Alternating Direction Method of Multipliers for Quantization

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Abstract

Quantization of the parameters of machine learning models, such as deep neural networks, requires solving constrained optimization problems, where the constraint set is formed by the Cartesian product of many simple discrete sets. For such optimization problems, we study the performance of the Alternating Direction Method of Multipliers for Quantization (ADMM-Q) algorithm, which is a variant of the widely-used ADMM method applied to our discrete optimization problem. We establish the convergence of the iterates of ADMM-Q to certain stationary points. To the best of our knowledge, this is the first analysis of an ADMM-type method for problems with discrete variables/constraints. Based on our theoretical insights, we develop a few variants of ADMM-Q that can handle inexact update rules, and have improved performance via the use of “soft projection” and “injecting randomness to the algorithm”. We empirically evaluate the efficacy of our proposed approaches.

1 Introduction

The fields of machine learning and artificial intelligence have experienced significant advancements in recent years. Despite this rapid growth, the extreme energy consumption of many existing machine learning models prevents their use in low-power devices. As a solution, quantized and binarized training of these models has been proposed in recent years [1, 2, 3, 4]. This procedure requires training a machine learning model that has low training/test error, and at the same time, the parameters of the machine learning model must lie in a discrete set. The goal is to improve the energy efficiency of the model by simplifying the required computations in the inference phase.

To obtain accurate quantized models, a wide range of training techniques have been proposed. Among them, Alternating Direction Method of Multipliers (ADMM) has recently gained popularity and results in training highly accurate machine learning models with low power consumption [5, 6, 7, 8]. Despite this empirical success, our theoretical understanding of ADMM for solving discrete optimization problems, such as training binarized neural networks, is almost non-existent. As a first step, in this paper we aim at gaining theoretical insights on the behavior of the ADMM algorithm in discrete optimization through answering the following simple yet fundamental questions.

Brief Problem Description. Assume ADMM algorithm is applied to a nonconvex discrete optimization problem such as training binarized/quantized neural networks.

• Is ADMM guaranteed to improve the objective function in this setting? Can we end up at a point that is worse than the initial point?
• What can we say about the “limit points” of the ADMM algorithm in this discrete context?
• Can ADMM tolerate inexact, randomized, or stochastic computations?
• Is ADMM better than simple algorithms such as projected gradient descent when applied to this discrete problem?
The answer to the above fundamental questions is non-trivial. This lack of understanding is due to the non-monotonic behavior of ADMM as well as the highly fragile relations between the primal and dual variables in this discrete optimization setting. In this paper, we (partially) answer the above questions by first showing that the ADMM-Q algorithm, which is a variant of ADMM in discrete setting, indeed improves the objective over iterations. We analyze the limit points of the iterates generated by ADMM-Q and show that every limit point of the iterates satisfies certain stationarity property. Then, we extend our analysis to inexact and randomized update rules that happen in many practical problems such as training binarized neural networks. Finally, we evaluate the performance of ADMM-Q and its extensions in our numerical experiments. The goal of our numerical experiments is not to obtain the best performance in a particular application or existing benchmark problems, but instead to better understand the behavior of ADMM-Q method. Notice that ADMM-Q has already been used in other papers and its efficiency (combined with other training heuristics) has been established in the literature for different problems [9, 10, 7, 8]. Moreover, to obtain a better understanding of ADMM-Q, we avoid using heuristics such as Straight-through Estimators, scaling factor, not binarizing last layer, playing with the architecture, which has been used in other papers [11, 3, 12, 13]. While these heuristics (combined with exact tuning of many parameters) can significantly improve the performance of the method. They make the scientific study of the core ADMM-Q algorithm almost impossible by bringing a lot of other not well-understood approaches to the table. Thus, in our numerical experiments, instead of aiming for the best possible performance, obtained by using multiple heuristics, we only focus on the empirical performance of the core quantization algorithm.

1.1 State of the art

This paper studies the behavior of the ADMM algorithm when applied to nonconvex discrete optimization problems. This is closely tied to the previous studies on the ADMM algorithm and training quantized machine learning models. Here we briefly review some of the existing works in each of these two categories:

Quantized machine learning models. In recent years, there have been numerous works on the quantization of machine learning models—specifically neural networks. One of the first works towards this was BinaryConnect [1] which used the “Straight Through Estimator” (STE) [11] to provide a “from-scratch” training method with binary weights. BinaryNet [2] extended upon this idea to binarize both weights and activations, replacing complex convolutions with simpler bit-wise operations and significantly reducing the computational complexity. These works performed very well on smaller datasets like MNIST, SVHN and CIFAR-10, and provided an important direction for compression of neural networks. However, their performance on ImageNet classification was poor. XNOR-Net [3] was one of the first works to improve binarized CNNs for ImageNet [14] classification by using scaling factors, that trade-off compression with accuracy. DoReFa-Net [15] further extended the idea of binarization (using the sign function) to gradients as well. They also generalized the method to create networks with arbitrary bit-widths for weights, activations and gradients. ABC-Net [16] improved upon the ideas from XNOR-Net by using multiple binary weights to approximate the full precision weights (instead of scaling factors) and using multiple binary activations. These changes showed that performance like that of XNOR-Net can be achieved without the scaling factors. Tang et al. (2017) [17] introduced seemingly small but impacting changes to improve accuracy, one of which was the use of a regularization function: $|1 - W^2|$ that carried on to further works. BNN+ [12] brought about yet another performance boost by careful regularization strategies and replacing the plain STE with a “SignSwish” activation, a modified version of the Swish-like activation [17]. Yin et al. (2019) [18] provide key theoretical justification to the use of STE by showing a positive correlation between the true and the estimated “coarse” gradient obtained through STE chain rule.

ADMM algorithm. ADMM is an optimization algorithm that combines the decomposability of dual ascent with the superior convergence guarantees of the method of multipliers. The algorithm, which is believed to be first introduced by Glowinski et al. (1975) [19] and Gabay et al. (1976) [20],
can be shown to be equivalent to the Douglas-Rachford splitting algorithm [21]. Boyd et al. (2011) [22] provide a comprehensive overview of the method. Recently, ADMM has sparked the interest of many researchers due to its simplicity, theoretical convergence rates, and parallelization capabilities. The extensibility of ADMM to inexact proximal updates and non-convex problems make it appealing for a lot of problems in machine learning. [23] is perhaps the first work that extended the analysis of ADMM to nonconvex problems and showed its convergence to first-order stationary points. This analysis is later strengthened in [24] by showing the convergence of ADMM iterates to second-order stationary points. Another interesting work by Wang et al. (2019) [25] analyzed the convergence of ADMM for nonconvex and possibly nonsmooth objectives and showed that ADMM, applied to many statistical problems, is guaranteed to converge. In the optimization society, the behavior of ADMM when applied to problems with nonconvex objective functions have also been studied in other regimes such as multi-affine constraints [26], dynamically changing convex constraints [27], finite-sum objective functions, inexact and asynchronous update rules [28] [29], to name just a few. ADMM has also been used as heuristics to solve mixed-integer quadratic programming. Takapouia et al. (2020) [30] proposed an ADMM based algorithm approximately solving convex quadratic functions over the intersection of affine and separable constraints. The Deep Learning community has been no exception to this increased interest in ADMM. [31] provided global convergence guarantees for an ADMM-based optimizer for deep neural networks. Ye et al. [6] used ADMM to devise an effective weight-pruning technique in DNNs for better compression.

With the rising interest in quantization for neural network compression, several works have tried ADMM-based approaches. Leng et al. (2018) [9] were among the first to use an ADMM formulation for weight quantization (not activations) and demonstrated extremely good results on ImageNet classification. Zhane et al. (2018) [32] proposed a systematic DNN weight pruning framework using ADMM. Ye et al. (2019) [9] extended the work [32] on progressive ADMM weight pruning by combining it with quantization in an extended one-shot ADMM learning framework. Lin et al. (2019) [10] further extended this idea by using a progressive multi-step approach for higher performance using only weight quantization. TP-ADMM [7] uses powerful practical improvements to decouple the training into optimized stages and extend the formulation for binarizing both weights and activations with state of the art results. Ren et al. (2019) [33] explored the idea of algorithm-hardware co-design framework using ADMM. Li et al. (2019) [34] showed ADMM based weight pruning achieved significant storage/memory reduction and speedup in mobile devices with negligible accuracy degradation. Yuan et al. (2019) [8] focused on hardware level experiments by combining weight pruning and quantization based on ADMM. Liu et al. (2020) [35] proposed an automatic structured pruning framework adopting ADMM based algorithm, which boosted the compression ratio to a even higher level. In spite of these promising empirical results, the theoretical understanding of ADMM with respect to quantization is still close to non-existent.

