Strong rainbow disconnection in graphs*

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Abstract

Let $G$ be a nontrivial edge-colored connected graph. An edge-cut $R$ of $G$ is called a rainbow edge-cut if no two edges of $R$ are colored with the same color. For two distinct vertices $u$ and $v$ of $G$, if an edge-cut separates them, then the edge-cut is called a $u$-$v$-edge-cut. An edge-colored graph $G$ is called strong rainbow disconnected if for every two distinct vertices $u$ and $v$ of $G$, there exists a both rainbow and minimum $u$-$v$-edge-cut (rainbow minimum $u$-$v$-edge-cut for short) in $G$, separating them, and this edge-coloring is called a strong rainbow disconnection coloring (srd-coloring for short) of $G$. For a connected graph $G$, the strong rainbow disconnection number (srd-number for short) of $G$, denoted by srd($G$), is the smallest number of colors that are needed in order to make $G$ strong rainbow disconnected.

In this paper, we first characterize the graphs with $m$ edges such that srd($G$) = $k$ for each $k \in \{1, 2, m\}$, respectively, and we also show that the srd-number of a nontrivial connected graph $G$ equals the maximum srd-number among the blocks of $G$. Secondly, we study the srd-numbers for the complete $k$-partite graphs, $k$-edge-connected $k$-regular graphs and grid graphs. Finally, we show that for a connected graph $G$, to compute srd($G$) is NP-hard. In particular, we show that it is already NP-complete to decide if srd($G$) = 3 for a connected cubic graph. Moreover, we show that for a given edge-colored (with an unbounded number of colors) connected graph $G$ it is NP-complete to decide whether $G$ is strong rainbow disconnected.

Keywords: edge-coloring; edge-connectivity; strong rainbow disconnection number; complexity; NP-hard (complete)

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a nontrivial connected graph with vertex-set $V(G)$ and edge-set $E(G)$. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and the neighborhood of $v$ in $G$ (or simply $d(v)$ and $N(v)$, respectively, when the graph $G$ is clear from the context). We use $\Delta(G)$ to denote the maximum degree of $G$. For any notation or terminology not defined here, we follow those used in [5].

Let $G$ be a graph with an edge-coloring $c : E(G) \to [k] = \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. When adjacent edges of $G$ receive different colors under $c$, the edge-coloring $c$ is called proper. The chromatic index of $G$, denoted by $\chi'(G)$, is the minimum number of colors needed in a proper edge-coloring of $G$. By a famous theorem of Vizing [14], one has that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

for every nonempty graph $G$. If $\chi'(G) = \Delta(G)$, then $G$ is said to be in Class 1; if $\chi'(G) = \Delta(G) + 1$, then $G$ is said to be in Class 2.

As we know that there are two ways to study the connectivity of a graph, one way is by using paths and the other is by using cuts. The rainbow connection using paths has been studied extensively; see for examples, papers [7, 11, 13] and book [12] and the references therein. So, it is natural to consider the rainbow edge-cuts for the colored connectivity in edged-colored graphs. In [6], Chartrand et al. first studied the rainbow edge-cut by introducing the concept of rainbow disconnection of graphs. In [4] we call all of them global colorings of graphs since they relate global structural property: connectivity of graphs.

An edge-cut of a connected graph $G$ is a set $F$ of edges such that $G - F$ is disconnected. The minimum number of edges in an edge-cut of $G$ is the edge-connectivity of $G$, denoted by $\lambda(G)$. For two distinct vertices $u$ and $v$ of $G$, let $\lambda_G(u, v)$ (or simply $\lambda(u, v)$ when the graph $G$ is clear from the context) denote the minimum number of edges in an edge-cut $F$ such that $u$ and $v$ lie in different components of $G - F$, and this kind of edge-cut $F$ is called a minimum $u$-$v$-edge-cut. A $u$-$v$-path is a path with ends $u$ and $v$. The following proposition presents an alternate interpretation of $\lambda(u, v)$ (see [8], [9]).

Proposition 1.1 For every two distinct vertices $u$ and $v$ in a graph $G$, $\lambda(u, v)$ is equal to the maximum number of pairwise edge-disjoint $u$-$v$-paths in $G$.

An edge-cut $R$ of an edge-colored connected graph $G$ is called a rainbow edge-cut if no two edges in $R$ are colored with the same color. Let $u$ and $v$ be two
distinct vertices of $G$. A rainbow $u$-$v$-edge-cut is a rainbow edge-cut $R$ of $G$ such that $u$ and $v$ belong to different components of $G - R$. An edge-colored graph $G$ is called rainbow disconnected if for every two distinct vertices $u$ and $v$ of $G$, there exists a rainbow $u$-$v$-edge-cut in $G$, separating them. In this case, the edge-coloring is called a rainbow disconnection coloring (rd-coloring for short) of $G$. The rainbow disconnection number (or rd-number for short) of $G$, denoted by $rd(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow disconnected. A rd-coloring with $rd(G)$ colors is called an optimal rd-coloring of $G$.

Remember that in the above Menger’s famous result of Proposition 1.1 only minimum edge-cuts play a role, however, in the definition of rd-colorings we only requested the existence of a $u$-$v$-edge-cut between a pair of vertices $u$ and $v$, which could be any edge-cut (large or small are both OK). This may cause the failure of a colored version of such a nice Min-Max result of Proposition 1.1. In order to overcome this problem, we will introduce the concept of strong rainbow disconnection in graphs, with a hope to set up the colored version of the so-called Max-Flow Min-Cut Theorem.

An edge-colored graph $G$ is called strong rainbow disconnected if for every two distinct vertices $u$ and $v$ of $G$, there exists a both rainbow and minimum $u$-$v$-edge-cut (rainbow minimum $u$-$v$-edge-cut for short) in $G$, separating them. In this case, the edge-coloring is called a strong rainbow disconnection coloring (srd-coloring for short) of $G$. For a connected graph $G$, we similarly define the strong rainbow disconnection number (srd-number for short) of $G$, denoted by $srd(G)$, as the smallest number of colors that are needed in order to make $G$ strong rainbow disconnected. An srd($G$)-coloring with $srd(G)$ colors is called an optimal srd-coloring of $G$.

