On Weak (Measure-Valued)–Strong Uniqueness for Compressible Navier–Stokes System with Non-monotone Pressure Law

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Abstract. In this paper our goal is to define a renormalized dissipative measure-valued (rDMV) solution of compressible Navier–Stokes system for fluids with non-monotone pressure–density relation. We prove existence of rDMV solutions and establish a suitable relative energy inequality. Moreover we obtain the weak (measure-valued)–strong uniqueness property of this rDMV solution with the help of relative energy inequality.

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1. Introduction

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ be a bounded domain with smooth boundary. We consider the compressible Navier–Stokes equation in time-space cylinder $Q_T = (0, T) \times \Omega$ describing the time evolution of the mass density $\rho = \rho(t, x)$ and the velocity field $u = u(t, x)$ of a compressible viscous fluid:

- Conservation of Mass:
  \[ \partial_t \rho + \text{div} (\rho u) = 0. \] (1.1)

- Conservation of Momentum:
  \[ \partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla_x p(\rho) = \text{div} S(\nabla_x u). \] (1.2)

- Constitutive Relation: Here $S(\nabla_x u)$ is Newtonian stress tensor defined by
  \[ S(\nabla_x u) = \mu \left( \frac{\nabla_x u + \nabla_x^T u}{2} - \frac{1}{d} (\text{div} u)^I \right) + \lambda (\text{div} u)^I, \] (1.3)
  where $\mu > 0$ and $\lambda > 0$ are the shear and bulk viscosity coefficients, respectively.

- Pressure Law: In an isentropic setting, the pressure $p$ and the density $\rho$ of the fluid are interrelated by
  \[ p(\rho) = h(\rho) + q(\rho), \] with $q \in C^1_c(0, \infty)$,
  \[ h \in C^1[0, \infty), \quad h(0) = 0, \quad h' > 0, \] in $(0, \infty),\]
  \[ \lim_{\rho \to \infty} \frac{h'(\rho)}{\rho^{\gamma-1}} = a > 0 \text{ and } \gamma \geq 1. \] (1.4)

Remark 1.1. In a simplified setting, $h$ and $\rho$ are interrelated by the isentropic equation of state, i.e. $h(\rho) = a\rho^{\gamma}$ with $\gamma \geq 1$ and $a > 0$. 

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We consider no-slip boundary conditions for the velocity i.e.
\[ u|_{\partial \Omega \times (0,T)} = 0. \]  

(1.5)

The compressible Navier–Stokes equations admit global-in-time weak solution(s) for general finite energy initial data and a large class of pressure–density constitutive relations. Considering \( q \equiv 0 \) in (1.4) and following the literatures of Antontsev, Kazhikov and Monakhov [1], Lions [15], Feireisl [8], Plotnikov and Weigant [18] and many others, we observe global-in-time weak solution for adiabatic exponent \( \gamma \geq 1 \) for \( d = 1, 2 \) and \( \gamma > \frac{3}{2} \) for \( d = 3 \). Even for non-monotone pressure, Feireisl [7] has proved a similar result and recent work by Bresch and Jabin [2] indicates that for \( p \in C^1([0, \infty)) \geq 0, p(0) = 0, \lim_{\rho \to \infty} p'(\rho) = a > 0 \) and \( \gamma \geq 2 \) the system admits a weak solution. So it may seem unnecessary to develop the theory of measure-valued solutions that extends the class of generalized solutions but in the following discussion we will try to justify why we still choose to consider it.

The concept of measure-valued solutions to partial differential equations, more precisely for hyperbolic conservation laws, was introduced by DiPerna [6]. The measure-valued solutions in the context of compressible Navier–Stokes solutions has been introduced by Neustupa [17], drawing inspiration from Malek et al. [16]. The topic has been revisited by Feireisl et al. [10], where a suitable form of energy inequality have been introduced in the definition of the dissipative measure-valued solutions.

Recently, the concept of measure-valued solutions has been studied again in the context of analysis of numerical schemes by Feireisl et al. [12,13]. The crucial result in the aforementioned articles is the weak (measure-valued)–strong uniqueness principle asserting that suitable (dissipative) measure-valued and a strong solution starting from the same initial data necessarily coincide on the life-span of the latter. Identification of dissipative measure-valued solutions as a limit of a given numerical scheme is easier and presence of weak (measure-valued)–strong uniqueness principle ensures the convergence of the scheme towards the strong solutions as long as the latter exists.

In the work [10], the corresponding Young measure describes oscillations of the density and velocity but handles the viscous term as a linear perturbation. In particular, the velocity gradient is not included in the Young measure. The weak–strong uniqueness principle can be established for a monotone pressure–density equation of state following the arguments used for the inviscid Euler system.

Weak–strong uniqueness principle for monotone pressure has been proved by Feireisl et al. [11,14] for weak solutions and in [10] for measure-valued solutions. Recently, weak–strong uniqueness principle in the class of weak solutions has been shown for the compressible Navier–Stokes system with a general non-monotone pressure density relation and/or the singular hard sphere pressure in [4,9]. To prove the above mentioned results the key tool is the presence of viscosity. Hence the above results cannot be extended to an inviscid system like Euler system.

To deal with the non-monotone pressure in Feireisl [4,9] the use of the renormalized version of the equation of continuity plays a crucial role. But that is non-linear with respect to the velocity gradient and density. Extension of these results to the class of measure-valued solutions therefore requires a new approach that incorporates the velocity gradient as an integral part of the associated Young measure in the spirit of Březina et al. [3].

