On the speed of the one-dimensional polymer in the large range regime

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1 Weakly self-avoiding walk
   • Self-avoiding walk
   • number of self-intersection

2 Discrete/Continuous setting
   • The model
   • LLN, CLT and LDP

3 New Hamiltonian
   • The model
   • Results and comparisons
Self-avoiding walk

- An $N$-step self-avoiding walk $\omega$ on $\mathbb{Z}^d$, beginning at the site $x$, is defined as a sequence of sites $(\omega(0), \omega(1), \ldots, \omega(N))$ with $\omega(0) = x$, satisfying $|\omega(j + 1) - \omega(j)| = 1$, and $\omega(i) \neq \omega(j) \ \forall i \neq j$.

- Let $c_N$ be the number of $N$-step self-avoiding walks beginning at the origin.

- Mean-square displacement

$$\langle |\omega(N)|^2 \rangle = \frac{1}{c_N} \sum_{\omega: |\omega| = N} |\omega(N)|^2.$$ 

$$\langle |\omega(N)|^2 \rangle \sim DN^{2\nu}.$$

Conjecture: $d = 2, \nu = \frac{3}{4}$; $d = 3, \nu = 0.588\ldots$; $d \geq 4, \nu = \frac{1}{2}$.

$d \geq 5$ proved by Hara and Slade 1992.
Weakly self-avoiding walk

A sequence of random variable \((S_n)_{n \in \mathbb{N} \cup 0}\) with \(S_0 = 0\) and \(S_n = \sum_{i=1}^{n} X_i\), where \((X_i)_{i \in \mathbb{N}}\) is a sequence of IID random variables. The distribution of \(X_i\)'s is

\[
P(X_1 = x) = \begin{cases} 
\frac{1}{2^d}, & x \in \mathbb{Z}^d \text{ with } ||x|| = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

The random process \((S_n)_{n \in \mathbb{N} \cup 0}\) is called the *simple symmetric random walk* (SSRW) on \(\mathbb{Z}^d\).

Fix \(n \in \mathbb{N}\) and a parameter \(\beta \in (0, \infty)\), we define the polymer measure \(P^\beta_n\) on \(S = (S_0, S_1, ..., S_{n-1})\) by

\[
P^H_n(S) := \frac{1}{Z^H_n} e^{-\beta H_n(S)} P(S),
\]

(1)

where

\[
Z^H_n := E(e^{-\beta H_n}) \quad \text{and} \quad H_n(S) := \sum_{i,j=0; i \neq j}^{n-1} 1_{S_i = S_j}.
\]

(2)
\[ P^H_n(S) := \frac{e^{-\beta H_n(S)}}{E(e^{-\beta H_n})} P(S), \]  

where

\[ H_n(S) := \sum_{i,j=0; i \neq j}^{n-1} 1_{S_i=S_j} = \sum_{x \in \mathbb{Z}^d} \ell^2_n(x) - n \]

is the self-intersection local time up to time \( n \), and

\[ \ell_n(x) = \#\{0 \leq i \leq n - 1 : S_i = x\}, \quad x \in \mathbb{Z}^d, \]

is the local time at site \( x \) up to time \( n \). \( \beta \) is called the strength of the self-repulsion. The path receives a penalty \( e^{2\beta} \) when the path self-intersects itself. This model is also called the \textit{weakly self-avoiding walk}. \( \beta = 0, \) SSRW; \( \beta = \infty, \) SAW.

\[ \hat{H}_n := \sum_{x \in \mathbb{Z}} \ell^2_n(x). \]
The heuristic of this model is that if the end-point of the path $S$ has the scale $\alpha_n$,

$$H_n \approx \sum_{x \in \mathbb{Z}^d} \left( \frac{n}{\alpha_n^d} \right)^2 \approx \alpha_n^d \times \left( \frac{n}{\alpha_n^d} \right)^2.$$  \hspace{1cm} (5)

On the other hand, by the local limit theorem of SSRW,

$$P(|S_n| = \alpha_n) \approx \exp(-C\alpha_n^2/n).$$  \hspace{1cm} (6)

