Dissipation and criticality in the lowest Landau level of graphene

Xun Jia, Pallab Goswami, and Sudip Chakravarty

Department of Physics and Astronomy, University of California Los Angeles, Los Angeles, CA 90095-1547

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The lowest Landau level of graphene is studied numerically by considering a tight-binding Hamiltonian with disorder. The Hall conductance $\sigma_{xy}$ and the longitudinal conductance $\sigma_{xx}$ are computed. We demonstrate that bond disorder can produce a plateau-like feature centered at $\nu = 0$, while the longitudinal conductance is nonzero in the same region, reflecting a band of extended states between $\pm E_c$, whose magnitude depends on the disorder strength. The critical exponent corresponding to the localization length at the edges of this band is found to be $2.47 \pm 0.04$. When both bond disorder and a finite mass term exist the localization length exponent varies continuously between $\sim 1.0$ and $\sim 7/3$.

The quantum Hall effect (QHE), either integer or fractional, is one of the most intriguing phenomena in physics that has drawn, and still continues to draw, enormous attention even after two decades since its discovery. In graphene unconventional QHE corresponding to $\sigma_{xy} = \pm (n + 1/2)4(e^2/h) = \pm 4\nu(e^2/h)$, $n = 0, 1, 2, \ldots$, where $\sigma_{xy}$ is the Hall conductance, has been addressed in terms of low lying excitations akin to relativistic Dirac fermions: the factor of 4 arises from the two-fold valley and spin degeneracies; $e$ is the electronic charge and $h$ the Planck’s constant. Yet, a complete theory of QHE requires an understanding of the localization-delocalization transitions within the Landau levels, which reflects a very special quantum phase transition, especially in graphene with its low energy Dirac spectra. It is indeed paradoxical that such a precise quantization requires material defects and disorder, and even the nature of disorder seems to matter for graphene.

A remarkable recent discovery in graphene is, what appears to be, a $\nu = 0$ plateau in a sufficiently large magnetic field, although the plateau in the measured $\rho_{xy}$ is difficult to decipher. The longitudinal resistivity $\rho_{xx}$, on the other hand, is observed to be nonzero ($\gtrsim h/e^2$) in the region of the plateau, in sharp contrast to the non-dissipative behavior of $\rho_{xx}$ in conventional QHE, crying out for a theoretical explanation. It is most peculiar because “a dissipative quantum Hall plateau” is an oxymoron, for, according to Laughlin, the precise quantization requires zero dissipation. An intriguing explanation of this paradoxical phenomenon is given in Ref. [10], where the removal of spin degeneracy plays a special role, and the nonzero longitudinal resistivity $\rho_{xx}$ is ascribed to a pair of gapless counter-propagating chiral edge modes carrying opposite spins, while a spin gap in the bulk protects the quantization of the Hall conductance. The role of interactions is crucial in this theory.

Surprisingly, in the present Letter we shall demonstrate by explicitly computing $\sigma_{xy}$, and the longitudinal conductance $\sigma_{xx}$, that the existence of a plateau-like feature at $\nu = 0$ and the non-zero $\rho_{xx}$ can be attributed to a single mechanism, an inter-valley coupling that is induced by bond disorder. Because $\sigma_{xy}$ is zero at $\nu = 0$, $\rho_{xx} = 1/\sigma_{xx}$ at the same point. The most remarkable feature is that there is a band of extended states centered at zero energy, whose extent depends on the strength of bond disorder. In contrast to Ref. [10], in our picture there is no quantization of $\sigma_{xy}$ at $\nu = 0$, only a sloping behavior as a function of energy. In addition, dissipation is not an edge phenomenon, but a bulk one.

Another remarkable aspect of the integer QHE in graphene is the possible existence of a continuously varying critical exponent of the divergence of the localization length within the Landau band for a class of disorder. We show: (1) If there is only bond disorder, the critical exponent is close to $\sim 7/3$, same as in the conventional integer QHE; (2) when a finite mass term is added in addition to bond disorder, the critical exponent depends on the ratio of the intensity of bond disorder to the finite mass, continuously varying from $\sim 1.0$ to $\sim 7/3$, as the system is tuned from weak to strong disorder, which correctly reaffirms the results of Ref. [2] obtained by an entirely different method; the present method is more powerful, because larger system sizes can be handled. A finite mass appears to be experimentally relevant. Although the removal of the nodal degeneracy in the lowest Landau level may be of many body origin, it can be approximately accounted for by including an explicit mass term in the Hamiltonian.

