Quantified Propositional Logspace Reasoning

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Abstract

In this paper, we develop a quantified propositional proof systems that corresponds to logarithmic-space reasoning. We begin by defining a class $\Sigma_{\text{CNF}}(2)$ of quantified formulas that can be evaluated in log space. Then our new proof system $GL^*$ is defined as $G_1^*$ with cuts restricted to $\Sigma_{\text{CNF}}(2)$ formulas and no cut formula that is not quantifier free contains a free variable that does not appear in the final formula.

To show that $GL^*$ is strong enough to capture log space reasoning, we translate theorems of $VL$ into a family of tautologies that have polynomial-size $GL^*$ proofs. $VL$ is a theory of bounded arithmetic that is known to correspond to logarithmic-space reasoning. To do the translation, we find an appropriate axiomatization of $VL$, and put $VL$ proofs into a new normal form.

To show that $GL^*$ is not too strong, we prove the soundness of $GL^*$ in such a way that it can be formalized in $VL$. This is done by giving a logarithmic-space algorithm that witnesses $GL^*$ proofs.

1 Introduction

Recently there has been a significant amount of research looking into the connection between computational complexity, bounded arithmetic, and propositional proof complexity. A recent survey on this topic can be found at [6]. The idea is that there is a hierarchy of complexity classes

$$AC^0 \subseteq TC^0 \subseteq NC^1 \subseteq L \subseteq NL \subseteq P.$$ 

The first class is the set of problems that can be solved by uniform, polynomial-size, constant depth circuits. This class is important because it can be shown that PARITY cannot be solved in $AC^0$. In fact, problems that involve counting cannot be solved in $AC^0$. The second class is $TC^0$. This set of problems is the same as $AC^0$ except that $TC^0$ circuits can use counting gates. The class $NC^1$ is the set of problems that can be solved using polynomial-size, logarithmic-depth circuits. This class can be thought of as the set of problems that can be solved very quickly when work is done in parallel. Evaluating boolean formulas is complete for this class. The class $L$ is the set of problems that can be solved in logarithmic space on a Turing machine. The class $NL$ is the set of problems that can
be solved in logarithmic space on a non-deterministic Turing machine. The reachability problem for directed graphs is complete for this class. The sequence finishes with $P$, the set of problems that can be solved in polynomial time on a deterministic Turing machine. Except for the first inclusion, is it unknown if any of these inclusions are proper.

Each of these complexity classes has a corresponding theory of arithmetic: $V^0$, $VTC^0$, $VC^1$, $VL$, $VNL$, and $TV^0$, respectively. Each of these theories can prove that the functions in their corresponding complexity class are total. As a consequence, any information we can obtain about the theory tells us something about the complexity class and vice versa.

There is also a connection with propositional proof complexity. Some of the theories mentioned above have a corresponding propositional proof system. As before, information about the proof systems tells us about the corresponding theory and complexity class. In this paper, we explore the proof systems. The goal is to try to understand how the strength of a proof system is affected by different restrictions.

Our focus will be on quantified propositional proof systems, but, to explain our method, we will use quantifier-free propositional proof systems. Start with a Frege proof system, sometimes called Hilbert Style Systems. These systems are described in standard logic text books. A Frege proof is a series of propositional formulas where each formula is an axiom or can be inferred from previous formulas using one of the rules of inference. There are two common ways of restricting this proof system. The first is to restrict all of the formulas in the proof. For example, one definition of bounded-depth Frege is to restrict every formula in the proof to formulas with a constant depth. This worked, but, if a proof system is defined this way, then there are formulas that cannot be proved simply because they are not allowed to appear in the proof. For example, bounded-depth Frege with formulas of depth $d$ cannot prove any formula of depth $d + 1$.

The other method is to restrict the formulas on which certain rules can be applied. This solves the problem of the first method and led to other definitions of bounded-depth Frege.

In this paper, we will look at restricting the cut rule in the tree-like sequent calculus for quantified propositional formulas. This systems is known as $G^*$.

The cut rule derives $\Gamma \rightarrow \Delta$ from $A, \Gamma \rightarrow \Delta$ and $\Gamma \rightarrow \Delta, A$. In $G^*$, $A$ can be any quantified propositional formulas. The proof system $G^0_0$ is defined by restricting $A$ to quantifier-free formulas. If we are given a $G^0_0$ proof of a $\Sigma^1_1$ formula ($\exists \vec{z}B(\vec{z})$, where $B$ is quantifier-free), then we can find a witness for existential quantifiers in this formula in uniform $NC^1$; moreover, this problem is complete for this class. The complexity class $NC^1$ is the set of problems that can be solved by polynomial-size, logarithmic-depth circuits with fan-in 2. The interesting observation is that evaluating quantifier-free formulas is also complete for $NC^1$. It is also possible to connect $G^0_0$ to $NC^1$ indirectly through bounded arithmetic. There is a theory of arithmetic $VNC^1$ that is known to correspond to $NC^1$ reasoning. Given a $VNC^1$ proof of a bounded formula it is possible to translate the proof into a family of polynomial-size $G^0_0$ proofs. This tells us that the reasoning power of $G^0_0$ is at least as strong as that of $VNC^1$ [5]. In the other direction, $VNC^1$ can prove that $G^0_0$ is sound when proving $\Sigma^1_1$ formulas. This means that, when proving $\Sigma^1_1$ formulas,
the reasoning power of $G^*_0$ is not stronger than that of $VNC^1$. So we say
that $G^*_0$ corresponds to $NC^1$ reasoning.

As well, if we restrict cut formulas to constant-depth, quantifier-free
formulas, we get a proof system that corresponds to $AC^0$ reasoning.
The complexity class $AC^0$ is the set of problems that can be solved by
polynomial-size, constant-depth circuits with unbounded fan-in. Again,
evaluating constant-depth formulas is complete for $AC^0$. We should note
we are talking about the proofs of quantifier-free formulas.

This gives us two proof systems whose reasoning power is the same as
the complexity of evaluating their cut formulas. This raises the question
of whether or not this holds in general. The quick answer is no. A counter-
example is $G^*_1$. Evaluating $\Sigma^1_1$ formulas is complete for $NP$, but the $\Sigma^0_1$
witnessing problem for $G^*_1$ is complete for $P$ [8]. Another counter-example
is $GPV^*$, where cut formulas are quantifier-free or formulas of the form
\( \exists x[x \leftrightarrow A] \), where $A$ is a quantifier-free formula that does not mention
$x$. Evaluating a cut formula for $GPV^*$ is complete for $NC^1$, but the
witnessing problem is complete for $P$ [14].

In this paper, we define a new proof system $GL^*$ that corresponds to
$L$ reasoning. The complexity class $L$ is the set of problems that can be
solved on a Turing Machine with a read-only input tape and a work tape
where the space used on the work tape is proportional to the logarithm
of the size of the input. Our proof system $GL^*$ is defined by restricting
cuts to $\Sigma^{CNF}(2)$ formulas, a set of formulas for which the evaluation
problem is complete for $L$. However, that is not enough. We also restrict
the free variables that appear in cut formulas with quantifiers to variables
that appear free in the final sequent. We then prove this proof system
corresponds to $L$ reasoning by connecting it with a theory of arithmetic
that is known to correspond to $L$ reasoning. This definition is meant to
demonstrate that the strength of a proof system is not related to the diffi-
culty of evaluating a single cut formula in the proof, but to the complexity
of witnessing the eigenvariables in the proof.

In Section 2, we give definitions of the important concepts. In partic-
ular, we define two-sorted computational complexity and bounded arith-
metic. As well, we define the standard proof systems and explain the
connection between proof systems and theories of bounded arithmetic in
more detail. In Section 3, we define $GL^*$. This includes the definition of
the $\Sigma^{CNF}(2)$ formulas. In Section 4, we change the theory $VL$ and prove
a normal-form that is necessary for our results. This is the most technical
section in the paper. In Section 5, we prove the translation theorem. In
Section 6, we prove that $GL^*$ is sound in the theory. This includes an
algorithm to evaluate $\Sigma^{CNF}(2)$ formulas in $L$.

This paper is an expanded version of the author’s earlier paper [13].

2 Basic Definitions And Notation

2.1 Two-Sorted Computational Complexity

In this paper, we use two-sorted computational complexity. The two sorts
are numbers and binary strings (aka finite sets). The numbers are intended

3
to range over the natural numbers and will be denoted by lower-case letters. For example, \( i, j, x, y, \) and \( z \) will often be used for number variables; \( r, s, \) and \( t \) will be used for number terms; and \( f, g, \) and \( h \) will be used for functions that return numbers. The strings are intended to be finite strings over \( \{0, 1\} \) with leading 0 removed. Since the strings are finite, they can be thought of as sets where the \( i \)th bit is 1 if \( i \) is in the set. The strings will be denoted by upper-case letters. The letters \( X, Y, \) and \( Z \) will often be used for string variables.

We focus on the complexity class \( L \). Let \( R(\vec{x}, \vec{X}) \) be a relation. If we are going to solve this relation on a Turing Machine \( M \), then the input to \( M \) will be \( \vec{x} \) in unary and \( \vec{X} \) as a series of binary strings. So the size of the input is \( \vec{x} + |\vec{X}| \). We say \( R \) is in \( L \) if \( R \) can be decided by a two-tape Turing Machine such that one tape is a read-only input tape, and less than \( O(\log(\vec{x} + |\vec{X}|)) \) squares are visited on the other tape.

For functions, we say a number function \( f(\vec{x}, \vec{X}) \) is in \( FL \) if there is a polynomial \( p \) such that \( f(\vec{x}, \vec{X}) < p(\vec{x}, |\vec{X}|) \), and the relation \( f(\vec{x}, \vec{X}) = y \) is in \( L \). A string function \( F(\vec{x}, \vec{X}) \) is in \( FL \) if the size of \( F(\vec{x}, \vec{X}) \) is bounded by a polynomial and if the relation

\[
R(i, \vec{x}, \vec{X}) \iff \text{the } i\text{th bit of } F(\vec{x}, \vec{X}) \text{ is 1}
\]

is in \( L \). This is equivalent to defining \( FL \) using a three-tape Turing Machine with a write-only output tape.

### 2.2 Two-Sorted Bounded Arithmetic

Besides two-sorted computational complexity, we also use the two-sorted bounded arithmetic. The sorts are the same. This notation was based on the work of Zambella in [15], but we follow the presentation of Cook and Nguyen from [4, 6].

