Electromagnetic Force and the Maxwell Stress Tensor in Condensed Systems

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While the electromagnetic force is microscopically simply the Lorentz force, its macroscopic form is more complicated, and given by expressions such as the Maxwell stress tensor and the Kelvin force. Their derivation is fairly opaque, at times even confusing, and their range of validity all but a well kept secret. These circumstances unnecessarily reduce the usefulness and trustworthiness of some key quantities in macroscopic electrodynamics.

This article presents a thorough yet pedagogical derivation of the Maxwell stress tensor and electromagnetic force in condensed media. It starts from universally accepted inputs: conservation laws, thermodynamics and the Maxwell equations. Simplifications are considered for various limits, especially the equilibrium, with a range of validity assigned to each expression. Some widespread misconceptions are scrutinized, and hidden ambiguities in popular notations revealed.

A number of phenomena typical of strongly polarizable systems, especially ferrofluids, are then considered. In addition to enhancing the appreciation of these systems, it helps to solidify the grasp of the introduced concepts and derived formulas, and it demonstrates the ease with which the Maxwell stress tensor can be handled, inviting theorists and experimentalists alike to embrace this useful quantity.

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I. INTRODUCTION

A. The Coarse-Grained Force

Studying polarizable and magnetizable materials, one central question concerns the electromagnetic force on matter. This is especially true for liquids (such as ferrofluids [2, 3, 4]), as these respond not only to the total force, but also to the spatially varying force density. In principle, one may take the Lorentz force as an exact microscopic expression and coarse-grain it to obtain the macroscopic force [1].
\[ \langle f_i \rangle = \langle \rho_e \mathbf{E} \rangle + \langle \mathbf{j}_e \times \mathbf{B} \rangle. \] (1)

Unfortunately, this is of little practical value, as we neither have, nor indeed are interested in, the detailed microscopic information of the fields \( \rho_e, \mathbf{j}_e, \mathbf{E}, \mathbf{B} \). On the other hand, although we do have the knowledge of the coarse-grained fields \( \langle \rho_e \rangle, \langle \mathbf{j}_e \rangle, \langle \mathbf{E} \rangle \) and \( \langle \mathbf{B} \rangle \), or may obtain them via the macroscopic Maxwell equations, it is clear that the difference between the above true expression and the “fake” Lorentz force,

\[ \langle \rho_e \rangle \langle \mathbf{E} \rangle + \langle \mathbf{j}_e \rangle \times \langle \mathbf{B} \rangle, \] (2)

may become quite important: Think of an iron nail in a magnetic field, which is obviously subject to an electromagnetic force, although the “fake” force vanishes, as \( \langle \rho_e \rangle, \langle \mathbf{j}_e \rangle \rightarrow 0 \).

The difference between the true and fake Lorentz force is frequently taken to be the Kelvin force, being \( P_i \nabla \langle E_i \rangle \) in the electric case \( \mathbb{B} \), where \( P_i \) denotes the polarization, with summation over repeated indices implied. Originally, the formula was microscopically derived: First write down the force exerted by an external field on a dipole, eg by taking the derivative of the electrostatic energy with respect to the position of the dipole for given field. Then, assuming that the dipoles are too far apart to interact, and to feed back to the given field, take the total force density as simply the sum of all forces exerted on the dipoles in a unit volume. Clearly, this result neglects nonlinear terms in \( P_i \) (which account for interaction) and is valid only in the dilute limit. This implies small polarization and therefore weak force, and one is free to add a second order term, writing the Kelvin force as

\[ P_i \nabla \langle E_i \rangle + P_i / \epsilon_0 \rightarrow P_i \nabla D_i / \epsilon_0. \] (3)

There is a more sophisticated macroscopic derivation of the Kelvin force that we shall discuss next. Stating that the same constraint applies here, and one may still write the Kelvin force either as \( P_i \nabla \langle E_i \rangle \) or as \( P_i \nabla D_i / \epsilon_0 \) is bound to raise a few eyebrows. But as will be shown in this review, it is really a rather straight-forward implication of the assumptions made during its derivation. By the same token, for linear constitutive relation, the Kelvin force is only valid for small susceptibility, \( \chi \ll 1 \), to linear order in \( \chi \).

The same holds for the magnetic case, where the corresponding force is much larger in praxis (as it is not subject to ionization or residual conductivity). The Kelvin force expression is again valid only when weak, to linear order in the magnetization \( M \ll \langle B \rangle \), or the magnetic susceptibility \( \chi^m \ll 1 \). And one may take its expression either in the usual form \( \mathbb{B} \mathbb{E} \mathbb{F} \mathbb{G} \), as

\[ M_i \nabla \langle (B_i) - \mu_0 M_i \rangle \equiv \mu_0 M_i \nabla H_i, \] (4)

or equally justified, as \( M_i \nabla \langle B_i \rangle \). Consequently and remarkably, the Kelvin force is not usually a valid formula in ferrofluids, where frequently \( M \approx H \) and \( \chi^m \approx 1 \) — though it is widely employed there.

The nowadays generally accepted method to derive the electromagnetic, or “ponderomotive” force \( f_i^P \), cf §15 of *Electrodynamics in Continuous Media* by Landau and Lifshitz \( \mathbb{B} \), is a definite improvement over the microscopic one given above. It is completely macroscopic and starts from identifying \( f_i^P \) as

\[ f_i^P = - \nabla_j [\Pi_{ij}^\text{tot} - P(T, \rho) \delta_{ij}], \] (5)

where \( -\Pi_{ij}^\text{tot} \) denotes the macroscopic Maxwell stress tensor, and \( P(T, \rho) \) the zero-field pressure — the pressure that is there for given temperature and density, but without the external field. The ponderomotive force \( f_i^P \) reduces to the Kelvin force, if the polarization and magnetization are (assuming the dilute limit) proportional to the density \( \rho \).

The idea behind the identification seems attractively simple: Since \( -\nabla_i \Pi_{ij}^\text{tot} \) is the total force on a volume element, and \( -\nabla_i P \) the force at zero field, their difference \( f_i^P \) must be the force due to the external field. Unfortunately, as we shall show in this review, the resultant force is rather ambiguous. The reason lies with the zero-field pressure \( P(T, \rho) \), which is not a unique quantity, because its value depends on the chosen variables, eg \( P(T, \rho) \neq P(s, \rho) \), where \( s \) is the entropy density. The conventionally defined Kelvin force is valid only under isothermal conditions with the density kept constant. Under adiabatic conditions, with \( s \) given, the temperature \( T \) is known to change with the field, so will \( P(T, \rho) \), the alleged zero-field pressure. Any field dependence implies that \( P(T, \rho) \) contributes to the electromagnetic force. Employing the force of Eq \( \mathbb{B} \) alone then leads to wrong predictions.

In physics, we tend to start from a robust base of broad applicability when studying a given situation, narrowing our focus along the way only when necessary. Clearly, while the universally valid Lorentz force is one such base, the fragile ponderomotive force, and even more specific, the Kelvin force are not. This makes it desirable to look for better alternatives, one of which is employing the Maxwell stress tensor directly.

### B. The Maxwell Stress Tensor

Given in terms of thermodynamic variables, the Maxwell tensor is an unequivocally macroscopic, coarse-grained quantity. It holds for all conceivable systems and is subject only to the validity of local equilibrium — a constraint in frequency but not in the type and strength of interaction. Besides, more frequently than not, in understanding the many facets of strongly polarizable systems, it is more convenient to employ the Maxwell tensor directly, rather than to follow the detour via pressure and...
the ponderomotive force $f^P_i$ — in spite of having to deal with quantities such as the chemical potential or the entropy.

Given the Maxwell tensor’s pivotal role for a solid understanding of electromagnetism in condensed systems, (not least as the starting point for forces), it is especially unfortunate that one is hard pressed to find a fully convincing derivation of its form. The classic reference is again to the otherwise excellent book by Landau and Lifshitz [2], the relevant §15 of which, however, leaves many readers unconvinced, even confused. The reasons for this are presented in the four sections below.

1. Three Obvious Objections

The three more obvious reasons for the frustration of the readers are (i) the gradient terms, (ii) the magnetic field dependence, and (iii) the concentration: First, the Maxwell tensor is derived considering a capacitor. Its geometry is so simple that terms proportional to the gradients of temperature and field may have been missed. Although these are asserted to be negligible in a footnote, one feels the need for convincing arguments.

Second, the magnetic terms are obtained, in §35 of [3], by the replacement $E \rightarrow H, D \rightarrow B$. This would be justified if the system under consideration does not contain sources, and the respective Maxwell equations are the same, $\nabla \times E = 0, \nabla \cdot D = 0$ versus $\nabla \times H = 0, \nabla \cdot B = 0$. In the geometry used to derive the electric part of the stress tensor, however, the charge on the capacitor plates is essential for producing the electric field. Therefore, an independent consideration for the magnetic part is necessary, with a current carrying coil replacing the charge laden capacitor. In §4 of his widely read book on ferrofluids [2], Rosensweig points to this gap, and aims to fill it. Unfortunately, the calculation starts from a faulty assumption, discrediting the derivation, cf section III E 2.

Finally, neither Landau and Lifshitz nor Rosensweig study the concentration dependence of the stress tensor, although it is an important thermodynamic and hydrodynamic variable in ferrofluids, liable to undergo much greater variations than the density, and hence of experimental importance.

2. The Range of Validity

Next, the lack of any discussion on the range of validity for the obtained expression further undermines the trust of the readers. The derivation is thermodynamic in nature and takes place in a stationary capacitor, so the result is apparently valid only in equilibrium and for vanishing liquid velocity. Consistent to this, the static Maxwell equations for the rest frame are repeatedly employed when deriving the electromagnetic force from the Maxwell tensor.

Nevertheless, it is customary to take for granted that the form of the Maxwell tensor and the electromagnetic force thus derived also hold in dynamic situations, at finite frequencies and nonvanishing liquid velocity. This may well be approximately true for some circumstances, but which and why? Should we not rather derive dynamically valid expressions for the Maxwell stress tensor and the electromagnetic force, and justify the employed approximations by estimating the magnitude of the terms neglected?

3. A Technical Flaw

Then there is a technical flaw that actually renders the derivation mathematically invalid: There is no question that one can always obtain a tensorial quantity $\Pi_{ij}^{{\text{tot}}}$ by evaluating the scalar $\Pi_{ij}^{{\text{tot}}} \xi_i n_j$ while varying the unit vectors $\xi_i$ and $n_j$. Taking each of the two unit vectors to successively point in all three directions, we obtain nine different results for $\Pi_{ij}^{{\text{tot}}} \xi_i n_j$, each a component of $\Pi_{ij}^{{\text{tot}}}$. In §35, $n_j$ is the surface normal, $\xi_i$ the direction of the virtual displacement, and the Maxwell tensor $-\Pi_{ij}^{{\text{tot}}}$ is denoted as $\sigma_{ij}$.

With the electric field being the only preferred direction, $\xi_i$ and $n_j$ need to be both along and perpendicular to the field. In §15 of [3], only the displacement of the capacitor plate is considered, implying that the surface normal $n_j$ is kept parallel to the E-field throughout. Therefore, if the direction of the electric field is taken as $\hat{e}_x$, only the components $\Pi_{xx}, \Pi_{yz}$ and $\Pi_{zx}$ could have been evaluated. A footnote states: It is not important that in this derivation $E$ is parallel to $n$, since $\Pi_{ij}$ can obviously depend only on the direction of $E$, not on that of $n$. This is incorrect, because although $\Pi_{ij}$ does not depend on $n$, the scalar $\Pi_{ij} \xi_i n_j$ does: Take for instance the term $D_i E_j$, which is part of the Maxwell stress, and which indeed does not depend on $n$. Yet with $D_i E_j \parallel \hat{e}_x$, we have $D_i E_j n_j \neq 0$ for $n \parallel \hat{e}_x$, and $D_i E_j n_j = 0$ if $n \perp \hat{e}_x$.

4. Conceptual Problems

Finally, there are conceptual problems of more fundamental nature. The Maxwell stress tensor is introduced in [2] by the terse sentences: It is well known that the forces acting on any finite volume in a body can be reduced to forces applied to the surface of that volume. This is a consequence of the law of conservation of momentum. The surface force density is subsequently taken as the stress tensor.

Momentum conservation is without further elaboration a confusing argument, because the very idea of force is something that alters the momentum of particles. Being accelerated by external fields, the momentum of a lump of material is not conserved, and there is no reason why the
momentum density $\rho \mathbf{v}$ should satisfy a continuity equa-
tion. On the other hand, what the electromagnetic field
does is to exchange momentum with the material, so the
total momentum of both remains conserved, which is, as
we shall see, $\mathbf{g}^{\text{tot}} = \rho \mathbf{v} + \mathbf{E} \times \mathbf{H}$. This momentum density
does satisfy a continuity equation, $\dot{\gamma}_i^{\text{tot}} + \nabla_i \Pi^{\text{tot}}_{ik} = 0$.
(Even in the presence of fields that accelerate particles,
the total system — including particles and the field pro-
ducing charge carriers — is still translational invariant,
with the associated total momentum a locally conserved
quantity, see §75 of [3], Ch 14 of [1], and [7].) The neg-
ative of the associated momentum flux, $-\Pi^{\text{tot}}_{ik}$, is the
properly defined, dynamically valid Maxwell tensor. It
holds its ground for finite frequencies, nonvanishing ve-
locities of the medium, and including dissipations. Un-
derstanding the conservation of $\mathbf{g}^{\text{tot}} = \rho \mathbf{v} + \mathbf{E} \times \mathbf{H}$ renders
the derivation of the Maxwell tensor cogent and coher-
ent, even though the second term is, under usual circum-
stances in condensed matter, much smaller than the first.

C. Content of this Article

Our aim in this paper is a careful and rigorous deriva-
tion of the Maxwell tensor and the dynamically valid elec-
 tromagnetic force, in a presentation that is detailed and
pedagogical. We also consider applications, especially for
experimentally relevant situations in ferrofluids, to give
hands-on illustrations on how to employ these quantities.

The paper may be divided into two parts. The first con-
 sists of Chapter I and concentrates on the fundamental
concepts. The hydrodynamic equations for a dense sys-
tem exposed to electromagnetic fields — including espe-
cially the form of the Maxwell tensor — are derived here.
The second part (Chapter II and IV) is more practically
concerned. It starts from the Maxwell tensor, derives the
electromagnetic force, considers different simplifica-
tions, and studies a number of experimentally relevant
situations. Although we believe that Chapter I provides
useful and necessary insights, the second part could if
necessary be read alone.

Chapter II starts by differentiating two types of the-
ories: First, the high-resolution theory for a low den-
sity system, with at most one particle per infinitesimal
volume element, or per grain (as in photographs); sec-
ond, the low resolution theory for a high density sys-
tem, with many particles per grain. The microscopic
Maxwell equations and the Newtonian equation of mo-
tion (including the Lorentz force) belong to the first type.
There are no hidden charges, polarization or magnetiza-
tion here, and we know the whereabouts of every single
particle. The second type is represented by the macro-
scopic Maxwell equations, or any thermodynamic and
hydrodynamic theories. The question about the coarse-
grained form of the electromagnetic force arises here.

We shall consider three systems with increasing densi-
ties, to be accounted for by theories of decreasing resolu-
tion and increasing complexities: The first system (sec-
tion IIA) is a dilute gas of charged particles, which is
well accounted for by the Newtonian equation of motion
and the microscopic Maxwell equations. The next sys-
tem (section IIB) is a slightly dissociated liquid of par-
ticles possessing negligible electric and magnetic dipole
moments. So there are many particles, but at most one
charge carrier, per grain. The appropriate theory here
is of a mixed type, being a combination of the hydrody-
namic theory and the microscopic Maxwell theory. The
former accounts for all particles, the latter accounts for
the spatially slowly varying field, either externally im-
posed or produced by the few charge carriers. Finally,
in section IIC, we consider a dense system, including
dipole moments and hidden sources. It is to be account-
ed for by the genuinely low-resolution, hydrodynamic-type
Maxwell theory that is our goal to derive and consider in
details.

In all these systems, there are a conserved energy $u^{\text{tot}}$
and a conserved momentum density, $\dot{\gamma}_i^{\text{tot}} \neq \rho \mathbf{v}_i$, both con-
 sist of material and field contributions. These we shall
choose as two of our dynamic variables, with the equa-
tions of motion $\dot{\gamma}_i^{\text{tot}} + \nabla_i Q_i^{\text{tot}} = 0$ and $\dot{\gamma}_i^{\text{tot}} + \nabla_i \Pi_{ik}^{\text{tot}} = 0$, where $-\Pi_{ik}^{\text{tot}}$ is the Maxwell stress tensor. We shall ob-
tain explicit expressions for $\dot{\gamma}_i^{\text{tot}}$ and $\Pi_{ik}^{\text{tot}}$ in all three sys-
tems — whereby finding the expressions in the genuinely
dense, third medium is rendered very much simpler, and
more transparent, by the considerations in the previous
two dilute systems.

In Chapter III, starting from the Maxwell tensor, we
shall first obtain the general expression for the electro-
magnetic force density, valid for arbitrarily strong fields,
moving medium and finite frequencies, given essentially
as $-(s \nabla T + \rho \nabla \xi)$, where $\xi$ is the chemical potential.
Although this expression is already valid without field,
it is the proper formula for the quantity usually written
as the sum of the zero-field pressure gradient and the
ponderomotive force. In contrast to the latter, however,$-(s \nabla T + \rho \nabla \xi)$ is valid for any sets of variables, and for
arbitrarily large polarization and magnetization.

In equilibrium, since both the temperature $T$ and
the chemical potential $\xi$ are uniform, this force density
is identically zero. Nevertheless, electric and magnetic
fields remain operative, may lift a polarizable system off
the ground, deform its shape and generally alter its hy-
drostatics — all signs that forces of electromagnetic ori-
gin are at work. In section IIIA, the responsible equi-
librium force is identified as a surface force. Its form
follows directly from the simplifications that occur for
the Maxwell stress if the system is in equilibrium (section
IIC). As most of the experiments considered in Chapter
IV reflect varying appearances of the equilibrium force,
this is a useful and concise result.

A thermodynamic rederivation of the Maxwell stress
tensor is given in section IIE, an approach which is sim-
ilar in spirit to that of Landau and Lifshitz [1, 2], yet
carefully freed of the flaws listed in section IIA above.
In comparison to the derivation given in Chapter III, this
here is narrower in focus and validity, but simpler. Chap-
tends section II, which points out the inherent ambiguity of the zero-field pressure \( P(T, \rho) \), as it is independent of external fields only if \( T \) and \( \rho \) are kept constant. We stress this point by repeating the separation of the zero-field pressure from the Maxwell stress for various combinations of variables, obtaining many forms of the ponderomotive force, all rather different from each other. Furthermore, a warning is issued on the unsuspected, narrow range of validity for various expressions in the notation containing the zero-field pressure, especially the Maxwell stress in the approximation \( (P + \frac{1}{2} H^2) \delta_{ij} - H_i B_j \), and the Kelvin force. Neither is valid for large magnetization and magnetic susceptibility, \( M \approx H \) and \( \chi_m \approx 1 \).

