A NOTE ON EXPANSION IN PRIME FIELDS

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ABSTRACT. Let $\beta, \epsilon \in (0, 1)$, and $k \geq \exp(122\max\{1/\beta, 1/\epsilon\})$. We prove that if $A, B$ are subsets of a prime field $\mathbb{Z}_p$, and $|B| \geq p^\delta$, then there exists a sum of the form

$$S = a_1B \pm \ldots \pm a_kB, \quad a_1, \ldots, a_k \in A,$$

with $|S| \geq 2^{-12}p^{-\epsilon}\min\{|A||B|, p\}$.

As a corollary, we obtain an elementary proof of the following sum-product estimate.

For every $\alpha < 1$ and $\beta, \delta > 0$, there exists $\epsilon > 0$ such that the following holds. If $A, B, E \subset \mathbb{Z}_p$ satisfy $|A| \leq p^\alpha$, $|B| \geq p^\beta$, and $|B||E| \geq p^\delta|A|$, then there exists $t \in E$ such that

$$|A + tB| \geq cp^\epsilon|A|,$$

for some absolute constant $c > 0$. A sharper estimate, based on the polynomial method, follows from recent work of Stevens and de Zeeuw.

1. INTRODUCTION

The work in this note was motivated by the following problem in fractal geometry:

**Conjecture 1.1.** Let $\alpha, \beta, \delta \in (0, 1)$. Assume that $A, B, E \subset [0, 1]$ are compact sets with $\dim A \leq \alpha$, $\dim B \geq \beta$ and $\dim B + \dim E \geq \dim A + \delta$. Then, there exists $t \in E$ such that

$$\dim(A + tB) \geq \dim A + \epsilon$$

for some $\epsilon > 0$ depending only on $\alpha, \beta, \delta$.

The conjecture follows from Bourgain’s work [2], if $0 < \dim A = \dim B < 1$. Problems such as Conjecture 1.1 often have natural, and easier, analogues in the setting of finite fields. The same is true here, and one encounters the following question:

**Question 1.** Let $p \in \mathbb{N}$ be prime. Let $A, B, E \subset \mathbb{Z}_p$ be sets such that $|A| \leq p^\alpha$, $|B| \geq p^\beta$ and $|B||E| \geq p^\delta|A|$ for some $\alpha < 1$ and $\beta, \delta > 0$. Does there exist $t \in E$ such that $|A + tB| \geq p^\epsilon|A|$ for some $\epsilon = \epsilon(\alpha, \beta, \delta) > 0$?

The following simple example motivates the requirements $\dim E + \dim B \geq \dim A + \delta$ and $|B||E| \geq p^\delta|A|$.

**Example 1.2.** Consider the sets

$$A = \left\{\frac{1}{n^{1/2}}, \frac{2}{n^{1/2}}, \ldots, \frac{n^{1/2}}{n^{1/2}}\right\}$$

and

$$B = \left\{\frac{1}{n^{1/4}}, \frac{2}{n^{1/4}}, \ldots, \frac{n^{1/4}}{n^{1/4}}\right\} = E.$$
for any integer \( n = m^4 \in \mathbb{N} \). Then \( |B||E| = |A| \) and \( BE \subset A \), so

\[
|A + tB| \leq |A + EB| \leq |A + A| = 2|A| - 1, \quad t \in E.
\]

Iterating the construction above, it is not difficult to produce compact sets \( A, B, E \subset [0,1] \) with \( \dim_H A = \frac{1}{2} \) and \( \dim_H B = \frac{1}{4} = \dim_H E \) such that \( \dim_H (A + tB) = \dim_H A \) for all \( t \in E \). In \( \mathbb{Z}_p \), an even easier example is given by \( A = \{1, \ldots, \lfloor p^{1/2} \rfloor \} \) and \( B = \{1, \ldots, \lfloor p^{1/4} \rfloor \} \) = \( E \).

