Two-loop unitarity constraints on the Higgs boson coupling

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Abstract

We use the results of Maher et al. (preceding paper) to construct the matrix of \( j = 0 \) partial-wave two-body and \( 2 \to 3 \) scattering amplitudes for the scattering of longitudinally polarized gauge bosons \( W^\pm_L, Z_L \) and Higgs bosons \( H \) correct to two loops in the high-energy, heavy-Higgs limit \( \sqrt{s} \gg M_H \gg M_W \).

We show explicitly that the energy dependence of the \( 2 \to 2 \) amplitudes can be completely absorbed into a running quartic Higgs coupling \( \lambda_s = \lambda_s(s, M_H^2) \) and factors which involve small anomalous dimensions and remain near unity.

After diagonalizing the matrix of partial-wave amplitudes, we use an Argand-diagram analysis to show that the elastic scattering amplitudes are approximately unitary and weakly interacting for \( \lambda_s \lesssim 2.3 \), but that three-loop corrections are necessary to restore unitarity for larger values of \( \lambda_s \). That is, the interactions in the Higgs sector of the standard model are effectively strong with respect to the perturbative expansion for \( \lambda_s \gtrsim 2.3 \). The bound \( \lambda_s \lesssim 2.3 \) for a weakly interacting theory translates to a physical Higgs mass \( M_H \lesssim 380 \) GeV if the bound is to hold for energies up to a few TeV, or \( M_H \leq 155 \) GeV in perturbatively unified theories with mass scales of order \( 10^{16} \) GeV.

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I. INTRODUCTION

In the absence of direct observations of the Higgs boson of the standard model of electroweak interactions, its mass $M_H$ and quartic coupling $\lambda = M_H^2/2v^2$ remain unknown parameters in the theory related through the known value of the vacuum expectation value $v = (\sqrt{2}G_F)^{-1/2} = 246$ GeV. It is therefore of interest to search for bounds on $M_H$, and to consider the possibility that $\lambda$ is large enough that the theory becomes strongly interacting in the Higgs sector \cite{1–3}. The only strict upper bound on $M_H$, $M_H \lesssim 650$–800 GeV, follows from the so-called triviality bound in theories with elementary scalar fields \cite{4} as implemented for the standard model in lattice calculations \cite{5}. A detailed analysis \cite{6} of two-body scattering in the Higgs sector of the theory shows that strong-interaction effects would be essentially invisible for $\sqrt{s} \ll M_H$ unless $M_H$ is very large, $M_H > 4$–5 TeV. Such a mass would violate the triviality bound, but could be allowed if the standard model is just a low-energy remnant of a more complete theory in which, for example, the Higgs boson is composite.

The suppression of the low-energy scattering is a general consequence of chiral symmetry and the known constraints on electroweak symmetry breaking \cite{3,7}. In contrast, high-energy scattering is not suppressed. The triviality limit on $\lambda$, $\lambda \lesssim 3.5$, is large enough that the standard model can be strongly interacting with the effects visible at energies $\sqrt{s} \gg M_H$ as judged by a detailed analysis of the scattering amplitudes in the Higgs sector calculated to one loop \cite{8,9}. That analysis indicated that strong coupling sets in substantially below the tree-level bound $\lambda \approx 8\pi/3 = 8.38$ (or $M_H = 1.007$ TeV) derived some time ago by Dicus and Mathur \cite{10} and Lee, Quigg, and Thacker \cite{1}. The question naturally arises as to whether the bound would be further strengthened—or substantially weakened—by extending the analysis to two loops. The question is especially pertinent because the analysis in \cite{8} and \cite{9} used renormalization-group-improved scattering amplitudes expressed as series in the running coupling $\lambda_s$. An effectively strongly interacting theory appears for $\lambda_s \gtrsim 2$–2.5 (see also \cite{11–17} for less restrictive analyses). The limits are less obvious in the expanded form
of the amplitudes restricted to terms of orders $\lambda$ and $\lambda^2$ with energy-dependent coefficients.

Our objective here is to extend the previous analyses to two loops using the results for the two-body scattering amplitudes for longitudinally polarized gauge bosons $W_L^\pm$, $Z_L$ and Higgs bosons $H$ obtained by Maher et al. [18] in the preceding paper. We explore the range of validity of the two-loop results with increasing $\lambda$ by making an Argand-diagram analysis of the diagonalized $j = 0$ partial-wave scattering amplitudes for two-body scattering in the neutral channels $W_L^+W_L^-$, $Z_LZ_L$, $HH$, and $Z_LH$. These amplitudes must lie on or inside the usual unitarity circle, and we may use major departures from the circle as evidence of a failure of the series to converge at order $\lambda^3$. Quantitative measures of convergence indicate that the renormalization-group-improved perturbation series converges very slowly, if at all, for running couplings $\lambda_s \gtrsim 2.3$. The standard model becomes in effect “electrostrong” in the Higgs sector for larger values of $\lambda_s$, even though the perturbative amplitudes are still quite small.

II. THE SCATTERING AMPLITUDES

A. Background

It is well known [1] that scattering amplitudes for processes involving longitudinally polarized gauge bosons $W_L^\pm$, $Z_L$ are enhanced in the limit $M_H \gg M_W$ by powers of $M_H^2/M_W^2$ relative to the corresponding amplitudes for transversely polarized gauge bosons. It is therefore natural to study the scattering of $W_L^\pm$, $Z_L$ and Higgs bosons when seeking an upper bound on $M_H$ or, equivalently, on the quartic Higgs coupling $\lambda = M_H^2/2v^2$. The necessary calculations can be greatly simplified in the limit of interest, $M_H \gg M_W$, by use of the Goldstone boson equivalence theorem [1, 3, 19–22]. This theorem states that the scattering amplitudes for processes which involve $n$ longitudinally polarized gauge bosons and any number of other external particles, are related to the corresponding scattering amplitudes for the scalar bosons $w^{\pm}, z$ to which $W_L^\pm, Z_L$ reduce in the limit of vanishing gauge couplings.
\( g, g' \) by

\[
T(W_L^\pm, Z_L, H, \ldots) = (iC)^n T(w^\pm, z, H, \ldots).
\]  

(1)

The corrections are of orders \( M_W/\sqrt{s} \) and \( g^2, g'^2 \). In the renormalization scheme used in [18], \( C = 1 + O(g^2, g'^2) \).

Maher et al. [18] used the equivalence theorem to calculate the complete matrix of two-body scattering amplitudes for the neutral channels \( w^+ w^-, zz, HH, zH \) in the high-energy, heavy-Higgs limit \( \sqrt{s} \gg M_H \gg M_W \). (The charged channels provide no extra information, as we will see.) The high-energy limit \( \sqrt{s} \gg M_H \), while not of immediate experimental interest, introduces the further simplification that only the dimension-four quartic couplings contribute to the two-body scattering graphs to leading order in \( s \). The dimension-three couplings contribute to the final results only through the renormalization constants, which contain contributions from the low-energy region \( \sqrt{s} \lesssim M_H \).

