Weil classes on abelian varieties
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1. Introduction. Let $X$ be a smooth projective variety over $\mathbb{C}$. We write $B^r(X) = \bigoplus_i B^i(X)$ for its Hodge ring (so $B^i(X) \subseteq H^{2i}(X, \mathbb{Q})$ is the subspace of Hodge classes). We call a class $c \in B^r(X)$ decomposable if it lies in the subalgebra $D^r(X) \subseteq B^r(X)$ generated by divisor classes. The non-decomposable Hodge classes are called exceptional classes.

As a consequence of the Lefschetz theorem on $(1,1)$ classes, the decomposable classes are algebraic; in particular, if $B^r(X) = D^r(X)$ then the Hodge conjecture for $X$ is true. Thus one is naturally led to the question whether there exist varieties $X$ for which $D^r(X) \subsetneq B^r(X)$.

That this question has a positive answer was shown in [8], where an example (due to Mumford) was given of an abelian fourfold $X$ of CM-type for which $D^2(X) \subsetneq B^2(X)$. A few years later, Weil [14] showed that the essential ingredient in Mumford’s example is the fact that $\text{End}^0(X)$ contains an imaginary quadratic subfield $k$ which acts with multiplicities $n_\sigma = n_\sigma' = 2$ on the tangent space of $X$. To the action of this field $k$ one associates a 2-dimensional subspace $W_k \subseteq H^4(X, \mathbb{Q})$ which, grace to the assumption $n_\sigma = n_\sigma'$, consists of Hodge classes. Moreover, Weil showed that for “generic” abelian fourfolds with complex multiplication by $k$ (subject to the condition $n_\sigma = n_\sigma'$), the space $W_k$ consists of exceptional Hodge classes. Later, Shioda [11] constructed an exceptional 2-dimensional algebraic Weil class on a non-simple abelian fourfold of Fermat type.

In this paper, we want to study to what extent Weil’s method to construct exceptional Hodge classes can be generalized. To make this more explicit, consider an abelian variety $X$ of dimension $g \geq 1$, and suppose $F$ is a subfield of $\text{End}^0(X)$, with $1 \in F$ acting as the identity on $X$. Write $V_X = H^1(X, \mathbb{Q})$ and let $r = 2g/[F : \mathbb{Q}]$. The 1-dimensional $F$-vector space

$$W_F = W_F(X) := \bigwedge_F^r V_X$$

can be identified in a natural manner with a subspace of $H^r(X, \mathbb{Q})$, see Lemma 12 (i). We call $W_F$ the space of Weil classes with respect to $F$.

The main questions that we are interested in are

**Q1:** under what conditions on $F$ does $W_F$ contain, or even consist of, Hodge classes?

**Q2:** if $W_F$ contains Hodge classes, under what conditions on $F$ are these exceptional?
Similar questions can be asked for Tate classes (in case $X$ is defined over a number field). We postpone this to section 16.

2. The multiplicative group $F^*$ acts on $\bigoplus_i H^i(X, \mathbb{Q}) \supset B^r(X) \supset D^r(X)$ and on the subspace $W_F \subseteq H^r(X, \mathbb{Q})$. If the latter action is given by $\rho: F^* \to \text{GL}(W_F)$ then its relation to the natural structure of $F$-vector space on $W_F$ is given by $\rho(f)(w) = f^r \cdot w$ for all $f \in F^*$, $w \in W$. Since $\dim_F(W_F) = 1$, it readily follows that either all elements of $W_F$ are Hodge classes, or $0 \in W_F$ is the only Hodge class, and in the first case, either $W_F \setminus \{0\}$ consists entirely of exceptional classes, or none of the classes in $W_F$ is exceptional.

