Some generic properties of non degeneracy for critical points of functionals and applications

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1 Introduction

In these last years there have been some theorems of existence and multiplicity of solutions of the following equation

\[
\begin{aligned}
- \varepsilon^2 \Delta_g u + u &= |u|^{p-2}u & \text{in } M \\
u &\in H^1_g(M).
\end{aligned}
\]  

(1)

Here \((M, g)\) is a smooth connected compact Riemannian manifold of dimension \(n \geq 3\) embedded in \(\mathbb{R}^N\). In [1, 9, 20] it is shown that the number of solutions is influenced by the topology of \(M\). In [3, 13, 14] there are some results about the effect of the geometry of \(M\) in finding solutions, more precisely the role of the scalar curvature \(S_g\) of \((M, g)\). In these results, a type of nondegeneracy on the critical points of \(S_g\) is assumed. In Section 2 we give a generic property (see Theorem 1 and [15]) of nondegeneracy of critical points of \(S_g\) with respect to the metric \(g\) and an application to the results of the papers [3, 13, 14] to obtain some theorems of existence and multiplicity of solutions of (1).

In Section 3 we consider the Neumann problem

\[
\begin{aligned}
- \varepsilon^2 \Delta_g u + u &= |u|^{p-2}u & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

(2)

This problem has similar feature with problem (1). Indeed there are some theorems about the existence of solutions of (2) in which the mean curvature of the boundary \(\partial \Omega\) of the domain \(\Omega\) plays the same role of the scalar curvature \(S_g\) of the manifold \((M, g)\) with respect to problem (1). We show (see Theorem 6 and [11]) a generic property for critical points of the mean curvature of the boundary \(\partial \Omega\) with respect to the deformation of the domain \(\Omega\). Thus the results in [2, 5, 7, 8, 10, 21, 22] can be applied.

In Section 4 we consider a Riemannian manifold \((M, g)\) embedded in \(\mathbb{R}^N\) invariant with respect to a given involution of \(\mathbb{R}^N\) and we show some results (see theorems 7 and 8 and [6]) of genericity for non degenerate sign changing solutions.

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of the problem (4) in which the metric $g$ is considered as a parameter. By these results we can use the Morse theory and we can give an estimate of the number of solutions which change sign exactly once. To obtain our generic properties of non degeneracy of critical point the main tool is an abstract transversality theorem of of Quinn [17], Uhlenbeck [19], Saut and Temam [18] which will be recalled in the appendix.

2 Genericity of nondegeneracy for critical points of the scalar curvature for a Riemannian manifold and applications

Let $M$ be a connected compact $C^\infty$ manifold of dimension $N \geq 2$, without boundary. Let $\mathcal{M}^k$ be the set of all $C^k$ Riemannian metrics on $M$. Any $g \in \mathcal{M}^k$ determines the scalar curvature $S_g$ of $(M,g)$. Our goal is to prove that for a generic Riemannian metric $g$ the critical points of the scalar curvature $S_g$ are nondegenerate. More precisely we can prove the following result (see [15])

**Theorem 1.** The set

$$A = \{g \in \mathcal{M}^k : \text{all the critical points of } S_g \text{ are nondegenerate} \}$$

is an open dense subset of $\mathcal{M}^k$. Here $k \geq 3$.

In the following we denote by $\mathcal{S}^k$ the space of all $C^k$ symmetric covariant 2-tensors on $M$. $\mathcal{S}^k$ is a Banach space equipped with the norm $\| \cdot \|_k$ defined in the following way. We fix a finite covering $\{V_\alpha\}_{\alpha \in L}$ of $M$ such that the closure of any $V_\alpha$ is contained by $U_\alpha$, where $(U_\alpha, \psi_\alpha)$ is an open coordinate neighborhood. If $h \in \mathcal{S}^k$, denoting $h_{ij}$ the components of $h$ with respect to local coordinates $(x_1, \ldots, x_n)$ on $V_\alpha$, we define

$$\|h\|_k = \sum_{\alpha \in L} \sum_{|\beta| \leq k} \sum_{i,j=1}^n \sup_{\psi_\alpha(V_\alpha)} \left| \frac{\partial^\beta h_{ij}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right|. \quad (3)$$

The set $\mathcal{M}^k$ of all $C^k$ Riemannian metrics on $M$ is an open subset of $\mathcal{S}^k$.

