EXTENDED REDUCED-FORM FRAMEWORK FOR NON-LIFE INSURANCE

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Abstract

In this paper we propose a general framework for modeling an insurance liability cash flow in continuous time, by generalizing the reduced-form framework for credit risk and life insurance. In particular, we assume a nontrivial dependence structure between the reference filtration and the insurance internal filtration. We apply these results for pricing and hedging non-life insurance liabilities in hybrid financial and insurance markets, while taking into account the role of inflation under the benchmarked risk-minimization approach. This framework offers at the same time a general and flexible structure, and an explicit and treatable pricing-hedging formula.

Keywords: Non-life insurance; reduced-form framework; marked point process; benchmark approach; filtration dependence; market-consistent valuation; risk mitigation

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1. Introduction

In this paper we propose a general framework for modeling an insurance liability cash flow in continuous time, by extending the classic reduced-form setting for credit risk and life insurance. In particular, we consider a nontrivial dependence structure between the reference filtration \( \mathcal{F} \) and insurance internal filtration \( \mathcal{H} \). The global information flow available to the insurance company is represented by \( \mathcal{G} = \mathcal{F} \vee \mathcal{H} \). In this way, we obtain for the first time a framework in continuous time for non-life insurance, where filtration dependence is taken into account. In view of the development of insurance-linked derivatives, which offer the possibility of transferring insurance risks to the financial market, this bottom-up modeling approach can be used for pricing and hedging both life and non-life insurance liabilities in hybrid financial and insurance markets. As an application of the general framework structure, we derive pricing and hedging formulas for non-life insurance claims by taking into account the role of inflation under the benchmarked risk minimization approach, as introduced in [34], [5], and [10].

Historically, the mathematical modeling of life and non-life insurance liabilities in continuous time is quite asymmetric, and risk mitigation of non-life portfolios via asset allocation is scarcely practiced. While there are many recent works concerning life insurance (see e.g. [28],...
non-life insurance is often studied in discrete time and/or state space (see e.g. [23], [24], and [18]). We refer to e.g. [36] for a unified framework for life and non-life insurance in discrete time. Mathematical frameworks for non-life insurance in continuous time can be found in e.g. [27], [16], [3], [32], [31], and [35]. However, these settings do not consider a nontrivial dependence structure between reference filtration and insurance internal filtration. In particular, in e.g. [32] and [31], the insurance internal filtration is not distinguished from the reference filtration, and in e.g. [27], [16], and [3], reference and insurance internal filtrations are assumed to be independent. The importance of considering a nontrivial dependence structure between filtrations, which represent different information flows in a hybrid market, is discussed in [4] in view of the recent introduction of insurance-linked derivatives. Derivatives based on occurrence intensity index, such as mortality derivatives, weather derivatives, etc., can play an important role in mitigating risks of insurance companies in the case of life and non-life insurance business. In particular, non-catastrophe non-life insurance (see e.g. [12] for the distinction between catastrophe and non-catastrophe insurance), which includes car insurance, theft insurance, home insurance, etc., as opposed to catastrophe non-life insurance, covers high-probability low-cost events, and is often neglected by the literature. This paper aims to fill this gap, i.e. to identify the role of non-life related hybrid products, to contribute to their design, and to provide analytical results which can be used for the non-life insurance reserving problem and the valuation and hedging of non-life insurance portfolios by trading hybrid non-life products, which are currently not common but may become attractive in the future.

The recent literature on non-life insurance in continuous time (see e.g. [3], [32], [31], and [35]) commonly assumes the insurance internal information flow as given by the natural filtration of a marked point process, which describes the insurance claim movement. Pricing and hedging formulas are then obtained by using the compensator of this marked point process. However, as we discuss in Section 4, this approach cannot be always followed in the case of multiple filtrations with nontrivial dependence. Indeed, with respect to a generic filtration, it is not always true that there exists a marked point process with a given compensator, and the compensator does not always uniquely determine the law of the process. To overcome these difficulties, we propose a new framework, which uses a direct approach as in Sections 5.1 and 9.1.2 of [11] and allows an explicit bottom-up construction to treat more general filtrations. We note that, when our general framework is reduced to the case of life insurance, the compensator approach and the direct modeling approach coincide; see the discussion in e.g. [11] for the classic reduced-form framework.

More precisely, in our new framework we consider a homogeneous insurance portfolio with \( n \) claims. We assume that the reference filtration \( \mathbb{F} \) includes information related to the financial market and to environmental, social, and economic indicators. Following the classic non-life insurance modeling approach as in e.g. [1] and [26], we assume the insurance internal filtration \( \mathbb{H} \), which represents the internal information of an insurance company given by the claim movements, to be generated by a family of marked point processes describing sequences of reporting times and associated losses. As is typical in the case of non-life insurance, accident times and their related damages are unknown until the moment of reporting. We are able to capture these features and at the same time to introduce a dependence structure between the filtrations \( \mathbb{F} \) and \( \mathbb{H} \) by providing a nontrivial extension of the classic reduced-form framework. In particular, we model accident times as \( \mathbb{F} \)-conditionally independent random variables with a common \( \mathbb{F} \)-adapted intensity process \( \mu \). We note that in this way, we assume that intensity of accident times \( \mu \) may be influenced by external factors and economic indicators. Random
delay between accident time and the first reporting is modeled in the first mark, and subsequent
development of the claim is modeled by a time shift of an independent marked point process
with respect to the first reporting. This structure includes the life insurance case and allows
us to obtain analytical valuation formulas, which can be expressed in term of the accident
intensity $\mu$, the delay distribution, and the updating distribution, as illustrated in the prelimi-
nary calculations in Section 3. We then apply these results for pricing and hedging insurance
liabilities in a hybrid market under the benchmarked risk-minimization approach of [34], [5],
and [10]. The hybrid nature of the combined market is given by the (hypothetical) presence
of derivatives related to the intensity process $\mu$ on the financial market and by the influence
of inflation and benchmark portfolios in the valuation and hedging of insurance liabilities.
This gives insight into the role of non-life-insurance-linked financial products, which might
be attractive for non-life insurance companies to mitigate and transfer non-life underwriting
risks, in a similar manner to the role of mortality or longevity derivatives for life insurance
companies.

In summary, the main contribution of this paper is to propose a general framework for
modeling an insurance liability cash flow in continuous time, taking into account filtration
dependence, which includes both the non-life and the life insurance case. This approach allows
us to obtain liability cash flow valuation formulas for non-life insurance portfolios in a general
setting, as well as for the pricing of (possible) hybrid financial products written as a bet on
non-life insurance events or portfolios. Although these hybrid products are uncommon in the
present market, our framework could help to identify their role and contribute to their design
and valuation.

This paper is organized as follows. In Section 2 we construct a general framework for insur-
ance liability cash flow in continuous time under a nontrivial dependence between the reference
and the insurance internal filtrations, applicable both to life and non-life insurance, and give
a brief comparison with the existing insurance frameworks in the literature. In Section 3 we
give some useful preliminary valuation results in this setting. In Section 4 we discuss the com-
penator approach. In Section 5 we describe the hybrid nature of the combined market and
derive the real-world pricing formula and benchmarked risk-minimizing strategy for non-life
insurance claims.

