Lamb shift calculated by simple noncovariant method

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The Lamb Shift (LS) of Hydrogenlike atom is evaluated by a simple method of quantum electrodynamics in noncovariant form, based on the relativistic stationary Schrödinger equation. An induced term proportional to $\mathbf{p}^4$ in the effective Hamiltonian is emphasized. Perturbative calculation of second order leads to the LS of $1S_{1/2}$ state and that of $2S_{1/2} - 2P_{1/2}$ states in H atom with the high accuracy within 0.1%

I. INTRODUCTION

The experimental discovery of Lamb Shift (LS) in 1947 and its theoretical explanations that followed are of great importance for the establishment of Quantum Electrodynamics (QED). (Ref. [1]-[3]). The experimental value of the energy difference between $2S_{1/2}$ and $2P_{1/2}$ states for Hydrogen atom reads (in unit of microwave frequency)

$$E(2S_{1/2}) - E(2P_{1/2}) = 1057.845 MHz$$  \hspace{1cm} (1)

while the absolute LS for $1S_{1/2}$ state is \[\Delta E_H(1S_{1/2}) = 8172.86 MHz\] \hspace{1cm} (2)

\[\Delta E_D(1S_{1/2}) = 8184.00 MHz\] \hspace{1cm} (3)

The LS only accounts for the order of $10^{-6}$ or $O(\alpha^3)$ of that of the binding energy for electron, i.e., that of Rydberg energy

$$R_y = R_H = \frac{1}{2} \alpha^2 \mu = 3.28805128 \times 10^9 MHz$$ \hspace{1cm} (4)

where $\mu$ is the reduced mass of electron and $\alpha = \frac{e^2}{4\pi\varepsilon_0} = 1/137.0359895$. ($\hbar = c = 1$).

The theoretical investigation over 50 years reveals that:

(a) The main contribution of LS comes from the difference of radiative correction (i.e., the perturbative energy stemming from emitting a virtual photon and then absorbing it) in different states of electron.

(b) The difference between the wave functions of $S$ states and $P$ states is important. The electron in $S$ states has more probability to move into the vicinity of nucleus. In other words, it has more high momentum components in the momentum representation of wave function for $S$ states.

In some literature, in the integration of momentum $k$ of virtual photon, the range of $k$ was often divided into two regions. For low $k$ from $k = 0$ to, say, $k = \alpha m_e = \alpha m$, the binding effect of electron is taken into account in noncovariant theory, whereas for high $k$ up from $k = \alpha m$ the covariant theory of QED is applied. This kind of treatment seems to us is difficult to avoid the double counting in virtual electron states conceptually. Moreover, the so-called long wave (i.e., $E1$) approximation was used in the noncovariant theory at low $k$ region as $e^{ikr} \sim 1$. But $\exp(ikr) \sim \exp(i\alpha m a) \sim \exp(i) \sim 1$ ($a = 1/\alpha \mu$ being the Bohr radius) is also doubtful to be a good approximation.

We wish to restudy the problem in noncovariant scheme. First of all, the mystery of LS is not only related to the small scale of energy shift shown at Eqs.(1)-(3) but also to the high accuracy of the noncovariant calculation based on the Stationary Schrödinger Equation (SSE):

$$H_0 \psi = \varepsilon \psi$$  \hspace{1cm} (5)

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with

\[ H_0 = \frac{p^2}{2\mu} - \frac{Z\alpha}{r} \]  

and

\[ \varepsilon = \frac{Z^2\alpha^2\mu}{2n^2} \]  

According to the theory of special relativity (SR), the energy of free electron reads

\[ E = \sqrt{\mu^2 + p^2} = \mu + \frac{p^2}{2\mu} - \frac{p^4}{8\mu^3} + \cdots \]  

The magnitude ratio of the third term to the second one is approximately \( \frac{\langle p^4 \rangle}{\langle p^2 \rangle^2} \sim \frac{|\chi|}{\mu} \sim O(\alpha^2) \) in a Hydrogenlike atom. So at first sight, the relativistic modification to \( H_0 \) would be expected to account for an energy decrease of \( \varepsilon \) to order of \( 10^5 MHz \). However, in fact, the LS shows an energy increase of S states to order of \( 10^3 MHz \) only. This is a mystery we should consider first before the LS could be understood in the noncovariant formalism of QED.

