NOETHER’S INVERSE SECOND THEOREM IN HOMOLOGY TERMS

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Abstract

A generic degenerate Lagrangian system of even and odd variables on an arbitrary smooth manifold is examined in terms of the Grassmann-graded variational bicomplex. Its Euler–Lagrange operator obeys Noether identities which need not be independent, but satisfy first-stage Noether identities, and so on. However, non-trivial higher-stage Noether identities are ill defined, unless a certain homology condition holds. We show that, under this condition, there exists the exact Koszul–Tate chain complex whose boundary operator produces all non-trivial Noether and higher-stage Noether identities of an original Lagrangian system. Noether’s inverse second theorem that we prove associates to this complex a cochain sequence whose ascent operator provides all gauge and higher-stage gauge supersymmetries of an original Lagrangian.

Introduction

Since Noether identities of a Lagrangian systems of even variables are parameterized by elements of a Grassmann algebra, we address from the beginning a generic degenerate Lagrangian system of even and odd variables on an arbitrary smooth manifold. It is described in terms of the Grassmann-graded variational bicomplex [2, 5, 15] generalizing the well-known variational bicomplex for even Lagrangian systems on fiber bundles [1, 14, 19] (Section 1). Theorem 1 provides its relevant cohomology.

Any Euler–Lagrange operator obeys trivial Noether identities which are defined as boundaries of a certain chain complex (Definition 2). A Lagrangian system is said to be degenerate if its Euler–Lagrange operator obeys non-trivial Noether identities given by homology of this complex. Noether identities need not be independent, but satisfy non-trivial first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. Thus, we have a hierarchy of reducible Noether identities. A problem is that trivial higher-stage Noether identities need not be boundaries.

The notion of reducible Noether identities has come from that of reducible constraints. Their Koszul–Tate complex has been invented by analogy with that of constraints under a rather restrictive regularity condition that field equations as well as Noether identities of arbitrary stage can be locally separated into the independent and dependent ones [9]. This condition has also come from the case of constraints locally given by a finite number of functions which the inverse mapping theorem is applied to. A problem is that, in contrast with constraints, Noether and higher-stage Noether identities are differential operators. They
are locally given by a set of functions and their jet prolongations on an infinite order jet manifold. Since the latter is a Fréchet, but not Banach manifold, the inverse mapping theorem fails to be valid.

In Section 2, we show that, if Noether and higher-stage Noether identities are finitely generated and iff a certain homology regularity condition (Definition 7) holds, one can associate to the Euler–Lagrange operator of a degenerate Grassmann-graded Lagrangian system the exact Koszul–Tate complex (54) whose boundary operator (55) produces all non-trivial Noether and higher-stage Noether identities (Theorem 2).

Noether’s second theorems in different formulations relate the Noether and higher-stage Noether identities to the gauge and higher-stage gauge symmetries and supersymmetries of a Lagrangian system [4, 5, 12]. In Section 3, we prove Noether’s inverse second theorem (Theorem 3) which associates to the above mentioned Koszul–Tate complex the cochain sequence (59), whose ascent operator (60) provides gauge and higher-stage gauge supersymmetries of an original Lagrangian system. This operator need not be nilpotent. Therefore, a formulation of Noether’s direct second theorem in cohomology terms meets difficulties. However, the ascent operator admits a nilpotent extension and the above mentioned cochain sequence is a complex if gauge and higher-stage gauge supersymmetries of an original Lagrangian system constitute an algebra (Remark 7).

In Section 4, an example of a reducible degenerate Lagrangian system coming from the topological BF theory is examined in detail.

1. Preliminary. Grassmann-graded Lagrangian systems

Smooth manifolds throughout are real, finite-dimensional, Hausdorff, second-countable (hence, paracompact) and connected. The symbols Λ, Σ, Ξ stand for symmetric multi-indices, e.g., Λ = (λ_1,...,λ_k), |Λ| = k, and λ + Λ = (λλ_1,...,λk).

Let Y → X, dim X = n, be a fiber bundle and J^rY, r ∈ N, the jet manifolds of its sections. The index r = 0 stands for Y. There is the inverse system of affine bundles

\[ \begin{array}{c}
X \leftarrow Y \leftarrow J^1Y \leftarrow \cdots \leftarrow J^{r-1}Y \leftarrow J^rY \leftarrow \cdots,
\end{array} \tag{1} \]

whose projective limit (J^∞Y; π^∞_r : J^∞Y → J^rY) is a paracompact Fréchet manifold [19]. A bundle atlas \((U_Y; x^λ, y^i)\) of Y → X induces the coordinate atlas

\[ ((\pi^∞_0)^{-1}(U_Y); x^λ, y^i, y^i_λ, \ldots, y^i_Λ, \ldots), \quad 0 \leq |Λ|, \tag{2} \]

of J^∞Y. The inverse system (1) yields the direct system

\[ \begin{array}{c}
\mathcal{O}^*X \xrightarrow{\pi^*} \mathcal{O}^*Y \xrightarrow{\pi^1_0} \mathcal{O}^*_1Y \longrightarrow \cdots \mathcal{O}^*_rY \xrightarrow{\pi^*_r} \mathcal{O}^*_Y \longrightarrow \cdots
\end{array} \tag{3} \]

of graded differential algebras (henceforth GDAs) \(\mathcal{O}^*_rY\) of exterior forms on jet manifolds \(J^rY\) with respect to the pull-back monomorphisms \(\pi^*_r\). Its direct limit is the GDA \(\mathcal{O}^*_∞Y\).
of all exterior forms on finite order jet manifolds modulo the pull-back identification. The GDA $\mathcal{O}_*^\infty Y$ is split into the above mentioned variational bicomplex describing Lagrangian systems on a fiber bundle $Y \to X$.

**Remark 1.** The GDA $\mathcal{O}_*^\infty Y$ is a subalgebra of the GDA considered in [19]. Let $\mathcal{G}_r^*$ be the sheaf of germs of exterior forms on $J^rY$ and $\mathcal{G}_r^*$ its canonical presheaf (we follow the terminology of [17]). There is the direct system of presheaves
$$\mathcal{G}^* \to \mathcal{G}_1^* \to \cdots \mathcal{G}_r^* \to \cdots,$$
whose direct limit is a presheaf of GDAs on $J^\infty Y$. Let $\mathcal{T}_r^*$ be the sheaf of germs of this presheaf. The structure module $\Gamma \mathcal{T}^*\infty Y$ of sections of this sheaf is a GDA such that, given an element $\phi \in \Gamma \mathcal{T}^*\infty Y$, there exist an open neighbourhood $U$ of each point of $J^\infty Y$ and an exterior form $\phi^{(k)}$ on some finite order jet manifold $J^kY$ so that $\phi|_U = \pi^*_k \phi^{(k)}|_U$. There is an obvious monomorphism $\mathcal{O}_*^\infty Y \to \Gamma \mathcal{T}_\infty^* Y$. Note that $J_\infty^* Y$ admits the partition of unity by elements of the ring $\Gamma \mathcal{T}_0^* Y$, but not $\mathcal{O}_0^\infty Y$. Therefore, one can obtain cohomology of $\Gamma \mathcal{T}_\infty^* Y$ by virtue the abstract de Rham theorem [1, 19]. The GDA $\mathcal{O}_*^\infty Y$ is proved to possesses the same cohomology as $\Gamma \mathcal{T}_\infty^* Y$ [13, 14]. The above mentioned Theorem 1 is similarly proved.

In order to describe Noether identities generated by elements of projective Grassmann-graded $C^\infty(X)$-modules of finite rank, we appeal to the well-known Serre–Swan theorem, extended to noncompact manifolds [16, 18], and to its following combination [6, 16] with the Batchelor theorem [3].

**Proposition 1.** Given a smooth manifold $Z$, the exterior algebra of a projective $C^\infty(Z)$-module of finite rank is isomorphic to the ring of graded functions on some graded manifold whose body is $Z$.

Let $(Z, \mathfrak{A})$ be a graded manifold with a body $Z$ and a structure sheaf $\mathfrak{A}$ of Grassmann $C_Z^\infty$-algebras of finite rank, where $C_Z^\infty$ is the sheaf of germs of smooth real functions on $Z$ [3]. The above mentioned Batchelor theorem states an isomorphism of $(Z, \mathfrak{A})$ to a graded manifold $(Z, \mathfrak{A}_Q)$ with the structure sheaf $\mathfrak{A}_Q$ of germs of sections of an exterior bundle
$$\wedge Q^* = \mathbb{R} \oplus Z Q^* \oplus Z^2 Q^* \oplus \cdots,$$
where $Q^*$ is the dual of some vector bundle $Q \to Z$. In our case, Batchelor’s isomorphism is fixed from the beginning. Let us call $(Z, \mathfrak{A}_Q)$ a graded manifold modelled over $Q$. Its structure ring $\mathcal{A}_Q$ of graded functions consists of global sections of the exterior bundle $\wedge Q^*$. Let $\mathfrak{d} \mathcal{A}_Q$ be the real Lie superalgebra $\mathfrak{d} \mathcal{A}_Q$ of graded derivations of the $\mathbb{R}$-ring $\mathcal{A}_Q$, i.e.,
$$u(ff') = u(f)f' + (-1)^{|u||f|}fu(f'), \quad f, f' \in \mathcal{A}_Q, \quad u \in \mathfrak{d} \mathcal{A}_Q,$$
where the symbol $[\cdot]$ stands for the Grassmann parity. Then the Chevalley–Eilenberg complex of $\mathfrak{d} \mathcal{A}_Q$ with coefficients in $\mathcal{A}_Q$ can be constructed [10]. Its subcomplex $S^*[Q; Z]$ of $\mathcal{A}_Q$-linear
morphisms is the Grassmann-graded Chevalley–Eilenberg differential calculus

\[ 0 \to \mathbb{R} \to A_Q \xrightarrow{d} S^1[Q; Z] \xrightarrow{d} \cdots \xrightarrow{d} S^k[Q; Z] \xrightarrow{d} \cdots \]

over \( A_Q = S^0[Q; Z] \). The graded exterior product \( \wedge \) and the Chevalley–Eilenberg coboundary operator \( d \) make \( S^*[Q; Z] \) into a bigraded differential algebra (henceforth BGDA)

\[
\phi \wedge \phi' = (-1)^{\|\phi\| + |\phi'|} \phi' \wedge \phi, \quad d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{|\phi|} \phi \wedge d\phi',
\]

where \(|.|\) denotes the form degree. Moreover, \( S^*[Q; Z] \) is a minimal differential calculus over \( A_Q \) generated by elements \( df, f \in A_Q \). There is a natural monomorphism \( O^*Z \to S^*[Q; Z] \).

Elements of the BGDA \( S^*[Q; Z] \) can be seen as graded exterior forms on a manifold \( Z \). Given an open subset \( U \subset Z \), let \( A_U \) be the Grassmann algebra of sections of the sheaf \( \mathfrak{A}_Q \) over \( U \), and let \( S^*[Q; U] \) be the Chevalley–Eilenberg differential calculus over \( A_U \). Given an open subset \( U' \subset U \), the restriction morphism \( A_U \to A_{U'} \) yields a homomorphism of BGDAs \( S^*[Q; U] \to S^*[Q; U'] \). Thus, we obtain the presheaf \( \{ U, S^*[Q; U] \} \) of BGDAs on a manifold \( Z \) and the sheaf \( \mathfrak{I}^*[Q; Z] \) of germs of this presheaf. Moreover, \( \{ U, S^*[Q; U] \} \) is a canonical presheaf of \( \mathfrak{I}^*[Q; Z] \). Hence, \( S^*[Q; Z] \) is the BGDA of global sections of the sheaf \( \mathfrak{I}^*[Q; Z] \), and there is a restriction morphism \( S^*[Q; Z] \to S^*[Q; U] \) for any open \( U \subset Z \). Due to this morphism, elements of \( S^*[Q; Z] \) take the following local form. Bundle coordinates \((z^A, q^a)\) on \( Q \) and the corresponding fiber basis \( \{ c^a \} \) for \( Q^* \) provide a local basis \((z^A, c^a)\) for the graded manifold \((Z, \mathfrak{A}_Q)\) such that graded functions on \( Z \) read

\[
f = \sum_{0 \leq k} \frac{1}{k!} f_{a_1\ldots a_k} c^{a_1} \cdots c^{a_k}, \quad f_{a_1\ldots a_k} \in C^\infty(Z).
\]

Owing to the isomorphism \( VQ = Q \times Q \), the fiber basis \( \{ \partial_a \} \) for the vertical tangent bundle \( VQ \to Q \) of \( Q \) is the dual of \( \{ c^a \} \). Then the \( A_Q \)-module \( \partial A_Q \) of graded derivations is locally generated by the elements \( \partial_A, \partial_a \) acting on graded functions \((5)\) by the rule

\[
\partial_A(f) = \sum_{0 \leq k} \frac{1}{k!} \partial_A(f_{a_1\ldots a_k}) c^{a_1} \cdots c^{a_k}, \quad \partial_a(f) = \sum_{0 \leq k} \frac{1}{k!} \sum_{1 \leq i \leq k} (-1)^{i-1} f_{a_{1\ldots i-1} a_k} c^{a_1} \cdots c^{a_i} \cdots c^{a_k}.
\]

Relative to the dual bases \( \{ dz^A \} \) for \( T^*Z \) and \( \{ dc^b \} \) for \( Q^* \), the BGDA \( S^*[Q; Z] \) is locally generated by graded one-forms \( dz^A \), \( dc^a \).

