Energy-Momentum Distribution: A Crucial Problem in General Relativity

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Abstract

This paper is aimed to elaborate the problem of energy-momentum in General Relativity. In this connection, we use the prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller to compute the energy-momentum densities for two exact solutions of Einstein field equations. The spacetimes under consideration are the non-null Einstein-Maxwell solutions and the singularity-free cosmological model. The electromagnetic generalization of the Godel solution and the Godel metric become special cases of the non-null Einstein-Maxwell solutions. It turns out that these prescriptions do not provide consistent results for any of these spacetimes. These inconsistence results verify the well-known proposal that the idea of localization does not follow the lines of pseudo-tensorial construction but instead follows from the energy-momentum tensor itself. These differences can also be understood with the help of the Hamiltonian approach.

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1 Introduction

Energy-momentum is an important conserved quantity in any physical theory whose definition has been under investigation for a long time from the General

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Relativity (GR) viewpoint. The problem is to find an expression which is physically meaningful. The point is that the gravitational field can be made locally vanish and so one is always able to find the frame in which the energy-momentum of gravitational field is zero while in the other frames, it is not true. Unfortunately, there is still no generally accepted definition of energy-momentum for gravitational field. The problem arises with the expression defining the gravitational field energy part.

In the theory of GR, the energy-momentum conservation laws are given by

\[ T_{ab}^b = 0, \quad (a, b = 0, 1, 2, 3), \]  

where \( T_{ab}^b \) denotes the energy-momentum tensor. In order to change the covariant divergence into an ordinary divergence so that global energy-momentum conservation, including the contribution from gravity, can be expressed in the usual manner as in electromagnetism, Einstein formulated [1] the conservation law in the following form

\[ \frac{\partial}{\partial x^b} (\sqrt{-g}(T_{ab}^b + t_{ab}^b)) = 0. \]  

Here \( t_{ab}^b \) is not a tensor quantity and is called the gravitational field pseudo-tensor. Schrodinger showed that the pseudo-tensor can be made vanish outside the Schwarzschild radius using a suitable choice of coordinates. There have been many attempts in order to find a more suitable quantity for describing the distribution of energy and momentum due to matter, non-gravitational and gravitational fields. The proposed quantities which actually fulfill the conservation law of matter plus gravitational parts are called gravitational field pseudo-tensors. The choice of the gravitational field pseudo-tensor is not unique. Because of this, quite a few definitions of these pseudo-tensors have been proposed.

In order to obtain a meaningful expression for energy, momentum and angular momentum for a general relativistic system, Einstein himself proposed an expression. After Einstein’s energy-momentum complex [2], many complexes have been found, for instance, Landau-Lifshitz [3], Tolman [4], Papapetrou [5], Möller [6,7], Weinberg [8] and Bergman [9]. Some of these definitions are coordinate dependent while others are not. Also, most of these expression can not be used to define angular momentum.

The lack of a generally accepted definition of energy-momentum in a curved spacetime has led to doubts regarding the idea of energy localization.
According to Misner et al. [10], energy is localizable only for spherical systems. Cooperstock and Sarracino [11] came up with the view that if energy is localizable for spherical system, then it can be localized for any system. Bondi [12] argued that a non-localizable form of energy is not allowed in GR. After this, an alternative concept of energy, called quasi-local energy, was developed. The use of quasi-local masses to obtain energy-momentum in a curved spacetime do not restrict one to use particular coordinate system. A large number of definitions of quasi-local masses have been proposed, those by Penrose and many others [13-15]. Although these quasi-local masses are conceptually very important, these definitions have serious problems. Bergqvist [16] considered seven different definitions of quasi-local masses and computed them for Reissner-Nordstrom and Kerr spacetimes. He concluded that no two of the seven definitions provide the same result. The seminal concept of quasi-local masses of Penrose cannot be used to handle even the Kerr metric [17]. The present quasi-local mass definitions still have inadequacies.

It is believed that different energy-momentum distribution would be obtained from different energy-momentum complexes. Virbhadra [18,19] revived the interest in this approach. He and his co-workers [19-23] considered many asymptotically flat spacetimes and showed that several energy-momentum complexes can give the same result for a given spacetime. They also carried out calculations in a few asymptotically non-flat spacetimes using different energy-momentum complexes and found encouraging results. Aguirregabiria et al. [24] proved that several energy-momentum complexes can provide the same result for any Kerr-Schild class metric. Chang et al. [25] showed that every energy-momentum complex can be associated with a particular Hamiltonian boundary term. Therefore, the energy-momentum complexes may also be considered as quasi-local. Xulu [26,27] extended this investigation and found that Melvin magnetic universe, Bianchi type I universe provided the same energy distribution.

Virbhadra, Xulu and others [28] provided the hope that some particular properties might give a basis to believe that some pseudo-tensors of energy-momentum density had a special meaning. Or equivalently that some coordinate exists which has a special meaning. However, some examples of spacetimes have been explored which do not support this viewpoint. In this regard, one of the authors [29,30] considered the class of gravitational waves and Gödel universe, and used the two definitions of energy-momentum. In a recent paper, the same author extended this procedure to Gödel-type metrics [31]. He concluded that both the definitions do not provide consistent results.
for these models. Ragab [32] also obtained similar results while dealing with Gödel-type metric using the prescriptions of Möller and Landau-Lifshitz. Contradictory results have also been obtained by Owen [33] for a regular MMaS-class black hole.

According to the Hamiltonian approach, the various energy-momentum expressions are each associated with distinct boundary conditions [25,34]. It is found that using homogeneous boundary conditions, the quasi-local energy vanishes for all Bianchi A but does not for B models. Energy-momentum is associated with a closed surface bounding a region. Energy can be identified as the value of the Hamiltonian. The Hamiltonian for a finite region includes a boundary term, which determines the quasi-local quantities and the boundary conditions. In this paper, we are extending this work to some more examples for the evaluation of energy-momentum density components by using different energy-momentum complexes. We would show that different prescriptions do not provide the same results for a given spacetime which can be expected.

The paper is organized as follows. In section 2, we shall briefly mention different prescriptions to evaluate energy-momentum distribution. Sections 3 and 4 are devoted for the evaluation of energy-momentum densities for the two particular spacetimes using the prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller. Finally, in the last section, we shall discuss and summarize all the results obtained.

2 Energy-Momentum Complexes

In this section, we shall elaborate four different approaches to evaluate the energy-momentum density components of different spacetimes. We shall briefly describe the prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller energy-momentum complexes.

2.1 Einstein Energy-Momentum Complex

The energy-momentum complex of Einstein [2] is given by

$$\Theta^b_a = \frac{1}{16\pi} H^{bc}_{a,c}, \quad (a, b, ... = 0, 1, 2, 3), \quad (3)$$
where

$$H^{bc}_a = \frac{g_{ad}}{\sqrt{-g}} \left[-g(g^{bd}g^{ce} - g^{cd}g^{be})\right]_e.$$  \hspace{1cm} (4)

It is to be noted that $H^{bc}_a$ is anti-symmetric in indices $b$ and $c$. $\Theta^0_0$ is the energy density, $\Theta^i_0$ ($i = 1, 2, 3$) are the components of momentum density and $\Theta^0_i$ are the energy current density components. The Einstein energy-momentum satisfies the local conservation laws

$$\frac{\partial \Theta^b_a}{\partial x^b} = 0.$$  \hspace{1cm} (5)

Einstein showed that the energy-momentum pseudo-complex $\Theta^b_a$ provides satisfactory expression for the total energy and momentum of closed system in the form of 3-dimensional integral.

