WEAKLY Z SYMMETRIC MANIFOLDS

CARLO ALBERTO MANTICA AND LUCA GUIDO MOLINARI

Abstract. We introduce a new kind of Riemannian manifold that includes weakly-, pseudo- and pseudo projective- Ricci symmetric manifolds. The manifold is defined through a generalization of the so called Z tensor; it is named weakly Z symmetric and denoted by \((WZS)_n\). If the Z tensor is singular we give conditions for the existence of a proper concircular vector. For non singular Z tensor, we study the closedness property of the associated covectors and give sufficient conditions for the existence of a proper concircular vector in the conformally harmonic case, and the general form of the Ricci tensor. For conformally flat \((WZS)_n\) manifolds, we derive the local form of the metric tensor.

1. Introduction

In 1993 Tamassy and Binh [31] introduced and studied a Riemannian manifold whose Ricci tensor\(^1\) satisfies the equation:

\[
\nabla_k R_{jl} = A_k R_{jl} + B_j R_{kl} + D_l R_{kj}.
\]

The manifold is called weakly Ricci symmetric and denoted by \((WRS)_n\). The covectors \(A_k, B_k\) and \(D_k\) are the associated 1-forms. The same manifold with the 1-form \(A_k\) replaced by \(2A_k\) was studied by Chaki and Koley [6], and called generalized pseudo Ricci symmetric. The two structures extend pseudo Ricci symmetric manifolds, \((PRS)_n\), introduced by Chaki [4], where \(\nabla_k R_{jl} = 2A_k R_{jl} + A_j R_{kl} + A_l R_{kj}\) (this definition differs from that of R. Deszcz [17]).

Later on, other authors studied the manifolds \([10, 20, 12]\); in [12] some global properties of \((WRS)_n\) were obtained, and the form of the Ricci tensor was found. In [10] generalized pseudo Ricci symmetric manifolds were considered, where the conformal curvature tensor

\[
C_{jkl,m} = R_{jkl}^m + \frac{1}{n-2}(\delta_j^m R_{kl,m} - \delta_k^m R_{jl,m} + R_j^m g_{kl} - R_k^m g_{jl})
\]

vanishes (for \(n = 3\): \(C_{jkl,m} = 0\) holds identically, [27]) and the existence of a proper concircular vector was proven. In [20] a quasi conformally flat \((WRS)_n\) was studied, and again the existence of a proper concircular vector was proven.

In [2] \((PRS)_n\) with harmonic curvature tensor (i.e. \(\nabla_m R_{jkl} = 0\)) or with harmonic conformal curvature tensor (i.e. \(\nabla_m C_{jkl,m} = 0\)) were considered.

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\(^1\)Here we define the Ricci tensor as \(R_{kl} = -R_{mkl}^m\) and the scalar curvature as \(R = g^{ij} R_{ij}\). \(\nabla_k\) is the covariant derivative with reference to the metric \(g_{kl}\). We also put \(\|\eta\| = \sqrt{\eta^k \eta_k}\).
Chaki and Saha considered the projective Ricci tensor $P_{kl}$, obtained by a contraction of the projective curvature tensor $P_{jk}^{mn}$ [13]:

$$P_{kl} = \frac{n}{n-1} \left( R_{kl} - \frac{R}{n} g_{kl} \right),$$

and generalized $(PRS)_n$ to manifolds such that

$$\nabla_k P_{jl} = 2A_k P_{jl} + A_j P_{kl} + A_l P_{kj},$$

The manifold is called pseudo projective Ricci symmetric and denoted by $(PRS)_n$ [8]. Recently another generalization of a $(PRS)_n$ was considered in [9] and [11], whose Ricci tensor satisfies the condition

$$\nabla_k R_{jl} = (A_k + B_k) R_{jl} + A_j R_{kl} + A_l R_{kj},$$

The manifold is called almost pseudo Ricci symmetric and denoted by $A(PRS)_n$. In ref.[11] the properties of conformally flat $A(PRS)_n$ were studied, pointing out their importance in the theory of General Relativity.

It seems worthwhile to introduce and study a new manifold structure that includes $(WRS)_n$, $(PRS)_n$ and $(PWRS)_n$ as special cases.

**Definition 1.1.** A $(0,2)$ symmetric tensor is a generalized $Z$ tensor if

$$Z_{kl} = R_{kl} + \phi \ g_{kl},$$

where $\phi$ is an arbitrary scalar function. The $Z$ scalar is $Z = g^{kl} Z_{kl} = R + n \phi$.

The classical $Z$ tensor is obtained with the choice $\phi = -\frac{1}{n} R$. Hereafter we refer to the generalized $Z$ tensor simply as the $Z$ tensor.

The $Z$ tensor allows us to reinterpret several well known structures on Riemannian manifolds.

1) If $Z_{kl} = 0$ the $(Z$-flat) manifold is an Einstein space, $R_{ij} = (R/n) g_{ij}$ [3].

2) If $\nabla_i Z_{kl} = \lambda_i Z_{kl}$, the $(Z$-recurrent) manifold is a generalized Ricci recurrent manifold [9] [26]: the condition is equivalent to $\nabla_i R_{kl} = \lambda_i R_{kl} + (n-1) \mu_i g_{kl}$ where $(n-1) \mu_i \equiv (\lambda_i - \nabla_i) \phi$. If moreover $0 = (\lambda_i - \nabla_i) \phi$, a Ricci Recurrent manifold is recovered.