2 Problem Formulation

Consider the following discrete optimization problem:

\[ \min_x f(x), \quad \text{s.t.} \quad x \in A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}^d \]  

(1)

where \( A \) is a discrete subset of \( \mathbb{R}^d \). One approach for solving this problem is to sweep across all values in \( A \) and find the optimum point. While this approach results in finding the global optimal solution(s), it is not practical in the quantization procedures of machine learning models. In particular, in this application, the set \( A \) is a discrete grid defined over the space of neural network parameters. Hence, \( n = |A| \) is exponential in the dimension \( d \) and it is computationally impossible to sweep over all values of \( A \). While the size of the set \( A \) can be very large, we make an assumption that the projection to the set \( A \) can be done efficiently. To state our assumption clearly, let us formally define the projection operator followed by two clarifying examples.
Definition 2.1. For any finite set $\mathcal{A}$, the projection of a point $x$, defined as $\mathcal{P}_\mathcal{A}(x)$, is a point $x_\mathcal{P} = \arg \min_{a \in \mathcal{A}} \|x-a\|^2$. If the set $\arg \min_{a \in \mathcal{A}} \|x-a\|^2$ is non-singleton, we choose an element in the set with the smallest lexicographical value.

Assumption 2.2. Projection to the set $\mathcal{A}$ can be done in a computationally efficient manner.

Example 2.3. Suppose $\mathcal{A} = \{-1, +1\}^d$ in $\mathbb{R}$ with $|\mathcal{A}| = 2^d$. One can verify that $\mathcal{P}_\mathcal{A}(x) = \text{sign}(x) = (\bar{x}_1, \ldots, \bar{x}_d) \in \mathbb{R}^d$ where $\bar{x}_i = +1$ if $x_i \geq 0$ and $\bar{x}_i = -1$ if $x_i < 0$. Thus, despite the exponential size of the set $\mathcal{A}$, the projection operator can be computed efficiently.

Example 2.4. Assume $\mathcal{A} = \{x \in \mathbb{Z}^d \mid a \leq x \leq b\}$ with $a,b \in \mathbb{R}^d$ and $\mathbb{Z}$ being the set of integer numbers. Due to the Cartesian product structure of the set $\mathcal{A}$, one can verify that $\mathcal{P}_\mathcal{A}(x) = (\bar{x}_1, \ldots, \bar{x}_d)$ with $\bar{x}_i = b_i$ if $x_i > b_i$, $\bar{x}_i = a_i$ if $x_i < a_i$, and $\bar{x}_i = \text{round}(x_i)$ if $a_i \leq x_i \leq b_i$. Thus, the projection operator can be computed efficiently despite the exponential size of $\mathcal{A}$.

The above two examples are the constraint sets that appear in the quantization/binarization of machine learning models. Next, we describe the ADMM algorithm for solving optimization problem (1).

3 Alternating Direction Method of Multipliers for Quantization (ADMM-Q)

3.1 Review of ADMM

ADMM aims at solving linearly constrained optimization problems of the form

$$\min_{w,z} \ h(w) + g(z) \quad \text{s.t.} \quad Aw + Bz = c,$$

where $w \in \mathbb{R}^{d_1}$, $z \in \mathbb{R}^{d_2}$, $c \in \mathbb{R}^k$, $A \in \mathbb{R}^{k \times d_1}$, and $B \in \mathbb{R}^{k \times d_2}$. By forming the augmented Lagrangian function

$$\mathcal{L}(w, z, \lambda) \triangleq h(w) + g(z) + \langle \lambda, Aw + Bz - c \rangle + \frac{\rho}{2} \|Aw + Bz - c\|^2,$$

each iteration of ADMM applies alternating minimization to the primal variables and gradient ascent to the dual variables. More precisely, at iteration $r$, ADMM uses the update rules:

**Primal Update:**

$$w^{r+1} = \arg \min_{w} \mathcal{L}(w, z^r, \lambda^r), \quad z^{r+1} = \arg \min_{z} \mathcal{L}(w^{r+1}, z, \lambda^r)$$

**Dual Update:**

$$\lambda^{r+1} = \lambda^r + \rho \left( Aw^{r+1} + Bz^{r+1} - c \right).$$

As discussed in section 1.1, this algorithm has been well-studied for continuous optimization. Next, we discuss how this algorithm can be used in the discrete optimization problem (1).

3.2 Description of ADMM-Q

In order to apply ADMM algorithm to the quantization problem (1), we first re-write (1) as

$$\min_{x} \ f(x) + \mathcal{I}_\mathcal{A}(y) \quad \text{s.t.} \quad x = y,$$

where $\mathcal{I}_\mathcal{A}(y) = 0$ if $y \in \mathcal{A}$, and $\mathcal{I}_\mathcal{A}(y) = +\infty$ if $y \notin \mathcal{A}$. Following the steps of regular ADMM in section 3.1, we can update the primal and dual variables alternately. The resulting algorithm, which is called Alternating Direction Method of Multipliers for Quantization (ADMM-Q), is summarized

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*We can break the tie in different ways. We can also pick one of the points in the set $\arg \min_{a \in \mathcal{A}} \|x-a\|^2$ uniformly at random. This choice will make our results to hold with probability one.*
in Algorithm 1. The details of the derivation of this algorithm can be found in Appendix A. Step 4 in this algorithm requires solving an unconstrained optimization problem. In our setting, as we will see later, when \( \rho \) is chosen large enough, the function \( \mathcal{L}(x, y^{r+1}, \lambda^r) \) is strongly convex in \( x \). Thus solving this problem is assumed to be possible for now. We later relax step 4 to an inexact update rule.

**Algorithm 1 ADMM-Q**

1. **Input**: Constant \( \rho > 0 \); initial points \( x^0 = y^0 \in A, \lambda^0 \in \mathbb{R}^d \)
2. for \( r = 0, 1, 2, \ldots \) do
3. \( \text{Update } y^r: \ y^{r+1} = \mathcal{P}_A(x^r + \rho^{-1} \lambda^r) \)
4. \( \text{Update } x^r: \ x^{r+1} = \arg \min_x \mathcal{L}(x, y^{r+1}, \lambda^r) \)
5. \( \text{Update } \lambda^r: \ \lambda^{r+1} = \lambda^r + \rho(x^{r+1} - y^{r+1}) \)
6. end for

### 3.3 Convergence Analysis of ADMM-Q

In order to analyze the behavior of ADMM-Q, we make the following assumptions on \( f \):

**Assumption 3.1.** The function \( f \) is lower bounded on \( A \). That is, \( -\infty < f_{\min} \triangleq \min_{a \in A} f(a) \).

**Assumption 3.2.** The function \( f \) is twice differentiable and its gradient is \( L_f \)-Lipschitz smooth, i.e.,
\[
\|\nabla f(x) - \nabla f(y)\| \leq L_f \|x - y\|, \quad \forall x, y \in \mathbb{R}^d.
\]

**Assumption 3.3.** There exist a constant \( \mu \geq 0 \) such that \( \nabla^2 f(x) \geq -\mu I \).

When \( f \) is twice continuously differentiable, it is easy to verify that \( \mu \leq L_f \). However, defining these two constants separately will allow us to get tighter bounds for the cases that these two constants are different. Let us also state a few useful lemmas that will help us understand the behavior of ADMM-Q. The proofs of these lemmas are relegated to Appendix B.

**Lemma 3.4.** If \( \rho \geq L_f \), we have \( \mathcal{L}(x^r, y^r, \lambda^r) \geq f(y^r) \geq f_{\min}, \quad \forall r \geq 1 \).

