An accurate effective action for ‘baby’ to ‘adult’ skyrmions

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Abstract

Starting with a Chern-Simons theory, we derive an effective action for interacting quantum Hall skyrmions that takes into account both large-distance physics and short-distance details as well. We numerically calculate the classical static skyrmion profile from this action and find excellent agreement with other, microscopic calculations over a wide range of skyrmion sizes including the experimentally relevant one. This implies that the essential physics of this regime might be captured by a continuum classical model rather than resorting to more microscopic approaches. We also show that the skyrmion energy closely follows the formula suggested earlier by Sondhi et al. for a broad parameter range of interest as well.

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I. INTRODUCTION

Quantum Hall systems near filling factor $\nu = 1/(2n + 1)$ should manifest a charged topologically stable object called a skyrmion. Various experiments have shown strong evidence for the existence of this exotic object in two dimensional electron systems. In particular the ground state of the quantum Hall system near filling factor $\nu = 1$ is believed to be a manybody state of weakly interacting skyrmions.

Just as the thermodynamics of certain superconductors can be well-described by their vortex degrees of freedom alone, skyrmions, the defects of the incompressible quantum Hall liquid, may well describe the essential physics of the quantum Hall system near $\nu = 1$. The thermodynamics of vortex systems can be derived from the phenomenological Ginzburg-Landau (GL) description of superconductivity. In traditional BCS-theory the spin-singlet Cooper-pairing mechanism removes the electronic spin degrees of freedom from the dynamics. Hence vortices have no spin structure and are entirely described by their integer charge. In contrast, the quantum Hall skyrmion is a topological spin-texture made of an intricate pattern of electron-spin orientations. It possesses internal degrees of freedom associated with this texture that allow for novel dynamics.

In analogy to GL theory in the superconductivity, the Chern-Simons theory of quantum Hall systems was derived by Zhang et al. and subsequently extended by Lee and Kane to describe the possible spin-unpolarized quantum Hall liquid. In superconductivity, it has been very useful to obtain a phenomenological theory of vortices based on GL theory. A similar theory for skyrmions should be invaluable for understanding transport, phase transitions, and spatial ordering.

In this paper, starting from the Chern-Simons theory we derive an action for the many skyrmion system which takes into account both large-distance physics and relatively short-distance details as well. Using this action we perform numerical studies of the classical skyrmion solution. Recently Abolfath et al. pointed out for small skyrmions, i.e. those typical in GaAs samples, the minimal effective field theory does not give a good quan-
titative agreement with the Hartree-Fock or exact-diagonalization study. We find that the various properties of a static skyrmion solution obtained from the above action exhibit an excellent agreement with the microscopic study for ‘baby’ skyrmions (textures containing from two to about 10 flipped spins) and larger. This implies that current experiments can be well modelled by a continuum theory, obviating the need for the more involved microscopic approaches.

In section II, we briefly summarize the Chern-Simons description of quantum Hall effect and then derive an action for many skyrmion system. In section III, the classical skyrmion profile is numerically calculated. The various properties of a static skyrmion solution are compared with the microscopic study. We show that the skyrmion energy follows the formula suggested earlier by Sondhi et al. very accurately for a broad parameter range. In section IV, we conclude with a summary.

II. CHERN-SIMONS THEORY

We briefly summarize the Chern-Simons description of quantum Hall effect which was subsequently extended by Lee and Kane in order to incorporate a possible spin-unpolarized quantum Hall liquid In the bosonic Chern-Simons theory, an electron is viewed as a composite object of a boson and a flux tube carrying an odd-multiple of flux quanta $\phi_0 = h/e$ attached via a Chern-Simons term, which correctly ensures fermionic statistics for the electron. We begin with the effective Chern-Simons Lagrangian introduced by Lee and Kane

\[
L[\psi, a\mu] = \bar{\psi}_\sigma (\partial_0 - ia_0) \psi_\sigma + \frac{1}{2m} \left| \left( \frac{\nabla}{i} - a - A_{ex} \right) \psi_\sigma \right|^2 \\
+ \frac{1}{2} \int dr' \left( \bar{\psi}_\sigma (r) \psi_\sigma (r) - \bar{\rho} \right) V(|r - r'|) \left( \bar{\psi}_\sigma (r') \psi_\sigma (r') - \bar{\rho} \right) + L_{cs}
\] 

