Abstract

The partition number $\pi(K)$ of a simplicial complex $K \subset 2^{[n]}$ is the minimum integer $\nu$ such that for each partition $A_1 \uplus \ldots \uplus A_\nu = [n]$ of $[n]$ at least one of the sets $A_i$ is in $K$. A complex $K$ is $r$-unavoidable if $\pi(K) \leq r$. Motivated by the problems of Tverberg-Van Kampen-Flores type, and inspired by the ‘constrained method’ of Blagojević, Frick, and Ziegler [BFZ], we study the combinatorics of $r$-unavoidable complexes.

1 Introduction

‘Unavoidable complexes’, introduced by Blagojević, Frick, and Ziegler in [BFZ], play the fundamental role in their ‘constraint method’ (Gromov-Blagojević-Frick-Ziegler reduction).

Inspired by [BFZ], and in part developing some ideas from the unpublished preprint [JVZ-3], here we begin a systematic study of ‘unavoidable complexes’, as combinatorial objects that may have some independent interest and which may deserve to be studied in their own right.

1.1 Intrinsically non-linear unavoidable complexes

The paper is focused on developing some new ideas and constructions which may provide additional insight into the general structure of ‘unavoidable simplicial complexes’. We begin with the observation that all known ‘key examples’ of $r$-unavoidable complexes, used in the seminal paper [BFZ] (Lemma 4.2), are ‘linearly realizable’ (threshold) complexes, or contain

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such a complex as a subcomplex, see Section 6.1 (and Definitions 28 and 30 in Section 5). For this reason one of the central themes in our paper is the construction of unavoidable complexes which are ‘non-linear’ in some, precisely defined, sense (see Sections 7 and 5.1).

We introduce two natural and possibly new invariants of a simplicial complex $K \subset \mathbb{2}^n$, the partition number $\pi(K)$ (Definition 3) and the threshold characteristic $\rho(K)$ (Definition 32). The ‘linearity’ or ‘non-linearity’ of a complex $K$ is then rephrased as a statement about mutual relationship of $\pi(K)$ and $\rho(K)$ (as exemplified by Proposition 34 and Example 35 in Section 6).

The fundamental relation between these two invariants is the inequality,

$$\pi(K) \leq \left\lfloor \frac{1}{\rho(K)} \right\rfloor + 1$$

(Proposition 34) where the equality holds precisely if $K$ is an ‘intrinsically linear complex’ (see Definition 30 and Corollary 51). We say that $K$ is ‘intrinsically non-linear’ (Definition 30) if the inequality in (1) is strict, and one of our central objectives is to generate and study these complexes. Especially interesting are ‘strongly non-linear complexes’ (Definition 52) which are both intrinsically non-linear and minimal in the sense that $\pi(L) > \pi(K)$ for each proper subcomplex $L \subset K$ of $K$.

1.2 An overview and summary of central new results

Theorem 53 in Section 7 describes a large class of strongly non-linear complexes, emphasizing the role of (highly symmetric) minimal triangulations of manifolds that ‘look like a projective plane’ (Section 6.3).

One of the highlights of the paper is an algorithmic procedure for evaluating $\rho(K)$ (see Propositions 38 and 39 in Section 6), based on the ideas from linear programming, which is quite efficient if $K$ has a reasonably large symmetry group. This connection with polyhedral combinatorics turns out to be very fruitful and a more complete report of (computer experiments and other results) will appear elsewhere.

In order to evaluate (estimate) the partition invariant $\pi(K)$, we rely on the assumption that $K = K_1 \ast \cdots \ast K_n$ has a join decomposition, where $K_i \subset V_i$ is $r_i$-unavoidable for some $r_i \geq 2$. One of our central results (Theorem 15) says that $K$, as a subcomplex of $2^V$ where $V = V_1 \uplus \cdots \uplus V_n$, is $r$-unavoidable provided $r \geq r_1 + \cdots + r_n - n + 1$.

Combined together, these ingredients (the join decomposition $K = K_1 \ast \cdots \ast K_n$ of carefully chosen complexes $K_i$ with known partition number and a large group of automorphisms) lead to the construction of ‘intrinsically non-linear’ unavoidable complexes and ‘strongly non-linear complexes’ (Section 7).

The observation that the large symmetry group may be instrumental in detecting non-linear examples of unavoidable complexes, shifted our attention and focus to known examples of 2-unavoidable (self-dual!) complexes with large group of automorphisms.

An immediate choice were the unique (6-vertex) minimal triangulation $\mathbb{R}P^2_6$ of the real projective plane, and the unique (9-vertex) minimal triangulation $\mathbb{C}P^2_9$ of the complex.
projective plane, together with the three element 0-dimensional complex $[3]$ and the known non-isomorphic simplicial complexes associated with the quaternionic projective plane $\mathbb{HP}^2$ \cite{BK92, BD-2}. The threshold characteristic of these combinatorial ‘projective planes’ is calculated in Section 6.3.

The paper is organized as follows. The partition number $\pi(K)$, together with its relatives and more general versions $\pi_{\mathcal{H}}(K)$ and $\pi_{\mathcal{H}}^{s,j}(K)$, is introduced in Section 2.1. In Section 3 we briefly review the ‘constraint method’ (Gromov-Blagojević-Frick-Ziegler reduction) and the reduction based on ‘Sarkaria’s inequality’. These two reduction procedures have been successfully applied to theorems of Tverberg-Van Kampen-Flores type, and they serve as our primary motivation for studying unavoidable complexes. Fundamental constructions (pushdown, join, etc.) and basic properties of $r$-unavoidable complexes, including the inequality $\pi(K_1 \ast \cdots \ast K_n) \leq \pi(K_1) + \cdots + \pi(K_n) - n + 1$ (Corollary 16), are presented in Section 4. Threshold complexes and their manyfold relations with unavoidable complexes are studied in Sections 5 and 6. The focus is on the ‘threshold characteristic’ $\rho(K)$ of a simplicial complex $K$ and its relationship with the partition number $\pi(K)$. Section 7 provides a summary and a list of our main examples of intrinsically non-linear and strongly non-linear simplicial complexes. Finally in Section 8 we offer a glimpse into some future developments, including a promising connection with cooperative game theory.

1.3 Examples of strongly non-linear $r$-unavoidable complexes

In the focus of our study are so called $r$-unavoidable complexes $K \subset 2[^n]$, characterized by the property that for each partition $[n] = A_1 \cup \cdots \cup A_r$ of $[n]$ into $r$ pieces, there exists $i \in [r]$ such that $A_i \in K$ (see \cite{BFZ, Definition 4.1} and \cite{JVZ-3, Section 2} as well as the Definition 4 in Section 2.1).

As a preview and illustration of properties analyzed in our paper, here is an example of an $r$-unavoidable simplicial complex which is strongly non-linear (Definition 52 in Section 7), in the sense that it is not $r$-unavoidable for ‘obvious reasons’ (it does not contain an $r$-unavoidable linear threshold complex), and which is minimal in the sense that if $L \subset K$ is a proper subcomplex of $K$ than $L$ is not $r$-unavoidable.

![Figure 1: Join of $(r - 1)$-copies of $\mathbb{RP}^2_6$.](image)

Let $\mathbb{RP}^2_6$ be the ‘hemi-icosahedron’ \cite{M03, Example 5.8.5], the simplicial complex on
6 vertices, arising as the unique, minimal triangulation of the real projective plane. Let \( J_r = (\mathbb{R}P^2)^{r-1}_6 \) be the join of \((r - 1)\) copies of \(\mathbb{R}P^2_6\) where \(r \geq 2\). As a complex on the set \([6] \times [r - 1]\) of cardinality \((6r - 6)\), the complex \(J_r\) is \(r\)-unavoidable (Corollary 17 to Theorem 15 in Section 4.2).

The complex \(\mathbb{R}P^2_6\) has a very large symmetry group inherited from the group of symmetries of the icosahedron. As a consequence the complex \(J_r\) also has a very large group of symmetries. This fact alone (Theorem 53 in Section 7) is the main reason why \(J_r\) is intrinsically non-linear, i.e. why there does not exist an \(r\)-unavoidable threshold complex \(K_{\mu \leq 1/r}\) (Section 5) such that \(K_{\mu \leq 1/r} \subset J_r\).

One way of relating topological and combinatorial complexity of an \(r\)-unavoidable simplicial complex \(K \subset 2^{|m|}\) is via the inequality (Corollary 9),

\[
\text{Ind}_{\mathbb{Z}/r} (K^r_\Delta) \geq m - r
\]

where \(r\) is a prime number, \(K^r_\Delta\) is the \(r\)-fold deleted join of \(K\) and \(\text{Ind}_{\mathbb{Z}/r}(\cdot)\) is the \(\mathbb{Z}/r\)-equivariant index function described in [M03, Section 6].

As a consequence we obtain the inequality,

\[
\text{Ind}_{\mathbb{Z}/r}((J_r)^r_\Delta) \geq 5r - 6,
\]

which in turn implies a Borsuk-Ulam type statement saying that there does not exist a \(\mathbb{Z}/r\)-equivariant map,

\[
\phi : (J_r)^r_\Delta \rightarrow Y
\]

where \(Y\) is a free \(\mathbb{Z}/r\)-complex of dimension \(\leq 5r - 7\).

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2 Partition invariant \(\pi_H(K, S)\)

In this section we collect the definitions of the key ‘partition invariants’ of simplicial complexes. The central are Definitions 3 and 4.

**Definition 1.** A simplicial complex \(K\) on a set \(S\) is a pair \(K = (K, S)\) of sets such that \(K \subset 2^S\) and,

(a) \(\emptyset \in K\) \hspace{1cm} (b) \(A \subset B \in K\) implies \(A \in K\).

In general the set \(\text{Vert}(K) = \{x \in S \mid \{x\} \in K\}\) of vertices of \(K\) is allowed to be a proper subset of the ambient set \(S\).
Remark 2. The set \( \{ \emptyset \} \) is not properly defined as a simplicial complex until we make explicit the ambient set \( S \). Similarly, a simplicial complex \( K \) (in the usual sense) is turned into a simplicial complex in the sense of Definition 1 by specifying a set \( S \) such that \( \text{Vert}(K) \subset S \) or equivalently \( K \subset 2^S \).

Caveat: Following the usual practise we will sometimes write \( K \) instead of \( K \) when the ambient set \( S \) is clear from the context.

2.1 The partition number

The partition number \( \pi(K) = \pi(K, S) \) of a simplicial complex \( K = (K, S) \) was introduced in \([JVZ-3]\) as a combinatorial invariant relevant for the study of results of Tverberg-Van Kampen-Flores type.

