A new method for the robust characterisation of pairwise statistical dependency between point processes

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Abstract

The robust detection of statistical dependencies between the components of a complex system is a key step in gaining a network-based understanding of the system. Because of their simplicity and low computation cost, pairwise statistics are commonly used in a variety of fields. Those approaches, however, typically suffer from one or more limitations such as lack of confidence intervals requiring reliance on surrogate data, sensitivity to binning, sparsity of the signals, or short duration of the records. In this paper we develop a method for assessing pairwise dependencies in point processes that overcomes these challenges. Given two point processes \(X\) and \(Y\) each emitting a given number of events \(m\) and \(n\) in a fixed period of time \(T\), we derive exact analytical expressions for the expected value and standard deviation of the number of pairs events \(X_i, Y_j\) separated by a delay of less than \(\tau\) one should expect to observe if \(X\) and \(Y\) were i.i.d. uniform random variables. We prove that this statistic is normally distributed in the limit of large \(T\), which enables the definition of a Z-score characterising the likelihood of the observed number of coincident events happening by chance. We numerically confirm the analytical results and show that the property of normality is robust in a wide range of experimental conditions. We then experimentally demonstrate the predictive power of the method using a noisy version of the common shock model. Our results show that our approach has excellent behaviour even in scenarios with low event density and/or when the recordings are short.

1 Introduction

Network-based modelling, whereby nodes denote components of the system and edges denote interactions between those components, has become a paradigm of choice for describing, understanding and controlling complex systems [25]. It is widely used in many fields including neuroscience [35], mathematical epidemiology [24] and social sciences [20] to name just a few. However, in many cases, the actual connectivity between individuals may not be directly available or may be incomplete. A substantial body of work has therefore focused on inferring links using the activity of the nodes (e.g. neural spikes, computer events, tweets, gene expression levels) and coupling measures [4, 29]. In this paper, we focus on the problem of inferring functional connectivity, i.e., the graph of pairwise statistical dependences between components of the system as opposed to causal connectivity where such dependences are assumed to be causal [30].
Further, we restrict ourselves to systems involving point processes. Although the discrete nature of such processes provides greater analytical tractability, reliable inference of functional coupling between components remains a challenging problem: the exact nature of the point processes may not be known [31]; non linearity, non stationarity and noise are often to be expected [22, 23, 27]; and finally, recordings may be short and repeated measurement unavailable [31].

A common way to quantify functional interaction between two point processes consists in computing Pearson’s cross-correlation and selecting its maximal value [19, 38]. Theoretically, after a hyperbolic transformation, the resulting value can then be compared to the theoretical value of the standard deviation and an interpretable Z-score may be extracted [12]. However, there are a number of issues with this approach: it assumes stationarity [1], it is bin dependent, it assumes that recordings are infinitely long, the Z-score is only approximately normal if the processes are bivariate normally distributed (hence making its interpretation somewhat hazardous) and the selection of the maximal value limits the ability to characterise different patterns of interaction, especially if a large range of lags is considered. Hence, in neuroscience and genetics, in particular, it is still very common to simply threshold the cross-correlation, where the threshold is arbitrarily chosen [6, 11] or based on some expected or desired [3, 16, 37] characteristics of the resulting network. Whilst such an approach removes the assumption of bivariate normal distribution, it is somewhat unsatisfactory because in general the characteristics of the network being inferred are not known. More recent studies rely on comparing the cross-correlation to that of surrogate data. For example, Smith et al. [34] create null data consisting of testing timeseries from different subjects, i.e., without causal connections between them. This enables the choice of a threshold and interpretation of the output. However, this does not solve the other aforementioned issues and also introduces the computational cost of generating the surrogate data.

A powerful alternative to the use of cross-correlation to infer functional connectivity is the frequency-domain concept of coherence (along with its somewhat less intuitive time-domain counterpart, the cumulant), applicable to both time series and point processes (or both) and for which rigorous confidence intervals have been derived [15]. Whilst this framework addresses many of the aforementioned issues (and we note recent work extending it to characterising directionality of interaction [14]), its reliance on long recordings can make analysis problematic for short recordings or when the data are sparse [17]. The issue of short and/or sparse data was addressed in much prior work when some sought to count the expected number of coincidental pair of events and derive its expected standard deviation [18]. This allowed the derivation of an interpretable Z-score and was successfully used to separate purely stimulus-induced correlations from intrinsic interneuronal correlation [33, 11, 2]. Similar results were independently found by Palm et al. who thoroughly analysed them [26]. Still, this approach did not address other issues such as bin dependency or the choice of a specific lag.

In recent years, the neuroscience community (in particular) has sought to move away from pairwise statistics.
tics and reveal higher order structures [29, 5]. And whilst some of those newer approaches do meet the aforementioned challenges and efficiently recover functional networks (see [5] for example), application to large systems can render them slow and hence not suitable to large networks such as those found in commercial computer networks [21] or social media networks. In addition, these Bayesian approaches typically see their performance decrease with the size of the network due to the increase in the number of parameters to be fitted [36].

In this paper, we derive a pairwise statistic of interaction between point processes that is computationally inexpensive and that overcomes all aforementioned problems: it is bin independent, it measures interactions at all lags, it does not require that the recording be infinitely long, it comes with confidence intervals. In Section 2 we first derive exact analytical expressions for the expected value and standard deviation of the statistic. We then prove that this statistic is normally distributed. In Section 3.1, we demonstrate the excellent agreement between analytical and experimental results. In Section 3.2, we experimentally confirm that the statistic converges to a normal distribution and therefore enables the construction of a Z-score. In Section 3.3, we use the delayed common shock model to demonstrate the effectiveness of our statistic in characterising both instantaneous and delayed interaction.

2 Methods

2.1 Graphical construction of the theoretical model

For any integer time horizon $T$ we construct two independent Bernoulli sequences, $\hat{X}_T$ and $\hat{Y}_T$ up to $T$. To be precise, $\hat{X}_T = (X_1, \ldots, X_T)$ and $\hat{Y}_T = (Y_1, \ldots, Y_T)$. Each coordinate $X_i$, $Y_j$ are i.i.d. Bernoulli($p_X$) and Bernoulli($p_Y$) respectively, and each realization is a sequence of 1’s and 0’s. Parameters $p_X$ (resp. $p_Y$) is the probability of seeing a 1 in the $\hat{X}_T$ (resp. $\hat{Y}_T$) sequence. On average, by the strong law of large numbers, one expects $p_X T$ many 1’s in the $\hat{X}_T$ sequence and $p_Y T$ in the $\hat{Y}_T$ sequence, for large $T$ values. From central limit theorem considerations, the actual number of 1’s is up to two leading orders $\hat{X}_T$ is $p_X T + c_Z \sqrt{T}$ where $c_Z$ will be a random normally distributed number, as long as $T$ is large enough.

We create a graphical arrangement of marks using the two independent sequences on $[1, T]^2$. A lattice square $(i, j)$ is considered marked if and only if $X_i = Y_j = 1$. We also define a lag $\delta$, $0 \leq \delta \leq T$; we are interested in the number of times the two processes obtained the value 1 in a time interval of size $\delta$ in either direction; in other words we want to know how many marks exist in a band of vertical height $2\delta + 1$ around the main diagonal $D = \{(i, i) : 1 \leq i \leq T\}$.

