SQUARE FUNCTION ESTIMATES FOR
THE BOCHNER-RIESZ MEANS

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Abstract. We consider the square function (known as Stein’s square function) estimate associated with the Bochner-Riesz means. The previously known range of sharp estimate is improved. Our results are based on vector valued extensions of Bennett-Carbery-Tao’s multilinear (adjoint) restriction estimate and adaptation of an induction argument due to Bourgain-Guth. Unlike the previous work by Bourgain-Guth on $L^p$ boundedness of the Bochner-Riesz means in which oscillatory operators associated to the kernel were studied, we take more direct approach by working on Fourier transform side. This enables us to obtain the correct order of smoothing which is essential for obtaining the sharp estimates for the square functions.

1. Introduction

We consider the Bochner-Riesz mean of order $\alpha$ which is defined by

$$\hat{R}_t^\alpha f(\xi) = \left(1 - \frac{|\xi|^2}{t^2}\right)^{\alpha} \hat{f}(\xi), \quad t > 0, \quad \xi \in \mathbb{R}^d, \quad d \geq 2.$$  

Let $1 \leq p \leq \infty$. The Bochner-Riesz conjecture is that the estimate

$$(1) \quad \|R_t^\alpha f\|_p \leq C\|f\|_p$$

holds (except $p = 2$) if and only if

$$(2) \quad \alpha > \alpha(p) = \max\left(d\left|\frac{1}{2} - \frac{1}{p}\right| - \frac{1}{2}, 0\right).$$

The Bochner-Riesz mean which is a kind of summability method has been studied in order to understand convergence properties of Fourier series and integrals. In fact, for $1 \leq p < \infty$, $L^p$ boundedness of $R_t^\alpha f$ implies $R_t^\alpha f \to f$ in $L^p$ as $t \to \infty$. The necessary condition (2) has been known for a long time ([25], [49, p. 389]).

When $d = 2$, the conjecture was verified by Carleson and Sjölin [19] (also see [25]). In higher dimensions $d \geq 3$ the problem is still open and partial results are known. The conjecture was shown to be true for $\max(p, p') \geq 2(d + 1)/(d - 1)$ by the argument due to Stein [24] (also see [19, Ch. 9]) and the sharp $L^2 \to L^{2(d+1)/(d-1)}$ restriction estimate (the Stein-Tomas theorem) for the sphere [60, 48]. It was Bourgain [7, 9] who first made progress beyond this result when $d = 3$. Since then, subsequent progress had been paralleled with those of restriction problem. Bilinear or multilinear generalizations under transversality assumptions have turned out to be most effective and fruitful tools. These results have propelled progresses in this area and there is a large body of literature on restriction estimates and related problems. See [58, 56, 63, 55, 39, 31, 32, 33] for bilinear restriction estimates and related results, [44, 43, 41, 40, 39, 31, 32, 33] for multilinear restriction estimates and their applications, and [25, 11, 29, 23, 63, 41] (also, references therein) for most recent developments related to polynomial partitioning method.
Concerning improved $L^p$ boundedness of the Bochner-Riesz means in higher dimensions, the sharp $L^p$ bounds for the Bochner-Riesz operator on the range $\max(p, p') \geq 2(d+2)/d$ were established by the author [31] making use of the sharp bilinear restriction estimate due to Tao [55]. When $d \geq 5$ further progress was recently made by Bourgain and Guth [13]. They improved the range of the sharp (linear) estimates for the oscillatory integral operators of Carleson-Sjölin type of which phases additionally satisfy elliptic condition (see [48], [8], [32] for earlier results) by using the multilinear estimates for oscillatory integral operators due to Bennett, Carbery and Tao [6] and a factorization theorem. Also see [19], [32], [35] and [49, Ch 11] for the relation between the Bochner-Riesz problem and the oscillatory integral operators of Carleson-Sjölin type.

The following is currently the best known result for the sharp $L^p$ boundedness of the Bochner-Riesz operator.

**Theorem 1.1** ([19] 31 13). Let $d \geq 2$, $p \in [1, \infty]$, and $p_\alpha$ be defined by

$$p_\alpha = p_\alpha(d) = 2 + \frac{12}{4d-3-k} \quad \text{if } d \equiv k \pmod{3}, \; k = -1, 0, 1.$$ 

If $\max(p, p') \geq p_\alpha$, then (1) holds for $\alpha > \alpha(p)$.

There are also results concerning the endpoint estimates at the critical exponent $\alpha = \alpha(p)$ (for example, see [22], [21], 45, [51]). It was shown by Tao [52] that the sharp $L^p$ bounds of $R_\alpha^p$ for $1 < p < p_\alpha < 2d/(d-1)$ imply the weak type bounds of $R_\alpha^p$ for $1 < p < p_\alpha$. We refer interested readers to [39] and references therein for variants and related problems.

**Square function estimate.** We now consider the square function $G^\alpha f$ which is defined by

$$G^\alpha f(x) = \left( \int_0^\infty \left| \frac{\partial}{\partial t} R_\alpha^p f(x) \right|^2 t dt \right)^{1/2}.$$ 

It was introduced by Stein [47] to study almost everywhere summability of Fourier series. Due to derivative in $t$ the square function behaves as if it is a multiplier of order $(\alpha-1)$ and the derivative $\partial/\partial t$ makes $L^p$ estimate possible by mitigating bad behavior near the origin. In this paper we are concerned with the estimate

$$\| G^\alpha f \|_p \leq C \| f \|_p.$$ 

The $L^p$ estimate for the square function has various consequences and applications. First of all, it is related to smoothing estimates for solutions to dispersive equations associated to radial symbols such as wave and Schrödinger operators. See [34], [35] for the details (also, Remark 35). The sharp square function estimate implies the sharp maximal bounds for Bochner-Riesz means, which is to be discussed below in connection to pointwise convergence. It also gives $L^p$ and maximal $L^p$ boundedness of general radial Fourier multipliers, especially the sharp $L^p$ boundedness result of Hörmander-Mikhlin type (see, Corollary 13 below, [10], [15] and [34]).

For $1 < p \leq 2$, the inequality (4) is well understood. In this range of $p$, $G^\alpha$ is bounded on $L^p$ if and only if $\alpha > d(1/p - 1/2) + 1/2$ (see [50] and [36]). Sufficiency can be shown by using the vector valued Calderón-Zygmund theory. In contrast with the case $1 < p \leq 2$, if $p > 2$, due to smoothing effect resulting from averaging in time the problem has more interesting features and may be considered as a vector valued extension of the Bochner-Riesz conjecture in that its sharp $L^p$ bound also implies that of Bochner-Riesz operator. The condition $\alpha > \max\{1/2, d(1/2 - 1/p)\}$ is known to be necessary for (4) (see, for example [36]) and it is natural to conjecture that this is also sufficient for $p > 2$. This conjecture in two dimensions was proven by Carbery [14], and in higher dimensions, $d \geq 3$, sharp estimates for $p > 2(d+1)/(d-1)$ were obtained by Christ [20].

*For the sharp bound for $\max(p, p') \geq p_\ast$, the numerology are related as follows: (bilinear) $p_\ast = 2+4/d$; (multilinear) $p_\ast = 3+3/d + O(d^{-2})$; (conjecture) $p_\ast = 2+2/d + O(d^{-2})$.\]
and Seeger [44] and it was later improved to the range of $p \geq 2(d + 2)/d$ by the author, Rogers, and Seeger [34]. There are also endpoint estimates at the critical exponent $\alpha = d/2 - d/p$ and weaker $L^{p, 2} \rightarrow L^{p}$ endpoint estimates were obtained in [50] for $2(d + 1)/(d - 1) < p < \infty$.

There are two notable approaches for the study of Bochner-Riesz problem. The one which may be called the spatial side approach is to prove the sharp estimates for the oscillatory integral operators of Carleson-Sjölin type [19, 30, 18]. These operators are natural variable coefficient generalizations of the adjoint restriction operators [8, 52, 61] for hypersurfaces with nonvanishing Gaussian curvature such as spheres, paraboloids, and hyperboloids. The other which we may call frequency side approach is more related to Fourier transform side, based on suitable decomposition in frequency side and orthogonality between the decomposed pieces [26, 14, 20, 21, 45, 52, 31]. As has been demonstrated in related works the latter approach makes it possible to carry out finer analysis and to obtain refined results such as the sharp maximal bounds, square function estimates, and various endpoint estimates.

The recently improved bound for the Bochner-Riesz operator in [13] was obtained from the sharp estimate for the oscillatory integral operators of Carleson-Sjölin type with additional elliptic assumption. However, this approach doesn’t seem appropriate for the study of the square function. Especially, there is an obvious difficulty when one tries to make use of disjointness of the singularity of Fourier transform of $R^t f$ which occurs as $t$ varies (for example, see (76)). This is where comes in the extra smoothing of order 1/2 for the square function estimate, which is most important for the sharp estimates for $G^a f$ (14, 20, 31, 34). This kind of smoothing can be seen clearly in the Fourier transforms of Bochner-Riesz means but is not easy to exploit in the oscillatory kernel side. As is already known [8, 61, 32, 13], the behavior of the oscillatory integral operators of Carleson-Sjölin type are more subtle and generally considered to be difficult to analyze when compared to their constant coefficient counterparts, the adjoint restriction operators. So, we take frequency side approach in which we directly handle the associated multiplier by working in frequency space rather than dealing with the oscillatory integral operator given by the kernel of the Bochner Riesz operator.

In this paper, we obtain the sharp square function estimates which are new when $d \geq 9$.

**Theorem 1.2.** Let us set $p_\alpha = p_\alpha(d)$ by

$$p_\alpha = 2 + \frac{12}{4d - 6 - k}, \quad d \equiv k \pmod{3}, \quad k = 0, 1, 2.$$  

Then, if $p \geq \min(p_\alpha, 2 \frac{(d + 2)}{d})$ and $\alpha > d/2 - d/p$, the estimate [4] holds.

The range here does not match with that of Theorem [1, 2]. This results from additional time average which increases the number of decomposed frequency pieces. (See Section 3.6)

**Maximal estimate and pointwise convergence.** A straightforward consequence of the estimate [4] is the maximal estimate

$$\| \sup_{t > 0} |R^t f| \|_p \leq C \| f \|_p$$

for $\alpha > \alpha(p)$, which follows from Sobolev imbedding and [4]. Hence, Theorem [1, 2] yields the sharp maximal bounds for $p \geq p_\alpha(d)$. When $p \geq 2$, it has been conjectured that [2] holds as long as [2] is satisfied. The sharp $L^2$ bound goes back to Stein [47]. The conjecture in $\mathbb{R}^2$ and the sharp bounds for $p > 2(d + 1)/(d - 1)$, $d \geq 3$ were verified by the square function estimates [26, 44]. The bounds were later improved to the range $p > 2(d + 2)/d$ by the author [31] using $L^p \rightarrow L^p(L^4)$ estimate. The inequality [G] has been studied in connection with almost everywhere convergence of Bochner-Riesz means. However, the problem of showing $R^t f \rightarrow f$ a.e. for $f \in L^p$, $p > 2$, $\alpha > \alpha(p)$ was settled by Carbery, Rubio de Francia and Vega [17]. Their result relies on weighted $L^2$ estimates. There are also results on pointwise convergence at the critical $\alpha = \alpha(p)$. See [57, 1].
When $1 < p < 2$, by Stein’s maximal theorem almost everywhere convergence of $R^d_a f \to f$ for $f \in L^p$ is equivalent to $L^p \to L^{p,\infty}$ estimate for the maximal operator and it was shown by Tao [62] that the stronger condition $\alpha \geq (2d-1)/(2p) - d/2$ is necessary for [14]. Except for $d = 2$ (54) little is known beyond the classical result which follows from interpolation between $L^2 (\alpha > 0)$ and $L^1 (\alpha > (d-1)/2)$ estimates.

**Radial multiplier.** Let $m$ be a function defined on $\mathbb{R}_+$. Combining the inequality due to Carbery, Gasper and Trebels [16] and Theorem 1.2 we obtain the following $L^p$ boundedness result of Hörmander-Mikhlin type, which is sharp in that the regularity assumption can not be improved. A similar result for the maximal function $f \to \sup_{t>0} |\mathcal{F}^{-1}(m(t) \cdot |\hat{f}|)|$ is also possible thanks to the inequality due to Carbery (see [15]).

**Corollary 1.3.** Let $d \geq 2$, and $\varphi$ be a nontrivial smooth function with compact support contained in $(0,\infty)$. If $\min(p, \frac{2(d+2)}{d}) \leq \max(p, p') < \infty$ and $\alpha > d|1/p - 1/2|$, then

$$\left\| \mathcal{F}^{-1}[m(|\cdot|)\hat{f}] \right\|_p \lesssim \sup_{t>0} \|\varphi m(t)\|_{L^\infty(\mathbb{R})} \|f\|_p.$$  

**About the paper.** In section 2, by working in frequency side we provide an alternative proof of Theorem 1.1. Although, this doesn’t give improvement over the current range, we include this because it has some new consequences, clarifies several issues, which were not clearly presented in [13], and provides preparation for Section 3 in which we work in vector valued setting. The proof in [13] is sketchy and doesn’t look readily accessible. Also the heuristic that a function with Fourier support in a ball of radius $\sigma$ behaves as if it is constant on balls of radius $1/\sigma$ is now widely accepted and has important role in the induction argument but it doesn’t seem justified at high level of rigor. We provide rigorous argument by making use of Fourier series (see Lemma 2.13 and Lemma 3.14). Another problem of the induction argument is that the primary object (the associated surfaces or phase functions) changes in the course of induction. However, these issues are not properly addressed before in literature. We handle this matter by introducing a stronger induction assumption (see Remark 2.4) and carefully handling stability of various estimates. We also use a different type multilinear decomposition which is more systematic, easier and efficient for dealing with multiplier operators (see Section 2.5, especially the discussion at the beginning of Section 2.5).

Section 3 is very much built on the frequency side analysis in Section 2 as it may be regarded a vector valued extension of Section 2. Consequently, the structure of Section 3 is similar to that of Section 2 and some of the arguments commonly work in both sections. In such cases we try to minimize repetition while keeping readability as much as possible. We first obtain vector valued extensions of multilinear estimates (Proposition 3.6, Proposition 3.10) which serve as basic estimates for the sharp square function estimate. Then, to derive linear estimate (Theorem 1.2) we adapt the frequency side approach in Section 2 to the vector valued setting and prove our main theorem.

Finally, oscillatory integral approach has its own limit to prove Bochner-Riesz conjecture. As is now well known (8 [61] 32 [13]), the sharp $L^p-L^q$ estimates for the oscillatory operators of Carleson-Sjölin type fail for $q < q_0$, $q_0 > \frac{2d}{d-1}$ even under the elliptic condition on the phase [61] 32 [13]. Fourier transform side approach may help further development in a different direction and thanks to its flexibility may have applications to related problems.

**Notations.** The following is a list of notation we frequently use for the rest of the paper.

- $C, c$ are constants which depend only on $d$ and may differ at each occurrence.
- For $A, B \geq 0$, $A \lesssim B$ if there is a constant $C$ such that $A \leq CB$.
- $I = [-1, 1]$ and $I^d = [-1, 1]^d \subset \mathbb{R}^d$. 

which is given by

\[ \epsilon \]  

Elliptic function.

2.1. \( \psi \).

Since \( \alpha > \alpha \) for \( 1 \leq \alpha \leq q \) and \( \parallel \alpha \parallel > \alpha \) if and only if \( \det H_\psi \neq 0 \), it is easy to see that the \( L^p \) boundedness of \( \tilde{\alpha} \) is equivalent to that of \( T^0 \) which is given by \( \psi(\zeta) = 1 - (1 - |\zeta|^2)^{1/2} \). A natural generalization of the Bochner-Riesz problem is as follows: If \( \det H_\psi \neq 0 \) on the support of \( \chi_\circ \) (here, \( H_\psi \) is the Hessian matrix of \( \psi \)), we may conjecture that, for \( 1 \leq p \leq \infty \), \( p \neq 2 \),

\[ \|T^0 f\|_p \leq C\|f\|_p \]

if and only if \( \alpha > \alpha(p) \). From explicit computation of the kernel of \( T^0 \) it is easy to see that the condition \( \alpha > \alpha(p) \) is necessary for \( (7) \). However, in this paper we only work with specific choices of \( \psi \).

2.1. Elliptic function. Let us set

\[ \psi_\circ(\zeta) = |\zeta|^2 / 2. \]

For \( 0 < \epsilon_0 \ll 1/2 \) and an integer \( N \geq 100d \) we denote by \( G(\epsilon_0, N) \) the collection of smooth function which is given by

\[ G(\epsilon_0, N) = \{ \psi : \|\psi - \psi_\circ\|_{C^N(I_n^d)} \leq \epsilon_0 \}. \]
If $\psi \in \mathfrak{G}(\epsilon_0, N)$ and $a \in \frac{1}{2} I_{d-1}$, $H\psi(a)$ has eigenvalues $\lambda_1, \ldots, \lambda_{d-1}$ close to 1 and we may write $H\psi(a) = P^{-1} D P$ for an orthogonal matrix $P$ while $D$ is a diagonal matrix with its diagonal entries $\lambda_1, \ldots, \lambda_{d-1}$. We denote by $\sqrt{H\psi(a)}$ the matrix $P^{-1} D' P$ where $D'$ is the diagonal matrix with its diagonal entries $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{d-1}}$. So, $(\sqrt{H\psi(a)})^2 = H\psi(a)$.

For $\psi \in \mathfrak{G}(\epsilon_0, N)$, $a \in \frac{1}{2} I_{d-1}$, and $0 < \epsilon \leq 1/2$, we define

$$\psi_\epsilon^a(\zeta) = \frac{1}{\epsilon^2} \left( \psi \left( \epsilon \sqrt{H\psi(a)}^{-1} \zeta + a \right) - \psi(a) - \epsilon \nabla \psi(a) \cdot \left[ \epsilon \sqrt{H\psi(a)} \right]^{-1} \zeta. \right).$$

Since $\psi \in \mathfrak{G}(\epsilon_0, N)$, by Taylor’s theorem it is easy to see that $\|\psi_\epsilon^a - \psi_0\|_{C^N(I_{d-1})} \leq C\epsilon$ for $\psi \in \mathfrak{G}(\epsilon_0, N)$. Hence we get the following.

**Lemma 2.1.** Let $\psi \in \mathfrak{G}(\epsilon_0, N)$ and $a \in \frac{1}{2} I_{d-1}$. Then there is a constant $\kappa = \kappa(\epsilon_0, N)$, independent of $a$, $\psi$, such that $\psi_\epsilon^a \in \mathfrak{G}(\epsilon_0, N)$ provided that $0 < \epsilon \leq \kappa$.

**Remark 2.2.** If $\psi$ is smooth and $H\psi(a)$ has $d-1$ positive eigenvalues, after finite decomposition and affine transformations we may assume $\psi \in \mathfrak{G}(\epsilon_0, N)$ for arbitrarily small $\epsilon_0$ and large $N$. Indeed, for given $\epsilon > 0$, decomposing the multiplier $\left( \tau - \psi(\zeta) \right)^{\alpha} \chi_\delta(\zeta)$ to multipliers supported in balls of small radius $\epsilon/C$ with some large $C$, one may assume that $\mathcal{F} \hat{f}$ is supported in $B((a, \psi(a)), \epsilon/C)$. Then, the change of variables $\tau \mapsto \psi_\epsilon(a)$ and give rise to a new multiplier operator $T^\alpha_\epsilon(\psi_\epsilon^a)$ and, as can be easily seen by a simple change of variables, the operator norm $\|T^\alpha_\epsilon(\psi_\epsilon^a)\|_{L^p} \rightarrow p$ remains same. (See the proof of Proposition 2.3.) By Lemma 2.1 we see $\psi_\epsilon^a \in \mathfrak{G}(\epsilon_0, N)$ if $\epsilon$ is small enough.

### 2.2. Multiplier operator with localized frequency

Let $\phi$ be a smooth function supported in $2I$. For $\delta > 0, \psi \in \mathfrak{G}(\epsilon_0, N)$, and $f$ of which Fourier transform is supported in $\frac{1}{2} I_d$ we define the (frequency localized) multiplier operator $T_\delta = T_\delta(\psi)$ by

$$\widehat{T_\delta f}(\xi) = \phi \left( \frac{\tau - \psi(\zeta)}{\delta} \right) \hat{f}(\xi).$$

As is wellknown, the $L^p$ bound for $T_\delta$ largely depends on curvature of the surface $\tau = \psi(\zeta)$. By decomposing the multiplier dyadically away from the singularity $\tau = \psi(\zeta)$, in order to prove $7$ for $p > 2d/(d-1)$ and $\alpha > \alpha(p)$, it is enough to show that, for any $\epsilon > 0$,

$$\|T_\delta f\|_p \leq C\delta^{\frac{d-1}{2} - \epsilon} \|f\|_p$$

whenever $\hat{f}$ is supported in $\frac{1}{2} I_d$. The following recovers the sharp $L^p$ bound up to the currently best known range in $13$.

**Proposition 2.3.** Let $\epsilon > 0$. If $p \geq p_0(d)$ and $\epsilon_0$ is small enough, there is an $N = N(\epsilon)$ such that $10$ holds uniformly provided that $\psi \in \mathfrak{G}(\epsilon_0, N)$ and $\operatorname{supp} \hat{f} \subset \frac{1}{2} I_d$.

It is possible to remove loss of $\delta^{-\epsilon}$ in $10$ by the $\epsilon$-removal argument in $52$ (in particular, see Section 4).

**Induction quantity.** To control $L^p$ norm of $T_\delta$, for $0 < \delta$, we define $A(\delta) = A_\delta(\delta)$ by

$$A(\delta) = \sup \{ \|T_\delta(\psi) f\|_{L^p} : \psi \in \mathfrak{G}(\epsilon_0, N), \|f\|_p \leq 1, \operatorname{supp} \hat{f} \subset \frac{1}{2} I_d \}.$$

**Remark 2.4.** Though the induction argument in $13$ heavily relies on stability of the multilinear estimates, such issue doesn’t seem properly addressed. In particular, after (multiscale) decomposition and rescaling the associated phase functions (or surfaces) are no longer fixed phase functions.
(or surfaces). This requires the induction quantity defined over a class of phase functions or surfaces. This leads us to consider \( A(\delta) \).

From the estimate for the kernel of \( T_\delta \) (see Lemma 2.9), it is easy to see that \( A(\delta) \leq C \) uniformly in \( \psi \in \mathcal{G}(\epsilon, N) \) if \( \delta \geq 1 \) and \( A(\delta) \leq C \delta^{-\frac{d+1}{2}} \) if \( 0 < \delta \leq 1 \), because \( L^1 \)-norm of the kernel is uniformly \( O(\delta^{-\frac{d+1}{2}}) \). To prove Proposition 2.8, we need to show \( A(\delta) \leq \delta^\beta - \frac{d+1}{2} - \tau \) for any \( \epsilon > 0 \). However, due to lack of monotonicity \( A(\delta) \) is not suitable to close induction. So, we need to modify \( A(\delta) \). For \( \beta, \delta > 0 \), we define

\[
A^\beta(\delta) = A^\beta(\delta) := \sup_{\delta < \epsilon \leq 1} s^{\frac{d+1}{2} + \beta} A_p(s).
\]

Hence, Proposition 2.8 follows if we show \( A^\beta(\delta) \leq C \) for any \( \beta > 0 \).

The following lemma makes precise the heuristic that the bound of \( T_\delta \) improves if it acts on functions of which Fourier transforms are supported a smaller set. However, this becomes less obvious for multiplier operator when it is compared to restriction (adjoint) operator (cf. [13]). This type of improvement is basically due to parabolic rescaling structure of the operator, and generally appears in \( L^p \)-\( L^q \) estimates for \( p, q \) satisfying \( (d+1)/q < (d-1)(1-1/p) \), \( p \leq q \), which are not invariant under the parabolic rescaling. The following is important for induction argument to work.

**Proposition 2.5.** Let \( 0 < \delta \ll 1 \), \( \psi \in \mathcal{G}(\epsilon, N) \), and \( (a, \mu, \epsilon) \in \mathbb{R}^{d-1} \times \mathbb{R} \). Suppose that \( \text{supp} \hat{f} \subset \mathcal{q}((a, \mu, \epsilon) \subset \frac{1}{\delta} I^d \), \( 0 < \epsilon < 1/2 \) and \( \delta \leq (10)^{-2} \epsilon^2 \). Then, there is a \( \kappa = \kappa(\epsilon, N) \) such that for \( 0 < \epsilon \leq \kappa \)

\[
\|(T_\delta f)\|_p \leq C A(\epsilon^{-2}\delta)\|f\|_p
\]

holds with \( C \), independent of \( \psi \) and \( \epsilon \).

**Proof.** Decomposing \( q(a, \epsilon) \) into as many as \( O(d^4) \), we may assume that \( \hat{f} \) is supported in \( q((a, \mu, \epsilon)/\delta^4) \).

Since \( \psi \in \mathcal{G}(\epsilon, N) \) and \( \text{supp} \hat{f} \subset q((a, \mu, \epsilon)/(10d^4)) \), by Taylor’s theorem we note that \( \phi((1-\psi(\xi))/\delta) \hat{f}(\xi) \) is supported in the set

\[
\{(\xi, \tau): |\tau - \psi(a) - \nabla \psi(a) \cdot (\zeta - a)| \leq \frac{1 + \epsilon_0}{2 \times 10^2} \frac{\epsilon^2}{2} \}
\]

Hence, we may write

\[
\phi((\tau - \psi(\xi))/\delta) \hat{f}(\xi) = \phi((\tau - \psi(\xi))/\delta) \hat{f}(\xi) \frac{(\tau - \psi(a) - \nabla \psi(a) \cdot (\zeta - a))}{\epsilon^2}
\]

where \( \hat{f} \) is a smooth function supported in \( \frac{1}{\delta} I \) such that \( \hat{f} = 1 \) on \( \frac{1}{\delta} I \). Let us set \( M = (\sqrt{H(\psi(a))})^{-1} \) and make the change of variables in the frequency domain

\[
(\zeta, \tau) \to L(\zeta, \tau) = (\epsilon M \zeta + a, \epsilon^2 \tau + \psi(a) + \epsilon \nabla \psi(a) \cdot M \zeta)
\]

Then it follows that

\[
\mathcal{F}(T_\delta(\psi)\hat{f})(L\zeta) = \phi((\tau - \psi^\delta(\xi))/\epsilon^2) \hat{f}(\xi) \frac{(\tau - \psi^\delta(\xi))/\epsilon^2}{\epsilon^2}
\]

Since \( L \) is an invertible affine transformation it is easy to see \( \|\mathcal{F}^{-1}(T(L))\|_p = \epsilon^{(d+1)(\frac{1}{p}-1)}\|g\|_p \) for any \( g \). We also note that \( \text{supp} \hat{f}(\tau f(L)) \subset \frac{1}{\delta} I^d \) and by Lemma 2.1 there exists a \( \kappa > 0 \) such that \( \psi^\delta \in \mathcal{G}(\epsilon, N) \) if \( 0 < \epsilon \leq \kappa \) whenever \( \psi \in \mathcal{G}(\epsilon, N) \). So, by the definition of \( A(\delta) \) it follows that, for \( 0 < \epsilon \leq \kappa \),

\[
\|(T_\delta(\psi)\hat{f})\|_p = \epsilon^{(d+1)(\frac{1}{p}-1)}\|\mathcal{F}^{-1}(\phi((\tau - \psi^\delta(\xi))/\epsilon^2) \hat{f}(\xi) \frac{(\tau - \psi^\delta(\xi))/\epsilon^2}{\epsilon^2})\|_p
\]

\[
\leq \epsilon^{(d+1)(\frac{1}{p}-1)} A(\epsilon^{-2}\delta)\|\mathcal{F}^{-1}(\hat{f}(\xi) \frac{(\tau - \psi^\delta(\xi))/\epsilon^2}{\epsilon^2})\|_p \leq C A(\epsilon^{-2}\delta)\|f\|_p.
\]

\(^4\)It is only true for the paraboloid.
For the last inequality we also use the trivial bound \( \|F^{-1}(\tilde{\chi}(\tau)\tilde{g})\|_p \leq C\|g\|_p \) for any \( 1 \leq p \leq \infty \).

