Symmetric bimonoidal intermuting categories and
\(\omega \times \omega\) reduced bar constructions

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Abstract
A new, self-contained, proof of a coherence result for categories equipped
with two symmetric monoidal structures bridged by a natural transfor-
mation is given. It is shown that this coherence result is sufficient for
\(\omega \times \omega\)-indexed family of iterated reduced bar constructions based on such
a category.

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spaces

1 Introduction
This is a paper in applied categorial proof theory. Pure syntactical methods
are used in order to prove a coherence result for a type of categories and to
show that this coherence is sufficient for a reduced bar construction based on
such a category. This reduced bar construction is usually the initial part of the
results connecting various monoidal categories with 1-fold, 2-fold, \(n\)-fold and
infinite loop spaces (see [15], [17], [1] and references therein). By the reduced
bar construction I mean a construction of a simplicial object based on a monoid
in a category whose monoidal structure is given by finite products (exactly the
same as the notion used in [17]), which in particular, for a special monoid in the
category \(\text{Cat}\), may be iterated in order to obtain a lax functor from an arbitrary
power of \((\Delta^+)^{op}\) to \(\text{Cat}\). This construction is based here on a category equipped
with two symmetric monoidal structures, given by the tensors \(\lor\) and \(\land\), and the
units \(\perp\) and \(\top\). These two structures are bridged by a natural transformation,
called intermutation in [4], given by the family of arrows

\[(A \land B) \lor (C \land D) \to (A \lor C) \land (B \lor D).\]

Such categories appeared under the name symmetric bimonoidal intermuting
categories in [4]. As a concrete example of a symmetric bimonoidal intermuting
category one can take any category with all finite coproducts and all finite products in which product of initial objects is initial and coproduct of terminal objects is terminal (see [4], Sections 13 and 15).

Since I am not a topologist, I will not go further in the procedure of delooping, which is very well traced by the work of Thomason in [17]. This procedure leads to an $\omega \times \omega$-indexed family of deloopings of the classifying space of a symmetric bimonoidal intermuting category. (According to this, one can make the following hierarchy of infinite loop spaces: simply symmetric monoidal structure corresponds to an infinite loop space with $\omega$-indexed family of deloopings, double symmetric monoidal structure without intermutation corresponds to an infinite loop space with $\omega + \omega$-indexed family of deloopings, double symmetric monoidal structure with intermutation corresponds to an infinite loop space with $\omega \times \omega$-indexed family of deloopings, etc.)

This paper gives, as a by-product, a complete formulation of a fragment of linear derivations in classical and intuitionistic propositional logic. Logic also helped us in [4] to find that something is inappropriate in the unbalanced treatment of units versus tensors in [11]. (Derivations of sequents of the form $A \lor B \vdash A \land B$ and $A \lor B \vdash B \land A$ are undesirable in logic.) I keep to the notation $\lor$, $\land$, $\bot$, $\top$ for tensors and units which is inspired by logic. This is partly because at one point (see Section 4, Lemma 4.1) there is a reference to a coherence result from [3] where this notation is primary. Also, some easy lemmata in Section 5 are taken over from [4]. Otherwise, this paper is self-contained.

I am aware that a result analogous to the coherence obtained in [11], but with relaxed treatment of units, would be stronger to a great extent than the coherence result given here. I will not give a refinement of the notion of $n$-fold monoidal category from [11]—one can see from the paper how the units should be treated in a relaxed notion and how to use a restricted coherence result to show that the iterated reduced bar construction delivers a lax functor (see Lemma 7.2). As it is noted in [11] (Remarks 1.6 and 1.10), braiding and symmetry appear in some special 2-fold and 3-fold monoidal categories. However, it is a miss-conclusion that symmetric bimonoidal intermuting categories are just special 6-fold monoidal categories. It is not only the case that the difference would appear in morphisms that involve the units, but the undesirable morphisms given in the preceding paragraph show that the axiomatization of $n$-fold monoidal categories given in [11] leads to a non-conservative extension of its fragment without the units. So, the categories would be different in their unit-free fragments too. Hence, to derive our coherence result, even for the unit-free fragment, from the coherence result of [11] would be as to derive Mac Lane’s symmetric monoidal coherence of [11] from the cartesian coherence, which has much simpler proof (see [10], p. 129, where the result is announced, [13], Theorem 2.2, [19], Theorem 8.2.3, p. 207, [14], Section 7 and [2]).

The first part of the paper is devoted to a coherence result for symmetric bimonoidal intermuting categories. At one point, for technical reasons, a strictification with respect to both associativity and symmetry is used, and since the latter is not so standard, although it is explained in details in [3], a sketch of a proof why it actually works is given in Section 4. In the second part of
the paper, this coherence result is used to establish that for every pair \((n, m)\)
of natural numbers one can iterate the reduced bar construction using first \(n\)times the monoidal structure given by \(\lor\) and \(\bot\), and then \(m\) times the monoidalstructure given by \(\land\) and \(\top\) of a symmetric bimonoidal intermuting category\(\mathcal{C}\) in order to obtain a lax functor mapping an \(n + m\)-tuple \((k_1, \ldots, k_{n+m})\)of natural numbers, regarded as objects of the simplicial category, to \(C^{k_1, \ldots, k_{n+m}}\).

The coherence result for symmetric bimonoidal intermuting categories is already present in [4]. Although that paper is not easy to read, this result, aswell as the other coherence results given there, is correct. The proof presentedhere is just more self-contained and because of that, by my opinion, easier forreading. However, the mathematical content remains the same. So, the correctreferring to this coherence result should go through [4].

Some parts of the paper may be skipped (Sections 4 and 6 are optional) andfor experts it is, perhaps, sufficient to see the definition of symmetric bimonoidalintermuting categories (Section 2), then the statement of a coherence result forthese categories (Section 3, Theorem 3.1) and eventually Section 7, especiallyLemma 7.2, which makes this coherence result sufficient for the construction ofa lax functor with desired properties. This paper is influenced very much bynicely written [1].

2 Symmetric bimonoidal intermuting categories

We say that a category \(\mathcal{C}\) is symmetric bimonoidal intermuting (SMI) categorywhen it has two symmetric monoidal structures given by \(\langle\mathcal{C}, \lor, \bot, \ddot{b}, \ddot{c}, \ddot{\delta}, \ddot{\sigma}\rangle\) and\(\langle\mathcal{C}, \land, \top, \hat{b}, \hat{c}, \hat{\delta}, \hat{\sigma}\rangle\) (here \(b\)'s, \(c\)'s, \(\delta\)'s and \(\sigma\)'s stay for associativity, symmetry, right and left identity natural isomorphisms; \(\ddot{b}_{A,B,C} : A \lor (B \lor C) \to (A \lor B) \lor C\) has theinverse \(\ddot{b}_{A,B,C}^{-1}\), etc.) together with a natural transformation \(\ddot{c}_k\) (note that \(k\)is not an index here) given by the family of arrows\(\ddot{c}_{A,B,C,D} : (A \land B) \lor (C \land D) \to (A \lor C) \land (B \lor D)\),two isomorphisms\(\ddot{w}_{\bot}^{-1} : \bot \to \bot \land \bot, \quad \ddot{w}_{\top} : \top \lor \top \to \top\),whose inverses are \(\ddot{w}_{\bot}^{-1}\) and \(\ddot{w}_{\top}^{-1}\) respectively, and an arrow \(\kappa : \bot \to \top\). Inaddition the following diagrams commute:

\[
\begin{array}{ccc}
(A \land (B \land C)) \lor (D \land (E \land F)) & \overset{\ddot{b}^{-1}\lor \ddot{b}^{-1}}{\longrightarrow} & ((A \land B) \land C) \lor ((D \land E) \land F) \\
\downarrow \ddot{c} & & \downarrow \ddot{c} \\
(A \lor D) \land ((B \lor E) \lor (C \lor F)) & \overset{1 \lor \ddot{c}}{\longrightarrow} & ((A \lor B) \lor (D \lor E)) \lor (C \lor F) \\
\end{array}
\]

\[
\begin{array}{ccc}
(A \land D) \lor ((B \land E) \lor (C \land F)) & \overset{\ddot{b}}{\longrightarrow} & ((A \lor D) \land (B \lor E)) \lor (C \lor F) \\
\downarrow \ddot{c} \land 1 & & \downarrow \ddot{c} \land 1 \\
(A \lor D) \land ((B \lor (C \lor F)) & \overset{1 \land \ddot{c}}{\longrightarrow} & ((A \lor D) \land (B \lor (C \lor F))) \\
\end{array}
\]
Two canonical arrows $f, g: A \to B$ of an SMI category are equal if either:

- the units $\bot$ and $\top$ do not “essentially” occur in $A$ and $B$, and $f$ and $g$ have the same graph (defined analogously to the Kelly-Mac Lane graphs in [9]), or
- $A$ and $B$ are isomorphic to $\bot$ or to $\top$.

Since this result has to say something about the canonical structure of an SMI category, and this structure is equationally presented, a precise formulation of our coherence result is given in terms of an SMI category freely generated by a set of objects.

### 3 Freely generated SMI category

Our category $\mathcal{M}$ (called $\mathbf{SC}^{k}_{\top, \bot}$ in [4]), which is an SMI category freely generated by an infinite set $\mathcal{P}$ of propositional letters, is constructed as follows:

The objects of $\mathcal{M}$ are propositional formulae of the language generated from $\mathcal{P}$, constants $\bot$ and $\top$, with the binary connectives $\lor$ and $\land$. The arrows of $\mathcal{M}$ are equivalence classes of arrow terms generated from primitive arrow terms $1_A, b^\wedge_{A,B,C}, \ldots, \kappa$, with the help of $\cdot$, $\lor$ and $\land$. These equivalence classes are taken with respect to the smallest equivalence relation on arrow terms which makes out of $\mathcal{M}$ an SMI category. So, this equivalence relation captures the equations of both symmetric monoidal structures, naturality of $\hat{c}$, isomorphism conditions for $w$’s, the equations brought by the commutative diagrams (1)-(13) of the preceding section, and it is congruent with respect to $\cdot$, $\lor$ and $\land$.

Throughout this section we use the following terminology. We say that an arrow term is a $b$-term if it is built from identities and one occurrence of $b$ with the help of $\lor$ and $\land$. For example, $1_A \land (b^\wedge_{B,C,D} \lor 1_E)$ is a $b$-term and we call
\( \hat{b}_{B,C,D} \) its head. We define analogously \( c, \delta, \sigma, \check{c}, w \) and \( \kappa \)-terms and their heads. Note that they are all composition free. We say that an arrow term \( f_n \circ \ldots \circ f_1 \circ 1_A \) is a developed arrow term if each \( f_i \) is a \( b, c, \delta, \sigma, \check{c}, w \) or \( \kappa \)-term. It is easy to see that every arrow term of \( M \) is equal to a developed one.

We say that an arrow term is defined by \( b \) if it is built from identities and \( b \)'s (both \( \check{b} \)'s and \( \hat{b} \)'s) with the help of \( \circ \), \( \lor \) and \( \land \). We say analogously that an arrow term is defined by \( c \), or \( b \) and \( c \), etc.

