Accelerating Stochastic Composition Optimization

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Abstract

Consider the stochastic composition optimization problem where the objective is a composition of two expected-value functions. We propose a new stochastic first-order method, namely the accelerated stochastic compositional proximal gradient (ASC-PG) method, which updates based on queries to the sampling oracle using two different timescales. The ASC-PG is the first proximal gradient method for the stochastic composition problem that can deal with nonsmooth regularization penalty. We show that the ASC-PG exhibits faster convergence than the best known algorithms, and that it achieves the optimal sample-error complexity in several important special cases. We further demonstrate the application of ASC-PG to reinforcement learning and conduct numerical experiments.

1 Introduction

The popular stochastic gradient methods are well suited for minimizing expected-value objective functions or the sum of a large number of loss functions. Stochastic gradient methods find wide applications in estimation, online learning, and training of deep neural networks. Despite their popularity, they do not apply to the minimization of a nonlinear function involving expected values or a composition between two expected-value functions.

In this paper, we consider the stochastic composition problem, given by

\[
\min_{x \in \mathbb{R}^n} \quad H(x) := \mathbb{E}_v(f_v(\mathbb{E}_w(g_w(x)))) + R(x)
\]

where \((f \circ g)(x) = f(g(x))\) denotes the function composition, \(g_w(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^m\) and \(f_v(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}\) are continuously differentiable functions, \(v, w\) are random variables, and \(R(x) : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}\) is an extended real-valued closed convex function. We assume throughout that there exists at least one optimal solution \(x^*\) to problem (1). We focus on the case where \(f_v\) and \(g_w\) are smooth, but we allow \(R\) to be a nonsmooth penalty such as the \(\ell_1\)-norm. We do not require either the outer function \(f_v\) or the inner function \(g_w\) to be convex or monotone. The inner and outer random variables \(w, v\) can be dependent.

Our algorithmic objective is to develop efficient algorithms for solving problem (1) based on random evaluations of \(f_v, g_w\) and their gradients. Our theoretical objective is to analyze the rate

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of convergence for the stochastic algorithm and to improve it when possible. In the online setting, the iteration complexity of stochastic methods can be interpreted as sample-error complexity of estimating the optimal solution of problem (1).

1.1 Motivating Examples

One motivating example is reinforcement learning [Sutton and Barto, 1998]. Consider a controllable Markov chain with states $1, \ldots, S$. Estimating the value-per-state of a fixed control policy $\pi$ is known as on-policy learning. It can be casted into an $S \times S$ system of Bellman equations:

$$\gamma P^\pi V^\pi + r^\pi = V^\pi,$$

where $\gamma \in (0,1)$ is a discount factor, $P^\pi_{ss'}$ is the transition probability from state $s$ to state $s'$, and $r^\pi_s$ is the expected state transition reward at state $s$. The solution $V^\pi$ to the Bellman equation is the value vector, with $V^\pi(s)$ being the total expected reward starting at state $s$. In the blackbox simulation environment, $P^\pi, r^\pi$ are unknown but can be sampled from a simulator. As a result, solving the Bellman equation becomes a special case of the stochastic composition optimization problem:

$$\min_{x \in \mathbb{R}^S} \| \mathbb{E}[A] x - \mathbb{E}[b] \|^2,$$  \hspace{1cm} (2)

where $A, b$ are random matrices and random vectors such that $\mathbb{E}[A] = I - \gamma P^\pi$ and $\mathbb{E}[b] = r^\pi$. It can be viewed as the composition of the square norm function and the expected linear function. We will give more details on the reinforcement learning application in Section 4.

Another motivating example is risk-averse learning. For example, consider the mean-variance minimization problem

$$\min_{x} \mathbb{E}_{a, b} [h(x; a, b)] + \lambda \text{Var}_{a, b}[h(x; a, b)],$$

where $h(x; a, b)$ is some loss function parameterized by random variables $a$ and $b$, and $\lambda > 0$ is a regularization parameter. Its batch version takes the form

$$\min_{x} \frac{1}{N} \sum_{i=1}^{N} h(x; a_i, b_i) + \frac{\lambda}{N} \sum_{i=1}^{N} \left( h(x; a_i, b_i) - \frac{1}{N} \sum_{i=1}^{N} h(x; a_i, b_i) \right)^2.$$  

Here the variance term is the composition of the mean square function and an expected loss function. Indeed, the stochastic composition problem (1) finds a broad spectrum of applications in estimation and machine learning. Fast optimization algorithms with theoretical guarantees will lead to new computation tools and online learning methods for a broader problem class, no longer limited to the expectation minimization problem.

1.2 Related Works and Contributions

Contrary to the expectation minimization problem, “unbiased” gradient samples are no longer available for the stochastic composition problem (1). The objective is nonlinear in the joint probability distribution of $(w, v)$, which substantially complicates the problem. In a recent work by Dentcheva et al. [2015], a special case of the stochastic composition problem, i.e., risk-averse optimization, has been studied. A central limit theorem has been established, showing that the
$K$-sample batch problem converges to the true problem at the rate of $O(1/\sqrt{K})$ in a proper sense. For the case where $R(x) = 0$, [Wang et al. 2016] has proposed and analyzed a class of stochastic compositional gradient/subgradient methods (SCGD). The SCGD involves two iterations of different time scales, one for estimating $x^*$ by a stochastic quasi-gradient iteration, the other for maintaining a running estimate of $g(x^*)$. Almost sure convergence and several convergence rate results have been obtained.

The idea of using two-timescale quasi-gradient traced back to the earlier work [Ermoliev 1976]. The incremental treatment of proximal gradient iteration has been studied extensively for the expectation minimization problem, see for examples Nedić and Bertsekas [2001], Bertsekas [2011], Nedić [2011], Wang and Bertsekas [2014], Beck and Teboulle [2009], Gurbuzbalaban et al. [2015], Rakhlin et al. [2012], Ghadimi and Lan [2013], Shamir and Zhang [2013]. However, except for [Wang et al. 2016], all of these works focus on the expectation minimization problem and do not apply to the stochastic composition problem

In this paper, we propose a new accelerated stochastic compositional proximal gradient (ASC-PG) method that applies to the more general penalized problem (1). We use a coupled martingale stochastic analysis to show that ASC-PG achieves significantly better sample-error complexity in various cases. We also show that ASC-PG exhibits optimal sample-error complexity in two important special cases: the case where the outer function is linear and the case where the inner function is linear.