Combine this with (5),

$$\log Z_n^H \approx -\beta \frac{n^2}{\alpha_n^d} - C \frac{\alpha_n^2}{n}.$$  

Let $\frac{n^2}{\alpha_n^d} = \alpha_n^2$, we get $\alpha_n = n^{\frac{3}{d+2}}$. It is expected that $E_n^H(|S_n|) \sim n^{\frac{3}{d+2}}$ for $d = 1, 2, 3$, and $E_n^H(|S_n|) \sim n^{1/2}$ for $d \geq 4$ with a logarithmic correction when $d = 4$. $d \geq 5$ proved by Brydges and Spencer 1985 (Lace expansion for weak interaction).
Known results in 1D in the discrete setting

\[ P_n^H(S) := \frac{e^{-\beta H_n(S)}}{Z_n^H} P(S), \quad Z_n^H := E(e^{-\beta H_n}). \]

**Theorem (Ballistic behavior Bolthausen 1990)**

For small \( \beta > 0 \), there exists a \( c(\beta) > 0 \) such that

\[
\lim_{n \to \infty} P_n^H \left( c \leq \left| \frac{S_n}{n} \right| \leq 1/c \right) = 1.
\]

**Theorem (LDP Geven and den Hollander 1993)**

\( \theta > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \log P_n^H(\{S_n/n \sim \theta\} | S_n > 0) = \begin{cases} 
J_\beta(\theta), & \theta^{**}(\beta) \leq \theta, \\
I_\beta(\theta), & \theta < \theta^{**}(\beta), 
\end{cases}
\]

\( \theta^{*}(\beta) \) is the unique zero of \( J_\beta \). \( 0 < \theta^{**}(\beta) < \theta^{*}(\beta) \).
Known results in 1D in the discrete setting

**Theorem (CLT Konig 1996)**

∀ \( C \in \mathbb{R} \), there exists \( \theta^*(\beta) > 0 \), \( \sigma^*(\beta) > 0 \) such that

\[
\lim_{n \to \infty} P_n^H \left( \frac{S_n - \theta^*(\beta)n}{\sigma^*(\beta)\sqrt{n}} \leq C|S_n > 0 \right) = \Phi(C),
\]

where
\[
\frac{1}{\sigma^2(\beta)} = \left. \frac{\partial^2}{\partial \theta^2} J_\beta(\theta) \right|_{\theta = \theta^*(\beta)}.
\]
Known results in the continuous setting

Let $B_t$ be the $d$–dimensional Brownian motion, the Hamiltonian is

$$H_t(B) := \int_0^t ds \int_0^t du \delta_0(B_s - B_u).$$

(7)

However, $H_t$ is infinity when the dimension is higher than one. Past results used truncations to obtain the polymer measure as a weak limit. $d = 2$, Varadhan 1969, $d=3$ Westwater 1984 and Bolthausen 1993. For $d=1$, LLN Westwater 1980; CLT van der Hofstad, den Hollander and Konig 1997; LDP van der Hofstad, den Hollander and Konig 2003.
New Hamiltonian

We discuss the model with a weaker Hamiltonian

\[ G_n := \frac{n^2}{R_n}, \quad (8) \]

where \( R_n \) is the number of sites occupied by the walk up to time \( n - 1 \), that is,

\[ R_n := \# \{ x : \exists i, S_i = x, 0 \leq i \leq n - 1 \}. \quad (9) \]

For “weaker” we mean that

\[
\left[ \sum_{x \in \mathbb{Z}^d} \ell_n^2(x) \right] \cdot \left[ \sum_{x \in \mathbb{Z}^d} 1_{\ell_n(x) > 0} \right] \geq \left[ \sum_{x \in \mathbb{Z}^d} \ell_n(x) \right]^2 = n^2. \quad (10)
\]

We have

\[
\hat{H}_n \geq \frac{n^2}{R_n} = G_n. \quad (11)
\]
Theorem (Hamana and Kesten 2002)

\[ I(x) = \lim_{n \to \infty} \frac{-1}{n} \log P\{R_n \geq xn\} \]  

exists in \([0, \infty]\) for all \(x\). \(I(x)\) is continuous on \([0, 1]\) and strictly increasing on \([\gamma_d, 1]\), and for \(d \geq 2\), \(I(x)\) is convex on \([0, 1]\). Furthermore,

\[ I(x) = 0 \quad \text{for} \quad x \leq \gamma_d, \]
\[ 0 < I(x) < \infty \quad \text{for} \quad \gamma_d < x \leq 1, \]
\[ I(x) = \infty \quad \text{for} \quad x > 1. \]

Note that \(I(1) = \log 2d\). When \(d = 1\) and \(S\) is the SSRW, \(I(x)\) can be found explicitly. For \(0 \leq x \leq 1\)

\[ I(x) = \frac{1}{2}(1 + x) \log(1 + x) + \frac{1}{2}(1 - x) \log(1 - x). \]
- $G_n := \frac{n^2}{R_n}$, $Z_n^G := E(\exp(-\beta G_n))$.
- The polymer measure is then defined by $P_n^G(S) := \frac{1}{Z_n^G} e^{-\beta G_n(S)} P(S)$.

