The $z$-matching problem on bipartite graphs

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The $z$-matching problem on bipartite graphs is studied with a local algorithm. A $z$-matching ($z \geq 1$) on a bipartite graph is a set of matched edges, in which each vertex of one type is adjacent to at most 1 matched edge and each vertex of the other type is adjacent to at most $z$ matched edges. The $z$-matching problem on a given bipartite graph concerns finding $z$-matchings with the maximum size. Our approach to this combinatorial optimization are of two folds. From an algorithmic perspective, we adopt a local algorithm as a linear approximate solver to find $z$-matchings on general bipartite graphs, whose basic component is a generalized version of the greedy leaf removal procedure in graph theory. From an analytical perspective, in the case of random bipartite graphs with the same size of two types of vertices, we develop a mean-field theory for the percolation phenomenon underlying the local algorithm, leading to a theoretical estimation of $z$-matching sizes on coreless graphs. We hope that our results can shed light on further study on algorithms and computational complexity of the optimization problem.

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I. INTRODUCTION

Graph theory \cite{11} and complex network theory \cite{2} provide a consistent framework to study the structure and the dynamics of complex connected systems. Combinatorial optimization problems \cite{3, 4} and their computational complexity \cite{5} are among the major themes in complex systems study. Among all the analytical options, the statistical physics of the spin glasses \cite{6, 8} contributes both message passing algorithms for finding solutions on graph instances and a principled framework to understand the transition behaviors in the solution space structure and their implications in an algorithmic sense. A frequently studied combinatorial optimization is the matching problem \cite{9}, which can be defined in various graphical contexts. In the simplest case on undirected graphs, the maximum matching (MM) problem is to find the maximum set of edges which shares no common vertex. The MM problem is studied with a local algorithm \cite{10, 11} and the cavity method \cite{12, 13}. In the case of directed graphs, the MM problem can be defined on their bipartite graph representations \cite{15}. This MM problem version is the core of the network controllability problem \cite{16}, in which finding the minimum driver node set to guide a dynamical system with directed interactions to any state in finite time is basically equivalent to finding the MMs on the undirected bipartite graph representation of the underlying directed network. The cavity method is adopted to analytically estimate MM sizes \cite{16–18}. The GLR procedure, leading to an analytical estimation of MM sizes on bipartite graphs without a core structure.

Among the local algorithms for the MM problem on undirected and directed graphs, the greedy leaf removal (GLR) procedure \cite{10}, an iterative process of removing leaves (any vertex with only one nearest neighbor) and their adjacent roots (the sole neighbor of any leaf), is frequently adopted as a constraint reduction method. The GLR procedure leads to the core percolation from a graph pruning perspective \cite{24, 25}, which at the same time signifies a phase transition in the structure of ground-state solution space of the MM problem \cite{12, 13}. Beside the MM problem, the original GLR procedure and its variants are adopted in other combinatorial optimization and satisfiability problems, such as the minimum vertex cover problem \cite{14, 24, 25}, the XOR-SAT problem and the p-spin problem \cite{27, 25}, the Boolean networks \cite{24}, the maximum independent set problem \cite{30}, and the minimum dominating set problem \cite{31, 32}.

In this paper, we study a specific matching problem on bipartite graphs with a local algorithm, whose basic idea is a generalized form of the GLR procedure. The $z$-matching problem on a bipartite graph \cite{33, 34} concerns finding the maximum size of $z$-matchings, which is a set of matched edges under the constraints that any vertex of one type is adjacent to at most 1 matched edge and any vertex of the other type is adjacent to at most $z$ matched edges. Our main contribution is in two parts. First, we generalize the GLR procedure as a local procedure to approximate $z$-matching solutions on general bipartite graphs. Second, in the case of equal sizes for two types of vertices on random bipartite graphs, a mean-field theory is developed to analyze the percolation phenomenon underlying the generalized GLR procedure, leading to an analytical estimation of $z$-matching sizes on bipartite graphs without a core structure.

Here is the structure of the paper. In section II we explain the $z$-matching problem and our local algorithm. In section III we develop an analytical theory for the percolation phenomenon underlying our local algorithm. In section IV we present results from simulation and theory on some random graph models. In section V we conclude the paper with some discussion.

II. MODEL

We first present some notations for bipartite graphs following an adopted way in the bipartite graph representation of directed networks \cite{16, 25}. A bipartite graph $B = \{V_+ \cup V_-, E\}$ consists of an out-vertex set $V_+$, an in-vertex set $V_-$, and an edge set $E = V_+ \times V_-$. As been denoted, edges are only between different types of vertices. We define the size ratio of $B$ as $f = |V_-|/|V_+|$. See an example in figure 1 (a). For any vertex $i^+ \in V_+$, all its nearest neighbors in $V_-$ consists of a set $\partial i^+$. Correspondingly, the degree of $i^+$ is $k_{i^+} = |\partial i^+|$. The same notation can be defined on any vertex $j^- \in V_-$ for its neighbor set $\partial j^-$ and its degree $k_{j^-} = |\partial j^-|$. The out-degree distribution $P_{\partial}(k_{i^+})$ is defined as the probability of finding a vertex with an out-degree $k_{i^+}$ in the out-vertex set. The mean out-degree of the out-vertex set is $c_+ = \sum k_{i^+}P_{\partial}(k_{i^+})$. The in-degree distribution $P_{\partial}(k_{j^-})$ and the mean in-degree $c_-$ for the in-vertex set can be defined in a similar way. It is easy to see that $|V_+|c_+ = |V_-|c_- = |E|$. With a random chosen edge $(i^+, j^-) \in B$, following $j^-$ to $i^+$, we define the excess out-degree distribution $Q_{\partial}(k_{i^+})$ as the probability that $i^+$ has a degree $k_{i^+}$; following $i^+$ to $j^-$, we define the excess in-degree distribution $Q_{\partial}(k_{j^-})$ as the probability that $j^-$ has a degree $k_{j^-}$. We can see that $Q_{\partial}(k_{i^+}) = k_{i^+}P_{\partial}(k_{i^+})/c_+$. For the ease of presenting the mean-field theory in the next section, we define the cavity graph based on a bipartite graph $B$. If a randomly chosen edge $(i^+, j^-) \in B$ is removed, the residual
subgraph is defined as the cavity graph $B_i\backslash (i^+,j^-)$; if a randomly chosen vertex $i^+$ ($j^-$) is removed along with all its adjacent edges, the residual subgraph is defined as the cavity graph $B_i\backslash i^+ (B\backslash j^-)$.

