MBM loci in families of hyperkähler manifolds

and centers of birational contractions

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Abstract

An MBM class on a hyperkähler manifold $M$ is a second cohomology class such that its orthogonal complement in $H^2(M)$ contains a maximal dimensional face of the boundary of the Kähler cone for some hyperkähler deformation of $M$. An MBM curve is a rational curve in an MBM class and such that its local deformation space has minimal possible dimension $2n - 2$, where $2n$ is the complex dimension of $M$. We study the MBM loci, defined as the subvarieties covered by deformations of an MBM curve within $M$. When $M$ is projective, MBM loci are centers of birational contractions. For each MBM class $z$, we consider the Teichmüller space $\text{Teich}^\text{min}_z$ of all deformations of $M$ such that $z^\perp$ contains a face of the Kähler cone. We prove that for all $I, J \in \text{Teich}^\text{min}_z$, the MBM loci of $(M, I)$ and $(M, J)$ are homeomorphic under a homeomorphism preserving the MBM curves, unless possibly the Picard number of $I$ or $J$ is maximal.

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1 Introduction

1.1 Teichmüller spaces in hyperkähler geometry

Let $M$ be a complex manifold. Recall that the Teichmüller space $\text{Teich}$ of complex structures on $M$ is the quotient $\text{Teich} := \text{Comp} / \text{Diff}_0$, where $\text{Comp}$ is the space of complex structures (with the topology of uniform convergence of all derivatives) and $\text{Diff}_0$ the connected component of the diffeomorphism group. In this paper we are interested in the action of the mapping class group $\Gamma$ (i.e. the finite-index subgroup of $\text{Diff} / \text{Diff}_0$ on $\text{Teich}$ (see Section 3).

In our case $M$ is a compact holomorphically symplectic manifold of Kähler type\footnote{By the Calabi-Yau theorem (Theorem 2.5), this is the same as a hyperkähler manifold.}. We assume that $M$ has maximal holonomy (Definition 2.6; such an $M$ is also called irreducible) and consider the Teichmüller space of all complex structures of hyperkähler type (Subsection 3.1). By a result of D. Huybrechts, this space has finitely many connected components, and we take the one containing the parameter point of our given complex structure; in other words we consider the Teichmüller space of all hyperkähler deformations of $M$. By abuse of notation, this space is also denoted $\text{Teich}$. The action of the mapping class group $\Gamma$ (i.e. the finite-index subgroup of $\text{Diff} / \text{Diff}_0$ preserving our connected component) on $\text{Teich}$ is ergodic, and its orbits are classified using the Ratner’s orbit classification theorem (Theorem 5.5).

In [Ma2], E. Markman has constructed the universal family

$\nu : \mathcal{U} \longrightarrow \text{Teich}$. 

The map $\nu$ is a smooth complex analytic submersion with fiber $(M, I)$ at a point $I \in \text{Teich}$ (throughout the paper, $(M, I)$ denotes a manifold $M$...
equipped with a complex structure $I$). In this paper we investigate the action of the mapping class group $\Gamma$ on this universal fibration. We are interested in applications to the geometry of rational curves $C \subset (M, I)$.

Fix a homology class $z \in H_2(M, \mathbb{Z})$. Let $\Gamma_z \subset \Gamma$ be the stabilizer of $z$ in $\Gamma$, and $\text{Teich}_z$ the Teichmüller space of all complex structures $I \in \text{Teich}$ such that $z$ is of Hodge type $(1,1)$ on $(M, I)$.

Recall that the second cohomology group of a hyperkähler manifold with maximal holonomy is equipped with a canonical bilinear symmetric pairing $q$, called Bogomolov-Beauville-Fujiki (BBF) form (Definition 2.10). This form is integral but in general not unimodular, so that it embeds $H_2(M, \mathbb{Z})$ into $H^2(M, \mathbb{Q})$ as an overlattice of $H^2(M, \mathbb{Z})$. It is often convenient to consider the homology classes of curves as second cohomology classes with rational coefficients, which we do throughout the paper. In the projective case, rational curves in the class $z$ satisfying $q(z, z) < 0$ can be contracted birationally by the Kawamata base point free theorem (Theorem 5.2). It is interesting to study the loci covered by such curves. In this paper we are concerned with the question of invariance of these loci by deformations.

It turns out, and is very helpful for such a study, that for any integral class $z \in H_2(M, \mathbb{Z})$ with $q(z, z) < 0$, the action of the group $\Gamma_z$ on $\text{Teich}_z$ is also ergodic on each connected component. Moreover we can classify, in the same way as for $\Gamma$ acting on $\text{Teich}$, the orbits of $\Gamma_z$-action on the space $\text{Teich}_{z}^{\text{min}}$, which is the same as $\text{Teich}_z$ up to inseparability issues (Theorem 5.6).

More precisely, the space $\text{Teich}_z$ is a smooth, non-Hausdorff manifold, equipped with a local diffeomorphism to the corresponding period space $\text{Per}_z := \frac{SO(3,b_2-4)}{SO(1,b_2-4) \times SO(2)}$ (alternatively, this is just the orthogonal of $z$ in the usual period space $\text{Per}$, seen as a subset of a quadric in the projective space $\mathbb{P}H^2(M, \mathbb{C})$), which becomes one-to-one if we glue together the inseparable points. Following E. Markman [Ma1], the set of preimages of $I \in \text{Per}_z$ (that is, the set of points inseparable from a given one $(M, I)$) is identified with the set of the Kähler chambers in the positive cone of $H^{1,1}(M, I)$, so that each Kähler chamber can be seen as the Kähler cone of the corresponding complex structure. For the classes relevant for us, the so-called MBM classes (Subsection 1.2 and Section 4), the orthogonal complement $z^\perp$ contains one of the walls of these Kähler chambers (Theorem 4.6). It is convenient to restrict oneself to the Kähler chambers adjacent to the $z^\perp$ wall (in other words, the complex structures where $z^\perp$ is actually part of the boundary of the Kähler cone): in this way we obtain the space $\text{Teich}_{z}^{\text{min}} \subset \text{Teich}_z$. Note that both spaces are non-Hausdorff even at their general points, since there are always at least two chambers adjacent to a given wall. We remark
that $z^\perp$ is co-oriented, and pick the set of the chambers adjacent to $z^\perp$ on the positive side. This last space, separated at its general point, is denoted $\text{Teich}_z^{\text{min}} \subset \text{Teich}_z$. This is precisely the space of complex structures $I \in \text{Teich}_z$ such that a positive multiple of $z$ is represented by an extremal rational curve: indeed, by a result of Huybrechts and Boucksom, the Kähler cone is characterized as the set of $(1,1)$-classes which are positive on all rational curves ([H1, H2], [Bou]).

