NON-COMMUTATIVE GEOMETRIC BROWNIAN MOTION EXHIBITS NONLINEAR CUTOFF STABILITY

GERARDO BARRERA, MICHAEL A. HÖGELE, AND JUAN CARLOS PARDO

Abstract. This article quantifies the asymptotic $\varepsilon$-mixing times, as $\varepsilon$ tends to 0, of a multivariate stable geometric Brownian motion with respect to the Wasserstein-2-distance. We study the cases of commutative, and first order non-commutative drift and diffusion coefficient matrices, respectively, in terms of the nilpotence of the respective nested Lie commutators.

1. Introduction

Geometric Brownian motion serves as an important class of model in mathematical finance (e.g. [27, 30, 32]), but it is also an important and well-studied mathematical object in its own right, see for instance [3, 14, 17, 19] and the references therein. In its simplest form a multivariate geometric Brownian motion with initial value $x \in \mathbb{R}^d$, $d \in \mathbb{N}$, has the following shape

\begin{equation}
X_t(x) = \exp(Y_t) x \quad \text{with} \quad Y_t = At + BW_t, \quad t \geq 0,
\end{equation}

where $(W_t)_{t \geq 0}$ is a standard scalar Wiener process and $A, B \in \mathbb{R}^{d \times d}$ are deterministic square matrices, whenever the coefficients $A$ and $B$ commute.

By the Stratonovich change of variables formula, it is well-known that in case of commuting $A$ and $B$ the process $(X_t(x))_{t \geq 0}$ is the unique strong solution of the Stratonovich stochastic differential equation

\begin{equation}
\mathrm{d}X_t(x) = A X_t(x) \, \mathrm{d}t + B X_t(x) \circ \mathrm{d}W_t, \quad t \geq 0, \quad X_0(x) = x.
\end{equation}

Note, that by [34] the solution $(X_t(x))_{t \geq 0}$ is given in closed form due to nilpotence of the Lie algebra generated by $B$.

Ultimately, this result goes back to the classical result that in a unital associative algebra of operators, the commutativity of two elements $U, V$ is necessary and sufficient for the following functional equality of the operator exponentials to be valid $\exp(U) \exp(V) = \exp(U + V)$, while in general the Baker-Campbell-Hausdorff-Dynkin (BCHD) formula gives additional correction summands in the exponent in terms of nested commutators, see [13] Chapter 5. In [21] the authors show a stochastic extension of this result, that is, a matrix exponential representation of type (1.1) for the solutions of (1.2) for non-commuting matrices $A$ and $B$ formulated for equations in the Itô sense. See also [16, 23, 28, 33, 35]. For comparison, (1.2) in the Itô sense reads as follows

\begin{equation}
\mathrm{d}X_t(x) = (A + (1/2)B^2) X_t(x) \, \mathrm{d}t + B X_t(x) \, \mathrm{d}W_t, \quad t \geq 0, \quad X_0(x) = x,
\end{equation}

see [22] Chapter 4, p. 159. The shape of (1.1) changes considerably when $A$ and $B$ do not commute, that is, $[A, B] := AB - BA \neq 0$. In this case following [25] the exponent (without
simplifications) is in general a nonlinear functional in $t$ and $W_t$:

\begin{align}
X_t(x) &= \exp(Y_t)x \quad \text{with} \quad t \geq 0, \quad \text{where} \\
Y_t &= (A + \frac{1}{2} B^2) t + BW_t + [B, A + \frac{1}{2} B^2] \left( \frac{1}{2} t W_t - \int_0^t W_s ds \right) - \frac{1}{2} B^2 t \\
&\quad + [[A + \frac{1}{2} B^2, B], B] \left( \frac{1}{2} \int_0^t W_s^2 ds - \frac{1}{2} W_t \int_0^t W_s ds + \frac{1}{2} t W_t^2 \right) \\
&\quad + [[A + \frac{1}{2} B^2, B], A + \frac{1}{2} B^2] \left( \int_0^t s W_s ds - \frac{1}{2} t \int_0^t W_s ds - \frac{1}{12} t^2 W_t \right) + \cdots .
\end{align}

The missing remainder terms in the exponent $Y_t$ of $X_t(x)$ contains only terms that includes higher order nested commutators of $A$ and $B$, see Section 3.1 in [21].

We prove quantitative abrupt convergence results between the law of the current state of the solution of (1.4) under certain (non-)commutative relations and its invariant distribution, such as a sort of abrupt convergence or cutoff phenomenon.

Recently, the concept of cutoff has been studied in dynamical systems in the life sciences [29] and machine learning [4]. Historically [1, 2, 12, 15, 23, 24] this dynamical feature emerged in the context of card-shuffling Markov chains, which implies a discrete setting, measured in terms of the total variation distance. However, in a continuous space and time setting, the total variation distance turns out to be cumbersome. Note that the total variation distance between two absolutely continuous distributions is proportional to the $L^1$-distance, which in the case of $A$ and $B$, for multivariate geometric Brownian motion by orders of magnitude, see Example 3.5 below.

In what follows, we embed our findings in the context of the so-called cutoff phenomenon mentioned before. For each $\varepsilon > 0$, let $X^{\varepsilon,x} := (X_t^\varepsilon(x))_{t \geq 0}$ be a stochastic process (including...
the degenerate deterministic case) with values in the Polish space $E_ε$ and with initial position $x_ε \in E_ε$. Consider $\mathcal{M}_1(E_ε)$ the space of probability measures on $E_ε$ equipped with the distance $d_ε$. Assume that for each $ε > 0$ there exists $μ^ε \in \mathcal{M}_1(E_ε)$ satisfying $\lim_{t \to \infty} d_ε(\text{Law}(X^ε_{t, x}), μ^ε) = 0$. We say that a system $(X^ε_{t, x}, μ^ε, d_ε)_{ε > 0}$ exhibits a profile cutoff phenomenon at a cutoff time $t_ε \to \infty$, as $ε \to 0$, and cutoff window $w_ε = o(t_ε)$, as $ε \to 0$, if the following limit exists

$$
\lim_{ε \to 0} d_ε(\text{Law}(X^ε_{t_ε + p \cdot w_ε(x_ε)}, μ^ε)) = \mathcal{P}(\rho) \quad \text{for all } \rho \in \mathbb{R},
$$

and additionally $\mathcal{P}(\infty) = 0$ and $\mathcal{P}(\infty) = D$, where

$$
D = \limsup_{ε \to 0} \text{Diameter}(\mathcal{M}_1(E_ε), d_ε) \in (0, \infty].
$$

This means that the time scale $t_ε$ is a temporal threshold in the sense that surfing ahead wave-front yields small values while lagging behind the threshold sees maximal values.