2. General framework

In this section we construct a general framework for modeling an insurance liability cash
flow. We consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, where $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$, $\mathcal{G} = \mathcal{G}_\infty$, and
$\mathbb{G}_0$ is trivial.

We assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{H} := (\mathcal{H}_t)_{t \geq 0}$ are filtrations represent-
ing respectively a reference information flow and the internal information flow only available
to the insurance company. Hence $\mathbb{G}$ describes the global information flow available to
the insurance company. The reference filtration $\mathbb{F}$ typically includes information related to
the financial market and to environmental, political, and social indicators. While we do not spec-
ify the structure of the reference filtration $\mathbb{F}$, we assume that the insurance internal filtration
$\mathbb{H}$ is generated by a family of marked point processes, representing the times and amounts of
losses of the insurance portfolio, as in e.g. [1], [22], [29], and [30]. The filtrations $\mathbb{F}$ and $\mathbb{H}$ are
not supposed be independent. Without loss of generality, we assume that all filtrations satisfy
the conditions of completeness and right-continuity. If not otherwise specified, all relations
in this paper hold in the $\mathbb{P}$-almost sure (a.s.) sense. For detailed background on marked point
processes we refer to e.g. [25], [15], and [20]. In the following we use the classic terminology of non-life insurance (see e.g. [37] and [33]), and specify the filtration $\mathbb{H}$ as follows.

We consider an insurance portfolio with $n$ policies. For the $i$th policy, with $i = 1, \ldots, n$, the insurance company is typically informed about the accident that occurred at a random time $\tau_i^0$ only after a random delay $\theta_i$, which may be very long, especially in the case of non-life insurance. Once the accident is reported at time $\tau_i^1$, where

$$\tau_i^1 := \tau_i^0 + \theta_i,$$

the accident time $\tau_i^0$, the reporting delay $\theta_i$, and the impact size of the accident, described by a nonnegative random variable $X_i^1$, all become available information. In particular, we assume that for all $i = 1, \ldots, n$, $\tau_i^0 > 0$ P-a.s.

Let $\mathbb{N}_0$ be the set of natural numbers without zero. We describe the $i$th insurance policy movement by a marked point process $(\tau_i^j, \Theta_i^j)_{j \in \mathbb{N}_0}$ with two-dimensional nonnegative marks. That is, the sequence $(\tau_i^j)_{j \in \mathbb{N}_0}$ is a point process, where

$$\tau_i^j : (\Omega, \mathcal{G}, P) \to (\mathbb{R}_+, B(\mathbb{R}_+)), \quad j \in \mathbb{N}_0,$$

and $(\Theta_i^j)_{j \in \mathbb{N}_0}$ is a sequence of two-dimensional nonnegative random variables, with

$$\Theta_i^j : (\Omega, \mathcal{G}, P) \to (\mathbb{R}_+^2, B(\mathbb{R}_+^2)), \quad j \in \mathbb{N}_0.$$ For every $j \in \mathbb{N}_0$, the random time $\tau_i^j$ describes the reporting time of the $j$th event related to the $i$th policy. The mark components $\Theta_i^j$ describe the reporting delay and the impact size of the corresponding event, respectively, which are known only if the event is reported. More precisely, we set

$$\tau_i^1 \quad \text{with mark} \quad \Theta_i^1 = (\theta_i, X_i^1),$$

and

$$\tau_i^{j+1} = \tau_i^j + \tilde{\tau}_i^j \quad \text{with mark} \quad \Theta_i^{j+1} = \left(0, X_i^{j+1}\right) := \left(0, \tilde{X}_i^j\right),$$

for $j \geq 1$, where $(\tilde{\tau}_i^j, \tilde{X}_i^j)_{j \in \mathbb{N}_0}$ is an auxiliary marked point process, which describes updating and development after the first reporting at $\tau_i^1$. For the sake of simplicity, we assume here that only the first reporting delay is different from zero, since in this paper we focus on modeling the first accident times $\tau_i^0$ and their relation to the reference filtration. However, our setting can be easily generalized by considering nonzero random delays in (2.3). We assume that the marked point process $(\tilde{\tau}_i^j, \tilde{X}_i^j)_{j \in \mathbb{N}_0}$ is simple, i.e.

$$\lim_{j \to \infty} \tilde{\tau}_i^j = \infty,$$

and $\tilde{\tau}_i^j < \tilde{\tau}_i^{j+1}$, if $\tilde{\tau}_i^j < \infty$, and satisfies the following integrability condition:

$$\mathbb{E} \left[ \sum_{j=1}^{\infty} 1_{\{\tilde{\tau}_i^j \leq t\}} \tilde{X}_i^j \right] < \infty \quad \text{for all} \ t \geq 0,$$

for $i = 1, \ldots, n$. In particular, the random times $(\tau_i^j)_{j \in \mathbb{N}_0}$ are strictly ordered in case of finite value:

$$\tau_i^1 < \tau_i^2 < \cdots < \tau_i^j < \tau_i^{j+1} < \cdots, \quad i = 1, \ldots, n.$$
Note that we may have $\infty = \tau^j = \tau^j_{j+1} = \ldots$; in such a case, an infinite value stands for an event which never happens. For the sake of simplicity we assume also the following.

**Assumption 1.**

1. Homogeneous delay: the random delays $\theta^i$, $i = 1, \ldots, n$, have the same distribution.

2. Homogeneous development: the marked point processes $\left(\tilde{\tau}^i_j, \tilde{X}^i_j\right)_{j \in \mathbb{N}_0}$, $i = 1, \ldots, n$, have the same distribution.

3. Independent first mark: the first marks $X^i_1$, $i = 1, \ldots, n$, are mutually independent and independent of $F_\infty \vee \sigma(\tau^1_1) \vee \ldots \vee \sigma(\tau^n_n)$.

4. Independent delay: the random delays $\theta^i$, $i = 1, \ldots, n$, are mutually independent and independent of $F_\infty \vee \sigma(\left(\tau^1_0, X^1_1\right)) \vee \ldots \vee \sigma(\left(\tau^n_0, X^n_n\right))$.

5. Independent development: the marked point processes $\left(\tilde{\tau}^i_j, \tilde{X}^i_j\right)_{j \in \mathbb{N}_0}$, $i = 1, \ldots, n$, are mutually independent and independent of $F_\infty \vee \sigma(\left(\tau^1_1, X^1_1\right)) \vee \ldots \vee \sigma(\left(\tau^n_n, \theta^n, X^n_n\right))$.