3. Write $\Sigma_F$ for the set of embeddings $F \to \mathbb{C}$, and for $\sigma \in \Sigma_F$, let $\sigma'$ denote its complex conjugate. The action of $F$ on $V_X$ gives a decomposition of $V_{\mathbb{C}} = V_X \otimes_{\mathbb{Q}} \mathbb{C}$ as

$$V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma_F} V_{\mathbb{C}, \sigma} = \bigoplus_{\sigma \in \Sigma_F} (V_{\mathbb{C}, \sigma}^{1,0} \oplus V_{\mathbb{C}, \sigma}^{0,1}).$$

The dimension $n_{\sigma}$ of $V_{\mathbb{C}, \sigma}^{1,0}$ is called the multiplicity of $\sigma$ on the tangent space of $X$; we have $n_{\sigma} + n_{\sigma'} = 2g/[F : \mathbb{Q}]$ for all $\sigma \in \Sigma_F$. Similar to the computation in the proof of [2, Prop. 4.4] (see also [3, Lemma 2.8]), we can write

$$W_F \otimes \mathbb{C} = (\bigwedge_F V_X) \otimes \mathbb{C} = \bigwedge_{\sigma \in \Sigma_F} V_{\mathbb{C}, \sigma} = \bigoplus_{\sigma \in \Sigma_F} (\bigwedge_{\mathbb{C}, \sigma}^{1,0} \otimes \bigwedge_{\mathbb{C}, \sigma}^{0,1}).$$

In view of the remarks in section 2 we obtain the following answer to question Q1.

4. **Criterion.** If $n_{\sigma} = n_{\sigma'}$ for all $\sigma \in \Sigma_F$ then $W_F$ consists entirely of Hodge classes; if $n_{\sigma} \neq n_{\sigma'}$ for some $\sigma \in \Sigma_F$ then the zero class is the only Hodge class in $W_F$.

5. **Remark.** In the case of an imaginary quadratic field $F$ this criterion is essentially due to Weil [14]; see also Ribet [3]. The case of arbitrary CM-fields $F$ was also treated in [12].

6. **Remark.** If all simple factors of $X$ are of type I, II or III in the Albert classification, then every subfield $F \subseteq \text{End}^0(X)$ satisfies the condition that $n_{\sigma} = n_{\sigma'}$ for all $\sigma \in \Sigma_F$. We can see this as follows: we have an inclusion $\text{Hg}(X) \subset \text{GL}_F(V_X)$ and $\text{Hg}(X)$ acts on $W_F$ through the $F$-linear determinant $\text{det}_F: \text{GL}_F(V_X) \to \text{Res}_{F/\mathbb{Q}}(\text{G}_m,F)$. If $X$ has no factors of type IV then the Hodge group $\text{Hg}(X)$ is semi-simple (see [1, Cor. 1.2.2]), hence contained in $\text{SL}_F(V_X)$, which means that $W_F$ consists of Hodge classes.
7. Up to isogeny we can decompose $X$ as

$$X \sim Y_1^{m_1} \times \cdots \times Y_k^{m_k},$$

where $Y_1, \ldots, Y_k$ ($k \in \mathbb{Z}_{\geq 1}$) are simple, mutually non-isogenous, abelian varieties, and $m_1, \ldots, m_k \in \mathbb{Z}_{\geq 1}$. The condition that $1 \in F$ acts as the identity implies that $F$ acts on each factor $Y_i^{m_i}$, and that the action of $F$ on $X$ is the “diagonal action” w.r.t. the decomposition (1).

Write $r_i = 2m_i \dim(Y_i)/[F : \mathbb{Q}]$, so that $r = r_1 + \cdots + r_k$. For each factor $Y_i^{m_i}$ we obtain a 1-dimensional $F$-subspace $W_F(Y_i^{m_i}) \subset H^r(Y_i^{m_i}, \mathbb{Q})$. We claim that the space $W_F(X)$ can be identified with the tensor product $W_F(Y_1^{m_1}) \otimes_F \cdots \otimes_F W_F(Y_k^{m_k})$, considered as a subspace (sic) of the Künneth component

$$H^r(Y_1^{m_1}, \mathbb{Q}) \otimes \mathbb{Q} \cdots \otimes \mathbb{Q} H^r(Y_k^{m_k}, \mathbb{Q}) \subset H^r(X, \mathbb{Q}).$$

The verification of this statement, which we leave to the reader, is a matter of linear algebra. Some of our identifications may seem a little unnatural; this is caused by our desire to view $W_F(X)$ as a subspace of $H^r(X, \mathbb{Q})$, rather than a quotient (cf. the proof of Lemma 12 (i.).)

