COUNTING REDUCIBLE MATRICES, POLYNOMIALS, AND SURFACE AND FREE GROUP AUTOMORPHISMS

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Abstract. We give upper bounds on the numbers of various classes of polynomials reducible over \( \mathbb{Z} \) and over \( \mathbb{Z}/p\mathbb{Z} \), and on the number of matrices in \( \text{SL}(n) \), \( \text{GL}(n) \) and \( \text{Sp}(2n) \) with reducible characteristic polynomials, and on polynomials with non-generic Galois groups. We use our result to show that a random (in the appropriate sense) element of the mapping class group of a closed surface is pseudo-Anosov, and that a random automorphism of a free group is irreducible with irreducible powers. We also give a necessary condition for all powers of an algebraic integers to be of the same degree, and give a simple proof (in the Appendix) that the distribution of cycle structures mod \( p \) for polynomials with a restricted coefficient is the same as that for general polynomials.

Introduction

In this paper we use simple algebraic, geometric, and probabilistic ideas to investigate the probability that a random (in a suitable sense) polynomial with integer coefficient is reducible (over \( \mathbb{Z} \)) and that a random (in a suitable sense) matrix in one of the classical groups \( \text{GL}(n, \mathbb{Z}) \), \( \text{SL}(n, \mathbb{Z}) \) or \( \text{Sp}(n, \mathbb{Z}) \) and also in \( M^{n\times n}(\mathbb{Z}) \) has irreducible characteristic polynomial. We use these results (following an idea of I. Kapovich) to show that for generating set of the mapping class

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\(^1\)To avoid encumbering the notation, we state the results for \( \text{SL}(n, \mathbb{Z}) \). The results and the proofs for \( \text{GL}(n, \mathbb{Z}) \) are essentially identical.
group, a sufficiently long random product of generators is almost certainly pseudo-Anosov\(^2\).

The plan of the rest of the paper is as follows: In Section 1, we discuss some generalities on elimination theory. In Section 2, we apply the results of Section 1 to gain insight into sets of polynomials with factors of certain types, in particular on the growth of the cardinality of these sets as a function of height. In Section 3, we apply the results of Sections 1 and 2 to the study of sets of matrices in \(M^{n \times n}(\mathbb{Z})\) whose characteristic polynomials are reducible, and again, to get estimates on the growth of these sets as a function of height (the size of coefficients). In Section 4, we will use a quite different method to show that the density of “reducible” elements in \(\text{Sp}(2n, \mathbb{Z})\) goes to zero as a function of the combinatorial distance of the elements to the identity. In Section 5, we use our results to show that a random element of the mapping class group of a closed surface of genus \(g\) is pseudo-Anosov. In Section 6, we show (using results of the Appendix) that the characteristic polynomial of a random matrix in \(\text{GL}(N, \mathbb{Z})\) has characteristic polynomial with Galois group \(S_N\), and also that all powers of such a random matrix have the same property. In Section 7, we apply our results to show that a random free group automorphism is strongly irreducible (what is more commonly known in the trade as “irreducible with irreducible powers”).

1. Generalities on elimination

Consider the following setup: we have a parametrized surface \(S\) in \(k^n\) (for \(k\) an algebraically closed field), that is:

\[
\begin{align*}
    x_1 &= f_1(s_1, \ldots, s_m), \\
    x_2 &= f_2(s_1, \ldots, s_m), \\
    & \vdots \\
    x_n &= f_n(s_1, \ldots, s_m),
\end{align*}
\]

where \(f_1, \ldots, f_n\) are polynomials in \(s_1, \ldots, s_m\). It is reasonable to believe that \(S\) is an algebraic \(m\)-dimensional variety in \(k^n\), that is, the simultaneous zero-set of \(n - m\) polynomial equations. That turns out to not be exactly true, but what is true is that the Zariski closure of \(S\) is an (at most) \(m\)-dimensional variety. For a proof of this Closure Theorem and plenty of examples see [4][Chapter 3].