Maher et al. [18] give the renormalized matrix \( \mathcal{M} \) of Feynmann transition amplitudes as

\[
\mathcal{M} = Z \mathcal{F}_R Z,
\]

(2)

where \( \mathcal{F}_R \) is a matrix of finite, partially renormalized amplitudes \( A_R(s, t, u) \), and \( Z \) is a finite diagonal matrix of ratios of renormalization constants. With the two-body channels taken in the order \( w^+ w^-, zz, HH, zH \), \( \mathcal{F}_R \) has the form

\[
\mathcal{F}_R = \begin{pmatrix}
A_R(s) + A_R(t) & A_R(s) & A_R(s) & 0 \\
A_R(s) & A_R(s) + A_R(t) + A_R(u) & A_R(s) & 0 \\
A_R(s) & A_R(s) & A_R(s) + A_R(t) + A_R(u) & 0 \\
0 & 0 & 0 & A_R(t)
\end{pmatrix},
\]

(3)

while

\[
Z = \text{diag} \left( 1, 1, Z_H/Z_w, (Z_H/Z_W)^{1/2} \right).
\]

(4)

The renormalization constants \( Z_w \) and \( Z_H \) are given in the Appendix in Eqs. (A4) and (A5).
We have indicated only the first variable in $A_R(s, t, u)$ in Eq. (3) since this function is symmetric under an interchange of the last two variables. Specifically [18],

$$A_R(s, t, u) = -2\lambda + \frac{\lambda^2}{16\pi^2} \left( -16\ln(-\hat{s}) - 4\ln(-\hat{t}) - 4\ln(-\hat{u}) + 2 + 6\sqrt{3}\pi \right)$$

$$+ \frac{\lambda^3}{(16\pi^2)^2} \left( -192\ln^2(-\hat{s}) + 176\ln(-\hat{s}) + 96\sqrt{3}\pi \ln(-\hat{s}) \\ -48\ln^2(-\hat{t}) + 80\ln(-\hat{t}) + 24\sqrt{3}\pi \ln(-\hat{t}) \\ -48\ln^2(-\hat{u}) + 80\ln(-\hat{u}) + 24\sqrt{3}\pi \ln(-\hat{u}) \\ +60\sqrt{5}\ln\left(\frac{\sqrt{5}+1}{2}\right) - 456\sqrt{3}C + 138\sqrt{3}\pi \\ +240\ln^2\left(\frac{\sqrt{5}+1}{2}\right) - \frac{3968}{9}\ln 2 - 180\pi^2 \\ -72K_2 - \frac{724}{3}\zeta(2) + 180\zeta(3) \\ -324K_5 + 24K_3 + \frac{3388}{27} - 162\zeta(2)\ln 2 \right)$$

(5)

where $\hat{s} = s/M_H^2$, $\hat{t} = t/M_H^2$, and $\hat{u} = u/M_H^2$. The phases are defined so that $-\hat{s} = e^{-i\pi}\hat{s}$, while the variables $-\hat{t}$ and $-\hat{u}$ are real and positive in the physical scattering region. $\zeta(n)$ is the Riemann zeta function, $C = 1.01494\ldots$ is the value of the Clausen function at argument $\pi/3$, and the $K$’s are certain constants evaluated by numerical integration [18], $K_2 = -0.86518$, $K_3 = -0.10666$, $K_5 = 0.92363$.

**B. The running coupling and anomalous dimensions**

A renormalization group analysis of the scattering amplitudes indicates that their entire energy dependence can be subsumed in the limit of interest ($s \gg M_H^2$) into a running coupling $\lambda_s = \lambda_s(s, M_H)$ and factors involving the anomalous dimensions $\gamma_w, \gamma_z$ and $\gamma_H$ associated with the $w^\pm, z$ and Higgs bosons. The SO(3) symmetry of the theory in the $w^\pm, z$ sector gives $\gamma_z = \gamma_w$ [18]. If all momenta are scaled by a factor $\sigma$ so that $s, t, u \to \sigma^2 s, \sigma^2 t, \sigma^2 u$, the scaled and original scattering amplitudes are related by [23]

$$\mathcal{M} \left( \{\sigma p_i\}, \lambda, M_H \right) = \Gamma \mathcal{M} \left( \{p_i\}, \lambda_s(\sigma^2 s, M_H), M_H \right) \Gamma.$$ 

(6)

Here $\Gamma$ is a diagonal matrix,
\[ \Gamma = \text{diag}(\Gamma_w^2, \Gamma_w^2, \Gamma_H^2, \Gamma_w \Gamma_H), \]  

(7)

with

\[ \Gamma_i = \exp \left( - \int_{\lambda_s(1)}^{\lambda_s(\sigma)} \frac{\gamma_i(\lambda)}{\beta(\lambda)} d\lambda \right), \quad i = w, H, \]  

(8)

and \( \lambda_s(\sigma) \equiv \lambda_s(\sigma^2 s, M_H) \). The evolution of \( \lambda_s \) is determined by the equation

\[ \ln \sigma = \int_{\lambda_s(1)}^{\lambda_s(\sigma)} \frac{d\lambda}{\beta(\lambda)}. \]  

(9)

It was demonstrated explicitly in [8,9] that the entire energy dependence of the scattering amplitudes calculated to one loop could be absorbed in a running coupling \( \lambda_s = \lambda_s(s, M_H^2) \). To that order, \( \beta^{(1)}(\lambda) = 24\lambda^2/16\pi^2 \) and the \( \gamma \)'s vanish. The running coupling \( \lambda_s \) was defined in those references, following the definition given by Sirlin and Zucchini [24] for the complete gauge theory, to incorporate some naturally occurring constants as well as the usual leading-logarithmic dependence on \( s \),

\[ \lambda_s(s, M_H) = \lambda \left[ 1 - \frac{\lambda}{16\pi^2} \left( 24 \ln \sqrt{s/M_H} + 25 - 3\sqrt{3}/\pi \right) \right]^{-1}. \]  

(10)

The constant \( \lambda \) is then related to the muon decay constant and the physical mass of the Higgs boson by [25]

\[ \lambda = M_H^2/2v^2 = G_\mu M_H^2/\sqrt{2}. \]  

(11)

The calculation of \( \gamma_w, \gamma_H, \) and \( \beta \) at two loops involves some unusual features because of our on-mass-shell renormalization conventions, and is considered in Appendix A. The results are:

\[ \gamma_w = -6 \left( \frac{\lambda}{16\pi^2} \right)^2 + O(\lambda^3), \]  

(12)

\[ \gamma_H = (150 - 24\pi\sqrt{3}) \left( \frac{\lambda}{16\pi^2} \right)^2 + O(\lambda^3), \]  

(13)

\[ \beta = 24 \frac{\lambda^2}{16\pi^2} \left( 1 - 13 \frac{\lambda}{16\pi^2} \right) + O(\lambda^3). \]  

(14)

Integration of Eq. (9) gives the running coupling \( \lambda_s \),
\[
\frac{1}{\lambda_s(\sigma)} = \frac{1}{\lambda_s(1)} - (\beta_0 + \beta_1 \lambda) \ln \sigma + O(\lambda^2),
\]
(15)
or with \(\sigma\) chosen as the ratio of \(\sqrt{s}\) to the energy scale at which \(\lambda_s = \lambda\),
\[
\lambda_s(s, M_H) = \lambda \left[ 1 - \frac{\lambda}{16\pi^2} \left( 1 - \frac{13\lambda}{16\pi^2} \right) \left( 24 \ln \frac{\sqrt{s}}{M_H} + 25 - 3\sqrt{3}\pi \right) \right]^{-1}.
\]
(16)
It is interesting to note that the Landau pole in \(\lambda_s\) apparently disappears for \(\lambda > \frac{16\pi^2}{13} \approx 12\) or \(M_H > 1214\) GeV when the second term in the denominator changes sign. However, as we will see later, such values of \(\lambda\) are far outside the range in which a two-loop calculation is reliable.