**Theorem**

(i) For $\beta > 0$, $\lim_{n \to \infty} \frac{1}{n} \log Z_n^G = g^*(\beta)$, where

$$g^*(\beta) := -\inf_{c \in [\tilde{c}(\beta), 1]} \left\{ \frac{\beta}{c} + I(c) \right\}$$

and $\tilde{c}(\beta) = \frac{\beta}{\beta + \log 2d}$.

(ii) $d=1$, the infimum is obtained at $c^*(\beta)$, where $c^*(\beta)$ is the solution of

$$\beta = c^2 l'(c) = \frac{c^2}{2} \log \left( \frac{1 + c}{1 - c} \right).$$

Note that $c^*$ is strictly monotone ($dc^*(\beta)/d\beta = \sigma^*(\beta)^2 / c^*(\beta)^2 > 0$), $\beta^{-1/3} c^* (\beta) \to 1$ as $\beta \to 0$ and $e^{2\beta} (1 - c^*(\beta)) \to 2$ as $\beta \to \infty$.

Furthermore, $\beta^{-2/3} g^*(\beta) \to -\frac{3}{2}$ as $\beta \to 0$. 
Theorem

(LLN and LDP) \( d = 1 \) and \( \theta > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \log P_n^G(\{ \frac{S_n}{n} \sim \theta \}|S_n > 0) = \begin{cases} 
-\frac{\beta}{\theta} - I(\theta) - g^*(\beta), & c^*(\frac{\beta}{2}) \leq \theta, \\
-\frac{\beta}{\tilde{r}} - I(2\tilde{r} - \theta) - g^*(\beta), & \theta < c^*(\frac{\beta}{2}), 
\end{cases}
\]

where \( \tilde{r} = \tilde{r}_\beta(\theta) \) is the solution of \( \beta = 2r^2 I'(2r - \theta) \).

Theorem

(CLT) \( d = 1 \), \( \forall C \in \mathbb{R} \),

\[
\lim_{n \to \infty} P_n^G \left( \frac{S_n - c^*(\beta)n}{\sigma^*(\beta)\sqrt{n}} \leq C | S_n > 0 \right) = \Phi(C),
\]

where \( \frac{1}{\sigma^{*2}(\beta)} = \left( \frac{\beta}{\theta} + I(\theta) \right)^{''} \bigg|_{\theta=c^*(\beta)} = \frac{2\beta}{c^*3(\beta)} + \frac{1}{1-c^*(\beta)^2}. \sigma^*(\beta) \to \frac{1}{\sqrt{3}} \) as \( \beta \to 0 \) and \( e^\beta \sigma^*(\beta) \to 2 \) as \( \beta \to \infty \).
\( G_t := \frac{t^2}{R_t^2}, \ Z_t^G := E(\exp (-\beta G_t)). \ R_t^\epsilon := \big| \bigcup_{s \leq t} B_\epsilon(B_t) \big|. \)

The polymer measure is then defined by \( dP_t^G := \frac{e^{-\beta G_t} Z_t^G}{Z_t^G} dP. \)

**Theorem**

(i) For \( \beta > 0 \) and any \( d \),

\[
\lim_{n \to \infty} \frac{1}{t} \log Z_t^G = g^{**}(\beta),
\]

where

\[
g^{**}(\beta) := - \inf_{c \in [\tilde{\beta}_d, \infty)} \left\{ \frac{\beta}{c} + J(c) \right\}
\]

and \( \tilde{\beta}_d := \frac{1}{2} \left( \frac{\beta}{w_{d-1}} \right)^{1/3} \). \( w_{d-1} \) is the volume of the unit ball in \((d - 1)\)-dimension. Set also \( w_0 = 1 \).

