On the Cauchy problem of a two-dimensional Benjamin-Ono equation.

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Abstract
In this work we shall show that the Cauchy problem

\[
\begin{align*}
(u_t + u^p u_x + \mathcal{H}\partial_x^2 u + \alpha \mathcal{H}\partial_y^2 u)_x - \gamma u_{yy} &= 0 & p \in \mathbb{N} \\
u(0; x, y) &= \phi(x, y)
\end{align*}
\]

is locally well-posed in the Sobolev spaces \(H^s(\mathbb{R}^2)\), \(X^s\) and weighted spaces \(X^s(w^2)\), for \(s > 2\).

1 Introduction
The purpose of this work is to show that the Cauchy problem

\[
(u_t + u^p u_x + \mathcal{H}\partial_x^2 u + \alpha \mathcal{H}\partial_y^2 u)_x - \gamma u_{yy} = 0,
\]

is locally well-posed in the Sobolev spaces \(H^s(\mathbb{R}^2)\) and \(X^s\), and in the weighted spaces \(X^s(w^2)\), for \(s > 2\) (see the Section 2 for the notations used here). We also prove global well-posedness for small enough initial data and examine the asymptotic behaviour of the solutions for these initial datas.

It should be noted that the equation (2) is the model of dispersive long wave motion in a weakly nonlinear two-fluid system, where the interface is subject to capillarity and bottom fluid is infinitely deep (see \cite{1}, \cite{2} and \cite{15}). For this equation, with \(\alpha = 0\), the local well-posedness was proven in \cite{4}. Also, the existence of solitary wave solution was proved in \cite{13} (for the case \(\alpha = 0\) in \cite{6} it was provided an incomplete proof).

Observe that (2) is a two-dimensional case of the Benjamin-Ono equation

\[
\partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0,
\]

which describes certain models in physics about wave propagation in a stratified thin regions (see \cite{8} and \cite{22}). This last equation shares with the equation KdV

\[
u_t + u_x + uu_x + u_{xxx} = 0
\]
many interesting properties. For example, they both have infinite conservation laws, they have solitary waves as solutions which are stable and behave like soliton (this last is evidenced by the existence of multisoliton type solutions) (see [1] and [19]). Also, the local and global well-posedness was proven in the Sobolev spaces context (in low regularity spaces inclusive, see, e.g., [8], [23], [13], [17] and [25]).

The plan of this paper is the following. In Section 2 we present the basic notations and results that we will need. In Section 3 we examine the local well-posedness in $H^s$ and $X^s$. To do so, we will use the abstract theory developed by Kato in [9] (see also [11]) to prove the local well-posedness of quasi-linear equations of evolution. Kato considered the problem

$$\begin{align*}
\partial_t u + A(t, u)u &= f(t, u) \in X, \ 0 < t, \\
u(0) &= u_0 \in Y,
\end{align*}$$

(5)

in a Banach space $X$ with initial data in a dense subspace $Y$ of $X$, where $A$ is a map from $\mathbb{R} \times X$ into the linear operators of $X$ with dense domain and $f(t, u)$ is a function from $\mathbb{R} \times Y$ to $X$, which satisfy the following conditions:

(X) There exists an isometric isomorphism $\mathcal{S}$ from $Y$ to $X$.

There exist $T_0 > 0$ and $W$ a open ball with center $w_0$ such that:

(A1) For each $(t, y) \in [0, T_0] \times W$, the linear operator $A(t, y)$ belongs to $G(X, 1, \beta)$, where $\beta$ is a positive real number. In other words, $-A(t, y)$ generates a $C_0$ semigroup such that

$$\|e^{-s A(t, y)}\|_{\mathcal{B}(X)} \leq e^{\beta s}, \text{ para } s \in [0, \infty).$$

It should be noted that if $X$ is a Hilbert space, $A \in G(X, 1, \beta)$ if, and only if,

a) $\langle Ay, y_X \rangle \geq -\beta \|y\|^2_X$ for all $y \in D(A),$

b) $(A + \lambda)$ is onto for all $\lambda > \beta.$

(See [12] or [24])

(A2) For all $(t, y) \in [0, T_0] \times W$ the operator $B(t, y) = [S, A(t, y)]S^{-1} \in \mathcal{B}(X)$ and is uniformly bounded, i.e., there exists $\lambda_1 > 0$ such that

$$\|B(t, y)\|_{\mathcal{B}(X)} \leq \lambda_1 \text{ for all } (t, y) \in [0, T_0] \times W,$$

In addition, for some $\mu_1 > 0$, it hat, for all $y$ and $z \in W$,

$$\|B(t, y) - B(t, z)\|_{\mathcal{B}(X)} \leq \mu_1 \|y - z\|_Y.$$  

(A3) $Y \subseteq D(A(t, y))$, for each $(t, y) \in [0, T_0] \times W$, (the restriction of $A(t, y)$ to $Y$ belongs to $\mathcal{B}(Y, X)$) and, for each fixed $y \in W$, $t \rightarrow A(t, y)$ is strongly continuous. Furthermore, for each fixed $t \in [0, T_0]$, it is satisfied the following Lipschitz condition,

$$\|A(t, y) - A(t, z)\|_{\mathcal{B}(Y, X)} \leq \mu_2 \|y - z\|_X,$$

where $\mu_2 \geq 0$ is a constant.

(A4) $A(t, y) w_0 \in Y$ for all $(t, y) \in [0, T] \times W$. Also, there exists a constant $\lambda_2$ such that

$$\|A(t, y) w_0\|_Y \leq \lambda_2, \text{ for all } (t, y) \in [0, T_0] \times W.$$
\( (f_1) \) \( f \) is a bounded function from \( [0, T_0] \times W \) in \( Y \), i.e., there exists \( \lambda_3 \) such that
\[
\|f(t,y)\|_Y \leq \lambda_3, \quad \text{for all } (t,y) \in [0, T_0] \times W,
\]
Besides, the function \( t \in [0, T_0] \mapsto f(t,y) \in Y \) is continuous with respect to \( X \) topology and, for all \( y \) and \( z \in Y \), we have that
\[
\|f(t,y) - f(t,z)\|_X \leq \mu_3 \|y - z\|_X,
\]
when \( \mu_3 \geq 0 \) is a constant.

