THE AVALANCHE POLYNOMIAL OF A GRAPH

DEMARA AUSTIN, MEGAN CHAMBERS, REBECCA FUNKE, LUIS DAVID GARCÍA PUENTE, AND LAUREN KEOUGH

Abstract. The (univariate) avalanche polynomial of a graph, introduced by Cori, Dar-tois and Rossin in 2006, captures the distribution of the length of (principal) avalanches in the abelian sandpile model. This polynomial has been used to show that the avalanche distribution in the sandpile model on a multiple wheel graph does not follow the expected power law function. In this article, we introduce the (multivariate) avalanche polynomial that enumerates the toppling sequences of all principal avalanches. This polynomial generalizes the univariate avalanche polynomial and encodes more information. In particular, the avalanche polynomial of a tree uniquely identifies the underlying tree. In this paper, the avalanche polynomial is characterized for trees, cycles, wheels, and complete graphs.

1. Introduction

The area of sandpile groups is a flourishing area that started in mathematical physics in 1987 with the seminal work of Bak, Tang and Wiesenfeld [4]. Since then it has found many, often unexpected, applications in diverse areas of mathematics, physics, computer science and even some applications in the biological sciences and economics. In thermodynamics, a critical point is the end point of a phase equilibrium curve. The most prominent example is the liquid-vapor critical point, the end point of the pressure-temperature curve at which the distinction between liquid and gas can no longer be made. In order to drive this system to its critical point it is necessary to tune certain parameters, namely pressure and temperature.

In nature, one can also observe different types of dynamical systems that have a critical point as an attractor. The macroscopic behavior of these systems displays the spatial and/or temporal scale-invariance characteristic of the critical point of a phase transition, but without the need to tune control parameters to precise values. Such a system is said to display self-organized criticality, a concept first introduced in 1987 by Bak, Tang and Wiesenfeld in their groundbreaking paper. This concept is thought to be present in a large variety of physical systems like earthquakes [31, 12], forest fires and in stock market fluctuations [2]. Self-organized criticality is considered to be one of the mechanisms by which complexity arises in nature [3] and has been extensively studied in the statistical physics literature during the last three decades [23, 30, 27].

In [4], Bak, Tang and Wiesenfeld conceived a cellular automaton model as a paradigm of self-organized criticality. This model is defined on a rectangular grid of cells as shown in Figure 1. The system evolves in discrete time such that at each time step a sand grain

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is dropped onto a random grid cell. When a cell amasses four grains of sand, it becomes unstable. It relaxes by toppling whereby four sand grains leave the site, and each of the four neighboring sites gets one grain. If the unstable cell is on the boundary of the grid then, depending on whether the cell is a corner or not, either one or two sand grains fall off the edge and disappear. As the sand percolates over the grid in this fashion, adjacent cells may accumulate four grains of sand and become unstable causing an avalanche. This settling process continues until all cells are stable. Then another cell is picked randomly, the height of the sand on that grid cell is increased by one, and the process is repeated.

**Figure 1.** The Bak, Tang and Wiesenfeld model of self-organized criticality.

Imagine starting this process on an empty grid. At first there is little activity, but as time goes on, the size (the total number of topplings performed) and extent of the avalanche caused by a single grain of sand becomes hard to predict. Figure 2 shows the distribution of avalanches in a computational experiment performed on a $20 \times 20$ grid. Starting with the maximal stable sandpile, i.e., the sandpile with three grains of sand at each site, a total of 100,000 sand grains were added at random, allowing the sandpile to stabilize in between. Many authors have studied the distribution of the sizes of the avalanches for this model showing that it obeys a power law with exponent $-1$ [16, 20, 24].

**Figure 2.** Distribution of avalanches $D(s)$ as a function of the size $s$.

In 1990, Dhar generalized the Bak, Tang and Wiesenfeld model replacing the rectangular grid with an arbitrary combinatorial graph [19]. In this model, known as the (abelian) sandpile model, the sand grains are placed at the vertices of the graph and the toppling threshold depends on the degree (outdegree) of each vertex. Certain conditions on the graph (the existence of a global sink vertex) ensure that any avalanche terminates after a finite
number of topplings. The sandpile model was also considered by combinatorialists as a game on a graph called the chip firing game \cite{11, 10, 9}.

The long-term behavior of the abelian sandpile model on a graph is encoded by the critical configurations, also known as recurrent sandpiles. These critical configurations have connections to parking functions \cite{7}, to the Tutte polynomial \cite{28}, and to the lattices of flows and cuts of a graph \cite{1}. Among other properties, the critical configurations have the structure of a finite abelian group. This group has been discovered in several different contexts and received many names: the sandpile group \cite{19, 22}, the critical group \cite{9}, the group of bicycles \cite{8}, the group of components \cite{26}, and the jacobian of a graph \cite{5}.

The fact that the distribution of avalanches on the grid follows a power law has been a focal point from the statistical physics perspective. A natural question is what type of distributions do we get in the more general setting introduced by Dhar. In this paper, we focus on this question. In fact, we go one step beyond just finding such distributions. We actually describe the combinatorial structure of each avalanche for certain families of graphs.

Experiments on the distribution of sizes of the avalanches have been mostly restricted to the cases of rectangular grids and some classes of regular graphs. However, very little is known for arbitrary graphs \cite{17}. In 2004, Cori, Dartois, and Rossin introduced the (univariate) avalanche polynomial that encodes the sizes of principal avalanches, that is, avalanches resulting from adding a single grain of sand to a recurrent sandpile \cite{13}. These authors completely describe the avalanche polynomials for trees, cycles, complete and lollipop graphs. Moreover, using these polynomials, they show that the resulting sandpile models on these graphs no longer obey the power law observed in the rectangular grid. In 2003, Dartois and Rossin obtained exact results for the avalanche distribution on wheel graphs \cite{17}. In 2009, Cori, Micheli and Rossin studied further properties of the avalanche polynomial on plane trees \cite{14}. In particular, they show that the avalanche polynomial of a tree does not uniquely characterize the tree. They also give closed formulas for the average and variance of the avalanche distribution on trees.

In this paper we introduce the (multivariate) avalanche polynomial, i.e., a multivariate polynomial encoding the toppling sequences of all principal avalanches. This polynomial generalizes the univariate avalanche polynomial and encodes more information. In Section 2 we describe the sandpile model on an undirected graph and introduce both the univariate and multivariate avalanche polynomials. We also present some particular evaluations of the latter polynomial. In particular, one such evaluation gives rise to the unnormalized distribution of burst sizes, that is, the number of grains of sand that fall into the sink in a principal avalanche. In Section 3 we characterize the avalanche polynomial of a tree. We also prove in Theorem 20 that this polynomial uniquely characterizes its underlying tree. In Section 4 we compute the multivariate avalanche polynomial for cycle graphs and in Section 5 we compute this polynomial for complete graphs using the bijection among recurrent sandpiles and parking functions. Our arguments fix some details in the proof of Proposition 5 in \cite{13} that enumerates the number of principal avalanches of positive sizes in the complete graph. In Section 6 we compute the avalanche polynomial for wheel graphs. Our methods simplify the arguments in \cite{17} where the authors use techniques from regular languages, automata and transducers to characterize the recurrent sandpiles and determine the exact distribution of avalanche lengths in the wheel graph.
2. The Abelian Sandpile Model

The abelian sandpile model is defined both on directed and undirected graphs, but here we focus on families of undirected multigraphs without loops.

Definition 1. An (undirected) graph $G$ is an ordered pair $(V, E)$, where $V$ is a finite set and $E$ is a finite multiset of the set of 2-element subsets of $V$. The elements of $V$ are called vertices and the elements of $E$ are called edges. Given an undirected graph $G = (V, E)$, the degree $d_v$ of a vertex $v \in V$ is the number of edges $e \in E$ with $v \in e$. For a pair of vertices $u, v \in V$, the weight $\text{wt}(u, v)$ is the number of edges between $u$ and $v$. We say $u$ and $v$ are adjacent if $\text{wt}(u, v) > 0$.

Here, we will always assume that our graphs are connected. Moreover, given a graph $G = (V, E)$ we will distinguish a vertex $s \in V$ and call it a sink. The resulting graph will be denoted $G = (V, E, s)$. We will also denote the set of all non-sink vertices by $\bar{V} = V \setminus \{s\}$.

Definition 2. A sandpile $c$ on $G = (V, E, s)$ is a function $c : \bar{V} \to \mathbb{Z}_{\geq 0}$ from the non-sink vertices of $G$ to the set of non-negative integers, where $c(v)$ represents the number of grains of sand at vertex $v$. We call $v$ unstable if $c(v) \geq d_v$. An unstable vertex $v$ can topple, resulting in a new sandpile $c'$ obtained by moving one grain of sand along each of the $d_v$ edges emanating from $v$; that is, $c'(w) = c(w) + \text{wt}(v, w)$ for all $w \neq v$ and $c'(v) = c(v) - d_v$. A sandpile is stable if $c(v) < d_v$ for every non-sink vertex $v$ and unstable otherwise.