We can apply Theorem 1 to study the following problem

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2} u & \text{in } M \\ u \in H^1_g(M) \end{cases} \quad (4)$$

where $p > 2$ if $N = 2$, $2 < p < \frac{2N}{N-2}$ if $N \geq 3$. Here $H^1_g(M)$ is the completion of $C^\infty$ with respect to

$$\|u\|_g^2 = \int_M (|\nabla_g u|^2 + u^2) d\mu_g.$$

There are some recent results on the effect of the geometry of $(M,g)$ on the number of solutions of (4). We will see that the scalar curvature $S_g$ relative to the metric $g$ is the geometric property which influences the number of solutions. Indeed we have the following results about the role of the scalar curvature.
Theorem 2. (See [14]) For any $C^1$-stable critical point $\bar{q}$ of scalar curvature $S_g$, there exists a positive single peak solution $u_\varepsilon$ of (4) such that the peak point $q_\varepsilon$ approaches point $\bar{q}$ as $\varepsilon$ goes to zero.

Theorem 3. (See [13]) Assume that the scalar curvature $S_g$ has $k \geq 2$, $C^1$-stable critical points: $q_1, q_2, \ldots, q_k$. Then, choosing $\varepsilon$ small enough, for any integer $j \leq k$, the problem (4) has a solution $u_\varepsilon$ with $j$ positive peaks $q_\varepsilon^1, \ldots, q_\varepsilon^j$ and $k-j$ negative peaks $q_\varepsilon^{j+1}, \ldots, q_\varepsilon^k$ such that $d_g(q_\varepsilon^i, q_i) \to 0$ as $\varepsilon \to 0$.

We now recall the definition of $C^1$-stable critical point.

Definition 4. Let $f \in C^1(M, \mathbb{R})$. The point $\bar{q}$ is a $C^1$-stable critical point of $f$ if $\bar{q}$ is a critical point such that, for any $\mu > 0$ there exists $\delta$ for which any $h \in C^1(M, \mathbb{R})$ with $\max_{d_g(x, \bar{q}) < \mu} |f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)| \leq \delta$ has at least one critical point $q$ with $d_g(q, \bar{q}) < \mu$.

Theorem 5. (See [3]) Let $\bar{q} \in M$ be an isolated local minimum point of $S_g$. For each positive integer $k$, choosing $\varepsilon$ small enough, there exists a positive $k$-peaked solution $u_\varepsilon$ of (4) such that the $k$-peaks $q_\varepsilon^1, \ldots, q_\varepsilon^k$ collapse to $\bar{q}$, that is $d_g(q_\varepsilon^i, \bar{q}) \to 0$ as $\varepsilon \to 0$ for $i = 1, \ldots, k$.

By Theorem 4 for a generic metric $g$, all the critical points of $S_g$ are nondegenerate, then $C^1$-stable, isolated and in a finite number. If $\nu$ is the number of critical points of $S_g$, by Theorem 2 we have: $\nu$ positive solutions with one peak, $\frac{\nu(\nu-1)}{2}$ solutions with two positive peaks, ..., one solution with $\nu$ positive peaks. Moreover, by Theorem 3 the problem (4) has some sign changing solutions: for example $\frac{\nu(\nu-1)}{2}$ pairs $(u_\varepsilon, -u_\varepsilon)$ of solutions with one positive and one negative peak. Finally, since the global minimum point of $S_g$ is isolated, the number of positive solutions of (4) goes to infinity as $\varepsilon$ goes to zero.