II. RELATIVISTIC SSE

In Refs. [5] - [6], it was argued that the SSE, Eq.(5), is essentially relativistic as long as the eigenvalue \( \varepsilon \) is related to the binding energy \( B \) as follows:

\[ B = Mc^2[1 - (1 + 2\varepsilon Mc^2)^{1/2}] \]  

where \( M = m + m_N \) is the total mass of Hydrogenlike atom with \( m_N \) being the mass of nucleus. Obviously, when \( \varepsilon \ll Mc^2 \), \( B \simeq -\varepsilon \) as expected.

Eqs.(5)-(7) together with Eq.(9) was derived from the time-dependent Schrödinger equation of two-body (say electron-proton) system in combination with a basic symmetry

\[ \theta(-r_e, r_p, -t) = \chi(r_e, r_p, t) \]  

The explanation is as follows. The electron and proton are all not pure. They not only have a particle state \( \theta(r_e, r_p, t) \), but also have a hiding antiparticle state \( \chi(r_e, r_p, t) \), \( \theta \) and \( \chi \) are coupled together according to the symmetry (10). For one body system, this symmetry leads to Klein-Gordon equation (without spin) or Dirac equation (with spin) (Refs. [1] [8]). The symmetry that “the space-time inversion is equivalent to particle-antiparticle transformation” shown as (10) is the essence of SR. The various strange effects of SR are nothing but the reflections of antimatter which is in a subordinate status (|\( \chi | < |\theta |)) and is just displaying its presence tenaciously [9].

Therefore, at the level of quantum mechanics, the Hamiltonian \( H_0 \) in SSE Eq.(5) is enough in the form of Eq.(6). No \( p^4 \) term like that in Eq.(8) is needed for relativistic correction. The latter is brought into consideration implicitly at the final stage \( \varepsilon \rightarrow -B \) as shown by Eq.(9).

III. SEMIEMPIRICAL CALCULATION OF LS

Based on SSE, we consider that in an effective Hamiltonian a term \( b_2p^4 \) will be induced at QED level by radiative correction. It will lead to an energy shift of Hydrogenlike atomic level as:

\[ \Delta E_{Z,nl}^{Rad} = (Znl | b_2p^4 | Znl) = \frac{8n}{(2l + 1)} - 3\frac{b_2Z^4}{n^4a^4} \]  

(\( a = \frac{1}{\alpha\mu} \))

The coefficient \( b_2 \) can be fixed from experiment by the LS between \( 2S_{1/2} - 2P_{1/2} \) states in Hydrogen:

\[ \frac{b_2}{a_H} = \frac{3}{2}(1057.845 - 0.087) MHz = 1586.637 MHz \]  

(12)
where 0.087 MHz is the small correction stemming from the finite nucleus radius.

To calculate the absolute LS of $1S$ state, Eq. (2), we should add three extra contributions:

(a) The vacuum polarization in QED induces a decrease in charge of electron: 

$$\Delta \alpha = -\frac{\alpha Z^4}{3 \pi n^2}$$  \hspace{1cm} (13)

which leads to an increase of energy

$$\Delta E_{Z, nl}^{VP} = \frac{2Z^4 \alpha^3}{3 \pi n^5} R_y$$  \hspace{1cm} (14)

For Hydrogen $1S$ state, it reads

$$\Delta E_{H, 1S}^{VP} = 271.140 MHz$$  \hspace{1cm} (15)

(b) The relativistic correction from Eq. (9) reads

$$\Delta E_{Rel} = -\frac{\varepsilon^2}{2M} = -\frac{1}{4} \alpha^2 \frac{\mu}{M} R_y n^3$$  \hspace{1cm} (16)

which yields a decrease in energy:

$$\Delta E_{H, 1S}^{Rel} = -23.814 MHz$$  \hspace{1cm} (17)