\(^2\)A closely related result on the mapping class group was shown by completely different methods by J. Maher in [12].
2. Applications to polynomials

Let $\mathcal{P}$ be the set of all monic polynomials in one variable of degree $d$ over a field $F$, which have a polynomial factor with constant term $\alpha$. Let us identify the set of all monic polynomials of degree $d$ with the affine space $F^d$. Then, we have the following:

**Theorem 1.** The set $\mathcal{P}$ is contained in an affine hypersurface of $F^d$.

*Proof.* Let

$$p(x) = x^d + \sum_{i=0}^{d-1} a_i x^i \in \mathcal{P}.$$  

By assumption, $p(x) = q(x)r(x)$. Assume that the degree of $q(x) = m$, while the constant term of $q(x)$ equals $\alpha$. Writing

$$q(x) = x^m + \sum_{j=1}^{m-1} b_j x^j,$$

and

$$r(x) = x^{d-m} + \frac{a_0}{\alpha} + \sum_{k=1}^{d-m-1} c_k x^k,$$

we find ourselves exactly in the setting of Section 1. The proof is almost complete, except for the fact that we do not know the degree of $q(x)$ *a priori*. However, each choice of $m$ gives us a polynomial $H_m$ vanishing at all the coefficient sequences of reducible polynomials with a factor of degree $m$, and so the product of $H_m$ over all $m$ vanishes at *all* the coefficient sequences of reducible polynomials. $\square$

3. Counting points on varieties

Let $S$ be a variety of dimension $m$ in $k^n$. Consider a reduction of $S$ modulo $p$.

**Theorem 2** (Lang-Weil, [11]). *The number of $F_p$ points on $S$ grows as $O(p^m)$. The implied constant is uniform (that is, it is a function of the dimension and codimension of the variety only).*

It should be noted that this gives an upper bound only. There might well be no $F_p$ points on $S$.

The following corollary is also classical (and easy):

**Corollary 3.** *Let $S$ be as above. Then the number of points of $S \cup \mathbb{Z}^n$ all of whose coordinates do not exceed $B$ in absolute value grows at most as $O(B^m)$.***
Proof. Pick $B$. By Bertrand’s postulate there is a prime $p$, such that $4B > p > 2B$. We know that every integer point of $S$ will give a (distinct) point on the reduction of $S$ modulo $p$ (the converse, of course, is not true). The result follows.

We have used

**Theorem 4** (Bertrand’s Postulate - proved by Chebyshev). *For any $N > 3$ there exists at least one prime $p$ between $n$ and $2n - 2$.*

### 4. More applications to polynomials

The results in Section 3 combined with the results in Section 1 immediately give the following results:

**Theorem 5.** Let $P_1(d, B)$ be the set of polynomials of degree $d$ with integer coefficients bounded in absolute value by $N$ and constant coefficient 1, and let $R_1(d, B)$ be the set of polynomials reducible over $\mathbb{Z}$ with the same coefficient bound. Then, $R_1$ lies on an algebraic hypersurface $C^{d-1}$ (where the coordinates are the coefficients), and consequently

$$\frac{R_1(B)}{P_1(B)} = O\left(\frac{1}{B}\right).$$

*Proof. A factor of a polynomial in $R_1(d, B)$ must have constant term $\pm 1$, The statement now follows immediately from the results in Sections 1 and 2.*

4.1. **Arbitrary polynomials.** What happens if we don’t require the constant coefficient to be 1? Consider the set $F(d, a)$ of all monic polynomials of degree $d$ and with constant term $a$. Clearly, the constant term of a divisor of such a polynomial must have constant term $d$ dividing $a$, and so for each $c \mid a$ we have a subvariety of $F(d, a)$ of polynomials having a factor with constant term $c$. The arguments above apply without change, and we see that the number of such polynomials modulo $p$ grows at most as $O(p^{d-2})$, where the constant is uniform. Denoting the number of divisors of $a$ by $\tau(a)$, it is not hard to see that $\tau(a) = o(a)$. Indeed, since the number of divisors is a multiplicative function,

$$\tau(n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = (\alpha_1 + 1) \cdots (\alpha_k + 1) < 2 \log_2 n,$$

whereupon the assertion follows easily.

So, it follows that for any $a$, the set of reducible polynomials is a union of $o(a)$ subvarieties of $F(d, a)$. To show that most polynomials with coefficients bounded by $B$ in absolute value are irreducible, we use Bertrand’s postulate to find a prime $p$, such that $2B < p <
Since the set of reducible polynomials lies on the union of $o(B^2)$ codimension two subvarieties, their total number is $o(p^d)$, while the total number of monic polynomials is $B^d \geq 2^{-d}p^d$, and we have our result.

