Affine Circle Geometry over Quaternion Skew Fields

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Abstract

We investigate the affine circle geometry arising from a quaternion skew field and one of its maximal commutative subfields.

1 Introduction

1.1

The present paper is concerned with the chain geometry Σ(K,L) (cf. [1]) on a field extension L/K, where K is a maximal commutative subfield of a quaternion skew field L. Thus L is not a K–algebra. This has many geometric consequences. Best known is probably that three distinct points do not determine a unique chain. As in ordinary Möbius–geometry, it is possible to obtain an affine plane by deleting one point, but a more sophisticated technique is necessary in order to define the lines of this plane. We take a closer look on this construction from two different points of view, starting either from a spread of lines associated to Σ(K,L) or the point model of this spread on the Klein quadric. The chains of Σ(K,L) yield the lines, degenerate circles and non–degenerate circles of such an affine plane. We establish some properties of these circles and show that degenerate circles are affine Baer subplanes. If K is Galois over the centre of L then each non–degenerate circle can be written as intersection of two affine Hermitian varieties.

We encourage the reader to compare our results with the survey article [9] on chain geometry over an algebra and [10]. There is an extensive literature on the real quaternions. A lot of references can be found, e.g., in [1, 2, 3, 15, 16].

1.2

Throughout this paper L will denote a quaternion skew field with centre Z and K will be a maximal commutative subfield of L. The following exposition follows [4, 8, 13, pp.168–171].

Choose any element a ∈ K \ Z with minimal equation, say

\[ a^2 + a\lambda_1 + \mu_1 = 0 \quad (\lambda_1, \mu_1 \in Z) \]

If K/Z is Galois then

\[ (\xi, \eta) \in Z \]

With the following definition:

\[ (\xi + a\eta) \rightarrow (\xi - (\lambda_1 + a)\eta) \]
is an automorphism of order 2 fixing $Z$ elementwise\footnote{By an appropriate choice of $a$ it would be possible to have $\lambda_1 = 0$ (Char$K \neq 2$) or $\lambda_1 = 1$ (Char$K = 2$).}. There exists an element $i \in L \setminus K$ such that 

\[ i^{-1}ui = \overline{\eta} \text{ for all } u \in K, \]

whence

\[ ui = i\overline{\eta} \text{ for all } u \in K. \tag{1} \]

If $K/Z$ is not Galois then, obviously, Char$K = 2$ and $\lambda_1 = 0$. The mapping

\[ D : K \to K, \quad u = \xi + a\eta \mapsto u^D := a\eta \quad (\xi, \eta \in Z) \]

is additive and satisfies $(uu')^D = u^D u' + uu'^D$ for all $u, u' \in K$, i.e., $D$ is a derivation of $K$. There exists an $i \in L \setminus K$ such that 

\[ a^{-1}ia = i + 1 \]

which leads to the rule

\[ ui = iu + u^D \text{ for all } u \in K. \tag{2} \]

In every case the element $i$ has a minimal equation over $Z$, say

\[ i^2 + i\lambda_2 + \mu_2 = 0 \quad (\lambda_2, \mu_2 \in Z). \]

If $K/Z$ is Galois then $i^2 \in Z$, whence $\lambda_2 = 0$. If $K/Z$ is not Galois then $i$ and $i + 1$ have the same minimal equation. This implies $\lambda_2 = 1$. The mapping

\[ A : L \to L, \quad u + iv \mapsto \begin{cases} u - iv : K/Z \text{ Galois}, \\ u + v + vi : K/Z \text{ not Galois}, \end{cases} \quad (u, v \in K) \tag{3} \]

is an involutory antiautomorphism of $L$ fixing $K$. The norm of $x \in L$ is given by $N(x) := x^Ax$.

1.3

The mappings (\(\cap\)) and $D$ allow, respectively, the following geometric interpretations:

Let $V$ be a right vector space over $Z$, dim $V \geq 2$. We are extending $V$ to $V \otimes_Z K$ with $v \in V$ to be identified with $v \otimes 1$. Then define a mapping $V \otimes_Z K \to V \otimes_Z K$ by

\[ \sum_{v \in V} v \otimes k_v \mapsto \begin{cases} \sum_{v \in V} v \otimes \overline{\eta}_v : K/Z \text{ Galois}, \\ \sum_{v \in V} v \otimes k_v^D : K/Z \text{ not Galois}, \end{cases} \quad (k_v \in K). \]

By abuse of notation, this mapping will also be written as (\(\cap\)) and $D$, respectively.

