Black Hole as a Point Radiator and Recoil Effect in the Brane World

Valeri Frolov and Dejan Stojković
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1

A small black hole attached to a brane in a higher dimensional space emitting quanta into the bulk may leave the brane as a result of a recoil. We construct a field theory model in which such a black hole is described as a massive scalar particle with internal degrees of freedom. In this model, the probability of transition between the different internal levels followed by emission of massless quanta is identical to the probability of thermal emission calculated for the Schwarzschild black hole. The discussed recoil effect implies that the thermal emission of the black holes, which might be created by interaction of high energy particles in colliders, could be terminated and the energy non-conservation can be observed in the brane experiments.

A common feature of all brane world models with large or infinite extra dimensions is that a lot of new interesting phenomena can be expected at the energy scale not much above the energy currently available in accelerators. Probably the most interesting and intriguing is the possibility of production of mini black holes in future collider and cosmic rays experiments. Preliminary calculations indicate that the probability for creation of a mini black hole in near future hadron colliders such as LHC (Large Hadron Collider) is so high that they can be called “black hole factories”. Also, mini black holes produced by ultra high energy cosmic rays could be observed at the Auger Observatory before LHC starts operating.

After the black hole is formed (either at LHC or the Auger Observatory), it decays by emitting Hawking radiation. Thermal Hawking radiation consists of particles propagating along the brane, and the bulk radiation. Usually the bulk radiation is neglected since the total number of species which are confined on the brane is quite large (∼ 60, see e.g. [3,6]). It should be noted that when the number of extra dimensions is greater than 1 this argument may not work. Really, the number of degrees of freedom of gravitons in the (N + 1)-dimensional space-time is \( N = (N + 1)(N - 2)/2 \). For example, for \( N + 1 = 7 \) (3 extra dimensions) \( N = 14 \).

It is also very important that mini black holes created in the high energy scattering are expected to have high angular momentum and to be extremal or close to them. For this reason, the effect of superradiance must be important. The superradiant emission is dominated by the particles with the highest spin, that is by gravitons. This effect was studied in \((3 + 1)\)-dimensional space-time by Don Page (see also [3]). He demonstrated that for an extremely rotating black hole the probability of emission of a graviton by an extremely rotating black hole is about 100 times higher than the probability of emission of a photon or neutrino. We may expect that this conclusion remains valid for higher-dimensional black hole so that the bulk radiation may be comparable with (or even dominate) the radiation along the brane.

But even for small number of extra dimensions the role of bulk graviton emission might be important. As a result of the emission of the graviton into the bulk space, the black hole recoil can move the black hole out of the brane. Black hole radiation would be terminated and an observer located on the brane would register the virtual energy non-conservation.

Since the problem in its complete scope is very complicated we make some simplifying assumptions. We assume that the characteristic size of extra dimensions is much larger than the Schwarzschild radius of the black hole, so that we can effectively describe the black hole by a higher dimensional Schwarzschild solution.

In our model, the center-of-mass motion of black hole of mass \( M \), in \( D = N + 1 \) space-time dimensions, is described by a scalar wave function \( \Phi \) with an action

\[ W = -\frac{1}{2} \int d^D x \left[ (\nabla \Phi)^2 + M^2 \Phi^2 \right], \tag{1} \]

and obeying the equation \( \Box \Phi - M^2 \Phi = 0 \).

We use the following mode decomposition for the quantum field \( \hat{\Phi}(X^A) = \int \frac{d^N P}{\sqrt{2\omega_P}} \frac{1}{(2\pi)^{N/2}} \left[ e^{-i\omega_P t + iP \cdot X} \hat{A}(P) + \text{h.c.} \right] \tag{2} \]

where h.c. stands for hermitian conjugate. The bulk energy is \( \omega_P = \sqrt{P^2 + M^2} \). Here and later we use the following notation: \( P^A \ (A = 1, ..., N) \) is total bulk momentum which has components \( p^i_{\parallel} \ (i = 1, 2, 3) \) along the brane, and \( p^a_{\perp} \ (a = 1, ..., N - 3) \) in the bulk direction. We also denote \( x^i \) coordinated along the brane, and \( y^a \) bulk coordinates. Thus, \( P \cdot X = p_{\parallel} x + p_{\perp} \cdot y \). In the case where there is no brane, all the space-like dimensions are equivalent.

