DYNAMICAL EMBEDDING IN CUBICAL Shifts & THE TOPOLOGICAL ROKHLIN AND SMALL BOUNDARY PROPERTIES

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Abstract. According to a conjecture of Lindenstrauss and Tsukamoto, a topological dynamical system $(X, T)$ is embeddable in the $d$-cubical shift $(([0,1]^d, shift)$ if both its mean dimension and periodic dimension are strictly bounded by $\frac{d}{2}$. We verify the conjecture for the class of systems admitting finite dimensional non-wandering sets (under the additional assumption of closed periodic points set). The main tool in the proof is the new concept of local markers. Continuing the investigation of (global) markers initiated in previous work it is shown that the marker property is equivalent to a topological version of the Rokhlin Lemma. Moreover new classes of systems are found to have the marker property, in particular, extensions of aperiodic systems with a countable number of minimal subsystems. Extending work of Lindenstrauss we show that for systems with the marker property vanishing mean dimension is equivalent to the small boundary property. Finally we answer a question by Downarowicz in the affirmative: the small boundary property is equivalent to admitting a zero-dimensional isomorphic extension.

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DYNAMICAL EMBEDDING IN CUBICAL SHIFTS, TRP & SBP

1. Introduction

The question under which conditions a topological dynamical system \((X, T)\) is embeddable in the \(d\)-cubical shift \(((\mathbb{Z}^d, \text{shift}))\) stems from Auslander’s 1988 influential book. According to Jaworski’s Theorem (1974), for \(X\) finite-dimensional and \(T\) aperiodic, embedding is possible with \(d = 1\). Auslander posed the question if for \(d = 1\), it is sufficient that \(X\) is minimal. The question was solved in the negative by Lindenstrauss and Weiss (2000), adroitly using the invariant of \textit{mean dimension} introduced by Gromov (1999). Around the same time Lindenstrauss (1999) showed that if \(X\) is an extension of a minimal system and \(\text{mdim}(X, T) < \frac{d}{36}\), then \((X, T)\) is embeddable in \(((\mathbb{Z}^d, \text{shift}))\). Recently Lindenstrauss and Tsukamoto (2012) have introduced a unifying conjecture and several cases of this conjecture have been verified. According to this conjecture the only obstructions for embeddability are given by the invariants of \textit{mean dimension} and \textit{periodic dimension}, the later quantifying the natural obstruction due to the set of periodic points. A precise statement of the conjecture is that \(\text{mdim}(X, T) < \frac{d}{2}\) and \(\text{perdim}(X, T) < \frac{d}{2}\) imply \((X, T)\) is embeddable in \(((\mathbb{Z}^d, \text{shift}))\). In Gutman and Tsukamoto (2012) the conjecture was verified for extensions of aperiodic subshifs and in Gutman (2012) the conjecture was verified for finite-dimensional systems. In the same article it was shown that for extensions of aperiodic finite-dimensional systems \(\text{mdim}(X, T) < \frac{d}{16}\) implies \((X, T)\) is embeddable in \(((\mathbb{Z}^{d+1}), \text{shift}))\).

A keen observer will notice that all embedding results mentioned above, involving infinite-dimensional systems, require the assumption of aperiodicity. This is due to a common device used in the proofs: existence of \textit{markers} (of all orders). Markers can be thought of as a suitable generalization of the familiar markers of symbolic dynamics, introduced by Krieger (1982), to the setting of arbitrary dynamical systems. As a necessary condition for the
existence of markers of all orders is the aperiodicity of the system, one is confined to the category of aperiodic systems.

In this article we resolve this difficulty by introducing the concept of *local markers*. It has the desired consequence of allowing us to treat some infinite-dimensional systems admitting periodic points. In particular we verify the Lindenstrauss-Tsukamoto Conjecture for the class of systems whose non-wondering set is finite dimensional (under the additional assumption of closed periodic points set) and discuss some examples. The result is a consequence of a more general embedding theorem stating that systems with the local marker property verifying $mdim(X,T) < \frac{d}{36}$ and $perdim(X,T) < \frac{d}{2}$ are embeddable in $([0,1]^d, shift)$.

Recognizing the importance of markers, both local and global, we continue the investigation of markers as carried out in Gutman (2012) which itself was a generalization of previous work by Bonatti and Crovisier (2004), and prove in particular that an aperiodic system with a countable number of minimal subsystems admits the marker property.

In Gutman (2011) the notion of the topological Rokhlin property was introduced. This is a dynamical topological analogue of the Rokhlin Lemma of measured dynamics. Here we show that the marker property is equivalent to a strong version of the topological Rokhlin property. Following closely Lindenstrauss (1999) this characterization results with several fruitful applications: Systems with the marker property admit a compatible metric with respect to which the *metric mean dimension* equals the (topological) mean dimension. Moreover for such systems, vanishing mean dimension is equivalent to the small boundary property and to being an inverse limit of finite entropy systems. Using a different method we show that the small boundary property is equivalent to admitting a zero-dimensional isomorphic extension. This answers a question by Downarowicz in the affirmative.
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2. Preliminaries

The following article is closely related to the article [Gut12] and we recommend the reader to familiarize herself or himself with the Introduction and Preliminaries sections of that article.

2.1. Conventions. Throughout the article with the exception of Section 9, a topological dynamical system (t.d.s) \((X, T)\) consists of a metric compact space \((X, d)\) and a homeomorphism \(T: X \to X\). \(P = P(X, T)\) denotes the set of periodic points and \(P_n\) denotes the set of periodic points of period \(\leq n\). In addition we use the notation \(H_n = P_n \setminus P_{n-1}\). \(\Delta = \{(x, x) | x \in X\}\) denotes the diagonal of \(X \times X\). If \(x \in X\) and \(\epsilon > 0\), let \(B_\epsilon(x) = \{y \in X | d(y, x) < \epsilon\}\) denote the open ball around \(x\). We denote \(\overline{B}_\epsilon(x) = \overline{B_\epsilon(x)}\). Note \(\overline{B}_\epsilon(x) \subseteq \{y \in X | d(y, x) \leq \epsilon\}\) but equality does not necessary hold.

For \(f, g \in (C(X, [0, 1]^d))\), we define \(\|f - g\|_\infty \triangleq \sup_{x \in X} |f(x) - g(x)|_\infty\).

2.2. The Non-Wandering Set. Let \((X, T)\) be a t.d.s. A point \(x \in X\) is said to be non-wandering if for every open set \(x \in U\), there is \(k \in \mathbb{Z}\) so that \(U \cap T^k U \neq \emptyset\). The non-wandering set \(\Omega(X)\) is the collection of all non-wandering points. Note \(\Omega(X)\) is a non-empty, closed and \(T\)-invariant set.

The following three subsections follow closely the corresponding subsections in [Gut12]:
2.3. **Dimension.** Let $\mathcal{C}$ denote the collection of open (finite) covers of $X$. For $\alpha \in \mathcal{C}$ define its order by $\text{ord}(\alpha) = \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1$. Let $D(\alpha) = \min_{\beta > \alpha} \text{ord}(\beta)$ (where $\beta$ refines $\alpha$, $\beta > \alpha$, if for every $V \in \beta$, there is $U \in \alpha$ so that $V \subset U$). The Lebesgue covering dimension is defined by $\dim(X) = \sup_{\alpha \in \mathcal{C}} D(\alpha)$.

2.4. **Periodic Dimension.** Let $P_m$ denote the set of points of period $\leq m$. Introduce the infinite vector $\rightarrow \text{perdim}(X, T) = (\dim(P_m))_{m \in \mathbb{N}}$. This vector is clearly a topological dynamical invariant. Let $d > 0$. We write $\rightarrow \text{perdim}(X, T) < d$, if for every $m \in \mathbb{N}$, $\rightarrow \text{perdim}(X, T)|_m < d$.

2.5. **Mean Dimension.** Define:

$$m\text{dim}(X, T) = \sup_{\alpha \in \mathcal{C}} \lim_{n \to \infty} \frac{D(\alpha^n)}{n}$$

where $\alpha^n = \bigvee_{i=0}^{n-1} T^{-i} \alpha$. Mean dimension was introduced by Gromov [Gro99] and systematically investigated by Lindenstrauss and Weiss in [LW00].

The following two Subsections follow closely the corresponding Subsections in [Gut11].

2.6. **The Topological Rokhlin Property.** The classical Rokhlin lemma states that given an aperiodic invertible measure-preserving system $(X, T, \mu)$ and given $\epsilon > 0$ and $n \in \mathbb{N}$, one can find $A \subset X$ so that $A, T A, \ldots, T^{n-1} A$ are pairwise disjoint and $\mu(\bigcup_{k=0}^{n-1} T^k A) > 1 - \epsilon$. It easily follows that given an aperiodic invertible measure-preserving system $(X, T, \mu)$ and given $\epsilon > 0$, one can find a measurable function $f : X \to \{0, 1, \ldots, n - 1\}$ so that if we define the exceptional set $E_f = \{x \in X \mid f(Tx) \neq f(x) + 1\}$, then $\mu(E_f) < \epsilon$.

