ON THE GEOMETRIC GENUS OF SUBVARIETIES OF GENERIC HYPERSURFACES

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ABSTRACT. We prove some lower bounds on certain twists of the canonical bundle of a subvariety of a generic hypersurface in projective space. In particular we prove that the generic sextic threefold contains no rational or elliptic curves and no nondegenerate curves of genus 2.

The geometry of a desingularization

\[ Y \]

of an arbitrary $k$—dimensional subvariety of a generic hypersurface

\[ X \]

in an ambient variety

\[ W \text{ (e.g. } W = \mathbb{P}^n) \]

has received much attention over the past decade or so. The first author ([C], see also [CKM], Lecture 21, for an exposition and amplification) has proved that for

\[ k = 1, n = 4, W = \mathbb{P}^4 \]

and $X$ of degree

\[ d \geq 7, \]

$Y$ has genus

\[ g \geq 1, \]

and conjectured that the same is true for $d = 6$ as well; this conjecture is sharp in the sense that hypersurfaces of degree $d \leq 5$ do contain rational

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curves. What the statements mean in plain terms is that, in the indicated range, a nontrivial function-field solution of a generic polynomial equation must have genus $> 0$. We shall refer to this as the sextic conjecture, to distinguish it from his other conjecture concerning rational curves on quintics.

The sextic conjecture was proved by Voisin [VE] who showed more generally that, for $X$ of degree $d$ in

$$\mathbb{P}^n, n \geq 4, k \leq n - 3,$$

$$p_g(Y) > 0 \text{ if } d \geq 2n - 1 - k$$

and $K_Y$ separates generic points if

$$d \geq 2n - k.$$  

In the case of codimension 1 in $X$, i.e. $k = n - 2$, Xu [X] gave essentially sharp geometric genus bounds. For $X$ a generic complete intersection of type $(d_1, \ldots, d_k)$ in any smooth polarized $(n + k)$-fold $M$, Ein [E] proved that

$$p_g(Y) > 0$$

if

$$d_1 + \ldots + d_k \geq 2n + k - m - 1$$

and $Y$ is of general type if

$$d_1 + \ldots + d_k \geq 2n + k - m + 2.$$  

Ein’s bounds are generally not sharp, e.g. they fail to yield the sextic conjecture for $M = \mathbb{P}^n$. The paper [CLR] gives some refinements and generalizations of the results of Ein and Xu by a method which seems to yield essentially sharp bounds in codimension 1 but not necessarily in general.

In this paper we give a result which improves and generalizes the sextic conjecture. It gives a lower bound on a twist of the canonical bundle of an arbitrary subvariety of a generic hypersurface in projective space. For the case of the sextic in $\mathbb{P}^4$ it shows that the minimal genus of a curve is at least 2, and at least 3 if the curve is nondegenerate. We proceed to state the result.

First, we fix a projective space

$$\mathbb{P} = \mathbb{P}^n.$$  

We denote by

$$\mathcal{L}_d$$

the space of homogeneous polynomials of degree $d$ on $\mathbb{P}$. If $Y$ is a variety with a given morphism

$$f : Y \to \mathbb{P}$$

and $A$ is a coherent sheaf on $Y$, the sheaf

$$A \otimes f^*(\mathcal{O}_\mathbb{P}(i))$$

will be denoted by $A(i)$. 
Theorem 0.1. Let 

\[ X \in \mathcal{L}_d \]

be generic with 

\[ d(d + 1)/2 \geq 3n - 1 - k, \quad d \geq n, \]

and 

\[ f : Y \to X \]

a desingularization of an irreducible subvariety of dimension \( k \). Set 

\[ t = \max(0, -d + n + 1 + \left[ \frac{n - k}{2} \right]). \]

Then either

(0.1) \[ h^0(\omega_Y(t)) > 0 \]

or \( f(Y) \) is contained in the union of the lines lying on \( X \).

In the case where \( k = n - 3 \) we have furthermore:

(i) if \( h^0(\omega_Y(t)) = 0 \), then \( f(Y) \) is ruled by lines;

(ii) if \( d \geq n + 2 \) and \( f(Y) \) is not ruled by lines, then

(0.2) \[ h^0(\omega_Y) + 1 \geq \min(\dim(\text{span}(f(Y))), 4). \]

Finally in the case in which \( Y \) is a curve on a generic sextic threefold, we have 

\[ g(Y) \geq 3, \]

except possibly if \( Y \) is a genus-2 curve such that \( f(Y) \) spans a hyperplane.

Remark 0.2. In case either \( k = n - 2 \) or \( k = n - 3 \), \( d > n + 2 \), our statement is weaker than the result of [CLR]. Therefore from now on we may assume that \( k \leq n - 3 \), and that if equality holds then \( d = n + 2 \).

Remark 0.3. If \( k = 2n - 2 - d \geq 1, \quad d \geq n + 3 \), our result implies that \( H^0(\omega_Y) \neq 0 \) unless \( Y \) is ruled by lines (which in this case means \( Y \) is a component of the union of lines in \( X \)). In this form the result was first obtained by Pacienza [P]. In fact in this paper we build on the methods of Voisin [V] and Pacienza [P], which play a very important role in our results.

If \( k > 2n - 2 - d \geq 0 \), then of course \( Y \) cannot be ruled by lines, so we conclude in this case that (0.1) (or, if \( k = n - 3, \quad d > n + 1 \), (0.2)) holds.

