Robust entanglement of an asymmetric quantum dot molecular system in a Josephson junction

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We demonstrate how robust entanglement of quantum dot molecular system in a voltage-controlled junction can be generated. To improve the quantum information characteristics of system, we propose an applicable protocol which contains the implementation of asymmetric quantum dots as well as engineering reservoirs. Quantum dots with tunable energy barriers can provide asymmetric coupling coefficients which can be tuned by gap voltages. Also by engineering reservoirs, superconductors can be used as leads in a biased-voltage junction. The high-controllability characteristics of system supplies the arbitrary entanglement by tuning the controlling parameters. Significantly in concurrence-voltage characteristics, perfect entanglement can be achieved in an asymmetric structure and it can be kept with near-unit magnitude in response to bias voltage increasing.

I. INTRODUCTION

Recent advancements in condensed matter physics and nanotechnology open new possibilities for the implementation of nanodevices in quantum information studies. Two-state qubit systems can be taken into account as promising candidates in these studies. Qubits would be realized with photons [1, 2], trapped ions and atoms [3–5], spins [6, 7], nitrogen vacancy centers in diamond [8, 9], superconducting qubits [10, 11] and quantum dots (QDs) [12, 13]. Particularly, QDs as artificial atoms play prominent roles in nanostructures for their tunable discrete energy levels and also for their easy controllability of barriers by gate voltages.

Also, quantum dot molecules (QDMs) consist of quantum dots which are coupled by tunneling and separated by barriers have received great attention theoretically [14] and experimentally [15, 16]. These quantum structures have been selected as the ideal candidates for studying the quantum information processing. Quantum information properties of a double quantum dot (DQD) system connected to the resonators was studied under the applied microwave-driven field [17, 18]. The analysis of entanglement dynamics between two electrons inside coupled quantum molecules showed that the role of Bell states were crucial in entanglement characteristics [14].

Theoretical [20] and experimental [21] studies showed that asymmetric structure of quantum molecules has enhanced the control of tunneling features. It was shown that in an asymmetric quantum dot molecular system, the fidelity of entangled photon pairs which are depending on temperature, electric field and tunnel coupling can be achieved near-unit theoretically [22]. In addition, the asymmetric quantum dot-lead couplings have been extensively implemented in electrical [23] and thermal [24] rectification devices to improve the electric and heat transport technologies.

Moreover, superconducting devices have found impressive interest in quantum information setups [25–27] because of their long intrinsic coherency with no dissipation characteristics. Recent years, employing the superconducting qubits and superconducting resonators have improved the exploring of entanglement [28], teleportation [29–31] and quantum computing [32–34] studies. Superconducting qubits namely phase [35], flux [36, 37] and charge [38, 39] qubits can be connected with microwave [40], electrical [41, 42], mechanical [43] and superconducting [14] resonators. According to the frequency range of superconducting devices, these nanostructures would be driven by microwave [17, 18] or optical [44, 45] fields. Also, QDs in normal biased-voltage junctions have extensively been used experimentally [47–49] and theoretically [50, 51]. Recently, the presence of QDs in the Josephson junctions (JJs) which act as the single transistors to filter the transfer of electrons have attracted a great deal of attention [52–54]. Particularly, quantum transport through the QDs system formed by a single QD [55, 56] and a DQD [57, 58] in interaction with superconducting leads have been studied.

In this study, we propose a QDM system in a conventional JJ with asymmetric tunneling coefficients to achieve the robust entanglement and also to keep its magnitude near-unit under the bias voltage control. To explore the quantum information processing of QDM system in a biased-voltage junction, we perform our analysis in Markovian regime. First, we obtain the quantum transport of molecular system to show the current-voltage characteristics (I-V) as one of the important properties of biased-voltage circuits. Then, we investigate the control of the entanglement with respect to bias voltage. We find that with only bias voltage control, the complete controllability to yield perfect entanglement is
not possible. Therefore, we apply the strategy of left-right asymmetric coupling strength to achieve robust entanglement. The dynamics of entanglement and its response to bias voltage in different situations of symmetric and asymmetric couplings demonstrate a wide flexibility of the proposed setup to provide a desired high entanglement. The main advantage of this molecular system includes the feasible controlling elements of easy-tunable bias voltage driving field and the manipulation of quantum dot couplings. Indeed by engineering reservoirs and the presence of superconducting leads, the performance of system is extensively influenced to provide robustly entangled states.

