Fiducial and Posterior Sampling

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Abstract

The fiducial coincides with the posterior in a group model equipped with the right Haar prior. This result is here generalized. For this the underlying probability space of Kolmogorov is replaced by a σ-finite measure space and fiducial theory is presented within this frame. Examples are presented that demonstrate that this also gives good alternatives to existing Bayesian sampling methods. It is proved that the results provided here for fiducial models imply that the theory of invariant measures for groups cannot be generalized directly to loops: There exist a smooth one-dimensional loop where an invariant measure does not exist.

KEYWORDS: Conditional sampling, Improper prior, Haar prior, Sufficient statistic, Quasi-group
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1 Introduction

A Bayesian posterior is said to be a fiducial posterior if it coincides with a fiducial distribution. The question of existence of a Bayesian prior such that the resulting posterior is a fiducial posterior has attracted interest since the introduction of the fiducial argument by Fisher [1930, 1935]. Cases where the fiducial is not a Bayesian posterior are interesting because the fiducial theory then brings truly new armory for the construction of new inference procedures. The cases where there is a fiducial posterior are interesting because the corresponding fiducial algorithm can be simpler to implement than the competitors based on the Bayesian theory.

The best result in the one-dimensional case was found by Lindley [1958]. He proved that, given appropriate smoothness conditions, a fiducial posterior exists if and only if the problem can be transformed by one-one transformations of the parameter and sample space into the standard location problem. The best result obtained so far in the multivariate case was found by Fraser [1961b,a]. For a group model where a right Haar measure exists, the fiducial coincides with the posterior from the right Haar measure as a Bayesian prior. The main result in this paper is Theorem 1 that contains both results as special cases.

2 Fiducial posteriors

The arguments in the following make it necessary to include improper priors in the considerations, and this will here be done based on the theory presented by Taraldsen and Lindqvist [2010]. A brief summary of the necessary ingredients from this theory is given next.

Definition 1 (The basic space). The basic space $\Omega$ is equipped with a $\sigma$-finite measure $P$ defined on the $\sigma$-field $E$ of events.

All definitions and results in the following will implicitly or explicitly rely on the existence of the underlying basic space. This is as in the theory of probability presented by Kolmogorov [1933], but the requirement $P(\Omega) = 1$ is here replaced by the weaker requirement that $P$ is $\sigma$-finite: There exist events $A_1, A_2, \ldots$ with $\Omega = \bigcup_i A_i$ and $P(A_i) < \infty$. The above can be summarized by saying that it is assumed throughout in this paper that the basic space $\Omega$ is a $\sigma$-finite measure space $(\Omega, E, P)$.

Definition 2 (Random element). A random element in a measurable space $(\Omega_Z, E_Z)$ is given by a measurable function $Z : \Omega \to \Omega_Z$. The law $P_Z$ of $Z$ is defined by

$$P_Z(A) = P \circ Z^{-1}(A) = P(Z \in A) = P\{\omega | Z(\omega) \in A\}. \quad (1)$$
The random element is $\sigma$-finite if the law is $\sigma$-finite.

Definition 2 corresponds to the definition $P_X = P \circ X^{-1}$ by Lamperti [1966, p.4, eq.2] and the definition $\mu_X = \mu \circ X^{-1}$ by Schervish [1995, p.607]. It also corresponds to the original definition given by Kolmogorov [1933, p.21, eq.1], but he used superscript notation instead of the above subscript notation. The law $P_Z$ will also be referred to as the distribution of $Z$. The term *random quantity* is used by Schervish [1995] and can be used as an alternative to the term *random element* used above and by Fréchet [1948]. The term *random variable* $X$ is reserved for the case of a random real number. This is given by a measurable $X : \Omega \to \Omega_X = \mathbb{R}$, where $\mathcal{E}_X$ is the $\sigma$-field generated by the open intervals.

Definition 2 of a random element is more general than any of the above given references since $(\Omega, \mathcal{E}, P)$ is not required to be a probability space, but it is assumed to be a $\sigma$-finite measure space. The space $\Omega$ comes, however, equipped with a large family of conditional distributions that are true probability distributions. This is exactly what is needed for the formulation of a statistical inference problem, and will be explained next.

Let $X$ and $Y$ be random elements, and assume that $Y$ is $\sigma$-finite. Existence of the conditional expectation $E(\phi(X) \mid Y = y) = E^y_X(\phi) = E_X(\phi \mid Y = y)$ and the factorization

$$P_{X,Y}(dx,dy) = P^y_X(dx)P_Y(dy)$$

(2)
can then be established. The proof follows from the Radon-Nikodym theorem exactly as in the case where the underlying space is a probability space [Taraldsen and Lindqvist, 2010]. The case $X(\omega) = \omega$ gives in particular $\{(\Omega, \mathcal{E}, P^y) \mid y \in \Omega_Y\}$ as a family of probability spaces. This last claim is not strictly true, but given appropriate regularity conditions there will exist a regular conditional law as claimed [Schervish, 1995, p.618].

A *statistical model* is usually defined to be a family $\{(\Omega_X, \mathcal{E}_X, P^\theta_X) \mid \theta \in \Omega_\Theta\}$ of probability spaces. This definition is also used here, but with an added assumption included in the definition: It is assumed that there exist a random element $X$, and a $\sigma$-finite random element $\Theta$ so that $P^\theta_X(A) = P_X(A \mid \Theta = \theta)$. It is in particular assumed that both the sample space $(\Omega_X, \mathcal{E}_X)$ and the model parameter space $(\Omega_\Theta, \mathcal{E}_\Theta)$ are measurable spaces. The law $P_\Theta$ is not assumed to be known and is not specified. Similarly, the functions $X : \Omega \to \Omega_X$ and $\Theta : \Omega \to \Omega_\Theta$ are assumed to exist, but they are also not specified. This is by necessity since the underlying space $\Omega$ is not specified. It is an abstract underlying space that makes it possible to formulate a consistent theory.