### 2.2 Landau-Lifshitz Energy-Momentum Complex

There were some drawbacks of Einstein energy-momentum complex. One main drawback was that it was not symmetric in its indices. As a result, this cannot be used to define conservation laws of angular momentum. However, Landau-Lifshitz energy-momentum complex is symmetric and they are able to develop a conserved angular momentum complex in addition to that of energy-momentum. They introduced a geodesic coordinate system at some particular point in spacetime in which all the first derivatives of the metric tensor vanish. The energy-momentum complex of Landau-Lifshitz [3] is given by

$$L^{ab} = \frac{1}{16\pi} \ell^{abcd},$$  \hspace{1cm} (6)

where

$$\ell^{abcd} = -g(g^{ab}g^{cd} - g^{ad}g^{cb}).$$  \hspace{1cm} (7)

$L^{00}$ represents the energy density of the whole system including gravitation and $L^{ai}$ represent the components of the total momentum density. $\ell^{abcd}$ has symmetries of the Riemann curvature tensor. It is clear from Eq.(7) that $L^{ab}$ is symmetric with respect to its indices. The energy-momentum complex of Landau-Lifshitz satisfies the local conservation laws

$$\frac{\partial L^{ab}}{\partial x^b} = 0.$$  \hspace{1cm} (8)
2.3 Papapetrou Energy-Momentum Complex

Papapetrou [5] energy-momentum complex is the least known among the four definitions under discussion, as a result, it has been re-discovered several times. The expression was found using the generalized Belinfante method. The symmetric energy-momentum complex of Papapetrou [5] is given as

\[ \Omega_{ab} = \frac{1}{16\pi} N_{abcd}^{\cd}, \]

where

\[ N_{abcd} = \sqrt{-g} \left(g^{ab} \eta^{cd} - g^{ac} \eta^{bd} + g^{cd} \eta^{ab} - g^{bd} \eta^{ac}\right), \]

and \( \eta^{ab} \) is the Minkowski spacetime. The quantities \( N_{abcd} \) are symmetric in its first two indices \( a \) and \( b \). The locally conserved quantities \( \Omega_{ab} \) contain contribution from the matter, non-gravitational and gravitational field. The quantity \( \Omega_{00} \) represents energy density and \( \Omega_{0i} \) are the momentum density components. The energy-momentum complex satisfies the local conservation laws

\[ \frac{\partial \Omega_{ab}}{\partial x^b} = 0. \]

2.4 Möller Energy-Momentum Complex

Although the Einstein energy-momentum complex provides useful expression for the total energy-momentum of a closed system. However, from the GR viewpoint, Möller [7] argued that it is unsatisfactory to transform a system into quasi-Cartesian coordinates. Möller tried to find out an expression of energy-momentum which is independent of the choice of particular coordinate system. His energy-momentum complex is given by

\[ M^b_a = \frac{1}{8\pi} K_{a,c}^{bc}, \]

where

\[ K_{a}^{bc} = \sqrt{-g} (g_{ad,e} - g_{ae,d}) g^{be} g^{cd}. \]

Here \( K_{a}^{bc} \) is antisymmetric, \( M^0_0 \) is the energy density, \( M^i_0 \) are the momentum density components and \( M^0_i \) are the energy current density components. The
local conservation laws for Möller energy-momentum complex are the following
\[ \frac{\partial M^b}{\partial x^b} = 0. \] (14)

After a critical analysis of Möller’s result, Kovacs [35] claimed that he found a mistake in Möller’s calculation. However, Novotny [36] showed that Möller [7] was right in concluding that \( p_a \) transforms like a four-vector under Lorentz transformation. Lessner [37] showed that the problem is with the interpretation of the result. He argued that energy-momentum four-vector can transform according to Special Relativity only if it is transformed to a reference system with the velocity constant everywhere. He also concluded that the Möller’s energy-momentum complex is a powerful expression of energy and momentum in GR.

3 Energy-Momentum Distribution in Non-Null Einstein-Maxwell Solutions

In this section, we calculate the energy-momentum densities for the non-null Einstein-Maxwell solutions by using the four different prescriptions given in the last section. Further, we consider the two special cases of this solution and evaluate the energy-momentum density components for these metrics. The non-null Einstein-Maxwell solution contains five classes of non-null electromagnetic field plus perfect fluid solutions which possesses a metric symmetry not inherited by the electromagnetic field and admits a homothetic vector field. Two of them contain electrovac solutions as special cases, while the other three necessarily contain fluid. This metric, representing the vacuum solution of the Einstein field equations, is generalized by Kramer et al. [38] and can be obtained by applying a complex invariance transformation.

The line element of the non-null Einstein-Maxwell solutions [39] is given by
\[ ds^2 = -(dt + A d\phi)^2 + F^2 d\phi^2 + e^{2K} (d\rho^2 + dz^2), \] (15)
where \( F = F(\rho) \), \( A = A(\rho, z) \) and \( K = K(\rho, z) \) are the functions satisfying
\[ A_{,1} = F V_{,3}, \quad A_{,3} = -F V_{,1}, \]
\[ K_{,1} = -\frac{1}{4} a F (V_{,1}^2 - V_{,3}^2), \]
\[ K_{33} = -\frac{1}{2} a F V_{13}, \]
\[ V_{11} + F_{1} F^{-1} V_{1} + V_{33} = 0. \quad (16) \]

In order to get meaningful results in the prescriptions of Einstein, Landau-Lifshitz and Papapetrou one needs to have the metric in Cartesian coordinates. For this reason, we transform the metric in Cartesian coordinates by using the following transformations
\[ x = \rho \cos \phi, \quad y = \rho \sin \phi. \quad (17) \]

The line element in \( t, x, y, z \) coordinates becomes
\[ ds^2 = dt^2 + (A^2 - F^2) \left( \frac{x dy - y dx}{\rho^2} \right)^2 - e^{2K} \left( \frac{x dx + y dy}{\rho} \right)^2 - e^{2K} dz^2 + 2 A dt \left( \frac{x dy - y dx}{\rho^2} \right). \quad (18) \]