With the results above, the factors \(\Gamma_i\) in Eq. (8) can be written as
\[
\Gamma_i = \exp \left( -\frac{\gamma_{0,i}}{\beta_0} \frac{\lambda_s(\sigma) - \lambda_s(1)}{16\pi^2} \right),
\]
(17)
where \(\gamma_{0,1}\) and \(\beta_0\) are the coefficients of \(\lambda^2\) in Eqs. (12)–(14). With the use of the one-loop running coupling in Eq. (10), this becomes
\[
\Gamma_i = \exp \left( -\gamma_{0,1} \left( \frac{\lambda_s(\sigma)}{16\pi^2} \right)^2 \ln \sigma + O(\lambda_s^3) \right).
\]
(18)
The \(\lambda_s\) which appears in this equation is to be interpreted as the running coupling in accord with the renormalization group analysis. The scale \(\sigma\) will be defined as above so that
\[
\beta_0 \ln \sigma = 24 \ln \frac{\sqrt{s}}{M_H} + 25 - 3\sqrt{3}\pi,
\]
(19)
or
\[
\sigma = \frac{\sqrt{s}}{M_H} \exp \left( \frac{25 - 3\sqrt{3}\pi}{24} \right) \approx 1.43 \frac{\sqrt{s}}{M_H}.
\]
(20)
With these conventions understood,
\[
\Gamma_i = \sigma^{-\gamma_i(\lambda_s)}.
\]
(21)

It is straightforward to check that all the energy dependence of the scattering amplitudes \(\mathcal{M}\) can indeed be absorbed in the running coupling and the anomalous-dimension factors \(\gamma_i\) as indicated in Eq. (8) [25]. Specifically,
\[ \mathcal{M}(s, \lambda) = \Gamma(s, \lambda_s) \hat{M}(\cos \theta, \lambda_s(s, M_H)) \Gamma(s, \lambda_s) \]  
\hspace{1cm} (22) \]

where \( \hat{M} \) depends on \( s \) only through the running coupling, and is otherwise a function only of the scattering angle. The anomalous dimensions \( \gamma_w \) and \( \gamma_H \) are quite small numerically for \( \lambda_s \) in the range of interest,

\[ \gamma_w = -0.000240 \lambda_s^2; \]  
\hspace{1cm} (23) \]

\[ \gamma_H = 0.000778 \lambda_s^2. \]  
\hspace{1cm} (24) \]

It will therefore be a good approximation in parts of the later analysis to replace the diagonal matrix \( \Gamma \) in Eq. (22) by 1.

**C. Partial-wave 2 → 2 amplitudes for \( j = 0 \)**

Our later analysis of (apparent) violations of unitarity in perturbation theory will be based on the known properties of partial-wave scattering amplitudes. The matrix \( a_{2 \to 2}^j \) of 2 → 2 partial-wave scattering amplitudes for angular momentum \( j \) is related to the matrix \( \mathcal{M} \) of Feynman amplitudes for the various channels by \[ a_{2 \to 2}^j(s) = N A_j N \]  
\hspace{1cm} (25) \]

where \( A_j \) is the properly normalized partial-wave projection of \( \mathcal{M} \),

\[ A_j(s) = \frac{1}{32\pi} \left( \frac{4p_ip_f}{s} \right)^{1/2} \int_{-1}^{1} d \cos \theta \mathcal{M}(s, \cos \theta) P_j(\cos \theta). \]  
\hspace{1cm} (26) \]

The momentum-dependent prefactor approaches unity for \( \sqrt{s} \gg M_H \). The matrix \( N \) incorporates the symmetry factors which must be inserted for each pair of identical particles in the initial and/or final state. It is given for channel labels \( w^+w^-, zz, HH, zH \) by

\[ N = \text{diag}(1, 1\sqrt{2}, 1/\sqrt{2}, 1). \]  
\hspace{1cm} (27) \]

corresponding to two-body states \( |w^+w^-\rangle, \frac{1}{\sqrt{2}}|zz\rangle, \frac{1}{\sqrt{2}}|HH\rangle, \text{ and } |zH\rangle \) normalized over the entire solid angle. The matrix \( a_{2 \to 2}^j \) is related to the \( S \) matrix by
\[ a_{j \rightarrow 2}^{2} = (S_{j \rightarrow 2}^{2} - 1)/2i. \]  

(28)

We will deal only with the \( j = 0 \) partial-wave amplitudes. The scattering amplitudes for \( j > 0 \) are quite small, and do not give useful unitarity constraints. The \( j = 0 \) amplitudes are easily calculated, but the results are too lengthy to record in detail. However, to illustrate their character, we give the result for the diagonal \( zH \) channel:

\[ \hat{a}^{zH}_{0}(s) = \sigma^{-2\gamma_{w} - 2\gamma_{H}} a^{zH}_{0}(s), \]  

(29)

where \( \hat{a}^{zH}_{0} \) is a cubic polynomial in \( \lambda_{s} \),

\[ \hat{a}^{zH}_{0}(s) = -\frac{2\lambda_{s}}{16\pi} + \frac{\lambda_{s}}{16\pi} \left( \frac{46 + 4\pi\sqrt{3} + 4\pi i}{16\pi^{2}} \right) \begin{pmatrix} -2980 + 360\zeta(3) - 356\zeta(2) - 648K_{5} \\ -144K_{2} - 640\ln 2 - 324\zeta(2)\ln 2 \\ +48\ln^{2} \frac{\sqrt{5} + 1}{2} + 672\pi\sqrt{3} - 1124\sqrt{3}C \\ +32\pi\sqrt{3}\ln 3 + 144\sqrt{5}\ln \frac{\sqrt{5} + 1}{2} \\ -162\pi^{2} - i(228\pi + 8\pi^{2}\sqrt{3}) \end{pmatrix}. \]  

(30)

D. The \( 2 \rightarrow 3 \) amplitudes and unitarity

The unitarity of the \( S \) matrix provides a nontrivial check on the calculation of the scattering amplitudes. The relation \( S^\dagger S = 1 \) reduces for the partial-wave amplitudes to the relation

\[ \text{Im} a_{j \rightarrow 2}^{2} = (a_{j \rightarrow 2}^{2})^\dagger a_{j \rightarrow 2}^{2} + \sum_{n>2} \left( a_{j \rightarrow n}^{2} \right)^\dagger a_{j \rightarrow n}^{2}, \]  

(31)

where the generalized sum in the last term includes an integration over the \( n \)-particle phase space. Expanding the scattering amplitudes in power series in \( \lambda \) and equating like powers to order \( \lambda^{3} \) gives the matrix relations

\[ \text{Im} a_{j}^{(0)} = 0, \]

\[ \text{Im} a_{j}^{(1)} = a_{j}^{(0)} a_{j}^{(0)} \]

\[ \text{Im} a_{j}^{(2)} = a_{j}^{(0)} \text{Re} a_{j}^{(1)} + \left( \text{Re} a_{j}^{(1)} \right)^\dagger a_{j}^{(0)} + \sum \left( a_{j,\text{tree}}^{2 \rightarrow n} \right)^\dagger a_{j,\text{tree}}^{2 \rightarrow n}. \]  

(32)
The last equation relates the imaginary part of the two-loop amplitude to the one-loop and tree level amplitudes for the $2 \to 2$ processes, and the tree-level contributions to the inelastic $2 \to 3$ processes. The diagrams for the $2 \to 3$ processes are shown in Fig. 1.

One would normally expect the contributions of the $2 \to 3$ processes to vanish for $s \to \infty$ since the relevant diagrams involve dimension-three operators. The graphs in Figs. 1a and 1b are suppressed as expected by the propagator of the exchanged particle (proportional to $1/t$ or $1/u$) and vanish for $s \to \infty$ as $\hat{s}^{-1} \ln \hat{s}$ when projected to any fixed $j$. However, the two-particle “jet” in the graph in Fig. 1c can have a low mass so that the extra propagator is nearly on shell. As a result, the square of this graph is not suppressed in the unitarity sum in Eq. (31) after the integration over the 3-body phase space. However, interference terms between jet graphs which differ by the exchange of a jet particle and the final particle from the 4-point vertex are still suppressed. The only finite contributions of the $2 \to 3$ processes to the unitarity sum are therefore from terms which have the topology of a cut scattering Eye graph in the terminology of [18]. The three such graphs for $zH$ scattering are shown in Fig. 2. We will consider this set as an example.

