Abstract

$L^p$ boundedness of the circular maximal function $M_{\mathbb{H}^1}$ on the Heisenberg group $\mathbb{H}^1$ has received considerable attentions. While the problem still remains open, $L^p$ boundedness of $M_{\mathbb{H}^1}$ on Heisenberg radial functions was recently shown for $p > 2$ by Beltran et al. (Ann Sc Norm Super Pisa Cl Sci. https://doi.org/10.2422/2036-2145.202001-006, 2021). In this paper we extend their result considering the local maximal operator $M^t_{\mathbb{H}^1}$ which is defined by taking supremum over $1 < t < 2$. We prove $L^p - L^q$ estimates for $M^t_{\mathbb{H}^1}$ on Heisenberg radial functions on the optimal range of $p, q$ modulo the borderline cases. Our argument also provides a simpler proof of the aforementioned result due to Beltran et al.

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1 introduction

For $d \geq 2$ the spherical maximal function is given by

$$M_{\mathbb{R}^d} f(x) = \sup_{t > 0} \left\{ \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} f(x - ty) d\sigma(y) \right\},$$

where $S^{d-1} \subset \mathbb{R}^d$ is the $(d - 1)$-dimensional sphere centered at the origin and $d\sigma$ is the surface measure on $S^{d-1}$. When $d \geq 3$, it was shown by Stein [21] that $M_{\mathbb{R}^d} f$...
is bounded on $L^p$ if and only if $p > \frac{d}{d-1}$. The case $d = 2$ was later settled by Bourgain [5]. An alternative proof of Bourgain's result was subsequently found by Mockenhaupt, Seeger, Sogge [11], who used a local smoothing estimate for the wave operator. We now consider the local maximal operator

$$M_{\mathbb{R}^d} f(x) = \sup_{1 < t < 2} \left| \int_{S^{d-1}} f(x - ty)d\sigma(y) \right|.$$

As is easy to see, the maximal operator $M_{\mathbb{R}^d}$ can not be bounded from $L^p$ to $L^q$ unless $p = q$. However, $M_{\mathbb{R}^d}$ is bounded from $L^p$ to $L^q$ for some $p < q$ thanks to the supremum taken over the restricted range $[1, 2]$. This phenomenon is called $L^p$ improving. Almost complete characterization of $L^p$ improving property of $M_{\mathbb{R}^2}$ was obtained by Schlag [17] except for the endpoint cases. A different proof which is based on $L^p-L^q$ smoothing estimate for the wave operator was also found by Schlag and Sogge [18]. They also proved $L^p-L^q$ boundedness of $M_{\mathbb{R}^d}$ for $d \geq 3$ which is optimal up to the borderline cases. Most of the left open endpoint cases were settled by the second author [8] but there are some endpoint cases where $L^p-L^q$ estimate remains unknown though restricted weak type bounds are available for such cases. There are results which extend the aforementioned results to variable coefficient settings, see [18, 19]. Also, see [1, 4, 14] and references therein for recent extensions of the earlier results.

The analogous spherical maximal operators on the Heisenberg group $H^n$ also have attracted considerable interests. The Heisenberg group $H^n$ can be identified with $\mathbb{R}^{2n} \times \mathbb{R}$ under the noncommutative multiplication law

$$(x, x_{2n+1}) \cdot (y, y_{2n+1}) = (x + y, x_{2n+1} + y_{2n+1} + x \cdot Ay),$$

where $(x, x_{2n+1}) \in \mathbb{R}^{2n} \times \mathbb{R}$ and $A$ is the $2n \times 2n$ matrix given by

$$A = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The natural dilation structure on $H^n$ is $t(x, x_{2n+1}) = (tx, t^2x_{2n+1})$. Abusing the notation, since there is no ambiguity, we denote by $d\sigma$ the usual surface measure of $S^{2n-1} \times \{0\}$. Then, the dilation $d\sigma_t$ of the measure $d\sigma$ is defined by $\langle f, d\sigma_t \rangle = \langle f(t \cdot), d\sigma \rangle$. Thus, the average over the sphere is now given by

$$f \ast_{H^n} d\sigma_t(x, x_{2n+1}) = \int_{S^{2n-1}} f(x - ty, x_{2n+1} - tx \cdot Ay)d\sigma(y).$$

We consider the associated local spherical maximal operator

$$M_{\mathbb{H}^n} f(x, x_{2n+1}) = \sup_{1 < t < 2} |f \ast_{\mathbb{H}^n} d\sigma_t(x, x_{2n+1})|.$$

Similarly, the global maximal operator $\mathcal{M}_{\mathbb{H}^n}$ is defined by taking supremum over $t > 0$. As in the Euclidean case, $L^p$ boundedness of $M_{\mathbb{H}^n}$ is essentially equivalent to
that of $\mathcal{M}_{\mathbb{H}^n}$ (for example, see [2] or Section 2.5). The spherical maximal operator on $\mathbb{H}^n$ was first studied by Nevo and Thangavelu [13]. It is easy to see that $M_{\mathbb{H}^n}$ is bounded on $L^p$ only if $p > \frac{2n}{2n-1}$ by using Stein’s example ([21]):

$$f(x, x_{2n+1}) = |x|^{1-2n} \log \frac{1}{|x|} \phi_0(x, x_{2n+1}),$$

where $\phi_0$ is a cutoff function supported near the origin. For $n \geq 2$, $L^p$ boundedness of $M_{\mathbb{H}^n}$ on the optimal range was independently proved by Müller and Seeger [10], and by Narayanan and Thangavelu [12]. Furthermore, for $n \geq 2$, Roos, Seeger and Srivastava [15] recently obtained the complete $L^p-L^q$ estimate for $M_{\mathbb{H}^n}$ except for some endpoint cases. Also see [7] for related results.

However, the problem still remains open when $n = 1$.

**Definition** We say a function $f : \mathbb{H}^1 \to \mathbb{C}$ is Heisenberg radial if $f(x, x_3) = f(Rx, x_3)$ for all $R \in \text{SO}(2)$.

Beltran, Guo, Hickman and Seeger [2] obtained $L^p$ boundedness of $\mathcal{M}_{\mathbb{H}^1}$ on the Heisenberg radial functions for $p > 2$. In the perspective of the results concerning the local maximal operators [8, 15, 17, 18], it is natural to consider $L^p-L^q$ estimate for $M_{\mathbb{H}^1}$. The main result of this paper is the following which completely characterizes $L^p$ improving property of $M_{\mathbb{H}^1}$ on Heisenberg radial function except for some borderline cases.

**Theorem 1.1** Let $P_0 = (0, 0, 0)$, $P_1 = (1/2, 1/2)$, and $P_2 = (3/7, 2/7)$, and let $T$ be the closed region bounded by the triangle $\Delta P_0 P_1 P_2$. Suppose $(1/p, 1/q) \in \{P_0\} \cup (T \setminus (P_1 P_2 \cup P_0 P_2))$. Then, the estimate

$$\|M_{\mathbb{H}^1} f\|_q \lesssim \|f\|_{L^p}$$

(1.1)

holds for all Heisenberg radial function $f$. Conversely, if $(1/p, 1/q) \notin T$, then the estimate fails.

Though the Heisenberg radial assumption significantly simplifies the structure of the averaging operator, the associated defining function of the averaging operator is still lacking of curvature properties. In fact, the defining function has vanishing rotational and cinematic curvatures at some points, see [2] for a detailed discussion. This increases the complexity of the problem. To overcome the issue of vanishing curvatures, Beltran et al. [2] used the oscillatory integral operators with two-sided fold singularities and the variable coefficient version of local smoothing estimate [3] combined with additional localization.

The approach in this paper is quite different from that in [2]. Capitalizing on the Heisenberg radial assumption, we make a change of variables so that the averaging operator on the Heisenberg radial function takes a form close to the circular average, see (2.1) below. While the defining function of the consequent operator still does not have nonvanishing rotational and cinematic curvatures, via a further change of variables we can apply the $L^p-L^q$ local smoothing estimate (see, Theorem 3.1 below)
in a more straightforward manner by exploiting the apparent connection to the wave operator (see (2.2) and (2.3)). Consequently, our approach also provides a simplified proof of the recent result due to Beltran et al. [2]. See Sect. 2.5.

Even though we utilize the local smoothing estimate, we do not need to use the full strength of the local smoothing estimate in \( d = 2 \) since we only need the sharp \( L^p - L^q \) local smoothing estimates for \((p, q)\) near \((7/3, 7/2)\). Such estimates can also be obtained by interpolation and scaling argument if one uses the trilinear restriction estimates for the cone and the sharp local smoothing estimate for some large \( p \) (for example, see [9]).

The estimate (1.1) remains open when \((1/p, 1/q) \in (P_1P_2 \cup P_0P_2)\setminus\{P_0, P_1\}\). However, we expect that those borderline cases should be true. Most of the corresponding endpoint estimates for the circular maximal function (in \( \mathbb{R}^2 \)) are known to be true [8], but to implement the approach in [8] we need the local smoothing estimate without \( \varepsilon \)-loss regularity, which we are not able to establish yet even under the Heisenberg radial assumption.

We close the introduction showing the necessity part of Theorem 1.1.

Optimality of \( p, q \) range. We show (1.1) implies \((1/p, 1/q) \in T\), that is to say,

\[
\begin{align*}
\text{(a)} & \ p \leq q, \\
\text{(b)} & \ 1 + 1/q \geq 3/p, \\
\text{(c)} & \ 3/q \geq 2/p.
\end{align*}
\]

To see (a), let \( f_R \) be the characteristic function of a ball of radius \( R \gg 1 \), centered at 0. Then, \( M_{\mathbb{H}^1}f_R \) is also supported in a ball \( B \) of radius \( \sim R \) and \( M_{\mathbb{H}^1}f_R \gtrsim 1 \) on \( B \). Thus, \( \sup_{R > 1} \|M_{\mathbb{H}^1}f_R\|_p / \|f_R\|_p \) is finite only if \( p \leq q \). For (b) let \( g_r \) be the characteristic function of a ball of radius \( r \ll 1 \) centered at 0. Then, \( |M_{\mathbb{H}^1}g_r(x, x_3)| \gtrsim r \) when \((x, x_3)\) is contained in a \( c_0r \) -neighborhood of \[ \{(x, x_3) : 1 < |x| < 2, x_3 = 0\} \] for a small constant \( c_0 > 0 \). Thus, (1.1) implies \( r^{1+1/q} \lesssim r^{-3/p} \), which gives \( 1 + 1/q \geq 3/p \) if we let \( r \to 0 \). Finally, to show (c) we consider \( h_r \) which is the characteristic function of an \( r \) -neighborhood of \[ \{(x, x_3) : |x| = 1, x_3 = 0\} \] with \( r \ll 1 \). Then, \[ |M_{\mathbb{H}^1}h_r(x, x_3)| \gtrsim c > 0 \] when \((x, x_3)\) is in an \( r \) -ball centered at 0. Thus, (1.1) gives \( r^{3/q} \lesssim r^{-2/p} \), which yields \( 3/q \geq 2/p \).

