Gravitational Radiation Assisted Capture

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It is shown that gravitational radiation can bind two initially unbound bodies; no third body is needed. Such captured bodies will almost always inspiral and merge due to further gravitational radiation on cosmologically negligible time scales (e.g., @ 5 years for GW150914). The capture cross-section \( \sigma \) for such “capture and inspiraling” is far larger, for initial relative speed of the two objects \( v_{\infty} \ll c \), than that \( \sigma_d \) for “direct capture”: \( \sigma \sim (c/v_{\infty})^{18/7} \), while \( \sigma_d \sim (c/v_{\infty})^2 \). Implications of these results for black hole binary mergers, and giant black holes at galactic centers, are discussed.

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The direct detection of gravitational radiation\(^1\),\(^2\), in addition to confirming one of the most important predictions of general relativity\(^3\), raises the question of the origin of the black hole binaries that have been the source of both definite detections, and one possible detection, so far. Prior work\(^4\) has focussed on three-body mechanisms which bind two previously gravitationally unbound bodies. Such mechanisms have the disadvantage that their rate is proportional to the cube of the density of stellar mass objects available to provide the third body, and is, therefore, very low when the density is low.

The purpose of this paper is to point out that gravitational radiation itself provides a very effective mechanism for two body capture, particularly of black holes\(^5\). I find that if two objects of masses \( m_1 \) and \( m_2 \) with total mass \( M \equiv m_1 + m_2 \) approach each other at asymptotic relative speed \( v_{\infty} \ll c \), they will lose enough energy to be captured into a highly elliptical orbit if their impact parameter \( b \) is less than a critical value \( b_c \) given by:

\[
b_c = C_b \left(f(1-f)\right)^{\frac{2}{7}} \left(\frac{c}{v_{\infty}}\right)^{\frac{9}{7}} r_S(M)
= 4.75 \times 10^5 \text{km} \left(f(1-f)\right)^{\frac{2}{7}} \left(\frac{M}{M_\odot}\right) \left(\frac{30 \text{ km}}{v_{\infty}}\right)^{\frac{9}{7}} \ (1)
\]

where \( M_\odot \) is the mass of the Sun, \( C_b \equiv \left(\frac{8\pi}{9\sqrt{3}}\right)^{\frac{2}{7}} \approx 1.157367, \ f \equiv \frac{m_1}{m_2} \), and the Schwarzschild radius \( r_S(M) = 2GM/c^2 \). For GW150914, \( M = 65M_\odot \) (which implies \( r_S(M) = 192\text{km} \)) and \( f = 29/65 \), assuming an initial relative velocity of \( v_{\infty} = 30 \text{ km/sec} \) (a typical relative velocity for stars in the neighborhood of the Sun), equation (1) then gives \( b_c = 2.53 \times 10^7 \text{km} = .17\text{AU} \).

The cross-section for capture is given by

\[
\sigma = \pi b_c^2 = C_\sigma \left[f(1-f)\right]^2 \left(\frac{c}{v_{\infty}}\right)^{18/7} r_S^2(M), \ (2)
\]

where \( C_\sigma \equiv \left(\frac{7225\pi}{9\sqrt{3}}\right)^{\frac{2}{7}} \approx 4.208 \). Again for GW150914 with \( v_{\infty} = 30 \text{ km/sec} \), this gives \( \sigma = 2 \times 10^{15} \text{km}^2 = .09\text{AU}^2 \).

Once captured in this way, the bodies will inspiral due to further gravitational radiation, until they merge. I find that the total inspiral time \( \tau \) of the pair after first periastron is

\[
\tau = \pi \left(\frac{c^2 r_S(M)}{v_{\infty}^3}\right) \left(\frac{b}{b_c}\right)^{21/2} \zeta \left(\frac{3}{2}, x\right) 
\approx (1\text{year}) \left(\frac{30 \text{ km}}{v_{\infty}}\right)^3 \left(\frac{M}{M_\odot}\right) \left(\frac{b}{b_c}\right)^{21/2} \zeta \left(\frac{3}{2}, x\right), \ (3)
\]

where I’ve defined \( x = 1 - \left(\frac{b}{b_c}\right)^{7} \), and

\[
\zeta(y, x) \equiv \sum_{n=0}^{\infty} \frac{1}{(x+n)^y}
\]

is the Hurwitz zeta function\(^6\).

Note that inspiral time \( \tau \) is much less than the age of the universe for any reasonable mass \( M \) and relative velocity \( v_{\infty} \); for example, for GW150914, assuming the impact parameter takes on its median value \( b = b_c/\sqrt{2} \), and, as before, taking \( v_{\infty} = 30 \text{ km/sec} \), I obtain \( \tau \approx 4.80 \) years. Therefore, virtually all pairs of masses captured in this way will inspiral essentially instantaneously on a cosmological timescale. As a result, the limit on the rate of mergers caused by this mechanism is the capture rate, not the subsequent inspiral. Only the extremely rare occurrence of an impact parameter \( b \) extremely close to \( b_c \) can lead to cosmologically significant inspiral times. For example, for GW150914 with all of the assumptions made earlier, achieving \( \tau \sim 10^9 \) years would require that \( 1 - \frac{b}{b_c} \leq 2.3 \times 10^{-6} \); the probability of this is only \( 4.6 \times 10^{-6} \). The distribution of inspiral times is very broad, however, as I will discuss further in the SM\(^7\).

One experimental signature of this mechanism of capture is that it would lead to inspiraling orbits of detectable eccentricity. For a given \( v_{\infty} \), once the bodies have inspiraled to an elliptical orbit with periastron \( r_p \ll r_p0 \), the distribution of eccentricities of the inspiraling pair implied by this mechanism is:

\[
p(e; r_p) = C_p \left(\frac{r_p}{r_S(M)}\right) [f(1-f)]^{-\frac{2}{7}} \left(\frac{v_{\infty}}{c}\right)^{\frac{4}{7}} e^{-\frac{2}{7}(\frac{v_{\infty}}{c})}, \ (4)
\]
for $e > e_{\text{min}}$ and $p(e; r_p) = 0$ for $e < e_{\text{min}}$, where
\[ e_{\text{min}} = C_{\text{em}} \left( \frac{r_p}{r_S(M)} \right)^{\frac{10}{19}} [f(1-f)]^{-\frac{19}{42}} \left( \frac{v_\infty}{c} \right)^{\frac{29}{7}} , \quad (5) \]
\[ C_{\text{p}} = \frac{6}{19} \left( \frac{425}{304} \right)^{\frac{85}{19}} \left( \frac{304}{85} \right)^{\frac{9}{2}} \approx 0.267622338..., \quad \text{and} \]
\[ C_{\text{em}} = \left( \frac{425}{304} \right)^{\frac{14}{9}} \left( \frac{304}{85} \right)^{\frac{10}{9}} 2^{-\frac{10}{9}} = 0.25678305711.... \]

To predict the actual distribution of observations, for which the asymptotic approach velocity $v_\infty$ will of course be unknown, this must be averaged over the distribution of approach velocities. Doing this for a Maxwellian distribution of speeds with variance $v_e^2$, I find [8]
\[ p(e) = \tilde{C} \left( \frac{1}{e_c} \right) \left( \frac{e_c}{e} \right)^{\frac{29}{7}} \gamma \left( \frac{25}{14} \frac{1}{2} \left( \frac{e}{e_c} \right) \right) , \quad (6) \]
where $\gamma(s, x)$ is the lower incomplete gamma function [7], I've defined the characteristic eccentricity scale
\[ e_c = C_{\text{em}} \left( \frac{r_p}{r_S(M)} \right)^{\frac{10}{19}} [f(1-f)]^{-\frac{19}{42}} \left( \frac{v_\infty}{c} \right)^{\frac{29}{7}} , \quad (7) \]
and the constant $\tilde{C} \equiv 2^{\frac{4}{7}} \left( \frac{24}{19\sqrt{\pi}} \right) = 0.8687429....$

The probability distribution $p(e)$ has the limiting forms:
\[ p(e) \approx \begin{cases} \frac{C_c}{e_c} \left( \frac{e_c}{e} \right)^{\frac{29}{7}}, & e \ll e_c, \\ \frac{C_c}{e_c} \left( \frac{e}{e_c} \right)^{\frac{29}{7}}, & e \gg e_c, \end{cases} , \quad (8) \]
where I've defined $C_c \equiv \frac{168}{475\sqrt{2\pi}} = 0.141099585...$ and $C_{\geq} \equiv CT \left( \frac{25}{19} \right) = 0.805906...$, where $T$ is the complete Gamma function. The probability $p(e)$ has a single maximum of $p_{\text{max}} = 1.9038586/\epsilon e_c$ at $e = 1.903676666c$.

