Interference of a Bose-Einstein condensate in a hard-wall trap: Formation of vorticity

J. Ruostekoski, B. Kneer, and W. P. Schleich
Abteilung für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany
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We theoretically study the coherent expansion of a Bose-Einstein condensate in the presence of a confining impenetrable hard-wall potential. The nonlinear dynamics of the macroscopically coherent matter field results in rich and complex spatio-temporal interference patterns demonstrating the formation of vorticity and solitonlike structures, and the fragmentation of the condensate into coherently coupled pieces.

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A landmark experiment demonstrated in a striking way the interference of two freely expanding Bose-Einstein condensates (BECs). In this paper we theoretically study the evolution of a BEC in a coherently reflecting hard-wall trap. The present situation is closely related to the recent experiments on an expanding BEC in an optically-induced ‘box’ by Ertmer and coworkers. Due to the macroscopic quantum coherence the BEC exhibits rich and complex self-interference patterns. We identify the formation of vorticity and solitonlike structures, and the dramatic fragmentation of an initially uniform parabolic BEC into coherently coupled pieces.

Atomic BECs exhibit a macroscopic quantum coherence in an analogy to the optical coherence of lasers. In the conventional reasoning the coherence of a BEC is introduced in the spontaneous symmetry breaking. Nevertheless, even two BECs with no phase information, could show relative phase correlations as a result of the back-action of quantum measurement. Moreover, the density-dependent self-interaction of a BEC demonstrates the analogy between nonlinear laser optics and nonlinear atom optics with BECs. BECs are predicted to exhibit dramatic coherence properties: The macroscopic coherent quantum tunneling and the formation of fundamental structures, e.g., vortices and solitons. Some basic properties of grey solitons have been recently addressed for harmonically trapped 1D BECs in Ref. Also optical solitons have been actively studied in the 1D homogeneous space.

In this paper we study the dynamics of a BEC confined to a hard-wall trap with potential \(V(r)\). Such walls can be realized, e.g., with a blue-detuned far-off-resonant light sheet. Throughout the paper we focus on repulsive interactions. When the BEC is released from a magneto-optical trap (MOT) inside the confining potential by suddenly turning off the MOT, the repulsive mean-field energy of the condensate transforms into kinetic energy and the BEC rapidly expands towards the walls. The reflections of the matter wave from the binding potential result in rich and complex spatio-temporal and interference patterns referred to in 1D as quantum carpets. They have been recently proposed as a thermometer for measuring the temperature of the BEC. In this paper we show that the nonlinear dynamics of a BEC displays dramatic local variations of the condensate phase including the formation of vorticity and solitonlike structures. The solitary waves could possibly be used as an experimental realization of the macroscopic coherent tunneling analogous to the Josephson effect.

The dynamics of a BEC follows from the Gross-Pitaevskii equation (GPE)

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2M} \nabla^2 + V(r) + \kappa |\psi(r; t)|^2 \right] \psi(r; t).
\]

Here \(M\) and \(\kappa \equiv 4\pi \hbar^2 a N / M\) denote the atomic mass and the coefficient of the nonlinearity, respectively, with the scattering length \(a\) and the number \(N\) of BEC atoms. Our initial distribution \(\psi(r; t = 0)\) is the stationary solution of the GPE with the potential \(V(r)\) replaced by the potential of the MOT.

We integrate GPE in one and two spatial dimensions. The projections from 3D into 1D or 2D require that the mean field \(\psi\) does not vary significantly as a function of time in the corresponding orthogonal directions. This condition can be satisfied, e.g., in the presence of a strong spatial confinement to these dimensions. Then we can approximate the position dependence in these directions by constants \(A\) and \(\ell\) resulting in \(\psi(r; t) \approx \psi_1(x)/A^{1/2}\) in 1D and \(\psi(r; t) \approx \psi_2(x,y)/\ell^{1/2}\) in 2D. This yields the strengths \(\kappa_1 = \kappa / A\) and \(\kappa_2 = \kappa / \ell\) of the nonlinearity in GPE for the mean fields in 1D and 2D \(\psi_1\) and \(\psi_2\). We emphasize that especially the 2D calculations may already contain the essential features of the full 3D coupling between the different spatial dimensions by the nonlinearity.