Similarly, we write the mode decomposition for the bulk massless scalar field \( \varphi \)

\[ \varphi(X^A) = \int \frac{d^N K}{\sqrt{2\omega}} \frac{1}{(2\pi)^{N/2}} \left[ e^{-i\omega t + i K \cdot X} \hat{a}(K) + \text{h.c.} \right]. \tag{3} \]
Here \( \omega = K = |K| \) is the bulk energy.

We choose the interaction action in the following form

\[
W_{\text{int}} = \sum_{I \neq J} \lambda_{IJ} \int dX^D \tilde{\Phi}_I(X) \tilde{\Phi}_J(X) \tilde{\varphi}(X).
\] (4)

\( \lambda_{IJ} \) is the coupling constant between the two different internal states. We characterize these states \( I \) by the value of black hole mass \( M_I \). Emission of quanta of a bulk field \( \varphi \) by the black hole changes its mass and hence provides a transition \( I \rightarrow J \) to the lower energy state \( J \). The amplitude of probability \( A_{JK,I} \) of the particle ("black hole") transition from the initial state \( I \) to the final state \( J \) with emission of a massless quantum is

\[
A_{JK,I} = i\langle \mathbf{P}_J, \mathbf{K} | W_{\text{int}} | \mathbf{P}_I \rangle = i\lambda_{IJ} \frac{2^{-3/2}}{(2\pi)^{N/2-1}} (\omega_{\mathbf{P}_I} \omega_{\mathbf{P}_J} \omega^{-1})^{1/2} \\
\times \delta^N (\mathbf{P}_J - \mathbf{P}_I - \mathbf{K}) \delta (\omega_{\mathbf{P}_I} - \omega_{\mathbf{P}_J} - \omega).
\] (5)

We assume that initially black hole is at rest, so that \( \mathbf{P}_I = 0 \) and \( \omega_{\mathbf{P}_J} = M_I \). The probability for the black hole to emit a quantum with energy \( \omega \) per unit time is

\[
p(\omega) = \frac{(2\pi)^N}{\Delta t V_N} \sum_j \int d^N \mathbf{P}_J \int d\mathbf{n}_K \omega^{N-1} |A_{JK,I}|^2.
\]

Here \( V_N \) is the space volume and \( \Delta t \) is the total time duration. We also denoted \( \mathbf{n}_K = \mathbf{K}/K \) so that \( \int d\mathbf{n}_K \) is the averaging over direction of emitted quanta \( \mathbf{K} \).

After integration (the detailed calculations are given in [9]), the final probability of emission of a massless particle of energy \( \omega \) per unit time by our "back hole" of mass \( M \) is

\[
p(\omega|M) = \frac{\Omega_{N-1}}{4 (2\pi)^{N/2-1}} \frac{\omega^{N-2}}{M} \Lambda^2 (M^2, M^2 - 2M\omega).
\] (6)

where \( \Omega_{N-1} \) is a volume of a \((N-1)\)-dimensional unit sphere and we omit the subscript "I", i.e. \( M_I \equiv M \). We used \( M_J^2 - M_I^2 = \epsilon (I - J) \) and \( \lambda_{IJ} = \sqrt{\lambda} (M_J^2, M_J^2) \) in order to provide the correct limit to the continuous mass spectrum case (\( \epsilon \rightarrow 0 \)).

We demonstrate now that, for a special choice of the function \( \Lambda \), the probability rate (6) coincides with the probability of emission of scalar massless quanta of energy \( \omega \) by a black hole of mass \( M \). For \((N+1)\)-dimensional non-rotating Schwarzschild black hole, in the low frequency approximation, we have (see e.g. [10] and references therein):

\[
P(\omega|M) = \frac{(\omega R_0)^{N-2}}{2^{N-2} \Gamma^2(N/2)} \frac{1}{e^{2\omega R_0} - 1},
\] (7)

where the Schwarzschild radius of \((N+1)\)-dimensional black hole \( R_0 = \left[ \frac{16\pi G_* M}{(N-1)\beta} \right]^{1/(N-2)} \). \( \beta = (TR_0)^{-1} = (4\pi)/(N-2) \) is a dimensionless temperature and \( G_* = 1/M^{N-1} \) is a fundamental gravitational constant determined by a fundamental energy scale \( M_* \).