### 3.1 Energy and Momentum in Einstein’s Prescription

In order to calculate the energy and momentum density components for the non-null Einstein-Maxwell solutions, we need to compute the components of \( H^{bc}_a \). The required non-zero components of \( H^{bc}_a \) are

\[ H^{01}_0 = -\frac{2x}{\rho^2} F_\rho - \frac{2x F}{\rho^2} K_\rho + \frac{x F}{\rho^3} + \frac{x A}{\rho^2 F} A_\rho + \frac{x}{\rho F} e^{2K}, \quad (19) \]
\[ H^{02}_0 = -\frac{2y}{\rho^2} F_\rho - \frac{2y F}{\rho^2} K_\rho + \frac{y F}{\rho^3} + \frac{y A}{\rho^2 F} A_\rho + \frac{y}{\rho F} e^{2K}, \quad (20) \]
\[ H^{03}_0 = \frac{A}{\rho F} A_z - \frac{2F}{\rho} K_z, \quad (21) \]
\[ H^{12}_0 = -H^{21}_0 = \frac{A_\rho}{F}, \quad (22) \]
\[ H^{13}_0 = -H^{31}_0 = \frac{A_\rho}{F} A_z, \quad (23) \]
\[ H^{23}_0 = -H^{32}_0 = -\frac{x}{\rho F} A_z, \quad (24) \]
\[ H^{01}_1 = \frac{2A xy}{\rho^4} F_\rho - \frac{A^2 xy}{\rho^4} A_\rho - \frac{F xy}{\rho^4} A_\rho, \quad (25) \]
\[ H^{02}_1 = \frac{2A y^2}{\rho^4} F_\rho - \frac{A^2 y^2}{\rho^4} A_\rho - \frac{F y^2}{\rho^4} A_\rho - \frac{A}{\rho F} e^{2K}, \quad (26) \]
\[
H_{03} = -\frac{A^2 y}{\rho^3 F} A_z - \frac{F y}{\rho^3} A_z, \tag{27}
\]
\[
H_{01} = -\frac{2A x^2}{\rho^4} F_p + \frac{A^2 x^2}{\rho^4} A_{\rho} + \frac{Fx^2}{\rho^4} A_{\rho} + \frac{A}{\rho^4} e^{2K}, \tag{28}
\]
\[
H_{02} = -\frac{2A xy}{\rho^4} F_p + \frac{A^2 xy}{\rho^4} A_{\rho} + \frac{F xy}{\rho^4} A_{\rho}, \tag{29}
\]
\[
H_{03} = \frac{A^2 x}{\rho^3 F} A_z + \frac{Fx}{\rho^3} A_z. \tag{30}
\]

Substituting Eqs.(19)-(30) in Eq.(3), we obtain the components of energy and momentum density in the prescription of Einstein as follows

\[
\Theta_0^0 = \frac{1}{16\pi\rho^3 F^2} [F^2(-2\rho^2 F_{\rho\rho} - 2\rho^2 F_\rho - 2\rho^2 F K_\rho - \rho F_\rho - F) + \rho^2 F A^2_{\rho\rho} - \rho^2 AA_\rho F_\rho + \rho^2 F e^{2K} + 2\rho^3 F K_\rho - \rho^3 e^{2K} F_\rho + \rho^2 A F A_{\rho\rho} + \rho^2 F(A^2_z + AA_{zz} - 2F^2 K_{zz})], \tag{31}
\]
\[
\Theta_0^1 = \frac{y}{16\pi\rho F^2} (F A_{\rho\rho} - A_\rho F_\rho + FA_{zz}), \tag{32}
\]
\[
\Theta_0^2 = \frac{x}{16\pi\rho F^2} (F A_{\rho\rho} - A_\rho F_\rho + FA_{zz}), \tag{33}
\]
\[
\Theta_1^0 = \frac{y}{16\pi\rho^3 F^2} [F^2(-2AF_\rho + \rho A F_\rho + 2\rho A F_{\rho\rho} + FA_\rho - \rho F A_{\rho\rho}) + \rho A^2 A_\rho F_\rho + A^2 F A_\rho - 2\rho A F A_{\rho\rho} - \rho A^2 F A_{\rho\rho} + \rho(\rho F_\rho + AF - \rho F A_{\rho\rho}) - \rho F A_\rho - 2\rho A F K_\rho) e^{2K} - \rho F(2AA^2_z + AA_{zz} + F^2 A_{zz})], \tag{34}
\]
\[
\Theta_2^0 = \frac{x}{16\pi\rho^4 F^2} [F^2(-2AF_\rho + \rho A F_\rho + 2\rho A F_{\rho\rho} + FA_\rho - \rho F A_{\rho\rho}) + \rho A^2 A_\rho F_\rho + A^2 F A_\rho - 2\rho A F A_{\rho\rho} - \rho A^2 F A_{\rho\rho} + \rho(\rho F_\rho + AF - \rho F A_{\rho\rho}) - \rho F A_\rho - 2\rho A F K_\rho) e^{2K} - \rho F(2AA^2_z + AA_{zz} + F^2 A_{zz})], \tag{35}
\]

and
\[
\Theta_3 = 0 = \Theta_0^3. \tag{36}
\]

If we choose the values of \(A, F, K\) such that
\[
A = \frac{m}{n} e^{\alpha \rho}, \quad F = e^{\alpha \rho}, \quad K = 0, \tag{37}
\]

where \(m, n\) are arbitrary constants, then the metric given by Eq.(15) reduces to the electromagnetic generalization of the Gödel solution [40] and is given
by
\[ ds^2 = -(dt + \frac{m}{n}e^{n\rho}d\phi)^2 + e^{2n\rho}d\phi^2 + d\rho^2 + dz^2. \] (38)

The corresponding energy-momentum density components turn out to be
\[ \Theta_0^0 = \frac{1}{16\pi\rho^4}[(\rho^2 - n\rho^3)e^{-n\rho} + (n\rho - 2n^2\rho^2 - 1 + m^2\rho^2)e^{n\rho}], \] (39)
\[ \Theta_1^0 = \frac{m\rho e^{2n\rho}}{16\pi n^2\rho^3}[2n^3\rho - n^2 + n\rho e^{-2n\rho} - 2m^2n\rho + m^2], \] (40)
\[ \Theta_2^0 = -\frac{m\rho e^{2n\rho}}{16\pi n^2\rho^3}[2n^3\rho - n^2 + n\rho e^{-2n\rho} - 2m^2n\rho + m^2]. \] (41)

The remaining momentum and energy current density components are zero.

When we choose the values of the metric functions \( A, F, K \) such that
\[ A = e^{ar}, \quad F = \frac{e^{ar}}{\sqrt{2}}, \quad K = 0, \] (42)

where \( a \) is an arbitrary constant, the original metric reduces to
\[ ds^2 = -(dt + e^{ar}d\phi)^2 + \frac{e^{2ar}}{2}d\phi^2 + d\rho^2 + dz^2. \] (43)

This metric is known as Gödel metric presented by K. Gödel in 1949 which represents one of the rotating spacetimes. When we replace these values in Eqs.(31)-(35), we obtain the same results as given in [30].