The new formulation allows us to generalize to the topological category. Indeed following [SW91], given a t.d.s $(X, T)$ and a set $E \subset X$, we define the orbit-capacity of a set $E$ in the following manner (the limit exists):
\[ \text{ocap}(E) = \lim_{n \to \infty} \frac{1}{n} \sup_{x \in X} \sum_{k=0}^{n-1} 1_E(T^k x) \]

\((X, T)\) is said to have the \textbf{topological Rokhlin property (TRP)} if and only if for every \(\epsilon > 0\) there exists a continuous function \(f : X \to \mathbb{R}\) so that for the \textit{exceptional set} \(E_f = \{x \in X \mid f(Tx) \neq f(x) + 1\}\), one has \(\text{ocap}(E_f) < \epsilon\).

\[ \text{mdim}_d(X, T) = \liminf_{\epsilon \to 0} \frac{s(\epsilon, d)}{\log(\epsilon)} \]

In [LW00] it was shown that \(\text{mdim}_d(X, T) \leq \text{mdim}(X, T)\). By a classical theorem of Bowen and Dinaburg, the topological entropy is given by \(h_{\text{top}}(X, T) = \lim_{\epsilon \to 0} s(\epsilon, d)\). Thus it was concluded in [LW00] that finite topological entropy implies mean dimension zero.
2.9. **Overview of the Article.** In Section 3 the definition of the marker property is recalled and new classes of systems admitting the marker property are exhibited, in particular, extensions of aperiodic systems with a countable number of minimal subsystems. Additionally some simple examples are discussed. In Section 4 the local marker property is defined and verified for systems of finite dimensional systems with closed sets of periodic points. In Section 5 the local and global strong topological Rokhlin properties are introduced and investigated. In particular it is shown that the marker property is equivalent to the (global) strong topological Rokhlin property. In Section 6 the following embedding theorem is proven: If \((X, T)\) has the local marker property, \(mdim(X, T) < \frac{d}{36}\) and \(perdim(X, T) < \frac{d}{2}\), then \((X, T)\) is generically embeddable in \((([0,1]^d)^\mathbb{Z}, \text{shift})\). In Section 7 various applications of the embedding theorem are given, in particular, the verification of the Lindenstrauss-Tsukamoto Conjecture for the class of systems admitting finite dimensional non-wandering sets (under the additional assumption of closed periodic points set). Additionally some examples are constructed. In Section 8 it is shown that systems with the marker property admit a compatible metric with respect to which the metric mean dimension equals the (topological) mean dimension. Moreover for systems with the marker property, vanishing mean dimension is equivalent to the having the small boundary property and to being an inverse limit of finite entropy systems. In Section 9 is is shown that the small boundary property is equivalent to admitting a zero-dimensional isomorphic extension. The Appendix contains auxiliary lemmas.

### 3. The Marker Property

**Definition 3.1.** A subset \(F\) of a t.d.s \((X, T)\) is called an \(n\)-**marker** \((n \in \mathbb{N})\) if:
(1) $F \cap T^i(F) = \emptyset$ for $i = 1, 2, \ldots, n - 1$.

(2) The sets $\{T^i(F)\}_{i=1}^m$ cover $X$ for some $m \in \mathbb{N}$.

The system $(X, T)$ is said to have the **marker property** if there exist open $n$-markers for all $n \in \mathbb{N}$.

**Remark 3.2.** Clearly the marker property is stable under extension, i.e. if $(X, T)$ has the marker property and $(Y, S) \to (X, T)$ is an extension, then $(Y, S)$ has the marker property.

**Remark 3.3.** By Lemma A.1 of [Gut12] $(X, T)$ has a closed $n$-marker iff $(X, T)$ has an open $n$-marker.

The marker property was first defined in [Dow06] (Definition 2), where one requires the $n$-markers to be clopen. In the same article it was proven that an extension of an aperiodic zero-dimensional (non necessarily invertible) t.d.s has the marker property. This was essentially based on the "Krieger Marker Lemma" (Lemma 2 of [Kri82]). In [Gut12] Theorem 6.1 it was proven that aperiodic finite dimensional t.d.s have the marker property. From [Lin99, Lemma 3.3] it follows that an extension of an aperiodic minimal system has the marker property. Given these results it is natural to ask the following question:

**Problem 3.4.** Does any aperiodic system have the marker property?

We do not know the answer of the previous problem. However we are able to prove two theorems establishing the existence of the marker property under natural assumptions. We also discuss examples.

**Theorem 3.5.** (Downarowicz & Gutman) If $(X, T)$ is an extension of an aperiodic t.d.s which has a countable number of minimal subsystems then it has the marker property.
Proof. We may assume w.l.o.g that \((X,T)\) is aperiodic and has a countable number of minimal subsystems. Let \(n \in \mathbb{N}\). We will construct inductively an open set \(U \subset X\) so that the sets \(\{T^i(U)\}_{i=1}^n\) are pairwise disjoint and \(\{T^i(U)\}_{i=1}^m\) cover \(X\) for some \(m\). Let \(M_1, M_2, \ldots\) be an enumeration of the minimal subsystems of \(X\). Using the fact there is only a countable number of minimal subsystems find \(m_1 \in M_1, r_1 > 0\) so that \(\{T^i B_{r_1}(m_1)\}_{i=-n}^n\) are pairwise disjoint and for all \(l \geq 2\), \(M_l \not\subset \bigcup_{i=-n}^n T^i \partial B_{r_1}(m_1)\) (here we use that \(\bigcup_{i=-n}^n T^i \partial B_{r_1}(m_1)\) is a uncountable collection of pairwise disjoint sets). Define \(U_1 = B_{r_1}(m_1)\). Assume one has defined an open set \(U_k \subset X\) so that:

1. For any \(i = 1, \ldots, k\) there exists \(j = j(i) \in \mathbb{Z}\) so that \(U_k \cap T^j M_i \neq \emptyset\).
2. For all \(l \geq k + 1\), \(M_l \not\subset \bigcup_{i=-n}^n T^i \partial U_k\).
3. \(\{T^i(U_k)\}_{i=1}^n\) are pairwise disjoint

If \(U_k \cap T^j M_{k+1} \neq \emptyset\) for some \(j \in \mathbb{Z}\), define \(U_{k+1} = U_k\). We now assume that \(U_k \cap T^j M_{k+1} = \emptyset\) for all \(j \in \mathbb{Z}\). By assumption \(M_{k+1} \not\subset \bigcup_{i=-n}^n T^i \partial U_k\). Conclude \(M_{k+1} \not\subset \bigcup_{i=-n}^n T^i \partial U_k\). Using the fact there is only a countable number of minimal subsystems, we can find \(m_{k+1} \in M_{k+1}\) and \(r_{k+1} > 0\) so that \(\{T^i B_{r_{k+1}}(m_{k+1})\}_{i=-n}^n\) are pairwise disjoint and so that it holds:

\[
B_{r_{k+1}}(m_{k+1}) \cap \bigcup_{i=-n}^n T^i U_k = \emptyset,
\]

\[
\forall l > k + 1 \quad M_l \setminus \bigcup_{i=-n}^n T^i \partial U_k \not\subset \bigcup_{i=-n}^n T^i \partial B_{r_1}(m_1)
\]

Define \(U_{k+1} = U_k \cup B_{r_{k+1}}(m_{k+1})\). We now verify that the desired properties hold:
(1) For any $i = 1, \ldots, k + 1$ there exists $j = j(i) \in \mathbb{Z}$ so that $U_{k+1} \cap T^j M_i \neq \emptyset$. Indeed if $i \leq k$, this follows from property (1) above. For $i = k + 1$ it is trivial.

(2) For all $l \geq k + 2$, $M_l \not\subset \bigcup_{i=-n}^{n} T^i \partial U_{k+1}$. Indeed if follows from $\partial U_{k+1} \subset \partial U_k \cup \partial B_{r_{k+1}}(m_{k+1})$ and (3.2).

(3) $\{T^i(U_{k+1})\}_{i=1}^{n}$ are pairwise disjoint. Indeed it is enough to show $T^{i_1} U_k \cap T^{i_2} B_{r_{k+1}}(m_{k+1}) = \emptyset$ for all $1 \leq i_1, i_2 \leq n$. This follows from (3.1).

Finally we define $U = \bigcup_{k=1}^{\infty} U_k$. As $U_1 \subset U_2 \subset \cdots$, it holds that $\{T^i(U)\}_{i=1}^{n}$ are pairwise disjoint. Clearly for any $i \in \mathbb{N}$ there exists $j = j(i) \in \mathbb{Z}$ so that $U \cap T^j M_i \neq \emptyset$. As $U$ is open and $M_i$ is compact this implies there exists $m(i) \in \mathbb{N}$ so that $M_i \subset \bigcup_{i=0}^{m(i)} T^i U$. Fix $x \in X$. There exists $i \in \mathbb{N}$ so that $\overline{\text{orb}(x)} \cap M_i \neq \emptyset$. Conclude there exists $k \in \mathbb{Z}$, so that $T^k x \in \bigcup_{i=0}^{m(i)} T^i U$.

By a simple compactness argument we deduce the existence of $m \in \mathbb{N}$ so that $\{T^i(U)\}_{i=1}^{m}$ cover $X$. \hfill \Box

**Example 3.6.** Clearly the previous theorem applies to every t.d.s which consists of a finite union of minimal systems. We now present a simple example of an aperiodic t.d.s with an infinite countable number of minimal systems. Let $C_r = \{(x,y) | x^2 + y^2 = r\} \subset \mathbb{R}^2$, be a circle of radius $r$ around the origin. Select a strictly decreasing sequence of positive numbers $r_1 > r_2 > \cdots$ with $r_i \to r_0 > 0$. Let $X = \bigcup_{i=0}^{\infty} C_{r_i} \subset \mathbb{R}^2$ and define $T : X \to X$ by rotating by $\alpha$ on each circle where $\alpha$ is some irrational number.

**Example 3.7.** Not every aperiodic system has a countable number of minimal subsystems. Indeed consider $X = \mathbb{T}^2$, the two-dimensional torus equipped with $T(x,y) = (x + \alpha, y)$ for some $\alpha$ irrational number.
Definition 3.8. Let \((X, T)\) be a t.d.s and denote by \(\mathcal{M}\) the collection of all minimal subspaces of \((X, T)\). \((X, T)\) has a **compact minimal subsystems selector** if there exists a compact \(L\) so that for every \(M \in \mathcal{M}\), \(|L \cap M| = 1\) and \(L \subset \bigcup \mathcal{M}\).

Theorem 3.9. (*Downarowicz*) If \((X, T)\) is an extension of an aperiodic t.d.s with a compact minimal subsystems selector than it has the marker property.