(2.1)

where $\psi_\sigma$ represents a bosonic field with spin $\sigma = \pm \frac{1}{2}$, $m$ the effective mass of boson, $A_{ex}$ a vector potential for the external magnetic field, $\bar{\rho}$ a mean density of boson, and $V(r) = e^2/\epsilon r$ Coulomb interaction between electrons. We will use the convention of units
where $h = c = e = 1$. The Chern-Simons term $L_{cs}$ can be written as

$$L_{cs} = \frac{i}{4\pi\alpha} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$  \hspace{1cm} (2.2)$$

where $a_\mu$ is a statistical gauge field with $\mu = 0, 1, 2$, (1 and 2 are spatial indices, 0 is time) and we impose a Coulomb gauge $\nabla \cdot a = 0$. The variable $\alpha$ is taken to be an odd integer, $\alpha = 2n + 1$, in order to describe a fermionic system.

In order to separate the amplitude and the spin degrees of freedom from the bosonic field $\Psi_\sigma$, we introduce the $CP^1$-field $z_\sigma$, which satisfies $|z_\sigma| = 1$ and is related to $\Psi_\sigma$ by

$$\Psi_\sigma = \sqrt{J_0} z_\sigma(x)$$  \hspace{1cm} (2.3)$$

where $J_0$ represents the boson density. The Lagrangian in terms of the fields $J_0$ and $z_\sigma$ is given by

$$L[z_\sigma, a_\mu] = \frac{1}{2} \int d\mathbf{r}' \left( J_0(\mathbf{r}) - \bar{\rho} \right) V(|\mathbf{r} - \mathbf{r}'|) \left( J_0(\mathbf{r}') - \bar{\rho} \right) + i J_0 \left( \bar{z}_\sigma \frac{\partial_0}{i} z_\sigma - a_0 \right)$$

$$+ \frac{\kappa}{2} \left| z_\sigma \nabla z_\sigma - \mathbf{a} - \mathbf{A}_{\text{ex}} \right|^2 + L_{NL\sigma} + L_{cs}$$  \hspace{1cm} (2.4)$$

where $\kappa = \bar{\rho}/m$ and the gradient term in $J_0(\mathbf{r})$ is neglected. The local spin orientation $\mathbf{m}$ is related to the field $z_\sigma$ via $\mathbf{m} = \bar{z}_\alpha \hat{\sigma}_{\alpha\beta} z_\beta$, so that the static non-linear sigma model (NL$\sigma$) term can be written by

$$L_{NL\sigma} = \frac{\kappa}{8} (\nabla \mathbf{m})^2.$$  \hspace{1cm} (2.5)$$

In order to decouple the quartic term, one introduces a Hubbard-Stratonovich field which represents a bosonic current $\mathbf{J}(x)$. After integrating out the statistical gauge field $a_\mu$, one obtains the following Lagrangian:

$$L[z_\sigma, A_\mu] = \frac{1}{2} \int d\mathbf{r}' \left( J_0(\mathbf{r}) - \bar{\rho} \right) V(|\mathbf{r} - \mathbf{r}'|) \left( J_0(\mathbf{r}') - \bar{\rho} \right) + \frac{1}{2\kappa} |\mathbf{J}|^2 + L_{NL\sigma}$$

$$+ i(A_\mu + A_\mu^{(0)}) J_s^\mu - \frac{i}{2\alpha} A_\mu (J_\mu - J_\mu^{(0)}) + \frac{i}{2\pi} A_0 \left( B - 2\pi \alpha \bar{\rho} \right)$$  \hspace{1cm} (2.6)$$

where $J_\mu = J_\mu^{(0)} + \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda$ and $J_\mu^{(0)} = (\bar{\rho}, 0, 0) = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda^{(0)}$. Here we used the fact that the bosonic current and density satisfies a continuity equation $\partial_\mu J_\mu = 0$. The skyrmion three-vector $J_s^\mu$ can be written in terms of $\mathbf{m}$.
\[ J^s_\mu = \frac{1}{8\pi\alpha} \epsilon_{\mu\nu\lambda} (\partial_\nu m \times \partial_\lambda m) \cdot m \]  

(2.7)

where we explicitly put the factor of \(1/\alpha\) in the definition of \(J^s_\mu\). The \(J^s_0(x)\) is the topological charge density of the spin texture, which is proportional to the electronic charge density.

