Stress distribution in elastic embankment using isogeometric analysis under Bézier extraction

Tan Nguyen i), Thirapong Pipatpongsa ii), Takafumi Kitaoka iii) and Hiroyasu Ohtsu iv)

i) Ph.D Student, Department of Urban Management, Kyoto University, Nishikyo-ku, Kyoto 615-8540, Japan.
ii) Associate Professor, Department of Urban Management, Kyoto University, Nishikyo-ku, Kyoto 615-8540, Japan.
iii) Assistant Professor, Department of Urban Management, Kyoto University, Nishikyo-ku, Kyoto 615-8540, Japan.
iv) Professor, Department of Urban Management, Kyoto University, Nishikyo-ku, Kyoto 615-8540, Japan.

ABSTRACT

This study expects to demonstrate that a) basal deformation and roughness have influence on stress distribution within embankments and b) implementing of Bézier Extraction in IsoGeometric Analysis (IGA) substantially improves the numerical robustness. Two-dimensional elastic solutions of embankment to which the prescribed basal deformation of either subsidence or uplift and different condition of either fully rough or fully smooth base has given were compared and discussed, focusing on the variation of pressure profiles. The numerical results obtained from IGA were validated with those of finite element method (FEM), revealing that the performance of IGA under Bézier extraction is superior to FEM by achieving the lower computational resources necessary to obtain the same accuracy.

Keywords: embankment, elastic solution, stress distribution, isogeometric analysis, Bézier extraction

1 INTRODUCTION

The variation of stress distribution beneath embankment subjected to basal deformation was investigated by using a series of 1g physical model in conjunction with lower bound limit analysis (Pipatpongsa et al. 2014a). These works had elucidated the arching action in embankments induced by basal subsidence. In this study, elastic solutions using IsoGeometric Analysis (IGA) were presented to calculate the stress distribution in an embankment subjected to basal deformation. IGA, a numerical method derived from the field of Computer Aid Design (CAD), plays a role as the particular discretization for Finite Element Analysis (FEA) by enabling to model the exact geometry with less number of variables; hence saving the computational resources (Cottrell et al. 2009). However, the parameter space of IGA consists of several elements with arbitrary dimension depending on knot span; therefore the implementation of IGA is more complicated than FEA. This study presents the Bézier extraction technique to decompose a set of B-spline basis functions to the Bernstein polynomials (Borden et al. 2011). This will allow for generation of C0 continuous Bézier extraction, giving a local representation of the basis functions. Consequently, element structure for IGA is similar to FEA, easing the implementation of IGA in finite element setting. It should be noted that B-spline is a special case of Non-Uniform Rational B-spline by setting weights of control points equal 1. Finally, this study discusses the advantage of IGA when compared with FEM (Finite Element Method) (Pipatpongsa et al. 2014b).

2 ISOGEOMETRIC ANALYSIS UNDER BÉZIER EXTRACTION

2.1 Isogeometric analysis

Though IGA was used to discretize the spatial domain similarly to isoparametric concept in FEA, IGA and FEM are totally different in the way of manipulating the interpolation functions. In FEM, the interpolation function is Lagrange polynomials while IGA employs the basis functions. In order to build any geometry by IGA, three components consisting of knot vectors, basis functions, and control points are required.

i) Control points

Playing the role as scaffold of physical space that controls its geometry.

ii) Knot vectors

A knot vector (defined for a given direction) is a sequence in ascending order of parameter values, $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$ where $\xi_i$ is the $i$th knot, $n$ is the number of basis functions, and $p$ is the polynomial order. The knot vector divides the parametric space into intervals, usually referred to as knot spans.
iii) Basis functions

When a knot vector is chosen, the basis functions are defined according to the Cox-de Boor recursion formula (Cottrell et al. 2009):

For \( p = 0 \):

\[
N_{i,p}(\xi) = \begin{cases} 
1 & \xi_i \leq \xi < \xi_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

(1)

For \( p \geq 0 \):

\[
N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p+1} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)
\]

(2)

The first derivatives of a B-spline basis function is given by:

\[
\frac{d}{d\xi} N_{i,p}(\xi) = p \left( \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p}(\xi) - \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p}(\xi) \right)
\]

(3)

iv) B-Spline curves

To construct an B-spline curve, a set of control points \( P_i, i = 1,2,\ldots,n \) is needed. If the curve is drawn in the two-dimensional space, \( P_i \in \mathbb{R}^d \). The B-spline curve is constructed as:

\[
C(\xi) = \begin{bmatrix} x \\ y \end{bmatrix}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)P_i = \sum_{i=1}^{n} N_{i,p}(\xi) \begin{bmatrix} x_i \\ y_i \end{bmatrix}
\]

(4)

The control points serve as the degrees of freedom in isogeometric analysis. These control points are analogous to nodal coordinates in FEA in the same manner as they are the coefficient of the basis functions.

v) B-Spline surfaces

Given two knot vectors (one for each parametric direction) \( \Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\} \) and \( H = \{\eta_1, \eta_2, \ldots, \eta_{m+q+1}\} \) and a control net \( P_{ij} \in \mathbb{R}^d \), a tensor-product B-spline surface is defined as:

\[
S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta)P_{ij}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta) \begin{bmatrix} x_{i,j} \\ y_{i,j} \end{bmatrix}
\]

(5)

where \( N_{i,p}(\xi) \) and \( M_{j,q}(\eta) \) are the univariate B-spline basis function of order \( p \) and \( q \) corresponding to knot vector \( \Xi \) and \( H \), respectively.

