A NOTE ON INTERMEDIATE SUBFACTORS OF KRISHNAN-SUNDER
SUBFACTORS

BINA BHATTACHARYYA

Abstract. A Krishnan-Sunder subfactor $R_U \subset R$ of index $k^2$ is constructed from a permutation
biunitary matrix $U \in M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, i.e. the entries of $U$ are either 0 or 1 and both $U$ and
its block transpose are unitary. The author previously showed that every irreducible Krishnan-
Sunder subfactor has an intermediate subfactor by exhibiting the associated Bisch projection. The
author has also shown in a separate paper that the principal and dual graphs of the intermediate
subfactor are the same as those of the subfactor $R_\Gamma \subset R_H$, where $H \subset \Gamma$ is an inclusion of finite
groups with an outer action on $R$. In this paper we give a direct proof that the intermediate
subfactor is isomorphic to $R_\Gamma \subset R_H$.

1. Background and Introduction

There is a well-known way of constructing subfactors of the hyperfinite II$_1$ subfactor $R$ from certain
squares of finite-dimensional C*-algebras algebras. Since we will use this construction repeatedly,
we review it briefly. Suppose we have a square of finite-dimensional C*-algebras,

\[(1) \quad B_0 \subset B_1 \cup \cup \quad \cup \quad A_0 \subset A_1\]

along with a nondegenerate trace on $B_1$. Given any inclusion of algebras $A \subset B$ with a nonde-
generate trace on $B$, let $E^B_A$ denote the unique trace preserving conditional expectations from $B$
to $A$. The square (1) is a commuting square if $E^B_A(B_0) = A_0$. The square is symmetric if $B_1$
is linearly spanned by $B_0A_1$. There are many equivalent conditions to these; see [7], [8] for details. An
inclusion of finite-dimensional C*-algebras $A \subset B$ is connected if its Bratteli diagram is connected
(equivalently, the centers of $B$ and $A$ have trivial intersection). Assume (1) is a symmetric com-
muting square with connected inclusions, and the trace is the unique Markov trace of the inclusion
$B_0 \subset B_1$. We can construct a ladder of symmetric commuting squares:

\[(2) \quad B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots \]

\[\cup \quad \cup \quad \cup \quad \cdots \quad \cup \quad \cdots \]

\[A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \]

by iterating the basic construction to the right, i.e. do the basic construction to the right on the top row
and adjoin the Jones projection to the bottom row (see [8], [7], [10]). Then we may complete
the inclusion of algebras $\bigcup_n A_n \subset \bigcup_n B_n$ with respect to the unique trace on $\bigcup_n B_n$ to obtain a
hyperfinite II$_1$ subfactor (8).

Fix integers $k$ and $p$. We will consider squares of the form (1). Index the rows and columns of
matrices in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$ by the set $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, k\}$ in the natural way. Following the
notation in [4], we denote elements of $\{1, 2, \ldots, p\}$ with Greek letters and elements of $\{1, 2, \ldots, k\}$
with Roman letters. For $F$ in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, denote the entry of $F$ in row $(\alpha, a)$ and column
$(\beta, b)$ by $F^{\beta b}_{\alpha a}$.

1991 Mathematics Subject Classification. Primary 46L37.
Let $U$ be a permutation matrix in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, i.e., its entries are either 0 or 1. $U$ is a permutation biunitary if the block transpose $\bar{U}$ of $U$, defined by

$$U_{\beta \alpha}^{\beta'} = U_{\alpha \alpha'}^{\beta \beta'} = U_{\alpha \beta}^{\alpha' \beta'} = U_{\beta \alpha}^{\beta' \alpha'}$$

is also a permutation matrix. Equivalently, $U$ is a permutation biunitary if

$$U(1 \otimes M_k(\mathbb{C}))U^* \subset M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$$

is a commuting square. We may then construct a subfactor as described above, which we denote $R_U \subset R$. In [9], Krishnan and Sunder list all the nonequivalent subfactors of this type with $k = p = 3$ and compute the principal graphs of all the finite depth ones.

## 2. The Intermediate Subfactor

In Proposition 2.2, we show that if $U$ is a permutation biunitary then the commuting square (4) may be decomposed into two adjacent symmetric commuting squares,

$$U(1 \otimes M_k(\mathbb{C}))U^* \subset M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$$

is a commuting square. Consequently, there are permutations $\nu, \lambda$ exist permutations $(\nu, \lambda)$ in Lemma 2.1 that

$$U(1 \otimes M_k(\mathbb{C}))U^* \subset M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$$

are symmetric commuting squares.

An essential ingredient in our analysis of Krishnan-Sunder subfactors is the following result of [9].

