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February 1995

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Stochastic Simulation of The Three Dimensional Quantum Vacuum

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Abstract. A complete solution to the long standing problem of basing Schrödinger quantum theory on standard stochastic theory is given. The solution covers all single particle three-dimensional Schrödinger theory linear or nonlinear and with any external potential present. The system is classical, set in a six-dimensional space and involves vacuum polarization as the background process. Basic vacuum polarization energy characterised oscillators are identified and then in assemblies are analysed in terms of energy occurrence frequencies. The orbits of polarization monopoles are given and shown to be elliptical on subspaces surfaces. The basic process takes place at the speed of light and is of a statistical zitterbewegung character. The orthodox quantum probability density bilinear quadratic form is derived from angular momentum consideration within the system which is shown to be a generalisation of the usual quantum structure. Two statistical assemblies are identified, a linear one associated with superposition of eigenfunctions and a quadratic one associated with interactions between eigenstates. It is suggested that this two-tier probabilistic system will remove some possible paradoxes that plague the orthodox theory. The relation of vacuum polarization in this work with its occurrence in other physical contexts and a connection with spins is discussed.

1. Introduction

In a series of earlier papers [12,13,14,15], the author has demonstrated that Schrödinger quantum theory can be alternatively expressed and constructed from classical physical concepts provided that the dimensionality of the background configuration space involved is increased and use is made of the idea of vacuum polarisation to explain and build the structure of the theory. These ideas were then taken further in reference [16], where it was shown that such an alternative theory based on classical physical concepts can be reformulated from a more fundamental level to solve the long standing problem of founding Schrödinger theory soundly on a classical stochastical basis. However, the study of this latter problem which firstly involves identifying the fundamental objects, in this case vacuum oscillators, which are to be subjected to the statistical analysis was carried out in terms of the equivalent of the one dimensional Schrödinger equation. This paper is concerned with a more general argument giving a solution for the important problem of providing a stochastical foundation for Schrödinger quantum theory in three dimensions. The step from the one dimensional case to the three dimensional case does involve some conceptual and technical difficulties. Not the least of which is the need to think in six dimensions as this is the least number of geometrical dimensions for the configuration space that are necessarily invoked in a classical alternative for three dimensional quantum theory. Some of the early work on fluid aspects of the quantum theory, attempts to reformulation the Schrödinger structure and
discussions of such attempts can be found in references [10,20,30,23–26,28]. Work emphasising classical aspects of quantum theory can be found in references [2,3]. The statistical features of the present work represent a substantial generalisation of an idea in reference [21]. Fluid aspects of the present work have been discussed in references [17,18,19].

2. OSCILLATORS

The six dimensional space in which the alternative theory is formulated and in which the physical events underlying the quantum process are considered to be taking place arises from the analytic continuation into complex planes of the three usual configuration coordinates which we shall take to be \(x_1, x_2, x_3\) in \(E^+_3\), the ordinary three dimensional configuration space of common experience. Where necessary, the subscripts on configuration coordinates such as the x’s will be represented by Greek letters. Thus by making the continuations

\[
x_\alpha \rightarrow z_\alpha = x_\alpha + iy_\alpha, \quad \alpha = 1, 2, 3
\]

into complex planes in functions originally of the \(x_\alpha\) variables only, an additional space \(E^-_3\) described by the three \(y_\alpha\) variables is generated. The following work is concerned with showing how events in the six dimensional space \(E_6\) produced by the continuation process (2.1) directly determine the form taken by the usual Schrödinger description, how it supports a sound statistical basis for that description and also makes possible explanations not available in the orthodox theory. It is assumed that the physics of this scheme is taking place in a real \(E_6\) space whilst the complex features of the Schrödinger structure is an aspect of a convenient representation as is the case in some fluid contexts [27]. However, if we start the theory in a real space as is perfectly possible we do eventually have to reform into a complex structure to recover the Schrödinger equation. Thus here we shall go straight to a complex form and note and discuss the possible real representation on the way. The discussion will be confined to quantum systems with energy states forming a discrete set as this type of quantum system is regarded as furthest removed from the classical continuum situation. However, systems with a mixed range of states both continuous and discrete could be covered by the formalism to be discussed with some elaboration. The device that is often used in stochastic systems under such circumstances is the employment of Stieltjes integrals. In this work, that would require some more elaborate discussion of how the energy states are distributed in energy space and the summations over the discrete state index would need be replaced by Stieltjes integrals over the full range of energy states both continuous and discrete.

In the case of a system with \(n\) discrete states of energy \(E_j, \quad j = 1 \text{ to } n\), a complex oscillator for each state \(j\) is introduced described by a polarisation vector \(q_j\), associated with a mass dipole moment \(q_j M_j\), where \(M_j\) is the mass equivalent of the energy of the state \(j\), \(E_j = M_j c^2\). The dipole moments are assumed formed from positive and negative mass monopoles resulting from vacuum polarisation. Such complex oscillators are essentially rotating complex units or unitary complex numbers. Thus we set up a complex oscillator for each state \(j\) with freedom of motion in \(E^+_3\), with phase varying with position in configuration space and also varying with direction of polarisation in configuration space. The prime on \(E^+_3\) is to indicate that complex expression in \(E^+_3\) are being used as we unfold the structure initially. Such an oscillatory process can be described by a complex configuration coordinate three-vector \(\bar{q}_j\),

\[
\bar{q}_j = q_{j,0} \exp(-i\omega_j t)\mathbf{e}_{jp}, \quad (2.2)
\]

where

\[
\mathbf{e}_{jp} = \exp(-i\alpha_j(x,y))\mathbf{e}_p \quad (2.3)
\]
and
\[ e_p = \sum_{\beta=1}^{3} \exp(-i\gamma_{\beta})e_{\beta}. \] (2.4)

e_{jp} is a polarisation phase dependent complex direction vector in \( E_3^+ \) including a state dependent and configuration dependent phase factor \( \exp(-i\alpha_j(x, y)) \) assumed to depend initially on the two unspecified vector parameters \( x \) and \( y \) in an unrestricted way. That is to say that initially any function of \( x \) and \( y \) can be used. \( e_p \) is a complex direction vector in \( E_3^+ \) depending on the three constant arbitrary phases \( \gamma_{\beta} \). \( b_p = e_p/\sqrt{3} \) is a unit vector in the sense that
\[ b_p \cdot b_p^* = 1 \] (2.5)
and consequently,
\[ e_{jp} \cdot e_{jp}^* = 3. \] (2.6)

