Geometric Study of Gas Behavior in a One-Dimensional Nozzle 
(the Case of the van Der Waals Gas)

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Abstract—We construct a three-component system of PDEs describing dynamics of van der Waals gas in one-dimensional nozzle. The group of conservation laws for this system is described. We also compute the Lie algebras of point symmetries and present group classification. Examples of exact invariant solutions are given.

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INTRODUCTION

Methods of differential geometry applied to the problems arising in analysis of partial differential equations (PDEs) allow one to construct a lot of important invariants, such as, e.g., symmetries and conservation laws (see [2,4,6]). In turn, these invariants may be used for qualitative study of equations, their classification, construction of exact solutions, etc.

In what follows, we use these methods to study gas behavior in a one-dimensional nozzle. In Section 1, we construct the model and derive the equations for the case of the van der Waals gas. Section 2 deals with basic constructions from the geometry of PDEs necessary for the exposition. Conservation laws of the obtained system are described in Section 3, while Section 4 deals with the computation of symmetries and classification of the equations at hand based on their symmetry algebras. Finally, we discuss invariant solutions in Section 5.

1. THE MODEL

We consider a gas flow in a wind tunnel (nozzle) of a variable radius. Assume that the $x$-axis is the symmetry axis of the tunnel, $A(x)$ is the area of section at the point $x$ and $(y, z)$ components of the gas flow velocity are negligibly small in comparison with the $x$-component $u(x, t)$. Then the Euler equations of the momentum conservation and the mass conservation laws take the form

$$\left( A\rho u \right)_t + \left( p + A\rho u^2 \right)_x = 0, \quad \left( A\rho \right)_t + \left( A\rho u \right)_x = 0,$$

where $\rho$ is the gas density and $p$ is the pressure. We also assume that the gas flow is adiabatic, i.e.,

$$s_t + u s_x = 0,$$

where $s(x, t)$ is the specific entropy.

The thermodynamic variables $s, v, e, p,$ and $T$ satisfy in addition the state equations

$$p = RT\Pi_v, \quad e = RT^2\Pi_T, \quad s = R(\Pi + T\Pi_T),$$

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where $v = \rho^{-1}$ is the specific volume, $e$ being the specific inner energy, $T$ temperature, $R$ the universal gas constant, while $\Pi = \Pi(v, T)$ is the Massieu–Plank potential (see, for example, [3, 7]).

Let us introduce potential functions $\varphi$ and $\psi$ for Equations (1):

$$A \rho u = -\varphi_x, \quad p + A \rho u^2 = \varphi_t; \quad A \rho = \psi_x, \quad A \rho u = -\psi_t.$$  

Then System (1) takes the form

$$\varphi_x - \psi_t = 0, \quad p = \varphi_t + \psi^2 \varphi_x^{-1}, \quad \rho = \psi_x A^{-1}, \quad u = -\psi_t \varphi_x^{-1}.$$  

Note the function $\psi(x, t)$ is the first integral of the vector field $\partial_t + u \partial_x$ and rewrite Equation (2) in the form

$$s = S(\psi),$$

for some function $S$. In order to eliminate the unknown function (3), we use Equation (2) in the form

$$s_v(v_t + uv_x) + s_T(T_t + uT_x) = 0,$$

assuming that the function $s = s(v, T)$ is given by the state equations. Then our system takes the form

$$\varphi_t = p - \varphi^2 \varphi_x^{-1}, \quad \psi_t = \varphi_x, \quad T_t = -uT_x - \frac{s_v}{s_T}(v_t + uv_x),$$

where $v = A\psi_x^{-1}, u = -\varphi_x \psi_x^{-1}$.

In the case of ideal gas flows we have the following state equations

$$p = Rv^{-1}T, \quad e = \frac{n}{2}RT, \quad s = \ln \left(\frac{T^{n/2}}{v}\right) + \text{const},$$

where $n$ is the degree of freedom. For the van der Waals gas the state equations read (see [5])

$$p = \frac{RTv^2 - a(v - b)}{v^2(v - b)}, \quad e = \frac{nRT}{2} - \frac{a}{b}, \quad s = \ln \left(\frac{T^{4n/3}(v - b)^{8/3}}{v}\right) + \text{const}.$$  

