We give the first examples of Fano manifolds with multiple optimal tori, i.e. we construct monotone Lagrangian tori \( L \), such that the weighted number of holomorphic Maslov index two discs with boundary on \( L \) equals the upper bound given by the symplectic invariant \( \lim \sup_n \left( \frac{m_0(L^n)}{n} \right) \), where \( m_0(L) \) is the Floer potential.

To every trivalent graph \( \gamma \) of genus \( g \) we associate an optimal torus \( L_\gamma \) on the celebrated symplectic Fano manifold \( N_g \) (of complex dimension \( 3g - 3 \)) given by the character variety of rank 2 on a genus \( g \) surface with prescribed odd monodromy at a puncture. We moreover show that all pairs \( (N_g, L_\gamma) \) are pairwise non-isotopic. In particular, we confirm a form of mirror symmetry between the A-model of the pairs \( (N_g, L_\gamma) \) and B-model of graph potentials, a family of Laurent polynomials we introduced in earlier work.

A crucial input from outside of symplectic geometry is an analysis of Manon’s toric degenerations of algebro-geometric models \( M_C(\mathcal{L}) \) for the spaces \( N_g \), as moduli spaces of stable rank 2 bundles on an algebraic curve with a fixed determinant, constructed using conformal field theory.

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1. Introduction

We discuss various aspects of the mirror symmetry for the moduli space $M_C(2, \mathcal{L})$ of rank 2 vector bundles with fixed determinant $\mathcal{L}$ of odd degree on an algebraic curve $C$ of genus $g \geq 2$. These are smooth projective Fano varieties of dimension $3g - 3$ in the algebro-geometric context, and monotone symplectic manifolds of dimension $6g - 6$ in the symplecto-geometric context equipped with the Atiyah-Goldman-Narasimhan symplectic form, where they are denoted $N_g$. They have been the subject of many works, spanning 50+ years of research.

In [11] we have introduced graph potentials, a class of Laurent polynomials with interesting symmetries, associated to (colored) trivalent graphs. The goal of this article is to relate these graph potentials to $M_C(2, \mathcal{L})$. We do this by considering the monotone Lagrangian ber $\mathcal{L}_\gamma$ of Nishinou–Nohara–Ueda’s integrable system [62, 60] associated with Manon’s toric degeneration for $M_C(2, \mathcal{L})$, and identify the graph potential $\mathcal{W}_\gamma \mathcal{C}$ with the Floer potential of $\mathcal{L}_\gamma$. This allows us to identify the classical period of the graph potential with the quantum period of $M_C(2, \mathcal{L})$. This is a manifestation of mirror symmetry for Fano varieties, and for an excellent survey on this subject one is referred to [44].

Graph potentials. Let $\gamma = (V, E, c)$ be a (colored) trivalent graph. For every trivalent vertex $v \in V$ with edges $a, b, c \in E$ we define the vertex potential as the Laurent polynomial $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ in the uncolored case, and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + abc$ in the colored case, with variables indexed by the edges. For the graph potential we take the sum over all vertices. For a trivalent graph of genus $g$ we obtain a Laurent polynomial in $3g - 3$ variables. For more details on the construction one is referred to Section 2.1.

Graph potentials have interesting symmetry properties, where the choice of different trivalent graphs of fixed genus $g$ gives rise to explicit mutations between the different potentials. It is moreover possible to combine series expansions of oscillating integrals of graph potentials for all $g$ simultaneously into a topological quantum field theory. These are the main results of [11]. By establishing mirror symmetry for graph potentials and $M_C(2, \mathcal{L})$ it becomes possible to use the efficient computational methods for classical periods of graph potentials in op. cit. and apply them to quantum periods of $M_C(2, \mathcal{L})$.

A monotone Lagrangian torus for $M_C(2, \mathcal{L})$. Our first goal is to obtain an integrable system $\Phi_{\gamma, c} : M_C(2, \mathcal{L}) \to \Delta_{\gamma, c}$ for every (colored) graph $\gamma$. We do this in Section 3.1. This crucially uses a degeneration of $M_C(2, \mathcal{L})$ into a toric variety constructed by Manon, as recalled in Section 2.2. Then we use symplectic parallel transport for the natural integrable system associated to a toric variety, and obtain the following

Proposition A (Proposition 3.9). For every trivalent graph $\gamma$ the Lagrangian torus

\[ L_\gamma := \Phi_{\gamma, c}^{-1}(0) \subset M_C(2, \mathcal{L}) \]

is monotone.

By Corollary 3.10 these tori are non-Hamiltonian isotopic to each other, for different choices of $\gamma$.

The next ingredient is motivated by a foundational result of Nishinou–Nohara–Ueda, [60, Theorem 1] and recalled in Theorem 4.2. It gives a recipe to compute the Floer potential of a smooth projective Fano variety using a toric degeneration. Our version gets rid of some of the technical conditions in the original result which could be hard to check in practice.
Theorem B (Theorem 4.4). Let $X_1$ be a smooth projective Fano variety which admits a degeneration into a Gorenstein Fano toric variety $X_0$. Assume that the degeneration preserves the second Betti number. Let $L_t(u_0)$ be a monotone Lagrangian torus obtained by symplectic parallel transport. Then the Newton polytope of $m_0(L)$ equal the polar dual $P$ of the moment polytope of the toric variety $X_0$.

In particular, if the polytope $P$ has non-zero lattice points other than the vertices i.e. $X_0$ has terminal singularities, then the Floer potential can be expressed as

$$m_0(L)(x) = \sum_{i=1}^m \exp(\langle u_i, x \rangle) T_{i}^{\ell_i(u_0)},$$

summing over the the generators of the polar dual of the moment polytope, where $\ell_i$ are the defining equations for the facets.

This is the key ingredient in establishing the relationship between $M_C(2, L)$ and graph potentials as mirror partners and we expect it can be used to study other instances.

Applying this to our setup we identify the Floer potential of the torus $L_\gamma$ with the graph potential, as in Theorem 4.8. In Remark 4.6 we will discuss a variation of the statement.

Classical periods and quantum periods. This identification allows us to interpret the constant term of the $d$th power of the graph potential as the number of maps from tropical trees to $\Delta_{g,c}$ in tropical descendant Gromov–Witten invariant enumeration [72], which in turn can be shown to agree with the descendant Gromov–Witten invariant of $M_C(2, L)$. In particular, this proves the relation between closed and open Gromov–Witten invariants found and proved in the context of toric degenerations independently by Bondal–Galkin [15] and Nishinou–Nohara–Ueda [61]. Recently Tonkonog proved a generalization of this relation for the case of an abstract monotone Lagrangian torus [79].

Such a correspondence between the classical period of a Laurent polynomial and the Gromov–Witten invariants of a Fano variety goes by the name of mirror partner [44, §3]. The main result we prove is the following

Theorem C. Let $(\gamma, c)$ be a trivalent colored graph of genus $g$ such that the total number of colored vertices is odd. Let $C$ be a smooth projective curve of genus $g$. Then the graph potential is a mirror partner of $M_C(2, L)$.

From this identification between the classical period and the regularized quantum period we can also deduce the regularity of regularized quantum periods, see Corollary 5.4.

Optimal monotone tori. Let $X$ be any Fano manifold and let $*_0$ denote the quantum multiplication in the small quantum cohomology ring of $X$. From Galkin–Golyshev–Iritani [29] consider

$$T_X := \max\{|u| : u \in \mathbb{C} \text{ is an eigenvalue of } c_1(X)*_0 \} \in \overline{\mathbb{C}}$$

The invariant $T_X$ is a symplectic invariant of the manifold $X$.

Let $L$ be a monotone Lagrangian torus in $X$. Consider the number $T_X^* := \limsup_{n} (|m_0(L)^n|)_{*0}^{\frac{1}{n}}$, where $|m_0(L)^n|_{*0}$ denotes the constant term of the $n$th power of the Laurent polynomial $m_0(L)(x)$. By definition, the weighted number $m_0(L)(1, \ldots, 1)$ of holomorphic Maslov index 2 discs with boundary on a monotone Lagrangian torus $L$ is bounded below by $T_X^*$. 
Observe that $m_0(L)(1, \ldots, 1)$ also is equal to the value $T_{\text{con}}(L)$ of the disk potential at the unique Morse point (also known as conifold point) of the Floer potential $m_0(L)$ in the domain $\mathbb{R}^{\dim L} \subset (\mathbb{C}^\times)^{\dim L}$. Since $T'_{\chi'}$ equals [29, 30] to the inverse of the radius of convergence of the regularized quantum period function of $X$, we get $T'_{\chi'} = T_{\chi}$. Hence we have an inequality

$$m_0(L)(x) = T_{\text{con}}(L) \geq T_{\chi}$$

and we say $L$ is optimal if this inequality is an equality.

Let $N_g$ be the symplectic manifold whose algebraic model is $M_{C^1}(2, \mathcal{L})$. This Fano symplectic manifold is also popularly known as the rank two character variety and it is equipped with the Atiyah–Bott–Goldman–Narasimhan symplectic form. Muñoz [57] has proved that $T_{N_g} = 8g - 8$. As a corollary to Theorem C combined with a Torelli-type theorem, we show that

**Corollary D.** *(Theorem 4.8 and Corollary 3.10)* For any trivalent colored $\gamma$ graph of genus $g$ with one colored vertex, the monotone Lagrangian torus $L_{\gamma}$ obtained as in Proposition A is optimal. Moreover for each pair of trivalent graphs $(\gamma, \gamma')$, the tori $L_{\gamma}$ and $L_{\gamma'}$ are not Hamiltonian isotopic.

**Remark 1.1.** Corollary D highlights a key feature of the graph potentials and hence of the toric degenerations and the monotone Lagrangian tori we obtained. We are not aware of any examples of Fano manifolds with multiple optimal monotone Lagrangian tori.

**On mirror symmetry for Fano varieties.** The mirror dual of $M_{C^1}(2, \mathcal{L})$ is expected to be a cluster-like variety equipped with a regular function, the so-called Landau–Ginzburg potential (closely related to the Floer potential), that will play the central rôle in this story. The graph potentials form a class of possible hierarchies of such functions adapted to the “collective” study of mirror symmetry for the moduli spaces of local systems on surfaces of all genera and all numbers of boundary components simultaneously, yet still keeping the rank fixed.

There are a couple of interdependent theoretical approaches to mirror symmetry for Fano varieties. These provide different methods, levels of rigour, constructivity, precision, predictive power, scope of interests and applications. But in what concerns experimental data, so far in any approach mirror symmetry was mostly studied either in low dimensions, or for (Schubert cycles in) homogeneous varieties, or for toric varieties, or for their products (and some fibrations), or for complete intersections therein, see e.g. [2, 52, 69, 70, 71].

On the other hand, moduli spaces of rank 2 bundles are Fano varieties which fall outside the usual scope of these methods, and which are notorious for their falsifications of naive conjectural extrapolations based on the data obtained from the examples above. For example, in order to apply techniques such as the abelian/non-abelian correspondence of [14, 17, 13, 47] to compute quantum periods (and Givental $J$-functions), the classification of Fano 3-folds was reworked in terms of GIT data in [19, Theorem A]. This theorem, which is proved in the first 105 sections of the paper, says that all fibers in a family of Fano threefolds (and del Pezzo surfaces) can be obtained as zero loci of regular sections of a fixed vector bundle on a fixed ”key variety”, which is isomorphic to a product of a toric variety with a Grassmannian. As a by-product this theorem implies that *moduli spaces of complex structures on such Fano manifolds are unirational.*
In contrast, the (last) Section 106 of op. cit. considers $M_{C^1_2}$: by the non-abelian Torelli theorem (see e.g. [48, Theorem E]) the moduli space of complex structures on $M_{C^1_2}$ has a dominant rational map to the Deligne–Mumford moduli space of stable curves. For $g \geq 23$ these have Kodaira dimension at least 0 by [39]. This indicates that the mirror map to the moduli space of complex structures on the Fano manifold $M_{C^1_2}$ from the complexified Kähler cone establishing mirror symmetry has to be a transcendental map, very different from all the known examples of mirror maps for Fano manifolds.

In [10] we will discuss aspects of homological mirror symmetry for $M_{C^1_2}$ and graph potentials, in order to understand certain (conjectural) decompositions of invariants of $M_{C^1_2}$ and graph potentials, in a motivic, symplectic and categorical setting.

Complements. The toric degeneration we have studied allows us to recover a result of Kiem–Li on the terminality of the singularities of $M_{C^1_2}$, at least for a generic curve $C$. We also state a conjecture on existence of a small resolution of the toric degenerations for $M_{C^1_2}$ and $M_{C^2_2}$. We refer the reader to Theorem A.3, Corollary A.8 and Conjecture A.10 for precise statements.

In Appendix B we will show how to compute classical periods of graph potentials via combinatorial means, or use algebro-geometric methods to compute some quantum periods. This shows how certain computations of periods can be performed on either side of the mirror.

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The frameworks that we use to establish mirror symmetry was set up in an unpublished joint work of Alexey Bondal and S.G. [15]. Our treatment of Maslov index computations and holomorphic discs uses constructions from joint works of S.G. and Grigory Mikhalkin, [32, 33, 31] slightly adapted for our more special situation.

This collaboration started in Bonn in January–March 2018 during the second author’s visit to the “Periods in Number Theory, Algebraic Geometry and Physics” Trimester Program of the Hausdorff Center for Mathematics (HIM) and the first and third author’s stay in the Max Planck Institute for Mathematics (MPIM), and the remaining work was done in the Tata Institute for Fundamental Research (TIFR) during the second author’s visit in December 2019–March 2020 and the first author’s visit in February 2020. We would like to thank HIM, MPIM and TIFR for the very pleasant working conditions. S.G. thanks Faculty of Mathematics of HSE University for providing a sabbatical leave in 2018 and Mathematics Department of PUC-Rio for summer leave in 2019-2020, which made possible the two visits above.

The initial progress on different parts of this work was reported at “Patchworking in Geometry and Topology” conference in Belalp (dedicated to the 70th anniversary of Oleg Viro) and “Geometry and Topology motivated by Physics” at the Monte Verità Conference center in Ascona (Ticino). We thank the participants for providing us with related references and context.

Several computations and experiments were performed using Pari/GP [66] and Sage [77] (in particular we used PALP [49] routines to analyze lattice polytopes).

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2. Toric degenerations of moduli spaces of vector bundles

Abstracting from his earlier work with Ciocan–Fontanine, Kim and van Straten [5, 4] on mirror symmetry for Grassmannians and partial flag varieties, Batyrev proposed in [3, 6] a passage from a "small" toric degeneration \( X \) of a Fano variety \( F \) to a Laurent polynomial \( W \), as a construction of a mirror dual to \( F \). See also [44] for more context.

The toric degeneration for \( M_{C}(2, \mathcal{L}) \) we are interested in is related to the graph potentials introduced in [11]. After recalling their construction and notation we will describe a class of toric degenerations of \( M_{C}(2, \mathcal{L}) \) introduced by Manon [54]. These two objects turn out to be closely related: the Newton polytope of the graph potential associated to a trivalent graph \( \gamma \) agrees with the moment polytope of Manon’s degeneration corresponding to the \( \gamma \).

2.1. Graph potentials. Let \( \gamma = (V, E) \) be an undirected trivalent graph of genus \( g \), possibly with loops. We have that \#\( V \) = 2g − 2 and \#\( E \) = 3g − 3. We will moreover assume it is connected.

We set

\[
\begin{align*}
\tilde{N}_{\gamma} &:= C^{1}(\gamma, \mathbb{Z}) \\
\tilde{M}_{\gamma} &:= C_{1}(\gamma, \mathbb{Z})
\end{align*}
\]

the free abelian groups of 1-(co)chains.