**Theorem 1.1** (Kato). Suppose that the conditions \((X), (A_1) - (A_4)\) and \((f_1)\) are satisfied. For \( u_0 \in Y \), there exist \( 0 < T < T_0 \) and a unique \( u \in C([0, T]; Y) \cap C^1((0, T); X) \) solution to \((5)\). Besides, the map \( u_0 \rightarrow u \) is continuous in the following sense: consider the following sequence of Cauchy problems,
\[
\begin{align*}
\partial_t u_n + A_n(t, u_n) u_n &= f_n(t, u_n) \quad t > 0 \\
u_n(0) &= u_{n_0} \quad n \in \mathbb{N}.
\end{align*}
\]
Assume that conditions \((X), (A_1) - (A_4)\) and \((f_1)\) hold for all \( n \geq 0 \) in \((6)\), with the same \( X, Y \) and \( S \), and the corresponding \( \beta, \lambda_1 - \lambda_3, \mu_2 - \mu_3 \) can be chosen independently from \( n \). Also assume that
\[
\begin{align*}
&\text{s-lim}_{n \to \infty} A_n(t, w) = A(t, w) \text{ in } B(X, Y) \\
&\text{s-lim}_{n \to \infty} B_n(t, w) = B(t, w) \text{ in } B(X) \\
&\lim_{n \to \infty} f_n(t,w) = f(t,w) \text{ in } Y \\
&\lim_{n \to \infty} u_{n_0} = u_0 \text{ in } Y,
\end{align*}
\]
where s-lim denotes the strong limit. Then, \( T \) can be so chosen in such a way that \( u_n \in C([0, T], Y) \cap C^1((0, T), X) \) and
\[
\lim_{n \to \infty} \sup_{[0, T]} \|u_n(t) - u(t)\|_Y = 0.
\]
A proof of this theorem can be seen in [9] and [10].

In the Section 4 is examined the local well-posedness in the weighted spaces \( X^s(w^2) \). For this, we use ideas of Milánés in [20] (see also [21]). Milánés, in her work, examines the local well-posedness of the problem
\[
\begin{align*}
\partial_t u + u^p u_y + \mathcal{H} u_{xy} &= 0 \quad p \in \mathbb{N} \\
 u(0; x, y) &= \phi(x, y)
\end{align*}
\]
in weighted Sobolev spaces, extending ideas developed by Iório in [7] and [8]. Finally, in the Section 5 we present the asymptotic behaviour of solutions with small initial data. This is obtained from \( L^p-L^q \) estimates of the group associated to the linear part of the equation \((2)\) analogous to those of the Schrödinger group \( e^{it\Delta} \) in dimension two, as it is done by Milánés in [20] for the equation \((6)\). Observe that this property is shared by generalized Benjamin-Ono equation (in one dimension, see [14]), from where this result is suggested. This, also, allows to prove the global existence for these small datas.
2 Preliminaries

In this paper we systematically use the following notations.

1. $S(\mathbb{R}^n)$ is the Schwartz space. If $n = 2$, we simply write $S$.

2. $S'(\mathbb{R}^n)$ is the space of tempered distributions. If $n = 2$, we simply write $S'$.

3. For $f \in S'(\mathbb{R}^n)$, $\hat{f}$ is the Fourier transform of $f$ and $\check{f}$ is the inverse Fourier transform of $f$. We recall that
   
   $$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{i(x,\xi)}dx,$$

   for all $\xi \in \mathbb{R}^n$, when $f \in S(\mathbb{R}^n)$.

4. $H^s = H^s(\mathbb{R}^2)$ is the Sobolev space of order $s$.

5. The inner product in $H^s$ is denoted as
   $$(f, g)_{L^2} = \int_{\mathbb{R}^2} (1 + |\xi|^2 + |\eta|^2)^{s/2} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.$$

6. $X^s(\rho)$ is the space $X^s(\rho) = X^s \cap L^2(\rho(x,y) dx dy)$.

7. $\Lambda^s = (1 - \Delta)^{s/2}$.

8. $L^s_p(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) \mid \Lambda^s f \in L^p(\mathbb{R}^n) \}.$

9. For $f \in L^s_p(\mathbb{R}^2)$, $|f|_{p,s} = \|\Lambda^s f\|_{L^p(\mathbb{R}^2)}$.

10. $[A, B]$ will denote the commutator of $A$ and $B$.

The following results about commutators of operators are part of the important stock of tools that are used in the analysis.

The first of them is given by the following proposition due to Kato (its proof can be found in [9]).

**Proposition 2.1 (Kato’s inequality).** Let $f \in H^s$, $s > 2$, $\Lambda = (1 - \Delta^2)^{1/2}$ and $M_f$ be the multiplication operator by $f$. Then, for $|\tilde{t}|$, $|\tilde{s}| \leq s - 1$, $\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f] \Lambda^{-\tilde{t}}$ is bounded in $B(L^2(\mathbb{R}^2))$ and

$$\left\| \Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f] \Lambda^{-\tilde{t}} \right\|_{B(L^2(\mathbb{R}^2))} \leq c\|\nabla f\|_{H^{s-1}}. \tag{8}$$
**Proposition 2.2** (Kato-Ponce’s inequality). Let $s > 0$, $1 < p < \infty$, $\Lambda = (1 - \Delta^2)^{1/2}$ and $M_f$ be the multiplication operator by $f$. Then,

$$
[[\Lambda^s, M_f]g]_p \leq c \left( |\nabla f|_\infty |\Lambda^{s-1}g|_p + |\Lambda^s f|_p |g|_\infty \right),
$$

for all $f$ and $g \in S$

**Corollary 2.3.** For $f$ and $g \in S$,

$$
|f, g|_{s,p} \leq c \left( |f|_\infty |\Lambda^s g|_p + |\Lambda^s f|_p |g|_\infty \right).
$$

The following theorem is due to A. P. Calderón (see [5]).

**Theorem 2.4** (Calderón’s commutator theorem). Let $A : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function. Then, for any $f \in S(\mathbb{R})$,

$$
\|[[H, A]f']_0 \leq C |A'|_\infty \|f\|_0.
$$

**Lemma 2.5.** Let $g, h \in S(\mathbb{R}^n)$ and $s \geq 0$. Then there exists a constant $C = C(s)$ such that

$$
\|gh\|_{[s]} \leq C \left[ \|g\|_{A} \|h\|_{[s]} + \|g\|_{[s]} \|h\|_{A} \right]
$$

where $\|\phi\|_{[s]} = \left\| (\Delta^2)^{\frac{s}{2}} \phi \right\|_o$, $\|\phi\|_{A} = \left\| \hat{\phi} \right\|_{L^1}$.