Our proof is based in the observation, already made by Voisin and, in a more limited context, by the second author, that the failure of the estimate (0.1) implies the existence of a line

\[ \ell = \ell(y) \]
through a general point

\[ y \in Y = Y_F \]

with the property that the space of first-order variations

\[ F' \]

of \( F \)

such that

\[ Y_{F'} \ni y \text{ and } F'|_\ell = F|_\ell \]

is larger than expected. This leads us to consider a distribution

\[ T' \]

on the space \( Y_x \) of polynomials \( F \) such that \( Y_F \ni x \). Inspired by Voisin, we show in \$6\) that \( T' \) is integrable (Lemmas 6.1-6.2). This in turn leads to the conclusion that \( \ell(y) \) is contained in the hypersurface \( F \) (Lemma 6.4). We then show in \$7\) that if \( k = n - 3 \) or, more generally, if a certain infinitesimal invariant \( s' \) vanishes, then in fact, \( \ell(y) \) is contained in \( Y_F \) itself, so that \( Y_F \) is ruled by lines.

The main argument is presented in \$6, 7\) which are preceded by 5 preliminary sections. In \$1\) we give precise definitions and statements. In \$2\) we recall some (probably well known) global generation results for some standard homogeneous vector bundles on projective spaces and Grassmanians. In \$3\) we study quotients of the space of homogeneous polynomials of degree \( d \) vanishing at a fixed point \( x \), and in particular the question of how much of such a quotient is generated by a generic collection of degree-\((d-1)\) polynomials. It turns out that if the answer is less than expected, this can sometimes be explained by the existence of some special line \( \ell \) through \( x \). In \$4\) we use the results of \$3\) to reduce the proof of the main theorem, in effect, to the study of the distribution \( T' \). \$5\) recalls, and reproves, some well-known results on canonical bundles of varieties parametrizing osculating lines to hypersurfaces.

This paper supersedes an earlier eprint by the second author entitled 'Beyond a conjecture of Clemens' \([\text{math.AG/9911161}]\).

1. The formal setting

We will fix integers \( d, n \) and denote by \( \mathcal{L} \) or \( \mathcal{L}_d \) the space of homogeneous forms of degree \( d \) on \( \mathbb{P}^n \) and set

\[ S = \mathcal{L} - \{0\}. \]

An element \( F \in \mathcal{L} \) is written in the form

\[ F = \sum_{|A| = d} a_A X^A. \]
We denote by
\[ \mathcal{X} \subset S \times \mathbb{P}^n \]
the universal hypersurface, which is smooth, as is the natural map
\[ s : \mathcal{X} \to \mathbb{P}^n, \]
whose fibre over \( x \in \mathbb{P}^n \) we denote by \( X_x \), while the fibre over \( F \in S \) is denote \( X_F \).

We will be studying a certain kind of versal families of \( k \) folds on \( \mathcal{X}/S \). By this we mean the following. Let \( T \) be an irreducible subvariety of the Chow variety (or Hilbert scheme) of \( \mathbb{P}^n \) whose general element corresponds to an irreducible \( k \)-dimensional variety. Suppose that
\[ \psi : T \to S \]
be a dominant morphism with the property that if we denote the universal cycle (or subscheme) over \( T \) by \( Z_T \), then
\[ Z_T \subset \mathcal{X}_T := \mathcal{X} \times_T T. \tag{1.1} \]
Clearly there is no loss of generality by assuming \( T \) is saturated with respect to projective equivalence, i.e. is stable under the natural action of \( \text{GL}(n+1) \). Note that \( \text{GL}(n+1) \) acts on (1.1) and choose a \( \text{GL}(n+1) \)-stable subvariety \( T' \subset T \) so that the map \( T' \to S \) is étale. Then choose a \( \text{GL}(n+1) \)-equivariant resolution of
\[ Z_{T'} \to T' \]
and denote this by \( \mathcal{Y}/S' \). Replacing \( S' \) by a \( \text{GL}(n+1) \)-stable open subset, we may assume \( \mathcal{Y}/S' \) is a smooth projective relatively \( k \)-dimensional family endowed with an \( S' \)-morphism
\[ f : \mathcal{Y} \to \mathcal{X} \times_S S'. \]
We will call this a \textit{versal family of \( k \)-folds}.

2. \textbf{Positivity results}

(2.0) A useful remark is the following. Let
\[ 0 \to A \to B \to C \to 0 \]
be a short exact sequence of vector bundles of respective ranks \( a, b, c \). Then for any \( k > 0 \), the exterior power
\[ \bigwedge^k B \]
is endowed with a descending filtration $F^i$ with

$$F^i/F^{i+1} \simeq \bigwedge^i C \otimes \bigwedge^i A, i = 0, \ldots, k.$$  

In particular, there is a surjection

$$c+1 \bigwedge B \to A \otimes \det(C) \to 0,$$

so that, e.g., $A \otimes \det(C)$ is globally generated if $B$ is.

Let $M_{\mathbb{P}^n}^d$ be the vector bundle on $\mathbb{P}^n$ defined by the natural exact sequence

$$0 \to M_{\mathbb{P}^n}^d \to \mathcal{L}_d \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$$  

Then it is well known that $M_{\mathbb{P}^n}^1 = \Omega_{\mathbb{P}^n}(1)$ and that $M_{\mathbb{P}^n}^1(1)$ is globally generated. This can be proved by taking 2nd exterior power of the Euler sequence. Similarly, taking $(i+1)$st exterior powers in the above sequence, we see that

$$(2.1) \quad (\bigwedge^i M_{\mathbb{P}^n}^d)(d) \text{ is globally generated } \forall d, i > 0.$$  

By surjectivity of the multiplication map $\mathcal{L}_{d-1} \otimes M_{\mathbb{P}^n}^1 \to M_{\mathbb{P}^n}^d$, it follows that $(\bigwedge^i M_{\mathbb{P}^n}^d)(i)$ is globally generated, hence

$$(2.2) \quad (\bigwedge^i M_{\mathbb{P}^n}^d)(\min(d, i)) \text{ is globally generated } \forall d, i > 0.$$  

