Distinct Partitions and Two $q$-Binomial Summation Identities

M.J. Kronenburg

Abstract

The partition functions $P(n, m, p)$, the number of integer partitions of $n$ into exactly $m$ parts with each part at most $p$, and $Q(n, m, p)$, the number of integer partitions of $n$ into exactly $m$ distinct parts with each part at most $p$, are related by two double summation identities which follow from their generating functions. From these identities and some identities from an earlier paper, some other identities involving distinct partitions and two $q$-binomial summation identities are proved, and from these follow two combinatorial identities.

Keywords: $q$-binomial coefficient, integer partition function.

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1 Introduction

The following two $q$-binomial summation identities are proved, of which the first one is known [5].

$$\sum_{k=0}^{\lfloor n/2 \rfloor} q^{(n-2k)(n-2k-1)/2} \binom{m+1}{n-2k} \binom{m+k}{m} q^2 = \binom{m+n}{m} q$$  \hspace{1cm} (1.1)

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k(k-1)} \binom{m+n-2k}{m} \binom{m+1}{k} q^2 = q^{n(n-1)/2} \binom{m+1}{n} q$$  \hspace{1cm} (1.2)

When $q = 1$ these give the following two combinatorial identities, of which the first one is known as (3.24) in [4].

$$\sum_{k=0}^{n} \binom{m+1}{2k} \binom{m+n-k}{m} = \binom{m+2n}{m}$$  \hspace{1cm} (1.3)

$$\sum_{k=0}^{n} (-1)^k \binom{m+2k}{m} \binom{m+1}{n-k} = (-1)^n \binom{m+1}{2n}$$  \hspace{1cm} (1.4)

2 Definitions and Basic Identities

Let the coefficient of a power series be defined as:

$$[q^n] \sum_{k=0}^{\infty} a_k q^k = a_n$$  \hspace{1cm} (2.1)
Let $P(n)$ be the number of integer partitions of $n$, let $Q(n)$ be the number of integer partitions of $n$ into distinct parts, let $P(n, m)$ be the number of integer partitions of $n$ into exactly $m$ parts, and let $Q(n, m)$ be the number of integer partitions of $n$ into exactly $m$ distinct parts. Let $P(n, m, p)$ be the number of integer partitions of $n$ into exactly $m$ parts, each part at most $p$, and let $P^*(n, m, p)$ be the number of integer partitions of $n$ into at most $m$ parts, each part at most $p$, which is the number of Ferrer diagrams that fit in a $m$ by $p$ rectangle:

$$P^*(n, m, p) = \sum_{k=0}^{m} P(n, k, p) \quad (2.2)$$

Let the following definition of the $q$-binomial coefficient, also called the Gaussian polynomial, be given.

**Definition 2.1.** The $q$-binomial coefficient is defined by [1, 3]:

$$\binom{m + p}{m}_q = \prod_{j=1}^{m} \frac{1 - q^{p+j}}{1 - q^{j}} \quad (2.3)$$

The $q$-binomial coefficient is the generating function of $P^*(n, m, p)$ [1]:

$$P^*(n, m, p) = [q^n] \binom{m + p}{m}_q \quad (2.4)$$

In the earlier paper [7] it was proved that:

$$P^*(n, m, p) = P(n + m, m, p + 1) \quad (2.5)$$

For the $q$-binomial coefficient there is the following symmetry identity [7]:

$$\binom{m + p}{m}_q = \binom{m + p}{p}_q \quad (2.6)$$

### 3 Formulas Involving Distinct Partitions

Let $Q(n, m, p)$ be the number of integer partitions of $n$ into exactly $m$ distinct parts with each part at most $p$. In the earlier paper [7] it was proved that:

$$Q(n, m, p) = P(n - m(m - 1)/2, m, p - m + 1) \quad (3.1)$$

The generating functions for $P(n, m, p)$ and $Q(n, m, p)$ are identities (7.3) and (7.4) in [2]:

$$\prod_{j=1}^{p} \frac{1}{1 - qz^j} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p) q^n z^m \quad (3.2)$$

$$\prod_{j=1}^{p} (1 + qz^j) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p) q^n z^m \quad (3.3)$$
Theorem 3.1.

\[ P(n, m, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} Q(n - 2k, m - 2l, p)P(k, l, p) \quad (3.4) \]

Proof. Using \((1 + zq^j)(1 - zq^j) = (1 - z^2q^{2j})\):

\[ \frac{1}{\prod_{j=1}^{p} (1 - zq^j)} = \frac{\prod_{j=1}^{p} (1 + zq^j)}{\prod_{j=1}^{p} (1 - z^2q^{2j})} \quad (3.5) \]

Substituting the generating functions \((3.2)\) and \((3.3)\):

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p)q^n z^m = (\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p)q^n z^m)(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p)q^{2n} z^{2m}) \]

\[ = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} Q(n_1, m_1, p)P(n_2, m_2, p)q^{n_1+2n_2} z^{m_1+2m_2} \quad (3.6) \]