The very useful magnetic Bernoulli equation by Rosensweig [2] is unfortunately also given in this notation. An formulation free of ambiguity is found, along with the generalization to include the variation of concentration.

In Chapter IV, we study a number of phenomena typical of strongly polarizable systems. Most of these experiments have already been given detailed theoretical analyses, especially by Rosensweig in his comprehensive book on ferrofluids [2]. The reason to repeat them here is threefold: First, to demonstrate that it is not necessary to introduce the zero field pressure, that experimental phenomena may be well and easily accounted for employing the Maxwell stress directly, using a generally valid, unambiguous notation. Second, we believe that our derivations are well streamlined and easy to follow. This is achieved by stressing the simplification of equilibrium, and by consistently using only the Maxwell stress and the associated boundary conditions. In contrast, confusingly many pressures are deemed necessary in [2]. Finally, we aim to amend some of the physics and especially include the concentration of magnetic particles as an independent variable. Being inhomogeneous where the field is, the concentration alters experimental outcomes and is something theoretical considerations need to include.

Since mesoscopic particles diffuse slowly, the equilibrium between concentration and field gradient needs time (up to days, even weeks) to establish. So for times much briefer, the concentration will remain constant and may indeed be taken as dependent. On the other hand, strong gradients, small geometry, large particles will all consid-
erably shorten the equilibrating time (down to fractions of a minute), then the spatially varying concentration will be an important feature in any system. Besides, long contacts with field gradients are also of interest (eg in magnetic O-rings). Although the non-uniform equilibrium distribution of magnetic particles is not a well studied subject, their diffusion under enforced non-equilibrium situation is, see [1, 3, 4].

Now a rundown of the considered experiments. In section IV A we study the variation of density and concentration in the presence of inhomogeneous fields: While the variation of density is a measured phenomenon, that of the concentration is not — although its variation is much more pronounced, by many orders of magnitude. A vertical, current-carrying wire that goes through a dish containing ferrofluid will drag up the ferrofluid along the wire. Our calculation in section IV B includes the surface tension and the variation of concentration. The following section IV C considers how far an external field will raise the level of a fluid column, and what a pressure gauge actually measures within the ferrofluid that is subject to external fields. Section IV D deals with rather similar physics and considers magnetic O-rings and scrap separation. Section IV E contains a simple calculation on the elliptical deformations a droplet of ferrofluid undergoes along the external magnetic field.

**D. Two Caveats**

There are two points we shall not address in this article. First, we shall not consider any dissipative terms, although the framework presented here is well suited for including them. This was a difficult decision, as the dissipative terms account for irreversibility, without which no macroscopic, coarse-grained theory is complete, and because they give rise to additional electromagnetic forces of comparable magnitude and experimental relevance. Yet, including these terms would have considerably lengthened an already lengthy article, and further complicated its formulas and arguments. We pledge amendments for the future, and for now direct the impatient readers to the literature [3, 4, 9].

Second, given the long and tortuous history of the struggle to come to terms with the macroscopic electromagnetic force, time and again forcing us to back up from blind alleys, any attempt by us on a comprehensive citation would bear historic rather than scientific interests. As we see it, the consistent thermodynamic treatment given first by Landau and Lifshitz [3], and later generalized to include nonlinearity by Rosensweig and others [2], cf the excellent summary by Byrne [10], is so much more superior than the many treatises preceding them, that this represents the only worthwhile starting point for any further considerations — and consequently these two are also the only ones subject to scrutiny and criticisms here.

**E. Notes on Units**

This paper will be in the SI units throughout, though with a little twist to render the display and manipulation of the formulas simple. We define, and from now on shall extensively use, the fields and sources

\[
H = \hat{H} \sqrt{\mu_0}, \quad B = \hat{B} / \sqrt{\mu_0}, \quad \varrho_c = \hat{\varrho}_c / \sqrt{\epsilon_0}, \\
E = \hat{E} \sqrt{\epsilon_0}, \quad D = \hat{D} / \sqrt{\epsilon_0}, \quad q = \hat{q} / \sqrt{\epsilon_0},
\]

\[
P = \hat{P} / \sqrt{\epsilon_0}, \quad M = \hat{M} / \sqrt{\mu_0},
\]

where (again starting from here) the quantities with hats are the usual ones, denominated in MKSA, or the SI-units. The advantage is, all new fields have the same
dimension, \( \sqrt{J/m^3} \), and sensibly, \( H = B \) and \( D = E \) in vacuum. (Electric charge \( q \) has the dimension \( \sqrt{J/im} \), and its density \( [\rho_e] = \sqrt{J/m^5} \).) Written in the new fields, all formulas are rid of the ubiquitous (and content-free) factors \( e_0 \) and \( \mu_0 \). To return to the more familiar MKSA-fields at any instance, simply employ Eqs (5). (These fields are sometimes referred to as being in the Heaviside-Lorentz, or the rational, units. But saying so immediately leads to heated and pointless discussions. We stick to the SI units throughout and use the above fields only as a convenient shorthand that may be abandoned whenever one chooses.)

Some literature, notably \( \text{\cite{2}} \), employs \( cgs \), or Gauss-units. To ease comparison with them, simply note that the Gauss-fields, with tilde, may be obtained any time via

\[
\begin{align*}
H &= \tilde{H}/\sqrt{4\pi}, & B &= \tilde{B}/\sqrt{4\pi}, & \rho_e &= \sqrt{4\pi} \rho_e, \\
E &= \tilde{E}/\sqrt{4\pi}, & D &= \tilde{D}/\sqrt{4\pi}, & j_e &= \sqrt{4\pi} j_e,
\end{align*}
\]

(7)

II. HYDRODYNAMIC THEORIES

A. Dilute Plasma

We first consider a rarefied system of charged particles, and choose a resolution that is high enough that each grain (or infinitesimal volume element) contains one or less particles. (This theory is meant as a starting point, to clarify a few concepts important for the more complex theories of the following chapters. So we shall simply discard the possibility that even in a rarefied gas, two particles will occasionally come close to each other.) The microscopic Maxwell equations account for the time evolution of the electromagnetic field for given sources,

\[
\begin{align*}
\nabla \cdot e &= \rho_e, & \nabla \cdot b &= 0, \\
\dot{e} &= c \nabla \times b - \rho_e \dot{v}, & \dot{b} &= -c \nabla \times e,
\end{align*}
\]

(8)

while the feedback of the field on the motion of the sources is given by the Newtonian equation of motion,

\[
m \ddot{v}^\alpha = q(e + v^\alpha \times b/c),
\]

(9)

one for each particle \( \alpha \). (To emphasize the fact that we are here dealing with the high-resolution, microscopically accurate fields, these are denoted by the lower case letters \( e \equiv \tilde{e}/\sqrt{\epsilon_0}, b \equiv \tilde{b}/\sqrt{\mu_0} \), while the coarse-grained fields below are denoted by the usual \( \text{D, E, H, B} \).) Eqs (8) represent a conceptually simple and complete theory, but it contains a notational inconsistency: The Maxwell equations are an Euler type theory, accounting for the time evolution of fields at a given point in space, while the Newtonian equation is of the Lagrange type, which concentrates on a given particle. (So the term \( \rho_e \dot{v} \) in Eq (4) denotes the electric current at a space point, while \( v^\alpha \) in Eq (10) is the velocity of a particle.) As only the Euler formulation lends itself to a scaling-up of the grains and a reduction of the resolution, we shall first find the Euler formulation for the Newtonian equation, before using it to draw a number of conclusions useful for dense systems.

1. Euler Version of Newtonian equation

Since there is at most one particle per grain, of volume \( V_G \), we may identify the velocity, mass and charge of a volume element with that of the particle occupying it at a given instance, and take all three to be zero if there is no particle. As a result, we obtain three (highly discontinuous) fields: \( v^\alpha \rightarrow \dot{v}(r,t), m/V_G \rightarrow \rho(r,t), q/V_G \rightarrow \rho_e(r,t) \). The many Newtonian equations of motion then reduce to one field equation,

\[
\rho \dot{\vec{v}} = \rho [\vec{v} + (\vec{v} \cdot \nabla)\vec{v}] = \rho_e (\vec{e} + \vec{v} \times \vec{b}/c).
\]

(11)

(The quantity \( \nabla \vec{v} \) is to be taken from the velocity of the same particle at two successive moments.) It is now essential to explicitly include the continuity equation,

\[
\dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0,
\]

(12)

that in the Lagrange version is implicit, nearly incidentally contained in the fact that one does not loose any of the many equations \( \text{\cite{1}} \). [The continuity equation for the charge density \( \rho_e \) is implied by the Maxwell equations Eq (5).]

The material contributions to the energy and momentum density are, respectively,

\[
u^M = \rho(\dot{v}^2 + v^2/2), \quad g^M = \rho \dot{v}.
\]

(13)

where the first comprises of the rest energy and the (non-relativistic) kinetic energy. Employing Eqs (14,15), we find

\[
\begin{align*}
\dot{\rho}^M + \nabla \cdot \dot{Q}^M &= \rho_e (\vec{v} \cdot \vec{e}), \\
\dot{g}^M + \nabla_k \Pi^\alpha_k &= \rho_e (\vec{e} + \vec{v} \times \vec{b}/c), \\
Q^M &= u^M \vec{v}, & \Pi^M_k &= g^M_k v_k.
\end{align*}
\]

(16)

2. The Field Contributions

The field contribution to the energy and momentum density are

\[
u^F = \frac{1}{2}(c^2 + b^2), \quad g^F = \vec{e} \times \vec{b}.
\]

(17)

From the Maxwell equations (5) we deduce

\[
\begin{align*}
\dot{u}^F + \nabla \cdot Q^F &= -\rho_e (\vec{v} \cdot \vec{e}), & Q^F &= \vec{e} \times \vec{b}, \\
\dot{g}^F + \nabla_k \Pi^\alpha_k &= - \rho_e (\vec{e} + \vec{v} \times \vec{b}/c), \\
\Pi^F_k &= (c^2 + b^2 - u^F) \delta_{ik} - c_i e_k - b_i b_k.
\end{align*}
\]

(18)

Note the relationship \( g^F = Q^F/c^2 \). This is far from accidental and derives from the symmetry of the relativistic energy-momentum 4-tensor, \( \Pi^F_{\alpha\beta} = \Pi^F_{\beta\alpha} \), because
\[ e g_k^F = \Pi_{0k}^F, \; Q_k^F/c = \Pi_{0k}^F. \] 
Less formally, \( g^F = Q^F/c^2 \) may also be seen as the conservation of the field angular momentum for \( p_e \to 0 \). The angular momentum density \( \mathbf{\hat{\ell}}^F \equiv \mathbf{r} \times g^F \) is a locally conserved quantity in neutral systems. Rewriting \( \text{Eq.} \) (with \( p_e \to 0 \)) as \( \partial \mathbf{\hat{\ell}}^F/\partial t + \nabla_m(\epsilon_{ijk} \mathbf{r}_j \Pi_{ik}^F) = \epsilon_{ijk} \Pi_{kj}^F \), we observe that the angular momentum \( \mathbf{\hat{\ell}}^F \) satisfies a continuity equation only if the stress tensor is symmetric. Although this argument seems to require merely the symmetry of the momentum 3-tensor, \( \Pi_{ik}^F \), we know that a nonvanishing \( \Pi_{ik}^F - \Pi_{ki}^F \) in one inertial system will foul up the symmetry of the 3-tensor in other systems, as the antisymmetric parts of any 4-tensors mix in a Lorentz transformation. Yet angular momentum is conserved in every inertial system.

Since this reasoning is so general, it also holds for the material part, \( g^M = Q^M/c^2 \). Hence the expression for the momentum density is actually \( g^M = \rho v(1 + v^2/(2c^2)) \), cf. Eqs (13-14), though we are quite justified to neglect the second term in the nonrelativistic limit. Later, when we have no prior knowledge of the form of the momentum density, we shall deduce it from that of the energy flux, as angular momentum is also conserved in dense systems.

We register the fact that while the expression for the energy density \( u^F = \frac{1}{2} (c^2 + b^2) \) is a genuine input independent of, and in addition to, the Maxwell equations, the formula \( g^F = e \times b/c \) is not, since \( Q^F \) is given once \( u^F \) is.

3. Energy and Momentum Conservation

The preceding two sections allow the simple and noteworthy conclusion that our starting equations imply local conservation of total energy, momentum and angular momentum even in the presence of charges, \( p_e \neq 0 \). Taking \( u^{\text{tot}} = u^F + u^M \) and \( g^{\text{tot}} = g^F + g^M \), we find

\[
\dot{u}^{\text{tot}} + \nabla \cdot Q^{\text{tot}} = 0, \quad \dot{g}^{\text{tot}} + \nabla_k \Pi_{0k}^{\text{tot}} = 0, \quad \Pi_{0k}^{\text{tot}} = \Pi_{0k}^F + \Pi_{0k}^M, \quad Q^{\text{tot}} = Q^F + Q^M, \quad \dot{Q}^{\text{tot}}/c^2 = g^{\text{tot}} \approx \rho v + e \times b/c.
\]

These results have been collected here because local conservation of these quantities is always true, independent of the above derivation tailored to a dilute and finely resolved system. So we may start from them as input next.

B. Weakly Dissociated Gas

Now we consider a dense macroscopic system in its hydrodynamic regime: To the above gas of dilute charge carriers we add a dense system of neutral particles with vanishing electric and magnetic dipole moments. One example of this composite system is a slightly ionized gas with negligible electric and magnetic susceptibility. This is still a comparatively simple system, as the highly resolving, vacuum Maxwell equations (13-14) remain valid—and with them every single of the equations of section II A 2 about the field contributions to energy and momentum. The equations in section II A 3 about the respective material contributions, must be modified, as these will now be accounted for by three hydrodynamic variables: the entropy density \( s \), and the coarse-grained mass density \( \rho \) and velocity \( v \) — all smooth and slowly varying fields. (The coarse-graining brackets are dropped below to retain the simplicity of display.)

1. The Material Contributions

The hydrodynamic theory of the material part of our system is given by two continuity and two balance equations,

\[
\dot{\rho} + \nabla \cdot (\rho v) = 0, \quad \dot{s} + \nabla \cdot (sv) = 0, \quad \dot{\rho}^M + \nabla \cdot (\rho^M v) = 0, \quad \dot{v}^M + \nabla \cdot (P v) = 0,
\]

\[
\rho^M = \rho (1 + v^2/c^2), \quad \Pi_{0k}^M = \Pi_{0k}^M, \quad Q^M = (u^M + P) v, \quad \dot{u}^M + \nabla \cdot (u^M v) = 0, \quad \dot{\Pi}_{ik}^M = g^M_{ik} v_k + P \delta_{ik}.
\]

For \( p_e \to 0 \), these are the well known hydrodynamic equations of a neutral, isotropic liquid (11). For finite \( p_e \), the right sides of Eqs (23-24) have the given form because the field source terms of Eqs (13-14) remain unchanged, and because summing up the respective right sides must yield nil, such that total energy and momentum are conserved, as given in Eqs (21). (Since \( v \) actually denotes \( \langle v \rangle \) now, the above formulas presume the validity of \( \dot{\mathbf{e}} = c \nabla \times \mathbf{b} - \rho_e \langle \mathbf{v} \rangle \), rather than the original Eqs (9). This is justified, however, because charge carriers do move with \( \langle v \rangle \) in the absence of dissipation.)

Comparing \( \dot{Q}^M \) and \( \dot{\Pi}_{ik}^M \) here to those of Eqs (13), the only apparent difference is the appearance of the pressure \( P \), clearly an expression of the present system being dense and interacting. However, there is more than meets the eyes, and the most important one is our ignorance about the explicit expression for the energy \( u^M \). (Also, being a hydrodynamic, coarse-grained theory, the equations of motion now contain dissipative terms to account for irreversibility and entropy production. As mentioned in the introduction, however, we shall neglect these in this article.)

In the co-moving, local rest frame of the liquid, (denoted by the subscript 0), the material energy density in local equilibrium is given by the thermodynamic expression,

\[
\dot{u}_0^M = T ds + \xi_0 dp, \quad \dot{\xi}_0 = \partial u_0^M/\partial s, \quad \dot{\xi}_0 = \partial u_0^M/\partial p.
\]

which states the simple fact that \( u_0^M \) is a function of \( s \) and \( p \), with the temperature \( T \) and the chemical potential \( \xi_0 \) defined respectively as \( T \equiv \partial u_0^M/\partial s \) and \( \xi_0 \equiv \partial u_0^M/\partial p \). (The dimension of the chemical potential \( \xi_0 \) is energy/mass.) The explicit form of \( u_0^M \) is usually unknown, it depends on microscopic specifics, and is in
general rather complicated. Yet, as we know from thermodynamics, and as we shall see below, Eq (28) is as such already very useful.

The second of Eqs (13), \( g^M = \rho v \), remains an excellent approximation, because the energy flux \( Q^M \) as given by Eq (22) is still dominated by the term \( \rho c^2 v \) (contained in \( u^v \)), from the transport of rest energy. All the other terms are (in comparison) relativistically small and can be neglected.

In nonrelativistic physics, it is customary to subtract \( d(c^2 \rho) = c^2 \rho d \rho \) from both sides of Eq (33), eliminating the rest energy from \( d \rho_0 \) on the left, redefining the chemical potential as \( \xi_0 = c^2 - \frac{1}{2} v^2 \rho \), on the right, and at the same time subtracting \( c^2 \rho v \) from the energy flux \( Q^M \), Eq (22).

This represents a shift to a different but equivalent set of independently conserved quantities: from \( u^M \) and \( \rho \) to \( (u^M - \rho c^2) \) and \( \rho \). However, this operation does alter the link between the momentum density and the energy flux, rendering it \( g^M = Q^M / c^2 + \rho v \approx \rho v \). We shall follow this convention, so \( u^M_0 \) and \( \xi_0 \), starting from this section, no longer include the rest energy and \( c^2 \), respectively.

To account for a liquid with a varying local velocity \( v \), we need to employ a globally valid “lab”-frame. The material energy is then given as \( u^M_0 = u^v_0 + \rho c^2 / 2 \), where \( d(u^v_0) = d((u^v_0 / 2 \rho) = v \cdot dg^M - \frac{1}{2} v^2 d \rho \).

So the thermodynamic expression of Eq (22) becomes \( du^M = Tds + \xi_0 d \rho + v \cdot dg^M \), with \( \xi_0 = -\frac{1}{2} v^2 \) the chemical potential of the lab frame. Clearly, this expression signals that \( u^M = u^M(s, \rho_0, g^M) \) is also a function of \( g^M \).