In general, when there is availability of data, we can count the number of ones in the available time series (say $n_X, n_Y$) and then infer an approximation to the (generally unknown) parameters $p_X \approx n_X T^{-1}$ and $p_Y \approx n_Y T^{-1}$. Then one can run the theoretical model, as we describe it above, up to time horizon $T$ and create two new independent time series, compare them with data and use them for predictions. This we use in Section 2.3 to derive an approximate central limit theorem (CLT) for the number of marks in a band, as $T \rightarrow \infty$. Part of the CLT is the expected value of marks of this completely independent model.
and approximate standard deviation. We use this result to construct a suitable Z-score for the number of marks in the band.

Moreover, in concrete applications where measurements can be done, the time horizon can in general be too short for the CLT to be a reasonable approximation, or the values $n_X T^{-1}$ and $n_Y T^{-1}$ can be far from their respective true values $p_X, p_Y$. In Section 2.2 we compute two statistics, namely the expected value and variance of the number of marks in finite band, conditional on the number of 1's, $n_X, n_Y$ that we counted in the given time horizon.

### 2.2 Expected value and standard deviation.

Fix two integers $n_X$ and $n_Y$, $n_X, n_Y \leq T$. These integers represent the given number of 1's in the $X$ and $Y$ arrangements. An arrangement now is a $T$-dimensional vector of 1's and 0's, under the constraint that the number of 1's is $n_X$ (for the $X = (X_1, \ldots, X_T)$ arrangement) and $n_Y$ for the $Y = (Y_1, Y_2, \ldots, Y_T)$ arrangement. The distribution of an arrangement is uniform among all possible ones, and there are $\binom{T}{n_X}$ possible ones for $X$ and $\binom{T}{n_Y}$ for $Y$. The $X$ arrangement is still independent of the $Y$.

Their one-dimensional marginals can be readily computed by

$$
P\{X_i = 1\} = 1 - P\{X_i = 0\} = \frac{n_X}{T}. $$

The marginals $X_i$ are identically distributed Bernoulli variables, however, they are negatively correlated, and as such, they are no longer independent. The covariance between any two one-dimensional marginals is

$$
\text{Cov}(X_i, X_j) = P\{X_i = 1, X_j = 1\} - (P\{X_i = 1\})^2 = \frac{T-2}{n_X - 1} - \frac{n_X^2}{T^2} = \frac{n_X(n_X - 1)}{T(T-1)} - \frac{n_X^2}{T^2} = \frac{n_X(n_X - T)}{T^2(T-1)}, \quad (2.1)
$$

for any indices $i \neq j$.

Consider a given lag $\delta$ and arrangements $X, Y$. We will find statistics for the number of marks in the band of distance $\delta$ around the main diagonal. The set of marks in the $\delta$-band $D_{T,\delta} = \{(i, j) \in [1, T]^2 : |i-j| \leq \delta\}$ is

$$
S_{T,\delta} = \{(i, j) \in [1, T]^2 : |i-j| \leq \delta, X_i = Y_j = 1\}. \quad (2.2)
$$

We denote by $|S_{T,\delta}|$ the cardinality of the set $S_{T,\delta}$; this is precisely the number of marks in $D_{T,\delta}$. In the next two subsections we show the calculations for the expected value, standard deviation and approximate normality for $|S_{T,\delta}|$. 

4
In all calculations that follow we adopt the standard notation of indicator functions. These are (Bernoulli) random variables that take values 1 or 0. For any \( \omega \) in a sample space \( \Omega \) and any event \( A \) we denote by \( \mathbb{1}\{A\} \) the random variable which satisfies
\[
\mathbb{1}\{A\}(\omega) = \begin{cases} 
1, & \omega \in A, \\
0, & \omega \in \Omega \setminus A.
\end{cases}
\]
The distribution of the indicators is Bernoulli, with probability of success \( \Pr\{A\} \) and therefore the expected value satisfies \( \mathbb{E}(\mathbb{1}\{A\}) = \Pr\{A\} \). We use these facts without any particular mention in the sequence. The computations rely on the decomposition
\[
|S_{T,\delta}| = \sum_{(i,j):|i-j|\leq\delta} \mathbb{1}\{X_i = Y_j = 1\}.
\]
With this we compute first the expected value of the number of marks in \( B_{T,\delta} \).
\[
\mathbb{E}(|S_{T,\delta}|) = \mathbb{E}\left( \sum_{(i,j):|i-j|\leq\delta} \mathbb{1}\{X_i = Y_j = 1\} \right) = \sum_{(i,j):|i-j|\leq\delta} \mathbb{E}(\mathbb{1}\{X_i = Y_j = 1\})
\]
\[
= \sum_{(i,j):|i-j|\leq\delta} \Pr\{X_i = 1, Y_j = 1\} = \sum_{(i,j):|i-j|\leq\delta} \Pr\{X_i = 1\} \Pr\{Y_j = 1\}
\]
\[
= \frac{n_X n_Y}{T^2} \left((2\delta T - \delta^2) + T - \delta\right) = n_X n_Y \left(1 - \left(1 - \frac{\delta}{T}\right)^2\right) + n_X n_Y \frac{T - \delta}{T^2}.
\]
(2.3)
At the last line of the computation above we used the number of terms in the sum. This can be calculated by adding the integer cells on the \( 2\delta + 1 \) diagonals,
\[
A_{T,\delta} = T + 2 \sum_{k=1}^{\delta} (T - k) = T(2\delta + 1) - \delta(\delta + 1).
\]
(2.4)
Observe that \( A_{T,0} = T \).

The variance is computed in a similar manner, with slightly more involved calculations. We need to separate into two cases, according to the size of the lag \( \delta \).
2.2.1 Case 1: $\delta + 1 \leq \frac{T}{2}$:

\[
\text{Var}(|S_{T,\delta}|) = \text{Cov}\left(\sum_{(i,j):|i-j|\leq \delta} \mathbb{I}\{X_i = Y_j = 1\}, \sum_{(k,\ell):|k-\ell|\leq \delta} \mathbb{I}\{X_k = Y_\ell = 1\}\right)
\]

\[
= \sum_{(i,j):|i-j|\leq \delta} \sum_{(k,\ell):|k-\ell|\leq \delta} \text{Cov}\left(\mathbb{I}\{X_i = 1\}\mathbb{I}\{Y_j = 1\}, \mathbb{I}\{X_k = 1\}\mathbb{I}\{Y_\ell = 1\}\right)
\]

\[
= \sum_{(i,j):|i-j|\leq \delta} \sum_{(k,\ell):|k-\ell|\leq \delta} \mathbb{E}\left(\mathbb{I}\{X_i = X_k = 1\}\mathbb{I}\{Y_j = Y_\ell = 1\}\right)
\]

\[= \mathbb{E}\left(\mathbb{I}\{X_i = 1\}\mathbb{I}\{Y_j = 1\}\right) - \mathbb{P}(X_i = 1)^2 \mathbb{P}(Y_j = 1)^2 A_{T,\delta}^2.
\]

The computation can be completed when we compute the double sum. Keep in mind that the $X$ arrangement and the $Y$ arrangement are independent and therefore

\[\mathbb{E}\left(\mathbb{I}\{X_i = X_k = 1\}\mathbb{I}\{Y_j = Y_\ell = 1\}\right) = \mathbb{P}\{X_i = X_k = 1\}\mathbb{P}\{Y_j = Y_\ell = 1\}.
\]

In order to complete the calculation we just need to find the number of terms in the sums with specific contributions. We separate into four cases, depending on the indices $(i, j), (k, \ell)$ and the four different types of terms in the sum.