We stress that such a profile cutoff phenomenon can occur even for systems without an intrinsic parameter $ε$, that is, $x_ε = x$, $X^ε_{t, x} = X^x$, $μ^ε = μ$ and $E_ε = E$ except in the renormalized distance $d_ε$. In the case of $d_ε = d_ε(\text{diameter } D = \infty)$ with a fixed distance $d$ on $\mathcal{M}_1(E)$ the parameter $ε$ plays the role of an external parameter which quantifies the abrupt convergence of the non-normalized distance in the following sense: for small $ε > 0$

$$
d(X^ε_{t_ε + p \cdot w_ε}, μ) \approx ε \cdot \mathcal{P}(\rho) \approx \begin{cases} ε \cdot 0 = 0, & \text{as } \rho \to \infty, \\ ε \cdot D = \infty, & \text{as } \rho \to -\infty. \end{cases}
$$

We denote this special type of profile cutoff phenomenon for non-parametrized systems as profile cutoff stability, since it gives a precise description of the asymptotics.

In the case that $(d_ε(\text{Law}(X^ε_{t_ε + p \cdot w_ε(x_ε)}, μ^ε)))_{ε > 0}$ has more than one accumulation point, which is generically the case (see [3, Theorem 3.2]), the natural generalization of the concept of profile cutoff phenomenon is the notion of the so-called window cutoff phenomenon, that is,

$$
\lim_{p \to -\infty} \liminf_{ε \to 0} d_ε(\text{Law}(X^ε_{t_ε + p \cdot w_ε(x_ε)}, μ^ε)) = D \quad \text{and}
\lim_{p \to +\infty} \limsup_{ε \to 0} d_ε(\text{Law}(X^ε_{t_ε + p \cdot w_ε(x_ε)}, μ^ε)) = 0.
$$

In other words, the time scale $t_ε$ still splits large values from small values in the sense that a growing backward deviation measured in $w_ε$-units from $t_ε$ yields a maximal distance while a growing forward deviation measured in $w_ε$-units from $t_ε$ yields small values. In the analogous setting of profile cutoff stability ([15]), the notion of window cutoff stability reads as follows: there exist functions $\bar{\mathcal{P}}, \hat{\mathcal{P}} : \mathbb{R} \to [0, \infty)$ such that for small $ε > 0$

$$
d(X^ε_{t_ε + p \cdot w_ε}, μ) \begin{cases} \approx ε \cdot \bar{\mathcal{P}}(\rho) = 0, & \text{as } \rho \to \infty, \\ \approx ε \cdot \hat{\mathcal{P}}(\rho) = \infty, & \text{as } \rho \to -\infty. \end{cases}
$$

We refer to the introductions [3, 9, 10, 11] for further details.

**Notation:** A matrix $U \in \mathbb{R}^{d \times d}$ is called Hurwitz stable ($U < 0$, for short), if its spectrum $\text{spec}(U) \subset \mathbb{C}_-$ for the open left complex half plane $\mathbb{C}_-$. For $U \in \mathbb{R}^{d \times d}$ let $U^*$ be the adjoint matrix of $U$ with respect to the standard Euclidean inner product. We define the Lie bracket or commutator by $[U, V] := UV - VU$ for $U, V \in \mathbb{R}^{d \times d}$.

**Lemma 1.1** (The asymptotics of stable matrix exponentials). For $Q \in \mathbb{R}^{d \times d}$ with $Q < 0$ we have the following. For any $y \in \mathbb{R}^d$, $y \neq 0$, there exist $q := q(y) > 0$, $\ell := \ell(y)$, $m := m(y) \in \{1, \ldots, d\}$, $θ_1 := θ_1(y), \ldots, θ_m := θ_m(y) \in \mathbb{R}$ and linearly independent vectors $v_1 := v_1(y), \ldots, v_m := v_m(y) \in \mathbb{C}^d$ such that

$$
\lim_{t \to \infty} \left| \frac{e^{Qt}}{t^{\ell-1}} \exp(tQ)y - \sum_{k=1}^m e^{iθ_k v_k} \right| = 0,
$$
and there are constants $K_0 := K_0(y) > 0$ and $K_1 := K_1(y) > 0$ such that

\begin{equation}
K_0 \leq \lim_{t \rightarrow \infty} \left| \sum_{k=1}^{m} e^{i\theta_k v_k} \right| \leq \lim_{t \rightarrow \infty} \left| \sum_{k=1}^{m} e^{i\theta_k v_k} \right| \leq K_1.
\end{equation}

The preceding lemma is the main tool of this article. The lemma is established as Lemma B.1 in [8], p. 1195-1196, and proved there.

2. Cutoff convergence for commuting matrices $A$ and $B$

**Hypothesis 2.1** (Normality of $B$). We assume $[B, B^*] = O$.

**Hypothesis 2.2** (Commutativity of $A$ and $B$). We assume $[A, B] = O$ and $[A, B^*] = O$.

**Theorem 2.3** (Window cutoff stability for commuting coefficients). Let Hypotheses 2.1 and 2.2 be satisfied and assume that $Q = A + (B + B^*)^2/4 < 0$. Then for any $x \in \mathbb{R}^d$, $x \neq 0$, there are $q > 0$ and $\ell \in \mathbb{N}$ such that for

\begin{equation}
t_{\varepsilon} := 2 \left( \frac{\ln(\varepsilon)}{q} + \frac{(\ell - 1) \ln(\ln(\varepsilon))}{q} \right), \quad \varepsilon > 0,
\end{equation}

and any $w > 0$ it follows the following window cutoff convergence

\begin{equation}
\lim_{\rho \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}[|X_{t_{\varepsilon} + \rho w}(x)|^2]}{\varepsilon^2} = 0 \quad \text{and} \quad \lim_{\rho \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}[|X_{t_{\varepsilon} + \rho w}(x)|^2]}{\varepsilon^2} = \infty.
\end{equation}

**Remark 2.4.** This result is the finite dimensional analogue of Theorem 5.1 for constant noise intensity in [8].

**Proof.** Hypothesis 2.2 implies by [20], Section 3.4 (iii) or [21], Theorem 1, the representation $X_t(x) = \exp(tA + W_t B)x$. Since $[A, B] = O$, it follows that $[tA, W_t B] = tW_t [A, B] = O$. Hence the BCHD formula, yields $X_t(x) = \exp(W_t B) \exp(tA)x$ such that

\begin{equation}
\mathbb{E}[|X_t(x)|^2] = x^* \exp(tA^*) \mathbb{E}[\exp(W_t B^*) \exp(W_t B)] \exp(tA)x.
\end{equation}

The self-similarity in law $W_t \overset{d}{=} \sqrt{t} W_1$, $W_1 \overset{d}{=} \mathcal{N}(0, 1)$ standard normal, the diagonalization of the symmetric matrix $(B + B^*)^2$ and Hypothesis 2.1 imply

\begin{equation}
\mathbb{E}[\exp(W_t B^*) \exp(W_t B)] = \mathbb{E}\left[\exp(\sqrt{t} W_1 (B^* + B))\right] = \exp\left(t \frac{(B^* + B)^2}{2}\right).
\end{equation}