We emphasize that the above assumptions cover sufficient generality. The homogeneity assumptions can be satisfied by appropriately subdividing the insurance portfolio into homogeneous groups of claims. The independence assumptions reflect the fact that reporting delays $\theta^i$, occurrences, and sizes of losses after the first reporting time, described by $\left(\tilde{\tau}^i_j, \tilde{X}^i_j\right)_{j \in \mathbb{N}_0}$, are typically idiosyncratic factors which are independent of each other and independent of the reference information. However, we introduce a dependence structure by modeling progressively measurable occurrence intensities of the accidents, as we will present in (2.13) and (2.14). This will reflect the assumption that the occurrence intensity of accidents can be deduced from the reference information flow represented by $F$, while further updates of accident events $\left(\tilde{\tau}^i_j, \tilde{X}^i_j\right)_{j \in \mathbb{N}_0}$ are typically insurance-portfolio-specific and are not available as third-party or reference information. We assume furthermore that the distribution of delay variables $\theta^i$, $i = 1, \ldots, n$, has the following structure.

**Assumption 2.** The common cumulative distribution function $G : [0, +\infty) \to [0, 1]$ of $\theta^i$, $i = 1, \ldots, n$, assigns probability $\alpha_0$ at 0 and has a density function $g$ for $x > 0$, i.e.,

$$G(x) = \alpha_0 + \int_0^x g(y)dy, \quad x \in \mathbb{R}_+.$$  \hfill (2.6)

According to the above assumption, the delays are nonnegative and may have a mixed distribution. In this way, we cover both the case of life insurance with $\theta^i \equiv 0$, i.e. $g = 0$, and the case of non-life insurance with non-null delays.

For every $i = 1, \ldots, n$, we define the marked cumulative process $N^i$ by

$$N^i(t, B)(\omega) := \sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}^i_j(\omega) \leq t\}} \mathbf{1}_{\{\theta^i_j(\omega) \in B\}},$$

for every $t \geq 0$, $B \in B(\mathbb{R}_+^2)$, $\omega \in \Omega$. The process $(N^i_t)_{t \geq 0}$ defined by

$$N^i_t := N^i(t, \mathbb{R}_+^2) = \sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}^i_j \leq t\}}, \quad t \geq 0,$$
is called the ground process associated to the marked point process. At any time $t \geq 0$, the random variable $N^i_t$ counts the number of occurrences of $\tau_j^i$ up to time $t$. In the literature, the name ‘marked point process’ sometimes refers to the process $N^i$. Indeed, there is a unique correspondence between the marked point process $\left( \tau_j^i, \Theta_j^i \right)_{j \in \mathbb{N}_0}$ and its marked cumulative process $N^i$. More precisely, $\tau_j^i$ can be uniquely defined by $N^i$ as

$$\{ \tau_j^i \leq t \} = \{ N^i_j \geq j \}, \quad (2.7)$$

for all $t \geq 0$, and $\Theta_j^i$ can be uniquely defined by $N^i$ as

$$\{ \Theta_j^i \in B \} = \bigcup_{k'=1}^{\infty} \bigcap_{K=K'}^{\infty} \bigcup_{k=1}^{\infty} \{ N^i_{(k-1)/2^K} = n - 1, N^i(k/2^K, B) - N^i((k - 1)/2^K, B) = 1 \}. \quad (2.8)$$

for all $B \in \mathcal{B} \left( \mathbb{R}^2_+ \right)$. See Equations (2.8), (2.9) of [20] and Lemma 2.2.2 of [25].

We consider the filtrations $\mathbb{H}_{i,1} := (\mathcal{H}^i_{t,1})_{t \geq 0}$ with

$$\mathcal{H}^i_{t,1} := \sigma \left( \mathbb{1}_{[\tau^i_{j \leq t}]} \mathbb{1}_{\{ (\theta^i, X^i_j) \in B \}}, 0 \leq s \leq t, \text{ for all } B \in \mathcal{B} \left( \mathbb{R}^2_+ \right) \right),$$

for all $t \geq 0$, and $\mathbb{H}_{i,j} := (\mathcal{H}^i_{t,j})_{t \geq 0}$, with

$$\mathcal{H}^i_{t,j} := \sigma \left( \mathbb{1}_{[\tau^i_{j \leq t}]} \mathbb{1}_{\{ X^i_j \in B \}}, 0 \leq s \leq t, \text{ for all } B \in \mathcal{B} \left( \mathbb{R}^2_+ \right) \right),$$

for all $t \geq 0$. It holds that

$$\mathcal{H}^i_{\infty} = \sigma (\tau^i_j) \lor \sigma (X^i_j) \text{ for } j > 1.$$

In particular, in view of (2.1) we have

$$\mathcal{H}^i_{\infty} = \sigma (\tau^i_0) \lor \sigma ((\theta^i, X^i_0)) = \sigma (\tau^i_0) \lor \sigma ((\theta^i, X^i_0)). \quad (2.9)$$

Let $\mathbb{H}^i := (\mathcal{H}^i_{t})_{t \geq 0}$ be the natural filtration of the marked cumulative process $N^i$, that is, for all $t \geq 0,$

$$\mathcal{H}^i_t = \sigma \left( N^i(s, B), 0 \leq s \leq t, \text{ for all } B \in \mathcal{B} \left( \mathbb{R}^2_+ \right) \right).$$

The internal information flow of the insurance company is described by the filtration $\mathbb{H} := (\mathcal{H}^i_{t})_{t \geq 0}$, where

$$\mathcal{H}^i_t := \mathcal{H}^i_{1,t} \lor \ldots \lor \mathcal{H}^i_{n,t}, \quad t \geq 0. \quad (2.10)$$

Similarly, for $i = 1, \ldots, n$, we call $\tilde{N}^i$ the corresponding marked cumulative processes associated to the marked point processes $\left( \tilde{\tau}_j^i, \tilde{X}_j^i \right)_{j \in \mathbb{N}_0}$ and $\tilde{\mathbb{H}}$ the corresponding filtration, respectively. Similarly, all other notation associated to these last processes will be marked by the symbol ‘$\sim$’.

**Lemma 1.** For every $i = 1, \ldots, n$, we have $\mathbb{H}^i = \bigvee_{j \in \mathbb{N}_0} \mathbb{H}^{i,j}$.

**Proof.** Clearly, we have

$$\mathcal{H}^i_t \subseteq \bigvee_{j \in \mathbb{N}_0} \mathcal{H}^{i,j}_t.$$
For the other inclusion, it is sufficient to show that for all $0 \leq s \leq t$ and $B \in \mathcal{B}(\mathbb{R}_+^2)$,

$$\{\tau_j^i \leq s\} \cap \{\Theta_j^i \in B\} \in \mathcal{H}_t^i.$$  

This follows directly from (2.7) and (2.8). □

We now introduce the following notation, which is useful in the sequel. For $i = 1, \ldots, n$, $j \in \mathbb{N}_0$, we define

$$\mathbb{H}^i_{\leq j} := \bigvee_{k \leq j} \mathbb{H}^i_{k}, \quad \mathbb{H}^i_{\geq j} := \bigvee_{k \geq j} \mathbb{H}^i_{k},$$

and similarly for $\mathbb{H}^i_{> j}$ and $\mathbb{H}^i_{< j}$. In particular, in the case of $j = 1$, we set $\mathcal{H}^i_{t, < 1} := \{\emptyset, \Omega\}$ for every $t \geq 0$. The following corollary is a direct consequence of Lemma 1.