1. $(i, j) = (k, \ell)$: Then the contribution coming from terms with this indices

\[\mathbb{P}\{X_i = X_k = 1\}\mathbb{P}\{Y_j = Y_\ell = 1\} = \mathbb{P}\{X_i = 1\}\mathbb{P}\{Y_j = 1\} = \frac{n_X n_Y}{T^2}.
\]

There are $N_1 = A_{T,\delta}$ terms like this in the double sum.

2. $i = k, j \neq \ell$. Then by (2.1)

\[\mathbb{P}\{X_i = 1\}\mathbb{P}\{Y_j = 1\} = \frac{n_X (n_Y - 1)}{T} \frac{n_Y}{T(T - 1)}.
\]

The number of terms with this contribution is

\[N_3 = 2 \sum_{i=1}^{\delta+1} (\delta + i)(\delta + i - 1) + (T - 2\delta - 2)(2\delta + 1)2\delta
\]

\[= \frac{2}{3}(\delta + 1)(\delta + 2)(7\delta - 3) - 4(\delta + 1)(\delta - 1) + (T - 2\delta - 2)(2\delta + 1)2\delta.
\] (2.5)

3. $i \neq k, j = \ell$. Again using (2.1) we compute

\[\mathbb{P}\{X_i = X_k = 1\}\mathbb{P}\{Y_j = 1\} = \frac{n_X (n_X - 1)}{T(T - 1)} \frac{n_Y}{T}.
\]

Now observe that the number of terms with this contribution is also $N_3$ by symmetry.
4. \( i \neq k, j \neq \ell \). Use (2.1) one last time,
\[
P\{X_i = X_k = 1\}P\{Y_j = Y_\ell = 1\} = \frac{n_X(n_X - 1) n_Y(n_Y - 1)}{T(T-1)} \cdot \frac{1}{T(T-1)}.
\]

Note that the above is positive. The number of terms that satisfy this condition is all the terms in the sums that remain, which are
\[
N_2 = A_{T,\delta}^2 - N_1 - 2N_3.
\] (2.6)

The overall variance is given by
\[
\text{Var}(|S_{T,\delta}|) = \frac{n_Xn_Y}{T^2} A_{T,\delta} + \frac{n_X(n_X - 1) n_Y(n_Y - 1)}{T(T-1)} N_2
\]
\[
+ \frac{n_Xn_Y(n_X + n_Y - 2)}{T^2(T-1)} N_3 - \frac{n^2_Xn^2_Y}{T^4} A_{T,\delta}^2,
\] (2.7)

where \( A_{T,\delta} \) is given by (2.4), \( N_3 \) by (2.5) and \( N_2 \) by (2.6).

2.2.2 Case 2: \( \delta + 1 > \frac{T}{2} \):

In this case, the strip of interest is quite fat, so the previous counting formulas don’t apply (since for example \( N_3 \) cannot have a term \( T - 2\delta - 2 \)). We compute the variance by counting a different way.

Consider the two left-over triangles on the bottom-right and upper-left of \([1, T]^2 \setminus D_{T,\delta} \) that their disjoint union is \([1, T]^2 \setminus D_{T,\delta} \). We denote these discrete triangles by
\[
\Delta_{T,\delta}^{\text{up}} = \{(i, j) \in [1, T]^2 : j > i + \delta\}, \quad \text{and} \quad \Delta_{T,\delta}^{\text{low}} = \{(i, j) \in [1, T]^2 : i > j + \delta\}.
\]

Since \( \delta + 1 > \frac{T}{2} \), there is no overlap between the \( X_i \) and \( Y_j \) variables involved for the arrangements of marks in each triangle. Moreover, since the size of the triangles is the same, the mark arrangements have the same distribution. Again, \( |\Delta_{T,\delta}^{\text{up}}|, |\Delta_{T,\delta}^{\text{low}}| \) denote the number of marks in these triangles, and since the marks in the two triangles are defined by disjoint \( X, Y \) marginals, they have the same distribution by symmetry. Keep in mind they are not independent since \( X \) and \( Y \) have correlated marginals anyway. Finally, note the relation
\[
|S_{T,\delta}| = n_Xn_Y - |\Delta_{T,\delta}^{\text{up}}| - |\Delta_{T,\delta}^{\text{low}}|.
\]

Using this relation we can begin the computation of the variance in this case. We have
\[
\text{Var}(|S_{T,\delta}|) = \text{Var}(n_Xn_Y - |\Delta_{T,\delta}^{\text{up}}| - |\Delta_{T,\delta}^{\text{low}}|)
\]
\[
= \text{Var}(|\Delta_{T,\delta}^{\text{up}}| + |\Delta_{T,\delta}^{\text{low}}|)
\]
\[
= \text{Var}(|\Delta_{T,\delta}^{\text{up}}|) + \text{Var}(|\Delta_{T,\delta}^{\text{low}}|) + 2\text{Cov}(|\Delta_{T,\delta}^{\text{up}}|, |\Delta_{T,\delta}^{\text{low}}|)
\]
\[
= 2\text{Var}(|\Delta_{T,\delta}^{\text{low}}|) + 2\text{Cov}(|\Delta_{T,\delta}^{\text{up}}|, |\Delta_{T,\delta}^{\text{low}}|). \quad (2.8)
\]
We first compute the variance of $|\Delta_{T,\delta}^{\text{low}}|$. We use the same calculation as in Case 1. The total number of integer pairs $(i, j)$ involved in this calculation is

$$B_{T,\delta} = \frac{(T - \delta - 1)(T - \delta)}{2}. \quad (2.9)$$

We utilise the indicator random variables once again, and proceed with the same methodology as for Case 1.

$$\text{Var}(|\Delta_{T,\delta}^{\text{low}}|) = \text{Cov} \left( \sum_{(i,j) \in \Delta_{T,\delta}^{\text{low}}} \mathbb{I}\{X_i = Y_j = 1\}, \sum_{(k,\ell) \in \Delta_{T,\delta}^{\text{low}}} \mathbb{I}\{X_k = Y_\ell = 1\} \right)$$

$$= \sum_{(i,j) \in \Delta_{T,\delta}^{\text{low}}} \sum_{(k,\ell) \in \Delta_{T,\delta}^{\text{low}}} \text{Cov} \left( \mathbb{I}\{X_i = 1\}\mathbb{I}\{Y_j = 1\}, \mathbb{I}\{X_k = 1\}\mathbb{I}\{Y_\ell = 1\} \right)$$

$$= \sum_{(i,j) \in \Delta_{T,\delta}^{\text{low}}} \sum_{(k,\ell) \in \Delta_{T,\delta}^{\text{low}}} \mathbb{E} \left( \mathbb{I}\{X_i = X_k = 1\}\mathbb{I}\{Y_j = Y_\ell = 1\} \right) - \mathbb{P}(X_i = 1)^2\mathbb{P}(Y_j = 1)^2B_{T,\delta}^2.$$

Once again, we have four cases (after shifting the point $(1,1)$ to be the first point in the low triangle) corresponding to the four different types of terms that appear in the double sum:

1. $(i, j) = (k, \ell)$: Then

$$\mathbb{P}\{X_i = X_k = 1\}\mathbb{P}\{Y_j = Y_\ell = 1\} = \mathbb{P}\{X_i = 1\}\mathbb{P}\{Y_j = 1\} = \frac{n_X n_Y}{T^2}.$$

There are $B_{T,\delta}$ terms like this in the double sum.