More generally, for any Grassmannian

$$G = G(r, \mathbb{P}^n),$$

we define a bundle $M_G^d$ by the natural exact sequence

$$0 \to M_G^d \to \mathcal{L}_d \otimes \mathcal{O}_G \to \text{Sym}^d(Q) \to 0,$$

where $Q$ is the tautological quotient bundle on $G$. Thus $M_G^1$ is just the tautological subbundle, which fits in an exact sequence

$$0 \to M_G^1 \to (n+1)\mathcal{O}_G \to Q \to 0.$$  

Taking $(n-r+1)$st exterior power of this sequence and using the fact that $\bigwedge^n Q = \det Q = \mathcal{O}_G(1)$, we conclude that $M_G^1(1)$ is globally generated, hence it follows as above that

$$(2.3) \quad M_G^d(1) \text{ is globally generated, } \forall d \geq 1.$$
3. Local linear systems

Fix $x \in \mathbb{P}^n$ and $d$ and set

$$M = M^d_x,$$

i.e. the vector space of homogeneous polynomials of degree $d$ zero at $x$. We will sometimes denote the same vector space by $\mathcal{L}_d(-x)$, and we similarly have $\mathcal{L}_d(-Z)$ for any subset $Z \subset \mathbb{P}^n$. A local linear system at $x$ is simply a subspace $T \subset M$, whose codimension we denote by $h$. We define the integer $s = s(T)$ as the smallest with the property that, for generic choice of $P_1, \ldots, P_s \in \mathcal{L}_{d-1}$, the natural map

$$\mu_s = \mu_{s,T} : \sum_{i=1}^s P_i M^1_x \to M/T$$

is surjective. More generally, suppose $Z$ is an irreducible variety, $g : Z \to \mathbb{P}^n$ a morphism and $T \subset g^*(M^d_{\mathbb{P}^n})$ a subhseaf. Then we define $s = s(T)$ to be smallest so that the map

$$\mu_s : \sum_{i=1}^s P_i g^*(M^1_{\mathbb{P}^n}) \to g^*(M^d_{\mathbb{P}^n})/T$$

defined as above is generically surjective for general choice of $P'_i$'s.

The interest in studying this integer comes from the following elementary observation.

**Lemma 3.1.** In the above situation, we have that

$$H^0(\det(g^*(M^d_{\mathbb{P}^n})/T) \otimes g^*(\mathcal{O}(s))) \neq 0.$$

**proof.** The above sheaf is generically a quotient of a sum of tensor products of sheaves of the form $g^*((\bigwedge^i M^1_{\mathbb{P}^n})(1))$. □

Returning to the above situation, set

$$\gamma(i) = \text{rk}(\mu_i)$$

and note that this is a strictly increasing function for $0 \leq i \leq s$, which is 'concave' in the sense that $\gamma(i+1) - \gamma(i)$ is non-increasing. Let $s' = s'_T$ be the smallest $i$ such that

$$\gamma(i+1) - \gamma(i) \leq 1.$$
Note that

\[(3.3) \quad s' = s \Rightarrow s \leq h/2\]

while if \(s' < s\) we have for generic \(P \in \mathcal{L}_{d-1}\)

\[(3.4) \quad \dim(PM^1_x \cap \text{im}(\mu_{s'})) = n - 1.\]

Of course by definition, we have

\[(3.5) \quad \text{rk}(\mu_{s'}) + s - s' = h.\]

Some interesting things happen when \(s - s' > 1:\)

**Lemma 3.2.** Suppose that \(s - s' > 1\). Then

(i) There is a line \(\ell = \ell_T\) through \(x\) such that

\[\mathcal{L}_d(-\ell) = M^d_{\ell} \subseteq T + \text{im}(\mu_{s'}).\]

(ii) \(\ell\) is uniquely determined independent of \(P_1, \ldots, P_{s'}\).

(iii) We have

\[(3.6) \quad \dim(T/(T \cap \mathcal{L}_d(-\ell))) = d - s,\]

\[(3.7) \quad \dim(\mathcal{L}_d(-\ell)/(T \cap \mathcal{L}_d(-\ell))) = \text{rk}(\mu_{s'}) - s'.\]

**proof.**

(i) Let \(\ell\) be the unique line through \(x\) so that \(M^1_{\ell} \subset M^1_x\) is the codimension-1 subspace in (3.4).

(ii) By assumption, we have for \(P, P_1, \ldots, P_{s'+1}\) generic

\[\dim(PM^1_x \cap (T + \sum_{1}^{s'+1} P_i M^1_x)) = \dim(PM^1_x \cap (T + \sum_{1}^{s'} P_i M^1_x)) = n - 1,\]

hence the two intersections are equal and hence they are independent of \(P_{s'+1}\); by symmetry, they are also independent of any \(P_i\), hence so is \(\ell\).

(iii) Let

\[r = \dim(\frac{M^d_x}{T + M^d_{\ell}}).\]

Note that the image of \(PM^1_x\) in this vector space is 1-dimensional, so if \(r \leq s'\) we have that \(\mu_{s'}\) is surjective, while if \(s' \leq r < s\) we have that \(\mu_r\) is surjective, both of which are impossible. Since in any case \(r \leq s\) we have \(r = s\), which easily implies our first assertion. The second assertion follows easily by dimension counting.
4. Reduction