The following proposition justifies the name “abelian sandpile model” and it was first proved by Dhar in [19].

Proposition 3. Given an unstable sandpile $c$ on a graph $G = (V, E, s)$, any sequence of topplings of unstable vertices will lead to the same stable sandpile.

Given a sandpile $c$, if $c'$ is obtained from $c$ after a sequence of sand additions and topplings, we say that $c'$ is accessible from $c$ and we call $c'$ a successor of $c$. We denote this by $c \leadsto c'$. Moreover, if $c'$ is obtained from $c$ by a sequence of topplings, then the toppling vector $f$ associated to the stabilization $c \leadsto c'$ is the integer vector indexed by the non-sink vertices of $G$ with $f(v)$ equal to the number of times vertex $v$ appears in the vertex toppling sequence that sends $c$ to $c'$. Finally, given a sandpile $c$, the unique stable sandpile obtained after a sequence of topplings is denoted by $c^\circ$ and is called the stabilization of $c$.

Let $G = (V, E, s)$ be a graph and $a, b$ be two sandpiles on $G$. Then $a + b$ denotes the sandpile obtained by adding the grains of sand vertex-wise, that is, $(a + b)(v) = a(v) + b(v)$ for each $v \in \bar{V}$. Note that even if $a$ and $b$ are stable, $a + b$ may not be. We denote the stabilization of $a + b$ by $a \oplus b$, that is, $a \oplus b = (a + b)^\circ$. The binary operator $\oplus$ is called stable addition.

Definition 4. A sandpile $c$ is recurrent if it is stable and given any sandpile $a$, there exists a sandpile $b$ such that

$$a \oplus b = c.$$  

As an example, given a graph $G = (V, E, s)$, the sandpile max defined by $\text{max}(v) = d_v - 1$ for each $v \in \bar{V}$ is recurrent. We call max the maximal stable sandpile on $G$. The following well-known proposition gives a simpler way to compute the recurrent sandpiles.

Proposition 5. A stable sandpile $c$ is recurrent if and only if there exists a sandpile $b$ with $\text{max} \oplus b = c$. 
As mentioned before, the set of recurrent sandpiles on a graph \( G \) under stable addition forms a finite abelian group denoted \( S(G) \), see [20]. Explicitly, the sandpile group of \( G \) is isomorphic to the cokernel of the reduced Laplacian matrix of \( G \). Moreover, this matrix can be used to compute the stabilization of a sandpile algebraically.

2.1. Graph Laplacians.

**Definition 6.** Let \( G \) be a graph with \( n \) vertices \( v_1, v_2, \ldots, v_n \). The Laplacian of \( G \), denoted \( L = L(G) \), is the \( n \times n \) matrix defined by

\[
L_{ij} = \begin{cases} 
-\text{wt}(v_i, v_j) & \text{for } i \neq j, \\
\text{d}_{v_i} & \text{for } i = j.
\end{cases}
\]

The reduced Laplacian of a graph \( G = (V,E,s) \), denoted \( \tilde{L} = \tilde{L}(G) \), is the matrix obtained by deleting the row and column corresponding to the sink \( s \) from the matrix \( L \). Kirchhoff’s Matrix-Tree Theorem implies that the number of recurrent sandpiles in \( G \), that is, the determinant of reduced Laplacian \( \tilde{L} \) equals the number of spanning trees of \( G \). From our definition we also have that if \( c \rightsquigarrow c' \) by toppling vertex \( v \), then \( c' = c - \tilde{L}1_v \), where \( 1_v \) denotes the (column) vector with \( 1_v(v) = 1 \) and \( 1_v(w) = 0 \), for all \( w \neq v \) in \( \tilde{V} \). The previous observation leads to the following result.

**Proposition 7.** Given a graph \( G = (V,E,s) \), and a sandpile \( c \). If \( c \rightsquigarrow c' \) by a sequence of topplings, then \( c' = c - \tilde{L}f \), where \( f \) is the (column) toppling vector associated to \( c \rightsquigarrow c' \).

The following result, known as Dhar’s Burning Criterion, gives an alternative and useful way to characterize recurrent sandpiles in an undirected graph. Given \( G = (V,E,s) \), let \( u \) denote the sandpile given by \( u_j = \text{wt}(v_j, s) \) for each \( v_j \in \tilde{V} \). We will refer to the sandpile \( u \) as the sandpile obtained by ‘firing the sink’.

**Proposition 8** ([15, Corollary 2.6]). The sandpile \( c \) is recurrent if and only if \( u \oplus c = c \). Moreover, the firing vector in the stabilization of \( u \oplus c \) is \((1,\ldots,1)\).

**Corollary 9.** Let \( G = (V,E,s) \) be a graph and \( c \) a recurrent sandpile on \( G \). When one grain of sand is added to a vertex adjacent to the sink then every vertex can topple at most once.

2.2. Avalanche Polynomials. The (univariate) avalanche polynomial was introduced in [13]. This polynomial enumerates the sizes of all principal avalanches.

**Definition 10.** Let \( G = (V,E,s) \) be a graph and let \( v \in \tilde{V} \). Let \( c \) be a recurrent sandpile \( c \) on \( G \) and \( v \in \tilde{V} \), the principal avalanche of \( c \) at \( v \) is the sequence of vertex topplings resulting from the stabilization of the sandpile \( c + 1_v \).

**Definition 11.** The avalanche polynomial for a graph \( G \) is defined as

\[
A_G(x) = \sum \lambda_m x^m,
\]

where \( \lambda_m \) is the number of principal avalanches of size \( m \).

**Example 12.** In this example, we consider the 3-cycle \( C_3 \). The recurrent sandpiles on \( C_3 \) are \((1,0)\), \((0,1)\), and \((1,1)\). The table in Figure 3 records the size of the avalanche for the corresponding recurrent and vertex. Therefore, \( A_{C_3}(x) = 2x^2 + 2x + 2 \).
Figure 3. Principal avalanche sizes on $C_3$.

2.3. Multivariate Avalanche Polynomial. The (univariate) avalanche polynomial does not contain any information regarding which vertices topple in a given principal avalanche. Here we introduce the multivariate avalanche polynomial that encodes this information.

Definition 13. Let $G = (V, E, s)$ be a graph on $n + 1$ vertices and let $\tilde{V} = \{v_1, \ldots, v_n\}$. Given $k \in \{1, \ldots, n\}$ and a recurrent sandpile $c$, the avalanche monomial of $c$ at $v_k$ is

$$
\mu_G(c, v_k) = x^{\nu(c, v_k)} = \prod_{i=1}^{n} x_i^{f_i}
$$

where $\nu(c, v_k) = (f_1, \ldots, f_n)$ is the toppling vector of the stabilization of $c + 1_{v_k}$.

Definition 14. Let $G = (V, E, s)$ be a graph. The multivariate avalanche polynomial of $G$ is defined by

$$
A_G(x_1, \ldots, x_n) = \sum_{c \in S(G)} \sum_{v \in \tilde{V}} \mu_G(c, v).
$$

Note that the multivariate avalanche polynomial is the sum of all possible avalanche monomials. In what follows the term "avalanche polynomial" will refer to the multivariate case.

Example 15. As in Example 12 we have recurrences $(1, 0), (0, 1), \text{ and } (1, 1)$ in $C_3$. The table in Figure 4 records the toppling vector $\nu(c, v_i)$ for the principal avalanche of $c$ at $v_i$.

Figure 4. Toppling vectors on $C_3$.

From this table we can easily read the avalanche monomials. For example, $\mu_{C_3}((1, 0), v_1) = x_1^1 x_2^0 = x_1$. Adding all avalanche monomials gives the avalanche polynomial

$$
A_{C_3}(x_1, x_2) = x_1^1 x_2^0 + 2x_1^0 x_2^0 + x_1^0 x_2^1 + 2x_1^1 x_2^1 = 2x_1 x_2 + x_1 + x_2 + 2.
$$

Note that the univariate avalanche polynomial for a graph $G$ can be recovered from the multivariate avalanche polynomial by substituting each $x_i$ by $x$, i.e.,

$$
A_G(x) = A_G(x, \ldots, x).
$$

It is a well-known fact that the sandpile group of an undirected graph is independent of the choice of sink [15, Proposition 1.1]. Nevertheless, the structure of the individual recurrent
sandpiles may differ. This implies that the avalanche polynomial of a graph is dependent on the choice of sink. For this reason, we will always fix a sink before discussing the avalanche polynomial of a graph. Nevertheless, for certain graphs like cycles and complete graphs, the avalanche polynomial does not depend on the choice of sink. For other families such as wheel and fan graphs there is a natural choice of sink, namely the dominating vertex. For trees, our method computes the avalanche polynomial for any choice of sink.