In this paper we are interested in subvarieties $Z_I \subset M_I$ covered by extremal or minimal curves. It is known since [R] (see also [AV1]) that a minimal curve in the cohomology class $z$ on a hyperkähler manifold $(M, I)$ deforms locally together with $(M, I)$ as long as $z$ remains of type $(1,1)$. Since $\text{Teich}_z$ is not Hausdorff, we cannot deduce from this that the curve deforms as long as the complex structure stays in $\text{Teich}_z \subset \text{Teich}$. However this is obviously true modulo this inseparability issue: for each complex structure $I \in \text{Teich}_z$ there exists an $I'$ unseparable from $I$ and carrying a deformation of our curve (which, however, does not have to be extremal anymore - it can, for instance, become reducible). It turns out that we can actually be more precise: over $\text{Teich}_z^{\text{min}}$, the homeomorphism class (and actually stratified diffeomorphism, and also bi-Lipschitz equivalence class) of the locus $Z_I \subset (M, I)$ defined as above does not depend on $I$, except possibly for the complex structures with maximal Picard number.

The proof of this is based on the following observation. Let $E \xrightarrow{\varphi} B$ be a proper holomorphic (or even real analytic) map, and assume that $B$ is obtained as a union of dense subsets, $B = \bigsqcup_{\alpha \in I} B_\alpha$, such that for any index $\alpha$, all fibers of $\varphi$ over $b \in B_\alpha$ are isomorphic. Then all fibers of $\varphi$ are homeomorphic, stratified diffeomorphic and bi-Lipschitz equivalent.

This observation is based on the classical results by Thom and Mather (the bi-Lipschitz case is due to Parusiński[3]). They proved that for any proper real analytic fibration $E \xrightarrow{\varphi} B$, there exists a stratification of $B$ such that the restriction of $\varphi$ to open strata is locally trivial in the category of topological spaces (or in bi-Lipschitz category). Since each $B_\alpha$ in the decomposition $B = \bigsqcup_{\alpha \in I} B_\alpha$ intersects the open stratum, this implies that all fibers of $E \xrightarrow{\varphi} B$ are homeomorphic and bi-Lipschitz equivalent.

The dense subsets $B_\alpha$ are in our case provided by the ergodicity of the mapping class group action.

We state our main results precisely in the subsection 1.3 after a brief digression on rational curves in the next subsection.

\[\footnote{[Pa1], [Pa2].}\]
1.2 MBM loci on hyperkähler manifolds

Let $C \subset M$ be a rational curve on a holomorphic symplectic manifold of dimension $2n$. According to a theorem of Ran [R], the irreducible components of the deformation space of $C$ in $M$ have dimension at least $2n - 2$.

**Definition 1.1:** A rational curve $C$ in a holomorphic symplectic manifold $M$ is called **minimal** if every component of its deformation space has dimension $2n - 2$ at $C$.

The dimension of a maximal irreducible uniruled subvariety of $M$ can take any value between $n$ and $2n - 1$. Such a subvariety is always coisotropic, and applying bend-and-break lemma one sees that there is always a minimal curve through a general point of such a subvariety ([AV1], Section 4).

The key property of a minimal curve is that such a curve deforms together with its cohomology class $[C]$. More precisely, any small deformation of $M$ on which $[C]$ is still of type $(1, 1)$, contains a deformation of $[C]$ ([AV1], Corollary 4.8). Taking closures in the universal family over $\text{Teich}_2$ gives a submanifold of $\text{Teich}_2$ of maximal dimension (which does not have to coincide with $\text{Teich}_2$, as it is not Hausdorff) such that every complex structure in this submanifold carries a deformation of $C$; this curve, however, can degenerate to a reducible curve, and one cannot in general say much about the cohomology classes of its components (Markman’s example on K3 surfaces is already enlightening, see [Ma3], Example 5.3).

In [AV1], we have defined and studied the MBM classes: these are classes $z \in H^2(M, \mathbb{Z})$ such that, up to monodromy and birational equivalence, $z$ contains a face of the Kähler cone. In other words, $z$ contains a face of some Kähler chamber (see [Ma1] for the definition of the latter, but it amounts to say that those are monodromy transforms of Kähler cones of the birational models of $M$). It is clear that the Beauville-Bogomolov square $q(z)$ is then negative; on the other hand, one can characterize MBM classes as negative classes such that some rational multiple $\lambda z$ is represented by a rational curve on a deformation of $M$ ([AV1], Theorem 5.11). For our purposes, it is convenient to extend the notion of MBM on the rational cohomology (or integral homology) classes in an obvious way.

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3 By convention, our “face”, or “wall”, shall always mean a face of maximal dimension, which is equal to $h^{1,1} - 1$. 
We would like to study rational curves whose class is MBM and prove certain deformation invariance statements related to such curves. The notion of MBM class is defined up to a rational multiple, whereas by the above discussion we want to restrict to “minimal” rational curves. The most straightforward definition of minimality, cf. [AV1], is probably “of minimal degree in the uniruled subvariety covered by its deformations”; then by bend-and-break the curve deforms within $M$ in a family of dimension at most $2n - 2$ and one deduces from this as in [R] or [AV1] that it deforms outside $M$ together with its cohomology class. However the uniruled subvariety in question might have several irreducible components. To keep track of this we have to consider the minimality with respect to those components, hence [Definition 1.1].

Note that it is apriori possible (though we don’t have any example) that the same rational curve $C$ is contained in two maximal irreducible uniruled subvarieties $Z_1$ and $Z_2$ of $M$, in such a way that the deformations of $C$ lying in $Z_1$ form a $2n - 2$-parameter family whereas those lying in $Z_2$ need more parameters. Such a $C$ is, by our definition, not minimal, but its generic deformation in $Z_1$ is.

**Definition 1.2:** An MBM curve is a minimal curve $C$ such that its class $[C]$ is MBM.

**Definition 1.3:** Let $C$ be an MBM curve on a hyperkähler manifold $(M,I)$, and $B$ an irreducible component of its deformation space in $M$ containing the parameter point for $C$. An MBM locus of $C$ is the union of all curves parameterized by $B$.

As mentioned in the beginning of this subsection, the MBM loci are coisotropic subvarieties which can have any dimension between $n$ and $2n - 1$, but the family of minimal rational curves in an MBM locus always has $2n - 2$ parameters.

**Definition 1.4:** Let $z$ be an MBM class in $H^2(M,\mathbb{Q})$. The full MBM locus of $z$ is the union of all MBM curves of cohomology class proportional to $z$ and their degenerations (in other words, the union of all MBM loci for MBM curves of cohomology class proportional to $z$).

**Remark 1.5:** If the complex structure on $M$ is in $\text{Teich}_z^{\text{min}}$, the full MBM locus has only finitely many irreducible components and is simply the union of all rational curves of cohomology class proportional to $z$. We sketch the
argument here and refer to Section 5 for details. The reason for the finiteness is that when \((M, I)\) is projective, this MBM locus is the exceptional set of a birational contraction. Indeed, for any rational point in the interior of the face of the Kähler cone associated with \([C]^{\perp}\), the corresponding line bundle is semiample by the Kawamata base-point-free theorem, and the corresponding birational contraction contracts exactly the curves which have cohomology class proportional to that of \(C\). The number of irreducible components of an exceptional set of a contraction is finite. One knows that these are uniruled and by bend-and-break lemma one finds a minimal rational curve in each.

On the other hand, any \((M, I)\) admits a projective small deformation. By semi-continuity one deduces that the number of irreducible components of a locus covered by rational curves of cohomology class proportional to \(z\) is always finite. Finally, there is a minimal curve in every uniruled component, and since we are in \(\text{Teich}_z^{\text{min}}\), the class of such an MBM curve remains proportional to \(z\), so that our uniruled components are MBM loci.