This paper is organized as follows: In Sec. II, we introduce the proposed model composed of a quantum dot molecular system in a JJ by describing the whole Hamiltonian. We compute the quantum transport of our molecular system in Sec. III. In Sec. IV, by introducing symmetric and asymmetric structures, we obtain the entanglement of QDM system under the bias voltage control. In Sec. V, we present the results of the entanglement behavior in bias voltage changes and its time evolution in constant bias voltages and also for specific order parameters. Finally, we conclude the results in Sec. VI. In Appendix A, we describe how to diagonalize the Hamiltonian of superconducting leads by applying Bogoliubov transformation. In Appendix B, we calculate the quantum master equation to study the dynamics of system.

II. MODEL

The proposed open quantum system consists of a QDM weakly coupled to the superconducting leads which is demonstrated in Fig. 1, schematically. Applying an external bias voltage between the leads L and R induces the electron transport from the left to the right. The Hamiltonian of the whole system can be written as:

\[ \hat{H} = \hat{H}_{QDM} + \hat{H}_{Leads} + \hat{H}_{int}, \]  

where \( \hat{H}_{QDM} \) describes the Hamiltonian of quantum dot molecule:

\[ \hat{H}_{QDM} = \sum_{\alpha,s} \frac{\varepsilon_\alpha}{2} \hat{\sigma}_\alpha^+ \hat{\sigma}_\alpha^s + \sum_{\alpha \neq s} t_{\alpha \alpha',ss'} \hat{\sigma}_\alpha^+ \hat{\sigma}_{\alpha'}^- \hat{\sigma}_{\alpha'}^s \hat{\sigma}_\alpha^s. \]

In this equation for \( \alpha = A, B \) and \( s = \uparrow, \downarrow \), \( \hat{H}_{QDM} \) introduces the Hamiltonian of the molecular system which consists of a double quantum dot with tunable energy levels of \( \varepsilon_A \) and \( \varepsilon_B \). Here under the Coulomb blockade regime, we assume the QDs in the single electron regime \([60, 61]\). Also, Pauli matrix operator in \( z \) direction is defined as \( \hat{\sigma}_z^\alpha = \frac{1}{2}(\ket{\alpha}\bra{\alpha} - \ket{\alpha}\bra{\alpha}) \). In the second term, \( t_{\alpha \alpha',ss'} \) describes the hopping strength between the quantum dot sites. In Eq. (1), \( \hat{H}_{Leads} \) corresponds to the left and right superconducting leads which are described by the mean-field Hamiltonian as \([62, 63]\):

\[ \hat{H}^{MF}_{Leads} = \sum_{k\nu\sigma} \xi_{k\nu}\hat{c}_{k\nu\sigma}^\dagger \hat{c}_{k\nu\sigma} + \sum_{k\nu} (\Delta_{\nu\uparrow\downarrow} \hat{c}_{k\nu\uparrow}^\dagger \hat{c}_{k\nu\downarrow} + \Delta_{\nu\downarrow\uparrow} \hat{c}_{k\nu\downarrow}^\dagger \hat{c}_{k\nu\uparrow}). \]

Here, \( \hat{c}_{k\nu\sigma}^\dagger (\hat{c}_{k\nu\sigma}) \) is the creation (annihilation) operator of an electron with momentum \( k \) and spin \( \sigma = \uparrow, \downarrow \) in lead \( \nu = L, R \). In this relation, \( \xi_{k\nu} = \varepsilon_k - \mu_{\nu} \) is the particle energy in which \( \xi_k \) denotes the single-particle energy with the electrochemical potential \( \mu_{\nu} \). Moreover, \( \Delta_{\nu\sigma} = |\Delta_{\nu}| e^{i\phi_{\nu}} \) remarks the superconducting energy gap of lead \( \nu \) with the superconducting phase, \( \phi_{\nu} \). The mean field Hamiltonian could be diagonalized by applying Bogoliubov transformation to obtain (Appendix A):

\[ \hat{H}_{Leads} = \hat{H}^{MF}_{Leads} + \sum_{k\nu\sigma} E_{k\nu\sigma} \hat{c}_{k\nu\sigma}^\dagger \hat{c}_{k\nu\sigma}, \]

where \( E_G \), the ground state energy, represents the Cooper pair condensate energy. The interaction Hamiltonian, \( \hat{H}_{int} \) in Eq. (1), corresponds to the tunneling between the QDs and electrodes which can be written as:

\[ \hat{H}_{int} = \sum_{k\nu\sigma\alpha\sigma'} \left(T_{k\nu\sigma\alpha\sigma'} \hat{c}_{k\nu\sigma} \hat{\sigma}_{\alpha\sigma'}^+ + T_{k\nu\sigma\alpha\sigma'}^* \hat{c}_{k\nu\sigma}^\dagger \hat{\sigma}_{\alpha\sigma'}^- \right). \]