It is the aim of the present paper to introduce a new concept of renormalized dissipative measure valued (rDMV) solutions for the compressible Navier–Stokes system that includes, in particular, the renormalized equation of continuity, and to show the weak–strong uniqueness principle in this class of a non-monotone pressure density state equations. The plan for the paper is as follows:

- **Definition.** In Sect. 2, we will introduce rDMV solutions.
- **Existence.** In Sect. 3, our goal is to show that an rDMV solution exists for any finite energy initial data.
- **Weak–strong uniqueness.** In Sect. 4, we prove that an rDMV solution coincides with the strong solution emanating from the same initial data on the life span of the latter.
2. Definition of Measure Valued Solution

2.1. Pressure Potential

- Let us define pressure potential $P$ as
  \[ P(\rho) = H(\rho) + Q(\rho) \]
  where
  \[ H(\rho) = \rho \int_1^\rho \frac{h(z)}{z^2} \, dz \quad \text{and} \quad Q(\rho) = \rho \int_1^\rho \frac{q(z)}{z^2} \, dz. \tag{2.1} \]

- As a trivial consequence of above we obtain
  \[ \rho H'(\rho) - H(\rho) = h(\rho) \quad \text{and} \quad \rho H''(\rho) = h'(\rho) \quad \text{for} \quad \rho > 0, \]
  \[ \rho Q'(\rho) - Q(\rho) = q(\rho) \quad \text{and} \quad \rho Q''(\rho) = q'(\rho) \quad \text{for} \quad \rho > 0. \tag{2.2} \]

2.2. Phase Space

We have discussed in the introduction that velocity gradient has been incorporated as a part of Young measure along with natural candidates for the phase space e.g. density and velocity $[\rho, u]$. Hence a suitable phase space framework for the measure-valued solution is therefore

\[ \mathcal{F} = \{ [s, \nu, \mathcal{D}_\nu] | s \in [0, \infty), \ \nu \in \mathbb{R}^d, \ \mathcal{D}_\nu \in \mathbb{R}^{d \times d} \}. \tag{2.3} \]

2.3. Notation

Let us assume $\mathcal{Q}$ be a locally compact Hausdorff metric space.

- The symbol $\mathcal{M}(\mathcal{Q})$ stands for the space of signed Borel measures on $\mathcal{Q}$.
- The symbol $\mathcal{M}^+(\mathcal{Q})$ denotes the cone of non-negative Borel measures on $\mathcal{Q}$.
- The symbol $\mathcal{P}(\mathcal{Q})$ indicates the space of probability measures, i.e. for $\nu \in \mathcal{P}(\mathcal{Q}) \subset \mathcal{M}(\mathcal{Q})$ we have $\nu[\mathcal{Q}] = 1$.
- The symbol $\mathcal{M}(\mathcal{Q}; \mathbb{R}^d)$ means for $\zeta = \{ \zeta_i \}_{i=1}^d \in \mathcal{M}(\mathcal{Q}; \mathbb{R}^d)$, $\zeta_i \in \mathcal{M}(\mathcal{Q})$, $\forall i = 1, 2, \ldots, d$, and the notation $\mathcal{M}(\mathcal{Q}; \mathbb{R}^{d \times d})$ stands for $\zeta = \{ \zeta_{i,j} \}_{i,j=1}^d \in \mathcal{M}(\mathcal{Q}; \mathbb{R}^{d \times d})$, $\zeta_{i,j} \in \mathcal{M}(\mathcal{Q})$, $\forall i, j = 1, 2, \ldots, d$.

**Definition 2.1.** We say that a parametrized measure $\{ \nu_{t,x} \}_{(t,x) \in (0,T) \times \Omega}$,

\[ \nu \in L^\infty_{\text{weak}}((0,T) \times \Omega; \mathcal{P}(\mathcal{F})) , \]

is a renormalised dissipative measure-valued (rDMV) solution of Navier–Stokes system (1.1)–(1.3) in $(0,T) \times \Omega$, with the initial condition $\nu_0 \in L^\infty_{\text{weak}}(\Omega; \mathcal{P}([0,\infty) \times \mathbb{R}^d))$ and dissipation defect $\mathcal{D}$,

\[ \mathcal{D} \in L^\infty(0,T), \ \mathcal{D} \geq 0, \]

if the following holds.

- **Equation of Continuity:** For a.e. $\tau \in (0,T)$ and $\psi \in C^1([0,T] \times \Omega)$, we have
  \[ \int_\Omega \langle \nu_{t,x}; s \rangle \psi(\tau, \cdot) \, dx - \int_\Omega \langle \nu_{0,x}; s \rangle \psi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left[ \langle \nu_{t,x}; s \rangle \partial_t \psi + \langle \nu_{t,x}; sv \rangle \cdot \nabla_x \psi \right] \, dx \, dt. \tag{2.4} \]

- **Renormalized equation of continuity:** For a.e. $\tau \in (0,T)$ and $\psi \in C^1([0,T] \times \Omega)$, we have
  \[ \int_\Omega \langle \nu_{t,x}; b(s) \rangle \psi(\tau, \cdot) \, dx - \int_\Omega \langle \nu_{0,x}; b(s) \rangle \psi(0, \cdot) \, dx = \int_0^\tau \int_\Omega \left[ \langle \nu_{t,x}; b(s) \rangle \partial_t \psi + \langle \nu_{t,x}; b(s)v \rangle \cdot \nabla_x \psi \right] \, dx \, dt. \]
where \( b \in C^1[0, \infty), \) \( \exists r_b > 0 \) such that \( b'(x) = 0, \forall x > r_b. \)

- **Momentum Equation:** There exists a measure \( \tau^M \in L^1(0, T; \mathcal{M}(\bar{\Omega}; \mathbb{R}^{d \times d})) \) and \( \xi \in L^1(0, T) \) such that for a.e. \( \tau \in (0, T) \) and every \( \varphi \in C^1([0, T] \times \bar{\Omega}; \mathbb{R}^d), \varphi|_{\partial \Omega} = 0, \) we obtain

\[
|\langle \tau^M(\tau); \nabla_x \varphi \rangle_{\mathcal{M}(\bar{\Omega}; \mathbb{R}^{d \times d}), C(\bar{\Omega}; \mathbb{R}^{d \times d})}| \leq \xi(\tau)D(\tau)\|\varphi\|_{C^1(\bar{\Omega})}
\]  

(2.6)

and

\[
\begin{align*}
\int_\Omega \langle \nabla_t; s\varphi \rangle \cdot \varphi(\tau, \cdot) \, dx & - \int_\Omega \langle \nabla_0; s\varphi \rangle \cdot \varphi(0, \cdot) \, dx \\
& = \int_0^T \int_\Omega \left[ \langle \nabla_t; s\varphi \rangle \cdot \partial_t \varphi + \langle \nabla_t; s(v \otimes v) \rangle : \nabla_x \varphi + \langle \nabla_t; p(s) \rangle \, \text{div}_x \varphi \right] \, dx \, dt \\
& - \int_0^T \int_\Omega \langle \nabla_t; S(Dv) \rangle : \nabla_x \varphi \, dx \, dt + \int_0^T \langle \tau^M; \nabla_x \varphi \rangle_{\mathcal{M}(\Omega; \mathbb{R}^{d \times d}), C(\Omega; \mathbb{R}^{d \times d})} \, dt.
\end{align*}
\]  