Our local algorithm for the $z$-matching problem is basically both a graph pruning process and a protocol of selecting edges into a $z$-matching. In a slight extension of the context of the $z$-matching problem on a bipartite graph $B$, we define a parameter of residual constraints $z_{j^-}$ for any in-vertex $j^- \in B$ as the maximum number of matched edges it can be adjacent to. If an edge, say $(i^+, j^-) \in B$ between an out-vertex $i^+$ and an in-vertex $j^-$, is matched, there are dual consequences leading to a pruning process of vertices and edges. For the out-vertex, $i^+$ is 'fully' matched, since it can only be adjacent to at most one matched edge. Correspondingly, $i^+$ is removed along with all its adjacent edges. For the in-vertex, $z_{j^-}$ is updated as $z_{j^-} \leftarrow z_{j^-} - 1$. If $z_{j^-} = 0$, $j^-$ is also 'fully' matched, since it is adjacent to the maximum size of matched edges it can admit, and consequently it is further removed along with all its adjacent edges. It is easy to see that, in the $z$-matching problem on $B$, we can simply set $z_{i^+} = 1$ for any out-vertex $i^+ \in B$ and $z_{j^-} = z$ for any in-vertex $j^- \in B$ as initial values. The case of $z = 1$ corresponds simply to the MM problem on bipartite graphs.

Our local algorithm consists of two parts. The first part is a generalized version of the GLR procedure to reduce constraints. On a bipartite graph $B$, we define four types of local graphical structure: (1) an in-root as any in-vertex, say $j^-$, having $\geq z_{j^-}$ nearest neighbors which all have $j^-$ as their sole nearest neighbor; (2) an out-leaf which have an in-root as its sole nearest neighbor; (3) an in-leaf as any in-vertex, say $j^-$, having a degree $\leq z_{j^-}$; (4) an out-root as any out-vertex which is a nearest neighbor of an in-leaf. In the generalized GLR procedure, a graph pruning process and a selection of matched edges into a $z$-matching are carried out at the same time in two elementary steps. (1) For any in-root, say $j^-$, its $z_{j^-}$ neighboring out-leaves are randomly chosen and their adjacent edges are further added into a $z$-matching; correspondingly, $j^-$ is removed along with all its adjacent edges; (2) For any in-leaf, all its adjacent edges are added into a $z$-matching; correspondingly, the in-leaf and its adjacent out-roots are all removed along with their adjacent edges. After the iterative removal of in-root and out-roots, the residual subgraph is named as the core subgraph.
of the bipartite graph. If there is no core, the generalized GLR procedure approximates a $z$-matching as an upper bound. We can see that the case of $z = 1$ reduces to the original GLR procedure. See figure (b) for an example.

The second part of our local algorithm is a randomized procedure to further extract edges for a $z$-matching on the core of a bipartite graph. This procedure is much like the one in the Karp-Sisper algorithm for the MM problem on undirected graphs [10]. On the core of a bipartite graph instance $B$, an edge, say $(i^+, j^-) \in B$, is randomly chosen and matched, which leads to the removal of $i^+$ and the update $z_{j^-} \leftarrow z_{j^-} - 1$. After a single randomized procedure, upon newly generated in-roots or out-roots, the generalized GLR procedure can be further applied on the subgraph. Taken together, in our local algorithm, the generalized GLR procedure and the randomized procedures are carried out iteratively, until all the edges in $B$ are removed. See figure (b) for an example.

Here we elaborate how the generalized GLR procedure serves as a reasonable local algorithm for constructing a $z$-matching on a bipartite graph $B$. For an optimization problem to find its maximum, the basic logic of a local algorithm is to remove the least possible edges to achieve a given size of matched edges among all the possible matching choices. First, we consider the removal of in-roots. We consider two matching choices based on a given in-root: matching an edge between the in-root and one of its out-leaves, and matching an edge between the in-root and one of its non-out-leaf nearest neighbors. The latter choice leads to more removed edges as the degree of a non-out-leaf vertex is surely $\geq 2$. Thus for an in-root (say $j^- \in B$) and its nearest neighbors, choosing randomly $z_{j^-}$ edges between the in-root and its neighboring out-leaves into a $z$-matching is a locally optimal procedure in constructing a $z$-matching. Once we select the $z_{j^-}$ edges for $j^-$ to match, $j^-$ is removed along with all its adjacent edges in the generalized GLR procedure. We take the in-root 1 and its nearest neighbors in figure (a) for an example. Compared with matching any edge in $\{(1^+, 1^-), (2^+, 1^-), (3^+, 1^-)\}$, matching $(4^+, 1^-)$ results in removing more edges with a size $|\partial 4^+| = 1(= 2)$. Thus matching randomly 2 of these 3 edges is an optimal choice for the local structure of the in-root 1 and its nearest neighbors. After this matching choice, 1 is removed along with all its adjacent edges in the generalized GLR procedure.