A generic Lagrangian system of even and odd variables on a smooth manifold \( X \) is defined in terms of composite graded manifolds whose bodies are a fiber bundle \( Y \to X \) and its jet manifolds \( J^rY \) [6] (see [5, 15] for a particular case of an affine bundle \( Y \to X \)). Let \( F \to X \) be a vector bundle. Let us consider the graded manifold \((J^rY, \mathfrak{A}_F)\) modelled over the product \( F_r = J^rY \times_X J^rF \). There is an epimorphism of graded manifolds \((J^{r+1}Y, \mathfrak{A}_{F_{r+1}}) \to (J^rY, \mathfrak{A}_F)\). It consists of the surjection \( \pi_{r+1} \) and the sheaf monomorphism \( \pi_{r+1}^* \mathfrak{A}_F \to \mathfrak{A}_{F_{r+1}} \), where \( \pi_{r+1}^* \mathfrak{A}_F \) is the pull-back of the topological fiber bundle \( \mathfrak{A}_F \to J^rY \) onto \( J^{r+1}Y \). This
monomorphism of sheaves yields a monomorphism of their canonical presheaves $\mathfrak{A}_F \to \mathfrak{A}_{F+1}$, which associates to every open subset $U \subset J^r Y$ the ring of sections of $\mathfrak{A}_F$ over $\pi^{r+1}_r(U)$. Accordingly, there is a monomorphism of graded commutative rings $A_F \to A_{F+1}$ which induces the monomorphism of BGDAs

$$S^*[F_r; J^r Y] \to S^*[F_{r+1}; J^{r+1} Y].$$

As a consequence, we have the direct system of BGDAs

$$S^*[Y \times F; Y] \longrightarrow S^*[F_1; J^1 Y] \longrightarrow \cdots S^*[F_r; J^r Y] \longrightarrow \cdots.$$  

Its direct limit $S^*_\infty[F; Y]$ is a BGD of all graded exterior forms on jet manifolds $J^r Y$ modulo monomorphisms (6). The relations (4) hold. Monomorphisms $O^*_Y \to S^*[F_r; J^r Y]$ provide a monomorphism of the direct system (3) to the direct system (7) and, thus, a monomorphism $O^*_Y \to S^*_\infty[F; Y]$ of their direct limits. Moreover, $S^*_\infty[F; Y]$ is an $O^0_\infty Y$-algebra.

Elements of the BGD $S^*_\infty[F; Y]$ can be seen as graded exterior forms on $J^\infty Y$. Indeed, let $S^*[F_r; J^r Y]$ be the sheaf of BGDAs on $J^r Y$ and $\mathfrak{G}^*[F_r; J^r Y]$ its canonical presheaf whose elements are the Chevalley–Eilenberg differential calculus over elements of the presheaf $\mathfrak{A}_F$. Then the presheaf monomorphisms $\mathfrak{A}_F \to \mathfrak{A}_{F+1}$ yield the direct system of presheaves

$$\mathfrak{G}^*[Y \times F; Y] \longrightarrow \mathfrak{G}^*[F_1; J^1 Y] \longrightarrow \cdots \mathfrak{G}^*[F_r; J^r Y] \longrightarrow \cdots.$$  

Its direct limit is a presheaf of BGDAs on $J^\infty Y$. Let $\mathfrak{T}^*_\infty[F; Y]$ be the sheaf of germs of this presheaf. The structure module $\Gamma \mathfrak{T}^*_\infty[F; Y]$ of sections of $\mathfrak{T}^*_\infty[F; Y]$ is a BGD such that, given an element $\phi \in \Gamma \mathfrak{T}^*_\infty[F; Y]$, there exist an open neighbourhood $U$ of each point of $J^\infty Y$ and a graded exterior form $\phi^{(k)}$ on some finite order jet manifold $J^r Y$ so that $\phi|_U = \pi^\infty_\phi \phi^{(k)}|_U$. There is a monomorphism $S^*_\infty[F; Y] \to \Gamma \mathfrak{T}^*_\infty[F; Y]$ (cf. that in Remark 1).

Due to this monomorphism, one can restrict $S^*_\infty[F; Y]$ to the coordinate chart (2), and say that $S^*_\infty[F; Y]$ as an $O^0_\infty Y$-algebra is locally generated by the elements

$$(c_\Lambda^0, dx^\lambda, \theta_\Lambda^a = dc_\Lambda^a - c_{\Lambda+1}^a dx^\lambda, \theta_\Lambda^i = dy^i - y_{\Lambda+1}^i dx^\lambda), \quad 0 \leq |\Lambda|.$$  

One calls $(y^i, c^a)$ the local basis for $S^*_\infty[F; Y]$. We further use the collective symbol $s^A$ for its elements, together with the notation $s_\Lambda^A, \theta_\Lambda^A = ds_\Lambda^A - s_{\Lambda+1}^A dx^\lambda$, and $[A] = [s^A]$. Let $\mathfrak{S}^0_\infty[F; Y]$ be the Lie superalgebra of graded derivations of the $\mathbb{R}$-ring $S^0_\infty[F; Y]$. Its elements read

$$\partial = \partial^\lambda \partial_\Lambda + \sum_{0 \leq |\Lambda|} \theta_\Lambda^A \partial_\Lambda^A, \quad \partial^\lambda, \partial^A_\Lambda \in S^0[F; Y],$$  

where $\partial^A_\lambda(s_B^C) = \delta^B_C s^A_\lambda$ up to permutations of multi-indices $\Lambda$ and $\Sigma$. The interior product $\partial| \phi$ and the Lie derivative $L_\phi, \phi \in S^*_\infty[F; Y]$, obey the relations

$$\partial| \phi = \partial^\lambda \phi_\lambda + \sum_{0 \leq |\Lambda|} (-1)^{|\phi_\Lambda||\Lambda|} \theta_\Lambda^A \phi_\Lambda^A, \quad \phi \in S^1_\infty[F; Y],$$

$$\partial| (\phi \wedge \sigma) = (\partial| \phi) \wedge \sigma + (-1)^{|\phi|+|\phi|} \phi \wedge (\partial| \sigma), \quad \phi, \sigma \in S^*_\infty[F; Y],$$

$$L_\phi = \partial| d\phi + d(\partial| \phi), \quad L_\phi (\phi \wedge \sigma) = L_\phi (\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge L_\phi (\sigma).$$
In particular, the total derivatives are defined as graded derivations
\[ \partial \mathcal{S}^0_\infty[F; Y] \ni d_\lambda = \partial_\lambda + \sum_{0 \leq |A|} s^A_{\lambda+\lambda} \partial_A^\lambda, \quad d_\lambda \phi = L_{d_\lambda} \phi, \quad d_A = d_{\lambda_1} \cdots d_{\lambda_k}. \]

The BGDA \( S^*_\infty[F; Y] \) is split into \( \mathcal{S}^0_\infty[F; Y] \)-modules \( S^{k,r}_\infty[F; Y] = h_k \circ h^r(S^*_\infty[F; Y]) \) of \( k \)-contact and \( r \)-horizontal graded forms. Accordingly, the graded exterior differential \( d \) on \( S^*_\infty[F; Y] \) falls into the sum \( d = d_H + d_V \) of the total and vertical differentials where
\[ d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi). \]

These differentials together with the graded projection endomorphism
\[ \varrho = \sum_{k > 0} \frac{1}{k} \varrho \circ h_k \circ h^n, \quad \varrho(\phi) = \sum_{0 \leq |A|} (-1)^{|A|} \theta_A \wedge [d_\lambda(\partial_A^\lambda) \phi], \quad \phi \in S^{0,0}_\infty[F; Y], \]
and the graded variational operator \( \delta = \varrho \circ d \) make the BGDA \( S^*_\infty[F; Y] \) into the above mentioned Grassmann-graded variational bicomplex. We restrict our consideration to its short variational subcomplex and the subcomplex of one-contact graded forms
\[ 0 \rightarrow \mathbb{R} \rightarrow \mathcal{S}^0_\infty[F; Y] \xrightarrow{d_{H,R}} \mathcal{S}^{0,1}_\infty[F; Y] \cdots \xrightarrow{d_{H,R}} \mathcal{S}^{0,n}_\infty[F; Y] \xrightarrow{\delta} \mathcal{E}_1 = \varrho(S^{1,n}_\infty[F; Y]), \quad (10) \]
\[ 0 \rightarrow \mathcal{S}^{1,0}_\infty[F; Y] \xrightarrow{d_{H,R}} \mathcal{S}^{1,1}_\infty[F; Y] \cdots \xrightarrow{d_{H,R}} \mathcal{S}^{1,n}_\infty[F; Y] \xrightarrow{\varrho} \mathcal{E}_1 \rightarrow 0. \quad (11) \]

One can think of their even elements
\[ L = L \omega \in S^{0,0}_\infty[F; Y], \quad \omega = dx^1 \wedge \cdots \wedge dx^n; \quad (12) \]
\[ \delta L = \theta_A \wedge \mathcal{E}_A \omega = \sum_{0 \leq |A|} (-1)^{|A|} \theta_A \wedge d_\lambda(\partial_A^\lambda L) \omega \in \mathcal{E}_1 \quad (13) \]
as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

**Theorem 1.** (i) Cohomology of the complex (10) equals the de Rham cohomology \( H^*(Y) \) of \( Y \). (ii) The complex (11) is exact.

**Proof.** The proof follows that of [15], Theorem 2.1 (see Section 5). q.e.d.

**Corollary 1.** A \( \delta \)-closed (i.e., variationally trivial) graded density \( L \in S^{0,n}_\infty[F; Y] \) reads
\[ L = h_0 \psi + d_H \xi, \quad \xi \in S^{0,n-1}_\infty[F; Y], \quad (14) \]
where \( \psi \) is a closed \( n \)-form on \( Y \). In particular, a \( \delta \)-closed odd graded density is \( d_H \)-exact.

**Corollary 2.** Any graded density \( L \) admits the decomposition
\[ dL = \delta L - d_H \Xi, \quad \Xi \in S^{1,n-1}_\infty[F; Y], \quad (15) \]
where \( L + \Xi \) is a Lepagean equivalent of \( L \).
A graded derivation $\vartheta$ (9) is called contact if the Lie derivative $L_{\vartheta}$ preserves the ideal of contact graded forms of the BGDA $S^*_\infty[F;Y]$. It reads

$$\vartheta = \vartheta_H + \vartheta_V = [\vartheta^A d_A] + [\nu^A \partial_A + \sum_{0<|A|} d_A \nu^A \partial^A], \quad \nu^A = \vartheta^A - s^A \vartheta^\lambda.$$

(16)

A contact graded derivation $\vartheta$ (16) is called a variational supersymmetry of a Lagrangian $L$ (12) if the Lie derivative $L_{\vartheta}L$ is $d_H$-exact. The following holds [15].

**Proposition 2.** A contact graded derivation $\vartheta$ (16) is a variational supersymmetry of $L$ iff its vertical part $\vartheta_V$, vanishing on $O^*X \subset S^*_\infty[F;Y]$, is well.

Therefore, we further restrict our consideration to vertical contact graded derivations

$$\vartheta = \nu^A \partial_A + \sum_{0<|A|} d_A \nu^A \partial^A.$$

(17)

Such a derivation is completely defined by its first summand $\nu = \nu^A \partial_A$.

**Proposition 3.** As a result of the splitting (15), the Lie derivative $L_{\vartheta}L$ of a Lagrangian $L$ along a vertical contact graded derivations $\vartheta$ (17) admits the decomposition

$$L_{\vartheta}L = v\delta L + d_H(\vartheta|\Xi)).$$

(18)

**Proposition 4.** An odd vertical contact graded derivations $\vartheta$ (17) is a variational supersymmetry of $L$ iff the odd density $v\delta L = \nu^A E_A \omega$ is $d_H$-exact.