### 1.3 Main results of this paper

Global deformations of rational curves, even minimal ones, in a given cohomology class \(z \in H^2(M, \mathbb{Q})\) (where \(M\) is the underlying differentiable manifold) are rather difficult to understand. As an illustration of what we still cannot do, consider a general projective K3 surface \(S\): by an argument due to Bogomolov and Mumford ([MM] and more recent work [BoHT], [LL]), \(S\) contains (singular) rational curves. It follows that the second punctual Hilbert scheme of \(S\) (often denoted by \(S^{[2]}\)) contains a rational surface (swept out by deformations of the diagonal in the symmetric square of a rational curve). A natural question is whether each projective deformation of \(S^{[2]}\) (the deformations of \(S^{[2]}\) are often called irreducible holomorphic symplectic fourfolds of K3 type) contains a rational surface. For the moment the answer seems to be still unknown. A natural idea would be to deform the surface together with the cohomology class of a minimal rational curve in it, however this does not always work. For instance, the surface may be contained in a uniruled divisor swept out by deformations of the same minimal curve, in such a way that only the divisor survives on the neighbouring manifolds.

The main point of the present paper is that it turns out to be much easier to understand the deformations of subvarieties swept out by minimal rational curves with negative Beauville-Bogomolov square. As we have mentioned before, the Beauville-Bogomolov quadratic form induces an embedding from
$H_2(M,\mathbb{Z})$ to $H^2(M, \mathbb{Q})$ so that we may view the homology classes of curves as rational cohomology classes of type $(1,1)$. Therefore, it makes sense to talk of the Beauville-Bogomolov square of a curve. By [AV1], the class $z$ of such a curve is MBM, meaning that up to monodromy action and birational equivalence its orthogonal supports a wall of the Kähler cone and this is also the case on each deformation where $z$ remains of type $(1,1)$.

We restrict ourselves to the space $\text{Teich}^{\text{min}}_z \subset \text{Teich}^{\text{min}} \subset \text{Teich}_z$ described in the first subsection. Recall that to construct $\text{Teich}^{\text{min}}_z \subset \text{Teich}_z$, we first take the complex structures where $z$ actually contains a wall of the Kähler cone obtaining $\text{Teich}_{\pm z}$, then take the “positive half” of it. On the space $\text{Teich}^{\text{min}}_z$, there is an action of the subgroup of the monodromy group preserving $z$, and it turns out, thanks to the negativity of $z$, that almost all orbits of this action are dense. This allows us to make conclusions such as the uniform behaviour of subvarieties swept out by curves of class $z$ on the manifolds represented by the points of $\text{Teich}^{\text{min}}_z$.

Our main result is as follows.

**Theorem 1.6:** Let $M$ be a hyperkähler manifold of maximal holonomy and $z \in H_2(M,\mathbb{Z}) \subset H^2(M, \mathbb{Q})$ a class of negative Beauville-Bogomolov square. Assume that $z$ is represented by a minimal rational curve in some complex structure $I$ on $M$ (this means that $z$ and the curve are MBM, see [AV1]). Let $Z = Z_I \subset (M, I)$ be the union of all rational curves in cohomology classes proportional to $z$ and degenerations of such curves. Then for all $I, I' \in \text{Teich}^{\text{min}}_z$ such that the Picard number of $(M, I)$ and $(M, I')$ is not maximal (that is, not equal to $h^{1,1}(M_I)$), there exists a homeomorphism $h : (M, I) \to (M, I')$ identifying $Z_I$ and $Z_{I'}$.

**Proof:** See [Theorem 5.4] □

**Remark 1.7:** In fact $h$ is more than just a homeomorphism: the spaces $Z_I$ and $Z_{I'}$ are naturally stratified by complex analytic subvarieties, and $h$ is a diffeomorphism on open strata.

**Remark 1.8:** Any complex variety $X$ can be locally embedded to $\mathbb{C}^n$. Consider the path metric on $X$ obtained from the usual metric on $\mathbb{C}^n$. It is not hard to see that the bi-Lipschitz class of this metric is independent from the choice of the local embedding. The homeomorphism $h$ constructed in [Theorem 5.4] is bi-Lipschitz (Subsection 5.4).

**Remark 1.9:** When the manifold $(M, I)$ is projective, the varieties $Z_I \subset (M, I)$ are exceptional loci of birational contractions. This observation fol-
lows directly from the Kawamata base point free theorem (Theorem 5.2). Kawamata base point free theorem in non-algebraic setting is unknown, but we conjecture that $Z_I$ are centers of bimeromorphic contractions for non-algebraic Kähler deformations of $(M, I)$ as well.

In Subsection 5.4 we explain the following two variants/strengthenings of Theorem 1.6.

**Theorem 1.10:** In assumptions of Theorem 1.6, let $B_I$ be the Barlet space of all rational curves of cohomology class proportional to $z$. Then the homeomorphism $h : Z_I \rightarrow Z_{I'}$ can be chosen to send any rational curve $C \in B_I$ to some rational curve $h(C) \in B_{I'}$, inducing a homeomorphism from $B_I$ to $B_{I'}$.

Recall that a compact Kähler manifold has a so-called MRC fibration ([Cam], [KMM]) whose fiber at a general point $x$ consists of all the points which can be reached from $x$ by a chain of rational curves. In particular, considering such a fibration on a desingularization of a component of $Z_I$ gives a rational map $Q : Z_I \dashrightarrow Q_I$.

**Theorem 1.11:** In assumptions of Theorem 1.6, consider the MRC fibrations $Q : Z_I \dashrightarrow Q_I$, $Q' : Z_{I'} \dashrightarrow Q_{I'}$. Then $h : Z_I \rightarrow Z_{I'}$ induces a biholomorphism between open dense subsets of the fibers of $Q$ and $Q'$.

**Remark 1.12:** Notice that a homeomorphism between normal complex analytic spaces which is holomorphic in a dense open set is holomorphic everywhere. This result follows from a version of Riemann removable singularities theorem, see e.g. [Mag, Theorem 1.10.3].

When the complex structure we consider is in $\Teich^\text{min}_z$, the set $Z$ defined as above is the full MBM locus of $z$. We also have a similar statement for MBM loci of curves, which is a straightforward consequence of Theorem 1.6 and Theorem 1.10.

**Theorem 1.13:** Let $z$ be an MBM class on a hyperkähler manifold, $C \subset (M, I)$ an MBM curve in this class and $Z_C$ its MBM locus. Then $C$ is deformed to an MBM curve $C_J \subset (M, J)$ for all $J \in \Teich^\text{min}_z$, and the corresponding MBM locus $Z_{C_J}$ is homeomorphic to the MBM locus $Z_C$, except possibly if the Picard number of $(M, I)$ or $(M, J)$ is maximal. This
homeomorphism can be chosen in such a way that all MBM deformations of \( C \) in \( Z_C \) are mapped to MBM deformations of \( C_J \) in \( Z_{C_J} \).

**Remark 1.14:** It is conceivable that two homology classes \( k_z \) and \( l_z \), where \( k \) and \( l \) are positive rational numbers, would both be represented by an MBM curve in the same manifold. Of course, the corresponding MBM loci would be different. So far no such example is known.