**Proof.** We may assume w.l.o.g that \((X, T)\) has a compact minimal subsystems selector \(L\). Denote by \(\mathcal{M}\) the collection of all minimal subspaces of \((X, T)\). If \(x, y \in L, x \neq y\), then there exists distinct \(M_x, M_y \in \mathcal{M}\) so that \(x \in M_x\) and \(y \in M_y\). This implies \(\text{orb}(x) \cap \text{orb}(y) = \emptyset\). As \((X, T)\) is aperiodic conclude the closed sets \(\{T^i L\}_{i=-\infty}^{\infty}\) are pairwise disjoint. Let \(n \in \mathbb{N}\). There exists \(\epsilon > 0\) so that \(T^i B_{\epsilon}(L) (1 \leq i \leq n)\) are pairwise disjoint. For every \(M \in \mathcal{M}\) there exists by minimality \(m = m(M)\) so that:

\[
M \subset \bigcup_{i=0}^{m} T^i B_{\epsilon}(L)
\]

For every \(z \in X\), there exists \(M_z \in \mathcal{M}\) so that \(\overline{\text{orb}(z)} \cap M_z \neq \emptyset\). We conclude by a compactness argument that \(B_{\epsilon}(L), T B_{\epsilon}(L), T^2 B_{\epsilon}(L), \ldots,\) eventually cover \(X\). □

Example 3.10. A simple example of an aperiodic system with a compact minimal subsystems selector is given by Example 3.7. A selector is given by \(\{0\} \times \mathbb{T}\). An aperiodic system with a compact minimal subsystems selector without a non trivial minimal factor is given by taking a disjoint union of the previous example \(X\) with a circle, \(Y = X \hat{\cup} \mathbb{T}\) where the circle is equipped with a rotation by \(\beta\), such that \(\alpha\) and \(\beta\) are incommensurable.

Example 3.11. Not all aperiodic t.d.s have a compact minimal subsystems selector. Indeed let \(X = \mathbb{T}^2\) be the two-dimensional torus and \(T : X \to X\) be
given by \( T(x, y) = (x + \frac{1}{2}, x + \alpha) \) for some \( \alpha \) irrational. Let \( \mathcal{M} \) the collection of all minimal subspaces of \( (X, T) \). Note \( \mathcal{M} = \{ \{ t, t + \frac{1}{2} \} \times \mathbb{T} | t \in [0, \frac{1}{2}) \} \).

Assume for a contradiction \( (X, T) \) has a compact minimal selector \( L \). Let \( L^2 \) be the projection of \( L \) on the first coordinate. \( L^2 \) is a closed set so that \( T = L^2 \cup (L^2 + \frac{1}{2}) \). Contradiction.

4. The Local Marker Property

**Definition 4.1.** Let \( Z, W \) be closed sets with \( Z \times W \subset (X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X)) \). A subset \( F \) of a t.d.s \( (X, T) \) is called a **local** \( n \)-**marker** (\( n \in \mathbb{N} \)) for \( Z \times W \) if:

1. \( F \cap T^i(F) = \emptyset \) for \( i = 1, 2, \ldots, n - 1 \).
2. The sets \( \{ T^i(F) \}_{i=1}^m \), \( i = 0, 1, \ldots, m - 1 \), cover \( Z \cup W \) for some \( m \).

We say \( Z \times W \) has **local markers** if it has **open** \( n \)-markers for all \( n \in \mathbb{N} \).

We say that a cover of \( Y \subset (X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X)) \) by a countable collection of products of closed sets \( \{ Z_i \times W_i \}_{i=1}^\infty \) has the **local marker property relatively to** \( Y \) if for every \( i \), \( Z_i \times W_i \) has local markers.

We say \( (X, T) \) has the **local marker property** if there is a cover with the local marker property relatively to \( (X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X)) \).

**Remark 4.2.** If \( (X, T) \) has the marker property than it has the local marker property.

**Remark 4.3.** Just as in the case of the marker property, \( Z \times W \) has local markers iff it has closed \( n \)-markers for all \( n \in \mathbb{N} \).

**Theorem 4.4.** If \( \dim(X) < \infty \) and \( P \) is closed, then \( (X, T) \) has the local marker property.

**Proof.** The proof follows closely the proof of Theorem 6.1 of [Gut12] where it is shown that an aperiodic finite dimensional t.d.s has the marker property.
Theorem 6.1 of [Gut12] is based on a certain generalization of Lemma 3.7 of [BC04] which is one of the building blocks in the proof of the Bonatti-Crovisier Tower Theorem for $C^1$-diffeomorphisms on manifolds [BC04, Theorem 3.1]. Let $\{Z_i \times V_i\}_{i=1}^\infty$ be an arbitrary countable cover of $(X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X))$ by a countable collection of products of closed sets so that for every $i$, $Z_i \cap P = \emptyset$ and $V_i \cap P = \emptyset$ (here we use that $P$ is closed) and $Z_i \cap V_i = \emptyset$. Fix $n, k \in \mathbb{N}$. For every $x \in Z_k \cup V_k$ choose an open set $U_x$ so that $x \in U_x$, $U_x \subset X \setminus P$ and $U_x \cap T^i U_x = \emptyset$ for $i = 1, 2, \ldots, m = (2\dim(X) + 2)n - 1$. Let $U_{x_1}, U_{x_2}, \ldots, U_{x_s}$ be a finite cover of $Z_k \cup V_k$. We now continue exactly as in the proof of Theorem 6.1 of [Gut12], to find a $W$, so that $W \cap T^i W = \emptyset$, $i = 1, 2, \ldots, n - 1$ and $Z_k \cup V_k \subset \bigcup_{i=1}^s U_{x_i} \subset \bigcup_{i=0}^m T^i(W)$ (we use the fact that $P$ is closed in order to invoke Lemma 6.2 of [Gut12]). The existence of an open $n$-marker for $Z_k \cup V_k$ follows easily. 

**Proposition 4.5.** If $(\Omega(X), T)$ has the local marker property then $(X, T)$ has the local marker property.

**Proof.** We will show there is a closed countable cover which has the local marker property relatively to $S = (X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X))$ by defining $S_1, S_2, S_3 \subset X \times X$ so that $S = S_1 \cup S_2 \cup S_3$ where $S_2^* = \{(y, x) | (x, y) \in S_2\}$, and exhibiting 3 closed countable covers which have the local marker property relatively to $S_1$, $S_2$ and $S_3$ respectively (the case of $S_2^*$ will follow from the case of $S_2$). As $(\Omega(X), T)$ has the local marker property, there exists a a countable cover $\{Z_i \times W_i\}_{i=1}^\infty$ which has the local marker property relatively to $(\Omega(X) \times \Omega(X)) \setminus (\Delta \cup (\Omega(X) \times P) \cup (P \times \Omega(X)))$, however it is important to note this is w.r.t the topology induced by $\Omega(X)$. Also note that for all $i$, $Z_i \cup W_i \subset \Omega(X)$. We now define $S_1, S_2, S_3:$
(1) $S_1 = (\Omega(X) \times \Omega(X)) \cap S = (\Omega(X) \times \Omega(X)) \setminus (\triangle \cup (\Omega(X) \times P) \cup (P \times \Omega(X)))$. We claim $\{Z_i \times W_i\}_{i=1}^{\infty}$ has the local marker property relatively to $S_1$ (w.r.t. the topology induced by $X$). Indeed fix $k$. Let $F$ be a closed (in $\Omega(X)$ and therefore in $X$) $n$-marker for $Z_k \times W_k$ in $\Omega(X)$. Clearly one can find an $\epsilon > 0$ so that $B_\epsilon(F) \subset X$ is an open $n$-marker for $Z_k \times W_k$ in $X$.

(2) $S_2 = ((X \setminus \Omega(X)) \times \Omega(X)) \cap S$. As $X \times X$ is second-countable, every subspace is Lindelöf, i.e. every open cover has a countable subcover. Let $\{U_i\}_{i=1}^{\infty}$ be an open cover of $X \setminus \Omega(X)$ such that for each $i$ there is an $\epsilon_i > 0$ so that $\{T^k B_{\epsilon_i}(U_i)\}_{k \in \mathbb{Z}}$ are pairwise disjoint. We claim the countable closed cover $\{\overline{U_i} \times W_k\}_{i,k=1}^{\infty}$ has the local marker property relatively to $S_2$. First observe that as $B_{\epsilon_i}(U_i) \subset X \setminus \Omega(X)$, $\overline{U_i} \cap W_k = \emptyset$. Now fix $i, k, n$. Let $F \subset \Omega(X)$ be a closed $n$-marker for $Z_k \times W_k$. Let $0 < \delta < d(\bigcup_{j=-(n-1)}^{n-1} T^j B_{\epsilon_i/2}(U_i), \Omega(X))$ so that $B_\delta(F)$ is still an open $n$-marker for $Z_k \times W_k$. We claim $B_\delta(F) \cup B_{\epsilon_i/2}(U_i)$ is an open $n$-marker for $\overline{U_i} \times W_k$. Indeed as $F \subset \Omega(X)$, the choice of $\delta$ guarantees $T^{j_1} B_\delta(F) \cap T^{j_2} B_{\epsilon_i/2}(U_i) = \emptyset$ for all $0 \leq j_1, j_2 \leq n - 1$.