**Lemma 2.1** (Krishnan-Sunder). If $U$ is a permutation biunitary in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, then there exist permutations $\lambda_1, \lambda_2, \ldots, \lambda_p$ in $S_k$ and permutations $\rho_1, \rho_2, \ldots, \rho_p$ in $S_p$, such that

$$U_{\alpha \alpha}^{\beta \beta} = \delta_{\beta, \rho_1(a)} \delta_{\beta, \lambda_1(a)}$$

Consequently, there are permutations $\nu_1, \nu_2, \ldots, \nu_p$ in $S_k$ and permutations $\theta_1, \theta_2, \ldots, \theta_k$ in $S_p$, such that

$$(U^*)_{\alpha \alpha}^{\beta \beta} = \delta_{\beta, \theta_1(a)} \delta_{\beta, \nu_1(a)}$$

(In Krishnan and Sunder’s notation [9], $\nu = \psi^{-1}$ and $\theta = \phi^{-1}$).

**Proposition 2.2.** If $U$ is a permutation biunitary in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, then $U(1 \otimes \Delta_k)U^* \subset M_p(\mathbb{C}) \otimes \Delta_k$, and the two small squares in (3) are symmetric commuting squares.

**Proof.** To simplify notation, in this proof we denote $\lambda_a(a)$ and $\rho_a(a)$ in Lemma 2.1 by $a(a)$ and $a(a)$, respectively. Let $\{e_{\alpha, \beta}\}$ and $\{f_{a,b}\}$ be the natural sets of matrix units for $M_p(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively. Then $\{1 \otimes f_{a,a}\}$ is a basis of $1 \otimes \Delta_k$, and

$$U(1 \otimes f_{a,a})U^* = U \left( \sum_{a=1}^{p} e_{\alpha,a} \otimes f_{a,a} \right) U^* = \sum_{a=1}^{p} e_{\alpha(a),a(a)} \otimes f_{\alpha(a),a(a)}$$

which is contained in $M_p(\mathbb{C}) \otimes \Delta_k$. Hence, $U(1 \otimes \Delta_k)U^* \subset M_p(\mathbb{C}) \otimes \Delta_k$.

Since (3) is a symmetric commuting square, it suffices to prove that the upper square is commuting and the lower square is symmetric.
A NOTE ON INTERMEDIATE SUBFACTORS OF KRISHNAN-SUNDER SUBFACTORS

The trace-preserving conditional expectation $E$ from $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$ to $M_p(\mathbb{C}) \otimes \Delta_k$, is given by

$$E(e_{\alpha,\beta} \otimes f_{a,b}) = \delta_{a,b}(e_{\alpha,\beta} \otimes f_{a,a})$$

So,

$$E(U(1 \otimes f_{a,b})U^*) = E \left( \sum_{\alpha=1}^{p} e_{\alpha(a),\beta(a)} \otimes f_{\alpha(a),\alpha(b)} \right)$$

$$= \delta_{a,b} \sum_{\alpha=1}^{p} (e_{\alpha(a),\alpha(a)} \otimes f_{\alpha(a),\alpha(a)})$$

$$= \delta_{a,b} U(1 \otimes f_{a,a})U^*$$

Therefore, the upper square of (3) is commuting.

It remains to prove that the lower square is symmetric. For any $b \in \{1, 2, \ldots, k\}$ and $\beta \in \{1, 2, \ldots, p\}$ there exists $a, \alpha$ such that $U^{\beta,\alpha}_{a,a} = 1$, because $U$ is a permutation matrix. By (1) and the fact that $\rho_a$ is a permutation,

$$e_{\alpha(a),\gamma} \otimes f_{\alpha(a),\alpha(a)} \in (U(1 \otimes \Delta_k)U^*)(M_p(\mathbb{C}) \otimes 1)$$

for all $\gamma \in \{1, 2, \ldots, p\}$. Since $\beta = a(\alpha)$ and $b = \alpha(\alpha)$ were chosen arbitrarily, $M_p(\mathbb{C}) \otimes \Delta_k$ is linearly spanned by $(U(1 \otimes \Delta_k)U^*)(M_p(\mathbb{C}) \otimes 1)$. □

Note that the inclusion $U(1 \otimes \Delta_k)U^* \subset M_p(\mathbb{C}) \otimes \Delta_k$ is not necessarily connected. Proposition 2.2 immediately implies,

**Corollary 2.3.** If $U$ is a permutation biunitary in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, then $R_U \subset R$ has an intermediate von Neumann subalgebra $R_U \subset P_U \subset R$, i.e. the subalgebra constructed from the lower symmetric commuting square in (3). In particular, if $R_U \subset R$ is irreducible then $R_U \subset P_U \subset R$ is an intermediate subfactor.