It is convenient to denote the state dependant part of the phase by the real functions \( \frac{1}{2}\phi_j(x, y, t) = \omega_j t + \alpha_j(x, y) \) so that
\[ E_j = \hbar \omega_j = \hbar \partial (\phi_j/2)/\partial t. \] (2.7)

The three space vector components of \( \bar{q}_j \), denoted by \( \bar{q}_{j,\beta} \) then take the form,
\[ \bar{q}_{j,\beta} = q_{j,0} \exp((\phi_j(x, y, t) + 2\gamma_{\beta})/2i)e_{\beta} \] (2.8)

with the corresponding complex scalar components being
\[ \bar{q}_{j,\beta} = q_{j,0} \exp((\phi_j(x, y, t) + 2\gamma_{\beta})/2i). \] (2.9)

All the components of \( \bar{q}_j \) are solutions of the equation of simple harmonic motion \( \partial^2 \bar{q}_j/\partial t = -\omega_j^2 \bar{q}_j \) with \( \omega_j = E_j/\hbar \). As the structure to be unfolded relies totally on vectors such as \( \bar{q}_j \) this will be seen to have the effect of building all Schrödinger wave motion on simple linear harmonic motion or equivalently planar rotational motion. The \( E_3^+ \) and \( E_3^- \) components of another vector \( q_{l,\beta} \) related to \( q_{j,\beta} \) can now be defined as
\[ q_{+j,\beta} = q_{j,0} \cos((\phi_j(x, y, t) + 2\gamma_{\beta})/2) \] (2.10)
and
\[ q_{-j,\beta} = q_{j,0} \sin((\phi_j(x, y, t) + 2\gamma_{\beta})/2). \] (2.11)

We then have \( q_{l,\beta} = q_{+j,\beta} + iq_{-j,\beta} \) and \( \bar{q}_{l,\beta} = q_{+j,\beta} - iq_{-j,\beta} \) which is why the bar was introduced over the \( q \) at the initial stage (2.2). In three dimensions, setting \( q_{j,0} = 2m_0c^2\hbar/\sqrt{3}E_j \) gives a version of the theory with the physical advantage that monopolar velocities lie in magnitude between \( \pm c \) in both \( E_3^+ \) and \( E_3^- \). This setting of \( q_{j,0} \) also represents the assumption that the planar vacuum polarisation processes are essentially the activation of a fundamental, state independent, energy dipole moment of magnitude \( 2m_0c^2\hbar/\sqrt{3} = hc/\sqrt{3} \) for each of the continuation planes with the state involved being determined by the positive and negative monopolar masses that constitute the activated dipole.

The real six dimensional space representation for an oscillator in the state \( j \) can now be defined as a parameterized vector \( q_j \) given in two sets of three coordinates by,
\[ q_{+j,\beta} = q_{j,0} \cos((\phi_j(x, y, t) + 2\gamma_{\beta})/2)e_{\beta} \] (2.12)
and
\[ \mathbf{q}_{-j,\beta} = q_{j,\beta} \sin((\phi_j(x,y,t) + 2\gamma_\beta)/2)\mathbf{e}_\beta', \tag{2.13} \]
where the real vectors \( \mathbf{e}' \) in \( E_6 \) are equivalent to using a real function \( f(x) \) to make the continuations and identifications,
\[ f(x)\mathbf{e}_\beta \to f(x + iy)\mathbf{e}_\beta \equiv f_1(x,y)\mathbf{e}_\beta + f_2(x,y)\mathbf{e}_\beta' \tag{2.14} \]
or
\[ i\mathbf{e}_\beta \equiv \mathbf{e}'_\beta. \tag{2.15} \]
It should be emphasised that the phase functions \( \phi_j \) are strictly real functions. This can be regarded as a stability condition on the fundamental dipolar oscillators that are being described and simply means that these objects, if part of the system, have a special permanence in that they do not degrade with the passage of time or with being found in displaced positions in space. The vector parameters \( x, y \) can now be identified as the position vectors in \( E^+_6 \) and \( E^-_6 \) respectively with the phase angles \( \phi_j(x,y,t) \) losing some of their arbitrariness. This results from identifying the \( y \) as having the components given by the analytic continuations (2.1) and because the \( \phi_j \) are strictly real they cannot in general be regular functions of the \( z \) and so will be the harmonic components of such functions. The real six dimensional representations for the vectors \( \mathbf{q}_j \) and \( \bar{\mathbf{q}}_j \) of the state \( j \) are given by,
\[ \mathbf{q}_j = (2m_0c^2l_0/\sqrt{3}E_j)\Sigma_{\beta=1}^3(\cos((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta + \sin((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta'), \tag{2.16} \]
\[ \bar{\mathbf{q}}_j = (2m_0c^2l_0/\sqrt{3}E_j)\Sigma_{\beta=1}^3(\cos((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta - \sin((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta'). \tag{2.17} \]
These are real phase point vectors. They are both indicative pointers locating an angular cycling of the system through its phase states as clock hands can be used as indicators in relation to periodicity in time. \( \bar{\mathbf{q}} \) as given in (2.17) also plays the important rôle of position of the positive monopole relative to position of the negative monopole of the oscillating dipole. Thus it is the polar position separation vector for the monopoles of the state \( j \). There are two important characteristics of these fundamental vacuum polarisation units or dipolar oscillators that have been introduced, one kinematical and the other dynamical. The kinematic one is that the magnitude of the velocities \( \beta_j \) and \( \bar{\beta}_j \) of the phase vectors on the continuation planes is always \( c/\sqrt{3} \), as can be seen by differentiating partially with respect to time thus,
\[ \beta_j = \partial\mathbf{q}_j/\partial t = -(c/\sqrt{3})\Sigma_{\beta=1}^3(\sin((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta - \cos((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta'), \tag{2.18} \]
\[ \bar{\beta}_j = \partial\bar{\mathbf{q}}_j/\partial t = -(c/\sqrt{3})\Sigma_{\beta=1}^3(\sin((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta + \cos((\phi_j + 2\gamma_\beta)/2)\mathbf{e}_\beta'), \tag{2.19} \]
where (2.7) has been used. It should be emphasised that the circular motions described by (2.16–19) is deterministic and takes place at the constant speed \( c/\sqrt{3} \) in each of the continuation planes but that at any moment has the total speed of light value \( c \) in \( E_6 \). However, the positions on the circles of the phase point is different in general for the three planes according to the values of the \( \gamma_\beta \) . It should also be noted that so far there are no statistical considerations involved in the construction. When such statistical considerations are involved and consequently there is uncertainty of which energy state the system may be in at any moment of time, the same uncertainty will apply to the local direction of motion at the speed \( c \) exhibited by the system. Under those conditions, it is appropriate to call the process zitterbewegung [7,8,31]. This later process has some resemblance to Brownian motion [20,1,28,25]. The important dynamical property of the basic dipolar oscillators concerns the angular momentum they induce in relation
to the three continuation planes. In each of the three continuation planes \( \alpha = 1, 2, 3 \) a relative angular momentum \( a_{ij,\alpha} \) can be defined using the dipole moments of the oscillators and the velocities (2.16–19) as follows,