The main theorem shall be proved in Subsection 5.3 and the Subsection 5.4 is devoted to the variants.

**Remark 1.15:** One cannot affirm that the same statements hold along the whole of \( \text{Teich}_z \), and this is false already for K3 surfaces. Indeed a \((-2)\)-curve on a K3 surface \( X \) can become reducible on a suitable deformation \( X' \). What we do affirm is that in \( \text{Teich}_z \) there is another point, nonseparable from the one corresponding to \( X' \), such that on the corresponding K3 surface \( X'' \) our curve remains irreducible. In this two-dimensional case, this easily follows from the description of the decomposition into the Kähler chambers in [Ma1]; Theorem 1.6 allows us to go further in the higher-dimensional case, unless possibly if \( X' \) is of maximal Picard rank.

### 2 Hyperkähler manifolds

#### 2.1 Hyperkähler and holomorphically symplectic manifolds

Here we remind basic results and definitions of hyperkähler and holomorphically symplectic geometry. Please see [Bes] and [Bea] for more details and reference.

**Definition 2.1:** A hyperkähler structure on a manifold \( M \) is a Riemannian structure \( g \) and a triple of complex structures \( I, J, K \), satisfying quaternionic relations \( I \circ J = -J \circ I = K \), such that \( g \) is Kähler for \( I, J, K \).

**Remark 2.2:** A hyperkähler (i.e. the one carrying a hyperkähler structure) manifold has three symplectic forms

\[
\omega_I := g(I \cdot, \cdot), \omega_J := g(J \cdot, \cdot), \omega_K := g(K \cdot, \cdot).
\]
Definition 2.3: A holomorphically symplectic manifold is a complex manifold equipped with nowhere degenerate holomorphic $(2,0)$-form.

Remark 2.4: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M, J)$.

Theorem 2.5: (Calabi-Yau, [Yau]; see [Bes]) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

For the rest of this paper, we call a compact Kähler complex manifold hyperkähler if it is holomorphically symplectic.

Definition 2.6: Such a manifold $M$ is moreover called of maximal holonomy, or simple, or IHS (irreducible holomorphically symplectic) if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Theorem 2.7: (Bogomolov’s decomposition, [Bo1], [Bea]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.

Remark 2.8: Further on, all hyperkähler manifolds are tacitly assumed to be of maximal holonomy.

2.2 Bogomolov-Beauville-Fujiki form

Theorem 2.9: (Fujiki) Let $\eta \in H^2(M)$, and $\dim M = 2n$, where $M$ is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, for some primitive integer quadratic form $q$ on $H^2(M, \mathbb{Z})$, and $c > 0$ a rational number.

Proof: [F].

Definition 2.10: This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville; see [Bea]):

$$
\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \bar{\Omega}^{n-1} - \frac{n-1}{n} \left( \int_X \eta \wedge \Omega^{n-1} \wedge \bar{\Omega}^{n} \right) \left( \int_X \eta \wedge \Omega^{n} \wedge \bar{\Omega}^{n-1} \right) - 11$$
where $\Omega$ is the holomorphic symplectic form, and $\lambda > 0$.

**Remark 2.11:** The form $q$ has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on $(\Omega, \overline{\Omega}, \omega)$, where $\omega$ is a Kähler form.

### 3 Teichmüller spaces and global Torelli theorem

In this section, we recall the global Torelli theorem for hyperkähler manifolds, and state some of its applications. We follow [V2] and [V1].

#### 3.1 Teichmüller spaces and the mapping class group

**Definition 3.1:** Let $M$ be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (the group of isotopies). Denote by $\text{Comp}$ the space of complex structures on $M$, and let $\text{Teich} := \text{Comp} / \text{Diff}_0(M)$. We call it the Teichmüller space of complex structures on $M$.

**Theorem 3.2:** (Bogomolov-Tian-Todorov) Suppose that $M$ is a Calabi-Yau manifold. Then $\text{Teich}$ is a complex manifold, possibly non-Hausdorff.

**Proof:** This statement is essentially contained in [Bo2]; see [Cat] for more details. ■

Working more specifically with hyperkähler manifolds, one usually takes for $\text{Teich}$ the Teichmüller space of all complex structures of hyperkähler type. It is an open subset in the Teichmüller space of all complex structures by Kodaira-Spencer Kähler stability theorem [KoSp].

**Definition 3.3:** Let $\text{Diff}(M)$ be the group of diffeomorphisms of $M$. We call $\Gamma = \text{Diff}(M) / \text{Diff}_0(M)$ the mapping class group.

**Remark 3.4:** The quotient $\text{Teich} / \Gamma$ is identified with the set of equivalence classes of complex structures.

If $M$ is IHS, the space $\text{Teich}$ has finitely many connected components by a result of Huybrechts ([H3]). We consider the subgroup $\Gamma_0$ of the mapping
class group which preserves the one containing the parameter point for our chosen complex structure.

**Theorem 3.5:** ([VI])

Let $M$ be a simple hyperkähler manifold, and $\Gamma_0$ as above. Then

(i) The image of $\Gamma_0$ in $\text{Aut} H^2(M, \mathbb{Z})$ is a finite index subgroup of the orthogonal lattice $O(H^2(M, \mathbb{Z}), q)$.

(ii) The map $\Gamma_0 \to O(H^2(M, \mathbb{Z}), q)$ has finite kernel.

**Definition 3.6:** We call the image of $\Gamma_0$ in $\text{Aut} H^2(M, \mathbb{Z})$ the monodromy group, denoted by $\text{Mon}(M)$.

**Remark 3.7:** From now on, by abuse of notation and since we are interested in deformations of $(M, I)$, we denote by $\text{Teich}$ the connected component of the Teichmüller space containing the parameter point for our given complex structure, and the mapping class group means the finite index subgroup of $\text{Diff}(M)/\text{Diff}_0(M)$ preserving this component.

### 3.2 The period map

**Remark 3.8:** For any $J \in \text{Teich}$, $(M, J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

**Definition 3.9:** Let $\text{Per} : \text{Teich} \to \mathbb{P}H^2(M, \mathbb{C})$ map $J$ to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $\text{Per} : \text{Teich} \to \mathbb{P}H^2(M, \mathbb{C})$ is called the period map.

**Remark 3.10:** From the properties of BBF form, it follows that $\text{Per}$ maps $\text{Teich}$ into an open subset of a quadric, defined by

$$\text{Per} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$ 

It is called the period space of $M$.

**Remark 3.11:** One has

$$\text{Per} = \frac{SO(b_2 - 3, 3)}{SO(2) \times SO(b_2 - 3, 1)} = \text{Gr}_+(H^2(M, \mathbb{R})).$$
Indeed, the group $SO(H^2(M, \mathbb{R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\text{Per}$, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point.