One-dimensional case – The linear Schrödinger equation for the 1D box of length \(L\) exhibits regular spatio-temporal patterns. These patterns consist of straight lines, so called traces, of different steepness corresponding to harmonics of a fundamental velocity, \(v_0\). The traces arise from the interferences between degenerate eigenmodes of the system. The eigenmodes of frequency \(\omega_n \equiv n^2 \omega_1 \equiv n^2 \hbar \pi^2 / (2ML^2)\) are a superposition of right and left propagating plane waves with wavenumbers \(k_n \equiv n \pi / L \equiv nk_1\). The probability density consists of the interferences between different eigenmodes. Therefore the lines of constant phase \(\pm (k_n x + \omega_n t)\) correspond to straight lines in space-time with velocities
The soliton is called dark, when it exhibits \[ v_{0} \equiv \hbar \pi / (2mL). \]

We now turn to the quantum carpet of Fig. 1 representing a 1D BEC trapped between two impenetrable steep Gaussian potentials approximating infinitely high walls at \( x = \pm L/2 \). At time \( t = 0 \) the BEC is released from a harmonic trap of frequency \( \Omega \). In the limit of strong confinement the initial state is well approximated by the Thomas-Fermi solution \( \psi_1(x; t = 0) = \theta(R_1 - |x|)[3(R_1^2 - x^2)/(2R_1^3)]^{1/2} \). Here \( R_1 \equiv [3\kappa_1/(M\Omega^2)]^{1/3} \) describes the 1D radius of the BEC wave function.

After the turn-off of the MOT the kinetic energy term in GPE becomes dominant and the matter wave expands towards, and eventually reflects from, the box boundaries. Due to the macroscopic quantum coherence of a BEC different spatial regions of the matter field generate a complex self-interference pattern that exhibits canals analogously to the quantum carpet structures of the linear Schrödinger equation \([12]\). From Fig. 1 we note that only destructive interference fringes, canals, appear in the nonlinear carpet. Constructive interference fringes, ridges, do not emerge.

For the single particle Schrödinger equation the resulting quantum carpet demonstrates the fundamental wave nature of the particle. Therefore, it is perhaps surprising that in the case of GPE, which represents the coherent matter field of the many-particle system, the intermode traces acquire properties that demonstrate dramatic particle nature. In particular, in the case of repulsive interparticle interactions the traces correspond to the evolution of solitonlike structures with an associated phase kink in Fig. 1a,c.

Grey solitons correspond to the propagation of ‘density holes’ in the matter field and the depth of the hole characterizes the greyness of the soliton. In homogeneous space the times of the phase profiles (b) and (c). Here time is scaled in units of the revival time \( T \equiv 2\pi/\omega_1 \equiv 4ML^2/\pi\hbar \) of the linear Schrödinger equation and the initial state is \( R/L \equiv 0.3 \) and \( k_1 = 190000/\pi \).

**Two-dimensional case** – We now turn to the evolution of a BEC in a circular box. Again we approximate the infinitely high walls by a steep Gaussian potential aligned along a circle of radius \( R = L/2 \). The initial wave function is the Thomas-Fermi solution of the symmetric MOT \( \psi_2(x, y; t = 0) = 2(R_2^2 - x^2 - y^2)/(\pi R_2^4) \), and \( R_2 \equiv 4\kappa_2/\pi \). Hence, the initial wave function is located at the center of the circular box and we can show that in this case the state remains symmetric also at later times.

Figure 2 shows the 2D density \( |\psi_2(x, y; t)|^2 \) and the phase profile \( |\varphi| \) of a BEC obtained from GPE at a later time. We note the formation of a regular interference patterns similar to solitary waves. The fragmentation of the BEC is even more dramatic than in the 1D case: The fringes exhibit a vanishing density at the center of the dip. The BEC forms coherently coupled loops and the resulting structures are similar to optical ring solitons [1].

When we now slightly displace the initial state of the BEC from the center of the circular box, the rotational symmetry is broken. In Figs. 3 and 4 we show the resulting 2D density (left column) and the phase profiles (right...
column) at three characteristic times. The reflections of the BEC from the hard-wall potential create solitonlike structures. At later times the stripes bend and eventually break up forming dark spots that correspond to vortices with associated phase windings around closed paths.