(c) The correction from finite nucleus radius ($r_N$) reads

$$\Delta E_{nl}^{Nu} = \frac{4}{3} \frac{1}{n^3} (\frac{r_N}{a})^2 R_y \delta_{l0}$$  \hspace{1cm} (18)

which contributes a small increase of energy

$$\Delta E_{H, 1S}^{Nu} = 0.697 MHz$$  \hspace{1cm} (19)

Altogether, we obtain theoretically

$$\Delta E_{H, 1S}^{Theory} = 8181.208 MHz$$  \hspace{1cm} (20)

The deviation between (20) and the experimental value, Eq. (2) is only 0.1%.

**IV. CALCULATION OF $B_2$ FROM THE FIRST PRINCIPLE**

We are now in a position to derive the value of coefficient $b_2$ from the first principle of QED in noncovariant form.

Consider an electron with charge $-e$ is moving with (three dimensional) momentum $\vec{p}$ in the center of mass system of Hydrogenlike atom and is coupled to the electromagnetic field via two kinds of interactions (1), (12)

$$H^{(1)} = \frac{e}{\mu c} \hat{A} \cdot \vec{p}$$  \hspace{1cm} (21)

$$H^{(2)} = \frac{e\hbar}{2\mu c} \vec{\sigma} \cdot \vec{\nabla} \times \hat{A}$$  \hspace{1cm} (22)

According to the perturbation theory in quantum mechanics, the energy shift due to (21) from original $\varepsilon_p = \frac{\vec{p}^2}{2\mu}$ will be

$$\Delta E_p^{(1)} = \sum_i \frac{|\langle i | H^{(1)} | \vec{p} \rangle|^2}{\varepsilon_p - \varepsilon_i}$$  \hspace{1cm} (23)
where the stationary state $|\vec{p}\rangle = \frac{1}{\sqrt{V}} e^{i\vec{p} \cdot \vec{r}}$ is normalized in a volume $V$. The intermediate (virtual) state $|i\rangle$ is composed of a plane wave eigenstate of SSE (Eq.(5)) $\vec{q}$ and a virtual photon with continuous momentum $\vec{k}$, see Fig. 1. So the $\varepsilon_i$ in the denominator of (23) reads $\varepsilon_i = \frac{\vec{q}^2}{2\mu} + \omega_k$, $(\omega_k = |\vec{k}| = k)$. The quantized potential $\hat{A}$ of electromagnetic field reads as usual:

$$\hat{A}(\vec{r}, t) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \sum_{\lambda=1,2} \vec{e}_{\lambda} \cdot \vec{k}, \lambda (\hat{a}_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{r}} + \hat{a}_{\vec{k}}^\dagger(t) e^{-i\vec{k} \cdot \vec{r}})$$  \hspace{1cm} (24)

After the integration of matrix element of $H^{(1)}$ with respect to space, we substitute one $\delta$ function $\delta(\vec{p} - \vec{q} - \vec{k})$ by $\frac{V}{(2\pi)^3}$ and then perform the integration with respect to $\vec{q}$, yielding

$$\Delta E^{(1)}_p = -\frac{\alpha p^2}{\pi \mu} \int_{-1}^{1} d\eta (1 - \eta^2) I$$  \hspace{1cm} (25)

$$I = \int_0^{\infty} \frac{dk}{k + \xi}$$  \hspace{1cm} (26)

where $\eta = \frac{\vec{p} \cdot \vec{k}}{p k}$ and $\xi = 2(\mu - p\eta)$.

**V. THE RENORMALIZATION IS A PROCEDURE TO RECONFIRM THE MASS.**

As in the calculation of QED in covariant form, we also encounter the divergent integral $I$, Eq.(26), here in the integration of three dimensional momentum $k$ of virtual photon. To treat the divergence, we will follow the spirit of a simple but effective method used in covariant quantum field theory, which evolved from the so-called differential renormalization in the literature [13]-[16], then was proposed by Ji-feng Yang [17] and applied extensively in Refs. [11], [18], [19] (see also the discussion in [20]). Here the trick is as follows.