Comparing (6) with (7) one can conclude that if we want a decaying massive particle \( M \) to emit massless quanta with the same probability as an evaporating black hole in the low frequency approximation one must choose

\[
\Lambda^2 (M^2, M^2 - 2M\omega) = \frac{(2\pi)^{N-1}}{\pi^{N/2} 2^{N-3} \Gamma(N/2)} \frac{M R_0^{N-2}}{e^{2\omega R_0} - 1}.
\] (8)

This result is valid for any \((N+1)\)-dimensional black hole which can be considered as a point radiator and is independent from the brane world models. Since \( R_0 \) can be expressed in terms of \( M \), the right hand side of eq. (8) is a function of \( M \) and \( \omega \) which allows us to determine \( \Lambda \).

We note that the results concerning the field theoretical model describing a black hole as a point radiator, are quite general and are not restricted to brane world models.

Suppose now that there exists a brane representing our physical world embedded in a higher dimensional universe. For simplicity we assume that the brane has a co-dimension 1, i.e. there exist only one extra dimension. The model can be easily generalized to any number of dimensions. As earlier, we neglect the effects connected with the spin and consider emission of the scalar massless field. Moreover we use a simplified model to take into account the effects connected with the effect of the black hole with the brane. Namely, we neglect the effects of the brane gravitational field on the Hawking radiation, and use one parameter, \( \mu \), similar to a chemical potential, to describe the interaction of the black hole with the brane.

The action is

\[
W = -\frac{1}{2} \int d^Dx \left( (\nabla \Phi)^2 + U \Phi^2 \right),
\] (9)

with \( U = M^2 - 2\mu \delta(y) \). The field \( \Phi \) obeys the equation \( \Box \Phi - U \Phi = 0 \).

It is easy to see that there is only one level with positive \( \lambda = \mu^2 \). A wave function of the corresponding bound state is \( \Phi^{(0)}(y) = \sqrt{\mu} e^{-\mu |y|} \).

For negative \( \lambda \) the spectrum is continuous. We denote \( \lambda = -p_+^2 \). Solving the scattering problem for the \( \delta \)-like potential one obtains the following set of solutions

\[
\Phi_{p_+}^{(+)}(y) = e^{ip_+ y} - \frac{\mu}{\mu + ip_+} e^{-ip_+ y}, \quad y < 0,
\]

\[
\Phi_{p_+}^{(-)}(y) = \frac{ip_+}{\mu + ip_+} e^{ip_+ y}, \quad y > 0,
\]

\[
\Phi_{p_+}^{(-)}(y) = \frac{\mu}{\mu + ip_+} e^{-ip_+ y}, \quad y < 0,
\]

\[
\Phi_{p_+}^{(-)}(y) = \frac{i p_+}{\mu + ip_+} e^{ip_+ y}, \quad y > 0.
\]
The complete set of modes consists of two types of solutions. The first type are solutions describing a black hole attached to the brane which can freely propagate only along it:

$$\Phi^{(\pm)}_p(X) = \frac{e^{-i\omega t}}{\sqrt{2\omega(2\pi)^{(N-1)/2}}} \Phi_0(y) e^{i[\mp \omega \|y\|]}$$

(10)

where \(\hat{\omega}^2 = \hat{M}^2 + p_\perp^2\) and \(\hat{M}^2 = M^2 - \mu^2\). The second type of modes are bulk modes of the form

$$\Phi^{(\pm)}_p(X) = \frac{e^{-i\omega t}}{\sqrt{2\omega(2\pi)^{(N-1)/2}}} \Phi_{p\perp}(y) e^{i[\mp \omega \|y\|]}$$

(11)

where \(\omega^2 = M^2 + p_\perp^2 + p_\parallel^2\).

