NORMAL BASES OF RAY CLASS FIELDS OVER IMAGINARY QUADRATIC FIELDS

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Abstract. We develop a criterion for a normal basis (Theorem 2.4), and prove that the singular values of certain Siegel functions form normal bases of ray class fields over imaginary quadratic fields other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ (Theorem 4.2). This result would be an answer for the Lang-Schertz conjecture on a ray class field with modulus generated by an integer ($\geq 2$) (Remark 4.3).

1. Introduction

Let $L$ be a finite Galois extension of a field $K$. From the normal basis theorem (17) we know that there exists a normal basis of $L$ over $K$, namely, a basis of the form \( \{ x^\gamma : \gamma \in \text{Gal}(L/K) \} \) for a single element $x \in L$.

Okada (9) showed that if $k$ and $q$ ($> 2$) are positive integers with $k$ odd and $T$ is a set of representatives for which \( (\mathbb{Z}/q\mathbb{Z})^\times = T \cup (-T) \), then the real numbers \( \left( \frac{1}{2} \right)^k (\cot \pi z) \big|_{z=a/q} \) for $a \in T$ form a normal basis of the maximal real subfield of $\mathbb{Q}(\sqrt{2\pi i/q})$ over $\mathbb{Q}$. Replacing the cotangent function by the Weierstrass $\wp$-function with fundamental period $i$ and $1$, he further obtained in [10] normal bases of class fields over Gauss’ number field $\mathbb{Q}(\sqrt{-1})$. This result was due to the fact that Gauss’ number field has class number one, which can be naturally extended to any imaginary quadratic field of class number one.

After Okada, Taylor (16) and Schertz (12) established Galois module structures of rings of integers of certain abelian extensions over an imaginary quadratic field, which are analogues to the cyclotomic case (8). They also found normal bases by making use of special values of modular functions. And, Komatsu (4) considered certain abelian extensions $L$ and $K$ of $\mathbb{Q}(\sqrt{2\pi i/5})$ and constructed a normal basis of $L$ over $K$ in terms of special values of Siegel modular functions.

For any pair $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ we define the Siegel function $g(r_1, r_2)(\tau)$ on $\mathcal{H}$ (the complex upper half-plane) by the following infinite product

$$g(r_1, r_2)(\tau) = -q^{(1/2)} B_2(r_1) e^{\pi i r_2(r_1 - 1)} (1 - q_z) \prod_{n=1}^{\infty} (1 - q^2_z q_z) (1 - q^2_z q_z^{-1}),$$

where $B_2(X) = X^2 - X + 1/6$ is the second Bernoulli polynomial, $q_\tau = e^{2\pi i \tau}$ and $q_z = e^{2\pi i z}$ with $z = r_1 \tau + r_2$. It is a modular unit in the sense of [4].

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Let $K (\neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3}))$ be an imaginary quadratic field of discriminant $d_K$ and $\mathcal{O}_K = \mathbb{Z}[\theta]$ be its ring of integers with

$$\theta = \begin{cases} \sqrt{d_K}/2 & \text{for } d_K \equiv 0 \pmod{4} \\ (-1 + \sqrt{d_K})/2 & \text{for } d_K \equiv 1 \pmod{4} \end{cases}$$

(1.1)

In what follows we denote the Hilbert class field and the ray class field modulo $N\mathcal{O}_K$ for an integer $N (\geq 2)$ by $H$ and $K(N)$, respectively. We showed in [3] that the singular value $x = g_{0,1/N}(\theta)^{12N/\gcd(6,N)}$ is a primitive generator of $K(N)$ over $K$. We achieved this result by showing that the absolute value of $x$ is the smallest one among those of all the conjugates.

In this paper we will show that the conjugates of a high power of $x$ form a normal basis of $K(N)$ over $K$ by applying a criterion for a normal basis (Theorems 2.4 and 4.2). As for the action of $\text{Gal}(K(N)/K)$ on $x$ in the process we adopt the idea of Gee-Stevenhagen ([2], [15]). Our result is also related to the Lang-Schertz conjecture on the Siegel-Ramachandra invariant to construct ray class fields $K(N)$ (Remark 4.3).