Note that $[B, B^*] = [A, B] = [A, B^*] = O$ and thus

$$
[A, (B + B^*)]^2 = (B + B^*)[A, (B + B^*)] + [A, (B + B^*)](B + B^*) = O
$$

and also $[A^*, (B + B^*)]^2 = O$. Hence, we obtain

\begin{equation}
\exp(tA^*) \mathbb{E}[\exp(W_t B^*) \exp(W_t B)] \exp(tA) = \exp(tQ^*) \exp(tQ).
\end{equation}

Consequently, $\mathbb{E}[|X_t(x)|^2] = |\exp(tQ)x|^2$. Since $Q < 0$, Lemma 1.1 yields for $\exp(tQ)x$ the $x$-dependent parameters $q > 0$, $\ell, m \in \{1, \ldots, d\}$, $\theta_1, \ldots, \theta_m \in \mathbb{R}$ and vectors $v_1, \ldots, v_m \in \mathbb{C}^d$ such that

\begin{equation}
\lim_{t \rightarrow \infty} \left| \frac{e^{qt}}{t^{\ell-1}} \exp(tQ)x - \sum_{j=1}^{m} e^{i\theta_j t} v_j \right| = 0, \quad \text{where}
\end{equation}

\begin{equation}
0 < K_0 \leq \lim_{t \rightarrow \infty} \left| \sum_{j=1}^{m} e^{i\theta_j t} v_j \right| \leq \lim_{t \rightarrow \infty} \left| \sum_{j=1}^{m} e^{i\theta_j t} v_j \right| \leq K_1 < \infty.
\end{equation}
Hence for \( t_\varepsilon \) given in (2.4) and for fixed \( \rho \in \mathbb{R} \), \( s_\varepsilon := t_\varepsilon + \rho \cdot w \), we have by a straightforward calculation that
\[
\lim_{\varepsilon \to 0} \frac{e^{-q\varepsilon s_\varepsilon} s_\varepsilon^{\ell-1}}{\varepsilon} = \frac{e^{-q\rho w}}{q^{\ell-1}}.
\]
Consequently, the right-hand side of (2.3) and an application of Lemma B.1 to \( e^{s_\varepsilon Q} \) imply
\[
\lim_{\rho \to \infty} \lim_{\varepsilon \to 0} E[|X_{t_\varepsilon + \rho \cdot w}(x)|^2] = \lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \left( \frac{e^{-q\varepsilon s_\varepsilon} s_\varepsilon^{\ell-1}}{\varepsilon} \right)^2 \exp(s_\varepsilon Q) x^2
\leq \lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \left( \frac{e^{-q\rho w}}{q^{\ell-1}} \sum_{j=1}^{m} e^{ij\theta t_\varepsilon v_j} \right)^2 = \lim_{\rho \to \infty} \left( e^{-q\rho w} \right)^2 \lim_{\varepsilon \to 0} \left( \sum_{j=1}^{m} e^{ij\theta t_\varepsilon v_j} \right)^2
\leq \lim_{\rho \to \infty} K_1^2 \left( \frac{e^{-q\rho w}}{q^{\ell-1}} \right)^2 = 0.
\]
The proof of the lower bound of (2.3) follows analogously changing \( \lim \) by \( \lim \) and using the left inequality of (2.3). This finishes the proof. \( \square \)

**Corollary 2.5** (Profile cutoff stability). Assume the hypotheses and notation of Theorem 2.3. Additionally, assume that \( A \) is diagonalizable. Then for any \( x \in \mathbb{R}^d \), \( x \neq 0 \), there are \( q > 0 \) and \( v \in \mathbb{R}^d \), \( v \neq 0 \), such that for any \( \rho \in \mathbb{R} \)
\[
(2.4) \quad \lim_{\varepsilon \to 0} \frac{E[|X_{t_\varepsilon + \rho \cdot w}(x)|^2]}{\varepsilon^2} = e^{-q\rho w}|v|.
\]

**Proof.** Note that \([A, BB^*] = B[A, B^*] + B^*[A, B] = 0\) and \([A, B^*] = B^*[A, B] + [B, A, B^*] = 0\). Hence \([A, (B + B^*)^2] = 0\) and \((B + B^*)^2\) is diagonalizable. Since \( A \) is diagonalizable, there is a joint base of eigenvectors for \( A \) and \((B + B^*)^2\) in (2.2). For given \( Q \) and \( x \neq 0 \), Lemma 1.1 yields the existence of \( x \)-dependent parameters \( q > 0 \) and \( v \in \mathbb{R}^d \), \( v \neq 0 \), such that \(|e^{rt}(tQ)x - v|\) tends to 0 as \( t \to \infty \), which implies the desired result in (2.4). \( \square \)

**Remark 2.6.** We stress that the Dirac measure at zero, \( \delta_0 \), is invariant for the dynamics \( (1.2) \) and hence \( E[|X_t(x)|^2] = W_2^2(X_t(x), \delta_0) \), where \( W_2 \) is the standard Wasserstein distance of order 2, see [35]. Moreover, the map
\[
t \mapsto W_2(X_t(x), \delta_0)
\]
is known to be non-increasing, see Lemma B.3 (Monotonicity) in [13].

The following result connects the cutoff stability with the notion of mixing times with respect to the Wasserstein-2-distance and the respective cutoff phenomenon in the sense of Levin, Peres and Wilmer given in Chapter 18 of [23], see the definition by (18.3) and Lemma 18.1.

**Corollary 2.7** (Asymptotic \( \varepsilon \)-mixing time). Assume the hypotheses and notation of Theorem 2.3. Given \( \delta \in (0, 1) \) we define the \( \delta \)-mixing time as follows.
\[
\tau_\varepsilon^\delta := \inf \left\{ t \geq 0 : \frac{E[|X_t(x)|^2]}{\varepsilon^2} \leq \delta \right\}.
\]
Then it follows that
\[
(2.6) \quad \lim_{\varepsilon \to 0} \frac{\tau_\varepsilon^\delta}{\tau_\varepsilon^\delta(1 - \delta)} = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\tau_\varepsilon^\delta}{t_\varepsilon^\delta} = 1.
\]