5. Reciprocal Polynomials

We say that a polynomial $p(x) \in P_1(d)$ is reciprocal if $x^d p(1/x) = p(x)$ – in other words, the list of coefficients of $p$ is the same read from left to right as from right to left. Reciprocal polynomials can also be defined as follows: A (monic) polynomial (of even degree $2n$) is reciprocal if it can be written as

$$\prod_{j=1}^{n} (x - r_j)(x - 1/r_j) = \prod_{j=1}^{n} j = 1^n(x^2 - (r_j + 1/r_j)x + 1).$$

Notice that this means that every reciprocal polynomial lies on our “factorization variety”3, and so the methods do not work directly. However, we can get around this with a trick.

Note that any reciprocal polynomial in $x$ of even degree $2n$ can be written (uniquely) as a multiple (by $x^n$) of a polynomial $g(y)$ in $y = x + 1/x$ of degree $n$. The proof is very simple: Dividing through by $x^n$, we write

$$f(x) = a_n + \sum_{i=0}^{n-1} a_i(x^{i-n} + x^{n-i}).$$

Note that $(x+1/x)^n$ is a reciprocal polynomial, and so is $f(x)-(x+1/x)^n$, which is also of lower degree than $f(x)$. The result now follows by induction (notice that the coefficients of $g$ are integer linear combinations (whose coefficients depend only on the degree of $f$) of the coefficients of $f$, and, obviously, vice versa.

Now, it is clear that in order for $f(x)$ to be reducible, $g(y)$ must be also. Indeed, suppose $f(x) = f_1(x) \ldots f_k(x)$, where the $f_i$ are irreducible. Since $x^2nf(1/x) = f(x)$, it follows that $f(x) = \prod_{j=1}^{k} x^{\deg f_j(x)}f_j(1/x)$. By the irreducibility of $f_j(x)$, it follows that either $f_j$ is a reciprocal polynomial, or $f_j$ is the reciprocal of some $f_j$, in which case $f_j(x)f_j(x)$ is a reciprocal polynomial. So, $f(x)$ has a reciprocal factorization, and so $g(y)$ is reducible.

We now reason as in Section 4B, but with polynomials $g(y)$ replacing $f(x)$.

3 the author thanks N. Katz for the suggestion of using this term
6. Applications to matrices

6.1. The special linear group. Consider first the matrix group $\text{SL}(n,k)$. Since the coefficients of the characteristic polynomial of a matrix $M$ are polynomials in the entries of $M$, and the dimension of $\text{SL}(n)$ is $n^2-1$, we see that

**Lemma 6.** The number of matrices in $\text{SL}(n,p)$ whose characteristic polynomial has a factor over $F_p$, with constant term 1, grows as $o(p^{n^2-1})$.

**Proof.** The proof requires one additional observation: that every monic polynomial $p(x)$ of degree $d$ with constant term 1 is the characteristic polynomial of some matrix in $\text{SL}(d)$ – namely the companion matrix of $p(x)$. It follows that the set of matrices whose characteristic polynomial satisfies the assumptions of the Lemma lies on an algebraic subvariety of $\text{SL}(n)$, and the result follows by Lang-Weil. □

**Corollary 7.** The probability that a matrix in $\text{SL}(n,p)$ satisfies the hypotheses of Lemma 6 goes to 0 as $p$ goes to infinity.

**Proof.** The order of $\text{SL}(n,p)$ is well known to be $p^{n^2(n-1)/2}(p^2-1)(p^3-1)\cdots(p^n-1) \sim p^{n^2-1}$, (see Newman’s book [13][VII.17]). The assertion of the corollary follows immediately. □

Unfortunately, since the number of integral points on $\text{SL}(n,\mathbb{Z})$ of height (absolute value) bounded by $B$ grows much slower than $B^{n^2-1}$ the above results do not imply the following

**Conjecture 8.** The probability that a matrix in $\text{SL}(n,\mathbb{Z})$ with coefficients bounded by $B$ has reducible characteristic polynomial goes to 0 as $B$ goes to infinity.\(^4\)

But since we know that the number of points on $M^{n \times n}(\mathbb{Z})$ of height bounded by $B$ grows like $B^{n^2}$, we do have

**Theorem 9.** The probability that a matrix in $M^{n \times n}(\mathbb{Z})$ with coefficients bounded by $B$ has reducible characteristic polynomial goes to 0 as $B$ goes to infinity.