Then, we consider the removal of out-roots. We consider two matching choices for a given out-root, say $i^+ \in B$, of a specific in-leaf, say $j^- \in B$: matching the edge between $i^+$ and $j^-$, and matching the edge between $i^+$ with one of its non-in-leaf nearest neighbors, say $l^- \in B$. If $l^-$ is also an in-leaf, matching $(i^+, j^-)$ and $(i^+, l^-)$ are both locally optimal choices. If $l^-$ is not an in-leaf, we have $k_l^- \geq z_{l^-} + 1$. We further consider the local perturbation from matching the two edges. As the perturbation of matching two edges is the same for the in-vertices of $V^- \setminus \{j^-, l^-\}$, we only consider their effect on $j^-$ and $l^-$. If $(i^+, j^-)$ is matched, for $j^-$ we have an updating as $(k_{j^+}, z_{j^-}) \leftarrow (k_{j^+} - 1, z_{j^-} - 1)$, and for $l^-$ we have an updating as $(k_{l^-}, z_{l^-}) \leftarrow (k_{l^-} - 1, z_{l^-})$. If $(i^+, l^-)$ is matched, for $j^-$ we have an updating as $(k_{j^-} - 1, z_{j^-}) \leftarrow (k_{j^-} - 1, z_{j^-})$, and for $l^-$ we have an updating as $(k_{l^-}, z_{l^-}) \leftarrow (k_{l^-} - 1, z_{l^-} - 1)$. After matching $(i^+, j^-)$, the maximum possible size of further matched edges is $k_{j^-} - 1$ for $j^-$ and $z_{l^-} - 1$ for $l^-$. After matching $(i^+, l^-)$, the maximum possible size of further matched edges is $k_{j^-} - 1$ for $j^-$ and $z_{l^-} - 1$ for $l^-$. Beware that $k_{j^-} \leq z_{j^-}$ for $j^-$ and $k_{l^-} \geq z_{l^-} + 1$ for $l^-$. We can see that locally speaking, compared with matching $(i^+, l^-)$, matching $(i^+, j^-)$ leads to 1 more possible edge further to match. Taken the above considerations together, for the local structure of an in-leaf and its neighboring out-roots, matching all the edges adjacent to the in-leaf is a locally optimal matching choice. After this matching procedure, all the edges adjacent to the in-leaf and its out-roots are removed in the generalized GLR procedure. We take the in-leaf 4 and all its neighboring out-roots in figure (b) for an example. Compared with matching $(7^+, 3^-)$, matching $(7^+, 4^-)$ leads to 1 more possible edge further to match for $3^-$. Thus in all, matching $(7^+, 4^-)$ and $(8^+, 4^-)$ is a locally optimal procedure for the local structure of the in-leaf 4 and its out-roots. Then 4, 7, 8 are removed along with all their adjacent edges in the generalized GLR procedure.

### III. METHOD

Our local algorithm (the generalized GLR procedure and the randomized procedure) can be applied to approximate a $z$-matching on any bipartite graph instance, thus the relative size of matched edges or equivalently matched out-vertices $y$, which is normalized by the out-vertex size of a bipartite graph, can be calculated. Yet in the specific case of random bipartite graphs with an equal size of two types of vertices, we can develop a mean-field theory to calculate the pertinent quantities of the generalized GLR procedure, such as the relative sizes of the matched edges revealed during the procedure and the residual subgraph structure after the procedure.

To present our mean-field theory, we basically follow a unified language of the cavity method [8], which has been widely employed in the analytical theory of percolation problems. The general procedure for the cavity method of a percolation problem is to define cavity probabilities and their self-consistent equations describing the state transitions of the vertices in the graph pruning process, to calculate the stable fixed solutions of the cavity probabilities, and finally to arrive at the relative sizes of graphical structure revealed by the graph pruning process. A quite similar derivation of the cavity method for the generalized GLR procedure in the context of the MM problem on undirected graphs and undirected bipartite graph representations of directed graphs can be found respectively in [14] [15].
For a random bipartite graph $B$ with an equal size of out- and in-vertices, a set of 4 cavity probabilities \{$\alpha^+, \alpha^-, \beta^+, \beta^-$\} are defined. On $B$, an edge $(i^+, j^-)$ with the out-vertex $i^+$ and the in-vertex $j^-$ is randomly chosen, and we define the cavity probabilities on the cavity graph $B \setminus (i^+, j^-)$. Following the edge to arrive at $i^+$, we define $\alpha^+$ as the probability that $i^+$ becomes an out-leaf of a certain in-root, $\beta^+$ as the probability that $i^+$ becomes an out-root of a certain in-leaf. Following the edge to arrive at $j^-$, we define $\alpha^-$ as the probability that $j^-$ becomes an in-leaf, $\beta^-$ as the probability that $j^-$ becomes an in-root. The self-consistent equations of the cavity probabilities for the pruning process of the generalized GLR procedure can be established based on the locally tree-like structure approximation on sparse random graph [8]. The local approximation here assumes that on a cavity graph, say $B$ for the pruning process of the generalized GLR procedure can be established based on the locally tree-like structure approximation on sparse random graph [8]. The local approximation here assumes that on a cavity graph, say $B$ with a vertex $i \in B$, the states of $j \in \partial i$ are independent of each other in the graph pruning process taken place on $B \setminus i$. The self-consistent equations can be derived as

$$\alpha^+ = \sum_{k_+ = 1}^{\infty} Q_+ (k_+) (\beta^-)^{k_+ - 1}, \quad (1)$$

$$\beta^+ = 1 - \sum_{k_+ = 1}^{\infty} Q_+ (k_+) (1 - \alpha^-)^{k_+ - 1}, \quad (2)$$

$$\alpha^- = \sum_{k_- = 1}^{\infty} Q_- (k_-) \sum_{s=0}^{z-1} \left( \frac{k_- - 1}{s} \right) (\beta^+)^{k_+ - 1 - s} (1 - \beta^+)^s, \quad (3)$$

$$\beta^- = 1 - \sum_{k_- = 1}^{\infty} Q_- (k_-) \sum_{s=0}^{z-1} \left( \frac{k_- - 1}{s} \right) (\alpha^+) s (1 - \alpha^+)^{k_- - 1 - s}. \quad (4)$$

