Relating decision and search algorithms for rational points on curves of higher genus

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1 Introduction

Let

\[ F(x, y) = 0 \quad (*) \]

be a polynomial equation over the rational numbers of degree \( d \) and assume that the complex solutions carve out an irreducible algebraic curve of geometric genus at least two. This condition will be satisfied by the ‘generic’ equation of degree \( \geq 4 \).

A theorem of Faltings [3] says that the equation has at most finitely many rational solutions. The nature of Falting’s proof, however, as well as the several other proofs that followed [12], [1], [9], does not provide, even in principle, an effective algorithm for actually finding all the solutions. This is because the proofs provide only a bound on size differences (or ratios), at least in principle, rather than on the absolute size of solutions. (Here, the size of a rational number can be taken as the larger of the absolute values of the numerator and the denominator.) The effective Mordell conjecture proposes one program whereby this deficiency can be remedied. Given a rational point in the plane \( P = (p/r, q/r) \) written in reduced form so that \( (p, q, r) = 1 \), define its *height* to be

\[ h(P) := \sup \{ |p|, |q|, |r| \} \]

Also, given the equation \((*)\), define

\[ M(F) := \sup \{ h(P) : f(P) = 0 \}, \]

the maximum height of a rational solution. The conjecture, in weakest form, says \( M(F) \) should be computable as a function of \( F \). More precise forms of the conjecture usually postulate arithmetic-geometric quantities that should appear in an effective bound for the maximum size \( [3] \).

At present, the conjecture appears to be very much out of reach and much effort is expended in relating one effective conjecture to another or in proving function field analogues. Because of the lack of palpable structure on curves of higher genus, as opposed to elliptic curves, for example, it seems hard to
come by techniques for producing solutions at all, let alone a complete set. We should note, however, that much progress has been made in the arena of effective algorithms for integral solutions of special families of equations, mostly building on Baker’s method \cite{10}.

As mentioned, the main application of the effective Mordell conjecture would be a search algorithm which would allow us to deterministically find all solutions to higher genus equations. On the other hand, it is conceivable that a search algorithm exists even without a priori bounds of the sort usually considered in Diophantine geometry. Put differently, an algorithm need not necessarily be that given by an elementary ‘formula’ in terms of geometric invariants, which is what occurs in the function field case, and therefore, informs research over number fields.

Even as far as a priori bounds are concerned, consider the following classical example of genus zero:

$$ax^2 + by^2 = c$$

with $a, b, c$ positive. A theorem of Holzer \cite{8} says that if a solution exists at all, then there is a solution $(p/r, q/r)$ in reduced form that satisfies the bound

$$\sup \{|p|, |q|, |r|\} \leq \sup \{\sqrt{|ab|}, \sqrt{|ac|}, \sqrt{|bc|}\}$$

Phrased differently, this is a bound for the minimum size of solutions. In this case, since the solution set can be infinite, there is no question of a bound on the maximum. But the utility of the bound shown is that it provides

(1) a decision algorithm for the existence of solutions; and

(2) the starting point for a purely geometric algorithm that allow us to generate all solutions in a precise sense, even when there are infinitely many.

Conjectures of Lang \cite{6} bounding the size of generators for the Mordell-Weil group of an elliptic curve are of similar nature.

The purpose of this paper is to express the suspicion that this inequality might be the correct paradigm to follow. That is, perhaps one should formulate conjectures on the minimum size of the solutions, even in the higher genus case, and in as elementary a form as possible. The problem of finding all solutions could then indeed be approached in a genuinely algorithmic fashion, rather than by seeking a simple (or even complicated) formula bounding the maximal size. Rather than investigate the plausibility of such conjectures here (cf. \cite{11}), we would like to provide motivation, by proving a theorem that illustrates the utility of this point of view in very elementary terms. Define

$$m(F) := \inf \{h(P) : f(P) = 0\},$$

the minimum height of solutions to (\*).

**Theorem 1** If $m(F)$ is computable, then $M(F)$ is computable.

This theorem is plainly equivalent to
Theorem 2 A decision algorithm for the existence of a rational solution to (*) actually provides a deterministic search algorithm for the full solution set.