Our contributions are summarized as follows:

1. We propose the first stochastic proximal-gradient method for the stochastic composition problem. This is the first algorithm that is able to address the nonsmooth regularization penalty $R(\cdot)$ without deteriorating the convergence rate.

2. We obtain a convergence rate $O(K^{-4/9})$ for smooth optimization problems that are not necessarily convex, where $K$ is the number of queries to the stochastic first-order oracle. This improves the best known convergence rate and provides a new benchmark for the stochastic composition problem.

3. We provide a comprehensive analysis and results that apply to various special cases. In particular, our results contain as special cases the known optimal rate results for the expectation minimization problem, i.e., $O(1/\sqrt{K})$ for general objectives and $O(1/K)$ for strongly convex objectives.

4. In the special case where the inner function $g(\cdot)$ is a linear mapping, we show that it is sufficient to use one timescale to guarantee convergence. Our result achieves the non-improvable rate of convergence $O(1/K)$. It implies that the inner linearity does not bring fundamental difficulty to the stochastic composition problem.

5. We show that the proposed method leads to a new on-policy reinforcement learning algorithm. The new learning algorithm achieves the optimal convergence rate $O(1/\sqrt{K})$ for solving Bellman equations based on $K$ observed state transitions.

In comparison with [Wang et al. 2016], our analysis is more succinct and leads to stronger results. To the best of our knowledge, results in this paper provide the best-known rates for the stochastic composition problem.

**Paper Organization.** Section 2 states the sampling oracle and the accelerated stochastic compositional proximal gradient algorithm (ASC-PG). Section 3 states the convergence rate results in the case of general nonconvex objective and in the case of strongly convex objective, respec-
tively. Section 4 describes an application of ASC-PG to reinforcement learning and gives numerical experiments.

**Notations and Definitions.** For \( x \in \mathbb{R}^n \), we denote by \( x' \) its transpose, and by \( \|x\| \) its Euclidean norm (i.e., \( \|x\| = \sqrt{x^T x} \)). For two sequences \( \{y_k\} \) and \( \{z_k\} \), we write \( y_k = O(z_k) \) if there exists a constant \( c > 0 \) such that \( \|y_k\| \leq c \|z_k\| \) for each \( k \). We denote by \( \text{I}_{\text{value}} \) the indicator function, which returns “value” if the “condition” is satisfied; otherwise 0. We denote by \( H^* \) the optimal objective function value for (1), denote by \( X^* \) the set of optimal solutions, and denote by \( P_S(x) \) the Euclidean projection of \( x \) onto \( S \) for any convex set \( S \). We let \( f(y) = \mathbb{E}_v[f_v(y)] \) and \( g(x) = \mathbb{E}_w[g_w(x)] \).

## 2 Algorithm

We focus on the blackbox sampling environment. Suppose that we have access to a stochastic first-order oracle, which returns random realizations of first-order information upon queries. This is a typical simulation oracle that is available in both online and batch learning. More specifically, assume that we are given a **Sampling Oracle (SO)** such that

- Given some \( x \in \mathbb{R}^n \), the SO returns a random vector \( g_w(x) \) and a noisy subgradient \( \nabla g_w(x) \).
- Given some \( y \in \mathbb{R}^m \), the SO returns a noisy gradient \( \nabla f_v(y) \).

Now we propose the Accelerated Stochastic Compositional Proximal Gradient (ASC-PG) algorithm, see Algorithm 1. ASC-PG is a generalization of the SCGD proposed by [Wang et al. 2016], in which a proximal step is used to replace the projection step.

**Algorithm 1** Accelerated Stochastic Compositional Proximal Gradient (ASC-PG)

**Require:** \( x_1 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m, \text{SO}, K, \) stepsize sequences \( \{\alpha_k\}_{k=1}^K \), and \( \{\beta_k\}_{k=1}^K \).

**Ensure:** \( \{x_k\}_{k=1}^K \)

1: for \( k = 1, \ldots, K \) do
2: Query the SO and obtain gradient samples \( \nabla f_v(y_k), \nabla g_w(z_k) \).
3: Update the main iterate by

\[
    x_{k+1} = \text{prox}_{\alpha_k R(\cdot)} \left( x_k - \alpha_k \nabla g_w^T(x_k) \nabla f_v(y_k) \right).
\]

4: Update auxiliary iterates by an **extrapolation-smoothing** scheme:

\[
    z_{k+1} = \left( 1 - \frac{1}{\beta_k} \right) x_k + \frac{1}{\beta_k} x_{k+1},
\]

\[
    y_{k+1} = \left( 1 - \beta_k \right) y_k + \beta_k g_{w_{k+1}}(z_{k+1}),
\]

where the sample \( g_{w_{k+1}}(z_{k+1}) \) is obtained via querying the SO.

5: end for

In Algorithm 1, the **extrapolation-smoothing scheme** (i.e., the \( (y,z) \)-step) is critical for convergence acceleration. The acceleration is due to the fast running estimation of the unknown quantity \( g(x_k) := \mathbb{E}_w[g_w(x_k)] \). At iteration \( k \), the running estimate \( \hat{y}_k \) of \( g(x_k) \) is obtained using a weighted smoothing scheme, corresponding to the \( y \)-step; while the new query point \( z_{k+1} \) is obtained through
extrapolation, corresponding to the $z$-step. The updates are constructed in a way such that $y_k$ is a nearly unbiased estimate of $g(x_k)$. To see how the extrapolation-smoothing scheme works, we define the weights as 

$$
\xi^{(k)}_t = \begin{cases} 
\beta_t \prod_{i=t+1}^k (1 - \beta_i), & \text{if } k > t \geq 0 \\
\beta_k, & \text{if } k = t \geq 0.
\end{cases} \tag{3}
$$

Then we can verify the following important relations:

$$
x_{k+1} = \sum_{t=0}^k \xi^{(k)}_t z_{t+1}, \quad y_{k+1} = \sum_{t=0}^k \xi^{(k)}_t g_{w_{t+1}}(z_{t+1}).
$$