**Corollary 2.6.** Let $g, h$ and $s$ be as in the Lemma 2.5 and $\frac{s}{2} < s_0$. Then there exists a constant $C = C(s)$ such that

$$
\|g \partial_x h\|_{s} \leq C \left( \|g\|_{s} \|h\|_{s} + \|g\|_{s_0} \|h\|_{s+1} \right).
$$

## 3 Local theory in Sobolev spaces

In this section we examine the local well-posedness of a Cauchy problem associated to a two-dimensional generalization of the Benjamin-Ono equation given in (2).

First, we consider the local well-posedness in $H^s(\mathbb{R}^2)$ when $\gamma = 0$.

**Theorem 3.1.** Let $s > 2$ and $p \in \mathbb{N}$. For $\phi \in H^s(\mathbb{R}^2)$, there exist $T > 0$, that depends only on $\|\phi\|_{s}$, and a unique $u \in C([0, T], H^s(\mathbb{R}^2)) \cap C^1([0, T], H^{s-2}(\mathbb{R}^2))$ solution to the Cauchy problem

$$
\begin{cases}
    u_t + \mathcal{H} \partial_x^2 u + \alpha \mathcal{H} \partial_y^2 u + u^p u_x = 0 \\
    u(0) = \phi.
\end{cases}
$$

Furthermore, the map $\phi \mapsto u$ from $H^s$ to $C([0, T], H^s)$ is continuous.

**Proof.** Without loss generality we can suposse $\alpha = 1$. In this case, $u$ is solution to (10) if and only if $v(t) = e^{t \mathcal{N}} u(t)$ is solution to

$$
\begin{cases}
    \frac{dv}{dt} + A(t, v) v = 0 \\
    v(0) = \phi,
\end{cases}
$$

(11)
Lemma 3.2. $A(t, v) = e^{t\Delta}(e^{-t\Delta}v)^{p}\partial_x e^{-t\Delta}$.

Let us see for this problem that each one of the conditions of the Kato’s theorem (Theorem 1.1) is satisfied. For the moment, let $X = L^2(\mathbb{R}^2)$ and $Y = H^s(\mathbb{R}^2)$, for $s > 2$. It is clear that $S = (1 - \Delta)^{s/2}$ is an isomorphism between $X$ and $Y$. In the following lemmas we verify that the problem (11) satisfies the conditions (A1)–(A4) of the Theorem 1.1.

Lemma 3.3. If $S = (1 - \Delta)^{s/2}$, then

$$SA(t, v)S^{-1} = A(t, v) + B(t, v),$$

where $B(t, v)$ is a bounded operator in $L^2$, for all $t \in \mathbb{R}$ and $v \in H^s$, and satisfies the inequalities

$$\|B(t, v)\|_{B(X)} \leq \lambda(v),$$

$$\|B(t, v) - B(t, v')\|_{B(X)} \leq \mu(v, v')\|v' - v\|,$$

for all $t \in \mathbb{R}$, and every $v$ and $v' \in H^s$, where $\lambda(v) = \sup_t C_s\|\nabla (e^{-t\Delta}v)^p\|_{L^\infty(\mathbb{R}^2)}$ and $\mu(v, v') = C_{p, s}(\|v\|_{s-1}^p + \|v'\|_{s-1}^p)$.

Proof. Since $\{e^{-t\Delta}\}$ is a strongly continuous group of unitary operators, and thanks to the observation immediately below of the condition (A1) of the Theorem 1.1 it follows the lemma.

Lemma 3.4. $H^s(\mathbb{R}^2) \subset D(A(t, v))$ and $A(t, v)$ is a bounded operator from $Y = H^s(\mathbb{R}^2)$ to $X = L^2(\mathbb{R}^2)$ with

$$\|A(t, v)\|_{B(X, Y)} \leq \lambda(v),$$

for all $v \in Y$, and where $\lambda$ is as in the Lemma 3.3. Also, the function $t \mapsto A(t, v)$ is strongly continuous from $\mathbb{R}$ to $B(Y, X)$, for all $v \in H^s$. Moreover, the function $v \mapsto A(t, v)$ satisfies the following Lipschitz condition

$$\|A(t, v) - A(t, v')\|_{B(Y, X)} \leq \mu(v, v')\|v - v'\|_X,$$

where $\mu$ is as in the lemma above.
Proof. Inasmuch as $e^{-t\mathcal{A}} = (e^{t\mathcal{A}})^{-1}$ is an unitary operator in $X = L^2(\mathbb{R}^2)$, from
the definition of $A(t,v)$, it follows $H^s(\mathbb{R}^2) \subset D(A(t,v))$. In fact,
$$
\|A(t,v)f\|_0 = \|e^{-t\mathcal{A}}f\|_0 \leq C_s\|e^{t\mathcal{A}}f\|_0 \leq \lambda(v)\|f\|_s,
$$
for all $f \in Y$.

Now, for all $t, t' \in \mathbb{R}$ and all $f, v \in Y$, we have
$$
\|A(t,v)f - A(t',v)f\|_0 \leq \|e^{t\mathcal{A}}f - e^{t'\mathcal{A}}f\|_0 + \|e^{t'\mathcal{A}}f - e^{t\mathcal{A}}f\|_0
$$
$$
+ \|e^{t\mathcal{A}}f - e^{t'\mathcal{A}}f\|_0 + \|e^{t'\mathcal{A}}f - e^{t\mathcal{A}}f\|_0
$$
$$
\leq C_p\|(e^{t\mathcal{A}}f)p - (e^{t'\mathcal{A}}f)p\|_0\|e^{t\mathcal{A}}f\|_0
$$
$$
\leq \mu(v,v')\|v' - v\|_0\|f\|_s.
$$

This completes the proof of the lemma. 

The immediately preceding lemmas show that the problem (11) satisfies the Theorem 1.1 conditions and, therefore, for each $\phi \in H^s$, $s > 2$, there exist $T > 0$, which depends on $\|\phi\|_s$, and an unique $v \in C([0,T], H^s(\mathbb{R}^2) \cap C([0,T], H^{s-1}(\mathbb{R}^2)))$ solution to problem (11). Also, the map $\phi \mapsto v$ is continuous from $H^s(\mathbb{R}^2)$ to $C([0,T], H^s(\mathbb{R}^2)$.