If we are dealing with a solution, a suspension, or any system with more than one conserved densities, we need to replace the term \( \xi_0 d \rho_0 \) with \( \xi_0 d \rho_0 \), introducing all conserved densities \( \rho_\alpha \) as independent thermodynamic variables,

\[
du^M = Tds + \xi_\alpha d \rho_\alpha + v \cdot dg^M, \tag{29}
\]

where summation over \( \alpha = 1, 2, \cdots \) is implied. Being conserved, all \( \rho_\alpha \) obey continuity equations, so the first of Eqs (29) is to be replaced by \( \rho_\alpha + \nabla \cdot (\rho_\alpha v) = 0 \). Usually, it is more convenient to retain the total density \( \rho \) as a variable. For a binary mixture with the solute density \( \rho_\alpha \) and the solvent density \( \rho_2 \), we therefore take \( \rho_1 \) and \( \rho = \rho_1 + \rho_2 \) as the variables, implying that \( \xi_\alpha d \rho_\alpha \) in Eq (29) is to be read as \( d \rho + \xi_\alpha d \rho_\alpha \).

The pressure, defined as the energy change if the volume changes at constant entropy and mass, is related to the variables and conjugate variables of \( u^M \) via the Duhem-Gibbs (or Euler) relation,

\[
P = -\partial (u^M / d^4 r) / \partial V = -u^M + T s + \xi_\alpha \rho_\alpha = -u^M + T s + \xi_\alpha \rho_\alpha + v \cdot g^M. \tag{30}
\]

Combining Eq (29) with (30) yields

\[
\nabla P = s \nabla T + \rho_\alpha \nabla \xi_\alpha + g_j^M \nabla v_j. \tag{31}
\]

Inserting this and the first of Eq (29) in (31), we find

\[
\rho_\alpha v = \rho_\alpha (e + v \times b) + (s \nabla T + \rho_\alpha \nabla \xi_\alpha + g_j^M \nabla v_j), \tag{32}
\]

an equation that will prove useful later. Note an interesting feature of the hydrodynamic equations of neutral systems, Eqs (24) with \( \rho_c \to 0 \). They are given in terms only of the quantities appearing in Eq (29): the energy \( u^M \), the thermodynamic variables \( s, \rho_\alpha, g^M \), and the conjugate variables \( T, \xi_\alpha, v \). Without an explicit expression for \( u^M \), the hydrodynamic theory is clearly necessarily written in these general and abstract quantities, and we may take this observation as an indication that the hydrodynamic theory contains no more input than conservation laws and thermodynamics, which is the basic reason of its general validity. We shall return to, and build upon, this point in the next section.

For a charged system, \( \rho_c \neq 0 \), we need to include the Maxwell equations (34) for a complete description. The independent variables now include \( e \) and \( b \), each with an equation of motion. All equations also depend on \( \rho_c \), which however only stands for \( \nabla \cdot e \) and is not independent.

2. “2-field” Theory

For our purpose of deriving the genuine low-resolution Maxwell theory, it is instructive to again change the notation to \( u^{tot} \) and \( g^{tot} \) by adding up the material and field contributions, as we did in section 1A. The new theory remains closed and complete, and is given by the Maxwell equations (34), the continuity equations (24) for mass and entropy, and Eqs (24) for \( u^{tot} \) and \( g^{tot} \), where the fluxes

\[
Q^{tot} = (T s + \xi_\alpha \rho_\alpha + v \cdot g^M) v + c e \times b, \tag{33}
\]

\[
\Pi_{ik}^{tot} = (-u^{tot} + T s + \xi_\alpha \rho_\alpha + v \cdot g^M + c^2 + b^2) \delta_{ik} + g_i^M v_k - e_i e_k - b_i b_k, \tag{34}
\]

are obtained by adding the fluxes of Eq (13) and (24) to that of (17). The total momentum density is still as in Eq (23), \( g^{tot} = \rho v + e \times b/c \), while the energy \( du^{tot} \) is obtained by adding Eq (24) to the differential of the first of Eq (17), yielding

\[
du^{tot} = T ds + \xi_\alpha d \rho_\alpha + v \cdot dg^M + e \cdot de + b \cdot db. \tag{35}
\]

\[
= T ds + \xi_\alpha d \rho_\alpha + v \cdot dg^M + e_0 \cdot de + b_0 \cdot db. \tag{36}
\]

Eq (36) is algebraically identical to Eq (35), because \( v \cdot (g^M - g^{tot})c = -v \cdot (d(e \times b) + (v \times e) \cdot b) \cdot de \), where \( e_0 = e + (v/c) \times b \), \( b_0 = b - (v/c) \times e \) are the respective rest frame fields. For reasons that will become clear soon, we shall refer to this set of equations as the “2-field” theory.

In the low-resolution theory of the next chapter, we shall be dealing with a dense system containing hidden charges and dipole moments. This entails coarse-graining operations, of which the most obvious result is the appearance of four fields, \( E, D, B, H \), replacing the two here. For the remaining of this chapter, however, we shall ignore this complication, while considering the less obvious, but not less important changes.
Of these, the most relevant one is that we no longer have an explicit expression for the field energy \( u^F \); moreover, it is no longer possible to distinguish \( u^F \) from \( u^M \), such that the former only depends on the field variables and the latter only on the hydrodynamic variables. (We may of course expand \( u^F \) in the low-field limit, and retain only the lowest order term, say \( \frac{1}{2} B^2 / \mu \).

Although this resembles the microscopic expression, \( \frac{1}{2} B^2 / \mu \), the coefficient \( \mu \) is a function of hydrodynamic variables.) Only \( u^\text{tot} = u^F + u^M \) retains a clear-cut significance, because it is conserved. This is relevant, as the form of \( u^\text{tot} \) was an important input of the electromagnetic part of our theory, of the last paragraph of section I A 2. On the other hand, this is not an entirely new circumstance, as we did also lose the knowledge about the explicit form of \( u^M \) when going over to the dense, second system, cf Eq (28).

So we shall proceed in two steps. First, we reconsider the hydrodynamic theory of a neutral system to understand its derivation, especially how the ignorance about the energy was overcome. Next, this method is generalized to include the \( e, b \)-fields. The result will be a low-resolution Maxwell theory — except that two instead of four fields are considered, or the 2-field theory.

In equilibrium, the entropy \( s \) is a function of all the conserved quantities in the system, because these are the only ones that do not change when the system is closed. In our case, the variables are \( u^M, g^M, \rho, \alpha \). Given some (usually satisfied) mathematical properties of \( s(u^M, g^M, \rho, \alpha) \), we may rewrite this as \( u^M(s, \rho, g^M) \), or Eq (29).

[Note that, at this stage, we do not yet know the relation between \( g^M \) and \( \nu = \partial u^M / \partial g^M \).

As remarked in the last section, the hydrodynamic equations are given in terms of the thermodynamic variables and the associated conjugate variables. This is closely related to the algebraic identity given by the time derivative of Eq (33),

\[
\dot{u}^M = T \dot{s} + \xi_\alpha \dot{\rho}_\alpha + v_i \dot{g}_i^M, \tag{37}
\]

which the equations of motion (32, 34) must satisfy.

\[
\nabla \cdot Q^M = T \nabla \cdot (s \nu) + \xi_\alpha \nabla \cdot (\rho_\alpha \nu) + v_i \nabla_j \Pi^M_{ij} \tag{38}
\]

This is a very confining expression, and although not all that obvious, does uniquely determine the fluxes \( Q^M \) and \( \Pi^M_{ij} \) with the help of some auxiliary thermodynamic considerations. (This is referred to as the hydrodynamic standard procedure.) So the hydrodynamics is indeed determined once the thermodynamics is; there is an one-to-one correspondence between them.

Given the explicit form of \( Q^M \), the equality \( g^M = \rho \nu \) may now be established via \( g^M = Q^M / c^2 \). Had \( Q^M \) been different for some reasons, so would \( g^M \), but the hydrodynamic theory — in terms of \( g^M \) and \( \nu \) — would remain formally unchanged.

Including charges and fields, the conserved quantities are \( u^\text{tot}, \rho, \) and \( g^\text{tot} \). So the entropy will depend on them and the two fields, \( s(u^\text{tot}, \rho, g^\text{tot}, e, b) \). Again, this is equivalent to \( u^\text{tot}(s, \rho, g^\text{tot}, e, b) \) — compare this to Eq (34) to find that we have already made the right choice of variables for the “2-field” theory. What is more, given the cogenate, one-to-one relationship between the thermodynamics and hydrodynamics, the associated set of differential equations also possesses the right structure of the low-resolution theory. This is fortunate, as all there is left to do is to find out how the two fields turn into four.

That is, we may now construct the genuine low-resolution theory by requiring it to reduce to the present one in the 2-field limit, \( E, D \to e \) and \( H, B \to b \), (implying also \( E_0, D_0 \to e_0 \) and \( H_0, B_0 \to b_0 \)).

One point remains to be settled: Given the algebraic identity,

\[
\dot{u}^\text{tot} = T \dot{s} + \xi_\alpha \dot{\rho}_\alpha + v_i \dot{g}_i^\text{tot} + e_0 \cdot \dot{e} + b_0 \cdot \dot{b}. \tag{39}
\]

the “2-field” theory must be given only in terms of the quantities in this equation. Unfortunately, the two fluxes, Eqs (33, 34), do not seem to conform to this rule and contain \( g^M \). This is alright at present, because \( g^M = \rho \nu \) is given in terms of two quantities that do appear in Eq (39), but it seems to spell disaster for the low-resolution theory, because \( g^M \) is without a clear meaning there.

Similar to the case for \( u^\text{tot} \), there is no unambiguous division of \( g^\text{tot} \) into \( g^M \) and \( g^F \) in systems with hidden charges (although it is being attempted over and over again with predictably rather contradictory results).

This is because \( g^M \) lacks any physical significance: First, \( g^M \) is devoid of its significance as the conserved material momentum density of a neutral system. Second, without a uniquely defined \( u^M \) and the associated energy current \( Q^M \)

never is \( g^M = Q^M / c^2 \) useful as a work-around. (Not to mention that this equality is invalid, since \( t^M \) is no longer conserved). The same holds for \( g^F \). Nevertheless, one cut-cut definition of \( g^M \) does survive: Because of the algebraic equivalence between Eqs (33) and (34), we may start from the second expression to obtain the first, \( \text{defining } g^M \equiv g^\text{tot} - e \times b / c \) as a shorthand. Clearly, this guarantee that the two fluxes, Eqs (33, 34), are still given by the quantities of Eq (39), even if \( g^M \) is no longer given by \( \rho \nu \).

Finally, a few remarks about the pressure. While \( T \) and \( \xi \), as indicated by Eq (30), remain well defined as conjugate variables, and hence retain their place in the low-resolution theory of the next section, the pressure does not. [This is the reason we have chosen to eliminate it from Eqs (33, 34).] One might think of taking \( P = -\partial / \partial u^\text{tot} \partial V / \partial V \), instead of Eq (39), only this derivative is ambiguous and ill-defined: A dense system in the presence of field is anisotropic, so the energy not only depends on the volume, as implied by this definition, but also on the shape. And the appropriate quantity to deal with here is not the pressure, but the stress tensor.

In isotropic liquids, the pressure encompasses many concepts that we find convenient, even intuitive: as the surface force density, as the momentum current, as a quantity that is continuous across interfaces, and as a function of \( T \) and \( \rho \). Hence there is widespread reluc-
tance to abandon the pressure at finite fields. Unfortunately, though there are numerous ways to generalize the pressure that will preserve some of these properties, none covers all. So one may either define many different pressures – an approach we eschew as it requires exceeding care and has led to considerable confusion in the literature – or face up to the Maxwell stress tensor, as we shall do here.

C. Hydrodynamic Maxwell Theory

The theory we are going to consider, and the expressions obtained hereby, are fairly general: They are valid for arbitrarily strong fields and nonlinear constitutive relations; the medium may be moving, and the electromagnetic field may depend on time. But we do assume the validity of local equilibrium, which needs the characteristic time $\tau$ to be established. Therefore, all frequencies are confined to $\omega \tau \ll 1$. (We emphasize that the theory is only complete after the dissipative terms have been included. Although the terms given here retain their form off global equilibrium, at finite frequencies, we need to exercise caution when applying them alone.)

To derive the general theory, we need, as in the previous two cases, the input of the Maxwell equations and some information on the energy content of the field. The macroscopic Maxwell equations are,

$$\nabla \cdot \mathbf{D} = \rho_e, \quad \nabla \cdot \mathbf{B} = 0, \quad \mathbf{D} = c \nabla \times \mathbf{H} - \rho_e \mathbf{v}, \quad \mathbf{B} = -c \nabla \times \mathbf{E}. \quad (40, 41)$$

Taking $D, B$ as the variables, the first two are constraints, and the next two their equations of motion. If $E$ and $H$ are not specified, Eqs (40) do not contain any information beyond charge conservation, as they may actually be derived from $\rho_e + \nabla \cdot (\rho_e \mathbf{v}) = 0$ and Eqs (41). Especially, no restriction whatever is implied for the two “fluxes” $\mathbf{E}$ and $\mathbf{H}$.

Taking the temporal derivative of the second constraint yields $\mathbf{D} \cdot \mathbf{v} = 0$, implying there is a vector field $c \mathbf{E}$ such that the divergence-free field $\mathbf{B}$ may be written as $\mathbf{B} = \nabla \times (-c \mathbf{E})$. Similarly, deriving the first constraint yields $\nabla \cdot (\mathbf{D} + \rho_e \mathbf{v}) = 0$, so this divergence-free field must relate to a field $c \mathbf{H}$, such that $\mathbf{D} + \rho_e \mathbf{v} = \nabla \times (c \mathbf{H})$.

The temporal Maxwell equations (40) are given content only by taking $E$ and $H$ as thermodynamic conjugate variables, $\partial u_0^{\text{tot}} / \partial D$ and $\partial u_0^{\text{tot}} / \partial B$, respectively. More precisely, we assert that, in the rest frame, the thermodynamic energy has the form

$$u_0^{\text{tot}} = T ds + (\xi_0) d\rho_\alpha + E_0 \cdot d\mathbf{D} + H_0 \cdot d\mathbf{B}. \quad (42)$$

Poynting vector is obtained by inserting the equations of motion (21, 24, 41), for $\mathbf{v} = 0$, into Eq (42), resulting in

$$\dot{u}_0^{\text{tot}} = -\nabla \cdot Q_0^{\text{tot}}, \quad \text{with } Q_0^{\text{tot}} = c_0 \mathbf{E} \times \mathbf{H}. \quad (43)$$

Knowing the rest frame quantities, it is not difficult in principle to deduce the corresponding ones for the lab frame of a moving liquid, $\mathbf{v} \neq 0$, although this does entail some algebra. Fortunately, having established the 2-field limit, we have prepared ourselves a shortcut. (The results of the dilute systems of course display the correct transformation behavior.) First, Eqs (33, 34) leave little choice other than

$$du_0^{\text{tot}} = T ds + \xi_0 d\rho_\alpha + v \cdot dg^{\text{tot}} + E_0 \cdot d\mathbf{D} + H_0 \cdot d\mathbf{B}. \quad (43)$$

[The algebraically minded reader may prefer to derive this expression directly. The appendix in section V is specifically written for him.] Second, since the rest frame fields are $\mathbf{E}_0 = \mathbf{E} + (v/c) \times \mathbf{B}$ and $\mathbf{H}_0 = \mathbf{H} - (v/c) \times \mathbf{D}$, we may rewrite Eq (43) as

$$du_0^{\text{tot}} = T ds + \xi_0 d\rho_\alpha + v \cdot dg^M + E \cdot d\mathbf{D} + H \cdot d\mathbf{B}, \quad (44)$$

$$g^M = g^{\text{tot}} - D \times B/c. \quad (45)$$

The stress tensor may be obtained from the Maxwell equations by inserting the respective equations of motion, Eqs (21, 24, 41), for $\mathbf{v} = 0$ into the Maxwell equations.

$$\mathbf{D} \cdot \mathbf{v} = 0, \quad \mathbf{B} = -c \nabla \times \mathbf{E}, \quad \mathbf{D} = c \nabla \times \mathbf{H} - \rho_e \mathbf{v}, \quad \mathbf{B} = -c \nabla \times \mathbf{E}. \quad (40, 41)$$

Eqs (44) and (45) give the generalization of Eq (23), with $g^M$ now given by Eq (45). As discussed, this does not imply a field contribution of $\mathbf{D} \times \mathbf{B}/c$, though it does tell us that this new $g^M$ replaces the old one in the fluxes of Eqs (40, 41) below. Note that all conjugate variables are now functions of all thermodynamic variables, eq $T$ is a function also of $\mathbf{D}, \mathbf{B}$, and conversely, $\mathbf{H}_0$ depends on $s, \rho$.

The rest frame Poynting vector $c \mathbf{E}_0 \times \mathbf{H}_0$ and Eq (43) makes it unambiguous that the energy flux, and hence the momentum density, now have the form

$$Q_{\text{tot}} = \left( T + \xi_0 \rho_\alpha + v \cdot g^M \right) v + c \mathbf{E} \times \mathbf{H}, \quad (46)$$

$$g^{\text{tot}} = pv + c \mathbf{E} \times \mathbf{H}/c, \quad (47)$$

where $[T + \xi_0 \rho_\alpha + v \cdot g^M] / c^2 \ll 1$ is again neglected in $g^{\text{tot}}$. (Being a term of zeroth order in the velocity, $\mathbf{E} \times \mathbf{H}/c$ may not be neglected with the same argument.)

To derive the stress tensor, the last unknown expression of the low resolution theory, we again take the temporal derivative of Eq (43), $\dot{u}_0^{\text{tot}} = T \dot{s} + \xi_0 \rho_\alpha + \cdot \cdot \cdot$ and insert the respective equations of motion, Eqs (21, 24, 41), and Eq (15), to arrive at

$$\nabla \cdot \left( \left[ T + \xi_0 \rho_\alpha + v \cdot g^M \right] v + c \mathbf{E} \times \mathbf{H} \right) \quad (48)$$

$$= -\nabla \cdot \left( s v + \xi_0 \nabla \cdot (\rho_\alpha \mathbf{v}) + v \nabla \cdot \mathbf{v} \right)$$

$$-E_0 \cdot (c \nabla \times \mathbf{H} - \rho_e \mathbf{v}) + cH_0 \cdot \nabla \times \mathbf{E}. \quad (48)$$

This equation is satisfied by the stress tensor

$$\Pi_{ij} = \Pi_{i1} = \tilde{g}^M \tilde{v}_k - E_i D_k - H_i B_k +$$

$$\left( -u^{\text{tot}} = T + \xi_0 \rho_\alpha + v \cdot g^M + E \cdot D + H \cdot B \right) \delta_{ik}. \quad (49)$$

Although this calculation only yields $\nabla_k \Pi_{i1}^{\text{tot}}$, so one can always add a term $\epsilon_{kmn} \nabla_m A_{in}$ (with $A_{in}$ arbitrary), the
The independent variables are: systems that are charged or subject to external fields. It is obvious, otherwise it is not.)

Consider the rotational invariance of the energy $u^{\text{tot}}$ to see that $\Pi^{n\ell}$ is symmetric: Rotating the system by an infinitesimal angle $d\theta$, the scalars are invariant, $du^{\text{tot}}, ds, d\rho_n = 0$, while the vectors change according to $dg^M = g^M \times d\theta, dD = D \times d\theta, dB = B \times d\theta$. Inserting these into Eq (44) yields,

$$
v \times g^M + E \times D + H \times B = 0,
$$

where the left side is equal to $\epsilon_{ijk} \Pi^{n\ell}_{ik}$. (For vanishing fluid velocity and linear constitutive relations, $v \equiv 0$, $\epsilon E = D$ and $\mu H = B$, the symmetry of the stress tensor is obvious, otherwise it is not.)