Zhang et al. have shown that when external magnetic field is tuned so that the number of flux quanta is commensurate with the mean boson density, \((B = 2\pi\alpha\bar{\rho}, \text{ so that, } \nu = 1/(2n + 1))\), the bosons condense and form a superfluid. The bosonic superfluidity in the Chern-Simons theory implies that a quantum Hall effect occurs in the corresponding two-dimensional electron system. The ground state has been shown to be a fully spin-polarized quantum ferromagnet. By noticing that the dynamics of quantum Hall system with a spin-polarized ground state will follow that of a quantum ferromagnet and that the skyrmion is a charged object of the system, Sondhi et al. proposed a phenomenological action, which is valid for the long-wavelength and small-frequency limit.

However we can explicitly integrate out the bosonic field in Eq.(2.6) and derive an action for skyrmions which takes fully into account the short- and long-distance physics. A similar exercise is standard in the GL theory for vortices. We proceed by integrating out the bosonic field \(A_\mu\) with a Coulomb gauge condition, \(\nabla \cdot A = 0\). In order to impose the Coulomb gauge, we introduce an auxiliary field \(\lambda(x)\) and introduce an additional term \(i\lambda(x)\nabla \cdot A\) into Eq.(2.7). Since the action is still quadratic in the field \(A_\mu\) and diagonal in the frequency-momentum space, one can exactly integrate out the bosonic field \(A_\mu\). The integration of the \(\mathbf{k} = 0\) mode leads to the following relations between the skyrmion density \(n_s\) and the external magnetic field \(B\), i.e. \(n_s = (2\pi\alpha\bar{\rho} - B)/(2\pi)\).

After lengthy but straightforward calculations, we finally obtain the following action

\[
S_E[m] = \frac{1}{2} \sum_{\mathbf{k},\omega} \frac{\kappa\alpha^2 V(k)}{P(k, \omega)} |J^s_0|^{2} + \frac{1}{2} \sum_{\mathbf{k},\omega} \frac{\alpha^2}{P(k, \omega)} |J^s|^2 + \int d\tau \int d\mathbf{r} \mathcal{L}_{NL\sigma} + \int d\tau \int d\mathbf{r} \mathcal{L}_z
\]

\[
+ i\alpha \sum_{\mathbf{k},\omega} A^{(0)}(-\mathbf{k}) \cdot J^s(k) - \sum_{\mathbf{k},\omega} \frac{2\pi\kappa\alpha^3}{P(k, \omega)k^2} J^s_0(-\mathbf{k}) \hat{z} \cdot \mathbf{k} \times J^s(k)
\]

\[
+ \sum_{\mathbf{k},\omega} \frac{Q(\omega)\alpha^2}{P^2(k, \omega) + Q^2(\omega)} \left\{ \frac{k \cdot J^s(-\mathbf{k}) (k \times J^s(k)) \cdot \hat{z}}{k^2} - \frac{1}{2} J^s(-\mathbf{k}) \times J^s(k) \cdot \hat{z} \right\}
\]