Recall $\tilde{U}$ in (3). We will show in Proposition 2.3 that $R_U \subset P_U$ and $P_U \subset R$ are dual (by symmetry so are $R_U \subset P_U$ and $P_U \subset R$) and can be constructed from a biunitary permutation matrix $\Lambda \in \Delta_R \otimes M_k(\mathbb{C})$.

Let the permutations $\lambda, \rho, \nu$, and $\theta$ be as in Lemma 2.1. Define permutation matrices $\Lambda \in \Delta_R \otimes M_k(\mathbb{C})$ and $P \in M_p(\mathbb{C}) \otimes \Delta_R$ by:

$$\Lambda^{ab} = \delta_{b,\lambda(a)} \delta_{\lambda(a),a} \quad P^{ab} = \delta_{b,\rho(a)} \delta_{\rho(a),a}$$

and permutation matrices $N \in \Delta_R \otimes M_k(\mathbb{C})$ and $\Theta \in M_p(\mathbb{C}) \otimes \Delta_R$ by:

$$N^{ab} = \delta_{b,\nu(a)} \delta_{\nu(a),a} \quad \Theta^{ab} = \delta_{b,\theta(a)} \delta_{\theta(a),a}.$$

It is easy to check that

$$U = \Theta^* \Lambda = N^* P$$

and

$$\tilde{U} = \Theta N^* = \Lambda P^*.$$

The following lemma shows that we may replace $U$ in the commuting squares that engender $R_U \subset P_U$ and $P_U \subset R$, by the simpler unitaries $N^*$ and $\Lambda$. 

Lemma 2.4. Let $\Lambda, N \in \Delta_p \otimes M_k(\mathbb{C})$ and $P, \Theta \in M_p(\mathbb{C}) \otimes \Delta_k$ be permutation matrices satisfying (7) and (3). Then

\[(9)\quad U(1 \otimes \Delta_k)U^* \subset M_p(\mathbb{C}) \otimes \Delta_k = N^*(1 \otimes \Delta_k)N \subset M_p(\mathbb{C}) \otimes \Delta_k\]

and

\[(10)\quad U(1 \otimes M_k(\mathbb{C}))U^* \subset M_p(\mathbb{C}) \otimes M_k(\mathbb{C}) \cong \Lambda(1 \otimes M_k(\mathbb{C}))\Lambda^* \subset M_p(\mathbb{C}) \otimes M_k(\mathbb{C})\]

Proof. Note that conjugation by any permutation matrix in $M_p(\mathbb{C}) \otimes \Delta_k$ (such as $P$) stabilizes $1 \otimes \Delta_k$. So substituting $N^*P$ for $U$ yields (9). Similarly, substitute $U = \Theta^*\Lambda$ in left-hand side of (10), and then conjugate the entire square by $\Theta$ to obtain the right-hand side. \(\Box\)

Proposition 2.5. $P_U \subset R$ and $R_{U^*} \subset P_U$ are dual inclusions.

Proof. Clearly $\tilde{U} = U$. So by symmetry, Lemma 2.4 implies

\[(11)\quad \tilde{U}(1 \otimes \Delta_k)\tilde{U}^* \subset M_p(\mathbb{C}) \otimes \Delta_k \cong \Lambda(1 \otimes \Delta_k)\Lambda^* \subset M_p(\mathbb{C}) \otimes \Delta_k\]

Let $e \in M_k(\mathbb{C})$ be the usual Jones projection of the Jones extension $\mathbb{C} \subset \Delta_k \subset M_k(\mathbb{C})$. Conjugation by $\Lambda$ fixes $1 \otimes e$, hence (10) is the upward basic construction of (11). Since both squares are symmetric, doing the basic contraction to the right yields dual inclusions. \(\Box\)

3. The Subgroup Subfactor

Fix a permutation biunitary $U$. Let $\nu_\alpha$, $1 \leq \alpha \leq p$, and $\theta_a$, $1 \leq a \leq k$, be the permutations defined before Lemma 2.4.

Let $\Gamma$ be the subgroup of $S_k$ generated by elements of the form $\nu_\alpha \nu_\beta^{-1}$, i.e.,

$\Gamma = \Gamma_U = \langle \nu_\alpha \nu_\beta^{-1} : \alpha, \beta \in \{1, 2, \ldots, p\} \rangle$

For each $a \in \{1, 2, \ldots, k\}$ let $H_a \subset \Gamma$ be the subgroup that fixes $a$. If $\Gamma$ acts transitively on $\{1, 2, \ldots, k\}$, then the subgroups $H_a$ are all conjugate. In this case set $H = H_1$. In general, let $\Omega$ be the set of orbits in $\{1, 2, \ldots, k\}$, and for each $r \in \Omega$ set $H_r = H_a$ for an arbitrary representative $a$ in $r$.