Remarks:
(1) If one has a strong enough decision algorithm, one easily has a search algorithm. For example, if one had a decision algorithm for any algebraic curve presented in whatever form, then simply removing solutions after we find them through an exhaustive search and applying the decision algorithm again would tell us whether we need to continue. To be more concrete, consider the case where we have a decision algorithm for systems of equations and inequations. Then one could construct a search algorithm as follows. Apply the decision algorithm to (*). If the algorithm says YES, search exhaustively for a solution by ordering the rational points in $\mathbb{A}^2$ by size. After finding a solution $(p, q)$ look for all solutions with $x = p$ (easy). Now apply the algorithm to 

$$F(x, y) = 0, \quad x \neq p$$

to see if we should stop.

Consider also the case where we have an algorithm for systems of equations in three or more variables. Then the system above could be replaced by

$$F(x, y) = 0, \quad (x - p)z = 1$$

The discerning reader will notice that, in fact, a decision algorithm for systems of two equations in $\mathbb{A}^3$ easily provides a search algorithm. That is, a trivial projection technique circumvents the necessity of going to progressively higher-dimensional spaces.

The point of our theorem is that it postulates a decision algorithm only for the simplest possible case of a single equation in two variables. The proof still uses a projection technique, but of a slightly more subtle nature (Hilbert irreducibility).

In this regard, notice that Matiyasevich’s theorem, for example, makes it generally desirable to decrease the number of variables when postulating the existence of a decision algorithm.

(2) Of course, the kind of decision algorithm we have in mind is the one mentioned: a bound for the minimum size of solutions. But from the viewpoint of recursion theory or computer science, one could imagine other strategies. The motivation for this theorem arose exactly from the notion that the problem of finding a decision algorithm for a single equation in two-variables has a decidedly elementary flavour that suggests approaches from disciplines other than Diophantine geometry.

2 Proof of theorem.

The idea is to send rational points off to infinity as we find them. Assume we are given a decision algorithm that answers YES or NO, given the equation.
Apply it to our equation. If the algorithm says NO, stop. If it says YES, search exhaustively until we find one, after ordering all the rational points in the plane by size, for example. Eventually, we will find a solution \((p, q)\). First investigate for any other solutions with \(x = p\) by solving the single variable equation

\[ F(p, y) = 0 \]

Now, construct a system of equations:

\[ F(x, y) = 0; \quad (x - p)z = 1 \]

Clearly, the solutions \((x, y, z)\) of this system are in 1-1 correspondence with the solutions \((x, y)\) of the original equation for which \(x \neq p\). We will project this curve back into the plane without creating any new rational points. That is, look at the corresponding equations in \(\mathbb{P}^3\):

\[ w^dF(x/w, y/w) = 0; \quad (x - pw)z = w^2 \]

and call the curve it defines \(C\).

**Lemma 1** There exists a rational point \(m\) in the plane \(H : \{w = 0\}\) such that projecting from \(m\) maps \(C\) birationally to a curve \(D\) in \(\mathbb{P}^2\) with the property that any rational point on \(D\) comes from a rational point on \(C\).

**Proof of Lemma.** If we project from a rational point \(m \in H\), a point in \(D\) will be rational iff the line through \(m\) that corresponds to it is rational. If this line meets the curve at just one point, then the rational point in \(D\) will have come from a rational point on our original curve \(C\). So the only way new rational points can be created is if there is a rational secant line to \(C\) passing through \(m\). Thus, we need to find a point \(m \in H\) with the property that all the lines through \(m\) that are secant to \(C\) are irrational. Here’s how we achieve this:

Consider the rational secant map

\[ [C \times C - \Delta] \times \mathbb{P}^1 \to \mathbb{P}^3 \]

which sends \((a, b, (t_0 : t_1))\) to the point \(t_0a + t_1b\). That is, it is sent to the corresponding point on the parametrized secant line through \(a\) and \(b\). Suppose the image is of dimension \(\leq 2\). The image cannot be the plane \(H\), because if it were, then all secant lines would lie in \(H\) and hence, \(C\) itself would have to lie in \(H\) (since \(w\) would have infinitely many zeros on \(C\)), which it doesn’t. Hence, the image meets \(H\) properly, so we will have plenty of rational points on \(H\) which lie on no secant line at all. Therefore, we need only consider the case where the image has dimension 3. In this case, the map is generically finite to one. Also, the locus of the points in the domain where it’s not finite-to-one has dimension \(\leq 2\), so its image has dimension \(\leq 1\). In particular, this image meets \(H\) at most in a curve. Let \(Z\) be the portion of \([C \times C - \Delta] \times \mathbb{P}^1\) lying above \(H\). \(Z\) consists of triples \((a, b, (t_0 : t_1))\) such that \(t_0a + t_1b \in H\). But a generic line in \(\mathbb{P}^3\) meets a plane in exactly one point and only finitely many secants lie on \(H\),
so given most $(a, b)$, $(t_0 : t_1)$ is uniquely determined. Therefore, the projection to $(a, b)$ defines a birational map $Z \to C \times C$. Now when viewed on $C \times C$, our rational map clearly factors through the symmetric product $\text{Sym}^2(C)$ which, in turn, maps birationally to $\text{Sec}(C)$, the secant variety to $C$.

Thus, we end up with a rational map from the secant variety: $\text{Sec}(C) \to H$ which is generically finite-to-one. Clearly, $\text{Sym}^2(C)$ is irreducible. Also, the symmetric product of a curve of genus $\geq 1$ is not birational to $\mathbb{P}^2$ (it has non-zero global differential forms), so the map has degree $> 1$. Recall Hilbert's irreducibility theorem \cite{4}, which says that given a map defined over $\mathbb{Q}$:

$$f : X \to U$$

where $U$ is a non-empty open subset of $\mathbb{P}^n$ and $X$ is an irreducible variety of dimension $n$, there exists a rational point of $U$ which does not ‘lift’ to a rational point of $X$, i.e., does not lie in $f(X(\mathbb{Q}))$. Therefore, there exists a rational point in $H$ which does not lie in the image of any rational point in $\text{Sec}(C)$ and with finite inverse image. Thus, projecting from the point will give a generically 1-1 map on $C$, and the points where its not 1-1 will not map to rational points, proving the lemma.

Now, search exhaustively for the point on $H$ whose existence is guaranteed by the lemma. One does this, for example on the affine coordinate chart where $z \neq 0$, by examining for each point $(0, a, b) \in H$ the set:

$$\{(w, x, y, w', x', y', t) \in \mathbb{A}^3 \times \mathbb{A}^3 \times \mathbb{A}^1 |$$

$$F(x, y) = 0, \quad x - wp = w, \quad F(x', y') = 0, \quad x' - w'p = w'$$

$$w + tw' = 0, \quad x + tx' = a, \quad y + ty' = b\}$$

The previous discussion says that this equation for seven unknowns in seven variables has finitely many solution for generic $(a, b)$. In fact, there is a Groebner basis algorithm for computing the dimension of the set for any fixed $(a, b)$ \cite{2}. If the application of this algorithm gives you dimension one move on to the next point. Whenever, the dimension turns out to be zero, find all the solutions and check for rationality. For this, recall that there is a Groebner basis algorithm which finds all the solutions to a zero-dimensional equation by elimination theory \cite{4}. Also, note that rationality of the secant line means that two points of intersection with $C$, $(w, x, y)$ and $(w', x', y')$ are either both rational or quadratically conjugate. This can be readily checked. If there is a rational secant for $(0, a, b)$ move on to the next point. Eventually and algorithmically one finds the point $m$ whose existence is guaranteed by the lemma. Now, project from $m$ and compute an equation for the image using elimination theory. Now dehomogenize by putting the image of $H$ (which is a line) at infinity. We have thereby arrived at another affine equation $F_i(x, y) = 0$ whose solution set has cardinality strictly less than that of $(\ast)$. Since this is birational to the original curve, it still has genus $\geq 2$. Now apply the decision algorithm again and iterate the process above if it says YES.
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