**Lemma 3.5.** Define \( \sigma(\rho) \triangleq \rho - \mu \). We have
\[
\mathcal{L}(x^{r+1}, y^{r+1}, \lambda^{r+1}) - \mathcal{L}(x^r, y^r, \lambda^r) \leq (\rho^{-1} L_f^2 - \frac{\sigma(\rho)}{2}) \|x^{r+1} - x^r\|^2. \tag{3}
\]

This lemma states that by choosing \( \rho \) large enough so that \( \rho^{-1} L_f^2 - \frac{\sigma(\rho)}{2} < 0 \), we ensure the decrease of the augmented Lagrangian function at each iteration. This property combined with Lemma 3.4 implies that \( f(y^r) \leq \mathcal{L}(x^r, y^r, \lambda^r) \leq \mathcal{L}(x^0, y^0, \lambda^0) = f(y^0) \). That is, ADMM-Q cannot output a point worse than the initial point. Next, we use these lemmas to analyze the limiting behavior of the iterates of ADMM-Q. To do that, let us first define the following stationarity concept.

**Definition 3.6.** We say a point \( \bar{x} \) is a \( \rho \)-stationary point of the optimization problem \( 1 \) if
\[
\bar{x} \in \arg \min_{a \in A} \|a - (\bar{x} - \rho^{-1} \nabla f(\bar{x}))\|.
\]

In other words, the point \( \bar{x} \) cannot be locally improved using projected gradient descent with step-size \( \rho^{-1} \). Unlike the usual definitions of stationarity for convex constraints, our definition of stationarity depends on the constant \( \rho \). Denoting the set of \( \rho \)-stationary solutions with \( \mathcal{T}_\rho \), it is easy to see that \( \mathcal{T}_{\rho_1} \subset \mathcal{T}_{\rho_2} \) when \( \rho_1 \geq \rho_2 \). Thus, in general we would want to have \( \rho \) as small as possible.

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\(^1\)When \( f \) is convex, \( \mu = 0 \) and hence \( \sigma(\rho) = \rho \). Thus choosing \( \rho > \sqrt{2} L_f \) suffices to ensure the decrease of the augmented Lagrangian function. For the general nonconvex twice differentiable functions, choosing \( \rho > 2L_f \) will imply that \( \rho^{-1} L_f^2 - \frac{\sigma(\rho)}{2} < 0 \), and hence the decrease is guaranteed by Lemma 3.5.
To resolve this issue, notice that the strong convexity of $I$-ADMM-Q of $x$ is a $\rho$–stationary point of the optimization problem (1).

While this theorem establishes its convergence, ADMM-Q is far from its inexact version implemented in practice. Next, we analyze the inexact version of ADMM-Q.

## 4 Inexact ADMM-Q (I-ADMM-Q)

Updating the variable $x$ in ADMM-Q requires finding the minimizer of $\mathcal{L}(\cdot, y^{r+1}, \lambda')$; see step 4 in Algorithm 1. Although $\mathcal{L}(\cdot, y^{r+1}, \lambda')$ is strongly convex when $\rho > \mu$, finding the exact minimizer might not be practically possible. In practice, we apply iterative methods such as (stochastic) gradient descent to obtain an approximate solution $x^{r+1} \approx \arg\min_x \mathcal{L}(x, y^{r+1}, \lambda')$. In this section, we show that ADMM-Q algorithm converges under such an inexact update rule. More precisely, instead of the exact update rule in step 4 of Algorithm 1, we choose a $\gamma$–approximate point $x^{r+1}$ that satisfies

$$
\|x^{r+1} - x^*\| \leq \gamma \min \{\|x^{r+1} - y^{r+1}\|, \|x^{r+1} - x^r\|\},
$$

for some positive constant $\gamma$. Here $x^{r+1} \triangleq \arg\min_x \mathcal{L}(x, y^{r+1}, \lambda')$ is the exact minimizer. The resulting inexact ADMM algorithm, dubbed I-ADMM-Q, is summarized in Algorithm 2. Notice that when $\gamma = 0$, this inexact algorithm reduces to the exact ADMM-Q algorithm.

### Algorithm 2 I-ADMM-Q

1: **Input:** Constants $\rho, \gamma > 0$; initial points $x^0 = y^0 \in A$, $\lambda^0 \in \mathbb{R}^d$
2: **for** $r = 0, 1, 2, \ldots$ **do**
3: **Update** $y$: $y^{r+1} = P_A(x^r + \rho^{-1}\lambda^r)$
4: **Update** $x$ by finding a point $x^{r+1}$ satisfying (4)
5: **Update** $\lambda$: $\lambda^{r+1} = \lambda^r + \rho(x^{r+1} - y^{r+1})$
6: **end for

Similar inexactness measures have previously been used in the literature; see, e.g., [36, 37]. Notice that since $\mathcal{L}(x, y, \lambda)$ is strongly convex in $x$, gradient descent algorithm requires only $O(\log(1/\gamma))$ iterations to find a $\gamma$–approximate solution. Hence, in practice, we do not need to run many iterations of GD. Next, we present our convergence result for I-ADMM-Q.

**Theorem 4.1.** Assume that $f$ satisfies Assumptions 3.1, 3.2, and 3.3. Also assume that the iterates of I-ADMM-Q are bounded, and the constant $\rho$ and $\gamma$ are chosen such that

$$
\left(\frac{2L_f^2}{\rho} + \frac{8(\rho + L_f)^2\gamma^2}{\rho} + \frac{\gamma^2(\rho + L_f)}{2} - \frac{(1 - \gamma)^2\sigma(\rho)}{2}\right) < 0,
$$

with $\sigma(\rho) = \rho - \mu$. Then, for any limit point $(\bar{x}, \bar{y}, \bar{\lambda})$ of the iterates, $\bar{x}$ is a $\rho$–stationary point of (1).

One can verify that the inequality in (5) always holds for $\rho = 6L_f$ and $\gamma \leq 0.1$. However, depending on various trade-offs, we may choose different values of $\gamma$ and $\rho$.

**Remark 4.2.** In practice, checking condition (4) may be impossible since $x^{r+1}$ is not known exactly. To resolve this issue, notice that the strong convexity of $\mathcal{L}(\cdot, y^{r+1}, \lambda')$ implies that $\sigma(\rho)\|x - x^{r+1}\| \leq \|\nabla_x \mathcal{L}(x, y, \lambda)\|$. Hence, we can use the following checkable sufficient condition instead of (4):

$$
\|\nabla_x \mathcal{L}(x^{r+1}, y^{r+1}, \lambda')\| \leq \rho\gamma \min \{\|x^{r+1} - y^{r+1}\|, \|x^{r+1} - x^r\|\}.
$$
5 Injecting Randomness to the Algorithm

The analyses in the previous sections only show that the algorithm converges to a stationary solution of the form defined in Definition 3.6. As mentioned earlier, our stationary set includes more points as $\rho$ increase. Thus, to obtain a point satisfying stronger stationary condition, we need to pick the smallest possible $\rho$. However, reducing the value of $\rho$ beyond certain value results in instability and divergence in ADMM-Q, as suggested by our theory and numerical experiments. Another approach that has been utilized in practice to escape spurious stationary solutions is the use of randomness/noise in the algorithm [38, 39, 40, 41, 42, 43]. In order to inject randomness to our algorithm, we propose the following step at each iteration $r$: draw a set of (potentially correlated) Bernoulli random variables $m' = \{m'_1, m'_2, \ldots, m'_d\}$. Each $m'_i$, corresponds to the coordinate $i$ in vector $y$ with $\text{Prob}(y'_i = 1) = p'_i > 0$. Then, we update $y_i$ in iteration $r$ if and only if $m'_i = 1$. This variant of ADMM-Q, which we denote by ADMM-R, is presented in Algorithm 3. The convergence result of this algorithm is stated in Theorem 5.1. The proof of this result follows the same steps as in the ones in Theorem 3.7 and hence we omit the proof here.

Algorithm 3 ADMM-R

1: Input: Constants $\rho, \gamma > 0$; initial points $x^0 = y^0 \in A$, $\lambda^0 \in \mathbb{R}^d$; the sequence $\{p'_i\}_{i,r} \geq \alpha > 0$.
2: for $r = 0, 1, 2, \ldots$ do
3: Generate $m$: $m' = \{m'_1, m'_2, \ldots, m'_d\}$
4: Compute $\hat{y}$: $\hat{y}^{r+1} = P_A(x^r + \rho^{-1}\lambda^r)$
5: Update $y$: $y_i^{r+1} = m'_i\hat{y}_i^{r+1} + (1 - m'_i)y_i^r$, $\forall i = 1, \ldots, d$
6: Update $x$: $x^{r+1} = \text{arg min}_x L(x, y^{r+1}, \lambda^r)$
7: Update $\lambda$: $\lambda^{r+1} = \lambda^r + \rho(x^{r+1} - y^{r+1})$
8: end for

Theorem 5.1. Assume that the constraint set $A$ in (1) is a Cartesian product of simple coordinate-wise sets of scalers. Then, under the same set of assumptions as in Theorem 3.7, every iterate of the ADMM-R algorithm is a $\rho$-stationary point of (1).