We conclude from this that, if $W_F(X)$ consists of Hodge classes, then

$$W_F(X) \text{ consists of decomposable Hodge classes} \iff \text{each of the spaces } W_F(Y_i^{m_i}) \text{ consists of decomposable Hodge classes}.$$  

8. Example. Let $X = Y_1 \times Y_2$ where the ratios $2 \dim(Y_1)/[F : \mathbb{Q}]$ and $2 \dim(Y_2)/[F : \mathbb{Q}]$ are odd. Then non-zero elements of $W_F(Y_1)$ and $W_F(Y_2)$ are not Hodge classes and therefore all non-zero elements of $W_F(X)$ are exceptional Hodge classes. Shioda’s example mentioned above is of this type. See also [13].

9. We may from now on restrict our attention to the case that $k = 1$. Since everything only depends on $X$ up to isogeny, we may even assume that $X = Y^m$ for some $m \geq 1$, where $Y$ is simple. Let $D = \text{End}^0(Y)$, let $E$ be the center of $D$, and let $E_0$ be the maximal totally real subfield of $E$. We write $e_0 = [E_0 : \mathbb{Q}]$, $e = [E : \mathbb{Q}]$, $d^2 = \dim_{\mathbb{Q}}(D)$, and we say that $X$ and $Y$ are of type I, II, III or IV if the algebra $D$ is of the corresponding type in the Albert classification.

10. Choose a polarization $\lambda$ of $X$, and write $\alpha \mapsto \alpha^\dagger$ for the associated Rosati involution of $\text{End}^0(X) \cong M_m(D)$. Let $S_{\lambda} \subseteq \text{End}^0(X)$ be the set of $\dagger$-symmetric elements. We define the algebra $B \subseteq \text{End}^0(X)$ as the $\mathbb{Q}$-subalgebra generated by $S_{\lambda}$. If $\dagger'$ is the Rosati involution
associated to another polarization, then †′ is conjugated to † by an element of $S_\lambda$. It follows that the algebra $B$ does not depend on the choice of the polarization $\lambda$.

For all possible types in the Albert classification one can determine the algebra $B$ and its center $K_B$. The results are listed in Table 1.

| Type | $m = 1$ | $m \geq 2$ |
|------|---------|-----------|
| I    | $K_B = E$, $B = M_m(E)$ |          |
| II   | $K_B = E$, $B = M_m(D)$ |          |
| III  | $K_B = B = E$, $K_B = E$, $B = M_m(D)$ |          |
| IV, $d = 1$ | $K_B = B = E_0$, $K_B = E$, $B = M_m(E)$ |          |
| IV, $d \geq 2$ | $K_B = E$, $B = M_m(D)$ |          |

Table 1

Let $\varphi = \varphi_X: V \times V \to \mathbb{Q}$ be the nondegenerate alternating bilinear form associated to $\lambda$. We define the algebraic group $G_{\text{div}}(X) \subseteq \text{Sp}(V, \varphi)$ as the centralizer of $B$ in $\text{Sp}(V, \varphi)$. More precisely,

$$G_{\text{div}}(X) := \text{GL}_B(V) \cap \text{Sp}(V, \varphi).$$

Of course, the main motivation for introducing this group $G_{\text{div}}(X)$ is the fact that it is the largest algebraic subgroup of $\text{GL}(V)$ defined over $\mathbb{Q}$ which leaves invariant all divisor classes in $H^2(X, \mathbb{Q}) = \bigwedge^2 V$. In fact, the divisor classes, viewed as alternating bilinear forms on $V$, are precisely the forms

$$\varphi_s: (v_1, v_2) \mapsto \varphi(s \cdot v_1, v_2)$$

for $s \in S_\lambda$.

The group $G_{\text{div}}(X)$ does not depend on the choice of $\lambda$. Without loss of generality we may therefore assume that there is a polarization $\mu$ on $Y$ such that $\lambda$ is the product polarization $\mu^m$ on $X = Y^m$. We write $\varphi_Y$ for the alternating form on $V_Y$ associated to $\mu$ and $d \mapsto d^*$ for the Rosati involution on $D = \text{End}^0(Y)$. (With $\text{End}^0(X) = M_m(D)$, the Rosati involutions $\ast$ and $\dagger$ are related by $\alpha^\dagger = (\alpha^*_j)^i$ for $\alpha = (\alpha_{ij}) \in M_m(D)$.)

