-trivial variations are

\[
\delta \eta_{12} = \epsilon, \quad \delta \eta_{13} = \epsilon \eta_{23}, \quad \delta \eta_{14} = \epsilon \eta_{24}, \quad \delta a_{13} = \epsilon a_{23}, \quad \delta a_{14} = \epsilon a_{24}.
\]

Similarly examining the remaining three generators we find

\[
\begin{align*}
\mathcal{D}_1 &= \partial_{\eta_{12}} + \eta_{23} \partial_{\eta_{13}} + a_{23} \partial_{a_{13}} + a_{24} \partial_{a_{14}} \\
\mathcal{D}_2 &= \partial_{\eta_{23}} + \eta_{34} \partial_{\eta_{24}} + a_{13} \partial_{a_{12}} + a_{34} \partial_{a_{24}} \\
\mathcal{D}_- &= \partial_{\eta_{34}} + a_{14} \partial_{a_{13}} + a_{24} \partial_{a_{23}} \\
\mathcal{D}_+ &= \partial_{a_{34}}.
\end{align*}
\]

Thus any irreducible representation of \( SO(8) \) is realized on the space of polynomials of 12 variables \( a, \eta \) subject to the constraints

\[
\begin{align*}
(\partial_{\eta_{12}} + \eta_{23} \partial_{\eta_{13}} + a_{23} \partial_{a_{13}} + a_{24} \partial_{a_{14}})^{\ell_1 + 1} f(a, \eta) &= 0 \\
(\partial_{\eta_{23}} + \eta_{34} \partial_{\eta_{24}} + a_{13} \partial_{a_{12}} + a_{34} \partial_{a_{24}})^{\ell_2 + 1} f(a, \eta) &= 0 \\
(\partial_{\eta_{34}} + a_{14} \partial_{a_{13}} + a_{24} \partial_{a_{23}})^{\ell_3 + 1} f(a, \eta) &= 0 \\
(\partial_{a_{34}})^{\ell_4 + 1} f(a, \eta) &= 0.
\end{align*}
\]

Note that the constant polynomial always satisfies (5.57) and corresponds to the highest weight state. Considering right multiplication it is not difficult to find explicit expressions for the generators of \( SO(8) \) as operators acting on the space of polynomials. For our later proposes let us specify how the diagonal part \( \Lambda \subset SO(8) \) is represented. Since

\[
z(a, \eta)\lambda = \lambda \lambda^{-1} z(a, \eta)\lambda = \lambda z(a', \eta'),
\]

where

\[
a'_{ij} = \lambda_j^{-1} \lambda_i^{-1} a_{ij}; \quad \eta'_{ij} = \lambda_j \lambda_i^{-1} \eta_{ij}
\]

we simply get

\[
\lambda \circ f(a, \eta) = \lambda_1^{m_1} \lambda_2^{m_2} \lambda_3^{m_3} \lambda_4^{m_4} f(a', \eta')
\]

Notice that the variable \( a_{ij} \) shifts the weights as \( m_i \to m_i - 1, m_j \to m_j - 1 \) while the variable \( \eta_{ij} \) shifts them as \( m_i \to m_i - 1, m_j \to m_j + 1 \).
Consider now the \( GL(4, C) \subset SO(8, C) \) subgroup whose off-diagonal blocks in \( 2 \times 2 \) block notation are zero. This subgroup does not mix the coordinates \( X^i \) with \( \tilde{X}^i \) and after restriction to the real sector it becomes the subgroup \( U(4) \subset SO(8) \).

In other words, for the reduction from \( S^7 \) to \( S^7 / \mathbb{Z}_k \lor \mathbb{C}P^3 \ltimes S^1 \) we are interested in, the decomposition \( SO(8) \rightarrow SO(6) \times SO(2) \) is given by the embedding
\[
8_v(1, 0, 0, 0) \rightarrow 4_{+1}[0, 1, 0] + 4^*_{-1}[0, 0, 1]
\]
where \( (\ell_1, \ell_2, \ell_3, \ell_4) \) and \([k, l, m]\) denote \( SO(8) \) and \( SO(6) \) Dynkin labels respectively. As a result, for the Adjoint representation one has
\[
28(0, 1, 0, 0) \rightarrow 15_0[0, 1, 1] + 1_0[0, 0, 0] + 6_{+2}[1, 0, 0] + 6_{-2}[1, 0, 0]
\]
while
\[
8_s(0, 0, 0, 1) \rightarrow 6_0[1, 0, 0] + 1_{+2}[0, 0, 0] + 1_{-2}[0, 0, 0]
\]
\[
8_c(0, 0, 1, 0) \rightarrow 4_{-1}[0, 1, 0] + 4^*_{+1}[0, 0, 1]
\]
for the spinorial representations.

Our goal is to identify the highest weight states of this subgroup inside the space of polynomials of a given representation of \( SO(8) \). It is evident from the decomposition (5.54, 5.53) that the right action by the raising operators of \( GL(4) \) subgroup \( e_1, e_2, e_3 \) (see eq. (5.49)) shifts the parameters \( \eta \) and leave the parameters \( a \) untouched. Thus, in order to be a highest weight state, a polynomial, besides satisfying the equations (5.57) should be independent of \( \eta \). The indicator system for the highest weight states becomes
\[
(a_{23}\partial_{a_{13}} + a_{24}\partial_{a_{14}})^{\ell_1+1}f(a) = 0
\]
\[
(a_{13}\partial_{a_{12}} + a_{34}\partial_{a_{24}})^{\ell_2+1}f(a) = 0
\]
\[
(a_{14}\partial_{a_{13}} + a_{24}\partial_{a_{23}})^{\ell_3+1}f(a) = 0
\]
\[
(\partial_{a_{24}})^{\ell_4+1}f(a) = 0.
\]
Solving these equations one can fully decompose KK harmonics on \( S^7 \) into KK harmonics of \( \mathbb{C}P^3 \ltimes S^1 \) which is our next task.

5.10 From \( S^7 \) to \( \mathbb{C}P^3 \ltimes S^1 \)

\( S^7 \) is a \( U(1) \) bundle over \( \mathbb{C}P^3 \). The \( \mathbb{C}P^3 \) solution of the \( d = 10 \) theory can be obtained from the \( S^7 \) solution of the \( d = 11 \) theory by Hopf fibration,
i.e. keeping only $U(1)$ invariant states [100]. The compactification on \( \mathbb{C}P^3 \) of the \( d = 10 \) theory yields a four dimensional theory with \( N = 6 \) supersymmetry and with gauge group \( SO(6) \times SO(2) \).

The truncation from \( S^7 \) to \( \mathbb{C}P^3 \ltimes S^1 \) cannot be thought of as spontaneous (super)symmetry breaking and one has to really discard the states that are projected out by \( \mathbb{Z}_k \) or \( SO(2) \) for \( k \to \infty \) even if it acts freely. In particular we will later check that no Higgsing can account for the breaking of \( SO(8) \) to \( SO(6) \times SO(2) \) but rather the coset vectors are dressed with monopole operators and become massive for \( k \neq 1, 2 \) [81, 82, 96, 102, 106].

Let us start with the KK towers of bosons. Using the procedure described in the previous section or otherwise, for scalar spherical harmonics with Dynkin labels \((\ell, 0, 0, 0)\) one finds as independent polynomials \( \{a^{m}_{14} \mid m = 0, \ldots, \ell\} \). Thus the following decomposition holds:

\[
N_1 : \quad (\ell, 0, 0, 0) \to \oplus [0, \ell - m, m]_{\ell-2m} \quad (5.66)
\]

where the subscript is the \( SO(2) \) charge \( Q \) of the appropriate representation.

For vector spherical harmonics with \( SO(8) \) Dynkin labels \((\ell - 2, 1, 0, 0)\) one gets \( \{a_{12}a^{m}_{14}, a_{24}a^{m}_{14}, (a_{13}a_{24} - a_{14}a_{23})a^{m}_{14}, a^{m}_{14} \mid m = 0, \ldots, \ell\} \) as independent polynomials. The \( SO(8) \) representation decomposes into \( SO(6) \) representations as:

\[
N_7 : \quad (\ell, 1, 0, 0) \to \oplus [0, \ell - m, m]_{\ell-2m} \oplus [0, \ell - m + 1, m + 1]_{\ell-2m} \\
\oplus [1, \ell - m, m]_{\ell-2m-2} \oplus [1, \ell - m, m]_{\ell-2m+2} \quad (5.67)
\]

One obtains the decomposition of the representation \((\ell - 2, 1, 0, 0)\) from the previous one by shifting \( \ell \) to \( \ell - 2 \). In what follows we will simply omit the decompositions which differ by shifts of the parameter \( \ell \).

For two-form spherical harmonics with \( SO(8) \) Dynkin labels \((\ell - 1, 0, 1, 1)\) one finds \( \{a^{m}_{14}, a_{23}a^{m}_{14}, a_{34}a^{m}_{14}, a_{23}a_{34}a^{m}_{14}, (a_{34}a_{12} - a_{13}a_{24})a^{m}_{14}, a_{23}(a_{23}a_{14} + a_{34}a_{12} - a_{13}a_{24})a^{m}_{14}, a_{13}a^{n}_{14}, a_{34}a_{13}a^{n}_{14}, (a_{34}a_{12} - a_{13}a_{24})a_{13}a^{n}_{14} \mid m = 0, \ldots, \ell - 1, n = 0, \ldots, \ell - 2\} \) as independent polynomials. One then finds the following
decomposition:

\[ N_{21} : \ (\ell - 1, 0, 1, 1) \rightarrow \bigoplus [0, \ell - m, m]_{\ell - 2m - 4} \bigoplus [0, \ell - m - 1, m + 1]_{\ell - 2m + 2} \]
\[ \bigoplus [1, \ell - m, m]_{\ell - 2m - 2} \bigoplus [1, \ell - m - 1, m + 1]_{\ell - 2m} \]
\[ \bigoplus [0, \ell - m, m]_{\ell - 2m} \bigoplus [0, \ell - m - 1, m + 1]_{\ell - 2m - 2} \]
\[ \bigoplus [1, \ell - n - 2, n]_{\ell - 2n - 4} \bigoplus [2, \ell - n - 2, n]_{\ell - 2n - 2} \]
\[ \bigoplus [1, \ell - n - 2, n]_{\ell - 2n} \]

The decomposition of the KK towers corresponding to \( 0^+_1 \) and \( 0^+_2 \) can be found from the decomposition of \( 2^+ \) via appropriate shifts.

For second rank symmetric traceless harmonics with Dynkin labels \((\ell - 2, 2, 0, 0)\) the polynomials are: \( \{ a_{14}^m, a_{12}a_{14}^m, a_{12}(a_{23}a_{14} - a_{13}a_{24})a_{14}^m, a_{12}^2a_{14}^m, a_{12}a_{24}a_{14}^2a_{14}a_{14}^m, a_{24}(a_{23}a_{14} - a_{13}a_{24})a_{14}^m, (a_{13}a_{24} - a_{14}a_{23})a_{14}^m, (a_{14}a_{23} - a_{13}a_{24})^2a_{14}^m, a_{12}^2a_{14}^m \mid m = 0, ..., \ell - 2 \} \). The \( SO(6) \) representations decomposed from \( SO(8) \)'s are:

\[ N_{27} : \ (\ell - 2, 2, 0, 0) \rightarrow \bigoplus [2, \ell - m - 2, m]_{\ell - 2m + 2} \bigoplus [1, \ell - m - 2, m]_{\ell - 2m} \]
\[ \bigoplus [1, \ell - m - 2, m]_{\ell - 2m - 4} \bigoplus [0, \ell - m - 1, m + 1]_{\ell - 2m - 2} \]
\[ \bigoplus [0, \ell - m - 2, m]_{\ell - 2m - 2} \bigoplus [1, \ell - m - 1, m + 1]_{\ell - 2m} \]
\[ \bigoplus [1, \ell - m - 1, m + 1]_{\ell - 2m - 4} \bigoplus [2, \ell - m - 2, m]_{\ell - 2m - 2} \]
\[ \bigoplus [2, \ell - m - 2, m]_{\ell - 2m - 6} \bigoplus [0, \ell - m, m + 2]_{q = \ell - 2m - 2} \]

For the three-form spherical harmonic with \( SO(8) \) Dynkin labels \((\ell, 0, 2, 0)\) one finds \( \{ (a_{14}^m + a_{23}a_{14}^m + a_{23}a_{14}^m), a_{13}(a_{14}^m + a_{23}a_{14}^m), a_{13}^2a_{14}^m \mid m = 0, ..., \ell, n = 0, ..., \ell - 1, p = 0, ..., \ell - 2 \} \) polynomials. The representation \((\ell, 0, 2, 0)\) decomposes as:

\[ N_{35} : \ (\ell, 0, 2, 0) \rightarrow \bigoplus [0, \ell - m, m + 2]_{\ell - 2m + 2} \bigoplus [0, \ell - m + 1, m + 1]_{\ell - 2m} \]
\[ \bigoplus [0, \ell - m + 2, m]_{\ell - 2m - 2} \bigoplus [1, \ell - n - 1, n + 1]_{\ell - 2n} \]
\[ \bigoplus [1, \ell - n, n]_{\ell - 2n - 2} \bigoplus [2, \ell - p - 2, p]_{\ell - 2p - 2} \]

For the three-form spherical harmonic with \( SO(8) \) Dynkin labels \((\ell - 2, 0, 0, 2)\) one has \( \{ (a_{14}^m, (a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24})a_{14}^m, (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2a_{14}^m, a_{34}a_{14}^m, a_{34}(a_{24}a_{13} - a_{34}a_{12} - a_{14}a_{23})a_{14}^m, a_{34}^2a_{14}^m \mid m = 0, ..., \ell - 2 \)
and the following decomposition:

\[
N_{35}': \quad (\ell - 2, 0, 0, 2) \rightarrow \bigoplus [0, \ell - m - 2, m]_{\ell - 2m - 2} \oplus [0, \ell - m - 2, m]_{\ell - 2m + 2} \\
\bigoplus [1, \ell - m - 2, m]_{\ell - 2m - 4} \oplus [1, \ell - m - 2, m]_{\ell - 2m} \\
\bigoplus [2, \ell - m - 2, m]_{\ell - 2m - 2} \oplus [0, \ell - m - 2, m]_{\ell - 2m - 6} \\
(5.71)
\]

Let us now consider the fermionic KK towers. There are two gravitini in the \(SO(8)\) representations \((\ell, 0, 0, 1)\) and \((\ell - 1, 0, 1, 0)\).