**Proof.** We start by noticing that Theorem 2.3 implies
\[
(2.7) \quad \lim_{\varepsilon \to 0} \frac{E[|X_{t_\varepsilon}(x)|^2]}{\varepsilon^2} = \begin{cases} \infty & \text{if } c \in (0, 1), \\ 0 & \text{if } c > 1. \end{cases}
\]
Let $\delta \in (0, 1)$ be fixed and choose $c > 1$. Then (2.7) yields the existence of $\varepsilon_0 := \varepsilon_0(\delta, c)$ such that for all $\varepsilon \in (0, \varepsilon_0)$ it follows

$$\mathbb{E}[|X_{\varepsilon t_\varepsilon}(x)|^2] \leq \delta.$$  

By (2.5) we infer $\tau^\varepsilon_\varepsilon(\delta) \leq c \cdot t_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$. Conversely, for $1/c \in (0, 1)$ there exists $\varepsilon_1 := \varepsilon_1(\delta, c)$ such that for all $\varepsilon \in (0, \varepsilon_1)$ it follows

$$\mathbb{E}[|X_{1(1/\varepsilon)} t_\varepsilon(x)|^2] \geq 1 - \frac{\delta}{\varepsilon^2}$$

and by (2.5) we infer $(1/c) \cdot t_\varepsilon \leq \tau^\varepsilon_\varepsilon(1 - \delta)$ for all $\varepsilon \in (0, \varepsilon_1)$. Therefore, we have

$$\limsup_{\varepsilon \to 0} \frac{\tau^\varepsilon_\varepsilon(\delta)}{\tau^\varepsilon_\varepsilon(1 - \delta)} \leq c^2$$

for all $c > 0$. Sending $c \to 1$ we obtain the upper bounds in (2.10). The lower bounds follow similarly. 

\[\square\]

3. Nonlinear cutoff stability for first order non-commutatilities

**Hypothesis 3.1** (Normality of $[A, B]$). We assume $[[A, B], [A, B]^*] = O$.

**Hypothesis 3.2** (First order non-commutativity). We assume

$[A, B] \neq O, \quad [A, B]^* \neq O \quad$ and \n
$[A, [A, B]] = [A, [A, B]^*] = [B, [A, B]] = [B, [A, B]^*] = O$.

**Theorem 3.3** (Window cutoff stability, first order non-commutativity).

Assume Hypotheses 2.1, 3.1 and 3.2 and fix an initial value $x \in \mathbb{R}^d$, $x \neq 0$. Further assume that $\Gamma := ([A, B] + [A, B]^*)^2/6 < 0$ and set

$$\gamma := \max \left\{ \frac{\gamma_j}{2} \cdot 1\{\langle x, v_j \rangle \neq 0\} \mid j = 1, \ldots, d, \gamma_j \in \text{spec}(\Gamma), \Gamma v_j = \gamma_j v_j \right\}.$$ 

(1) If, in addition, $A < 0$ and the leading eigenvalue of $A$ is real, then there exist numbers $a, b \in \mathbb{R}$ depending on $x$ such that for $\varepsilon_0 > 0$ sufficiently small $(t_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ is the unique real solution of

$$\gamma t^3_\varepsilon + bt^2_\varepsilon + at_\varepsilon + \ln(\varepsilon) = 0.$$ 

By Tartaglia-Cardano’s classical formula for polynomials of degree $3$

$$t_\varepsilon = \sqrt[3]{\frac{-q_2}{2} + \sqrt{\left(\frac{-q_2}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{-q_2}{2} - \sqrt{\left(\frac{-q_2}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{b}{3\gamma},$$

where $p = \frac{a}{\gamma} - \frac{b^2}{3\gamma^2}$ and $q_2 = \frac{2b^3}{3\gamma^3} - \frac{ab}{\gamma} + \frac{\ln(\varepsilon)}{\gamma}$, $w_\varepsilon = \frac{1}{t^2_\varepsilon}$, and the solution $(X_t(x))_{t \geq 0}$ satisfies the following cutoff convergence

$$\lim_{t \to \infty} \lim_{\varepsilon \to 0} \mathbb{E}[|X_{t_\varepsilon + p w_\varepsilon}(x)|^2] = 0 \quad \text{and} \quad \lim_{t \to \infty} \lim_{\varepsilon \to 0} \mathbb{E}[|X_{t_\varepsilon + p w_\varepsilon}(x)|^2] = \infty.$$ 

(2) In general, there are numbers $a, b \in \mathbb{R}$ and $\ell_* \in \mathbb{N}_0$ such that for $\varepsilon_0 > 0$ sufficiently small $(T_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ is given as the unique real solution of

$$\gamma_\varepsilon t^3_\varepsilon + bt^2_\varepsilon + at_\varepsilon - \ell_* \ln(T_\varepsilon) + \ln(\varepsilon) = 0$$

and $w_\varepsilon = \frac{1}{t^2_\varepsilon}$, and we have the cutoff convergence (3.2), where $t_\varepsilon$ in (3.2) is replaced by $T_\varepsilon$. The solution $T_\varepsilon$ of (3.3) is not given explicitly in general, but there is an approximate time scale $\tau_\varepsilon$ with $\lim_{\varepsilon \to 0} T_\varepsilon/\tau_\varepsilon \to c \in (0, \infty)$ satisfying the cutoff convergence (3.2), if
t_ε in (3.2) is replaced by τ_ε and w_ε := 1/τ_ε. The time scale τ_ε, however, is explicit, τ_ε := t_ε + r_ε, where t_ε is given by (3.1) and r_ε is the unique real solution of

\[ \gamma r_ε^3 + (3\gamma t_ε + b)r_ε^2 + (3\gamma t_ε^2 + 2bt_ε + a)r_ε - \ell_ε \ln(t_ε) = 0, \]

which by Tartaglia-Cardano’s formula satisfies the explicit formula

\[ r_ε = \frac{3}{\sqrt[3]{2}} \sqrt{\frac{\tilde{q}_ε}{2} + \sqrt{\left(\frac{\tilde{q}_ε}{2}\right)^2 + \left(\frac{\tilde{p}_ε}{3}\right)^3}} + \frac{3}{\sqrt[3]{2}} \sqrt{\frac{-\tilde{q}_ε}{2} - \sqrt{\left(\frac{\tilde{q}_ε}{2}\right)^2 + \left(\frac{\tilde{p}_ε}{3}\right)^3} - \left(\frac{3\gamma t_ε + b}{3\gamma}\right)}, \]

with coefficients \( \tilde{p}_ε = \frac{(3\gamma t_ε^2 + 2bt_ε + a)}{\gamma} - \frac{(3\gamma t_ε + b)^2}{3\gamma^2} \) and

\[ \tilde{q}_ε = \frac{2(3\gamma t_ε + b)^3}{27\gamma^3} - \frac{(3\gamma t_ε^2 + 2bt_ε + a)(3\gamma t_ε + b)}{3\gamma^2} - \frac{\ell_ε}{\gamma t_ε}. \]

The constants \( a, b \) and \( \ell_ε \) are tracked completely in the proof.

Remark 3.4. For small values of \( \varepsilon \), \( t_ε \) in (3.1) is of leading order \( |\ln(\varepsilon)|^{1/3} \), which differs strongly from the commutative case of Theorem 2.3 i.e. non-commutativities accelerate the cutoff convergence by higher order roots. In comparison to analogous \( \varepsilon \)-small noise results for the Ornstein-Uhlenbeck process [5, 10] and for the stochastic heat and the damped wave equation [8], the “cutoff-window”, that is, the “units” \( w_ε \) in which the instability of the cutoff time scale \( t_ε \) is “measured”, the \( \infty/0 \)-threshold effect is bound to happen in an almost instantaneously short time window of length order \( t_ε^2 \).