It turns out that the answer to Question 1 is positive, and a good estimate follows easily from the recent incidence bound of Stevens and de Zeeuw, [9, Theorem 4]:

**Proposition 1.3.** Assume that \( A, B, E \subset \mathbb{Z}_p \) are sets with \( |B| \leq |A| \) and \( |B||E| \leq p \). Then, there exists \( t \in E \) such that

\[
|A + tB| \gtrsim \min\{|A|^{2/3}(|B||E|)^{1/3}, |A||B|, p\}.
\]

For \( A, B, E \subset \mathbb{R} \), by comparison, the Szemerédi-Trotter theorem gives \( |A + tB| \gtrsim \min\{|(|B||E|)^{1/2}, |A||B|\} \) for some \( t \in E \). Proposition 1.3 is also closely related to [5, Theorem 3]. Proposition 1.3 certainly answers Question 1. However, the proofs in [5, 9] are based on the polynomial method, more precisely on a point-plane incidence bound in \( \mathbb{Z}_p^2 \) by Rudnev [6]. It is not clear how to apply similar ideas to the continuous problem, Conjecture 1.1, so a more elementary approach to Question 1 seemed desirable. Here is the main result of this note:

**Theorem 1.4.** Let \( A, B \subset \mathbb{Z}_p \) and \( \beta, \epsilon \in (0, 1] \). Assume that \( |B| > p^{\beta} > 4 \). Then, there exists an integer

\[
k \leq \exp(C \max\{1/\beta, 1/\epsilon\}),
\]

and elements \( a_1, \ldots, a_k \in A \) such that

\[
|a_1B \pm \ldots \pm a_kB| \geq cp^{-\epsilon} \min\{|A||B|, p\} \tag{1.5}
\]

for certain choices of signs \( \pm \in \{-, +\} \). Here \( c \geq 2^{-12} \) and \( C \leq 122 \) are absolute constants.

A positive answer to Question 1 follows easily from Theorem 1.4, applied to \( B \) and \( E \), and combined with the Plünnecke-Ruzsa inequalities. The proof of Theorem 1.4 is elementary and does not use polynomials; instead, it consists of a reduction to a sum-product estimate of Bourgain, [1, Lemma 2], stating briefly that

\[
|8AB - 8AB| \gtrsim \min\{|A||B|, p\}. \tag{1.6}
\]

Unfortunately, while the proof of (1.6) is elementary as well, it does not easily generalise to a "continuous" setting. So, at the end of the day, we are not much closer to proving Conjecture 1.1.

The paper is organised so that Theorem 1.4 is proven in Section 3. The application to Question 1, as well as the proof or Proposition 1.3, is discussed in Section 4.

2. ACKNOWLEDGEMENTS

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3. PROOF OF THE MAIN THEOREM

Before starting the proof of Theorem 1.4, we record the following lemma. It is quite likely well-known, and at least we extracted the argument from a paper of Bourgain, see [2, (7.20)].

**Lemma 3.1.** Let \((G, +)\) be an Abelian group, and assume that \(A, B \subset G\) are sets with \(|A + A| \leq C_1|A|\) and \(|B + B| \leq C_2|B|\). Assume moreover that there exists \(G \subset A \times B\) with \(|G| \geq |A||B|/C_3\) such that

\[
|\pi_1(G)| \leq C_4|A|,
\]

where \(\pi_1(x, y) = x + y\). Then \(|A + B| \leq C|A|\) with \(C = C_1C_2C_3C_4\).

**Proof.** We start by observing that

\[
\chi_{A + B}(t) \leq \frac{1}{|G|} \sum_{(x,y) \in (A + A) \times (B + B)} \chi_{\pi_1(-G + (x,y))}(t), \quad t \in G. \tag{3.2}
\]

Indeed, if \(t \in A + B = \pi_1(A \times B)\), then \(t = \pi_1(a, b)\) for some \((a, b) \in A \times B\). We then note that \((a, b) \in -G + (x, y) -\) and hence \(t = \pi_1(a, b) \in \pi_1(-G + (x, y))\) for all \((x, y) \in G + (a, b) \subset (A \times B) + (A \times B) = (A + A) \times (B + B)\).

