VISCOSITY PRESCRIPTION FOR GRAVITATIONALLY UNSTABLE ACCRETION DISKS

ROMAN R. RAFIKOV
Department of Astrophysical Sciences, Princeton University, Ivy Lane, Princeton, NJ 08540, USA; rr@astro.princeton.edu

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ABSTRACT

Gravitationally unstable accretion disks emerge in a variety of astrophysical contexts—giant planet formation, FU Orioni outbursts, feeding of active galactic nuclei, and the origin of Pop III stars. When a gravitationally unstable disk is unable to cool rapidly, it settles into a quasi-stationary, fluctuating gravitoturbulent state, in which its Toomre $Q$ remains close to a constant value $Q_0 \sim 1$. Here we develop an analytical formalism describing the evolution of such a disk, which is based on the assumptions of $Q = Q_0$ and local thermal equilibrium. Our approach works in the presence of additional sources of angular momentum transport (e.g., MRI), as well as external irradiation. Thermal balance dictates a unique value of the gravitoturbulent stress $\alpha_{gt}$ driving disk evolution, which is a function of the local surface density and angular frequency. We compare this approach with other commonly used gravitoturbulent viscosity prescriptions, which specify the explicit dependence of stress $\alpha_{gt}$ on Toomre $Q$ in an ad hoc fashion, and identify the ones that provide consistent results. We nevertheless argue that our $Q = Q_0$ approach is more flexible, robust, and straightforward and should be given preference in applications. We illustrate this with a couple of analytical calculations—locations of the snow line and of the outer edge of the dead zone in a gravitoturbulent protoplanetary disk—which clearly show the simplicity and versatility of the $Q = Q_0$ approach.

Key words: accretion, accretion disks – instabilities – protoplanetary disks – quasars: general

1. INTRODUCTION

Astrophysical accretion disks can be prone to gravitational instability (hereafter GI) when their temperature is low and surface density is high (Safronov 1960; Toomre 1964). In particular, GI is conceivable in the outer parts of protoplanetary disks (Cameron 1978; Boss 1998) at the early stages, when the disks are still massive because of ongoing infall. Accretion outbursts of FU Orioni stars can be driven by the gravito-magnetic cycle in the dense, gravitationally unstable parts of the protoplanetary disk (Audard et al. 2014). Outer parts of quasar disks accreting at high $M$ are also expected to become gravitationally unstable far from the black hole (Paczynski 1978a, 1978b; Kozlowski et al. 1979; Kolykalov & Syunyaev 1980; Goodman 2003). Turk et al. (2009), Clark et al. (2011), and Greif et al. (2012) find gravitationally unstable disks around young Population III stars in their simulations of star formation at redshifts of $z \sim 20–30$.

Possibility of the GI in a gaseous disk is characterized by the so-called Toomre $Q$ parameter, defined as (Toomre 1964)

$$Q \equiv \frac{\Omega c_s}{\pi G \Sigma},$$

where $\Sigma$ and $c_s = (k_B T/\mu)^{1/2}$ are the surface density and sound speed of the disk, respectively, and $\Omega = (GM/r^3)^{1/2}$ is the angular frequency (assuming Keplerian rotation around a central object with mass $M$). In the linear regime GI sets in as $Q \rightarrow Q_0$ from above, where $Q_0 \approx 1–1.5$ is the threshold value suggested by numerical experiments (Boss 2002; Cossins et al. 2009, 2010).

Operation of the GI is accompanied by enhanced angular momentum transport in the disk driven by the non-axisymmetric gravitational torques. As a result, GI results in mass redistribution in the disk, driving its evolution. Energy dissipation caused by the application of gravitational stresses heats up the disk and tends to oppose the GI.

Because of this feedback, the nonlinear outcome of the GI sensitively depends on another dimensionless parameter—the product of the local cooling time in the disk $t_{cool}$ and $\Omega$. Gammie (2001) demonstrated that when cooling is fast and $\Omega_{cool} \lesssim \Omega_{crit} \sim 1$, the disk disintegrates into a number of bound self-gravitating structures. Such fragmentation has been invoked by Boss (1998) to explain the origin of giant planets (see Rafikov 2005, 2007 for constraints on this scenario). Goodman & Tan (2004) and Levin (2007) suggested that fragmentation of gravitationally unstable quasar disks should result in formation of massive stars migrating through the disk. Clark et al. (2011), Greif et al. (2012), and Latif & Schleicher (2014a) propose that fragmentation of protostellar disks around Pop III stars can produce low-mass extremely metal-poor stars that can survive until present days.

In the opposite limit of long cooling time $\Omega_{cool} \gtrsim \Omega_{crit}$ it was shown by Gammie (2001) that the disk settles into a quasi-stationary state of gravitoturbulence. In this regime surface density fluctuates in time, but the disk state averaged over the period longer than the dynamical timescale $\Omega^{-1}$ remains the same. The time-averaged value of the Toomre $Q$ parameter in a gravitoturbulent disk is around $Q_0$, i.e., the disk maintains itself in a marginally stable state. This general picture has been confirmed with global simulations by different groups (Rice et al. 2003, 2005; Durisen et al. 2007; Cossins et al. 2009, 2010).

The critical value of the cooling time $\beta_{crit} \Omega^{-1}$, corresponding to the transition between the gravitoturbulent and fragmenting regimes, depends on a variety of factors. One of the key determinants is the equation of state (EOS) of the gas, with softer EOS promoting fragmentation and resulting in higher $\beta_{crit}$ (Rice et al. 2005; Jiang & Goodman 2011). Opacity transitions, e.g., due to dust grain evaporation (Johnson & Gammie 2003), as well as other forms of the temperature dependence of opacity (Cossins et al. 2010), also affect critical $t_{cool}$. External irradiation
(Rice et al. 2011) and the details of the disk structure (Meru & Bate 2011a) may also affect the value of $\beta_{\text{crit}}$.

Some concerns have been raised regarding the convergence of gravitoturbulent disk simulations and the existence of the well-defined $\beta_{\text{crit}}$, as its value has been claimed (Meru & Bate 2011b; Paardekooper 2012) to vary with the grid resolution. However, later this non-convergence has been traced mainly to the numerical effects (Paardekooper et al. 2011; Meru & Bate 2012; Rice et al. 2014).

It has also been debated whether fragmentation ensues because rapid cooling facilitates collapse of unstable fragments, or because the disk can withstand only a certain amount of stress and fragments when the dimensionless stress parameter $\alpha$ (Shakura & Sunyaev 1973) exceeds a threshold value $\alpha_{\text{crit}}$. Since in thermal equilibrium in the absence of external energy inputs (Gammie 2001)

$$\alpha = \frac{4}{9\gamma(\gamma-1)}(\Omega_{\text{cool}})^{-1},$$

where $\gamma$ is the adiabatic index of gas, the values of $\alpha_{\text{crit}}$ and $\beta_{\text{crit}}$ are directly connected. Comparing simulations with different $\gamma$ (different EOS), Rice et al. (2005) noted that $\alpha_{\text{crit}}$ is essentially independent of $\gamma$ and is about 0.06. This led them to suggest that the primary reason for fragmentation is the maximum stress that can be sustained by the disk. However, simulations with external irradiation (Rice et al. 2011) suggest that $\alpha_{\text{crit}}$ does depend on the level of irradiation and is thus non-universal.

The main goal of this work is to explore disk characteristics in the gravitoturbulent state, which can persist for a long time over a significant range of radii. For that reason it is important to understand how the disk evolves in this regime under the action of the non-axisymmetric gravitational torques. Currently direct 2D and 3D simulations are too numerically expensive to permit such exploration, and one often has to resort to azimuthally averaged, one-dimensional (in radius) disk models. To evolve them properly, one must provide a description of the angular momentum transport by the gravitoturbulence. Formulating such a description is the focus of the present work.