Example 3.5. Note that \( X_t = \exp(-t^3 - t^2) \) exhibits the cutoff stability in the sense of (3.2). In addition, one can check with help of Tartaglia-Cardano’s formula that \( (X_t)_{t \geq 0} \) satisfies the nonlinear scalar ODE

\[ \dot{X}_t = f(X_t), \quad X_0 = 1, \]

where \( f(x) = -x(3g^2(x) + 2g(x)) \) with

\[ g(x) = \frac{1}{3} \sqrt{\frac{-27}{2} \ln(x) + \frac{3\sqrt{3}}{2} \sqrt{27 \ln^2(x) + 4 \ln(x) - 1}} + \frac{1}{3} \sqrt{\frac{-27}{2} \ln(x) + \frac{3\sqrt{3}}{2} \sqrt{27 \ln^2(x) + 4 \ln(x) - 1}} - \frac{1}{3}. \]

In other words, in multidimensional non-commutative linear dynamics behaves in terms of abrupt thermalization times and windows as a fully nonlinear scalar ODE.

Remark 3.6 (Profile cutoff stability). Under the hypotheses and notation of Theorem 3.3 a step by step inspection of the proof of Theorem 3.3 yields that the analogue “Dynamical profile cutoff characterization” of Theorem 2 in [7] or Theorem 3.1 and Theorem 3.2 in [5] can be shown with some technical effort.

We establish the same connection with the cutoff phenomenon as in Corollary 2.7 for the modified \( t_ε \) given in Theorem 3.3 The proof is analogous to the proof of Corollary 2.7 and hence omitted.

Corollary 3.7 (Asymptotic \( \varepsilon \)-mixing time). Assume the hypotheses and notation of Theorem 3.3. Given \( \delta \in (0, 1) \) we define the \( \delta \)-mixing time as follows.

\[ \tau_ε^\delta := \inf \left\{ t \geq 0 : \frac{E[|X_t(x)|^2]}{\varepsilon^2} \leq \delta \right\}. \]
Then it follows that
\[ \lim_{\varepsilon \to 0} \frac{\tau^r_\varepsilon(\delta)}{\tau^r_\varepsilon(1 - \delta)} = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\tau^r_\varepsilon(\delta)}{t_\varepsilon} = 1. \]

**Proof of Theorem 3.3 I: Solution representation:** By Hypothesis 3.2 and Section 3.1 in [21] the solution of (1.2) is given by \( X_t(x) := \exp(Y_t)x \), where

\[ Y_t := tA + W_tB + \left( \frac{1}{2} tW_t - \int_0^t W_s ds \right) C, \quad C := [B, A]. \]

In the sequel, we compute \(|X_t(x)|^2\). Integration by parts yields

\[ \int_0^t W_s ds = tW_t - \int_0^t s dW_s \quad \text{such that} \quad Y_t = tA + J_t, \]

where

\[ J_t := \int_0^t (B - (t/2 - s)C)dW_s. \]

Combining the linearity of the Lie bracket with Hypothesis 3.2 yields

\[ [tA, J_t] = [tA, W_tB - \int_0^t (t/2 - s)dW_sC] = -tW_tC - t \int_0^t (t/2 - s)dW_s[A, C] = -tW_tC. \]

By the linearity of the Lie bracket, (3.7) and Hypothesis 3.2 we have

\[ [tA, [tA, J_t]] = -[tA, tW_tC] = -t^2W_t[A, C] = O \quad \text{and} \quad [J_t, tW_tC] = O. \]

By (3.7) we have \([-tA, J_t] = tW_tC\). Since \([-U, V] = [V, U]\) for any \(U, V \in \mathbb{R}^{d \times d}\), we obtain

\[ [J_t, tA] = tW_tC. \]

Hence (3.4)–(3.8) with the BCHD formula and the relation \(-\frac{1}{2}tW_tC + J_t = \int_0^t (B - (t - s)C)dW_s\), which is a direct consequence of (3.6), yields

\[ X_t(x) = \exp(J_t + tA)x = \exp(-\frac{1}{2}[J_t, tA]) \exp(J_t) \exp(tA)x \]

\[ = \exp(-\frac{1}{2}tW_tC + J_t) \exp(tA)x = \exp(t) \int_0^t (B - (t - s)C)dW_s \exp(tA)x. \]

**II. Mean square representation:** The bilinearity of the brackets combined with the Hypotheses 2.1, 3.1 and 3.2 yields

\[ \int_0^t (B - (t - s)C)dW_s, \int_0^t (B^* - (t - s)C^*)dW_s = O. \]

We fix the notation \( \hat{B} := B + B^* \) and \( \hat{C} := C + C^* \). Hence by (3.9), (3.10), Hypothesis 3.1 and the BCHD formula we obtain

\[ \mathbb{E}[|X_t(x)|^2] = x^* \exp(tA^*) \mathbb{E}[\exp(\int_0^t (\hat{B} - (t - s)\hat{C})dW_s)] \exp(tA)x. \]

We recall the integration by parts formula (3.5) and calculate the exponent

\[ \int_0^t (\hat{B} - (t - s)\hat{C})dW_s = W_t\hat{B} - (tW_t - \int_0^t s dW_s)\hat{C} = W_t\hat{B} - \int_0^t W_s ds \hat{C}. \]
By diagonalization we have \( \hat{B} = U \text{diag}(\mu_j) U^* \), where \( \mu_j \in \mathbb{R} \) are the eigenvalues of \( \hat{B} \) and an orthogonal \( U \in \mathbb{R}^{d \times d} \). Due to Hypothesis 3.1 we have the simultaneous diagonalization \( \hat{C} = U \text{diag}(\nu_j) U^* \), where \( \nu_j \in \mathbb{R} \) are the eigenvalues of \( \hat{C} \). Then

\[
\exp(- \int_0^t (t - s) dW_s \hat{C}) = \exp(- \int_0^t W_s ds \hat{C}) = U \text{diag}(e^{-\nu_j \int_0^s W_s ds}) U^*
\]

and \( \exp(W_t \hat{B}) = U \text{diag}(e^{\mu_j W_t}) U^* \). Next, taking expectation and using Itô’s isometry we obtain

\[
\mathbb{E}[\exp(W_t \hat{B}) \exp(- \int_0^t (t - s) dW_s \hat{C})] = U \text{diag}(\mathbb{E}[e^{\mu_j W_t} e^{-\nu_j \int_0^s W_s ds}]) U^*
\]

\[
= U \text{diag}(\mathbb{E}[e^{\mu_j W_t} e^{-\nu_j (t-s) W_s ds}]) U^* = U \text{diag}(\mathbb{E}[e^{\nu_j (t-s) W_s ds}]) U^*
\]

\[
= \exp \left( \frac{1}{2} \int_0^t (\hat{B} - (t-s)\hat{C})^2 ds \right).
\]