**Corollary 1.** For every $i = 1, \ldots, n$, $j \in \mathbb{N}_0$, we have

$$\mathbb{H}^i = \mathbb{H}^i_{\leq j} \vee \mathbb{H}^i_{\geq j} = \mathbb{H}^i_{< j} \vee \mathbb{H}^i_{> j}.$$  

Similarly to the reduced-form setting for credit risk and life insurance, we now model the accident times $\tau_0^i$, $i = 1, \ldots, n$, and their relation to the reference filtration in the following way. We assume that random times $(\tau_j^i)_{i \in \mathbb{N}}$, $i = 1, \ldots, n$, are not $\mathcal{F}$-stopping times, i.e. for every $i$ and $j$, there is $t$ such that $\{\tau_j^i \leq t\} \notin \mathcal{F}_t$. As in Section 9.1.2 of [11], we set that accident times $\tau_0^i$, $i = 1, \ldots, n$, are such that for $t \in [0, \infty)$ and $s \in [0, t] \cap [0, \infty)$,

$$\mathbb{P}\left(\tau_0^i > s \mid \mathcal{F}_t\right) = \mathbb{P}\left(\tau_0^i > s \mid \mathcal{F}_s\right), \quad (2.11)$$

and for $l, k = 1, \ldots, n$ with $l \neq k$, $\tau_0^l$ and $\tau_0^k$ are $\mathcal{F}$-conditionally independent, i.e. if $t \in [0, \infty)$ and $r, s \in [0, t] \cap [0, \infty)$, we have

$$\mathbb{P}\left(\tau_0^l > r, \tau_0^k > s \mid \mathcal{F}_t\right) = \mathbb{P}\left(\tau_0^l > r \mid \mathcal{F}_t\right) \mathbb{P}\left(\tau_0^k > s \mid \mathcal{F}_t\right). \quad (2.12)$$

**Remark 1.** If we define $\mathcal{H}^i_{t, < 0} := \sigma\left(1_{\{\tau_0^i \leq s\}} : 0 \leq s \leq t\right)$, $i = 1, \ldots, n$, then the condition (2.11) is equivalent to

$$\mathbb{E}[X \mid \mathcal{F}_t] = \mathbb{E}[X \mid \mathcal{F}_s],$$

for each integrable $\mathcal{H}^i_{t, < 0}$-measurable random variable $X$. The condition (2.12) is equivalent to the $\mathcal{F}_t$-conditional independence between the $\sigma$-algebras $\mathcal{H}^i_{t, < 0}$ and $\mathcal{H}^k_{t, < 0}$.

Furthermore, if $F^i := (F_t^i)_{t \geq 0}$ is the $\mathcal{F}$-conditional cumulative process of $\tau_0^i$,

$$F_t^i := \mathbb{P}(\tau_0^i \leq t \mid \mathcal{F}_t), \quad t \geq 0,$$

we assume that there exists a locally integrable and $\mathcal{F}$-progressively measurable process $\mu^i := (\mu_t^i)_{t \geq 0}$ such that

$$e^{-\int_0^t \mu_s^id{ds}} = 1 - F_t^i \quad \text{for all } t \geq 0. \quad (2.13)$$

We define $\Gamma^i := (\Gamma_t^i)_{t \geq 0}$ as

$$\Gamma_t^i := \int_0^t \mu_s^id{ds}, \quad t \geq 0. \quad (2.14)$$
The process $\mu^i$ is called the intensity process of the random jump time $\tau^i_0$, and the process $\Gamma^i$ is called the hazard process of $\tau^i_0$. An explicit construction in Example 9.1.5 of [11] shows that for a given family of locally integrable $\mathbb{F}$-progressively measurable process $\mu^i$, $i = 1, \ldots, n$, it is always possible to construct random times $\tau^i_0$, $i = 1, \ldots, n$, such that $\Gamma^i$ is the hazard process of $\tau^i_0$ for every $i = 0, \ldots, n$, and all the assumptions above are satisfied. For the sake of simplicity, we assume that the insurance portfolio is homogeneous.

**Assumption 3.** The accident times $\tau^i_0$, $i = 1, \ldots, n$, have the same intensity process $\mu$.

Under this homogeneity condition, we denote the common $\mathbb{F}$-conditional cumulative function and hazard process respectively by $\mathbb{F}$ and $\Gamma$. The above assumption reflects the fact that, while the policy developments may not have a direct link to the information flow $\mathbb{F}$, the accident occurrences $\tau^i_0$, $i = 1, \ldots, n$, are influenced by some common external systematic risk factors described by the occurrence intensity $\mu$ and observable from filtration $\mathbb{F}$.

We now show how the general framework described above comprehends in a synthetic way both life and non-life insurance modeling, and compare our setting with the existing literature.

### 2.1. Life insurance

Life insurance policies typically do not have reporting delays and depend only on $\tau^i_0$, $i = 1, \ldots, n$, which actually represent the decease times. This can be included in our framework by setting $\theta^i \equiv 0$, $\tau^j_0 \equiv \infty$ for all $j > 1$, and $X^i_j \equiv 1$ for all $j \in \mathbb{N}_0$, and interpreting $\tau^i_0$ as the decease time of person $i$, where $i = 1, \ldots, n$. The filtration $\mathcal{G}$ is hence reduced to

$$
\mathcal{G} = \mathbb{F} \lor \mathcal{H}^1 \lor \ldots \lor \mathcal{H}^n,
$$

where

$$
\mathcal{H}^i_t = \sigma \left( 1 \{ \tau^i_0 \leq s \}, 0 \leq s \leq t \right), \quad t \geq 0, \quad i = 1, \ldots, n.
$$

In particular, the $\mathbb{F}$-progressively measurable process $\mu$ is interpreted as mortality intensity in this context. The financial market is typically assumed to include mortality- or longevity-linked derivatives, such as longevity bonds, which pay off the longevity index value $e^{-\int_0^T \mu_s ds}$ at maturity $T$.

Life insurance within a hybrid market in this setting has been intensively studied in the literature; see e.g. [2], [9], [7], and [10].

### 2.2. Non-life insurance

The framework in Section 2 in its full generality describes the case of non-life insurance. Indeed, non-life insurance policies typically have reporting delay, i.e. $\theta^i \neq 0$, which can sometimes come to several years. For the $i$th policy, we interpret $X^i_j$ as the payment amount at the $j$th random time $\tau^i_j$; the exact accident time $\tau^i_0$ and first payment amount $X^i_1$ are known only after reporting at time $\tau^i_1$. Further developments may occur after the first reporting and before the settlement of the claim. The total number of developments $(\tau^i_j)_{j \in \mathbb{N}_0}$ is unknown, as well as the amount of the corresponding payments $(X^i_j)_{j \in \mathbb{N}_0}$. The accident time $\tau^i_0$ admits an $\mathbb{F}$-progressively measurable intensity process $\mu$ related to the underlying risk. If liquidly traded derivatives related to the $\mu$ process are available on the financial market, they could be used for hedging systematic risks related to non-life portfolios.