The inequality is valid for any \( \psi \in \mathcal{G}(\epsilon, N) \). This gives the desired bound. \( \square \)

We will need the following estimate which is easy to show by making use of Rubio de Francia’s one dimensional inequality \cite{rubio}.

**Lemma 2.6.** Let \( \{q\} \) be a collection of (distinct) dyadic cubes of the same side length \( \sigma \). Let \( 2 \leq p < \infty \). Then, there is a constant \( C \), independent of the collection \( \{q\} \), such that

\[
\left( \sum_q \|F^{-1}(\tilde{f}_q\chi_q)\|_p^p \right)^{\frac{1}{p}} \leq C\|f\|_p.
\]

2.3. **Multilinear estimates.** In this subsection we consider various multilinear estimates which are basically consequences of multilinear restriction and Kakeya estimates in [6].

For \( \psi \in \mathcal{G}(\epsilon, N) \) let us set

\[
\Gamma = \Gamma(\psi) = \{(\zeta, \psi(\zeta)) : \zeta \in \frac{1}{2}I^{d-1}\}.
\]

Let \( 2 \leq k \leq d \), and let \( U_1, U_2, \ldots, U_k \) be compact subsets of \( I^{d-1} \). For \( i = 1, \ldots, k, \) and \( \lambda > 0 \), set

\[
\Gamma_i = \{ (\zeta, \psi(\zeta)) : \zeta \in U_i \}, \quad \Gamma_i(\lambda) = \Gamma_i + O(\lambda).
\]

For \( \xi = (\zeta, \psi(\zeta)) \in \Gamma(\psi) \), let \( N(\xi) \) be the upward unit normal vector at \( (\zeta, \psi(\zeta)) \).

For \( v_1, \ldots, v_k \in \mathbb{R}^d \), denote by \( Vol(v_1, \ldots, v_k) \) the \( k \)-dimensional volume of the parallelepiped given by \( \{s_1v_1 + \cdots + s_kv_k : s_i \in [0,1], 1 \leq i \leq k\} \). Transversality among the surfaces \( \Gamma_1, \ldots, \Gamma_k \) is important for the multilinear estimates. Degree of transversality is quantitatively stated as follows:

\[
Vol(N(\xi_1), N(\xi_2), \ldots, N(\xi_k)) \geq \sigma
\]

for some \( \sigma > 0 \) whenever \( \xi_i \in \Gamma_i, \ i = 1, \ldots, k \). Since \( \psi \in \mathcal{G}(\epsilon, N), \) \( \nabla \psi \) is a diffeomorphism which is close to the identity map. The condition \cite{rubio} may be replaced by a simpler one that \( Vol(\Gamma_1, \xi_2, \ldots, \xi_k) \geq \sigma \) whenever \( \xi_i \in U_i, \ i = 1, \ldots, k \). The following is due to Bennett, Carbery and Tao [6].

**Theorem 2.7.** Let \( 0 < \delta \ll \sigma \ll 1 \) and \( \psi \in \mathcal{G}(\epsilon, N) \). Suppose that \( \Gamma_1, \ldots, \Gamma_k \) are given as in the above and \cite{rubio} is satisfied whenever \( \xi_i \in \Gamma_i, \ i = 1, \ldots, k \), and suppose that \( \tilde{F}_i \subset \Gamma_i(\delta) \), \( i = 1, \ldots, k \). Then, if \( p \geq 2k/(k-1) \) and \( \epsilon_0 \) is sufficiently small, for \( \epsilon > 0 \) there are constants \( N = N(\epsilon) \) such that, for \( x \in \mathbb{R}^d \),

\[
\left\| \prod_{i=1}^k F_i \right\|_{L^p(\mathbb{R}^d, \delta^{-1})} \leq C\sigma^{-C_\epsilon} \delta^{-\epsilon} \prod_{i=1}^k \delta^{-\frac{1}{2}} \|F_i\|_2
\]

holds with \( C, C_\epsilon \), independent of \( \psi \).

Besides stability issue this estimate is essentially the same as the multilinear restriction estimate in \cite{guth}. (See \cite{guth} Theorem 1.16 for the case \( k = d \) (also see Lemma 2.2) and see \cite{guth} Section 5) for the case of lower linearity \( 2 \leq k < d \). Though we are considering only the surfaces which are the graphs of \( \psi \in \mathcal{G}(\epsilon, N) \), but theorem remains true for surfaces even with vanishing curvature as long as the transversality condition is satisfied. Uniformity of the estimate follows from the fact that the multilinear Kakeya and restriction estimates are stable under perturbation of the associated surfaces. The estimate is conjectured to be true without \( \delta^{-\epsilon} \) loss (this is equivalent with the endpoint \( k \)-linear restriction estimate) but it remains open when \( k \geq 3 \) even though the corresponding endpoint case for the multilinear Kakeya estimate is obtained by Guth \cite{guth}. 
Remark 2.8. The proof of Theorem 2.7 is based on the multilinear Kakeya estimate and induction on scale argument which involves iteration of induction assumption to reduce the exponent of $\delta^{-1}$. Such improvement of exponent is possible at the expense of extra loss of bounds in terms of $\sigma^{-c}$. By following the argument in [1], one can easily see that one may take $C_\ast \leq C \log \frac{1}{\epsilon}$. (See the paragraph below (20)). Hence, the bound becomes less efficient when $\sigma$ gets as small as $\delta^k$ for some $c > 0$. In $\mathbb{R}^3$ the sharp bound depending on $\sigma$ was recently obtained by Ramos [42]. However, the argument of Bourgain-Guth avoids such problem by keeping Fourier supports of functions largely separated while being decomposed. In contrast with the conventional approach in which functions are usually decomposed into finer frequency pieces this was achieved by decomposing the input functions into those of relatively large frequency supports.

Lemma 2.9. Let $\varphi \in C_c^\infty(2I)$ and $\eta \in C_c^\infty(I^d)$ which satisfies $1/2 \leq \eta \leq 2$. Let $0 < \delta \ll \sigma \leq 1$. Set

$$K_\delta = F^{-1} \left( \varphi \left( \eta(\xi)(\tau - \psi(\xi)) \right) \tilde{\chi}(\xi) \right),$$

and $\mathcal{R}_M(x) = (1 + \delta|x|)^{-M}$. Suppose $\tilde{\chi}$ is supported in a cube of sidelength $C\sigma$ and $|\partial^\alpha_x \tilde{\chi}| \lesssim \sigma^{-|\alpha|}$ for any $\alpha$. Then, for any $M$, there is an $N = N(M)$ such that

$$|K_\delta(x)| \leq C\delta\sigma^{d-1}\mathcal{R}_M(x)$$

with $C$ depending only on $\|\psi\|_{C^N(1^d-1)}$.

Proof. Changing variables $\tau \to \delta \tau + \psi(\xi)$, we write

$$K_\delta(x) = (2\pi)^{-d} \delta^{d+1} \int e^{i\delta \xi d} \int e^{i(x'\cdot \xi + x_d \psi(\xi))} \tilde{\varphi}(\xi) d\xi d\tau,$$

where

$$\tilde{\varphi}(\xi) = \varphi \left( \eta(\xi) \frac{\delta \tau + \psi(\xi)}{C} \right) \tilde{\chi}(\xi, \delta \tau + \psi(\xi)).$$

We note that $|\partial^\alpha_x \tilde{\varphi}| \lesssim \sigma^{-|\alpha|}(\|\psi\|_{C^\alpha} + \|\eta\|_{C^\alpha}).$ Then, if $|x'|/100 \geq |x_d|$, by integration by parts it follows that

$$\left| \int e^{i(x'\cdot \xi + x_d \psi(\xi))} \tilde{\varphi}(\xi) d\xi \right| \leq C\delta^{d-1}(\|\psi\|_{C^M(1^d-1)} + \|\eta\|_{C^M(1^d)})(1 + \sigma|x'|)^{-M}.$$ 

Note that $\tilde{\varphi}(\xi) = 0$ if $|\tau| \geq 5C$ since $1/2 \leq \eta \leq 1$. This gives the desired inequality (14) by taking integration in $\tau$ since $\delta \ll \sigma$. On the other hand, if $|x'|/100 < |x_d|$, we integrate in $\tau$ first. Since $|\partial^\alpha_x \tilde{\varphi}| \lesssim (\|\psi\|_{C^\alpha} + \|\eta\|_{C^\alpha})$, by integration by parts again we have $|\int e^{i\delta \tau d} \tilde{\varphi}(\xi) d\tau| \leq C(\|\psi\|_{C^M(1^d-1)} + \|\eta\|_{C^M(1^d)})(1 + \sigma|x_d|)^{-M}$. This and taking integration in $\xi$ yield (14). \qed

From Theorem 2.7 and Lemma 2.9 we can obtain the sharp multilinear $L^p$ estimate for $T_\delta$ under transversality condition without localizing the multilinear operator on a ball of radius $1/\delta$. In fact, since $T_\delta f = K_\delta \ast f$ and the kernel $K_\delta$ (from Lemma 2.9) is rapidly decaying outside of $B(0, C/\delta)$, one may handle $f$ as if $f$ were supported in a ball $B$ of radius $\delta^{-1-\epsilon}$. This type of localization and Hölder’s inequality make it possible to lift $L^2$ estimate to that of $L^p$, $p \geq 2$, with sharp bound. Such idea of deducing $L^p$ estimates from $L^2$ ones goes back to Stein [49] p. 442-443 [24, 26], and in [31, 33] the similar idea was used to make use of $L^2$ bilinear restriction estimate. The same argument also works with the multilinear estimates with a little modification. We make it precise in what follows.

Proposition 2.10. Let $0 < \delta \ll \sigma \ll \bar{\sigma} \ll 1$ and $\psi \in \mathcal{G}(\epsilon_0, N)$, and let $Q_1, \ldots, Q_k \in 1/2 I^d$ be dyadic cubes of sidelength $\bar{\sigma}$. Suppose that (13) is satisfied whenever $\xi_i \in \Gamma \cap Q_i$, $i = 1, \ldots, k$, and supp $f_i \subset Q_i$, $i = 1, \ldots, k$. Then, if $p \geq 2k/(k-1)$ and $\epsilon_0$ is small enough, for $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$\left\| \prod_{i=1}^k T_{\delta} f_i \right\|_{L^p(\mathbb{R}^d)} \leq C\sigma^{-C}, \delta^{-\epsilon} \prod_{i=1}^k \left\| \tilde{\mathcal{R}}_{\sigma^{d-2}} f_i \right\|_p$$

for some $C \geq 1$.
holds with $C, C_\epsilon$, independent of $\psi$.

Proof. Set $\tilde{Q}_i = \{ \xi : \text{dist}(\xi, Q_i) \leq \tilde{c}\sigma \}$, and let $\tilde{\chi}_i$ be a smooth function supported in $\tilde{Q}_i$ which satisfies $\tilde{\chi}_i = 1$ on $Q_i$ and $|\partial^\alpha \tilde{\chi}_i| \leq \sigma^{-|\alpha|}$. Let us define $K_i$ by

$$F(K_i)(\xi) = \phi\left(\frac{\tau - \psi(\xi)}{\delta}\right)\tilde{\chi}_i(\xi).$$

Since $\hat{f}_i$ is supported in $Q_i$, we have $T_{\delta} f_i = K_i * f_i$.

Let $\{ B \}$ be the collection of boundedly overlapping balls of radius $\delta^{-1}$ which cover $\mathbb{R}^d$. For $\epsilon > 0$ we denote by $\tilde{B}$ the balls $B(a, \delta^{-1-\epsilon})$ if $B = B(a, \delta^{-1})$. By decomposing $f_i = \chi_{\tilde{B}} f_i + \chi_{\tilde{B}^c} f_i$, we bound the $p/k$-th power of the left hand side of (15) by

$$\sum_{B} \int_{B} \prod_{i=1}^{k} |T_{\delta} f_i(x)|^{\frac{p}{k}} \, dx = \sum_{B} \int_{B} \prod_{i=1}^{k} |K_i * f_i(x)|^{\frac{p}{k}} \, dx \lesssim I + II,$

where

$$I = \sum_{B} \int_{B} \prod_{i=1}^{k} |K_i * (\chi_{\tilde{B}} f_i)(x)|^{\frac{p}{k}} \, dx, \quad II = \sum_{B} \left( \sum_{g_i = \chi_{\tilde{B}} f_i \text{ for some } i} \int_{B} \prod_{i=1}^{k} |K_i * g_i(x)|^{\frac{p}{k}} \, dx \right).$$

The second sum in $II$ is summation over all possible choices of $g_i$ with $g_i = \chi_{\tilde{B}} f_i$ or $\chi_{\tilde{B}^c} f_i$, and $g_i = \chi_{\tilde{B}^c} f_i$ for some $i$. So, in the product $\prod_{i=1}^{k} K_i * g_i(x)$ there is at least one $g_i$ which satisfies $g_i = \chi_{\tilde{B}^c} f_i$. Since $F(K_i * (\chi_{\tilde{B}} f_i) \subset \Gamma(\delta) \cap \tilde{Q}_i$, taking a sufficiently small $\tilde{c} > 0$, from continuity it is easy to see that $F_1 = K_1 * (\chi_{\tilde{B}} f_1), \ldots, F_k = K_k * (\chi_{\tilde{B}} f_k)$ satisfy the assumption of Theorem 2.7. So, by Theorem 2.7 and Plancherel’s theorem we see

$$I \lesssim \sigma^{-C_\epsilon} \left(\frac{1}{\delta}\right)^{\epsilon} \sum_{B} \prod_{i=1}^{k} \delta^{\frac{d}{p}} \|K_i * (\chi_{\tilde{B}} f_i)\|_2^{\frac{p}{k}} \leq \sigma^{-C_\epsilon} \left(\frac{1}{\delta}\right)^{\epsilon} \sum_{B} \prod_{i=1}^{k} \delta^{\frac{d}{p}} \|\chi_{\tilde{B}} f_i\|_2^{\frac{p}{k}}$$

for $\psi \in \mathcal{G}(\epsilon_o, N)$ and $\epsilon_o$ small enough. Since $p > 2$, by applying Hölder’s inequality twice we have

$$I \lesssim \sigma^{-C_\epsilon} \left(\frac{1}{\delta}\right)^{\epsilon} \prod_{i=1}^{k} \delta^{\frac{d}{p}} \left(\frac{d}{d+1} + \frac{1}{2}\right) \left(\frac{d}{d+1} + \frac{1}{2}\right)^{\frac{1}{2}} \left(\frac{d}{d+1} + \frac{1}{2}\right)^{\frac{1}{2}} \|\chi_{\tilde{B}} f_i\|_p^p \lesssim \sigma^{-C_\epsilon} \left(\frac{1}{\delta}\right)^{\epsilon} \prod_{i=1}^{k} \delta^{\frac{d}{p}} \|f_i\|_p^p.$$

For $II$, we use Lemma 2.9. There is a constant $C = C(\|\psi\|_{C^N (I^{d-1})})$ such that $|K_i * (\chi_{\tilde{B}} f_i)(x)| \leq C\delta^{\epsilon} \delta^{(M-d-1)} \mathcal{R}_{d+1} * |f_i(x)|$ if $x \in B$, and $|K_i * g_i(x)| \leq C\delta \mathcal{R}_{d+1} * |f_i(x)|$. Thus, we get

$$II \lesssim \delta^{\frac{k-1}{p}} \epsilon(N-d-1)^{\frac{1}{2}} \prod_{i=1}^{k} \mathcal{R}_{d+1} * |f_i(x)|^{\frac{p}{k}} \lesssim \delta^{\epsilon} \|\mathcal{R}_{d+1} * f\|_p \lesssim \delta^{\epsilon} \|f\|_p$$

for some $c_1, c_2 > 0$ because $\|\mathcal{R}_{d+1} * f\|_p \leq C\delta^{-d} \|f\|_p$ for $1 \leq p \leq \infty$ by Young’s convolution inequality. Combining two estimates for $I$ and $II$ with $N$ large enough, we see that for $\epsilon > 0$ there is an $N$ such that

$$\left\|\prod_{i=1}^{k} T_{\delta} f_i\right\|_{L^\infty} \lesssim C \sigma^{-C_\epsilon} \left(\frac{1}{\delta}\right)^{\epsilon} \prod_{i=1}^{k} \delta^{\frac{d}{p}} \|f_i\|_p$$

for $\psi \in \mathcal{G}(\epsilon_o, N)$ and $\epsilon_o$ small enough. Therefore, choosing $\epsilon = \epsilon/c$, we get the desired bound (15).

In what follows we show that if the normal vectors of the surfaces are confined in $C\delta$-neighborhood of a $k$-plane in Proposition 2.11, then the associated multilinear restriction estimate has improved bound. In particular, if one takes $p = \frac{2k}{k-1}$, the bound in (14) is $\sim \delta^{-\epsilon} \delta^{\frac{d}{k}}$, which is better than the
corresponding bound $\sim \delta^{-c} \delta^2$ in Proposition 2.10. However, it seems difficult to cooperate on such improvement to get a better linear bound without using the square sum function (see Proposition 2.12 below).

Proposition 2.11. Let $0 < \delta \ll \sigma \ll 1$, $\psi \in \mathcal{G}(\epsilon_0, N)$, and $\Pi$ be a $k$-plane containing the origin. Suppose that $\Gamma(\psi)$, $\Gamma_1, \ldots, \Gamma_k$ are given as in the above and (13) is satisfied whenever $\xi_i \in \Gamma_i$, $i = 1, \ldots, k$. Suppose that

$$\text{supp} \tilde{F}_i \subset \Gamma_i(\delta) \cap \Delta^{-1}(\Pi + O(\delta)), \ i = 1, \ldots, k.$$  

Then, if $2 \leq p \leq 2k/(k - 1)$ and $\epsilon_0$ is sufficiently small, for $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$\left\| \prod_{i=1}^{k} F_i \right\|_{L^2_x(B(\delta^{-1}))} \leq C \sigma^{-c} \delta^{-\epsilon} \delta^{k(\frac{1}{2} - \frac{1}{p})} \prod_{i=1}^{k} \| F_i \|_2$$

holds with $C, C_\epsilon$, independent of $\psi$.

If $p/k$ were bigger than equal to $\geq 1$, the inequality could be shown by using H"older’s inequality and $k$ linear multilinear restriction estimate in [6]. However, this is not true in general and we prove Proposition 2.11 by making use of the induction on scale argument and multilinear Kakeya estimate. The following is a consequence of Proposition 2.11.

Corollary 2.12. Suppose that the same assumptions in Proposition 2.11 hold. Let $\{q\}, q \subset \frac{1}{2} I^d$, be the collection of dyadic cubes of side length $\ell$, $2^{-2} \theta < 2^{-1} \delta$. Then, if $2 \leq p \leq 2k/(k - 1)$, for $\epsilon > 0$ there is an $N = N(\epsilon)$ such that, for $x \in \mathbb{R}^d$,

$$\left\| \prod_{i=1}^{k} F_i \right\|_{L^2_x(B(\delta^{-1}))} \leq C \sigma^{-c} \delta^{-\epsilon} \prod_{i=1}^{k} \| F_i \|_2$$

holds with $C, C_\epsilon$, independent of $\psi \in \mathcal{G}(\epsilon_0, N)$.

This may be compared with a discrete formulation of multilinear inequality in [13] (see (1.1), p. 1250). The inequality (18) can be easily deduced from Proposition 2.11 by the standard argument using Plancherel’s theorem and orthogonality (cf. Proof of Corollary 2.11). So, we omit the proof.

Proof of Proposition 2.11. For $p = 2$ the estimate (17) follows from H"older’s inequality and Plancherel’s theorem. Hence, in view of interpolation, it is enough to show (17) for $p = 2k/(k - 1)$.

We prove (17) by adapting the proof of multilinear restriction estimate in [6]. By translation we may assume $x = 0$. We make the following assumption that, for $0 < \delta \ll \sigma$ and some $\alpha > 0$,

$$\left\| \prod_{i=1}^{k} F_i \right\|_{L^2_x(B(\delta^{-1}))} \leq \delta^{-\alpha} \delta^2 \prod_{i=1}^{k} \| F_i \|_2$$

holds uniformly for $\psi \in \mathcal{G}(\epsilon_0, N)$ whenever (16) holds and (13) is satisfied for $\xi_i \in \Gamma_i$, $i = 1, \ldots, k$. It is clearly true with a large $\alpha > 0$ as can be seen by making use of Lemma 2.9. We show (19) implies that, for $\epsilon > 0$, there is an $N$ such that

$$\left\| \prod_{i=1}^{k} F_i \right\|_{L^2_x(B(\delta^{-1}))} \leq C \sigma^{-c} \delta^{-\epsilon} \delta^2 \prod_{i=1}^{k} \| F_i \|_2$$

holds uniformly for $\psi \in \mathcal{G}(\epsilon_0, N)$. In what follows we set $R = \delta^{-1}$.

Iteration of implication from (16) to (20) allows us to suppress $\alpha$ as small as $\sim \epsilon$. In fact, since the implication remains valid as long as $\psi \in \mathcal{G}(\epsilon_0, N)$, by fixing an $\epsilon$ and iterating the implication (19) $\rightarrow$ (20) $l$ times we have the bound

$$C_\epsilon \sigma^{-c} R^{2^{-1} \alpha + \epsilon (1 + 2^{-1} \epsilon + \ldots + 2^{-l-1})} \leq C_\epsilon \sigma^{-c} R^{2^{-1} \alpha + 2 \epsilon}.$$
Choosing $l$ such that $2^{-l} \alpha \sim \varepsilon$ gives the bound $\hat{\beta}_\varepsilon \sigma^{Ck} \log R^{C\varepsilon}$. Hence, taking $\varepsilon = \epsilon/C$, we get the desired bound.

Let $\{q\}$ be the collection of dyadic cubes (hence essentially disjoint) of sidelength $\ell$, $\ell < R^{-1/2} \leq 2\ell$, so that $\mathbb{R}^d = \bigcup q$. Since the Fourier transform of $\rho_{B(z, \sqrt{\pi})} F_i$ is supported in $\Gamma(\delta_x^2) \cap N^{-1}(\Pi + O(\delta_x^2))$, by the assumption it follows that

$$\| \prod_{i=1}^k F_i \|_{L^\infty(B(z, R^{1/2}))} \leq \| \prod_{i=1}^k \rho_{B(z, \sqrt{\pi})} F_i \|_{L^\infty(B(z, R^{1/2}))} \leq \delta^{-\frac{d}{2}} \delta_x^2 \prod_{i=1}^k \| \rho_{B(z, \sqrt{\pi})} F_i \|_2 \leq \delta^{-\frac{d}{2}} \delta_x^2 \prod_{i=1}^k \| \rho_{B(z, \sqrt{\pi})} \left( \sum_q |F_i q|^2 \right)^{\frac{1}{2}} \|_2^2.$$

Here $F_i q$ is given by $\mathcal{F}(F_i q) = \hat{F}_i q$. Since the supports of $\mathcal{F}(\rho_{B(z, \sqrt{\pi})} F_i q)$ are boundedly overlapping, the last inequality follows from Plancherel's theorem. By rapid decay of $\rho$, we have, for a large $M > 0$,

$$\left( \sum_{q} \left| \frac{1}{\rho(L(i,q) \mathcal{K}_M)} F_i q \right| \right)^2 \leq \frac{1}{\rho(L(i,q) \mathcal{K}_M)} F_i q \subset \frac{1}{2} p(\xi_\alpha) \quad \text{with a sufficiently large } C_1 > 0.$$

For $(i, q)$ satisfying $\text{supp} \ F_i \cap q \neq \emptyset$ let us denote by $\xi_{i,q}$ the $\xi_\alpha$ which satisfies (22) (if there are more than one, we simply choose one of them). We also denote by $L(i, q)$ the bijective affine map from $\frac{1}{2} p(\xi_{i,q})$ to $q(0, 1)$. Then we define $\bar{F}_{i,q}$ by

$$\mathcal{F}(\bar{F}_{i,q})(\xi) = \frac{1}{\rho(L(i,q) \mathcal{K}_M)} F_i q(\xi).$$

We also set $P_{i,q} = P(\xi_{i,q})$ and $K_{i,q} = F^{-1}(\rho(L(i,q) \mathcal{K}_M) \cdot )$. By $RP_{i,q}$ we denote the rectangle which is $R$ times dilation of $P_{i,q}$ from the center of $P_{i,q}$. Also denote by $P(\hat{F}_{i,q})$ the set $R^{1+\varepsilon} P_{i,q}$ which is the $R^{1+\varepsilon}$ times dilation of $P_{i,q}$ from its center. Since $K_{i,q} * \bar{F}_{i,q} = F_i q$ and $|K_{i,q}| \leq \frac{C}{|P_{i,q}|}$, we have, for $y \in B(z, 2R^{1+\varepsilon})$ and some $c > 0$,

$$|F_i q(y)|^2 = |K_{i,q} * \bar{F}_{i,q}^2(y) \lesssim \frac{\chi_{RP_{i,q}}}{|RP_{i,q}|} |K_{i,q}^2(y) \lesssim R^{c} \frac{\chi_{RP_{i,q}}}{|P_{i,q}|} * |F_{i,q}|^2(x).$$

The last inequality is trivial since $|P_{i,q}| \sim R^{c} \left| R P_{i,q} \right|$ for some $c > 0$. Hence, for $x, y \in B(z, R^{1+\varepsilon})$ we have

$$\sum_q |F_i q|^2(y) \lesssim R^{c} \sum_q \frac{\chi_{RP_{i,q}}}{|P_{i,q}|} * |F_{i,q}|^2(x).$$
Taking integration in $y$ over $B(z, R^{\frac{1}{2}+\varepsilon})$ for each $1 \leq i \leq k$, we see that, for $x \in B(z, R^{\frac{1}{2}+\varepsilon})$,

\[(24) \quad \prod_{i=1}^{k} \left\| \chi_{B(z,R^{\frac{1}{2}+\varepsilon})} \left( \sum_{\mathbf{q}} |F_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \right\|_{L^2} \leq R^{c_\varepsilon} R^{\frac{d+1}{2}} \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \frac{\chi_{P_{i,\mathbf{q}}}}{|P_{i,\mathbf{q}}|} * |\widehat{F}_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} (x). \]