For every object \( A \) of \( M \) let \( \nu(A) \) be obtained by iterated replacing in \( A \) every subformulae of the form \( B \lor \perp, \perp \lor B, B \land \top \) and \( \top \land B \) by \( B \) and every subformulae \( \perp \land \perp \) by \( \perp \) and \( \top \lor \top \) by \( \top \) (c.f. the formal definition of \( \nu(A) \) in [4], Section 5). We say that \( A \) reduces by \( \nu \) to \( \nu(A) \). If no letter occurs in \( A \), then \( \nu(A) \) is either \( \perp \) or \( \top \). It is clear that for every \( A \) there is an isomorphism \( i_A : A \rightarrow \nu(A) \) defined by \( \delta, \sigma \) and \( w \) with superscripts \( \rightarrow \). We call the arrow terms defined by \( \delta, \sigma \) and \( w \), \( N_{\top,\perp} \)-terms as in [4], and when all the superscripts are \( \rightarrow \) we call them directed. It can be easily shown that for directed arrow terms \( f, g : A \rightarrow \nu(A) \) we have \( f = g \), hence every definition of the isomorphism \( i_A : A \rightarrow \nu(A) \) leads to the same arrow. From this result the following lemma is derived.

**Lemma 3.1.** Every diagram of \( N_{\top,\perp} \)-terms is commutative.

This is established in the same way as the coherence result for monoidal categories in [11]. Then we can prove the following.

**Lemma 3.2.** If no letter occurs in \( A \) and \( B \) then for every pair \( f, g : A \rightarrow B \) of arrow terms defined by \( \delta, \sigma, w \) and \( \kappa \), we have \( f = g \).

**Proof.** We establish first that every arrow term defined by \( \delta, \sigma, w \) and \( \kappa \) is either equal to an \( N_{\top,\perp} \)-term or it is equal to a term of the form \( h'' \circ \kappa \circ h' \) for \( N_{\top,\perp} \)-terms \( h' \) and \( h'' \). To do this, we rely on the equations (9), (10), the following naturality conditions

\[
\kappa \lor 1_\perp = \hat{\delta}_\top \circ \kappa \circ \hat{\delta}_\perp, \quad 1_\perp \lor \kappa = \check{\sigma}_\top \circ \kappa \circ \check{\sigma}_\perp,
\]

\[
\kappa \land 1_\top = \check{\delta}_\top \circ \kappa \circ \check{\delta}_\perp, \quad 1_\top \land \kappa = \hat{\sigma}_\top \circ \kappa \circ \hat{\sigma}_\perp,
\]

and the fact that there are no arrow terms of the form \( \kappa \circ h \circ \kappa \). If \( f \) is equal to an \( N_{\top,\perp} \)-term then \( A \) is isomorphic to \( B \) and so \( g \) must be equal to an \( N_{\top,\perp} \)-term too, and vice versa. It only remains to apply Lemma 3.1.

**Lemma 3.3.** If no letter occurs in \( A \) and \( B \) then every arrow \( f : A \rightarrow B \) may be defined by \( \delta, \sigma, w \) and \( \kappa \).

**Proof.** We rely on the equations of symmetric monoidal categories, the equations (5), (6), (7), (8), (11), (12), (13), the naturality conditions, and the fact that the arrow terms \( i_A \) and \( i_A^{-1} \) are \( N_{\top,\perp} \)-terms, to eliminate the presence of \( b \)'s, \( c \)'s and \( \check{c} \)'s. For example, we have

\[
\hat{b}_{B,C,D} \]
As a direct consequence of Lemmata 3.2 and 3.3 we have:

**Lemma 3.4.** If no letter occurs in $A$ and $B$ then for every $f, g : A \rightarrow B$ we have $f = g$.

Here is the explanation what we meant by not “essential” occurrence of the units in an object. We say that an object $A$ of $\mathcal{M}$ is $\bot$-pure when there is no occurrence of $\bot$ in $\nu(A)$. It is easy to see that $A$ is not $\bot$-pure iff either $\nu(A) = \bot$ or there is a conjunction in $A$ (by a conjunction in $A$ we mean a subformula of the form $B \land C$) such that one of its conjuncts reduces by $\nu$ to $\bot$ and a letter occurs in the other. We define analogously a $\top$-pure object of $\mathcal{M}$ and derive an analogous characterization. An object of $\mathcal{M}$ is pure when it is both $\bot$-pure and $\top$-pure.

**Lemma 3.5.** Let $f : A \rightarrow B$ be an arrow of $\mathcal{M}$. If $A$ is $\bot$-pure, then $B$ is $\bot$-pure, and if $B$ is $\top$-pure, then $A$ is $\top$-pure.

**Proof.** Since $f$ may be represented by a developed term it is sufficient to verify the lemma for $b, c, \delta, \sigma, \check{c}, w$ and $\kappa$-terms. The only interesting case is when $f$ is a $\check{c}$-rem.

Suppose $B$ is not $\bot$-pure. By using the above-mentioned characterization of such objects of $\mathcal{M}$, we have two possibilities. If $\nu(B) = \bot$ then we easily conclude that $\nu(A) = \bot$ too. If there is a conjunction in $B$ such that one of its conjuncts is reduced by $\nu$ to $\bot$ and a letter occurs in the other conjunct, then we obviously have the same situation in $A$, except in the case when this conjunction is the target of the head $\check{c}^{E,F,G,H} : (E \land F) \lor (G \land H) \rightarrow (E \lor G) \land (F \lor H)$ of $f$. If $\nu(E \lor G)$ is $\bot$ and there is a letter in $F \lor H$, then $\nu(E) = \nu(G) = \bot$ and there is a letter in either $F$ or $G$. So, $A$ is not $\bot$-pure. This is sufficient for the first implication and we proceed analogously for the second implication of the lemma.

**Corollary.** If $f : A \rightarrow B$ and $g : B \rightarrow C$ are arrows of $\mathcal{M}$ such that $A$ and $C$ are pure, then $B$ is pure.

**Lemma 3.6.** If $f : A \rightarrow B$ is an arrow term such that $A$ and $B$ are pure, then there is an arrow term $f' : \nu(A) \rightarrow \nu(B)$ such that $\delta, \sigma, w, \kappa, \top$ and $\bot$ do not occur in $f'$ and

$$f = i_B^{-1} \circ f' \circ i_A.$$
PROOF. By the corollary of lemma 3.5, it is sufficient to verify the lemma for $b$, $c$, $\delta$, $\sigma$, $\delta^k$, $w$ and $\kappa$-terms. If $f$ is an $N_{\top,\bot}$-term then $\nu(A) = \nu(B)$ and by Lemma 3.1 we have $f = i_B^{-1} \circ i_A$.

If $f$ is a $b$-term whose head is $\hat{b}_{C,D,E}$, then by the following naturality diagram

\[
\begin{array}{c}
\text{nat} \\
\nu(X) \land (\nu(Y) \land \nu(Z)) \\
\end{array} \xrightarrow{\hat{b}_{C,D,E}} \begin{array}{c}
(X \land Y) \land Z \\
\end{array}
\]

we may assume that the indices $C$, $D$ and $E$ are already reduced by $\nu$. By the assumption that $A$ and $B$ are pure we have the following cases:

(1) the units do not occur in $C$, $D$ and $E$; hence we are already done,

(2) one of $C$, $D$ or $E$ is $\top$; we are done by the following commutative diagram delivered by the second monoidal structure (here we assume $C = \top$ and we proceed analogously when $D = \top$ or $E = \top$),

\[
\begin{array}{c}
\nu(X) \land (\nu(Y) \land \nu(Z)) \\
\end{array} \xrightarrow{\hat{b}_{C,D,E}} \begin{array}{c}
(X \land Y) \land Z \\
\end{array}
\]

(3) $C = D = E = \bot$; we are done by the commutative diagram (8).

The situation is quite similar with the other $b$ and $c$-terms.

If $f$ is a $\delta^k$-term then again by naturality we may assume that all the indices of the head of $f$ are reduced by $\nu$. It is not possible that only one of its indices is reduced to $\bot$ or to $\top$ since then $A$ or $B$ is not pure. If two of its indices are $\bot$ or $\top$ while the units do not occur in the remaining two indices, then by the assumption that $A$ and $B$ are pure, we may apply the equations (5) or (6) to eliminate this $\delta^k$. Situation is analogous when three indices of $\delta^k$ are $\bot$ or $\top$ and the forth is not. If all the indices of $\delta^k$ are $\bot$ or $\top$ then we have two cases: either we apply the equations (5) or (6) to eliminate $\delta^k$, or we apply the equation (11) to reduce $\delta^k$ to $\kappa$ and we deal with the new occurrence of $\kappa$ as in the following last case for $f$.

If $f$ is a $\kappa$-term. Since $A$ and $B$ are pure, $f$ is not just $\kappa$, so the head of $f$ is in the immediate scope of $\lor$ or $\land$. If $1_E \lor \kappa$ is a subterm of $f$, then since $A$ and $B$ are pure, no letter occurs in $E$ and again we may assume that $E$ is already reduced by $\nu$ to $\bot$ or $\top$. If $E$ is $\top$ then we use the equation (9) to eliminate $\kappa$. If $E$ is $\bot$ then we apply the naturality equation $1_\bot \lor \kappa = \check{\sigma}_\bot \circ \kappa \circ \check{\sigma}_\bot$, mentioned in the proof of Lemma 3.2. This equation does not eliminate $\kappa$ but it replaces a $\kappa$-term of a greater complexity by a $\kappa$-term of lower complexity and by induction $\kappa$ will be eliminated.
We proceed analogously in all the other possible cases for a $\kappa$ term $f$ relying on equations (9), (10) or the remaining naturality conditions mentioned in the proof of Lemma 3.2.

In order to complete the formulation of our coherence result, we have to define graphs corresponding to the arrow terms of $\mathcal{M}$. However, we avoid the graphs and give an equivalent formulation of the result by introducing the following notion. We say that an object of $\mathcal{M}$ is diversified if every letter occurs in it at most once. So, our coherence for $SMI$ categories (called Restricted Symmetric Bimonoidal Intermuting Coherence in [4]) is the following:

**Theorem 3.1.** If $A$ and $B$ are either pure and diversified or no letter occurs in them, then there is at most one arrow $f : A \to B$ in $\mathcal{M}$.

One part of the theorem is established by Lemma 3.4. By Lemma 3.6 we have reduced the rest of the theorem to the case when the units do not occur in $A$ and $B$, and $f$ and $g$ are defined by $b$, $c$ and $c^k$. So, to complete the proof of Theorem 3.1 it is sufficient to prove a coherence result for categories like $SMI$ categories but without units, which we call as in [4], symmetric biassociative intermuting ($SAI$) categories. The canonical structure of $SAI$ categories is given by two biendofunctors $\lor$ and $\land$, natural isomorphisms given by associativities $b$ and symmetries $c$ that satisfy Mac Lane’s pentagonal and hexagonal conditions, and a natural transformation $c^k$ satisfying the coherence conditions given by the diagrams (1), (2), (3) and (4).

This coherence result is formulated in terms of the category $A$ which is freely generated $SAI$ category by the same set $P$ of generators as $\mathcal{M}$. The construction of $A$ is analogous to the construction of $\mathcal{M}$ given at the beginning of this section. So, our auxiliary coherence result is the following:

**Theorem 3.2.** If $A$ and $B$ are diversified, then there is at most one arrow $f : A \to B$ in $A$.

The following two sections contain a proof of this theorem.