2. A criterion for a normal basis

In this section we let $L$ be a finite abelian extension of a number field $K$ with $G = \text{Gal}(L/K) = \{\gamma_1 = \text{id}, \cdots, \gamma_n\}$. Furthermore, we denote by $|\cdot|$ the usual absolute value defined on $\mathbb{C}$.

**Lemma 2.1.** A set of elements $\{x_1, \cdots, x_n\}$ in $L$ is a $K$-basis of $L$ if and only if

$$\det(x_i^\gamma - 1)_{1 \leq i,j \leq n} \neq 0.$$  

**Proof.** Straightforward. □

By $\hat{G}$ we mean the character group of $G$.

**Lemma 2.2** (Frobenius determinant relation). If $f$ is any $\mathbb{C}$-valued function on $G$, then

$$\prod_{\chi \in \hat{G}} \sum_{1 \leq i \leq n} \chi(\gamma_i^{-1})f(\gamma_i) = \det(f(\gamma_i\gamma_j^{-1}))_{1 \leq i,j \leq n}.$$  

**Proof.** See [7] Chapter 21 Theorem 5. □

Combining Lemmas 2.1 and 2.2 we derive the following proposition.

**Proposition 2.3.** The conjugates of an element $x \in L$ form a normal basis of $L$ over $K$ if and only if

$$\sum_{1 \leq i \leq n} \chi(\gamma_i^{-1})x^{\gamma_i} \neq 0 \quad \text{for all } \chi \in \hat{G}.$$  

**Proof.** For an element $x \in L$, set $x_i = x^{\gamma_i}$ ($i = 1, \cdots, n$). We establish that

the conjugates of $x$ form a normal basis of $L$ over $K$

$\iff$ \{ $x_1, \cdots, x_n$ \} is a $K$-basis of $L$ by the definition of a normal basis

$\iff$ \det($x_i^{\gamma_i^{-1}}$)_{1 \leq i,j \leq n} $\neq$ 0 \quad by Lemma 2.1

$\iff$ \sum_{1 \leq i \leq n} \chi(\gamma_i^{-1})x_i \neq 0 \quad \text{for all } \chi \in \hat{G} \quad \text{by Lemma 2.2 with } f(\gamma_i) = x_i.$

□
Now we present a useful criterion which enables us to determine whether the conjugates of an element \( x \in L \) form a normal basis of \( L \) over \( K \).

**Theorem 2.4.** Assume that there exists an element \( x \in L \) such that
\[
| x^{\gamma_i} / x | < 1 \quad \text{for } 1 < i \leq n. \tag{2.1}
\]
Then the conjugates of a high power of \( x \) form a normal basis of \( L \) over \( K \).

**Proof.** By the hypothesis (2.1) we can take a suitably large integer \( m \) such that
\[
| x^{\gamma_i} / x |^m \leq 1 / \# G \quad \text{for } 1 < i \leq n, \tag{2.2}
\]
where \( \# G \) is the cardinality of \( G \). Then, for any \( \chi \in \hat{G} \) we derive that
\[
\left| \sum_{1 \leq i \leq n} \chi(\gamma_i^{-1})(x^m)^{\gamma_i} \right| \geq | x^m | (1 - |(x^m)^{\gamma_i} / x^m |) \quad \text{by the triangle inequality}
\geq | x^m | (1 - (1 / \# G)(\# G - 1)) = | x^m | / \# G > 0 \quad \text{by (2.2)}.
\]
Therefore the conjugates of \( x^m \) form a normal basis of \( L \) over \( K \) by Proposition 2.3. \( \square \)

3. Actions of Galois groups

We shall investigate an algorithm to get all conjugates of the singular value of a modular function.

For each positive integer \( N \), let \( F_N \) be the field of modular functions of level \( N \) defined over \( \mathbb{Q}(\zeta_N) \) with \( \zeta_N = e^{2\pi i / N} \). Then, \( F_N \) is a Galois extension of \( F_1 = \mathbb{Q}(j) \) (\( j \) = the elliptic modular function) whose Galois group is isomorphic to \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) / \{\pm 1\} \) (7 or 14).

Throughout this section we let \( K \) be an imaginary quadratic field of discriminant \( d_K \) and \( \theta \) be as in (1.1).