**Theorem 3.12:** (Bogomolov)
For any hyperkähler manifold, period map is locally a diffeomorphism.

**Proof:** [Bo2].

### 3.3 Birational Teichmüller moduli space

**Definition 3.13:** Let $M$ be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

By a result of Huybrechts ([H1]), any two non-separable points $I, I'$ in the Teichmüller space correspond to birational complex manifolds $(M, I)$ and $(M, I')$. The birational map in question, though, might be biregular: indeed the Teichmüller space is non-separated even for K3 surfaces. The precise description of non-separable points of $\text{Teich}$ can be found in [Ma1] and is as follows. Consider the positive cone $\text{Pos}(M, I)$ which is one of the two connected components of the set of positive vectors $\{x \in H^{1,1}(M, I) | q(x, x) > 0\}$ containing the Kähler cone. By a result of Huybrechts and Boucksom, the Kähler classes are those elements of $\text{Pos}(M, I)$ which are positive on all rational curves ([H1 H2], [Bou]). It turns out that $\text{Pos}(M, I)$ is decomposed into chambers which are Kähler cones of all hyperkähler birational models of $(M, I)$ and their transforms by monodromy. The points of $\text{Teich}$ nonseparable from $I$ correspond to the chambers of this decomposition of $\text{Pos}(M, I)$. We shall return to this in more detail in Section [3]

In particular if there is no rational curve on $(M, I)$, then the Kähler cone of $(M, I)$ is equal to the positive cone and $I$ is a separated point of $\text{Teich}$. Note that a very general hyperkähler manifold has no curves at all; the ones which contain rational curves belong to a countable union of divisors in $\text{Teich}$. Therefore $\text{Teich}$ is separated “almost everywhere”.

**Definition 3.14:** The space $\text{Teich}_b := \text{Teich} / \sim$ is called the **birational Teichmüller space** of $M$, or the **Hausdorff reduction** of $\text{Teich}$.

**Theorem 3.15:** (Torelli theorem for hyperkähler manifolds, [V1])
The period map \( \text{Teich}_b \rightarrow \text{Per} \) is a diffeomorphism, for each connected component of \( \text{Teich}_b \).

**Definition 3.16:** Let \( z \) be a class of negative square in \( H^2(M, \mathbb{Z}) \). We call \( \text{Teich}_z \) the part of \( \text{Teich} \) consisting of all complex structures on \( M \) where \( z \) of type \((1,1)\).

The following proposition is well-known (see e. g. [AV1]).

**Proposition 3.17:** \( \text{Teich}_z = \text{Per}^{-1}(z^\perp) \), where \( z^\perp \) is the set of points corresponding to lines orthogonal to \( z \) in \( \mathbb{P} \text{Per} \subset \mathbb{P}H^2(M, \mathbb{C}) \).

On \( \text{Teich}_z \), we have a natural action of the stabilizer of \( z \) in \( \Gamma \), denoted by \( \Gamma_z \subset \Gamma \).

### 3.4 Ergodicity of the mapping class group action

**Definition 3.18:** Let \( M \) be a complex manifold, \( \text{Teich} \) its Teichmüller space, and \( \Gamma \) the mapping class group acting on \( \text{Teich} \). An **ergodic complex structure** is a complex structure with dense \( \Gamma \)-orbit.

This term comes from the following definition and facts.

**Definition 3.19:** Let \( (M, \mu) \) be a space with measure, and \( G \) a group acting on \( M \) preserving the measure. This action is **ergodic** if all \( G \)-invariant measurable subsets \( M' \subset M \) satisfy \( \mu(M') = 0 \) or \( \mu(M \setminus M') = 0 \).

The following claim is well known.

**Claim 3.20:** Let \( M \) be a manifold, \( \mu \) a Lebesgue measure, and \( G \) a group acting on \( M \) ergodically. Then the set of non-dense orbits has measure 0.

**Proof:** Consider a non-empty open subset \( U \subset M \). Then \( \mu(U) > 0 \), hence \( M' := G \cdot U \) satisfies \( \mu(M \setminus M') = 0 \). For any orbit \( G \cdot x \) not intersecting \( U, x \in M \setminus M' \). Therefore, the set \( Z_U \) of such orbits has measure 0.

**Step 2:** Choose a countable base \( \{U_i\} \) of topology on \( M \). Then the set of points in dense orbits is \( M \setminus \bigcup_i Z_{U_i} \).

**Definition 3.21:** A **lattice** in a Lie group is a discrete subgroup \( \Gamma \subset G \) such that \( G/\Gamma \) has finite volume with respect to Haar measure.
Theorem 3.22: (Calvin C. Moore, [Mo]) Let $\Gamma$ be a lattice in a non-compact simple Lie group $G$ with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of $\Gamma$ on $G/H$ is ergodic. ■

Theorem 3.23: Let $\text{Per}$ be a component of the birational Teichmüller space identified with the period domain, and $\Gamma$ its monodromy group. Let $\text{Per}_e$ be a set of all points $L \in \text{Per}$ such that the orbit $\Gamma \cdot L$ is dense. Then $Z := \text{Per} \setminus \text{Per}_e$ has measure 0.

**Proof. Step 1:** Let $G = SO(3, b_2 - 3), \ H = SO(2) \times SO(1, b_2 - 3)$. Then $\Gamma$-action on $G/H$ is ergodic, by Moore’s theorem.

**Step 2:** Ergodic orbits are dense, because the union of all non-ergodic orbits has measure 0. ■

In [V2] and [V2bis], a more precise result has been established using Ratner theory. For reader’s convenience we recall the idea of proof as well.

Theorem 3.24: Assume $b_2(M) \geq 5$. Let $\text{Per}$ and $\Gamma$ be as above, then there are three types of $\Gamma$-orbits on $\text{Per}$:

1) closed orbits which are orbits of those period planes

$$V_I := \text{Re}(H^{2,0}(M, I) \oplus H^{0,2}(M, I)) \in \text{Per}$$

which are rational in $H^2(M, \mathbb{R})$ (equivalently, the corresponding complex structures $I$ are of maximal Picard number, as $H^{1,1}(M, I)$ is then also a rational subspace);

2) dense orbits which are orbits of period planes $V_I$ containing no non-zero rational vectors;

3) “intermediate orbits”: orbits of period planes containing a single rational vector $v$. The orbit closure then consists of all period planes containing $v$.

**Idea of proof:** Let $G = SO^+(3, b_2 - 3)$ and $H = SO^+(1, b_2 - 3)$, so that $G/H$ is fibered in $SO(2)$ over $\text{Per}$. The group $H$ is generated by unipotents (this is why this time we take $H = SO^+(1, b_2 - 3)$ rather than $H = SO(2) \times SO^+(1, b_2 - 3)$, so that Ratner theory applies to the action of $H$ on $\Gamma \setminus G$, where $\Gamma$ remains the same as above). Ratner theory describes orbit closures of this action: $\overline{\Gamma H}$ is again an orbit under a closed intermediate subgroup $S$, also generated by unipotents and in which $\Gamma \cap S$ is a lattice.
From the study of Lie group structure on $G$ we derive that the subgroup must be either $H$ itself (the orbit is closed), or the whole of $G$ (the orbit is dense), or the stabilizer of an extra vector $\cong SO^+(2, b_2 - 3)$ (the third case). One concludes passing in an obvious way (via the double quotient) from an $H$-action on $\Gamma \backslash G$ to a $\Gamma$-action on $G/H$. 

A useful point for us is the following observation from [V2bis].

**Proposition 3.25:** In the last case, the orbit closure is a fixed point set of an antiholomorphic involution, in particular, it is not contained in any complex submanifold nor contains any positive-dimensional complex submanifold.