The contribution of any of the diagrams in Fig. 2 to the unitarity sum is given up to vertex and symmetry factors by

$$
\left. \frac{|p_a|}{4\sqrt{s}} \int \frac{dP_3}{4\pi} \left( i \frac{i}{s_{12} - m^2 + i\epsilon} \right)^* \frac{i}{s_{12} - m^2 + i\epsilon} \right|_{s \gg M_H^2} \frac{1}{(16\pi)^2} \frac{1}{\pi} \int_{(m_1 + m_2)^2}^{\infty} ds_{12} \frac{\Delta(s_{12}, m_1^2, m_2^2)}{16\pi s_{12} \left[ (s_{12} - m^2)^2 + \epsilon^2 \right]}.
$$

Here $\Delta$ is the triangle function,

$$
\Delta(a, b, c) = \left[ (a - b - c)^2 - 4bc \right]^{1/2},
$$

and $m_1, m_2, \sqrt{s_{12}},$ and $m$ are, respectively, the masses of the two particles in the jet, the jet itself, and the intermediate particle. The kinematic factor $|p_a|/4\sqrt{s}$ in Eq. (33) normalizes the $2 \to 3$ jet amplitudes (which automatically have $j = 0$ at $O(\lambda^3)$) to the standard $2 \to 2$ partial-wave amplitudes used above [27].
The calculations for the graphs in Figs. 2a and 2b are straightforward. Summing over the independent diagrams and taking account of the symmetry factors for identical particles in the final state we obtain the value

\[ \frac{\lambda}{(4\pi)^2} \left| a_0^{(0)}(zH) \right|^2 = -Z_w^{(1)} \left| a_0^{(0)}(zH) \right|^2 \] (35)

for the graph in Fig. 2a and

\[ \frac{\lambda}{(4\pi)^2} (-9 + 2\pi \sqrt{3}) \left| a_0^{(0)}(zH) \right|^2 \] (36)

for the graph in Fig. 2b, where \( a_0^{(0)}(zH) = -2\lambda/16\pi \) is the tree-level amplitude for \( zH \) scattering.

The graph in Fig. 2c is singular for \( \epsilon = 0 \) (that is, for \( \text{Im}s_{12} \to 0^+ \)) since the decays \( H \to w^+w^- \) and \( H \to zz \) are allowed and the intermediate propagator can be on mass shell. A direct calculation retaining the \( \epsilon \) in the denominator in Eq. (32) gives

\[ \frac{3\lambda}{(4\pi)^2} \left( \frac{\pi M_H^2}{\epsilon} - 1 \right) \left| a_0^{(0)}(zH) \right|^2 \] (37)

where \( \epsilon \) can be identified in the narrow-width approximation for the \( H \) decay as \( M_H \Gamma_H \).

The tree-level decay width is \( \Gamma_H = 3\lambda M_H / 16\pi \), so the first term in Eq. (37) is just equal to the square of the tree-level \( zH \) scattering amplitude. This has already been included in the calculation in the first term on the right hand side of Eq. (31) since we treated the Higgs boson as stable, and should be dropped here—the \( H \) always decays. (Alternatively, had we recognized from the start that the \( H \) does not appear as an asymptotic state, we would have had no tree-level amplitude for \( zH \) scattering; it would appear as here when the \( H \) resonance is treated as narrow in intermediate states.)

With this specification, the combined result for Eqs. (36) and (37) reduces to

\[ \frac{\lambda}{(4\pi)^2}(-12 + 2\pi \sqrt{3}) \left| a_0^{(0)}(zH) \right|^2 = -Z_H^{(1)} \left| a_0^{(0)}(zH) \right|^2 \] (38)

where \( Z_H^{(1)} \) is the one-loop contribution to \( Z_H \). Upon substituting the results in Eqs. (35) and (38) into Eq. (32), we find that the unitarity condition is satisfied in the channel \( zH \to zH \).
Similar results hold in other channels for the remaining non-suppressed $2 \rightarrow 3$ contributions. In each case the relevant graphs have the topology of cut scattering $Eye$ graphs, possibly connecting different 2-body states, and give results which are just the negative of the first-order renormalization constant $Z^{(1)}$ for the intermediate particle in the jet, multiplied by a product of a tree-level amplitudes. This is not accidental: after the tree-level amplitudes are extracted, the integrals which remain are closely related to dispersion relations for the renormalization constants. In particular, the self-energy functions satisfy once-subtracted dispersion relations

$$\text{Re}\Pi(p^2) - \text{Re}\Pi(m^2) = \frac{p^2 - m^2}{\pi} \mathcal{P} \int_{(m_1 + m_2)^2}^{\infty} ds' \frac{\text{Im}\Pi(s')}{(s' - p^2)(s' - m^2)}.$$  

Dividing by $(p^2 - m^2)$, taking the limit $p^2 \rightarrow m^2$, and using the definition of $Z$ in terms of the physical fields and self-energy functions gives

$$Z - 1 = \frac{1}{\pi} \mathcal{P} \int_{(m_1 + m_2)^2}^{\infty} ds' \frac{\text{Re}\Pi(s')}{(s - m^2)^2}.$$  

The integral in Eq. (33) is already in the form of a dispersion relation for $m^2 < (m_1 + m_2)^2$ since $\epsilon$ can then be set equal to zero. If $m^2 > (m_1 + m_2)^2$, the integration contour is trapped between poles at $m^2 \pm i\epsilon$, but by moving the contour outside the poles, once above and once below and averaging, we pick up a term proportional to $\epsilon^{-1}\text{Im}\Pi(m^2)$ and a principal value integral. Once the former is deleted, the integrals which remain give the renormalization constants. In particular, the product of the 3-point vertex factors in Fig. 2a and the factor $\Delta/16\pi s_{12}$ in Eq. (33) is equal to $-\text{Im}\Pi_w^{(1)}(s_{12})$, where the minus sign arises from the difference between the factor $(2i\lambda v)^2$ which appears in $\Pi_w^{(1)}$ and the factor $|2i\lambda v|^2$ which appears in the absolute square of the $2 \rightarrow 3$ amplitude. The sum of the corresponding contributions from Figs. 2b and 2c gives $-\text{Im}\Pi_H^{(1)}$. This analysis clearly generalizes to higher orders.

The final result for the $2 \rightarrow 3$ contributions to the last of Eqs. (32)—with the $H$ decay term eliminated—is of the form

$$a_0^{(0)T} \left( -\sum Z^{(1)} \right) a_0^{(0)}.$$  

(41)
where $\sum Z^{(1)}$ is a diagonal matrix composed of the sums of the first-order renormalization constants for the final two particles in the amplitude $a_{(0)}$, e.g., $Z_w + Z_H$ for the $zH$ channel. With this specification, Eqs. (32) are satisfied identically as matrix equations, a useful and nontrivial check on the calculations.

III. ANALYSIS AND CONCLUSIONS

A. Diagonalization of the partial-wave amplitudes

The general unitarity relation for the $S$ matrix, $S^\dagger S = 1$, reduces in a basis in which the matrix $a_j^{2\to2}$ of $2 \to 2$ partial-wave amplitudes is diagonal to the condition

$$\left| a_j^{2\to2} - \frac{i}{2} \right|^2 + \sum_{n>2} \left( a_j^{2\to n} \right)^\dagger a_j^{2\to n} = \frac{1}{4}. \quad (42)$$

This gives the familiar constraint that the exact two-body elastic scattering amplitudes lie on or inside a circle of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}\right)$ in the complex plane, a constraint which we will use to examine the breakdown of low-order perturbation theory for large values of $\lambda_s$.