In terms of the projective spaces $P_Z(V)$ and $P_K(V \otimes_Z K)$ the first projective space is being embedded in the second one as a Baer subspace. If $xK$ is a point of $P_K(V \otimes_Z K) \setminus P_Z(V)$ then through this point there is a unique line of $P_K(V \otimes_Z K)$ containing more than one point of $P_Z(V)$. That line is given by 

\[ xK \vee \overline{\eta}K \text{ and } xK \vee (x^D)K, \]
respectively. Note that defining a mapping by setting \( xK \mapsto (x^D)K \) is ambiguous, since

\[(xu)^D = x^Du + xu^D \quad \text{for all } x \in V \otimes_{Z} K, u \in K.\]

We give a second interpretation in terms of affine planes.

**Lemma 1** Let \( W \) be a right vector space over \( K \), \( \dim W = 2 \), and let \( \{u, v\} \) be a basis of \( W \). Then

\[
\{uk + v\bar{k} | k \in K\} : K/Z \text{ Galois,} \\
\{uk + vk^D | k \in K\} : K/Z \text{ not Galois,}
\]

is an affine Baer subplane (over \( Z \)) of the affine plane on \( W \).

**Proof.** If \( u' \) and \( v' \) are linearly independent vectors of \( W \) then the set of all linear combinations of \( u' \) and \( v' \) with coefficients in \( Z \) is an affine Baer subplane over \( Z \). Write \( k = \xi + a\eta \) with \( \xi, \eta \in Z \).

If \( K/Z \) is Galois then

\[
u k + v\bar{k} = (u + v)\xi + (ua - v(\lambda_1 + a))\eta.
\]

The vectors \( u + v \) and \( ua - v(\lambda_1 + a) \) are linearly independent, since otherwise we would have the contradiction \( \pi = -\lambda_1 - a = a \).

If \( K/Z \) is not Galois then

\[
u k + vk^D = u\xi + (u + v)a\eta.
\]

The vectors \( u \) and \( (u + v)a \) are linearly independent. \( \blacksquare \)

## 2 Projective Chain Geometry on \( L/K \)

### 2.1

Let \( L/K \) be given as before. Following [I p.320ff.] we obtain an incidence structure \( \Sigma(K, L) \) as follows: The points of \( \Sigma(K, L) \) are the points of the projective line over \( L \), viz. \( \mathcal{P}_L(L^2) \), the blocks, now called chains, are the \( K \)-sublines of \( \mathcal{P}_L(L^2) \). However, in contrast to [I], we shall regard \( L^2 \) as right vector space over \( K \) rather than \( L \). Each \( s \)-dimensional subspace of \( L^2 \) (over \( L \)) is \( 2s \)-dimensional over \( K \), whence \( \mathcal{P}_K(L^2) =: \mathcal{P}_K \) is 3-dimensional. The points of \( \Sigma(K, L) \) now appear as lines of a spread of \( \mathcal{P}_K \), say \( S_{L/K} \); cf. [I]. If \( t \) is a line of \( \mathcal{P}_K \) not contained in \( S_{L/K} \) then through each point of \( t \) there goes exactly one line of \( S_{L/K} \). The subset \( C \) of \( S_{L/K} \) arising in this way is a chain of \( \Sigma(K, L) \). We call \( t \) a transversal line of the chain \( C \). If \( L/K \) is not Galois then each chain has exactly one transversal line, otherwise exactly two transversal lines that are interchanged under the non-projective collineation

\[\iota: \mathcal{P}_K \rightarrow \mathcal{P}_K, \quad (l_0, l_1)K \mapsto (l_0i, l_1i)K.\]

Cf. [II, Theorem 2], [III].

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2 At least in the first case this is very well known.