The maximal estimate (1.1) for general \( L^p \) functions has a smaller range of \( p, q \). Let \( h_r \) be a characteristic function of the set \[ \{(x, x_3) : |x_1 - 1| < r^2, |x_2| < r, |x_3| < r\} \] for a sufficiently small \( r > 0 \). Then \( M_{\mathbb{H}^1}h_r \approx r \) if \( -1 \leq x_1 \leq 0, |x_2| < cr, |x_3| < cr \) for a small constant \( c > 0 \) independent of \( r \). Thus, (1.1) implies \( r^{1+2/q} \lesssim r^{-4/p} \).

It seems to be plausible to conjecture that (1.1) holds for general \( f \) modulo some endpoint cases as long as \( 1 + 2/q - 4/p \geq 0, 3/q \geq 2/p, \) and \( 1/q \leq 1/p \). The range of \( p, q \) is properly contained in \( T \).

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1 while assuming Proposition 2.1 and Proposition 2.2 (see below), which we show in the next section.
2.1 Heisenberg radial function

Since $f$ is a Heisenberg radial function, we have $f(x, x_3) = f_0(|x|, x_3)$ for some $f_0$. Let us set

$$g(s, z) = f_0(\sqrt{2s}, z), \quad s \geq 0.$$  

Then, it follows $f(x, x_3) = g(|x|^2/2, x_3)$. Since $f \ast_{\mathbb{H}} d\sigma_t(r, 0, x_3) = \int f(r - ty_1, -ty_2, x_3 - try_2) d\sigma(y) = \int g(r^2 + t^2, x_3) d\sigma(y)$, we have

$$f \ast_{\mathbb{H}} d\sigma_t(r, 0, x_3) = g \ast d\sigma_{tr}\left(\frac{r^2 + t^2}{2}, x_3\right).$$  

(2.1)

Let us define an operator $A_t$ by

$$A_t g(r, x_3) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(r^2 + t^2) \xi_1 + x_3 \xi_2} \hat{d\sigma}(tr\xi) \hat{g}(\xi) d\xi. $$  

(2.2)

Using Fourier inversion, we have

$$f \ast_{\mathbb{H}} d\sigma_t(r, 0, x_3) = A_t g(r, x_3).$$  

(2.3)

Since $f \ast_{\mathbb{H}} d\sigma_t$ is also Heisenberg radial, $\|M_{\mathbb{H}}^f f\|_q = \int |M_{\mathbb{H}}^f f(r, 0, x_3)|^q r dr dx_3$. A computation shows $\|f\|_{L^p_{r,x_3}} = \|g\|_{L^p_{r,x_3}}$. Therefore, we see that the estimate (1.1) is equivalent to

$$\|r^{1/q} \sup_{1 < t < 2} |A_t g|\|_{L^q_{r,x_3}} \leq C\|g\|_p.$$  

(2.4)

In what follows we show (2.4) holds for $p, q$ satisfying

$$p \leq q, \quad 3/p - 1/q < 1, \quad 1/p + 2/q > 1.$$  

(2.5)

Then, interpolation with the trivial $L^\infty$ estimate proves Theorem 1.1.

2.2 Decomposition

Let $\phi$ denote a positive smooth function on $\mathbb{R}$ supported in $[1 - 10^{-3}, 2 + 10^{-3}]$ such that $\sum_{j=\infty}^{\infty} \phi(s/2^j) = 1$ for $s > 0$. We set $\phi_j(s) = \phi(s/2^j)$. To show (2.4) we decompose $A_t$ as follows:

$$A_t g(r, x_3) = \sum_{k \in \mathbb{Z}} \phi_k(r) A_t g(r, x_3).$$

1 This is true because SO(2) is an abelian group. However, SO(n) is not commutative in general, so the property is not valid in higher dimensions.
We break $g$ via the Littlewood–Paley decomposition and try to obtain estimates for each decomposed pieces. For the purpose we denote $\phi_{<j} = \sum_{\ell < j} \phi_{\ell}$ and $\phi_{\geq j} = \sum_{\ell \geq j} \phi_{\ell}$ and define the projection operators

$$\hat{P}_j g(\xi) := \phi_j(|\xi|) \hat{g}(\xi), \quad \hat{P}_{<j} g(\xi) := \phi_{<j}(|\xi|) \hat{g}(\xi).$$

Our proof of (2.4) mainly relies on the following two propositions, which we prove in Sect. 3.

**Proposition 2.1** Let $|k| \geq 2$ and $j \geq -k$. Suppose

$$p \leq q, \quad 1/p + 1/q \leq 1, \quad 1/p + 3/q \geq 1.$$ (2.6)

Then, for $\epsilon > 0$ we have

$$\left\| \sup_{1 < t < 2} |\phi_k(r) A_t \hat{P}_j g| \right\|_{L^q_t L^p_{r,x^3}} \lesssim 2^{j(3/p - 1/q - 1/2 + \epsilon j)} \|g\|_{L^p}, \quad k \geq 2,$$

$$2^{j(3/p - 1/q - 1/2 + \epsilon j)} \frac{2}{q} \|\hat{g}\|_{L^p}, \quad k < -2.$$ (2.7)

The estimate (2.7) continues to be valid for the case $k = -1, 0, 1$. However, the range of $p, q$ for which (2.7) holds gets smaller.

**Proposition 2.2** Let $j \geq -1$ and $k = -1, 0, 1$. Suppose $p \leq q$, $1/p + 1/q < 1$ and $1/p + 2/q > 1$. Then, for $\epsilon > 0$ we have

$$\left\| \sup_{1 < t < 2} |\phi_k(r) A_t \hat{P}_j g| \right\|_{L^p_t L^p_{r,x^3}} \lesssim 2^{j(3/p - 1/q - 1/2)} \|\hat{g}\|_{L^p}.$$

We frequently use the following elementary lemma (for example, see [8]) which plays the role of the Sobolev imbedding.

**Lemma 2.3** Let $I$ be an interval and let $F$ be a smooth function defined on $\mathbb{R}^n \times I$. Then, for $1 \leq p \leq \infty$,

$$\left\| \sup_{t \in I} |F(x, t)| \right\|_{L^p(\mathbb{R}^n)} \lesssim |I|^{-\frac{1}{p}} \|F\|_{L^p(\mathbb{R}^n \times I)} + \|F\|_{L^p(\mathbb{R}^n \times I)}^{(p-1)} \|\partial_t F\|_{L^p(\mathbb{R}^n \times I)}^{\frac{1}{p}}.$$

2.3 Proof of (2.4)

We prove (2.4) handling the three cases $k \leq -2$, $|k| \leq 1$, and $k \geq 2$, separately. We first consider a change of variables

$$(r, x_3, t) \rightarrow (y_1, y_2, \tau) := \left( \frac{r^2 + t^2}{2}, x_3, rt \right).$$ (2.8)
which plays an important role in what follows. Note that

$$\det \frac{\partial (y_1, y_2, \tau)}{\partial (r, x_3, t)} = r^2 - t^2. \quad (2.9)$$

In order to show (2.4), we shall use the change of variables (2.8) to apply the local smoothing estimate to the averaging operator $A_t$ (see Sect. 3.1). Since $1 < t < 2$, $|\det \partial (y_1, y_2, \tau)/\partial (r, x_3, t)| = |r^2 - t^2| \sim \max(2^{2k}, 1)$ for $|k| \geq 2$. Thus, the cases $|k| \geq 2$ can be handled directly by using local smoothing estimates for the half wave propagator. However, the determinant of the Jacobian may vanish when $|k| \leq 1$. This requires further decomposition away from the set $\{r = t\}$. See Sect. 3.3. This is why we need to consider the three cases separately.

Let us set $g_k = \mathcal{P}_{< k} g$ and $g^k = g - \mathcal{P}_{< k} g$ so that $g = g_k + g^k$. Then, we break

$$\phi_k(r) A_t g = \phi_k(r) A_t g_k + \phi_k(r) A_t g^k. \quad (2.10)$$

We use Propositions 2.1 and 2.2 to obtain the estimate for $\phi_k(r) A_t g^k$, whereas we show the estimate for $\phi_k(r) A_t g_k$ by elementary means using (2.2).

**Case $k \leq -2$**

We claim that

$$\left\| \frac{1}{r^q} \sum_{k \leq -2} \sup_{1 < t < 2} |\phi_k(r) A_t g| \right\|_{L^p_{r,x}} \lesssim \|g\|_{L^p} \quad (2.11)$$

holds provided that $p, q$ satisfy $2/p < 3/q, 3/p - 1/q < 1$, and (2.6). Thus (2.11) holds for $p, q$ satisfying (2.5).