(2.7)

- **Momentum Compatibility:** The following compatibility condition remains true:

\[
- \int_0^T \int_\Omega \langle \nabla_t; \nu \rangle \cdot \text{div}_x \mathbb{M} \, dx \, dt = \int_0^T \int_\Omega \langle \nabla_t; Dv \rangle : \mathbb{M} \, dx \, dt
\]  

for any \( \mathbb{M} \in C^1(\bar{Q}_T; \mathbb{R}_{\text{sym}}^{d \times d}). \)  

(2.8)

- **Energy Inequality:** The energy inequality

\[
\begin{align*}
\int_\Omega \left( \frac{1}{2} s|v|^2 + P(s) \right) \, dx & + \int_0^T \int_\Omega \langle \nabla_t; S(Dv) \rangle : Dv \, dx \, dt + D(\tau) \\
& \leq \int_\Omega \left( \rho_0 \left( \frac{1}{2} s|v|^2 + P(s) \right) \right) \, dx
\end{align*}
\]  

holds for a.e. \( \tau \in (0, T). \)

- **Generalized Korn–Poincaré inequality:** Let

\[
\mathcal{T}(A) = A + A^T - \frac{2}{d} \text{tr}(A)I.
\]  

(2.10)

For \( \tilde{u} \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^d)), \) the following inequality is true,

\[
\int_0^T \int_\Omega \langle \nabla_t; |v - \tilde{u}|^2 \rangle \, dx \, dt \leq c_P \int_0^T \int_\Omega \langle \nabla_t; |T(Dv) - T(\nabla_x \tilde{u})|^2 \rangle \, dx \, dt.
\]  

(2.11)

**Remark 2.2.** In all the above expressions, \( \nu_{0,x} = \nu_0(x) \) for a.e. \( x \in \Omega. \)

**Remark 2.3.** Since here in this article our goal is to prove the weak–strong uniqueness, instead of considering initial condition as measure \( \nu_0 \) we can consider finite energy initial data. That means \( \langle \nu_0; s \rangle, \langle \nu_0; sv \rangle \) = \( \langle \varrho_0, \varrho u \rangle \) are functions with \( \varrho_0 \geq 0, \langle \varrho u \rangle = 0 \) on the set \( \{ x \in \Omega | \varrho_0(x) = 0 \} \) and

\[
\int_\Omega \left( \frac{1}{2} \left| \frac{\langle \varrho u \rangle_0}{\varrho_0} \right|^2 + P(\varrho_0) \right)(t, \cdot) \, dx < \infty.
\]  

(2.12)

**Remark 2.4.** As a consequence of the above definition, we have

\[
\left[ \int_\Omega \langle \nabla_t; Q(s) \rangle(t, \cdot) \, dx \right]_{t=0}^{t=T} = - \int_0^T \int_\Omega \langle \nabla_t; q(s) \text{tr}(Dv) \rangle \, dx \, dt.
\]  

(2.13)

**Remark 2.5.** From the definition of \( \mathcal{T} \) it follows,

\[
\mathcal{T}(A) : \mathcal{T}(A) = 2\mathcal{T}(A) : A, \ \ A \in \mathbb{R}^{d \times d}.
\]  

(2.14)
3. Existence of Solution

From Feireisl [8], we have existence of finite energy weak solutions for large adiabatic exponent. Hence, this motivates the following approximate problem,

\begin{align}
\partial_t \varrho + \text{div}_x (\varrho \mathbf{u}) &= 0, \\
\partial_t (\varrho \mathbf{u}) + \text{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) + \delta \nabla_x q^\Gamma &= \text{div}_x \mathbf{S}(\nabla_x \mathbf{u}), \\
u|_{\partial \Omega} &= 0,
\end{align}

(3.1)

(3.2)

(3.3)

where \( \delta > 0 \) is a small parameter, \( \Gamma > 1 \) is large enough to ensure the existence of weak solution and \( p \) follows (1.4). Further we assume that for the above mentioned problem, initial conditions \( \{ \varrho_{0\delta}, (\varrho \mathbf{u})_{\delta_0} \} \) belongs to a certain regularity class for which weak solution exists. As an additional assumption we have,

\[
\frac{1}{2} \varrho_{0\delta,0} |\mathbf{u}_{\delta,0}|^2 + P(\varrho_{0\delta,0}) + \frac{\delta}{\Gamma - 1} \varrho_{0\delta,0}^\Gamma \to \frac{1}{2} \left(\frac{\varrho_{0\delta,0}}{\varrho_0}\right)^2 + P(\varrho_0) \text{ in } L^1(\Omega),
\]

(3.4)

when \( \delta \to 0 \).

Thus we obtain

\[
\int_\Omega \left( \frac{1}{2} \varrho_{0\delta,0} |\mathbf{u}_{\delta,0}|^2 + P(\varrho_{0\delta,0}) + \frac{\delta}{\Gamma - 1} \varrho_{0\delta,0}^\Gamma \right) \, dx \leq c,
\]

where \( c \) is independent of \( \delta \).

In addition we consider constitutive relation (1.3), boundary condition (1.5) and initial data satisfying (3.4). Thus, For each \( \delta > 0 \), existence of finite energy weak solution \( \{ \varrho_{\delta}, \mathbf{u}_{\delta} \} \) for (3.1) and (3.2) follows directly from Feireisl [8] for some \( \Gamma \geq \frac{d}{4} \).

Our goal is to verify that the family of weak solutions \( \{ \varrho_{\delta}, \mathbf{u}_{\delta}, \nabla_x \mathbf{u}_{\delta} \} \) for \( \delta > 0 \) generates a renormalized dissipative measure-valued solution (rDMV) as defined in (2.1).

3.1. Apriori Estimates

From the definition of finite energy weak solution as in [8] we have the following estimates,

\[
\sup_{t \in [0, T]} \int_\Omega H(\varrho_{\delta})(t, \cdot) \, dx \leq c,
\]

\[
\sup_{t \in [0, T]} \int_\Omega \varrho_{\delta}|\mathbf{u}_{\delta}|^2(t, \cdot) \, dx \leq c,
\]

\[
\int_0^T \int_\Omega \mathbf{S}(\nabla_x \mathbf{u}_{\delta}) : \nabla_x \mathbf{u}_{\delta} \, dx \, dt \leq c,
\]

\[
\sup_{t \in [0, T]} \frac{\delta}{\Gamma - 1} \int_\Omega \varrho_{0\delta}^\Gamma(t, \cdot) \, dx \leq c.
\]

(3.5)

By Korn inequality and Poincaré inequality we have that \( \mathbf{u}_{\delta} \) is bounded in \( L^2(0, T; W_0^{1,2}(\Omega)) \). Further from (2.1), \( \{ \varrho_{\delta} \} \) is bounded in \( L^\infty(0, T; L^\gamma(\Omega)) \) for \( \gamma > 1 \) and \( \{ \varrho_{\delta} \log \varrho_{\delta} \} \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \) for \( \gamma = 1 \).