Aside from the dissipative terms that we have consistently neglected, the expressions of this section represent the complete and closed hydrodynamic theory of dense systems that are charged or subject to external fields. The independent variables are: $s, \rho_n, g^{\text{tot}}, D, B$, with $u^{\text{tot}}$ and $g^{\text{tot}}$ given by Eqs (44) and (47), respectively. The attendant equations of motion are the Maxwell equations Eqs (43), and

$$
\dot{\rho}_n + \nabla \cdot (\rho_n v) = 0, \quad \dot{s} + \nabla \cdot (sv) = 0, \quad \dot{u}^{\text{tot}} + \nabla \cdot Q^{\text{tot}} = 0, \quad \dot{g}^{\text{tot}} + \nabla \Pi^{\text{tot}}_{ik} = 0,
$$

where $Q^{\text{tot}}$ and $\Pi^{\text{tot}}_{ik}$ are given by Eqs (46,49).

In comparison to the considerations in available textbooks, the main advantage of the above derivation is the proof of consistency of the equations of motion with general principles: Especially thermodynamics, transformation behavior, and conservation laws. At the same time, the range of validity of these equations are clarified. The advantage does not lie so much in the additional, frequency and velocity dependent terms, which are frequently small – although this will change once dissipative terms are included.

**D. Boundary Conditions**

To solve the derived set of hydrodynamic equations, boundary conditions are needed. In mathematics, boundary conditions represent information that is extrinsic to the differential equations. They are provided to choose one special solution from the manifold of all functions satisfying the differential equations. In physics, circumstances are frequently different. An example are the Maxwell equations, for which the boundary conditions are actually derived by integrating the differential equations across the interface, yielding

$$
\Delta D_n, \Delta B_n, \Delta E_l, \Delta H_l = 0
$$

(provided surface charges and currents are absent). The subscripts $n$ and $l$ denote the components normal or tangential to the interface, and $\Delta A \equiv A_L - A_R$ denotes the discontinuity of any quantity $A$, from left to right of the interface, toward positive spatial coordinates. Note that since the integration has to take place in a frame in which the interface is stationary, we have $v_n = 0$ in all the boundary conditions derived in this section.

The results of the integration are better termed “connecting conditions”, since they connect the behavior of the variables on both sides of the interface. This is possible because the Maxwell equations are valid also on the other side of the boundary. In contrast, mathematical boundary conditions make no supposition on the validity of the differential equations outside the considered region – in a sense, they shield the region from outside influence. An analogous integration of the energy and momentum conservation, Eqs (42), yields

$$
\Delta Q_n^{\text{tot}} = 0, \quad \Delta \Pi^{\text{tot}}_{in} = 0, \quad \Delta \Pi^{\text{tot}}_{nn} = \alpha (R_1^{-1} + R_2^{-1}).
$$

The first two connecting conditions do not contain much independent information: Assuming $v = 0$ for both sides of the interface, and inserting the expressions for $Q_n, \Pi_{in}$ from Eqs (44,49), we find them to be satisfied automatically in equilibrium if the connecting conditions of the fields are: Defining $n_l$ and $t_l$ as unit vectors that are normal and tangential to the interface, $\Pi^{\text{tot}}_{ln} = \Pi^{\text{tot}}_{ik} n_k t_i$, $H_l = H_l t_l$, $E_l = E_l t_l$, $B_n = B_k n_k$, $D_n = D_k n_k$, and with $H_l, E_l, B_n$ and $D_n$ continuous, cf Eqs (43), we have $\Delta Q^{\text{tot}}_n = 0$ and

$$
\Delta \Pi^{\text{tot}}_{nn} = -\Delta (H_l B_n + E_l D_n) = 0.
$$

Being a generalization of $\Delta P = \alpha (R_1^{-1} + R_2^{-1})$, the third boundary condition for $\Pi^{\text{tot}}_{nn} = \Pi^{\text{tot}}_{ik} n_k n_i$ is rather more useful and will be employed below repeatedly. ($\alpha > 0$ is the surface tension, and $R_1, R_2$ the principle radii of curvature. The term $\alpha (R_1^{-1} + R_2^{-1})$ is the equilibrium surface current for the momentum density, a singular sink or source for the bulk current.)

This completes the derivation of the hydrodynamic theory. Strictly speaking, this is all we need to predict the behavior of polarizable systems, as these partial differential equations will deliver unique solutions if a sufficient number of appropriate initial and boundary conditions are provided. This renders our initial question concerning the force of electromagnetic origin, although as yet unanswered, seemingly irrelevant. However, physics does not comprise solely of mathematics, and clear thinking about the valid expressions for the electromagnetic force will give us considerable heuristic power in prediction — without having to solve the set of partial differential equations each and every time.
III. STRESS, FORCE AND PRESSURE

To find an expression for the force in the presence of fields, let us first remind ourselves of its form without fields, a quantity that revolves around the pressure:

- Its gradient, $\nabla P = s \nabla T + \rho \nabla \xi + g_j M \nabla v_j$, is the bulk force density, the quantity that accelerates a neutral volume element in Eq (54). $\nabla P$ vanishes in equilibrium if we neglect gravitation.

- The pressure $P$ itself is a surface force density, which contributes to force equilibrium for instance in hydrostatics. $P$ remains finite in equilibrium.

Since the Maxwell stress $\Pi_{ik}^{\text{tot}}$ is the generalization of the pressure for polarizable media, we expect its gradient to be a bulk force density, see section III[A]. Presumably, as in the field free case, the bulk force density vanishes in equilibrium if gravitation is neglected, while the surface force density remains finite.

Generally speaking, the magnetic ponderomotive force is up to five orders of magnitude stronger than the electric one. This is the fact to the their respective, easily attainable values are similar in SI units: $E \approx 10^7 \text{V/m}$, $H \approx 10^2 \text{A/m}$ (ie $E \approx 30$, $H \approx 10^4$, in $\sqrt{\text{F/m}}$). Also, both susceptibilities are similar in magnitude, and do not usually exceed $10^4$, hence we have $\mu_0 H^2 \approx \epsilon_0 E^2 \times 10^5$. In addition to the greater ease and safety of handling, this frequently makes magnetic fields the preferred ones. In almost all the formulas of this and the next chapter (except in sections III[A] and III[B]), the electric terms are completely analogous to the magnetic ones, and the former may be obtained from the latter simply by employing the replacements

$$B \rightarrow D, H \rightarrow E, M \rightarrow P.$$ (56)

For linear constitutive relations (abbreviated hereafter as LCR), they imply $\mu \rightarrow \epsilon$, $\chi^m \rightarrow \chi$. To render the formulas simple, we shall therefore usually display only the magnetic terms.

A. The Bulk Force Density

In this section, we shall calculate $\nabla_k \Pi_{ik}^{\text{tot}}$ and look for all terms in the momentum conservation that alter the acceleration $\frac{d}{dt} \mathbf{v}$ of a volume element, starting from the energy density Eq (54),

$$\text{du}_x^{\text{tot}} = T ds + \xi_\alpha d\rho_\alpha + \mathbf{v} \cdot d\mathbf{g} + E \cdot d\mathbf{D} + H \cdot d\mathbf{B}. (57)$$

As discussed in the last chapter, the subscript $\alpha$ enumerates the conserved densities of the system. If there are more than one, a summation over $\alpha$ is implied: $\xi_\alpha d\rho_\alpha = \xi d\rho$ for a one-component fluid, and $\xi_\alpha d\rho_\alpha = \xi d\rho + \xi_1 d\rho_1$ for a two-component one such as ferrofluids, where $\rho_1$ is the density of magnetic particles, $\rho_2$ that of the fluid, and $\rho = \rho_1 + \rho_2$ the total density. (A ferrofluid is a suspension of magnetic particles, typically of 10 nm diameter. Note that $\rho_1$ is the mass of particles over the total volume. Its variation does not arise from compressing each particle individually, but from increasing the number of the particles in a given volume element. This appears to be a recurrent misunderstanding in the ferrofluid literature.) It is useful to introduce a number of potentials,

$$F(T, \rho_\alpha, v_i, B_i, D_i) = u_i^{\text{tot}} - T s - g_j^M v_j, \quad (58)$$
$$\tilde{F}(T, \rho_\alpha, v_i, H_i, E_i) = F - \tilde{H}_i B_i - E_i D_i, \quad (59)$$
$$G(T, \xi_\alpha, v_i, B_i, D_i) = F - \xi_\alpha \rho_\alpha, \quad (60)$$
$$\tilde{G}(T, \xi_\alpha, v_i, H_i, E_i) = \tilde{F} - \xi_\alpha \rho_\alpha. \quad (61)$$

The gradient of $\tilde{G}$ is

$$\nabla_k \tilde{G} = -s \nabla_k T - \rho_\alpha \nabla_k \xi - g_j^M \nabla_k v_j - B_i \nabla_k H_i - D_i \nabla_k E_i. \quad (62)$$

Writing the Maxwell stress, Eq (19), as

$$\Pi_{ik}^{\text{tot}} = -\tilde{G} \delta_{ik} + g_j^M v_k - H_i B_k - E_i D_k, \quad (63)$$
we find

$$\nabla_k \Pi_{ik}^{\text{tot}} = s \nabla_k T + \rho_\alpha \nabla_k \xi + B_i (\nabla_k H_i - \nabla_k H_i) + D_k (\nabla_i E_k - \nabla_k E_i) + \rho_\alpha \xi + g_j^M \nabla_i v_k + \nabla_k (g_j^M v_k) \quad (64)$$

which, in addition to the Maxwell equations (51), are to be inserted into the momentum conservation,

$$\frac{d}{dt} \mathbf{v} + (\mathbf{E} \times \mathbf{H} - \mathbf{D} \times \mathbf{B})/c \rho_e = \mathbf{v} + \mathbf{v} \times \mathbf{B}^e/c \rho_e$$
$$- (s \nabla T + \rho_\alpha \nabla \xi + g_j^M \nabla v_j) - \rho g \mathbf{e}_z. \quad (66)$$

Compare this to Eq (52) and register the great similarity — but do also remember that the temperature $T$ and chemical potentials $\xi_\alpha$ are now functions of the fields. All terms except $\rho \frac{dv}{dt}$ may be taken as various force densities: The frequency-dependent, so-called Abraham-force, $-\rho_\alpha (\mathbf{E} \times \mathbf{H} - \mathbf{D} \times \mathbf{B})/c \rho_e$, not usually a large term, the “fake” Lorentz force, $\rho_e (\mathbf{E} + \mathbf{v} \times \mathbf{B})/c$, only significant if the system is macroscopically charged, $\rho_e \neq 0$, and the bulk force density

$$\mathbf{f}^{\text{bulk}} = -(s \nabla T + \rho_\alpha \nabla \xi + g_j^M \nabla v_j) - \rho g \mathbf{e}_z, \quad (67)$$
which includes both the gravitational and the electromagnetic force. Note $\mathbf{f}^{\text{bulk}} = -\nabla_k \Pi_{ik}^{\text{tot}} - \rho g \mathbf{e}_z$ in the neutral, stationary limit, for $\mathbf{v}, \nabla \times \mathbf{E}, \nabla \times \mathbf{H} = 0$, cf Eq (64). $\mathbf{f}^{\text{bulk}}$ is the proper macroscopic, coarse-grained force, valid as long as the hydrodynamic, macroscopic...
Maxwell theory is. Together with the Abraham force, it accounts for the difference between the true and the “fake” Lorentz force discussed in the Introduction. It consists only of thermodynamically well defined quantities, either variables or conjugate variables of Eq (57). This is no longer the case if we follow a widespread convention to write \( f^{\text{bulk}} \) as a sum of the zero-field pressure gradient and the ponderomotive (or Kelvin) force, as this introduces thermodynamically ill-defined quantities and unwelcome ambiguities, cf section III F 3. In equilibrium, we have
\[
\nabla T, \nabla \xi_1 = 0, \quad \nabla \xi = -ge_z. \tag{68}
\]
Inserting these in Eq (57), we find, for stationary fluids in equilibrium, \( f^{\text{bulk}} = 0 \).

The considerations of the last two chapters make abundantly clear that the conserved momentum density is the sum of material and field contribution, \( g_{\text{tot}}^\rho = \rho v + E \times H/c, \) and that the corresponding flux is the Maxwell tensor. Nevertheless, in the context of condensed matter, we have \( \rho v \gg E \times H/c, \) so the second term may usually be neglected — and since \( D \times B \) and \( E \times H \) are of the same order of magnitude, so is the Abraham force. [Taking \( \rho v \approx 1 \text{ g/cm}^3, \) \( v \approx 1 \text{ cm/s}, \) \( H \approx 10^4 \) and \( E \approx 30, \) both in \( \sqrt{J/M^2}, \) (ie \( \bar{H} = 10^7 \text{ A/m, } E = 10^5 \text{ V/m,} \)) we have \( \rho vc/EH \approx 3000. \)]

### B. The Incompressible Limit

It is noteworthy that the incompressible limit in ferrofluids does not imply \( \dot{\rho}, \nabla \rho = 0, \) and hence \( \nabla \cdot v = 0. \) This is because incompressibility means the constancy of the two local, actual densities, \( \rho_M \) of magnetic particles and \( \rho_F \) of the fluid matrix (ie \( \rho_1 = \langle \rho_M \rangle \) and \( \rho_2 = \langle \rho_F \rangle, \) with the averaging taken over a volume containing many particles). Yet since the particles are usually denser than the fluid, \( \rho_M \approx 5\rho_F, \) an increase of particle concentration will also increase the total density \( \rho = \rho_1 + \rho_2. \) More quantitatively, \( \rho_1/\rho_M + \rho_2/\rho_F = 1 \) because \( \rho_1/\rho_M \) is the fraction of volume occupied by the particles, and \( \rho_2/\rho_F \) that occupied by the fluid. Taking \( \rho_M \) and \( \rho_F \) as constant in the incompressible limit, we have
\[
\rho = \gamma \rho_1, \quad \gamma = 1 - \rho_F/\rho_M. \tag{69}
\]
Inserting this expression into \( \xi d\rho + \xi_1 d\rho_1, \) we have \( \xi^{\text{in}} d\rho_1 \) with \( \xi^{\text{in}} = \xi_1 + \gamma \xi, \) and the modified equilibrium conditions
\[
\nabla T = 0, \quad \nabla \xi^{\text{in}} = -g\gamma e_z. \tag{70}
\]
Ferrofluids may frequently be approximated as being in the incompressible limit considered here, because the variations of \( \rho_F \) and \( \rho_M, \) say as a result of field inhomogeneities, are usually negligible. The particle density \( \rho_1, \) on the other hand, may vary greatly, and so does \( \rho \) via Eq (69). The parameter controlling this behavior is the osmotic compressibility, \( \kappa_{\text{os}} = \rho_1^2 \partial \rho_1/\partial \xi^{\text{in}}, \) which is larger by around 6 orders of magnitude than the compressibility of an ordinary liquid, cf section IV A.

### C. The Maxwell Stress in Equilibrium

We consider the simplifications that occur for the Maxwell stress tensor if the system is stationary (\( \dot{\rho} \approx 0 \)) and in equilibrium. [Starting from this section, we shall no longer display the electric terms explicitly, \( \mathbf{E}, \mathbf{D} = 0. \) And the subscript \( 0 \) denoting rest frame quantities will also be eliminated.]

#### 1. Total Equilibrium

With the equilibrium conditions as given in Eqs (68), \( \tilde{G} \) is nonuniform only due to the inhomogeneities in the fields, both gravitational and electromagnetic, cf Eq (52). This we may utilize to obtain a more handy expression for the stress. Inserting Eq (58) in (52) and integrating the resulting expression from 1 to 2, two arbitrary points in the medium, we have
\[
\Delta(\tilde{G} - \tilde{\rho}g + \tilde{\rho}_0 \tilde{B}_i dH_i) = 0, \tag{71}
\]
where \( \Delta \) denotes the difference between the two points of the quantity behind it, eg \( \Delta G \equiv \tilde{G}_2 - \tilde{G}_1. \) The first in the above equation is from \( \Delta G = \int_1^2 \nabla_k \tilde{G} dr_k; \) the magnetic term is from the integral \( \int_1^2 \langle B_i \nabla_k H_i \rangle dr_k: \) Writing it as \( \int_1^2 B_i dH_i, \) we note that it is in fact a difference of purely local quantities, with no reference to the path: \( \int_1^2 B_idH_i = \int_0^2 B_i dH_i - \int_0^1 B_i dH_i \equiv \Delta \int B_i dH_i, \) where \( 0 \) denotes a (possibly virtual) spot of vanishing field, \( H_i = 0, \) and 1, 2 respectively the field values at the spatial points 1, 2. The integration is unambiguous because it is to be carried out in equilibrium, for constant \( T, \xi_1 \) and \( \xi + \varphi, \) or for constant \( T, \xi^{\text{in}} + \gamma \varphi, \) in the incompressible approximation. To emphasize this, the subscript \( \tilde{\rho}_0 \) is added. With \( M_i = B_i - H_i, \) we may also write this term as \( \frac{1}{2} H^2 + \int \tilde{\rho}_0 M_i dH_i. \) The gravitational term comes from integrating the chemical potential, \( -\int_1^2 \rho \nabla_k \xi d\rho_k = g \int_1^2 \rho d\xi \) \( = \int_0^1 \rho d\xi, \) which is the chemical potential of the quantity in the bracket, call it \( -K, \)
\[
-K \equiv \tilde{G} - \tilde{\rho}g + \frac{1}{2} H^2 + \int \tilde{\rho}_0 M_i dH_i. \tag{72}
\]
This enables us to write the stress tensor, Eq (53), as
\[
\Pi_{ij}^{\text{tot}} = (K + \frac{1}{2} H^2 + \int M_i dH_i + \tilde{\rho}g \delta_{ij}) 
- H_i B_j, \tag{73}
\]
the announced handy expression. As we shall see, it is rather useful, and even includes the information contained in the magnetic Bernoulli equation (52), see section IV F 3. More generally, we may calculate \( \Pi_{ij}^{\text{tot}} \) for
an arbitrary point within the medium—if we know the field everywhere and the value of the constant $K$ (usually via the boundary conditions to be discussed below).