The constant $a$ takes into account the intermolecular forces and the constant $b$ is the molecular volume, while $n$ is the degree of freedom. As the result, we arrive to the following system of evolutionary equations

$$\varphi_t = -\varphi_x(aA\varphi_x + ab\varphi_x^2 + A^3 \varphi_x^2 + bA^2 \varphi_x^2 \varphi_x + R\tau A^2), \quad \psi_t = -\varphi_x,$$

$$\tau_t = -\frac{A(2\tau \varphi_x \varphi_{xx} + n\varphi_x \tau_x \varphi_x - 2\tau \varphi_x \varphi_{xx}) + 2\tau A \varphi_x \varphi_x \varphi_x + nb \varphi_x \tau_x \varphi_x^2}{\psi_x^2(A + b\psi_x)},$$

where the temperature $T$ is relabeled to $\tau$ and $\psi$ to $-\psi$. We study this system in the forthcoming sections. Note that the case $a, b = 0$ corresponds to the ideal gas.

2. PRELIMINARY FACTS AND NOTATION

We consider the space $J^{\infty}(2, 3)$ with the coordinates $x, t, \varphi_{kl}, \psi_{kl}, \tau_{kl}, k, l = 0, 1, \ldots$, where the subscript corresponds to the partial derivative $\partial^{k+l}/\partial x^k \partial t^l$, and the subspace $\mathcal{E} \subset J^{\infty}(2, 3)$ defined by Equations (4) and all their differential consequences. The functions $x, t, \varphi_k = \varphi_{k0}, \psi_k = \psi_{k0}$, and $\tau_k = \tau_{k0}$ may be chosen for internal coordinates in $\mathcal{E}$. Denote by $r_x$ and $r_\tau$ the right-hand sides of the first and third equations in (4), respectively. Then the total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} \left( \varphi_{k+1} \frac{\partial}{\partial \varphi_k} + \psi_{k+1} \frac{\partial}{\partial \psi_k} + \tau_{k+1} \frac{\partial}{\partial \tau_k} \right),$$

$$D_t = \frac{\partial}{\partial t} - \sum_{k \geq 0} \left( D_x(r_\varphi) \frac{\partial}{\partial \varphi_k} + \varphi_{k+1} \frac{\partial}{\partial \psi_k} + D_x(r_\tau) \frac{\partial}{\partial \tau_k} \right)$$

are vector fields on $\mathcal{E}$. For any function $Y$ on $\mathcal{E}$ define its linearization

$$\ell_Y(f, g, h) = \sum_{k \geq 0} \left( \frac{\partial Y}{\partial \varphi_k} D_x(f) + \frac{\partial Y}{\partial \psi_k} D_x(g) + \frac{\partial Y}{\partial \tau_k} D_x(h) \right).$$
A symmetry of $\mathcal{E}$ is a vector field that commutes with $D_x$ and $D_t$. Any symmetry is of the form

$$E_S = \sum_{k \geq 0} \left( D^k_x(S_\varphi) \frac{\partial}{\partial \varphi_k} + D^k_x(S_\psi) \frac{\partial}{\partial \psi_k} + D^k_x(S_\tau) \frac{\partial}{\partial \tau_k} \right),$$

where the vector-function $S = (\Phi, \Psi, \Theta)$ must satisfy the linear system

$$D_t(\Phi) = \ell_{r_\varphi}(\Phi, \Psi, \Theta), \quad D_t(\Psi) = -D_x(\Phi), \quad D_t(\Theta) = \ell_{r_\psi}(\Phi, \Psi, \Theta). \quad (5)$$

We identify symmetries with the corresponding functions $S$.

A conservation law of $\mathcal{E}$ is a differential form $\omega = X dx + T dt$ such that $D_t(X) = D_x(T)$. It is called trivial if there exists a potential $Z$ such that $X = D_x(Z)$, $T = D_t(Z)$. The generating function of $\omega$ is the triple $G = (G_\varphi, G_\psi, G_\tau)$, where

$$G_\varphi = \frac{\delta X}{\delta \varphi}, \quad G_\psi = \frac{\delta X}{\delta \psi}, \quad S_\tau = \frac{\delta X}{\delta \tau} \quad (6)$$

and

$$\frac{\delta X}{\delta \varphi} = \sum_{k \geq 0} (-1)^k D^k_x \left( \frac{\partial X}{\partial \varphi_k} \right),$$