3) If $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ (i.e. $Z$ is a Codazzi tensor, [10]) then $\nabla_k R_{jl} - \nabla_j R_{kl} = (g_{kl} \nabla_j - g_{jl} \nabla_k) \phi$. By transvecting with $g^{jl}$ we get $\nabla_k [R + 2(n-1) \phi] = 0$ and, finally,

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)} (g_{jl} \nabla_k - g_{kl} \nabla_j) R.$$

This condition defines a nearly conformally symmetric manifold, $(NCS)_n$. The condition was introduced and studied by Roter [29]. Conversely a $(NCS)_n$ has a Codazzi $Z$ tensor if $\nabla_k [R + 2(n-1) \phi] = 0$.

4) Einstein’s equations [14] with cosmological constant $\Lambda$ and energy-stress tensor $T_{kl}$ may be written as $Z_{kl} = k T_{kl}$, where $\phi = -\frac{1}{2} R + \Lambda$, and $k$ is the gravitational constant. The $Z$ tensor may be thought of as a generalized Einstein gravitational tensor with arbitrary scalar function $\phi$.

Conditions on the energy-momentum tensor determine constraints on the $Z$ tensor: the vacuum solution $Z = 0$ determines an Einstein space with $\Lambda = \frac{-2}{2n} R$; conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla^l Z_{kl} = 0$ and $\nabla_k (\frac{1}{2} R + \phi) = 0$; the condition $\nabla_i Z_{kl} = 0$ describes a space-time with conserved energy-momentum.
Several cases accommodate in a new kind of Riemannian manifold:

**Definition 1.2.** A manifold is *Weakly Z symmetric*, and denoted by \((WZS)_n\), if the generalized \(Z\) tensor satisfies the condition:

\[
∇_kZ_{jl} = A_kZ_{jl} + B_jZ_{kl} + D_lZ_{kj}.
\]

If \(φ = 0\) we recover a \((WRS)_n\) and its particular case \((PRS)_n\). If \(φ = -R/n\) (classical \(Z\) tensor) and if \(A_k\) is replaced by \(2A_k\), \(B_k = D_k = A_k\), then \(Z_{jl} = P_{jl}/n\) and the space reduces to a \((PWRS)_n\).

In sect.2 we obtain general properties of \((WZS)_n\) that descend directly from the definition and strongly depend on \(Z_{ij}\) being singular or not. The two cases are examined in sections 3 and 4. In sect.3 we study \((WZS)_n\) that are conformally or pseudo conformally harmonic with \(B−D \neq 0\); we show that \(B−D\), after normalization, is a proper concircular vector. Sect.4 is devoted to \((WZS)_n\) with non-singular \(Z\) tensor, and gives conditions for the closedness of the 1-form \(A−B\) that involve various generalized curvature tensors. In sect.5 we study conformally harmonic \((WZS)_n\) and obtain the explicit form of the Ricci tensor. In the conformally flat case we also give the local form of the metric.

2. General properties

From the definition of a \((WZS)_n\) and its symmetries we obtain

\[
0 = η_jZ_{kl} − η_lZ_{kj},
\]

\[
∇_kZ_{jl} − ∇_jZ_{kl} = ω_kZ_{jl} − ω_jZ_{kl},
\]

with covectors \(ω_k = A_k − B_k\) and \(η_k = B_k − D_k\) that will be used throughout.

Let’s consider eq.(8) first, it implies the following statements:

**Proposition 2.1.** In a \((WZS)_n\), if the \(Z\) tensor is non-singular then \(η_k = 0\).

**Proof.** If the \(Z\) tensor is non singular, there exists a \((2,0)\) tensor \(Z^{-1}\) such that \((Z^{-1})^{kh}Z_{kl} = δ^h_l\). By transvecting eq.(8) with \((Z^{-1})^{kh}\) we obtain \(η_jδ^h_l = η_lδ^h_j\); put \(h = l\) and sum to obtain \((n−1)η_j = 0\).

**Proposition 2.2.** If \(η_k \neq 0\) and the scalar \(Z \neq 0\), then the \(Z\) tensor has rank one:

\[
Z_{ij} = Z \frac{η_iη_j}{η^kη_k}
\]

**Proof.** Multiply eq.(8) by \(η^j\) and sum: \(η^jη_jZ_{kl} = η_lη^jZ_{kj}\). Multiply eq.(8) by \(g^{jk}\) and sum: \(η^kZ_{kl} = Zη_l\). The two results imply the assertion.

The result translates to the Ricci tensor, whose expression is characteristic of *quasi Einstein* Riemannian manifolds [7], and generalizes the results of [12].

**Proposition 2.3.** A \((WZS)_n\) with \(η_k \neq 0\), is a quasi Einstein manifold:

\[
R_{ij} = -φg_{ij} + ZT_iT_j, \quad T_i = \frac{η_i}{||η||}.
\]

Next we consider eq.(9). If \(Z_{ij}\) is a Codazzi tensor, then the l.h.s. of the equation vanishes by definition, and the above discussion of eq.(8) can be repeated. We merely state the result:
Proposition 2.4. In a $WZS_n$ with a Codazzi $Z$ tensor, if $Z$ is singular then $\omega_k \neq 0$. Conversely, if rank $[Z_{kl}] > 1$ then $\omega_k = 0$.