(2.8)
where \( k \) stands for \((\omega, k)\), \( P(k, \omega) = \kappa \alpha^2 + V(k)k^2/(4\pi^2) + \omega^2/(4\pi^2\kappa) \), and \( Q(\omega) = \alpha \omega/(2\pi) \) and the Zeeman term \( \mathcal{L}_z \) is given by
\[
\mathcal{L}_z = \frac{t}{2\pi\ell^2} \left( 1 - m_z(r) \right)
\]
(2.9)
where the magnetic length \( \ell = (\hbar c/|e|B)^{1/2} \) and \( t = (1/2)g\mu_B B \). In the above action, the first term represents a charge-density interaction between skyrmions including the self-energy contribution. The function \( V(k) \) is the Fourier transform of the Coulomb interaction; due to the additional term \( V(k)k^2/(4\pi^2) \propto k \) in the denominator, the interaction is modified by the short-range fluctuations of the gauge field from a Coulombic one \( \sim 1/r \) to \( \ln(1/r) \) at short-distances. The next term represents the kinetic energy for skyrmion. In the limit of long-wavelengths and small-frequencies, we obtain the following action:
\[
\mathcal{S} = \frac{1}{2} \sum_{k,\omega} V(k) |J_0^s|^2 + \sum_{k,\omega} \frac{1}{2\kappa} |\mathbf{J}|^2 + \int d\tau \int d\mathbf{r} \mathcal{L}_{NL\sigma} + \int d\tau \int d\mathbf{r} \mathcal{L}_z + i\alpha \sum_{k,\omega} A^{(0)}(-k) \cdot \mathbf{J}^s(k) - \frac{2\pi\alpha}{\ell^2} \int d\mathbf{r} \mathcal{L}_z.
\]
(2.10)
The term \( i\alpha A^{(0)}(\mathbf{r}_i) \cdot \mathbf{J}^s_i \) indicates that the skyrmion views the original boson as a magnetic flux tube. The last term in Eq.(2.10) contains the exchange-statistics of skyrmion. It can be re-written into the more suggestive form
\[
i\alpha A_{sk} \cdot \mathbf{J}^s \]
where
\[
\nabla \times A_{sk} = 2\pi\alpha J_0^s.
\]
(2.11)
In order to see the exchange-statistics of a skyrmion, suppose that all the other skyrmions are at rest while one moves around the static skyrmion configuration. When a skyrmion traverses around a closed loop, this term generates a phase proportional to the number of skyrmions enclosed in the loop:
\[
\int d\mathbf{r} \int_0^T dt \ A_{sk} \cdot \mathbf{J}^s = 2\pi\alpha q_{sk} \int_S d\mathbf{r} \ J_0^s
\]
(2.12)
where \( S \) stands for the space enclosed by the closed skyrmion loop. Using the fact that the skyrmion charge \( q_{sk} \) is equal to \( e/\alpha \), one can show that skyrmion picks up a phase \((2\pi/\alpha)N_{enc}\), where \( N_{enc} \) is the number of skyrmions enclosed. Now consider a process which exchanges
two skyrmions: in the rest frame of one of the skyrmions, the exchange corresponds to the other skyrmion moving about the first in a half circle; and hence it picks up a phase $\pi/\alpha$. Since $\alpha$ is $(2n + 1)$, the statistical phase of a skyrmion is $\pi/(2n + 1)$. For $n = 0$, skyrmion is a fermion, while for $n \neq 0$, it’s an anyon.

III. NUMERICAL SOLUTION OF CLASSICAL SKYRMION

We calculate the classical skyrmion solution with varying Zeeman energy and make a comparison to the microscopic result obtained by Hartree-Fock and exact diagonalization studies. We begin with the static energy functional $E[m]$ derived from Eq.(2.8)

$$E[m] = \frac{1}{2} \sum_k \frac{\kappa \alpha^2 V(k)}{\kappa \alpha^2 + V(k)k^2/(4\pi^2)} |J_0|^2 + \frac{\kappa}{8} \int d\mathbf{r} (\nabla m)^2 + \frac{t}{2\pi \ell^2} \int d\mathbf{r} (1 - m_z(r)).$$

Note that the charge-density interaction changes from a Coulombic one $\sim 1/r$ to $\ln(1/r)$ at short-distances. Since the skyrmion size is determined by balancing the Zeeman energy and the Coulomb interaction, the size of skyrmion will be reduced from the estimates of the minimal field theory.

By using the fact that the skyrmion solution is azimuthally-symmetric, we choose the form $m = (\sin \theta(r) \cos \phi, \sin \theta(r) \sin \phi, \cos \theta(r))$. In order to solve for the classical skyrmion profile, we extremize the energy functional $E[m]$ with respect to $\theta(r)$ and obtain the following equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) - \frac{\sin 2\theta}{2r^2} - \sqrt{\frac{32}{\pi}} \tilde{g} \sin \theta + \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \frac{\sin \theta}{r} f(r) = 0$$

where $\tilde{g} = 2t/(e^2/\epsilon \ell)$ and $f(r)$ is given by

$$f(r) = \int dr' \left( \frac{d}{dr} U(r, r') \right) \sin \theta(r') \frac{d\theta}{dr'}$$

where the azimuthally-averaged interaction potential, $U(r, r')$, is given by $U(r, r') = 4 \int_{0}^{\pi/2} d\phi \tilde{V}(\sqrt{(r-r')^2 + 4rr\sin^2 \phi})$ with $\tilde{V}(r) = \int_{0}^{\infty} dq J_0(qr)/(1 + bq)$, $b = 2\sqrt{2/\pi}$, and $J_0$ the Bessel function of the zeroth order. If instead we set $b = 0$, we recover the case of a pure Coulomb interaction, and then Eq.(3.2) agrees with the one obtained by Abolfath et al.
al. We have used the fact that $\kappa$ is equal to $4\rho_s$ where $\rho_s = e^2/(16\sqrt{2}\pi\ell)$ as shown by Sondhi et al. and by Moon et al. We impose the boundary conditions: $\theta(r = 0) = \pi$ and $\theta(r \to \infty) = 0$.