\[
a_{ij,\alpha} = (M_j q_{+j,\alpha} - M_i q_{-i,\alpha}) \wedge (\beta_{+j,\alpha} - \beta_{-i,\alpha}),
\]

(2.20)

with

\[
a_{ji,\alpha} = (M_i q_{+i,\alpha} - M_j q_{-j,\alpha}) \wedge (\beta_{+i,\alpha} - \beta_{-j,\alpha}),
\]

(2.21)

when two different states \( i \) and \( j \) are employed or as,

\[
a_{jj,\alpha} = (M_j q_{+j,\alpha} - M_j q_{-j,\alpha}) \wedge (\beta_{+j,\alpha} - \beta_{-j,\alpha}),
\]

(2.22)

when the single state \( j \) is employed. These are relative angular momenta in the sense that they all involve the positive mass monopolar field relative to the negative mass monopolar field.

In evaluating the angular momenta (2.22) from a single dipole or (2.20) and (2.21) from interactions between dipoles of different energies a difficulty associated with working in six dimensions is encountered. In three dimensional space, a pseudo vector product such as \( e \wedge e' \) has a clean interpretation as a vector \( k \) say, orthogonal to the plane of \( e \) and \( e' \). In six dimensions, any two-dimensional plane has a four-dimensional space orthogonal to it. Thus a product such as \( e \wedge e' \) does not have a simple unique vector interpretation in \( E_6 \). This confronts us with some technical difficulties and also with the possibility that there may be advantages in expressing this work in more than the basic six dimensions that are being employed here. However, the dimension issue will be avoided and left open in the present work by employing a self consistent definition for such products that can be used in the two contexts that they are needed without giving them a physical vector identification. Thus a product such as \( e_\alpha \wedge e'_\alpha \) will be denoted by \( k_\alpha \) and assumed to have the properties,

\[
k_\alpha = e_\alpha \wedge e'_\alpha = -e'_\alpha \wedge e_\alpha,
\]

(2.23)

\[
k_\alpha \cdot e_\alpha = k_\alpha \cdot e'_\alpha = 0,
\]

(2.24)

\[
|k_\alpha| = 1.
\]

(2.25)

Besides the need to use the \( k_\alpha \) in evaluating the angular momenta (2.20–22) they are also necessary for the construction of a curl on each of the continuation planes. These curls are then used when statistical considerations are introduced in writing down an average Maxwell equation for each plane which in turn are used to find the electric current densities on the planes. Thus anticipating introduction of the statistics, a special curl \( \nabla_\alpha \wedge \) only applicable to vectors that are restricted by the condition of being parallel with \( k_\alpha \) is defined by the usual three dimensional determinantal form,

\[
\nabla_\alpha \wedge (\chi k_\alpha) = \begin{vmatrix} e_\alpha & e'_\alpha & k_\alpha \\ \partial/\partial x_\alpha & \partial/\partial y_\alpha & \partial/\partial \zeta_\alpha \\ 0 & 0 & \chi \end{vmatrix} = e_\alpha \partial \chi/\partial y_\alpha - e'_\alpha \partial \chi/\partial x_\alpha,
\]

(2.26)

where \( \zeta_\alpha \) is an unspecified extension into the \( k_\alpha \) direction. Thus without identifying the \( k_\alpha \), (2.26) can be used to evaluate curls of quantities of the form \( \chi(x, y, t)k_\alpha \) which are like vectors pointing in the direction of \( k_\alpha \) and are related to the plane \( \alpha \). This is a simple way of handling the aforementioned difficulty. Probably a better, but likely more abstruse scheme can be constructed to deal with this type of difficulty.
The expressions (2.12–13) or (2.16–17) describe the six vector parts of a real deterministic oscillatory dipole associated with the energy state \( j \) located at position \((x, y)\) in \( E_6 \). It is these expressions that describe the behaviour of what will be regarded as a basic type of stable physical unit system. The actual state of the vacuum at any place and time will be described by assemblies of such basic dipolar oscillators according to the number of oscillators for any given energy state \( j \) present in the assembly. Thus the statistical considerations are introduced by studying assemblies of deterministic objects. The physical interpretation for (2.12–13) and (2.16–17) is, apart from the increased number of degrees of freedom and the additional phases \( \gamma_{\alpha} \), exactly the same as for expressions (2.2) and (2.3) in reference [16]. The positive mass monopoles are only free to move in the normal three space of common experience and experiment, \( E_3^+ \), whilst the negative mass monopoles are only free to move in the extra space arising from analytic continuation, \( E_3^- \). Thus \( q_{j,\beta} \) in (2.12) is interpreted as the extension into \( E_3^+ \) of a dipole of mass moment \( M_j q_{j,\beta} \) whilst \( q_{-j,\beta} \) is interpreted as the extension into \( E_3^- \) of a dipole of mass moment \( -M_j q_{-j,\beta} \).