The base language is

\[
L^2_A = \{0, 1, +, \times, <, =, \in, \|\}
\]

The constants 0 and 1 are number constants. The functions \( + \) and \( \times \) take two numbers as input and return a number—the intended meanings are the obvious ones. The language also includes two binary predicates that take two numbers: \( < \) and \( = \). The predicate \( =_2 \) is meant to be equality between strings, instead of numbers. In practice, the \( 2 \) will not be written because which equality is meant is obvious from the context. The membership predicate \( \in \) takes a number \( i \) and a string \( X \). It is meant to be true if the \( i \)th bit of \( X \) is 1 (or \( i \) is in the set \( X \)). This will also be written as \( X(i) \). The final function \( |X| \) takes a string as input and returns a number. It is intended to be the number of bits needed to write \( X \) when leading zeros are removed (or the least upper bound of the set \( X \)). The set of axioms \( 2\text{BASIC} \) is the set of defining axioms for \( L^2_A \).
∀
i > 0

For formulas whose only quantifiers are bounded number quantifiers. For \( \Phi \) is a set of formulas and \( \phi \)

The theory \( \Phi \) is: Given a graph with edge relation \( \mathcal{E} \) and nodes \( \{0, \ldots, a\} \), where every vertex in the graph has out-degree at least 1, find a path of length \( b \). This is expressed using the \( \Sigma^B_0 \)-rec axiom:

\[\forall x \leq a \exists y \leq a \phi(x, y) \supset \exists Z, \forall w \leq b \phi(f(a, w, Z), f(a, w + 1, Z)) \text{ (} \Sigma^B_0 \text{-rec)\]
where \( f(a, w, Z) = \min_x (Z(w, x) \lor x = a) \) and \( \phi \) is a \( \Sigma^B_0 \) formula. The idea is that the function \( f(a, w, Z) \) extracts the \( w \)th node in the path that \( Z \) encodes.

**Definition 2.2.** The theory \( VL \) is the theory axiomatized by \( V^0 \) plus \( \Sigma^B_0 \)-rec.

The \( \Sigma^B_0 \)-rec axiom has the disadvantage that the path can start at any node. However, as Zambella pointed out in [16], it is possible to prove that there is a path of length \( b \) starting at a particular node \( a \).

**Lemma 2.3.** Let \( E \) be the edge relation for a directed graph on the nodes \( \{0, \ldots, n - 1\} \). Then for all \( a < n \) and \( b \), \( VL \) proves, if \( \forall i < n \exists j < n \ E(i, j) \), then there is a path of length \( b \) starting at node \( a \).

**Proof.** Define \( \phi((w, i), (w', j)) \) as
\[
\phi((w, i), (w', j)) \equiv (w' = w + 1 \mod b + 1) \land (w' \neq 0 \lor E(i, j)) \land (w' = 0 \lor j = a).
\]
Take a path of length \( 2b \) in the graph of \( \phi \). At some point in the first half of that path, the path passes through the node \( (0, a) \). Starting from there we can extract a path of length \( b \) in \( E \) that starts at node \( a \).

### 2.3 A Universal Theory For L Reasoning

Another way to get a theory for \( L \) is to define a universal theory with a language that contains a function symbol for every function in \( FL \). Then, we get a theory for \( L \) by taking the defining axioms for these functions. This is the idea behind other universal theories like \( PV \) and \( V^0 \). In our case, we characterize the \( FL \) functions using Lind’s characterization [10] adjusted for the two-sort setting.

In the next definition, we define the set of function symbols in \( L_{FL} \) and give their intended meaning.

**Definition 2.4.** The language \( L_{FL} \) is the smallest language satisfying
1. \( L^A_2 \cup \{pd, \text{min}\} \) is a subset of \( L_{FL} \) and have defining axioms 2BASIC, and the axioms
   \[
   pd(0) = 0 \quad (2.1)
   \]
   \[
   pd(x + 1) = x \quad (2.2)
   \]
   \[
   \text{min}(x, y) = z \iff (z = x \land x \leq y) \lor (z = y \land y \leq x) \quad (2.3)
   \]
2. For every open formula \( \alpha(i, \vec{x}, \vec{X}) \) over \( L_{FL} \) and term \( t(\vec{x}, \vec{X}) \) over \( L^A_2 \), there is a string function \( F_{\alpha, t} \) in \( L_{FL} \) with bit defining axiom
   \[
   F_{\alpha, t}(\vec{x}, \vec{X})(i) \iff i < t(\vec{x}, \vec{X}) \land \alpha(i, \vec{x}, \vec{X}) \quad (2.4)
   \]
3. For every open formula \( \alpha(z, \vec{x}, \vec{X}) \) over \( L_{FL} \) and term \( t(\vec{x}, \vec{X}) \) over \( L^A_2 \), there is a number function \( f_{\alpha, t} \) in \( L_{FL} \) with defining axioms
   \[
   f_{\alpha, t}(\vec{x}, \vec{X}) \leq t(\vec{x}, \vec{X}) \quad (2.5)
   \]
   \[
   z < t(\vec{x}, \vec{X}) \land \alpha(z, \vec{x}, \vec{X}) \lor \alpha(f_{\alpha, t}(\vec{x}, \vec{X}), \vec{x}, \vec{X}) \quad (2.6)
   \]
   \[
   z < f_{\alpha, t}(\vec{x}, \vec{X}) \lor \lnot \alpha(z, \vec{x}, \vec{X}) \quad (2.7)
   \]
4. For all number functions $g(\vec{x}, \vec{X})$ and $h(p, y, \vec{x}, \vec{X})$ in $L_{FL}$ and term $t(y, \vec{x}, \vec{X})$ over $L^2_A$, there is a number function $f_{g,h,t}(y, \vec{x}, \vec{X})$ with defining axioms

$$f_{g,h,t}(0, \vec{x}, \vec{X}) = \min(g(\vec{x}, \vec{X}), t(\vec{x}, \vec{X})) \quad (2.8)$$
$$f_{g,h,t}(y + 1, \vec{x}, \vec{X}) = \min(h(f(y, \vec{x}, \vec{X}, y, \vec{x}, \vec{X})), t(\vec{x}, \vec{X})) \quad (2.9)$$

The last scheme is called $p$-bounded number recursion. The $p$-bounded number recursion is equivalent to the log-bounded string recursion given in [10]. The other schemes come from the definition of $L_{FAC0}$ in [4].

It is not difficult to see every function in $L_{FL}$ is in $FL$. The only point we should note is that the intermediate values in the recursion are bounded by a polynomial in the size of the input. This means, if we store intermediate values in binary, the space used is bounded by the log of the size of the input. So the recursion can be simulated in $L$. To show that every $FL$ function has a corresponding function symbol in $L_{FL}$, note that the $p$-bounded number recursion can be used to traverse a graph where every node has out-degree at most one.

**Definition 2.5.** $VL$ is the theory over the language $L_{FL}$ with B1-B11, SE, plus 2.1; 2.2; 2.3; axiom 2.4 for each string function $F_{a,t}$ in $L_{FL}$; axioms 2.5, 2.6, and 2.7 for each number function $f_{a,t}$ in $L_{FL}$; and axioms 2.8 and 2.9 for each number function $f_{g,h,t}$ in $L_{FL}$.

An open($\mathcal{L}$) formula is a formula over the language $\mathcal{L}$ that does not have any quantifiers. The important part of this theory is that it really is a universal version of $VL$.

**Theorem 2.6.** $VL$ is a conservative extension of $VL$.

**Proof.** First to prove that $VL$ is an extension of $VL$. All that is required is to prove the $\Sigma^B_0$-COMP and $\Sigma^B_0$-rec axioms. To prove $\Sigma^B_0$-COMP, note that every $\Sigma^B_0$ formula $\phi$ is equivalent to an open formula $\phi'$. For example,

$$VL \models \exists z < b \psi(z, \vec{x}, \vec{X}) \leftrightarrow \psi(f_{\psi,b}(\vec{x}, \vec{X}), \vec{x}, \vec{X})$$

when $\psi$ is an open formula. Then the function $F_{\psi', t}$ is the witness for

$$\exists Z \leq t \forall i < t[Z(i) \leftrightarrow \phi(i)].$$

To prove the $\Sigma^B_0$-rec axiom, we can define a function $f(i, a, E)$ that returns the $i$th node in the path the axiom says exists. The function $f$ can be defined using $p$-bounded number recursion. From there, a function witnessing the $\Sigma^B_0$-rec axiom can be defined.

To prove that the extension is conservative, we show how to take any model $M$ of $VL$ and find an expansion that is a model of $VL$. The idea is to expand the model one function at a time. We can order the functions in $L_{FL}$ such that each function is defined in terms of the previous functions.

Let $L_i$ be the language $L^2_i$ plus the first $i$ functions in $L_{FL}$. Let $M_i$ be the model obtained by expanding $M$ to the functions in $L_i$. We will show that the model $M_{\infty} = \bigcup M_i$ is a model $VL$. A similar proof can be found in Chapter 9 of [6] and we will not repeat it here.

$\square$
2.4 Quantified Propositional Calculus

We are also interested in quantified propositional proof systems. The proof systems we use were originally defined in [9], and then they were redefined in [5, 11], which is the presentation we follow.

The set of connectives are \{∧, ∨, ¬, ∃, ∀, ⊤, ⊥\}, where \(\top\) and \(\bot\) are constants for true and false, respectively. Formulas are built using these connectives in the usual way. We will often refer to formulas by the number of quantifier alternations.

**Definition 2.7.** The set of formulas \(\Sigma^0_q = \Pi^0_q\) is the set of quantifier-free propositional formulas. For \(i > 0\), the set of \(\Sigma^i_q (\Pi^i_q)\) formulas is the smallest set of formulas that contains \(\Pi^{i-1}_q\) \((\Sigma^{i-1}_q)\) and is closed under \(∧, ∨\), existential (universal) quantification, and if \(A \in \Pi^i_q (A \in \Sigma^i_q)\) then \(¬A \in \Sigma^i_q (¬A \in \Pi^i_q)\).

The first proof system, from which all others will be defined, is the proof system \(G\). This proof system is a sequent calculus based on Gentzen’s system \(LK\). The system \(G\) is essentially the DAG-like, propositional version of \(LK\). We will not give all of the rules, but will mention a few of special interest.

The cut rule is

\[
\frac{A, \Gamma \rightarrow \Delta \quad \Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta}
\]

In this rule, we call \(A\) the cut formula. There are also four rules that introduce quantifiers:

\[
\begin{align*}
\exists\text{-left} & : \frac{A(x), \Gamma \rightarrow \Delta}{\exists z A(z), \Gamma \rightarrow \Delta} & \exists\text{-right} & : \frac{\Gamma \rightarrow \Delta, A(B)}{\Gamma \rightarrow \Delta, \exists z A(z)} \\
\forall\text{-left} & : \frac{\Gamma \rightarrow \Delta, A(x)}{\Gamma \rightarrow \Delta, \forall z A(z)} & \forall\text{-right} & : \frac{A(B), \Gamma \rightarrow \Delta}{\forall z A(z), \Gamma \rightarrow \Delta}
\end{align*}
\]

These rules have conditions on them. In \(\exists\)-left and \(\forall\)-right, the variable \(x\) must not appear in the bottom sequent. In these rules, \(x\) is called the eigenvariable. In the other two rules, the formula \(B\) must be a \(\Sigma^0_q\) formula, and no variable that appears free in \(B\) can be bound in \(A(x)\).