The remainder of this paper will be organized as follows. In Section 2, we first obtain some basic results for the srd-numbers of graphs. In Section 3, we study the srd-numbers for some well-known classes of special graphs. In Section 4, we show that for a connected graph $G$, to compute $srd(G)$ is NP-hard. In particular, we show that it is already NP-complete to decide if $srd(G) = 3$ for a connected cubic graph. Moreover, we show that for a given edge-colored (with an unbounded number of colors) connected graph $G$ it is NP-complete to decide whether $G$ is strong rainbow disconnected.

2 Some basic results

Let $G$ be a connected graph. Recall that for a pair of distinct vertices $x$ and $y$ of $G$, we say that an edge-cut $\partial(X)$ separates $x$ and $y$ if $x \in X$ and $y \in V \setminus X$. 

We denote by $C_G(x, y)$ the minimum cardinality of such an edge-cut in $G$. Let $X$ be a vertex subset of $G$, and let $\overline{X} = V(G) \setminus X$. Then the graph $G/X$ is obtained from $G$ by shrinking $X$ to a single vertex. A trivial edge-cut is one associated with a single vertex. A block of a graph is a maximal connected subgraph of $G$ containing no cut-vertices. The block decomposition of $G$ is the set of blocks of $G$. From definitions, the following inequalities are obvious.

**Proposition 2.1** If $G$ is a nontrivial connected graph with edge-connectivity $\lambda(G)$, upper edge-connectivity $\lambda^+(G)$ and number $e(G)$ of edges, then

\[\lambda(G) \leq \lambda^+(G) \leq \text{rd}(G) \leq \text{sr}(G) \leq e(G).\]

Our first question is that the new parameter srd-number is really something new, different from rd-number? However, we have not found any connected graph $G$ with $sr(G) \neq rd(G)$. So, we pose the following conjecture.

**Conjecture 2.2** For any connected graph $G$, $sr(G) = rd(G)$.

In the rest of the paper we will show that for many classes of graphs the conjecture is true.

In this section, we characterize all those nontrivial connected graphs with $m$ edges such that $sr(G) = k$ for each $k \in \{1, 2, m\}$, respectively. We first characterize the graphs with $sr(G) = m$. The following are two lemmas which we will be used.

**Lemma 2.3** ([10]) Let $\partial(X)$ be a minimum edge-cut in a graph $G$ separating two vertices $x$ and $y$, where $x \in X$, and let $\partial(Y)$ be a minimum edge-cut in $G$ separating two vertices $u$ and $v$ of $X \setminus \overline{X}$, where $y \in Y$. Then every minimum $u-v$-edge-cut in $G/X$ ($G/X$) is a minimum $u-v$-edge-cut in $G$.

It follows from Lemma 2.3 that we have the following result.

**Lemma 2.4** Let $G$ be a connected graph of order at least 3. Then $sr(G) \leq e(G) - 1$.

**Proof.** We distinguish the following two cases.

**Case 1.** There exists at least one pair of vertices having nontrivial minimum edge-cut.

Let $C_G(x, y)$ be a nontrivial minimum $u-v$-edge-cut of $G$, where $x, y \in V(G)$, and let $\partial(X) = \min\{C_G(x, y)|x, y \in V(G)\}$. Suppose that $\partial(X)$ is a nontrivial minimum $x_0-y_0$-edge-cut in graph $G$, where $x_0 \in X$, and let $\partial(Y)$ be a minimum $u-v$-edge-cut
in \( G \), where \( u, v \in X \) and \( y_0 \in Y \). By Lemma 2.3, we get that every minimum \( u-v \)-edge-cut in \( G/X \) is a minimum \( u-v \)-edge-cut in \( G \). Now we give an edge-coloring \( c \) for \( G \) by assigning different colors for each edge of \( G[X] \) using colors from \( [e(G[X])] \) and assigning different colors for each edge of \( G[X] \) using colors from \( [e(G[X])] \), respectively, and assigning \( |\partial(X)| \) new colors for \( \partial(X) \). Note that the set \( E_w \) of edges incident with \( w \) is rainbow for each vertex \( w \) of \( G \), and \( |c| = \max\{e(G[X]), e(G[X])\} + |\partial(X)| \leq e(G) - 1 \) since \( e(G[X]), e(G[X]) \geq 1 \).

We can verify that the coloring \( c \) is an srd-coloring of \( G \). Let \( p \) and \( q \) be two vertices of \( G \). If \( p \) and \( q \) have a nontrivial minimum edge-cut \( C_{G}(p, q) \) in \( G \), then \( |C_{G}(p, q)| \geq |\partial(X)| \). Suppose that \( p \in X \) and \( q \in \overline{X} \). Without loss of generality, let \( d(p) \leq d(q) \). If \( d(p) < |\partial(X)| \), then the set \( E_p \) of edges incident with \( p \) is a rainbow minimum \( p-q \)-edge-cut in \( G \) under the coloring \( c \); if \( |\partial(X)| \leq d(p) \leq d(q) \), then the \( \partial(X) \) is a rainbow minimum \( p-q \)-edge-cut in \( G \) under the coloring \( c \). If \( p, q \in X \) \( (X) \), then the minimum \( p-q \)-edge-cut in \( G/X \) (\( G/X \)) is a rainbow minimum \( p-q \)-edge-cut in \( G \) since the colors of the edges in graph \( G/X \) (\( G/X \)) are different from each other under the restriction of coloring \( c \).