From Table 1 we see that in all cases there is a simple $\mathbb{Q}$-subalgebra $\Delta \subseteq D = \text{End}^0(Y)$ such that $\ast$ restricts to a positive involution on $\Delta$, and such that $G_{\text{div}}(X)$ is the centralizer of $\Delta$ in $\text{Sp}(V_Y, \varphi_Y)$, embedded diagonally into $\text{Sp}(V, \varphi)$. (In fact, if $m \geq 2$ or if $X$ is of type I or II, then we simply have $\Delta = D$. If $m = 1$ then $\Delta = B$.) This means that $G_{\text{div}}(X)$ is an algebraic group of the type that is usually studied in the context of moduli problems of
PEL-type. We refer the reader to [5], [4] or [7] for more information; here we only give a description of \( G_{\text{div}}(X) \otimes \mathbb{C} \).

Write \( \Sigma_{E_0} \) for the set of complex (in fact real) embeddings of \( E_0 \). We have

\[
\text{Sp}(V_Y, \varphi_Y) \otimes \mathbb{C} = \prod_{\tau \in \Sigma_{E_0}} \text{Sp}(V_{\tau Y}, \varphi_{\tau Y})
\]

and

\[
\Delta \otimes \mathbb{C} = \prod_{\tau \in \Sigma_{E_0}} \Delta_{\mathbb{C}}^{(\tau)},
\]

where \( \Delta_{\mathbb{C}}^{(\tau)} \) is a semi-simple \( \mathbb{C} \)-subalgebra of \( \text{End}(V_{\tau Y}) \). We thus see that \( G_{\text{div}}(X) \otimes \mathbb{C} \) splits as the direct product of \( e_0 \) factors \( G_{\text{div}}^{(\tau)} \). Writing \( k = \dim_{\Delta}(V_Y) \), we have the following description of these factors and their representations \( V_{\tau Y}^{(\tau)} \). In each case \( St \) denotes the standard representation of the group in question.

| Type | \( k \) | \( G_{\text{div}}^{(\tau)} \) and \( V_{\tau Y}^{(\tau)} \) |
|------|------|-------------------|
| I    | \( 2g/me \) | \( G_{\text{div}}^{(\tau)} \cong \text{Sp}_{2k} \), \( V_{\tau Y}^{(\tau)} \cong St \) |
| II   | \( g/2me \) | \( G_{\text{div}}^{(\tau)} \cong \text{Sp}_{2k} \), \( V_{\tau Y}^{(\tau)} \cong St \oplus St \) |
| III, \( m = 1 \) | \( 2g/e \) | \( G_{\text{div}}^{(\tau)} \cong \text{Sp}_{2k} \), \( V_{\tau Y}^{(\tau)} \cong St \) |
| III, \( m \geq 2 \) | \( g/2me \) | \( G_{\text{div}}^{(\tau)} \cong \text{O}_{2k} \), \( V_{\tau Y}^{(\tau)} \cong St \oplus St \) |
| IV, \( d = 1, m = 1 \) | \( 2g/e_0 \) | \( G_{\text{div}}^{(\tau)} \cong \text{Sp}_{2k} \), \( V_{\tau Y}^{(\tau)} \cong St \) |
| IV, \( d \geq 2 \) or \( m \geq 2 \) | \( 2g/med^2 \) | \( G_{\text{div}}^{(\tau)} \cong \text{GL}_{dk} \), \( V_{\tau Y}^{(\tau)} \cong St \oplus St^\vee \) |

Table 2

11. Lemma. (i) The center of \( G_{\text{div}}(X) \) is the group \( U_{K_B} \) given by

\[
U_{K_B}(R) = \{ a \in (K_B \otimes \mathbb{Q})^* \mid aa^t = 1 \}. 
\]

For \( X \) of type IV with either \( d \geq 2 \) or \( m \geq 2 \) this is a connected torus of rank \( e_0 \); in all other cases it is finite.

(ii) If \( X \) is not of type III with \( m \geq 2 \), then \( G_{\text{div}}(X) \) is geometrically connected; for \( X \) of type III and \( m \geq 2 \), the group \( \pi_0(G_{\text{div}}(X)) \) has (geometrically) order \( 2^{e_0} \).

(iii) \( \text{End}(V_X)^{G_{\text{div}}(X)} = B; \ (\bigwedge^2 V_X)^{G_{\text{div}}(X)} = B^1(X), \) and \( (\oplus_i \bigwedge^i V_X)^{G_{\text{div}}(X)} = \mathcal{D}^*(X) \).