A *Bayesian model* is given by a statistical model and the additional specification of the law $P_\Theta$ of $\Theta$. This prior law $P_\Theta$ can be improper in the theory as just described,
and discussed in more detail by Taraldsen and Lindqvist [2010]. The posterior law $P_\Theta$ is well defined if $X$ is $\sigma$-finite. The result of Bayesian inference is given by the posterior law, and Bayesian inference is hence trivial except for the practical difficulties involved in the calculation of the posterior and derived statistics. The most difficult part from a theoretical perspective is to justify the choice of statistical model and the prior in concrete modeling cases.

Fiducial arguments were invented by Fisher [1930, 1935] to tackle cases without a prior law, but with the aim to obtain a result similar to the posterior distribution. The resulting distribution from the fiducial argument is called a fiducial distribution. The following definition [Taraldsen and Lindqvist, 2013] will be used here. It should be noted that the definition uses concepts that rely on existence of the underlying basic space $\Omega$.

**Definition 3** (Fiducial model and distribution). Let $\Theta$ be a $\sigma$-finite random element in the model parameter space $\Omega_\Theta$. A fiducial model $(U, \zeta)$ is defined by a random element $U$ in the Monte Carlo space $\Omega_U$ and a measurable function $\zeta : \Omega_U \times \Omega_\Theta \rightarrow \Omega_Z$ where $\Omega_Z$ is the sample space. The model is conventional if the conditional law $P^\theta_U$ does not depend on $\theta$. The model is simple if the fiducial equation $\zeta(u, \theta) = z$ has a unique solution $\theta^\varepsilon(u)$ for all $u, z$. If the model is both conventional and simple, then the fiducial distribution corresponding to an observation $z \in \Omega_Z$ is the distribution of $\Theta^z = \theta^\varepsilon(U)$ where $U \sim P^\theta_U$.

A fiducial model $(U, \zeta)$ is a fiducial model for the statistical model $\{P^\theta_Z | \theta \in \Omega_\Theta\}$ if

$$\langle \zeta(U, \Theta) | \Theta = \theta \rangle \sim (Z | \Theta = \theta)$$

(3)

The fiducial model gives a method for simulation from the statistical model: If $u$ is a sample from the known Monte Carlo law $P^\theta_U$, then $z = \zeta(u, \theta)$ is a sample from $P^\theta_Z$. Sampling from the fiducial follows likewise, but involves solving the fiducial equation $\tau(u, \theta) = t$ to obtain the sample $\theta = \theta^t(u)$. This, and related definitions in the literature, are discussed in more detail by Taraldsen and Lindqvist [2013].

We have now presented the necessary ingredients for the formulation of the main theoretical results here. The first result gives conditions that ensure that the fiducial coincides with a Bayesian posterior.

**Theorem 1.** Assume that $(U, \tau)$ is a conventional simple fiducial model for the statistical model $\{P^\theta_T | \theta \in \Omega_\Theta\}$. If the Bayesian prior $P_\Theta$ implies that distribution of $\tau(u, \Theta)$ does not depend on $u$, then the Bayesian posterior distribution $P^\theta_\Theta$ is well defined and identical with the fiducial distribution of $\Theta^t$. 


It should in particular be observed that the required σ-finiteness of \( T = \tau(U, \Theta) \) is a part of the conclusion in the previous theorem. This ensures that the Bayesian posterior exists.

The next result gives a recipe for posterior sampling based on a fiducial model.

**Theorem 2.** Assume that \( (U, \tau) \) is a conventional fiducial model and that \( T = \tau(U, \Theta) \) is σ-finite for a given prior \( P_\Theta \). Assume furthermore that \( \tau(u, \Theta) \sim w(t, u)\mu(dt) \) for some σ-finite measure \( \mu \) and jointly measurable \( w \). If \( u \) is a sample from a probability distribution proportional to \( w(t, u)P^\theta_U(du) \) and \( \theta \) is a sample from the conditional law \( (\Theta | \tau(u, \Theta) = t) \), then \( \theta \) is a sample from the Bayesian posterior distribution of \( \Theta \) given \( T = t \).

The proofs of Theorem 1 and Theorem 2 are postponed until Section 7. We choose to discuss examples and consequences of these results next.

### 3 The location problem

Assume that \( t \) is the observed realization of a random variable where

\[
    t = \tau(u, \theta) = u + \theta
\]

where \( u \) is a sample from the conditional law \( P^\theta_U \). It is assumed that \( P^\theta_U \) is known and does not depend on \( \theta \). The pair \( (U, \tau) \) is then a fiducial model for the statistical model \( P^\theta_T \). The problem is to make statistical inference regarding the model parameter \( \theta \in \Omega_\Theta = \mathbb{R} \) based on the model and the observation \( t \in \Omega_T = \mathbb{R} \).

Consider first fiducial inference. The fiducial distribution is determined by the solution \( \theta^f(u) = t - u \) of the fiducial equation \( t = u + \theta \). Monte Carlo sampling \( u \) from the known law \( P^\theta_U \) gives corresponding samples \( t - u \) from the fiducial distribution. The mean and standard deviation can then be calculated with a precision depending on the choice of Monte Carlo size and the random number generator. This can then be reported as an estimate of \( \theta \) and a standard error respectively. A more complete report can be given by a direct Monte Carlo estimate of the fiducial distribution itself in the form of a graph. This represents then the state of knowledge regarding \( \theta \) based on the observation and the fiducial model.