Now, from the properties of group $Q(t) = e^{-t\mathcal{A}}$ can be verified that $u(t) = Q(t)v(t)$ is solution to (10) and satisfies the properties enunciated in Theorem 3.1.

Theorem 3.5. The time of existence of the solution to (10) can be chosen independently from $s$ in the following sense: if $u \in C([0,T], H^s)$ is the solution to (10) with $u(0) = \phi \in H^r$, for some $r > s$, then $u \in C([0,T], H^r)$. In particular, if $\phi \in H^\infty$, $u \in C([0,T], H^\infty)$.

Proof. The proof of this result is essentially the same as part (c) of the Theorem 1 in [10]. We will briefly outline this. Let $r > s$, $u \in C([0,T], H^s)$ be the solution to (10) and $v = e^{t\mathcal{A}}u$. Let us suppose that $r \leq s + 1$. Applying $\partial_x^2$ in both sides of the differential equation in (11), we arrive at the following linear evolution equation in $w(t) = \partial_x^2 v(t)$,
$$
\frac{dw}{dt} + A(t)w + B(t)w = f(t)
$$
(15)

where
$$
A(t) = \partial_x e^{t\mathcal{A}}(u(t))p e^{-t\mathcal{A}}
$$
(16)
$$
B(t) = 2e^{t\mathcal{A}}[p(u(t))p^{-1}u_x(t)]^{-1}e^{-t\mathcal{A}}
$$
(17)
$$
f(t) = -e^{t\mathcal{A}}[p(p-1)u^{p-2}(t)]u_x(t)
$$
(18)
Since \( v \in C([0,T),H^s) \) we have that \( w \in C([0,T),H^{s-2}) \). Also, \( w(0) = \phi_{xx} \in H^{r-2}, \) because \( \phi \in H^r \). Let us prove that \( w \in C([0,T],H^{r-2}) \). To do this, we shall prove that the Cauchy problem associated to the linear equation \((15)\) is well-posed for \( 1-s \leq k \leq s-1 \). In this direction we have the following lemma whose proof is completely similar to that of Lemma 3.1 in [10].

**Lemma 3.6.** The family \( \{A(t)\}_{0 \leq t \leq T} \) has an unique family of evolution operators \( U(t,\tau)_{0 \leq t \leq \tau \leq T} \) in the spaces \( X = H^k, \ Y = H^k \) (in the Kato sense), where

\[
-s \leq h \leq s-2 \quad 1-s \leq k \leq s-1 \quad k+1 \leq h
\]

In particular, \( U(t,\tau) : H^r \to H^r \) for \(-s \leq r \leq s-1\).

Then, the last lemma allows us to show that \( w \) satisfies the equation

\[
w(t) = U(t,0)\phi_{xx} + \int_0^t U(t,\tau)[-B(\tau)w(\tau) + f(\tau)]d\tau.
\]

Now, since \( w(0) = \phi_{xx} \in H^{r-2} \), by \((15)\), \( f \) is in \( C([0,T),H^{s-1}) \subset C([0,T),H^{r-2}) \) (if \( r \leq s+1 \)) and \( B(t) \), given in \((17)\), is a family of operators in \( \mathcal{B}(H^{r-2}) \) strongly continuous for \( t \) in the interval \([0,T]\) (if \( r \leq s+1 \), from Lemma 3.6 the solution to \((20)\) is in \( C([0,T],H^{r-2}) \)).

If \( w_1(t) = \partial_x \partial_y v(t) \), we have

\[
\frac{dw_1}{dt} + A(t)w_1 + B_1(t)w_1 = f_1(t)
\]

where

\[
B_1(t) = e^{t\Delta}[p(u(t))^{p-1}u_x(t)e^{-t\Delta} = \frac{1}{2}B(t)
\]

\[
f_1(t) = -e^{t\Delta}((p(p-1)u^{p-2}(t)[u_x(t)]^2 + p(u(t))^{p-1}u_{xx}(t))u_y(t))
\]

As above, we have

\[
w_1(t) = U(t,0)\phi_{xy} + \int_0^t U(t,\tau)[-B_1(\tau)w_1(\tau) + f_1(\tau)]d\tau.
\]

Inasmuch as \( u_{xx} \in C([0,T],H^{r-2}) \), \( f_1 \in C([0,T],H^{r-2}) \). Since, also, \( B_1(t) \in \mathcal{B}(H^{r-2}) \) is strongly continuous in the interval \([0,T]\), arguing as before, we have that \( w_1 \in C([0,T],H^{r-2}) \) or, equivalently, \( u_{xy} \in C([0,T],H^{r-2}) \).

Analogously, if \( w_2(t) = \partial_y^2 v(t) \), we have

\[
\frac{dw_2}{dt} + A(t)w_2 = f_2(t)
\]

where

\[
f_2(t) = -e^{t\Delta}((p(p-1)u^{p-2}(t)u_x(t)u_y(t) + 2p(u(t))^{p-1}u_{xy}(t))u_y(t))
\]
Then,
\[ w_2(t) = U(t, 0)\phi_{yy} + \int_0^t U(t, \tau)f_2(\tau)d\tau. \] \hspace{1cm} (27)

Since \( u_{xy} \in C([0, T], H^{r-2}) \), \( f_2 \in C([0, T], H^{r-2}) \). Repeating the argument above, we can conclude that \( w_1 \in C([0, T], H^{r-2}) \) or, equivalently, \( \partial_x^2 u \in C([0, T], H^{r-2}) \).

Then, we have proved that, if \( s < r \leq s + 1 \) and \( \phi \in H^r \), \( u \in C([0, T], H^r) \). To the case \( r > s + 1 \), as \( \phi \in H^s' \), for \( s' < r \), using a bootstrapping argument can be shown that \( u \in C([0, T], H^r) \). \( \square \)

Now we examine the local well-posedness of \( [2] \) in \( X^s(\mathbb{R}^2) \) without any restriction on the parameters.