We are now ready to apply the results of the last section to our versal family. So let
\[ f : \mathcal{Y}/S' \rightarrow \mathcal{X}' = \mathcal{X} \times_S S' \]
be as in §1. Then we have the normal sheaf \( N_f \) which fits in an exact sequence
\[ 0 \rightarrow T_{\mathcal{Y}} \rightarrow f^*T_{\mathcal{X}'} \rightarrow N_f \rightarrow 0. \]
By GL\((n + 1)\)-equivariance, clearly \( \mathcal{Y} \) and \( \mathcal{X}' \) are smooth over \( \mathbb{P}^n \), so we have another exact
\[ 0 \rightarrow T_{\mathcal{Y}/\mathbb{P}^n} \rightarrow f^*T_{\mathcal{X}'/\mathbb{P}^n} \rightarrow N_f \rightarrow 0. \]
Now we have
\[ f^*T_{\mathcal{X}'/\mathbb{P}^n} \simeq f^*M_d^{\mathbb{P}^n}, \]
therefore we may apply Lemma 3.1 with \( T = T_{\mathcal{Y}/\mathbb{P}^n} \) to conclude that
\[ H^0(\det(N_f) \otimes f^*\mathcal{O}(s)) \neq 0. \]
Now use the generic smoothness of \( \mathcal{Y} \) and \( \mathcal{X}' \) over \( S' \) to conclude that for generic \( F \in S' \) we have
\[ \det(N_f)|_{\mathcal{Y}_F} = \omega_{\mathcal{Y}_F} \otimes f^*\omega_{\mathcal{X}_F}^{-1} = \omega_{\mathcal{Y}_F} \otimes f^*\mathcal{O}(-d + n + 1). \]
Therefore by Lemma 3.1 we have
\[ (4.1) \quad H^0(\omega_{\mathcal{Y}_F}(-d + n + 1 + s)) := H^0(\omega_{\mathcal{Y}_F} \otimes f^*\mathcal{O}(-d + n + 1 + s)) \neq 0. \]
The case where \( s - s' > 1 \) will be analyzed at length below; for now we just note that, in that case, the uniquely determined line \( \ell \) going through \( f(y) \) for general \( y \in \mathcal{Y} \) gives rise to a rational mapping
\[ g : Y \rightarrow G = G(1, \mathbb{P}^n). \]
Putting things together we get a lifting of \( f \) to a map
\[ \mathcal{Y} \rightarrow \Delta := \{ (\ell, x, F) : x \in \ell \cap X_F \} \subset G \times \mathbb{P}^n \times S'. \]
Viewing \( T_{\mathcal{Y}/\mathbb{P}^n} \) as a subsheaf of \( L_d \otimes \mathcal{O}_{\mathcal{Y}} \), we have another subsheaf of \( L_d \otimes \mathcal{O}_{\mathcal{Y}} \), which we denote by
\[ L_d(-\ell), \]
whose fibre at $y$ is the $d$-forms vanishing on $\ell(y)$; we set
\[ T' = T_{Y/P^n} \cap \mathcal{L}_d(-\ell). \]

By Lemma 3.1, we have
\[ \text{(4.2)} \quad \text{rk}(T_{Y/P^n}/T') = d - s \]
\[ \text{(4.3)} \quad \text{rk}(\mathcal{L}_d(-\ell)/T') = \text{rk}(\mu_{s'}) - s'. \]

Now suppose that
\[ s - s' \leq 1. \]

Recall that
\[ n - k - 1 = \text{rk}(\mu_{s'}) + s - s' \geq s + s'. \]

Going through the various possibilities we see easily that in all cases where
\[ s - s' \leq 1 \]
we have
\[ \text{(4.4)} \quad s \leq \left\lceil \frac{n - k}{2} \right\rceil. \]

Therefore we conclude, using (4.1),

**Alternative 4.1.** In the situation of the main theorem, we have either
\[ \text{(4.5)} \quad H^0(\omega_{Y_P}(-d + n + 1 + \left\lceil \frac{n - k}{2} \right\rceil)) \neq 0 \]
or
\[ \text{(4.6)} \quad s - s' > 1. \]

Note that when $n - 1 - k = 2$, the only possibilities are $s' = 0, s = 2$ and
$s = s' = 1$. Let us now analyze generally the case $s = 1$. Let $Y$ be a general fibre $Y = Y_P$ and $N$ the normal sheaf for the map $Y \to \mathbb{P}^n$, so we have an exact sequence
\[ 0 \to N_f \to N \to \mathcal{O}_Y(d) \to 0. \]

Then choosing a general polynomial $P \in \mathcal{L}_{d-1}$, we get a diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \to & f^*\mathcal{M}_{\mathbb{P}^n}^1 & \to & \mathcal{L}_1 \otimes \mathcal{O}_Y & \to & \mathcal{O}_Y(1) & \to & 0 \\
0 & \to & f^*\mathcal{M}_{\mathbb{P}^n}^d & \to & \mathcal{L}^d \otimes \mathcal{O}_Y & \to & \mathcal{O}_Y(d) & \to & 0 \\
0 & \to & N_f & \to & N & \to & \mathcal{O}_Y(d) & \to & 0
\end{array}
\]
Now the assumption that \( s = 1 \) means that the left column composite is generically surjective, hence the same is true for the middle column. Thus we get a generically surjective map

\[
\psi : L_1 \otimes O_Y \rightarrow N
\]

which drops rank on \( \text{Zeroes}(P) \) and also clearly factors through \( H^0(O_Y(1)) \otimes O_Y \) whose rank we write as \( p + 1 \). Set

\[
N' = \text{im}(\psi) / (\text{torsion}).
\]

From the bottom row in the above display we conclude

\[
c_1(N') \leq K_Y + (n + 2 - d)H
\]

where \( A \leq B \) means \( B - A \) is effective and \( H \) is a hyperplane. Now the torsion-free quotient \( N' \) of the trivial bundle corresponds to a rational map to a Grassmannian, which by blowing up we may assume is a morphism

\[
\gamma : Y \rightarrow G := G(k + 1, p + 1),
\]

such that

\[
\bigcup_{y \in Y} \gamma(y) \text{ spans } \mathbb{C}^{p+1}
\]

(because \( \gamma(y) \ni f(y) \)) and that

\[
c_1(N') = \gamma^*(O_G(1)).
\]

Thus clearly \( c_1(N') \) is effective and if \( p > k \) (i.e. \( Y \) itself is not a \( \mathbb{P}^k \)), then \( \gamma \) is nonconstant, so

\[
h^0(c_1(N')) \geq 2.
\]