Definition 3. The partition number or \( \pi \)-invariant \( \pi(K) = \pi(K, S) \) of a simplicial complex \( K \subset 2^S \) is the minimum integer \( \nu \) such that for each partition \( A_1 \cup \ldots \cup A_{\nu} = S \) of \( S \) at least one of the sets \( A_i \) is in \( K \). More generally, if \( \mathcal{H} \subset 2^S \) is an arbitrary set (hypergraf) then the associated \( \mathcal{H} \)-partition number \( \pi_{\mathcal{H}}(K) \) of \( K = (K, S) \) is defined as the minimum integer \( \nu \) such that for each subpartition \( A_1 \cup \ldots \cup A_{\nu} \subseteq S \) in \( S \), where \( A_i \in \mathcal{H} \) for each \( i \in [\nu] \), at least one of the sets \( A_i \) is in \( K \).

Definition 4. (\( r \)-unavoidable complexes, \([BFZ, JVZ-3]\)) Let \( r \geq 2 \) be an integer. Suppose that \( K = (K, S) \) is a simplicial complex with vertices in \( S \). If \( \pi(K) = \pi(K, S) \leq r \) we say that the complex \( K \) is \( r \)-unavoidable, or equivalently that \( K \) is \( r \)-unavoidable on \( S \). More explicitly \( K \) is \( r \)-unavoidable on \( S \) if,

\[
\forall A_1, \ldots, A_r \in 2^S, \quad A_1 \cup \ldots \cup A_r = S \Rightarrow (\exists i) A_i \in K.
\]

More generally \( K \subset 2^S \) is \( r \)-unavoidable relative to a hypergraph \( \mathcal{H} \subset 2^S \) if \( \pi_{\mathcal{H}}(K) \leq r \).

If \( K \subset L \subset 2^S \) and the complex \( K \) is \( r \)-unavoidable then \( L \) is \( r \)-unavoidable as well. This is the reason why it may be sometimes useful to focus on ‘minimal \( r \)-unavoidable complexes’.

Definition 5. A complex \( K \subset 2^S \) is minimal \( r \)-unavoidable on \( S \) if it is \( r \)-unavoidable on \( S \) and if \( L \subsetneq K \) is a proper subcomplex of \( K \) then \( L \) is not \( r \)-unavoidable on \( S \).

2.2 Auxiliary definitions

Here we collect, for completeness and further reference, some auxiliary definitions. The following natural extension of Definition 3 is also useful. Recall that the covering multiplicity of a finite family \( \mathcal{A} = \{ A_i \}_{i \in I} \) of sets is the cardinality of the largest subset \( J \subset I \) such that \( \bigcap_{j \in J} A_j \neq \emptyset \).
Definition 6. Suppose that s and j are integers and let \( H \subseteq 2^S \). The extended partition number \( \pi_{s,j}^H(K) \) of \( K = (K,S) \) is defined as the minimum integer \( \nu \) such that for each collection \( A = \{A_i\}_{i \in I} \subseteq H \subseteq 2^S \) of size \( |I| = \nu \) and covering multiplicity at most \( j \), the set \( \{i \in I \mid A_i \in K\} \) has at least \( s \) elements. In particular, \( \pi_H(K) = \pi_{1,1}^H(K) \).

The property of being \( r \)-unavoidable is an intrinsic, combinatorial property of a simplicial complex \( K \). Here is a natural generalization.

Definition 7. \((r,s,j)\)-unavoidable complexes\) Choose integers \( m, r, s \) and \( j \) and let \( H \subseteq 2^m \). A simplicial complex \( K \subseteq 2^m \) is called \((r,s,j)\)-unavoidable on \( [m] \) (relative to \( H \)) if \( \pi_{s,j}^H(K) \leq r \). More explicitly, \( K \) is \((r,s,j)\)-unavoidable on \( [m] \) (relative to \( H \)) if for each collection \( A = \{A_i\}_{i \in I} \subseteq 2^m \) of size \( |I| = r \) and covering multiplicity at most \( j \), where \( A_i \in H \) for each \( i \), the cardinality of the set \( \{i \in I \mid A_i \in K\} \) is at least \( s \). We say that a complex is \((r,s)\)-unavoidable (relative to \( H \)) if it is \((r,s,1)\)-unavoidable (relative to \( H \)).

3 Constraint method and unavoidable complexes

The ultimate reason for studying various classes of unavoidable complexes is their central role in reduction methods for proving results of Tverberg-Van Kampen-Flores type. There exist two main reduction procedures, the ‘constraint method’ or the Gromov-Blagojević-Frick-Ziegler reduction, and the method based on Sarkaria’s inequality.

3.1 Gromov-Blagojević-Frick-Ziegler reduction

The Gromov-Blagojević-Frick-Ziegler reduction, or the ‘constraint method’, is an elegant and powerful method for proving results of Tverberg-van Kampen-Flores type. In its basic form the method can be summarized as follows.

\[
\begin{array}{ccc}
K & \xrightarrow{f} & \mathbb{R}^d \\
\downarrow e & & \downarrow i \\
L & \xrightarrow{F} & \mathbb{R}^{d+1}
\end{array}
\]

Suppose that \( K \subseteq L \subseteq 2^m \) is a pair of simplicial complexes. Assume that an analogue of the continuous Tverberg theorem holds for the triple \((L, r, \mathbb{R}^{d+1})\) in the sense that for each continuous map \( F : L \to \mathbb{R}^{d+1} \) there exists a collection of \( r \) disjoint faces \( \Delta_1, \ldots, \Delta_r \) of \( L \) such that \( f(\Delta_1) \cap \ldots \cap f(\Delta_r) \neq \emptyset \). Suppose that \( K \) is \( r \)-unavoidable relative to \( L \) in the sense that \( \pi_L(K) \leq r \) (Definitions 3 and 4). More explicitly we assume that for each subpartition \( A_1 \uplus \ldots \uplus A_r \subseteq [n] \), where \( A_i \in L \) for each \( i \in [r] \), at least one of the simplices \( A_i \) is in \( K \).

Under these assumptions the conclusion is that \((K, r, \mathbb{R}^d)\) is also a Tverberg triple in the sense that for each continuous map \( f : K \to \mathbb{R}^d \) there exist disjoint simplices \( \sigma_1, \ldots, \sigma_r \in K \) such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \).
Indeed, let \( \tilde{f} \) be an extension \((\tilde{f} \circ e = f)\) of the map \( f \) to \( L \). Suppose that \( \rho : L \to \mathbb{R} \) is the function \( \rho(x) := \text{dist}(x, K) \), measuring the distance of the point \( x \in L \) from \( K \). Define \( F = (\tilde{f}, \rho) : L \to \mathbb{R}^{d+1} \) and assume that \( \Delta_1, \ldots, \Delta_r \) is the associated family of vertex disjoint faces of \( L \), such that \( F(\Delta_1) \cap \ldots \cap F(\Delta_r) \neq \emptyset \). More explicitly suppose that \( x_i \in \Delta_i \) such that \( F(x_i) = F(x_j) \) for each \( i, j = 1, \ldots, r \). Since \( K \) is \( r \)-unavoidable relative to \( L \), \( \Delta_i \in K \) for some \( i \). As a consequence \( \rho(x_i) = 0 \), and in turn \( \rho(x_j) = 0 \) for each \( j = 1, \ldots, r \). If \( \sigma_i \) is the minimal face in \( L \) containing \( x_i \) then \( \sigma_i \in K \) for each \( i = 1, \ldots, r \) and \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \).

For a more complete exposition and numerous examples of applications of the ‘constraint method’ the reader is referred to [BFZ], see also [Gr10, Section 2.9(c)] and [L02, Proposition 2.5].

### 3.2 Reduction based on Sarkaria’s inequality

The preprint [JVZ-3] offers a different reduction procedure, based on Sarkaria’s index inequality [M03] and [Živ98], illustrating a different use of the \( r \)-unavoidability of the complex \( K \).

As in Section 3.1 we assume that \((L, r, \mathbb{R}^D)\) is a Tverberg triple in the sense that for each continuous map \( F : L \to \mathbb{R}^D \) there exists a collection of \( r \) disjoint faces \( \Delta_1, \ldots, \Delta_r \) of \( L \) such that \( f(\Delta_1) \cap \ldots \cap f(\Delta_r) \neq \emptyset \). Assume that \( D > d \) and suppose that \( K \) is \( r \)-unavoidable relative to \( L \) (Definition 4). We want to show that \((K, r, \mathbb{R}^d)\) is a Tverberg triple as well.

So far the set up is the same as in Section 3.1 and in particular we have the following commutative diagram (where \( d + 1 \) is replaced by \( D \)).

\[
\begin{array}{ccc}
K & \xrightarrow{f} & \mathbb{R}^d \\
\downarrow e & \ & \downarrow i \\
L & \xrightarrow{F} & \mathbb{R}^D
\end{array}
\] (7)

Following the standard ‘Configuration Space - Test Map’ reduction scheme [Živ98], a map \( f : K \to \mathbb{R}^d \) induces an \( \mathbb{Z}_r \)-equivariant map \( \hat{f} : K^*_\Delta \to (\mathbb{R}^d)^r / \mathbb{R}^d \), where \( K^*_\Delta \) is the \( r \)-fold deleted join of \( K \) [M03] and \( \mathbb{R}^d \subset (\mathbb{R}^d)^r \) is the diagonal subspace.

The problem is to show that this map must have a zero, which is directly related to the problem of estimating the equivariant index \( \text{Ind}_G(K^*_\Delta) \) where \( G = \mathbb{Z}_r \) is a cyclic group and \( r \) is a prime number. This transition is functorial and in particular one obtains from (7) the following commutative diagram,

\[
\begin{array}{ccc}
K^*_\Delta & \xrightarrow{\hat{f}} & (\mathbb{R}^d)^r / \mathbb{R}^d \\
\downarrow e & \ & \downarrow i \\
L^*_\Delta & \xrightarrow{F} & (\mathbb{R}^D)^r / \mathbb{R}^D
\end{array}
\] (8)
where $D = d + 1$ (or more generally some other integer $D > d$). In this setting the constraint method, based on the diagram (6), can be replaced by the index calculation (Sarkaria’s inequality) (based on the diagram (8)). (In diagram (8) we tacitly assume that the target spaces are subspaces of the corresponding (quotient) vector spaces.)

Recall that the Sarkaria’s inequality [M03, Proposition 6.2.4] is the index inequality,

$$\text{Ind}_G(Q) \geq \text{Ind}_G(Q_0) - \text{Ind}_G(\Delta(Q_0 \setminus Q)) - 1$$

where $Q_0$ is a free $G$-complex, $Q \subset Q_0$ is $G$-invariant, and $\Delta(Q_0 \setminus Q)$ is the order complex of the poset $(Q_0 \setminus Q, \subseteq)$. By specializing $Q_0 = L_{\Delta}^r$ and $Q = K_{\Delta}^r$ one obtains a tool for estimating the equivariant index of $K_{\Delta}^r$.