3 Cf. the concept of ‘Minimalkoordinaten’ described, e.g., in [II] p.35
2.2

Write \( \mathcal{L} \) for the set of lines of \( \mathcal{P}_K \) and \( \gamma : \mathcal{L} \rightarrow \hat{\mathcal{P}}_K \) for the Klein mapping. Here \( \hat{\mathcal{P}}_K \) is the ambient space of the Klein quadric \( \mathcal{Q} := \mathcal{L}^\gamma \). The underlying vector space of \( \hat{\mathcal{P}}_K \) is \( L^2 \wedge L^2 \) (over \( K \)). In [8, Theorem 1] it is shown that there is a unique 5–dimensional Baer subspace \( \Pi_Z \) (over \( Z \)) of \( \hat{\mathcal{P}}_K \) such that

\[
\mathcal{S}_{L/K} = \Pi_Z \cap \mathcal{Q}.
\]

With respect to \( \Pi_Z \) the set \( \mathcal{S}_{L/K} \) is an oval quadric, i.e. a quadric without lines. A subset \( C \) of \( \mathcal{S}_{L/K} \) is a chain if, and only if, there exists a 3–dimensional subspace \( X \) of \( \hat{\mathcal{P}}_K \) such that

\[
X \cap \Pi_Z \text{ is an elliptic quadric of } X \cap \Pi_Z \text{ (over } Z),
\]

\[
X \cap Q \text{ contains a line of } \hat{\mathcal{P}}_K; \tag{7}
\]

\[
f X \cap \Pi_Z \text{ is in } \Pi_Z \text{ pointwise. See [8, Theorem 4].}
\]

2.3

The automorphism group of \( \Sigma(K, L) \) is formed by all bijections of \( \mathcal{S}_{L/K} \) taking chains to chains in both directions. If \( \kappa \) is a collineation or a duality of \( \mathcal{P}_K \) with \( \mathcal{S}_{L/K}^\kappa = \mathcal{S}_{L/K} \) then \( \kappa \) is yielding an automorphism of \( \Sigma(K, L) \). Conversely, according to [12] and [8, Theorem 4], each automorphism of \( \Sigma(K, L) \) can be induced by an automorphic collineation or duality of \( \mathcal{S}_{L/K} \), say \( \kappa \). This \( \kappa \) is uniquely determined for \( K/Z \) not being Galois, otherwise the product of \( \iota \) (cf. formula (5)) and \( \kappa \) is the only other solution.

Transferring these results to \( \hat{\mathcal{P}}_K \) establishes that an automorphic collineation \( \mu \) of the Klein quadric is the \( \gamma \)–transform of an automorphic collineation or duality of \( \mathcal{S}_{L/K} \) if, and only if, \( \Pi_Z \) is invariant under \( \mu \). If \( K/Z \) is Galois, then the \( \gamma \)–transform of the collineation \( \iota \) (cf. (5)) is the Baer involution of \( \hat{\mathcal{P}}_K \) fixing \( \Pi_Z \) pointwise. See [8, Theorem 4].

2.4

Let \( C_0 \) and \( C_1 \) be two chains with a common element, say \( p \in \mathcal{S}_{L/K} \). We say that \( C_0 \) is tangent to \( C_1 \) at \( p \) if there exist transversal lines \( t_i \) of \( C_i \) \( (i = 0, 1) \) such that \( p, t_0, t_1 \) are in one pencil of lines. This is a reflexive and symmetric relation.

If \( K/Z \) is Galois then there is also an orthogonality relation on the set of chains: If \( C_i \) \( (i = 0, 1) \) are chains with transversal lines \( t_i, t_i^\iota \), respectively, then \( C_0 \) is said to be orthogonal to \( C_1 \) if \( t_0 \) intersects both \( t_1 \) and \( t_1^\iota \). This relation is symmetric, since \( \iota \) is an involution. Given two orthogonal chains their transversal lines form a skew quadrilateral.

The two definitions above are not given in an intrinsic way. However, both relations are invariant under automorphic collineations and dualities of \( \mathcal{S}_{L/K} \) and hence invariant under automorphisms of \( \Sigma(K, L) \).