The determination of the diagonal $2 \to 2$ scattering amplitudes is straightforward. The matrix $\mathcal{M}$ of Feynman amplitudes defined in Eqs. (2)–(4) would be SO(4) symmetric and could be diagonalized in an SO(4) basis except for the difference between the renormalization constraints $Z_w$ and $Z_H$ in the matrix $Z$. However $\mathcal{M}$ and the partial-wave scattering matrices $a_j^{2\to2}$ retain the exact SO(3) symmetry of the Higgs Lagrangian $[18]$. We will therefore follow $[9]$ and diagonalize the scattering matrices using states based on the SO(3) decomposition of SO(4). The sixteen possible two-body combinations of $w^{\pm}, z, H$ states break up under SO(4) as $4 \otimes 4 = 9 \oplus 6 \oplus 1$. Because of Bose symmetry, only the symmetric representations 9 and 1 are allowed for even angular momenta $j$, and only the antisymmetric representation 6 for odd $j$. The SO(4) representations decompose under SO(3) as $9 \to 5 \oplus 3 \oplus 1, 6 \to 3 \oplus 3'$, and $1 \to 1$. The states we will need for even $j$ are $\mathbb{1}$.
\[ 9, 5 = \begin{cases} 
  w^+w^- \\
  w^+z \\
  \frac{1}{\sqrt{3}}(w^+w^- - zz) \\
  w^-z \\
  w^-w^- 
\end{cases}, \quad 9, 3 = \begin{cases} 
  w^+H \\
  zH \\
  w^-H 
\end{cases} \]

\[ 9, 1 = \frac{1}{\sqrt{24}}(2w^+w^- + zz - 3HH), \]

\[ 1, 1 = \frac{1}{\sqrt{8}}(2w^+w^- + zz + HH), \]  \hspace{1cm} (43)

while for \( j \) odd the relevant states are

\[ 6, 3 = \begin{cases} 
  w^+z \\
  w^+w^- \\
  w^-z 
\end{cases}, \quad 6, 3' = \begin{cases} 
  w^+H \\
  zH \\
  w^-H 
\end{cases} \]  \hspace{1cm} (44)

Because the SO(3) symmetry is exact, the \((9, 5), (9, 3), \) and two \((6, 3)\) representations give eigenstates of \(a_j^{2\rightarrow2}\) to any order in \(\lambda\). However, the two identity representations of \(\text{SO}(3), (9, 1)\) and \((1, 1)\), mix through the \(\text{SO}(4)\)-breaking contributions of the \(HH\) channel.

The eigenvalues of \(a_j^{2\rightarrow2}\) for any \(j\) are easily determined by diagonalizing the \(4 \times 4\) scattering matrix for the neutral channels. The charged channels add no new information as is evident from the decompositions above. We will concentrate here on \(j = 0\), and will use basis states normalized over the entire solid angle, specifically the initial states \(w^+w^-, zz/\sqrt{2}, HH/\sqrt{2}, zH\), and the \(\text{SO}(3)\) basis states \(\chi_i, i = 1, \ldots, 4\) defined as

\[ \chi_1 = \chi_{1, 1} = \frac{1}{\sqrt{8}}(2w^+w^- + zz + HH), \]

\[ \chi_2 = \chi_{9, 1} = \frac{1}{\sqrt{24}}(2w^+w^- + zz - 3HH), \]

\[ \chi_3 = \chi_{9, 5} = \frac{1}{\sqrt{3}}(w^+w^- - zz), \]

\[ \chi_4 = \chi_{9, 3} = zH. \]  \hspace{1cm} (45)

The transformation \(O^T a O\) with
splits off diagonal 9, 5 and 9, 3 amplitudes to any order in \( \lambda \), leaving a 2 \( \times \) 2 matrix to be diagonalized in the \( \chi_1, \chi_2 \) sector.

We can determine the eigenvalues of the 2 \( \times \) 2 matrix rather easily by transforming to the states which diagonalize the matrix to order \( \lambda^2 \), that is, at one loop \([9]\),

\[
\chi_1' = N(\chi_2 - \sqrt{3}\Delta \chi_2),
\]

\[
\chi_2' = N(\chi_2 + \sqrt{3}\Delta \chi_1).
\]

(47)

Here \( \Delta = \frac{1}{2} \left( Z_H^{(1)} - Z_w^{(1)} \right) \) and \( N = (1 + 3\Delta)^{-1/2} \). In this basis, the off-diagonal elements of the 2 \( \times \) 2 matrix are of minimum order \( \lambda^3 \) and do not contribute to the eigenvalues to the accuracy of the two-loop calculation. We conclude, therefore, that the diagonal scattering amplitudes \( a_i \) are given to \( O(\lambda^2) \) by the diagonal elements of the transformed matrix,

\[
a_i = \left( O'^T O'^T a O O' \right)_{ii}.
\]

(48)

Here \([9]\)

\[
O' = \begin{pmatrix}
1 - \frac{3}{2}\Delta^2 & \sqrt{3}\Delta & 0 & 0 \\
-\sqrt{3}\Delta & 1 - \frac{3}{2}\Delta^2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(49)

where we have expanded the normalization factor \( N \) to the relevant order.

The diagonal elements \( a_1 \) and \( a_2 \) obtained by this construction appear initially as sums of terms which involve different factors \( \Gamma(s) \) from the anomalous dimensions. Since these factors are close to unity for the values of \( s \) we will consider, it is convenient to expand the \( \Gamma \)'s and resum the results to obtain overall effective \( \Gamma \)'s for these two scattering eigenstates.
The anomalous dimension factors for $a_3$ and $a_4$ appear just as given below. The final results are best presented numerically:

$$
a_1(s) = \sigma \gamma_1 \frac{\lambda_s}{16\pi} \left[ -6 + \frac{\lambda_s}{16\pi^2} (185.6 + 113.1i) - \left( \frac{\lambda_s}{16\pi^2} \right)^2 (8330.9 + 6892.3i) + O(\lambda_s^3) \right],$$

$$
a_2(s) = \sigma \gamma_2 \frac{\lambda_s}{16\pi} \left[ -2 + \frac{\lambda_s}{16\pi^2} (65.65 + 12.57i) - \left( \frac{\lambda_s}{16\pi^2} \right)^2 (3590.7 + 839.7i) + O(\lambda_s^3) \right],$$

$$
a_3(s) = \sigma \gamma_3 \frac{\lambda_s}{16\pi} \left[ -2 + \frac{\lambda_s}{16\pi^2} (72 + 12.57i) - \left( \frac{\lambda_s}{16\pi^2} \right)^2 (4752.5 + 879.6i) + O(\lambda_s^3) \right],$$

$$
a_4(s) = \sigma \gamma_4 \frac{\lambda_s}{16\pi} \left[ -2 + \frac{\lambda_s}{16\pi^2} (67.77 + 12.57i) - \left( \frac{\lambda_s}{16\pi^2} \right)^2 (3980.2 + 853.0i) + O(\lambda_s^3) \right],$$

(50)

where

$$
\gamma_1 = -3\gamma_w - \gamma_H = -24 \left( \frac{\lambda_s}{16\pi^2} \right)^2 \left( \frac{11}{2} - \sqrt{3\pi} \right) \approx -6 \times 10^{-5} \lambda_s^2,
$$

$$
\gamma_2 = -\gamma_w - 3\gamma_H = -24 \left( \frac{\lambda_s}{16\pi^2} \right)^2 \left( \frac{37}{2} - 3\sqrt{3\pi} \right) \approx -2 \times 10^{-3} \lambda_s^2,
$$

$$
\gamma_3 = -4\gamma_w = 24 \left( \frac{\lambda_s}{16\pi^2} \right)^2 \approx 1 \times 10^{-3} \lambda_s^2,
$$

$$
\gamma_4 = -2\gamma_w - 2\gamma_H = -24 \left( \frac{\lambda_s}{16\pi^2} \right)^2 \left( 12 - 2\sqrt{3\pi} \right) \approx -1 \times 10^{-3} \lambda_s^2.
$$

(51)

The relative weights with which $\gamma_w$ and $\gamma_H$ appear in the effective anomalous dimensions $\gamma_1$ and $\gamma_2$ can be read off from the probabilities with which the normalized states $w^+w^-$, $zz/\sqrt{2}$, and $HH/\sqrt{2}$ appear in $\chi_1$ and $\chi_2$, Eq. (45). The parameter $\sigma$ is defined in Eq. (20). The $a_1$ amplitude arises primarily from the SO(4) singlet state, and is consistently about three times larger than the SO(4) nonet amplitudes. We will use these amplitudes in the analysis in the following section.
B. Argand diagram analysis

The diagonalized 2 → 2 scattering amplitudes $a_1$ and $a_3$ from Eq. (50) are plotted in Figs. 3a and 3b as functions of $\lambda_s$. The anomalous dimension factors $\sigma_\gamma$ have been omitted as they are close to unity for $\sqrt{s} < 100$ TeV and the values of $\lambda_s$ which will be of interest. The corresponding curves for $a_2$ and $a_4$ are quite close to that for $a_3$, Fig. 3b, and are not shown.