We study the tight-binding model of graphene subject to a constant perpendicular magnetic field and disorder, using real space transfer matrix and exact diagonalization methods, to explore the peculiar properties of the lowest Landau level discussed above. The Hamiltonian defined on a honeycomb lattice of dimension $L_x \times L_y$, shown in Fig. 4 is

$$H = \sum_{\mathbf{n}} \left[ (\epsilon_{\mathbf{n},A} + m)c^{\dagger}_{\mathbf{n},A}c_{\mathbf{n},A} + (\epsilon_{\mathbf{n},B} - m)c^{\dagger}_{\mathbf{n},B}c_{\mathbf{n},B} \right]$$

$$- \sum_{\mathbf{n}} \sum_{k=1}^{3} (t_{\mathbf{n},k} e^{i\alpha_{\mathbf{n},k}} c^{\dagger}_{\mathbf{n},A} c_{\mathbf{n}+\delta_k,B} + \text{h.c.}),$$

where the summation of $\mathbf{n}$ ranges over all unit cells, and
The spatially defined hopping elements connecting the slices $i$ and $i+1$, and $H_i$ is the Hamiltonian within the slice. All positive Lyapunov exponents of the transfer matrix $[13]$, $\gamma_1 > \gamma_2 > \ldots > \gamma_{2M}$, are computed by iterating Eq. (2) and performing frequent orthonormalizations. The convergence of this algorithm is guaranteed by the well known Osledec theorem [14]. The conductance per square, $\sigma_{xx}$, is given by the Landauer formula [15] [16] [17] [18] (note the special factor of $\sqrt{3}$ in the argument of cosh):

$$\sigma_{xx} = \frac{\hbar^2}{4} \sum_{i=1}^{2M} \frac{1}{\cosh^2 (2\sqrt{3}M \gamma_i)}. \quad (3)$$

The localization length in the quasi-1D system of width $L_y$ is given by $\lambda_M = 1/\gamma_{2M}$. Assuming single parameter scaling, $\lambda_M/M = f(|E - E_c|/M^{1/\nu})$, the data collapse yields the critical exponent $\nu$ and the critical energy $E_c$.

To compute the Hall conductance $\sigma_{xy}$, we impose periodic boundary conditions in both directions of the system. The Hamiltonian (1) is diagonalized to obtain a set of energy eigenvalues $E_\alpha$ and the corresponding set of eigenstates $|\alpha\rangle$ for $\alpha = 1, \ldots, 2M \times N$. Then $\sigma_{xy}$ is computed using the Kubo formula [12]:

$$\sigma_{xy}(E) = \frac{ie^2}{L_x L_y} \sum_{E_{\alpha} < E < E_\beta} \frac{(\alpha | v_x | \beta) (\beta | v_y | \alpha) - (x \leftrightarrow y)}{(E_\alpha - E_\beta)^2}, \quad (4)$$

where $v_x = [H, x]/ih$ is the velocity operator along the $x$ direction and similarly for $v_y$ in the $y$ direction. Note that the bonds in Fig. 1 that are not parallel to the $y$ direction contribute to both $v_x$ and $v_y$. The summation corresponds to sum over the states below and above the energy $E$. Finally, the expression is disorder averaged.

In the language of Dirac fermions [17] random hopping gives rise to both intranode and internode scattering between states on different sublattices. Intranode scattering appears as a random abelian gauge field, and the two inequivalent nodes have opposite charges corresponding to this gauge field. When projected to the lowest Landau level, the abelian gauge field leaves it unaffected. However, the internode scattering mixes the degenerate states corresponding to the two inequivalent nodes and produces extended states at $\pm E_c$. The existence of extended states at energies symmetric about $E = 0$ is the consequence of the sublattice symmetry of the disorder (often referred to as the chiral or the particle-hole symmetry). It is this special symmetry that leads to a divergent density of states and delocalized states at $E = 0$ [7] [19]. However, the calculated finite $\sigma_{xx}$ and a linear
FIG. 2: (Color online) The longitudinal conductance $\sigma_{xx}$ as a function of energy $E$. The magnetic flux through the hexagonal plaquette $\phi = 1/200$ and $g_T = 0.5$.

