Dissipative Properties of $\omega$-Order Preserving Partial Contraction Mapping in Semigroup of Linear Operator

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Abstract

This paper consists of dissipative properties and results of dissipation on infinitesimal generator of a $C_0$-semigroup of $\omega$-order preserving partial contraction mapping (semigroup of linear operator) which is an example of $C_0$-semigroup. Furthermore, let $Mm(\mathbb{N})$ be a matrix, $L(X)$ a bounded linear operator on $X$, $P_n$ a partial transformation semigroup, $\rho(A)$ a resolvent set, $F(x)$ a duality mapping on $X$ and $A$ is a generator of $C_0$-semigroup. Taking the importance of the dissipative operator in a semigroup of linear operators into cognizance, dissipative properties characterized the generator of a semigroup of linear operator which does not require the explicit knowledge of the resolvent.

This paper will focus on results of dissipative operator on $\omega$-OCP$_n$ on Banach space as an example of a semigroup of linear operator called $C_0$-semigroup.

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1. Introduction

Suppose $X$ is a Banach space, $X_n \subseteq X$ a finite set, $\left\{ T(t) \right\}_{t \geq 0}$ the $C_0$-semigroup that is strongly continuous one-parameter semigroup of bounded linear operator in $X$. Let $\omega$-OCP$_n$ be $\omega$-order-preserving partial contraction mapping (semigroup of linear operator) which is an example of $C_0$-semigroup. Furthermore, let $Mm(\mathbb{N})$ be a matrix, $L(X)$ a bounded linear operator on $X$, $P_n$ a partial transformation semigroup, $\rho(A)$ a resolvent set, $F(x)$ a duality mapping on $X$ and $A$ is a generator of $C_0$-semigroup. Taking the importance of the dissipative operator in a semigroup of linear operators into cognizance, dissipative properties characterized the generator of a semigroup of linear operator which does not require the explicit knowledge of the resolvent.

This paper will focus on results of dissipative operator on $\omega$-OCP$_n$ on Banach space as an example of a semigroup of linear operator called $C_0$-semigroup.

Yosida [1] proved some results on differentiability and representation of one-parameter semigroup of linear operators. Miyadera [2], generated some
strongly continuous semigroups of operators. Feller [3], also obtained an unbounded semigroup of bounded linear operators. Balakrishnan [4] introduced fractional powers of closed operators and semigroups generated by them. Lumer and Phillips [5], established dissipative operators in a Banach space and Hille & Phillips [6] emphasized the theory required in the inclusion of an elaborate introduction to modern functional analysis with special emphasis on functional theory in Banach spaces and algebras. Batty [7] obtained asymptotic behaviour of semigroup of operator in Banach space. More relevant work and results on dissipative properties of \( \omega \)-Order preserving partial contraction mapping in semigroup of linear operator could be seen in Engel and Nagel [8], Vrabie [9], Laradji and Umar [10], Rauf and Akinyele [11] and Rauf et al. [12].

2. Preliminaries

**Definition 2.1** (\( C_0 \)-Semigroup) [9]

\( C_0 \)-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

**Definition 2.2** (\( \omega \)-OCP\(_n\)) [11]

Transformation \( \alpha \in P_n \) is called \( \omega \)-order-preserving partial contraction mapping if \( \forall x, y \in \text{Dom} \alpha : x \leq y \Rightarrow \alpha x \leq \alpha y \) and at least one of its transformation must satisfy \( \alpha y = y \) such that \( T(t+s) = T(t)T(s) \) whenever \( t, s > 0 \) and otherwise for \( T(0) = I \).

**Definition 2.3** (Subspace Semigroup) [8]

A subspace semigroup is the part of \( A \) in \( Y \) which is the operator \( A^* \) defined by \( A^* y = Ay \) with domain \( D(A^*) = \{ y \in D(A) \cap Y : Ay \in Y \} \).

**Definition 2.4** (Duality set)

Let \( X \) be a Banach space, for every \( x \in X \), a nonempty set defined by \( F(x) = \{ x^* \in X^* : (x, x^*) = \| x \|^2 = \| x^* \|^2 \} \) is called the duality set.

**Definition 2.5** (Dissipative) [9]

A linear operator \( (A, D(A)) \) is dissipative if each \( x \in X \), there exists \( x^* \in F(x) \) such that \( \text{Re}(Ax, x^*) \leq 0 \).

