Unified Statistical Description of Quasithermodynamic Systems in
and out of Equilibrium

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Abstract

We propose the method of statistical description of broad class of dynamic
systems (DS) whose equations of motion are determined by two state depending
functions: 1) "energy" - the quantity which conserves in time and 2) "entropy"
- the quantity which does not decrease in time. It is demonstrated that the
behavior of such systems in the equilibrium state reduces to the thermodynamic
laws in particular the Le Chatelier principle is satisfied and so on. Taking into
account the interaction system of interest with ergometer – the device which
continuously measures its energy one can possible to find the system distribution
function in arbitrary non-equilibrium stationary state (NESS). Some general
relations for mean values of certain quantities in NESS which can be compared
with experimental data are obtained.
1 Introduction

The dynamic equations method is the base mathematical tool for investigation of the behavior of complex systems. Such method of description is widely used not only in classical physics where it appeared for the first time but also in chemistry, biology, ecology and other sciences. The types of equations describing sufficiently broad classes of systems (and the processes taking place in them) on the one hand and having some general features which do not depend on the nature of the studying objects on the other hand are especially important. In this connection we should point out such classes of equations: 1) Hamiltonian ones and 2) equations of gradient type. Hamiltonian equations of motion have the form: \( \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \), \( \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \), where \( H(x_i, p_i) \) is the Hamilton function of the system. They are used for describing the evolution of conservative DS with even-dimensional phase space whose behavior is reversible in time. For describing the nonreversible systems with the strong dissipation the equations of gradient type are usually used: \( \frac{dx_i}{dt} = -\frac{\partial \Phi}{\partial x_i} \), where \( \Phi(x_i) \) - Lyapunov function of the system, the quantity which monotonically decreases in time.

Let us now call attention to the study of evolution of closed system with the large number of degrees of freedom. It is well known that such system comes to the equilibrium state without fail at the condition \( t \to \infty \) (zero law of thermodynamics). In his seminal paper of 1865, R. Clausius stated the laws determining the behavior of such systems in the form of the following two propositions (see [1], p. 194):

1) Die Energie der Welt ist constant

2) Die Entropie der Welt strebt einem Maximum zu

We’ll consider the class of DS which satisfy to this Clausius conditions. In this connection we want to ask two questions: 1) What equations of motion with necessity provide such behavior of the system? 2) What results related to the statistical behavior of such systems in and out of equilibrium can be obtained from such equations of motion? The attempt to answer these questions is the main goal of our paper. Note that in phenomenological approach which is used in this paper the terms "energy" and "entropy" will be used only as conventional labels for two quantities satisfying the Clausius conditions. Now we describe briefly the content of the paper. It consists of five sections. In Sec.1 we introduce the notion of quasithermodynamic system (QTS) and give the concrete recipes to construct the equations of motion for such systems. In Sec.2 on the base of these equations we investigate the conditions of equilibrium of two QTS. We prove that in the case when an interaction between two QTS is absent our conditions of equilibrium are reduced to the standard thermodynamic relation: \( \frac{\partial S_1}{\partial E_1} = \frac{\partial S_2}{\partial E_2} \). Besides using the condition of equilibrium state stability the Le Chatelier principle for QTS is derived directly from equations of motion. In Sec.3 we propose the method of statistical description of QTS behavior. In order to avoid the appearance of singular distribution functions we introduce
the interaction between the system of interest and the ergometer-device which continuously measures its energy. Such method gives us the possibility to formulate the simple Fokker-Planck equation for QTS distribution function whose stationary solution can be obtained. This solution can be expressed via state functions $S(x_i)$ and $E(x_i)$ and additional parameter $\beta$. In the limiting case of equilibrium state the solution obtained reduces to the ordinary Gibbs distribution: $f(x_i) \sim \exp(-\beta E(x_i))$. In Sec. 4 we investigate the non-equilibrium stationary states of QTS more detail and formulate some general relations for them. In the last part of the paper we summarize the results and briefly discuss the possible applications of them. Now let us pass to the detailed account.