Under the properly equivalence relation, primitive positive definite binary quadratic forms \( aX^2 + bXY + cY^2 \) of discriminant \( d_K \) determine a group \( C(d_K) \), called the form class group of discriminant \( d_K \). We identify \( C(d_K) \) with the set of all reduced quadratic forms, which are characterized by the conditions
\[
- a < b \leq a < c \quad \text{or} \quad 0 \leq b \leq a = c \tag{3.1}
\]
together with the discriminant relation
\[
b^2 - 4ac = d_K. \tag{3.2}
\]
It is well-known that \( C(d_K) \) is isomorphic to \( \text{Gal}(H/K) \) (11). For a reduced quadratic form \( Q = aX^2 + bXY + cY^2 \) in \( C(d_K) \) we define a CM-point
\[
\theta_Q = (-b + \sqrt{d_K})/2a. \tag{3.3}
\]
Furthermore, we define \( \beta_Q = (\beta_p) \in \prod_p : \text{primes} \, \text{GL}_2(\mathbb{Z}_p) \) as

Case 1 : \( d_K \equiv 0 \pmod{4} \)
\[
\beta_p = \begin{cases} 
\left( \begin{array}{cc}
a & b/2 \\
0 & 1 \\
\end{array} \right) & \text{if } p \nmid a \\
\left( \begin{array}{cc}
-b/2 & -c \\
1 & 0 \\
\end{array} \right) & \text{if } p \mid a \text{ and } p \nmid c \\
\left( \begin{array}{cc}
-a - b/2 & -c - b/2 \\
1 & -1 \\
\end{array} \right) & \text{if } p \mid a \text{ and } p \mid c \\
\end{cases} \tag{3.4}
\end{cases}
\]
Proposition 3.1. Let $\text{GL}_2(\mathbb{Q})$ be a reduced quadratic form in $C(d_K)$, then the value $h^{\beta_Q}(\theta_Q)$ belongs to $K_{(N)}$. Here, we note that there exists $\beta \in \text{GL}_2^+(\mathbb{Q}) \cap M_2(\mathbb{Z})$ such that $\beta \equiv \beta_p \pmod{NJ_p}$ for all primes $p$ dividing $N$ by the Chinese remainder theorem. Then the action of $\beta_Q$ on $\mathcal{F}_N$ is understood as that of $\beta$ which is an element of $\text{GL}_2(\mathbb{Z}/NJ)/\{\pm 1\}$ ($\simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$). Furthermore, we have an isomorphism

$$
C(d_K) \longrightarrow \text{Gal}(H/K)
$$

$$
Q \mapsto (\bar{h}(\theta) \mapsto h^{\beta_Q}(\theta_Q))|_H,
$$

where $h \in \mathcal{F}_N$ is defined and finite at $\theta$.

Proof. See [2] or [15].

Proposition 3.2. Let $\min(\theta, \mathbb{Q}) = X^2 + BX + C \in \mathbb{Z}[X]$. For each positive integer $N$, the matrix group

$$
W_{N,\theta} = \left\{ \begin{pmatrix} t - Bs & -Cs \\ s & t \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/NJ) \right\}
$$

gives rise to a surjection

$$
W_{N,\theta} \longrightarrow \text{Gal}(K_{(N)}/H)
$$

$$
\alpha \mapsto (h(\theta) \mapsto h^\alpha(\theta)),
$$

where $h \in \mathcal{F}_N$ is defined and finite at $\theta$. The action of $\alpha$ on $\mathcal{F}_N$ is the action as an element of $\text{GL}_2(\mathbb{Z}/NJ)/\{\pm 1\}$ ($\simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$). If $K \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$, then the kernel is $\{\pm 1\}$.

Proof. See [2] or [15].

Combining the above two propositions we achieve the next result.

Proposition 3.3. Let $K$ be an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, and $N$ be a positive integer. Then we have a bijective map

$$
W_{N,\theta}/\{\pm 1\} \times C(d_K) \longrightarrow \text{Gal}(K_{(N)}/K)
$$

$$
\alpha \times Q \mapsto (h(\theta) \mapsto h^{\alpha}(\theta_Q)),
$$

where $h \in \mathcal{F}_N$ is defined and finite at $\theta$.