(3) $S_3 = (X \setminus \Omega(X))^2 \cap S$. Let $\{U_i\}_{i=1}^{\infty}$ be the open cover of the previous case. We can assume w.l.o.g that for each $x, y \in (X \setminus \Omega(X))^2$ with $x \neq y$ there are $i \neq k$ so that $x \in U_i$ and $y \in U_k$ and $\overline{U_i} \cap \overline{U_k} = \emptyset$. Note $\mathcal{C} = \{\overline{U_i} \times \overline{U_k}\} \subset \Omega(X)$ is a closed cover of $(X \setminus \Omega(X))^2 \cap S$. A countable closed cover which has the local marker property relatively to $S_3$, will be achieved by splitting each member of the cover $\mathcal{C}$ to an union of at most a countable number of closed products. Fix $i \neq k$ so that $\overline{U_i} \times \overline{U_k} \in \mathcal{C}$. If for all $j \in \mathbb{Z}$, $T^j B_{\epsilon_i}(U_i) \cap B_{\epsilon_k}(U_k) = \emptyset$, then $B_{\epsilon_i}(U_i) \cup B_{\epsilon_k}(U_k)$ is an open $n$-marker of $\overline{U_i} \times \overline{U_k}$ for all $n$. Assume this is not the case. Note that $\{\overline{U_i} \times (\overline{U_k} \cap T^j B_{\epsilon_i/2}(U_i))\}_{j \in \mathbb{Z}} \cup \{\overline{U_i} \times (\overline{U_k} \setminus \bigcup_{j \in \mathbb{Z}} T^j B_{\epsilon_i/2}(U_i))\}$
is a countable closed cover of $\overline{U}_i \times \overline{U}_k$. Let $j \in \mathbb{Z}$. Note that $B_{\epsilon_i}(U_i)$ is an open $n$-marker of $\overline{U}_i \times (\overline{U}_k \cap T^j B_{\epsilon_i/2}(U_i))$ for all $n$. Fix $n$. As $d(\bigcup_{j=-(n-1)}^{n-1} T^k(U_i), \overline{U}_k \setminus \bigcup_{j \in \mathbb{Z}} T^j B_{\epsilon_i/2}(U_i)) > 0$, there is $\delta > 0$ so that $T^j B_{\delta}(U_i) \cap T^{j+1} B_{\delta}(U_k \setminus \bigcup_{j \in \mathbb{Z}} T^j B_{\epsilon_i/2}(U_i)) = \emptyset$ for all $0 \leq j_1, j_2 \leq n - 1$. Taking $\delta < \min\{\epsilon_i, \epsilon_k\}$, guarantees that $B_{\delta}(U_i) \cup B_{\delta}(U_k \setminus \bigcup_{j \in \mathbb{Z}} T^j B_{\epsilon_i/2}(U_i))$ is an open $n$-marker for $\overline{U}_i \times \overline{U}_k \setminus \bigcup_{j \in \mathbb{Z}} T^j B_{\epsilon_i/2}(U_i)$.

\[\square\]

5. The Strong Topological Rokhlin Property

In [Gut11 Subsection 1.9] the topological Rokhlin property was introduced (see also Subsection 2.6). Here is a stronger variant, originating in [Lin99]:

**Definition 5.1.** We say that $(X, T)$ has the (global) strong topological Rokhlin property if for every $n \in \mathbb{N}$ there exists a continuous function $f : X \to \mathbb{R}$ so that if we define the exceptional set $E_f = \{x \in X \mid f(Tx) \neq f(x) + 1\}$, then $T^{-i}(E_f)$, $i = 0, 1, \ldots, n - 1$, are pairwise disjoint.

**Remark 5.2.** Assume $b - a \leq n - 1$. Under the above conditions consider $x \in X$. Then there exists at most one index $a \leq l \leq b$ so that $f(T^{l+1}x) \neq f(T^lx) + 1$. Indeed if $T^{l}x, T^{l}x' \in E_f$ for $a \leq l < l' \leq b$, then $E_f \cap T^{l-l'}E_f \neq \emptyset$ contradicting the definition.

**Definition 5.3.** We say that $(X, T)$ has the local strong topological Rokhlin property if one can cover $(X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ by a countable collection of products of closed sets $\{Z_i \times W_i\}_{i=1}^{\infty}$, where for every $k$, $Z_k \cap W_k = \emptyset$ and for every $m \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ with $a < b$ there exists a continuous function $f : \bigcup_{i=a}^{b} T^i(Z_k \cup W_k) \to \mathbb{R}$ so that if we define the exceptional set $E_f = \{x \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k) \mid f(Tx) \neq f(x) + 1\}$, then
$T^{-i}(E_f), i = 0, \ldots, m - 1$, are pairwise disjoint (note $x \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k)$ implies that both $f(x)$ and $f(Tx)$ are defined). In this context $\{Z_i \times W_i\}_{k=1}^\infty$ is also said to have the **local strong topological Rokhlin property**.

**Remark 5.4.** Assume $m \geq b - a - 2$. Under the above conditions consider $x \in (Z_k \cup W_k)$. Then there exists at most one index $a \leq l \leq b - 1$ so that $f(T^{l+1}x) \neq f(T^lx) + 1$. Indeed $T^lx \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k)$ and $f(T^{l+1}x) \neq f(T^lx) + 1$ imply $T^lx \in E_f$. If $T^lx, T^{l'}x \in E_f$ for $a \leq l < l' \leq b - 1$, then $E_f \cap T^{l-l'} E_f \neq \emptyset$ contradicting the definition.

The following proposition is based on Lemma 3.3 of [Lin99], which is the statement that a system with an aperiodic minimal factor has the strong topological Rokhlin property:

**Proposition 5.5.** If $(X,T)$ has the local marker property then $(X,T)$ has the local strong topological Rokhlin property.

**Proof.** By assumption one can cover $(X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X))$ by a countable collection of products of closed sets $\{Z_i \times W_i\}_{i=1}^\infty$, with the local marker property. We will now show that $\{Z_i \times W_i\}_{i=1}^\infty$ has the local strong topological Rokhlin property. Fix $m, k \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ with $a < b$. Let $F$ be an open $m$ marker for $Z_k \cup W_k$. It will be convenient to write

$Z_k \cup W_k \subset \bigcup_{i=-q}^{b-a} T^i(F)$

for some $q > b - a$. Choose a closed $R \subset F$ so that $Z_k \cup W_k \subset \bigcup_{i=-q}^{b-a} T^i R$. Conclude:

\[ T^i(Z_k \cup W_k) \subset \bigcup_{i=0}^q T^i R \quad \text{(5.1)} \]

Let $\omega : X \to [0,1]$ be a continuous function so that $\omega|_R \equiv 1$ and $\omega|_{F^c} \equiv 0$. We define a random walk for $z \in X$. At any point $p$ we arrive during the random walk, the walk terminates with probability $\omega(p)$ and moves to $T^{-1}p$ with probability $1 - \omega(p)$. Notice that for every point $z \in \bigcup_{i=a}^b T^i(Z_k \cup W_k)$
the random walk will terminate after a finite number of steps. Indeed by (5.1) there is an \( i \in \{0, \ldots, q\} \) so that \( z \in T^i R \), which implies the walk terminates in at most \( q \) steps (when the point hits \( R \)). Conclude there is a finite number of possible walks starting at \( z \) and we denote by \( f(z) \) the expected length of the walk starting at \( z \). As there is a uniform bound on the length of walks, \( f : \bigcup_{i=0}^q T^i(Z_k \cup W_k) \to \mathbb{R}_+ \) is continuous. Note that if \( y \notin F \) and \( y \in \bigcup_{i=a+1}^q T^i(Z_k \cup W_k) \), then \( f(T^{-1}y) = f(y) - 1 \). Notice that \( x \in \bigcup_{i=a+1}^q T^i(Z_k \cup W_k) \) and \( f(Tx) \neq f(x) + 1 \) implies \( y \triangleq Tx \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k) \) and \( f(T^{-1}y) = f(TT^{-1}x) = f(x) \neq f(Tx) - 1 = f(y) - 1 \). Therefore in such a case one must have \( Tx = y \in F \). Conclude \( E_f = \{ x \in \bigcup_{i=a}^{b-1} T^i(U \cup V) \mid f(Tx) \neq f(x) + 1 \} \subset T^{-1}F \). We therefore have \( T^{-1}(E_f) \cap E_f = \emptyset \) \( i = 1, \ldots, m-1 \).

\[ \square \]

The following question is interesting:

**Problem 5.6.** Does the the local strong topological Rokhlin property imply the local marker property?

The question can be answered assuming the global strong topological Rokhlin property:

**Theorem 5.7.** \((X,T)\) has the strong topological Rokhlin property iff \((X,T)\) has the marker property.

**Proof.** The fact that the marker property implies the strong topological Rokhlin property follows by a similar argument to the proof of Proposition 5.5. To prove the other direction assume \((X,T)\) has the strong topological Rokhlin property. Fix \( n \in \mathbb{N} \) and let \( f : X \to \mathbb{R} \) be a continuous function such that for the open set \( E_f = \{ x \in X \mid f(Tx) \neq f(x) + 1 \} \), \( T^{-i}(E_f), i = 0, 1, \ldots, n-1 \), are pairwise disjoint. We claim that the iterates
$E_f, T^{-1}E_f, \ldots$ eventually cover $X$. Indeed as $f$ is bounded from above for any $x \in X$, the series $f(T^i x), i = 1, 2, \ldots$ cannot increase indefinitely.

\[ \square \]

6. An Embedding Theorem

6.1. The Baire Category Framework. We are interested in the question under which conditions a topological dynamical system $(X, T)$ is embeddable in the $d$-cubical shift $(([0, 1]^d)^{\mathbb{Z}}, \text{shift})$ for some $d \in \mathbb{N}$. Notice that a continuous function $f : X \to [0, 1]^d$ induces a continuous $\mathbb{Z}$-equivariant mapping $I_f : (X, T) \to (([0, 1]^d)^{\mathbb{Z}}, \text{shift})$ given by $x \mapsto (f(T^k x))_{k \in \mathbb{Z}}$, also known as the orbit-map. Conversely, any $\mathbb{Z}$-equivariant continuous factor map $\pi : (X, T) \to (([0, 1]^d)^{\mathbb{Z}}, \text{shift})$ is induced in this way by $\pi_0 : X \to [0, 1]^d$, the projection on the zeroth coordinate. We therefore study the space of continuous functions $C(X, [0, 1]^d)$. Instead of explicitly constructing a $f \in C(X, [0, 1]^d)$ so that $I_f : (X, T) \hookrightarrow (([0, 1]^d)^{\mathbb{Z}}, \text{shift})$ is an embedding, we show that the property of being an embedding $I_f : (X, T) \hookrightarrow (([0, 1]^d)^{\mathbb{Z}}, \text{shift})$ is generic in $C(X, [0, 1]^d)$ (but without exhibiting an explicit embedding). To make this precise introduce the following definition:

**Definition 6.1.** Let $K \subset (X \times X) \setminus \Delta$ be a compact set and $f \in C(X, [0, 1]^d)$.

We say that $I_f$ is $K$-compatible if for every $(x, y) \in K$, $I_f(x) \neq I_f(y)$, or equivalently if for every $(x, y) \in K$, there exists $n \in \mathbb{Z}$ so that $f(T^n x) \neq f(T^n y)$. Define:

$$D_K = \{ f \in C(X, [0, 1]^d) | I_f \text{ is K-compatible} \}$$

By Lemma A.2 of [Gut12], $D_K$ is open in $C(X, [0, 1]^d)$. Under the assumptions of the theorem below we will show there exists a closed countable cover $K$ of $(X \times X) \setminus \Delta$ so that $D_K$ is dense for all $K \in \mathcal{K}$. By the Baire Category Theorem $(C(X, [0, 1]^d), \| \cdot \|_\infty)$, is a Baire space, i.e., a topological space
where the intersection of countably many dense open sets is dense. This implies \( \bigcap_{K \in \mathcal{K}} D_K \) is dense in \((C(X, [0, 1]^d), \| \cdot \|_\infty)\). Any \( f \in \bigcap_{K \in \mathcal{K}} D_K \) is \( K \)-compatible for all \( K \in \mathcal{K} \) simultaneously and therefore induces an embedding \( I_f : (X, T) \hookrightarrow ([0, 1]^d)^\mathbb{Z} \), \( shift \). A set in a topological space is said to be comeagre or generic if it is the complement of a countable union of nowhere dense sets. A set is said to be \( G_\delta \) if it is the countable intersection of open sets. As a dense \( G_\delta \) set is comeagre, the above argument shows that the set \( A \subset C(X, [0, 1]^d) \) for which \( I_f : (X, T) \hookrightarrow ([0, 1]^d)^\mathbb{Z} \) is an embedding is comeagre, or equivalently, that the fact of \( I_f \) being an embedding is generic in \((C(X, [0, 1]^d), \| \cdot \|_\infty)\).

6.2. The Embedding Theorem.

**Theorem 6.2.** Assume \((X, T)\) has the local marker property. Let \( d \in \mathbb{N} \) be such that \( m\dim(X, T) < \frac{d}{36} \) and \( p\dim(X, T) < \frac{d}{27} \), then the collection of continuous functions \( f : X \to [0, 1]^d \) so that \( I_f : (X, T) \hookrightarrow ([0, 1]^d)^\mathbb{Z} \), \( shift \) is an embedding is comeagre in \( C(X, [0, 1]^n) \).

**Proof.** As explained in Subsection 6.1 we need to exhibit a closed countable cover \( \mathcal{C} \) of \((X \times X) \setminus \Delta\) so that \( D_C \) is dense for all \( C \in \mathcal{C} \). By Proposition 5.5 \((X, T)\) has the local strong topological Rokhlin property. By Proposition 6.3 below one can cover \((X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X))\) by a closed countable cover \( \mathcal{W} \) so that for all \( W \in \mathcal{W}, D_W \) is dense in \( C(X, [0, 1]^d) \). Let \( P_n \) denote the set of points of period \( \leq n \) and define \( H_n = P_n \setminus P_{n-1} \). By Proposition 6.4 below for every \( n \in \mathbb{N} \) there is a countable closed cover \( \mathcal{K}_n \) of \((X \setminus P) \times H_n \cup (H_n \times (X \setminus P))\) so that for all \( K \in \mathcal{K}_n, D_K \) is dense in \( C(X, [0, 1]^d) \). By the proof of Theorem 4.1 of [Gut12] there is closed countable cover \( \mathcal{P} \) of \((P \times P) \setminus \Delta\) so that \( D_K \) is dense for all \( K \in \mathcal{K} \). Let \( \mathcal{C} = \mathcal{W} \cup \mathcal{P} \cup \bigcup_n \mathcal{K}_n \). As \((X \times X) \setminus \Delta\) is the union of \((X \times X) \setminus (\Delta \cup (X \times P) \cup (P \times X))\), \( \bigcup_n ((X \setminus P) \times H_n \cup (H_n \times (X \setminus P)) = ((X \setminus P) \times P) \cup (P \times (X \setminus P)) \)
and \((P \times P) \setminus \Delta\), clearly \(C\) has the desired properties. We now proceed to prove Proposition 6.3 and Proposition 6.4. Throughout, it turns out to be convenient to define:

\[
mdim = \begin{cases} 
\frac{1}{72} & \text{if } mdim(X, T) = 0 \\
mdim(X, T) & \text{otherwise}
\end{cases}
\]

Notice it holds \(36mdim < d\).  

**Proposition 6.3.** Let \(K\) be a countable closed cover of \(X \times X \setminus (\Delta \cup (X \times P) \cup (P \times X))\) which has the local marker property, then for all \(K \in K\), \(D_K\) is dense in \(C(X, [0, 1]^d)\).

**Proof.** This proof is heavily influenced by the proof of Theorem 5.1 of [Lin99]. Fix \(K \in K\) with \(K = Z \times W\) with \(Z, W\) closed and \(Z \cap W = \emptyset\). Fix \(\epsilon > 0\). Let \(\tilde{f} : X \to [0, 1]^d\) be a continuous function. We will show that there exists a continuous function \(f : X \to [0, 1]^d\) so that \(\|f - \tilde{f}\|_{\infty} < \epsilon\) and \(I_f\) is \(K\)-compatible. We start by a general construction and then relate it to \(\tilde{f}\). Let \(\delta = dist(Z, W) > 0\). Let \(\alpha\) be a cover of \(X\) with \(\max_{U \in \alpha} \text{diam}(\tilde{f}(U)) < \frac{\epsilon}{2}\) and \(\max_{U \in \alpha} \text{diam}(U) < \delta\). Let \(\epsilon' > 0\) be such that \(36mdim(1 + 2\epsilon') < d\). Let \(N \in \mathbb{N}\) be such that it holds \(\frac{1}{N}D(\alpha^{N+1}) < (1 + \epsilon')mdim\) (here we use \(mdim > 0\) and Remark 2.5.1 of [Gut12]), \(36(1 + 2\epsilon') < \epsilon'N\) and \(N\) is divisible by 36. Let \(\gamma \succ \alpha^{N+1}\) be an open cover so that \(D(\alpha^{N+1}) = \text{ord}(\gamma)\). We have thus \(\text{ord}(\gamma) < N(1 + \epsilon')mdim\). Let \(M = \frac{2}{9}N\) and \(\Delta = \frac{M}{N} - 1\). Notice \(M, \Delta \in \mathbb{N}\). Notice \(\Delta d > (\frac{N}{90} - 1)36mdim(1 + 2\epsilon') = Nmdim(1 + \epsilon') + mdim(N\epsilon' - 36(1 + 2\epsilon')) > Nmdim(1 + \epsilon')\). Conclude:

\[
\text{ord}(\gamma) < \Delta d
\]

For each \(U \in \gamma\) choose \(q_U \in U\) so that \(\{q_U\}_{U \in \gamma}\) is a collection of distinct points in \(X\), and define \(\tilde{v}_U = (\tilde{f}(T^i q_U))_{i=0}^{N-1}\). According to Lemma 5.6 of...
one can find a continuous function $F : X \to ([0,1]^d)^N$, with the following properties:

1. $\forall U \in \gamma$, $\|F(q_U) - \tilde{v}_U\|_\infty < \frac{\epsilon}{2}$,
2. $\forall x \in X$, $F(x) \in \text{co}\{F(q_U) | x \in U \} \in \gamma$,
3. If for some $0 \leq l, j < N - 4\Delta$ and $\lambda, \lambda' \in (0,1]$ and $x, y, x', y' \in X$ so that:

$$
\lambda F(x)_{l+1}^{l+4\Delta-1} + (1 - \lambda)F(y)_{l+1}^{l+4\Delta} = \lambda' F(x')_{j+1}^{j+4\Delta-1} + (1 - \lambda')F(y')_{j+1}^{j+4\Delta}
$$

then there exist $U \in \gamma$ so that $x, x' \in U$ and $l = j$ (note the statement $l = j$ is missing from Lemma 5.6 of [Lin99] but follows from the proof).