If the system under consideration (say the atmosphere) is non-magnetic, $M_i = 0$, $B_i = H_i$ and $u^\text{tot} = u^\text{em} + \frac{1}{2} H^2$ hold. Inserting these into Eq (72) and employing Eqs (84), we have

$$K = \rho gz + P_\text{atm},$$

with $K$ clearly the pressure at $z = 0$. If the system is magnetic, we may still write $\tilde{G}(T, \xi, H_i) = \tilde{G}(0) + \tilde{G}_\text{em}$, where the first denotes $\tilde{G}$ for vanishing $H$, while the second the contribution from the field,

$$-\tilde{G}_\text{em} = \int_{\text{Eq}} B_i dH_i = \frac{1}{2} H^2 + \int_{\text{Eq}} M_i dH_i,$$

and identify

$$K = \rho gz - \tilde{G}(0).$$

For obvious reasons, $\tilde{G}(0)$ is frequently referred to as the “zero-field pressure”. This is unfortunate, as $\tilde{G}(0) \equiv \tilde{G}$ 

$\tilde{G}_\text{em}$ is a function of $T, \xi, \xi$—these are the natural variables of $\tilde{G}$, and the value of the integral in Eq (77) explicitly depends on this choice of variables. On the other hand, if the zero-field pressure were a physically sensible quantity, it should remain the same whichever variables it is taken to depend on. It does not, of course, as it is equal to $\tilde{G}(0)$. This renders the associated concept of pressure ambiguous, ill defined, and best avoided, see section 111F for more details.

Combining Eq (72) and (75)——ignore gravitation and the electric field—we find

$$\nabla \int B_i dH_i = B_i \nabla H_i.$$  

(77)

For a function of one variable, this relation always holds: Defining $F = \int f(y) dy$, we have $\nabla F = (dF/dy) \nabla y = \int \nabla g$. For a function that depends on more variables, this relation holds only if the additional variables are spatially constant. In the present case, $B$ does depend on additional variables, $T, \xi, \xi + g z$. Fortunately, all three are constant. [T may also, as in next section, depend on constant $T, \rho$ and $\rho_1$—which is the reason the subscript in $\int_{\text{Eq}}$ is eliminated from Eq (77).] Considering LCR, we find $\nabla \int B_i dH_i = \nabla (\frac{1}{2} \mu H^2)$, which is equal to $B_i \nabla H_i$ if $\mu$ (say as a function of $T$) is constant.

Turning our attention now to boundary conditions, we find that the expression $\Delta \Pi^\text{tot}_{\text{en}}$ [which occurs in the boundary condition Eqs (54)] may be written as

$$\Delta \Pi^\text{tot}_{\text{en}} = \Delta (K + \int_{\text{Eq}} M_i dH_i + \frac{1}{2} M_i^2 - \rho gz)$$

(81)

in equilibrium. Two remarks: (i) The equality $\Delta (\frac{1}{2} H^2 - H_n B_n) = \Delta (\frac{1}{2} M_i^2 + \frac{1}{2} H^2 - \frac{1}{2} B_i^2)$ was used to derive Eq (78), where $\Delta (H^2)$, $\Delta (B_i^2)$ vanish, cf. Eq (53). (ii) If appropriate, substitute $\int_{\text{Eq}}$ with $\int_{\text{uni}}$, and $\rho$ with $\rho$, cf. Eq (82) below.

Inserting Eq (78) into the third of Eqs (54) assuming atmosphere on one side, employing Eq (79) while neglecting the gravitational term $\rho gz$ of the atmosphere, the boundary condition reduces to

$$K + \int_{\text{Eq}} M_i dH_i + \frac{1}{2} M_i^2 - \rho gz$$

$$= P_{\text{atm}} + \sigma (R_1^{-1} + R_2^{-1}),$$

(79)

a useful formula that we shall frequently refer to below.

2. Quasi-Equilibrium

The considerations of the last section is not confined to total equilibrium, and may be generalized to include quasi-equilibria. Establishing equilibrium with respect to the distribution of magnetic (or electric) particles is a slow process, because mesoscopic particles diffuse much more slowly than atomic ones. Depending on the field gradient, the relevant volume and the size of the particles, this may take days, even weeks [4, 8]. For a rough estimate, we equate the Stokes with the Kelvin force to calculate the velocity $v$ with which a magnetic particle moves: $6\pi \eta R v = (4\pi R^3/3) \chi_\mu \nabla H^2/2$. Taking the particle radius as $R = 10\text{nm}$, the viscosity as $\eta = 10^{-3}\text{kg}/\text{ms}$, the susceptibility as $\chi \approx 1$, the field as $B = 0.1\text{T}$, and the field gradient $\nabla B$ as $1\text{T/mm}$, the velocity is around $10^{-2}\text{mm/s}$, and the time the particles needs to achieve equilibrium is $\tau \approx 1\text{mm}/v \approx 10^3\text{s}$. On the other hand, particles $10^2$ times larger, of the size of $1\mu$, lead to velocities $10^4$ times larger, and a characteristic time of $10^4\text{s}$. For time scales much smaller than $\tau$, after a uniform ferrofluid is brought into contact with an inhomogeneous magnetic field, the following conditions hold instead of Eqs (88).

$$\nabla \rho_1 = 0, \nabla T = 0, \rho_1 \nabla \xi + \rho \nabla \xi = -\rho g e_z.$$

(80)

This is because both the diffusion of heat and the establishment of mechanical equilibrium are in comparison fast processes. The last of Eq (81) states $\Phi^\text{bulk} = 0$, cf Eq (67), which is due to the density $\rho$ quickly turning slightly nonuniform to compensate for the field inhomogeneity. Inserting Eq (80) in (81), we find $\nabla \tilde{G} = \rho ge_z - B_i \nabla H_i$, an expression that we can integrate directly, from a point 1 to a point 2, if we approximate the variables $T, \rho, \rho_1$ as constant, yielding

$$\Delta (\tilde{G} - \rho gz + \int_{\text{uni}} B_i dH_i) = 0,$$

(81)

where the subscript $\int_{\text{uni}}$ indicates that the integral is evaluated at constant $T, \rho$, and $\rho_1$. Again calling the spatially constant quantity in the bracket $-K$, the stress tensor is

$$\Pi_{ij} = (K + \frac{1}{2} H^2 + \int_{\text{uni}} M_i dH_i - \rho gz) \delta_{ij} - H_i B_j,$$

(82)

where the subscript of the integral and the lacking bar over $\rho$ are the only differences to Eq (84). Integrating as
here the magnetization while holding constant $T$ and $\rho_1$ yields the electromagnetic contribution to the free energy,

$$-\tilde{F}_{\text{en}}(T, \rho_1) = \int_{\text{uni}} B_i \, dH_i = \frac{1}{2} H^2 + \int_{\text{uni}} M_i \, dH_i,$$  \hspace{1cm} (83)

so $K$ may be written as

$$K = \rho g z - \left[ \tilde{F}(0) - \xi_\rho \rho_1 \right],$$  \hspace{1cm} (84)

while Eq (74) still holds for nonmagnetic media.

Note that $M_i(T, \rho_1, H_i)$ is the appropriate function for evaluating $\int_{\text{uni}} M_i \, dH_i$, while $M_i(T, \xi_1^0, H_i)$ is the proper one for $\int_{\text{ext}} M_i \, dH_i$ – both in the incompressible approximation. The first is given by an experiment that quickly measures the magnetization while varying the external field in a closed system, such that $\rho_1$ stays constant – though contact with a heat bath is necessary to maintain the temperature. The latter, $M_i(T, \xi_1^0, H_i)$, is to be measured in an open system that is connected to a particle reservoir, which itself is not subject to a varying field, so its chemical potentials $\xi_1^0$ remains constant. Increasing the field of the system now to measure the magnetization, magnetic particles will enter the system from the reservoir, resulting in a stronger magnetization than in the closed case.

Either experiment is sufficient to determine both magnetizations, as one can be calculated from the other. This is accomplished by employing the thermodynamic relation,

$$\left[ \frac{\partial M}{\partial H} \right]_{\xi_1^0} = \left[ \frac{\partial M}{\partial H} \right]_{\rho_1} + \frac{\rho_1^2}{\kappa_\osm} \left[ \frac{\partial M}{\partial \rho_1} \right]_H^2,$$  \hspace{1cm} (85)

where the inverse osmotic pressure is given as $\kappa_\osm^{-1} = \rho_1^2 \times \left( \frac{\partial \xi_1^0}{\partial \rho_1} \right)_H$. Eq (85) may be verified by combining a thermodynamic identity with a Maxwell relation,

$$\left[ \frac{\partial M}{\partial H} \right]_{\xi_1^0} = \left[ \frac{\partial M}{\partial H} \right]_{\rho_1} + \left[ \frac{\partial M}{\partial \rho_1} \right]_H \left[ \frac{\partial \rho_1}{\partial H} \right]_{\xi_1^0},$$  \hspace{1cm} (86)

$$\left[ \frac{\partial \rho_1}{\partial H} \right]_{\xi_1^0} = \left[ \frac{\partial B}{\partial \xi_1^0} \right]_H = \left[ \frac{\partial M}{\partial \rho_1} \right]_H \left[ \frac{\partial \rho_1}{\partial H} \right]_{\xi_1^0}.$$  \hspace{1cm} (87)

[We are assuming that $M||H$, so all $M$ and $H$ here are to be understood as the respective magnitude. In addition, $T$ is held constant and $\rho$ is taken to satisfy the incompressibility condition Eq (66) throughout.] For LCR, Eq (83) reduces to

$$\chi^m(T, \xi_1^0, \rho) = \chi^m(T, \rho_1, \rho) + \left( \rho_1 \frac{\partial \chi^m}{\partial \rho_1} \right) \frac{H^2}{\kappa_\osm},$$  \hspace{1cm} (88)

showing that the difference between the two susceptibilities is of higher order in the field, and may be neglected if one strictly adheres to LCR. On the other hand, this calculation also shows when the second term may no longer be neglected: Assuming for an order of magnitude estimate that $\rho_1(\partial \chi^m / \partial \rho_1) \approx \chi^m \approx 1$, and taking $\kappa_\osm^{-1} \approx 10^3$ Pa (see the calculation in section [III A] below), we find that a field of $H \approx 300$ Oe suffices to render $H^2 / \kappa_\osm \approx 1$.

**D. Equilibrium Surface Force**

Given a solid, polarizable body, or one that is (although not polarizable) submerged in ferrofluid, we may lift the body off the ground electromagnetically, against the gravitational force, or balance it against some elastic force exerted by a string or spring. The same applies to a vessel containing ferrofluid. All these happen even in equilibrium, when the bulk force density, Eq (57), vanishes. This force is

$$\mathbf{F}^\text{elm} = \Delta \int \left[ \frac{1}{2} M_n^2 + \int_{\text{Eq}} M_k \, dH_k \right] \, dA$$  \hspace{1cm} (89)

$$= \Delta \int \left[ \int_{\text{Eq}} M_t \, dH_t + \int_{\text{Eq}} M_n \, dB_n \right] \, dA,$$  \hspace{1cm} (90)

with the electric counter terms as usual given by Eq (56). $\Delta$ now specifically denotes: internal - external, and the two integrals are to be taken along the surface of the body, one right inside and the other right outside of it. As in the last section, $\int_{\text{Eq}}$ is to be substituted by $\int_{\text{uni}}$, and $\rho$ by $\rho$, where appropriate.

Before deriving Eq (55), we shall first consider the ramifications of this formula in two examples: (i) A magnetizable solid body (or a nonmagnetic vessel containing ferrofluid) in atmosphere, (ii) a nonmagnetic solid body in ferrofluid.

In the first example, the external magnetization is zero. Employing the Gauss law, we change the surface integral of Eq (53) into a volume integral: With $\nabla \int_{\text{Eq}} M_t \, dH_t = M_t \nabla H_t$ and $\nabla \int_{\text{Eq}} M_n \, dB_n = M_n \nabla B_n$, cf Eq (57), the electromagnetic force is

$$\mathbf{F}^\text{elm} = \int (M_t \nabla H_t + M_n \nabla B_n) \, d^3r.$$  \hspace{1cm} (91)

Consider a plate with the field gradient normal to its surface: If the field is predominantly tangential to its surface, the electromagnetic force is $\int M_k \nabla H_k \, d^3r$; if the field is normal to the plate’s surface, it is $\int M_k \nabla B_k \, d^3r$. So interestingly, the magnetic force density interpolates between both forms of the Kelvin force as discussed in the introduction – and in greater details in section [III F 3] below – though with the difference that these here are exact results, as no assumption was made with respect to the constitutive relation, especially the density dependence of the magnetization (or susceptibility).

If the geometry is more complicated than in a plate, say if the magnetizable body is an ellipsoid, it is less clear what the normal and tangential component of $M, H$ and $B$ are in the bulk, and Eq (53) appears ambiguous. Fortunately, this does not matter, because the integral of Eq (51) has (by virtue of the Gauss law) a unique value, as long as $\int M_t \, dH_t$ and $\int M_n \, dB_n$ are given at the surface, cf Eq (51), and the fields $M_t, H_t, M_n$, and $B_n$ being continuous in the bulk. [An analogous situation is given by the simple integral $\int_1^2 f \, dx$, $F' = f$, and $F(1), F(2)$ fixed. Then $\int_1^2 f \, dx = F(2) - F(1)$ is unique irrespective of how $f$ varies in the interval between 1 and 2.]
Turning our attention now to the second system, a non-polarizable body submerged in ferrofluid, we take the internal magnetization as zero and find

\[ F_{\text{elm}} = - \int \left[ \frac{1}{2} M_i^2 + \int_{\text{Eq}} M_k dH_k \right] dA \quad (92) \]

\[ = - \int \left[ \int_{\text{Eq}} M_i dH_i + \int_{\text{Eq}} M_n dB_n \right] dA. \quad (93) \]

It may also here be useful to employ the Gauss law for a conversion of the surface integral into one over the volume of the body. However, as the physical fields in the volume are not continuous with their respective surface values, virtual fields have to be defined which do. They need not be healthy fields, may violate the Maxwell equations, or have a non-physical susceptibility.

We now derive Eqs (89) by first considering a solid body submerged in fluid and attached to a string. Both the solid and fluid may be magnetizable — though differently, with the Maxwell stress \( \Pi_{ij}^\text{tot} \) respectively denoted as \( \Pi_{ij}^\text{elast} \) and \( \Pi_{ij}^\text{f} \). One of the boundary conditions at the interface is then

\[ \sigma_{ij}^s + \Pi_{ij}^\text{f} = \Pi_{ij}^\text{elast} + \sigma_{ij}^s, \quad (94) \]

where \( \sigma_{ij}^s \equiv \partial u_i / \partial n_j \) is the elastic stress tensor of the solid, given by deriving the thermodynamic energy with respect to the elastic strain \( u_{ij} \). In equilibrium, \( \nabla_j \sigma_{ij}^s = 0 \). Being a liquid, \( \sigma_{ij}^f = 0 \) except where the surface integral cuts across the string, which also has elasticity we need to account for. Same as Eqs (4), this boundary condition states the continuity of the total momentum current \( \sigma_{ij} + \Pi_{ij} \). [The surface tension \( \alpha \) is irrelevant for a solid surface of given shape and therefore eliminated.]

Each of the four terms of Eq (4) stands for a surface force density. So the integral over the closed surface of the solid body must yield the equation of force equilibrium, between the gravitational, electromagnetic and elastic force,

\[ F_i^{\text{elast}} + F_i^{\text{grav}} + F_i^{\text{elm}} = 0. \quad (95) \]

The first term of Eq (94) vanishes, because \( \int \sigma_{ij}^s dA_i = \int \nabla_j \sigma_{ij}^s d^3r = 0 \). The last term yields the elastic force exerted by the string, which is given by the normal component, \( F_i^{\text{elast}} = - \int \sigma_{nn}^f dA_i \), because the string cannot sustain a shear stress, \( \sigma_{tn}^e = 0 \). Clearly, this implies

\[ F_i^{\text{elm}} + F_i^{\text{grav}} = \int \Delta \Pi_{ij} dA_j. \quad (96) \]

Because \( \int \Delta \Pi_{ij} dA_j = \int \Delta \Pi_{ij} n_j dA = \int \delta_{ik} \Delta \Pi_{kj} n_j dA = \int \left( t_k n_k n_j \Delta \Pi_{ijk} n_j + t_k \Delta \Pi_{kn} + t_k \Delta \Pi_{jn} dA \right) \), and because \( \Delta \Pi_{kn} \) vanishes identically, cf Eq (53), this reduces to

\[ F_i^{\text{elm}} + F_i^{\text{grav}} = \int \Delta \Pi_{nn} dA_i. \quad (97) \]

Inserting Eq (78) for \( \Delta \Pi_{kn} \), the Gauss law is employed to find \( \int \vec{F} K dA_j = \int \nabla K d^3r = 0 \). As a result, only terms that depend explicitly on fields are left. From these, the gravitational force is found to be

\[ F_i^{\text{grav}} = - \int g z \partial A dA = - \int \nabla (\Delta \rho g) d^3r = -g e \Delta \rho d^3r, \]

while the electromagnetic force is found to be given by Eq (89).

Because Eq (94) is valid for any interface, and not confined to the considered geometry, so are the formulas Eqs (89), (94), and (97). Clearly, there is an electromagnetic surface force density of the given form whenever the Maxwell stress is discontinuous, i.e., when the magnetization (or polarization) changes abruptly.

To confirm this, consider a further example: Ferrofluid contained in a nonmagnetic vessel, and the whole system hung on a string in atmosphere. It is slightly more complicated geometry, as there are two interfaces to consider: ferrofluid-solid, and solid-air. Each has a boundary condition of the type of Eq (94). We have \( \sigma_{ij}^s = \sigma_{ij}^s \) at the solid-air interface, because the Maxwell stress is continuous there. \( \Pi_{ij}^s = \Pi_{ij}^a \). An integration yields, as before,

\[ \int \Pi_{ij}^a dA_i = \int \sigma_{nn}^a dA_i = - F_i^{\text{elm}}. \]

The boundary at the ferrofluid-solid interface is \( \Pi_{ij}^f = \Pi_{ij}^s + \sigma_{ij}^s \). Integrating over the interface, we find \( \int (\Pi_{ij}^f - \Pi_{ij}^s) dA_j = F_i^{\text{elm}} = 0 \), or again that Eq (95), and therefore Eqs (97) and (96) to be valid.

In the literature [2, 16, 23], instead of Eq (90), the electromagnetic force is usually given as

\[ F_i^{\text{elm}} = \int (H_i B_j - B_k dH_k \delta_{ij}) dA_j. \quad (98) \]

This expression may also be obtained by integrating over Eq (14). First note \( \int \sigma_{ij}^a dA_i = \int \nabla_j (\sigma_{ij}^a + \Pi_{ij}^a) d^3r = 0 \), because the integrand, as the total momentum flux density, has vanishing divergence in equilibrium (neglect, for simplicity, gravitation). Writing the force equilibrium as \( \int (\Pi_{ij}^f + \sigma_{ij}^f) dA_j = 0 \), again identifying \( F_i^{\text{elm}} = - \int \sigma_{ij}^f dA_j \), we find \( F_i^{\text{elm}} = - \int \Pi_{ij}^a dA_j \) by comparison with Eq (95). Now, inserting Eq (78) for \( \Pi_{ij}^f \) and because \( \int g K dA_j = 0 \), we obtain Eq (98).

Although algebraically equivalent, Eqs (89) and (98) may behave quite differently when being evaluated for a concrete geometry. For instance, considering the force on a plate starting from Eq (89), it is not easy to find the result of Eq (14). This is because Eq (98) is a rather non-local expression, which does not even explicitly depend on the susceptibility of the magnetizable body.