Consider next Bayesian inference. Assume for simplicity that \( P^\theta_U(du) = f(u)du \), where \( du \) is Lebesgue measure on the real line. If \( P_\Theta(d\theta) = \pi(\theta) d\theta \) is the prior law, then the posterior law is given by a density \( \pi(\theta | t) = C_if(t - \theta)\pi(\theta) \) where \( C_i \) is a normalization constant. This normalization is generally possible if \( T = \tau(U, \Theta) \) is σ-finite, and this happens exactly when \( \int f(t - \theta)\pi(\theta) d\theta < \infty \) for (almost) all \( t \).
It is always possible if $\pi$ is a probability density, but an alternative sufficient condition is that $\pi$ is bounded. The particular case $\pi(\theta) = 1$ gives the result $\pi(\theta \mid t) = f(t - \theta)$. A simple calculus exercise shows directly that this coincides with the fiducial law derived above. The reporting of the result can be done as in the case of fiducial inference.

The previous result can also be inferred from Theorem 1 since the law of $\tau(u, \Theta) = u + \Theta$ is the Lebesgue measure when the law of $\Theta$ is the Lebesgue measure. More generally the assumption $P_{\Theta}(d\theta) = \pi(\theta) d\theta$ gives $\tau(u, \Theta) \sim \pi(t - u) dt$. Theorem 2 and the assumption of $\sigma$-finiteness of $T$ can then be used for Bayesian sampling more generally as follows: Sample $u$ from a law proportional to the measure $\pi(t - u) P_{\Theta}(du)$ and return $\theta = t - u$. The latter proof does not rely on the existence of a density $f$ for $P_{\Theta}$ with respect to Lebesgue measure. This argument is used in Section 4.1 to provide a concrete example where a traditional Bayesian sampling recipe fails, but the fiducial algorithm from Theorem 2 can be used.

Consider finally frequentist inference. If $P_{\Theta}(du) = f(u) du$ where $f$ has a unique maximum at $m_u$, then $\hat{\theta} = t - m_u$ is the maximum likelihood estimator. Assume that the expected value $E^\theta U = \mu_u$ exists. It follows then that the expected value $\bar{\theta} = t - \mu_u$ of the fiducial distribution is the shift equivariant estimator with smallest mean square error [Taraldsen and Lindqvist, 2013]. It is in particular better than the maximum likelihood estimator when both exist, it is unbiased, and the standard error is given by the standard deviation of $U$.

The fiducial distribution is also a confidence distribution since $U = T - \theta$ is a pivotal. Consequently an expanded uncertainty can be found corresponding to 95% confidence intervals. Symmetric, shortest, or uniformly most powerful limits can be calculated. The most powerful limits follow with reference to the likelihood ratio test as exemplified for the exponential by Taraldsen [2011]. This reference also gives the route for the inclusion of the effect of finite resolution into the analysis.

The previous analysis with the assumption $\Omega_U = \Omega_T = \Omega_\Theta = \mathbb{R}$ can be generalized verbatim to the case $\Omega_U = \Omega_T = \Omega_\Theta = V$ where $V$ is a finite dimensional real or complex vector space. The property $\tau(u, \Theta) = u + \Theta \sim \Theta$ holds for the finite dimensional Lebesgue distribution for $\Theta$. The further generalization to the case where $V$ is an infinite dimensional Hilbert space gives an example where the Bayesian algorithm fails to produce optimal frequentist inference. The fiducial argument given above holds also for the infinite dimensional case, and gives optimal inference as stated above [Taraldsen and Lindqvist, 2013].

The analysis can be generalized further to the case $\Omega_U = \Omega_T = \Omega_\Theta = V^n$. This includes in particular the case of a random sample of size $n$ from the original model given in equation (4), but the independence assumption is not required in the
following argument. Equation (4) must be replaced by the equation \( T_1 = U_1 + \theta \) corresponding to the first component of the random element \( T \) in \( V^n \). The law \( \mathcal{P}_U^\theta \) must be replaced by the conditional law \( (U_1 \mid \Theta = \theta, U_2 - U_1 = t_2 - t_1, \ldots, U_n - U_1 = t_n - t_1) \). Except for the practical difficulties related to this conditional law, the analysis proceeds as before. Optimal frequentist inference procedures including confidence distributions follow from the resulting fiducial also in this case [Taraldsen and Lindqvist, 2013].

4 Location examples

4.1 A singular example

The purpose of this example is to demonstrate that Theorem 2 can be used to calculate the Bayesian posterior in certain cases where the traditional Bayesian recipe fails.

Let the Monte Carlo law \( \mathcal{P}_U^\theta \) give probability \( p_i \) to the value \( u_i \) for \( i = 1, 2 \). The model given by \( X = U + \theta \) with \( \theta \in \Omega_\Theta = \mathbb{R} \) gives a law \( \mathcal{P}_X^\theta \) which is concentrated on \( \{u_1 + \theta, u_2 + \theta\} \subset \Omega_X = \mathbb{R} \). The traditional Bayesian posterior would usually be calculated by \( \pi(\theta \mid x) \propto f(x \mid \theta) \pi(\theta) \), but this fails here since the density \( f(x \mid \theta) \) fails to exist for the case considered.