The coefficients on both sides must be equal, so \(n_1 + 2n_2 = n\) and \(m_1 + 2m_2 = m\), which is equivalent to \(n_1 = n - 2n_2\) and \(m_1 = m - 2m_2\):

\[ P(n, m, p) = \sum_{n_1=0}^{\lfloor n/2 \rfloor} \sum_{n_2=0}^{\lfloor m/2 \rfloor} \sum_{m_1=0}^{n_1+2n_2=n} \sum_{m_2=0}^{m_1+2m_2=m} Q(n_1, m_1, p)P(n_2, m_2, p) \]

\[ = \sum_{n_2=0}^{\lfloor n/2 \rfloor} \sum_{m_2=0}^{\lfloor m/2 \rfloor} Q(n - 2n_2, m - 2m_2, p)P(n_2, m_2, p) \quad (3.7) \]

Let \(Q^*(n, m, p)\) be the number of integer partitions of \(n\) into at most \(m\) distinct parts with each part at most \(p\), which is defined like \(P^*(n, m, p)\) in \((2.2)\):

\[ Q^*(n, m, p) = \sum_{k=0}^{m} Q(n, k, p) \quad (3.8) \]

From the previous theorem a relation between \(Q^*(n, m, p)\) and \(P(n, m, p)\) can be derived.

Theorem 3.2.

\[ P(n + m, m, p + 1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} Q^*(n - 2k, m - 2l, p)P(k, l, p) \quad (3.9) \]
Proof. Using the previous theorem and (2.5):

\[ P^*(n, m, p) = P(n + m, m, p + 1) = \sum_{h=0}^{m} P(n, h, p) \]
\[ = \sum_{h=0}^{m} \sum_{k=0}^{[n/2]} \sum_{l=0}^{[h/2]} Q(n - 2k, h - 2l, p)P(k, l, p) \]
\[ = \sum_{k=0}^{[n/2]} \sum_{l=0}^{[m/2]} \sum_{h=2l}^{m} Q(n - 2k, h - 2l, p)P(k, l, p) \]
\[ = \sum_{k=0}^{[n/2]} \sum_{l=0}^{[m/2]} Q^*(n - 2k, m - 2l, p)P(k, l, p) \]

(3.10)

Let \( P_{\text{most}}(n, p) \) be the number of integer partitions of \( n \) with each part at most \( p \). From (2.2), (2.5) and conjugation of Ferrer diagrams [7]:

\[ P_{\text{most}}(n, p) = \sum_{k=0}^{n} P(n, k, p) = P^*(n, n, p) = P(2n, n, p + 1) = P(n + p, p) \]  
(3.11)

Obviously \( P^*(n, m, p) = P_{\text{most}}(n, p) \) when \( m \geq n \). Let \( Q_{\text{most}}(n, p) \) be the number of integer partitions of \( n \) into distinct parts with each part at most \( p \), for which \( Q^*(n, m, p) = Q_{\text{most}}(n, p) \) when \( m \geq n \). When taking \( m = n \) in this theorem for nonzero summands \( l \leq k \) and therefore \( n - 2l \geq n - 2k \):

\[ P(n + p, p) = \sum_{k=0}^{[n/2]} Q_{\text{most}}(n - 2k, p)P(p + k, p) \]  
(3.12)

From this identity follows as a special case when taking \( p = n \), and using \( P(2n, n) = P(n) \) and from [6] \( P(n, m) = P(n - m) \) if \( 2m \geq n \) and therefore \( P(n + k, n) = P(k) \) if \( n \geq k \):

\[ P(n) = \sum_{k=0}^{[n/2]} Q(n - 2k)P(k) \]  
(3.13)

Theorem 3.3.

\[ Q(n, m, p) = \sum_{k=0}^{[n/2]} \sum_{l=0}^{[m/2]} (-1)^l P(n - 2k, m - 2l, p)Q(k, l, p) \]  
(3.14)

Proof. Using \((1 + izq^j)(1 - izq^j) = (1 + z^2q^{2j})\):

\[ \prod_{j=1}^{p}(1 + izq^j) = \frac{\prod_{j=1}^{p}(1 + z^2q^{2j})}{\prod_{j=1}^{p}(1 - izq^j)} \]  
(3.15)
Substituting the generating functions (3.2) and (3.3):

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p) i^m q^n z^m = (\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p) i^m q^n z^m)(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p) q^{2n} z^{2m})
\]

\[
= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} P(n_1, m_1, p) Q(n_2, m_2, p) i^{m_1+2n_2} z^{m_1+2m_2}
\]

The coefficients on both sides must be equal, so \(n_1 + 2n_2 = n\) and \(m_1 + 2m_2 = m\), which is equivalent to \(n_1 = n - 2n_2\) and \(m_1 = m - 2m_2\):

\[
i^n Q(n, m, p) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} i^{m_1} P(n_1, m_1, p) Q(n_2, m_2, p)
\]

\[
= \sum_{n_2=0}^{n/2} \sum_{m_2=0}^{m/2} i^{m-2m_2} P(n - 2n_2, m - 2m_2, p) Q(n_2, m_2, p)
\]

With \(i^{-2m_2} = (-1)^{m_2}\) the theorem is proved.