By Proposition 5.5, $(X, T)$ has the local strong topological Rokhlin property. By Definition 5.3 one can find a continuous function $n : \bigcup_{-4M}^{M-1} T^i(Z \cup W) \to \mathbb{R}$ so that for $E_n = \{x \in \bigcup_{-4M}^{M-2} T^i(Z \cup W) | n(Tx) \neq n(x) + 1\}$ one has $E_n \cap T^i(E_n) = \emptyset$ for $1 \leq i \leq \frac{\omega}{2}M - 1$. Let $\underline{n}(x) = [n(x)] \mod M$, $\overline{n}(x) = [n(x)] \mod M$, $n'(x) = \{n(x)\}$. Let $A = \bigcup_{-4M}^{M-1} T^i(Z \cup W)$. Define:

(6.1) $f'(x) = (1 - n'(x)F(T^{-\underline{n}(x)}x)\underline{n}(x) + n'(x)F(T^{-\overline{n}(x)}x))\overline{n}(x)$ \quad $x \in A$

$f'$ is continuous by the argument appearing on p. 241 of [Lin99]. By the argument of Claim 1 on p. 241 of [Lin99], as $\max_{U \in \gamma} \|F(q_U) - \tilde{v}_U\|_\infty < \frac{\epsilon}{2}$ we have $\|f - \tilde{f}\|_{A, \infty} < \epsilon$. By Lemma A.5 of [Gut12] there is $f : X \to [0,1]^d$ so that $f|_A = f'|_A$ and $\|f - \tilde{f}\|_\infty < \epsilon$. We now show that $f \in D_K$. Fix $x' \in Z$ and $y' \in W$. Assume for a contradiction $f(T^a x') = f(T^a y')$ for all $a \in \mathbb{Z}$. Notice that by Remark 5.3 for both $x', y'$ there is at most one index $-4M + 1 \leq j_{x'}, j_{y'} \leq \frac{M}{2}$ for which $n(T_{j_{x'}+1} x') \neq n(T_{j_{x'}+1} x') + 1$, $n(T_{j_{y'}+1} y') \neq n(T_{j_{y'}+1} y') + 1$ respectively. By Lemma A.1 one can find an index $-4M \leq r \leq 0$ so that for $r \leq s \leq r + \frac{M}{2}$,
for $z' = x', y'$, $\bar{n}(T^n z') = \bar{n}(T^n z') + s - r$, $\overline{\pi}(T^n z') = \pi(T^n z') + s - r$ and $\bar{n}(T^n z') \leq \frac{M}{r}$. Denote $\lambda = n'(T^n x')$, $\lambda' = n'(T^n y')$, $a = n(T^n x') \leq \frac{M}{n}$ and $a' = n(T^n y')$. Substituting $T^n x', T^n y'$ for $r \leq s \leq r + 4\Delta - 1 = r + \frac{M}{2} - 5$ in equation (6.1) (note $T^n x', T^n y' \in A$), we conclude from the equality $I_f(x')|_{r + \frac{M}{2} - 5} = I_f(y')|_{r + \frac{M}{2} - 5}$:

$$(1 - \lambda) F(T^{r-a} x')|_{a+4\Delta-1}^{a+4\Delta} + \lambda F(T^{r-a} y')|_{a'+4\Delta-1}^{a'+4\Delta} = (1 - \lambda') F(T^{r-a'} y')|_{a'+4\Delta-1}^{a'+4\Delta}$$

E.g. notice that for $0 \leq i \leq \frac{M}{2} - 5$ it holds that $T^{-a}(T^{r+i} x')T^{r+i} x' = T^{-a} x'$. As the conditions of Lemma 5.6 of [Lin99] are fulfilled then by condition (3), one has that $a = a'$ and that there exist $U \in \gamma > \alpha^{N+1}$ so that $T^{r-a} x', T^{r-a} y' \in U$. As $N = -4M - \frac{M}{2} \leq r - a \leq 0$ we can find $V \in \alpha$, so that $x', y' \in V$. This is a contradiction to $\max_{U \in \alpha} \text{diam}(U) < \text{dist}(Z, W) = \delta$.

\[\square\]

**Proposition 6.4.** Assume $(X, T)$ has the local strong topological Rokhlin property and let $n \in \mathbb{N}$, then there is a countable closed cover $K$ of $(X \setminus P) \times H_n$ so that for $K \in \mathcal{K}$, $D_K$ is dense in $C(X, [0, 1]^d)$.

**Proof.** Let $\mathcal{C}$ be a cover of $(X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ with the local strong topological Rokhlin property. Cover $H_n$ by a countable collection. Let $W$ be an open set in $H_n$ (not necessarily open in $X$) with $y \in W \subset \overline{W} \subset H_n$. Let $Z \times R \in \mathcal{C}$. Fix $\epsilon > 0$. Let $\tilde{f} : X \to [0, 1]^d$ be a continuous function.

We will show that there exists a continuous function $f : X \to [0, 1]^d$ so that $\|f - \tilde{f}\|_{\infty} < \epsilon$ and $I_f$ is $K$-compatible for $K = Z \times \overline{W}$. Let $\alpha$ be a cover of $X$ with $\max_{U \in \alpha} \text{diam}(\tilde{f}(U)) < \min\{\frac{\epsilon}{2}, d(Z, P_n)\}$. Let $\epsilon' > 0$ be such that $36m_{\text{dim}}(1 + 2\epsilon') < d$. We will see it is enough to assume $18m_{\text{dim}}(1 + 2\epsilon') < d$ (actually it is enough to assume $8m_{\text{dim}}(1 + 2\epsilon') < d$ but we will not use this fact). Let $N \in \mathbb{N}$, divisible by 18, be such that it holds...
\[ \frac{1}{N} D(\alpha^N) < (1 + \epsilon') m_{\dim} \text{ and } N\epsilon' - 9n(1 + 2\epsilon') > \frac{1}{m_{\dim}}. \]

Let \( \gamma \succ \alpha^N \) be an open cover so that \( D(\alpha^N) = \ord(\gamma) \). Let \( M = \frac{2}{3} N \) and \( S = \frac{M}{3} \).

Notice
\[
(S - \frac{n}{2})d > (\frac{N}{3} - \frac{n}{2})18 m_{\dim}(1 + 2\epsilon') = N m_{\dim}(1 + \epsilon') + m_{\dim}(N\epsilon' - 9n(1 + 2\epsilon')) > 1 + N m_{\dim}(1 + \epsilon').
\]

As \( \ord(\gamma) < N(1 + \epsilon') m_{\dim} \), conclude:

\[
\ord(\gamma) + 1 < (S - \frac{n}{2})d
\]

For each \( U \in \gamma \) choose \( q_U \in U \) so that \( \{q_U\}_{U \in \gamma} \) is a collection of distinct points in \( X \), and define \( \tilde{v}_U = (\tilde{f}(T^i q_U))_{i=0}^{N-1}. \) According to Lemma \( \ref{lem:continuous_function} \) one can find a continuous function \( F : X \to ([0,1]^d)^N \), with the following properties:

1. \( \forall U \in \gamma, \| F(q_U) - \tilde{v}_U \|_{\infty} < \frac{\epsilon}{2} \).
2. \( \forall x \in X, F(x) \in \co \{ F(q_U) | x \in U \in \gamma \} \).
3. For any \( 0 \leq l,j < N - 2S \), and \( \lambda \in (0,1] \) and \( x,y \in X \) with \( d(x,y) > \max_{W \in \gamma} \text{diam}(W) \) it holds:

\[
(1 - \lambda) F(x)_{l}^{l+2S-1} + \lambda F(y)_{l+1}^{l+2S} \notin V_{2S}^n
\]

where,

\[
V_{2S}^n \triangleq \{ w = (w_0, \ldots, w_{2\Delta - 1}) \in ([0,1]^d)^{2\Delta} | \forall 0 \leq a,b \leq 2S - 1, a = b \mod n \Rightarrow w_a = w_b \}.
\]

By Definition \( \ref{def:continuous_function} \) one can find a continuous function \( n : \bigcup_{-4M}^{M-1} T^i Z \to \mathbb{R} \) so that for \( E_n = \{ x \in \bigcup_{-4M}^{M-1} T^i Z | n(Tx) \neq n(x) + 1 \} \) one has \( E_n \cap T^i( E_n ) = \emptyset \) for \( 1 \leq i \leq \frac{M}{2} - 1 \). Let \( \bar{n}(x) = \lfloor n(x) \rfloor \mod M, \bar{m}(x) = \lceil n(x) \rceil \mod M, \)
\( n'(x) = \{ n(x) \}. \) Let \( A = \bigcup_{-4M}^{M-1} T^i Z, \) Define:

\[
(6.2) \quad f'(x) = (1 - n'(x)F(T^{-\bar{n}(x)} x)|_{\bar{n}(x)} + n'(x)F(T^{-\bar{m}(x)} x)|_{\bar{m}(x)}) x \in A
\]
$f'$ is continuous by the argument appearing on p. 241 of [Lin99]. By the argument of Claim 1 on p. 241 of [Lin99], as $\max_{U \in \alpha} \|F(q_U) - \tilde{v}_U\|_\infty < \epsilon$ and $\max_{U \in \gamma} \|f' - \tilde{f}\|_\infty < \epsilon$ we have $\|f' - \tilde{f}\|_A < \epsilon$. By Lemma A.5 of [Gut12] there is $f : X \to [0,1]^d$ so that $f|_A = f'|_A$ and $\|f - \tilde{f}\|_\infty < \epsilon$. We now show that $f \in D_K$. Fix $x' \in Z$ and $y' \in \overline{W}$. Assume for a contradiction $f(T^{a}x') = f(T^{a}y')$ for all $a \in \mathbb{Z}$. Notice that by Remark [5.4] there is at most one index $-4M \leq j_{x'} \leq \frac{M}{2} - 2$ for which $n(T^{j_{x'}+1}x') \neq n(T^{j_{x'}x'}) + 1$. By Lemma [A.1] one can find an index $-4M \leq r \leq 0$ so that for $r \leq s \leq r + \frac{M}{2} - 1$, $n(T^{s}x') = n(T^{r}x') + s - r$ and $\pi(T^{s}x') = \pi(T^{r}x') + s - r$. Denote $\lambda = n'(T^{r}x')$ and $a = (T^{r}x')$. Substituting $T^{s}x'$ for $r \leq s \leq r + 2S - 1 = r + \frac{M}{2} - 1$ in equation (6.2) (note $T^{s}x' \in A$), we conclude from the equality $I_f(x')^{r + \frac{M}{2} - 1} = I_f(y')^{r + \frac{M}{2} - 1}$ (compare with the analogue part in the proof of Proposition [6.3]):

$$(1 - \lambda)F(T^{r-a}x')^{a+2S-1} + \lambda F(T^{r-a-1}x')^{a+2S} = I_f(y')^{r+2S-1}$$

As $y' \in \overline{W} \subset H_n$, one clearly has $I_f(y')^{r+2S-1} \in V^n_{2S}$. This is a contradiction to property $(3)$. 

\[\Box\]

7. Applications

**Theorem 7.1.** Assume $(X,T)$ is an extension of an aperiodic t.d.s which either is finite-dimensional or has a countable number of minimal subsystems or has a compact minimal subsystems selector. Then $(X,T)$ has the strong Rokhlin property. If in addition $d \in \mathbb{N}$ is such that $\overline{d} \dim(X,T) < \frac{d}{36}$, then the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X,T) \hookrightarrow (([0,1]^d)^Z, \text{shift})$ is an embedding is comeagre in $C(X,[0,1]^d)$.  