### 3.2 Energy and Momentum in Landau-Lifshitz’s Prescription

The following non-vanishing components of \( \ell^{abcd} \) are required to find energy-momentum densities in this prescription
\[ \ell^{0101} = \frac{e^{2K}}{\rho^4}(A^2x^2 - F^2x^2 - \rho^2y^2e^{2K}), \] (44)
\[ \ell^{0202} = \frac{e^{2K}}{\rho^4}(A^2y^2 - F^2y^2 - \rho^2x^2e^{2K}), \] (45)
\[ \ell^{0102} = \frac{xye^{2K}}{\rho^4}(A^2 - F^2 + \rho^2e^{2K}), \] (46)
\[
\ell^{0303} = \frac{e^{2K}}{\rho^2} (A^2 - F^2), \quad (47)
\]
\[
\ell^{0112} = \frac{Ax}{\rho^2} e^{2K}, \quad (48)
\]
\[
\ell^{0212} = \frac{Ay}{\rho^2} e^{2K}, \quad (49)
\]
\[
\ell^{0313} = \frac{Ay}{\rho^2} e^{2K}, \quad (50)
\]
\[
\ell^{0323} = -\frac{Ax}{\rho^2} e^{2K}. \quad (51)
\]

Substituting these values in Eq.(6), we obtain the energy and momentum density components as follows

\[
L^{00} = \frac{e^{2K}}{8\pi\rho^4} [A^2 - F^2 - 2\rho(\rho A_{\rho} - FF_{\rho}) - 2\rho K_{\rho}(A^2 - F^2) \\
+ \rho^2 K_{\rho\rho}(A^2 - F^2) + 2\rho^2 K_{\rho}^2(A^2 - F^2) + 4\rho^2 K_{\rho}(A A_{\rho} \\
- FF_{\rho}) + \rho^2(A_{\rho}^2 - F_{\rho}^2) + \rho^2(2AA_{\rho\rho} - 2FF_{\rho\rho}) + \rho^2\{AA_{zz} \\
+ A_{z}^2 + 4AA_{z}K_{z} + K_{zz}(A^2 - F^2) + 2K_{z}^2(A^2 - F^2)\}]. \quad (52)
\]
\[
L^{01} = \frac{ye^{2K}}{16\pi^2\rho^4}\left[-A_{\rho} + \rho(A_{\rho\rho} + A_{zz}) + 4\rho(A_{\rho}K_{\rho} + A_{z}K_{z}) \\
+ 2\rho A(K_{\rho\rho} + K_{zz}) + 4\rho A(K_{\rho}^2 + K_{z}^2) - 2AK_{\rho}\right], \quad (53)
\]
\[
L^{02} = -\frac{x e^{2K}}{16\pi^2\rho^4}\left[-A_{\rho} + \rho(A_{\rho\rho} + A_{zz}) + 4\rho(A_{\rho}K_{\rho} + A_{z}K_{z}) \\
+ 2\rho A(K_{\rho\rho} + K_{zz}) + 4\rho A(K_{\rho}^2 + K_{z}^2) - 2AK_{\rho}\right], \quad (54)
\]
\[
L^{03} = 0. \quad (55)
\]

The energy and momentum density components for the metric given by Eq.(38) are

\[
L^{00} = \frac{e^{2n\rho}}{8\pi n^2\rho^4}[m^2 - 2m^2n\rho + 2m^2n^2\rho^2 - 2n^4\rho^2 + 2n^3\rho - n^2], \quad (56)
\]
\[
L^{01} = \frac{my e^{n\rho}}{16\pi^2\rho^4}(n\rho - 1), \quad (57)
\]
\[
L^{02} = -\frac{mx e^{n\rho}}{16\pi^2\rho^4}(n\rho - 1), \quad (58)
\]
\[
L^{03} = 0. \quad (59)
\]

11
The energy and momentum density components in the prescription of Landau-Lifshitz for the G"odel metric take the form

\[ L^{00} = \frac{e^{2ar}}{16\pi r^4}[1 + 2ar(ar - 1)], \quad (60) \]
\[ L^{01} = \frac{ay}{16\pi r^3}(ar - 1)e^{ar}, \quad (61) \]
\[ L^{02} = -\frac{ax}{16\pi r^3}(ar - 1)e^{ar}, \quad (62) \]
\[ L^{03} = 0. \quad (63) \]

3.3 Energy and Momentum in Papapetrou’s Prescription

In this prescription, the required non-zero components of \( N^{abcd} \), given by Eq.(10), are

\[ N^{0011} = \frac{A^2}{\rho F} e^{2K} - \frac{F}{\rho} e^{2K} - \frac{F x^2}{\rho^3} - \frac{y^2}{\rho F} e^{2K}, \quad (64) \]
\[ N^{0022} = \frac{A^2}{\rho F} e^{2K} - \frac{F}{\rho} e^{2K} - \frac{F y^2}{\rho^3} - \frac{x^2}{\rho F} e^{2K}, \quad (65) \]
\[ N^{0033} = \frac{A^2}{F \rho} e^{2K} - \frac{F}{\rho} e^{2K} - \frac{F}{\rho}, \quad (66) \]
\[ N^{0012} = \frac{xy}{\rho F} e^{2K} - \frac{F xy}{\rho^3}, \quad (67) \]
\[ N^{0121} = \frac{Ax}{\rho F} e^{2K}, \quad (68) \]
\[ N^{0122} = \frac{Ay}{\rho F} e^{2K}, \quad (69) \]
\[ N^{0133} = \frac{Ay}{\rho F} e^{2K}, \quad (70) \]
\[ N^{0233} = -\frac{Ax}{\rho F} e^{2K}. \quad (71) \]

Making use of the Eqs.(64)-(71) in Eq.(9), we obtain energy and momentum densities in Papapetrou’s prescription

\[ \Omega^{00} = \frac{1}{16\pi \rho^3 F^3}[2A^2\rho^2 F^2(A_{\rho\rho} + A_{zz}) + 2\rho^2 F^2(A_{\rho}^2 + A_z^2)] \]
+ 8Aρ²F²(AρKρ + AｚKｚ) − 4ρ²AFFρFρ + 4ρ²A²F²(Kρ² + Kｚ²) + 2ρ²A²F²(Kρρ + Kzz) − 4ρ²A²FFFρKρ − ρ²A²FFρρ + 2ρ２A²F²ρ 
+ F²(A² − F²) + 2ρF²(F² − A²)Kρ − ρA²FFρ − ρ²F³Fρρ 
= 2ρ²F⁴(Kρρ + Kzz) − 2ρ²F³FρKρ − 4ρ²F⁴(Kρ² + Kｚ²) − ρ³FFρ 
+ 2ρ³F²Kρ + ρ²F²}^{e²K − ρ²F³Fρρ}, (72)

\[ \Omega^{01} = \frac{ye^{2K}}{16\pi\rho^3F^3}[ρF²Aρ − AF² + ρ²F²(Aρρ + Azz)] \]

\[- 2ρ²FAρFρ + 4ρ²F²(AρKρ + AｚKｚ) + 4ρ²AFF²Kρ² − 4ρ²AFFρKρ + 2ρ²AF²(Kρρ + Kzz) \]

\[ − ρAFFρ − ρ²AFFρρ + 2ρ²AF² + 2ρAF²Kρ], (73)

\[ \Omega^{02} = − \frac{xe^{2K}}{16\pi\rho^3F^3}[ρF²Aρ − AF² + ρ²F²(Aρρ + Azz)] \]

\[- 2ρ²FAρFρ + 4ρ²F²(AρKρ + AｚKｚ) + 4ρ²AF²Kρ² − 4ρ²AFFρKρ + 2ρ²AF²(Kρρ + Kzz) \]

\[ − ρAFFρ − ρ²AFFρρ + 2ρ²AF² + 2ρAF²Kρ], (74)

\[ \Omega^{03} = 0. \] (75)

For the electromagnetic generalization of the Gödel solution, substituting the values of A, F, K in the above expressions, we obtain

\[ \Omega^{00} = \frac{e^{np}}{16\pi n²\rho^3}[m² − m²nρ + m²n²ρ² − 2n⁴ρ^² + n³ρ − n² + (n²ρ² − n³ρ³)e^{-2nρ}], (76) \]

\[ \Omega^{01} = - \frac{my}{16\pi n\rho^3}, (77) \]

\[ \Omega^{02} = \frac{mx}{16\pi n\rho^3}, (78) \]

\[ \Omega^{03} = 0. \] (79)

If we substitute the values of the metric functions given by Eq.(42), we obtain the same energy-momentum density given in [30].