A vertical contact graded derivation $\vartheta$ (17) is called nilpotent if $L_{\vartheta}(L_{\vartheta}\phi) = 0$ for any horizontal graded form $\phi \in S^0_\infty[F;Y]$. It is nilpotent only if it is odd and iff

$$\vartheta(v) = \vartheta(\nu^A \partial_A) = \sum_{0\leq|\Sigma|} v^B \partial_B^\Sigma(\nu^A) \partial_A = 0.$$

(19)

For the sake of simplicity, the common symbol $v$ further stands for $\vartheta$ (17), its summand $\nu$, and the Lie derivative $L_{\vartheta}$. We agree to call $v$ the graded derivation of the BGDA $S^*_\infty[F;Y]$.

**Remark 2.** Right contact graded derivations $\tilde{\vartheta} = \tilde{\vartheta}_A \nu^A$ of the BGDA $S^*_\infty[F;Y]$ are also involved in the sequel. They act on graded forms $\phi$ on the right by the rule

$$\tilde{\vartheta}(\phi) = \tilde{d}(\phi)[\tilde{\vartheta} + \tilde{d}(\phi)\tilde{\vartheta}], \quad \tilde{\vartheta}(\phi \wedge \phi') = (-1)^{|\phi'||\tilde{\vartheta}|} \tilde{\vartheta}(\phi) \wedge \phi' + \phi \wedge \tilde{\vartheta}(\phi').$$

For instance, $\tilde{\vartheta}_A(\phi) = (-1)^{(|\phi|+1)|A|} \partial_A(\phi)$, $\tilde{d}_A = d_A$ and $\tilde{d}_H(\phi) = (-1)^{|\phi|} d_H(\phi)$. With right graded derivations, we have the right Euler–Lagrange operator

$$\tilde{\delta} L = \tilde{E}_A \omega \wedge \theta^A, \quad \tilde{E}_A = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|} d_A(\tilde{\vartheta}^A_A(L)).$$
An odd right graded derivation $\tilde{\nu}$ is a variational supersymmetry of a graded Lagrangian $L$ iff the odd graded density $\tilde{E}_A \nu^A \omega$ is $d_H$-exact.

**Remark 3.** Any local graded functions $f', f^A$, $0 \leq |A| \leq k$, and a graded exterior form $\phi$ obey the equalities

\[
\sum_{0 \leq |A| \leq k} f^A d_A f' \omega = \sum_{0 \leq |A| \leq k} (-1)^{|A|} d_A (f^A) f' \omega + d_H \sigma, \quad \text{(20)}
\]

\[
\sum_{0 \leq |A| \leq k} (-1)^{|A|} d_A (f^A \phi) = \sum_{0 \leq |A| \leq k} \eta(f)^A d_A \phi, \quad (\eta \circ \eta)(f)^A = f^A, \quad \text{(21)}
\]

\[
\eta(f)^A = \sum_{0 \leq |\Sigma| \leq k - |A|} (-1)^{|\Sigma+|A|} C_{|\Sigma+|A|}^{d|\Sigma+|A|} f^{\Sigma+|A|}, \quad C^a_b = \frac{b!}{a!(b-a)!}. \quad \text{(22)}
\]

In particular, the decomposition (18) takes the local form (20), but Corollary 2 states that the second term in its right-hand side is globally $d_H$-exact.

## 2. The Koszul–Tate complex of Noether identities

Given a degenerate Grassmann-graded Lagrangian system $(S^*_\infty[F;Y], L)$, let us associate to the Euler–Lagrange operator $\delta L$ (13) of a graded Lagrangian $L$ (12) the exact Koszul–Tate chain complex with the boundary operator whose nilpotency conditions provide all non-trivial Noether and higher-stage Noether identities for $\delta L$.

**Remark 4.** We introduce the following notation. Let $E \to X$ be a vector bundle and $E^*$ its dual. The bundle product

\[
\bar{E}^* = E^* \otimes_X T^* X
\]

is called the density-dual of $E$. Given the pull-back $E_Y$ of $E$ onto $Y$, let us consider the BGDA $S^*_\infty[F;E_Y]$. There are monomorphisms of $\mathcal{O}^0_Y$-algebras $S^*_\infty[F;Y] \to S^*_\infty[F;E_Y]$ and $\mathcal{O}^*_\infty E \to S^*_\infty[F;E_Y]$ whose images contain the common subalgebra $\mathcal{O}^*_\infty Y$. We consider: (i) the subring $P^0_\infty E_Y \subset \mathcal{O}^0_\infty E_Y$ of polynomial functions in fiber coordinates of the vector bundles $J^r E_Y \to J^r Y$, (ii) the corresponding subring $P^0_\infty [F;E_Y] \subset S^*_\infty[F;E_Y]$ of graded functions with polynomial coefficients belonging to $P^0_\infty E_Y$, (iii) the subalgebra $P^*_\infty [F;Y;E]$ of the BGDA $S^*_\infty[F;E_Y]$ over the subring $P^0_\infty [F;E_Y]$. Given vector bundles $V, V', E, E'$ over $X$, let us denote

\[
P^*_\infty [V'V;F;Y;EE'] = P^*_\infty [V' \times_X V \times_X F;Y;E \times E']. \quad \text{(23)}
\]

The BGDA $P^*_\infty [F;Y;E]$ and, similarly, the BGDA (23) possess the same cohomology as $S^*_\infty[F;Y]$ in Theorem 1. Since $H^*(Y) = H^*(E_Y)$, this cohomology of the BGDA $S^*_\infty[F;Y]$ equals that of the BGDA $S^*_\infty[F;E_Y]$. Furthermore, one can replace the BGDA $S^*_\infty[F;E_Y]$
with $\mathcal{P}_\infty^0[F; Y; E]$ in the condition of Theorem 1 due to the fact that sheaves of $\mathcal{P}_\infty^0 E_Y$-modules are also sheaves of $\mathcal{O}_\infty^0 Y$-modules.

Remark 5. For the sake of simplicity, we assume that the vertical tangent bundle $V Y$ of a fiber bundle $Y \to X$ admits the splitting $V Y = Y \times_X W$, where $W \to X$ is some vector bundle. In this case, there are no fiber bundles under consideration whose transition functions can vanish on the shell $\mathcal{E}_A = 0$. Let $\overline{Y}$ denote the density-dual of $W$ in this splitting.

Let us enlarge the BGDA $S^\ast_\infty[F; Y]$ to the BGDA $\mathcal{P}_\infty^\ast[Y^\ast; F; Y; F^\ast]$ whose local basis is

$$\{ s^A, \overline{s}_A \}, \quad [\overline{s}_A] = ([A] + 1) \text{mod } 2.$$

Following the physical terminology [2], we agree to call $\overline{s}_A$ the antifields of antifield number $\text{Ant}[\overline{s}_A] = 1$. The BGDA $\mathcal{P}_\infty^\ast[Y^\ast; F; Y; F^\ast]$ is provided with the nilpotent right graded derivation

$$\overline{\delta} = \partial^A \mathcal{E}_A,$$

where $\mathcal{E}_A$ are the graded variational derivatives (13). We call $\overline{\delta}$ the Koszul–Tate differential.

Definition 1. One says that an element of the BGDA $\mathcal{P}_\infty^\ast[Y^\ast; F; Y; F^\ast]$ or its extension vanishes on the shell if it is $\overline{\delta}$-exact.

With the Koszul–Tate differential (24), the module $\mathcal{P}_\infty^{0,n}[Y^\ast; F; Y; F^\ast]$ of graded densities is split into the chain complex

$$0 \leftarrow S^{0,n}_\infty[F; Y] \leftarrow \mathcal{P}^{0,n}_\infty[Y^\ast; F; Y; F^\ast]_1 \cdots \leftarrow \mathcal{P}^{0,n}_\infty[Y^\ast; F; Y; F^\ast]_k \cdots$$

graded by the antifield number. Let us consider its subcomplex

$$0 \leftarrow \text{Im } \overline{\delta} \leftarrow \mathcal{P}^{0,n}_\infty[Y^\ast; F; Y; F^\ast]_1 \leftarrow \mathcal{P}^{0,n}_\infty[Y^\ast; F; Y; F^\ast]_2. \quad (25)$$

It is exact at $\text{Im } \overline{\delta}$. Let us examine its first homology $H_1(\overline{\delta})$.

Remark 6. If there is no danger of confusion, elements of homology of a chain complex are identified to its representatives. A chain complex is called $r$-exact if its homology of degree $k \leq r$ is trivial.

A generic one-chain of the complex (25) takes the form

$$\Phi = \sum_{0 \leq |A|} \Phi^A \overline{s}_A \omega, \quad \Phi^A \in S^0_\infty[F; Y]. \quad (26)$$

The cycle condition reads

$$\overline{\delta}\Phi = \sum_{0 \leq |A|} \Phi^A d_A \mathcal{E}_A \omega = 0. \quad (27)$$
This equality is a Noether identity which the graded variational derivatives $E_A$ (13) satisfy. Conversely, any equality of the form (27) comes from some cycle (26). A Noether identity (27) is trivial if a cycle is a boundary

$$\Phi = \sum_{0 \leq |\Lambda|} T^{(A\Lambda)(B\Sigma)} d_\Sigma E_B \bar{s}_{A\Lambda} \omega, \quad T^{(A\Lambda)(B\Sigma)} = (-1)^{|A||B|} T^{(B\Sigma)(A\Lambda)}.$$  

**Definition 2.** Noether identities which the Euler–Lagrange operator $\delta L$ (13) satisfies are one-cycles of the chain complex (25). Trivial Noether identities are boundaries. Non-trivial Noether identities considered modulo the trivial ones are non-zero elements of the first homology $H_1(\delta)$ of the chain complex (25).

One can say something more if the $S^0_\infty[F; Y]$, module $H_1(\delta)$ is finitely generated. Namely, there exists a projective Grassmann-graded $C^\infty(X)$-module $C(0) \subset H_1(\delta)$ of finite rank such that any element $\Phi \in H_1(\delta)$ factorizes via elements of $C(0)$ as

$$\Phi = \sum_{0 \leq |\Xi|} G^{r,\Xi} d_\Xi \Delta_r \omega, \quad G^{r,\Xi} \in S^0_\infty[F; Y],$$  

$$\Delta_r = \sum_{0 \leq |\Lambda|} \Delta_r^{A\Lambda} \bar{s}_{A\Lambda}, \quad \Delta_r^{A\Lambda} \in S^0_\infty[F; Y],$$

where $\{\Delta_r\}$ is a local basis for $C(0)$. This means that any Noether identity (27) is a corollary of the Noether identities

$$\bar{\delta} \Delta_r = \sum_{0 \leq |\Lambda|} \Delta_r^{A\Lambda} d_\Lambda E_A = 0.$$  

Clearly, the factorization (28) is independent of specification of local bases $\{\Delta_r\}$. By virtue of the Serre–Swan theorem, the module $C(0)$ is isomorphic to a module of sections of the product $V^* \times_X E^*$, where $V^*$ and $E^*$ are the density-duals of some vector bundles $V \to X$ and $E \to X$.

**Definition 3.** If the first homology $H_1(\delta)$ of the chain complex (25) is finitely generated, its generating elements $\Delta_r \in C(0)$ (29) and the corresponding equalities (30) are called the complete Noether identities.

For instance, let $L$ (12) be a variationally trivial Lagrangian. Its Euler–Lagrange operator $\delta L = 0$ obeys the Noether identities which are finitely generated by the Noether identities $\Delta_A = \bar{s}_A$.

**Proposition 5.** If the homology $H_1(\delta)$ of the chain complex (25) is finitely generated, this complex can be extended to the one-exact chain complex (32) with a boundary operator whose nilpotency conditions are equivalent to the complete Noether identities (30).