3. Harmonic conformal or quasi conformal $(WZS)_n$ with $\eta \neq 0$

In this section we consider manifolds $(WZS)_n$ ($n > 3$) with $\eta \neq 0$, and the property $\nabla_mC_{jkl}^m = 0$ (i.e. harmonic conformal curvature tensor [3]) or $\nabla_mW_{jkl}^m = 0$ (i.e. harmonic quasi conformal curvature tensor [34]). We provide sufficient conditions for $\eta/\Vert \eta \Vert$ to be a proper concircular vector [28, 32].

We begin with the case of harmonic conformal tensor. From the expression for the divergence of the conformal tensor,

\begin{equation}
\nabla_mC_{jkl}^m = \frac{n-3}{n-2} \left[ \nabla_kR_{jl} - \nabla_jR_{kl} + \frac{1}{2(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R \right]
\end{equation}

we read the condition $\nabla_mC_{jkl}^m = 0$:

\begin{equation}
\nabla_kR_{jl} - \nabla_jR_{kl} = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)R.
\end{equation}

We need the following theorem, whose proof given here is different from that in [13] (see also [10]):

**Theorem 3.1.** Let $M$ be a $n > 3$ dimensional manifold, with harmonic conformal curvature tensor, and Ricci tensor $R_{kl} = \alpha g_{kl} + \beta T_kT_l$, where $\alpha$, $\beta$ are scalars, and $T^kT_k = 1$. If

\begin{equation}
(T_j\nabla_k - T_k\nabla_j)\beta = 0,
\end{equation}

then $T_k$ is a proper concircular vector.

**Proof.** Since $M$ is conformally harmonic, eq.(13) gives:

\begin{equation}
\beta[\nabla_k(T_jT_l) - \nabla_j(T_kT_l)] = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)S,
\end{equation}

where $S = -(n-2)\alpha + \beta$, and condition (14) was used. The proof is in four steps.

1) We show that $T^i\nabla_iT_k = 0$: multiply eq.(15) by $g^{il}$ to obtain: a) $-\beta\nabla^l(T_kT_l) = \frac{1}{2}\nabla_kS$. The result a) is multiplied by $T^k$ to give: b) $-\beta\nabla^lT^k = \frac{1}{2}T^k\nabla_lS$. a) and b) combine to give: c) $-\beta T^k\nabla_lT_k = \frac{1}{2}\nabla_k - T_kT^l\nabla_l)S$. Finally multiply eq.(16) by $T^kT^l$ and use the property $T^i\nabla_iT_k = 0$ to obtain:

\begin{equation}
\beta T^k\nabla_kT_j = \frac{1}{2(n-1)}(T_jT^k\nabla_k - \nabla_j)S
\end{equation}

which, compared to c) shows that d) $T^i\nabla_iT_k = 0$ and $(T_jT^k\nabla_k - \nabla_j)S = 0$.

2) We show that $T$ is a closed 1-form: multiply eq.(15) by $T^l$.

\begin{equation}
\beta[\nabla_kT_j - \nabla_jT_k] = \frac{1}{2(n-1)}(T_j\nabla_k - T_k\nabla_j)S.
\end{equation}

$T$ is a closed form if the r.h.s. is null. This is proven by using identity a) to write:

\begin{equation}
(T_j\nabla_k - T_k\nabla_j)S = -2\beta [T_j\nabla^l(T_kT_l) - T_k\nabla^l(T_jT_l)] = 0
\end{equation}

by property d).

3) With condition d) in mind, transvect eq.(15) with $T^k$ and obtain:

\begin{equation}
-\beta \nabla_jT_l = \frac{1}{2(n-1)}(g_{jl}T^k\nabla_k - T_l\nabla_j)S
\end{equation}
Use d) to replace $T|\nabla jS$ with $T_iT_jT^k\nabla_kS$. Then:

$$\nabla_jT_i = f(T_jT_i - g_{jl}), \quad f = \frac{T^k\nabla_kS}{2\beta(n-1)}$$

which means that $T_k$ is a concircular vector.

4) We prove that $T_k$ is a proper concircular vector, i.e. $fT_k$ is a closed 1-form: from d) by a covariant derivative we obtain $\nabla_j\nabla_kS = (\nabla_jT_k)(T^l\nabla_lS) + T_k\nabla_j(T^l\nabla_lS)$; subtract same equation with indices $k$ and $j$ exchanged. Since $T_k$ is a closed 1-form we obtain: $T_k\nabla_j(T^l\nabla_lS) = T_j\nabla_k(T^l\nabla_lS)$. Multiply by $T^k$:

$$(T^kT^l\nabla_k - \nabla_j)(T^l\nabla_iS) = 0$$

From the relation (14), one obtains: $(T^kT^l\nabla_k - \nabla_j)\beta = 0$. It follows that the scalar function $f$ has the property $\nabla_j\beta = \mu T_j$ where $\mu$ is a scalar function. Then the 1-form $fT_k$ is closed.

With the identifications $\alpha = -\phi$ and $\beta = Z$, $T_i = \eta_i/|\eta|$ (see Prop. 2.3) the condition (14) is $\eta_j\nabla_k - \eta_k\nabla_j) = 0$. Since $Z = S - (n-2)\phi$ and $(\eta_j\nabla_k - \eta_k\nabla_j) S = 0$, the condition can be rewritten as $(\eta_j\nabla_k - \eta_k\nabla_j)\phi = 0$. Thus we can state the following:

**Theorem 3.2.** In a $(WZS)_n$ manifold with $\eta_k \neq 0$ and harmonic conformal curvature tensor, if

$$\nabla_j\nabla_k - \eta_k\nabla_j)\phi = 0$$

then $\eta_i/|\eta|$ is a proper concircular vector.