For the \(SO(8)\) representation \((\ell, 0, 0, 1)\) one finds \(\{a_{14}^m, (a_{14}a_{23} + a_{12}a_{24} - a_{13}a_{24})a_{14}^m, a_{34}a_{14}^m | m = 0, \ldots, \ell\}\) as polynomials and the following decomposition holds

\[
(\ell, 0, 0, 1) \rightarrow \bigoplus [0, \ell - m, m]_{\ell - 2m + 2} \oplus [0, \ell - m, m]_{\ell - 2m - 2} \oplus [1, \ell - m, m]_{\ell - 2m} \\
(5.72)
\]

For the \(SO(8)\) representation \((\ell - 1, 0, 1, 0)\) the independent polynomials are \(\{a_{14}^m, a_{23}a_{14}^m, a_{13}a_{14}^m | m = 0, \ldots, \ell - 1, n = 0, \ldots, \ell - 2\}\) and is decomposed as:

\[
(\ell - 1, 0, 1, 0) \rightarrow \bigoplus [0, \ell - m - 1, m + 1]_{\ell - 2m} \oplus [0, \ell - m, m]_{\ell - 2m - 2} \\
\bigoplus [1, \ell - n - 2, n]_{\ell - 2n - 2} \\
(5.73)
\]

There are other fermions in the representations \((\ell + 1, 0, 1, 0), (\ell - 2, 0, 0, 1), (\ell - 1, 1, 1, 0)\) and \((\ell - 2, 1, 0, 1)\).

For the \(SO(8)\) representation \((\ell - 1, 1, 1, 0)\) the polynomials have the form

\[
\{a_{14}^m, a_{23}a_{14}^m, a_{23}(a_{13}a_{24} - a_{14}a_{23})a_{14}^m, a_{24}a_{14}^m, a_{13}a_{24}a_{14}^m, a_{23}a_{24}a_{14}^m, a_{13}a_{14}^m, a_{14}a_{14}^m, a_{12}a_{14}a_{14}^m, a_{12}a_{13}a_{14}^m | m = 0, \ldots, \ell - 1, n = 0, \ldots, \ell - 2\}\] and one has the following decomposition:

\[
(\ell - 1, 1, 1, 0) \rightarrow \bigoplus [1, \ell - m - 1, m + 1]_{\ell - 2m + 2} \oplus [1, \ell - m, m]_{\ell - 2m} \\
\bigoplus [1, \ell - m - 1, m + 1]_{\ell - 2m} \oplus [0, \ell - m, m + 2]_{\ell - 2m} \\
\bigoplus [1, \ell - m - 1, m + 1]_{\ell - 2m - 2} \oplus [0, \ell - m + 1, m + 1]_{\ell - 2m - 2} \\
\bigoplus [0, \ell - m - 1, m + 1]_{\ell - 2m} \oplus [0, \ell - m, m]_{\ell - 2m - 2} \\
\bigoplus [2, \ell - n - 2, n]_{\ell - 2n} \oplus [2, \ell - n - 2, n]_{\ell - 2n - 4} \\
\bigoplus [1, \ell - n - 2, n]_{\ell - 2n - 2} \\
(5.74)
\]
5.11. A CLOSER LOOK AT THE KK SPECTRUM

Finally for the $SO(8)$ representation $(\ell - 2, 1, 0, 1)$ the polynomials have the form $\{a^m_{14}, (a_{14}a_{23} - a_{13}a_{24})a^m_{14}, (a_{13}a_{24} - a_{12}a_{34} - a_{14}a_{23})(a_{14}a_{23} - a_{13}a_{24})a^m_{14}, a_{12}(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})a^m_{14}, a_{24}(a_{12}a_{34} - a_{13}a_{24})a^m_{14} + a_{14}a_{23})a^m_{14}, a_{34}(a_{13}a_{24} - a_{14}a_{23})a^m_{14}, a_{34}a_{24}a^m_{14}, a_{34}a_{12}a^m_{14} | m = 0, ..., \ell - 2\}$ and the decomposition reads

$$(\ell - 2, 1, 0, 1) \rightarrow \bigoplus [1, \ell - m - 2, m]_{\ell - 2m + 2} \bigoplus [1, \ell - m - 2, m]_{\ell - 2m - 2}$$
$$\bigoplus [0, \ell - m - 2, m]_{\ell - 2m - 6} \bigoplus [0, \ell - m - 2, m]_{\ell - 2m - 6}$$
$$\bigoplus [0, \ell - m - 1, m + 1]_{\ell - 2m - 4} \bigoplus [2, \ell - m - 2, m]_{\ell - 2m - 4}$$
$$\bigoplus [2, \ell - m - 2, m]_{\ell - 2m - 4} \bigoplus [1, \ell - m - 1, m + 1]_{\ell - 2m - 2}$$
$$\bigoplus [1, \ell - m - 2, m]_{\ell - 2m - 2}$$

(5.75)

The relevant $SO(8) \rightarrow SO(6) \times SO(2)$ decomposition is given by the embedding (5.61), (5.63), (5.64). In particular this implies

$$35_v(2, 0, 0, 0) \rightarrow 15_0[0, 1, 1] + 10^*[2, 0, 0] + 10^*[-2][0, 0, 2]$$
$$35_e(0, 0, 2, 0) \rightarrow 15_0[0, 1, 1] + 10^*[2, 0, 0] + 10^*[-2][0, 0, 2]$$
$$35_s(0, 0, 2, 2) \rightarrow 20'_{0}[2, 0, 0] + 6_{+2}[1, 0, 0] + 6_{-2}[1, 0, 0] + 1_{0}[0, 0, 0] + 1_{+4}[0, 0, 0] + 1_{-4}[0, 0, 0]$$

(5.76)

that are necessary to analyze the spectrum of scalars.

The zero charge spectrum i.e. the states which constitute the KK spectrum of Type IIA supergravity on $\mathbb{CP}^3$ can be easily identified in the above decompositions. For completeness and comparison with the original literature [100], we collect the relevant formulae in an Appendix.

5.11 A closer look at the KK spectrum

As already observed, the $\mathbb{Z}_k$ orbifold projection from $S^7$ to $S^7/\mathbb{Z}_k \approx \mathbb{CP}^3 \times S^1$ cannot be thought of as spontaneous (super)symmetry breaking. ‘Untwisted’ states that are projected out do not simply become ‘massive’ but are rather eliminated from the spectrum. In particular in the large $k$ limit only $SO(2)$ singlets survive. It is amusing to observe that only states with $\ell$ even on $S^7$ give rise to neutral states. This suggests that the parent
theory could be either a compactification on $S^7$ or on $\mathbb{RP}^7 = S^7/\mathbb{Z}_2$. Indeed both lead to $SO(8)$ gauged supergravity corresponding to the ‘massless’ multiplet
\begin{equation}
\{g_{\mu\nu}, 8\psi_\mu, 28A_\mu, 56\lambda, 35^+ + 35^- \phi\} \quad (5.77)
\end{equation}

Massless scalars, corresponding to marginal operators with $\Delta = 3$ on the boundary, only appear in higher KK multiplets, \textit{i.e.} in the $840' = (2,0,0,2)$ and $1386 = (6,0,0,0)$. None of these can play the role of St"uckelberg field for the 12 coset vectors in the $6_{+2} + 6_{-2}$ of $SO(8)/SO(6) \times SO(2)$.

Indeed, using the group theory techniques described in Section 5.9 or otherwise, the decomposition of $840' = (2,0,2,0)$ under $SO(8) \rightarrow SO(6) \times SO(2)$ reads
\begin{equation}
840_{\text{ve}}(2,0,2,0) \rightarrow 84_{+4}[0,2,2] + 70_{+2}[0,3,1] + 70_{+2}[0,1,3] + 64_{+2}[1,1,1] + 84_0[0,2,2] + 45_0[1,2,0] + 45_0[1,0,2] + 35_0[0,4,0] + 35_0[0,0,4] + 20_0[2,0,0] + 84_{-4}[0,2,2] + 70_{-2}[0,3,1] + 70_{-2}[0,1,3] + 64_{-2}[1,1,1] \quad (5.78)
\end{equation}

This means that the massless scalars in the $840_{\text{ve}}(2,0,2,0)$ cannot account for the ‘needed’ St"uckelberg fields in the $6_{+2} + 6_{-2}$. Yet one can recognize massless scalars neutral under $SO(2)$ that survive in $k \rightarrow \infty$ limit and transform non-trivially under $SO(6)$. Turning them on in the bulk, \textit{e.g.} in domain-wall solutions, should trigger RG flows to theories with lower supersymmetry on the boundary.

The same applies to the other massless scalars in the $1386(6,0,0,0)$, the totally symmetric product of 6 $8_\nu \rightarrow 4_{+1} + 4^*_{-1}$. The relevant decomposition reads
\begin{equation}
1386(6,0,0,0) \rightarrow 84_{+6}[0,6,0] + 189_{+4}[0,5,1] + 270_{+2}[0,4,2] + 300_0[0,3,3] + 84_{-6}[0,0,6] + 189_{-4}[0,1,5] + 270_{-2}[0,2,4] \quad (5.79)
\end{equation}

Once again there are no $6_{+2} + 6_{-2}$. In this case, ‘neutral’ fields appear in the $300$ representation of $SO(6)$.

In the KK spectrum, neutral (wrt to $SO(2)$) singlets (of $SO(6)$) appear in the decomposition of $35_s$ parity odd scalars $0_2$ with $M^2L^2_{\text{AdS}} = 10$ that
They correspond to boundary operators with dimension $\Delta = 5$. The only other neutral singlets arise from the $SO(8)$ singlet parity even scalar with $M^2L^2_{AdS} = 18$, i.e. $\Delta = 6$. Neither ones belongs in the supergravity multiplet. They correspond to the ‘stabilized’ complexified Kähler deformation $\mathcal{J} + iB$ and as such couple to the Type IIA world-sheet instanton recently identified in [107]. Indeed the bosonic action schematically reads $S_{\text{wsi}} = \int \mathcal{J} + iB = L^2/\alpha'$ since $B = 0$ in the ABJM model, while $B = l/k$ with $l = 1, \ldots, k - 1$ for the ABJ model involving fractional M2-branes. Effects induced by world-sheet instantons in Type IIA on $\mathbb{CP}^3$ should be dual to the non-perturbative corrections discussed in [108]. It may be worth to observe that in ‘ungauged’ $\mathcal{N} = 6$ supergravity, arising from freely acting asymmetric orbifolds of Type II superstrings on tori, world-sheet and other asymmetric brane instantons [109,110] should correct $\mathcal{R}^4$ terms very much as in their parents with $\mathcal{N} = 8$ local supersymmetry.