2.4. Burst Size. There are other evaluations of the multivariate avalanche polynomial $A_G(x_1, \ldots, x_n)$ that are relevant in the larger field of sandpile groups. In [25], Levine introduces the concept of burst size to prove a conjecture of Poghosyan, Poghosyan, Priezzhev, Ruelle [29] on the relationship between the threshold state of the fixed-energy sandpile and the stationary state of Dhar’s abelian sandpile.

Definition 16. Let $G = (V, E, s)$ be a graph and $c$ be a recurrent sandpile on $G$. Given $v \in \tilde{V}$, define the burst size of $c$ at $v$ as

$$av(c, v) := |c'| - |c| + 1,$$

where the sandpile $c'$ is defined such that $c' \oplus 1_v = c$ and $|c|$ denotes the number of grains of sand in $c$. Equivalently, $av(c, v)$ is the number of grains that fall into the sink $s$ during the stabilization of $c' + 1_v \sim c$.

Let $G = (V, E, s)$ be a simple graph and $A_G(x_1, \ldots, x_n)$ be its multivariate avalanche polynomial. Now, let $x_i = 1$ for each vertex $v_i$ that is not adjacent to the sink $s$. Also, for each vertex $v_j$ adjacent to $s$, let $x_j = x$. The resulting univariate polynomial $B(x) = \sum_k b_k x^k$ satisfies the condition that $b_k$ is the number of principal avalanches with burst size $k$.

3. Toppling Polynomials of Trees

Let $T$ be a tree on $n + 1$ vertices labelled $v_1, \ldots, v_n, s$. Assume further that $T$ is rooted at the sink $s$. It is a basic observation that $T$ has only one recurrent, namely $\max_T$. So

$$A_T(x_1, \ldots, x_n) = \sum_{v \in \tilde{V}} \mu_T(\max_T, v).$$

As noted in [13] any tree can be constructed from a single vertex using two operations $\phi$ and $+$ defined below.

Definition 17. For two trees $T$ and $T'$ rooted at $s$ and $s'$ respectively, the operation $+$, called tree addition, identifies $s$ and $s'$. For a tree $T$ rooted at $s$, the operation $\phi$, called grafting or root extension, refers to adding an edge from $s$ to a new root $s'$.

The operations $\phi$ and $+$ can be seen in Figure 5. Theorem 18 explains what happens to the toppling polynomial of a tree under these operations.

Theorem 18. Let $A_T$, $A_{T_1}$, and $A_{T_2}$ be the avalanche polynomials of trees $T$, $T_1$, and $T_2$, respectively. Then

1. $A_{T_1 + T_2} = A_{T_1} + A_{T_2},$
2. $A_{\phi(T)} = x_1 x_2 \cdots x_n (A_T + 1)$, where $n = |\tilde{V}(\phi(T))|$. 
Proof. Under tree addition, trees \( T_1 \) and \( T_2 \) are only connected at the sink \( s \). Since the sink never topples, a principal avalanche at a vertex in \( T_1 \) will never affect the vertices in \( T_2 \), and vice versa. Therefore, \( A_{T_1+T_2} = A_{T_1} + A_{T_2} \). For the second part, let \( T \) be a tree on \( n \) vertices with sink \( s \). Let \( \max_{\phi(T)} \) and \( \max_T \) be the maximum stable sandpile on \( \phi(T) \) and \( T \), respectively. First we consider \( \max_{\phi(T)} \oplus 1_s \). By Proposition 8,

\[
\mu_{\phi(T)}(\max_{\phi(T)}, s) = x_1x_2 \cdots x_n.
\]

Now consider \( \max_{\phi(T)} \oplus 1_{v_k} \) where \( v_k \neq s \). Note that when we apply the toppling sequence \( \nu_T(\max_T, v_k) \) to \( \max_{\phi(T)} + 1_{v_k} \), we get the sandpile \( \max_{\phi(T)} + 1_s \). Thus, each vertex will now topple once more. So for each \( v_k \) with \( v_k \neq s \),

\[
\mu_{\phi(T)}(\max_{\phi(T)}, v_k) = (x_1x_2 \cdots x_n) \cdot \mu_T(\max_T, v_k).
\]

Therefore,

\[
A_{\phi(T)} = (x_1x_2 \cdots x_n) + (x_1x_2 \cdots x_n) \cdot A_T = x_1x_2 \cdots x_n(A_T + 1).
\]

\[\Box\]

Note that using Theorem 18 we can compute the multivariate avalanche polynomial of any tree. Furthermore, as noted in [13], it is possible for two non-isomorphic trees to have the same univariate avalanche polynomial. In contrast, the multivariate avalanche polynomial distinguishes between labeled trees.

Figure 6 gives an example, first presented in [13], of two non-isomorphic trees with the same univariate avalanche polynomial. The vertices are labeled with the size of the principal avalanche starting at that vertex. One can clearly see that \( T_1 \) and \( T_2 \) have the same univariate avalanche polynomial. However, they have different multivariate avalanche polynomials.

The avalanche polynomial of \( R_1 \) is \( x_6 + x_5x_7(x_5 + 1) + x_8 + x_9 + x_{10} \) and the avalanche polynomial of \( R_2 \) is \( x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \). Note that the first polynomial has total degree 3 and the second polynomial has total degree 1. The avalanche polynomials of the left subtrees of \( T_1 \) and \( T_2 \) will be on a disjoint set of variables. Thus, the avalanche polynomials for \( T_1 \) and \( T_2 \) must be distinct.

Corollary 19. Let \( T \) and \( T' \) be two trees on \( n+1 \) vertices. Then, \( A_T(x_1, \ldots, x_n) = A_{T'}(x_1, \ldots, x_n) \) if and only if \( A_{\phi(T)}(x_1, \ldots, x_{n+1}) = A_{\phi(T')}(x_1, \ldots, x_{n+1}) \).
THE AVALANCHE POLYNOMIAL OF A GRAPH

Figure 6. $\mathcal{A}_{T_1}(x) = \mathcal{A}_{T_2}(x) = x^{10} + x^9 + 6x^8 + 2x^7 + x^4$

Figure 7. Labelled right subtrees of $T_1$ and $T_2$.

Proof. Suppose that $\mathcal{A}_T = \mathcal{A}_{T'}$. Theorem 18 implies

$$\mathcal{A}_{\phi(T)} = x_1x_2 \cdots x_n(\mathcal{A}_T + 1) = x_1x_2 \cdots x_n(\mathcal{A}_{T'} + 1) = \mathcal{A}_{\phi(T')}.$$  

Now assume $\mathcal{A}_{\phi(T)} = \mathcal{A}_{\phi(T')}$. Then $x_1x_2 \cdots x_n(\mathcal{A}_T + 1) = x_1x_2 \cdots x_n(\mathcal{A}_{T'} + 1)$. This clearly implies $\mathcal{A}_T = \mathcal{A}_{T'}$.

\textbf{Theorem 20.} Let $T$ be a tree on $n + 1$ vertices. If $\mathcal{A}_T(x_1, \ldots, x_n) = \mathcal{A}_{T'}(x_1, \ldots, x_n)$ for some tree $T'$, then $T = T'$.

Proof. We use induction on the height of $T$. Recall that the \textit{height} of a tree is the number of edges in the longest path between the root and a leaf. Suppose $T$ has height 0, that is, $T$ consists of one vertex. Then $\mathcal{A}_T = 0$. Clearly, if $T'$ has two or more vertices then $\mathcal{A}_{T'} \neq 0$. Since $\mathcal{A}_T = \mathcal{A}_{T'}$, then $T'$ must also consist of one vertex and $T = T'$. Now suppose that $T$ has height $h > 0$. In this case, the sink $s$ of $T$ must have at least one child. Assume $s$ has degree $d$. Deleting $s$ creates $d$ trees $T_1, T_2, \ldots, T_d$. We have that $T = \phi(T_1) + \phi(T_2) + \cdots + \phi(T_d)$. So $\mathcal{A}_T$ is the sum of $d$ multivariate polynomials with pairwise disjoint supports

$$\mathcal{A}_T = \mathcal{A}_{\phi(T_1)} + \mathcal{A}_{\phi(T_2)} + \cdots + \mathcal{A}_{\phi(T_d)}.$$  

Since $\mathcal{A}_T = \mathcal{A}_{T'}$, then $\mathcal{A}_{T'}$ must also satisfy the same condition. Hence the sink of $T'$ must also have degree $d$ and $T' = \phi(T'_1) + \phi(T'_2) + \cdots + \phi(T'_d)$ for some trees $T'_1, T'_2, \ldots, T'_d$. Since the supports are pairwise disjoint, we must also have that $\mathcal{A}_{\phi(T_i)} = \mathcal{A}_{\phi(T'_i)}$ for some $j$. Corollary 19 implies $\mathcal{A}_{T_1} = \mathcal{A}_{T'_1}$. But $T_1$ is a tree of height $h - 1$. By induction, $T_1 = T'_1$. Therefore, after relabeling the subtrees in $T'$, we must have $T_i = T'_i$ for all $1 \leq i \leq d$ and $T = T'$.  \qed
4. Toppling Polynomials of Cycles

Now, we will compute the avalanche polynomial of the cycle graph $C_{n+1}$ on $n+1$ vertices. Unless otherwise stated, we will label the vertices $s, v_1, v_2, \ldots, v_n$ in a clockwise manner. As shown in Example 12, $A_{C_3}(x_1, x_2) = 2x_1 x_2 + x_1 + x_2 + 2$. We will denote by $C_2$ the graph with two vertices and two edges between these vertices. It is clear that $A_{C_2}(x_1) = x_1 + 1$.