**Remark 1.** If $\phi = 0$ or $\nabla_k\phi = 0$, the condition (17) is fulfilled automatically. In the case $\phi = 0$ we recover a $(WRS)_n$ manifold (and the results of refs 10, 12).

Now we consider the case of a $(WZS)_n$ manifold with harmonic quasi conformal curvature tensor. In 1968 Yano and Sawaki [34] defined and studied a tensor $W_{jkl}^m$ on a Riemannian manifold of dimension $n > 3$, which includes as particular cases the conformal curvature tensor $C_{jkl}^m$, eq. (2), and the concircular curvature tensor

$$C_{jkl}^m = R_{jkl}^m + R \frac{\delta^m_{jl}}{n(n-1)}(\delta^m_{kl}g_{jl} - \delta^m_{jl}g_{kl}).$$

The tensor is known as the quasi conformal curvature tensor:

$$W_{jkl}^m = -(n-2)b^C_{jkl} + [a + (n-2)b]C_{jkl}^m;$$

$a$ and $b$ are nonzero constants. From the expressions (12) and (32) we evaluate

$$\nabla_mW_{jkl}^m = (a+b)^{\nabla_mR_{jkl}^m + 2a - b(n-1)(n-4) \frac{2n(n-1)}{2n(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R.$$

A manifold is quasi conformally harmonic if $\nabla_mW_{jkl}^m = 0$. By transvecting the condition with $g^{jk}$ we get:

$$\nabla_j[(1 - 2/n)(a+b)^{n-2}] \nabla_jR = 0,$$

which means that either $a + b(n-2) = 0$ or $\nabla_jR = 0$. The first condition implies $W = C$, and gives back the harmonic conformal case. If $\nabla_jR = 0$ it is $\nabla_mR_{jkl}^m = 0$ by (20), and the equations in the proof of Theorem 3.1 simplify and we can state the following (analogous to Theorem 3.2):
Theorem 3.3. Let \((WZS)_n\) be a quasi conformally harmonic manifold, with \(\eta_k \neq 0\). If \((\eta_j \nabla_k - \eta_k \nabla_j)\phi = 0\), then \(\eta/\|\eta\|\) is a proper concircular vector.

4. \((WZS)_n\) with non-singular Z tensor: conditions for closed \(\omega\)

In this section we investigate in a \((WZS)_n\) \((n > 3)\) the conditions the 1-form \(\omega_k\) to be closed: \(\nabla_i \omega_j - \nabla_j \omega_i = 0\). We need:

Lemma 4.1 (Lovelock’s differential identity, \([23, 24]\)). In a Riemannian manifold the following identity is true:

\[
\nabla_i \nabla_m R_{ijkl}^m + \nabla_j \nabla_m R_{kil}^m + \nabla_k \nabla_m R_{ijl}^m = -R_{km} R_{jkl}^m - R_{jm} R_{kil}^m - R_{km} R_{ijl}^m
\]

and also the contracted second Bianchi identity in the form

\[
\nabla_m R_{ijkl}^m = \nabla_k Z_{jl} - \nabla_j Z_{kl} + (g_{kl} \nabla_j - g_{lj} \nabla_k) \phi.
\]

Now we prove the relevant theorem (see also \([24]\)):

Theorem 4.2. In a \((WZS)_n\) \((n > 3)\) with non-singular Z tensor, \(\omega_k\) is a closed 1-form if and only if:

\[
R_{im} R_{jkl}^m + R_{jm} R_{kil}^m + R_{km} R_{ijl}^m = 0.
\]

Proof. The covariant derivative of eq. \([23]\) and eq. \([24]\) give:

\[
\nabla_i \nabla_m R_{ijkl}^m + \nabla_j \nabla_m R_{kil}^m + \nabla_k \nabla_m R_{ijl}^m
\]

\[
= (\nabla_i \omega_k - \nabla_k \omega_i) Z_{jl} + (\nabla_j \omega_l - \nabla_l \omega_j) Z_{kl} + (\nabla_k \omega_j - \nabla_j \omega_k) Z_{il}
\]

\[
+ \omega_j (\nabla_k Z_{il} - \nabla_i Z_{kl}) + \omega_k (\nabla_i Z_{jl} - \nabla_j Z_{il}) + \omega_l (\nabla_j Z_{il} - \nabla_i Z_{jl}).
\]

Cancellations occur by eq. \([21]\). By lemma \([41]\) one obtains:

\[
-R_{im} R_{jkl}^m - R_{jm} R_{kil}^m - R_{km} R_{ijl}^m
\]

\[
= (\nabla_i \omega_k - \nabla_k \omega_i) Z_{jl} + (\nabla_j \omega_l - \nabla_l \omega_j) Z_{kl} + (\nabla_k \omega_j - \nabla_j \omega_k) Z_{il}.
\]