The initial sequents of \(G\) are sequents of the form \(\rightarrow \top\), \(\bot \rightarrow \), or \(x \rightarrow x\), where \(x\) is any propositional variable. A \(G\) proof is a series of sequents such that each sequent is either an initial sequent or can be derived from previous sequents using one of the rules of inference. The proof system \(G_i\) is \(G\) with cut formulas restricted to \(\Sigma^i_q\) formulas.

We define \(G^*\) as the treelike version of \(G\). So, a \(G^*\) proof is a \(G\) proof where each sequent in \(G\) is used as an upper sequent in an inference at most once. A \(G^*_i\) proof is a \(G^*\) proof in which cut formulas are prenex \(\Sigma^i_q\). In [11], it was shown that, for treelike proofs, it did not matter if the cut formulas in \(G^*_i\) were prenex or not. So when we construct \(G^*_i\) proofs, the cut formulas will not always be prenex, but that does not matter.

To make proofs simpler, we assume that all treelike proofs are in free-variable normal form.
Definition 2.8. A parameter variable for a $G^*_i$ proof $\pi$ is a variable that appears free in the final sequent of $\pi$. A proof $\pi$ is in free-variable normal form if (1) every non-parameter variable is used as an eigenvariable exactly once in $\pi$, and (2) parameter variables are not used as eigenvariables.

Note that, if a proof is treelike, we can always put it in free-variable normal form by simply renaming variables. In fact, VPV proves that every treelike proof can be put in free-variable normal form.

A useful property of these proof systems is the subformula property. It can be shown in $VL$ that every formula in a $G^*_i$ proof is an ancestor (and therefore a subformula) of a cut formula or a formula in the final sequent. This is useful because it tells us that any non-$\Sigma^q_0$ formula in a $G^*_i$ proof must be an ancestor of a final formula.

2.5 Truth Definitions

In order to reason about the proof systems in the theories, we must be able to reason about quantified propositional formulas. We follow the presentation in [8, 9, 5].

Formally formulas will be coded as strings, but we will not distinguish between a formula and its encoding. So if $F$ is a formula, we will use $F$ as the string encoding the formula as well. The method of coding a formula can be found in [5].

In this paper, we are only interested in $\Sigma^0_0$ formulas and prenex $\Sigma^q_1$ formulas. For $\Sigma^0_0$ formulas, we are able to give an $\Sigma^0_0(L_{FL})$ functions that evaluates the formula. This formula will be referred to using $A \models \_\_0 F$, where $A$ is an assignment and $F$ is a formula. We leave the precise definition to the readers.

Given a prenex $\Sigma^q_1$ formula $F$, the truth definition is a formula that says there is an assignment to the quantified variables that satisfies the $\Sigma^0_0$ part of the formula. This formula will be referred to as $A \models \_\_1 F$.

Valid formulas (or tautologies) are defined as

$$TAUT_i(F) \equiv \forall A, (\"A is an assignment to the variables of $F$" \supset A \models \_\_i F)$$

This truth definition can be extended to define the truth of a sequent. So, if $\Gamma \rightarrow \Delta$ is a sequent of $\Sigma^q_i \cup \Pi^q_i$ formulas, then

$$(A \models \_\_i \Gamma \rightarrow \Delta) \equiv \"there exists a formula in $\Gamma$ that $A$ does not satisfy" \lor \"there exists a formula in $\Delta$ that $A$ satisfies"$$

Another important formula we will use is the reflection principle for a proof system. We define the $\Sigma^q_i$ reflection principle for a proof system $P$ as

$$(\Sigma^q_i-RFN(P)) \equiv \forall F \forall \pi, (\"\pi is a P proof of $F$" \land F \in \Sigma^q_i) \supset TAUT_i(F)$$

This formula essentially says that, if there exists a $P$ proof of a $\Sigma^q_i$ formula $F$, then $F$ is valid. Another way of putting it is to say that $P$ is sound when proving $\Sigma^q_i$ formulas.
2.6 Propositional Translations

There is a close connection between the theory $V^1$ and the proof system $G^*_1$. You can think of $G^*_1$ as the non-uniform version of $V^1$. This idea might not make much sense at first until you realize you can translate a $V^1$ proof into a polynomial-size family of $G^*_1$ proofs. The translation that we use is described in [4, 5]. It is a modification of the Paris-Wilkie translation [12]. Given a $\Sigma^0_i$ formula $\phi(\vec{x}, \vec{X})$ over the language $L^2_3$, we want to translate it into a family of propositional formulas $\langle \phi(\vec{x}, \vec{X}) \rangle |\vec{m}; \vec{n}|$, where the size of the formulas is bounded by a polynomial in $\vec{m}$ and $\vec{n}$. The formula $\langle \phi(\vec{x}, \vec{X}) \rangle |\vec{m}; \vec{n}|$ is meant to be a formula that is a tautology when $\phi(\vec{x}, \vec{X})$ is true in the standard model whenever $x_i = m_i$ and $|X_i| = n_i$. Then if $\phi(\vec{x}, \vec{X})$ is true in the standard model for all $\vec{x}$ and $\vec{X}$, then every $\langle \phi(\vec{x}, \vec{X}) \rangle |\vec{m}; \vec{n}|$ is a tautology.

The variables $\vec{m}$ and $\vec{n}$ will often be omitted since they are understood. The free variables in the propositional formula will be $p_i^{X_i}$ for $j < n_i - 1$. The variable $p_i^{X_i}$ is meant to represent the value of the $j$th bit of $X_i$; we know that the $n_i$th bit is 1, and for $j > n_i$, we know the $j$th bit is 0. The definition of the translation proceeds by structural induction on $\phi$.

Suppose $\phi$ is an atomic formula. Then it has one of the following forms: $s = t$, $s < t$, $X_i(t)$, or one of the trivial formulas $\bot$ and $\top$, for terms $s$ and $t$. Note that the terms $s$ and $t$ can be evaluated immediately. This is because the exact value of every number variable and the size of each string variable is known. Let $val(t)$ be the value of the term $t$.

In the first case, we define $\langle s = t \rangle$ as the formula $\top$, if $val(s) = val(t)$, and $\bot$, otherwise. A similar construction is done for $s < t$. If $\phi$ is one of the trivial formulas, then $\langle \phi \rangle$ is the same trivial formula. So now, if $\phi \equiv X_i(t)$, let $j = val(t)$. Then the translation is defined as follows:

$$\langle \phi \rangle = \begin{cases} p_i^{X_i} & \text{if } j < n_i - 1 \\ 1 & \text{if } j = n_i - 1 \\ 0 & \text{if } j > n_i - 1 \end{cases}$$

Now for the inductive part of the definition. Suppose $\phi \equiv \alpha \land \beta$. Then

$$\langle \phi \rangle = \langle \alpha \rangle \land \langle \beta \rangle.$$ 

When the connective is $\lor$ or $\neg$, the definition is similar. If the outermost connective is a number quantifier bound by a term $t$, let $j = val(t)$. Then the translation is defined as

$$\langle \exists y \leq t, \alpha(y) \rangle \equiv \bigvee_{i=0}^{j} \langle \alpha(y) \rangle |[s]$$

$$\langle \forall y \leq t, \alpha(y) \rangle \equiv \bigwedge_{i=0}^{j} \langle \alpha(y) \rangle |[s]$$

$$\langle \exists Y \leq t, \alpha(Y) \rangle \equiv \exists p_0^Y, \ldots, \exists p_{m-2}^Y, \bigvee_{i=0}^{j} \langle \alpha(Y) \rangle |[s]$$

$$\langle \forall Y \leq t, \alpha(Y) \rangle \equiv \forall p_0^Y, \ldots, \forall p_{m-2}^Y, \bigwedge_{i=0}^{j} \langle \alpha(Y) \rangle |[s]$$
Now we are able to state the translation theorem for $V^i$ and $G_1^*$.

**Theorem 2.9.** Suppose $V^i \vdash \phi(\vec{x},\vec{X})$, where $\phi$ is a bounded formula. Then there are polynomial-size $G_1^*$ proofs of the family of tautologies $\|\phi(\vec{x},\vec{X})\|[\vec{m};\vec{n}]$.

This type of theorem is the standard way of proving that the reasoning power of the proof system is as least as strong as that of the theory.

## 3 Definition of $GL^*$

In this section, we will define the proof system we wish to explore. As was stated in the introduction, this proof system is defined by restricting cut formulas to a set of formulas that can be evaluated in $L$. Alone that is not enough to change the strength of the proof system, so we also restrict the use of eigenvariables.

The first step is to define a set of formulas that can be evaluated in $L$. These formulas will be based on $CNF(2)$ formulas. A $CNF(2)$ formula is a $CNF$ formula where no variable has more than two occurrences in the entire formula. It was shown in [7] that determining whether or not a given $CNF(2)$ formula is satisfiable is complete for $L$. Based on this we get the following definition:

**Definition 3.1.** The set of formulas $\Sigma_{CNF}(2)$ is the smallest set

1. containing $\Sigma^0_0$,

2. containing every formula $\exists \vec{z}. \phi(\vec{x},\vec{z})$ where (1) $\phi$ is a quantifier-free $CNF$ formula $\bigwedge_{i=1}^{m} C_i$ and (2) existence of a $z$-literal $l$ in $C_i$ and $C_j$, $i \neq j$, implies existence of an $x$-variable $x$ such that $x \in C_i$ and $\neg x \in C_j$ or vice versa, and

3. closed under substitution of $\Sigma^0_0$ formulas that contain only $x$-variables for $x$-variables.

**Definition 3.2.** The idea behind this definition is that any assignment to the variables $\vec{x}$ reduces the quantifier-free portion to a $CNF(2)$ formula in $\vec{z}$. $GL^*$ is the propositional proof system $G_1^*$ with cuts restricted to $\Sigma_{CNF}(2)$ formulas and no restriction on the free variables.

The restriction on the free variables in the cut formula might seem strange, but it is necessary. If we did not have this restriction, then the proof system would be as strong as $G_1^*$. We will not give a full proof of this, but the interested reader can see information on $GPV^*$ in [14]. What we will show is that, if the restriction on the variables is not present, then the proof system can simulate $G_1^*$ for $\Sigma^0_1$ formulas.