The above lengthy discussion should not obscure the point that these equilibrium surface forces are no more than interpretation and visualization of boundary conditions, especially Eqs (78, 94). In fact, when considering the experiments in Chapter 1 we shall usually employ the boundary conditions, to solidify the algebraic, safe, albeit slightly more tedious approach. But we shall also frequently point out how the surface forces considered here would have yielded the same results. The latter serves to demonstrate that these forces represent a
heuristic concept of considerable power, and to give further reassurance, if one is needed, of the validity of the formulas in this section.

E. Thermodynamic Derivation of the Stress

It is worthwhile to rederive our central result, the Maxwell tensor, Eq (63), via a more direct path, though with a rather narrower focus and ensured range of validity. This is done in the form of an expanded Landau/Lifshitz consideration, separately for the electric and magnetic contributions. We start from the energy expression Eq (57), or from one of its potentials, Eqs (58, 59, 60, 61), and shall take pains in eliminating the flaws mentioned in the introduction.

Deforming an isolated system at given entropy, mass and electric charge, the energy change is

\[ \delta U = \delta \int u^\text{tot} dV = - \oint (\Delta \Pi^\text{tot}_k \delta r_i) dA_k, \]

where the energy density \( u^\text{tot} \) is given by Eq (57) with \( v = 0 \). Notations: \( \delta r_i \) is the infinitesimal (or virtual) displacement of the surface, and \( dA_k \equiv n_k dA \) with \( dA \) the surface element and \( n_k \) the surface normal pointing outwards. The validity of this equation is connected to \( -\Delta \Pi^\text{tot}_k \) being the electromagnetic force density needed to deform the system, see Eq (94). All gravitational terms are neglected in this section.

If systems are considered for which the external stress tensor vanishes, because neither material nor field lies outside the considered volume, we may substitute \( \Pi^\text{tot}_k \) into \( \Pi^\text{tot}_k \) in Eq (94). Only these systems will be considered below. In the absence of fields, \( \Pi^\text{tot}_k \rightarrow P \delta r_i \) with \( P \) uniform in thermodynamic equilibrium, so \( \delta U \) reduces to \(-P \oint \delta r_i dA_k = -P \delta V \), or Eq (10). Note that if the projection of the surface force \( \Pi^\text{tot}_k \equiv \Pi^\text{tot}_{ik} n_i n_k \) is positive (ie positive pressure if without field), the volume tends to expand, if \( \Pi^\text{nn} < 0 \), it tends to contract.

In simpler geometries, if \( u^\text{tot} \) and \( \Pi^\text{tot}_k \delta r_i \) are uniform, Eq (99) reduces to

\[ \delta U = \delta (u^\text{tot} V) = u^\text{tot} \delta V + V \delta u^\text{tot} = -A_k \Pi^\text{tot}_k \delta r_i, \]

Since \( u^\text{tot} \) is known, we shall evaluate \( A_k \Pi^\text{tot}_k \delta r_i \) while taking \( A_k \) and \( \delta r_i \) to point in all three directions, perpendicular and parallel to the field, and hereby obtain all nine components of \( \Pi^\text{tot}_k \).

1. The Electric Part

We consider a parallel-plate capacitor that is filled with a dielectric medium. Denoting the three linear dimensions of the capacitor as \( x, y, z \), with \( x \ll y, z \), the six surfaces \( S^+_x, S^-_x, S^+_y, S^-_y, S^+_z, S^-_z \) (with the outward pointing normal \( \pm \hat{e}_x, \pm \hat{e}_y, \pm \hat{e}_z \)) have the areas \( A_x = yz, A_y = xz, A_z = xy \), respectively; and the volume is \( V = xyz \). Taking the two metal plates as \( S^+_x \) the electric fields \( E, D \) are along \( \hat{e}_x \), see Fig 1. (We neglect the small stray fields at the edges.) The capacitor is placed in a vacuum, so there is neither electric field nor material outside.

We now successively displace the three surfaces \( S^+_x, S^+_y, S^+_z \), in all three directions, \( \delta r_i = \delta x, \delta y, \delta z \) (which is why the capacitor has to be finite), while holding constant the quantities: entropy \( sV \), masses \( \rho \alpha \), and electric charges \( q = \pm DA_\alpha \) (from \( \nabla \cdot D = \rho_e \)). Because of the simple geometry, Eq (100) holds and will be evaluated.

First, take the surface to be displaced as \( S^+_x \). When the displacement is \( \delta x \), we have

\[ \delta V = A_x \delta x, \delta s/s = \delta \rho \alpha /\rho \alpha = -\delta x/x, \delta D = 0; \] (101)

and we have \( \delta V, \delta s, \delta \rho \alpha, \delta D = 0 \) if the displacement is \( \delta y \) or \( \delta z \) (implying a shear motion of \( S^+_x \)). Inserting all three into Eq (100), we obtain

\[ \Pi^\text{tot}_x \delta x = (Ts + \xi \rho \alpha - u^\text{tot}) \delta x, \quad \Pi^\text{tot}_y = \Pi^\text{tot}_x = 0. \] (102)

If the surface is \( S^+_y \) and the displacement \( \delta z \), we have

\[ \delta V = A_x \delta z, \delta s/s = \delta \rho \alpha /\rho \alpha, \delta D/D = -\delta z/z. \] If the displacement is \( \delta x \) or \( \delta y \), we have \( \delta V, \delta s, \delta \rho \alpha, \delta D = 0 \). Hence

\[ \Pi^\text{tot}_z \delta z = (Ts + \xi \rho \alpha + E_z D_x - u^\text{tot}) \delta z, \] (103)

and \( \Pi^\text{tot}_y = \Pi^\text{tot}_z = 0 \). As the directions \( \hat{e}_y, \hat{e}_z \) are equivalent, we know without repeating the calculation that a displacement of \( S^+_y \) yields \( \Pi^\text{tot}_x = \Pi^\text{tot}_y \) and \( \Pi^\text{tot}_x = \Pi^\text{tot}_z = 0 \). (The term \( E_z D_x \) is a result of the metal plates being squeezed, compressing the surface charges, \( \delta q/q = \delta D/D = -\delta z/z \). The compressibility of the metal is taken to be infinite. Otherwise, it would contribute an additional term in the stress tensor that we are not interested in here.)

These considerations have yielded all nine components of \( \Pi^\text{tot}_k \) for a special coordinate system. Because the stress tensor of Eq (57), for \( D, E \parallel \hat{e}_x \) and \( B, v = 0 \) produces exactly these components, it is the correct, coordinate-independent expression.
This conclusion may appear glib, but is in fact quite solid. Consider first a vector: If two vectors are shown to be equal in a special coordinate system, we know that they remain equal in any other system – as long as we are sure that they are indeed vectors. The same also holds for tensors. (The careful reader may notice an ambiguity with respect to the off-diagonal part, as both $E_i D_k$ and $E_k D_i$ yields the same nine components derived here. Fortunately, there is no difference between these two expressions, because $\mathbf{E} \times \mathbf{D} = 0$ for $\mathbf{B}$, $\mathbf{v} = 0$, cf Eq (54).) This concludes a thermodynamic derivation of the electric part of the Maxwell stress tensor.

Substituting the dielectric medium with vacuum, $\Pi_{\text{tot}}^{\text{xx}}$, $\Pi_{\text{tot}}^{\text{zz}}$ reduce to $\mp e^2/2$, respectively, implying a tendency to contract along $\hat{e}_x$ and expand along $\hat{e}_y$ and $\hat{e}_z$ — as it should in the considered case: The differently charged plates want to come closer, while the charge in each plate would like to expand. If the medium is one with negligible susceptibilities, $\Pi_{\text{tot}}^{\text{xx}}, \Pi_{\text{tot}}^{\text{zz}}$ are given as $P \mp e^2/2$, so the same electric force must now contend with the pressure in deforming the system.

Consider the same capacitor, now held at a constant voltage $\phi$. The modified capacitor must lead to the same stress tensor, because the stress tensor is a local expression and may depend only on the local field. The calculation is rather similar, though one needs to replace $u^{\text{tot}}$ in Eq (100) with the potential $\tilde{u} \equiv u^{\text{tot}} - \mathbf{E} \cdot \mathbf{D}$, as the system is no longer electrically isolated. And the constraint for $E$ as the new variable is $Ex = \phi$ (which replaces $DA_x = q$). Connecting the capacitor in addition to a heat bath necessitates the employment of the free energy, $\tilde{F} = u^{\text{tot}} - Ts - \mathbf{E} \cdot \mathbf{D}$, and changes the constraint (from constant $sV$) to $\delta T = 0$. ($\tilde{F}$ is the potential used in [3].) For the explicit calculation please of the magnetic case below, Eqs (104, 105, 106, 107, 108), with the replacement $\mathbf{B} \rightarrow \mathbf{D}$, $\mathbf{H} \rightarrow \mathbf{E}$ implemented. The final result is the same, given by Eq (63).

2. The Magnetic Part

To obtain the magnetic part of the stress tensor, consider a rod along $\hat{e}_x$, of square cross section, made of a magnetizable material and placed in a vacuum. The surfaces $S_{yz}^\pm, S_{zx}^\pm$ are covered with a sheet of metal that carries a current $J \perp \hat{e}_x$. With $A_x \ll A_y, A_z$, the magnetic field will be essentially along $\hat{e}_z$ and confined to the interior of the rod, see Fig 2. So again, there is neither field nor material outside.

If the system is isolated, the metal needs to be superconducting to sustain the current, and the constraint on the variable $B$ during a deformation is constant flux, $BA_x = \Phi$. (Compare this to the isolated electric case with $DA_x = q$.) If the current $J$ is held constant by a battery, the constraint is $Hx = J/c$ (from $\mathbf{V} \times \mathbf{H} = j_0/c$ and analogous to $Ex = \phi$), and the attendant potential is $\tilde{u} \equiv u^{\text{tot}} - \mathbf{H} \cdot \mathbf{B}$.

The calculation of the isolated magnetic case repeats the isolated electric one, with all equations – both the displayed Eqs (100, 101, 102, 103) and the others around these – remaining valid after the displacement $\mathbf{D} \rightarrow \mathbf{B}$, $\mathbf{E} \rightarrow \mathbf{H}$ has taken place. (When considering compressional displacements, the elastic energy of the metal sheet is again neglected.)

We shall now consider the rod being deformed at constant current and temperature, so the energy needed to deform the system is

$$\delta(\tilde{F}V) = \tilde{F}\delta V + V \delta \tilde{F} = -A_k \Pi_{\text{tot}}^{\text{xx}} \delta r_k, \quad (104)$$

$$\delta \tilde{F} = -s\delta T + \xi_\alpha \delta \rho_\alpha - \mathbf{B} \cdot \delta \mathbf{H}, \quad (105)$$

where $\tilde{F} = u^{\text{tot}} - Ts - \mathbf{H} \cdot \mathbf{B}$. We again successively displace the three surfaces $S_{xz}^+, S_{yz}^+, S_{zx}^+$, in all three directions, $\delta r_i = \delta x, \delta y, \delta z$, while holding constant the quantities: temperature, masses, and the current, i.e under the conditions, $\delta T = 0$, $\delta (\rho_\alpha V) = 0$ and $\delta (Hx) = 0$.

The first surface to be displaced is $S_{xz}^+$. When the displacement is $\delta x$, we have

$$\delta V = A_x \delta x, \quad \delta H/H = \delta \rho_\alpha/\rho_\alpha = -\delta x/x; \quad (106)$$

and we have $\delta V, \delta \rho_\alpha, \delta H = 0$ if the displacement is $\delta y$ or $\delta z$ (implying a shear motion of $S_{xy}^+$). Inserting these into Eq (104), we obtain

$$\Pi_{\text{tot}}^{\text{xx}} \delta x = (Ts + \xi_\alpha \rho_\alpha - u^{\text{tot}}) \delta x, \quad \Pi_{\text{y}x}^{\text{tot}} = \Pi_{\text{z}x}^{\text{tot}} = 0. \quad (107)$$

If the surface is $S_{yz}^+$ and the displacement $\delta z$, we have $\delta V = A_z \delta z, \delta H = 0$, and $\delta \rho_\alpha/\rho_\alpha = -\delta z/x$. If the displacement is $\delta x$ or $\delta y$, we have $\delta V, \delta \rho_\alpha, \delta H = 0$. Hence

$$\Pi_{\text{tot}}^{\text{zz}} \delta z = (Ts + \xi_\alpha \rho_\alpha + Hx B_x - u^{\text{tot}}) \delta z, \quad (108)$$

and $\Pi_{\text{zy}}^{\text{tot}} = \Pi_{\text{yz}}^{\text{tot}} = 0$. Since the directions $\hat{e}_y$ and $\hat{e}_z$ are equivalent, we have $\Pi_{\text{zy}}^{\text{tot}} = \Pi_{\text{yz}}^{\text{tot}}$ and $\Pi_{\text{zz}}^{\text{tot}} = \Pi_{\text{yy}}^{\text{tot}}$.
This consideration has yielded nine components of $\Pi^{\text{tot}}_{i\pm}$ for a special coordinate system. Because the stress tensor of Eq (32), for $B, H \parallel \hat{e}_x$ and $D = 0$ produces these components, it is the correct, coordinate-independent expression. This concludes the thermodynamic derivation of the magnetic part of the Maxwell stress tensor.

Although we have only employed the potentials $F$ and $\tilde{F}$ above, it should be clear by now that we could just as well have employed $G$ and $\tilde{G}$ from Eqs (45-46), assuming that the polarizable medium is connected to particle reservoirs, which keep the chemical potentials $\xi_a$ constant. This changes the constraint from $\delta \rho_a / \rho_a = -\delta x/x, -\delta y/y$ to simply $\delta \xi_a = 0$. The derived expression for the stress tensor remains unchanged.

As a stand-alone, the thermodynamic consideration of the last two sections gives us a fairly clear idea on the form of the macroscopic Maxwell stress tensor $\Pi_{ik}^{\text{tot}}$, in equilibrium, for $v = 0$, and with either the electric or the magnetic field present. The most important information it withholds is that about $\eta_{ik}^{\text{tot}}$, without which $\Pi_{ik}^{\text{tot}}$ is not unambiguously defined at finite frequencies. Also, $\eta_{ik}^{\text{tot}}$ is needed to complete and close the set of differential equations given previously, which alone is capable of providing a consistent and comprehensive description of polarizable systems. In addition, relying solely on this consideration, one would perhaps need to be more careful with the stray fields, especially when thinking about the possibilities of terms such as $\nabla_i E_j, \nabla_j H_j$.

Rosensweig has also considered and derived the magnetic part of the stress tensor in [8]. His result is the same as the one here, though his algebra is rather more complicated. More problematically, one of his basic, starting assumptions does not hold up: His geometry is a slab with $A_x, A_x \ll A_z$ and current-carrying wires along the surfaces $S^+_{1x}, S^+_{2x}$, see his Fig 4.1. The winding of the wires is oblique, the currents flow along the two larger plates $S^+_z$, but has a component along $\pm \hat{e}_z$ in the two narrow side walls $S^\pm_y$ – take them to be along $\pm \hat{m}$, a vector in the $xz$-plane. Rosensweig maintains that the resultant field is uniform and perpendicular to the surface given by the winding, i.e by $\hat{e}_y$ and $\hat{m}$.

Unfortunately, the field is neither uniform nor mainly oblique, rendering large parts of the ensuing consideration invalid. First the qualitative idea: If the two much larger plates $S^+_z$ were infinite, the field would be strictly parallel to $\hat{e}_z$. This basic configuration should not change much if the plates are made finite, and supplemented with the two narrow side walls $S^\pm_y$ – irrespective of the currents’ direction there. This argument is born out by a calculation to superpose the fields from various portions of the currents. First, divide all currents along $\pm \hat{m}$ into two components, along $\pm \hat{e}_z$ and $\pm \hat{e}_x$. Next, combine the first with the currents along $\pm \hat{e}_z$, such that the four sections of the four surfaces form a closed loop at the same $x$-coordinate. The resultant field of all loops is clearly the main one, and strictly along $\hat{e}_z$. The leftover currents are those at $S^+_z$ along $\pm \hat{e}_x$ and their total effect is a small dipole field.

F. Ambiguous Notations

In this section, we shall consider some frequently employed expressions and notations that we shall see are rather misleading, and therefore perhaps best avoided. On the other hand, many of us have so accustomed ourselves to this notation that we tend to think along the categories it provides. But even then, or especially then, one should welcome the opportunity to realize all its pitfalls which, as discussed in the introduction, lie mainly in the ambiguity of the zero-field pressure, and of $\int M \cdot dH$. The expressions one arrives at are for instance $(P + \frac{1}{2} H^2) \delta_{ik} - H_i B_k$ for the stress, or $M_i \nabla H_i$ for the Kelvin force. Scrutinizing their derivation, we shall in addition conclude that these formulas are only valid for small susceptibility $\chi^m \ll 1$, or $M \ll H$ — a range of validity dramatically smaller than taken for granted usually. So if $\chi^m$ is of order unity, and $M \approx H$, as is frequently the case in strongly polarizable systems such as ferrofluids, these formulas are invalid.

1. Different Zero Field Pressures

We shall take an approach here that is somewhat broader than usual, and simultaneously work with different potentials. This will lead to rather flagrant appearing contradictions, the understanding of which should lend us a sharpened view of the adopted notation. To simplify the formulas, we shall neglect gravitation, and consider the incompressible limit of ferrofluids, $\rho, \xi \rightarrow \rho_1, \xi^1$, cf section II B. (Replacing $\rho_1, \xi^1$ with $\rho, \xi$ yields the results for a one-component, compressible liquid.)

We start by separating the energy $u^{\text{tot}}(s, \rho_1, B)$, the free energy $F(T, \rho_1, B)$, and the potentials $\tilde{F}(T, \rho_1, H)$, $G(T, \xi^1, B)$, and $\tilde{G}(T, \xi^1, H)$, cf Eqs (58 - 61), into their zero-field and electromagnetic contributions,

$$u^{\text{tot}} = u(0) + u_{\text{em}}(B), \quad F = F(0) + F_{\text{em}}(B), \quad (109)$$

with analogously defined $G_{\text{em}}(B)$, $\tilde{F}_{\text{em}}(H)$, $\tilde{G}_{\text{em}}(H)$. Adhering to convention, we write the potentials that are functions of $B$ as

$$u_{\text{em}}, F_{\text{em}}, G_{\text{em}} = \int H \cdot dB = \frac{1}{2} B^2 - \int M \cdot dH, \quad (110)$$

and the potentials with tildes that are functions of $H$ as

$$\tilde{G}_{\text{em}}, \tilde{F}_{\text{em}} = -\int B \cdot dH = -\frac{1}{2} H^2 - \int M \cdot dH. \quad (111)$$

Note that the respective integrals is to be taken at constant $s, \rho_1$ for $u_{\text{em}}$, at constant $T, \rho_1$ for $F_{\text{em}}, \tilde{F}_{\text{em}}$, and at constant $T, \xi_1$ for $G_{\text{em}}, \tilde{G}_{\text{em}}$. So $u_{\text{em}}, F_{\text{em}}, G_{\text{em}}$ are in fact not equal, neither are $G_{\text{em}}, \tilde{F}_{\text{em}}$. The situation is similar to evaluating $\int T dS$ when considering a Carnot process: Since $T = T(s, \rho)$, we need to specify what the density $\rho$ does when $s$ varies. We have already introduced $-\tilde{G}_{\text{em}} = \frac{1}{2} H^2 + \int Eq M_i dH_i$ and $-\tilde{F}_{\text{em}} = \frac{1}{2} H^2 + \int Eq M_i dH_i$. 


\[\frac{1}{2}H^2 + \int_{\text{tot}} M \cdot dH,\]
given each a specifying index, and discussed why the difference between these expressions is irrelevant as long as LCR holds, cf Eq \([\text{33}]\).