Other non-perturbative effects are induced by E5-brane instantons that should mediate the process of annihilation of $k$ D0-branes into $N$ D4-branes wrapping $\mathbb{CP}^2$ [81,96]. In order to determine the action of such an instanton it is worth recalling that the pseudo-scalar mode $B_2 = \beta(x)J_2(y)$ is eaten by the vector field $A^{\mu}_H = kA^{D4}_\mu - NA^{D0}_\mu$ that becomes massive. The complete E5-brane instanton action should be $S_{E5} = L^6/g_s^4(\alpha')^3 + i\beta$ that indeed shifts under $U(1)_H$ gauge transformations and as such can compensate for the ‘charge’ violation in the above process as in similar cases with unoriented D-brane instantons [1].
Chapter 6

Singleton, partition functions and Higher Spins

In this section, we would like to discuss the higher spin (HS) extension of \( \mathcal{N} = 6 \) gauged supergravity. Higher spin extensions of various supergravity theories in \( AdS_4 \) have been studied in [111–113] but to the best of our knowledge the case of \( \mathcal{N} = 6 \) has been overlooked.

Let us start by briefly recalling some basic features of higher spin theories in \( AdS_4 \). In the non supersymmetric case the HS algebra represents an extension of the conformal group \( SO(3, 2) \) that admits two singleton representations \( D(1/2, 0) \) (free boson) and \( D(1, 1/2) \) (free fermion). The two labels denote conformal dimension \( \Delta \) and spin \( s \). Indeed the maximal compact subgroup of \( SO(3, 2) \) is \( SO(3) \times SO(2) \approx SU(2) \times U(1) \) while ‘Lorentz’ transformations and dilatations commute and generate \( SO(2, 1) \times SO(1, 1) \subset SO(3, 2) \). We will continue and call \( \Delta \) the dimension and \( s \) or \( j \) spin. In ‘radial’ quantization the ‘Hamiltonian’ \( \mathcal{H} \) has eigenvalues \( \Delta \).

For later use let us collect here the partition functions of the two singletons that take into account their conformal descendants \( i.e. \) non vanishing derivatives. For free bosons such that \( \partial^2 X = 0 \) one has

\[
Z_B(q) = \text{Tr} q^{2\mathcal{H}} = \frac{q - q^5}{(1 - q^2)^3} = \frac{q + q^3}{(1 - q^2)^2}
\]  

(6.1)

For free fermions \( \partial \Psi = 0 \) one has

\[
Z_F(q) = \text{Tr} q^{2\mathcal{H}} = 2q^2 q^4 \frac{q^2 - q^4}{(1 - q^2)^3} = 2q^2 \frac{q^2}{(1 - q^2)^2}
\]  

(6.2)

\(^1\)See \( e.g. \) [114–117] for recent reviews of both Vasiliev’s and geometric approaches.
Combining \( n_b = 8_v \) free bosons and \( n_f = 8_c \) free fermions one finds the singleton representation of \( Osp(8|4) \supset SO(8) \times SO(3,2) \), whose Witten index reads
\[
Z(q) = Tr(-)^F q^{2H} = 8_v Z_B(q) - 8_c Z_F(q)
\]
(6.3)
One can also keep track of the spin of the states in the spectrum by including a chemical potential \( y = e^{i\alpha} (y^J_3 = e^{i\alpha J_3}) \) and find
\[
Z_B(q, \alpha) = \frac{q(1 - q^4)}{(1 - q^2)(1 - e^{i\alpha} q^2)(1 - e^{-i\alpha} q^2)} = \frac{q(1 + q^2)}{1 - 2q^2 \cos \alpha + q^4}
\]
(6.4)
\[
Z_F(q, \alpha) = \frac{q^2(1 - q^2)\chi_\frac{1}{2}(\alpha)}{(1 - q^2)(1 - 2q^2 \cos \alpha + q^4)}
\]
(6.5)
where
\[
\chi_\frac{1}{2}(\alpha) = 2 \cos \alpha = tr_{1/2} e^{i\alpha J_3}
\]
(6.6)
is the character of the fundamental representation of the ‘Lorentz’ group \( SU(2) \).

Before switching to higher spins, notice that \( Z_k \) acts on the singleton simply as
\[
8_v \rightarrow 4\omega + 4^*\bar{\omega} \quad 8_c \rightarrow 4\bar{\omega} + 4^*\omega \quad 8_s \rightarrow 6 + \omega^2 + \bar{\omega}^2
\]
(6.7)
with \( \omega = e^{2\pi i/k} \) playing the role of chemical potential or rather fugacity for the \( SO(2) \approx U(1)_B \) charge \( Q \) commuting with \( SO(6) \) R-symmetry. One can introduce another three chemical potentials \( \beta_i \) or fugacities \( x_i = e^{i\beta_i} \) in order to keep track of the three Cartan’s of \( SO(6) \approx SU(4) \). We refrain from doing so here.

### 6.1 Doubleton and higher spin gauge fields

*Doubleton* representations can be obtained as tensor products of two singletons [118–120].
\[
\mathcal{D}(1/2, 0) \otimes \mathcal{D}(1/2, 0) = \bigoplus_{s=0}^{\infty} \mathcal{D}(\Delta = s + 1, s)
\]
(6.8)
or
\[
\mathcal{D}(1, 1/2) \otimes \mathcal{D}(1, 1/2) = \mathcal{D}(\Delta = 2, s = 0) + \bigoplus_{s \neq 0}^{\infty} \mathcal{D}(\Delta = s + 1, s)
\]
(6.9)
A consistent truncation, giving rise to minimal HS theories with even spins only, stems from restricting to symmetric tensors for bosons

\[
[D(1/2, 0) \otimes D(1/2, 0)]_S = \bigoplus_{k=0}^{\infty} D(\Delta = 2k + 1, s = 2k)
\]  

(6.10)

or anti-symmetric for fermions

\[
[D(1, 1/2) \otimes D(1, 1/2)]_A = D(\Delta = 2, s = 0) + \bigoplus_{k \neq 0}^{\infty} D(\Delta = 2k + 1, s = 2k)
\]  

(6.11)

Odd spin states appear in the product with opposite symmetry

\[
[D(1/2, 0) \otimes D(1/2, 0)]_A = \bigoplus_{k=0}^{\infty} D(\Delta = 2k + 2, s = 2k + 1)
\]  

(6.12)

for bosons and

\[
[D(1, 1/2) \otimes D(1, 1/2)]_S = \bigoplus_{k=0}^{\infty} D(\Delta = 2k + 2, s = 2k + 1)
\]  

(6.13)

for fermions. Generators of the HS symmetry algebra can be realized as polynomials of bosonic oscillators \(y_\alpha, y_\beta = (y_\alpha)^\dagger\) satisfying \([y_\alpha, y_\beta] = i\varepsilon_{\alpha\beta}\) and \([y_\alpha^\dagger, y_\beta^\dagger] = i\varepsilon_{\alpha^\dagger\beta^\dagger}\).

The supersymmetric extensions require the introduction of fermionic oscillators \(\xi^i\) with \(i = 1, ..., N\), satisfying \(\{\xi^i, \xi^j\} = \delta^{ij}\). The resulting HS superalgebra denoted by \(shs^E(N|4)\) contains \(Osp(N|4)\) whose bosonic generators span \(SO(3, 2) \cong Sp(4, R)\) (conformal group) and \(SO(N)\) R-symmetry [111–113].

In particular for \(N = 8\), with \(SO(8)\) R-symmetry, \(Osp(8|4)\) is the maximal finite dimensional subalgebra of the HS gauge algebra \(shs^E(8|4)\), which is a Lie superalgebra. The relevant super-singleton consists in²

\[
\hat{D}_{N=8} = D(1/2, 0; 8_v) \oplus D(1, 1/2; 8_c)
\]  

(6.14)

The (graded) symmetric product of two singletons \(\hat{D}_{N=8} \otimes \hat{D}_{N=8}\) yields

\[
\{(D(1/2, 0; 8_v) \oplus D(1, 1/2; 8_c)) \otimes (D(1/2, 0; 8_v) \oplus D(1, 1/2; 8_c))\}_S =
D(1, 0; 1 + 35_v) \oplus D(2, 0; 1 + 35_c) \oplus_k D(k + \frac{3}{2}, k + \frac{1}{2}; 8_s + 56_s)
\]

\[
\oplus_{k \neq 0} D(2k + 1, 2k; 1 + 35_v + 1 + 35_c) \oplus_k D(2k + 2, 2k + 1; 28 + 28)
\]  

(6.15)

²Different conventions for the \(SO(8)\) representations of bosons and fermions appear in the literature which are related to the present one, chosen for compatibility with our previous analysis, by \(SO(8)\) triality.
It is reassuring to recognize above the ‘massless’ states of $\mathcal{N} = 8$ gauged supergravity on $AdS_4$. The remaining states with spin $s \leq 2$ belong to the ‘short’ Konishi multiplet and a ‘semishort’ multiplet with spin ranging from 2 to 6 [121–123]. Holography allows to relate AdS compactifications of supergravity and superstring theories to singleton field theories on the 3-d boundary. As a first step, these field theories can be constructed on the boundary of AdS as free superconformal theories. A remarkable property of singletons is that the symmetric product of two super-singletons gives an infinite tower of massless higher spin states. In the limit $\lambda \to 0$, all higher spin states become massless. After turning on interactions, a pantagruelic Higgs mechanism, named *Grande Bouffe* in [124–127], takes place. All but a handful of HS gauge fields become massive after ‘eating’ lowest spin states. The boundary counterpart of this phenomenon is the appearance of anomalous dimensions for HS currents and their superpartners. One should keep in mind that genuinely massive states are already present in the spectrum at $\lambda \to 0$ and arise in the product of three and more singletons.

Interacting theories for massless HS gauge fields, thus only describing the doubleton, have been proposed by Vasiliev [116] that capture some aspects of the holographic correspondence in the extremely stringy (high AdS curvature) regime. Only vague glimpses of an interacting theory incorporating the *Grande Bouffe* have been offered so far [124–127].

Barring these subtle issues, let us discuss how to perform a $\mathbb{Z}_k$ projection of the spectrum giving rise to an $\mathcal{N} = 6$ HS supergravity in $AdS_4$. In the limit $k \to \infty$ only $SO(2)$ singlets survive

\[
\{ [\mathcal{D}(1/2, 0; 8_v) \oplus \mathcal{D}(1, 1/2; 8_v)] \}^{\otimes 2}_{SO(2)\text{ singlets}} = \\
\mathcal{D}(1, 0; 1 + 15) \oplus \mathcal{D}(2, 0; 1 + 15) \oplus _k \mathcal{D}(k + \frac{3}{2}, k + \frac{1}{2}; 6 + 6 + 10 + 10^*) \\
\oplus _{s \neq 0} \mathcal{D}(s + 1, s; 1 + 15 + 1 + 15)
\]  

(6.16)

where indicated in bold-face are the surviving representations of the $SO(6)$ R-symmetry. Candidate bosonic HS operators on the boundary in the $1 + 15$ of $SO(6)$ are

\[
\mathcal{J}_{\mu_1...\mu_s}^i = X^i \partial_{\mu_1} \partial_{\mu_2} ... \partial_{\mu_s} \bar{X}_j + \bar{\Psi}^i \gamma_{\mu_1} \partial_{\mu_2} ... \partial_{\mu_s} \Psi_j + ...
\]

(6.17)

where dots stand for symmetrization and subtraction of the traces and the coefficients of the linear combination are to be chosen appropriately.
At finite $k$ and $\lambda$, states with $SO(2)$ charges $Q = kn$ survive. One can exploit orbifold technique to deduce the ‘free’ spectrum$^3$.

The partition function or rather Witten index for the super-singlet of $OSp(8|4)$ reads:

$$Z_\square = \frac{8q}{(1 + q)^2}$$

(6.18)

the $\mathbb{Z}_k$ projection reads

$$Z_\square^{\mathbb{Z}_k} = \frac{1}{k} \sum_{r=0}^{k-1} Z_r^{(r)}$$

(6.19)

where

$$Z_r^{(r)} = \frac{(4\omega^r + 4\bar{\omega}^r)q}{(1 + q)^2}$$

(6.20)

with $\omega = e^{2\pi i/k}$. Clearly $Z_\square^{\mathbb{Z}_k} = 0$ since $\sum_{r=0}^{k-1} \omega^r = 0$.

A non-trivial spectrum arises from the doubleton partition function. Prior to the $\mathbb{Z}_k$ projection one has

$$Z_\square = \frac{1}{2} (Z_\square^2(q) + Z_\square(q^2)) = 4q^2(8(1 + q)^{-4} + (1 + q^2)^{-2})$$

(6.21)

for the (graded) symmetric doubleton, giving rise to precisely the spectrum of $hs(8|4)$ discussed above.