**Case 2.** For any two vertices of \( G \), there are only trivial minimum edge-cut.

If \( G \) is a tree, then \( \text{srd}(G) = 1 \). Obviously, \( \text{srd}(G) \leq e(G) - 1 \) since \( G \) is a connected graph with \( n \geq 3 \). Otherwise, we give a proper edge-coloring for \( G \) using \( n - 1 \) colors. Since \( G \) is not a tree, we have \( n - 1 \leq e(G) - 1 \). For any two vertices \( p, q \) of \( G \), without loss of generality, let \( d(p) \leq d(q) \), the set \( E_p \) of edges incident with \( p \) is a rainbow minimum \( p-q \)-edge-cut in \( G \). □

By Lemma 2.3, we immediately obtain the following result.

**Corollary 2.5** Let \( G \) be a connected graph. Then \( \text{srd}(G) = e(G) \) if and only if \( G = P_2 \).

Next, we further characterize the graphs \( G \) with \( \text{srd}(G) = 1 \) and \( 2 \), respectively. We first restate two results as lemmas which characterize the graphs with \( \text{rd}(G) = 1 \) and \( 2 \), respectively.

**Lemma 2.6** [6] Let \( G \) be a nontrivial connected graph. Then \( \text{rd}(G) = 1 \) if and only if \( G \) is a tree.

**Lemma 2.7** [6] Let \( G \) be a nontrivial connected graph. Then \( \text{rd}(G) = 2 \) if and only if each block of \( G \) is either \( K_2 \) or a cycle and at least one block of \( G \) is a cycle.

Furthermore, we obtain the following two results.
Theorem 2.8  Let $G$ be a nontrivial connected graph. Then $srd(G) = 1$ if and only if $rd(G) = 1$.

Proof. First, if $srd(G) = 1$, then we have $1 \leq rd(G) \leq srd(G)$ by Proposition 2.1. Next, if $rd(G) = 1$, then the graph $G$ has no cycle, namely, the $G$ is a tree. We give one color for all edges of $G$. Obviously, the coloring is an optimal srd-coloring of $G$, and so $srd(G) = 1$ by Proposition 2.1. □

Theorem 2.9  Let $G$ be a nontrivial connected graph. Then $srd(G) = 2$ if and only if $rd(G) = 2$.

Proof. First, if $srd(G) = 2$, then $G$ has no cycle with a chord by Proposition 2.1. Furthermore, if $G$ is a tree, we showed $srd(G) = 1$. Therefore, each block of $G$ is either a $K_2$ or a cycle and at least one block of $G$ is a cycle. By Lemma 2.7, we get $rd(G) = 2$.

Conversely, suppose $rd(G) = 2$. Then each block of $G$ is either a $K_2$ or a cycle and at least one block of $G$ is a cycle. We can give a 2-edge-coloring $c$ for $G$ as follows. Choose one edge from each cycle to give color 1. The remaining edges are assigned color 2. One can easily verify that the coloring $c$ is strong rainbow disconnected. Combined with Proposition 2.1, we have $srd(G) = 2$. □

By Lemmas 2.6 and 2.7, and Theorems 2.8 and 2.9, we immediately get the following corollary.

Corollary 2.10  Let $G$ be a nontrivial connected graph. Then

(i) $srd(G) = 1$ if and only if $G$ is a tree.

(ii) $srd(G) = 2$ if and only if each block of $G$ is either a $K_2$ or a cycle and at least one block of $G$ is a cycle.

Furthermore, we get $srd(G) = srd(B)$, where $srd(B)$ is maximum among all blocks of $G$. It implies that the study of srd-numbers can be restricted to 2-connected graphs.

Lemma 2.11  If $H$ is a block of a graph $G$, then $srd(H) \leq srd(G)$.

Proof. Let $c$ be an optimal srd-coloring of $G$, and let $u, v$ be two vertices of $H$. Suppose $R$ is a rainbow minimum $u-v$-edge-cut in $G$. Then $R \cap E(H)$ is a rainbow minimum $u-v$-edge-cut in $H$. Assume that there exists a smaller $u-v$-edge-cut $R'$ in $H$. Then there is no $u-v$-path in $G \setminus R'$, which is a contradiction with definition of $R$.  

since $|R'| < |R|$. Hence, the coloring $c$ restricted to $H$ is an srd-coloring of $H$. Thus, $\text{srd}(H) \leq \text{srd}(G)$.

\begin{proof}
Let $\{B_1, B_2, \ldots, B_t\}$ be the block decomposition of $G$, and let $k = \max\{\text{srd}(B_i) : i \in [t]\}$. If $G$ has no cut-vertex, then $G = B_1$ and the result follows. Hence, we may assume that $G$ has at least one cut-vertex. By Lemma 2.11, we have $k \leq \text{srd}(G)$.

Let $c_i$ be an optimal srd-coloring of $B_i$. We define the edge-coloring $c: E(G) \to [k]$ of $G$ by $c(e) = c_i(e)$ if $e \in E(B_i)$. Let $u$ and $v$ be two vertices of $G$. If $u, v \in B_i (i \in [t])$, let $C_G(u, v) = C^r_{B_i}(u, v)$, where $C^r_{B_i}(u, v)$ is the rainbow minimum $u-v$-edge-cut in $B_i$. Obviously, $C_G(u, v)$ is rainbow under the coloring $c_i$. Moreover, it is minimum $u-v$-edge-cut in $G$. Otherwise, assume that $R$ is a smaller $u-v$-edge-cut in $G$. Then $R \cap E(B_i)$ is also a $u-v$-edge-cut in $B_i$, which contradicts to the definition of $C^r_{B_i}(u, v)$ since $|R \cap E(B_i)| < |C_{B_i}(u, v)|$. Hence, the $C_G(u, v)$ is a rainbow minimum $u-v$-edge-cut in $G$. Suppose that $u \in B_i$ and $v \in B_j$, where $i < j$ and $i, j \in [t]$. Let $B_i x_iB_{i+1}x_{i+1} \ldots x_{j-1}B_j$ be a unique $B_iB_j$-path in the block-tree of $G$, and let $x_i$ be the cut-vertex between blocks $B_i$ and $B_{i+1}$. If $u = x_i$ and $v = x_{j-1}$, let $C_G(u, v) = \min\{C^r_{B_{i+1}}(x_i, x_{i+1}), \ldots, C^r_{B_{j-1}}(x_{j-2}, x_{j-1})\}$. If $u = x_i$ and $v \neq x_{j-1}$, let $C_G(u, v) = \min\{C^r_{B_{i+1}}(x_i, x_{i+1}), \ldots, C^r_{B_{j-1}}(x_{j-2}, x_{j-1}), C^r_{B_j}(x_{j-1}, v)\}$. If $u \neq x_i$ and $v = x_{j-1}$, let $C_G(u, v) = \min\{C^r_{B_i}(u, x_i), C^r_{B_{i+1}}(x_i, x_{i+1}), \ldots, C^r_{B_{j-1}}(x_{j-2}, x_{j-1})\}$. If $u \neq x_i$ and $v \neq x_{j-1}$, let $C_G(u, v) = \min\{C^r_{B_i}(u, x_i), C^r_{B_{i+1}}(x_i, x_{i+1}), \ldots, C^r_{B_j}(x_{j-1}, v)\}$.