The above-described setting gives a nontrivial extension of the underlying frameworks in e.g. [16], [3], [32], and [31]. In e.g. [16] and [3], the reference filtration $\mathbb{F}$ is assumed to be
independent of the insurance internal filtration $\mathcal{H}$ generated by the non-life portfolio movement. The interaction between the financial and the insurance markets is thus captured only by means of interest rate and/or inflation risk. By contrast, in e.g. [32] and [31], it is assumed that $\mathcal{G} = \mathcal{H} = \mathcal{F}$. Financial products used for hedging purposes are in these cases liquidly traded catastrophe derivatives and/or reinsurance contracts, which share similar risk structures of the target non-life insurance portfolio. Considering a more general setting, where $\mathcal{F}$ and $\mathcal{H}$ are not necessarily independent or equal, is technically challenging, as we discuss in Section 4. However, the extended reduced-form framework proposed in this paper allows us to consider a nontrivial dependence structure between the filtrations $\mathcal{F}$ and $\mathcal{H}$ and still to derive analytical pricing formulas for non-life insurance liabilities. Furthermore, besides the financial instruments used in e.g. [16], [3], [32], and [31], it is possible to use intensity-related derivatives as a hedging instrument; see the discussion in Section 5. This last type of derivative is still not common but is potentially attractive for covering systematic risks arising from non-catastrophe non-life insurance.

3. Valuation formulas

In this section, we state several results within the framework presented in Section 2, i.e. under the structure of Assumptions 1, 2, and 3. We follow [11, Section 5.1] for the presentation. These preliminary calculations are fundamental for providing the pricing formulas for non-life insurance claims in Section 5.1.

We start with an extension of the relation (2.11) and the $\mathcal{F}$-independence (2.12) of $\tau^l_0, i = 1, \ldots, n$. In particular, if these relations hold for the filtrations $\mathcal{H}^i_0, i = 1, \ldots, n$, then they also hold for the filtrations $\mathcal{H}^i, i = 1, \ldots, n$.

**Lemma 2.** For any $t \in [0, \infty)$ and $l, k = 1, \ldots, n$ with $l \neq k$, the $\sigma$-algebras $\mathcal{H}^l_t$ and $\mathcal{H}^k_t$ are $\mathcal{F}_t$-independent. Furthermore, for any $0 \leq s \leq t < \infty$ and $i = 1, \ldots, n$, if $X$ is $\mathcal{H}^i_s$-measurable, then $\mathbb{E}[X | \mathcal{F}_t] = \mathbb{E}[X | \mathcal{F}_s]$.

**Proof.** The proof is straightforward in view of Lemma 1, Remark 1, and the independence conditions in Assumption 1. We outline the main steps for the first part of the lemma for completeness; the second part can be shown in a similar manner. See also the proofs of Lemma 3.2.1 and Lemma 3.2.2 in [38].

In view of Lemma 1, (2.2), and (2.3), for any $t \in [0, \infty)$ and $l, k = 1, \ldots, n$ with $l \neq k$, the $\sigma$-algebras $\mathcal{H}^l_t$ and $\mathcal{H}^k_t$ are $\mathcal{F}_t$-independent if and only if

$$
\mathbb{E}\left[\mathbf{1}\{\tau^l_0 + \theta^l + \bar{\tau}^l_\theta \leq s\} \mathbf{1}\{\tilde{X}^l_\theta \in B^l\} \mathbf{1}\{\tau^k_0 + \theta^k + \bar{\tau}^k_\theta \leq r\} \mathbf{1}\{\tilde{X}^k_\theta \in B^k\} | \mathcal{F}_t\right] = \mathbb{E}[X | \mathcal{F}_t | \mathcal{F}_s],
$$

where $s, r \in [0, t] \cap [0, \infty)$ (note that $t$ may assume the value $\infty$), $B^l, B^k \in \mathcal{B}(\mathbb{R}_+)$, and $p, q \in \mathbb{N}_0$. Without loss of generality, we take $p \neq 1$ and $q \neq 1$. We consider the following deterministic functions:

$$
f^l(x) := \mathbb{E}\left[\mathbf{1}\{\theta^l + \bar{\tau}^l_\theta \leq x - s\} \mathbf{1}\{\tilde{X}^l_\theta \in B^l\}\right], \quad f^l(x) = f^l(x) \mathbf{1}_{[x \leq s]},
$$

$$
f^k(x) := \mathbb{E}\left[\mathbf{1}\{\theta^k + \bar{\tau}^k_\theta \leq r - x\} \mathbf{1}\{\tilde{X}^k_\theta \in B^k\}\right], \quad f^k(x) = f^k(x) \mathbf{1}_{[x \leq r]}.
$$
Hence, \( f^l(\tau_0^l) \) and \( f^k(\tau_0^k) \) are respectively \( \mathcal{H}_t^{l,0} \) - and \( \mathcal{H}_t^{k,0} \)-measurable. This together with Remark 1 and the independence conditions in Assumption 1 leads to the \( \mathcal{F}_t \)-independence of \( \mathcal{H}_t^l \) and \( \mathcal{H}_t^k \).

\[
\mathbb{E}
\left[
\mathbf{1}_{\{\tau_0^l+\theta^l+\tau_0^k \leq s\}}
\mathbf{1}_{\{\tilde{x}_0^l \in B^l\}}
\mathbf{1}_{\{\tau_0^k+\theta^k+\tau_0^l \leq r\}}
\mathbf{1}_{\{\tilde{x}_0^k \in B^k\}}
\middle| \mathcal{F}_t
\right]
= \mathbb{E}
\left[
\mathbf{1}_{\{\tau_0^l+\theta^l+\tau_0^k \leq s\}}
\mathbf{1}_{\{\tilde{x}_0^l \in B^l\}}
\mathbf{1}_{\{\tau_0^k+\theta^k \leq r\}}
\mathbf{1}_{\{\tilde{x}_0^k \in B^k\}}
\middle| \mathcal{F}_t
\right]
\mathbb{E}
\left[
\mathbf{1}_{\{\tau_0^k+\theta^k+\tau_0^l \leq r\}}
\mathbf{1}_{\{\tilde{x}_0^k \in B^k\}}
\middle| \mathcal{F}_t
\right].
\]

As a consequence of Lemma 3, the \( \mathbb{G} \)-conditional expectation can be reduced to \( (\mathbb{F} \lor \mathbb{H}^{(l)}) \)-conditional expectation in most cases.