**Proof.** Let us enlarge the BGDA $P^*_\infty[Y^*; F; Y; \bar{F}^*]$ to the BGDA

$$P^*_\infty[E^*Y^*; F; Y; \bar{F}^*\bar{Y}^*],$$

(31)
possessing the local basis
\[ \{ s^A, \overline{s}_A, \tau_\nu \}, \quad \overline{[\tau_\nu]} = ([\Delta_\nu] + 1) \text{mod } 2, \quad \text{Ant}[\tau_\nu] = 2. \]

The BGDA (31) is provided with the nilpotent right graded derivation
\[ \delta_0 = \delta + \partial \Delta_\nu, \]
called the zero-stage Koszul–Tate differential. It is readily observed that its nilpotency conditions (19) are equivalent to the complete Noether identities (30). Then the module \( \mathcal{P}_{\infty}^{0,n}[E^\nu Y^\nu; F; Y; F^\nu V^\nu] \leq 3 \) of graded densities of antifield number \( \text{Ant}[\phi] \leq 3 \) is split into the chain complex
\[
0 \leftarrow \text{Im} \delta \leftarrow \mathcal{P}_{\infty}^{0,n}[E^\nu Y^\nu; F; Y; F^\nu V^\nu]_1 \leftarrow \mathcal{P}_{\infty}^{0,n}[E^\nu Y^\nu; F; Y; F^\nu V^\nu]_2 \leftarrow \mathcal{P}_{\infty}^{0,n}[E^\nu Y^\nu; F; Y; F^\nu V^\nu]_3. \tag{32}
\]
Let \( H_*(\delta_0) \) denote its homology. We have \( H_0(\delta_0) = H_0(\delta) = 0 \). Furthermore, any one-cycle \( \Phi \) up to a boundary takes the form (28) and, therefore, it is a \( \delta_0 \)-boundary
\[
\Phi = \sum_{0 \leq |\Sigma|} G^{r,\Xi} d_\Xi \Delta_\nu \omega = \delta_0(\sum_{0 \leq |\Sigma|} G^{r,\Xi} \tau_\Xi \omega). \]
Hence, \( H_1(\delta_0) = 0 \), i.e., the complex (32) is one-exact. \( \square \)

Let us examine the second homology \( H_2(\delta_0) \) of the complex (32). A generic two-chain reads
\[
\Phi = G + H = \sum_{0 \leq |A|} G^{r,A} \tau_\nu \omega + \sum_{0 \leq |A|, |\Sigma|} H^{(A,A)(B,\Sigma)} \pi_{ABA} \pi_{\Sigma A} \omega. \tag{33}
\]
The cycle condition takes the form
\[
\delta_0 \Phi = \sum_{0 \leq |A|} G^{r,A} d_\Delta \omega_\nu + \delta H = 0. \tag{34}
\]
This is a first-stage Noether identity which the complete Noether identities (29) satisfy. Conversely, let
\[
\Phi = \sum_{0 \leq |A|} G^{r,A} \tau_\nu \omega \in \mathcal{P}_{\infty}^{0,n}[E^\nu Y^\nu; F; Y; F^\nu V^\nu]_2
\]
be a graded density such that the first-stage Noether identity (34) hold. This identity is obviously a cycle condition of the two-chain (33).

**Definition 4.** The first-stage Noether identities which the complete Noether identities satisfy are two-cocycles of the one-exact chain complex (32).

The first-stage Noether identity (34) is trivial either if a two-cycle \( \Phi \) (33) is a boundary or its summand \( G \), linear in antifields, vanishes on the shell. Because of the second requirement,
trivia first-stage Noether identities need not be two-boundaries, unless the following condition is satisfied.

**Definition 5.** One says that the chain complex (32) obeys the two-homology regularity condition if any \( \delta \)-cycle \( \phi \in \mathcal{P}^{0,n}_{\infty} [\nabla^*; F; Y; F^*] \) is a \( \delta_0 \)-boundary.

**Proposition 6.** Non-trivial first-stage Noether identities are identified to non-zero elements of the second homology \( H_2(\delta_0) \) of the complex (32) iff the two-homology regularity condition hold.

**Proof.** It suffices to show that, if the summand \( G \) of a two-cycle \( \Phi \) (33) is \( \delta \)-exact, \( \Phi \) is a boundary. If \( G = \delta \Psi \), then

\[
\Phi = \delta_0 \Psi + (\delta - \delta_0) \Psi + H. \tag{35}
\]

The cycle condition reads

\[
\delta_0 \Phi = \delta ((\delta - \delta_0) \Psi + H) = 0.
\]

Then \( (\delta - \delta_0) \Psi + H \) is \( \delta_0 \)-exact since any \( \delta \)-cycle \( \phi \in \mathcal{P}^{0,n}_{\infty} [\nabla^*; F; Y; F^*] \), by assumption, is a \( \delta_0 \)-boundary. Consequently, \( \Phi \) (35) is \( \delta_0 \)-exact. Conversely, let \( \Phi \in \mathcal{P}^{0,n}_{\infty} [\nabla^*; F; Y; F^*] \) be an arbitrary \( \delta \)-cycle. The cycle condition reads

\[
\delta \Phi = 2\Phi^{(A, \Lambda)(B, \Sigma)} \sigma_{A, \Lambda, \Sigma} = 2\Phi^{(A, \Lambda)(B, \Sigma)} \sigma_{A, \Lambda, \Sigma} \delta \omega = 0. \tag{36}
\]

It follows that \( \Phi^{(A, \Lambda)(B, \Sigma)} \sigma_{A, \Lambda, \Sigma} = 0 \) for all indices \((A, \Lambda)\). We obtain

\[
\Phi^{(A, \Lambda)(B, \Sigma)} \sigma_{A, \Lambda, \Sigma} = G^{(A, \Lambda)(r, \Xi)} d_{\Xi} \Delta_r + \delta S^{(A, \Lambda)}.
\]

Hence, \( \Phi \) takes the form

\[
\Phi = G^{(A, \Lambda)(r, \Xi)} d_{\Xi} \Delta_r \sigma^{(A, \Lambda)}_{\Lambda, \omega} + \delta S^{(A, \Lambda)} \sigma^{(A, \Lambda)}_{\Lambda, \omega}. \tag{37}
\]

We can associate to \( \Phi \) (37) the three-chain

\[
\Psi = G^{(A, \Lambda)(r, \Xi)} \tau_{\Xi \Lambda} \sigma^{(A, \Lambda)}_{\Lambda, \omega} + S^{(A, \Lambda)} \sigma^{(A, \Lambda)}_{\Lambda, \omega}
\]

such that

\[
\delta_0 \Psi = \Phi + \sigma = \Phi + G^{(A, \Lambda)(r, \Xi)} d_{\Lambda} E_{\Lambda} \sigma^{(A, \Lambda)}_{\Lambda, \omega} + S^{(A, \Lambda)} \delta \sigma^{(A, \Lambda)}_{\Lambda, \omega}.
\]

Owing to the equality \( \delta \Phi = 0 \), we have \( \delta_0 \sigma = 0 \). Since the term \( G'' \) of \( \sigma \) is \( \delta \)-exact, then \( \sigma \) by assumption is \( \delta_0 \)-exact, i.e., \( \sigma = \delta_0 \psi \). It follow that \( \Phi = \delta_0 \Psi - \delta_0 \psi \). q.e.d.

If the two-homology regularity condition is satisfied, let us suppose that the second homology \( H_2(\delta_0) \) of the complex (32) is finitely generated as follows. There exists a projective
Grassmann-graded $C^\infty(X)$-module $\mathcal{C}_{(1)} \subset H_2(\delta_0)$ of finite rank such that any element $\Phi \in H_2(\delta_0)$ factorizes via elements of $\mathcal{C}_{(1)}$ as

$$
\Phi = \sum_{0 \leq |x|} \Phi^{r_x} d_x \Delta_{r_x} \omega, \quad \Phi^{r_x} \in S^0_{\infty}[F; Y],
$$

(38)

$$
\Delta_{r_1} = G_{r_1} + h_{r_1} = \sum_{0 \leq |x|} \Delta^{r_x} \omega + h_{r_1}, \quad h_{r_1} \omega \in \mathcal{P}^{0, n}_{\infty}[Y^*; F; Y; F'],
$$

(39)

where $\{\Delta_{r_x}\}$ is a local basis for $\mathcal{C}_{(1)}$. Thus, any first-stage Noether identity (34) results from the equalities

$$
\sum_{0 \leq |x|} \Delta_{r_1}^{r_x} d_x \Delta_{r_x} + \delta h_{r_1} = 0.
$$

(40)

By virtue of the Serre–Swan theorem, $\mathcal{C}_{(1)}$ is isomorphic to the module of sections of the product $\overline{V}'_1 \times \overline{E}'_1$, where $\overline{V}'_1$ and $\overline{E}'_1$ are the density-duals of some vector bundles $V_1 \to X$ and $E_1 \to X$.

**Definition 6.** (i) If the chain complex (32) obeys the two-homology regularity condition and its second homology $H_2(\delta_0)$ is finitely generated, the generating elements $\Delta_{r_1} \in \mathcal{C}_{(1)}$ (39) of $H_2(\delta_0)$ and the corresponding equalities (40) are called the complete first-stage Noether identities. (ii) A degenerate Lagrangian system is said to be one-stage reducible if it possesses complete Noether and first-stage Noether identities.

In other words, a degenerate Lagrangian system $(\mathcal{S}^*_{\infty}[F; Y], L)$ is first-stage reducible if one associates to it a one-exact chain complex (32) which obeys the two-homology regularity condition and whose second homology is finitely generated.

**Proposition 7.** The one-exact chain complex (32) associated to a first-stage reducible degenerate Lagrangian system can be extended to the two-exact chain complex (41) with a boundary operator whose nilpotency conditions are equivalent to complete Noether and first-stage Noether identities.

**Proof.** Let us consider the BGDA $\mathcal{P}^*_{\infty}[\overline{E}'_1 \overline{E}'_1^*; F; Y; F' \overline{V}'_1]$ possessing the local basis

$$
\{ s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1} \}, \quad [\overline{c}_{r_1}] = ([\Delta_{r_1}] + 1) \mathrm{mod} 2 \quad \mathrm{Ant}[\overline{c}_{r_1}] = 3.
$$

It can be provided with the nilpotent graded derivation

$$
\delta_1 = \delta_0 + \overline{\vartheta} \overline{r_1} \Delta_{r_1},
$$

called the first-stage Koszul–Tate differential. It is easily seen that its nilpotency conditions (19) are equivalent to the complete Noether identities (30) and complete first-stage Noether identities (40). Then the module $\mathcal{P}^{0, n}_{\infty}[\overline{E}'_1 \overline{E}'_1^*; F; Y; F' \overline{V}'_1]_{\leq 4}$ of graded densities of antifield number $\mathrm{Ant}[\phi] \leq 4$ is split into the chain complex

$$
0 \leftarrow \mathrm{Im} \overline{\delta} \leftarrow \mathcal{P}^{0, n}_{\infty}[\overline{Y}^*; F; Y; F']_1 \leftarrow \mathcal{P}^{0, n}_{\infty}[\overline{E} \overline{Y}^*; F; Y; F' \overline{V}_1]_2 \leftarrow \mathcal{P}^{0, n}_{\infty}[\overline{E}'_1 \overline{E}'_1^*; F; Y; F' \overline{V}_1]_3 \leftarrow \mathcal{P}^{0, n}_{\infty}[\overline{E}'_1 \overline{Y}^*; F; Y; F' \overline{V}_1]_4.
$$

(41)
Let $H_*(\delta_1)$ denote its homology. It is readily observed that

$$H_0(\delta_1) = H_0(\delta), \quad H_1(\delta_1) = H_1(\delta_0) = 0.$$ 

By virtue of the expression (38), any two-cycle of the complex (41) is a boundary

$$\Phi = \sum_{0 \leq |\xi|} \Phi^{r_1, \xi}_s A_\xi \Delta_{r_1} \omega = \delta_1(\sum_{0 \leq |\xi|} \Phi^{r_1, \xi}_s A_\xi) \omega.$$

It follows that $H_2(\delta_1) = 0$, i.e., the complex (41) is two-exact. q.e.d.

If the third homology $H_3(\delta_1)$ of the chain complex (41) is not trivial, there are non-trivial second-stage Noether identities which the first-stage ones satisfy, and so on. Iterating the arguments, we come to the following.

Given a first-stage reducible degenerate Lagrangian system $(S^*_\infty[F; Y], L)$ in accordance with Definition 6, let us assume the following.

(a) Given an integer $N \geq 1$, there are vector bundles $V_1, \ldots, V_N, E_1, \ldots, E_N$ over $X$, and the BGDA $S^*_\infty[F; Y]$ is enlarged to the BGDA

$$\overline{\mathcal{P}}^*\infty\{N\} = \mathcal{P}^*\infty[E_N \cdots E_1 E_0; F; Y; F' Y' V_1 \cdots V_N]$$

with a local basis $\{s^A, \sigma_A, \varphi, \varphi_1, \ldots, \varphi_N\}$ graded by antifield numbers $\text{Ant}[\varphi_N] = k + 2$. Let the indexes $k = -1, 0$ further stand for $\sigma_A$ and $\varphi$, respectively.