Remark 3.1. Krishnan and Sunder use the group $\langle \nu_\alpha^{-1} \nu_\beta : \alpha, \beta \in \{1, 2, \ldots, p\} \rangle$ instead of $\Gamma$ in [9]. However, the two groups, as well as their fixed point subgroups, are equivalent via conjugation by $\nu_\gamma$ for any $\gamma \in \{1, 2, \ldots, p\}$. We depart from [9] for notational convenience in the proof of the following theorem.
**Theorem 3.2.** Let \( \Gamma \) act on \( R \) by outer automorphisms. There is a canonical isomorphism of \( Z(P_U) \) with \( \mathbb{C}^\Omega \). If \( q_r \) is the minimal projection in \( Z(P_U) \) corresponding to \( r \in \Omega \), then \( q_r R_U \subset q_r P_U \) is isomorphic to \( R^r \subset R^{H^r} \). In particular, if \( \Gamma \) acts transitively on \( \{1, 2, \ldots, k\} \) then \( R_U \subset P_U \) is a subfactor, and \( R_U \subset P_U \) is isomorphic to \( R^U \subset R^{H^U} \).

**Proof.** By construction of \( \Gamma \), the cosets \( \nu^{-1}_a \Gamma \) are the same for all \( \alpha \in \{1, 2, \ldots, p\} \). Denote this coset by \( \Gamma' \). Given a set \( S \), let \( \Delta_S \) denote the algebra of functions \( S \to \mathbb{C} \) with pointwise multiplication and the trace \( f \mapsto \frac{1}{|S|} \sum_{s \in S} f(s) \). Denote the characteristic function of \( s \in S \) by \( x_s \). Define an inclusion map \( i: \Delta_\Gamma \to M_p(\mathbb{C}) \otimes \Delta_{\Gamma'} \) by \( i(x_g) = \sum_\alpha x_\alpha \otimes x_{\nu^{-1}_a g} \). It is straightforward to check that

\[
\Delta_\Gamma \subset i \ M_p(\mathbb{C}) \otimes \Delta_{\Gamma'} \quad U \quad U
\]

\[
\mathbb{C} \subset \ M_p(\mathbb{C}) \cup \ 2 \ 3 \ \cdots \ B
\]

is a symmetric commuting square with connected inclusions, if we take the trace on \( M_p(\mathbb{C}) \otimes \Delta_{\Gamma'} \) to be the product trace.

Then we can construct a hyperfinite \( \Pi_1 \) subfactor \( B \subset A \) by iterating the basic construction to the right in the usual way.

\[
\Delta_\Gamma \subset^G M_p(\mathbb{C}) \otimes \Delta_{\Gamma'} \subset^{G'} A_2 \subset^{G'} A_3 \subset^{G'} \cdots \ A
\]

\[
\mathbb{C} \subset \ M_p(\mathbb{C}) \cup \ 2 \ 3 \ \cdots \ B
\]

Note that the Bratteli diagram \( G \) as marked in \( \boxed{[13]} \) is the bipartite graph with even vertices labeled by \( \Gamma \), odd vertices labeled by \( \Gamma' \), and an edge for each pair \( (g, \alpha) \in \Gamma \times \{1, 2, \ldots, p\} \) going from \( g \) to \( \nu^{-1}_a g \). We denote the reflection of \( G \) by \( G' \). For each \( n \), \( B_n \subset A_n \) is isomorphic to \( M_{p^n}(\mathbb{C}) \otimes 1 \subset M_{p^n}(\mathbb{C}) \otimes \Delta_\Gamma \), where \( \tilde{\Gamma} = \Gamma \) or \( \Gamma' \) according to whether \( n \) is even or odd. We claim that \( B \subset A \) is irreducible. By Ocneanu compactness, \( B' \cap A = M_p(\mathbb{C})' \cap \Delta_\Gamma = \Delta_\Gamma \cap (1 \otimes \Delta_{\Gamma'}) \). Suppose \( \sum_{g \in \Gamma} k_g x_g \in 1 \otimes \Delta_{\Gamma'} \), where \( k_g \in \mathbb{C} \). Then

\[
\sum_{g \in \Gamma} k_g x_g = \sum_{g' \in \Gamma'} \sum_{\nu^{-1}_a g = g'} k_g (f_{\alpha, \alpha} \otimes x_{g'})
\]

Since \( \sum k_g x_g \in 1 \otimes \Delta_{\Gamma'} \), we must have that \( k_g \) is constant over all \( g \in \{\nu^{-1}_a g\}_a \). Since this is holds for all \( g' \), it follows that \( k_g \) is constant over all \( g \in \Gamma' = \Gamma \). Therefore, \( \sum k_g x_g \in \mathbb{C} \). This proves the claim.