For every vertex \( v \in V \) we have the 3 edges \( e_{uv}, e_{vq}, e_{vm} \in E \) adjacent to it. These span a sublattice \( \tilde{N}_{v} \) of \( \tilde{N}_{\gamma} \), generated by the cochains \( x_{i}, x_{j}, x_{k} \) where we take \( x_{i}(e_{uv}) = \delta_{i,v} \). For every \( v \) we can then consider the sublattice \( N_{v} \) spanned by the cochains \( \{ \pm x_{i} \pm x_{j} \pm x_{k} \} \), and we set

\[
N_{\gamma} := \text{im}
\left( \bigoplus_{v \in V} N_{v} \to \tilde{N}_{\gamma} \right).
\]

and

\[
M_{\gamma} := N_{\gamma}^{\vee}.
\]

Finally we set

\[
A_{\gamma} := (\tilde{N}_{\gamma}/N_{\gamma})^{\vee} \cong M_{\gamma}/\tilde{M}_{\gamma},
\]

which by [11, Lemma 2.1] is isomorphic to \( H_{1}(\gamma, \mathbb{Z}) \cong \mathbb{Z}^{\oplus g} \). These are the relevant lattices, relative to which all constructions in toric geometry are to be considered.

Associated to the different lattices we have algebraic tori

\[
\begin{align*}
\tilde{T}_{\gamma}^{\vee} &:= \text{Spec } \mathbb{C}[N_{\gamma}] \\
\tilde{T}_{\gamma} &:= \text{Spec } \mathbb{C}[\tilde{N}_{\gamma}]
\end{align*}
\]

Hence \( N_{\gamma} \) and \( \tilde{N}_{\gamma} \) are the character lattice of \( T_{\gamma}^{\vee} \) and \( \tilde{T}_{\gamma}^{\vee} \), and the cocharacter lattices of \( T_{\gamma} \) and \( \tilde{T}_{\gamma} \). The group \( A_{\gamma} \) is the kernel of the isogeny \( \tilde{T}_{\gamma}^{\vee} \to T_{\gamma}^{\vee} \). 
The next ingredient is a coloring, which is a function \( c : V \rightarrow \mathbb{F}_2 \). In the graph (co)homology language used above this corresponds to a 0-chain on \( y \) with values in \( \mathbb{F}_2 \). If \( c(v) = 0 \) we will say that \( v \) is uncolored, and if \( c(v) = 1 \) we say that \( v \) is colored.

**Definition 2.1.** Let \( (y, c) \) be a colored trivalent graph, with an enumeration of the edges given by \( e_1, \ldots, e_{3g-3} \), and let \( x_i \) be the coordinate function in \( \mathbb{Z}[\tilde{N}_y] \) corresponding to \( e_i \).

- The **vertex potential** for a vertex \( v \in V \) adjacent to the edges \( e_i, e_j, e_k \) is defined as the Laurent polynomial

\[
\overline{W}_{v,c} := \sum_{(x_i, x_j, x_k) \in \mathbb{F}_2^3} x_i^{(-1)^{x_i}} x_j^{(-1)^{x_j}} x_k^{(-1)^{x_k}} \in \mathbb{Z}[\tilde{N}_e].
\]

- The **graph potential** of \( (y, c) \) is defined as the Laurent polynomial

\[
\overline{W}_y,c := \sum_{v \in V} \overline{W}_{v,c} \in \mathbb{Z}[\tilde{N}_y].
\]

By [11, Lemma 2.4] the regular function \( \overline{W}_{y,c} \) on the torus \( \overrightarrow{T}_y \) descends to a function \( W_{y,c} \) on the torus \( T_y \). We will study toric varieties with cocharacter lattice \( N_y \) whose moment polytope is the polar dual of the Newton polytope of \( W_{y,c} \). However the lattice \( N_y \) has no natural basis, hence we will not express \( W_{y,c} \) in terms of a basis of \( N_y \). We refer to [11, §2] for more details and examples.

By [11, Corollary 2.9] the graph potential depends only on the homology class of \( [c] \) in \( H_0(y, \mathbb{F}_2) \), up to biregular automorphism of the torus (hence under the connectedness assumption only on the parity).

**Elementary transformations of graph potentials.** The second invariance result corresponds to the relation between graph potentials for different choices of trivalent graphs (of fixed genus). As explained in [11, §2.2] these correspond to different pair of pants decompositions of a surface of genus \( g \), and these are related via Hatcher–Thurston moves. We are only interested in the operation from [11, Figure 2], and call this an **elementary transformation**. This is a procedure that transforms a colored trivalent graph \( (y, c) \) into a new colored trivalent graph \( (y', c) \).

We will now briefly recall the formulas for an elementary transformation of graph potentials, for more details the reader is referred to [11, §2.2]. Because a transformation is applied to an edge \( e \in E \) corresponding to a variable \( x \) we can always split the graph potential as

\[
\overline{W}_{y,c} = \overline{W}^{\text{mut}}_{y,c} + \overline{W}^{\text{frozen}}_{y,c}
\]

with the mutated part involving the variables attached to the vertices \( v_1 \) and \( v_2 \) incident to the edge \( e \). There are (with possible repetitions) five variables involved: the variables \( a, b, x \) for the vertex \( v_1 \) and the variables \( c, d, x \) for the vertex \( v_2 \), see also [11, Figure 3]. The frozen part of the graph potential is not changed, and can be ignored.
If the parity of the coloring of the vertices $v_1$ and $v_2$ is odd, then we have

$$\tilde{W}^{\text{mut}}_{\gamma,c} = xcd + \frac{x}{cd} + \frac{c}{dx} + \frac{d}{cx} + \frac{1}{abcd} + \frac{ax}{x} + \frac{bx}{b} + \frac{ax}{a}$$

(13)

$$= \frac{1}{x} \left( ab + \frac{c}{d} + \frac{d}{c} \right) + x \left( cd + \frac{1}{bd} + \frac{a}{c} + \frac{b}{a} \right).$$

If the parity of the coloring of the vertices $v_1$ and $v_2$ is even, then we have

$$\tilde{W}^{\text{mut}}_{\gamma,c} = x'db + \frac{x'}{bd} + \frac{b}{dx'} + \frac{d}{bx'} + \frac{1}{ac} + \frac{c}{x'} + \frac{x'}{ac}$$

(14)

$$= \frac{1}{x'} \left( ac + \frac{b}{d} + \frac{d}{b} \right) + x' \left( bd + \frac{a}{c} + \frac{c}{a} \right).$$

By [11, Theorems 2.12 and 2.13] the graph potentials $\tilde{W}^{\text{mut}}_{\gamma,c}$ and $\tilde{W}^{\text{mut}}_{\gamma',c'}$ are identified after a rational change of coordinates, which is moreover invariant under the actions of the finite groups $A_\gamma$ and $A_{\gamma'}$. We will come back to elementary transformations in the discussion surrounding the nature of mutations of Lagrangian tori, see Remark 3.12 and discussions preceeding it.

Recall that given any Laurent polynomial $W$, the $m$-th classical period $c_m$ is defined to be the coefficient of the constant term of $W^m$. The following theorem relates the period of the graph potential $(\gamma, c)$ and $(\gamma', c')$

**Theorem 2.2.** (Graph potential TQFT) Let $(\gamma, c)$ and $(\gamma', c')$ be colored trivalent graphs of genus $g$ with same parity of the coloring, then for any positive integer $m$, the classical periods of the associated graph potentials agree.

2.2. Manon’s degeneration. In [34] Manon studied the homogeneous coordinate rings of the moduli stack $M_{C,p_1,\ldots,p_n}(SL_2)$ of quasi-parabolic bundles of rank 2. The Picard group of this stack is $X(B)^n \times \mathbb{Z}$ where $X(B) = \mathbb{Z}_{\omega_1}$ is the character group of the Borel subgroup of $SL_2$. Manon studies the algebraic properties of the various graded algebras one obtains by choosing a line bundle on the stack.

For us it will suffice to take $n \leq 1$, and hence we are interested in the moduli stacks $M_{C}(SL_2)$ and $M_{C,p}(SL_2)$. Then we denote

$$R_C(b) := \bigoplus_{\ell \geq 0} H^\ell(M_C(SL_2), \mathcal{L}(b)^{\oplus \ell})$$

(15)

$$R_{C,p}(a, b) := \bigoplus_{\ell \geq 0} H^\ell(M_{C,p}(SL_2), \mathcal{L}(a\omega_1, b)^{\oplus \ell})$$

the graded algebras associated to the corresponding line bundles.
Using the theory of conformal blocks and the Rees algebra construction Manon defines a flat degeneration of these graded algebras to the boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ of the Deligne–Mumford compactification of the stack of pointed curves. This stack has a stratification by closed substacks, indexed by the weighted dual graphs of the stable curves. The most degenerate curves one gets (with only rational components) correspond to trivalent graphs of genus $g$ and $n$ half-edges with all weights being zero, corresponding to pair of pants decompositions we discussed earlier.

The algebras obtained in this way are semigroup algebras associated to polytopes, hence we get toric varieties. To understand the relationship between these algebras and the varieties $M_C(2, O_C)$ and $M_C(2, L)$, recall that by Drézet-Narasimhan [24, Théorème B] the Picard groups of these varieties are generated by the class of the Theta divisor. By Beauville and Laszlo [8, Theorem 8.5 and Theorem 9.4] (and more generally the works [68, 50, 56]) we have identifications

$$R_C(2) \cong \bigoplus_{\ell \geq 0} H^0(M_C(2, O_C), \Theta^{\otimes 2\ell})$$

(16)

$$R_{C,p}(2, 2) \cong \bigoplus_{\ell \geq 0} H^0(M_C(2, L), \Theta^{\otimes \ell}).$$

Hence we get a toric degeneration of the moduli spaces we are interested in, for every choice of trivalent graph $\gamma$ with at most one half-edge.

**Graphs from stable curves.** Let $(C; p_1, \ldots, p_n)$ be a stable nodal curve with $n$ marked points. The dual graph associated to $(C; p_1, \ldots, p_n)$ is the weighted graph $\gamma = (V, E, g)$ with $n$ half-edges defined as

- each vertex $v_i \in V$ corresponds to an irreducible component $C_i$ of $C$, with weight $g(v_i)$ the genus of $C_i$;
- the number of edges between $v_i$ and $v_j$ is $\#C_i \cap C_j$;
- for each marked point $p$ in the component $C_i$ we add a half-edge to the vertex $v_i$.

The arithmetic genus of the curve $C$ is then $\#E - \#V + 1 + \sum_{v \in V} g(v)$.

We will only consider nodal curves with marked points $(C; p_1, \ldots, p_n)$, for which the dual graph $\gamma$ (containing half-edges) is trivalent (and all weights are zero). These correspond to the zero-dimensional strata in the stratification of $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ in terms of the type of the dual graph. Equivalently, all components $C_i$ are required to be rational.

We can rephrase the definition of Manon’s polytope from [54, Definition 1.1] as follows.

**Definition 2.3.** Let $\gamma$ be a trivalent weighted graph of genus $g$ with at most one half-edge. The polytope $P_\gamma \subseteq \mathbb{R}^{\#E}$ is the set of non-negative real weightings of the edges of $\gamma$, such that

- the weights of half-edges are precisely 2;
- for every vertex $v \in V$, if $w_1(v), w_2(v), w_3(v)$ are the weights of the edges incident to it, then

$$2 \max\{w_1(v), w_2(v), w_3(v)\} \leq w_1(v) + w_2(v) + w_3(v) \leq 4.$$

(17)

We define the lattice $\mathcal{M}_\gamma$ using the lattice of integer points in $\mathbb{R}^{\#E}$ and the condition that $w_1(v) + w_2(v) + w_3(v) \in 2\mathbb{Z}$ for every $v \in V$. 

The following theorem then describes the sought-after toric degeneration, which is a special case of [54, Theorem 1.2] and [55, Theorem 1.3].

**Theorem 2.4** (Manon). Let $C$ be a smooth projective curve of genus $g \geq 2$. Let $γ$ be a trivalent graph of genus $g$, with at most one half-edge. Then

- if there are no half-edges in $γ$, then the homogeneous coordinate ring $R_{C}(2)$ for $M_{C}(2, \mathcal{O}_{C})$ can be flatly degenerated to the semigroup algebra associated to the polytope $P_{γ}$ in the lattice $\mathcal{M}_{γ}$;
- if there is one half-edge in $γ$, then the homogeneous coordinate ring $R_{C}(2, 2)$ for $M_{C}(2, \mathcal{L})$ can be flatly degenerated to the semigroup algebra associated to the polytope $P_{γ}$ in the lattice $\mathcal{M}_{γ}$.

**Definition 2.5.** The toric variety from Theorem 2.4 will be denoted $Y_{P_{γ}, M_{γ}}$. Observe that $P_{γ}$ is the moment polytope and $M_{γ}$ is the character lattice.

We will rewrite the inequality in Definition 2.3 so that the origin becomes an internal point of the polytope and the polytope becomes full-dimensional in the character lattice of the toric variety. Setting $u_{i}(v) := w_{i}(v) - 1$ for the weight of the $i$th edge at the vertex $v$ we obtain the following lemma.

**Lemma 2.6.** Let $γ$ be a trivalent graph of genus $g \geq 2$ with at most one half-edge. The polytope $P_{γ}$ is defined by the inequalities

$$
\begin{align*}
-u_{1}(v) - u_{2}(v) - u_{3}(v) & \geq -1 \\
-u_{1}(v) + u_{2}(v) + u_{3}(v) & \geq -1 \\
u_{1}(v) - u_{2}(v) + u_{3}(v) & \geq -1 \\
u_{1}(v) + u_{2}(v) - u_{3}(v) & \geq -1;
\end{align*}
$$

(18)

- if $v$ is a vertex with a (necessarily unique) half-edge, then $u_{1} = -u_{2}$ for the weights of the other two edges.

**Removing half-edges.** To make the link to the setup of colored trivalent graphs from Section 2.1 we will explain how to reduce trivalent graphs with one half-edge to colored trivalent graphs. A more general procedure can be constructed, but in this paper we are only interested in the case with one half-edge.

Let $γ = (V, E)$ be a trivalent graph of genus $g \geq 2$ with precisely one half-edge. Let $v \in V$ be the vertex to which the half-edge $e$ is attached. Let $e_{1}$ and $e_{2}$ be the other edges in $v$, which are incident to the vertices $v_{1}$ and $v_{2}$. Then we construct a colored trivalent graph $(γ′, c)$ as follows:

- the vertices $V′$ are $V \setminus \{v\}$;
- the edges $E′$ are $(E \cup \{e_{1,2}\}) \setminus \{e, e_{1}, e_{2}\}$, where $e_{1,2}$ is an edge between $v_{1}$ and $v_{2}$
- we color precisely one vertex from $\{v_{1}, v_{2}\}$.

Graphically we have the situation from Figure 1.

With this procedure we define a new polytope as follows.
We can also describe the polar dual

\[ P \subseteq \mathbb{R}^{gE} \]

of the polytope \( P \) obtained by removing the half-edge, where \( v \in V' \) is the colored vertex. The polytope \( P' \subseteq \mathbb{R}^{gE} \) is defined by the same inequalities for all \( v \in V' \setminus \{ v \} \), and in \( v \) we consider the inequalities

\[
\begin{align*}
&u_1(v_1) + u_2(v_1) - u_3(v_1) \geq -1 \\
&u_1(v_1) - u_2(v_1) - u_3(v_1) \geq -1 \\
&-u_1(v_1) + u_2(v_1) - u_3(v_1) \geq -1 \\
&-u_1(v_1) - u_2(v_1) + u_3(v_1) \geq -1
\end{align*}
\]

Hence this polytope is defined in the subspace of \( \mathbb{R}^{gE} \) given by the equality \( u_1(v_1) = u_2(v_2) \) where \( u_1 \) and \( u_2 \) refer to the edges \( e_1 \) and \( e_2 \) (recall that half-edges have a constant value assigned to them).