(b) The BGDA $\overline{\mathcal{P}}^\infty\infty\{N\}$ (42) is provided with a nilpotent graded derivation

$$\delta_N = \overline{\delta} A e_A + \sum_{0 \leq |\Lambda|} \overline{\delta} r_\Lambda^A \Lambda A + \sum_{0 \leq |\Lambda|} \overline{\delta} r_k \Lambda_1$$

$$= \sum_{0 \leq |\Lambda|} \overline{\delta} r_\Lambda^A \Lambda A + \sum_{0 \leq |\Lambda|} \overline{\delta} r_k \Lambda_1$$

of antifield number $-1$.

(c) With $\delta_N$, the module $\overline{\mathcal{P}}^{0, n}_{\infty}\{N\}_{\leq N+3}$ of graded densities of antifield number $\text{Ant}[\phi] \leq N + 3$ is split into the $(N+1)$-exact chain complex

$$0 \leftarrow \text{Im} \overline{\delta} \overline{\Phi}^0_{\infty}\{N-1\}_{N+1} \leftarrow \overline{\mathcal{P}}^{0, n}_{\infty}\{N\}_{N+1}$$

which satisfies the following $(N+1)$-homology regularity condition.

**Definition 7.** One says that the chain complex (45) obeys the $(N+1)$-homology regularity condition if any $\delta_{k<N-1}$-cycle $\phi \in \overline{\mathcal{P}}^{0, n}_{\infty}\{k\}_{k+3} \subset \overline{\mathcal{P}}^{0, n}_{\infty}\{k+1\}_{k+3}$ is a $\delta_{k+1}$-boundary.

Note that the $(N+1)$-exactness of the complex (45) implies that any $\delta_{k<N-1}$-cycle $\phi \in \overline{\mathcal{P}}^{0, n}_{\infty}\{k\}_{k+3}, k < N$, is a $\delta_{k+2}$-boundary, but not necessary a $\delta_{k+1}$-boundary.

If $N = 1$, the complex $\overline{\mathcal{P}}^{0, n}_{\infty}\{1\}_{\leq 4}$ (45) is the chain complex (41). Therefore, we agree to call $\delta_N$ (43) the $N$-stage Koszul–Tate differential. Its nilpotency implies the complete
Noether identities (30), first-stage Noether identities (40), and the complete \((k \leq N)\)-stage Noether identities

\[
\sum_{0 \leq |\Lambda|} \Delta^{r_{k-1};A}_{r_k} a_A( \sum_{0 \leq |\Sigma|} \Delta^{r_{k-2};\Sigma}_{r_k-1} c_{\Sigma r_{k-2}} ) + \delta( \sum_{0 \leq |\Sigma|, |\Xi|} h^{(r_{k-2};\Sigma)}_{r_k}(A,\Xi) c_{\Sigma r_{k-2}} = 0, \tag{46}
\]

which the complete \((k - 1)\)-stage Noether identities \(\Delta_{r_{k-1}} \tag{44}\) satisfy.

**Definition 8.** If the above mentioned assumptions (a) – (c) hold, a degenerate Grassmann-graded Lagrangian system \(S_{\infty}[F; \Sigma], L\) is called \(N\)-stage reducible.

If the \((N+2)\)-homology of the complex (45) is not trivial, an \(N\)-stage reducible Lagrangian system is \((N + 1)\)-stage reducible under the following conditions.

**Theorem 2.** Given an \(N\)-stage reducible Lagrangian system in accordance with Definition 8, let us suppose that the \((N + 2)\)-homology \(H_{N+2}(\delta_N)\) of the associated chain complex (45) is not trivial. Then the following holds.

(i) The \((N + 1)\)-stage Noether identities which the complete \(N\)-stage Noether identities satisfy are the \((N + 2)\)-cycles of the complex (45), and vice versa.

(ii) The trivial \((N + 1)\)-stage Noether identities are \((N + 2)\)-boundaries iff the \((N + 2)\)-homology regularity condition holds. In this case, non-trivial \((N + 1)\)-stage Noether identities modulo the trivial ones are identified to non-zero elements of the homology \(H_{N+2}(\delta_N)\).

(iii) If the homology \(H_{N+2}(\delta_N)\) is finitely generated, the complex (45) admits an \((N + 2)\)-exact extension. The nilpotency of its boundary operator implies the complete Noether and \((k \leq N + 1)\)-stage Noether identities.

**Proof.** (i) A generic \((N + 2)\)-chain \(\Phi \in \overline{P}_{\infty}^0 \{N\}_{N+2}\) takes the form

\[
\Phi = G + H = \sum_{0 \leq |\Lambda|} G^{r_{N};A} c_{\Lambda r_{N}} \omega + \sum_{0 \leq |\Sigma|, |\Xi|} (H^{(A,\Xi)(r_{N-1};\Sigma)} c_{\Xi A} c_{\Sigma r_{N-1}} + ...) \omega. \tag{47}
\]

The cycle condition \(\delta_N \Phi = 0\) implies the equality

\[
\sum_{0 \leq |\Lambda|} G^{r_{N};A} a_A( \sum_{0 \leq |\Sigma|} \Delta^{r_{N-1};\Sigma}_{r_{N-1}} c_{\Sigma r_{N-1}} ) + \delta( \sum_{0 \leq |\Sigma|, |\Xi|} H^{(A,\Xi)(r_{N-1};\Sigma)} c_{\Xi A} c_{\Sigma r_{N-1}} ) = 0, \tag{48}
\]

which is an \((N + 1)\)-stage Noether identity. Conversely, let

\[
\Phi = \sum_{0 \leq |\Lambda|} G^{r_{N};A} c_{\Lambda r_{N}} \omega \in \overline{P}_{\infty}^0 \{N\}_{N+2}
\]

be a graded density such that the condition (48) holds. Then this condition can be extended to a cycle one as follows. It is brought into the form

\[
\delta_N( \sum_{0 \leq |\Lambda|} G^{r_{N};A} c_{\Lambda r_{N}} + \sum_{0 \leq |\Sigma|, |\Xi|} H^{(A,\Xi)(r_{N-1};\Sigma)} c_{\Xi A} c_{\Sigma r_{N-1}} ) = - \sum_{0 \leq |\Lambda|} G^{r_{N};A} a_A h_{r_{N}} + \sum_{0 \leq |\Sigma|, |\Xi|} H^{(A,\Xi)(r_{N-1};\Sigma)} c_{\Xi A} d_{\Sigma} \Delta_{r_{N-1}}.
\]
A glance at the expression (44) shows that the term in the right-hand side of this equality belongs to $\overline{P}^{b,n}_{\infty}\{N-2\}_{N+1}$. It is a $\delta_{N-2}$-cycle and, consequently, a $\delta_{N-1}$-boundary $\delta_{N-1}\Psi$ in accordance with the $(N+1)$-homology condition. Then the $(N+1)$-stage Noether identity (48) is a $\overline{\delta}_{N}r_{N-1}$-dependent part of the cycle condition

$$
\delta_{N}(\sum_{0 \leq |A|} G^{r_{N},A}\overline{c}_{A}r_{N} + \sum_{0 \leq |\Sigma|,|\Xi|} H^{(A,\Xi)}(r_{N-1},\Sigma)\overline{c}_{A}\overline{c}_{\Sigma r_{N-1}} - \Psi) = 0,
$$

but $\delta_{N}\Psi$ does not make a contribution to this identity.

(ii) Being a cycle condition, the $(N + 1)$-stage Noether identity (48) is trivial either if a cycle $\Phi$ (47) is a $\delta_{N}$-boundary or its summand $G$ is $\overline{\delta}$-exact. The $(N + 2)$-homology regularity condition implies that any $\delta_{N-1}$-cycle $\Phi \in \overline{P}^{b,n}_{\infty}\{N - 1\}_{N+2} \subset \overline{P}^{n}_{\infty}\{N\}_{N+2}$ is a $\delta_{N}$-boundary. Therefore, if $\Phi$ (47) is a representative of a non-trivial element of $H_{N+2}(\delta_{N})$, its summand $G$ linear in $\overline{c}_{A}r_{N}$ does not vanish. Moreover, it is not a $\overline{\delta}$-boundary. Indeed, if $G = \overline{\delta}\Psi$, then

$$
\Phi = \delta_{N}\Psi + (\overline{\delta} - \delta_{N})\Psi + H.
$$

(49)

The cycle condition takes the form

$$
\delta_{N}\Phi = \delta_{N-1}(\overline{\delta} - \delta_{N})\Psi + H = 0.
$$

Hence, $(\overline{\delta} - \delta_{N})\Psi + H$ is $\delta_{N}$-exact since any $\delta_{N-1}$-cycle $\phi \in \overline{P}^{b,n}_{\infty}\{N - 1\}_{N+2}$ is a $\delta_{N}$-boundary. Consequently, $\Phi$ (49) is a boundary. If the $(N + 2)$-homology regularity condition does not hold, trivial $(N + 1)$-stage Noether identities (48) also come from non-trivial elements of the homology $H_{N+2}(\delta_{N})$.

(iii) Let the $(N + 1)$-stage Noether identities be finitely generated. Namely, there exists a projective Grassmann-graded $C^{\infty}(X)$-module $C_{(N + 1)}$ of finite rank $C_{(N + 1)} \subset H_{N+2}(\delta_{N})$ such that any element $\Phi \in H_{N+2}(\delta_{N})$ factorizes via elements of $C_{(N + 1)}$ as

$$
\Phi = \sum_{0 \leq |\Xi|} \Phi^{r_{N+1},\Xi} d_{\Xi}\Delta_{r_{N+1}} + \sum_{0 \leq |A|} \Phi^{r_{N+1},A} \overline{c}_{A}r_{N+1} + \Psi_{r_{N+1}} \in S^{0}_{\infty}[F, Y],
$$

(50)

$$
\Delta_{r_{N+1}} = G_{r_{N+1}} + h_{r_{N+1}} = \sum_{0 \leq |A|} \Delta_{r_{N+1}}^{r_{N+1},A}\overline{c}_{A}r_{N+1} + h_{r_{N+1}},
$$

(51)

where $\{\Delta_{r_{N+1}}\}$ is local basis for $C_{(N + 1)}$. Clearly, this factorization is independent of specification of this local basis. By virtue of the Serre–Swan theorem, $C_{(N + 1)}$ is isomorphic to a module of sections of the product $V_{N+1}^{*} \times E_{N+1}^{*}$, where $V_{N+1}$ and $E_{N+1}$ are the density-duals of some vector bundles $V_{N+1} \to X$ and $E_{N+1} \to X$. Let us extend the BGDA $\overline{P}^{*}_{\infty}\{N\}$ (42) to the BGDA $\overline{P}^{*}_{\infty}\{N + 1\}$ possessing the local basis

$$
\{s^{A}, \overline{s}_{A}, \overline{c}_{r}, \overline{c}_{r_{1}}, \ldots, \overline{c}_{r_{N}}, \overline{c}_{r_{N+1}}\}, \quad \text{Ant}[\overline{c}_{r_{N+1}}] = N + 3, \quad [\overline{c}_{r_{N+1}}] = ([\Delta_{r_{N+1}}] + 1) \bmod 2.
$$

It is provided with the nilpotent graded derivation

$$
\delta_{N+1} = \delta_{N} + \overline{\delta}^{r_{N+1}} \Delta_{r_{N+1}}
$$

16
Following the physical terminology [2], we agree to call \( c \) the nilpotency conditions of the graded derivation of antifield number \(-1\). With this graded derivation, the module \( \mathcal{P}_{\infty}^{0,n}\{N + 1\}_{\leq N+4} \) of graded densities of antifield number \( \text{Ant}[\phi] \leq N + 4 \) is split into the chain complex

\[
0 \leftarrow \text{Im} \delta \leftarrow \mathcal{P}_{\infty}^{0,n}[Y'; F; Y; F']_1 \leftarrow \mathcal{P}_{\infty}^{0,n}\{0\}_2 \leftarrow \mathcal{P}_{\infty}^{0,n}\{1\}_3 \cdots \tag{52}
\]

It is readily observed that this complex is \((N + 2)\)-exact. In this case, the \((N + 1)\)-stage Noether identities (48) come from the complete \((N + 1)\)-stage Noether identities

\[
\sum_{0 \leq |A|} \Delta_{r_{N+1}}^{\text{rN}} d_A \Delta_{r_{N}} \omega + \bar{\delta} h_{r_{N+1}} \omega = 0, \tag{53}
\]

which are reproduced as the nilpotency conditions of the graded derivation \( \delta_{N+1} \). q.e.d.