The field operator decomposition takes the form

$$\hat{\Phi}(X) = \int d^{N-1}p_\parallel \left[ \Phi^{(0)}_p(X) \hat{B}(p_\parallel) + \text{h.c.} \right] + \sum_{\pm} \int d^Np \left[ \Phi^{(\pm)}_p(X) \hat{A}_\pm(p) + \text{h.c.} \right].$$

For the scalar massless field we shall use the decomposition \(\Phi = \Phi^{(0)}\). This means that we neglect possible interaction of scalar quanta with the brane.

The amplitude of probability \(A_{JK,I}\) of the particle (“black hole”) to stay on the brane after emitting a massless quantum \(K\) is \(A^{\text{on}}_{JK,I} = i\langle P_J, K|W_{\text{in}}|P_I\rangle\), where \(W_{\text{in}}\) is given by (8). We choose as the initial state \(|I\rangle\) a state of the black hole at rest at the brane, \(|I\rangle = \hat{B}^\dagger(p_\parallel = 0)|0\rangle\). We are interested in those final states of a black hole when as a result of the recoil it leaves the brane and moves in the bulk space, i.e. \(|J, K\rangle = \hat{A}_\pm(p) \hat{a}^\dagger(K)|0\rangle\).

Thus, we have

$$A^{\text{on}}_{JK,I} = i\lambda_{JJ} \frac{2^{2-3/2}}{(2\pi)^{(N-1)/2}} (\hat{M}_I \omega_{P_J} \omega)^{-1/2}$$

$$\delta^{N-1}(p_\parallel + k_\parallel) \delta(\hat{M}_I - \omega_{P_J} - \omega_k) C_\pm,$$

where \(C_\pm = \int dy \Phi^{(0)}(y) e^{i(k_\parallel \cdot y)} \Phi^{(\pm)}_{p\perp}(y)\).

The total probability per unit time of the black hole emission which results in the recoil taking the black hole away from the brane is

$$w_{\text{off}} = \frac{(2\pi)^{N-1}}{\Delta t V_{N-1}} \sum_J \int d^Np_J \int dK |A^{\text{on}}_{JK,I}|^2.$$ (12)

Performing integrations using (8) and the fact that the main contribution to the integrals comes from the low frequency modes (the high frequency ones are suppressed by a thermal factor) give the following result for \(w_{\text{off}}\) in \(N = 4\):

$$w_{\text{off}}(\nu) = \frac{\mu}{8\pi^2 \nu} \int_0^\infty d\chi \frac{\chi^2(\sqrt{\chi^2 + \nu^2} - \nu)}{\sqrt{\chi^2 + \nu^2}} \frac{1}{\exp(\chi) - 1}.$$ (13)

in terms of dimensionless variables \(\chi = ak\) and \(\nu = 2\mu a = 8\sqrt{\frac{a^2}{\pi}} \sqrt{G/M}\), where \(a = 4\sqrt{\frac{2\pi}{\sqrt{3}}} \sqrt{G/M}\) and \(k = \sqrt{k_\parallel^2 + k_\perp^2}\) is the absolute value of the total momentum of the emitted quantum \(K\). Also, \(\hat{M}_I = \hat{M}\).

Calculation of the probability that the black hole remains on the brane is analogous to the previous calculation. The amplitude of probability \(A_{JK,I}\) for the particle (“black hole”) to stay on the brane after emitting a massless quantum \(K\) is \(A^{\text{on}}_{JK,I} = i\langle P_J, K|W_{\text{in}}|P_I\rangle\), where \(W_{\text{in}}\) is again given by (8). We choose as the initial state \(|I\rangle\) a state of the black hole at rest at the brane, \(|I\rangle = \hat{B}^\dagger(p_\parallel = 0)|0\rangle\). We are interested in those final states of a black hole when as a result of the recoil it remains on the brane, i.e. \(|J, K\rangle = \hat{A}_\pm(p) \hat{a}^\dagger(K)|0\rangle\).

Repeating calculations we obtain (for \(N = 4\)):

$$w_{\text{on}}(\nu) = \frac{8\mu}{\pi} \int_0^\infty d\chi \frac{\chi^2}{\sqrt{\chi^2 + \nu^2}} \frac{1}{\exp(\chi) - 1}.$$ (13)

in the same dimensionless variables as for \(w_{\text{off}}\).