Take the derivative of integral with respect to the parameter $\xi$ having a mass dimension:

$$\frac{\partial I}{\partial \xi} = -\int_0^{\infty} \frac{dk}{(k + \xi)^2} = -\frac{1}{\xi}$$  \hspace{1cm} (27)

which is convergent now. Then we reintegrate Eq.(27) with respect to $\xi$ for returning back to $I$

$$I = -\ln \xi + C_1$$  \hspace{1cm} (28)

where an arbitrary constant $C_1$ appears. Substituting (28) into (27), one obtains

$$\Delta E^{(1)}_p = \frac{\alpha \mu}{\pi} \left( \frac{2}{3} \frac{p^2}{\mu^2} + \frac{2}{3} \frac{p - \mu}{3p} \ln(1 + \frac{p}{\mu}) + \frac{2}{3} \left( \frac{p}{\mu} - \frac{p}{3\mu} \right) \ln\left( 1 - \frac{p}{\mu} \right) \right)$$

$$\frac{16}{9} \left( \frac{p}{\mu} \right)^2 + \frac{2}{3} \ln 2 + \frac{4}{3} \ln \mu - \frac{4}{3} C_1 (\frac{p}{\mu})^2$$

$$= b_1^{(1)} p^2 + b_2^{(1)} p^4 + \cdots$$  \hspace{1cm} (29)

$$b_1^{(1)} = \frac{\alpha}{\pi \mu} \left( \frac{4}{3} \ln 2 + \frac{4}{3} \ln \mu - \frac{4}{3} C_1 \right)$$  \hspace{1cm} (30)

$$b_2^{(1)} = \frac{\alpha}{\pi \mu^3} \left( -\frac{2}{15} \right)$$  \hspace{1cm} (31)

Note that, however, the term $b_1^{(1)} p^2$ will be combined into the kinetic energy term in original $H_0$. They are indistinguishable. The appearance of arbitrary constant $C_1$ precisely reflects the fact that we can not calculate the
reduced mass of an electron via the perturbation calculation of $\Delta E_p^{(1)}$. So the value of $C_1$ is chosen such that $b_1^{(1)} = 0$, implying that the value of reduced mass $\mu$ (yet not $\mu_{\text{obs}}$, see below) is reconfirmed as an observed mass which can only be fixed by the experiment (not by theory).

Next turn to $H^{(2)}$, which induces the spin flip process between $|\vec{p}, \pm \frac{1}{2}\rangle$ and $|\vec{q}, \pm \frac{1}{2}\rangle$ states, yielding

$$\Delta E_p^{(2)} = \frac{1}{2} \sum_{i,s_i=\pm \frac{1}{2}} |\langle i | H^{(2)} | \vec{p}, s_z \rangle|^2 \frac{\varepsilon_\nu - \varepsilon_i}{\varepsilon_p - \varepsilon_i}$$

$$= -\frac{\alpha}{2\pi\mu} \int_{-1}^{1} d\eta J$$

$$J = \int_0^\infty \frac{k^2 dk}{k + \xi}$$

For this divergent integral, derivative of third order is needed to render it convergent:

$$\frac{\partial^3 J}{\partial \xi^3} = -\frac{2}{\xi}$$

So after reintegrating with respect to $\xi$, we have:

$$J = -\xi^2 \ln \xi + C_2 \xi^2 + C_3 \xi + C_4$$

$$\Delta E_p^{(2)} = \frac{\alpha \mu}{\pi} \left( \frac{2\mu}{3p} \ln(1 + \frac{p}{\mu}) - (1 - \frac{p}{\mu})^3 \ln\left(1 - \frac{p}{\mu}\right) \right) - \frac{22}{9} \left( \frac{p}{\mu} \right)^2 - \frac{4}{3}$$