By the connectivity of $G$, we know that $\lambda_G(u, v) = |C_G(u, v)|$, and $C_G(u, v)$ is rainbow. Then $C_G(u, v)$ is a rainbow minimum $u-v$-edge-cut in $G$. Hence, $\text{srd}(G) \leq k$, and so $\text{srd}(G) = k$.

\end{proof}

\begin{remark}
As one has seen that all the above results for the srd-number behave the same as for the rd-number. This supports Conjecture 2.2.
\end{remark}

\section{The srd-numbers of some classes of graphs}

In this section, we investigate the srd-numbers of complete graphs, complete multipartite graphs, regular graphs and grid graphs. Again, we will see that the srd-number behaves the same as the rd-number. At first, we restate several results as lemmas which will be used in the sequel.
Lemma 3.1 [1] Let \( G \) be a connected graph. If every connected component of \( G_\Delta \) is a unicyclic graph or a tree, and \( G_\Delta \) is not a disjoint union of cycles, then \( G \) is in Class 1.

Lemma 3.2 [6] For each integer \( n \geq 4 \), \( \text{rd}(K_n) = n - 1 \).

Lemma 3.3 [2] If \( G = K_{n_1, n_2, \ldots, n_k} \) is a complete \( k \)-partite graph with order \( n \), where \( k \geq 2 \) and \( n_1 \leq n_2 \leq \cdots \leq n_k \), then
\[
\text{rd}(K_{n_1, n_2, \ldots, n_k}) = \begin{cases} 
n - n_2, & \text{if } n_1 = 1, \\
n - n_1, & \text{if } n_1 \geq 2.
\end{cases}
\]

Lemma 3.4 [2] If \( G \) is a connected \( k \)-regular graph, then \( k \leq \text{rd}(G) \leq k + 1 \).

Lemma 3.5 [1] The \( \text{rd} \)-number of the grid graph \( G_{m,n} \) is as follows.
(i) For all \( n \geq 2 \), \( \text{rd}(G_{1,n}) = \text{rd}(P_n) = 1 \).
(ii) For all \( n \geq 3 \), \( \text{rd}(G_{2,n}) = 2 \).
(iii) For all \( n \geq 4 \), \( \text{rd}(G_{3,n}) = 3 \).
(iv) For all \( 4 \geq m \geq n \), \( \text{rd}(G_{m,n}) = 4 \).

First, we get the \( \text{srd} \)-number for complete graphs.

Theorem 3.6 For each integer \( n \geq 2 \), \( \text{srd}(K_n) = n - 1 \).

Proof. By Proposition 2.1 and Lemma 3.2, \( n - 1 \leq \text{rd}(K_n) \leq \text{srd}(K_n) \). It remains to show that there exists an \( \text{srd} \)-coloring for \( K_n \) using \( n - 1 \) colors. Suppose first that \( n \geq 2 \) is even. Let \( u \) and \( v \) be two vertices of \( K_n \), and let \( c \) be a proper edge-coloring of \( K_n \) using \( n - 1 \) colors. Since \( \lambda(K_n) = n - 1 \), the set \( E_u \) of edges incident with \( u \) is a rainbow minimum \( u-v \)-edge-cut in \( G \). Next suppose \( n \geq 3 \) is odd. We give the same edge-coloring for \( G \) as the coloring in Lemma 3.2. We now restate it as follows.

Let \( x \) be a vertex of \( K_n \) and \( K_{n-1} = K_n - x \). Then \( K_{n-1} \) has a proper edge-coloring \( c \) using \( n - 2 \) colors since \( n - 1 \) is even. Now we extend an edge-coloring \( c \) of \( K_{n-1} \) to \( K_n \) by assigning color \( n - 1 \) for each edge incident with vertex \( x \). We show that the \( c \) is an \( \text{srd} \)-coloring of \( G \). Let \( u \) and \( v \) be two vertices of \( K_n \), say \( u \neq x \). Then the set \( E_u \) of edges incident with \( u \) is a rainbow minimum \( u-v \)-edge-cut in \( G \) since \( \lambda(K_n) = n - 1 \). \( \square \)

Then, we give the \( \text{srd} \)-number for complete multipartite graphs.
Theorem 3.7 If \( G = K_{n_1,n_2,\ldots,n_k} \) is a complete \( k \)-partite graph with order \( n \), where \( k \geq 2 \) and \( n_1 \leq n_2 \leq \cdots \leq n_k \), then
\[
srd(K_{n_1,n_2,\ldots,n_k}) = \begin{cases} 
    n - n_2, & \text{if } n_1 = 1, \\
    n - n_1, & \text{if } n_1 \geq 2.
\end{cases}
\]

Proof. It remains to prove that \( srd(G) \leq n - n_2 \) for \( n_1 = 1 \), and \( srd(G) \leq n - n_1 \) for \( n_1 \geq 2 \) by Proposition 2.1 and Lemma 3.3. Let \( V_1, V_2, \ldots, V_k \) be the \( k \)-partition of the vertices of \( G \), and \( V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\} \) for each \( i \in [k] \). We distinguish the following two cases.