If \(\omega_k\) is a closed 1-form then eq. \([21]\) is fulfilled. Conversely, suppose that eq. \([24]\) holds: if the Z tensor is non singular, there is a \((2, 0)\) tensor such that \(Z_{kl} (Z^{-1})^{km} = \delta^m_i\). Multiply the last equation by \((Z^{-1})^h_l\):

\[
(\nabla_i \omega_k - \nabla_k \omega_i) \delta^h_i + (\nabla_j \omega_l - \nabla_l \omega_j) \delta^h_j + (\nabla_k \omega_j - \nabla_j \omega_k) \delta^h_k = 0.
\]

Set \(h = j\) and sum: \((n - 2)(\nabla_i \omega_k - \nabla_k \omega_i) = 0\). Since \(n > 2\), \(\omega_k\) is a closed 1-form. \(\square\)

Remark 2. By Lovelock’s identity, the condition \([23]\) is obviously true if \(\nabla_m R_{ijkl}^m = 0\), i.e. the \((WZS)_n\) is a harmonic manifold. However, we have shown in ref. \([24]\) that there is a broad class of generalized curvature tensors for which the case \(\nabla_m K_{ijkl}^m = 0\) implies the same condition. This class includes several well known curvature tensors, and is the main subject of this section.

Definition 4.3. A tensor \(K_{ijkl}^m\) is a generalized curvature tensor\(^2\) if:

1) \(K_{ijkl}^m = -K_{jikl}^m\);

2) \(K_{ijkl}^m + K_{iklj}^m + K_{ijlk}^m = 0\).

\(^2\)The notion was introduced by Kobayashi and Nomizu \([22]\), but with the further antisymmetry in the last pair of indices.
The second Bianchi identity does not hold in general, and is modified by a tensor source $B_{ijkl}^m$ that depends on the specific form of the curvature tensor:
\begin{equation}
\nabla_i K_{jkl}^m + \nabla_j K_{kli}^m + \nabla_k K_{lij}^m = B_{ijkl}^m
\end{equation}

**Proposition 4.4** ([24]). If $K_{ijkl}^m$ is a generalized curvature tensor such that
\begin{equation}
\nabla_m K_{ijkl}^m = A\nabla_m R_{jkl}^m + B(a_{ik} \nabla_j - a_{ij} \nabla_k)\psi,
\end{equation}
where $A \neq 0, B$ are constants, $\psi$ is a scalar field, and $a_{ij}$ is a symmetric $(0, 2)$ Codazzi tensor (i.e. $\nabla_i a_{kl} = \nabla_k a_{ij}$), then the following relation holds:
\begin{equation}
\nabla_i \nabla_j K_{mkl}^m + \nabla_j \nabla_m K_{kli}^m + \nabla_k \nabla_m K_{lij}^m
= -A(R_{im} R_{jkl}^m + R_{jm} R_{kli}^m + R_{km} R_{ijl}^m).
\end{equation}

**Remark 3.** In [16] it is proven that any smooth manifold carries a metric such that $(M, g)$ admits a non trivial Codazzi tensor (i.e. proportional to the metric tensor) and the deep consequences on the structure of the curvature operator are presented (see also [25]).

Given a Codazzi tensor it is possible to exhibit a $K$ tensor that satisfies the condition
\begin{equation}
K_{ijkl}^m = A R_{ijkl}^m + B \psi (\delta_j^m a_{kl} - \delta_k^m a_{jl}).
\end{equation}
Its trace is: $K_{kl} = -K_{mkl}^m = A R_{kl} - B(n-1)\psi a_{kl}$. Note that for $a_{kl} = g_{kl}$ the tensor $K_{kl}$ is up to a factor a $Z$ tensor. Thus $Z$ tensors arise naturally from the invariance of Lovelock’s identity.

**Remark 4.** In the literature one meets generalized curvature tensors whose divergence has the form ([20]), with trivial Codazzi tensor:
\begin{equation}
\nabla_m K_{ijkl}^m = A \nabla_m R_{jkl}^m + B (g_{kl} \nabla_j - g_{jl} \nabla_k) R.
\end{equation}
They are the projective curvature tensor $P_{ijkl}^m$ [18], the conformal curvature tensor $C_{ijkl}^m$ [27], the concircular tensor $C_{ijkl}^m$ [28, 32], the conharmonic tensor $N_{ijkl}^m$ [26, 30] and the quasi conformal tensor $W_{ijkl}^m$ [24].

**Definition 4.5.** A manifold is $K$-harmonic if $\nabla_m K_{ijkl}^m = 0$.

**Proposition 4.6.** In a $K$-harmonic manifold, if $K$ is of type ([20]) and $A \neq 2(n-1)B$, then $\nabla_j R = 0$.

**Proof.** By transvecting eq.([29]) with $g^{kl}$ and by the second contracted Bianchi identity, we obtain $\frac{1}{2}[A - 2(n-1)B] \nabla_j R = 0$. □

Hereafter, we specialize to $(WZS)_n$ manifolds with non singular $Z$ tensor, and with a generalized curvature tensor of the type ([20]). From eqs. ([20]) and ([6]) we obtain:
\begin{equation}
\nabla_m K_{ijkl}^m = A(\omega_k Z_{jl} - \omega_j Z_{kl}) + (g_{kl} \nabla_j - g_{jl} \nabla_k)(A\phi + B R).
\end{equation}
Then, the manifold is $K$-harmonic if:
\begin{equation}
A(\omega_k Z_{jl} - \omega_j Z_{kl}) = (g_{kl} \nabla_j - g_{jl} \nabla_k)(A\phi + B R).
\end{equation}

**Lemma 4.7.** In a $K$-harmonic $(WZS)_n$ with non singular $Z$ tensor:
1) $\omega_k = 0$ if and only if $\nabla_k (A\phi + B R) = 0$;
2) If $A \neq 2(n-1)B$, then $\omega_k = 0$ if and only if $\nabla_k \phi = 0$. 
Harmonic conformal curvature:

Theorem 4.8 applies.