The $r$-unavoidability of $K$ relative to $L$, emerges in the form of the following simple lemma (see also the proof of Theorem 3.6. in [JVZ-3]).

**Lemma 8.** Let $r \geq 2$ be a prime. Suppose that $K \subset L \subset 2^{[m]}$ and assume that $K$ is $r$-unavoidable relative to $L$ (Definition [7]). Let $Q_0 = L_{\Delta}^r$ and $Q = K_{\Delta}^r$ and let $\Delta(Q_0 \setminus Q)$ be the ordered complex of the set $Q_0 \setminus Q$ ordered by the inclusion. Then,

$$\text{Ind}_G(\Delta(Q_0 \setminus Q)) \leq r - 2.$$  

**Proof:** By construction a simplex in $Q_0 = L_{\Delta}^r$ is an $r$-tuple $\tau = (A_1, \ldots, A_r)$ of disjoint simplices $A_i \in L$. By definition $\tau \in Q_0 \setminus Q$ if and only if $\phi(\tau) := \{i \in [r] \mid A_i \notin K\}$ is a non-empty set. Since $K$ is $r$-unavoidable relative to $L$, we observe that $\emptyset \neq \phi(\tau) \neq [r]$ for each $\tau \in Q_0 \setminus Q$.

The map $\phi : Q_0 \setminus Q \to 2^r \setminus \{\emptyset, [r]\}$ is monotone so it induces a $\mathbb{Z}_r$-equivariant map of the associated order complexes,

$$\hat{\phi} : \Delta(Q_0 \setminus Q) \to \Delta(2^r \setminus \{\emptyset, [r]\}) \cong S^{r-2}.$$  

This completes the proof of the lemma in light of the monotonicity property of the index function $\text{Ind}_G$, [M03, Section 6].

**Corollary 9.** Suppose that $K \subset 2^{[m]}$ is an $r$-unavoidable complex with vertices in $[m]$. Assume that $r = p$ is a prime and let $G = \mathbb{Z}_p$ be the associated cyclic group. Let $K_{\Delta}^r$ be the $r$-fold deleted join of $K$. Then,

$$\text{Ind}_G(K_{\Delta}^r) \geq m - r.$$  

**Proof:** By assumption $K$ is $r$-unavoidable relative to $L = \Delta([m]) = 2^{[m]}$. In this case $Q_0 = L_{\Delta}^r \cong [r]^m$ is a $\mathbb{Z}_r$-free, $(m - 1)$-dimensional, $(m - 2)$-connected simplicial complex. It follows that $\text{Ind}_G(Q_0) = m - 1$ which, together with Lemma 8 and inequality (9), implies the inequality (11).
4 Fundamental constructions on unavoidable complexes

In this section we collect some general procedures for constructing new unavoidable complexes from the old ones. We start with very simple and basic constructions which pave the way for discovering less obvious examples of unavoidable complexes, including the construction of intrinsically non-linear (Definition 30) and strongly non-linear unavoidable complexes (Section 7).

4.1 Pushforward of unavoidable complexes

Definition 10. A map \((K, S) \xrightarrow{f} (K', S')\) of simplicial complexes \((K, S)\) and \((K', S')\) is a function \(f : S \to S'\) such that \(\forall A \in 2^S, A \in K \Rightarrow f(A) \in K'\). If \((K, S)\) is a simplicial complex and \(f : S \to S'\) an arbitrary map then \(K' = f^*(K)\) is the simplicial complex on \(S'\) defined by \(f^*(K) = \{ B \in 2^{S'} \mid f^{-1}(B) \in K \}\). The complex \((K', S')\) is referred to as the pushforward of \((K, S)\) or the complex induced on \(S'\) by the map \(f\).

Proposition 11. Let \(f : S \to S'\) be an arbitrary map and let \(K = (K, S)\) be a simplicial complex on \(S\). Then,
\[
\pi(f^*(K)) \leq \pi(K) \tag{12}
\]
where \(f^*(K) = f^*(K, S) = f^*(K)\) is the complex on \(S'\) induced by \(f\) (Definition 10). As a consequence if \(K\) is \(r\)-unavoidable on \(S\) then \(K' = f^*(K)\) is \(r\)-unavoidable on \(S'\).

Example 12. In the following examples \(f\) is an inclusion map.

1. The complex \(\{\emptyset\}\) is \(r\)-unavoidable on \([r - 1]\). As a consequence the complex \(\{ A \subset [m] \mid A \cap [r - 1] = \emptyset \}\) is \(r\)-unavoidable on \([m]\) where \(m \geq r - 1\).

2. The complex \(\binom{[2r - 1]}{\leq 1}\) = \(\{ A \subset [2r - 1] \mid |A| \leq 1 \}\) is \(r\)-unavoidable on \([2r - 1]\). It follows that for each \(m \geq 2r - 1\) the complex \(K = \{ A \subset [m] \mid |A \cap [2r - 1]| \leq 1 \}\) is \(r\)-unavoidable on \([m]\).

4.2 Join of unavoidable complexes

The central result in this section is Theorem 15 which allows us to construct new unavoidable complexes as joins of known unavoidable complexes (typically with smaller \(\pi\)-invariant). We begin with the observation that the classes of \((r + s - 1, s)\)-unavoidable and \(r\)-unavoidable complexes coincide if \(\mathcal{H} = 2^V\). More generally this is true for \(\cup\)-closed hypergraphs, and in particular this holds for hypergraphs \(\mathcal{H} = \mathcal{J} \cup \{\emptyset\}\) where \(\mathcal{J} \subset 2^V\) is an order ideal.

Definition 13. A hypergraph \(\mathcal{H} \subset 2^V\) is \(\cup\)-closed if,

- \(\emptyset \in \mathcal{H}\);
- If \(A, B \in \mathcal{H}\) then \(A \cup B \in \mathcal{H}\).
Recall (Definition 7) that a complex \( K \subset 2^V \) is \((n, s)\)-unavoidable, relative to \( H \subset 2^V \), if for each collection \( \mathcal{A} = \{A_i\}_{i \in [n]} \subset H \) of pairwise disjoint sets, the cardinality of the set \( \{j \in [n] \mid A_j \in K\} \) is at least \( s \).

**Proposition 14.** Suppose that \( H \) is a \( \cup \text{-closed} \) hypergraph on \( V \). A complex \( K \subset 2^V \) is \((r + s - 1, s)\)-unavoidable (relative \( H \)) if and only if it is \( r \)-unavoidable (relative \( H \)).

**Proof:** Since \( \emptyset \in H \), each disjoint \( r \)-family \( \{A_i\}_{i=1}^r \subset H \) can be completed to a disjoint family \( \{A_i\}_{i=1}^{r+s-1} \subset H \) where \( A_j = \emptyset \) for \( j \geq r + 1 \). From here it immediately follows that if \( K \) is \((r + s - 1, s)\)-unavoidable it must be \( r \)-unavoidable as well.

For the opposite implication let us assume that \( K \subset 2^V \) is an \( r \)-unavoidable simplicial complex (relative \( H \)). Let \( n = r + s - 1 \) (hence \( s = n - r + 1 \)) and let \( \mathcal{A} = \{A_j\}_{j=1}^n \) be a (pairwise) disjoint family of sets in \( H \). We claim that the cardinality of the set \( \{j \in [n] \mid A_j \in K\} \) is at least \( n - r + 1 \).

For a given set \( I \subset [n] \) of cardinality \( n - r + 1 \) let \( \mathcal{A}_I = \{A_j\}_{j \in I} \cup \{A_I\} \) where \( A_I := \bigcup_{j \in I} A_j \). Since \( H \) is \( \cup \text{-closed} \) the family \( \mathcal{A}_I \) consists of \( r \) pairwise disjoint sets which are all elements in \( H \).

There are two possibilities:

1) There exists some \( I \subset [n] \) of cardinality \( n - r + 1 \) such that \( A_I \in K \). In this case \( A_i \in K \) for each \( i \in I \) which concludes the proof.

2) In the opposite case we observe that for each set \( I \subset [n] \) of cardinality \( n - r + 1 \) there exists \( i \in [n] \setminus I \) such that \( A_i \in K \). It follows that there can be at most \((r - 2)\) elements \( j \in [n] \) such that \( A_j \notin K \), hence \( A_i \in K \) for at least \((n - r + 2)\) distinct indices \( i \in [n] \).

\( \square \)

**Theorem 15.** Let \( K_1, \ldots, K_n \) be a collection of simplicial complexes where \( K_j \subset 2^{V_j} \). Assume that \( K_j \) is \( r_j\)-unavoidable on \( V_j \) for some integers \( r_j \geq 1 \) and \( j = 1, \ldots, n \). Then the join \( K = K_1 \ast \cdots \ast K_n \) is an \( r \)-unavoidable complex on \( V = V_1 \uplus \cdots \uplus V_n \) provided,

\[ r > r_1 + \cdots + r_n - n. \quad (13) \]

**Proof:** By definition \( K \subset 2^V \), where \( V = V_1 \uplus \cdots \uplus V_n \). Let \( V = \{A^{(1)}, \ldots, A^{(r)}\} \) be an arbitrary family of pairwise disjoint sets in \( V \) where by assumption \( r > r_1 + \cdots + r_n - n \). Each \( A^{(i)} \) can be uniquely represented as a disjoint union \( A^{(i)} = \bigcup_{j=1}^n A_j^{(i)} \) where \( A_j^{(i)} = A^{(i)} \cap V_j \) for each \( i \in [r] \) and \( j \in [n] \).

For each \( j \in [n] \) let \( S_j = \{i \in [r] \mid A_j^{(i)} \in K_j\} \). In order to show that \( A^{(i)} \in K \) for some \( i \in [r] \) we are supposed to prove that,

\[ \bigcap_{j=1}^n S_j \neq \emptyset \quad \text{or equivalently,} \quad \bigcup_{j=1}^n S_j^c \neq [r]. \quad (14) \]
Corollary 16. If \( K = K_1 \star \cdots \star K_n \) is a join of \( n \geq 1 \) simplicial complexes then,
\[
\pi(K) = \pi(K_1 \star \cdots \star K_n) \leq \pi(K_1) + \cdots + \pi(K_n) - n + 1.
\]

(15)

Here are some special cases and consequences of Corollary 16.

Corollary 17.

1. Let \( K \) be an \( r \)-unavoidable complex. Then the join of \( n \) copies of \( K \) is \((n(r-1)+1)\)-unavoidable.