2. $i \neq k, j \neq \ell$. Then by (2.1)

$$\mathbb{P}\{X_i = X_k = 1\}\mathbb{P}\{Y_j = Y_\ell = 1\} = \frac{n_X (n_X - 1) n_Y (n_Y - 1)}{T(T-1)}.$$

Note that the above is non-negative. The number of terms that satisfy this condition is

$$N_2 = \sum_{i=1}^{T-\delta-1} \sum_{j=1}^{i} (B_{T,\delta} - i - (T - \delta - 1 - j))$$

$$= 1/6(T - \delta)(T - \delta - 1)(3B_{T,\delta} - 4T + 4\delta + 5). \quad (2.10)$$

3. $i = k, j \neq \ell$. Then by (2.1)

$$\mathbb{P}\{X_i = 1\}\mathbb{P}\{Y_j = Y_\ell = 1\} = \frac{n_X n_Y (n_Y - 1)}{T^2}.$$
The number of terms with this contribution is

\[ N_3 = \sum_{i=1}^{T-\delta-1} i(i-1) = \frac{B_{T,\delta}}{3}(2(T-\delta) - 1) - B_{T,\delta}. \]  

(2.11)

4. \( i \neq k; j = \ell \). Then by (2.1).

\[ \mathbb{P}\{X_i = X_k = 1\} \mathbb{P}\{Y_j = 1\} = \frac{n_X(n_X - 1) n_Y}{T(T-1)}. \]

The number of terms with this contribution is also \( N_3 \) by symmetry.

Overall the variance term in (2.8) is given by

\[ 2\text{Var}(|\Delta_{T,\delta}^{\text{low}}|) = 2\frac{n_X n_Y}{T^2}B_{T,\delta} + 2\frac{n_X(n_X - 1) n_Y(n_Y - 1)}{T(T-1)} N_2 \]

\[ + 2\frac{n_X n_Y(n_X + n_Y - 2)}{T^2(T-1)} N_3 - 2\frac{n_X^2 n_Y^2}{T^4} B_{T,\delta}^2, \]

(2.12)

where \( B_{T,\delta} \) is given by (2.9), \( N_2 \) by (2.10) and \( N_3 \) by (2.11).

Now for the covariance term in (2.8):

\[ \text{Cov}\left( \sum_{(i,j) \in \Delta_{T,\delta}^{\text{low}}} \mathbb{1}\{X_i = Y_j = 1\}, \sum_{(k,\ell) \in \Delta_{T,\delta}^{\text{up}}} \mathbb{1}\{X_k = Y_\ell = 1\} \right) \]

\[ = \sum_{(i,j) \in \Delta_{T,\delta}^{\text{low}}} \sum_{(k,\ell) \in \Delta_{T,\delta}^{\text{up}}} \text{Cov}\left( \mathbb{1}\{X_i = 1\} \mathbb{1}\{Y_j = 1\}, \mathbb{1}\{X_k = 1\} \mathbb{1}\{Y_\ell = 1\} \right) \]

\[ = \sum_{(i,j) \in \Delta_{T,\delta}^{\text{low}}} \sum_{(k,\ell) \in \Delta_{T,\delta}^{\text{up}}} \mathbb{E}\left( \mathbb{1}\{X_i = X_k = 1\} \mathbb{1}\{Y_j = Y_\ell = 1\} \right) - \mathbb{P}(X_i = 1)^2 \mathbb{P}(Y_j = 1)^2 B_{T,\delta}^2 \]

\[ = \left( \frac{n_X(n_X - 1) n_Y(n_Y - 1)}{T(T-1)} - \frac{n_X^2 n_Y^2}{T^4} \right) B_{T,\delta}^2. \]

(2.13)

Observe that the above is negative, as it should.

Then finally, substitute everything back in (2.8), to obtain

\[ \text{Var}(|S_{T,\delta}|) = 2\text{Var}(|\Delta_{T,\delta}^{\text{low}}|) + 2\text{Cov}(|\Delta_{T,\delta}^{\text{up}}|, |\Delta_{T,\delta}^{\text{low}}|) = \text{(2.12)} + 2 \times \text{(2.13)}. \]

(2.14)

We summarise the results we just proved above in the following proposition, for a quick reference.
Proposition 2.1. Fix a horizon $T$, integers $n_X$ and $n_Y$ and a lag $\delta$. Let $|S_{T,\delta}|$ denote the number of marks in the $\delta$-band $D_{T,\delta}$, defined in equation (2.2). Then the mean and variance of $|S_{T,\delta}|$ can be explicitly computed and are given by:

$$E(|S_{T,\delta}|) = n_X n_Y \left(1 - \left(1 - \frac{\delta}{T}\right)^2\right) + n_X n_Y \frac{T - \delta}{T^2} = \text{Equation (2.3)},$$

and

$$\text{Var}(|S_{T,\delta}|) = \begin{cases} \text{Equation (2.7)}, & \text{if } \delta + 1 \leq T/2, \\
\text{Equation (2.14)}, & \text{if } \delta + 1 > T/2. \end{cases}$$

2.3 Central limit theorem for number of marks in $\delta$-band as $T \to \infty$

For the purposes of this section, fix parameters $p_X$ and $p_Y$ to denote the probability of success for two independent Bernoulli sequences $\{X_i\}_{i \geq 1}$ and $\{Y_j\}_{j \geq 1}$ so that each sequence is i.i.d. with marginal distributions $X_i \sim \text{Ber}(p_X)$ and $Y_j \sim \text{Ber}(p_Y)$.

For any $T \in \mathbb{N}$, the random variable under consideration is still

$$\tilde{S}_{T,\delta} = \{(i,j) \in [1,T]^2 : |i-j| \leq \delta, X_i = Y_j = 1\}. \quad (2.15)$$

The cardinality $|\tilde{S}_{T,\delta}|$ gives the total number of marks in a band of size $\delta$ around the main diagonal, and using an identical calculation as for (2.3), we obtain

$$E(|\tilde{S}_{T,\delta}|) = p_X p_Y \left(T - (T - \delta)^2\right) + p_X p_Y (T - \delta). \quad (2.16)$$

Notice that (2.16) and (2.3) coincide when $p_X = n_X T^{-1}$, and $p_Y = n_Y T^{-1}$.

We now present a different way to count the marks in the band, that will lend itself into the application of an ergodic CLT. For any $\delta + 1 \leq i \leq T - \delta - 1$ consider the vector random variable of dimension $d = 2\delta + 1$

$$Y_i = [Y_{i-\delta}, Y_{i-\delta+1}, \ldots, Y_{i+\delta-1}, Y_{i+\delta}].$$

Then the total number of marks between $\delta + 1 \leq i \leq T - \delta - 1$ is given by

$$L_{T,\delta} = \sum_{i=\delta+1}^{T-\delta-1} X_i (Y_i \cdot 1_d), \quad (2.17)$$

where $1_d$ is the $d$-dimensional vector $(1, 1 \ldots 1)$. For each $i$, a direct calculation gives that

$$E(X_i(Y_i \cdot 1_d)) = E(X_i) E(Y_i \cdot 1_d) = p_X p_Y (2\delta + 1).$$

Define $W_i = X_i(Y_i \cdot 1_d)$ for notational convenience.
Before stating the theorem, two observations follow. First we have the immediate inequality

\[ L_{T,\delta} \leq S_{T,\delta} \leq L_{T,\delta} + (\delta + 1)^2. \]

Thus, as \( T \to \infty \), asymptotically \( L_{T,\delta} \sim S_{T,\delta} \). The same inequality holds for expectations. Scaled by the same quantity, both variables satisfy the same limiting law as long as \( \delta \) does not depend on \( T \). If \( \delta \) depends on \( T \), one needs more refined arguments to show limiting results, and they will depend on this relation as well.