Proof. See [3] Theorem 3.4.

Proposition 3.3 and the following transformation formula of Siegel functions enable us to find all conjugates of the singular value $g_{(0,1/N)}(\theta)^{-12N/gcd(6,N)}$, which will be used to prove our main theorem.
Proposition 3.4. Let \( N \geq 2 \). For \((v, w) \in \mathbb{Z}^2 - N\mathbb{Z}^2\), the function \( g_{(v/N, w/N)}(\tau)^{-12N/\gcd(6, N)} \) is determined by \( \pm(v/N, w/N) \mod \mathbb{Z}^2 \). It belongs to \( F_N \), and \( \alpha \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm1\} \sim \text{Gal}(F_N/F_1) \) acts on the function by
\[
(g_{(v/N, w/N)}(\tau)^{-12N/\gcd(6, N)})^\alpha = g_{(v/N, w/N)\alpha}(\tau)^{-12N/\gcd(6, N)}.
\]

Proof. See [5] Proposition 2.4 and Theorem 2.5. □

4. Normal bases of ray class fields

The following lemma is a slight modification of a result in [9].

Lemma 4.1. Let \( d_K \leq -7 \) be the discriminant of an imaginary quadratic field and \( \theta \) be as in \( \{L\} \). Let \( Q = aX^2 + bXY + cY^2 \) be a reduced quadratic form in \( \mathbb{Z}/K \) and the main theorem of complex multiplication [7] or [14]. It suffices to show by Theorem 2.4 that
\[
|\theta|^N/K \leq 1 \quad \text{for all } \theta \in \mathbb{Z}/K.
\]

Proof. For simplicity we set \( x = g_{(0,1/N)}(\theta)^{-12N/\gcd(6, N)} \), which belongs to \( K(N) \) by Proposition 3.4 and the main theorem of complex multiplication [7] or [14]. It suffices to show by Theorem 2.4 that
\[
|x^\gamma/x| < 1 \quad \text{for all } \gamma \in \text{Gal}(K(N)/K), \quad \gamma \neq \text{id}.
\]

To this end we consider by Propositions 3.3 and 3.4 a conjugate \( x^\gamma \) of \( x \) which is of the form
\[
x^\gamma = (g_{(0,1/N)}(\tau)^{-12N/\gcd(6, N)})^{\beta_Q} = g_{(s/N,t/N)}(\theta_Q)^{-12N/\gcd(6, N)},
\]
for some \( \alpha = \pm(t-Bs/Cx) \in W_{N, \theta}/\{\pm1\} \), where \( \min(\theta, \mathbb{Q}) = X^2 + BX + C \), and \( Q = aX^2 + bXY + cY^2 \) is a reduced quadratic form in \( C(d_K) \). If \( a \geq 2 \), then the inequality (4.1) holds by Lemma 4.1(i). If \( a = 1 \), then we derive from the conditions (3.1) and (3.2) for a reduced quadratic form that
\[
Q = \left\{ \begin{array}{ll}
X^2 - (d_K/4)Y^2 & \text{for } d_K \equiv 0 \pmod{4} \\
X^2 + XY + ((1 - d_K)/4)Y^2 & \text{for } d_K \equiv 1 \pmod{4},
\end{array} \right.
\]
which yields \( \beta_Q \equiv 12 \pmod{N} \) by the definitions (3.3) and (3.5). Thus \( x^\gamma = g_{(s/N,t/N)}(\theta_Q)^{-12N/\gcd(6, N)} \). Moreover, if \( (s, t) \neq (0, \pm1) \pmod{N} \), then the inequality (4.1) is also true by Lemma 4.1(ii) and (iii). It is not necessary to consider the remaining case when \( a = 1 \) and \( (s, t) \equiv (0, \pm1) \pmod{N} \), because \( \gamma = \text{id} \in \text{Gal}(K(N)/K) \) in this case. Therefore (4.1) holds true for all \( \gamma \in \text{Gal}(K(N)/K) \), \( \gamma \neq \text{id} \), as desired. This completes the proof. □
Remark 4.3. Let $K$ be an imaginary quadratic field and $\mathfrak{f}$ be a nontrivial integral ideal of $K$. We denote by $\text{Cl}(f)$ the ray class group modulo $\mathfrak{f}$ and write $C_0$ for its unit class. By definition the ray class field $K_f$ modulo $\mathfrak{f}$ is a finite abelian extension of $K$ whose Galois group is isomorphic to $\text{Cl}(f)$ via the Artin map. For $C \in \text{Cl}(f)$ we take an integral ideal $\mathfrak{c}$ in $C$ so that $\mathfrak{f}^{-1} = [z_1, z_2]$ with $z = z_1/z_2 \in \mathfrak{f}$. We define the Siegel-Ramachandra invariant $g_f(C)$ by