Consider next the algorithm given by Theorem 2. The relation \( u + \Theta \sim \pi(x - u) \, dx \) gives the following recipe: Sample \( u \) from a law that gives relative probability \( q_i = \pi(x - u_i)p_i \) to the values \( u_1 \) and \( u_2 \). The resulting \( \theta = x - u \) is a sample from the Bayesian posterior \( \mathcal{P}_\Theta^x \). The conclusion is that the posterior gives probability \( q_i/(q_1 + q_2) \) to the two values \( \theta_i = x - u_i \) for \( i = 1, 2 \).

The uniform prior case \( \pi = 1 \) gives that the posterior equals the fiducial which gives probability \( p_i \) to \( \theta_i \).

4.2 Normal distribution

Assume that \( X_i = \chi_i(U, \theta) = \theta + \sigma_0 U_i \) where the Monte Carlo law of \( U \) corresponds to a random sample of size \( n \) from the standard normal distribution. The \((U, \chi)\) is then a fiducial model for a random sample of size \( n \) from a Normal distribution \((\theta, \sigma_0^2)\) where the variance \( \sigma_0^2 \) is assumed known.

This gives \( T = \bar{X} = \chi(U, \theta) = \tau(V, \theta) = \theta + \sigma V \) with \( \sigma = \sigma_0/\sqrt{n} \) and \( V = \sqrt{nU} \) has a standard normal distribution. The \((V, \tau)\) is then a fiducial model for the sufficient statistic \( T \) which is a Normal distribution \((\theta, \sigma^2)\) statistical model.
The fiducial based on the sufficient statistic is the law of \( \Theta^t = t - \sigma V \) which is Normal\((t, \sigma^2)\). This gives the optimal equivariant estimator \( t = \bar{x} \), the standard error \( \sigma = \sigma_0 / \sqrt{n} \), and the expanded error \( k\sigma \) where the coverage factor \( k = 1.96 \) gives the level 95%.

The Bayesian conclusion with the uniform law as prior is given by the same numbers since the fiducial coincides with the posterior in this case.

### 4.3 Gamma distribution I

The example here is a generalization of the case given by a random sample from the exponential distribution [Taraldsen and Lindqvist, 2013]. Let \( \chi_i(u, \theta) = \theta F^{-1}(u_i; \alpha) \), where \( F^{-1} \) is the inverse CDF of the gamma distribution with scale \( \beta = 1 \). If \( (U_1, \ldots, U_n | \Theta = \theta) \sim U(0,1) \) independent, then the inversion method gives that \( (U, \chi) \) is a fiducial model for a random sample \( X = (X_1, \ldots, X_n) \) from the gamma density:

\[
f_{X_i}(x_i | \theta) = \left\{ \theta^\alpha \Gamma(\alpha) \right\}^{-1} x_i^{\alpha-1} e^{-x_i/\theta}, \quad \text{shape } \alpha > 0, \text{scale } \theta > 0
\]

It follows from this density that \( T = \bar{X} \) is sufficient. A fiducial model from the above fiducial model is then \( T = \theta V \), where the Monte Carlo variable \( V = F^{-1}(U; \alpha) \) has a Gamma\((n\alpha, 1/n)\) distribution.

The fiducial equation \( t = \theta v \) gives the fiducial \( \Theta^t = t/V \) with an InvGamma\((n\alpha, nt)\) distribution. This is a confidence distribution for \( \theta \), and also the Bayesian posterior corresponding to a uniform prior for \( \log \theta \). The mean

\[
\bar{\theta} = t/(\alpha - 1/n)
\]

is the best Bayesian estimator for the quadratic loss. It can be seen as a sample size adjustment of the likelihood estimate \( \hat{\theta} = t/\alpha \).

The scale model transforms to the location model \( \ln t = \ln(\theta) + \ln(v) \). The best equivariant estimator for \( \ln(\theta) \) is then \( E^\theta(\ln \Theta^t) \), and this integral equals \( \ln(nt) - \psi(n\alpha) \) where \( \psi \) is the digamma function.

The best equivariant estimate for \( \theta \) is

\[
\hat{\theta} = (t/\alpha) \exp(\ln(n\alpha) - \psi(n\alpha))
\]

This is best with respect to the squared distance \( |\ln(\theta_1) - \ln(\theta_2)|^2 \) from the Fisher metric as explained in more detail by Taraldsen and Lindqvist [2013].

The reason for the choice of the above formulation of equation (7) is that \( t \) is the uniformly minimum variance estimator of \( \alpha \beta \), and the \( \exp(\cdot) \) term can be
seen as a correction of this. The following asymptotic and divergent series $\psi(x) - \ln(x) \sim 1/(2x) - \sum_{n\geq 1} B_{2n}/(2nx^{2n})$ for $x \to \infty$ shows in particular consistency of the estimator in equation (7) with the more common estimator $t/\alpha$ in the limit of infinite sample size $n \to \infty$.

The main reason for the inclusion of this example is not the possibly novel result given by equation (7), but rather demonstration purposes. We consider the arguments as given above as a competitive alternative to the arguments given by a Bayesian calculation. The Bayesian calculation is of course possible in this case, but it seems more cumbersome to us. The claims on optimality can indeed also be proved directly without any mention of a Bayesian prior [Taraldsen and Lindqvist, 2013].

5 One-dimensional fiducial inference

Fiducial inference was first considered in the one-dimensional case. This is discussed here, and the connection between the original definition and the more general Definition 3 is in particular explained.

5.1 Lindley’s result

Lindley [1958] considered the one-dimensional case. His sufficient and necessary condition for a fiducial posterior is equivalent with the conditions given in Theorem 1. The result is only valid by consideration of a more restrictive definition of the fiducial distribution defined directly and uniquely by an absolutely continuous cumulative distribution function. This is explained next.