The tunneling coefficient, \( T_{k\nu\sigma\alpha\sigma'} \), describes the coupling strength depending on \( k \), the momentum of an electron in lead \( \nu \), the site of quantum dot \( \alpha \) and spin \( \sigma \). Here, \( \hat{\sigma}_{\alpha\sigma} = \ket{\alpha\sigma}\bra{\alpha\sigma} \) and \( \hat{\sigma}_{\alpha\sigma}^\dagger = \ket{\alpha\sigma}\bra{\alpha\sigma}^\dagger \) define the lowering and rising operators of QDM respectively.

In this study, the temperature of system is assumed in the order of \( mK \) and all energy scales are in \( meV \) units. To investigate the time evolution of system, firstly the quantum master equation (QME) and the density matrix are obtained (Appendix B) and then we calculate the current and entanglement in the following sections.
In steady state of the system, by taking so long time it reaches the platform for the large enough level of system is increased by rising the magnitude of energy gaps, it illustrates the current formula, Eq. (6), as [59, 65]:

\[ I = \frac{e}{\hbar} \sum_{\sigma} \left( T_{k\sigma\sigma} \tilde{c}_{k\sigma}^\dagger \tilde{c}_{k\sigma} - T_{k\sigma\sigma}^* \tilde{c}_{k\sigma} \tilde{c}_{k\sigma}^\dagger \right), \tag{6} \]

where \( \tilde{c}_c \) shows the contribution of lead \( \nu \) in matrix \( \tilde{M} \). In steady state of the system, by taking so long time \( t \to \infty \), the stationary transport is shown in Fig. 2 containing the plots of normal junction \( (\Delta = 0) \) and JJ with different energy gaps.

According to I-V characteristics which is shown in Fig. 2, the magnitude of current is growing by the increase of bias voltage. Only in energies equal to the quantum dots’ energies, the current hits the peaks in delta type for the superconducting leads while it illustrates the smooth steps for the normal leads. Although, the current level of system is increased by rising the magnitude of energy gaps, it reaches the platform for the large enough bias voltage.

### IV. CONCURRENCE

It is convenient to apply the concurrence as a measure of entanglement for systems consist of two qubits in both pure and mixed states [66, 67]. This measure of entanglement is defined as:

\[ C(\rho) = \text{Max}[0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4], \tag{8} \]

in which \( \lambda_i, (i = 1, 2, 3, 4) \), represents the non-negative eigenvalues of a matrix \( \hat{R} \) in decreasing order \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \). The matrix \( \hat{R} \) is defined as:

\[ \hat{R} = \sqrt{\hat{\rho} \hat{\rho}^*}, \tag{9} \]

where \( \hat{\rho} \) denotes the density matrix of system and also \( \hat{\rho} = (\hat{\sigma}_y \otimes \hat{\sigma}_y) \hat{\rho}^*(\hat{\sigma}_y \otimes \hat{\sigma}_y) \). In this relation, \( \hat{\sigma}_y \) describes the \( y \) element of Pauli matrices and \( \hat{\rho}^* \) represents the complex conjugate of the density matrix.

Here, we discuss about the influence of energy-dependent coefficients on the entanglement of quantum dot molecular system. The energy contributions which can be taken into account asymmetrically originated from the QD-reservoir couplings. Indeed, the unequal left and right superconducting energy gaps of reservoirs \( (\Delta_L \neq \Delta_R) \) can intensively influence the behavior of entanglement. According to Eq. (17) and Eq. (18) the energy-dependent of molecular system is affected by a set of elements: distribution function, density of states and coupling coefficients. To observe the prominent role of asymmetry on the entanglement of QDM system, we consider the coupling coefficients and superconducting energy gaps in right-left asymmetric configuration. For this purpose, the strength of coupling coefficients which strongly depends on the properties of QDs can easily be tuned left-right asymmetrically by mean of the relevant gap voltages. Also, the superconducting energy gaps of left and right reservoirs can be simply chosen unequally in the arrangement of setup. To present the effect of asymmetric coupling coefficients, we define the asymmetric factor as a function of coupling contributions:

\[ \kappa = \frac{T_{AL} - T_{AR}}{T_{AL} + T_{AR}} = \frac{T_{BR} - T_{BL}}{T_{BR} + T_{BL}}, \tag{10} \]

in which \( T_{AL} \) denotes the coupling of \( QDA \) to the near-lead(Left Lead) and \( T_{AR} \) shows the coupling of this \( QD \) to the far-lead(Right Lead) illustrated in Fig. 1. Similarly, the coupling of \( QDB \) with the far-lead(Left Lead) is shown by \( T_{BL} \) and the coupling coefficient with the near-lead(Right Lead) is indicated by \( T_{BR} \).