From our assumption \( q \in C^1_+ [0, \infty) \), we have \( Q(\varrho) \approx \varrho \). Hence we can conclude that

\[
\left[ \frac{1}{2} \varrho_{\delta} |\mathbf{u}_{\delta}|^2 + P(\varrho_{\delta}) \right](t, \cdot) \in \mathcal{M}(\Omega) \text{ is bounded uniformly for } t \in (0, T),
\]

\[
[\mu |\nabla_x \mathbf{u}_{\delta}|^2 + (\lambda - \frac{\mu}{d}) |\text{div}_x \mathbf{u}_{\delta}|^2] \text{ is bounded in } \mathcal{M}^+(0, T) \times \Omega),
\]

\[
\delta \varrho_{0\delta}^\Gamma(t, \cdot) \in \mathcal{M}^+(\Omega) \text{ is bounded uniformly for } t \in (0, T).
\]

(3.6)
Thus passing to a subsequence, we obtain
\[
\left[ \int_\Omega \left( \frac{1}{2} \rho_\delta |u_\delta|^2 + P(\rho_\delta) \right) (t, \cdot) \to E \text{ weakly-}^* \text{ in } L^\infty_{\text{weak}}(0, T; \mathcal{M}(\Omega)) \right],
\]
\[
\left[ \mu |\nabla U\nabla u_\delta|^2 + (\lambda - \frac{\mu}{d}) \text{div}_x u_\delta |^2 \to \sigma \text{ weakly-}^* \text{ in } M'^{+}([0, T] \times \Omega) \right),
\]
\[
\delta g_\delta^T(t, \cdot) \to \zeta \text{ weakly-}^* \text{ in } L^\infty_{\text{weak}}(0, T; \mathcal{M}^+(\Omega)).
\] (3.7)

Let \( V \) be a Young measure generated by \( \{ \rho_\delta, u_\delta, D_{u_\delta} = \nabla_x u_\delta + \frac{\nabla^T u_\delta}{2} \}_{\delta > 0} \).

Now we introduce two non-negative measures \( E_\infty = E - \langle \mathcal{V}_{t,x}; s | \mathcal{V} |^2 + P(s) \rangle \, dx \), \( \sigma_\infty = \sigma - \langle \mathcal{V}_{t,x}; S(\mathcal{D}_\mathcal{V}) : \mathcal{D}_\mathcal{V} \rangle \, dx \, dt \).

### 3.2. Passage to Limit

#### 3.2.1. Passage to Limit in Energy Inequality.

For the approximate problem (3.1)–(3.3) we have
\[
\left[ \int_\Omega \left( \frac{1}{2} g_\delta |u_\delta|^2 + P(g_\delta) + \frac{\delta}{\Gamma - 1} g_\delta^T \right) (t, \cdot) \, dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega S(\nabla x u_\delta) : \nabla_x u_\delta \, dx \, dt \leq 0.
\] (3.8)

We perform the passage of limit in the energy inequality and, we obtain
\[
\int_\Omega \left( V_{t,x}; \left( \frac{1}{2} s | \mathcal{V} |^2 + P(s) \right) \right) \, dx + \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; S(\mathcal{D}_\mathcal{V}) : \mathcal{D}_\mathcal{V} \rangle \, dx \, dt
\]
\[
+ E_\infty(\tau) |\Omega| + C_\zeta(\tau) |\Omega| + \sigma_\infty([0, \tau] \times \Omega) \leq \int_\Omega \left( V_{0,x}; \left( \frac{1}{2} s | \mathcal{V} |^2 + P(s) \right) \right) \, dx,
\] (3.9)

here \( C > 0 \) is a constant. We consider
\[
\mathcal{D}(\tau) = E_\infty(\tau) |\Omega| + C_\zeta(\tau) |\Omega| + \sigma_\infty([0, \tau] \times \Omega).
\] (3.10)

#### 3.2.2. Passage to Limit in Renormalised Continuity Equation.

We have,
\[
\left[ \int_\Omega (g_\delta + b(\rho_\delta)) \varphi \, dx \right]_{t=0}^{t=\tau}
\]
\[
= \int_0^\tau \int_\Omega \left[ (\rho_\delta + b(\rho_\delta)) \partial_t \varphi + (\rho_\delta + b(\rho_\delta)) \mathcal{V} \cdot \nabla x \varphi + (b(\rho_\delta) - \rho_\delta b'(\rho_\delta)) \text{div}_x u_\delta \varphi \right] \, dx \, dt,
\] (3.11)

where, \( b \in C^1([0, \infty), \exists r_b > 0 \) such that \( b'(x) = 0, \forall x > r_b \). This choice of \( b \) implies that,
\[
(b(\rho_\delta) - \rho_\delta b'(\rho_\delta)) \text{div}_x u_\delta \in L^1((0, T) \times \Omega) \text{ is uniformly bounded}.
\] (3.12)

Hence we obtain,
\[
\int_\Omega \langle \mathcal{V}_{t,x}; s \rangle \psi(\tau, \cdot) \, dx - \int_\Omega \langle \mathcal{V}_{0,x}; s \rangle \psi(0, \cdot) \, dx
\]
\[
= \int_0^\tau \int_\Omega \left[ \langle \mathcal{V}_{t,x}; s \rangle \partial_t \psi + \langle \mathcal{V}_{t,x}; sv \rangle \cdot \nabla x \psi \right] \, dx \, dt,
\] (3.13)

and
\[
\int_\Omega \langle \mathcal{V}_{t,x}; b(s) \rangle \psi(\tau, \cdot) \, dx - \int_\Omega \langle \mathcal{V}_{0,x}; b(s) \rangle \psi(0, \cdot) \, dx
\]
\[
= \int_0^\tau \int_\Omega \left[ \langle \mathcal{V}_{t,x}; b(s) \rangle \partial_t \psi + \langle \mathcal{V}_{t,x}; b(s)v \rangle \cdot \nabla x \psi \right] \, dx \, dt
\]
\[
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (sb'(s) - b(s)) \text{tr}(\mathcal{D}_\mathcal{V}) \rangle \psi \, dx \, dt.
\] (3.14)
3.2.3. Passage to Limit in Momentum Equation. We have,

\[
\left[ \int_{\Omega} g_\delta u_\delta(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx \right]_{t=0}^{t=T} = \int_0^T \int_{\Omega} \left[ g_\delta u_\delta \cdot \partial_t \varphi + ( g_\delta u_\delta \otimes u_\delta) : \nabla_x \varphi + \left( p(\theta_\delta) + \delta \theta_\delta \right) \text{div}_x \varphi - S(\nabla_x u_\delta) : \nabla_x \varphi \right] \, dx \, dt.
\]