In this section, we will write sandpiles and toppling vectors as strings instead of vectors. For example, the string $1p^{-1}01^{n-p}$ denotes the sandpile with no grains of sand at vertex $v_p$ and 1 grain of sand at every other vertex. We will also make the convention that a bit raised to the 0 power does not appear in the string, e.g., $0^21^02 = 0^4$. In the previous section, we saw that a tree has exactly one recurrent sandpile. The cycle graph $C_{n+1}$ has exactly $n+1$ recurrent sandpiles, namely, max = $1^n$ and $b_p = 1^{n-1}01^{-p}$ for $p = 1, 2, \ldots, n$, see [13].

4.1. Toppling Sequence for the Maximal Stable Sandpile. We first focus our attention on understanding the toppling sequences for $1^n + 1 v_i$ for $1 \leq i \leq n$.

Example 21. Figure 8 shows that $\mu_{C_6}(1^5, v_2) = x_1 x_2^2 x_3^2 x_4 x_5$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{The principal avalanche created by adding a grain of sand to $v_2$.}
\end{figure}

Lemma 22. Let $n \geq 1$. Then

$$
\mu_{C_{n+1}}(1^n, v_1) = \mu_{C_{n+1}}(1^n, v_n) = x_1 x_2 \cdots x_n.
$$

Proof. Because $v_1$ and $v_n$ are adjacent to the sink, each vertex topples at most once by Corollary 9. Now consider $\text{max} + 1 v_1$. Toppling $v_1$ results in $v_2$ being unstable. Inductively, for $i \geq 2$, if $v_i$ becomes unstable it will topple and $v_{i+1}$ will become unstable. Thus each vertex topples exactly once. A similar argument works for $\mu_{C_{n+1}}(1^n, v_n)$. □

Observe that the reduced Laplacian of the cycle $C_{n+1}$ is the $n \times n$ matrix

$$
\tilde{L} = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & 0 & 0 & -1 & 2
\end{bmatrix},
$$
Lemma 23. Let $n \geq 3$, $2 \leq i \leq n - 1$ and consider the sandpile $1^n + 1_v$ on $C_{n+1}$. We can legally topple each vertex once and the resulting sandpile is $01^{i-2}21^{n-i-1}0$.

Proof. The sequence $v_i, v_{i-1}, v_{i-2}, \ldots, v_1, v_{i+1}, v_{i+2}, \ldots, v_n$ is a legal toppling sequence. By Proposition 7, $1^n + 1_v$ accesses the sandpile

$$(1^n + 1_v) - \tilde{L} \cdot 1^n = (1^n + 1_v) - 10^{n-2}1 = 01^{i-2}21^{n-i-1}0.$$
The toppling sequence for recurrants $1^{p-1}01^{n-p}$. Theorem 25 gives us the avalanche monomials for the maximal stable sandpile at all vertices. Now we find the avalanche monomials for recurrants of the form $1^{p-1}01^{n-p}$. We will see that these monomials are closely related to the avalanche monomials arising from $1^n$.

Example 26. In Example 21 we saw $\mu_{C_6}(1^2, v_2) = x_1 x_2 x_3^2 x_4^2 x_5$. Figure 9 shows that $\mu_{C_{10}}(1^301^5, v_6) = x_5 x_6 x_7^2 x_8^2 x_9$. Notice that the structure of these monomials. The only difference is that there is a relabeling of the variables $x_i \rightarrow x_{i+4}$.

Based on what we’ve seen in Example 26, we may guess that the toppling monomials associated to $1^{p-1}01^{n-p}$ are related to the toppling monomials of $C_p$ and $C_{n-p+1}$. This turns out to be true and will now formally state and prove this idea. First, we must introduce some useful notation.

Definition 27. Let $q$ be an integer. Then $C_{n+1}^q$ will denote the cycle graph on $n + 1$ vertices labeled $v_{q+1}, \ldots, v_{q+n}, s$.

Theorem 28. Let $b_p = 1^{p-1}01^{n-p}$ be a recurrent on $C_{n+1}$ such that $1 \leq p \leq n$.

1. If $1 \leq i \leq p - 1$, then $\mu_{C_{n+1}}(b_p, v_i) = \mu_{C_p}(1^{p-1}, v_i)$.
2. If $p + 1 \leq i \leq n$, then $\mu_{C_{n+1}}(b_p, v_i) = \mu_{C_{n-p+1}}(1^{n-p}, v_i)$.
3. $\mu_{C_{n+1}}(b_p, v_p) = 1$.

Proof. Let $1 \leq i \leq p - 1$, this implies $p \geq 2$. If $p = 2$, then $i = 1$, and $b_p + 1 v_1 = 201^{n-2} \sim 021^{n-2}$. So $\mu_{C_{n+1}}(b_p, v_1) = x_1 = \mu_{C_2}(1, v_1)$ and the result holds.

Assume $p \geq 3$. If $i = 1$, then $v_1, \ldots, v_{p-1}$ is a legal toppling sequence and $b_p + 1 v_1 \sim 1^{p-2}01^{n-p+1}$. Lemma 26 implies $\mu_{C_{n+1}}(b_p, v_1) = x_1 x_2 \cdots x_{p-1} = \mu_{C_p}(1^{p-1}, v_1)$. Similarly, if $i = p - 1$, then $v_{p-1}, v_{p-2}, \ldots, v_1$ is a legal toppling sequence and $b_p + 1 v_{p-1} \sim 01^{n-1}$. So again $\mu_{C_{n+1}}(b_p, v_{p-1}) = x_1 x_2 \cdots x_{p-1} = \mu_{C_p}(1^{p-1}, v_{p-1})$.

If $p \geq 3$ and $2 \leq i \leq p - 2$, we have $b_p + 1 v_i = 1^{p-1}01^{n-p+1} v_i = 1^{i-2}21^{p-i-1}01^{n-p}$. Note that $v_1, v_{i-1}, \ldots, v_1, v_{i+1}, \ldots, v_{p-1}$ is a legal toppling sequence. So $b_p + 1 v_i \sim 01^{i-2}21^{p-i-2}01^{n-p+1}$.
From this computation we can deduce two things. First, during the stabilization of \( b_p + 1_{v_i} \), none of the vertices \( v_p, \ldots, v_n \) will topple. Second, the sandpile \( 01^{i-2}21^{p-i-2}01^{n-p+1} \) is in fact the sandpile \( c_1 \) defined in Lemma \([24]\) when \( n + 1 = p \) with the string \( 1^{n-p+1} \) concatenated at the end. Therefore, from these two observations we conclude that the principal avalanche resulting from \( b_p + 1_{v_i} \) in \( C_{n+1} \) follows the pattern described in Lemma \([24]\) for \( C_p \). Hence 
\[
\mu_{C_{n+1}}(b_p, v_i) = \mu_{C_n}(1^{p-1}, v_i).
\]

A similar argument works for the case \( p + 1 \leq i \leq n \) with the exception that now the vertices that topple are \( v_{p+1}, \ldots, v_n \). So \( \mu_{C_{n+1}}(b_p, v_i) = \mu_{C_{n-p+1}}(1^{n-p}, v_i) \). Finally, it is clear that \( b_p + 1_{v_p} \) is stable. So \( \mu_{C_{n+1}}(1^{p-1}01^{n-p}, v_p) = 1 \). \( \square \)

The following result follows immediately from Theorem \([25]\) and Theorem \([28]\).

**Corollary 29.** The avalanche polynomial for \( C_{n+1} \) for \( n \geq 1 \) is

\[
\sum_{i=1}^{n} \mu_{C_{n+1}}(1^n, v_i) + \sum_{p=2}^{n} \sum_{i=1}^{p-1} \mu_{C_p}(1^{p-1}, v_i) + \sum_{p=1}^{n} \sum_{i=p+1}^{n} \mu_{C_{n-p+1}}(1^{n-p}, v_i) + n,
\]

where \( \mu_{C_{q+1}}(1^q, v_i) = \prod_{j=1}^{m} x_j \cdots x_{q-j+1}, 1 \leq i \leq q, \) and \( m = \min\{i, q - i + 1\} \).