Second, notice that the random numbers \( \{W_i\}_i \) have the same distribution for each fixed \( i \) and they are stationary and ergodic. Moreover, as long as \( |i-k| > 2\delta + 1 \), variables \( W_i = X_i(Y_i \cdot 1_d) \) and \( W_k = X_k(Y_k \cdot 1_d) \) are completely independent. Therefore we have just defined a stationary ergodic sequence that is \( 2\delta + 1 \)-dependent only. In particular this implies that the stationary sequence is **strongly mixing**.

We introduce some notation. First, let

\[ Z_i = X_i(Y_i \cdot 1_d) - p_X p_Y (2\delta + 1) = W_i - \mathbb{E}(W_i). \]

Therefore \( \mathbb{E}(Z_i) = 0 \). Also note that \( \mathbb{E}(Z_i^2) < \infty \) for any \( \ell \) (high moment hypotheses are crucial for CLT for ergodic sequences). Then define (the value does not depend on the index \( \ell \) below as long as \( \ell \geq \delta + 1 \))

\[ \sigma_\delta^2 = \mathbb{E}(Z_\ell^2) + 2 \sum_{k=1}^{2\delta+1} \mathbb{E}(Z_\ell Z_{\ell+k}) = \text{Var}(W_\ell) + 2 \sum_{k=1}^{2\delta+1} \text{Cov}(W_\ell, W_{\ell+k}). \quad (2.18) \]

**Lemma 2.2.** The constant \( \sigma_\delta^2 \) in (2.18) is given by

\[ \sigma_\delta^2 = (2\delta + 1)p_X p_Y (1 - p_X p_Y + 2\delta p_Y (1 - p_X)) + 2p_X^2 p_Y (1 - p_Y) \frac{2\delta (2\delta + 1) (4\delta + 1)}{6}. \quad (2.19) \]

**Proof.** We are going to compute each of the terms appearing in the last expression of (2.18). First,

\[
\text{Var}(W_\ell) = \text{Cov}(W_\ell, W_\ell) = \text{Cov} \left( \sum_{j=-\delta}^{\delta} X_j Y_{\ell+j}, \sum_{k=-\delta}^{\delta} X_k Y_{\ell+k} \right) = \sum_{j=-\delta}^{\delta} \sum_{k=-\delta}^{\delta} \text{Cov}(X_j Y_{\ell+j}, X_k Y_{\ell+k})
\]

\[
= \sum_{j=-\delta}^{\delta} \sum_{k=-\delta}^{\delta} \left( \mathbb{E}(X_j^2 Y_{\ell+j} Y_{\ell+k}) - \mathbb{E}(X_j Y_{\ell+j}) \mathbb{E}(X_k Y_{\ell+k}) \right)
\]

\[
= \sum_{i=-\delta}^{\delta} \left( \mathbb{E}(X_i^2 Y_{\ell+i}^2) - p_X^2 p_Y^2 \right) + \sum_{j=-\delta}^{\delta} \sum_{k=-\delta, k\neq j}^{\delta} \left( \mathbb{E}(X_j^2 Y_{\ell+j} Y_{\ell+k}) - p_X^2 p_Y^2 \right)
\]

\[
= \sum_{i=-\delta}^{\delta} \left( p_X p_Y - p_X^2 p_Y^2 \right) + \sum_{j=-\delta}^{\delta} \sum_{k=-\delta, k\neq j}^{\delta} \left( \mathbb{E}(X_j^2 Y_{\ell+j} Y_{\ell+k}) - p_X^2 p_Y^2 \right)
\]
Proof.

There exists a constant \( \sigma \) such that

\[
\sum_{i=-\delta}^{\delta} (p_x p_Y - p_x^2 p_Y^2) + \sum_{j=-\delta}^{\delta} \sum_{k=-\delta, k\neq j}^{\delta} (p_x p_Y^2 - p_x^2 p_Y) = (2\delta + 1)p_x p_Y (1 - p_x p_Y + 2\delta p_Y (1 - p_x)).
\]

The covariance for the other terms in (2.18) is computed in a similar way. First note that the variables common between \( W_\ell \) and \( W_{\ell+k} \) are the \( Y_{\ell+j} \) for which \( -\delta + k \leq j \leq \delta \) which precisely equal the values for \( Y_{\ell+k+i} \) when \( -\delta \leq i \leq \delta - k \). Then

\[
\text{Cov}(W_\ell, W_{\ell+k}) = \text{Cov} \left( \sum_{j=-\delta}^{\delta} X_\ell Y_{\ell+j}, \sum_{i=-\delta}^{\delta} X_{\ell+k} Y_{\ell+k+i} \right) = \text{Cov} \left( \sum_{j=-\delta+k}^{\delta} X_{\ell+j}, \sum_{i=-\delta}^{\delta} X_{\ell+k} Y_{\ell+k+i} \right) = \sum_{j=-\delta+k}^{\delta} \sum_{j=-\delta+k}^{\delta} \text{Cov} \left( X_{\ell+j}, X_{\ell+k} Y_{\ell+j} \right) = \sum_{j=-\delta+k}^{\delta} \sum_{j=-\delta+k}^{\delta} \mathbb{E}(X_{\ell+j}) \mathbb{E}(X_{\ell+k}) \mathbb{E}(Y_{\ell+j})^2 - \mathbb{E}(X_{\ell+j}) \mathbb{E}(X_{\ell+k}) \mathbb{E}(Y_{\ell+j})^2 = (2\delta - k + 1)^2 p_x^2 p_Y (1 - p_Y).
\]

Then, overall

\[
\sigma_3^2 = (2\delta + 1)p_x p_Y (1 - p_x p_Y + 2\delta p_Y (1 - p_x)) + 2p_x^2 p_Y (1 - p_Y) \sum_{k=1}^{2\delta + 1} (2\delta - k + 1)^2 = (2\delta + 1)p_x p_Y (1 - p_x p_Y + 2\delta p_Y (1 - p_x)) + 2p_x^2 p_Y (1 - p_Y) \frac{2\delta(2\delta + 1)(4\delta + 1)}{6}.
\]

Then we can directly apply the central limit theorem for strongly mixing variables and obtain the following

**Theorem 2.3.** Let \( S_{T, \delta} \) be the number of marks as defined by (2.15). Then, (with the notation introduced above), the following limit holds in distribution

\[
\lim_{T \to \infty} \frac{|S_{T, \delta} \mid - \mathbb{E}(|S_{T, \delta}|)}{\sqrt{T}} \equiv Z \sim \mathcal{N}(0, \sigma_3^2),
\]

where \( \sigma_3^2 \) is given by (2.19).