Now, integration in $x$ over $B(z, R^{\frac{1}{2}+\varepsilon})$ yields

\[(25) \quad \prod_{i=1}^{k} \left\| \chi_{B(z,R^{\frac{1}{2}+\varepsilon})} \left( \sum_{\mathbf{q}} |F_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{\infty}} \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \frac{\chi_{P_{i,\mathbf{q}}}}{|P_{i,\mathbf{q}}|} * |\widehat{F}_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \left\|_{L^{\infty R}(B(z,R^{\frac{1}{2}+\varepsilon}))}. \]

Combining this with (21) we have, for any large $M > 0$,

\[(26) \quad \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \frac{\chi_{P_{i,\mathbf{q}}}}{|P_{i,\mathbf{q}}|} * |\widehat{F}_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \left\|_{L^{\infty R}(B(z,R^{\frac{1}{2}+\varepsilon}))} \right\| \leq \delta^{\frac{d}{2}} - c_\varepsilon \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} |F_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \frac{\chi_{P_{i,\mathbf{q}}}}{|P_{i,\mathbf{q}}|} * |\widehat{F}_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \left\|_{L^{\infty R}(B(z,R^{\frac{1}{2}+\varepsilon}))} + \delta^M \prod_{i=1}^{k} \left\| F_{i} \right\|_{L^2}. \]

We now cover $B(0, R)$ with boundedly overlapping balls $B(z, \sqrt{R})$ and use the above inequality for each of them. Then we get

\[\prod_{i=1}^{k} \left\| F_{i} \right\|_{L^{\infty R}(B(0,R))} \right\| \leq \delta^{\frac{d}{2}} - c_\varepsilon \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \frac{\chi_{P_{i,\mathbf{q}}}}{|P_{i,\mathbf{q}}|} * |\widehat{F}_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{\infty R}(B(0,2R))} + \delta^{M-C} \prod_{i=1}^{k} \left\| F_{i} \right\|_{L^2}. \]

Here we have an increased $c$ because of overlapping of the balls $B(z, R^{\frac{1}{2}+\varepsilon})$ in the right hand side. Since $\sum_{\mathbf{q}} \left\| F_{i,\mathbf{q}} \right\|_{L^2}^2 \sim \left\| F_{i} \right\|_{L^2}$, for (20) it is sufficient to show

\[\prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \frac{\chi_{P_{i,\mathbf{q}}}}{|P_{i,\mathbf{q}}|} * |\widehat{F}_{i,\mathbf{q}}|^2 \right)^{\frac{1}{2}} \right\|_{L^{\infty R}(B(0,2R))} \right\| \leq \sigma^{-\kappa} \delta^{\frac{d}{2}} - c_\varepsilon \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \left\| F_{i,\mathbf{q}} \right\|_{L^2}^2 \right)^{\frac{1}{2}}. \]

By rescaling this is equivalent to

\[(27) \quad \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \frac{\chi_{P_{i,\mathbf{q}}}}{|P_{i,\mathbf{q}}|} * f_{i,\mathbf{q}} \right)^{\frac{1}{2}} \right\|_{L^{\infty R}(B(0,2R))} \right\| \leq \sigma^{-\kappa} \delta^{\frac{d}{2}} - c_\varepsilon \prod_{i=1}^{k} \left( \sum_{\mathbf{q}} \left\| f_{i,\mathbf{q}} \right\|_{L^2} \right)^{\frac{1}{2}}. \]

Let $I_\theta = \{ \mathbf{q} : \text{supp} \hat{F}_i \cap \mathbf{q} \neq \emptyset \}$, $I_i \subset I_\theta$ and $T_{i,\mathbf{q}}$ be a finite subset of $\mathbb{R}^d$. Allowing the loss of $(\log R)^C$ in bound, by a standard reduction with pigeonholing it suffices to show

\[(28) \quad \prod_{i=1}^{k} \left( \sum_{\mathbf{q} \in I_i} \sum_{\tau \in T_{i,\mathbf{q}}} \chi_{P_{i,\mathbf{q}}+\tau} \right) \right\|_{L^{\infty R}(B(0,2R))} \right\| \leq \sigma^{-\kappa/2} R^{c_\varepsilon} \prod_{i=1}^{k} \left( \sum_{\mathbf{q} \in I_i} \sum_{\tau \in T_{i,\mathbf{q}}} |P_{i,\mathbf{q}}+\tau| \right). \]

We write $x = (u, v) \in \Pi \times \Pi^\perp$ ( = $\mathbb{R}^d$). Then the left hand side is clearly bounded by

\[\sup_{v \in \Pi^\perp} \prod_{i=1}^{k} \left( \sum_{\mathbf{q} \in I_i} \sum_{\tau \in T_{i,\mathbf{q}}} \chi_{P_{i,\mathbf{q}}+\tau}(u, v) \right) \right\|_{L^{\infty R}(B(0,2))}, \]

where $\tilde{B}(0, \rho) \subset \mathbb{R}^k$ is the ball of radius $\rho$ which is centered at the origin.

For $v \in \Pi^\perp$ let us set

\[(P_{i,\mathbf{q}} + \tau)^v = \{ u : (u, v) \in P_{i,\mathbf{q}} + \tau \}. \]

Then $(P_{i,\mathbf{q}} + \tau)^v$ is contained in a tube of length $\sim 1$ and width $CR^{-1/2}$ of which axes are parallel with $N(\xi_{i,\mathbf{q}})$. This is because the longer sides of $P_{i,\mathbf{q}}$ except the one parallel to $N(\xi_{i,\mathbf{q}})$ are transversal to $\Pi$. More precisely, we can show that if $\epsilon_0$ is sufficiently small and $N$ is large enough, there a constant $c > 0$, independent of $\psi \in \mathcal{S}(\epsilon_0, N)$, such that, for $w \in (T_{\xi_{i,\mathbf{q}}}(N^{-1}(\Pi)) \oplus \text{span}\{N(\xi_{i,\mathbf{q}})\})^\perp$,

\[(29) \quad \angle(w, \Pi) \geq c > 0. \]
Since (12) is satisfied whenever $\xi_i \in \Gamma_i$, $i = 1, \ldots, k$, $N(\xi_i, q_i)$, and $N(\xi_i, q_i)$ which are, respectively, parallel to the axes of tubes $(P_{1, \theta} + \tau)^v, \ldots, (P_{k, \theta} + \tau)^v$ satisfy $|\text{Vol}(N(\xi_i, q_i))| > \sigma$. Also note that $|P_{1, \theta} + \tau| \sim |P_{1, \theta}|$. Hence, by the multilinear Kakeya estimate in $\mathbb{R}^d$ (Theorem 3.7) it follows that

$$\left\| \prod_{i=1}^k \left( \sum_{a_i} \varrho^{P_{i, \theta} + \tau\xi'_i} \right) \right\|_{L^1(\mathbb{R}^d)} \lesssim \sigma^{-1} \prod_{i=1}^k \left( \sum_{a_i} |P_{i, \theta} + \tau| \right).$$

This gives the desired inequality (28).

Now it remains to show (29). By continuity, taking sufficiently small $\epsilon_0$, we only need to show (29) when $\psi = \psi_0$ since $\|\psi - \psi_0\|_{C^\infty(\mathbb{R}^d)} \leq \epsilon_0$. Though it is easy to show and intuitively obvious, we include a proof for clarity. By rotation we may assume $\Pi \cap \{ x_d = -1 \} = \{(y, a, -1) : y \in \mathbb{R}^{d-1} \}$ for some $a \in \mathbb{R}^{d-k}$. Since $\Pi$ contains the origin, $\Pi$ can parametrized (except $\Pi \cap \{ x_d = 0 \}$) as follows:

$$s(y, a, -1), s \in \mathbb{R}, y \in \mathbb{R}^{d-k}. $$

We may assume $\Gamma_i(\theta) \cap (N^{-1}(\Pi) + O(\theta)) \neq \emptyset$ because otherwise $F_i = 0$ and there is nothing to prove. Since $N(\Gamma) \cap \Pi = \emptyset$ if $|a|$ is large, so we may assume that $|a| \leq C$ for some $C > 0$ and note that $\xi_i, q_i \in \Gamma(\psi)$. Furthermore, it suffices to show that

$$\Pi \cap \left( T_{\xi, q_i}(N^{-1}(\Pi)) \oplus \text{span}(N(\xi_i, q_i)) \right)^\perp = \{0\},$$

which implies $\angle(w, \Pi) > 0$ if $w \in (T_{\xi, q_i}(N^{-1}(\Pi)) \oplus \text{span}(N(\xi_i, q_i)))^\perp$. Then, by continuity and compactness (29) follows. We now verify (29) with $\psi = \psi_0$. By rotation we may assume $a = (0, \ldots, 0, a) = (0, a) \in \mathbb{R}^{d-k-1} \times \mathbb{R}$. Using the above parametrization of $\Pi$, we see that $\Pi = \text{span}(e_1, \ldots, e_{k-1}, (0, \ldots, 0, a, -1)).$

The normal vector at $(x', |x'|^2/2) \in \mathbb{R}^{d-1} \times \mathbb{R}$ is parallel to $(x', -1)$. Hence, if $(x', |x'|^2/2) \in N^{-1}(\Pi)$, that is, $(x', -1) \in \Pi$, then $x'$ takes the form $x' = (y, a)$ because of (30). Hence, it follows that $N^{-1}(\Pi) = \{(y, 0, a, \frac{1}{2}(|y|^2 + |a|^2)) \}$. Then, if $\xi_i, q_i = (y, 0, a, \frac{1}{2}(|y|^2 + |a|^2))$, $T_{\xi, q_i}(N^{-1}(\Pi))$ is spanned by $y_1 = (1, 0, \ldots, 0, y_1), y_2 = (0, 1, \ldots, 0, 0, y_2), \ldots, y_{k-1} = (0, 0, \ldots, 1, 0, 0, y_{k-1}).$ For (31) it is sufficient to show that $\mathcal{P} := \Pi \cap (\text{span}(y, 0, a, -1), y_1, \ldots, y_{k-1})^\perp = \{0\}$. Let $w \in \mathcal{P}$. Then, since $w \in \text{span}(e_1, \ldots, e_{k-1}, (0, \ldots, 0, a, -1))$, we may write $w = (e_1, \ldots, e_{k-1}, 0, c_k a, -c_k).$ Also, $w \cdot y_1 = \cdots = w \cdot y_{k-1} = w \cdot (0, a, -1) = 0$ gives $c_1 = \cdots = c_k = 0$. So, $v = 0$ and, hence, we get (31). This completes the proof.

## 2.4. Scattered modulation sum of scale $\sigma$.

When the Fourier transform of a given function $f$ is supported in a ball of radius $\sigma$, then $f$ behaves as though it were constant on balls of radius $\sigma^{-1}$. This observation has important role in Bourgain-Guth’s argument [3] and is widely taken for granted without being made rigorous. There seems to be several ways which make this heuristic rigorous (see [57, 53]). For this purpose we make use of Fourier series expansion.

Fix $\sigma > 0$ and large positive constants $M = M(d) \geq 100d$ and $C_M$ which are to be chosen to be large. For $l \in \sigma^{-1} \mathbb{Z}^d$ we set

$$A_l = A_l(\sigma) = C_M(1 + |\sigma l|)^{-M}, \quad \tau f(x) = f(x - l).$$

For $\sigma > 0$, we define $[F]_\sigma$, $[F]_\sigma$ (scattered modulation sum of $\sigma$-scale) by

$$[F]_\sigma(x) = \sum_{l \in \sigma^{-1} \mathbb{Z}^d} A_l |\tau F(x)|, \quad [F]_\sigma(x) = \sum_{l_1, l_2 \in \sigma^{-1} \mathbb{Z}^d} A_{l_1} A_{l_2} |\tau_{l_1 + l_2} F(x)|. $$

We have the following lemma.

**Lemma 2.13.** Let $\xi_0, x_0 \in \mathbb{R}^d$. Suppose that $F$ is a function with $\hat{F}$ supported in $q(\xi_0, \sigma)$. Then, if $x \in q(x_0, 1/\sigma)$,

$$|F(x)| \leq [F]_\sigma(x_0) \leq [F]_\sigma(x).$$
It should be noted that the inequality holds regardless of $\xi_0, x_0,$ and $\sigma$.

**Proof.** Let $a$ be a smooth function supported in $[-\pi, \pi]^d$ and $a(x) = 1$ if $|x_i| \leq 1, i = 1, \ldots, d$. Let us set

$$A(x, \xi) = a(x)a(\xi)e^{ix\cdot\xi}. $$

Since $|\partial^\alpha\xi A| \leq C_\alpha$ for any multi-indices $\alpha$, by expanding $A$ into Fourier series in $\xi$ we have

$$a(x)a(\xi)e^{ix\cdot\xi} = \sum_{l \in \mathbb{Z}^d} a_l(x)e^{-i\xi\cdot l}, \quad x, \xi \in [-\pi, \pi]^d$$

while $a_l$ satisfies $|a_l(x)| \leq C_M(1 + |l|)^{-M}$ for any large $M > 0$. On the other hand, from the inversion formula we have

$$F(x) = (2\pi)^{-d} \int e^{i(x-x_0)\cdot \xi_0}e^{i(x-x_0)\cdot(\xi-\xi_0)}e^{ix_0\cdot \xi} \widehat{F}(\xi)d\xi.$$ 

Hence, since $x \in \mathcal{q}(x_0, \frac{1}{\sigma})$, inserting the harmless bump function $a$, we may write

$$F(x) = (2\pi)^{-d} e^{i(x-x_0)\cdot \xi_0} \int A(\sigma(x-x_0), \frac{\xi-\xi_0}{\sigma}) e^{ix_0\cdot \xi} \widehat{F}(\xi)d\xi.$$ 

Using (34) we have

$$F(x) = (2\pi)^{-d} e^{i(x-x_0)\cdot \xi_0} \sum_{l \in \mathbb{Z}^d} a_l(\sigma(x-x_0)) \int e^{-i\frac{(x-x_0)}{\sigma}\cdot l} e^{ix_0\cdot \xi} \widehat{F}(\xi)d\xi.$$ 

Then it follows that

$$|F(x)| \leq \sum_{l \in \mathcal{q}^{-1}\mathbb{Z}^d} |A_l| |\tau_l F(x_0)| \leq \sum_{l_1, l_2 \in \mathcal{q}^{-1}\mathbb{Z}^d} A_{l_1}A_{l_2} |\tau_{l_1+l_2} F(x)|.$$ 

The second inequality follows by applying the first one to each $\tau_l F$ with the roles of $x, x_0$ interchanged. \hfill $\Box$

### 2.5. Multi-scale decomposition

We now attempt to bound part of $T_{\delta} f$ with a sum of products which satisfy the transversality assumption while the remaining parts are given by a sum of functions which have relatively small Fourier supports. The first is rather directly estimated by making use of the multilinear estimates and the latter is to be handled by Proposition 2.5, the induction assumption and Lemma 2.6.

In what follows, we basically adapt the idea in [13]. However, concerning the decomposition in [13], reappearance of many small scale functions in large scale decomposition becomes problematic when one attempts to sum up resulting estimates. For the adjoint restriction estimates this can be overcome by using $L^\infty$ function as was done in [13]. But such an argument doesn’t work for the multiplier operators and leads loss in its bound. To get over this, unlike the decomposition in [13] where one starts to decompose with $d$-linear products and proceeds by reducing the degree multi-linearity based on dichotomy, we decompose the multiplier operator by increasing the degree of multi-linearity in order to avoid small scale functions appearing inside of large scale ones. This has a couple of advantages. First, this allows us to keep the function relatively intact in the course of decomposition so that we can easily add up decomposed pieces to obtain the sharp $L^p$ bound. Secondly, the decomposition makes it possible to obtain directly obtain $L^p$-estimate. Hence we don’t need to rely on the factorization theorem to deduce $L^p-L^p$ from $L^\infty-L^p$. (The same is also true for the adjoint restriction operators.) Hence, we can obtain the sharp $L^p$ bounds for multiplier operators of Bochner-Riesz type which lacks symmetry.
2.5.1. **Spatial and frequency dyadic cubes.** Let $0 < \varepsilon_0 < 1$, $N, \psi \in \mathfrak{S}(\varepsilon_0, N)$, and $T_\delta$ be given by (37). Let $\kappa = \kappa(\varepsilon_0, N)$ be the number given in Proposition 2.12 so that (11) holds whenever

$$0 < \varepsilon \leq \kappa \quad \text{and} \quad \psi \in \mathfrak{S}(\varepsilon_0, N).$$

Let $m$ be an integer such that $2 \leq m \leq d - 1$, and $\sigma_1, \ldots, \sigma_m$ be dyadic numbers such that

$$\delta \ll \sigma_m \ll \cdots \ll \sigma_1 \ll \min(\kappa, 1).$$

These numbers will be specified to terminate induction. We call $\sigma_i$ $i$-th scale.

Let us denote by $\{q^i\}$ the collection of the dyadic cubes $q^i$ of sidelength $2\sigma_i$ which are contained in $I^d$ (so, $q^i$ denotes the member of $\{q^i\}$ and the cubes $q^i$ are essentially disjoint). Rather than introducing new notation to denote each collection of $q^i$, we take the convention that $\{q^i\}$ denotes the collection of all dyadic cubes of sidelength $2\sigma_i$ contained in $I^d$ if it is not specified otherwise. For each $i$-th scale there is a unique collection so that there will be no ambiguity, and we also use $q^i$ as indices which run over the set $\{q^i\}$. Thus, we may write

$$\bigcup_{q^i} q^i = I^d.$$  

For the rest of this section, we assume that

$$\supp \hat{f} \subset \frac{1}{2} I^d.$$  

Since $f = \sum_{q^i} f_{q^i}$, for $i = 1, \ldots, m$, we write

$$T_\delta f = \sum_{q^i} T_\delta f_{q^i}.$$  

Clearly, we may assume that $q^i$ is contained in $C\sigma_i$- neighborhood of the surface $\Gamma(\psi)$ because $T_\delta f_{q^i} = 0$ otherwise. So, in what follows, $q^1, q^2, \ldots, q^{m+1}$ and $q^i$ denote the elements of $\{q^i\}$.

For convenience we extend in a trivial way the map $N$ defined on $\Gamma(\psi)$ to the cube $I^d$ by setting, for $\xi = (\zeta, \tau) \in I^d$,

$$n(\zeta, \tau) = N(\zeta, \psi(\zeta)).$$  

This extension is not necessarily needed in what follows because we only consider a small neighborhood of $\Gamma(\psi)$. However, this allows us to define normal vector for any point in $I^d$ and makes exposition simpler. This definition of $n$ becomes coherent with the one given in the next section.

**Definition 2.14.** Let $k$ be an integer such that $1 \leq k \leq m$ and fix a constant $c > 0$. Let $q^i_1, \ldots, q^i_{k+1} \in \{q^i\}$ ($k$-th scale cubes). We say $q^i_1, q^i_2, \ldots, q^i_{k+1}$ are $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ transversal if

$$\Vol(n(\xi_1), n(\xi_2), \ldots, n(\xi_{k+1})) \geq c\sigma_1\sigma_2\cdots\sigma_k,$$

whenever $\xi_i \in q^i_i$, $i = 1, \ldots, k+1$. And we simply denote this by $q^i_1, q^i_2, \ldots, q^i_{k+1}$: trans omitting dependence on $\sigma_1, \sigma_2, \ldots, \sigma_k$.

Let us set

$$M_i = \frac{1}{\sigma_i}, \quad i = 1, \ldots, m.$$  

We denote by $\{\Omega^i\}$ the collection of the dyadic cubes of sidelength $2M_i$, which covers $\mathbb{R}^d$ (so, $\Omega^i$ again denotes a member of the sets $\{\Omega^i\}$). We write

$$\bigcup_{\Omega^i} \Omega^i = \mathbb{R}^d \mathbb{1}$$

Since the Fourier support of $T_\delta f_{q^i}$ is contained $q^i$, it may be thought of as a constant on $\Omega^i$ by invoking Lemma 2.13 with $\sigma = \sigma_i$. Since the scale $\sigma_i$ is clear from the side length of the cube $q^i$, we simply set

$$[T_\delta f_{q^i}] := [T_\delta f_{q^i}]_{\sigma_i}, \quad [T_\delta f_{q^i}] := [T_\delta f_{q^i}]_{\sigma_i}.$$  

\footnote{Here we take the same convention for $\{\Omega^i\}$ as we do for $\{q^i\}$.}
2.5.2. $\sigma_1$-scale decomposition. Bilinear decomposition is rather elementary. Fix $x \in \mathbb{R}^d$. From (38) note that

$$|T_{\delta} f(x)| \leq \sum_{q^1} |T_{\delta} f_{q^1}(x)|.$$ 

We denote by $q^1_i = q^1_i(x)$ a cube $q^1_i \in \{q^1_i\}$ such that $|T_{\delta} f_{q^1_i}(x)| = \max_{q^1_i} |T_{\delta} f_{q^1_i}(x)|$. (There may be many such cubes but $q^1_i$ denotes just one of them.) Then we consider the following two cases separately:

$$\sum_{q^1} |T_{\delta} f_{q^1}(x)| \leq 100^d |T_{\delta} f_{q^1}(x)|, \quad \sum_{q^1} |T_{\delta} f_{q^1}(x)| > 100^d |T_{\delta} f_{q^1}(x)|.$$

For the second case $\sum_{\text{dist}(q^1_i, q^1_j) < 10\sigma_1} |T_{\delta} f_{q^1_i}(x)| < 50^d |T_{\delta} f_{q^1}(x)| \leq 2^{-d} \sum_{q^1} |T_{\delta} f_{q^1_i}(x)|$. Hence there is $q^1_i \in \{q^1_i\}$ such that $\text{dist}(q^1_i, q^1_j) \geq 10\sigma_1$ and

$$\sum_{q^1} |T_{\delta} f_{q^1_i}(x)| \leq \sigma_1^{-(d-1)} |T_{\delta} f_{q^1_i}(x)| \leq \sigma_1^{-(d-1)} |T_{\delta} f_{q^1_i}(x)|T_{\delta} f_{q^1_i}(x)|^{\frac{\beta}{2}}.$$

From these two cases we get

$$(41) \quad \sum_{q^1} |T_{\delta} f_{q^1_i}(x)| \leq \max_{q^1_i} |T_{\delta} f_{q^1_i}(x)| + C\sigma_1^{-(d-1)/2} \max_{\text{dist}(q^1_i, q^1_j) \geq \sigma_1} |T_{\delta} f_{q^1_i}(x)|T_{\delta} f_{q^1_i}(x)|^{\frac{\beta}{2}}.$$ 

Using imbedding $\ell^p \subset \ell^\infty$, Proposition 2.5 and Lemma 2.6 give

$$(42) \quad \|T_{\delta} f_p\|_p \leq \left( \sum_{q^1} \|T_{\delta} f_{q^1_i}\|_p^p \right)^{\frac{1}{p}} \leq \left( \sum_{q^1} A(\sigma_1^{-2}\delta)^{p} \|f_{q^1_i}\|_p^p \right)^{\frac{1}{p}} \lesssim A(\sigma_1^{-2}\delta)\|f\|_p.$$ 

Hence, combining this with (11), we have

$$(43) \quad \|T_{\delta} f\|_p \lesssim A(\sigma_1^{-2}\delta)\|f\|_p + \sigma_1^{-C} \max_{\text{dist}(q^1_i, q^1_j) \geq \sigma_1} \|T_{\delta} f_{q^1_i}T_{\delta} f_{q^1_j}\|_p^{\frac{\beta}{2}}.$$ 

We now proceed to decompose the bilinear expression appearing in the left hand side.

In the following section we explain how one can achieve trilinear decomposition out of (42) before we inductively obtain the full $k$ linear decomposition which we need for the proof of Theorem 1.1. Once one gets familiar with it, extension to higher degree of multi-linearity becomes more or less obvious.

2.5.3. $\sigma_2$-scale decomposition. Suppose that we are given two cubes $q^1_i$ and $q^2_j$ of 1st scale such that $\text{dist}(q^1_i, q^2_j) \geq \sigma_1$. For $i = 1, 2$, we denote by $\{q^2_i\}$ the collection of dyadic cubes $q^2_i$ of sidelength $\sigma_2$ contained in $q^1_i$ such that

$$(44) \quad q^1_i = \bigcup_{q^2_i} q^2_i, \quad i = 1, 2.$$ 

We also denote by $\{q^2\}$ the set $\{q^2_1\} \cup \{q^2_2\}$. Then it follows that

$$(45) \quad T_{\delta} f_{q^1_i} = \sum_{q^2_i} T_{\delta} f_{q^1_i}, \quad i = 1, 2.$$ 

We may also assume that $q^2_1$, $q^2_2$ are contained in the $C\sigma_2$-neighborhood of $\Gamma(\psi)$ because $T_{\delta} f_{q^2_1}$, $T_{\delta} f_{q^2_2}$ are zero otherwise.

Decomposition from this stage is no longer simple as in the $\sigma_1$-scale case. We need to use spatial localization in order to compare the values of the decomposed pieces. This makes it possible to bounds large part of the operator with transversal products.