**Corollary 2.** If \( 0 \leq t \leq T < \infty \), and \( Y \) is an integrable \( (\mathcal{F}_T \lor \mathcal{H}_T^{l,}) \)-measurable random variable, then

\[
\mathbb{E}[Y | \mathcal{G}_t] = \mathbb{E}[Y | \mathcal{F}_t \lor \mathcal{H}_t^l].
\]

**Proof.** It is sufficient to prove the statement for the indicator functions of the form \( Y = A_1B \) where \( A \in \mathcal{F}_T \) and \( B \in \mathcal{H}_T^l \). Given \( C \in \mathcal{F}_t \), \( D \in \mathcal{H}_T^l \), \( j = 1, \ldots, n \), it follows from Lemma 2 that

\[
\int_{C \cap D_1 \cap \ldots \cap D^n} A_1B dP = \int_{C \cap D_1 \cap \ldots \cap D^n} \mathbb{E}[A_1B | \mathcal{F}_t \lor \mathcal{H}_T^l] dP.
\]

Details can also be found in the proof of Corollary 3.2.3 in [38].

Another important consequence of Lemma 2 is the so-called \( H \)-hypothesis between the filtrations \( \mathbb{F} \) and \( \mathbb{G} \), i.e. the property that every \( \mathbb{F} \)-martingale is also a \( \mathbb{G} \)-martingale.

**Corollary 3.** The \( H \)-hypothesis holds between the filtrations \( \mathbb{F} \) and \( \mathbb{G} \).

**Proof.** By Lemma 6.1.1 of [11], it is equivalent to the \( H \)-hypothesis between two filtrations \( \mathbb{F} \subseteq \mathbb{G} \) that for any \( t \geq 0 \) and any bounded, \( \mathcal{G}_t \)-measurable random variable \( \eta \), it holds that

\[
\mathbb{E}[\eta | \mathcal{F}_\infty] = \mathbb{E}[\eta | \mathcal{F}_t].
\]

It is sufficient to prove (3.1) for indicator functions of the form \( A_1B_1^1 \ldots B_1^n \), where \( A \in \mathcal{F}_t \), \( B_i^e \in \mathcal{H}_T^l \), \( i = 1, \ldots, n \). This can be achieved by applying Lemma 2 multiple times to get

\[
\mathbb{E}
\left[
A_1B_1^1 \ldots B_1^n | \mathcal{F}_\infty
\right] = \mathbb{E}
\left[
A_1B_1^1 \ldots B_1^n | \mathcal{F}_t
\right].
\]

Now we would like to derive some more explicit representations. We note that for every integrable random variable \( Y \), \( t \geq 0 \), \( i = 1, \ldots, n \), and \( j \in \mathbb{N}_0 \), we have the decomposition

\[
\mathbb{E}[Y | \mathcal{H}_T^l \lor \mathcal{F}_t] = \mathbb{E}
\left[
\mathbf{1}_{\{\tau_i^l > t\}} Y | \mathcal{H}_T^l \lor \mathcal{F}_t
\right] + \mathbb{E}
\left[
\mathbf{1}_{\{\tau_i^l \leq t\}} Y | \mathcal{H}_T^l \lor \mathcal{F}_t
\right].
\]
In the following we will evaluate separately the two components on the right-hand side of (3.2). The following lemma is important for deriving a representation of the first component.

**Lemma 3.** For any $t \geq 0$, $i = 1, \ldots, n$, and $j \in \mathbb{N}_0$, we have

$$\mathcal{H}_t^i \vee \mathcal{F}_t \subseteq G_t^{i,j},$$

where

$$G_t^{i,j} := \left\{ A \in \mathcal{G} : \exists C \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t, A \cap \{ \tau_j^{i} > t \} = C \cap \{ \tau_j^{i} > t \} \right\}. \quad (3.3)$$

**Proof.** By Corollary 1, it holds that

$$\mathcal{H}_t^i = \mathcal{H}_t^{i,<j} \vee \mathcal{H}_t^{i,j}.$$

Hence, it is sufficient to show that both $\mathcal{H}_t^{i,j}$ and $\mathcal{H}_t^{i,j} \vee \mathcal{F}_t$ belong to $G_t^{i,j}$. In the first case, if $i > 1$ and $A = \left\{ \tau_k^i \leq s \right\} \cap \{ X_k^i \in B \}$ for some $k \geq j, 0 \leq s \leq t$, and $B \in \mathcal{B}(\mathbb{R})$, we take $C = \emptyset$; and similarly for $i = 1$ and $A = \left\{ \tau_k^1 \leq s \right\} \cap \{ (\theta_k, X_k^1) \in B \}$ for $k \geq j, 0 \leq s \leq t$, and $B \in \mathcal{B}(\mathbb{R}^2)$. In the second case, if $A \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t$, we take $C = A$. \(\Box\)

The following proposition gives two representations of the first component on the right-hand side of (3.2). The representation (3.4) is analogous to Lemma 5.1.2 in [11]; the representation (3.5) is new and will be used for our further discussion.

**Proposition 1.** For any $t \geq 0$, $i = 1, \ldots, n$, $j \in \mathbb{N}_0$ and any integrable $\mathcal{G}$-measurable random variable $Y$, we have

$$\mathbb{E}\left[ 1_{\{\tau_j^i > t\}} Y \mid \mathcal{H}_t^i \vee \mathcal{F}_t \right] = 1_{\{\tau_j^i > t\}} \frac{\mathbb{E}\left[ 1_{\{\tau_j^i > t\}} Y \mid \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t \right]}{P(\tau_j^i > t \mid \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t)} \quad (3.4)$$

$$= 1_{\{\tau_j^i > t\}} \mathbb{E}\left[ Y \mid \mathcal{H}_t^{i,j} \vee \mathcal{F}_t \right]. \quad (3.5)$$

**Proof.** The equality (3.4) is equivalent to

$$\mathbb{E}\left[ 1_{\{\tau_j^i > t\}} Y \mid \mathcal{H}_t^{i,j} \vee \mathcal{F}_t \right] = 1_{\{\tau_j^i > t\}} \mathbb{E}\left[ Y \mid \mathcal{H}_t^{i,j} \vee \mathcal{F}_t \right].$$

We note that the right-hand side is $(\mathcal{H}_t^i \vee \mathcal{F}_t)$-measurable. Hence, it suffices to show that for any $A \in \mathcal{H}_t^i \vee \mathcal{F}_t$,

$$\int_A 1_{\{\tau_j^i > t\}} Y \cdot P(\tau_j^i > t \mid \mathcal{H}_t^{i,j} \vee \mathcal{F}_t) \, dP = \int_A 1_{\{\tau_j^i > t\}} \mathbb{E}\left[ 1_{\{\tau_j^i > t\}} Y \mid \mathcal{H}_t^{i,j} \vee \mathcal{F}_t \right] \, dP.$$