(3.11)

Note that \( B \) and \( C^* \), \( B \) and \( C^* \) commute by Hypothesis 3.2 and hence

\[
(\hat{B}\hat{C})^* = \hat{C}^*\hat{B}^* = \hat{C}\hat{B} = \hat{B}\hat{C}.
\]

(3.12)

Hence, the integrand in the exponent of right-hand side of (3.11) reads

\[
\hat{B}^2 - 2s\hat{B}\hat{C} + s^2\hat{C}^2.
\]

The resulting first partial integral in the exponent of the right-hand side of (3.11) is equal to \( t\hat{B}^2/2 \), while the second and third terms equal

\[
\frac{1}{2} \hat{B}\hat{C} \int_0^t s ds = \frac{t^2}{2} \hat{B}\hat{C} \quad \text{and} \quad \frac{1}{2} \hat{C}^2 \int_0^t s^2 ds = \frac{t^3}{6} \hat{C}^2.
\]

Consequently, by (3) and Hypothesis 3.1 we have

\[
\mathbb{E}[|X_t(x)|^2] = x^* \exp(tA^*) \exp(t\alpha - t^2\beta + t^3\Gamma) \exp(tA)x
\]

\[
= |\exp(tA) \exp(t\alpha - t^2\beta + t^3\Gamma)x|^2 \quad \text{with}\n\]

\[
\alpha := \frac{\hat{B}^2}{2}, \quad \beta := \frac{\hat{B}\hat{C}}{2} \quad \text{and} \quad \Gamma = \frac{\hat{C}^2}{6}.
\]

(3.13)

III. Commutativity relations: \( [\alpha, \beta] = [\alpha, \Gamma] = [\beta, \Gamma] = O \) and \( [A, \Gamma] = O \).

\[
16[\alpha, \beta] = [B^2 + BB^* + B^*B + (B^*)^2, BC + BC^* + B^*C + B^*C^*]
\]

\[
= [B^2, BC] + [BB^*, BC] + [B^*B, BC] + [(B^*)^2, BC]
\]

\[
+ [B^2, BC^*] + [BB^*, BC^*] + [B^*B, BC^*] + [(B^*)^2, BC^*]
\]

\[
+ [B^2, B^*C] + [BB^*, B^*C] + [B^*B, B^*C] + [(B^*)^2, B^*C]
\]

\[
+ [B^2, B^*C^*] + [BB^*, B^*C^*] + [B^*B, B^*C^*] + [(B^*)^2, B^*C^*].
\]

(3.14)

For \( M, N, K \in \mathbb{R}^{d \times d} \) satisfying Hypotheses 2.1 and 3.2 the product rule \( [M, NK] = [M, N]K + N[M, K] \) yields \( [M, N^n] = nN^{n-1}[M, N], \ n \in \mathbb{N} \), and hence

\[
[B^2, BC] = 2B[B, BC] = 2B([B, B]C + B[B, C]) = O.
\]

In addition, Hypothesis 3.1 implies

\[
[BB^*, BC] = [BB^*, B]C + B[BB^*, C] = -[B, BB^*]C - B[C, BB^*]
\]

\[
= -([B, B]B^* + B[B, B^*])C - B([C, B]B^* + B[C, B^*]) = O.
\]
All other terms in (3.14) are structurally identical (up to changing $B$ by $B^*$ or $C$ by $C^*$) and vanish due to Hypotheses 3.1 and 3.2. We continue by

$$48[\alpha, \Gamma] = [B^2 + BB^* + B^*B + (B^*)^2, C^2 + CC^* + C^*C + (C^*)^2].$$

Due to the bilinearity, the product rules and Hypotheses 2.1, 3.2 and 3.1 it is easy to see that $[\alpha, \Gamma] = 0$. Finally,

$$48[\Gamma, \beta] = [C^2 + CC^* + C^*C + (C^*)^2, BC + BC^* + B^*C + B^*C^*]
+ [C^2, BC] + [CC^*, BC] + [C^*C, BC] + [(C^*)^2, BC]
+ [C^2, B^*C^*] + [CC^*, BC^*] + [(C^*)^2, B^*C^*]
+ [C^2, B^*C^*] + [CC^*, B^*C^*] + [C^*C, B^*C^*] + [(C^*)^2, B^*C^*].$$

(3.15)

Analogously, each of the terms on the right-hand side of (3.15) vanishes. Note that $\alpha$ and $\Gamma$ are symmetric by construction, and $\beta$ is symmetric by (3.12). Moreover

$$12[A, \Gamma] = [A, C^2] + [A, CC^*] + [A, C^*C] + [A, (C^*)^2] = O.$$

### IV. Simultaneous diagonalization:

By hypothesis $\Gamma < 0$. Thus there is a number $p_T > 0$ sufficiently large such that $\tilde{A} := A + \frac{\Gamma}{2} \Gamma < 0$. By the commutativities of III. we have that (3.13) can be written as

$$\mathbb{E}[[X_t(x)]^2] = |\exp(t\tilde{A}) \exp(\frac{1}{2}(t^3 - t^2\beta + (t^3 - p_T t)\Gamma))x|^2.$$

(3.16)

Note that for $A < 0$ (item 1.) we have $p_T = 0$. The mutual commutativities and the symmetries of III. are inherited by $\exp(t\alpha)$, $\exp(-t^2\beta)$ and $\exp(t\Gamma)$ for all $t \geq 0$. Hence all three matrices are diagonalizable by the same orthogonal matrix $\tilde{U}$, there is an orthonormal basis $v_1, \ldots, v_d$ of $\mathbb{R}^d$, such that

$$\exp(t\alpha) \exp(-t^2\beta) \exp((t^3 - p_T t)\Gamma)x = \tilde{U}^* \text{diag}(e^{-a_j t - b_j t^2 - \gamma_j(t^3 - p_T t)})\tilde{U}x$$

$$= \sum_{j=1}^{d} e^{-a_j t - b_j t^2 - \gamma_j(t^3 - p_T t)} \langle x, v_j \rangle v_j,$$

where $-a_j$, $b_j$ and $-\gamma_j$ are the eigenvalues of $\alpha$, $\beta$ and $\Gamma$, respectively. Consequently, for $p(t) := t^3 - p_T t$, $t \geq p_T$, (3.16) reads as follows

$$\mathbb{E}[[X_t(x)]^2] = |\exp(t\tilde{A}) \exp(\frac{t}{2} \alpha) \exp(-\frac{t^2}{2} \beta) \exp(\frac{p(t)}{2} \Gamma)x|^2$$

(3.17)

$$= \sum_{j=1}^{d} e^{-\frac{a_j t}{2} - \frac{b_j t^2}{2} - \frac{\gamma_j(t^3 - p_T t)}{2}} \langle x, v_j \rangle \exp(t\tilde{A})v_j|^2.$$