**Theorem 3.7.** Let \( s > 2 \) and \( p \in \mathbb{N} \). For \( \phi \in X^s(\mathbb{R}^2) \), there exist \( T > 0 \), that depends only on \( \|\phi\|_s \), and a unique \( u \in C([0, T], X^s(\mathbb{R}^2)) \cap C^1([0, T], H^{s-2}(\mathbb{R}^2)) \) solution to the Cauchy problem

\[
\begin{aligned}
& u_t + 3\partial_x^2 u + \alpha 3\partial_y^2 u - \gamma \partial^{-1}_x \partial_y^2 u + u^p u_x = 0 \\
& u(0) = \phi.
\end{aligned}
\] \hspace{1cm} (28)

Furthermore, the map \( \phi \mapsto u \) from \( X^s \) to \( C([0, T], X^s) \) is continuous.

**Proof.** The proof is basically the same as the Theorem 3.1. Let \( A = 3\partial_x^2 + \alpha 3\partial_y^2 - \gamma \partial^{-1}_x \partial_y^2 \). It is easy to check that \( A \) generates a strongly continuous group in \( H^s \). Therefore, the local well-posedness in \( H^s \) of the Cauchy problem

\[
\begin{aligned}
& \frac{dv}{dt} + A(t, v)v = 0 \\
& v(0) = \phi.
\end{aligned}
\] \hspace{1cm} (29)

follows from lemmas completely analogous to the Lemmas 3.2, 3.3 and 3.4 with which we proved the local well-posedness of the Cauchy problem \( [11] \).

Now, let \( v \) be the solution to Cauchy problem \( [29] \) and \( u = e^{-tA}v \). Let us prove that if \( \phi \in X^s \), \( u \in C([0, T], X^s(\mathbb{R}^2)) \) and is solution to \( [28] \). From \( [29] \) it can be easily proved that

\[
\begin{aligned}
& u = e^{-tA}\phi + \int_0^t e^{-(t-\tau)A} \partial_x \left( \frac{u^{p+1}(\tau)}{p+1} \right) d\tau \\
& = e^{-tA}\phi + \partial_x \int_0^t e^{-(t-\tau)A} \left( \frac{u^{p+1}(\tau)}{p+1} \right) d\tau.
\end{aligned}
\] \hspace{1cm} (30)

Indeed, \( u \in C([0, T], H^s(\mathbb{R}^2)) \) is solution to the last equation if only if \( v = e^{tA}u \) is solution to \( [29] \). Since \( H^s \) is a Banach algebra, \( t \mapsto u^{p+1}(t) \) is continuous from
\([0, T]\) to \(H^s\). In particular, \(\int_0^t e^{-(t-\tau)A} \left( u^{p+1} (\tau) \right) d\tau\) is a continuous function in \(t\) with values in \(H^s\). Hence, if \(\phi \in X^s\),

\[
\partial_x^{-1} u = e^{-tA} \partial_x^{-1} \phi + \int_0^t e^{-(t-\tau)A} \left( \frac{u^{p+1} (\tau)}{p+1} \right) d\tau \in C([0, T], H^s).
\]

Therefore \(u \in C([0, T], X^s(\mathbb{R}^2))\) and \(u\) is solution to (28). Also, by (14)

\[
\sup_{t \in [0, T]} \| \partial_x^{-1} (u - \tilde{u})(t) \|_s \leq \| \partial_x^{-1} (\phi - \tilde{\phi}) \|_s + C_{p,s} \sup_{t \in [0, T]} (\| u \|_s^{p-1} + \| \tilde{u} \|_s^{p-1}) \sup_{t \in [0, T]} \| (u - \tilde{u})(t) \|_s,
\]

where \(\tilde{\phi} \in X^s\) and \(\tilde{u} \in C([0, T], X^s(\mathbb{R}^2))\) is solution to

\[
\tilde{u} = e^{-tA} \tilde{\phi} + \int_0^t e^{-(t-\tau)A} \partial_x \left( \frac{\tilde{u}^{p+1} (\tau)}{p+1} \right) d\tau.
\]

Therefore, the local well-posedness of (28) is equivalent to the local well-posedness of (29). This finishes the proof.

The following theorem is totally analogous to the Theorem 3.5

**Theorem 3.8.** The time of existence of the solution to (28) can be chosen independently from \(s\) in the following sense: if \(u \in C([0, T], X^s)\) is the solution to (28) with \(u(0) = \phi \in X^r\), for some \(r > s\), then \(u \in C([0, T], X^r)\). In particular, if \(\phi \in X^\infty\), \(u \in C([0, T], X^\infty)\).

**Proof.** Suppose \(u \in C([0, T], X^s)\) is the solution to (28) with \(u(0) = \phi \in X^r\) with \(r > s\). To see that \(u \in C([0, T], X^r)\) with \(u(0) = \phi \in X^r\), we repeat the same arguments that we used in the proof of Theorem 3.5; it is just to replace the operator \(H \Delta\) with \(A\), the operator defined in the proof of the immediately above theorem. Since

\[
\partial_x^{-1} u = e^{-tA} \partial_x^{-1} \phi + \int_0^t e^{-(t-\tau)A} \left( \frac{u^{p+1} (\tau)}{p+1} \right) d\tau,
\]

we have that \(u \in C([0, T], X^r)\).

\[ \square \]

## 4 Local theory in weighted Sobolev spaces

In this section we shall examine the local well-posedness of the Cauchy problem (2) in some weighted Sobolev spaces. We use ideas developed in [17], [8] and [20].

First, we consider the case \(\gamma = 0\).

**Theorem 4.1.** Assume that \(w\) is a weight with its first and second derivatives bounded and, for some \(\lambda^*\), there exist \(C_\lambda > 0\) such that

\[
|w(x, y)| \leq C_\lambda e^{\lambda (x^2 + y^2)},
\]

for all \((x, y)\).
for all \((x, y) \in \mathbb{R}^2\) and all \(\lambda \in (0, \lambda^*)\). Let

\[ X^s(w^2) = \{ f \in X^s \mid wf \in L^2 \}. \]

This is a Hilbert space with the inner product \(\langle \cdot, \cdot \rangle_{w, s} = \langle \cdot, \cdot \rangle_{X^s} + \langle \cdot, \cdot \rangle_{L^2(w^2)}\). Then, for \(s > 2\), the Cauchy problem is local well-posed in \(X^s(w^2)\).