Let $H^*$ be the proof system $G_1^*$ with cuts restricted to $\Sigma_{CNF}(2)$ formulas and no restriction on the free variables.

**Definition 3.3.** An extension cedent $\Lambda$ is a sequence of formulas

$$\Lambda \equiv y_1 \leftrightarrow B_1, y_2 \leftrightarrow B_2, \ldots, y_n \leftrightarrow B_n$$

(3.1)

where $B_i$ is a $\Sigma^0_0$ formula that does not mention any of the variables $y_1, \ldots, y_n$. We call the variables $y_1, \ldots, y_n$ extension variables.
Based on a lemma in [8], Cook and Nguyen proved the following lemma in [6].

**Lemma 3.4.** If $\pi$ is a $G^*_1$ proof of $\exists \mathbf{z} A(\mathbf{z}, \mathbf{x})$, where $A$ is a $\Sigma_0^q$ formula, then there exists a $PK$ proof $\pi'$ of

$$
\Lambda \rightarrow A(\mathbf{y}, \mathbf{x})
$$

where $\Lambda$ is as in 3.1 and $|\pi'| \leq p(|\pi|)$, for some polynomial $p$.

The proof guaranteed by this lemma is also an $H^*$ proof since every $PK$ proof is also an $H^*$ proof. Extending this proof with a number of applications of $\exists$-right, we get an $H^*$ proof of

$$
\Lambda \rightarrow \exists \mathbf{z} A(\mathbf{z}, \mathbf{x}). \quad (3.2)
$$

So now we need to find a way to remove the extension cedent $\Lambda$. This is done one formula at a time. Suppose $y \leftrightarrow B$ is the last formula in $\Lambda$. The key observation is that $\exists y[y \leftrightarrow B]$ is a $\Sigma CNF(2)$ formula because the formula can be express as $\exists y[(y \lor \neg B) \land (\neg y \lor B)]$. So we can apply $\exists$-left with $y$ as the eigenvariable to (3.2). The eigenvariable restriction is met because $y$ is the last eigenvariable, and, therefore, cannot appear anywhere else the extension cedent. Then we cut $\exists y[y \leftrightarrow B]$ after deriving $\rightarrow \exists y[y \leftrightarrow B]$. We can then do this for every formula is $\Lambda$ starting at the end. This proves the following theorem.

**Theorem 3.5.** $H^*$ $p$-simulates $G^*_1$ for $\Sigma_1^q$ formulas.

This proof is not always a $GL^*$ proof because the extension variables are not parameter variables, yet they appear in cut formulas.

## 4 Adjusting $VL$

In order to prove the translation theorem, we start with the theory $VL$, which corresponds to $L$ reasoning. This theory was defined in Section 2.2. The proof of the translation theorem is similar to other proofs of its type. We take an anchored (or free-cut free) proof. Then the cut formulas in this proof will translate into the cut formulas in the propositional proof. If we use $VL$ for this, there are two problem: (1) not all of the axioms of $VL$ translate into $\Sigma CNF(2)$ formulas and (2) the restriction of the free variables in cut formulas may not be met. In the first subsection, we take care of the first problem. The second problem in taken care of in Section 4.2.

### 4.1 A New Axiomatization For $VL$

We want to reformulate the axioms of $VL$ so they translate into $\Sigma CNF(2)$ formulas. All of the 2BASIC axioms are $\Sigma_0^B$, so they translate into $\Sigma_0^q$ formulas, which are $\Sigma CNF(2)$, so they do not create any problems. We only need to consider $\Sigma_0^B$-COMP and $\Sigma_0^B$-rec. We handle $\Sigma_0^B$-COMP the same way Cook and Morioka did in [5]. That is, if the proof system is asked to cut the translation of an instance of the $\Sigma_0^B$-COMP axiom, then the propositional proof is changed so that the cut becomes $\bigwedge_{i=0}^l[||\phi(i)|| \leftrightarrow$
theory that is equivalent to $\mathcal{VL}$. To take care of $\Sigma_0^B$-rec, we define a new theory that is equivalent to $\mathcal{VL}$ by replacing the $\Sigma_0^B$-rec axiom.

Informally the new axiom says that there exists a string $Z$ that gives a specific pseudo-path of length $b$ in the graph with $a$ nodes and edge relation $\phi(i, j)$. This path starts at node 0. If $(i, j)$ is an edge in this path, then $j$ is the smallest number with an edge from $i$ to $j$, or $j = a$ when there are no outgoing edges. Note that the edge may not exist in the original graph when $j = a$. This is why we call it a pseudo-path. If $(i, j)$ is the $eth$ edge in the path, then $Z(w, i, j)$ is true, and $Z(w, i', j')$ is false for every other pair. This is described by the $\Sigma_0^B$-edge-rec scheme:

$$\exists Z \leq 1 + (b, a, a)[\rho_1 \land \rho_2 \land \rho_3 \land \rho_4 \land \rho_5 \land \rho_6 \land \rho_7 \land \rho_8], \quad (\Sigma_0^B\text{-edge-rec})$$

where

$$\begin{align*}
\rho_1 &\equiv \forall j < a, \neg Z(0, 0, j) \lor \phi(0, j) \lor \exists l < j \phi(0, l)) \\
\rho_2 &\equiv \forall j \leq a \lor k < j, \neg Z(0, 0, j) \lor \neg \phi(0, k) \lor \exists l < k \phi(0, l)) \\
\rho_3 &\equiv \forall i \leq a \lor j \leq a, i = 0 \lor \neg Z(0, i, j) \\
\rho_4 &\equiv \forall w < b, \forall j < a, \neg Z(w + 1, i, j) \\
&\lor \exists h \leq a Z(w, h, i) \lor \neg \phi(i, j) \lor \exists l < j \phi(i, l) \\
\rho_5 &\equiv \forall w < b, \forall i \leq a \lor j < a, \neg Z(w + 1, i, j) \lor \phi(i, j) \lor \exists l < j \phi(i, l) \\
\rho_6 &\equiv \forall w < b, \forall i \leq a \lor j < a, \neg Z(w + 1, i, j) \lor \neg \phi(i, k) \lor \exists l < k \phi(i, l) \\
\rho_7 &\equiv \exists i \leq a Z(b, i, j) \\
\rho_8 &\equiv \forall (w, i, j) \leq (b, a, a), [w > b \lor i > a \lor j > a] \lor \neg Z(w, i, j)
\end{align*}$$

and $\phi(i, j)$ is a $\Sigma_0^B$ formula that does not mention $Z$, but may have other free variables. It is not immediately obvious that the axiom says what it is suppose to, so we will take a closer look.

Let $Z$ be a string that witnesses the axiom. We want to make sure $Z$ is the path described above. Looking at $\rho_3$, we see the path starts at 0. Suppose $Z(0, 0, j)$ is true. We must show that $j$ is the first node adjacent to 0. This follows from $\rho_1$, which guarantees $\phi(i, j)$ is true when $j < a$, and $\rho_2$, which guarantees $\phi(i, k)$ is false when $k < j$. A similar argument can be made with $\rho_5$ and $\rho_6$ to show that every node is the smallest node adjacent to its predecessor. To make sure the path is long enough, we have $\rho_7$, which says there is a $bth$ edge, and $\rho_4$, which says if there is a $(w + 1)th$ edge there is a $uth$. As you may have noticed, there are parts of this formula that semantically are not needed. For example, the $\exists l < j \phi(0, l)$ in $\rho_1$ is not needed. It is used to make sure the axiom translates into a $\Sigma CNF(2)$ formula. We add $\rho_8$ to make sure there is a unique $Z$ that witnesses this axiom.

Notation 1. For simplicity, $\psi_0$ is the $\Sigma_0^B$ part of the $\Sigma_0^B$-edge-rec axiom instantiated with $\phi$. Note this includes the bound on the size of $Z$. So the axiom can be written as $\exists Z \psi_0$.

Definition 4.1. $\mathcal{VL}'$ is the theory axiomatized by the axioms of $\mathcal{V}_0$, the $\Sigma_0^B$-edge-rec axioms, and Axiom (4.1). The language of $\mathcal{VL}'$ is the language of $\mathcal{V}_0$ plus a string constant $C$ with defining axiom

$$|C| = 0 \quad (4.1)$$

13
We add the string constant to the language so we can put \( VL' \) proofs in free variable normal form (below). We do not use the constant for any other reason. Also, in the translation, we can treat \( C \) as a string variable with \( n = 0 \).

**Lemma 4.2.** The theory \( VL \) is equivalent to \( VL' \).

**Proof.** To prove the two theories are equivalent, we must show that \( VL \) proves the \( \Sigma^B_0 \)-edge-rec axiom and that \( VL' \) proves the \( \Sigma^B_0 \)-rec axiom. Since the two axioms express similar ideas, this is not surprising.

To show that \( VL \) proves the \( \Sigma^B_0 \)-edge-rec axiom, let \( \phi(i, j) \) be any \( \Sigma^B_0 \) formula. Then let \( Y \) be the string such that \( Y(i, j) \Leftrightarrow (j < a \supset \phi(i, j)) \land \forall k < j \neg \phi(i, k) \). This \( Y \) exists by \( \Sigma^B_0 \)-COMP. We can think of \( Y \) as the graph that contains only the edges the \( \Sigma^B_0 \)-edge-rec axiom would use. Since \( VL \) proves the \( X - MIN \) formula, it follows that \( VL \) proves \( \forall i \leq a, \exists j \leq a, Y(i, j) \). This means there exists a path of length \( b \) in \( Y \) that starts at node \( 0 \) Lemma 2.3. It is a simple task to verify the \( b \) edges in this path satisfy the \( \Sigma^B_0 \)-edge-rec axiom for \( \phi \).

To show that \( VL' \) proves the \( \Sigma^B_0 \)-rec axiom, let \( \phi(i, j) \) be a \( \Sigma^B_0 \) formula such that \( \forall i \leq a \exists j \leq a, \phi(i, j) \). By the \( \Sigma^B_0 \)-edge-rec axiom, there is a pseudo-path of length \( b \) in the graph \( \phi \). We need to show that this is a real path. Suppose \( (i, j) \) is an edge in the path. If \( j < a \), then \( (i, j) \) is in the graph by \( \rho_1 \) and \( \rho_3 \). Otherwise, \( j = a \), and \( \forall k < j \neg \phi(i, k) \). This implies \( \phi(i, j) \) since every node has out-degree at least 1. This means every edge in the pseudo-path exists, and there exists a path of length \( b \).

The next step is to be sure the translation of the \( \Sigma^B_0 \)-edge-rec axiom is a \( \Sigma CNF(2) \) formula. This is done by a careful inspection of the formula.

