Lower Complexity Bounds of Finite-Sum Optimization Problems: The Results and Construction

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Abstract

The contribution of this paper includes two aspects. First, we study the lower bound complexity for the minimax optimization problem whose objective function is the average of $n$ individual smooth component functions. We consider Proximal Incremental First-order (PIFO) algorithms which have access to gradient and proximal oracle for each individual component. We develop a novel approach for constructing adversarial problems, which partitions the tridiagonal matrix of classical examples into $n$ groups. This construction is friendly to the analysis of incremental gradient and proximal oracle. With this approach, we demonstrate the lower bounds of first-order algorithms for finding an $\varepsilon$-suboptimal point and an $\varepsilon$-stationary point in different settings. Second, we also derive the lower bounds of minimization optimization with PIFO algorithms from our approach, which can cover the results in [35] and improve the results in [41].

1 Introduction

We consider the following optimization problem

$$\min_{x \in X} \max_{y \in Y} f(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x, y),$$

where the feasible sets $X \subseteq \mathbb{R}^{d_x}$ and $Y \subseteq \mathbb{R}^{d_y}$ are closed and convex. This formulation contains several popular machine learning applications such as matrix games [7, 8, 17], regularized empirical risk minimization [40, 32], AUC maximization [18, 38, 31], robust optimization [4, 36] and reinforcement learning [14, 12].

A popular approach for solving minimax problems is the first order algorithm which iterates with gradient and proximal point operation [9, 10, 24, 25, 33, 22]. Along this line, Zhang et al. [39] and Ibrahim et al. [17] presented tight lower bounds for solving strongly-convex-strongly-concave minimax problems by first order algorithms. Ouyang and Xu [28] studied a more general case that the objective function is possibly not strongly-convex or strongly-concave. However, these analyses [28, 39, 17] do not consider the specific finite-sum structure as in Problem (1). They only consider the deterministic first order algorithms which are based on the full gradient and exact proximal point iteration.

In big data regimes, the number of components $n$ in Problem (1) could be very large and we would like to devise stochastic optimization algorithms that avoid accessing the full gradient

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frequently. For example, Palaniappan and Bach [29] used stochastic variance reduced gradient algorithms to solve (1). Similar to convex optimization, one can accelerate it by catalyst [20, 37] and proximal point techniques [13, 22]. Although stochastic optimization algorithms are widely used for solving minimax problems, the study of their lower bounds complexity is still open. All of the existing lower bound analysis for stochastic optimization focuses on convex or nonconvex minimization problems [1, 35, 6, 5, 19, 15, 3].

This paper focuses on stochastic first order methods for solving Problem (1), which have access to the Proximal Incremental First-order Oracle (PIFO); that is,

$$h_{f_i}(\mathbf{x}, \gamma) \triangleq [f_i(\mathbf{x}, y), \nabla f_i(\mathbf{x}, y), \text{prox}_{\gamma} f_i(\mathbf{x}, y), \mathcal{P}_X(\mathbf{x}), \mathcal{P}_Y(y)],$$

where $$i \in \{1, \ldots, n\}$$, $$\gamma > 0$$, the proximal operator is defined as

$$\text{prox}_{\gamma} f_i(\mathbf{x}, y) \triangleq \arg \min_{u \in \mathbb{R}^d_x, v \in \mathbb{R}^d_y} \left\{ f_i(u, v) + \frac{1}{2\gamma} \| x - u \|^2 - \frac{1}{2\gamma} \| y - v \|^2 \right\},$$

and the projection operators are defined as

$$\mathcal{P}_X(\mathbf{x}) = \arg \min_{u \in X} \| u - x \|_2 \quad \text{and} \quad \mathcal{P}_Y(y) = \arg \min_{v \in Y} \| v - y \|_2.$$

We also define the Incremental First-order Oracle (IFO)

$$g_{f_i}(\mathbf{x}, y, \gamma) \triangleq [f_i(\mathbf{x}, y), \nabla f_i(\mathbf{x}, y), \mathcal{P}_X(\mathbf{x}), \mathcal{P}_Y(y)].$$

PIFO provides more information than IFO and it would be potentially more powerful than IFO in first order optimization algorithms. In this paper, we consider the general setting where $$f(\mathbf{x}, y)$$ is $$L$$-smooth and $$(\mu_x, \mu_y)$$-convex-concave, i.e., the function $$f(\cdot, y) - \frac{\mu_y}{2} \| \cdot \|^2$$ is convex for any $$y \in \mathcal{Y}$$ and the function $$-f(\mathbf{x}, \cdot) - \frac{\mu_x}{2} \| \cdot \|^2$$ is convex for any $$\mathbf{x} \in \mathcal{X}$$. When $$\mu_x, \mu_y \geq 0$$, our goal is to find an $$\varepsilon$$-suboptimal solution $$(\hat{\mathbf{x}}, \hat{y})$$ to Problem (1) such that the primal dual gap is less than $$\varepsilon$$, i.e.,

$$\max_{y \in \mathcal{Y}} f(\hat{\mathbf{x}}, y) - \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}, \hat{y}) < \varepsilon.$$

On the other hand, when $$\mu_x < 0, \mu_y > 0$$, $$f(\mathbf{x}, y)$$ is called a nonconvex-strongly-convex function, which has been widely studied in [30, 21, 27, 23]. In this case, our goal is instead to find an $$\varepsilon$$-stationary point $$\hat{x}$$ of $$\phi_f(\mathbf{x}) \triangleq \max_{y \in \mathcal{Y}} f(\mathbf{x}, y)$$, which is defined as

$$\| \nabla \phi_f(\hat{x}) \|_2 < \varepsilon.$$

In this paper we propose a novel framework to analyze lower complexity bounds for finite-sum optimization problems. Our construction decomposes Nesterov [26]'s classical tridiagonal matrix into $$n$$ groups and it facilitates the analysis for both the IFO and PIFO algorithms. In contrast, previous work is based on an aggregation method [19, 41] or a very complicated adversarial construction [35]. Their results do not cover the minimax problems. Moreover, we can also establish the tight lower bounds for finite-sum minimization problems [35, 19, 41] by the proposed decomposition framework with concise proofs. More details on our lower bound results refer to Tables 1 and 2.

1.1 Related Work

In this section, we review some upper bounds of PIFO Algorithms for minimax optimization Problem (1).
Table 1: Lower Bounds with the assumption that \( f_i \) is \( L \)-smooth and \( f \) is \((\mu_x, \mu_y)\)-convex-concave. When \( \mu_x \geq 0 \) and \( \mu_y \geq 0 \), the goal is to find an \( \varepsilon \)-suboptimal solution with \( \text{diam}(\mathcal{X}) \leq 2R_x, \text{diam}(\mathcal{Y}) \leq 2R_y \). And when \( \mu_x < 0 \), the goal is to find an \( \varepsilon \)-stationary point of the function \( \phi_f(x) \triangleq \max_{y \in \mathcal{Y}} f(\cdot, y) \) with \( \Delta = \phi_f(x_0) - \min_x \phi_f(x) \) and \( \mathcal{X} = \mathbb{R}^d, \mathcal{Y} = \mathbb{R}^d \).

| Cases                  | PIFO Lower Bounds                                                                 |
|------------------------|-----------------------------------------------------------------------------------|
| \( \mu_x > 0, \mu_y > 0 \) | \( \Omega \left( n + \frac{L}{\mu_x} \right) \left( n + \frac{L}{\mu_y} \right) \log(1/\varepsilon) \) |
| \( \mu_x = 0, \mu_y > 0 \) | \( \Omega \left( n + \frac{L}{\mu_x} \right) \left( n + \frac{L}{\mu_y} \right) \log(1/\varepsilon) \) |
| \( \mu_x = 0, \mu_y = 0 \)  | \( \Omega \left( n + \frac{LR_x R_y}{\varepsilon} + \left( R_x + R_y \right) \sqrt{nL} \right) \) |
| \( \mu_x < 0, \mu_y > 0 \) | \( \Omega \left( n + \frac{\Delta L}{\varepsilon} \min \left\{ \sqrt{\frac{L}{\mu_x}}, \sqrt{\frac{L}{\mu_y}} \right\} \right) \) |

**Convex-Concave Cases**  
Zhang and Xiao [40] considered a specific bilinear case of Problem (1) with \( \mathcal{X} = \mathbb{R}^d \) and \( \mathcal{Y} = \mathbb{R}^n \). Each individual component function has the form of

\[
  f_i(x, y) = h(x) + y_i \langle a_i, x \rangle - J_i(y_i),
\]

where \( h \) is \( \mu_x \)-strongly-convex, \( J_i \) is \( \mu_y \)-strongly-convex and \( \|a_i\|_2 \leq L \). They proposed a stochastic primal-dual coordinate (SPDC) method which can find \( \mathcal{O}(\varepsilon) \)-suboptimal solution with at most \( \mathcal{O}\left( n + \sqrt{nL} \log(1/\varepsilon) \right) \) PIFO queries.

Furthermore, Lan and Zhou [19] considered another specific bilinear case where \( \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_n \) and \( y = (y_1; y_2; \ldots; y_n) \) for any \( y_i \in \mathcal{Y}_i \), \( i = 1, \ldots, n \). And each individual component function has the form of

\[
  f_i(x, y) = h(x) + \langle y_i, x \rangle - J_i(y_i),
\]

where \( h \) is \( L \)-smooth and \( \mu_x \)-strongly-convex, and \( J_i \) is \( \mu_y \)-strongly-convex. They developed a similar upper bound of \( \mathcal{O}\left( n + \sqrt{nL} \log(1/\varepsilon) \right) \) with a randomized primal-dual gradient (RPDG) method. We remark that the SPDC method requires the proximal oracle related to \( h \) while the RPDG method only need the gradient oracle with respect to \( h \).

In the general strongly-convex-strongly-concave case, if each component \( f_i \) is \( L \)-smooth, the best known upper bound complexity for IFO/PIFO algorithms is \( \mathcal{O}\left( n + \sqrt{n(L/\mu_x + L/\mu_y)} \right) \log(1/\varepsilon) \) [7, 22]. For the case where \( \{f_i\}_{i=1}^n \) is \( L' \)-average smooth, the best known upper bound complexity is \( \mathcal{O}\left( n + \sqrt{n(L'/\mu_x + L'/\mu_y)} \right) \log(1/\varepsilon) \) [22]. Furthermore, if each component function \( f_i \) has \( L \)-cocoercive gradient, which is a stronger assumption than \( L \)-smooth, Chavdarova et al. [11] provided an upper bound of \( \mathcal{O}\left( n + \tilde{L}/\mu_x + \tilde{L}/\mu_y \right) \log(1/\varepsilon) \). Recent studies on deterministic algorithm for minimax optimization [21, 37, 34] implies that the term \( (\tilde{L}/\mu_x + \tilde{L}/\mu_y) \) in these upper bounds can be improved to be \( \frac{L}{\sqrt{\mu_x \mu_y}} \) by Catalyst framework [24].

Recently, for the convex-strongly-concave case, Yang et al. [37] demonstrated that employing SVRG/SAGA [29] with Catalyst framework can achieve an upper bound of \( \bar{\mathcal{O}}\left( n + \frac{L^2}{\sqrt{\mu_x \mu_y}} + \frac{\epsilon^{3/4}L^2}{1/\sqrt{\varepsilon}} \right) \).
Table 2: Lower Bounds with the assumption that \( \{f_i\}_{i=1}^n \) is \( L' \)-average smooth and \( f \) is \( (\mu_x, \mu_y) \)-convex-concave. When \( \mu_x \geq 0 \) and \( \mu_y \geq 0 \), the goal is to find an \( \varepsilon \)-suboptimal solution with \( \text{diam}(\mathcal{X}) \leq 2R_x \), \( \text{diam}(\mathcal{Y}) \leq 2R_y \). And when \( \mu_x < 0 \), the goal is to find an \( \varepsilon \)-stationary point of the function \( \phi_f(x) \triangleq \max_{y \in \mathcal{Y}} f(\cdot, y) \) with \( \Delta = \phi_f(x_0) - \min_x \phi_f(x) \) and \( \mathcal{X} = \mathbb{R}^{d_x}, \mathcal{Y} = \mathbb{R}^{d_y} \).

Moreover, Alacaoglu and Malitsky \cite{2} considered a more general case where \( f \) is convex-concave and \( \{f_i\}_{i=1}^n \) is \( L' \)-average smooth. They developed an upper bound of \( \tilde{O}(n + \sqrt{n}L'\bar{R}_x^2 + \bar{R}_y^2) \) for several methods.

**Nonconvex-Concave Cases.** In the nonconvex-strongly-concave case, Luo et al. \cite{23} proposed an upper bound of \( \tilde{O}(n + \min\{L'^2\mu_y^{-2}n^{1/2}, L'\mu_y^{-2} + L'\mu_y^{-1}n\} \varepsilon^{-2}) \), while Yang et al. \cite{37} developed an upper bound of \( \tilde{O}(n + n^{3/4}L^2\varepsilon^{-3}) \) for nonconvex-concave case.

## 2 Preliminaries

We first introduce the preliminaries used in this paper.

**Definition 2.1.** For a differentiable function \( \varphi(x) \) from \( \mathcal{X} \) to \( \mathbb{R} \) and \( L > 0 \), \( \varphi \) is said to be \( L \)-smooth if its gradient is \( L \)-Lipschitz continuous; that is, for any \( x_1, x_2 \in \mathcal{X} \), we have

\[
\|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2 \leq L \|x_1 - x_2\|_2.
\]

**Definition 2.2.** For a class of differentiable functions \( \{\varphi_i(x) : \mathcal{X} \to \mathbb{R}\}_{i=1}^n \) and \( L > 0 \), \( \{\varphi_i\}_{i=1}^n \) is said to be \( L \)-average smooth if for any \( x_1, x_2 \in \mathcal{X} \), we have

\[
\frac{1}{n} \sum_{i=1}^n \|\nabla \varphi_i(x_1) - \nabla \varphi_i(x_2)\|_2^2 \leq L^2 \|x_1 - x_2\|_2^2.
\]

**Definition 2.3.** For a differentiable function \( \varphi(x) \) from \( \mathcal{X} \) to \( \mathbb{R} \), \( \varphi \) is said to be convex if for any \( x_1, x_2 \in \mathcal{X} \), we have

\[
\varphi(x_2) \geq \varphi(x_1) + \langle \nabla \varphi(x_1), x_2 - x_1 \rangle.
\]

**Definition 2.4.** For a constant \( \mu \), if the function \( \tilde{\varphi}(x) = \varphi(x) - \frac{\mu}{2} \|x\|_2^2 \) is convex, then \( \varphi \) is said to be \( \mu \)-strongly-convex if \( \mu > 0 \) and \( \varphi \) is said to be \( \mu \)-weakly-convex if \( \mu < 0 \).
Especially, if $\varphi$ is $L$-smooth, then it can be checked that $\varphi$ is $(-L)$-weakly-convex. If $\varphi$ is $\mu$-weakly-convex, in order to make the operator $\text{prox}_\gamma \varphi$ valid, we set $\frac{1}{\gamma} > -\mu$ to ensure the function

$$
\hat{\varphi}(u) = \varphi(u) + \frac{1}{2\gamma} \|x - u\|_2^2
$$

is a convex function.

**Definition 2.5.** For a differentiable function $\varphi(x)$ from $\mathcal{X}$ to $\mathbb{R}$, we call $x$ an $\varepsilon$-stationary point of $\varphi$ if

$$
\| \nabla \varphi(x) \|_2 < \varepsilon.
$$

**Definition 2.6.** For a differentiable function $f(x, y)$ from $\mathcal{X} \times \mathcal{Y}$ to $\mathbb{R}$, $f$ is said to be convex-concave, if the function $f(\cdot, y)$ is convex for any $y \in \mathcal{Y}$ and the function $-f(x, \cdot)$ is convex for any $x \in \mathcal{X}$. Furthermore, $f$ is said to be $(\mu_x, \mu_y)$-convex-concave, if the function $f(x, y) - \frac{\mu_x}{2} \|x\|_2^2 + \frac{\mu_y}{2} \|y\|_2^2$ is convex-concave.

**Definition 2.7.** We call a minimax optimization problem $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y)$ satisfying the strong duality condition if $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} f(x, y)$.

By Sion’s minimax theorem, if $\varphi(x, y)$ is convex-concave and either $\mathcal{X}$ or $\mathcal{Y}$ is a compact set, then the strong duality condition holds.

**Definition 2.8.** We call $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ the saddle point of $f(x, y)$ if

$$
f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*)
$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

**Definition 2.9.** Suppose the strong duality of Problem (1) holds. We call $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$ an $\varepsilon$-suboptimal solution to Problem (1) if

$$
\max_{y \in \mathcal{Y}} f(\hat{x}, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}) < \varepsilon.
$$

## 2.1 A Concentration Inequality about Geometric Distributions

In the following part of this section, we introduce a concentration inequality about geometric distributions. We first give the formal definition of the geometric distribution.

**Definition 2.10.** For a nonnegative, integer-valued random variable $Y$, it is said to follow the geometric distribution with success probability $p$, if

$$
P[Y = k] = (1 - p)^k p \quad \text{for } k \in \{0, 1, 2, \ldots\},
$$

where $0 < p \leq 1$. The geometric distribution with success probability $p$ is denoted by $\text{Geo}(p)$.

The concentration inequality about geometric distributions is as follows.

**Lemma 2.11.** Let $\{Y_i\}_{i=1}^m$ be independent random variables, and $Y_i$ follows a geometric distribution with success probability $p_i$. Then for $m \geq 2$, we have

$$
P\left[ \sum_{i=1}^m Y_i > \frac{m^2}{4(\sum_{i=1}^m p_i)} \right] \geq \frac{1}{9}.
$$
We can view the probability \( P \left( \sum_{i=1}^{m} Y_i > j \right) \) as a function of \( m \) variables \( p_1, p_2, \ldots, p_m \):

\[
f_{m,j}(p_1, p_2, \ldots, p_m) \triangleq P \left( \sum_{i=1}^{m} Y_i > j \right). \tag{3}
\]

Before proving Lemma 2.11, we first provide the following useful result about the function \( f_{m,j} \).

**Lemma 2.12.** For \( m \geq 2 \) and \( j \geq 1 \), we have that

\[
f_{m,j}(p_1, p_2, \ldots, p_m) \geq f_{m,j}\left( \frac{\sum_{i=1}^{m} p_i}{m}, \ldots, \frac{\sum_{i=1}^{m} p_i}{m} \right).
\]

The proof of Lemma 2.12 is given in Appendix Section A.

With Lemma 2.12 in hand, we give the proof of Lemma 2.11.

**Proof of Lemma 2.11.** Let \( p = \frac{\sum_{i=1}^{m} p_i}{m} \) and \( \{ Z_i \sim \text{Geo}(p) \}_{i=1}^{m} \) be independent geometric random variables. Then we have

\[
P \left( \sum_{i=1}^{m} Y_i > \frac{m^2}{4(\sum_{i=1}^{m} p_i)} \right) > P \left( \sum_{i=1}^{m} Z_i > \frac{m}{4p} \right).
\]

Denote \( \sum_{i=1}^{m} Z_i \) by \( \tau \). It is easily checked that

\[
E[\tau] = \frac{m}{p} \quad \text{and} \quad \text{Var}(\tau) = \frac{m(1-p)}{p^2}.
\]

Hence, we have

\[
P \left( \tau > \frac{1}{4} E\tau \right) = P \left( \tau - E\tau > -\frac{3}{4} E\tau \right)
\]

\[
= 1 - P \left( \tau - E\tau \leq -\frac{3}{4} E\tau \right) \geq 1 - P \left( |\tau - E\tau| \geq \frac{3}{4} E\tau \right)
\]

\[
\geq 1 - \frac{16\text{Var}(\tau)}{9(E\tau)^2} = 1 - \frac{16m(1-p)}{9m^2} \geq 1 - \frac{16}{9m} \geq \frac{1}{9},
\]

which completes the proof.

\[\square\]

### 3 Lower Complexity Bounds for the Minimax Problems

In this section, we consider the following minimax problem

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x, y), \tag{4}
\]

where each component \( f_i(x, y) \) is \( L \)-smooth or the function class \( \{f_i(x, y)\}_{i=1}^{n} \) is \( L' \)-average smooth, and the feasible sets \( \mathcal{X} \) and \( \mathcal{Y} \) are closed and convex. In addition, \( f(x, y) \) is convex in \( x \) and concave in \( y \) or \( f(x, y) \) is non-convex in \( x \) and strongly-concave in \( y \).

In Section 3.1, we formally provide the definition of PIFO algorithms for solving Problem (4), function classes that we focus on, and optimization complexity which we want to lower bound. In
Section 3.2, we present our lower bound results for different function classes. In Section 3.3, we briefly summarize our framework for construction. The details on the construction for the smooth cases are in Sections 3.4, 3.5, 3.6, and 3.7. In Section 3.4, the objective function $f(x, y)$ is strongly-convex in $x$ and strongly-concave in $y$. In Section 3.5, $f(x, y)$ is convex in $x$ and strongly-concave in $y$ but not strongly-convex in $x$. In Section 3.6, $f(x, y)$ is convex in $x$ and concave in $y$. In Section 3.7, $f(x, y)$ is strongly-concave in $y$ but non-convex in $x$. The details on the construction for the average smooth cases are in Section 3.8.

### 3.1 The Setup

We study the PIFO algorithms to solve Problem (4), which we set up formally in this section. Define $\phi_f(x) = \max_{y \in Y} f(x, y)$ and $\psi_f(y) = \min_{x \in X} f(x, y)$.

**Algorithms** We define PIFO algorithms for minimization problem as follows.

**Definition 3.1.** Consider a stochastic optimization algorithm $A$ to solve Problem (4). Denote $(x_t, y_t)$ to be the point obtained by $A$ at time-step $t$. A PIFO algorithm consists of a categorical distribution $\mathcal{D}$ over $[n]$ and obtains $(x_t, y_t)$ by the following linear span protocol

$$(\tilde{x}_t, \tilde{y}_t) \in \text{span} \left\{ (x_0, y_0), \ldots, (x_{t-1}, y_{t-1}), \nabla f_{i_0}(x_0, y_0), \ldots, \nabla f_{i_{t-1}y_{t-1}}, \right\},$$

$$x_t = \mathcal{P}_x(\tilde{x}_t), \quad y_t = \mathcal{P}_y(\tilde{y}_t),$$

where $i_t \sim \mathcal{D}$ is drawn a single time at the beginning of the protocol. We denote by $\mathcal{A}$ the class of all PIFO algorithms.

We remark some details in our definition of PIFO algorithms.

1. Note that simultaneous queries $[15, 14, 23]$ are allowed in our definition of PIFO algorithms. At time-step $t$, the algorithm has the access to observe $\nabla f_{i_t}(x_0, y_0), \ldots, \nabla f_{i_{t-1}y_{t-1}}$ with shared $i_t$.

2. Without loss of generality, we assume that the PIFO algorithm $A$ starts from $(x_0, y_0) = (0_d, 0_d)$ to simplify our analysis. Otherwise, we can take $\{f_i(x, y) = f_i(x + x_0, y + y_0)\}_{i=1}^n$ into consideration.

3. The uniform distribution over $[n]$ and the distributions based on the smoothness of the component functions, e.g., the distribution which satisfies $P_{Z \sim \mathcal{D}}[Z = i] \propto L_i$ or $P_{Z \sim \mathcal{D}}[Z = i] \propto L_i^2$ for $i \in [n]$, are widely used in algorithm design for the categorical distribution $\mathcal{D}$, where $L_i$ is the smoothness of $f_i$.

4. Let $p_i = P_{Z \sim \mathcal{D}}[Z = i]$ for $i \in [n]$. We can assume that $p_1 \leq p_2 \leq \cdots \leq p_n$ by rearranging the component functions $\{f_i\}_{i=1}^n$. Suppose that $p_{s_1} \leq p_{s_2} \leq \cdots \leq p_{s_n}$ where $\{s_i\}_{i=1}^n$ is a permutation of $[n]$. We can consider $\{f_i\}_{i=1}^n$ and categorical distribution $\mathcal{D}'$ such that the algorithm draws $f_i \propto p_{s_i}$ with probability $p_{s_i}$ instead.

**Function class** We develop lower bounds for PIFO algorithms that find a suboptimal solution to the problem in the following four sets

$$\mathcal{F}_{\text{CC}}(R_x, R_y, L, \mu_x, \mu_y) = \left\{ f(X, Y) = \frac{1}{n} \sum_{i=1}^n f_i(X, Y) \bigm\vert f : X \times Y \to \mathbb{R}, \text{diam}(X) \leq 2R_x, \right\}$$
In this subsection, we present the our lower bound results for PIFO algorithms.

### 3.2 Main Results

For a function \( f \) where \( \phi \),

\[
\bar{F}_{\text{NCC}}(\Delta, L', \mu_x, \mu_y) = \left\{ f(x, y) = \frac{1}{n} \sum_{i=1}^{n} f_i(x, y) \left| f: \mathcal{X} \times \mathcal{Y} \to \mathbb{R}, \text{diam}(\mathcal{X}) \leq 2R_x, \right. \right. \\
\left. \left. \text{diam}(\mathcal{Y}) \leq 2R_y, \{f_i\}_{i=1}^{n} \right. \right. \text{is L'-average smooth, } f \text{ is } (\mu_x, \mu_y)\)-convex-concave \right\}.
\]

Optimization complexity  
We formally define the optimization complexity as follows.

**Definition 3.2.** For a function \( f \), a PIFO algorithm \( \mathcal{A} \) and a tolerance \( \varepsilon > 0 \), the number of queries needed by \( \mathcal{A} \) to find an \( \varepsilon \)-suboptimal solution to Problem (4) or an \( \varepsilon \)-stationary point of \( \phi_f(x) \) is defined as

\[
T(\mathcal{A}, f, \varepsilon) = \begin{cases} 
\inf \left\{ T \in \mathbb{N} \mid \mathbb{E}\phi_f(x_{\mathcal{A}, T}) - \mathbb{E}\psi_f(y_{\mathcal{A}, T}) < \varepsilon \right\}, & \text{if } f \in \mathcal{F}_{\text{CC}}(R_x, R_y, L, \mu_x, \mu_y) \cup \bar{F}_{\text{CC}}(R_x, R_y, L', \mu_x, \mu_y), \\
\inf \left\{ T \in \mathbb{N} \mid \mathbb{E}\|\nabla \phi_f(x_{\mathcal{A}, T})\|_2 < \varepsilon \right\}, & \text{if } f \in \mathcal{F}_{\text{NCC}}(\Delta, L, \mu_x, \mu_y) \cup \bar{F}_{\text{NCC}}(\Delta, L', \mu_x, \mu_y), 
\end{cases}
\]

where \((x_{\mathcal{A}, T}, y_{\mathcal{A}, T})\) is the point obtained by the algorithm \( \mathcal{A} \) at time-step \( T \).

Furthermore, the optimization complexity with respect to the function class \( \mathcal{F}(\Delta, R, L, \mu) \) and \( \bar{F}(\Delta, R, L', \mu) \) is defined as

\[
m^\varepsilon_{\text{CC}}(R_x, R_y, L, \mu_x, \mu_y) \triangleq \inf_{\mathcal{A} \in \mathcal{A}} \sup_{f \in \mathcal{F}_{\text{CC}}(R_x, R_y, L, \mu_x, \mu_y)} T(\mathcal{A}, f, \varepsilon),
\]

\[
m^\varepsilon_{\text{NCC}}(\Delta, L, \mu_x, \mu_y) \triangleq \inf_{\mathcal{A} \in \mathcal{A}} \sup_{f \in \mathcal{F}_{\text{NCC}}(\Delta, L, \mu_x, \mu_y)} T(\mathcal{A}, f, \varepsilon),
\]

\[
m^\varepsilon_{\text{CC}}(R_x, R_y, L', \mu_x, \mu_y) \triangleq \inf_{\mathcal{A} \in \mathcal{A}} \sup_{f \in \mathcal{F}_{\text{CC}}(R_x, R_y, L', \mu_x, \mu_y)} T(\mathcal{A}, f, \varepsilon),
\]

\[
m^\varepsilon_{\text{NCC}}(\Delta, L', \mu_x, \mu_y) \triangleq \inf_{\mathcal{A} \in \mathcal{A}} \sup_{f \in \bar{F}_{\text{NCC}}(\Delta, L', \mu_x, \mu_y)} T(\mathcal{A}, f, \varepsilon).
\]

### 3.2 Main Results

In this subsection, we present the our lower bound results for PIFO algorithms.
3.2.1 Smooth Cases

We first focus on the cases where each component function is $L$-smooth. When the objective function is strongly-convex in $x$ and strongly-concave in $y$, we have the following lower bound.

**Theorem 3.3.** Let $n \geq 2$ be a positive integer and $L, \mu_x, \mu_y, R_x, R_y, \varepsilon$ be positive parameters. Assume additionally that $\kappa_x = L/\mu_x \geq 2$, $\kappa_y = L/\mu_y \geq 2$, $\kappa_x \leq \kappa_y$ and $\varepsilon \leq \min \left\{ n^2 \mu_x R_x^2 / 1000 \kappa_x \kappa_y, \mu_y R_y^2 / 1000, \frac{L R_x^2}{4} \right\}$. Then we have

$$m^\text{CC}_\varepsilon(R_x, R_y, L, \mu_x, \mu_y) = \begin{cases} \Omega \left( \left(n + \sqrt{\frac{\kappa_x \kappa_y}{\mu_y}}\right) \log \left(\frac{1}{\varepsilon}\right) \right), & \text{for } \kappa_x, \kappa_y = \Omega(n), \\ \Omega \left( \left(n + \sqrt{\frac{\kappa_x \kappa_y}{\mu_y}}\right) \log \left(\frac{1}{\varepsilon}\right) \right), & \text{for } \kappa_y = \Omega(n), \kappa_x = O(n), \\ \Omega(n), & \text{for } \kappa_x, \kappa_y = O(n). \end{cases}$$

The best known upper bound complexity in this case for IFO/PIFO algorithms is $O \left( \left(n + \frac{\sqrt{\mu_y} L}{\min \{\mu_x, \mu_y\}}\right) \log(1/\varepsilon) \right) [22]$. There still exists a $\sqrt{n}$ gap to our lower bound.

Next we give the lower bound when the objective function is not strongly-convex in $x$.

**Theorem 3.4.** Let $n \geq 2$ be a positive integer and $L, \mu_y, R_x, R_y, \varepsilon$ be positive parameters. Assume additionally that $L/\mu_y \geq 2$ and $\varepsilon \leq \min \left\{ \frac{L R_x^2}{4}, \frac{\mu_y R_y^2}{30} \right\}$. Then we have

$$m^\text{CC}_\varepsilon(R_x, R_y, L, 0, \mu_y) = \Omega \left( n + \frac{R_x \sqrt{n L}}{\varepsilon} + \frac{R_y L}{\sqrt{\mu_y} \varepsilon} \right).$$

For the general convex-concave case, we have the following lower bound.

**Theorem 3.5.** Let $n \geq 2$ be a positive integer and $L, R_x, R_y, \varepsilon$ be positive parameters. Assume additionally that $\varepsilon \leq \frac{L}{\Delta} \min \{R_x^2, R_y^2\}$. Then we have

$$m^\text{CC}_\varepsilon(R_x, R_y, L, 0, 0) = \Omega \left( n + \frac{R_x R_y}{\varepsilon} + \frac{R_x + R_y}{\Delta} \sqrt{\frac{n L}{\varepsilon}} \right).$$

Finally, we give the lower bound when the objective function is not convex in $x$ but strongly-concave in $y$.

**Theorem 3.6.** Let $n \geq 2$ be a positive integer and $L, \mu_x, \mu_y, \Delta, \varepsilon$ be positive parameters. Assume additionally that $\varepsilon^2 \leq \frac{\Delta L^2 \alpha}{2 \mu_x \mu_y}$, where $\alpha = \min \left\{ 1, \frac{8(\sqrt{3}+1)n^2 \mu_x \mu_y}{45L^2}, \frac{n^2 \mu_y}{90L} \right\}$. Then we have

$$m^\text{NCC}_\varepsilon(\Delta, L, \mu_x, \mu_y) = \Omega \left( n + \frac{\Delta L^2 \sqrt{\alpha}}{n \mu_y \varepsilon^2} \right).$$

**Remark 3.7.** For $\kappa_y = L/\mu \geq n^2/90$, we have

$$\Omega \left( n + \frac{\Delta L^2 \sqrt{\alpha}}{n \mu_y \varepsilon^2} \right) = \Omega \left( n + \frac{\Delta L}{\varepsilon^2} \min \left\{ \sqrt{\kappa_y}, \sqrt{\frac{\mu_y}{\mu}} \right\} \right).$$
3.2.2 Average Smooth Cases

Then we extend our results to the weaker assumption: the function class \( \{f_i\}_{i=1}^n \) is \( L' \)-average smooth \([11]\). We start with the case where the objective function \( f \) is strongly-convex in \( x \) and strongly-concave in \( y \).

**Theorem 3.8.** Let \( n \geq 4 \) be a positive integer and \( L', \mu_x, \mu_y, R_x, R_y, \varepsilon \) be positive parameters. Assume additionally that \( \kappa'_y = L'/\mu_y \geq 2 \), \( \kappa'_y = L'/\mu_y \geq 2 \), \( \kappa'_x \leq \kappa'_y \) and \( \varepsilon \leq \min \left\{ \frac{\mu_y R_y^2}{1000}, \frac{L'R_x^2}{4} \right\} \).

Then we extend our results to the weaker assumption: the function class \( \{f_i\}_{i=1}^n \) is \( L' \)-average smooth. We start with the case where the objective function \( f \) is strongly-convex in \( x \) and strongly-concave in \( y \).

Then we have

\[
m'_\varepsilon^{CC}(R_x, R_y, L', \mu_x, \mu_y) = \begin{cases} 
\Omega \left( (n+\sqrt{\kappa'_x R_y^2 n}) \log(1/\varepsilon) \right), & \text{for } \kappa'_x, \kappa'_y = \Omega(\sqrt{n}), \\
\Omega \left( (n+n^{3/4} \sqrt{\kappa'_y}) \log(1/\varepsilon) \right), & \text{for } \kappa'_y = \Omega(\sqrt{n}), \kappa'_x = \mathcal{O}(\sqrt{n}), \\
\Omega(n), & \text{for } \kappa'_x, \kappa'_y = \mathcal{O}(\sqrt{n}).
\end{cases}
\]

We remark that the upper bound of Accelerated SVRG/SAGA \([29]\) is \( \mathcal{O} \left( (n + \frac{\sqrt{nL}}{\min\{\mu_x, \mu_y\}}) \log(1/\varepsilon) \right) \).

Finally, we give the lower bound when the objective function is not strongly-convex in \( x \).

**Theorem 3.9.** Let \( n \geq 4 \) be a positive integer and \( L', \mu_x, \mu_y, R_x, R_y, \varepsilon \) be positive parameters. Assume additionally that \( L'/\mu_y \geq 2 \) and \( \varepsilon \leq \min \left\{ \frac{L'R_x^2}{3}, \frac{\mu_y R_y^2}{36} \right\} \). Then we have

\[
m'_\varepsilon^{CC}(R_x, R_y, L', 0, \mu_y) = \Omega \left( n + \frac{R_x n^{3/4}}{\varepsilon} + R_x L' \sqrt{\frac{n}{\mu_y \varepsilon}} \right).
\]

For the general convex-concave case, we have the following lower bound.

**Theorem 3.10.** Let \( n \geq 4 \) be a positive integer and \( L', R_x, R_y, \varepsilon \) be positive parameters. Assume additionally that \( \varepsilon \leq \frac{L'}{4} \min\{R_x^2, R_y^2\} \). Then we have

\[
m'_\varepsilon^{CC}(R_x, R_y, L', 0, 0) = \Omega \left( n + \frac{\sqrt{n} L' R_y R_x}{\varepsilon} + (R_x + R_y) n^{3/4} \sqrt{\frac{L'}{\varepsilon}} \right).
\]

For \( \varepsilon = \mathcal{O} \left( \frac{L' R_y^2}{\sqrt{n} (R_x + R_y)^2} \right) \), our lower bound is \( \Omega \left( n + \frac{\sqrt{n} L' R_y R_x}{\varepsilon} \right) \), which matches the upper bound \( \mathcal{O} \left( n + \frac{\sqrt{n} L' R_y R_x}{\varepsilon} \right) \) of Alacaoglu and Malitsky \([2]\) in terms of \( n \), \( L' \) and \( \varepsilon \).