Notice that the convergence of this algorithm requires that the constraint set $A$ to be of the Cartesian product form. This assumption is necessary since the coordinates of $y$ is updated separately; see [44, 45, 46] for necessity of such an assumption in the presence of coordinate-wise update rule. Having said that, the constraint sets in the quantization context satisfy this assumption as illustrated in Example 2.3 and Example 2.4.

6 ADMM-Q with Soft Projection (ADMM-S)

Step 3 in ADMM-Q algorithm requires projection to the discrete set $A$. Such a projection is non-continuous which may result in instabilities in the algorithm. As a solution, we can use "soft projection" in ADMM algorithm. To obtain such soft projections, we start by replacing the indicator function $I_A(\cdot)$ in the objective function with a soft indicator function defined below.

Definition 6.1. Given a finite set $A \subseteq \mathbb{R}^d$, we define the Soft Indicator Function $S_A : \mathbb{R}^d \to \mathbb{R}$ as

$$S_A(x) = \min_{a \in A} \|x - a\|_2.$$  

Replacing the indicator function $I_A(\cdot)$ with the soft indicator function $S_A$ in (2), we obtain

$$\min_x f(x) + \beta S_A(y) \quad \text{s.t.} \quad x = y,$$

7
where \( \beta > 0 \) is some given constant. Following the steps of ADMM, we obtain the ADMM algorithm with soft projections (ADMM-S), which is summarized in Algorithm 4. The details of the derivation of this algorithm is summarized in Appendix E.

Algorithm 4 ADMM-S

1: **Input:** Constant \( \rho > 0 \), \( \beta > 0 \); initial points \( x^0 = y^0 \in A \), \( \lambda^0 \in \mathbb{R}^d \)
2: for \( r = 0, 1, 2, \ldots \) do
3: \[ z^{r+1} = x^r + \rho^{-1} \lambda^r \quad \text{and} \quad \tilde{z}^{r+1} = \mathcal{P}_A(z^{r+1}) \]
4: \[ y^{r+1} = \begin{cases} z^{r+1} + \rho^{-1} \beta (\tilde{z}^{r+1} - z^{r+1}) & \text{if} \quad \rho^{-1} \beta \leq ||z^{r+1} - \tilde{z}^{r+1}||_2 \\ \tilde{z}^{r+1} & \text{if} \quad \rho^{-1} \beta > ||z^{r+1} - \tilde{z}^{r+1}||_2 \end{cases} \]
5: \[ x^{r+1} = \arg\min_x \mathcal{L}(x, y^{r+1}, \lambda^r) \]
6: \[ \lambda^{r+1} = \lambda^r + \rho(x^{r+1} - y^{r+1}) \]
7: end for

As shown in Appendix E, this algorithm coincides with ADMM-Q if \( \beta \) is chosen large enough. However, for small values of \( \rho \), this algorithm results in a different trajectory. In this case, while the iterates of the algorithm does not necessarily converge to the set \( A \), the \( y \) iterates are kept close to set \( A \). Moreover, as shown in Appendix E, the augmented Lagrangian function is monotonically decreasing and converges. Finally, we would like to mention that similar hard and soft indicators have been used before for sparse signal recovery through soft and hard thresholding operators [47, 48].

7 Numerical Experiments

We empirically evaluate the performance of the proposed algorithms in the following two problems: training simple neural networks and solving quantized quadratic optimization instances. Since the ability of ADMM in training neural networks has been demonstrated in the past [9, 10, 7, 8], we defer the presentation of our empirical results on training neural networks to Appendix F. In short, the results on training neural networks show that we can achieve the same performance as the state of the art on simple networks without using heuristics. More importantly, the proposed algorithm enjoys theoretical convergence guarantees. Having said that, in the main body of the manuscript we only focus on presenting numerical experiments on solving discrete quadratic optimization problems.

In this experiment, we use our algorithms to solve the optimization problem

\[
\min_x \frac{1}{2} x^\top Q x + b^\top x \quad \text{s.t.} \quad x \in A \triangleq v \mathbb{Z}^d, \tag{6}
\]

for some given \( Q \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d, \) and \( v \in \mathbb{Z}^+ \). Here, the constraint set says that any feasible point should be an integer number which is a multiple of \( v \). We generate matrix \( Q \) via the rule \( Q = \tilde{Q}^\top \tilde{Q} + \tilde{q} \tilde{q}^\top \), where \( \tilde{Q}_{ij} \sim N(0,1), \tilde{q}_i \sim N(0, \sigma_q^2) \), \( 1 \leq i, j \leq d \). Note that the Lipschitz constant \( L_f \) can be adjusted through \( \sigma_q^2 \). We compare the performance of projected gradient gradient (PGD), GD+Proj, ADMM-Q, ADMM-S, and ADMM-R over different combinations of \( d \) and \( \sigma_q^2 \) (see appendix E). The PGD algorithm is defined through the iterative update rule \( x^{r+1} = \mathcal{P}_A(x^r - \rho^{-1} \nabla f(x^r)) \). The “GD+Proj” algorithm, runs gradient descent to find the global optimum of unconstrained problem, then it projects the final solution onto the feasible set \( A \).

For each problem instance, we run each algorithm starting at the same random generated point for 30,000 iterations (except 100,000 iterations for PGD). The best objective value over the last 50 iterations will be recorded as the result of each run. We repeat this procedure for 50 different initializations, and compute the median, 25% quartile and 75% quartile over 50 runs. We use the best hyper-parameter for each algorithm by median, and report the median 25% quartile and 75% quartile. The list of hyper parameters used can be found in Appendix E.
Results: We only report our results for $(v, d, \sigma^2_q) = (8, 16, 30)$ here. More simulations are detailed in Appendix. Figure 1 shows the performance of the studied algorithms for five different problem instances. Each point on x-axis represents one problem instance; and y-axis is the final obtained objective value. As expected, ADMM-Q outperforms PGD and GD+Proj with large margins. We also observe that both ADMM-S and ADMM-R have better median final objective values than ADMM-Q. In addition, the final objective value has a smaller variance in these two algorithms. More importantly, the median tends to overlap with the 25% quantile, showing that the objective of at least 25 runs are exactly the same as the minimal objective over 50 runs.

To better understand the performance gap between different algorithm for the same initialization, we conduct one additional experiment: we generate 5 instance of $Q$ and $b$, and for each instance we run 50 different random initialization, resulting in 250 total runs. We record the final obtained objective value by each algorithm. Then we compute the differences between the objective values obtained by two algorithms for the same initialization. We plot the histograms of these differences in Figure 2. In this plot, $f(x_{ADMM-Q})$ denotes the final objective value obtained by ADMM-Q algorithm (similar notation is used for other algorithms). Our histogram plot suggests that ADMM-S and ADMM-R outperform ADMM-Q for almost all 250 runs. It shows the same result that PGD performs much worse than ADMM-Q or any ADMM-Q variants. We also observe that ADMM-R slightly outperforms ADMM-S.
Figure 2: Histogram of the difference of obtained objective values for different algorithm pairs
Broader Impact

While in this paper we only discuss the application of ADMM in quantization, the tools and theories developed in this work can potentially be used to solve more general discrete optimization problems. This could be the first step toward the understanding of continuous optimization algorithms in generic discrete optimization context. Clearly, such a development will be useful in a wide range of applications that discrete optimization problems appear.

Another potential impact of this work is on the development of distributed algorithms for solving discrete optimization problems. ADMM and other methods based on dual decomposition have been extensively used for solving continuous optimization problems in a distributed fashion; see, [49, 50, 51, 22, 23, 52] for surveys and examples in continuous convex and non-convex settings. The theories and tools developed in this work can serve as a stepping stone to the invention of new distributed algorithms based on dual decomposition for solving discrete optimization problems.

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A On the update rules of ADMM-Q

Consider the following optimization problem mentioned in section 3.2:

$$\min_x f(x) + I_A(y) \quad \text{s.t.} \quad x = y.$$ 

Following the steps of regular ADMM in section 3.1, we have:

$$L(x, y, \lambda) \triangleq f(x) + I_A(y) + \langle \lambda, x - y \rangle + \frac{\rho}{2} \|x - y\|^2_2,$$

In regular ADMM, the order of updating variables does not matter for convergence. When we extend its use to quantization, we update $y$, $x$ and $\lambda$ in sequence at each iteration, which is convenient for analyzing its convergence.