5. Avalanche Polynomials of Complete Graphs

In this section we will compute the avalanche polynomial of the complete graph \( K_{n+1} \) on \( n + 1 \) vertices \( v_1, v_2, \ldots, v_n, s \). In Example we computed \( A_{K_3}(x_1, x_2) \). Using SageMath \([18]\), we can compute the avalanche polynomial of \( K_4 \):

\[
A_{K_4}(x_1, x_2, x_3) = 9x_1x_2x_3 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + 3x_1 + 3x_2 + 3x_3 + 24.
\]

Note that \( A_{K_4}(x) = 9x^3 + 6x^2 + 9x + 24 \). So the set of principal avalanches of size \( m \) is evenly partitioned into \( \binom{3}{m} \) subsets for \( 0 \leq m \leq 3 \). Moreover, \( A_{K_4}(x_1, x_2, x_3) \) is a linear combination of elementary symmetric polynomials. We will show that this characterizes the avalanche polynomial of \( K_{n+1} \).

**Definition 30.** Let \( m \) be an integer such that \( 0 \leq m \leq n \). The **elementary symmetric polynomial** of degree \( m \) on variables \( x_1, x_2, \ldots, x_n \) is

\[
e_m(x_1, \ldots, x_n) = \sum_{A \subseteq [n], |A| = m} \prod_{i \in A} x_i.
\]

Observe that \( e_0(x_1, \ldots, x_n) = 1 \) and the number of terms in \( e_m(x_1, \ldots, x_n) \) is \( \binom{n}{m} \).

As mentioned in Section \([2]\) the number of recurrent sandpiles of a graph \( G \) equals the number of spanning trees of \( G \). Cayley’s formula implies that \( K_n \) has \( n^{n-2} \) recurrent sandpiles.

In order to study the principal avalanches in this graph, we will use a beautiful result first proved in \([15]\) that establishes a bijection between recurrent sandpiles in \( K_{n+1} \) and \( n \)-parking functions.

**Definition 31.** Given a function \( p : \{0, 1, \ldots, n - 1\} \rightarrow \{0, 1, \ldots, n - 1\}, \) let \( a_0 \leq a_1 \leq \cdots \leq a_{n-1} \) be the non-decreasing rearrangement of \( p(0), \ldots, p(n-1) \). We say that \( p \) is an \( n \)-parking function provided that \( a_i \leq i \) for \( 0 \leq i \leq n - 1 \).

Note that the parking function \( p \) can be represented by the vector \((p(0), p(1), \ldots, p(n-1))\).
Proposition 32 ([15 Proposition 2.8]). The sandpile $c$ is recurrent on $K_{n+1}$ if and only if $\max_{K_{n+1}} - c$ is an $n$-parking function.

It is clear from the definition that any permutation of a parking function is also a parking function. We can concatenate parking functions to obtain new parking functions.

Lemma 33. Let $p = (p_0, p_1, \ldots, p_{m-1})$ and $q = (q_0, q_1, \ldots, q_{n-1})$ be two parking functions. Then $(p_0, p_1, \ldots, p_{m-1}, q_0 + m, q_1 + m, \ldots, q_{n-1} + m)$ is also a parking function.

Proof. Let $a_0 \leq a_1 \leq \cdots \leq a_{m-1}$ and $b_0 \leq b_1 \leq \cdots \leq b_{n-1}$ be non-decreasing rearrangements of $p$ and $q$, respectively. Note $b_0 + m \leq b_1 + m \leq \cdots \leq b_{n-1} + m$ and $b_i + m \leq i + m$ for each $i = 0, \ldots, n - 1$, since $q$ is a parking function. Moreover, $a_{m-1} \leq m - 1 = m = b_0 + m$. So

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} < b_0 + m \leq b_1 + m \leq \cdots \leq b_{n-1} + m$$

and each term is less than or equal to its index. \hfill $\square$

5.1. The Avalanche Polynomial of $K_{n+1}$. The following lemma gives a partial description of a sandpile $c$ given the size of a principal avalanche.

Lemma 34. Let $c$ be a sandpile on $K_{n+1}$. Suppose that the principal avalanche resulting from stabilizing $c + 1_v$ has length $m \geq 1$. Let $w_0, w_1, \ldots, w_{m-1}$ be the associated toppling sequence and let $\{u_0, \ldots, u_{m-1}\}$ be the set of vertices that do not topple. Assume further, perhaps after relabeling, that $c(u_0) \geq c(u_1) \geq \cdots \geq c(u_{m-1})$. The following are true:

1. $c(w_0) = c(v_k) = n - 1$.
2. $n - i \leq c(w_i) \leq n - 1$, for $i = 1, \ldots, m - 1$.
3. $n - m - i - 1 \leq c(u_i) \leq n - m - 1$, for $i = 0, \ldots, n - m - 1$.

Proof. First note that since $c$ is stable then $c(v) \leq n - 1$. Since $m \geq 1$, then $w_0 = v_k$ must topple. Thus, $c(w_0) = n - 1$. Corollary 9 implies that each $w_j$ appears exactly once in the toppling sequence. Moreover, when a vertex topples, it adds one grain of sand to every other non-sink vertex of $K_{n+1}$. Thus, toppling vertices $w_0, \ldots, w_{i-1}$ adds $i$ grains of sand to $w_i$. Since this vertex must topple next, then $c(w_i) + i \geq n$. So, $n - i \leq c(w_i) \leq n - 1$. On the other hand, since $u_i$ does not topple, then $c(u_i) = n - m - 1$ for $0 \leq i \leq n - m - 1$. By Proposition 32 $p = \max_{K_{n+1}} - c$ is an $n$-parking function. Let $p'$ be its non-decreasing rearrangement. Note that the first $m$ entries in $p'$ correspond to the $m$ vertices that topple and the last $n - m$ entries correspond to the vertices that do not topple. So $p'(m + i) = n - 1 - c(u_i)$. Since $p'(m + i) \leq m + i$, then $c(u_i) \geq n - m - i$. So, $n - m - i - 1 \leq c(u_i) \leq n - m - 1$. \hfill $\square$

Proposition 35. Let $\lambda_m$ denote the number of principal avalanches of size $m$ in $K_{n+1}$. Then

$$A_{K_{n+1}}(x_1, \ldots, x_n) = \sum_{m=0}^{n} \frac{\lambda_m}{\binom{n}{m}} e_m(x_1, \ldots, x_n).$$

Proof. First note that each vertex can topple at most once in any principal avalanche by Corollary 9. So every monomial in $A_{K_{n+1}}$ is square-free and completely characterized by its support. Fix an integer $m$ with $1 \leq m \leq n$. Let $A \subseteq [n]$ with $|A| = m$ and let $\mu_A = \prod_{i \in A} x_i$. Consider the sandpile $c$ defined by

$$c(v_i) = \begin{cases} 
  n - 1 & \text{if } i \in A, \\
  n - 1 - m & \text{if } i \notin A. 
\end{cases}$$
Note that \( \max - c(v_i) = 0 \) if \( i \in A \) and \( \max - c(v_i) = m \) if \( i \notin A \). The non-decreasing rearrangement of \( \max - c \) is \( (0, 0, \ldots, 0, m, m, \ldots, m) \) which is a parking function. Thus, by Proposition 32, the sandpile \( c \) is recurrent. By Lemma 34, \( \mu(c, v_i) = \mu_A \) for any vertex \( v_i \) with \( i \in A \). To complete the proof, we must show that if \( A = (i_1, \ldots, i_m) \) and \( A' = (i'_1, \ldots, i'_m) \) are two ordered subsets of \([n]\) of cardinality \( m \), then the number of principal avalanches that produce the toppling sequence \((v_{i_1}, \ldots, v_{i_m}) \) equals the number of principal avalanches that produce the toppling sequence \((v_{i'_1}, \ldots, v_{i'_m}) \). However, this follows directly from the symmetry of \( K_{n+1} \). Explicitly, if \( \pi \) denotes the permutation that sends \( v_{i_j} \to v_{i'_j} \) for each \( j = 1, \ldots, m \). Then \( \mu_{K_{n+1}}(c, v_{i_1}) = \mu_A \) if and only if \( \mu_{K_{n+1}}(c', v_{i'_1}) = \mu_{A'} \), where \( c' \) is the sandpile obtained by permuting the entries of \( c \) according to \( \pi \). The explicit form of the coefficient of \( e_m(x_1, \ldots, x_n) \) follows from the fact that this polynomial has \( \binom{n}{m} \) monomials.

The coefficients \( \lambda_m \) in Proposition 35 were computed in [13, Propositions 4 and 5]. Explicitly, \( \lambda_0 = n(n - 1)(n + 1)^{n-2} \) and

\[
\lambda_m = \binom{n}{m} m^{-1} (n - m + 1)^{n-m-1}, \quad 1 \leq m \leq n.
\]

We include a proof of the latter result in order to correct a mistake in their original argument. However, we also want to point out that the coefficient \( \lambda_m \) is also the number of principal avalanches with burst size \( m \) since every non-sink vertex in \( K_{n+1} \) is adjacent to the sink.