It may happen that the iteration procedure based on Theorem 2 is infinite. We restrict our consideration to the case of a finitely \((N\text{-stage})\) reducible Lagrangian system possessing the finite \((N + 2)\)-exact chain complex, called the Koszul–Tate complex,

\[
0 \leftarrow \text{Im} \delta \leftarrow \mathcal{P}_{\infty}^{0,n}[Y'; F; Y; F']_1 \leftarrow \mathcal{P}_{\infty}^{0,n}\{0\}_2 \leftarrow \mathcal{P}_{\infty}^{0,n}\{1\}_3 \cdots \tag{54}
\]

where \( \Delta_{r_k} \) (44) and the corresponding equalities (46) are the complete \( k \)-stage Noether identities. The

### 3. Noether’s inverse second theorem

Given the BGDA \( \overline{\mathcal{P}}_{\infty}\{N\}\) (42), let us consider the BGDA

\[
\mathcal{P}_{\infty}^{*}\{N\} = \mathcal{P}_{\infty}^{*}[V_{N} \cdots V_{1} V; F; Y; E E_{1} \cdots E_{N}] \tag{56}
\]

possessing the local basis

\[
\{ s^{A}, c^{r}, c^{r_{1}}, \ldots, c^{r_{N}} \}, \quad [c^{r_k}] = ([\overline{c} r_k] + 1) \mod 2, \quad \text{Ant}[c^{r_k}] = -(k + 1),
\]

and the BGDA

\[
P_{\infty}^{*}\{N\} = \mathcal{P}_{\infty}^{*}[\overline{E}_{N} \cdots \overline{E}_{1} \overline{F} \overline{V} V_{N} \cdots V_{1} V; F; Y; E E_{1} \cdots E_{N} \overline{F} \overline{V} \overline{V}_{1} \cdots \overline{V}_{N}] \tag{57}
\]

with the local basis

\[
\{ s^{A}, c^{r}, c^{r_{1}}, \ldots, c^{r_{N}}, \overline{s}_{A}, \overline{c}_{r}, \overline{c}_{r_{1}}, \ldots, \overline{c}_{r_{N}} \}.
\]
of the BGDA $P^*_\infty\{N\}$ (57). The Koszul–Tate differential $\delta_N$ (55) is naturally extended to a graded derivation of the BGDA $P^*_\infty\{N\}$ (57).

**Theorem 3.** With the Koszul–Tate complex (54), the graded commutative ring $\mathcal{P}^0_\infty\{N\} \subset \mathcal{P}^*_\infty\{N\}$ (56) is split into the cochain sequence

$$0 \rightarrow \mathcal{S}^{0}_\infty[F;Y] \xrightarrow{u_e} \mathcal{P}^0_\infty\{N\}_1 \xrightarrow{u_e} \mathcal{P}^0_\infty\{N\}_2 \xrightarrow{u_e} \cdots,$$

graded in a ghost number, where $u$ (66), $u(1)$ (68) and $u(k)$ (70), $k = 2, \ldots, N$, are the gauge, first-stage and higher-stage gauge supersymmetries of an original Grassmann-graded Lagrangian.

**Proof.** Let us extend an original graded Lagrangian $L_e$ to the even graded density

$$L_e = L + L_1 = L + \sum_{0 \leq k \leq N} c^{r_k} \Delta_{r_k} \omega = L + \delta_N(\sum_{0 \leq k \leq N} c^{r_k} \Delta_{r_k} \omega),$$

whose summand $L_1$ is linear in ghosts and possesses the zero antifield number. It is readily observed that $\delta_N(L_e) = 0$, i.e., $\delta_N$ is a variational supersymmetry of the graded Lagrangian $L_e$ (61). Using the formulas (21) – (22), we obtain

$$\left[\frac{\delta}{\delta \bar{S}_A} L_e + \sum_{0 \leq k \leq N} \frac{\delta}{\delta \bar{c}_{r_k}} c^{r_k} \Delta_{r_k}\right] \omega = \left[\frac{\delta}{\delta \bar{S}_A} L_e + \sum_{0 \leq k \leq N} \frac{\delta}{\delta \bar{c}_{r_k}} \delta L_e \delta c^{r_k}\right] \omega = \left[u^A \delta L_e \delta \bar{s}_A + \sum_{0 \leq k \leq N} u^{r_k} \delta L_e \delta c^{r_k}\right] \omega = d_H \sigma,$$

$$u^A = \frac{\delta L_e}{\delta \bar{s}_A} = u^A + w^A = \sum_{0 \leq |A|} c_A^r \eta(\Delta^A) + \sum_{1 \leq i \leq \infty} \sum_{0 \leq |A|} c_A^r \eta(\partial^A(h_{r_i})),$$

$$u^{r_k} = \frac{\delta L_e}{\delta \bar{c}_{r_k}} = u^{r_k} + w^{r_k} = \sum_{0 \leq |A|} c^{r_{k+1}} \eta(\Delta^{r_{k+1}}) + \sum_{k+1 \leq i \leq \infty} \sum_{0 \leq |A|} c_A^r \eta(\partial^{r_k}(h_{r_i})).$$

The equality (62) falls into the set of equalities

$$\frac{\delta}{\delta \bar{s}_A} (c^n \Delta_{r_k}) \omega = u^A \delta A \omega = d_H \sigma_0,$$

$$\left[\frac{\delta}{\delta \bar{s}_A} (c^n \Delta_{r_k}) \omega + \sum_{0 \leq k < i} \frac{\delta}{\delta \bar{c}_{r_k}} (c^n \Delta_{r_k}) \Delta_{r_k}\right] \omega = d_H \sigma_i, \quad i = 2, \ldots, N.$$

A glance at the equality (63) shows that, by virtue of the decomposition (18), the graded derivation

$$u = u^A \frac{\partial}{\partial s_A}, \quad u^A = \sum_{0 \leq |A|} c_A^r \eta(\Delta^A),$$

18
is a variational supersymmetry of an original graded Lagrangian $L$. Parameterized by ghosts $c'$, it is a gauge supersymmetry of $L$ [5, 15].

The equality (64) takes the form
\[
\left[ \frac{\delta}{\delta \bar{\sigma}^A_c} (c'^r h^{(B, \Sigma)}(A, \Xi)_{\bar{\sigma}^A_c \bar{\Xi} B \bar{\Xi} A}) \right] \mathcal{E}_A + \left[ \frac{\delta}{\delta \bar{\sigma}^r} (c'^r \sum_{0 \leq |\Xi|} \Delta^{r, \Sigma}_{r_1} \bar{\sigma}^A_c) \sum_{0 \leq |\Xi|} \Delta^{B, \Xi}_{B \Xi B} \right] \omega = \left[ \sum_{0 \leq |\Xi|} (-1)^{|\Xi|} d_\Xi (c'^r \sum_{0 \leq |\Xi|} 2h^{(B, \Sigma)}(A, \Xi)_{\bar{\sigma}^A_c \bar{\Xi} B \bar{\Xi} A}) + u'^r \sum_{0 \leq |\Xi|} \Delta^{B, \Xi}_{B \Xi B} \right] \omega = d_H \sigma'_1.
\]

Using the relation (20), we obtain
\[
\left[ \sum_{0 \leq |\Xi|} c'^r \sum_{0 \leq |\Xi|} 2h^{(B, \Sigma)}(A, \Xi)_{\bar{\sigma}^A_c \bar{\Xi} B \bar{\Xi} A} d_\Xi \mathcal{E}_A + u'^r \sum_{0 \leq |\Xi|} \Delta^{B, \Xi}_{B \Xi B} \right] \omega = d_H \sigma'_1.
\]

The variational derivative of the both sides of this equality with respect to the antifield $\bar{\sigma}^A_c$ leads to the relation
\[
\sum_{0 \leq |\Xi|} \eta(h^{(B, \Xi)}(A, \Xi))_{\Sigma} d_\Sigma (2c'^r d_\Xi \mathcal{E}_A) + \sum_{0 \leq |\Xi|} u'^r \eta(\Delta^{(B, \Xi)}_{r_1}) \Sigma = 0,
\]
which is brought into the form
\[
\sum_{0 \leq |\Sigma|} d_\Sigma u'^r \frac{\partial}{\partial c'^r} u^B = \bar{\sigma}(\alpha^B), \quad \alpha^B = - \sum_{0 \leq |\Xi|} \eta(2h^{(B, \Xi)}(A, \Xi))_{\Sigma} d_\Sigma (c'^r \bar{\sigma}^A_c A).
\]

Therefore, the odd graded derivation
\[
u_{(1)} = u'^r \frac{\partial}{\partial c'^r}, \quad u'^r = \sum_{0 \leq |\Xi|} c'^r \eta(\Delta^{(r_1)}_{r_1})^\Lambda,
\]
is the first-stage gauge supersymmetry of a reducible Lagrangian system [5].

Every equality (65) is split into a set of equalities with respect to the polynomial degree in antifields. Let us consider the one, linear in antifields $\bar{\sigma}^A_{r_{i-2}}$ and their jets. We have
\[
\left[ \frac{\delta}{\delta \bar{\sigma}^A_r} (c'^r \sum_{0 \leq |\Xi|} h^{(r_{i-2}, \Sigma)}(A, \Xi)_{\bar{\sigma}^A_r \bar{\Xi} r_{i-2} \bar{\Xi} A}) \mathcal{E}_A + \left[ \frac{\delta}{\delta \bar{\sigma}^r} (c'^r \sum_{0 \leq |\Xi|} \Delta^{(r_{i-1}, \Xi)}_{r_{i-1}} \bar{\sigma}^A_r \bar{\Xi} r_{i-1} \bar{\Xi} A) \right] \omega = \left[ \sum_{0 \leq |\Xi|} (-1)^{|\Xi|} d_\Xi (c'^r \sum_{0 \leq |\Xi|} h^{(r_{i-2}, \Sigma)}(A, \Xi)_{\bar{\sigma}^A_r \bar{\Xi} r_{i-2} \bar{\Xi} A}) \mathcal{E}_A + u'^r \sum_{0 \leq |\Xi|} \Delta^{r_{i-2}, \Xi}_{r_{i-2} \Xi r_{i-2}} \bar{\sigma}^A_{r_{i-2}} \bar{\Xi} r_{i-2} \bar{\Xi} A) \right] \omega = d_H \sigma_i.
\]

It is brought into the form
\[
\left[ \sum_{0 \leq |\Xi|} (-1)^{|\Xi|} d_\Xi (c'^r \sum_{0 \leq |\Xi|} h^{(r_{i-2}, \Sigma)}(A, \Xi)_{\bar{\sigma}^A_r \bar{\Xi} r_{i-2} \bar{\Xi} A}) \mathcal{E}_A + u'^r \sum_{0 \leq |\Xi|} \Delta^{r_{i-2}, \Xi}_{r_{i-2} \Xi r_{i-2}} \bar{\sigma}^A_{r_{i-2}} \bar{\Xi} r_{i-2} \bar{\Xi} A) \right] \omega = d_H \sigma_i.
\]

Using the relation (20), we obtain
\[
\left[ \sum_{0 \leq |\Xi|} c'^r \sum_{0 \leq |\Xi|} h^{(r_{i-2}, \Sigma)}(A, \Xi)_{\bar{\sigma}^A_r \bar{\Xi} r_{i-2} \bar{\Xi} A} \mathcal{E}_A + u'^r \sum_{0 \leq |\Xi|} \Delta^{r_{i-2}, \Xi}_{r_{i-2} \Xi r_{i-2}} \bar{\sigma}^A_{r_{i-2}} \bar{\Xi} r_{i-2} \bar{\Xi} A) \right] \omega = d_H \sigma_i'.
\]
The variational derivative of the both sides of this equality with respect to the antifield $\bar{r}_{r_{i-2}}$ leads to the relation

$$\sum_{0 \leq |\Sigma|} \eta(h^{(r_{i-2})(A,\Xi)}) d_{\Sigma}(c^r_{\Xi} d_{\Xi} E_{A}) + \sum_{0 \leq |\Sigma|} u^{r_{i-1}}_\Sigma \eta(\Delta^{r_{i-2}}_{r_{i-1}})^{\Sigma} = 0,$$

which takes the form

$$\sum_{0 \leq |\Sigma|} d_{\Sigma}u^{r_{i-1}} \frac{\partial}{\partial c^{r_{i-1}}_{\Sigma}} u^{r_{i-2}} = \delta(\alpha^{r_{i-2}}), \quad \alpha^{r_{i-2}} = - \sum_{0 \leq |\Sigma|} \eta(h^{(r_{i-2})(A,\Xi)}) d_{\Sigma}(c^r_{\Xi} \bar{r}_{r_{i-1}} A), \quad (69)$$

Therefore, the odd graded derivations

$$u_{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}}, \quad u^{r_{k-1}} = \sum_{0 \leq |A|} c^r_A \eta(\Delta^{r_{k-1}}_{r_k}) A, \quad k = 2, \ldots, N, \quad (70)$$

are the $k$-stage gauge supersymmetries [5]. The graded derivations $u (66), u_{(1)} (68), u_{(k)} (70)$ are assembled into the ascent operator (60) of ghost number 1. It provides the cochain sequence (59).