With the stable fixed solutions of the above equations, we can calculate quantities resulting from the generalized GLR procedure, such as the relative size of the out-vertices in the core structure $n^+$ (normalized by $|V^+|$), the relative size of the in-vertices in the core structure $n^-$ (normalized by $|V^-|$), the relative size of the edges in the core structure $l$ (normalized by $|V^+|$), and the relative size of the matched out-vertices or equivalently matched edges (normalized by $|V^+|$). We further have

$$n^+ = \sum_{k_+ = 2}^{\infty} P_+ (k_+) \sum_{s=2}^{k_+} \left( \frac{k_+}{s} \right) (1 - \alpha^- - \beta^-)^s (\beta^-)^{k_+ - s}, \quad (5)$$

$$n^- = \sum_{k_- = z+1}^{\infty} P_- (k_-) \sum_{s=z+1}^{k_-} \left( \frac{k_-}{s} \right) (1 - \alpha^+ - \beta^+)^s (\beta^+)^{k_- - s}, \quad (6)$$

$$l = c_+ (1 - \alpha^+ - \beta^+) (1 - \alpha^- - \beta^-), \quad (7)$$

$$w = [1 - \sum_{k_+ = 0}^{\infty} P_+ (k_+) (1 - \alpha^-)^{k_+}]$$

$$+ z [1 - \sum_{k_- = 0}^{\infty} P_- (k_-) \sum_{s=0}^{z-1} \left( \frac{k_-}{s} \right) (\alpha^+) s (1 - \alpha^+)^{k_- - s}]$$

$$- z \sum_{k_- = z}^{\infty} P_- (k_-) \left( \frac{k_-}{z} \right) (\alpha^+) s (\beta^+)^{k_- - s}. \quad (8)$$

It is easy to see that with $z = 1$, equations (1) - (8) reduce to the mean-field theory for the MM problem in [13]. We should mention that equations (1) - (4) are previously derived as equations (23) and (24) in [33] from a different framework, which is based on the self-consistent equations of coarse-grained cavity messages in the cavity method at the zero temperature limit, even though the paper [33] does not explicitly show an equivalent definition of the generalized GLR procedure.

We explain briefly the above equations. We assume that $(i^+, j^-)$ with the out-vertex $i^+$ and the in-vertex $j^-$ is a randomly chosen edge on a random bipartite graph $B$. We first move from a cavity graph $B \setminus i^+$ to another cavity graph $B \setminus (i^+, j^-)$ after the corresponding edges are added. If $i^+$ is an out-leaf in $B \setminus (i^+, j^-)$, the in-vertices $\partial i^+ \setminus j^-$ should all be in-roots in $B \setminus i^+$. Thus we have equation (1). If $i^+$ is an out-root in $B \setminus (i^+, j^-)$, there should be at least
one in-leaf among the in-vertices $\partial i^+ \setminus j^-$ in $B \setminus i^+$. Thus we have equation (2). We then move from a cavity graph $B \setminus j^-$ to another cavity $B \setminus (i^+, j^-)$ after the corresponding edges are added. If $j^-$ is an in-leaf in $B \setminus (i^+, j^-)$, there should be at most $z - 1$ residual neighbors among the out-vertices $\partial j^- \setminus i^+$ in $B \setminus j^-$. Thus we have equation (3). If $j^-$ is an in-root in $B \setminus (i^+, j^-)$, there should be at least $z$ out-leaves among the out-vertices $\partial j^- \setminus i^+$ in $B \setminus j^-$. Thus we have equation (4). We further move from a cavity graph $B \setminus j^+$ to the original graph $B$ after the corresponding edges are added. If $i^+$ is in the core of $B$, there should be no in-leaf and also at least two in-vertices among the in-vertices $\partial i^+$ also in the core in $B \setminus i^+$. Thus we have equation (5). We further move from a cavity graph $B \setminus j^-$ to $B$ after the corresponding edges are added. If $j^-$ is in the core of $B$, there should be no out-leaf and also at least $z + 1$ out-vertices among the out-vertices $\partial j^-$ also in the core in $B \setminus j^-$. We also move from a cavity graph $B \setminus (i^+, j^-)$ to $B$ after the edge $(i^+, j^-)$ is added. If the edge $(i^+, j^-)$ is in the core of $B$, the vertices $i^+$ and $j^-$ should be both in the core of $B \setminus (i^+, j^-)$. Thus we have equation (6). Finally, we consider the derivation of equation (6). If an out-vertex $i^+$ is an in-root in $B$, there should be at least one in-leaves among the in-vertices $\partial i^+$ in $B \setminus i^+$. Thus we have its first term. If an in-vertex $j^-$ is an in-root in $B$, there should be at least $z$ out-leaves among the out-vertices $\partial j^-$ in $B \setminus j^-$. Besides, each in-root contributes $z$ matched out-leaves into a $z$-matching. Thus we have its second term. Yet we have an recounting case in the first two terms: an in-vertex with exactly $z$ residual nearest neighbors all as out-leaves, can be considered both as an in-leaf and an in-root. The recounting case is subtracted in the third term.