In contrast with \( E_3 \) geometry, it is a feature of the \( E_6 \) geometry as used in this system that for any point \( P \) in \( E_6 \), three pairs of vectors such as \((e_{\beta}, e'_{\beta})\), \( \beta = 1, 2, 3 \) represent three mutually orthogonal planes intersecting at only the one point \( \bar{P} \). On these planes, the projections of the oscillator motion lie on the circular orbits,

\[
q_{j,\beta}^2 + q_{-j,\beta}^2 = (2m_0c^2l_0/\sqrt{E_j})^2, \quad \beta = 1, 2, 3
\]

of radius \(|2m_0c^2l_0/\sqrt{E_j}|\). However, the phase of these circular motions differ from plane to plane according to the values of the constant phases \( \gamma_{\beta}, \beta = 1, 2, 3 \). To analyse the monopolar motions in \( E_3^+ \) and \( E_3^- \) it is convenient to introduce new configuration variables \( X_{\beta} = q_{j,\beta}/q_{j,0} \) for the positive mass monopoles and \( Y_{\beta} = q_{-j,\beta}/q_{j,0} \) for the negative mass monopoles. The analysis using these \( X \) and \( Y \) vectors scales the motions to the same value for all energies so that the \( j \) subscript is temporarily dropped. We find that for all states the motions are confined to two-dimensional planes in \( E_3^+ \) and \( E_3^- \) given by,

\[
X_1 \sin(\gamma_3 - \gamma_2) + X_2 \sin(\gamma_1 - \gamma_3) + X_3 \sin(\gamma_2 - \gamma_1) = 0
\]

(2.28)

and

\[
Y_1 \sin(\gamma_3 - \gamma_2) + Y_2 \sin(\gamma_1 - \gamma_3) + Y_3 \sin(\gamma_2 - \gamma_1) = 0
\]

(2.29)

respectively. On the plane (2.28), the motion of a positive mass monopole lies on an ellipse with projections onto the three coordinate planes of \( E_3^+ \) given by,

\[
X_1^2 + X_2^2 - 2X_1X_2 \cos(\gamma_1 - \gamma_2) = \sin^2(\gamma_1 - \gamma_2),
\]

(2.30)

\[
X_1^2 + X_3^2 - 2X_1X_3 \cos(\gamma_1 - \gamma_3) = \sin^2(\gamma_1 - \gamma_3),
\]

(2.31)

\[
X_2^2 + X_3^2 - 2X_2X_3 \cos(\gamma_2 - \gamma_3) = \sin^2(\gamma_2 - \gamma_3),
\]

(2.32)

On the plane (2.29), the motion of a negative mass monopole lies on an ellipse with projections onto the three coordinate planes of \( E_3^- \) given by,

\[
Y_1^2 + Y_2^2 - 2Y_1Y_2 \cos(\gamma_1 - \gamma_2) = \sin^2(\gamma_1 - \gamma_2),
\]

(2.33)

\[
Y_1^2 + Y_3^2 - 2Y_1Y_3 \cos(\gamma_1 - \gamma_3) = \sin^2(\gamma_1 - \gamma_3),
\]

(2.34)

\[
Y_2^2 + Y_3^2 - 2Y_2Y_3 \cos(\gamma_2 - \gamma_3) = \sin^2(\gamma_2 - \gamma_3),
\]

(2.35)
In the equations for these orbits, the differences relating to different energies have been scaled away by introducing the new variables $X$ and $Y$. However, it is clear from the scale factor $2m_0c^2l_0/\sqrt{3}E_j$ that the larger energy orbits are nested within the smaller energy orbits.

The projections of the motions onto the three continuation planes spanned by the pairs of vectors $(e_\beta, e'_\beta)$, $\beta = 1, 2, 3$ given by the circles (2.27) play a very important part in the development of the theory as these motions generate the internal angular momentum (2.20–22) additional to any angular momentum arising from the motion (2.30–35) that might be present in $E^+_3$ or indeed present in $E^-_3$. This internal angular momentum arises in discrete units of the size of one third of one Planck’s constant on each of the three continuation planes for every single oscillator that might be present at the location $(x, y)$ as can be seen by using the vector products (2.23–25) to evaluate the angular momenta $a_{ij}$ defined at (2.20–22). We find that the interaction angular momentum for two distinct oscillators in energy states $i$ and $j$ involves the two contributions given by

$$a_{ij,\beta} = a_{ji,\beta} = -\left(2m_0cl_0/3\right)\cos((\phi_i - \phi_j)/2)k_\beta$$

(2.36)

while the contribution from one oscillator in self interaction is,

$$a_{ij,\beta} = -(h/3)k_\beta,$$

(2.37)

with the last quantity having the magnitude of one third of one Planck unit $\hbar$. (2.36) and (2.37) both hold for the three continuation planes $\beta = 1, 2, 3$. Thus the basic objects with their characteristics about which in assembled form a statistical or stochastical analysis is to be conducted are identified. These basic objects are dipoles though not of the limiting form of vanishing length and infinite pole strength sometimes employed in other contexts but rather dipoles with finite separation of finite mass weighted pole and antipole. The positive mass pole in motion on one of the ellipses described above in normal configuration space with the negative mass antipole in motion on the corresponding ellipse in the three space generated by analytic continuation. The relative angular motion of the constituent monopolar pair generates an angular momentum in units of $\hbar/3$ as three vectors orthogonal to the projected motions on the three continuation planes. Any one such dipole has a definite energy associated with it given by the formula (2.7), where $\phi_j(x, y, t)/2$ is the phase angle of the oscillating dipole. The restrictions on how the functions $\phi_j/2$ depend on the variables $x, y$ and $t$, are, so far, firstly the condition (2.7) relating them to the energy $E_j$ involving time and secondly the condition that they should be harmonic if the $y$ parameter is a vector with components identified with the imaginary parts of the analytic continuation variables. However, it should be remarked that the class of harmonic functions is a very large class of functions so that the harmonic condition is not very restrictive. Clearly, dipolar objects such as we have defined via their energy and kinematic properties can be conceived as arising out of the zeroeness of the vacuum by polarisation. The nature and properties and statistical aspects of assemblies of such objects spread about configuration space and their relation with three dimensional Schrödinger theory will be discussed in the next sections.