Using a similar derivation as in theorem 3.2 gives:

\[
Q^*(n, m, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l P(n + m - 2(k + l), m - 2l, p + 1) Q(k, l, p)
\]

Using a similar reasoning as above:

\[
Q(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor n/2 \rfloor} (-1)^l P(n - 2k) Q(k, l)
\]

4 Two q-Binomial Summation Identities

From theorem 3.1 and theorem 3.3 the following two q-binomial summation identities are proved, of which the first is known [5].

**Theorem 4.1.** For nonnegative integer \(n, m\):

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} q^{(n-2k)(n-2k-1)/2} \binom{m+1}{n-2k} q^{m+k} \binom{m+k}{m} q^2 = \binom{m+n}{m} q
\]

*Proof.* From theorem 3.1 using (2.3) and (3.1):

\[
P^*(n-m, m, p-1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} P^*(n-(m-2l)l-2l+1)/2-2k, m-2l, p-m+2l) P^*(k-l, l, p-1)
\]
Using (2.4):

\[
[q^{n-m}] \binom{m+p-1}{m} q = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} [q^{n-(m-2l)(m-2l+1)/2-2k}] \binom{p}{m-2l} \cdot [q^{k-l}] \binom{p+l-1}{l} q
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} [q^{n-2k}] q^{(m-2l)(m-2l+1)/2} \binom{p}{m-2l} q \cdot [q^k] q^l \binom{p+l-1}{l} q
\]

(4.3)

In theorem [3.1] it was used that:

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} a_{n-2k} b_k = [q^n] \sum_{k=0}^{\infty} a_k q^k \sum_{k=0}^{\infty} b_k q^{2k}
\]

(4.4)

so the summation over \( k \) can be done:

\[
[q^n] q^m \binom{m+p-1}{m} q = [q^n] \sum_{l=0}^{\lfloor m/2 \rfloor} q^{(m-2l)(m-2l+1)/2} \binom{p}{m-2l} q^l \binom{p+l-1}{l} q^2
\]

(4.5)

Because all coefficients \([q^n]\) are equal, the polynomials must be equal, and cancelling some powers of \( q \):

\[
\binom{m+p-1}{m} q = \sum_{l=0}^{\lfloor m/2 \rfloor} q^{(m-2l)(m-2l-1)/2} \binom{p}{m-2l} q \binom{p+l-1}{l} q^2
\]

(4.6)

Replacing \( m \) by \( n \) and \( l \) by \( k \) and \( p \) by \( m+1 \) and using (2.6) gives the theorem.

In this theorem replacing \( n \) by \( 2n \) and \( k \) by \( n-k \) gives:

\[
\sum_{k=0}^{n} q^{k(2k-1)} \binom{m+1}{2k} q \left( \binom{m+n-k}{m} q^2 \right) = \left( \binom{m+2n}{m} \right) q
\]

(4.7)

When \( q = 1 \) this is combinatorial identity (3.24) in [3]:

\[
\sum_{k=0}^{n} \binom{m+1}{2k} \binom{m+n-k}{m} = \binom{m+2n}{m}
\]

(4.8)

Theorem 4.2. For nonnegative integer \( n, m \):

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k(k-1)} \binom{m+n-2k}{m} \binom{m+1}{k} q^2 = q^{n(n-1)/2} \binom{m+1}{n} q
\]

(4.9)

Proof. From theorem [3.3] using (2.5) and (3.1):

\[
P^*(n-m(m+1)/2, m, p-m)
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l P^*(n-m-2k+2l, m-2l, p-1) P^*(k-l(l+1)/2, l, p-l)
\]

(4.10)
Using (2.4) and (2.6):

\[
[q^{n-m(m+1)/2}] \binom{p}{m}_q = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l [q^{n-2l}] \binom{m+p-2l-1}{m-2l}_q \cdot [q^{k-l(l+1)/2}] \binom{p}{l}_q
\]

As in theorem 4.1 the sum over \(k\) can be done:

\[
[q^n] q^{m(m+1)/2} \binom{p}{m}_q = [q^n] \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l q^{m-2l} \binom{m+p-2l-1}{p-1}_q q^{l(l+1)/2} \binom{p}{l}_q^2
\]

Because all coefficients \([q^n]\) are equal, the polynomials must be equal, and cancelling some powers of \(q\):

\[
q^{m(m-1)/2} \binom{p}{m}_q = \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l q^{l(l-1)} \binom{m+p-2l-1}{p-1}_q q^{l(l+1)/2} \binom{p}{l}_q^2
\]

Replacing \(m\) by \(n\) and \(l\) by \(k\) and \(p\) by \(m+1\) gives the theorem. \(\square\)

In this theorem taking \(q = 1\) and replacing \(n\) by \(2n\) and \(k\) by \(n-k\) gives the combinatorial identity:

\[
\sum_{k=0}^{n} (-1)^k \binom{m+2k}{m} \binom{m+1}{n-k} = (-1)^n \binom{m+1}{2n}
\]

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