**Proof.** In this proof we use following lemma.

**Lemma 4.2.** For \(w\) as in the theorem. Let \(w_\lambda(x, y) = w(x, y)e^{-\lambda(x^2+y^2)}\), there exist constants \(c_1, c_2, c_3\) and \(c_4\) independent of \(\lambda\) and such that

\[ |\nabla w_\lambda|_\infty \leq c_1 |\nabla w|_\infty + c_2 \]

and

\[ |D^\alpha w_\lambda|_\infty \leq c_3 |\nabla w|_\infty + |D^\alpha w|_\infty + c_4, \]

for any multiindex \(\alpha = (\alpha_1, \alpha_2)\) with \(|\alpha| = 2\).

In view of the local well posedness in \(X^s\), it is enough with examining some estimates of \(L^2(w^2)\) norm. Well, with this purpose let \(w_\lambda(x, y) = w(x, y)e^{-\lambda(x^2+y^2)}\). It is clear that \(||w_\lambda u(t)||_0 < \infty\) and \(||w_\lambda u(t)||_0 < \infty\), for all \(t \in [0, T]\) and all \(\lambda > 0\). Hence, multiplying on both sides of the equation \((\text{II})\) by \(w_\lambda^2 u\) and integrating we obtain

\[ \frac{1}{2} \frac{d}{dt} ||w_\lambda u||_0^2 = \langle w_\lambda u, w_\lambda (\partial^x \partial_x^2 u - \alpha \partial^x \partial_y^2 u - w_\lambda u) \rangle. \]

The first two terms in the sum on the right hand of the last equation satisfy

\[ \langle w_\lambda u, w_\lambda \partial^x \partial_x^2 u \rangle = \langle w_\lambda u, \partial^x \partial_x^2 u \rangle_0 + \langle w_\lambda u, \partial^x \partial_y^2 u \rangle_0 \]

\[ \langle w_\lambda u, w_\lambda \partial^x \partial_y^2 u \rangle_0 = \langle w_\lambda u, \partial^x \partial_y^2 u \rangle_0 + \langle w_\lambda u, \partial^x \partial_y^2 u \rangle_0 \]

The Cauchy-Schwarz inequality, the Calderón’s commutator theorem and the lemma above imply that

\[ \langle w_\lambda u, [w_\lambda, \partial^x \partial_y^2 u] \rangle_0 \leq ||w_\lambda u||_0 ||[w_\lambda, \partial^x \partial_y^2 u]||_0 \leq C_1 ||\partial_x w_\lambda||_\infty ||w_\lambda u||_0 ||\partial_x \partial_y^2 u||_0 \leq C_2 ||w_\lambda u||_0 ||u||_{X^s}. \]

On the other hand,

\[ \langle w_\lambda u, [w_\lambda, \partial^x] \rangle_0 \leq ||w_\lambda u||_0 ||[w_\lambda, \partial^x] u||_0 \leq C_1 ||w_\lambda u||_0 \left(||\partial^2 \partial_y^2 w_\lambda|| + ||w_\lambda u||_0 2||\partial_y^2 w_\lambda|| \right) \leq C_2 ||w_\lambda u||_0 ||u||_{X^s}. \]

In an entirely similar way we obtain

\[ \langle w_\lambda u, [w_\lambda, \partial^x \partial_y] \rangle_0 \leq C ||w_\lambda u||_0 ||u||_{X^s}. \]
Thanks to the Lebesgue's monotone convergence, it follows that

\[ \langle w_\lambda u, \mathcal{H}^{(x)}[w_\lambda, \partial^2_x u] \rangle_0 \leq C \|w_\lambda u\|_0 \|u\|_{X_s}. \]

Also,

\[ \|w_\lambda u^p u_x\|_0 \leq |w^{p-1}u_x|_\infty \|w_\lambda u\|_0 \leq C_s \|w_\lambda u\|_0. \]

With the help of the estimates above we can infer

\[ \frac{d}{dt}\|w_\lambda u\|_0^2 \leq A \|u\|_{X_s}^\lambda + B \|w_\lambda u\|_0^2, \]

where \( A \) and \( B \) are constants that do not depend on \( \lambda \). From the Gronwall inequality it is concluded that

\[ \|w_\lambda u\|_0^2 \leq e^{BT} (\|w_\lambda \phi\|_0^2 + TA \|u\|_{X_s}^\lambda). \]

Thanks to the Lebesgue's monotone convergence, it follows that

\[ \|wu\|_0^2 \leq e^{BT} (\|w\phi\|_0^2 + TA \|u\|_{X_s}^\lambda) \]

Therefore, \( u(t) \in X_s(w^2) \), for all \( t \in [0, T] \). By proceeding in the same way it is deduced that

\[ \|w(u - v)\|_0^2 \leq e^{BT} (\|w(\phi - \psi)\|_0^2 + TA \|u - v\|_{X_s}^\lambda), \]

where \( \psi \in X_s(w^2) \) and \( v \) is the solution to (10), with \( \psi \) instead of \( \phi \). Remains to be seen that \( t \mapsto u(t) \) is continuous from \([0, T] \) in \( X_s(w^2) \). But this is immediate from dominated convergence theorem, from the continuity of \( u \) in \( X^\ast \) and from the equation

\[ \|w(u(t) - u(t'))\|_0 \leq \|(w - w_\lambda)u(t)\|_0 + \|w_\lambda(u(t) - u(t'))\|_0 + \|(w_\lambda - w)u(t')\|_0. \]

This terminates the proof of the theorem. \( \square \)

Remark 4.3. The weights \( w_\vartheta(x, y) = (1 + x^2 + y^2)^{\vartheta/2}, \) for \( \vartheta \in [0, 1] \), satisfy the conditions of the previous theorem.

For \( \gamma \neq 0 \) we have the following result

Theorem 4.4. Assume that \( w \) in the Theorem 4.1 depends only on \( y \). Then, in this case the Cauchy problem 28 is local well-posed in \( X^\ast(w^2) \).