Now consider the special case where $g_w(\cdot)$ is always a linear mapping $g_w(z) = A_wz + b_z$ and $\beta_k = \xi^{(k)}_t = 1/(k+1)$. Then we have

$$
g(x_{k+1}) = \frac{1}{k+1} \sum_{t=0}^k E[A_wz_{t+1} + E[b_w], \quad y_{k+1} = \frac{1}{k+1} \sum_{t=0}^k A_wz_{t+1} + \frac{1}{k+1} \sum_{t=0}^k b_w].
$$

In this way, we can see that the scaled error

$$
k(y_{k+1} - g(x_{k+1})) = \sum_{t=0}^k (A_wz_{t+1} - E[A_w])z_{t+1} + \sum_{t=0}^k (b_w - E[b_w])
$$

is a zero-mean and zero-drift martingale. Under additional technical assumptions, we have

$$
E[||y_{k+1} - g(x_{k+1})||^2] \leq O \left( \frac{1}{k} \right).
$$

Note that the zero-drift property of the error martingale is the key to the fast convergence rate. The zero-drift property comes from the near-unbiasedness of $y_k$, which is due to the special construction of the extrapolation-smoothing scheme. In the more general case where $g_w$ is not necessarily linear, we can use a similar argument to show that $y_k$ is a nearly unbiased estimate of $g(x_k)$. As a result, the extrapolation-smoothing ($y, z$)-step ensures that $y_k$ tracks the unknown quantity $g(x_k)$ efficiently.

### 3 Main Results

We present our main theoretical results in this section. Let us begin by stating our assumptions. Note that all assumptions involving random realizations of $v, w$ hold with probability 1.

**Assumption 1.** The samples generated by the SO are unbiased in the following sense:

1. $E_{w_k, v_k}(\nabla g^T_{w_k}(x) \nabla f_k(y)) = \nabla g^T(x) \nabla f(y) \forall k = 1, 2, \cdots, K, \forall x, \forall y.$

2. $E_{w_k}(g_{w_k}(x)) = g(x) \forall x.$

Note that $w_k$ and $v_k$ are not necessarily independent.

**Assumption 2.** The sample gradients and values generated by the SO satisfy

$$
E_{w}(||g_w(x) - g(x)||^2) \leq \sigma^2 \forall x.
$$
Assumption 3. The sample gradients generated by the SO are uniformly bounded, and the penalty function $R$ has bounded gradients.

\[ \|\nabla f_v(x)\| \leq \Theta(1), \|\nabla g_w(x)\| \leq \Theta(1), \|\partial R(x)\| \leq \Theta(1) \quad \forall x, \forall w, \forall v \]

Assumption 4. There exists $L_F, L_f, L_g > 0$ such that the inner and outer functions satisfying the following Lipschitzian conditions

1. $F(z) - F(x) \leq \langle \nabla F(x), z - x \rangle + \frac{L_f}{2} \|z - x\|^2 \quad \forall x, \forall z.$
2. $\|\nabla f_v(y) - \nabla f_v(w)\| \leq L_f \|y - w\| \quad \forall y, \forall w, \forall v.$
3. $\|g(x) - g(z) - \nabla g(z)^T (x - z)\| \leq \frac{L_g}{2} \|x - z\|^2 \quad \forall x, \forall z.$

Our first main result concerns with general optimization problems which are not necessarily convex.

Theorem 1 (Smooth Optimization). Let Assumptions 1, 2, 3, and 4 hold. Denote by $F(x) := (E_v(f_v) \circ E_w(g_w))(x)$ for short and suppose that $R(x) = 0$ in 1 and $E(F(x_k))$ is bounded from above. Choose $\alpha_k = k^{-a}$ and $\beta_k = 2k^{-b}$ where $a \in (0, 1)$ and $b \in (0, 1)$ in Algorithm 1. Then we have

\[ \sum_{k=1}^{K} \frac{E(\|\nabla F(x_k)\|^2)}{K} \leq O(K^{a-1} + L_f^2 L_g K^{4b-4a} + L_f^2 K^{-b} + K^{-a}). \tag{4} \]

If $L_g \neq 0$ and $L_f \neq 0$, choose $a = 5/9$ and $b = 4/9$, yielding

\[ \sum_{k=1}^{K} \frac{E(\|\nabla F(x_k)\|^2)}{K} \leq O(K^{-4/9}). \tag{5} \]

If $L_g = 0$ or $L_f = 0$, then the optimal $a$ and $b$ can be chosen to be $a = b = 1/2$, yielding

\[ \sum_{k=1}^{K} \frac{E(\|\nabla F(x_k)\|^2)}{K} \leq O(K^{-1/2}). \tag{6} \]

The result of Theorem 1 strictly improves the corresponding results in Wang et al. [2016]. First the result in 5 improves the convergence rate from $O(k^{-2/7})$ to $O(k^{-4/9})$ for the general case. This improves the best known convergence rate and provides a new benchmark for the stochastic composition problem.

Our second main result concerns strongly convex objective functions. We say that the objective function $H$ is optimally strongly convex with parameter $\lambda > 0$ if

\[ H(x) - H(P_{X^*}(x)) \geq \lambda \|x - P_{X^*}(x)\|^2 \quad \forall x. \tag{7} \]

(see Liu and Wright [2015]). Note that any strongly convex function is optimally strongly convex, but the reverse does not hold. For example, the objective function in on-policy reinforcement learning is always optimally strongly convex (even if $E(A)$ is a rank deficient matrix), but not necessarily strongly convex.
Theorem 2. (Strongly Convex Optimization) Suppose that the objective function $H(x)$ in (1) is optimally strongly convex with parameter $\lambda > 0$ defined in (7). Set $\alpha_k = C_\alpha k^{-a}$ and $\beta_k = C_\beta k^{-b}$ where $C_\alpha > 4\lambda$, $C_\beta > 2$, $a \in (0,1]$, and $b \in (0,1]$ in Algorithm 2. Under Assumptions 1, 2, 3, and 4, we have

$$\mathbb{E} (\|x_k - P_{X^*} (x_k)\|^2) \leq O \left( k^{-a} + L_f^2 k^{-(4a+4b)} + L_g^2 k^{-b} \right).$$

(8)