### 4 A note on strictification

In order to provide an easier record of equations of arrow terms in the proof of Theorem 3.2 we will replace our category $A$ by a symmetric biassociative intermuting category in which associativity and symmetry arrows are identities. Strictification under associativity is a standard procedure in coherence results. For example, this is how Mac Lane reduced his proof of symmetric monoidal coherence in [11] to the standard presentation of symmetric groups by generators and relations. However, strictification under symmetry is not so standard and it may cause a suspicion. (A reference where it is used implicitly is [8].) Although various strictifications, including this with respect to symmetry, are thoroughly investigated in [3], Chapter 3 and §§4.7, 7.6-8, 8.4, we briefly pass through such a strictification of our category $A$. 


Note first that if we factor the arrow terms of $\mathcal{A}$ by the new equations

$$\hat{c}_{A,A} = 1_{A \vee A}, \quad \hat{c}_{A,A} = 1_{A \wedge A}$$

obtaining a new category $\mathcal{A}'$ with the same objects as $\mathcal{A}$, the full subcategories of $\mathcal{A}$ and $\mathcal{A}'$ on diversified objects are the same. This is because we can easily establish that for every pair of arrow terms $f, g : A \to B$, if $f = g$ in $\mathcal{A}'$ and $f \neq g$ in $\mathcal{A}$, then $A$ and $B$ are not diversified. Since the objects $A$ and $B$ are diversified in Theorem 3.2, we can replace the category $\mathcal{A}$ in the formulation of that theorem by the category $\mathcal{A}'$ without loosing its strength. We use this fact later on.

Let the arrow terms defined by associativities $b$ and symmetries $c$ (c.f. the beginning of the preceding section) be called $S$-terms. Then we have the following result from [3], §6.5.

**Lemma 4.1.** Every diagram of $S$-terms commutes in $\mathcal{A}'$.

This fact together with the property that every $S$-term represents an isomorphism of $\mathcal{A}'$ is sufficient for our strictification of $\mathcal{A}'$ with respect to its associative and symmetric structures. Roughly speaking, we can further factor the arrow terms so that associativity and symmetry natural transformations become identity natural transformations. Of course, this makes some identifications among the objects of $\mathcal{A}'$ too.

We define a relation $\equiv$ on the set of objects of $\mathcal{A}'$ (which are the same as the objects of $\mathcal{A}$) in the following way. Let $A \equiv B$ iff there is an $S$-term $f : A \to B$. Since $1_A$ is an $S$-term, every $S$-term represents an isomorphism whose inverse may be represented by an $S$-term, and the composition of two $S$-terms is an $S$-term, we have that $\equiv$ is an equivalence relation. Let $[A]$ denotes the equivalence class with respect to $\equiv$ of an object $A$ of $\mathcal{A}'$. We can also denote (not in a unique way) the equivalence class $[A]$ by deleting from the formula a parenthesis tied to $\vee$ in the immediate scope of another $\vee$ and the same for $\wedge$. For example, the equivalence class $[(p \wedge q) \wedge ((p \vee r) \vee p)]$ is denoted by $p \wedge q \wedge (p \vee r \vee p)$, and the same equivalence class may be denoted by $q \wedge (r \vee p \vee p) \wedge p$ or by $(p \vee p \vee r) \wedge p \wedge q$, etc. We call such equivalence classes of formulas form multisets (see [3], §7.7), in particular, when $A$ is diversified we call $[A]$ a form set. We use $S$, $T$, $U$, $V$, $W$, $X$, $Y$ and $Z$, possible with indices, for form multisets and form sets.

Note that if $A_1 \equiv A_2$ and $B_1 \equiv B_2$, then $A_1 \wedge B_1 \equiv A_2 \wedge B_2$ and $A_1 \vee B_1 \equiv A_2 \vee B_2$, hence we may define the operations $\vee$ and $\wedge$ on form multisets as

$$[A] \vee [B] =_{df} [A \vee B], \quad [A] \wedge [B] =_{df} [A \wedge B].$$

Let $\mathcal{A}^{st}$ be a category built out of syntactical material, starting from the same set $\mathcal{P}$ of generators as in the case of $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{A}'$, whose objects are the form multisets. The only primitive arrow terms of $\mathcal{A}^{st}$ are of the form

$$1_S : S \to S, \quad or \quad \hat{c}_{[S,T,U,V]} : (S \wedge T) \vee (U \wedge V) \to (S \vee U) \wedge (T \vee V),$$

10
where $[S, T, U, V]$ is an abbreviation for the set $\{\{S, T\}, \{U, V\}, \{S, U\}, \{T, V\}\}$.

(It is straightforward to check that $[S, T, U, V] = [W, X, Y, Z]$ iff $(W, X, Y, Z) \in \{(S, T, U, V), (T, S, V, U), (U, V, S, T), (V, U, T, S)\}$.)

Hence, $c^k_{[S,T,U,V]}, c^k_{[T,S,V,U]}, c^k_{[U,V,S,T]}, c^k_{[V,U,T,S]}$ are the same primitive arrow term which prevents us for having many primitive arrow terms representing the same arrow of $\mathcal{A}^\text{st}$. Moreover, the strictified versions of the equations (3) and (4) are now incorporated in our notation, and when we draw the arrow $c^k: (S \wedge T) \vee (U \wedge V) \rightarrow (S \vee U) \wedge (T \vee V)$ in a diagram, one can form the index of $c^k$ in a unique way.

The arrows of $\mathcal{A}^\text{st}$ are equivalence classes of arrow terms generated from primitive arrow terms with the help of $\circ, \vee$ and $\wedge$. These equivalence classes are taken with respect to the smallest equivalence relation on arrow terms which makes out of $\mathcal{A}^\text{st}$ a strict associative and strict symmetric SAI category. So, this equivalence relation is congruent with respect to $\circ, \vee$ and $\wedge$ are biendofunctors, the following equations

$$
s \vee (t \wedge u) = (s \vee t) \wedge u,
$$

$$
s \wedge (t \vee u) = (s \wedge t) \vee u,
$$

which are the rudiments of naturality conditions for associativity and symmetry, naturality of $c^k$, and the equations brought by the following commutative diagrams:

$$(U \vee V \wedge W) \wedge (X \wedge Y \wedge Z)$$

$$(U \wedge X) \vee ((V \wedge W) \vee (Y \wedge Z))$$

$$1 \wedge (U \vee X) \wedge (V \vee Y) \wedge (W \vee Z)$$

$$(U \wedge X) \vee ((V \vee W) \wedge (Y \wedge Z))$$

$$(U \vee V \wedge W) \wedge (X \vee Y \wedge Z)$$

$$(U \wedge X) \vee ((V \vee W) \wedge (Y \wedge Z))$$

$$(U \vee V \wedge W) \wedge (X \wedge Y \wedge Z)$$

$$1 \vee (U \vee V) \wedge (X \vee Y) \wedge (W \wedge Z)$$

This concludes the definition of $\mathcal{A}^\text{st}$.

The categories $\mathcal{A}'$ and $\mathcal{A}^\text{st}$ are equivalent via functors that preserve the SAI structure. Here is just a sketch of the proof. We define two functors $H_G: \mathcal{A}' \rightarrow \mathcal{A}^\text{st}$ and $H: \mathcal{A}^\text{st} \rightarrow \mathcal{A}'$ in the following way. Let $H_G A := [A]$, and let $H_G f$ be obtained from the arrow term $f$ by replacing every $S$-term in it by $1$ indexed by the equivalence class of the source and the target of this term, and
by replacing every $c_A, B, C, D$ in it by $c_{[A], [B], [C], [D]}$. It is not difficult to verify that $H_G$ is indeed a functor, i.e. that if $f = g$ in $A'$ then $H_G f = H_G g$ in $A_{st}$.

On the other hand, to define $H : A_{st} \to A'$ we have first to choose a formula $A_H$ in each equivalence class $[A]$. By Lemma 4.1, there is a unique arrow $\varphi_A : A_H \to A$ of $A'$ represented by an $S$-term. We define

\[
H[A] = df A_H,
\]

\[
H[1] = df 1_{HS},
\]

\[
H(t \circ s) = df H_t \circ H_s,
\]

\[
H_{[S, T, U, V]} = df \varphi^{-1}_{(HS, HT, HU, HV)} \circ \varphi_{(HS \wedge HT) \vee (HU \wedge HV)},
\]

\[
H(s \vee t) = \varphi^{-1}_{HS_2 \vee HT_2} \circ (Hs \vee Ht) \circ \varphi_{HS_1 \vee HT_1},
\]

for $s : S_1 \to S_2, t : T_1 \to T_2$, and the same for $\vee$ replaced by $\wedge$.

It can be easily checked that this definition is correct and that so defined $H$ is indeed a functor. It is straightforward that $H_G \circ H$ is the identity functor on $A_{st}$ and one can verify that $\varphi$, defined as above, is a natural isomorphism from $H \circ H_G$ to the identity functor on $A'$. (Details of the proof, but in more general context, are given in [3], §3.2.) Hence, $A'$ and $A_{st}$ are equivalent via $H_G$. Following the terminology of [3], functor $H_G$ strictly preserves $SAI$ structure and $H$ is just strong with respect to this structure.

As a consequence of this equivalence and the fact that $A$ and $A'$ have the same full subcategories on diversified objects, we have that the following coherence result is sufficient for Theorem 3.2.

**Proposition.** If $X$ and $Y$ are form sets, then there is at most one arrow $t : X \to Y$ in $A_{st}$.

As we said at the beginning of this section, the strictification of $A$ enables us to record our derivations in the proof of the Proposition, and there are no other reasons, except these technical, for this step. Note that one can always decorate the arrow terms of $A_{st}$ (using the functor $H$) by lengthy compositions of $S$-terms to get back into a rather natural environment given by the category $A$.

## 5 Proof of the Proposition

In this section we are interested only in form sets (i.e. the equivalence classes of diversified formulae) as objects of $A_{st}$. We are going to establish a normalization procedure for arrow terms of $A_{st}$ that eventually delivers our coherence result. For this we use a sequence of definitions and lemmata. We say that a form set $S$ is a subformset of a form set $T$ if there is a formula $A$ in $S$ and a formula $B$ in $T$ ($S$ and $T$ are equivalence classes) such that $A$ is a subformula of $B$. For example $p \wedge (q \vee r)$ is a subformset of $(r \vee q) \wedge s \wedge p$. We use freely for form sets the terminology which is standard for formulae and say, for example, that
$r \lor q$ and $(r \lor q) \land p$ are conjuncts of the form set $(r \lor q) \land s \land p$ whose main connective is $\land$. We say that a conjunct $X$ of a form set is prime if $\land$ is not the main connective in $X$. For example $r \lor q$ is a prime conjunct of $(r \lor q) \land s \land p$ but $(r \lor q) \land p$ is not. Also when $\land$ is not the main connective of a form set, we treat this form set as the prime conjunct of itself. We use the same conventions for $\lor$ and, for example, $(r \lor q) \land s \land p$ is the prime disjunct of itself. We denote by $\text{let}(X)$ the set of letters in a form set $X$.

Every arrow term of $\mathcal{A}^{st}$ is equal to a developed arrow term of the form

$$s_n \circ \ldots \circ s_1 \circ 1$$

where every $s_i$ (if there is any) is a $c^k$-term. We tacitly use developed form of arrow terms throughout the proofs of lemmata given below. We take over the following lemma from [4].