For our present purposes we need the following variant of Theorem 3.24.

**Theorem 3.26:** Assume $b_2(M) > 5$. Let $z \in H^2(M, \mathbb{Z})$ be a negative class and $\mathbb{P}er_z = z^+ \subset \mathbb{P}er$ be the locus of period points of complex structures where $z$ is of type $(1, 1)$. Let $\Gamma_z$ be the subgroup of $\Gamma$ fixing $z$. Then the same conclusion as in Theorem 3.24 holds, namely an orbit of $\Gamma$ is either closed and consists of points with maximal Picard number, or dense, or the orbit closure consists of period planes containing a rational vector and in this last case it is not contained in any complex subvariety nor contains one.

**Proof:** It is exactly the same as the proof of Theorem 3.24 once one interprets $\mathbb{P}er_z$ as the Grassmannian of positive 2-planes in $V = z^+ \subset H^2(M, \mathbb{R})$, takes $G = SO^+(3, b_2 - 4)$, $H = SO^+(1, b_2 - 4)$ and replaces $\Gamma$ by $\Gamma_z$. 

4 MBM curves and the Kähler cone

The notion of MBM classes was introduced in [AV1] and studied further in [AV2]. We recall the setting and some results and definitions from these papers.

**Definition 4.1:** The BBF form on $H^{1,1}(M, \mathbb{R})$ has signature $(1, b_2 - 3)$. This means that the set $\{\eta \in H^{1,1}(M, \mathbb{R}) \mid \langle \eta, \eta \rangle > 0\}$ has two connected components. The component which contains the Kähler cone $Kah(M)$ is called the positive cone, denoted $Pos(M)$.

The starting point is the following theorem.
Theorem 4.2: (Huybrechts [H1, H2], Boucksom [Bou])
The Kähler cone of $M$ is the set of all $\eta \in \text{Pos}(M)$ such that $(\eta, C) > 0$ for all rational curves $C$.

Remark that it is sufficient to consider the curves of negative square (as only these have orthogonals passing through the interior of the positive cone) and moreover extremal, i.e. such that their cohomology class cannot be decomposed as a sum of classes of other curves. An extremal curve is minimal in the sense of our Definition 1.1, though apriori the converse needs not be true.

Remark 4.3: Here, using the BBF form, we identify $H_2(M, \mathbb{Q})$ and $H^2(M, \mathbb{Q})$ for any hyperkähler manifold $M$. This allows us to talk of the BBF square of a homology class of a curve.

The Kähler cone is thus locally polyhedral in the interior of the positive cone (with some round pieces in the boundary), and its faces are supported on the orthogonal complements to the extremal curves.

The notion of an extremal curve is however not adapted to the deformation-invariant context. In order to put the theory in this context we have defined the MBM (monodromy birationally minimal) classes in [AV1]. Here we recall several equivalent definitions (we refer to Section 5 of [AV1] for proofs of equivalences).

Definition 4.4: A negative class $z$ in the image of $H_2(M, \mathbb{Z})$ in $H^2(M, \mathbb{Q})$ is called MBM if $\text{Teich}_z$ contains no twistor curves. This is equivalent to saying that a rational multiple of $z$ is represented by a curve in some complex structure where the Picard group is generated by $z$ over the rationals, and also to saying that in some complex structure $X = (M, I)$ where $z$ is of type $(1, 1)$, the orthogonal complement $\gamma(z)^\perp$ contains a face of the Kähler cone of a birational model $X' = (M, I')$ of $X$ (whence the terminology). Moreover in these two equivalent definitions, “some” may be replaced by ”all” without changing the content.

Remark 4.5: Let $M$ be a hyperkähler manifold. We call a codimension 1 face, or just a face of the Kähler cone if there is no risk of confusion, a subset of its boundary with nonempty interior obtained as the intersection of this boundary and a hyperplane in $H^{1,1}(M, \mathbb{R})$ - in other words, for us a face is always of maximal dimension unless otherwise specified.
**Theorem 4.6:** ([AV1], Theorem 6.2) The Kähler cone is a connected component of the complement, in $\text{Pos}(M)$, of the union of hyperplanes $z^\perp$ where $z$ ranges over MBM classes of type $(1,1)$.

**Definition 4.7:** (cf. [Ma1]) The Kähler chambers are other connected components of this complement.

Moreover we have the following connection between the Kähler chambers and the inseparable points of the Teichmüller space (note that the decomposition of Pos into the Kähler chambers is an invariant of a period point rather than of the complex structure itself, since it is determined by the position of $H^{1,1}$ in $H^2(M, \mathbb{R})$):

**Theorem 4.8:** ([Ma1], theorem 5.16) The points of a fiber of $\text{Per}$ over a period point are in bijective correspondence with the Kähler chambers of the decomposition of the positive cone of the corresponding Hodge structure.

**Definition 4.9:** The space $\text{Teich}_{\pm z}^\text{min} \subset \text{Teich}_z$ is obtained by removing the complex structures where $z$ does not support a wall of the Kähler cone.

In other words, at a general point of $\text{Teich}_z$, where the Picard group is generated by $z$ over the rationals, $\text{Teich}_{\pm z}^\text{min}$ coincides with $\text{Teich}_z$, whereas at special points of $\text{Teich}_z$ where we have other MBM classes as well, we remove those complex structures where e.g. $z$ becomes a sum of two effective classes, and rational curves representing $z$ thus cease to be extremal.

Notice that the space $\text{Teich}_{\pm z}^\text{min}$ is not separated even at its general point, since $z^\perp$ divides the positive cone in at least two chambers. In order to avoid working with such generically non-separated spaces we divide $\text{Teich}_{\pm z}^\text{min}$ in two halves:

**Definition 4.10:** The space $\text{Teich}_z^\text{min}$ is the part of $\text{Teich}_{\pm z}^\text{min}$ where $z$ has non-negative intersection with Kähler classes (that is, $z$ is pseudo-effective).

Now at a general point $\text{Teich}_z^\text{min}$ coincides with $\text{Per}_z$ (but at special points it is still non-separated).
5 Loci of MBM curves in families

Proposition 5.1: Let $z$ be an MBM class in some complex structure $I \in \Teich_{	ext{min}}$. The full MBM locus $Z$ is the union of all rational curves $C$ such that $[C]$ is proportional to $z$.

Proof: We have defined the full MBM locus as the union of subvarieties swept out by minimal rational curves of cohomology class proportional to $z$, so clearly the full MBM locus is included in the union of all rational curves of cohomology class proportional to $z$. On the other hand, take any component of the latter. By bend-and-break one can find a minimal rational curve through a general point of this component (see for example [AV1], theorem 4.4, corollary 4.6), so this is also a component of $Z$.

These loci are interesting since, at least in the projective case, these are centers of elementary birational contractions (Mori contractions). Indeed, recall the following partial case of Kawamata base-point-freeness theorem.

Theorem 5.2: (Kawamata bpf theorem, [K1])
Let $L$ be a nef line bundle on a projective manifold $M$ such that $L^\otimes a \otimes \mathcal{O}(-K_M)$ is big for some $a$. Then $L$ is semiample.