Using \( g_\delta u_{\delta,i} u_{\delta,j} \leq \theta_\delta |u_\delta|^2 \), \( p(\theta_\delta) \lesssim P(\theta_\delta) \) and Lemma 2.1 from Feireisl et al. [10] we obtain

\[
\int_\Omega \langle \varphi_t, s \varphi \rangle \, dx - \int_\Omega \langle \varphi_0, s \varphi \rangle \, dx = \int_0^T \int_\Omega \left[ \langle \varphi_t, s \varphi \rangle + \langle \varphi_t, s(\varphi \otimes \varphi) \rangle : \nabla_x \varphi + \langle \varphi_t, s(p) \rangle \text{div}_x \varphi \right] \, dx \, dt - \int_0^T \int_\Omega \langle \varphi_t, S(\nabla \varphi) \rangle : \nabla_x \varphi \, dx \, dt + \int_0^T \langle r^M; \nabla \varphi \rangle \, dt + \int_0^T \langle r^L; \text{div}_x \varphi \rangle \, dt.
\]

Here, \( r^M = \{ r^M_{i,j} \}_{i,j=1}^d \), \( r^M \in L^\infty_{\text{weak}}(0, T; M(\Omega)) \) and \( r^L \in L^\infty_{\text{weak}}(0, T; M(\Omega)) \) such that

\[
| r^M_{t_i} |(\tau) \leq E_{\infty}(\tau) \quad \text{and} \quad | r^L(\tau) | \leq \zeta(\tau).
\]

3.2.4. Verification of Momentum Compatibility. Since \( u_\delta \) is bounded in \( L^2(0, T; W_0^{1,2}(\Omega)) \), in this case we can check the relation easily.

3.2.5. Verification of Generalized Korn–Poincaré Inequality. It can be proved along similar lines as in Březina et al. [3].

3.3. Main Theorem

We conclude this section with the following theorem,

**Theorem 3.1.** Suppose \( \Omega \) is a regular bounded domain in \( \mathbb{R}^d \) with \( d = 1, 2, 3 \) and suppose the pressure satisfies (1.4). If \( (\varrho_0, (\varrho u)_0) \) satisfies (2.12), then there exists a renormalized dissipative measure-valued solution \( (rDMV) \) as defined in Definition 2.1 with initial data \( V_0 = \delta_{\{\varrho_0, (\varrho u)_0\}} \).

4. Relative Energy and Weak–Strong Uniqueness

Relative energy was first introduced by Dafermos [5] in the context of hyperbolic conservation laws. In the context of compressible Navier–Stokes it had been introduced by Feireisl et al. [11,14]. Motivated from the relative energy mentioned in those articles for weak solutions to barotropic Navier–Stokes system, i.e.

\[
E(t) = E(\varrho, u|r, U)(t) := \int_\Omega \frac{1}{2} \varrho |u - U|^2 + (H(\varrho) - H(r) - H'(r)(\varrho - r))(t, \cdot) \, dx,
\]

we define,

\[
E_{mv}(\varrho, u|r, U)(t) := \int_\Omega \left( \langle \varphi_t, \frac{1}{2} s |v - U|^2 + H(s) - H(r) - H'(r)(s - r) \rangle \right) \, dx,
\]

where \( r, U \) are smooth test functions and \( \{\varrho, u\} \) in (4.1) is finite energy weak solution of (1.1)–(1.3), while in (4.2) \( V \) is a solution as defined in Definition 2.1.
Lemma 4.1. Let $(\mathcal{V}, \mathcal{D})$ be a renormalized measure-valued solution of (1.1)–(1.3) for initial data $\mathcal{V}_0$ by Definition 2.1. Then for smooth compactly supported $r, U$, we have the following relative energy inequality:

$$
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \left( \langle \mathcal{V}_{\tau,x}; s \mathbb{D} \mathcal{V} \rangle : \mathbb{D} \mathcal{V} \rangle \right) \ dx \ dt - \int_0^\tau \int_\Omega \left( \langle \mathcal{V}_{\tau,x}; s \mathbb{D} \mathcal{V} \rangle : \nabla_x U \right) \ dx \ dt + D(\tau)
$$

\[ \leq \int_\Omega \left[ \langle \mathcal{V}_{0,x}; \frac{1}{2} |v - U_0|^2 + H(s) - H(r_0) - H'(r_0)(s - r_0) \rangle \right] \ dx 
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau,x}; s \rangle \cdot \partial_t U \ dx \ dt 
- \int_0^\tau \int_\Omega \left[ (\mathcal{V}_{\tau,x}; s v) : \nabla_x U + \langle \mathcal{V}_{\tau,x}; h(s) \rangle \right] \ dx \ dt 
+ \int_0^\tau \int_\Omega \left[ (\mathcal{V}_{\tau,x}; s U) \cdot \partial_t U + \langle \mathcal{V}_{\tau,x}; s v \rangle \cdot (U \cdot \nabla_x U) \right] \ dx \ dt 
+ \int_0^\tau \int_\Omega \left[ (\mathcal{V}_{\tau,x}; s (1 - \frac{s}{r_0})) h'(r) \partial_t r - \langle \mathcal{V}_{\tau,x}; s v \rangle \cdot \frac{h'(r)}{r} \nabla_x r \right] \ dx \ dt 
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau,x}; q(s) \rangle \ n \ dx \ dt + \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau,x}; q(s) tr(\mathbb{D} \mathcal{V}) \rangle \ dx \ dt
- \int_0^\tau \langle r^M ; \nabla_x U \rangle \ dx ,
\]  

4.3

Here, $U_0(x) = U(0, x)$ and $r_0(x) = r(0, x)$ for $x \in \Omega$.