For each \( g \in \Gamma \), let \( \mu_g \) be the automorphism of \( G \) that maps each vertex \( g' \in \Gamma \cup \Gamma' \) to \( g' g^{-1} \) and each edge \( (g', \alpha) \) to the edge \( (g' g^{-1}, \alpha) \). The morphism is well defined since the endpoints of the edge \( (g', \alpha) \) are mapped to the endpoints of \( (g' g^{-1}, \alpha) \). Moreover, \( \mu_g \) obviously preserves the trace weights of \( G \). Clearly \( g \mapsto \mu_g \) is a group action of \( G \) on \( \mathbb{K} \). Now extend \( \mu \) to the chain of Bratteli diagrams of the top row of inclusions in \( \boxed{[13]} \). For each \( n \), \( \mu \) implements an action \( \mu^n \) of \( \Gamma \) on \( A_n \) by trace preserving automorphisms. The family of actions \( \{\mu^n : \Gamma \to \text{Aut}(A_n)\}_n \) are consistent, i.e. \( \mu^n |_{A_{n-1}} = \mu^{n-1} \), and thus extend to an action of \( \Gamma \) on \( A \). We denote this action again by \( \mu \). Note that the action of \( \Gamma \) on \( A \) is outer since \( B \subset A \) is irreducible. Therefore \( B \subset A \) is isomorphic to \( R \subset R^H \) as in the statement of the theorem.

Let \( E^\Gamma \) be the group averaging maps from \( A \) onto the fixed point algebra \( A^\Gamma \). The action of \( \mu_g \) on \( A_n = M_{p^n}(\mathbb{C}) \otimes \Delta_\Gamma \) is given by \( \mu_g (F \otimes x'_g) = F \otimes x_{g' g^{-1}} \), hence \( E^\Gamma(A_n) = B_n \) for each \( n \). Therefore, \( A^\Gamma = B \).

We first assume that \( \Gamma \) acts transitively on \( \{1, 2, \ldots, k\} \). Define an inclusion \( \Delta_{\Gamma/H} \to \Delta_\Gamma \) by \( x_{gH} \mapsto \sum_{g' \in gH} x_{g'} \). Similarly define \( \Delta_{\Gamma'/H} \to \Delta_{\Gamma'} \). For each \( n \), let \( C_n = A_n^H \). Clearly \( C_n = B_n \otimes \Delta_{\Gamma/H} \)
where $\tilde{\Gamma}$ is $\Gamma$ or $\Gamma'$ according to whether $n$ is even or odd. Thus we have an intermediate chain of algebras

\[
\begin{array}{cccccccc}
\Delta_{\Gamma} & \subset & M_p(\mathbb{C}) & \otimes & \Delta_{\Gamma'} & \subset & A_2 & \subset & A_3 & \subset & \cdots & \subset & A \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cdots & & \cup \\
\Delta_{\Gamma/H} & \subset & M_p(\mathbb{C}) & \otimes & \Delta_{\Gamma'/H} & \subset & C_2 & \subset & C_3 & \subset & \cdots & \subset & C \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cdots & & \cup \\
\mathbb{C} & \subset & M_p(\mathbb{C}) & \subset & B_2 & \subset & B_3 & \subset & \cdots & \subset & B \\
\end{array}
\]

where $C = A^H$. Since the group averaging map $E^H$ from $A$ onto $A^H$ is the conditional expectation from $A_n$ onto $C_n$ for each $n$, it follows that the upper-left-most square of (14) is commuting. It is straightforward to verify that the lower-left-most square of (14) is symmetric, hence both of the left-most squares are symmetric commuting squares. For $n \geq 2$, $C_n$ contains the Jones projection of the inclusion $A_{n-2} \subset A_{n-1}$, hence the chain $(C_n)_n$ contains the Jones tower of $C_0 \subset C_1$. Moreover, the Bratteli diagram of $C_{n-1} \subset C_n$ is the transpose of $C_{n-2} \subset C_{n-1}$ for $n \geq 2$, hence by dimension considerations, the chain $(C_n)_n$ is no more than the Jones tower. Therefore, both the upper and the lower ladders are the ones obtained by iterating the basic construction in the usual way from the left-most square.

Now consider the lower left square $(\ast)$ of (14). We claim that $(\ast)$ is isomorphic to (14) via the identification of $\Gamma/H$ and $\Gamma'/H$ with $\{1,2,\ldots,k\}$, by $gH \mapsto g(1)$. Let $(\hat{a})_{\alpha}^{p} \subset \Delta_{\Gamma}$ be the minimal projections in $\Delta_{\Gamma}$, and define an isomorphism $M_p(\mathbb{C}) \otimes \Delta_{\Gamma'/H} \to M_p(\mathbb{C}) \otimes \Delta_{\Gamma}$ by $F \otimes x_{gH} \to F \otimes g(1)$. Fix $a$ and choose $f \in \Gamma$ such that $f(1) = a$. Then $N^* (1 \otimes \hat{a})N \subset M_p(\mathbb{C}) \otimes \Delta_{\Gamma}$ is the image of $\sum_{\alpha} c_{a,\alpha} \otimes x_{\nu_{\alpha}^{-1}(a)}$ is the image of $\sum_{\alpha} c_{a,\alpha} \otimes x_{\nu_{\alpha}^{-1}(a)}$. Therefore, $(\ast)$ is isomorphic to (14), and $R_U \subset P_U$ is isomorphic to $A^F \subset A^H$. This proves $R_U \subset P_U$ is isomorphic to $R^F \subset R^H$, as in the statement of the theorem.