Finally, we give the lower bound when the objective function is not convex in \( x \) but strongly-concave in \( y \).

**Theorem 3.11.** Let \( n \geq 4 \) be a positive integer and \( L', \mu_x, \mu_y, R_x, R_y, \varepsilon \) be positive parameters. Assume additionally that \( \varepsilon^2 = \frac{\Delta L'^2 \alpha}{435456 n \mu_y} \), where \( \alpha = \min \left\{ 1, \frac{128(\sqrt{3}+1)n \mu_x \mu_y}{45 L^2}, \frac{32 n \mu_y}{135 L^2} \right\} \). Then we have

\[
m'_\varepsilon^{NCC}(\Delta, L', \mu_x, \mu_y) = \Omega \left( n + \frac{\Delta L'^2 \sqrt{\alpha}}{\mu_y \varepsilon^2} \right).
\]

**Remark 3.12.** For \( \kappa'_y = L'/\mu_y \geq 32n/135 \), we have

\[
\Omega \left( n + \frac{\Delta L'^2 \sqrt{\alpha}}{\mu_y \varepsilon^2} \right) = \Omega \left( n + \frac{\Delta L' \sqrt{n}}{\varepsilon^2} \min \left\{ \sqrt{\kappa'_y}, \sqrt{\mu_x / \mu_y} \right\} \right).
\]

10
3.3 Framework of Construction

To demonstrate the construction of adversarial functions, we first introduce the following class of matrices, which is also used in proof of lower bounds in deterministic minimax optimization [28, 39]:

\[
\mathbf{B}(m, \omega, \zeta) = \begin{bmatrix}
\omega & -1 & & \\
1 & -1 & & \\
& & \ddots & \\
& & & 1 & -1 \\
& & & & \zeta
\end{bmatrix} \in \mathbb{R}^{(m+1) \times m}.
\]

Denote the \(l\)-th row of the matrix \(\mathbf{B}(m, \omega, \zeta)\) by \(\mathbf{b}_{l-1}(m, \omega, \zeta)^T\). We will partition the row vectors \(\{\mathbf{b}_l(m, \omega, \zeta)^T\}_{l=0}^m\) by index sets \(\mathcal{L}_1, \ldots, \mathcal{L}_n\), where \(\mathcal{L}_i = \{l : 0 \leq l \leq m, l \equiv i - 1 \pmod{n}\}\). For the general convex-concave case and the nonconvex-strongly-concave case, the constructions are slightly different. So the following analysis is divided into two parts referred to Sections 3.3.1 and 3.3.2.

3.3.1 Convex-Concave Case

The adversarial problem for the convex-concave case is constructed as

\[
\min_{\mathbf{x} \in \mathcal{X}} \max_{\mathbf{y} \in \mathcal{Y}} \tilde{r}(\mathbf{x}, \mathbf{y}; m, \zeta, \bar{c}) \triangleq \frac{1}{n} \sum_{i=1}^n \tilde{r}_i(\mathbf{x}, \mathbf{y}; m, \zeta, \bar{c}),
\]

where \(\bar{c} = (\bar{c}_1, \bar{c}_2)\), \(\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_2 \leq R_x\}\), \(\mathcal{Y} = \{\mathbf{y} \in \mathbb{R}^m : \|\mathbf{y}\|_2 \leq R_y\}\),

\[
\tilde{r}_i(\mathbf{x}, \mathbf{y}; m, \zeta, \bar{c}) = \begin{cases}
\frac{1}{n} \sum_{l \in \mathcal{L}_i} \mathbf{e}_l \mathbf{b}_l(m, 0, \zeta)^T \mathbf{x} + \frac{\bar{c}_1}{2} \|\mathbf{x}\|_2^2 - \frac{\bar{c}_2}{2} \|\mathbf{y}\|_2^2 - n (\mathbf{e}_1, \mathbf{x}), & \text{for } i = 1, \\
\frac{1}{n} \sum_{l \in \mathcal{L}_i} \mathbf{e}_l \mathbf{b}_l(m, 0, \zeta)^T \mathbf{x} + \frac{\bar{c}_1}{2} \|\mathbf{x}\|_2^2 - \frac{\bar{c}_2}{2} \|\mathbf{y}\|_2^2, & \text{for } i = 2, 3, \ldots, n,
\end{cases}
\]

and \(\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}\) is the standard basis of \(\mathbb{R}^m\). And we remark that \(\mathbf{b}_0(m, 0, \zeta) = \mathbf{0}_m\). Then we can determine the smooth and strongly-convex coefficients of \(\tilde{r}_i\) as follows.

**Proposition 3.13.** For \(\bar{c}_1, \bar{c}_2 \geq 0\) and \(0 \leq \zeta \leq \sqrt{2}\), we have that \(\tilde{r}_i\) is \(L\)-smooth and \((\bar{c}_1, \bar{c}_2)\)-convex-concave, and \(\{\tilde{r}_i\}_{i=1}^n\) is \(L'\)-average smooth, where

\[
L = \sqrt{4n^2 + 2 \max\{\bar{c}_1, \bar{c}_2\}^2}, \quad \text{and} \quad L' = \sqrt{8n + 2 \max\{\bar{c}_1, \bar{c}_2\}^2}.
\]

Define the subspaces \(\{\mathcal{F}_k\}_{k=0}^m\) as

\[
\mathcal{F}_k = \begin{cases}
\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}, & \text{for } 1 \leq k \leq m, \\
\{\mathbf{0}_m\}, & \text{for } k = 0.
\end{cases}
\]

The following technical lemma plays a crucial role in our proof.

**Lemma 3.14.** Suppose that \(n \geq 2\) and \(\mathcal{F}_{-1} = \mathcal{F}_0\). Then for \((\mathbf{x}, \mathbf{y}) \in \mathcal{F}_k \times \mathcal{F}_{k-1}\) and \(0 \leq k < m\), we have that

\[
\nabla \tilde{r}_i(\mathbf{x}, \mathbf{y}), \text{prox}^\gamma_{\tilde{r}_i}(\mathbf{x}, \mathbf{y}) \in \begin{cases}
\mathcal{F}_{k+1} \times \mathcal{F}_k, & \text{if } i \equiv k + 1 \pmod{n}, \\
\mathcal{F}_k \times \mathcal{F}_{k-1}, & \text{otherwise},
\end{cases}
\]

where we omit the parameters of \(\tilde{r}_i\) to simplify the presentation.
Figure 1: An illustration of the process of solving the Problem (5) with a PIFO algorithm $A$.

The proofs of Proposition 3.13 and Lemma 3.14 are given in Appendix Section C.

When we apply a PIFO algorithm $A$ to solve the Problem (5), Lemma 3.14 implies that $x_t = y_t = 0_m$ will hold until algorithm $A$ draws the component $f_1$. Then, for any $t < T_1 = \min \{ t : i_t = 1 \}$, we have $x_t, y_t \in F_0$ while $x_{T_1} \in F_1$ and $y_{T_1} \in F_0$ hold. The value of $T_1$ can be regarded as the smallest integer such that $x_{T_1} \in F_1$ could hold. Similarly, for $T_1 \leq t < T_2 = \min \{ t > T_1 : i_t = 2 \}$ there holds $x_t \in F_1$ and $y_t \in F_0$ while we can ensure that $x_{T_2} \in F_2$ and $y_{T_2} \in F_1$. Figure 1 illustrates this optimization process.

We can define $T_k$ to be the smallest integer such that $x_{T_k} \in F_k$ and $y_{T_k} \in F_{k-1}$ could hold.

We give the formal definition of $T_k$ recursively and connect it to geometrically distributed random variables in the following corollary.

**Corollary 3.15.** Assume we employ a PIFO algorithm $A$ to solve the Problem (5). Let

$$T_0 = 0, \quad \text{and } T_k = \min_t \{ t : t > T_{k-1}, i_t \equiv k \pmod{n} \} \quad \text{for } k \geq 1.$$

Then we have

$$(x_t, y_t) \in F_{k-1} \times F_{k-2}, \quad \text{for } t < T_k, k \geq 1.$$

Moreover, the random variables $\{ Y_k \}_{k \geq 1}$ such that $Y_k \triangleq T_k - T_{k-1}$ are mutual independent and $Y_k$ follows a geometric distribution with success probability $p_k$ where $k' \equiv k \pmod{n}$ and $l \in [n]$.

**Proof.** Assume that $(x_t, y_t) \in F_{k-1} \times F_{k-2}$ for some $k \geq 1$ and $t < T$. Following from Lemma 3.14, then for any $t < T$, we have

$$\nabla r_{i_T}^2(x_t, y_t), \text{prox}^2_{r_{i_T}}(x_t, y_t) \in \begin{cases} F_k \times F_{k-1}, & \text{if } i_T \equiv k \pmod{n}, \\ F_{k-1} \times F_{k-2}, & \text{otherwise.} \end{cases}$$

Hence we know that

$$\text{span} \left\{ (x_0, y_0), \ldots, (x_{T-1}, y_{T-1}), \nabla r_{i_T}^2(x_0, y_0), \ldots, \nabla r_{i_T}^2(x_{T-1}, y_{T-1}), \right.$$  

$$\left. \text{prox}^2_{r_{i_T}}(x_0, y_0), \ldots, \text{prox}^2_{r_{i_T}}(x_{T-1}, y_{T-1}) \right\}$$

$$\subseteq \begin{cases} F_k \times F_{k-1}, & \text{if } i_T \equiv k \pmod{n}, \\ F_{k-1} \times F_{k-2}, & \text{otherwise.} \end{cases}$$

Therefore, by the definition of PIFO algorithm and Lemma B.2 related to projection operator, it is clear that

$$(x_T, y_T) \in \begin{cases} F_k \times F_{k-1}, & \text{if } i_T \equiv k \pmod{n}, \\ F_{k-1} \times F_{k-2}, & \text{otherwise.} \end{cases}$$

Consequently, when $t < T' \triangleq \min \{ t : t \geq T, i_t \equiv k \pmod{n} \}$, there also holds $(x_t, y_t) \in F_{k-1} \times F_{k-2}$. Moreover, we can ensure that $(x_{T'}, y_{T'}) \in F_k \times F_{k-1}$. Based on this fact, the desired result just follows from induction and $(x_0, y_0) = (0_m, 0_m) \in F_0 \times F_{-1}$.
Next, note that
\[
\mathbb{P} [T_k - T_{k-1} = s] = \mathbb{P} [i_{T_{k-1}+1} \not\equiv k \pmod{n}, \ldots, i_{T_{k-1}+s-1} \not\equiv k \pmod{n}, i_{T_{k-1}+s} \equiv k \pmod{n}] = \mathbb{P} [i_{T_{k-1}+1} \not\equiv k', \ldots, i_{T_{k-1}+s-1} \not\equiv k', i_{T_{k-1}+s} = k'] = (1 - p_k)^s - 1 p_{k'},
\]
where $k' \equiv k \pmod{n}$, $1 \leq k' \leq n$. So $Y_k = T_k - T_{k-1}$ is a geometric random variable with success probability $p_{k'}$. The independence of $\{Y_k\}_{k \geq 1}$ is just according to the independence of $\{i_t\}_{t \geq 1}$.

The basic idea of our analysis is that we guarantee that the $\varepsilon$-suboptimal solution or $\varepsilon$-stationary point of Problem (3) does not lie in $F_k \times F_k$ for $k < m$ and assure that the PIFO algorithm extends the space of span$\{(x_0, y_0), (x_1, y_1), \ldots, (x_t, y_t)\}$ slowly with $t$ increasing. By Corollary 3.15, we know that span$\{(x_0, y_0), (x_1, y_1), \ldots, (x_{T_k-1}, y_{T_k-1})\} \subseteq F_{k-1} \times F_{k-1}$. Hence, $T_k$ is just the quantity that measures how span$\{(x_0, y_0), (x_1, y_1), \ldots, (x_t, y_t)\}$ expands. Note that $T_k$ can be written as the sum of geometrically distributed random variables. Recalling Lemma 2.11, we can obtain how many PIFO calls we need.

**Lemma 3.16.** If $M$ satisfies $1 \leq M < m$,
\[
\min_{x \in X \cap F_M \atop y \in Y \cap F_M} \left( \max_{v \in Y} \tilde{r}(x, v) - \min_{u \in X} \tilde{r}(u, y) \right) \geq 9 \varepsilon
\]
and $N = n(M + 1)/4$, then we have
\[
\min_{t \leq N} \mathbb{E} \left( \max_{v \in Y} \tilde{r}(x_t, v) - \min_{u \in X} \tilde{r}(u, y_t) \right) \geq \varepsilon,
\]
where $X, Y$ are arbitrary convex sets.

**Proof.** For $t \leq N$, we have
\[
\mathbb{E} \left( \max_{v \in Y} \tilde{r}(x_t, v) - \min_{u \in X} \tilde{r}(u, y_t) \right) \geq \mathbb{E} \left( \max_{v \in Y} \tilde{r}(x_t, v) - \min_{u \in X} \tilde{r}(u, y_t) \right) \mathbb{P} [N < T_{M+1}] \mathbb{P} [N < T_{M+1}] \geq 9 \varepsilon \mathbb{P} [N < T_{M+1}],
\]
where $T_{M+1}$ is defined in (6), and the second inequality follows from Corollary 3.15 (if $N < T_{M+1}$, then $x_t \in F_M$ and $y_t \in F_{M-1} \subset F_M$ for $t \leq N$).

By Corollary 3.15 $T_{M+1}$ can be written as $T_{M+1} = \sum_{t=1}^{M+1} Y_t$, where $\{Y_t\}_{1 \leq t \leq M+1}$ are independent random variables, and $Y_t$ follows a geometric distribution with success probability $q_t = p_{l'}$ ($l' \equiv l \pmod{n}$, $1 \leq l' \leq n$). Moreover, recalling that $p_1 \leq p_2 \leq \cdots \leq p_n$, we have $\sum_{t=1}^{M+1} q_t \leq \frac{M+1}{n}$. Therefore, by Lemma 2.11, we have
\[
\mathbb{P} [T_{M+1} > N] = \mathbb{P} \left[ \sum_{t=1}^{M+1} Y_t > \frac{(M+1)n}{4} \right] \geq \frac{1}{9},
\]
which implies our desired result.
Figure 2: An illustration of the process of solving the Problem (7) with a PIFO algorithm A.

### 3.3.2 Nonconvex-Strongly-Concave Case

For the nonconvex-strongly-concave case, the adversarial problem is constructed as

$$\min_{\mathbf{x} \in \mathbb{R}^m} \max_{\mathbf{y} \in \mathbb{R}^m} \hat{r}(\mathbf{x}, \mathbf{y}; m, \omega, \hat{c}) \triangleq \frac{1}{n} \sum_{i=1}^{n} \hat{r}_i(\mathbf{x}, \mathbf{y}; m, \omega, \hat{c})$$  \hspace{1cm} (7)

where $\hat{c} = (\hat{c}_1, \hat{c}_2, \hat{c}_3)$,

$$\hat{r}_i(\mathbf{x}, \mathbf{y}; m, \omega, \hat{c}) = \begin{cases} \sum_{t \in L_i} \mathbf{y}^T \mathbf{e}_{i+1} \mathbf{b}_i(m, \omega, 0)^T \mathbf{x} - \frac{c_1}{2} \| \mathbf{y} \|_2^2 + \hat{c}_2 \sum_{i=1}^{m-1} \Gamma(\hat{c}_3 x_i) - n (\mathbf{e}_1, \mathbf{y}) \text{, for } i = 1, \\ \sum_{t \in L_i} \mathbf{y}^T \mathbf{e}_{i+1} \mathbf{b}_i(m, \omega, 0)^T \mathbf{x} - \frac{c_1}{2} \| \mathbf{y} \|_2^2 + \hat{c}_2 \sum_{i=1}^{m-1} \Gamma(\hat{c}_3 x_i), \text{ for } i = 2, 3, \ldots, n, \end{cases}$$

and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ is the standard basis of $\mathbb{R}^m$. The non-convex function $\Gamma : \mathbb{R} \to \mathbb{R}$ is

$$\Gamma(x) \triangleq 120 \int_{1}^{x} \frac{t^2(t-1)}{1+t^2} dt,$$

which was introduced by Carmon et al. [5]. We remark that $\mathbf{b}_m(m, \omega, 0) = \mathbf{0}_m$, and $\mathbf{e}_{m+1}$ is indifferent in the definition of $\hat{r}$. Then we can determine the smooth and strongly-convex coefficients of $\hat{r}_i$ as follows.

**Proposition 3.17.** For $\hat{c}_1 \geq 0$, $\hat{c}_2, \hat{c}_3 > 0$ and $0 \leq \omega \leq \sqrt{2}$, we have that $\hat{r}_i$ is $L$-smooth and $(-45(\sqrt{3} - 1)\hat{c}_2\hat{c}_3^2, \hat{c}_1)$-convex-concave, and $\{\hat{r}_i\}_{i=1}^{n}$ is $L'$-average smooth, where

$$L = \sqrt{4n^2 + 2\hat{c}_1^2 + 180\hat{c}_2^2\hat{c}_3^2} \text{ and } L' = 2\sqrt{4n + \hat{c}_1^2 + 16200\hat{c}_3^2\hat{c}_3^4}.$$ 

The following technical lemma plays a crucial role in our proof.

**Lemma 3.18.** Suppose that $n \geq 2$, $\hat{c}_2, \hat{c}_3 > 0$ and $\gamma < \frac{\sqrt{2} + 1}{600\hat{c}_3}$, $\mathbf{F}_k \times \mathcal{F}_k$ and $0 \leq k < m-1$, we have that

$$\nabla \hat{r}_i(\mathbf{x}, \mathbf{y}), \text{ prox}_\hat{r}_i(\mathbf{x}, \mathbf{y}) \in \begin{cases} \mathcal{F}_{k+1} \times \mathcal{F}_{k+1}, & \text{if } i \equiv k + 1 \text{ (mod } n), \\ \mathcal{F}_k \times \mathcal{F}_k, & \text{otherwise}, \end{cases}$$

where we omit the parameters of $\hat{r}_i$ to simplify the presentation.

The proofs of Proposition 3.17 and Lemma 3.18 are given in Appendix Section C.

When we apply a PIFO algorithm to solve the Problem (7), the optimization process is similar to the process related to the Problem (5). We demonstrate the optimization process in Figure 2 and present a formal statement in following corollary.
Corollary 3.19. Assume we employ a PIFO algorithm $A$ to solve the Problem (7). Let
\[ T_0 = 0, \quad \text{and} \quad T_k = \min_t \{ t : t > T_{k-1}, i_t \equiv k \mod n \} \quad \text{for} \ k \geq 1. \]

Then we have
\[ (x_t, y_t) \in F_{k-1} \times F_{k-1}, \quad \text{for} \ t < T_k, k \geq 1. \]

Moreover, the random variables \( \{Y_k\}_{k \geq 1} \) such that \( Y_k \overset{\text{d}}{=} T_k - T_{k-1} \) are mutually independent and \( Y_k \) follows a geometric distribution with success probability \( p_k \) wherever \( k \equiv k \mod n \) and \( l \in [n] \).

The proof of Corollary 3.19 is similar to that of Corollary 3.15. Furthermore, the prime-dual gap in Lemma 3.10 can be replaced with the gradient norm in the nonconvex-strongly-concave case.

Lemma 3.20. Let \( \phi_f(x) \overset{\text{d}}{=} \max_{y \in \mathbb{R}^m} \tilde{r}(x, y) \). If \( M \) satisfies \( 1 \leq M < m \) and
\[ \min_{x \in F_M} \| \phi_f(x) \|_2 \geq 9\varepsilon \]
and \( N = n(M + 1)/4 \), then we have
\[ \min_{\tilde{t} \leq N} \mathbb{E} \| \phi_f(x_{\tilde{t}}) \|_2 \geq \varepsilon. \]

### 3.4 Construction for the Strongly-Convex-Strongly-Concave Case

We first consider the finite-sum minimax problem where the objective function is strongly-convex in \( x \) and strongly-concave in \( y \).

Without loss of generality, we assume \( \mu_x \geq \mu_y \). Denote \( \kappa_x = L/\mu_x \) and \( \kappa_y = L/\mu_y \). Then we have \( \kappa_y \geq \kappa_x \). The construction can be divided into three parts referred to Sections 3.4.1, 3.4.2 and 3.4.3.

#### 3.4.1 \( \kappa_x, \kappa_y = \Omega(n) \)

For the case \( \kappa_x, \kappa_y = \Omega(n) \), the analysis depends on the following construction.

**Definition 3.21.** For fixed \( L, \mu_x, \mu_y, R_x, R_y \) and \( n \) such that \( \mu_x \geq \mu_y \), \( \kappa_x = L/\mu_x \geq 2 \) and \( \kappa_y = L/\mu_y \geq 2 \), we define \( f_{SCSC,i} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) as follows
\[ f_{SCSC,i}(x, y) = \lambda \tilde{r}_i \left( \frac{x}{\beta}, \frac{y}{\beta}; m, \sqrt{\frac{2}{\alpha + 1}}, \tilde{c} \right), \quad \text{for} \ 1 \leq i \leq n, \]

where
\[ \alpha = \sqrt{\frac{(\kappa_x - 2/\kappa_x) \kappa_y}{n^2}} + 1, \quad \tilde{c} = \left( \frac{2n}{\sqrt{\kappa_x^2 - 2}}, \frac{2n\kappa_x}{\kappa_y \sqrt{\kappa_x^2 - 2}} \right), \]
\[ \beta = \min \left\{ \frac{2nR_x}{\kappa_x^2 - 2}, \frac{2nR_x}{\alpha + 1}, \frac{2\alpha R_y}{\kappa_y^2 - 2}, \frac{2\alpha R_y}{\alpha - 1} \right\} \quad \text{and} \quad \lambda = \frac{\beta^2}{2n} \sqrt{L^2 - 2\mu_x^2}. \]

Consider the minimax problem
\[ \min_{x \in X} \max_{y \in Y} f_{SCSC}(x, y) \overset{\Delta}{=} \frac{1}{n} \sum_{i=1}^{n} f_{SCSC,i}(x, y). \quad (8) \]

where \( X = \{ x \in \mathbb{R}^m : \| x \|_2 \leq R_x \} \) and \( Y = \{ y \in \mathbb{R}^m : \| y \|_2 \leq R_y \} \). Define \( \phi_{SCSC}(x) = \max_{y \in Y} f_{SCSC}(x, y) \) and \( \psi_{SCSC}(y) = \min_{x \in X} f_{SCSC}(x, y) \).
Then we have the following proposition.

**Proposition 3.22.** For any \( n \geq 2, m \geq 2, f_{SCSC,i} \) and \( f_{SCSC} \) in Definition 3.21 satisfy:

1. \( f_{SCSC,i} \) is \( L \)-smooth and \((\mu_x, \mu_y)\)-convex-concave. Thus, \( f_{SCSC} \) is \((\mu_x, \mu_y)\)-convex-concave.

2. The saddle point of Problem (8) is

\[
\begin{align*}
\{\begin{array}{l}
x^* = \frac{2n\beta \mu_y}{(1-q)\sqrt{L^2-2\mu_x^2}}(q, q^2, \ldots, q^m)^\top, \\
y^* = \beta \left(q, q^2, \ldots, q^{m-1}, \sqrt{\frac{\alpha+1}{2}q} \right)^\top,
\end{array}
\end{align*}
\]

where \( q = \frac{\alpha-1}{\alpha+1} \). Moreover, \( \|x^*\|_2 \leq R_x, \|y^*\|_2 \leq R_y \).

3. For \( 1 \leq k \leq m - 1 \), we have

\[
\min_{x \in X \cap F_k} \phi_{SCSC}(x) - \max_{y \in Y \cap F_k} \psi_{SCSC}(y) \geq \frac{\beta^2 \left(L^2 - 2\mu_x^2\right) q^{2k}}{4n^2(\alpha + 1)\mu_x}.
\]

The proof of Proposition 3.22 is given in Appendix Section C.

We can now prove the lower bound complexity for finding \( \varepsilon \)-suboptimal point of Problem (8) by PIFO algorithms.

**Theorem 3.23.** Consider the minimax problem (8) and \( \varepsilon > 0 \). Let \( \kappa_x = L/\mu_x, \kappa_y = L/\mu_y \) and \( \alpha = \sqrt{\frac{(\kappa_x - 2/\kappa_x)\kappa_x}{n^2}} + 1 \). Suppose that

\[
n \geq 2, \kappa_y \geq \kappa_x \geq \sqrt{n^2 + 2}, \varepsilon \leq 1600 \min \left\{ \frac{n^2 \mu_x R_x^2}{\kappa_x \kappa_y}, \mu_y R_y^2 \right\},
\]

and \( m = \left\lceil \frac{\alpha}{4} \log \left( \max \left\{ \mu_x R_x^2, \mu_y R_y^2 \right\} \right) \right\rfloor + 1 \).

In order to find \((\hat{x}, \hat{y}) \in X \times Y \) such that \( \mathbb{E}\phi_{SCSC}(\hat{x}) - \mathbb{E}\psi_{SCSC}(\hat{y}) < \varepsilon \), PIFO algorithm \( A \) needs at least \( N \) queries, where

\[
N = \Omega \left( (n + \sqrt{\kappa_x \kappa_y}) \log \left( \frac{1}{\varepsilon} \right) \right).
\]

**Proof.** Let \( q = \frac{\alpha-1}{\alpha+1} \). For \( \kappa_y \geq \kappa_x \geq \sqrt{n^2 + 2} \), we have \( \alpha = \sqrt{\frac{(\kappa_x - 2/\kappa_x)\kappa_x}{n^2}} + 1 \geq \sqrt{2}, \ k = \frac{\alpha-1}{\alpha+1} \geq \frac{\sqrt{2} - \frac{1}{\sqrt{2} + 1}}{\sqrt{2} + 1} \) and \( \kappa_x - 2/\kappa_x \geq \kappa_x/2 \).

Denoting \( M = \left\lceil \frac{\log(9(\alpha+1)\mu_x \varepsilon / \beta^2 \xi^2)}{2 \log q} \right\rceil \) where \( \xi = \sqrt{\frac{L^2-2\mu_x^2}{2n}} \), we have

\[
\min_{x \in X \cap F_M} \phi_{SCSC}(x) - \max_{y \in Y \cap F_M} \psi_{SCSC}(y) \geq \frac{\beta^2 \xi^2}{(\alpha + 1)\mu_x} q^{2M} \geq 9\varepsilon.
\]

where the first inequality follows from the third property of Proposition 3.22.

First, we need to ensure \( 1 \leq M < m \). Note that \( M \geq 1 \) is equivalent to \( \varepsilon \leq \frac{\beta^2 \xi^2}{9(\alpha+1)\mu_x} \). Recall that

\[
\beta = \min \left\{ \frac{2n R_x}{\sqrt{\kappa_x^2 - 2}}, \frac{2n R_x}{\sqrt{\kappa_y^2 - 2}}, \frac{2\alpha}{\alpha + 1}, \frac{\sqrt{2\alpha} R_y}{\alpha - 1} \right\}.
\]
When $\beta = 2nR_x \sqrt{\frac{\alpha}{\kappa - 2}}$, we have

$$\frac{q^2 \beta^2 \xi^2}{9(\alpha + 1)\mu_x} = \frac{\alpha(\alpha - 1)^2}{9(\alpha + 1)^3} \mu_x R_x^2 \geq \frac{\sqrt{2}(\sqrt{2} - 1)^5}{9} \mu_x R_x^2.$$

When $\beta = \frac{2nR_x}{\alpha + 1} \sqrt{\frac{2\alpha}{\kappa_x - 2}}$, recalling that $\alpha^2 - 1 = \frac{(\kappa_x - 2)\kappa_y}{\kappa^2} \leq \frac{\kappa_x \kappa_y}{\kappa^2}$, we have

$$\frac{q^2 \beta^2 \xi^2}{9(\alpha + 1)\mu_x} = \frac{2\alpha(\alpha - 1)^3}{9(\alpha + 1)^5} \mu_x R_x^2 \geq \frac{2\sqrt{2}(\sqrt{2} - 1)^7}{9 n^2 \mu_x R_x^2} \frac{n^2 \mu_x R_x^2}{\kappa_x \kappa_y}.$$

When $\beta = \frac{\sqrt{2} \alpha R_x}{\alpha - 1}$, recalling that $\frac{\mu_x \mu_y}{\xi^2} = \frac{4}{\alpha^2 - 1}$, we have

$$\frac{q^2 \beta^2 \xi^2}{9(\alpha + 1)\mu_x} = \frac{\alpha(\alpha - 1)}{18(\alpha + 1)^3} \mu_y R_y^2 \geq \frac{\sqrt{2}(\sqrt{2} - 1)^3}{18} \mu_y R_y^2.$$

Thus, $\varepsilon \leq 1000 \min \left\{ \frac{n^2 \mu_x R_x^2}{\kappa_x \kappa_y}, \frac{n^2 \mu_y R_y^2}{\kappa_x \kappa_y} \right\}$ is a sufficient condition for $M \geq 1$. On the other hand, we have

$$\frac{q^2 \beta^2 \xi^2}{9(\alpha + 1)\mu_x \varepsilon} \leq \min \left\{ \frac{\alpha(\alpha - 1)}{9(\alpha + 1)^3} \mu_x R_x^2, \frac{2\alpha(\alpha - 1)^3}{9(\alpha + 1)^5} \mu_x R_x^2, \frac{\alpha(\alpha - 1)}{18(\alpha + 1)^3} \mu_y R_y^2 \right\}
\leq \frac{1}{9} \min \{ \mu_x R_x^2, \mu_y R_y^2 \}.$$

Note that the function $h(\beta) = \frac{1}{\log\left(\frac{\alpha + 1}{\beta - 1}\right)} - \frac{\beta}{2}$ is increasing when $\beta > 1$ and $\lim_{\beta \to +\infty} h(\beta) = 0$. Thus there holds

$$\frac{\alpha}{2} + h(\sqrt{2}) \leq -\frac{1}{\log q} \leq \frac{\alpha}{2}.$$

Then we have

$$m = \left\lfloor \frac{\alpha}{4} \log \left( \frac{\max \{ \mu_x R_x^2, \mu_y R_y^2 \}}{9 \varepsilon} \right) \right\rfloor + 1 \geq \left\lfloor -\frac{\log \left( \frac{q^2 \beta^2 \xi^2}{9(\alpha + 1)\mu_x \varepsilon} \right)}{2 \log q} \right\rfloor + 1 > M.$$

By Lemma 3.16, for $M \geq 1$ and $N = (M + 1)n/4$, we have

$$\min_{t \leq N} \mathbb{E}[\phi_{\text{SCSC}}(x_t)] - \min_{t \leq N} \mathbb{E}[\psi_{\text{SCSC}}(y_t)] \geq \varepsilon.$$

Therefore, in order to find $(\hat{x}, \hat{y}) \in X \times Y$ such that $\mathbb{E}[\phi_{\text{SCSC}}(\hat{x})] - \mathbb{E}[\psi_{\text{SCSC}}(\hat{y})] \geq \varepsilon$, $A$ needs at least $N$ PIFO queries.

At last, we can estimate $N$ by

$$\frac{1}{\log(q)} = \frac{1}{\log\left(\frac{\alpha + 1}{\beta - 1}\right)} \geq \frac{\alpha}{2} + h(\sqrt{2})
\geq \frac{1}{2} \sqrt{\frac{(\kappa_x - 2)\kappa_y}{\kappa^2}} + 1 + h(\sqrt{2})
\geq \frac{1}{2} \sqrt{\frac{\kappa_x \kappa_y}{2n^2}} + 1 + h(\sqrt{2}).$$
\[ \geq \frac{\sqrt{2}}{4} \left( \sqrt{\frac{\kappa x \kappa y}{2n^2}} + 1 \right) + h \left( \sqrt{2} \right) \]
\[ \geq \frac{\kappa x \kappa y}{4n} + \frac{\sqrt{2}}{4} + h \left( \sqrt{2} \right), \]

and

\[ N = (M + 1)n/4 \]
\[ \geq \frac{n}{4} \left( \frac{1}{\log(q)} \right) \log \left( \frac{\beta^2 \xi^2}{9(\alpha + 1)\mu_x \varepsilon} \right) \]
\[ \geq \frac{n}{4} \left( \frac{\kappa x \kappa y}{4n} + \frac{\sqrt{2}}{4} + h \left( \sqrt{2} \right) \right) \log \left( \frac{\min \{ n^2 \mu_x R_x^2 / \kappa x \kappa y, \mu_y R_y^2 \} \right) \]
\[ = \Omega \left( (n + \sqrt{n \kappa y}) \log \left( \frac{1}{\varepsilon} \right) \right). \]

This completes the proof.

\[ \square \]

### 3.4.2 \( \kappa_x = O(n), \kappa_y = \Omega(n) \)

For the case \( \kappa_x = O(n), \kappa_y = \Omega(n) \), we have the following result.

**Theorem 3.24.** For any \( L, \mu_x, \mu_y, n, R_x, R_y, \varepsilon \) such that

\[ n \geq 2, \kappa_y \geq \sqrt{n^2 + 2} \geq \kappa_x \geq 2, \varepsilon \leq \frac{1}{120} \mu_y R_y^2, \]

and \( m = \left\lceil \frac{1}{4} \left( \sqrt{\frac{2(\kappa_y - 1)}{n}} + 1 \right) \log \left( \frac{\mu_y R_y^2}{9\varepsilon} \right) \right\rceil + 1, \)

where \( \kappa_x = L/\mu_x \) and \( \kappa_y = L/\mu_y \), there exist \( n \) functions \( \{ f_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \}_{i=1}^n \) such that \( f_i(x, y) \) is \( L \)-smooth and \( f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x, y) \) is \( (\mu_x, \mu_y) \)-convex-concave. Let \( \mathcal{X} = \{ x \in \mathbb{R}^m : \|x\|_2 \leq R_x \} \) and \( \mathcal{Y} = \{ y \in \mathbb{R}^m : \|y\|_2 \leq R_y \} \). In order to find \( (\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \) such that \( \mathbb{E} \max_{y \in \mathcal{Y}} f(\hat{x}, y) - \mathbb{E} \min_{x \in \mathcal{X}} f(x, \hat{y}) < \varepsilon \), PIFO algorithm \( A \) needs at least \( N \) queries, where

\[ N = \Omega \left( (n + \sqrt{n \kappa y}) \log \left( \frac{1}{\varepsilon} \right) \right). \]

**Proof.** Let \( \alpha = \sqrt{\frac{2(\kappa_y - 1)}{n}} + 1. \) Consider the functions \( \{ f_{SC,i} \}_{i=1}^n \) and \( f_{SC} \) defined in Definition 4.20 with \( \mu \) and \( R \) replaced by \( \mu_y \) and \( R_y \). We construct \( \{ G_{SCSC,i} \}_{i=1}^n, G_{SCSC} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) as follows

\[ G_{SCSC,i}(x, y) = \frac{\mu_x}{2} \|x\|_2^2 - f_{SC,i}(y), \]
\[ G_{SCSC}(x, y) = \frac{1}{n} \sum_{i=1}^n G_{SCSC,i}(x, y) = \frac{\mu_x}{2} \|x\|_2^2 - f_{SC}(y). \]

By Proposition 4.15 and Lemma B.1, we can check that each component function \( G_{SCSC,i} \) is \( L \)-smooth and \( (\mu_x, \mu_y) \)-convex-concave. Then \( G_{SCSC} \) is \( (\mu_x, \mu_y) \)-convex-concave. Moreover, we have

\[ \max_{y \in \mathcal{Y}} G_{SCSC}(x, y) = \frac{\mu_x}{2} \|x\|_2^2 - \min_{y \in \mathcal{Y}} f_{SC}(y) \quad \text{and} \quad \min_{x \in \mathcal{X}} G_{SCSC}(x, y) = f_{SC}(y). \]
Moreover, we have component function \( H \). Note that for \( \kappa \)

\[
H \kappa
\]

and satisfies \( \kappa \geq \sqrt{n^2 + 2} \geq n/2 + 1 \). By Theorem 4.22, for

\[
\varepsilon \leq \frac{\mu_y R^2}{18} \left( \frac{\alpha - 1}{\alpha + 1} \right)^2
\]

and \( m = \left[ \frac{1}{4} \left( \sqrt{2(\kappa y - 1) n} + 1 \right) \log \left( \frac{\mu_y R^2}{9 \varepsilon} \right) \right] + 1 \),

in order to find \( (\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \) such that \( \mathbb{E} (\max_{y \in \mathcal{Y}} G_{SCSC}(\hat{x}, y) - \min_{x \in \mathcal{X}} G_{SCSC}(x, \hat{y})) < \varepsilon \), PIFO algorithm \( \mathcal{A} \) needs at least \( N = \Omega \left( (n + \sqrt{n\kappa y}) \log \left( \frac{1}{\varepsilon} \right) \right) \) queries.