The $e \gg e_c$ asymptotic scaling $p(e) \propto e^{-\frac{29}{7}}$ holds for all distributions of the asymptotic speed $v_\infty$, with the replacements $v_\infty \to v_c$ in [11], and $C_{\geq} \to \frac{12}{19}$ in [13], where $v_c \equiv \left( \frac{8\pi^2}{9} \right)^{\frac{7}{4}}$ with the brackets denoting an average over speeds.

Since these results ignore relativistic effects, for comparison with observational data, they should be used at a value of $r_p$ sufficiently large compared to $r_S$ (say, $r_p \sim 10r_S(M)$) that relativistic effects are negligible, and then use that eccentricity as an initial condition for a numerical solution for the final stages of the inspiral.

Note that the typical eccentricities $e_c$ (eqn. (7)) will be very small if the rms velocity variance $v_\sigma \ll c$. For example, taking $v_\sigma = 30 \text{ km/sec}$, $f = 29/65$ (the value for GW150914), and $r_p = 10r_S(M)$ gives $e_c = 4.45 \times 10^{-3}$. Nonetheless, if, as is anticipated [11], LIGO eventually detects hundreds of black hole binary mergers, my result [10] implies that some of these will have appreciable eccentricities: e.g., 7% of all mergers will have an eccentricity greater than 100$e_c$, which is $\sim 0.445$ for the parameter values just assumed. This should be detectable.

This gravitational radiation assisted capture mechanism (hereafter ”GRAC”) may dominate the creation of both binary black hole mergers, and supermassive black holes (hereafter ”GBH’s”) at galactic centers.

For both processes, there are well-defined limits in which GRAC becomes infinitely more effective than the other two competing mechanisms: direct capture (that is, the two objects plunging directly into each other on their first passage), and three body capture. The cross-section for direct capture for $f \ll 1$ is $\sigma_d = 4\pi r_S^2(M) \left( \frac{v_\infty}{c} \right)^2$. While calculating the direct capture cross-section for objects of comparable mass would require numerical solution of the full equations of general relativity, it is presumably of this order of magnitude. Therefore, the ratio of this direct capture cross-section to that of the gravitational wave assisted mechanism I consider here (given by equation (5)) is $\sim \left( \frac{v_\infty}{c} \right)^d \frac{1}{f} - \frac{1}{f}$; hence, direct capture is much less common, in the limit $v_\infty \to 0$, than GRAC.

Note, however, that because its cross-section does not vanish as $f \to 0$, direct capture surpasses GRAC for $f \lesssim \left( \frac{v_\infty}{c} \right)^2$. This is clearly the case for, e.g., the capture of subatomic dark matter particles by giant black holes at galactic centers [10]. On the other hand, for the capture of stars by a giant black hole, GRAC is more effective even if the black hole is enormous; for example, for the giant black hole at the center of our own galaxy [11], $M \sim 4 \times 10^6 M_\odot$, even stars of mass $\sim M_\odot/10$ (i.e., the mass of a typical star), can satisfy the condition $f \gtrsim \left( \frac{v_\infty}{c} \right)^2$ for relative asymptotic speeds $v_\infty \sim 30 \text{ km/sec}$. So although dark matter is far more common, the principle component of the diet of giant black holes may be stars.

Note also that GRAC is actually much more effective in the early stage of the growth of such a giant black hole, since $f$ is much smaller at that stage (when the GBH is much lighter). This could potentially explain how giant black holes grow [12] from intermediate mass black holes.

For the formation of BH binaries, as noted earlier, this mechanism is always favored over three body mechanisms as the number density $\rho \to 0$, since three body rates vanish like $\rho^3$, whereas the rate for two body processes like GRAC vanish like $\rho^2$.

Note, however, that the rate for three body mechanisms can scale like $\rho^3$ if an $O(1)$ fraction of the black holes are formed in bound pairs [4]. Furthermore, in high density regions, not only is the three body rate faster, but GRAC becomes less effective, since the highly elliptical orbits created by this capture are quite delicate, and easily gravitationally perturbed by a third body.

These are clearly quantitative questions which should be investigated to determine how important a role GRAC plays in the creation of binary BH mergers.

I’ll now derive the above results. Detailed calculations are given in the Supplemental Materials [8]; here I will give simple rough arguments that recover the above results up to numerical factors of $O(1)$.

Consider two bodies approaching each other at non-
relativistic speeds \(v_\infty \ll c\) with impact parameter \(b\). For Newtonian motion, conservation of energy and angular momentum imply\[^8\] that the distance of closest approach \(r_{p0}\) of the two bodies on their first passage is:

\[
r_{p0} = \frac{v_\infty^2 b^2}{2GM} = \left(\frac{v_\infty}{c}\right)^2 \frac{b^2}{r_S(M)} \tag{9}
\]

where I have assumed, and will verify \textit{a posteriori}, that \(b \gg r_{p0}\) for all captured orbits. This condition also implies that the relative speed \(v(r_{p0})\) of the pair at closest approach is nearly the escape velocity at that distance:

\[
v(r_{p0}) \approx \sqrt{\frac{2GM}{r_{p0}}} \tag{10}
\]

With the parameters of the orbit in hand, we can now calculate the energy emitted by gravitational radiation on the first passage. To do so, I begin with the general expression\[^3\] for the power \(P\) emitted by a weak, slow-moving \((v \ll c)\) gravitational wave source:

\[
P = \frac{G}{6c^3} \sum_{ij} Q_{ij} \ddot{Q}_{ij},
\]

where

\[
Q_{ij} = \sum_\alpha m_\alpha \left( r_{i}^\alpha r_{j}^\alpha - \frac{1}{3} \delta_{ij} |r_\alpha|^2 \right) \tag{12}
\]

is the usual mass quadrupole tensor of a set of masses labeled by \(\alpha\). Here there are only two masses \(m_1\) and \(m_2\), which, in center of mass coordinates, are located at \(r_1 = -\frac{m_1}{m_2} r\) and \(r_2 = \frac{m_2}{m_1} r\) respectively, where \(r \equiv r_2 - r_1\) is the relative displacement of the two masses.

I will verify \textit{a posteriori} that the assumptions of slow motion \((i.e., v \ll c)\) and weak gravitational fields are valid for the initial capture, and most of the inspiral process, for almost all pairs captured by GRAC. This means the orbits are nearly Newtonian\[^1\], which makes it possible to do all calculations analytically.