### 3.4 Energy and Momentum in Möller’s Prescription

Since the Möller’s prescription is not restricted to use the Cartesian coordinates and hence the original metric given by Eq.(15) can be used to find
energy-momentum distribution. The required non-vanishing components of $K^a_{bc}$ are

\[ K^0_{01} = \frac{A}{F} A_{\rho}, \]
\[ K^0_{03} = \frac{A}{F} A_{z}, \]
\[ K^2_{01} = -\frac{A_{\rho}}{F}, \]
\[ K^2_{03} = -\frac{A_{z}}{F}, \]
\[ K^0_{01} 2 = F A_{\rho} + A^2_{\rho} - 2 AF_{\rho}, \]
\[ K^0_{03} \ 2 = F A_{z} + A^2_{z}. \]

Using the above results in Eq.(12), we get

\[ M^0_0 = \frac{1}{8\pi F^2} [AF(A_{\rho\rho} + F_{zz}) + F(A^2_{\rho} + A^2_{z}) - AA_{\rho} F_{\rho}], \]
\[ M^2_0 = \frac{1}{8\pi F^2} [A_{\rho} F_{\rho} - F(A_{\rho\rho} + A_{zz})], \]
\[ M^0_2 = \frac{1}{8\pi F^2} [F^3(A_{\rho\rho} + A_{zz}) + A^2 F(A_{\rho\rho} + A_{zz}) + 2AF(A^2_{\rho} + A^2_{z}) - 2AF^2 F_{\rho\rho} - (A^2 + F^2) A_{\rho} F_{\rho}], \]

and

\[ M^0_1 = 0 = M^0_3 = M^1_0 = M^3_0. \]

The corresponding components of the energy-momentum density components for the metric (38) are

\[ M^0_0 = \frac{m^2}{8\pi} e^{\rho}, \]
\[ M^2_0 = \frac{mc^{2n}}{4\pi n} (m^2 - n^2). \]

The rest of the components are zero.

The energy and momentum densities for the Gödel solution are

\[ M^0_0 = \frac{a^2 e^{ar}}{4\sqrt{2\pi}}. \]
\[ M_2^0 = \frac{a^2 e^{2ar}}{4\sqrt{2\pi}}, \quad (93) \]
\[ M_1^0 = 0 = M_3^0 = M_0^i. \quad (94) \]

4 Energy-Momentum Distribution in Singularity-Free Cosmological Model

In this section, we extend the same procedure, applied in the previous section, to another spacetime which is also cylindrically symmetric. We consider a cosmological model representing perfect fluid solution of EFEs which is non-separable in co-moving coordinates and has non-singular scalar curvature invariants. This corresponds to a cylindrical symmetric spacetime filled with an isotropic radiation perfect fluid. This model is different from the model investigated by Senovilla [40] in 1990. Also, it is geodesically complete and globally hyperbolic. It fulfills the energy, generic and causual conditions.

The line element for a spacetime that admits an abelian two-dimensional orthogonal transitive group of isometries acting on spacelike surfaces can be written in the form [41]
\[ ds^2 = e^{2K}(-dt^2 + dr^2) + \rho^2 e^{2U} d\phi^2 + e^{-2U} dz^2 \quad (95) \]
where \( K \) and \( U \) are functions of \( t \) and \( r \). In Cartesian coordinates, it becomes
\[ ds^2 = e^{2K} [dt^2 - \left( \frac{xdx + ydy}{r} \right)^2] - \rho^2 e^{2U} \left( \frac{x dy - y dx}{r^2} \right)^2 - e^{-2U} dz^2. \quad (96) \]

4.1 Einstein’s Prescription

The required components of \( H_{abc}^l \) are the following

\[ H_{01}^{01} = -H_{0}^{10} = \frac{x \rho}{r^3} - \frac{2x}{r^2} \rho r + \frac{x}{pr} e^{2(K-U)}, \quad (97) \]
\[ H_{0}^{02} = -H_{0}^{20} = \frac{y \rho}{r^3} - \frac{2y}{r^2} \rho r + \frac{y}{pr} e^{2(K-U)}, \quad (98) \]
\[ H_{1}^{01} = -\frac{2x^2}{r^3} \rho t - \frac{2x^2 \rho}{r^3} (K_t - U_t), \quad (99) \]
\[ H_{1}^{02} = H_{2}^{01} = -\frac{2xy}{r^3} \rho t + \frac{2xy \rho}{r^3} (K_t - U_t), \quad (100) \]
\[ H_{2}^{02} = -\frac{2y^2}{r^3} \rho t - \frac{2x^2 \rho}{r^3} (K_t - U_t). \quad (101) \]
Substituting Eqs. (97)-(101) in Eq. (3), we obtain the components of energy and momentum density

\[
\Theta^0_0 = \frac{1}{16\pi \rho^2 r^3} \left[ r \rho^2 \rho_r - \rho^3 - 2r^2 \rho^2 \rho_{rr} + \rho r^2 \right. \\
\left. + 2 \rho r^3 (K_r - U_r) - r^3 \rho_r \right] e^{2(K-U)}, \tag{102}
\]

\[
\Theta^1_0 = \frac{x}{16\pi \rho^2 r^3} \left[ 2r \rho^2 \rho_t - \rho^3 \rho_t + \{r^2 \rho_t - 2 \rho r^2 (K_t - U_t)\} e^{2(K-U)} \right], \tag{103}
\]

\[
\Theta^2_0 = \frac{y}{16\pi \rho^2 r^3} \left[ 2r \rho^2 \rho_t - \rho^3 \rho_t + \{r^2 \rho_t - 2 \rho r^2 (K_t - U_t)\} e^{2(K-U)} \right], \tag{104}
\]

\[
\Theta^0_1 = \frac{x}{8\pi r^3} [\rho (K_t - U_t) - r \rho_t], \tag{105}
\]

\[
\Theta^0_2 = \frac{y}{8\pi r^3} [\rho (K_t - U_t) - r \rho_t], \tag{106}
\]

and

\[
\Theta^0_3 = 0 = \Theta^3_0. \tag{107}
\]