2. Let \( K \) be a 2-unavoidable complex. Then the join of \( n \) copies of \( K \) is \((n+1)\)-unavoidable.

3. The complex \( \mathbb{R}P_6^2 \) is self-dual (minimal 2-unavoidable). It follows that the complex \( J_r = (\mathbb{R}P_6^2)^{\ast(r-1)} = \mathbb{R}P_6^2 \ast \cdots \ast \mathbb{R}P_6^2 \) is \( r \)-unavoidable.

4.3 Minimal \( r \)-unavoidable complexes

Definition 18. (Minimal \( r \)-unavoidable complex) Let \( K \subset 2^V \) be an \( r \)-unavoidable simplicial complex. We say that \( K \) is a minimal \( r \)-unavoidable complex if for every maximal simplex \( A \in K \), the complex \( K \setminus \{ A \} \subset 2^V \) is not \( r \)-unavoidable.

Lemma 19. Let \( K \subset 2^V \) be an \( r \)-unavoidable simplicial complex on \( V \). Let \( A \in K \) be a maximal simplex in \( K \). Then,
\[
K \setminus \{ A \} \text{ is } r \text{-unavoidable on } V \text{ if and only if } K|_{V \setminus A} \text{ is } (r-1)\text{-unavoidable on } V \setminus A,
\]
where \( K|_{V \setminus A} := \{ C \in K \mid C \subset V \setminus A \} \) is the restriction of \( K \) to the set of vertices \( V \setminus A \).

Proof:

(\( \Rightarrow \)) Suppose that \( K \setminus \{ A \} \) is an \( r \)-unavoidable complex on \( V \). If \( V \setminus A = B_1 \uplus \cdots \uplus B_{r-1} \) is a partition of \( V \setminus A \) then \( A \uplus B_1 \uplus \cdots \uplus B_{r-1} \) is a partition of \( V \). By assumption \( B_i \in K \setminus \{ A \} \) for some \( i \) and as a consequence \( B_i \in K|_{V \setminus A} \).

(\( \Leftarrow \)) Suppose that \( K|_{V \setminus A} \) is \((r-1)\)-unavoidable on \( V \setminus A \). Let \( B_1 \uplus \cdots \uplus B_r = V \) be a partition of \( V \). Since \( K \) is \( r \)-unavoidable then \( B_i \in K \) for some \( i \in [r] \). If \( B_i \neq A \) then \( B_i \in K \setminus \{ A \} \) and we are done. If \( B_i = A \) then \( B_1 \uplus \cdots \uplus \hat{B}_i \uplus \cdots \uplus B_r \) is a partition of \( V \setminus A \). By assumption \( B_j \in K|_{V \setminus A} \) for some \( j \neq i \) which implies \( B_j \in K \setminus \{ A \} \).
Corollary 20. (Criterion for minimal $r$-unavoidability) Let $K \subset 2^{V}$ be an $r$-unavoidable complex. Then $K$ is minimal $r$-unavoidable if and only if for every maximal simplex $A \in K$, the complex $K_{|V \setminus A}$ is not $(r-1)$-unavoidable on $V \setminus A$.

Corollary 21. If $K \subset 2^{[n]}$ is a minimal $r$-unavoidable complex then $\pi(K) = r$.

Proof: Since $K$ is $r$-unavoidable we know that $\pi(K) \leq r$. By minimality of $K$ for a given facet $A \in K$ there exists a partition $B_1 \cup \cdots \cup B_{r-1} = [n] \setminus A$ such that $B_i \notin K$ for each $i \in [r-1]$. In turn $(A \cup B_1) \cup B_2 \cup \cdots \cup B_{r-1} = [n]$ is a partition of $[n]$ into $(r-1)$ sets proving that $\pi(K) \geq r$. \hfill \Box

Proposition 22. Let $K_1, \ldots, K_n$ be a family of complexes such that for each $j \in [n]$ the complex $K_j \subset 2^{V_j}$ is a minimal $r_j$-unavoidable complex on $V_j$. Then the join $K = K_1 \star \cdots \star K_n$ is a minimal $r$-unavoidable complex on $V = V_1 \cup \cdots \cup V_n$ where $r = r_1 + \cdots + r_n - n + 1$.

Proof: By Theorem [13] we know the $K$ is $r$-unavoidable on $V$. In order to prove its minimality, it is sufficient (by Lemma [19] and Corollary [20]) to prove that for every maximal simplex $A \in K$ the complex $K_{|V \setminus A}$ is not $(r-1)$-unavoidable on $V \setminus A$. 

The simplex $A$, as a maximal simplex in $K$, is the join of maximal simplices $A = A_1 \star \cdots \star A_n$, where $A_i$ is a maximal simplex in $K_i$ for each $i$.

By Lemma [19] we know that for each $i \in \{1, \ldots, n\}$, $K_i_{|V \setminus A_i}$ is not $(r_i - 1)$-unavoidable, which implies that there exists a partition,

\[ V_i \setminus A_i = B^{(1)}_i \cup \cdots B^{(r_i - 1)}_i, \tag{16} \]

such that for each $i$ and $j$, $B^{(j)}_i \notin K_i$. From here we obtain a partition of $V \setminus A = \bigcup_{i=1}^{n} (V_i \setminus A_i)$ by taking the union of all partitions (16),

\[ V \setminus A = B^{(1)}_1 \cup \cdots B^{(r_1 - 1)}_1 \cup \cdots \cup B^{(1)}_n \cup \cdots B^{(r_n - 1)}_n. \]

This way we obtain a partition of $V \setminus A$ into $\sum_{i=1}^{n} (r_i - 1) = r - 1$ subsets such that none of the pieces is a simplex in $K_{|V \setminus A}$. It follows that the complex $K_{|V \setminus A}$ is not $(r-1)$-unavoidable, which completes the proof of the theorem. \hfill \Box

5 Unavoidable threshold complexes

For a given function $f : E \rightarrow \mathbb{R}$ and a ‘threshold’ $\alpha \in \mathbb{R}$ the associated sub-level set is $E_{f \leq \alpha} := \{ x \in E \mid f(x) \leq \alpha \}$. Suppose that $B = 2^S$ is a Boolean partially ordered set (poset) and let $f : B \rightarrow \mathbb{R}_+$ be a monotone function which satisfies the property $f(\emptyset) = 0$. Then the associated sub-level set $B_{f \leq \alpha}$ is a simplicial complex. This simplicial complex is referred to as the $(f, \alpha)$-threshold complex and often denoted by $K_{f \leq \alpha}$. In this section we collect some of the most fundamental properties of $r$-unavoidable threshold complexes.
5.1 Linearly realizable $r$-unavoidable complexes

Definition 23. A (positive) weight distribution on $[m]$ is a vector $\mu = (\alpha_1, \ldots, \alpha_m)$ where $\alpha_i \geq 0$ for each $i$ and $\alpha := \alpha_1 + \ldots + \alpha_m > 0$. The associated positive measure on $2^m$ (also denoted by $\mu$) is defined by $\mu(A) = \sum_{i \in A} \alpha_i$. If $\mu([m]) = 1$ then the measure $\mu$ is referred to as a probability measure.

More generally a superadditive measure $\nu$ on $2^m$ is a non-negative function $\nu : 2^m \to \mathbb{R}_+$ such that $\nu(\emptyset) = 0$, $\nu([m]) > 0$, $A \subset B \Rightarrow \nu(A) \leq \nu(B)$ and,

$$\nu(A \cup B) \geq \nu(A) + \nu(B) \text{ if } A \cap B = \emptyset.$$  \hspace{1cm} (17)

Example 24. Suppose that $\{\mu_t\}_{t \in T}$ is a finite collection of positive measures on $[m]$. Then the function defined by,

$$\nu(A) := \text{Min} \{\mu_t(A) \mid t \in T\}$$ \hspace{1cm} (18)

is a (positive) superadditive measure on $[m]$. For the reasons explained in Section 5.2 the function $\nu$ is referred to as a geometric superadditive measure.

Definition 25. Suppose that $\mu = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m$ is a positive measure on $[m]$ and let $\beta \geq 0$. The associated sub-level simplicial complexes $K_{\mu \leq \beta}$ and $K_{\mu < \beta}$, are defined by,

$$K_{\mu \leq \beta} = \{A \subset [m] \mid \mu(A) \leq \beta\}, \quad K_{\mu < \beta} = \{A \subset [m] \mid \mu(A) < \beta\}. \hspace{1cm} (19)$$

More generally, if $\nu$ is a positive superadditive measure on $[m]$ then the associated sub-level sets,

$$K_{\nu \leq \beta} = \{A \subset [m] \mid \nu(A) \leq \beta\}, \quad K_{\nu < \beta} = \{A \subset [m] \mid \nu(A) < \beta\}$$

are also simplicial complexes.

Proposition 26. Suppose that $\nu$ is a positive superadditive measure on $[m]$ and let $\nu([m]) = \alpha$. Then the sub-level complex $K_{\nu \leq \alpha/r}$ is $r$-unavoidable.

Proof: Suppose that $[m] = A_1 \cup \ldots \cup A_r$ is a partition. Then at least one of the sets $A_i$ is in $K_{\nu \leq \alpha/r}$, otherwise we would have a contradiction,

$$\alpha = \nu([m]) = \nu(A_1 \cup \ldots \cup A_r) \geq \nu(A_1) + \ldots + \nu(A_r) > r \frac{\alpha}{r} = \alpha.$$

Corollary 27. Suppose that $\nu$ is a positive superadditive measure on $[m]$ and let $\nu([m]) = \alpha$. Then for each $0 < \beta \leq \alpha$,

$$\pi(K_{\nu \leq \beta}) \leq \lfloor \alpha/\beta \rfloor.$$ \hspace{1cm} (20)

Definition 28. An $r$-unavoidable simplicial complex $K \subset 2^m$ is called ‘linear’ or ‘linearly realizable’ if $K = K_{\mu \leq 1/r}$ for some probability measure $\mu$ on $[m]$. The $r$-unavoidable complexes described in Proposition 26 are for similar reasons sometimes referred to as ‘concave’ or ‘concavely realizable’ complexes.
5.2 Geometric interpretation of threshold complexes

The map \( \chi : 2^m \to \{0,1\}^m \), identifying a subset \( A \in [m] \) with its characteristic function \( \chi_A \in Q_m = \{0,1\}^m \), allows us to give a simple and useful geometric interpretation of threshold complexes \( K_{f \leq \alpha} \) in the situation when the threshold function \( f : 2^m \to \mathbb{R}_+ \) has a natural extension \( \hat{f} : \mathbb{R}^m \to \mathbb{R}_+ \) (or \( \hat{f} : [0,1]^m \to \mathbb{R}_+ \)).