**Lemma 4.3.** The formula \( \exists Z_\psi(a, b, Z) \) is a \( \Sigma CNF(2) \) formula.

**Proof.** First we assume \( \phi(i, j) \equiv X(i, j) \) for some variable \( X \). It is easy to see that \( \exists Z_\psi(a, b, Z)[a, b; t, a \ast a] \), where \( t \) is the bound on \( Z \) given in the \( \Sigma^B_0 \)-edge-rec axiom, is a CNF formula. Note that we assigned \( |Z| = t \) and \( |X| = a \ast a \). We now need to make sure the clauses have the correct form. This is done by examining each occurrence of a bound literal. To verify this, the proof will require a careful inspection of the definition of the axiom. The only bound variables are those that come from \( Z \). These are \( p^2_{u, i, j} \), which we will refer to as \( z_{w, i, j} \). The only free variables are those corresponding to \( X \). These variables will be referred to as \( x_{i, j} \).

We will first look at the positive occurrences of \( z_{w, i, j} \). On inspection, we can observe that, when \( w < b \), every occurrence of \( z_{w, i, j} \) must be in clauses that are part of the translation of \( \rho_4 \). We want to show that every clause that is part of the translation of \( \rho_4 \) has conflicting free variables. This is true since \( \neg X(i, j_1) \) will conflict with one of the variables from \( \exists l < j_2, X(i, l) \) when \( j_1 < j_2 \). When \( w = b \), the variable \( z_{0, i, j} \) appears once in \( \rho_2 \). Now we turn to the negative occurrences. When \( w = 0 \), the variable \( z_{0, i, j} \) will appear negatively in the clauses corresponding to \( \rho_1 \), \( \rho_2 \), and \( \rho_3 \). If \( i > 0 \), it will appear only in the clauses corresponding to \( \rho_3 \) and will appear only once. If \( i = 0 \), the variable \( z_{0, 0, j} \) will not appear in the translation of \( \rho_3 \) because the \( i = 0 \) part will satisfy the clause. It is
easy to observe that every occurrence of the variable in the translation of \( \rho_1 \) and \( \rho_2 \) will have a conflicting free variable. Examine the construction \( X(0, j) \lor \exists l < j X(0, l) \) at the end of \( \rho_1 \) and \( \neg X(0, k) \lor \exists l < k X(0, l) \) at the end of \( \rho_2 \). A similar argument can be made with \( \rho_4, \rho_5, \) and \( \rho_6 \) when \( w > 0 \). This implies that the translation is a \( \Sigma CNF(2) \) formula when \( \phi(i, j) \equiv X(i, j) \). When \( \phi \) is a more general formula, the translation is the formula in the first case with the free variables substituted with the translation of \( \phi \), which will be \( \Sigma q_0 \). Since \( \Sigma CNF(2) \) formulas are closed under this type of substitution, the formula is \( \Sigma CNF(2) \) in all cases.

### 4.2 Normal Form For VL

In this section, we want to find a normal form for \( VL' \) proofs that makes sure the translation of \( VL' \) proofs satisfy the variable restriction for \( GL' \).

The normal form we want is cut variable normal form (CVNF) and is defined in the following.

**Definition 4.4.** A formula \( \phi(Y) \) is bit-dependent on \( Y \) if there is an atomic sub-formula of \( \phi \) of the form \( Y(t) \), for some term \( t \).

**Definition 4.5.** A proof is in free variable normal form if (1) every non-parameter free variable \( y \) or \( Y \) that appears in the proof is used as an eigenvariable exactly once and (2) parameter variables are never used as eigenvariables.

Note that if a proof is in free variable normal form we can assume that every instance of the non-parameter variable \( Y \) (or \( y \)) is in an ancestor of the sequent where \( Y \) is used as an eigenvariable. If it is not, we can replace \( Y \) with the constant \( C \) in all those sequents.

**Definition 4.6.** A cut in a proof is anchored if the cut formula is an instance of an axiom.

**Definition 4.7.** A \( VL' \) proof \( \pi \) is in cut variable normal form if \( \pi \) is (1) in free variable normal form, (2) every cut with a non-\( \Sigma B_1 \) cut formula is anchored, and (3) no cut formula that is an instance of the \( \Sigma B_0 \)-edge-rec axiom is bit-dependent on a non-parameter free string variable.

It is known how to find a proof with the first two properties \([6, 2]\), and this part will not be repeated here. Instead we focus on how to find a proof satisfying the third property.

**Theorem 4.8.** For every \( \Sigma_1^B \) theorem of \( VL' \) there exists a \( VL' \)-proof of that formula in CVNF.

The proof of this theorem is the most technical in this paper. At a high level, it amounts to showing \( \Sigma_0^B \)-edge-rec is closed under substitution of strings defined by \( \Sigma_0^B \)-edge-rec and \( \Sigma_0^B \)-COMP. We begin with an anchored proof that is in free variable normal form. We want to change every cut that violates condition (3) in the definition of CVNF. Consider the proof given in Figure 1. This is a simple example of what can go wrong. The general case is handled in the same way, so we will only consider this case.

Since all \( \Sigma_1^B \) cut formulas are anchored and the \( \exists Y \gamma(Y) \) must eventually be cut, it is be an instance of \( \Sigma_0^B \)-COMP or \( \Sigma_0^B \)-edge-rec. So you
can think of $\gamma$ as a formula that completely defines $Y$. Then we want to change $\phi(Y)$ so that it does not mention $Y$ explicitly, but instead uses the definition of $Y$ given by $\gamma$. Note that, for this to be true, the final formula must be $\Sigma^B_1$; otherwise, $Y$ could have been used as an eigenvariable in a $\forall$-right inference and would not be well defined.

**Lemma 4.9.** For any $\Sigma^B_0$ formula $\phi(Y)$, there exist $\Sigma^B_0$ formulas $\phi_1$ and $\phi_2$ such that $\phi_1$ is not bit-dependent on $Y$ and $V^B$ proves the sequent

$$\gamma(Y), \psi_1(Z), \forall i < t[Z(i) \leftrightarrow \phi_2(Z)] \rightarrow \psi_2(Y)(Z').$$

**Proof.** This proof is divided into two cases. In the first case, we assume $\gamma(Y) \equiv |Y| \leq t \land \forall i < t[Y(i) \leftrightarrow \phi'(i)].$ (4.2)

That is, $\exists Y \gamma(Y)$ is an instance of $\Sigma^B_0$-COMP. We know $Y$ must appear in that position because it eventually gets quantified. In this case, $\phi_1$ is $\phi$ with every atomic formula of the form $Y(s)$ replaced by $s < t \land \phi'(s)$, and $\phi_2$ is the formula $Z(i)$. We can prove that there exists a $V^B$ proof of (4.2) by structural induction on $\phi$.

For the second case, we assume $\gamma(Y) \equiv \psi_\phi(Y)$. That is, $Y$ is the pseudo-path in the graph of $\phi'$. The first step is to define branching programs that compute $Y$ and $Z'$ (the pseudo-path in the graph of $\phi$) using $Y$. Then $\phi_1$ is the $\Sigma^B_0$ description of the composition of these branching programs, and $\phi_2$ is the $\Sigma^B_0$ formula that extracts $Z'$ from the run of this last branching program.

**Definition 4.10.** A branching program is a nonempty set of nodes labeled with triples $(\alpha, i, j)$, where $\alpha$ is a $\Sigma^B_0$ formula over some set of variables and $0 \leq i, j \leq t$ for some term $t$ that depends only on the inputs to the program. Semantically, if a node $u$ is labeled with $(\alpha, i, j)$, then, when the branching program is at node $u$, it will go to node $i$, if $\alpha$ is true, or node $j$, otherwise. The initial node is always 0.

Note that a branching program is essentially a graph with a special form, and, as with graphs, we use families of branching programs that can be described by $\Sigma^B_0$ formulas. However, we will not give the explicit construction of the formula; we leave it to the reader.

The first step is to introduce the initial branching program $BP_0$ that computes $Z'$. The nodes of $BP_0$ are interpreted as triples $(w, i, j)$. An invariant for this branching program is that, if we reach the node $(w, i, j)$,
then the \(w\)th node of \(Z'\) is \(i\) and \(\forall k < j \neg \phi(i, k)\). At each node, we check if \(j\) is the next node. Let \(a\) be the maximum value of a node and \(b\) be the length of the path. This means the number of nodes in \(BP_0\) is bound by \(\langle b, a, a \rangle\). So now to define the labels. If \(j < a\), then \(\langle w, i, j \rangle\) is labeled with \((\phi(i, j), \langle w + 1, j, 0 \rangle, \langle w, i, j + 1 \rangle)\). If \(j = a\), then \(\langle w, i, j \rangle\) is labeled with \((T, \langle w + 1, j, 0 \rangle, 0)\). It is easy to see that the invariants hold and that \(Z'\) can be obtained from a path in \(BP_0\) using \(\Sigma_0\)-COMP.

The branching program that computes \(Y\) is constructed the same way except \(\phi'\) is used instead of \(\phi\). Let this branching program be \(BP\).

Moving on to the second step, we now want to simplify \(BP_0\) so that every node whose label is bit-dependent on \(Y\) is labeled with an atomic formula. This is done to simplify the construction of the composition. We start with \(BP_0\). Then, given \(BP\), we define \(BP_{i+1}\) by removing one connective in a node of \(BP_i\) that is not in the right form. Let node \(n\) in \(BP\) be labeled with \(\langle n, u_1, u_2 \rangle\). The construction is divided into five cases: one for each possible outer connective.

- **Case \(\alpha \equiv \neg \beta\):** \(BP_{i+1}\) is the same as \(BP_i\) except node \(n\) is now labeled with \(\langle \beta, u_2, u_1 \rangle\).

- **Case \(\alpha \equiv \beta_1 \land \beta_2\):** The nodes of \(BP_{i+1}\) are interpreted as pairs \(\langle u, v \rangle\). The node \(\langle u, 0 \rangle\) corresponds to node \(u\) in \(BP_i\). The label for \(\langle n, 1 \rangle\) becomes \((\beta_1, \langle n, 1 \rangle, \langle v_2, 0 \rangle)\) and the label for \(\langle n, 1 \rangle\) is \((\beta_2, \langle u_1, 0 \rangle, \langle u_2, 0 \rangle)\). Notice that \(\langle n, 1 \rangle\) is used as an intermediate node while evaluating \(\alpha\).

- **Case \(\alpha \equiv \beta_1 \lor \beta_2\):** \(BP_{i+1}\) is defined as in the previous case, with a few minor modifications. This case is left to the reader.