Using the center of mass coordinates for the two masses in \((12)\), a typical component of the mass quadrupole tensor \(Q\) can be estimated entirely in terms of \(r\):

\[
Q_{ij} \sim \frac{m_1 m_2}{M} r^2 = \mu r^2, \tag{13}
\]

where \(\mu \equiv \frac{m_1 m_2}{M}\) is the usual reduced mass. Taking three time derivatives of this expression near periastron, where most of the gravitational radiation occurs, essentially amounts to multiplying it by \(\omega^4\), where \(\omega \equiv \sqrt{\frac{G}{r_{p0}}} \varepsilon(\omega) \) is the angular velocity of the pair at periastron. Using this and \((10)\) gives, near periastron,

\[
\ddot{Q}_{ij} \sim \frac{G^2 m_1 m_2 M^2}{r_{p0}^7}. \tag{14}
\]

Using this in the general expression \((11)\) gives, for the emitted power at periastron:

\[
P_p \sim \frac{G^4 m_1^2 m_2^2 M}{c^7 r_{p0}^8}. \tag{15}
\]

This power is emitted for a time \(\delta t\) of order \(\delta t \sim \frac{r_{p0}}{v_\infty}\); hence the total energy emitted on the first passage is:

\[
\Delta E = P_p \delta t \times O(1) = C_E E \frac{G^2 m_1^2 m_2^2 M^2}{c^7 r_{p0}^8}. \tag{16}
\]

The detailed analysis given in the supplemental materials\[^8\] recovers precisely this result, with a numerical prefactor of \(C_E \equiv 85\pi^2/24 \approx 15.73521\). Using my earlier expression \((9)\) for the distance of closest approach \(r_{p0}(b)\) in this estimate of \(\Delta E\) gives

\[
\Delta E = D_E G^7 m_1^2 m_2^2 M^4 \frac{1}{c^9 b^6 v_\infty^2}, \tag{17}
\]

where a precise calculation\[^3\] gives the numerical prefactor \(D_E = 170\pi/3 \approx 178\). When this energy loss is greater than the total original Newtonian energy of the system, which is just the center of mass kinetic energy at infinity, the two masses will become bound. The largest impact parameter \(b_0\) that satisfies this condition therefore obeys

\[
\Delta E(b_0) = \frac{m_1 m_2}{2M} v_\infty^2. \tag{18}
\]

Combining \((17)\) with \((18)\), using the fact that \(r_S(M) = 2GM/c^2\), and solving for \(b_0\), gives equation \((14)\). Using the fact that the capture cross section \(\sigma = \pi b_0^2\) then immediately gives my principal result, equation \((2)\).

I can now verify \textit{a posteriori} my earlier assumption of slow motion \((i.e., v \ll c)\) and weak gravitational fields by noting that both of these assumptions are satisfied if \(r \gg r_S(M)\) throughout the orbit, which is clearly true if the periastron distance on first passage \(r_{p0} \gg r_S(M)\). This is readily verified using \((1)\) for the maximum impact parameter \(b_0\) and the relation \((3)\) between \(r_{p0}\) and \(b_0\), which, taken together, imply

\[
r_{p0} = \left( \frac{c}{v_\infty} \right)^{4/7} \left( \frac{85\pi f(1-f)}{96} \right)^{1/7} \frac{\sqrt{b_0}}{b_0} r_S(M), \tag{19}
\]

from which it is clear that \(r_{p0} \gg r_S(M)\) if \(v_\infty \ll c\), unless \(f \ll 1\) or \(b_0 \ll b_c\). The latter condition will rarely happen. Hence, the motion will be nearly Newtonian if \(v_\infty \ll c\sqrt{f}\). This condition will be satisfied by any objects of roughly equal mass for \(v_\infty \ll c\), and by stars approaching giant black holes for relative velocities \(\lesssim 30\) km/s.

The inspiral time can now be calculated by assuming that each subsequent return of the pair to periastron will occur at almost exactly the same periastron distance:

\[
r_{pn} \approx r_\infty^n \quad \text{for } n \gg 1.
\]

Furthermore, the orbit during this “constant \(r_{pn}\)” phase of the inspiral is nearly parabolic near periastron. Finally, almost of of the inspiral time is spent in this “constant \(r_{pn}\)” phase. These statements will all be verified \textit{a posteriori} in the SM.

Since the periastron distances \(r_{pn} \approx r_{p0}\) and the orbit remains nearly parabolic near periastron, the energy loss on each return will be nearly the same as that on the first passage. Hence, the energy after \(n\) orbits is given by

\[
E_n = E_0 - n\Delta E, \tag{20}
\]
where $E_0$ is the energy after the first passage. My result \[17\] for $\Delta E$ can be rewritten as

$$\Delta E = \frac{\mu v_\infty^2}{2} \left(\frac{b_c}{b}\right)^7.$$  \ (21)

Using the standard relation \[10\] between semi-major axis $a$ and energy then gives

$$a_n = \frac{G m_1 m_2}{2E_n} = \frac{G m_1 m_2}{2(n \Delta E - E_0)} = \frac{G m_1 m_2}{2(n + x) \Delta E},$$ \ (22)

where I've defined

$$x = -\frac{E_0}{\Delta E} = \left(\frac{\mu v_\infty^2}{2} - \Delta E\right) = \left(1 - \left(\frac{b}{b_c}\right)^7\right).$$  \ (23)

Using my expression \[21\] for the energy loss per orbit $\Delta E$, I obtain

$$a_n = \frac{1}{2} \left(\frac{c}{v_\infty}\right)^2 \left(\frac{1}{n + x}\right) \left(\frac{b}{b_c}\right)^7 r_S(M),$$ \ (24)

where I’ve used $r_S(M) = 2GM/e^2$ again. Using the standard relation $T_n = 2\pi\sqrt{\frac{r_S^3}{GM}}$ for the period $T_n$ of the $n$th orbit, and summing this from $n = 0$ to infinity gives the total inspiral time equation \[3\].

The results that $r_{pn} \approx r_{p0}$ and \[22\] imply that a very large number of orbits will have $r_{pn} \ll a_n$; i.e., their eccentricities will be very close to $1$. Indeed, I show in the SM\[8\] that this will be the case until

$$n \sim n_c = \left(\frac{c}{v_\infty}\right)^{10/7} [f(1 - f)]^{-2/7} \left(\frac{b}{b_c}\right)^5,$$  \ (25)

which is $\gg 1$ for $v_\infty \ll c$ unless $b \ll b_c$, which is highly unlikely, or $f \ll \left(\frac{\mu v_\infty^2}{2}\right)^{5}$, i.e., widely disparate masses. This large value of $n_c$ justifies extending the sum to $n = \infty$ in the calculation of the total inspiral time \[3\].

To derive the final eccentricity distribution law \[4\], I begin with the equations for the evolution of the eccentricity $e$ and semi-major axis $a$ of an inspiraling, nearly Newtonian orbit derived by Peters\[13\]:

$$\frac{da}{dt} = -\frac{64K(1 + \frac{24}{25}e^2 + \frac{16}{125}e^4)}{5a^3(1 - e^2)^{7/2}} = -\frac{64Kf(e)}{5a^3(1 - e^2)^{7/2}},$$  \ (26)

$$\frac{de}{dt} = -\frac{K(e(304 + 121e^2))}{15a^4(1 - e^2)^{7/2}} = -\frac{Kg(e)}{15a^4(1 - e^2)^{7/2}},$$  \ (27)

where I’ve defined $K \equiv G^3m_1 m_2 M/e^5$.

These equations were derived by Peters\[13\] in the approximation that the parameters $a$ and $e$ undergo only small percentage changes on each orbit. This is clearly not the case for $a$ for the first few orbits after capture, as inspection of my expression \[22\] for the semi-major axis $a_n$ of the $n$th orbit makes clear.

However, as noted earlier, the distance of closest approach of the $n$th orbit $r_{pn}$ does not vary appreciably from orbit to orbit (until the very last stages of the inspiral, which contribute negligibly to the total inspiral time). Furthermore, as detailed in the SM\[8\], by combining \[26\], \[27\], and the elementary relation $r_p = a(1 - e)$, I obtain a differential equation describing the evolution of $r_p$ as a function of eccentricity $e$:

$$\frac{d \ln r_p}{de} = \frac{d \ln a}{de} - \frac{1}{1 - e} = \frac{y(e)}{e},$$  \ (28)

where $y(e)$ is a rational function of $e$, given explicitly in the SM\[8\], that is finite and $O(1)$ for all $e$ in the range $0 \leq e \leq 1$. Hence, I can use this differential equation out to $e = 1$. Doing so, I find \[8\] that the solution of \[28\] with the initial conditions $e = 1$ and $r_p = r_{p0}$ implies that by the time $e \ll 1$,

$$e(r_p) = C_e \left(\frac{r_p}{r_{p0}}\right)^{\frac{19}{19}}.$$  \ (29)

where $C_e = (\frac{425}{19})^{\frac{142}{19}} / 2^{\frac{16}{19}} = 40790 \ldots$. Using the relation \[9\] between the minimum first passage distance $r_{p0}$ and the impact parameter $b$, I can rewrite this as a relation between the eccentricity and the impact parameter:

$$e(b) = C_e \left(\frac{r_{p0}r_p}{b^2} \left(\frac{c}{v_\infty}\right)^2\right)^{\frac{19}{19}}.$$  \ (30)