So, (3.2) follows from \(|G + (a, b)| = |G|\). Finally,

\[
|A + B| = \sum_{t \in G} \chi_{A + B}(t) \\
\leq \frac{1}{|G|} \sum_{(x,y) \in (A + A) \times (B + B)} \sum_{t \in G} \chi_{\pi_1(-G + (x,y))}(t) \\
\leq \frac{|\pi_1(G)||A + A||B + B|}{|G|} \leq \frac{C_1C_2C_3C_4|A|^2|B|}{|A||B|} = C|A|,
\]

as claimed. \(\square\)

Now, we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** We may assume that \(A, B \neq \emptyset\). Fix \(n \in \mathbb{N}\) such that

\[
\max \left\{ \frac{4}{\beta}, \frac{58}{e} \right\} \leq n \leq 60 \max \left\{ \frac{4}{\beta}, \frac{1}{2} \right\}, \tag{3.3}
\]

and write \(\delta := 1/n\). In particular

\[
\frac{p^{-2\delta}}{2}|B| \geq \frac{p^{-2(n+\beta)}}{2} > \frac{p^{\delta/2}}{2} > 1. \tag{3.4}
\]

Write \(A_1 := A\), and inductively

\[A_{j+1} := A_j + A_j, \quad j \geq 1.\]

We note that there exists \(1 \leq j \leq n + 1\) such that

\[|A_j + A_j| = |A_{j+1}| \leq p^\delta|A_j|, \tag{3.5}\]

since otherwise

\[p \geq |A_{n+2}| \geq p^\delta|A_{n+1}| \geq \ldots \geq p^{\delta(n+1)}|A_1| > p,\]

a contradiction. We define \(A := A_j\) for some index \(j \leq n + 2\) satisfying (3.5).
Next, in a similar spirit, we define a sequence of sets $H_k$, as follows. Start by setting $H_1 := a_1 B$ for any $a_1 \in \bar{A}$. Next, assume that $H_l$ has already been defined for some $l \geq 1$. Choose $2^l - 1$ elements $a'_1, \ldots, a'_{2^l - 1} \in \bar{A}$, and $(2^l - 1)$ signs $\pm \in \{+, -\}$ such that

$$H_{l+1} := H_l \pm a'_1 B \pm \ldots \pm a'_{2^l - 1} B$$

has maximal cardinality (among all such choices of $a'_1, \ldots, a'_{2^l - 1}$, and choices of signs). As before, there exists $1 \leq l \leq n + 1$ such that

$$|H_{l+1}| \leq p^\delta |H_l|.$$  

Now, we set $H := H_l$ for such an index $1 \leq l \leq n + 1$. We note that

$$|H + H| \leq |H_{l+1}| \leq p^\delta |H_l| = p^\delta |H|,$$  

by the maximality of $|H_{l+1}|$, since $H = H_l$ can be written as a sum of $2^l - 1$ terms of the form $a_j B$, $a_j \in \bar{A}$. It is even clearer that

$$|H \pm a B| \leq |H_{l+1}| \leq p^\delta |H_l| = p^\delta |H|, \quad a \in \bar{A}.$$  

Evidently $H$ is a set of the kind appearing on the left hand side of (1.5); more precisely $H$ is a sum of at most

$$2^{2(n+1)} \leq \exp(122 \max\{1/\beta, 1/\epsilon\})$$

terms of the form $a_j B$ with $a_j \in A$. It remains to show that $H$ satisfies (1.5).

We start the proof by showing that there exists an element $b_0 \in B$, and subset $B' \subset B$ of cardinality $|B'| \geq p^{-2\delta} |B|/2$ such that

$$|H + (b_0 - b) \bar{A}| \leq 2p^{4\beta} |H|, \quad b \in B'.$$  

To prove (3.8), we consider the following set $P \subset \mathbb{Z}_p^2$

$$P := \{(a, r) \in \mathbb{Z}_p^2 : a \in \bar{A} \text{ and } r \in aB + H\},$$

and we note that

$$|\bar{A}||H| \leq |P| \leq p^\delta |\bar{A}||H|$$  

by (3.7). Consider also the following family of lines: $L := \{\ell_{h,b}\}_{(h,b) \in H \times B}$, where

$$\ell_{h,b} = \{(x, y) \in \mathbb{Z}_p^2 : y = xb + h\}.$$  

We note that every line in $L$ contains exactly $|\bar{A}|$ points in $P$. Indeed:

$$P \cap \ell_{h,b} = \{(a, ab + h) : a \in \bar{A}\}, \quad (h, b) \in H \times B.$$  