Mostly, in the study of QDs system for simplification, the coupling of QD with the far-lead is ignored [68]. However, we assume the both coupling of each QD to the near-lead and far-lead non-zero with only different strengths which are involved in the asymmetric factor definition, (Eq. (10)).

According to the definition of asymmetric factor, (Eq. (10)), we investigate the entanglement of quantum dot molecular system in two configurations, namely symmetric and asymmetric structures as follows.
A. Symmetric Structure

In symmetric structure, the left coupling coefficient of each QD is similar to the right one ($T_{AL} = T_{AR}$ and $T_{BL} = T_{BR}$) which means the left-right symmetric coupling coefficients. This situation supplies the minimum magnitude of the asymmetric factor, $\kappa = 0$. Also, this situation corresponds to equal superconducting energy gaps of the left and right reservoirs ($\Delta_L = \Delta_R$). In these conditions, the entanglement of QDM system is obtained only for the initial entangled states. For this purpose, we consider the following Bell state as an initial state with the highest degree of entanglement:

$$\rho(0) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0.5 & -0.5i & 0 \\
0 & 0.5i & 0.5 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}. \tag{11}$$

B. Asymmetric Structure

The asymmetric structure is defined for the left-right different coupling coefficients with $0 < \kappa \leq 1$ magnitude and the unequal order parameters of reservoirs, $\Delta_L \neq \Delta_R$. In this group, the ideal asymmetry element is achieved for the maximum amount of asymmetric factor $\kappa \simeq 1$. The situation of ideal asymmetry is available when one of the left or right coupling coefficient is much larger than the other one. To apply the ideal asymmetry properties in physically rational considerations, we assume that each QD is coupled to the near-lead with much larger strength than the far-lead. In other words, we consider $\Gamma_{AL} \gg \Gamma_{AR}$ and $\Gamma_{BR} \gg \Gamma_{BL}$ to realize the most magnitude of asymmetric factor. An interesting feature in the composed systems is the realization of entanglement from the initial unentangled states. This important point would be accomplished in the asymmetric structure. To investigate this significant situation in the present system, we assume an appropriate separated initial state as:

$$\rho(0) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}. \tag{12}$$

In the next section, we present the concurrence behavior of the present QDM system for both symmetric and asymmetric structures.

V. RESULTS

In this section, we investigate the concurrence behavior of molecular system firstly in response to bias voltage, secondly by the time evolution in constant voltage and finally through the dynamics for specific superconducting energy gaps.

![FIG. 3. The concurrence-voltage characteristics for the symmetric structure in panel (a) and for the asymmetric structure in panel (b). Normal leads: Solid line $\Delta = 0$, Superconducting leads: panel (a), Dashed line $\frac{\Delta}{\Gamma_0} = 2.5$, Dotted-dashed $\frac{\Delta}{\Gamma_0} = 3.5$ ($\Delta_L = \Delta_R = \Delta$) and panel (b), Dashed line $\frac{\Delta_L}{\Gamma_0} = 2.5$, $\frac{\Delta_R}{\Gamma_0} = 3.5$; Dotted-dashed $\frac{\Delta_L}{\Gamma_0} = 3$, $\frac{\Delta_R}{\Gamma_0} = 6$. $\Gamma_0 = \pi N_F |T|^2$.](image)