IV. RESULTS

Here we test our local algorithm and analytical theory for the $z$-matching problem on some random bipartite graph models with equal sizes of out- and in-vertices. Our concerned results include the fraction of matched out-vertices $y$ from the local algorithm, the core fractions $n^\pm$ and the fraction of matched out-vertices $w$ from the generalized GLR procedure.

We first consider the diluted regular random bipartite graphs with a degree heterogeneity. On an original bipartite graph with an integer out- and in-degree $K \geq 2$, after a fraction $1 - \rho$ ($0 < \rho \leq 1$) of its edges are randomly chosen and removed, we have a diluted bipartite graph with degree distributions $P_\pm(k_\pm) = C_k^\pm \rho^k(1 - \rho)^K-k_\pm$ with $0 \leq k_\pm \leq K$ and mean out-degree and in-degree $c_\pm = \rho K_\pm$. We then consider the bipartite graphs with both Poisson distributions [35] for out- and in-vertices as $P_\pm(k_\pm) = e^{-c_\pm}c_\pm^{k_\pm}/k_\pm!$ in which $c_+ (c_-)$ is the mean degree for out-(in-)vertices. We also consider the bipartite graphs with both power-law distributions [37] for out- and in-vertices as $P_\pm(k_\pm) \propto k_\pm^{-\gamma_\pm}$ in which $\gamma_+$ ($\gamma_-$) is the degree exponent for out-(in-)vertices. We further consider bipartite graphs with different forms of degree distributions for the out- and in-vertices. Specifically, we consider a type of bipartite graphs with a Poisson out-degree distribution and a power-law in-degree distribution with an in-degree exponent $\gamma_-$. The power-law degree distributions above can be asymptotically realized with the static model [38-40]. Details of related equations on random bipartite graphs are left in Appendix.

Figure 2 (a) and (d) show the results of $y$ with our local algorithm. Figure 2 (b), (c), (e), and (f) show the results of $n^\pm$ and $w$ from the generalized GLR procedure. For $z = 1$, the generalized GLR procedure reduces to the original GLR procedure, and from [18, 24, 25] we can see that the behaviors of $n^\pm$ and $w$ are continuous. While for $z = 2$ and 3, their transitions experience abrupt changes. These sudden changes can be explained by the behaviors of the stable fixed solutions of $(\alpha^\pm, \beta^\pm)$. We define the right-hand sides of equations (1), (2), (3), and (4) as $F_\pm(\beta^-)$, $G_+ (\alpha^-)$, $F_- (\beta^+)$, and $G_- (\alpha^+)$, respectively. Then we define $f_\pm (\alpha^\pm) \equiv -\alpha^\pm + F_\pm (G_\pm (\alpha^\pm))$. Finding the stable fixed solutions of equations (1) - (4) is equivalent to finding the smallest fixed solutions of $f_\pm (\alpha^\pm)$ for $f_\pm (\alpha^\pm) = 0$ and then calculating the corresponding stable fixed $\beta^\pm$ with $\beta^\pm = G_\pm (\alpha^\pm)$. It is easy to prove that $f_\pm (0) \geq 0$ and $f_\pm (1) \leq 0$, then there are only one odd size of fixed points for $f_\pm (\alpha^\pm) = 0$ with $\alpha^\pm \in [0,1]$. For equations (1) - (4), if we make a substitution as $\alpha^\pm \rightarrow 1 - \beta^\pm$, we retain their same forms. Thus we have a trivial solution of the equations as $\alpha^\pm = 1 - \beta^\pm = F_\pm (G_\pm (\alpha^\pm))$. Equivalently, we have $1 - \alpha^\pm - \beta^\pm = 0$ and then $n^\pm = 0$ from equations (5) and (6). Yet, with $c^*$ tuning, nontrivial stable solutions emerge as $1 - \alpha^\pm - \beta^\pm > 0$, leading to nontrivial $n^\pm$. See figure 3 for an example on bipartite graphs with both Poisson out- and in-degree distributions. In figure 3 (a) and (b), when $c^* < c^* \approx 6.495$ with $c^*$ as the critical mean out-degree, $f_\pm (\alpha^\pm) = 0$ both has only one fixed solution as $1 - \alpha^\pm - \beta^\pm = 0$, which is also the stable fixed solution; when $c^* > c^*$, another two fixed solutions emerge. Yet there are different scenarios for stable solutions of $\alpha^+$ and $\alpha^-$. For $f_\pm (\alpha^+)$, the newly emerged fixed solutions are both larger than the trivial fixed solution, which means the branching of fixed solutions of $\alpha^+$ brings no discontinuous change to its stable fixed solution. For $f_\pm (\alpha^-)$, the newly emerged fixed solutions are both smaller than the trivial fixed solution, leading to a discontinuous drop of its stable fixed solution and further $1 - \alpha^\pm - \beta^\pm > 0$. From figure 3 (c), we can see that the stable fixed solutions of $(\alpha^+, \beta^-)$ behavior continuously, yet the stable fixed solutions $(\alpha^-, \beta^+)$ decrease discontinuously at $c^* \approx 6.495$, resulting in both a sudden emergence of core structure and an abrupt decrease of matched out-vertex size. This discontinuity analysis applies in the cases of $z \geq 2$. A simple intuitive understanding is that the percolation phenomenon on the random bipartite graphs with equal out- and in-vertex sizes is in-leaf
FIG. 2. The relative sizes of matching size $y$ from the local algorithm, the relative sizes of the out- and in-vertices in the core structure $n^\pm$ and the matched out-vertices $w$ from the generalized GLR procedure are calculated on random bipartite graphs with equal size of out- and in-vertices. Results on the diluted regular random bipartite graphs (RRRR), the bipartite graphs with both Poisson out- and in-degree distributions (ERER), the bipartite graphs with a Poisson out-degree distribution and a power-law in-degree distribution (ERSM), and the bipartite graphs with both power-law out- and in-degree distributions (SMSM) are listed from left to right. For RRRR, the mean out-degree $c_\perp = \rho K_\perp$, in which $\rho$ is the fraction of remained edges in the edge dilution procedure and $K_\perp = 20$ is the integer degree of the original regular random bipartite graphs. For ERSM, the in-degree exponent $\gamma\perp = 3.0$. For SMSM, the out- and in-degree exponents $\gamma\parallel = \gamma\perp = 3.0$. The solid lines are for the analytical results on infinitely large random graphs. The dashed lines between solid line segments are for the discontinuous change in the analytical results. Each sign is for a simulation result on a random graph instance with a vertex size $|V_\perp| = 10^5$. In (a) and (d), solid lines are for the result of $w$ only before percolation transitions. In (a), (b), and (c), the case of $z = 2$ is considered. In (d), (e), and (f), the case of $z = 3$ is considered.
FIG. 3. The behavior of the stable fixed solutions of \((\alpha^+, \beta^-)\) and \((\alpha^-, \beta^+)\) for the generalized GLR procedure on random bipartite graphs for the 2-matching problem. The bipartite graphs considered here have both Poisson distributions and equal sizes for out- and in-vertices. Results are for the analytical results on infinitely large random graphs. (a) shows the behavior of \(f_+ (\alpha^+)\). (b) shows the behavior of \(f_- (\alpha^-)\). (c) shows the stable fixed solution pairs of \((\alpha^+, \beta^-)\) and \((\alpha^-, \beta^+)\). Dotted lines are for discontinuous changes in stable fixed solutions of \((\alpha^-, \beta^+)\).