If $L_g \neq 0$ and $L_f \neq 0$, choose $a = 1$ and $b = 4/5$, yielding

$$\mathbb{E} (\|x_k - P_{X^*} (x_k)\|^2) \leq O (k^{-4/5}).$$

(9)

If $L_g = 0$ or $L_f = 0$, choose $a = 1$ and $b = 1$, yielding

$$\mathbb{E} (\|x_k - P_{X^*} (x_k)\|^2) \leq O (k^{-1}).$$

(10)

Let us discuss the results of Theorem 2. In the general case where $L_f \neq 0$ and $L_g \neq 0$, the convergence rate in (9) is consistent with the result of [Wang et al. 2016]. Now consider the special case where $L_g = 0$, i.e., the inner mapping is linear. This result finds an immediate application to Bellman error minimization problem (2) which arises from reinforcement learning problem in (and with $\ell_1$ norm regularization). The proposed ASC-PG algorithm is able to achieve the optimal rate $O(1/K)$ without any assumption on $f_v$. To the best of our knowledge, this is the best (also optimal) sample-error complexity for on-policy reinforcement learning.

Remarks Theorems 1 and 2 give important implications about the special cases where $L_f = 0$ or $L_g = 0$. In these cases, we argue that our convergence rate (10) is “optimal” with respect to the sample size $k$. To see this, it is worth pointing out the the $O(1/K)$ rate of convergence is optimal for strongly convex expectation minimization problem. Because the expectation minimization problem is a special case of problem (1), the $O(1/K)$ convergence rate must be optimal for the stochastic composition problem too.

- Consider the case where $L_f = 0$, which means that the outer function $f_v(\cdot)$ is linear with probability 1. Then the stochastic composition problem (1) reduces to an expectation minimization problem since $(\mathbb{E}_{x,f_v \circ g_w}(x)) = \mathbb{E}_{x} (f_v(\mathbb{E}_{w} g_w(x))) = \mathbb{E}_{x} \mathbb{E}_{w} (f_v \circ g_w)(x)$. Therefore, it makes a perfect sense to obtain the optimal convergence rate.

- Consider the case where $L_g = 0$, which means that the inner function $g(\cdot)$ is a linear mapping. The result is quite surprising. Note that even $g(\cdot)$ is a linear mapping, it does not reduce problem (1) to an expectation minimization problem. However, the ASC-PG still achieves the optimal convergence rate. This suggests that, when inner linearity holds, the stochastic composition problem (1) is not fundamentally more difficult than the expectation minimization problem.

The convergence rate results unveiled in Theorems 1 and 2 are the best known results for the composition problem. We believe that they provide important new result which provides insights into the complexity of the stochastic composition problem.

4 Application to Reinforcement Learning

In this section, we apply the proposed ASC-PG algorithm to conduct policy value evaluation in reinforcement learning through attacking Bellman equations. Suppose that there are in total $S$
states. Let the policy of interest be \( \pi \). Denote the value function of states by \( V^\pi \in \mathbb{R}^S \), where \( V^\pi(s) \) denotes the value of being at state \( s \) under policy \( \pi \). The Bellman equation of the problem is

\[
V^\pi(s_1) = \mathbb{E}_\pi \{ r_{s_1,s_2} + \gamma \cdot V^\pi(s_2) | s_1 \} \text{ for all } s_1, s_2 \in \{1, ..., S\},
\]

where \( r_{s_1,s_2} \) denotes the reward of moving from state \( s_1 \) to \( s_2 \), and the expectation is taken over all possible future state \( s_2 \) conditioned on current state \( s_1 \) and the policy \( \pi \). We have that the solution \( V^* \in \mathbb{R}^S \) to the above equation satisfies that \( V^* = V^\pi \). Here a moderately large \( S \) will make solving the Bellman equation directly impractical. To resolve the curse of dimensionality, in many practical applications, we approximate the value of each state by some linear map of its feature \( \phi_s \in \mathbb{R}^m \), where \( d < S \) to reduce the dimension. In particular, we assume that \( V^\pi(s) \approx \phi_s^T w^* \) for some \( w^* \in \mathbb{R}^m \).

To compute \( w^* \), we formulate the problem as a Bellman residual minimization problem that

\[
\min_w \sum_{s=1}^S (\phi_s^T w - q^\pi,s^*(w))^2,
\]

where \( q^\pi,s^*(w) = \mathbb{E}_\pi \{ r_{s,s'} + \gamma \cdot \phi_{s'} w \} = \sum_{s'} \mathbb{P}_{\pi,ss'}(\{ r_{s,s'} + \gamma \cdot \phi_{s'} w \}); \gamma < 1 \) is a discount factor, and \( r_{s,s'} \) is the random reward of transition from state \( s \) to state \( s' \). It is clearly seen that the proposed ASC-PG algorithm could be directly applied to solve this problem where we take

\[
g(w) = (\phi_1^T w, q^\pi,1(w), ..., \phi_S^T w, q^\pi,S(w)) \in \mathbb{R}^{2S},
\]
\[ f\left((\phi_1^T w, q_{\pi,1}(w), ..., \phi_S^T w, q_{\pi,S}(w))\right) = \sum_{s=1}^{S} (\phi_{s} w - q_{\pi,s}(w))^2 \in \mathbb{R}. \]

We point out that the \( g(\cdot) \) function here is a linear map. By our theoretical analysis, we expect to achieve a faster \( O(1/k) \) rate of convergence, which is justified empirically in our later simulation study.

We consider three experiments, where in the first two experiments, we compare our proposed accelerated ASC-PG algorithm with SCGD algorithm [Wang et al., 2016] and the recently proposed GTD2-MP algorithm [Liu et al., 2015]. Also, in the first two experiments, we do not add any regularization term, i.e. \( R(\cdot) = 0 \). In the third experiment, we add an \( \ell_1 \)-penalization term \( \lambda \|w\|_1 \).

In all cases, we choose the step sizes via comparison studies as in Dann et al. [2014]:

- **Experiment 1:** We use the Baird’s example [Baird et al., 1995], which is a well-known example to test the off-policy convergent algorithms. This example contains \( S = 6 \) states, and two actions at each state. We refer the readers to [Baird et al., 1995] for more detailed information of the example.