etc., are the variational derivatives. To find generating functions, one needs to solve the system

$$D_t(G_\varphi) = \sum_{k \geq 0} (-1)^{k+1} \left( D^k_x \left( \frac{\partial G_\varphi}{\partial \varphi_k} \right) + D^k_x \left( \frac{\partial G_\psi}{\partial \psi_k} \right) + D^k_x \left( \frac{\partial G_\tau}{\partial \tau_k} \right) \right),$$

$$D_t(G_\psi) = \sum_{k \geq 0} (-1)^{k+1} \left( D^k_x \left( \frac{\partial G_\varphi}{\partial \psi_k} \right) + D^k_x \left( \frac{\partial G_\psi}{\partial \psi_k} \right) + D^k_x \left( \frac{\partial G_\tau}{\partial \tau_k} \right) \right),$$

$$D_t(G_\tau) = \sum_{k \geq 0} (-1)^{k+1} \left( D^k_x \left( \frac{\partial G_\varphi}{\partial \tau_k} \right) + D^k_x \left( \frac{\partial G_\psi}{\partial \tau_k} \right) + D^k_x \left( \frac{\partial G_\tau}{\partial \tau_k} \right) \right) \quad (7)$$

adjoint to (5). A conservation law is trivial if and only if its generating function vanishes.

### 3. Conservation Laws

Solving System (7) in the case when $G$ depends on $x$, $t$, $\varphi_k$, $\psi_k$, $\tau_k$, $k \leq 2$, we get the solutions $G^0 = (0, 1, 0)$, $G^\infty = (0, G_\psi, G_\tau)$, where

$$G_\psi = -\frac{1}{n \tau \psi_x^2 (b \psi_x + A)^2} \left( 2A^2 \tau \left( \psi_x H - \tau^2 \frac{\partial}{\partial y} \right) \psi_{xx} \right.$$

$$\left. + \psi_x^2 \left( 2\tau A \frac{\partial}{\partial x} + n \tau_x (b \psi_x + A)^2 \right) H - 2A \tau \psi_x^3 (b \psi_x + A) \frac{\partial}{\partial y} \right),$$

$$- A \tau^2 \psi_x (b \psi_x + A) \left( 2\tau A \frac{\partial}{\partial x} + n \tau_x (b \psi_x + A) \frac{\partial}{\partial y} \right),$$

$$G_\tau = \frac{H \psi_x}{\tau},$$

$H = H(y, \psi)$ is an arbitrary smooth function and $y = \frac{b \psi_x + A}{\psi_x}$. Using relations (6), one can reconstruct the corresponding conservation laws and obtain $\omega_0 = \psi dx + \varphi dt$, $\omega_H = \frac{2H \ln y}{n} (\psi_x dx - \varphi_x dt)$. 

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4. SYMMETRIES

We compute here point symmetries, i.e., such that the function $S$ depends on $x$, $t$ and $\varphi_k$, $\psi_k$, $\tau_k$ for $k\leq 2$. Note first that the second equation in (5) means that the form $\Psi dx - \Phi dt$ is a conservation law. Due to the results of Section 3, we have

$$
\Phi = -\lambda \varphi + \frac{2H \ln(y)}{n} \varphi_x - D_t(P), \quad \Psi = \lambda \psi + \frac{2H \ln(y)}{n} \psi_x + D_x(P),
$$

where $\lambda = \text{const}$ and $P = P(x, t, \varphi, \psi, \tau, \varphi_x, \psi_x, \tau_x)$ is an arbitrary smooth function. Denote by $\mathfrak{g}$ the Lie algebra of symmetries. The structure of $\mathfrak{g}$ depends on the values of the parameters that enter the basic equations.\(^{1)}\) Note that in all the cases below the algebra $\mathfrak{g}$ contains the 3-dimensional Abelian ideal $\mathfrak{g}_0$ spanned by the symmetries

$$
S_1 = (1, 0, 0), \quad S_2 = (0, 1, 0), \quad S_3 = (\varphi_t, \psi_t, \tau_t).
$$

Describing the algebras, we present only those generators that do not belong to $\mathfrak{g}_0$. In the description of Lie algebra structures we indicate nonzero commutators only.

4.1. $b = 0, a \neq 0$

There are eight subcases.