3 ASSEMBLIES

Having identified the oscillatory dipolar basic objects, it is now possible to consider assemblies of them involving collections locally with numbers of individual members at various energies $E_j$ and with different phases at different positions in $E_j$. This approach represents an uncomplicated, unambiguous and indeed a standard way to introduce probabilistic considerations into the analysis of an initially deterministic situation. Firstly, let us consider a local collection of oscillators composed of $n_i$ members in the energy state $E_i$ for $i = 1$ to $n$. The total internal angular
momentum on the plane $\beta$ generated by all the $n^2$ interactions in the assembly between the oppositely signed fields from different energy oscillators and including self interactions is

$$a_\beta = \Sigma_{ij} n_i n_j a_{ij,\beta} = -(\hbar/3) k_\beta (\Sigma_i n_i \exp(\phi_i/2i))(\Sigma_i n_i \exp(-\phi_i/2i)), \tag{3.1}$$

where $(2.36–37)$ have been used. Denoting the factors in $(3.1)$ by $\Psi$ and $\Psi^*$ which are in this paper dimensionless functions gives,

$$\Psi(x, y, t) = \Sigma_i n_i \exp(\pm \phi_i(x, y, t)/2i), \tag{3.2}$$
$$\Psi^*(x, y, t) = \Sigma_i n_i \exp(-\phi_i(x, y, t)/2i), \tag{3.3}$$

and

$$a_\beta = -(\hbar/3) k_\beta \Psi^*(x, y, t)\Psi(x, y, t) = -(\hbar/3) k_\beta \rho^{(0)}(x, y, t), \tag{3.4}$$

having introduced the useful dimensionless function $\rho^{(0)} = \Psi^*\Psi$. The dependence of the functions $\Psi^*$ and $\Psi$ on $x$ and $y$ only arising through the harmonic functions $\phi_i(x, y, t)$ involved in their structure. It is interesting to consider the total internal angular momentum $a$ contributed by the motions on all the continuation planes. Using $(3.4)$, this can be written in the form $a = -(\hbar k_\beta/\sqrt{2}) N_0$, where the unit vector $k = \Sigma_{\beta=1}(1/\sqrt{3}) k_\beta$ has been introduced and $N_0 = \sqrt{2/3} \rho^{(0)}$. The unit vector $k$ can have components in $E^+_3$ and $E^-_3$. Denoting these components by $\cos(\chi)$ and $\sin(\chi)$ respectively we see that $a$ can be written in the form, $a = -(\hbar/\sqrt{2})(k_+ \cos(\chi) - k_- \sin(\chi)) N_0$. Confining our attention to the one special case when the angle $\chi$ has the value $\pi/4$, the total relative internal angular momentum becomes $a = -(\hbar/2)(k_+ - k_-) N_0$. Thus in this case the total relative internal angular momentum $a$ is composed of $N_0$ differences of half integral spin units. If the spins in the $E^+_3$ direction and the $E^-_3$ direction are regarded as coming from the positive and negative monopoles respectively then it becomes apparent that the polarisation process under consideration in this paper is responsible for the generation of the equal magnitude spin vectors $(\hbar/2) k_+$ and $(\hbar/2) k_-$. However, it should be noted that being in orthogonal spaces, the directions of these spins are not opposite in the way that opposite monopolar charges or masses are opposite. The other three special cases given by $\chi = 3\pi/4, 5\pi/4$ and $7\pi/4$ which are related to spin flip through an angle of $\pi/2$ or spin reversal can be analysed in a similar manner to the $\pi/4$ case. Thus spin and vacuum polarisation appear to be very closely related and the vacuum polarisation structure used in this theory implies the existence of spin.

The derivation given in this paper, of the form $(3.4)$ from the energy state assembly contains the solution to the long standing conceptual problem of why, when in quantum mechanics state functions are superposed as in $(3.2)$ and $(3.3)$, probability takes on a bilinear form such as is contained in $(3.4)$. Such a structure is not found in what is called classical probability theory. It emerged in quantum theory in the form of an ad hoc rule for constructing probabilities from wave functions. It is clear that the bilinear form $N_0 = \sqrt{2/3} \rho^{(0)}$ gives the number of Planck half angular momentum units arising from all of the three continuation planes generated by all the interactions between dipolar oscillators in the assembly of states of varying energy contributed by the positive mass monopoles. It similarly gives the number of subtracted half spin units contributed by the negative mass monopoles. Thus the number of Planck units depends directly on the higher assembly of interactions rather than as one might at first expect on the assembly of energy states. In this version of the quantum structure, the roles of and relations between these two assemblies is totally transparent. However, in the orthodox theory there is no explanation for the relation between the bilinear form and the superposition principle other than the ad hoc rule of thumb for constructing probabilities from wave functions.
The quantity $N_0$ is essentially the number of Planck half units contributed by monopoles activated by the planar motions and generated by the angular momentum interactions between the energy states. It is also effectively the number measure of various other physical quantities induced or present in the $E_6$ space. Assuming that such quantities are produced in or reside in some standard constant volume $v_0$ which according to context represents six, three or the two dimensional volume we call area, then a quantity $\rho = N_0/v_0$ can be defined that will apply throughout the space and represent number density for whatever dynamical quantity is under discussion be it angular momenta units, electric dipoles, magnetic dipoles or indeed some other local characteristic quantity. Such a number density is, of course, not a probability density but often it can be converted to a probability density by suitable normalisation. Thus if it is desired to keep strictly within the remit of probabilistic stochastic theory the number frequencies $n_j$ for energy states present in an assembly and the interaction number frequencies $n_in_j$ for interactions between energy states can be used to define the two-level linear and quadratic sets of probabilities $P_j$ and $P_{ij}$,

$$P_j = n_j/N, \quad N = \Sigma n_i$$

(3.5)

and

$$P_{ij} = n_in_j/N_f, \quad N_f = (\Sigma n_i)^2.$$  

(3.6)