Case 1. \( n_1 = 1 \).

First, we have \( V_1 = \{v_{1,1}\} \) and \( d(v_{1,1}) = n - 1 \). Let \( H = G - \{v_{1,1}\} \). Then \( \Delta(H) = n - n_2 - 1 \). Then, we construct a proper edge-coloring \( c_0 \) of \( H \) using colors from \( [\Delta(H) + 1] \). For each vertex \( x \in V(H) \), since \( d_H(x) \leq \Delta(H) \), there is an \( a_x \in [\Delta(H) + 1] \) such that \( a_x \) is not assigned to any edge incident with \( x \) in \( H \). Since \( E(G) = E(H) \cup \{v_{1,1}x \mid x \in N_G(v_{1,1})\} \), we now extend the edge-coloring \( c_0 \) of \( H \) to an edge-coloring \( c \) of \( G \) by assigning \( c(v_{1,1}x) = a_x \) for every vertex \( x \in N_G(v_{1,1}) \). Note that the set \( E_x \) of edges incident with \( x \) is a rainbow set for each vertex \( x \in V(G) \setminus v_{1,1} \) in \( G \). Suppose \( p \) and \( q \) are two vertices of \( G \). If \( p \in V_i \) and \( q \in V_j \) (\( 1 \leq i < j \leq t \)), then the set \( E_q \) of edges incident with \( q \) is a rainbow minimum \( p-q \)-edge-cut in \( G \) since \( \lambda_G(p,q) = n - n_j \). If \( p, q \in V_i \), then the set \( E_q \) of edges incident with \( q \) is a rainbow minimum \( p-q \)-edge-cut in \( G \) since \( \lambda_G(p,q) = n - n_i \). Hence, we obtain \( srd(G) \leq \Delta(H) + 1 = n - n_2 \).

Case 2. \( n_1 \geq 2 \).

Pick a vertex \( u \) of \( V_1 \) and let \( F = G - u \). Then \( \Delta(F) = n - n_1 \) since \( n_1 \geq 2 \) and \( F_\Delta = G[V_1-u] \). It follows from Lemma 3.1 that \( F \) is in Class 1, and so \( \chi'(F) = n - n_1 \). Furthermore, for each vertex \( x \in N_G(u) \), we know \( d_F(x) \leq \Delta(F) - 1 = n - n_1 - 1 \). Similar to the argument of Case 1, we can construct an edge-coloring \( c \) for \( G \) such that the set \( E_x \) of edges incident with \( x \) is a rainbow set for each vertex \( x \in V(G) \setminus u \) using \( n - n_1 \) colors. Suppose \( p \) and \( q \) are two vertices of \( G \). If \( p \in V_i \) and \( q \in V_j \) (\( 1 \leq i < j \leq t \)), then the set \( E_q \) of edges incident with \( q \) is a rainbow minimum \( p-q \)-edge-cut in \( G \) since \( \lambda_G(p,q) = n - n_j \). If \( p, q \in V_i \) (\( i \in [t] \)), say \( q \neq u \), then the set \( E_q \) of edges incident with \( q \) is a rainbow minimum \( p-q \)-edge-cut in \( G \) since \( \lambda_G(p,q) = n - n_i \). Hence, \( srd(G) \leq n - n_1 \). \( \square \)

For regular graphs, we only study the \( srd \)-number of \( k \)-edge-connected \( k \)-regular graphs. Moreover, we obtain that \( srd(G) = k \) if and only if \( \chi'(G) = k \) for a \( k \)-edge-connected \( k \)-regular graph \( G \), where \( k \) is odd.
Lemma 3.8 [3] Let $k$ be an odd integer, and $G$ a $k$-edge-connected $k$-regular graph of order $n$. Then $\chi'(G) = k$ if and only if $rd(G) = k$.

Theorem 3.9 Let $G$ be a $k$-edge-connected $k$-regular graph. Then $k \leq srd(G) \leq \chi'(G)$.

Proof. It follows from Proposition 2.1 that $srd(G) \geq k$. Let $u, v$ be two vertices of $G$. Using the fact that $G$ is a $k$-edge-connected $k$-regular graph, one may easily verify that the set $E_v$ of edges incident with $v$ is a rainbow minimum $u$-$v$-edge-cut under a proper edge-coloring of $G$. Hence, $srd(G) \leq \chi'(G)$.

Theorem 3.10 Let $k$ be an odd integer, $G$ a $k$-edge-connected $k$-regular graph. Then $srd(G) = k$ if and only if $rd(G) = k$.

Proof. First, suppose $srd(G) = k$. Since $\lambda(G) = k$, we have $rd(G) = k$ by Proposition 2.1. Conversely, if $rd(G) = k$, then we have $srd(G) = k$ by Proposition 2.1 and Lemma 3.8 and Theorem 3.9.

Corollary 3.11 Let $k$ be an odd integer, $G$ a $k$-edge-connected $k$-regular graph. Then $srd(G) = k$ if and only if $\chi'(G) = k$.

The Cartesian product $G \square H$ of two vertex-disjoint graphs $G$ and $H$ is the graph with vertex-set $V(G) \times V(H)$, where $(u, v)$ is adjacent to $(x, y)$ in $G \square H$ if and only if either $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$. We consider the $m \times n$ grid graph $G_{m,n} = P_m \square P_n$, which consists of $m$ horizontal paths $P_n$ and $n$ vertical paths $P_m$. Now we determine the srd-number for grid graphs.

Theorem 3.12 The srd-number of the grid graph $G_{m,n}$ is as follows.