**The polytope** \( P' \). Let \( (\gamma, c) \) be a colored trivalent graph with no half-edges. We use \( M' \) and \( N' \) as in Section 2.1.

Consider the half-spaces \( H^{c_1}_i \) (for \( i = 1, 2, 3, 4 \)) defined by the inequalities

\[
\begin{align*}
(-1)^{c_1} (-u_1 - u_2 - u_3) &\geq -1, \\
(-1)^{c_1} (-u_1 + u_2 + u_3) &\geq -1, \\
(-1)^{c_1} (u_1 - u_2 + u_3) &\geq -1, \\
(-1)^{c_1} (u_1 + u_2 - u_3) &\geq -1,
\end{align*}
\]

where \( u_1, u_2, u_3 \) correspond to the three edges incident at the vertex \( v \). Observe that the inequalities (20) specialize to the inequalities (18) (resp. (19)) if \( c(v) = 0 \) (resp. \( c(v) = 1 \)).

**Definition 2.8.** Let \( (\gamma, c) \) be a colored trivalent graph with no half-edges. Let \( V \) be the set of vertices and \( E \) be the set of edges of \( \gamma \). Consider the polytope \( P'_{\gamma, c} \) defined by the intersection of half-spaces in \( \tilde{M}' \)

\[
P'_{\gamma, c} = \bigcap_{v \in V} \left( H_1^{c(v)} \cap H_2^{c(v)} \cap H_3^{c(v)} \cap H_4^{c(v)} \right).
\]

We can also describe the polar dual \( P''_{\gamma, c} \) in \( \tilde{N}' \) as follows. For \( v \in V \) a vertex with coloring \( c(v) \), and \( e_1, e_2, e_3, e_4 \) the edges adjacent to \( v \) we can consider the subgraph \( \gamma_v \), and define the vector space

\[
C_{\gamma, c} := \{ s \in C_1(\gamma_v, E_2) \mid d(s) = c(v) \}.
\]

For \( v \in V \) and \( s \in C_{\gamma, c} \) we can then define the point

\[
p(v, s) := (0, \ldots, 0, (-1)^{k_v}, 0, \ldots, 0, (-1)^{k_v}, 0, \ldots, 0, (-1)^{k_v}, 0, \ldots, 0) \in N' \subset \mathbb{R}^{gE}
\]

with \(+1\) in positions \( i, j, k \).
The following standard lemma then makes the description as a convex hull of the polytope we are interested in explicit.

**Lemma 2.9.** Let \((γ, c)\) be a colored trivalent graph of genus \(g \geq 2\) with no half-edges. The polar dual \(P^θ_{γ,c}\) of the polytope \(P_{γ,c}\) is the convex hull of the points \(p(v,s)\), where \(v \in V\) and \(s \in C_{γ,c}\).

This allows us to make the next definition.

**Definition 2.10.** Let \((γ, c)\) be a colored trivalent graph of genus \(g\) with no half-edges. The graph potential toric varieties \(X_{P^θ_{γ,c},M_γ}\) (respectively \(X_{P^θ_{γ,c},M_γ}^e\)) are defined as the toric varieties with moment polytope \(P_{γ,c}\) in the character lattice \(M_γ\) (respectively \(M_γ^e\)).

We will often refer to the polar dual \(P^θ_{γ,c}\) as the fan polytope of the toric variety \(X_{P^θ_{γ,c},M_γ}\).

We can now compare Manon’s toric degeneration to the toric varieties obtained from the graph potentials, and show that they are the same.

**Proposition 2.11.** Let \(γ, c\) be a trivalent graph of genus \(g\), with at most one half-edge. Let \((γ', c)\) be the associated colored trivalent graph obtained from removing half-edges. There exists an isomorphism

\[ Y_{P^θ_{γ,c},M_γ} \cong X_{P^θ_{γ',c},M_γ}^e. \]

**Proof.** By [20, Proposition 2.3.9], for any \(ω\) in the character lattice of a toric variety, the moment polytope \(P\) and the polytope \(ω + P\) have the same normal fan, and in turn give isomorphic toric varieties.

Faust–Manon [25, §3] translates the moment polytope of \(Y_{P^θ_{γ,c},M_γ}\) by \(ω = (2, \ldots, 2) \in M_γ\) to get the origin as the unique interior point. Observe that we have the identification of the lattices \(M_γ = 2M_γ^e\). Hence, we have to scale down by a factor of 2. Starting from Manon’s equations for the polytope \(P^θ_{γ,c}\), we translate it by the vector \((1, 1, \ldots, 1)\) in \(M_γ\) to get the equations of the reflexive polytope \(P^θ_{γ',c}\). The result follows.

In particular have we found another realization of Manon’s toric degeneration using an explicit full-dimensional reflexive polytope.

As an application of the description of the toric degeneration in terms of the combinatorics of trivalent graphs we will in Appendix A discuss when these toric degenerations have terminal singularities or admit a small resolution. These properties will depend on the choice of trivalent graph used for the construction of the degeneration. Terminality will be crucially used in Section 4.

### 3. Monotone Lagrangian tori from graph potentials

We will now consider \(M_C(2, L)\) and the toric degenerations considered in Section 2.2 as symplectic manifolds, and use symplectic transport to construct monotone Lagrangian tori in \(M_C(2, L)\) in Proposition 3.9. Every choice of trivalent graph leads to a toric degeneration, and as explained in Corollary 3.10 we obtain as many monotone Lagrangian tori which are not Hamiltonian-isotopic to each other as we have trivalent graphs of genus \(g\). The case \(M_C(2, O_C)\) is similar.
3.1. **Reminder on integrable systems.** Let us recall the notion of an integrable system on a symplectic manifold \((M, \omega)\) of real dimension \(2n\).

**Definition 3.1.** An **integrable system** on \((M, \omega)\) is a collection of \(n\) functionally independent real-valued \(C^\infty\) functions \(\{H_1, \ldots, H_n\}\) on the manifold \(M\) which are pairwise Poisson-commutative, i.e. \(\{H_i, H_j\} = 0\) for all \(i, j = 1, \ldots, n\), where \(\{-, -\}\) is the Poisson bracket induced by the symplectic form \(\omega\). We will denote it \(\Phi : M \to \mathbb{R}^n\).

By definition, an integrable system induces a Hamiltonian \(\mathbb{R}^n\)-action on \(M\). By the Arnold–Liouville theorem we have that any regular, compact, connected orbit of this action gives a Lagrangian torus in \(M\).

We will work with integrable systems on singular toric varieties. In that set-up we want the real-valued \(C^1\) functions to Poisson-commute on the smooth locus.

Natural examples of integrable systems are found in the theory of toric varieties. Here we let \(X_0\) be a toric variety of complex dimension \(n\), and consider the torus action on \(X_0\). The moment map for the torus action with respect to any torus-invariant Kähler form gives an integrable system.

**Integrable systems from toric degenerations.** Nishinou–Nohara–Ueda and Harada–Kaveh \([62, 38]\) construct integrable systems from a toric degeneration of a smooth projective variety satisfying some additional conditions. We briefly recall their construction.

Let \(B\) be a manifold containing two points 0 and 1 and let \(\pi : \mathcal{X} \to B\) be a flat family of complex projective varieties of dimension \(n\), such that \(X_0 := \pi^{-1}(0)\) is a toric variety, whilst the general fiber \(X_t := \pi^{-1}(t)\) for \(t \neq 0\) is smooth. Assume moreover that

- the singular locus of the total space \(\mathcal{X}\) is contained in the singular locus of \(X_0\);
- the regular part of \(\mathcal{X}\) has a Kähler form which restrict to a torus-invariant Kähler form on the regular part \(X_0^{\text{reg}}\) of \(X_0\).

The choice of a piecewise smooth curve \(I : [0, 1] \to B\) from 0 to 1 in \(B\) gives by symplectic parallel transport along \(I\) a symplectomorphism \(\tilde{I} : X_0^{\text{reg}} \to X_1^{\text{reg}}\), where \(X_1^{\text{reg}}\) is an open subset of \(X_1\).

The singular toric variety has a natural map \(\Phi_0 : X_0 \to \mathbb{R}^n\) which makes it an integrable system. It is well-known that the image of \(\Phi_0\) is the moment polytope \(P\) associated to the toric variety \(X_0\).

Using the symplectomorphism \(\tilde{I}\) we can transport the integrable system \(\Phi_0 : X_0^{\text{reg}} \to \mathbb{R}^n\) to the integrable system

\[
(25) \quad \Phi_1 := \Phi_0 \circ \tilde{I}^{-1} : X_1^{\text{reg}} \to \mathbb{R}^n
\]

on \(X_1\). Observe that the same flow gives a \(C^\infty\) map from the compact manifold \(X_1\) to the compact space \(X_0\), however over the preimage of the singularities, one cannot extend the torus action. A suitable choice of the form \(\omega\) and path \(I\) makes the map real-analytic, cf. [51]. Hence \(\Phi_1\) extends as a map between \(X_1 \to \mathbb{R}^n\) but the functions only Poisson-commute on an open dense subset of \(X_1\).

Let \(P\) denote the moment polytope of the toric variety \(X_0\). Let

\[
(26) \quad \ell_i(u) := \langle v_i, u \rangle - \tau_i
\]
for \( i \in \{1, \ldots, m\} \) denote the affine equations defining the polytope, i.e.
\begin{equation}
P := \Phi_0(X_0) = \{ u \in \mathbb{R}^n \mid \forall i = 1, \ldots, m: \ell_i(u) \geq 0 \}.
\end{equation}
The convex hull of the \(-u_i/\ell_i\)'s as in (26) is the polar dual \( P^\circ \) of the moment polytope \( P \). The polytope \( P^\circ \) will be referred to be as the fan polytope.

Further observe that we have a natural inclusion \( \iota : X_0 \to \mathcal{X} \) and the map
\begin{equation}
c : \mathcal{X} \to X_0
\end{equation}
given by inverse of the symplectic transport on each fiber \( X_t \). For each \( t \in I \), we can consider maps
\begin{equation}
c_t := c \circ \iota_t : X_t \to X_0,
\end{equation}
where \( \iota_t : X_t \to \mathcal{X} \) is the natural inclusion. In particular \( \Phi_1 = \Phi_0 \circ c_t \).

Let \( u \in P \) be an interior point of \( P \) and let \( L_0(u) \) be a Lagrangian torus in \( X_0 \). By symplectic parallel transport via the map \( c_t \), we get a Lagrangian torus
\begin{equation}
L_t(u) := (\Phi_0 \circ c_t)^{-1}(L_0(u)) \subset X_t^{\reg}.
\end{equation}
These tori are all isomorphic for a fixed \( u \in P \) and for any \( t \in I \).

In Section 2 we have discussed a toric degeneration of \( M_C(2, \mathcal{L}) \), and we have now recalled the construction of an integrable system. We will use the following notation when we apply this construction with \( X_1 \) as \( M_C(2, \mathcal{L}) \).

**Definition 3.2.** Let \((\gamma, c)\) be a trivalent graph with one colored vertex. Then
\begin{equation}
\Phi_{\gamma, c} : M_C(2, \mathcal{L}) \to \mathbb{R}^{3g-3}
\end{equation}
is the map obtained by applying the Nishinou–Nohara–Ueda construction to Manon’s toric degeneration. \( \Phi_{\gamma, c} \) gives an integrable system when restricted to \( M_C(2, \mathcal{L})^{\reg} \). Its image will be denoted \( \Delta_{\gamma, c} \).

### 3.2. Construction of monotone Lagrangian tori in \( M_C(2, \mathcal{L}) \)

Now to every trivalent graph \( \gamma \) of genus \( g \) without leaves we will associate a monotone Lagrangian torus \( L_\gamma \) on the moduli space \( M_C(2, \mathcal{L}) \). In Corollary 3.10 we will comment on how the monotone Lagrangian tori for different choices of \( \gamma \) can be compared to each other.

First we recall the definition of Maslov index. Let \( L \) be a Lagrangian submanifold in a symplectic manifold \((M, \omega)\), and let \( D \) be a two-dimensional disk with boundary a circle \( \partial D \cong \mathbb{S}^1 \). For a continuous map \( f : (D, \partial D) \to (M, L) \) its restriction \( \partial f : \partial D \to L \) is a loop on \( L \). Denote the class of \( f \) in the relative homotopy or homology group as \([D, \partial D] = [f] \in \pi_2(M, L) \to \mathbb{H}_2(L, \mathbb{Z})\) and the boundary class as
\begin{equation}
[\partial D] = [\partial f] = \partial [f] \in \pi_1(L) \cong \mathbb{H}_1(L, \mathbb{Z}).
\end{equation}
The bundle \( \text{LGr}(TM) \) of Lagrangian Grassmannians \( \text{LGr}(T_mM) \cong \text{LGr}(\mathbb{R}^{\text{dim}_M}) =: \text{LGr} \) trivializes when pulled back to the contractible space \( D \).

**Definition 3.3.** The trivialization of the pullback of \( \text{LGr}(TM) \) to \( D \) via \( f \) gives the tangential section \( p \to T_{f(p)}L \in \text{LGr}(TM) \) of \((\partial f)^*\text{LGr}(TM) \cong \partial D \times \text{LGr} \) a class in \( \pi_1(L) \cong \mathbb{H}_1(L, \mathbb{Z}) \) known as the Maslov index\(^1 \) \( \mu(D) \). We will further call such maps disks of Maslov index \( \mu \).

\(^1\)This notation hides the mapping \( f \) from a pair \((D, \partial D)\) to \((M, L)\).
For an orientable $L$ the lifting of $L$ to the orientable Lagrangian Grassmannian $LGr^*$ guarantees that the Maslov index takes even values, i.e. $\mu(D) \in \pi_1(LGr^*) = 2\mathbb{Z} \subset \mathbb{Z} = \pi_1(LGr)$.

Recall the following definition.

**Definition 3.4.** Let $L$ be a Lagrangian submanifold of a symplectic manifold $(M, \omega)$. We say that $L$ is (positive) monotone if the symplectic area $H_2(M, L; \mathbb{Z}); \mathbb{R}$ and the Maslov index $H_2(M, L; \mathbb{Z}) \to \mathbb{Z}$ are (positively) proportional to each other.

We now discuss a general result which is implicit in the works of Nishinou–Nohara–Ueda and Ritter \[62, 60, 74\].

**Lemma 3.5.** Let $X_1$ be a simply connected smooth variety of dimension $n$ admitting a degeneration to a normal toric variety $X_0$. For any interior point $u \in P$, let $L = L_1(u)$ be a Lagrangian torus in $X_1$ constructed via symplectic parallel transport. Identify $H_1(L; \mathbb{Z})$ with the cocharacter lattice $N$ of the toric variety $X_0$. Assume that the vertices of $P^\circ$ generate $N$. Then there is a isomorphism

\[(33) \quad H_2(X_1, L; \mathbb{Z}) \cong N \oplus H_2(X_1, \mathbb{Z}).\]

**Proof.** Consider the long exact sequence for the homology of the pair $(X_1, L)$,

\[(34) \quad H_2(L, \mathbb{Z}) \to H_2(X_1, \mathbb{Z}) \to H_2(X_1, L; \mathbb{Z}) \to H_1(L, \mathbb{Z}) \to H_1(X_1, \mathbb{Z}).\]

The last term vanishes, as $X_1$ is simply connected. Moreover by Lemma 3.6, the image of $H_2(L, \mathbb{Z})$ in $H_2(X_1, \mathbb{Z})$ is zero. Hence we obtain a short exact sequence from (34), which allows us to identify

\[(35) \quad H_2(X_1, L; \mathbb{Z}) \cong H_1(L; \mathbb{Z}) \oplus H_2(X_1; \mathbb{Z})
\]

\[\cong N \oplus H_2(X_1; \mathbb{Z}).\]

\[\square\]

The following lemma is used in the proof of Lemma 3.5, however it does not require the assumption that $X_1$ is projective.