Let us fix a cube $Q^2$ and $x_0$ be the center of $Q^2$. Let $q^1_i \in \{q^1_i\}$, $q^2_{i*} \in \{q^2_i\}$ be the cubes such that

$$[T_{\delta} f_{q^1_i}](x_0) = \max_{q^1_i} [T_{\delta} f_{q^1_i}](x_0), \quad [T_{\delta} f_{q^2_{i*}}](x_0) = \max_{q^2_{i*}} [T_{\delta} f_{q^2_{i*}}](x_0).$$
Let us define \( \Lambda_i \subset \{ q_i \}, i = 1, 2, \) by
\[
\Lambda_i = \{ q_i^2 : \langle T_{\delta} f_{q_i^2} \rangle (x_0) \geq \sigma_2^{2d} \max (\langle T_{\delta} f_{q_i^2} \rangle (x_0), \langle T_{\delta} f_{q_i^2} \rangle (x_0)) \}.
\]
Using (44), we split the summation to get
\[
T_{\delta} f_{q_1^2} \cdot T_{\delta} f_{q_2^2} = \sum_{(q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2} T_{\delta} f_{q_1^2} \cdot T_{\delta} f_{q_2^2}^\prime + \sum_{(q_1^2, q_2^2) \not\in \Lambda_1 \times \Lambda_2} T_{\delta} f_{q_1^2} \cdot T_{\delta} f_{q_2^2}^\prime.
\]
Since there are at most \( O(\sigma_2^{-2(d-1)}) \) \( (q_1^2, q_2^2) \), the second sum in the right hand side is bounded by
\[
\sum_{(q_1^2, q_2^2) \not\in \Lambda_1 \times \Lambda_2} |T_{\delta} f_{q_1^2} (x)||T_{\delta} f_{q_2^2} (x)| \leq \sigma_2^d \max ([T_{\delta} f_{q_2^2} (x)]^2).
\]
For a cube \( q \) we denote by \( c(q) \) the center of \( q \). Let \( \Pi = \Pi (q_1^2, q_2^2) \) be the 2-plane which is spanned by \( n_1 = n (c(q_1^2)), n_2 = n (c(q_2^2)) \), and define
\[
\mathfrak{N} = \mathfrak{N} (\Omega^2, q_1^2, q_2^2) = \{ q^2 \in \Lambda_1 \cup \Lambda_2 : \text{dist}(n(q^2), \Pi) \leq C \sigma_2 \}.
\]
Clearly, \( \text{Vol}(n_1, n_2) \gtrsim \sigma_1 \) and \( \text{dist}(n(q^2), \Pi) \gtrsim \sigma_2 \) if \( q^2 \not\in \mathfrak{N} \). Since \( \sigma_1 \gg \sigma_2 \), if \( q^2 \not\in \mathfrak{N} \), then \( \text{Vol}(n_1, n_2) \gtrsim \sigma_1 \sigma_2 \) for \( \xi \in q^2 \). Also, \( n(q_1^2, q_2^2) \subset n_i + O(\sigma_2), i = 1, 2. \) So, it follows that
\[
\text{Vol}(n(q_1^2), n(q_2^2)) \gtrsim \sigma_1 \sigma_2
\]
if \( q_1^2 \in q_1^2, q_2^2, \) and \( q \not\in \mathfrak{N} \). That is, \( q_1^2, q_2^2 \), \( q^2 \) are transversal. Hence, we split \( \sum_{(q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2} T_{\delta} f_{q_1^2} \cdot T_{\delta} f_{q_2^2} \) into
\[
\sum_{(q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2 : q_1^2, q_2^2 \in \mathfrak{N}} T_{\delta} f_{q_1^2} (x) T_{\delta} f_{q_2^2} (x) + \sum_{(q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2 : q_1^2 \not\in \mathfrak{N}} T_{\delta} f_{q_1^2} (x) T_{\delta} f_{q_2^2} (x).
\]
Each term appearing in the second sum can be bounded by a product of three operators which satisfy transversality condition. Indeed, suppose that \((q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2 \) and \( q_3^2 \not\in \mathfrak{N} \). The case that \( q_3^2 \not\in \mathfrak{N} \) can be handled similarly by symmetry. Since \( \langle T_{\delta} f_{q_2^2} \rangle (x_0) \geq \sigma_2^d \langle T_{\delta} f_{q_2^2} \rangle (x_0) \), we have
\[
|T_{\delta} f_{q_1^2} (x)| T_{\delta} f_{q_2^2} (x) \leq \left( \langle T_{\delta} f_{q_1^2} \rangle (x_0) \langle T_{\delta} f_{q_2^2} \rangle (x_0) \right)^{\frac{1}{2}} \leq \sigma_2^{-2d/3} \left( \langle T_{\delta} f_{q_1^2} \rangle (x_0) \langle T_{\delta} f_{q_2^2} \rangle (x_0) \right)^{\frac{1}{2}}.
\]
Hence, from this and (49) it follows that
\[
\sum_{(q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2 : q_3^2 \not\in \mathfrak{N}} T_{\delta} f_{q_1^2} (x) T_{\delta} f_{q_2^2} (x) \leq \sigma_2^{-d} \sum_{q_1^2, q_2^2, q_3^2 \text{trans}} \left( \prod_{i=1}^{3} \langle T_{\delta} f_{q_i^2} \rangle (x_0) \right)^{\frac{1}{2}}.
\]
We combine (48), (50), (51) and (50) to get, for \( x \in \Omega^2 \),
\[
|T_{\delta} f_{q_1^2} (x) T_{\delta} f_{q_2^2} | \leq \sigma_2^d \left( \max (\langle T_{\delta} f_{q_2^2} \rangle (x)) \right)^{2} + \sum_{(q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2 : q_3^2 \not\in \mathfrak{N}} T_{\delta} f_{q_1^2} (x) T_{\delta} f_{q_2^2} (x) + \sigma_2^{-d} \sum_{q_1^2, q_2^2, q_3^2 \text{trans}} \left( \prod_{i=1}^{3} \langle T_{\delta} f_{q_i^2} \rangle (x_0) \right)^{\frac{1}{2}}.
\]
Using Lemma 2.13 again, we have, for \( x \in \Omega^2 \),
\[
|T_{\delta} f_{q_1^2} (x) T_{\delta} f_{q_2^2} | \leq \sigma_2^d \left( \max (\langle T_{\delta} f_{q_2^2} \rangle (x)) \right)^{2} + \sum_{(q_1^2, q_2^2) \in \Lambda_1 \times \Lambda_2 : q_3^2 \not\in \mathfrak{N}} T_{\delta} f_{q_1^2} (x) T_{\delta} f_{q_2^2} (x) + \sigma_2^{-d} \sum_{q_1^2, q_2^2, q_3^2 \text{trans}} \left( \prod_{i=1}^{3} \langle T_{\delta} f_{q_i^2} \rangle (x_0) \right)^{\frac{1}{2}}.
\]
Taking $L^{p/2}$ on both sides of inequality (integrating on each of $\Omega^2$), summing along $\Omega^2$, and using Proposition \ref{prop:boundedness} and Lemma \ref{lem:boundedness}, we get
\[
\norm{T_\delta f_{q1^2} T_\delta f_{q2^2}}_p \lesssim (A(\sigma_2^{-2} \delta))^{2/2} \norm{f}_p + \left( \sum_{\Omega^2} \norm{T_\delta f_{q1^2} T_\delta f_{q2^2}}_{L^p(\Omega^2)} \right)^{1/2}
\]
(53)
\[+ \sigma_2^{-C} \sup_{\tau_1, \tau_2, q1^2, q2^2 \in \trans} \max \norm{T_\delta (\tau_1 f_{q1^2}) T_\delta (\tau_2 f_{q2^2}) T_\delta (\tau_3 f_{q3^2})}_{L^p(\Omega^2)}^{1/2},\]
where $[\mathcal{N}(\Omega^2, q1^2, q2^2)$ is a subset of $\mathcal{N}(\Omega^2, q1^2)$. Here, for simplicity we now denote $\tau_i f$ by $\tau_i f$ just to indicate translation by a vector. Precise value of $l_i$ is not significant in the overall argument. To show (53), for the first term in the right hand side of (52) we may repeat the same argument as in (52). In fact, by (33) and rapid decay of $A_i^k$ combined with Hölder's inequality to summation along $l, l'$, and using Proposition \ref{prop:boundedness} and Lemma \ref{lem:boundedness} we have
\[
\norm{\max_{q^2} \sum_{\tau_2} T_\delta f_{q2^2}}_p \lesssim \sum_{q^2} \norm{T_\delta (\tau_2 f_{q2^2})}_{L^p(\Omega^2)} \lesssim A(\sigma_1^{-2} \delta) \norm{f}_p.
\]
For the third term of the right hand side of (52), thanks to (33) and rapid decay of $A_i^k$, it is enough to note that there are as many as $O(\sigma_2^{-C})$ $q1^2, q2^2, q3^2 : trans$.

We combine (53) with (53) to get
\[
\norm{T_\delta f}_p \leq A(\sigma_1^{-2} \delta) \norm{f}_p + \sigma_1^{-C} A(\sigma_2^{-2} \delta) \norm{f}_p
\]
(54)
\[+ \sigma_2^{-C} \sup_{\tau_1, \tau_2, q1^2, q2^2 \in \trans} \max \left( \sum_{\Omega^2} \norm{T_\delta f_{q1^2} T_\delta f_{q2^2}}_{L^p(\Omega^2)} \right)^{1/2}
\]
\[+ \sigma_2^{-C} \sup_{\tau_1, \tau_2, \tau_3, q1^2, q2^2, q3^2 \in \trans} \max \norm{T_\delta (\tau_1 f_{q1^2}) T_\delta (\tau_2 f_{q2^2}) T_\delta (\tau_3 f_{q3^2})}_{L^p(\Omega^2)}^{1/2}.
\]
Here $[\mathcal{N}(\Omega^2, q1^2, q2^2)$ also depends on $\tau_1, \tau_2$. We keep decomposing the trilinear transversal part in order to achieve higher level of multilinearity.

2.5.4. From $k$-transversal to $(k+1)$-transversal. Now we proceed inductively. Suppose that we are given dyadic cubes $q1^k, q2^k, \ldots, q_{k-1}^k$ of $(k-1)$-th scale which are transversal:
\[
\text{Vol}(n(x1), n(x2), \ldots, n(x_k)) \geq c_1 \sigma_1 \sigma_2 \cdots \sigma_{k-1}
\]
whenever $x_i \in q_{k-1}^k$, $i = 1, \ldots, k$. As before, we denote by $\{q_i^k\}$ the collection of dyadic cubes of sidelength $2\sigma_k$ contained in $q_{k-1}^k$ such that
\[
\bigcup_{q_i^k} q_i^k = q_i^{k-1}, \ i = 1, \ldots, k,
\]
and we also denote by $\{q_i^k\}$ the set $\bigcup_{i=1}^k \{q_i^k\}$. Hence,
\[
\prod_{i=1}^k T_\delta f_{q_{i-1}^k} = \prod_{i=1}^k \left( \sum_{q_i^k} T_\delta f_{q_i^k} \right) = \sum_{q_{k-1}^k} \prod_{i=1}^k T_\delta f_{q_i^k}.
\]
(55)

Fix a dyadic cube $\Omega^k$ of sidlelength $2M_i$ and let $x_0$ be the center of $\Omega^k$. For $i = 1, \ldots, k$, let us denote by $q_i^k \in \{q_i^k\}$ such that
\[
[T_\delta f_{q_i^k}] (x_0) = \max_{q_i^k} [T_\delta f_{q_i^k}] (x_0)
\]
and we set, for $i = 1, \ldots, k$,
\[
\Lambda_i^k = \{q_i^k : [T_\delta f_{q_i^k}] (x_0) \geq (\sigma_k)^{kd} \max_{i = 1, \ldots, k} [T_\delta f_{q_i^k}] (x_0) \}.
\]

*Note that the sequence is independent of $\Omega^2$. 
Then, it follows that

\begin{equation}
\sum_{(q_1^i, \ldots, q_k^i) \in \Pi^k \setminus \Lambda^i} T_{\delta f_{q_i^k}}(x_0) \leq \max [T_{\delta f_{q_i^k}}(x_0)].
\end{equation}

Let \( n_1, \ldots, n_k \) denote the normal vectors \( n(c(q_1^k)), \ldots, n(c(q_k^k)) \), respectively, and let \( \Pi^k = \Pi^k(\Omega^k, q_1^k, q_2^k, \ldots, q_k^k) \) be the \( k \)-plane spanned by \( n_1, \ldots, n_k \). Now, for a sufficiently large constant \( C > 0 \), we define

\begin{equation}
\mathcal{N} = \mathcal{N}(\Omega^k, q_1^k, q_2^k, \ldots, q_k^k) = \{ q^k : \text{dist}(n(q^k), \Pi^k) \leq C \sigma_k \}.
\end{equation}

By (55) it follows that if \( q_i^k \not\in \mathcal{N} \), (39) holds whenever \( \xi_1 \in q_1^k, \ldots, \xi_k \in q_k^k \) and \( \xi_{k+1} \in q_i^k \). Hence, \( q_1^k, \ldots, q_k^k \) are transversal.

We write

\begin{equation}
\prod_{i=1}^k T_{\delta f_{q_i^k}} = \sum_{(q_1^i, \ldots, q_k^i) \in \Pi^k \setminus \Lambda^i} T_{\delta f_{q_i^k}} + \sum_{(q_1^i, \ldots, q_k^i) \in \mathcal{N} \setminus \Lambda^i} T_{\delta f_{q_i^k}}.
\end{equation}

Consider a \( k \)-tuple \( (q_1^i, \ldots, q_k^i) \) which appears in the second sum. There is a \( q_i^k \not\in \mathcal{N} \). By the same manipulation as before, we get

\begin{equation}
\prod_{i=1}^k T_{\delta f_{q_i^k}}(x_0) \leq \sigma_k^{-\frac{d-2}{d+1}} \prod_{i=1}^k \left( [T_{\delta f_{q_i^k}}](x_0) \right)^{\frac{1}{d+1}}.
\end{equation}

Since \( q_1^k, \ldots, q_k^k, q_i^k \) are transversal, by Lemma 2.13 we have, for \( x \in \Omega^k \),

\begin{equation}
\prod_{i=1}^k T_{\delta f_{q_i^k}}(x) \lesssim \sigma_k^{-C} \sum_{q_1^k, \ldots, q_{k+1}^k : \text{trans}} \prod_{i=1}^{k+1} \left( [T_{\delta f_{q_i^k}}](x_0) \right)^{\frac{1}{d+1}}.
\end{equation}

Combining (58) and (61) with (57) and (60), and applying Lemma 2.13 yield, for \( x \in \Omega^k \),

\begin{equation}
\prod_{i=1}^k T_{\delta f_{q_i^k}}(x) \lesssim \left( \max_{q_k^k} \| T_{\delta f_{q_i^k}} \|_p \right)^k + \sigma_k^{-C} \sum_{q_1^k, \ldots, q_{k+1}^k : \text{trans}} \prod_{i=1}^{k+1} \left( [T_{\delta f_{q_i^k}}](x) \right)^{\frac{1}{d+1}}.
\end{equation}

where \( \mathcal{N}(\Omega^k, q_1^k, q_2^k, \ldots, q_k^k) \) is a subset of \( \mathcal{N}(\Omega^k, q_1^k, q_2^k, \ldots, q_k^k) \). After taking \( p/k \)-th power of both sides of inequality, we integrate on \( \mathbb{R}^d \), and use Lemma 2.7 and Lemma 2.6 along with (39) to get

\begin{equation}
\prod_{i=1}^k T_{\delta f_{q_i^k}}(x) \lesssim \left( \max_{q_k^k} \| T_{\delta f_{q_i^k}} \|_p \right)^k + \sigma_k^{-C} \sup_{\tau_1, \ldots, \tau_{k+1}} \max_{q_1^k, \ldots, q_{k+1}^k : \text{trans}} \prod_{i=1}^{k+1} \left( [T_{\delta f_{q_i^k}}](x) \right)^{\frac{1}{d+1}}.
\end{equation}

(62)
2.5.5. Multi-scale decomposition. For \( k = 2, \ldots, d - 1 \), let us set
\[
\mathcal{M}^k f = \sup_{\tau_1, \ldots, \tau_k} \max_{q^k_{i-1}, \ldots, q^k_{k-1}, \text{trans}} \left( \sum_{\Omega^k} \left| \sum_{q^k_i \in q^k_{i-1}, q^k_{i+1} \in [\mathcal{N}] (\Omega^k)} \prod_{i=1}^k T_\beta(\tau_i f_{q^k_i}) \right|_{L^p (\Omega^k)} \right)^{\frac{1}{p}}.
\]
Here \([\mathcal{N}] (\Omega^k)\) depends on \( \tau_1, \ldots, \tau_k \), and \( q^k_{i-1}, \ldots, q^k_{k-1} \), but \( n(q^k) \), \( q^k \in [\mathcal{N}] (\Omega^k) \) is contained in a \( k \)-plan. Starting from (54) we iteratively apply (52) to the transversal products to get
\[
\| T_\beta f \|_p \lesssim \sum_{k=1}^m \sigma_k^{-C} A(\sigma_k^{-2} \delta) \| f \|_p + \sum_{k=2}^m \sigma_k^{-C} \mathcal{M}^k f
\]
(63)
\[+ \sigma_k^{-C} \sup_{\tau_1, \ldots, \tau_{m+1}} \max_{q^m_{i-1}, \ldots, q^m_{m+1}, \text{trans}} \left( \sum_{\Omega^k} \left| \prod_{i=1}^{m+1} T_\beta(\tau_i f_{q^m_i}) \right|_{L^p (\Omega^k)} \right)^{\frac{1}{p}} \]

2.6. Proof of Proposition 2.3. For given \( \beta > 0 \), we need to show that \( A^\beta(s) \leq C \) for \( 0 < s \leq 1 \) if \( p \geq p_3(d) \). Let \( \epsilon > 0 \) be small enough such that \((100d)^{-1} \beta \geq \epsilon \), and choose small \( \epsilon_0 > 0 \) and \( N = N(\epsilon) \) large enough such that Proposition 2.10 and Corollary 2.12 hold uniformly for \( \psi \in \mathcal{G}(\epsilon_0, N) \).

Let \( 0 < s < \delta \leq 1 \), and let \( \sigma_1, \ldots, \sigma_m \) be dyadic numbers satisfying (59). Since \( A(\delta) \leq C \) for \( \delta \geq 1 \) and \( s \leq \sigma_k^{-2} \delta \), we see
\[
A(\sigma_k^{-2} \delta) \leq A(\sigma_k^{-2} \delta)\chi_{(0,1 - \tau_i)}(\sigma_k^{-2} \delta) + C \leq (\sigma_k^{-2} \delta)^{-\frac{d+1}{2} + \frac{1}{d} - \beta} A^\beta(s) + C.
\]
By Proposition 2.11 and Lemma 2.6 we have, for \( p \geq 2(m + 1)/m \),
\[
\sup_{\tau_1, \ldots, \tau_{m+1}} \max_{q^m_{i-1}, \ldots, q^m_{m+1}, \text{trans}} \left( \prod_{i=1}^{m+1} T_\beta(\tau_i f_{q^m_i}) \right) \| f \|_p \lesssim (\sigma_1 \cdots \sigma_m)^{-C} \delta^{-\epsilon} \delta^{\frac{d}{2} - \frac{d+1}{2}} \| f \|_p,
\]
which uniformly holds for \( \psi \in \mathcal{G}(\epsilon_0, N) \).

We have two types of estimate for \( \mathcal{M}^k f \). Since \( q^k_{k-1}, \ldots, q^k_{1} \) are already transversal,
\[
\left| \prod_{q^k_i \in q^k_{i-1} \cap \mathcal{N} (\Omega^k)} \prod_{i=1}^k T_\beta(\tau_i f_{q^k_i}) \right| \leq \sum_{q^k_i \in q^k_{i-1} \cap \mathcal{N} (\Omega^k)} \left| \prod_{i=1}^k T_\beta(\tau_i f_{q^k_i}) \right|.
\]
Here we slightly abuse the definition ‘trans’ and \( q^k_1, \ldots, q^k_k \) : trans means that (55) holds if \( \xi_i \in q^k_i \), \( i = 1, \ldots, k \). Since there are as many as \( O(\sigma_k^{-C}) \) \( (q^k_1, \ldots, q^k_k) \) and the above inequality holds regardless of \( \Omega^k \), we get
\[
\mathcal{M}^k f \lesssim \sigma_k^{-C} \sup_{\tau_1, \ldots, \tau_k} \max_{q^k_{i-1}, \ldots, q^k_{k-1}, \text{trans}} \left( \prod_{i=1}^k T_\beta(\tau_i f_{q^k_i}) \right)^{\frac{1}{p}}.
\]
Since \( q^k_1, \ldots, q^k_k \) are transversal, by Proposition 2.10 (also see Remark 2.3) and Lemma 2.6 we get, for \( p \geq \frac{2k}{k-1} \),
\[
\left( \prod_{i=1}^k T_\beta(\tau_i f_{q^k_i}) \right)^{\frac{1}{p}} \lesssim (\sigma_1 \cdots \sigma_{k-1})^{-C} \delta^{\frac{d}{2} - \frac{d+1}{2} - \epsilon} \left( \prod_{i=1}^k \| T_\beta(\tau_i f_{q^k_i}) \|_p \right)^{\frac{1}{p}} \lesssim \sigma_k^{-C} \delta^{\frac{d}{2} - \frac{d+1}{2} - \epsilon} \| f \|_p.
\]
Hence, for \( p \geq \frac{2k}{k-1} \), we have the uniform estimate for \( \psi \in \mathcal{G}(\epsilon_0, N) \)
\[
\mathcal{M}^k f \lesssim \sigma_k^{-C} \delta^{\frac{d}{2} - \frac{d+1}{2} - \epsilon} \| f \|_p.
\]
On the other hand, fixing $\tau_1, \ldots, \tau_k$, $q_1^{-1}, \ldots, q_k^{-1}$; $\text{trans}$, and $\Omega^k$, we consider the integrals appearing in the definition of $\mathfrak{M}^k f$. Let us write $\Omega^k = q(z, 1/\sigma_k)$. Using Corollary 2.12 for $2 \leq p \leq 2k/(k - 1)$, we have

\begin{equation}
\left\| \sum_{q_i^k \subset q_i^{k-1}} \prod_{i=1}^k T_\delta(\tau_i f_{q_i^k}) \right\|_{L^p\left(\Omega^k\right)} \lesssim \sigma_k^{-\epsilon} \prod_{i=1}^k \left\| \left( \sum_{q_i^k \in [0,1]} |T_\delta(\tau_i f_{q_i^k})|^2 \right)^{\frac{1}{2}} \rho_B(z, \sigma_k^2) \right\|_p.
\end{equation}

Since $[\mathfrak{M}]_p(\Omega^k) \subset \mathfrak{M}(\Omega^k, q_1^{-1}, \ldots, q_k^{-1})$, it is clear that if $q_i^k \in [\mathfrak{M}]_p(\Omega^k)$, $q_i^k \subset N^{-1}(\Pi) + O(\sigma_k)$ for a $k$-plane $\Pi$. Since $q_i^k$, $i = 1, \ldots, k$, $\sum_{q_i^k \in [0,1]} T_\delta(\tau_i f_{q_i^k})$, satisfy the assumptions of Corollary 2.12 (Proposition 2.11) with $\delta = \sigma_k$ and $\sigma = \sigma_1 \cdots \sigma_{k-1}$. Hence, Corollary 2.12 gives (67).

Recalling that $q_i^k$ are contained in $C\sigma_k$-neighborhood of $\Gamma(\psi)$, we see that $\# \mathfrak{M}(\Omega^k) \lesssim \sigma_k^{-\epsilon}$. So, by H"older’s inequality we have

\begin{equation}
\left\| \sum_{q_i^k \subset q_i^{k-1}} \prod_{i=1}^k T_\delta(\tau_i f_{q_i^k}) \right\|_{L^p\left(\Omega^k\right)} \lesssim \sigma_k^{-\epsilon} \sup_{\delta \geq 1} \left\| \left( \sum_{q_i^k} |T_\delta(\tau_i f_{q_i^k})|^p \right)^{\frac{1}{p}} \rho_B(z, \sigma_k^2) \right\|_p.
\end{equation}

Here we bound $\sigma_1, \ldots, \sigma_{k-1}$ with $\sigma_1$ using (66) and replace $C$ with a larger constant $C$, since $\epsilon$ is fixed. By using rapid decay of $\rho$ we sum the estimates along $\Omega^k$ to get

\begin{equation}
M^k f \lesssim C \sigma_k^{-\epsilon} \left( \sum_{q_i^k} |T_\delta(\tau_i f_{q_i^k})|^p \right)^{\frac{1}{p}}.
\end{equation}

By Proposition 2.5, Lemma 2.6, and (64) we get, for $2 \leq p \leq 2k/(k - 1)$,

\begin{equation}
M^k f \lesssim \left( \sigma_k^{-\epsilon} \right)^{\frac{\beta + 2d - k - 1}{2d - k + 1}} \delta^{-\frac{1}{2d - k + 1} + \frac{d - \beta}{p} - \beta} A^\beta(s) + \sigma_k^{-\epsilon} \right) \|f\|_p.
\end{equation}

Here we also use (100d$^{-1} \beta \geq \epsilon$. So, if $p \geq \frac{2d - k + 1}{2d - k + 1}$, \(M^k f \lesssim \left( \sigma_k^{-\epsilon} \sigma_k^\beta \right)^{\frac{\beta + 2d - k - 1}{2d - k + 1}} \delta^{-\frac{1}{2d - k + 1} + \frac{d - \beta}{p} - \beta} A^\beta(s) + \sigma_k^{-\epsilon} \right) \|f\|_p$ for some $a > 0$. Combining this with (66), we have for some $a > 0$

\begin{equation}
M^k f \lesssim \left( \sigma_k^{-\epsilon} \sigma_k^\beta \right)^{\frac{\beta + 2d - k - 1}{2d - k + 1}} \delta^{-\frac{1}{2d - k + 1} + \frac{d - \beta}{p} - \beta} A^\beta(s) + \sigma_k^{-\epsilon} \right) \|f\|_p
\end{equation}

provided that $p \geq \min\left( \frac{2d - k + 1}{2d - k + 1}, \frac{2k}{k - 1} \right)$.

Since $\left(100d\right)^{-1} \beta \geq \epsilon$ and $p_0 > \frac{2d}{\beta}$, from (64) we note that $A(\sigma_k^{-\epsilon} \delta) \lesssim \sigma_k^\beta \delta^{-\frac{1}{2d - k + 1} + \frac{d - \beta}{p} - \beta} A^\beta(s)$. Thus, by (66), the above inequality, and (64) we obtain

\begin{equation}
\|T_\delta f\|_p \lesssim \sum_{k=1}^m \left( \sigma_k^{-\epsilon} \sigma_k^\beta \right) \delta^{-\frac{1}{2d - k + 1} + \frac{d - \beta}{p} - \beta} \|f\|_p + \sigma_k^{-\epsilon} \delta^{-\frac{1}{2d - k + 1} + \frac{d - \beta}{p} - \beta} \|f\|_p
\end{equation}

for some $a > 0$ provided that

\begin{equation}
p \geq \min\left( \frac{2d - k + 1}{2d - k + 1}, \frac{2k}{k - 1} \right), \quad k = 2, \ldots, m, \quad p \geq \frac{2(m + 1)}{m}.
\end{equation}

Since the estimates (65)–(68) hold uniformly for $\psi \in \mathfrak{G}(\varepsilon, N)$, so does (69). Taking sup along $\psi$ and $f$, we have

\begin{equation}
A(\delta) \leq \left( \sum_{k=1}^m C\sigma_k^{-\epsilon} \sigma_k^\beta A^\beta(s) + C\sigma_k^{-\epsilon} \right) \delta^{-\frac{1}{2d - k + 1} + \frac{d - \beta}{p} - \beta}.
\end{equation}
By multiplying $\delta^{\frac{d+1}{4} - \frac{d}{2} - \beta}$ to both sides, $\delta^{\frac{d+1}{4} - \frac{d}{2} + \beta} A(\delta) \leq \sum_{k=1}^{m} C_{\sigma_{k-1}^{C}} \sigma_{k}^{\alpha} A^{\beta}(s) + C_{\sigma_{m}^{C}}$. This is valid as long as $s < \delta \leq 1$. Hence, taking sup for $s < \delta \leq 1$ yields

$$A^{\beta}(s) \leq \sum_{k=1}^{m} C_{\sigma_{k-1}^{C}} \sigma_{k}^{\alpha} A^{\beta}(s) + C_{\sigma_{m}^{C}}$$

if (70) is satisfied. Therefore, choosing $\sigma_{1} \ll \cdots \ll \sigma_{m}$, successively, we can make $\sum_{k=1}^{m} C_{\sigma_{k-1}^{C}} \sigma_{k}^{\alpha} \leq \frac{1}{2}$. This gives the desired $A^{\beta}(s) \leq C_{\sigma_{m}^{C}}$ provided that (70) holds.

Finally, we only need to check that the minimum of

$$\mathcal{P}(m) = \max \left( \frac{2(m + 1)}{m}, \max_{k=2, \ldots, m} \min \left( \frac{2(2d - k + 1)}{2d - k - 1}, \frac{2k}{k - 1} \right) \right), \quad 2 \leq m \leq d - 1$$

is $p_{\alpha}(d)$ as can be done by routine computation. This completes proof. \qed

Remark 2.15. The minimum of $\mathcal{P}$ is achieved when $m$ is near $2d/3$. So, it doesn’t seem that the argument makes use of the full strength of the multilinear restriction estimates.