The monotonicity of a simple fiducial model has as a consequence monotonicity in $\theta$ of the cumulative distribution function $F(t \mid \theta)$ of $P_{\theta}^T$. In the following it is furthermore assumed that $\theta \mapsto F(t \mid \theta)$ is absolutely continuous and onto $(0, 1)$. The relation $u = \hat{u}(t, \theta) = F(t \mid \theta)$ can be inverted to give $\theta = \hat{\theta}(u, t)$ and $t = \tau(u, \theta) = F^{-1}(u \mid \theta)$. The well known inversion method gives that $\{U, \tau\}$ with $P_{\theta}^U$ the uniform law on $(0, 1)$ is a fiducial model for $P_{\theta}^T$. This is the Fisher fiducial model, and it is a simple conventional fiducial model. It can be shown that the corresponding Fisher fiducial distribution coincides with the fiducial distribution of the original fiducial model [Dawid and Stone, 1982]. Fiducial inference is hence unique in this case. If the cumulative distribution $F(t \mid \theta)$ is decreasing in $\theta$, then $1 - F(t \mid \theta)$ is the cumulative fiducial distribution.

The result of Lindley is that a fiducial posterior is obtained if and only if the fiducial model $(U, \tau)$ is a transformation of a fiducial model $(V, \eta)$ where $\eta(v, \varsigma) =$
The prior for \( \varsigma \) is Lebesgue measure on \( \mathbb{R} \) and the resulting fiducial model \( (V, \eta) \) is the location model. The transformation assumption is that \( \tau(u, \theta) = \phi_3(\phi_1(u) + \phi_2(\theta)) \) with \( v = \phi_1(u) \) and \( \varsigma = \phi_2(\theta) \).

The if part of the claim is a special case of the results discussed in Section 3 since both Bayesian and fiducial inference behave consistently under transformations. The if part does not require existence of densities, and this result here is then an extension of the results of Lindley.

An example which is more general than the Lindley case is obtained by choosing a \( \phi_3 \) which is strictly increasing, but nowhere differentiable. The result is then a singular continuous fiducial posterior, and this is not covered by the proof of Lindley. Another class of examples is given by choosing an arbitrary probability distribution \( P_\theta \) which does not need to be absolutely continuous. A third class of examples not covered by Lindley is given by countable \( \Omega_U = \Omega_\Theta = \Omega_T \subset \mathbb{R} \) equipped with a possibly non-commutative group or loop operation.

It remains to prove the only if part of the fiducial posterior claim given the above restrictions on the cumulative distribution. The necessary parts of the argument of Lindley is reproduced next.

Assume that the fiducial model has a fiducial posterior in the sense that the fiducial density \(-\partial_\theta F(t | \theta)\) equals the posterior density \(\partial_t F(t | \theta)h(\theta)g(t)^{-1}\). This gives the following generalization of the one-way wave equation

\[-h(\theta)^{-1}\partial_\theta F(t | \theta) = g(t)^{-1}\partial_t F(t | \theta)\]  \hspace{1cm} (8)

A general solution is given by \( F = S(G(t) - H(\theta)) \), where \( G' = g \) and \( H' = h \). Consequently, the family of conditional distributions for \( G(T) \) is a location family with location parameter \( H(\theta) \). The one-one correspondence \( G(T) \mapsto G^{-1}(G(T)) = T \) proves that \( T \) is given by a transformation of the location group model.

A particularly nice aspect of the above proof is that it gives explicitly the required transformation to a standard location model. The function \( G \) is the cumulative distribution of the (marginal) law of \( T \), and a fiducial posterior is obtained if and only if the variable \( G(T) \) corresponds to a standard location model.

It is also the explicit transformation that ensures that the law of \( H(\Theta) \) is the uniform law on the real line:

\[
P(a < H(\Theta) < b) = \int_{H^{-1}(a)}^{H^{-1}(b)} h(\theta) \, d\theta = b - a
\]  \hspace{1cm} (9)

A particular consequence is that the prior law of \( \Theta \) is always improper when the posterior coincides with the fiducial.
5.2 The correlation coefficient

Let \( F(r | \rho) \) be the cumulative distribution function of the empirical correlation coefficient \( r \) of a random sample from the bivariate normal distribution. The parameter \( \theta = \rho \) is the correlation coefficient. A fiducial model \((U, \chi)\) is given by a uniform law \( P^\theta_U \) and the fiducial relation \( \chi(u, \theta) = F^{-1}(u | \theta) \). It is possible to sample from the fiducial based on \( F \), but a much simpler algorithm is described in section 6.2 below.

In this case it is known that there exists no prior on \( \rho \) that gives the fiducial as a posterior \( P^\rho \) \cite[2008, p.966]{BergerSun}. The proof is not trivial. The fiducial for the correlation coefficient gives the very first example \cite[1930]{Fisher} of a derivation of a fiducial distribution \cite[1973, p.176]{Fisher}. The fiducial for the correlation coefficient is, however, a Bayesian posterior from the multivariate normal model considered in section 6.2.

5.3 Gamma distribution II

Consider a random sample from the gamma density

\[
f_{X_i}(x_i | \theta) = \left\{\beta \Gamma(\theta)\right\}^{-1} x_i^{\theta - 1} e^{-x_i/\beta}, \quad \text{shape } \theta > 0, \text{ scale } \beta > 0
\]

The case with a general scale \( \beta \) can be reduced to the case of a scale \( \beta = 1 \) by consideration of \( x_i/\beta \). It will hence initially be assumed that \( \beta = 1 \).