It is elementary and well known that any linear \( \mathbb{P}^1 \) in \( G \) consists of the pencil subspaces contained in a fixed \( k + 2 \)-dimensional subspace \( A \subseteq \mathbb{C}^{p+1} \) and containing a fixed \( k \)-dimensional one \( B \subseteq A \). Now if \( k < p - 1 \) then \( A \not\subseteq \mathbb{C}^{p+1} \) and since \( \bigcup_{y \in Y} \gamma(y) \) spans \( \mathbb{C}^{p+1} \) it follows that \( \gamma(Y) \) cannot be contained in such a pencil, so

\[
h^0(c_1(N')) \geq 3.
\]

Thus we conclude
Lemma 4.2. Assuming $s = 1$ and that the general $f(Y)$ spans a $\mathbb{P}^p$, we have

$$h^0(\omega_Y(n + 2 - d)) \geq \min(p - 1, 3).$$

Now suppose moreover that $n = 4, d = 6, k = 1$ (still assuming $s = 1$). We claim that if $Y$ is planar, i.e. $p = 2$, then $g(Y) \geq 2$ (in fact, we will show $g(Y) \geq 4$). Let

$$I \subset \mathcal{L}_6 \times G(2, \mathbb{P}^4)$$

denote the (open) set of pairs $(X, B)$ where $X$ is a smooth sextic hypersurface in $\mathbb{P}^4$ and $B$ is a plane in $\mathbb{P}^4$, and let $J$ denote the set of pairs $(D, B)$ where $B$ is a plane in $\mathbb{P}^4$ and $D$ is a sextic curve in $B$, not necessarily reduced or irreducible. Since a smooth sextic cannot contain a plane, there is a natural morphism

$$\pi : I \to J,$$

$$(X, B) \mapsto (X \cap B, B).$$

Clearly $\pi$ is a fibre bundle. Now a fundamental fact of plane geometry [AC] is that the family of reduced irreducible plane curves of geometric genus $g$ and degree $d$ is of dimension $3d + g - 1$. It follows easily from this that the locus $J_0 \subset J$ consisting of pairs $(D, B)$ such that $D$ is the target of a nonconstant map from a curve of genus $\leq 3$ is of codimension $> 6$, hence $I_0 := \pi^{-1}(J_0) \subset I$ is also of codimension $> 6$. Since $G(2, \mathbb{P}^4)$ is 6-dimensional, $I_0$ cannot dominate $\mathcal{L}_6$. Thus we conclude

Lemma 4.3. If $s = 1, n = 4, d = 6, k = 1$, then $Y$ is a curve of genus at least 3, except possibly if $Y$ is genus-2 curve spanning a hyperplane.

5. Line osculation hierarchy

Before continuing with the proof of the main theorem, we digress to discuss canonical bundles of 'line osculation’ varieties, i.e. the varieties

$$\Delta_r := \{(\ell, x, F) : \ell.X_F \geq r.x\} \subset G \times \mathbb{P}^n \times S$$

where $G = G(1, \mathbb{P}^n)$. We begin with a formal definition. Let $Q$ be the tautological quotient bundle on $G$ and

$$\mathcal{O}_S(-1) \subset \mathcal{L}_d \otimes \mathcal{O}_S$$

the tautological subbundle on $S$ (whose fibre at $F$ is $\mathbb{C}F$); of course, this bundle admits a canonical nowhere vanishing section, so it is canonically
trivial. Then $\Delta = \Delta_1$ is the common zero-locus on $G \times \mathbb{P}^n \times S$ of the two natural maps

(5.1) \[ O_{\mathbb{P}^n}(-1) \to Q \]

(5.2) \[ O_S(-1) \to O_{\mathbb{P}^n}(d) \]

where to save notation we have suppressed the various pullbacks. From this it is easy to see by the adjunction formula that $\Delta$, which is obviously smooth, has canonical bundle

(*) \[ \omega_{\Delta} = \omega_{G \times \mathbb{P}^n \times S} \otimes \det(Q) \otimes O_{\mathbb{P}^n}(n - 1 + d) = O_G(-n) \otimes O_{\mathbb{P}^n}(d - n + 1). \]

Let $S$ denote the tautological subbundle on $G$. Then because (5.1) vanishes on $\Delta$, we have on $\Delta$ a natural injection

\[ O_{\mathbb{P}^n}(-1) \to S \]

whose (rank-1) quotient we denote by $R^\vee$. Thus we have an exact sequence on $\Delta$

(5.3) \[ 0 \to R \to S^\vee \to O_{\mathbb{P}^n}(1) \to 0 \]

which yields

(5.4) \[ R = \det(S^\vee) \otimes O_{\mathbb{P}^n}(-1) = O_G(1) \otimes O_{\mathbb{P}^n}(-1). \]

The sequence (5.3) induces a descending filtration $F^\cdot$ on $\text{Sym}^d(S^\vee)$ with quotients

\[ F^i / F^{i+1} = O_{\mathbb{P}^n}(d - i) \otimes R^i = O_G(i) \otimes O_{\mathbb{P}^n}(d - 2i), 0 \leq i \leq d, \]

\[ F^{d+1} = 0. \]

Note that the vanishing of 5.2 implies that the natural map

\[ L_d \to \text{Sym}^d(S^\vee) \]

induces a map

\[ O_S(-1) \to F^1. \]

We define $\Delta_r$ as the zero scheme of the induced map

\[ O_S(-1) \to F^1 / F^r. \]
In particular, $\Delta_{d+1}$, the zero-scheme of the natural map

$$\mathcal{O}_S(-1) \to F^1,$$

parametrizes the triples

$$(\ell, x, F)$$

such that

$$x \in \ell \subset X_F.$$  

By considering the projection to (the incidence subvariety of) $G \times \mathbb{P}^n$, it is easy to see that $\Delta_r$ is smooth for $r \leq d + 1$. Using the adjunction formula again, we see that

$$\omega_{\Delta_r} = \omega_{\Delta} \otimes \det(F^1/F^r) = \omega_{\Delta} \otimes \bigotimes_{i=1}^{r-1} (\mathcal{O}_{\mathbb{P}^n}(d-2i) \otimes \mathcal{O}_G(i)).$$