(i) For all $n \geq 2$, $srd(G_{1,n}) = srd(P_n) = 1$.

(ii) For all $n \geq 3$, $srd(G_{2,n}) = 2$.

(iii) For all $n \geq 4$, $srd(G_{3,n}) = 3$.

(iv) For all $4 \geq m \geq n$, $srd(G_{m,n}) = 4$.

Proof. First, it follows from Proposition 2.1 and Lemma 3.5 that the lower bounds on $srd(G_{m,n})$ in (i)-(iv) hold. It remains to show that the upper bound on $srd(G_{m,n})$ in each of (i)-(iv) also holds.
(i) We get $\text{srd}(G_{1,n}) = \text{srd}(P_n) = 1$ by Corollary 2.10.

For the rest of the proof, the vertices of $G_{m,n}$ are regarded as a matrix. Let $x_{i,j}$ be the vertex in row $i$ and column $j$, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

(ii) We give the same edge-coloring $c$ for $G_{2,n}$ ($n \geq 3$) using colors from the elements of $Z_3$ of the integer modulo 3 as in Lemma 3.5 (ii). Define the edge-coloring $c$ for $G_{2,n}$: $c \rightarrow Z_3$, and we now restate it as follows.

- $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 2$ and $1 \leq j \leq n - 1$;
- $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n - 1$.

One can verify that the coloring $c$ is an srd-coloring for $G_{2,n}$. Let $u$ and $v$ be two vertices of $G_{2,n}$. If $u$ and $v$ are in different columns, then two parallel edges between $u$ and $v$ joining vertices in the same two columns form a rainbow minimum $u$-$v$-edge-cut in $G_{2,n}$ since $\lambda(u, v) = 2$. Suppose $u$ and $v$ are in the same column. Because the set $E_u$ of edges incident with $u$ is rainbow and $\lambda(u, v) = d(u) = d(v)$, the set $E_u$ of edges incident with $u$ is a rainbow minimum $u$-$v$-edge-cut in $G_{2,n}$.

(iii) Give the same edge-coloring $c$ as for the graph $G_{3,n}$ ($n \geq 3$) in Lemma 3.5 (iii). Again we use the elements of $Z_3$ as the colors here. Define the edge-coloring $c$ for $G_{3,n}$: $c \rightarrow Z_3$ as follows.

- $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 3$ and $1 \leq j \leq n - 1$;
- $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n - 1$;
- $c(x_{2,j}x_{3,j}) = j + 2$ for $1 \leq j \leq n - 1$.

Now we show that the coloring $c$ is an srd-coloring of $G_{3,n}$. Observe that the set $E_x$ of edges incident with $x$ is rainbow for each vertex $x$ with $d(x) \leq 3$ in $G_{3,n}$ under the coloring $c$. Let $u$ and $v$ be two vertices of $G_{3,n}$. If $u$ and $v$ have at most one vertex with degree 4, without loss of generality, $2 \leq d(u) \leq d(v) \leq 4$, then the set $E_u$ of edges incident with $u$ is a rainbow minimum $u$-$v$-edge-cut in $G_{3,n}$ since $\lambda(u, v) = d(u)$. If $d(u) = d(v) = 4$, then three parallel edges between $u$ and $v$ joining vertices in the same two columns form a rainbow minimum $u$-$v$-edge-cut in $G_{3,n}$ since $\lambda(u, v) = 3$.

(iv) For the graph $G_{m,n}$ ($4 \leq m \leq n$), because $G_{m,n}$ is bipartite and $\Delta(G_{m,n}) = 4$, there exists a proper edge-coloring $c$ using 4 colors. Now we show that the $c$ is an srd-coloring of $G_{m,n}$. Let $u$ and $v$ be two vertices of $G_{m,n}$. Suppose $d(u) \leq d(v)$. Then the set $E_u$ of edges incident with $u$ is a rainbow minimum $u$-$v$-edge-cut in $G_{m,n}$ ($4 \leq m \leq n$) since $\lambda(u, v) = d(u)$. □
4 Hardness results

First, we show that our problem is in NP for any fixed integer $k$.

**Lemma 4.1** For a fixed positive integer $k$, given a $k$-edge-colored graph $G$, deciding whether $G$ is a strong rainbow disconnected under the coloring is in P.

**Proof.** Let $n$, $m$ be the number of the vertices and edges of $G$, respectively. Let $u$, $v$ be two vertices of $G$. Because $G$ has at most $k$ colors, we have at most $\sum_{l=1}^{k} \binom{m}{l}$ rainbow edge subsets in $G$, denoted the set of the subsets by $\mathcal{S}$. One can see that this number is upper bounded by a polynomial in $m$ when $k$ is a fixed integer (say $km^k$, roughly speaking). Given a rainbow subset of edges $S \in \mathcal{S}$, it is checkable in polynomial time to decide whether $S$ is a $u$-$v$-edge-cut of $G$, just to see whether $u$ and $v$ lie in different components of $G \setminus S$, and the number of components is a polynomial in $n$. If each rainbow subset in $\mathcal{S}$ is not a $u$-$v$-edge-cut in $G$, then the coloring is not an srd-coloring of $G$, which can be checked in polynomial time since the number of such subsets is polynomial many in $m$. Otherwise, let the integer $l_0(\leq k)$ be the minimum size of a $u$-$v$-edge-cut in $G$, and this $l_0$ can be computed in polynomial time. Then, if one of the rainbow subsets of $\mathcal{S}$ is a $u$-$v$-edge-cut of $G$ with size $l_0$, then it is a rainbow minimum $u$-$v$-edge-cut of $G$, which can be done in polynomial time since the number of such subsets is polynomial many in $m$. Otherwise, the coloring is not an srd-coloring. Moreover, there are at most $\binom{n}{2}$ pairs of vertices in $G$. Since $k$ is an integer, we can deduce that deciding wether a $k$-edge-colored graph $G$ is strong rainbow disconnected can be checked in polynomial time. □

In particular, it is NP-complete to determine whether $\text{srd}(G) = 3$ for a cubic graph. We first restate the following result as a lemma.