It is clear from Fig. 3 that $a_1$ and $a_3$ move away from the unitarity circle rather quickly as $\lambda_s$ is increased. The contributions of the 2 → 3 processes to the unitarity sum in Eq. (32) are quite small,

$$\sum \left( a_{0}^{2 \rightarrow 3} + a_{0}^{2 \rightarrow 3} \right) = \begin{cases} \frac{\lambda_s}{16\pi^2} \left( \pi\sqrt{3} - \frac{9}{2} \right) |a_1|^2 \approx 8.5 \times 10^{-5} \lambda_s^3 \\ \frac{2\lambda_s}{16\pi^2} |a_3|^2 \approx 2.0 \times 10^{-5} \lambda_s^3 \end{cases},$$

for scattering from the initial states $\chi'_1$ and $\chi_3$. The contributions from 2 → 4 processes are similarly small, less than $8.0 \times 10^{-5} \lambda_s^4$ and $3.2 \times 10^{-5} \lambda_s^4$ for the $\chi_1$ and $\chi_3$ channels for $\sqrt{s} < 100$ TeV. As a result, the exact 2 → 2 amplitudes $a_1$ and $a_3$ must lie essentially on the unitarity circle for $\lambda_s$ not too large [28]. The deviations of the calculated amplitudes from the circle give a measure of the range of $\lambda_s$ in which the calculated amplitudes are reliable.

The convergence—or lack of convergence—of the perturbation series is illustrated for the channel $\chi'_1 \rightarrow \chi'_1$ in the vector diagram for $a_1$ in Fig. 4. In this figure, we show the zero-, one-, and two-loop amplitudes for $\lambda_s = 2.5$ as vectors in the Argand diagram. The complete two-loop amplitude $a^{(0)} + a^{(1)} + a^{(2)}$ is the sum of these vectors. It must lie essentially on the unitarity circle if the perturbative approximation is to be valid. It is immediately evident from the figure that the series is not converging well for $\lambda_s = 2.5$: $|a^{(2)}|$ is nearly as large as $|a^{(1)}|$ which is as large as $|a^{(0)} + a^{(1)}|$. Furthermore, the imaginary part of the amplitude becomes negative for $\lambda_s \gtrsim 2.6$. It must be positive in the exact result.

We can quantify the incipient breakdown of the perturbation series using several tests discussed in [9]. We will limit our attention here to familiar ratio tests which quantify the observations above about the vector diagram in Fig. 4 and its analogs for the other channels.
Further less-general but sometimes more restrictive tests are discussed in [31].

In Figs. 5a and 5b, we show the ratios $|a_i^{(2)}/(a_i^{(0)} + a_i^{(1)})|$ and $|a_i^{(2)}/a_i^{(1)}|$ for the four amplitudes $a_i$. For values of $\lambda_s$ greater than 2.3 to 3.2 (depending on the ratio and channel considered), the ratios exceed unity and there is no evidence of convergence of the series from either the ratio of the two-loop amplitude to the previous partial sum, or from the ratio of successive terms in the series. The strongest limits, $\lambda_s < 2.3–2.4$ for ratios less than unity, come from the $\chi_3$ channel. This channel is associated with the 5 representation of SO(3), and does not involve $HH$ scattering. We also note that Im $a_3$ is negative for $\lambda_s > 2.26$. 

We will adopt the value $\lambda_{s,\text{max}} = 2.3$ as the maximum value of $\lambda_s$ for which the perturbation series in $\lambda_s$ may reasonably be said to converge at two loops for energies $\sqrt{s} < 100$ TeV. At higher energies, the anomalous-dimension factors in Eq. (50) begin to affect the magnitudes of the $a_i$. These factors can be treated in two ways. If the expression for $\mathcal{M}$ is used in the form in Eq. (22) given by the renormalization group, the anomalous-dimension factors $\Gamma$ multiply each of $a^{(0)}$, $a^{(1)}$, $a^{(2)}$ and divide out in the ratios considered above, exactly for $a_3$ and $a_4$, and approximately for the average factors in $a_1$ and $a_2$. In this approach, the limits above are unchanged. An alternative procedure is to expand the anomalous dimension factors. This adds an energy-dependent term to $a^{(2)}$. In this case the ratio bounds from $a_3$ and $a_4$ are strengthened and those from $a_1$ and $a_2$ are weakened. For example, $|a_3^{(2)}/a_3^{(1)}|$ now reaches unity for $\lambda_s = 1.85$ for $\sqrt{s} = 10^{16}$ GeV and $M_H = 155$ GeV. The change from $\lambda_s = 2.3$ becomes irrelevant when $M_H$ is determined by inverting Eq. (16) at the high energy scale. A general limit $\lambda_{s,\text{max}} = 2.3$ therefore seems reasonable.

C. Conclusions

The restriction $\lambda_s < \lambda_{s,\text{max}} = 2.3$ is a quantitative condition on the running coupling which must be satisfied if low-order perturbation theory in $\lambda_s$ is to give a good description of high-energy $w^\pm, z, H$ scattering. As illustrated above, the perturbation series for the $a_i$ show little or no sign of converging at two loops for $\lambda_s > \lambda_{s,\text{max}}$. This of course does not
preclude at least asymptotic convergence for large $\lambda_s$. It is simply that perturbation theory breaks down as a useful tool, and the theory becomes in that sense strongly interacting even though the scattering amplitudes are quite small. Nonperturbative methods are then needed to investigate the problem. However, it is quite useful to bound the region in which low-order perturbation theory can be trusted since that is the method used for most phenomenological calculations.

The limiting value $\lambda_{s,\text{max}} = 2.3$ for the running coupling in a weakly interacting or perturbative theory translates directly into an upper bound on the mass of the Higgs boson in such a theory once it is decided at what energy the bound is to be applied. The running coupling $\lambda_s = \lambda_s(s, M_H^2)$ is plotted in Fig. 6 as a function of $M_H$ for various values of $\sqrt{s}$. Different choices for the upper bound on the perturbative region correspond to different horizontal lines on this plot. The line for $\lambda_{s,\text{max}} = 2.3$ is shown.

Since there is no experimental evidence at the present time for any breakdown of the standard model, it is plausible—and is generally assumed—that it will remain valid up to at least a few TeV. If it remains valid up to a (minimal) value $\sqrt{s} = 5$ TeV and we apply the bound $\lambda_s < 2.3$ there, Fig. 6 shows that the Higgs sector of the standard model will be approximately unitary and weakly interacting at two loops only for $M_H < 380$ GeV. This is significantly below the values $M_H \approx 1$ TeV used in a number of phenomenological studies done using tree-level amplitudes. These results should be re-examined. The limit is also below the nonperturbative “triviality” bound $M_H \lesssim 650$–800 GeV obtained in [5]. The existence of a Higgs boson with a mass in the nonperturbative or strongly interacting sector between our bound and the triviality limit is not precluded, and would be quite interesting. Calculations would be difficult.

The bound on $M_H$ in a perturbative theory becomes much stronger, $M_H \lesssim 155$ GeV, if the standard model is assumed to hold up to a typical unification energy of order $10^{16}$ GeV. This result may be altered slightly when other couplings are included in the renormalization group equations, but the Higgs boson will still remain light.