It follows that if $b \in B$ is fixed, the (disjoint) lines $\{\ell_{h,b}\}_{h \in H}$ cover $|\bar{A}||H|$ points of $P$ in total. In other words, the sets

$$P_b := \bigcup_{h \in H} (P \cap \ell_{h,b}) \subset P, \quad b \in B,$$

satisfy

$$|P_b| = |\bar{A}||H| \geq p^{-\delta} |P|, \quad b \in B,$$  

recalling (3.9). Using Cauchy-Schwarz in a standard way, see for example [3, Lemma 4.2], it follows from (3.10) that there exists $b_0 \in B$, and a subset $B' \subset B$ with $|B'| \geq p^{-2\delta} |B|/2$ such that

$$|P_{b_0} \cap P_{b_0}| \geq \frac{p^{-2\delta}}{2} |P|, \quad b \in B'.$$  


We record here that
\[ \pi_{-b}(P_b \cap P_{b_0}) \subset \pi_{-b}(P_b) \subset H, \quad b \in B, \]  
(3.12)
where \( \pi_c(x, y) = xc + y \). Indeed, if \( p = (a, ab + h) \in P_b \) for some \( a \in \bar{A} \) and \( h \in H \), then
\[ \pi_{-b}(p) = -ab + ab + h = h \in H. \]

Now, given such a point \( b_0 \in B \), we define the following bijective linear map:
\[ T(x, y) := (x, y - b_0x). \]

It is immediate that
\[ \pi_{b_0-b}(T(x, y)) = \pi_{-b}(x, y), \quad (x, y) \in \mathbb{Z}_p^2. \]
(3.13)
Moreover, \( T(P_{b_0}) \subset \bar{A} \times H \). Indeed, if \( p = (a, ab + h) \in P_{b_0} \) for some \( a \in \bar{A} \) and \( h \in H \), then
\[ T(p) = (a, ab + h - ab_0) = (a, h) \in \bar{A} \times H, \]
as claimed. We write
\[ G_b := T(P_b \cap P_{b_0}) \subset \bar{A} \times H, \quad b \in B', \]
and conclude from (3.11) that
\[ |G_b| \geq \frac{p^{-2\delta}}{2}|P| \geq \frac{p^{-2\delta}}{2}|ar{A}|H|. \]
(3.14)
From (3.13) and (3.12), we conclude that
\[ \pi_{b_0-b}(G_b) = \pi_{-b}(P_b \cap P_{b_0}) \subset H, \quad b \in B', \]
so in particular \( |\pi_{b_0-b}(G_b)| \leq |H| \). Now, Lemma 3.1 applied to the sets \( H, (b_0-b)\bar{A} \) and \( \{(y, (b_0-b)x) : (x, y) \in G_b \} \subset H \times (b_0-b)\bar{A} \) implies, recalling (3.5), (3.6) and (3.14), that
\[ |H + (b_0-b)\bar{A}| \leq 2p^{4\delta}|H|, \quad b \in B', \]
as claimed in (3.8).

Set \( B := b_0 - B' \). Then \( |B| \geq p^{-2\delta}|B|/2 > 1 \) by (3.4), and
\[ |H \pm b\bar{A}| \leq 2p^{5\delta}|H| \quad \text{and} \quad |H \pm a\bar{B}| \leq p^{\delta}|H|, \quad a \in \bar{A}, \ b \in \bar{B}, \]
(3.15)
combining (3.7) and (3.8). The "-" inequality \( |H - b\bar{A}| \leq 2p^{5\delta}|H| \) moreover uses (3.5) and Ruzsa’s triangle inequality:
\[ |H - b\bar{A}| \leq \frac{|H + b\bar{A}| |b\bar{A} + b\bar{A}|}{|A|} \leq 2p^{5\delta}|H|, \quad b \in \bar{B}. \]