Performing the $\mathbb{Z}_k$ projection on the symmetric doubleton one finds

$$Z_\square^{\mathbb{Z}_k} = \frac{1}{2k} \sum_r (Z_r^{(r)}(q, \omega)^2 + Z_r^{(r)}(q^2, \omega^2))$$

$$= 4q^2 \left[ 4 \left( 1 + \sum_r \frac{\omega^{2r} + \bar{\omega}^{2r}}{2k} \right)^2 (1 + q)^{-4} + \sum_r \frac{\omega^{2r} + \bar{\omega}^{2r}}{2k} (1 + q^2)^{-2} \right]$$

(6.22)

for the (graded) symmetric doubleton, giving rise to precisely the ‘massless’ HS gauge fields of $hs(6|4)$ for $k \neq 2$ and $hs(8|4)$ for $k = 1, 2$, as expected $Z_{HS} = Z_\square^{\mathbb{Z}_k}$! Indeed

$$Z_{HS} = \frac{36q^2(q^2 + q^4) + 72 \sum_{s=2k+1} F_s(q) + 56 \sum_{s=2k+1} F_s(q) - 64 \sum_{s=k+1} F_s(q)}{(1 - q^2)^3}$$

(6.23)

$^3$Although $k$ is finite, one can take $k \gg N$, so that $\lambda \ll 1$, in order to identify states that eventually become massive.
with $F_s(q) = (2s + 1)q^{2(s+1)} - (2s - 1)q^{2(s+1)+2}$ taking into account the presence of null descendants for conserved spin $s$ currents of dimension $\Delta = s + 1$. The relevant characters read

$$X_{s}^{\Delta=s+1} = \frac{q^{2\Delta}(2s + 1) - q^{2(\Delta+1)}(2s - 1)}{(1 - q^2)^3} = \frac{q^{2\Delta}[\chi_s(\alpha) - q^2\chi_{s-1}(\alpha)]}{(1 - q^2)(1 - 2q^2 \cos \alpha + q^4)}$$

(6.24)

up to some $SO(8)$ multiplicity $d_{\ell,...}^{SO(8)}$.

The situation is summarized in the following Tables, where $s$ denotes spin and $h$ the ‘string’ level.

The decomposition into charged sectors reads

$$Z_{\Delta} = \frac{1}{(1 - q^2)(1 - 2q^2 \cos \alpha + q^4)} \left\{ 10 \left( \omega^2 + \omega_c^2 \right) + 16 \left[ (q^2 + q^4) \chi_0(y) + 12 \left( \omega^2 + \omega_c^2 \right) + 32 \right] \chi_j(y)q^{2(j+1)} - \chi_{j-1}(y)q^{2(j+1)+2} \right\} + \sum_{j \in 2,4,...} 20 \left( \omega^2 + \omega_c^2 \right) + 32 \left[ (q^2 + q^4) \chi_0(y) + 12 \left( \omega^2 + \omega_c^2 \right) + 32 \right] \chi_j(y)q^{2(j+1)} - \chi_{j-1}(y)q^{2(j+1)+2} \right\} - \sum_{j \in 1/2,3/2,...} 16(\omega + \omega_c)^2 \left( \chi_j(y)q^{2(j+1)} - \chi_{j-1}(y)q^{2(j+1)+2} \right) \right\} \right\} \right\}.$$

(6.25)
### 6.1. Doubleton and Higher Spin Gauge Fields

#### Table 6.2: SO(2) neutral HS for $\mathcal{N} = 6$: $hs(6|4) \supset Osp(6|4)$

| $s \backslash h$ | 0     | 1     | 2     | 3     |
|----------------|-------|-------|-------|-------|
| 0              | 15+15 | 1+1   |       |       |
| $\frac{1}{2}$  | 10 + 10* + 6 | 6     |       |       |
| 1              | 15+1  | 15+1  |       |       |
| $\frac{5}{2}$  | 6     | 10 + 10* + 6 | 6     |       |
| 2              | 1     | 15+15 | 1     |       |
| $\frac{9}{2}$  | 6     | 10 + 10* + 6 | 6     |       |
| 3              | 15+1  | 15+1  |       |       |
| $\frac{11}{2}$ | 6     | 10 + 10* + 6 | 6     |       |
| 6              | 1+1   | 15+15 |       |       |
| ...            | ...   | ...   | ...   | ...   |

#### Table 6.3: Charged HS for $\mathcal{N} = 6$: $hs(8|4)/hs(6|4) \supset Osp(8|4)/Osp(6|4)$

| $s \backslash h$ | 0     | 1     | 2     |
|----------------|-------|-------|-------|
| 0              | (10 + 10*)$_{\pm 2}$ |       |       |
| $\frac{1}{2}$  | 15$_{\pm 2}$  | 1$_{\pm 2}$ |       |
| 1              | 6$_{\pm 2}$  | 6$_{\pm 2}$ |       |
| $\frac{3}{2}$  | 1$_{\pm 2}$  | 15$_{\pm 2}$ |       |
| 2              | (10 + 10*)$_{\pm 2}$ |       |       |
| $\frac{5}{2}$  | 15$_{\pm 2}$  | 1$_{\pm 2}$ |       |
| 3              | 6$_{\pm 2}$  | 6$_{\pm 2}$ |       |
| $\frac{7}{2}$  | 1$_{\pm 2}$  | 15$_{\pm 2}$ |       |
| 4              | (10 + 10*)$_{\pm 2}$ |       |       |
| ...            | ...   | ...   | ...   |
6.2 Tripletons and higher \( n \)-plets

For higher multipletons one has to resort to Polya theory [124–127]. Consider a set of ‘words’ \( A, B, \ldots \) of \( n \) ‘letters’ chosen within the alphabet \( \{ a_i \} \) with \( i = 1, \ldots, p \). When \( p \to \infty \), let us denote by \( Z_1(q) \) the single letter ‘partition function’. Let also \( G \) be a group action defining the equivalence relation \( A \sim B \) for \( A = gB \) with \( g \) an element of \( G \subseteq S_n \). Elements \( g \in S_n \) can be divided into conjugacy classes \( [g] = (1)^{b_1}(n)^{b_n} \), according to the numbers \( \{ b_k(g) \} \) of cycles of length \( k \). Polya theorem states that the set of inequivalent words are generated by the formula

\[
Z^G_n = \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{n} Z_1(q^k)^{b_k(g)}
\]  

(6.26)

In particular, for the cyclic group \( G = \mathbb{Z}_n \), conjugacy classes are \( [g] = (d)^{n/d} \) for each divisor \( d \) of \( n \). The number of elements in a given conjugacy class labelled by \( d \) is given by Eulers totient function \( \mathcal{E}(d) \), equal to the number of integers relatively prime to \( d \). For \( d = 1 \) one defines \( \mathcal{E}(1) = 1 \).

\[
Z^C_n = \frac{1}{n} \sum_{d|n} \mathcal{E}(d) Z_1(q^d)^{n/d}
\]  

(6.27)

For the full symmetric group one has

\[
Z^{S_n} = \frac{1}{n!} \sum_{n_r} n! \prod_{r} r^{n_r} n_r! \prod_{r} Z_1(q^r)^{n_r}
\]  

(6.28)

Let us consider the product of three singletons.

\[
Z^3 = Z^\square \times Z^\square \times Z^\square \to Z^\square + Z^\square + Z^\square
\]  

(6.29)

There are thus three kinds of tri-pletions.

The totally symmetric triplet is coded in the partition function

\[
Z^\square = \frac{1}{6} (Z^3(u) + 3Z^\square(u)Z^\square(u^2) + 2Z^\square(u^3))
\]  

(6.30)

where \( u \) collectively denotes the ‘fugacities’ \( q, y = e^{i\alpha}, \omega \approx t, \ldots \).

For the cyclic triplet one has

\[
Z_{cyc} = Z^\square + Z^\square = \frac{1}{3} (Z^3(u) + 2Z^\square(u^3))
\]  

(6.31)
For totally anti-symmetric tripletons one finds
\[
Z = Z_{\text{cycl}} - Z = \frac{1}{6}(Z^3(u) + 2Z(u^3) - 3Z(u)Z(u^2))
\]
(6.32)
while for mixed symmetry, incompatible with the cyclicity of the trace, one eventually finds
\[
Z = Z^3(u) - \frac{1}{3}Z^3(u) - \frac{2}{3}Z(u^3) = \frac{2}{3}(Z^3(u) - Z(u^3))
\]
(6.33)

Recalling the singleton partition function
\[
Z(q, \alpha, \omega) = \left(4\omega + 4^*\bar{\omega}\right)q(1 + q^2) - \left(4\bar{\omega} + 4^*\omega\right)q^2\chi_s(\alpha)
\]
\[
= \frac{4(\omega + \bar{\omega})q}{(1 - 2q^2\cos\alpha + q^4)}[1 + q^2 - \chi_s(\alpha)]
\]
(6.34)
where \(\omega = e^{2\pi i/k}\) and \(\chi_s(\alpha) = \text{tr}_{1/2}\exp(i\alpha J_3)\), one eventually finds
\[
Z = \frac{1}{6}\left(\frac{4^3(\omega + \bar{\omega})^3q^3(1 + q^2 - q\chi_s(\alpha))^3}{(1 - 2q^2\cos\alpha + q^4)^3} + \frac{4^4(\omega^2 + \bar{\omega}^2)q^2(1 + q^2 - q\chi_s(\alpha))(1 + q^4 - q^2\chi_s(2\alpha))}{(1 - 2q^2\cos\alpha + q^4)(1 - 2q^4\cos\alpha + q^8)} + \frac{4(\omega^3 + \bar{\omega}^3)q^3(1 + q^6 - q^3\chi_s(3\alpha))}{(1 - 2q^6\cos\alpha + q^{12})}\right)
\]
(6.35)
for the totally symmetric triplet. Let us analyze the spectrum arising in this case. Except for the 1/2 BPS states, we will consider later on, only ‘massive’ representations above the unitary bound, whose characters read
\[
\chi_{s}^{\Delta \neq s+1} = \frac{q^{2\Delta}\chi_s(\alpha)}{(1 - q^2)(1 - 2q^2\cos\alpha + q^4)} \quad \rightarrow \quad \frac{q^{2\Delta}(2s + 1)}{(1 - q^2)^3}
\]
(6.36)
appear in the decomposition
\[
Z(q, \alpha, \omega) = \sum_{s, \Delta, Q} c(s, \Delta, Q) \frac{q^{2\Delta}\chi_s(\alpha)\omega^Q}{(1 - q^2)(1 - 2q^2\cos\alpha + q^4)}
\]
(6.37)
Indeed it is easy to see that no current like (twist \(\tau = 1\)) fields appear beyond the double-ton, since the twist
\[
\tau = \Delta - s = \frac{n_X}{2} + n\Phi + n\Psi - n\bar{\Phi} - \frac{n\Psi}{2} = \frac{n_X}{2} + \frac{n\Psi}{2} > 1
\]
(6.38)
whenever \( n_X + n_\Psi > 2 \).

Using orthogonality of the \( SU(2) \) characters

\[
\frac{1}{\pi} \int_0^{2\pi} \chi_s(\alpha) \chi_{s'}(\alpha) \sin^2 \frac{\alpha}{2} d\alpha = \delta_{2s+1,2s'+1} \tag{6.39}
\]

one can decompose the partition function according to

\[
\sum_{Q,\Delta} \frac{c(s, \Delta, Q) \omega^Q q^{2\Delta}}{1 - q^2} = \frac{1}{\pi} \int_0^{2\pi} (1 - 2q^2 \cos \alpha + q^4) \sin^2 \frac{\alpha}{2} \chi_s(\alpha) \mathcal{Z}_{\Delta}^Q(q, \alpha, \omega) d\alpha \tag{6.40}
\]