2.1. Properties of Dissipative Operator

For dissipative operator \( A : D(A) \subseteq X \rightarrow X \), the following properties hold:

a) \( \lambda - A \) is injective for all \( \lambda > 0 \) and

\[
\| (\lambda - A)^{-1} \| \leq \frac{1}{\lambda} \| y \| \tag{2.1}
\]

for all \( y \) in the range \( \text{rg}(\lambda - A) = (\lambda - A)D(A) \).

b) \( \lambda - A \) is surjective for some \( \lambda > 0 \) if and only if it is surjective for each \( \lambda > 0 \). In that case, we have \( (0, \infty) \subset \rho(A) \), where \( \rho(A) \) is the resolvent of the generator \( A \).

c) \( A \) is closed if and only if the range \( \text{rg}(\lambda - A) \) is closed for some \( \lambda > 0 \).

d) If \( \text{rg}(A) \subseteq D(A) \), that is if \( A \) is densely defined, then \( A \) is closable. Its closure \( A \) is again dissipative and satisfies \( \text{rg}(\lambda - A) = \text{rg}(\lambda - A) \) for all \( \lambda > 0 \).
Example 1

2 × 2 matrix \[ M_m \left( \mathbb{N} \cup \{0\} \right) \]

Suppose 

\[ A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \]

and let \( T(t) = e^{it} \), then

\[ e^{it} = \begin{pmatrix} e^t & e^{2t} \\ e^{2t} & e^{2t} \end{pmatrix} \]

3 × 3 matrix \[ M_m \left( \mathbb{N} \cup \{0\} \right) \]

Suppose 

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ -2 & 3 \end{pmatrix} \]

and let \( T(t) = e^{it} \), then

\[ e^{it} = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & e^{2t} & e^{3t} \\ I & e^{2t} & e^{3t} \end{pmatrix} \]

Example 2

In any 2 × 2 matrix \( M_m \left( \mathbb{C} \right) \), and for each \( \lambda > 0 \) such that \( \lambda \in \rho(A) \) where \( \rho(A) \) is a resolvent set on \( X \).

Also, suppose

\[ A = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \]

and let \( T(t) = e^{it} \), then

\[ e^{it} = \begin{pmatrix} e^{it} & e^{2it} \\ e^{it} & e^{2it} \\ I & e^{2it} \end{pmatrix} \]

Example 3

Let \( X = C_m \left( \mathbb{N} \cup \{0\} \right) \) be the space of all bounded and uniformly continuous function from \( \mathbb{N} \cup \{0\} \) to \( \mathbb{R} \), endowed with the sup-norm \( \| \cdot \|_\infty \) and let \( \{ T(t); t \geq 0 \} \subseteq L(X) \) be defined by

\[ [T(t)f](s) = f(t+s) \]

For each \( f \in X \) and each \( t,s \in \mathbb{R}_+ \), it is easily verified that \( \{ T(t); t \geq 0 \} \) satisfies Examples 1 and 2 above.

Example 4

Let \( X = C[0,1] \) and consider the operator \( Af = -f' \) with domain \( D(A) = \{ f \in C'[0,1]: f(0) = 0 \} \). It is a closed operator whose domain is not dense. However, it is dissipative, since its resolvent can be computed explicitly as

\[ R(\lambda,A)f(t) = \int_0^t e^{-\lambda(t-s)}f(s) \, ds \]
for \( t \in [0,1] \), \( f \in C[0,1] \). Moreover, \( \|R(\lambda, A)\| \leq \frac{1}{\lambda} \) for all \( \lambda > 0 \). Therefore \((A, D(A))\) is dissipative.

2.2. Theorem (Hille-Yoshida [9])

A linear operator \( A : D(A) \subseteq X \to X \) is the infinitesimal generator for a \( C_0 \)-semigroup of contraction if and only if

1) \( A \) is densely defined and closed,
2) \( (0, +\infty) \subseteq \rho(A) \) and for each \( \lambda > 0 \)

\[
\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda} \quad (2.2)
\]

2.3. Theorem (Lumer-Phillips [5])

Let \( X \) be a real, or complex Banach space with norm \( \| \| \), and let us recall that the duality mapping \( F : X \to 2^* \) is defined by

\[
F(x) = \left\{ x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2 \right\} \quad (2.3)
\]

for each \( x \in X \). In view of Hahn-Banach theorem, it follows that, for each \( x \in X \), \( F(x) \) is nonempty.