3. Square function estimates

In this section we prove Theorem 1.2. We firstly obtain multi-(sub)linear square function estimates which are vector valued extensions of multilinear restriction estimates. Then, we modify the argument in Section 2.6 to obtain the sharp square function estimate from these multilinear estimates. Although basic strategy here is similar to the one in the previous section, due to the additional integration in $t$ we need to handle a family of surfaces. This argument in this section is very much in parallel with that of the previous section.

3.1. One parameter family of elliptic functions. As before, for $0 < \epsilon_{\alpha} \ll 1/2$ and an integer $N \geq 100d$, we denote by $\mathfrak{S}(\epsilon_{\alpha}, N)$ the class of smooth functions defined on $I^{d-1} \times I$ which satisfy the following:

(71) \[ \| \psi - \psi_{0} - t \|_{C^{N}(I^{d-1} \times I)} \leq \epsilon_{\alpha}. \]

This clearly implies that, for all $(x, t) \in I^{d-1} \times I$,

(72) \[ \partial_{t} \psi(x, t) \in [1 - \epsilon_{\alpha}, 1 + \epsilon_{\alpha}]. \]

For $\psi \in \mathfrak{S}(\epsilon_{\alpha}, N)$ and $z_{0} = (\zeta_{0}, t_{0}) \in \frac{1}{2}I^{d}$, define

$$\psi_{z_{0}}(\zeta, t) = e^{-2} \left( \psi(\zeta_{0} + t \psi_{\zeta_{0}}(\zeta_{0}, t_{0} + \frac{c^{2}t}{\partial_{t}\psi(z_{0})}) - \psi(z_{0}) - \epsilon \nabla_{\zeta} \psi(z_{0}) \mathcal{H}_{\zeta_{0}}^{\psi} \right),$$

where $\mathcal{H}_{\zeta_{0}}^{\psi} = (\sqrt{H(\psi(\cdot, t_{0}))(\zeta_{0})})^{-1}$. Then we have the following.

Lemma 3.1. Let $z_{0} \in \frac{1}{2}I^{d}$ and $\psi \in \mathfrak{S}(\epsilon_{\alpha}, N)$. There is a $\kappa = \kappa(\epsilon_{\alpha}, N) > 0$, independent of $\psi, \zeta_{0}, t_{0}$, such that $\psi_{z_{0}}$ is contained in $\mathfrak{S}(\epsilon_{\alpha}, N)$ if $0 < \epsilon \leq \kappa$.

Proof. It is sufficient to show that $|\partial_{\zeta}^{\alpha} \partial_{t}^{\beta} (\psi_{z_{0}}(\zeta, t) - \psi_{\zeta}(\zeta) - t)| \leq C\epsilon$, with $C$ independent of $\psi \in \mathfrak{S}(\epsilon_{\alpha}, N)$, if $|\alpha| + |\beta| \leq N$ and $(\zeta, t) \in I^{d}$.

Let $0 < \epsilon \leq 1/4$. If $(\zeta, t) \in I^{d}$ and $|\alpha| + 2\beta > 2$, trivially $|\partial_{\zeta}^{\alpha} \partial_{t}^{\beta} (\psi_{z_{0}}(\zeta, t) - \psi_{\zeta}(\zeta, t) - t)| \leq C\epsilon$ because $z_{0} = (\zeta_{0}, t_{0}) \in \frac{1}{2}I^{d}$. Thus, it is sufficient to consider the cases $\beta = 1, |\alpha| = 0; \beta = 0, 0 \leq |\alpha| \leq 2$. The first case is easy to handle. Indeed, from Taylor’s theorem and (72) \[ \partial_{t}(\psi_{z_{0}}(\zeta, t) - \psi_{\zeta}(\zeta) - t) = (\partial_{t}\psi(z_{0}))^{-1} (\partial_{t}\psi(\zeta_{0} + t \psi_{\zeta_{0}}(\zeta_{0}, t_{0} + \frac{c^{2}t}{\partial_{t}\psi(z_{0})}) - \partial_{t}\psi(z_{0})) = O(\epsilon). \]
To handle the second case, we consider Taylor’s expansion of $\psi$ in $t$ with integral remainder:

$$\psi(\zeta, t) = \psi(\zeta, t_0) + \partial_t \psi(\zeta, t_0)(t - t_0) + R_1(\zeta, t),$$

where $R_1(\zeta, t) = (t - t_0)^2 \int_0^1 (1 - s) \partial^2_s \psi(\zeta, (t - t_0)s + t_0) ds$. The change of variables $t \to t_0 + \epsilon^2(\partial_t \psi(\zeta_0))^{-1}t$, $\zeta \to \zeta_0 + \epsilon \mathcal{H}_0^\psi \zeta$ gives

$$\psi(\zeta_0 + \epsilon \mathcal{H}_0^\psi \zeta, t_0 + \frac{\epsilon^2 t}{\partial_t \psi(\zeta_0)}) = \epsilon^2 \psi(\cdot, t_0)^{\psi}(\zeta) + \psi(\zeta_0) + \epsilon \nabla_\zeta \psi(\zeta_0) \mathcal{H}_0^\psi \zeta$$

$$+ \epsilon^2 \partial_t \psi(\zeta_0 + \epsilon \mathcal{H}_0^\psi \zeta, t_0) + \epsilon^2 \partial_t \psi(\zeta_0)^{-1}t + \tilde{R}(\zeta, t)$$

where $\psi(\cdot, t_0)^{\psi}$ is defined by \eqref{eq:psi} and $\tilde{R}(\zeta, t) = R_1(\zeta_0 + \epsilon \mathcal{H}_0^\psi \zeta, t_0 + \epsilon^2(\partial_t \psi(\zeta_0))^{-1}t)$. Hence, it follows that

$$\psi(\cdot, t_0)^{\psi} = \psi(\cdot, t_0)^{\psi}(\zeta) - \psi_0 + \partial_t \psi(\zeta_0 + \epsilon \mathcal{H}_0^\psi \zeta, t_0) - \partial_t \psi(\zeta_0)^{-1}t + \epsilon^{-2} \tilde{R}(\zeta, t).$$

Since $\psi(\cdot, t_0) - t_0 \in \Phi(\epsilon_0, N)$ and $|\psi(\cdot, t_0)^{\psi} - \psi_0| \leq C \epsilon$ on $I^d$ for $|\alpha| = 0, 1, 2$ (similarly to the proof of Lemma \eqref{lem:translation}). By \eqref{eq:bound} and mean value theorem we also have $(\partial_t \psi(\zeta_0))^{-1} \partial^\alpha_t (\partial_t \psi(\zeta_0 + \epsilon \mathcal{H}_0^\psi \zeta, t_0) - \partial_t \psi(\zeta_0)) \leq O(\epsilon)$ in $C^{N}(I^{d-1})$ for $|\alpha| = 0, 1, 2$. Note that

$$\epsilon^{-2} \tilde{R}(\zeta, t) = \epsilon^{-2} \int_0^1 (1 - s) \partial^2_s \psi(\zeta_0 + \epsilon \mathcal{H}_0^\psi \zeta, \epsilon^2(\partial_t \psi(\zeta_0))^{-1}ts + t_0) ds.$$

Thus, again by \eqref{eq:bound} it is easy to see that $\partial^\alpha_t (\epsilon^{-2} \tilde{R}) = O(\epsilon^{2+|\alpha|})$ for any $\alpha$. Therefore, combining the two together we have $|\partial^\alpha_t (\psi(\cdot, t) - \psi_0)| \leq C \epsilon$ on $I^d$ for $|\alpha| = 0, 1, 2$.

3.2. Square function with localized frequency. Abusing the conventional notation we denote by $m(D)f$ the multiplier operator given by $m(D)f(\xi) = m(\xi)f(\xi)$, and we also write $D = (D^k, D_d)$ where $D^k, D_d$ correspond to the frequency variables $\zeta$, $\tau$, respectively.

In order to show \eqref{eq:Phi}, by Littlewood-Paley decomposition, scaling, and further finite decompositions, it is sufficient to show

$$\left\| \left( \int_0^1 \frac{\partial}{\partial t} \mathcal{R}_t f(x)^2 \, dt \right)^{1/2} \right\|_p \leq C \| f \|_p$$

for some small $\varepsilon > 0$. And by decomposing $\hat{f}$ which may now be assumed to be supported in $S^{d-1} + O(\varepsilon^2)$ and rotation we may assume $\hat{f}$ is supported in $B(-\varepsilon_d, \varepsilon^2 \varepsilon^2)$ with some $c > 0$. Hence, by discarding harmless smooth multiplier the matter reduces to showing

$$\left\| \left( \int_0^1 \frac{\partial}{\partial t} \mathcal{R}_t f(x)^2 \, dt \right)^{1/2} \right\|_p \leq C \| f \|_p.$$

By changing variables in frequency domain, $D_d \to D_d + 1$, $D^k \to D^k$, $D_d \to (\varepsilon D^k, \varepsilon^2 D_d)$ and $t \to \varepsilon^2 t + 1,$ this is equivalent to

$$\left\| \left( \int_0^1 \frac{\partial}{\partial t} \mathcal{R}_t f(x)^2 \, dt \right)^{1/2} \right\|_p \leq C \| f \|_p.$$

where $\psi_{br}(\zeta, t) = \varepsilon^{-2}(1 - \sqrt{1 + 2\varepsilon^2 t + \varepsilon^4 t^2 - \varepsilon^2 |\zeta|^2})$ and $\chi_0$ is a smooth function supported in a small neighborhood of the origin. Clearly, $\psi_{br}$ satisfies \eqref{eq:psi} with $\epsilon_0 = C \varepsilon^2$ for some $C > 0$. Consequently, we are led to consider general $\psi \in \Phi(\epsilon_0, N)$ rather than the specific $\psi_{br}$.

Let us define the class $\mathcal{E}(N)$ of smooth functions by setting

$$\mathcal{E}(N) = \{ \eta \in C^\infty(I^d \times I): \| \eta \|_{C^N(I^d \times I)} \leq 1, 1/2 \leq \eta \leq 1 \}.$$

Let $\psi \in \Phi(\epsilon_0, N)$ and $\eta \in \mathcal{E}(N)$. For $0 < \delta$ and $f$ with $\hat{f}$ supported in $\frac{1}{\delta} I^d$, we define $S_\delta = S_\delta(\psi, \eta)$ by

$$S_\delta f(x) = \left\| \phi \left( \frac{\eta(D)(D_d - \psi(D^k, t))}{\delta} \right) f \right\|_{L^p_{\text{loc}}(I^d)}.$$
Compared to \( \psi \), the role of \( \eta \) is less significant but this enables us to handle more general square functions (in particular, see Remark 3.3). By dyadic decomposition away from the singularity \( \psi \) is reduced to obtaining the sharp bound

\[
\|S_\delta f\|_p \leq C\delta^{\frac{d}{2} - \frac{d+2}{4} - \epsilon}\|f\|_p, \quad \epsilon > 0,
\]

when \( \hat{f} \) is supported in a small neighborhood of the origin. This is currently verified for \( \delta \geq \frac{2(d+2)}{d} \) by making use of bilinear restriction estimate for the elliptic surfaces. The following is our main result concerning the estimate \( \psi \).

**Proposition 3.2.** Let \( p_s = p_s(d) \) be given by \( \psi \) and \( \supp \hat{f} \subset \frac{1}{2}I^d \). If \( p \geq \min(p_s(d), \frac{2(d+2)}{d}) \) and \( \epsilon_0 \) is sufficiently small, for \( \epsilon > 0 \) there is a \( N = N(\epsilon) \) such that \( \psi \) holds uniformly for \( \psi \in \mathcal{E}(\epsilon_0, N), \eta \in \mathcal{E}(N) \).

**Proof of Theorem 1.2** By choosing small \( \epsilon > 0 \) in the above, we can make \( \psi_{br} \) be in \( \mathcal{E}(\epsilon_0, N) \) for any \( \epsilon_0 \) and \( N \). Hence, Proposition 3.2 gives \( \psi \) for any \( \epsilon > 0 \) if \( p \geq \min(p_s(d), \frac{2(d+2)}{d}) \). Hence, dyadic decomposition of the multiplier operator in \( \psi \) and using \( \psi \) followed by summation along dyadic pieces gives \( \psi \) for \( \alpha > d/2 - d/p \). This proves Theorem 1.2. \( \square \)

**Remark 3.3.** As has been shown before, for the proof of Theorem 1.2 it suffices to consider an operator which is defined without \( \eta \) but by allowing \( \eta \) in \( \psi \) we can handle the square function estimates for the operator \( f \rightarrow \phi(\frac{1-|D|/t}{\delta})f \) which is closely related to smoothing estimates for the solutions to the Schrödinger and wave equations (for example, see \( \psi \)). In fact, Proposition 3.2 implies, for \( \epsilon > 0 \),

\[
\left\| \left( \int_{1/2}^2 \left| \phi \left( \frac{1-|D|/t}{\delta} \right) f \right|^2 dt \right)^{1/2} \right\|_p \leq \delta^{\frac{d}{2} - \frac{d+2}{4} - \epsilon} C\|f\|_p
\]

if \( p \geq p_s(d) \). Indeed, by finite decompositions, rotation and scaling, as before, it is sufficient to consider time average over the interval \( I_s = (1 - \epsilon^2, 1 + \epsilon^2) \) and we may assume that \( \hat{f} \) is supported in \( B(-\epsilon_d, \epsilon^2) \). Writing \( 1 - |\xi|/t = t^{-2}(t+|\xi|)^{-1}(\tau - \sqrt{\tau^2 - |\xi|^2} \tau + \sqrt{\tau^2 - |\xi|^2}) \) for \( \xi \in B(-\epsilon_d, \epsilon^2) \), the same change of variables \( D_d \rightarrow D_d + 1, (D', D_d) \rightarrow (\epsilon D', \epsilon D_d) \) and \( t \rightarrow \epsilon^2 t + 1 \) transforms \( \phi(\frac{1-|D|/t}{\delta}) \) to \( \phi(\frac{2(\epsilon_d^2)}{(\tau-\epsilon^2 \phi_{br})}) \) with a smooth \( \eta \) which satisfies \( \eta \in (1 - \epsilon_c/2, 1 + \epsilon_c/2) \). Hence, we now apply Proposition 3.2 with sufficiently small \( \epsilon \) to get \( \psi \).

Similarly as before, in order to control \( L^p \) norm of \( S_\delta \) we define \( B(\delta) = B_p(\delta) \) by

\[
B(\delta) \equiv \sup \left\{ \|S_\delta(\psi, \eta)f\|_p : \psi \in \mathcal{E}(\epsilon_0, N), \eta \in \mathcal{E}(N), \|f\|_p \leq 1, \supp \hat{f} \subset \frac{1}{2}I^d \right\}.
\]

As before, using Lemma 2.9 it is easy to see that \( B(\delta) \leq C \) if \( \delta \geq 1 \), and \( B(\delta) \leq C\delta^{-c} \) for some \( c > 0 \), otherwise (for example, see the paragraph below Proposition 3.6). We also define for \( \beta > 0 \) and \( \delta \in (0, 1) \),

\[
B^\beta(\delta) = B_p^\beta(\delta) \equiv \sup_{\delta < s \leq 1} s^{\frac{d-2}{2} - \frac{d+2}{4} + \beta} B_p(s).
\]

Thus, Theorem 1.2 follows if we show \( B^\beta(\delta) \leq C \) for any \( \beta > 0 \). As observed in the previous section the bound for \( S_\delta f \) improves if the Fourier transform of \( f \) is contained in a set of smaller diameter. The following plays a crucial role in the induction argument (see Section 3.6).

**Proposition 3.4.** Let \( 0 < \delta \ll 1, \psi \in \mathcal{E}(\epsilon_0, N), \) and \( \eta \in \mathcal{E}(N) \). Suppose that \( \hat{f} \) is supported in \( q(a, \epsilon), 10\sqrt{\delta} \leq \epsilon \leq 1/2, \) and \( a \in \frac{1}{2}I^d \). Then, if \( \epsilon_0 > 0 \) is small enough, there is a \( \kappa = \kappa(\epsilon_0, N) \) such that

\[
\|S_\delta(\psi, \eta)f\|_p \leq C\epsilon^{\frac{d}{2} + \frac{1}{2}} B_p(\epsilon^{-2}\delta)\|f\|_p
\]

holds with \( C \), independent of \( \psi, \psi, \text{whenever} \epsilon \ll \kappa.\)
Proof. By breaking the support of \( \hat{f} \) into a finite number of dyadic cubes, we may assume that \( \hat{f} \) is supported in \( q(a, \nu) \) for a small constant \( \nu > 0 \) satisfying \( \nu^2 d^2 \in [2^{-5}, 2^{-4}] \). This only increases the bound by a constant multiple. Since \( \hat{f} \) is supported in \( q(a, \nu) \) and \( a = (a', a_d) \in \frac{1}{2} I^d \), from (72) and the fact that \( 1/2 \leq \eta \leq 1 \) it is clear that \( \phi \left( \frac{\eta(D, t)}{\delta} - \psi(D', t) \right)^1 \hat{f}(\xi) \) is supported in \( O(\delta) \)-neighborhood of \( \tau = \psi(\zeta, t) \).

Let \( \alpha = t_0 < t_1 < \ldots < t_l = \beta, \ l \leq O(\varepsilon^{-1}) \), such that \( t_{k+1} - t_k \leq \nu^2 \varepsilon^2 \). Since \( \delta \leq 10^{-2} \varepsilon^2 \), by (71) and (72) it follows that if \( t \in [t_k, t_{k+1}] \), then \( \phi \left( \frac{\eta(\zeta, t)}{\delta} - \psi(\zeta, t) \right)^1 \hat{f}(\xi) \) is supported in the parallelepiped

\[
P_k = \left\{ (\zeta, \tau) : \max_{i=1, \ldots, d-1} |\zeta_i - a'_i| < \nu \varepsilon, |\tau - \psi(a', t_k) - \nabla_\zeta \psi(a', t_k)(\zeta - a')| \leq 2d^2 \nu^2 \varepsilon^2 \right\}.
\]

This follows from Taylor's theorem since \( \psi \in \overline{E}(\varepsilon, N) \). By (72) it is easy to see that \( \{P_k\}_{k=1}^l \) are overlapping boundedly. In fact, \( \phi \left( \frac{\eta(\zeta, t)}{\delta} - \psi(\zeta, t) \right)^1 \hat{f}(\xi), t \in [t_k, t_{k+1}] \), is supported in

\[
P_k = \{ \xi \in q(a, \varepsilon \varepsilon) : |\tau - \psi(\zeta, t_k)| \leq C \varepsilon^2 \}, \ k = 0, \ldots, l - 1,
\]

with \( C \geq 3d^2 \nu^2 \varepsilon^2 \) and \( \{P_k\} \) are boundedly overlapping because of (72), and by Taylor's expansion it is easy to see that \( P_k \subset \tilde{P}_k \) because the 2nd remainder is uniformly \( O(\varepsilon^2) \) for \( \psi \in \overline{E}(\varepsilon, N) \).

Let \( \varphi \) be a smooth function supported in \( 2I^d \) and \( \varphi = 1 \) on \( I^d \). Let \( L_{P_k} \) be the affine map which bijectively maps \( P_k \) to \( I^d \), and set \( \varphi_{P_k} = \varphi(L_{P_k} \cdot) \) so that \( \varphi_{P_k} \) vanishes outside of \( 2P_k \) and equals 1 on \( P_k \). Here \( 2P_k \) denotes the parallelepiped which is given by dilating \( P_k \) twice from the center of \( P_k \). Then we have

\[
\left( S_\delta f(x) \right)^2 = \sum_k \int_{I_k} \left| \phi \left( \frac{\eta(D, t)(D_d - \psi(D', t))}{\delta} \right)^1 \varphi_{P_k}(D) f(x) \right|^2 dt.
\]

Since \( p \geq 2 \), by Hölder’s inequality it follows that

\[
S_\delta f(x) \leq C \varepsilon^{ \frac{1}{p} - \frac{1}{2}} \left( \sum_k \left\| \phi \left( \frac{\eta(D, t)(D_d - \psi(D', t))}{\delta} \right)^1 \varphi_{P_k}(D) \right\|_{L^p_{\delta}(I_k)}^p \right)^{\frac{1}{p}}.
\]

Hence it is sufficient to show that

\[
\left\| \phi \left( \frac{\eta(D, t)(D_d - \psi(D', t))}{\delta} \right)^1 \varphi_{P_k}(D) \right\|_{L^p_{\delta}(I_k)} \leq C \varepsilon B_p(\varepsilon^{-2} \delta) \| \varphi_{P_k}(D) \|_p.
\]

Because \( \sum_k \| \varphi_{P_k}(D) \|_p^{\frac{1}{p}} \leq C \| f \|_p \) for \( 2 \leq p \leq \infty \). This follows by interpolation between the estimates for \( p = 2 \) and \( p = \infty \). The first is an easy consequence of Plancherel’s theorem because \( \{2P_k\} \) are boundedly overlapping and the latter is clear since \( F^{-1}(\varphi_{P_k}) \in L^1 \) uniformly.

Now we make the change of variables

\[
t \rightarrow \varepsilon^2 (\partial_\psi \psi(a', t_k))^{-1} t + t_k, \quad \xi \rightarrow L(\xi) = (L'(\xi), L_d(\xi)),
\]

where

\[
L'(\xi) = \varepsilon H^\psi_{(a', t_k)} \xi + a', \quad L_d(\xi) = \varepsilon^2 \tau + \psi(a', t_k) + \varepsilon \nabla_\zeta \psi(a', t_k) H^\psi_{(a', t_k)} \xi,
\]

and

\[
\varepsilon^2 x_d \rightarrow x_d, \quad \varepsilon H^\psi_{(a', t_k)} (x' + x_d \nabla_\zeta \psi(a', t_k)) \rightarrow x'.
\]

Then, (78) follows if we show

\[
\left\| \phi \left( \frac{\eta(L(D), t)(D_d - \psi_{(a', t_k)}(D', t))}{\varepsilon^2} \right)^1 \right\|_{L^p_{\varepsilon^2}(0, 2 \varepsilon^2)} \leq C B_p(\varepsilon^{-2} \delta) \| f \|_p
\]

when the support \( \hat{f} \) is contained in \( L^{-1}(2P_k) \). Clearly, \( \eta(L(\xi), t) \in E(N) \) and \( L^{-1}(2P_k) \) is contained in the set \( \{ (\zeta, \tau) : |\zeta| \leq 4 \varepsilon, |\tau| \leq 8d^2 \nu^2 \} \subset \frac{1}{2} I^d \). From Lemma 8.11 there exists \( \kappa > 0 \) such that
\( \psi^{\varepsilon}_{u,t} \in \overline{B}(\varepsilon_0, N) \) if \( 0 < \varepsilon \leq \kappa \). Hence, using the definition of \( B_p(\delta) \) we get the desired inequality for \( \varepsilon \leq \kappa \).

3.3. Multi-(sub)linear square function estimates. Let \( \psi \in \overline{B}(\varepsilon_0, N) \) and set

\[
\Gamma^t = \Gamma^t(\psi) := \{ (\zeta, \psi(\zeta, t)) : \zeta \in \frac{1}{2} I^d \}.
\]

As before we denote by \( \Gamma^t(\delta) \) the \( \delta \)-neighborhood \( \Gamma^t + O(\delta) \). Clearly, from (72) it follows that, for \( \delta > 0 \),

\[
\Gamma^t(\delta) \cap \Gamma^s(\delta) = \emptyset, \quad \text{if } |t - s| \geq C\delta
\]

for some \( C > 0 \). We also denote by \( N^t \) the (upward) normal map from the surface \( \Gamma^t \) to \( \mathbb{S}^{d-1} \).

**Definition 3.5** (Normal vector field \( n = n(\psi) \)). The map \( (\zeta, t) \rightarrow (\zeta, \psi(\zeta, t)) \) is clearly one to one and we may assume that the image of this map contains \( I^d \) by extending \( \psi(\zeta, t) \) to a larger set \( I^d \times CI \), while (74) is satisfied. Hence, for each \( \xi = (\zeta, \tau) \in I^d \) there is a unique \( t \) such that \( \xi = (\zeta, \psi(\zeta, t)) \). Then we define \( n(\xi) \) to be the normal vector to \( \Gamma^t \) at \( \xi \), which forms a vector field on \( I^d \).

A natural attempt for multilinear generalization of \( S_\delta \) is to consider \( \prod_{i=1}^k S_\delta f_i \) under transversality condition between \( \operatorname{supp} f_i \). But, induction on scale argument does not work well with this naive generalization and it doesn’t seem easy to obtain the sharp multilinear square function estimates directly. We get around the difficulty by considering a vector valued extension in which we discard the exact structure of the operator \( S_\delta \). As is clearly seen in its proof, the estimate in Proposition 3.6 is not limited to the surfaces given by \( \psi \in \overline{B}(\varepsilon_0, N) \) but it holds for more general class of surfaces as long as the transversality is satisfied.

**Proposition 3.6.** Let \( 2 \leq k \leq d \) be an integer and \( 0 < \sigma \ll 1 \), and let \( \Gamma^t \) be given by \( \psi \in \overline{B}(\varepsilon_0, N) \), and the functions \( G_i, 1 \leq i \leq k, \) be defined on \( \mathbb{R}^d \times I \). Suppose that, for each \( t \in I, \ G_1(\cdot,t), \ldots, G_k(\cdot,t) \) satisfy that, for \( 0 < \delta \ll \sigma, \)

\[
supp \hat{G}_i(\cdot, t) \subset \Gamma^t(\delta), \quad t \in I,
\]

and suppose that

\[
Vol(n(\xi_1), n(\xi_2), \ldots, n(\xi_k)) \geq \sigma,
\]

whenever \( \xi_i \in \supp \hat{G}_i(\cdot, t) + O(\delta) \) for some \( t \in I \). Then, if \( p \geq 2k/(k - 1) \) and \( \varepsilon_0 > 0 \) is small enough, for \( \varepsilon > 0 \) there is an \( N = N(\varepsilon) \) such that

\[
\left\| \prod_{i=1}^k \| G_i \|_{L^p(I)} \right\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^{-\sigma} \delta^{-\varepsilon} \prod_{i=1}^k (\delta^{-\varepsilon} \| G_i \|_{L^2(I)})
\]

holds with \( C, C_\varepsilon \), independent of \( \psi \).

Without being concerned about the optimal \( \alpha \) for a while, we first observe that, for \( p \geq 2 \), there is an \( \alpha \) such that

\[
\left\| \| G_i \|_{L^p(I)} \right\|_{L^p(\mathbb{R}^d)} \leq C\delta^{-\alpha} \| G_i \|_{L^2(I)}
\]

holds uniformly if \( \psi \in \overline{B}(\varepsilon_0, N) \) and \( N \) is large enough \( (N \geq 100d) \). (It is enough to keep \( \| \psi \|_{CN(1)^d} \) uniformly bounded.) To see this, let \( \varphi \) be a smooth function supported in \( 2I \) and \( \varphi = 1 \) on \( I \), and we set \( K' \varphi = F^{-1}(\varphi(\frac{\psi(x)}{|\psi(x)|} \tilde{\chi}(x))) \). Then, by Lemma 2.9, \( |K' \varphi(x)| \leq C\delta \mathcal{R}_M(x) \) for a large \( M \) with \( C \), depending only on \( \| \psi \|_{CN(1)^d} \). Since \( \supp F(G_i(\cdot,t)) \subset \Gamma^t(\delta), \ G_i(\cdot,t) = K' \varphi \ast G_i(\cdot,t) \). So, \( |G_i(x,t)| \leq C\delta \mathcal{R}_M \ast |G_i(\cdot,t)|, t \in I \) and by Minkowski’s inequality we get

\[
\| G_i(x,t) \|_{L^2(I)} \leq C\delta \mathcal{R}_M \ast (\| G_i(\cdot,t) \|_{L^2(I)})(x).
\]
Young’s convolution inequality gives the inequality (81), namely with $\alpha = d - 1$, if taking sufficiently large $M$.