**Proof.** The probability that such a matrix factors modulo a large prime $B < p < 2B$ (factors having constant terms equal to the divisors of the constant term of the characteristic polynomial mod $p$) already goes to 0, □

\(^4\)It has been suggested by Peter Sarnak that the methods of [5] can be extended to prove this conjecture. This is the subject of a forthcoming paper by the author.
6.2. Lower bounds and asymptotics. Theorem 9 gives an estimate of \(O(B^{n-1} \log B)\) on the number of matrices in \(M_{n \times n}(\mathbb{Z})\) with reducible characteristic polynomial. To get a lower bound, we recall the following theorem of Yonatan Katznelson:

**Theorem 10** (Y. Katznelson, [8]). The number of \(n \times n\) singular integral matrices with entries bounded by \(B\) is asymptotic to \(c_n B^{n^2-n} \log B\).

The following Corollary is quite easy:

**Corollary 11.** The number of \(n \times n\) matrices whose characteristic polynomial has a linear factor over \(\mathbb{Z}\) is bounded below by \(c'_n B^{n^2-n+1} \log B\).

**Proof.** For every singular matrix \(M\), the matrices \(M + kI_n, \ k \in \mathbb{Z}\) have characteristic polynomial which has a linear factor over \(\mathbb{Z}\). \(\square\)

So, it follows that if \(N_{n,B}\) is the number of reducible integer matrices with coefficients bounded by \(B\), we have, for some non-zero constants \(c_1, c_2\):

\[
(5) \quad c_1 B^{n^2-n+1} \log B \leq N(n, B) \leq c_2 B^{n^2-n} \log B.
\]

Note that for \(n = 2\), the upper and lower bounds grow at the same rate, so we now the order of growth (which can be sharpened to an asymptotic result without too much difficulty). Otherwise, there is a considerable gap between the upper and the lower bounds, We conjecture that the lower bound is the truth:

**Conjecture 12.**

\[
N(n, B) \approx c_n B^{n^2-n+1} \log B.
\]

7. Random products of matrices in the symplectic and special linear groups

In the preceding section we defined the size of a matrix by (in essence) its \(L^1\) norm (any other Banach norm will give the same results). However, it is sometimes more natural to measure size differently: In particular, if we have a generating set \(\gamma_1, \ldots, \gamma_l\) of our lattice \(\Gamma\) (which might be \(\text{SL}(n, \mathbb{Z})\) or \(\text{Sp}(2n, \mathbb{Z})\)) we might want to measure the size of an element by the length of the (shortest) word in \(\gamma_i\) equal to that element – this is the combinatorial measure of size. The relationship between the size of elements and combinatorial length is not at all clear, so the results in this section are proved quite differently from the results in the preceding section. We will need the following results: First a result of this author.
Theorem 13 (Rivin [14]). Let \( G \) be a graph whose vertices are labeled by generators of a finite group \( \Gamma \). Consider the set of \( S_N \) elements of \( \Gamma \) obtained by multiplying elements along walks of length \( n \). Then, \( S_N \) becomes equidistributed over \( \Gamma \) as \( N \) goes to infinity.

We will also need the following results of Nick Chavdarov and Armand Borel.

Theorem 14 (Chavdarov, A. Borel [3]). Let \( q > 4 \), and let \( R_q(n) \) be the set of \( 2n \times 2n \) symplectic matrices over the field \( F_q \) with reducible characteristic polynomials. Then

\[
\frac{|R_q(n)|}{|\text{Sp}(2n, F_q)|} < 1 - \frac{1}{3^n}.
\]

Theorem 15 (Chavdarov, A. Borel [3]). Let \( q > 4 \), and let \( G_q(n) \) be the set of \( n \times n \) matrices with determinant \( \gamma \neq 0 \) over the field \( F_q \) with reducible characteristic polynomials. Then

\[
\frac{|G_q(n)|}{|\text{SL}(n, F_q)|} < 1 - \frac{1}{2^n}.
\]