Proof. Using (2.2) we have

$$
\mathcal{E}_{mv}(\tau) = \int_\Omega \left( \langle \mathcal{V}_{\tau,x}; \frac{1}{2} s |v|^2 + H(s) \right) \ dx - \int_\Omega \langle \mathcal{V}_{\tau,x}; s v \rangle \cdot U \ dx 
+ \int_\Omega \frac{1}{2} \langle \mathcal{V}_{\tau,x}; s |U|^2 \rangle \ dx - \int_\Omega \langle \mathcal{V}_{\tau,x}; s H'(r) \rangle \ dx + \int_\Omega \ h(r) \ dx = \Sigma_{i=1}^5 K_i .
$$  

Now we look for the terms $K_i$ for $i = 1(1)5$. We have some bound for $K_1$ from (2.9). To estimate $K_2$ we use (2.7) and for $K_3, K_4$ we use (2.4). We rewrite $K_5 = \int_0^\tau \int_\Omega \partial_t h(r) \ dx \ dt + \int_\Omega \ h(r(0, \cdot)) \ dx$. Following Feireisl et al. [11], we obtain the desired result.

4.1. Main Theorem

Now we state the main theorem.

Theorem 4.2. Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a smooth bounded domain. Suppose the pressure $p$ satisfies (1.4). Let $\{\mathcal{V}_{\tau,x}, \mathcal{D}\}$ be a dissipative measure-valued solution to the barotropic Navier–Stokes system (1.1)–(1.3) in $(0, T) \times \Omega$, with initial state represented by $\mathcal{V}_0$, as mentioned in Definition 2.1. Let $\{r, U\}$ be a strong solution to (1.1)–(1.3) in $(0, T) \times \Omega$ with initial data $\{r_0, U_0\}$ satisfying $r_0 > 0$ in $\Omega$. We assume that the solution belongs to the class

$$
r, \nabla_x r, U, \nabla_x U \in C([0, T] \times \Omega), \partial_t U \in L^2(0, T; C(\bar{\Omega}; \mathbb{R}^d)), r > 0, U |_{\partial \Omega} = 0 .
$$

Then there is a constant $\Lambda = \Lambda(T)$, depending only on the norms of $r$, $r^{-1}$, $U$, and $\xi$ in the aforementioned spaces, such that

$$
\int_\Omega \left[ \langle \mathcal{V}_{\tau,x}; \frac{1}{2} s |v - U|^2 + H(s) - H(r) - H'(r)(s - r) \rangle \right] \ dx + D(\tau)
\leq \Lambda(T) \int_\Omega \left[ \langle \mathcal{V}_{0,x}; \frac{1}{2} s |v - U_0(x)|^2 + H(s) - H(r_0(x)) - H'(r_0(x))(s - r_0(x)) \rangle \right] \ dx ,
$$

4.6
for a.e. \( \tau \in (0, T) \), \( U_0(x) = U(0, x) \) and \( r_0(x) = r(0, x) \) for \( x \in \Omega \). In particular, if the initial states coincide, i.e.

\[
\mathcal{V}_0.x = \delta_{\{r_0(x), U_0(x)\}}, \text{ for a.e. } x \in \Omega
\]

then \( \mathcal{D} = 0 \), and

\[
\mathcal{V}_{\tau, x} = \delta_{\{r(\tau, x), U(\tau, x), \nabla_x U(\tau, x)\}} \text{ for a.e. } \tau \in (0, T) \text{ for a.e. } x \in \Omega.
\]

From now on our goal is to prove the aforementioned theorem. We assume that \( \{r, U\} \) solves (1.1)–(1.3) and belongs to the regularity class (4.5). Further to simplify the calculation, we assume

\[
\partial_t U \in C(\bar{Q}_{T}; R^d).
\]  

(4.7)

Then we rewrite (4.3) as,

\[
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; \mathbb{S}(\nabla \mathbb{V}) : \mathbb{D} \mathbb{V} \rangle \, dx \, dt + \mathcal{D}(\tau)
\]

\[
- \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; \mathbb{S}(\nabla \mathbb{V}) \rangle : \nabla_x U \, dx \, dt
\]

\[
- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x U) : \langle \mathcal{V}_{\tau, x}; (\mathbb{D} \mathbb{V} - \nabla_x U) \rangle \, dx \, dt
\]

\[
\leq \int_\Omega \left[ \left\langle \mathcal{V}_{0, x}; \frac{1}{2} |\mathbb{V} - U_0|^2 + H(s) - H(r_0) - H'(r_0)(s - r_0) \right\rangle \right] \, dx
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (s(\mathbb{V} - U) \cdot \nabla_x U) \cdot (U - \mathbb{V}) \rangle \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (s - r)(U - \mathbb{V}) \rangle \cdot \frac{1}{r} (\text{div}_x \mathbb{S}(\nabla_x U) - \nabla_x q(r)) \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (-h(s) + h(r) + h'(r)(s - r)) \rangle \text{div}_x U \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (q(s) - q(r)) (\text{tr}(\mathbb{D} \mathbb{V}) - \text{div}_x U) \rangle \, dx \, dt
\]

\[
+ \|U\|_{C^1([0, T] \times \Omega; R^N)} \int_0^\tau \xi(t) \mathcal{D}(t) \, dt.
\]  

(4.8)

We have,

\[
\mathcal{E}_{mv}(\tau) + \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; \mathbb{S}(\nabla \mathbb{V} - \nabla_x U) : \mathbb{D} \mathbb{V} - \nabla_x U \rangle \, dx \, dt + \mathcal{D}(\tau)
\]

\[
\leq \int_\Omega \left[ \left\langle \mathcal{V}_{0, x}; \frac{1}{2} |\mathbb{V} - U_0|^2 + H(s) - H(r_0) - H'(r_0)(s - r_0) \right\rangle \right] \, dx
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (s(\mathbb{V} - U) \cdot \nabla_x U) \cdot (U - \mathbb{V}) \rangle \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (s - r)(U - \mathbb{V}) \rangle \cdot \frac{1}{r} (\text{div}_x \mathbb{S}(\nabla_x U) - \nabla_x q(r)) \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (-h(s) + h(r) + h'(r)(s - r)) \rangle \text{div}_x U \, dx \, dt
\]

\[
+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{\tau, x}; (q(s) - q(r)) (\text{tr}(\mathbb{D} \mathbb{V}) - \text{div}_x U) \rangle \, dx \, dt
\]

\[
+ \|U\|_{C^1([0, T] \times \Omega; R^N)} \int_0^\tau \xi(t) \mathcal{D}(t) \, dt.
\]  