Solving for $b(e)$ gives

$$b(e) = C_e^{\frac{1}{19}} \sqrt{r_{p0}r_p} \left(\frac{c}{v_\infty}\right) e^{-\frac{16}{19}}.$$  \ (31)

The probability distribution for the final eccentricity can now be obtained from that for the impact parameter $b$ via simple statistics, which imply:

$$p(e; v_\infty) = p(b; v_\infty) \left|\frac{db}{de}\right|.$$  \ (32)

Since the impact parameters of captured pairs should be uniformly distributed over a circle of radius $b_c$, I have

$$p(b; v_\infty) = \frac{2b}{b_c^2(v_\infty)}.$$  \ (33)

Using this and \[31\] in \[32\] gives the probability distribution \[4\] for the final eccentricity. In the SM\[8\], I show that averaging this over a Maxwellian speed distribution gives \[6\]. I also show in the SM that the $e \gg e_c$ limit of the velocity averaged distribution of final eccentricities (i.e., the second line of equation \[5\]) is universal for all speed distributions, in the sense described earlier.
I. SUPPLEMENTAL MATERIALS

A. Distance of closest first approach \( r_{p0} \)

I will treat the motion as Newtonian, which can be justified \textit{a posteriori} by showing that the distance of closest approach on first passage \( r_{p0} \) obeys \( r_{p0} \gg r_S(M) \). I will also assume that \( r_{p0} \ll b \), the impact parameter. I’ll later verify \textit{a posteriori} that this holds for all pairs that are captured by GRAC.

Given this, it is clear from conservation of angular momentum, which implies

\[
v(r_p0)r_{p0} = v_\infty b,
\]

that the speed \( v(r_p0) \gg v_\infty \). This in turn implies that most of the kinetic energy of the pair at periastron is obtained from their potential energy. Of course, an orbit on which the kinetic energy is equal to the potential energy is parabolic. Hence, the orbit near periastron, which is where most of the gravitational radiation will take place (as we’ll see below), is nearly parabolic. The velocity at periastron is therefore very close to the escape velocity at that radius; hence

\[
v(r_p0) = \sqrt{\frac{2GM}{r_p0}}.
\]

Using this in (34) and solving for \( r_{p0} \) gives

\[
r_{p0} = \frac{v_\infty^2 b^2}{2GM} = \left(\frac{v_\infty}{c}\right)^2 \frac{b^2}{r_S(M)}.
\]

which is just equation (35).

I’ll now verify my two \textit{a posteriori} assumptions above. First, to see that \( r_{p0} \ll b \), I take the ratio \( \frac{r_{p0}}{b} \) using (36), which gives \( \frac{r_{p0}}{b} = \left(\frac{v_\infty}{c}\right)^2 \frac{b}{r_S(M)} \). Since this ratio grows with increasing impact parameter \( b \), its largest possible value for a captured pair occurs when \( b = b_c \). Hence, \( \frac{r_{p0}}{b} \leq \left(\frac{v_\infty}{c}\right)^2 \frac{b}{r_S(M)} = C_b f(1 - f) \left(\frac{v_\infty}{c}\right)^2 \), where in the second equality I have used (1) of the main text for \( b_c \).

Note that this ratio is clearly much less than 1 if \( v_\infty \ll c \).

(Recall that \( C_b = \left(\frac{8GM}{3b_c^2}\right)^{\frac{1}{2}} \approx 1.157367 \) and \( f(1 - f) \leq 1/4 \).)

To see that \( r_{p0} \gg r_S(M) \), I consider the ratio \( \frac{r_{p0}}{r_S(M)} = \left(\frac{v_\infty}{c}\right)^2 \left(\frac{b}{r_S(M)}\right)^2 = \left(\frac{v_\infty}{c}\right)^2 \left(\frac{b_c}{r_S(M)}\right)^2 \left(\frac{b}{b_c}\right)^2 \). Again using my result (1) for the maximum impact parameter for capture \( b_c \), I find \( \frac{r_{p0}}{r_S(M)} = \left(\frac{v_\infty}{c}\right)^2 C_b^2 \left(1 - f\right) \frac{b}{b_c} \), which will always be much greater than 1 for \( v_\infty \ll c \) unless \( b \ll b_c \), which will very rarely happen, or \( f \ll \left(\frac{v_\infty}{c}\right)^2 \), which can only happen if the two bodies are of extremely disparate masses; even a brown dwarf with \( M \sim M_{\odot}^{\frac{1}{10}} \) encountering the GBH at the center of our galaxy (\( M_{GBH} \sim 4 \times 10^6 M_{\odot} \)) will violate this condition.
B. Constancy of \( r_{p1} \) for most of the inspiral time

I begin by considering the first passage. In the main text, I have already estimated the energy loss \( [17] \) on this passage. The angular momentum loss rate \( \dot{L} \) can also be expressed in terms of the mass quadrupole tensor of the pair \( \mathbf{Q} \):

\[
\dot{L}_i = -\frac{G}{5c^5} \epsilon_{ijk} \ddot{Q}_{jm} \ddot{Q}_{km}. \tag{37}
\]

As I did in the main text for the energy, I can estimate the total change \( \delta L \) in the magnitude of the angular momentum on the first passage by replacing each of the five time derivatives in \( [37] \) with the angular velocity \( \omega \sim \frac{v}{r} \), estimating \( Q \) itself by \( \mu r^2 \), where \( \mu \) is the reduced mass, and multiplying the resultant rate by the rough estimate \( \delta t \sim \frac{r \rho_0}{v(r \rho_0)} \) of the time spent near periastron on the first passage. Doing this gives

\[
\delta L \sim -\frac{G}{c^5} \left( \frac{v(r \rho_0)}{r \rho_0} \right)^5 \mu^2 r^4 \left( \frac{r \rho_0}{v(r \rho_0)} \right) \frac{\mu^2 v^4 (r \rho_0)}{c^5} = -\frac{G \mu^2 v^4 (r \rho_0)}{c^5}. \tag{38}
\]

Using my earlier result \( [35] \) for \( v(r \rho_0) \) in this expression gives

\[
\frac{|\delta L|}{L_0} \sim (f(1 - f))^{\frac{5}{2}} \left( \frac{v_{\infty}}{c} \right)^{\frac{10}{3}} \left( \frac{b_c}{b} \right)^{\frac{5}{3}}. \tag{42}
\]

Since \( f(1 - f) < 1 \), this ratio will always be much less than 1 if \( v_{\infty} \ll c \), unless the impact parameter \( b \lesssim b_c \left( \frac{v_{\infty}}{c} \right) \frac{5}{3} \), which will rarely happen (indeed, the probability of it happening is \( \left( \frac{v_{\infty}}{c} \right)^2 \sim \left( \frac{v_{\infty}}{c} \right)^4 = 5 \times 10^{-3} \) for \( v_{\infty} = 30 \frac{\text{km}}{\text{sec}} \)).

So the magnitude of the angular momentum \( L \) after the first passage is almost the same as that before the first passage.