Since $\tilde{A} < 0$, Lemma 1.1 yields for each of the vectors $v_j$ the existence of $m_j$, $\ell_j$, $\lambda_j$ and $\theta_{k,j}$ and $w_{k,j}$ such that for

$$R_j(t) := \frac{e^{\lambda_j t}}{t \ell_j - 1} \exp(t\tilde{A})v_j - \sum_{k=1}^{m_j} e^{i \theta_{k,j} t} w_{k,j} \quad \text{we have} \quad \lim_{t \to \infty} |R_j(t)| = 0.$$  

(3.18)
Then the argument vector on the right-hand side of (3.17) reads $S_1(t) + S_2(t)$, where

$$S_1(t) := \sum_{j=1}^{d} e^{-\left(\frac{a_j}{2} + \gamma_j - \frac{b_j}{2} \gamma_j \right)t - \frac{b_j}{2} \gamma_j \ell_j t^2 - \frac{b_j}{2} \gamma_j^\ell_j t^3} \langle x, v_j \rangle R_j(t) \quad \text{and}$$

$$S_2(t) := \sum_{j=1}^{d} e^{\left(\frac{a_j}{2} + \gamma_j - \frac{b_j}{2} \gamma_j \right)t - \frac{b_j}{2} \gamma_j \ell_j t^2 - \frac{b_j}{2} \gamma_j^\ell_j t^3} \langle x, v_j \rangle \sum_{k=1}^{\ell_j} e^{i \theta k, j \ell_j w_{k, j}}.$$

V. Cutoff convergence: Let $J_0 := \{ j \in \{1, \ldots, d\} \mid \langle x, v_j \rangle \neq 0 \}$ and set

$$j^+_1 := \arg \min_{j \in J_0} \gamma_j, \quad J_1 := \{ j \in J_0 \mid \gamma_j = \gamma_{j^+_1} \},$$

$$j^+_2 := \arg \min_{j \in J_1} b_j, \quad J_2 := \{ j \in J_1 \mid b_j = b_{j^+_2} \}, \quad \tilde{a}_j := (a_j/2 + \lambda_j - (p_t/2)\gamma_j),$$

$$j^+_3 := \arg \min_{j \in J_2} \tilde{a}_j, \quad J_3 := \{ j \in J_2 \mid \tilde{a}_j = \tilde{a}_{j^+_3} \},$$

$$j^+_4 := \arg \max_{j \in J_3} \ell_j, \quad J_4 := \{ j \in J_3 \mid \ell_j = \ell_{j^+_4} \}.$$

It is easy to see that for $\gamma = \gamma_{j^+_2}/2$, $b = b_{j^+_2}/2$, $a := \tilde{a}_{j^+_3}$ and $\ell_* = \ell_{j^+_4} - 1$, and for any $\varepsilon > 0$ sufficiently small there is a unique real solution $t_\varepsilon > 0$ of

$$e^{-at_\varepsilon - bt_{\varepsilon}^2 - \gamma t_{\varepsilon}^3} \ell_{\varepsilon} = \varepsilon,$$

which is also the unique solution of (3.3). For fixed $\rho \in \mathbb{R}$ let $s_\varepsilon = s_\varepsilon(\rho) := t_\varepsilon + \rho w_\varepsilon$, $w_\varepsilon = t_{\varepsilon}^{-2}$. Hence (3.18) yields $\lim_{\varepsilon \to 0} S_1(s_\varepsilon) = 0$ and

$$\lim_{\varepsilon \to 0} \frac{e^{-as_\varepsilon - bs_\varepsilon^2 - \gamma s_\varepsilon^3} s_\varepsilon^{\ell_*}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{e^{-a(t_\varepsilon + \rho w_\varepsilon) - b(t_{\varepsilon}^2 + 2\rho t_\varepsilon w_\varepsilon + \rho^2 w_\varepsilon^2) - \gamma(t_{\varepsilon}^3 + 3t_{\varepsilon}^2 \rho w_\varepsilon + 3t_\varepsilon \rho^2 w_\varepsilon^2 + \rho^3 w_\varepsilon^3)} s_\varepsilon^{\ell_*}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} e^{-a\rho w_\varepsilon - b(2\rho t_\varepsilon w_\varepsilon + \rho^2 w_\varepsilon^2) - \gamma(3t_\varepsilon^2 \rho w_\varepsilon + 3t_\varepsilon \rho^2 w_\varepsilon^2 + \rho^3 w_\varepsilon^3)} \left(\frac{t_\varepsilon + \rho w_\varepsilon}{t_\varepsilon}\right)^{\ell_*}$$

$$= e^{-3\gamma \rho}.$$

In virtue of (1.6) in Lemma 1.1 we obtain

$$\lim_{\varepsilon \to 0} \frac{\mathbb{E}[|X_{s_\varepsilon}(x)|^2]}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{e^{-as_\varepsilon - bs_\varepsilon^2 - \gamma s_\varepsilon^3} s_\varepsilon^{\ell_*}}{\varepsilon} \langle x, v_j \rangle \sum_{k=1}^{m_j} e^{i \theta k, j s_\varepsilon w_{k, j}} |w_{k, j}|^2$$

$$= e^{-3\gamma \rho} \langle x, v_j \rangle^2 \lim_{\ell \to \infty} \left| \sum_{k=1}^{m_j} e^{i \theta k, j s_\varepsilon} |w_{k, j}|^2 \right| \leq K_1 e^{-3\gamma \rho} \langle x, v_j \rangle^2,$$

such that

$$\lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \frac{\mathbb{E}[|X_{s_\varepsilon}(x)|^2]}{\varepsilon^2} \leq \lim_{\rho \to \infty} K_1 e^{-3\gamma \rho} \langle x, v_j \rangle^2 = 0.$$

Likewise keeping the notation we have the left-hand estimate of (1.6)

$$\lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \frac{\mathbb{E}[|X_{s_\varepsilon}(x)|^2]}{\varepsilon^2} \geq K_0 \lim_{\rho \to -\infty} e^{-3\gamma \rho} \langle x, v_j \rangle^2 = \infty.$$

This proves (3.2) and finishes the proof.\qed
Acknowledgments

GB thanks the Academy of Finland, via the Matter and Materials Profi4 University Profiling Action, the Academy project No. 339228 and project No. 346306 of the Finnish Centre of Excellence in Randomness and STructures. The research of MAH was supported by the project INV-2019-84-1837 of Facultad de Ciencias at Universidad de los Andes.

Availability of data and material Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interests The authors declare that they have no conflict of interest.

Authors' contributions All authors have contributed equally to the paper.