One evaluates Theorem 4.8.

A covariant derivative and the second contracted Bianchi identity give:

If \( A \neq 2B(n-1) \) then \( \nabla_k R = 0 \) and part 1) applies.

**Theorem 4.8.** In a K-harmonic \((WZS)_n\) with non-singular Z tensor and K of type \([29]\), if \( \omega \neq 0 \) then \( \omega \) is a closed 1-form.

This theorem extends theorem [12] (where \( K = R \)), and has interesting corollaries according to the various choices \( K = C, W, \bar{C}, N \).

**Corollary 4.9.** Let \( (WZS)_n \) have non-singular Z tensor and \( \omega \neq 0 \). If \( \nabla_m K_{jkl} = 0 \), and \( K = P, \bar{C}, N \), then \( \omega \) is a closed 1-form.

**Proof.**

1) Harmonic conformal curvature: \( \nabla_m C_{jkl} = 0 \).

Note that in this case \( A = 2B(n-1) \); theorem [13] applies.

2) Harmonic quasi conformal curvature: \( \nabla_m W_{jkl}^m = 0 \):

Eq. (21) gives either \( \nabla_j R = 0 \) or \( a + b(n-2) = 0 \). If \( \nabla_j R = 0 \) then \( \nabla_m R_{jkl}^m = 0 \) and theorem [12] applies.

If \( a + b(n-2) = 0 \) then \( \nabla_m C_{jkl} = 0 \) and theorem 1) applies.

3) Harmonic projective curvature: \( \nabla_m P_{jkl}^m = 0 \).

The components of the projective curvature tensor are [18] [30]:

\[
P_{jkl}^m = R_{jkl}^m + \frac{1}{n-1} (\delta_j^m R_{klt} - \delta_k^m R_{jlt}).
\]

One evaluates \( \nabla_m P_{jkl}^m = \frac{n-2}{n-1} \nabla_m R_{jkl}^m \), and theorem [12] applies.

4) Harmonic concircular curvature: \( \nabla_m \tilde{C}_{jkl}^m = 0 \).

The concircular curvature tensor is given in eqs. [18], [28] [32]. Its divergence is

\[
\nabla_m \tilde{C}_{jkl}^m = \nabla_m R_{jkl}^m + \frac{1}{n(n-1)} (g_{jl} \nabla_j - g_{jl} \nabla_k) R
\]

Theorem [13] applies.

5) Harmonic conharmonic curvature: \( \nabla_m N_{jkl}^m = 0 \).

The conharmonic curvature tensor [26] [30] is:

\[
N_{jkl}^m = R_{jkl}^m + \frac{1}{n-2} (\delta_j^m R_{klt} - \delta_k^m R_{jlt} + R^m g_{jl} - R_j^m g_{klt}).
\]

A covariant derivative and the second contracted Bianchi identity give:

\[
\nabla_m N_{jkl}^m = \frac{n-3}{n-2} \nabla_m R_{jkl}^m + \frac{1}{2(n-2)} (g_{jl} \nabla_j - g_{jl} \nabla_k) R.
\]

Theorem [13] applies.

There are other cases where the 1-form \( \omega_k \) is closed for a \((WZS)_n\) manifold.

**Definition 4.10** ([21] [21]). A \( n \)-dimensional Riemannian manifold is \( K \)-recurrent, \((KR)_n\), if the generalized curvature tensor is recurrent, \( \nabla_i K_{jkl} = \lambda_i K_{jkl} \), for some non zero covector \( \lambda_i \).

**Theorem 4.11** ([21]). In a \((KR)_n\), if \( \lambda_i \) is closed then:

\[
R_{im} R_{jkl}^m + R_{jm} R_{klt}^m + R_{km} R_{ijl}^m = -\frac{1}{A} \nabla_m B_{ijkl}^m.
\]

where \( B \) is the source tensor in eq. [29]. In particular, for \( K = C, P, \bar{C}, N, W \) the tensor \( \nabla_m B_{ijkl}^m \) either vanishes or is proportional to the l.h.s. of eq. [33].
Theorem 4.14. Let \((WZS)_n\) have non-singular \(Z\) tensor, and be \(K\) recurrent with closed \(\lambda_i\). If \(K = C, P, \bar{C}, N, W\), then \(\omega\) is a closed 1-form.

Definition 4.13. A Riemannian manifold is pseudosymmetric in the sense of R. Deszcz if the following condition holds:

\[
(\nabla_s \nabla_i - \nabla_i \nabla_s)R_{jklm} = L_R (g_{js}R_{sklm} - g_{ji}R_{sklm} + g_{ks}R_{jilm} - g_{ki}R_{jilm} + g_{ks}R_{jkim} - g_{ki}R_{jkim} + g_{ms}R_{jklm} - g_{mi}R_{jklm}),
\]

where \(L_R\) is a non null scalar function.