$$g_f(C) = g((a/N, b/N)(z)^{12N},$$

where $N$ is the smallest positive integer in $\mathfrak{f}$ and $a, b$ are integers such that $1 = (a/N)z_1 + (b/N)z_2$. This value depends only on the class $C$ and belongs to $K_f$.

Ramachandra showed in [11] that $K_f$ can be generated over $K$ by certain elliptic unit, but his invariant involves overly complicated product of high powers of singular values of the Klein forms and singular values of the $\Delta$-function for practical use. Thus, Lang proposed in his book ([7] p.292) to find a simpler one by utilizing Siegel-Ramachandra invariants, and Schertz ([13]) showed that $g_f(C_0)$ (so any $g_f(C)$) generates $K_f$ over $K$ under some conditions on $\mathfrak{f}$. He further conjectured that $g_f(C_0)$ is always a ray class invariant.

On the other hand, the modulus $\mathfrak{f}$ can be enlarged by class field theory so that we may assume $\mathfrak{f} = N\mathcal{O}_K$ for an integer $N \geq 2$. By definition we see that $g_f(C_0) = g((0,1/N)(\theta)^{12N})$. If $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, then Theorem 1.2 indicates that $g_f(C_0)$ is indeed a primitive generator of $K_f$ over $K$, which would be an answer for the Lang-Schertz conjecture.

Example 4.4. Let $K = \mathbb{Q}(\sqrt{-5})$ and $N = 6$, then $d_K = -20$ and $\theta = \sqrt{-5}$. We get

$$C(d_K) = \{Q_1 = X^2 + 5Y^2, Q_2 = 2X^2 + 2XY + 3Y^2\}$$

$$\theta_{Q_1} = \sqrt{-5}, \quad \theta_{Q_2} = (-1 + \sqrt{-5})/2, \quad \beta_{Q_1} = (1, 0), \quad \beta_{Q_2} = (1, 2)$$

$$W_{N, \theta}/\{(\pm 1)\} = \{(1, 0), (0, 1), (3, 2), (2, 3)\}.$$  

By Propositions 3.3 and 3.4 the conjugates of $x = g((6,1/6)(\sqrt{-5})^{-12}$ are

$x_1 = x,$

$x_3 = g(3/6, 2/6)(\sqrt{-5})^{-12}, \quad x_4 = g(3/6, 3/6)(\sqrt{-5})^{-12}$

$x_5 = g(3/6, 2/6)((-1 + \sqrt{-5})/2)^{-12}, \quad x_6 = g(1/6, 5/6)((-1 + \sqrt{-5})/2)^{-12}$

$x_7 = g(3/6, 1/6)((-1 + \sqrt{-5})/2)^{-12}, \quad x_8 = g(5/6, 4/6)((-1 + \sqrt{-5})/2)^{-12}$

possibly with multiplicity. By using a computer program (such as MAPLE) one can find that

$$|x_i/x_1| < 10^{-4} < 1/\#\text{Gal}(K(6)/K) = 1/8 \quad \text{for } i = 2, \ldots, 8.$$  

Therefore $\{x_1, \ldots, x_8\}$ becomes a normal basis of $K(6)$ over $K$ by Theorem 2.4. Moreover, one can show that the minimal polynomial of $x$ would be

$$(X - x_1)\cdots(X - x_8) = X^8 - 1263840X^7 + 42016796X^6 + 7289440X^5$$

$$+ 1505640X^4 - 4525280X^3 + 167196X^2 - 1280X + 1$$

with integer coefficients ([5] §3).

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