If $\Gamma$ does not act transitively, then $N^* (1 \otimes \Delta_{\Gamma})N \subset M_p(\mathbb{C}) \otimes \Delta_{\Gamma}$ is not a connected inclusion; its connected components correspond to the orbits of $\Gamma$ in $\{1,2,\ldots,k\}$. Given an orbit $r \in \Omega$, let $q_r = N^* \sum_{\alpha \in \Omega} \hat{a}_\alpha$. Clearly $q_r$ is central in $P_U$ and $q_r R_U \subset q_r P_U$ can be obtained by iterating the basic construction on $q_r N^* (1 \otimes \Delta_{\Gamma})N \subset q_r (M_p(\mathbb{C}) \otimes \Delta_{\Gamma})$ by $\cup$. By an identical above argument (using the group $H_r$ instead of $H$), $q_r R_U \subset q_r P_U$ is isomorphic to $A^F \subset A^{H_r}$. This proves $q_r R_U \subset q_r P_U$ is isomorphic to $R^F \subset R^{H_r}$, as in the statement of the theorem. Then, $q_r R_U \subset q_r P_U$ is a subfactor for each $r$, which implies that $Z(P_U) = \bigoplus_{r \in \Omega} C_q_r$.

\[\square\]

**Corollary 3.3.** Let $H \subset \Gamma$ be any inclusion of finite groups, and let $\Gamma$ act on the hyperfinite $\Pi_1$ factor $R$ by an outer action. Let $k = |\Gamma/H|$. Suppose the action of $\Gamma$ on $\Gamma/H$ can be generated by $p'$ elements of $\Gamma$. Then there exists a permutation biunitary $U \in \Delta_{\Gamma}^{p'} \otimes M_k(\mathbb{C})$ such that $R_U \subset P_U$ (as defined in Corollary 2.3) is isomorphic to $R^F \subset R^H$.

**Proof.** Let $U_1, U_2, \ldots, U_{p'}$ be $k \times k$ permutation matrices that generate the action of $\Gamma$ on $\Gamma/H$. Set $U_0$ to be the identity matrix. Let $\{e_{\alpha}\}_{0 \leq \alpha \leq p'}$ be a basis of $\Delta_{\Gamma}^{p'+1}$, and set $U = \sum_{\alpha} e_{\alpha} \otimes U_{\alpha}$. Obviously $G_U = \Gamma$ and the fixed point subgroup of $\Gamma$’s action on $\{1,2,\ldots,k\}$ is $H$. By Theorem 3.2, this $U$ does the job. \[\square\]
4. The Bisch Projection

We now show that the Bisch projection defined in [1] (see also [3]) corresponds to the intermediate sub-von Neumann algebra $R_U \subset P_U$.

The upward basic construction of $M_p(C) \otimes 1 \subset M_p(C) \otimes M_k(C)$ in [4] is $M_p(C) \otimes M_{k^2}(C)$. The matrix rows and columns of its subalgebra $1 \otimes M_{k^2}(C)$ are indexed naturally by the set $\{1, 2, \ldots, k\} \times \{1, 2, \ldots, k\}$ ([4] ([3]). Given $x \in 1 \otimes M_{k^2}(C)$, denote by $x_{\alpha \beta}^{cd}$ the entry of $x$ in row $\{\alpha, \beta\}$ and column $\{c, d\}$. The first relative commutant of $R_U \subset R$ is the subalgebra of $1 \otimes M_{k^2}(C)$ satisfying the Ocneanu compactness condition [3].

Define $p \in 1 \otimes M_{k^2}(C)$ by

$$ p_{ab}^{cd} = \begin{cases} 1, & \text{if } a = b = c = d; \\ 0, & \text{otherwise} \end{cases} $$

Proposition 4.1. The projection $p$ defined above is contained in the first relative commutant of $R_U \subset R$; and $p$ is a Bisch projection, that is, $pndp$ implements the conditional expectation from $R$ to $(p) \cap R$ with respect to the trace.