**Definition 36.** Let \( c \in \mathcal{S}(K_{n+1}) \) and \( v_i \in \tilde{V} \) such that when a grain of sand is added to \( v_i \), an avalanche of size \( m \geq 1 \) occurs. Define the function

\[
\phi : \mathcal{S}(K_{n+1}) \times \tilde{V} \to \tilde{V} \times \left( \tilde{V} \setminus \{v_i\} \right) \times \mathcal{S}(K_m) \times \mathcal{S}(K_{n-m+1}),
\]

such that \( \phi(c, v_i) = (v_i, J, c_1, c_2) \), where \( J = \{w_1, \ldots, w_{m-1}\} \) is the set of \( m - 1 \) vertices that topple other than \( w_0 = v_i \). The sandpile \( c_1 \) in \( K_m \) is defined by

\[
c_1 = (c(w_1) - (n - m + 1), \ldots, c(w_{m-1}) - (n - m + 1)),
\]

and the sandpile \( c_2 \) in \( K_{n-m+1} \) is defined by the values \( c(v_k) \) for \( v_k \notin J \cup \{v_i\} \).

**Example 37.** Consider the recurrent sandpile \( c = (8, 7, 8, 1, 0, 3, 7, 2, 4) \) on \( K_{10} \). Note that adding a grain of sand at \( v_1 \) causes an avalanche of size \( m = 4 \). In this case \( J = \{v_2, v_3, v_7\} \), \( c_1 = (7 - 6, 8 - 6, 7 - 6) = (1, 2, 1) \) and \( c_2 = (1, 0, 3, 2, 4) \). In [13], the authors define the sandpile \( c_1 \) by substracting \( m - 2 \) instead of \( n - m + 1 \). In here, this would result in the sandpile \((5, 6, 5)\) that is not even a stable sandpile on \( K_4 \).

**Lemma 38.** The map \( \phi \) described in Definition 36 is a bijection.

**Proof.** First we need to show that the map \( \phi \) above is well-defined, that is, we need to show that \( c_1 \) and \( c_2 \) are, in fact, recurrent sandpiles on \( K_m \) and \( K_{n-m+1} \), respectively. To show \( c_1 \) is recurrent, let \( J = \{v_1, \ldots, w_{m-1}\} \) such that \( c(w_i) \leq c(w_{i+1}) \) for \( 1 \leq i \leq m - 2 \). By Lemma 34, for \( 1 \leq i \leq m - 1 \), we have that \( n - i - c(w_i) \leq n - 1 \). So,

\[
n - i - (n - m + 1) \leq c(w_i) - (n - m + 1) \leq n - 1 - (n - m + 1)
\]

and \( m - i - 1 \leq c_1(w_i) \leq m - 2 \). This implies \( c_1 \) is a stable sandpile on \( K_m \). Consider \( p_1 = \max_{K_m} - c_1 \). For \( 1 \leq i \leq m - 1 \), \( 0 \leq p_1(w_i) \leq i - 1 \), so \( p_1 \) is a parking function and
c_1 is recurrent. To show c_2 is recurrent, let \( \{u_0, \ldots, u_{n-m-1}\} = \tilde{V} \setminus (J \cup \{v_1\}) \) such that 
\[ c(u_i) \leq c(u_{i+1}) \quad \text{for} \quad 0 \leq i \leq n - m - 2. \]
By Lemma 34
\[ n - m - i - 1 \leq c(u_i) \leq n - m - 1. \]
Since \( c_2(u_i) = c(u_i) \) then \( c_2 \) is stable in \( K_{n-m+1} \). Consider \( p_2 = \max_{K_{n-m+1}} - c_2 \). For \( 0 \leq i \leq n - m - 2 \), we have \( 0 \leq p_2(i) \leq i \). Since \( p_2 \) is a parking function, \( c_2 \) is recurrent.

The fact that the map \( \phi \) is injective follows immediately from the definition of \( c_1 \) and \( c_2 \). Finally, we will show that \( \phi \) is onto. Given \((v, J, c_1, c_2)\) we define \( c \) as follows. First, let \( c(v) = n - 1 \). Now, for each \( w_i \in J \), define \( c(w_i) \) by adding \( n - m + 1 \) to the \( i \)th entry in \( c_1 \). The remaining \( n - m \) entries in \( c \) are filled with the entries in \( c_2 \). Since \( c_1 \) and \( c_2 \) are recurrent, Proposition 32 implies \( p_1 = \max_{K_m} - c_1 \) and \( p_2 = \max_{K_{n-m+1}} - c_2 \) are \((m - 1)\) and \((n - m)\)-parking functions, respectively. By Lemma 33, concatenating \( p_1 \) and \( p_2 + m \) defines an \((n - 1)\)-parking function \( p' \), where \( m = (m, \ldots, m) \). Furthermore, concatenating \( 0 \) and \( p' \) gives an \( n \)-parking function \( p \). Moreover, \( \max_{K_{n+1}} - p \) is a rearrangement of \( c \), so \( c \) is a recurrent sandpile on \( K_{n+1} \). Clearly, \( \phi(c, v) = (v, J, c_1, c_2) \) and this completes the proof. \( \square \)

From the bijection \( \phi \) we are able to compute the number \( \lambda_m \) of principal avalanches of size \( m > 0 \). Given \( A \subseteq [n] \) with \( |A| = m \), Proposition 35 states that \( \lambda_m / \binom{n}{m} \) is the number of principal avalanches with avalanche monomial \( \mu_A = \prod_{i \in A} x_i \). The bijection \( \phi \) implies that this number equals the number of four-tuples \((v_i, J, c_1, c_2)\) with \( J \cup \{v_i\} = A \). There are \( m \) ways to pick \( v_i \). Cayley’s formula implies that the number of recurrents on \( K_m \) and \( K_{n-m+1} \) is \( m^{m - 2} \) and \((n - m + 1)^{n-m-1} \), respectively. Therefore,
\[ \lambda_m = \binom{n}{m} \cdot m \cdot m^{m - 2}(n - m + 1)^{n-m-1} = \binom{n}{m} m^{m-1}(n - m + 1)^{n-m-1}. \]

6. The Avalanche Polynomial of the Wheel

The wheel graph, denoted \( W_n \), is a cycle on \( n \geq 3 \) vertices with an additional dominating vertex. Throughout, the vertices in the cycle will be labeled clockwise as \( v_0, \ldots, v_{n-1} \), where the indices are taken modulo \( n \). The dominating vertex, denoted \( s \), will always be assumed to be the sink.

The sandpile group of \( W_n \) was first computed by Biggs in [9]:
\[ S(W_n) = \begin{cases} \mathbb{Z}_l \oplus \mathbb{Z}_s & \text{if } n \text{ is odd} \\ \mathbb{Z}_f \oplus \mathbb{Z}_5 f_s & \text{if } n \text{ is even} \end{cases} \]
where \( \{l_n\} \) is the Lucas sequence and \( \{f_n\} \) is the Fibonacci sequence. These sequences are defined by initial conditions \( l_0 = 2, l_1 = 1 \) and \( f_0 = 1, f_1 = 1 \), respectively, and the recursion \( x_n = x_{n-1} + x_{n-2} \). There are many relationships among these numbers. For example, \( l_n = f_{n-1} + f_{n+1} \). Moreover, the order of \( S(W_n) \) equals the number of spanning trees \( \tau(W_n) \) in \( W_n \). This number equals \( \tau(W_n) = l_{2n} - 2 \), see [21, 9].

We have already computed the avalanche polynomial of \( W_3 \) since \( W_3 = K_4 \). In this case,
\[ A_{W_3}(x_0, x_1, x_2) = 9x_0x_1x_2 + 2(x_0x_1 + x_1x_2 + x_2x_0) + 3(x_0 + x_1 + x_2) + 24. \]
Observe that the set of principal avalanches of size \( 0 < m < 3 \) is evenly partitioned into \( n = 3 \) subsets. Moreover, \( A_{W_3}(x_0, x_1, x_2) \) is a linear combination of cyclic polynomials. We will show that this characterizes \( A_{W_n}(x_0, \ldots, x_{n-1}) \).
Lemma 42. This is not the case for avalanches of size $n$ integer with $m$ of size $\leq 1$.\n
W Burning Criterion (Proposition 8), Cori and Rossin [15] showed that a sandpile on this graph can be written as a word of length $n$.\n
First note that for each $1 \leq m \leq n - 1$, $w_m$ is the sum of $n$ terms of degree $m$. For example, $w_1 = x_0 + \cdots + x_{n-1}$. For the case $m = n$, the above definition would give $w_n = \sum_{i=0}^{n-1} x_i x_{i+1} \cdots x_{i+n-1} = nx_0 \cdots x_{n-1}$. However, we will remove the coefficient $n$ and define $w_n(x_0, \ldots, x_{n-1}) = x_0 \cdots x_{n-1}$.