FIG. 3: (Color online) Longitudinal conductance $\sigma_{xx}$ at $E = 0$ as a function of the transverse size $M$ for $g_T = 0.5$ and $g_V = 0$. The inset shows $\sigma_{xx}(E = 0)$ as a function of $g_T$ and close to $e^2/\pi h$; the errors bars are about the size of the scatter in the numerical data. The near universality disappears if $g_V$ is added.

To the study of the critical behavior in the presence of bond disorder in the massless case, we set $g_T = 0.5$. The critical exponent is expected to be independent of the value $g_T$. The renormalized localization lengths $\lambda_M/M$ as a function of $E$ for various $M$ are plotted in the insert of Fig. 4(a). The critical energy $E_c$ is located at a non-zero value $E_c = 0.0167$ where $\lambda_M/M$ is independent of $M$. A successful data collapse based on the data with $E > E_c$ leads to a critical exponent of $\nu = 2.47 \pm 0.04$, close to conventional integer QHE. The

dependent of $M$ and $g_T$. This behavior of $\sigma_{xx}$ implies an unusual dissipative nature. We have also checked the existence of the dissipative behavior when $g_V \neq 0$ in addition.
scaling form is depicted in Fig. 4(a).

For the massive case, we vary bond disorder with a fixed \( m = 0.2 \) for the purpose of illustration. An example of data collapse is shown in Fig. 4(b) with \( g_T = 0.2 \).

The critical exponent \( \nu_t \approx 2.08 \pm 0.01 \) is different from conventional QHE; \( \nu_t \) varies continuously from \( 1.1 \pm 0.05 \) to \( 2.36 \pm 0.07 \) as the system is tuned from \( g_T = 0.02 \) to \( g_T = 1.5 \). This behavior agrees with the previous results [7]. When \( g_T \gg m \), the effect of the finite mass is negligible, and the critical exponent of \( \nu_t \approx 7/3 \) is recovered, as before.

![Graph showing the Hall conductance \( \sigma_{xy} \) as a function of energy \( E \) with bond disorder intensities \( g_T \). The \( \nu = 0 \) plateau emerges due to the bond disorders.](image)

**FIG. 5:** (Color online) Hall conductance \( \sigma_{xy} \) as a function of energy \( E \) with bond disorder intensities \( g_T \). The \( \nu = 0 \) plateau emerges due to the bond disorders.

In computing the energy dependence of \( \sigma_{xy} \) a relatively large flux \( \phi = 1/20 \) is chosen, because smaller values of flux involve many Landau bands in the diagonalization calculations, which are hard to track accurately. The chosen system size was \( N = M = 40 \), and an average over 1000 disorder realizations was performed. The results are shown in Fig. 5. Although there is not a strict plateau at \( \nu = 0 \), there is a break in the slope in the rise between \( \nu = -1 \) to \( \nu = 1 \), as the energy sweeps past the band center at \( E = 0 \). This can be construed as a plateau-like feature. Since the lowest Landau level splitting is expected to increase with \( g_T \), so is the extent of the region around \( E = 0 \) with a smaller slope.

The striking results here are the band of extended states in the region \(-E_c < E < E_c\) for bond disorder and the vindication of a continuously varying localization length exponent when, in addition, there is a finite uniform mass present in the Dirac spectrum. The situation is a bit more subtle, however. We had previously observed that the density of states diverges very weakly at \( E = 0 \) [7]. In particular, for the Lorentzian distribution of disorder this divergence was found to be exactly logarithmic. A \( \log^2 E \) divergence was predicted in Ref. [19] for Gaussian disorder corresponding to a Hamiltonian which in fact is formally identical when projected to the lowest Landau level. Such a weak divergence at \( E = 0 \) may give rise to fluctuations responsible for dissipation leading to a finite \( \sigma_{xy} \) [19]. The slightly sloping profile of \( \sigma_{xy} \) centered at \( E = 0 \) is harder to explain analytically. The dissipative behavior at \( \nu = 0 \) is consistent with experiments. However, in contrast to Ref. [10], this dissipative behavior is a bulk phenomenon, not an edge phenomenon. It is possible to test this experimentally by varying the aspect ratio of the sample. But we only have a sloping plateau-like feature of \( \sigma_{xy} \) at \( \nu = 0 \) unlike Ref. [10]. It is possible that the difference between the two pictures depends on the relative size of the spin splitting compared to the width of the extended band of states. Clearly further experimental and theoretical work would be very helpful to elucidate the precise nature of this exciting new development.

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