$$+ 4 (\ln 2 + \ln \mu) - 4C_2 - \frac{2C_3}{\mu} - \frac{C_4}{\mu^2} + \left( \frac{4}{3} \ln 2 + 2 + \frac{4}{3} \ln \mu - \frac{4}{3} C_2 \right) \left( \frac{p}{\mu} \right)^2$$

$$= b_0^{(2)} + b_1^{(2)} p^2 + b_2^{(2)} p^4 + \cdots$$

$$b_0^{(2)} = \frac{\alpha \mu}{\pi} \left( 4 \ln 2 + \ln \mu - 4C_2 - \frac{2C_3}{\mu} - \frac{C_4}{\mu^2} \right)$$

$$b_1^{(2)} = \frac{\alpha}{\pi \mu} \left( \frac{4}{3} \ln 2 + 2 + \frac{4}{3} \ln \mu - \frac{4}{3} C_2 \right)$$

$$b_2^{(2)} = \frac{\alpha}{\pi \mu^3} \left( -\frac{1}{15} \right)$$

We shall fix three arbitrary constants $C_2$, $C_3$ and $C_4$ carefully. First look at the $b_1^{(2)} p^2$ term which should be combined with the term $\frac{2\mu}{3p}$ with $\mu$ already fixed. Any modification on $\mu$ must be finite and fixed. So the only possible choice of $C_2$ is to cancel the ambiguous term $\frac{4}{3} \ln \mu$ in $b_1^{(2)}$, leaving

$$b_1^{(2)} = \frac{\beta}{2\mu} \quad \beta = \frac{2\alpha}{\pi} \left( \frac{4}{3} \ln 2 + 2 \right)$$

The constants $C_3$ and $C_4$ must be chosen such that $b_0^{(2)} = 0$, which means that we reconfirm the SSE as our starting point. There is always no rest energy term in SSE.

Hence, the nonzero contribution of $b_1^{(2)} p^2$ does bring a finite and fixed modification on $\mu$ so that

$$\mu_{\text{obs}} = \frac{\mu}{1 + \beta}$$
where $\mu_{\text{obs}}$ is the reduced mass eventually observed in experiment. However, as discussed in Eq.(8), there is no term like $-\frac{p^4}{8\mu_{\text{obs}}^3}$ in SSE. The relativistic correction is left to the modification of $\varepsilon$ to $B$ at the final stage. So an unobservable term is also modified, the difference $-\frac{1}{8} \left( \frac{1}{\mu_{\text{obs}}} - \frac{1}{\mu} \right) p^4$ should be treated also as an unobservable background of modification in $p^4$ term, which should be subtracted off from the observable one. Hence while $b_1 = b_1^{(1)} + b_1^{(2)} = \frac{1}{\pi \mu_{\text{obs}}} \left( -\frac{1}{8} \right)$, we should have a renormalized $b_2^R$ as

$$b_2^R = b_2 + \frac{1}{8\mu_{\text{obs}}^3} (3\beta + 3\beta^2 + \beta^3) = \frac{\alpha}{\pi \mu_{\text{obs}}^3} (1.942816878)$$

(42)

VI. COMPARISON WITH THE EXPERIMENTAL VALUES

Let us use Eq.(42) to evaluate the LS of H atom:

$$\Delta E_{\text{Rad}}^{\text{H} \cdot 1S} = \langle b_2^R p^4 \rangle_{1S} = \frac{5b_2^R}{a_H^2} = 7901.629 \text{MHz}$$

(43)

Adding the contributions from Eqs.(15), (17) and (19), we get

$$\Delta E_{\text{Theory}}^{\text{H} \cdot 1S} = 8149.653 \text{MHz}$$

(44)

which is smaller than the experimental value Eq.(2) up to 0.28%.

For the LS between $2S_{1/2}$ and $2P_{1/2}$ states,

$$\Delta E_{\text{Rad}}^{2S_{1/2} \rightarrow 2P_{1/2}} = \frac{2b_2^R}{3a_H^2} = 1053.551 \text{MHz}$$

(45)

After adding a small contribution due to the finite nucleus radius, we have

$$\Delta E_{\text{Theory}}^{2S_{1/2} \rightarrow 2P_{1/2}} = 1053.638 \text{MHz}$$

(46)

which is also smaller than the experimental value Eq.(3) up to 0.40%.