We first consider $\phi_k(r) A_t g_k$. We shall show that

$$\left\| \frac{1}{r^q} \sup_{1 < t < 2} |\phi_k(r) A_t g_k| \right\|_{L^q_{r,x}} \lesssim 2^{\frac{3k}{q} - \frac{2k}{p}} \|g\|_{L^p} \quad (2.12)$$

holds for $1 \leq p \leq q \leq \infty$. We recall (2.2) and note that $\partial_t (d\sigma (r \xi))$ is uniformly bounded because $|r \xi| \lesssim 1$. Since supp $\hat{g}_k \subset \{\xi : |\xi| \leq C 2^{-k}\}$ and $\partial_t e^{\frac{2|\xi|^2 + t}{\xi^2}} = t \xi_1 e^{\frac{2|\xi|^2 + t}{\xi^2}}$, we have $\|\phi_k(r) \partial_t A_t g_k\|_q \lesssim 2^{-k} \|\phi_k(r) A_t g_k\|_q$ by the Mikhlin multiplier theorem. Applying Lemma 2.3 to $\phi_k(r) A_t g_k$, we see that (2.12) follows if we show

$$\|\phi_k(r) A_t g_k\|_{L^q_{r,x}} \lesssim 2^{\frac{3k}{q} - \frac{2k}{p}} \|g\|_{L^p}. \quad (2.13)$$

We now make use of the change of variables (2.8). Since $k \leq -2$ and $t \in [1, 2]$, we have $|\det \frac{\partial (y_1, y_2, \tau)}{\partial (r, x_3, t)}| \sim 1$. Thus the left hand side of (2.13) is bounded by

$$C \left\| \phi_k(r (y_1, y_2, \tau)) \int e^{i \hat{\xi} \cdot \hat{y}} \hat{g}(\xi) d\sigma (\tau \xi) \phi_{< k}(\xi) d\xi \right\|_{L^q_{y,\tau}} \lesssim 2^{\frac{3k}{q} - \frac{2k}{p}} \|g\|_{L^p}. \quad (2.13)$$
Changing variables $\xi \to 2^{-k} \xi$ and $(y, \tau) \to (2^k y, 2^k \tau)$ gives

$$
\| \phi_k(r) A_t g_k \|_{L^q_{r,x_3}} \lesssim 2^{\frac{3k}{q}} \left( \sum_{k \leq -2} 2^{\frac{3k}{q} - \frac{2k}{p}} \right) \| g \|_p \lesssim \| g \|_p
$$

where $m(\xi) = \hat{d}(r \xi) \phi_{<0}(\xi)$. Since $r \sim 1$ and $\phi_{<0}(\xi)$ is a smooth function supported in the set $\{ \xi : |\xi| \lesssim 1 \}$, $m(\xi)$ is a smooth multiplier whose derivatives are uniformly bounded. So, the multiplier operator given by $m$ is uniformly bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for $r \in [2^{-1}, 2^2]$. Thus, via scaling we obtain (2.13) and, hence, (2.12).

Using the triangle inequality and (2.12), we have

$$
\left\| r^{-\frac{3}{q}} \sup_{1 < t < 2} \left( \sum_{k \leq -2} |\phi_k(r) A_t g_k| \right) \right\|_{L^q_{r,x_3}} \lesssim \left( \sum_{k \leq -2} 2^{\frac{3k}{q} - \frac{2k}{p}} \right) \| g \|_p \lesssim \| g \|_p
$$

because $2/p < 3/q$. We now consider $\phi_k(r) A_t g^k$ for which we use Proposition 2.1. Since

$$
\left\| r^{-\frac{3}{q}} \sup_{1 < t < 2} \left( \sum_{k \leq -2} |\phi_k(r) A_t g^k| \right) \right\|_{L^q_{r,x_3}} \leq \sum_{k \leq -2} \sum_{j \geq -k} \left\| r^{-\frac{3}{q}} \sup_{1 < t < 2} |\phi_k(r) A_t P_j g^k| \right\|_{L^q_{r,x_3}}
$$

and since $p, q$ satisfy $3/p - 1/q < 1, 2/p < 3/q$, and (2.6), using the estimate (2.7), we get

$$
\left\| r^{-\frac{3}{q}} \sup_{1 < t < 2} \left( \sum_{k \leq -2} |\phi_k(r) A_t g^k| \right) \right\|_{L^q_{r,x_3}} \lesssim \left( \sum_{k \leq -2} 2^{\frac{3k}{q} - \frac{2k}{p}} \right) \| g \|_p \lesssim \| g \|_p.
$$

Combining this with the above estimate for $g \to \phi_k(r) A_t g^k$ gives (2.11) and this proves the claim.

**Case $k \geq 2$**

In this case we show

$$
\left\| r^{-\frac{3}{q}} \sup_{k \geq 2} \left( \sum_{1 < t < 2} |\phi_k(r) A_t g^k| \right) \right\|_{L^q_{r,x_3}} \lesssim \| g \|_{L^p} \tag{2.14}
$$

if $p \leq q, 3/p - 1/q < 1$, and (2.6) holds. So, we have (2.14) if (2.5) holds.

In order to prove (2.14) we first prove the following.

**Lemma 2.4** Let $k \geq -1$. If $|t| \lesssim 1$ and $0 \leq s \lesssim 2^k$, then

$$
|A_t P_{<k} g|((\sqrt{2} s, x_3) \lesssim \mathcal{E}_{\ell}^N * |g| (s, x_3), \tag{2.15}
$$

where $\mathcal{E}_{\ell}^N(y) = 2^{-2\ell} (1 + 2^{-\ell} |y|)^{-N}$. 

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Proof We note that

\[ A_t \mathcal{P}_{s-k} g(\sqrt{2s}, x_3) = K \ast g(s + 2^{-1}t^2, x_3), \]

where

\[ K(y) = \frac{1}{(2\pi)^2} \int e^{iy \cdot \xi} \hat{\phi}_{s-k}(\xi) \hat{d}\sigma(t \sqrt{2s} \xi) d\xi. \]

We note \( \partial_\xi^r [\phi_{s-k}(2^{-k} \xi) \hat{d}\sigma(2^{-k} t \sqrt{2s} \xi)] = O(1) \) since \( s \lesssim 2^k \). Thus, changing variables \( \xi \to 2^{-k} \xi \), by integration by parts we have \( |K| \lesssim \mathcal{E}^N_k \) for any \( N > 0 \). Since \( |t| \lesssim 1 \) and \( k \geq -1 \), we see \( \mathcal{E}^N_k (y_1 + 2^{-1}t^2, y_2) \lesssim \mathcal{E}^N_k (y_1, y_2) \). Therefore, we get (2.15).

Proof of 2.14 We begin by observing a localization property of the operator \( A_t \). From (2.1) we note that

\[ \frac{r^2 + t^2}{2} - t r y_1 \subset I_k := [2^{2k-1}(1 - 10^{-2}), 2^{2k+1}(1 + 10^{-2})] \]

for \( r \in \text{supp} \phi_k \) if \( k \) is large enough, i.e., \( 2^{-k} \lesssim 10^{-3} \). Thus, from (2.1) and (2.3) we see that

\[ \phi_k(r) A_t g(r, x_3) = \phi_k(r) A_t ([g]_k)(r, x_3) \]

(2.16)

where \([g]_k(r, x_3) = \chi_{I_k}(r) g(r, x_3)\). Clearly, the intervals \( I_k \) are finitely overlapping and so are the supports of \( \phi_k \). Since \( p \leq q \), by a standard localization argument it is sufficient for (2.14) to show

\[ \left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r) A_t g| \right\|_{L^q_t, x_3} \lesssim \|g\|_{L^p} \]

(2.17)

for \( k \geq 2 \).

Using the decomposition (2.10), we first consider \( \phi_k(r) A_t g_k \). Changing variables \( r \mapsto \sqrt{2s} \), we have

\[ \left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r) A_t g_k| \right\|_{L^q_t, x_3} \lesssim \int \phi_k(\sqrt{2s}) \left( \sup_{1 < t < 2} |A_t g_k(\sqrt{2s}, x_3)| \right)^q d s d x_3. \]

Since \( 1 < t < 2, k \geq 2 \), and \( g_k = \mathcal{P}_{s-k} g \), by Lemma 2.4 \( |A_t g_k(\sqrt{2s}, x_3)| \lesssim \mathcal{E}^N_k * |g|(s, x_3) \). Hence,

\[ \left\| r^{\frac{1}{q}} \sup_{1 < t < 2} |\phi_k(r) A_t g_k| \right\|_{L^q_t, x_3} \lesssim \|\mathcal{E}^N_k * |g|\|_{L^q_t, x_3} \lesssim 2^{2k(1/q - 1/p)} \|g\|_p \leq \|g\|_p. \]
The second inequality follows by Young’s convolution inequality and the third is clear because $k \geq 2$ and $p \leq q$. We now handle $\phi_k(r)A_tg^k$. Since

$$\left\| \frac{1}{t} \sup_{1 < t < 2} |\phi_k(r)A_tg^k| \right\|_{L^q_{t,x3}} \leq \sum_{j \geq -k} \left\| \frac{1}{t} \sup_{1 < t < 2} |\phi_k(r)A_tP_jg| \right\|_{L^q_{t,x3}}$$

(2.18)

and since $3/p - 1/q < 1$, $p \leq q$, and (2.6) holds, using the estimate (2.7), we get

$$\left\| \frac{1}{t} \sup_{1 < t < 2} |\phi_k(r)A_tg^k| \right\|_{L^q_{t,x3}} \lesssim 2^{\frac{2k}{p} - \frac{2k}{q}} \|g\|_p \lesssim \|g\|_p.$$

Therefore, we get (2.17).

2.4 Case $|k| \leq 1$

To complete the proof of (2.4), the matter is now reduced to obtaining

$$\left\| \frac{1}{t} \sup_{1 < t < 2} |\phi_k(r)A_tg| \right\|_{L^q_{t,x3}} \lesssim \|g\|_p, \quad k = -1, 0, 1$$

if $p, q$ satisfy (2.5). In order to show this we use Proposition 2.2. Using the decomposition (2.10), we first consider $\phi_k(r)A_tg_k$. Since $1 < t < 2$ and $|k| \leq 1$, by Lemma 2.4 we have $\phi_k(r)|A_tg_k| \lesssim E_0^N \ast |g|$. Hence, it follows that

$$\left\| \frac{1}{t} \sup_{1 < t < 2} |\phi_k(r)A_tg^k| \right\|_{L^q_{t,x3}} \lesssim \|g\|_p$$

for $1 \leq p \leq q \leq \infty$.

We now consider $\phi_k(r)A_tg^k$. Note that (2.6) is satisfied if (2.5) holds. Since $3/p - 1/q < 1$, by (2.18) and Proposition 2.2 we see

$$\left\| \frac{1}{t} \sup_{1 < t < 2} |\phi_k(r)A_tg^k| \right\|_{L^q_{t,x3}} \lesssim \sum_{j \geq -k} 2^{\frac{j}{q} \left(\frac{3}{p} - 1\right) + \epsilon j} \|g\|_{L^p} \lesssim \|g\|_p$$

taking a small enough $\epsilon > 0$. Therefore we get the desired estimate.