Balbus & Papaloizou (1999) argued that due to the long-range nature of the gravitational interaction the angular momentum transport due to the non-axisymmetric gravitational perturbations is inherently non-local. However, Gammie (2001) showed that in cold, geometrically thin disks angular momentum transport by the gravitational torques can still be described as a local process. Lodato & Rice (2004, 2005) and Cossins et al. (2009) confirmed this conclusion numerically, certainly for the low-mass disks.

In this work we adopt the latter point of view and will explicitly assume the gravitoturbulent transport to be local and characterized by the effective viscosity parameter $\alpha_{\text{gt}}$. Different explicit and implicit prescriptions for $\alpha_{\text{gt}}$ have been proposed, which can be classified into two general categories.

The first class of $\alpha_{\text{gt}}$ prescriptions relies on the fact that in local dynamical and thermal equilibrium the angular momentum transport in the disk is intimately related to its thermal state. This allows one to directly express $\alpha_{\text{gt}}$ via the disk temperature and density. Using constant $M$ disk without external energy inputs as an example, Rafikov (2009) has demonstrated that this property, when coupled with the defining characteristic of the gravitoturbulence—the condition $Q \approx Q_0$—allows one to directly relate $\alpha_{\text{gt}}$ to $\Sigma$, providing a complete description of the disk evolution. This approach has also been implicitly featured in several numerical studies (Terquem 2008; Clarke 2009; Zhu et al. 2009a).

A different way of describing the gravitoturbulent disk evolution does not explicitly assume $Q \approx Q_0$. Instead, it specifies the explicit dependence of $\alpha_{\text{gt}}$ on $Q$ (Kratter et al. 2008; Zhu et al. 2009b, 2010a, 2010b; Martin & Lubow 2011, 2013, 2014). This approach dates back to the work of Lin & Pringle (1987), who suggested that $\alpha_{\text{gt}} \approx Q^{-2}$ based on a phenomenological description of transport in gravitationally unstable disks. However, in most cases such prescriptions do not naturally follow from physical arguments. Instead, they are designed to replicate certain qualitative features of $\alpha_{\text{gt}}$ behavior.

The goal of this work is to formulate, analyze, and compare different approaches to characterizing $\alpha_{\text{gt}}$. First, in Section 2 we discuss equations used to describe evolution of gravitationally unstable disks. Then, in Section 3 we highlight the differences between the two aforementioned approaches to closing the system of evolution equations. After considering the steady, constant $M$ models in Section 4, we contrast the performance of different gravitoturbulent viscosity prescriptions in Section 5. We discuss our results in Section 6, where we show, in particular, how the gravitoturbulent prescription based on the $Q \approx Q_0$ condition can be naturally used to obtain simple analytical expressions for the locations of the ice lines and dead zone edges in the gravitoturbulent protoplanetary disks.

2. BASIC EQUATIONS

Provided that gravitoturbulent stress can be described as local $\alpha$-viscosity, it is convenient (Rafikov 2013) to characterize the disk structure via the angular momentum flux $F_j$, defined (in a Keplerian disk) as

$$F_j \equiv 3 \pi \nu \Sigma l = 3 \pi \alpha c_s^2 \Sigma r^2,$$

where $l = \Omega r^2$ is the specific angular momentum and $\nu$ is the kinematic viscosity expressed through the dimensionless parameter $\alpha$ as (Shakura & Sunyaev 1973)

$$\nu = \frac{\alpha c_s^2}{\Omega}.$$

As always, we will assume the gas sound speed $c_s$ to be determined by the midplane temperature of the disk. Angular momentum transport is effected both by gravitoturbulence and by any other background sources of effective viscosity, such as MRI.

Mass accretion rate through the disk $\dot{M}$ (defined to be positive for inflow) is related to $F_j$ via a simple relation (Rafikov 2013)

$$\dot{M} = \frac{\partial F_j}{\partial l}.$$
Coupled with the continuity equation, this results in the following evolution equation:
\[
\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r} \frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r} r^2 \frac{\partial F_l}{\partial r} \right].
\] (6)

Expressing \( F_l \) via Equation (3), one reproduces the classical equation for the viscous disk evolution (Lightman & Eardley 1974; Lin & Papaloizou 1996)
\[
\frac{\partial \Sigma}{\partial t} = 3 \frac{\partial}{\partial r} \left[ r^{1/2} \frac{\partial}{\partial r} \left( r^{1/2} \Sigma \right) \right].
\] (7)

Solving either of these equations requires the knowledge of the thermodynamical properties of the disk since both \( F_l \) and \( \nu \) are functions of gas temperature \( T \); see Equations (3) and (4).

We now address the thermal state of the disk. We assume the disk to be in thermal equilibrium, so that energy sources and sinks are in local balance. The former include viscous dissipation and external irradiation intercepted by the disk. The latter is effected by radiative cooling. For local angular momentum transport the rate of viscous energy dissipation per unit radial distance in the disk is \( dE/dr = -F_l d\Omega/dr \). Then the rate of energy production by viscous stresses per unit area of the disk is
\[
\dot{Q} = \frac{1}{2\pi} \frac{dE}{dr} = 3 \frac{F_l \Omega}{4\pi r^2}.
\] (8)

Gravitationally unstable disks are often studied in the limit of a self-luminous disk (Gammie 2001; Rafikov 2009), in which viscous heating of any nature dominates over the energy input due to external irradiation. This assumption should be reasonable for, e.g., Class 0 T Tauri stars, which are characterized by intense mass accretion and relatively low luminosity of a very young central star, which is also heavily obscured.

However, in general, the disk, especially its outer parts, may also be heated appreciably by the external radiation field with temperature \( T_{hr} \) (Rice et al. 2011; Zhu et al. 2012); \( T_{hr} \) can be a function of \( r \) if irradiation is due to the central object. To account for this possibility, we describe the local thermal balance, which ultimately determines the midplane disk temperature \( T \), via the following approximate relation:
\[
\dot{Q} = 3 \frac{F_l \Omega}{4\pi r^2} = \frac{2\sigma}{f(\tau)} \left( T^4 - T_{hr}^4 \right).
\] (9)

Details of the vertical transport of radiation in the disk are specified here via an explicit function \( f(\tau) \) of the optical depth
\[
\tau \approx \kappa(T) \Sigma.
\] (10)

To zeroth order the disk optical depth is determined predominantly by the midplane value of the temperature \( T \) (\( \kappa \) can also be a function of gas density).

The explicit form of \( f(\tau) \) depends on whether the disk is optically thick or thin and on the mechanism of the vertical energy transport (Rafikov 2007). Here we assume radiative energy transfer. Rafikov (2007) has derived \( f(\tau) \) for convective transport of energy with dust-dominated opacity. In this case \( f(\tau) \) can be reasonably well approximated (up to constant factors of order unity) by
\[
f(\tau) \approx \tau + \tau^{-1}.
\] (11)

This expression does not discriminate between the Rosseland and Planck mean opacities and is approximately valid (up to constant factors of order unity) in both the optically thick and thin regimes (Goodman 2003). It is easy to see that the condition (9) supplemented with Equation (11) properly describes the disk thermal balance in different asymptotic regimes of \( \tau \). Indeed, in the optically thick case (\( \tau \gg 1 \)) one finds \( \sigma T^4 \approx \sigma T_{hr}^4 + (\dot{Q}/2)\tau \), while in the optically thin regime (\( \tau \ll 1 \)) one has \( \sigma T^4 \approx \sigma T_{hr}^4 + (\dot{Q}/2)\tau^{-1} \). Both expressions are valid up to constant factors in the appropriate limits.