2.4. Theorem (Hahn-Banach Theorem [2])

Let \( V \) be a real vector space. Suppose \( p : [0, +\infty] \) is mapping satisfying the following conditions:

1) \( p(0) = 0 \);
2) \( p(tx) = tp(x) \) for all \( x \in V \) and real of \( t \geq 0 \); and
3) \( p(x + y) \leq p(x) + p(y) \) for every \( x, y \in V \).

Assume, furthermore that for each \( x \in V \), either both \( p(x) \) and \( p(-x) \) are \( \infty \) or that both are finite.

3. Main Results

In this section, dissipative results on \( \omega-OCP_n \) as a semigroup of linear operator were established and the research results (Theorems) were given and proved appropriately:

**Theorem 3.1**

Let \( A \in \omega-OCP_n \) where \( A : D(A) \subseteq X \to X \) is a dissipative operator on a Banach space \( X \) such that \( \lambda - A \) is surjective for some \( \lambda > 0 \). Then

1) the part \( A \), of \( A \) in the subspace \( X_0 = D(A) \) is densely defined and generates a constrain semigroup in \( X_0 \), and
2) considering \( X \) to be a reflexive, \( A \) is densely defined and generates a contraction semigroup.

**Proof**

We recall from Definition 2.3 that

\[
A_x = Ax \quad (3.1)
\]
for
\[ x \in D(\mathcal{A}) = \{ x \in D(A) : Ax \in X_0 \} = R(\lambda, A)X_0 \quad (3.2) \]

Since \( R(\lambda, A) \) exists for \( \lambda > 0 \), this implies that \( R(\lambda, A)_x = R(\lambda, A)_x \), hence
\[ (0, \infty) \subset \rho(\mathcal{A}) \]

we need to show that \( D(\mathcal{A}) \) is dense in \( X_0 \).

Take \( x \in D(A) \) and set \( x_n = nR(n, A)x \). Then \( x_n \in D(A) \) and
\[ \lim x_n = \lim R(n, A)Ax + x = x, \]

since \( \|R(n, A)\| \leq \frac{1}{n} \). Therefore the operators \( nR(n, A) \) converge pointwise on \( D(A) \) to the identity. Since \( \|nR(n, A)\| \leq 1 \) for all \( n \in \mathbb{N} \), we obtain the convergence of \( y_n = nR(n, A)y \to y \) for all \( y \in X_0 \). If for each \( y_n \) in \( D(A) \), the density of \( D(\mathcal{A}) \) in \( X_0 \) is shown which proved (i).

To prove (ii), we need to obtain the density of \( D(\mathcal{A}) \).

Let \( x \in X \) and define \( x_n = nR(n, A)x \in D(A) \). The element \( y = nR(1, A)x \), also belongs to \( D(A) \). Moreover, by the proof of (i) the operators \( nR(n, A) \) converges towards the identity pointwise on \( X_0 = D(\mathcal{A}) \). It follows that
\[ y_n = R(1, A)x_n = nR(n, A)R(1, A)x \to y \quad \text{for } n \to \infty \]

Since \( X \) is reflexive and \( \{ x_n : n \in \mathbb{N} \} \) is bounded, there exists a subsequence, still denoted by \( (x_n)_{(n \in \mathbb{N})} \), that converges weakly to some \( z \in X \). Since \( x_n \in D(A) \), implies that \( z \in D(\mathcal{A}) \).

On the other hand, the elements \( x_n = (1 - A)y_n \) converges weakly to \( z \), so the weak closedness of \( A \) implies that \( y \in D(A) \) and \( x = (1 - A)y = z \in D(A) \) which proved (ii).

**Theorem 3.2**

The linear operator \( A : D(\mathcal{A}) \subseteq X \to X \) is a dissipative if and only if for each \( x \in D(A) \) and \( \lambda > 0 \), where \( A \in \omega\text{-OCP}_n \), then we have
\[ \| (\lambda_i - A)x \| \geq \lambda \| x \| \quad (3.3) \]

**Proof**

Suppose \( A \) is dissipative, then, for each \( x \in D(A) \) and \( \lambda > 0 \), there exists \( x^* \in F(x) \) such that \( \text{Re}(\lambda x - Ax, x^*) \leq 0 \). Therefore
\[ \|x\|\|\lambda x - Ax\| \geq |(\lambda x - Ax, x)| \geq \text{Re}(\lambda x - Ax, x) \geq \lambda \|x\|^2 \]

and this completes the proof. Next, let \( x \in D(A) \) and \( \lambda > 0 \).