It is easy to see that the sum \(w = w_{\text{off}}(\nu) + w_{\text{on}}(\nu)\) does not depend on \(\nu\). In fact \(w\) coincides with the total probability of emission of a massless field quantum by the 5-dimensional black hole.

$$w = \int_0^\infty d\omega P(\omega|M),$$ (14)

where \(P(\omega|M)\) is given by (7). The plots of the functions \(w_{\text{off}}(\nu)/w\) and \(w_{\text{on}}(\nu)/w\) are shown at Fig. \(\text{FIG. 1}\). Analysis shows that, for large \(\nu\) (or \(\mu\)), the probability \(w_{\text{off}}\) falls off as \(1/\nu\), while for small \(\nu\) (or \(\mu\)) the (normalized) probability \(w_{\text{off}}\) goes to 1.

![FIG. 1. Functions $w_{\text{off}}(\nu)/w$ and $w_{\text{on}}(\nu)/w$.](image-url)

In literature, the process of evaporation of mini black holes in brane world models is well studied. The pattern of such a black hole decay is markedly different from any other standard model event. There exist very detailed calculations estimating the energy spectrum and ratio between emitted particles (leptons, photons, hadrons etc.) \(\text{[9]}\). It is also claimed that, because of very little missing energy, the determination of the mass and the temperature of the black hole may lead to a test of Hawking’s
radiation. The recoil effect discussed above may change some of these predictions. Certainly the most important observable effect of a black hole recoil is a suddenly disrupted evaporation and local non-conservation of energy.

We developed a phenomenological model for description of the recoil effect. This model contains an important parameter $\mu$ which play the role of the chemical potential. In the general case, $\mu$ depends on the mass of a black hole $M$ and the tension $\sigma$ of the brane, $\mu = \mu(M, \sigma)$. Unfortunately, to determine this dependence is not an easy task. One can expect that in the limit $\sigma \to 0$, that is for a test brane, $\mu \to 0$. In this case, it is very likely that the black hole leaves the brane as soon as emits first quanta with non-zero bulk momentum (see Fig. 4).

We can also estimate $\mu$ for small $\sigma$ as follows. Consider two different states: first, a brane with a black hole of radius $R_0$, and the second, when the same black hole is out of the brane. The second configuration has extra energy $\Delta E \sim \sigma R_0^3$ (for 1 extra dimension). One can identify $\Delta E$ with $M - \tilde{M}$. This gives $\mu^2 \approx 2M\Delta E$, (see eq. (10) or $\mu \sim \sigma^{1/2} R_0^{5/2} \sim \sigma^{1/2} M_0^{5/2}$.

In the other limit of infinitely heavy brane, or a brane with $Z_2$ symmetry, the process of a black hole leaving the brane reminds to a black hole splitting into two symmetric black holes in the “mirror” space. Classically this process is forbidden in a higher dimensional space-time for the same reason as in (3 + 1)-dimensional space-time in connection with non-decreasing property of the entropy. In the presence of cosmological constant, such an effect may become possible as a tunneling process. These arguments show that in this case $\mu \to \infty$ or is exponentially large, and the recoil effect is suppressed. This feature could help in distinguishing between the two different scenarios of compact and infinite extra dimensions. Also, if the general conclusion that the recoil effect may be important for branes of small tension is correct, it opens an interesting possibility of using the experiments with decay of mini black holes to put restrictions on the brane tension.

The above order of magnitude estimates of the parameter $\mu$ do not follow from the exact calculations and therefore further investigation is required.

Acknowledgments: The authors are grateful to Don Page and Glenn Starkman for stimulating discussions. This work was partly supported by the Natural Sciences and Engineering Research Council of Canada. The authors are grateful to the Killam Trust for its financial support.

*Our attention was drown by Don Page who pointed out that more detailed analysis shows that the estimate for $\mu$ given above is quite valid in the case of the brane of a co-dimension 3 and higher (three or more extra spatial dimensions). For co-dimensions 1 and 2 result might depend on a regime in which we extract the black hole, i.e. whether it is an adiabatic process or not.