Moreover, \( \kappa y \geq n/2 + 1 \) implies \( \alpha \geq \sqrt{2} \). Then we have \( \left( \frac{n - 1}{\alpha + 1} \right)^2 \geq \left( \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right)^2 \geq \frac{1}{\varepsilon^2} \). This completes the proof. \( \square \)

3.4.3 \( \kappa_x, \kappa_y = \mathcal{O}(n) \)

For the case \( \kappa_x, \kappa_y = \mathcal{O}(n) \), we can apply the following lemma.

**Lemma 3.25.** For any \( L, \mu_x, \mu_y, n, R_x, R_y, \varepsilon \) such that \( n \geq 2, L \geq \mu_x, L \geq \mu_y, \mu_x \geq \mu_y \) and \( \varepsilon \leq L R^2 \), there exist \( n \) functions \( \{ f_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \}_{i=1}^n \) such that \( f_i(x, y) \) is \( L \)-smooth and \( f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x, y) \) is \( (\mu_x, \mu_y) \)-convex-concave. Let \( \mathcal{X} = \{ x \in \mathbb{R} : |x| \leq R_x \} \) and \( \mathcal{Y} = \{ y \in \mathbb{R} : |y| \leq R_y \} \). In order to find \( (\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \) such that \( \mathbb{E} \max_{y \in \mathcal{Y}} f(\hat{x}, y) - \mathbb{E} \min_{x \in \mathcal{X}} f(x, \hat{y}) < \varepsilon \), PIFO algorithm \( \mathcal{A} \) needs at least \( N = \Omega(\sqrt{n}) \) queries.

**Proof.** Consider the functions \( \{ H_{SCSC,i} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \}_{i=1}^n \) where

\[
H_{SCSC,i}(x, y) = \begin{cases} 
\frac{L}{2} (x^2 - y^2) - n L R_x x, & \text{for } i = 1, \\
\frac{L}{2} (x^2 - y^2), & \text{otherwise},
\end{cases}
\]

and \( H_{SCSC}(x, y) = \frac{1}{n} \sum_{i=1}^n H_{SCSC,i}(x, y) = \frac{L}{2} (x^2 - y^2) - L R_x x \). It is easy to check that each component function \( H_{SCSC,i} \) is \( L \)-smooth and \( (\mu_x, \mu_y) \)-convex-concave for any \( 0 \leq \mu_x, \mu_y \leq L \). Moreover, we have

\[
\max_{|y| \leq R_y} H_{SCSC}(x, y) = \frac{L}{2} x^2 - L R_x x \quad \text{and} \quad \min_{|x| \leq R_x} H_{SCSC}(x, y) = -\frac{L R^2}{2} - \frac{L}{2} y^2.
\]

Note that for \( i \geq 2 \), it holds that

\[
\nabla_x H_{SCSC,i}(x, y) = L x \quad \text{and} \quad \text{prox}_{\beta H_{SCSC,i}}(x, y) = \left( \frac{x}{L \gamma + 1}, \frac{y}{L \gamma + 1} \right).
\]

This implies \( x_t = x_0 = 0 \) will hold till the PIFO algorithm \( \mathcal{A} \) draws \( H_{SCSC,1} \). Denote \( T = \min \{ t : i_t = 1 \} \). Then, the random variable \( T \) follows a geometric distribution with success probability \( p_1 \), and satisfies

\[
\mathbb{P} \left[ T \geq n/2 \right] = (1 - p_1) \left( (n - 1)/n \right) \geq (1 - 1/n) (n - 1)/n \geq 1/2,
\]

where the last inequality is according to that \( h(\beta) = (\beta / (\beta + 1))^{\beta/2} \) is a decreasing function and \( \lim_{\beta \rightarrow \infty} h(\beta) = 1/\sqrt{2} \geq 1/2 \). Consequently, for \( N = n/2 \) and \( t < N \), we know that

\[
\mathbb{E} \left( \max_{|y| \leq R_y} H_{SCSC}(x_t, y) - \min_{|x| \leq R_x} H_{SCSC}(x, y_t) \right)
\]
\[ \geq \mathbb{E} \left( \max_{|y| \leq R_y} H_{\text{SCSC}}(x_t, y) - \min_{|x| \leq R_x} H_{\text{SCSC}}(x, y_t) \middle| t < T \right) \mathbb{P}[T > t] \]
\[ = \mathbb{E} \left( \max_{|y| \leq R_y} H_{\text{SCSC}}(0, y) - \min_{|x| \leq R_x} H_{\text{SCSC}}(0, y_t) \middle| t < T \right) \mathbb{P}[T > t] \]
\[ \geq \frac{L R_x^2}{2} \mathbb{P}[T \geq N] \geq L R_x^2 / 4 \geq \varepsilon. \]

Therefore, in order to find \((\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}\) such that
\[ \mathbb{E} \left( \max_{|y| \leq R_y} H_{\text{SCSC}}(\hat{x}, y) - \min_{|x| \leq R_x} H_{\text{SCSC}}(x, \hat{y}) \right) < \varepsilon, \]
PIFO algorithm \(A\) needs at least \(N = \Omega(n)\) queries. \(\square\)

### 3.5 Construction for the Convex-Strongly-Concave Case

We now consider the finite-sum minimax problem where the objective function is strongly-concave in \(y\) but possibly non-strongly-convex in \(x\). Our analysis is based on the following functions.

**Definition 3.26.** For fixed \(L, \mu, n, R_x, R_y\) such that \(L / \mu_y \geq 2\), we define \(f_{\text{CSC}, i} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}\) as follows
\[ f_{\text{CSC}, i}(x, y) = \lambda \tilde{r}_i(x / \beta, y / \beta; m, 1, \tilde{c}), \]
where
\[ \tilde{c} = \left( 0, \frac{2n}{\sqrt{L^2 / \mu_y^2 - 2}} \right), \quad \beta = \min \left\{ \frac{R_x \sqrt{L^2 / \mu_y^2 - 2}}{2n(m + 1)^{3/2}}, \frac{R_y}{\sqrt{m}} \right\} \quad \text{and} \quad \lambda = \frac{\beta^2 \sqrt{L^2 - 2 \mu_y^2}}{2n}. \]

Consider the minimax problem
\[ \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f_{\text{CSC}}(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_{\text{CSC}, i}(x, y), \quad (9) \]
where \(\mathcal{X} = \{x \in \mathbb{R}^m : \|x\|_2 \leq R_x\}\) and \(\mathcal{Y} = \{y \in \mathbb{R}^m : \|y\|_2 \leq R_y\}\). Define \(\phi_{\text{CSC}}(x) = \max_{y \in \mathcal{Y}} f_{\text{CSC}}(x, y)\) and \(\psi_{\text{CSC}}(y) = \min_{x \in \mathcal{X}} f_{\text{CSC}}(x, y)\).

Then we have the following proposition.

**Proposition 3.27.** For any \(n \geq 2, m \geq 2, f_{\text{CSC}, i}\) and \(f_{\text{CSC}}\) in Definition 3.26 satisfy:

1. \(f_{\text{CSC}, i}\) is \(L\)-smooth and \((0, \mu_y)\)-convex-concave. Thus, \(f_{\text{CSC}}\) is \((0, \mu_y)\)-convex-concave.

2. For \(1 \leq k \leq m - 1\), we have
\[ \min_{x \in \mathcal{X} \cap \mathcal{F}_k} \phi_{\text{CSC}}(x) - \max_{y \in \mathcal{Y} \cap \mathcal{F}_k} \psi_{\text{CSC}}(y) \geq -\frac{k \mu_y \beta^2}{2} + \frac{R_x \beta \sqrt{L^2 - 2 \mu_y^2}}{2n \sqrt{k + 1}}, \]
where \(\beta = \min \left\{ \frac{R_x \sqrt{L^2 / \mu_y^2 - 2}}{2n(m + 1)^{3/2}}, \frac{R_y}{\sqrt{m}} \right\} \).
The proof of Proposition 3.27 is given in Appendix Section C.

We can now prove the lower bound complexity for finding $\varepsilon$-suboptimal point of Problem (9) by PIFO algorithms.

**Theorem 3.28.** Consider the minimax problem (9) and $\varepsilon > 0$. Suppose that

$$n \geq 2, \frac{L}{\mu_y} \geq 2, \varepsilon \leq \min \left\{ \frac{L^2 R_x^2}{2592 n^2 \mu_y}, \frac{\mu_y R_y^2}{36} \right\}$$

and $m = \left\lceil \frac{R_x}{6n} \sqrt{\frac{L^2 - 2 \mu_y^2}{\mu_y \varepsilon}} \right\rceil - 2$.

In order to find $(\hat{x}, \hat{y}) \in X \times Y$ such that $\mathbb{E} \phi_{CSC}(\hat{x}) - \mathbb{E} \psi_{CSC}(\hat{y}) < \varepsilon$, PIFO algorithm $A$ needs at least $N$ queries, where

$$N = \Omega \left( n + \frac{R_x L}{\sqrt{\mu_y \varepsilon}} \right).$$

**Proof.** Since $L/\mu_y \geq 2$, we have $L^2 - 2 \mu_y^2 \geq L^2/2$. Then $\varepsilon \leq \frac{L^2 R_x^2}{2592 n^2 \mu_y} \leq \frac{(L^2 - 2 \mu_y^2) R_x^2}{1296 n^2 \mu_y}$, which implies that $m \geq 4$ and $\frac{R_x}{6n} \sqrt{\frac{L^2 - 2 \mu_y^2}{\mu_y \varepsilon}} - 2 \geq \frac{R_x}{12n} \sqrt{\frac{L^2 - 2 \mu_y^2}{\mu_y \varepsilon}} + 1$. It follows that $m \geq \frac{R_x}{12n} \sqrt{\frac{L^2 - 2 \mu_y^2}{\mu_y \varepsilon}}$. Then we have

$$\frac{R_x}{2n(m+1)^{3/2}} \leq \frac{R_x}{2n(m+1)^{3/2}} \leq 6 \sqrt{\frac{\varepsilon}{\mu_y m}} \leq \frac{R_y}{\sqrt{m}},$$

which implies that $\beta = \min \left\{ \frac{R_x}{2n(m+1)^{3/2}}, \frac{R_y}{\sqrt{m}} \right\} = \frac{R_x}{2n(m+1)^{3/2}}$. Following Proposition 3.27, for $1 \leq k \leq m - 1$, we have

$$\min_{x \in \mathcal{X}\cap \mathcal{F}_k} \phi_{CSC}(x) - \max_{y \in \mathcal{Y}\cap \mathcal{F}_k} \psi_{CSC}(y) \geq -k \mu_y \beta^2 \frac{2}{2n \sqrt{k+1}} + \frac{R_x \beta \sqrt{L^2 - 2 \mu_y^2}}{2n \sqrt{k+1}}$$

$$= \frac{(L^2 - 2 \mu_y^2) R_x^2}{8n^2 \mu_y} \frac{2(m+1)^{3/2} - k \sqrt{k+1}}{(m+1)^{3/2} \sqrt{k+1}}.$$

Denote $M \triangleq \left\lceil \frac{m}{2} \right\rceil$. Then we have $M = \left\lceil \frac{R_x}{12n} \sqrt{\frac{L^2 - 2 \mu_y^2}{\mu_y \varepsilon}} \right\rceil - 1 \geq 2$ and $M < m$.

Since $2(M+1) = 2 \left\lceil \frac{m}{2} \right\rceil + 2 \geq m + 1$ and $h(\beta) = \frac{2 \beta^3/2 - \beta^3/2}{\beta^3}$ is a decreasing function when $\beta > \beta_0$, we have

$$\min_{x \in \mathcal{X}\cap \mathcal{F}_M} \phi_{CSC}(x) - \max_{y \in \mathcal{Y}\cap \mathcal{F}_M} \psi_{CSC}(y) \geq \frac{(L^2 - 2 \mu_y^2) R_x^2}{8n^2 \mu_y} \frac{2(M+1)^{3/2} - k(k+1)}{(M+1)^{3/2} \sqrt{k+1}}$$

$$\geq \frac{9 \varepsilon}{16n^2 \mu_y (M+1)^2} \geq 9 \varepsilon,$$

where the last inequality is due to $M + 1 \leq \frac{R_x}{12n} \sqrt{\frac{L^2 - 2 \mu_y^2}{\mu_y \varepsilon}}$. By Lemma 3.16 for $N = n(M + 1)/4$, we know that

$$\min_{t \leq N} \mathbb{E} (\phi_{CSC}(x_t) - \psi_{CSC}(y_t)) \geq \varepsilon.$$
Therefore, in order to find suboptimal solution $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$ such that \( \mathbb{E} (\phi_{\text{CSC}}(\hat{x}) - \psi_{\text{CSC}}(\hat{y})) < \varepsilon \), algorithm \( \mathcal{A} \) needs at least \( N \) PIFO queries, where

\[
N = \frac{n}{4} \left( \frac{R_x}{12n} \sqrt{\frac{L^2 - 2\mu_y^2}{\mu_y \varepsilon}} \right) = \Omega \left( n + \frac{R_x L}{\sqrt{\mu_y \varepsilon}} \right).
\]

This completes the proof. \( \square \)

When \( L/\mu_y = O(n) \), we can provide a better lower bound as follows.

**Theorem 3.29.** For any \( L, \mu_y, n, R_x, R_y, \varepsilon \) such that \( n \geq 2, L \geq \mu_y, \varepsilon \leq \frac{R_x^2 L}{384n} \) and \( m = \left\lfloor \sqrt{\frac{R_x^2 L}{24n\varepsilon}} \right\rfloor - 1 \), there exist \( n \) functions \( \{f_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}\}_{i=1}^n \) such that \( f_i(x, y) \) is \( L \)-smooth and \( f(x, y) = \frac{1}{n} \sum_{i=1}^n f_i(x, y) \) is \((0, \mu_y)\)-convex-concave. Let \( \mathcal{X} = \{x \in \mathbb{R}^m : \|x\|_2 \leq R_x\} \) and \( \mathcal{Y} = \{y \in \mathbb{R}^m : \|y\|_2 \leq R_y\} \). In order to find \((\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \) such that \( \mathbb{E} \max_{y \in \mathcal{Y}} f(\hat{x}, y) - \mathbb{E} \min_{x \in \mathcal{X}} f(x, \hat{y}) < \varepsilon \), PIFO algorithm \( \mathcal{A} \) needs at least \( N = \Omega \left( n + R_x \sqrt{\frac{mL}{\varepsilon}} \right) \) queries.

**Proof.** Consider the functions \( \{f_{C,i}\}_{i=1}^n \) and \( f_C \) defined in Definition 4.25 with \( R \) replaced by \( R_x \). We construct \( \{G_{\text{CSC},i} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}\}_{i=1}^n \) as follows

\[
G_{\text{CSC},i}(x, y) = f_{C,i}(x) - \frac{\mu_y}{2} \|y\|^2_2,
\]

\[
G_{\text{CSC}}(x, y) = \frac{1}{n} \sum_{i=1}^n G_{\text{CSC},i}(x, y) = f_C(x) - \frac{\mu_y}{2} \|y\|^2_2.
\]

By Proposition 4.15 and Lemma B.1 we can check that each component function \( G_{\text{CSC},i} \) is \( L \)-smooth and \((0, \mu_y)\)-convex-concave. Then \( G_{\text{CSC}} \) is \((0, \mu_y)\)-convex-concave. Moreover, we have

\[
\max_{y \in \mathcal{Y}} G_{\text{CSC}}(x, y) = f_C(x) \quad \text{and} \quad \min_{x \in \mathcal{X}} G_{\text{CSC}}(x, y) = \min_{x \in \mathcal{X}} f_{C}(x) - \frac{\mu_y}{2} \|y\|^2_2.
\]

It follows that for any \((\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \), we have

\[
\max_{y \in \mathcal{Y}} G_{\text{CSC}}(\hat{x}, y) - \min_{x \in \mathcal{X}} G_{\text{CSC}}(x, \hat{y}) \geq f_C(\hat{x}) - \min_{x \in \mathcal{X}} f_C(x).
\]

By Theorem 4.27 for

\[
\varepsilon \leq \frac{R_x^2 L}{384n} \quad \text{and} \quad m = \left\lfloor \sqrt{\frac{R_x^2 L}{24n\varepsilon}} \right\rfloor - 1,
\]

in order to find \((\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \) such that \( \mathbb{E} (\max_{y \in \mathcal{Y}} G_{\text{CSC}}(\hat{x}, y) - \min_{x \in \mathcal{X}} G_{\text{CSC}}(x, \hat{y})) < \varepsilon \), PIFO algorithm \( \mathcal{A} \) needs at least \( N = \Omega \left( n + R_x \sqrt{\frac{mL}{\varepsilon}} \right) \) queries. \( \square \)

We can also provide the lower bound \( \Omega(n) \) if \( \varepsilon \leq LR_x^2/4 \) (see Lemma 3.25). Note that if \( \varepsilon \geq \frac{LR_x^2}{2592n\mu_y} \), \( \Omega(n) = \Omega \left( n + \frac{R_x L}{\sqrt{\mu_y \varepsilon}} \right) \). And if \( \varepsilon \geq \frac{R_x^2 L}{384n} \), \( \Omega(n) = \Omega \left( n + R_x \sqrt{\frac{mL}{\varepsilon}} \right) \). Then we obtain Theorem 3.34.
3.6 Construction for the Convex-Concave Case

The analysis for general convex-concave case is similar to that of Section 3.5. We consider the following functions.

Definition 3.30. For fixed $L, n, R_x, R_y$ such that $n \geq 2$, we define $f_{CC,i} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ as follows

$$f_{CC,i}(x, y) = \lambda \tilde{r}_i \left( \frac{x}{\beta}, \frac{y}{\beta}; m, 1, 0_2 \right),$$

where $\lambda = \frac{LR_x^2}{2nm}$ and $\beta = \frac{R_y}{\sqrt{m}}$. Consider the minimax problem

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f_{CC}(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_{CC,i}(x, y),$$

where $\mathcal{X} = \{ x \in \mathbb{R}^m : \| x \|_2 \leq R_x \}$ and $\mathcal{Y} = \{ y \in \mathbb{R}^m : \| y \|_2 \leq R_y \}$. Define $\phi_{CC}(x) = \max_{y \in \mathcal{Y}} f_{CC}(x, y)$ and $\psi_{CC}(y) = \min_{x \in \mathcal{X}} f_{CC}(x, y)$.

Then we have the following proposition.

Proposition 3.31. For any $n \geq 2$, $m \geq 3$, $f_{CC,i}$ and $f_{CC}$ in Definition 3.30 satisfy:

1. $f_{CC,i}$ is $L$-smooth and convex-concave. Thus, $f_{CC}$ is convex-concave.

2. For $1 \leq k \leq m - 1$, we have

$$\min_{x \in \mathcal{X} \cap F_k} \phi_{CC}(x) - \max_{y \in \mathcal{Y} \cap F_k} \psi_{CC}(y) \geq \frac{LR_x R_y}{2n \sqrt{m(k + 1)}}.$$

The proof of Proposition 3.31 is given in Appendix Section C.

Then, we obtain a PIFO lower bound complexity for general finite-sum convex-concave minimax problem.

Theorem 3.32. Consider minimax problem (10) and $\varepsilon > 0$. Suppose that

$$n \geq 2, \varepsilon \leq \frac{LR_x R_y}{36 \sqrt{2} m}, \text{ and } m = \left\lfloor \frac{LR_x R_y}{9 \sqrt{2} \varepsilon n} \right\rfloor - 1.$$

In order to find $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$ such that $\mathbb{E}\phi_{CC}(\hat{x}) - \mathbb{E}\psi_{CC}(\hat{y}) < \varepsilon$, PIFO algorithm $A$ needs at least $N = \Omega \left( \frac{n + LR_x R_y}{\varepsilon} \right)$ queries.

Proof. The assumption on $\varepsilon$ implies $m \geq 3$. Let $M \triangleq \lfloor (m - 1)/2 \rfloor = \left\lfloor \frac{LR_x R_y}{18 \sqrt{2} \varepsilon n} \right\rfloor - 1$. Then we have $M \geq 1$ and $m/2 \leq M + 1 \leq (m + 1)/2$. By Proposition 3.31, we have

$$\min_{x \in \mathcal{X} \cap F_M} \phi_{CC}(x) - \max_{y \in \mathcal{Y} \cap F_M} \psi_{CC}(y) \geq \frac{LR_x R_y}{2n \sqrt{m(M + 1)}} \geq \frac{LR_x R_y}{2 \sqrt{2n} (m + 1)} \geq \frac{LR_x R_y}{\sqrt{2n} (m + 1)} \geq 9 \varepsilon.$$

Hence, by Lemma 3.16 for $N = n(M + 1)/4$, we know that

$$\min_{i \leq N} \mathbb{E} (\phi_{CC}(x_i) - \psi_{CC}(y_i)) \geq \varepsilon.$$

Therefore, in order to find an approximate solution $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$ such that $\mathbb{E} (\phi_{CC}(\hat{x}) - \psi_{CC}(\hat{y})) < \varepsilon$, the PIFO algorithm $A$ needs at least $N$ queries, where

$$N = \frac{n}{4} \left( \left\lfloor \frac{LR_x R_y}{18 \sqrt{2} \varepsilon n} \right\rfloor \right) = \Omega \left( \frac{n + LR_x R_y}{\varepsilon} \right).$$
Note that Theorem 3.28 requires the condition \( \varepsilon \leq O(L/n) \) to obtain the desired lower bound. For large \( \varepsilon \), we can apply the following lemma.

**Lemma 3.33.** For any positive \( L, n, R_x, R_y, \varepsilon \) such that \( n \geq 2 \) and \( \varepsilon \leq \frac{1}{4} LR_x R_y \) there exist \( n \) functions \( \{f_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\}_{i=1}^{n} \) such that \( f_i(x, y) \) is \( L \)-smooth. Let \( \mathcal{X} = \{x \in \mathbb{R} : |x| \leq R_x\} \) and \( \mathcal{Y} = \{y \in \mathbb{R} : |y| \leq R_y\} \). In order to find \( (\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y} \) such that \( \mathbb{E} \max_{y \in \mathcal{Y}} f(\hat{x}, y) - \mathbb{E} \min_{x \in \mathcal{X}} f(x, \hat{y}) < \varepsilon \), PIFO algorithm \( A \) needs at least \( N = \Omega(n) \) queries.

**Proof.** Consider the functions \( \{H_{CC,i} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\}_{i=1}^{n} \) where

\[
H_{CC,i}(x, y) = \begin{cases} 
Lxy - nLR_x y, & \text{for } i = 1, \\
Lxy, & \text{otherwise},
\end{cases}
\]

and \( H_{CC}(x, y) = \frac{1}{n} \sum_{i=1}^{n} H_{CC,i}(x, y) = Lxy - LR_x y \). Consider the minimax problem

\[
\min_{|x| \leq R_x} \max_{|y| \leq R_y} H_{CC}(x, y).
\]

It is easy to check that each component function \( H_{CC,i} \) is \( L \)-smooth and convex-concave. Moreover, we have

\[
\max_{|y| \leq R_y} H_{CC}(x, y) = LR_y |x - R_x|, \quad \text{and} \quad \min_{|x| \leq R_x} H_{CC}(x, y) = -LR_x (|y| + y) \leq 0,
\]

and it holds that

\[
\min_{|x| \leq R_x} \max_{|y| \leq R_y} H_{CC}(x, y) = \max_{|y| \leq R_y} \min_{|x| \leq R_x} H_{CC}(x, y) = 0.
\]

Note that for \( i \geq 2 \), we have

\[
\nabla_x H_{CC,i}(x, y) = Ly, \quad \nabla_y H_{CC,i}(x, y) = Lx, \quad \text{and} \quad \text{prox}_{H_{CC,i}}(x, y) = \left( \frac{L \gamma x + y}{L^2 \gamma^2 + 1}, \frac{x - L \gamma y}{L^2 \gamma^2 + 1} \right),
\]

which implies \( x_t = y_t = x_0 = y_0 = 0 \) will hold till the PIFO algorithm \( A \) draws \( H_{CC,1} \).

Denote \( T = \min\{t : i_t = 1\} \). Then, the random variable \( T \) follows a geometric distribution with success probability \( p_1 \), and satisfies

\[
\mathbb{P} [T \geq n/2] = (1 - p_1)^{(n-1)/2} \geq (1 - 1/n)^{(n-1)/2} \geq 1/2, \quad (11)
\]

where the last inequality is according to that \( h(\beta) = (\frac{\beta}{\beta+1})^{\beta/2} \) is a decreasing function and \( \lim_{\beta \to \infty} h(\beta) = 1/\sqrt{\varepsilon} \geq 1/2 \).

For \( N = n/2 \) and \( t < N \), we know that

\[
\mathbb{E} \left( \max_{|y| \leq R_y} H_{CC}(x_t, y_t) - \min_{|x| \leq R_x} H_{CC}(x, y_t) \right)
\]

\[
\geq \mathbb{E} \left( \max_{|y| \leq R_y} H_{CC}(x_t, y_t) - \min_{|x| \leq R_x} H_{CC}(x, y_t) \mid t < T \right) \mathbb{P} [T > t]
\]

\[
= \mathbb{E} \left( \max_{|y| \leq R_y} H_{CC}(0, y) - \min_{|x| \leq R_x} H_{CC}(x, 0) \mid t < T \right) \mathbb{P} [T > t]
\]

\[
= \frac{LR_x R_y}{2} \mathbb{P} [T \geq N] \geq LR_x R_y / 4 \geq \varepsilon.
\]
Therefore, in order to find \((\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}\) such that
\[
\mathbb{E}\left( \max_{|y| \leq R_y} H_{CC}(\hat{x}, y) - \min_{|x| \leq R_x} H_{CC}(x, \hat{y}) \right) < \varepsilon,
\]
the algorithm \(\mathcal{A}\) needs at least \(N = \Omega(n)\) PIFO queries.

Note that for \(\varepsilon \geq \frac{L R_x R_y}{36 \sqrt{2n}}\), we have \(\Omega\left(n + \frac{L R_x R_y}{\varepsilon}\right) = \Omega(n)\). Then for \(\varepsilon \leq LR_x R_y/4\), we obtain the lower bound \(\Omega\left(n + \frac{L R_x R_y}{\varepsilon}\right)\).

Moreover, note that \(H_{SCSC}\) defined in the proof of Lemma 3.25 and \(G_{CSC}\) defined in the proof of Theorem 3.29 are also convex-concave. And \(\varepsilon \geq \frac{R^2 L}{384 n}\) implies \(\Omega(n) = \Omega\left(n + R_x \sqrt{\frac{n L}{\varepsilon}}\right)\). Then for \(\varepsilon \leq LR_x^2/4\), we can obtain the lower bound \(\Omega\left(n + R_x \sqrt{\frac{n L}{\varepsilon}}\right)\). Similarly, for \(\varepsilon \leq LR_y^2/4\), we can obtain the lower bound \(\Omega\left(n + \frac{L R_x R_y}{\varepsilon} + (R_x + R_y) \sqrt{\frac{n L}{\varepsilon}}\right)\).

In summary, for \(\varepsilon \leq \frac{L}{4} \min\{R_x^2, R_y^2\}\), the lower bound is \(\Omega\left(n + \frac{L R_x R_y}{\varepsilon} + (R_x + R_y) \sqrt{\frac{n L}{\varepsilon}}\right)\).

### 3.7 Construction for the Nonconvex-Strongly-Concave Case

In this subsection, we consider the finite-sum minimax problem where the objective function is strongly-concave in \(y\) but nonconvex in \(x\). The analysis is based on the following construction.

**Definition 3.34.** For fixed \(L, \mu_x, \mu_y, \Delta, n\), we define \(f_{NCSC,i} : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \rightarrow \mathbb{R}\) as follows
\[
f_{NC,i}(x, y) = \lambda \hat{r}_i (x/\beta, y/\beta; m + 1, \sqrt{\alpha}, \hat{c}), \quad \text{for } 1 \leq i \leq n,
\]
where \(\alpha = \min \left\{ 1, \frac{n^2 \mu_y}{90 L}, \frac{8(\sqrt{3} + 1)n^2 \mu_x \mu_y}{45 L^2} \right\}\), \(\hat{c} = \left(\frac{4n \mu_y}{L}, \frac{\sqrt{\alpha} L}{4n \mu_y}, \frac{\sqrt{\alpha}}{L^3} \right)\), \(\lambda = \frac{32944 n^2 \mu_y^2 \varepsilon^2}{L^3 \alpha}\), \(\beta = 2 \sqrt{\lambda n/L}\) and \(m = \left\lfloor \frac{\Delta L^2 \sqrt{\alpha}}{217728 n^2 \varepsilon^2 \mu_y} \right\rfloor\).

Consider the minimax problem
\[
\min_{x \in \mathbb{R}^{m+1}} \max_{y \in \mathbb{R}^{m+1}} f_{NCSC}(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_{NCSC,i}(x, y). \tag{12}
\]

Define \(\phi_{NCSC}(x) = \max_{y \in \mathbb{R}^{m+1}} f_{NCSC}(x, y)\).

Then we have the following proposition.

**Proposition 3.35.** For any \(n \geq 2, L/\mu_y \geq 4\) and \(\varepsilon^2 \leq \frac{\Delta L^2 \alpha}{433456 n^2 \mu_y}\), the following properties hold:

1. \(f_{NCSC,i}\) is \(L\)-smooth and \((-\mu_x, \mu_y)\)-convex-concave.
2. \(\phi_{NCSC}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} \phi_{NCSC}(x^*) \leq \Delta\).
3. \(m \geq 2\) and for \(M = m - 1\), \(\min_{x \in \mathcal{F}_M} \|\nabla \phi_{NCSC}(x)\|_2 \geq 9\varepsilon\).
The proof of Proposition 3.35 is given in Appendix Section C. Now we give the proof of Theorem 3.6.

Proof of Theorem 3.6. By Proposition 3.35, \( f_{\text{NCSC}}(x, y) \in F_{\text{NCC}}(\Delta, L, \mu_x, \mu_y) \). Combining Lemma 3.20 and the third property of Proposition 3.35, for \( N = nm/4 \), we have

\[
\min_{t \leq N} \mathbb{E} \| \nabla \phi_{\text{NCSC}}(x_t) \|_2 \geq \varepsilon.
\]

Thus, in order to find \((\hat{x}, \hat{y})\) such that \( \mathbb{E} \| \nabla \phi_{\text{NCSC}}(\hat{x}) \|_2 < \varepsilon \), \( A \) needs at least \( N \) PIFO queries, where

\[
N = \frac{nm}{4} = \Omega \left( \frac{\Delta L^2 \sqrt{\alpha}}{n \mu_y^2} \right).
\]

Since \( \varepsilon^2 \leq \frac{\Delta L^2 \alpha}{4n^2 \mu_y^2} \) and \( \alpha \leq 1 \), we have \( \Omega \left( \frac{\Delta L^2 \sqrt{\alpha}}{n \mu_y^2} \right) = \Omega \left( n + \frac{\Delta L^2 \sqrt{\alpha}}{n \mu_y^2} \right) \).

3.8 Construction for the Average Smooth Case

In this subsection, we consider the lower bounds of PIFO complexity under the the average smooth assumption.

3.8.1 Strongly-Convex-Strongly-Concave Case

We first consider the minimax problem where the objective function is strongly-convex in \( x \) and strongly-concave in \( y \).

Without loss of generality, we assume \( \mu_x \geq \mu_y \). For fixed \( L', \mu_x, \mu_y, R_x, R_y, n, \varepsilon \) such that \( L/\mu_x \geq 2 \), we set \( L = \sqrt{\frac{n(L'^2 - \mu_y^2)}{2} + 2\mu_x^2} \), and consider \( \{ f_{\text{SCSC},i} \}_{i=1}^n \) and \( f_{\text{SCSC}} \) defined in Definition 3.21. Let \( \kappa'_x = L/\mu_x \) and \( \kappa'_y = L/\mu_y \). We have the following proposition.

Proposition 3.36. For any \( n \geq 4 \) and \( \kappa'_x = \frac{L'}{\mu_x} \geq 2 \), we have that

1. \( f_{\text{SCSC}}(x, y) \) is \((\mu_x, \mu_y)\)-convex-concave and \( \{ f_{\text{SCSC},i} \}_{i=1}^n \) is \( L' \)-average smooth.
2. \( \frac{\sqrt{n}}{2} L' \leq L \leq \sqrt{n} L' \) and \( \kappa_x = \frac{L}{\mu_x} \geq 2 \).

Proof. 1. Clearly, \( f_{\text{SCSC}}(x, y) \) is \((\mu_x, \mu_y)\)-convex-concave. By Proposition 3.13 and Lemma B.1 \( \{ f_{\text{SCSC},i}(x, y) \} \) is \( \hat{L} \)-average smooth where

\[
\hat{L} = \frac{\sqrt{L'^2 - 2\mu_x^2}}{2n} \sqrt{8n + \frac{8n^2}{L'^2/\mu_x^2 - 2}} = \sqrt{\frac{L'^2 - 2\mu_x^2}{8n}} \sqrt{8n + \frac{16n}{L'^2/\mu_x^2 - 2}} = L'.
\]

2. It is easy to check the second inequality. For the first inequality, we find that

\[
L^2 - \frac{n}{4} L'^2 = \frac{n}{4} L'^2 - (n - 2) \mu_x^2 = \mu_x^2 \left( \frac{n}{4} \kappa_x^2 - n + 2 \right) \geq \mu_x^2 (n - n + 2) \geq 0.
\]

Since \( n \geq 4 \), we have \( \kappa_x = \frac{L}{\mu_x} \geq \frac{\sqrt{n} L'}{2 \mu_x} \geq 2 \).

This completes the proof. □

Now we give the proof of Theorem 3.8.
Proof of Theorem 3.3. 1. For \( \kappa' \geq \kappa_x = \Omega(\sqrt{n}) \), consider the minimax problem \( 8 \) where \( L = \sqrt{n(L^2 - \mu_x^2)} + \mu_x^2 \). By Theorem 3.3 and Proposition 3.36, we have

\[
m_{CC}(x, R_y, L, \mu_x, \mu_y) = \Omega \left( \frac{n + \sqrt{\kappa_x \kappa_y \mu_x}}{\sqrt{n}} \right).
\]

2. For \( \kappa' = \Omega(\sqrt{n}) \) and \( \kappa_x = O(\sqrt{n}) \), we set \( L = \sqrt{n(L^2 - \mu_x^2)} + \mu_x^2 \) and consider \( \{G_{SCSC,i}\}_{i=1}^{n} \), \( G_{SCSC} \), \( X \) and \( Y \) defined in the proof of Theorem 3.24. By Proposition 4.31, we know that \( G_{SCSC} \) is \((\mu_x, \mu_y)\)-convex-concave, \( \{G_{SCSC,i}\}_{i=1}^{n} \) is \( L' \)-average smooth, \( \sqrt{\frac{n}{2}} L' \leq L \leq \sqrt{\frac{7}{2}} L' \) and \( \kappa_y = \frac{L}{\mu_y} \geq 2 \). Following the proof of Theorem 3.24, we can obtain

\[
m_{CC}(x, R_y, L', \mu_x, \mu_y) = \Omega \left( \left( \frac{n + \sqrt{\kappa_x \kappa_y \mu_x}}{\sqrt{n}} \right) \right).
\]

3. For \( \kappa', \kappa_y = O(\sqrt{n}) \), note that \( \{H_{SCSC,i}\}_{i=1}^{n} \) defined in Lemma 3.25 is also \( L \)-average smooth. Then we have \( m_{CC}(x, R_y, L, \mu_x, \mu_y) = \Omega(n) \). This completes the proof.

3.8.2 Convex-Strongly-Concave Case

We now consider the minimax problem where the objective function is convex in \( x \) and strongly-concave in \( y \).

For fixed \( L', \mu_y, R_x, R_y, n, \varepsilon \) such that \( L'/\mu_y \geq 2 \), we set \( L = \sqrt{n(L^2 - \mu_x^2)} + \mu_x^2 \), and consider \( \{f_{SCSC,i}\}_{i=1}^{n} \) and \( f_{SCSC} \) defined in Definition 3.26. Similar to Proposition 3.36, we have the following result.