For simplicity in this work we will consider the following behavior of \( \kappa \) appropriate for dust opacity in certain temperature regimes (Bell & Lin 1994):
\[
\kappa = \kappa_0 T^\beta,
\] (12)

with \( \kappa_0 \) and \( \beta \) being constants. For example, at low temperatures, below 170 K the opacity is dominated by ice grains and is characterized by (Bell & Lin 1994)
\[
\beta = 2, \ \kappa_0 = 5 \times 10^{-4} \text{ cm}^2 \text{ g}^{-1} \text{ K}^{-2}.
\] (13)

Our results can be trivially extended to other, more complicated forms of \( \kappa \).

Equations (9)–(13) provide the sought relation between the disk temperature and surface density, which is used to determine the viscosity behavior in Equation (7).

Our final note on disk thermodynamics concerns the case when cooling is not due to dust emission. In particular, Latif & Schleicher (2014a, 2014b) considered the structure of gravitationally unstable disks around Pop III stars. Such disks are believed to be metal-free (i.e., no dust opacity), with cooling provided by molecular hydrogen. In this case, the right-hand side of the energy balance Equation (9) should be modified to \( 2(c_V/\Omega)\Lambda \), where \( c_V/\Omega \) is the disk scale height and \( \Lambda \) is the volumetric cooling rate. This expression is valid in the optically thin regime, with certain modifications required when line cooling switches to the optically thick regime (Ripamonti & Abel 2004; Latif & Schleicher 2014a).

3. CLOSURE IN THE GRAVITOTURBULENT STATE

Definition (3) and thermal balance condition (9) supplemented with Equations (10) and (11) allow one to uniquely relate \( T \) to \( \Sigma \) if the behavior of \( \alpha \) is known. This situation is typical for the disk regions dominated by the background viscosity (e.g., due to MRI), where one usually knows (or, more often, postulates) some behavior of \( \alpha \). Knowing the \( T(\Sigma) \) dependence, one can express \( F_l \) and \( \nu \) as a function of \( \Sigma \) only, leaving \( \Sigma \) as the only dependent variable in Equation (7).

This provides a closure of the system of equations for the density, angular momentum, and thermal balance and allows one to self-consistently evolve \( \Sigma \) in time and space.

In gravitoturbulent state the behavior of \( \alpha \) is not known a priori and this approach does not apply. To close the system of evolution equations, one needs to make additional assumptions. Two conceptually different approaches to this problem are described next.

3.1. Closure via the \( Q = Q_0 \) Condition

Rafikov (2009) pointed out that a necessary closure is naturally provided by the basic property of a gravitoturbulent
disk following directly from simulations— that the disk hovers on the margin of gravitational stability with
\[ Q = Q_0, \]  
(14)

where \( Q_0 \sim 1 \).

Indeed, with definition (1) this condition predicts an explicit relation between \( T \) and \( \Sigma \) in the form
\[ T_Q(r, \Sigma) = \frac{\mu}{k_B} \left( \frac{\pi G Q_0}{\Omega(r)} \right)^2 \Sigma^2. \]  
(15)

Plugging this expression into the thermal balance condition (9) gives the following expression for the gravitoturbulent stress:
\[ F_{\text{J,gt}}(r, \Sigma) = \frac{8\pi}{3} r^2 \frac{\sigma(T_Q^4 - T_{\text{irr}}^4)}{\Sigma \Omega f(\tau)}. \]  
(16)

The dependence on surface density \( \Sigma \) arises here because \( T_Q \), which enters this expression both explicitly and also through
\[ \tau(r, \Sigma) = \Sigma \psi(T_Q(r, \Sigma)), \]  
(17)
is a function of \( \Sigma \).

Combining Equations (3)–(16), one also comes up with the explicit expression for the effective \( \alpha \)-parameter due to gravitoturbulence:
\[ \alpha_{\text{gt}}^R(r, \Sigma) = \frac{8}{9} \frac{\sigma(\pi G Q_0)^6}{f(\tau)} \left( \frac{\mu}{k_B} \right)^4 \Sigma^5 \frac{1 - T_{\text{irr}}^4}{T_Q^4}. \]  
(18)

The dependence on \( r \) comes only through \( \Omega(r) \). This result was first obtained by Rafikov (2009) in the non-irradiated case (\( T_{\text{irr}} = 0 \)) when studying constant \( M \) gravitoturbulent disks. However, it clearly also holds more generally, for evolving and irradiated gravitoturbulent disks, because thermal balance gets established faster than the viscous evolution proceeds. Note that Rice et al. (2011) found a different (linear) scaling with \( T_{\text{irr}} \) in their expression for the gravitoturbulent \( \alpha \) because they used a different cooling model, namely, assuming a constant cooling time. Expression (18) for \( \alpha_{\text{gt}}^R \) completes the closure and makes evolution Equation (7) self-consistent.

On the other hand, gravitoturbulent disk evolution can be described entirely without introducing the \( \alpha \)-parameter. Indeed, substituting \( F_J = F_{\text{J,gt}}(r, \Sigma) \) given by expression (16) into Equation (6), one finds
\[ \frac{\partial \Sigma}{\partial t} = \frac{8}{3 r} \frac{\partial}{\partial r} \left[ \frac{1}{\Omega} \frac{\partial}{\partial r} \left( r^2 \frac{\sigma(T_Q^4 - T_{\text{irr}}^4)}{\Omega f(\tau(T_Q))} \right) \right], \]  
(19)

where, again, \( T_Q = T_Q(r, \Sigma) \), and \( T_{\text{irr}}(r) \) behavior is specified. This (in general nonlinear) equation is a closed-form, fully self-contained evolution equation for \( \Sigma \) as a function of \( t \) and \( r \). It represents one of the main results of this work.

In the parts of the disk where external irradiation plays the dominant role this equation adopts a simple time-independent solution \( T_Q \rightarrow T_{\text{irr}} \), so that \( \Sigma(r) = \Omega(r) c_1(T_{\text{irr}}(r))/\rho G Q_0 \) (Rafikov 2009).

### 3.1.1. Additional Sources of Viscosity

Angular momentum transport in the disk may be affected not only by gravitoturbulence but also by additional stresses. This would be the case, for example, if the disk is both sufficiently ionized for the MRI to operate and massive enough for being gravitationally unstable.

To account for the possibility of additional, non-gravitoturbulent viscosity parametrized by \( \alpha \)-parameter \( \alpha_m \), one can simply write
\[ F_J = \psi F_{\text{J,gt}}(r, \Sigma) + F_{\text{J,m}}(r, \Sigma, T). \]  
(20)

Here \( F_{\text{J,gt}}(r, \Sigma) \) is still given by expression (16) with the switch function \( \psi \) introduced as described below. The non-gravitational viscous angular momentum flux is
\[ F_{\text{J,m}}(r, \Sigma, T) = 3\pi \alpha_m c_s^2(T) \Sigma r^2, \]  
(21)

and the behavior of \( \alpha_m \) is specified. Thermal balance relation (9) now gives
\[ \psi F_{\text{J,gt}}(r, \Sigma) + F_{\text{J,m}}(r, \Sigma, T) = \frac{8\pi}{3} r^2 \frac{\sigma(T_Q^4 - T_{\text{irr}}^4)}{\Sigma \Omega f(\tau(\Sigma, T))}, \]  
(22)

from which we determine \( T \) as a function of \( r \) and \( \Sigma \). Then Equation (20) yields \( F_J \) as a function of \( r \) and \( \Sigma \), allowing the evolution of \( \Sigma \) to be followed using Equation (6).