**Lemma 3.6.** In the situation of Lemma 3.5 the map $H_2(L, \mathbb{Z}) \to H_2(X_1, \mathbb{Z})$ is zero.

**Proof.** Let $P^{reg}$ be the image of the moment map from the smooth part $X_0^{reg}$ of the toric variety $X_0$. Let $X_0^{1-\text{cosk}}$ be the toric variety whose fan is the 1-skeleton of the fan of $X_0$ and let $P^{1-\text{cosk}}$ be the image of $X_0^{1-\text{cosk}}$ under the moment map. In particular $P^{1-\text{cosk}}$ is a union of open faces of codimension at most one. Let $P^{\text{int}}$ be the interior of $P$. 
Let $X^\text{int}_0$ denote the inverse image of $P^\text{int}$. Similarly define $X^\text{int}_1$. Both $X^\text{int}_1$ and $X^\text{int}_0$ contain a torus isomorphic to $L$. Consider the following diagram.

![Diagram](image)

Since the tori $L_1(u)$ and $L_0(u)$ are isomorphic by construction, we denote both of them by $L$ in this section.

By construction $P^\text{reg} = P \setminus P^\text{singular}$, where $P^\text{singular}$ is the singular locus of $P$. To show that the image of $H_2(L, \mathbb{Z})$ in $H_2(X_1, \mathbb{Z})$ is zero it would suffice to show that the image of $H_2(L, \mathbb{Z})$ is zero in $H_2(X^\text{reg}_0, \mathbb{Z})$ while going up along the middle column of the diagram Eq. (36).

Now let $V = \{v_1, \ldots, v_m\}$ be the vertices of $P^\circ$. Each $v_i$ corresponds to a unit sphere $S^1$ in $L$. The toric variety $T_V$ whose fan is $\bigcup_{v_i \in V} \mathbb{R}_{\geq 0} v_i$ is homotopy equivalent to $L \cup \bigcup_{v_i \in V} D_{v_i}$, where $D_{v_i}$ is a disk whose boundary is the one-cycle corresponding to $v_i$. By our notation $T_V$ is nothing but $X^\text{1-\cosk}_0$. We have the following diagram:

![Diagram](image)

Hence $H_1(T_V, \mathbb{Z}) = L / \bigoplus_{v_i \in V} \mathbb{Z} v_i$. But by assumption $V$ generates $L$. Hence $H_1(T_V, \mathbb{Z})$ is zero. Moreover the natural embedding of

\[(38)\quad S^1 \times S^1 \times P \hookrightarrow L \hookrightarrow D_{v_i} \times (\mathbb{C}^\times)^{n-1}\]

induces the zero map between $H_2(S^1 \times S^1, \mathbb{Z}) \rightarrow H_2(D_{v_i} \times (\mathbb{C}^\times)^{n-1}, \mathbb{Z})$. Thus we are done. \hfill \Box

**Remark 3.7.** The toric variety $T_V$ and its homotopy retract to the union of the Lagrangian torus $L$ with attached holomorphic disks $D_{v_i}$ is borrowed from Galkin–Mikhalkin [32], where it is used to compute the (non-abelian) relative second homotopy group $\pi_2(T_V, L)$ responsible for a “quantization” of the Floer potential of surfaces.

The following proposition gives us a useful criterion to test whether the Lagrangian torus $L_1(u)$ in Lemma 3.5 is monotone.

**Proposition 3.8.** Let $u \in P$ be an interior point as in Lemma 3.5 such that that $\ell_i(u) = \ell > 0$, where $\ell_i$ are the length functions (26) defining the polytope $P$. Further assume that $(X_1, \omega)$
is monotone and the degeneration preserves second Betti numbers. Then the Lagrangian torus \( L_1(u) \) is monotone.

Proof. Consider the (holomorphic) disks \( D_i \) of Maslov index 2 (Eq. (47) and Remark 4.6) enumerated by the vertices \( v_i \) of the polytope \( P^p \). By [62, Theorem 10.1] or [16, Theorem 8.1] they have symplectic area

\[
\int_{D_i} \omega = \ell_i(u) = \ell > 0.
\]

Non-negative linear combinations of the classes of their boundaries \( [\partial D_i] \in H_1(L, \mathbb{Z}) = N \) generate a finite index sublattice. Hence any element \( \beta = \beta_1 + \sum b_i [D_i] \), a linear combination of \([D_i]\) and some \( \beta_1 \in H_2(X, \mathbb{Q}) \). Moreover there exists \( a_i \geq 0 \) such that \( \sum a_i > 0 \) such that \( \sum a_i [D_i] \in H_2(X, \mathbb{Z}) \).

Then we have that

\[
\int_{\beta} \omega = \left( \sum b_i \right) \ell + \int_{\beta_1} \omega,
\]

and

\[
\int_{\sum a_i [D_i]} \omega = \left( \sum a_i \right) \ell > 0
\]

where \( \mu(\sum a_i [D_i]) = 2 \sum a_i > 0 \). Monotonicity of \((X, \omega)\) implies that

\[
\int_{\beta_1} \omega = \mu(\beta_1) \int_{\sum a_i [D_i]} \omega = \mu(\beta_1)/2 \cdot \ell,
\]

hence \( \int_{\beta} \omega = (\ell/2) \mu(\beta) \).

Now we apply Proposition 3.8 to produce a Lagrangian torus in \( M_\mathcal{L}(2, \mathcal{L}) \), using Manon’s toric degeneration and the integrable system described in Definition 3.2.

**Proposition 3.9.** The Lagrangian torus \( L := \Phi^{-1}_Y(0) \subset M_\mathcal{L}(2, \mathcal{L}) \) is monotone.

Proof. For each vertex \( v \) of the graph \((\gamma, e)\) and \( s \in C_{\gamma, e} \), consider the equations \( \ell_{a, s} \) from (19) used in Section 2.2 to describe the polytope \( P_{\gamma, e} \) and the polar dual \( P_{\gamma, e}^o \). Rewrite \( \ell_{a, s} \) as in (26)

\[
\langle p(v, s), u \rangle - \tau_{a, s}
\]

where \( p(v, s) \) is as in (23) and \( \tau_{a, s} = -1 \). Setting \( u = 0 \) we obtain that \( \ell_{a, s}(0) = 1 \). Observe that the lattice points \( p(a, s) \) of the polytope \( P_{\gamma, e}^o \) are non-zero, that the vertices form a subset by Theorem A.3, and furthermore span the lattice \( N_\gamma \). The statement then follows directly from Proposition 3.8.

There is a combinatorial non-abelian Torelli theorem for these toric degenerations, see [9]. This allows one to reconstruct the trivalent graph from the degeneration, and thus we can obtain the following corollary.

**Corollary 3.10.** Let \( \gamma \) and \( \gamma' \) be different trivalent graphs of genus \( g \). Let \( L_\gamma \) and \( L_{\gamma'} \) be the monotone Lagrangian tori in \( M_\mathcal{L}(2, \mathcal{L}) \) constructed above. Then they are not Hamiltonian isotopic to each other.
Remark 3.11. In the case of $M_C(2, O_C)$ symplectic transport gives a symplectomorphism from the non-singular locus of the toric fiber to an open subset in non-singular locus of $M_C(2, O_C)$. The corresponding Nishinou–Nohara–Ueda integrable system seems to be quite different from Goldman–Jeffrey–Weitsman integrable system from [42], despite sharing the base $P_Y$. In the former case the preimage of the interior of the polytope is the cotangent bundle of the torus and in the latter it is singular.

On the nature of mutations. We now briefly speculate on the relationship between the different Lagrangian tori $L_\tau$, constructed for different choices of trivalent graphs, i.e. what role of elementary transformations from [11, §2.2] play. In the context of Batyrev’s small toric degeneration approach to Fano mirror symmetry cluster mutations in the sense of Berenstein–Fomin–Zelevinsky [12] and Fock–Goncharov [26] were rediscovered independently

- for spherical varieties using representation theory, by Rusinko [76],
- for del Pezzo surfaces and Fano threefolds using deformation theory, by Galkin [28].

It was observed that the Laurent polynomials associated by a mirror construction to different toric degenerations of a Fano manifold are related to each other by volume-preserving birational changes of coordinates.

An algebraic theory of mutations of mirror potentials was constructed in joint works of Galkin with Usnich and Cruz Morales [21, 34], and the respective excessive Laurent phenomenon was proved in this context:

- in [34] from a more algebro-geometric point of view (similar to that of Gross–Hacking–Keel–(Kontsevich) [36, 37]),
- and then in [21] using the relation to cluster algebras and their upper bounds.

In this way a framework free from (commutative) algebraic geometry was constructed, with a view towards quantization. Amongst others, the notions of exchange collections, upper bounds, and mutation rules for them are introduced, and the Laurent phenomenon was proved algebraically and geometrically.

Ten equivalence classes of exchange collections in dimension 2 were constructed in [34, 21]. It was conjectured that each respective upper bound written in all possible seeds is generated by a single Laurent polynomial with positive integer coefficients, which is equal to a Floer potential for a monotone Lagrangian torus in an integrable system associated by the Nishinou–Nohara–Ueda construction to the respective toric degeneration of a smooth del Pezzo surface. Moreover it was conjectured that these birational transformations coincide with Lagrangian wall-crossing formulae, and that Floer potentials for all embedded monotone Lagrangian tori are equal to one of the constructed functions.

Most of these were settled positively by Vianna, Galkin–Mikhalkin, Pascaleff–Tonkonog and Ekholm–Rizell–Tonkonog [80, 31, 67, 75, 33]. However in higher dimensions (apart from the holomorphic symplectic case) our knowledge is much less complete. Already at the most basic level: for mirror symmetry of del Pezzo surfaces Cruz Morles, Galkin and Usnich give a recursive procedure to construct infinitely many different Laurent polynomials from just one was constructed, but we do not know any general higher-dimensional recursive
rule of generalized “mutations” that would enjoy a Laurent phenomenon and the same
time produce enough potentials.

Pascaleff–Tonkonog [67] introduced the idea that the birational mutation of potentials
of Cruz Moralez–Galkin [21] (which itself is a geometric lift in the sense of Berenstein–
Zelevinsky of a piecewise-linear homeomorphism of polytopes) can be lifted further to
a Lagrangian mutation, i.e. a symplectic surgery between the respective Lagrangian tori
with the help of auxiliary Lagrangian disc, such that class of its boundary belongs to the
exchange collection.

Dimitroglou–Eckholm–Tonkonog [75] provide further insight into mutations of two-
dimensional Lagrangian tori by interpreting it as two deformations of an immersed
Lagrangian sphere. For the latter they propose one unifying Floer-like potential with
values in a (non-commutative) multiplicative preprojective algebra, out of which the Floer
potentials for Lagrangian tori are obtained by specialization. Finally, Hong–Kim–Lau
[40] also study study Floer potentials for immersed Lagrangian subvarieties, which are
isomorphic to a product of an embedded torus with an immersed two-sphere, from the
perspective of Abouzaid–Auroux–Katzarkov [1], based on Strominger–Yau–Zaslow [78].

This leads us to the following observation.

Remark 3.12. One naturally expects that the rational changes of coordinates corre-
sponding to elementary transformations of graph potentials are similarly associated with a (new
kind of) Lagrangian mutations between the monotone Lagrangian tori \( L_\gamma \) in \( \mathcal{M}_C(2, \mathcal{L}) \).

One new and peculiar feature is that elementary transformations of graph potentials
naturally equate a sum of two monomials to a sum of two monomials as in [11, Equation
(43)]. This is in contrast to (various generalizations of) cluster algebras where a single
mutation involves an equality between a monomial and a sum of two monomials. A
limiting procedure, discussed in [11, Remark 4.19] specializes these new transformations
to the transformations of Nohara–Ueda [65, 63, 64] between Floer potentials of Euclidean
polygon spaces. In [64] they show how the specialized transformations produce a covering
for a total space of Rietsch’s Lie-theoretic mirror potential from [73] for the Grassmannian
of planes, which generally is an open subspace in a Langlands dual homogeneous variety. In
fact, Nohara–Ueda [64] show that formulae for mutations of their potentials are equivalent
to Plücker relations of the Grassmannian, so in some sense [11, Equation (43)] is related to
Plücker relations as the group \( SU(2) \) is related to Lie algebra \( su(2) \). It would be interesting
to understand this behavior better.

4. Disks of Maslov index two and disk potentials

We will now introduce the disk potential, for a Lagrangian torus inside a symplectic
manifold \((X, \omega)\). The next ingredient is the choice of an almost complex structure \(J\). We
say that an almost complex structure \(J\) is \(\omega\)-tame if \(\omega(v, Jv) > 0\) for all non-zero tangent
vectors \(v\). For a fixed \(\omega\) the space of \(\omega\)-tame almost complex structures is contractible,
hence the Chern classes of \(J\) are symplectic invariants. In this situation the Maslov index of
a \(J\)-holomorphic map \(f\) equals twice its first Chern number. In particular a \(J\)-holomorphic
disk with boundary on \(L\) that intersects an effective anti-canonical divisor transversally in
a unique point has Maslov index 2. More generally, one can show that the Maslov index
controls the dimension of the (virtual) fundamental cycle on the (Kuranishi) moduli space
of \( \mathcal{J} \)-holomorphic maps

\[
(44) \quad f : (C \ni P_1, \ldots, P_a; \partial C \ni Q_1, \ldots, Q_b) \to (M, L)
\]

from a Riemann surface \( C \) with \( a \) marked points in its interior and \( b \) marked points on the boundary \( \partial C \).

**Definition 4.1.** For a fixed the disk/Floer potential of \( L \), denoted \( m_0(L) \) (or \( \Psi \mathcal{O} \) or \( W_L \)) is the generating function for the \( \mathcal{J} \)-holomorphic disks \( \sum \cdots \) of Maslov index 2, whose boundary passes through a specified point \( P \) on \( L \) (one can also fix an interior point and ask it to lie on a fixed effective anti-canonical divisor).

In general, it is defined on the moduli space of weakly bounded cochains as

\[
(45) \quad m_0(L)(x) := \sum_{(D \ni \partial D \ni P) \subset (M \ni L \ni P)} \exp(\langle [\partial D], x \rangle) T^{\mu} \omega
\]

where

- \( \int \omega : H_2(M, L; \mathbb{Z}) \to \mathbb{R} \) is the symplectic area;
- \( \mu : H_2(M, L; \mathbb{Z}) \to \mathbb{Z} \) is the Maslov index;
- \( x \in H^1(L, \Lambda_0) \) is a cohomology class valued in Novikov ring \( \Lambda_0 \);
- \( \Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \right\} | a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lim_{i \to +\infty} \lambda_i = +\infty \} \) and \( \Lambda_0 \) is its subring of elements with non-negative valuation

\[
\nu \left( \sum a_i T^{\lambda_i} \right) := \min \{ \lambda_i | a_i \neq 0 \} \in \mathbb{R}
\]

The Novikov ring is used to deal with symplectic areas \( T^{\mu} \omega \) when the class \([\omega] \in H^2(M, \mathbb{R})\) of the symplectic form is not an integer, or when the surface \( \Sigma \) is not closed. However, under assumptions of integrality and monotonicity, the generating functions can be rewritten as Taylor series. For similar results on monotone toric manifolds and the passage between critical points in Archimedean and Novikov contexts, see the work of Ritter [74, Section 7.9] and Judd–Rietsch [43].