By Lemma 3, there is an event $C \in \mathcal{H}_t^{i,j} \vee \mathcal{F}_t$ such that

$$A \cap \{\tau_j^i > t\} = C \cap \{\tau_j^i > t\};$$
hence
\[
\int_A \mathbf{1}_{\{\tau_j^t > t\}} Y \, P \left( \tau_j^t > t \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee \mathcal{F}_t \right) \, dP \\
= \int_C \mathbf{1}_{\{\tau_j^t > t\}} Y \, P \left( \tau_j^t > t \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee \mathcal{F}_t \right) \, dP \\
= \int_C \mathbb{E} \left[ \mathbf{1}_{\{\tau_j^t > t\}} Y \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee \mathcal{F}_t \right] \mathbb{E} \left[ \mathbf{1}_{\{\tau_j^t > t\}} \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee \mathcal{F}_t \right] \, dP \\
= \int_C \mathbb{E} \left[ \mathbf{1}_{\{\tau_j^t > t\}} Y \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee \mathcal{F}_t \right] \, dP \\
= \int_A \mathbf{1}_{\{\tau_j^t > t\}} \mathbb{E} \left[ \mathbf{1}_{\{\tau_j^t > t\}} Y \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee \mathcal{F}_t \right] \, dP.
\]

The equality (3.5) can be proved in the same way. We only need to observe that
\[
\mathcal{G}^i_{t,\leq j} \subseteq \left\{ A \in \mathcal{G} : \exists C \in \mathcal{H}_{t}^{i,\leq j} \vee \mathcal{F}_t, A \cap \{ \tau_j^t > t \} = C \cap \{ \tau_j^t > t \} \right\}.
\]

Hence, the \( \sigma \)-algebra \( \mathcal{H}_{t}^{i,\leq j} \) in (3.4) can be replaced by \( \mathcal{H}_{t}^{i,\leq j} \). This concludes the proof. \( \square \)

Now we focus on the second component on the right-hand side of (3.2). The following lemma gives a slightly more general result.

**Lemma 4.** For any \( t \geq 0, \, i = 1, \ldots, n, \, j \in \mathbb{N}_0 \), any \( \sigma \)-algebra \( A \subseteq \mathcal{G} \), and any integrable \( \mathcal{G} \)-measurable random variable \( Y \), we have
\[
\mathbb{E} \left[ \mathbf{1}_{\{\tau_j^t \leq t\}} Y \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee A \right] = \mathbb{E} \left[ \mathbf{1}_{\{\tau_j^t \leq t\}} Y \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee A \right].
\]

**Proof.** We note that the left-hand side is \( \left( \mathcal{H}_{t}^{i,\leq j} \vee A \right) \)-measurable. Since the marked point process \( (\tau_j^t, \Theta_j^t)_{j \in \mathbb{N}_0} \) is simple, i.e. the strict monotonicity (2.5) holds, if \( A \in \mathcal{H}_{t}^{i,\leq j} \vee A \), then \( A \cap \{ \tau_j^t \leq t \} \in \mathcal{H}_{t}^{i,\leq j} \vee A \), and
\[
\int_A \mathbf{1}_{\{\tau_j^t \leq t\}} Y \, dP = \int_{A \cap \{\tau_j^t \leq t\}} Y \, dP = \int_{A \cap (\tau_j^t \leq t)} \mathbb{E} \left[ Y \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee A \right] \, dP \\
= \int_A \mathbb{E} \left[ \mathbf{1}_{\{\tau_j^t \leq t\}} Y \, \mid \, \mathcal{H}_{t}^{i,\leq j} \vee A \right] \, dP.
\]

This concludes the proof. \( \square \)

**Remark 2.** Since we have
\[
\mathcal{H}_{t}^{i,\leq j} = \sigma (\tau_j^t, \, h = 1, \ldots, j),
\]
Lemma 4 shows that, if $\tau^i_j$ has occurred before time $t$, then partial information about $\tau^i_j$ up to $t$ is equivalent to full information about all the random times $\tau^i_j$, $h = 1, \ldots, j$. In particular, if $Y$ is a function of $\tau^1_j, \ldots, \tau^j_j$, i.e. $Y = f(\tau^1_j, \ldots, \tau^j_j)$, then the conditional expectation is simply

$$
\mathbb{E}\left[1_{\{\tau^i_j \leq t\}} Y | \mathcal{H}^{i \leq j} \vee \mathcal{A} \right] = 1_{\{\tau^i_j \leq t\}} Y.
$$

We summarize the above results in the following representation theorem.

**Theorem 4.** For any $t \geq 0$, $i = 1, \ldots, n$, $j \in \mathbb{N}_0$ and any integrable $\mathcal{G}$-measurable random variable $Y$, we have

$$
\mathbb{E}[Y | \mathcal{H}^i_t \vee \mathcal{F}_t] = 1_{\{\tau^i_j \leq t\}} \mathbb{E}[Y | \mathcal{H}^{i \leq j} \vee \mathcal{H}^i_{t \wedge j} \vee \mathcal{F}_t] + 1_{\{\tau^i_j > t\}} \mathbb{E}[Y | \mathcal{H}^{i \leq j} \vee \mathcal{F}_t].
$$

If furthermore $Y$ is $(\mathcal{H}^i_t \vee \mathcal{F}_t)$-measurable, then

$$
\mathbb{E}[Y | \mathcal{G}_t] = 1_{\{\tau^i_j \leq t\}} \mathbb{E}[Y | \mathcal{H}^{i \leq j} \vee \mathcal{H}^i_{t \wedge j} \vee \mathcal{F}_t] + 1_{\{\tau^i_j > t\}} \mathbb{E}[Y | \mathcal{H}^{i \leq j} \vee \mathcal{F}_t].
$$

**Proof.** Since

$$
\mathbb{E}[Y | \mathcal{H}^i_t \vee \mathcal{F}_t] = \mathbb{E}\left[1_{\{\tau^i_j \leq t\}} Y | \mathcal{H}^i_t \vee \mathcal{F}_t \right] + \mathbb{E}\left[1_{\{\tau^i_j > t\}} Y | \mathcal{H}^i_t \vee \mathcal{F}_t \right],
$$

the first part is a straightforward consequence of Proposition 1 and Lemma 4. For the second part, it suffices to apply Corollary 2. \qed

We now prove some results which we will use to solve the reserve estimation problem in Section 5.1. For this purpose, our approach allows us to obtain analytical formulas in a general setting in continuous time, where filtration dependence is taken into account. As we illustrate in Section 4, this is not possible in such generality when using more classical approaches. Let $0 \leq t \leq T < \infty$, and let $Z := (Z_t)_{t \in [0, T]}$ be a continuous, bounded, and $\mathbb{F}$-adapted process. For $i = 1, \ldots, n$, we now consider

$$
Y = \sum_{j=N^i_j}^{N^i_T} X^i_j Z^j_{\tau^i_j} = \sum_{j=1}^{\infty} 1_{\{t < \tau^i_j \leq T\}} X^i_j Z^j_{\tau^i_j},
$$

and compute

$$
\mathbb{E}[Y | \mathcal{G}_t] = \mathbb{E}\left[ \sum_{j=N^i_j}^{N^i_T} X^i_j Z^j_{\tau^i_j} \Bigg| \mathcal{G}_t \right].
$$