4If \( L \) is the skew field of real quaternions then \( K \) is a field of complex numbers and \( Z \) the field of real numbers. Here conditions (6) and (7) are already sufficient to characterize the \( \gamma \)–images of chains.
The proofs of the following results are left to the reader: Chains \( C_0, C_1 \) are tangent at \( p \in C_0 \cap C_1 \) if, and only if, their images under the Klein mapping are quadrics with the same tangent plane at the point \( p^\gamma \). A chain \( C_0 \) is orthogonal to a chain \( C_1 \) if, and only if, the subspace of \( \mathcal{P}_K \) spanned by \( C_0^\gamma \) contains the orthogonal subspace (with respect to the Klein quadric) of \( C_1^\gamma \).

3 Affine Circle Geometry on \( L/K \)

3.1 With the notations introduced in section 2, select one line of \( S_{L/K} \) and label it \( \infty \). Let \( \tilde{A} \) be a (projective) plane of \( \mathcal{P}_K \) through \( \infty \) and write \( A := \tilde{A} \setminus \infty \). Then \( A \) can be viewed as an affine plane with \( \infty \) as line at infinity. The mapping

\[
\rho : S_{L/K} \setminus \{\infty\} \to A, \quad s \mapsto A \cap s
\]

is well-defined and bijective. A chain \( C \) containing \( \infty \) yields an affine line \((C \setminus \{\infty\})^\rho \) if, and only if, \( C \) has a transversal line in \( \tilde{A} \). Two chains with transversal lines in \( \tilde{A} \) yield parallel lines if, and only if, the chains are tangent at \( \infty \).

If \( \tilde{A}' \) is any plane through \( \infty \) then, with \( A' := \tilde{A}' \setminus \infty \), the mapping

\[
\beta : A \to A', \quad A \cap s \mapsto A' \cap s \quad (s \in S_{L/K} \setminus \{\infty\})
\]

is a well-defined bijection. This \( \beta \) is an affinity if either \( \tilde{A}' = \tilde{A} \) or \( \tilde{A}' = \tilde{A} \) [5]; the second alternative is only possible when \( K/Z \) is Galois.

3.2 The group of automorphic collineations of \( S_{L/K} \) operates 3-fold transitively on the lines of \( S_{L/K} \) [11 p.322]. Thus we may transfer \( \infty \) to the line given by \((0,1)L\). Moreover, for all \( c \in L, c \neq 0 \)

\[
(l_0, l_1)K \mapsto (cl_0, cl_1)K \quad ((0,0) \neq (l_0, l_1) \in L^2)
\]

is an automorphic collineation of \( S_{L/K} \) fixing \( \infty \). Hence, without loss of generality, we may assume in the sequel that

\[
\infty = \mathcal{P}_K((0,1)L) \text{ and } \tilde{A} = (1,0)K \lor \infty.
\]

Then the mapping \( \rho \) becomes

\[
\mathcal{P}_K((l_0, l_1)L) \mapsto (1, l_1 l_0^{-1})K.
\]

We shall identify \( A \) with \( L \) via \( (1, l)K \equiv l. \) Thus \( L \) gets the structure of an affine plane over \( K. \) We shall emphasize this by writing \( A_K(L) \) rather than \( L. \)

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5One could also select some point \( A \in \infty \) and then obtain an affine plane by a dual construction.

6This is accordance with the inhomogeneous notation used in [4].
Theorem 1  Let $\kappa$ be an automorphic collineation or duality of $\mathcal{S}_{L/K}$ fixing $\infty$. Then there exist elements $m_0, m_1, m \in L, m_0, m_1 \neq 0$ and an automorphism or antiautomorphism $J$ of $L$ with $K^J = K$ such that
\[ x_{\rho^{-1}\kappa} = m_1 x^J m_0 + m \quad \text{for all } x \in L. \] (11)

The additional conditions
\[ J \text{ is an automorphism of } L, \] (12)
\[ m_0 \in K \text{ or, only if } K/Z \text{ is Galois, } m_0^{-1} \in K \] (13)

are necessary and sufficient for $\rho^{-1}\kappa$ to be an affinity of $A_K(L)$. 

Proof. The assertion in formula (11) is obviously true.