Primal Update: \(y^{r+1} = \arg \min_y L(x^r, y, \lambda^r)\), \(x^{r+1} = \arg \min_x L(y^{r+1}, x, \lambda^r)\)

Dual Update: \(\lambda^{r+1} = \lambda^r + \rho(x^{r+1} - y^{r+1})\).

The update rule of $x$ and $\lambda$ is clear. We only derive the update rule of $y$ here:

\[
y^{r+1} = \arg \min_y L(x^r, y, \lambda^r)
\]

\[
= \arg \min_y f(x^r) + I_A(y) + \langle \lambda^r, x^r - y \rangle + \frac{\rho}{2} \|x^r - y\|^2_2
\]

\[
= \arg \min_y I_A(y) + \langle \lambda^r, x^r - y \rangle + \frac{\rho}{2} \|x^r - y\|^2_2
\]

\[
= \arg \min_y I_A(y) + \|y - x^r - \rho^{-1} \lambda^r\|^2_2
\]

\[
= P_A(x^r + \rho^{-1} \lambda^r)
\]

B Proofs in Section 3.3

Lemma B.1. For any \(r \geq 1\) we have \(\lambda^r = -\nabla_x f(x^r)\).

Proof. based on the algorithm updates and the optimality condition for \(x^{r+1}\) we can easily verify that:

\[
\nabla_x f(x^{r+1}) + \lambda^{r+1} = 0.
\]

Lemma 3.4. If \(\rho \geq L_f\), we have \(L(x^r, y^r, \lambda^r) \geq f(y^r) \geq f_{\min}, \forall r \geq 1\).

Proof. Note that based on Lemma 3.1, we have

\[
L(x^r, y^r, \lambda^r) = f(x^r) + \langle \nabla f(x^r), y^r - x^r \rangle + \frac{\rho}{2} \|x^r - y^r\|^2
\]

\[
\geq f(y^r) \geq f_{\min}
\]

where the last two inequalities are due to Assumptions 3.2 and 3.1 respectively.

Lemma 3.5. Define \(\sigma(\rho) \triangleq \rho - \mu\). We have

\[
L(x^{r+1}, y^{r+1}, \lambda^{r+1}) - L(x^r, y^r, \lambda^r) \leq \left(\rho^{-1} L_f^2 - \frac{\sigma(\rho)}{2}\right) \|x^{r+1} - x^r\|^2. \tag{9}
\]
Proof. Let us re-write (9) as
\[
\mathcal{L}(x^{r+1}, y^{r+1}, \lambda^{r+1}) - \mathcal{L}(x^r, y^r, \lambda^r) = \underbrace{\mathcal{L}(x^{r+1}, y^{r+1}, \lambda^{r+1}) - \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r)}_{(A)} + \underbrace{\mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^r, \lambda^r)}_{(B)}.
\]
We want to show that \((A) + (B) \leq 0\). First of all note that
\[
(A) = \langle \lambda^{r+1}, x^{r+1} - y^{r+1} \rangle - \langle \lambda^r, x^{r+1} - y^{r+1} \rangle = \rho^{-1} \|\lambda^{r+1} - \lambda^r\|^2.
\]
By the optimality condition of \(x^{r+1}\), we have:
\[
\nabla_x f(x^{r+1}) + \underbrace{\lambda^{r+1}}_{\lambda^{r+1}} = 0,
\]
showing that \(\nabla_x f(x^{r+1}) = -\lambda^{r+1}\), or \(\nabla_x f(x^r) = -\lambda^r\).
Furthermore, by the lipschitz assumption of \(f(\cdot)\), we have
\[
\|\nabla_x f(x^{r+1}) - \nabla_x f(x^r)\|^2 \leq L_f^2 \|x^{r+1} - x^r\|^2,
\]
showing that
\[
\|\lambda^{r+1} - \lambda^r\|^2 \leq L_f^2 \|x^{r+1} - x^r\|^2.
\]
Therefore,
\[
(A) \leq \rho^{-1} L_f^2 \|x^{r+1} - x^r\|^2.
\]
On the other hand:
\[
(B) = \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^r, \lambda^r)
= \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^{r+1}, \lambda^r) + \underbrace{\mathcal{L}(x^r, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^r, \lambda^r)}_{\leq 0}
\leq \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^{r+1}, \lambda^r)
\leq -\frac{\sigma(\rho)}{2} \|x^{r+1} - x^r\|^2,
\]
where \(\sigma(\rho)\) is the strong convex modulus of \(\mathcal{L}(\cdot, y^{r+1}, \lambda^r)\) (note that \(\sigma(\rho) = \rho - \mu\)).

**Theorem 3.7** Assume \((\bar{x}, \bar{y}, \bar{\lambda})\) is a limit point of the ADMM-Q algorithm. Then \(\bar{x}\) is a \(\rho\)-stationary point of the optimization problem \(\{1\}\).

**Proof.** Consider a sub-sequence \((x^{t_r}, y^{t_r}, \lambda^{t_r})\), for \(t = 0, \ldots\) which converges to \((\bar{x}, \bar{y}, \bar{\lambda})\). First of all due to the decrease lemma \(3.5\) and lower boundedness of augmented Lagrangian, Lemma \(3.4\) we know that \(\lim_{t \to \infty} \|x^{t_r+1} - x^{t_r}\| = 0\). Thus,
\[
\lim_{t \to \infty} x^{t_r+1} = \bar{x}
\]
(10)
Now based on Lemma \(B.1\), we also know that
\[
\bar{\lambda} = \lim_{t \to \infty} \lambda^{t_r} = \lim_{t \to \infty} \nabla f(x^{t_r}) = \nabla f(\bar{x})
\]
(11)
\[
\lim_{t \to \infty} \lambda^{t_r+1} = \lim_{t \to \infty} \nabla f(x^{t_r+1}) = \nabla f(\bar{x})
\]
(12)
Thus, $\lim_{t \to \infty} \lambda^{t+1} = \bar{\lambda}$.

Moreover, due to strong convexity we also know that: $\lambda^{t+1} = \lambda^t + \rho(\bar{y} - x^t)$. Taking the limits from both sides, we get

$$\bar{y} = \bar{x}.$$  \hfill (14)

Combining the above with (13) we can easily see that

$$\|\bar{x} - (x^t + \rho^{-1}\lambda^t)\| \leq \|a_i - (x^t + \rho^{-1}\lambda^t)\|, \quad i = 0, \ldots, N$$

Taking the limits $\lim_{t \to \infty}$ from both hand sides of the inequality for all the points $a_i$ we have

$$\|\bar{x} - (\bar{x} + \rho^{-1}\bar{\lambda})\| \leq \|a_i - (\bar{x} + \rho^{-1}\bar{\lambda})\|, \quad i = 0, \ldots, N.$$  \hfill (15)

Thus, $\bar{x} \in \arg\min_{a \in A} \|a - (\bar{x} + \rho^{-1}\nabla f(\bar{x}))\|$, \hfill (17)

where we used the fact that $\bar{\lambda} = -\nabla f(\bar{x})$.

\section{Convergence Analysis for I-ADMM-Q}

In order to prove the main convergence results, we need a few definitions and helper lemmas. Throughout this section we re-state all the theoretical results and prove them in the order we need them. For a reference of the steps in the algorithm see Algorithm 2.