I can determine the distance of closest approach \( r_{p1} \) on the second passage using the fact that energy and angular momentum will be conserved until the next close passage. This implies that

\[
E_0 = \frac{\mu v^2}{2} \frac{GM \mu}{r_{p1}} \tag{43}
\]

and

\[
\mu v(r_{p1}) r_{p1} = L_{p1} \approx L_0 = \mu v_{\infty} b, \tag{44}
\]

Using my earlier result \( [35] \) for \( v(r_{p0}) \) in this expression gives

\[
\delta L \sim \frac{G^3 \mu^2 M^2}{c^5 r_{p0}^2}. \tag{39}
\]

Now using eqn. \( [40] \) to relate \( r_{p0} \) to the impact parameter \( b \), and using \( \mu = f(1 - f)M \) and \( \frac{G \mu}{c^2} = r_S(M) \), I can rewrite this as

\[
\delta L \sim -Mr_S(M) c(f(1 - f))^2 \left( \frac{c}{v_{\infty}} \right)^4 \left( \frac{r_S(M)}{b} \right)^4. \tag{40}
\]

This is a small fraction of the initial center of mass angular momentum \( L_0 = \mu v_{\infty} b = f(1 - f)M v_{\infty} b \) of the pair, as can be seen by taking the ratio:

\[
\frac{|\delta L|}{L_0} = f(1 - f) \left( \frac{c}{v_{\infty}} \right)^5 \left( \frac{r_S(M)}{b} \right)^5 = f(1 - f) \left( \frac{b_c}{b} \right)^5 \left( \frac{c}{v_{\infty}} \right)^5 \left( \frac{r_S(M)}{b_c} \right)^5. \tag{41}
\]

where in the second, approximate equality, I have used the result just derived that the angular momentum hardly changes between the first and the second passage.

If I assume, as I’ll verify \textit{a posteriori}, that \( E_0 \) is negligible compared to \( \frac{GM \mu}{r_{p1}} \), and solve \( [43] \) and \( [44] \) for \( r_{p1} \), I get

\[
r_{p1} = \frac{v_{\infty}^2 b_c^2}{2GM} = \left( \frac{v_{\infty}}{c} \right)^2 \frac{b_c^2}{r_S(M)} = r_{p0}. \tag{45}
\]

Thus, the periastron distance \( r_{p1} \) of the second passage is, as I claimed in the main text, almost exactly equal to that of the first passage, provided I can verify my \textit{a posteriori} assumption about the negligibility of the energy \( E_0 \).

To verify my assumption that \( E_0 \ll \frac{GM \mu}{r_{p1}} \), I take the ratio

\[
\Gamma = \frac{|E_0|}{\frac{GM \mu}{r_{p1}}}. \tag{46}
\]

The magnitude \( |E_0| \) of \( E_0 \) is bounded above by \( \Delta E \), eqn. \( [17] \), the energy loss on the first passage, since the initial energy \( E_0 \) of the objects before the first passage (i.e., as they approach from infinity) is positive, and \( E_0 = E_0 - \Delta E \) is negative (since the captured orbit is bound).
Hence

$$\Gamma \leq \frac{\Delta E}{\left(\frac{2GM}{v_\infty}\right)^5},$$  \hspace{1cm} (47)$$

Using (36) and (17) to relate \(r_{\rho 0}\) and \(\Delta E\) to the impact parameter \(b\), this expression can trivially be rewritten in terms of the maximum impact parameter for capture \(b_c\) as

$$\Gamma \leq \frac{D_E}{2} f(1-f) \left(\frac{GM}{cbv_\infty}\right)^5 \leq \frac{D_E}{2} \left(\frac{r_S(M)c}{b\nu_\infty}\right)^5 f(1-f) < 3 \left(\frac{r_S(M)c}{b_c\nu_\infty}\right)^5 \left(\frac{b_c}{b}\right)^5 (f(1-f))^2,$$  \hspace{1cm} (48)$$

which is clearly \(\ll 1\) until \(n \gtrsim n_c\), where

$$n_c = \left(\frac{c}{\nu_\infty}\right)^{10/7} [f(1-f)]^{-2/7} \left(\frac{b}{b_c}\right)^5,$$  \hspace{1cm} (52)$$

which is very large unless either \(f \ll \left(\frac{\nu_\infty}{c}\right)^5\), which is not even satisfied for relative velocities at infinity \(\nu_\infty = 30 \text{ km sec}^{-1}\) for very small stars falling into the GBH at the center of our galaxy, or \(\frac{b}{b_c} \ll \left(\frac{\nu_\infty}{c}\right)^{-2/7}\), which will very rarely occur.

Since the sum over orbit number \(n\) that enters the calculation of the inspiral time in the main text (i.e., the sum in the Hurwitz zeta function) converges as \(n \to \infty\), and the value \(n_c\) of \(n\) at which my approximations break down is so large, it is quite accurate to extend this sum all the way out to \(n = \infty\), as I have done in writing (3).

C. Precise calculation of the the energy loss \(\Delta E\) per passage

Using the center of mass coordinates for the two masses in (12), I can express the mass quadrupole tensor \(Q\) entirely in terms of \(r\):

$$Q_{ij} = \frac{m_1m_2}{M} \left(r_i r_j - \frac{1}{3} \delta_{ij} |\mathbf{r}|^2\right).$$  \hspace{1cm} (53)$$

Taking two time derivatives of this expression gives

$$\dot{Q}_{ij} = \frac{m_1m_2}{M} \left[ a_i r_j + a_j r_i + 2 v_i v_j - \frac{2}{3} \delta_{ij} (\mathbf{a} \cdot \mathbf{r} + v^2)\right],$$  \hspace{1cm} (54)$$

where \(\mathbf{v} \equiv \dot{\mathbf{r}}\) and \(\mathbf{a} \equiv \ddot{\mathbf{r}}\) are the relative velocity and acceleration of the two masses. Using the equation of motion \(a = \frac{2}{M} \mathbf{r}\) for \(\mathbf{r}\), I can rewrite this expression as
\[ \dot{Q}_{ij} = -\frac{2Gm_1m_2}{r^3} \left( r_ir_j - \frac{1}{3} \delta_{ij} |r|^2 \right) + \frac{2m_1m_2}{M} \left( v_iv_j - \frac{1}{3} \delta_{ij} |v|^2 \right). \] (55)

Now taking one further time derivative to obtain \( \ddot{Q} \), I obtain, after using the equation of motion for \( r \) again,

\[ \ddot{Q}_{ij} = \frac{2Gm_1m_2}{r^3} \left[ -2(v_ir_j + v_jr_i) + \mathbf{v} \cdot \mathbf{r} \left( \frac{3v_ir_j}{r^2} + \frac{\delta_{ij}}{3} \right) \right]. \] (56)

Inserting this into the general expression (11) for the emitted power \( P \) gives, after a little (!) algebra,

\[ P = \frac{8G^2m_1^2m_2^2}{5c^5r^6} \left[ 4v^2r^2 - \frac{11(\mathbf{v} \cdot \mathbf{r})^2}{3} \right]. \] (57)

To proceed further, I use the fact that, for a Newtonian orbit, the velocity vector \( \mathbf{v} \) at any point on the orbit can be written as

\[ \mathbf{v} = \mathbf{B} + \frac{GM}{h} \mathbf{\hat{y}} \] (58)

where the "binormal" vector \[\hat{y}\] \( \mathbf{B} \) is a constant of Newtonian motion which lies in the plane of the orbit perpendicular to the line from the origin to periastron, \( h \) is the angular momentum about the center of mass divided by the reduced mass \( \mu \equiv \frac{m_1m_2}{m_1 + m_2} = f(1 - f)M \), and \( \mathbf{\hat{y}} \) is the unit vector orthogonal to \( \mathbf{r} \) in the plane of the orbit.