- **Experiment 2:** We generate a Markov decision problem (MDP) using similar setup as in White and White [2016]. In each instance, we randomly generate an MDP which contains \( S = 100 \) states, and three actions at each state. The dimension of the Given one state and one action, the agent can move to one of four next possible states. In our simulation, we generate the transition probabilities for each MDP instance uniformly from \([0, 1]\) and normalize the sum of transitions to one, and we generate the reward for each transition also uniformly in \([0, 1]\).

- **Experiment 3:** We generate the data same as Experiment 2 except that we have a larger \( d = 100 \) dimensional feature space, where only the first 4 components of \( w^* \) are non-zeros. We add an \( \ell_1 \)-regularization term, \( \lambda \|w\|_1 \), to the objective function.

Denote by \( w_k \) the solution at the \( k \)-th iteration. For the first two experiments, we report the empirical convergence performance \( \|w_k - w^*\| \) and \( \|\Phi w_k - \Phi w^*\| \), where \( \Phi = (\phi_1, ..., \phi_S)^T \in \mathbb{R}^{S \times d} \) and \( \Phi w^* = V \), and all \( w_k \)'s are averaged over 100 runs, in the first two subfigures of Figures 1 and 2. It is seen that the ASC-PG algorithm achieves the fastest convergence rate empirically in both experiments. To further evaluate our theoretical results, we plot \( \log(t) \) vs. \( \log(\|w_k - w^*\|) \) (or \( \log(\|\Phi w_k - \Phi w^*\|) \)) averaged over 100 runs for the first two experiments in the second two subfigures of Figures 1 and 2. The empirical results further support our theoretical analysis that \( \|w_k - w^*\|^2 = O(1/k) \) for the ASC-PG algorithm when \( g(\cdot) \) is a linear mapping.

For Experiment 3, as the optimal solution is unknown, we run the ASC-PG algorithm for one million iterations and take the corresponding solution as the optimal solution \( \hat{w}^* \), and we report \( \|w_k - \hat{w}^*\| \) and \( \|\Phi w_k - \Phi \hat{w}^*\| \) averaged over 100 runs in Figure 3. It is seen the the ASC-PG algorithm achieves fast empirical convergence rate.

## 5 Conclusion

We develop a proximal gradient method for the penalized stochastic composition problem. The algorithm updates by interacting with a stochastic first-order oracle. Convergence rates are established under a variety of assumptions, which provide new rate benchmarks. Application of the ASC-PG to reinforcement learning leads to a new on-policy learning algorithm, which achieves faster convergence than the best known algorithms. For future research, it remains open whether or under what circumstances the current \( O(K^{-4/9}) \) can be further improved. Another direction is to customize
and adapt the algorithm and analysis to more specific problems arising from reinforcement learning and risk-averse optimization, in order to fully exploit the potential of the proposed method.
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Supplemental Materials

Lemma 3. Under Assumption 3, two subsequent iterates in Algorithm 1 satisfy
\[ \|x_k - x_{k+1}\|^2 \leq \Theta(\alpha_k^2). \]

Proof. From the definition of the proximal operation, we have
\[
x_{k+1} = \text{prox}_{\alpha_k R}(x_k - \alpha_k \nabla g^T w_k(x_k) \nabla f_v(y_k))
\]
\[ = \arg\min_x \frac{1}{2}\|x - x_k + \alpha_k \nabla g^T w_k(x_k) \nabla f_v(y_k)\|^2 + \alpha_k R(x). \]

The optimality condition suggests the following equality:
\[ x_{k+1} - x_k = -\alpha_k (\nabla g^T w_k(x_k) \nabla f_v(y_k) + s_{k+1}) \]
where \( s_{k+1} \in \partial R(x_{k+1}) \) is some vector in the sub-differential set of \( R(\cdot) \) at \( x_{k+1} \). Then apply the boundedness condition in Assumption 3 to yield
\[ \|x_{k+1} - x_k\| = \alpha_k \| (\nabla g^T w_k(x_k) \nabla f_v(y_k) + s_{k+1}) \| \]
\[ \leq \alpha_k \| (\nabla g^T w_k(x_k) \nabla f_v(y_k) + \|s_{k+1}\|) \]
\[ \leq \alpha_k \| (\nabla g^T w_k(x_k) \nabla f_v(y_k) + \|s_{k+1}\|) \]
\[ \leq \Theta(1) \alpha_k, \]
which implies the claim.

Lemma 4. Under Assumptions 3 and 4, we have
\[ \|\nabla g^T w(x) \nabla f(g(x)) - \nabla g^T w(x) \nabla f_v(y)\| \leq \Theta(L_f \| y - g(x) \|). \]

Proof. We have
\[ \|\nabla g^T w(x) \nabla f(g(x)) - \nabla g^T w(x) \nabla f_v(y)\| \leq \|\nabla g^T w(x)\| \|\nabla f_v(g(x)) - \nabla f_v(y)\| \]
\[ \leq \Theta(1) \|\nabla f_v(g(x)) - \nabla f_v(y)\| \]
\[ \leq \Theta(L_f) \| y - g(x) \|. \]

It completes the proof.

Lemma 5. Given a positive sequence \( \{w_k\}_{k=1}^\infty \) satisfying
\[ w_{k+1} \leq (1 - \beta_k + C_1 \beta_k^2) w_k + C_2 k^{-a} \]
where \( C_1 \geq 0, C_2 \geq 0, \) and \( a \geq 0. \) Choosing \( \beta_k \) to be \( \beta_k = C_3 k^{-b} \) where \( b \in (0,1] \) and \( C_3 > 2, \) the sequence can be bounded by
\[ w_k \leq C_4 k^{-c}. \]
where \( C \) and \( c \) are defined as
\[
C := \max_{k \leq (C_1 C_2^2)^{1/b} + 1} w_k k^c + \frac{C_2}{C_3 - 2} \quad \text{and} \quad c := a - b.
\]

In other words, we have
\[
w_k \leq \Theta(k^{-a+b}).
\]