The linear probabilities $P_j$ are directly related to the linear superposition principle of quantum mechanics that is generally used to combine eigenstates as in (3.2). The quadratic probabilities $P_{ij}$ are closely related to the probabilities usually employed in quantum mechanics and they measure the weight carried by the interactions between different states and include self interactions of single states as in the angular momentum structure discussed earlier. Thus two different expectation values can be defined; a linear one $E_L[.]$ for use with averaging vector like quantities such as a set of unitary exponential oscillator amplitudes $u = \{\exp i\phi_i\}$ and a quadratic one $E_Q[.]$ for use with averaging matrix like quantities such as the set of angular momentum interactions between vector like components $a = \{a_{ij}\}$. Thus using the probabilistic formulation the wave function can be expressed as,

$$\Psi = NE_L[u],$$  

(3.7)

while the number function $\rho^{(0)}$ can be expressed as,

$$\rho^{(0)} = -\sqrt{3}N_fE_Q[a \cdot k/\hbar].$$  

(3.8)

The clear separation of these different statistical measures should prevent the intrusion and the assuming of undue importance by those paradoxical issues \[23,29,5\] that arise from confusing the linear and the quadratic assembly. The connection between the linear superposition principle and the number of Planck units to be found at any position on the configuration planes is an aspect of a wider issue of the significance of the nonlinear exponential transformation that the wave function represents. The insight used here with regard to how the number of Planck units depends on the higher assembly of interactions and its significance for quantum theory, was a direct result of a conversation between the Author and Professor M. Atiyah on nonlinearity in 1986.

It has been implicitly assumed so far that given the harmonic functions $\phi_j(x, y, t)$, the assembly of energy states is constructed with the two vector parameters $x, y$ fixed. When these two parameters are taken to represent a position in $E_6$ this implicit assumption is equivalent to setting up the energy assembly at the six-point $(x, y)$. The space dependence of the assembly can be further extended by allowing the assembly frequencies $n_j$ to themselves depend on position.
This has the effect of setting up differing energy assemblies from point to point in $E_n$. Taking this step, we now introduce a set of arbitrary assembly frequencies $n_i$ as follows,

$$n_j = N_j n_{ci}(x, y), \quad j = 1 \text{ to } n.$$  \hfill (3.9)

The functions $N_j$ are absolute constants and the function $n_{ci}(x, y)$ depend only on $x$ and $y$. The function $\Psi$ introduced at (3.2) now assumes the form,

$$\Psi(x, y, t) = \sum_i N_i n_{ci}(x, y) \exp(\phi_i(x, y, t)/2i).$$  \hfill (3.10)

If the $y$ vector in these functions is identified as arising from the analytic continuation process (2.1) or $\Psi(x, y, t)$ is taken to be of the form $\Psi(x + iy, t)$, it then follows that $\ln(\Psi(x, y, t))$ can be taken to be a regular function of the complex vector $z = x + iy$. Uniformity of structure and the fact that the $\phi_i$ are harmonic then implies that the same argument can be applied to every term in the finite sum in (3.10) to give the result that the eigen complex fluid potential functions,

$$w_i(z) = 2\nu(\phi_i(x, y, t)/2 + i\ln(n_{ci}(x, y))),$$  \hfill (3.11)

are regular functions of $z$. This is a dramatic conclusion because it implies that although the basic oscillators are stable against degradation in time or space their phase is soft in the sense that their distribution information in space as given by $\ln(n_{ci}(x, y))$ affects their phase angles as given by $\phi_i(x, y, t)/2$ because these two function are related by harmonic conjugacy. Thus it is impossible to set up an arbitrary space distribution of assembled energy oscillators with preset harmonic phases in the structure being examined here. The conclusion is dramatic because it has been arrived at without any use of a Schrödinger or equivalent equation that would mould the spatial form of dependence of the ensembles. It leads to the significant conjecture that at some fundamental level information and phase are linked each being obtainable from the other by methods such as the Hilbert transform. This, of course, is reminiscent of the causality context and its attendant dispersion relations [32].

4 CURRENTS

The angular momenta (3.4) arising from the oscillators and associated with the continuation planes is a consequence of a relative rotational motion of the state monopoles as given by the circular orbits (2.27). As the state monopoles carry charges proportional to their masses it is to be expected that the same rotational motion will produce a magnetic moment which can be obtained from the angular momentum $a$ using the classical gyromagnetic ratio, $-|e|/2m_0$. This gives for the magnetic moment density induced on the continuation planes,

$$g_{mag,\beta} = +(|e|c_0/3)k_\beta \rho_0(x, y, t)/v_0 = (|e|c/\sqrt{6})k_\beta \rho(x, y, t),$$  \hfill (4.1)

where the dipole density function $\rho$ has been introduced via a standard constant comparison volume $v_0$ as $\rho = N_0/v_0$. The equivalent magnetic induction vector for the plane can then be written as,

$$B_{0,\beta} = \mu_0 g_{mag,\beta}.$$  \hfill (4.2)

This quantity is a characteristic of the assembly depending as it does on the frequencies $n_j$ of the distribution of energy states. If $B_{0,\alpha}$ is used in a Maxwell equation on the plane $\alpha$ to determine the surface electric currents that it may be regarded as having induced or that it may have been
induced by, then these currents themselves will be characteristic of the assembly. That is to say the currents are in some sense average currents. The three Maxwell equations to be used are,

$$\rho_0 \mathbf{J}_\alpha = \nabla \times (\mathbf{B}_{0,\alpha}), \quad \alpha = 1, 2, 3. \quad (4.3)$$