For example if \( K_{\mu \leq \beta} \) is the linear threshold complex associated to a measure \( \mu = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_m^+ \) then,

\[
K_{\mu \leq \beta} = \{0,1\}^m \cap H^{-}(\mu, \beta) = \{A \in 2^m \mid \langle \mu, \chi_A \rangle \leq \beta\}
\]

where \( H^{-}(\mu, \beta) = \{x \in \mathbb{R}^m \mid \langle \mu, x \rangle \leq \beta\} \) is the negative half-space associated to \((\mu, \beta)\). A similar interpretation can be given to the 'concave' threshold complex \( K_{\nu \leq \beta} \) where, in agreement with (18), the concave function \( \nu \) is defined on \( \mathbb{R}^m \) by the formula,

\[
\nu(x) := \text{Min} \{\langle \mu_t, x \rangle \mid t \in T\}.
\]

Figure 2: Linear vs. non-linear threshold complex.

The question when a general superadditive function \( \nu \) has a concave extension to \( \mathbb{R}^m \) is more subtle. This is one of the reasons why we pay a special attention to 'geometric superadditive functions' (Example 41 and Proposition 29).

If \( K \subset 2^m \) is an arbitrary simplicial complex, identified with its geometric image \( K = \chi(K) = \{\chi_A \in Q_m \mid A \in K\} \subset [0,1]^m \), we can check if it is a linear threshold complex as follows.

Let \( L = 2^m \setminus K \) be the complementary set or the associated 'simple game' (in the language of cooperative game theory [1-Z]). We also identify \( L \) with its image \( \chi(L) \subset [0,1]^m \). Then \( K \) is a linear threshold complex if and only if Conv\((K) \cap \text{Conv}(L) = \emptyset \) (Figure 2). Moreover the size (volume, number of integer points, etc.) of the convex set \( \eta(K) = \text{Conv}(K) \cap \text{Conv}(L) \) can be used as a measure of its geometric non-linearity.
5.3 Complexes arising from geometric concave measures

Proposition 29. Suppose that \( \nu \) is a geometric superadditive measure on \([m]\) (Example 24). Suppose that \( \nu([m]) = \alpha \) and let \( K = K_{\nu \leq \alpha/r} \) be the associated sub-level \( r \)-unavoidable complex. Then \( K \) contains a linearly realizable, \( r \)-unavoidable complex, \( K_{\mu \leq 1/r} \subset K_{\nu \leq \alpha/r} \).

Proof: Let \( 1 = (1, \ldots, 1) \in \mathbb{R}^m \) and \( 1_r = (1/r)(1, \ldots, 1) = (1/r, \ldots, 1) \). By assumption \( \nu \) admits an extension \( \nu : \mathbb{R}^m \to \mathbb{R} \), which in turn implies the convexity of the set,

\[
D = K_{\nu > \alpha/r} = \{ v \in \mathbb{R}^m \mid \nu(v) > \alpha/r \}.
\]

(22)

If \( C = \{1_r\} - \mathbb{R}^m_+ = \{ v \in \mathbb{R}^m \mid v \leq 1_r \} \) then \( C \cap D = \emptyset \). Indeed, if \( u \in C \) then \( \nu(u) \leq \nu(1_r) = \alpha/r \) while \( u \in D \) implies \( \nu(u) > \alpha/r \). Note that \( C \) is closed while \( D \) is an open convex set in \( \mathbb{R}^m \). Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a linear function that separates convex sets \( C \) and \( D \),

\[
C \subset f^{-1}(-\infty, c] \quad \text{and} \quad D \subset f^{-1}(c, +\infty).
\]

(23)

From the first inclusion in (23) we deduce that \( f(x) \geq 0 \) for each \( x \in \mathbb{R}^m_+ \). Moreover if \( f(1) = \beta \) (where \( \beta > 0 \)) then \( \beta/r = f(1/r) \leq c \). From the second inclusion in (23) we deduce that \( \mathbb{R}^m_{f \leq \beta/r} \subset \mathbb{R}^m_{\nu \leq \alpha/r} \) which implies \( K_{f \leq \beta/r} \subset K_{\nu \leq \alpha/r} \). By normalizing we observe that the probability measure \( \mu = (\alpha_1, \ldots, \alpha_m) \) defined by \( \langle \mu, x \rangle = (1/\beta)f(x) \) has the properties claimed by the proposition. \( \square \)

![Figure 3: Unavoidable linear complex dominated by an unavoidable concave complex.](image)

5.4 Domination of \( r \)-unavoidable complexes

Definition 30. Suppose that \( \mathcal{K} = (K, S) \) and \( \mathcal{L} = (L, S) \) are \( r \)-unavoidable complexes on the same set \( S \) of vertices. If \( K \subseteq L \) we say that \( K \) is dominated by \( L \) (or that \( L \) dominates...
We also say that a probability measure $\mu$ on $S$ is dominated by $K$ if $K_{\mu \leq 1/r} \subseteq K$. By definition an $r$-unavoidable complex $K$ is intrinsically linear if $K_{\mu \leq 1/r} \subseteq K$ for some probability measure $\mu$ on $S$, otherwise we call it intrinsically non-linear.

**Proposition 31.** Let $K \subseteq 2^{[n]}$ be an $r$-unavoidable complex on $[n]$. Then the set of all probability measures on $[n]$ dominated by $K$,

$$\mathcal{M}_K = \{ \mu \in \mathbb{R}^n_+ \mid \mu([n]) = 1 \text{ and } K_{\mu \leq 1/r} \subseteq K \}$$

is a convex subset of the simplex $\Delta_{n-1}$ of all probability measures on $[n]$. Moreover, the group $G$ of symmetries of $K$ acts on $\mathcal{M}_K$ and in particular if $\mathcal{M}_K \neq \emptyset$ then there exists a $G$-invariant measure $\mu$ dominated by $K$.

**Proof:** A probability measure $\mu \in \Delta_{n-1}$ is in $\mathcal{M}_K$ if and only if $\mu(C) > 1/r$ for each non-simplex $C \notin K$. The set $O_{C, \frac{1}{r}} = \{ x \in \mathbb{R}^n \mid \langle \chi_C, x \rangle > 1/r \}$ is an open half-space in $\mathbb{R}^n$ and $\mathcal{M}_K$ is convex since it can be represented as the intersection,

$$\mathcal{M}_K = \Delta_{n-1} \cap \bigcap_{C \notin K} O_{C, \frac{1}{r}}. \quad (24)$$

The set $\mathcal{M}_K$ is clearly invariant with respect to the (permutation) group $G$ which acts on $\mathcal{M}_K$ from the right. A $G$-invariant measure is obtained from any initial measure $\mu_0$ as the average $1/|G| \sum_{g \in G} \mu_0 \cdot g$. \qed

### 6 Threshold characteristic of a simplicial complex

It may be interesting to study how large can be a threshold complex which is dominated by a simplicial complex $K$ (which is not necessarily assumed to be $r$-unavoidable). In this section we introduce the *threshold characteristic* of the complex $K$ as the unique number $\rho(K)$ such that the equivalence,

$$\alpha < \rho(K) \iff K_{\mu \leq \alpha} \subseteq K, \quad (25)$$

holds for some probability measure $\mu$.

**Definition 32.** Let $K \subseteq 2^{[n]}$ be a simplicial complex and let $\Delta_{n-1} \subseteq \mathbb{R}^n_+$ be the simplex of probability measures on $[n]$. The threshold characteristic $\rho(K)$ of $K$ is defined by,

$$\rho(K) = \sup \{ \alpha \in [0, +\infty) \mid (\exists \mu \in \Delta_{n-1}) \ K_{\mu \leq \alpha} \subseteq K \} = \max \{ \alpha \in [0, +\infty) \mid (\exists \mu \in \Delta_{n-1}) \ K_{\mu < \alpha} \subseteq K \}. \quad (26)$$

**Remark 33.** The equivalence of two definitions of $\rho(K)$ in (26) can be checked directly, or deduced (more geometrically) from (the proof of) Proposition 38. By Definition 32 $\rho(K) = +\infty$ if and only if $K = 2^{[n]}$. The reason the supremum (maximum) is evaluated in the interval $[0, +\infty]$ (rather than in $[0, 1]$) is the inequality (28) which (by our definition) reduces to an equality in the case $K = 2^{[n]}$ (see also Proposition 50).
The following consequence of (25) is a fundamental relation between invariants $\pi(K)$ and $\rho(K)$ that holds unconditionally for all simplicial complexes $K$.

**Proposition 34.** If $K \subset 2^{[n]}$ is a simplicial complex on $[n]$ and $r \geq 1$ is an integer then,

$$\frac{1}{r} < \rho(K) \implies \pi(K) \leq r.$$  \hspace{1cm} (27)

The implication (27) is equivalent to either of the following equivalent inequalities,

$$\pi(K) - 1 \leq \left\lfloor \frac{1}{\rho(K)} \right\rfloor \iff \rho(K) \pi(K) \leq \rho(K) + 1.$$  \hspace{1cm} (28)

**Example 35.** Let $n = r(k + 1) - 1$ where $r, k \geq 2$ are integers. Let $K = \binom{[n]}{k}$. Then $\pi(K) = r$ and $\rho(K) = \frac{k+1}{r(k+1)-1}$ (see Example 40). It follows that the inequalities (28) cannot be improved in general.

**Remark 36.** Observe that if $K = \{\emptyset\} \subset 2^{[n]}$ then $\rho(K) = 1/n$. Note also that if $K \subset 2^{[n]}$ then $\rho(K) = +\infty$ if and only if $K = 2^{[n]}$. Since $\rho(K) \geq 1 \implies \rho(K) = +\infty$ we could have restricted thresholds $\alpha$ to the interval $[0, 1]$ in Definition 32. Our choice was motivated by the inequality (28) and the relation $\pi(K) = 1 \iff \rho(K) = +\infty$.

The following geometric interpretation of the invariant $\rho(K)$ is also useful. For a given non-simplex $C \notin K$ and $\alpha \geq 0$ let,

$$\mathcal{O}_{C,\alpha} := \{x \in \mathbb{R}^n_+ \mid \langle x, \chi_C \rangle > \alpha\} \quad \text{and} \quad \overline{\mathcal{O}}_{C,\alpha} := \{x \in \mathbb{R}^n_+ \mid \langle x, \chi_C \rangle \geq \alpha\}.$$  

Observe that $\mathcal{O}_{C,\alpha} = \alpha \cdot \mathcal{O}_C$ and $\overline{\mathcal{O}}_{C,\alpha} = \alpha \cdot \overline{\mathcal{O}}_C$ where $\mathcal{O}_C := \mathcal{O}_{C,1}$ (respectively $\overline{\mathcal{O}}_C := \overline{\mathcal{O}}_{C,1}$).

**Definition 37.** The convex set,

$$AD(K) = \bigcap_{C \notin K} \overline{\mathcal{O}}_C$$  \hspace{1cm} (29)

is referred to as the anti-dual polyhedron of $K$.