**Proposition 3.37.** For any \( n \geq 4 \) and \( \kappa_y' = \frac{L'}{\mu_y} \geq 2 \), we have that

1. \( f_{SCSC}(x, y) \) is \((0, \mu_y)\)-convex-concave and \( \{f_{SCSC,i}\}_{i=1}^{n} \) is \( L' \)-average smooth.

2. \( \sqrt{\frac{7}{2}} L' \leq L \leq \sqrt{\frac{2}{7}} L' \) and \( \kappa_y = \frac{L}{\mu_y} \geq 2 \).

Then we give the proof of Theorem 3.9.

**Proof of Theorem 3.9.** Consider the minimax problem \( 0 \). By Theorem 3.28 and Proposition 3.37, for \( \varepsilon \leq \min \left\{ \frac{L'^2 R_x^2}{10368 \mu_y}, \frac{R_x^2 R_y^2}{36} \right\} \), we have

\[
m_{CC}(x, R_y, L', 0, \mu_y) = \Omega \left( \frac{n + R_x L'}{\mu_y \varepsilon} \right).
\]

Moreover, consider \( \{G_{SCSC,i}\}_{i=1}^{n} \), \( G_{SCSC} \), \( X \) and \( Y \) defined in the proof Theorem 3.29 with \( L = \sqrt{n(L^2 - \mu_x^2)} + \mu_x^2 \). By Theorem 3.29 and Proposition 4.31, for \( \varepsilon \leq \frac{R_x^2 L'}{768 \sqrt{n}} \), we have

\[
m_{CC}(x, R_y, L', 0, \mu_y) = \Omega \left( \frac{n + R_x L' n^{3/4}}{\mu_y \varepsilon} \right).
\]

Note that \( \{H_{SCSC,i}\}_{i=1}^{n} \) defined in the proof of Lemma 3.25 is also \( L \)-average smooth. Then for \( \varepsilon \leq L'R_x^2/4 \), we can get the lower bound \( \Omega(n) \). Since \( \varepsilon \geq \frac{R_x^2 L'}{10368 \mu_y} \) implies \( \Omega(n) = \Omega \left( \frac{n + R_x L'}{\mu_y \varepsilon} \right) \) and \( \varepsilon \geq \frac{R_x^2 L'}{768 \sqrt{n}} \) implies that \( \Omega(n) = \Omega \left( \frac{n + R_x n^{3/4} \sqrt{L'}}{\varepsilon} \right) \), we obtain the desired result.
3.8.3 Convex-Concave Case

For the general convex-concave case, we set $L = \sqrt{\frac{\mu}{2}} L'$, and consider $\{f_{CC,i}\}_{i=1}^n$, $f_{CC}$ and Problem \ref{eq:3.30} defined in Definition 3.30. By Proposition 3.13 and Lemma B.1, $\{f_{CC,i}\}_{i=1}^n$ is $L'$-average smooth. Then Theorem 3.32 implies that for $\varepsilon \leq \frac{L' R_x R_y}{12 \sqrt{n}}$, the lower bound is $\Omega \left( n + \frac{\sqrt{n} L' R_x R_y}{\varepsilon} \right)$.

Note that $\{H_{CC,i}\}_{i=1}^n$ defined in the proof of Lemma 3.33 is also $L'$-average smooth. Then for $\varepsilon \leq L' R_x R_y / 4$, we get the lower bound $\Omega(n)$. Since $\varepsilon \geq \frac{L' R_x R_y}{12 \sqrt{n}}$ implies that $\Omega(m) = \Omega \left( n + \frac{\sqrt{n} L' R_x R_y}{\varepsilon} \right)$, we obtain the lower bound $\Omega \left( n + \frac{\sqrt{n} L' R_x R_y}{\varepsilon} \right)$ when $\varepsilon \leq L' R_x R_y / 4$.

Following the proof of Theorem 3.4 for $\varepsilon \leq L' R_x^2 / 4$, we can also get the lower bound $\Omega \left( n R_x n^{3/4} \sqrt{L'/\varepsilon} \right)$.

Similarly, for $\varepsilon \leq L' R_y^2 / 4$, we get the lower bound $\Omega \left( n R_y n^{3/4} \sqrt{L'/\varepsilon} \right)$.

In summary, we obtain the result of Theorem 3.10.

3.8.4 Nonconvex-Strongly-Concave Case

The analysis for the nonconvex-strongly-concave case under the average smooth assumption is similar to that under the smooth assumption. It is based on the following construction.

Definition 3.38. For fixed $L', \mu_x, \mu_y, \Delta, n$, we define $\tilde{f}_{NCSC,i} : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ as follows

$$\tilde{f}_{NCSC}(x, y) = \lambda \hat{\nu}_i \left( \frac{x}{\beta}, \frac{y}{\beta}; m+1, \sqrt{\alpha}, \hat{c} \right), \text{ for } 1 \leq i \leq n,$$

where

$$\alpha = \min \left\{ 1, \frac{32 \mu_y}{135 L'}, \frac{128 (\sqrt{3} + 1) n \mu_x \mu_y}{45 L'^2} \right\}, \hat{c} = \left( \frac{16 \sqrt{n} \mu_y}{L'}, \frac{\sqrt{\alpha} L'}{16 \sqrt{n} \mu_y}, \sqrt{\alpha} \right),$$

$$\lambda = \frac{5308416 n^{3/2} \mu_y^2 \varepsilon^2}{L^3 \alpha}, \beta = 4 \sqrt{\lambda \sqrt{n} / L'} \text{ and } m = \left\lfloor \frac{\Delta L^2 \sqrt{\alpha}}{3483648 n \varepsilon^2 \mu_y} \right\rfloor.$$

Consider the minimax problem

$$\min_{x \in \mathbb{R}^{m+1}} \max_{y \in \mathbb{R}^{m+1}} \tilde{f}_{NCSC}(x, y) \triangleq \frac{1}{n} \sum_{i=1}^n \tilde{f}_{NCSC,i}(x, y). \tag{13}$$

Define $\tilde{\phi}_{NCSC}(x) = \max_{y \in \mathbb{R}^{m+1}} \tilde{f}_{NCSC}(x, y)$.

Then we have the following proposition.

Proposition 3.39. For any $n \geq 2$, $L'/\mu_y \geq 4$ and $\varepsilon^2 \leq \frac{L'^2 \alpha}{6967296 n \mu_y}$, the following properties hold:

1. $\tilde{f}_{NCSC,i}$ is $(-\mu_x, \mu_y)$-convex-concave and $\{\tilde{f}_{NCSC,i}\}_{i=1}^n$ is $L'$-average smooth.

2. $\tilde{\phi}_{NCSC}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} \tilde{\phi}_{NCSC}(x^*) \leq \Delta$.

3. $m \geq 2$ and for $M = m-1$, $\min_{x \in F_M} \left\| \nabla \tilde{\phi}_{NCSC}(x) \right\|_2 \geq 9 \varepsilon$.

The proof of Proposition 3.39 is given in Appendix Section C.

Now we give the proof of Theorem 3.11.
Proof of Theorem 3.11. By Proposition 3.35, \( f_{NCSC}(x, y) \in \tilde{F}_{NCC}(\Delta, L', \mu_x, \mu_y) \). Combining Lemma 3.20 and the third property of Proposition 3.39 for \( N = nm/4 \), we have
\[
\min_{t \leq N} \mathbb{E} \left\| \nabla \phi_{NCSC}(x_t) \right\|_2 \geq \epsilon.
\]
Thus, in order to find \((\hat{x}, \hat{y})\) such that \( \mathbb{E} \left\| \nabla \phi_{NCSC}(\hat{x}) \right\|_2 < \epsilon \), \( A \) needs at least \( N \) PIFO queries, where
\[
N = \frac{nm}{4} = \Omega \left( \frac{\Delta L'^2 \sqrt{\alpha}}{\epsilon^2 \mu_y} \right).
\]
Since \( \epsilon^2 \leq \frac{\Delta L'^2 \alpha}{6967296 m \mu_y} \) and \( \alpha \leq 1 \), we have \( \Omega \left( \frac{\Delta L'^2 \sqrt{\alpha}}{\epsilon^2 \mu_y} \right) = \Omega \left( n + \frac{\Delta L'^2 \sqrt{\alpha}}{\epsilon^2 \mu_y} \right) \).

4 Lower Complexity Bounds for the Minimization Problems

In this section, we provide a new proof of the results of Woodworth and Srebro [35] and Zhou and Gu [41] by our framework. Zhou and Gu [41] proved lower bound complexity for IFO algorithms, while our framework also applies to PIFO algorithms.

Consider the following minimization problem
\[
\min_{x \in \mathcal{X}} f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),
\]
where each individual component \( f_i(x) \) is \( L \)-smooth or the function class \( \{f_i(x)\}_{i=1}^{n} \) is \( L' \)-average smooth, the feasible set \( \mathcal{X} \) is closed and convex such that \( \mathcal{X} \subseteq \mathbb{R}^d \).

In Section 4.1, we formally provide the definition of PIFO algorithms for solving Problem (14), function classes that we focus on, and optimization complexity which we want to lower bound. In Section 4.2, we present our lower bound results for different function classes. In Section 4.3, we briefly summarize our framework for construction. The details on the construction for the smooth cases are in Sections 4.4, 4.5 and 4.6. In Section 4.4, the objective function \( f(x) \) is strongly-convex in \( x \). In Section 4.5, \( f(x) \) is convex in \( x \) but not strongly-convex in \( x \). In Section 4.6, \( f(x) \) is non-convex in \( x \). The details on the construction for average smooth cases are in Section 4.7.

4.1 The Setup

We study the PIFO algorithms to solve Problem (14), which we set up formally in this section.

Algorithms We define PIFO algorithms for minimization problem as follows.

Definition 4.1. Consider a stochastic optimization algorithm \( A \) to solve Problem (14). Denote \( x_t \) to be the point obtained by \( A \) at time-step \( t \). A PIFO algorithm consists of a categorical distribution \( \mathcal{D} \) over \([n]\) and obtains \( x_t \) by following linear span protocol
\[
\begin{align*}
\tilde{x}_t &\in \text{span} \left\{ x_0, \ldots, x_{t-1}, \nabla f_{i_1}(x_0), \ldots, \nabla f_{i_1}(x_{t-1}), \text{prox}_{\gamma_{i_1}} f_{i_1}(x_0), \ldots, \text{prox}_{\gamma_{i_1}} f_{i_1}(x_{t-1}) \right\}, \\
x_t &= P_{\mathcal{X}}(\tilde{x}_t),
\end{align*}
\]
where \( i_t \sim \mathcal{D} \) is drawn a single time at the beginning of the protocol. We denote \( \mathcal{A} \) to be the class of all PIFO algorithms.
We remark some details in our definition of PIFO algorithms.

1. Note that simultaneous queries is allowed in our definition of PIFO algorithms. At time-step $t$, the algorithm have the access to observe $\nabla f_{i_t}(x_0), \ldots, \nabla f_{i_t}(x_{t-1})$ with shared $i_t$. The algorithm SPIDER [15] and SNVRG [42] are examples of employing simultaneous queries for finding suboptimal stationary points.

2. Without loss of generality, we assume that the PIFO algorithm $\mathcal{A}$ starts from $x_0 = 0_d$ to simplify our analysis. Otherwise, we can take $\{\hat{f}_i(x) = f_i(x + x_0)\}_{i=1}^n$ into consideration.

3. The uniform distribution over $[n]$ and the distributions based on the smoothness of the component functions, e.g. the distribution which satisfies $\mathbb{P}_{Z \sim \mathcal{D}}[Z = i] \propto L_i$ or $\mathbb{P}_{Z \sim \mathcal{D}}[Z = i] \propto L_i^2$ for $i \in [n]$, are widely-used in algorithm designing for the categorical distribution $\mathcal{D}$, where $L_i$ is the smoothness of $f_i$.

4. Let $p_i = \mathbb{P}_{Z \sim \mathcal{D}}[Z = i]$ for $i \in [n]$. We can assume that $p_1 \leq p_2 \leq \cdots \leq p_n$ by rearranging the component functions $\{f_i\}_{i=1}^n$. Suppose that $p_{s_1} \leq p_{s_2} \leq \cdots \leq p_{s_n}$ where $\{s_i\}_{i=1}^n$ is a permutation of $[n]$. We can consider $\{\hat{f}_i\}_{i=1}^n$ and categorical distribution $\mathcal{D}'$ such that the algorithm draw $\hat{f}_i \triangleq f_{s_i}$ with probability $p_{s_i}$ instead.

**Function class** We develop lower bounds for PIFO algorithms that find suboptimal solution to the problems in the following four sets

\[
\mathcal{F}_C(R, L, \mu) = \left\{ f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \mid f : \mathcal{X} \to \mathbb{R}, \text{diam}({\mathcal{X}}) \leq 2R, \quad f_i \text{ is } L\text{-smooth, } f \text{ is } \mu\text{-strongly convex} \right\},
\]

\[
\mathcal{F}_C(R, L', \mu) = \left\{ f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \mid f : \mathcal{X} \to \mathbb{R}, \text{diam}({\mathcal{X}}) \leq 2R, \quad \{f_i\}_{i=1}^n \text{ is } L'\text{-average smooth, } f \text{ is } \mu\text{-strongly convex} \right\},
\]

\[
\mathcal{F}_{NC}(\Delta, L, \mu) = \left\{ f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \mid f : \mathcal{X} \to \mathbb{R}, f(0) - \inf_{x \in \mathcal{X}} f(x) \leq \Delta, \quad f_i \text{ is } L\text{-smooth, } f \text{ is } (-\mu)\text{-weakly convex} \right\},
\]

\[
\mathcal{F}_{NC}(\Delta, L', \mu) = \left\{ f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \mid f : \mathcal{X} \to \mathbb{R}, f(0) - \inf_{x \in \mathcal{X}} f(x) \leq \Delta, \quad \{f_i\}_{i=1}^n \text{ is } L'\text{-average smooth, } f \text{ is } (-\mu)\text{-weakly convex} \right\}.
\]

**Optimization complexity** We formally define the optimization complexity as follows.

**Definition 4.2.** For a function $f$, a PIFO algorithm $\mathcal{A}$ and a tolerance $\varepsilon > 0$, the number of queries needed by $\mathcal{A}$ to find $\varepsilon$-suboptimal solution to the Problem (14) or the $\varepsilon$-stationary point of
\( f(x) \) is defined as
\[
T(A, f, \varepsilon) = \begin{cases} 
\inf \{ T \in \mathbb{N} | E f(x_{A,T}) - \min_{x \in X} f(x) < \varepsilon \} & \text{if } f \in \mathcal{F}(R, L, \mu) \cup \mathcal{F}(R, L', \mu) \\
\inf \{ T \in \mathbb{N} | E \|\nabla f(x_{A,T})\|_2 < \varepsilon \} & \text{if } f \in \mathcal{F}(\Delta, L, \mu) \cup \mathcal{F}(\Delta, L', \mu) 
\end{cases}
\]

where \( x_{A,T} \) is the point obtained by the algorithm \( A \) at time-step \( T \).

Furthermore, the optimization complexity with respect to the function class \( \mathcal{F}(\Delta, R, L, \mu) \) and \( \bar{\mathcal{F}}(\Delta, R, L, \mu) \) is defined as
\[
m_{\varepsilon}^C(R, L, \mu) \triangleq \inf_{A \in \mathcal{A}} \sup_{f \in \mathcal{F}(R, L, \mu)} T(A, f, \varepsilon),
\]
\[
m_{\varepsilon}(R, L', \mu) \triangleq \inf_{A \in \mathcal{A}} \sup_{f \in \mathcal{F}(R, L', \mu)} T(A, f, \varepsilon),
\]
\[
m_{\varepsilon}^{NC}(\Delta, L, \mu) \triangleq \inf_{A \in \mathcal{A}} \sup_{f \in \mathcal{F}(\Delta, L, \mu)} T(A, f, \varepsilon),
\]
\[
m_{\varepsilon}^{NC}(\Delta, L', \mu) \triangleq \inf_{A \in \mathcal{A}} \sup_{f \in \mathcal{F}(\Delta, L', \mu)} T(A, f, \varepsilon).
\]

### 4.2 Main Results

In this subsection, we present the our lower bound results for PIFO algorithms.

#### 4.2.1 Smooth Cases

We first start with smooth and strongly-convex setting.

**Theorem 4.3.** Let \( n \geq 2 \) be a positive integer and \( L, \mu, R, \varepsilon \) be positive parameters. Assume additionally that \( \kappa = L/\mu \geq 2 \) and \( \varepsilon \leq LR^2/4 \). Then we have
\[
m_{\varepsilon}^C(R, L, \mu) = \begin{cases} 
\Omega \left( (n+\sqrt{\kappa n}) \log \left( \frac{1}{\varepsilon} \right) \right), & \text{for } \kappa = \Omega(n), \\
\Omega \left( n + \left( \frac{n}{1+(\log(n/\kappa))^+} \right) \log \left( \frac{1}{\varepsilon} \right) \right), & \text{for } \kappa = O(n).
\end{cases}
\]

**Remark 4.4.** In fact, the lower bound in Theorem 4.3 perfectly matches the upper bound of the PIFO algorithm Point SAGA \[13\] in \( n = O(\kappa) \) case and matches the the upper bound of the IFO algorithm \[11\] prox-SVRG \[16\] in \( \kappa = O(n) \) case. Hence, the lower bound in Theorem 4.3 is tight, while Woodworth and Srebro \[33\] only provided lower bound \( \Omega \left( (n+\sqrt{\kappa n}) \log \left( \frac{1}{\varepsilon} \right) \right) \) in \( n = O(\kappa) \) case. The theorem also shows that the PIFO algorithm can not be more powerful than the IFO algorithm in the worst case, because Hannah et al. \[16\] proposed a same lower bound for IFO algorithms.

Next we give the lower bound when the objective function is not strongly-convex.

**Theorem 4.5.** Let \( n \geq 2 \) be a positive integer and \( L, R, \varepsilon \) be positive parameters. Assume additionally that \( \varepsilon \leq LR^2/4 \). Then we have
\[
m_{\varepsilon}^C(R, L, 0) = \Omega \left( n + R \sqrt{nL/\varepsilon} \right)
\]

**Remark 4.6.** The lower bound in Theorem 4.5 is the same as the one of Woodworth and Srebro’s result. However, from the analysis in Section 4.3, our construction only requires the dimension to be \( O \left( 1 + R \sqrt{L/(n\varepsilon)} \right) \), which is much smaller than \( O \left( \frac{L^2 R^4}{\varepsilon^2} \log \left( \frac{nLR^2}{\varepsilon} \right) \right) \) in \[33\].

\[1\] IFO algorithm is apparently also a PIFO algorithm.
Finally, we give the lower bound when the objective function is non-convex.

**Theorem 4.7.** Let \( n \geq 2 \) be a positive integer and \( L, \mu, \Delta, \varepsilon \) be positive parameters. Assume additionally that \( \varepsilon^2 \leq \frac{\Delta L}{81648n} \), where \( \alpha = \min \left\{ 1, \frac{\sqrt{3}+1)\mu}{30L}, \frac{n}{180} \right\} \). Then we have

\[
m^\text{NC}_\varepsilon(\Delta, L, \mu) = \Omega \left( n + \frac{\Delta L\sqrt{\alpha}}{\varepsilon^2} \right)
\]

**Remark 4.8.** For \( n > 180 \), we have

\[
\Omega \left( n + \frac{\Delta L\sqrt{\alpha}}{\varepsilon^2} \right) = \Omega \left( n + \frac{\Delta}{\varepsilon^2} \min \{ L, \sqrt{\mu L} \} \right).
\]

Thus, our result is comparable to the one of Zhou and Gu's result (their result only related to IFO algorithms, so our result is stronger). However, from the analysis in Section 4.6, our construction only requires the dimension to be \( O \left( 1 + \frac{\Delta}{\varepsilon^2} \min \{ L/n, \sqrt{\mu L/n} \} \right) \), which is much smaller than \( O \left( \frac{\Delta}{\varepsilon^2} \min \{ L, \sqrt{\mu L} \} \right) \) in [41].

### 4.2.2 Average Smooth Case

Then we extend our results to the weaker assumption: the function class \( \left\{ f_i \right\}_{i=1}^n \) is \( L' \)-average smooth [41]. We start with the case where \( f \) is strongly-convex.

**Theorem 4.9.** Let \( n \geq 4 \) be a positive integer and \( L', \mu, R, \varepsilon \) be positive parameters. Assume additionally that \( \kappa' = \frac{L'}{\mu} \geq 2 \) and \( \varepsilon \leq \frac{L'\mu R^2}{4} \). Then we have

\[
\bar{m}^\text{C}_\varepsilon(R, L', \mu) = \begin{cases} 
\Omega \left( (n+n^{3/4}\sqrt{\kappa'}) \log \left( \frac{1}{\varepsilon} \right) \right), & \text{for } \kappa' = \Omega(\sqrt{n}), \\
\Omega \left( n + \left( \frac{n}{\log(\sqrt{n}/\kappa')} \right) \log \left( \frac{1}{\varepsilon} \right) \right), & \text{for } \kappa' = O(\sqrt{n}).
\end{cases}
\]

**Remark 4.10.** Compared with Zhou and Gu's lower bound \( \Omega \left( n + n^{3/4}\sqrt{\kappa}\log \left( 1/\varepsilon \right) \right) \) for IFO algorithms, Theorem 4.9 shows tighter dependency on \( n \) and supports PIFO algorithms additionally.

We also give the lower bound for general convex case under the \( L' \)-average smooth condition.

**Theorem 4.11.** Let \( n \geq 2 \) be a positive integer and \( L', R, \varepsilon \) be positive parameters. Assume additionally that \( \varepsilon \leq \frac{L'\mu R^2}{4} \). Then we have

\[
\bar{m}^\text{C}_\varepsilon(R, L', 0) = \Omega \left( n + Rn^{3/4}\sqrt{L'/\varepsilon} \right)
\]

**Remark 4.12.** The lower bound in Theorem 4.11 is comparable to the one of Zhou and Gu's result [41].

Finally, we give the lower bound when the objective function is non-convex.

**Theorem 4.13.** Let \( n \geq 2 \) be a positive integer and \( L', \mu, \Delta, \varepsilon \) be positive parameters. Assume additionally that \( \varepsilon^2 \leq \frac{\Delta L'}{435456\sqrt{n}} \), where \( \alpha = \min \left\{ 1, \frac{8(\sqrt{3}+1)\mu}{45L}, \frac{\sqrt{270}}{\mu L} \right\} \). Then we have

\[
m^\text{NC}_\varepsilon(\Delta, L', \mu) = \Omega \left( n + \frac{\Delta L'\sqrt{\alpha}}{\varepsilon^2} \right)
\]

**Remark 4.14.** For \( n > 270 \), we have

\[
\Omega \left( n + \frac{\Delta L'\sqrt{\alpha}}{\varepsilon^2} \right) = \Omega \left( n + \frac{\Delta}{\varepsilon^2} \min \left\{ \sqrt{nL'}, \sqrt{n^{3/4}\mu L'} \right\} \right).
\]

Thus, our result is comparable to the one of Zhou and Gu's result [41]. Their result only related to IFO algorithms, so our result is stronger.
4.3 Framework of Construction

In this subsection, we present our framework of construction. Recall the following class of matrices defined in Section 3.3:

\[ B(m, \omega, \zeta) = \begin{bmatrix} \omega & -1 & \cdots & 1 & -1 \\ 1 & -1 & \cdots & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & \cdots & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(m+1) \times m}. \]

Then we define

\[ A(m, \omega, \zeta) \triangleq B(m, \omega, \zeta)^\top B(m, \omega, \zeta) = \begin{bmatrix} \omega^2 + 1 & -1 & \cdots & -1 \\ -1 & 2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \zeta^2 + 1 \end{bmatrix}. \]

The matrix \( A(m, \omega, \zeta) \) is widely used in the analysis of lower bounds for convex optimization [26, 1, 19, 6, 41].

Denote the \( l \)-th row of the matrix \( B(m, \omega, \zeta) \) by \( b_{l-1}(m, \omega, \zeta)^\top \). Partition the row vectors \( \{b_l(m, \omega, \zeta)^\top\}_{l=1}^m \) by index sets \( L_1, \ldots, L_n \), where \( L_i = \{l : 0 \leq l \leq m, l \equiv i-1 \ (\text{mod} \ n)\} \). Then the adversarial problem is constructed as

\[
\min_{x \in \mathcal{X}} r(x; m, \omega, \zeta, \mathbf{c}) \triangleq \frac{1}{n} \sum_{i=1}^n r_i(x; m, \omega, \zeta, \mathbf{c}),
\]

where \( \mathbf{c} = (c_1, c_2, c_3) \), \( \mathcal{X} = \{x \in \mathbb{R}^m : \|x\|_2 \leq R_x\} \) or \( \mathbb{R}^m \),

\[
r_i(x; m, \omega, \zeta, \mathbf{c}) = \begin{cases} \frac{n}{2} \sum_{l \in L_i} \|b_l(m, \omega, \zeta)^\top x\|_2^2 + \frac{c_2}{2} \|x\|_2^2 + c_2 \sum_{i=1}^{m-1} \Gamma(x_i) - c_3 n \langle e_1, x \rangle, & \text{for } i = 1, \\
\frac{n}{2} \sum_{l \in L_i} \|b_l(m, \omega, \zeta)^\top x\|_2^2 + \frac{c_2}{2} \|x\|_2^2 + c_2 \sum_{i=1}^{m-1} \Gamma(x_i), & \text{for } i = 2, 3, \ldots, n, \end{cases}
\]

and \( \{e_1, e_2, \ldots, e_m\} \) is the standard basis of \( \mathbb{R}^m \). The non-convex function \( \Gamma : \mathbb{R} \to \mathbb{R} \) is

\[
\Gamma(x) \triangleq 120 \int_1^x \frac{t^2(t-1)}{1+t^2} dt.
\]

We can determine the smooth and strongly-convex coefficients of \( r_i \) as follows.

**Proposition 4.15.** Suppose that \( 0 \leq \omega, \zeta \leq \sqrt{2} \) and \( c_1 \geq 0 \).

1. **Convex case.** For \( c_2 = 0 \), we have that \( r_i \) is \((2n + c_1)\)-smooth and \( c_1\)-strongly-convex, and \( \{r_i\}_{i=1}^n \) is \( L' \)-average smooth where

\[
L' = \sqrt{\frac{4}{n} \left[(n + c_1)^2 + n^2\right] + c_1^2}.
\]
2. **Non-convex case.** For $c_1 = 0$, we have that $r_i$ is $(2n + 180c_2)$-smooth and $[-45(\sqrt{3} - 1)c_2]$-weakly-convex, and $\{r_i\}_{i=1}^n$ is $4\sqrt{n + 4050c_2^2}$-average smooth.

Recall the subspaces $\{F_k\}_{k=0}^m$ which are defined as

$$
F_k = \begin{cases} 
\text{span}\{e_1, e_2, \ldots, e_k\}, & \text{for } 1 \leq k \leq m, \\
\{0\}, & \text{for } k = 0.
\end{cases}
$$

The following technical lemma plays a crucial role in our proof.

**Lemma 4.16.** Suppose that $n \geq 2$, $c_1 \geq 0$ and $x \in F_k$, $0 \leq k < m$.

1. **Convex case.** For $c_2 = 0$ and $\omega = 0$, we have that

$$
\nabla r_i(x), \, \text{prox}_{\gamma}^i(x) \in \begin{cases} 
F_{k+1}, & \text{if } i \equiv k + 1 (\text{mod } n), \\
F_k, & \text{otherwise}.
\end{cases}
$$

2. **Non-convex case.** For $c_1 = 0$, $c_2 > 0$, $\zeta = 0$ and $\gamma < \frac{\sqrt{r_i+1}}{60c_2}$, we have that

$$
\nabla r_i(x), \, \text{prox}_{\gamma}^i(x) \in \begin{cases} 
F_{k+1}, & \text{if } i \equiv k + 1 (\text{mod } n), \\
F_k, & \text{otherwise}.
\end{cases}
$$

We omit the parameters of $r_i$ to simplify the presentation.

The proofs of Proposition 4.15 and Lemma 4.16 are given in Appendix Section D.

In short, if $x \in F_k$, then there exists only one $i \in \{1, \ldots, n\}$ such that $h_f(x, i, \gamma)$ could (and only could) provide additional information in $F_{k+1}$. The “only one” property is important to the lower bound analysis for first order stochastic optimization algorithms [19, 41], but these prior constructions only work for IFO rather than PIFO.

When we apply a PIFO algorithm $A$ to solve the Problem (15), Lemma 4.16 implies that $x_t = 0_n$ will hold until algorithm $A$ draws the component $r_1$. Then, for any $t < T_1 = \min_t \{t : i_t = 1\}$, we have $x_t \in F_0$ while $x_{T_1} \in F_1$ holds. The value of $T_1$ can be regarded as the smallest integer such that $x_{T_1} \in F_1$ could hold. Similarly, for $T_1 \leq t < T_2 = \min_t \{t > T_1 : i_t = 2\}$, there holds $x_t \in F_1$ while we can ensure that $x_{T_2} \in F_2$.

We can define $T_k$ to be the smallest integer such that $x_{T_k} \in F_k$ could hold. We give the formal definition of $T_k$ recursively and connect it to geometrically distributed random variables in the following corollary.

**Corollary 4.17.** Assume we employ a PIFO algorithm $A$ to solve the Problem (15). Let

$$
T_0 = 0, \quad \text{and} \quad T_k = \min_t \{t : t > T_{k-1}, i_t \equiv k (\text{mod } n)\} \quad \text{for } k \geq 1.
$$

Then we have

$$
x_t \in F_{k-1}, \quad \text{for } t < T_k, k \geq 1.
$$

Moreover, the random variables $\{Y_k\}_{k \geq 1}$ such that $Y_k \triangleq T_k - T_{k-1}$ are mutual independent and $Y_k$ follows a geometric distribution with success probability $p_k$, where $k' \equiv k (\text{mod } n)$ and $k' \in [n]$. 

34
The proof of Corollary 4.17 is similar to that of Corollary 3.15.

The basic idea of our analysis is that we guarantee that the minimizer of \( r \) does not lie in \( F_k \) for \( k < m \) and assure that the PIFO algorithm extends the space of \( \text{span}\{x_0, x_1, \ldots, x_t\} \) slowly with \( t \) increasing. We know that \( \text{span}\{x_0, x_1, \ldots, x_t\} \subseteq F_{k-1} \) by Corollary 4.17. Hence, \( T_k \) is just the quantity that measures how \( \text{span}\{x_0, x_1, \ldots, x_t\} \) expands. Note that \( T_k \) can be written as the sum of geometrically distributed random variables. Recalling Lemma 2.11, we can obtain how many PIFO calls we need.

**Lemma 4.18.** Let \( H_r(x) \) be a criterion of measuring how \( x \) is close to solution to the Problem (15). If \( M \) satisfies \( 1 < M < m \), \( \min_{x \in X \cap F_M} H_r(x) \geq 9\varepsilon \) and \( N = n(M + 1)/4 \), then we have

\[
\min_{t \leq N} \mathbb{E} H_r(x_t) \geq \varepsilon
\]

**Remark 4.19.** If \( r(x) \) is convex in \( x \), we set \( H_r(x) = r(x) - \min_{x \in X} r(x) \). If \( r(x) \) is nonconvex, we set \( H_r(x) = \|\nabla r(x)\|_2 \).

The proof of Lemma 4.18 is similar to that of Lemma 3.16.

### 4.4 Construction for the Strongly-Convex Case

The analysis of lower bound complexity for the strongly-convex case depends on the following construction.

**Definition 4.20.** For fixed \( L, \mu, R, n \) such that \( L/\mu \geq 2 \), let \( \alpha = \sqrt{2(L/\mu - 1)/n} + 1 \). We define \( f_{SC,i} : \mathbb{R}^m \to \mathbb{R} \) as follows

\[
f_{SC,i}(x) = \lambda r_i \left( \frac{x}{\beta}; m, 0, \sqrt{\frac{2}{\alpha + 1}}, c \right), \text{ for } 1 \leq i \leq n,
\]

where

\[
c = \left( \frac{2n}{L/\mu - 1}, 0, 1 \right), \quad \lambda = \frac{2\mu R^2 \alpha n}{L/\mu - 1} \text{ and } \beta = \frac{2R\sqrt{\alpha n}}{L/\mu - 1}.
\]

Consider the minimization problem

\[
\min_{x \in \mathcal{X}} f_{SC}(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_{SC,i}(x) \tag{17}
\]

where \( \mathcal{X} = \{x \in \mathbb{R}^m : \|x\|_2 \leq R\} \).

Then we have the following proposition.

**Proposition 4.21.** For any \( n \geq 2 \), \( m \geq 2 \), \( f_{SC,i} \) and \( f_{SC} \) in Definition 4.20 satisfy:

1. \( f_{SC,i} \) is \( L \)-smooth and \( \mu \)-strongly-convex. Thus, \( f_{SC} \) is \( \mu \)-strongly-convex.

2. The minimizer of the function \( f_{SC} \) is

\[
x^* = \arg \min_{x \in \mathbb{R}^m} f_{SC}(x) = \frac{2R \sqrt{\alpha}}{\alpha - 1} (q^1, q^2, \ldots, q^m)^	op,
\]

where \( \alpha = \sqrt{2(L/\mu - 1)/n} + 1 \) and \( q = \frac{\alpha - 1}{\alpha + 1} \). Moreover, \( f_{SC}(x^*) = -\frac{\mu R^2 \alpha}{\alpha + 1} \) and \( \|x^*\|_2 \leq R. \)
3. For $1 \leq k \leq m - 1$, we have

$$\min_{x \in X \cap F_k} f_{SC}(x) - \min_{x \in X} f_{SC}(x) \geq \frac{\mu R^2}{\alpha + 1} q^{2k}.$$ 

The proof of Proposition 4.21 is given in Appendix Section D.

Next we show that the functions $\{f_{SC,i}\}_{i=1}^n$ are “hard enough” for any PIFO algorithm $A$, and deduce the conclusion of Theorem 4.3.

**Theorem 4.22.** Consider the minimization problem (17) and $\varepsilon > 0$. Suppose that $n \geq 2$, $\varepsilon \leq \frac{\mu R^2}{18} \left(\frac{\alpha - 1}{\alpha + 1}\right)^2$ and $m = \left[\frac{1}{4} \left(\sqrt{2 \frac{L/\mu - 1}{n}} + 1\right) \log \left(\frac{\mu R^2}{9\varepsilon}\right)\right] + 1$,

where $\alpha = \sqrt[2]{\frac{2(L/\mu - 1)}{n} + 1}$. In order to find $\hat{x} \in X$ such that $\mathbb{E} f_{SC}(\hat{x}) - \min_{x \in X} f_{SC}(x) < \varepsilon$, PIFO algorithm $A$ needs at least $N$ queries, where

$$N = \left\{ \begin{array}{ll}
\Omega \left( n \sqrt{\frac{nL}{\mu}} \log \left(\frac{1}{\varepsilon}\right) \right), & \text{for } \frac{L}{\mu} \geq \frac{n}{2} + 1, \\
\Omega \left( n + \left( \frac{n}{1+(\log(n\mu/L))} \right) \log \left(\frac{1}{\varepsilon}\right) \right), & \text{for } 2 \leq \frac{L}{\mu} < \frac{n}{2} + 1.
\end{array} \right.$$ 

**Proof.** Let $\Delta = \frac{\mu R^2}{\alpha + 1}$. Since $\alpha > 1$, we have $\frac{\mu R^2}{2} < \Delta < \mu R^2$. Let $M = \left[\log(9\varepsilon/\Delta)\right]$, then we have

$$\min_{x \in X \cap F_M} f_{SC}(x) - \min_{x \in X} f_{SC}(x) \geq \Delta q^M \geq 9\varepsilon,$$

where the first inequality is according to the third property of Proposition 4.21.

By Lemma 4.18 if $1 \leq M < m$ and $N = (M + 1)n/4$, we have

$$\min_{t \leq N} \mathbb{E} f_{SC}(x_t) - \min_{x \in X} f_{SC}(x) \geq \varepsilon.$$ 

Therefore, in order to find $\hat{x} \in X$ such that $\mathbb{E} f_{SC}(\hat{x}) - \min_{x \in X} f_{SC}(x) < \varepsilon$, $A$ needs at least $N$ queries.

We estimate $-\log(q)$ and $N$ in two cases.