The form of the density shows that \( T = \ln(X) \) is sufficient. A fiducial model is given by \( T = \tau(U, \theta) = \ln(F^{-1}(U | \theta)) \) where \( F^{-1} \) is the inverse CDF of the gamma distribution with scale \( \beta = 1 \) and \((U_1, \ldots, U_n | \Theta = \theta) \sim U(0, 1)\) independent. Each \( \ln(F^{-1}(u_i; \theta)) \) is increasing in \( \theta \), since \( F(u | \theta) \) is increasing.

An alternative fiducial model is given by \( T = G^{-1}(V, \theta) \) where \( G(t | \theta) \) is the CDF of \( T \) and \((V | \Theta = \theta) \sim U(0, 1)\). An explicit expression for \( G \) can be given in terms of the Meijer G-function using results by \cite{Nadarajah}. Both models are simple, and give the same fiducial distribution. We conjecture that this fiducial is not obtainable as a Bayesian posterior, but do not attempt a proof.

The fiducial is a confidence distribution for the shape \( \theta \), and both of the previous fiducial models give sampling algorithm. Reasonable estimators for \( \theta \) are given by \( E^t,\Theta \) and \( \exp \left( E^t, \log(\Theta) \right) \) corresponding to a squared distance loss on the direct and logarithmic scale respectively. Alternatives are given by the Fisher information metric or an entropy distance. Natural competitors are the maximum-likelihood and the Jeffreys prior Bayesian versions of the previous fiducial estimators. A detailed discussion of this will not be give here.
6 Group and loop models

It will next be explained, as promised in the abstract, that Fraser’s result on fiducial posteriors follows as a special case of Theorem 1.

6.1 A generalized location-scale model

Let \( \chi \) be defined by

\[
    x_i = \chi_i(u, \theta) = \theta u_i = [\mu, L] u_i = \mu + Lu_i, \quad i = 1, \ldots, n
\]

(11)

where \( \mu, u_i \in \mathbb{R}^p \) are columns of length \( p \) and \( L \) is a lower triangular \( p \times p \) matrix with positive diagonal. The case \( p = 1 \) gives the standard location-scale model \( x_i = \mu + \sigma u_i \) with \( \sigma = L \), and equation (11) can be seen as a natural generalization.

The generalized location-scale group \( G = \Omega \Theta \) with elements \( \theta = (\mu, L) \) is discussed in more detail by Fraser [1979, p.175] in the context of structural inference. Multiplication is defined by \( [\mu_1, L_1][\mu_2, L_2] = [\mu_1 + L_1 \mu_2, L_1 L_2] \), the inverse is \( [\mu, L]^{-1} = [-L^{-1} \mu, L^{-1}] \), and the identity is \( e = [0, I] \). The group may also be identified with the group of lower triangular matrices on the \( 2 \times 2 \) block form

\[
    g = \begin{pmatrix} L & 0_p \\ \mu^* & 1 \end{pmatrix}
\]

(12)

which gives the previous calculation rules from matrix multiplication directly.

A Monte Carlo law \( P_{\theta}^U \) gives that \( (U, \chi) \) is a fiducial model for the conditional law \( P_{\theta}^X \) of \( X = \chi(U, \Theta) \). It will be assumed that the \( U_i \) are independent and corresponds to a random sample of size \( n \) from a known probability distribution on \( V = \mathbb{R}^p \). The columns \( X_i = \mu + LU_i \) corresponds then also to a random sample of size \( n \) from a distribution on \( V \) with \( E^\theta(X_i) = \mu \) and \( \text{Cov}^\theta X_i = E^\theta(X_i - \mu)(X_i - \mu)^* = LL^* = \Sigma \), where it is assumed that \( \text{Cov}^\theta U_i = I \) and \( E^\theta U_i = 0 \).

The result so far is a fiducial model where the model parameter space corresponds to the mean \( \mu \) and covariance \( \Sigma \) of some multivariate law on \( V = \mathbb{R}^p \). It should be observed that the Cholesky decomposition \( \Sigma = LL^* \) determines \( L \) uniquely, and it is hence a matter of choice if \( L \) or \( \Sigma \) is considered as a model parameter.

The model given by equation (11) is not a simple fiducial model, but it can be reduced to a simple fiducial model by conditioning similarly to how the location model was treated. The general recipe for this is explained by Taraldsen and Lindqvist [2013], but the details of this will not be give here.
6.2 The multivariate normal

The possibly most important group model is given from the previous discussion and assuming that $P_{\theta}^T$ is the law of a $p \times n$ matrix of independent standard normal variables. The result is then a fiducial model $(U, \chi)$ corresponding to a random sample of size $n$ from the multivariate normal $\text{Normal}_{p}(\mu, \Sigma)$. This is not the only possible fiducial model for this case, but other possibilities will not be discussed.

A simple fiducial model is then obtained from the sufficient statistic $T = (X, L_x)$ where $L_x L_x^*$ is the Cholesky decomposition of the empirical covariance matrix of $X$. The fiducial model $(U, \chi)$ from equation (11) gives then a fiducial model $(V, \tau)$ for $P_{\theta}^T$ where

$$t = \tau(u, \theta) = \theta v, \quad t, \theta, v \in G$$

and $v = [\overline{v}, L_v]$. This model is simple, and the fiducial as given by Definition 3 is the law of $\Theta^t = tV^{-1}$.

Let $P_\Theta$ be the right Haar prior on $G$. The explicit form for this is not needed in the following argument. The right invariance gives $\tau(v, \Theta) = \Theta v \sim \Theta$, and Theorem 1 gives that the Bayesian posterior coincides with the fiducial.

Sampling from the posterior can be done by alternative methods, but it seems that the algorithm that follows from the fiducial argument is the simplest possible that generate independent samples. It involves only standard matrix calculations including solving lower triangular linear systems, and calculation of Cholesky decompositions. This gives in particular a simple sampling algorithm for the fiducial distribution of the correlation coefficient considered in section 5.2.