(4.9)
Using the relation between $S$ and $T$, we obtain

$$
E_{mv}(r) + \frac{\mu}{4} \int_0^T \int_\Omega \langle \nu_{t,x}; \mathbb{T}(\nabla \nu - \nabla U) \rangle \, dx \, dt \\
+ \lambda \int_0^T \int_\Omega \langle \nu_{t,x}; |\text{tr}(\nabla \nu) - \text{div} \nu| \rangle \, dx \, dt + D(\tau) \\
\leq \int_0^T \left[ \langle \nu_{0,x}; \frac{1}{2} |\nu - U_0|^2 + H(s) - H(r_0) - H'(r_0)(s - r_0) \rangle \right] \, dx \\
+ \int_0^T \int_\Omega \langle \nu_{t,x}; (s(\nu - U) \cdot \nabla x)U \cdot (U - v) \rangle \, dx \, dt \\
+ \int_0^T \int_\Omega \langle \nu_{t,x}; (s-r)(U - v) \rangle \cdot \frac{1}{r} (\text{div} x S(\nabla_x U) - \nabla x q(r)) \, dx \, dt \\
+ \int_0^T \int_\Omega \langle \nu_{t,x}; (s-r)(U - v) \rangle \cdot (\text{tr}(\nabla \nu) - \text{div} \nu) \, dx \, dt \\
+ \Vert U \Vert_{C^1([0,T] \times \Omega; \mathbb{R}^N)} \int_0^T \xi(t) D(t) \, dt = \Sigma_1^0 I_i. 
$$

(4.10)

From pressure and density relation (1.4), we have

**Lemma 4.3.** Suppose $H$ is defined as in (2.1) and $r$ lies on a compact subset $[r_1, r_2]$ of $(0, \infty)$. Then we have,

$$
H(q) - H(r) - H'(r)(q - r) \geq c(r) \begin{cases} 
(q - r)^2 & \text{for } r_1 \leq q \leq r_2, \\
(1 + q^\gamma) & \text{otherwise}, 
\end{cases} 
$$

(4.11)

where $c(r)$ is uniformly bounded for values of $r$ belonging to compact subsets of $(0, \infty)$.

As a direct consequence of above we obtain

**Lemma 4.4.** For $q \geq 0$,

$$
|h(q) - h(r) - h'(r)(q - r)| \leq C(r)(H(q) - H(r) - H'(r)(q - r)),
$$

(4.12)

where $C(r)$ is uniformly bounded for values of $r$ belonging to compact subsets of $(0, \infty)$.

**Remark 4.5.** Since $\{r, U\}$ are strong solution, we choose $p_1$ and $p_2$ such that $p_1 = \inf_{(t,x) \in (0,T)} r(t,x) > 0$ and $p_2 = 2 \sup_{(t,x) \in (0,T)}$ satisfying $1 + q^\gamma = \max \{q, q^2\}, \forall q \geq p_2$.

Let supp$(q) = [q_1, q_2]$. Finally we consider $r_1$ and $r_2$ such that

$$
0 < r_1 < \min \left\{ \frac{q_1}{2}, \frac{p_1}{2} \right\} \quad \text{and} \quad r_2 > \max \{2q_2, 2 \times \sup r\}.
$$

**Remark 4.6.** From now on we use the generic constant $c = c(r, U, q)$.

**Remark 4.7.** Now we introduce a function $\psi \in C^\infty_c(0, \infty)$, $0 \leq \psi \leq 1$, such that

$$
\psi(s) = 1 \quad \text{for } s \in (r_1, r_2).
$$

Next we will estimate $I_i$ for $i = 2, 3, 4, 5, 6$ of (4.10).

- **Remainder term $I_2$**:
  
  We have

$$
|I_2| \leq \Vert U \Vert_{C^1([0,T] \times \Omega; \mathbb{R}^N)} \int_0^T E_{mv}(t) \, dt.
$$

(4.13)
• **Remainder term $I_4$:**

Similarly using lemma we get

$$|I_4| \leq C \int_0^T \mathcal{E}_{mv}(t) \, dt .$$  \hfill (4.14)

• **Remainder term $I_3$:**

We rewrite

$$\langle \mathcal{V}_{t,x}; (s-r)(U-v) \rangle = \langle \mathcal{V}_{t,x}; \psi(s)(s-r)(U-v) \rangle + \langle \mathcal{V}_{t,x}; (1-\psi(s))(s-r)(U-v) \rangle.$$ 

Consequently we obtain

$$\langle \mathcal{V}_{t,x}; \psi(s)(s-r)(U-v) \rangle \leq \frac{1}{2} \left\langle \mathcal{V}_{t,x}; \frac{\psi^2(s)}{\sqrt{s}}(s-r)^2 \right\rangle + \frac{1}{2} \left\langle \mathcal{V}_{t,x}; \frac{\psi^2(s)}{\sqrt{s}}|U-v|^2 \right\rangle .$$  \hfill (4.15)

Now using that $\psi$ is compactly supported in $(0, \infty)$ and Lemma 4.3 we conclude that,

$$\int_0^T \int_{\Omega} \langle \mathcal{V}_{t,x}; \psi(s)(s-r)(U-v) \rangle \cdot \frac{1}{r} \left\langle \text{div}_x \mathcal{S}(\nabla_x U) - \nabla_x q(r) \right\rangle \, dx \, dt$$

$$\leq \left\| \frac{1}{r} \left( \text{div}_x \mathcal{S}(\nabla_x U) - \nabla_x q(r) \right) \right\|_{C([0,T] \times \Omega; R^N)} \int_0^T \mathcal{E}_{mv}(t) \, dt .$$  \hfill (4.16)

We rewrite $1-\psi(s) = w_1(s) + w_2(s)$, where $\text{supp}(w_1) \subset [0, r_1)$ and $\text{supp}(w_2) \subset (r_2, \infty)$,

$$\langle \mathcal{V}_{t,x}; (1-\psi(s))(s-r)(U-v) \rangle = \langle \mathcal{V}_{t,x}; (w_1(s) + w_2(s))(s-r)(U-v) \rangle .$$