V. DISCUSSION AND CONCLUSIONS

In this paper, we study the z-matching problem on bipartite graphs as a combinatorial optimization problem with a local algorithm. The basic component of our local algorithm is a generalized version of the GLR procedure, which is both a set of local procedures in choosing edges for constructing a z-matching configuration and a graph pruning process. If the generalized GLR procedure on a bipartite graph leaves a residual subgraph, a randomized matching procedure and the generalized GLR procedure can be applied iteratively until a z-matching is constructed. The generalized GLR procedure, as a graph pruning process, leads to a core percolation phenomenon. On random bipartite graphs with equal sizes of out- and in-vertices, we further theoretically analyze the core percolation, presenting an analytical approximation and also an upper bound for the z-matching problem. We hope that our algorithm and theory help to illustrate some numerical issues of the z-matching problem.

There are still lots of questions to resolve. The first question is the computational complexity of the z-matching problem, which is fundamental to any optimization problem. The case of \(z = 1\) leads to the MM problem, which is a proved polynomial combinatorial optimization problem [9]. Yet, the computational complexity in the general case of \(z \geq 2\) are unknown. Probably, the z-matching problems with \(z \geq 2\) is ‘harder’ than just being polynomial. Yet a rigorous evaluation is for further study.

The second question is about the effectiveness region of our local algorithm. Our local algorithm and theory only give an estimation to the energy (relative size) of the z-matching problem. Whether the percolation threshold of the generalized GLR procedure corresponds to the limit of any local algorithm to be globally optimal is much involved to the entropy of the ground states and the configurational complexity of their solution clusters. A good starting point is to study the phase transition behaviors in the solution space of the problem with the cavity method at finite temperature presented in [33].

The third question is a missing mean-field theory for the generalized GLR procedure on general random bipartite graphs with unequal sizes of out- and in-vertices. A natural extension of the existing framework in this paper is to substitute \(z\) with \(z f\) in equations [8] in which \(f\) is the size ratio. Unfortunately, both the behaviors of the stable fixed solutions of \((\alpha^\pm, \beta^\mp)\) and the relative sizes \(n^\pm\) and \(w\) suffer from a systematic discrepancy between the theoretical prediction and simulation on the random bipartite graph models considered in the main text. Thus, the general case of unequal sizes of out- and in-vertices needs a new set of mean-field equations. Simulation results also indicate that, with varying \(c_+\) and fixed \(c_-\), a core exists only in an intermediate range of \(c_+\), whose emergence and disappearance...
are out-leaf driven and in-leaf driven respectively. Further study seeking a general analytical framework will be continued.

The fourth question is related to the perfect matching idea in the matching theory. Once we have a core after the generalized GLR procedure on a bipartite graph, a naive extension of the original perfect matching is to assume that the matching size is estimated as \( y = w + \min\{n^+, zn^-\} \). We test on random bipartite graphs with Poisson out- and in-degree distributions, and we see a drop of the estimated matching size \( y \) of 0.00953 from \( c = 6.49 \) to \( c = 6.50 \), while the critical mean out-degree here is \( c^* \approx 6.495 \). A possible reason for this unphysical result is that the generalized GLR procedure is probably not a globally optimal procedure for constructing a maximum \( z \)-matching with \( z \geq 2 \) even on a coreless bipartite graph, or the perfect matching cannot be extended in such a naive way.