Theorem 15 follows easily from the following result of A. Borel:

Theorem 16 (A. Borel). Let \( F \) be a monic polynomial of degree \( N \) over \( \mathbb{Z}/p\mathbb{Z} \) with nonzero constant term. Then, the number \( \#(F, p) \) of matrices in \( \text{GL}(N, p) \) with characteristic polynomial equal to \( F \) satisfies

\[
(p - 3)^{N^2-N} \leq \#(F, p) \leq (p + 3)^{N^2-N}.
\]

Theorem 16 will be used in Section 8. A result we will need in Section 9 and might as well state here, is:

Theorem 17 (D. Kirby, [9]). Any reciprocal polynomial is the characteristic polynomial of a symplectic matrix.

We now have our results:

Theorem 18. Let \( G \) and \( S_N \), be as in the statement of Theorem 13 but with \( \Gamma = \text{Sp}(2n, \mathbb{Z}) \), or \( \Gamma = \text{SL}(2, \mathbb{Z}) \). Then the probability that a matrix in \( S_N \) has a reducible characteristic polynomial goes to 0 as \( N \) tends to infinity.

Proof. Let \( \Gamma_i \) be the set of matrices in \( \Gamma \) reduced modulo \( l \) – it is known (see [13]) that \( \Gamma_i \) is \( \text{SL}(n, l) \) or \( \text{Sp}(2n, l) \) (depending on which \( \Gamma \) we took. Let \( p_1, \ldots, p_k \) be distinct primes, let \( K = p_1 \ldots p_k \). We know that:

\[
\Gamma_K = \Gamma_{p_1} \times \cdots \times \Gamma_{p_k}.
\]

(see [13] for the proof of the last equality). A generating set of \( \text{Sp}(2n, \mathbb{Z}) \) projects via reduction modulo \( K \) to a a generating set of
Remark 19. Using Lemma 6 instead of Theorem 15 for SL() gives a sharper result, as well as a more elementary argument.

An example of a graph $G$ is a bouquet of circles. In this case, we are just taking random products of generators or their inverses. Another is the graph (studied in [14]) where a generator is never followed by its inverse (so only reduced words in generators are allowed), and so on.

8. Stronger irreducibility

We might ask if something stronger than irreducibility of the characteristic polynomial can be shown. The answer is in the affirmative. Indeed, the methods of the preceding sections combined with the results of the Appendix give immediately:

**Theorem 20.** The probability that a random word of length $L$ in a generating set of $\text{SL}(N, \mathbb{Z})$ has characteristic polynomial with Galois group $S_N$ goes to 1 as $L$ goes to infinity.

Aside from its intrinsic interest, Theorem 20 implies the following:

**Theorem 21.** The probability that a random word $w$ of length $L$ in a generating set of $\text{SL}(N, \mathbb{Z})$ and all proper powers $w^k$ have irreducible characteristic polynomials goes to 1 as $L$ goes to infinity.

Theorem 21 will follow easily from Theorem 20 together with the following Lemma:

**Lemma 22.** Let $M \in \text{SL}(n, \mathbb{Z})$ be such that the characteristic polynomial of $M^k$ is reducible for some $k$. Then the Galois group of the characteristic polynomial of $M$ is imprimitive, or the characteristic polynomial of $M$ is cyclotomic.