**Lemma 4.2** [2] It is NP-complete to determine whether the rd-number of a cubic is 3 or 4.

**Theorem 4.3** It is NP-complete to determine whether the srd-number of a cubic is 3 or 4.

**Proof.** The problem is in NP from Lemma 4.1. Furthermore, we get that it is NP-hard to determine whether the srd-number of a 3-edge-connected cubic is 3 or 4 by Theorem 3.10 and the proof of Lemma 4.2. □

Lemma 4.1 tells us that deciding whether a given $k$-edge-colored graph $G$ is strong rainbow disconnected for a fixed integer $k$ is in P. However, it is NP-complete to decide...
whether a given edge-colored (with an unbounded number of colors) graph is strong rainbow disconnected.

**Theorem 4.4** Given an edge-colored graph \( G \) and two vertices \( s, t \) of \( G \), deciding whether there is a rainbow minimum \( s-t \)-edge-cut is NP-complete.

**Proof.** Clearly, the problem is in NP, since for a graph \( G \) checking whether a given set of edges is a rainbow minimum \( s-t \)-edge-cut in \( G \) can be done in polynomial time, just to see whether it is an \( s-t \)-edge-cut and it has the minimum size \( \lambda_G(s, t) \) by solving the maximum flow problem. We exhibit a polynomial reduction from the problem 3SAT. Given a 3CNF for \( \phi = \bigwedge_{i=1}^{m} c_i \) over variables \( x_1, x_2, \ldots, x_n \), we construct a graph \( G_{\phi} \) with two special vertices \( s, t \) and an edge-coloring \( f \) such that there is a rainbow minimum \( s-t \)-edge-cut in \( G_{\phi} \) if and only if \( \phi \) is satisfiable.

We define \( G_{\phi} \) as follows:

\[
V(G_{\phi}) = \{ s, t \} \cup \{ x_{i,0}, x_{i,1} | i \in [n] \} \cup \{ c_{i,j} | i \in [m], j \in \{0, 1, 2, 3\} \} \\
\cup \{ p_{i,j}, q_{i,j} | i \in [n], j \in [\ell_i] \} \cup \{ y_i | i \in [5m + 1] \},
\]

where \( \ell_i \) is the number of times of each variable \( x_i \) appearing among the clauses of \( \phi \).

\[
E(G_{\phi}) = \{ sp_{i,l}, sq_{i,l} | i \in [n], l \in [\ell_i] \} \\
\cup \{ p_{i,l}x_{i,0}, q_{i,l}x_{i,1} | i \in [n], l \in [\ell_i] \} \\
\cup \{ x_{j,0}c_{i,0}, c_{i,0}c_{i,k}, c_{i,k}x_{j,1} | i \in [m], j \in [n], k \in \{1, 2, 3\} \} \\
\cup \{ x_{j,1}c_{i,0}, c_{i,0}c_{i,k}, c_{i,k}x_{j,0} | i \in [m], j \in [n], k \in \{1, 2, 3\} \} \\
\cup \{ E(K_{6m+2}) | V(K_{6m+2}) = \{ c_{1,0}, \ldots, c_{m,0}, y_1, \ldots, y_{5m+1}, t \} \}.
\]

The edge-coloring \( f \) is defined as follows (see Figure 1):

- The edges \( \{ sp_{i,l}, p_{i,l}x_{i,0}, sq_{i,l}, q_{i,l}x_{i,1} | i \in [n], l \in [\ell_i] \} \) are colored with a special color \( r_0^0 \).
- The edge \( x_{j,0}c_{i,0} \) or \( x_{j,1}c_{i,0} \) is colored with a special color \( r_{i,k} \) when \( x_j \) is the \( k \)-th literal of clause \( c_i \), \( i \in [m], j \in [n], k \in \{1, 2, 3\} \).
- The edge \( c_{i,k}x_{j,0} \) or \( c_{i,k}x_{j,1} \) is colored with a special color \( r_{i,4} \), \( i \in [m], j \in [n], k \in \{1, 2, 3\} \).

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• The edge $c_{i,k}c_{i,0}$ is colored with a special color $r_{i,5}$, $i \in [m], k \in \{1, 2, 3\}$.
• The remaining edges are colored with a special color $r_0$.

![Diagram](image)

Figure 1: The clause $c_1 = (x_1, \overline{x_2}, x_3)$ and the variable $x_3$ is in clause $c_1$ and $c_2$.

Now we verify that there is a rainbow minimum $s$-$t$-edge-cut in $G_\phi$ if and only if $\phi$ is satisfiable.

Assume that there exists a rainbow minimum $s$-$t$-edge-cut $S$ in $G_\phi$ under the coloring $f$, and let us show that $\phi$ is satisfiable. Note that for each $j \in [n]$, $l \in I_j$, if $S$ has an edge in $\{sp_{j,l}, p_{j,l}x_{j,0}\}$ (or $\{sq_{j,l}, q_{j,l}x_{j,0}\}$), then a rainbow $s$-$x_{j,0}$ (or $s$-$x_{j,1}$)-edge-cut in $G[s \cup x_{j,0} \cup \{p_{j,l}[l \in I_j]\}]$ is in $S$, and no edge of $\{sq_{j,l}, q_{j,l}x_{j,1}[l \in I_j]\}$ (or $\{sp_{j,l}, p_{j,l}x_{j,0}[l \in I_j]\}$) is in $S$. Otherwise, it contradicts to the assumption that $S$ is a rainbow minimum $s$-$t$-edge-cut in $G_\phi$. For each $j \in [n]$, if a rainbow $s$-$x_{j,0}$-edge-cut in $G[s \cup x_{j,0} \cup \{p_{j,l}[l \in I_j]\}]$ is in $S$ under the coloring $f$, then set $x_j = 0$; if a rainbow $s$-$x_{j,1}$-edge-cut in $G[s \cup x_{j,1} \cup \{q_{j,l}[l \in I_j]\}]$ is in $S$ under the coloring $f$, then set $x_j = 1$.