Proof. This is proved in somewhat different notation in Lemma 6.4.1 of [1]. For the convenience of the reader we give a proof here. We first show that $p$ is in the first relative commutant using Jones’ diagrammatic formulation of the higher relative commutants of $R_U \cap R$ [8].

We claim that for $a, b \in \{1, 2, \ldots, k\}$ and $\alpha, \beta \in \{1, 2, \ldots, p\}$:

$$
\begin{array}{c}
\text{b} \\
\downarrow \\
\text{a}
\end{array}
\begin{array}{c}
\text{\beta} \\
\downarrow \\
\text{\alpha}
\end{array}
= \begin{cases} 
\delta_{\alpha, \beta} \cdot \delta_{b, b'} \cdot \delta_{a, \nu_\beta(b)}, & \text{if } a = a' \\
\delta_{\alpha, \beta} \cdot \delta_{a, a'} \cdot \delta_{a, \nu_\beta(b)}, & \text{if } b = b'
\end{cases}
$$

The claim is obvious from Section 5 of [3], but here is a direct proof. The left-hand side of (11) is by definition $\sum_{\gamma \in \{1, 2, \ldots, p\}} U_{\gamma a}^{\beta b} \gamma_{\alpha b}$. Note that in our case entries of $U$ are either 0 or 1, so $U = U$.

We have:

$$
\sum_{\gamma \in \{1, 2, \ldots, p\}} U_{\gamma a}^{\beta b} \gamma_{\alpha b} = \sum_{\gamma} \delta_{\beta, \rho_\alpha(\gamma)} \delta_{b, \lambda_\gamma(\alpha)} \delta_{\alpha, \rho_\alpha(\gamma)} \delta_{b', \lambda_\gamma(a')} = \delta_{\alpha, \beta} \delta_{b, b'} \delta_{a, \nu_\beta(a)} = \delta_{\alpha, \beta} \delta_{b, b'} \delta_{a, \nu_\beta(a)} (\text{Lemma 5 of [3]})
$$

and

$$
\sum_{\gamma \in \{1, 2, \ldots, p\}} U_{\gamma a}^{\beta b} \gamma_{\alpha b} = \sum_{\gamma} (U^*)_{\gamma a}^{\beta b} (U^*)_{\alpha b}^{\gamma a'} = \sum_{\gamma} \delta_{\gamma, \beta(\beta)} \delta_{a, \nu_\beta(b)} \delta_{\gamma, \theta_\gamma(\alpha)} \delta_{a', \nu_\beta(b')} = \delta_{\alpha, \beta} \delta_{a, a'} \delta_{a, \nu_\beta(b)} (\text{Lemma 5 of [3]})
$$

This proves the claim. Using (10), it is easy to verify that $p$ satisfies the diagrammatic condition for $p$ to be in $R_U \cap R_1$ (see Theorem 6.1.4 and the preceding discussion in [3]).

Let $q \in M_{k^2}(C)$ be the projection identified with $p$, that is, $p = 1 \otimes q \in M_p(C) \otimes M_{k^2}(C)$. Define $p_n = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes q \in M_p(C) \otimes M_{k^{n+2}}(C)$, for $n = 0, 1, 2, \ldots$. The same argument as above shows that $p_n$ is contained in $(n - 1)$st relative commutant of $R_U \subset R$. It is easy to verify that the sequence of projections $(p_n)$ along with the sequence of Jones projections $(e_n)$ of $R_U \subset R$ satisfies

the Bisch-Jones relations of the Fuss-Catalan algebras with $\alpha = \beta = k$. It follows by \[\] that $p$ implements the conditional expectation from $R$ to $\{p\}' \cap R$ with respect to the trace $\square$

**Proposition 4.2.** Let $p$ be the Bisch projection defined above and let $E_p$ be the conditional expectation from $R$ to $\{p\}' \cap R$ implemented by $p$ (by Proposition [\ref{prop:bisch_projection}]). Then $P_U = E_p(R)$.

**Proof.** Let $e$ be the Jones projection of the extension $R_U \subset R \subset R_1$. Let $(A_m)$, $(C_m)$, and $(B_m)$, $m = 0, 1, 2, \ldots$, be the chains of algebras obtained by iterating the basic construction on $\text{\ref{example:bratteli_diagram}}$ to the right, and let $D_m = (B_m, e)$:

\begin{equation}
D_0 \subset D_1 \subset D_2 \subset D_3 \subset \cdots \subset R_1
\end{equation}

\begin{equation}
U(1 \otimes M_k(\mathbb{C}))U^* \subset M_p(\mathbb{C}) \otimes M_k(\mathbb{C}) \subset A_2 \subset A_3 \subset \cdots \subset R
\end{equation}

\begin{equation}
U(1 \otimes \Delta_k)U^* \subset M_p(\mathbb{C}) \otimes \Delta_k \subset C_2 \subset C_3 \subset \cdots \subset P_U
\end{equation}

\begin{equation}
\mathbb{C} \subset M_p(\mathbb{C}) \subset B_2 \subset B_3 \subset \cdots \subset R_U
\end{equation}