By (*) we get the formula

$$\omega_{\Delta_r} = \mathcal{O}_G\left(\frac{r(r-1)}{2} - n\right) \otimes \mathcal{O}_{\mathbb{P}^n}(r(d-r+1) - n + 1)$$

6. The case $s - s' > 1$

We will assume from now on, for the remainder of this paper, that

$$s' + 1 < s$$

and that

$$H^0(\omega_{Y_F}) = 0$$

for generic $F \in S$. With no loss of generality we may also assume that

$$s > \left\lfloor \frac{n-k}{2} \right\rfloor,$$

and

$$-d + n + 1 + s > 0,$$

else the Main Theorem’s assertion follows from (4.1). This means

$$2s > n - k, d - n - s - 1 < 0.$$
Our aim is to show in this case that

\[ Y = Y_F \]

is contained in the locus ruled by lines in \( X_F \). We fix a point \( x_0 \in \mathbb{P}^n \) which we assume has coordinates \([1, 0, ..., 0]\), and let

\[ G_1 < GL(n + 1) \]

be the stabilizer of \( x_0 \), that is, the parabolic subgroup

\[
\left\{ \begin{bmatrix}
  * & * & \cdots & * \\
  0 & * & \cdots & * \\
  \vdots & \cdots & \cdots & \vdots \\
  0 & * & \cdots & * 
\end{bmatrix} \right\}
\]

and let

\[ G_0 \simeq GL(n) < G_1 \]

be the subgroup fixing \( X_0 \), i.e.

\[
G_0 = \left\{ \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & * & \cdots & * \\
  \vdots & \cdots & \cdots & \vdots \\
  0 & * & \cdots & * 
\end{bmatrix} \right\}.
\]

Note that the fibre \( Y_0 \) of \( Y \) over \( x_0 \) is invariant under the natural action of \( G_1 \) on the space of polynomials.

Now a homogeneous polynomial of degree \( d \) in \( X_0, ..., X_n \) may be written in the form

\[ (6.5) \quad F = \sum_{|A| \leq d} a_A X_0^{d-|A|} X^A \]

where

\[ A = (A(1), ..., A(n)) \]

is a multi-index referring to the variables \( X_1, ..., X_n \). Thus the \( X_0^{d-|A|} X^A \) form a basis of the space \( \mathcal{L}_d \) of such polynomials and the \( a_A \) is the associated coordinate system or dual basis and give rise to differentiation operators

\[ (6.6) \quad \partial_A = \partial/\partial a_A. \]

Note that the action of \( GL(n + 1) \) on \( Y \) induces an action of \( G_1 \) on the fibre \( Y_0 \) of \( Y \) over

\[ x_0 = [1, 0, ..., 0] \in \mathbb{P}^n. \]
Working in an analytic neighborhood $Y_{00}$ of a general point $y_0 \in Y_0$

we may identify it via the map $f$ with an open subset of $S$. Of course, $S$
being an open subset of $L_d$, we have trivializations

$$T_S \simeq L_d \otimes \mathcal{O}_S, T_S \otimes \mathcal{O}_{Y_0} \simeq L_d \otimes \mathcal{O}_{Y_0},$$

with a frame being provided by the commuting vector fields $\partial_A$ for $|A| \leq d$,
which correspond to the respective monomials $X_0^{d-|A|}X^A$. Then we get an embedding

$$T_{Y_0} \subset L_d \otimes \mathcal{O}_{Y_0}.$$  

Recall the subbundle

$$T' = T_{Y_0} \cap L_d(-\ell).$$

Let

$$L_{d,m}(\ell) \subset L_d(l)$$

denote the subsheaf of elements of degree $m$ in $X_1, \ldots, X_n$. We assume that
for $y \in Y_{00}$, the line $\ell(y)$ is given by

$$X_r = b_r(y)X_1, r = 2, \ldots, n,$$

and that at our 'initial' (general) point $y_0 \in Y_0$ we have

$$b_2(y_0) = \ldots = b_n(y_0) = 0.$$  

Note that generators for $L_d(-\ell)$ are given under the identification (6.7)
by the vector fields

$$\partial_A r - b_r \partial_{A1} \in T_S \otimes \mathcal{O}_{Y_{00}}, |A| \leq d - 1, r = 2, \ldots n,$$

corresponding to the polynomials

$$X_0^{d-1-|A|}X^A(X_r - b_r X_1).$$

**Lemma 6.1.** $T'$ annihilates $b_r$ for all $r \geq 2$.

**proof.** It suffices to show this at the point $y_0$. Set

$$T'' = T' \cap L_d(-2\ell).$$
We begin by showing that $T''$ annihilates the $b_r$. Note that sections of $T''$ can be written as linear combinations of the vector fields

$$\partial_{Ar} - b_r \partial_{A1r} - b_s \partial_{A1s} + b_r b_s \partial_{A11}, \quad |A| \leq d - 2, r, s \geq 2,$$

from which it follows that

$$[T'', T''] \subseteq T'.$$

We thus have a well-defined bracket pairing

(6.9) \[ T'' \times (T'/T'') \rightarrow T/T' \]

and we are claiming this vanishes. $T'/T''$ is generated by sections the form

$$\tau' = \sum_{r=2}^{n} \sum_{\alpha=1}^{d-1} c_{r\alpha}(\tau')(\partial_{r1\alpha} - b_r \partial_{1\alpha + 1})$$

and the assignment

$$\tau' \mapsto (c_{r\alpha}(\tau'))$$

identifies $T'/T''$ with a subsheaf, which we may assume is locally free,

$$T'/T'' \subseteq M_{(n-1) \times d}(O)$$

of $O_{Y_{00}}$-valued $(n - 1) \times d$ matrices, and the corank of this subsheaf is at most the corank of $T''$ in $L_d(-\ell)$. Hence by (4.3) this corank is at most $n - k - 1 - s$. On the other hand a section