A. Concurrence-Voltage characteristics

The concurrence-voltage (C-V) characteristics demonstrates the response of concurrence to the bias voltage as an external easy-tunable driving field. Fig.3 shows C-V characteristics for normal reservoirs with $\Delta = 0$ and superconducting ones with specific order parameters in the conditions of symmetric structure (Panel(a)) and asymmetric one (Panel(b)). In panel (a) of Fig.3, the concurrence shows degradation for the symmetric structure while in panel (b) of this figure, the concurrence indicates rising for the asymmetric conditions by the increase of bias voltage. In both panels of Fig.3, the concurrence changes in the energy levels of QDs with the step shapes for the normal leads and with the delta peaks for the superconducting reservoirs. The presence of superconductors as reservoirs provides stronger response than the normal leads. This effect is obvious in Fig.3 when in panel (a) the concurrence is decreased with higher values of potential and in panel (b) concurrence shows increment in higher magnitude for the Josephson junction than the normal one. Also, the influence of superconducting reservoirs is displayed more clearly when by increasing the amount of superconducting energy gaps, the concurrence has lower magnitude for the symmetric structure (panel (a)) and inversely for the asymmetric group (panel (b)).
It is interesting that our proposed setup is able to support two different fundamental concepts of physics which are quantum transport and quantum entanglement simultaneously in response to bias voltage changes. There are two different responses to the increase of bias voltage in I-V characteristics (Fig. 2) and C-V one (Fig. 3)-(a)) which are increasing for the former and decreasing for the later in the symmetric structure. This various behavior can be interpreted as the response of electrons to the external bias voltage. As a consequence of the bias voltage increasing, electrons are accelerated which cause a rising current across the symmetric junction. This faster movement of carriers means that the electrons with the initial entangled states(Eq.(11)) in the symmetric coupling situation can be present in a shorter period of time which leads to have less time to be entangled. Therefore, the entanglement degradation in response to bias voltage rising does make sense for the left-right symmetric conditions.

The behavior of concurrence originates from the moving manner of electrons. So, it would be physically reasonable that carriers can move more quickly through the Josephson junction than the normal one which means the lower value of entanglement in JJ for the left-right symmetric situation. However, the electrons of asymmetric structure with the initial unentangled states(Eq.(12)) have different conditions. The asymmetric coupling coefficients provide a bounded-like situation for the unentangled electrons which give them an opportunity to be well entangled. It means that although electrons can move faster by the increase of bias voltage, the left-right asymmetric situation arranges the possibility of being robustly entangled for them. Therefore, it would be logical that C-V characteristics shows increasing in response to bias voltage rising for asymmetric group.

It is a crucial point that although the concurrence of asymmetric structure(panel (b) of Fig. 3) behaves differently from the symmetric one(panel (a) of Fig. 3), their origins are the same. The main reason for this variety behavior of concurrence refers to the powerful strength of our proposed setup in controlling the features to obtain the desired results. In addition, the quantum correlation between the localized sites (QDs), energies which are in resonant with the energy levels of QDs, \(eV = \varepsilon_i + \Delta \) \((i = A, B)\), in symmetric structure (panel (a)) and asymmetric one (panel (b)). These resonant points are illustrated as peaks with respect to bias voltage in Fig. 3. Due to the proximity effect of superconducting reservoirs, the concurrence behaves differently in two sides of each resonant points. For the left-right symmetric group, concurrence shows longer elapsed time for the left side of the resonant point \((eV = \varepsilon_i + \Delta - 0.01; i = A, B; \Delta_L = \Delta_R = \Delta)\) than the right side \((eV = \varepsilon_i + \Delta + 0.01)\) which is illustrated in panel (a) of Fig. 3. However for high bias voltage \(eV \gg 0\), the dynamics of concurrence decays in mod-

**B. Dynamics of concurrence in constant bias voltage**

The time evolution of concurrence for given values of superconducting energy gap in a constant bias voltage is demonstrated for symmetric and asymmetric structures in panels (a) and (b) of Fig. 4, respectively. According to Eq. (11) and Eq. (12) which express the initial states of symmetric and asymmetric group conditions, the concurrence of these structures are increased and decreased through the time, respectively. In Fig. 4, the concurrence of both left-right symmetric and asymmetric situations decay faster for JJs than the normal ones to receive the ultimate magnitudes only with opposite manner. Indeed, the decay rate of concurrence is speeded up by increasing the superconducting energy gap for them.

**C. Dynamics of concurrence for specific superconducting energy gaps**

Fig. (5) indicates the time evolution of entanglement for energies which are in resonant with the energy levels of QDs, \(eV = \varepsilon_i + \Delta \) \((i = A, B)\), in symmetric structure (panel (a)) and asymmetric one (panel (b)). These resonant points are illustrated as peaks with respect to bias voltage in Fig. 3. Due to the proximity effect of superconducting reservoirs, the concurrence behaves differently in two sides of each resonant points. For the left-right symmetric group, concurrence shows longer elapsed time for the left side of the resonant point \((eV = \varepsilon_i + \Delta - 0.01; i = A, B; \Delta_L = \Delta_R = \Delta)\) than the right side \((eV = \varepsilon_i + \Delta + 0.01)\) which is illustrated in panel (a) of Fig. 3. However for high bias voltage \(eV \gg 0\), the dynamics of concurrence decays in mod-