To evaluate the stress \(\Pi_{\text{tot}}\), Eq \([\text{13}]\), in terms of these potentials, we start with its diagonal part \(-\tilde{G}\), given respectively as

\[-\tilde{G} = (s \frac{\partial}{\partial \rho} + \rho_1 \frac{\partial}{\partial \rho_1} - 1) u(0) = P(s, \rho_1) + \frac{1}{2}H^2 - \frac{1}{2}M^2 + \int d\mathbf{B} \cdot (1 - s \frac{\partial}{\partial \rho} - \rho_1 \frac{\partial}{\partial \rho_1}) \mathbf{M}, (112)\]

\[-\tilde{G} = (\rho_1 \frac{\partial}{\partial \rho_1} - 1) F = P(T, \rho_1) + \frac{1}{2}H^2 - \frac{1}{2}M^2 + \int d\mathbf{B} \cdot (1 - \rho_1 \frac{\partial}{\partial \rho_1}) \mathbf{M}, (113)\]

\[-\tilde{G} = (\rho_1 \frac{\partial}{\partial \rho_1} - 1) \tilde{F} = P(T, \rho_1) + \frac{1}{2}H^2 + \int d\mathbf{H} \cdot (1 - \rho_1 \frac{\partial}{\partial \rho_1}) \mathbf{M}, (114)\]

\[-\tilde{G} = P(T, \xi_1^{in}) + \frac{1}{2}H^2 + \int M \cdot d\mathbf{H}, (115)\]

where depending on the variables, the zero-field pressures are \(P(s, \rho_1) = (s \frac{\partial}{\partial \rho} + \rho_1 \frac{\partial}{\partial \rho_1} - 1)u(0), P(T, \rho_1) = (\rho_1 \frac{\partial}{\partial \rho_1} - 1)F(0), \) and \(P(T, \xi_1^{in}) = -\tilde{G}(0), \) cf Eq \([\text{30}]\) with \(u(0) = u^M, v_1 = 0.\) These four differing \(\tilde{G},\) with them the attendant stress \(\Pi_{\text{tot}}^{\text{em}} = -\tilde{G}\delta_{ik} - H_i B_k,\) certainly look disturbingly contradictory. Yet all must be 0. These four differing \(\tilde{G}\) are then interpreted as the electromagnetic force, referred to as ponderomotive or Helmholz force. But all this is obviously only valid if the temperature and density are indeed kept constant when the field is switched on. Under adiabatic circumstances, when \(s\) is kept constant instead of \(T,\) the pressure \(P(s, \rho_1)\) is the one that is field independent. Consequently, the above Helmholz force is no longer the correct expression for the electromagnetic force. Rather, one must take the field dependent terms from Eq \([\text{112}]\) instead.

The ambiguity of \(P\) arises from the fact that the dependent variables change when the field is switched on. For instance, the chemical potential, \(\xi_1^{in} = \partial \tilde{F}(0)/\partial \rho_1 + \partial F_{\text{em}}/\partial \rho_1 = \xi_1^{in}(0) + \Delta \xi_1^{in},\) changes by the amount \(\Delta \xi_1^{in} = \partial F_{\text{em}}/\partial \rho = -\int d\mathbf{H} \cdot \frac{\partial}{\partial \rho_1} \mathbf{M}.\) Therefore

\[P(T, \xi_1^{in}) = P(T, \xi_1^{in}(0)) + \frac{\partial P}{\partial \rho_1} \Delta \xi_1^{in} = P(T, \rho_1) - \int d\mathbf{H} \cdot \rho_1 \frac{\partial}{\partial \rho_1} \mathbf{M}, (116)\]

explaining the difference between Eqs \([\text{114}]\) and \([\text{113}]\). The ill behavior of \(P\) takes one by surprise, as proper thermodynamic expressions do not depend on the variables chosen. This stems from the fact that \(P\) is not a thermodynamically defined quantity in the presence of fields: It does not appear in Eq \([\text{14}]\), as all bona fide thermodynamic variables do. Since Eq \([\text{19}]\) or \([\text{33}]\) consist only of quantities which do appear in Eq \([\text{14}]\) — both with or without field — the stress tensor \(\Pi_{\text{tot}}^{\text{em}}\) given there holds for any set of variables. These are good reasons to make this the expression of choice.

Frequently, a further approximation is employed for the diagonal part of the stress. In dilute, one-component systems, the magnetization is usually proportional to the density, or \(M = \rho \partial \mathcal{M}/\partial \rho.\) Similarly, in magnetically diluted ferrofluids, we may assume that the magnetization is proportional to the particle density \(\rho_1,\) as \(M = \rho_1 \partial \mathcal{M}/\partial \rho_1.\) Inserting this into Eqs \([\text{113}]\) and \([\text{114}]\), we find, respectively,

\[-\tilde{G} = P + \frac{1}{2}(H^2 - M^2), \quad -\tilde{G} = P + \frac{1}{2}H^2. (117)\]

The difference between these two expressions are not due to a different set of variables, as \(P\) is a function of \(T\) and \(\rho_1\) in both cases. So we are dealing with a different pitfall here, one that we shall discuss in details in section \([\text{II F 3}]\) below. The gist of it is: When we assume \(M \sim \rho_1,\) this is meant as an approximation, implying the neglect of square and higher order terms \(\sim \rho_1^2, \rho_1^3 \ldots\) But consistency then dictates that we must neglect all higher order terms, including \(M^2 \sim \rho_1^2,\) This implies especially that the dilute limit is only given if \(M \ll H\) and \(\chi \ll 1\) hold. Therefore, the term \(\frac{1}{2}M^2\) in Eqs \([\text{117}]\) must be neglected, and the popular form for the Maxwell stress tensor, \(\Pi_{ik} = (P + \frac{1}{2}H^2)\delta_{ik} - H_i B_k,\) is to be taken with a large grain of salt, as it is valid only for \(M \ll H,\) and quite useless if \(M \approx H,\) or \(\chi \approx 1.\)

2. Magnetic Bernoulli Equation

The magnetic Bernoulli equation by Rosensweig is a very useful relation. It has been extensively employed in his book \([\text{2}],\) and in the literature on ferrofluids. We shall include the variation of concentration, which he did not consider, and in addition, free this relation from the ambiguous notation of the last section \([\text{III F 3}].\) The point is, the information in the magnetic Bernoulli equation is contained in Eq \([\text{7}].\) Combine it with Eq \([\text{113}]\) to yield

\[\Delta [P(T, \xi_1^{in}) + \rho g z] = 0, (118)\]

an expressions of the magnetic Bernoulli equation if the system is in equilibrium with respect to particle distribution. (Note that of all the \(\int M_i dH_i\) in Eqs \([\text{112}],[\text{113}],\) only that in Eq \([\text{113}]\) has the same variables as \(\int M_i dH_i,\) hence these two expressions cancel each other.) Before arriving at total equilibrium, and as long as the concentration is uniform, we may start from Eq \([\text{117}],\) and
combine it with Eq (114). The result is

$$\Delta[P(T, \rho_1) + \rho g z - \rho_1 \frac{2}{\eta_p} \int M_dH_i] = 0. \quad (119)$$

Substitute $\rho$ for $\rho_1$ to arrive at the expression as given by Rosensweig.

The velocity dependent terms in the original magnetic Bernoulli equation have not been included here, because considerations of mass currents in ferrofluids need to include viscosities. Besides, some of the velocity dependent terms in the stress tensor, Eq (49), are missing in $\alpha$.

3. Kelvin and Helmholtz Force

As in section III F 1 above, we may separate out the zero-field pressure $\bar{P}$ from the hydrodynamic bulk force density of Eq (67), $f^{\text{bulk}} = s \nabla T + \rho_o \nabla \xi_o$. (We again neglect gravitation, and consider a stationary medium, $\nu = 0.$) The remaining terms are referred to as the ponderomotive force $f^p$,

$$f^{\text{bulk}} = \nabla P(T, \rho_o) - f^p, \quad (120)$$

$$f^p = (\partial F_{em}/\partial T) \nabla T - \rho_o \nabla (\partial F_{em}/\partial \rho_o) \quad (121)$$

$$= \nabla (F_{em} - \rho_o \partial F_{em}/\partial \rho_o) - H_i \nabla B_i. \quad (122)$$

[The relation $\nabla F_{em} = (\partial F_{em}/\partial T) \cdot \nabla T + (\partial F_{em}/\partial \rho_o) \cdot \nabla \rho_o + H_i \nabla B_i$ was used for the last equal sign.]

In this section, we shall only employ the two potentials $F(T, \rho_o, H_i)$ and $F(T, \rho_o, B_i)$. So the zero field pressure depends on temperature and densities, and the derived ponderomotive force is valid only under isothermal condition of constant densities, cf the discussion of section III F 1.

We now proceed to derive the Kelvin force by incorporating some specific simplifications. For LCR, $-\partial F_{em}/\partial T = \frac{1}{2} H^2 \chi^m/\partial T$, and similarly $\partial F_{em}/\partial \rho_o$, so $f^p$ can be cast as

$$f^p = \nabla (\frac{1}{2} H^2 \rho_o \partial \chi^m/\partial \rho_o) - \frac{1}{2} H^2 \nabla \chi^m, \quad (123)$$

usually referred to as the Helmholtz force in the literature. If the system is magnetically dilute, we have $\chi^m \sim \rho$ in a one-component gas, and $\chi^m \sim \rho_1$ for $d\rho = \gamma d\rho_1$ in ferrofluids (with no additional dependence on $\rho$). Both imply $\rho_o \partial \chi/\partial \rho_o = \chi$. Then the ponderomotive force reduces to $\frac{1}{2} \chi \nabla (H^2)$, or

$$f^p = M_i \nabla H_i, \quad M = \chi^m H. \quad (124)$$

Assuming in addition a static field: $\nabla \times H = 0$, the force $f^p$ may also be written as $(M_i \nabla)_i H$, a form one often encounters.

This seems a satisfactory derivation of the Kelvin force, as both the linear constitutive relation and the proportionality to the density are frequently valid approximations. Unfortunately, the obviously different expression

$$f^p = M_i \nabla B_i \quad (125)$$

may be obtained by essentially the same derivation. We start by defining a slightly different susceptibility: $M = \chi^m B$. Although different from the more usual convention of Eq (124), the new susceptibility is undoubtedly physically equivalent to the old one, and we have no a priori reason to prefer either. Both susceptibilities are related via $\chi^m = 1 - 1/\mu = \chi^m/(1 + \chi^m)$, or $d\chi^m = d\chi^m/\mu^2$. So Eq (123) may be rewritten as

$$f^p = \nabla (\frac{1}{2} B^2 \rho_o \partial \chi^m/\partial \rho_o) - \frac{1}{2} B^2 \nabla \chi^m. \quad (126)$$

This time, assuming $\chi^m$ is proportional to one of the densities, we obtain Eq (123).

Which is the correct one, $f^p$ of $\chi^m$ or $f^p$ of $\chi^m$? Since Eq (123) and (126) are algebraically equivalent, the difference must stem from the two assumptions, $\chi^m \sim \rho$ and $\chi^m \sim \chi^m/(1 + \chi^m) \sim \rho$. Reviewing the above derivations, it is obvious that if one of the two assumptions were strictly correct, the other would be wrong, and only the associated force expression is applicable. Generically, however, both $\chi^m$ or $\chi^m$ are power series of $\rho$. Assuming either susceptibility is linear in $\rho$ is only an approximation. And consistency dictates that all quadratic terms are then to be discarded. This implies (i) in the dilute approximation, both susceptibilities may only retain the term linear in $\rho$ and are therefore equal; (ii) all other terms quadratic in the two susceptibilities are also to be discarded, as $(\chi^m)^2, (\chi^m)^2 \sim \rho^2 \to 0$. We conclude: the Kelvin force is only valid for $\chi^m \ll 1$ or $M \ll H$, to linear order in $\chi^m$ or $M$. But then $M_i \nabla B_i$ and $M_i \nabla H_i$ are equally valid. The same of course also holds for $P_i \nabla D_i$ and $P_i \nabla E_i$ in the electric case.

The following example aims to illustrate this conclusion [12]. Consider a thin slab of ferrofluid exposed to a homogeneous, external magnetic field $B$, oriented normal to the slab. An enforced temperature gradient within the slab ensures an inhomogeneous susceptibility, $\chi^m(T)$. The internal $B$-field is uniform, but the not the internal $H$-field, as $H = B/(1 + \chi^m)$. The ponderomotive force of Eq (125) is zero in the given circumstance, but not that of Eq (124), which yields $M_i \nabla H_i = -\chi^m H^2 (\nabla \chi^m)/(1 + \chi^m)$, an apparent contradiction.

A proper analysis of the force should start from the simple expression of Eq (67). If not, the contribution of the zero-field pressure, $P(T, \rho_1)$, in the presence of a temperature and concentration gradient, needs to be included. In addition, one must include higher order terms in the density, take $\chi = \alpha \rho + \beta \rho^2$, and $\chi = \chi/(1 - \chi) \approx \alpha \rho + (\beta + \alpha^2) \rho^2$. Inserting these respectively into Eq (123) and (124), we find, for both cases and a constant $B$-field, $f^p = \frac{1}{2} B^2 \nabla [\beta \rho^2]$. (127)

Assuming $\chi \sim \rho$ or $\chi \sim \rho$ to hold strictly is equivalent to taking $\beta = 0$ or $\beta = -\alpha^2$, resulting respectively in $f^p = 0$ and $f^p = -\chi B^2 \nabla \chi$, or equivalently, $M_i \nabla B_i$ or $M_i \nabla H_i$.

Next we go on to consider nonlinear constitutive relations, and shall convince ourselves that the two Kelvin
force expressions Eqs. \([24,23]\) remain valid — though (in a generalization of the above conclusion) only to linear order in the magnetization and polarization. We insert Eq. \([110]\) in Eq. \([22]\), and equate \(-\partial F_{em}/\partial \rho_a\) with \(\int (\partial M_i/\partial \rho_a) dB_i\), to yield

\[ f^P = M_i \nabla B_i + \nabla \int (\rho_a \partial M_i/\partial \rho_a - M_i) dB_i. \]  

(128)

Assuming that \(\rho_a \partial M_i/\partial \rho_a = M_i\) at constant \(B\) — equivalent to \(\rho_a \partial \chi^m/\partial \rho_a = \chi^m\) for LCR — the integral vanishes, and we retain Eq. \([125]\). If we start from \(\tilde{F}_{em}\) and \(\tilde{F}_{em}/\partial T\) to evaluate

\[ f^P = (\partial \tilde{F}_{em}/\partial T) \nabla T - \rho_a \nabla (\partial \tilde{F}_{em}/\partial \rho_a) \]  

(129)

\[ = \nabla \tilde{F}_{em} - \rho_a \partial \tilde{F}_{em}/\partial \rho_a \]  

(130)

we find \(f^P = M_i \nabla H_i\) if \(\rho_a \partial M_i/\partial \rho_a = M_i\) for constant \(H\).

IV. EXPERIMENTS

Having been derived from thermodynamics and conservation of total energy, momentum and angular momentum, the theory presented in the preceding chapters is fairly general, and valid for all magnetizable and polarizable liquids, from single-component paramagnetic fluid to ferrofluids, and for their respective electric counterparts. In the case of ferrofluids, which are suspensions of ferromagnetic particles, one is tempted to think that the properties of the particles are important, especially their orientation (given by the magnetic moment), and their internal angular momentum. This is indeed true if a high-resolution, mesoscopic account of the system is the prescribed goal. However, on a coarser scale, with many particles per grain and relevant for most experiments, the theory derived above is quite adequate, even uniquely appropriate for being not unnecessarily detailed and complicated.

As in the last chapter, we shall only display the magnetic terms, accounting for magnetic effects, as the analogous electric ones are easily obtained via Eqs. \([90]\).

A. Field Induced Variation in Densities

We consider the change in densities, \(\Delta \rho\) and \(\Delta \rho_1\), of a magnetizable liquid in equilibrium, from a region of low (or no) field to one of high field. In equilibrium, Eqs. \([83]\) holds. For a one-component liquid, they reduce to \(\nabla T = 0\) and

\[ \nabla \xi = \left[ \frac{\partial \xi}{\partial \rho} \right]_H \nabla \rho + \left[ \frac{\partial \xi}{\partial H} \right]_\rho \nabla H = -g e_z, \]  

(132)

where the chemical potential has been taken as a function of \(\rho, T\) and the field magnitude \(H\). It is useful to rewrite the two thermodynamic derivatives, the second as \([\partial \xi/\partial H]_\rho = -\partial B/\partial \rho_H = -\partial M/\partial \rho_H = -H [\partial \chi^m/\partial \rho]\), where the last equal sign is valid only for LCR. The first may be approximated: Without field, the inverse isothermal compressibility \(\rho^2 [\partial \xi/\partial \rho]_T = \rho [\partial \rho/\partial \rho]_T \equiv \kappa_1^{-1}\) is usually a large enough quantity that one may neglect the field related corrections, which is \(-1/2 (\rho H)^2 (\partial^2 \chi^m/\partial \rho^2)\) for LCR. Employing this approximation and assuming LCR, we integrate Eq. \([134]\) to yield the variation in density in response to the gradient of magnetic field strength and the gravitational potential,

\[ \Delta \rho = \rho^2 \kappa_1 \Delta (\frac{1}{2} H^2 \partial \chi^m/\partial \rho - g z), \]  

(133)

where the boldfaced \(\Delta\) denotes (as before) the difference of any quantity at two different points in the liquid, \(\Delta A \equiv A_2 - A_1\). The electric terms may be obtained, as usual, according to Eqs. \([56]\). (Frequently, this effect — referred to as the electro- or magnetostriiction — is calculated using the ponderomotive force of section \(\text{III F} 3\). The above calculation shows that there is no difficulty at all to avoid the ponderomotive force, and the ambiguity associated with its notation.) Electrostriction has been verified \([13]\), using the refractive index to measure the density change.