2 Section 1

We define as quasitermodynamic (of N order) the DS whose state is described by $N$ variables $x_1,..,x_N$ and whose behavior in time satisfies two Clausius conditions (1). Meanwhile we suppose that variables $x_1,..,x_N$ unlike from coordinates and momenta define the ”coarse-grained” description of macroscopic system evolution and therefore the number of them can be arbitrary. Let us establish now the form of equations of motion for given state functions: energy $H(x_1,..,x_N)$ and the entropy $S(x_1,..,x_N)$ providing the realization of the distinctive properties of QTS:

$$\frac{dH}{dt} = \frac{dH}{dx_i} x_i = 0, \quad (2)$$

$$\frac{dS}{dt} = \frac{dS}{dx_i} x_i \geq 0. \quad (3)$$

(In relations (2), (3) and later on we use the usual agreement about the summation on repeating indexes).

We start from the simplest case of QTS of second order ($N = 2$) and suppose that required system of equations of motion is an autonomous system of differential equations: $\frac{dx_i}{dt} = \Phi_i(x_1,x_2)$. By direct verification one can be convinced that taking functions $\Phi_i(x_1,x_2)$ in the form: $\Phi_i \equiv \varepsilon_{ik} \frac{\partial H}{\partial x_k} \{S,H\}$, where $\varepsilon_{ik} \equiv \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ is antisymmetric tensor of 2 rank and $\{S,H\}$ is Poisson bracket of functions $S(x_1, x_2)$ and $H(x_1, x_2)$, i.e. by the definition $\{S,H\} = \frac{\partial S}{\partial x_1} \frac{\partial H}{\partial x_2} - \frac{\partial S}{\partial x_2} \frac{\partial H}{\partial x_1}$, we obtain the equations of motion

$$\frac{dx_i}{dt} = \varepsilon_{ik} \frac{\partial H}{\partial x_k} \{S,H\}, \quad (4)$$

from which one can easily obtain properties (2) and (3).

Similarly in the case $N = 3$ one can be convinced that the system of equations of motion with required properties can be presented in the form:

$$\frac{dx_i}{dt} = \varepsilon_{ikl} \frac{\partial H}{\partial x_k} A_l, \quad (5)$$
where \( \varepsilon_{ikl} \) is the antisymmetric tensor of 3 rank and

\[
A_l \equiv \varepsilon_{lmn} \frac{\partial S}{\partial x_m} \frac{\partial H}{\partial x_n}.
\] (6)

In addition we give the form of equations of motion for the case \( N = 4 \):

\[
\frac{dx_i}{dt} = \varepsilon_{iklm} \frac{\partial H}{\partial x_k} A_{lm},
\] (7)

where \( \varepsilon_{iklm} \) is the antisymmetric tensor of 4 rank and

\[
A_{lm} \equiv \varepsilon_{lmnp} \frac{\partial S}{\partial x_n} \frac{\partial H}{\partial x_p}.
\] (8)

The generalization of equations of motion in the case of arbitrary \( N \) is obvious. We will not discuss here the rigorous mathematical justification of the uniqueness of equations obtained. About this problem one can see the paper [2] and references in it. In this paper we are interested in the properties of QTS implied from equations of motion (4), (5), (7) and the possibility of statistical description of such systems.

3 Section 2

We start our analysis of equations of motion for QTS from the simplest case \( N = 2 \).

Let us rewrite the equations (2) into another, more convenient form. For this purpose we introduce the "phase" \( \varphi \) which is variable canonically conjugated to the energy i.e. \( \{ \varphi, H \} = 1 \). Now one can easily see that the system (2) is equivalent to the following system of equations for \( H \) and \( \varphi \):

\[
\frac{dH}{dt} = 0, \quad \frac{d\varphi}{dt} = \frac{\partial S}{\partial \varphi},
\] (9)

where QTS entropy \( S \) now must be expressed via \( \varphi \) and \( H \). It follows from (9) that in the case when \( S(\varphi, H) \) has maximum on \( \varphi \) there is the line i.e. the one-parameter set of equilibrium points \( \varphi = \varphi_0(E) \) corresponding to the solution of equation \( \frac{\partial S(\varphi,E)}{\partial \varphi} = 0 \) when \( \varphi = \varphi_0(E) \). Similar equilibrium condition is valid in general case also. It is easy to show that equilibrium conditions for QTS of \( N \)-order reduce to the next system of equations (it follows from equality to zero the right parts of (4), (5), (7)):

\[
\frac{\partial S}{\partial \varphi_1} = \frac{\partial S}{\partial \varphi_2} = \ldots = \frac{\partial S}{\partial \varphi_N} \equiv \lambda.
\] (10)