For a parabolic orbit,

\[ v = \sqrt{\frac{2GM}{r}} \] (59)

everywhere. Applying this at periastron \( (r = r_p) \), where \( \mathbf{v} \) is perpendicular to \( \mathbf{r} \), so that \( h = |\mathbf{v} \times \mathbf{r}| = vr \), I obtain

\[ h = r_p \sqrt{\frac{2GM}{r_p}} = \sqrt{2GMr_p}. \] (60)

Using this and (58) in (59), again applied at periastron, gives

\[ \sqrt{\frac{2GM}{r_p}} \mathbf{y} = \mathbf{B} + \frac{GM}{2r_p} \mathbf{y}, \] (61)

where I’ve defined my \( x \) and \( y \) axes to lie in the plane of the orbit along and perpendicular to the line to the periastron, respectively. Equation (61) can of course easily be solved for \( \mathbf{B} \):

\[ \mathbf{B} = \frac{GM}{2r_p} \mathbf{y}. \] (62)

Using polar coordinates in the orbital plane with \( \theta = 0 \) along the \( x \)-axis, and using (58) and (62) for \( \mathbf{v} \) and \( \mathbf{B} \) respectively, I can write

\[ \mathbf{v} \cdot \mathbf{r} = \mathbf{B} \cdot \mathbf{r} = \sqrt{\frac{GM}{r_p}} r sin \theta. \] (63)

Using this and (59) for the speed \( \mathbf{v} \) in my expression (57) for the power, I get

\[ P = \frac{8G^4m_1^2m_2^2M}{5c^5r^3} \left[ 8 - \frac{11r}{6r_p} \sin^2 \theta \right]. \] (64)

I can now get the total power lost on the first passage by integrating this over all time:

\[ \Delta E = \int_{-\infty}^{\infty} P(t) dt = \frac{8G^4m_1^2m_2^2M}{5c^5r^3} \int_{-\infty}^{\infty} \frac{dt}{r^3} \left[ 8 - \frac{11r}{6r_p} \sin^2 \theta \right]. \] (65)

To evaluate the integral, I change variables of integration from time \( t \) to angle \( \theta \) using \( d\theta/dt = h/r^2 \), which follows from conservation of angular momentum. Using this in (65), and using the fact that, for a parabolic orbit,

\[ r(\theta) = \frac{2r_p}{1 + \cos \theta}, \] (66)

together with my earlier expression (60) for the angular momentum per unit reduced mass \( h \), I obtain
\[ \Delta E = \frac{G^2 m_1^2 m_2^2 M^4}{5\sqrt{2}c^5 r_p^2} \int_{-\pi}^{\pi} d\theta \left[ 8(1 + \cos \theta)^3 - \frac{11}{3}(1 + \cos \theta)^2 \sin^2 \theta \right]. \] (67)

The angular integral in this expression is elementary. Evaluating it, I get
\[ \Delta E = C_E \frac{G^2 m_1^2 m_2^2 M^4}{c^5 r_p^2} \] (68)
where \( C_E \equiv 85\pi\sqrt{2}/24 \approx 15.73521 \). Using (60) to rewrite this in terms of the impact parameter \( b \) gives
\[ \Delta E = D_E \frac{G^2 m_1^2 m_2^2 M^4}{c^5 b^5 v_{\infty}^2}, \] (69)
where \( D_E = 170\pi/3 \approx 178 \). When this energy loss is greater than the total original energy of the system, which is just the kinetic energy at infinity, the two masses will become bound. The largest impact parameter \( b_c \) that will satisfy this condition is therefore given by solving
\[ \Delta E(b_c) = \frac{m_1 m_2 v_{\infty}^2}{2M}. \] (70)
Combining (69) with (70), using the fact that \( r_S(M) = 2GM/c^2 \), and solving for \( b_c \) gives equation (11). Using the fact that the capture cross section \( \sigma = \pi b_c^2 \) then immediately gives my principal result, equation (12).

### D. Derivation of the final eccentricity distribution law

Finally, I turn to the derivation of the final eccentricity distribution law (13). This begins with the equations for the evolution of the eccentricity and semi-major axis \( a \) of an inspiraling, nearly Newtonian orbit derived by Peters (13):

\[
\frac{da}{dt} = -\frac{64K f(e)}{5a^3(1 - e^2)^2}, \tag{71}
\]
and the eccentricity \( e \):

\[
\frac{de}{dt} = -\frac{K g(e)}{15a^4(1 - e^2)^2}, \tag{72}
\]
where I’ve defined \( K \equiv G^3 m_1 m_2 M/c^5 \),

\[ f(e) = 1 + \frac{73}{24}e^2 + \frac{37}{96}e^4, \tag{73} \]
and

\[ g(e) \equiv e(304 + 121e^2). \tag{74} \]

These equations were derived by Peters (13) using a sort of adiabatic approximation, in which it is assumed that the parameters \( a, e, \) and \( \epsilon \equiv 1 - e \) undergo only small percentage changes on each orbit. This is clearly not the case for \( a \) for the first few orbits after capture, as inspection of my expression

Taking the ratio of (71) and (72) gives a differential equation for the evolution of the semimajor axis \( a \) with the eccentricity \( e \):

\[
\frac{da}{de} = \frac{\dot{a}}{\dot{e}} = \frac{192af(e)}{(1 - e^2)g(e)}, \tag{75}
\]
a result also first obtained by Peters (13). Since \( a \) changes substantially between one orbit and the next for the first few orbits, I cannot actually use this differential equation for those orbits. It is therefore more useful to rewrite this expression in terms of the periastron distance

\[ r_p = a(1 - e), \tag{76} \]
which does not change substantially between orbits, even initially.

I can rewrite (75) in terms of the periastron distance by first recasting it as an equation for \( \ln a \):

\[
\frac{d\ln a}{de} = \frac{192f(e)}{e(1 - e^2)g(e)}, \tag{77}
\]
and then using (76) to write

\[
\frac{d\ln r_p}{de} = \frac{d\ln a}{de} - \frac{1}{1 - e} = \frac{y(e)}{e}, \tag{78}
\]
where I’ve defined

\[ y(e) \equiv \frac{192 - 112e + 168e^2 + 47e^3}{(1 + e)(304 + 121e^2)}. \tag{79} \]

The right hand side of equation (78) can be rewritten

\[ \frac{y(e)}{e} = \frac{y(0)}{e} + \frac{y(e) - y(0)}{e} = \frac{12}{19} - z(e). \tag{80} \]
where I’ve used the fact that \( y(0) = 0 \) and...
is, by construction, finite as \( e \to 0 \). It is easy to check that \( z(e) \) is in fact finite and \( O(1) \) throughout the range \( 0 \leq e \leq 1 \), including the endpoints \( e = 0 \) and \( e = 1 \). I will make use of this fact in a moment.

Using (80) in (78), and integrating from the initial orbit to some later orbit gives

\[
\ln \left( \frac{r_p}{r_{p0}} \right) = \frac{12}{19} \ln \left( \frac{e}{e_1} \right) - \int_{e_1}^{e} \frac{z(e')}{e'} de'.
\]

(82)

For most captured orbits, the initial eccentricity \( e_1 \) is very close to 1. Furthermore, the initial periastron distance \( r_{p0} \) is much greater than the Schwarzschild radius \( r_s \). Hence, if I wish to know the eccentricity when the pair has inspiraled enough to emit detectable gravitational radiation, which only occurs when \( r_p \approx r_s \), I need only consider \( r_p \ll r_{p0} \). It can be seen from (82) that this implies that \( e \ll 1 \). Using \( e_1 \approx 1 \) and \( e \ll 1 \) in (82), and reversing the order of limits in the integral on the right hand side so that the smaller value of \( e \) is the lower limit, I see that, to an excellent approximation for most captured orbits,

\[
\ln \left( \frac{r_p}{r_{p0}} \right) = \frac{12}{19} \ln e(r_p) + \int_{0}^{1} \frac{z(e')}{e'} de'.
\]

(83)

(Note that I can extend the limits on the integral to 0 and 1 with impunity because of the aforementioned fact that \( z(e) \) is well behaved at those limits.)