The derivatives of bivariate B-spline basis functions with respect to the parametric coordinates are:

\[
\frac{\partial N_{i,p}(\xi, \eta)}{\partial \xi} = \frac{d}{d\xi} \left( N_{i,p}(\xi) \right) M_{j,q}(\eta)
\]

\[
\frac{\partial N_{i,p}(\xi, \eta)}{\partial \eta} = \frac{d}{d\eta} \left( M_{j,q}(\eta) \right) N_{i,p}(\xi)
\]

(6)

Unlike isoparametric element of FEM, IGA utilizes, physical, parameter, and parent domains as outlined in Fig.1. Since the dimension of each element in parameter domain is different, surface integral in each element of parameter domain cannot be taken.

Therefore, the parent domain has added to fulfill the integral. The mapping from the parametric domain to the physical domain is given by Eq.(7) and the displacement field is approximated by Eq.(8).

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^{n} N_{i}(\xi, \eta)P_i = \sum_{i=1}^{n} N_{i}(\xi, \eta) \begin{bmatrix} x_i \\ y_i \end{bmatrix}
\]

(7)

\[
\begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=1}^{n} N_{i}(\xi, \eta)u_i = \sum_{i=1}^{n} N_{i}(\xi, \eta) \begin{bmatrix} u_i \\ v_i \end{bmatrix}
\]

(8)

Where \( N(\xi, \eta) \) is B-spline basic functions, \( n \) is the number of control points, \( [u_i v_i]^T \) denotes the value of the displacement field at the control point \( P_i \).

![Physical domain, parameter domain, and parent domain.](image)

2.2 Bézier extraction technique

By manipulating knot insertion operation in mesh refinement technique, the set of B-spline basis functions are decomposed to its Bézier elements, namely, Bézier decomposition. In this decomposition, each of interior knot needs to repeat \( p \) times, in which \( p \) is the polynomial order.

To illustrate Bézier decomposition, commencing with knot vector \( \Xi = \{0,0,0,1,2,3,3,3,3\} \), the set of B-spline basis function is decomposed to its Bézier
elements by inserting the knots \{1,1\},\{2,2\}. The number of basis function increases from 6 to 8, and up to 10 as shown in Figs.2, 3 and 4, respectively. Thus, the new control points corresponding their additional basis function is inserted by:

\[
P^{j+1} = \left( C^j \right)^T P^j
\]

In which \( C \) is called the Bézier extraction operator.

\[
C^j = \begin{bmatrix}
\alpha_1 & 1 - \alpha_2 & 0 & \cdots & 0 \\
0 & \alpha_2 & 1 - \alpha_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_{n+j-1} & 1 + \alpha_{n+j}
\end{bmatrix}
\]

Where the value of \( \alpha_i \) was defined as:

\[
\alpha_i = \begin{cases} 
1 & i \leq k - p \\
\frac{-\xi - \xi_i}{\xi_i + p - \xi_i} & k - p + 1 \leq i \leq k \\
0 & i \geq k + 1
\end{cases}
\]

3 NUMERICAL MODEL

Constitutive equation for linear elastic body under plane strain condition is given below:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{1}{2}(1-2\nu)
\end{bmatrix} \begin{bmatrix}
e_x \\
e_y \\
r_{xy}
\end{bmatrix}
\]

Fig. 5 shows the sand heap inclined at an angle 30º with a width \( B=2 \) m and a height \( H=\tan30º \times B/2=0.5774 \) m, supporting by hinges and rollers as shown in Fig.5. Material parameters are the unit weight of sand \( \gamma=20 \) kN/m³, Young’s modulus \( E=4000 \) kPa, Poisson ratio \( \nu=0.3 \). Three cases of the vertical stress distribution in a triangle embankment were considered: no basal deformation, basal uplift \( \Delta H=1\% \) and basal subsidence \( \Delta H=-1\% \) and where \( \Delta \) is the maximum deformation of the parabolic profile particularly imposed by the equation \( Y=\pm(H/100)\cdot4X/B\cdot(1-X/B) \) for demonstration as shown in Fig.6. The stress components are scaled by \( \gamma H \). Soil mechanics sign convention taken compression as positive is employed.
4 RESULT AND DISCUSSION

The numerical results regarding hinge and roller supports show different stress distribution in both magnitude and pattern. Dimensionless contours of scaling vertical pressure ($\sigma_y/\gamma H$) for a case of hinge supports are displayed in Figs. 7-9 under conditions $\Delta H = 0$, 1% and -1%, respectively and Figs. 10-12 for a case of roller in the similar manner. To clarify the differences among each condition, the vertical stress beneath the base are plotted in Fig. 13.