It is clear that only states with charge \( Q = \pm 3, \pm 1 \) are present in the triplet spectrum. Setting \( y = e^{i\alpha} \), for states with \( Q = \pm 1 \) one finds

\[
\mathcal{Z}_{\Delta}^{Q=\pm1} = \sum_{k=0}^{\infty} \left[ (40 + 256k)q^{4k+3} + (104 + 256k)q^{4k+5} \right] \chi_{2k}(y)
- \left[ (104 + 256k)q^{4k+4} + (152 + 256k)q^{4k+6} \right] \chi_{2k+\frac{3}{2}}(y)
+ \left[ (152 + 256k)q^{4k+5} + (216 + 256k)q^{4k+7} \right] \chi_{2k+1}(y)
- \left[ (216 + 256k)q^{4k+6} + (296 + 256k)q^{4k+8} \right] \chi_{2k+\frac{5}{2}}(y) \tag{6.41}
\]

these states are always projected out by \( \mathbb{Z}_k \) since \( \pm 1 \neq nk \). For states with \( Q = \pm 3 \) one finds instead

\[
\mathcal{Z}_{\Delta}^{Q=\pm3} = \sum_{k=0}^{\infty} \left[ (20 + 256k)q^{12k+3} + (40 + 256k)q^{12k+5} \right] \chi_{6k}(y)
- \left[ (40 + 256k)q^{12k+4} + (44 + 256k)q^{12k+6} \right] \chi_{6k+\frac{3}{2}}(y)
+ \left[ (44 + 256k)q^{12k+5} + (68 + 256k)q^{12k+7} \right] \chi_{6k+1}(y)
- \left[ (68 + 256k)q^{12k+6} + (104 + 256k)q^{12k+8} \right] \chi_{6k+\frac{5}{2}}(y)
+ \left[ (104 + 256k)q^{12k+7} + (124 + 256k)q^{12k+9} \right] \chi_{6k+2}(y)
- \left[ (124 + 256k)q^{12k+8} + (132 + 256k)q^{12k+10} \right] \chi_{6k+\frac{7}{2}}(y)
+ \left[ (132 + 256k)q^{12k+9} + (152 + 256k)q^{12k+11} \right] \chi_{6k+3}(y)
- \left[ (152 + 256k)q^{12k+10} + (188 + 256k)q^{12k+12} \right] \chi_{6k+\frac{9}{2}}(y)
+ \left[ (188 + 256k)q^{12k+11} + (212 + 256k)q^{12k+13} \right] \chi_{6k+4}(y)
- \left[ (212 + 256k)q^{12k+12} + (216 + 256k)q^{12k+14} \right] \chi_{6k+\frac{11}{2}}(y)
+ \left[ (216 + 256k)q^{12k+13} + (236 + 256k)q^{12k+15} \right] \chi_{6k+5}(y)
- \left[ (236 + 256k)q^{12k+14} + (276 + 256k)q^{12k+16} \right] \chi_{6k+\frac{13}{2}}(y) \tag{6.42}
\]
6.3. KK EXCITATIONS

These states survive only for \( k = 3 \), i.e. \( \mathbb{Z}_3 \) projection. It is amusing to observe how the number of representations of given spin \( s = 6k + \frac{n}{2} \) grows with \( k \) at the rate \( 256k \) for any \( n \). This is due to the possible distributions of derivatives among three fields up to symmetries and total derivatives and to the structure of higher spin supermultiplets [128].

For higher multi-pletons the analysis is similar. It is clear that only states with charge \( Q = \pm n, \pm(n-2), \ldots \) are present in the n-pleton spectrum. In particular \( Q = 0 \) states are only present when \( n \) is even as already observed. We defer a detailed analysis to the future. For the time being let us only display the partition functions for the cyclic tetrapleton

\[
\mathcal{Z}_{4, \text{cycl}} = \frac{1}{4} (\mathcal{Z}_4(q) + \mathcal{Z}_2^2(q^2) + 2\mathcal{Z}_4(q^4)) \tag{6.43}
\]

and for the totally symmetric tetrapleton

\[
\mathcal{Z}_{4, \text{tot}} = \frac{1}{4!} (\mathcal{Z}_4(q) + 6\mathcal{Z}_2^2(q)\mathcal{Z}_2(q^2) + 3\mathcal{Z}_2^2(q^2) + 8\mathcal{Z}_3(q^3)\mathcal{Z}_1(q) + 6\mathcal{Z}_4(q^4)) \tag{6.44}
\]

The \( \mathbb{Z}_k \) projection on n-pletons reads

\[
\mathcal{Z}_{n, \mathbb{Z}_k} = \frac{1}{k} \sum_r \mathcal{Z}_n^{(r)}(q, \omega^r) \tag{6.45}
\]

and corresponds to keeping only states with \( Q = kn \) i.e. integer multiples of \( k \).

6.3 KK excitations

Let us now focus on the KK excitations, which deserve a separate treatment. One can indeed write down the single-particle partition function on \( S^7 \), decompose it into super-characters and identify the \( SO(2) \) charge sectors, relevant for the subsequent \( \mathbb{Z}_k \) projection i.e. compactification on \( \mathbb{C}P^3 \).

Introducing a chemical potential for the charge \( Q \) \((t^Q)\), the super-
character of an ultra-short 1/2 BPS representation of $Osp(8|4)$ reads:

\[
\mathcal{X}_{\ell}^{1/2\text{BPS}}(q,t) = \frac{t^{-2-\ell}q^{2+\ell}}{6(1-t^2)^3(1+q)^3} \left[ t^3(-1 + t^2)^2(-1 + q)^3 \right. \\
\times \left. \left( t^{6+2\ell}(t^2 - q)^2 - (1 + t^2q)^2 \right) - 6t^2(-1 + t^2)(-1 + q)^2 \right.
\times \left. \left( t^{6+2\ell}(t^2 - q)^2(-3 + 2t^2 + q) + (2 + t^2(-3 + q))(-1 + t^2q)^2 \right) \right.
+ \left. 6t^{6+2\ell}(t^2 - q)^2(-35 + q(35 + (-9 + q)q) + 2t^4(-5 + q^2) \right.
\times \left. t^2(35 + q(-13 + (-7 + q)q)) \right) - \left( 2(-5 + q^2) \right. \right)
\times \left. \left[ t^{4}(-35 + q(35 + (-9 + q)q)) + t^2(35 + q(-13 + (-7 + q)q)) \right]
\times \left. \left( t^{6+2\ell}(t^2 - q)^2(-107 + (70 - 11q)q \right. \right.
\times \left. t^{4}(-47 + (-2 + q)q) + 2t^2(-71 + q(22 + q)) + (-1 + t^2q)^2 \right.
\times \left. \left( 47 - (-2 + q)q + 2t^2(-71 + q(22 + q)) + t^4(107 + q(-70 + 11q))) \right) \right]\right]
\]

(6.46)

For $\ell = 0$, corresponding to the gauged supergravity multiplet, there is further shortening (null descendants) due to the presence of conserved ‘currents’ \textit{i.e.} stress-tensor, $SO(8)$ vector currents and $8_s$ supercurrents. Taking this into account one finds the following super-character

\[
\mathcal{X}_{\ell=0}^{1/2\text{BPS}}(q) = \frac{1}{(1 - q^2)^3} \left[ (10t^2 + 15 + 10t^{-2})q^2 - \right.
\left. 2(15t^2 + 10 + 6 + 10 + 15t^{-2})q^3 + \right.
\left. (10t^2 + 15 + 10t^{-2} + 3(6t^2 + 15 + 1 + 6t^{-2}))q^4 - \right.
\left. 4(t^2 + 6 + t^{-2})q^5 - (6t^2 + 15 + 1 - 5 + 6t^{-2})q^6 + \right.
\left. 2(t^2 + 6 + t^{-2})q^7 - 3q^8 \right]\]

(6.47)

the denominator takes into account derivatives (descendants). Quite remarkably this formula coincides with the previous one when $\ell = 0$.

After some algebra, putting $t = 1$, one finds

\[
\mathcal{X}_{\ell=0}^{1/2\text{BPS}}(q) = \frac{q^2(3q^3 - 7q^2 - 7q + 35)}{(1 + q)^3}
\]

(6.48)

a factor $(1 - q)^2$ cancels between numerator and denominator meaning that not only $n_b = n_f$ and the sum with $\Delta^1$ vanishes but also the sum with $\Delta^2$ should vanish. This should be related to the absence of quantum corrections to the negative vacuum energy, \textit{i.e.} cosmological constant in the bulk.
6.3. KK EXCITATIONS

The 1/2 BPS partition function is given by

\[ Z_{1/2\text{BPS}}^{N=8} = \sum_\ell X_\ell^{1/2\text{BPS}} = \frac{35q^2}{(1-q^2)^2} \]  \hspace{1cm} (6.49)

The simplicity of the result is due to ‘miraculous’ cancellations between bosonic and fermionic operators with the same scaling dimensions in different KK multiplets i.e. with different \( \ell \)'s. This does not happen in AdS\(_5\)/CFT\(_4\) holography, whereby (protected) bosonic operator have integer dimensions and (protected) fermionic operators have half-integer dimensions [114, 115, 124–128].

In order to perform the \( Z_k \) projection it is useful to decompose into \( SO(2) \) charge sectors according to

\[ Z_{1/2\text{BPS}}^{N=8\rightarrow N=6}(q,t) = \frac{q^2[(1+q^6)P_2(t) - (q+q^5)P_3(t) + (q^2+q^4)P_4(t) - q^3P_5(t)]}{(1-qt)^4(1-qt^{-1})^4(1+q)^2} \]  \hspace{1cm} (6.50)

where

\[
\begin{align*}
P_2(t) &= 10t^2 + 15 + 10t^{-2} \\
P_3(t) &= 20t^3 + 10t^2 + 64t + 22 + 64t^{-1} + 10t^{-2} + 2t^{-3} \\
P_4(t) &= 15t^4 + 8t^3 + 104t^2 + 48t^1 + 175 + 48t^{-1} + 104t^{-2} + 8t^{-3} + 15t^{-4} \\
P_5(t) &= 4t^5 + 2t^4 + 64t^3 + 40t^2 + 196t^1 + 88 + 196t^{-1} + 40t^{-2} + 64t^{-3} + 2t^{-4} + 4t^{-5}
\end{align*}
\]  \hspace{1cm} (6.51)

Depending on the choice of \( k \) one can recognize the surviving 1/2 BPS states as those with \( Q = kn \). In formulae one has to replace \( t \) with \( \omega^r \) and sum over \( r = 0, ..., k - 1 \).
Chapter 7

Instantons in $\mathbb{CP}^3$

An interesting characteristic of $AdS_4 \times \mathbb{CP}^3$, which $AdS_5 \times S^5$ is lacking, is the existence of stringy instantons in $\mathbb{CP}^3$ which arise from the string worldsheet which wraps a topologically non-trivial two-cycle $\mathbb{CP}^1 \simeq S^2$ of $\mathbb{CP}^3$ in the Wick rotated theory. This is a stringy counterpart of the instantons of two dimensional $\mathbb{CP}^n$ sigma-models. The two-cycle corresponds to the closed Kähler two-form $J_2$ on $\mathbb{CP}^3$. The consistent gauge fixing of kappa-symmetry does not allow to reduce the string action to the supercoset sigma model, i.e. to eliminate the eight fermionic modes corresponding to the broken supersymmetries. This stringy instanton has twelve fermionic zero modes all corresponding to unbroken supersymmetries of the background and there are no zero modes associated with broken supersymmetries. These twelve zero modes are divided into eight massive fermionic zero modes, which are four copies of the two component Killing spinor on $S^2$, and four other modes, which are two copies of massless chiral and anti-chiral fermion on $S^2$ electrically coupled to the electromagnetic potential created on $S^2$ by a monopole placed in the center of $S^2$. The presence of the stringy instanton and its fermionic zero modes may generate non-perturbative corrections to the string effective action, which may affect its properties and if so should be taken into account in studying, e.g. the $AdS_4/CFT_3$ correspondence. The instantons may contribute to the worldsheet S-matrix. Due to the presence of the fermionic zero modes, the non-perturbative amplitude should contain operator insertions up all fermionic zero modes.