Proof. We proceed as in the proof of Theorem 4.1. Here, the fact that \( w \) is not dependent on \( y \) make our work easier. Is clear that

\[ \frac{1}{2} \frac{d}{dt}\|w_\lambda u\|_0^2 = \langle w_\lambda u, w_\lambda \left( -3\mathcal{H}^{(x)} \partial^2_x u - \alpha \mathcal{H}^{(x)} \partial^2_y u + \gamma \partial^{-1}_x \partial^2_y u - w^p u_x \right) \rangle_0. \]

In this case the estimates of the linear terms are

\[ \langle w_\lambda u, w_\lambda \mathcal{H}^{(x)} \partial^2_x u \rangle_0 = 0 \]
\[ \langle w_\lambda u, w_\lambda \mathcal{H}^{(x)} \partial^2_y u \rangle_0 = \langle w_\lambda u, w_\lambda \mathcal{H}^{(x)} \partial^2_y u \rangle_0 \]
\[ \langle w_\lambda u, w_\lambda \partial^{-1}_x \partial^2_y u \rangle_0 = \langle w_\lambda u, \left[ w_\lambda, \partial^2_y \right] \partial^{-1}_x u \rangle_0 \]

Henceforth, the proof follows the same steps of the proof of Theorem 4.1. \( \square \)
Remark 4.5. Observe that \( w(y) = y \) is a particular case of weights considered in the above theorem. In reality, this theorem is valid even for the following Cauchy problem
\[
\begin{align*}
    u_t + u^p u_x + \delta \partial_y^2 u + \mathcal{H} \partial_x^2 u + \alpha \mathcal{H} \partial_y^2 u - \gamma \partial_x^{-1} u_{yy} &= 0, \\
    u(0) &= \phi,
\end{align*}
\]
which represents an improvement of Theorem 2.4 in [4].

5 Asymptotic behaviour of solutions with small initial data

For \( \gamma = 0 \), in this section we show that the solution to (2) (in other words, the solution to (10)) is global if it is taken an small enough initial data, in a sense which will be made precise later on. Also we show that the solution, at a time sufficiently large, behaves as the solution to the linear equation associated. These last are often called scattering states.

For \( \phi \in H^s(\mathbb{R}^2) \), let \( P(-t)\phi = e^{-tA} \phi \) (A as in the proof of Theorem 3.7, with \( \gamma = 0 \)) the solution to the linear problem associated to the Cauchy problem (10), i.e., if \( u(t) = P(-t)\phi \), \( u \) satisfy the equation
\[
\frac{du}{dt} + \mathcal{H} \partial_x^2 u + \alpha \mathcal{H} \partial_y^2 u = 0.
\]
Without loss generality we can assume \( \alpha = 1 \).

If \( \phi \in \mathcal{S} \) then
\[
P(-t)\phi(x, y) = \frac{1}{2\pi} \int e^{i(sgn(\xi)(\xi^2 + \eta^2)t + x\xi + y\eta)} \hat{\phi}(\xi, \eta)d\xi d\eta = \frac{1}{2\pi} I(t) \ast \phi(x, y)
\]
where \( I(t) = (e^{i sgn(\xi)(\xi^2 + \eta^2)t})^\vee \).

Lemma 5.1. For any \( x, y \) and \( t \neq 0 \) real numbers,
\[
I(t)(x, y) = \frac{c}{t} e^{-\frac{i}{4}(x^2 + y^2)} \int_0^\infty e^{i s^2} ds + \frac{\overline{c}}{t} e^{\frac{i}{4}(x^2 + y^2)} \int_0^\infty e^{-i s^2} ds,
\]
where \( c = (1 + i)/2 \).

Proof. Is clear that
\[
2\pi I(1)(x, y) = \int_\mathbb{R} \int_0^\infty e^{i(\xi^2 + \eta^2 + x\xi + y\eta)} d\xi d\eta + \int_\mathbb{R} \int_0^\infty e^{i(-\xi^2 - \eta^2 + x\xi + y\eta)} d\xi d\eta
\]
\[
= e^{-\frac{i}{4}(x^2 + y^2)} \left( \int_\mathbb{R} e^{i(\eta + y/2)^2} d\eta \right) \left( \int_0^\infty e^{i(\xi + x/2)^2} d\xi \right) +
\]
\[
+ e^{\frac{i}{4}(x^2 + y^2)} \left( \int_\mathbb{R} e^{-i(\eta - y/2)^2} d\eta \right) \left( \int_0^\infty e^{-i(\xi - x/2)^2} d\xi \right).
\]
A simple change of variable prove the theorem for \( t = 1 \). Using the homogeneity property of the Fourier transform, the theorem follows for any \( t \neq 0 \). \( \square \)
The last lemma implies the following $L^p - L^q$ estimate for the group $P(t)$.

**Proposition 5.2.** For any $f \in L^1 \cap L^2$, it has that

$$|P(-t)f|_{\frac{q}{q-p}} \leq c|t|^{-\theta}|f|_{\frac{q}{q-p}},$$

for $\theta \in [0,1].$

**Proof.** We obtain the result by using the Young’s inequality for convolution, the lemma above and interpolation. □

From Sobolev imbedding theorem it follows the following proposition.

**Proposition 5.3.** For $s > 1$ and $f \in L^1 \cap H^s$, we have

$$|P(-t)f|_{\infty} \leq c(1 + |t|)^{-1}(|f|_1 + \|f\|_s)$$

Now, we are ready to prove the following theorem.

**Theorem 5.4.** Let $p \geq 3$ and $s > 3$. Then, there exist $\delta > 0$ and $R = R(\delta) > 0$ such that if $\phi \in L^1 \cap H^s$ satisfies

$$|\phi|_{1,1} + \|\phi\|_s < \delta,$$

the solution $u$ to (10) belongs to $C_b(\mathbb{R}, H^s)$ and satisfies

$$\sup_{t \in \mathbb{R}} (1 + |t|)|u(t)|_{1,\infty} \leq R. \quad (32)$$

**Proof.** For this proof we need the following lemma whose proof can be found in [21] (Lemma 3.0.52)

**Lemma 5.5.** For $t \geq 0$, let

$$J(t) = (1 + t) \int_0^t \frac{1}{(1 + t - \tau)(1 + \tau)^{p-1}} d\tau.$$

Then,

1. $J(t) = O(1)$ as $t \to \infty$, if $p \geq 3$
2. $J(t) \to \infty$ as $t \to \infty$, if $p = 1, 2$.

Let us see first

$$\|u(t)\|_s \leq \|\phi\|_s \exp \left( c \int_0^t |u_x|_{\infty} |u|^{p-1}_{\infty} d\tau \right). \quad (33)$$

Making the inner product in $H^s$ by $u$ in both sides of the equation we obtain

$$\frac{d}{dt}\|u\|_s^2 = -2\langle u, u_p u_x \rangle_s.$$
By virtue of the Kato-Ponce inequality and its corollary (Corollary 2.3), we get
\[
\frac{d}{dt} \| u \|_2^2 \leq C |u|^{p-1}_\infty \| u \|_2^2.
\]
The inequality (33) follows from this last and the Gronwall inequality.