In [17], Dartois and Rossin gave exact results on the distribution of avalanches on $W_n$. Their approach consisted in showing that the recurrents on $W_n$ can be seen as words of a regular language. They built an automaton associated to this language and used the concept of transducers to determine the exact distribution of avalanche lengths in this graph. Here we take a different approach focused solely on the structure of the recurrent sandpiles.

Note that the degree of every non-sink vertex in $W_n$ is 3. So any stable sandpile on this graph can be written as a word of length $n$ in the alphabet $\{0, 1, 2\}$. Applying Dhar’s Burning Criterion (Proposition 8), Cori and Rossin [15] showed that a sandpile on $W_n$ is recurrent if and only if there is at least one vertex with 2 grains of sand and between any two vertices with 0 grains, there is at least one vertex with 2 grains.

Definition 40. Let $m$ be an integer with $1 \leq m \leq n - 1$. A sandpile $c$ in $W_n$ has a maximal 2-string of length $m$ if there are vertices $v_i, v_{i+1}, \ldots, v_{i+m-1}$, such that $c(v_i) = \cdots = c(v_{i+m-1}) = 2$ and $c(v_{i-1}) \neq 2 \neq c(v_{i+m})$.

Note that $\max_{W_n} = 2^n$ is the unique recurrent with a maximal 2-string of length $n$.

Lemma 41. Let $c \in S(W_n)$. The principal avalanche of $c$ at a non-sink vertex $v$ has size $1 \leq m \leq n - 2$ if and only if $v$ is part of a maximal 2-string of length $m$.

Proof. Suppose a grain is added to a vertex $v$ that is part of a maximal 2-string of length $m$. Since $m < n - 1$, the two non-sink vertices adjacent to the ends of the 2-string are distinct. Thus, exactly the $m$ vertices in the maximal 2-string will topple. On the other hand, if $v$ is part of a longer or shorter maximal 2-string, the avalanche will not have size $m$. \qed

Lemma 41 implies that for each $1 \leq m \leq n - 2$, the number $\lambda_m$ of principal avalanches of size $m$ equals $m$ times the number of maximal 2-strings of length $m$ over all recurrents. This is not the case for avalanches of size $n - 1$ or $n$ as the following simple lemma shows.

Lemma 42. For any non-sink vertex $v$ in $W_n$, $\mu(2^n, v) = x_0 \cdots x_{n-1}$. Also, let $p$ be an integer with $0 \leq p \leq n - 1$. For any non-sink vertex $v$ with $v \neq v_p$ we have $\mu_{\lambda_p}(2^p 12^{n-p-1}, v) = x_0 \cdots x_{n-1}$, $\mu_{\lambda_p}(2^p 02^{n-p-1}, v) = \frac{x_0 \cdots x_{n-1}}{x_p}$.

This implies $\lambda_n = n^2$ and $\lambda_{n-1} = n(n - 1)$.\n
THE AVALANCHE POLYNOMIAL OF A GRAPH 17

Definition 39. Let $m$ be an integer such that $1 \leq m \leq n - 1$. We will denote by $w_m(x_0, \ldots, x_{n-1})$ the cyclic polynomial of degree $m$ on variables $x_0, \ldots, x_{n-1}$ defined as

$$w_m(x_0, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} x_i x_{i+1} \cdots x_{i+m-1}.$$
Proof. Clearly the avalanche monomials for the given recurrents satisfy the above claims. Note also that the only avalanches of size \( n \) occur on recurrents of the form \( 2^p12^{n-p-1} \) and \( 2^n \). So there are \( n(n-1)+n=n^2 \) avalanches of size \( n \). The avalanches of size \( n-1 \) occur on recurrents of the form \( 2^p02^{n-p-1} \). Hence there are \( n(n-1) \) avalanches of size \( n-1 \). \( \square \)

For each \( 1 \leq m \leq n-2 \), we will count the maximal 2-strings of length \( m \) by establishing a map from the set of recurrents on \( W_n \) with a given maximal 2-string of length \( m \) into the set of recurrents on the fan graph \( F_{n-m} \). Let \( k \geq 2 \), the fan graph on \( k+1 \) vertices, denoted \( F_k \), is a path on \( k \) vertices, plus an additional dominating vertex \( s \).

**Proposition 43.** For each \( 1 \leq m \leq n-2 \), there is a bijection between the set of recurrents on \( W_n \) with a maximal 2-string of length \( m \) starting at \( v_0 \) and the set of recurrents on \( F_{n-m} \).

Proof. Let \( c \) be a recurrent sandpile on \( F_{n-m} \). Dhar’s Burning Criterion (Proposition 8) implies that adding 1 grain of sand to each vertex must result in an avalanche where every vertex topples exactly once. This implies that at least one of the endpoint vertices has 1 grain of sand or both endpoints have 0 grains of sand and there is an internal vertex with 2 grains of sand. Moreover, if a vertex has 0 grains of sand, then its neighbors must topple before it, hence there are no consecutive vertices with 0 grains of sand. For the same reason, between any two vertices with 0 grains of sand there cannot be a sequence of 1’s. In summary, \( c \) is a recurrent on \( F_{n-m} \) if and only if between any two vertices with 0 grains of sand there is a vertex with 2 grains of sand. Hence \( c \) is a recurrent sandpile on \( F_{n-m} \) if and only if the sandpile obtained by prepending a string of \( m \) 2’s to \( c \) is recurrent on \( W_n \). \( \square \)

It is well-known that the number of spanning trees in the fan graph \( F_k \) is precisely the Fibonacci number \( f_{2k} \), see [21]. So Proposition 43 implies that for each \( 1 \leq m \leq n-2 \), there are \( f_{2(n-m)} \) recurrent sandpiles that have a maximal 2-string of length \( m \) starting at \( v_0 \).

**Theorem 44.** Given \( n \geq 3 \), the avalanche polynomial of the wheel graph \( W_n \) is

\[
\mathcal{A}_{W_n} = n^2 w_n(x_0, \ldots, x_{n-1}) + \sum_{m=1}^{n-1} m \cdot f_{2(n-m)} w_m(x_0, \ldots, x_{n-1}) + 2n (f_{2n-1} - 1).
\]

Proof. In Lemma 42 we showed that \( \lambda_n = n^2 \). This lemma also shows that the avalanches of size \( n-1 \) are caused by adding a grain of sand at any vertex with 2 grains in any recurrent of the form \( 2^p02^{n-p-1} \) with \( 0 \leq p \leq n-1 \). Since \( \mu_{W_n}(2^p02^{n-p-1}, v) = x_0 \cdots x_{n-1}/x_p \), for any \( v \neq v_p \). Then the degree \( n-1 \) part of \( \mathcal{A}_{W_n} \) equals \( (n-1)w_{n-1}(x_0, \ldots, x_{n-1}) \). Note that when \( m = n-1 \) we have \( f_{2(n-m)} = f_2 = 1 \).

Now let \( 1 \leq m \leq n-2 \). Proposition 43 implies that there are \( f_{2(n-m)} \) recurrents on \( W_n \) with a maximal 2-string of length \( m \) starting at \( v_0 \). So by Lemma 41, there are \( mf_{2(n-m)} \) principal avalanches with avalanche monomial \( x_0 \cdots x_{m-1} \). This lemma also shows that any avalanche of size \( m \) must occur at a maximal 2-string of length \( m \). So the only possible avalanche monomials of degree \( m \) are the monomials occurring in the cyclic polynomial \( w_m \).

Moreover, the cyclic symmetry of \( W_n \) implies that the number of principal avalanches that produce the toppling sequence \( (v_0, v_1, \ldots, v_{m-1}) \) equals the number of principal avalanches that produce the toppling sequence \( (v_i, v_{i+1}, \ldots, v_{i+m-1}) \) for any \( 0 \leq i \leq n-1 \). Therefore, for any \( 1 \leq m \leq n-2 \), the degree \( m \) part of \( \mathcal{A}_{W_n} \) equals \( mf_{2(n-m)} w_m(x_0, \ldots, x_{n-1}) \).