Now suppose that $\rho^{-1}\kappa$ is an affinity of $A_K(L)$. Then $\kappa$ has to take each chain with a transversal line in $\tilde{A}$ to a chain with a transversal line in $\tilde{A}$. Hence $\tilde{A}^c = \tilde{A}$ or, only if $K/Z$ is Galois, $\tilde{A}^c = \tilde{A}^\delta$. Therefore $\kappa$ cannot be a duality, so that $J$ cannot be an antiautomorphism [8, Theorem 4]. Consequently, $g : x \mapsto m_1 x^J m_0$ has to be a semilinear mapping of the right vector space $L$ over $K$. We infer from
\[ x \mapsto (m_1 x^J m_0)(m_0^{-1} k^J m_0) \quad \text{for all } x \in L, k \in K \]
that $m_0^{-1} K m_0 = K$. There are two possibilities: If
\[ m_0^{-1} k m_0 = k \quad \text{for all } k \in K \]
then $m_0$ is a non–zero element of $K$, since $K$ is a maximal commutative subfield of $L$. On the other hand, however only if $K/Z$ is Galois, also
\[ m_0^{-1} k m_0 = \overline{k} \quad \text{for all } k \in K \]
is possible. Now, again using that $K$ is maximal commutative, it follows from (11) that $m_0^{-1} \in K$.

The proof of the converse is a straightforward calculation. ■

3.3

If $C$ is a chain such that $(C \setminus \{\infty\})^\rho$ is not a line of $A_K(L)$ then $(C \setminus \{\infty\})^\rho$ will be named a circle. There are two kinds of circles: If $\infty \in C$ then the circle is called degenerate, otherwise non–degenerate. The following Lemma shows that distinct chains cannot define the same circle. In addition it establishes that a circle cannot be degenerate and non–degenerate at the same time:

Lemma 2  Let $C_0$ and $C_1$ be two chains such that $C_0 \setminus \{\infty\} = C_1 \setminus \{\infty\}$. Then $C_0 = C_1$.

Proof. According to [4, 7, 5] there exists a 3–dimensional subspace $X_0$ of $\tilde{P}_K$ with
\[ C_0^\gamma = X_0 \cap \Pi_Z \cap \mathcal{Q}. \]
Since $C_0^\gamma$ is an oval quadric of $X_0 \cap \Pi_Z$ and $Z$ is infinite, $(C_0 \setminus \{\infty\})^\gamma$ is still spanning $X_0$. Repeating this, mutatis mutandis, for $C_1$ gives $X_0 = X_1$, whence $C_0 = C_1$, as required. ■
3.4

By Lemma 2 we may unambiguously speak of a line being \textit{tangent} to a circle at some point \( P \in A_K(L) \) or of circles touching at \( P \) if they arise from chains that are tangent at \( P^{ho^{-1}} \).

A degenerate circle has no tangent lines. A point \( P \) of a non–degenerate circle is called \textit{regular} if there exists a tangent line of that circle at \( P \). If such a circle is given as \( C^\rho, \mathcal{C} \) a chain, then \( P \in C^\rho \) is regular if, and only if, \( P \) (regarded as point of \( A \)) is incident with a transversal line of \( C \). Thus a non–degenerate circle has either one or two regular points.

3.5

If \( K/Z \) is Galois then call two lines, or a circle and a line, or two circles of \( A_K(L) \) \textit{orthogonal} if they arise from orthogonal chains.

By virtue of the collineation \( \iota \) (cf. formula (5)), a line \( lK + m \ (l, m \in L, \ l \neq 0) \) is orthogonal to all lines being parallel to \( liK \).

We introduce a unitary scalar product \( * \) on the right vector space \( L \) over \( K \) by setting

\[
(u + iv) \ast (u' + iv') := uu' + \mu_2 vv' \quad \text{for all} \ u, u', v, v' \in K. \tag{14}
\]

This scalar product is describing the orthogonality relation on lines from above. Moreover, \((u + iv) \ast (u + iv) = N(u + iv)\), whence the norm is a Hermitian form \( \text{7} \) on \( L \) over \( K \).

It is easily seen that there exists no line orthogonal to a degenerate circle. The join of the two regular points of a non–degenerate circle is the only line being orthogonal to that circle. It will be called the \textit{midline} of the circle. The midline is orthogonal to both tangent lines.