Proof of Proposition 2.11. Since $F(G_i(\cdot, t)) = \varphi \left( \frac{x - \gamma(t)}{\delta(t)} \right) \overline{\xi} F(G_i(\cdot, t))$, by Schwarz’s inequality and Plancherel’s theorem, $|G_i(x, t)| \lesssim \delta^{-\alpha} \|G_i(\cdot, t)\|_2$. So, this gives (83) for $p = \infty$. Thus, by interpolation it is sufficient to show (83) with $p = \frac{2k}{k-1}$.

Let us set $R = \delta^{-1}$ and we may set $x = 0$. Following the same argument as in the proof of Proposition 2.11 we start with the assumption that, for $0 < \delta < \sigma$,

\begin{equation}
\left\| \prod_{i=1}^{k} \|G_i\|_{L^2_t(L^2_x)} \right\|_{L^\infty_{t, x}(B(0, R))} \lesssim R^\alpha R^\frac{\beta}{2} \prod_{i=1}^{k} \|G_i\|_{L^2_t, x},
\end{equation}

holds uniformly $\psi \in \mathcal{T}(\epsilon_2, N)$ whenever (81) and (82) are satisfied. By (83) and Hölder’s inequality, this is true for a large $\alpha > 0$. Hence, it is sufficient to show (86) implies that for $\epsilon > 0$ there is an $N = N(\epsilon)$ such that, for some $\kappa > 0$,

\begin{equation}
\left\| \prod_{i=1}^{k} \|G_i\|_{L^2_t(L^2_x)} \right\|_{L^\infty_{t, x}(B(0, R))} \lesssim C_\epsilon \alpha^{-\kappa} R^\frac{\beta}{2} + \epsilon R^{-\frac{\beta}{2}} \prod_{i=1}^{k} \|G_i\|_{L^2_t, x},
\end{equation}

holds uniformly for $\psi \in \mathcal{T}(\epsilon_2, N)$. Then, iterating this implication from (86) to (87) gives the desired inequality. (See the paragraph below (20).)

Since $\hat{\rho} \in \mathcal{T}(\epsilon_2, N)$ is supported in a ball of radius $\sim R^{-\frac{\beta}{2}}$, the Fourier transform of $\rho \in \mathcal{T}(\epsilon_2, N)$ is contained in $\Gamma^{\frac{\beta}{2}} + O(R^{-1/2})$ for each $t$ and (82) holds with $\delta = R^{-\frac{\beta}{2}}$ since $\delta \ll \sigma$. Hence, by the assumption (86), it follows that

\begin{equation}
\left\| \prod_{i=1}^{k} \|\rho \in \mathcal{T}(\epsilon_2, N) G_i\|_{L^2_t(L^2_x)} \right\|_{L^\infty_{t, x}(B(0, R))} \lesssim C R^\frac{\beta}{2} R^{-\frac{\beta}{2}} \prod_{i=1}^{k} \|\rho \in \mathcal{T}(\epsilon_2, N) G_i\|_{L^2_t, x}.
\end{equation}

We now decompose $G_i(\cdot, t)$ into $\{G_i, q, \cdot, t\}$ which is defined by

\begin{equation}
\mathcal{F}(G_i, q, \cdot, t) = \chi \mathcal{F}(G_i(\cdot, t)).
\end{equation}

Here $\{q\}$ are the dyadic cubes of sidelength $l$, $R^{-1/2} < l \leq 2R^{-1/2}$, which we already used in the Proof of Proposition 2.11. We write

\begin{equation}
G_i(x, t) = \sum_{q} G_{i, q}(x, t).
\end{equation}

In what follows we may assume $G_{i, q} \neq 0$. By (81) it follows that, for each $t$, the cubes $\{q\}$ appearing in the sum are contained in $\Gamma^{\frac{\beta}{2}}(R^{-\frac{\beta}{2}})$ because $G_{i, q}(\cdot, t) = 0$, otherwise. We also note from (72) that there is an interval $I_{i, q}$ of length $CR^{1/2}$ such that $G_{i, q}(\cdot, t) = 0$ if $t \not\in I_{i, q}$. Hence we may multiply the characteristic function of $\chi_{I_{i, q}}$ so that

\begin{equation}
G_{i, q} = G_{i, q}(\cdot, t) \chi_{I_{i, q}}(t).
\end{equation}

Since the Fourier supports of $\{\rho \in \mathcal{T}(\epsilon_2, N) G_{i, q}(\cdot, t)\}$ are boundedly overlapping, by Plancherel’s theorem it follows that

\begin{equation}
\prod_{i=1}^{k} \|\rho \in \mathcal{T}(\epsilon_2, N) G_{i, q}\|_{L^2_t(L^2_x)} \leq C \prod_{i=1}^{k} \left\| \left( \sum_{q} |\rho \in \mathcal{T}(\epsilon_2, N) G_{i, q}|^2 \right)^{\frac{1}{2}} \right\|_{L^2_t, x}.
\end{equation}

Combining this with (88) we have

\begin{equation}
\left\| \prod_{i=1}^{k} \|\rho \in \mathcal{T}(\epsilon_2, N) G_{i}\|_{L^2_t(L^2_x)} \right\|_{L^\infty_{t, x}(B(0, R))} \leq C R^\frac{\beta}{2} R^{-\frac{\beta}{2}} \prod_{i=1}^{k} \left\| \left( \sum_{q} |\rho \in \mathcal{T}(\epsilon_2, N) G_{i, q}|^2 \right)^{\frac{1}{2}} \right\|_{L^2_t, x}.
\end{equation}
Since $\rho_{B(z, \sqrt{R})}$ is rapidly decaying outside of $B(z, \sqrt{R})$, we have for any large $M > 0$
\[
\left\| \prod_{i=1}^{k} \rho_{B(z, \sqrt{R})} G_i \right\|_{L^{2, \infty}_x} \lesssim R^{\frac{d}{2} - \frac{k}{4}} \prod_{i=1}^{k} \left\| \chi_{B(z, R^{\frac{d}{2} + \varepsilon})} \left( \sum_{q_i} |G_i(q)|^2 \right)^{\frac{1}{2}} \right\|_{L^2_x} + R^{-M} \prod_{i=1}^{k} \|G_i\|_{L^2_x},
\]
(92)

We now partition the interval $I_{i,q}$ further into intervals $I^l_{i,q} = [t_l, t_{l+1}]$, $l = 1, \ldots, \ell_0$, of length $\sim R^{-1}$. Then the Fourier support of $G_{i,q}(\cdot, t)$, $t \in I^l_{i,q}$ is contained in $O(R^{-1})$ neighborhood of $\Gamma^l$. Let $(\zeta_q, \tau_q)$ be the center of $q$ and we define a set $r^l_{i,q}$ by
\[
\{ (\zeta, \tau) : |\zeta - \zeta_q| \leq C\delta^{\frac{1}{2}}, |\tau - \psi(\zeta_q, t_l) - \nabla_\zeta \psi(\zeta_q, t_l) \cdot (\zeta - \zeta_q)| \leq C\delta \}
\]
with a constant $C > 0$ large enough. It follows that Fourier transform of $G_{i,q}(\cdot, t)$, $t \in I^l_{i,q}$ is supported in $r^l_{i,q}$. This is easy to see from 2nd order Taylor approximation because $\psi \in C^2(\epsilon, N)$.

Also define $m^l_{i,q}$ by
\[
m^l_{i,q} = \rho \left( \frac{\zeta - \zeta_q}{C\sqrt{\delta}}, \frac{\tau - \psi(\zeta_q, t_l) - \nabla_\zeta \psi(\zeta_q, t_l) \cdot (\zeta - \zeta_q)}{C\delta} \right)
\]
with a suitable $C > 0$ such that $m^l_{i,q}$ is comparable to 1 on $r^l_{i,q}$. Now, we set
\[
\mathcal{F}(G^l_{i,q}(\cdot, t)) = (m^l_{i,q})^{-1} (m^l_{i,q}) \mathcal{F}(G_{i,q}(\cdot, t)) \chi_{I^l_{i,q}}(t).
\]
Denoting by $n^l_{i,q}$ the normal vector $n(\zeta_q, \psi(\zeta_q, t_l))$, we also set with a large $C > 0$
\[
T^l_{i,q} = \{ x : |x \cdot n^l_{i,q}| \leq C, |x - (x \cdot n^l_{i,q})n^l_{i,q}| \leq CR^{-\frac{d}{2}} \}.
\]

Let us set $K^l_{i,q} = \mathcal{F}^{-1}(m^l_{i,q})$ so that $G_{i,q}(\cdot, t) = G_{i,q}(\cdot, t) * K^l_{i,q}$ if $t \in I^l_{i,q}$. Since $\rho$ is supported in $q(0, 1)$, $K^l_{i,q} \lesssim R^{-\frac{d+1}{2}} \mathcal{F}^{l}_{i,q}$. By (90) it follows that $\sum_{q_i} \|G_{i,q}\|_{L^2_{t,x}}^2 = \sum_{q_i} \|G_{i,q}\|_{L^2_{t,x}}^2 \lesssim \sum_{q_i} \|G_{i,q}\|_{L^2_{t,x}}^2 * |K^l_{i,q}|$
\[
\lesssim \sum_{q_i} \|G_{i,q}\|_{L^2_{t,x}}^2 \lesssim \left( R^{-\frac{d+1}{2}} \mathcal{F}^{l}_{i,q} \right) (x).
\]
(96)

We denote by $\tilde{T}^l_{i,q}$ the tube $R^{1+\varepsilon}T^l_{i,q}$, which is an $R^{1+\varepsilon}$ times dilation of $T^l_{i,q}$ from its center. So, from (96) we have, for $x, y \in B(z, R^{1/2+\varepsilon})$
\[
\sum_{q_i} \|G_{i,q}(y, \cdot)\|_{L^2_{t,x}}^2 \lesssim \left( R^{1+\varepsilon} \mathcal{F}^{l}_{i,q} \right) (x).
\]

Once we have this equality we can repeat the argument from (24) to (26) which is in Proof of Proposition 2.11 and also using (92), we have
\[
\left\| \prod_{i=1}^{k} |G_i(\cdot)|_{L^2_{t,x}}^2 \right\|_{L^{\frac{2}{\varepsilon+2}}(B(0, R))} \lesssim R^{\varepsilon_1 + \frac{d}{2}} \prod_{i=1}^{k} \left( \sum_{q_i} \|G_{i,q}(\cdot, t)\|_{L^2_{t,x}}^2 \right) \left( \mathcal{F}^{l}_{i,q} \right) (x) + \varepsilon,
\]
(97)

\[
\left\| \prod_{i=1}^{k} |G_i(\cdot)|_{L^2_{t,x}}^2 \right\|_{L^{\frac{2}{\varepsilon+2}}(B(0, R))} + \varepsilon,
\]
(97)
where $\mathcal{E} = R^{-M} \prod_{i=1}^{k} \|G_i\|_{L_{R^k}^2}$ for any large $M > 0$. Hence, for (87) it suffices to show that

$$\left\| \prod_{i=1}^{k} \left( \sum_{q_i} \|G_{i,q}^l\|_{L^2(I_{i,q}^l)}^2 \right)^{\frac{1}{2}} \right\|_{L^\infty(B(0,2R))} \lesssim \sigma^{-\kappa} R^{\kappa} R^{-\frac{d-k}{2}} \prod_{i=1}^{k} \|G_i\|_{L_{R^k}^2}.$$

Since $\|G_{i,q}^l\|_{L^2(I_{i,q}^l)} = \|G_{i,q}\|_{L^2(I_{i,q}^l)}$ by (96), making use of disjointness of $I_{i,q}^l$ and the supports of $\mathcal{F}(G_{i,q}^l, t)$, and by Plancherel’s theorem, $\sum_{q_i} \|G_{i,q}^l\|_{L^2(I_{i,q}^l)} = \sum_{q_i} \|G_{i,q}\|_{L^2(I_{i,q}^l)} = \sum_{q_i} \|G_{i,q}\|_{L^2(I_{i,q}^l)}$. Hence, the above inequality follows from

$$\left\| \prod_{i=1}^{k} \sum_{q_i} f_{i,q}^l \ast \left( \frac{\chi_{T_i^l}}{T_i^l} \right) \right\|_{L^\infty(B(0,2R))} \leq C \sigma^{-\kappa} R^{\kappa} R^{-\frac{d-k}{2}} \prod_{i=1}^{k} \sum_{q_i} \|f_{i,q}^l\|_1.$$

Let $I_i = \{ (q,l) : G_{i,q} \neq 0 \}$, $I_i \subset I_i$ and $T_i^l$ be a finite subset of $\mathbb{R}^d$. By scaling and pigeonholing, losing (log $R$) in its bound, this reduces to

$$\left( \prod_{i=1}^{k} \sum_{\langle q,l \rangle \in I_i} \sum_{\tau \in T_i^l} \chi_{T_i^l + \tau} \right)_{L^\infty(B(0,2))} \leq C \sigma^{-\kappa} R^{\kappa} R^{-\frac{d-k}{2}} \prod_{i=1}^{k} \sum_{\langle q,l \rangle \in I_i} \sum_{\tau \in T_i^l} |T_i^l + \tau|.$$

Here we note that if $G_{i,q} = 0$, then $G_i \in \supp \mathcal{F}(G_i^l, t) + O(\sqrt{\delta})$ for some $t$. So, by (82) we have $\text{Vol}(n_1, \ldots, n_k) \geq \sigma$ whenever $n_i \in \{ n_i^l, G_{i,q} \neq 0 \}$, $i = 1, \ldots, k$. Therefore, the estimate follows from the multilinear Kakeya estimate which is stated below in Theorem 3.7. This completes the proof.

**Theorem 3.7** (0 [27] [18]). Let $2 \leq k \leq d$, $1 \ll R$ and $\Theta_i, i = 1, 2, \ldots, k$ be collections of tubes of width $R^{-1/2}$ (possibly with infinite length), of which major axes are parallel to the vectors in $\Theta_i \subset S^{d-1}$. Suppose $\text{Vol}(\Theta_1, \Theta_2, \ldots, \Theta_k) \geq \sigma$ holds whenever $\Theta_i \subset \Theta_i$, $i = 1, \ldots, k$, then there is a constant $C$ such that, for any subset $T_i \subset T_i$, $i = 1, \ldots, k$,

$$\left\| \prod_{i=1}^{k} \left( \sum_{T_i \subset T_i} \chi_{T_i} \right) \right\|_{L^\infty(B(0,1))} \leq C R^{-\frac{d-k}{2}} \sigma^{-1} \prod_{i=1}^{k} \left( \sum_{T_i \subset T_i} |T_i| \right).$$

This is a rescaled version of the estimate due to Guth [27] (the case $d = k$) and Carbery-Valdimarsson [18] (also see [0]). However, we don’t need the endpoint estimate for our purpose and the estimate in [0] is actually enough because we allow $\delta^{-\varepsilon}$ loss in our estimate.

**Corollary 3.8.** Let $\psi \in \mathcal{B}(\varepsilon, N)$, $\eta \in \mathcal{E}(N)$, and $0 < \delta \ll \sigma$. Suppose that (82) holds whenever $\xi_i \in \supp \tilde{f}_i + O(\delta)$, $i = 1, 2, \ldots, k$. Then, if $p \geq 2k/(k-1)$ and $\varepsilon$ is small enough, for $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that the following estimate holds with $C, C_\varepsilon$, independent of $\psi$ and $\eta$:

$$\left\| \prod_{i=1}^{k} S_\delta(\psi, \eta) f_i \right\|_{L^p(B(x, \delta^{-1}))} \leq C \sigma^{-C_\varepsilon \delta^{-\varepsilon}} \prod_{i=1}^{k} \left( \delta \|f_i\|_2 \right).$$

To show this we need only to replace $G_i$ with $\phi(\delta(D_{i,t} - \Delta_{i,t})) f_i$ and apply Proposition 3.6. The assumptions in Proposition 3.7 are satisfied with $G_1, \ldots, G_k$. Thus, the estimate is straightforward because $\|\phi(\delta(D_{i,t} - \Delta_{i,t})) f_i\|_{L^2_{R^k}} \lesssim \delta^{\frac{d}{2}} \|f\|_2$, which follows by Plancherel’s theorem and taking $t$-integration first.

The following is a consequence of Corollary 3.8 and localization argument in the proof of Proposition 2.10.

**Proposition 3.9.** Let $0 < \delta \ll \sigma \ll \sigma \ll 1$ and $\psi \in \mathcal{B}(\varepsilon, N)$, $\eta \in \mathcal{E}(N)$ and let $Q_1, \ldots, Q_k \subset \frac{1}{2} I^d$ be dyadic cubes of sidelength $\bar{\sigma}$. Suppose that (82) is satisfied whenever $\xi_i \in Q_i$, $i = 1, \ldots, k$, and
suppose that $\text{supp} \tilde{f}_i \subset Q_i$, $i = 1, \ldots, k$. Then, if $p \geq 2k/(k - 1)$ and $\epsilon_0$ is small enough, for $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$\left\| \prod_{i=1}^k S_\delta(\psi, \eta) f_i \right\|_{L^\infty} \leq C \sigma^{-C_\epsilon} \delta^{-\epsilon} \prod_{i=1}^k \left( \delta^{\frac{d-2}{2}} \| f_i \|_p \right).$$

holds with $C, C_\epsilon$, independent of $\psi$ and $\eta$.

Proof. The proof is similar to that of Proposition 2.10. So, we shall be brief. Let $\varphi, \tilde{Q}_i, \tilde{x}_i, \{B\}$, and $\{\mathcal{B}\}$ be the same as in the proof of Proposition 2.11. We set

$$K_i^t = F^{-1} \left( \phi \left( \frac{\eta(t_1)(\tau - \psi(t_1))}{\delta} \right) \tilde{x}_i(t) \right).$$

Then $S_\delta(\psi, \eta) f_i = \| K_i^t * f_i \|_{L^2(t)}$. The $p/k$-th power of the left hand side of (98) is bounded by

$$\sum_B \int_B \prod_{i=1}^k \| K_i^t * f_i \|_{L^2(t)}^p \, dx \lesssim I + II,$$

where

$$I = \sum_B \int_B \prod_{i=1}^k \| K_i^t * (\chi_B f_i) \|_{L^2(t)}^p \, dx, \quad II = \sum_B \left( \sum_{\gamma_i = \chi_B f_i} \text{for some } i \int_B \prod_{i=1}^k \| K_i^t * g_i \|_{L^2(t)}^p \, dx \right).$$

As before, the second sum is taken over all choices with $g_i = \chi_B f_i$ or $\chi_B f_i$, and $g_i = \chi_B f_i$ for some $i$. By choosing $\epsilon > 0$ small enough, we see that $\chi_i(D)(\chi_B f_i)$, $\ldots$, $\chi_k(D)(\chi_B f_k)$ satisfy the assumption of Corollary 3.8. Since $K_i^t * (\chi_B f_i) = \phi \left( \frac{\eta(D,t)(\tau - \psi(D,t))}{\delta} \right) \chi_i(D)(\chi_B f_i)$, by Corollary 3.8 and Hölder’s inequality

$$I \lesssim \sigma^{-C_\epsilon} \left( \frac{1}{\delta} \right)^\epsilon \sum_{i=1}^k \| \chi_B f_i \|_{L^2}^p \lesssim \sigma^{-C_\epsilon} \left( \frac{1}{\delta} \right)^{\epsilon k} \left( \prod_{i=1}^k \delta^{\frac{d-2}{2}} \| f_i \|_p \right)^{\frac{p}{2p}}.$$

To handle $II$ we note from Lemma 2.9 that $\| K_i^t(x) \| \leq C\delta \mathcal{R}_M(x)$ with $C$, depending only on $\| \psi \|_{C_N(I^{d-1})}$, $\| \eta \|_{C_N(I^{d})}$. Thus, $\| K_i^t * (\chi_B f_i)(x) \|_{L^2} \leq C\delta \mathcal{R}_M(x) * | f_i(x) |$ if $x \in B$, and $\| K_i^t * f_i(x) \|_{L^2(t)} \leq C\delta \mathcal{R}_{d,k} * | f_i(x) |$. The rest of the proof is the same as before. We omit the details. \qed

3.4. Multilinear square function estimate with confined direction sets. From the point view of Proposition 2.11, we may expect a better estimate thanks to smallness of supports of Fourier transforms of the input functions when they are confined in a small neighborhood of a $k$-dimensional submanifold. The following is a vector valued generalization of Proposition 2.11.

**Proposition 3.10.** Let $k$, $2 \leq k \leq d$, be an integer, $0 < \sigma \ll 1$ be fixed, and $\Pi \subset \mathbb{R}^d$ be a $k$-plane containing the origin. Let $\psi \in \mathcal{S}(\epsilon_0, N)$ and $\Gamma^t$ be defined by (79). For $0 < \sigma \ll \epsilon$, suppose that the functions $G_1, \ldots, G_k$ defined on $\mathbb{R}^d \times I$ satisfy \(\mathbf{S1}\) for $t \in I$ and \(\mathbf{S2}\) whenever $\xi_i \in \text{supp} \mathcal{F}(G_i(\cdot, t)) + O(\delta)$, $i = 1, 2, \ldots, k$, for some $t \in I$. Additionally we assume that, for all $t \in I$,

$$n(\text{supp} \tilde{G}_1(\cdot, t)), \ldots, n(\text{supp} \tilde{G}_k(\cdot, t)) \subset S^{d-1} \cap (\Pi + O(\delta)).$$

Then, if $2 \leq p \leq 2k/(k - 1)$ and $\epsilon_0$ is sufficiently small, for $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$\left\| \prod_{i=1}^k \| G_i \|_{L^2(t)} \right\|_{L^p(B(x, \delta^{-1}))} \lesssim \sigma^{-C_\epsilon} \delta^{dk \left( \frac{d}{2} - \frac{1}{2} \right) - \epsilon \prod_{i=1}^k \| G_i \|_{L^2}}$$

holds uniformly for $\psi \in \mathcal{S}(\epsilon_0, N)$. 

The following is an easy consequence of (100).
Corollary 3.11. Let \( \{q_i\} \), \( q_i \subset \frac{1}{2} I^d \), be the collection of dyadic cubes of side length \( l \), \( \delta < l \leq 2\delta \). Define \( G_{i,q} \) by \( \mathcal{F}(G_{i,q}(\cdot,t)) = \chi_q \mathcal{F}(G_i(\cdot,t)) \) and set \( R = 1/\delta \). Suppose that the same assumptions as in Proposition 3.10 are satisfied. Then, if \( 2 \leq p \leq 2k/(k-1) \) and \( \epsilon_o \) is small enough, for \( \epsilon > 0 \) there is an \( N = N(\epsilon) \) such that

\[
(101) \quad \left\| \prod_{i=1}^{k} \|G_i\|_{L^p(B_{(x,R)})} \right\|_{L^\frac{p}{p}} \lesssim \sigma^{-C_\epsilon} \delta^{-\epsilon} \prod_{i=1}^{k} \left( \sum_{q} \|G_{i,q}\|_{L^p(B_{(x,R)})}^2 \right)^{\frac{1}{2}} \rho_B(x,R)_{p}
\]

holds uniformly for \( \psi \in \mathcal{O}(\epsilon_o, N) \).

Proof. Observe that \( \left\| \prod_{i=1}^{k} \|G_i\|_{L^p(B_{(x,R)})} \right\|_{L^\frac{p}{p}} \leq \left\| \prod_{i=1}^{k} \|\rho(-x\cdot/R)G_i\|_{L^p(B_{(x,R)})} \right\|_{L^\frac{p}{p}} \). Then, the functions \( \rho(-x\cdot/R)G_i, i = 1, \ldots, k \), satisfy the assumption in Proposition 3.10 because \( \text{supp} \mathcal{F}(\rho(-x\cdot/R)G_i(\cdot,t)) = \text{supp} G_i(\cdot,t) + O(R^{-1}) \). So, from Proposition 3.10 we get

\[
\left\| \prod_{i=1}^{k} \|G_i\|_{L^p(B_{(x,R)})} \right\|_{L^\frac{p}{p}} \lesssim \sigma^{-C_\epsilon} R \prod_{i=1}^{k} \left\| \|\rho(-x\cdot/R)G_i\|_{L^p} \right\|_{L^\frac{p}{p}}.
\]

Since \( G_i = \sum_q G_{i,q} \) and supports of \( \mathcal{F}(\rho(-x\cdot/R)G_{i,q}(\cdot,t)) \) are boundedly overlapping, by Plancherel's theorem it follows that \( \left\| \|\rho(-x\cdot/R)G_i\|_{L^p} \right\|_{L^\frac{p}{p}} \lesssim \left( \sum_q \|\rho(-x\cdot/R)G_{i,q}\|_{L^p}^2 \right)^{\frac{1}{2}} \). Combining this with the above inequality, we get

\[
\left\| \prod_{i=1}^{k} \|G_i\|_{L^p(B_{(x,R)})} \right\|_{L^\frac{p}{p}} \lesssim \sigma^{-C_\epsilon} R \prod_{i=1}^{k} \left\| \|\rho(-x\cdot/R)G_i\|_{L^p} \right\|_{L^\frac{p}{p}}.
\]

Now Hölder's inequality gives the desired estimate (101). \( \square \)

As an application of Corollary 3.11 we obtain the following.

Corollary 3.12. Let \( \psi \in \mathcal{O}(\epsilon_o, N) \), \( \eta \in \mathcal{E}(N) \), \( 0 < \delta \ll \bar{\sigma} \ll \sigma \), and \( S_\delta = S_\delta(\psi, \eta) \) be defined by (21). Let \( \Pi \) be a \( k \)-plane which contains the origin. Suppose (82) holds whenever \( \xi_i \in \text{supp} \tilde{f}_i + O(\bar{\sigma}), i = 1, 2, \ldots, k \), and

\[
(102) \quad n(\text{supp} \tilde{f}_i) \subset \Pi + O(\bar{\sigma}), \quad i = 1, 2, \ldots, k.
\]

Let \( \{q_i\} \), \( q_i \subset \frac{1}{2} I^d \), be the collection of dyadic cubes of side length \( l \), \( \bar{\sigma} < l \leq 2\bar{\sigma} \). Define \( f_{i,q} \) by \( \mathcal{F}(f_{i,q}) = \chi_q \mathcal{F}(f_i) \). Then, if \( 2k/(k-1) \leq p \leq 2 \) and \( \epsilon_o \) is sufficiently small, for \( \epsilon > 0 \) there is an \( N = N(\epsilon) \) such that

\[
\left\| \prod_{i=1}^{k} S_{\delta} f_i \right\|_{L^\frac{p}{p}(B_{(x,1/\bar{\sigma})})} \lesssim \sigma^{-C_\epsilon} \bar{\sigma}^{-\epsilon} \prod_{i=1}^{k} \left( \sum_{q} \|S_{\delta} f_{i,q}\|^2 \right)^{\frac{1}{2}} \rho_B(x,1/\bar{\sigma})_{p}
\]

holds uniformly for \( \psi \) and \( \eta \).