**Remark 23.** For the definition of imprimitive see, for example, [16, 7].

**Proof.** Assume that the characteristic polynomial $\chi(M)$ is irreducible (otherwise the conclusion of the Lemma obviously holds, since the Galois group of $\chi(M)$ is not even transitive). Let the roots of $\chi(M)$ (in the algebraic closure of $\mathbb{Q}$) be $\alpha_1, \ldots, \alpha_n$. The roots of $\chi(M^k)$ are...
\[ \beta_1, \ldots, \beta_n, \text{ where } \beta_j = \alpha_k^j. \] Suppose that \( \chi(M^k) \) is reducible, and so there is a factor of \( \chi(M^k) \) whose roots are \( \beta_1, \ldots, \beta_l \), for some \( l < n \). Since \( \text{Gal}(\chi(M)) \) acts transitively on \( \alpha_1, \ldots, \alpha_n \), it must be true that for every \( i \in \{1, \ldots, n\} \), \( \alpha_k^i = \beta_j \), for some \( j \in \{1, \ldots, l\} \). Let \( B_j \) be those \( i \) for which \( \alpha_k^i = \beta_j \). This defines a partition of \( \{1, \ldots, n\} \) into blocks, which is stabilized by the Galois group of \( \chi(M) \), and so \( G \) is an intransitive subgroup of \( S_n \), unless \( l = 1 \). In that case, the characteristic polynomial of \( M^k \) equals \( (x - \beta)^n \), and since \( M^k \in \text{SL}(n, \mathbb{Z}) \) it follows that \( \beta = 1 \), and all the eigenvalues of \( M \) are \( n \)-th roots of unity, so that \( M^k = 1 \). \( \square \)

9. The mapping class group

Let \( S_g \) be a closed surface of genus \( g \), and let \( \Gamma_g \) be the mapping class group of \( S_g \). The group \( \Gamma_g \) admits a homomorphism \( s \) onto \( \text{Sp}(2g, \mathbb{Z}) \) (we associate to each element its action on homology; the symplectic structure comes from the intersection pairing). The following result can be find in [2]:

**Theorem 24.** For \( \gamma \in \Gamma_g \) to be pseudo-Anosov, it is sufficient that \( g = \gamma \) satisfy all of the following conditions:

1. The characteristic polynomial of \( g \) is irreducible.
2. The characteristic polynomial of \( g \) is not cyclotomic.
3. The characteristic polynomial of \( g \) is not of the form \( g = h(x^k) \), for some \( k > 1 \).

The following is a corollary of our results on matrix group:

**Theorem 25.** Let \( g_1, \ldots, g_k \) be a generating set of \( \text{Sp}(2n, \mathbb{Z}) \). The probability that a random product of length \( N \) of \( g_1, \ldots, g_k \) satisfies the conditions of Theorem 24 goes to 1 as \( N \) goes to infinity.

**Proof.** We prove that the probability that the random word \( w_N \) not satisfy the conditions goes to 0. By Theorem 18 the probability that \( w_N \) has reducible characteristic polynomial goes to 0. In order for the characteristic polynomial to be of the form \( g = h(x^k) \) it is necessary that the linear term (the trace) vanish. This is a proper subvariety of \( \text{Sp}(2g) \), and so the number of elements of any \( \text{Sp}(2g, p) \) satisfying this condition is of order of \( p^{2g^2 + g - 1} \). Since the number of elements in \( \text{Sp}(2g, p) \) is of order of \( p^{2g^2 + g} \) (Dickson’s Theorem, see [13]), the proof of Theorem 18 goes through verbatim (but needs Theorem 17) to show that this is an asymptotically negligible condition. Finally, since the set of cyclotomic polynomials of a given degree is finite, the set of symplectic matrices with those characteristic polynomials
is also a subvariety of the full group (again, needing Theorem 17), and the same result holds.

10. Free Group Automorphisms

An automorphism of $\phi$ of a free group $F_n$ is called strongly irreducible\(^5\) if no (positive) power of $\phi$ sends a free factor $H$ of $F_n$ to a conjugate. This concept was introduced by M. Bestvina and M. Handel \(\text{[1]}\), and many of the results of the theory of automorphisms of free groups are shown for such automorphisms. By passing to the action of $\phi$ on homology, Section 8\(\text{[2]}\) shows the following:

**Theorem 26.** Let $f_1, \ldots, f_k$ be a generating set of the automorphism group of $F_n$. Consider all words of length $L$ in $f_1, \ldots, f_k$. Then, for any $n$, the probability that such a word is irreducible tends to 1 as $L$ tends to infinity and also the probability that such a word is strongly irreducible tends to 1 as $L$ tends to infinity.