First, let us define:

$$e^r = \nabla_x \mathcal{L}(x^r, y^r, \lambda^r) = \nabla f(x^r) + \lambda^r - \rho(x^r - y^r) = \nabla f(x^r) + \lambda^r$$  \hfill (18)

\begin{lemma}
Due to $\sigma(\rho)$-strong convexity and $(L_f + \rho)$-smoothness of $\mathcal{L}(., y^r, \lambda^r)$, we know that

$$\sigma(\rho)\|x^r - x^r_\ast\| \leq \|e^r\| \leq (\rho + L_f)\|x^r - x^r_\ast\|$$  \hfill (19)

Moreover, due to strong convexity we also know that:

$$\langle e^r, x^r - x^r_\ast \rangle \geq \sigma(\rho)\|x^r - x^r_\ast\|^2$$  \hfill (20)

\begin{lemma}
If $\rho \geq L_f$ and we also assume that the iterates $x^r$ stay bounded. Then there exists a non-negative number $\bar{D}$ s.t. $\|x^r - y^r\| \leq \bar{D}$. With this definition,

$$\mathcal{L}(x^r, y^r, \lambda^r) \geq f_{\min} - \gamma(\rho + L_f)\bar{D}^2$$  \hfill (21)

\end{lemma}

Proof. Note that

$$\mathcal{L}(x^r, y^r, \lambda^r) = f(x^r) + \langle \lambda^r, x^r - y^r \rangle + \frac{\rho}{2}\|x^r - y^r\|^2$$

$$= f(x^r) + \langle \nabla f(x^r), y^r - x^r \rangle + \frac{\rho}{2}\|x^r - y^r\|^2 + \langle e^r, x^r - y^r \rangle$$

$$\geq f(y^r) - \|e^r\|\|x^r - y^r\| + \frac{\gamma}{2}\|x^r - y^r\|^2$$

$$\geq f_{\min} - \gamma(\rho + L_f)\bar{D}^2$$  \hfill (25)

where the last inequality is due to the assumptions and Lemma C.1.
are chosen such that \( \alpha = \left( \frac{2L_f^2}{\rho} + \frac{4(\rho + L_f)^2 \gamma^2}{\rho} + \gamma^2 (\rho + L_f) \right) - \frac{(1 - \gamma)^2 \sigma(\rho)}{2} \) \quad (26) 
and \( \beta = \frac{4(\rho + L_f)^2 \gamma^2}{\rho} \). Note that \( \sigma(\rho) = \rho - \mu \geq 0 \). Furthermore, assume that the parameters \( \rho \) and \( \gamma \) are chosen such that \( \alpha + \beta < 0 \). Then, have

\[
\lim_{r \to \infty} \| x^{r+1} - x^r \| = 0. \quad (27)
\]

**Proof.** Let us re-write (9) as

\[
\mathcal{L}(x^{r+1}, y^{r+1}, \lambda^{r+1}) - \mathcal{L}(x^r, y^r, \lambda^r) = \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^{r+1}) - \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r)
\]

\[
+ \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^r, \lambda^r). \quad (A)
\]

(B) We want to show that \( (A) + (B) \leq 0 \).

\[
(A) = \langle \lambda^{r+1}, x^{r+1} - y^{r+1} \rangle - \langle \lambda^r, x^{r+1} - y^{r+1} \rangle = \rho^{-1} \| \lambda^{r+1} - \lambda^r \|^2.
\]

Using our definitions, we have

\[
(A) = \rho^{-1} \| \lambda^{r+1} - \lambda^r \|^2
\]

\[
= \rho^{-1} \| \nabla f(x^{r+1}) - \nabla f(x^r) + e^r - e^{r+1} \|^2
\]

\[
\leq \frac{2}{\rho} \left( \| \nabla f(x^{r+1}) - \nabla f(x^r) \|^2 + \| e^{r+1} - e^r \|^2 \right)
\]

\[
\leq \frac{2}{\rho} \left( L_f^2 \| x^{r+1} - x^r \|^2 + 2 \| e^r \|^2 + 2 \| e^{r+1} \|^2 \right)
\]

\[
= \frac{2}{\rho} \left( L_f^2 \| x^{r+1} - x^r \|^2 + 2(\rho + L_f)^2 \gamma^2 \left( \| x^{r+1} - x^r \|^2 + \| x^r - x^{r-1} \|^2 \right) \right),
\]

where the last inequality is due to Lemma C.1 and the way \( x^r \) is chosen in Algorithm (2).

On the other hand:

\[
(B) = \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^r, \lambda^r)
\]

\[
= \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^{r+1}, \lambda^r) + \mathcal{L}(x^r, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^r, \lambda^r)
\]

\[
\leq 0 \quad \text{(due to update of } y) \quad \text{(33)}
\]

\[
= \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^{r+1}, \lambda^r) + \mathcal{L}(x^r, y^{r+1}, \lambda^r) - \mathcal{L}(x^r, y^{r+1}, \lambda^r)
\]

\[
\leq L_f \| x^{r+1} - x^r \|^2 - \frac{\sigma(\rho)}{2} \| x^{r+1} - x^r \|^2.
\]

Now note that \( \| x^r - x^{r+1} \|^2 \geq (1 - \gamma) \| x^{r+1} - x^r \|^2 \) and \( \| x^{r+1} - x^r \|^2 \leq \gamma \| x^{r+1} - x^r \|^2 \) because of the update rules of Algorithm (2). Plugging in these, we get

\[
(B) \leq \left( \gamma^2 (\rho + L_f) - \frac{(1 - \gamma)^2 \sigma(\rho)}{2} \right) \| x^{r+1} - x^r \|^2
\]
Now combining the inequalities for (A) and (B), we have
\[
\mathcal{L}(x^{r+1}, y^{r+1}, \lambda^{r+1}) - \mathcal{L}(x^r, y^r, \lambda^r) 
\leq \left( \frac{2L_f^2}{\rho} + \frac{4(\rho + L_f)^2\gamma^2}{\rho} + \frac{\gamma^2(\rho + L_f)}{2} - \frac{(1 - \gamma)^2\sigma(\rho)}{2} \right) \|x^{r+1} - x^r\|^2 + \frac{4(\rho + L_f)^2\gamma^2}{\beta} \|x^r - x^{r-1}\|^2
\]
(34)

Now for any $T$:
\[
f_{\min} - \gamma(\rho + L_f)\bar{D}^2 \leq \mathcal{L}(x^{T+1}, y^{T+1}, \lambda^{T+1})
\]
\[
= \mathcal{L}(x^0, y^0, \lambda^0) + \sum_{r=0}^{T} \mathcal{L}(x^{r+1}, y^{r+1}, \lambda^{r+1}) - \mathcal{L}(x^r, y^r, \lambda^r)
\]
\[
\leq (\alpha + \beta) \sum_{r=0}^{T-1} \|x^{r+1} - x^r\|^2 + \alpha\|x^{T+1} - x^T\|^2 + \mathcal{L}(x^0, y^0, \lambda^0)
\]
\[
\leq (\alpha + \beta) \sum_{r=0}^{T} \|x^{r+1} - x^r\|^2 + \mathcal{L}(x^0, y^0, \lambda^0),
\]
(35)

where the last inequality is due to the fact the $\beta \geq 0$. Now if the parameters are chosen appropriately such that $\alpha + \beta < 0$, then the right hand side of the above inequality is decreasing as $T$ increases, while the left hand side is constant. Therefore, we have $\lim_{T \to \infty} \sum_{r=0}^{T} \|x^{r+1} - x^r\|^2 < \infty$. Thus, $\lim_{r \to \infty} \|x^{r+1} - x^r\| = 0$. \hfill \Box

**Theorem C.4.** Assume that all the assumptions of Lemma C.3 is satisfied. Then, For any limit point $(\bar{x}, \bar{y}, \bar{\lambda})$ of the Algorithm 2, $\bar{x}$ is a stationary solution of the problem.

**Proof.** Consider a sub-sequence $(x^{r_t}, y^{r_t}, \lambda^{r_t})$, for $t = 0, \cdots$ which converges to $(\bar{x}, \bar{y}, \bar{\lambda})$. First of all due to Lemma C.3, we know that $\lim_{t \to \infty} \|x^{r_t+1} - x^{r_t}\| = 0$ and $\lim_{t \to \infty} \|x^{r_t-1} - x^{r_t}\| = 0$. Thus,
\[
\lim_{t \to \infty} x^{r_t+1} = \bar{x} \quad \& \quad \lim_{t \to \infty} x^{r_t-1} = \bar{x}
\]
(40)

Moreover, due to the updates of the algorithm
\[
\lim_{t \to \infty} \|x^{r_t+1} - x^{r_t}\| \leq \lim_{t \to \infty} \gamma \|x^{r_t+1} - x^{r_t}\| = 0 \quad \& \quad \lim_{t \to \infty} \|x^{r_t-1} - x^{r_t}\| \leq \lim_{t \to \infty} \gamma \|x^{r_t} - x^{r_t-1}\| = 0
\]
(41)

Thus, $\lim_{t \to \infty} e^{r_t} = \lim_{t \to \infty} e^{r_t+1} = 0$, which means
\[
\bar{\lambda} = \lim_{t \to \infty} \lambda^{r_t} = - \lim_{t \to \infty} (\nabla f(x^{r_t}) - e^{r_t}) = -\nabla f(\bar{x})
\]
(42)