A subtle point in this prescription is that Equation (20) always includes the gravitoturbulent transport in the form (16), with \( F_{\text{J,gt}} \) independent of the actual disk temperature \( T \) and the value of \( Q \). As a result, our prescription formally has non-zero gravitoturbulent stress \( F_{\text{J,gt}} \) even in the gravitationally stable part of the disk, which is not expected.

For this reason we introduce a switch function \( \psi \) in Equation (20), which takes care of this issue. There are several different ways in which this can be done. One method would be to adopt \( \psi \) that quickly goes to zero as soon as \( Q \gtrsim Q_0 \). However, in the absence of a microscopic theory of gravitoturbulence such a factor would necessarily be introduced in an ad hoc fashion.

Instead, we have chosen to use
\[ \psi = \theta(f_{\text{J,gt}}), \]  
(23)

where \( \theta(z) \) is the Heaviside step function \( \theta(z) = 1 \) for \( z \gtrsim 0 \); \( \theta(z) = 0 \) for \( z < 0 \). With this approach we simply keep \( F_{\text{J,gt}} \) in the form (16) in Equation (20), as long as it is positive. This is not a problem, since, as demonstrated in Rafikov (2009), as soon as \( Q \) exceeds the threshold value \( Q_0 \), the transport is guaranteed to be dominated by the viscous stress, i.e., \( F_{\text{J,m}} \gg F_{\text{J,gt}} \). In other words, \( \alpha_{\text{gt}}^R \ll \alpha_m \) in gravitationally stable disks, even if \( T_{\text{irr}} = 0 \). And vice versa, \( F_{\text{J,m}} \ll F_{\text{J,gt}} \) and \( \alpha_{\text{gt}}^R \gg \alpha_m \) when the disk is gravitationally unstable and \( Q \rightarrow Q_0 \). This situation is further discussed in Section 4 and is illustrated in Figure 5, where we display the actual run of the viscous and gravitoturbulent transport contributions for a particular disk model.
Under certain circumstances one may find that $T_Q < T_{\text{int}}$ in the gravitationally stable part of the disk, so that $F_{\text{gt}, Q} < 0$ according to Equation (16). In this case our choice (23) of $\psi$ simply guarantees that the gravitoturbulent part of the angular momentum flux vanishes and $\alpha = \alpha_m$ exactly.

Evolution Equation (6) based on the prescription (20) with $\psi$ given by Equation (23) allows us to explore the transition between the disk regions dominated by gravitoturbulence and background viscosity. In the appropriate limits its solution reduces to either the gravitoturbulent disk solution (for large $r$) following from Equation (19) or the conventional viscous disk solution (for small $r$, where $F_{\text{gt}, Q} \ll F_{\text{m}, Q}$). Alternatively, one can evolve the disk structure using Equation (7) with $\nu$ determined by

$$\alpha = \psi \alpha_{\text{gt}}^R(r, \Sigma) + \alpha_m. \quad (24)$$

It is easy to show that this approach results in almost the same disk properties, except for the region where $\alpha_{\text{gt}}^R \sim \alpha_m$ and the disk is close to marginal gravitational stability.

In summary, the prescription (20) with $F_{\text{gt}, Q}$ given by (16) should be applicable for any value of $Q$ (even though only approximately in the parts of the disk just transitioning to the marginally gravitationally unstable regime).

### 3.2. Closure via the Explicit $\alpha_{\text{gt}}(Q)$ Dependence

An alternative closure scheme that has been broadly used in the literature postulates some dependence of the gravitoturbulent viscosity $\alpha_{\text{gt}}$ on $Q$. In this case closure is analogous to a regular viscous disk anzatz: Equations (1) and (3) with $\alpha = \alpha_m + \alpha_{\text{gt}}(Q)$ and (9)–(11) are combined to yield a unique $T(r, \Sigma)$ relation. It is then plugged into the relation (4) for viscosity, resulting in the expression for $\nu(r, \Sigma)$ and allowing Equation (7) to be evolved in time.

Note that the condition (14) is not used explicitly in this approach and thus in general there is no guarantee that this prescription would result in a truly gravitoturbulent disk structure, with the disk hovering at the edge of instability with $Q \approx Q_0$.

In the absence of a microscopic model of gravitoturbulence that could motivate a possible dependence of $\alpha_{\text{gt}}$ on $Q$, several ad hoc prescriptions for $\alpha_{\text{gt}}(Q)$ have been suggested. Their main feature is the rapid increase of $\alpha_{\text{gt}}$ as $Q$ approaches some critical value $\sim 1$ from above.

In particular, Zhu et al. (2010a, 2010b) used

$$\alpha_{\text{gt}}^Z(Q) = e^{-Q^4}. \quad (25)$$

Martin & Lubow (2011, 2013, 2014) adopted

$$\alpha_{\text{gt}}^M(Q) = \max \left\{ \alpha_m \left( \frac{Q_{\text{crit}}}{Q} \right)^2 - 1, 0 \right\}, \quad (26)$$

with $\alpha_m = 10^{-2}$ and $Q_{\text{crit}} = 2$. The $Q^{-2}$ dependence is motivated by the work of Lin & Pringle (1987). Note that this prescription explicitly relates the value of gravitoturbulent viscosity to the background viscosity $\alpha_m$. This recipe was also used in Owen & Jacquet (2014) to explore the chemical evolution of a protoplanetary disk undergoing accretion outbursts.

Kratter et al. (2008) have used

$$\alpha_{\text{gt}}^K(Q) = \max \left\{ 0.14 \left[ \frac{1.3}{\max(Q, 1)} \right]^2 - 1, 0 \right\}. \quad (27)$$

One can see that $\alpha_{\text{gt}}^M$ reduces to $\alpha_{\text{gt}}^K$ if one uses 0.14 and 1.3 instead of $\alpha_m$ and $Q_{\text{crit}}$. Also, $\alpha_{\text{gt}}^K$ saturates at a constant value $\approx 0.1$ for $Q \leq 1$, unlike $\alpha_{\text{gt}}^M$, which can be arbitrarily large for small $Q$.

These $\alpha_{\text{gt}}(Q)$ prescriptions are illustrated in Figure 1. We summarize and analyze their properties in Section 5.

### 4. CONSTANT $M$ DISKS

One is often interested in knowing the structure of a steady-state accretion disk, which necessarily has $M$ independent of $r$. This limiting case provides a nice point of comparison between the different types of viscosity prescriptions and will be used in Section 5.

If $M$ is independent of $r$ (and, correspondingly, $l = \Omega r^2$) and there are no external torques acting on the disk, then Equation (5) naturally yields $F_l = \dot{M}$ (Rafikov 2013). We now illustrate the difference in approaches between the various closure philosophies when calculating the constant $M$ disk structure.