Proof. To prove this, we can first extend scalars to \( \mathbb{C} \), and (i) and (ii) then readily follow from Table 3. The last statement is based on results from classical invariant theory; both the statement and its proof are in fact easy variants of [5, Lemma 3.6] (see also [4]). \( \square \)

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12. Lemma. (i) The space $W_F = \bigwedge_F^r V$ can naturally be identified with a subspace of $\bigwedge_Q^r V$.
(ii) If $g \in \text{GL}_Q(V)$ acts as the identity on $W_F$ then $g$ is $F$-linear, hence $g \in \text{SL}_F(V)$.

Proof. There is a canonical isomorphism $\text{Tr}_{F/Q} : \text{Hom}_F(V_X, F) \cong \text{Hom}_Q(V_X, Q)$ and we simply write $V_X^\vee$ for this space. There is a canonical surjective map $\bigwedge_Q^r V_X^\vee \twoheadrightarrow \bigwedge_F^r V_X^\vee$. It induces an injective linear map

$$\text{Alt}_F^r(V_X) \cong \text{Hom}_F(\bigwedge_F^r V_X^\vee, F) \cong \text{Hom}_Q(\bigwedge_F^r V_X^\vee, Q) \hookrightarrow \text{Hom}_Q(\bigwedge_Q^r V_X^\vee, Q) \cong \text{Alt}_Q^r(V_X),$$

where all isomorphisms are canonical. Since our fields are of characteristic 0, there are identifications $\bigwedge_F^r V_X \cong \text{Alt}_F^r(V_X)$ and $\bigwedge_Q^r V_X \cong \text{Alt}_Q^r(V_X)$, and statement (i) follows.

(ii) Choose an $F$-basis for $V_X$. This gives an isomorphism $F \cong \text{Hom}_F(\bigwedge_F^r V_X^\vee, F)$ by sending $f \in F$ to the functional $t_1 \wedge_F \ldots \wedge_F t_r \mapsto f \cdot \text{det}_F(t_1, \ldots, t_r)$. Suppose $g \in \text{GL}_Q(V_X)$ acts trivially on $W_F$. This means that for all $f \in F$ and all $r$-tuples $t_1, \ldots, t_r \in V_X^\vee$ we have the identity

$$\text{Tr}_{F/Q}(f \cdot \text{det}_F(t_1, \ldots, t_r)) = \text{Tr}_{F/Q}(f \cdot \text{det}_F(g \cdot t_1, \ldots, g \cdot t_r)).$$

The trace form being non-degenerate it follows that

$$\text{det}_F(t_1, \ldots, t_r) = \text{det}_F(g \cdot t_1, \ldots, g \cdot t_r) \quad \text{for all } t_1, \ldots, t_r \in V_X^\vee.$$

For $f \in F$ this gives the identity

$$\text{det}_F(g \cdot (ft_1), g \cdot t_2, \ldots, g \cdot t_r) = f \cdot \text{det}_F(g \cdot t_1, \ldots, g \cdot t_r) = \text{det}_F(f \cdot t_1, g \cdot t_2, \ldots, g \cdot t_r),$$

and (ii) readily follows from this. \qed

13. Criterion. Suppose $X$ is isogenous to a power $Y^m$ of a simple abelian variety $Y$ (see section \[\mathcal{H}\]). Suppose $F \hookrightarrow \text{End}^0(X)$ is a subfield such that $W_F = \bigwedge^r V_X$ consists of Hodge classes (see section \[\mathcal{H}\] and recall that this assumption implies that $r = \text{dim}_F(V_X)$ is even). Then either all classes in $W_F$ are decomposable, or all non-zero classes in $W_F$ are exceptional; this last possibility occurs precisely in the following cases:

- $Y$ is of Type III, $m = 1$ and $F \subsetneq E$,
- $Y$ is of Type III, $m \geq 2$ and the integer $2m \cdot [E : Q]/[F : Q]$ is odd,
- $Y$ is of Type IV, $d = 1$, $m = 1$ and $F \subsetneq E_0$,
- $Y$ is of Type IV with $d \geq 2$ or $m \geq 2$ and

the map $\theta : E_+ \rightarrow \text{End}_F(V_X) \xrightarrow{\text{Tr}_F} F$ is non-zero. \hfill (2)
Proof. First assume that $X$ is either of type I, II or of type III with $m = 1$, or that $X$ is of type IV with $d = 1$ and $m = 1$. We claim that, in these cases, $G_{\text{div}}(X)$ acts as the identity on $W_F$ if and only if $F \subseteq B$. In the “only if” direction, this follows from Lemmas 12 (ii) and 11 (iii). Conversely, suppose that $F \subseteq B$, so that $G_{\text{div}}(X) \subseteq \text{GL}_F(V_X)$. In the cases we are considering, the group $G_{\text{div}}(X)$ is connected and semi-simple, so $G_{\text{div}}(X) \subseteq \text{SL}_F(V_X)$, hence $G_{\text{div}}(X)$ acts trivially on $W_F$.

Next assume that $X$ is of type IV with either $m \geq 2$ or $d \geq 2$. We have $F \subseteq B = \text{End}^0(X)$. Since in this case $G_{\text{div}}(X)$ is connected (see 11), it acts trivially on $W_F$ if and only if the composition

$$U_E = Z(G_{\text{div}}) \subset G_{\text{div}}(X) \hookrightarrow \text{GL}_F(V_Y) \xrightarrow{\text{det}_F} F^*$$

is trivial. The torus $U_E$ being connected, this is the case if and only if the induced map on Lie algebras

$$\theta: E_- \longrightarrow \text{End}_F(V_X) \xrightarrow{\text{Tr}_F} F$$

is zero.

Finally we consider the case where $X$ is of type III with $m \geq 2$. We have $B = M_m(D)$, hence $F \subseteq B$. The centralizer $C$ of $B$ in $\text{End}_E(V_X)$ is isomorphic to $M_k(D')$, where $D'$ is the opposite algebra of $D$ and $k = \dim_E(V_X)/(m \cdot \dim_E(D)) = g/2me$. Let $\text{Nrd}: C \rightarrow E$ denote the reduced norm.

Let us first assume that $F$ contains the field $E$. The restriction of the $F$-linear determinant map $\text{det}_F: \text{End}_F(V_X) \rightarrow F$ to the subalgebra $C \subseteq \text{End}_F(V_X)$ coincides with a power $\text{Nrd}^q$ of the reduced norm, and comparing the degrees we find that $q = \dim_F(V_X)/2k = 2m/[F : E]$.

Let us now show that, in this case, $W_F$ consists of decomposable classes if and only if $2m/[F : E]$ is even. By the same arguments as in the type I and II case, we see that the connected component of the identity $G^0_{\text{div}}$ acts trivially on $W_F$. Next consider an embedding $\tau: E \rightarrow \mathbb{C}$, and an element $g \in G^0_{\text{div}}(\mathbb{C})$ which is not in the connected component $(G^0_{\text{div}})^0$. The $\mathbb{C}$-linear extension of the reduced norm to $C \otimes_{E, \tau} \mathbb{C} \cong M_{2k}(\mathbb{C})$ is nothing but the $\mathbb{C}$-linear determinant, so $\text{Nrd}_C(g) = \text{det}_C(g) = -1$. (Recall that $G^0_{\text{div}} \cong O_{2k}$.) It then follows from the preceding remarks that $g$ acts on $W_F \otimes \mathbb{C}$ as multiplication by $(-1)^{2m/[F: E]}$, which proves the assertion.

Next let us do the general case (i.e., no longer assuming that $E \subseteq F$). The compositum $EF \subseteq \text{End}^0(X)$ is a product of fields, say $EF = K_1 \times \cdots \times K_t$. Correspondingly, we can
decompose $V_X$ as a direct sum $V_1 \oplus \cdots \oplus V_t$, and we have
\[ W_F = \det_F(V_X) = \det_F(V_1) \otimes_F \cdots \otimes_F \det_F(V_t), \]
where $F$ acts on $V_i$ through its embedding into $K_i$. For $g \in G_{\text{div}}$ we have
\[ \det_F(g; V_i) = \text{Nm}_{K_i/F}(\det_{K_i}(g; V_i)). \]