**Proposition 38.** Let $\mathbf{m}$ be the minimum value on the anti-dual polyhedron $AD(K)$ of $K$ of the functional $\phi : \mathbb{R}^n \to \mathbb{R}$, defined by $\phi(x) = \langle x, 1 \rangle = x_1 + \cdots + x_n$. Then,

$$\rho(K) = 1/\mathbf{m}.$$  

**Proof:** By Definition 32 $\rho(K)$ is the largest $\alpha$ such that,

$$\Delta_{n-1} \cap \alpha AD(K) \neq \emptyset,$$

or equivalently it is equal to the reciprocal of the smallest $\mathbf{m}$ such that,

$$\mathbf{m} \Delta_{n-1} \cap AD(K) \neq \emptyset.$$

It immediately follows that $\mathbf{m} = \min \{\phi(x) \mid x \in AD(K)\}$.  \hspace{1cm} \(\square\)

The following proposition is easily deduced from Proposition 38 (cf. the proof of Proposition 31).
Proposition 39. Let $G$ be the group of all permutations of $[n]$ which keep $K$ invariant and let $V = (\mathbb{R}^n)^G$ be the associated invariant subspace of $\mathbb{R}^n$. Then,

$$\min\{\phi(x) \mid x \in AD(K)\} = \min\{\phi(x) \mid x \in AD(K) \cap V\}.$$  \hfill (30)

Our first application of Proposition 39 is to complexes $K \subset 2^{[n]}$ which are invariant with respect to the full symmetry group $\Sigma_n$.

Example 40. If $K = \binom{[n]}{\leq k} \subset 2^{[n]}$ then $\rho(K) = \frac{k+1}{n}$. Indeed, the invariant subspace $V = (\mathbb{R}^n)^{\Sigma_n}$ is the diagonal $D = \{\bar{\alpha} = (\alpha, \ldots, \alpha) \in \mathbb{R}^n \mid \alpha \in \mathbb{R}\} \cong \mathbb{R}$. The intersection $D \cap AD(K)$ is identified as the interval $\{\bar{\alpha} \mid (k+1)\alpha \geq 1\}$ while the restriction of $\phi$ on $D$ is given by the formula $\phi(\bar{\alpha}) = n\alpha$. It follows that $m = \frac{n}{k+1}$ and $\rho(K) = \frac{k+1}{n}$.

6.1 Key examples from [BFZ]

The first three ‘key examples’ from [BFZ, Lemma 4.2] are easily seen to be linearly realizable, $r$-unavoidable (threshold) complexes. The last example in this lemma (example (iv)) is in general not a linear threshold complex. As an application of Proposition 39 (and in agreement with Proposition 29) we show (Proposition 42) that the complex constructed in this example always contains a linearly realizable $r$-unavoidable threshold complex as a subcomplex. According to our classification (Definition 30 and Corollary 51 to Proposition 50) this complex is intrinsically linear.

Example 41. The key example (iv) from [BFZ, Lemma 4.2] says that if $k \geq 0$ and $s$ are integers such that $r(k+1)+s > N+1$ with $0 \leq s \leq r$, then the complex,

$$K = \Delta^{(k-1)}_N \cup \Delta^{(k)}_{N-(r-s)} = \left[\left\lfloor\frac{N+1}{k}\right\rfloor \leq k\right] \cup \left[\left\lfloor\frac{N+1-(r-s)}{k+1}\right\rfloor \leq k+1\right] \subset 2^{[N+1]}$$  \hfill (31)

is $r$-unavoidable on $[N+1]$.

Proposition 42. The complex $K = \Delta^{(k-1)}_N \cup \Delta^{(k)}_{N-(r-s)}$ described in Example 41 is intrinsically linear, $r$-unavoidable complex.

Proof: Let $n = N+1$ and $n = p+q$ where $p = N+1-(r-s)$ and $q = r-s$. Following the algorithm described in Proposition 39 we calculate $\rho(K)$ where (in agreement with (31)),

$$K = \left[\left\lfloor\frac{n}{k}\right\rfloor \leq k\right] \cup \left[\left\lfloor\frac{p}{k+1}\right\rfloor \leq k+1\right] \subset 2^{[n]}.$$  \hfill (32)

The complex $K$ is invariant with respect to the action of the group $G = \Sigma_p \times \Sigma_q$ of all permutations of $[n]$ which preserve the partition $[n] = [p] \sqcup ([n] \setminus [p])$. It follows that $V = (\mathbb{R}^n)^G = \{x \in \mathbb{R}^n \mid \exists \alpha, \beta \in \mathbb{R}, x_1 = \cdots = x_p = \alpha \text{ and } x_{p+1} = \cdots = x_n = \beta\}$ so the restriction of the linear function $\phi$ (Proposition 39) to $V$ has the form $\phi(\alpha, \beta) = p\alpha + q\beta$.

The polyhedron $AD(K) \cap V$ is described by specializing the inequalities defining the anti-dual polyhedron (29) to the plane $V$. After some reflection (assuming $p \geq k$) we
observe that the inequalities describing $AD(K) \cap V$ (as a polyhedral subset of $\mathbb{R}^2$) are the following,

\begin{equation}
\begin{aligned}
k\alpha + \beta &\geq 1, \quad (k-1)\alpha + 2\beta \geq 1, \ldots, \\
(k-1)\alpha + 2\beta &\geq 1, \quad (k+1-j)\alpha + j\beta, \quad (k+2)\alpha \geq 1
\end{aligned}
\end{equation}

where $j = \text{Min}\{q, k\}$. The intersection of lines $(k+2)\alpha = 1, k\alpha + \beta = 1$ is the point $A(1/(k+2), 2/(k+2))$ (Figure 4). It is not difficult to check that $A$ is a feasible point for the polyhedron $AD(K) \cap V$ (i.e. it satisfies all the inequalities (33)). As a consequence the minimum $m$ of the function $\phi$ on $AD(K) \cap V$ satisfies the inequality $m \leq \phi(A) = \frac{n+q}{k+2}$. From here we immediately deduce (Proposition 39) that $ho(K) = \frac{1}{m} \geq \frac{k+2}{n+q}$. Finally, the condition that $K_{\mu \leq 1/r} \subset K$ for some $\mu \in \Delta_{n-1}$ is equivalent to the initial inequality (on the left hand side),

\[
\frac{1}{r} < \rho(K) \iff \frac{1}{r} < \frac{k+2}{n+q} \iff n < r(k+2) - q,
\]

while the last one is precisely the inequality $N + 1 < r(k+1) + s$.

**Remark 43.** It is interesting to check when the minimum $m$ of $\phi$ is attained precisely at the point $A(1/(k+2), 2/(k+2))$. For example if $k \leq q$ then the defining equations for $AD(K) \cap V$ are the inequalities $(k+2)\alpha \geq 1, k\alpha + \beta \geq 1, \alpha + k\beta \geq 1$. In this case $m = \phi(A)$ if and only if (Figure 4) $\phi(A) \leq \phi(B)$ which is equivalent to the inequality $kq \leq p$.

**Remark 44.** The significance of Propositions 38 and 39 (as demonstrated by Proposition 42) is that they provide a quite efficient algorithm for calculating $\rho(K)$ for a complex $K \subset 2^{[n]}$ with a reasonably large group of automorphisms. In light of the implication \( \frac{1}{r} < \rho(K) \implies \pi(K) \leq r \) (Proposition 34), they provide a method for a systematic generation of examples of $r$-unavoidable (threshold) complexes. For example if $1 < p < q < n$ one can try this method on the complex,

\[
K = \left( \begin{array}{c} \left[ n \right] \\ \leq k \end{array} \right) \cup \left( \begin{array}{c} \left[ q \right] \\ \leq k + 1 \end{array} \right) \cup \left( \begin{array}{c} \left[ p \right] \\ \leq k + 2 \end{array} \right)
\]
or on some other, more general (‘rank selected’) simplicial complex. A more detailed presentation of computer experiments and other related results will be documented in a separate publication.

6.2 Threshold characteristic of joins of complexes

In this section we evaluate (or estimate) the threshold characteristic of joins of simplicial complexes. Before we formulate more general statements, let us test Proposition 39 on some important special cases.

Example 45. If $K = [2]^n = [2] \ast \cdots \ast [2] = \partial({\triangle}_n)$ is the boundary complex of the $n$-dimensional cross-polytope ${\triangle}_n = \text{Conv}\{e_i, -e_i\}_{i=1}^n \subset \mathbb{R}^n$, then $\rho(K) = 1/n$. Indeed, the group of symmetries of $K$ is transitive on its vertices so $\rho(K)$ is attained if $\mu$ is the uniform measure where $\mu(\pm e_i) = 1/2n$ for each $i \in [n]$. The minimal non-simplices $\{e_i, -e_i\}$ are all of the same cardinality and $\rho(K) = \mu(\{e_i, -e_i\}) = 1/n$.

This example shows that a complex $K$ can have a small $\rho$-characteristic, and at the same time non-trivial homology in the top dimension.

Example 46. Generalizing Example 45, it is easily checked that $\rho([k]^n) = \frac{2}{kn}$ for each $k \geq 2$ and $n \geq 1$. The case $k = 3$ is especially interesting since [3] (as a complex on [3] as a set of vertices) is self-dual (minimal 2-unavoidable). In light of Proposition 22 $\pi([3]^n) = n + 1$ and $\pi(L) \geq n + 2$ for each proper subcomplex $L \subset [3]^n$. The associated $\rho$-invariant is $\rho([3]^n) = \frac{2}{3n}$. It follows that in the case $K = [3]^n$ the difference,

$$\epsilon(K) := [1/\rho(K)] + 1 - \pi(K) = [3n/2] - n = [n/2]$$

can be as large as desired. Hence, for $n \geq 2$ we obtain examples of strongly non-linear $(n + 1)$-unavoidable complexes in the sense of Definition 52.

The following proposition allows us to estimate the threshold characteristic of a join of complexes even in the case when the constituent complexes are asymmetrical.

Proposition 47. Let $K_1, \ldots, K_n$ be a family of complexes $K_i \subset 2^V_i$ and let $\alpha \in [0, 1]$. Let $K = K_1 \ast \cdots \ast K_n \subset 2^V$ be the associated join where $V = V_1 \uplus \cdots \uplus V_n$. If there is a positive probability measure $\mu$ on $V$ such that $K_{\mu \leq \alpha} \subset K$, then there exists some $j \in [n]$, and a positive probability measure $\nu$ on $V_j$, such that $K_{\nu \leq \alpha} \subset K_j$.

Proof: Let $\mu_i$ be the restriction of the measure $\mu$ on $V_i$ and let $\mu(V_i) = \mu_i(V_i) = t_i$. Since $t_1 + \cdots + t_n = 1$ there exists $j \in [n]$ such that $t_j \leq 1/n$. Let $\nu = (1/t_j)\mu_j$ be the probability measure on $V_j$ induced by $\mu_j$.