2.5 Global maximal estimate

Using the estimates in this section, one can provide a simpler proof of the result due to Beltran et al. [2], i.e.,

$$\left\| \frac{1}{t} \sup_{0 < t < \infty} |A_tg| \right\|_{L^p_{t,x3}} \leq C \|g\|_p$$

(2.19)

for $2 < p \leq \infty$. In order to show this we use the following lemma which is a consequence of Propositions 2.1 and 2.2.
Lemma 2.5  Let $2 \leq p \leq 4$. Then, for some $c > 0$ we have

$$
\| r^{\frac{1}{p}} \sup_{1 < t < 2} |A_t \mathcal{P} j g| \|_{L^p_{r x_3}} \leq C 2^{-c j} \|g\|_p. 
$$

(2.20)

Proof  We briefly explain how one can show (2.20). In fact, similarly as before, we decompose

$$
A_t \mathcal{P} j g = S_1 + S_3 + S_3 + S_4,
$$

where

$$
S_1 := \sum_{k < -j} \phi_k(r) A_t \mathcal{P} j g, \quad S_2 := \sum_{-j \leq k \leq -2} \phi_k(r) A_t \mathcal{P} j g, \quad S_3 := \sum_{-1 \leq k \leq 1} \phi_k(r) A_t \mathcal{P} j g,
$$

and $S_4 = A_t \mathcal{P} j g - S_1 - S_2 - S_3$. Then, the estimate (2.20) follows if we show

$$
\| r^{\frac{1}{p}} \sup_{1 < t < 2} |S_\ell| \|_{L^p_{r x_3}} \leq C 2^{-c j} \|g\|_p, \quad \ell = 1, 2, 3, 4 \text{ for some } c > 0.
$$

The estimate for $S_1$ follows from (2.12) and summation over $k < -j$. Using the estimate of the second case in (2.7), one can easily get the estimate for $S_2$. The estimate for $S_3$ is obvious from Proposition 2.2. By Proposition 2.1 combined with the localization property (2.16) we can obtain the estimate for $S_4$. However, due to the projection operator $\mathcal{P} j$ we need to modify the previous argument slightly.

From (2.1) and (2.3) we see

$$
A_t \mathcal{P} j g(r, x_3) = \int \int g(z_1, z_2) K_j \left( \frac{r^2 + t^2}{2} - z_1 - t r y_1, x_3 - z_2 - t r y_2 \right) d\sigma(y) d z,
$$

(2.21)

where $K_j = \mathcal{F}^{-1}(\phi(2^{-j} \cdot \cdot \cdot ))$. Note that $|K_j| \lesssim E_{-j}^N$ for any $N$ and $k \geq 2$. If $r \in \text{supp} \phi_k, \sqrt{2z_1} \notin I_k,$ and $k$ is large enough, then we have

$$
|K_j \left( \frac{r^2 + t^2}{2} - t r y_1 - z_1, x_3 - t r y_2 - z_2 \right)| \lesssim 2^{-(2k+j)N} \left(1 + 2^j |r^2 - 2z_1| + 2^{-k} |x_3 - z_2| \right)^{-N}
$$

for any $N$ since $|2^{-1} r^2 - z_1| \gtrsim 2^{2k}$ and $|t r y| \lesssim 2^k$. Hence it follows that

$$
\| r^{\frac{1}{p}} \phi_k(r) A_t \mathcal{P} j (1 - \chi_{I_k}) g \|_p \leq C 2^{-(k+j)N} \|g\|_p, \quad 1 \leq p \leq \infty
$$

for any $N$. We break $A_t \mathcal{P} j g = A_t \mathcal{P} j \chi_{I_k} g + A_t \mathcal{P} j (1 - \chi_{I_k}) g$. Using the last inequality and then Proposition 2.1, we obtain
\[ \|S_4\|_p \leq \left( \sum_{k \geq 2} \| r^{\frac{1}{p}} \phi_k(r) A_p P_j \chi_{L_k} g \|_p^p \right)^{\frac{1}{p}} + \sum_{k \geq 2} 2^{-(k+j)N} \| g \|_p \lesssim 2^{-cj} \| g \|_p \]

for some \( c > 0 \) by taking an \( N \) large enough. \( \square \)

Once we have (2.20), using a standard argument which relies on the Littlewood–Paley decomposition and rescaling (for example, see [2, 5, 16]) one can easily show (2.19). Indeed, we break the maximal function into high and lower frequency parts:

\[ \sup_{0 < t < \infty} |A_t g| \leq A_{low} g + A_{high} g, \]

where

\[ A_{low} g = \sup_l \sup_{2^l \leq t < 2^{l+1}} |A_t P_{<-2l} g|, \]
\[ A_{high} g = \sum_{k \geq 0} \sup_l \sup_{2^l \leq t < 2^{l+1}} |A_t P_{k-2l} g|. \]

For \( A_{low} g \) we claim

\[ \sup_{2^l \leq t < 2^{l+1}} |A_t P_{<-2l} g(r, x_3)| \lesssim M_{\mathbb{R}^2} g(2^{-1}r^2, x_3). \tag{2.22} \]

This gives \( A_{low} g(r, x_3) \lesssim M_{\mathbb{R}^2} g(2^{-1}r^2, x_3) \). Since \( M_{\mathbb{R}^2} \) is bounded on \( L^p \) for \( p > 2 \), for \( 2 < p \leq \infty \) we get

\[ \|r^{\frac{1}{p}} A_{low} g\|_{L^p_{r^2, x_3}} \leq C \|g\|_p. \]

We now proceed to prove (2.22). Note that \( \sum_{j \leq 2l} \phi(2^{-j} \cdot | \cdot |) = \phi_{<1}(2^{2l} \cdot | \cdot |) \) and \( \phi_{<1} \) is a smooth function supported on \([-2^2, 2^2]\). Thus, similarly as in (2.21) we note that

\[ A_t P_{<-2l} g(r, x_3) = \int \int g(z_1, z_2) \tilde{K}_l \ast d\sigma_{tr}(2^{-1}(r^2 + t^2) - z_1, x_3 - z_2) dz \]

where \( \tilde{K}_l = F^{-1}(\phi_{<1}(2^{2l} \cdot | \cdot |)) \). Since \( \tilde{K}_l \lesssim \mathcal{E}_{2l}^N \) for any \( N \), for \( 2^l \leq t < 2^{l+1} \) we see

\[ |A_t P_{<-2l} g(r, x_3)| \lesssim \int \int |g(z_1, z_2)| \mathcal{E}_{2l}^N \ast d\sigma_{tr}(2^{-1}r^2 - z_1, x_3 - z_2) dz \tag{2.23} \]

because \( 2^{2l}t^2 \lesssim 1 \) and \( \mathcal{E}_{2l}^N = 2^{-4l}(1 + 2^{-2l}|y|)^{-2N} \). Hence, taking an \( N \) large enough, we note that

\[ \mathcal{E}_{2l}^N \ast d\sigma_{tr}(x) \lesssim \begin{cases} (2^{2l}tr)^{-1}(1 + 2^{-2l}|x| - tr)^{-N}, & 2^{2l} \ll tr, \\
2^{-4l}(1 + 2^{-2l}|x|)^{-N}, & 2^{2l} \gtrsim tr, \end{cases} \tag{2.24} \]

provided that \( 2^l \leq t < 2^{l+1} \). Indeed, to show this we only have to consider the case \( 2^{2l} \ll tr \) since the other case is trivial. By scaling \( x \to trx \) we may assume that
\(tr = 1\). Thus, it is enough to show \(\int L^{-2}(1 + L^{-1}|x - y|)^{-2N} d\sigma(y) \lesssim L^{-1}(1 + L^{-1}|x - y|)^{-N}\) for \(L \ll 1\) with an \(N\) large enough. However, this is easy to see since \(|x - y| \geq |x| - 1|\) and \(\int L^{-1}(1 + L^{-1}|x - y|)^{-N} d\sigma(y) \lesssim 1\).

Therefore, combining (2.23) and (2.24), one can see

\[
\sup_{2^l \leq r < 2^{l+1}} |A_r \mathcal{P}_{-2l} g(r, x_3)| \lesssim \mathcal{M}_{\mathbb{R}^2} g(2^{-1} r^2, x_3) + \mathcal{M}_{\mathbb{R}^2} g(-2^{-1} r^2, x_3).
\]

Here \(\mathcal{M}_{\mathbb{R}^2}\) denotes the Hardy-Littlewood maximal function on \(\mathbb{R}^2\). This proves the claim (2.22) since \(\mathcal{M}_{\mathbb{R}^2} g \lesssim \mathcal{M}_{\mathbb{R}^2} g\).

So we are reduced to showing \(\|r^p A_{\text{high}} g\|_{L^p_{r, x_3}} \leq C \|g\|_p\) for \(p > 2\). For the purpose it is sufficient to show

\[
\| \sup_{2^l \leq r < 2^{l+1}} |A_r \mathcal{P}_{k-2l} g| \|_p \lesssim 2^{-ck} \|g\|_p
\]

(2.25)

because \(A_{\text{high}} g \leq \sum_{k \geq 0} (\sum_{l} |A_r \mathcal{P}_{k-2l} | g|^{p})^{1/p} \) and \((\sum_{l} \| \mathcal{P}_{k-2l} g\|_p^{p})^{1/p} \lesssim \|g\|_p\). By scaling, using (2.2), we can easily see the inequality (2.25) is equivalent to (2.20) while \(j\) replaced by \(k\). So, we have (2.25) and this completes the proof of (2.19).

3 Proof of Propositions 2.1 and 2.2

In order to prove Propositions 2.1 and 2.2, we are led by (2.2) to consider \(\widehat{\sigma}(tr\xi)\) for which we use the following well known asymptotic expansion (see, for example, [20]):

\[
\widehat{\sigma}(\xi) = \sum_{j=0}^{N} C_j ||\xi||^{-1-j} e^{\pm i|\xi|} + E_N(||\xi||), \quad ||\xi|| \gtrsim 1
\]

(3.1)

where \(E_N\) is a smooth function satisfying

\[
|\frac{d^\ell}{dr^\ell} E_N(r)| \lesssim r^{-N}
\]

(3.2)

for \(0 \leq \ell \leq 4\) if \(r \gtrsim 1\). The expansion (3.1) relates the operator \(A_t\) to the wave propagator. After changing variables, to prove Propositions 2.1 and 2.2 we can use the local smoothing estimate for the wave operator (see Proposition 3.1 below).