To calculate $\det_{K_i}(g; V_i)$ we can argue as in the preceding case (replacing $F$ by $K_i$ in the argument). If $g$ does not lie in the connected component of $G_{\text{div}}$ we find that $\det_{K_i}(g; V_i \otimes \C) = (-1)^q_i$, where $q_i = 2m \cdot \dim_{K_i}(V_i) / \dim_{E}(V_X)$. In total we therefore get
\[ \det_F(g; V_X \otimes \C) = \prod_{i=1}^{t} (-1)^{[K_i:Q]q_i} = (-1)^{\sum_{i=1}^{t} 2m \cdot [K_i:F] \cdot \dim_{K_i}(V_i) / \dim_{E}(V_X)}, \]
and since
\[ \sum_{i=1}^{t} \frac{2m \cdot [K_i:F] \cdot \dim_{K_i}(V_i)}{\dim_{E}(V_X)} = \frac{2m}{\dim_{E}(V_X)} \sum_{i=1}^{t} \dim_{F}(V_i) = 2m \cdot \frac{\dim_{F}(V_X)}{\dim_{E}(V_X)} = 2m \cdot \frac{[E:Q]}{[F:Q]}, \]
we see that $W_F$ consists of decomposable Hodge classes if and only if the integer $2m[E:Q]/[F:Q]$ is even. This proves the criterion. \hfill \Box

14. Remark. The existence of exceptional Weil classes on simple abelian varieties of type III was proven by V. K. Murty \[7\].

15. In practice, the condition (2) in the case that $Y$ is of Type IV with $d \geq 2$ or $m \geq 2$, is not as easy to verify as the conditions on $F$ in the other cases. Let us add some remarks to illustrate what can happen. Here, as before, we assume that $X$ is isogenous to $Y^m$, where $Y$ is simple and $m \geq 1$.

(i) Suppose we have two subfields $F \subseteq F' \subseteq \text{End}^0(X)$. As remarked before, if $X$ is of type I, II or III, then the spaces $W_F$ and $W_{F'}$ always consist of Hodge classes. In the type IV case this is not true in general. However, it follows directly from Criterion 3 that
\[ W_{F'} \text{ consists of Hodge classes } \implies W_F \text{ consists of Hodge classes}. \]

Moreover, if $F' \supseteq E$, then the converse is true. To see this, let us recall that the center $Z(Hg)$ of the Hodge group is contained in the torus $U_E$, and that the action of $Hg$ on $W_F$ is given by the $F$-linear determinant. Therefore, if $E \subseteq F'$, then $W_{F'}$ consists of Hodge classes if and only if $Hg$ is semi-simple. If this holds, then for any other subfield $F \subseteq \text{End}^0(X)$, the space $W_F$ also consists of Hodge classes.
Furthermore, if \( F \subseteq F' \) then we have the implication
\[
W_{F'} \text{ consists of decomposable Hodge classes } \implies W_F \text{ consists of decomposable Hodge classes}.
\]
This is a direct consequence of Criterion \([13]\). It can also be seen more directly, by using that \( W_F \) is contained in the vector subspace generated by exterior products of elements of \( W_{F'} \).

(ii) Assume \( X \) is of type IV with either \( d \geq 2 \) or \( m \geq 2 \). If the map \( \theta: E_- \hookrightarrow \text{End}_F(V_X) \xrightarrow{\text{Tr}_F} F \) is zero, then the intersection \( E \cap F \) is a totally real subfield of \( E \), i.e., \( E \cap F \subseteq E_0 \). In fact, if \( E \cap F \) is not totally real, then (being a CM-subfield of \( E \)) it must contain totally imaginary elements \( 0 \neq \alpha \in E_- \), which then obviously have a non-zero trace over \( F \).

(iii) Let us show that the converse of (ii) holds if either \( E \) or \( F \) is Galois over \( E \cap F \). So, we assume that \( E \cap F \) is totally real and that \( E \) is Galois over \( E \cap F \). The compositum \( EF \subseteq \text{End}^0(X) \) is a product of fields, say \( EF = K_1 \times \cdots \times K_t \). Correspondingly, we can decompose \( H^1(X, \mathbb{Q}) \) as a direct sum \( V_1 \oplus \cdots \oplus V_t \). It suffices to show that each of the maps \( \theta_i: E_- \hookrightarrow \text{End}_F(V_i) \xrightarrow{\text{Tr}_F} F \) is zero.