Now let us consider the composite system consisting of two subsystems: A with variables \( x_1, \ldots, x_{N_1} \) and B with coordinates \( y_1, \ldots, y_{N_2} \). Besides we suppose that subsystems A and B are non-interacting i.e. relations \( H = E_A(x_1, \ldots, x_{N_1}) + \)
$E_B(y_1,..,y_{N_2})$ and $S = S_A(x_1,..,x_{N_1}) + S_B(y_1,..,y_{N_2})$. In this case equilibrium conditions reduce to the form:

$$\frac{\partial S_A}{\partial E_A} = \ldots = \frac{\partial S_A}{\partial x_{N_1}} = \frac{\partial S_B}{\partial y_{N_1}} = \ldots = \frac{\partial S_B}{\partial x_{N_2}}. \quad (11)$$

Equations (11) can be satisfied by setting $S_A = S_A(E_A)$ and $S_B = S_B(E_B)$ after that equations reduce to standard condition of thermodynamic equilibrium of two subsystems:

$$\frac{\partial S_A}{\partial E_A} = \frac{\partial S_B}{\partial E_B} \quad (12)$$

Now we demonstrate how using only equilibrium equations (10) one can deduce the Le Chatelier principle. We remind that Le Chatelier principle claims that response of the system in the equilibrium state on the external influence always results in the decreasing of this influence. For proving this principle in the simplest case of 2-order QTS with variables $x$ and $y$ let us consider three states of such system: I - $(x_0, y_0)$ - the initial equilibrium state, II - $(x_0, y_0 + \delta y)$ - nonequilibrium state enforced by small influence on the variable $y$, III - $(x_0 + \tilde{x}, y_0 + \tilde{y})$ - the final equilibrium state. It is obvious that the final equilibrium state III differs from initial equilibrium state I. Because the energy of the system have been changed. The first condition we write down is equality of energies of states II and III:

$$H_{II} = H_{III} = H(x_0, y_0) + \delta H, \quad \delta H = \left(\frac{\partial H}{\partial y}\right)_0 \delta y = \left(\frac{\partial H}{\partial x}\right)_0 \tilde{x} + \left(\frac{\partial H}{\partial y}\right)_0 \tilde{y}. \quad (13)$$

Besides both states I and III are equilibrium states the following conditions must be satisfied:

$$\left(\frac{\partial S}{\partial x}\right)_0 = \left(\frac{\partial S}{\partial y}\right)_0 \equiv \lambda \quad (14)$$

and

$$\frac{S_x + S_x \tilde{x} + S_{xy} \tilde{y}}{H_x + H_{xx} \tilde{x} + H_{xy} \tilde{y}} = \frac{S_y + S_{yx} \tilde{x} + S_{yy} \tilde{y}}{H_y + H_{yx} \tilde{x} + H_{yy} \tilde{y}}, \quad (15)$$

where for simplicity we used notations: $S_x = \left(\frac{\partial S}{\partial x}\right)_0 = \left(\frac{\partial S}{\partial x}\right)_{(x_0,y_0)}$; $S_y = \left(\frac{\partial S}{\partial y}\right)_0$ and so on. We can write $\tilde{x} = A\delta y$ and $\tilde{y} = B\delta y$, where $A$ and $B$ are yet unknown functions of $x_0$ and $y_0$. Equations (13) and (15) now are reduced to the form:

$$H_y = H_y B + H_x A, \quad (16)$$
\[
\frac{(S_{xx}A + S_{xy}B)}{H_x^2} H_x - S_x (H_{xx}A + H_{xy}B) =
\]
\[
= \frac{(S_{yx}A + S_{yy}B)}{H_y^2} H_y - S_y (H_{yx}A + H_{yy}B).
\]

We can simplify the form of equations (16) and (17) by using the quantity \( F = S - \lambda H \). Then taking into account (14) for unknown functions \( A \) and \( B \) we obtain the following system

\[
H_x A + H_y B = H_y, \quad \left( \frac{F_{xx}}{H_x} - \frac{F_{xy}}{H_y} \right) A + \left( \frac{F_{xy}}{H_x} - \frac{F_{yy}}{H_y} \right) B = 0.
\]

Solving the system (18) we obtain

\[
A = \frac{H_y}{\Delta} (H_x F_{yy} - H_y F_{xy}), \quad B = \frac{H_x}{\Delta} (H_y F_{xx} - H_x F_{xy}),
\]

where \( \Delta \equiv H_x^2 F_{xx} - 2H_y H_x F_{xy} + H_y^2 F_{yy} \).