**Proof.** There exists a constant \( C = C(\delta, p_x, p_y) < \infty \) so that

\[
\left| \frac{|S_{T, \delta}| - \mathbb{E}(|S_{T, \delta}|)}{\sqrt{T}} - \frac{L_{T, \delta} - \mathbb{E}(L_{T, \delta})}{\sqrt{T}} \right| \leq C/\sqrt{T}.
\]
This is a \( \mathbb{P} \)-a.s. statement, so as long as \( T \to \infty \) the two fractions have the same distributional limit. Now focus on

\[
\frac{L_{T,\delta} - \mathbb{E}(L_{T,\delta})}{\sqrt{T}} = \frac{\sum_{\delta+1}^{T-\delta-1} W_i - (T - 2\delta - 2)\mathbb{E}(W)}{\sqrt{T}} = \frac{\sum_{\delta+1}^{T-\delta-1} Z_i}{\sqrt{T}} = \frac{\sqrt{T - 2\delta - 1}}{\sqrt{T}} \cdot \frac{\sum_{\delta+1}^{T-\delta-1} Z_i}{\sqrt{T - 2\delta - 2}}.
\]

Now, as \( T \to \infty \), the fraction multiplying the variable tends to 1 while the second fraction converges weakly to \( \mathcal{N}(0, \sigma^2) \) by the CLT for strongly mixing ergodic sequences. \( \square \)

3 Results

3.1 Agreement between analytical and empirical estimates

Given the strong dependence of the analytical expressions on the product of the rates \( n_X n_Y \) and the density of events \( n_X n_Y / T^2 \), we examined the agreement between analytical and empirical estimates for i.i.d. uniform point processes along two dimensions – event density and rate homogeneity – considering both average and limit cases. Concretely, we considered 4 pairs of point processes emitting respectively \( (1,1), (1,500), (500,500), ([500^{\frac{1}{2}}],[500^{\frac{3}{4}}]) \) events uniformly distributed within \( [1;T] \), \( T=1,000 \). As shown by Figure 1, agreement between analytical and empirical estimates is excellent across all scenarios, even in the most extreme case of spurious correlation where each node only emits 1 event in the period. It should be noted that whereas the expected value of the statistic is insensitive to rate inhomogeneity, i.e., the expected value for one pair of point processes emitting \( (1,100) \) events and that of a pair emitting \( (10,10) \) events will be identical, the variance is not. In general (i.e., provided none of the rates are exactly \( T/2 \)), for a given product of rates, the more heterogeneous the rates, the higher the standard deviation.

Next, we investigated the extent to which agreement between analytical and empirical estimates holds when considering point processes that depart from the assumptions underlying our analytical derivation. First, we considered the case of sampled Poisson processes, which is consistent with most real-world scenarios, e.g., in physiology. Next, we introduced auto-correlations in the Poisson processes as this has been shown to induce biases when measuring statistical dependence between point processes [28].

3.1.1 Impact of sampling frequency

For a given time horizon and rates, homogeneous Poisson processes will emit events that satisfy our assumptions provided the sampling frequency is large enough and therefore we should expect excellent agreement between analytical and empirical estimates. However, if the sampling frequency is reduced (or put differently, the rates increase in relation to the sampling frequency), then the likelihood that two events fall within the same time bin increases. For a Poisson process of rate \( \lambda \) the probability that there
Figure 1: Analytical (solid lines) and empirical (dotted lines) expected value (left) and standard deviation (right) as a function of the lag for the four scenarios given in the legend. For ease of presentation, both quantities were normalised by $n_Xn_Y$ and $\sqrt{n_Xn_Y}$ respectively. As expected from Eq. 2.3, this leads to all analytical estimates of the expected value being identical. Only three distinct curves can be seen in the right panel. The standard deviations for scenarios (500,1) and (500,500) are almost identical due to one of the rates being exactly $T/2$. Error bars were computed over 50 sets of 10,000 pairs of point processes.

are $n$ events within any bin of duration $b$ is:

$$P(N(t+b) - N(t) = n) = \frac{(\lambda b)^n}{n!} e^{-\lambda b}$$ (3.1)

And therefore, the expected number of events that are 'lost' to binning over the given time horizon $T$ will be:

$$\mathbb{E}(N_L) = \frac{T}{b} \sum_{n=2}^{\infty} (n-1) \frac{(\lambda b)^n}{n!} e^{-\lambda b} = \lambda T - \frac{T}{b} (1 - e^{-\lambda b})$$ (3.2)

This shows that for a given time horizon $T$ and known number of events, our statistic should be more accurate if the sampling frequency is high (i.e., the bin size $b$ is small). For a fixed time horizon $T = 1000$, and a number of rates $\lambda \in \{0.01, 0.02, 0.05, 0.1, 0.2\}$, we systematically investigated agreement when the fraction of events lost to binning varied between 0.05 and 0.5. For the sake of brevity, we only considered processes with homogeneous rates. To prevent fluctuations in the number of events produced by each process over the time horizon $T$ (for a given rate) being a confound in the analysis, we ensured that only realisations with an identical number of events were included. Agreement was quantified in terms of dimensionless estimation errors of the expected value and standard deviation of the statistic. As shown by Figure 2, both estimation errors increase with the proportion of events lost to binning. Across all rates, these errors remain small even when up to one third of the events are lost. This is not unexpected given that in a Poisson process about half of the events occur will tend to have a waiting time greater
or equal than the mean waiting time and that the distribution of these (surviving) events will satisfy the assumptions of the analytical result. See Appendix A for a proper explanation. The degradation in performance shows a clear dependence to the rates themselves, with lower rates leading to higher errors, and, importantly, higher variance of these errors. This is merely a result of the number of bins becoming too small for the asymptotic properties mentioned in Appendix A to hold. For example, for the rate $\lambda = 0.01$, an event loss of 0.5 occurs when there are only 6 bins. For most real-world applications, however, we can expect our statistic to be fairly robust to the choice of sampling frequency.

![Figure 2](image-url)  

**Figure 2**: Average error in the estimation of the standard deviation as a function of the proportion of events lost to binning for the 5 rates shown in the legend. The shaded areas denote the standard deviations of this average error and were calculated over 50 sets of 10,000 estimations.

### 3.1.2 Auto-correlation

We used INAR(1) processes to generate auto-correlated waiting times for the events of the point processes. The INAR(1) process is defined as:

$$X_t = \alpha \circ X_{t-1} + \epsilon_t$$  \hspace{1cm} (3.3)

where $X$ is an integer variable, $\epsilon_t$ is a sequence of uncorrelated non-negative integer valued random variables and the $\circ$ operator is defined as:

$$\alpha \circ X = \sum_{i=0}^{X} Y_i$$  \hspace{1cm} (3.4)

where $Y_i \sim Ber(\alpha)$ are iid variables and $\alpha \in [0; 1]$.

Put simply, this model states that the elements at time $t$, $X_t$, are the sum of (1) the survivors at time $t-1$, $X_{t-1}$, each with probability of survival $\alpha$ and of (2) elements from the innovation process $\epsilon_t$. For any non negative integer $k$, the auto-covariance at lag $k$ is given by $\gamma(k) = Cov(X_t, X_{t-k}) = \alpha^k \gamma(0)$, which
means that the amount of auto-correlation is controlled by the value of the parameter $\alpha$ (with $\alpha$ being the auto-correlation at lag 1 in particular).

Figure 3: Top row: Average error in the estimation of the expected value (left panel) and standard deviation (right panel) as a function of $\alpha$, the auto-correlation at lag 1 for the 5 rates shown in the legend. The shaded areas denote the standard deviations of this average error and were calculated over 50 sets of 10,000 estimations. Bottom row: Analytical and empirical standard deviations for the highest-error case ($\lambda = 0.005$) and $\alpha = 0.2$ (arbitrarily) as a function of the lag. All processes emitted exactly 4 events (the expected value for this scenario). The error bars are computed over 50 sets of 10,000 estimates.