**Remark 7.** The ascent operator (60) need not be nilpotent. We say that gauge and higher-stage gauge supersymmetries of a Lagrangian system form an algebra on the shell if the graded derivation (60) can be extended to a graded derivation $\nu$ of ghost number 1 by means of terms of higher polynomial degree in ghosts such that $\nu$ is nilpotent on the shell. Namely, we have

$$\nu = \nu_e + \xi = u^A \partial_A + \sum_{1 \leq k \leq N} (u^{r_{k-1}} + \xi^{r_{k-1}}) \partial_{r_{k-1}} + \xi^{r_N} \partial_{r_N}, \quad (71)$$

where all the coefficients $\xi^{r_{k-1}}$ are at least quadratic in ghosts and $(\nu \circ \nu)(f)$ is $\delta$-exact for any graded function $f \in P_0^0 \{ N \} \subset P_0^0 \{ N \}$. This nilpotency condition falls into a set of equalities with respect to the polynomial degree in ghosts. Let us write the first and second of them involving the coefficients $\xi^{r_{k-1}}$ quadratic in ghosts. We have

$$\sum_{0 \leq |\Sigma|} d_{\Sigma}u^r \partial_{r}^{\Sigma} u^B = \delta(\alpha^1_B), \quad \sum_{0 \leq |\Sigma|} d_{\Sigma}u^{r_{k-1}} \partial_{r_{k-1}}^{\Sigma} u^{r_{k-2}} = \delta(\alpha^{r_{k-2}}_1), \quad 2 \leq k \leq N, \quad (72)$$

$$\sum_{0 \leq |\Sigma|} [d_{\Sigma}u^A \partial_{r}^{\Sigma} u^B + d_{\Sigma} \xi^{r}_{2} \partial_{r}^{\Sigma} u^B] = \delta(\alpha^2_B), \quad (73)$$

$$\sum_{0 \leq |\Sigma|} [d_{\Sigma}u^A \partial_{r}^{\Sigma} u^{r_{k-1}} + d_{\Sigma} \xi^{r}_{2} \partial_{r_{k}}^{\Sigma} u^{r_{k-1}} + d_{\Sigma}u^{r_{k-1}} \partial_{r_{k-1}}^{\Sigma} \xi^{r_{k-1}}] = \delta(\alpha^{r_{k-1}}_2), \quad (74)$$

$$\xi^r = \xi^{r_{k-1}} \xi^{r_{k}} \partial_{r}^{\Sigma} \xi^{r_{k}}, \quad \xi^{r}_{2} = \xi^{r_{k-1}} \xi^{r_{k}} \partial_{r_{k}}^{\Sigma} \xi^{r_{k-1}}, \quad 2 \leq k \leq N. \quad (75)$$

The equalities (72) reproduce the relations (67) and (69) in Theorem 3. The equalities (73) – (74) provide the generalized commutation relations on the shell between gauge and
higher-stage gauge supersymmetries, and one can think of the coefficients \( \xi_2 \) (75) as being sui generis generalized structure functions [5, 11].

4. Example

We address the topological BF theory of two exterior forms \( A \) and \( B \) of form degree \(|A| + |B| = \dim X - 1\) on a smooth manifold \( X \) [7], but restrict our consideration to its simplest variant where \( A \) is a function [4, 6].

Let us consider the fiber bundle

\[
Y = \mathbb{R} \times X^{n-1} \wedge T^* X,
\]

coordinated by \((x^\lambda, A, B_{\mu_1...\mu_{n-1}})\) and provided with the canonical \((n-1)\)-form

\[
B = \frac{1}{(n-1)!} B_{\mu_1...\mu_{n-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n-1}}.
\]

The Lagrangian and the Euler–Lagrange operator of the topological BF theory read

\[
L_{BF} = \frac{1}{n} Ad_H B, \quad \delta L = dA \wedge \mathcal{E} \omega + dB_{\mu_1...\mu_{n-1}} \wedge \mathcal{E}^{\mu_1...\mu_{n-1}} \omega, \quad \mathcal{E} = \epsilon^{\mu_1...\mu_{n-1}} d_{\mu} B_{\mu_1...\mu_{n-1}}, \quad \mathcal{E}^{\mu_1...\mu_{n-1}} = -\epsilon^{\mu_1...\mu_{n-1}} d_{\mu} A, \quad (77)
\]

where \( \epsilon \) is the Levi–Civita symbol. Let consider the BGDA \( \mathcal{P}_\infty^*[Y^*; Y] \) where

\[
VY = Y \times Y, \quad Y^* = (\mathbb{R} \times X^{n-1} \wedge T^* X) \otimes X^{n-1} \wedge T^* X.
\]

It possesses the local basis \( \{A, B_{\mu_1...\mu_{n-1}}, s, s_{\mu_1...\mu_{n-1}}\} \), where \( s, s_{\mu_1...\mu_{n-1}} \) are odd antifields of antifield number 1. With the nilpotent Koszul–Tate differential

\[
\delta = \frac{\partial}{\partial s} \mathcal{E} + \frac{\partial}{\partial s_{\mu_1...\mu_{n-1}}} \mathcal{E}^{\mu_1...\mu_{n-1}},
\]

we have the complex (25):

\[
0 \leftarrow \text{Im} \delta \leftarrow \mathcal{P}_\infty^{0,n}[Y^*; Y]_1 \leftarrow \mathcal{P}_\infty^{0,n}[Y^*; Y]_2.
\]

A generic one-chain reads

\[
\Phi = \sum_{0 \leq |\Lambda|} (\Phi^\Lambda s_\Lambda + \Phi^\Lambda_{\mu_1...\mu_{n-1}} s_{\mu_1...\mu_{n-1}}) \omega,
\]

and the cycle condition takes the form

\[
\delta \Phi = \Phi^\Lambda \mathcal{E}_\Lambda + \Phi^\Lambda_{\mu_1...\mu_{n-1}} \mathcal{E}^{\mu_1...\mu_{n-1}} = 0. \quad (78)
\]
If $\Phi^\Lambda$ and $\Phi^\Lambda_{\mu_1...\mu_{n-1}}$ are independent of the variational derivatives (77) (i.e., $\Phi$ is a nontrivial cycle), the equality (78) is split into the following ones

$$\Phi^\Lambda E^\Lambda = 0, \quad \Phi^\Lambda_{\mu_1...\mu_{n-1}} E^\Lambda_{\mu_1...\mu_{n-1}} = 0.$$  

The first equality holds iff $\Phi^\Lambda = 0$, i.e., there is no Noether identity involving $E$. The second one is satisfied iff

$$\Phi^\Lambda_{\mu_1...\mu_{n-1}} E^\Lambda_{\mu_1...\mu_{n-1}} = -\Phi^\Lambda_{\mu_1...\mu_{n-1}} E^\Lambda_{\mu_1...\mu_{n-1}}.$$  

It follows that $\Phi$ factorizes as

$$\Phi = \sum_{0 \leq |\Xi|} C^\Xi_{\nu_2...\nu_{n-1}} d^\Xi_{\nu_2...\nu_{n-1}} \omega$$  

via local graded densities

$$\Delta^\nu_{\nu_2...\nu_{n-1}} = \Delta^\nu_{\nu_2...\nu_{n-1}} \delta^\alpha_{\alpha_1...\alpha_{n-1}} = \delta^\lambda_{\alpha_1} \delta^\nu_{\alpha_2} \cdots \delta^\nu_{\alpha_{n-1}} \delta^\alpha_{\alpha_1...\alpha_{n-1}} = d^\nu \nu^\nu_{\nu_2...\nu_{n-1}},$$  

which provide the complete Noether identities

$$d^\nu E^\nu_{\nu_2...\nu_{n-1}} = 0.$$  

The local graded densities (79) form the basis for a projective $C^\infty(X)$-module of finite rank which is isomorphic to the module of sections of the vector bundle

$$\nabla^s = \wedge^{n-2} T^s X \otimes_n T^* X, \quad V = \wedge^{n-2} T^* X.$$  

Therefore, let us enlarge the BGDA $P^*_\infty[\nabla^s; Y]$ to the BGDA $\overline{P}^*_\infty\{0\} = P^*_\infty[\nabla^s; Y; V]$ possessing the local basis $\{A, B_{\mu_1...\mu_{n-1}}, \overline{\alpha}, \overline{\nu}^\alpha_{1...\mu_{n-1}}, \overline{\sigma}^{\mu_2...\mu_{n-1}}\}$, where $\overline{\sigma}^{\mu_2...\mu_{n-1}}$ are even antifields of antifield number 2. We have the nilpotent graded derivation

$$\delta_0 = \overline{\alpha} + \frac{\partial}{\partial \overline{\sigma}^{\mu_2...\mu_{n-1}}} \Delta^{\mu_2...\mu_{n-1}}$$  

of $P^*_\infty\{0\}$. Its nilpotency is equivalent to the complete Noether identities (80). Then we obtain the one-exact complex

$$0 \leftarrow \text{Im} \overline{\delta} \leftarrow P^0,1_{\infty}[\nabla^s; Y]_1 \leftarrow \text{Im} \delta_0_2 \leftarrow P^0,2_{\infty}\{0\}_2 \leftarrow \text{Im} \delta_0_3.$$  

Iterating the arguments, we come to the $(N + 1)$-exact complex (45) for $N \leq n - 3$ as follows. Let us consider the corresponding BGDA

$$\overline{P}^*_{\infty}\{N\} = \overline{P}^*[...V_3 V_1 \nabla^s; Y; V V_2 V_4 ...], \quad V_k = \wedge^{n-k-2} T^* X, \quad k = 1, \ldots, N,$$
possessing the local basis
\[ \{ A, B_\mu, \ldots, \mu_{n-1}, \bar{s}, \bar{\sigma}^{i_1 \cdots i_{n-1}}, \bar{\sigma}^{j_2 \cdots j_{n-1}}, \ldots, \bar{\sigma}^{j_{n+2} \cdots j_{n-1}} \}, \]
\[ [\bar{\sigma}^{j_{k+2} \cdots j_{n-1}}] = (k+1) \text{mod } 2, \quad \text{Ant}[\bar{\sigma}^{j_{k+2} \cdots j_{n-1}}] = k + 3. \]

It is provided with the nilpotent graded derivation
\[ \delta_N = \delta_0 + \sum_{1 \leq k \leq N} \frac{\partial}{\partial \bar{\sigma}^{\mu_{k+2} \cdots \mu_{n-1}}} \Delta^{\mu_{k+2} \cdots \mu_{n-1}}, \quad \Delta^{\mu_{k+2} \cdots \mu_{n-1}} = d_{\mu_{k+1}} \bar{\sigma}^{\mu_{k+1} \mu_{k+2} \cdots \mu_{n-1}}, \]

of antifield number -1. Its nilpotency results from the Noether identities (80) and equalities
\[ d_{\mu_{k+2}} \Delta^{\mu_{k+2} \cdots \mu_{n-1}} = 0, \quad k \in \mathbb{N}, \]

which are \( k \)-stage Noether identities [4]. Then the manifested \((N+1)\)-exact complex reads
\[ 0 \leftarrow \text{Im} \bar{\sigma} \leftarrow \mathcal{P}_\infty^{0,n-1} \{ Y \} \leftarrow \left[ \delta_0 \mathcal{P}_\infty^{0,n} \{ 0 \} \right] \leftarrow \mathcal{P}_\infty^{0,n} \{ 1 \} \leftarrow \left[ \delta_{N-1} \mathcal{P}_\infty^{0,n} \{ N \} \right] \leftarrow 0 \]

It obeys the following \((N+2)\)-homology regularity condition.

**Lemma 1.** Any \((N+2)\)-cycle \( \Phi \in \mathcal{P}_\infty^{0,n} \{ N-1 \} \) up to a \( \delta_{N-1} \)-boundary is
\[ \Phi = \sum_{k_1 + \cdots + k_i + 3i = N+2} \sum_{0 \leq |A_1| \leq |A_i|} C_{\mu_1 \cdots \mu_{n-1}}^{A_1 \cdots A_i} \Delta^{\mu_1 \cdots \mu_{n-1}} \Delta \]
where \( \bar{\sigma}^{\mu_1 \cdots \mu_{n-1}} = \bar{\sigma}^{\mu_1 \cdots \mu_{n-1}} \) and \( \Delta^{\mu_1 \cdots \mu_{n-1}} = \mathcal{E}^{\mu_1 \cdots \mu_{n-1}}. \) It follows that \( \Phi \) is a \( \delta_N \)-boundary.