By assumption we known that for each subset $A \subset V_j$ if $\mu(A) = \mu_j(A) \leq \alpha$ then $A \in K \cap 2^{V_j} = K_j$. Since $\mu_j(A) = t_j\nu(A)$ we observe that for each $A \subset V_j$ if $\nu(A) \leq (1/t_j)\alpha$ then $A \in K_j$. Since $n\alpha \leq (1/t_j)\alpha$ we have the implication,

$$A \subset V_j \text{ and } \nu(A) \leq n\alpha \implies A \in K_j,$$

which proves that $K_{\nu \leq \alpha} \subset K_j$. □
**Corollary 48.** Under the assumptions of Proposition 47 there is an inequality,
\[ \rho(K) = \rho(K_1 \ast \cdots \ast K_n) \leq \frac{1}{n} \max\{\rho(K_i)\}_{i=1}^n. \] (34)

**Proof:** As a consequence of Proposition 47 there is an implication,
\[ \alpha < \rho(K) \implies \alpha < \frac{1}{n} \max\{\rho(K_i)\}_{i=1}^n. \] (35)

**Corollary 49.** Assume that \( K \subset 2^V \) is a simplicial complex and let \( \alpha \in [0, 1] \). Let \( K^{\ast n} = K \ast \cdots \ast K \) be the \( n \)-fold join of \( K \) which is assumed to be a simplicial complex on \( V \cup \cdots \cup V \cong V \times [n] \). If there is a positive probability measure \( \mu \) on \( V \times [n] \) such that \( K_{\mu \leq \alpha} \subset K^{\ast n} \), then there exists a positive probability measure \( \nu \) on \( V \), such that \( K_{\nu \leq n \alpha} \subset K \). As a consequence there is an implication,
\[ K_{\mu \leq \alpha} \subset K^{\ast n} \implies K_{\nu \leq n \alpha} \subset K \] (36)
and an inequality,
\[ \rho(K^{\ast n}) \leq \frac{1}{n} \cdot \rho(K). \] (37)

### 6.3 Threshold characteristic of ‘projective planes’

In this section we compute the threshold characteristic of complexes which are minimal triangulations of classical projective planes \( \mathbb{R}P^2, \mathbb{C}P^2 \) and \( \mathbb{H}P^2 \). Together with the 0-dimensional ‘projective plane’ \( [3] = \{1, 2, 3\} \), these are (following [EK6]) manifolds which are ‘like a projective plane’ in the sense that they allow a Morse function with exactly three critical points. Recall that all these complexes are self-dual (minimal 2-avoidable) in the sense that \( A \in K \) if and only if \( A^c \notin K \), see [M03, KB83, BD-1, BK92].

Following the algorithm described in Section 6 (Proposition 39) we need the information about (a) the number and size of orbits of the associated symmetry groups and (b) the minimum size of non-simplices in these complexes.

\( \mathbb{R}P^2_6 \)

The minimum triangulation of the real projective plane has six elements (Figure 1). The group of simplicial automorphisms of \( \mathbb{R}P^2_6 \) is vertex transitive. The size of the smallest non-simplex is 3. It follows that \( \rho(\mathbb{R}P^2_6) = 1/2 \).

\( \mathbb{C}P^2_9 \)

The minimum triangulation of the complex projective plane has 9 elements. The size of the smallest non-simplex is 4. The group of simplicial automorphisms of \( \mathbb{C}P^2_9 \) is also vertex transitive. This can be easily deduced from the description of this group given in [KB83] (see also [BD-1, Section 2]). It follows that \( \rho(\mathbb{C}P^2_9) = 4/9 \).

\( \mathbb{H}P^2 = M^8_{15} = \mathbb{H}P^2_{15} \)

Brehm and Kühnel constructed in [BK92] \( PL \)-isomorphic simplicial complexes \( M^8_{15}, \widetilde{M}^8_{15}, \tilde{M}^8_{15} \) on 15 vertices and conjectured that they triangulate the quaternionic projective plane \( \mathbb{H}P^2 \).
(this conjecture was recently confirmed by Gorodkov in [Go]). The most symmetric among them is $M_{16}^8$, which is invariant under vertex transitive action of the group $A_5$ (see the Theorem on page 169 in [BK92]). Moreover the size of the minimum size non-simplex in $M_{16}^8$ is 6. From here we deduce that $\rho(M_{16}^8) = 6/15$.

7 Intrinsically non-linear unavoidable complexes

The inequality (28), which provides a fundamental relation between invariants $\pi(K)$ and $\rho(K)$, can be rewritten as follows,

$$\pi(K) \leq \frac{1}{\rho(K)} + 1 \iff \pi(K) \leq \lfloor 1/\rho(K) \rfloor + 1 \quad (38)$$

The following proposition clarifies when the inequality in (38) on the right is actually an equality.

**Proposition 50.** Let $K \subset 2^{[n]}$ be a simplicial complex. Then $r = \pi(K) = \lfloor 1/\rho(K) \rfloor + 1$ if and only if $K_{\mu \leq 1/r} \subset K$ for some probability measure $\mu \in \Delta_{n-1}$.

**Proof:** If $\pi(K) = r$ and $K_{\mu \leq 1/r} \subset K$ then $\frac{1}{r} < \rho(K) \leq \frac{1}{r-1}$ and in (28) we have the equality $\pi(K) = \lfloor 1/\rho(K) \rfloor + 1$. Conversely, if $\pi(K) = \lfloor 1/\rho(K) \rfloor + 1 = r$ then $\lfloor 1/\rho(K) \rfloor = r - 1$ and,

$$\frac{1}{r} < \rho(K) \leq \frac{1}{r-1}$$

which implies $K_{\mu \leq 1/r} \subset K$ for some probability measure $\mu \in \Delta_{n-1}$.

The following corollary of Proposition 50 gives a characterization of *intrinsically linear* and *intrinsically non-linear* complexes introduced in Definition 30.

**Corollary 51.** A simplicial complex $K \subset 2^{[n]}$ is intrinsically linear (respectively intrinsically non-linear) if and only if $\pi(K) = \lfloor 1/\rho(K) \rfloor + 1$ (respectively $\pi(K) < \lfloor 1/\rho(K) \rfloor + 1$).

**Definition 52.** A simplicial complex $K \subset 2^V$ is strongly non-linear if it is both intrinsically non-linear and minimal in the sense of Definition 18.

The following theorem collects some of the central and the most interesting examples of strongly non-linear unavoidable complexes which can be studied by the methods developed in previous sections.

**Theorem 53.** Let $K$ be one of the following complexes:

1. The 3-element, 0-dimensional complex $[3] \subset 2^{[3]}$;
2. The unique, minimal 6-vertex triangulation $\mathbb{R}P_2^2 \subset 2^{[6]}$ of the real projective plane;
3. The minimal 9-vertex triangulation $\mathbb{C}P_2^2 \subset 2^{[6]}$ of the complex projective plane;
(4) The 15-element triangulation $M^8_{15}$ of the quaternionic projective plane $\mathbb{H}P^8_{16}$ [BK92].

Let $K_n = K^{*n}$ be the join of $n$ copies of $K$. Given a complex $L \subset 2^S$, let $\epsilon(L) := [1/\rho(L)] + 1 - \pi(L)$ and $\nu(L) = |S|$ (the number of vertices of $L$). The following table summarizes our knowledge about the invariants $\pi, \rho, \epsilon$ and $\nu$ of the complex $K_n$.

$$
\begin{array}{|c|c|c|c|c|}
\hline
K & \pi(K^{*n}) & \mathbb{R}P^2_6 & \mathbb{C}P^2_9 & \mathbb{H}P^2_{15} \\
\hline
\pi(K^{*n}) & n + 1 & n + 1 & n + 1 & n + 1 \\
\rho(K^{*n}) & 2/3n & 1/2n & 4/9n & 6/15n \\
\epsilon(K^{*n}) & [n/2] & n & [5n/4] & [3n/2] \\
\nu(K^{*n}) & 3n & 6n & 9n & 15n \\
\hline
\epsilon/\nu & \approx 1/6 & 1/6 & \approx 5/36 & \approx 1/10 \\
\hline
\end{array}
$$

Proof: All complexes $K$ listed in the theorem are self-dual [M03, KB83, BD-1, BK92] which implies that in all these cases $\pi(K) = 2$. In follows from the inequality (15) (Corollary [16]) that $\pi(K^{*n}) \leq 2n - n + 1 = n + 1$. As a consequence of Proposition [22] the complex $K_n = K^{*n}$ is minimal $(n+1)$-unavoidable. From here and Corollary [21] we immediately deduce that $\pi(K^{*n}) = n + 1$. (Note that the existence of the partition $Vert(K^{*n}) = Vert(K) \cup \cdots \cup Vert(K)$ provides a more direct proof of the inequality $\pi(K^{*n}) \geq n + 1$.)

The second row in the table (evaluating the invariant $\rho$) is computed by the method developed in Section 6 which uses Proposition [39] as the key ingredient (see Example [46] for a more detailed calculation in the case $K = [3]$). Indeed, if $G$ acts transitively on the vertices of the complex $K$ then the wreath product $G \wr S_n$ acts transitively of the set of vertices of $K_n = K^{*n}$. In particular this holds for the triangulations of ‘projective planes’ $\mathbb{R}P^2_6$, $\mathbb{C}P^2_9$ and $\mathbb{H}P^2_{15}$. So the numbers in the second row are obtained as the quotients of the size of a minimal non-simplex in $K^{*n}$ (which is the same as the size of the minimal non-simplex in $K$) and the number $\nu(K^{*n})$ of vertices in $K^{*n}$. The rest of the table is completed by simple calculation. 

7.1 More examples of intrinsically non-linear complexes

The join operation $\text{Join}_n(K) = K^{*n}$ was in previous sections our key tool for constructing intrinsically non-linear complexes. Here we show that it may be applied even to complexes which do not have a large symmetry group with a similar effect. It turns out that even if the complex $K$ is intrinsically linear ($\epsilon(K) := [1/\rho(K)] + 1 - \pi(K) = 0$) than the resulting complex is by a rule intrinsically non-linear, $\epsilon(\text{Join}_n(K)) > 0$, provided $n \geq 2$.