For generic \( \omega \)-tame \( \mathcal{J} \) the potential gives a well-defined element in the Novikov ring thanks to the Gromov compactness theorem.

We now discuss several methods to compute the disk potential for the moduli space \( M_C(2, \mathcal{L}) \) with respect to a monotone Lagrangian toric \( L \) obtained via symplectic parallel transport as in Definition 3.2. We discuss the general set-up of toric degenerations and analyse the various special cases starting with the existence of small resolutions that connect to Conjecture A.10 and then the case of terminal singularities and ending with the most general case. These results improve upon the results of Nishinou–Nohara–Ueda [60].

**The special case of small resolutions.** The symplectic parallel transport for integrable systems is used to show the following [60, Theorem 1].

**Theorem 4.2** (Nishinou–Nohara–Ueda). Consider a toric degeneration \( \mathfrak{X} \to B \) as in Section 3.1 and let \( \Phi := \Phi_1 : \mathfrak{X}_{\text{reg}}^{\text{reg}} \to \mathbb{R}^n \) be the integrable system obtained as in Section 3.1. Assume further that \( X_0 \) is Fano, and admits a small resolution of singularities.
Then for any \( u \in \mathbb{P}^{\text{reg}} \subseteq \mathbb{R}^n \) in the interior of the moment polytope, the Floer potential of the Lagrangian torus fiber \( L(u) := \Phi^{-1}(u) \) equals the disk potential and is given by

\[
m_0(L)(x) = \sum_{i=1}^{\ell} \exp(\langle v_i, x \rangle) T^i(u),
\]

where \( v_1, \ldots, v_\ell \) are the generators of the polar dual \( P^\circ \) and \( t_i \)'s are the equations defining the facets of the moment polytope \( P \) and valuations of \( x \) belong to the moment polytope \( P \).

In particular it says that if the degenerating toric Fano variety admits a small resolution of singularities, then the holomorphic disks with Lagrangian boundary that contribute to the Floer potential of the smoothing are transversal to toric strata.

**Remark 4.3.** By Conjecture A.10, it is expected that we can always find a graph \( \gamma \), such that the corresponding toric degeneration \( Y_{\gamma,C,M_{\gamma,C}} \) of \( M_\mathbb{C}(2, \mathcal{L}) \) has a small resolution.

Given a small resolution Theorem 4.2 can be applied to relate graph potentials to the Floer potential of \( M_\mathbb{C}(2, \mathcal{L}) \).

**A generalization of Nishinou–Nohara–Ueda without small resolution.** Lacking a proof of Conjecture A.10 we will rather generalise Theorem 4.2 to Theorem 4.4

**Some curves in \( \pi_2(X_0, L_0) \).** Before we discuss Theorem 4.4, let us construct a system of curves in \( \pi_2(X_0, L_0) \). We will later show that these curves form a generating set for \( \pi_2(X_0, L_0) \). We denote \( L_0 := L_0(u) \).

For each irreducible component \( D_i \) of the torus-invariant divisor \( D \) on \( X_0 \) consider the complement of \( X \) in the union of all other irreducible components of \( D \), this is an affine variety isomorphic to \( \mathbb{C} \times (\mathbb{C}^*)^{n-1} \) with coordinates \( x_1, x_2, \ldots, x_n \) and \( L_0(u) \) is given by \( |x_k| = u_k \), where \( u_k \) are the components of \( u \).

For any point \( P \in L_0(u) \) with coordinates \( z_1, \ldots, z_n \) there is a disk \( C_i = C_i(u, P) \) given by a system of \( n - 1 \) equalities \( x_k = P_k, k = 2, \ldots, n - 1 \), and one inequality \( |x_1| \leq u_1 \). Also the fibers of \( c \) over \( c(D_i^P) \) are \( n - 1 \)-dimensional tori, where \( c \) is the map in (28) and \( D_i^P = D_i \cup [i] D_i \). There is a map from generic \( n \)-dimensional torus \( L_0 = L_0(u) \) to these tori that collapses one direction corresponding to the boundary of the disk \( C_i \); in the local coordinates from above just vary \( u_1 \) to 0.

So for each \( i \) the homotopy class \( [C_i] \in \pi_2(X_0, L_0) \) has the boundary

\[
\partial[C_i] = [\partial C_i] \in \pi_1(L_0) = N
\]

and under the last identification \( \partial[C_i] = \rho_i \), where \( \rho_i \in N \) is the primitive ray corresponding to the divisor \( D_i \). We will use these curve classes in Proposition 4.9 which is a key step in the proof of the following main result.

**Theorem 4.4.** Let \( X_t \) be a smooth projective Fano variety and let \( \mathfrak{X} \rightarrow B \) be a degeneration of \( X_t \) into a Gorenstein Fano toric variety \( X_0 \) such that \( X_t \) is smooth for \( t \neq 0 \). Moreover assume that the degeneration preserves the second Betti number. Let \( L_t(u) \) be a monotone Lagrangian torus obtained by the symplectic parallel transport of a Lagrangian torus \( L_0(u) \) in \( X_0^{\text{reg}} \) for some point \( u \in \mathbb{P}^{\text{reg}} \).

Then the Newton polytope of the disk potential \( m_0(L_0(u)) \) equals the fan polytope \( P^\circ \) of the toric variety \( X_0 \). Moreover the coefficients of \( m_0(L_0(u)) \) corresponding to the rays of \( P^\circ \) are one.
Remark 4.5. We are removing the need for a small resolution but with an assumption that the second Betti number is preserved by the degeneration. This assumption is enough for our application as Manon’s toric degeneration preserve the second Betti numbers.

Remark 4.6. The condition on the preservation of the second Betti number in Theorem 4.4 can be substituted by the more general notion of $Q_{\Gamma}$-smoothing due to Galkin–Mikhalkin \[33\]. We refer the reader to \[33, Theorem 1\] for a precise result. In particular, for us it is enough to assume that the canonical class $K_{X_0} \in H^2(X_0, \mathbb{R})$ vanishes on the kernel of the map $H^2(X_0, \mathbb{R}) \to H^2(X_0, \mathbb{R})$. These notions are only used in defining the Maslov index and monotonicity for singular symplectic spaces \[33, Section 1.2\]. The condition on the preservation of Betti numbers in the statement of Theorem B already holds in the setup of Nishinou–Nohara–Ueda \[62\].

Now consider the special case where the only non-zero lattice point of $P^o$ are the rays (i.e. $X_0$ is terminal c.f. Appendix A.1). Then Theorem 4.4 directly implies

Corollary 4.7.

$$m_0(L_0(u)) = \sum_{i=1}^{m} \exp(\langle v_i, x \rangle)T^{\ell_i(u)}$$

where $v_i$ are the generators of the polar dual $P^o$ and $\ell_i$’s are the equations defining the facets of the moment polytope $P$ and valuations of $x$ belong to the moment polytope $P$.

We postpone the proof of Theorem 4.4 and focus on applications of the theorem for $X = M_C(2, \mathcal{L})$. Corollary 4.7 combined with Manon’s construction of a toric degeneration has the following important consequence in our setting that computes the disk potential for $M_C(2, \mathcal{L})$.

Theorem 4.8. Let $(y, c)$ be a trivalent graph with one colored vertex. Then the following are equal:

- the graph potential $W_{y,c}$,
- the disk/Floer potential of the Lagrangian torus $L_{y,c} \subset M_C(2, \mathcal{L})$,

where $L_{y,c} = \Phi^{-1}_{y,c}(0)$ is a fiber of the integrable system $\Phi_{y,c} : M_C(2, \mathcal{L})^{\reg} \to P_{y,c}$ associated with Manon’s toric degeneration of $M_C(2, \mathcal{L})$ to $Y_{y,c,M_{y,c}}$. Moreover the monotone torus $L_{y,c}$ is optimal i.e.

$$T_{\text{con}}(L_{y,c}) = 8g - 8 = T_{M_C(2, \mathcal{L})}.$$ 

Proof. We apply Theorem 4.4 to Manon’s toric degenerations associated to a trivalent graph and observe that for $u = 0$ the Lagrangian torus $L_1(0)$ is monotone by Lemma 3.5. Now we consider the following two cases and without loss of generality we assume that the genus of the graph is at least 3.

Case I. Assume that the graph $y$ has no separating edges. This assumption combined with Theorem A.3 implies that the lattice points of the polytope $P_{y,c}$ come only from the rays and hence by Theorem 4.4, we get the required result from Corollary 4.7.
**Case II.** We now consider the case of a general graph. First we observe that for any trivalent graph \((y, c)\), the classical periods of \(m_0(L_{y,c})\) are equal. On the other hand the classical periods of the graph potentials \(W_{y,c}\) for trivalent graphs are also equal by Theorem 2.2.

Thus we know that the classical periods of \(m_0(L_{y,c})\) equal the classical periods of \(W_{y,c}\). Now by Theorem 4.4, we get that in order to determine \(m_0(L_{y,c})\), we need to only focus on the lattice points on the polytope \(P^0_{y,c}\) which do not come from rays. By Lemma A.7, those are exactly the lattice points \(\pm e_b\), where \(b\) is a bridge.

Consider the difference \(Z_{y,c} := m_0(L_{y,c}) - W_{y,c}\), which is supported on the monomials associated to lattice points \(\pm e_b\) where \(b\) is a bridge. Let \(z_b^+\) be the coefficients of the monomials associated to the lattice points \(\pm e_b\). Now we have equations

\[
(W_{y,c} + Z_{y,c})^m = [W_{y,c}]_0 \text{ for all } m \geq 0.
\]

Thus we have infinitely many equations in as many variables as there are bridges in the graph. Thus it follows that \(Z_{y,c} = 0\) is the unique solution. Thus we are done with determining the Floer/disk potentials.

The graph potential evaluated at the unique Morse point \((1, \ldots, 1)\) in the domain \(\mathbb{R}^\dim L_{y,c}\) gives the value \(8g - 8\) which agrees with that of \(T_{(\mathbb{C},e,C)}\) as computed in [57]. Thus optimality follows. \(\square\)

Recall that for each irreducible components \(D_i\) of the torus invariant divisor \(D\) of \(X_0\), we have constructed curves \((C_i, \partial C_i)\) such that \([\partial C]\) are the primitive rays \(\rho_j \in N\) corresponding to the divisor \(D_i\). We have the following result that implies Theorem B.

**Proposition 4.9.** If \((C, \partial C)\) is the class of an irreducible curve of Maslov index two, then it lies in the convex hull of the classes \([C_i]\). Moreover, the class \((C, \partial C)\) (under the identification \([\partial C] \in \pi_1(L_0) = N\) is at the vertex (which is the extremal case) of the fan polytope, only if the limit curve lies inside one of these charts of \(X_0\) given by one ray.

**Proof of Proposition 4.9.** We will prove Proposition 4.9 first in the special case of terminal singularities, which is in fact enough for our intended application involving graph potentials, and then we will give the general case which is of independent interest. By Theorem A.3, we know that the toric varieties associated to trivalent graphs constructed by Manon have Gorenstein terminal singularities.

**The special case of terminal singularities.** Let \(X_0\) be a \(\mathbb{Q}\)-Gorenstein, terminal, Fano toric variety with \(D = \sum_{i \in S} D_i\) the boundary anti-canonical divisor (the complement of the open orbit) and let \(C\) be an irreducible curve not contained in \(D\). Let \(P\) be a point such that the local intersection index of \(C\) and \(D\) in \(P\) equals 1.

We have the following lemma.

**Lemma 4.10.** \(\#\{i \in S \mid P \in D_i\}\) = 1.

**Proof.** Let \(f : \tilde{X}_0 \to X_0\) be any toric resolution of singularities and consider the unique irreducible curve \(C'\) such that \(f\) maps to \(C\). Thus \(f_*(C') = C\) as 1-cycles. On the other hand

\[
f^*(-K_{\tilde{X}_0}) = -K_{\tilde{X}_0} + \sum_j a_j E_j,
\]

where \(a_j\) are the coefficients of the monomials associated to lattice points in the convex hull of the classes \([C_i]\).

**References:**

[1] A.3
[2] 4.4
[3] 4.7
[4] 4.10
[5] 4.9
[6] 2.2
[7] 57
where $E_j$ are exceptional divisors and since $\tilde{X}_0$ is toric, we get
\begin{equation}
(52) \quad f^*D = \sum_i D'_i + \sum_j (1 + a_j)E_j,
\end{equation}
where $D'_i$ denotes the proper preimages of $D_i$. Now by the projection formula we get
\begin{equation}
(53) \quad C \cdot D = C' \cdot f^*D.
\end{equation}
By assumption the left-hand side of (53) is one. Expanding the right-hand side of (53) gives
\begin{equation}
(54) \quad \sum_i (C' \cdot D'_i) + \sum_j (1 + a_j)(C' \cdot E_j) = 1
\end{equation}
Observe that since $\tilde{X}_0$ is smooth and the divisors $D'_i$ and $E_j$’s are all Cartier divisors, the intersection numbers are all integers. Moreover all the intersection numbers are non-negative. Thus for (54) to hold we either have that
- there exists exactly one $i$ such that $C \cdot D'_i = 1$ and all other terms are zero, which forces $C$ to be disjoint from $D_k$ for $k \neq i$
- there exists an index $j$ such that $C' \cdot E_j \neq 0$ and thus $C' \cdot f^*D \geq 1 + a_j$.
Now since $X$ has terminal singularities we have that $1 + a_j > 1$. Thus the second possibility cannot happen. Thus we are done. □

Proof of Proposition 4.9 in the terminal case. Let $u$ be a interior point of $P^{\text{int}}$ as in the statement of Theorem 4.4 and let $L_0(u)$ be a monotone Lagrangian torus in $X_0$. Moreover, we have a family of pairs $(X_t, L_t)_{t \in \mathbb{R}}$ of smooth Fano varieties along with monotone Lagrangian tori $L_t = L_t(u) \subset X_t$ as in (30). Recall that $B$ has two special points 0 and 1 and $X_1 = X$.
We further assume that $X_0$ is a $\mathbb{Q}$-Gorenstein, terminal, Fano toric variety.
Let $u$ be an interior point in the moment polytope and $L_1(u)$ be a monotone Lagrangian torus in $X_0^{\text{reg}}$. We denote its symplectic transport to $X_t$ by $L_t$ as in (30). We refer the reader to (35).
For a family $C_t$ of curves in $X_t$, we denote the limiting curve as $t \to 0$ by $C \in X_0$. Now since the total space $\mathfrak{X}$ is $\mathbb{Q}$-Gorenstein, by the adjunction formula the Maslov index of $C$ equals the Maslov index of $C_t$.
We are interested in counting disks of Maslov index two whose boundary lies on the monotone Lagrangian torus $L = L_1(u)$ of $X$. By deforming the varieties $X$, we have a family of disks of Maslov index two in the fibers $X_t$ with boundary $L_t$. Using the condition that the Maslov index is two, these disks will intersect the anti-canonical divisor $D$ in the toric variety $X_0$ with total intersection index one.
Now Lemma 4.10 tells us that the limiting curves $C$ intersects exactly one boundary divisor say $D_l$, thus it lies in the complement of $D - D_l$ in $X_0$. This complement is isomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{\dim X_0 - 1}$. Conversely for every chart, we have exactly one curve by direct computation. Now chasing the torus in (35) we can transport the disk back to $(X_t, L_t)$. This is possible since Lemma 4.10 allows us to ignore the 2-coskeleton in the toric variety $X_0$. This completes the proof in the terminal case. □
The general case. We now consider the general case as in the statement of Theorem 4.4. We first find generators for \( \pi_2(X_0, L) \).

Generators for \( \pi_2(X_0, L) \). We first show that the curves \( [C_i] \) constructed before generate \( \pi_2(X_0, L_0) \).