![Image 1](image1.png)

**FIG. 2.** Formation of loops of solitonlike structures in a BEC expanding in a 2D circular box. The symmetric condensate has started from the center of the circular box, has expanded and reflected from the boundary. We show the density (left) and phase (right) profiles of the BEC at time $t = 4.9 \times 10^{-3} ML^2/\hbar$. In the phase profile we have chosen a continuous phase $|\varphi|$, where $-\pi \leq \varphi \leq \pi$. The initial state is $R_2/L \simeq 0.28$ and $\kappa_2 \equiv 1500\hbar/(ML^2)$.

We note that we can use the present situation of an expanding BEC in a circular box to create vortices in some particular spatial location by simply introducing a static potential dip and letting the expanding BEC flow across it. This is a simplified version of the suggestion by Ref. [1] that a moving potential barrier through a BEC can create vorticity in the vicinity of the potential.

As a final example and to demonstrate the effect of the symmetry of the hard-wall trap we consider the evolution of a BEC in the 2D square box. We generate the boundary by steep Gaussians approximating infinitely high walls at $x = \pm L/2$ and $y = \pm L/2$.

The square has the symmetry of rotations of $\pi/2$. Hence, for a symmetric nonrotating initial state the minimum number of vortices conserving the total angular momentum is eight. In Fig. 3 we show the density profile at two characteristic times. The reflections of the BEC from the boundary generate dark solitonlike structures with an amazingly regular square shape that start bending, break up, and form vorticity.

![Image 2](image2.png)

**FIG. 3.** Formation of vorticity in a BEC expanding in a 2D circular box with displaced initial state. We show the density profiles (left column) and the corresponding phase profiles (right column) of the BEC at three characteristic times $t = 2.5 \times 10^{-3}$, $4.9 \times 10^{-3}$, and $6.1 \times 10^{-3}$ in units of $ML^2/\hbar$. The reflections from the boundary generate solitonlike structures that bend, break up, and form vorticity. The velocity of the BEC is proportional to the gradient of the phase.

![Image 3](image3.png)

The density profiles of the BECs could be directly measured via absorption imaging, if the necessary spatial resolution could be obtained, e.g., via ballistic expansion of the atomic cloud. Vortices may also be detected by interfering a BEC with and without vorticity. Then the phase slip in the interference fringes would be the signature of the vorticity. The phase slip between two pieces of the BEC has a dramatic effect on the dynamical structure factor of the two-component system [17], which may be observed, e.g., via the Bragg scattering [18].

In conclusion, we studied the generation of vorticity and solitonlike structures of a BEC in a hard-wall trap. The nonlinear evolution of GPE dramatically divides the initially uniform parabolic BEC into coherently coupled pieces. We showed that the density profile of the BEC can be a direct manifestation of the macroscopic quantum coherence. Obviously it could also be a sensitive measure for the decoherence rate of the BEC [19]. Unlike the typical coherence measurement that detects the relative macroscopic phase between two well-distinguishable BECs [14], the present set-up probes the self-interference of an initially uniform matter field.

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![Image 4](image4.png)

**FIG. 4.** The magnification of the last density and phase profile of Fig. 3 displaying vorticity. For example, the dark spot in the center represents a vortex with a unit circular quantization of $2\pi$. 

In recent experiments Bongs et al. [2] studied the coherent reflections of a BEC by atom-optical mirrors and the evolution of a BEC in an atom-optical waveguide closely related to the present theoretical study.
FIG. 5. Expanding BEC in a 2D square box with a symmetric initial state at the center of the square. At \( t = 2.9 \times 10^{-3} ML^2/\hbar \) (left) the density profile of the BEC exhibits a rectangular pattern of solitonlike structures. At a later time \( t = 3.9 \times 10^{-3} ML^2/\hbar \) (right) the pattern is distorted and we see the beginning of the formation of vorticity symmetrically around the diagonals. The initial radius of the BEC is \( R_2/L \simeq 0.28 \) and \( \kappa_2 = 5000 \hbar^2/(ML^2) \).

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