The integral in this expression is elementary, and is

\[
\int_{0}^{1} \frac{z(e')}{e'} de' = \ln 2 - \frac{870}{2299} \ln \left( \frac{425}{304} \right) = 0.5663514... .
\]

(84)

Using this result in (83) and solving for \( e(r_p) \) gives

\[
e(r_p) = C_e \left( \frac{r_p}{r_{p0}} \right)^{\frac{12}{19}},
\]

(85)

where \( C_e = (\frac{425}{304})^{\frac{12}{19}} / 2^{\frac{12}{19}} = 0.40790... \). Using the relation \( r_{p0} \) between the minimum first passage distance \( r_{p0} \) and the impact parameter \( b \), I can rewrite this as a relation between the eccentricity and the impact parameter:

\[
e(b) = C_e \left( \frac{r_{p0}^2}{b^2} \left( \frac{e_{\infty}}{v_{\infty}} \right)^2 \right)^{\frac{12}{19}}.
\]

(86)

Solving for \( b(e) \) gives

\[
b(e) = C_e^{\frac{19}{12}} \sqrt{\frac{r_{p0}}{b}} \left( \frac{e_{\infty}}{v_{\infty}} \right) e^{-\frac{12}{19}}.
\]

(87)

The probability distribution for the final eccentricity can now be obtained from that for the impact parameter \( b \) via simple statistics, which imply:

\[
p(e; v_\infty) = p(b; v_\infty) \left| \frac{db}{de} \right|.
\]

(88)

Since the impact parameters of captured pairs should be uniformly distributed over a circle of radius \( b_c \), I have

\[
p(b; v_\infty) = \frac{2b}{b_c^2(v_\infty)}.
\]

(89)

Using this and my expressions equation (87) for \( b(e) \) and (11) for \( b_c(v_\infty) \) in (82) gives equation (11). The lower limit \( e_{min} \) on \( e \) follows from recognizing that we must have \( b < b_c \) if the pair are to be captured; therefore replacing \( b \) with \( b_c \) in equation (80) gives \( e_{min} \), equation (15) of the main text. I reiterate both of these equations here for completeness:

\[
p(e; r_p) = \begin{cases} 
C_p \left( \frac{r_p}{r_S(M)} \right) \left[ f(1 - f) \right]^{-\frac{12}{19}} \left( \frac{v_\infty}{c} \right)^{\frac{42}{19}} e^{-\frac{12}{19}}, & e_{min} < e < 1, \\
0, & e < e_{min},
\end{cases}
\]

(90)

where

\[
C_p = \frac{12C_e^{\frac{19}{12}}}{19C_b^{\frac{19}{12}}} = \frac{6}{19} \left( \frac{425}{304} \right)^{\frac{870}{2299}} \left( \frac{96}{85\pi} \right)^{\frac{12}{19}} = 0.267622338...
\]
and

\[ C_{em} = \frac{C_e}{C_{b}} = \left( \frac{425}{304} \right)^{\frac{19}{2}} \left( \frac{96}{85\pi} \right)^{\frac{12}{2}} 2^{-\frac{19}{12}} = .25678305711... \]

To predict the actual distribution of observations, for which the asymptotic approach velocity \( v_\infty \) will of course be unknown, this must be averaged over the distribution of approach velocities. That is,

\[ p(e) = \int_0^{v_{max}} dv_\infty \ p(v_\infty)p(e; v_\infty) . \quad (92) \]

There is no contribution to \( p(e) \) from velocities that are so large that \( e_{min}(v_\infty) > e \). Equivalently, this means that the integral over \( v_\infty \) in (92) has an upper limit \( v_{max} \) determined by \( e_{min}(v_{max}) = e \). Solving this condition for \( v_{max} \) implies that

\[
p(e) = \int_0^{v_{max}(e)} dv_\infty \ p(v_\infty)p(e; v_\infty) = \sqrt{\frac{2}{\pi}} C_p \left( \frac{r_p}{r_S(M)} \right) [f(1-f)]^{-\frac{7}{19}} \left( \frac{1}{v_\sigma^3} \right) e^{-\frac{42}{19}} \int_0^{v_{max}(e)} dv \ v^{18/7} \exp \left( -\frac{v^2}{2v_\sigma^2} \right) . \quad (95)\]

Changing variables of integration to \( u \equiv \frac{v^2}{2v_\sigma^2} \), and defining the characteristic eccentricity scale

\[ e_c = C_{em} \left( \frac{r_p}{r_S(M)} \right)^{\frac{12}{19}} [f(1-f)]^{-\frac{42}{19}} \left( \frac{v_\sigma}{c} \right)^{\frac{12}{19}} , \quad (96)\]

I obtain

\[ p(e) \approx \tilde C \left( \frac{1}{e_c} \right) \left( \frac{e}{e_c} \right)^{31/19} \int_0^{v_{max}(e)/2v_\sigma} du \ u^{11/14} e^{-u} = \tilde C \left( \frac{1}{e_c} \right) \left( \frac{e}{e_c} \right)^{31/19} \left( \frac{25}{14} \right) \left( \frac{1}{2} \right) \left( \frac{e}{e_c} \right)^{\frac{29}{19}} \], \quad (97)\]

where \( \gamma(s, x) \) is the lower incomplete gamma function, and I’ve defined \( \tilde C \equiv 2^{\frac{12}{19}} \left( \frac{24}{18\sqrt{\pi}} \right) = .8687429... \). This has the limiting forms:

\[ p(e) \approx \begin{cases} \frac{C_c}{e_c} \left( \frac{e}{e_c} \right)^{\frac{29}{19}} , & e \ll e_c , \\ \frac{C_\infty}{e_c} \left( \frac{e}{e_c} \right)^{\frac{12}{19}} , & e \gg e_c , \end{cases} \quad (98)\]

where I’ve defined \( C_c \equiv \frac{168}{4722\sqrt{2\pi}} = .141099585... \) and \( C_\infty \equiv \tilde C \Gamma(\frac{23}{19}) = .805906... \), where \( \Gamma \) is the complete Gamma function.

Of course, we do not really know what the distribution of asymptotic approach speeds \( v_\infty \) is. Fortunately, the asymptotic form of the final eccentricity distribution for final eccentricities large compared to the typical eccentricities can be calculated with no knowledge of that distribution, as I’ll now show.

For an arbitrary distribution \( p(v_\infty) \) of asymptotic speeds, I have

\[ p(e) = \int_0^{v_{max}(e)} dv_\infty \ p(v_\infty)p(e; v_\infty) . \quad (99) \]

For \( e \gg e_{\text{char}}, v_m(e) \gg v_{\text{char}} \), therefore, I can take the upper limit on the integral in (99) to infinity. Doing so,
and using my expression (4) for \( p(e; \nu) \), I get

\[
p(e \gg e_{\text{char}}) \approx C_p \left( \frac{r_p}{r_S(M)} \right) [f(1 - f)]^{-\frac{5}{2}} e^{-\frac{2}{7} e_\text{char}} \int_0^\infty d\nu \, p(\nu) \nu^{-\frac{4}{7}}. \tag{100}
\]

The integral in this expression is simply \( \langle \nu^{\frac{4}{7}} \rangle \). Hence, defining \( \nu_c \equiv \left( \langle \nu^{\frac{4}{7}} \rangle \right)^{\frac{7}{4}} \), I obtain

\[
p(e \gg e_{\text{char}}) \approx C_p \left( \frac{r_p}{r_S(M)} \right) [f(1 - f)]^{-\frac{5}{2}} \left( \frac{\nu_c}{c} \right)^{\frac{7}{4}} e^{-\frac{2}{7} e_\text{char}}, \tag{101}
\]

which is the general asymptotic result claimed in the main text.

E. Distribution of Inspiral times

I note here that the distribution of inspiral times is extremely broad, as illustrated in the following table.

| \( b/b_c \) | \( P_<(b) \) | \( P_>(b) \) | \( T_{\text{inspiral}} \) (years) |
|---|---|---|---|
| .1 | .01 | .99 | \( 5.37 \times 10^{-9} = (1/6) \text{sec} \) |
| .2 | .04 | .96 | \( 7.776 \times 10^{-6} = 4 \text{ minutes} \) |
| .3 | .09 | .91 | \( 5.493 \times 10^{-4} = 4.8 \text{ hours} \) |
| .4 | .16 | .84 | \( 1.12753 \times 10^{-2} = 4.12 \text{ days} \) |
| .5 | .25 | .75 | \( 0.118 = 43 \text{ days} \approx 6 \text{ weeks} \) |
| .6 | .36 | .64 | \( 0.813 \) |
| .7 | .49 | .51 | \( 4.3 \) |
| .8 | .64 | .36 | \( 19.7 \) |
| .8715 | .76 | .24 | \( 60 \) |
| .9 | .81 | .19 | \( 100 \) |
| .934 | .87 | .13 | \( 200 \) |
| .9771 | .955 | .045 | \( 1000 \) |
| .985617 | .97 | .03 | \( 2000 \) |
| .9921895 | .984 | .016 | \( 5000 \) |
| .9950703 | .99 | .01 | \( 10,000 \) |
| .9989314326 | .998 | .002 | \( 100,000 \) |
| .999769241 | .9995 | .0005 | \( 1,000,000 \) |

Here \( P_<(b) = (b/b_c)^2 \) and \( P_>(b) = 1 - P_<(b) \) denote the probabilities of impact parameters less than \( b \), and greater than \( b \), respectively.