3.1 Local smoothing estimate

Let us denote

\[
e^{it\sqrt{-\Delta}} f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x-\xi+t|\xi|)} \widehat{f}(\xi) d\xi.
\]
We make use of \( L^p - L^q \) local smoothing estimate for the wave equation in \( \mathbb{R}^2 \).

**Theorem 3.1** Let \( j \geq 0 \). Suppose (2.6) holds. Then, for \( \epsilon > 0 \) we have

\[
\left\| e^{it\sqrt{-\Delta}} P_j f \right\|_{L^q_{t,r}(\mathbb{R}^2 \times [1,2])} \lesssim 2^{\frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) j + \epsilon j} \left\| f \right\|_{L^p} \quad (3.3)
\]

This follows by interpolating the estimates (3.3) with \((p, q) = (2, 2), (1, \infty), \) and \((4, 4)\). The estimate (3.3) with \((p, q) = (2, 2)\) is a straightforward consequence of Plancherel's theorem and (3.3) with \((p, q) = (1, \infty)\) can be shown by the stationary phase method (for example, see [8]). The case \((p, q) = (4, 4)\) is due to Guth et al. [6].

From Theorem 3.1 we can deduce the following estimate via simple rescaling argument.

**Corollary 3.2** Let \( j \geq -\ell \). Suppose (2.6) holds. Then, for \( \epsilon > 0 \) we have

\[
\left\| e^{it\sqrt{-\Delta}} P_j f \right\|_{L^q_{t,r}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} \lesssim 2^{\frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) (\ell + j) + \epsilon (\ell + j)} \left\| f \right\|_{L^p}.
\]

**Proof** Changing variables \((x, t) \rightarrow 2^\ell (x, t)\), we see

\[
\left\| e^{i\ell\sqrt{-\Delta}} P_j f \right\|_{L^q_{t,r}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} = 2^\frac{3\ell}{q} \left\| e^{it\sqrt{-\Delta}} P_{\ell + j} f (2^\ell \cdot) \right\|_{L^q_{t,r}(\mathbb{R}^2 \times [1,2])}.
\]

Thus, using (3.3) we have

\[
\left\| e^{i\ell\sqrt{-\Delta}} P_j f \right\|_{L^q_{t,r}(\mathbb{R}^2 \times [2^\ell, 2^{\ell+1}])} \lesssim 2^{\frac{3\ell}{q} + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) (\ell + j) + \epsilon (\ell + j)} \left\| f (2^\ell \cdot) \right\|_{L^p}.
\]

So, rescaling gives the desired inequality. \( \square \)

### 3.2 Proof of Proposition 2.1

We now recall (2.2) and (3.1). To show Proposition 2.1 we first deal with the contribution from the error part \( E_N \). Let us set

\[
E_t g(r, x_3) = \int e^{i \left( \frac{r^2}{2} + \xi_1^2 + x_3^2 \right)} E_N(tr|\xi|) \hat{g}(\xi) d\xi.
\]

**Lemma 3.3** Let \( j \geq -k \). Suppose (2.6) holds. Then, we have

\[
\left\| \sup_{1 < t < 2} |\phi_k(r) E_t P_j g| \right\|_{L^q_{t,x_3}} \lesssim \begin{cases} 2^{-(N-3)(j+k)} 2^k \left( \frac{1}{p} - \frac{1}{q} \right) \| g \|_{L^p}, & k \geq -2, \\ 2^{-(N-3)(j+k)} 2^k \left( \frac{3}{q} - \frac{2}{p} \right) \| g \|_{L^p}, & k < -2. \end{cases} \quad (3.4)
\]

**Proof** We first consider the case \( k \geq -2 \). Using Lemma 2.3, we need to estimate \( \phi_k(r) E_t P_j g \) and \( \phi_k(r) \partial_t E_t P_j g \) in \( L^q_{t,x_3} \((\mathbb{R}^2 \times [1, 2]\)) \). For simplicity we denote
\( L_{r,x_3,t}^q = L_{r,x_3,t}^q(\mathbb{R}^2 \times [1, 2]) \). We first consider \( \phi_k(r) \mathcal{E}_t \mathcal{P}_j g \). Changing variables \( r^2 \to s \), we note that

\[
\phi_k(\sqrt{2s}) \mathcal{E}_t \mathcal{P}_j g(\sqrt{2s}, x_3) = \phi_k(\sqrt{2s}) \int \mathcal{K}(s - y_1 + 2^{-1}t^2, x_3 - y_2) g(y_1, y_2) dy,
\]

where

\[
\mathcal{K}(s, u) = 2^{2j} \int e^{i(2j(s\xi_1 + u\xi_2))} \phi_0(\xi) E_N(2^j t \sqrt{2s} |\xi|) d\xi.
\]

Since \( s \sim 2^{2k} \), using (3.2), we have \( |\mathcal{K}(s, u)| \lesssim 2^{2j} (1 + 2^j |(s, u)|)^{-M 2^{-N(j+k)}} \) for \( 1 \leq M \leq 4 \) via integration by parts. Thus, we have \( \| \phi_k(\sqrt{2s}) \mathcal{K}(s + \frac{t^2}{2}, u) \|_{L^p_{s,u}} \leq C 2^{-N(j+k)} 2^{2j(1 - \frac{1}{p})} \| \mathcal{K}(s, u) \|_{L^p_{s,u}} \) for \( 1 < t < 2 \) with a positive constant \( C \). Young’s convolution inequality gives \( \| \phi_k(\sqrt{2s}) \mathcal{E}_t \mathcal{P}_j g(\sqrt{2s}, x_3) \|_{L^q_{r,x_3,t}} \lesssim 2^{-N(j+k)} 2^{2j(1 - \frac{1}{p})} \| g \|_{L^p_r} \). Thus, reversing \( s \to r^2/2 \), after a simple manipulation we get

\[
\| \phi_k(r) \mathcal{E}_t \mathcal{P}_j g \|_{L^q_{r,x_3,t}} \lesssim 2^{-(N-2)(j+k)-2(k(\frac{1}{q} - \frac{2}{p})} \| g \|_{L^p_r} \tag{3.5}
\]

for \( 1 \leq p \leq q \leq \infty \). Indeed, we need only note that \( 2j(\frac{1}{p} - \frac{1}{q}) - k \frac{1}{q} \leq 2(j + k) + k(\frac{1}{q} - \frac{2}{p}) \) because \( j \geq -k \) and \( \frac{1}{p} - \frac{1}{q} - 1 < 0 \).

We now consider \( \phi_k(r) \partial_t \mathcal{E}_t \mathcal{P}_j g \). Note that

\[
\partial_t \mathcal{E}_t g(r, x_3) = \int e^{i((r^2 + t^2)\xi_1 + x_3\xi_2))} ((t\xi_1 E_N(tr|\xi|) + r\xi_2 |E_N'(tr\xi_2)|) \hat{g}(\xi) d\xi. \tag{3.6}
\]

Using (3.2), we can handle \( \phi_k(r) \partial_t \mathcal{E}_t \mathcal{P}_j g \) similarly as before. In fact, since \( |t\xi_1| \lesssim 2^j \) and \( r|\xi| \sim 2^{k+j} \), we see

\[
\| \phi_k(r) \partial_t \mathcal{E}_t \mathcal{P}_j g \|_{L^q_{r,x_3}} \lesssim 2^{-(N-2)(j+k)+2k(\frac{1}{q} - \frac{2}{p})} (2j+k+2j) \| g \|_{L^p_r}.
\]

Hence, combining this and (3.5) with Lemma 2.3, we get (3.4) for \( k \geq -2 \).

We now consider the case \( k < -2 \). We first claim that

\[
\| \phi_k(r) \mathcal{E}_t \mathcal{P}_j g \|_{L^q_{r,x_3,t}} \lesssim 2^{-(N-2)(j+k)+2k(\frac{2}{q} - \frac{2}{p})} \| g \|_{L^p_r}. \tag{3.7}
\]

We use the transformation (2.8). By (2.9) we have \( |\frac{\partial(y_1,y_2,\tau)}{\partial(r,x_3,t)}| \sim 1 \). Therefore,

\[
\| \phi_k(r) \mathcal{E}_t \mathcal{P}_j g \|_{L^q_{r,x_3,t}} \lesssim \left( \int \left| \phi_k(r(y, \tau)) \tilde{K}(\cdot, \tau) \ast g(y) \right|^q dyd\tau \right)^{\frac{1}{q},}
\]
where

$$\tilde{K}(y, \tau) = \int e^{iy\cdot\xi} \phi_j(\xi) E_N(\tau |\xi|) d\xi.$$ 

Note that \(\tau \sim 2^k\). Changing \(\tau \mapsto 2^k \tau\) and \(\xi \mapsto 2^j \xi\), using (3.2) and integration by parts, we have \(|\tilde{K}(y, 2^k \tau)| \leq C 2^{2j} (1 + 2^j |y|)^{-M} 2^{-(j+k)}\) for \(1 \leq M \leq 4\) and \(1 < \tau < 2\). Young's convolution inequality gives

$$\|\phi_k(r) E_{\ell} P_j g\|_{L^q_{\varepsilon_1 \varepsilon_2}, t} \lesssim 2^{-N(j+k)} 2^{2j} 2^{j \left(\frac{1}{p} - \frac{1}{q}\right)} \|g\|_{L^p_{\varepsilon_1 \varepsilon_2}}.$$ 

Thus, we get (3.7). As for \(\phi_k(r) \partial_t E_{\ell} P_j g\), we use (3.6) and repeat the same argument to see \(\|\phi_k(r) \partial_t E_{\ell} P_j g\|_{L^q_{\varepsilon_1 \varepsilon_2}, t} \lesssim 2^{-N(j+k)} 2^{2j} 2^{j \left(\frac{1}{p} - \frac{1}{q}\right)} \|g\|_{L^p_{\varepsilon_1 \varepsilon_2}}\) since \(|t \varepsilon_1| \lesssim 2^j, r |\varepsilon_2| \sim 2^{k+j}, \) and \(k < -2\). Thus, we get

$$\|\phi_k(r) \partial_t E_{\ell} P_j g\|_{L^q_{\varepsilon_1 \varepsilon_2}, t} \lesssim 2^{-(N-2)(j+k)} 2^j 2^{k \left(\frac{1}{p} - \frac{1}{q}\right)} \|g\|_{L^p_{\varepsilon_1 \varepsilon_2}}.$$ 

Putting (3.7) and this together, by Lemma 2.3 we obtain (4.4) for \(k < -2\).