![Figure 4](image-url)

**FIG. 4.** The dynamics of concurrence for the symmetric structure in panel (a) and for the asymmetric structure in panel (b). Normal leads: Solid line \(\Delta = 0\), Superconducting leads: panel (a), Dashed line \(\Delta = 1.6\), Dotted-dashed \(\Delta = 3\) \((\Delta_L = \Delta_R = \Delta)\) and panel (b), Dashed line \(\Delta = 2.5\), \(\Delta = 3.5\); Dotted-dashed \(\Delta = 3\), \(\Delta = 6\). \(\Gamma_0 = \pi N_F|T|^2\).
FIG. 5. The dynamics of concurrence for bias voltages in resonant with QD’s energy levels, panel (a): for the symmetric structure and panel (b) for the asymmetric structure. Solid line: left side of the first resonant point, Dashed line: high bias. Γ of the second resonant point, Dotted line: right side of the first resonant point, Dot-dashed line: left side, Dashed line: right side of this point(\(\Gamma_0\)). In summery, we proposed a protocol to obtain perfect entanglement for two coupled QDs molecular structure in a voltage-controlled junction. In this strategy, we focused on the arrangement of different controlling elements to enhance the quantum information characteristics of system. First by engineering the reservoirs, we applied superconductors as leads using the significant properties of Josephson junction under the bias voltage control. Second, we utilize the energy couplings of QD-reservoirs asymmetrically. The main advantage of this hybrid quantum system refers to its wide strength of controllability due to the easy tuning driven bias voltage and also control of coupling coefficients by manipulating the quantum dot barriers with respect to the required results. In concurrence-voltage characteristics, applying the asymmetric coupling energy conditions can provide high degree of entanglement while for the symmetric situation, entanglement shows degradation.

APPENDIX A: DIAGONALIZING THE HAMILTONIAN OF SUPERCONDUCTING LEADS

It would be possible to diagonalize the superconducting Hamiltonian. The mean-field Hamiltonian of superconducting leads is mostly diagonalized by Bogoliubov transformation. To this end, we consider the following Bogoliubov transformation [69]:

\[
\hat{c}_{-\nu \sigma} = u_{\nu \sigma} \hat{\gamma}_{-\nu \sigma} + v_{\nu \sigma} \hat{\gamma}_{+\nu \sigma} - \text{v.c.}, \\
\hat{c}_{+\nu \sigma} = u_{\nu \sigma} \hat{\gamma}_{+\nu \sigma} + v_{\nu \sigma} \hat{\gamma}_{-\nu \sigma} - \text{v.c.},
\]

where \(\hat{\gamma}_{\nu \sigma} (\hat{\gamma}_{\nu \sigma}^\dagger)\) denotes the creation (annihilation) operator of Bogoliubov fermionic quasiparticle excitation. Bogoliubov quasiparticles follow the fermionic anticommutation relation \(\{\hat{\gamma}_{\nu \sigma}, \hat{\gamma}_{\nu' \sigma'}^\dagger\} = \delta_{\nu \nu'} \delta_{\sigma \sigma'}\). The complex number parameters \(u_{\nu \sigma}\) and \(v_{\nu \sigma}\) adopting the relation \(|u_{\nu \sigma}|^2 + |v_{\nu \sigma}|^2 = 1\) are defined as:

\[
u_{\nu \sigma} = e^{-i\Phi_{\nu}} \sqrt{\frac{1}{2} \left( 1 + \frac{\xi_{\nu \sigma}}{|E_{\nu \sigma}|} \right)},
\]

\[
u_{\nu \sigma} = \sqrt{\frac{1}{2} \left( 1 - \frac{\xi_{\nu \sigma}}{|E_{\nu \sigma}|} \right)}.
\]

Here, \(E_{\nu \sigma} = \sqrt{\xi_{\nu \sigma}^2 + |\Delta_e|^2}\) indicates the quasiparticle energy. Inserting the Bogoliubov transformation into the mean-field Hamiltonian, the diagonalized Hamiltonian is achieved:

\[
\hat{H}_{\text{leads}} = E_G + \sum_{\nu \sigma} E_{\nu \sigma} \hat{\gamma}_{\nu \sigma}^\dagger \hat{\gamma}_{\nu \sigma},
\]

in which the ground state energy \(E_G\) shows the Cooper pair condensate energy.