1. If $L/\mu \geq n/2 + 1$, then $\alpha = \sqrt{2 \frac{L/\mu - 1}{n} + 1} \geq \sqrt{2}$. Observe that function $h(\beta) = \frac{1}{\log(\frac{n+1}{\beta+1})} - \frac{\beta}{2}$ is increasing when $\beta > 1$. Thus, we have

$$-\frac{1}{\log(q)} = \frac{1}{\log \left(\frac{n+1}{\beta+1}\right)} \geq \frac{\alpha}{2} + h(\sqrt{2})$$

$$= \frac{1}{2} \sqrt{2 \frac{L/\mu - 1}{n} + 1 + h(\sqrt{2})}$$

$$\geq \frac{\sqrt{2}}{4} \left( \sqrt{2 \frac{L/\mu - 1}{n} + 1} + h(\sqrt{2}) \right)$$

$$\geq \frac{1}{2} \sqrt{\frac{L/\mu - 1}{n} + \frac{\sqrt{2}}{4} + h(\sqrt{2})},$$

36
\[ N = (M + 1)n/4 = \frac{n}{4} \left( \frac{\log(9\varepsilon/\Delta)}{2\log q} \right) + 1 \]
\[ \geq \frac{n}{8} \left( -\frac{1}{\log(q)} \right) \log \left( \frac{\Delta}{9\varepsilon} \right) \]
\[ \geq \frac{n}{8} \left( \frac{1}{2} \sqrt{\frac{L/\mu - 1}{n}} + \frac{\sqrt{2}}{4} + h(\sqrt{2}) \right) \log \left( \frac{\mu R^2}{18\varepsilon} \right) \]
\[ = \Omega \left( \left( n + \frac{nL}{\mu} \right) \log \left( \frac{1}{\varepsilon} \right) \right). \]

2. If \( 2 \leq L/\mu < n/2 + 1 \), then we have

\[-\log(q) = \log \left( \frac{\alpha + 1}{\alpha - 1} \right) = \log \left( 1 + \frac{2(\alpha - 1)}{\alpha^2 - 1} \right) \]
\[ = \log \left( 1 + \frac{\sqrt{2L/\mu - 1}}{n} + 1 - 1 \right) \leq \log \left( 1 + \frac{\sqrt{2} - 1}{L/\mu - 1} \right) \]
\[ \leq \log \left( \frac{\sqrt{2} - 1/2}{L/\mu - 1} \right) \leq \log \left( \frac{2\sqrt{2} - 1}{L/\mu} \right), \]

where the first inequality and second inequality follow from \( L/\mu - 1 < n/2 \) and the last inequality is according to \( \frac{1}{x - 1} \leq \frac{2}{x} \) for \( x \geq 2 \).

Note that \( n \geq 2 \), thus \( \frac{n}{L/\mu - 1} \leq 2 \leq \frac{n}{L/\mu - 1} \), and hence \( n \geq L/\mu \), i.e. \( \log(n\mu/L) \geq 0 \).

Therefore,

\[ N = (M + 1)n/4 \geq \frac{n}{8} \left( -\frac{1}{\log(q)} \right) \log \left( \frac{\mu R^2}{18\varepsilon} \right) \]
\[ = \Omega \left( \left( n + \frac{nL}{\mu} \right) \log \left( \frac{1}{\varepsilon} \right) \right). \]

Recalling that we assume that \( q^2 \geq \frac{18\varepsilon}{\mu R^2} > \frac{9\varepsilon}{\Delta} \), thus we have

\[ N \geq \frac{n}{8} \left( -\frac{1}{\log(q)} \right) \log \left( \frac{\Delta}{9\varepsilon} \right) \geq \frac{n}{8} \left( -\frac{1}{\log(q)} \right) (-2\log(q)) = \frac{n}{4}. \]

Therefore, \( N = \Omega \left( n + \left( \frac{n}{1 + \log(n\mu/L)} \right) \log \left( \frac{1}{\varepsilon} \right) \right). \)

At last, we must ensure that \( 1 \leq M < m \), that is

\[ 1 \leq \frac{\log(9\varepsilon/\Delta)}{2\log q} < m. \]

Note that \( \lim_{\beta \to +\infty} h(\beta) = 0 \), so \( -1/\log(q) \leq \alpha/2 \). Thus the above conditions are satisfied when

\[ m = \left\lfloor \frac{\log(\mu R^2/(9\varepsilon))}{2(-\log q)} \right\rfloor + 1 \leq \frac{1}{4} \left( \sqrt{2L/\mu - 1} + 1 \right) \log \left( \frac{\mu R^2}{9\varepsilon} \right) + 1 = \mathcal{O} \left( \sqrt{L/n\mu} \log \left( \frac{1}{\varepsilon} \right) \right), \]

37
Proof. Consider the following functions

\[ N = \Omega(\varepsilon) \]

\[ f(x) = \frac{1}{2} \sum_{i=1}^{n} f_i(x) \]

\[ \text{in order to find } x_0 \]

\[ \text{Thus, for } t \leq T \]

\[ \text{such that } G(x) \]

\[ \text{and } \]

\[ \varepsilon \leq \frac{1}{9} \left( \frac{\alpha - 1}{\alpha + 1} \right)^2. \]

For larger \( \varepsilon \), we can apply the following Lemma.

**Lemma 4.23.** For any \( \mu, n, R, \varepsilon \) such that \( n \geq 2 \) and \( \varepsilon \leq LR^2/4 \), there exist \( n \) functions \( \{f_i : \mathbb{R} \rightarrow \mathbb{R}\}_{i=1}^{n} \) such that \( f_i(x) \) is \( L \)-smooth and \( f(x) = \frac{1}{2} \sum_{i=1}^{n} f_i(x) \) is \( \mu \)-strongly-convex. In order to find \( |x| \leq R \) such that \( \mathbb{E} f(\hat{x}) - \min_{|x| \leq R} f(x) < \varepsilon \), PIFO algorithm \( A \) needs at least \( N = \Omega(n) \) queries.

**Proof.** Consider the following functions \( \{G_{SC,i}\}_{i=1}^{n}, G_{SC} : \mathbb{R} \rightarrow \mathbb{R} \), where

\[ G_{SC,i}(x) = \frac{L}{2} x^2 - nLRx, \quad \text{for } i = 1, \]

\[ G_{SC,i}(x) = \frac{L}{2} x^2, \quad \text{for } i = 2, 3, \ldots, n, \]

\[ G_{SC}(x) = \frac{1}{n} \sum_{i=1}^{n} G_{SC,i}(x) = \frac{L}{2} x^2 - LRx. \]

Note that \( \{G_{SC,i}\}_{i=1}^{n} \) is \( L \) smooth and \( \mu \)-strongly-convex for any \( \mu \leq L \). Observe that

\[ x^* = \arg \min_{x \in \mathbb{R}} G_{SC}(x) = R, \]

\[ G_{SC}(0) - G_{SC}(x^*) = \frac{LR^2}{2}, \]

and \( |x^*| = R \). Thus \( x^* = \arg \min_{|x| \leq R} G_{SC}(x) \).

For \( i > 1 \), we have \( \frac{dG_{SC,i}(x)}{dx} \bigg|_{x=0} = 0 \) and \( \text{prox}_{G_{SC,i}}(0) = 0 \). Thus \( x_t = 0 \) will hold till our first-order method \( A \) draws the component \( G_{SC,1} \). That is, for \( t < T = \arg \min \{t : i_t = 1\} \), we have \( x_t = 0 \).

Hence, for \( t \leq \frac{1}{2p_1} \), we have

\[ \mathbb{E} G_{SC}(x_t) - G_{SC}(x^*) \geq \mathbb{E} \left[ G_{SC}(x_t) - G_{SC}(x^*) \bigg| \frac{1}{2p_1} < T \right] \mathbb{P} \left[ \frac{1}{2p_1} < T \right] \]

\[ = \frac{LR^2}{2} \mathbb{P} \left[ \frac{1}{2p_1} < T \right]. \]

Note that \( T \) follows a geometric distribution with success probability \( p_1 \leq 1/n, \) and

\[ \mathbb{P} \left[ T > \frac{1}{2p_1} \right] = \mathbb{P} \left[ T > \left\lfloor \frac{1}{2p_1} \right\rfloor \right] = (1 - p_1) \left( \frac{1}{2p_1} \right)^{\left\lfloor \frac{1}{2p_1} \right\rfloor} \]

\[ \geq (1 - p_1) \frac{1}{2p_1} \geq (1 - 1/n)^n/2 \geq \frac{1}{2}, \]

where the second inequality follows from \( h(z) = \frac{\log(1-z)}{2z} \) is a decreasing function.

Thus, for \( t \leq \frac{1}{2p_1} \), we have

\[ \mathbb{E} G_{SC}(x_t) - G_{SC}(x^*) \geq \frac{LR^2}{4} \geq \varepsilon. \]

Thus, in order to find \( |\hat{x}| \leq R \) such that \( \mathbb{E} G_{SC}(\hat{x}) - G_{SC}(x^*) < \varepsilon \), \( A \) needs at least \( \frac{1}{2p_1} \geq n/2 = \Omega(n) \) queries.
Now we explain that the lower bound in Lemma 4.23 is the same as the lower bound in Theorem 4.22 for $\varepsilon > \frac{\mu R^2}{18} \left(\frac{\alpha - 1}{\alpha + 1}\right)^2$.

**Remark 4.24.** Suppose that

$$\frac{\varepsilon}{\mu R^2} > \frac{1}{18} \left(\frac{\alpha - 1}{\alpha + 1}\right)^2, \quad \alpha = \sqrt{\frac{2}{n}} \frac{n - 1}{n + 1} \kappa = \frac{L}{\mu}$$

1. If $\kappa \geq n/2 + 1$, then we have $\alpha \geq \sqrt{2}$ and

$$\left(n + \sqrt{\kappa n}\right) \log \left(\frac{\mu R^2}{18\varepsilon}\right) \leq 2 \left(n + \sqrt{\kappa n}\right) \log \left(\frac{\alpha + 1}{\alpha - 1}\right)$$

$$\leq \frac{4}{\alpha - 1} \left(n + \sqrt{\kappa n}\right) = O(n) + \frac{4\sqrt{\kappa n}}{(1 - \sqrt{\kappa/2})\alpha}$$

$$\leq O(n) + \frac{4\sqrt{\kappa n}}{\sqrt{2} - 1} \sqrt{\kappa/n} = O(n),$$

where the second inequality follows from $\log(1 + x) \leq x$ and the last inequality is according to $\alpha \geq \sqrt{2\kappa/n}$. That is

$$\Omega(n) = \Omega \left(\left(n + \sqrt{\kappa n}\right) \log \left(\frac{1}{\varepsilon}\right)\right).$$

2. If $2 \leq L/\mu < n/2 + 1$, then we have

$$\left(\frac{n}{1 + (\log(n\mu/L))^+}\right) \log \left(\frac{\mu R^2}{18\varepsilon}\right) \leq \left(\frac{n}{1 + (\log(n\mu/L))^+}\right) \left(2 \log \left(\frac{\alpha + 1}{\alpha - 1}\right)\right)$$

$$\leq \left(\frac{n}{1 + (\log(n\mu/L))^+}\right) \left(2 \log \left(\frac{2\sqrt{2} - 1}{n} \frac{n}{L/\mu}\right)\right) = O(n),$$

where the second inequality follows from (18). That is

$$\Omega(n) = \Omega \left(\left(\frac{n}{1 + (\log(n\mu/L))^+}\right) \log \left(\frac{1}{\varepsilon}\right) + n\right).$$

In summary, we obtain Theorem 4.3.

### 4.5 Construction for the Convex Case

The analysis of lower bound complexity for the convex case depends on the following construction.

**Definition 4.25.** For fixed $L, R, n$, we define $f_{C,i} : \mathbb{R}^m \to \mathbb{R}$ as follows

$$f_{C,i}(x) = \lambda r_i(x/\beta; m, 0, 1, c), \quad \text{for } 1 \leq i \leq n,$$

where

$$c = (0, 0, 1), \quad \lambda = \frac{3LR^2}{2n(m + 1)^3} \quad \text{and} \quad \beta = \frac{\sqrt{3}R}{(m + 1)^{3/2}}.$$

Consider the minimization problem

$$\min_{x \in \mathcal{X}} f_c(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_{C,i}(x).$$

where $\mathcal{X} = \{x \in \mathbb{R}^m : \|x\|_2 \leq R\}$. 

39
Then we have the following proposition.

**Proposition 4.26.** For any \( n \geq 2, m \geq 2 \), the following properties hold:

1. \( f_{C,i} \) is \( L \)-smooth and convex. Thus, \( f_C \) is convex.

2. The minimizer of the function \( f_C \) is

\[
x^* = \arg \min_{x \in \mathbb{R}^m} f_C(x) = \frac{2\xi}{L} (m, m-1, \ldots, 1)^\top,
\]

where \( \xi = \frac{\sqrt{3}}{2} \frac{RL}{(m+1)^{3/2}} \). Moreover, \( f_C(x^*) = -\frac{m\xi^2}{nL} \) and \( \|x^*\|_2 \leq R \).

3. For \( 1 \leq k \leq m \), we have

\[
\min_{x \in X \cap F_k} f_C(x) - \min_{x \in X} f_C(x) = \frac{\xi^2}{nL}(m-k).
\]

The proof of Proposition 4.26 is given in Appendix Section D.

Next we show the lower bound for functions \( f_{C,i} \) defined above.

**Theorem 4.27.** Consider the minimization problem (19) and \( \varepsilon > 0 \). Suppose that \( n \geq 2, \varepsilon \leq \frac{R^2L}{384n} \) and

\[
m = \left\lfloor \sqrt{\frac{R^2L}{24n\varepsilon}} \right\rfloor - 1.
\]

In order to find \( \hat{x} \in X \) such that \( \mathbb{E} f_C(\hat{x}) - \min_{x \in X} f_C(x) < \varepsilon \), PIFO algorithm \( A \) needs at least \( N \) queries, where

\[
N = \Omega \left( n + R\sqrt{nL/\varepsilon} \right).
\]

**Proof.** Since \( \varepsilon \leq \frac{R^2L}{384n} \), we have \( m \geq 3 \). Let \( \xi = \frac{\sqrt{3}}{2} \frac{RL}{(m+1)^{3/2}} \).

For \( M = \left\lfloor \frac{m-1}{2} \right\rfloor \geq 1 \), we have \( m - M \geq (m+1)/2 \), and

\[
\min_{x \in X \cap F_M} f_C(x) - \min_{x \in X} f_C(x) = \frac{\xi^2}{nL}(m-M) = \frac{3R^2L}{4n} \frac{m-M}{(m+1)^3} \geq \frac{3R^2L}{8n} \frac{1}{(m+1)^2} \geq 9\varepsilon,
\]

where the first equation is according to the 3rd property in Proposition 4.26 and the last inequality follows from \( m + 1 \leq R\sqrt{L/(24n\varepsilon)} \).

Similar to the proof of Theorem 4.22 by Lemma 4.18 we have

\[
\min_{t \leq N} \mathbb{E} f_C(x_t) - \min_{x \in X} f_C(x) \geq \varepsilon.
\]

In other words, in order to find \( \hat{x} \in X \) such that \( \mathbb{E} f_C(\hat{x}) - \min_{x \in X} f_C(x) < \varepsilon \), \( A \) needs at least \( N \) queries.

At last, observe that

\[
N = (M+1)n/4 = \frac{n}{4} \left\lfloor \frac{m+1}{2} \right\rfloor \geq \frac{n(m-1)}{8} \geq \frac{n}{8} \left( \sqrt{\frac{R^2L}{24n\varepsilon}} - 2 \right) = \Omega \left( n + R\sqrt{\frac{nL}{\varepsilon}} \right),
\]

where we have recalled \( \varepsilon \leq \frac{R^2L}{384n} \) in last equation. \( \square \)
To derive Theorem 4.5, we also need the following lemma in the case \( \epsilon > \frac{R^2 L}{384 n} \).

**Lemma 4.28.** For any \( L, n, R, \epsilon \) such that \( n \geq 2 \) and \( \epsilon \leq LR^2/4 \), there exist \( n \) functions \( \{f_i : \mathbb{R} \to \mathbb{R}\}_{i=1}^n \) such that \( f_i(x) \) is \( L \)-smooth and \( f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \) is convex. In order to find \( |\hat{x}| \leq R \) such that \( \mathbb{E} f(\hat{x}) - \min_{|x| \leq R} f(x) < \epsilon \), PIFO algorithm \( A \) needs at least \( N = \Omega(n) \) queries.

It is worth noting that Lemma 4.28 follows from Lemma 4.23 and if \( \epsilon > \frac{R^2 L}{384 n} \), then \( \Omega(n) = \Omega\left(n + R\sqrt{\frac{\alpha}{\epsilon}}\right) \). Thus combining Theorem 4.27 and Lemma 4.28, we obtain Theorem 4.5.

### 4.6 Construction for the Nonconvex Case

The analysis of lower bound complexity for the nonconvex case depends on the following construction.

**Definition 4.29.** For fixed \( L, \mu, \Delta, n \), we define \( f_{NC,i} : \mathbb{R}^{m+1} \to \mathbb{R} \) as follows

\[
f_{NC,i}(x) = \lambda r_i \left( \frac{x}{\beta} ; m+1, \sqrt{\alpha}, 0, c \right), \quad \text{for } 1 \leq i \leq n,
\]

where

\[
\alpha = \min \left\{ 1, \frac{(\sqrt{3} + 1)n\mu}{30L}, \frac{n}{180} \right\}, \quad c = (0, \alpha, \sqrt{\alpha}),
\]

\[
m = \left\lfloor \frac{\Delta L \sqrt{\alpha}}{40824 n \epsilon^2} \right\rfloor, \quad \lambda = \frac{3888 n \epsilon^2}{L \alpha^{3/2}} \text{ and } \beta = \sqrt{3} \frac{\lambda n}{L}.
\]

Consider the minimization problem

\[
\min_{x \in \mathbb{R}^{m+1}} f_{NC}(x) \triangleq \frac{1}{n} \sum_{i=1}^n f_{NC,i}(x).
\] (20)

Then we have the following proposition.

**Proposition 4.30.** For any \( n \geq 2 \) and \( \epsilon^2 \leq \frac{\Delta L \alpha}{81648 n} \), the following properties hold:

1. \( f_{NC,i} \) is \( L \)-smooth and \((-\mu)\)-weakly-convex. Thus, \( f_{NC} \) is \((-\mu)\)-weakly-convex.
2. \( f_{NC}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} f_{NC}(x) \leq \Delta \).
3. \( m \geq 2 \) and for \( M = m - 1 \), \( \min_{x \in F_M} \|\nabla f_{NC}(x)\|_2 \geq 9 \epsilon \).

The proof of Proposition 4.30 is given in Appendix Section D.

Next we prove Theorem 4.7.

**Proof of Theorem 4.7.** By Lemma 4.18 and the third property of Proposition 4.30, in order to find \( \hat{x} \in \mathbb{R}^{m+1} \) such that \( \mathbb{E} \|\nabla f_{NC}(\hat{x})\|_2 < \epsilon \), PIFO algorithm \( A \) needs at least \( N = \Omega \left( \frac{\Delta L \sqrt{\alpha}}{\epsilon^2} \right) \) queries, where

\[
N = nm/4 = \Omega \left( \frac{\Delta L \sqrt{\alpha}}{\epsilon^2} \right).
\]

Since \( \epsilon^2 \leq \frac{\Delta L \alpha}{81648 n} \) and \( \alpha \leq 1 \), we have \( \Omega \left( \frac{\Delta L \sqrt{\alpha}}{\epsilon^2} \right) = \Omega \left( n + \frac{\Delta L \sqrt{\alpha}}{\epsilon^2} \right) \).

\(\blacksquare\)
4.7 Construction for the Average Smooth Case

Zhou and Gu [41] established the lower bounds of IFO complexity under the average smooth assumption. Here we demonstrate that our technique can also develop lower bounds of PIFO algorithm under this assumption.

4.7.1 strongly-convex Case

We first consider the minimization problem where the objective function is strongly-convex in \( x \).

For fixed \( L', \mu, R, n, \varepsilon \) such that \( L'/\mu \geq 2 \), we set \( L = \sqrt{n(L'^2 - \mu^2) - \mu^2} \), and consider \( \{f_{SC,i}\}_{i=1}^n \), \( f_{SC} \) and Problem (17) defined in Definition 4.20. We have the following proposition.

**Proposition 4.31.** For \( n \geq 4 \) and \( \kappa' = \frac{L'}{\mu} \geq 2 \), we have that

1. \( f_{SC}(x) \) is \( \mu \)-strongly-convex and \( \{f_{SC,i}\}_{i=1}^n \) is \( L' \)-average smooth.
2. \( \sqrt{\frac{\pi}{2}} L' \leq L \leq \sqrt{\frac{\pi}{4}} L' \) and \( \kappa = \frac{L}{\mu} \geq 2 \).

**Proof.**

1. It is easy to check that \( f_{SC}(x) \) is \( \mu \)-strongly-convex. By Proposition 4.15 and Lemma B.1 \( \{f_{SC,i}\}_{i=1}^n \) is \( L' \)-average smooth, where

\[
\hat{L} = \frac{L - \mu}{2n} \left( \frac{4}{n} \left( \frac{n L/\mu + n}{L/\mu - 1} \right)^2 + n^2 \right) + \left( \frac{2n}{L/\mu - 1} \right)^2 = \sqrt{\frac{2(L^2 + \mu^2)}{n}} + \mu^2 = L'.
\]

2. Clearly, \( L = \sqrt{n(L'^2 - \mu^2) - \mu^2} \leq \sqrt{\frac{\pi}{2}} L' \).

Furthermore, according to \( \kappa' \geq 2 \) and \( n \geq 4 \), we have

\[
L^2 - \frac{n}{4} L'^2 = \frac{n}{4} L'^2 - \frac{n}{2} \mu^2 - \mu^2 = \mu^2 \left( \frac{n}{4} \kappa'^2 - \frac{n}{2} - 1 \right) \geq \mu^2 \left( \frac{n}{2} - 1 \right) \geq 0.
\]

and \( \kappa = \frac{L}{\mu} \geq \frac{\sqrt{\pi L'}}{2\mu} \geq \kappa' \geq 2 \).

This completes the proof. \( \square \)

Recalling Theorem 4.22 we have the following result.

**Theorem 4.32.** Consider the minimization problem (17) and \( \varepsilon > 0 \). Suppose that \( \kappa' = L'/\mu \geq 2 \), \( n \geq 4 \) and \( \varepsilon \leq \frac{\mu^2}{18} \left( \frac{\alpha+1}{\alpha-1} \right)^2 \) where \( \alpha = \sqrt{\frac{2(L/\mu-1)}{n} + 1} \) and \( L = \sqrt{n(L'^2 - \mu^2) - \mu^2} \). In order to find \( \hat{x} \in X \) such that \( \mathbb{E} f_{SC}(\hat{x}) - \min_{x \in X} f_{SC}(x) < \varepsilon \), PIFO algorithm \( A \) needs at least \( N \) queries, where

\[
N = \left\{ \begin{array}{ll}
\Omega \left( \left( n + \frac{n^{3/4} \sqrt{\kappa}}{1+ \log (\sqrt{n/\kappa})} \right) \log \left( \frac{1}{\varepsilon} \right) \right), & \text{for } \kappa' = \Omega(\sqrt{n}) \\
\Omega \left( n + \frac{n}{1+ \log (\sqrt{n/\kappa})} \right) \log \left( \frac{1}{\varepsilon} \right), & \text{for } \kappa' = O(\sqrt{n}).
\end{array} \right.
\]

For large \( \varepsilon \), we can apply the following lemma.

**Lemma 4.33.** For any \( L', \mu, n, R, \varepsilon \) such that \( n \geq 2 \) and \( \varepsilon \leq L'R^2/4 \), there exist \( n \) functions \( \{f_i : \mathbb{R} \rightarrow \mathbb{R}\}_{i=1}^n \) such that \( \{f_i(x)\}_{i=1}^n \) is \( L' \)-average smooth and \( f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \) is \( \mu \)-strongly-convex. In order to find \( |\hat{x}| \leq R \) such that \( \mathbb{E} f(\hat{x}) - \min_{|x| \leq R} f(x) < \varepsilon \), PIFO algorithm \( A \) needs at least \( N = \Omega(n) \) queries.

42
Proof. Note that \( \{G_{SC,i}\}_{i=1}^n \) defined in proof of Lemma 4.23 is also \( L \)-average smooth, so Lemma 4.33 holds for the same reason.

Similar to Remark 4.24, we can show that the lower bound in Lemma 4.33 is the same as the lower bound in Theorem 4.32 for \( \epsilon > \frac{\mu R^2}{18} (\frac{\alpha-1}{\alpha+1})^2 \). Then we obtain Theorem 4.9.

4.7.2 Convex Case

We now consider the minimization problem where the objective function is not strongly-convex in \( x \).

For fixed \( L', R, n, \epsilon \), we set \( L = \sqrt{\frac{n}{2}} L' \), and consider \( \{f_{C,i}\}_{i=1}^n \), \( f_C \) and Problem (19) defined in Definition 4.25. It follows from Proposition 4.15 and Lemma B.1 that \( f_C \) is convex and \( \{f_{C,i}\}_{i=1}^n \) is \( L' \)-average smooth. By Theorem 4.27, we have the following conclusion.

**Theorem 4.34.** Consider the minimization problem (19) and \( \epsilon > 0 \). Suppose that

\[
n \geq 2, \quad \epsilon \leq \frac{\sqrt{2} R L'}{768 \sqrt{n}} \quad \text{and} \quad m = \left\lfloor \frac{\sqrt{18}}{12} R n^{-1/4} \sqrt{\frac{L'}{\epsilon}} \right\rfloor - 1.
\]

In order to find \( \hat{x} \in \mathcal{X} \) such that \( \mathbb{E} f_C(\hat{x}) - \min_{x \in \mathcal{X}} f_C(x) < \epsilon \), PIFO algorithm \( A \) needs at least \( N = \Omega(n + R n^{3/4} \sqrt{\frac{L'}{\epsilon}}) \) queries, where

\[
N = \Omega(n + R n^{3/4} \sqrt{\frac{L'}{\epsilon}}).
\]

Similar to Lemma 4.28, we also need the following lemma for the case \( \epsilon > \frac{\sqrt{2} R L'}{768 \sqrt{n}} \).

**Lemma 4.35.** For any \( L', n, R, \epsilon \) such that \( n \geq 2 \) and \( \epsilon \leq L'R^2/4 \), there exist \( n \) functions \( \{f_i : \mathbb{R} \to \mathbb{R}\}_{i=1}^n \) such that \( \{f_i(x)\}_{i=1}^n \) is \( L' \)-average smooth and \( f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x) \) is convex. In order to find \( |\hat{x}| \leq R \) such that \( \mathbb{E} f(\hat{x}) - \min_{|x| \leq R} f(x) < \epsilon \), PIFO algorithm \( A \) needs at least \( N = \Omega(n) \) queries.

Proof. Note that \( \{G_{SC,i}\}_{i=1}^n \) defined in proof of Lemma 4.23 is also \( L \)-average smooth, so Lemma 4.33 holds for the same reason.

Note that if \( \epsilon > \frac{\sqrt{2} R L'}{768 \sqrt{n}} \), then \( \Omega(n) = \Omega(n + R n^{3/4} \sqrt{\frac{L'}{\epsilon}}) \). In summary, we obtain Theorem 4.11.

4.7.3 Nonconvex Case

The analysis of lower bound complexity for the non-convex case under the average smooth assumption depends on the following construction.

**Definition 4.36.** For fixed \( L', \mu, \Delta, n \), we define \( \tilde{f}_{NC,i} : \mathbb{R}^{m+1} \to \mathbb{R} \) as follows

\[
\tilde{f}_{NC,i}(x) = \lambda r_i \left( x/\beta; m + 1, \sqrt{\alpha}, 0, c \right), \quad \text{for } 1 \leq i \leq n,
\]

where

\[
\alpha = \min \left\{ 1, \frac{8(\sqrt{3} + 1)\sqrt{\mu}}{45L'}, \sqrt{\frac{n}{270}} \right\}, \quad c = (0, \alpha, \sqrt{\alpha}),
\]
\[ m = \left\lfloor \frac{\Delta L' \sqrt{\alpha}}{217728 \sqrt{n \varepsilon^2}} \right\rfloor, \quad \lambda = \frac{20736 \sqrt{n \varepsilon^2}}{L' \alpha^{3/2}} \quad \text{and} \quad \beta = 4 \sqrt{\lambda \sqrt{n}/L'}. \]

Consider the minimization problem

\[
\min_{\mathbf{x} \in \mathbb{R}^{m+1}} \bar{f}_{\text{NC}}(\mathbf{x}) \triangleq \frac{1}{n} \sum_{i=1}^{n} \bar{f}_{\text{NC},i}(\mathbf{x}).
\]

Then we have the following proposition.

**Proposition 4.37.** For any \( n \geq 2 \) and \( \varepsilon^2 \leq \frac{\Delta L' \alpha}{435456 \sqrt{n}} \), the following properties hold:

1. \( \bar{f}_{\text{NC},i} \) is \((-\mu)\)-weakly-convex and \( \{\bar{f}_{\text{NC},i}\}_{i=1}^{n} \) is \( L' \)-average smooth. Thus, \( f_{\text{NC}} \) is \((-\mu)\)-weakly-convex.

2. \( f_{\text{NC}}(0_{m+1}) - \min_{\mathbf{x} \in \mathbb{R}^{m+1}} f_{\text{NC}}(\mathbf{x}) \leq \Delta \).

3. \( m \geq 2 \) and for \( M = m - 1 \), \( \min_{\mathbf{x} \in \mathcal{F}_M} \|\nabla f_{\text{NC}}(\mathbf{x})\|_2 \geq 9 \varepsilon \).

The proof of Proposition 4.37 is given in Appendix Section D.

Next we prove Theorem 4.13.

**Proof of Theorem 4.13.** By Lemma 4.18 and the third property of Proposition 4.37, in order to find \( \hat{\mathbf{x}} \in \mathbb{R}^{m+1} \) such that \( \mathbb{E} \|\nabla f_{\text{NC}}(\hat{\mathbf{x}})\|_2 < \varepsilon \), PIFO algorithm \( \mathcal{A} \) needs at least \( N \) queries, where

\[ N = nm/4 = \Omega \left( \frac{\Delta L' \sqrt{n \alpha \varepsilon^2}}{\varepsilon^2} \right). \]

Since \( \varepsilon^2 \leq \frac{\Delta L' \alpha}{435456 \sqrt{n}} \) and \( \alpha \leq 1 \), we have \( \Omega \left( \frac{\Delta L' \sqrt{n \alpha \varepsilon^2}}{\varepsilon^2} \right) = \Omega \left( n + \frac{\Delta L' \sqrt{n \alpha \varepsilon^2}}{\varepsilon^2} \right). \)

## 5 Conclusion and Future Work

In this paper, we have proved the lower bounds of PIFO complexity for first-order algorithms to find \( \varepsilon \)-suboptimal solutions or \( \varepsilon \)-approximate stationary points of finite-sum minimax optimization, where the objective function is the average of \( n \) individual functions. There still remain some open problems. In the cases where each component \( f_i \) is \( L \)-smooth or in the nonconvex-strongly-concave case, there is no stochastic optimization algorithm that could match our lower bounds. Moreover, it would be interesting to apply our construction approach to address the lower bound for general nonconvex-concave cases.

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A Results of the Sum of Geometric Distributions

In this section, we present the approach to prove Lemma 2.12.

We first present some results about $f_{2,j}$, which is defined in Equation 3.

Lemma A.1. The following properties hold for the function $f_{2,j}$.

1. For $j \geq 1$, $p_1, p_2 \in (0, 1]$, it holds that

$$f_{2,j}(p_1, p_2) = \begin{cases} 
jp_1(1 - p_1)^{j-1} + (1 - p_1)^j, & \text{if } p_1 = p_2, \\
\frac{p_2(1 - p_2)^j - p_1(1 - p_1)^j}{p_2 - p_1}, & \text{otherwise.}
\end{cases}$$

2. For $j \geq 2, p_1 \neq p_2$, we have

$$f_{2,j}(p_1, p_2) > f_{2,j} \left( \frac{p_1 + p_2}{2}, \frac{p_1 + p_2}{2} \right).$$

Proof. 1. Let $Y_1 \sim \text{Geo}(p_1), Y_2 \sim \text{Geo}(p_2)$ be two independent random variables. Then

$$\mathbb{P} [Y_1 + Y_2 > j] = \sum_{l=1}^{j} \mathbb{P} [Y_1 = l] \mathbb{P} [Y_2 > j - l] + \mathbb{P} [Y_1 > j]$$

$$= \sum_{l=1}^{j} (1 - p_1)^{l-1} p_1 (1 - p_2)^{j-l} + (1 - p_1)^j$$

$$= p_1 (1 - p_2)^{j-1} \sum_{l=1}^{j} \left( \frac{1 - p_1}{1 - p_2} \right)^{l-1} + (1 - p_1)^j.$$

If $p_1 = p_2$, then $\mathbb{P} [Y_1 + Y_2 > j] = jp_1 (1 - p_1)^{j-1} + (1 - p_1)^j$; and if $p_1 < p_2$, we have

$$\mathbb{P} [Y_1 + Y_2 > j] = p_1 \frac{(1 - p_1)^j - (1 - p_2)^j}{p_2 - p_1} + (1 - p_1)^j$$

$$= \frac{p_2 (1 - p_1)^j - p_1 (1 - p_2)^j}{p_2 - p_1}.$$

2. Now we suppose that $p_1 + p_2 = c$ and $p_1 < p_2$. Consider

$$h(p_1) \triangleq f_{2,j}(p_1, c - p_1) = \frac{(c - p_1)(1 - p_1)^j - p_1 (1 + p_1 - c)^j}{c - 2p_1},$$

where $p_1 \in (0, c/2)$. It is clear that

$$h(c/2) \triangleq \lim_{p_1 \to c/2} h(p_1) = f_{2,j} \left( c/2, c/2 \right).$$

If $h'(p_1) < 0$ for $p_1 \in (0, c/2)$, then there holds $h(p_1) > h(c/2)$, i.e.,

$$f_{2,j}(p_1, p_2) > f_{2,j} \left( \frac{p_1 + p_2}{2}, \frac{p_1 + p_2}{2} \right).$$

Note that

$$h'(p_1) = \frac{-(1 - p_1)^j - j(c - p_1)(1 - p_1)^{j-1} - (1 + p_1 - c)^j - j p_1 (1 + p_1 - c)^{j-1}}{c - 2p_1}.$$
\[+2 \frac{(c-p_1)(1-p_1)^j - p_1(1+p_1-c)^j}{(c-2p_1)^2}\]
\[= \frac{[c(1-p_1) - j(c-p_1)(c-2p_1)](1-p_1)^{j-1} - [c(1+p_1-c) + jp_1(c-2p_1)](1+p_1-c)^{j-1}}{(c-2p_1)^2}.\]

Hence \( h'(p_1) < 0 \) is equivalent to
\[\frac{c(1-p_1) - j(c-p_1)(c-2p_1)}{c(1+p_1-c) + jp_1(c-2p_1)} < \left(\frac{1+p_1-c}{1-p_1}\right)^{j-1}.\] \hspace{1cm} (22)

Observe that
\[\frac{c(1-p_1) - j(c-p_1)(c-2p_1)}{c(1+p_1-c) + jp_1(c-2p_1)} = 1 - \frac{(1-p_1)(1+p_1-c)}{c(1+p_1-c) + jp_1(c-2p_1)} = 1 - \frac{j-1}{c-2p_1 + jp_1/c}.
\]

Denoting \( x = \frac{1+p_1-c}{c-2p_1} \), inequality \( \text{(22)} \) can be written as
\[1 - \frac{j-1}{x + jp_1/c} < \left(\frac{x}{x+1}\right)^{j-1}.
\]

Note that
\[(x+1)^j - j/2(x+1)^{j-1} = x^j + \sum_{l=0}^{j-1} \left(\frac{j}{l} \right) \frac{1}{2} \left(\frac{j-1}{l}\right) x^l
\]
\[= x^j + \sum_{l=0}^{j-1} \left[\left(\frac{j}{l} - \frac{j-1}{l}\right) \left(\frac{j-1}{l}\right)\right] x^l
\]
\[\leq x^j + j/2x^j - 1 = x^{j-1}(x+j/2).
\]

That is
\[(x+1)^{j-1}(x+j/2) - (j-1)(x+1)^{j-1} \leq x^{j-1}(x+j/2).
\]

Consequently, we have
\[\left(\frac{x}{x+1}\right)^{j-1} \geq 1 - \frac{j-1}{x+j/2} > 1 - \frac{j-1}{x+jp_1/c},
\]
which is the result we desired.