Finally, as noted elsewhere [3,26], the limit on the range of validity of perturbation theory
can become a real upper bound on $M_H$ if the standard model is embedded in a specific unified or dynamical model, which must remain perturbative up to some unification or dynamical energy. If this is the actual situation, the Higgs boson cannot be much more massive than 380 GeV in theories such as technicolor with a low mass scale, and will be much lighter in typical unified theories. It will then be accessible in future experiments.

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**APPENDIX: CALCULATION OF $\gamma_w$, $\gamma_H$, AND $\beta$ AT TWO LOOPS**

Because of our on-mass-shell convention for the renormalization of the scalar theory, we will determine $\beta$ and the $\gamma$’s to two loops using the Callan-Symanzik definitions for these quantities [23],

\[
\beta = \lim_{\epsilon \to 0} \left( 2M_H^2 \frac{\partial \lambda}{\partial M_H^2} \right)_{\lambda_0, \epsilon},
\]

\[
\gamma_i = \lim_{\epsilon \to 0} \left( M_H^2 \frac{\partial Z_i}{\partial M_H^2} \right)_{\lambda_0, \epsilon}.
\]

Here $M_H$ and $\lambda$ are the renormalized or physical Higgs boson mass and coupling. $M_0$ and $\lambda_0$ are the corresponding bare quantities. Since the Goldstone boson mass $m_w$ is fixed at zero, $M_H$ is the only mass in the physical theory.

The bare coupling $\lambda_0$ is given in [13] as

\[
\lambda_0 = \lambda + \frac{\lambda^2 \xi}{16\pi^2} \left( \frac{12}{\epsilon} + 25 - 12\gamma - 3\sqrt{3}\pi + \epsilon [3\pi \sqrt{3}\ln 3 - 12\sqrt{3}\zeta(3)] - 2\xi^2 - 6\gamma^2 - 25\gamma + \frac{99}{2} + O(\epsilon^2) \right) + \frac{\lambda^3 \xi^2}{(16\pi)^2} \left( \frac{144}{\epsilon^2} + \frac{18}{\epsilon} (29 - 16\gamma - 4\sqrt{3}\pi) \right) + \frac{32906}{27} + 12K_3 + 162K_5 + 36K_2 - 90\zeta(3)
\]
to two loops. Here $\xi = 4\pi \mu^2 / M_H^2$, and $\mu$ is the mass scale introduced to keep the coupling $\lambda$ dimensionless in $d = 4 - 2\epsilon$ dimensions. The renormalization constants $Z_w$ and $Z_H$ are

$$Z_w = 1 - \frac{\lambda}{16\pi^2} \xi^\epsilon \left[ 1 + \epsilon \left( \frac{3}{2} - \gamma \right) + O(\epsilon^2) \right]$$

$$+ \frac{\lambda^2}{(16\pi^2)^2} \xi^{2\epsilon} \left( -\frac{3}{\epsilon} + 42\sqrt{5} \ln \left( \frac{\sqrt{5} + 1}{2} \right) - 96 \ln^2 \left( \frac{\sqrt{5} + 1}{2} \right) \right)$$

$$+ \frac{400}{9} \ln 2 + 6\gamma - \frac{38}{3} \zeta(2) - 12K_3 - \frac{1525}{54} + 3\sqrt{3}\pi + O(\epsilon) \right), \quad (A4)$$

and

$$Z_H = 1 + \frac{\lambda}{16\pi^2} \xi^\epsilon \left[ 12 - 2\pi\sqrt{3} \right.$$

$$+ \epsilon \left( 24 - 12\gamma - 3\pi\sqrt{3}(1 - \frac{2}{3}\gamma) + 2\pi\sqrt{3} \ln 3 - 8\sqrt{3}C \right) + O(\epsilon^2) \right]$$

$$+ \frac{\lambda^2}{(16\pi^2)^2} \xi^{2\epsilon} \left( -\frac{3}{\epsilon} + 144 \ln 2 + 42\pi^2 + 81\zeta(2) \ln 2 \right.$$  

$$+ 36K_2 + 162K_5 + 6\gamma + 33\zeta(2) - 90\zeta(3)$$

$$- 16\sqrt{3}\pi \ln 3 + 334\sqrt{3}C + \frac{273}{2} - 292\sqrt{3}\pi + O(\epsilon) \right). \quad (A5)$$

Finally, $M_0$, which we will not need explicitly, is given in terms of the self-energy functions $\Pi^0_w$ and $\Pi^0_H$ for the bare fields calculated in [18] by [29]

$$M_0^2 = M_H^2 - \Re \Pi^0_H(M_H^2) + \Pi^0_w(0). \quad (A6)$$

We have calculated $\beta$ and $\gamma$'s using two different approaches. The first is based on the expressions for the $Z$'s and $\lambda_0$ above. These give the first few terms in series of the form

$$\lambda_0 = \mu^{2\epsilon} \lambda \left( 1 + \sum_{j=1}^{\infty} \sum_{k=0}^{j} a_{jk} \frac{\lambda^k \xi^j}{\epsilon^k} \right), \quad (A7)$$

$$Z = 1 + \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} c_{jk} \frac{\lambda^k \xi^j}{\epsilon^k}. \quad (A8)$$
$M_H$ and $\lambda$ are regarded in Eqs. (A1) and (A2) as implicit functions of the independent variables $M_0$ and $\lambda_0$; $\mu$ and $\epsilon$ are fixed. Differentiation of the equation for $\lambda_0$ gives

$$\frac{d\lambda_0}{dM_0^2} = 0 = \frac{\partial \lambda_0}{\partial \lambda} \frac{\partial}{\partial M_0^2} - \frac{\xi}{M_H^2} \frac{\partial \lambda_0}{\partial \xi} \frac{\partial}{\partial M_0^2},$$

or from Eq. (A11),

$$\beta = \lim_{\epsilon \to 0} \frac{2\xi}{\epsilon} \frac{\partial \lambda_0}{\partial \lambda} \frac{\partial}{\partial \xi},$$

where $\lambda_0$ is defined in the last expressions by the series in Eq. (A7). Substituting the series into Eq. (A11), expanding to order $\lambda^3$, and taking the limit $\epsilon \to 0$ gives

$$\beta = \lambda^2(2a_{1,1} + 4\lambda a_{2,1} - 8\lambda a_{1,1} a_{1,0})$$

$$= 24 \frac{\lambda^2}{16\pi^2} \left(1 - 13 \frac{\lambda}{16\pi^2}\right)$$

$$= \beta_0 \lambda^2 + \beta_1 \lambda^3.$$

A potentially singular term of the form

$$\frac{\lambda^2 \xi^2 \epsilon}{\epsilon} \left(a_{2,2} - a_{1,1}^2\right)$$

vanishes identically in this construction because $a_{2,2} \equiv a_{1,1}^2$. The result is therefore finite and independent of the arbitrary mass $\mu$, as expected.