Here a line bundle $L$ is said to be nef if $c_1(L)$ is in the closure of the Kähler cone, and big if the dimension of the space of global sections of its tensor powers has maximal possible growth. For the nef line bundles this last condition is equivalent to $c_1(L)^{\dim M} > 0$ (that is, the maximal self-intersection number of $L$ being positive). A semiample line bundle is a line bundle $L$ such that $L^\otimes n$ is base point free for some $n$; then for $n$ big enough the linear system of sections of $L$ defines a projective morphism with connected fibers $\varphi : M \to M_0$. The bigness of $L$ in fact implies that $\varphi$ is birational. Clearly, for a curve $C$, $\varphi(C)$ is a point if and only if $L \cdot C = 0$.

If $M$ is a holomorphic symplectic manifold, the canonical divisor $K_M$ is zero and any big and nef line bundle is semiample. Let $z$ be a class in $H^2(M, \mathbb{Z})$ such that $z^+$ contains a face of the Kähler cone. If $M$ is projective, there is an integral point in the interior of this face, and this point is the Chern class of a nef and big line bundle $L$. By Kawamata base point freeness $L$ is semiample and the morphism associated with the space of sections of $L^\otimes n$ contracts exactly the curves with cohomology class proportional to $z$. It is well-known that the exceptional set of a birational contraction on a
hyperkähler manifold is covered by rational curves (one can deduce this for instance from [K2, Theorem 1]). It follows that the exceptional set of $\varphi$ is exactly the locus $Z$.

**Corollary 5.3:** $Z$ has finitely many irreducible components.

**Proof:** Minimal rational curves survive on the small deformations of $M$ provided that their cohomology class stays of type $(1,1)$ (see [AV1], corollary 4.8), and their loci can only collide over closed subsets of the parameter space. So if $Z$ has infinitely many components, so does the “neighbouring” full MBM locus $Z_{I'}, I' \in \text{Teich}_z^{\text{min}}$. But one can always find an $I'$ close to $I$ such that $(M, I')$ is projective, and in this case it is an exceptional set of a holomorphic birational contraction.

The purpose of this section is to prove the following result (Theorem 1.6 from the Introduction).

**Theorem 5.4:** Let $M$ be a simple hyperkähler manifold and $z$ an MBM class. For any complex structure $I \in \text{Teich}_z^{\text{min}}$, let $M_I$ denote the complex manifold $M$ equipped with $I$ and $Z_I \subset M_I$ the subvariety covered by rational curves of cohomology class proportional to $z$. Then for all $I \in \text{Teich}_z^{\text{min}}$ such that the Picard number of $M_I$ is not maximal, the subvarieties $Z_I$ are homeomorphic.

The proof uses two main ingredients: ergodicity of monodromy action on $\text{Teich}_z^{\text{min}}$ and Whitney stratification.

### 5.1 Mapping class group action on $\text{Teich}_z^{\text{min}}$

The group $\Gamma_z \subset \Gamma$ obviously acts on $\text{Teich}_z^{\text{min}}$. Indeed the action of any $\gamma \in \Gamma$ is just the transport of the complex structure; if $z^\perp$ contains a wall of the Kähler cone in a complex structure $I$, then so does $\gamma z$ in the complex structure $\gamma I$. Notice that the same remark applies to rational curves: $\gamma C$ is a rational curve in the structure $\gamma I$ and the minimality is preserved. So the locus $Z \subset X = (M, I)$ is sent by an element of $\Gamma_z$ to $Z_{\gamma I} \subset X' = (M, \gamma I)$.

It turns out that the results on the mapping class group action on $\text{Per}$ “lift” to those on the action on $\text{Teich}$, but if we want to work on a subspace where $z$ remains of type $(1,1)$ this has to be $\text{Teich}_z^{\text{min}}$ rather than $\text{Teich}$.

The following theorem from [V2, V2bis] strengthens [Theorem 3.24]
Theorem 5.5: Assume $b_2(M) \geq 5$. Let $\Gamma$ denote the mapping class group. Then there are three types of $\Gamma$-orbits on $\text{Teich}$: closed (where the period planes are rational, thus the complex structures have maximal Picard number), dense (where the period planes contain no rational vectors) and such that the closure is formed by points whose period planes contain a fixed rational vector $v$. In the last case, the orbit closure $C_v$ is totally real, so that no neighbourhood of a point $c \in C_v$ in $C_v$ is contained in a proper complex subvariety of $\text{Teich}$.

The proof proceeds by establishing that the period map commutes with taking orbit closures, in the following way. Introduce the space $\text{Teich}_K$ which consists of pairs $(I, \omega)$ where $I \in \text{Teich}$ and $\omega \in \text{Kah}(I)$ is of square 1. Calabi-Yau theorem (Theorem 2.5) immediately implies that this is the Teichmüller space of pairs (complex structure, hyperkähler metric compatible with it). As shown in [AV3], the period map is injective on the space of hyperkähler metrics; therefore, it is injective on $\text{Teich}_K$. In other words, $\text{Teich}_K$ is naturally embedded in $\text{Per}_K$, the homogeneous manifold of all pairs consisting of a period point $I \in \text{Per}$ and an element $\omega$ of square one in its positive cone (which indeed depends only on the period point, not on the complex structure itself). The latter is a homogeneous space, so we can try to apply Ratner theory to prove the following result, which clearly implies what we need: for any $I$, the closure of the $\Gamma$-orbit of $(I, \text{Kah}(I)) \subset \text{Teich}_K \subset \text{Per}_K$ contains the orbit of $(\text{Per} I, \text{Pos}(I))$ (here by an orbit of the subset we mean the union of its translates). Now one can construct orbits of one-parameter subgroups which are entirely contained in $(I, \text{Kah}(I))$ and such that the closure of their projection to $\text{Per}_K/\Gamma$ contains the projection of the positive cone. Indeed, one deduces from the non-maximality of the Picard number that $\text{Kah}(I)$ has a “round part”, for instance in the following sense: in the intersection of $\text{Kah}(I)$ with a general 3-dimensional subspace $W$ in $H^{1,1}(I)$, of signature $(1, 2)$. This is used to find many horocycles in $\text{Kah}(I)$ tangent to the round part of the boundary. The horocycle is an orbit of a one-parameter unipotent subgroup. Applying Ratner theory to a sufficiently general horocycle of this type, one sees that the closure of its image in $\text{Per}_K/\Gamma$ contains an entire $\text{SO}(H^{1,1}(I))$-orbit ([V2bis], Proposition 3.5), which is the positive cone $\text{Pos}(I)$.

The analogue of Theorem 5.5 in our setting is as follows.

Theorem 5.6: Assume $b_2(M) > 5$ and let $z \in H^2(M, \mathbb{Z})$ be an MBM class
and \( \Gamma_z \) the subgroup of the mapping class group consisting of all elements whose action on the second cohomology fixes \( z \). Then \( \Gamma_z \) acts on \( \text{Teich}^{\text{min}}_z \) ergodically, and there are the same three types of orbits of this action as in Theorem 5.5.