All affinities described in Theorem 1 are preserving orthogonality.

3.6

Let \( \mathcal{C} \) be a chain such that \( \Delta := (\mathcal{C} \setminus \{\infty\})^\rho \) is a degenerate circle. Then either there are two points or there is one point on the line \( \infty \) incident with transversal lines of \( \mathcal{C} \). We call these points at infinity of \( A_K(L) \) the \textit{absolute points} or the \textit{absolute directions} of \( \Delta \). This terminology will be motivated in 3.10.

The group \( \text{AGL}(1, L) \) of all transformations \( (11) \) with \( m_0 = 1 \) operates sharply 2-fold transitively on \( A_K(L) \). Thus each degenerate circle can be transferred under \( \text{AGL}(1, L) \) to a degenerate circle through 0 and 1. Write

\[
L^\circ := \left\{ \begin{array}{ll}
L \setminus (K \cup Ki) & : K/Z \text{ Galois}; \\
L^\rho := L \setminus K & : K/Z \text{ not Galois}.
\end{array} \right.
\]

Then, by \[ \text{II} \text{ p.329] and (13), the degenerate circles through 0 and 1 are exactly the sets}

\[ eKc^{-1} \quad \text{with} \ c \in L^\circ. \] (15)

From now on assume that a degenerate circle \( \Delta \) is given by \( (15) \). Let \( \mathcal{C} \) be the chain with transversal line \( (c, 0)K \lor (0, c)K \). Then \( \Delta = (\mathcal{C} \setminus \{\infty\})^\rho \), whence

\[ \text{7If} \ K/Z \text{ is not Galois then the norm does not seem to be a quadratic or Hermitian form on} \ L \text{ over} \ K. \]
cK is an absolute direction of $\Delta$. Each affinity of $AGL(1, L)$ (cf. formula (11)) with $m_1, m \in cK^{-1}$ ($m_1 \neq 0, m_0 = 1$ as before) takes $\Delta$ onto $\Delta$.

**Theorem 2** Each degenerate circle of $A_K(L)$ is an affine Baer subplane of $A_K(L)$ with the centre of $L$ as underlying field.

**Proof.** It is sufficient to show this for a degenerate circle given by (15). Set $c^{-1} =: d + ie$ with $d, e \in K$. Then, by (1) and (2),

$$cKc^{-1} = \left\{ \left\{ (cd + k)K \mid k \in K \right\} : K/Z \text{ Galois}, \right. \left\{ (k + ce)K \mid k \in K \right\} : K/Z \text{ not Galois}. \right.$$}

Now the assertion follows by Lemma 1. □

### 3.7

Next we turn to non–degenerate circles.

**Theorem 3** All non–degenerate circles of the affine plane $A_K(L)$ are in one orbit of $AGL(1, L)$.

**Proof.** Let $C_0$ be the chain with transversal line

$$(1, 0)K \lor (i, i)K.$$ (16)

Then $\Gamma_0 := C_0\rho$ is a non–degenerate circle with regular point 0.

Let $K/Z$ be Galois. Then 1 is the other regular point of $\Gamma_0$. If $\Gamma_1$ is a non–degenerate circle then there exists an affinity $\alpha \in AGL(1, L)$ taking the regular points of $\Gamma_1$ to 0 and 1, respectively. Hence $\Gamma_1\rho^{-1}$ is a chain with one transversal line through $(1, 0)K$ and the other transversal line through $(1, 1)K$. Applying the collineation $\iota$ on $(1, 1)K$ establishes that (16) is a transversal line of this chain, whence $\Gamma_0 = \Gamma_1\alpha$.