### 4.2 Landau-Lifshitz’s Prescription

The required non-vanishing components of \( \ell^{abcd} \) are

\[
\ell^{0101} = -\frac{1}{r^4} \left( y^2 r^2 e^{2(K-U)} + x^2 \rho^2 \right), \tag{108}
\]

\[
\ell^{0202} = -\frac{1}{r^4} \left( x^2 r^2 e^{2(K-U)} + y^2 \rho^2 \right), \tag{109}
\]

\[
\ell^{0102} = \frac{xy}{r^4} \left( r^2 e^{2(K-U)} - \rho^2 \right), \tag{110}
\]

\[
\ell^{0110} = \frac{1}{r^4} \left( y^2 r^2 e^{2(K-U)} + x^2 \rho^2 \right), \tag{111}
\]

\[
\ell^{0210} = -\frac{xy}{r^4} \left( r^2 e^{2(K-U)} - \rho^2 \right), \tag{112}
\]

\[
\ell^{0220} = \frac{1}{r^4} \left( x^2 r^2 e^{2(K-U)} + y^2 \rho^2 \right). \tag{113}
\]

Using the above results in Eq. (6), the energy and momentum density components become

\[
L^{00} = \frac{1}{8\pi \rho^4} \left[ r^3 e^{2(K-U)} (K_r - U_r) - \rho^2 + 2r \rho \rho_r - r^2 \rho \rho_{rr} - r^2 \rho_r^2 \right], \tag{114}
\]

\[
L^{01} = \frac{x}{8\pi \rho^4} \left[ \rho \rho_{rt} + r \rho_t \rho_r - \rho \rho_t - r^2 e^{2(K-U)} (K_r - U_r) \right], \tag{115}
\]
\[ L^{02} = \frac{y}{8\pi \rho^4} [r \rho \rho_t + r \rho_t \rho_r - \rho \rho_t - r^2 e^{2(K-U)} (K_r - U_r)], \]  
\[ L^{03} = 0. \]  

4.3 Papapetrou’s Prescription

We require the following non-vanishing components of \( N^{abcd} \) to find the energy-momentum density components in the prescription of Papapetrou:

\[ N^{0011} = -\frac{y^2}{r \rho} e^{2(K-U)} - \frac{x^2 \rho}{r^3} - \frac{\rho}{r}, \]  
\[ N^{0022} = -\frac{x^2}{r \rho} e^{2(K-U)} - \frac{y^2 \rho}{r^3} - \frac{\rho}{r}, \]  
\[ N^{0012} = \frac{xy}{r \rho} e^{2(K-U)} - \frac{xy \rho}{r^3}, \]  
\[ N^{0101} = \frac{y^2}{r \rho} e^{2(K-U)} + \frac{x^2 \rho}{r^3} + \frac{\rho}{r}, \]  
\[ N^{0202} = \frac{x^2}{r \rho} e^{2(K-U)} + \frac{y^2 \rho}{r^3} + \frac{\rho}{r}, \]  
\[ N^{0102} = \frac{xy \rho}{r^3} - \frac{xy}{r \rho} e^{2(K-U)}. \]

Making use of the Eqs.(118)-(123) in Eq.(9), we obtain energy and momentum densities as follows:

\[ \Omega^{00} = \frac{1}{16\pi r^3 \rho^2} [r \rho^2 \rho_r - r^2 \rho^2 \rho_{rr} - \rho^3 \\
+ \{r^2 \rho + 2r^3 \rho (K_r - U_r) - r^3 \rho_r \} e^{2(K-U)}], \]  
\[ \Omega^{01} = \frac{x}{16\pi r^3 \rho^2} [2r \rho^2 \rho_{tt} - \rho^2 \rho_t + \{r^2 \rho_t - r^2 \rho (K_t - U_t) \} e^{2(K-U)}], \]  
\[ \Omega^{02} = \frac{y}{16\pi r^3 \rho^2} [2r \rho^2 \rho_{tt} - \rho^2 \rho_t + \{r^2 \rho_t - r^2 \rho (K_t - U_t) \} e^{2(K-U)}], \]  
\[ \Omega^{03} = 0. \]

4.4 Möller’s Prescription

The required non-vanishing components of \( K^a_{bc} \) are:

\[ K^{01}_0 = 2\rho K_r, \]  
\[ K^{01}_1 = 2\rho K_t. \]
Using these values in Eq.(12), we get

\[ M_0^0 = \frac{1}{4\pi} [\rho_r K_r + \rho K_{rr}], \]  
(130)

\[ M_0^1 = -\frac{1}{4\pi} [\rho K_{rt} + \rho_t K_r], \]  
(131)

\[ M_1^0 = \frac{1}{4\pi} [\rho K_{tr} + \rho_r K_t], \]  
(132)

and

\[ M_2^0 = 0 = M_3^0 = M_0^2 = M_0^3. \]  
(133)

5 Summary and Discussion

The problem of energy-momentum localization has been a subject of many researchers but still remains un-resolved. Numerous attempts have been made to explore a quantity which describes the distribution of energy-momentum due to matter, non-gravitational and gravitational fields. Many energy-momentum complexes have been found [2-9] and the problem associated with the energy-momentum complexes leads to the doubts about the idea of energy localization. This problem first appeared in electromagnetism which turns out to be a serious matter in GR due to the non-tensorial quantities. Many researchers considered different energy-momentum complexes and obtained encouraging results. Virbhadra et al. [18-23] explored several spacetimes for which different energy-momentum complexes show a high degree of consistency in giving the same and acceptable energy-momentum distribution.

This paper continues the investigation of comparing various distributions presented in the literature. It is devoted to discuss the burning problem of energy-momentum in the frame work of GR and four different energy-momentum complexes have been used to find the energy-momentum distribution. These prescriptions turn out to be a powerful tool to evaluate energy-momentum for various physical systems. However, this tool is not proved to be the best for some systems. Keeping this point in mind, we have applied the prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Möller to investigate energy-momentum distribution for various spacetimes.

We have obtained energy-momentum densities for the non-null Einstein-Maxwell solutions using the above prescriptions. This solution reduces to the electromagnetic generalization of the Gödel solution and Gödel metric
for particular values of the metric functions. We have extended the same procedure of evaluating the energy-momentum distribution for these special solutions and also for the singularity-free cosmological model. The summary of the results (only non-zero quantities) can be given in the form of tables in the following:

Table 1(a) Non-null Einstein-Maxwell Solutions: Einstein’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $\Theta_0^1$              | $\frac{1}{16\pi\rho^2F^2}[F^2(-2\rho^2 F_{\rho\rho} - 2\rho^2 F_{\rho\rho} K_{\rho} - 2\rho^2 F K_{\rho\rho} + \rho F_{\rho} - F) + \rho^2 F A_{\rho}^2 - \rho^2 AA_{\rho} F_{\rho} + \rho^2 F e^{2K} + 2\rho^3 F K_{\rho} - \rho^4 e^{2K} F_{\rho} + \rho^2 A F A_{\rho\rho} + \rho^2 F (A_{\rho}^2 + AA_{\rho\rho} - 2F^2 K_{\rho\rho})] - \rho F (2AA_{\rho\rho}^2 + AA_{\rho\rho}^2 + F^2 A_{\rho\rho}^2)]$ |
| $\Theta_0^2$              | $\frac{1}{16\pi\rho^2F^2}(F A_{\rho\rho} - A_{\rho} F_{\rho} + FA_{\rho\rho})$ |
| $\Theta_1^0$             | $\frac{1}{16\pi\rho^2F^2}[F^2(-2AF_{\rho} + \rho A_{\rho} F_{\rho} + 2\rho AF_{\rho\rho} + FA_{\rho} - \rho F A_{\rho\rho}) + \rho^2 A^2 A_{\rho}^2 F_{\rho} + AF - \rho F A_{\rho} - 2\rho AF K_{\rho}) e^{2K} - \rho F (2AA_{\rho\rho}^2 + A^2 A_{\rho\rho}^2 + F^2 A_{\rho\rho}^2)]$ |
| $\Theta_0^2$             | $-\frac{1}{16\pi\rho^2F^2}[F^2(-2AF_{\rho} + \rho A_{\rho} F_{\rho} + 2\rho AF_{\rho\rho} + FA_{\rho} - \rho F A_{\rho\rho}) + \rho^2 A^2 A_{\rho}^2 F_{\rho} + AF - \rho F A_{\rho} - 2\rho AF K_{\rho}) e^{2K} - \rho F (2AA_{\rho\rho}^2 + A^2 A_{\rho\rho}^2 + F^2 A_{\rho\rho}^2)]$ |
Table 1(b) Non-null Einstein-Maxwell Solutions: Landau-Lifshitz’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $L^{00}$                  | \[
\frac{2\kappa}{8\pi\rho^4}[(A^2 - F^2) - 2\rho(2A_A^2 - F_F)]
-2\rho K_p(A^2 - F^2) + \rho^2 K_{pp}(A^2 - F^2)
+2\rho^2 K_p^2(A^2 - F^2) + 4\rho^2 K_p(A A_p^2 - F_F)
-FF_{pp} + \rho^2(A A_{zz} + A_z^2 + 4AA_z K_z)
+(A^2 - F^2)K_{zz} + 2(A^2 - F^2)K_z^2\]
| $L^{01}$                  | \[
\frac{y\pi}{16\pi\rho^4}[-A_\rho + \rho(A_{pp} + A_{zz})
+4\rho(A_p K_p + A_z K_z) + 2\rho A(K_{pp})
+K_{zz} + 4\rho A(K_p + K_z^2) - 2AK_p]\]
| $L^{02}$                  | \[
-\frac{x\pi}{16\pi\rho^4}[-A_\rho + \rho(A_{pp} + A_{zz})
+4\rho(A_p K_p + A_z K_z) + 2\rho A(K_{pp})
+K_{zz} + 4\rho A(K_p + K_z^2) - 2AK_p]\]
Table 1(c) Non-null Einstein-Maxwell Solutions: Papapetrou’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $\Omega^{00}$             | \[ \frac{16\pi \rho^2 F^2}{\rho} \{2A^2 \rho^2 F^2 \rho (A_{\rho\rho} + A_{zz}) + \rho^2 F^2 (A_{\rho\rho}^2 + A_{zz}^2) + 8A^2 \rho^2 F^2 (A_{\rho\rho} K_{\rho} + A_{zz} K_{z}) - 4\rho^2 A F A_{\rho} F_{\rho} + 4\rho^2 A^2 F^2 (K_{\rho}^2 + K_{z}^2) + 2\rho^2 A F^2 (K_{\rho\rho} + K_{zz}) - 4\rho^2 A^2 F F_{\rho} K_{\rho} - \rho^2 A^2 F F_{\rho\rho} + 2\rho^2 A^2 F^2 F^2 (A - F^2) + 2\rho F^2 (A - F^2) K_{\rho} - \rho F^2 F F_{\rho} - \rho F^2 F F_{\rho\rho} - 2\rho F^2 F^2 F F_{\rho\rho} - 2\rho F^2 F^2 F F_{\rho\rho} - 2\rho F^2 F^2 K_{\rho} + \rho^2 F^2 e^{2K} - \rho^2 F^2 F F_{\rho\rho} \} \]

| $\Omega^{01}$             | \[ \frac{16\pi \rho^2 F^2}{\rho} \{\rho F^2 A_{\rho} - A F^2 + \rho^2 F^2 (A_{\rho\rho} + A_{zz}) - 2\rho^2 F A_{\rho} F_{\rho} + 4\rho^2 F^2 (A_{\rho} K_{\rho} + A_{zz} K_{z}) + 4\rho^2 A F^2 K_{\rho}^2 - 4\rho^2 A F F_{\rho} K_{\rho} + 2\rho^2 A F^2 (K_{\rho\rho} + K_{zz}) - \rho A F F_{\rho} - \rho^2 A F F_{\rho\rho} + 2\rho^2 A F^2 F_{\rho\rho} \} \]

| $\Omega^{02}$             | \[ \frac{16\pi \rho^2 F^2}{\rho} \{\rho F^2 A_{\rho} - A F^2 + \rho^2 F^2 (A_{\rho\rho} + A_{zz}) - 2\rho^2 F A_{\rho} F_{\rho} + 4\rho^2 F^2 (A_{\rho} K_{\rho} + A_{zz} K_{z}) + 4\rho^2 A F^2 K_{\rho}^2 - 4\rho^2 A F F_{\rho} K_{\rho} + 2\rho^2 A F^2 (K_{\rho\rho} + K_{zz}) - \rho A F F_{\rho} - \rho^2 A F F_{\rho\rho} + 2\rho^2 A F^2 F_{\rho\rho} \} \]

Table 1(d) Non-null Einstein-Maxwell Solutions: Möller’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $M^0_{00}$                | \[ \frac{1}{8\pi F^2} \{AF(A_{\rho\rho} + A_{zz}) + F(A_{\rho\rho}^2 + A_{zz}^2) - AA_{\rho} F F_{\rho}\} \]
| $M^2_{00}$                | \[ \frac{1}{8\pi F^2} \{AF(A_{\rho\rho} + A_{zz}) + F(A_{\rho\rho}^2 + A_{zz}^2) - AA_{\rho} F F_{\rho}\} \]
| $M^0_{20}$                | \[ \frac{1}{8\pi F^2} \{AF(A_{\rho\rho} + A_{zz}) + F(A_{\rho\rho}^2 + A_{zz}^2) - AA_{\rho} F F_{\rho}\} \]

21
Table 2(a) Electromagnetic Generalization of the Gödel solutions: Einstein’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $\Theta_0^0$              | $\frac{1}{16\pi}\rho\left[(\rho^2 - n\rho^3)e^{-n\rho} + (n\rho - 2n^2\rho^2 - 1 + m^2\rho^2)e^{n\rho}\right]$ |
| $\Theta_1^0$              | $\frac{m_{\gamma}c^2}{16\pi\rho^3}\left[2n^3\rho - n^2 + n\rho e^{-2n\rho} - 2m^2n\rho + m^2\right]$ |
| $\Theta_2^0$              | $-\frac{m_{\gamma}c^2}{16\pi\rho^3}\left[2n^3\rho - n^2 + n\rho e^{-2n\rho} - 2m^2n\rho + m^2\right]$ |