#### 4.1. Constant $M$ Disk: $Q = Q_0$ Closure

When we explicitly demand a constant $M$ disk to be marginally unstable with condition (14), Equation (20) implies

![Figure 1. Illustration of different explicit $\alpha_{\text{gt}}(Q)$ prescriptions given by Equations (25)–(27). Curves show the runs of $\alpha_{\text{gt}}^M(Q)$ of Zhu et al. (2010a, 2010b; green, short-dashed), $\alpha_{\text{gt}}^K(Q)$ of Kratter et al. (2008; red, dotted), Equation (27), and $\alpha_{\text{gt}}^M(Q)$ of Martin & Lubow (2011; long-dashed), Equation (26). The latter is displayed for two values of the background viscosity: $\alpha_m = 10^{-4}$ (black) and $\alpha_m = 10^{-2}$ (blue).](image-url)
a relation between $\Sigma$, $T$, $r$ in the form

$$\psi F_{j,gt}(r, \Sigma) + F_{j,m}(r, \Sigma, T) = M\Omega(r).$$

(28)

The second algebraic relation between these variables is provided by Equations (9)–(12), thus fully specifying the behavior of $\Sigma(r)$ and $T(r)$. Rafikov (2009) and Clarke (2009) used this approach to understand the properties of gravitoturbulent disks with constant $M$.

### 4.2. Constant $M$ Disk: Explicit $\alpha_{gt}(Q)$ Closure

When one uses an explicit $\alpha_{gt}(Q)$ prescription, the approach to determining disk structure is different from that in Section 4.1. Assuming a non-zero background viscosity, Equation (3) now yields for the constant $M$ disk

$$3\pi c_s^2 \Sigma [\alpha_{gt}(Q) + \alpha_m] = M\Omega(r).$$

(29)

Here both $c_s$ and $Q$ explicitly depend on temperature. Thus, $\alpha_{gt}(Q)$ is a function of $T$ in this approach, unlike the case considered in Section 3.1.

Again, Equations (9)–(12) provide a second relation between $\Sigma$, $T$, $r$, allowing one to fully determine the constant $M$ disk structure (see, e.g., Zhu et al. 2010a, 2010b; Martin & Lubow 2011).

### 5. COMPARISON OF DIFFERENT $\alpha_{gt}$ PRESCRIPTIONS

We now compare the performance of different prescriptions for $\alpha_{gt}$ described in Sections 3.1 and 3.2. To that effect we construct steady-state (constant $M$) models of gravitoturbulent protoplanetary disks using the results of Section 4. We then compare the behaviors of various disk characteristics obtained with different $\alpha_{gt}$ recipes.

Our calculations adopt the opacity behavior (12), (13) typical for low temperatures $T \lesssim 200$ K (Bell & Lin 1994). For simplicity we will assume this behavior to extend also to higher temperatures; this should not be a problem as our present goal is to explore the differences between the various $\alpha_{gt}$ prescriptions. We take the mass of the central star to be $M_* = M_\odot$ and set the threshold for gravitational stability at $Q_0 = 1.5$. In this and subsequent calculations we assume that fragmentation ensues when $\alpha$ exceeds $\alpha_{crit} = 0.1$ (rather than unity, as, e.g., is assumed in Equations (A1)–(A3)). Numerical studies suggest that a lower value of $\alpha_{crit}$ is more realistic (Rice et al. 2005; Paardekooper et al. 2011).

Rafikov (2009) showed that gravitoturbulent state results in a set of fiducial values of various disk variables: surface density $\Sigma_f$, midplane temperature and sound speed $T_f$ and $c_s,f$, mass accretion rate $\dot{M}_f$, and angular frequency $\Omega_f$. They are chosen such that at the radius corresponding to the angular frequency $\Omega_f$ a steady gravitoturbulent disk with the mass accretion rate equal to $\dot{M}_f$ simultaneously (1) has midplane temperature and sound speed equal to $T_f$ and $c_{s,f}$, (2) fragments, and (3) has optical depth $\tau = 1$. In the Appendix we provide explicit expressions and numerical estimates for these variables for the case of interest to us (opacity $\kappa \propto T^2$). These fiducial quantities provide characteristic values of the important parameters of gravitoturbulent disks and will be used in our subsequent comparisons.

![Figure 2](image_url)

**Figure 2.** Comparison of gravitoturbulent disk models computed using $Q = Q_0$ closure (black solid curve) and several different $\alpha_{gt}(Q)$ prescriptions: $\alpha_{gt}(Q)$ of Zhu et al. (2010a, 2010b; green, short-dashed), $\alpha_{gt}(Q)$ of Martin & Lubow (2011; blue, long-dashed), and $\alpha_{gt}(Q)$ of Kratter et al. (2008; red, dotted). Plotted are: (a) optical depth, (b) Toomre $Q$, (c) $\alpha$-parameter, (d) surface density, (e) midplane temperature. Opacity in the form (12), (13) is assumed. Background viscosity is high, $\alpha_m = 10^{-2}$. A high-$M$ case is considered ($M = 10M_\odot \approx 7 \times 10^{-5} M_\odot$ yr$^{-1}$) with the disk remaining optically thick all the way out to $\approx 150$ AU. Beyond this radius it inevitably fragments as $\alpha_{gt}$ exceeds critical value $\alpha_{crit} = 0.1$.

For a given mass of a central object $M$, characteristic angular frequency $\Omega_f$ determines a fiducial distance $r_f$ according to the formula

$$r_f = \left(\frac{GM_*}{\Omega_f^2}\right)^{1/3} \approx 130\text{ AU}\left(\frac{M_*}{M_\odot}\right)^{1/3},$$

(30)

where we took the numerical value of $\Omega_f$ from Equation (A2) in the Appendix. This is the radius beyond which an optically thick disk with opacity $\kappa \propto T^2$ inevitably fragments (Matzner & Levin 2005; Clarke 2009; Rafikov 2009).

In all our models we assume that the disk is immersed in a uniform radiation field with the radially independent temperature $T_{int} = T_f \approx 12$ K. We have checked that all our conclusions remain the same for other values of $T_{int}$, in particular for the self-luminous disks with $T_{int} = 0$.

In Figure 2 we show the radial profiles of the midplane temperature $T$, surface density $\Sigma$, optical depth $\tau$, $\alpha$-parameter, and Toomre $Q$ for a gravitoturbulent disk that has a high mass accretion rate, $M = 10M_\odot \approx 7 \times 10^{-5} M_\odot$ yr$^{-1}$ (leaving aside the question of how realistic such $M$ is). We also use a relatively high value of the background viscosity, $\alpha_m = 10^{-2}$, which is often adopted in the literature (Zhu et al. 2009a; Martin & Lubow 2011). Different curves correspond to $\alpha_{gt}$ prescriptions given by Equations (18) and (25)–(27), as indicated in the figure.
The shaded region outside of \( \sim 130 \) AU corresponds to a fragmenting part of the disk where \( \alpha > \alpha_{\text{crit}} \), and the behavior of disk parameters in this region is irrelevant (many curves are very discrepant there). As expected (Rafikov 2009), at the fragmentation edge our disk model has \( \Sigma > \Sigma_f, T > T_f \), and is optically thick (\( \tau \sim 10 \)).

One can see that all \( \alpha_{\text{gt}} \) prescriptions reliably reproduce the radius at which the transition from the gravitationally stable to the gravitoturbulent state occurs (around the vertical \( Q = Q_0 \) line). Interior to this radius background viscosity dominates, \( \alpha \approx \alpha_m \), and different curves overlap. But outside this radius, in the region where \( \alpha \gtrsim \alpha_m \), some differences between \( \alpha_{\text{gt}} \) prescriptions start emerging. In particular, at the fragmentation edge (boundary of the shaded region) \( \alpha_{\text{gt}}^R \) computed through \( Q = Q_0 \) closure is different from \( \alpha_{\text{gt}}^M \) of Martin & Lubow (2011) by about a factor of 2. Similar differences at the same \( r \approx 130 \) AU are seen in the values of \( \Sigma \) and \( \tau \). Other explicit \( \alpha_{\text{gt}} \) prescriptions show better agreement (within tens of percent) for \( \Sigma, \alpha, \) and \( \tau \) with the results of \( Q = Q_0 \) closure. On the other hand, the behavior of the midplane temperature (Figure 2(e)) shows good agreement between different \( \alpha_{\text{gt}} \) prescriptions. As expected, \( T \) never drops below about 10 K as it is limited by \( T_{\text{vir}} \) far from the star; however, for this disk model this is true only beyond the fragmentation edge so that in practice irradiation with the adopted \( T_{\text{vir}} \) is irrelevant here.