In ref. \[24\] the following theorem is proven:

Theorem 4.14. In a Riemannian manifold which is pseudosymmetric in the sense of R. Deszcz, it is \(R^{im}R_{jkl}{}^m + R^{jm}R_{kil}{}^m + R^{km}R_{ijl}{}^m = 0\).

Then we can state the following:

Proposition 4.15. In a \((WZS)_n\) which is pseudosymmetric in the sense of R. Deszcz, if the \(Z\) tensor is non-singular then \(\omega_k\) is a closed 1-form.

Definition 4.16. A Riemannian manifold is generalized Ricci pseudosymmetric in the sense of R. Deszcz, if the following condition holds:

\[
(\nabla_s \nabla_i - \nabla_i \nabla_s)R_{jklm} = L_S (R_{jkl}R_{sklm} - R_{jlm}R_{sklm} + R_{kil}R_{jilm} - R_{kij}R_{jilm} + R_{klm}R_{jklm} - R_{kij}R_{jklm} + R_{klm}R_{jklm} - R_{kij}R_{jklm}),
\]

where \(L_S\) is a non null scalar function.

Theorem 4.17. In a generalized Ricci pseudosymmetric manifold in the sense of R. Deszcz, it is either \(L_S = -\frac{1}{3}\), or \(R^{im}R_{jkl}{}^m + R^{jm}R_{kil}{}^m + R^{km}R_{ijl}{}^m = 0\).

Proof. Equation \((35)\) is transvectioned with \(g^{mj}\) to obtain

\[
(\nabla_s \nabla_i - \nabla_i \nabla_s)R_{kl} = L_S [R^{im}R_{sklm} + R_{skm}{}^m] - R^{sm}R_{sklm} + R^{im}R_{skm}{}^m].
\]

Then:

\[
\begin{align*}
(\nabla_i \nabla_k - \nabla_k \nabla_i)R_{jl} &+ (\nabla_j \nabla_i - \nabla_i \nabla_j)R_{kl} + (\nabla_k \nabla_j - \nabla_j \nabla_k)R_{il} \\
&= 3L_S (R^{im}R_{jkl}{}^m + R^{jm}R_{kil}{}^m + R^{km}R_{ijl}{}^m)
\end{align*}
\]

By Lovelock’s identity \((1.3)\), the l.h.s. of the previous equation is:

\[
\begin{align*}
\partial_i \partial_m R_{jkl}{}^m &+ \partial_j \partial_m R_{kil}{}^m + \partial_k \partial_m R_{jlm}{}^m \\
&= (\nabla_i \nabla_k - \nabla_k \nabla_i)R_{jl} + (\nabla_j \nabla_i - \nabla_i \nabla_j)R_{kl} + (\nabla_k \nabla_j - \nabla_j \nabla_k)R_{il} \\
&= -R^{im}R_{jkl}{}^m - R^{jm}R_{kil}{}^m - R^{km}R_{ijl}{}^m.
\end{align*}
\]

Compare the two results and conclude that either \(L_S = -\frac{1}{3}\), or \(R^{im}R_{jkl}{}^m + R^{jm}R_{kil}{}^m + R^{km}R_{ijl}{}^m = 0\).

Finally we state:

Proposition 4.18. In a \((WZS)_n\) which is also a generalized Ricci pseudosymmetric manifold in the sense of R. Deszcz, if the \(Z\) tensor is non-singular and \(L_S \neq -\frac{1}{3}\), then \(\omega_k\) is a closed 1-form.
5. Conformally harmonic (WZS)\(_n\): Form of the Ricci Tensor

In this section we study conformally harmonic (WZS)\(_n\) in depth. We show the existence of a proper concircular vector in such manifolds, and obtain the form of the Ricci tensor. The proof only requires the Z tensor to be non singular. For the conformally flat case, in particular, we give the explicit local form of the metric tensor.

The condition \(\nabla_m C_{jkl} = 0\) is eq. (13) which, by using \(R_{ij} = Z_{ij} - g_{ij} \phi\) and the property eq. (9), becomes:

\[
\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{2(n-1)}(g_{jl} \nabla_k - g_{kl} \nabla_j)[R + 2(n-1)\phi].
\]

This is the starting point for the proofs. By prop 4.7, since \(Z\) is non singular, \(\omega_k \neq 0\) if and only if \(\nabla_k[R + 2(n-1)\phi] \neq 0\).

Remark 5.1) The condition \(\nabla_m C_{jkl} = 0\) implies that the manifold is a (NCS)\(_n\).

2) If \(\nabla_k[R + 2(n-1)\phi] = 0\) the \(Z\) tensor is a Codazzi tensor.

The following theorem generalizes a result in [11] for A(PRS)\(_n\):

**Theorem 5.1.** In a conformally harmonic (WZS)\(_n\) the 1-form \(\omega\) is an eigenvector of the \(Z\) tensor.

**Proof.** By transvecting eq. (36) with \(g_{kl}\) we obtain

\[
\omega_j Z - \omega^m Z_{jm} = \frac{1}{2} \nabla_j[R + 2(n-1)\phi];
\]

the result is inserted back in eq. (36),

\[
\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{(n-1)}[(\omega_k Z - \omega^m Z_{km})g_{jl} - (\omega_j Z - \omega^m Z_{jm})g_{kl}],
\]

and transvected with \(\omega^j \omega^l\) to obtain \(\omega_k (\omega^j \omega^l Z_{jl}) = (\omega_j \omega^l \omega^j Z_{kl})\). The last equation can be rewritten as: \(Z_{kl} \omega^j = \zeta \omega^j\).