The type of construction above is well-known (\[\text{\ref{example:bratteli_diagram}},\] also see Ocneanu compactness in \[\text{\ref{example:ocneanu_compactness}}\]), and in particular has the property that the chain $B_m \subset C_m \subset A_m \subset D_m$ has the same Bratteli diagram for all odd $m$. In other words, $B_m \subset C_m \subset A_m \subset D_m$ is isomorphic to $B_m \subset B_m \otimes \Delta_k \subset B_m \otimes M_k(\mathbb{C}) \subset B_m \otimes M_k(\mathbb{C})$ for all odd $m$.

By definition, $p$ is a projection in $1 \otimes M_{k^2}(\mathbb{C}) \subset D_1 = M_p(\mathbb{C}) \otimes M_{k^2}(\mathbb{C})$. Let $q \in M_{k^2}(\mathbb{C})$ be the projection identified with $p$, that is, $p = 1 \otimes q \in M_p(\mathbb{C}) \otimes M_{k^2}(\mathbb{C})$. Using (\[\ref{example:ocneanu_compactness}\]), it is easy to check:

\begin{equation}
q(M_k(\mathbb{C}) \otimes 1)q = (\Delta_k \otimes 1)q.
\end{equation}

It follows that $p \in C_1' \cap D_1$ and $pA_1p = C_1p$, hence $C_1 = E_p(A_1)$.

By Proposition \[\text{\ref{example:flat}}\], $p$ is contained in $R_1' \cap R_1$, hence $p$ is flat \[\text{\ref{example:flat}}\]. By flatness, $p$, as an element of $B_m \otimes M_{k^2}(\mathbb{C}) \cong D_m$, for $m$ odd, is identified with $1_{B_m} \otimes q \in B_m \otimes M_{k^2}(\mathbb{C})$. Recalling that $B_m \subset C_m \subset A_m \subset D_m$ is isomorphic to $B_m \subset B_m \otimes \Delta_k \subset B_m \otimes M_k(\mathbb{C}) \subset B_m \otimes M_{k^2}(\mathbb{C})$, we have by \[\text{\ref{example:ocneanu_compactness}}\] and \[\text{\ref{example:flat}}\] that $p \in C_m' \cap D_m$ and $pA_m p = C_m p$ for all odd $m$. Hence, $C_m = E_p(A_m)$ for all odd $m$. Then by weak continuity of $E_p$, $P_U = E_p(R)$. $\square$

We restate Proposition 4.2 and Proposition 3.2 as follows:

**Corollary 4.3.** The intermediate sub-von Neumann algebra of $R_U \subset R$ corresponding to the Bisch projection $p$ is $R_U \subset P_U$, as defined in Corollary \[\ref{example:corollary}\.\]

**References**

[1] Bina Bhattacharyya, *Krishnan-Sunder subfactors and a new countable family of subfactors related to trees*, Ph.D. thesis, University of California at Berkeley, 1998.

[2] ______, *Group actions on graphs related to Krishnan-Sunder subfactors*, Trans. Amer. Math. Soc. 355 (2003), no. 2, 433–463.

[3] Bina Bhattacharyya and Zeph Landau, *Intermediate standard invariants and intermediate planar algebras*, to appear.

[4] D. Bisch, *A note on intermediate subfactors*, Pacific J. Math. 163 (1994), no. 2, 201–216.

[5] D. Bisch and V. F. R. Jones, *Algebras associated to intermediate subfactors*, Invent. Math. 128 (1997), no. 1, 89–157.
[6] A. Ocneanu (Lecture Notes by Y. Kawahigashi), *Quantum symmetry, differential geometry of finite graphs and classification of subfactors*, 1990, University of Tokyo Seminar Notes.

[7] F. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter graphs and towers of algebras*, MSRI Publications, vol. 14, Springer, 1989.

[8] V. F. R. Jones and V. S. Sunder, *Introduction to subfactors*, London Mathematical Society Lecture Note Series, vol. 234, Cambridge University Press, 1997.

[9] U. Krishnan and V. S. Sunder, *On biunitary permutation matrices and some subfactors of index 9*, Trans. Amer. Math. Soc. **348** (1996), no. 12, 4691–4736.

[10] S. Popa, *Orthogonal pairs of $\ast$-subalgebras in finite von Neumann algebras*, J. Operator Theory **9** (1983), no. 2, 253–268.

Deephaven, 4699 Old Ironsides Dr., #210, Santa Clara, CA 94054, US

E-mail address: Bina\\_Bhattacharyya\\_91@post.harvard.edu