Now we give the proof of Lemma 2.12.

Proof of Lemma 2.12 We first prove continuity of the function \( f_{m,j} \). Actually, we can prove that
\[|f_{m,j}(p_1, p_2, \ldots, p_m) - f_{m,j}(p'_1, p_2, \ldots, p_m)| \leq j|p_1 - p'_1|.\] \hspace{1cm} (23)

Recall that
\[f_{m,j}(p_1, p_2, \ldots, p_m) \triangleq P\left[\sum_{i=1}^{m} Y_i > j\right].\]
Lemma A.1 implies the desired result holds for $m$ we have $\xi$ where

Moreover, by symmetry of the function $f$ with the domain $f$ which implies that $\bar{\epsilon}_l |p_1 - p'_1| \leq j |p_1 - p'_1|$, where we have used $P$ independent of each $\{\xi_l\}_{l=1}^{m}$, then by mean value theorem for $1 \leq l \leq j - 1$ there holds

$$\left|P[Y_1 > l] - P[Y'_1 > j]\right| = \left|(1 - p_1)^l - (1 - p'_1)^l\right|$$

$$= |l(1 - \xi)^{l-1}| |p_1 - p'_1|$$

$$\leq l |p_1 - p'_1| \leq j |p_1 - p'_1|,$$

where $\xi$ lies on the interval $[p_1, p'_1]$. Consequently, with $Z \triangleq \sum_{i=2}^{m} Y_i$, we conclude that

$$\left|f_{m,j}(p_1, p_2, \ldots, p_m) - f_{m,j}(p'_1, p_2, \ldots, p_m)\right|$$

$$= \left|P[Y_1 + Z > j] - P[Y'_1 + Z > j]\right|$$

$$= \sum_{l=1}^{j-1} P[Z = l] \left|P[Y_1 > j - l] + P[Z > j - 1] - \sum_{l=1}^{j-1} P[Z = l] P[Y'_1 > j - l] + P[Z > j - 1]\right|$$

$$\leq \sum_{l=1}^{j-1} P[Z = l] \left|P[Y_1 > j - l] - P[Y'_1 > j - l]\right|$$

$$\leq j |p_1 - p'_1| \sum_{l=1}^{j-1} P[Z = l]$$

$$= j |p_1 - p'_1| \sum_{l=1}^{j-1} P[Z = l] \leq j |p_1 - p'_1|,$$

where we have used $P[Y_1 > 0] = 1$ in the second equality.

Following from Equation (23) and symmetry of the function $f_{m,j}$, we know that

$$|f_{m,j}(p_1, p_2, \ldots, p_m) - f_{m,j}(p'_1, p_2, \ldots, p'_m)| \leq j \sum_{i=1}^{m} |p_i - p'_i|,$$

which implies that $f_{m,j}$ is a continuous function.

Furthermore, following the way we obtain the Equation (23) and the fact that

$$|(1 - p_1)^l - 1| \leq lp_1, \quad l = 1, 2, \ldots, j - 1,$$

we have

$$|f_{m,j}(p_1, p_2, \ldots, p_m) - 1| \leq j p_1.$$

Moreover, by symmetry of the function $f_{m,j}$, it holds that

$$1 - f_{m,j}(p_1, p_2, \ldots, p_m) \leq j \min\{p_1, p_2, \ldots, p_m\}.$$

For $1 \leq j \leq m - 1$, we have $f_{m,j}(p_1, p_2, \ldots, p_m) \equiv 1$ and the desired result is apparent. Then Lemma A.1 implies the desired result holds for $m = 2$.

For $m \geq 3$, $j \geq m$ and $c \in (0, m)$, our goal is to find the minimal value of $f_{m,j}(p_1, p_2, \ldots, p_m)$ with the domain

$$\mathcal{B} = \left\{(p_1, p_2, \ldots, p_m) \mid \sum_{i=1}^{m} p_m = c, \; p_i \in (0, 1) \text{ for } i \in [m]\right\}.$$
For $j \geq m$, note that

$$f_{m,j}(c/m,c/m, \ldots, c/m) = \mathbb{P} \left[ \sum_{i=1}^{m} Z_i > j \right] \leq \mathbb{P} \left[ \sum_{i=1}^{m} Z_i > m \right]$$

$$= 1 - \mathbb{P} \left[ \sum_{i=1}^{m} Z_i \leq m \right] = 1 - \mathbb{P} [Z_1 = 1, Z_2 = 1, \ldots, Z_m = 1]$$

$$= 1 - \left( \frac{c}{m} \right)^m < 1,$$

where $\{Z_i \sim \text{Geo}(c/m)\}_{i=1}^{m}$ are independent random variables, and we have used that $\mathbb{P} [Z_i \geq 1] = 1$ for $i \in [m]$.

By Equation $24$, if there is an index $i$ satisfies $p_i < \delta \triangleq \frac{1-f_{m,j}(c/m,c/m, \ldots, c/m)}{2} > 0$, then we have

$$f_{m,j}(p_1, p_2, \ldots, p_m) \geq 1 - j p_i > f_{m,j}(c/m,c/m, \ldots, c/m).$$

Therefore, we just need to find the minimal value of $f_{m,j}(p_1, p_2, \ldots, p_m)$ with the domain

$$B' = \left\{ (p_1, p_2, \ldots, p_m) \mid \sum_{i=1}^{m} p_m = c, \ p_i \in [\delta, 1] \text{ for } i \in [m] \right\},$$

which is a compact set. Hence, by continuity of $f_{m,j}$, we know that there exists $(q_1, q_2, \ldots, q_m) \in B'$ such that

$$\min_{(p_1, p_2, \ldots, p_m) \in B'} f_{m,j}(p_1, p_2, \ldots, p_m) = f_{m,j}(q_1, q_2, \ldots, q_m).$$

Suppose that there are indexes $k, l \in [m]$ such that $q_k < q_l$. By symmetry of the function $f_{m,j}$, we assume that $q_1 < q_2$.

Let $\{X'_1, X'_2\} \cup \{X_i\}_{i=1}^{m}$ be independent geometric random variables and $X'_1, X'_2 \sim \text{Geo}(\frac{q_1+q_2}{2})$, $X_i \sim \text{Geo}(q_i)$ for $i \in [m]$. Denoting $Z' = \sum_{i=3}^{m} X_i$, we have

$$f_{m,j}(q_1, q_2, \ldots, q_m)$$

$$= \mathbb{P} [X_1 + X_2 + Z' > j]$$

$$= \sum_{l=1}^{j-1} \mathbb{P} [Z' = l] \mathbb{P} [X_1 + X_2 > j - l] + \mathbb{P} [Z' > j - 1]$$

$$\geq \sum_{l=1}^{j-1} \mathbb{P} [Z' = l] \mathbb{P} [X'_1 + X'_2 > j - l] + \mathbb{P} [Z' > j - 1]$$

$$= \mathbb{P} [X'_1 + X'_2 + Z' > j]$$

$$= f_{m,j} \left( \frac{q_1 + q_2}{2}, \frac{q_1 + q_2}{2}, \ldots, q_m \right),$$

where the inequality is according to Lemma $A.1$.

However, for $l = m-2$, it holds that $\mathbb{P} [Z' = m-2] = 1-\prod_{i=2}^{m} q_i > 0$ and $\mathbb{P} [X'_1 + X'_2 > j - m + 2] > \mathbb{P} [X'_1 + X'_2 > j - m + 2]$ by Lemma $A.1$ which implies that

$$f_{m,j}(q_1, q_2, \ldots, q_m) > f_{m,j} \left( \frac{q_1 + q_2}{2}, \frac{q_1 + q_2}{2}, \ldots, q_m \right).$$

51
Note that \( \frac{q_1 + q_2}{2} + \frac{q_1 + q_2}{2} + \sum_{i=2}^{m} q_i = c \) and \( \frac{q_1 + q_2}{2} \in [\delta, 1] \). Hence we have

\[
\left( \frac{q_1 + q_2}{2}, \frac{q_1 + q_2}{2}, \ldots, q_m \right) \in \mathcal{B}',
\]

which contradicts the fact that \((q_1, q_2, \ldots, q_m)\) is the optimal point in \(\mathcal{B}'\).

Therefore, we can conclude that

\[
f_{m,j}(p_1, p_2, \ldots, p_m) \geq f_{m,j}\left( \frac{\sum_{i=1}^{m} p_i}{m}, \frac{\sum_{i=1}^{m} p_i}{m}, \ldots, \frac{\sum_{i=1}^{m} p_i}{m} \right).
\]

\[
\square
\]

## B Technical Lemmas

In this section, we present some technical lemmas.

**Lemma B.1.** Suppose \( f(x, y) \) is \((\mu_x, \mu_y)\)-convex-concave and \(L\)-smooth, then the function \( \hat{f}(x, y) = \lambda f(x/\beta, y/\beta) \) is \((\lambda \mu_x/\beta, \lambda \mu_y/\beta)\)-convex-concave and \(\lambda L/\beta^2\)-smooth. Moreover, if \( \{f_i(x, y)\}_{i=1}^{n} \) is \(L'\)-average smooth, then the function class \( \{\hat{f}_i(x, y) = \lambda f_i(x/\beta, y/\beta)\}_{i=1}^{n} \) is \(\lambda L'/\beta^2\)-average smooth.

**Lemma B.2.** Suppose that \(X = \{x \in \mathbb{R}^d : \|x\|_2 \leq R_x\}\), then we have

\[
\mathcal{P}_X(x) = \begin{cases} 
  x, & \text{if } x \in X, \\
  \frac{R_x}{\|x\|_2}x, & \text{otherwise}.
\end{cases}
\]

**Remark B.3.** By Lemma B.2, vectors \(\mathcal{P}_X(x)\) and \(x\) are always collinear.

**Proposition B.4** (Lemmas 2,3,4, [5]). Let \(G_{NC} : \mathbb{R}^{m+1} \rightarrow \mathbb{R} \) be

\[
G_{NC}(x; \omega, m+1) = \frac{1}{2} \|B(m+1, \omega, 0)x\|_2^2 - \omega^2 \langle e_1, x \rangle + \omega^4 \sum_{i=1}^{m} \Gamma(x_i).
\]

For any \(0 < \omega \leq 1\), it holds that

1. \(\Gamma(x)\) is 180-smooth and \([-45(\sqrt{3} - 1)]\)-weakly convex.
2. \(G_{NC}(0_{m+1}; \omega, m+1) - \min_{x \in \mathbb{R}^{m+1}} G_{NC}(x; \omega, m+1) \leq \omega^2/2 + 10\omega^4 m.\)
3. For any \(x \in \mathbb{R}^{m+1}\) such that \(x_m = x_{m+1} = 0\), \(G_{NC}(x; \omega, m)\) is \((4 + 180\omega^4)\)-smooth and \([-45(\sqrt{3} - 1)\omega^4]\)-weakly convex and

\[
\|\nabla G_{NC}(x; \omega, m)\|_2 \geq \omega^3/4.
\]

**Lemma B.5.** Suppose that \(0 < \lambda_2 < (2 + 2\sqrt{2})\lambda_1\), then \(z = 0\) is the only real solution to the equation

\[
\lambda_1 z + \lambda_2 \frac{z^2(z - 1)}{1 + z^2} = 0.
\]

(25)
Proof. Since $0 < \lambda_2 < (2 + 2\sqrt{2})\lambda_1$, we have
\[
\lambda_2^2 - 4\lambda_1(\lambda_1 + \lambda_2) < 0,
\]
and consequently, for any $z$, $(\lambda_1 + \lambda_2)z^2 - \lambda_2z + \lambda_1 > 0$.

On the other hand, we can rewrite Equation (25) as
\[
z((\lambda_1 + \lambda_2)z^2 - \lambda_2z + \lambda_1) = 0.
\]
Clearly, $z = 0$ is the only real solution to Equation (25). \qed

Lemma B.6. Suppose that $0 < \lambda_2 < (2 + 2\sqrt{2})\lambda_1$ and $\lambda_3 > 0$, then $z_1 = z_2 = 0$ is the only real solution to the equation
\[
\begin{align*}
\lambda_1z_1 + \lambda_3(z_1 - z_2) + \lambda_2\frac{z_1^2(z_1 - 1)}{1 + z_1^2} &= 0. \\
\lambda_1z_2 + \lambda_3(z_2 - z_1) + \lambda_2\frac{z_2^2(z_2 - 1)}{1 + z_2^2} &= 0.
\end{align*}
\]

Proof. If $z_1 = 0$, then $z_2 = 0$. So let assume that $z_1z_2 \neq 0$. Rewrite the first equation of Equations (26) as
\[
\frac{\lambda_1 + \lambda_3}{\lambda_3} + \frac{\lambda_2}{\lambda_3} \frac{z_1(z_1 - 1)}{1 + z_1^2} = \frac{z_2}{z_1}.
\]
Note that
\[
\frac{1 - \sqrt{2}}{2} \leq \frac{z(z - 1)}{1 + z^2}.
\]
Thus, we have
\[
\frac{\lambda_1 + \lambda_3}{\lambda_3} + \frac{\lambda_2}{\lambda_3} \frac{1 - \sqrt{2}}{2} \leq \frac{z_2}{z_1}.
\]
Similarly, it also holds
\[
\frac{\lambda_1 + \lambda_3}{\lambda_3} + \frac{\lambda_2}{\lambda_3} \frac{1 - \sqrt{2}}{2} \leq \frac{z_1}{z_2}.
\]
By $0 < \lambda_2 < (2 + 2\sqrt{2})\lambda_1$, we know that $\lambda_1 + \frac{1 - \sqrt{2}}{2}\lambda_2 > 0$. Thus
\[
\frac{\lambda_1 + \lambda_3}{\lambda_3} + \frac{\lambda_2}{\lambda_3} \frac{1 - \sqrt{2}}{2} > 1.
\]
Since $z_1/z_2 > 1$ and $z_2/z_1 > 1$ can not hold at the same time, so we get a contradiction. \qed

Lemma B.7. Define the function
\[
J_{k,\beta}(y_1, y_2, \ldots, y_k) \triangleq y_k^2 + \sum_{i=2}^{k} (y_i - y_{i-1})^2 + (y_1 - \beta)^2.
\]
Then we have $\min J_{k,\beta}(y_1, \ldots, y_k) = \frac{\beta^2}{k+1}$.

Proof. Letting the gradient of $J_{k,\beta}$ equal to zero, we get
\[
2y_k - y_{k-1} = 0, \quad 2y_1 - y_2 - \beta = 0, \quad \text{and} \quad y_{i+1} - 2y_i + y_{i-1} = 0, \quad \text{for} \quad i = 2, 3, \ldots, k - 1.
\]
That is,
\[
y_i = \frac{k - i + 1}{k + 1}\beta \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{27}
\]
Thus by substituting Equation (27) into the expression of $J_{k,\beta}(y_1, y_2, \ldots, y_k)$, we achieve the desired result. \qed
C Proofs for Section 3

In this section, we present some omitted proofs in Section 3.

C.1 Proofs of Proposition 3.13 and Lemma 3.14

Let $\tilde{B}(m, \zeta)$ denote the last $m$ rows of $B(m, 0, \zeta)$ and $\tilde{b}_l(m, \zeta) = b_l(m, 0, \zeta)$ for $0 \leq l \leq m$. Note that $\tilde{b}_0(m, \zeta) = 0_m$. For simplicity, we omit the parameters of $B$, $b_l$ and $\tilde{r}_i$. Then we have $\tilde{B} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m)^\top$.

Recall that $L_i = \{l : 0 \leq l \leq m, l \equiv i - 1(\text{mod} n)\}, i = 1, 2, \ldots, n$.

For $1 \leq i \leq n$, let $\tilde{B}_i$ be the submatrix of $\tilde{B}$ whose rows are $\{\tilde{b}_l\}_{l \in L_i}$. Note that $\tilde{B} = \sum_{l=1}^m e_l \tilde{b}_l^\top$ and $\tilde{B}_i = \sum_{l \in L_i} e_l \tilde{b}_l^\top$. Then $\tilde{r}_i$ can be written as

$$\tilde{r}_i(x, y) = n \langle y, \tilde{B}_i x \rangle + \frac{c_1}{2} \|x\|_2^2 - \frac{\tilde{c}_2}{2} \|y\|_2^2 - n \langle e_1, x \rangle 1_{\{i=1\}}.$$

**Proof of Proposition 3.13.** Firstly, it is clear that $\tilde{r}_i$ is $(\tilde{c}_1, \tilde{c}_2)$-convex-concave.

Next, note that for $l_1, l_2 \in L_i$ and $l_1 \neq l_2$, we have $|l_1 - l_2| \geq n \geq 2$, thus $\tilde{b}_{l_1}^\top \tilde{b}_{l_2} = 0$. Since $\zeta \leq 2$, $\tilde{b}_l^\top \tilde{b}_l \leq 2$, it follows that

$$\left\| \sum_{l \in L_i} \tilde{b}_l e_l^\top y \right\|^2_2 = \sum_{l \in L_i} y^\top e_l \tilde{b}_l^\top \tilde{b}_l e_l^\top y \leq \sum_{l \in L_i} \left( e_l^\top y \right)^2 \leq 2 \|y\|_2^2,$$

$$\left\| \sum_{l \in L_i} e_l \tilde{b}_l^\top x \right\|^2_2 = \sum_{l \in L_i} \left( \tilde{b}_l^\top x \right)^2 \leq \sum_{l \in L_i \backslash \{m\}} 2 (x_l^2 + x_{l+1}^2) + \zeta^2 x_m^2 1_{\{m \in L_i\}} \leq 2 \|x\|_2^2.$$

Note that

$$\nabla_x \tilde{r}_i(x, y) = n \tilde{B}_i^\top y + \tilde{c}_1 x - n e_1 1_{\{i=1\}};$$

$$\nabla_y \tilde{r}_i(x, y) = n \tilde{B}_i x - \tilde{c}_2 y.$$

With $u = x_1 - x_2$ and $v = y_1 - y_2$, we have

$$\|\nabla \tilde{r}_i(x_1, y_1) - \nabla \tilde{r}_i(x_2, y_2)\|_2^2$$

$$= \|\nabla_x \tilde{r}_i(x_1, y_1) - \nabla_x \tilde{r}_i(x_2, y_2)\|_2^2 + \|\nabla_y \tilde{r}_i(x_1, y_1) - \nabla_y \tilde{r}_i(x_2, y_2)\|_2^2$$

$$= \|\tilde{c}_1 u + n \sum_{l \in L_i} \tilde{b}_l e_l^\top v\|_2^2 + \|\tilde{c}_2 v - n \sum_{l \in L_i} e_l \tilde{b}_l^\top u\|_2^2$$

$$\leq 2 \left( \frac{\tilde{c}_1^2}{2} \|u\|_2^2 + \frac{\tilde{c}_2^2}{2} \|v\|_2^2 \right) + 2n^2 \left\| \sum_{l \in L_i} \tilde{b}_l e_l^\top v \right\|^2_2 + 2n^2 \left\| \sum_{l \in L_i} e_l \tilde{b}_l^\top u \right\|^2_2$$

$$\leq 2 \left( \frac{\tilde{c}_1^2}{2} \|u\|_2^2 + \frac{\tilde{c}_2^2}{2} \|v\|_2^2 \right) + 4n^2 \sum_{l \in L_i} \left( e_l^\top v \right)^2 + 2n^2 \sum_{l \in L_i} \left( \tilde{b}_l^\top u \right)^2$$

$$\leq (2 \max\{\tilde{c}_1, \tilde{c}_2\}^2 + 4n^2) \left( \|u\|_2^2 + \|v\|_2^2 \right).$$
where the first inequality follows from \((a + b)^2 \leq 2(a^2 + b^2)\). In addition,

\[
\frac{1}{n} \sum_{i=1}^{n} \|\nabla \tilde{r}_i(x_1, y_1) - \nabla \tilde{r}_i(x_2, y_2)\|^2_2 \\
\leq 2 \left( \hat{c}_1 \|u\|^2_2 + \hat{c}_2 \|v\|^2_2 \right) + 4n \sum_{i=1}^{m} (e_i^\top v)^2 + 2n \sum_{i=1}^{m} (\tilde{b}_i^\top u)^2 \\
\leq 2 \left( \hat{c}_2 \|u\|^2_2 + \hat{c}_2 \|v\|^2_2 \right) + 4n \|v\|^2_2 + 8n \|u\|^2_2 \\
\leq (2 \max\{\hat{c}_1, \hat{c}_2\})^2 + 8n \left( \|u\|^2_2 + \|v\|^2_2 \right).
\]

Thus, \(\tilde{r}_i\) is \(\sqrt{4n^2 + 2\max\{\hat{c}_1, \hat{c}_2\}^2}\)-smooth, and \(\{\tilde{r}_i\}_{i=1}^{n}\) is \(\sqrt{8n + 2\max\{\hat{c}_1, \hat{c}_2\}^2}\)-average smooth. \(\square\)

**Proof of Lemma 3.14** Note that

\[ e_i \tilde{b}_i^\top x = \begin{cases} (x_l - x_{l+1})e_l, & 1 \leq l < m, \\ \zeta(x_m) e_m, & l = m, \end{cases} \]

and

\[ \tilde{b}_i e_i^\top y = \begin{cases} y_l (e_l - e_{l+1}), & 1 \leq l < m, \\ \zeta y_m e_m, & l = m. \end{cases} \]

For \(x \in F_k\) with \(1 \leq k < m\), we have

\[ e_i \tilde{b}_i^\top x \in \begin{cases} F_k, & l = k, \\ F_{k-1}, & l \neq k. \end{cases} \quad (28) \]

For \(y \in F_k\) with \(1 \leq k < m\), we have

\[ \tilde{b}_i e_i^\top y \in \begin{cases} F_{k+1}, & l = k, \\ F_k, & l \neq k. \end{cases} \quad (29) \]

Recall that

\[ \nabla_x \tilde{r}_i(x, y) = n \sum_{l \in \mathcal{L}_i} \tilde{b}_i e_i^\top y + \hat{c}_1 x - ne_1 \mathbb{1}_{\{i=1\}}, \]

\[ \nabla_y \tilde{r}_i(x, y) = n \sum_{l \in \mathcal{L}_i} e_l \tilde{b}_i^\top x - \hat{c}_2 y. \]

By Inclusions (28) and (29), we have the following results.

1. Suppose that \(x, y \in F_0\). It holds that \(\nabla_x \tilde{r}_1(x, y) = ne_1 \in F_1\), \(\nabla_x \tilde{r}_j(x, y) = 0_m\) for \(j \geq 2\) and \(\nabla_y \tilde{r}_j(x, y) = 0_m\) for any \(j\).

2. Suppose that \(x \in F_1\) and \(y \in F_0\) and \(1 \in \mathcal{L}_i\). It holds that \(\nabla_x \tilde{r}_j(x, y) = \hat{c}_1 x + ne_1 \mathbb{1}_{\{i=1\}} \in F_1\) for any \(j\), \(\nabla_y \tilde{r}_i(x, y) \in F_1\) and \(\nabla_y \tilde{r}_j(x, y) = 0_m\) for \(j \neq i\).

3. Suppose that \(x \in F_{k+1}\), \(y \in F_k\), \(1 \leq k < m\) and \(k + 1 \in \mathcal{L}_i\). It holds that \(\nabla_x \tilde{r}_j(x, y) \in F_{k+1}\) for any \(j\), \(\nabla_y \tilde{r}_i(x, y) \in F_{k+1}\) and \(\nabla_y \tilde{r}_j(x, y) \in F_k\) for \(j \neq i\).
Now we turn to consider \((u_i, v_i) = \text{prox}_\gamma^2(x, y)\). We have
\[
\nabla_x \tilde{r}_i(u_i, v_i) + \frac{1}{\gamma} (u_i - x) = 0_m,
\]
\[
\nabla_y \tilde{r}_i(u_i, v_i) - \frac{1}{\gamma} (v_i - y) = 0_m,
\]
that is
\[
\begin{bmatrix}
\tilde{c}_1 + \frac{1}{\gamma} & \frac{n \tilde{B}_i^\top}{\gamma} \\
- \frac{n}{\gamma} \tilde{B}_i & \tilde{c}_2 + \frac{1}{\gamma}
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_i
\end{bmatrix}
= \begin{bmatrix}
x_i \\
y_i
\end{bmatrix},
\]
where \(\tilde{x}_i = x/\gamma + ne_1\mathbb{1}_{\{i=1\}}\) and \(\tilde{y} = y/\gamma\). Recall that for \(l_1, l_2 \in \mathcal{L}_i\) and \(l_1 \neq l_2\), \(\tilde{b}_{l_1}^\top \tilde{b}_{l_2} = 0\). It follows that
\[
\tilde{B}_i \tilde{B}_i^\top = \left(\sum_{l \in \mathcal{L}_i} e_l \tilde{b}_l^\top\right) \left(\sum_{l \in \mathcal{L}_i} \tilde{b}_l e_l^\top\right)
= \sum_{l \in \mathcal{L}_i} e_l \tilde{b}_l^\top \tilde{b}_l e_l^\top,
\]
which is a diagonal matrix. Assuming that
\[
D_i \triangleq \left(\tilde{c}_2 + \frac{1}{\gamma}\right) I_m + \frac{n^2}{\tilde{c}_1 + 1/\gamma} \tilde{B}_i \tilde{B}_i^\top = \text{diag}(d_{i,1}, d_{i,2}, \ldots, d_{i,m}),
\]
we have
\[
\begin{bmatrix}
u_i \\
v_i
\end{bmatrix}
= \begin{bmatrix}
\tilde{c}_1 + \frac{1}{\gamma} & \frac{n \tilde{B}_i^\top}{\gamma} \\
- \frac{n}{\gamma} \tilde{B}_i & \tilde{c}_2 + \frac{1}{\gamma}
\end{bmatrix}^{-1}
\begin{bmatrix}
x_i \\
y_i
\end{bmatrix}
= \begin{bmatrix}
1/(\tilde{c}_1 + 1/\gamma) I_m - \frac{n^2}{(\tilde{c}_1 + 1/\gamma)^2} \tilde{B}_i \tilde{B}_i^\top D_i^{-1} \tilde{B}_i \\
- \frac{n}{\tilde{c}_1 + 1/\gamma} D_i^{-1} \tilde{B}_i
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i
\end{bmatrix}
= \begin{bmatrix}
\tilde{x}_i \\
\tilde{y}_i
\end{bmatrix}.
\]
Note that for \(1 \leq k \leq m\), \(y \in \mathcal{F}_k\) implies \(D_k^{-1} \tilde{y} \in \mathcal{F}_k\) and \(x \in \mathcal{F}_k\) implies \(\tilde{x}_i \in \mathcal{F}_k\). And recall that
\[
\tilde{b}_l \tilde{b}_l^\top x = \begin{cases}
(x_l - x_{l+1})(e_l - e_{l+1}), & l < m, \\
\zeta^2 x_m e_m, & l = m.
\end{cases}
\]
Then for \(x \in \mathcal{F}_k\) with \(1 \leq k \leq m\), we have
\[
\tilde{b}_l \tilde{b}_l^\top x \in \begin{cases}
\mathcal{F}_{k+1}, & l = k, \\
\mathcal{F}_k, & l \neq k.
\end{cases}
\]
By Inclusions (28), (29), (31) and Equations (30), we have the following results.

1. Suppose that \(x, y \in \mathcal{F}_0\). It holds that \(\tilde{x}_1 \in \mathcal{F}_1\) and \(\tilde{x}_j = 0_m\) for \(j \geq 2\), which implies \(u_1 \in \mathcal{F}_1\) and \(u_j = 0_m\) for \(j \geq 2\). Moreover, \(v_j = 0\) for any \(j\).

2. Suppose that \(x \in \mathcal{F}_1\), \(y \in \mathcal{F}_0\) and \(1 \in \mathcal{L}_i\). It holds that \(u_i \in \mathcal{F}_2\), \(v_i \in \mathcal{F}_1\) and \(u_j \in \mathcal{F}_1\), \(v_j \in \mathcal{F}_0\) for \(j \neq i\).

3. Suppose that \(x \in \mathcal{F}_{k+1}\), \(y \in \mathcal{F}_k\), \(1 \leq k < m - 1\) and \(k + 1 \in \mathcal{L}_i\). It holds that \(u_i \in \mathcal{F}_{k+2}\), \(v_i \in \mathcal{F}_{k+1}\) and \(u_j \in \mathcal{F}_{k+1}\), \(v_j \in \mathcal{F}_k\) for \(j \neq i\).

This completes the proof. □

56
C.2 Proofs of Proposition 3.17 and Lemma 3.18

Let \( \hat{\mathbf{B}}(m, \omega) \) denote the first \( m \) rows of \( \mathbf{B}(m, \omega, 0) \) by and \( \hat{\mathbf{b}}_l(m, \omega) = \mathbf{b}_l(m, \omega, 0) \) for \( 0 \leq l \leq m \). Note that \( \mathbf{b}_m(m, \omega) = 0_m \). For simplicity, we omit the parameters of \( \hat{\mathbf{B}}, \mathbf{b}_l \) and \( \hat{r}_i \). Then we have \( \hat{\mathbf{B}} = (\hat{\mathbf{b}}_0, \hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_{m-1})^\top \).

Let \( G(x) \triangleq \sum_{i=1}^{m-1} \Gamma(x_i) \). Recall that

\[
\mathcal{L}_i = \{ l : 0 \leq l \leq m, l \equiv i - 1(\text{mod} n) \}, \quad i = 1, 2, \ldots, n.
\]

For \( 1 \leq i \leq n \), let \( \hat{\mathbf{B}}_i \) be the submatrix whose rows are \( \{ \hat{\mathbf{b}}_l \}_l \in \mathcal{L}_i \). Note that \( \hat{\mathbf{B}} = \sum_{l=0}^{m-1} e_{l+1} \hat{\mathbf{b}}_l^\top \) and \( \hat{\mathbf{B}}_i = \sum_{l \in \mathcal{L}_i} e_{l+1} \hat{\mathbf{b}}_l^\top \). Then \( \hat{r}_i \) can be written as

\[
\hat{r}_i(x, y) = n \langle y, \hat{\mathbf{B}}_i x \rangle - \frac{c_1}{2} \| y \|_2^2 + c_2 G(\hat{\mathbf{c}}_3 x) - n \langle e_1, y \rangle 1_{\{i=1\}}.
\]

Proof of Proposition 3.17. Denote \( s_i(x, y) = \hat{r}_i(x, y) - c_2 G(\hat{\mathbf{c}}_3 x) \). Similar to the proof of Proposition 3.13 we can establish that for any \( x_1, x_2, y_1, y_2 \),

\[
\| \nabla s_i(x_1, y_1) - \nabla s_i(x_2, y_2) \|_2^2 \leq \left( 4n^2 + 2c_1^2 \right) \left( \| x_1 - x_2 \|_2^2 + \| y_1 - y_2 \|_2^2 \right),
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} \| \nabla s_i(x_1, y_1) - \nabla s_i(x_2, y_2) \|_2^2 \leq \left( 8n + 2c_1^2 \right) \left( \| x_1 - x_2 \|_2^2 + \| y_1 - y_2 \|_2^2 \right).
\]

By Proposition 3.4 and the inequality \((a+b)^2 \leq 2(a^2 + b^2)\), we conclude that \( \hat{r}_i \) is \((-45(\sqrt{3} - 1)\hat{c}_2\hat{c}_3^2, \hat{c}_1)\)-convex-concave,

\[
\| \nabla \hat{r}_i(x_1, y_1) - \nabla \hat{r}_i(x_2, y_2) \|_2 \leq \left( 4n^2 + 2c_1^2 + 180\hat{c}_2\hat{c}_3^2 \right) \sqrt{\| x_1 - x_2 \|_2^2 + \| y_1 - y_2 \|_2^2},
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} \| \nabla \hat{r}_i(x_1, y_1) - \nabla \hat{r}_i(x_2, y_2) \|_2^2 \leq \left( 16n + 4c_1^2 + 64800\hat{c}_2\hat{c}_3^2 \right) \left( \| x_1 - x_2 \|_2^2 + \| y_1 - y_2 \|_2^2 \right).
\]

Now we prove the Lemma 3.18

Proof of Lemma 3.18. Note that

\[
e_{l+1} \hat{\mathbf{b}}_l^\top x = \begin{cases} \omega x_1 e_1, & l = 0, \\ (x_l - x_{l+1}) e_{l+1}, & 1 \leq l < m. \end{cases}
\]

and

\[
\hat{\mathbf{b}}_l e_{l+1}^\top y = \begin{cases} \omega y_1 e_1, & l = 0, \\ y_{l+1} (e_l - e_{l+1}), & 1 \leq l < m. \end{cases}
\]

57
For \( x \in \mathcal{F}_k \) with \( 1 \leq k < m \), we have

\[
e_{l+1}^T b_l^T x = \begin{cases} \mathcal{F}_{k+1}, & l = k, \\ \mathcal{F}_k, & l \neq k. \end{cases}
\]

(32)

For \( y \in \mathcal{F}_k \) with \( 1 \leq k < m \), we have

\[
b_l e_{l+1}^T y = \begin{cases} \mathcal{F}_k, & l = k - 1, \\ \mathcal{F}_{k-1}, & l \neq k - 1. \end{cases}
\]

(33)

Recall that

\[
\nabla_x \hat{r}_i(x, y) = n \sum_{l \in \mathcal{L}_i} \hat{b}_l e_{l+1}^T y + \hat{c}_2 \hat{c}_3 \nabla G(\hat{c}_3 x),
\]

\[
\nabla_y \hat{r}_i(x, y) = n \sum_{l \in \mathcal{L}_i} e_{l+1} \hat{b}_l^T x - \hat{c}_1 y + ne_1 \mathbb{1}_{\{i=1\}}.
\]

By Inclusions (32) and (33), we have the following results.

1. Suppose that \( x, y \in \mathcal{F}_0 \). It holds that \( \nabla_x \hat{r}_j(x, y) = 0_m \) for any \( j \), \( \nabla_y \hat{r}_i(x, y) = ne_1 \in \mathcal{F}_1 \) and \( \nabla_y \hat{r}_j(x, y) = 0_m \) for \( j \geq 2 \).

2. Suppose that \( x, y \in \mathcal{F}_k \), \( 1 \leq k < m \) and \( k \in \mathcal{L}_i \). It holds that \( \nabla_x \hat{r}_j(x, y) \in \mathcal{F}_k \) for any \( j \), \( \nabla_y \hat{r}_i(x, y) \in \mathcal{F}_{k+1} \) and \( \nabla_y \hat{r}_j(x, y) \in \mathcal{F}_k \) for \( j \neq i \).