The Green-Schwarz superstring action in a generic Type IIA super-
CHAPTER 7. INSTANTONS IN CP\(^3\)

Gravity background has the following form:

\[ S = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-h} h^{IJ} \mathcal{E}_I^A \mathcal{E}_J^B \eta_{AB} - \frac{1}{2\pi\alpha'} \int B_2 \]  

(7.1)

where \(\xi^I (I = 0, 1)\) are the worldsheet coordinates, \(h_{IJ}(\xi)\) is an intrinsic worldsheet metric, \(\mathcal{E}_I^A\) are worldsheet pullbacks of target superspace vielbeins and \(B_2\) is the pullback of the NS-NS two-form. One has to substitute the expressions for the vielbeins and the NS-NS two-form up to second order in fermions in the above action and keep only terms up to quadratic order in fermions. After Wick rotation the action takes the following form:

\[ S_E = e^{\frac{2\phi_0}{4\pi\alpha'}} \int d^2\xi \sqrt{h} h^{IJ} (e^a_I e^b_J \delta_{ab} + e^{a'}_{I} e^{b'}_{J} \delta_{a'b'}) \]  

(7.2)

\[ + e^{\frac{2\phi_0}{2\pi\alpha'}} \int d^2\xi \sqrt{h} h^{IJ} (e^a_I e^b_J \delta_{ab} + e^{a'}_{I} e^{b'}_{J} \delta_{a'b'}) \]

and the kappa-symmetry matrix is

\[ \Gamma = -\frac{i}{2\sqrt{\det G_{IJ}}} \varepsilon^{IJ} \mathcal{E}_I^A \mathcal{E}_J^B \Gamma_{AB} \Gamma_{11}, \quad \Gamma^2 = 1 \]  

(7.3)

where \(\Gamma_{11}\) is given in terms of \(D = 4\) gamma matrices \(\Gamma_{11} = \gamma^5 \otimes \gamma^7\), \(\Gamma^A\) are \(D = 10\) gamma matrices, \(e^{\frac{2\phi_0}{4\pi\alpha'}}\) is the vacuum expectation value of the dilaton, where \(R\) is the radius of the \(S^7\) sphere whose base is \(\mathbb{CP}^3\) and \(l_p\) is the eleven dimensional Planck length related to the string tension as \(l_p = e^{\frac{2\phi_0}{4\pi\alpha'}} \sqrt{\alpha'}\). The 32-component fermionic variable \(\Theta^\alpha\) is split by projectors \(P_6\) and \(P_2\) into the 24-component spinors \(\psi^{a\alpha'}\) (\(\alpha = 1, \ldots, 4; a' = 1, \ldots, 6\)) corresponding to the 24 supersymmetries of the \(AdS_4 \times \mathbb{CP}^3\) solution and the 8-component spinors \(\psi^{aq}\) (\(q = 1, 2\)) corresponding to the broken supersymmetries. \(e^a(x)\) and \(e^{a'}(y)\) are the vielbein for \(AdS_4\) of radius \(R/2\) and for \(\mathbb{CP}^3\), respectively. The induced metric on the worldsheet is \(G_{IJ} = \mathcal{E}_I^A \mathcal{E}_J^B \eta_{AB}\). To identify the stringy instanton in \(\mathbb{CP}^3\) it is convenient to consider the Fubini-Study metric on \(\mathbb{CP}^3\) \([133]\)

\[ ds^2 = R^2 \left( \frac{1}{4} (d\theta^2 + \sin^2 \theta (d\varphi + \frac{1}{2} \sin^2 \alpha \sigma_3)^2) + \sin^2 \theta d\alpha^2 \right. \]

\[ + \left. \frac{1}{4} \sin^2 \theta \sin^2 \alpha (\sigma_1^2 + \sigma_2^2 + \cos^2 \alpha \sigma_3^2) \right) \]  

(7.4)

where \(0 \leq \theta \leq \pi\), \(0 \leq \varphi \leq 2\pi\) and \(0 \leq \alpha \leq \frac{\pi}{2}\), and \(\sigma_1, \sigma_2, \sigma_3\) are three left-invariant one forms on \(SU(2)\) and obey \(d\sigma_1 = -\sigma_2 \sigma_3\) etc. Notice that now \(\theta\) and \(\varphi\) parametrize a two-sphere of radius \(R/2\), which is
topologically non-trivial and associated to the Kähler form on $\mathbb{CP}^3$. If a stringy instanton wraps this sphere once then $\theta$ and $\varphi$ can be identified with the string worldsheet coordinates, while all the other $\mathbb{CP}^3$ and $AdS_4$ coordinates are worldsheet constants for the instanton solution. Thus the pullback on the stringy instanton of the metric (7.4) of $\mathbb{CP}^3$ of radius $R$ is the metric of radius $R/2$:

$$ds^2 = \frac{R^2}{4}(d\theta^2 + \sin^2 \theta d\varphi^2)$$ (7.5)

In this coordinate system the $S^2$ vielbein $e^i$ and the spin connection $w_S^{ij}$ ($i, j = 1, 2$) can be chosen to have the following form:

$$e^1 = \frac{R}{2} d\theta, \quad e^2 = \frac{R}{2} \sin \theta d\varphi, \quad w_S^{12} = \cos \theta d\varphi$$ (7.6)

and the $S^2$ curvature 2-form is

$$R^{ij} = dw_S^{ij} = \frac{4}{R^2} e^i e^j.$$ (7.7)

Let us consider the bosonic part of the Wick rotated action [7.2], which is:

$$S_E = \frac{T}{2} \int d^2 \xi \sqrt{h} h^{IJ} e_i^i e_j^j \delta_{ij}$$ (7.8)

where $T = \frac{e^{3\phi_0}}{2\pi \alpha'}$ and $e^i$ are the vielbeins on $S^2$. To discuss the instanton it is convenient to introduce complex coordinates both on the worldsheet and in the target space. In the $(z, \bar{z})$ coordinate system on the worldsheet the action takes the form

$$S_E = \frac{T}{2} \int d^2 z e_z^i e_{\bar{z}}^j \delta_{ij}$$ (7.9)

where it is taken into account that in the conformal gauge $\sqrt{h} h^{IJ} = \delta^{IJ}$. The Fubini-Study metric on $\mathbb{CP}^1$ is

$$ds_{\mathbb{CP}^1}^2 = \frac{d\zeta d\bar{\zeta}}{(1 + |\zeta|^2)^2}$$ (7.10)

If we choose

$$\zeta = \tan \frac{\theta}{2} e^{i\varphi}$$ (7.11)

then the Fubini-Study metric on $\mathbb{CP}^1$ gets the form of the metric on $S^2$ of radius $\frac{1}{2}$:

$$ds^2 = \frac{1}{4}(d\theta^2 + \sin^2 \theta d\varphi^2)$$ (7.12)
In the $\zeta, \bar{\zeta}$ coordinate system the string action takes the following form:

$$S_E = \frac{TR^2}{4} \int d^2 z \frac{\partial \zeta |^2 + \bar{\partial} \zeta |^2}{(1 + |\zeta|^2)^2}$$  \hspace{1cm} (7.13)

The local minimum is at $\bar{\partial} \zeta = 0$ or at $\partial \zeta = 0$, which means that the embedding is given by a holomorphic function $\zeta = \zeta(z)$ for the instanton or by an anti-holomorphic function $\zeta = \zeta(\bar{z})$ for the anti-instanton. This is the classical instanton solution of the two-dimensional $O(3)$ sigma-model [134] or rather its extension to $\mathbb{CP}^3$ [135–137]. The remaining part of the action can be shown to be a topological invariant. The Virasoro constraints should also be satisfied by the classical string solution. The Virasoro constraints in the conformal gauge have the form

$$\frac{\partial \zeta \bar{\partial} \zeta}{(1 + |\zeta|^2)^2} = 0$$  \hspace{1cm} (7.14)

which obviously are satisfied by the (anti)instanton solution.

To discuss fermionic sector it will be convenient to choose the $\mathbb{CP}^3$ gamma matrices as:

$$\gamma^a = (\rho^i \otimes 1, \rho^3 \otimes \gamma^a), \quad \gamma_7 = \rho^3 \otimes \gamma_5, \quad \gamma_5 = \frac{1}{4!} \varepsilon_{\tilde{a} \tilde{b} \tilde{c} \tilde{d}} \gamma_{\tilde{a} \tilde{b} \tilde{c} \tilde{d}}$$  \hspace{1cm} (7.15)

$\gamma_a$, $\tilde{a} = 3, 4, 5, 6$, are $4 \times 4$ Dirac gamma matrices corresponding to the four-dimensional subspace of $\mathbb{CP}^3$ orthogonal to $\mathbb{CP}^1$ and $\gamma_5^2 = 1$. In the fermionic sector it is natural to impose on the fermionic fields the conventional kappa-symmetry gauge-fixing condition, which is the following:

$$\frac{1}{2}(1 + \Gamma)\Theta = \frac{1}{2}(1 - \gamma_5 \gamma_5)\Theta = 0$$  \hspace{1cm} (7.16)

This means that the fermions split into two sectors according to their chiralities in $AdS_4$ and in the four dimensional subspace of $\mathbb{CP}^3$ orthogonal to $\mathbb{CP}^1$.

$$\Theta_+ : \quad \gamma_5 \Theta_+ = \gamma_5 \Theta_+ = \Theta_+$$  \hspace{1cm} (7.17)

$$\Theta_- : \quad \gamma_5 \Theta_- = \gamma_5 \Theta_- = -\Theta_-$$  \hspace{1cm} (7.18)

The supersymmetry projection matrices $\mathcal{P}_2$ and $\mathcal{P}_6$ act on these two sets as:

$$\mathcal{P}_6 \Theta_+ = \Theta_+ = \vartheta_+$$  \hspace{1cm} (7.19)

$$\mathcal{P}_2 \Theta_+ = \vartheta_+ = 0$$  \hspace{1cm} (7.20)

$$\mathcal{P}_6 \Theta_- = \frac{1}{2}(1 + \rho^3 \bar{J})\Theta_- = \vartheta_-$$  \hspace{1cm} (7.21)

$$\mathcal{P}_2 \Theta_- = \frac{1}{2}(1 + \rho^3 \bar{J})\Theta_- = \vartheta_-$$  \hspace{1cm} (7.22)
It follows that all the eight $\vartheta_+$ are fermions corresponding to unbroken supersymmetries of the $AdS_4 \times \mathbb{CP}^3$ background. In the $\Theta_-$ sector four fermions $\vartheta_-$ correspond to unbroken supersymmetries and the other four $\nu$ to the broken ones. Since the kappa-symmetry projector commutes with the supersymmetry projectors, it is not possible to choose the kappa-symmetry gauge-fixing condition in such a way to put to zero all the eight broken supersymmetry fermions.

Examining fermionic equations for instanton configuration one concludes that in the $\Theta_+$ sector the stringy instanton has eight fermionic zero modes. In the spherical coordinates they have the following form:

$$\vartheta_+ = e^{-\frac{i\rho_1}{2}\vartheta} e^{\frac{i\rho_3^*}{2}\varphi^*} \epsilon_+ = \left( \cos \frac{\vartheta}{2} - i \rho_1 \sin \frac{\vartheta}{2} \right) \left( \cos \frac{\varphi}{2} + i \rho_3 \sin \frac{\varphi}{2} \right) \epsilon_+ \quad (7.23)$$

where $\rho_1 = \sigma_1$, $\rho_2 = \sigma_3$ and $\rho_3 = -\sigma^2$ are Pauli matrices and $\epsilon_+$ is an arbitrary constant spinor satisfying the chirality conditions $\gamma_5 \epsilon_+ = \gamma_5 \tilde{\epsilon}_+ = \epsilon_+$. The other equation of motion requires to put $v = 0$. In the $\vartheta_-$ sector the stringy instanton has four zero modes which have the form:

$$\vartheta_- = \frac{1}{2} e^{-\frac{i\rho_3^*}{2}\varphi^*} \left[ (1 + \rho_3^*) \lambda_-(\zeta) + (1 - \rho_3^*) \mu_-(\bar{\zeta}) \right] \quad (7.24)$$

where $\lambda_-(\zeta)$ and $\mu_-(\bar{\zeta})$ are holomorphic and anti-holomorphic spinors in the projective coordinates $\zeta$ and $\bar{\zeta}$ of $S^2 \simeq \mathbb{CP}^1$. They are anti-chiral in the directions transverse to the instanton. For the anti-instanton the solution takes the same form but with $\lambda_-(\bar{\zeta})$ and $\mu_-(\zeta)$. These fermionic zero modes do not contribute to the bosonic equations.

Thus, the stringy instanton wrapping the non-trivial two-cycle inside $\mathbb{CP}^3$ has twelve fermionic zero modes. The eight fermionic fields $\vartheta_+$ and four $\vartheta_-$ correspond to twelve (of the twenty four) supersymmetries of the $AdS_4 \times \mathbb{CP}^3$ background which are linearly realized on the string worldsheet. Thus, the fermionic zero modes play the role similar to goldstinos, which break supersymmetry.
Conclusions

In this thesis we have discussed non-perturbative phenomena emerging in gauge and in string/supergravity theories. We discussed instantons in supersymmetric Yang-Mills theories. We computed the partition function of 5D minimal supersymmetric $U(1)$ gauge theory with extra adjoint matter in general $\Omega$-background and have shown in particular that unlike the case with no extra matter, the partition function with extra adjoint at some special values of the parameters directly reproduces the generating function for the Poincaré polynomial of the moduli space of instantons. We discussed instantons and their effects in string theories. In particular ‘gauge’ instantons may generate a VY-ADS-like superpotential of the form

$$W \approx \frac{\Lambda^\beta}{\phi^{\beta-3}}$$

where $\beta$ is the one-loop coefficient in the expansion of the $\beta$ function and $\Lambda^\beta = M_s^\beta e^{-T(C)}$. ‘Exotic’ instantons may generate a non-perturbative superpotential of the form

$$W \approx M_s^{\beta-n} e^{-S_{EDP}(C')} \phi^n \quad (n = 0, 1, ... )$$

Combining the two kinds of superpotentials one can achieve (partial) moduli stabilization and SUSY breaking! The same may happen when only one kind of superpotential is generated in the presence of fluxes, which we discussed in details, and in another dynamical effects, such as FI terms [60]. When extra zero-modes are present, threshold corrections to (higher-derivative) terms may arise. We illustrated this possibility for a compactification to $D = 6$ on $T^6/\mathbb{Z}_2$, where a fully non-perturbative four hyperini amplitude (Fermi interaction) can be computed exploiting Heterotic - Type I duality. Threshold corrections to gauge couplings in freely acting orbifolds $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ were computing by similar means. A by-product of the analysis in $D = 6$, an economical mechanism of moduli
stabilization can be exploited whereby non-anomalous $U(1)$'s in $D = 4$ eat would-be hypers due to anomalies in $D = 6$. The behaviour of D-brane instanton effects in the presence of fluxes or under wall crossing and the reformulation of (unoriented) D-brane instanton calculus in terms of localization are extremely active subjects. The part of the vast literature on the subject are [23], [71], [26], [29], [36], [60], [4].