To investigate these differences further, we recomputed the disk structure for the same high mass accretion rate \( 7 \times 10^{-5} M_\odot \text{yr}^{-1} \) but decreasing the background viscosity by two orders of magnitude, to \( \alpha_m = 10^{-4} \). This lower value of \( \alpha_m \) may be more realistic for the cold protoplanetary disks, where the MRI is weakened by the non-ideal effects such as ambipolar diffusion (Bai & Stone 2011), or may not be operating at all, e.g., in a dead zone (Gammie 1996; Fleming & Stone 2003).

Figure 3 shows the results of this calculation. Its inspection reveals good quantitative agreement between the models computed via \( Q = Q_0 \) closure (black solid), which rely on \( \alpha_{\text{gt}}^R \) given by Equation (18), and those using explicit viscosity in the form proposed by Kratter et al. (2008), \( \alpha_{\text{gt}}^K(Q) \), and Zhu et al. (2010a, 2010b), \( \alpha_{\text{gt}}^X(Q) \). They typically agree within several tens of percent for all variables in the whole gravitoturbulent region of the disk, between the fragmentation edge and the \( Q = Q_0 \) line.

It is also obvious that the use of \( \alpha_{\text{gt}}^M(Q) \) suggested by Martin & Lubow (2011) and \( \alpha_m \) this low leads to rather poor quantitative agreement with all other prescriptions: \( \Sigma, \alpha, \) and \( \tau \) show a discrepancy of more than an order of magnitude in the outer disk, at the edge of the fragmentation zone. The value of \( Q \) at this radius plunges to \( \sim 0.2 \) for \( \alpha_{\text{gt}}^M(Q) \), in contrast to \( Q \approx Q_0 \) maintained by all other prescriptions in agreement with simulations. The lower value of \( \alpha_m \) used in this calculation pushes the gravitoturbulent zone down to \( \sim 30 \) AU, accentuating the deviations that were already visible at higher \( \alpha_m \), with a less extended gravitoturbulent region (see Figure 2).

This general situation holds for other disk models. For example, in Figure 4 we compare different \( \alpha_{\text{gt}} \) recipes in a low-\( M \) disk with \( M = 0.1 M_\odot \approx 7 \times 10^{-3} M_\odot \text{yr}^{-1} \), while keeping \( \alpha_m = 10^{-4} \). This disk is optically thin in its outer parts, with the \( \tau = 1 \) transition happening \textit{interior} to \( r_f \). In this case, as shown in Rafikov (2009), fragmentation can be pushed out to very large radii and the disk can extend beyond \( 10^7 \) AU in the optically thin, gravitoturbulent regime. Moreover, external irradiation is very important in this case: \( T_{\text{vir}} \approx 12 \) K sets the temperature floor outside 100 AU, in the part of the disk that is stable against fragmentation; see Figure 4(e).

Again, with such low \( \alpha_m \) disk models using \( \alpha_{\text{gt}}^M(Q) \) suggested by Martin & Lubow (2011) are considerably
different from all other models, which tend to agree with each other. For \( \Sigma \) the discrepancy can be as high as an order of magnitude. From this we can conclude that the \( \alpha_{gt}^M(Q) \) prescription underperforms at low values of \( \alpha_m \). We discuss the reasons for this next.

6. DISCUSSION

Our interpretation of the behavior found in models using \( \alpha_{gt}^M(Q) \) is that it is due to the rather gradual dependence of the former on \( Q \). Indeed, Figure 1 shows that when \( \alpha_{gt} \) is between \( \alpha_m \) and the critical value for fragmentation (\( \alpha_{crit} = 0.1 \) in our case) \( \alpha_{gt}^M(Q) \) grows with decreasing \( Q \) much slower than either \( \alpha_{gt}^K(Q) \) or \( \alpha_{gt}^Z(Q) \). This is especially obvious in the \( \alpha_m = 10^{-4} \) case, when \( Q \) has to go down to \( Q \approx 0.1 \) for \( \alpha \) to reach \( \alpha_{crit} \). This explains a significant drop in \( Q \) toward the fragmentation edge in Figures 3(b) and 4(b). Thus, we conclude that \( \alpha_{gt}^M(Q) \) does not properly capture the main feature of the gravitoturbulent disk—almost constant and close to unity value of the Toomre \( Q \). The explicit dependence of \( \alpha_{gt}^M(Q) \) on \( \alpha_m \) may be another feature that causes it to underperform compared to \( \alpha_{gt}^K(Q) \) and \( \alpha_{gt}^Z(Q) \).

At the same time, we must emphasize that we are not aware of the existing gravitoturbulent disk calculations using low \( \alpha_m = 10^{-4} \)—most of them assume \( \alpha_m = 10^{-2} \). Thus, Figures 3 and 4 are not providing comparison with any published results, but merely urging caution in choosing the \( \alpha_{gt} \) recipe when working with low \( \alpha_m \). And at higher \( \alpha_m = 10^{-2} \) calculations using \( \alpha_{gt}^M(Q) \) can still be acceptable (see Figure 2), depending on the required accuracy.

Explicit prescriptions suggested in Kratter et al. (2008) and Zhu et al. (2010a, 2010b) \( \alpha_{gt}^R(Q) \) and \( \alpha_{gt}^Z(Q) \) do rather well at reproducing the \( Q = Q_0 \) calculation using \( \alpha_{gt}^R(Q) \). Based on this, we expect that effectively any explicit \( \alpha_{gt}(Q) \) scheme that exhibits very sharp rise in the vicinity of \( Q_0 \) as \( Q \) decreases should be suitable for practical calculations of the gravitoturbulent disk properties. In fact, the sharper, the better, and something as simple as \( \alpha_{gt}(Q) = \exp[(Q_0 - Q)/\delta Q] \) with \( \delta Q \lesssim 10^{-2} \) will maintain \( Q \approx Q_0 \) in the gravitoturbulent state quite well (conditional upon \( \alpha < \alpha_{crit} \)).

Nevertheless, we strongly believe that there are significant benefits to using the \( Q = Q_0 \) approach outlined in Section 3.1. First, the only assumption that it employs is that the disk is able to maintain itself in a state of marginal gravitational stability \( Q = Q_0 \). This stipulation is in agreement with the results of direct numerical simulations (Cossins et al. 2009, 2010). Explicit \( \alpha_{gt}(Q) \) prescriptions also make this assumption, even though implicitly. But in addition they have to postulate some actual functional form of the \( \alpha_{gt}(Q) \) dependence, which does not have physical justification and is always introduced in an ad hoc fashion. This makes the explicit \( \alpha_{gt}(Q) \) approach rather arbitrary, which may result in certain problems, as we have demonstrated in Section 5.

Second, deep in the gravitoturbulent regime, when one can neglect the background viscosity \( \alpha_m \) compared to the gravitoturbulent contribution \( \alpha_{gt}^R \), the explicit \( Q = Q_0 \) approach results in a self-contained Equation (19) for the surface density evolution. This is allowed by the fact that the expression (18) already implicitly accounts for the thermal balance of the disk, even in the presence of external irradiation. Disk temperature is uniquely defined in this case by relation (15) after the disk surface density has been solved for. This is not the case for the explicit \( \alpha_{gt}(Q) \) recipes, which always need to explicitly relate \( T \) to \( \Sigma \) via the equation of thermal balance (even deep in the gravitoturbulent state, when \( \alpha_{gt} \gg \alpha_m \)), because both enter definition (1) of the Toomre \( Q \). Thus, our \( Q = Q_0 \) prescription allows a more efficient numerical exploration of the gravitoturbulent disk structure.