For further improvement, we keep all $p^n$ ($n \geq 4$) terms. So we manage to evaluate the renormalized radiation correction as follows

$$\Delta E_{\text{Rad}}^{(p^4)} = \Delta E_{p}^{(1)} + \Delta E_{p}^{(2)} - b_2 = \frac{p^2}{2\mu_{\text{obs}}} \left( \sqrt{p^2 + \mu_{\text{obs}}^2} - \mu_{\text{obs}} \right) - \left( \sqrt{p^2 + \mu^2} - \mu - \frac{p^2}{2\mu} \right)$$

(47)

with $\mu_{\text{obs}} - \mu = d\mu = -\beta \mu_{\text{obs}}$ and $\sqrt{p^2 + \mu_{\text{obs}}^2} - \sqrt{p^2 + \mu^2} = \frac{\mu_{\text{obs}} d\mu}{\sqrt{p^2 + \mu_{\text{obs}}^2}}$ as we wish to keep the explicit dependence on $\alpha$ throughout Eq. (47) being of order $O(\alpha)$. Then we calculate the expectation value of $\Delta E_{\text{Rad}}^{(p^4)}$ for a fixed state numerically, yielding

$$\Delta E_{\text{Rad}}^{H \cdot 1S} = 7920.533 \text{MHz}$$

(48)

$$\Delta E_{\text{Rad}}^{H \cdot 2S_{1/2} \rightarrow 2P_{1/2}} = 1057.550 \text{MHz}$$

(49)

$$\Delta E_{\text{Rad}}^{D \cdot 1S} = 7922.688 \text{MHz}$$

(50)

$$\Delta E_{\text{Rad}}^{D \cdot 2S_{1/2} \rightarrow 2P_{1/2}} = 1057.838 \text{MHz}$$

(51)

After adding the other three corrections Eqs.(14)-(18), we are pleased to see the results

$$\Delta E_{\text{Theory}}^{1S} = 8168.557 \text{MHz}$$

(52)
\[ \Delta E_{H,2S_{1/2} - 2P_{1/2}}^{\text{Theory}} = 1057.637 \text{MHz} \]  
(53)

\[ \Delta E_{D,1S}^{\text{Theory}} = 8186.181 \text{MHz} \]  
(54)

\[ \Delta E_{D,2S_{1/2} - 2P_{1/2}}^{\text{Theory}} = 1058.363 \text{MHz} \]  
(55)

coinciding with the experimental data to a high accuracy (\(\lesssim 0.1\%\)).

**VII. SUMMARY AND DISCUSSION**

(a) We propose a simple but effective method for calculating the LS in a noncovariant form. Two crucial observations are as follows:

(i) The SSE is essentially relativistic as long as its eigenvalue \(\varepsilon\) is related to the binding energy \(E\) by Eq.(1).

(ii) The main contribution to LS is coming from the different mean values of operator \(\vec{p}^4\) in \(S\) and \(P\) states as shown in the semiempirical calculation. The analysis convinced ourselves that the use of three dimensional momentum \(\vec{p}\) is much suitable than that of four dimensional one. Actually, a simple one-loop calculation in covariant form of QED gave us the value of LS for H atom with accuracy only 5\% for \(2S_{1/2} - 2P_{1/2}\) states and even worse (\(\sim 22\%)\) for \(1S_{1/2}\) state.[11]

(b) The subtlety of SSE can be seen further by the following comparison. At free moving condition, the relativistic effect is contained in the definition \(\varepsilon = \frac{\vec{p}^2}{2\mu}\) with rest (reduced) mass being a constant containing all the radiative correction. On the other hand, when the electron is bound in a Hydrogenlike atom, its Hamilton \(H_0\) contains no rest mass and no explicit (negative) \(\vec{p}^4\) term either. The relativistic effect is contained in the definition \(\varepsilon = \frac{\vec{E}^2-M^2}{2M}\) (with \(E\) and \(M\) being the total energy and mass of the system) and Eq.(3) implicitly. However, the radiative correction does induce a small (positive) term of \(\vec{p}^4\) at the level of QED.