Then we have discussed $AdS/CFT$ correspondence and the role of instantons particularly in $AdS_5/CFT_4$. We discussed another interesting application of the correspondence, $AdS_4/CFT_3$. This allows to investigate the worldvolume theory of M2-branes, the basic objects of M-theory. In this context we considered $\mathcal{N} = 8$ supergravity on $AdS_4 \times S^7$, which is the low energy limit of M-theory compactified on $S^7$. We have re-analyzed the KK spectrum of $d = 11$ supergravity on $S^7$ and $S^7/\mathbb{Z}_k$. The latter includes monopole operators dual to charged states in Type IIA on $\mathbb{CP}^3$. To this end we have presented some group theoretic methods for the decomposition of the $SO(8)$ into $SO(6) \times SO(2)$ valid also for other cosets [129–131] where resolution of the mixing among various fluctuations should be possible on the basis of symmetry arguments. In particular, massless vectors associated to Killing vectors in generic flux vacua with isometries have been recently discussed in [132]. We have then considered higher spin symmetry enhancement. We have displayed the partition functions for singletons, doubletons and tripletons and discussed in details higher spin fields and 1/2 BPS states corresponding to KK excitations of $\mathcal{N} = 6$ gauged supergravity. It would be worth pursuing the analysis to higher $n$-pletions and to cases with lower supersymmetry, yet based on internal coset manifolds. It would be also nice to try to get semi-realistic $AdS_4$ compactifications in the both cases when the Chern-Simons levels sum up to zero or not using orientifold projections [96], [87]. We discussed also instantons in $\mathbb{CP}^3$. 
Appendix A

General formula

\[ d_{(\ell_1, \ell_2, \ell_3, \ell_4)}^{SO(8)} = \frac{1}{4320} \times \frac{(1 + \ell_1)(1 + \ell_2)(1 + \ell_3)(1 + \ell_4)}{(2 + \ell_1 + \ell_2)(2 + \ell_2 + \ell_3)(2 + \ell_2 + \ell_4)} \]
\[ (3 + \ell_1 + \ell_2 + \ell_3)(3 + \ell_1 + \ell_2 + \ell_4)(3 + \ell_2 + \ell_3 + \ell_4) \]
\[ (4 + \ell_1 + \ell_2 + \ell_3 + \ell_4)(5 + \ell_1 + 2\ell_2 + \ell_3 + \ell_4) \quad (A.1) \]

Specific cases (KK harmonics)

\[ d_{(\ell,0,0,0)}^{SO(8)} = \frac{1}{360}(1 + \ell)(2 + \ell)(3 + \ell)^2(4 + \ell)(5 + \ell) \quad \leftrightarrow \quad Y_{N_1} \]
\[ d_{(\ell,1,0,0)}^{SO(8)} = \frac{1}{60}(1 + \ell)(3 + \ell)(4 + \ell)^2(5 + \ell)(7 + \ell) \quad \leftrightarrow \quad Y_{N_7} \]
\[ d_{(\ell,0,1,1)}^{SO(8)} = \frac{1}{24}(1 + \ell)(2 + \ell)(4 + \ell)^2(6 + \ell)(7 + \ell) \quad \leftrightarrow \quad Y_{N_{21}} \]
\[ d_{(\ell,2,0,0)}^{SO(8)} = \frac{1}{18}(1 + \ell)(4 + \ell)(5 + \ell)^2(6 + \ell)(9 + \ell) \quad \leftrightarrow \quad Y_{N_{37}} \quad (A.2) \]
\[ d_{(\ell,0,2,0)}^{SO(8)} = d_{(\ell,0,0,2)}^{SO(8)} = \frac{1}{36}(1 + \ell)(2 + \ell)(3 + \ell)(5 + \ell)(6 + \ell)(7 + \ell) \quad \leftrightarrow \quad Y_{N_{35}} \]
\[ d_{(\ell,0,1,0)}^{SO(8)} = d_{(\ell,0,0,1)}^{SO(8)} = \frac{1}{90}(1 + \ell)(2 + \ell)(3 + \ell)(4 + \ell)(5 + \ell)(6 + \ell) \]
\[ d_{(\ell,1,0,0)}^{SO(8)} = d_{(\ell,1,0,1)}^{SO(8)} = \frac{1}{18}(1 + \ell)(3 + \ell)(4 + \ell)(5 + \ell)(6 + \ell)(8 + \ell) \]
Appendix B

Zero Charge states

In this Appendix we list states with $Q = 0$ in the KK towers of $S^7$ after
the decomposition of $SO(8)$ into $SO(6) \times SO(2)$.

Bosons:

$(\ell, 0, 0, 0)_{\ell \geq 0} \rightarrow \left[ 0, \frac{\ell}{2}, \frac{\ell}{2} \right]$ (B.1)

$(\ell, 1, 0, 0)_{\ell \geq 0} \rightarrow \left[ 0, \frac{\ell}{2}, \frac{\ell}{2} \right] + \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} + 1 \right]$
$+ \left[ 1, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right] + \left[ 1, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right]$ (B.2)

$(\ell - 1, 0, 1, 1)_{\ell \geq 1} \rightarrow \left[ 0, \frac{\ell}{2} + 2, \frac{\ell}{2} - 2 \right] + \left[ 0, \frac{\ell}{2} - 2, \frac{\ell}{2} + 2 \right] + \left[ 1, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right]$
$+ \left[ 1, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right] + \left[ 0, \frac{\ell}{2}, \frac{\ell}{2} \right] + \left[ 0, \frac{\ell}{2}, \frac{\ell}{2} \right]$
$+ \left[ 1, \frac{\ell}{2} - 2 \right] + \left[ 1, \frac{\ell}{2} - 2, \frac{\ell}{2} \right] + \left[ 2, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right]$ (B.3)

$(\ell - 2, 2, 0, 0)_{\ell \geq 2} \rightarrow \left[ 2, \frac{\ell}{2} - 3, \frac{\ell}{2} + 1 \right] + \left[ 1, \frac{\ell}{2} - 2, \frac{\ell}{2} \right] + \left[ 1, \frac{\ell}{2} - 2, \frac{\ell}{2} \right]$
$+ \left[ 0, \frac{\ell}{2}, \frac{\ell}{2} \right] + \left[ 0, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] + \left[ 1, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right]$
$+ \left[ 1, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right] + \left[ 2, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] + \left[ 2, \frac{\ell}{2} + 1, \frac{\ell}{2} - 3 \right] + \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} + 1 \right]$ (B.4)
\( (\ell, 0, 2, 0)_{\ell \geq 0} \rightarrow \left[ 0, \frac{\ell}{2} - 1, \frac{\ell}{2} + 3 \right] + \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} + 1 \right] + \left[ 0, \frac{\ell}{2} + 3, \frac{\ell}{2} - 1 \right] \\
+ \left[ 1, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right] + \left[ 1, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right] + \left[ 2, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] \) (B.5)

\( (\ell - 2, 0, 0, 2)_{\ell \geq 2} \rightarrow \left[ 0, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] + \left[ 0, \frac{\ell}{2} - 3, \frac{\ell}{2} + 1 \right] + \left[ 1, \frac{\ell}{2}, \frac{\ell}{2} - 2 \right] \\
+ \left[ 1, \frac{\ell}{2} - 2, \frac{\ell}{2} \right] + \left[ 2, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] + \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} - 3 \right] \) (B.6)

Fermions:

\( (\ell, 0, 0, 1)_{\ell \geq 0} \rightarrow \left[ 0, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right] + \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right] + \left[ 1, \frac{\ell}{2}, \frac{\ell}{2} \right] \) (B.7)

\( (\ell - 1, 0, 1, 0)_{\ell \geq 1} \rightarrow \left[ 0, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right] + \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right] + \left[ 1, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] \) (B.8)

\( (\ell - 1, 1, 0, 0)_{\ell \geq 1} \rightarrow \left[ 1, \frac{\ell}{2} - 2, \frac{\ell}{2} + 2 \right] + 2 \left[ 1, \frac{\ell}{2}, \frac{\ell}{2} \right] + \left[ 1, \frac{\ell}{2} + 2, \frac{\ell}{2} - 2 \right] \\
+ \left[ 0, \frac{\ell}{2}, \frac{\ell}{2} + 2 \right] + \left[ 0, \frac{\ell}{2} + 2, \frac{\ell}{2} \right] + \left[ 0, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right] \\
+ \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right] + \left[ 2, \frac{\ell}{2} - 2, \frac{\ell}{2} \right] + \left[ 2, \frac{\ell}{2}, \frac{\ell}{2} - 2 \right] + \left[ 1, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] \) (B.9)

\( (\ell - 2, 1, 0, 1)_{\ell \geq 2} \rightarrow \left[ 1, \frac{\ell}{2} - 3, \frac{\ell}{2} + 1 \right] + 2 \left[ 1, \frac{\ell}{2} - 1, \frac{\ell}{2} - 1 \right] + \left[ 1, \frac{\ell}{2} + 1, \frac{\ell}{2} - 3 \right] \\
+ \left[ 0, \frac{\ell}{2} - 2, \frac{\ell}{2} \right] + \left[ 0, \frac{\ell}{2}, \frac{\ell}{2} - 2 \right] + \left[ 0, \frac{\ell}{2} - 1, \frac{\ell}{2} + 1 \right] \\
+ \left[ 0, \frac{\ell}{2} + 1, \frac{\ell}{2} - 1 \right] + \left[ 2, \frac{\ell}{2} - 2, \frac{\ell}{2} \right] + \left[ 2, \frac{\ell}{2}, \frac{\ell}{2} - 2 \right] + \left[ 1, \frac{\ell}{2}, \frac{\ell}{2} \right] \) (B.10)
Appendix C

Generating functions for $SO(8)$ representations

The generating function for multiplicities of the scalar spherical harmonics on $S^7$ is given by

$$F_{N_1}(q) = \frac{1 + q}{(1 - q)^7} \quad (C.1)$$

The coefficient of $q^\ell$ gives the dimension of the $SO(8)$ representation with Dynkin label $(\ell, 0, 0, 0)$.

The generating function for vector spherical harmonics with $SO(8)$ Dynkin label $(\ell - 1, 1, 0, 0)$ reads:

$$F_{N_7}(q) = \frac{(28 - 36q + 35q^2 - 21q^3 + 7q^4 - q^5)q}{(1 - q)^7} \quad (C.2)$$

For two-form spherical harmonics with $SO(8)$ Dynkin label $(\ell - 1, 0, 1, 1)$ the generating function is:

$$F_{N_{21}}(q) = \frac{(56 - 42q + 22q^2 - 7q^3 + q^4)q^2}{(1 - q)^7} \quad (C.3)$$

For second rank symmetric traceless harmonics the $SO(8)$ Dynkin index is $(\ell, 2, 0, 0)$ and the generating function is given by the following formula:

$$F_{N_{27}}(q) = \frac{4(75 - 175q + 203q^2 - 133q^3 + 47q^4 - 7q^5)q^2}{(1 - q)^7} \quad (C.4)$$

Finally, for three-form spherical harmonics with $SO(8)$ Dynkin label $(\ell - 1, 0, 2, 0)$ (or $(\ell - 1, 0, 0, 2)$) one has

$$F_{N_{35}}(q) = \frac{(35 - 21q + 7q^2 - q^3)q^2}{(1 - q)^7} \quad (C.5)$$
Let us complete the description with the spectrum of spinor spherical harmonics.

For gravitini with Dynkin labels \((\ell, 0, 0, 1)_{\ell \geq 0}\) and \((\ell - 1, 0, 1, 0)_{\ell \geq 1}\), the generating function is:

\[
F_{\text{gravitini}}(q) = \frac{8q}{(1 - q)^7}. \tag{C.6}
\]

For spinors with Dynkin labels \((\ell - 1, 1, 1, 0)_{\ell \geq 1}\) and \((\ell - 2, 1, 0, 1)_{\ell \geq 2}\) one has

\[
F_{\text{spinor}}(q) = \frac{8q^2(20 - 35q + 35q^2 - 21q^3 + 7q^4 - q^5)}{(1 - q)^7}. \tag{C.7}
\]
Appendix D

Generating functions for $SO(6)$ representations

In this Appendix we present the decomposition of the $SO(8)$ generating functions under $SO(6) \times SO(2)$. Below a factor of $(1 - qt^{-1})^{-4}(1 - qt)^{-4}$ is always understood.