Appendix A. Galois groups of generic restricted polynomials

Let $P_{N,d}(\mathbb{Z})$ be the set of monic polynomials of degree $d$ with integral coefficients bounded by $N$ in absolute value. It is a classical result of B. L. van der Waerden that the probability that the Galois group of $p \in P_{N,d}(\mathbb{Z})$ is the full symmetric group $S_d$ tends to 1 as $N$ tends to infinity. The argument is quite elegant: First, it is observed that a subgroup $H < S_d$ is the full symmetric group if and only if $H$ intersects every conjugacy class of $S_d$. This means that $H$ has an element with every possible cycle type. It is further noted that there is a cycle type $(n_1, \ldots, n_k)$ in the Galois group of $p$ over $\mathbb{Z}/p\mathbb{Z}$ if and only if $p$ factors over $\mathbb{Z}/p\mathbb{Z}$ into irreducible polynomials of degrees $n_1, \ldots, n_k$. Using Dedekind’s generating function for the number of irreducible polynomials over $\mathbb{Z}/p\mathbb{Z}$ of a given degree, it is shown that the probability of a fixed partition is is bounded below by a constant (independent of the prime $p$), and the proof is finished by an application of a Chinese Remainder Theorem.

In this note, we ask the following simple-sounding question: Let $P_{N,d,a}(\mathbb{Z})$ be the set of all polynomials in $P_{N,d}(\mathbb{Z})$ where the coefficient of $x^k$ equals $a$. Is it still true that the Galois group of a random such polynomial is the full symmetric group? The result would obviously

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\(^5\)This terminology, with strong support from this author, has been introduced by L. Mosher and M. Handel for what was previously known as irreducible with irreducible powers

\(^6\)We need to change $\text{SL}(n, \mathbb{Z})$ to $\text{GL}(n, \mathbb{Z})$ throughout
follow if the probability that the Galois group of a random general polynomial is “generic” were to go to 1 sufficiently fast with $N$. In fact, the probability that an element of $P_{N,d}$ is reducible (which means that its Galois group is not transitive, hence not $S_n$) is of the order of $1/N$, so that approach does not work.

Mimicking the proof of van der Waerden’s result does not appear to work (at least not easily): Dedekind’s argument enumerates all irreducible polynomials, and the result is not “graded” by specific coefficients. It is certainly possible that the argument can be pushed through, but this appears to be somewhat involved.

Given this sad state of affairs, we first use a simple trick and Dirichlet’s theorem on primes on arithmetic progressions to show first the following technical result:

**Theorem 27.** The probability that a random element of $P_{N,d,a,k}(\mathbb{Z}/p\mathbb{Z})$ has a prescribed splitting type $s$ approaches the probability that a random unrestricted polynomial of degree $d$ has the splitting type $s$, as long as $p - 1$ is relatively prime to $(d - k)!$, and as $p$ becomes large. relatively prime to $d - k$.

which implies (by van der Waerden’s sieve argument):

**Theorem 28.** The probability that a random element of $P_{N,d,a,k}(\mathbb{Z})$ has $S_d$ as the Galois group tends to 1 as $N$ tends to infinity,

It should be noted that the (multivariate) Large Sieve (as used by P. X. Gallagher in [6]) can be used to give an effective estimate on the probability in the statement of Theorems 28: that is: $p(N) \ll N^{-1/2}$.

**A.1. Proof of Theorem 28.** We will need two ingredients other than van der Waerden’s original idea. The first of these is A. Weil’s estimate on the number of $\mathbb{F}_p$ points on a curve defined over $\mathbb{F}_p$:

**Theorem 29 (A. Weil,[15]).** Let $f \in \mathbb{F}_p[X, y]$ be an absolutely irreducible (that is, irreducible in $\mathbb{F}_p[X, Y]$) polynomial of degree $d$. Then if

$$C = \{(x, y) \in \mathbb{F}_p^2 | f(x, y) = 0\},$$

we have the estimate

$$|C| - p \leq 2g \sqrt{p} + d^2,$$

where $g$ is the genus of the curve defined $f$ (which satisfies $g \leq (d-1)(d-2)$.

This estimate is optimal.

The other classical result we shall need is the following:
Lemma 32. Consider a polynomial $f$ of degree $d$ over $k$. It is not possible for $g(a, b)(x) = ax + b$ to permute $x_1, \ldots, x_k$ such that the coefficient of $x^d$ does not vanish, and such that the coefficient of $x^{d-1}$ does not vanish. Then there is no pair $(a, b) \neq (1, 0)$, such that $f(ax + b) = a^d f(x)$, for all $x \in \mathbb{F}_p$.