Lastly, note that an avalanche of size 0 is produced by adding a grain of sand to a vertex with 0 or 1 grains of sand. So \( \lambda_0 \) equals the number of 0’s and 1’s in every recurrent. Since there are \( l_{2n-2} \) recurrents on \( W_n \), then \( \lambda_0 \) equals \( n(l_{2n-2}) \) minus the total number of 2’s
in every recurrent. Recall that for $1 \leq m \leq n - 2$, the number $\lambda_m$ of principal avalanches of size $m$ equals $m$ times the number of maximal 2-strings of length $m$ over all recurrences, that is, $\lambda_m$ equals the total number of 2’s in every maximal 2-string of length $m$. Moreover, $\lambda_{n-1} + \lambda_n = n^2 + n(n - 1) = 2n^2 - n$ equals the number of principal avalanches of size $\geq n - 1$. But this number also equals the number of 2’s in every recurrent with a maximal 2-string of size $\geq n - 1$. Therefore, $\lambda_1 + \cdots + \lambda_n$ equals the number of 2’s in every recurrent. Hence

$$
\lambda_0 = n(l_{2n} - 2) - (\lambda_1 + \cdots + \lambda_n) = n(l_{2n} - 2) - 2n^2 + n - \sum_{m=1}^{n-2} nm f_2(n-m)
$$

$$
= n\left[l_{2n} - 2n - 1 - \sum_{m=1}^{n-2} mf_2(n-m)\right] = n\left[l_{2n} - 2n - 1 - \sum_{m=2}^{n-1} (n-m)f_2m\right]
$$

$$
= n\left[l_{2n} - n - 2 - \sum_{m=1}^{n-1} (n-m)f_2m\right] = n\left[l_{2n} - n - 2 - \sum_{m=1}^{n-1} \sum_{k=1}^{m} f_2k\right]
$$

$$
= n\left[l_{2n} - n - 2 - \sum_{m=1}^{n-1} (f_2m+1 - 1)\right] = n\left[l_{2n} - 3 - \sum_{m=1}^{n-1} f_2m+1\right]
$$

$$
= n(l_{2n} - 2 - f_{2n}) = n(f_{2n+1} + f_{2n-1} - f_{2n} - 2) = n(2f_{2n-1} - 2) = 2n(f_{2n-1} - 1).
$$

□

In this case, $\lambda_m$ is also the number of principal avalanches with burst size $m$ since every non-sink vertex in $W_n$ is adjacent to the sink. Note also that as $n \to \infty$, the proportion of avalanches of size 0 is

$$
\lim_{n \to \infty} \frac{2n(f_{2n-1} - 1)}{n(l_{2n} - 2)} = 1 - \frac{1}{\sqrt{5}}.
$$

Thus, recovering the last result in [17, Section 2].

7. Conclusions

In this paper, we introduce the multivariate avalanche polynomial of a graph $G$. This new combinatorial object enumerates the toppling sequences of all principal avalanches generated by adding a grain of sand to any recurrent sandpile on $G$. We also give explicit descriptions of the multivariate avalanche polynomials for trees, cycles, complete, and wheel graphs. Furthermore, we show that certain evaluations of this polynomial recover some important information. In particular from this polynomial we can compute the distribution of the size of all principal avalanches, that is, we recover the (univariate) avalanche polynomial first introduced in [13]. Moreover, a different evaluation gives rise to the unnormalized distribution of burst sizes, that is, the number of grains of sand that fall into the sink in a principal avalanche. The burst size, introduced by Levine in [25], is an important statistic related to the relationship between the threshold state of the fixed-energy sandpile and the stationary state of Dhar’s abelian sandpile. Of special interest is a description of the avalanche polynomial for grids and the family of multiple wheel graphs introduced in [17].

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References

[1] Roland Bacher, Pierre de la Harpe, and Tatiana Nagnibeda. The lattice of integral flows and the lattice of integral cuts on a finite graph. Bull. Soc. Math. France, 125(2):167–198, 1997.
[2] P. Bak. How Nature Works. Copernicus, New York, 1996. The Science of Self-organized Criticality.
[3] P Bak and M Paczuski. Complexity, contingency, and criticality. Proceedings of the National Academy of Sciences, 92(15):6689–6696, 1995.
[4] P. Bak, C. Tan, and K. Wiesenfeld. Self-organized criticality. Phys. Rev. A (3), 38(1):364–374, Jul 1988.
[5] Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. Adv. Math., 215(2):766–788, 2007.
[6] Arthur T. Benjamin and Carl R. Yerger. Combinatorial interpretations of spanning tree identities. Bull. Inst. Combin. Appl., 47:37–42, 2006.
[7] Brian Benson, Deeparnab Chakrabarty, and Prasad Tetali. G-parking functions, acyclic orientations and spanning trees. Discrete Math., 310(8):1340–1353, 2010.
[8] Kenneth A. Berman. Bicycles and spanning trees. SIAM J. Algebraic Discrete Methods, 7(1):1–12, 1986.
[9] N. L. Biggs. Chip-firing and the critical group of a graph. J. Algebraic Comb., 9:25–45, January 1999.
[10] A. Björner and L. Lovász. Chip-firing games on directed graphs. J. Algebraic Combin., 1(4):305–328, December 1992.
[11] A. Björner, L Lovász, and P. Shor. Chip-firing games on graphs. European J. Combin., 12(4):283–291, 1991.
[12] Kan Chen, Per Bak, and S. P. Obukhov. Self-organized criticality in a crack-propagation model of earthquakes. Phys. Rev. A, 43:625–630, Jan 1991.
[13] R. Cori, A. Dartois, and D. Rossin. Avalanche polynomials of some families of graphs. In Mathematics and computer science. III, Trends Math., pages 81–94. Birkhäuser, Basel, 2004.
[14] R. Cori, A. Micheli, and D. Rossin. Avalanche polynomials. ArXiv e-prints, May 2009.
[15] R. Cori and D. Rossin. On the sandpile group of dual graphs. European J. Combin., 21(4):447–459, 2000.
[16] Michael Creutz. Abelian sandpiles. Nuclear Physics B - Proceedings Supplements, 20(0):758 – 761, 1991.
[17] Arnaud Dartois and Dominique Rossin. Analysis of the distribution of the length of avalanches on the sandpile group of the (n, k)-wheel. In Linusson S. Eriksson K., Björner A., editor, Formal Power Series and Algebraic Combinatorics - 15th+ Conference (FPSAC’03), 2003. ap.
[18] The Sage Developers. SageMath, the Sage Mathematics Software System (Version 7.1), 2016. http://www.sagemath.org.
[19] D. Dhar. Self-organized critical state of sandpile automaton models. Phys. Rev. Lett., 64(14):1613–1616, Apr 1990.
[20] D. Dhar, P. Ruelle, S. Sen, and D. Verma. Algebraic aspects of sandpile models. Journal of Physics A, 28:805–831, 1995.
[21] A. J. W. Hilton. The number of spanning trees of labeled wheels, fans and baskets. In Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), pages 203–206. Inst. Math. Appl., Southend-on-Sea, 1972.
[22] A. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. Wilson. Chip-firing and rotor-routing on directed graphs. In In and out of equilibrium. 2, volume 60 of Progress in Probability, pages 331–364. Birkhäuser, Basel, 2008.
[23] Henrik Jeldtoft Jensen. Self-organized criticality: emergent complex behavior in physical and biological systems. Cambridge lecture notes in physics. Cambridge University Press, Cambridge, 1998.
[24] D. V. Kritarev, S. Lübeck, P. Grassberger, and V. B. Priezzhev. Scaling of waves in the bak-tang-wiesenfeld sandpile model. Phys. Rev. E, 61:81–92, Jan 2000.
[25] L. Levine. Threshold state and a conjecture of Poghosyan, Poghosyan, Priezzhev and Ruelle. Comm. Math. Phys., 335(2):1003–1017, 2015.
[26] D. Lorenzini. Arithmetical graphs. Math. Ann., 285(3):481–501, 1989.
[27] Dimitrije Marković and Claudius Gros. Power laws and self-organized criticality in theory and nature. *Physics Reports*, 536(2):41 – 74, 2014. Power laws and Self-Organized Criticality in Theory and Nature.

[28] Criel Merino. The chip-firing game. *Discrete Math.*, 302(1-3):188–210, 2005.

[29] Su. S. Poghosyan, V. S. Poghosyan, V. B. Priezzhev, and P. Ruelle. Numerical study of the correspondence between the dissipative and fixed-energy abelian sandpile models. *Phys. Rev. E*, 84:066119, Dec 2011.

[30] Gunnar Pruessner. *Self-organised criticality : theory, models, and characterisation*. Cambridge ; New York : Cambridge University Press, 2012. Formerly CIP.

[31] A. Sornette and D. Sornette. Self-organized criticality and earthquakes. *EPL (Europhysics Letters)*, 9(3):197, 1989.

(D. Austin) **Department of Mathematics, Kansas State University, Manhattan, KS 66506**

E-mail address: draustin@math.ksu.edu

(M. Chambers) **Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205**

E-mail address: mjchambe@ncsu.edu

(R. Funke) **Department of Math & Computer Science, University of Richmond, VA 23173**

E-mail address: becca.funke@gmail.com

(L. D. García Puente) **Department of Mathematics and Statistics, Sam Houston State University, Huntsville, TX 77341-2206**

E-mail address: lgarcia@shsu.edu

URL: http://www.shsu.edu/ldg005

(L. Keough) **Department of Mathematics, Grand Valley State University, Allendale, MI 49401**

E-mail address, Corresponding author: keough.lauren@gmail.com