This follows from Corollary 3.11. Indeed, it suffices to check that \( G_i = \rho(\bar{\sigma}(\cdot - x)) \phi(D\cdot) f_i \) satisfies the assumption of Corollary 3.11 with \( \delta = \bar{\sigma} \) as long as \( \sigma \ll \bar{\sigma} \). This is clear because \( G_i(\cdot,t) = \bar{\sigma}^{-d} (e^{(1-i)\cdot/\bar{\sigma}} \rho(\cdot/\bar{\sigma}) \cdot (\phi(-x\cdot/R) f_i) \).

Proof of Proposition 3.10. The argument here is similar to the proof of Proposition 3.6. The estimate for \( p = 2 \) follows from Hölder's inequality and Plancherel's theorem. So, by interpolation it is sufficient to show (100) for \( p = 2k/(k-1) \).
Let us set $R = 1/\delta \gg 1$ and we may set $x = 0$. As usual we start with the assumption that, for $0 < \delta \ll \sigma$,

$$\left\| \prod_{i=1}^{k} \left\| G_i \right\|_{L^{2}(\Gamma, t)} \right\|_{L^{2}(\mathbb{R}^{d})} \leq CR^\alpha R^{-\frac{\alpha}{2}} \prod_{i=1}^{k} \left\| G_i \right\|_{L^{2}, t}$$

holds uniformly for $\psi \in \mathcal{F}(\epsilon_0, N)$ whenever $G_1, \ldots, G_k$ satisfy (31), (32) and (33). By (34) and Hölder’s inequality (103) is true with some large $\alpha$. As before it is sufficient to show that (103) implies for any $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$\left\| \prod_{i=1}^{k} \left\| G_i \right\|_{L^{2}(\mathbb{R}^{d})} \right\|_{L^{2}(\mathbb{R}^{d})} \leq C\sigma^{-\kappa} R^{\frac{\alpha}{2} + \varepsilon} R^{-\frac{\alpha}{4}} \prod_{i=1}^{k} \left\| G_i \right\|_{L^{2}, t}$$

holds uniformly for $\psi \in \mathcal{F}(\epsilon_0, N)$. Then iteration of this implication gives the desired estimate (100).

Fix $z \in \mathbb{R}^d$ and consider $\rho_{B(z, \sqrt{R})} G_1(\cdot, t), \ldots, \rho_{B(z, \sqrt{R})} G_k(\cdot, t)$. Then it is clear from (34) and (39) that $\operatorname{supp} \mathcal{F}(\rho_{B(z, \sqrt{R})} G_i(\cdot, t))$ is contained in $\Gamma^\delta + O(R^{-1/2})$ and $n(\operatorname{supp} \mathcal{F}(\rho_{B(z, \sqrt{R})} G_i(\cdot, t))) \subset \Pi + O(R^{-1/2})$. Also, since $\delta \ll \sigma$, (32) holds if $\xi_i \in \operatorname{supp} \mathcal{F}(\rho_{B(z, \sqrt{R})} G_i(\cdot, t))$. Hence, by the assumption (103) we get

$$\left\| \prod_{i=1}^{k} \left\| \rho_{B(z, \sqrt{R})} G_i \right\|_{L^{2}(\mathbb{R}^{d})} \right\|_{L^{2}(\mathbb{R}^{d})} \lesssim R^{\frac{\alpha}{2} - \frac{\delta}{4}} \prod_{i=1}^{k} \left\| \rho_{B(z, \sqrt{R})} G_i \right\|_{L^{2}, t}.$$

Now we proceed in the same way as in the proof of Proposition 3.6 and we keep using the same notations. As before, let $\{q\}$ be the collection of dyadic cubes (hence essentially disjoint) of sidelength $\sim R^{-1/2}$ such that $I^d = \bigcup q$. We decompose the function $G_i(\cdot, t)$ into $G_{i,q}(\cdot, t)$ which is defined by (39), and get (41), which is clear. Then, combining (41) and (104), we have

$$\left\| \prod_{i=1}^{k} \left\| \rho_{B(z, \sqrt{R})} G_i \right\|_{L^{2}(\mathbb{R}^{d})} \right\|_{L^{2}(\mathbb{R}^{d})} \leq C R^{\frac{\alpha}{2} - \frac{\delta}{4}} \prod_{i=1}^{k} \left\| \rho_{B(z, \sqrt{R})} G_i \right\|_{L^{2}, t}.$$\hspace{1cm} (105)

Then this gives

$$\left\| \prod_{i=1}^{k} \left\| \rho_{B(z, \sqrt{R})} G_i \right\|_{L^{2}(\mathbb{R}^{d})} \right\|_{L^{2}(\mathbb{R}^{d})} \lesssim R^{\frac{\alpha}{2} - \frac{\delta}{4}} \prod_{i=1}^{k} \left\| \rho_{B(z, \sqrt{R})} G_i \right\|_{L^{2}, t}$$

where $\mathcal{E} = R^{-M} \prod_{i=1}^{k} \left\| G_i \right\|_{L^{2}, t}$ for any large $M$.

We also denote by $(N^d)^{-1}$ (defined from $N^d(I^{-1})$ to $I^{-1}$) the inverse of $N^d : \Gamma^t \rightarrow \mathbb{R}^{d-1}$ which is well defined because $\psi \in \mathcal{F}(\epsilon_0, N)$. Since $\partial \psi \in (1 - \epsilon_0, 1 + \epsilon_0)$, there is an interval $I_{t,q}$ of length $CR^{-1/2}$ such that $G_{i,q}(\cdot, t) = 0$ if $t \notin I_{t,q}$ (see (38)). As in the proof of Proposition 3.6 we partition $I_{t,q}$ into intervals $I_{t,q}^l = [t_l, t_{l+1}]$, $l = 1, \ldots, l_0$, of sidelength $\sim R^{-1}$. Since the Fourier transform of $G_i(\cdot, t)$ is supported in $\Gamma^\delta + O(\delta)$ if $t \in I_{t,q}^l = [t_l, t_{l+1}]$ and the normal vectors are confined in $\Pi + O(\delta)$, it follows that

$$\operatorname{supp} \mathcal{F}(G_{i,q}(\cdot, t)) \subset \Gamma^\delta(\delta) \cap ((N^d)^{-1}(\Pi) + O(\delta)), \quad t \in [t_l, t_{l+1}].$$

Fix $t_l$, and let us set

$$\xi_{t_l,q} = (\xi_{t_l,q}, \pi_{t_l,q}) \in ((N^d)^{-1}(\Pi) \cap \Gamma^\delta) \cap (\operatorname{supp} \mathcal{F}(G_{t_l,q}(\cdot, t_l)) + O(\delta)).$$

(As before, we may assume that this set is nonempty, otherwise the associated function $G_{t_l,q} = 0$. See below.) Let $v_1, \ldots, v_{k-1}$ be an orthonormal basis for the tangent space $T_{\xi_{t_l,q}} \left( (N^d)^{-1}(\Pi) \right)$ at $\xi_{t_l,q}$, and $u_1, \cdots, u_{d-k}$ be a set of orthonormal vectors such that $\{N^d(\xi_{t_l,q}), v_1, \ldots, v_{k-1}, u_1, \ldots, u_{d-k}\}$
forms an orthonormal basis for $\mathbb{R}^d$. Let us set
\[
\mathbf{r}_{t_i, q}^{l_i} = \{ \xi : |\langle \xi - \xi_{t_i, q}^{l_i}, N^{l_i} \rangle (\xi_{t_i, q}^{l_i}) \rangle | \leq C\delta, |\langle \xi - \xi_{t_i, q}^{l_i}, v_i \rangle | \leq C\sqrt{\delta}, i = 1, \ldots, k - 1, |\langle \xi - \xi_{t_i, q}^{l_i}, u_i \rangle | \leq C\delta, i = 1, \ldots, d - k \}
\]
and
\[
\mathbf{P}_{t_i, q}^{l_i} = \{ \xi : |\langle \xi, N^{l_i} (\xi_{t_i, q}^{l_i}) \rangle | \leq C, |\langle \xi, v_i \rangle | \leq C\sqrt{\delta}, i = 1, \ldots, k - 1, |\langle \xi, u_i \rangle | \leq C, i = 1, \ldots, d - k \}
\]
with a sufficiently large $C > 0$. Then $F(G(t_i, \cdot), t) \in [t_i, t_{i+1}]$ is supported in $\mathbf{r}_{t_i, q}^{l_i}$.

The rest of the proof is similar to that of Proposition 3.6 and so we shall be brief. Let $m_{t_i, q}^{l_i}$ be a smooth function naturally adapted to $\mathbf{r}_{t_i, q}^{l_i}$ such that $m_{t_i, q}^{l_i} \sim 1$ on $\mathbf{r}_{t_i, q}^{l_i}$ and $F^{-1}(m_{t_i, q}^{l_i})$ is supported in $R\mathbf{P}_{t_i, q}^{l_i}$. This can be done by using $\phi$ and composition with it an appropriate affine map (for example, see [23]). As before we define $G_{t_i, q}^{l_i}(\cdot, t)$ by (29) and let $K_{t_i, q}^{l_i} = F^{-1}(m_{t_i, q}^{l_i})$ so that $G_{t_i, q}^{l_i}(\cdot, t) = G_{t_i, q}^{l_i}(\cdot, t) * K_{t_i, q}^{l_i}$ if $t \in I_{t_i, q}^{l_i}$. Hence, \[ \sum G_{t_i, q}^{l_i} = \sum_{l_i} G_{t_i, q}^{l_i}(\cdot, t) * K_{t_i, q}^{l_i} \mid K_{t_i, q}^{l_i} \mid < |R\mathbf{P}_{t_i, q}^{l_i}|^{-1} \chi_{R\mathbf{P}_{t_i, q}^{l_i}}. \]

Let us set $\tilde{\mathbf{P}}_{t_i, q}^{l_i} = R^{1+\epsilon} \mathbf{P}_{t_i, q}^{l_i}$ Hence, from the same lines of inequalities as in (39) and repeating the similar argument in the proof of Proposition 3.9 we have, for $x \in B(y, R^{1/2+\epsilon})$,
\[
\prod_{i=1}^{k} \left( \sum_{q} \left\| G_{t_i, q}^{l_i} \right\|_{L^2(I_{t_i}^{l_i})}^2 \right)^{1/2} \leq R^{\epsilon+2} \prod_{i=1}^{k} \left( \sum_{q} \left\| G_{t_i, q}^{l_i} \right\|_{L^2(I_{t_i}^{l_i})}^2 \right)^{1/2} \left( \frac{\chi_{\tilde{\mathbf{P}}_{t_i, q}^{l_i}}}{\mathbf{P}_{t_i, q}^{l_i}} \right)^{1/2} \right)^{1/2} L_{(B(0, 2R)} \mathcal{E}.
\]

Now, we use the lines of argument from (23) to (26), and combine this with (105) to get
\[
\prod_{i=1}^{k} \left| G_{t_i, q}^{l_i} \right|_{L^2(I_{t_i}^{l_i})} \leq R^{\epsilon+2} \prod_{i=1}^{k} \left( \sum_{q} \left\| G_{t_i, q}^{l_i} \right\|_{L^2(I_{t_i}^{l_i})}^2 \right)^{1/2} \left( \frac{\chi_{\tilde{\mathbf{P}}_{t_i, q}^{l_i}}}{\mathbf{P}_{t_i, q}^{l_i}} \right)^{1/2} \right)^{1/2} L_{(B(0, 2R)} \mathcal{E}.
\]

Since $\sum_{q, l} \left\| G_{t_i, q}^{l_i} \right\|_{L^2(I_{t_i}^{l_i})}^2 \sim \sum_{q} \left\| G_{t_i, q}^{l_i} \right\|_{L^2(I_{t_i}^{l_i})}^2 \sim \left\| G_{t_i, q} \right\|_{L^2, t_i}^2$, the proof is completed if we show
\[
\prod_{i=1}^{k} \left( \sum_{q, l} f_{q, l} * \frac{\chi_{\tilde{\mathbf{P}}_{t_i, q}^{l_i}}}{\mathbf{P}_{t_i, q}^{l_i}} \right) \left\| L_{(B(0, 2R)} \mathcal{E} \right\| \leq CR^{\epsilon+1} R^{-d} \prod_{i=1}^{k} \left( \sum_{q, l} f_{q, l} \right)^{1/2}.
\]

Finally, to show the above inequality we may repeat the argument in the last part of the proof of Proposition 3.11. In fact, we need only to show the associated Kakeya estimate (for example, see (25), (27)). Using the coordinates $(u, v) \in \Pi \times \Pi = \mathbb{R}^2$, it is sufficient to show that the longer sides of $\mathbf{P}_{t_i, q}^{l_i}$ are transverse to $\Pi$. More precisely, if $c_0$ is sufficiently small and $N$ is large enough, there is a constant $C > 0$, independent of $q \in \mathbb{R}(\epsilon, N)$, such that, for $w \in \left( \bigwedge_{\xi_{q, i}} (N^{-1} / 2) \right) \otimes \left( \bigwedge_{\xi_{q, i}} (N^{-1} / 2) \right)$, (29) holds. Since $\psi(\xi, t) = \frac{1}{2} |\xi|^2 + t + R$ with $\| R \|_{C^N / (t^4 \times t)} \leq c_0$, by the same perturbation argument it is sufficient to consider $\psi(\xi, t) = \frac{1}{2} |\xi|^2 + t$. For this case (29) clearly holds for $w \in \left( \bigwedge_{\xi_{q, i}} (N^{-1} / 2) \right) \otimes \left( \bigwedge_{\xi_{q, i}} (N^{-1} / 2) \right)$ because translation by $t$ doesn’t have any effect. The same argument works without modification. This completes the proof.

### 3.5. Multi-scale decomposition for $Sd$.

In this section we obtain multi-scale decomposition for the square function, which is to be combined with multilinear square function estimates to prove Proposition 3.2. This is will be carried out in the similar way that we obtain the decomposition in Section 2 though we need to take care of the additional $t$ average.

Let $0 < \epsilon_0 \ll 1$, $N \ll \epsilon_0, N \in \mathbb{N}, N \in \mathbb{N}, N \in \mathbb{N}$, and $Sd$ be given by (74). Let $N$, $\mathbf{n}$ be given by Definition 3.3. Let $\kappa = \kappa(\epsilon_0, N)$ be the number given in Proposition 3.3, so that (77) holds whenever $0 < \kappa \leq \kappa$, $\psi \in \mathbb{R}(\epsilon_0, N)$, and $\psi \in \mathbb{N}(N)$. As before, let $\sigma_1, \ldots, \sigma_m$, and $M_1, \ldots, M_m$ be dyadic numbers such that
\[
(106) \quad \delta \leq \sigma_{d-1} \leq \cdots \leq \sigma_1 \leq \min(\kappa, 1), \quad M_1 = 1/\sigma_1.
\]
We assume that $f$ is Fourier supported in $\frac{1}{2} f_d$. We keep using the same notation as in Section 2.\textsuperscript{13} In particular, $\{q^i\}, \{\Omega^i\}$ are the collection of (closed) dyadic intervals of sidelength $2\sigma_i, 2M_i$, respectively, so that \textsuperscript{11} and \textsuperscript{10} holds.

3.5.1. Decomposition by normal vector sets. Let $\{\theta^i\}$ be a discrete subset of $\mathbb{S}^{d-1}$ of which elements are separated by distance $\sim \sigma_i$. Let $\vartheta^i$ be disjoint subsets of $\{q^i\}$ which satisfies, for some $\theta^i$,

\begin{equation}
\vartheta^i \subset \{q^i : \text{dist}(n(q^i), \theta^i) \leq C\sigma_i\}
\end{equation}

and

\begin{equation}
\bigcup_{\vartheta^i} \vartheta^i = \{q^i\}, \quad i = 1, \ldots, m.
\end{equation}

Obviously, such a partitioning of $\{q^i\}$ is possible. Disjointness between $\vartheta^i$ will be useful later for decomposing the square function. Then we also define an auxiliary operator by

\[ \mathcal{G}_{\vartheta^i} f = \left( \sum_{q^i \in \vartheta^i} |S_{\delta} f_{q^i}|^2 \right)^{\frac{1}{2}}. \]

Similarly, as before, $\vartheta^i, \vartheta^i, \vartheta^i, \vartheta^i$ denote the elements in $\{\vartheta^i\}$ for the rest of this section.

**Definition 3.13.** We define $n(\vartheta^i)$ to be a vector $1 \in \{\theta^i\}$ such that $\text{dist}(n(q^i), \theta) \leq C\sigma_i$ whenever $q^i \in \vartheta^i$. Particularly, we may set $n(\vartheta^i) = \theta$ if \textsuperscript{11} holds.

Since the map $N^i$ is injective for each $t$, the elements of $\vartheta^i$ are contained in a $O(\sigma_i)$ neighborhood of the curve $\{\xi : n(\xi) = \theta^i\}$ with $\theta^i = n(\vartheta^i)$. From \textsuperscript{12} we observe that for any interval $J$ of length $\sigma_i$ there are as many as $O(1)$ $q^i \in \vartheta^i$ such that $\phi(\frac{D_{n(\theta^i)}}{\delta}) f_{q^i} \neq 0$ if $t \in J$. Hence, dividing $I$ intervals of length $\sim \sigma_i$ and taking integration in $t$ we see that

\begin{equation}
S_\delta \left( \sum_{q^i \in \vartheta^i} f_{q^i} \right) \lesssim \left( \sum_{q^i \in \vartheta^i} |S_{\delta} f_{q^i}|^2 \right)^{\frac{1}{2}} = \mathcal{G}_{\vartheta^i} f
\end{equation}

with the implicit constant independent of $\vartheta^i$. Since $S_{\delta} f \lesssim \sum_{\vartheta^i} S_{\delta} (\sum_{q^i \in \vartheta^i} f_{q^i})$, $i = 1, \ldots, m$, we also have

\begin{equation}
S_{\delta} f \lesssim \sum_{\vartheta^i} \left( \sum_{q^i \in \vartheta^i} |S_{\delta} f_{q^i}|^2 \right)^{\frac{1}{2}} = \sum_{\vartheta^i} \mathcal{G}_{\vartheta^i} f.
\end{equation}

3.5.2. $\sigma_1$-scale decomposition. Decomposition at this stage is similar with that of $T_{\delta}$ in Section 2. So, we shall be brief. Fix $x \in \mathbb{R}^d$ and let us denote by $\vartheta^i \in \{\vartheta^i\}$ such that

\[ \mathcal{G}_{\vartheta^i} f(x) = \max_{\vartheta^i} \mathcal{G}_{\vartheta^i} f(x). \]

Considering the cases $\sum_{\vartheta^i} \mathcal{G}_{\vartheta^i} f(x) \leq 100^d \mathcal{G}_{\vartheta^i} f(x)$ and $\sum_{\vartheta^i} \mathcal{G}_{\vartheta^i} f(x) > 100^d \mathcal{G}_{\vartheta^i} f(x)$ separately, we have

\[ S_{\delta} f(x) \lesssim \sum_{\vartheta^i} \mathcal{G}_{\vartheta^i} f(x) \lesssim \mathcal{G}_{\vartheta^i} f(x) + \sigma_1^{-1} \max_{\vartheta^i : |n(\vartheta^i)| \geq \sigma_1} \left( \mathcal{G}_{\vartheta^i} f(x) \mathcal{G}_{\vartheta^i} f(x) \right)^{\frac{1}{2} - \frac{1}{2}} \]

\[ \lesssim \mathcal{G}_{\vartheta^i} f(x) + \sigma_1^{-1} \max_{\vartheta^i, \vartheta^i : |n(\vartheta^i)| \geq \sigma_1} \left( \mathcal{G}_{\vartheta^i} f(x) \mathcal{G}_{\vartheta^i} f(x) \right)^{\frac{1}{2} - \frac{1}{2}}. \]

Since $\# \vartheta^i \lesssim \sigma_1^{-1}$ and $\mathcal{G}_{\vartheta^i} f \mathcal{G}_{\vartheta^i} f = \left( \sum_{q^i \in \vartheta^i : q^i \in \vartheta^i} (S_{\delta} f_{q^i} S_{\delta} f_{q^i}) \right)^{1/2}$,

\[ S_{\delta} f(x) \lesssim \sigma_1^{-1} \left( \sum_{q^i \in \vartheta^i} |S_{\delta} f_{q^i}|^2 \right)^{\frac{1}{2}} + \sigma_1^{-1} \sum_{\vartheta^i, \vartheta^i : |n(\vartheta^i)| \geq \sigma_1} \left( S_{\delta} f_{q^i} S_{\delta} f_{q^i} \right)^{\frac{1}{2}}. \]

\textsuperscript{\|}Possibly, there are more than one $\theta$. In the case we simply choose one of them. Ambiguity of the definition does not cause any problem in what follows.
Taking $L^p$ norm on both side of the inequality yields
\[
\|S_\delta f\|_p \lesssim \sigma_1^{1-\frac{2}{p}} \left( \sum_{q^i} \|S_\delta f_{q^i}\|_p^2 \right)^{\frac{1}{2}} + \sigma_1^{-C} \left( \sum_{q^i, q^j, l, q^k, \text{trans}} \|S_\delta f_{q^i} S_\delta f_{q^j} S_{\delta f_{q^k}, l}\|_2^2 \right)^{\frac{1}{2}}.
\]

Hence, using Lemma 3.3 and Lemma 2.6 we have
\[
(111) \quad \|S_\delta f\|_p \lesssim \sigma_1^{1-\frac{2}{p}} B_p(\sigma_1^{-2} \delta) \|f\|_p + \sigma_1^{-C} \max_{q^i, q^j, \text{trans}} \|S_\delta f_{q^i} S_\delta f_{q^j}\|_2^2.
\]

We proceed to decompose those terms appearing in the bilinear expression.

3.5.3. $\sigma_k$-scale decomposition, $k \geq 2$. Fixing $\sigma$, for $l \in \sigma^{-1}\mathbb{Z}^d$, let $A_l$ and $\tau_l$ be given by (32). The following is a slight modification of Lemma 2.13.

**Lemma 3.14.** Let $\mathcal{D}$ be a subset of $\{q^i\}$. Set $\mathcal{G}_\mathcal{D} f = \left( \sum_{q^i \in \mathcal{D}} |S_\delta f_{q^i}|^2 \right)^{1/2}$, and set
\[
[\mathcal{G}_\mathcal{D} f] = \sum_{l \in M, M, \mathbb{Z}^d} A_l \mathcal{G}_\mathcal{D} (\tau_l f), \quad [\mathcal{G}_\mathcal{D} f] = \sum_{l, l' \in M, \mathbb{Z}^d} (A_l A_{l'})^{\frac{1}{2}} \mathcal{G}_\mathcal{D} (\tau_l f \tau_{l'} f).
\]

If $x, x_0 \in \Omega^i$, the following inequality holds with the implicit constants independent of $\mathcal{D}$:
\[
(112) \quad \mathcal{G}_\mathcal{D} f(x) \lesssim [\mathcal{G}_\mathcal{D} f](x_0) \lesssim \|\mathcal{G}_\mathcal{D} f\|(x).
\]

**Proof.** Note that $q^i$ is a cube of sidelength $2\sigma_l$. Since $x, x_0 \in \Omega^i$, using (35) and Cauchy-Schwarz inequality, we get
\[
\left| \phi \left( \frac{D - \psi(D', t)}{\delta} \right) f_{q^i}(x) \right|^2 \lesssim \sum_{l \in M, \mathbb{Z}^d} A_l \left| \phi \left( \frac{\eta(D, t) (D - \psi(D', t))}{\delta} \right) \tau_l f_{q^i}(x_0) \right|^2.
\]

By taking integration in $t$ we get
\[
(113) \quad (S_\delta f_{q^i}(x))^2 \lesssim \sum_{l \in M, \mathbb{Z}^d} A_l (S_\delta (\tau_l f_{q^i})(x_0))^2.
\]

Summation in $q^i \in \mathcal{D}$ gives
\[
\left( \sum_{q^i \in \mathcal{D}} (S_\delta f_{q^i}(x))^2 \right)^{\frac{1}{2}} \lesssim \sum_{l \in M, \mathbb{Z}^d} A_l \left( \sum_{q^i \in \mathcal{D}} (S_\delta (\tau_l f_{q^i})(x)) \right)^{\frac{1}{2}},
\]

by which we get the first inequality of (112). By interchanging the roles of $x$ and $x_0$ in (113) and summation in $q^i \in \mathcal{D}$ it follows that
\[
\sum_{q^i \in \mathcal{D}} (S_\delta (\tau_{q^i} f_{q^i})(x_0))^2 \lesssim \sum_{l \in M, \mathbb{Z}^d} A_l \sum_{q^i \in \mathcal{D}} (S_\delta (\tau_{l+q^i} f_{q^i})(x))^2
\]

Putting this in the right hand side of the above inequality and repeating the same argument, we get the second inequality of (112). \qed

Now we have bilinear decomposition (111) on which we build higher degree of multilinear decomposition.
3.5.4. From \(k\)-transversal to \(k+1\)-transversal. Let us be given cubes \(q_1^{k-1}, q_2^{k-1}, \ldots, q_n^{k-1}\) of sidelength \(\sigma_{k-1}\) which satisfy (55). Though we use the same notations as in the multiplier estimate case, it should be noted that the normal vector field \(n\) is defined on \(I^{d-1} \times CI\) (see Definition 3.3). As before, we denote by \(\{q_i^k\}\) the collection of dyadic cubes of sidelength \(\sigma_k\) contained in \(q_i^{k-1}\) (see (56)), which are partitioned into the subsets of \(\{d^k\}\) so that

\[
\bigcup_{d^k_i} \bigcup_{q_i^k \in d^k_i} q_i^k = q_i^{k-1}, i = 1, \ldots, k.
\]

So, we can write

\[
\prod_{i=1}^{k} S_\delta(\sum_{q_i^k \in q_i^{k-1}} f_{q_i^{k-1}}) = \prod_{i=1}^{k} S_\delta(\sum_{q_i^k \in \overline{\Omega}} f_{q_i^k})
\]

and recall the definition \(\mathcal{S}_{\sigma_k} F_{q_i^{k-1}} := (\sum_{q_i^k \in \overline{\Omega}} |S_\delta F_{q_i^k}|^2)^{1/2}\). Fix \(\Omega^k\) and let \(x_0\) be the center of \(\Omega^k\). Let \(d^k_i \in \{d^k\}\) be an angular partition such that

\[
\mathcal{S}_{\sigma_k} f_{q_i^{k-1}}(x_0) = \max_{d^k_i} \mathcal{S}_{\sigma_k} f_{q_i^{k-1}}(x_0).
\]

Let us set

\[
(114) \quad \overline{\Lambda}_i^k = \{d^k_i : [\mathcal{S}_{\sigma_k} f_{q_i^{k-1}}(x_0) > (\sigma_k)^{kd} \max_{1 \leq j \leq k} |\mathcal{S}_{\sigma_k} f_{q_j^{k-1}}(x_0)|, \ 1 \leq i \leq k\).
\]

We split the sum to get

\[
(115) \quad \prod_{i=1}^{k} S_\delta(\sum_{d^k_i \in \overline{\Omega}} f_{d^k_i}) \leq \prod_{i=1}^{k} S_\delta(\sum_{d^k_i \in \overline{\Lambda}_i^k} f_{d^k_i}) + \sum_{(d^k_1, \ldots, d^k_k) \in \prod_{i=1}^{k} \overline{\Lambda}_i^k} \prod_{i=1}^{k} [\mathcal{S}_{\sigma_k} f_{d^k_i}](x_0)\]

Thus, if \(x \in \Omega^k\), by (112) and (119), the second term in the right hand side is bounded by

\[
(116) \quad \sum_{(d^k_1, \ldots, d^k_k) \in \prod_{i=1}^{k} \overline{\Lambda}_i^k} \prod_{i=1}^{k} [\mathcal{S}_{\sigma_k} f_{d^k_i}](x_0) \leq \sum_{(d^k_1, \ldots, d^k_k) \in \prod_{i=1}^{k} \overline{\Lambda}_i^k} \prod_{i=1}^{k} \max_{1 \leq j \leq k} |\mathcal{S}_{\sigma_k} f_{q_i^{k-1}}(x_0)| \lesssim \left( \max_{1 \leq j \leq k} \mathcal{S}_{\sigma_k} f_{q_j^{k-1}}(x_0) \right)^k \lesssim \left( \max_{1 \leq j \leq k} \mathcal{S}_{\sigma_k} f_{q_j^{k-1}}(x) \right)^k.
\]

Here \(\{d^k\} = \bigcup_{1 \leq i \leq k} \{d^k_i\}\) and the third inequality follows from the definition of \(\mathcal{S}_{\sigma_k} f\) because \(q_i^k \subset q_i^{k-1}\). Since (116) holds for each \(\Omega^k\), integrating over all \(\Omega^k\), using Lemma 3.14 Proposition 3.4, and Lemma 2.6, we get

\[
(117) \quad \left\| \sum_{(d^k_1, \ldots, d^k_k) \in \prod_{i=1}^{k} \overline{\Lambda}_i^k} \prod_{i=1}^{k} S_\delta(\sum_{q_i^k \in \overline{\Lambda}_i^k} f_{d^k_i}) \right\|_{L^p} \lesssim \left\| \max_{d^k} \mathcal{S}_{\sigma_k} f \right\|_p \lesssim \sup_h \left\| \mathcal{S}_{\sigma_k} (\tau_h f) \right\|_p \lesssim \sup_{h} \left( \sum_{q_i^k} \|\mathcal{S}_{\sigma_k} (\tau_h f)\|_p^{p/2} \right)^{p/2} \lesssim \sup_{(\sigma_k^2, \delta, \Pi_k^k)} \left( \sum_{q_i^k} \|\mathcal{S}_{\sigma_k} (\tau_h f)\|_p \right)^{1/p} \lesssim \sigma_k^{\frac{d}{2}} B_p(\sigma_k^{-2} \delta) \|f\|_p.
\]

The inequality before the last one follows from the definition of \(\mathcal{S}_{\sigma_k} f\) and Hölder’s inequality since there are as many as \(O(\sigma_k^{-1}) q^k \subset \overline{d^k}\).