APPENDIX B: DYNAMICS OF SYSTEM

To study the dynamics of system, we start from the Liouville-von Neumann equation of the complete system.
According to the BCS theory, the superconducting density of states in the interaction picture \[70\]. A comparison between the characteristics time scales of the system, the QD relaxation time and the superconducting coherence time as the environment time scale, implies that the present system would be studied under the Markovian approximation \[58, 71, 72\]. After partial tracing out the lead degrees of freedom and applying the Born-Markov approximation, the quantum master equation (QME) for the reduced density matrix is obtained:

\[
\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_I, \hat{\rho}(t)] - \frac{1}{\hbar^2} \int_0^\infty dt' T_R\{[\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}(t')]],
\]

where \(\hat{\rho}\) denotes the reduced density matrix of system in the interaction picture. The first term shows the Lamb shift which is ignored in the present study and the second one represents the dissipation of system.

In general case, the interaction Hamiltonian can be considered as \(\hat{H}_I = \sum_\alpha A_\alpha B_\alpha\) with the operators \(A_\alpha\) and \(B_\alpha\) which satisfy the commutation relation \([A_\alpha, B_\beta] = 0\) and act on the system and leads Hilbert spaces, respectively. Finally, the master equation for the central system of QDM in the presence of superconducting leads is derived as:

\[
\frac{d\hat{\rho}_s(t)}{dt} = \frac{1}{\hbar^2} \sum_\omega \sum_{\alpha,\beta} \left(\Gamma^-_{\alpha,\beta}(\omega) \left(\hat{A}_\beta(\omega)\hat{\rho}_s(t)\hat{A}_\alpha^\dagger(\omega) - \hat{A}_\alpha(\omega)\hat{A}_\beta(\omega)\hat{\rho}_s(t)\hat{A}_\alpha^\dagger(\omega)\right) + \Gamma^+_{\alpha,\beta}(\omega) \left(\hat{A}_\alpha(\omega)\hat{\rho}_s(t)\hat{A}_\beta^\dagger(\omega) - \hat{A}_\alpha^\dagger(\omega)\hat{A}_\beta(\omega)\hat{\rho}_s(t)\hat{A}_\alpha(\omega)\right)\right),
\]

where \{\} denotes the anticommutation relation. The dissipation coefficients \(\Gamma_{\alpha,\beta}(\omega) = \int_0^\infty ds e^{i\omega s} \langle \hat{B}_\alpha(t)\hat{B}_\beta^\dagger(t-s) \rangle_{B}\) and \(\Gamma^+_{\alpha,\beta}(\omega) = \int \int_0^\infty ds e^{i\omega s} \langle \hat{B}_\beta^\dagger(t)\hat{B}_\alpha(t-s) \rangle_{B}\) are related to the bath correlation function. For superconducting leads, the distribution function is defined as \(\langle \hat{G}_{k,\nu}^\dagger(\nu)\hat{G}_{k,\nu}(\nu) \rangle_B = 1 - f^+(E_{k,\nu})\) and \(\langle \hat{G}_{k,\nu}^\dagger(\nu)\hat{G}_{k,\nu}(\nu) \rangle_B = (1 - f^+(E_{k,\nu})) = f^-(E_{k,\nu})\). So, we have

\[
\Gamma^+_{\alpha,\beta}(\omega) = 2\pi \sum_{k\nu\sigma\sigma'} T_{k\nu\sigma\sigma'} T_{k'\nu'\sigma'}^* \langle \hat{G}_{k\nu}^\dagger(\nu)\hat{G}_{k'\nu'}(\nu) \rangle

\]

\[
\Gamma^-_{\alpha,\beta}(\omega) = 2\pi \sum_{k\nu\sigma\sigma'} T_{k\nu\sigma\sigma'} T_{k'\nu'\sigma'}^* \langle \hat{G}_{k'\nu'}^\dagger(\nu')\hat{G}_{k\nu}(\nu) \rangle
\]

According to the BCS theory, the superconducting density of states \(R_{k\nu}(E)\) is defined \[57, 59\]:

\[
R_{k\nu}(E) = N_F \frac{|E_{k\nu}|}{\sqrt{E_{k\nu}^2 - \Delta_\nu^2}},
\]

in which \(N_F\) denotes the density of states for normal reservoirs which is assumed as a constant parameter close to the Fermi level of energy. We define \(\Gamma_{\alpha,\beta}(\omega) = 2\pi N_F|\langle E_{\nu'k}\rangle|\) so, Eq.\(15\) can be written as:

\[
\Gamma^+_{\alpha,\beta}(\omega) = \Gamma_0 \int dE f^+(E_{k,\nu}) |E_{k\nu}|\sqrt{E_{k\nu}^2 - \Delta_\nu^2}^2
\]

\[
\Gamma^-_{\alpha,\beta}(\omega) = \Gamma_0 \int dE f^-(E_{k,\nu}) |E_{k\nu}|\sqrt{E_{k\nu}^2 - \Delta_\nu^2}^2
\]