**Proof.** We prove it by induction. First it is easy to verify that the claim holds for \( k \leq (C_1 C_2^2)^{1/b} \) from the definition for \( C \). Next we prove from “\( k \)” to “\( k + 1 \)”, that is, given \( w_k \leq C k^{-c} \) for \( k > (C_1 C_2^2)^{1/b} \), we need to prove \( w_{k+1} \leq C(k + 1)^{-c} \).
\[
\begin{align*}
w_{k+1} &\leq (1 - \beta_k + C_1 \beta_k^2)w_k + C_2 k^{-a} \\
&\leq (1 - C_3 k^{-b} + C_1 C_3^2 k^{-2b}) C k^{-c} + C_2 k^{-a} \\
&= C k^{-c} - C C_3 k^{-b - c} + C C_1 C_3^2 k^{-2b - c} + C_2 k^{-a}.
\end{align*}
\]

To prove that (13) is bounded by \( C(k + 1)^{-c} \), it suffices to show that
\[
\Delta := (k + 1)^{-c} - k^{-c} + C_3 k^{-b - c} - C C_1 C_3^2 k^{-2b - c} > 0 \quad \text{and} \quad C \geq \frac{C_2 k^{-a}}{\Delta}.
\]

From the convexity of function \( h(t) = t^{-c} \), we have the inequality \( (k + 1)^{-c} - k^{-c} \geq (-c) k^{-c-1} \). Therefore we obtain
\[
\Delta \geq \begin{cases} (b \leq 1, k > (C_1 C_2^2)^{1/b}) & \frac{-ek^{-c-1} + C_3 k^{-b - c} - C C_1 C_3^2 k^{-2b - c}}{(C_3 > 2)} \\
(C_3 > 2) & (C_3 - 2)(k^{-b - c}) > 0.
\end{cases}
\]

To verify the second one, we have
\[
\frac{C_2 k^{-a}}{\Delta} \leq \frac{C_2}{C_3 - 2} k^{-a + b + c} \leq \frac{C_2}{C_3 - 2} \leq C
\]
where the last inequality is due to the definition of \( C \). It completes the proof. \( \square \)

**Lemma 6.** Choose \( \beta_k \) to be \( \beta_k = C_b k^{-b} \) where \( C_b > 2 \), \( b \in (0, 1] \), and \( \alpha_k = C_a k^{-a} \). Under Assumptions 4 and 8 we have
\[
\mathbb{E}||y_k - g(x_k)||^2 \leq L_g \Theta(k^{-4a+4b}) + \Theta(k^{-b}). \tag{14}
\]

**Proof.** Denote by \( m_{k+1} \)
\[
m_{k+1} := \sum_{t=0}^{k} \xi_t^{(k)} ||x_{k+1} - z_{t+1}||^2
\]
and \( n_{k+1} \)
\[
n_{k+1} := \sum_{t=0}^{k} \xi_t^{(k)} (g_{w_{t+1}}(z_{t+1}) - g(z_{t+1}))
\]
for short.

From Lemma 10 in [Wang et al., 2016], we have
\[
\|y_k - g(x_k)\|^2 \leq \left( \frac{L_g}{2} m_k + n_k \right)^2 \leq L_g m_k^2 + 2 n_k^2. \tag{15}
\]

From Lemma 11 in [Wang et al., 2016], \(m_{k+1}\) can be bounded by
\[
m_{k+1} \leq (1 - \beta_k) m_k + \beta_k q_k + \frac{2}{\beta_k} \|x_k - x_{k+1}\|^2 \tag{16}
\]
where \(q_k\) is bounded by
\[
q_{k+1} \leq (1 - \beta_k) q_k + \frac{4}{\beta_k} \|x_{k+1} - x_k\|^2 \tag{Lemma 3}
\]
\[
\leq (1 - \beta_k) q_k + \Theta(k^{-2a+b}).
\]

Use Lemma 5 and obtain the following decay rate
\[
q_k \leq \Theta(k^{-2a+2b}).
\]
Together with (16), we have
\[
m_{k+1} \leq (1 - \beta_k) m_k + \beta_k q_k + \frac{2}{\beta_k} \|x_k - x_{k+1}\|^2 \leq (1 - \beta_k) m_k + \Theta(k^{-2a+b}) + \Theta(k^{-2a+b}) \leq (1 - \beta_k) m_k + \Theta(k^{-2a+b}),
\]
which leads to
\[
m_k \leq \Theta(k^{-2a+2b}) \quad \text{and} \quad m_k^2 \leq \Theta(k^{-4a+4b}). \tag{17}
\]

by using Lemma 5 again. Then we estimate the upper bound for \(\mathbb{E}(n_k^2)\). From Lemma 11 in [Wang et al., 2016], we know \(\mathbb{E}(n_k^2)\) is bounded by
\[
\mathbb{E}(n_{k+1}^2) \leq (1 - \beta_k)^2 \mathbb{E}(||n_k||^2) + \beta_k^2 \sigma_y^2 = (1 - 2 \beta_k + \beta_k^2) \mathbb{E}(||n_k||^2) + \beta_k^2 \sigma_y^2.
\]
By using Lemma 5 again, we have
\[
\mathbb{E}(n_k^2) \leq \Theta(k^{-b}). \tag{18}
\]

Now we are ready to estimate the upper bound of \(\|y_{k+1} - g(x_{k+1})\|^2\) by following (15)
\[
\mathbb{E}(||y_k - g(x_k)||^2) \leq L_g \mathbb{E}(m_k^2) + 2 \mathbb{E}(n_k^2) \leq L_g \Theta(k^{-4a+4b}) + \Theta(k^{-b}).
\]

It completes the proof.

\textbf{Proof to Theorem 1}
Proof. From the Lipschitzian condition in Assumption \textbf{4} we have
\[
F(x_{k+1}) - F(x_k) 
\leq \langle \nabla F(x_k), x_{k+1} - x_k \rangle + \frac{L_F}{2} \|x_{k+1} - x_k\|^2
\]
(Lemma \textbf{5})
\[
\leq -\alpha_k \langle \nabla F(x_k), \nabla g_{w_k}(x_k)\nabla f_{v_k}(y_k) \rangle + \Theta(\alpha_k^2)
\]
\[
= -\alpha_k \|\nabla F(x_k)\|^2 + \alpha_k \left( \langle \nabla F(x_k), \nabla F(x_k) - \nabla g_{w_k}(x_k)\nabla f_{v_k}(y_k) \rangle \right)
\]
\[
\leq \alpha_k \|\nabla F(x_k)\|^2 + \Theta(\alpha_k^2)
\]
(19)