**Lemma 5.1** ([4], Section 14, Lemma 1). If $u: X \to Y$ is an arrow of $\mathcal{A}^{st}$, and $P$ is a set of letters such that for every subformset $U \land V$ of $X$

$$\text{let}(U) \subseteq P \quad \text{iff} \quad \text{let}(V) \subseteq P,$$

then this equivalence holds for every subformset $U \land V$ of $Y$.

As a corollary (taking $P = \text{let}(X_1)$) we have the following:

**Lemma 5.2.** If $u: X_1 \lor X_2 \to Y$ is an arrow of $\mathcal{A}^{st}$, then for every subformset $U \land V$ of $Y$ we have that

$$\text{let}(U) \subseteq \text{let}(X_1) \quad \text{iff} \quad \text{let}(V) \subseteq \text{let}(X_1).$$

(Since $X_1 \lor X_2$ is the same form set as $X_2 \lor X_1$, it is not necessary to mention that the same holds when we replace $X_1$ by $X_2$ in the conclusion of this lemma.) By induction on the complexity of a developed arrow term we can easily show:

**Lemma 5.3.** Every arrow term $t: X' \land X'' \to Y$ of $\mathcal{A}^{st}$ is equal to $t' \land t''$ for some arrow terms $t': X' \to Y'$ and $t'': X'' \to Y''$.

Since the primitive equations of $\mathcal{A}$ and $\mathcal{A}^{st}$ are such that the number of occurrences of $c^k$ is the same on the both sides, we have:

**Lemma 5.4.** All the arrow terms representing the same arrow of $\mathcal{A}$ or $\mathcal{A}^{st}$ have the same number of occurrences of $c^k$.

We introduce now a procedure of deleting letters from form sets. Roughly speaking, to delete a letter $p$ from a form set (which includes some other letters) means to take a formula in this form set, delete the letter $p$ together with its connective and associated brackets from this formula, and then form its equivalence class. It is not difficult to see that this does not depend on the choice of the formula in a form set. In terms of our notation for form sets we define $X^{-p}$, for a form set $X$ different from $p$, in the following way:
If \( p \) is not in \( X \), then \( X^{-p} \) is \( X \);
if \( X \) is of the form \( Y \lor p \) or \( Y \land p \), then \( X^{-p} \) is \( Y \);
if \( X \) is of the form \( Y \lor Z \) for \( Y \) and \( Z \) different from \( p \), then \( X^{-p} \) is \( Y^{-p} \lor Z^{-p} \), and the same holds when we replace \( \lor \) by \( \land \).

If \( \text{let}(X) \setminus \{p, q\} \neq \emptyset \) then it is easy to see that
\[
(X^{-p})^{-q} = (X^{-q})^{-p},
\]
and we can define, for a finite set \( P = \{p_1, \ldots, p_n\} \) of letters such that \( \text{let}(X) \setminus P \neq \emptyset \),
\[
X^{-P} = \text{df} \ (\ldots(X^{-p_1})^{-p_2} \ldots)^{-p_n}.
\]
This can be extended to a procedure of letter deletion from the arrow terms of \( \mathcal{A}^* \).

Let \( u : X \to Y \) be an arrow term of \( \mathcal{A}^* \), and let \( P \) be a finite set of letters such that \( \text{let}(X) \setminus P \neq \emptyset \) (hence \( \text{let}(Y) \setminus P \neq \emptyset \), since \( \text{let}(Y) = \text{let}(X) \)) and such that, as in Lemma 5.1, for every subformset \( U \lor V \) of \( X \) we have \( \text{let}(U) \subseteq P \) iff \( \text{let}(V) \subseteq P \). We define inductively the arrow term \( u^{-P} : X^{-P} \to Y^{-P} \) in the following way:

if \( u \) is \( 1_X \), then \( u^{-P} \) is \( 1_{X^{-p}} \);
if \( u \) is \( \text{df}^{k}_{[S,T,U,V]} \) then
\[
u^{-P} = \text{df} \begin{cases} 
1_{X^{-p}}, & \text{when } \text{let}(S \lor T) \subseteq P \text{ or } \text{let}(U \land V) \subseteq P \\
\text{df}^{k}_{[S^{-p}, T^{-p}, U^{-p}, V^{-p}]} , & \text{otherwise};
\end{cases}
\]
if \( u \) is \( s \lor t \) for \( s : S_1 \to S_2 \) and \( t : T_1 \to T_2 \), then
\[
u^{-P} = \text{df} \begin{cases} 
s^{-P}, & \text{let}(T_1) \subseteq P \\
t^{-P}, & \text{let}(S_1) \subseteq P \\
s^{-P} \lor t^{-P}, & \text{otherwise};
\end{cases}
\]
and we have the same clause when we replace \( \lor \) by \( \land \);
if \( u \) is \( u_2 \circ u_1 \), then by Lemma 5.1, both \( u_1^{-P} \) and \( u_2^{-P} \) are defined and \( u^{-P} \)
is \( u_2^{-P} \circ u_1^{-P} \).

Let \( X_1 \) and \( X_2 \) be form sets. We say that \( \text{df}^{k}_{[S,T,U,V]} \) is \( (X_1, X_2) \)-splitting when one of \( \text{let}(S \lor T) \), \( \text{let}(U \land V) \) is a subset of \( \text{let}(X_1) \) while the other is a subset of \( \text{let}(X_2) \). We say that an arrow term of \( \mathcal{A}^* \) is \( (X_1, X_2) \)-splitting when every occurrence of \( c^k \) in it is \( (X_1, X_2) \)-splitting, and we say that it is \( (X_1, X_2) \)-nonsplitting when every occurrence of \( c^k \) in it is not \( (X_1, X_2) \)-splitting. For example, \( \text{df}^{k}_{[p,q,s,t]} \land 1_{r \lor u} \circ \text{df}^{k}_{[p,q,r,s \land t,u]} \) is a \( (p \land q \land r, s \land t \land u) \)-splitting arrow term.

One can easily check that if \( f = g \) and \( f \) is \( (X_1, X_2) \)-splitting, then \( g \) is \( (X_1, X_2) \)-splitting, too. This is not the case when we replace “splitting” by
By Lemma 5.6, we have \( Y - X \) be (\( Y - X \)) contradiction. We proceed in the other cases quite similar. \( \sqsubseteq \)

**Proof.**

Suppose letters we have that

Note that for \( X_1, X_2, X'_1 \) and \( X'_2 \) as in Lemma 5.5, an arrow term is \( (X_1, X_2) \)-splitting if and only if it is \( (X'_1, X'_2) \)-splitting, which we will use later on.

**Lemma 5.6 (\([4], \) Section 14, Lemma 6).** If \( u: X_1 \vee X_2 \to Y \) is \( (X_1, X_2) \)-splitting, then \( Y - X_1 \) is \( X_2 \) and \( Y - X_2 \) is \( X_1 \).

**Lemma 5.7 (\([4], \) Section 14, Lemma 7).** If \( u: X_1 \vee X_2 \to Y_1 \wedge Y_2 \) is \( (X_1, X_2) \)-splitting, then the main connective in \( X_1 \) and \( X_2 \) is \( \wedge \).

Let \( X_1 = S \wedge T \) and \( X_2 = U \wedge V \) and let \( u = [S,T,U,V] : X_1 \vee X_2 \to Y \) be \( (X_1, X_2) \)-splitting. By Lemma 5.3, the main connective in \( Y \) is \( \wedge \) and by Lemma 5.2, the deletions \( -X_1 \) and \( -X_2 \) are defined for every conjunct of \( Y \).

By Lemma 5.6, we have \( Y - X_2 = X_1 = S \wedge T \) and hence \( Y \) is of the form \( Y_S \wedge Y_T \) for \( Y_S \) and \( Y_T \) such that \( Y_S - X_2 = S \) and \( Y_T - X_2 = T \). Analogously, since \( Y - X_1 = X_2 = U \wedge V \), we have \( Y = Y_U \wedge Y_V \) for \( Y_U - X_1 = U \) and \( Y_V - X_1 = V \). We can then prove the following.

**Lemma 5.8.** For \( u = [S,T,U,V] \) as above, we have \( Y_S = Y_U \) and \( Y_T = Y_V \).

**Proof.** Suppose \( Y_S = Y_U \wedge Z \), and hence, \( Y_V = Y_T \wedge Z \). We have

By Lemma 5.3, \( u \) is of the form \( s \wedge t \) for \( t: Y_T - X_2 \wedge (Y_T - X_1 \wedge Z - X_1) \to W \), where \( W \) is a conjunct of \( Y \). Since the source and the target of \( t \) share the same letters we have that \( let(W) = let(Y_T) \cup let(Z - X_1) \). Hence \( W \) is of the form \( Y_T \wedge W' \) for \( W' \) such that \( let(W') = let(Z - X_1) \subseteq X_2 \). Since \( W' \) is a conjunct of \( Y \), by Lemma 5.2, we have \( let(Y) \subseteq X_2 \) which means that \( let(X_1) = \emptyset \), i.e. a contradiction. We proceed in the other cases quite similar.

In the sequel, for \( u = [S,T,U,V] \) as above, we denote by \( Y_{SU} \) both \( Y_S \) and \( Y_U \) (which are equal by the preceding lemma), and by the same reasons we denote by \( Y_{TV} \) both \( Y_T \) and \( Y_V \).

**Lemma 5.9.** If \( u: X_1 \vee X_2 \to Y_1 \wedge Y_2 \) is \( (X_1, X_2) \)-splitting, then \( u \) factors as:

\[ u = e^{[S,T,U,V]}_{X_1,X_2} \]
Proof. We proceed by induction on number \( n \geq 1 \) of occurrences of \( c^k \) in \( u \). First we prepare a ground for this induction. By relying on the remark after the definition of \((X_1,X_2)\)-splitting arrow term and on Lemma 5.7, \( u \) is equal to an arrow term of the form \( v = c^k_{[S,T,U,V]} \) for \( X_1 = S \land T, X_2 = U \lor V \), and for \( v \), which by Lemma 5.4 has \( n - 1 \) occurrences of \( c^k \), being \((X_1,X_2)\)-splitting.

If we denote \( Y_1 \land Y_2 \) by \( Y \), then by Lemma 5.8 we have \( Y = Y_{SU} \land Y_{TU} \) such that \( Y_{SU}^{-X_2} = S, Y_{SU}^{-X_1} = U, Y_{TV}^{-X_2} = T \) and \( Y_{TV}^{-X_1} = V \). There are several possibilities how the “partition” of \( Y \) given by the conjuncts \( Y_1 \) and \( Y_2 \) may be related to the one given by the conjuncts \( Y_{SU} \) and \( Y_{TV} \), among which the following three cases make all the essentially different situations.

\[
\begin{align*}
(0) \{Y_1, Y_2\} &= \{Y_{SU}, Y_{TV}\}, \text{ when we are done;} \\
(1) Y_1 \land Z &= Y_{SU} \land Y_{TV} \land Z; \\
(2) Y_1 &= Z_1 \land W_1, Y_2 = Z_2 \land W_2, Y_{SU} = Z_1 \land Z_2 \land Y_{TV} = W_1 \land W_2.
\end{align*}
\]

We can start now with the induction.

If \( n = 1 \), then \( v = 1_{(SU) \land (TU \lor V)} \) and \( \{S \lor U, T \lor V\} = \{Y_1, Y_2\} \). Hence we are in case (0) and we are done.