**Lemma 4.11.** The classes \( [C_i] \) generate \( \pi_2(X_0, L_0) \).

**Proof.** First we consider the case that \( X_0 \) is smooth (or \( \mathbb{Q} \)-factorial/simplicial). In the case of a smooth (or \( \mathbb{Q} \)-factorial/simplicial) toric variety the curves \( C_i \) are dual to divisors \( D_i \).

We can associate to \( D_i \) classes in \( H^2(X_0, \mathbb{Z}) \) and there is a short exact sequence

\[
0 \to M \to \bigoplus_i \mathbb{Z}D_i \to H^2(X_0, \mathbb{Z}) \to 0
\]

Here \( M \) is the lattice of monomials dual to the lattice \( N \), and we also consider the dual short exact sequence

\[
0 \to H_2(X_0, \mathbb{Z}) \to \pi_2(X_0, L_0) \to N \to 0
\]

Equivalently in the basis of curves

\[
0 \to H_2(X_0, \mathbb{Z}) \to \bigoplus_i \mathbb{Z}C_i \to N \to 0
\]

Comparing (56) with (57) we get \( \pi_2(X_0, L_0) = \bigoplus_i \mathbb{Z}C_i \). Thus we are done when \( X_0 \) is either smooth or simplicial.

For any \( \mathbb{Q} \)-factorialization \( \tilde{X}_0 \) (i.e. a triangulation of a fan of \( X_0 \)), the proper preimages of \( C_i \) and \( D_i \) will form a pair of dual bases of \( H_2(\tilde{X}_0, L_0) \) and \( H^2(\tilde{X}_0, L_0) \). Also note that the pair \( (X_0, L_0) \) is homotopy equivalent to the pair \( (X_0, (\mathbb{C}^*)^n) \). So for non-\( \mathbb{Q} \)-factorial \( X_0 \) we have more generators than needed for a base, and the map \( \bigoplus_i \mathbb{Z}C_i \to \pi_2(X_0, L) \) has a kernel. \( \square \)

Proof of Proposition 4.9 in the general case. Let us analyze a holomorphic irreducible curve \( (C, \partial C) \subset (X_0, L_0) \) of Maslov index two, for which \( C \cdot D = 1 \), where \( D \) is the anti-canonical divisor in \( X_0 \).

Choose any small \( \mathbb{Q} \)-factorisation \( r: X'_0 \to X_0 \) so that \( r^*D = \sum D'_i \), and lift a curve \( (C', \partial C') \subset (X'_0, L_0) \) to \( (C', \partial C') \subset (X'_0, L) \). So we have \( H_2(r)([C']) = [C] \in H_2(X_0, L_0) \). On \( X'_0 \) all intersection indices \( C' \cdot D'_i \) are well-defined non-negative rational numbers, say \( p_i = C' \cdot D'_i \), and \( \sum p_i = 1 \).

Note that the curve (or, rather, effective relative cycle) \( \sum p_i C'_i \) has exactly the same intersection indices with all \( D_i \), thus we have an equality

\[
[C'] = \sum p_i [C'_i] \in H_2(X'_0, L_0; \mathbb{Q})
\]

to which we can apply \( H_2(r) \) to obtain an equality

\[
H_2(r)([C']) = \sum p_i H_2(r)([C'_i]) \in H_2(X_0, L_0; \mathbb{Q}),
\]

i.e.

\[
[C] = \sum p_i [C_i] \in H_2(X_0, L_0; \mathbb{Q}).
\]
This implies that the class of any irreducible curve of Maslov index two lies in the convex hull of the classes \([C_i]\). By considering the boundary (which we also could have applied at the level of \(X_0')\) we obtain
\[
[\partial C] = \sum p_i [\partial C_i] \in H_1(L_0; \mathbb{Q})
\]
so the class of \([\partial C]\) has to be an integer point inside the fan polytope of \(X_0\) (the convex hull of the primitive rays \(\rho_i\)).

Now we also see that if \(C\) is an irreducible curve of Maslov index two with \([\partial C] = \rho_i\) for some \(i\), then \([C'] = [C_i'] \in H_2(X_0', L_0)\), hence \(C' \cdot D' = 0\) for \(k \neq i\), thus \(C'\) is disjoint from \(D_k'\) for \(k \neq i\), so \(C = r(C')\) is disjoint from \(D_k = r(D_k')\) for \(k \neq i\), i.e. \(C\) lies in an affine chart \(\mathbb{C} \times (\mathbb{C}^*)^{n-1}\) associated with \(D_i\) (so \(C\) is the basic disk described above). This completes the proof. \(\square\)

5. Mirror partners for moduli of vector bundles

We can now establish the proof of Theorem C. It is given as Corollary 5.2 and Proposition 5.3.

Descendant invariants. Let \(X\) be a Fano variety. We will denote by \(X_{0,1,\beta}\) the moduli space of stable maps \(f: (C; x) \to X\) where \(C\) is a rational curve with a marked point \(x \in C\) such that \([f(C)] = \beta \in H_2(X, \mathbb{Z})\) is a fixed homology class. It has a virtual fundamental class in degree \(\int_{[X_{0,1,\beta}]} c_1(X) - 2 + \dim X\), and comes with the evaluation and the forgetful morphisms
\[
\begin{align*}
ev: X_{0,1,\beta} &\to X = X_{0,1,0}, \\
p: X_{0,1,\beta} &\to X_{0,0,\beta}.
\end{align*}
\]
Denote by \(\psi\) the first Chern class of the universal cotangent line bundle, which is also the relative dualizing sheaf \(\omega_{X/\mathbb{C}}\). We can use it to define the descendant Gromov–Witten invariants for \(\beta \in H_2(X, \mathbb{Z})\). These count count rational curves of degree \(d\) passing through a fixed generic point with some kind of tangency condition, and are defined as
\[
\begin{align*}
p_\beta := (\nu^{d-2} X)_{0,1,\beta} &\to \int_{[X_{0,1,\beta}]} \nu^*([pt]) \cup \nu^{d-2}, \\
p_d(X) &:= \sum_{\mu(\beta) = 2d} p_\beta, \quad d \geq 2,
\end{align*}
\]
where \([pt]\) is the class of the point, i.e. a homogeneous class such that \(\int_X [pt] = 1\). We moreover set
\[
\begin{align*}
p_0(X) &:= 1, \quad p_1(X) := 0,
\end{align*}
\]
and
\[
\begin{align*}c_d := d! \cdot p_d,
\end{align*}
\]
Remark 5.1. Under the assumption that \(X\) has a smooth anti-canonical divisor \(D^2\), Mandel [33, Theorem 1.2] shows that \(c_d\) is the naive number of maps from \(\mathbb{P}^1\) to \(X\) passing through a fixed generic point on \(X\) such that the preimage of the divisor \(D\) is a fixed generic \(d\)-tuple of points on \(\mathbb{P}^1\). In particular the number \(c_d\) is a non-negative integer.

\(^2\)Which by the Bertini theorem holds for the varieties we consider in this paper, but fails for some other Fano varieties such as the squares of degree one del Pezzo surfaces.
Graph potentials as mirror partners. The next result, demonstrates how the combinatorics of graph potentials controls the enumerative geometry of $M_{C^1,2} \cdot L$º. The equality in this corollary corresponds to the original condition of enumerative mirror symmetry for Fano manifolds proposed by Eguchi–Hori–Xiong.

Let us briefly recall the notion of quantum periods for Fano varieties, as discussed in Coates–Corti–Galkin–Kasprzyk and Galkin–Iritani [18, 30]. The (unregularized and regularized) quantum periods are the generating power series for descendant Gromov–Witten invariants $p_d$ and $c_d = d! \cdot p_d$ defined in (63):

\[
G_X(t) := 1 + \sum_{d=2}^{\infty} p_d(X) t^d, \\
\tilde{G}_X(\kappa) := 1 + \sum_{d=2}^{\infty} c_d(X) \kappa^d = \frac{1}{\kappa} \int_0^\infty G_X(t) \exp(t/\kappa) dt
\]

**Corollary 5.2.** Let $(\gamma, c)$ be a colored trivalent graph of genus $g \geq 2$ with one colored vertex. Let $d \geq 2$. We have the equality

\[
c_d(M_C(2, L)) = [W^d_{Y,C}]_{2^n}
\]

of the Gromov–Witten invariant $c_d$ and the constant term of $d$th power of the Laurent polynomial $W_{Y,C}$.

In particular we obtain that $W_{Y,C}$ is a mirror partner for $M_C(2, L)$, in the sense of [44, §3].

**Proof.** By Theorem 4.8 we have that the graph potential agrees with the Floer potential for any trivalent graph with no separating edges. Hence their classical periods agree.

By applying [79, Theorem 1.1] (see also Bondal–Galkin [15] and Rau [72]) to the monotone Lagrangian torus constructed in Proposition 3.9 we obtain that the classical period of the Floer potential agrees with the quantum period of $M_C(2, L)$. The conclusion about the mirror partner now follows for trivalent graph with no separating edges.

Now since by Theorem 2.2 the periods of the graph potential associated to any trivalent graph of genus $g$ with one colored vertex are the same, the general case follows. □

Consequences. The unregularized quantum period $G_X(t)$ is equal to the fundamental term of Givental’s $J$-series $J_X(t)$ from [35]. This implies that $G_X(t)$ satisfies the quantum differential equation $L(t, \partial_t)$ with an irregular singularity. Its Fourier–Laplace transform $\tilde{G}_X(\kappa)$ satisfies the so-called regularized quantum differential equation $\tilde{L}(\kappa, \partial_\kappa)$ that is expected to be regular.

Under the assumption that all eigenvalues of the quantum multiplication operator $\star_0 c_1(X)$ are distinct the respective regularized connection is Fuchsian. However for $X = M_C(2, L)$ only two eigenvalues have multiplicity one, as can be computed by result due to Muñoz in [57] (see [10, §3.1] for additional discussion). But under the identification of the period sequences we can show that the principle irreducible component of the regularized quantum connection is a summand of a Picard–Fuchs equation and, as a corollary, indeed has regular singularities.

Summing (67) with weights $\frac{\kappa^d}{d!}$ or $\kappa^d$ we get an equivalent reformulation of Corollary 5.2 in terms of the equality of the respective period series.
Proposition 5.3. Let \((γ, c)\) be a colored trivalent graph of genus \(g \geq 2\) with one colored vertex.

\[
G_{M_C(2, ℳ)}(t) = \int_{|z|=1} \exp(t \cdot W_{γ,c}(z)) \frac{dz}{z},
\]

(68)

\[
\hat{G}_{M_C(2, ℳ)}(κ) = π_{W_{γ,c}}(κ) \quad \text{for } |κ| \leq \frac{1}{8g - 8},
\]

of the unregularized (resp. regularized) quantum period of \(M_C(2, ℳ)\) and the oscillating integral (resp. period) of the graph potential \(W_{γ,c}\).

Corollary 5.4. The regularized quantum period is a solution of an ordinary differential equation with regular singularities.

Proof. By Proposition 5.3 we have the identification of the regularized quantum period with the period of the graph potential. As discussed in [18, §3], the period of a Laurent polynomial \(f\) satisfies an ordinary differential equation, given by the Picard–Fuchs operator. The local system of solutions is a summand of a polarised variation of Hodge structures, which by [22, Theorem 4.5] has regular singularities. □

Remark 5.5. This also allows us to use methods of convex geometry and random walks to estimate the asymptotics of the coefficients, see e.g. [30, Lemma 3.13]. In particular, the limit \(\lim_{k \to \infty} (|W^{4k}|)_x^{1/4k}\) exists and is equal to \(8(g - 1) = T_{M_C(2, ℳ)}\).

In Tables 1 and 2 we give the period sequences for the even and odd graph potentials. In the odd case this in turn coincides with the quantum period sequence for \(M_C(2, ℳ)\) by Proposition 5.3. These numbers were computed using the method introduced in [11] for computing periods of graph potentials using the graph potential TQFT. This efficient method for computing periods from [11] could be useful for algorithmically determining the quantum differential equation for the quantum periods of \(M_C(2, ℳ)\).

In Appendix B we will give some explicit calculations for the quantum period sequence, highlighting some of its patterns which one can read off from the tables.

Remark 5.6. We have mostly focused on \(M_C(2, ℳ)\), and only briefly commented on some aspects of \(M_C(2, ℰ_C)\) in Remark 3.11. There are further alternative points of view on \(M_C(2, ℰ_C)\), namely as a Poisson variety (by the work of Huebschmann, see e.g. [41]) or as a stack, which could take care of the singularities present in \(M_C(2, ℰ_C)\) for \(g \geq 3\), and the discrepancy between the period sequence of the uncolored graph potential for \(g = 2\) and the quantum period of \(M_C(2, ℰ_C) \cong \mathbb{P}^3\). Indeed, by [11, Tables 1 and 2] which are recalled in Table 2 we have that the non-zero terms \(p_{4k}\) in the classical period sequence of the uncolored graph potential for \(g = 2\) are

\[
1, 384, 645120, 1513881600, 4132896768000, \ldots
\]

whereas the non-zero terms \(p_{4k}\) in the regularised quantum period sequence for \(\mathbb{P}^3\) (see e.g. [19, §1]) are

\[
1, 24, 2520, 369600, 63063000, \ldots
\]

We can make two comments:

- To fix the discrepancy for \(g = 2\), it suffices to rescale the graph potential described in [11, Example 2.6] by \(1/2\) to get the correct period sequence.
For $g \geq 3$ and a connected graph $\gamma$ without separating edges the Laurent polynomial $W_{\gamma}$ is supported in the vertices of its Newton polytope. There are $8g - 8$ non-vanishing coefficients, each equal to one, which are in bijection with facets of $\Delta_{\gamma}$ and with irreducible torus-invariant Weil divisors on $X_{0}$. A fiber of a moment map bounds $8g - 8$ holomorphic disks, each transversally intersecting exactly one of toric divisors, and if $X_{0}$ has a small resolution of singularities this shows that there are no other holomorphic disks of Maslov index 2.

We observe that the graph potential $W_{\gamma,0}$ reflects some aspects of the symplectic geometry of the singular variety $M_{C}(2,O_{C})$. Since Fukaya categories of singular varieties are not so well established, it seems reasonable to start with categories of matrix factorizations of the graph potentials $W_{\gamma,0}$ instead of the not so well-defined categories $\text{Fuk} M_{C}(2,O_{C})$, in order to gain further insight into their (conjecturally common) properties. This gives further evidence for the speculation on the Atiyah–Floer conjecture and Donaldson–Floer theories in [11, §1.3].

**Appendix A. Analyzing the singularities**

Let $(\gamma',c)$ be the colored trivalent graph constructed from a trivalent graph $\gamma$ with at most one half-edge in Section 2.2. We wish to study the singularities of the toric variety $Y_{P_{\gamma},M_{\gamma}} \cong X_{P_{\gamma},c,M_{\gamma}}$.

Before we discuss the general picture, let us discuss what happens for the toric degenerations of $M_{C}(2,L)$ and $M_{C}(2,O_{C})$ for $g = 2$. There are two graphs, and as will be clear from Theorem A.3 and Remark A.11, the distinction between the behavior of the toric degenerations for $M_{C}(2,L)$ depending on the choice of graph is representative of what happens in general.

**A.1. Terminal singularities.** Recall from [20, Proposition 11.4.12(ii)] that for a $\mathbb{Q}$-Gorenstein toric variety, having terminal singularities is equivalent to the only non-zero lattice points in the fan polytope being its vertices. We can use this criterion, as by [25, Theorem 12] the homogeneous coordinate rings of the toric degenerations $Y_{P_{\gamma},M_{\gamma}}$ are all arithmetically Gorenstein and in particular the toric degenerations themselves are Gorenstein.