Also, as $\mathcal{A}$ is finite, there exists a large enough $T$, such that $y^{r_T} = \hat{y}$ for $t \geq T$. Again due to the fact that $\mathcal{A}$ is finite, we can re-fine the sub-sequence such that $y^{r_{1+1}} = \hat{y}$. Thus, without loss of generality assume that these two conditions hold, i.e. $y^{r_T} = \hat{y}$ and $y^{r_{1+1}} = \hat{y}$ for all $t$ for an appropriately refined sub-sequence. This means that
\[
\hat{y} \in \arg \min_a \|a - (x^{r_t} + \rho^{-1}\lambda^{r_t})\|
\]
(44)

Moreover, $\lambda^{r_{1+1}} = \lambda^{r_T} + \rho(x^{r_{1+1}} - \hat{y})$. Taking the limit from both sides, we get
\[
\hat{y} = \bar{x}.
\]
(45)
Combining the above with \[ (44) \] we can easily see that
\[
\| \tilde{x} - (x^r + \rho^{-1}\tilde{\lambda}^r) \| \leq \| a_i - (x^r + \rho^{-1}\tilde{\lambda}^r) \|, \quad i = 0, \cdots, N
\] (46)

Taking the limits \( \lim_{t \to \infty} \) from both hand sides of the inequality for all the points \( a_i \), we have
\[
\| \tilde{x} - (\tilde{x} + \rho^{-1}\tilde{\lambda}) \| \leq \| a_i - (\tilde{x} + \rho^{-1}\tilde{\lambda}) \|, \quad i = 0, \cdots, N.
\] (47)

Thus,
\[
\tilde{x} \in \arg \min_{a \in \mathcal{A}} \| a - (\tilde{x} + \rho^{-1}\nabla f(\tilde{x})) \|
\] (48)
where we used the fact that \( \tilde{\lambda} = -\nabla f(\tilde{x}) \).

\[ \Box \]

D On the update rules of \( ADMM-S \) and its behavior

The update rules of \( x \) and \( \lambda \) variables are similar to the \( ADMM-Q \) algorithm. Here we only present the \( y \) update rule. Let us define \( \beta' = \beta \rho^{-1} \), \( z^r+1 = x^r + \rho^{-1}\lambda^r \) and \( \tilde{z}^r+1 = \mathcal{P}_\Lambda(z^r+1) \). Following the steps of regular ADMM, the update rule of \( y \) can be written as
\[
y^{r+1} = \arg \min_y \mathcal{L}(x^r, y, \lambda^r)
\]
\[
= \arg \min_y f(x^r) + \langle \lambda^r, x^r - y \rangle + \frac{\rho}{2} \| x^r - y \|^2_2 + \beta \mathcal{S}_\Lambda(y)
\]
\[
= \arg \min_y \frac{1}{2} \| y - x^r - \rho^{-1}\lambda^r \|^2 + \beta \rho^{-1} \mathcal{S}_\Lambda(y)
\]
\[
= \arg \min_y \frac{1}{2} \| y - z^{r+1} \|^2 + \beta \rho^{-1} \| y - \mathcal{P}_\Lambda(z^{r+1}) \|^2_2
\]
\[
= \arg \min_y \frac{1}{2} \| y - z^{r+1} \|^2 + \beta \| y - \tilde{z}^{r+1} \|^2_2
\]
(49)

If \( y^{r+1} \neq \tilde{z}^{r+1} \), then we can take the derivative of the above function and set it to 0 to get the update rule of \( y \):
\[
(y^{r+1} - z^{r+1}) + \beta' \frac{y^{r+1} - \tilde{z}^{r+1}}{\| y^{r+1} - \tilde{z}^{r+1} \|^2_2} = 0
\]
\[
\Rightarrow y^{r+1} = z^{r+1} + \beta' \frac{z^{r+1} - \tilde{z}^{r+1}}{\| z^{r+1} - \tilde{z}^{r+1} \|^2_2}
\] (50)

Now we need to find out when the solution is \( y^{r+1} = \tilde{z}^{r+1} \) and when it is given by equation (50).

Using the sub-gradient of the function \( \| y - z^{r+1} \|^2 \) at the point \( y = \tilde{z}^{r+1} \), we obtain that
\[
y^{r+1} = \tilde{z}^{r+1} \quad \text{if} \quad \| z^{r+1} - \tilde{z}^{r+1} \|^2 \leq \beta'
\]

Combining this equation with (50), we obtain the following update rule for \( y \):
\[
y^{r+1} = \begin{cases} 
  z^{r+1} + \frac{\beta'(z^{r+1} - \tilde{z}^{r+1})}{\| z^{r+1} - \tilde{z}^{r+1} \|^2_2}, & \beta' \leq \| z^{r+1} - \tilde{z}^{r+1} \|^2 \\
  \tilde{z}^{r+1}, & \beta' > \| z^{r+1} - \tilde{z}^{r+1} \|^2_2
\end{cases}
\]

Notice that this update rule would keep \( y^{r+1} \) very close to the set \( \mathcal{A} \), especially when \( \beta \) is large. In fact in the extreme case where \( \beta \) is large enough, i.e. when \( \beta' = \frac{\beta}{\rho} \geq \sup_z \| z - \mathcal{P}_\Lambda(z) \| \), the update rule of \( y \) in \( ADMM-S \) coincide with the update rule of \( y \) in \( ADMM-Q \) algorithm. Obviously due to the fact that \( y^{r} \) is not in \( \mathcal{A} \), we cannot expect the \( ADMM-S \) to converge to a stationary solution defined in
Definition 3.6. But in what follows we show that under assumptions similar to what we used for ADMM-Q, we can actually show that the Lagrangian function converges in ADMM-S.

Most of the proofs follow the same steps as in the convergence analysis of ADMM-Q. Thus, they are mostly omitted and we only focus on the overall steps and the results here. First of all it is easy to verify that the result of Lemma B.1 is also true for ADMM-S, i.e. \( \lambda^r = -\nabla_x f(x^r) \). Moreover, Let us assume that the \( y^r \) iterates stay bounded, i.e. \( y^r \in A' \), where \( A' \) is a compact set. Note that this is a reasonable assumption due to the proximity of \( y^r \) to the bounded set \( A \). As \( f \) is continuous, we can assume there exists a \( f_{\min} \) such that \( f(y) \geq f_{\min} \) for all \( y \in A' \). Under these assumptions we have the following lemma, which states that the Lagrangian function is lower bounded.

**Lemma D.1.** If \( \rho \geq L f \), we have \( L(x^r, y^r, \lambda^r) \geq f(y^r) \geq f_{\min}, \forall r \geq 1 \).

The proof is similar to the proof of Lemma 3.4 and is omitted. Moreover, we have the following result which is similar to Lemma 3.5 for ADMM-Q.

**Lemma D.2.** Define \( \sigma(\rho) \triangleq \rho - \mu \). We have

\[
L(x^{r+1}, y^{r+1}, \lambda^{r+1}) - L(x^r, y^r, \lambda^r) \leq (\rho^{-1}L_f^2 - \frac{\sigma(\rho)}{2}) \| x^{r+1} - x^r \|^2.
\] (51)

The proof of this lemma also follows the same arguments provided in the proof of Lemma 3.5.

Based on these two lemmas, we have that augmented Lagrangian function is decreasing and lower bounded when \( \rho \) is chosen appropriately. Thus, it has to converge:

**Proposition D.3.** If \( \rho \) is chosen such that \( \rho^{-1}L_f^2 - \frac{\sigma(\rho)}{2} < 0 \), then \( L(x^r, y^r, \lambda^r) \) is decreasing and lower bounded. Thus, it converges.

### E Simulations on Convex Quadratic Case

Recall in section 7, we solve the following problem:

\[
\min_x \frac{1}{2} x^\top Q x + b^\top x \quad \text{s.t.} \quad x \in A \triangleq v\mathbb{Z}^d,
\] (52)

for some given \( Q \in \mathbb{R}^{d \times d} \), \( b \in \mathbb{R}^d \), and \( v \in \mathbb{Z}^+ \). We generate matrix \( Q \) via the rule \( Q = \hat{Q}^\top \hat{Q} + \hat{q}q^\top \), where \( \hat{Q}_{ij} \sim N(0,1) \), \( \hat{q}_i \sim N(0,\sigma_q^2) \), \( 1 \leq i,j \leq d \). We follow the same procedure as discussed in section 7; see Table 2 for the hyper-parameters used in ADMM-Q, ADMM-S and ADMM-R. We report the results for the following combinations of \( v \), \( d \) and \( \sigma_q^2 \), as seen in Table 1.