By (3.1) and Lemma 3.3, to prove Propositions 2.1 and 2.2 we only have to consider contributions from the remaining \(C_j^{r} |tr \xi|^{-\frac{1}{2}} e^{\pm i |tr \xi|}, j = 0, \ldots, N\). To this end, it is sufficient to consider the major term \(C_0^{r} |tr \xi|^{-\frac{1}{2}} e^{\pm i |tr \xi|}\) since the other terms can be handled similarly. Furthermore, by reflection \(t \mapsto -t\) it is enough to deal with \(|tr \xi|^{-\frac{1}{2}} e^{\pm i |tr \xi|}\) since the estimate (3.3) clearly holds with the interval \([1, 2]\) replaced by \([-2, -1]\).

Let us set

$$U_t g(r, x_3) = \int e^{i \left(\frac{r^2}{2} + r \xi_1 + x_3 \xi_2 + tr |\xi_1|\right) |r \xi_2|^{-\frac{1}{2}} g(\xi)} d\xi.$$ 

(3.8)

To complete the proof of Proposition 2.1, we need to show

$$\| \text{sup}_{1 < t < 2} |\phi_k(r) U_t P_j g| \|_{L^q_{\varepsilon_1 \varepsilon_2}, t} \lesssim \begin{cases} 2^{(j+k)(\frac{3}{2p} - \frac{1}{2q} - \frac{1}{2} + \epsilon) + \frac{k}{q} - \frac{2k}{p}} \|g\|_{L^p_{\varepsilon_1 \varepsilon_2}}, & k \geq 2, \\ 2^{(j+k)(\frac{3}{2p} - \frac{1}{2q} - \frac{1}{2} + \epsilon) + \frac{k}{q} - \frac{2k}{p}} \|g\|_{L^p_{\varepsilon_1 \varepsilon_2}}, & k \leq -2. \end{cases}$$

(3.9)

Using Lemma 2.3, the matter is reduced to obtaining estimates for \(\phi_k(r) U_t P_j g\) and \(\phi_k(r) \partial_t U_t P_j g\) in \(L^q_{\varepsilon_1 \varepsilon_2, t}\). Note that

$$\partial_t U_t P_j g(r, x_3, t) = \int e^{i \left(\frac{r^2}{2} + r \xi_1 + x_3 \xi_2 + tr |\xi_1|\right) |r \xi_2|^{-\frac{1}{2}} \partial_j g(\xi)} \frac{t \xi_1 + r |\xi_1|}{|r \xi_2|^{1/2}} d\xi.$$ 

(3.10)

By the Mikhlin multiplier theorem one can easily see

$$\|\phi_k(r) \partial_t U_t P_j g\|_{L^q_{\varepsilon_1 \varepsilon_2, t}} \lesssim \begin{cases} 2^{j+k} \|\phi_k(r) U_t P_j g\|_{L^q_{\varepsilon_1 \varepsilon_2, t}}, & k \geq 0, \\ 2^j \|\phi_k(r) U_t P_j g\|_{L^q_{\varepsilon_1 \varepsilon_2, t}}, & k < 0, \end{cases}$$

Theorem 2.4 and Lemma 2.5 imply that...
where $L^q_{r,x_3,t}$ denotes $L^q_{r,x_3,t}([\mathbb{R}^2 \times [1, 2])$. Therefore, by Lemma 2.3 it is sufficient for (3.9) to prove that

$$
\|\phi_k(r)U_j P_j g\|_{L^q_{r,x_3,t}} \lesssim \begin{cases} 
2(j+k)(\frac{1}{2^p} - \frac{3}{2^q} - \frac{1}{2} + \epsilon) + \frac{k}{q} - \frac{2k}{p} \|g\|_{L^p_r}, & k \geq 2, \\
2(j+k)(\frac{1}{2^p} - \frac{3}{2^q} - \frac{1}{2} + \epsilon) + \frac{k}{q} - \frac{2k}{p} \|g\|_{L^p_r}, & k \leq -2.
\end{cases}
$$

We first consider the case $k \geq 2$. As before, we use the change of variables (2.8). Since $|\det \frac{\partial (y_1,y_2,\tau)}{\partial (r,x_3,t)}| \sim 2^{2k}$ from (2.9) and since $\tau = rt$ and $1 < t < 2$, we have

$$
\|\phi_k(r)U_j P_j g\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{2k}{p} - \frac{i+k}{2} + \frac{j}{q}} \|e^{i\tau \sqrt{-\Delta}} P_j g\|_{L^q_y([\mathbb{R}^2 \times [2^{-1},2^{1+2}])}
$$

since $|r\xi| \sim 2^{j+k}$. Thus, Corollary 3.2 gives the desired estimate (3.9) for $k \geq 2$.

The case $k \leq -2$ can be handled in the exactly same manner. The only difference is that $|\det \frac{\partial (y_1,y_2,\tau)}{\partial (r,x_3,t)}| \sim 1$. Thus, the desired estimate (3.9) immediately follows from Corollary 3.2.

### 3.3 Proof of Proposition 2.2

As mentioned already, the determinant of the Jacobian $\partial (y_1,y_2,\tau) / \partial (r,x_3,t)$ may vanish when $|k| \leq 1$. So, we need additional decomposition depending on $|r - t|$. We also make decomposition in $\xi$ depending on $|\xi|^{-1} \xi_1 + 1$ to control the size of the multiplier $|r\xi_1 + r|\xi||$ in a more accurate manner (for example, see (3.22)).

For $m \geq 0$ let us set

$$
\psi_m(\xi) = \phi(2^m |\xi|^{-1} \xi_1 + 1),
\psi^m(\xi) = 1 - \sum_{0 \leq j < m} \psi_j(\xi),
$$

so that $\sum_{0 \leq k < m} \psi_k + \psi^m = 1$. We additionally define

$$
P_{j,m} g = (\phi_j \psi^m \hat{g})^\vee, \quad P_{j}^m g = (\phi_j \psi^m \hat{g})^\vee.
$$

So it follows that

$$
P_j = \sum_{0 \leq k < m} P_{j,k} + P_{j}^m. \quad (3.11)
$$

**Proposition 3.4** Let us set $\phi_k(t) = \phi_k(t)\phi(2^t|t - t|)$. Let $j \geq -1$ and $k = -1, 0, 1$. Suppose (2.6) holds. Then, for $\epsilon > 0$ we have

$$
\|\phi_k U_j P_{j,m} g\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{j}{2} + \frac{1}{p} + \frac{3}{2} - \frac{1}{2} + \epsilon} \|g\|_{L^p_r}. \quad (3.12)
$$

In order to prove Proposition 3.4, we make the change of variables (2.8). Since $|k| \leq 1$, we need only to consider $(r, t)$ contained in the set $[2^{-1} - 10^{-2}, 2^2 + 10^2] \times [1, 2]$. 

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Set
\[ S_l = \{(y_1, y_2, \tau) : 2^{-2l-1} \leq |y_1 - \tau| \leq 2^{-2l+1}, \ y_1, \tau \in [2^{-3}, 2^{3}]\}. \]

By (2.8) \( y_1 - \tau = (r - t)^2/2 \). From (2.9) we note \( |\det \frac{\partial(y_1, y_2; \tau)}{\partial(r; x_3, t)}| \sim 2^{-l} \) if \( (y_1, \tau) \in S_l \). Thus, changing variables \((r, x_3, t) \to (y_1, y_2, \tau)\) we obtain
\[
\|\hat{\phi}_{k,l} U_t P_j h\|_{L^q_{r,x_3,t}} \lesssim 2^{-\frac{l}{2}}2^\frac{l}{q} \|e^{i\tau \sqrt{-\Delta}} P_j h\|_{L^q_{y,,\tau}(S_l)}. \tag{3.13}
\]

Therefore, for (3.12) it is sufficient to show
\[
\|e^{i\tau \sqrt{-\Delta}} P_j m g\|_{L^q_{y,,\tau}(S_l)} \lesssim 2^{\left(\frac{m}{p} - l\right)(\frac{1}{p} + \frac{1}{q} - 1) + \frac{3j}{2} (\frac{1}{p} - \frac{1}{q}) + \epsilon} \|g\|_{L^p} \tag{3.14}
\]
for \( p, q \) satisfying (2.6). For the purpose we need the following lemma, which gives an improved \( L^2 \) estimate thanks to restriction of the integral over \( S_l \). Indeed, one can remove the localization \( y_1, \tau \in [2^{-3}, 2^{3}] \).

**Lemma 3.5** Let \( D_l = \{(x_1, x_2, t) : 2^{-2l} \leq |x_1 - t| \leq 2^{-2l+1}\}. \) Then, we have
\[
\left\| \int e^{i(x \cdot \xi + t|\xi|)} \hat{g}(\xi) \psi_m(\xi) d\xi \right\|_{L^2_{x,t}(D_l)} \lesssim 2^{\frac{m}{p}} \|g\|_{L^2}. \tag{3.15}
\]

**Proof** We write \( x \cdot \xi + t|\xi| = x_1(\xi_1 + |\xi|) + x_2\xi_2 + (t - x_1)|\xi| \). Then, changing variables \((x, t - x_1) \to (x, t)\) and \( \xi \to \eta := L(\xi) = (\xi_1 + |\xi|, \xi_2) \), we see
\[
\left\| \int e^{i(x \cdot \xi + t|\xi|)} \hat{g}(\xi) \psi_m(\xi) d\xi \right\|_{L^2_{x,t}(D_l)} \leq \left\| \int e^{i(x \cdot \eta + t|\eta|) L^{-1}\eta)} \hat{h}(L^{-1} \eta)} d\eta \right\|_{L^2_{x,t}(\mathbb{R}^2 \times I_l)}
\]
where \( \hat{h}(\xi) = \hat{g}(\xi) \psi_m(\xi) \) and \( I_l = [-2^{-2l+1}, 2^{-2l}] \cup [2^{-2l}, 2^{-2l+1}] \). By Plancherel’s theorem in the \( x \)-variable and integrating in \( t \), we have
\[
\left\| \int e^{i(x \cdot \xi + t|\xi|)} \hat{g}(\xi) \psi_m(\xi) d\xi \right\|_{L^2_{x,t}(D_l)} \leq C \left\| \hat{h}(L^{-1}) \right\|_{L^2} \frac{1}{|\det J L|}.
\]

A computation shows \( |\det J L| = 1 + |\xi|^{-1} |\xi_1| \), so \( |\det J L| \sim 2^{-m} \) on the support of \( \hat{h} \). Thus, by changing variables and Plancherel’s theorem we get (3.15). \( \square \)

We also use the following elementary lemma.