If the fluid has two components — such as when it is a solution or suspension — we need to consider both chemical potentials, \(\nabla \xi_1 = 0\) and \(\nabla \xi = -g e_z\). More conveniently, however, in the incompressible limit of section \(\text{III B}\) we may consider Eqs. \([70]\) alone,

\[ \nabla \xi_1^{in} = \left[ \frac{\partial \xi_1^{in}}{\partial \rho_1} \right]_H \nabla \rho_1 + \left[ \frac{\partial \xi_1^{in}}{\partial H} \right]_\rho \nabla H = -g e_z, \]  

(134)

\[ \nabla \rho_1 = [\partial \xi_1^{in}/\partial \rho_1]_H^{-1} ([\partial M/\partial \rho_1]_H \nabla H - g z e_z). \]  

(135)

(Note that \(\partial M/\partial \rho_1\) is connected to \(\partial \xi_1^{in}/\partial \rho_1\) via a Maxwell relation.) To our knowledge, field-induced equilibrium variation in the solute or particle density has not yet been measured in any two-component liquids. This is unfortunate especially in ferrofluids, where the variation in particle density should be rather pronounced. Quantitatively, this experiment yields the thermodynamic derivative \(\partial \xi_1^{in}/\partial \rho_1 = [\rho_1]^2 \kappa_{os}^{-1}\), with \(\kappa_{os}\) the osmotic compressibility. Since this is a diagonal derivative, its significance in characterizing the ferrofluid rank with that of the compressibility, specific heat and magnetic susceptibility.

Let us estimate the magnitude of this effect: Writing Eq. \([135]\) as \(\Delta \rho_1/\rho_1 = \kappa_{os} (\rho_1 \partial \chi^m/\partial \rho_1) \Delta (\mu_0 1/2 H^2)\), we approximate \((\rho_1 \partial \chi^m/\partial \rho_1) \approx 1\), estimate \(\kappa_{os} \approx 10^{-3}/\text{Pa}\), and find \(\Delta \rho_1/\rho_1 \approx 0.1\) for \(B = 10^{-2} \text{T}\). [The value for \(\kappa_{os}\) is obtained by considering a ferrofluid with 10% of its volume occupied by magnetic particles of the radius \(r = 10\mu\text{m}\), so the particle density is \(n_1 = 0.1/4\pi r^3/3\).] Approximating these particles as ideal gas, the inverse osmotic compressibility \(\kappa_{os}^{-1}\) is equal to the osmotic pressure, \(P_{os} = n_1 k_B T\), so \(\kappa_{os} = 10^{-3}/\text{Pa}\) if \(T = 300\text{K}\).] Compare this with tiny change of the total density,
\[ \Delta \rho / \rho = 5 \cdot 10^{-8} \] at the same field, cf Eq (133), a result of the small total compressibility, \[ \kappa = 5 \cdot 10^{-10} / \text{Pa}. \]

B. Current Carrying Vertical Wire

We consider a vertical wire that goes through a dish filled with ferrofluid. Feeding the wire with electric current will drag the ferrofluid toward the wire (located at \( r = 0 \) and along \( z \) in cylindrical coordinates). The surface of the ferrofluid column is given by \( z(r) \), with \( z \) at which the radius diverges (obtained by extrapolation) as the origin, i.e. \( z(\infty) = 0 \). The boundary condition Eq (79) is evaluated for two points, \( z(r) \) and 0. Because the magnetization vanishes and the curvature radii diverge at 0, the attendant result is \( K = P_{\text{atm}} \). Inserting this into the boundary condition at \( z(r) \), we obtain

\[ \int_{\text{Eq}} M_i dH_i = \alpha (R_1^{-1} + R_2^{-1}) + \bar{\rho} g z, \quad (136) \]

because \( M_n = 0 \) for the given geometry. The integral is to be evaluated for \( H = J/2(2\pi r) \), and at constant \( T, \xi \) if equilibrium has time to be established. For times much briefer after the current has been applied, we need to substitute \( \rho \) for \( \bar{\rho} \), and \( \int_{\text{uni}} \) for \( \int_{\text{Eq}} \), and evaluated the integral at constant \( T, \rho_1 \), cf the discussion leading to Eq (81). Neglecting the surface tension, \( \alpha = 0 \), and taking the subscript \( \int_{\text{uni}} \) lead to the result given in (136).

In the spirit of the last paragraph of section III D, we remark that Eq (136) may be considered as an expression for force equilibrium, between gravitation, surface tension and the magnetic surface force.

For LCR, \( M_i = \chi m H_i \), the left hand side reduces to \( \frac{1}{2} \chi \rho H^2 \), and we are left with a quadratic, hyperbolic profile of the interface, \( 8 \pi^2 \bar{\rho} g z = J^2 \chi m / r^2 \) if the surface tension \( \alpha \) is neglected. The effect of \( \alpha \) is more important for weak currents, \( J \) small. It may be neglected in any case for \( z \to 0 \), where both curvature radii are large enough to be ignored. For \( z \) large, one curvature radius is simply \( r \), and the other \( \infty \). So this part of the ferrofluid column is accounted for by \( 8 \pi^2 \bar{\rho} g z + \alpha / r = J^2 \chi m / r^2 \), with the term \( \sim r^{-2} \) being asymptotically \( (r \to 0) \) the dominant one. In between, where the actual bend from the horizontal to the vertical takes place, both curvature radii are finite and need to be included for an understanding of the surface – note, however, that they have different signs.

C. Hydrostatics in the Presence of Fields

In a system of two connected tubes, with only the second subject to a magnetic field, we expect the ferrofluid column to be higher in this tube, as ferrofluid is attracted to the region of stronger fields, see Fig 3. To calculate the level difference, we employ the boundary condition Eq (73) for the (flat) liquid-air interface in both tubes. Since the field vanishes in the first tube (denoted as 1), the boundary condition simply states \( P_{\text{atm}} = K - \bar{\rho} g z \).

Inserting this into the boundary condition for tube 2, we obtain

\[ \bar{\rho} g (z_2 - z_1) = \int_{\text{Eq}} M_i dH_i + \frac{1}{2} M_n^2. \quad (137) \]

Note again the force balance between the electromagnetic surface force Eq (53), operational at the surface in tube 2, and the gravitational force from the disparity in height. As in section III C, the integral is to be evaluated at constant \( T \) and \( \xi \) if equilibrium has time to be established. For much briefer times after the field has been applied, \( \bar{\rho} \) is to be replaced by \( \rho \), and the integral by \( \int_{\text{uni}} \), evaluated at constant \( T, \rho_1 \).)

If the field is either predominantly tangential to the liquid surface \( (H = H_i, B = B_i \) and \( H_n, B_n = 0) \), or predominantly normal to the liquid surface \( (H = H_n, B = B_n \) and \( H_i, B_i = 0) \), we have respectively,

\[ g \bar{\rho} (z_2 - z_1) = \int_{\text{Eq}} M_i dH_i = \frac{1}{2} H^2 \chi n \]
\[ g \bar{\rho} (z_2 - z_1) = \int_{\text{Eq}} M_i dB_i = \frac{1}{2} H^2 \chi m \mu_i \quad (138) \]
\[ \int_{\text{Eq}} M_i dH_i = \int_{\text{Eq}} M_i dH_i = \frac{1}{2} H^2 \chi n \]
\[ \int_{\text{Eq}} M_i dB_i = \int_{\text{Eq}} M_i dH_i = \frac{1}{2} H^2 \chi n \mu_i \quad (139) \]

with the last equal signs in both equations only valid for LCR.

Next we consider the quantity that a pressure gauge measures in a ferrofluid exposed to a magnetic field. As emphasized, it is not the ordinary pressure, yet as it will give some value, the question is what this is. Think of the gauge as an enclosed volume of air, at the pressure \( P_{\text{atm}} \), see Fig 4. One side of this volume is an elastic membrane, which is displaced if the external stress tensor deviates from the internal one. A finite displacement \( d \) stores up the elastic energy \( k d^2 / 2 \) per unit area of the membrane. (Take the membrane to be stiff, i.e. \( k \) large and \( d \) small, then we need not worry about the pressure change inside.) The elastic energy implies a force density \( k d \), rendering the boundary condition across the membrane as \( \Delta \Pi_{\text{tot}} = k d \), or via Eq (73), \( K - P_{\text{atm}} + \int_{\text{Eq}} M_i dH_i + \frac{1}{2} M_n^2 - \bar{\rho} g z = kd \). (\( d \) is taken to be positive when the membrane protrudes into the
and subtracting one from the other, we find:

\[ \Delta P = P_1 - P_2 = \Delta \rho g d. \]

ploying the above boundary condition for both points, we may compare the elastic membrane below the surface – take this as point 2. Moving down the liquid column, to an arbitrary point 3, the membrane moves the distance \( d \) to maintain force equilibrium. Employing the above boundary condition for both points, and subtracting one from the other, we find:

\[ kd = \Delta \left( \int_{ Eq} M_i dH_i + \frac{1}{2} M^2_n - \bar{\rho} gz \right). \]

(140)

with \( \Delta A = A_3 - A_2 \). Note \( \Delta M_n \equiv M^3_n - M^2_n \), where \( M^3_n \) is the magnetization at point 3 normal to the pressure gauge membrane, and \( M^2_n \) the field at point 2 normal to the liquid surface – both components are not necessarily parallel. Eq (140) again states a force balance, between the elastic, magnetic surface and gravitational force.

The displacement \( d \) is the readout of the pressure gauge: If the field is uniform, and if the membrane of the gauge is parallel to the liquid surface, Eq (140) reduce to \( \rho g (z_2 - z_3) = kd \), the zero-field hydrostatic relation; otherwise, field contributions abound – even if the pressure gauge is simply rotated at point 3 in the presence of a uniform field. This amply demonstrates the system’s anisotropy. A further complication is that all fields are the actual ones, disturbed by the presence of the pressure gauge – though this is an effect that may be minimized.

D. Magnetic O-Rings and Scrap Separation

In this section, we address the physics of some technical applications: magnetic O-rings, self-levitation and scrap separation. Consider scrap separation first. An inhomogeneous magnetic field which becomes weaker with increasing height may lift non-polarizable bodies submerged in ferrofluids off the ground, and hold them at specific heights which depend on the shape and density of the bodies.

The calculation is already given in section [III]. Balancing the gravitation with magnetic force, we have

\[ \mathbf{F}^{E+M} + \mathbf{F}^{grav} = 0, \]

where \( \mathbf{F}^{grav} = -\nabla (\rho - \rho^f) \mathbf{g} \hat{e}_z \), and \( \mathbf{F}^{E+M} \) is given by Eqs (89, 90), and especially by Eqs (92, 93) if the solid is completely nonmagnetic. (Again, depending on the time scale, the integral \( \int_{uni} \) may be the appropriate one, cf section [III].)

Because scrap separation is an equilibrium phenomenon, we may compare energies instead of computing forces. This is a simpler and more qualitative approach to understand the behavior of polarizable systems. Consider for instance the fact that ferrofluid is attracted to the region of high magnetic field. Take the field \( B \) as given, the field energy per unit volume is \( B^2/2 \) in vacuum (or air), and \( B^2/(2\mu) \) in ferrofluids. Since \( \mu > 1 \) in any paramagnets, the second expression is smaller. Given the choice, a volume element of ferrofluid will therefore always occupy a region with the highest possible field, to reduce the field energy. Conversely, a small, non-magnetizable object, submerged in ferrofluid, will on the other hand tend to occupy the region of lowest field strength. If a difference in height is involved, all these of course happen only as long as the gain in field energy is larger than the loss in gravitational energy levitating the object. (If the field \( H \) is given instead of \( B \), we need to consider \( \tilde{F}_{em} = F_{em} - H B \). And again, it is larger in vacuum than in the ferrofluid: \( -H^2/2 \).)

\[ -H^2/2 > -\mu H^2/2. \]

FIG. 5: Self-levitation of a magnetized body submerged in ferrofluid.

Similarly, a permanently magnetized body submerged in ferrofluid tends to collect as much liquid around itself as possible, in the space occupied by its stray field – even at the price of levitating itself off the bottom, a phenomenon that is usually referred to as “self-
levitation”. Frequently, the magnetized body consists of a periodic array of north and south poles, with periodicity $\lambda$, so the stray field extends one or two $\lambda$ into the ferrofluid. Levitated approximately that far from the bottom, the body will usually have reached its equilibrium position, as no further gain in field energy may be achieved by pushing more ferrofluid below it, and levitating itself yet higher, Fig 5.

Generally speaking, although energetic considerations are a useful heuristic tool, the actual minimization of total energy or free energy is often rather cumbersome, as the field does change considerably when a volume element of ferrofluid is displaced. For quantitative calculations, it is therefore frequently more convenient to consider boundary conditions involving the Maxwell stress tensor, as we have done throughout this review article.

This also pertains to magnetic O-rings, which may be found in most computer hard disk drives and are perhaps the most widely deployed ferrofluid device. In these, some ferrofluid is positioned as an O-ring between a highly permeable rotating shaft and the pole of a permanent magnet, Fig 6. Serving as a pressure seal, it enables the rotary component to work in vacuum. In contrast to the above circumstances, we are dealing with a metastable state here, as it is always energetically more favorable to remove the ferrofluid seal first, have the pressure equalized, and then return the seal to its original place at the poles. So the quantity of interest is the lowest energy barrier that must be overcome, which in any realistic problems is a quantity notoriously difficult to find. The correct force balance, on the other hand, is easy to write down:

$$A \Delta P = \Delta \int \left( \frac{1}{2} M_n^2 + \int_E M_k dH_k \right) dA,$$  \hspace{1cm} (141)

expressing the balance between the difference in the electromagnetic surface force [cf Eq (88)] on the two free surfaces (area $A$) of the ferrofluid and the difference in external pressure. The equation is obtained by considering the boundary condition $\Delta \Pi_{nn}^{\text{tot}} = 0$, for both free surfaces, call them 1 and 2. Use Eq (78), forget gravitation, and remember that $P$ is different for 1 and 2 to arrive at Eq (141), with $\Delta P = P_0(1) − P_0(2)$.

The field is strongest in the middle of the O-ring and decays towards both ends. If $\Delta P$ were zero, the force $\mathbf{F}_{\text{m}}$ would be the same on both surfaces, and the ferrofluid stays in the middle of the O-ring. Increasing the pressure on the left (surface 1) pushes the ferrofluid towards right, such that surface 1 is in the region of higher, and surface 2 is in the region of weaker, fields. Equilibrium is achieved when the difference in $\mathbf{F}_{\text{m}}$ balances $\Delta P$. The strongest pressure difference maintainable is when one surface is at the region of highest field, and the other is field-free. Assuming for simplicity that the magnetic field is predominantly tangential, and that LCR holds, we have $\Delta P = \frac{1}{2} \lambda m H^2$. With $\lambda m \approx 1$, $H^2 = \mu_0 H^2$, and taking $H$ as of order $10^5 \text{A/m}$, this pressure difference is about $10^5 \text{N/m}^2$, approximately the atmospheric pressure.

E. Elliptical Deformation of Droplets

A droplet of ferrofluid exposed to an external magnetic field along $\mathbf{e}_z$ is distorted, cf eg [14]. If the field is not too strong, it turns from a sphere of radius $r$ to an ellipsoid of essentially the same volume, $y^2/a^2 + (x^2 + z^2)/b^2 = 1$, with $a$, $b$ the semimajor and semiminor axes, $b < a$, see Fig 7. Given a uniform applied field, $B_{ex}$, the internal field of a magnetizable ellipsoid is also uniform, and along the applied field, even for nonlinear constitutive relation, cf § 8 of [5]. Once these facts are assumed, it is fairly simple to calculate the distortion, even nonlinearly, as a function of the field, because again, only the boundary condition, $\Delta \Pi_{nn}^{\text{tot}} = \alpha (R_1^{-1} + R_2^{-1})$ from Eqs (14), needs to be evaluated, at the points $(x, y, z) = (0, a, 0)$ and $(b, 0, 0)$. They are respectively

$$\Delta \Pi_{yy}^{\text{tot}} = \alpha (2a/b^2), \quad \Delta \Pi_{xx}^{\text{tot}} = \alpha (1 + b/a^2).$$  \hspace{1cm} (142)

Employing Eq (78) to evaluate the respective left side of these two equations, we find $\Delta \Pi_{yy}^{\text{tot}} - \Delta \Pi_{xx}^{\text{tot}} = \frac{1}{2} M^2$, because all fields are constant within the ellipsoid, and because $M_n = M$ at $(0, a, 0)$, $M_n = 0$ at $(b, 0, 0)$. (The extent of the droplet is presumed small enough for us to neglect the gravitation.) Taking the difference also of the right hand side, we arrive at

$$\frac{1}{2} M^2 = (\alpha/r)(2\eta^{-2/3} - \eta^{-1/6} - \eta^{5/6}),$$  \hspace{1cm} (143)

where the parameter $\eta$ is related to the eccentricity $e$: $\eta = 1 - e^2 = (b/a)^2$. (With $r^3 = ab^2$, we have $a = \eta^{-1/3}$, $b = \eta^{1/6}$.) This result is the same as that given in [5]. Both sides of this equation may be approximated independently: For small eccentricity, the right side reduces to $(\alpha/r)2e^2$, while the left side reduces to $\frac{1}{2}(\mu - 1)^2 H^2$ for LCR. In addition, we have $H_{ex}/H = 1 + n(\mu - 1)$, where $H_{ex}$ is the applied field, and $n = (1 - 2e^2/5)/3$ for small eccentricity, cf §8 in [5].
V. APPENDIX

The validity of Eq (43) is shown here directly by transforming the rest frame expression. More specifically, we demonstrate $\partial(u - v g^\text{tot})/\partial D = E_0$, holding $s$, $p_0$, $v$, and $B$ constant. We start with

$$u = u_0(D_0 \rightarrow D, B_0 \rightarrow B) + \frac{1}{2} \rho v^2. \quad (144)$$

This pleasingly simple expression is a result of the accidental cancellation of the terms from the Galilean-Lorentz transformation with that of the Tailor expansion,

$$\begin{align*}
u(D, B) &= u_0(D_0, B_0) + 2 v \cdot g_0^\text{tot} + \frac{1}{2} \rho v^2 \\
&= u_0(D_0, B_0) + 2 v \cdot (E \times H)/c + \frac{1}{2} \rho v^2 \\
&= u_0(D, B) + \frac{1}{2} \rho v^2.
\end{align*}$$

Assuming LCR, or $u = \frac{1}{2}(D^2/\epsilon + \rho v^2)$ with $D_0 = eE_0$, we have

$$\begin{align*}
u - v \cdot g^\text{tot} &= \frac{1}{2}(D^2/\epsilon - \rho v^2) - v \cdot (E \times H)/c \\
&= \frac{1}{2}(D^2/\epsilon - \rho v^2) - v \cdot (D \times H)/c + O(v^2/c^2). \quad (145)
\end{align*}$$

and deduce

$$\begin{align*}
\partial(u - v \cdot g^\text{tot})/\partial D &= (D + v \times H/c)/\epsilon = D_0/\epsilon = E_0. \quad (146)
\end{align*}$$

Higher order terms (such as one $\sim D^4$ in the energy $u$) do not invalidate this result. The terms in the magnetic field behave analogously.

In Eq (145), the explicit form of $g^\text{tot}$ in the lab frame was employed to deduce the lab frame energy, Eq (143), from which then the lab frame energy flux, Eq (146), is deduced. This may appear as an inconsistency, but is not, because with the rest frame expression for $g^\text{tot}$ given, we already know that the term $\sim v$ is from the rest mass. No detailed information about the energy flux is necessary here.

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