4.1.1. $A = p = \text{const}, n = 2$. The algebra $\mathfrak{g}$ is 6-dimensional. Its generators are

$$
\Phi_4 = \frac{\psi_2^2 at + pt(p \varphi_2^2 + R \tau) \psi_x - \varphi p^2}{p^2}, \quad \Psi_4 = t \varphi_x,
$$

$$
\Theta_4 = -\frac{\tau t \varphi_x \psi_{xx}}{\psi_x^2} + \frac{\tau t \varphi_{xx}}{\psi_x} + \frac{t \varphi_x \tau_x}{\psi_x} - 2\tau - \frac{2a \psi_x}{pR};
$$

$$
\Phi_5 = \varphi_x, \quad \Psi_5 = \psi_x, \quad \Theta_5 = \tau_x; \Phi_6 = x \varphi_x - 2 \varphi, \quad \Psi_6 = x \psi_x, \quad \Theta_6 = x \tau_x - 2\tau - \frac{2a \psi_x}{pR}
$$

with the commutators

$$
[S_1, S_4] = -S_1, \quad [S_1, S_6] = -2S_1, \quad [S_2, S_6] = -S_2, \quad [S_3, S_4] = S_3, \quad [S_5, S_6] = -S_5.
$$

4.1.2. $A = p = \text{const}, n \neq 2$. We have $\dim \mathfrak{g} = 5$ with the generators

$$
\Phi_4 = \frac{\psi_2^2 at + pt(p \varphi_2^2 + R \tau) \psi_x - \varphi p^2}{p^2} (\varphi x - \varphi), \quad \Psi_4 = t \varphi_x - \psi_x x + \psi,
$$

$$
\Theta_4 = -\frac{2\tau t \varphi_x \psi_{xx} + 2\tau t \varphi_{xx} \psi_x + (t \varphi_x - \psi_x x) \tau_x \psi_x n}{\psi_x^2 n}; \quad \Phi_5 = \varphi_x, \quad \Psi_5 = \psi_x, \quad \Theta_5 = \tau_x.
$$

The commutators are $[S_2, S_4] = S_2, [S_2, S_5] = 0, [S_3, S_4] = S_3, [S_4, S_5] = -S_5$.

4.1.3. $A = pe^{q x}, p, q \neq 0, n = 2$. One has $\dim \mathfrak{g} = 5$. The generators are

$$
\Phi_4 = \frac{e^{-2q x} \psi_2^2 at + e^{-q x} \psi_x R p t \tau + \psi_2^2 p^2 t - \varphi p^2}{p^2}, \quad \Psi_4 = t \varphi_x,
$$

$$
\Theta_4 = \frac{\tau t \varphi_x \psi_x p R - \tau \varphi_x \psi_x p R + \psi_x (-2 \psi_2^2 a e^{-q x} + (-2 \psi_x t + \varphi \psi \Psi (q t + \tau_x) p R)}{\psi_x^2 p R};
$$

$$
\Phi_5 = \varphi_x, \quad \Psi_5 = \psi_x, \quad \Theta_5 = \frac{-2 e^{-q x} q a \psi_x - p R (q t - \tau_x)}{p R}.
$$

The commutators read $[S_1, S_4] = -S_1, [S_3, S_4] = S_3$.

4.1.4. $A = pe^{q x}, p, q \neq 0, n \neq 2$. The algebra of symmetries is 4-dimensional with the generators

$$
\Phi_4 = \frac{e^{-2q x} \psi_2^2 a q t + e^{-q x} \psi_x R p q t + \psi_2^2 p^2 q t - \varphi p^2 q - \varphi_x p^2}{(p q)}, \quad \Psi_4 = t \varphi_x - \frac{\psi_x}{q},
$$

\(^{1)}\)The full classification includes the case $n = -2$, which is physically senseless and we omit it. So, everywhere below $n \neq -2$. 

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\[
\Theta_4 = \frac{2\tau \varphi_x \psi_x q - 2t \varphi_x \psi_x q + (-n(q\tau + \tau_x) \psi_x + qt \varphi_x(n\tau_x + 2q\tau))\psi_x}{n\psi^2 xq}.
\]

and commutators \([S_1, S_3] = -S_1, [S_3, S_4] = S_3\).