- **Case \(\alpha \equiv \exists z \leq t \beta(z)\):** The nodes become pairs as in the previous case, but this time the labels are different. The node \(\langle n, i \rangle\) is labeled with \((\beta(i), \langle u_1, 0 \rangle, \langle n, i + 1 \rangle)\), when \(i < t\). If \(i = t\), the node is labeled with \((\beta(i), \langle u_1, 0 \rangle, \langle u_2, 0 \rangle)\). In this case, the branching program is looking for an \(i\) that falsifies \(\beta(i)\).

- **Case \(\alpha \equiv \forall z \leq t \beta(z)\):** This case is similar to the previous case. The only difference is the branching program is looking for an \(i\) that falsifies \(\beta(i)\).

Let \(BP_n\) be the final branching program in this construction above. We now construct a branching program \(BP'\) that is the composition of \(BP_n\) and \(BP\). The nodes of \(BP'\) are pairs \(\langle u_1, u_2 \rangle\) where the first element corresponds to a node in \(BP_n\) and the second element corresponds to a node in \(BP\).

Suppose node \(u_1\) in \(BP_n\) is labeled with \(\langle \alpha, v_1, v_2 \rangle\). If \(\alpha\) is not bit-dependent on \(Y\), then the node \(\langle u_1, 0 \rangle\) is labeled with \(\langle \alpha, \langle v_1, 0 \rangle, \langle v_2, 0 \rangle \rangle\).

It is also possible that \(\alpha\) is bit-dependent on \(Y\); in which case, \(\alpha\) is of the form \(Y(w, i, j)\). Let \(\langle \beta, u_1, u_2 \rangle\) be the label for node \(u_2\) in \(BP\). Then the node \(\langle u_1, u_2 \rangle\) is labeled as follows:

\[
\begin{align*}
(\beta, \langle u_1, w_1 \rangle, \langle u_1, w_2 \rangle), & \quad \text{if } w_2 \leq \langle w, a, a \rangle \text{ and } u_2 \neq \langle w, i, j \rangle, \\
(\beta, \langle v_1, 0 \rangle, \langle v_2, 0 \rangle), & \quad \text{if } w_2 = \langle w, i, j \rangle, \text{ and} \\
(T, \langle v_2, 0 \rangle, \langle v_2, 0 \rangle), & \quad \text{otherwise.}
\end{align*}
\]

In this case, we are using the second element to run \(BP\) and determine if the \(s\)th edge in the path is \((i, j)\). If it is, we move on to \(\langle v_1, 0 \rangle\), and if it is not, we move on to \(\langle v_2, 0 \rangle\). In the labels above, the first line corresponds...
Figure 2: Modification of the proof in Figure 1. The formula \( \tau(Z') \) is used to replace running \( BP \). The second line corresponds to a check if \((i,j)\) is the \( w \)th edge. The third line is used when we have already found the \( w \)th edge and it is not \((i,j)\).

It is not difficult to see that it is possible to construct \( \phi_1 \) (a \( \Sigma^B_0 \) formula describing \( BP' \)), and \( \phi_2 \) (a formula extracting \( Z' \) from a run of \( BP' \)). Moreover, \( V^0 \) proves that this construction works.

Using this lemma, we are able to change the proof in Figure 1 into the proof in Figure 2. In that proof, \( P' \) is the proof \( P \) with the rules that introduced \( \exists Z \) ignored (renaming variables if necessary), and \( Q \) is an anchored \( V^0 \) proof, which we know exists by the lemma above. This gives us a new proof of the same formula that still satisfies properties (1) and (2) in Definition 4.7 and it contains one less cut that is bit-dependent on \( Y \).

Using this manipulation, we prove Theorem 4.8.

**Proof of Theorem 4.8.** It would be nice to be able to simply say we can repeatedly apply the manipulations above and eventually the proof will be in CVNF, but this is not obvious. In the manipulation, if \( \gamma(Y) \) is bit-dependent on a string variable other than \( Y \), then the new \( \Sigma^B_0 \)-edge-rec cut formula is bit-dependent on that variable. This includes non-parameter string variables. So we need to state our induction hypothesis more carefully.

Let \( Y_1, \ldots, Y_n \) be all of the non-parameter free string variables that appear in \( \pi \) ordered such that the variable \( Y_i \) is used as a eigenvariable before \( Y_j \) for \( i < j \). This implies \( Y_i \) does not appear in \( \gamma(Y_j) \) in the manipulations above. So now suppose no \( \Sigma^B_0 \)-edge-rec cut formula is bit-dependent on the variables \( Y_1, \ldots, Y_k \), for some \( k < n \). Then we can
manipulate $\pi$ such that the same holds for the variables $Y_1, \ldots, Y_{k+1}$.

To accomplish this, we simply manipulate every $\Sigma^B_0$-edge-rec cut formula that is bit-dependent on $Y_{k+1}$ as described above. Since $Y_1, \ldots, Y_k$ cannot appear in $\gamma(Y_{k+1})$, those variables will not violate the condition. So by induction, we can get a proof that is in CVNF.

5 Translation Theorem

We are now prepared to prove the translation theorem. The proof is done by induction on the length of the proof. For the base case, we need to prove the translation of the axioms of $VL'$. We know the $\Sigma^B_0$-COMP and the 2BASIC axioms have polynomial-size $G^*_0$ proofs from other translation theorems [5]. This means they also have polynomial-size $GL^*$ proofs. Axiom (4.1) is easy to prove since it translates to $\vdash \top$. We still need to show how to prove the $\Sigma^B_0$-edge-rec axiom in $GL^*$. Recall that we write the axiom as $\exists Z \psi \phi(a,b,Z)$. Note that the axiom does have a bound on $Z$, but it has been omitted since the specific bound is not important.

Lemma 5.1. The formula $||\exists Z \psi \phi(a,b,Z)||$ has a $GL^*$ proof of size $p(a,b)$ for some polynomial $p$.

Proof. The proof is done by a brute force induction. We prove, in $GL^*$, that, if there exists a pseudo-path of length $b$, then there exists a pseudo-path of length $b+1$. It is easy to prove there exists a pseudo-path of length 0. Then with repeated cutting we get our final result. The entire path is quantified, so we do not cut formulas with non-parameter free variables.

Given variables that encode a path of length $b$, we can define $\Sigma^0_q$ formulas that determine the next edge. Let $A_{i,j} \equiv ||\phi(i,j)||$. Since $\phi$ is a $\Sigma^B_0$ formula, $A_{i,j}$ is a $\Sigma^0_q$ formula. To prove that there is an edge that starts the path, consider the formula

$$ B_{0,0,j} \equiv A_{0,j} \land \bigwedge_{k=0}^{j-1} \neg A_{0,k}, $$

when $j < a$, and

$$ B_{0,0,a} \equiv \bigwedge_{k=0}^{a-1} \neg A_{0,k}. $$

It is easy to see $B_{0,0,j}$ is true for exactly one $j \leq a$. This is also provable in $GL^*$. This shows that $GL^*$ has a polynomial-size proof of

$$ ||\exists Z \psi \phi(a,1,Z)||. $$

For the inductive step, if there is a path of length $b$ and the path is given by the variables $z_{w,i,j}$, then the witnesses for the next edge are defined as follows:

$$ B_{b+1,i,j} \equiv \bigvee_{k=0}^{a} z_{b,k,i} \land A_{i,j} \land \bigwedge_{k=0}^{j-1} \neg A_{i,k}, $$

19
when \( j < a \), and

\[
B_{b+1, i, a} \equiv \bigvee_{k=0}^{a} z_{b,k,i} \land \bigwedge_{k=0}^{a-1} \neg A_{i,k}.
\]

Using the fact that exactly one \( z_{b, i, j} \) is true, we can prove in \( GL^* \) that exactly one \( B_{b+1, i, j} \) is true. This shows that \( GL^* \) has a polynomial-size proof of

\[
||\exists Z \psi(a, b, Z)|| \rightarrow ||\exists Z \psi(a, b+1, Z)||.
\]

So now we are able to prove \( ||\exists Z \psi(a, b, Z)|| \) for any \( b \) by successive cutting. Recall that \( ||\exists Z \psi(a, b, Z)|| \) is a \( \Sigma_q^B \) formula, and note that the free variables in \( ||\exists Z \psi(a, b, Z)|| \) do not change as \( b \) changes. This means we are allowed to do the cut.

This can be used to prove the translation theorem.

**Theorem 5.2 (VL-GL* Translation Theorem).** Suppose \( VL \) proves \( \exists Z < t\phi(\vec{x}, \vec{X}, Z) \), where \( \phi \) is a \( \Sigma_q^B \) formula. Then there are polynomial-size \( GL^* \) proofs of \( ||\exists Z \psi(a, b, Z)|| \rightarrow ||\exists Z \psi(a, b+1, Z)|| \).

**Proof.** By Theorem 4.2 and Theorem 4.8, there exists a \( VL' \) proof \( \pi \) of \( \exists Z < t\phi(\vec{x}, \vec{X}, Z) \) that is in CVNF.

We proceed by induction on the depth of \( \pi \). The base case follows from Lemma 5.1 and the comments that precede it. The inductive step is divided into cases: one for each rule. With the exception of cut, every rule can be handled the same way it is handled in the \( V^1_G \) Translation Theorem (Theorem 7.51, [6]), and will not be repeated here.

When looking at the cut rule, there are three cases. If the cut formula is \( \Sigma_q^B \), then we simply cut the corresponding \( \Sigma_q^B \) formula in the \( GL^* \) proof. If the cut formula is not \( \Sigma_q^B \), then it must be anchored since the proof is in CVNF. This means the cut formula is an instance of \( \Sigma_q^B \)-edge-rec or an instance of \( \Sigma_q^B \)-COMP. First suppose it is an instance of \( \Sigma_q^B \)-edge-rec.

Then we are able to cut the corresponding formula in the \( GL^* \) proof. This is because the axiom translates into a \( \Sigma_q^B \) formula, and the free variables in the translation are parameter variables since the formula is not bit-dependent on non-parameter string variables.

When the cut formula is an instance of \( \Sigma_q^B \)-COMP, we apply the same transformation as in the proof of the \( VNC^1_G \) translation theorem [5]. That is, we remove the quantifiers by replacing the variables with \( \Sigma_q^B \) formulas that witness the quantifiers. This change does not effect other cuts since their free variables are parameter variables or they are \( \Sigma_q^B \) formulas and remain \( \Sigma_q^B \) after the substitution. The current cut formula becomes a \( \Sigma_q^B \) formula, which can be cut. Note that, since there are a constant number of cuts of this axiom, the substitution does not cause an exponential increase in the size of the formulas.