Note that equilibrium stability conditions for state I require inequalities \( F_{xx} < 0, F_{yy} < 0 \) and \( F_{xx} F_{yy} - F_{xy}^2 > 0 \) are valid. In particular this case follows that \( \Delta < 0 \).

Now for proving the Le Chatelier principle it is necessary to introduce the measure of QTS reaction on the exterior influence. As such measure it is natural to choose the magnitude of thermodynamic force \( Y \equiv -\left( \frac{\partial F}{\partial y} \right) \) that is equal to zero in unperturbed state I. Then for state II we have following relation:

\[
\Delta Y \equiv Y_{II} - Y_I = \left( \frac{\partial Y}{\partial y} \right)_0 \delta y = -F_{yy} \delta y \quad \text{and for equilibrium state III} \quad \Delta \tilde{Y} \equiv Y_{III} - Y_I = \left( \frac{\partial Y}{\partial y} \right)_0 \delta y = -(F_{xy} A + F_{yy} B) \delta y.
\]

The Le Chatelier principle will be valid under the condition

\[
F_{xy} A + F_{yy} B \geq F_{yy}.
\]

Substituting \( A \) and \( B \) from (19) into (20) after some simple algebra we came to the inequality \( F_{yy}^2 H_x^2 + H_y^2 F_{xy}^2 \geq 2H_x H_y F_{xy} F_{yy} \), which is obviously valid. Thus the validity of the Le Chatelier principle is proved.

4 Section 3

Now let us pass to the statistical description of QTS and for simplicity we start from the system of 2-order. Our main goal is to determine the distribution function for nonequilibrium states of such system. Note that under pure dynamical description the distribution function \( f(x_i, t) \) must satisfy the continuity equation:

\[
\frac{\partial f}{\partial t} + \text{div}(f \xi_i) = 0.
\]

Because of attractors existence in QTS such function necessarily must be singular. To get the smooth distribution function we will
use the following method. We introduce the interaction of QTS of interest with ergometer - large exterior system which measures its energy From the definition of ergometer it follows that during the measurement process mean energy of QTS system remains the same. As we have showed in [3] such method results in Fokker-Planck equation for \( f(x_i, t) \) of the form

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i} (f \dot{x}_i) = \frac{\partial}{\partial x_j} (D_{ik} \frac{\partial f}{\partial x_k}),
\]  

(21)

where diffusion tensor \( D_{ik} \) representing the influence of ergometer on the system behavior must be chosen in the form:

\[
D_{ik} = \varepsilon_{il} \varepsilon_{km} \frac{\partial H}{\partial x_l} \frac{\partial H}{\partial x_m},
\]  

(22)

Here \( H(x_1, x_2) \) is the energy of QTS. It is obviously that tensor \( D_{ik} \) defined in such a way possesses the symmetry property: \( D_{ik} = D_{ki} \) and the nonnegative property: \( D_{11} > 0, D_{22} > 0 \) and \( D_{11} D_{22} - D_{12}^2 > 0 \). Having the Fokker-Planck equation (15) in hand we can search its stationary solutions corresponding to equations of motion: \( \dot{x}_i = \varepsilon_{ik} \frac{\partial H}{\partial x_k} \{S, H\} \). It is easy to show that all such solutions can be represented in the form:

\[
f(x_1, x_2) = A(H) e^{S(x_1, x_2)},
\]  

(23)

where \( S \) is the entropy of QTS and \( A(H) \) is arbitrary function of its energy. We define function \( A(H) \) from the following physical reasons. First of all let us pay attention to the fact that QTS entropy is defined up to arbitrary function because the transformation: \( S(x_1, x_2) \rightarrow S(x_1, x_2) + C(H) \) does not change the form of equations of motion. Besides we must require the distribution function (17) transforms into well-known Gibbs distribution in the limiting case of equilibrium state. It is easy to verify these conditions impose by the only possible choice of distribution function in the form:

\[
f(x_1, x_2) = N e^{S(x_1, x_2) - S_{eq}(x_1^0, x_2^0) - \beta H(x_1, x_2)},
\]  