Third, when the background viscosity cannot be neglected compared to \( \alpha_{gt} \), the explicit \( Q = Q_0 \) prescription (18) does not make the task of computing the disk properties more complicated than the explicit \( \alpha_{gt}(Q) \) prescriptions; see Section 3.1.1. Dealing with this case using our prescription (24), which results in a non-zero \( \alpha_{gt} \) even in the gravitationally stable part of the disk, is not a problem.

Indeed, in Figure 5 we demonstrate that \( \alpha_{gt} \) computed in our disk models via Equation (18) rapidly becomes subdominant as the disk transitions into the regime dominated by the background viscosity \( \alpha_m \). Thus, even though in general \( \alpha_{gt}^R \) is not precisely zero\(^3\) in this part of the disk, it still does not affect the disk properties. The only place where our approach may be quantitatively less accurate is around the transition \( \alpha_{gt} \approx \alpha_m \). However, this is the location where all \( \alpha_{gt} \) prescriptions have some degree of arbitrariness.

\(^3\) Gravitoturbulent contribution does vanish inside of 25 AU in Figure 5 for \( \alpha_m = 10^{-2} \) because of our use of \( \psi \) in the form (23) and the fact that \( T_Q \) drops below \( T_m \) inside this radius.
Fourth, $Q = Q_0$ prescription allows a much more transparent analytical derivation of certain gravitoturbulent disk characteristics. We demonstrate this next when we compare the derivations of the snow line location (Section 6.1) and of the dead zone edge (Section 6.2) in gravitoturbulent disks using the two classes of $\alpha_{\text{gt}}$ prescriptions.

6.1. Snow Line Location

Snow (or ice) lines for different volatile materials ($\text{H}_2\text{O}$, CO, etc.) arise in protoplanetary disks at the locations where these materials undergo a transition from solid to gaseous phase. It is conventional to identify snow lines with the disk radii where the midplane temperature $T$ is equal to some characteristic value $T_{\text{snow}}$ for sublimation of a particular material at the typical midplane pressure. Following common wisdom, we will adopt this as a working definition of a snow line.

Martin & Livio (2013) derived an analytical expression for the snow line radius $R_{\text{snow}}$ in a gravitoturbulent, optically thick, self-luminous protoplanetary disk with constant $M$. Here we compare their calculation with the derivation of $R_{\text{snow}}$ using our favored $\alpha_{\text{gt}}$ recipe (18) based on the condition $Q = Q_0$. To facilitate comparison, we employ a common set of assumptions: (1) neglect background opacity $\alpha_0$ and external irradiation (i.e., $T_{\text{eff}} = 0$), (2) describe optically thick radiation transfer by

$$ f(\tau) = \frac{3}{8} \tau $$

in place of our Equation (11), with $\tau = (1/2)\kappa \Sigma$ (instead of our definition (17)), and (3) take opacity at the icy grain sublimation radius to be given by Equation (12) with $\beta = -0.01$ and $\kappa_0 = 3$ (in appropriate CGS units).

6.1.1. $R_{\text{snow}}$ via the $Q = Q_0$ Prescription

In our approach we first express $\Sigma$ via $R_{\text{snow}}$ and $T_{\text{snow}}$ by setting $T_0 = T_{\text{snow}}$ in Equation (15). Then we plug this $\Sigma(R_{\text{snow}})$ into expression (18) for $\alpha_{\text{gt}}^R$. Finally, inserting all this into Equation (28), written in the form $3\pi \kappa_0^2 \Sigma \alpha_{\text{gt}}^R = M \Omega$ with all variables expressed through $R_{\text{snow}}$ and $T_{\text{snow}}$, and resolving it for $R_{\text{snow}}$, we get

$$ R_{\text{snow}} = \left[ \frac{9}{128 \pi^2} \frac{\kappa_0 \Sigma_0^{3/2}}{\sigma Q_0} \left( \frac{G k_B}{\mu} \right) M_{\text{snow}}^{\beta - 7/2} \right]^{2/9} $$

$$ \approx (5.78 \text{ AU}) Q_0^{-20} \left( \frac{M}{10^{-8} M_\odot \text{ yr}^{-1}} \right)^{2/9} \times \left( \frac{M}{M_\odot} \right)^{1/3} \left( \frac{T_{\text{snow}}}{170 \text{ K}} \right)^{-0.78}. $$

Note that this derivation does not use thermodynamic balance condition (9) as it has already been used in deriving expression (18) for $\alpha_{\text{gt}}^R$.

6.1.2. $R_{\text{snow}}$ via the Explicit $\alpha_{\text{gt}}(Q)$

Now we outline the steps used in Martin & Lubow (2013) to derive $R_{\text{snow}}$. They adopted an explicit $\alpha_{\text{gt}}(Q) = \alpha_0 \exp(-Q^4)$ with $\alpha_0 = 10^{-2}$. This expression is similar to that in Zhu et al. (2009b; see our Equation (25)) in spirit, but not in detail, as the maximum value of $\alpha_{\text{gt}}(Q)$ is case. Note that this ansatz would never allow a gravitationally unstable disk to fragment unless the critical value of $\alpha_{\text{gt}}$ for fragmentation $\alpha_{\text{crit}}$ were very low, below $\alpha_0$.

With this prescription the constant $M$ assumption, embodied in Equation (29), allowed Martin & Livio (2013) to relate $Q$ at the snow line to the disk midplane temperature (assumed equal to $T_{\text{snow}}$) and $R_{\text{snow}}$. Because of the adopted form of $\alpha_{\text{gt}}(Q)$, this relation depended on $\alpha_0$ and involved a non-elementary function (Lambert function), complicating the analysis. The latter problem was overcome by using an asymptotic relation for the Lambert function, valid for relatively low values of $M \lesssim 10^{-7} M_\odot$ yr$^{-1}$.

The additional (algebraic) relation between $T_{\text{snow}}$, $R_{\text{snow}}$, and $\Sigma$ at the snow line comes from the thermodynamical relation (9) with $F_0 = M \Omega r_*^2$. Eliminating $\Sigma(R_{\text{snow}})$ with the aid of these two relations, Martin & Lubow (2013) obtained a scaling for $R_{\text{snow}}$ in terms of powers of $T_{\text{snow}}$, $M$, and the central star mass $M$, given by Equation (19) of their paper.

6.1.3. Comparison of Approaches

Comparing the result for $R_{\text{snow}}$ in Martin & Lubow (2013) with Equation (32) shows that the two are essentially identical, with all the scalings being the same and the numerical estimates different at the 1% level for $Q_0 = 1$. However, it is clear that our derivation of expression (32) is much more straightforward and flexible.

First, it involves only elementary functions. Second, it does not involve ad hoc factors, such as $\alpha_0$ used by Martin & Lubow (2013) in their definition of $\alpha_{\text{gt}}(Q)$, which are confusing and do not appear in the final result anyway. Third, our result (32) is clearly valid regardless of the value of $M$, while the asymptotic representation of the Lambert function used by Martin & Lubow (2013) works only for relatively low $M \lesssim 10^{-7} M_\odot$ yr$^{-1}$.