3 Genericity of nondegeneracy for critical points for the mean curvature of the boundary of a domain and applications

In the following we denote by $E^k$ the vector space of all $C^k$ maps $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$
\|\Psi\|_k = \sup_{x \in \mathbb{R}^N} \max_{0 \leq |\alpha| \leq k} \max_{i = 1, \ldots, N} \left| \frac{\partial^\alpha \Psi_i(x)}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \right| < +\infty.
$$

$E^k$ is a Banach space equipped with the norm $\| \cdot \|_k$. Let $B_\rho$ the ball in $E^k$ centered at zero with radius $\rho$. Fixed an open bounded subset $\Omega \subset \mathbb{R}^N$ of class
$C^k$, we have that the map

$$I + \Psi : \bar{\Omega} \to (I + \Psi)\bar{\Omega}$$

is a diffeomorphism of class $C^k$ when $\Psi \in \mathcal{B}_\rho$ with $\rho$ small enough. We are interested in studying the nondegeneracy of the critical points of the mean curvature of the boundary of the domain $(I + \Psi)\Omega$ with respect to the parameter $\Psi$. More precisely we can prove the following result (see [11]).

**Theorem 6.** Given a domain $\Omega \subset \mathbb{R}^N$ of class $C^k$ with $k \geq 3$, the set

$$A = \left\{ \psi \in \mathcal{B}_\rho \subset E^k : \text{all the critical points of the mean curvature of } (I + \Psi)\Omega \text{ are not degenerate} \right\}$$

is an open dense subset of $\mathcal{B}_\rho$.

We can apply Theorem 6 to study the following Neumann problem

$$\begin{cases}
-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}$$

(5)

Here $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$, $p > 2$ if $N = 2$.

For Neumann problem (5) we have some results about existence of solutions in which the mean curvature $H$ of the boundary $\partial \Omega$ of the domain $\Omega$ plays the same role of the scalar curvature $S_g$ of the manifold $(M, g)$ in the problem (4). We recall these results. Wei [21] and Del Pino, Felmer, Wei [5] proved that any $C^1$ stable critical point of the mean curvature of $\partial \Omega$ gives a single peaked solution. Gui [7], Li [10], Wei and Winter [22] proved the existence of multipeak solutions if the mean curvature $H$ of $\partial \Omega$ has multiple $C^1$ stable critical points. Dancer Yan [2] and Gui, Wei and Winter [8] proved the existence of clustered positive solutions such that the peaks collapse to an isolated local minimum point of the mean curvature of $\partial \Omega$. As far as it concerns the existence of sign changing solutions there are results by Noussair and Wei [16], Micheletti and Pistoia [12], Wei and Weth [23], D’Aprile and Pistoia [3]. All these results require a sort of non degeneracy of critical points of the mean curvature. By Theorem 6 for a generic deformation $\Psi \in \mathcal{B}_\rho$, all the critical points of the mean curvature of the boundary of the domain $(I + \Psi)\Omega$ are non degenerate, thus we can apply all the previous results.

## 4 Generic properties for singularly perturbed non-linear elliptic problems on symmetric Riemannian manifolds

We are now interested in studying the nondegeneracy of changing sign solutions when the Riemannian manifold $(M, g)$ is symmetric. We consider the problem

$$\begin{cases}
-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u & u \in H^1_g(M) \\
u(\tau x) = -u(x) & \forall x \in M
\end{cases}$$

(6)
where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal linear transformation such that $\tau \neq \text{Id}$ and $\tau^2 = \text{Id}$, $\text{Id}$ being the identity on $\mathbb{R}^N$. Here the compact connected Riemannian manifold $(M, g)$ of dimension $n \geq 2$ is a regular submanifold of $\mathbb{R}^N$ invariant with respect to $\tau$. In the following we consider the space $H^*_g = \{ u \in H^1_g : \tau^* u = u \}$, where the linear operator $\tau^* : H^1 \to H^1$ is $\tau^* u = -u(\tau x)$.

We obtain the following results (see [3]) about the nondegeneracy of changing sign solutions of (6) with respect the symmetric metric $g$ considered as a parameter.

**Theorem 7.** Given the metric $g_0$ on $M$ and the positive number $\varepsilon_0$. The set

$$
D = \left\{ h \in \mathcal{B}_\rho : \text{any } u \in H^*_g, \text{ solution of } -\varepsilon_0^2 \Delta g_0 + h u + u = |u|^{p-2} u \text{ is not degenerate} \right\}
$$

is a residual subset of $\mathcal{B}_\rho$.