6.3 General group case

Assume that $t = \tau(v, \theta) = \theta v$ is given by group multiplication. Let $P_\Theta$ be a right Haar measure on the group $G = \Omega_\Theta = \Omega_V = \Omega_T$. It follows from the right invariance that $\tau(u, \Theta) = \Theta v \sim P_\Theta$ for all $v$. Theorem 1 can now be applied, and it follows that the distribution of $tV^{-1}$ conditional on $\Theta = \theta$ equals both the fiducial and the posterior. This case can be referred to as the fiducial group model case. The fiducial model for $T$ is pivotal, and the right Haar distribution is a matching prior: The posterior is fiducial and also a confidence distribution since $v = \theta^{-1} t$ gives a pivotal quantity. This result is the result obtained by Fraser [1961b], but he obtained it by a different argument.

The previous group case is important since it gives a multitude of non-trivial examples where the assumptions in Theorem 1 are fulfilled. It is in particular noteworthy that the required $\sigma$-finiteness of $T$ and $(U, T)$ follows as consequences in the
Existence of a $\sigma$-finite random quantity $\Theta$ such that the distribution $\tau(u, \Theta)$ does not depend on $u$ is a non-trivial problem in general. It is a generalization of the existence and uniqueness problem for Haar measure on a group. This is ensured in the fiducial group case if it is assumed that $G$ is a locally compact group [Halmos, 1950].

A more general family of examples can be constructed as follows. Let $\phi_3 : G \to \Omega_T$, $\phi_2 : \Omega_\Theta \to G$, $\phi_1 : \Omega_U \to G$, and $\tau(u, \theta) = \phi_3(\phi_2(\theta)\phi_1(u))$ where the product is the group multiplication in a group $G$ equipped with a right Haar measure $\mu$. Assume that $\phi_2$ is such that $\phi_2(\Theta) \sim \mu$. It follows then that $\tau(u, \Theta) \sim \tau(U, \Theta) \sim \mu_{\phi_3}$. If $\phi_3$ and $\phi_2$ are invertible, then the fiducial model is simple and the fiducial posterior is distributed like $\phi_3^{-1}(\phi_2^{-1}(t)(\phi_1(U))^{-1})$ conditional on $\Theta = \theta$. It can be observed that the functions $\phi_2$ and $\phi_3$ can be used to identify $G$ and $\Omega_\Theta$ respectively $\phi_3(G) \subset \Omega_T$ both as sets and as groups. The previous model is hence essentially reducible to the group case by a change of variables.

The reduction can alternatively be formulated as follows. Define a new parameter $\zeta = \phi_2(\theta)$, a new variable $V = \phi_1(U)$, and let $\eta(v, \zeta) = \zeta v$. It follows that $(V, \eta)$ is a fiducial group model for $S = \phi_3^{-1}(T)$, and inference can be based on this. The result is the same as in the previous paragraph. It can not be concluded that the original fiducial model $(U, \tau)$ is a group model, but the model is transformed into a fiducial group model $(V, \eta)$.

Let $G$ be a group with an invariant measure, and let $\tau(u, \theta) = \phi_3(\phi_2(\theta)\phi_1(u))$ with $\phi_i$ one-one on $G$. This is a special case of the case considered in the previous two paragraphs. This defines a binary operation on $G$ which need not be a group since the associative law may fail. An example is given by $\tau(u, \theta) = (u + \theta)/2$ with addition on the real line $G = \mathbb{R}$. It is however a quasi-group [Smith, 2006], but in the context here it is essentially reduced to the group case by relabeling as explained in the previous paragraphs.

### 6.4 Loop models

A quasigroup $(G, \circ)$ is a set $G$ equipped with a binary operation $\circ$ such that for each $a, b \in G$, there exist unique elements $x, y \in G$ such that $a \circ x = b$ and $y \circ a = b$. A loop $(G, \circ, e)$ is a quasigroup with an identity element, that is, an element $e$ such that $a \circ e = a = e \circ a$ for all $a \in G$. A group $(G, \circ, e)$ is a loop so that the associative law $(a \circ b) \circ c = a \circ (b \circ c)$ holds for all $a, b, c \in G$. The concept of a loop within abstract algebra as just defined is probably less familiar to most readers than the concept of a group. On an intuitive level it can be considered to be an object similar to a group,
but without the associative law. It will next be explained that loops occur naturally in the context of fiducial theory.

Consider the case \( t = \tau(u, \theta) \) where \( \tau \) is a bijection separately in both arguments. The fiducial model is then said to be a pivotal and simple model. The bijections defined by \( \tau \) can be used to define a change of variables so that it may be assumed that \( \Omega_T = \Omega_U = \Omega_\Theta = G \). The result is a set \( G \) equipped with a binary operation \( \tau(u, \theta) \) with inverse \( \hat{\theta}(u, t) \) and inverse \( \hat{u}(t, \theta) \). \( G \) is then a quasi-group. The notation \( \tau(u, \theta) = \theta u \), \( \hat{\theta}(u, t) = t / u \), and \( \hat{u}(t, \theta) = \theta \setminus t \) with right and left division is standard. The change of variables can also be chosen so that there is an identity element \( e \) such that \( ge = eg = g \) for all \( g \). \( G \) is then a loop. The conclusion is that a change of variables reduces a pivotal simple model to a loop model.