First, we have $|S| = 6m$ and $S \subseteq G[V(G_\phi) \setminus \{y_1, \ldots, y_{5m+1}, t\}]$. Moreover, for given $c_{i,0}$ ($i \in [m]$), we know that $S$ has at most two edges from three paths of length two between $c_{i,0}$ and $\{x_{j,0}, x_{j,1}\}$ in $c_i$ and $j \in [n]$ under the coloring $f$ of $G_\phi$. Suppose, without loss of generality, that the path of length two between $x_{j,0}$ (or $x_{j,1}$) and $c_{i,0}$ has no edge belonging to $S$ for some $j \in [n]$. If $x_j$ in $c_i$ is positive, then there exists a rainbow $s$-$x_{j,1}$-edge-cut with size $\ell_j$ in $G[s \cup x_{j,1} \cup \{q_{j,l}[l \in I_j]\}]$ belonging to $S$, where $i \in [m], j \in [n]$. Then $x_j = 1$ and $c_i$ is satisfiable. If $x_j$ in $c_i$ is negative, then there exists a rainbow $s$-$x_{j,0}$-edge-cut with size $\ell_j$ in $G[s \cup x_{j,1} \cup \{p_{j,l}[l \in I_j]\}]$ belonging to $S$, where $i \in [m], j \in [n]$. Then $x_j = 0$ and $c_i$ is satisfiable. Since this is true for each $c_i$ ($i \in [m]$), we get that $\phi$ is satisfiable.
Now suppose $\phi$ is satisfiable, and let us construct a rainbow minimum $s$-$t$-edge-cut in $G_\phi$ under the coloring $f$. First, there exists a satisfiable assignment of $\phi$. If $x_j = 0$, we put the rainbow $s$-$x_{j,0}$-edge-cut in $G[s \cup x_{j,0} \cup \{p_l \mid l \in l_j\}]$ into $S$ for each $j \in [n]$. If the vertex $x_{j,0}$ is adjacent to $c_{i,0}$, then let one edge of $c_{i,k}x_{j,1}, c_{i,k}c_{i,0}$ be in $S$ for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If the vertex $x_{j,0}$ is adjacent to $c_{i,k}$, then let the edge $x_{j,1}c_{i,0}$ be in $S$ for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If $x_j = 1$, we put the rainbow $s$-$x_{j,1}$-edge-cut in $G[s \cup x_{j,1} \cup \{q_l \mid l \in l_j\}]$ into $S$ for each $j \in [n]$. If the vertex $x_{j,1}$ is adjacent to $c_{i,0}$, then let one edge of $c_{i,k}x_{j,0}, c_{i,k}c_{i,0}$ be in $S$ for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. If the vertex $x_{j,1}$ is adjacent to $c_{i,k}$, then let the edge $x_{j,0}c_{i,0}$ be in $S$ for each $i \in [m], j \in [n], k \in \{1, 2, 3\}$. Now we verify that $S$ is indeed a rainbow minimum $s$-$t$-edge-cut. First, we can verify that $|S| = 6m$ and it is a minimum $s$-$t$-edge-cut. In fact, if a literal of $c_i$ is false, then one edge colored with $r^4_i$ or $r^5_i$ is in $S$. Since the three literals of $c_i$ cannot be false at the same time, we can find a rainbow minimum $s$-$t$-edge-cut in $G_\phi$ under the coloring $f$. □

5 Concluding remarks

In this paper we defined a new colored connection parameter srd-number for connected graphs. We hope that with this new parameter, avoiding the drawback of the parameter rd-number, one could get a colored version of the famous Menger’s Min-Max Theorem. We do not know if this srd-number is actually equal to the rd-number for every connected graph, and then posed a conjecture to further study on the two parameters. The results in the last sections fully support the conjecture.

References

[1] S. Akbari, D. Cariolaro, M. Chavooshi, M. Ghanbari, S. Zare, Some criteria for a graph to be in Class 1, Discrete Math. 312(2012), 2593–2598.

[2] X. Bai, R. Chang, Z. Huang, X. Li, More on rainbow disconnection in graphs, Discuss. Math. Graph Theory, in press. doi:10.7151/dmgt.2333.

[3] X. Bai, Z. Huang, X. Li, Bounds for the rainbow disconnection number of graphs, arXiv: 2003.13237[math.CO].

[4] X. Bai, X. Li, Graph colorings under global structural conditions, arXiv: 2008.07163 [math.CO].
[5] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 224, Springer, 2008.

[6] G. Chartrand, S. Devereaux, T.W. Haynes, S.T. Hedetniemi, P. Zhang, Rainbow disconnection in graphs, Discuss. Math. Graph Theory 38(2018), 1007–1021.

[7] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem. 133(2008), 85–98.

[8] P. Elias, A. Feinstein, C.E. Shannon, A note on the maximum flow through a network, IRE Trans. Inform. Theory, IT 2(1956), 117–119.

[9] L.R. Ford Jr., D.R. Fulkerson, Maximal flow through a network, Canad. J. Math. 8(1956), 399–404.

[10] R.E. Gomory, T.C. Hu, Multi-terminal network flows, J. Soc. Indust. Appl. Math. 9(1961), 551–570.

[11] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, Graphs Combin. 29(2013), 1–38.

[12] X. Li, Y. Sun, Rainbow Connections of Graphs, Springer Briefs in Math., Springer, New York, 2012.

[13] X. Li, Y. Sun, An updated survey on rainbow connections of graphs - a dynamic survey, Theo. Appl. Graphs 0(2017), Art. 3, 1–67.

[14] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Anal. 3(1964), 25–30, in Russian.