As the weight of the portion of sand heap under the gravitational force is partially supported by the upward shear stresses, the maximum vertical pressure under the apex of sand heap is less than the apparent geo-static pressure $\gamma H$ in both cases of hinge and roller supports. It is clear that the larger basal shear stress due to the fixity of the supports, the lower central pressure. The basal roughness also affects to the stress redistribution if the basal support deforms. Ideally, hinge supports represent a rough base while roller supports represent a smooth base. Under basal settlement, the downward displacement at the center is relatively larger than those of the toes. As a result, arch action formed in the sand heap across the basal sag transfers the central weight to neighboring zones a certain distance away from the center, causing the central pressure drop as shown in Fig. 13 when $\Delta H = -1\%$. The pronounced dip appears in a case of hinge supports because the basal shear stresses help promote arch action. Accordingly, basal roughness of sand heaps plays an important role in arching effect. Since uplift pressure required to deform the base upward is more than the weight transfer to the toes of sand heap, the negative vertical pressure is partly appeared in Fig. 13 when $\Delta H = 1\%$. 

Fig. 7. Contour of $\sigma_y/\gamma H$ when $\Delta H = 0$ with hinge supports.

Fig. 8. Contour of $\sigma_y/\gamma H$ when $\Delta H = 1\%$ with hinge supports.

Fig. 9. Contour of $\sigma_y/\gamma H$ when $\Delta H = -1\%$ with hinge supports.

Fig. 10. Contour of $\sigma_y/\gamma H$ when $\Delta H = 0$ with roller supports.

Fig. 11. Contour of $\sigma_y/\gamma H$ when $\Delta H = 1\%$ with roller supports.

Fig. 12. Contour of $\sigma_y/\gamma H$ when $\Delta H = -1\%$ with roller supports.
Different patterns of horizontal stresses between cases of hinge and roll supports are showed in Fig.14. Lack of horizontal constraint at the base as shown in Fig.15 induces much lower magnitude of horizontal stress in a case of roller supports than that of hinge supports. The horizontal stress for a case of roller supports when $\Delta H = -1\%$ becomes negative, indicating tension due to negative bending moment. In fact, tension is not allowed to occur in any part of sand heap; therefore, some results are unrealistic because of elastic models.

For sake of verification of the presented numerical solutions, the stress profiles at base with no basal deformation in a case of hinge supports are compared with results obtained by DACSAR software (Pipatpongsa et al. 2014b). The comparison shown in Fig.16 confirms the good agreement between the presented solutions and DACSAR software. Thus, this comparison justifies the accuracy of the elastic model based IGA.

Table 1. Comparison between IGA and FEM.

| No. elements | 200  | 600  | 1200 | 3000 |
|--------------|-----|------|------|------|
| **IGA**     |     |      |      |      |
| $\sigma_y/\gamma H$ | 0.8076 | 0.812 | 0.8135 | 0.8147 |
| Time (s)     | 0.91 | 2.34 | 4.6  | 26.42 |
| **FEM**     |     |      |      |      |
| $\sigma_y/\gamma H$ | 0.8039 | 0.8083 | 0.8098 | Out of memory |
| Time (s)     | 1.5 | 21.03 | 148.94 |      |

To verify the efficiency of IGA, the computing performances of IGA and FEM are evaluated by comparing the central vertical pressure at the base of the sand heap with hinge supports using the same operation system of computer (MS Windows 7 – 64 bit, Intel @ CoreTM 2 duoProcessor p8700 – 2.53GHz). From Table 1, both IGA and FEM can solve the accurate value of $\sigma_y$; however, IGA utilized much less time than FEM when increasing the number of elements. Because in IGA method, control points play a role as degree of freedoms, and each of which is employed to interpolate the elements neighboring to it as showed in Fig.17.
Therefore, IGA used fewer variables than FEM. For example, to discrete the analyzed domain into 800 elements, IGA requires 1848 variables while FEM requires 6642 variables.

5 CONCLUSION

The elastic solutions of vertical stress in a triangular sand heap subjected to three particular basal deformations were numerically analyzed in this study by using the isogeometric analysis under Bézier extraction. This study have highlighted the effect of basal roughness on the arch action and convinced that IGA solution can effectively solve the problem of finite element analysis in the same accuracy with FEM but less time than FEM.