Now, we apply a result of Bourgain, namely [1, Lemma 2]. It states that if
\[ (A - \bar{A}) \cap (\bar{B} - \bar{B}) \neq \{0\}, \]
(3.16)
then there exist subsets \( \bar{B}_1 \subset \bar{B}, Z \subset (\bar{A} - \bar{A}) \cap (\bar{B} - \bar{B}) \), and elements \( a_1, \ldots, a_6 \in \bar{A}, b_1, \ldots, b_6 \in \bar{B} \) such that
\[ |(b_1-b_2)A + (a_1-a_2+a_3-a_4)\bar{B}_1 + (a_5-a_6+b_3-b_4+b_5-b_6)Z| \geq \frac{1}{2} \min\{|A||B|, p-1\}. \]
(3.17)
The condition (3.16) is not automatically satisfied, but in any case we can proceed as follows. Since Theorem 1.4 is trivial for \( |A| \leq 1 \), we may assume that \( |\bar{A}| \geq |A| \geq 2 \). Since
also $|\bar{B}| \geq 2$ by the choice of $\delta$ in (3.4), there exist $a, a' \in \bar{A}$ and $b, b' \in \bar{B}$ with $a \neq a'$ and $b \neq b'$. Then, writing $\xi := (a - a')/(b - b') \neq 0$, we have
\[(\bar{A} - \bar{A}) \cap (\xi \bar{B} - \xi \bar{B}) \supset \{a - a'\},\]
and so Bourgain’s result is applicable to $\bar{A}$ and $\xi \bar{B}$. Then, (3.17) implies the existence of $a_1, \ldots, a_6 \in \bar{A}$ and $b_1, \ldots, b_6 \in \bar{B}$ such that the sum
\[\xi (b_1 - b_2) \bar{A} + (a_1 - a_2 + a_3 - a_4) \bar{B} + (a_5 - a_6) (\xi \bar{B} - \xi \bar{B}) + \xi (b_3 - b_4 + b_5 - b_6) (\bar{A} - \bar{A})
\]
has cardinality at least $\min\{|A||B|, p - 1\}/2$. Then, the same conclusion follows automatically for the sum
\[(b_1 - b_2) \bar{A} + (a_1 - a_2 + a_3 - a_4) \bar{B} + (a_5 - a_6) (\bar{B} - \bar{B}) + (b_3 - b_4 + b_5 - b_6) (\bar{A} - \bar{A})
\]
\[\subset b_1 \bar{A} - b_2 \bar{A} + b_3 \bar{A} - b_4 \bar{A} + b_5 \bar{A} - b_6 \bar{A} = \bar{A} \bar{A} - \bar{A} \bar{A} + b_4 \bar{A} - b_5 \bar{A} + b_6 \bar{A} + a_1 \bar{B} - a_2 \bar{B} + a_3 \bar{B} - a_4 \bar{B} + a_5 \bar{B} - a_6 \bar{B} - a_5 \bar{B} + a_6 \bar{B} =: \sum_{i=1}^{b_6} \sum_{j=a_6}^{b_6} (\bar{A}, \bar{B}).
\]
(In fact, Bourgain uses the same argument to prove the second part of [1, Lemma 2].) Finally, using (3.15) and the Plünnecke-Ruzsa inequalities for different summands (see [7] or [8, Theorem 6.1]), we conclude that
\[\frac{1}{2} \min\{|A||B|, p - 1\} \leq |\sum_{i=1}^{b_6} \sum_{j=a_6}^{b_6} (\bar{A}, \bar{B})| \leq 2^{10} p^{586} |H| \leq 2^{10} p^\epsilon |H|.
\]
The final inequality follows from (3.3). Hence,
\[|H| \geq 2^{-12} p^{-586} \min\{|A||B|, p\} \geq 2^{-12} p^{-\epsilon} \min\{|A||B|, p\}
\]
as required. The proof of Theorem 1.4 is complete. \qed

4. LOWER BOUNDS FOR $|A + tB|$

We start by applying Theorem 1.4 to Question 1. Then, we recall the result of Stevens and de Zeeuw from [9] and apply it to prove Proposition 1.3.