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VII. APPENDIX: EQUATIONS ON RANDOM BIPARTITE GRAPHS

A. Regular random graphs

A regular random undirected graph has a uniform degree distribution \( P(k) = \delta(k, K) \) with \( K \geq 2 \). A simple way to construct a diluted regular random graphs with a heterogeneous degree distribution is to randomly remove \( 1 - \rho \) (\( 0 \leq \rho \leq 1 \)) of all its edges. Correspondingly, the degree distribution \( P(k) \) and the excess degree distribution \( Q(k) \equiv kP(k)/\rho K \) are

\[
P(k) = \binom{K}{k} \rho^k (1 - \rho)^{K-k},
\]

\[
Q(k) = \binom{K-1}{k-1} \rho^{k-1} (1 - \rho)^{K-k}.
\]

We further have

\[
\sum_{k=s}^{K} P(k) \binom{k}{s} x^{k-s} = \frac{\rho^s}{s!} \prod_{t=0}^{s-1} (K-t)(1-\rho(1-x))^{K-s},
\]

\[
\sum_{k=s+1}^{K} Q(k) \binom{k-1}{s} x^{k-1-s} = \frac{\rho^s}{s!} \prod_{t=0}^{s-1} (K-1-t)(1-\rho(1-x))^{K-1-s}.
\]

In the case of bipartite graphs, we can apply the same edge removal procedure on an initial bipartite graph with an integer out- and in-degree \( K_\pm \) with \( K_+ + K_- \geq 2 \). The diluted graph has a mean out- and in-degree \( c_\pm = \rho K_\pm \). With the above summation equations, we have simplified equations as
\[ \alpha^+ = (1 - \rho(1 - \beta^-))^{K^+ - 1}, \]
\[ \beta^+ = 1 - (1 - \rho \alpha^-)^{K^+ - 1}, \]
\[ \alpha^- = \sum_{s=0}^{z-1} \frac{(\rho(1 - \beta^+))^s}{s!} (1 - \rho(1 - \beta^+))^{K^- - 1 - s} \prod_{t=0}^{s-1} (K^- - 1 - t), \]
\[ \beta^- = 1 - \sum_{s=0}^{z-1} \frac{(\rho \alpha^+)^s}{s!} (1 - \rho \alpha^+)^{K^- - 1 - s} \prod_{t=0}^{s-1} (K^- - 1 - t), \]
\[ n^+ = (1 - \rho \alpha^-)K^+ - (1 - \rho(1 - \beta^-))K^+ - \rho K^+ \alpha^+(1 - \alpha^- - \beta^-), \]
\[ n^- = (1 - \rho \alpha^+)K^- - \sum_{s=0}^{z} \frac{(\rho(1 - \alpha^+ - \beta^+))^s}{s!} (1 - \rho(1 - \beta^+))^{K^- - s} \prod_{t=0}^{s-1} (K^- - t), \]
\[ w = [1 - (1 - \rho \alpha^-)K^+] + z[1 - \sum_{s=0}^{z-1} \frac{(\rho \alpha^+)^s}{s!} (1 - \rho \alpha^+)^{K^- - s} \prod_{t=0}^{s-1} (K^- - t)] \]
\[ - z \frac{(\rho \alpha^+)^z}{z!} (1 - \rho(1 - \beta^+))^{K^- - 1 - z} \prod_{t=0}^{z-1} (K^- - t). \]

**B. Erdős-Rényi random graphs**

An Erdős-Rényi random undirected graph with a mean degree \( c \) has a Poisson degree distribution \( P(k) \) and an excess degree distribution \( Q(k) \equiv kP(k)/c \) as

\[ P(k) = e^{-c} \frac{c^k}{k!}, \]
\[ Q(k) = e^{-c} \frac{c^{k-1}}{(k-1)!}. \]

We further have

\[ \sum_{k=s}^{\infty} P(k) \binom{k}{s} x^{k-s} = \frac{c^s}{s!} e^{-c(1-x)}, \]
\[ \sum_{k=s+1}^{\infty} Q(k) \binom{k-1}{s} x^{k-1-s} = \frac{c^s}{s!} e^{-c(1-x)}. \]

In the case of bipartite graphs, with the above summation equations, we have simplified equations as

\[ \alpha^+ = e^{-c+ (1 - \beta^-)}, \]
\[ \beta^+ = 1 - e^{-c+ \alpha^-}, \]
\[ \alpha^- = \sum_{s=0}^{z-1} \frac{(c_+ (1 - \beta^+))^s}{s!} e^{-c_+ (1 - \beta^+)}, \]
\[ \beta^- = 1 - \sum_{s=0}^{z-1} \frac{(c_+ \alpha^+)^s}{s!} e^{-c_+ \alpha^+}, \]
\[ n^+ = 1 - \alpha^+ - \beta^+ - c_+ \alpha^+(1 - \alpha^- - \beta^-), \]
\[ n^- = e^{-c_+ \alpha^-} - \sum_{s=0}^{z} \frac{(c_+ (1 - \alpha^+ - \beta^+))^s}{s!} e^{-c_+ (1 - \beta^+)}, \]
\[ w = \beta^+ + z \beta^- - z \frac{(c_+ \alpha^+)^z}{z!} e^{-c_+ (1 - \beta^+)} \].
C. Asymptotical scale-free networks

The scale-free undirected networks follow a power-law degree distribution as \( P(k) \propto k^{-\gamma} \) as \( \gamma \) is the degree exponent. There are various ways to construct scale-free networks, for example, with the configurational model or the static model. In this paper we construct asymptotical scale-free networks with the latter model whose degree distribution has an analytical expression. Correspondingly, we define \( \xi \equiv 1/(\gamma - 1) \). We have the degree distributions as

\[
P(k) = \frac{(c(1-\xi))^{1/\xi} \Gamma(k-1/\xi, c(1-\xi))}{\xi \Gamma(k+1)}
\]

\[
= \frac{(c(1-\xi))^k}{\xi k!} E_{-k+1+1/\xi}(c(1-\xi)),
\]

\[
Q(k) = \frac{1}{\xi} \frac{(c(1-\xi))^{k-1}}{(k-1)!} E_{-k+1+1/\xi}(c(1-\xi)).
\]