If \( n > 1 \), then if we are in case (0), then we are done again. If we are in case (1), then by Lemma 5.3 we have \( v = s \land t \) for \((X_1,X_2)\)-splitting arrow terms \( s : S \lor U \rightarrow Y_1 \land Z \) and \( t : T \lor V \rightarrow Y_{TV} \) with less than \( n \) occurrences of \( c^k \) in them. Since \( \text{let}(S) \subseteq \text{let}(X_1) \) and \( \text{let}(U) \subseteq \text{let}(X_2) \), we have that \( s \) is \((S,U)\)-splitting and we may apply the induction hypothesis in order to obtain that \( s \) is equal to an \((S,U)\)-splitting (and hence \((X_1,X_2)\)-splitting) arrow term of the form

\[
(Y_1^{-U} \land Z^{-U}) \lor (Y_1^{-S} \land Z^{-S}) \xrightarrow{c^k} (Y_1^{-U} \lor Y_1^{-S}) \land (Z^{-U} \lor Z^{-S}) \xrightarrow{s_1 \land s_2} Y_1 \land Z.
\]

Consider the following commutative diagram whose upper part is an instance of (1s):

\[
\begin{array}{ccc}
(Y_1^{-X_2} \land Y_2^{-X_2}) \lor (Y_1^{-X_1} \land Y_2^{-X_1}) & \xrightarrow{(Y_1^{-X_2} \lor Y_1^{-X_1}) \land (Y_2^{-X_2} \lor Y_2^{-X_1})} & Y_1 \land Y_2 \\
\end{array}
\]

where \( u_1 : Y_1^{-X_2} \lor Y_1^{-X_1} \rightarrow Y_1 \) is \((Y_1^{-X_2}, Y_1^{-X_1})\)-splitting and \( u_2 : Y_2^{-X_2} \lor Y_2^{-X_1} \rightarrow Y_2 \) is \((Y_2^{-X_2}, Y_2^{-X_1})\)-splitting.
\[(Y_{1}^{-U} \land Z^{-U} \land T) \lor (Y_{1}^{-S} \land Z^{-S} \land V) = X_{1} \lor X_{2}\]

\[((Y_{1}^{-U} \land Z^{-U}) \lor (Y_{1}^{-S} \land Z^{-S})) \land (T \lor V)\]

\[(Y_{1}^{-U} \lor Y_{1}^{-S}) \land ((Z^{-U} \lor T) \lor (Z^{-S} \lor V))\]

\[(Y_{1}^{-U} \lor Y_{1}^{-S}) \lor (Z^{-U} \lor Z^{-S}) \land (T \lor V)\]

\[s_{1} \land s_{2} \land t\]

\[Y_{1} \land Z \land Y_{7} \lor V = Y_{1} \land Y_{2}\]

Since \(Y_{1}^{-U} \land Z^{-U} = S\) and \(Y_{1}^{-S} \land Z^{-S} = U\), the left leg of this diagram is equal to \(u\). Also, we have \(Y_{1}^{-U} = Y_{1}^{-X_{2}}, Z^{-U} = Z^{-X_{2}}\) (hence \(Z^{-U} \land T = Y_{2}^{-X_{2}}\)), \(Y_{1}^{-S} = Y_{1}^{-X_{1}},\) and \(Z^{-S} = Z^{-X_{1}}\) (hence \(Z^{-S} \land V = Y_{2}^{-X_{1}}\)). So, the right leg of this diagram is in the desired form.

If we are in case (2), then we use the induction hypothesis twice and appeal to an instance of the following commutative diagram of \(A^{st}\) obtained by pasting instances of (18):

\[(S \land T \land U \land V) \lor (W \land X \land Y \land Z)\]

\[((S \land T) \lor (W \land X)) \land ((U \land V) \lor (Y \land Z))\]

\[((S \land U) \lor (W \lor Y)) \land ((T \land V) \lor (X \land Z))\]

\[(S \lor W) \land (T \lor X) \land (U \lor Y) \land (V \lor Z)\]

We have also the following three lemmata.

**Lemma 5.10.** If \(u: X_{1} \times X_{2} \rightarrow Y' \lor Y''\) is \((X_{1}, X_{2})\)-splitting and \(Y'\) is a prime disjunct of \(Y\) (i.e. \(\lor\) is not the main connective in \(Y'\)), then \(u = u' \lor u''\) for \(u': X' \rightarrow Y'\), where either

for \(i = 1\) or \(i = 2\), \(X'\) is a prime disjunct of \(X_{i}\) and \(u' = 1_{X_{i}'}\), or

\(X' = X_{1}' \lor X_{2}'\) for \(X_{1}'\) and \(X_{2}'\) being prime disjuncts of \(X_{1}\) and \(X_{2}\) respectively and \(u'\) is \((X_{1}', X_{2}')\)-splitting.

**Proof.** By the dual of Lemma 5.3, \(u\) is equal to \(u' \lor u''\), for \(u'\) having \(Y'\) as the target. If the source \(X'\) of \(u'\) is a disjunct of \(X_{1}\), by the assumption that \(u\) is \((X_{1}, X_{2})\)-splitting there are no occurrences of \(\land\) in \(u'\) and hence it must be \(1_{Y'}\), and \(X'\), which is equal to \(Y'\), must be a prime disjunct of \(X_{1}\).

If the source \(X'\) of \(u'\) is of the form \(X_{1}' \lor X_{2}'\) for \(X_{1}'\) a disjunct of \(X_{1}\) and \(X_{2}'\) a disjunct of \(X_{2}\), then \(Y'\) cannot be a letter and hence its main connective is \(\land\). Also, \(u'\) is \((X_{1}', X_{2}')\)-splitting and by Lemma 5.7, \(X_{1}'\) and \(X_{2}'\) are prime disjuncts of \(X_{1}\) and \(X_{2}\), respectively.
Lemma 5.11. Let \( v \cdot u : X_1 \lor X_2 \rightarrow Z \) be such that \( v \) is a \( e^k \)-term that is not \((X_1, X_2)\)-splitting and \( u \) is an \((X_1, X_2)\)-splitting arrow term. Then there exist an arrow term \( w \) and a \( e^k \)-term \( v' \), which is not \((X_1, X_2)\)-splitting, such that \( v \cdot u = w \cdot v' \).

Proof. Let \( X_1 \lor X_2 \xrightarrow{u} Y \xrightarrow{v} Z \), and let \( e^k_{[S, T, U, V]} \) be the head of \( v \). We proceed by induction on “depth” of \((S \land T) \lor (U \land V)\) in \( Y \).

For the base of this induction we have the case when \( S \land T \) and \( U \land V \) are prime disjuncts of \( Y \). If \( Y = (S \land T) \lor (U \land V) \lor Y'' \), then by Lemma 5.10 we have

\[
v \cdot u = (e^k_{[S, T, U, V]} \circ (u' \lor u'')) \lor u''
\]

for \( u' : X' \rightarrow S \land T \) and \( u'' : X'' \rightarrow U \land V \) satisfying the conditions given by that lemma. (The arrow term \( u'' : X'' \rightarrow Y'' \) is out of our interest and it does not exist when \( Y = (S \land T) \lor (U \land V) \).

We have several different situations depending on whether \( X' \) or \( X'' \) are prime disjuncts of \( X_1 \) or of \( X_2 \) or they are of the form \( X'_1 \lor X'_2 \) or \( X''_1 \lor X''_2 \) for \( X'_1, X'_2 \) prime disjuncts of \( X_1 \) and \( X''_1, X''_2 \) prime disjuncts of \( X_2 \). The following three cases represent essentially different situations:

(0) For \( i = 1 \) or \( i = 2 \), \( X' \) and \( X'' \) are prime disjuncts of \( X_i \). (By the assumption that \( v \) is not \((X_1, X_2)\)-splitting, \( X' \) and \( X'' \) cannot be prime disjuncts one of \( X_1 \) and the other of \( X_2 \).) By Lemma 5.10, \( u' \) and \( u'' \) are identities and we are done.

(1) \( X' = X'_1 \lor X'_2 \) for \( X'_1 \) and \( X'_2 \) being prime disjuncts of \( X_1 \) and \( X_2 \) respectively, and \( X'' \) is a prime disjunct of \( X_1 \). By Lemma 5.10, \( u'' = 1_{X''} = 1_{U \land V} \) and we may apply Lemma 5.9 to \( u' : X'_1 \lor X'_2 \rightarrow S \land T \) which is \((X'_1, X'_2)\)-splitting. So, \( e^k_{[S, T, U, V]} \circ (u' \lor u'') \) is equal to the left leg of the following commutative diagram whose upper part is an instance of \((2s)\) and whose lower part is a naturality diagram for \( e^k \).

\[
\begin{align*}
&(S \land T) \lor (U \land V) \\
&((S \land T) \lor (U \land V)) \lor (S \land T) \lor (U \land V) \\
&S \land T \lor (U \land V) \\
&((S \land T) \lor (U \land V)) \lor (S \land T) \lor (U \land V) \\
&((S \land T) \lor (U \land V)) \lor (S \land T) \lor (U \land V) \\
&((S \land T) \lor (U \land V)) \lor (S \land T) \lor (U \land V)
\end{align*}
\]

The right leg of this diagram is of the desired form since it starts with \( 1 \lor e^k \) which is not \((X_1, X_2)\)-splitting.
(2) \( X' = X'_1 \lor X'_2 \) and \( X'' = X''_1 \lor X''_2 \) for \( X'_1 \) and \( X''_1 \) being prime disjuncts of \( X_1 \), and \( X'_2 \) and \( X''_2 \) being prime disjuncts of \( X_2 \). Then we apply Lemma 5.9 to \( u' \colon X'_1 \lor X'_2 \to S \land T \) and to \( u'' \colon X''_1 \lor X''_2 \to U \land V \), and proceed as in case (1) relying on the following commutative diagram of \( A^\ast \) obtained by pasting instances of (2s):

\[
\begin{array}{c}
(S \land W) \lor (T \land X) \lor (U \land Y) \lor (V \land Z) \\
\downarrow \phi \lor \phi' \\
((S \lor T) \land (W \lor X)) \lor ((U \lor V) \land (Y \lor Z)) \\
\downarrow \phi \\
(S \lor T \lor U \lor V) \land (W \lor X \lor Y \lor Z)
\end{array}
\]

For the induction step, we proceed as follows. If \( Y \) is of the form \( Y_1 \land Y_2 \) where \( Y_1 \) is a prime conjunct of \( Y \) whose subformset is \( (S \land T) \lor (U \land V) \), then, by Lemma 5.9, \( v \circ u \) factors as:

\[
\begin{array}{c}
(Y_1^{-X_2} \land Y_2^{-X_2}) \lor (Y_1^{-X_1} \land Y_2^{-X_1}) \\
\downarrow \phi \\
(Y_1^{-X_2} \lor Y_1^{-X_1}) \land (Y_2^{-X_2} \lor Y_2^{-X_1}) \\
\downarrow \phi \\
Z_1 \land Y_2 \\
\downarrow v_1 \land 1_{Y_2}
\end{array}
\]

where \( u_1 \) is \( (Y_1^{-X_2}, Y_1^{-X_1}) \)-splitting and \( v_1 \) is a \( \phi \)-term that is not \( (X_1, X_2) \)-splitting, and hence it is not \( (Y_1^{-X_2}, Y_1^{-X_1}) \)-splitting. By the induction hypothesis \( v_1 \circ u_1 \) is equal to an arrow term of the form \( w_1 \circ v_1' \) for \( v_1' \) a \( \phi \)-term that is not \( (Y_1^{-X_2}, Y_1^{-X_1}) \)-splitting. By Lemma 5.5, we may assume that \( v_1' \) is of the form \( v_1' \lor 1_{Y_1^{-X_1}} \lor 1_{Y_1^{-X_2}} \lor v_1'' \). In both cases we just apply the naturality of \( \phi \) and we are done.