A similar calculation gives

$$\gamma_i = \lim_{\epsilon \to 0} \left(\xi \frac{\partial Z_i}{\partial \xi} + \frac{\xi}{\partial \lambda_0} \frac{\partial}{\partial \xi} \frac{\partial Z}{\partial \lambda}\right)$$

$$= \lambda^2(a_{1,1} c_{1,0} - 2c_{2,1}),$$

hence,

$$\gamma_w = -6 \left(\frac{\lambda}{16\pi^2}\right)^2$$

$$\gamma_H = (150 - 24\pi \sqrt{3}) \left(\frac{\lambda}{16\pi^2}\right)^2.$$

We note finally that the mass dimension $\gamma_M$, defined as
\[ \gamma_M = 1 - \frac{M^2_H/M^2_0}{\partial M^2_H/\partial M^2_0}, \quad (A16) \]

is connected to \( \beta \) and \( \gamma_w \) through the relation \( \lambda_0 = (M^2_0/M^2_H Z_w) \lambda \). This gives

\[ \gamma_M = \frac{\beta}{2\lambda} - \gamma_w \]
\[ = 12 \frac{\lambda}{16\pi^2} - 150 \left( \frac{\lambda}{16\pi^2} \right)^2. \quad (A17) \]

A second approach to the calculations above uses the observation that the introduction of the mass \( \mu \) in the course of dimensional regularization is unnecessary. With our on-shell renormalization scheme, the theory contains a single mass \( M_H \) which can be used to scale the coupling in the transition to \( 4 - 2\epsilon \) dimensions. With the choice \( \mu^2 = M^2_H/4\pi \), the quantity \( \xi \) disappears from the theory, and

\[ \lambda_0 = \lambda \left( \frac{M^2_H}{4\pi} \right)^\epsilon \left( 1 + \sum_{j=1}^{\infty} \sum_{k=0}^{j} a_{jk} \frac{\lambda^j}{\epsilon^k} \right). \quad (A18) \]
\[ Z = 1 + \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} c_{jk} \frac{\lambda^j}{\epsilon^k}. \quad (A19) \]

Using the Callan-Symanzik definitions, we find that

\[ \beta = \lim_{\epsilon \to 0} \beta(\epsilon) \equiv \lim_{\epsilon \to 0} \frac{-2\epsilon\lambda_0}{\partial \lambda_0/\partial \lambda} \quad (A20) \]

and

\[ \gamma_i = \lim_{\epsilon \to 0} \frac{1}{2} \beta(\epsilon) \frac{\partial Z_i}{\partial \lambda}. \quad (A21) \]

It is straightforward to show that the limits of these expressions are identical to all orders in \( \lambda \) to those obtained from Eqs. (A10) and (A13). They are also formally identical to the expressions obtained in mass-independent subtraction schemes with dimensional regularization [30], and imply in the same way recurrence relations for the coefficients of powers of \( 1/\epsilon \) greater than the first [23].
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processes.

[28] It is curious that the residual $2 \rightarrow 3$ contributions to the unitarity sums for scattering from the initial states $\chi'_{2}$ and $\chi_{4}$ which have large Higgs components are negative after the leading Higgs pole term is extracted approximately using the narrow-width approximation [see the discussion in the paragraphs following Eq. (37)]. This would apparently allow the $2 \rightarrow 2$ amplitude to lie very slightly outside the unitarity circle. This unphysical peculiarity is clearly connected with the instability of the Higgs boson which we are treating as a real asymptotic state, with $M_{H}$ defined in terms of the renormalized self-energy function by $M_{H}^{2} = \text{Re} \Pi_{R}(M_{H}^{2})$. A $2 \rightarrow 2$ amplitude involving unstable Higgs bosons does not exist in the usual sense, and must be defined in terms of intermediate resonance poles in multiparticle processes which involve only $w$’s and $\zeta$’s. The complex Higgs boson mass is then defined as the solution to the equation $M^{2} = M_{H}^{2} + iM_{H} \Gamma_{H} = \Pi_{R}(M^{2})$. With the definition $M_{H}^{2} = \text{Re} \Pi_{R}(M_{H}^{2})$ used here, the two-body unitarity constraints are disrupted, but by amounts which are unimportant in our analysis.

[29] The definition of $M_{0}^{2}$ follows from the discussion of the renormalized Lagrangian given in [18] when the mass terms given in Eq. (7) in that paper are rewritten in the form

$$-\frac{1}{2}M_{0}^{2}H_{0}^{2} - \frac{1}{2}\Pi_{0}(0)\left(w_{0}^{2} + H_{0}^{2}\right).$$

The last SO(4)-symmetric term guarantees that the bare fields $w_{0}$ and $H_{0}$ are massless for $M_{0}^{2} = 0$, with tadpole contributions suppressed, and are not part of the bare mass.

[30] For mass-independent subtraction schemes, $\lambda_{0}$ and $Z$ are given by series identical in form to those in Eqs. (A18) and (A19) with $M_{H}^{2}/4\pi$ replaced in Eq. (A18) by the scale parameter $\mu^{2}$, while $\beta$ and $\gamma$ are defined by

$$\beta = \lim_{\epsilon\rightarrow 0} \frac{\mu}{\partial \mu} \frac{\partial \lambda}{\partial \mu}, \quad \gamma = \lim_{\epsilon\rightarrow 0} \frac{1}{2} \frac{\partial Z}{\partial \mu}.$$

Use of the equation $\partial \lambda_{0}/\partial \mu = 0$ gives expressions identical to those in Eqs. (A20) and
The only difference between the on-mass-shell and mass-independent renormalization schemes is in the specific values of the coefficients $a_{jk}$ and $c_{jk}$.

[31] P.N. Maher, University of Wisconsin-Madison PhD dissertation, 1991 (unpublished). Several misprints and errors in the results given in the dissertation are corrected here.
FIGURES

FIG. 1. The three distinct topologies for the tree-level \(2 \rightarrow 3\) processes. The exchange graphs (a) and (b) are suppressed in the \(2 \rightarrow 3\) cross section or unitarity sum for \(s \rightarrow \infty\). The “jet” graph (c) is not suppressed because of the contributions of low-mass pairs to the jet.

FIG. 2. Inelastic \(2 \rightarrow 3\) diagrams with the topology of cut Eye graphs for initial \(zH\) states. The terminology is that of Maher et al. [18]. We show: (a), the cut \(E_2\) graph which contributes to \(-Z_w^{(1)}|a_{zH}^{(0)}|^2\); (b), (c), the cut \(E_4\) and \(E_2^*\) graphs which contribute to \(-Z_H^{(1)}|a_{zH}^{(0)}|^2\).

FIG. 3. Behavior of the \(j = 0\) eigenamplitudes for \(w^\pm, z, H\) scattering shown as functions of the running coupling \(\lambda_s = \lambda_s(s, M_H^2)\): (a) \(a_1\) and (b) \(a_3\). The factors associated with the anomalous dimensions are very close to unity for \(\sqrt{s}\) in the TeV range and have been neglected. The curves for \(a_2\) and \(a_4\) are very similar to that for \(a_3\) shown in (b).

FIG. 4. Vector diagram showing the decomposition of the eigenamplitude \(a_1\) into tree-level, one-loop, and two-loop contributions \(a^{(0)}, a^{(1)}, \) and \(a^{(2)}\). The dashed curve is a segment of the unitarity circle. Corresponding values of the running couplings \(\lambda_s\) are shown along the real axis for \(a^{(0)}\), along the upper solid curve for \(a^{(0)} + a^{(1)}\), and along the lower solid curve for \(a^{(0)} + a^{(1)} + a^{(2)}\).

FIG. 5. (a) The ratio of the magnitude \(|a^{(2)}|\) of the two-loop contribution to the \(j = 0\) scattering amplitude \(a_i = a^{(0)} + a^{(1)} + a^{(2)} + \cdots\) to the magnitude \(|a^{(0)} + a^{(1)}|\) of the complete amplitude at one loop, shown as a function of the running coupling \(\lambda_s\). (b) The ratio \(|a^{(2)}|/|a^{(1)}|\) of the magnitudes of the two- and one-loop contributions to the scattering amplitude as a function of \(\lambda_s\). Requiring that each term of the perturbation series be smaller than the previous term (or previous partial sum), gives a limit \(\lambda_s \lesssim 2.3\) for the \(\chi_3 \rightarrow \chi_3\) channel.

FIG. 6. The relation between the running coupling \(\lambda_s\) and the physical mass of the Higgs boson \(M_H\) for various values of the energy at which the unitarity constraint could be applied. The horizontal line is at \(\lambda_s = 2.3\) which we adopt as the maximum value of \(\lambda_s\) for which the perturbation series in \(\lambda_s\) may reasonably be said to converge.