4.1.5. \(A = (px + r)^q, p, q \neq 0, n = 2\). The symmetry algebra is 5-dimensional and generated by

\[
\Phi_4 = \psi_x^2 (px + r)^{-a} p t + \psi_x (px + r)^{-q} R t + \varphi_x^2 \psi_x t - \varphi_x, \quad \Psi_4 = t \varphi_x,
\]

\[
\Theta_4 = \frac{-t \varphi_x \psi_x \psi_x + t \varphi_x \psi_x \psi_x + \frac{\varphi p t \varphi_x}{p} (px + r)^{-q} \psi_x - 2 \tau - 2(px + r)^{-q} a \psi_x}{R}
\]

\[
\Phi_5 = -2 \psi + \frac{(px + r) \varphi_x}{p}, \quad \Psi_5 = -\psi + \frac{(px + r) \varphi_x}{p},
\]

\[
\Theta_5 = \frac{(-2apq \psi_x - 2 \psi_x)(px + r)^{-q} a p + (\tau_x(px + r) - p\tau(q + 2)) R}{pR}
\]

with the commutators \([S_1, S_4] = -S_1, [S_1, S_5] = -2S_1, [S_2, S_5] = -S_2, [S_3, S_4] = S_3\).

4.1.6. \(A = (px + r)^q, p, q \neq 0, n \neq 2\). Here \(\dim g = 4\). Generators are

\[
\Phi_4 = -2 \varphi + (q + 1) \varphi - (q + 1) t \psi_x(R(px + r)^{-q} + \psi_x(px + r)^{-2a} + \varphi_x^2) + \frac{(px + r) \varphi_x}{p},
\]

\[
\Psi_4 = -\psi - (q + 1)t \varphi_x + \frac{(px + r) \psi_x}{p},
\]

\[
\Theta_4 = \frac{1}{(px + r)^2 p R} \left(-2px^3 anp((px + r)^{-q} + 2(px + r)(px + r)^{-q})
\right.
\]

\[
- 2R((px + r) \tau + (q + 1)(px + r) \tau \psi_x - \tau \varphi_x t (q + 1)(px + r) \psi_x
\]

\[
+ \left(\frac{1}{2}((px + r) \tau + p q \tau)(px + r) n \psi_x + \left(\frac{1}{2} \tau x n (px + r) + p q t \right) \psi_x
\right)
\]

while the brackets read \([S_1, S_4] = (q - 1) S_1, [S_2, S_4] = -S_2, [S_3, S_4] = -(q + 1) S_3\).

4.1.7. A general, \(n = 2\). The algebra is 4-dimensional with the generators

\[
\Phi_4 = \frac{(\varphi_x^2 a t + At(\varphi_x^2 + R \tau) \psi_x - \varphi_x^2 A^2)}{A^2}, \quad \Psi_4 = t \varphi_x,
\]

\[
\Theta_4 = -\frac{t \varphi_x \psi_x \psi_x}{\psi_x^2} + \frac{\tau t \varphi_x \psi_x}{\psi_x^2} + \frac{\varphi_x \tau \psi_x}{\psi_x} + \frac{A_x \tau \varphi_x}{\psi_x A} - 2 \tau - \frac{2a \psi_x}{A R}
\]

and the commutator \([S_3, S_4] = S_3\).

4.1.8. A general, \(n \neq 2\). Here \(g\) coincides with \(g_0\).

4.2. \(b = 0, a = 0\) (the Ideal Gas)

We have four cases.

4.2.1. \(A = p = \text{const}\). The 6-dimensional algebra is generated by

\[
\Phi_4 = \frac{-\varphi p + t \psi_x(\varphi_x^2 p + R \tau)}{p}, \quad \Psi_4 = t \varphi_x,
\]

\[
\Theta_4 = \frac{-2t \varphi_x \psi_x}{\psi_x^2 n} + \frac{2t \varphi_x \psi_x}{\psi_x^2 n} + \frac{t \varphi_x \tau \psi_x}{\psi_x} - 2 \tau;
\]

\[
\Phi_5 = \varphi_x, \quad \Psi_5 = \psi_x, \quad \Theta_5 = \tau_x;
\]

\[
\Phi_6 = x \varphi_x - 2 \varphi, \quad \Psi_6 = x \psi_x - \psi, \quad \Theta_6 = x \tau_x - 2 \tau.
\]