If \( Y \) is of the form \( Y' \lor Y'' \) where \( Y' \) is the prime disjunct containing \( (S \land T) \lor (U \land V) \) as a subformset, then by Lemma 5.10, \( v \circ u \) is of the form

\[
X' \lor X'' \quad u' \lor u'' \quad Y' \lor Y'' \quad v' \lor 1_{Y''} \quad Z' \lor Y''
\]

and if for \( i = 1 \) or \( i = 2 \), \( X' \) is a prime disjunct of \( X_i \), then \( u' \) is \( 1_{X_i} \) and we are done. If \( X' = X'_1 \lor X'_2 \), then we just apply the induction hypothesis to \( v' \circ u' \).

\[\text{Lemma 5.12.} \quad \text{For every arrow term } t : X_1 \lor X_2 \to Y \text{ there are arrow terms } v_1 : X_1 \to Y^{-X_2}, v_2 : X_2 \to Y^{-X_1} \text{ and an } (X_1, X_2) \text{-splitting arrow term } u : Y^{-X_2} \lor Y^{-X_1} \to Y \text{ such that } t = u \circ (v_1 \lor v_2).\]

\[\text{Proof.} \quad \text{We proceed by induction on the number } n \text{ of occurrences of } \phi \text{ in } t. \text{ If } n = 0, \text{ then since identities are at the same time } (X_1, X_2) \text{-splitting and } (X_1, X_2) \text{-nonsplitting we are done.}\]
For the induction step, take a developed arrow term equal to \( t \). If every \( \delta^k \)-term in it is \((X_1, X_2)\)-splitting, then by Lemma 5.6 we are done. Otherwise, by Lemma 5.11 (applied to this developed arrow term from its right-hand side end up to the rightmost \( \delta^k \)-term in it that is not \((X_1, X_2)\)-splitting) we have \( t = t' \ast v' \) where \( v' \) is not \((X_1, X_2)\)-splitting \( \delta^k \)-term. By Lemma 5.5, \( v' = v'_1 \lor v'_2 \) for \( v'_1 : X_1 \to X'_1 \) and \( v'_2 : X_2 \to X'_2 \). By Lemma 5.4, \( t' \) has \( n - 1 \) occurrences of \( \delta^k \) and since \( \text{let}(X_1) = \text{let}(X'_1) \) and \( \text{let}(X_2) = \text{let}(X'_2) \), we may apply the induction hypothesis to it.

We conclude this section with the following proof.

**Proof of the Proposition.** Let \( t : X \to Y \) be an arrow of \( A^{st} \). To prove that \( t \) is unique, we proceed by induction on the complexity of \( X \) and \( Y \). If \( X \) is a letter \( p \), then \( Y \) must be \( p \) too, and \( t : p \to p \) must be \( 1_p \).

If \( X = X' \land X'' \), then by Lemma 5.3 and the induction hypothesis, \( t = t' \land t'' \) for unique arrows \( t' : X' \to Y' \) and \( t'' : X'' \to Y'' \). We reason analogously when \( Y = Y' \lor Y'' \).

Suppose \( X = X_1 \lor X_2 \) and \( Y = Y_1 \land Y_2 \). Then by Lemmata 5.12 and 5.9, and the induction hypothesis, \( t \) is equal to the following composition

\[
X_1 \lor X_2 \xrightarrow{v_1 \lor v_2} Y - X_2 \lor Y - X_1 \xrightarrow{\delta^k} (Y - X_2 \lor Y - X_1) \land (Y - X_2 \lor Y - X_1) \xrightarrow{u_1 \land u_2} Y_1 \land Y_2
\]

for unique arrows \( u_1, u_2, v_1 \) and \( v_2 \). (Note that all the sources and the targets above are completely determined by \( X_1, X_2, Y_1 \) and \( Y_2 \).) So, \( t \) is the unique arrow with the source \( X \) and the target \( Y \).

\( \square \)

### 6 A note on reduced bar construction

This section is optional. Its aim is to give an analysis of a reduced bar construction based on a monoid in a category whose monoidal structure is given by finite products. Such a reduced bar construction was used by Thomason in [117]. I believe this analysis is not new, but I couldn’t find (or just couldn’t recognize) a reference which covers it completely, especially in its graphical approach I intend to use.

Let \( \Delta \) be algebraist’s simplicial category defined as in [112], VII.5, for whose arrows we take over the notation used in that book. Let \( \Delta^+ \) be the topologist’s simplicial category which is the full subcategory of \( \Delta \) with objects all nonempty ordinals. In order to use geometric dimension, the objects of \( \Delta^+ \) are decreased by \( 1 \). So, \( \Delta^+ \) has all finite ordinals as objects, and in this category the source of \( \delta^n_i \), for \( n \geq 1 \) and \( 0 \leq i \leq n \), is \( n - 1 \) and the target is \( n \), while the source of \( \sigma^n_i \), for \( n \geq 1 \) and \( 0 \leq i \leq n - 1 \) is \( n \) and the target is \( n - 1 \). When we speak of \((\Delta^+)^{op}\), then we denote its arrows \((\delta^n_i)^{op} : n \to n - 1\) by \( d^n_i \) and \((\sigma^n_i)^{op} : n - 1 \to n\) by \( s^n_i \).
It is known (see, for example, [18], [5], Section 6, [6], Section 6) that the functor \( \mathcal{J} : \Delta^{\text{op}} \to \Delta \) defined on objects as \( \mathcal{J}(n) = n+1 \), and on arrows by the clauses

\[
\mathcal{J}(\delta^n_i)^{\text{op}} = \sigma_i^{n+1}
\]

\[
\mathcal{J}(\sigma^n_i)^{\text{op}} = \delta_{i+1}^{n+1}
\]

is faithful and obviously injective on objects. Intuitively, this functor is given by taking complements (pictures on the right-hand sides) of the standard graphical presentations for the arrows of \( \Delta^{\text{op}} \) (pictures on the left-hand sides). So, we may regard of \( \Delta^{\text{op}} \) as a subcategory of \( \Delta \). From now on we restrict \( \mathcal{J} \) to \( (\Delta^+)^{\text{op}} \) taking into account that, this time, it is defined on objects by the clause \( \mathcal{J}(n) = n+2 \).

Let \( \Delta_2 \) be the subcategory of \( \Delta \) whose objects are finite ordinals greater or equal to 2 and whose arrows are the arrows of \( \Delta \), i.e. order-preserving functions, which preserve, moreover, the first and the last element. The category \( \Delta_2 \) is the image of \( (\Delta^+)^{\text{op}} \) under the functor \( \mathcal{J} \). So, \( \Delta_2 \) is isomorphic to \( (\Delta^+)^{\text{op}} \) and in the sequel we will represent the arrows of \( (\Delta^+)^{\text{op}} \) by the standard graphical presentations for the corresponding arrows of \( \Delta_2 \).

Let \( \Delta_p \) be the category whose objects are again finite ordinals and whose arrows are order preserving partial functions. Beside the arrows \( \delta^n_i \) and \( \sigma^n_i \), to generate \( \Delta_p \) we need also the arrows \( \rho^n_i : n+1 \to n \) for \( n \geq 0 \) and \( 0 \leq i \leq n \), which are partial functions graphically presented as

\[
\begin{array}{cccccccc}
0 & i-1 & i & i+1 & \cdots & n \\
\cdots & & & & & \\
0 & i-1 & i & i+1 & \cdots
\end{array}
\]

The standard list of equations that satisfy \( \delta \)'s and \( \sigma \)'s should be extended by the following equations:

\[
\rho_j \rho_i = \rho_i \rho_{j+1} \quad i \leq j
\]

\[
\begin{align*}
\rho_j \delta_i &= \begin{cases} 
\delta_{i-1} \rho_j & i > j \\
1 & i = j \\
\delta_i \rho_{j-1} & i < j
\end{cases} \\
\rho_j \sigma_i &= \begin{cases} 
\sigma_{i-1} \rho_j & i > j \\
\rho_i \rho_i & i = j \\
\sigma_i \rho_{j+1} & i < j
\end{cases}
\end{align*}
\]

A counital monad \( \langle T, \eta, \mu, \varepsilon \rangle \) in a category \( X \) consists of a functor \( T : X \to X \) and three natural transformations

\[
\eta : \mathcal{I}_X \Rightarrow T, \quad \mu : T^2 \Rightarrow T \quad \text{and} \quad \varepsilon : T \Rightarrow \mathcal{I}_X
\]
such that \( \langle T, \eta, \mu \rangle \) is a monad in \( X \), and moreover,
\[
\varepsilon \circ \eta = 1_{I_X}, \quad \varepsilon \circ \mu = \varepsilon \circ \varepsilon \circ T.
\]

It is not difficult to see that \( \Delta_p \) is a counital monad freely generated by a single object.

We have that \( \Delta_2 \) is a subcategory of \( \Delta_p \) and also we have a functor \( \mathcal{H} : \Delta_2 \to \Delta_p \) defined on objects as \( \mathcal{H}(n) = n - 2 \), and on arrows, for \( f : n \to m \), as
\[
\mathcal{H}(f) = \rho_{m-2}^{m-1} \circ f \circ \delta_{n-1}^{n-2} \circ \delta_0^{n-2}.
\]

(Intuitively, \( \mathcal{H}(f) \) is obtained by omitting points 0, \( n-1 \) from the source, and 0, \( m-1 \) from the target in the graphical presentation of \( f \) together with all edges including them.) By using essentially the property that the arrows of \( \Delta_2 \) may be built free of \( \delta_0^n \) and \( \delta_n^0 \), it is not difficult to check that \( \mathcal{H} \) so defined is indeed a functor. (Note that \( \mathcal{H} \) is not a functor from \( \Delta \) to \( \Delta_p \).)

The composition \( \mathcal{H} \circ \mathcal{J} \) is a functor from \((\Delta^+)^{op} \to \Delta_p \) which is identity on objects. In this way \( d_0^n \) and \( d_n^n \) are mapped to the partial functions graphically presented as

\[
\begin{array}{c}
d_0^n \mapsto \begin{array}{c}
0 \\
0
\end{array}
\begin{array}{cccc}
1 & \cdots & n-1 \\
0 & & 0
\end{array}
\end{array}
\]

while \( d_i^n \), for \( 0 < i < n \), and \( s_i^n \) are mapped to the functions graphically presented as

\[
\begin{array}{c}
d_i^n \mapsto \begin{array}{cccc}
0 & i-1 & i & n-1 \\
0 & i-1 & i & 0
\end{array}
\end{array}
\begin{array}{c}
s_i^n \mapsto \begin{array}{cccc}
0 & i-1 & i & n-2 \\
0 & i-1 & i + 1 & n-2
\end{array}
\end{array}
\]

However, \( \mathcal{H} \circ \mathcal{J} \) is not faithful. For example, \( d_0^1 : 1 \to 0 \) and \( d_1^1 : 1 \to 0 \) are both mapped to the empty partial function from 1 to 0. So, \( \mathcal{H} \) cannot be used for the constructions like, for example, the functor nerve is, where \( d_0^0 \) and \( d_1^1 \) should be mapped to the source and the target function, respectively.