**Results.** Most of the observations in section 7 carry over here regardless of the values of \( d \) and \( \sigma_q^2 \). More precisely, ADMM-Q outperforms PGD and GD+Proj with large margins. Both ADMM-S and ADMM-R not only have better median final objective values, but also smaller variance as compared with ADMM-Q. More importantly, the median tends to overlap with the 25\% quantile, see Figure 7. It means the objective of at least 25 runs are exactly the same as the minimal objective over 50 runs. We also observe that ADMM-S or ADMM-R is not always better than ADMM-Q. As we conduct more experiments, we observed cases that ADMM-S yields large objective value; see, e.g., instance 3 in Figure 1 and compare with Figure 7. Having said that, we observe that ADMM-S and ADMM-R outperform ADMM-Q in most instances.
| Parameter Pairs |
|-----------------|
| $v$ | $d$ | $\sigma_q^2$ |
| 8   | 8   | 30         |
| 8   | 16  | 30         |
| 8   | 32  | 30         |
| 8   | 64  | 30         |
| 8   | 16  | 10         |
| 8   | 16  | 50         |
| 8   | 16  | 70         |

Table 1: Parameter pairs used in the experiment

Figure 3: Performance of ADMM-Q, ADMM-S, ADMM-R and PGD on different problem instances with $d = 8$, $\sigma_q^2 = 30$
Figure 4: Performance of ADMM-Q, ADMM-S, ADMM-R and PGD on different problem instances with $d = 16$, $\sigma^2_\theta = 30$, note the difference compared with Figure 1.
Figure 5: Performance of ADMM-Q, ADMM-S, ADMM-R and PGD on different problem instances with $d = 32, \sigma_q^2 = 30$
Figure 6: Performance of ADMM-Q, ADMM-S, ADMM-R and PGD on different problem instances with $d = 64, \sigma^2 = 30$
Figure 7: Performance of ADMM-Q, ADMM-S, ADMM-R and PGD on different problem instances with $d = 16$, $\sigma^2_q = 10$
Figure 8: Performance of ADMM-Q, ADMM-S, ADMM-R and PGD on different problem instances with $d = 16$, $\sigma_q^2 = 50$
Table 2: Hyper-parameters used for ADMM-Q, ADMM-S and ADMM-R

| Algorithm | Hyper-parameters |
|-----------|------------------|
| ADMM-Q   | None             |
| ADMM-S   | $\beta = 10^{-5}, 10^{-4.5}, 10^{-4}, \ldots, 10^{-4.5}, 10^{-5}$ $\rho = 10^{-k}, k \in \mathbb{Z}, -6 \leq k \leq 2$ |
| ADMM-R   | $p_r^i = 0.01, 0.1, 0.3, 0.5, 0.7, 0.9, 0.99$ $\rho = 10^{-k}, k \in \mathbb{Z}, -6 \leq k \leq 2$ |

Figure 9: Performance of ADMM-Q, ADMM-S, ADMM-R and PGD on different problem instances with $d = 16$, $\sigma^2_q = 70$

F Simulations on Neural Networks

Binarization of neural networks has been tried in various applications before, especially in the field of image recognition. Here we present training binarized neural networks on training over MNIST dataset. The MNIST dataset consists of $28 \times 28$ arrays of grayscale pixel images classified into 10 handwritten digits. It includes 60,000 training images and 10,000 testing images. The task here is build an binary-weighted classifier to recognized digits, which can be formulated as

$$\min_W \frac{1}{N} \sum_{i=1}^{N} f(f(W; x_i), y_i) \quad \text{s.t.} \quad W \in \{-1, +1\}^d$$

(53)

where $(x_i, y_i)$ is the $i$-th training sample; $x_i$ is the input image; $y_i$ is the label; $W$ represents the weights of the network. The work in [1] used “Straight Through Estimator” to binarize the network and reached the accuracy level of the full-precision network. We repeat the experiment
with the same network as [1] and apply ADMM-Q and its variants. Similar to the quadratic case, we also compare the performance with PGD and GD+Proj. We conduct two sets of experiments: with pretraining and without pretraining. To the best of our knowledge, all the ADMM-based approaches [5, 6, 7, 8, 9, 10, 32, 33, 34] start from a pre-trained full-precision network. We also observed in our simulations that pre-training (with non-binarized weights) in fact improves the performance of ADMM-based methods. However, in order to solely study the performance of ADMM-based methods (and not additional modules around it), we avoid using pre-training in some of our experiments. We also did not use any popular heuristics and we relies on implementing our plain ADMM-based algorithms. We run our algorithms with and without pre-training. Notice that this optimization problem is not convex. To remove the effect of random initialization, we run each algorithm for 5 times and record the mean and standard deviation of the testing accuracy. For algorithms with pre-training, we pre-train the model and then apply the algorithm and repeat this entire procedure for 5 times. Training parameters and network structures be found in Table 4 and Table 5. Adam optimizer is used for all algorithms. Note that in step 3 of algorithm 1, it is required to solve a minimization problem, which is not always tractable in practice. Thus, here we apply 5 epochs of Adam update on $W$.

**Results.** Table 3 shows that ADMM-Q and its variants have comparable results with BinaryConnect [1]. One substantial difference between [1] and the proposed work is that we do not use any heuristics and the proposed algorithm enjoys theoretical guarantees. Note that for ADMM-Q without pre-training, we pick a small $\rho$ and keep it until the end of the training process. We observed that one can indeed use “scheduling” for parameter $\rho$, i.e., increasing it gradually, to shorten the training time. It is worth mentioning that, with pre-training, ADMM-Q and its variants and even PGD algorithm are converging extremely fast, sometimes within as few as 3 epochs. While in the presence of pre-training, and PGD and ADMM-based algorithms all work reasonably well, PGD is much more sensitive to initialization. In particular, omitting the pre-training phase drops the performance of PGD much more than the performance of ADMM-based methods.

| Algorithm               | Accuracy       |
|-------------------------|----------------|
| BinaryConnect [1]       | 98.71%         |
| GD+Proj                 | 74.92 ± 4.83%  |
| PGD                     | 92.73 ± 0.23%  |
| ADMM-Q                  | 98.21 ± 0.16%  |
| PGD with pre-training   | 98.55 ± 0.05%  |
| ADMM-Q with pre-training| 98.55 ± 0.04%  |
| ADMM-R with pre-training| 98.61 ± 0.06%  |

Table 3: Testing accuracies for MNIST dataset
| Algorithm            | Parameter          |
|----------------------|--------------------|
| GD (+ Proj)          | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 80 40  
|                      | Batch-size 512 512  |
| PGD                  | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 80 40  
|                      | Batch-size 512 512  |
| ADMM-Q               | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 80 40  
|                      | Batch-size 512 512  |
|                      | $\rho$ $10^{-5}$  |
| ADMM-Q with pre-training | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 20 20  
|                      | Batch-size 512 512  |
|                      | $\rho$ $10^{-3}$
| PGD with pre-training | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 20 20  
|                      | Batch-size 512 512  |
|                      | $\rho$ $10^{-3}$
| Binarization         | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 20 20  
|                      | Batch-size 512 512  |
| ADMM-Q with pre-training | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 20 20  
|                      | Batch-size 512 512  |
|                      | $\rho$ $10^{-3}$
| ADMM-R with pre-training | Learning rate $10^{-2}$ $10^{-3}$  
|                      | Epoch 20 20  
|                      | Batch-size 512 512  |
|                      | $\rho$ $10^{-3}$
|                      | $p_{i}$ 0.3

Table 4: Training parameters for MNIST dataset.

| Layer Type            | Shape   |
|-----------------------|---------|
| Dropout               | 0.2     |
| Fully Connected + BatchNorm + ReLU | 4096   |
| Dropout               | 0.5     |
| Fully Connected + BatchNorm + ReLU | 4096   |
| Dropout               | 0.5     |
| Fully Connected + BatchNorm + ReLU | 4096   |
| Dropout               | 0.5     |
| Fully Connected + BatchNorm | 10      |

Table 5: Model architecture for MNIST dataset.