Now assume that $K/Z$ is not Galois. If $\Gamma_1$ is a non–degenerate circle then there exists an affinity $\alpha \in AGL(1, L)$ taking the only regular point of $\Gamma_1$ to 0. The chain $\Gamma_1\rho^{-1}$ has a unique transversal line through $(1, 0)K$ and some point of the plane $(i, 0)K \lor \infty$, say

$$(id, e + if)K \quad \text{with } d, e, f \in K, \quad d, e + if \neq 0.$$ Then there exists an element $m_1 \in L \setminus \{0\}$ such that $m_1(e + if) = id$. The collineation $\kappa$ of $P_K$ given by $(l_0, l_1)K \mapsto (l_0, m_1l_1)K$ leaves $S_{L/K}$ invariant, fixes the point $(1, 0)K$ as well as the line $\infty$ and takes $(id, e + if)K$ to $(i, i)K$. Hence the induced affinity $\rho^{-1}\kappa\rho$ of $A_K(L)$ carries $\Gamma_1\alpha$ over to $\Gamma_0$. □

### 3.8

The non–degenerate circle $\Gamma_0$ arising from the chain $C_0$ with transversal line (16) has the parametric representation

$$\{ik_1(k_0 + ik_1)^{-1} \mid (0, 0) \neq (k_0, k_1) \in K^2\};$$ (17)

cf. also [1] Satz 3.2. Next we establish an equation for $\Gamma_0$:
Theorem 4  The non–degenerate circle $\Gamma_0$ given by (17) equals the set of all points $u + iv$ $(u, v \in K)$ satisfying

$$u = N(u + iv).$$  \hspace{1cm} (18)

Proof. The term $ik_1(k_0 + ik_1)^{-1}$ in formula (17) can be rewritten as follows: If $K/Z$ is Galois then

$$ik_1(k_0 + ik_1)^{-1} = ik_1(k_0 - ik_1) \left( (k_0 + ik_1)(k_0 - ik_1) \right)^{-1} = (\mu_2 k_1 k_0 + \mu_2 k_1 k_1)^{-1},$$

otherwise

$$ik_1(k_0 + ik_1)^{-1} = ik_1(k_0 + k_1 + k_1 i) \left( (k_0 + ik_1)(k_0 + k_1 + k_1 i) \right)^{-1} = (\mu_2 k_1^2 + ik_0 k_1) \left( k_0^2 + k_0 k_1 + (k_0 k_1)^D + \mu_2 k_1^2 \right)^{-1}.$$

Now, since $N(u + iv) = \begin{cases} \frac{u + \mu_2 iv}{u^2 + uv + (uv)^D + \mu_2 v^2} : & K/Z \text{ Galois,} \\ u^2 + uv + (uv)^D + \mu_2 v^2 : & K/Z \text{ not Galois,} \end{cases}$

it is easily seen that all points of $\Gamma_0$ are satisfying equation (18).

Conversely, let $q + ir$ $(q, r \in K)$ be a solution of (18). If $q = 0$ then $r = 0$, whence we have a point of $\Gamma_0$. Otherwise set

$$k_0 := \begin{cases} \mu_2 r^{-1} : & K/Z \text{ Galois,} \\ \mu_2 r^{-1} : & K/Z \text{ not Galois,} \end{cases} \text{ and } k_1 := 1.$$

The point of $\Gamma_0$ with these parameters equals $q + ir.$ \hfill \blacksquare

3.9

We are able to say a little bit more about non–degenerate circles provided that $K/Z$ is Galois. Formula (18) becomes

$$N(u + iv) - u = (u - 1 + iv) * (u + iv) = 0.$$  \hspace{1cm} (19)

Thus, if we intersect each line through 0 with its orthogonal line through 1 then the set of all such points of intersection equals $\Gamma_0$. This is a nice analogon to a well–known property of opposite points on a Euclidean circle.

Theorem 5  Let $K/Z$ be Galois. Write $E := \{ y \in K \mid y + \overline{y} = 1 \}$ and $\mathcal{H}_e \ (e \in E)$ for the affine Hermitian variety formed by all points $u + iv$ $(u, v \in K)$ subject to the equation

$$N(u + iv) = eu + \overline{ea}.$$  

Then the non–degenerate circle $\Gamma_0$ given by (17) can be written as

$$\Gamma_0 = \mathcal{H}_e \cap \mathcal{H}_f \text{ for all } e, f \in E \text{ with } e \neq f.$$  \hspace{1cm} (20)

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8In the elementary plane of complex numbers the same kind of equation gives a circle through 0 and 1.