Let now \( C \) be a monoid in a category \( \mathcal{K} \) whose monoidal structure is given by finite products. Then the functor \( C \times \mathcal{K} : \mathcal{K} \to \mathcal{K} \) together with \( \eta \) and \( \mu \) given by the unit and the multiplication of \( C \) and \( \varepsilon \) given by projections, form a counital monad. Since \( \Delta_p \) is a counital monad freely generated by a single object, we have a functor \( \mathcal{F} : \Delta_p \to \mathcal{K} \) such that \( \mathcal{F}(0) \) is a terminal object \( I \) of \( \mathcal{K} \) and \( \mathcal{F}(n+1) = C \times \mathcal{F}(n) \). (If we regard \( \Delta_p \) as a monoidal category with + as a tensor, then this functor is not strictly monoidal if \( \mathcal{K} \) is not strict monoidal category, however, it is always strong monoidal functor.)

It is not difficult to see that \( \mathcal{W}C = \mathcal{F} \circ \mathcal{H} \circ \mathcal{J} \) underlies the reduced bar construction of \( [17] \) when \( \mathcal{K} \) is strict monoidal. To avoid permanent decoration with associativity, and right and left identity isomorphisms of the monoidal structure of \( \mathcal{K} \), we will always consider this monoidal structure to be strict, which is supported by the strictification given by \([12], XI.3, Theorem 1\). So we will write \( \mathcal{F}(n) = C^n \) and in this case we have:
$\overline{W}C(d^n_0) = pr_2: C \times C^{n-1} \to C^{n-1}, \quad \overline{W}C(d^n_n) = pr_1: C^{n-1} \times C \to C^{n-1},$

and for $1 \leq i \leq n-1$ and $0 \leq j \leq n-1$,

$\overline{W}C(d^n_i) = 1^{i-1} \times \mu \times 1^{n-i-1}, \quad \overline{W}C(s^n_j) = 1^j \times \eta \times 1^{n-j-1}.$

In particular, we are interested in cases when $K$ is the category $\text{Cat}$ (regarded again as strict monoidal) and when $C$ is a strict monoidal category $C$, hence a monoid in $\text{Cat}$. For example, if we take the arrow $f: 4 \to 3$ of $(\Delta^+)^{op}$ graphically presented as

```
0 1 2 3 4 5
```

then for $\overline{W}C: (\Delta^+)^{op} \to \text{Cat}$, we have that $\overline{W}C(f)$, denoted by $f^*: C^4 \to C^3$, is a functor such that

$f^*(A, B, C, D) = (I, I, C \otimes D),$

where $I$ is the unit and $\otimes$ is the tensor of the strict monoidal category $C$. Note that one needs just a part of the cartesian structure of $K$ that provides “counits”, for such a reduced bar construction. The construction may work, for example, in any category with a bialgebra object.

7 Iterated reduced bar construction

Let now $C$ be an SMI category which is strict monoidal with respect to both $\lor$, $\bot$ and $\land$, $\top$. For $n, m$ such that $n+m \geq 1$ we define, along the lines of [1], a lax functor (c.f. [16]), which is an ordinary functor when $n+m = 1$

$\overline{W}C_{n,m}: (\Delta^+)^{op} \times \ldots \times (\Delta^+)^{op} \to \text{Cat}.$

It is defined on objects as $\overline{W}C_{n,m}(k_1, \ldots, k_{n+m}) =_{df} C^{k_1 \ldots k_{n+m}}$, and for the arrows we have the following. First, for $1 \leq i \leq n+m$ and $f_i: k_i \to l_i$ an arrow of $(\Delta^+)^{op}$ let

$\overline{W}C_{n,m}(1_{k_1}, \ldots, 1_{k_{i-1}}, f_i, 1_{k_{i+1}}, \ldots, 1_{k_{n+m}}) =_{df} [\overline{W}D(f_i)]^{k_1 \ldots k_{i-1},}$

where $D$ is the category $C^{k_{i+1} \ldots k_{n+m}}$ whose monoidal structure is defined componentwise in terms of $\lor$, $\bot$ when $i \leq n$ and in terms of $\land$, $\top$ when $n < i \leq n+m$. With this in mind, we define $\overline{W}C_{n,m}(f_1, \ldots, f_{n+m})$ to be the following composition:

$\overline{W}C_{n,m}(1_{l_1}, \ldots, f_{n+m}) \ldots \overline{W}C_{n,m}(f_1, 1_{k_2}, \ldots, 1_{k_{n+m}}).$

We call this construction of $\overline{W}C_{n,m}$, the $(n, m)$-reduced bar construction based on $C$. (When $n+m = 0$ we may define it to be the functor mapping the object
and the arrow of the trivial category \(((\Delta^+)^{op})^0\) to \(\mathcal{C}\) and to the identity functor on \(\mathcal{C}\), respectively.)

**Example** Let \(n = 2\), \(m = 1\), and let \(f = (f_1, f_2, f_3): (1, 2, 2) \rightarrow (2, 1, 2)\) be the arrow of \((\Delta^+)^{op} \times (\Delta^+)^{op} \times (\Delta^+)^{op}\) graphically presented as

\[
\begin{array}{ccc}
0 & 1 & 2 \\
- & - & - \\
0 & 1 & 2 & 3 \\
\end{array}
\]

and \(g = (g_1, g_2, g_3): (2, 1, 2) \rightarrow (2, 2, 2)\) be the arrow of \((\Delta^+)^{op} \times (\Delta^+)^{op} \times (\Delta^+)^{op}\) graphically presented as

\[
\begin{array}{ccc}
0 & 1 & 2 & 3 \\
\circ & \circ & \circ & \circ \\
0 & 1 & 2 & 3 \\
\end{array}
\]

With abbreviation \(h^*\) for \(\mathbf{W}C_{2,1}(h)\), we have that \(f^*\) is the functor from \(C^{1\cdot2^2}\) to \(C^{2\cdot1^2}\) defined by

\[
(A, B, C, D) \mapsto (A, B, C, D, \bot, \bot, \bot, \bot) \mapsto (A, B, \bot, \bot) \mapsto (A \land B, \top, \bot, \bot, \top),
\]

and \(g^*\) is the functor from \(C^{2\cdot1^2}\) to \(C^{2\cdot2^2}\) defined by

\[
(A, B, C, D) \mapsto (A \lor C, B \lor D, \bot, \bot) \mapsto (A \lor C, B \lor D, \bot, \bot, \bot, \bot, \bot).
\]

That \(\mathbf{W}C_{2,1}\) is not a functor could be seen from the fact that \(g^* \cdot f^*\), which is defined by

\[
(A, B, C, D) \mapsto ((A \land B) \lor (\bot \land \bot), \top \lor \top, \bot, \bot, \bot, \bot, \bot, \bot),
\]

is different from \((g \cdot f)^*\). Here \(g \cdot f = (g_1 \cdot f_1, g_2 \cdot f_2, g_3 \cdot f_3): (1, 2, 2) \rightarrow (2, 2, 2)\) is graphically presented as

\[
\begin{array}{ccc}
0 & 1 & 2 \\
\circ & \circ & \circ \\
0 & 1 & 2 & 3 \\
\end{array}
\]

and \((g \cdot f)^*: C^{1\cdot2^2} \rightarrow C^{2\cdot2^2}\) is defined by

\[
(A, B, C, D) \mapsto (A, B, C, D, \bot, \bot, \bot, \bot) \mapsto (A, B, \bot, \bot, \bot, \bot, \bot, \bot) \mapsto
\]

\[
(A \land B, \top \land \bot, \top, \bot, \bot, \bot, \bot, \bot, \top).
\]

However, we have a natural transformation from \(g^* \cdot f^*\) to \((g \cdot f)^*\) whose components are \(\hat{\varphi}_{A, B, \bot, \bot, \bot, \bot}, \hat{\varphi}_{A, B, \bot, \bot, \bot, \bot}\) (three times) and \(\kappa\) (three times). This natural transformation acts as \(\omega_{g, f}\) from the definition of lax functor and since
$\overline{\mathcal{C}}_{n,m}$ preserves identity arrows, there is no need for the natural transformation $\omega_A$.

To show that $\overline{\mathcal{C}}_{n,m}$, for $n + m \geq 2$, is indeed a lax functor, we have to find for every composable pair of arrows $f, g$ of $(\Delta^+)^{op} \times \ldots \times (\Delta^+)^{op}$, a natural transformation $\omega_{g,f} : g^* \circ f^* \Rightarrow (g \circ f)^*$, such that the following diagram commutes

\[
\begin{array}{ccc}
\omega_{h,g}(f^*) & \xrightarrow{h^*(\omega_{g,f})} & h^*(g \circ f)^* \\
\downarrow \omega_{h,g} \circ f^* & & \downarrow h^*(g \circ f)^* \\
(h \circ g)^* & \rightarrow & (h \circ g)^*
\end{array}
\]

For this, we rely on the category $\mathcal{M}^{st}$, which is strict monoidal SMI category freely generated by the same infinite set $\mathcal{P}$ of generators we have used for the category $\mathcal{M}$ in Section 3. The category $\mathcal{M}^{st}$ is obtained from our category $\mathcal{M}$ by factoring its objects through the smallest equivalence relation $\equiv$ satisfying

\[
A \lor (B \land C) \equiv (A \lor B) \land C, \quad A \land (B \lor C) \equiv (A \land B) \lor C, \\
A \equiv A \lor \top \equiv \top \land A \equiv A \lor \bot \equiv \bot \land A,
\]

which is congruent with respect to $\lor$ and $\land$, and by further factoring its arrow terms according to the new equations

\[
\begin{align*}
\delta_A^+ &= \delta_A^c = \delta_A^t = \delta_A^r = \delta_A^s = \delta_A^o = \delta_A^i = \delta_A^a = 1_A, \\
\delta_A^{-} &= \delta_A^c = \delta_A^t = \delta_A^r = \delta_A^s = \delta_A^o = \delta_A^i = \delta_A^a = 1_A.
\end{align*}
\]

(Hence, in writing objects of $\mathcal{M}^{st}$, we may omit parentheses tied to $\lor$, and the constant $\bot$ in the immediate scope of another $\lor$ and the same for $\land$ and $\top$.) An object of $\mathcal{M}^{st}$ is pure and diversified when it, as an equivalence class, consists of formulae that are pure and diversified. As a direct consequence of Theorem 3.1 we have

**Corollary of Theorem 3.1.** If $A$ and $B$ are either pure and diversified or no letter occurs in them, then there is at most one arrow $f : A \rightarrow B$ in $\mathcal{M}^{st}$.

The following lemma serves to reduce our problem to the category $\mathcal{M}^{st}$.