(24)

where \( N \) is normalizing coefficient defined by the condition:

\[
\int f(x_1, x_2) dx_1 dx_2 = 1,
\]

and \( S_{eq}(x_1^0, x_2^0) \) is the entropy in the equilibrium state what is the function of the QTS energy and \( \beta \) is the parameter having sense of inverse temperature of system in the equilibrium state. The relation (24) for distribution function of QTS is valid both in equilibrium and nonequilibrium states and it is a main result of this section of the paper. Note that relation of type (24) for distribution function can be derived also on the base of information-theoretic Jaynes method [4] but we suppose these two methods are essentially different. The main difference that in our method the "entropy" is actually the dynamic quantity defined by equations of motion. So it can be used for describing both equilibrium and
nonequilibrium states of QTS. Also it is worth to note that for deriving the distribution function (24) from Fokker-Planck equation the Markov approach of system evolution is required. This fact is ignored in Jaynes method.

5 Section 4

In this section using the explicit form of distribution function (24) we find the number of relations which are connecting the different characteristics of QTS. First of all we point to simple relation between relaxation time \( \tau(E) \) and entropy of QTS in the equilibrium state. For this purpose we write down the equation of motion for phase close to equilibrium line \( \varphi = \varphi_0(E) \). In the lowest approximation it has a form: \( \dot{\varphi} = -\frac{[\varphi - \varphi_0(E)]}{\tau(E)} \). Comparing this equation with (9) we derive:

\[
\tau(E) = -\frac{1}{\frac{d^2 S}{d\varphi^2}}_{\varphi = \varphi_0(E)}.
\]  

(25)

It is obviously not far from equilibrium line QTS entropy may be written in the following form up to terms of second order:

\[
S(\varphi, E) = S_{eq}(E) - \frac{(\varphi - \varphi_0(E))^2}{2\tau(E)}.
\]  

(26)

Comparing (20) with (18) we derive the distribution function for NESS close to equilibrium:

\[
f(\varphi, E) = N e^{-\frac{(\varphi - \varphi_0(E))^2}{2\tau(E)}} e^{-\beta E}.
\]  

(27)

We denote by \( \Delta \varphi \) the deviation of system phase from its equilibrium value: \( \Delta \varphi = \varphi - \varphi_0(E) \). Let us evaluate the mean value \( \langle (\Delta \varphi)^2 \rangle \) in NESS. From the definition of mean value we obtain

\[
\langle (\Delta \varphi)^2 \rangle = \int \frac{(\Delta \varphi)^2}{\tau(E)} e^{-\frac{(\Delta \varphi)^2}{2\tau(E)} - \beta E} d\varphi dE \cdot \int \frac{e^{-\frac{(\Delta \varphi)^2}{2\tau(E)} - \beta E} d\varphi dE}{\sqrt{\tau(E) e^{-\beta E}}} = 1.
\]  

(28)

Relation \( \langle (\Delta \varphi)^2 \rangle \) \text{NESS} = 1 is valid close to equilibrium state as one can see from its derivation. Now we derive the relation which is valid not only close to equilibrium. Let us consider the mean value \( \langle \frac{\partial(S - S_{eq})}{\partial E} \rangle \) \text{NESS}. Using the definition of mean value and integrating by parts in numerator we obtain:

\[
\langle \frac{\partial(S - S_{eq})}{\partial E} \rangle \big|_{NESS} = \int \frac{\partial(S - S_{eq})}{\partial E} e^{S - S_{eq} - \beta E} d\varphi dE \cdot \int e^{S - S_{eq} - \beta E} d\varphi dE = \beta.
\]  

(29)
The obtained relation (29) connects mean value of $\frac{\beta(S - S_{eq})}{\partial E}$ in nonequilibrium stationary state with inverse temperature of equilibrium state of the same mean energy. And finally let us set the connection between the relaxation time $\tau(E)$ and inverse temperature $\beta$. For this purpose we consider the value

$$Z(\beta) = \int_0^\infty \sqrt{\tau(E)} e^{-\beta E} dE. \quad (30)$$

As follows from (22) the function $\sqrt{\tau(E)} e^{-\beta E}$ has the sense of non-normalized distribution function of QTS close to equilibrium. Now we find the mean energy of the system $- U$. By the definition we have:

$$U = \langle E \rangle = \frac{\int E \sqrt{\tau(E)} e^{-\beta E} dE}{\int \sqrt{\tau(E)} e^{-\beta E} dE} = -\frac{\partial}{\partial \beta} \ln Z(\beta). \quad (31)$$

We suppose $\tau(E)$ to be monotonically increasing function of $E$ and also suppose that the function $\sqrt{\tau(E)} e^{-\beta E}$ possesses the sharp maximum on $E$.