(ii) For \( d=1 \), the infimum is obtained at \( c^{**}(\beta) = \beta^{1/3} \) and \( g^{**}(\beta) = -\frac{3}{2} \beta^{2/3} \). Moreover, \( Z_t^G \sim \frac{8}{\sqrt{3}} eg^{**}(\beta)t \).
Theorem (Hamana and Kesten 2002)

\[ J(x) = \lim_{n \to \infty} \frac{-1}{t} \log P\{ R^1_t \geq xt \} \quad (19) \]

exists in \([0, \infty)\) for all \(x\). \(J(x)\) is continuous on \([0, \infty)\) and strictly increasing on \([C_d, \infty)\), and for \(d \geq 2\), \(J(x)\) is convex on \([0, \infty)\).

Furthermore,

\[ J(x) = 0 \quad \text{for} \quad x \leq C_d, \]
\[ 0 < J(x) < \infty \quad \text{for} \quad C_d < x. \quad (20) \]

For \(d = 1\), \(J(x) = \frac{x^2}{2}\) for \(x \geq 0\). \(C_d\) is the heat capacity of the unit ball for the \(d\)-dimension Brownian motion.
Theorem

(LLN and LDP) \( d = 1 \) and \( \theta > 0 \),

\[
\lim_{t \to \infty} \frac{1}{t} \log P^G_t(\{B_t / t \sim \theta\} | B_t > 0) = \begin{cases} 
-\frac{\beta}{\theta} - \frac{\theta^2}{2} - g^{**}(\beta), & 3\sqrt{\frac{\beta}{2}} \leq \theta, \\
-\frac{\beta}{\bar{r}} - \frac{(2\bar{r} - \theta)^2}{2} - g^{**}(\beta), & \theta < 3\sqrt{\frac{\beta}{2}}, 
\end{cases}
\]

where \( \bar{r} = \bar{r}_\beta(\theta) \) is the solution of \( \beta = 2r^2(2r - \theta) \).

Theorem

(CLT) \( d = 1 \), \( \forall \ C \in \mathbb{R} \),

\[
\lim_{t \to \infty} P^G_t \left( \frac{B_t - c^{**}(\beta)t}{\sigma^{**}(\beta)\sqrt{t}} \leq C | B_t > 0 \right) = \Phi(C), \tag{21}
\]

where \( c^{**}(\beta) = \beta^{1/3} \) and \( \sigma^{**}(\beta) = \frac{1}{\sqrt{3}} \).
In the case $d = 1$, we have the explicit formula of the density of

$$R_t := \max_{0 \leq s \leq t} B_s - \min_{0 \leq s \leq t} B_s = R_t^1 - 2.$$

From Feller 1951,

$$P(R_t \in dr) = \frac{8}{\sqrt{t}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi \left( \frac{kr}{t^{1/2}} \right),$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$$Ee^{-\beta \frac{t^2}{R_t}} \approx \exp(-\beta \frac{t^2}{r}) \exp(-\frac{k^2r^2}{t}).$$
We can also compute the joint distribution of $B_t$ and $R_t$ under the condition $B_t > 0$. For $0 < x < r$,

$$P(B_t \in dx, R_t \in dr, B_t > 0) = \frac{r - x}{t \sqrt{t}} \cdot \left\{ \sum_{k=-\infty}^{\infty} 4k^2 \left[-1 + \left(\frac{2kr - x}{\sqrt{t}}\right)^2\right] \phi\left(\frac{2kr - x}{\sqrt{t}}\right) \right\}$$

$$+ \sum_{k=1}^{\infty} \left\{ 4k(k-1) \left(\frac{2kr - x}{t \sqrt{t}}\right) \phi\left(\frac{2kr - x}{t \sqrt{t}}\right) - 4k(k+1) \left(\frac{2kr + x}{t \sqrt{t}}\right) \phi\left(\frac{2kr + x}{t \sqrt{t}}\right) \right\}.$$
van der Hofstad 1998 had numerical results for the speed and the variance of the speed in Edwards model, that is, $c^{**}(\beta)\beta^{-1/3} \in [1.104, 1.124]$ and $\sigma^{**}(\beta) \in [0.60, 0.66]$, while we have $1$ and $1/\sqrt{3} \approx 0.577$ in our model.

- Our Hamiltonian is well-defined.
- Recently, SAW in $d=2,3$ is sub-ballistic Duminil-Copin and Hammond 2013.
- Recently, WSAW in $d=4$ is diffusive with $\sqrt{T}(\log T)^{1/8}$ Bauerschmidt, Brydges and Slade.