Proof. In those cases, by Theorems 3.5, 3.9 as well as Theorem 6.1 of \cite{Gut12}, $(X,T)$ has the marker property. We can therefore conclude by Theorem 5.7 that $(X,T)$ has the strong Rokhlin property. By Theorem 6.2, as $(X,T)$ is aperiodic the second part of the theorem holds.\qed

Lemma 7.2. $\text{mdim}(X,T) = \text{mdim}(\Omega(X), T)$

Proof. Clearly $\text{mdim}(X,T) \geq \text{mdim}(\Omega(X), T)$. To see the reversed inequality let $Y = X/\Omega(X)$ (i.e. the quotient space where the closed and $T$-invariant subspace $\Omega(X)$ is identified with a point) and let $\pi : (X,T) \to (Y,T')$ be the quotient map, where $T'$ is the induced transformation. Note that $\pi_{X\setminus\Omega(X)}$ is injective. We can therefore use Theorem 4.6 of \cite{Tsu08} in order to conclude $\text{mdim}(X,T) \leq \text{mdim}(Y,T') + \text{mdim}(\Omega(X), T)$. As $\pi(\Omega(X)) \simeq \{\bullet\}$ is the only closed invariant subsystem of $Y$, it holds $h_{top}(Y,T') = 0$ which implies $\text{mdim}(Y,T') = 0$ (see Subsection 2.8). We therefore conclude $\text{mdim}(X,T) \leq \text{mdim}(\Omega(X), T)$ as desired.\qed

Theorem 7.3. Let $(X,T)$ be a t.d.s so that $\Omega(X)$ is finite dimensional, the set of periodic points $P(X,T)$ is closed and $\text{perdim}(X,T) < \frac{d}{2}$. Then the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X,T) \hookrightarrow (([0,1]^d)^Z, \text{shift})$ is an embedding is comeagre in $C(X,[0,1]^d)$.

Proof. By Lemma 7.2 as $\Omega(X)$ is finite dimensional, $\text{mdim}(X,T) = \text{mdim}(\Omega(X), T) = 0$. By Theorem 4.4 $(\Omega(X),T)$ has the local marker property. By Proposition 4.5 $(X,T)$ has the local marker property. Combining all of these facts, we conclude by Theorem 6.2 that the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X,T) \hookrightarrow (([0,1]^d)^Z, \text{shift})$ is an embedding is comeagre in $C(X,[0,1]^d)$.\qed

Recall the Lindenstrauss-Tsukamoto Conjecture from the Introduction.
Corollary 7.4. Let \((X,T)\) be a t.d.s so that \(\Omega(X)\) is finite dimensional and the set of periodic points \(P(X,T)\) is closed, then the Lindenstrauss-Tsukamoto Conjecture holds for \((X,T)\).

Proof. It is sufficient to notice that \(mdim(X,T) = 0\) (as pointed out in the proof of Theorem 7.3) and apply Theorem 7.3. □

Example 7.5. We now construct a family of examples for which the previous theorem is applicable. Let \(R : [0,1] \to [0,1]\) be a continuous invertible map such that \(R(0) = 0, R(1) = 1\) and such there are no other fixed points. It easily follows that for all \(0 < x < 1\), \(\lim_{n \to \infty} R^n(x) = 1\), \(\lim_{n \to -\infty} R^n(x) = 0\) or \(\lim_{n \to \infty} R^n(x) = 0, \lim_{n \to -\infty} R^n(x) = 1\), e.g. \(R(x) = \sqrt{x}, x^2\). Let \(Q = [0,1]^\mathbb{N}\), be the Hilbert cube, equipped with the product topology. Define \(R : Q \to Q\), by \(R((x_i)_{i=1}^\infty) = (R(x_i))_{i=1}^\infty\). It is easy to see \(\Omega(Q,R) = \{0,1\}^\mathbb{N}\).

Let \((Y,S)\) be a finite dimensional t.d.s with a closed set of periodic points. It follows easily that \(\Omega(Y \times Q, S \times R) = \Omega(Y,S) \times \Omega(Q,R) = \Omega(Y,S) \times \{0,1\}^\mathbb{N}\). As \(\{0,1\}^\mathbb{N}\) is zero-dimensional and \(\Omega(Y,S) \subset Y\), we conclude \(\Omega(Y \times Q, S \times R)\) is finite-dimensional. Moreover \(P(Y \times Q, S \times R) = P(Y,S) \times \{0,1\}^\mathbb{N}\), which is closed. We have thus verified all prerequisites that enable us to apply the previous theorem for the infinite-dimensional system \((Y \times Q, S \times R)\). Additionally notice that as \(\{0,1\}^\mathbb{N}\) consists of fixed points of \((Q,R)\) and is zero-dimensional, \(\overrightarrow{perdim}(Y \times Q, S \times R) = \overrightarrow{perdim}(Y,S)\).

8. The Equivalence of SBP and Vanishing Mean Dimension Under the Marker Property

Recall the definition of \(mdim_d(X,T)\) in Subsection 2.8

Theorem 8.1. If \((X,T)\) has the marker property then there is a compatible metric \(d'\) such that \(mdim(X,T) = mdim_{d'}(X,T)\).
Proof. This is a straightforward generalization of Theorem 4.3 of \cite{Lin99}, which is the statement that the conclusion of the theorem holds if the system has an aperiodic minimal factor. \qed

As a corollary of the previous theorem we have the following theorem:

**Theorem 8.2.** Assume \((X, T)\) is an extension of an aperiodic t.d.s which either is finite dimensional or has a countable number of minimal subsystems or has a compact minimal subsystems selector then there is a compatible metric \(d'\) such that \(mdim(X, T) = mdim_{d'}(X, T)\).

**Theorem 8.3.** If \((X, T)\) has the marker property then the following conditions are equivalent:

1. \(mdim(X, T) = 0\)
2. \((X, T)\) has the small boundary property (SBP)
3. \((X, T) = \lim_{i \in \mathbb{N}} (X_i, T_i)\) where \(h_{top}(X_i, T_i) < \infty\) for \(i \in \mathbb{N}\).

Proof. (a) \(\Rightarrow\) (b) is straightforward generalization of Theorem 6.2 of \cite{Lin99}, which is the statement that \((a) \Rightarrow (b)\) holds if the system has an aperiodic minimal factor. (c) \(\Rightarrow\) (a) follows from Proposition 2.8 of \cite{LW00} (this implication is true for any system). (b) \(\Rightarrow\) (a) is Theorem 5.4 of \cite{LW00} (this implication is true for any system). (a) \(\Rightarrow\) (c) is straightforward generalization of Proposition 6.14 of \cite{Lin99}, which is the statement that \((a) \Leftrightarrow (c)\) holds if the system has an aperiodic minimal factor. \qed

As a corollary of the previous theorem we have the following theorem:

**Theorem 8.4.** Assume \((X, T)\) is an extension of an aperiodic t.d.s which either is finite dimensional or has a countable number of minimal subsystems or has a compact minimal subsystems selector then then \(mdim(X, T) = 0\) if\(f\) \((X, T)\) has the small boundary property if\(f\) \(f\) \((X, T) = \lim_{i \in \mathbb{N}} (X_i, T_i)\) where \(h_{top}(X_i, T_i) < \infty\) for \(i \in \mathbb{N}\).
9. A CHARACTERIZATION OF THE SMALL BOUNDARY PROPERTY
THROUGH ISOMORPHIC EXTENSIONS

The following section answers a question of Tomasz Downarowicz in the affirmative (see Theorem 9.3). In this section (and only in this section) a topological dynamical system (t.d.s) \((X,T)\) consists of a compact metric space \(X\) and a continuous transformation (not necessarily invertible) \(T : X \to X\). We denote by \(M_T(X)\) the set of \(T\)-invariant Borel (probability) measures and by \(\mathcal{B}_\mu\) the \(\sigma\)-algebra of Borel sets completed with respect to \(\mu\). A measure preserving system (m.p.s) is a quadruple \((Y,\mathcal{C},\nu,T)\) where \((Y,\mathcal{C},\nu)\) is a probability space and \(T : (Y,\mathcal{C},\nu) \to (Y,\mathcal{C},\nu)\) is a measure preserving transformation.

**Definition 9.1.** \(\pi : (Z,S) \to (X,T)\) is an isomorphic extension iff for any \(\mu \in M_S(Z)\), \(\pi : (Z,S,\mathcal{B}_\mu,\mu) \to (X,T,\mathcal{B}_{\pi_*\mu},\pi_*\mu)\) is an m.p.s isomorphism.

The following trivial proposition is included for completeness.

**Proposition 9.2.** If \(D\) is closed then \(\partial \overset{o}{D} \subset \partial D\).

**Proof.** Note \(\partial \overset{o}{D} = \overline{D} \cap (\overset{o}{D})^c\) and \(\partial D = \overline{D} \cap \overline{D^c}\). Therefore it is enough to show \((\overset{o}{D})^c \subset \overline{D^c}\). Indeed if \(y \in (\overset{o}{D})^c\) and \(U\) is an open set such that \(y \in U\), then \(U \cap D^c = \emptyset\) would imply \(y \in U \subset D\) which would imply \(y \in U \subset \overset{o}{D}\), contradicting \(y \in (\overset{o}{D})^c\). \(\square\)

**Theorem 9.3.** Let \((X,T)\) be a t.d.s. \((X,T)\) has a zero-dimensional isomorphic extension iff \((X,T)\) has the small boundary property.