**Example A.1.** If $\gamma$ is the Theta graph from Figure 2(a) and there is one colored vertex, then the toric variety we obtain is the singular intersection of quadrics in $\mathbb{P}^{5}$ given by $x_{0}x_{5} - x_{1}x_{2} = 0$ and $x_{3}x_{4} - x_{1}x_{2} = 0$ [59].
The polar dual of the moment polytope is a cube with vertices \((\pm 1, \pm 1, \pm 1)\). This variety has 6 ordinary double points and hence has a small resolution. In particular, the variety also has terminal singularities. This case is studied by Galkin [28] and also by Nishinou–Nohara–Ueda [59, 60]. The results in this section and the next can be seen as a generalization from \(g = 2\) to \(g \geq 3\).

On the other hand, if \(y\) is the dumbbell graph from Figure 2(b) with one colored vertex, the toric variety is given by the equations \(x^2 = zw\) and \(zw = u^2 v\) in \(\mathbb{P}^3\) [59]. The dual of the moment polytope of the toric variety is the convex hull of the points

\[(71) \quad (1, 2, 0), (1, -2, 0), (-1, 0, 0), (-1, 0, -2), (1, 0, 0), (-1, 0, 2).\]

The polytope contains both the lattice points \((-1, 0, -2)\) and \((-1, 0, 2)\), but then it also contains \((-1, 0, 0)\), and hence it cannot be terminal. The distinction between the toric varieties obtained from these two graphs is explained by Theorem A.3.

The case of \(M_C(2, \mathcal{O}_C)\) for \(g = 2\) is a bit pathological, as we by [58, Theorem 1] have an isomorphism \(M_C(2, \mathcal{O}_C) \cong \mathbb{P}^3\). Moreover the locus of semistable bundles up to \(S\)-equivalence is identified with the Kummer surface in \(\mathbb{P}^3\).

**Example A.2.** If \(y\) is the Theta graph and there are no colored vertices, then the toric variety is just \(\mathbb{P}^3\). Clearly it is smooth and hence terminal.

If \(y\) is the dumbbell graph and there are no colored vertices, then the dual of the moment polytope is the convex hull of the points

\[(72) \quad (1, 2, 0), (1, -2, 0), (-1, 0, 0), (1, 0, 2), (1, 0, 0), (1, 0, -2).\]

The polytope contains the lattice points \((1, 0, 0)\) which is exactly the half sum of the lattice points \((1, 0, 2)\) and \((1, 0, -2)\). Hence the toric variety is non-terminal.

Then the dichotomy observed for \(g = 2\) from Example A.1 in the case of 1 colored vertex is illustrative for what happens for \(g \geq 3\).

**Theorem A.3.** Let \((y, c)\) be a colored trivalent graph of genus \(\geq 3\). The toric variety \(X_{P_{y,c}, M_y}\) has terminal singularities if and only if \(y\) has no separating edges.

Before we prove this statement we give some preliminary lemmas. The following two lemmas work for an arbitrary integer \(n\), even though we are mostly interested in the case \(n = 3g - 3\).

**Lemma A.4.** Let \(n \geq 3\) and consider \(\mathbb{Z}^n\). Consider the set \(S\) of \(\binom{n}{3}\) points of the form \(\pm e_i \pm e_j \pm e_k\) for \(1 \leq i, j, k \leq n\) distinct. Let \(p = s_1e_i + s_je_j + s_ke_k\) be an element of \(S\), then \(p\) does not lie in the convex hull of \(S \setminus \{p\}\).

**Proof.** Consider the function \(h = s_1e_i + s_je_j + s_ke_k\). We have that \(h(s_1e_i + s_je_j + s_ke_k) = 3\), and \(h(q) < 3\) for \(q \in S \setminus \{p\}\). \(\square\)

We will denote \(\Pi_n\) the polytope given by the convex hull of the set \(S\) from Lemma A.4.

**Lemma A.5.** With respect to the standard lattice \(\mathbb{Z}^n\) the polytope \(\Pi_n\) contains precisely the following lattice points:

- \(0 \in \Pi_n\)
• $\pm e_i \in \Pi_n$ for $1 \leq i \leq n$
• $\pm e_i \pm e_j \in \Pi_n$ for $1 \leq i, j \leq n$ distinct
• $\pm e_i \pm e_j \pm e_k \in \Pi_n$ for $1 \leq i, j, k \leq n$ distinct.

Moreover,
• for all $1 \leq i \leq n$ the lattice points $\pm e_i$ lie in the convex hull of $\pm e_i \pm e_j \pm e_k$ for $1 \leq j, k \leq n$ distinct and $i \neq j, k$;
• for all $1 \leq i, j \leq n$ distinct the lattice points $\pm e_i \pm e_j$ lie in the convex hull of $\pm e_i \pm e_j \pm e_k$ for $1 \leq k \leq n$ such that $k \neq i, j$.

**Proof.** The function $h = \sum_{i=1}^{n} (e_i^2) = 2$ is convex, and $h(\pm e_i \pm e_j \pm e_k) = 3$, hence all these lattice points lie on a sphere of radius 1 and they cannot be contained in each other convex hulls. This function takes on the values $0, 1, 2, 3$ on the lattice points listed in the statement of the lemma.

To see that there are no other, observe that for two vectors $u, u'$ whose coefficients in the standard basis belong to $\{-1, 0, 1\}$, we have that $(u', u) \leq (u, u)$ in the standard bilinear pairing. If we write $u = \sum_{i=1}^{n} \alpha_i \gamma_i$ where $\gamma_i$ are the vertices of $\Pi_n$, such that $\alpha_i \geq 0$ for all $i$ and $\sum_{i=1}^{n} \alpha_i = 1$, we can apply this observation to $u' = e_i$ for all $i$ and we get

\[
(u, u) = \sum_{i=1}^{2} \alpha_i (e_i, u) \leq \sum_{i=1}^{2} \alpha_i (u, u) = (u, u)
\]

hence either $\alpha_i = 0$ or $(\gamma_i, u) = (u, u)$.

The final claim is immediate. \qed

We need one more lemma, giving a homological interpretation to the notion of a separating edge, using the notation of Section 2.1.

**Lemma A.6.** Let $\gamma$ be a trivalent graph. Let $e \in E$ be an edge, and consider the associated indicator function $e^V \in C^1(\gamma, \mathbb{Z}) = \mathbb{N}_r$. The vectors $e^V$ and $-e^V$ are always inside the polytope. Moreover $e^V \in N_r \subseteq \mathbb{N}_r$ if and only if $e$ is separating.

**Proof.** Assume that $e$ is separating. i.e. by removing it we get two connected components which we will denote $\gamma'$ and $\gamma''$. Choose an orientation of the edges of $\gamma$, and assign 1 to an incoming half-edge, and $-1$ to an outgoing half-edge. For $v \in \gamma'$ we consider the sublattice $N_o$, and in there consider the sum

\[
\sum_{e \in \gamma'} P_{o, s(v, o)} \in N_r
\]

where $o$ denotes the orientation scheme we choose on the graph $\gamma$, $s(v, 0) \in C_1(\gamma, \mathbb{F}_2)$ defined by $o$ and $P_{o, s(v, 0)}$ be as in (23). By construction the sum is, up to a sign, equal to $e^V$, hence $e^V \in N_r$.

Assume that $e^V \in N_r$, and for the sake of contradiction assume that $e$ is non-separating. Then there exists a cycle $Z$ in $\gamma$ (without repetition of vertices), defining an element of $C_1(\gamma, \mathbb{Z})$ by assigning 1 to all edges of the graph.
On the other hand, there exists a chain of inclusions
\[(75) \quad 2\tilde{N}_\gamma \subseteq N_\gamma \subseteq \tilde{N}_\gamma\]
induced by the identity \(\pm 2x_i = (\pm x_i \pm x_j \pm x_k) + (\pm x_i \mp x_j \mp x_k)\), where the \(x_i\)'s are as in Section 2.1. These induce an isomorphism
\[(76) \quad \tilde{N}_\gamma / 2\tilde{N}_\gamma \cong C^1(\gamma, \mathbb{Z}/2\mathbb{Z})\]
which then induces an isomorphism
\[(77) \quad N_\gamma / 2\tilde{N}_\gamma \cong d(C^0(\gamma, \mathbb{Z}/2\mathbb{Z}))\]
For \(e^\gamma\) to be in \(N_\gamma\) (and not just \(\tilde{N}_\gamma\)) it must evaluate to zero on cycles since
\[(78) \quad N_\gamma / 2\tilde{N}_\gamma \cong d(C^0(\gamma, \mathbb{Z}/2\mathbb{Z})).\]
But by construction \(e^\gamma(Z)\) is odd. This contradiction shows that \(e^\gamma\) cannot be an element of \(N_\gamma\).

**Proof of Theorem A.3.** We give the following general Lemma describing the lattice points of the Newton polytope of a graph potential. The proof of Theorem A.3 is an immediate corollary.

**Lemma A.7.** Let \(\gamma, c\) be a connected trivalent colored graph of genus \(g > 2\) and consider the Newton polytope \(P_{\gamma,c}^\circ\) of the graph potential. The non-zero integer points are the following:

- rays (vertices of the polytope)
  - with every non-loop vertex of the graph \(\gamma\) there are 4 associated rays.
  - with every loop vertex of graph \(\gamma\) there are 2 associated rays.

- non-rays lattice points of the form \(e_b\) and \(-e_b\) associated to every separating edge \(b\) in the graph.

In particular total number of rays equals \(8g - 8 - 2\#\text{loops}\) and the number of lattice points other than zero and rays is \(2\#\text{bridges}\).

**Proof.** Assume that \(\gamma\) has no separating edges. Then in particular it does not contain loops, and hence the set of vertices of \(P_{\gamma,c}^\circ\) is a subset of \(\{\pm e_i \pm e_j \pm e_k\}\), where \(e_i\) are standard vectors in \(\tilde{N}_\gamma \cong \mathbb{Z}^E\). By Lemma A.4 we have that all the points \(p(v, s)\) given by (23) are vertices. It remains to show that there are no other lattice points on the boundary of this polytope.

For this, by Lemma A.5 it suffices to show that no points of the form \(\pm e_i\) and \(\pm e_i \pm e_j\) is a lattice point in the convex hull of the vertices of \(P_{\gamma,c}^\circ\). Since the graph \(\gamma\) has no separating edges, it follows that from Lemma A.6 that \(\pm e_i\) cannot be a lattice point in \(N_\gamma\).

Now say for example that \(e_i + e_j\) would be a lattice point of the polytope, and let \(v\) be a vertex adjacent to \(e_i\) and \(e_j\). Then we can write
\[(79) \quad e_i + e_j = \frac{1}{2} \left((e_i + e_j + e_k) + (e_i + e_j - e_k)\right)\]
where \(e_k\) is the third edge adjacent to \(v\). However, the only way in which we can do this if \(e_i, e_j\) and \(e_k\) have the same end points, but then we obtain a subgraph of genus 2 in \(\gamma\) which is a contradiction.
The converse follows from Lemma A.6 and (74).

As an application of Theorem A.3, we can give an alternative proof of the following result due to Kiem–Li [46, Corollary 5.4] (albeit only for a generic curve, rather than for every curve). Their proof is based on understanding the discrepancy between the canonical bundle of $M_C(2, O_C)$ and its Kirwan desingularization. In our setup it rather follows from Kawamata’s inversion of adjunction for terminal singularities.

**Corollary A.8** (Kiem–Li). Let $C$ be a generic smooth projective curve of genus $g \geq 2$. Then $M_C(2, O_C)$ has at most terminal singularities.

**Proof.** By Kawamata [45, Theorem 1.5] we know that having at most terminal singularities in the central fiber implies that a general fiber has at most terminal singularities.

Manon’s construction provides us with a family of varieties whose special fiber is toric and whose general fibers are of the form $M_C(2, O_C)$, where $C$ runs over smooth projective curves of genus $g$ in the smooth locus of the family.

By Theorem A.3 it suffices to choose a trivalent graph $\gamma$ of genus $g$ without separating edges, so that the toric special fiber has at most terminal singularities. For this we can consider for instance the ladder graph as in Figure 3. Then in the family given by the toric degeneration the central fiber has at most terminal singularities, hence so does $M_C(2, O_C)$ in a Zariski open neighborhood of the special fiber.

---

**A.2. Small resolutions of singularities.** In the remainder of this section, we further discuss when the toric degenerations constructed by Manon admit a small resolution of singularities. We will not use results of this section in the rest of the article. Toric degenerations admitting a small resolution of singularities are useful in the study of mirror symmetry, as e.g. showcased in [5, 3, 62]. We refer the reader to Section 4 for a more precise result on how existence of small resolution implies a direct way of computing descendant Gromov-Witten invariants.

Recall that (torically) resolving the singularities of a toric variety $X$ amounts to subdividing each cone in the fan of the toric variety into a union of cones generated by a basis of the lattice. A toric resolution is small if we can find such a subdivision without adding new rays. This is in general a non-trivial combinatorial problem.

We consider the following class of trivalent graphs.

**Definition A.9.** Let $\gamma$ be a trivalent graph. We say it is **maximally edge-connected** if one needs to remove at least 3 edges to make it disconnected.

Maximally connected trivalent graphs $\gamma$ have a very interesting positivity property. Let $C_0$ be a nodal curve whose dual graph is a maximally edge-connected trivalent graph, then by [7, Proposition 2.5] the canonical bundle of the curve $C_0$ is very ample. We do not precisely
know which role maximally edge-connectedness plays in the combinatorial description of the cones of toric varieties \( X_{\Gamma_c,M} \), but motivated by the genus two case and experimental evidence, we conjecture the following.

**Conjecture A.10.** Let \((\gamma, c)\) be a maximally edge-connected trivalent graph. Then the graph potential toric variety \( X_{\Gamma_c,M} \) admits a small resolution of singularities. Combinatorially, we conjecture that there exists a refinement of the fan of \( X_{\Gamma_c,M} \) into a simplicial fan such that the resulting toric variety is smooth.

**Remark A.11.** For \( g = 2 \) this follows from the discussion in Example A.1 for the Theta graph, which is the unique maximally connected trivalent graph of genus 2.

Using computer algebra we have checked Conjecture A.10 also in genus 3 for the tetrahedron graph (depicted in Figure 4(a)), and in genus 4 for the bipartite graph (see Figure 4(b)) with an odd number of vertices colored, by explicitly constructing a simplicial subdivision of the fan polytope. Moreover, in the genus three, four and five cases, we can find non-maximally connected graphs with no separating edges such that the corresponding toric varieties do not admit a small resolution of singularities.

### Appendix B. Some explicit calculations for periods

In this section we collect some observations about the (non-)vanishing and values of the (quantum) periods studied in the main body of the paper. These explain some of the patterns which one can observe in Tables 1 and 2.

First we will consider the second and fourth (quantum) period, in Appendix B.3 we discuss some patterns which hold more generally.

**B.1. Second quantum period.** The first interesting quantum period of \( M_{C}(2, \mathcal{L}) \) is \( p_2 \). We will show in Corollary B.7 using algebro-geometric methods that it is always zero, as soon as \( g \geq 3 \). This is suggested by the combinatorial calculation in Proposition B.1 for the classical period \( \pi_2 \).

In Appendix B.3 we will extend this combinatorial vanishing result for the period \( \pi_2 \) to the vanishing of \( \pi_{4n+2} \) for all \( n \geq 0 \) and \( g \gg 0 \).
Proposition B.1. Let \( \gamma \) be a trivalent graph of genus \( g \). The second period of \( \overline{W}_{r,1} \) associated to \( M_C(2, \mathcal{L}) \) is given by

\[
\pi_2(\overline{W}_{r,1}) = \begin{cases} 
8 & g = 2 \\
0 & g \geq 3.
\end{cases}
\]

Proof. For \( g = 2 \) this is immediate from taking the square of the graph potential for the colored Theta graph in [11, Example 2.6] and pairing inverse monomials together.