To quantify the effect of auto-correlations on the agreement between analytical and empirical estimates, we generated point processes systematically varying $\alpha$ (thus controlling the auto-correlation at lag 1) for processes emitting $n$ events with $n \in \{5, 10, 20, 50, 100\}$ within $[1; T]$, $T=1,000$. Agreement was once again quantified in terms of the previously used dimensionless estimation error terms. As shown by Figure 3 (top row), the error in both expected value and standard deviation increases with the degree of auto-correlation, as expected from the fact that the data increasingly depart from the assumption of independence underlying the derivation of our statistic. Estimation errors increase as the rates decrease.
This can be explained as follows. For a fixed horizon $T$, the larger the rate, the more constrained the distribution of inter-event intervals is. Auto-correlations introduce structure in the sequencing of the inter-event intervals which exacerbates the covariance term $\frac{\lambda}{2}$. As a result, empirical estimates show less variability around the mean than in the un-correlated case. In any case, estimation errors remain low for all rates. As shown by the bottom row of Figure 3, even in the worst case scenario of $\lambda = 0.005$ (where the expected number of events observed is 4), there is still good agreement between analytical and empirical estimates for both expected value and standard deviation. Importantly, it is observed (in this scenario and all other scenarios that we examined) that the analytical standard deviation overestimates the empirical standard deviation at every lag (again due to how auto-correlations constrain the distribution of inter-event intervals). This observation has a positive implication on the applicability of our statistic to real-world events. Indeed, the analytical standard deviation can be seen as a high bound for the empirical standard deviation which makes any inference based on it less prone to yield false positives.

3.2 Normal convergence of the empirical estimates

In Section 2.3, we have shown that in the limit of large horizon times $T$, the distribution of our statistic converges to a normal distribution with parameters the analytical results provided in (2.3) for the expected value and (2.7) and (2.14) for the standard deviation. By way of empirical validation, we systematically varied rates and lags and showed that there was always a data duration after which the distribution of the empirical estimates could be said to be normally distributed based on the evidence of a KS test with $p > .05$ (with 1000 estimates). Figure 4 shows those times for the rates and lags considered. The range of lags was specified in terms of fraction of the maximal possible duration for the data, $T_{\text{max}} = 10,000$ (which was assumed to be long enough for asymptotic properties to hold in most cases), specifically, $\tau \in \{\frac{T_{\text{max}}}{100}, \frac{2T_{\text{max}}}{100}, \ldots, \frac{T_{\text{max}}}{2}\}$. Since the expected value is only dependent on the product of the rates, we used homogeneous rates, $\lambda_x = \lambda_y = \frac{n}{T} \in \{0.01, 0.02, \ldots, 0.5\}$. Focusing on homogeneous rates is justified by the fact that convergence is only marginally affected by heterogeneity, as demonstrated by panel D. The gradual convergence of the empirical distributions to a normal distribution is illustrated in panels B and C when the lag and recording duration increase, all other factors being kept constant.

Empirical convergence to a normal distribution is significantly impacted by the fact that our statistic takes integer values (it is a count of co-occurrent events). First, normal approximation can only be achieved if there is a wide range of values the statistic can take. This is clearly evidenced by panels B and C whereby the p-value increases (the distribution becomes more normal) as recording duration (respectively lag) increases leading to a wider range of values. Using kernel density estimation (green curve) provides a smoother estimation of the density which matches closely the analytical one (red), see $T = 200$ or $\tau = 10$ for example, however, the empirical distribution is step-like and therefore not assessed as being normal. Second, if the statistic at a lag is close to 0 or close to its maximum value ($n_x n_y$), the integer-valued nature of the statistic will once again prevent the empirical distribution to be approximated as normal. For example, the distribution of the statistic in the first plot in panel B is not symmetric because of its lower bound. This was previously noted by [26]. More generally, this issue is illustrated by the top row of Figure 4 whereby estimates for both low rate and high rate processes require longer durations for the
Figure 4: **A:** Minimal recording duration after which the empirical distribution is approximately normal as a function of rates and lags (left). Zoom on low lags (right). **B:** Empirical distributions as a function of the lag for given total duration and rates. **C:** Empirical distributions as a function of the total duration for given lag and rates. **D:** Empirical distributions as a function of rate heterogeneity for given total duration, lags and rate products. All panels: (red) the theoretical distribution for the scenario; (green) the kernel density estimation of the empirical distribution.
empirical distributions to become approximately normal. For all rates, it can be seen that the minimal duration required for the empirical distribution to be approximately normal first decreases with the lag (this is particularly acute for low lag and low rate, see right panel in top row), then increases with the lag since it must be at least as large as the lag considered.

The asymptotic convergence of the statistic around its expected value justifies the construction and use of a Z-score to measure deviation from the assumption of independence. For two time series \(X\) and \(Y\) emitting \(n_X\) and \(n_Y\) events within \([1; T]\), it is defined as:

\[
Z_{X,Y}(\tau) = \frac{|S_{T,\tau}| - \mathbb{E}(|S_{T,\tau}|)}{\sigma_{\delta} \sqrt{T}}
\]

(3.5)

where \(|S_{T,\tau}|\) is the statistic at lag \(\tau\), \(\mathbb{E}(|S_{T,\tau}|)\) is the analytical expected value (2.3) and \(\sigma_{\delta}\) the analytical standard deviation (2.18) where we are estimating parameters \(p_X, p_Y\) with their empirical estimators \(\frac{n_X}{T}\) and \(\frac{n_Y}{T}\) respectively.

### 3.3 Application to a delayed version of the Common Shock Model

#### 3.3.1 Description of model

The bivariate Poisson Common Shock Model provides a flexible platform to validate the effectiveness of our statistic in quantifying pairwise coupling between two point processes. Indeed it is often used as a benchmark for bivariate counts [13]. In its classic form, and following [7], it takes two Poisson point processes \(X_1 \sim \mathcal{P}(\lambda_{X_1})\) and \(X_2 \sim \mathcal{P}(\lambda_{X_2})\) and assumes that each of them can be written as the sum of two independent Poisson processes respectively \(Z\) and \(Y_1\), and \(Z\) and \(Y_2\), where \(Z \sim \mathcal{P}(\lambda_Z)\), \(Y_1 \sim \mathcal{P}(\lambda_{X_1} - \lambda_Z)\) and \(Y_2 \sim \mathcal{P}(\lambda_{X_2} - \lambda_Z)\). Whilst the choice of rates \(\lambda_Z\), \(\lambda_{Y_1}\) and \(\lambda_{Y_2}\) enables to adjust the strength of the interaction, in this form, the model only allows for instantaneous interaction. Here, we use the delayed version [9] such that, instead of \(Z\), we consider two processes \(Z_1\) and \(Z_2\) that are delayed versions of \(Z\), with delays taken from some probability distribution function \(F_i(\cdot), i \in \{1, 2\}\) (see Figure 5 for a graphical representation). This process has a number of possible interpretations but a very prominent one in the neurophysiology literature is the notion of common drive [32]. Since \(Z\) is not actually observed and we are not interested in recovering it, without loss of generality, it is possible to consider that only one of the process, e.g. \(Z_2\), is a delayed version of \(Z\), the other, \(Z_1\) being equal to \(Z\). We thus denote \(Z^*\) the delayed version of \(Z\). To model the delay we used a normal distribution of parameters \(\mu_\delta\) and \(\sigma_\delta\), allowing us to control both average lag between events from \(Z\) and \(Z^*\) and lag jitter.