Table 2(b) Electromagnetic Generalization of the Gödel solutions: Landau-Lifshitz’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $L^0^0$                   | $\frac{c^2}{8\pi\rho^3}\left[m^2 - 2m^2n\rho + 2m^2n^2\rho^2 - 2n^4\rho^2 + 2n^3\rho - n^2\right]$ |
| $L^0^1$                   | $\frac{m_{\gamma}c^2}{16\pi\rho^3}(n\rho - 1)$ |
| $L^0^2$                   | $-\frac{m_{\gamma}c^2}{16\pi\rho^3}(n\rho - 1)$ |

Table 2(c) Electromagnetic Generalization of the Gödel solutions: Papapetrou’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $\Omega^0^0$              | $\frac{c^2}{16\pi\rho^3}\left[m^2 - m^2n\rho + m^2n^2\rho^2 - 2n^4\rho^2 + n^3\rho - n^2 + (n^2\rho^2 - n^3\rho^3)e^{-2n\rho}\right]$ |
| $\Omega^0^1$              | $-\frac{m_{\gamma}}{16\pi\rho^3}$ |
| $\Omega^0^2$              | $\frac{m_{\gamma}}{16\pi\rho^3}$ |

Table 2(d) Electromagnetic Generalization of the Gödel solutions: Möller’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| $M^0^0$                   | $\frac{m_{\gamma}c^2}{8\pi\rho^3}$ |
| $M^0^2$                   | $\frac{2m_{\gamma}c^2}{16\pi\rho^3}(m^2 - n^2)$ |
Table 3(a) Gödel Metric: Landau-Lifshitz’s Prescription

| Energy-Momentum Densities | Expression                                      |
|---------------------------|-------------------------------------------------|
| $L_0^0$                   | $\frac{e^{ar}}{16\pi r^2}[1 + 2ar(1 - ar)]$  |
| $L_0^1$                   | $\frac{e^{ar}}{16\pi r^2}(ar - 1)e^{ar}$      |
| $L_0^2$                   | $-\frac{e^{ar}}{16\pi r^2}(ar - 1)e^{ar}$      |

Table 3(b) Gödel Metric: Möller’s Prescription

| Energy-Momentum Densities | Expression                                      |
|---------------------------|-------------------------------------------------|
| $M_0^0$                   | $\frac{a^2 e^{ar}}{\sqrt{2\pi}}$               |
| $M_2^0$                   | $\frac{a^2 e^{ar}}{4\sqrt{2\pi}}$              |

Table 4(a) Singularity-Free Cosmological Model: Einstein’s Prescription

| Energy-Momentum Densities | Expression                                      |
|---------------------------|-------------------------------------------------|
| $\Theta_0^0$              | $\frac{1}{16\pi \rho^2 r^2} [r \rho^2 \rho_r - \rho^4 - 2r^2 \rho^2 \rho_{rr} + \{r^2 + 2r \rho r(\rho_t - \rho_r) - r^3 \rho_r\} e^{2(K-U)}]$ |
| $\Theta_0^1$              | $\frac{x}{16\pi \rho^2 r^2} [2r \rho^2 \rho_{rt} - \rho^2 \rho_t + \{r^2 \rho_t - 2r \rho r^2 (\rho_t - \rho_r)\} e^{2(K-U)}]$ |
| $\Theta_2^2$              | $\frac{y}{16\pi \rho^2 r^2} [2r \rho^2 \rho_{rt} - \rho^2 \rho_t + \{r^2 \rho_t - 2r \rho r^2 (\rho_t - \rho_r)\} e^{2(K-U)}]$ |
| $\Theta_0^1$              | $\frac{x}{8\pi r^3} [\rho (\rho_t - \rho_r) - r \rho_{rr}]$ |
| $\Theta_2^2$              | $\frac{y}{8\pi r^3} [\rho (\rho_t - \rho_r) - r \rho_{rr}]$ |
Table 4(b) Singularity-Free Cosmological Model: Landau-Lifshitz’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| \( L^{00} \)             | \( \frac{1}{8\pi r^4} r^3 e^{2(K-U)}(K_r - U_r) - \rho r + 2r \rho r_t - r^2 \rho \rho_t - r^2 \rho^2 \) |
| \( L^{01} \)             | \( \frac{x}{8\pi r^4} [r \rho r_t + r \rho_t r - \rho r_t - r^2 e^{2(K-U)}(K_r - U_r)] \) |
| \( L^{02} \)             | \( \frac{y}{8\pi r^4} [r \rho r_t + r \rho_t r - \rho r_t - r^2 e^{2(K-U)}(K_r - U_r)] \) |

Table 4(c) Singularity-Free Cosmological Model: Papapetrou’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| \( \Omega^{00} \)        | \( \frac{1}{16\pi r^2} [r \rho^2 r_t - r^2 \rho^2 \rho_{rr} - \rho'] + \{r^2 \rho + 2r^3 \rho(K_r - U_r) - r^3 \rho r_t\} e^{2(K-U)} \) |
| \( \Omega^{01} \)        | \( \frac{x}{16\pi r^2} [2r^2 \rho^2 \rho_{rr} + r^2 \rho_t + r^2 e^{2(K-U)}(K_t - U_t)] \) |
| \( \Omega^{02} \)        | \( \frac{y}{16\pi r^2} [2r^2 \rho^2 \rho_{rt} + r^2 \rho_t + r^2 e^{2(K-U)}(K_t - U_t)] \) |

Table 4(d) Singularity-Free Cosmological Model: Möller’s Prescription

| Energy-Momentum Densities | Expression |
|---------------------------|------------|
| \( M^0_0 \)              | \( \frac{1}{4\pi} [\rho r K_r + \rho K_{rr}] \) |
| \( M^0_1 \)              | \( -\frac{1}{4\pi} [\rho K_{rt} + \rho K_t] \) |
| \( M^1_1 \)              | \( \frac{1}{4\pi} [\rho K_{tr} + \rho K_t] \) |

From these results, it can be seen that the energy-momentum density components turn out to be finite and well-defined in the above mentioned prescriptions. The four prescriptions of the energy-momentum complexes
do not provide the same results for any of these spacetimes. The energy-
momentum densities for the non-null Einstein-Maxwell solutions reduce to
the energy-momentum densities for the electromagnetic generalization of the
Gödel solution and Gödel metric for particular values of the metric functions.
We have also applied the same procedure to the singularity-free cosmological
model which also gives different results in each prescriptions.

It is worth mentioning that the results of energy-momentum distribution
for different spacetime are not surprising rather they justify that different
energy-momentum complexes, which are pseudo-tensors, are not covariant
objects. This is in accordance with the equivalence principle [10] which im-
plies that the gravitational field cannot be detected at a point. These exam-
pies indicate that the idea of localization does not follow the lines of pseudo-
tensorial construction but instead it follows from the energy-momentum ten-
sor itself. This supports the well-defined proposal developed by Cooperstock
[42] and verified by many authors [29-33,43]. In GR, many energy-momentum
expressions (reference frame dependent pseudo-tensors) have been proposed.
There is no consensus as to which is the best. Hamiltonian’s principle helps
to solve this enigma. Each expression has a geometrically and physically
clear significance associated with the boundary conditions.

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