Let $A, B, E \subset \mathbb{Z}_p$ be as in Question 1, with $|A| \leq p^\alpha$, $|B| \geq p^\beta$ and $|B||E| \geq p^\delta |A|$, and assume that $|A + tB| \leq p^\epsilon |A|$ for all $t \in E$, and for some $\epsilon > 0$. Then, for any $t_1, \ldots, t_k \in E$, it follows from the Plünnecke-Ruzsa inequalities for different summands (see [7] or [8, Theorem 6.1]) that there exists a non-empty subset $X \subset A$ with the property that
\[|X + t_1 B + \ldots + t_k B| \lesssim_k p^{k\epsilon} |X|.
\]
By another application of the Plünnecke-Ruzsa inequalities,
\[|(t_1 B + \ldots + t_k B) - (t_1 B + \ldots + t_k B)| \lesssim_k p^{2k\epsilon} |A|.
\]
Now, write $\epsilon' := \min\{\delta/2, (1 - \alpha)/2\}$, and use Theorem 1.4 to choose $t_1, \ldots, t_k \in E$, with $k \leq 2^c \max(1/\beta, 1/\epsilon')$, so that
\[|t_1 B \pm \ldots \pm t_k B| \geq p^{\epsilon'} \min\{|B||E|, p\} \geq \min\{p^{\delta/2}, p^{(1-\alpha)/2}\} |A|.
\]
for certain signs $\pm \in \{+, -\}$. It follows from (4.1) that $p^{2k\epsilon} |A| \gtrsim \min\{p^{\delta/2}, p^{(1-\alpha)/2}\} |A|$, which gives a lower bound for $\epsilon$, depending only on $\alpha, \beta, \delta$.

To prove Proposition 1.3, we recall the statement of [9, Theorem 4]:
Theorem 4.2 (Stevens-de Zeeuw). Let $A, B \subset \mathbb{Z}_p$ be sets, and let $\mathcal{L}$ be a collection of lines in $\mathbb{Z}_p^2$, with
\[ |B| \leq |A|, \quad |A^2|B| \leq |\mathcal{L}|^3, \quad \text{and} \quad |B||\mathcal{L}| \leq p^2. \]
Then, the set of incidences $I(A \times B, \mathcal{L}) := \{(a, b, \ell) : (a, b) \in (A \times B) \cap \ell \And \ell \in \mathcal{L}\}$ has cardinality at most
\[ |I(A \times B, \mathcal{L})| \leq |A|^{1/2}|B|^{3/4}|\mathcal{L}|^{3/4} + |\mathcal{L}|. \]

Proof of Proposition 1.3. Recall that $|B| \leq |A|$ and $|B||E| \leq p$. Assume that $|A + tB| \leq N$, $t \in E$.

The aim is to prove that $N \gtrsim \min\{|A|^{2/3}|B||E|^{1/3}, |A||B|, p\}$. Consider the family of lines
\[ \mathcal{L}_t := \{(x, y) \in \mathbb{Z}_p^2 : x = r - ty\}_{r \in A + tB}. \]
Write also $\mathcal{L} := \bigcup\{\mathcal{L}_t : t \in E\}$, so that $|\mathcal{L}| \leq |E|N$. Assuming that $N \leq p$, as we may, we see that the hypotheses of Theorem 4.2 are valid:
\[ |A^2|B| \leq |A|^3 \leq |\mathcal{L}|^3 \quad \text{and} \quad |B||\mathcal{L}| \leq |B||E|N \leq p^2, \]
Note that every point $(a, b) \in A \times B$ is incident to $|E|$ lines in $\mathcal{L}$, since
\[ (a, b) \in \{(x, y) \in \mathbb{Z}_p^2 : x = (a + tb) - ty\} \in \mathcal{L}_t, \quad t \in E. \]
It follows from this, and Theorem 4.2, that
\[ |A||B||E| \leq |I(A \times B, \mathcal{L})| \leq |A|^{1/2}|B|^{3/4}(|E|N)^{3/4} + |E|N. \]
If the second term is larger, we have $N \gtrsim |A||B|$, and the proof is complete. If the first term is larger, re-arranging the inequality gives
\[ N \gtrsim |A|^{2/3}(|B||E|)^{1/3}, \]
as desired. The proof of Proposition 1.3 is complete. \hfill \Box

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