Now, in light of (33) it is enough to prove (32). Indeed, from the hypotheses, we have
\[
\int_0^t |u_x|_\infty |u|_\infty^{p-1} d\tau \leq \int_0^t |u|_1^{p} d\tau \leq R^p \int_0^t (1 + |\tau|)^{-p} d\tau \leq C.
\]
So let us prove (32). We take \( T \in (0,T_s) \) and let \( K(T) = \sup_{t \in [0,T]} \{(1 + |t|)|u(t)|_1\} \). From the Proposition 5.3, the Lemma 5.5, (33) and the integral equation (30), we obtain
\[
(1 + t)|u(t)|_{1,\infty} \leq c\delta + c(1 + t) \int_0^t (1 + \tau)^{-1} |u(\tau)|_\infty^{p-1} \| u(\tau) \|_s^2 d\tau
\]
\[
\leq c\delta + c\delta^2 K(T)^{p-1} e^{cK(t)^p},
\]
for \( t \in [0,T] \).

We choose \( \delta > 0 \), small enough, such that the function \( x \mapsto c\delta + c\delta^2 x^{p-1} e^{cx^p} - x \), has a positive zero. Let \( R = R(\delta) \) the first positive zero of this function. Then, the estimates shown above imply that \( K(T) \leq R \). From the fact that the set of solutions is invariant under transformation \( (t,x,y) \rightarrow (-t,-x,-y) \) and using an extension argument the theorem is obtained. \( \square \)

As corollary one has the following interesting theorem.

**Theorem 5.6.** Under the hypotheses of the preceding theorem, there exists \( \phi_\pm \in H^s \) such that
\[
\| u(t) - P(-t)\phi_\pm \|_r \rightarrow 0,
\]
as \( t \rightarrow \pm \infty \), for \( r \in [s - 1, s) \).

**Proof.** See [21]. \( \square \)

**References**

[1] Ablowitz, M. J., and Clarkson, P. A. *Solitons, nonlinear evolution equations and inverse scattering*, vol. 149 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1991.

[2] Ablowitz, M. J., and Segur, H. Long internal waves in fluids of great depth. *Stud. Appl. Math.* 62, 3 (1980), 249–262.

[3] Benjamin, T. B. Internal waves of permanent form in fluids of great depth. *J. Fluid Mech.* 29 (1967), 559–592.
[4] Boling, G., and Yongqian, H. Remarks on the generalized Kadomtsev-
Petviashvili equations and two-dimensional Benjamin-Ono equations. Proc. Roy.
Soc. London Ser. A 452, 1950 (1996), 1585–1595.

[5] Calderón, A.-P. Commutators of singular integral operators. Proc. Nat. Acad.
Sci. U.S.A. 53 (1965), 1092–1099.

[6] Esfahani, A. Remarks on solitary waves of the generalized two dimensional
benjamin-ono equation. Appl. Math. Comput. 218, 2 (2011), 308–323.

[7] Íorio, Jr., R. J. On the Cauchy problem for the Benjamin-Ono equation.
Comm. Partial Differential Equations 11, 10 (1986), 1031–1081.

[8] Íorio, Jr., R. J. The Benjamin-Ono equation in weighted Sobolev spaces. J.
Math. Anal. Appl. 157, 2 (1991), 577–590.

[9] Kato, T. Quasi-linear equations of evolution, with applications to partial dif-
f erential equations. In Spectral theory and differential equations (Proc. Sympos.,
Dundee, 1974; dedicated to Konrad Jörgens), vol. 448 of Lecture Notes in Math.
Springer, Berlin, 1975, pp. 25–70.

[10] Kato, T. On the Korteweg-de Vries equation. Manuscripta Math. 28, 1-3 (1979),
89–99.

[11] Kato, T. On the Cauchy problem for the (generalized) Korteweg-de Vries
 equation. In Studies in applied mathematics, vol. 8 of Adv. Math. Suppl. Stud.
Academic Press, New York, 1983, pp. 93–128.

[12] Kato, T. Perturbation theory for linear operators. Classics in Mathematics.
Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[13] Kenig, C. E., and Koenig, K. D. On the local well-posedness of the Benjami-
ono and modified Benjamin-Ono equations. Math. Res. Lett. 10, 5-6 (2003),
879–895.

[14] Kenig, C. E., Ponce, G., and Vega, L. On the generalized Benjamin-Ono
equation. Trans. Amer. Math. Soc. 342, 1 (1994), 155–172.

[15] Kim, B. Three-dimensional solitary waves in dispersive wave systems. PhD
thesis, Massachusetts Institute of Technology. Dept. of Mathematics., Cambridge,
MA, 2006.

[16] Kobayasi, K. On a theorem for linear evolution equations of hyperbolic type.
J. Math. Soc. Japan 31, 4 (1979), 647–654.

[17] Koch, H., and Tzvetkov, N. On the local well-posedness of the Benjamin-
onequation in $H^s(\mathbb{R})$. Int. Math. Res. Not., 26 (2003), 1449–1464.

[18] López, G. P., and Soriano, F. H. On the existence and analycity of solitary
waves solutions to a two-dimesional Benjamin-Ono equation. To be submitted.
[19] Matsuno, Y. *Bilinear transformation method*, vol. 174 of *Mathematics in Science and Engineering*. Academic Press Inc., Orlando, FL, 1984.

[20] Milanês, A. Some results about a bidimensional version of the generalized BO. *Commun. Pure Appl. Anal.* 2, 2 (2003), 233–250.

[21] Milanês, A. *On some bidimensional versions of the generalized Benjamin-Ono equation*. PhD thesis, IMPA, Rio de Janeiro, Brasil, 2002.

[22] Ono, H. Algebraic solitary waves in stratified fluids. *J. Phys. Soc. Japan* 39, 4 (1975), 1082–1091.

[23] Ponce, G. On the global well-posedness of the Benjamin-Ono equation. *Differential Integral Equations* 4, 3 (1991), 527–542.

[24] Reed, M., and Simon, B. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[25] Tao, T. Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$. *J. Hyperbolic Differ. Equ.* 1, 1 (2004), 27–49.