Now eq. (37) simplifies: \(\omega_j(\zeta - Z) = -\frac{1}{2} \nabla_j[R + 2(n-1)\phi]\). The result is a natural generalization of a similar one given in ref. [11] for A(PRS)\(_n\).

**Theorem 5.2.** Let \(M\) be a conformally harmonic (WZS)\(_n\). Then:
1) \(M\) is a quasi Einstein manifold;
2) if the \(Z\) tensor is non singular and if \((\omega_j \nabla_k - \omega_k \nabla_j)\phi = 0\), then:

\[
(\omega_j \nabla_k - \omega_k \nabla_j) \left[ \frac{n \zeta - Z}{n-1} \right] = 0,
\]

and \(M\) admits a proper concircular vector.

**Proof.** Eq. (36) is transvect ed with \(\omega^j\) and theorem 5.1 is used to show that

\[
R_{kl} = \left[ \frac{Z - \zeta - \phi}{n-1} \right] g_{kl} + \left[ \frac{n \zeta - Z}{n-1} \right] \omega_k \omega_l \omega_j,
\]

i.e. \(R_{kl}\) has the structure \(\alpha g_{kl} + \beta T_k T_l\) and the manifold is quasi Einstein [7]. By transvecting eq. (23) with \(g^{jl}\) we obtain

\[
\frac{1}{2} \nabla_k Z + \frac{n-2}{2} \nabla_k \phi = \omega_k Z - \omega^j Z_{kl}.
\]
This and theorem 5.1 imply:

\[ \frac{1}{2} \nabla_k Z + \frac{n - 2}{2} \nabla_k \phi = \omega_k (Z - \zeta). \]  

A covariant derivative gives \( \frac{1}{2} \nabla_j \nabla_k Z + \frac{n - 2}{2} \nabla_j \nabla_k \phi = \nabla_j [\omega_k (Z - \zeta)] \). Subtract the equation with indices \( k \) and \( j \) exchanged:

\[ (Z - \zeta) (\nabla_j \omega_k - \nabla_k \omega_j) + (\omega_k \nabla_j - \omega_j \nabla_k) (Z - \zeta) = 0. \]

According to corollary 4.3 in a conformally harmonic (WZS) manifold with non-singular \( Z \) the 1-form \( \omega_k \) is closed. Then

\[ (\omega_k \nabla_j - \omega_j \nabla_k) (Z - \zeta) = 0 \]  

Multiply eq. (39) by \( \omega_j \) and subtract from it the equation with indices \( k \) and \( j \) exchanged: \( (\omega_j \nabla_k - \omega_k \nabla_j) Z + (n - 2) (\omega_j \nabla_k - \omega_k \nabla_j) \phi = 0 \). Suppose that \( \omega_k \), besides being a closed 1-form, has the property \( (\omega_j \nabla_k - \omega_k \nabla_j) \phi = 0 \), then one obtains the further equation:

\[ (\omega_k \nabla_j - \omega_j \nabla_k) Z = 0. \]

Eqs. (40) imply the assertion eq. (38). The existence of a proper concircular vector follows from Theorem 5.1. 

Let us specialize to the case \( C_{ijk}^m = 0 \) (conformally flat (WZS) manifold). It is well known [1] that if a conformally flat space admits a proper concircular vector, then the space is subprojective in the sense of Kagan.

From theorem 5.2 we state the following:

**Theorem 5.3.** Let \( \text{(WZS)}_n (n > 3) \) be conformally flat with nonsingular \( Z \) tensor and \( (\omega_j \nabla_k - \omega_k \nabla_j) \phi = 0 \), then the manifold is a subprojective space.

In [33] K. Yano proved that a necessary and sufficient condition for a Riemannian manifold to admit a concircular vector, is that there is a coordinate system in which the first fundamental form may be written as:

\[ ds^2 = (dx_1)^2 + e^{\eta(x_1)} g_{\alpha \beta}(x_2, \ldots, x_n) dx^\alpha dx^\beta, \]

where \( \alpha, \beta = 2, \ldots, n \). Since a conformally flat \( \text{(WZS)}_n \) manifold with non-singular \( Z \) tensor admits a proper concircular vector field, this space is the warped product \( 1 \times e^\eta M^* \), where \( (M^*, g^*) \) is a \((n - 1)\)-dimensional Riemannian manifold. Gebarosky [11] proved that the warped product \( 1 \times e^\eta M^* \) has the metric structure [12] if and only if \( M^* \) is Einstein. Thus the following theorem holds:

**Theorem 5.4.** Let \( M \) be a \( n \)-dimensional conformally flat \( \text{(WZS)}_n (n > 3) \). If \( Z_{kl} \) is non singular and \( (\omega_j \nabla_k - \omega_k \nabla_j) \phi = 0 \), then \( M \) is the warped product \( 1 \times e^\eta M^* \), where \( M^* \) is Einstein.

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Physics Department, Università degli Studi di Milano and I.N.F.N., Via Celoria 16, 20133, Milano, Italy., Present address of C. A. Mantica: I.I.S. Lagrange, Via L. Modignani 65, 20161, Milano, Italy

E-mail address: luca.molinari@mi.infn.it, carloalberto.mantica@libero.it