Next we estimate the upper bound for $\mathbb{E}(T)$:
\[
\mathbb{E}(T) \leq \mathbb{E}(\langle \nabla F(x_k), \nabla F(x_k) - \nabla g_{w_k}(x_k)\nabla f_{v_k}(y_k) \rangle)
\]
\[
+ \mathbb{E}(\langle \nabla F(x_k), \nabla g_{w_k}(x_k)\nabla f_{v_k}(y_k) \rangle)
\]
(Assumption \textbf{1})
\[
\leq \mathbb{E}(\langle \nabla F(x_k), \nabla g_{w_k}(x_k)\nabla f_{v_k}(y_k) \rangle)
\]
\[
\leq \mathbb{E}(\|\nabla F(x_k)\|^2) + \frac{1}{2} \mathbb{E}(\|\nabla g_{w_k}(x_k)\nabla f_{v_k}(y_k) \|^2)
\]
(Lemma \textbf{4})
\[
\leq \mathbb{E}(\|\nabla F(x_k)\|^2) + \Theta(L^2_f)\mathbb{E}(\|y_k - g(x_k)\|^2).
\]

Take expectation on both sides of \textbf{19} and substitute $\mathbb{E}(T)$ by its upper bound:
\[
\frac{\alpha_k}{2} \|\nabla F(x_k)\|^2
\]
\[
\leq \mathbb{E}(F(x_k)) - \mathbb{E}(F(x_{k+1})) + \Theta(L^2_f \alpha_k)\mathbb{E}(\|y_k - g(x_k)\|^2) + \Theta(\alpha_k^2)
\]
(Lemma \textbf{6})
\[
\leq \mathbb{E}(F(x_k)) - \mathbb{E}(F(x_{k+1})) + L_g \Theta(L^2_f \alpha_k)\Theta(k^{-4a+4b}) + \Theta(L^2_f \alpha_k k^{-b}) + \Theta(\alpha_k^2)
\]
\[
\leq \mathbb{E}(F(x_k)) - \mathbb{E}(F(x_{k+1})) + L^2_f L_g \Theta(k^{-5a+4b}) + L^2_f \Theta(k^{-a-b}) + \Theta(k^{-2a})
\]
which suggests that
\[
\mathbb{E}(\|\nabla F(x_k)\|^2)
\]
\[
\leq 2\alpha_k^{-1} \mathbb{E}(F(x_k)) - 2\alpha_k^{-1} \mathbb{E}(F(x_{k+1})) + L^2_f L_g \Theta(k^{-4a+4b}) + L^2_f \Theta(k^{-b}) + \Theta(k^{-a})
\]
\[
\leq 2k^{-a} \mathbb{E}(F(x_k)) - 2k^{-a} \mathbb{E}(F(x_{k+1})) + L^2_f L_g \Theta(k^{-4a+4b}) + L^2_f \Theta(k^{-b}) + \Theta(k^{-a})
\]
(20)

Summarize Eq. \textbf{20} from $k = 1$ to $K$ and obtain
\[
\sum_{k=1}^{K} \mathbb{E}(\|\nabla F(x_k)\|^2) \leq 2K^{-1} \alpha_1^{-1} F(x_1) + K^{-1} \sum_{k=2}^{K} ((k+1)^a - k^a) \mathbb{E}(F(x_k))
\]
\[
+ K^{-1} \sum_{k=1}^{K} L^2_f L_g \Theta(k^{-4a+4b}) + K^{-1} L^2_f \sum_{k=1}^{K} \Theta(k^{-b}) + K^{-1} \sum_{k=1}^{K} \Theta(k^{-a})
\]
\[
\leq 2K^{-1} F(x_0) + K^{-1} \sum_{k=2}^{K} ak^{a-1} \mathbb{E}(F(x_k))
\]

16
\[ +K^{-1} \sum_{k=1}^{K} L_f^2 L_q \Theta(k^{-4\alpha} + \beta) + K^{-1} L_f^2 \sum_{k=1}^{K} \Theta(k^{-b}) + K^{-1} \sum_{k=1}^{K} \Theta(k^{-a}) \]
\[ \leq O(K^{-a-1} + L_f^2 L_q k^{-4\alpha} \log K + L_f^2 K^{-b} + K^{-a}), \]

where the second inequality uses the fact that \( h(t) = t^a \) is a concave function suggesting \((k + 1)^a \leq k^a + ak^{a-1}\), and the last inequality uses the condition \( \mathbb{E}(F(x_k)) \leq \Theta(1) \).

The optimal \( a^* = 5/9 \) and the optimal \( b^* = 4/9 \), which leads to the convergence rate \( O(K^{-4/9}) \).

\[ \square \]

**Proof to Theorem 2**

Proof. Following the line of the proof to Lemma 3, we have
\[ x_{k+1} - x_k = -\alpha_k (\nabla g_{\mu_k}(x_k) \nabla f_{\nu_k}(y_k) + s_{k+1}) \tag{21} \]