Let \( y_j^* \in F(\lambda x - Ax) \) and let us observe that, by virtue of (3.3), \( \lambda x - Ax = 0 \) \( \Rightarrow \) \( x = 0 \).

So, in this case, we clearly have \( \text{Re}(x^*, \lambda x - Ax) = 0 \). Therefore, by assuming that \( \lambda x - Ax \neq 0 \) As a consequence, \( y_j^* \neq 0 \), and thus
\[ z^*_x = \frac{y_j^*}{\|y_j^*\|^2} \]
lies on the unit ball, i.e. \( \|z\| = 1 \). We have \( (\lambda x - Ax, z^*) = \|\lambda x - Ax\| \geq \lambda \|x\| \Rightarrow Re(x, z) - Re(Ax, z^*) \leq \lambda \|x\| - Re(Ax, z^*) \) hence 
\[
Re(Ax, z^*) \leq 0
\]

and \( Re(z^*, x) \geq \|x\|^{-1} \|Ax\| \). Now, let us recall that the closed unit ball in \( X^* \) is weakly-star compact. Thus, the net \( (z^*_{\lambda})_{\lambda > 0} \) has at least one weak-star cluster point \( z^* \in X^* \) with 
\[
\|z^*\| \leq 1
\]

From (3.4), it follows that \( Re(Ax, z^*) \leq 0 \) and \( Re(x, z^*) \geq \|x\| \). Since \( Re(x, z^*) \leq \|x\| \leq \|x\| \), it follows that \( (x, z^*) = \|x\| \). Hence \( x^* = \|x\|z^* \in F(x) \) and \( Re(Ax, x^*) \leq 0 \) and this completes the proof.

**Proposition 3.3**

Let \( A : D(A) \subseteq X \to X \) be infinitesimal generator of a \( C_0 \)-semigroup of contraction and \( A \in \omega-OCP_n \). Suppose \( X_* = D(A) \) is endowed with the graph-norm \( \|u\|_{D(A)} : X_* \to \mathbb{N} \cup \{0\} \) defined by \( \|u\|_{D(A)} = \|u - Au\| \) for \( u \in X_* \). Then operator \( A : D(A) \subseteq X_* \to X_* \) defined by
\[
\begin{align*}
D(A) &= \{x \in X_* ; Ax \in X_*\} \\
A_x = Ax, \quad &\text{for } x \in D(A)
\end{align*}
\]
is the infinitesimal generator of a \( C_0 \)-semigroup of contractions on \( X_* \).

**Proof**

Let \( \lambda > 0 \) and \( f \in X_* \) and let us consider the equation \( \lambda u - Au = f \). Since \( A \) generates a \( C_0 \)-semigroup of contraction [6], it follows that this equation has a unique solution \( u \in D(A) \).

Since \( f \in X_* \), we conclude that \( Au \in D(A) \) and thus \( u \in D(A) \).

Thus \( \lambda u - Au = f \). On the other hand, we have
\[
\begin{align*}
\| (\lambda I - A)^{-1} f \|_{D(A)} &= \| (I - A)(\lambda I - A)^{-1} f \| \\
&= \| (\lambda I - A)^{-1} (I - A) f \| \leq \frac{1}{\lambda} \| f - Af \| = \frac{1}{\lambda} \| f \|_{D(A)}
\end{align*}
\]
which shows that \( A \) satisfies condition (ii) in Theorem 2.2. Moreover, it follows that \( A \) is closed in \( X_* \).

Indeed, as \( (\lambda I - A)^{-1} \in L(X_*) \), it is closed, and consequently \( \lambda I - A \) enjoys the same property which proves that \( A \) is closed.

Now, let \( x \in X_* \), \( \lambda > 0 \), \( A \in \omega-OCP_n \) and let \( x_\lambda = \lambda x - Ax \). Clearly \( x_\lambda \in D(A) \), and in addition \( \lim_{\lambda \to 0} \| x_\lambda - x \|_{D(A)} = 0 \). Thus, \( D(A) \) is dense in \( X_* \) by virtue of Theorem 2.2, \( A \) generates a \( C_0 \)-semigroup of contraction on \( X_* \). Hence the proof.

**4. Conclusion**

In this paper, it has been established that \( \omega-OCP_n \) possesses the properties of dissipative operators as a semigroup of linear operator, and obtaining some dis-
sipative results on $\omega$-OCP$_n$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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