Now we turn to consider \( (u_i, v_i) = \text{prox}_\gamma^G(x, y) \). We have

\[
\nabla_x \hat{r}_i(u_i, v_i) + \frac{1}{\gamma}(u_i - x) = 0_m,
\]

\[
\nabla_y \hat{r}_i(u_i, v_i) - \frac{1}{\gamma}(v_i - y) = 0_m,
\]

that is

\[
\begin{bmatrix}
\frac{1}{\gamma} I_m \\
-n \hat{b}_i^T \\
\hat{c}_1 + \frac{1}{\gamma} I_m
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_i
\end{bmatrix} = \begin{bmatrix}
x - \hat{u}_i \\
\hat{y}_i
\end{bmatrix},
\]

where \( \hat{x} = x/\gamma \), \( \hat{y}_i = y/\gamma + ne_1 \mathbb{1}_{\{i=1\}} \) and \( \hat{u}_i = \hat{c}_2 \hat{c}_3 \nabla G(\hat{c}_3 u_i) \). Recall that for \( l_1, l_2 \in \mathcal{L}_i \) and \( l_1 \neq l_2 \), \( \hat{b}_{l_1}^T \hat{b}_{l_2} = 0 \). It follows that

\[
\hat{b}_i \hat{b}_i^T = \left( \sum_{l \in \mathcal{L}_i} e_{l+1} b_l^T \right) \left( \sum_{l \in \mathcal{L}_i} b_l e_{l+1}^T \right) = \sum_{l \in \mathcal{L}_i} e_{l+1} b_l^T b_l e_{l+1}^T,
\]

which is a diagonal matrix. Denote

\[
D_i \triangleq \left( \hat{c}_1 + \frac{1}{\gamma} \right) I_m + \gamma n^2 \hat{b}_i \hat{b}_i^T = \text{diag}(d_{i,1}, d_{i,2}, \ldots, d_{i,m}).
\]

For \( 0 < l < m \), \( l \in \mathcal{L}_i \) implies \( d_{i,l+1} = \hat{c}_1 + \frac{1}{\gamma} + 2\gamma n^2 \). Then we have

\[
\begin{bmatrix}
u_i \\
v_i
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\gamma} I_m \\
-n \hat{b}_i \\
\hat{c}_1 + \frac{1}{\gamma} I_m
\end{bmatrix}^{-1} \begin{bmatrix}
x - \hat{u}_i \\
\hat{y}_i
\end{bmatrix}
\]

58
Recalling that \( w \) Setting \( z \) apply Lemma B.6 with \( \gamma \) Applying Lemma B.5 with \( z \) We can establish the following claims.

\[
\begin{align*}
\mathbf{u}_i &+ \gamma \hat{\mathbf{u}}_i - \gamma^2 n^2 \sum_{l \in \mathcal{L}_i} d_{i,l}^{-1} \mathbf{b}_l \mathbf{b}_l^\top \hat{\mathbf{u}}_i = \gamma \mathbf{x} - \gamma^2 n^2 \sum_{l \in \mathcal{L}_i} d_{i,l}^{-1} \mathbf{b}_l \mathbf{b}_l^\top \hat{\mathbf{x}} - \gamma \sum_{l \in \mathcal{L}_i} \mathbf{b}_l \mathbf{e}_l^\top \mathbf{D}_i^{-1} \hat{\mathbf{y}}_i, \\
\mathbf{v}_i &+ \gamma \sum_{l \in \mathcal{L}_i} d_{i,l}^{-1} \mathbf{e}_l^\top \mathbf{D}_i^{-1} \mathbf{b}_l^\top (\mathbf{x} - \hat{\mathbf{u}}_i) + \mathbf{D}_i^{-1} \hat{\mathbf{y}}_i.
\end{align*}
\]

(34)

We first focus on Equations (34). Recall that \( \hat{\mathbf{u}}_i = \hat{c}_2 \hat{c}_3 \nabla G(\hat{c}_3 \mathbf{u}_i) \) and

\[
\mathbf{b}_l \mathbf{b}_l^\top x = \begin{cases} \omega^2 x_1 \mathbf{e}_1, & l = 0, \\ (x_l - x_{l+1})(\mathbf{e}_l - \mathbf{e}_{l+1}), & 0 < l < m. \end{cases}
\]

For simplicity, let \( \mathbf{u}_i = (u_1, u_2, \ldots, u_m)^\top \) and \( \hat{\mathbf{u}}_i = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_m)^\top \), and denote the right hand side of Equations (34) by \( w \). Recalling the definition of \( G(\mathbf{x}) \), we have \( \hat{u}_l = 12 \hat{c}_2 \hat{c}_3 \frac{\hat{c}_3 u_l (\hat{c}_3 u_{l+1} - 1)}{1 + \hat{c}_3 u_l^2} \) for \( l < m \) and \( \hat{u}_m = 0 \). We can establish the following claims.

1. If \( 0 < l < m - 1 \) and \( l \in \mathcal{L}_i \), we have

\[
\begin{align*}
u_l &+ \left( \gamma - \gamma^2 n^2 d_{i,l+1}^{-1} \right) \hat{u}_l + \gamma^2 n^2 d_{i,l+1}^{-1} \hat{u}_{l+1} = w_l, \\
u_{l+1} &+ \gamma^2 n^2 d_{i,l+1}^{-1} \hat{u}_l + \left( \gamma - \gamma^2 n^2 d_{i,l+1}^{-1} \right) \hat{u}_{l+1} = w_{l+1}.
\end{align*}
\]

(36)

Setting \( w_l = w_{l+1} = 0 \) yields

\[
\begin{align*}
(1 - 2 \gamma^2 n^2 d_{i,l+1}^{-1}) u_l + \gamma n^2 d_{i,l+1}^{-1} (u_l - u_{l+1}) + \left( \gamma - 2 \gamma^2 n^2 d_{i,l+1}^{-1} \right) \hat{u}_l = 0, \\
(1 - 2 \gamma^2 n^2 d_{i,l+1}^{-1}) u_{l+1} + \gamma n^2 d_{i,l+1}^{-1} (u_{l+1} - u_l) + \left( \gamma - 2 \gamma^2 n^2 d_{i,l+1}^{-1} \right) \hat{u}_{l+1} = 0.
\end{align*}
\]

Recalling that \( d_{i,l+1} = \hat{c}_1 + 1/\gamma + 2 \gamma n^2 \), we find \( 1 - 2 \gamma^2 n^2 d_{i,l+1}^{-1} > 0 \). Since \( \gamma < \frac{\sqrt{2} + 1}{6 \omega c_3} \), we can apply Lemma [13.1] with \( z_1 = \hat{c}_3 u_l \) and \( z_2 = \hat{c}_3 u_{l+1} \) and conclude that \( u_l = u_{l+1} = 0 \).

2. If \( m - 1 \in \mathcal{L}_i \), we have

\[
\begin{align*}
u_{m-1} &+ \left( \gamma - \gamma^2 n^2 d_{i,m}^{-1} \right) \hat{u}_{m-1} = u_{m-1}, \\
u_m &+ \gamma n^2 d_{i,m}^{-1} \hat{u}_{m-1} = w_m.
\end{align*}
\]

(37)

Setting \( w_{m-1} = w_m = 0 \) yields

\[
\begin{align*}
u_{m-1} &+ \left( \gamma - \gamma^2 n^2 d_{i,m}^{-1} \right) \hat{u}_{m-1} = 0, \\
\gamma n^2 d_{i,m}^{-1} u_{m-1} - (1 - \gamma n^2 d_{i,m}^{-1}) u_m = 0.
\end{align*}
\]

Recalling that \( d_{i,m+1} = \hat{c}_1 + 1/\gamma + 2 \gamma n^2 \) and \( \gamma < \frac{\sqrt{2} + 1}{6 \omega c_3^2} \), we have \( 0 < \gamma - \gamma^2 n^2 d_{i,m}^{-1} < \gamma < \frac{\sqrt{2} + 1}{6 \omega c_3^2} \).

Applying Lemma [13.2] with \( z = \hat{c}_3 u_{m-1} \), we conclude that \( u_{m-1} = 0 \). It follows that \( u_m = 0 \).
3. If $0 < l < m$ and $l, l - 1 \not\in \mathcal{L}_i$, we have
\[ u_l + \gamma \hat{u}_l = w_l. \] (38)

Setting $w_l = 0$ and applying Lemma 13.5 with $z = \hat{c}_3 u_l$, we conclude that $u_l = 0$.

Note that for $1 \leq k \leq m$, $x \in \mathcal{F}_k$ implies $\hat{x} \in \mathcal{F}_k$ and $y \in \mathcal{F}_k$ implies $D_i^{-1} \hat{y}_i \in \mathcal{F}_k$. And for $x \in \mathcal{F}_k$ with $1 \leq k < m$, we have
\[ \hat{b}_l \hat{b}_l^\top x \in \begin{cases} \mathcal{F}_{k+1}, & l = k, \\ \mathcal{F}_k, & l \neq k. \end{cases} \] (39)

Then we can provide the following analysis.

1. Suppose that $x, y \in \mathcal{F}_0$. Note that $0 \in \mathcal{L}_1$.

For $j = 1$, we have $\hat{x} = 0_m$ and $\hat{y}_1 \in \mathcal{F}_1$. Since $0 \in \mathcal{L}_1$, Inclusion 32 implies $w \in \mathcal{F}_1$. Then we consider the solution to Equations (34). Since $n \geq 2$, we have $1 \not\in \mathcal{L}_1$. If $2 \in \mathcal{L}_1$, we can consider the solution to Equations (36) or (37) and conclude that $u_2 = 0$. If $2 \not\in \mathcal{L}_1$, we can consider the solution to Equation (38) and conclude that $u_2 = 0$. Similarly, we obtain $u_l = 0$ for $l \geq 2$, which implies $u_1 \in \mathcal{F}_1$. Since $1 \not\in \mathcal{L}_1$, by Inclusion (32) and Equations (35), we have $v_1 \in \mathcal{F}_1$.

For $j \neq 1$, we have $\hat{x} = \hat{y}_j = 0_m$. It follows that $w = 0_m$. Note that $0 \not\in \mathcal{L}_j$. If $1 \in \mathcal{L}_j$, we can consider the solution to Equations (36) or (37) and conclude that $u_1 = 0$. If $1 \not\in \mathcal{L}_j$, we can consider the solution to Equation (38) and conclude that $u_1 = 0$. Similarly, we obtain $u_l = 0$ for $l \geq 2$, which implies $u_j = 0_m$. By Equations (35), we have $v_j = 0_m$.

2. Suppose that $x, y \in \mathcal{F}_k$, $1 \leq k < m$ and $k \in \mathcal{L}_i$.

For $j = i$, we have $\hat{x}, \hat{y}_i \in \mathcal{F}_k$. If $k = m - 1$, clearly $u_i, v_i \in \mathcal{F}_m$. Now we assume $k < m - 1$. Inclusions (33) and (34) imply $w \in \mathcal{F}_{k+1}$. Then we consider the solution to Equations (34). Since $n \geq 2$, we have $k + 1 \not\in \mathcal{L}_i$. If $k + 2 \in \mathcal{L}_i$, we can consider the solution to Equations (36) or (37) and conclude that $u_{k+2} = 0$. If $k + 2 \not\in \mathcal{L}_i$, we can consider the solution to Equation (38) and conclude that $u_{k+2} = 0$. Similarly, we obtain $u_l = 0$ for $l \geq k + 2$, which implies $u_i \in \mathcal{F}_{k+1}$. Since $k + 1 \not\in \mathcal{L}_i$, by Inclusion (32) and Equations (35), we have $v_i \in \mathcal{F}_{k+1}$.

For $j \neq i$, we also have $\hat{x}, \hat{y}_i \in \mathcal{F}_k$. Since $k \not\in \mathcal{L}_j$, by Inclusions (33) and (39), we have $w \in \mathcal{F}_k$. If $k + 1 \in \mathcal{L}_j$, we can consider the solution to Equations (36) or (37) and conclude that $u_{k+1} = 0$. If $k + 1 \not\in \mathcal{L}_j$, we can consider the solution to Equation (38) and conclude that $u_{k+1} = 0$. Similarly, we obtain $u_l = 0$ for $l \geq k + 1$, which implies $u_j \in \mathcal{F}_k$. Since $k \not\in \mathcal{L}_j$, by Inclusion (32) and Equations (35), we have $v_j \in \mathcal{F}_k$.

This completes the proof. □

C.3 Proof of Proposition 3.22

1. Just recall Proposition 3.13 and Lemma B.1

2. It is easy to check
\[ f_{SCSC}(x, y) = \frac{\sqrt{L^2 - 2\mu^2}}{2n} \left\langle y, B \left( m, \sqrt{\frac{2}{\alpha + 1}} \right) x \right\rangle + \frac{\mu_x}{2} \|x\|_2^2 - \frac{\mu_y}{2} \|y\|_2^2 \]
\[- \frac{\beta \sqrt{L^2 - 2\mu_x^2}}{2n} \langle e_1, x \rangle.\]

Set \( \zeta = \sqrt{\frac{2}{\alpha + 1}} \) and \( \xi = \frac{\sqrt{L^2 - 2\mu_x^2}}{2n} \). Letting the gradient of \( f_{\text{SCSC}}(x, y) \) be zero, we obtain

\[
\begin{aligned}
\xi \mathbf{B} & (m, \zeta)^\top y + \mu_x x - \beta \xi e_1 = 0, \\
\xi \mathbf{B} & (m, \zeta) x - \mu_y y = 0,
\end{aligned}
\]

which implies

\[
y = \frac{\xi}{\mu_y} \mathbf{B}(m, \zeta)x, \quad (40)
\]

\[
\left( \mu_x \mathbf{I} + \frac{\xi^2}{\mu_y} \mathbf{B}(m, \zeta)^\top \mathbf{B}(m, \zeta) \right) x = \beta \xi e_1. \quad (41)
\]

The Equations (41) are equivalent to

\[
\begin{bmatrix}
1 + \frac{\mu_x \mu_y}{\xi^2} & -1 & -1 & \cdots & -1 \\
-1 & 2 + \frac{\mu_x \mu_y}{\xi^2} & -1 & \cdots & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 & \cdots & -1 & 2 + \frac{\mu_x \mu_y}{\xi^2} & -1 \\
-1 & \cdots & -1 & -1 & \xi^2 + 1 + \frac{\mu_x \mu_y}{\xi^2}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{m-1} \\
x_m
\end{bmatrix}
= \begin{bmatrix}
\frac{\beta \mu_x}{\zeta} \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix},
\quad (42)
\]

Note that

\[
\frac{\mu_x \mu_y}{\xi^2} = \frac{4n^2 \mu_x \mu_y}{L^2 - 2\mu_x^2} = \frac{4n^2 \mu_y}{(\kappa_x^2 - 2)\mu_x} = \frac{4n^2}{(\kappa_x - 2/\kappa_x) \kappa_y} = \frac{4}{\alpha^2 - 1}.
\]

It is easy to check \( q \) is a root of the equation

\[
z^2 - \left( 2 + \frac{\mu_x \mu_y}{\xi^2} \right) z + 1 = 0.
\]

Then, we can check that the solution of (42) equation is

\[
x^* = \frac{\beta \mu_y}{(1 - q)\xi} (q, q^2, \ldots, q^m)^\top.
\]

Substituting above result into (40), we have

\[
y^* = \beta \left( q, q^2, \ldots, q^{m-1}, \frac{q^m}{\zeta} \right)^\top.
\]

Moreover, from the definition of \( \beta \), we have

\[
\|x^*\|^2 = \frac{\beta^2 \mu_y^2 (q^2 - q^{2m+2})}{(1 - q)^2 (1 - q^2) \xi^2} \leq \frac{\beta^2 \mu_y^2 q^2}{(1 - q)^2 (1 - q^2) \xi^2} \leq \frac{\beta^2 (\kappa_x^2 - 2)}{4n^2 \alpha} \leq R_x^2,
\]

and

\[
\|y^*\|^2 = \beta^2 \left( \frac{q^2 - q^{2m}}{1 - q^2} + \frac{q^{2m}}{\xi^2} \right) \leq \beta^2 \frac{2q^2}{1 - q^2} = \frac{\beta^2 (\alpha - 1)^2}{4\alpha} \leq R_y^2.
\]
3. Define \( \tilde{\phi}_{\text{SCSC}}(x) = \max_{y \in \mathbb{R}^m} f_{\text{SCSC}}(x, y) \) and \( \tilde{\psi}_{\text{SCSC}}(y) = \min_{x \in \mathbb{R}^m} f_{\text{SCSC}}(x, y) \). We first show that
\[
\min_{x \in \mathcal{F}_k} \tilde{\phi}_{\text{SCSC}}(x) - \max_{y \in \mathcal{F}_k} \tilde{\psi}_{\text{SCSC}}(y) \geq \frac{\beta^2 \xi^2}{(\alpha + 1) \mu_x} q^{2k},
\]
where \( \xi = \frac{\sqrt{1 - 2 \mu^2}}{2n} \). Recall that
\[
f_{\text{SCSC}}(x, y) = \xi \left( y, \tilde{B}(m, \zeta) x \right) + \frac{\mu_x}{2} \|x\|_2^2 - \frac{\mu_y}{2} \|y\|_2^2 - \beta \xi \left( e_1, x \right),
\]
where \( \zeta = \sqrt{\frac{2}{\alpha + 1}} \). Note that we can write \( f_{\text{SCSC}}(x, y) \) as
\[
f_{\text{SCSC}}(x, y) = -\frac{\mu_y}{2} \left\| y - \frac{\xi}{\mu_y} \tilde{B}(m, \zeta) x \right\|_2^2 + \frac{\xi^2}{2 \mu_y} \left\| \tilde{B}(m, \zeta) x \right\|_2^2 + \frac{\mu_x}{2} \|x\|_2^2 - \beta \xi \left( e_1, x \right). \tag{43}
\]
Thus \( \tilde{\phi}_{\text{SCSC}}(x) = \frac{\xi^2}{2 \mu_y} \left\| \tilde{B}(m, \zeta) x \right\|_2^2 + \frac{\mu_x}{2} \|x\|_2^2 - \beta \xi \left( e_1, x \right) \). For \( x \in \mathcal{F}_k \), let \( \bar{x} \) be the first \( k \) coordinates of \( x \). Then we can rewrite \( \tilde{\phi}_{\text{SCSC}} \) as
\[
\tilde{\phi}_k(x) \triangleq \tilde{\phi}_{\text{SCSC}}(x) = \frac{\xi^2}{2 \mu_y} \left\| \tilde{B}(k, 1) \bar{x} \right\|_2^2 + \frac{\mu_x}{2} \|\bar{x}\|_2^2 - \beta \xi \left( \bar{e}_1, \bar{x} \right),
\]
where \( \bar{e}_1 \) is the first \( k \) coordinates of \( e_1 \). Letting \( \nabla \tilde{\phi}_k(\bar{x}) = 0_k \), we obtain
\[
\frac{\xi^2}{\mu_y} \tilde{B}(k, 1)^\top \tilde{B}(k, 1) \bar{x} + \mu_x \bar{x} = \beta \xi \bar{e}_1,
\]
that is
\[
\begin{pmatrix}
1 + \frac{\mu_x \mu_y}{\xi^2} & -1 & \cdots & -1 \\
-1 & 2 + \frac{\mu_x \mu_y}{\xi^2} & \cdots & -1 \\
\vdots & \ddots & \ddots & \vdots \\
-1 & \cdots & -1 & 2 + \frac{\mu_x \mu_y}{\xi^2}
\end{pmatrix}
\begin{pmatrix}
\frac{\mu_x \mu_y}{\xi^2} \\
0 \\
\vdots \\
0
\end{pmatrix}
= \begin{pmatrix}
\bar{x} \\
0 \\
\vdots \\
0
\end{pmatrix} \tag{44}
\]
Recall that \( \frac{\mu_x \mu_y}{\xi^2} = \frac{4}{\alpha + 1} \) and \( q = \frac{\alpha - 1}{\alpha + 1} \). \( q \) and \( 1/q \) are two roots of the equation
\[
z^2 - \left( 2 + \frac{\mu_x \mu_y}{\xi^2} \right) z + 1 = 0.
\]
Then, we can check that the solution to Equations (44) is
\[
\bar{x}^* = \frac{\beta \mu_y (\alpha + 1) q^{k+1}}{2 \xi (1 + q^{2k+1})} \left( q^{-k} - q^{-k+1} + q^{k+1} - q^k, \ldots, q^{k+1} - q^k \right)^\top,
\]
and the value of \( \min_{x \in \mathcal{F}_k} \tilde{\phi}_{\text{SCSC}}(x) \) is
\[
\min_{x \in \mathcal{F}_k} \tilde{\phi}_{\text{SCSC}}(x) = -\frac{\beta^2 \mu_y (\alpha + 1)}{4} \frac{q - q^{2k+1}}{1 + q^{2k+1}}.
\]
On the other hand, observe that
\[
 f_{\text{SCSC}}(x, y) = \frac{\mu_x}{2} \left[ x + \frac{\xi}{\mu_x} \tilde{B}(m, \zeta) y - \frac{\beta \xi}{\mu_x} e_1 \right]^2 - \frac{\xi^2}{2 \mu_x} \left[ \tilde{B}(m, \zeta) y - \beta e_1 \right]^2 - \frac{\mu_y}{2} \| y \|^2.
\]

It follows that
\[
 \tilde{\psi}_{\text{SCSC}}(y) = -\frac{\xi^2}{2 \mu_x} \left[ \tilde{B}(m, \zeta)^\top y - \beta e_1 \right]^2 - \frac{\xi^2}{2 \mu_x} \left( \tilde{e}_k, \tilde{y} \right)^2 - \frac{\mu_y}{2} \| \tilde{y} \|^2.
\]

For \( y \in F_k \), let \( \tilde{y} \) be the first \( k \) coordinated of \( y \). Then we can rewrite \( \tilde{\psi}_{\text{SCSC}} \) as
\[
 \tilde{\psi}_k(\tilde{y}) \triangleq \tilde{\psi}_{\text{SCSC}}(y) = -\frac{\xi^2}{2 \mu_x} \left[ \tilde{B}(k, 1)^\top \tilde{y} - \tilde{\beta} \tilde{e}_1 \right]^2 - \frac{\xi^2}{2 \mu_x} \left( \tilde{e}_k, \tilde{y} \right)^2 - \frac{\mu_y}{2} \| \tilde{y} \|^2,
\]
where \( \tilde{e}_1, \tilde{e}_k \) are the first \( k \) ordinates of \( e_1 \) and \( e_k \) respectively. Letting \( \nabla \tilde{\psi}_k(\tilde{y}) = 0_k \), we obtain
\[
 \frac{\xi^2}{\mu_x} \left( \tilde{B}(k, 1)^\top + \tilde{e}_k \tilde{e}_k^\top \right) \tilde{y} + \mu_y \tilde{y} = \frac{\beta \xi^2}{\mu_x} \tilde{B}(k, 1) \tilde{e}_1,
\]
that is
\[
 \begin{bmatrix}
2 + \frac{\mu_y}{\mu_x} \\
-1 & 2 + \frac{\mu_y}{\xi} \\
& \ddots & \ddots \\
& -1 & 2 + \frac{\mu_y}{\xi} \\
-1 & 2 + \frac{\mu_y}{\xi} & -1 & 2 + \frac{\mu_y}{\xi} 
\end{bmatrix}
\begin{bmatrix}
\tilde{y} \\
\\vdots \\
\\vdots \\
\\vdots \\
\\vdots 
\end{bmatrix}
= \begin{bmatrix}
\beta \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

Then, we can check that the solution to the above equations is
\[
 \tilde{y}^* = \frac{\beta q^{k+1}}{1 - q^{2k+2}} (q^{-k} - q^k, q^{-k+1} - q^{k-1}, \ldots, q^{-1} - q)^\top,
\]
and the optimal value of \( \tilde{\psi}_{\text{SCSC}}(y) \) is
\[
 \min_{y \in F_k} \tilde{\psi}_{\text{SCSC}}(y) = -\frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \frac{1 + q^{2k+1}}{1 - q^{2k+2}}.
\]
It follows that
\[
 \min_{x \in F_k} \tilde{\psi}_{\text{SCSC}}(x) - \max_{y \in F_k} \tilde{\psi}_{\text{SCSC}}(y)
\]
\[
= -\frac{\beta^2 \mu_y (\alpha + 1)}{4} \frac{q - q^{2k+1}}{1 + q^{2k+1}} + \frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \frac{1 + q^{2k+1}}{1 - q^{2k+2}}
\]
\[
= -\frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \frac{\mu_x \mu_y (\alpha + 1)^2 q}{4 \xi^2} \frac{1 - q^k}{1 + q^{2k+1}} + \frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \frac{1 + q^{2k+1}}{1 - q^{2k+2}}
\]
\[
= \frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \left( \frac{1 + q^{2k+1}}{1 - q^{2k+2}} \frac{1 - q^k}{1 + q^{2k+1}} \right)
\]
\[
= \frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \frac{2 q^{2k+1} + q^k + q^{2k+2}}{(1 - q^{2k+2})(1 + q^{2k+1})}
\]
\[
= \frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \frac{2 q^{2k+1} + q^k + q^{2k+2}}{(1 - q^{2k+2})(1 + q^{2k+1})}
\]
\[
= \frac{\beta^2 \xi^2}{\mu_x (\alpha + 1)} \frac{2 q^{2k+1} + q^k + q^{2k+2}}{(1 - q^{2k+2})(1 + q^{2k+1})}
\]
Clearly, we have
\[
\min_{x \in X \cap F_k} \phi_{SCSC}(x) - \max_{y \in Y \cap F_k} \psi_{SCSC}(y) \geq \min_{x \in F_k} \phi_{SCSC}(x) - \max_{y \in F_k} \psi_{SCSC}(y).
\]

It remains to show that \( \min_{x \in F_k} \phi_{SCSC}(x) = \min_{x \in F_k} \tilde{\phi}_{SCSC}(x) \) and \( \max_{y \in F_k} \psi_{SCSC}(y) = \max_{y \in F_k} \tilde{\psi}_{SCSC}(y) \). Recall the expressions (43) and (45). It suffices to prove
\[
\|\hat{x}\|_2 \leq R_x \text{ and } \|\hat{y}\|_2 \leq R_y
\]
where
\[
\hat{x} = -\frac{\xi}{\mu_x} \mathbf{B}(m, \zeta) \top \begin{bmatrix} \tilde{y}^* \\ 0_{m-k} \end{bmatrix} + \frac{\beta \xi}{\mu_x} e_1,
\]
\[
\hat{y} = \frac{\xi}{\mu_y} \mathbf{B}(m, \zeta) \begin{bmatrix} \tilde{x}^* \\ 0_{m-k} \end{bmatrix},
\]
that is
\[
\hat{x} = \frac{\beta \xi (1 - q)}{\mu_x (1 - q^{2k+2})} \begin{bmatrix} 1 + q^{2k+1}, q + q^{2k}, \ldots, q^k, 0, \ldots, 0 \end{bmatrix} \top,
\]
\[
\hat{y} = \frac{\beta}{1 + q^{2k+1}} \begin{bmatrix} q + q^{2k}, q^2 + q^{2k-1}, \ldots, q^k, q^{k+1}, 0, \ldots, 0 \end{bmatrix} \top.
\]

Then we have
\[
\|\hat{x}\|_2^2 = \frac{\beta^2 \xi^2 (1 - q)^2}{\mu_x^2 (1 - q^{2k+2})^2} \left( \frac{1 - q^{4k+2}}{1 - q^2} + 2(k + 1)q^{2k+1} \right),
\]
\[
\|\hat{y}\|_2^2 = \frac{\beta^2}{(1 + q^{2k+1})^2} \left( \frac{q^2 - q^{4k+2}}{1 - q^2} + 2kq^{2k+1} \right).
\]

Note that \( \max_{x > 0} x q^x = \log \frac{1}{q} e^{- \left( \log \frac{1}{q} \right)^2} \) and \( \log r - r^2 \leq -r \) for any \( r > 0 \). It follows that \( \max_{x > 0} x q^x \leq e^{- \log \frac{1}{q}} = q \). Then we have
\[
\|\hat{x}\|_2^2 \leq \frac{\beta^2 \xi^2 (1 - q)^2}{\mu_x^2 (1 - q^2)^2} \left( \frac{1}{1 - q^2} + 1 \right)
\]
\[
\leq \frac{2 \beta^2 \xi^2}{\mu_x^2 (1 - q^2)} \leq \frac{2 \beta^2 (L^2 - 2 \mu_x^2) (\alpha + 1)^2}{16n^2 \mu_x^2 \alpha} \leq R_x^2
\]
and
\[
\|\hat{y}\|_2^2 \leq \beta^2 \left( \frac{q^2}{1 - q^2} + q^2 \right)
\]
\[
\leq \frac{2 \beta^2 q^2}{1 - q^2} \leq \frac{\beta^2 (\alpha - 1)^2}{2 \alpha} \leq R_y^2.
\]

This completes the proof.
C.4 Proof of Proposition 3.27

1. Just recall Proposition 3.13 and Lemma B.1.

2. It it easy to check

\[ f_{\text{CSC}}(x, y) = \sqrt{L^2 - 2\mu_y^2} \left( y, \tilde{B}(m, 1)x \right) - \frac{\mu_y}{2} ||y||^2 - \frac{\beta \sqrt{L^2 - 2\mu_y^2}}{2n} \langle e_1, x \rangle. \]

Define \( \xi = \sqrt{L^2 - 2\mu_y^2} \) and \( \tilde{\phi}_{\text{CSC}}(x) = \max_{y \in \mathbb{R}^m} f_{\text{CSC}}(x, y) \). We first show that

\[ \min_{x \in X \cap F_k} \tilde{\phi}_{\text{CSC}}(x) = \max_{y \in \mathbb{R}^m} \psi_{\text{CSC}}(y) \geq -\frac{k\mu_y \beta^2}{2} + \frac{\beta \xi}{\sqrt{k + 1}}. \]

On one hand, we have

\[ \tilde{\phi}_{\text{CSC}}(x) = \max_{y \in \mathbb{R}^m} \left( \xi \langle y, \tilde{B}(m, 1)x \rangle - \frac{\mu_y}{2} ||y||^2 - \beta \xi \langle e_1, x \rangle \right) \]

\[ = \max_{y \in \mathbb{R}^m} \left( -\frac{\mu_y}{2} ||y - \frac{\xi}{\mu_y} \tilde{B}(m, 1)x||^2 + \frac{\xi^2}{2\mu_y} ||\tilde{B}(m, 1)x||^2 - \beta \xi \langle e_1, x \rangle \right) \]

\[ = \frac{\xi^2}{2\mu_y} ||\tilde{B}(m, 1)x||^2 - \beta \xi \langle e_1, x \rangle. \]

For \( x \in F_k \), let \( \tilde{x} \) be the first \( k \) coordinates of \( x \). We can rewrite \( \tilde{\phi}_{\text{CSC}}(x) \) as

\[ \tilde{\phi}_k(\tilde{x}) \triangleq \tilde{\phi}_{\text{CSC}}(x) = \frac{\xi^2}{2\mu_y} ||\tilde{B}(k, 1)\tilde{x}||^2 - \beta \xi \langle \hat{e}_1, \tilde{x} \rangle, \]

where \( \hat{e}_1 \) is the first \( k \) coordinates of \( e_1 \). Letting \( \nabla \tilde{\phi}_k(\tilde{x}) = 0_k \), we get

\[ \tilde{B}(k, 1)^T \tilde{B}(k, 1)\tilde{x} = \frac{\beta \mu_y}{\xi} \hat{e}_1. \]

The solution is \( \tilde{x}^* = \frac{\beta \mu_y}{\xi} (k, k - 1, \ldots, 1)^T \). Noting that

\[ ||\tilde{x}^*||^2_2 = \frac{\beta^2 \mu_y^2 k(k+1)(2k+1)}{6} \leq \frac{4n^2 \beta^2}{L^2/\mu_y^2 - 2}(m+1)^3 \leq R_x^2, \]

we obtain

\[ \min_{x \in X \cap F_k} \tilde{\phi}_{\text{CSC}}(x) = -\frac{k\mu_y \beta^2}{2}. \]

On the other hand,

\[ \min_{x \in X} \left\langle x, \tilde{B}(m, 1)^T y - \beta e_1 \right\rangle \geq \min_{||x||_2 \leq R_x} -||x||_2 \left\| \tilde{B}(m, 1)^T y - \beta e_1 \right\|_2 \]

\[ \geq -R_x \left\| \tilde{B}(m, 1)^T y - \beta e_1 \right\|_2, \]

(48)
where the equality will hold when either 

\[ x = -\frac{R_y}{B(m,1)\top y - \beta e_1} \left( \tilde{B}(m,1)\top y - \beta e_1 \right) \] 

or 

\[ \tilde{B}(m,1)\top y - \beta e_1 = 0. \]

It follows that 

\[
\psi_{\text{CSC}}(y) = \min_{x \in X} \left( \xi \left( x, \tilde{B}(m,1)\top y - \beta e_1 \right) - \frac{\mu_y}{2} \|y\|_2^2 \right)
\]

\[
= -R_x \xi \left\| \tilde{B}(m,1)\top y - \beta e_1 \right\|_2 - \frac{\mu_y}{2} \|y\|_2^2. 
\] (49)

We can upper bound \( \max_{y \in Y \cap F} \psi_{\text{CSC}}(y) \) as 

\[
\max_{y \in Y \cap F} \psi_{\text{CSC}}(y) = \max_{y \in Y \cap F} \left( \xi \left\| \tilde{B}(m,1)\top y - \beta e_1 \right\|_2 - \frac{\mu_y}{2} \|y\|_2^2 \right)
\]

\[
\leq \max_{y \in F} \left( \xi \left\| \tilde{B}(m,1)\top y - \beta e_1 \right\|_2 \right)
\]

\[
= -R_x \xi \sqrt{J_k,\beta(y_1, y_2, \ldots, y_k)} \leq -\frac{R_x \xi \beta}{\sqrt{k+1}},
\]

where the last inequality follows from Lemma B.7.

It remains to prove \( \min_{x \in X \cap F} \phi_{\text{CSC}}(x) = \min_{x \in X \cap F} \tilde{\phi}_{\text{CSC}}(x) \). Recall the expression (47). It suffices to show that 

\[
\hat{y} = \frac{\xi}{\mu_y} \tilde{B}(m,1) \begin{bmatrix} \hat{x}^* \\ 0_{m-k} \end{bmatrix},
\]

this is 

\[
\hat{y} = \beta \begin{bmatrix} 1_k \\ 0_{m-k} \end{bmatrix}.
\]

Since \( \beta \leq \frac{R_y}{\sqrt{m}} \), \( \|\hat{y}\|_2 \leq R_y \). This completes the proof.

C.5 Proof of Proposition 3.31

1. Just recall Proposition 3.13 and Lemma B.1

2. It is easy to check

\[
f_{\text{CC}}(x, y) = \frac{L}{2n} \left( y, \tilde{B}(m,1) x \right) - \frac{LR_y}{2n \sqrt{m}} (e_1, x).
\]

By similar analysis from Equation (48) to Equation (49) of the proof of Proposition 3.27, we can conclude that 

\[
\phi_{\text{CC}}(x) = \frac{LR_y}{2n} \left\| \tilde{B}(m,1) x \right\|_2 - \frac{LR_y}{2n \sqrt{m}} (e_1, x),
\]

\[
\psi_{\text{CC}}(y) = -\frac{LR_x}{2n} \left\| \tilde{B}(m,1)\top y - \frac{R_y e_1}{\sqrt{m}} \right\|_2.
\]

Note that 

\[
\phi_{\text{CC}}(x) = \max_{y \in Y} f_{\text{CC}}(x, y) \geq \max_{y \in Y} \min_{x \in X} f_{\text{CC}}(x, y) = \max_{y \in Y} \psi(y) \geq \psi(y^*) = 0,
\]

66
where $y^* = \frac{R_y}{\sqrt{m}}1_m \in \mathcal{Y}$. Therefore, we have

$$\min_{x \in \mathcal{X} \cap \mathcal{F}_k} \phi_{\text{CC}}(x) = \phi_{\text{CC}}(0_m) = 0.$$ On the other hand, following from Lemma B.7, we can obtain

$$\max_{y \in \mathcal{Y} \cap \mathcal{F}_k} \psi_{\text{CC}}(y) = \max_{y \in \mathcal{Y} \cap \mathcal{F}_k} -LR_x \frac{R_y}{2n} \left\| \frac{1}{\sqrt{m}} \mathbf{1} - \mathbf{e}_1 \right\|_2 = -LR_x \frac{R_y}{2n} \sqrt{m(k+1)},$$

where the optimal point is $\tilde{y}^* = \frac{R_y}{(k+1)\sqrt{m}}(k, k-1, \ldots, 1, 0, \ldots, 0)^\top$, which satisfies

$$\|\tilde{y}^*\|_2 = \frac{R_y}{(k+1)\sqrt{m}} \frac{k(k+1)(2k+1)}{6} \leq R_y.$$

Finally, note that $k + 1 \geq m/2$. Thus we obtain

$$\min_{x \in \mathcal{X} \cap \mathcal{F}_k} \phi_{\text{CC}}(x) - \max_{y \in \mathcal{Y} \cap \mathcal{F}_k} \psi_{\text{CC}}(y) \geq \frac{LR_x R_y}{2n \sqrt{m(k+1)}}.$$

### C.6 Proof of Proposition 3.35

1. By Proposition 3.17 and Lemma B.1, $f_{\text{NCSC},i}$ is $(-\mu_1, \mu_2)$-convex-concave and $l$-smooth where

$$\mu_1 = \frac{45(\sqrt{3} - 1)L^2\alpha}{16n^2\mu_y} \leq \mu_x,$$

$$\mu_2 = \mu_y,$$

$$l = \frac{L}{4n} \sqrt{4n^2 + \frac{32n^2\mu_y^2}{L^2} + \frac{45L^2\alpha}{4n^2\mu_y}} \leq \frac{L}{4n} \left(2n + 4\sqrt{2n\mu_y}\right) + \frac{L}{8} \leq L.$$ Thus each component $f_{\text{NCSC},i}$ is $(-\mu_x, \mu_y)$-convex-concave and $L$-smooth.