A brief explanation of how the non-local term $f(r)$ is handled is appropriate. In order to calculate the function $f(r)$, we first need to obtain the explicit form of a modified skyrmion interaction $\tilde{V}(r)$ following the integration over momenta $q$ by standard numerical integration methods. The interaction potential $\tilde{V}(r)$ varies as $1/r$ for $r \gg b$ and $(1/b) \ln(1 + b/r)$ for $r \ll b$. By virtue of this asymptotic behaviour, we accurately approximate $\tilde{V}(r)$ by $(1/b) \ln(1 + b/r) + 1/(a_0 + a_1 r + a_2 r^2 + a_3 r^3)$, where the best fit parameters are $a_0 = 13.222$, $a_1 = 6.158$, $a_2 = 1.223$, and $a_3 = 0.0004$. After differentiating $\tilde{V}(\sqrt{(r - r')^2 + 4rr'\sin^2\phi})$ with respect to $r$, the function $R(r, r') \equiv dU(r, r')/dr$ can be written as follows

$$R(r, r') = 4 \int_0^{\pi/2} d\phi \frac{r - r' + 2r' \sin^2\phi}{(r - r')^2 + 4rr'\sin^2\phi} \ln(x)$$

where $x \equiv \sqrt{(r - r')^2 + 4rr'\sin^2\phi}$ and $H(x) \equiv x(d\tilde{V}/dx)$ satisfies $\lim_{x \to 0} H(x) = -1/b$. We can now perform the integration over $\phi$. By noticing that for $r \approx r'$, then $R(r, r')$ as a function of $r'$ asymptotically approaches a step-function $-(\pi/4r)\text{sgn}(r' - r)$, we see that $R(r, r')$ can be decomposed into a regular and a discontinuous part:

$$R(r, r') = \pi/2r \Theta(r - r') + R_{\text{reg}}(r, r')$$

where $\Theta(x)$ is the Heaviside step function and the $R_{\text{reg}}$ is a smooth, continuous function. Then $f(r)$ can be decomposed as well yielding

$$f(r) = \pi/2r(1 + \cos\theta(r)) + \int_0^\infty dr' R_{\text{reg}}(r, r') \frac{d}{dr} \cos\theta(r').$$

The integration in the second term can be accurately done by standard numerical methods.

In order to solve the differential equation numerically, we discretized the equation by $N_0$ segments with uniform spacing $\Delta r$ in units of the magnetic length $\ell$. First, the differential equation is solved for a typical experimental value of $\tilde{g} = 0.015$, which gives a suitable set of parameters for the number of discretization $N_0$ and the spacing $\Delta r$. The set of parameters
are chosen to be $N_0 = 500$ and $\Delta r = 0.1$; the results are not sensitive to reasonable choice of $N_0$ and $\Delta r$. For other values of $\tilde{g}$, we re-scale $\Delta r$ by the length scale $\xi \propto 1/\sqrt{\tilde{g}}$ set by the Zeeman energy, which controls the size of skyrmion: $\Delta r = 0.1\sqrt{0.015/\tilde{g}}$. The differential equation is solved using an iterative method on the finite number of grid $N_0$.