The \( \mathbf{J}_\alpha \) are three two-dimensional current densities for the three planes. We see from (4.1) that the density of Bohr magneton units \(|e|c_0\rho_0 \) generated on the continuation planes is \( \rho/\sqrt{6} \). Thus for consistency we use a Maxwell equation to relate this to a current density \( \mathbf{J}_\alpha \) with the numerically same density of electric charges \(|e| \). This is where the special curls (2.26) are needed and when used in (4.3) with (4.2) the currents \( \mathbf{J}_\alpha \) are found to be

$$\mathbf{J}_\alpha = -(|e|/\sqrt{6})\rho(x,y,t)\tilde{\beta}_\alpha, \quad (4.4)$$

where

$$\tilde{\beta}_\alpha = \mathbf{u}_\alpha - \mathbf{v}_\alpha \quad (4.5)$$

or in terms of \( \rho^{(0)} \)

$$\tilde{\beta}_\alpha = -c_0(\partial \ln \rho^{(0)}(x,y,t)/\partial y_\alpha \mathbf{e}_\alpha - \partial \ln \rho^{(0)}(x,y,t)/\partial x_\alpha \mathbf{e}'_\alpha). \quad (4.6)$$

\( \tilde{\beta} \) given by (4.5) is an average local velocity field of the positive monopoles relative to the negative monopoles on the plane. So far, the only basic concepts introduced are vacuum polarisation, analytic continuation and assemblies with their statistical connotations. However, the structure that has emerged looks much like the usual Schrödinger theory. In fact, a generalisation of orthodox three-dimensional linear Schrödinger theory has been produced. All that is needed to recover the subcases of linear or indeed nonlinear Schrödinger theory is the important classical constraint of some type of continuity equation for density describing its transport about the configuration space and possible input or output by external interference. It turns out that the continuity equation is the source of both the Schrödinger equation for the system and also of the space time governing stochastic differential equation. Equivalent equations of flow continuity can be written down in \( E_3^+ \), \( E_3^- \) or in \( E_6 \). In the latter case, the appropriate form is,

$$\partial \rho/\partial t = -\nabla \cdot (\rho \tilde{\beta}^{(c)}(x,y,t) + \Gamma + \Gamma_{fb}) \quad (4.7)$$

where \( \nabla \cdot \) is the six-dimensional divergence in \( E_6 \) and \( \tilde{\beta}^{(c)} \) is the six dimensional centroidal monopolar velocity field \( \beta^{(c)} = (\mathbf{u} + \mathbf{v})/2 \) for monopolar pairs and is obtainable from the \( \mathbf{u}, \mathbf{v} \) components of (4.5) or (4.6). The source term \( \Gamma \) which is of the form \( \rho V_2(x,y)/\rho_0 c_0 \) is always necessary if there is an external field present but is generally zero on the subspace \( E_3^+ \) where \( y = 0 \) as \( V_2 \) is the imaginary part of the usually real external potential \( V_1(x) \). That is to say \( V_2(x,y) \) is chosen to be the harmonic conjugate of some specific function \( V_1(x,y) \) so that together they form a regular function \( V(\mathbf{z}) = V_1(x,y) + iV_2(x,y) \) of \( \mathbf{z} = x + iy \). This choice of \( V_2 \) is important for the derivation of the Schrödinger equation when \( V_1 \) is recognised as the usual external potential. The source term \( \Gamma_{fb}(x,y,t) \) will be taken to be zero in this paper but is an additional feedback contribution involving some functional form of the density that is necessary if the structure is used to analyse nonlinear Schrödinger systems [17]. If the expressions for \( \mathbf{u} \) and \( \mathbf{v} \) are substituted into (4.7), the equation of centrifugal diffusion in \( E_6 \),

$$\partial \rho/\partial t = \nu \Sigma_{\alpha=1}^3 \partial^2 \rho/\partial x_\alpha \partial y_\alpha + \Gamma, \quad (4.8)$$

is obtained with centrifugal diffusion constant \( \nu = c_0 \). The feedback term is omitted from (4.8) onwards while we concentrate on linear Schrödinger theory. Equation (4.8) is the basic
stochastical differential equation that governs the process described by the distribution as it unfolds in the six dimensional configuration space against time. This equation fully explains the structure of orthodox linear Schrödinger theory as a rotatory diffusive dynamical equilibrium of the dipolar density $\rho$ in $E_6$ [19].

The Schrödinger equation is obtained from (4.8) by substituting into it the bilinear form for the density $\rho = \sqrt{2/3} \rho^{(0)}(x, y, t)/\nu_0$ formed from $\Psi^*$ and $\Psi$ in (3.4). The quickest route to the Schrödinger equation is to replace the second order differential operators in (4.8) by their $z, \bar{z}$ equivalents,

$$\frac{\partial^2}{\partial x_\alpha \partial y_\alpha} = i(\partial^2/\partial z^2_\alpha - \partial^2/\partial \bar{z}^2_\alpha)$$

with

$$\rho^{(0)}(x, y, t) = \Psi^*(\bar{z}, t)\Psi(z, t).$$

The asterisk superscript on $\Psi$ denotes the complex conjugate of the functional form only. Thus the full complex conjugate denoted by the top bar is given by $\bar{\Psi}(z, t) = \Psi^*(\bar{z}, t)$. We find that

$$V_2(x, y) = 2Im.((m_0 \nu/\Psi(z, t))(i\partial \Psi(z, t)/\partial t + \nu \Sigma_{\alpha=1}^3 \partial^2 \Psi(z, t)/\partial z^2_\alpha)).$$

(4.11)

However, $V_2(x, y)$ was chosen to be the harmonic conjugate of some definite function $V_1(x, y)$. It follows that there is a regular function $V(z)$ of the complex vector $z$ such that

$$V(z) = V_1(x, y) + iV_2(x, y) = (\hbar/\Psi(z, t))(i\partial \Psi(z, t)/\partial t + \hbar/2m_0 \Sigma_{\alpha=1}^3 \partial^2 \Psi(z, t)/\partial z^2_\alpha).$$

(4.12)

From (4.12), the analytically continued three dimensional linear Schrödinger equation with complex external potential $V(z)$ follows,

$$i\hbar \partial \Psi(z, t)/\partial t = -(\hbar^2/2m_0) \Sigma_{\alpha=1}^3 \partial^2 \Psi(z, t)/\partial z^2_\alpha + V(z)\Psi(z, t)$$

(4.13)

and this clearly reduces to the usual $E^+_3$ Schrödinger equation,

$$i\hbar \partial \Psi(x, t)/\partial t = -(\hbar^2/2m_0) \nabla^2_x \Psi(x, t) + V_1(x)\Psi(x, t).$$

(4.14)

of the orthodox theory on the boundary $y = 0$, where usually $V_2 = 0$.