**Lemma 7.1.** If for $\overline{\mathcal{M}}_{n,m}^{st}$ the following holds:

1. for every pair of arrows $f : (k_1, \ldots, k_{n+m}) \rightarrow (l_1, \ldots, l_{n+m})$ and $g : (l_1, \ldots, l_{n+m}) \rightarrow (j_1, \ldots, j_{n+m})$ of $(\Delta^+)^{op} \times \ldots \times (\Delta^+)^{op}$, and every $k_1, \ldots, k_{n+m}$-tuple of different letters $\rho = p_{11}, \ldots, p_{1k_1} \ldots, p_{2k_2} \ldots, p_{nk_n}$, there is an arrow $\omega_{f,g}(\rho) : g^* \circ f^*(\rho) \rightarrow (g \circ f)^*(\rho)$ of $(\mathcal{M}^{st})^{j_1 \ldots j_{n+m}}$, and

2. for every sequence of composable arrows $f_1 \ldots f_u$ of $(\Delta^+)^{op} \times \ldots \times (\Delta^+)^{op}$,
each coordinate of \( f_u^* \cdots f_1^* (\vec{p}) \), is either pure and diversified or no letter occurs in it.

then for every strict monoidal SMI category \( \mathcal{C} \), we have that \( \mathcal{WC}_{n,m} \) is a lax functor.

PROOF. Using the freedom of \( \mathcal{M}^{st} \) and (1) we define for every \( k_1 \cdots k_{n+m} \) tuple \( \bar{A} = (A_{11\cdots1}, A_{11\cdots2}, \ldots, A_{k_1\cdots k_{n+m}}) \) of objects of \( \mathcal{C} \) the arrow \( \omega_{f,g}(\bar{A}) \) as the image of \( \omega_{f,g}(\vec{p}) \) under the functor that extends the function sending generator \( p_{11\cdots i} \) of \( \mathcal{M}^{st} \) to the object \( A_{i_1\cdots i_{n+m}} \) of \( \mathcal{C} \). From (2) (\( u = 1 \) and \( u = 3 \) are the only interesting cases), appealing again to the freedom of \( \mathcal{M}^{st} \), and to Corollary of Theorem 3.1, we have that the diagram (lax) commutes.

To prove that (1) holds, we reason as in [1]. Since for every \( i \in \{1, \ldots, n+m\} \) we have that

\[
(1_{k_1}, \ldots, 1_{k_{i-1}}, f_i, 1_{k_{i+1}}, \ldots, 1_{k_{n+m}})^* \]

is a functor, it is sufficient to show that for every \( 1 \leq i < j \leq n+m \), and \( f_i: k_i \to l_i \) and \( f_j: k_j \to l_j \) arrows of \( (\Delta^+)^{op} \) there is an arrow of \( (\mathcal{M}^{st})_{k_1\cdots k_{i-1} l_i \cdots k_{j-1} l_j \cdots k_{n+m}} \) whose source is

\[
(1_{k_1}, \ldots, f_i, \ldots, 1_{k_i}, \ldots, f_j, \ldots, 1_{k_{n+m}})^* \circ (1_{k_1}, \ldots, f_i, \ldots, f_j, \ldots, 1_{k_{n+m}})^* (\vec{p}),
\]

and whose target is

\[
(1_{k_1}, \ldots, f_i, \ldots, 1_{l_i}, \ldots, f_j, \ldots, 1_{k_{n+m}})^* \circ (1_{k_1}, \ldots, f_i, \ldots, f_j, \ldots, 1_{k_{n+m}})^* (\vec{p}).
\]

Since it is sufficient to find each coordinate of this arrow, we may assume that all numbers except \( k_i \) and \( k_j \) are 1, and we write \( k_i \cdot k_j \) tuple \( \vec{p} \) as \( p_{11}, p_{12}, \ldots, p_{k_i k_j} \).

Let \( f_i: k_i \to 1 \) and \( f_j: k_j \to 1 \) be the arrows of \( (\Delta^+)^{op} \) graphically presented as

\[
0 \quad \cdots \quad t \quad \cdots \quad t + u + 1 \quad k_i + 1
\]

\[
0 \quad 1 \quad 2
\]

\[
0 \quad v \quad v + w + 1 \quad k_j + 1
\]

\[
0 \quad 1 \quad 2
\]

So, in the case when \( i < j \leq n \) we need an arrow

\[
\begin{array}{ccc}
\vee & \vee & \vee \\
\times = u + 1 & \times = t + 1 & \times = v + 1 \\
\end{array}
\]

\[
p_{xy} \to \bigvee \bigvee
\]

which is \( 1_1 \) when either \( u \) or \( w \) is 0, or it is built out of \( \cdot \), otherwise. In the case when \( i \leq n < j \) we need an arrow

\[
\begin{array}{ccc}
\vee & \vee & \vee \\
\times = u + 1 & \times = t + 1 & \times = v + 1 \\
\end{array}
\]

\[
p_{xy} \to \bigwedge \bigwedge
\]

which is built out of \( \bar{w} \), \( \bar{w} \) and \( \kappa \) when \( u \) or \( v \) is 0, or it is built out of \( \cdot \), otherwise. In the case when \( n < i < j \) we proceed as in the first case relying on \( 1_1 \) and \( \cdot \). So, (1) is proved.
To prove that (2) holds, note first that the equivalence relation used to factor the objects of \( M \) in order to obtain the objects of \( M^{st} \) is congruent with respect to the function \( \nu \) defined in Section 3. So, \( \nu \) may be considered as a function on the objects of \( M^{st} \). We say that an object \( \bar{A} = (A_{11 \ldots 1}, A_{11 \ldots 2}, \ldots, A_{k_1 k_2 \ldots k_{n+m}}) \) of \( (M^{st})^{k_1 \ldots k_{n+m}} \) is \((n, m)\)-coherent when the following holds for \( 1 \leq i, j \leq k_1 \):

\[
(*) \text{ Every } A_{i_1 i_2 \ldots i_{n+m}} \text{ is either pure and diversified or no letter occurs in it, and let } (A_{i_1 i_2 \ldots i_{n+m}}) \cap (A_{j_1 j_2 \ldots j_{n+m}}) = \emptyset, \text{ when } i_1 i_2 \ldots i_{n+m} \neq j_1 j_2 \ldots j_{n+m};
\]

\[
(**) \text{ For every } m\text{-tuple } i_{n+1} \ldots i_{n+m}, \text{ if for some } n\text{-tuple } i_1, \ldots, i_n \text{ we have that } \nu(A_{i_1 \ldots i_{n+1} \ldots i_{n+m}}) = \top, \text{ then for every } n\text{-tuple } j_1, \ldots, j_n \text{ we have that } \nu(A_{j_1 \ldots j_{n+1} \ldots j_{n+m}}) \text{ is } \top \text{ or } \bot;
\]

\[
(***) \text{ For every } n\text{-tuple } i_1, \ldots, i_n, \text{ if for some } m\text{-tuple } i_{n+1}, \ldots, i_{n+m} \text{ we have that } \nu(A_{i_1 \ldots i_{n+1} \ldots i_{n+m}}) = \bot, \text{ then for every } m\text{-tuple } j_{n+1}, \ldots, j_{n+m} \text{ we have that } \nu(A_{j_1 \ldots j_{n+1} \ldots j_{n+m}}) \text{ is } \top \text{ or } \bot.
\]

The following lemma has (2) as an immediate corollary.

**Lemma 7.2.** For every \( 1 \leq i \leq n+m \) and every arrow \( f_i : k_i \to l_i \) of \((\Delta^+)^{op}\), if \( \bar{A} \) is \((n, m)\)-coherent object of \( (M^{st})^{k_1 \ldots k_{n+m}} \), then

\[
\bar{B} = (1_{k_1}, \ldots, 1_{k_{i-1}}, f_i, 1_{k_{i+1}}, 1_{k_{n+m}})^*(\bar{A})
\]

is \((n, m)\)-coherent object of \( (M^{st})^{l_1 \ldots l_{n+m}} \).

**Proof.** Since for every \( i \in \{1, \ldots, n+m\} \) we have that

\[
(1_{k_1}, \ldots, 1_{k_{i-1}}, f_i, 1_{k_{i+1}}, \ldots, 1_{k_{n+m}})^*\]

is a functor, it is sufficient to prove the lemma for \( f_i \) being \( d_{k_i}^0 : k_i \to k_i - 1 \) or \( s_{k_i + 1}^0 : k_i \to k_i + 1 \). One can use the following table to verify that \( \bar{B} \) satisfies \((*)\), \((**)\) and \((***)\). In this table the index \( \alpha \) is an \( n+m \) sequence of natural numbers, \( \alpha_i \in [1, l_i] \) is its \( i\)-th component, \( e_i \) is the \( n+m \) sequence with 1 as the \( i\)-th component and 0 everywhere else, and the addition-subtraction is componentwise.

| \( f_i \) | \( B_{\alpha} \) |
|---------|----------------|
| \( d_{k_i}^0 \) | \( A_{\alpha + e_i} \) |
| \( d_{k_i}^0 \) | \( A_{\alpha} \) |
| \( d_{j}^k \) | \( A_{\alpha} \lor A_{\alpha + e_i} \) | \( \alpha_i < j \) |
| \( 0 < j < k_i \) | \( A_{\alpha} \land A_{\alpha + e_i} \) | \( \alpha_i = j \land 1 \leq i \leq n \) \((**)\) |
| \( A_{\alpha} \land A_{\alpha + e_i} \) | \( \alpha_i = j \land n < i \leq n+m \) \((***)\) |
| \( A_{\alpha + e_i} \) | \( \alpha_i > j \) |
| \( s_{k_i + 1}^0 \) | \( A_{\alpha} \) | \( \alpha_i < j+1 \) |
| \( \bot \) | \( \alpha_i = j+1 \land 1 \leq i \leq n \) |
| \( \top \) | \( \alpha_i = j+1 \land n < i \leq n+m \) |
| \( A_{\alpha - e_i} \) | \( \alpha_i > j+1 \) |
In the case marked by (∗∗) we use essentially the property (∗∗) of \( \tilde{A} \) to establish that (∗) holds for \( B \), and analogously for (∗ ∗ ∗). This is the reason why the properties (∗∗) and (∗ ∗ ∗) occur in the definition of \((n, m)\)-coherent object.

Now (2) follows immediately since every \( k_1 \cdot \ldots \cdot k_{n+m} \)-tuple \( \vec{p} \) of different letters is obviously \((n, m)\)-coherent and one has just to iterate Lemma 7.2 through the definition of \( f_0^+ \circ \ldots \circ f_1^*(\vec{p}) \), and eventually, to use the property (∗) of the obtained object. Hence, we conclude from Lemma 7.1 that every \((n, m)\)-reduced bar construction based on a strict monoidal SMI category \( \mathbb{C} \) produces a lax functor.

8 Two questions

The following questions, to which I have no answer, may come to mind to a careful reader of this paper:

1) Do we have an unrestricted coherence for SMI categories, i.e. whether all diagrams (with diversified objects in the nodes) commute in \( \mathcal{M} \)?

2) Since there is no need for \( \tilde{w}^+ \leftarrow \top \) and \( \hat{w} \rightarrow \bot \) in the construction of \( \omega_{f,g} \), is it possible to omit the assumptions that \( \tilde{w}^+ \) and \( \hat{w} \) are isomorphisms from the definition of SMI categories, without loss of coherence necessary for \((n, m)\)-reduced bar construction?

The first question is of lower interest, at least for the \((n, m)\)-reduced bar construction, since we managed to work without unrestricted coherence for SMI categories. Some serious doubts about holding of such a coherence result may be found in [7], Section 7. However, the second question may be quite interesting for the matters of \((n, m)\)-reduced bar construction. An affirmative answer says that this construction may be based on every category with finite coproducts and products without restriction to those categories having initial object as the product of initial objects, and terminal object as the coproduct of terminal objects.

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