where \( s_{k+1} \in \partial R(x_{k+1}) \) is some vector in the sub-differential set of \( R(\cdot) \) at \( x_{k+1} \). Then we consider \( \|x_{k+1} - P_{X^*}(x_{k+1})\|^2 \):
\[
\|x_{k+1} - P_{X^*}(x_{k+1})\|^2 \\
\leq \|x_{k+1} - x_k + x_k - P_{X^*}(x_k)\|^2 \\
= \|x_k - P_{X^*}(x_k)\|^2 - \|x_{k+1} - x_k\|^2 + 2\langle x_{k+1} - x_k, x_{k+1} - P_{X^*}(x_k) \rangle \\
\overset{(21)}{=} \|x_k - P_{X^*}(x_k)\|^2 - \|x_{k+1} - x_k\|^2 - 2\alpha_k \langle \nabla g_{\mu_k}(x_k) \nabla f_{\nu_k}(y_k) + s_{k+1}, x_{k+1} - P_{X^*}(x_k) \rangle \\
= \|x_k - P_{X^*}(x_k)\|^2 - \|x_{k+1} - x_k\|^2 + 2\alpha_k \langle \nabla g_{\mu_k}(x_k) \nabla f_{\nu_k}(y_k), P_{X^*}(x_k) - x_{k+1} \rangle \\
+ 2\alpha_k \langle s_{k+1}, P_{X^*}(x_k) - x_{k+1} \rangle \\
\leq \|x_k - P_{X^*}(x_k)\|^2 - \|x_{k+1} - x_k\|^2 + 2\alpha_k \langle \nabla g_{\mu_k}(x_k) \nabla f_{\nu_k}(y_k), P_{X^*}(x_k) - x_{k+1} \rangle \\
+ 2\alpha_k \langle R(P_{X^*}(x_k)) - R(x_{k+1}) \rangle \quad \text{(due to the convexity of } R(\cdot)) \]
\[
\leq \|x_k - P_{X^*}(x_k)\|^2 - \|x_{k+1} - x_k\|^2 + 2\alpha_k \langle \nabla F(x_k), P_{X^*}(x_k) - x_{k+1} \rangle \\
+ 2\alpha_k \langle \nabla g_{\mu_k}(x_k) \nabla f_{\nu_k}(y_k) - \nabla F(x_k), P_{X^*}(x_k) - x_{k+1} \rangle \\
+ 2\alpha_k \langle R(P_{X^*}(x_k)) - R(x_{k+1}) \rangle \\
\overset{T_1}{=} \langle \nabla F(x_k), x_k - x_{k+1} \rangle + \langle \nabla F(x_k), -x_k + P_{X^*}(x_k) \rangle \\
\leq \|x_k - x_{k+1}\|^2 + \frac{L_F}{2} \|x_{k+1} - x_k\|^2 + F(P_{X^*}(x_k)) - F(x_k) \quad \text{(due to Assumption 4)} \\
\overset{T_2}{=} F(P_{X^*}(x_k)) - F(x_{k+1}) + \frac{L_F}{2} \|x_{k+1} - x_k\|^2 \\
\leq F(P_{X^*}(x_k)) - F(x_{k+1}) + \Theta(\alpha_k^2),
\]

where the last inequality uses Lemma 3.
where the last line is due to the inequality \((a, b) \leq \frac{1}{2\phi_k}||a||^2 + \frac{\alpha_k}{L_f}||b||^2\). For \(T_{2,1}\), we have \(\mathbb{E}(T_{2,1}) = 0\) due to Assumption I. For \(T_{2,2}\), we have

\[
T_{2,2} \leq \alpha_k \frac{L_f^2}{2\phi_k} \|\nabla f_v(x_k)\| \|y_k - g(x_k)\|^2 + \frac{2\alpha_k}{L_f^2} \|x_k - P_{\mathcal{X}_c}(x_k)\|^2 + \frac{\phi_k}{L_f^2} \|x_k - x_{k+1}\|^2.
\]

\(T_{2,3}\) can be bounded by a constant

\[
T_{2,3} \leq 2\|\nabla F(x_k)\|^2 + 2\|\nabla g_{w_k}^T \nabla f_v(y_k)\|^2 \overset{\text{Assumption I}}{\leq} \Theta(1).
\]

Take expectation on \(T_2\) and put all pieces into it:

\[
\mathbb{E}(T_2) \leq \Theta\left(\frac{L_f^2 \alpha_k}{2\phi_k}\right) \|\nabla f_v(y_k)\|^2 + \frac{\phi_k}{L_f^2} \|x_k - x_{k+1}\|^2 + \Theta(\alpha_k).
\]

Taking expectation on both sides of (22) and plugging the upper bounds of \(T_1\) and \(T_2\) into it, we obtain

\[
2\alpha_k(\mathbb{E}(H(x_{k+1})) - H^*) + \mathbb{E}(\|x_{k+1} - P_{\mathcal{X}_c}(x_{k+1})\|^2)
\leq (1 + \phi_k)\mathbb{E}(\|x_k - P_{\mathcal{X}_c}(x_k)\|^2) + \Theta(\alpha_k^3) + \Theta(L_f^2 \alpha_k^2 / \phi_k)\mathbb{E}(\|y_k - g(x_k)\|^2) + \Theta(\alpha_k^2).
\]

Using the optimally strong convexity in (7), we have

\[
(1 + 2\lambda \alpha_k)\mathbb{E}(\|x_{k+1} - P_{\mathcal{X}_c}(x_{k+1})\|^2)
\leq (1 + \phi_k)\mathbb{E}(\|x_k - P_{\mathcal{X}_c}(x_k)\|^2) + \Theta(\alpha_k^3) + \Theta(L_f^2 \alpha_k^2 / \phi_k)\mathbb{E}(\|y_k - g(x_k)\|^2) + \Theta(\alpha_k^2).
\]

It follows by dividing \(1 + 2\lambda \alpha_k\) on both sides

\[
\mathbb{E}(\|x_{k+1} - P_{\mathcal{X}_c}(x_{k+1})\|^2)
\leq \frac{1 + \phi_k}{1 + 2\lambda \alpha_k} \mathbb{E}(\|x_k - P_{\mathcal{X}_c}(x_k)\|^2) + \Theta(\alpha_k^3) + \Theta(L_f^2 \alpha_k^2 / \phi_k)\mathbb{E}(\|y_k - g(x_k)\|^2) + \Theta(\alpha_k^2).
\]

Choosing \(\phi_k = \lambda \alpha_k - 2\lambda^2 \alpha_k^2 \geq 0.5\lambda \alpha_k\) yields

\[
\mathbb{E}(\|x_{k+1} - P_{\mathcal{X}_c}(x_{k+1})\|^2)
\]

18
\begin{align*}
\leq & \quad (1 - \lambda) \mathbb{E}(\|x_k - P_{X^*}(x_k)\|^2) + \Theta(\alpha_k^2) + \Theta(L_{g}^2 \alpha_k) \mathbb{E}(\|g(x_k) - y_k\|^2) \\
\leq & \quad (1 - \lambda) \mathbb{E}(\|x_k - P_{X^*}(x_k)\|^2) + \Theta(k^{-2a}) + \Theta(L_{g}^2 L_j^2 k^{-a+b} + L_j^2 k^{-a-b}).
\end{align*}

Apply Lemma 5 to obtain the first claim in (8)

\[ \mathbb{E}(\|x_k - P_{X^*}(x_k)\|^2) \leq O(k^{-a} + L_j^2 L_g^2 k^{-a+b} + L_j^2 k^{-b}). \]

The followed specification of $a$ and $b$ can easily verified. \qed