#### 3.3.2 Brief validation of the statistic

Since previous results from Section 3.1.1 showed excellent agreement between empirical and analytical estimates of the expected value and standard deviation of the statistic for independent Poisson processes, application of the Z-score to the delayed common shock model in the no-interaction case (i.e., \(\lambda_z = 0\)) should lead to a Z-score taking an expected value of 0 and a standard deviation of 1. This behaviour is confirmed by the left panel of Figure 6 with consistent empirical estimates (expected value and standard
As representative example of the delayed interaction case, we considered the case where all 3 processes had the same rate $\lambda = 0.01$, an average lag $\mu_\delta = 100$ and jitter $\sigma_\delta = 10$ (chosen to be significantly smaller than the average lag, as expected in most real-world scenarios). The behaviour of the Z-score is shown in right panel of Figure 6. At first sight, the interpretation of this behaviour is not straightforward. First, it is observed that whilst the Z-score for value of lags $\tau \ll \mu_\delta$ is close to zero, it is not zero. This is a direct implication of the delay introduced by the model. The analytical expected value of the statistic provides the expected number of co-incident events at each lag conditional to the processes emitting a given number of events over the time horizon $T$. With delayed interaction, and depending on the relationship between the delay and the rate of the processes, the actual number of co-incident events observed for lags less than the delay will diverge from that expected at random. In this case, with each process only emitting (on average) 10 events (i.e., 1 every 100 time steps), a delay of 100 is likely to reduce the likelihood of co-occurrence at very small lags. The Z-score then shows a steady increase peaking at around $\mu_\delta + 1.5\sigma_\delta$ before steadily decreasing. The fact that the Z-score does not peak at the average lag but later is a natural result of the cumulated nature of our statistic. To illustrate this, we calculated the derivative of the Z-score (orange line) as a proxy for the cross-correlation at each lag and superimposed the distribution of delays used in the model (green line). As the lag approaches the mean lag, the number of co-incident events begins to exceed that expected at random and therefore, the Z-score increases. The rate of increase is maximum where the distribution of delays peaks. As the lag increases from the mean delay, the ratio between the number of observed co-incident events to the number of co-incident events expected at random reduces and with it the Z-score. Once again, because the Z-score is a cumulative statistic (measuring the number of co-incident events up to the lag considered), the statistic will not return to 0. In other words, the Z-score at high lags takes into account the fact there has been substantial interaction at smaller lags. There are two implications to this. First, the Z-score will not show great sensitivity to interactions that have large (in relation to the duration of the record) average lags. Second, a positive value of the Z-score at a given lag does not indicate that significant interaction is actually taking place at that lag. Our results are based on calculating the statistic at each of all lags considered. Note that the alternative approach of extracting the peak value of the statistic over those lags for each realisation would not permit comparison to a $\mathcal{N}(0, 1)$

![Figure 5: The Delayed Common Shock Model](image)
distribution as noted in [19] for cross-correlation analysis. For such comparison to be made, one needs to look at each lag. However, given the cumulated nature of the Z-score, any substantial interaction will lead to significant Z-score values over a wide range of lags (greater than the mean delay) such that any prior knowledge and/or hypothesis as to the nature of the delay (e.g., conduction velocity in neurophysiology) could be usefully exploited.

A comprehensive sensitivity analysis of the statistic to the various parameters of the delayed common shock model will be provided in a forthcoming update of this manuscript, along with a comparison with coherence, a state of the art, frequency-domain measure of coupling.

4 Discussion

In this paper, we have proposed a statistic that enables the robust assessment of the presence of (instantaneous or delayed) statistical dependence between two point processes. This statistic extends rigorous work on Pearson’s correlation [18] by providing rigorous results when considering interaction over multiple lags. The identification of statistical dependence is based on an exact derivation of the number (and standard deviation) of co-incident events expected to be observed over a time period given two discrete independent point processes. This statistic depends only on parameters that can be readily estimated by counting the number of events observed over a given period of time and is valid for every value of those parameters, even extreme cases (e.g., only one event over the entire record), unlike its frequency-domain counterpart, coherence. A central limit theorem for this number was derived that demonstrates convergence of the distribution of the statistic to a normal distribution of known parameters and permits the construction of a Z-score quantifying the likelihood of a given number of co-incident events in a time period being observed if the processes were independent.
Agreement between analytical and empirical values was verified and the statistic was shown to be well behaved even when departing from the assumptions of the model. For example, it was shown to be robust to sampling (in most reasonable cases) as well as when auto-correlated point processes were considered (within limit).

The statistic has two main limitations. First, from an interpretation viewpoint, it is somewhat less intuitive than other methods due to its cumulative nature. A significant Z-score at a particular lag does not imply significant underlying dependence at that lag, merely, that there was dependence at one or many shorter lags. This means some care must be taken when selecting the lag(s) over which to make an inference. Nevertheless, we have shown that in some cases, it will be possible to make inference about the nature of the interaction through differencing of the method. Second, in its basic form, and like most such techniques, the statistic assumes stationarity of the signals. Adaptations of the statistic to handle non-stationarity (e.g., via optimal filtering or windowing) is left for future work.

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A The case of Bins for Poisson processes

Suppose that $N_X$, $N_Y$ are independent Poisson processes with rates $\lambda_X$, and $\lambda_Y$ respectively. We will discretise these processes using bins of size $b > 0$, with a time horizon $T$.

Partition the time interval $[0, T]$ into disjoint intervals of size $b$. In other words, interval $k$ (denoted by $I_k$) is the interval $[(k-1)b, kb)$, for $1 \leq k \leq \lceil Tb^{-1} \rceil$. The Poisson processes are independent in each interval, so we define a sequence of independent Bernoulli variables

$$X_i = \mathbb{1}\{N_X((i-1)b, ib]) \geq 1\}, \quad \text{and} \quad Y_j = \mathbb{1}\{N_Y((i-1)b, ib]) \geq 1\} \quad (A.1)$$

Their respective probabilities of success are

$$p_X = 1 - e^{-\lambda_X b} \quad \text{and} \quad p_Y = 1 - e^{-\lambda_Y b}. \quad (A.2)$$

Therefore, using the bin size $b$, we reduce this to our discrete model, with discrete time horizon $T_d(b) = \lceil Tb^{-1} \rceil$ and success probabilities for the Bernoulli variables. The lag value $\delta$ from the continuous model is also scaled by the bin size; since it needs to be an integer, by convention we take it to be $\delta_d(b) = \lfloor \delta b^{-1} \rfloor$.

Using this information, the expected number of marks will be $\mathbb{E}(|\tilde{S}_{T_d(b), \delta_d(b)}|)$ and the variance needed for the CLT will be of leading order $\sigma_\delta^2 T_d(b)$.

Here is one way that can act as a good approximation to find the correct size $b$: Suppose for convenience that $\lambda_X \geq \lambda_Y$. The probability $p_{\lambda_X, \delta}$ of seeing 2 or more Poisson points for $N_X$ in $I_k$ is

$$p_{\lambda_X, \delta} = 1 - e^{-\lambda_X b} (1 + \lambda_X b) \leq 1 - (1 - \lambda_X b)(1 + \lambda b) = (\lambda_X b)^2.$$ 

Therefore, for this probability to be small, one needs $b \ll \lambda_X^{-1}$ as to not discard too many points.

If it furthermore can be arranged that $\lambda_X b \leq CT^{-\alpha}$ for some $\alpha > 1/2$, a Taylor expansion of the expected value of points lost because of binning shows that $\mathbb{E}(N_L) \ll \sqrt{T}$ which is not enough to alter the leading order of $\sigma_\delta^2 \sqrt{T}$ and $\mathbb{E}(|\tilde{S}_{T, \delta}|)$. In this case the Poisson model and the discrete model obtained by decreasing bin sizes will coincide.