**Lemma 3.6** For any \( 1 \leq p \leq \infty \), \( j \), and \( m \), we have
\[
\| (\phi_j \psi_m \hat{g})^\gamma \|_{L^p} \lesssim \|g\|_{L^p}, \quad \| (\phi_j \psi_m \hat{g})^\gamma \|_{L^p} \lesssim \|g\|_{L^p}.
\]
**Proof** Since $\psi^m - \psi^{m+1} = \psi_m$, it suffices to prove the second inequality only. By Young’s inequality we need only to show $\|(\phi_j \psi^m)'\|_{L^1} \lesssim 1$. By scaling it is clear that $\|(\phi_j (\xi) \psi^m (\xi))'\|_{L^1} = \|(\phi_0 (\xi) \psi^m (\xi))'\|_{L^1}$. Note that $m(\xi) := \phi_0 (\xi) \psi^m (\xi)$ is supported in a rectangular box with dimensions $2^{-m} \times 1$. So, $m(\xi_1, 2^{-m} \xi_2)$ is supported in a cube of side length $\sim 1$ and it is easy to see $\partial_x^\alpha (m(\xi_1, 2^{-m} \xi_2))$ is uniformly bounded for any $\alpha$. This gives $\|(m(-, 2^{-m} \cdot))'\|_1 \lesssim 1$. Therefore, after scaling we get $\|(\phi_0 (\xi) \psi^m (\xi))'\|_{L^1} \lesssim 1$. 

**Proof of 3.14** In view of interpolation the estimate (3.14) follows for $p, q$ satisfying (2.6) if we show the next three estimates:

$$
\|e^{i \sqrt{-\Delta} P_{j,m} g}\|_{L^2_y, \tau (S_j)} \lesssim 2^{\frac{m}{2} - l} \|g\|_{L^2}, \quad (3.16)
$$

$$
\|e^{i \sqrt{-\Delta} P_{j,m} g}\|_{L^\infty_y, \tau (S_j)} \lesssim 2^j \|g\|_{L^1}, \quad (3.17)
$$

$$
\|e^{i \sqrt{-\Delta} P_{j,m} g}\|_{L^4_y, \tau (S_j)} \lesssim 2^{\frac{3}{2} j} \|g\|_{L^4}. \quad (3.18)
$$

The first estimate follows from Lemma 3.5, Corollary 3.2 and Lemma 3.6 give the other two estimates. 

It is possible to improve the estimate (3.12) when $j > m$.

**Proposition 3.7** Let $j \geq -1$ and $k = -1, 0, 1$. Suppose $1 \leq p \leq q$, $1/p + 1/q \leq 1$, and $j > m$, then

$$
\|\phi_k U_t P_{j,m} g\|_{L^q_y, \tau^3, \tau} \lesssim 2^{-\frac{1}{2} j} 2 \sqrt{2} 2^{\frac{1}{2} (\frac{m}{2} - l) + \frac{j-m}{2} (1 - \frac{1}{p} - \frac{1}{q}) + \frac{3}{2} j (1 - \frac{1}{p} - \frac{1}{q})} \|g\|_{L^p}. \quad (3.19)
$$

**Proof** By (3.13) it is sufficient to show

$$
\|e^{i \sqrt{-\Delta} P_{j,m} g}\|_{L^q_y, \tau (S_j)} \lesssim 2^\frac{3}{2} (\frac{m}{2} - l) + \frac{j-m}{2} (1 - \frac{1}{p} - \frac{1}{q}) + \frac{3}{2} j (1 - \frac{1}{p} - \frac{1}{q}) \|g\|_{L^p}
$$

for $p, q$ satisfying $1 \leq p \leq q$, $1/p + 1/q \leq 1$. In fact, by interpolation with the estimates (3.16) and (3.17) we only have to show

$$
\|e^{i \sqrt{-\Delta} P_{j,m} g}\|_{L^\infty_y, \tau (S_j)} \lesssim 2^{\frac{j-m}{2}} \|g\|_{L^\infty}. \quad (3.18)
$$

Let us set

$$
K_{\tau}^{j,m} (x) = \frac{1}{(2\pi)^2} \int e^{i(x - \xi + t|\xi|)} \phi_j (|\xi|) \psi_m (\xi) d\xi.
$$

Then $e^{i \sqrt{-\Delta} P_{j,m} g} = K_{\tau}^{j,m} * g$. Therefore, (3.18) follows if we show

$$
\|K_{\tau}^{j,m}\|_{L^1} \lesssim 2^{\frac{j-m}{2}}. \quad (3.19)
$$
when $t \sim 1$. Note that $|\xi_2|/|\xi| = \sqrt{1 - \xi_1/|\xi|} \sqrt{1 + \xi_1/|\xi|} \lesssim 2^{-m/2}$ if $\xi \in \text{supp } \psi_m$. So, supp $\psi_m$ is contained in a conic sector with angle $\sim 2^{-m/2}$. Let $S$ be a sector centered at the origin in $\mathbb{R}^2$ with angle $\sim 2^{-m/2}$ and $\phi_S$ be a cut-off function adapted to $S$. Then, by integration by parts it follows that

$$\left\| \int e^{i(x\cdot \xi + t/|\xi|)} \phi_j(|\xi|) \phi_S(\xi) d\xi \right\|_{L_1^3} \lesssim 1$$

if $t \sim 1$. (See, for example, [8]). Now (3.19) is clear since the support of $\psi_m$ can be decomposed into as many as $C 2^{-m/2}$ such sectors.

Finally, we prove Proposition 2.2 making use of Propositions 3.4 and 3.7. We recall (2.2) and (3.1). As mentioned before, by Lemma 3.3 we need only to consider $\mathcal{U}_t$ (see (3.8)) and it is sufficient to show

$$\left\| \sup_{1 < t < 2} |\phi_k(r)\mathcal{U}_t \mathcal{P}_j g| \right\|_{L_q^q} \lesssim 2^{1/2} \left( \frac{3}{p} - \frac{1}{q} - 1 \right) j + \epsilon \|g\|_{L^p} \quad (3.20)$$

for $p, q$ satisfying $p \leq q$, $1/p + 1/q < 1$ and $1/p + 2/q > 1$.

**Proof of 3.20** Let us set $\phi^j(\cdot) = 1 - \sum_{j=0}^{l-1} \phi(2^j \cdot)$ and $\phi^j_k(r, t) = \phi_k(r) \phi^j(|r - t|)$. Then, we decompose

$$\phi_k(r) = \sum_{0 \leq l \leq j/2} \phi_{k, l}(r, t) + \sum_{j/2 < l < j} \phi_{k, l}(r, t) + \phi^j_k(r, t).$$

Combining this with (3.11) and using $\sum_{j/2 < l < j} \phi_{k, l} \leq \phi^j_k \lesssim j$, by the triangle inequality we have

$$\left\| \sup_{1 < t < 2} |\phi_k(r)\mathcal{U}_t \mathcal{P}_j g| \right\|_{L_q^q} \leq 5 \sum_{i=1}^{5} S_i,$$

where

$$S_1 = \sum_{0 \leq l \leq j/2} \sum_{0 \leq m \leq l-1} \left\| \sup_{1 < t < 2} \phi_{k, l}[\mathcal{U}_t \mathcal{P}_j, m] g \right\|_{L_q^q}, \quad S_2 = \sum_{0 \leq l \leq j/2} \left\| \sup_{1 < t < 2} \phi_{k, l}[\mathcal{U}_t \mathcal{P}_j, m] g \right\|_{L_q^q},$$

$$S_3 = \sum_{j/2 < l < j} \sum_{0 \leq m \leq l-1} \left\| \sup_{1 < t < 2} \phi_{k, l}[\mathcal{U}_t \mathcal{P}_j, m] g \right\|_{L_q^q}, \quad S_4 = \sum_{0 \leq m \leq j-1} \left\| \sup_{1 < t < 2} \phi^j_k[\mathcal{U}_t \mathcal{P}_j, m] g \right\|_{L_q^q},$$

$$S_5 = \left\| \sup_{1 < t < 2} \phi^{[j/2]-1}_k[\mathcal{U}_t \mathcal{P}_j] g \right\|_{L_q^q}.$$

The proof of (3.20) is now reduced to showing

$$S_i \lesssim 2^{1/2} \left( \frac{3}{p} - \frac{1}{q} - 1 \right) j + \epsilon \|g\|_{L^p}, \quad 1 \leq i \leq 5, \quad (3.21)$$
for $p, q$ satisfying $p \leq q, 1/p + 1/q < 1$ and $1/p + 2/q > 1$.

Before we start the proof of (3.21), we briefly comment on the decomposition $S_i, i = 1, \ldots, 5$. As for $S_4$ and $S_5$, which are easier to handle, the sizes of $r - t$ and $|\xi|^{-1} \xi_1 + 1$ are sufficiently small on the supports of the associated multipliers, so we can remove the dependence of $t$ by an elementary argument. For $S_1, S_2,$ and $S_3$, we use Lemma 2.3 combined with (3.10) to control the maximal operators. Different magnitudes of contribution come from $\partial_t \phi_{k,l} = O(2^l)$ and $|t \xi_1 + r| |\xi||$, so we need to compare them. Writing $t \xi_1 + r |\xi| = t (|\xi|^{-1} \xi_1 + 1) + (r - t)$, we note

$$|t \xi_1 + r| |\xi|| \lesssim 2^l \max\{2^{-m}, 2^{-l}\}. \quad (3.22)$$

The decompositions in $S_1, S_2,$ and $S_3$ are made according to comparative sizes of $\partial_t \phi_{k,l} = O(2^l)$ and $|t \xi_1 + r| |\xi||$ in terms of $l, m,$ and $j$.