6 Proving Reflection Principles

In this section, we show that \( GL^* \) does not capture reasoning for a higher complexity class. This is done by proving, in \( VL \), that \( GL^* \) is sound.
This idea comes from [3], where Cook showed that $PV$ proves extended-Frege is sound, and [9], where Krajicek and Pudlak showed $T^i_2$ proves $G_i$ is sound for $i > 0$.

We will actually show that $VL$ proves $GL^*$ is sound. The idea behind the proof is to give an $L_{FL}$ function that witnesses the quantifiers in the proof. Then we prove, by $\Sigma^B_0 (L_{FL})$-IND, that this function witnesses every sequent, including the final sequent. Therefore the formula is true.

We start by giving an algorithm that witnesses $\Sigma CNF(2)$ formulas in $L$ when the formula is true. This algorithm is the algorithm given in [7] with a few additions to find the satisfying assignment. We describe an $L_{FL}$ function that corresponds to this algorithm and prove it correct in $VL$. We then use this function to find an $L_{FL}$ function that witnesses $GL^*$ proofs, and prove it correct in $VL$.

6.1 Witnessing $\Sigma CNF(2)$ Formulas

Let $\exists A(\vec{x}, \vec{z})$ be a $\Sigma CNF(2)$ formula. We will describe how to find a witness for this formula. We assume that $A$ is a $CNF$ formula. That is, the substitution of the $\Sigma^2$ formulas has not happened. The general case is essentially the same.

The first thing to take care of is the encoding of $A$. We will not go through this in detail. Suffice it to say that parsing a formula can be done in $TC^0$ [5], and, as long as we are working in a theory that extends $TC^0$ reasoning, we can use any reasonable encoding. We will refer to the $i$th clause of $A$ as $C^i_A$. A clause will be viewed as a set of literals. A literal is either a variable or its negation. So we will write $l \in C^i_A$ to mean that the literal $l$ is in the $i$th clause of $A$. Since the parsing can be done in $TC^0$, these formulas can be defined by $\Sigma^B_0 (L_{FL})$ formulas. An assignment will also be viewed as a set of literal. If a literal is in the set, then that literal is true. So an assignment $X$ satisfies a clause $C$ if and only in $X \cap C \neq \emptyset$.

Given values for $\vec{x}$, we first simplify $A$ to get a $CNF(2)$ formula. We will refer to the simplified formula as $F$. This can be done using the $L_{FL}$ function defined by the following formula:

$$l \in C^i_F \iff l \in C^i_A \land X \cap C^i_A = \emptyset,$$

where $X$ is the assignment to the free variables. From the definition of a $\Sigma CNF(2)$ formula, $VL$ can easily prove that $F$ now encodes a $CNF(2)$ formula. In fact, it can be shown that no literal appears more than once. A satisfying assignment to this formula is the witness we want. Mark Bravermen gave an algorithm for finding this assignment [1], but we use a different algorithm that is easier to formalize.

Before we describe the algorithm that finds this assignment, we go through a couple definitions. First, a pure literal is a literal that appears in the formula, but its negation does not. Next the formula imposes an order on the literals. We say a literal $l_1$ follows a literal $l_2$ if the clause that contains $l_1$ also contains $l_2$, and $l_1$ is immediately to the right of $l_2$. 

21
circling to the beginning if \( l_2 \) is the last literal. More formally:

\[
\text{follows}(l_1, l_2, F) \iff \exists i, l_1 \in C^F_i \land l_2 \in C^F_i \land \forall l_3 (l_2 < l_3 < l_1 \lor l_3 \not\in C^F_i) \\
\land \forall l_3 (l_3 < l_1 < l_2 \lor l_3 \not\in C^F_i) \\
\land \forall l_3 (l_1 < l_2 < l_3 \lor l_3 \not\in C^F_i)
\]

Note that if a clause contains a single literal then that literal follows itself. Also, note that literals are coded by numbers and \( l_1 < l_2 \) means the number coding \( l_1 \) is less than the number coding \( l_2 \).

To find the assignment to \( F \), we will go through the literals in the formula in a very specific order. Starting with a literal \( l \) that is not a pure literal, the next literal is the literal that follows \( F \):

\[
\text{next}(l_1, F) = l_2 \iff \text{follows}(l_2, l_1, F).
\]

Note that if \( l_1 \) is a pure literal, then there is no next literal, so we simply define it to be itself. The important distinction is that \( \text{next} \) gives an ordering of the literals in a formula, and \( \text{follows} \) orders the literal in a clause. When \( F \) is understood, we will not mention \( F \) in \( \text{next} \) and \( \text{follows} \).

The algorithm that finds the assignment works in stages. At the beginning of stage \( i \), we have an assignment that satisfies the first \( i - 1 \) clauses. Then, in the \( i \)th stage, we make local changes to this assignment to satisfy the \( i \)th clause as well. At a high level, to satisfy the \( i \)th clause, we start with the first literal in the \( i \)th clause, and assign that literal to true. The clause that contains this literal’s negation may be have gone from being satisfied to being unsatisfied. So we now go to the next literal, which is in this other clause. We continue this until we get to a point where we know the other clause is satisfied. We need to be able to do this in \( L \).

Algorithm 1 shows how to do this. At any point in the algorithm, the only

Algorithm 1 Algorithm for Stage \( i \)

Set \( l_1 \) to the first literal in clause \( i \).

repeat

Assign true to \( l_1 \).

set \( l_2 := \text{next}(l_1) \)

while \( l_2 \) is not the complement of \( l_1 \) do

Assign true to \( l_2 \).

set \( l_2 := \text{next}(l_2) \)

If \( l_2 \) is a pure literal, assign true to \( l_2 \), and stage \( i \) is done.

If \( l_1 \) and \( l_2 \) are in the same clause, stage \( i \) is done.

end while

Assign true to \( l_1 \). \{This statement is redundant, but it is included to emphasis that \( l_1 \) is true.\}

set \( l_1 := \text{next}(l_1) \)

until \( l_1 \) is the first literal in clause \( i \)

At this point we know the formula is unsatisfiable.

information we need are the values of \( l_1 \) and \( l_2 \), so this is in \( L \). Note that
we do not store the assignment on the work tape, but on a write-only, output tape. What is not obvious is why this algorithm works.

The next lemma can be used to show that the both loops will eventually finish.

**Lemma 6.1.** For all literals $l$, there exists a $t > 0$ such that after $t$ applications of next to $l$, we get to $l$ or a pure literal.

**Proof.** Let $\text{next}^0(l) = l$ and $\text{next}^t(l) = \text{next}(\text{next}^{t-1}(l))$. Since next has a finite range, there exist a minimum $i$ and $t$ such that $\text{next}^i(l) = \text{next}^i(l)$. Suppose this is not a pure literal. If $i > 0$, then $\text{next}(\text{next}^{i-1}(l)) = \text{next}(\text{next}^{i-1}(l))$. However, this implies $\text{next}^{i-1}(l) = \text{next}^{i-1}(l)$ since next is one-to-one when not dealing with pure literals. This violates our choice of $i$. Therefore $i = 0$, and $l = \text{next}^0(l) = \text{next}^t(l)$. \hfill \Box

The implies the inner loop will halt, because, if it does not end earlier, $l_2$ will eventually equal $l_1$ which both will be in the same clause. For the outer loop, if the algorithm does not halt for any other reason, $l_1$ will eventually return to the first literal in the $i$th clause.

The next lemma plays a small role in the proof of correctness.

**Lemma 6.2.** Suppose the algorithm fails at stage $i$ and that $\text{next}^i(l') = l$, where $l'$ is the first literal in clause $i$. Then, for every literal in the same clause as $l$, there is a $t'$ such that $\text{next}^{t'}(l')$ equals that literal.

**Proof.** To prove this lemma, we will show that there exists a $t'$ that equals the literal that follows $l$. Then by continually applying this argument, you get that every literal in the clause is visited.

Let $l'$ be the first literal in the $i$th clause. Then, after going through the outer loop $t$ times, $l_1 = l$. Since the algorithm fails, the inner loop will finish because $l_2 = 7_i$. This means there is a $t'$ such that $\text{next}^{t'}(l') = 7_i$. Then $\text{next}^{t'+1}(l')$ is the literal that follows $l$. \hfill \Box

**Theorem 6.3.** If the algorithm fails, the formula is unsatisfiable.

**Proof.** This is proved by contradiction. Let $F$ be a $CNF(2)$ formula and $A$ be an assignment that satisfies it. Assume that the algorithm fails. From this we can defined a function from the set of variables to the set of clauses as follows:

$$f(i) = j \iff (x_i \in C_j^P \land x_i \in A) \lor (\neg x_i \in C_j^P \land \neg x_j \in A).$$

Informally, if $f(i) = j$ then clause $C_j$ is true because of the variable $x_i$. Since the formula is satisfied, this function is onto the set of clauses. Also, since $F$ is $CNF(2)$, no literal appear more than once. So $f$ is indeed a function because if $f(i) = j$ and $f(i) = j'$ then the literal $x_i$ or $\neg x_i$ is in both $C_j^P$ and $C_j'^P$.

Now we will use the assumption that the algorithm fails to find a way to restrict $f$ so that it violates the PHP. Suppose the algorithm fails at stage $i$. Let $l$ be first literal in clause $i$. We then define sets of variables $V^n$ as follows:

$$V^n = \{x_n : \exists b < a \text{next}^b(l) = x_n \lor \text{next}^b(l) = \neg x_n\}.$$
We also defined sets of clauses $W^a$ as follows:

$$W^a = \{ C_n : \exists x \in V^n (x \in C_n \lor \neg x \in C_n) \}.$$  

Note that for a large enough $a$, say $|F|$, if $C_n$ is in $W^a$, then every variable that appears in $C_n$ is in $W^a$ by 6.2. We show by induction on $a$ that $|V^n| < |W^n|$.

For $a = 1$, $|V^1| = 1$. If $l$ is a pure literal or $l$ and $\neg l$ are in the same clause, then the algorithm would succeed. Otherwise $|W^1| = 2$.

For the inductive case, suppose $|V^n| < |W^n|$. Let $l' = \text{next}^{a+1}(l)$. If $l'$ is not a new variable, then $|V^{a+1}| = |V^n| < |W^n| = |W^{a+1}|$. If $l'$ is a new variable, then $l'$ must be in a new clause. For, if this was not the case, the algorithm would succeed. To see this, let $l_2$ be the most recent literal in the same clause as $l'$. We know $l_1$ is not $l'$ since $l'$ is a new variable. Then eventually $l_2$ will become $\text{next}(l')$, which is in the same clause as $l_1$. The inner loop will not end because $l_2$ becomes the complement of $l_1$ since that would mean $\text{next}(l_1)$ is more recent.

This gives $|V^{a+1}| = |V^n| + 1 < |W^n| + 1 = |W^{a+1}|$.