Proof. There are two distinct cases to analyze. The first is when $a = 1$. In that case, $f(x + b) = f(x)$ for all $x \in \mathbb{F}_p$, and since $p > d$, $f(x + b) = f(x)$, for all $x$ in the algebraic closure of $\mathbb{F}_p$. Let $r$ be a root of $f$. Then, so are $r + a, r + 2a, \ldots, r + a(p - 1)$, but since $p$ is greater than $d$ that means that $f$ is identically 0.

The second case is when $a \neq 1$. In that case, $x_0 = b/(1 - a)$ is fixed under the substitution $x \to ax + b$, and changing of variables to $z = x - x_0$, sends $f(z)$ to $f(az)$. By the same argument as above, $f(az) = a^d f(z)$, and so the corresponding coefficients of the right and the left hand polynomials must be equal. Since the coefficient of $x^{d-1}$ does not vanish, it follows that $a = 1$, which contradicts our assumption.

Theorem 30 goes essentially back to N. H. Abel’s foundational memoir.

We will need an additional observation:

Lemma 31. Let $q = p^k$, and let $x_1, \ldots, x_k \in \mathbb{F}_q$. Let $a, b \in \mathbb{F}_p$, with $(a, b) \neq (1, 0)$, and let $g(a, b)(x) = ax + b$ be a transformation of $\mathbb{F}_q$ to itself. Then, it is not possible for $g(a, b)$ to permute $x_1, \ldots, x_k$, if $k!$ is coprime to $p - 1$.

Lemma 32. Consider a polynomial $f$ of degree $d$ over $\mathbb{F}_p$, such that $d < p$, and such that the coefficient of $x^{d-1}$ does not vanish. Then there is no pair $(a, b) \neq (1, 0)$, such that $f(ax + b) = a^d f(x)$, for all $x \in \mathbb{F}_p$.

Proof. There are two distinct cases to analyze. The first is when $a = 1$. In that case, $f(x + b) = f(x)$ for all $x \in \mathbb{F}_p$, and since $p > d$, $f(x + b) = f(x)$, for all $x$ in the algebraic closure of $\mathbb{F}_p$. Let $r$ be a root of $f$. Then, so are $r + a, r + 2a, \ldots, r + a(p - 1)$, but since $p$ is greater than $d$ that means that $f$ is identically 0.

The second case is when $a \neq 1$. In that case, $x_0 = b/(1 - a)$ is fixed under the substitution $x \to ax + b$, and changing of variables to $z = x - x_0$, sends $f(z)$ to $f(az)$. By the same argument as above, $f(az) = a^d f(z)$, and so the corresponding coefficients of the right and the left hand polynomials must be equal. Since the coefficient of $x^{d-1}$ does not vanish, it follows that $a = 1$, which contradicts our assumption.

The argument now proceeds as follows. First, we note that if the polynomial $f(x)$ of degree $d$ has a certain splitting type (hence Galois group) over $\mathbb{F}_p$, then so does $f(ax + b)/a^d$, for any $a \neq 0, b \in \mathbb{F}_p$. The set of all linear substitutions forms a group $\mathbb{A}$, which acts freely on the set of polynomials of degree $d$, except for the (small) exceptional set of polynomials with a vanishing coefficient of $x^{d-1}$ as long as $d < p$ (by Lemma 32), so the distribution of splitting types among the $\mathbb{A}$ orbits is the same as among all of the polynomials of degree $d$. Now, consider polynomials with constant term 1. How many of them are there in the $\mathbb{A}$ orbit of $f(x)$? It is easy to see that the number is equal to the number of solutions to

$$f(b) = a^d.$$

If the curve $C_f$ given by $f(x) - y^d$ is absolutely irreducible, that number is $p + O(\sqrt{p})$, by Theorem 29. By Theorem 30 in order for $C_f$ to not be
absolutely irreducible, we must either have that \( f(x) = g^q(x) \), for some \( q \mid d \), or \( f(x) = -4h^4(x) \), in case \( 4 \mid d \). But the number of such polynomials is bounded by \( O(p^{d/2}) \), which is asymptotically negligible. So, we see that the distribution of splitting types amongst polynomials of degree \( d \) with constant term 1 is the same as for all polynomials, as long as \( d < p \).

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