Then we can find the value $Z(\beta)$ using the saddle point method: $Z(\beta) \approx \tau(U) e^{-\beta U}$, where mean energy $U$ is defined by the condition of integrand function maximum in (30) $\frac{\partial}{\partial U} \ln \tau(U) = 0$. From that we find the required connection between $\tau$ and $\beta$:

$$\beta(U) = \frac{1}{2} \frac{d}{dU} \ln \tau(U). \quad (32)$$

Note though we investigated only the simplest case of QTS of 2-order nevertheless all results obtained in Sec.3 and Sec.4 may be generalized with the small modifications to the case of arbitrary $N$. For example for $N = 3$ the diffusion tensor corresponding to the interaction with ergometer must be chosen in the following form: $D_{ik} = \varepsilon_{ilm} \varepsilon_{klm} \frac{\partial H}{\partial x_m} \frac{\partial H}{\partial x_n}$. One can see the form of stationary distribution function (24) derived as the solution of Fokker-Planck equation (21) remains the same. We give also the analog of (32) for QTS of $N$-order without derivation. In this case system has the collection of relaxation times: $\tau_1(E), \tau_2(E), ... \tau_{N-1}(E)$. They are connected with the inverse temperature of the equilibrium state by the relation:

$$\beta(E) = \frac{1}{2} \frac{d}{dE} \left[ \ln \prod_{i=1}^{N-1} \tau_i(E) \right]. \quad (33)$$

### 6 Conclusion

Let us sum up the results of the paper. Our main goal was to demonstrate how using Clausius conditions (1) one can successively construct classes of DS of increasing complexity exactly satisfying to these conditions. We demonstrated also that properties of such systems in the equilibrium coincide with properties
of usual thermodynamic systems. That’s why the term “quasithermodynamic system” was proposed. Besides using the interaction of QTS of interest with ergometer we could give the statistical description of nonequilibrium stationary states and obtain universal distribution function for them. Note though we investigated the case of classical systems only the proposed method can be extended to the quantum realm. Speaking more accurately the equations of motion for QTS (4) admit the semiclassical quantization, for example by using the method of paper [5]. Detailed investigation of quantum-mechanical variant of QTS description will be given in future publications. Concerning to the possible application of obtained results we must point out that they are highly diverse and may be applicable not only in physics. We bring here only one such example concerning to the chemical kinetics. It is known (see e.g. [6]) that for closed chemical systems where the chemical reactions not higher then second-order on concentrations of reagents \( x_i \geq 0 \) take place one can write down the following system of kinetic equations:

\[
\frac{dx_i}{dt} = \beta_{ik} x_k + \gamma_{ikl} x_k x_l.
\]  

(34)

We suppose the absence of autocatalytic reactions in such system and condition \( \sum_i \rho_i x_i = M \) where \( \rho_i \) is molecular mass of reagent \( x_i \) is satisfied. Then it is easy to demonstrate that all such systems are quasithermodynamic. Let us illustrate this statement on the simplest case of two reagents \( x_1 \) and \( x_2 \) with coinciding molecular masses: \( \rho_1 = \rho_2 \).

In this case kinetic equations have the form:

\[
\begin{align*}
\frac{dx_1}{dt} &= -\alpha x_1 + \beta x_2 - \gamma x_1 x_2, \\
\frac{dx_2}{dt} &= \alpha x_1 - \beta x_2 + \gamma x_1 x_2.
\end{align*}
\]  

(35)

One can easy see by the direct verification that using functions

\[
\begin{align*}
H &= x_1 + x_2, \\
S &= -\frac{\alpha x_1^2}{2} - \frac{\beta x_2^2}{2} - \frac{\gamma x_1^3}{6} - \frac{\gamma x_1^2 x_2}{2}.
\end{align*}
\]  

(36)

we can rewrite the system (35) in a standard form: \( \dot{x}_i = \varepsilon_{ik} \partial H \partial x_k \{S, H\} \). This fact means that all results of the paper, in particular the possibility of description of reagents concentrations fluctuations, become applicable to the closed chemical systems. From this example we can see also that the nature of functions \( H \) and \( S \) is defined exclusively by character of the considered system that is by its equations of motion and may be essentially different from their thermodynamic analogs.

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