We first calculate the number of spin-flips as a function of $\tilde{g}$ which is defined as follows:\textsuperscript{19}

$$K = \frac{1}{4\pi \ell^2} \int dr (1 - \cos \theta(r)) - \frac{1}{2}. \quad (3.7)$$

In Fig.(1), the solid curve is our field-theoretical result for $K \geq 2$, which shows a very good agreement with the Hartre-Fock calculation by Fertig et al.\textsuperscript{20} Since the Hartree-Fock and exact diagonalization method can not easily calculate the single skyrmion energy due to the long-range nature of Coulomb interaction, the energy cost for creating a charge-neutral object is calculated by Abolfath et al.\textsuperscript{19} However it has been pointed out by Abolfath et al. that the energy difference $\Delta(K)$ between the skyrmion with $K$ and $K + 1$ flipped spins can be still obtained. $\Delta(K)$ corresponds to $2t$ at which an energy level crossing occurs between a skyrmion with $K$ spin-flips and $K + 1$ spin-flips.\textsuperscript{19} Since $K$ is not quantized in the continuum field theory, $\Delta(K)$ roughly corresponds to the value of $2t$ at the field where the number of flipped spins equals $K + 1/2$. In Fig.(2), $\Delta(K)$ is plotted with respect to $K$. The solid curves are obtained from our field-theoretical calculation. Our field-theoretical calculation gives good overall agreement with the microscopic calculations ranging from ‘baby’ skyrmions ($2 < K < 10$) to full-fledged skyrmions. For ‘infant’ skyrmions, i.e. $K \leq 2$, the quantum fluctuations about the classical skyrmion solution will become important.\textsuperscript{8} Since the number of spin flips in typical experiments with GaAs samples\textsuperscript{11,4,20} is $K \sim 3$, our field-theoretical calculation produces a reasonable result over the whole parameter range of interest. For relatively large (‘adult’) skyrmions, Sondhi et al. have suggested the following asymptotic formula for the skyrmion energy\textsuperscript{8,24}: $E[\tilde{g}]/(4\pi \rho_s) = 1 + A/(4\pi)[\tilde{g} \ln(\tilde{g})]^{1/3}$ with $A = (3\pi^2/4)(72/\pi)^{1/6} \approx 24.9$. One can see that in the absence of Zeeman energy, i.e. $\tilde{g} \to 0$, the skyrmion energy correctly approaches to the NL$\sigma$ model result. The above formula is not completely on rigorous footing yet as commented by Sondhi et al. in their
recent publication\textsuperscript{2}. In Fig.(3), the skyrmion energy is plotted with respect to $[\tilde{g} \ln(\tilde{g})]^{1/3}$, which shows a nice linear behaviour for small $\tilde{g}$. We estimate the line slope by a linear fit and $A$ is obtained to be $24.9 \pm 0.1$. Hence we have numerically confirmed that the skyrmion energy follows the formula quite accurately over a very broad parameter range. For example, for typical experimental value of $\tilde{g} = 0.015$ where $K \sim 3.7$, the energy obtained from the above formula is within $0.2\%$ of our numerical estimates. Since the Hartree-Fock calculation is numerically limited to rather small skyrmion sizes\textsuperscript{19} we believe that our field theoretical calculation incorporating short-distance physics will be a valuable tool to get a quantitative information over a wide range of skyrmion sizes.

IV. SUMMARY

We have obtained an effective Skyrmion action which incorporates short-distance physics as well as large-distance physics based on the Chern-Simons theory. We have numerically calculated the classical skyrmion profile and shown that for $K \geq 2$, our field theoretical results exhibit an excellent agreement with the microscopic study. We have also demonstrated that the skyrmion energy very closely follows the formula suggested earlier by Sondhi \textit{et al.} over a broad parameter range of interest. We believe that our field theoretical calculation incorporating short-distance physics will be a valuable tool to get a quantitative information for skyrmion over a wide parameter range of skyrmion sizes with reasonable numerical effort.

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FIGURES

FIG. 1. The number of spin flip as a function of $\tilde{g}$: The solid curve corresponds to our field theoretical result and the open circles the Hartree-Fock calculation by Fertig et al.[20].

FIG. 2. $\Delta(K)$ as a function of the number of spin-flip $K$. The solid curve is calculated from the field theory. The filled circle represents the Hartre-Fock data and the open circles the exact diagonalization result by Abolfath et al.[19].

FIG. 3. The skyrmion energy as a function of $[\tilde{g} \ln(\tilde{g})]^{1/3}$. The line is a linear fit to our numerical data for small $\tilde{g}$. 
K. Moon and K. Mullen  Fig.2
K. Moon and K. Mullen  Fig. 3

\[ E/(4\pi\rho_s) = 1 + A/(4\pi) [\tilde{g} \ln(\tilde{g})]^{1/3} \]

\[ A = 24.9 \]
K. Moon and K. Mullen  Fig. 1