$$\tau \in T/T'$$

can be written in the form

$$\tau = \sum_{\beta=1}^{d} c_{\beta}(\tau) \partial_{1\beta}$$

and this identifies $T/T'$ with a subsheaf

$$T/T' \subseteq dO_{Y_{00}}$$

whose corank by (4.2) is exactly $s$. Now for $\tau' \in T'/T''$, $\tau'' \in T''$, the pairing (6.9) is given by

$$[\tau', \tau''] = \sum c_{r\alpha} \tau''(b_r) \partial_{1\alpha + 1}.$$
Now suppose $\tau''(b_r) \neq 0$ for some $r$. Then the map

$$M_{(n-1) \times d}(\mathcal{O}) \rightarrow d\mathcal{O}$$

given by multiplication by the vector

$$\beta(\tau'') := (\tau''(b_2), \ldots, \tau''(b_r))$$

is surjective, hence its restriction

$$T'/T'' \rightarrow T'/T'$$

can only lower corank, i.e. has image of corank at most

$$n - k - 1 - s$$

in $d\mathcal{O}$. But this contradicts the fact that $T/T'$ has corank $s$ in $d\mathcal{O}$, while by (6.2) we have $s > n - k - 1 - s$.

This proves that $T''$ annihilates all the $b_r$, which in turn implies that we have a well-defined bracket pairing

$$T'/T'' \times T'/T'' \rightarrow T'/T',$$

which has the form

$$[\tau_1, \tau_2] = \sum_{s, \beta} \tau_1(b_s)c_s\beta(\tau_2)\partial_1^{\beta+1} - \sum_{r, \alpha} \tau_2(b_s)c_r\alpha(\tau_1)\partial_1^{\alpha+1}.$$  

(6.10)

Now suppose that

$$\tau_1(b_s) \neq 0$$

for some

$$\tau_1 \in T'/T'', s \geq 2.$$

Then the following linear algebra observation (compare with [P]) shows that the set of elements

$$[\tau_1, \tau_2]$$

ranges over a subspace of $T/T'$ of codimension at most

$$1 + n - k - 1 - s,$$

which by (6.2) is $> s$, so we have a contradiction as above. $\square$
Sublemma 6.1.1. Let
\[ U, V \]
be finite-dimensional vector spaces,
\[ H < \text{Hom}(U, V) \]
a subspace of codimension \( c \), and
\[ \beta : H \rightarrow U \]
a nonzero homomorphism. Define a pairing
\[ I(\beta) : \bigwedge^2 H \rightarrow V \]
by
\[ (6.11) \quad A \wedge B \mapsto A\phi(B) - B\phi(A). \]

Then the image of \( I(\beta) \) is of codimension at most \( c + 1 \) in \( V \).

proof. Pick \( c \) rank-1 elements in \( \text{Hom}(U, V) \) that are linearly independent modulo \( H \), and extend \( \beta \) linearly to be zero on these. It is clear that this extension increases the rank of \( I(\beta) \) by at most \( c \). Therefore it suffices to prove the Sublemma in case \( c = 0 \). In this case, pick any \( A \) of rank 1 with \( \beta(A) \neq 0 \). Then as \( B \) ranges over \( H = \text{Hom}(U, V) \), the first term in (6.11) ranges over a 1-dimensional subspace while the second ranges over all of \( V \), hence the difference ranges at least over a codimension-1 subspace. □

As an immediate consequence of Lemma 6.1, we conclude

Lemma 6.2. The distribution \( T' \) is integrable

proof. For \( \tau_1, \tau_2 \in T' \), write
\[ \tau_i = \sum c_{Ar}(\tau_i)(\partial_{Ar} - b_r\partial_{A1}), \quad i = 1, 2. \]

Then we calculate
\[ [\tau_1, \tau_2] \equiv \sum (c_{r1\alpha}(\tau_1)\tau_2(b_r) - c_{r1\alpha}(\tau_2)\tau_1(b_r))\partial_{1\alpha+1} \quad \text{mod } T' \]
and by Lemma 6.1 the latter vanishes. □
Lemma 6.3. For general $y \in Y$, the line $\ell(y)$ is either contained in the hypersurface $X_F(y)$ or meets it set-theoretically in at most 2 points.

*proof. emoproof* The natural map

$$\ell : Y_0 \to G$$

is clearly $G_1$–equivariant, hence generically onto the locus of lines through $x_0$. Let $Z$ be its fibre over

$$\ell_0 = \{X_2 = \ldots = X_n = 0\},$$

which we may assume is a general fibre, hence of codimension $n - 1$ in $Y_0$.

There is a natural map

$$R : Z \to H^0(\mathcal{O}_{\ell_0}(d)(-x_0)),$$

$$R(F) = F|_{\ell_0}.$$  

Clearly $R$ is equivariant with respect to the stabilizer $G_{11}$ of $(x_0, \ell_0)$ in $GL(n)$, so its image is a cone invariant under the stabilizer $H$ of $x_0$ in $GL(\ell_0)$.

Note that our distribution $T'$ is none other than the vertical tangent space for the map $R$. From (4.2) it follows that

$$\dim R(Z) = d - s - n + 1$$

and by our assumptions (6.3) this is $\leq 1$. As $H$ acts doubly transitively on $\ell - x_0$, it follows easily that $R(Z)$ is contained in the set of multiples of $X_1^d$.

Therefore the lemma holds. $\square$

*Remark 6.3.1.* Note that if we assume instead of (6.3) the stronger condition that

$$-d + n + 1 + s > 1,$$

then we conclude in the same way that

$$\ell(y) \subset X_F(y).$$

In the next Lemma we will see that this holds anyway, without the extra hypothesis.