The special function \( E_a(x) \) is the general exponential integral function as \( E_a(x) \equiv \int_1^\infty dt e^{-xt}t^{-a} \) with \( a, x > 0 \). For large \( k \), we have \( P(k) \propto k^{-\gamma} \). We further have

\[
\sum_{k=s}^{\infty} P(k) \left( \frac{k}{s} \right) x^{k-s} = \frac{(c(1-\xi))^s}{\xi s!} E_{-s+1+1/\xi}(c(1-\xi)(1-x)),
\]

\[
\sum_{k=s+1}^{\infty} Q(k) \left( \frac{k-1}{s} \right) x^{k-1-s} = \frac{1-\xi}{\xi} \frac{(c(1-\xi))^s}{s!} E_{-s+1+1/\xi}(c(1-\xi)(1-x)).
\]

In the case of bipartite graphs, with the above summation equations and the definition \( \xi_\pm \equiv 1/(\gamma_\pm - 1) \), we have simplified equations as

\[
\alpha^+ = \frac{1-\xi_+}{\xi_+} E_{1/\xi_+} (c_+(1-\xi_+)(1-\beta^-)),
\]

\[
\beta^+ = 1 - \frac{1-\xi_+}{\xi_+} E_{1/\xi_+} (c_+(1-\xi_+)^\alpha^-),
\]

\[
\alpha^- = \sum_{s=0}^{z-1} \frac{1-\xi_-}{\xi_-} \frac{(c_-(1-\xi_-)(1-\beta^+))^s}{s!} E_{-s+1+1/\xi_-} (c_-(1-\xi_-)(1-\beta^+))
\]

\[
\beta^- = 1 - \sum_{s=0}^{z-1} \frac{1-\xi_-}{\xi_-} \frac{(c_-(1-\xi_-)^\alpha^+)^s}{s!} E_{-s+1+1/\xi_-} (c_-(1-\xi_-)^\alpha^+)
\]

\[
n^+ = \frac{1}{\xi_+} E_{1+1/\xi_+} (c_+(1-\xi_+)^\alpha^-) - \frac{1}{\xi_+} E_{1+1/\xi_+} (c_+(1-\xi_+)(1-\beta^-)) - c_+\alpha^+ (1-\alpha^- - \beta^-)
\]

\[
n^- = \frac{1}{\xi_-} E_{1+1/\xi_-} (c_-(1-\xi_-)^\alpha^+)
\]

\[
- \sum_{s=0}^{z-1} \frac{1}{\xi_-} \frac{(c_-(1-\xi_-)(1-\alpha^+ - \beta^+))^s}{s!} E_{-s+1+1/\xi_-} (c_-(1-\xi_-)(1-\beta^+))
\]

\[
w = [1 - \frac{1}{\xi_+} E_{1+1/\xi_+} (c_+(1-\xi_+)^\alpha^-)]
\]

\[
+z[1 - \sum_{s=0}^{z-1} \frac{1}{\xi_-} \frac{(c_-(1-\xi_-)^\alpha^+)^s}{s!} E_{-s+1+1/\xi_-} (c_-(1-\xi_-)^\alpha^+)]
\]

\[
-z \frac{1}{\xi_-} \frac{(c_-(1-\xi_-)^\alpha^+)^z}{z!} E_{-z+1+1/\xi_-} (c_-(1-\xi_-)(1-\beta^+))
\]
D. Random graphs with different forms of out- and in-degree distributions

Here we consider a specific type of random bipartite graphs, which have a Poisson out-degree distribution and a power-law in-degree distribution with an in-degree exponent $\gamma^-$. The power-law distribution is also generated with the static model. Correspondingly we define $\xi^- \equiv 1/(\gamma^- - 1)$. With the above summation equations, we have simplified equations as

\[ \alpha^+ = e^{-c_+(1-\beta^-)}, \]
\[ \beta^+ = 1 - e^{-c_+\alpha^-}, \]
\[ \alpha^- = \sum_{s=0}^{z-1} \frac{1 - \xi^-}{\xi^-} \frac{(c_-(1 - \xi^-)(1 - \beta^+))^s}{s!} E_{-s+\frac{1}{\xi^-}}(c_-(1 - \xi^-)(1 - \beta^+)), \]
\[ \beta^- = 1 - \sum_{s=0}^{z-1} \frac{1 - \xi^-}{\xi^-} \frac{(c_-(1 - \xi^-)\alpha^+)^s}{s!} E_{-s+\frac{1}{\xi^-}}(c_-(1 - \xi^-)\alpha^+), \]
\[ n^+ = 1 - \alpha^+ - \beta^+ - c_+\alpha^+(1 - \alpha^- - \beta^-), \]
\[ n^- = \frac{1}{\xi^-} E_{1+\frac{1}{\xi^-}}(c_-(1 - \xi^-)\alpha^+) \]
\[ - \sum_{s=0}^{z} \frac{1}{\xi^-} \frac{(c_-(1 - \xi^-)(1 - \alpha^+ - \beta^+))^s}{s!} E_{-s+1+\frac{1}{\xi^-}}(c_-(1 - \xi^-)(1 - \beta^+)), \]
\[ w = \beta^+ + z[1 - \sum_{s=0}^{z-1} \frac{1}{\xi^-} \frac{(c_-(1 - \xi^-)\alpha^+)^s}{s!} E_{-s+1+\frac{1}{\xi^-}}(c_-(1 - \xi^-)\alpha^+) \]
\[ - z \frac{1}{\xi^-} \frac{(c_-(1 - \xi^-)\alpha^+)^z}{z!} E_{-z+1+\frac{1}{\xi^-}}(c_-(1 - \xi^-)(1 - \beta^+)). \]
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