We first consider $S_1$. Using Lemma 2.3, we need to estimate $\phi_{k,l} U_t P_j, m g$ and $\partial_t (\phi_{k,l} U_t P_j, m g)$ in $L^q_r(x_3,t)$. Note that $\partial_t \phi_{k,l} = O(2^l)$ and $2^l \lesssim 2^{j-m}$. Thus, recalling (3.10), we apply Lemma 2.3 and the Mikhlin multiplier theorem to get

$$S_1 \lesssim \sum_{0 \leq l \leq j/2} \sum_{m=0}^{l-1} 2^{\frac{j-m}{q}} \|\phi_{k,l} U_t P_j, m g\|_{L^q_r}. \quad (3.22)$$

Thus, by Proposition 3.4 it follows that

$$S_1 \lesssim 2^{-\frac{j}{4} + \frac{j}{q} + \frac{3j}{2} \left(\frac{1}{p} - \frac{1}{q}\right) + \epsilon} \sum_{0 \leq l \leq j/2} 2^{l(1 - \frac{1}{p}) - \frac{2}{q}} \sum_{m=0}^{l-1} 2^{\frac{m}{2} \left(\frac{1}{p} + \frac{1}{q} - 1\right)} \|g\|_{L^p}. \quad (3.22)$$

Since $1/p + 1/q - 1 < 0$ and $1/p + 2/q > 1$, we obtain (3.21) with $i = 1$.

We can show the estimate (3.21) with $i = 2$ in the same manner. As before, since $\partial_t \phi_{k,l} = O(2^l)$ and $2^l \lesssim 2^{j-l}$, using (3.22), Lemma 2.3, and the Mikhlin multiplier theorem, we have

$$S_2 \lesssim \sum_{0 \leq l \leq j/2} 2^{-\frac{j-l}{q}} \|\phi_{k,l} U_t P_j, m g\|_{L^q_r}. \quad (3.22)$$

Thus, by (3.13) and Theorem 3.1, we have $S_2 \lesssim \sum_{0 \leq l \leq j/2} 2^{-\frac{j}{4} + \frac{j}{q} + \frac{3j}{2} \left(\frac{1}{p} - \frac{1}{q}\right) + \epsilon} \|g\|_{L^p}$, which gives (3.21) with $i = 2$.

We now consider $S_3$, which we handle as before. Since $j < 2l, 2^j \max\{2^{-m}, 2^{-l}\} \leq 2^l$ if $l + m \geq j$. Similarly, $2^{j-m} \geq 2^j \max\{2^{-m}, 2^{-l}\}$ and $2^{j-m} \geq 2^l$ if $l + m < j$. Using (3.22) and (3.10), we see

$$S_3 \lesssim \sum_{j/2 < l < j} \left( \sum_{j-l \leq m \leq j-1} 2^{\frac{l}{q}} \|\phi_{k,l} U_t P_j, m g\|_{L^q_r} + \sum_{0 \leq m < j-l} 2^{\frac{j-m}{q}} \|\phi_{k,l} U_t P_j, m g\|_{L^q_r} \right) \quad (3.22)$$
Since $1/p + 2/q > 1$, using Proposition 3.7, we get (3.21) for $i = 3$.

We handle $S_4$ and $S_5$ in an elementary way without relying on Lemma 2.3. Instead, we can control $S_4$ and $S_5$ more directly. Concerning $S_4$ we claim that

$$S_4 \lesssim 2^{\frac{j}{p} - \frac{1}{q} - 1} j \|g\|_{L^p} \tag{3.23}$$

if $5/q > 1 + 1/p$ and $2 \leq p \leq q \leq \infty$. This clearly gives (3.21) with $i = 4$ for $p, q$ satisfying $p \leq q$, $1/p + 1/q < 1$ and $1/p + 2/q > 1$. We note that

$$|\phi_k(U^j \mathcal{P}_{j,m} g(r, x_3))| \lesssim 2^{-\frac{1}{2} j} \|\phi_k^j\| e^{i 2 j (r^2 \xi_1 + x_3 \xi_2 + r^2 \xi_3)} m(\xi) \hat{\phi}_0(\xi) \hat{\psi}_m(\xi) g(2^{-j} \xi) d\xi,$$

where

$$m(\xi) = e^{i 2 j (\frac{r^2}{2} \xi_1 + (r-r) \xi_2 + r^2 \xi_3)} |\xi|^{-\frac{1}{2}} \hat{\phi}_0(\xi),$$

and $\hat{\phi}_0$ is a smooth function supported in $[-\pi, \pi]^2$ such that $\hat{\phi}_0 \hat{\phi}_0 = 1$. If $(r, t) \in$ supp $\phi_k^j$, then $|t-r| \lesssim 2^{-j}$. Thus, $|\partial^\alpha m(\xi)| \lesssim 1$ for any $\alpha$. We remove the dependence on $t$ by using a bound on the coefficient of Fourier series, not the Sobolev embedding. Expanding $m$ into Fourier series on $[-\pi, \pi]^2$ we have $m(\xi) = \sum_{k \in \mathbb{Z}^2} C_k(r, t) e^{i k \xi}$ while $|C_k(r, t)| \lesssim (1 + |k|)^{-N}$. Since $1 < t < 2$, the estimate (3.23) follows after scaling $\xi \to 2^j \xi$ if we obtain

$$\|\mathcal{R} \mathcal{P}_{j,m} g\|_{L^q_{r,x}(2^{-2},2^3) \times \mathbb{R}} \lesssim 2^{\frac{j}{p} - \frac{1}{q} - 1} j \|g\|_{L^p},$$

where

$$\mathcal{R} g(r, x_3) = \int e^{i (r^2 \xi_1 + x_3 \xi_2 + r^2 \xi_3)} \hat{g}(\xi) d\xi.$$

When $q = 2$, changing variables $r^2 \to r$ and following the argument in the proof of Lemma 3.5 we have $\|\mathcal{R} \mathcal{P}_{j,m} g\|_{L^2_{r,x}(2^{-2},2^3) \times \mathbb{R}} \lesssim 2^{m/2} \|g\|_{L^2}$. On the other hand, (3.18) gives $\|\mathcal{R} \mathcal{P}_{j,m} g\|_{L^\infty_{r,x}(2^{-2},2^3) \times \mathbb{R}} \lesssim 2^{(j-m)/2} \|g\|_{L^\infty}$. Interpolation between these two estimates gives

$$\|\mathcal{R} \mathcal{P}_{j,m} g\|_{L^q_{r,x}(2^{-2},2^3) \times \mathbb{R}} \lesssim 2^{\frac{m}{q} + \frac{j-m}{2} \left(1 - \frac{2}{q}\right)} \|g\|_{L^q}$$

for $2 \leq q \leq \infty$. Since the support $\widehat{\mathcal{P}_{j,m} g(\xi)}$ is contained in a rectangular region of dimensions $2^j \times 2^{j - \frac{m}{2}}$, by Bernstein’s inequality we have

$$\|\mathcal{R}^j_m g\|_{L^q_{r,x}(2^{-2},2^3) \times \mathbb{R}} \lesssim 2^{j \left(\frac{2}{p} - \frac{1}{q}\right) + m \left(\frac{2}{q} - \frac{1}{2} - \frac{1}{2p}\right)} \|g\|_{L^p}$$

for $2 \leq p \leq q \leq \infty$. Since $5/q > 1 + 1/p$, this proves the claimed estimate (3.23).
Finally, we show (3.21) with \( i = 5 \). Changing variables \( (\xi_1, \xi_2) \to (2^j \xi_1, \xi_2) \), we observe

\[
\phi_k^{[j/2]-1}[U_j \mathcal{P}_j g(r, x_3)] \lesssim 2^j \phi_k^{[j/2]-1} \left| \int e^{i \frac{(r-x_3)^2}{2}} \mathcal{P}_j g(2^j \xi_1, \xi_2) d\xi \right|
\]

where

\[
\tilde{m}(\xi) = e^{i 2^{j-1} r t [(\xi_1, 2^{-j} \xi_2)]} |(\xi_1, 2^{-j} \xi_2)|^{-\frac{1}{2}} \tilde{\phi}_0 (|\xi|, 2^{-j} \xi_2) \psi^{-1}(2^j \xi_1, \xi_2).
\]

Note that \( \text{supp} \tilde{m} \subset \{ \xi_1 \in [2^{-1}, 2^2], |\xi_2| \leq 2^2 \} \). Since \( |\partial^\alpha \tilde{m}(\xi)| \lesssim 1 \) for any \( \alpha \), expanding \( \tilde{m} \) into Fourier series on \([-2\pi, 2\pi]^2\) we have

\[
\tilde{m}(\xi) = \sum_{k \in \mathbb{Z}^2} C_k (r, t) e^{i \frac{r t}{N} \cdot k \xi} \text{ while } |C_k(r, t)| \lesssim (1 + |k|)^{-N}.
\]

Hence, similarly as before, changing variables \( (\xi_1, \xi_2) \to (2^{-j} \xi_1, \xi_2) \), to show (3.21) with \( i = 5 \) it is sufficient to obtain

\[
\left\| \sup_{1 < t < 2} |P_j g(r - t, x_3)| \right\|_{L^{q}_{r, x_3}([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{\frac{1}{2} (\frac{3}{p} - \frac{1}{q}) j} \|g\|_{L^p}
\]

for \( 1 \leq p \leq q \leq \infty \). Clearly, the left hand side is bounded by \( \|P_j g(x_1, x_3)\|_{L^{q}_{x_3}([2^{-1}, 2^2])} \). The Fourier transform of \( P_j g \) is supported on the rectangle \( \{ \xi_1 \in [2^{-1}, 2^{j+2}], |\xi_2| \leq 2^{j+2} \} \). Thus, using Bernstein’s inequality in \( x_1 \), we get

\[
\left\| \sup_{1 < t < 2} |P_j g(r - t/2, x_3)| \right\|_{L^{q}_{r, x_3}([2^{-2}, 2^3] \times \mathbb{R})} \lesssim 2^{-\frac{j}{4} + \frac{j}{q}} \|P_j g\|_{L^q}
\]

for \( 1 \leq q \leq \infty \). Another use of Bernstein’s inequality gives (3.24) for \( 1 \leq p \leq q \leq \infty \). This completes the proof of (3.20).

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