2. We first give a closed form expression of $\phi_{\text{NCSC}}$. For simplicity, we omit the parameters of $\hat{\mathbf{B}}$.

It is easy to check

$$f_{\text{NCSC}}(x, y) = \frac{L}{4n} \left\langle y, \hat{\mathbf{B}}x \right\rangle - \frac{\mu_y}{2} \|y\|_2^2 + \sqrt{\alpha\lambda L} \frac{m}{4n\mu_y} \sum_{i=1}^m \Gamma \left(\frac{1}{2} \sqrt{\frac{\alpha L}{\lambda n}} x_i \right) - \frac{1}{2} \sqrt{\frac{\lambda L}{n}} \langle \mathbf{e}_1, y \rangle.$$ Then we can rewrite $f_{\text{NCSC}}(x, y)$ as

$$f_{\text{NCSC}}(x, y) = -\frac{\mu_y}{2} \left\| y - \frac{1}{\mu_y} \left( \frac{L}{4n} \hat{\mathbf{B}}x - \frac{1}{2} \sqrt{\frac{\lambda L}{n}} \mathbf{e}_1 \right) \right\|_2^2$$

$$+ \frac{1}{2\mu_y} \left\| \frac{L}{4n} \hat{\mathbf{B}}x - \frac{1}{2} \sqrt{\frac{\lambda L}{n}} \mathbf{e}_1 \right\|_2^2 + \frac{\sqrt{\alpha\lambda L}}{4n\mu_y} \sum_{i=1}^m \Gamma \left(\frac{1}{2} \sqrt{\frac{\alpha L}{\lambda n}} x_i \right).$$

It follows that

$$\phi_{\text{NCSC}}(x) = \frac{1}{2\mu_y} \left\| \frac{L}{4n} \hat{\mathbf{B}}x - \frac{1}{2} \sqrt{\frac{\lambda L}{n}} \mathbf{e}_1 \right\|_2^2 + \frac{\sqrt{\alpha\lambda L}}{4n\mu_y} \sum_{i=1}^m \Gamma \left(\frac{1}{2} \sqrt{\frac{\alpha L}{\lambda n}} x_i \right).$$
\[
L = \frac{L^2}{32n^2\mu_y}\left\| \hat{B}x \right\|^2_2 - \frac{L}{8n\mu_y} \sqrt{\frac{\alpha \lambda L}{n}} \langle x, e_1 \rangle + \frac{\sqrt{\alpha \lambda L}}{4n\mu_y} \sum_{i=1}^{m} \Gamma \left( \frac{1}{2} \sqrt{\frac{\alpha \lambda L}{\lambda_n}} x_i \right) + \frac{\lambda L}{8n\mu_y}.
\]

Letting \( \hat{x} = \frac{1}{2} \sqrt{\frac{\alpha L}{\lambda_n}} x \), we have

\[
\tilde{\phi}_{\text{NCSC}}(\hat{x}) \triangleq \phi_{\text{NCSC}}(x) = \frac{\lambda L}{4n\mu_y \sqrt{\alpha}} \left( \frac{1}{2} \left\| \hat{B}x \right\|^2_2 - \sqrt{\alpha} \langle \hat{x}, e_1 \rangle + \alpha \sum_{i=1}^{m} \Gamma(\hat{x}_i) \right) + \frac{\lambda L}{8n\mu_y}.
\]

By Proposition \[B.4\]

\[
\phi_{\text{NCSC}}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} \phi_{\text{NCSC}}(x) = \tilde{\phi}_{\text{NCSC}}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} \phi_{\text{NCSC}}(\hat{x})
\]

\[
= \frac{\lambda L}{4n\mu_y \sqrt{\alpha}} \left( \frac{\sqrt{\alpha}}{2} + 10\alpha m \right)
\]

\[
\leq \frac{10368n^2\mu_y^2\varepsilon^2}{L^2\alpha} + \frac{207360n^2\mu_y^2\varepsilon^2}{L^2\alpha}
\]

\[
\leq \frac{10368}{217728} \Delta + \frac{207360}{217728} \Delta \leq \Delta.
\]

3. Since \( \alpha \leq 1 \), we have \( \frac{\Delta L^2\sqrt{\alpha}}{217728n^2\varepsilon^2\mu_y} \geq \frac{\Delta L^2\alpha}{217728n^2\varepsilon^2\mu_y} \geq 2 \) and consequently \( m \geq 2 \). By Proposition \[B.4\]

\[
\min_{x \in \mathcal{F}_M} \left\| \nabla \phi_{\text{NCSC}}(x) \right\|_2 = \frac{1}{2} \sqrt{\frac{\alpha L}{\lambda_n}} \min_{x \in \mathcal{F}_M} \left\| \tilde{\phi}_{\text{NCSC}}(x) \right\|_2
\]

\[
\geq \frac{1}{2} \sqrt{\frac{\alpha L}{\lambda_n}} \frac{\lambda L}{4n\mu_y \sqrt{\alpha}} \frac{\alpha^{3/4}}{4} \geq 9\varepsilon.
\]

C.7 Proof of Proposition \[3.39\]

1. By Proposition \[3.17\] and Lemma \[B.1\] \( \tilde{f}_{\text{NCSC}, i} \) is \((-\mu_1, \mu_2)\)-convex-concave and \( \{\tilde{f}_{\text{NCSC}, i}\}_{i=1}^{n} \) \( l \)-average smooth where

\[
\mu_1 = \frac{45(\sqrt{3} - 1)L^2\alpha}{256n\mu_y} \leq \mu_x,
\]

\[
\mu_2 = \mu_y,
\]

\[
l = \frac{L'}{8\sqrt{n}} \left( \sqrt{4n + \frac{256n\mu_y^2}{L'^2} + 16200\frac{\alpha^2L'^2}{256n\mu_y^2}} \right) \leq \frac{L'}{8\sqrt{n}} \left( 2\sqrt{n} + 10\sqrt{n}\mu_y + \frac{45\sqrt{2}\alpha L'}{8\sqrt{n}\mu_y} \right) \leq L'.
\]

Thus each component \( \tilde{f}_{\text{NCSC}, i} \) is \((-\mu_x, \mu_y)\)-convex-concave and \( \{\tilde{f}_{\text{NCSC}, i}\}_{i=1}^{n} \) is \( L' \)-smooth.

2. We first give a closed form expression of \( \tilde{\phi}_{\text{NCSC}} \). For simplicity, we omit the parameters of \( \hat{B} \). It is easy to check

\[
\tilde{f}_{\text{NCSC}}(x, y) = \frac{L'}{16\sqrt{n}} \langle y, \hat{B}x \rangle - \frac{\mu_y}{2} \left\| y \right\|^2_2 + \frac{\sqrt{\alpha \lambda L'} L'}{10\sqrt{n} \mu_y} \sum_{i=1}^{m} \Gamma \left( \frac{1}{4} \sqrt{\frac{\alpha \lambda L'}{\lambda_n}} x_i \right) - \frac{1}{4} \sqrt{\frac{\lambda L'}{\sqrt{n}}} \langle e_1, y \rangle.
\]
Then we can rewrite $\tilde{f}_{\text{NCSC}}(x, y)$ as

$$
\tilde{f}_{\text{NCSC}}(x, y) = -\frac{\mu_y}{2} \left\| y - \frac{1}{\mu_y} \left( \frac{L'}{16\sqrt{n}} \hat{B}x - \frac{1}{4} \frac{\lambda L'}{\sqrt{n}} e_1 \right) \right\|^2_2 \\
+ \frac{1}{2\mu_y} \left\| \frac{L'}{16\sqrt{n}} \hat{B}x - \frac{1}{4} \frac{\lambda L'}{\sqrt{n}} e_1 \right\|^2_2 + \frac{\sqrt{\alpha} \lambda L'}{16\sqrt{n} \mu_y} \sum_{i=1}^{m} \Gamma \left( \frac{1}{4} \frac{\sqrt{\alpha} L'}{\lambda \sqrt{n} x_i} \right).
$$

It follows that

$$
\tilde{\phi}_{\text{NCSC}}(x) = \frac{1}{2\mu_y} \left\| \frac{L'}{16\sqrt{n}} \hat{B}x - \frac{1}{4} \frac{\lambda L'}{\sqrt{n}} e_1 \right\|^2_2 + \frac{\sqrt{\alpha} \lambda L'}{16\sqrt{n} \mu_y} \sum_{i=1}^{m} \Gamma \left( \frac{1}{4} \frac{\sqrt{\alpha} L'}{\lambda \sqrt{n} x_i} \right) \\
= \frac{L'^2}{512n\mu_y} \left\| \hat{B}x \right\|^2_2 - \frac{L'}{64\mu_y} \sqrt{\frac{\alpha}{\lambda^3}} \langle x, e_1 \rangle + \frac{\sqrt{\alpha} \lambda L'}{16\sqrt{n} \mu_y} \sum_{i=1}^{m} \Gamma \left( \frac{1}{4} \frac{\sqrt{\alpha} L'}{\lambda \sqrt{n} x_i} \right) + \frac{\lambda L'}{32\sqrt{n} \mu_y}.
$$

Letting $\tilde{x} = \frac{1}{4} \sqrt{\frac{\alpha L'}{\lambda \sqrt{n}}} x$, we have

$$
\hat{\phi}_{\text{NCSC}}(\tilde{x}) = \frac{\lambda L'}{16\mu_y \sqrt{\alpha n}} \left( \frac{1}{2} \left\| \hat{B}x \right\|^2_2 - \sqrt{\alpha} \langle \tilde{x}, e_1 \rangle + \alpha \sum_{i=1}^{m} \Gamma (\tilde{x}_i) \right) + \frac{\lambda L'}{32\sqrt{n} \mu_y}.
$$

By Proposition [B.3],

$$
\hat{\phi}_{\text{NCSC}}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} \hat{\phi}_{\text{NCSC}}(x) = \hat{\phi}_{\text{NCSC}}(0_{m+1}) - \min_{\tilde{x} \in \mathbb{R}^{m+1}} \hat{\phi}_{\text{NCSC}}(\tilde{x}) \\
= \frac{\lambda L'}{16\mu_y \sqrt{\alpha n}} \left( \frac{\sqrt{\alpha}}{2} + 10\alpha m \right) \\
\leq \frac{165888n\mu_y \varepsilon^2}{L'^2} + \frac{331760n\mu_y \varepsilon^2 m}{L'^2} \\
\leq \frac{3483648\Delta}{3483648} + \frac{331760\Delta}{3483648} \Delta \leq \Delta.
$$

3. Since $\alpha \leq 1$, we have $\frac{\Delta L'^2 \sqrt{\alpha}}{3483648n\mu_y \varepsilon^2 \mu_y} \geq \frac{\Delta L'^2 \mu_y}{3483648n\mu_y \varepsilon^2 \mu_y} \geq 2$ and consequently $m \geq 2$. By Proposition [B.3],

$$
\min_{x \in \mathcal{F}_M} \left\| \nabla \hat{\phi}_{\text{NCSC}}(x) \right\|^2_2 = \frac{1}{4} \sqrt{\frac{\alpha L'}{\lambda \sqrt{n}}} \min_{x \in \mathcal{F}_M} \left\| \nabla \hat{\phi}_{\text{NCSC}}(x) \right\|^2_2 \\
\geq \frac{1}{4} \sqrt{\frac{\alpha L'}{\lambda \sqrt{n}}} \frac{\lambda L'}{16\mu_y \sqrt{\alpha n}} \frac{\alpha^{3/4}}{4} \geq 9\varepsilon.
$$

**D  Proofs for Section 4**

In this section, we present some omitted proofs in Section 4.
D.1 Proofs of Proposition 4.15 and Lemma 4.16

We use $\|A\|$ to denote the spectral radius of $A$. Recall that $b_{l-1}^T$ is the $l$-th row of $B$, $G(x) = \sum_{i=1}^{m-1} \Gamma(x_i)$ and

$$\mathcal{L}_i = \{l: 0 \leq l \leq m, l \equiv i - 1 \pmod{n}\}, i = 1, 2, \ldots, n.$$ 

For simplicity, we omit the parameters of $B$, $b_l$ and $r_l$. For $1 \leq l \leq n$, let $B_l$ be the submatrix whose rows are $\{b_{l}^T\}_{l \in \mathcal{L}_i}$. Then $r_l$ can be written as

$$r_l(x) = \frac{n}{2} \|B_l x\|^2_2 + \frac{c_l}{2} \|x\|^2_2 + c_2 n \langle e_1, x \rangle \mathbf{1}_{l \{i=1\}}.$$ 

Proof of Proposition 4.15

1. For the convex case,

$$r_l(x) = \frac{n}{2} \|B_l x\|^2_2 + \frac{c_l}{2} \|x\|^2_2 - c_3 n \langle e_1, x \rangle \mathbf{1}_{l \{i=1\}}.$$ 

Obviously, $r_l$ is $c_1$-strongly convex. Note that

$$\langle u, B_l^T B_l u \rangle = \|B_l u\|^2_2$$

$$= \sum_{l \in \mathcal{L}_i} (b_l^T u)^2$$

$$= \sum_{l \in \mathcal{L}_i \setminus \{0, m\}} (u_l - u_{l+1})^2 + \omega^2 u_1^2 \mathbb{I}_{\{0 \in \mathcal{L}_i\}} + \zeta^2 u_m^2 \mathbb{I}_{\{m \in \mathcal{L}_i\}}$$

$$\leq 2 \|u\|^2_2,$$

where the last inequality is according to $(x + y)^2 \leq 2(x^2 + y^2)$, and $|l_1 - l_2| \geq n \geq 2$ for $l_1, l_2 \in \mathcal{L}_i$. Hence, $\|B_l^T B_l\| \leq 2$, and

$$\|\nabla^2 r_l(x)\| = \left\|nB_l^T B_l + c_1 I\right\| \leq 2n + c_1.$$ 

Next, observe that

$$\|\nabla r_l(x_1) - \nabla r_l(x_2)\|^2_2 = \left\|(nB_l^T B_l + c_1 I)(x_1 - x_2)\right\|^2_2$$

Let $u = x_1 - x_2$. Note that

$$b_l b_l^T u = \begin{cases} (u_l - u_{l+1})(e_l - e_{l+1}), & 0 < l < m, \\
\omega^2 u_1 e_1, & l = 0, \\
\zeta^2 u_m e_m, & l = m. \end{cases}$$

Thus,

$$\left\|(nB_l^T B_l + c_1 I)u\right\|^2_2$$

$$= \left\|n \sum_{l \in \mathcal{L}_i \setminus \{0, m\}} (u_l - u_{l+1})(e_l - e_{l+1}) + n \omega^2 u_1^2 \mathbb{I}_{\{0 \in \mathcal{L}_i\}} + n \zeta^2 u_m^2 \mathbb{I}_{\{m \in \mathcal{L}_i\}} + c_1 u \right\|^2_2$$

70
\sum_{l \in \mathcal{L} \setminus \{0,m\}} 
\left( (n(u_l - u_{l+1}) + c_1 u_l)^2 + (-n(u_l - u_{l+1}) + c_1 u_{l+1})^2 \right) 
\leq 2 \left[ (n + c_1)^2 + n^2 \right] 
\left[ \sum_{l \in \mathcal{L} \setminus \{0,m\}} (u_l^2 + u_{l+1}^2) \right] 
+ c_1^2 \|u\|_2^2,

where we have used \((2n + c_1)^2 \leq 2 \left( (n + c_1)^2 + n^2 \right)\).

Therefore, we have

\frac{1}{n} \sum_{i=1}^n \| \nabla r_i(x_1) - \nabla r_i(x_2) \|_2^2 
\leq \frac{1}{n} \sum_{i=0}^m 4 \left[ (n + c_1)^2 + n^2 \right] u_i^2 + c_1^2 \|u\|_2^2 
\leq \frac{4}{n} \left[ (n + c_1)^2 + n^2 \right] \|u\|_2^2 + c_1^2 \|u\|_2^2,

In summary, we get that \(\{r_i\}_{i=1}^n\) is \(L'\)-average smooth, where

\[ L' = \sqrt{\frac{4}{n} \left[ (n + c_1)^2 + n^2 \right] + c_1^2}. \]

2. The results of the non-convex case follow from the above proof, Proposition B.4 and the inequality \((a + b)^2 \leq 2(a^2 + b^2)\).

\[ \square \]

Proof of Lemma 4.16

1. For the convex case,

\[ r_j(x) = \frac{n}{2} \|B_jx\|_2^2 + \frac{c_1}{2} \|x\|_2^2 - c_3 n \langle e_1, x \rangle \mathbb{1}_{\{j=1\}}. \]

Recall that

\[ b_l^\top x = \begin{cases} 
(x_l - x_{l+1})(e_l - e_{l+1}), & 0 < l < m, \\
\omega^2 x_1 e_1, & l = 0, \\
\zeta^2 x_m e_m, & l = m.
\end{cases} \]

For \(x \in \mathcal{F}_0\), we have \(x = 0_m\), and

\[ \nabla r_1(x) = c_3 n e_1 \in \mathcal{F}_1, \quad \nabla r_j(x) = 0_m \quad (j \geq 2). \]

For \(x \in \mathcal{F}_k \quad (1 \leq k < m)\), we have

\[ b_l^\top x \in \begin{cases} 
\mathcal{F}_k, & l \neq k, \\
\mathcal{F}_{k+1}, & l = k.
\end{cases} \]
Moreover, we suppose \( k \in \mathcal{L}_i \). Since
\[
\nabla r_j(x) = nB_j^T B_j x + c_1 x - c_3 n e_1 1_{\{j=1\}} \\
= n \sum_{l \in \mathcal{L}_j} b_l b_l^T x + c_1 x - c_3 n e_1 1_{\{j=1\}},
\]
it follows that \( \nabla r_i(x) \in \mathcal{F}_{k+1} \) and \( \nabla r_j(x) \in \mathcal{F}_k \) (\( j \neq i \)).

Now, we turn to consider \( u = \text{prox}_r^\gamma(x) \). We have
\[
\left( nB_j^T B_j + \left( c_1 + \frac{1}{\gamma} \right) I \right) u = c_3 n e_1 1_{\{j=1\}} + \frac{1}{\gamma} x,
\]
i.e.,
\[
u = d_1 (I + d_2 B_j^T B_j)^{-1} y,
\]
where \( d_1 = \frac{1}{c_1 + 1/\gamma} \), \( d_2 = \frac{n}{c_1 + 1/\gamma} \), and \( y = c_3 n e_1 1_{\{j=1\}} + \frac{1}{\gamma} x \).

Note that
\[
(I + d_2 B_j^T B_j)^{-1} = I - B_j^T \left( \frac{1}{d_2} I + B_j B_j^T \right)^{-1} B_j.
\]
If \( k = 0 \) and \( j > 1 \), we have \( y = 0_m \) and \( u = 0_m \).
If \( k = 0 \) and \( j = 1 \), we have \( y = c_3 n e_1 \). Since \( \omega = 0 \), \( B_1 e_1 = 0_m \), so \( u = c_1 y \in \mathcal{F}_1 \).

For \( k \geq 1 \), we know that \( y \in \mathcal{F}_k \). And observe that if \( |l - l'| \geq 2 \), then \( b_l^T b_{l'} = 0 \), and consequently \( B_j B_j^T \) is a diagonal matrix, so we can assume that \( \frac{1}{d_2} I + B_j B_j^T = \text{diag}(\beta_{j,1}, \ldots, \beta_{j,|\mathcal{L}_j|}) \).

Therefore,
\[
u = d_1 y - d_1 \sum_{s=1}^{n} \beta_{j,s} b_{j,s} b_{j,s}^T y,
\]
where we assume that \( \mathcal{L}_j = \{l_{j,1}, \ldots, l_{j,|\mathcal{L}_j|}\} \).

Thus, we have \( \text{prox}_r^\gamma(x) \in \mathcal{F}_{k+1} \) for \( k \in \mathcal{L}_i \) and \( \text{prox}_r^\gamma(x) \in \mathcal{F}_k \) (\( j \neq i \)).

2. For the non-convex case,
\[
r_j(x) = \frac{n}{2} \|B_j x\|_2^2 + c_2 G(x) - c_3 n (e_1, x) 1_{\{j=1\}}.
\]
Let \( \Gamma'(x) \) be the derivative of \( \Gamma(x) \). First note that \( \Gamma'(0) = 0 \), so if \( x \in \mathcal{F}_k \), then
\[
\nabla G(x) = (\Gamma'(x_1), \Gamma'(x_2), \ldots, \Gamma'(x_{m-1}), 0)^T \in \mathcal{F}_k.
\]

For \( x \in \mathcal{F}_0 \), we have \( x = 0_m \), and
\[
\nabla r_1(x) = c_3 n e_1 \in \mathcal{F}_1,
\]
\[
\nabla r_j(x) = 0_m \ (j \geq 2).
\]
For $x \in \mathcal{F}_k$ ($1 \leq k < m$), recall that

$$b_l^\top b_l^\top x = \begin{cases} (x_l - x_{l+1})(e_l - e_{l+1}), & 0 < l < m, \\ \omega^2 x_1 e_1, & l = 0, \\ \zeta^2 x_m e_m, & l = m. \end{cases}$$

Suppose $k \in \mathcal{L}_i$. Since

$$\nabla r_j(x) = nB_j^\top B_j x + c_2 \nabla G(x) - c_3 n e_1 \mathbb{1}_{\{j=1\}} = n \sum_{l \in \mathcal{L}_j} b_l b_l^\top x + c_2 \nabla G(x) - c_3 n e_1 \mathbb{1}_{\{j=1\}},$$

it follows that $\nabla r_i(x) \in \mathcal{F}_{k+1}$ and $\nabla r_j(x) \in \mathcal{F}_k$ ($j \neq i$).

Now, we turn to consider $u = \text{prox}_r^j(x)$. We have

$$\nabla r_j(u) + \frac{1}{\gamma}(u - x) = 0_m,$$

that is

$$\left(n \sum_{l \in \mathcal{L}_j} b_l b_l^\top + \frac{1}{\gamma}I\right) u + c_2 \nabla G(u) = y,$$

where $y = c_3 n e_1 \mathbb{1}_{\{j=1\}} + \frac{1}{\gamma} x$. Since $\gamma < \frac{\sqrt{2}+1}{60 c_2}$, we have the following claims.

(a) If $0 < l < m - 1$ and $l \in \mathcal{L}_j$, we have

$$n(u_l - u_{l+1}) + \frac{1}{\gamma} u_l + 120 c_2 \frac{u_l^2 (u_l - 1)}{1 + u_l^2} = y_l$$

$$n(u_{l+1} - u_l) + \frac{1}{\gamma} u_{l+1} + 120 c_2 \frac{u_{l+1}^2 (u_{l+1} - 1)}{1 + u_{l+1}^2} = y_{l+1}. \quad (50)$$

By Lemma B.6, $y_l = y_{l+1} = 0$ implies $u_l = u_{l+1} = 0$.

(b) If $m - 1 \in \mathcal{L}_j$, we have

$$n(u_{m-1} - u_m) + \frac{1}{\gamma} u_{m-1} + 120 c_2 \frac{u_{m-1}^2 (u_{m-1} - 1)}{1 + u_{m-1}^2} = y_{m-1}$$

$$n(u_m - u_{m-1}) + \frac{1}{\gamma} u_m = y_m. \quad (51)$$

If $y_{m-1} = y_m = 0$, we obtain

$$\frac{1 + 2\gamma n}{\gamma(1 + \gamma n)} u_{m-1} + 120 c_2 \frac{u_{m-1}^2 (u_{m-1} - 1)}{1 + u_{m-1}^2} = 0$$

$$\left(n + \frac{1}{\gamma}\right) u_m - \frac{1}{\gamma} u_{m-1} = 0.$$

By Lemma B.5, $u_{m-1} = u_m = 0$. 

73
(c) If \( m \in \mathcal{L}_j \), we have
\[
n\zeta^2 u_m + \frac{1}{\gamma} u_m = y_m.
\]
(52)
y_m = 0 implies \( u_m = 0 \).

(d) If \( l > 0 \) and \( l - 1, l \notin \mathcal{L}_j \), we have
\[
\frac{1}{\gamma} u_l + 120 c_2 \frac{u_l^2 (u_l - 1)}{1 + u_l^2} 1_{\{l < m\}} = y_l.
\]
(53)
By Lemma B.5, \( y_l = 0 \) implies \( u_l = 0 \).

For \( x \in F_0 \) and \( j = 1 \), we have \( x = 0_m \) and \( y = n \omega^2 e_1 \). Since \( n \geq 2 \), we have \( 1 \notin \mathcal{L}_1 \). If \( 2 \in \mathcal{L}_1 \), we can consider the solution to Equations (50), (51) or (52) and conclude that \( u_2 = 0 \). If \( 2 \notin \mathcal{L}_1 \), we can consider the solution to Equation (53) and conclude that \( u_1 = 0 \). Similarly, we can obtain \( u_l = 0 \) for all \( l \geq 2 \), which implies \( u \in F_1 \).

For \( x \in F_0 \) and \( j > 1 \), we have \( y = 0_m \) and \( 0 \notin \mathcal{L}_j \). If \( 1 \in \mathcal{L}_j \), we can consider the solution to Equations (50) or (51) and conclude that \( u_1 = 0 \). If \( 1 \notin \mathcal{L}_j \), we can consider the solution to Equation (53) and conclude that \( u_1 = 0 \). Similarly, we can obtain \( u_l = 0 \) for all \( l \geq 1 \), which implies \( u \in F_j \).

This completes the proof.

D.2 Proof of Proposition 4.21

1. Just recall Proposition 4.15 and Lemma B.1.

2. It is easy to check
\[
f_{SC}(x) = \frac{L - \mu}{4n} \parallel B \left( m, 0, \sqrt{\frac{2}{\alpha + 1}} \right) x \parallel^2 + \frac{\mu}{2} \parallel x \parallel^2 - \mu R \sqrt{\alpha} \langle e_1, x \rangle.
\]
Set \( \xi = \frac{2R \sqrt{\alpha}}{\alpha - 1} \) and \( \Delta = \frac{\mu^2 \alpha}{\alpha + 1} \). Let \( \nabla f_{SC}(x) = 0_m \), that is
\[
\left( \frac{L - \mu}{2n} A \left( m, 0, \sqrt{\frac{2}{\alpha + 1}} \right) + \mu \mathbf{I} \right) x = \frac{L - \mu}{n(\alpha + 1)} \xi e_1.
\]

74
or
\[
\begin{bmatrix}
1 + \frac{2n\mu}{L - \mu} & -1 & & & & \\
-1 & 2 + \frac{2n\mu}{L - \mu} & -1 & & & \\
& \ddots & \ddots & \ddots & & \\
& & -1 & 2 + \frac{2n\mu}{L - \mu} & -1 & \\
& & & & -1 & \zeta^2 + 1 + \frac{2n\mu}{L - \mu}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{m-1} \\
x_m
\end{bmatrix}
= 
\begin{bmatrix}
\frac{2\xi}{\alpha + 1} \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix},
\tag{54}
\]

where \( \zeta = \sqrt{\frac{2}{\alpha + 1}} \).

Note that \( q = \frac{\alpha - 1}{\alpha + 1} \) is a root of the equation
\[
z^2 - \left(2 + \frac{2n\mu}{L - \mu}\right) z + 1 = 0,
\]
and
\[
\zeta^2 + 1 + \frac{2n\mu}{L - \mu} = \frac{1}{q},
\]
\[
\frac{2}{\alpha + 1} = 1 - q = -q^2 + \left(1 + \frac{2n\mu}{L - \mu}\right) q.
\]

Hence, it is easy to check that the solution to Equations (54) is
\[
x^* = \xi(q^1, q^2, \ldots, q^m)^\top,
\]
and
\[
f_{SC}(x^*) = -\frac{L - \mu}{2n(\alpha + 1)} \xi^2 q = -\Delta.
\]

Moreover, we have
\[
\|x^*\|_2^2 = \zeta^2 \frac{q^2 - q^{2m+2}}{1 - q^2} \leq \zeta^2 \frac{q^2}{1 - q^2} = \zeta^2 \frac{(\alpha - 1)^2}{4\alpha} \leq R^2.
\]

3. If \( x \in \mathcal{F}_k, 1 \leq k < m \), then \( x_{k+1} = x_{k+2} = \cdots = x_m = 0 \).

Let \( y \) be the first \( k \) coordinates of \( x \) and \( A_k \) be first \( k \) rows and columns of the matrix in Equation (54). Then we can rewrite \( f_{SC}(x) \) as
\[
f_k(y) \triangleq f_{SC}(x) = \frac{L - \mu}{4n} y^\top A_k y - \frac{L - \mu}{n(\alpha + 1)} \xi \langle \hat{e}_1, y \rangle,
\]
where \( \hat{e}_1 \) is the first \( k \) coordinates of \( e_1 \). Let \( \nabla f_k(y) = 0_k \), that is
\[
\begin{bmatrix}
1 + \frac{2n\mu}{L - \mu} & -1 & & & & \\
-1 & 2 + \frac{2n\mu}{L - \mu} & -1 & & & \\
& \ddots & \ddots & \ddots & & \\
& & -1 & 2 + \frac{2n\mu}{L - \mu} & -1 & \\
& & & & -1 & \zeta^2 + 1 + \frac{2n\mu}{L - \mu}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_{m-1} \\
y_m
\end{bmatrix}
= 
\begin{bmatrix}
\frac{2\xi}{\alpha + 1} \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}.
\]
By some calculation, the solution to above equation is
\[ \frac{\xi q^{k+1}}{1 + q^{2k+1}} \left( q^{-k} - q^{k} - q^{-k+1} - q^{k-1}, \ldots, q^{-1} - q^{1} \right)^{\top}. \]

Thus
\[ \min_{x \in \mathcal{F}_k} f_{SC}(x) = \min_{y \in \mathbb{R}^k} f_k(y) = -\frac{L - \mu}{2n(\alpha + 1)} \xi^2 \frac{1 - q^{2k}}{1 + q^{2k+1}} = \Delta \frac{1 - q^{2k}}{1 + q^{2k+1}}, \]
and
\[ \min_{x \in \mathcal{X} \cap \mathcal{F}_k} f_{SC}(x) - \min_{x \in \mathcal{X}} f_{SC}(x) \geq \min_{x \in \mathcal{F}_k} f_{SC}(x) - f_{SC}(x^*) = \Delta \left( 1 - \frac{1 - q^{2k}}{1 + q^{2k+1}} \right) = \Delta q^{2k} \frac{1 + q}{1 + q^{2k+1}} \geq \Delta q^{2k}. \]

**D.3 Proof of Proposition 4.26**

1. Just recall Proposition 4.15 and Lemma B.1.

2. It is easy to check
\[ f_C(x) = \frac{L}{4n} \|B(m, 1)x\|^2 - \frac{\sqrt{3}}{2} \frac{BL}{(m + 1)^{3/2}n} (e_1, x). \]

Denote \( \xi = \frac{\sqrt{3}}{2} \frac{BL}{(m + 1)^{3/2}n}. \) Let \( \nabla f_C(x) = 0_m, \) that is
\[ \frac{L}{2n} A(m, 0, 1)x = \frac{\xi}{n} e_1, \]
or
\[
\begin{bmatrix}
  2 & -1 \\
  -1 & 2 & -1 \\
  \vdots & \vdots & \ddots & \ddots \\
  -1 & 2 & -1 \\
  -1 & 1
\end{bmatrix}
\begin{bmatrix}
  x
\end{bmatrix}
= \begin{bmatrix}
  \frac{2\xi}{L} \\
  0 \\
  \vdots \\
  0 \\
  0
\end{bmatrix}. \tag{55}
\]

Hence, it is easily to check that the solution to Equations (55) is
\[ x^* = \frac{2\xi}{L} (m, m - 1, \ldots, 1)^{\top}, \]
and
\[ f_C(x^*) = -\frac{m\xi^2}{nL}. \]
Moreover, we have
\[ \|x^*\|^2_2 = \frac{4\xi^2}{L^2} \frac{m(m + 1)(2m + 1)}{6} \leq \frac{4\xi^2}{3L^2} (m + 1)^3 = \frac{R^2}{L^2}. \]
3. The second property implies $\min_{x \in X} f_C(x) = -\frac{m\xi^2}{nL}$. By similar calculation to above proof, we have

$$\arg \min_{x \in X \cap F_k} f_C(x) = \frac{2\xi}{L}(k, k-1, \ldots, 1, 0, \ldots, 0)^\top,$$

and

$$\min_{x \in X \cap F_k} f_C(x) = -\frac{k\xi^2}{nL}.$$  

Thus

$$\min_{x \in X \cap F_k} f_C(x) - \min_{x \in X} f_C(x) = \frac{\xi^2}{nL}(m - k).$$

### D.4 Proof of Proposition 4.30

1. By Proposition 4.15 and Lemma B.1, $f_{NC,i}$ is $(-l_1)$-weakly convex and $l_2$-smooth where

$$l_1 = \frac{45(\sqrt{3} - 1)\alpha\lambda}{\beta^2} = \frac{45(\sqrt{3} - 1)L}{3n} \leq \frac{45(\sqrt{3} - 1)(\sqrt{3} + 1)n\mu}{30L} = \mu,$$

$$l_2 = \frac{(2n + 180\alpha)\lambda}{\beta^2} = \frac{L}{3n}(2n + 180\alpha) \leq L.$$

Thus each $f_i$ is $L$-smooth and $(-\mu)$-weakly convex.

2. By Proposition B.4, we know that

$$f_{NC}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} f_{NC}(x) \leq \lambda(\sqrt{\alpha}/2 + 10\alpha m) = \frac{1944n\varepsilon^2}{L\alpha} + \frac{38880n\varepsilon^2}{L\sqrt{\alpha}m} \leq \frac{1944}{40824}\Delta + \frac{38880}{40824}\Delta = \Delta.$$

3. Since $\alpha \leq 1$, we have $\frac{\Delta L^2\sqrt{\alpha}}{40824n^2 \varepsilon^2} \geq \frac{\Delta L^2\alpha}{40824n^2 \varepsilon^2}$ and consequently $m \geq 2$. By Proposition B.4, we know that

$$\min_{x \in F_M} \|\nabla f_{NC}(x)\|_2 \geq \frac{\alpha^{3/4}\lambda}{4\beta} = \frac{\alpha^{3/4}\lambda}{4\sqrt{3}\alpha n/L} = \sqrt{\frac{L\lambda\alpha^{3/4}}{3n^4 / 4}} = 9\varepsilon.$$

### D.5 Proof of Proposition 4.37

1. By Proposition 4.15 and Lemma B.1, $\tilde{f}_{NC,i}$ is $(-l_1)$-weakly convex and $\{\tilde{f}_{NC,i}\}_{i=1}^n$ is $l_2$-average smooth where

$$l_1 = \frac{45(\sqrt{3} - 1)\alpha\lambda}{\beta^2} = \frac{45(\sqrt{3} - 1)L'}{16\sqrt{n}} \alpha \leq \frac{45(\sqrt{3} - 1)L'}{16\sqrt{n}} \frac{8(\sqrt{3} + 1)\sqrt{n}\mu}{45L'} = \mu,$$

$$l_2 = 4\sqrt{n} + 4050\alpha^2 \frac{\lambda}{\beta^2} = \frac{L'}{4\sqrt{n}} \sqrt{n + 4050\alpha^2} \leq L'.$$

2. By Proposition B.4, we know that

$$f_{NC}(0_{m+1}) - \min_{x \in \mathbb{R}^{m+1}} f_{NC}(x) \leq \lambda(\sqrt{\alpha}/2 + 10\alpha m) = \frac{10368\sqrt{n}\varepsilon^2}{L\alpha} + \frac{207360\sqrt{n}\varepsilon^2}{L\sqrt{\alpha}m} \leq \frac{10368}{217728}\Delta + \frac{207360}{217728}\Delta = \Delta.$$
3. Since $\alpha \leq 1$, we have $\frac{\Delta L'}{211728 \sqrt{n\varepsilon}} \geq \frac{\Delta L'}{211728 \sqrt{n\varepsilon}}$ and consequently $m \geq 2$. By Proposition [B.4], we know that

$$\min_{x \in \mathcal{F}_M} \|\nabla f_{NC}(x)\|_2 \geq \frac{\alpha^{3/4} \lambda}{4\beta} = \frac{\alpha^{3/4} \lambda}{4\sqrt{16\lambda \sqrt{n} / L'}} = \frac{\sqrt{L'}}{\sqrt{n}} \frac{\alpha^{3/4}}{16} = 9\varepsilon.$$