**Proof.** Let us choose some basis element \( \bar{\sigma}^{\mu_1 \cdots \mu_{n-1}} \) and denote it simply by \( \bar{\sigma} \). Let \( \Phi \) contain a summand \( \phi_1 \bar{\sigma} \), linear in \( \bar{\sigma} \). Then the cycle condition reads
\[ \delta_{N-1} \Phi = \delta_{N-1} (\Phi - \phi_1 \bar{\sigma}) + (-1)^{[\bar{\sigma}]} \delta_{N-1} (\phi_1 \bar{\sigma}) + \phi \Delta = 0, \quad \Delta = \delta_{N-1} \bar{\sigma}. \]

It follows that \( \Phi \) contains a summand \( \psi \Delta \) such that
\[ (-1)^{[\bar{\sigma}]+1} \delta_{N-1} (\psi) \Delta + \phi \Delta = 0. \]

This equality implies the relation
\[ \phi_1 = (-1)^{[\bar{\sigma}]+1} \delta_{N-1} (\psi) \]
because the reduction conditions (82) involve total derivatives of \( \Delta \), but not \( \Delta \). Hence,
\[ \Phi = \Phi' + \delta_{N-1} (\psi \bar{\sigma}), \]

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where $\Phi'$ contains no term linear in $\tau$. Furthermore, let $\tau$ be even and $\Phi$ has a summand $\sum \phi_r \tau^r$ polynomial in $\tau$. Then the cycle condition leads to the equalities

$$\phi_r \Delta = -\delta_{N-1} \phi_{r-1}, \quad r \geq 2.$$  

Since $\phi_1$ (85) is $\delta_{N-1}$-exact, then $\phi_2 = 0$ and, consequently, $\phi_{r>2} = 0$. Thus, a cycle $\Phi$ up to a $\delta_{N-1}$-boundary contains no term polynomial in $\tau$. It reads

$$\Phi = \sum_{k_1 + \cdots + k_i + 3i = N+2} \sum_{0 < |A_1|, \ldots, |A_i|} G^A_{\mu_{k_1+2}\cdots\mu_{n-1} \cdots \mu_{n-1}} \tau^A_{\lambda_{k_1+2} \cdots \lambda_{n-2} \cdots \lambda_{n-1}, \sigma_{k_2+2} \cdots \sigma_{n-2} \cdots \sigma_{n-1} \cdot \omega}. \quad (86)$$

However, the terms polynomial in $\tau$ may appear under general covariant transformations

$$\tau^{\mu_{k+2} \cdots \mu_{n-1}} = \det \left( \frac{\partial x^\alpha}{\partial y^\beta} \right) \frac{\partial x^{\mu_{k+2}}}{\partial y^\beta} \cdots \frac{\partial x^{\mu_{n-1}}}{\partial y^\beta} \tau_{\mu_{k+2} \cdots \mu_{n-1}}$$

of a chain $\Phi$ (86). In particular, $\Phi$ contains the summand

$$\sum_{k_1 + \cdots + k_i + 3i = N+2} F_{\nu_{k_1+2} \cdots \nu_{n-1} \cdots \nu_{n-1}} \tau^\nu_{k_1+2 \cdots \nu_{n-1} \cdots \nu_{n-1}}$$

which must vanish if $\Phi$ is a cycle. This takes place only if $\Phi$ factorizes through the graded densities $\Delta_{\mu_{k+2} \cdots \mu_{n-1}}$ (81) in accordance with the expression (84).

q.e.d.

Following the proof of Lemma 1, one can also show that any $(N+2)$-cycle $\Phi \in \overline{P}_{N+2} \{N\}_{n+2}$ up to a boundary takes the form

$$\Phi = \sum_{0 \leq |A|} G^A_{\mu_{N+2} \cdots \mu_{n-1}} d_A \Delta_{\mu_{N+2} \cdots \mu_{n-1}} \omega,$$

i.e., the homology $H_{N+2}^{\delta} \delta_N)$ of the complex (83) is finitely generated by the cycles $\Delta_{\mu_{N+2} \cdots \mu_{n-1}}$. Thus, the complex (83) admits the $(N+2)$-exact extension (52).

The iteration procedure is prolonged till $N = n - 3$. We have the BGDA $\overline{P} \{n-2\}$, where $V_{n-2} = X \times \mathbb{R}$. It possesses the local basis

$$\{A, B_{\mu_1 \cdots \mu_{n-1}}, \tau, \overline{\tau}^{\mu_1 \cdots \mu_{n-1}}, \overline{\tau}^{\mu_2 \cdots \mu_{n-1}}, \ldots, \overline{\tau}^{\mu_{n-1}}, \tau\},$$

where $[\tau] = (n-1) \mod 2$ and $\text{Ant} [\tau] = n+1$. The corresponding Koszul–Tate complex reads

$$0 \leftarrow \overline{\delta}_{n-2} \overline{\delta}_{n-3} \overline{\delta}_0 \overline{\delta}_1 \overline{\delta}_2 \overline{\delta}_3 \cdots$$

$$\overline{\delta}_{n-3} \overline{\delta}_{n-4} \overline{\delta}_{n-5} \overline{\delta}_6 \overline{\delta}_{n-6} \overline{\delta}_7 \overline{\delta}_{n-7} \overline{\delta}_8 \cdots$$

$$\overline{\delta}_{n-2} = \overline{\delta}_0 + \sum_{1 \leq k \leq n-3} \overline{\delta}_{\nu_{k+2} \cdots \nu_{n-1}} \Delta_{\mu_{k+2} \cdots \mu_{n-1}} + \overline{\delta}_0 \Delta, \quad \Delta = \overline{\delta}_{\nu_{n-1}} \overline{\tau}^{\mu_{n-1}}.$$

Let us enlarge the BGDA $\overline{P}_{N+2} \{n-2\}$ to the BGDA $P_{n-2} \{n-2\}$ (57) with the local basis

$$\{A, B_{\mu_1 \cdots \mu_{n-1}}, c_{\mu_2 \cdots \mu_{n-1}}, \ldots, c_{\mu_{n-1}}, c, \overline{\tau}^{\mu_1 \cdots \mu_{n-1}}, \overline{\tau}^{\mu_2 \cdots \mu_{n-1}}, \ldots, \overline{\tau}^{\mu_{n-1}}, \tau\},$$

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where \( c_{\mu_2 \cdots \mu_{n-1}}, \ldots, c_{\mu_n}, c \) are the corresponding ghosts, and let us consider the BGDA \( \mathcal{P}_*(n-2) \) with the local basis \( \{ A, B_{\mu_1 \cdots \mu_{n-1}}, c_{\mu_2 \cdots \mu_{n-1}}, \ldots, c_{\mu_n}, c \} \). By virtue of Theorem 3, the graded commutative ring \( \mathcal{P}_\infty^0(n-2) \) is split into the cochain sequence

\[
0 \to \mathcal{O}_\infty^0 Y \xrightarrow{u_e} \mathcal{P}_\infty^0(n-2)_1 \xrightarrow{u_e} \mathcal{P}_\infty^0(n-2)_2 \xrightarrow{u_e} \cdots,
\]

(87)

\[
u = u + \sum_{1 \leq k \leq n-2} u_{(k)},
\]

(88)

\[
u = -d_{\mu_1} c_{\mu_2 \cdots \mu_{n-1}} \frac{\partial}{\partial B_{\mu_1 \cdots \mu_{n-1}}},
\]

(89)

\[
u_{(k)} = -d_{\mu_k+1} c_{\mu_{k+2} \cdots \mu_{n-1}} \frac{\partial}{\partial c_{\mu_{k+1} \cdots \mu_{n-1}}}, \quad 1 \leq k \leq n-3,
\]

(90)

\[
u_{(n-2)} = -d_{\mu_n} \frac{\partial}{\partial c_{\mu}},
\]

(91)

where \( u \) (89) and \( u_{(k)} \) (90) – (91) are the gauge and higher-stage gauge supersymmetries of the Lagrangian (76) [4]. It is readily observed that the ascent operator (88) is nilpotent, i.e., the sequence (87) is a cochain complex.

5. Appendix

The proof of Theorem 1 follows that of [15], Theorem 2.1 when \( Y \) is an affine bundle.

Lemma 2. If \( Y = \mathbb{R}^{n+m} \to \mathbb{R}^n \), the complex (10) at all the terms, except \( \mathbb{R} \), is exact, while the complex (11) is exact.

Proof. This is the case of an affine bundle \( Y \), and the above mentioned exactness has been proved when the ring \( \mathcal{O}_\infty^0 Y \) is restricted to the subring \( \mathcal{P}_\infty^0 Y \) of polynomial functions (see [15], Lemmas 4.2 – 4.3). The proof of these lemmas is straightforwardly extended to \( \mathcal{O}_\infty^0 Y \) if the homotopy operator (4.5) in [15], Lemma 4.2 is replaced with that (4.8) in [15], Remark 4.1.

We first prove Theorem 1 for the above mentioned BGDA \( \Gamma(\mathcal{S}_\infty^*[F; Y]) \). Similarly to \( \mathcal{S}_\infty^*[F; Y] \), the sheaf \( \mathcal{S}_\infty^*[F; Y] \) and the BGDA \( \Gamma(\mathcal{S}_\infty^*[F; Y]) \) are split into the variational bicomplexes, and we consider their subcomplexes

\[
0 \to \mathcal{S}_\infty^0[F; Y] \xrightarrow{d_{\mu_1}} \mathcal{S}_\infty^{0,1}[F; Y] \cdots \xrightarrow{d_{\mu_1}} \mathcal{S}_\infty^{0,n}[F; Y] \xrightarrow{\delta} \mathcal{E}_1,
\]

(92)

\[
0 \to \mathcal{S}_\infty^{1,0}[F; Y] \xrightarrow{d_{\mu_1}} \mathcal{S}_\infty^{1,1}[F; Y] \cdots \xrightarrow{d_{\mu_1}} \mathcal{S}_\infty^{1,n}[F; Y] \xrightarrow{\theta} \mathcal{E}_1 \to 0,
\]

(93)

\[
0 \to \mathcal{S}_\infty^{1,0}[F; Y] \xrightarrow{d_{\mu_1}} \Gamma(\mathcal{S}_\infty^{1,0}[F; Y]) \cdots \xrightarrow{d_{\mu_1}} \Gamma(\mathcal{S}_\infty^{1,n}[F; Y]) \xrightarrow{\delta} \Gamma(\mathcal{E}_1),
\]

(94)

\[
0 \to \Gamma(\mathcal{S}_\infty^{1,0}[F; Y]) \xrightarrow{d_{\mu_1}} \Gamma(\mathcal{S}_\infty^{1,1}[F; Y]) \cdots \xrightarrow{d_{\mu_1}} \Gamma(\mathcal{S}_\infty^{1,n}[F; Y]) \xrightarrow{\theta} \Gamma(\mathcal{E}_1) \to 0,
\]

(95)

where \( \mathcal{E}_1 = \mathcal{O}(\mathcal{S}_\infty^{1,n}[F; Y]) \). By virtue of Lemma 2, the complexes (92) – (93) at all the terms, except \( \mathbb{R} \), are exact. The terms \( \mathcal{S}_\infty^*[F; Y] \) of the complexes (92) – (93) are sheaves of \( \Gamma(\mathcal{S}_\infty^0) \)-modules. Since \( J^\infty Y \) admits a partition of unity just by elements of \( \Gamma(\mathcal{S}_\infty^0) \), these sheaves
are fine and, consequently, acyclic. By virtue of the abstract de Rham theorem (see [15], Theorem 8.4, generalizing [17], Theorem 2.12.1), cohomology of the complex (94) equals the cohomology of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$ and, consequently, the de Rham cohomology of $Y$, which is the strong deformation retract of $J^\infty Y$. Similarly, the complex (95) is proved to be exact. It remains to prove that cohomology of the complexes (10) – (11) equals that of the complexes (94) – (95). The proof of this fact straightforwardly follows the proof of [15], Theorem 2.1, and it is a slight modification of the proof of [15], Theorem 4.1, where graded exterior forms on the infinite order jet manifold $J^\infty Y$ of an affine bundle are treated as those on $X$.

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