5 CONCLUSIONS

In this paper, it has been shown that three dimensional Schrödinger quantum theory can be soundly based on classical stochastic theory by setting up a simple probabilistic structure using assemblies of oscillating mass dipoles structured according the frequencies of occurrence of discrete dipolar energy values. Classical physical ideas only are employed in this generalisation and reformulation of quantum theory within a framework of not greatly advanced or abstruse mathematics. The term classical is emphasised here and as far as the physics is concerned, it means essentially the type of system and formalism that is presented in books on dynamics, hydrodynamics and electro-fluids such as contained in references [22], [9] and [27] where operators are not used to represent dynamical quantities. The classical stochastic theory can be found in reference [1] and is characterised by the involvement of firmly established probability theory or standard statistical method without involving any new forms of logic introduced to patch up paradoxes [29] real or apparent. With regard to the mathematics employed in this new foundation for quantum theory, it has been the author’s policy to use only straightforward or hesitatingly what one might call simple mathematical methods such as is used in the very clear treatment of complex variable theory that can be found in reference [4]. However, it is likely that the
formalism might well be better expressed and further advanced conceptually by the use of higher level techniques such as differential forms [6].

In this work, heavy use is made of the idea of vacuum polarisation by employing the negative energy states that first assumed importance in classical relativity theory when Einstein introduced the energy equation \((E/c)^2 = p^2 + (mc)^2\) for the free relativistic particle with its two energy states \(E = \pm |E|\). However, vacuum polarisation has assumed great importance even in the orthodox theory in the development of the theory of quantized fields [31] where it is deeply involved in the rôle of enabling the divergent mathematics of that area to be patched up by renormalisation so as to yield finite measurable information. Thus vacuum polarisation has to be used to rescue orthodox theory from very severe internal mathematical difficulties. Dirac [11] made a drastic contribution to the topic of vacuum polarisation when he introduced the postulate that the sea of negative energy states were occupied. This was to eliminate difficulties with the negative energy concept such as state evolution in reversed time and it led eventually to the real particle, the anti-electron, or positron being identified in the orthodox theory. However, it had the effect of drenching the negative energy states almost out of existence except that their contribution had to be taken into account in the form of effectively minute corrections to measurable quantities calculated by the renormalisation process introduced to render quantum electrodynamics a viable theory. Certainly at the technical level Dirac’s postulate made possible great progress with the structural edifice of quantized fields. From the point of view of obtaining philosophical understanding of the basis of quantum theory at the analytic level, Dirac’s postulate disposed of the baby with the bath water. In the scheme described in the present paper, the negative energy states are dealt with in a way that is not entirely different from that suggested by Dirac. Here negative energy states are used in a balanced way with the positive energy states so that together they form the statistical and dynamical net of the quadratic assembly and consequently, while in a sense mutually cancelling as positive and negative monopoles, they together cradle the actual measurable energy of the system. An Additional interesting result that is produced by this work is the classical explanation for spin as being a type of vector monopolar unit attached to the basic mass monopoles and generated by vacuum polarisation. This connection has emerged as an unexpected bonus from this hyperspace alternative to Schrödinger theory. The work seems to imply that spin as a kinematic rotational polarisation effect is at the foundation of all quantum theory.

Confirmation that this stochastically founded structure is a generalisation of Schrödinger theory has now been demonstrated [33] by its application to obtain the very simple formula for the fine structure constant, \(\alpha = \cos(\pi/N)/N\). This formula gives the value of \(\alpha\) to very great accuracy when \(N\) is set equal to 137. \(\alpha = \cos(\pi/137)/137 \approx 7.297351 \times 10^{-3}\). The numerical value given by this formula has an error in relation to the latest experimentally determined value [34] that is of the order of 2 parts in \(10^9\).

It has been shown that the relativistic negative energy states can be used to form a completely sound stochastic basis for three dimensional Schrödinger theory with the form of that basis having a high geometrical and structural visualisability in terms of classical assemblies of orbiting monopolar particles. This is in striking contrast to the deadpan visage of the admittedly otherwise technically highly efficient orthodox theory. The author believes that the accessibility to the underlying quantum process that the present theory achieves, both in terms of structure recognition and picturability of the processes taking place in the quantum vacuum background, will lead to progress with quantum fundamentals and also lead to a better understanding of space time structure in general. Certainly, the work puts the quantum process into a clearer perspective with regard to its relation to precursor theories, showing it to be of the same family and not an alien mutation with its own logic. One of the important conclusion that stands out from the present work is that although the orthodox theory does involve physical jumps and discontinuities
in energy and other physical variables the extrapolation to the idea that quantum mechanics is a philosophical jump away from the older classical ideas is false. This holds the implication that if there are paradoxes [29,5] present in the quantum theory that are real rather than apparent, the same paradoxes must already be present in the classical ways of thinking about the physical world.

ACKNOWLEDGEMENTS: I am very grateful to Professor M.S. Bartlett for the remarks that many years ago set me off on the search for a sound stochastic basis for quantum theory and to Professor Sir Michael Atiyah who more recently drew my attention to the central importance of the nonlinear relation between the quantum fluid complex potential and the space of wave function linear superposition.

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