The commutators are

\[
[S_1, S_4] = -S_1, \quad [S_1, S_6] = -2S_1, \quad [S_2, S_5] = -2S_2, \quad [S_3, S_4] = S_3, \quad [S_5, S_6] = -S_5.
\]
4.2.2. \( A = pe^{qx}, p, q \neq 0 \). One has \( \dim g = 5 \). The algebra is generated by
\[
\Phi_4 = \psi_x e^{-qx} R \tau + p(\psi_x t \varphi_x^2 - \varphi), \quad \Psi_4 = t \varphi_x,
\]
\[
\Theta_4 = \frac{2 \tau t \varphi_x \psi_x - 2t \varphi_x \psi_x}{n \psi_x^2}, \quad \eta_4 = \frac{2t \varphi_x \psi_x^2 + \psi_x}{n \psi_x^2}.
\]
\[
\Phi_5 = \varphi_x, \quad \Psi_5 = \psi_x, \quad \Theta_5 = -q \tau + \tau_x.
\]

The commutators are \([S_1, S_4] = -S_1, [S_3, S_4] = S_3\).

4.2.3. \( A = (px + r)^a, p, q \neq 0 \). One has \( \dim g = 5 \) and the algebra is generated by
\[
\Phi_4 = \psi_x(px + r)^{-a} R \tau + \psi_x \varphi_x^2 t - \varphi, \quad \Psi_4 = t \varphi_x,
\]
\[
\Theta_4 = \frac{2t \varphi_x \tau \psi_x^2}{\psi_x^2 n} + \frac{2 \tau t \varphi_x}{\psi_x n} + \frac{t \varphi_x \psi_x}{\psi_x} + \frac{2q \tau \psi_x - 2 \tau}{(px + r) \psi_x n}, \quad \Theta_5 = \frac{(px + r) \tau_x - p \tau (q + 2)}{p}.
\]

The commutator relations are \([S_1, S_1] = -S_1, [S_1, S_3] = -2S_1, [S_2, S_5] = -S_2, [S_3, S_4] = S_3\).

4.2.4. A general. The 4-dimensional algebra is generated by
\[
\Phi_4 = \frac{t \psi_x (\varphi_x^2 A + R \tau) - \varphi A}{A}, \quad \Psi_4 = t \varphi_x, \quad \Theta_4 = \frac{2t \varphi_x \tau \psi_x^2}{\psi_x^2 n} + \frac{2 \tau t \varphi_x}{\psi_x n} + \frac{t \varphi_x \psi_x}{\psi_x} + \frac{2t A \tau \varphi_x - 2 \tau}{\psi_x A n}.
\]

and the brackets are \([S_1, S_4] = -S_1, [S_3, S_4] = S_3\).

4.3. \( b \neq 0 \)

The four subcases here are as follows.

4.3.1. \( a = 0, A = p = \text{const.} \) We have \( \dim g = 6 \). The algebra is generated by
\[
\Phi_4 = \varphi_x^2 \psi_x^2 b t + (p t \varphi_x^2 + R \tau - \varphi b) \psi_x - p \varphi, \quad \Psi_4 = t \varphi_x,
\]
\[
\Theta_4 = \frac{2 \tau \varphi_x \tau \psi_x}{\psi_x^2 n(b \psi_x + p)} \left( \frac{2 \tau \varphi_x \psi_x}{\psi_x^2 n(b \psi_x + p)} \right) - \frac{2t \varphi_x \tau \psi_x}{\psi_x^2 n(b \psi_x + p)}.
\]
\[
\Phi_5 = \varphi_x, \quad \Psi_5 = \psi_x, \quad \Theta_5 = \tau_x;
\]
\[
\Phi_6 = x \varphi_x - 2 \varphi, \quad \Phi_6 = x \varphi_x - \psi, \quad \Theta_6 = x \tau_x - 2 \tau
\]

with the commutators
\[
[S_1, S_4] = -S_1, \quad [S_1, S_6] = -2S_1, \quad [S_2, S_6] = -S_2, \quad [S_3, S_4] = S_3, \quad [S_5, S_6] = -S_5.
\]

4.3.2. \( a = 0, A \text{ general} \). The symmetry algebra has 4 generators
\[
\Phi_4 = \varphi_x^2 \psi_x^2 b t + (A t \varphi_x^2 + R \tau - \varphi b) \psi_x - A \varphi, \quad \Psi_4 = \varphi_x t,
\]
\[
\Theta_4 = \frac{1}{\psi_x^2 n(b \psi_x + A)} \left( 2A \tau \varphi_x \psi_x - 2t A \tau \varphi_x \psi_x 
\]
\[
+ (-2b \tau \varphi_x^2 + (\varphi_x \tau_x bn - 2A n) \psi_x + \varphi_x t (A n + 2A \tau)) \psi_x \right).
\]

that enjoy the relations \([S_1, S_4] = -S_1, [S_3, S_4] = S_3\).