Proposition 54. Let $K^{*n} \subset 2^{V×[n]}$ be the $n$-fold join of a complex $K \subset 2^V$. Then,

$$\epsilon(K^{*n}) := [1/\rho(K^{*n})] + 1 - \pi(K^{*n}) \geq [n/\rho(K)] - n\pi(K) + n \geq n\epsilon(K) - \{n/\rho(K)\}. \quad (39)$$

Proof: By Corollary [16] and Corollary [49]

$$\pi(K^{*n}) \leq n\pi(K) - n + 1 \quad \text{and} \quad \rho(K^{*n}) \leq (1/n)\rho(K). \quad (40)$$
From the second inequality in (40) we deduce,

\[
\frac{1}{\rho(K^*n)} \geq \frac{n}{\rho(K)} \implies \left\lfloor \frac{1}{\rho(K^*n)} \right\rfloor \geq \left\lfloor \frac{n}{\rho(K)} \right\rfloor
\]

and the inequality (39) follows as an easy consequence. \hfill \square

8 Appendix

In this section we collect some ideas and facts not directly connected with the main themes and central results of the paper. In Section 8.1 we briefly describe a classification of unavoidable complexes based on weighted hypergraphs. Section 8.2 establishes a link of unavoidable complexes with Ramsey theory, while in concluding remarks we point to some promising future developments.

8.1 \(WH\)-realizable \(r\)-unavoidable complexes

‘Weighted hypergraph’ realizable or \(WH\)-realizable \(r\)-unavoidable complexes is a class of complexes considerably larger than the class of linearly realizable complexes. Here again we use a weight distribution but the weights may be associated to sets which are not necessarily singletons. A hypergraph \(F \subset 2^\left[ m \right]\) is by convention an arbitrary, non-empty family of sets of not necessarily the same cardinality. The \(k\)-uniform hypergraph on \(\left[ m \right]\) is the sets \(\left[ m \right](k) = \left( \left[ m \right] \atop k \right) = \{A \subset \left[ m \right] | |A| = k\} \).

**Definition 55.** Let \(\emptyset \neq F \subset 2^\left[ m \right]\) and let \(\omega : F \to \mathbb{R}^+ = [0, +\infty)\) be a distribution of non-negative weights on \(F\). Define the associated superadditive measure on \(\left[ m \right]\) by,

\[
\nu_\omega(A) = \text{Sup}\{\omega(A_1) + \ldots + \omega(A_k) | A_1 \cup \ldots \cup A_k \subseteq A \text{ where } A_i \in F \text{ for each } i\}
\]

(41)

If \(A \subset \left[ m \right]\) does not admit a (sub)partition \(A \supseteq A_1 \cup \ldots \cup A_k\) by elements from \(F\) we set \(\nu_\omega(A) = 0\) which is in agreement with \(\text{Sup}(\emptyset) = 0 = \text{Min}(\left[ 0, +\infty \right))\).

The following proposition is an immediate consequence of the definition.

**Proposition 56.** The measure \(\nu_\omega\) is superadditive, i.e.

\[
\nu_\omega(A \cup B) \geq \nu_\omega(A) + \nu_\omega(B)
\]

for each pair \(A, B \subset \left[ m \right]\) such that \(A \cap B = \emptyset\).

**Definition 57.** An \(r\)-unavoidable complex \(K \subset 2^\left[ m \right]\) is \(WH\)-realizable or ‘weighted hypergraph realizable’ if for some hypergraph \(F \subset 2^\left[ m \right]\), an associated weight distribution \(\omega : F \to \mathbb{R}^+\), and the corresponding superadditive measure \(\nu_\omega\),

\[
K = K_{\nu_\omega \leq \alpha/r}
\]

where \(\alpha = \nu_\omega(\left[ m \right])\). The pair \((F, \omega)\) is referred to as the \(WH\)-realization of the \(r\)-unavoidable complex \(K\).
Example 58. Suppose that $\mathcal{F} = \{ \{i\} \mid i \in [m] \}$ is the collection of all one-element subsets (singletons) in $[m]$. Then for each weight distribution $\omega : \mathcal{F} \to \mathbb{R}^+$ the associated superadditive measure $\nu_\omega$ is additive and we recover the class of linearly realizable, $r$-unavoidable complexes introduced in Section 5.1.

In light of Example 58 it is quite natural to ask for examples of $WH$-realizable, $r$-unavoidable complexes which are not linearly realizable. It is quite clear that this class is considerably bigger than the class of linearly realizable complexes. The following proposition shows that this is true already in the case of $2$-unavoidable complexes.

Proposition 59. Each self-dual complex $K \subset 2^{[m]}$ is $WH$-realizable.

Proof: Let $\mathcal{F} = 2^{[m]}$ be the collection of all subsets of $[m]$. Define the weight distribution $\omega = \omega_K : 2^{[m]} \to \mathbb{R}^+$ by,

$$
\omega(A) = 0 \text{ (respectively 1) if and only if } A \in K \text{ (respectively } A \in K^c).$

Since $K$ is self-dual, for each $B \subset [m]$ either $B \in K$ or $B^c \in K$ (but not both). It follows from here that $\nu_\omega = \omega$ and in turn $K = K_{\omega=0} = K_{\omega \leq 1/2}$. □

Proposition 60. Suppose that for a given $\mathcal{F} \subset 2^{[m]}$ and a weight distribution $\omega : \mathcal{F} \to \mathbb{R}^+$ there exists $X \in \mathcal{F}$ such that $\omega(X) = 0$. Then $\nu_\omega = \nu_{\omega'}$ where $\omega' : \mathcal{F} \setminus \{X\}$ is the restriction of $\omega$.

Proposition 60 is elementary however it has some important consequences. The following corollary suggests that there is an increasing hierarchy of $WH$-realizable, $r$-unavoidable complexes. In particular, for a given $r$-realizable complex $K$, it may be desirable to find its $(\mathcal{F}, \omega)$-realization with $\mathcal{F}$ of the smallest size (or minimizing some other complexity).

Corollary 61. For a given $m$ and $\mathcal{F} \subset 2^{[m]}$, let $R_r(\mathcal{F})$ be the class of all $WH$-realizable, $r$-unavoidable subcomplexes of $2^{[m]}$. Then $\mathcal{F} \subset \mathcal{F}'$ implies $R_r(\mathcal{F}) \subset R_r(\mathcal{F}')$.

8.2 Other classes of $r$-unavoidable complexes

Here we describe a procedure for generating intrinsically non-linear, $r$-unavoidable simplicial complexes based on Ramsey’s theorem and its relatives. The method is quite general and links in an unusual way topological combinatorics with Ramsey theory. The following example explains ‘in a nutshell’ the central idea of the construction.

Example 62. The simplest Ramsey type theorem says that for each graph $\Gamma$ on six vertices either $\Gamma$ or its complement contain a triangle as a subgraph (triangle = clique with three vertices).

We describe an intrinsically non-linear, 2-unavoidable complex $L$ on the set $V = \binom{[6]}{2}$ of all 2-element subsets of $[6]$. By definition $S \subset V$ is in $L$ if and only if there is a triangle in the complement of $S$. It is clear that $L$ is a simplicial complex. Moreover, Ramsey’s theorem $R(3, 3) = 6$ implies that $L$ is 2-unavoidable ($\pi(L) = 2$).
Let us compute the threshold characteristic $\rho(L)$ of $L$. Since the group of automorphisms of $L$ acts transitively on the set $V = \binom{[n]}{2}$ it is sufficient to determine the size of the minimal non-simplex in $L$. Figure 5 on the left displays a graph without triangles which means that the complementary graph (on the right) is not in $L$. The size of this graph is 6 and it is not difficult to see that this is precisely the minimum size of a non-simplex in $L$.

![Figure 5: A maximal graph without triangles and its complement.](image)

As an immediate consequence we obtain that $\rho(L) = 6/15$. It follows that $L$ is intrinsically non-linear since $\epsilon(L) = 1 + [1/\rho(L)] - \pi(L) = 1 > 0$. Note that the ratio $\epsilon(L)/v(L) = 1/15$ is relatively small (compared to the examples listed in Theorem 53).

Suppose that $P$ is a graph property which is monotone in the sense that if $\Gamma \in P$ and $\Gamma \subset \Gamma'$ then $\Gamma' \in P$. For example ‘having triangles’ (or cliques of given size) is such a property. If the set of vertices is fixed, we frequently identify a graph with its set of edges.

**Definition 63.** Choose $r \geq 2$ and denote by $\binom{[n]}{2}$ the set of all 2-element subsets of $[n]$. We say that an integer $n$ is $(P, r)$-admissible if for each partition (coloring)

$$\binom{[n]}{2} = A_1 \uplus A_2 \uplus \ldots \uplus A_r$$  \hspace{1cm} (42)

of $\binom{[n]}{2}$ into $r$ disjoint sets, for some $i$ the graph $\bigcup_{j \neq i} A_j$ has the property $P$.

For example if $P$ is the property that the graph has a clique of size $k$ then $n$ is $(k, r)$-admissible if for each coloring (42) there exists a set $C \subset [n]$ of size $k$ such that $\binom{C}{2}$ is colored by at most $(k - 1)$ colors (at least one of the colors is missed).

The following proposition extends the idea of Example 62 to the case of $r$-unavoidable complexes.

**Proposition 64.** Suppose that $n$ is $(P, r)$-admissible. Let $L$ be the simplicial complex defined on the set $V = \binom{[n]}{2}$ (as the set of vertices) where $S \in L$ if and only if the complement $S^c = \binom{[n]}{2} \setminus S$ of $S$ has the property $P$. Then the simplicial complex $L$ is $r$-unavoidable.

**Proof:** If (42) is a partition of the set of vertices $\binom{[n]}{2}$ of $L$ then at least one of the sets $\bigcup_{j \neq i} A_j$ has the property $P$ (Definition 63) which means that $A_i \in L$. \hfill $\square$
Remark 65. It is expected that the $r$-unavoidable complexes described in Proposition 64 are often intrinsically non-linear. Indeed, in the opposite case we should be able (in principle) to prove Ramsey type statements by a simple counting argument (using a system of non-negative weights $x_{i,j}$ on edges $\{i, j\}$).

8.3 Concluding remarks

Both threshold complexes and 2-unavoidable complexes can be directly linked to some classes of simplicial complexes which (often independently) emerged and attracted attention of researchers in game theory, combinatorial topology, social choice theory, reliability theory, geometry of moduli spaces of polygonal linkages, and other areas. In topology [M03] they appear as the Alexander self-dual complexes, and provide key examples of $n$-dimensional complexes non-embeddable in $\mathbb{R}^{2n}$. In social choice theory (reliability theory) [vNM44, R90, PS07] they arise as the complexes describing the winning (losing) coalitions in game theory (‘simple games’ of Von Neumann and Morgenstern [vNM44]). In geometry of configuration spaces [Ga-Pa] they arise as the complexes of ‘short sets’, characterizing the configuration spaces of polygonal linkages in Euclidean spaces. It may be expected that the invariant $\pi(K)$ and the associated class of $r$-unavoidable complexes are as important and interesting for other fields as their 2-unavoidable counterparts.

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