**Proof:** It proceeds along the same lines. We introduce the spaces \( \text{Per}_{K,z} \) consisting of pairs

\[
\{ (\text{Per}(I), \omega \in \text{Pos}(I) \cap z^\perp), q(\omega, \omega) = 1 \}
\]

and \( \text{Teich}_{K,z} \) consisting of pairs \( (I \in \text{Teich}^{\text{min}}_z, \omega) \in \text{Per}_{K,z} \) where \( \omega \) belongs to the wall of \( \text{Kah}(I) \) given by \( z^\perp \). We denote such a wall by \( \text{Kah}(I)_z \), though of course its elements are not Kähler forms on \( I \), but rather semi-positive limits of those. Since the complex structures in \( \text{Teich}^{\text{min}}_z \) which have the same period point are in one-to-one correspondence with the walls of the Kähler chambers in which the other MBM classes partition \( z^\perp \), \( \text{Teich}_{K,z} \) again embeds naturally in \( \text{Per}_{K,z} \). We fix a complex structure \( I \) with non-maximal Picard number. We need to prove that the closure of the \( \Gamma_z \)-orbit of the subset \( (I, \text{Kah}(I)_z) \) contains the orbit of \( (\text{Per}(I), \text{Pos}(I) \cap z^\perp) \). This is done exactly in the same way as in Theorem 5.5. We take a general three-dimensional subspace \( W \) in \( z^\perp \), the intersection of \( W \) with our wall \( \text{Kah}(I)_z \) contains horocycles, and we deduce from Ratner orbit closure theorem and Proposition 3.5 of [V2bis] that the closure of the projection of such a horocycle to \( \text{Per}_{K,z} / \Gamma_z \) is large, containing an \( \text{SO}(H^{1,1}(I) \cap z^\perp) \)-orbit, which is the projection of \( \text{Pos}(I) \cap z^\perp \).

### 5.2 Stratification

Consider the universal family \( \mathcal{X} \) over \( \text{Teich}^{\text{min}}_z \) ([M2]). We are interested in the family \( \mathcal{Z} \subset \mathcal{X} \) with the fiber over \( I \in \text{Teich}^{\text{min}}_z \) obtained as the full MBM locus of \( z \) on the complex manifold \( X = (M, I) \). This family can be constructed, for instance, by taking the image of the evaluation map for all components of the relative Barlet space corresponding to cohomology classes proportional to \( z \) and dominating \( \text{Teich}^{\text{min}}_z \). As \( \text{Teich}^{\text{min}}_z \) is not Hausdorff, we shall, whenever necessary, restrict both families to a small neighbourhood \( U \) of some point \( x \), or to a small compact \( K \) within \( U \), and denote by \( \mathcal{X}_U, \mathcal{Z}_U \) the restrictions of these families.

It is well-known that an analytic subset \( W \) of a complex manifold \( Y \) admits a “nice” stratification ([Whitney stratification](#) or [Thom-Mather stratification](#); see [M]). Recall also the following first isotopy lemma by Thom (we refer to [M] for precise definitions and proofs).
Lemma 5.7: ([M], Proposition 11.1) Let $f : Y \to B$ be a smooth mapping of smooth manifolds and $W$ a closed subset of $Y$ admitting Whitney stratification, such that $f : W \to B$ is proper. If the restriction of $f$ to each stratum of $W$ is a submersion then $W$ is locally trivial over $B$.

The idea is that $W$ acquires a structure of a stratified set so that $f$ is a “controlled submersion”, meaning that one can mimick Ehresmann’s construction of diffeomorphism between the fibers of a smooth, proper submersion in this setting.

In our situation, we can clearly stratify $U$ by complex analytic subsets and choose a stratification of $Z_U$ in such a way that the condition of the lemma is satisfied above the strata. We obtain that the family $Z_U$ is locally trivial over a complement to a (lower-dimensional) analytic subset in $\text{Teich}_z^\text{min}$, over a complement to an analytic subset in that analytic subset, etc.

5.3 Proof of the main result

We know that the family $Z$ is locally trivial on the complement to a union (possibly countable, but finite in a neighbourhood of any point in the base) of proper analytic subsets $P \subset \text{Teich}_z^\text{min}$. First we pick a point $x \in \text{Teich}_z^\text{min}$ which is not in $P$ and whose $\Gamma$-orbit is dense. Then $x$ has a neighbourhood $U_x$ over which all fibers $Z_b$ are homeomorphic. Moreover the union $\bigcup_{\gamma \in \Gamma} \gamma(U_x)$ is a dense open subset of $\text{Teich}_z^\text{min}$ and all fibers $Z_b$ over this union are homeomorphic.

Take another point $x' \in \text{Teich}_z^\text{min}$ with dense $\Gamma$-orbit (we call such a complex structure ergodic, see [V2]). Then the orbit of $x'$ hits $\bigcup_{\gamma \in \Gamma} \gamma(U_x)$ and therefore $Z_{x'}$ is homeomorphic to $Z_b$ for $b \in P$.

Now take $y \in \text{Teich}_z^\text{min}$ such that the corresponding complex structure is not ergodic but does not have maximal Picard number either (“the intermediate orbit” of Theorem 3.24 and Theorem 3.26). If $Z_y$ is not homeomorphic to $Z_b$ for $b \in P$, the orbit of $y$ should not hit $P$. As $P$ is open, the orbit closure does not hit $P$ either. As the orbit closure is irreducible it must be contained in an irreducible complement of the complement to $P$, but this is an analytic subvariety. However, the closure of an intermediate orbit is not contained in a proper analytic set, even locally [Proposition 3.25].
5.4 Closing remarks

Basically the same argument proves Theorem 1.10. Indeed, it suffices to consider, instead of the family $Z$ over $\text{Teich}_{\min}^z$, the families $B$ whose fiber over $I \in \text{Teich}_{\min}^z$ is the Barlet space of rational curves of cohomology class $z$, or more generally $\lambda z$ (with a fixed rational $\lambda$), and the incidence family $J \subset X \times_{\text{Teich}_{\min}^z} B$. Notice that this is not quite the situation of the first Whitney’s lemma as the families are not naturally embedded in other families smooth over $\text{Teich}_{\min}^z$, but its generalizations to the singular case do exist in the literature (see e.g. [Ver], Théorème 4.14, Corollaire 5.1). One thus gets the topological triviality of both families.

To prove Theorem 1.11 notice that by Theorem 1.10 $h$ respects the MRC fibrations of the components of $Z_I$ resp. $Z_{I'}$, so that the open parts of the fibers are homeomorphic. Moreover, since $h$ is a stratified diffeomorphism, locally at a general point $x$ of a general fiber $F_I$, the map $h$ restricts to a diffeomorphism $h_F : F_I \to F_{I'}$ with some MRC fiber of a component of $Z_{I'}$. But in every holomorphic tangent direction at $x$ there is a rational curve, and it is sent to a rational curve through its image by $h_F$. Therefore $h_F$ maps the holomorphic tangent space at $x$ into the holomorphic tangent space at $h_F(x)$, and vice versa, in other words, it is holomorphic at $x$.

In Theorem 1.10 and Theorem 1.6 we prove that the fibers of natural families associated with rational curves are homeomorphic and stratified diffeomorphic. However, there is a version of Thom-Mather theory which gives bi-Lipschitz equivalence of the fibers over open strata of Thom-Mather stratification ([Pa1], [Pa2]). Then the same arguments as above prove that the homeomorphisms constructed in Theorem 1.10 and Theorem 1.6 are bi-Lipschitz.

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