(c) The definition of \(\varepsilon\) and Eq.(1) also indicate that there is no any negative energy eigenstate in SSE. The hiding antiparticle state (which is the essence of special relativity) is already taken into account in deriving SSE with Eq.(3). In other words, no further explicit virtual positron state should be considered in our calculation. Actually, we had struggled for years before eventually realizing that only the simple formulas (23) and (32) with Fig.1 are needed in the perturbative calculation of second order.

(d) The effective Hamiltonian of Hydrogenlike atom for evaluating the LS can be summarized as

\[ H_{\text{eff}} = \frac{p^2}{2\mu} - \frac{Ze}{r} + \Delta E^{\text{Rad}}(\vec{p}) + \frac{\alpha^4 Z^3}{3\pi n^2 r} \]  
(56)

while the third term (\(\simeq \hbar R p^4\)) is the QED modification to the first term, the fourth term could be seen as that to the second term (see Eqs.(13), (14)).

Eq.(56) is applicable to \(j = 1\) states. The \(P_{3/2}\) state is pushed up by extra spin-orbit coupling to form the fine structure. For Hydrogen, \(E(2P_{3/2}) - E(2P_{1/2}) = 1.09691 \times 10^4 \text{MHz}\) about 10 times of LS. Furthermore, the hyperfine structure (hfs) in Hydrogen, stemming from the interaction between the magnetic moments of electron and proton, is smaller than the LS. The energy splitting of 1S\(_{1/2}\) states is well known as \(\Delta E_{1S_{1/2}}(\text{hfs}) = 1420.406 \text{MHz}\). As we don’t take the magnetic moment of nuclei into account, the hfs is not considered in this paper.

In deriving the \(H_{\text{eff}}\), the Pauli interaction term \(H^{(2)}\) between electron spin and the external magnetic field has to be added to the RSSE so that the spin-orbital coupling (hyperfine structure) and LS can be calculated quantitatively. This is the price we must pay for the use of RSSE. On the other hand, though the Dirac equation in external field can predict the electron spin with \(g = 2\), it fails to take the difference of reduced mass of electron into account because it is a one-body equation. Furthermore, it predicts a too low ground state (1S) and a too large splitting of 2P\(_{3/2}\) and 2P\(_{1/2}\) states. The weakness of Dirac equation seems to us is due to its overestimation of the antiparticle ingredient in the electron as discussed in Ref. [5]. Actually, only after long time study on Dirac equation with its difficulty in calculating LS, especially the absolute LS of 1S states, could we believe in the advantage of using RSSE as the starting point.

(e) The reasonable result in this paper again shows the correctness and effectiveness of the renormalization method used in Refs. [17]-[20]. For treating the divergence, which warns us of the lack of our knowledge about the parameters (mass, charge, etc.), the renormalization is a procedure to reconfirm the parameters step by step rigorously. Here it is interesting to see an example of finite and fixed renormalization. Note that, however, we must consider the
contribution of $H^{(1)}$ first to define the parameter $\mu$ before to consider that of $H^{(2)}$ for bringing $\mu$ to $\mu_{\text{obs}}$. This is the only reasonable logic. The inverse logic, i.e., to consider $H^{(2)}$ first and then $H^{(1)}$ next, would lead to inconsistent and wrong result.

ACKNOWLEDGMENTS

We thank Prof. D. Zwanziger, Mr. Hailong Li and Mr. Haibin Wang for discussions. We also thank Mr. Sangtian Liu in NYU who gave us a lot of help in $\LaTeX$ and figure of this paper. This work was supported in part by the NSF of China.

FIG. 1. The only one loop diagram calculated for radiation correction in the noncovariant formalism of QED. At the two vertices, either $H^{(1)}$ or $H^{(2)}$ is used (no interference between them would occur due to no polarization in the plane wave $p$ state).

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