For $\delta > 0$ we obtain,

$$\langle \mathcal{V}_{t,x}; w_1(s)(s-r)(U-v) \rangle \leq c(\delta) \langle \mathcal{V}_{t,x}; w_1^2(s)(s-r)^2 \rangle + \delta \langle \mathcal{V}_{t,x}; |U-v|^2 \rangle .$$

The first term on the right hand side is bounded by $\mathcal{E}_{mv}$ while the second term can be absorbed in the left hand side of (4.8) by virtue of generalised Korn–Poincaré inequality as in (2.11). Then we have,

$$\int_0^T \int_{\Omega} \langle \mathcal{V}_{t,x}; w_1(s) \rangle \, dx \, dt$$

$$\leq C \int_0^T \mathcal{E}_{mv}(t) \, dt + \delta c \int_0^T \int_{\Omega} \langle \mathcal{V}_{t,x}; |\text{div}_x \mathcal{S}(\nabla_x U) - \nabla_x q(r) \rangle \, dx \, dt .$$

We know $w_2(s) > 0$ and, $|s-r| \leq 2s$ if $s > 2r_2$. Now using standard inequality of arithmetic and geometric means for real numbers we obtain

$$\langle \mathcal{V}_{t,x}; w_2(s)(s-r)(U-v) \rangle \leq c \langle \mathcal{V}_{t,x}; w_2(s)(s+s)|U-v|^2 \rangle .$$

In this inequality both integrals can be controlled by $\mathcal{E}_{mv}$.

We take $\delta$ small enough and combine all the above terms to obtain,

$$|I_3| \leq c \int_0^T \mathcal{E}_{mv}(t) \, dt + \delta c \int_0^T \int_{\Omega} \langle \mathcal{V}_{t,x}; |\text{div}_x \mathcal{S}(\nabla_x U) - \nabla_x q(r) \rangle \, dx \, dt .$$  \hfill (4.17)

• **Remainder term $I_5$:**

From our choice of $q$ and $\psi$ we have,

$$q(s) - q(r) = \psi(s)(q(s) - q(r)) - (1-\psi(s))q(r).$$

Since $q$ is compactly supported $C^\infty$ function so we have,

$$|q(s) - q(r)| \leq c(\psi(s)|s-r| + (1-\psi(s))).$$
For $\epsilon > 0$ and $a, b \in \mathbb{R}$ we obtain $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$. Thus we conclude
\[
\langle \mathcal{V}_{t,x}; (q(s) - q(r)) (\text{tr}(Dv) - \text{div}_x U) \rangle
\]
\[
\leq \frac{c}{4\epsilon} \langle \mathcal{V}_{t,x}; (\psi(s)(s - r)^2 + (1 - \psi(s))) \rangle + \epsilon \langle \mathcal{V}_{t,x}; |\text{tr}(Dv) - \text{div}_x U|^2 \rangle.
\]
Further using Lemma 4.3 we obtain
\[
|\mathcal{I}_3| \leq c \int_{0}^{r} \mathcal{E}_{mv}(t) \, dt + \frac{\lambda}{2} \int_{0}^{r} \int_{\Omega} \langle \mathcal{V}_{t,x}; |\text{tr}(Dv) - \text{div}_x U|^2 \rangle \, dx \, dt.
\]

**Remainder term $\mathcal{I}_6$:**
From our Definition 2.1 we conclude
\[
|\mathcal{I}_6| \leq c \int_{0}^{r} \xi(t) \mathcal{D}(t) \, dt.
\]

4.1. Proof of the Theorem 4.2. Considering the above discussion, additional assumption (4.7) and combining all estimates of $\mathcal{I}_i$ for $i = 2, 3, 4, 5, 6$, we have
\[
\mathcal{E}_{mv}(\tau) + \frac{H}{8} \int_{0}^{r} \int_{\Omega} \langle \mathcal{V}_{t,x}; T(Dv - \nabla_x U) : T(Dv - \nabla_x U) \rangle \, dx \, dt
\]
\[
+ \frac{\lambda}{2} \int_{0}^{r} \int_{\Omega} \langle \mathcal{V}_{t,x}; |\text{tr}(Dv) - \text{div}_x U|^2 \rangle \, dx \, dt + \mathcal{D}(\tau)
\]
\[
\leq \int_{\Omega} \left[ \langle \mathcal{V}_{0,x}; \frac{1}{2} |v - U_{0}|^2 + H(s) - H(r_0) - H'(r_0)(s - r_0) \rangle \right] \, dx
\]
\[
+ C(r, U, q) \int_{0}^{r} \mathcal{E}_{mv}(t) \, dt + \int_{0}^{r} \xi(t) \mathcal{D}(t) \, dt.
\]
Now applying Grönwall’s lemma, we conclude
\[
\int_{\Omega} \left[ \langle \mathcal{V}_{\tau,x}; \frac{1}{2} |v - U|^2 + H(s) - H(r) - H'(r)(s - r) \rangle \right] \, dx + \mathcal{D}(\tau)
\]
\[
\leq \Lambda(T) \int_{\Omega} \left[ \langle \mathcal{V}_{0,x}; \frac{1}{2} |v - U_{0}(x)|^2 + H(s) - H(r_{0}(x)) - H'(r_{0}(x))(s - r_{0}(x)) \rangle \right] \, dx,
\]
for a.e. $\tau \in [0, T]$.

**Remark 4.8.** For simplicity of the proof we assume (4.7). If we stick to only (4.5), then we have $\int_{0}^{r} \eta(t, U, q) (t) \mathcal{E}_{mv}(t) \, dt$, where $\eta(t, U, q) \in L^1(0, T)$ instead of the term $c(r, U, q) \int_{0}^{r} \mathcal{E}_{mv}(t) \, dt$ in (4.20).

5. Concluding Remarks

Instead of considering $q \in C^1_{0}(0, \infty)$ if we assume $q \in C^1$ with $q'(\varrho) \approx a \varrho$ as $\varrho \to \infty$, then we can obtain weak (measure-valued)–strong uniqueness principle if $\alpha + 1 \leq \frac{2}{2}$. Even if $q$ is a globally Lipschitz function in $[0, \infty)$ we have the same principle for $\gamma \geq 2$. Further existence of such an rDMV solution can be generated by limit of weak solutions of the approximate problem (3.1) and (3.2) whose existence can be guaranteed by the work of Bresch and Jabin [2].
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