Similarly, for \( g \geq 3 \) we are pairing inverse monomials together by squaring and taking the constant term. But because we assume that the graph \( \gamma \) is connected, the monomials contributed by the unique colored vertex have no corresponding inverse monomial in the graph potential.

The second quantum period on the other hand counts the number of lines (in the polarization given by the ample generator \( \Theta \) of \( \text{Pic} \, M_C(2, \mathcal{L}) \)) through a generic point of the variety \( M_C(2, \mathcal{L}) \). We wish to explain the vanishing for \( g \geq 3 \) geometrically, by using the explicit geometry in the hyperelliptic case.

Remark B.2. For \( g = 2 \) we have that \( M_C(2, \mathcal{L}) \) is a Fano 3-fold, and its second (regularized) quantum period can be read off in [19, §6]. More generally, in [11, Corollary 4.10] we have obtained an expression of (the inverse Laplace transform of) the period sequence which agrees with that of the quantum period sequence from §6 of op. cit.

Geometric setup. In [23] the relationship between a hyperelliptic curve \( C \), the geometry of the intersection of 2 quadrics (given by quadratic forms \( q_1 \) and \( q_2 \), determined by the hyperelliptic curve \( C \)) in \( \mathbb{P}^{2g+1} \), and the moduli space \( M_C(2, \mathcal{L}) \) is described. We will use this to sketch a geometric argument for the vanishing of the second quantum period for \( g \geq 3 \).

By Theorem 1 of op. cit. \( M_C(2, \mathcal{L}) \) is isomorphic to the closed subvariety of \( \text{Gr}(g-1, 2g+2) \) parametrizing \( \mathbb{P}^{g-2} \subseteq \mathbb{P}^{2g+1} \) contained in the quadrics. Likewise, by Theorem 2 of op. cit. the Jacobian of \( C \) (or rather, the torsor \( \text{Pic}^0 C \)) is isomorphic to a closed subvariety of \( \text{Gr}(g, 2g+2) \) parametrizing \( \mathbb{P}^{g-1} \subseteq \mathbb{P}^{2g+1} \) contained in the quadrics. These two closed subvarieties are given as the intersection \( \text{OGr}(k, 2g+2; q_j) \cap \text{OGr}(k, 2g+2; q_i) \), where \( \text{OGr}(k, 2g+2; q_i) \) is the orthogonal Grassmannian embedded in \( \text{Gr}(k, 2g+2) \), for \( k = g - 1, g \).

Now consider the incidence correspondence

\[
\text{Fl}(g-1, g, 2g+2) \longrightarrow \text{Gr}(g-1, 2g+2) \quad \longrightarrow \quad \text{Gr}(g, 2g+2).
\]

The vertical map is a \( \mathbb{P}^{g-1} \)-bundle, the horizontal map is a \( \mathbb{P}^{g+3} \)-bundle.

The restriction of the vertical map to \( \text{Pic}^0(C) \) corresponds to the interpretation of the incidence correspondence from [23, Remark 5.10(I)]. For this interpretation we will consider \( M_C(2, \mathcal{L}) \) for a line bundle \( \mathcal{L} \) of degree \( 2g + 1 \), and consider non-split extensions

\[
0 \rightarrow j \rightarrow \mathcal{V}_j \rightarrow j^* \otimes \mathcal{L} \rightarrow 0
\]

for \( j \in \text{Pic}^0 C \), which are always stable.
The $g - 1$-dimensional projective space $\mathbb{P}_j := \mathbb{P}(H^1(C, j^2 \otimes L)^\vee)$ parametrizing these extensions gives rise to a $\mathbb{P}^{g-1}$-bundle $\mathcal{P}$. If we let $\mathcal{M}$ be the Poincaré bundle on $C \times \text{Pic}^g(C)$, then $\mathcal{P} \cong \mathcal{P}_C(\mathcal{E})$ where $\mathcal{E} := R^1 p_{2*}(\mathcal{M} \otimes^L (-1))$. Then [23, Remark 5.10(I)] gives the following

**Lemma B.3.** We have that $\mathcal{P}$ fits into the fiber product diagram

\[
\begin{array}{ccc}
\mathcal{P} & \hookrightarrow & \text{Fl}(g - 1, g, 2g + 2) \\
\downarrow & & \downarrow \\
\text{Pic}^g(C) & \hookrightarrow & \text{Gr}(g, 2g + 2).
\end{array}
\]  

(83)

The horizontal map in the incidence correspondence gives a morphism $f : \mathcal{P} \to M_C(2, \mathcal{L})$.

We can consider other incidence correspondences. Before proving the interpretation in Proposition B.5, we will give an easier description for the Fano variety of lines on $M_C(2, \mathcal{L})$. This is a natural continuation of the discussion in [23, Remark 5.10].

**Proposition B.4.** The Fano variety of lines $F(M_C(2, \mathcal{L}))$ fits in the fiber product diagram

\[
\begin{array}{ccc}
F(M_C(2, \mathcal{L})) & \hookrightarrow & \text{Fl}(g - 2, g, 2g + 2) \\
\downarrow & & \downarrow \\
\text{Pic}^g(C) & \hookrightarrow & \text{Gr}(g, 2g + 2).
\end{array}
\]  

(84)

In particular, it is a $\mathbb{P}^{g-1}$-bundle over $\text{Pic}^g(C)$.

**Proof.** The image of $\text{Pic}^g(C)$ in $\text{Gr}(g, 2g + 2)$ consists of $g$-dimensional subspaces which are isotropic for the quadratic forms $q_1$ and $q_2$ defined on the $2g + 2$-dimensional vector space $V$. Hence we can describe the fiber product as

\[
\{ E_2 \subset E_1 \subset V \mid \dim E_2 = g - 2, \dim E_1 = g, q_1(E_1) = q_2(E_1) = 0 \}.
\]  

(85)

Recall that $F(\text{OGr}(g - 1, 2g + 2; q_1))$ is again a homogeneous variety, namely the quotient $\text{SO}_{2g+2}/P_{g-2g}$, where $P_{g-2g}$ denotes the parabolic subgroup

\[
\text{SO}_{2g+2}/P_{g-2g},
\]  

(86)

Now observe that

\[
F(M_C(2, \mathcal{L})) = F(\text{OGr}(g - 1, 2g + 2; q_1) \cap \text{OGr}(g - 1, 2g + 2; q_2))
\]  

\[
\subset F(\text{Gr}(g - 1, 2g + 2)) \cong \text{Fl}(g - 2, g, 2g + 2)
\]  

allows us to describe the Fano variety of lines using the homogeneous spaces $\text{SO}_{2g+2}/P_{g-2g}$, giving the description from (85). \qed

From Section 5 we recall that to compute the second quantum period, we are interested in the moduli space $M_C(2, \mathcal{L})_{b,1,\beta}$ for $\beta = 2$ (because the Fano index of $M_C(2, \mathcal{L})$ is 2). Again we wish to use the geometry of the ambient partial flag varieties, now using the isomorphism $\text{Gr}(g - 1, 2g + 2)_{b,1,2} \cong \text{Fl}(g - 2, g - 1, g, 2g + 2)$. This allows us to obtain the following identification, whose proof is similar to that of Proposition B.4.
**Proposition B.5.** The variety $M_{C}(2, \mathcal{L})_{0,1,2}$ fits in the fiber product diagram

\[
\begin{array}{c}
M_{C}(2, \mathcal{L})_{0,1,2} \xrightarrow{\text{ev}} \text{Fl}(g - 2, 2g + 2) \\
\downarrow \text{ev} \\
\text{Pic}^s(C) \xrightarrow{\text{f}} \text{Gr}(g - 1, 2g + 2) \\
\end{array}
\]

Summarizing, we have the following picture

\[
\begin{array}{c}
M_{C}(2, \mathcal{L})_{0,1,2} \xrightarrow{\text{f}} \text{Fl}(g - 2, 2g + 2) \\
\downarrow \text{f} \\
M_{C}(2, \mathcal{L}) \xrightarrow{\text{f}} \text{Gr}(g - 1, 2g + 2) \\
\downarrow \text{Pic}^s(C) \xrightarrow{\text{f}} \text{Gr}(g, 2g + 2).
\end{array}
\]

To prove the vanishing of the second quantum period for $g \geq 3$, we need the following property of the evaluation morphism.

**Proposition B.6.** The morphism $\text{ev} : M_{C}(2, \mathcal{L})_{0,1,2} \to M_{C}(2, \mathcal{L})$ factors through the natural map $f : \mathcal{P} \to M_{C}(2, \mathcal{L})$ from Lemma B.3.

**Proof.** It suffices to observe that the image of $f_2 \circ f_1$ is contained in $\mathcal{P} \subset \text{Fl}(g - 1, 2g + 2)$, which can be proven along the lines of Proposition B.4.

Hence we obtain the following corollary, which is an independent proof of the vanishing result from Proposition B.1, without referring to the agreement of classical and quantum periods.

**Corollary B.7.** Let $g \geq 3$. Then $p_2 = 0$.

**Proof.** Because the image of $M_{C}(2, \mathcal{L})_{0,1,2} \to M_{C}(2, \mathcal{L})$ under the evaluation morphism is contained in the image of $\mathcal{P}$, we have that the codimension of the image is at least $g - 2$. Hence the second quantum period vanishes for $g \geq 3$, by the definition of the quantum period.

**B.2. Fourth period.** The next interesting period is $\pi_4$. In the case we can give a closed formula for it.

**Proposition B.8.** The fourth coefficient of the period of the potential $\tilde{W}_{r,1}$ associated to $M_{C}(2, \mathcal{L})$ is given by

\[
\pi_4(\tilde{W}_{r,1}) = \begin{cases} 
216 & g = 2 \\
192(g - 1) & g \geq 3.
\end{cases}
\]
Proof. The total degree of a monomial in the graph potential is either 3 or −1, except for two monomials which are contributed by the (by assumption) unique colored vertex, whose total degrees are 1 and −3. Let us moreover consider a graph potential associated to a graph without loops, so that all coefficients are 1.

To understand the constant term of the fourth power of the graph potential, we consider the monomials of degree 3. To obtain a constant term, we need to multiply this monomial with either a monomial of degree 3, or with 3 monomials of degree 1. If \( g = 2 \), then the colored vertex and uncolored vertex share the same variables, so it is possible to multiply monomials of degree 3 and get a constant term. But this is impossible if \( g \geq 3 \), and hence we assume from now on that we only multiply monomials of degree 3 with monomials of degree ±1.

There are \( 2g - 2 \) monomials of degree ±3, one for each vertex. There are 4 ways to cancel this monomial: one way is by using the 3 distinct monomials contributed by the same vertex in the vertex potentials (see (10) or [11, Example 2.5]). The other three ways are by considering the edge potential for the 3 edges going out of the vertex. This is an alternative way of writing the graph potential as a sum over the set \( E \) of edges instead of the set \( V \) of vertices, see [10, §3.3]. One can cancel the monomial by multiplying it with one monomial from the same vertex and two monomials from the neighbor. There are \( 4! \) choices for doing this, hence the statement follows. \( \square \)

B.3. General patterns. The following result is completely general, and immediate from the definition (63). Because \( M_C(2, L) \) has index 2, we have that \( \langle \beta, -K_X \rangle \) is always divisible by 2.

Lemma B.9. The odd-indexed coefficients of the quantum period of \( M_C(2, L) \) vanish:

\[
P_{2k+1}(M_C(2, L)) = c_{2k+1}(M_C(2, L)) = 0.
\]

On the other hand, for the periods of graph potentials we have the following result. It follows immediately from the observation that the total degrees of the monomials in the graph potentials are ±1, ±3, so that no product of an odd number of them can have degree 0.

Let us write \( [W]_a \) for the constant coefficient of a Laurent polynomial \( W \).

Lemma B.10. For any odd or even coloring \( c \) of a graph \( \gamma \) the odd-indexed coefficients of the period of the graph potential vanish:

\[
[W_{\gamma,c}(z)]_{2k} = 0.
\]

This explains why in Tables 1 and 2 at the end of this appendix we have not listed the odd periods. For the case of an odd coloring this is of course consistent with Proposition 5.3 which gives the equality of the periods from Lemmas B.9 and B.10.

We also have the following vanishing result.

Lemma B.11. Let \( g \geq 2 \). Then 4 | \( [W_{\gamma,0}(z)]_{2k} \).

Proof. All monomials of the even graph potential \( W := W_{\gamma,0} \) have total degree 3 and −1, so \( W^k \) is homogeneous of degree −k ∈ \( \mathbb{Z}/4\mathbb{Z} \). On the other hand, the constant term \( z^0 \) is homogeneous of degree 0 ∈ \( \mathbb{Z}/4\mathbb{Z} \). \( \square \)
Finally we have the following general result, explaining why the lower triangular parts of the tables agree.

**Proposition B.12.** Let \( g \geq 2 \) and \( k < 2g - 2 \). Then the \( k \)th coefficient of the period of the graph potential for the even coloring equals that of the odd coloring.

Observe that if \( k \) is odd then we already knew that the \( k \)th period is zero in both cases.

**Proof.** By [11, Corollary 2.18] we can consider a single graph \( \gamma \) of genus \( g \), with an even (resp. odd) coloring. This gives the graph potentials \( W_0 \) (resp. \( W_1 \)). Moreover by [11, Corollary 2.9] we have that the action of \( C_1(\gamma, \mathbb{Z}/2\mathbb{Z}) \) preserves the graph potential up to biregular automorphism of the torus.

By assumption we have that \( k \) is strictly less than the number of vertices in the graph \( \gamma \). If \( m_1 \cdots m_k \) is a product of monomials (which can also be considered as a sum of vectors in \( N_\gamma \)) contributing to the constant term of \( W_1^k \), taken from the product expansion of \( W_1^k \), it cannot use monomials from every vertex of the graph \( \gamma \).

Let \( v \) be an unused vertex. By the action of \( C_1(\gamma, \mathbb{Z}/2\mathbb{Z}) \) we can assume that the colored vertex is \( v \), without changing the value of \( m_1 \cdots m_k \), as by assumption this is a constant and hence invariant under biregular automorphisms of the torus. Therefore this is the same contribution as obtained from \( W_0^k \), and we are done. \( \square \)

Combining this with Lemma B.11 we get

**Corollary B.13.** Let \( k \geq 0. \) Let \( g \geq 2 \). If \( 4k + 2 < 2g - 2 \) then the \( 4k + 2 \)th period of the graph potential in the odd case vanishes.
Table 2. Period sequence for the even graph potential

|   | 0   | 0   | 0   | 0   | 0   | 0   | 10  | 1  | 0   | 0   | 0   | 0   | 0   | 0   | 1728 | 95987176 | 0   |
|---|-----|-----|-----|-----|-----|-----|-----|---|----|-----|-----|-----|-----|-----|-----|-----|--------|-----|
|   | 0   | 0   | 0   | 0   | 0   | 0   | 1536 | 74995200 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 1440 | 2322432 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 22115260 | 768 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 10659793 | 88672 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 22115260 | 4096 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 8 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 960   | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 0   |

See also [11, Table 2].

Table 1. Period sequence for the odd graph potential

|   | 0   | 0   | 0   | 0   | 0   | 0   | 1728 | 95987176 | 0   |
|---|-----|-----|-----|-----|-----|-----|-----|--------|-----|
|   | 0   | 0   | 0   | 0   | 0   | 0   | 1536 | 74995200 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 1440 | 2322432 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 22115260 | 768 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 10659793 | 88672 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 22115260 | 4096 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 8 | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 960   | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 0   |
|   | 0   | 0   | 0   | 0   | 0   | 0   | 512   | 0   |

See also [11, Table 1].
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