To parameterize the effect of left-right asymmetric coefficients in Eq.\(17\), we define \(T_{k\nu\sigma\sigma'} = \gamma_{\alpha,\nu} T_0\) in which \(\gamma_{\alpha,\nu} = \alpha, A, B\) denotes the asymmetric constant parameter and \(T_0\) shows the symmetric coupling coefficient.

In Eq.\(17\) \(\hat{A}_\alpha(\omega)\) denotes the projection superoperator which acts on the eigenoperator of system with eigenvalue of \(\omega\). Here, as we encounter with a bipartite central system, we introduce the eigenoperator as \(|\nu\rangle = |\epsilon_A\rangle_A \otimes |\epsilon_B\rangle_B\) with eigenvalue \(\omega = \{\omega_A, \omega_B\}\). Therefore, we define the superoperator:

\[
\hat{A}(\omega) = \hat{A}(\epsilon_A, \epsilon_B) = \sum_{i,j} |\epsilon_i\rangle \langle \epsilon_i| \hat{A} |\epsilon_i\rangle \langle \epsilon_j|.
\]

The present system includes four on-site energies \(\epsilon_A, \epsilon_B\), \(\epsilon_A, \epsilon_B\), \(g_A, \epsilon_B\) and \(g_A, \epsilon_B\) where \(g_A\) and \(g_B\) represent the ground and excited states of quantum dots respectively (\(A = A, B\)). The projection superoperators \(\sigma^-_{\alpha,\beta} = (\alpha = A, B; \delta = 1, 2)\) used in our calculation can be categorized:

\[
\sigma^-_{A,1}(\epsilon_A, 0) = |g_A, \epsilon_B > \epsilon_A, g_B|,
\]

\[
\sigma^-_{A,2}(\epsilon_A, 0) = |g_A, \epsilon_B > \epsilon_A, g_B|,
\]

\[
\sigma^-_{B,1}(\epsilon_B, 0) = |g_A, \epsilon_B > \epsilon_A, g_B|,
\]

\[
\sigma^-_{B,2}(\epsilon_B, 0) = |g_A, \epsilon_B > \epsilon_A, g_B|.
\]
Regards to the relations \(18\) and \(22\), we should consider the dissipation coefficient relations as \(\Gamma_{\alpha,\beta}^{\pm} = \Gamma_{A}^{\pm} \Gamma_{B}^{\pm} \Gamma_{AB}^{\pm} \Gamma_{BA}^{\pm}\). In the master equation \(17\), the projection super-operators which can be used are \(\sigma_{\alpha,\delta}^{\pm} = \sigma_{A1}^{\pm} \sigma_{A2}^{\pm} \sigma_{B1}^{\pm} \sigma_{B2}^{\pm}\). In the present system, the bias voltage would possess important roles. As this system is under bias voltage from the left side, it can influence the energy of the leads by replacing \(\varepsilon_{L} \rightarrow \varepsilon + \mu_{L} (\mu_{L} = \mu_{0} + eV)\) in our calculations. According to Fig. \(6\), the electrochemical energy level of the left lead is shifted by applying the asymmetric bias voltage. This shift makes the quasiparticle energy of the left lead higher than the energy levels of the QDM and the right lead.

In order to have the quasiparticle current and entanglement through the dots, the energy levels of system must satisfy this condition \(E_{L} > \varepsilon_{B} > \varepsilon_{A} > E_{R}\). In addition, we assume all energy levels are far enough from the order parameter of leads. This allows us dealing with only the quasiparticle transport and ignoring the Cooper pair current. We consider the weak-coupling regime to have the energies of system under this relation \(\Gamma_{\alpha,\beta}^{\pm} < \varepsilon_{A}, \varepsilon_{B}, |\Delta|\).

In our calculations, we apply the computational basis: \(|1\rangle = |gg\rangle\), \(|2\rangle = |ge\rangle\), \(|3\rangle = |eg\rangle\) and \(|4\rangle = |ee\rangle\). Here, \(|ij\rangle = |i\rangle_{A} |j\rangle_{B}\) denotes the excited or ground states of the QDM system.

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