**Theorem 8.** Given the metric $g_0$ on $M$ and the positive number $\varepsilon_0$. If there exists $\mu > m^*_{\varepsilon_0, g_0}$ not a critical value of the functional $J_{\varepsilon_0, g_0}$, then the set

$$
D = \left\{ h \in \mathcal{B}_\rho : \text{any } u \in H^*_g, \text{ solution of } -\varepsilon_0^2 \Delta g_0 + h u + u = |u|^{p-2} u \text{ is not degenerate} \right\}
$$

is an open subset of $\mathcal{B}_\rho$.

Here we set

$$
J_{\varepsilon_0, g_0}(u) = \frac{1}{2} \int_M \left( \varepsilon_0^2 |\nabla g_0 u|^2 + u^2 \right) d\mu_{g_0} - \frac{1}{p} \int_M |u|^p d\mu_{g_0}
$$

$$
N^*_\varepsilon = \left\{ u \in H^*_g \setminus \{0\} : J'_{\varepsilon_0, g_0}(u) [u] = 0 \right\}
$$

$$
m^*_{\varepsilon_0, g_0} = \inf_{N^*_\varepsilon} J_{\varepsilon_0, g_0}.
$$

$\mathcal{B}_\rho$ is the ball centered at 0 with radius $\rho$, with $\rho$ small enough, in the Banach space $\mathcal{S}^k$, $k \geq 3$ of all $C^k$ symmetric covariant 2-tensors $h(x)$ on $M$ such that $h(x) = h(\tau x)$ for $x \in M$ with $\| \cdot \|_k$ defined in (3).

If we choose the involution $\tau = -\text{Id}$ and the symmetric manifold such that $0 \notin M$, using the previous theorems and the Morse theory we obtain a generic result on the number of solutions which change sign exactly once.

**Theorem 9.** Given the metric $g_0$ and $\varepsilon_0 > 0$ the set

$$
D = \left\{ h \in \mathcal{B}_\rho : \text{the equation } -\varepsilon_0^2 \Delta g_0 + h u + u = |u|^{p-2} u \text{ has at least } P_1(M/G) \text{ pairs } (u, -u) \text{ of nontrivial solutions which change sign exactly once} \right\}
$$

(7)

is a residual subset of $\mathcal{B}_\rho$.

Here $P_1(M/G)$ is the Poincarè polynomial of $M/G$ and $G = \{ \text{Id}, -\text{Id} \}$

**Theorem 10.** Given $g_0$ and $\varepsilon_0 > 0$, if there exists $\mu > m^*_{\varepsilon_0, g_0}$ not a critical value of the functional $J_{\varepsilon_0, g_0}$, then the set $D$ defined in (7) is an open dense subset of $\mathcal{B}_\rho$. 

5
5 Appendix: A Transversality Theorem

We recall transversality results which have been used to prove our genericity theorems. For the proofs we refer to [17, 18, 19].

**Theorem 11.** Let $X,Y,Z$ be three real Banach spaces and let $U \subset X, V \subset Y$ be two open subsets. Let $F$ be a $C^1$ map from $V \times U$ into $Z$ such that

(i) For any $y \in V$, $F(y, \cdot) : x \to F(y,x)$ is a Fredholm map of index 0.

(ii) 0 is a regular value of $F$, that is $F'(y_0, x_0) : Y \times X \to Z$ is onto at any point $(y_0, x_0)$ such that $F(y_0, x_0) = 0$.

(iii) The map $\pi \circ i : F^{-1}(0) \to Y$ is proper, where $i$ is the canonical embedding form $F^{-1}(0)$ into $Y \times X$ and $\pi$ is the projection from $Y \times X$ onto $Y$.

Then the set

$$\theta = \{ y \in V : 0 \text{ is a regular value of } F(y, \cdot) \}$$

is a dense open subset of $V$.

**Theorem 12.** If $F$ satisfies (i) and (ii) and

(iv) The map $\pi \circ i$ is $\sigma$-proper, that is $F^{-1}(0) = \bigcup_{s=1}^{+\infty} C_s$ where $C_s$ is a closed set and the restriction $\pi \circ i|_{C_s}$ is proper for any $s$

then the set $\theta$ is a residual subset of $V$, that is $V \setminus \theta$ is a countable union of closed subsets without interior points.

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