4.3.3. \( a \neq 0, A = p = \text{const.} \) Dimension of \( g \) is live here. The generators are
\[
\Phi_4 = \psi_x^3 ab t + pt (bp \varphi_x + a) \psi_x^2 + p^2 (pt \varphi_x^2 + R \tau - bx \varphi_x + b \varphi) \psi_x - p^3 (\varphi_x x - \varphi),
\]
\[
\Psi_4 = t \varphi_x - \psi_x x + \psi,
\]
\[
\Theta_4 = \frac{2pt \varphi_x \psi_x - 2t \varphi_x pt \psi_x}{\psi_x^2 n(b \psi_x + p)}.
\]
They are subject to the relations $[S_1, S_4] = S_1$, $[S_2, S_4] = S_2$, $[S_3, S_4] = S_3$, $[S_4, S_5] = -S_5$.

4.3.4. $a \neq 0$, A general. One has $\Phi = \Phi_x$, $\Psi = \Psi_x$, $\Theta = \Theta_x$.

5. EXAMPLES OF EXACT SOLUTIONS

Let us now describe two types of exact solutions: stationary ones, i.e., independent of $t$, and traveling waves in the case $A = \text{const}$.

5.1. Stationary Solutions

Consider the stationary case, i.e., assume $\psi_t = \varphi_t = \tau_t = 0$ in Equations (4). Then the second equation reads $\psi_x = 0$, i.e., $\psi = \text{const}$, the entire system reduces to the sole equation

$$\psi_x (aA\psi_x + ab\psi_x^2 + RA^2\tau) = 0.$$ 

Under the assumption $ab \neq 0$, we obtain three solutions

$$\psi_x = \frac{-(a + \sqrt{a^2 - 4abR\tau})A}{2ab}, \quad \psi_x = \frac{-(a + \sqrt{a^2 - 4abR\tau})A}{2ab}, \quad \psi_x = 0$$

of which only the first one has physical meaning, because by definition $\psi_x = -\rho A < 0$ (recall that in the final system we relabeled $\psi$ to $-\psi$). Thus, we obtain

$$\rho = \frac{a + \sqrt{a^2 - 4abR\tau}}{2ab}, \quad \tau = \frac{a\rho(1 - b\rho)}{R},$$

which gives a simple dependence between temperature and density.

5.2. Traveling-Wave Solutions

If $A = \text{const}$ then System (4) becomes invariant with respect to $x$-translation symmetry $\mathcal{X} = (\varphi_x, \psi_x, \tau_x)$, and one can consider traveling-wave solutions, i.e., solutions invariant with respect to $\mathcal{X} + \nu T$, where $T = (\varphi_t, \psi_t, \tau_t)$, $\nu$ being the velocity. Such solutions are functions of the variable $z = x - \nu t$ and one has

$$\frac{\partial}{\partial x} = \frac{d}{dz}, \quad \frac{\partial}{\partial t} = -\nu \frac{d}{dz}.$$ 

Consequently, (4) reduces to the two equations $\dot{\varphi} = \nu \dot{\psi}$ and

$$(\nu^2 A^2 b\psi^3 + (ab + \nu^2 A^3)\dot{\psi})^2 + (aA - \nu^2 A^2 b)\dot{\psi} - \nu^2 A^3 + R\tau A^2) \dot{\psi} = 0,$$ 

where ‘dot’ denotes the $z$-derivative, while the definitions of the potentials (see Section 1) take the form

$$\rho = -\frac{\dot{\psi}}{A}, \quad u = \nu, \quad p = -\nu(1 + \nu)\dot{\psi}.$$ 

Now, solving (8) with respect to $\tau$ we obtain dependence of temperature

$$\tau = \frac{(bp - 1)(\nu^2 A^3 p^2 - a\rho - \nu^2 A)}{R}$$

on density.

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