We note that vectors \(n(d^k_i), \ldots, n(d^k_k)\) are linearly independent because \(q_1^{k-1}, q_2^{k-1}, \ldots, q_k^{k-1}\) : trans. We also denote by \(\Pi_k^k = \Pi_k^k(q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k)\) the \(k\) plane spanned by the vectors \(n(d^k_i), \ldots, n(d^k_k)\). Let us set

\[
\overline{\Pi} = \overline{\Pi}(q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k) = \{d^k : dist(n(d^k), \Pi_k^k) \leq C \sigma_k\}.
\]
We split the sum and use the triangle inequality so that
\[
\prod_{i=1}^{k} S_{d}(\sum_{\delta_{i}^{k} \subseteq \mathcal{D}_{i}^{k}} f_{\delta_{i}^{k}}(x)) \leq \prod_{i=1}^{k} S_{d}(\sum_{\delta_{i}^{k} \subseteq \mathcal{D}_{i}^{k}} \sum_{q_{i} \in \mathcal{Q}_{i}} f_{q_{i}}(x)) + \sum_{i=1}^{k} S_{d}(\sum_{q_{i} \in \mathcal{Q}_{i}} f_{q_{i}}(x)).
\]

For the \(k\)-tuples \((\delta_{1}^{k}, \ldots, \delta_{k}^{k})\) appearing in the second summation of the right hand side, there is exactly \(\delta_{i}^{k}\) for which \(\mathbf{n}(\delta_{i}^{k})\) is not contained in \(\Pi_{i}^{k} + O(\sigma_{k})\). In particular, suppose that \(\mathbf{n}(\delta_{i}^{k}) \notin \Pi_{i}^{k} + O(\sigma_{k})\).

Then, by \((112)\) and \((113)\) we have
\[
\prod_{i=1}^{k} S_{d}(\sum_{q_{i} \in \mathcal{Q}_{i}} f_{q_{i}}(x)) \lesssim \prod_{i=1}^{k} (S_{\Theta_{i}^{k}} + f_{q_{i}^{k-1}}(x)) \leq \sigma_{k}^{-C} \prod_{i=1}^{k} (S_{\Theta_{i}^{k}} f_{q_{i}^{k-1}}(x)) \frac{\|\tau_{1} \cdots \tau_{k} f_{q_{i}}(x)\|_{\text{trans}}}{\|\tau_{1} \cdots \tau_{k} f_{q_{i}}(x)\|_{\text{trans}}}.
\]

Recall that \(\text{Vol}(\mathbf{n}(\xi_{1}), \mathbf{n}(\xi_{2}), \ldots, \mathbf{n}(\xi_{k})) \gtrsim \sigma_{1} \cdots \sigma_{k-1}\) if \(\xi_{i} \in q_{i}^{k-1}, i = 1, \ldots, k\). From the definition of \(\mathcal{M}\) it follows that \(\text{dist} (\mathbf{n}(q_{i}), \Pi_{i}^{k}) \gtrsim \sigma_{i}\) if \(q_{i}^{k} \in \mathcal{D}_{i}^{k}\) and \(\mathbf{n}(\mathcal{Q}_{i}^{k}) \notin \mathcal{M}\). Hence \(\text{Vol}(\mathbf{n}(\xi_{1}), \mathbf{n}(\xi_{2}), \ldots, \mathbf{n}(\xi_{k})) \gtrsim \sigma_{1} \cdots \sigma_{k}\) if \(\xi_{i} \in q_{i}^{k}\) and \(q_{i}^{k} \in \mathcal{D}_{i}^{k}, i = 1, \ldots, k\), and \(\xi_{k+1} \in q_{k+1}^{k}\) and \(q_{k+1}^{k} \in \mathcal{D}_{k}^{k}\). Hence these cubes are transversal. Since there are only \(O(\sigma_{k}^{-C})\) \(\sigma_{k}\)-scale cubes, by \((112)\) and H"older’s inequality
\[
\prod_{i=1}^{k} S_{d}(\sum_{q_{i} \in \mathcal{Q}_{i}} f_{q_{i}}(x)) \lesssim \sigma_{k}^{-C} \sum_{l_{1}, l_{1}', \ldots, l_{k+1}, l_{k+1}' \in \mathcal{M}_{k}^{d}} \prod_{i=1}^{k+1} A_{l_{i}} A_{l_{i}'} \left( \sum_{q_{i} \in \mathcal{Q}_{i}^{k+1}: \text{trans}} \left( \prod_{i=1}^{k+1} S_{d}(\tau_{l_{i}, l_{i}'} f_{q_{i}}(x)) \right)^{\frac{\sigma_{k}}{\sigma_{k}}} \right)^{\frac{1}{k}}.
\]

Here \(A_{l_{i}}, A_{l_{i}'}\) are rapidly decaying sequences. The same is true for any \(\mathcal{D}_{1}^{k}, \ldots, \mathcal{D}_{k}^{k}\) satisfying \(\mathcal{D}_{i}^{k} \in \mathcal{D}_{i}^{k}\), \(1 \leq i \leq k\), and \(\mathcal{D}_{k}^{k} \notin \mathcal{M}\) for some \(i\) and this holds regardless of \(\mathcal{Q}_{k}\). So, we have, for any \(x\),
\[
\prod_{i=1}^{k} S_{d}(\sum_{q_{i} \in \mathcal{Q}_{i}} f_{q_{i}}(x)) \lesssim \sigma_{k}^{-C} \sum_{l_{1}, l_{1}', \ldots, l_{k+1}, l_{k+1}' \in \mathcal{M}_{k}^{d}} \prod_{i=1}^{k+1} \tilde{A}_{l_{i}} A_{l_{i}'} \left( \sum_{q_{i} \in \mathcal{Q}_{i}^{k+1}: \text{trans}} \left( \prod_{i=1}^{k+1} S_{d}(\tau_{l_{i}, l_{i}'} f_{q_{i}}(x)) \right)^{\frac{\sigma_{k}}{\sigma_{k}}} \right)^{\frac{1}{k}}.
\]

Since \(\tilde{A}_{l_{i}}, A_{l_{i}'}\) are rapidly decaying, taking \(L^{p/k}\) norm and a simple manipulation give
\[
\prod_{i=1}^{k} S_{d}(\sum_{q_{i} \in \mathcal{Q}_{i}} f_{q_{i}}(x)) \lesssim \frac{\sigma_{k}^{-C}}{\sigma_{k}^{-C}} \sum_{l_{1}, l_{1}', \ldots, l_{k+1}, l_{k+1}' \in \mathcal{M}_{k}^{d}} \prod_{i=1}^{k+1} S_{d}(\tau_{l_{i}, l_{i}'} f_{q_{i}}(x)) \frac{\|\tau_{1} \cdots \tau_{k} f_{q_{i}}(x)\|_{\text{trans}}}{\|\tau_{1} \cdots \tau_{k} f_{q_{i}}(x)\|_{\text{trans}}}.
\]

We now combine the inequalities \((115), (116), (117), (118)\) to get
\[
\prod_{i=1}^{k} S_{d}(\sum_{\delta_{i}^{k} \in \mathcal{D}_{i}^{k}} f_{\delta_{i}^{k}}(x)) \lesssim (\max_{\delta_{i}^{k}} |S_{\Theta_{i}^{k}} f(x)|)^{k} + \chi_{\mathcal{Q}_{i}^{k}} \prod_{i=1}^{k+1} S_{d}(\sum_{\delta_{i}^{k} \notin \mathcal{D}_{i}^{k} \subseteq \mathcal{D}_{i}^{k}} f_{\delta_{i}^{k}}(x)) + \sigma_{k}^{-C} \sum_{l_{1}, l_{1}', \ldots, l_{k+1}, l_{k+1}' \in \mathcal{M}_{k}^{d}} \prod_{i=1}^{k+1} \tilde{A}_{l_{i}} A_{l_{i}'} \left( \sum_{q_{i} \in \mathcal{Q}_{i}^{k+1}: \text{trans}} \left( \prod_{i=1}^{k+1} S_{d}(\tau_{l_{i}, l_{i}'} f_{q_{i}}(x)) \right)^{\frac{\sigma_{k}}{\sigma_{k}}} \right)^{\frac{1}{k}}.
\]
Here \( \mathcal{M} \) depends on \( q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k \). By taking \( 1/k \)-th power, integrating on \( \mathbb{R}^d \) and using (117) and (120) we get

\[
\left\| \left( \prod_{i=1}^k S_{\delta_i} \left( \sum_{q_i^{k-1} \subset q_i^{k-1}} f_{q_i^{k-1}} \right) \right)^{\frac{1}{p}} \right\|_p \lesssim \sigma_k^{\frac{2}{p}} B_p(\sigma_k^{-2}\delta) \|f\|_p + \sigma_k^{-C} \sup_{\tau_1, \ldots, \tau_k+1, q_1^{k-1}, \ldots, q_k^{k-1}} \max_{\text{trans}} \left\| \prod_{i=1}^{k+1} S_{\delta_i} (\tau_i f_{q_i^k}) \right\|_{L^{\frac{p}{m+1}}(\Omega^k)}^{\frac{m+1}{p}},
\]

(121)

\[
+ \left( \sum_{\Omega^k} \left\| \prod_{i=1}^k S_{\delta_i} \left( \sum_{q_i^{k-1} \subset q_i^{k-1}} f_{q_i^{k-1}} \right) \right\|_{L^{\frac{p}{m+1}}(\Omega^k)} \right)^{\frac{1}{p}},
\]

where \( \mathcal{M}(q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k) \) denotes a subset of \( \mathcal{M}(q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k) \) which depends on \( q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k \).

3.5.5. Multi-scale decomposition. For \( k = 2, \ldots, m \), let us set

\[
\mathcal{M}^k = \sup_{\tau_1, \ldots, \tau_k, q_1^{k-1}, \ldots, q_k^{k-1}} \max_{\text{trans}} \left( \sum_{\Omega^k} \left\| \prod_{i=1}^k S_{\delta_i} \left( \sum_{q_i^{k-1} \subset q_i^{k-1}} \sum_{q_i^{k-1} \subset q_i^{k-1}} \tau_i f_{q_i^k} \right) \right\|_{L^{\frac{p}{m+1}}(\Omega^k)} \right)^{\frac{1}{p}}.
\]

Here \( \mathcal{M}(q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k) \) also depends on \( \tau_1, \ldots, \tau_k \) but this doesn’t affect the overall bound. Starting from (111) we successively apply (121) to \( k \)-scale transversal products (given by \( q_1^{k-1}, \ldots, q_k^{k-1} : \text{trans} \)). After decomposition up to \( m \)-th scale we get

\[
\|S_{\delta} f\|_p \lesssim m \sum_{k=1}^m \sigma_{k-1}^{-C} \sigma_k^{\frac{2}{p}} B_p(\sigma_k^{-2}\delta) \|f\|_p + m \sum_{k=2}^m \sigma_k^{-C} \mathcal{M}^k f
\]

(122)

\[
+ \sigma_m^{-C} \sup_{\tau_1, \ldots, \tau_{m+1}, q_1^{m}, \ldots, q_{m+1}^{m}} \max_{\text{trans}} \left\| \prod_{i=1}^{m+1} S_{\delta_i} \tau_i f_{q_i^m} \right\|_{L^{\frac{m+1}{m+1}}(\Omega^{m+1})}^{\frac{m+1}{m+1}},
\]

3.6. Proof of Proposition 3.22 We may assume \( d \geq 9 \) since \( p_s \geq 2(d+2)/d \) for \( d < 9 \) and the sharp bound for \( p \geq 2(d+2)/d \) is verified in [34]. So, we have \( p_s(d) \geq \frac{2(d+1)}{d} \). The proof is similar to that of Proposition 2.3. Let \( \beta > 0 \) and we aim to show that \( B^\beta(s) \leq C \) for any \( s \leq 1 \) if \( p \geq p_s(d) \). We choose \( \epsilon > 0 \) such that \( (100d)^{-1} \beta \geq \epsilon \). Fix \( \epsilon_0 > 0 \) and \( N = N(\epsilon) \) such that Corollaries 3.8, 3.11 and 3.12 hold uniformly for \( \psi \in \mathcal{S}(\epsilon_0, N) \).

Let \( s < \delta \leq 1 \). Obviously, \( (\sigma_k^{-2}\delta)^{\frac{2}{d+2}-\frac{d}{p}+\beta} B(\sigma_k^{-2}\delta) \leq B^\beta(s) + \sigma_k^{-C} \) because \( s \leq \sigma_k^{-2}\delta \) and \( B(\delta) = B_p(\delta) \leq C \) for \( \delta \geq 1 \). Hence, it follows that

\[
\sigma_k^{\frac{2}{p}} B(\sigma_k^{-2}\delta) \lesssim \sigma_k^{\frac{2}{d+2}-\frac{d}{p}+\beta} \delta^{-\frac{d-2}{d+2}+\frac{d}{p}-\beta}(B^\beta(s) + \sigma_k^{-C}).
\]

(123)

We first consider the \((m+1)\)-product in (122). By Corollary 3.8 we have, for \( p \geq 2(m+1)/m \),

\[
\sup_{\tau_1, \ldots, \tau_{m+1}, q_1^{m}, \ldots, q_{m+1}^{m}} \max_{\text{trans}} \left\| \prod_{i=1}^{m+1} S_{\delta_i} \tau_i f_{q_i^m} \right\|_{L^{\frac{m+1}{m+1}}(\Omega^{m+1})} \leq C \sigma_m^{-C} \delta^{-\frac{d-2}{d+2}+\frac{d}{p}-\epsilon} \|f\|_p.
\]

(124)

For \( \mathcal{M}^k \), as before we have two types of estimates. The first one follows from Corollary 3.8 while the second one is a consequence of the square function estimates in Corollary 3.12. From the definition of \( \mathcal{M}^k \), we note that \( q_1^k, q_2^k, \ldots, q_k^k \) are contained, respectively, in \( q_1^{k-1}, q_2^{k-1}, \ldots, q_k^{k-1} \).
which are transversal. Hence, we have
\[
\prod_{i=1}^{k} S_{\delta} (\sum_{q_{i} \in \mathcal{M}(q_{i-1}, \ldots, q_{1}, \Omega^{k})} \sum_{q_{i} \in \mathcal{G}_{i}^{k}} (k_{i}) \sum_{q_{i} \in q_{i}^{k-1}} \tau_{i} f_{q_{i}^{k}}) (x) \leq \prod_{i=1}^{k} S_{\delta} (\tau_{i} f_{q_{i}^{k}}) (x).
\]
Here $q_{1}^{k}, q_{2}^{k}, \ldots, q_{k}^{k} : \text{trans}$ means $\text{Vol}(n(\xi_{1}), \ldots, n(\xi_{k})) \geq \sigma_{1} \ldots \sigma_{k-1}$ provided $\xi_{1} \in q_{1}^{k}, i = 1, \ldots, k$. Since there are as many as $O(\sigma_{k-1})$ $q_{1}^{k}, \ldots, q_{k}^{k}$ and the above holds regardless of $\Omega^{k}$, by Corollary 3.12 we have, for $p \geq 2k/(k-1)$,
\[
(125) \quad \mathfrak{Mf} \lessapprox_{\mathfrak{C}} \sigma_{k}^{-C} \sup_{\tau_{1}, \ldots, \tau_{k}} \sum_{q_{1}^{k}, \ldots, q_{k}^{k}} \left\| \prod_{i=1}^{k} S_{\delta} (\tau_{i} f_{q_{i}^{k}}) \right\|_{L^{p} (\Omega^{k})} \lessapprox \sigma_{k}^{-C} \delta^{-\frac{d-2}{2} + \frac{2}{p} - \epsilon} \left\| f \right\|_{p}.
\]
**Estimates for $\mathfrak{Mf}$ via Corollary 3.12.** By fixing $\tau_{1}, \ldots, \tau_{k}$, and $(q_{1}^{k-1}, \ldots, q_{k}^{k-1})$ satisfying $q_{1}^{k-1}, \ldots, q_{k}^{k-1} : \text{trans}$, we first handle the integral over $\Omega^{k}$ which is in the definition of $\mathfrak{Mf}$. For $i = 1, \ldots, k$, set
\[
f_{i} = \sum_{q_{i} \in \mathcal{M}(q_{i-1}, \ldots, q_{1}, \Omega^{k})} \left( \sum_{q_{i}^{k} \in q_{i}^{k-1}} \tau_{i} f_{q_{i}^{k}} \right).
\]
Since $q_{1}^{k-1}, \ldots, q_{k}^{k-1} : \text{trans}$, (122) holds with $\sigma = \sigma_{1} \ldots \sigma_{k-1}$ whenever $\xi \in \operatorname{supp} f_{1} + O(\sigma_{k})$, $i = 1, 2, \ldots, k$. Also note that $n(q_{i}^{k}) \subseteq \Pi^{k}(q_{1}^{k-1}, \ldots, q_{k}^{k-1}, \Omega^{k})$. Hence, it follows that (102) holds with $\bar{\sigma} = \sigma_{k}$. Let us set
\[
Q(q_{1}^{k-1}, \ldots, q_{k}^{k-1}, \Omega^{k}) = \{ q^{k} : \{ q^{k} \in \mathcal{M}(q_{1}^{k-1}, \ldots, q_{k}^{k-1}, \Omega^{k}) \}
\]
Let $\Omega^{k} = (z, 1/\sigma_{k})$. Then, by Corollary 3.12 we have, for $2 \leq p \leq 2k/(k-1)$,
\[
\left( \prod_{i=1}^{k} S_{\delta} f_{i} \right)^{\frac{1}{p}} \|_{L^{p} (\Omega^{k})} \lessapprox \sigma_{k}^{-C} \sigma_{k}^{-\epsilon} \left( \sum_{q_{i} \in q_{i}^{k-1}} \right) \left( \prod_{i=1}^{k} \left| S_{\delta} \tau_{i} f_{q_{i}^{k}} \right|^{\frac{1}{p}} \right) \rho_{\Omega^{k}}^{\frac{1}{p}}.
\]
The dyadic cubes of sidelength $\sigma_{k}$ in $Q(q_{1}^{k-1}, \ldots, q_{k}^{k-1}, \Omega^{k})$ are contained in $O(\sigma_{k})$-neighborhood of $n^{k-1}(\Pi^{k})$ which is a smooth $k$-dimensional surface. Thus, $\# \{ q_{i}^{k} \subset q_{i}^{k-1} : q_{i}^{k} \in Q(q_{1}^{k-1}, \ldots, q_{k}^{k-1}, \Omega^{k}) \} \leq C \sigma_{k}^{-k}$. Now, by Hölder’s inequality we get
\[
\left( \prod_{i=1}^{k} S_{\delta} f_{i} \right)^{\frac{1}{p}} \|_{L^{p} (\Omega^{k})} \lessapprox \sigma_{k}^{-C} \sigma_{k}^{-\epsilon} \left( \sum_{q_{i} \in q_{i}^{k-1}} \right) \left( \prod_{i=1}^{k} \left| S_{\delta} \tau_{i} f_{q_{i}^{k}} \right|^{\frac{1}{p}} \right) \rho_{\Omega^{k}}^{\frac{1}{p}}.
\]
Summation along $\Omega^{k}$ using rapid decay of Schwartz function $\rho$ gives
\[
\left( \prod_{i=1}^{k} S_{\delta} f_{i} \right)^{\frac{1}{p}} \|_{L^{p} (\Omega^{k})} \lessapprox \sigma_{k}^{-C} \sigma_{k}^{-\epsilon} \left( \sum_{q_{i} \in q_{i}^{k-1}} \right) \left( \prod_{i=1}^{k} \left| S_{\delta} \tau_{i} f_{q_{i}^{k}} \right|^{\frac{1}{p}} \right) \rho_{\Omega^{k}}^{\frac{1}{p}}.
\]
Hence, using Proposition 3.4, Lemma 2.3 and (123), for $2 \leq p \leq \frac{2k}{k-1}$, we have
\[
\left( \prod_{i=1}^{k} S_{\delta} f_{i} \right)^{\frac{1}{p}} \|_{L^{p} (\Omega^{k})} \lessapprox \sigma_{k}^{-C} \sigma_{k}^{-\epsilon} \left( \sum_{q_{i} \in q_{i}^{k-1}} \right) \left( \prod_{i=1}^{k} \left| S_{\delta} \tau_{i} f_{q_{i}^{k}} \right|^{\frac{1}{p}} \right) \rho_{\Omega^{k}}^{\frac{1}{p}}.
\]
with some $\alpha > 0$ if $p \geq \frac{2(2d-k-1)}{2d-\frac{k}{k-3}}$. Here we have used $(100d)^{-1} \geq \epsilon$. We note that the right hand side of the above is independent of $\tau_{1}, \ldots, \tau_{k}$ and there are only $O(\sigma_{k}^{-C})$ many $k$-tuples $(q_{1}^{k-1}, \ldots, q_{k}^{k-1})$ satisfying $q_{1}^{k-1}, \ldots, q_{k}^{k-1} : \text{trans}$. Thus, recalling the definition of $\mathfrak{Mf}$, we have for $2 \leq p \leq \frac{2k}{k-1}$
\[
\mathfrak{Mf} \lessapprox \sigma_{k}^{-C} \delta^{-\frac{d-2}{2} + \frac{2}{p} - \beta} \left( \sigma_{k}^{-C} + \sigma_{k}^{\alpha} B^{\beta} (s) \right) \left\| f \right\|_{p}.
\]
with some $\alpha > 0$ provided that $p \geq \frac{2(2d-k-1)}{2d-k-3}$. Combining this and (125) we have, for some $\alpha > 0$, (126)

$$\frac{M^2}{p} f \leq C \delta^{-\frac{4s}{d-2} + \frac{d}{2} - \beta} \left( \sigma_k^{-C} + \sigma_k^\alpha B^\beta(s) \right) \|f\|_p,$$

provided that $p \geq \min\left(\frac{2(2d-k-1)}{2d-k-3}, \frac{2k}{k-1}\right)$.

**Closing induction.** Let us set

$$p(m) = \max\left( \max_{1 \leq k \leq m} \min\left( \frac{2(2d-k-1)}{2d-k-3}, \frac{2k}{k-1}\right), \frac{2(m+1)}{m}\right).$$

Since $p \geq p_n > \frac{2(d-1)}{d-2}$ and $(100d)^{-1} \beta \geq \epsilon$, we have $\sigma_k^\beta B(\sigma_k^{-2} \delta) \lesssim \sigma_k^\beta \delta^{-\frac{d-2}{2} + \frac{d}{2} - \beta} (B^\beta(s) + \sigma_k^{-C})$ for some $\alpha > 0$. Using (122), we combine the estimates (123), (124), and (126) to get

$$\|S_\delta f\|_p \leq C \sum_{k=1}^m \left( \sigma_k^{-C} + \sigma_k^{-C} \sigma_k^\alpha B^\beta(s) \right) \delta^{-\frac{4s}{d-2} + \frac{d}{2} - \beta} \|f\|_p + C \sigma_m^{-C} \delta^{-\frac{4s}{d-2} + \frac{d}{2} - \beta} \|f\|_p$$

for some $\alpha > 0$ as long as $p \geq p(m)$. The rest of the proof is similar to that in Section 2.6. So, we intend to be brief. By using stability of the estimates along $\psi \in \mathcal{F}(\epsilon_a, N)$, $\eta \in \mathcal{E}(N)$, multiplying by $\delta^{-\frac{4s}{d-2} + \frac{d}{2} - \beta}$ on both sides and taking supremum along $\psi, \eta$ and $f$, and taking supremum along $\delta$, $s < \delta \leq 1$, we get

$$B^\delta(s) \leq C \left( \sum_{k=1}^m \sigma_k^{-C} \sigma_k^\alpha \right) B^\delta(s) + C \sum_{k=1}^m \sigma_k^{-C}$$

for some $\alpha > 0$ provided that $p \geq p(m)$. Choosing $\sigma_1, \ldots, \sigma_m$ so that $C \left( \sum_{k=1}^{m-1} \sigma_{k-1}^{-C} \sigma_k^\alpha \right) \leq 1/2$, gives $B^\delta(s) \leq C \sigma_m^{-C}$ for $p \geq p(m)$. Therefore, to complete the proof we need only to check that the minimum of $p(m)$, $2 \leq m \leq d-1$, is $p_s$. This can be done by a simple computation.

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