9The points 0 and 1 are, however, the only points of $\Gamma_0$ with this property.
Proof. A straightforward calculation yields
\[ E = \begin{cases} \frac{1}{2} + (\lambda_1 + 2a)Z : \text{Char}K \neq 2, \\ a\lambda_1^{-1} + Z : \text{Char}K = 2, \end{cases} \]
whence \( E \) is infinite. Given \( q + ir \in \Gamma_0 \ (q, r \in K) \) then \( q \in Z \) implies
\[ \Gamma_0 \subset \bigcap_{e \in E} H_e. \]
Choose distinct elements \( e, f \in E \) and \( q + ir \in H_e \cap H_f \ (q, r \in K) \). Then
\[ N(q + ir) - N(q + ir) = eq + eq -fq -fq = 0. \]
But
\[ \frac{e - f}{f - e} = 1, \]
so that \( q = \overline{q} \) and therefore \( q + ir \in \Gamma_0 \). \( \blacksquare \)

### 3.10

There is an alternative approach to \( A_K(L) \) via the point model of \( \Sigma(K, L) \) on the Klein quadric \( Q \).

Write \( I := \infty \) and \( Z \) for the \( \gamma \)–image of the ruled plane on \( \tilde{A} \); this \( Z \) is a plane on the Klein quadric. Furthermore let \( \tilde{F} \) be any plane of \( \tilde{P}_K \) skew to \( Z \) and write
\[ \pi : \tilde{P}_K \setminus Z \rightarrow \tilde{F} \]  \hspace{1cm} \text{(21)}
for the projection with centre \( Z \) onto the plane \( \tilde{F} \). It is well known from descriptive line geometry that there exists a collineation \( \psi \) of \( \tilde{A} \) onto \( \tilde{F} \) such that
\[ (p \cap \tilde{A})^\psi = p^\pi \]
for all lines \( p \) of \( \tilde{P}_K \) not contained in \( \tilde{A} \). Cf., e.g., [3]. We turn \( \tilde{F} \) into an affine plane \( F \), say, by regarding \( \tilde{F} \cap I^\perp \) as its line at infinity; here \( I^\perp \) denotes the tangent hyperplane of the Klein quadric at \( I \). Then \( \infty^\psi = F \cap I^\perp \).

The bijectivity of \( \rho \) implies that \( S_{L/K}^\gamma \setminus \{I\} \) is mapped bijectively under \( \pi \) onto the affine plane \( F \). The restriction
\[ \pi | S_{L/K}^\gamma \setminus \{I\} \]
can be seen as a **generalized stereographic projection** of the oval quadric \( S_{L/K}^\gamma \) of \( \Pi_Z \) onto the affine plane \( F \).

Let \( C \) be a chain. Then \( C^\gamma = X \cap Q \cap \Pi_Z \) for some 3-dimensional subspace \( X \) of \( \tilde{P}_K \). We leave it to the reader to show that \( (C \setminus \{\infty\})^\gamma \) is an affine line if \( X \cap Z \) is a line through \( I \), a degenerate circle if \( X \cap Z = \{I\} \) and a non–degenerate circle if \( X \cap Z \) is some point other than \( I \).

Using the mapping \( \gamma \pi \psi^{-1} \) instead of \( \rho \) is very convenient to establish results on the images of traces [11 p.327], since their \( \gamma \)–images are just the regular conics on \( S_{L/K}^\gamma \) [3, 3.4]. We sketch just one result without proof:

10A ‘usual’ stereographic projection would map onto a 4–dimensional affine space over \( Z \) rather than an affine plane over \( K \).
Let $C$ be a chain through $\infty$ such that $(C \setminus \{\infty\})^\rho =: \Delta$ is a degenerate circle of $A_K(L)$. Then the $\rho$-images of traces in $C$ are on one hand the lines of the affine plane $\Delta$ and on the other hand certain ellipses of $\Delta$. If these ellipses are extended to conics of $A_K(L)$ then the absolute directions of $\Delta$ determine their points at infinity. This is the well-known concept of absolute circular points. $\Delta$ is a Euclidean plane representing the extension $K/Z$. Cf. [14].

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11There is only one such point if $K/Z$ is not Galois.
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