References

[1] D. Aldous, P. Diaconis. Shuffling cards and stopping times. Amer. Math. Monthly 93 (5), 1986, 333–348.
[2] D. Aldous, P. Diaconis. Strong uniform times and finite random walks. Adv. Appl. Math. 8 (1), 1987, 69–97.
[3] J. Appleby, X. Mao, M. Riedle. Geometric Brownian motion with delay: mean square characterisation. Proc. Amer. Math. Soc. 137 (1), 2009, 339–348.
[4] B. Avelin, A. Karlsson. Deep limits and a cut-off phenomenon for neural networks. J. Mach. Learn. Res. 23 , 2022, 1–29.
[5] G. Barrera, M.A. H¨ ogele, J.C. Pardo. Cutoff thermalization for Ornstein-Uhlenbeck systems with small Lévy noise in the Wasserstein distance. J. Stat. Phys. 184 (27), 2021, 54 pp.
[6] G. Barrera, M.A. H¨ ogele, J.C. Pardo. The cutoff phenomenon in total variation for nonlinear Langevin systems with small layered stable noise. Electron. J. Probab. 26 (119), 2021, 1–76.
[7] G. Barrera, M.A. H¨ ogele, J.C. Pardo. The cutoff phenomenon in Wasserstein distance for nonlinear stable Langevin systems with small Lévy noise. J. Dyn. Diff. Equat. 2002, 28 pp.
[8] G. Barrera, M.A. H¨ ogele, J.C. Pardo. The cutoff phenomenon for the stochastic heat and the wave equation subject to small Lévy noise. Stoch. Partial Differ. Equ. Anal. Comput. 2022, 39 pp.
[9] G. Barrera, M. Jara. Thermalisation for small random perturbation of hyperbolic dynamical systems. Ann. Appl. Probab. 30 (3), 2020, 1164–1208.
[10] G. Barrera, J.C. Pardo. Cut-off phenomenon for Ornstein-Uhlenbeck processes driven by Lévy processes. Electron. J. Probab. 25 (15), 2020, 1–33.
[11] J. Barrera, O. Bertoncini, R. Fernández. Abrupt convergence and escape behavior for birth and death chains. J. Stat. Phys. 137 (4), 2009, 595–623.
[12] R. Basu, J. Hermen, Y. Peres. Characterization of cutoff for reversible Markov chains. Ann. Probab. 45 (3), 2017, 1448–1487.
[13] J. Boursier, D. Chafai, C. Labbé. Universal cutoff for Dyson Ornstein Uhlenbeck process. Probab. Theory Related Fields 2022, 64 pp.
[14] T. Byczkowski, M. Ryznar. Hitting distributions of geometric Brownian motion. Studia Math. 173 (1), 2006, 19–38.
[15] G. Chen, L. Saloff-Coste. The cutoff phenomenon for ergodic Markov processes. Electron. J. Probab. 13 (3), 2008, 26–78.
[16] H. Dietz. On the solution of matrix-valued linear stochastic differential equations driven by semimartingales. Stochastics Stochastics Rep. 34 (3-4), 1991, 127–147.
[17] D. Dufresne. The integral of geometric Brownian motion. Adv. in Appl. Probab. 33 (1), 2001, 223–241.
[18] B. Hall. Lie groups, Lie algebras, and representations. An elementary introduction. Second edition, Springer Graduate Texts in Mathematics 222, 2015.
[19] Y. Hu. Multi-dimensional geometric Brownian motions, Onsager-Machlup functions, and applications to mathematical finance. Acta Math. Sci. Ser. B 20 (3), 2020, 341–358.
[20] I. Ibragimov, Y. Linnik. Independent and stationary sequences of random variables. Wolters-Noordhoff Publishing, Groningen, 1971.
[21] K. Kamm, S. Pagliarani, A. Pascucci. On the stochastic Magnus expansion and its application to SPDEs. J. Sci. Comput. 89 (56), 2021, 31 pp.
[22] P. Kloeden, E. Platen. Numerical solution of stochastic differential equations. Applications of Mathematics New York 23. Springer-Verlag, Berlin, 1992.
[23] C. Lancia, F. Nardi, B. Scoppola. Entropy-driven cutoff phenomena. J. Stat. Phys. 149, (1), 2012, 108–141.
[24] D. Levin, Y. Peres, E. Wilmer. Markov chains and mixing times. With a chapter by J. Propp and D. Wilson. Amer. Math. Soc., Providence 2009.
[25] W. Magnus. On the exponential solution of differential equations for a linear operator. *Comm. Pure Appl. Math.* 7, 1954, 649–673.

[26] X. Mao. *Stochastic differential equations and applications*. Second edition. Horwood Publishing Limited, Chichester, 2008.

[27] R. Marathe, S. Ryan. On the validity of the geometric Brownian motion assumption. *Eng. Econ.* 50 (2), 2005, 159–192.

[28] M. Muniz, M. Ehrhardt, M. Günther, R. Winkler. Higher strong order methods for linear Itô SDEs on matrix Lie groups. *BIT Numer. Math.* 62 (4), 2022, 1095–1119.

[29] R. Murray, R. Pego. Cutoff estimates for the linearized Becker-Döring equations *Commun. Math. Sci.* 15 (6), 2017, 1685–1702.

[30] F. Postali, P. Picchetti. Geometric Brownian motion and structural breaks in oil prices: a quantitative analysis. *Energy Econ.* 28 (4), 2006, 506–522.

[31] S. Rachev, L. Klebanov, S. Stoyanov, F. Fabozzi. *The methods of distances in the theory of probability and statistics*. Springer, New York, 2013.

[32] V. Stojkoski, T. Sandev, L. Basnarkov, L. Kocarev, R. Metzler. Generalised geometric Brownian motion: theory and applications to option pricing. *Entropy* 22 (12), 2020, Paper no. 1432, 34 pp.

[33] Z. Wang, Q. Ma, Z. Yao, X. Ding. The Magnus expansion for stochastic differential equations. *J. Nonlinear Sci.* 30 (1), 2020, 419–447.

[34] Y. Yamato. Stochastic differential equations and nilpotent Lie algebras. *Z. Wahrsch. Verw. Gebiete* 47 (2), 1979, 213–229.

[35] G. Yang, K. Burrage, Y. Komori, P. Burrage, X. Ding. A class of new Magnus-type methods for semi-linear non-commutative Itô stochastic differential equations. *Numer. Algorithms* 88 (4), 2021, 1641–1665.

[36] C. Villani. *Optimal transport. Old and new*. Grundlehren der mathematischen Wissenschaften 338. Springer-Verlag, Berlin, 2009.

**University of Helsinki, Department of Mathematical and Statistical Sciences.** PL 68, Pietari Kalmin katu 5. Postal Code: 00560, Helsinki, Finland.

*Email address*: gerardo.barreravargas@helsinki.fi (Corresponding author)

**Departamento de Matemáticas, Universidad de los Andes, Bogotá, Colombia.**

*Email address*: ma.hoegele@uniandes.edu.co

**CIMAT. Jalisco S/N, Valenciana, CP. 36240. Guanajuato, México.**

*Email address*: jcpardo@cimat.mx