If we restrict $f$ to $V^{(F)}$, then $f$ is a function from $V^{(F)}$ that is onto $W^{(F)}$ violating the PHP.

**Theorem 6.4.** If the algorithm succeeds, then, for all $i$, the assignment after given at the end of stage $i$ satisfies the first $i$ clauses of $F$.

**Proof.** The proof is done by induction on $i$. For $i = 0$, the statement holds since there are no clauses to satisfy. As an induction hypothesis, suppose the statement holds for $i$. Then we will show if the algorithm ever visits one of the literals in clause $n$, then that clause is satisfied.

Consider clause $n$, where $n \leq i + 1$. Find the last point in the algorithm that either $l_1$ or $l_2$ was in clause $n$, and let $l$ be that literal. First, it is possible that when the algorithm ends $l_2$ is in clause $n$. If $l_2$ is a pure literal, then $l_2$ is set to true, satisfying the clause. Otherwise, $l_1$ and $l_2$ are in the same clause. In this case, $l_1$ is true since it was assigned true. If $l_2$ ever became $\neg l_1$, the algorithm would exit the inner loop, so $\neg l_1$ could never have been assigned true.

Second, we consider the possibility that $l_2$ was not in clause $n$ when the algorithm ended. Then we claim that $l$ is true, and, therefore, clause $n$ is satisfied. Suppose for a contradiction that it is not. Then at some later point $\neg l$ was assigned true. This could happen in one of three places. First is if $l_1 = l$ and we are at the beginning of the outer loop. However, $l_2$ would be set to $\text{next}(\neg l)$ right after, which is in clause $n$. This means we did not find the last occurrence of a literal in clause $n$ as we should have. A similar argument can be used in the other two places.

We now turn to formalizing this algorithm. For this, we define an $L_{FL}$ function $f(i, t)$ that will return the value of $l_1$ and $l_2$ after $t$ steps in stage $i$.  

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24
This is done using number recursion. In the following let \( f(c, t) = \langle l_3, l_4 \rangle \):

\[
\begin{align*}
f(i, 0) &= \min l \in C^F \land l_2 = \text{next}(l_1) \\
f(i, t + 1) &= \langle l_1, l_2 \rangle \iff \\
\phi_1 &\land \phi_2 \land (l_1 = l_3 \land l_2 = l_4) \\
\neg \phi_1 &\land \neg \phi_2 \land (l_1 = l_3 \land l_2 = \text{next}(l_4))
\end{align*}
\]

where

\[
\phi_1 \equiv l_3 = l_4 \\
\phi_2 \equiv (\text{sameClause}(l_3, l_4) \lor \text{pureLiteral}(l_4))
\]

The formulas \( \phi_1 \) and \( \phi_2 \) are the conditions that are used to recognize when the inner loop ends. The first formula is when the loop ends and we have to continue with the outer loop. The second formula is when the stage is finished. In the formula version, we do not stop if the algorithm fails. Instead we view the algorithm as failing if after \(|F|^2\) steps, \( \phi_2 \) was never true. We use this value since \(|F|\) is an upper bound on the number of literals in \( F \) and current state of the algorithm is determined by a pair of literal. In the following, any reference to time has the implicit bound of \(|F|^2\).

The final step is to extract the assignment. The assignment is done by finding the last time a variable is assigned a value. This means we must be able to determine when a variable is assigned a value. To do this, observe that a literal is assigned true just before the \( \text{next} \) function is applied to that literal. With this in mind we get the following:

\[
\text{Assigned}(i, t, l) \iff \exists t', f(i, t) = \langle \text{next}(l), t' \rangle \lor f(i, t) = \langle l, \text{next}(l) \rangle
\]

So \( \text{Assigned}(i, t, l) \) means that \( l \) was assigned true during the \( t \)th step of stage \( i \). Then we can get the assignment as follows:

\[
l \in \text{Assignment}(i, F) \iff c = \max_c \exists t \text{ Assigned}(c, t, l) \\
\land t = \max_t \text{ Assigned}(c, t, l) \\
\land c' = \max_{c'} \exists t' \text{ Assigned}(c', t', l) \\
\land t' = \max_{t'} \text{ Assigned}(c', t', l) \\
\land (c > c' \lor (c = c' \land t > t'))
\]

The idea is the value of a variable is the last value that was assigned to it.

The \( \text{VL} \) proof that this algorithm is correct is the essentially the same as the proofs of Theorem 6.3 and Theorem 6.4, which can be formalized in \( \text{VL} \). This gives the following.

**Theorem 6.5.** \( \text{VL} \) proves that, if the algorithm fails, the formula is unsatisfiable.

**Theorem 6.6.** \( \text{VL} \) proves that, if the algorithm succeeds, then, for all \( i \), \( \text{Assignment}(i, F) \) gives a satisfying assignment to the first \( i \) clauses of \( F \).
6.2 Witnessing GL* Proofs

Let $\pi$ be a GL* proof of a $\Sigma^0_1$ formula $\exists \vec{z} P(\vec{x}, \vec{z})$, and let $A$ be an assignment to the parameter variables. We assume $\pi$ is in free variable normal form (Definition 2.8).

Let $\Gamma_i \rightarrow \Delta_i$ be the $i$th sequent in $\pi$. We will prove by induction that for any assignment to all of the free variables of $\Gamma_i$ and $\Delta_i$, a function $\text{Wit}(i, \pi, A)$ will find at least one formula that satisfies the sequent.

There are two things to note. By the subformula property, every formula in $\Gamma_i$ is $\Sigma_{\text{CNF}}(2)$, which means it can be evaluated. Also, we need an assignment that gives appropriate values to the non-parameter free variables that could appear. To take care of this second point, we extend $A$ to an assignment $A'$ as follows:

1: Given a non-parameter free variable $y$, find the $\exists$-left inference in $\pi$ that uses $y$ as an eigenvariable. Let $z$ be the new bound variable and let $F$ be the principal formula.
2: Find the descendant of $F$ that is used as a cut formula. Let $F'$ be the cut formula. Note that $F$ is a subformula of $F'$, and, because of the variable restriction on cut formulas, every free variable in $F'$ is a parameter variable.
3: Assign $y$ the value that $\text{Assignment}(F', A)$ assigns $z$.

The reason for this particular assignment will become evident in the proof of Lemma 6.7.

We can now define $\text{Wit}(i, \pi, A')$, which witnesses $\Gamma_i \rightarrow \Delta_i$. $\text{Wit}$ will go through each formula in the sequent to find a formula that satisfies the sequent. $\Sigma_{\text{CNF}}(2)$ formulas are evaluated using the algorithm described in the previous section. We will now focus our attention on other $\Sigma^0_1$ formulas, which must appear in $\Delta_i$. Each $\Sigma^0_1$ formula $F \equiv \exists \vec{z} F^*(\vec{z})$ in $\Delta$ is evaluated by finding a witness to the quantifiers as follows:

1: Find a formula $F'$ in $\pi$ that is an ancestor of $F$, is satisfied by $A'$, and is a $\Sigma^0_1$ formula of the form $F^*(\vec{z}_1/B_1, \ldots, \vec{z}_n/B_n)$, where each $B_i$ is $\Sigma^0_1$.
2: $z_i$ is assigned $\top$ if $A'$ satisfies $B_i$, otherwise it is assigned $\bot$.
3: if no such $F'$ exists, then every bound variable is assigned $\bot$.

Lemma 6.7. For every sequent $\Gamma_i \rightarrow \Delta_i$ in $\pi$, $\text{Wit}(i, \pi, A')$ finds a false formula in $\Gamma_i$ or a witness for a formula in $\Delta_i$.

Proof. We prove the theorem by induction on the depth of the sequent. For the base case, the sequent is an axiom, and the theorem obviously holds. For the inductive step, we need to look at each rule. We can ignore $\forall$-left and $\forall$-right since universal quantifiers do not appear in $\pi$.

We will now assume all formulas in $\Gamma_i$ are true and all $\Sigma_{\text{CNF}}(2)$ formulas in $\Delta_i$ as false. So we need to find a $\Sigma^0_1$ formula in $\Delta_i$ that is true.

Consider cut. Suppose the inference is

$$\frac{F, \Gamma \rightarrow \Delta, F}{\Gamma \rightarrow \Delta}$$
First suppose $F$ is true. By induction, with the upper left sequent, $Wit$ witnesses one of the formulas in $\Delta$. Then the corresponding formula in the bottom sequent is witnessed by $Wit$. This is because the ancestor of the formula in the upper sequent that gives the witness is also an ancestor of the corresponding formula in the lower sequent. If $F$ is false, it cannot be the formula that was witnessed in the upper right sequent, and a similar argument can be made.

Consider $\exists$-right. Suppose the inference is

$$
\frac{\Gamma \rightarrow \Delta, F(B)}{\Gamma \rightarrow \Delta, \exists z F(z)}
$$

First suppose $F(B)$ is $\Sigma^q_0$. If it is false, we can apply the inductive hypothesis, and, by an argument similar to the previous case, prove one of the formulas in $\Delta$ must be witnessed. If $F(B)$ is true, then $Wit$ will witness $\exists z F(z)$ since $F(B)$ is the ancestor that gives the witness. If $F(B)$ is not $\Sigma^q_0$, then we can apply the inductive hypothesis, and, by the same argument, find a formula that is witnessed.

The last rule we will look at is $\exists$-left. Suppose the inference is

$$
\frac{F(y), \Gamma \rightarrow \Delta}{\exists z F(z), \Gamma \rightarrow \Delta}
$$

To be able to apply the inductive hypothesis, we need to be sure that $F(y)$ is satisfied. If $\exists z F(z)$ is true, then we know $F(y)$ is satisfied by the construction of $A'$: the value assigned to $y$ is chosen to satisfy $F(y)$ if it is possible. Otherwise, $\exists z F(z)$ is false, and we do not need induction.

For the other rules the inductive hypothesis can be applied directly and the witness found as in the previous cases.

**Theorem 6.8.** $\mathcal{VL}$ proves $GL^*$ is sound for proofs of $\Sigma^q_1$ formulas.

**Proof.** The functions $Assignment$ and $Wit$ are in $FL$ and can be formalized in $\mathcal{VL}$. A function that finds $A'$, given $A$, can also be formalized since it in $\mathcal{VL}$. The final thing to note is that the proof of Lemma 6.7 can be formalized in $\mathcal{VL}$ since the induction hypothesis can be express as a $\Sigma^q_0 (L_{FL})$ formula and the induction carried out.

The reason this proof does not work for a larger proof system, say $G^*_1$, is because $Assignment$ cannot be formalized for the larger class of cut formulas. Also, if the variable restriction was not present, we would not be able to find $A'$ in $L$, and the proof would, once again, break down.

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