**Proof.** Assume \((X,T)\) has SPB. For every \(n \in \mathbb{N}\) one can choose a partition of \(X\) (into measurable sets), \(\mathcal{P}^n = \{P^1_n, \ldots, P^n_m\}\) so that \(\text{diam}(P^1_n) < \frac{1}{n}\) and \(\text{o cap}(\partial P^n_k) = 0\) for all \(k\). Let \(M\) denote the following symbolic matrix system (this terminology is introduced in Section 4 of [Dow01]): Any element
$m \in M$ consists of a two dimensional infinite matrix $m = [m(n,i)]_{n,i \in \mathbb{N}}$, where for each $n$ the sequence $m_n = [m(n,i)]_{i \in \mathbb{N}}$, which is called the $n$-th row of $M$, is an element in $\{1, \ldots, m_n\}^\mathbb{N}$. $M$ is equipped with product topology which makes it a compact metric space. One defines the following left-shift action $S : M \to M$ by $Sm = [m(n,i+1)]_{n,i \in \mathbb{N}}$. For any $x \in X$, we associate an element $m_x \in M$, given by the formula $m_x(n,i) = \mathcal{P}^n(T^i x)$, where $\mathcal{P}^n(T^i x) = k$ iff $\mathcal{T}^i x \in P_k^n$. Let $Z = \{m_x \vert x \in X\} \subset M$. Notice $Z$ is closed and $S$-invariant. There is a natural morphism $\pi : (Z,S) \to (X,T)$. Indeed for any $z = [z(n,i)]_{n,i \in \mathbb{N}} \in Z$ define $\pi(z) = \bigcap_{n,i \in \mathbb{N}} T^{-i} \mathcal{P}^n z(n,i)$. As $\text{diam}(\mathcal{P}_k^n) < \frac{1}{n}$ for all $n,k$ the intersection can be at most one point. For $x \in X$, $\pi(m_x) = x$ and by choosing $x_k \to z$ for any $z \in Z$, we see that the intersection defining $\pi(z)$ is non-empty for all $z \in Z$. $Z$ is clearly zero-dimensional. We claim $\pi : (Z,S) \to (X,T)$ is an isomorphic extension. Fix $\mu \in M_S(Z)$. Let $O = \{x \in X \vert |\pi^{-1}(x)| > 1\}$. It is enough to show $\pi_*^\mu(O) = 0$ because this implies $\pi$ is injective on a set of $\mu$-measure 1. Note that $O \subset \bigcup_{n \in \mathbb{N}, 1 \leq i \leq m_n} \mathcal{T}^{-i} \partial \mathcal{P}_i^n$. As $\pi_*^\mu(\partial \mathcal{P}_i^n) = 0$ for all $n,i$ (see Subsection 2.7), the result follows.

Now Assume $\pi : (Z,S) \to (X,T)$ is a zero-dimensional isomorphic extension. Let $C\text{lop}(Z)$ be the collection of clopen sets of $Z$. Let $\mathcal{U} = \{\pi(D) \}_{D \in C\text{lop}(Z)}$. We claim $\mathcal{U}$ is a basis. Indeed let $U \subset X$ be open and $x \in U$. Using normality of $X$ choose $x \in V \subset \overline{V} \subset U$, so that $V$ is open. For any $z \in \pi^{-1}(V)$ choose a clopen set $B_z$, such that $z \in B_z \subset \pi^{-1}(U)$. Choose a finite subcover of $\pi^{-1}(V)$, $B_{z_1}, B_{z_2}, \ldots, B_{z_n}$ and define $D = \bigcup B_{z_i}$. Notice $\pi^{-1}(V) \subset D \subset \pi^{-1}(U)$ which implies $\overline{V} \subset \pi(D) \subset U$ which implies $x \in V \subset \pi(D) \subset U$.

We now show that the members of $\mathcal{U}$ have small boundaries. Let $D \in C\text{lop}(Z)$. As $\partial \pi(D)$ is closed it is enough to show for every $\mu \in M_S(Z)$, $\pi_*^\mu(\partial \pi(D)) = 0$. As $\pi : (Z,S,B_Z,\mu) \to (X,T,B_X,\pi_*^\mu)$ is an isomorphism,
it is enough to show $x \in \partial \pi(D) \Rightarrow |\pi^{-1}(x)| > 1$. Indeed notice $\pi(D) \cup \pi(D^c) = X$. Conclude $\pi(D)^c \subset \pi(D^c)$. As $x \in \partial \pi(D) = \pi(D) \cap \overline{\pi(D)^c}$ in particular, $x \in \overline{\pi(D)^c} \subset \pi(D^c)$, which implies $\pi^{-1}(x) \cap D^c \neq \emptyset$, i.e. $|\pi^{-1}(x)| > 1$ (as also $\pi^{-1}(x) \cap D^c \neq \emptyset$).

\[\square\]

Appendix.

Lemma A.1. Let $A \subset X, M \in \mathbb{N}$ be an even integer and $n : \bigcup_{i=-4M}^{M-1} T^n A \to \mathbb{R}$ a function. Assume there are $x_1, x_2 \in A$ so that for each $x_i, i = 1, 2$ there is at most one index (depending on $x_i$) $-4M \leq j_i \leq \frac{M}{2} - 2$ for which $n(T^{j_i+1}x_i) \neq n(T^{j_i}x_i) + 1$. Then one can find an index $-4M \leq r \leq 0$ so that $\lfloor n(T^r x_i) \rfloor \mod M \leq \frac{M}{2}$ and for $r \leq s \leq r + \frac{M}{2} - 2, i = 1, 2$:

(A.1) $\lceil n(T^s x_i) \rceil \mod M = \lceil n(T^r x_j) \rceil \mod M + s - r$

(A.2) $\lfloor n(T^s x_i) \rfloor \mod M = \lfloor n(T^r x_j) \rfloor \mod M + s - r$

Proof. By the proof of Lemma 5.7 of [Lin99] one can find an index $-4M \leq r \leq 0$ so that for $r \leq s \leq r + \frac{M}{2} - 1$ one has for $i = 1, 2$:

$\lfloor n(T^s x_i) \rfloor \mod M = \lfloor n(T^r x_j) \rfloor \mod M + s - r$

In particular for $r \leq s \leq r + \frac{M}{2} - 2, (n(T^s x_i) \mod M) \in [0, \frac{M}{2} + s - r + 1) \subset [0, M - 1)$ and $\lfloor n(T^r x_i) \rfloor \mod M \leq \frac{M}{2}$. We therefore conclude (A.1) and (A.2) hold for this range of indices. \[\square\]

The following Lemma is closely related to Lemma 6.5 of [Lin99].

Lemma A.2. Let $\epsilon > 0$. Let $N, n, d, S \in \mathbb{N}$ with $N > 2S$. Let $\gamma$ be an open cover of $X$ with $\text{ord}(\gamma) + 1 \leq (S - \frac{n}{2})d$. Assume $\{q_U\}_{U \in \gamma}$ is a collection of
distinct points in $X$ and $\tilde{v}_U \in (\mathbb{S})^N$ for every $U \in \gamma$, then there exists a continuous function $F : X \to ([0, 1]^d)^N$, with the following properties:

1. $\forall U \in \gamma, ||F(q_U) - \tilde{v}_U||_\infty < \frac{\epsilon}{2}$,
2. $\forall x \in X, F(x) \in \text{co}\{F(q_U) | x \in U \in \gamma\}$,
3. For any $0 \leq l < N - 2S$, and $\lambda \in (0, 1]$ and $x_0, x_1 \in X$ with $d(x_0, x_1) > \max_{W \in \gamma} \text{diam}(W)$ it holds:

\begin{equation}
(1 - \lambda)F(x_0)|_{l+2S-1} + \lambda F(x_1)|_{l+1} \notin V_{2S}^n
\end{equation}

where,

$V_{2S}^n \Delta \{ y = (y_0, \ldots, y_{2S-1}) \in ([0, 1]^d)^{2S} | \forall 0 \leq a, b \leq 2S - 1, (a = b \mod n) \to y_a = y_b \}$.

Proof. Let $\{ \psi_U \}_{U \in \gamma}$ be a partition of unity subordinate to $\gamma$ so that $\psi_U(q_U) = 1$.

Let $\tilde{v}_U \in ([0, 1]^d)^N, U \in \gamma$ be vectors that will be specified later. Define:

$F(x) = \sum_{U \in \gamma} \psi_U(x) \tilde{v}_U$

For $x \in X$ define $\gamma_x = \{ U \in \gamma | \psi_U(x) > 0 \}$. Let $\lambda_0 = 1 - \lambda, \lambda_1 = \lambda$. Write (A.3) explicitly as:

\begin{equation}
\sum_{j=0}^{1} \sum_{U \in \gamma_x_j} \lambda_j \psi_U(x_j) \tilde{v}_U|_{l+j}^{l+2S-1+j} \notin V_{2S}^n
\end{equation}

Note that $\text{dim}(V_{2S}^n) = nd$ and the left-hand side of (A.4) is a convex combination of at most $2(\text{ord}(\gamma) + 1)$ vectors. Note $d(x_1, x_2) > \max_{W \in \gamma} \text{diam}(W)$ implies $\gamma_{x_1} \cap \gamma_{x_2} = \emptyset$. As $2(\text{ord}(\gamma) + 1) + nd \leq 2Sd$ then by Lemma A.6 in [Gut12], almost surely in $([0, 1]^d)^{2(\text{ord}(\gamma) + 1)}$, (A.4) holds. As there is a finite number of constraints of the form (A.4), we can choose $\tilde{v}_U \in ([0, 1]^d)^N, U \in \gamma$ so that property (1) holds. Finally property (2) holds trivially as $F(q_U) = \tilde{v}_U$. 

\[\square\]
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