We have \( E_- \hookrightarrow K_i \subset \text{End}_{K_i}(V_i) \), and it follows that for \( \alpha \in E_- \),
\[
\theta_i(\alpha) = \text{Tr}_{K_i/F}(\dim_{K_i}(V_i) \cdot \alpha) = \dim_{K_i}(V_i) \cdot \text{Tr}_{K_i/F}(\alpha).
\]
On the other hand, if either \( E \) or \( F \) is Galois over \( E \cap F \), then the trace of \( \alpha \) (considered as an element of \( K_i \)) over \( F \) lies in \( E \cap F \), hence
\[
\text{Tr}_{K_i/F}(\alpha) = \frac{[K_i : E]}{[F : E \cap F]} \cdot \text{Tr}_{E/E \cap F}(\alpha) = 0.
\]

(iv) By means of an example, we can show that in general the converse of (ii) does not hold. For this, suppose that we have a field \( K \), containing a CM-subfield \( E \) and a subfield \( F \) such that \( E \cap F \) is totally real and such that the map \( \theta: E_- \subset K \xrightarrow{\text{Tr}_F} F \) is non-zero. Then we obtain an example of the kind we are looking for by using the constructions and results of \([11]\). First we choose a central simple algebra \( D \) over \( E \) containing \( K \) as a subfield, and then we take an abelian variety \( X \) with \( \text{End}^0(X) = D \) and such that \( n_{\sigma} = n_{\sigma'} \) for all \( \sigma \in \Sigma_E \) (which is possible, see \([11]\) Thm. 5)). Notice that this \( X \) is of type IV with \( m = 1 \) and \( d \geq 2 \) and that \( W_F \) consists of Hodge classes (using remark (i) above).

The construction of the fields \( K, E \) and \( F \) can be done using Galois theory. For example, we can start with a CM-field \( K \) which is Galois over \( \mathbb{Q} \) with group \( \{\pm 1\}^n \rtimes S_n \) (complex conjugation corresponding to \( (-1, \ldots, -1) \rtimes \text{Id} \)), take \( E \) to be the CM-subfield of elements that are invariant under \( \{\pm 1\}^{n-1} \rtimes S_{n-1} \), and let \( F \) be the fixed field of the transposition \((1, \ldots, 1) \rtimes (n-1 \ n)\).
(v) In the opposite direction, we can also use the results of [10] to construct examples where the space $W_F$ consists of exceptional Hodge classes. In particular, we see that whenever we have number fields $E \subset K \supset F$ with $E$ a CM-field and $E \cap F$ not totally real, then there exists a simple abelian variety $X$ of type IV with $d \geq 2$, such that $F \rightarrow \text{End}^0(X)$ and such that the associated space $W_F$ consists of exceptional Hodge classes.

16. To conclude, let us consider the questions Q1 and Q2 (see the introduction) in the context of Tate classes. So, let $X$ be an abelian variety defined over a number field $K$, assumed to be large enough, as always. Write $V_\ell = V_{\ell,X}$ for the first étale cohomology $H^1_\text{ét}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. If $F$ is a subfield of $\text{End}^0(X)$, then the space

$$W_{\ell,F} = \bigwedge^r_{\bigotimes F \otimes \mathbb{Q}_\ell} V_{\ell,X}$$

can be identified with a subspace of $H^1_\text{ét}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

We claim that the criteria [3] and [13] (replacing “Hodge classes” by “Tate classes” and $W_F$ by $W_{F,\ell}$) are valid in this context as well. The proof of Criterion [4] is essentially the same; the main difference is that one uses a Hodge-Tate decomposition to replace the Hodge decomposition (cf. the proof of [4], Lemma 2.8).

Criterion [13] then follows from the fact that the rings of divisor classes are “the same” in the “Hodge” and the “Tate” context, i.e., $(D(X_{\overline{\mathbb{Q}}})) \otimes \mathbb{Q}_\ell = D^t_{\ell}(X_{\overline{\mathbb{Q}}})$. This, of course, is an immediate consequence of the fact that $(B^1(X_{\overline{\mathbb{Q}}})) \otimes \mathbb{Q}_\ell = B^1_{\ell}(X_{\overline{\mathbb{Q}}})$ (by the Lefschetz theorem on $(1,1)$ classes and a special case of the Tate conjecture, as proven by Faltings in [3]).

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