Two examples of loops which do not seem to be essentially reduced to the group case are given next. One example is given by \( \tau(u, \theta) = (\theta u^2 \theta)^{\frac{1}{2}} \) where \( u \) and \( \theta \) are positive definite matrices. This gives an example of a Bruck loop. Another example is \( \tau(u, \theta) = \theta u \) where the multiplication is the multiplication of the invertible octonions, which is a Moufang loop. It is not known to the authors if there exist invariant loop measures for certain classes of loops, or for these concrete examples. This would provide examples beyond the group case.

The finite and countable loop cases are trivial in that counting measure is the unique invariant measure, but they can otherwise be quite exotic objects. They do, however, provide examples where the fiducial equals the Bayesian posterior, and this does not follow from the results of Lindley and Fraser.

The one-dimensional case considered in Section 5.2 provides in particular an example of a loop on the real line that does not possess an invariant measure. This could be of independent interest, and is hence stated separately here as a Theorem.

**Theorem 3.** There exist a smooth loop where an invariant right measure does not exist.

**Proof.** This follows from the correlation coefficient example in section 5.2 which provides a loop model where the fiducial can not be a Bayesian posterior. An invariant right measure would provide a Bayesian posterior as a special case of Theorem 1. \( \square \)

The result of Lindley can be reformulated to give an alternative characterization of loops with an invariant measure: A smooth loop on the real line has an invariant measure if and only if it can be reduced to a group by a change of variables. It is unclear if this can be generalized to more general loops. The term smooth is here interpreted to mean infinitely differentiable with continuous derivatives.
7 A fundamental lemma

The following Lemma has Theorems 1-2 as direct consequences.

**Lemma 1.** Assume that $\Theta$ is $\sigma$-finite with $(U \mid \Theta = \theta) \sim f(u)\nu(du)$ and $\tau(u, \Theta) \sim w(t, u)\mu(dt)$ for fixed $u$, where $\nu$ and $\mu$ are $\sigma$-finite measures and $w$ is jointly measurable. Let $T = \tau(U, \Theta)$. It follows then that

$$\langle U, T \rangle \sim f(u)w(t, u)\nu(du)\mu(dt) \quad (14)$$

**Proof.** Change variables from $(U, \Theta)$ to $(U, T)$

$$E\phi(U, T) = \iint \phi(u, \tau(u, \theta))f(u)\nu(du)\Theta(d\theta)$$

$$= \iint \phi(u, t)f(u)w(t, u)\mu(dt)\nu(du)$$

The key ingredients in the above proof are the Fubini theorem together with the general change-of-variables theorem $\int \psi(z)\mathbb{P}_Z(dz) = \int \psi(\phi(y))\mathbb{P}_Y(dy)$ when $Z = \phi(Y)$. This theorem is usually proved in the context of probability spaces [Halmos, 1950, p.163, Theorem C], but the proof is also valid for the more general case where $\mathbb{P}$ is assumed to be $\sigma$-finite.

The main point of Lemma 1 is that it provides an explicit expression for the density $h$ of $(U, T)$, and it follows in particular that this density exists. It follows from the proof that $(U, T)$ is $\sigma$-finite, and that $T$ is $\sigma$-finite if and only if $\int h(u, t) \nu(du) < \infty$ for $\mu$-a.e. $t$. This condition can be checked in applications.

**Proof. Theorem 1** The assumption gives $\tau(u, \Theta) \sim \mu(dt)$ for a $\sigma$-finite measure $\mu$. The $\sigma$-finiteness follows since $\theta \mapsto \tau(u, \theta)$ is a bijection. Lemma 1 gives that $h(u, t) = f(u)$ is the joint density of $(U, T)$, and then also that $T \sim \mu(dt)$ is $\sigma$-finite. A sample $\theta$ from $\Theta \mid T = t$ can generally be obtained by sampling $u$ from $(U \mid T = t)$ followed by sampling $\theta$ from $(\Theta \mid T = t, U = u)$. The result is identical with the result $\theta^2(u)$ from the fiducial as defined in Definition 3 since $h(u, t) = f(u)$ is the density of $(U \mid T = t)$.

**Proof. Theorem 2** The proof is as the previous, but the density of $(U \mid T = t)$ is now given by a density proportional to $h(u, t) = f(u)w(t, u)$. The required normalization is possible since it is assumed that $T$ is $\sigma$-finite.
8 Closing remarks

As explained in the introduction it is important to establish cases where the fiducial equals a Bayesian posterior, and also the cases where the fiducial is not a Bayesian posterior. In general and special cases this is a difficult task. Theorem 1 shows that existence of a law $P_\Theta$ such that $\tau(u, \Theta)$ has a law that does not depend on $u$ implies that the fiducial equals the resulting Bayesian posterior. The proof of existence of an invariant law $P_\Theta$ is difficult, but in the case of groups the theory is well established. The results of Fraser on fiducial posteriors follow then as corollaries of Theorem 1 as explained in section 6.3.

The correlation coefficient case gives an example where the fiducial is not equal to a Bayesian posterior from the statistical model for the empirical correlation coefficient. We believe that the gamma with known scale, and the gamma where both scale and shape are unknown give two more examples where the fiducial is not a Bayesian posterior, but we do not have a proof of this.

The use of sufficient statistics for the gamma model gives examples of respectively a one- and a two-dimensional loop model. The question of existence of invariant measures for quasi-groups or loops has here been shown to be related to the question of fiducial posteriors. Unfortunately, it seems that the question of existence of invariant measures for quasi-groups is an open and difficult question. A byproduct of the discussion given here is Theorem 3 that shows existence of a smooth loop where an invariant right measure does not exist.

Theorem 2 has a more direct application. It gives an alternative algorithm for Bayesian posterior sampling based on a fiducial model.

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