For $(\ell, 0, 0, 0)$ one has:

$$\hat{F}_{graviton}(q) = 1 - q^2$$  \hspace{1cm} (D.1)

For $(\ell, 1, 0, 0)$ one has:

$$\hat{F}_{gb1}(q, t) = 6t^2 - 4tq - 4t^3q + q^2 + t^4q^2$$
$$\hat{F}_{gb2}(q, t) = 1 - q^2$$  \hspace{1cm} (D.2)
$$\hat{F}_{gb3}(q, t) = 15 + 36q^2 - 4q^3t^{-3} - 4t^3q^3 + 16q^4 + q^6 + (16q^2 + 6q^4)t^{-2} + t^2(16q^2 + 6q^4) - (24q + 24q^3 + 4q^5)t^{-1} - t(24q + 24q^3 + 4q^5)$$
$$\hat{F}_{gb4}(q, t) = 6t^{-2} - 4qt^{-3} - 4qt^{-1} + q^2 + q^2t^{-4}$$

For $(\ell - 1, 0, 1, 1)$ one has:

$$\hat{F}_{gb1}(q, t) = 4t^3q - 6t^2q^2 - t^4q^2 + 4tq^3 - q^4$$
$$\hat{F}_{gb2}(q, t) = 4tq - q^2 - 6t^2q^2 + 4t^3q^3 - t^4q^4$$
$$\hat{F}_{gb3}(q, t) = -35q^2 + 4t^3q^3 - 16q^4 - 6q^4t^{-2} - q^6 - t^2(16q^2 + 6q^4) + (24q^5 + 4q^5)t^{-1} + t(20q + 24q^3 + 4q^5)$$
For $(\ell - 2, 2, 0, 0)$ one has:

\[
\tilde{F}_{gb}^{\ell}(q, t) = -35q^2 + 4q^3t^{-3} - 16q^4 - 6t^2q^4 - q^6 - (16q^2 + 6q^4)t^{-2} + \\
t(24q^3 + 4q^5) + (20q + 24q^3 + 4q^5)t^{-1}
\]

\[
\tilde{F}_{gb}^{\ell}(q, t) = 4qt^{-1} - q^2 - 6q^2t^{-2} + 4q^3t^{-3} - q^4t^{-4}
\]

\[
\tilde{F}_{gb}^{\ell}(q, t) = 4qt^{-3} - q^2t^{-4} - 6q^2t^{-2} + 4q^3t^{-1} - q^4
\]

\[
\tilde{F}_{gb}^{\ell}(q, t) = 6t^2q^2 - 4tq^3 - 4q^3q^3 + 4q + t^4q^4
\]

\[
\tilde{F}_{gb}^{\ell}(q, t) = 6q^2t^{-2} + 6t^2q^4 - q^2(20q + 4q^3)t^{-1} - tq^2(20q + 4q^3) + \\
q^2(20 + 15q^2 + q^4)
\]

\[
\tilde{F}_{gb}^{\ell}(q, t) = 6q^2t^{-2} - 4q^3t^{-3} - 4q^3t^{-1} + q^4 + q^4t^{-4}
\]
For \((\ell, 0, 2, 0)\) one has:

\[
\begin{align*}
\hat{F}_{sc2}^1(q, t) &= -4t^3q + 15q^2 - 4q^3t^{-1} + q^4 + t^2(10 + 6q^2) - t(20q + 4q^3) \\
\hat{F}_{sc2}^2(q, t) &= 15 + 36q^2 - 4q^3t^{-3} - 4t^3q^3 + 16q^4 + q^6 + (16q^2 + 6q^4)t^{-2} + t^2(16q^2 + 6q^4) - (24q + 24q^3 + 4q^5)t^{-1} - t(24q + 24q^3 + 4q^5) \\
\hat{F}_{sc2}^3(q, t) &= -4qt^{-3} + 15q^2 - 4tq^3 + q^4 + (10 + 6q^2)t^{-2} - (20q + 4q^3)t^{-1} \\
\hat{F}_{sc2}^4(q, t) &= 4t^3q^3 - 6q^4t^{-2} - t^2q(16q + 6q^3) + q(24q^2 + 4q^4)t^{-1} + tq(20 + 24q^2 + 4q^4) - q(35q + 16q^3 + q^5) \\
\hat{F}_{sc2}^5(q, t) &= 4q^3t^{-3} - 6t^2q^4 - q(16q + 6q^3)t^{-2} - q(20 + 24q^2 + 4q^4)t^{-1} + tq(24q^2 + 4q^4) - q(35q + 16q^3 + q^5) \\
\hat{F}_{sc2}^6(q, t) &= 6q^4t^{-2} + 6t^2q^4 - q^2(20q + 4q^3)t^{-1} - tq^2(20q + 4q^3) + q^2(20 + 15q^2 + q^4)
\end{align*}
\]

For \((\ell - 2, 0, 0, 2)\) one has:

\[
\begin{align*}
\hat{F}_{sc3}^1(q, t) &= t^4(q^2 - q^4) \\
\hat{F}_{sc3}^2(q, t) &= q^2 - q^4 \\
\hat{F}_{sc3}^3(q, t) &= (q^2 - q^4)t^{-4} \\
\hat{F}_{sc3}^4(q, t) &= 6t^2q^2 - 4tq^3 - 4t^3q^3 + q^4 + t^4q^4 \\
\hat{F}_{sc3}^5(q, t) &= 6q^2t^{-2} - 4q^3t^{-3} - 4q^3t^{-1} + q^4 + q^4t^{-4} \\
\hat{F}_{sc3}^6(q, t) &= 6q^4t^{-2} + 6t^2q^4 - q^2(20q + 4q^3)t^{-1} - tq^2(20q + 4q^3) + q^2(20 + 15q^2 + q^4)
\end{align*}
\]

For \((\ell, 0, 0, 1)\) one has:

\[
\begin{align*}
\hat{F}_{gr1}^1(q, t) &= t^2(1 - q^2) \\
\hat{F}_{gr1}^2(q, t) &= t^{-2}(1 - q^2) \\
\hat{F}_{gr1}^3(q, t) &= 6 - 4qt^{-1} - 4tq + q^2t^{-2} + t^2q^2
\end{align*}
\]

For \((\ell - 1, 0, 1, 0)\) one has:

\[
\begin{align*}
\hat{F}_{gr2}^1(q, t) &= 4tq - 6q^2 - t^2q^2 + 4q^3t^{-1} - q^4t^{-2} \\
\hat{F}_{gr2}^2(q, t) &= 4qt^{-1} - 6q^2 - q^2t^{-2} + 4tq^3 - t^2q^4 \\
\hat{F}_{gr2}^3(q, t) &= 6q^2 - 4q^3t^{-1} - 4tq^3 + q^4t^{-2} + t^2q^4
\end{align*}
\]
For $(\ell = 1, 1, 1, 0)$ one has:

\[
\begin{align*}
\hat{F}^1_{j_1}(q, t) &= 4t^5q^3 - 6q^4 - t^4q(16q + 6q^3) + tq(24q^2 + 4q^4) + t^3q(20 + 24q^2 + 4q^4) - t^2q(35q + 16q^3 + q^5), \\
\hat{F}^2_{j_1}(q, t) &= 4q^3t^{-1} - 6t^4q^4 - q(16q + 6q^3) + tq(20 + 24q^2 + 4q^4) + t^3q(24q^2 + 4q^4) - t^2q(35q + 16q^3 + q^5), \\
\hat{F}^3_{j_1}(q, t) &= 4q^3t^{-5} - 6q^4 - q(16q + 6q^3)t^{-4} + q(20 + 24q^2 + 4q^4)t^{-3} + q(24q^2 + 4q^4)t^{-1} - q(35q + 16q^3 + q^5)t^{-2}, \\
\hat{F}^4_{j_1}(q, t) &= -15q^4t^{-2} - 10t^4q^4 + q(56q^2 + 24q^4)t^{-1} - t^3q(40q^2 + 20q^4) + t^2q(60q + 80q^3 + 15q^5) - q(74q + 90q^3 + 16q^5) + tq(36 + 120q^2 + 60q^4 + 4q^6), \\
\hat{F}^5_{j_1}(q, t) &= 4q^3 - 6q^4t^{-4} - q(16q + 6q^3) + q(24q^2 + 4q^4)t^{-3} + q(20 + 24q^2 + 4q^4)t^{-1} - q(35q + 16q^3 + q^5)t^{-2}, \\
\hat{F}^6_{j_1}(q, t) &= -10q^4t^{-4} - 15t^2q^4 + t^3q(56q^2 + 24q^4) + t^2q(40q^2 + 20q^4)t^{-3} - q(60q + 80q^3 + 15q^5)t^{-2} - q(74q + 90q^3 + 16q^5) + q(36 + 120q^2 + 60q^4 + 4q^6)t^{-1}, \\
\hat{F}^7_{j_1}(q, t) &= 4tq - 6q^2 - t^2q^2 + 4q^3t^{-1} - q^4t^{-2}, \\
\hat{F}^8_{j_1}(q, t) &= 4qt^{-1} - 6q^2 - q^2t^{-2} + 4tq^3 - t^2q^4, \\
\hat{F}^9_{j_1}(q, t) &= 6q^4 + 6t^4q^4 - t^2q(20q + 4q^3) - t^3q^2(20q + 4q^3) + t^2q^2(20 + 15q^2 + q^4), \\
\hat{F}^{10}_{j_1}(q, t) &= 6q^4 + 6q^4t^{-1} - q^2(20q + 4q^3)t^{-3} - q^2(20q + 4q^3)t^{-1} + q^2(20 + 15q^2 + q^4)t^{-2}, \\
\hat{F}^{11}_{j_1}(q, t) &= 6q^2 - 4q^3t^{-1} - 4t^3q^4 + q^4t^{-2} + t^2q^4.
\end{align*}
\]

Finally, for $(\ell = 2, 1, 0, 1)$ one has:

\[
\begin{align*}
\hat{F}^1_{j_2}(q, t) &= 6t^4q^2 - 4t^3q^3 - 4t^5q^3 + t^2q^4 + t^6q^4, \\
\hat{F}^2_{j_2}(q, t) &= 6q^2 - 4q^3t^{-1} - 4tq^3 + q^4t^{-2} + t^2q^4, \\
\hat{F}^3_{j_2}(q, t) &= 6q^2t^{-4} - 4q^3t^{-5} - 4q^3t^{-3} + q^4t^{-6} + q^4t^{-2}, \\
\hat{F}^4_{j_2}(q, t) &= t^2(q^2 - q^4), \\
\hat{F}^5_{j_2}(q, t) &= (q^2 - q^4)t^{-2}, \\
\hat{F}^6_{j_2}(q, t) &= 4tq + 4t^3q - 6q^2 - 6t^4q^2 + 4q^3t^{-1} + 4t^5q^3 - q^4t^{-2} - t^6q^4 - t^2(1 + q^2).
\end{align*}
\]
\[
\hat{F}_{f_2}(q, t) = 4qt^{-3} + 4qt^{-1} - 6q^2 - 6q^2 t^{-4} + 4q^3 t^{-5} + 4tq^3 - \\
q^4 t^{-6} - t^2 q^4 - (1 + q^2)t^{-2}
\]

\[
\hat{F}_{f_2}(q, t) = 6q^4 + 6t^4 q^4 - t q^2 (20q + 4q^3) - t^3 q^2 (20q + 4q^3) + \\
t^2 q^2 (20 + 15q^2 + q^4)
\]

\[
\hat{F}_{f_2}(q, t) = 6q^4 + 6q^4 t^{-4} - q^2 (20q + 4q^3) t^{-3} - q^2 (20q + 4q^3) t^{-1} + \\
q^2 (20 + 15q^2 + q^4) t^{-2}
\]

\[
\hat{F}_{f_2}(q, t) = -6 - 32q^2 - 12q^4 - 6q^4 t^{-1} - 6q^4 + (24q^3 + 4q^5) t^{-3} + \\
t^3 (24q^3 + 4q^5) + (24q + 28q^3 + 4q^5) t^{-1} + t (24q + 28q^3 + 4q^5) - \\
(36q^2 + 16q^4 + q^6) t^{-2} - t^2 (36q^2 + 16q^4 + q^6)
\]

\[
\hat{F}_{f_2}(q, t) = 6q^2 - 4q^3 t^{-1} - 4tq^3 + q^4 t^{-2} + t^2 q^4 \quad \text{(D.10)}
\]
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