Analogous problems would arise if one were to use other explicit $\alpha_{\text{gt}}(Q)$ prescriptions for analytical calculations. This provides strong motivation for adopting the $Q = Q_0$ approach advocated in this work rather than the explicit $\alpha_{\text{gt}}(Q)$ recipes, when studying gravitoturbulent disks.

6.2. Outer Edge of the Dead Zone

In a very similar vein we can estimate the location of the outer edge of a dead zone (Gammie 1996) in a gravitoturbulent protoplanetary disk. This edge should be located at the radius $R_d$, where the disk surface density is twice the surface density of the active layer $\Sigma_a \approx 100$ g cm$^{-2}$, down to which external ionizing radiation can penetrate into the disk and keep it fully ionized.

For this estimate Equation (15) of the $Q = Q_0$ approach allows one to express the midplane temperature through $R_d$ and

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4 This is what Equations (4) and (5) of Martin & Livio (2013) effectively reduce to.

5 Martin & Livio (2013) used $\alpha_{\text{cm}}$ rather than $\alpha_0$ to denote $\alpha_{\text{gt}}(Q)$ in their Equation (9) for $\alpha_{\text{gt}}(Q)$, but this notation would cause confusion with the background viscosity in our paper.
while $\Sigma_\ast$, while $\alpha^{R}_{\ast}$ is already a function of the two; see Equation (18). Then, repeating the steps in Section 6.1.1 and eliminating $T(R_{d})$, one obtains

$$R_{d} = (GM)^{1/3} \left[ \frac{9}{128\pi} \frac{\kappa_{0}M}{\sigma\Sigma_{\ast}^{2}} \left( \frac{k_{B}/\mu}{\pi G Q_{0}} \right)^{4(4-\beta)} \right]^{1/(15-3\beta)}$$

$$\approx (21 \text{ AU})Q_{0}^{-4/9} \left( \frac{M}{10^{-8}M_{\odot} \text{ yr}^{-1}} \right)^{1/9}$$

$$\left( \frac{\Sigma_{\ast}}{10^{2} \text{ g cm}^{-2}} \right)^{-1/3}.$$  (33)

In making the estimate we again adopted the opacity in the form (13), which is perfectly reasonable since $R_{d} > R_{\text{snow}}$ and ice grains dominate opacity.

It is easy to see that deriving $R_{d}$ via some explicit $\alpha_{\ast}(Q)$ prescription would again require going through the rather contrived procedure described in Section 6.1.2, involving non-elementary functions, their asymptotic expansions, and so on. Thus, similar to Section 6.1.3, we can again argue that the $Q = Q_{0}$ approach provides the shortest and clearest path to finding $R_{d}$.

7. CONCLUSIONS

One of the main outcomes of this work is the development of a self-consistent analytical formalism for the evolution of a gravitoturbulent accretion disk, including the possibility of external irradiation. It explicitly enforces Toomre $Q$ in the disk to stay close to the value $Q_{0}$ needed for marginal gravitational stability. Implicit numerical implementations of this approach do exist (e.g., Terquem 2008; Clarke 2009; Zhu et al. 2009a), but a complete analytical formalism was lacking until now. Our work fills this gap with the prescriptions described in Section 3.1, in particular the explicit expression (18) for the gravitoturbulent viscosity $\alpha^{R}_{\ast}$, which depends only on $r$ and $\Sigma$ and fully accounts for the thermodynamic equilibrium of the disk. Our formalism includes the possibility of the non-zero background viscosity $\alpha_{m}$ (Section 3.1.1) provided by sources other than the gravitoturbulence (e.g., MRI) and external irradiation of the disk. This work thus generalizes the models of steady, constant $M$ gravitoturbulent disks explored in Rafikov (2009).

In the fully gravitoturbulent state, when $\alpha^{R}_{\ast} \gg \alpha_{m}$ we derive an explicit Equation (19) for the surface density evolution, which is fully self-contained—it involves only $r$ and $\Sigma$—and is non-linear in general. This equation provides a powerful tool for exploring gravitoturbulent disk evolution and is thus an important result of our work.

We then contrasted our $Q = Q_{0}$ approach with the calculations specifying an explicit dependence of the gravitoturbulent viscosity $\alpha_{\ast}$ on the Toomre $Q$, which is often done in the literature. We clearly demonstrated that our $Q = Q_{0}$ approach provides a much faster and more natural route to deriving analytical expressions for various important characteristics of the gravitoturbulent disks, such as the locations of the snow lines (Section 6.1) and the edge of the dead zone (6.2). We also compared disk models computed using different $\alpha_{\ast}(Q)$ recipes (Section 5) and found that some explicit $\alpha_{\ast}(Q)$ prescriptions are able to reproduce the results obtained using the $Q = Q_{0}$ approach reasonably well. This is the case only for those prescriptions that cause $\alpha_{\ast}(Q)$ to increase very steeply as $Q$ decreases in the vicinity of $Q_{0}$ (e.g., Kratter et al. 2008; Zhu et al. 2010a, 2010b). Certain disagreement may arise if this requirement is violated (Martin & Lubow 2011), especially when the background (non-gravitoturbulent) viscosity in the disk is low (see our discussion in Section 6). Thus, although it is often stated in the literature that the precise form of the $\alpha_{\ast}(Q)$ dependence used for building gravitoturbulent disk models is unimportant, we find this to be only partly true.

Finally, we emphasize that our formalism has the minimal number of imposed assumptions—namely, that $Q = Q_{0}$ is approximately maintained in the gravitoturbulent state, in agreement with simulations. It thus provides a natural and robust pathway to building efficient time-dependent models of gravitoturbulent disks around objects such as young stars, quasars, and Pop III stars.

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APPENDIX

CHARACTERISTIC DISK PARAMETERS

For our adopted opacity behavior (12)–(13), assuming that fragmentation happens at $\alpha_{\text{crit}} = 1$, the fiducial values of the surface density $\Sigma_{f}$, midplane temperature and sound speed $T_{f}$ and $\epsilon_{f}$, mass accretion rate $M_{f}$, and angular frequency $\Omega_{f}$ are

$$\Sigma_{f} = \left[ \frac{8}{9} \frac{\sigma}{\pi G Q_{0}} \right]^{4} \left( \frac{\mu}{k_{B}} \right)^{2} \left( \frac{\kappa_{0}}{G} \right)^{-7} \approx 14 \text{ g cm}^{-2},$$

$$T_{f} = \left[ \frac{9}{8} \frac{\pi G Q_{0}^{2}}{\sigma \kappa_{0}} \right]^{1/15} \approx 12 \text{ K},$$

$$\epsilon_{f} = \left[ \frac{9}{8} \frac{\pi G Q_{0}^{2}}{\sigma \kappa_{0}^{2}} \right]^{8/15} \approx 0.22 \text{ km s}^{-1},$$

$$\Omega_{f} = \left[ \frac{8}{9} \frac{\sigma}{\pi G Q_{0}} \right]^{3/15} \approx 1.4 \times 10^{-10} \text{ s}^{-1},$$

$$M_{f} = 3 \pi \left[ \frac{8}{9} \frac{\sigma}{\pi G Q_{0}^{2}} \right]^{1/3} \left( \frac{\mu}{k_{B}} \right)^{-1/15}$$

$$\approx 7.2 \times 10^{-6} M_{\odot} \text{ yr}^{-1}.$$  (A3)

The numerical estimates are for $Q_{0} = 1$. More general expressions for these variables for different opacity behavior ($\beta \neq 2$) and $\alpha_{\text{crit}}$, as well as the coefficient in the expression (18) for $\alpha_{\ast}$, can be found in Rafikov (2009).

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