Note that inspirals in the bottom four percentile take less than four minutes, while those in the top four percentile take more than 1000 years! Note, however,
that none of these times is cosmologically significant.

It is not possible to get a closed form analytic expression for the distribution of inspiral times, due to the impossibility of analytically inverting the Hurwitz zeta function in my expression (3) of the main text for the inspiral time. It is, however, possible to obtain this distribution in the limits of inspiral times \( \tau \ll \tau_{med} \) and \( \tau \gg \tau_{med} \).

In the former limit, which corresponds to \( b \ll b_c \), the argument \( x \) of the Hurwitz zeta function goes to 1, and the Hurwitz zeta function goes to a constant, namely, the Riemann zeta function \( \zeta(\frac{3}{2}) \). My expression (3) of the main text for the inspiral time then reduces to

\[
\tau \approx \pi \left( \frac{c^2 r_S(M)}{v^3_\infty} \right) \left( \frac{b}{b_c} \right)^{21/2} \zeta \left( \frac{3}{2} \right) .
\]  

(102)

The cumulative probability \( P_\prec (\tau) \) that the inspiral time is less than some specified \( \tau \) is just \( \left( \frac{b}{b_c} \right)^2 \); solving \( 103 \) for that quantity gives

\[
P_\prec (\tau \ll \tau_{med}) = \left( \frac{b}{b_c} \right)^2 \left( \frac{\tau v^3_\infty}{\pi c^2 r_S(M) \zeta \left( \frac{3}{2} \right)} \right)^{\frac{1}{3}}.
\]  

(103)

This can conveniently be written in terms of the median inspiral time \( \tau_{med} \), which is just \( \tau \) evaluated from (3) at the median impact parameter \( b_{med} = b_c / \sqrt{2} \); this gives

\[
\tau_{med} = C_{med} \left( \frac{\pi c^2 r_S(M)}{v^3_\infty} \right)
\]  

(104)

where I’ve defined

\[
C_{med} = 2^{-21/4} \zeta \left( \frac{3}{2} , 1 - \frac{2}{3} \right) \approx .073802 .
\]  

(105)

Using this in \( 103 \) gives

\[
P_\prec (\tau) = C_{r\prec} \left( \frac{\tau}{\tau_{med}} \right)^{\frac{1}{3}},
\]  

(106)

where I’ve defined

\[
C_{r\prec} \equiv \left( \frac{C_{med}}{\zeta \left( \frac{3}{2} \right)} \right)^{\frac{1}{3}} = \frac{1}{2} \left( \frac{\zeta \left( \frac{3}{2} , 1 - \frac{2}{3} \right)}{\zeta \left( \frac{3}{2} \right)} \right)^{\frac{1}{3}} = .506942...
\]  

For \( \tau \gg \tau_{med} \), which corresponds to \( b \to b_c \), the argument \( x \) of the Hurwitz zeta function in (3) is well approximated by \( x \approx 7 \epsilon \), where \( \epsilon \equiv 1 - \frac{r}{r_c} \), which goes to 0 as \( b \to b_c \). In this limit, the Hurwitz zeta function approaches \( x^{-\frac{3}{2}} \), while \( \left( \frac{\epsilon}{r_c} \right)^{\frac{1}{3}} \to 1 \). Putting these facts together in (3), I get a good approximation to \( \tau \) for \( \tau \gg \tau_{med}: \)

\[
\tau \approx \frac{\pi}{7^{\frac{3}{2}}} \left( \frac{c^2 r_S(M)}{v^3_\infty} \right) \epsilon^{-3/2}.
\]  

(107)

In this limit, the cumulative probability \( P_\succ (\tau) \) that the inspiral time is greater than some specified \( \tau \) is just \( 1 - \left( \frac{k}{b} \right)^2 \approx 2 \epsilon \); solving \( 107 \) for \( \epsilon \) then gives

\[
P_\succ (\tau) = \frac{2 \left( \frac{\pi c^2 r_S(M)}{7 v^3_\infty} \right)^{\frac{2}{3}}}{7 \left( \frac{\zeta \left( \frac{3}{2} , 1 - \frac{2}{3} \right)}{\zeta \left( \frac{3}{2} \right)} \right)^{\frac{2}{3}}} = C_{r\succ} \left( \frac{\tau_{med}}{\tau} \right)^{\frac{2}{3}},
\]  

(108)

where I’ve used my earlier result \( 104 \) for the median time \( \tau_{med} \), and I’ve defined

\[
C_{r\succ} \equiv \frac{2^{\frac{2}{3}}}{7 \left( \frac{\zeta \left( \frac{3}{2} , 1 - \frac{2}{3} \right)}{\zeta \left( \frac{3}{2} \right)} \right)^{\frac{2}{3}}} = 1.623877...
\]  

(109)

The small \( \tau \) limit equation \( 106 \) is accurate to 2\% up to \( \tau = .9 \tau_{med} \), while the large \( \tau \) limit \( 108 \) is accurate to 2\% down to \( \tau = 200 \tau_{med} \); these two ranges contain more than half of all captures.

F. Connection to earlier work of Walker and Will

The first recognition of the possibility of GRAC was by Walker and Will[6]. In this subsection, I recover their result for the maximum incoming eccentricity of a pair that are captured; that is, in my terminology, the incoming eccentricity of two bodies approaching with impact parameter \( b = b_c \).

Denoting the periastron distance of the two objects on their first passage as \( r_p \), the maximum value \( r_p^c \) that \( r_p \) can take on (which occurs when the two objects approach each other with exactly the maximum impact parameter \( b_c \) for capture) is given by

\[
r_p^c = \left( \frac{85 \pi b^2 f(1 - f)}{96 v^2_\infty(M)} \right)^{\frac{1}{2}} r_s(M).
\]  

(110)

This result can be used to determine the eccentricity of the orbit before capture, which can be compared with the result of [6]. To do so, I begin with the simple geometrical observation that

\[
b_c = \lim_{\theta \to \theta_c} \frac{p_c \sin(\theta_c - \theta)}{1 + \epsilon \cos(\theta)} = 2 r_p^c \lim_{\theta \to \theta_c} \frac{\theta - \theta_c}{\sin(\theta_c)} = 2 r_p^c \frac{\theta_c - \theta_c}{\sin(\theta_c)} = 2 r_p^c \theta_c ,
\]  

(111)

where \( \theta_c = \arccos \left( -\frac{1}{\epsilon} \right) \) is the angle at which \( r \to \infty \), with \( \epsilon \) the eccentricity of the orbit and \( p \) its semilatus rectum. In writing (111), I have used the fact that, for a nearly parabolic orbit, \( p \approx 2 r_p \). Another property of a nearly parabolic orbit is that \( e = 1 + \epsilon \) with \( \epsilon \ll 1 \), which is the case for any two bodies that are captured by GRAC when \( v_\infty \ll c \). In this case, it is straightforward to show that \( \sin(\theta_c) \approx \sqrt{2} \). Using this in (111), I find

\[
b_c = r_p^c \sqrt{\frac{2}{\epsilon}} .
\]  

(112)
Now solving (110) to write $b_c$ in terms of $r_c^2$ on the right hand side of this expression, and solving the resultant equation for $\epsilon$ gives

$$\epsilon = \left(\frac{170\pi}{3}\right) \left(\frac{\mu}{M}\right) \left(\frac{r_S(M)}{2p_c}\right)^{\frac{2}{3}}, \quad (113)$$

where $\mu \equiv \frac{m_1 m_2}{M} = f(1 - f)M$ is the reduced mass.

Equation (113) is the final (unnumbered) equation of reference[6] (which uses "natural" units in which $c = 1 = G$; in those units, $r_S(M) = 2M$). In writing (113), I have again used the fact that for a nearly parabolic orbit $p = 2r_p$. 