On the other hand if we assume the weaker condition that

$$(*)\quad -d + n + 1 + s > -1,$$

then the same argument, using double transitivity of $H$ shows that $\ell(y)$ meets $X_F(y)$ in at most 2 points set-theoretically, hence is either contained in $X_F(y)$, or is $d$–fold tangent to it at $x(y)$, or has contact of order $r$ with it at $x(y)$ and of order $d - r$ at another point $x'$, for some $1 \leq r \leq d - 1$.

Note that, as above, the inequality $(*)$ may be assumed to hold if

$$h^0(Y_F(-1)) = 0.$$
Lemma 6.4. For general \( y \in Y \), the line \( \ell(y) \) is contained in the hypersurface \( X_{F(y)} \).

proof. Denote by

\[ \pi : Y \to I \]

the natural map of \( Y \) to the incidence variety of pairs (point on line), which is clearly generically submersive (e.g. by \( GL(n+1) \)-equivariance) and let \( T_\pi \) denote the vertical tangent space for this map. Note that

\[ F \in T_\pi \]

(at \( (x, l, F) \)): indeed this is immediate from the fact that our universal hypersurface \( X/S \) together, we may assume, with its subvariety \( Y/S \) come from analogous families defined over the projectivization \( \mathbb{P}(S) \) so we may assume the map \( \pi \) descends accordingly; on the other hand \( F \) as tangent vector dies in the map \( S \to \mathbb{P}(S) \).

Now let

\[ M \]

be the vector bundle on \( Y \) whose fibre at

\[ y \mapsto (x, \ell, F) \]

is

\[ \mathcal{L}(-\ell), \]

the polynomials of degree \( d \) (on \( \mathbb{P}^n \)) vanishing on \( \ell \), considered as a subbundle of the tangent bundle to \( S \) (pulled back to \( Y \)). Suppose first that \( T_\pi \cap M \) has generic corank \( \leq c - 1 \) in \( M \), where

\[ c := 2n - 1 - d - k. \]

Then the rank of \( T_\pi \cap M \) is at least

\[ N - (d + c) = N + k - (2n - 1). \]

But owing to the generic submersivity of \( \pi \) the latter is precisely the rank of \( T_\pi \). Hence

\[ T_\pi \subseteq M, \]

and therefore \( F \in M \), i.e. \( F \) vanishes on \( \ell \) as claimed.

Now suppose that \( T_\pi \cap M \) has (generic) corank \( c \) in \( M \), which clearly implies that the natural map

\[ M \to N \]
of $M$ to the normal bundle of $Y$ in $\Delta_d$ is generically surjective, hence so is

$$\bigwedge^c M \to \bigwedge^c N = \det N.$$  

Now assume that the line $\ell$ has $d$--fold contact if $X_F$ at $x_0$. Since we have $d > n, d(d-1)/2 - n \geq c$, the results of §2 and §5,(5.5) imply that $\omega_{\Delta_d} \otimes \bigwedge^c M$ is globally generated and it follows that

$$\omega_{\Delta_d} \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \otimes \det N \subseteq \omega_Y(2 - d)$$

has a nonzero section, hence so does $\omega_Y(2 - d)$ for generic $F$, which is a contradiction.

\[\square\]

7. The case $s' = 0$

Throughout this section we continue to assume the hypotheses of the previous section, and assume additionally that

$$s' = 0.$$  

As has been remarked before, the latter assumption holds automatically whenever

$$s > 1, k = n - 3.$$  

Our aim is to show, under these assumptions, that a general $Y_F$ is ruled by lines. This will complete the proof of the Main Theorem.

Note that the condition $s' = 0$ is equivalent to

$$T' = \mathcal{L}_d(-\ell),$$

i.e. to the assumption that

$$(7.1) \quad \mathcal{L}_d(-\ell) \subseteq T_{Y/\mathbb{P}^n}.$$  

**Lemma 7.1.** $\ell(y)$ is tangent to $Y_F$ for general $y \in Y_F$.

**proof.** Using notations as in the previous section, let

$$v \in gl(n + 1)$$

be the vector field

$$v = X_0 \partial / \partial X_1,$$
which at $y_0$ is nonzero and points in the direction of $\ell_0$, and set

$$G = v(F).$$

Considering $G$ as an element of $T_F S$, note that by $GL(n+1)$-equivariance, the normal field

$$\bar{v}$$

to $Y = Y_F$ corresponding to $G$ is just the natural image of $v$. On the other hand since

$$G|_{\ell} = 0,$$

our assumption $s' = 0$, in the form (7.1), yields that $(G, 0)$ is tangent to $\mathcal{Y}$, hence

$$\bar{v}(y_0) = 0.$$

Thus $v$ is tangent to $Y_F y_0$, hence so is $\ell_0$, as claimed. □

**Lemma 7.2.** $Y_F$ is ruled by lines.

**proof.** In the above notations, note that since the action of $v$ on the Grassmannian fixes $\ell_0$, it follows by equivariance that on $\mathcal{Y}$,

$$\ell = \ell(y)$$

is fixed to first order as $y$ moves out of $y_0$ in the direction

$$v(y_0) \in T_{y_0} \mathcal{Y}.$$ 

Viewing the latter as a subspace of $T_F S \times T_{y_0} Y_F$, write

$$v(y_0) = (G, v_Y),$$

where

$$G = v(F) \in L_d, v_Y \in T_{y_0} Y_F.$$ 

As we have seen, $G$ is in fact in $L_d(-\ell)$ which under our assumption coincides with $T'(y_0)$ (more precisely, $(G, 0) \in T'(y_0)$). Therefore by Lemma 6.1, $\ell$ is constant to 1st order in the direction $(G, 0)$. Therefore it is also constant to 1st order in the direction $(0, v_Y)$. Now let

$$u$$

be a vector field locally on $Y_F$ whose direction at any $y$ is the same as $\ell(y)$. Then along any integral arc $U$ of $u$, $\ell$ is infinitesimally constant, hence constant, hence $U$ itself in on a line. Thus $Y_F$ is ruled by lines. □

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