(3.7)

In particular, as before, we study separately the two components of the decomposition of (3.7) with respect to the first reporting time $\tau^i_1$, i.e.

$$
\mathbb{E}\left[ \sum_{j=N^i_j}^{N^i_T} X^i_j Z^j_{\tau^i_j} \bigg| \mathcal{G}_t \right] = \mathbb{E}\left[ 1_{\{\tau^i_1 > t\}} \sum_{j=N^i_j}^{N^i_T} X^i_j Z^j_{\tau^i_j} \bigg| \mathcal{G}_t \right] + \mathbb{E}\left[ 1_{\{\tau^i_1 \leq t\}} \sum_{j=N^i_j}^{N^i_T} X^i_j Z^j_{\tau^i_j} \bigg| \mathcal{G}_t \right],
$$

(3.8)
and derive more explicit formulas in terms of the intensity process $\mu$, the distribution of delay $\theta^i$, and the distribution of development $N^i$ after the first reporting. We start with the $\mathbb{F}$-conditional expectation of $\tau^i_1$.

**Lemma 5.** For any $i = 1, \ldots, n$ and $t \geq 0$, we have

$$\mathbb{P}(\tau^i_1 \leq t \mid \mathcal{F}_t) = \int_0^t G(t - s)e^{-\int_0^s \mu\,d\nu}\mu ds,$$

(3.9)

where $G$ is the cumulative distribution function of $\theta^i$ given in (2.6).

**Proof.** Note that by Assumption 1, $\theta^i$ is independent of $\mathcal{F}_t \vee \sigma(\tau^i_0)$. Furthermore, both $\theta^i$ and $\tau^i_0$ are $\mathbb{P}$-a.s. nonnegative. Therefore, we have

$$\mathbb{P}(\tau^i_1 \leq t \mid \mathcal{F}_t) = \mathbb{E}[1\{\tau^i_0 + \theta^i \leq t\} \mid \mathcal{F}_t]$$

= $\mathbb{E}[\mathbb{E}[1\{\tau^i_0 \leq t\} \mathbb{1}\{\tau^i_0 + \theta^i \leq t\} \mid \mathcal{F}_t \vee \sigma(\tau^i_0)] \mid \mathcal{F}_t]$

= $\mathbb{E}[1\{\tau^i_0 \leq t\} \mathbb{E}[1\{\theta^i \leq t - \tau^i_0\} \mid x = \tau^i_0] \mid \mathcal{F}_t]$

= $\mathbb{E}[1\{\tau^i_0 \leq t\} G(t - \tau^i_0) \mid \mathcal{F}_t]$.

To conclude we only need to show

$$\mathbb{E}[1\{\tau^i_0 \leq t\} G(t - \tau^i_0) \mid \mathcal{F}_t] = \int_0^t G(t - s)e^{-\int_0^s \mu\,d\nu}\mu ds.$$

(3.10)

This can be done in the same way as for Proposition 5.1.1 of [11], in view of the relation (2.11) and the fact that $G$ is continuous according to Assumption 2. First the relation (3.10) can be shown for a piecewise constant function $G$; it is then obtained as a limit for continuous $G$. □

**Remark 3.** Note that from (3.10) we can derive the conditional probability that the accident event has been incurred, but not yet reported (IBNR, in the terminology used in the insurance sector).

In the expression (3.9) of Lemma 5, the parameter $t$ also appears in the integrand. The following corollary improves the relation (3.9) and shows that the process of conditional expectation $(\mathbb{P}(\tau^i_1 \leq t \mid \mathcal{F}_t))_{t \geq 0}$ is absolutely continuous with respect to the Lebesgue measure.

**Corollary 4.** For any $i = 1, \ldots, n$, we have

$$\mathbb{P}(\tau^i_1 \leq t \mid \mathcal{F}_t) = \int_0^t \left(\alpha_0 e^{-\int_0^u \mu\,d\nu}\mu_s + \int_0^u g(s - u)e^{-\int_0^u \mu\,d\nu}\mu du\right) ds,$$

(3.11)

where $\alpha_0$ and $g$ are defined in (2.6).

**Proof.** This follows immediately from Assumption 2 and the relation (3.9). □

Note that in the following lemma we do not assume that $Z$ is $\mathbb{F}$-adapted and the boundedness condition can be generalized.
Lemma 6. If the process \( Z := (Z_u)_{u \in [t, T]} \) is left-continuous and bounded and \( Z_t \) is \( F_T \)-measurable for all \( t \geq 0 \), then we have

\[
\mathbb{E} \left[ 1_{\{ t < \tau^i \leq T \}} Z_{\tau^i} \mid F_t \right] = \mathbb{E} \left[ \int_t^T Z_u d\mathbb{P}(\tau^i \leq u \mid F_u) \mid F_t \right],
\]

for \( i = 1, \ldots, n \) and \( t \in [0, T] \).

Proof. The argument is similar to Proposition 5.1.1 of [11], which we outline for completeness. We assume first that both \( Z \) and \( \tilde{Z} \) are stepwise constant, i.e. without loss of generality,

\[
Z_u = \sum_{j=0}^n Z_j 1_{\{ t_j < u \leq t_{j+1} \}}, \quad \tilde{Z}_u = \sum_{j=0}^n \tilde{Z}_j 1_{\{ t_j < u \leq t_{j+1} \}},
\]

for \( t < u \leq T \), where \( t_0 = t < \ldots < t_{j+1} = T \), \( Z_j \) is \( F_T \)-measurable, and \( \tilde{Z}_j \) is independent from \( F_T \cap \sigma(\tau^i_j) \) for all \( j = 0, \ldots, n \). By Lemma 2, it holds that

\[
\mathbb{E} \left[ 1_{\{ t < \tau^i \leq T \}} \tilde{Z}_{\tau^i} Z_{\tau^i} \mid F_t \right] = \mathbb{E} \left[ \sum_{j=0}^n \mathbb{E} \left[ \tilde{Z}_j Z_j 1_{\{ t_j < \tau^i \leq t_{j+1} \}} \mid F_T \right] \mid F_t \right]
\]

\[
= \mathbb{E} \left[ \sum_{j=0}^n \mathbb{E} \left[ \tilde{Z}_j \right] Z_j \left( \mathbb{E} \left[ 1_{\{ \tau^i \leq t_{j+1} \}} \mid F_{t_{j+1}} \right] - \mathbb{E} \left[ 1_{\{ \tau^i \leq t_j \}} \mid F_t \right] \right) \mid F_t \right]
\]

\[
= \mathbb{E} \left[ \int_t^T \mathbb{E} \left[ \tilde{Z}_u \right] Z_u d\mathbb{P}(\tau^i \leq u \mid F_u) \mid F_t \right].
\]

(3.12)

(3.13)

In the general case, it is sufficient to find stepwise constant approximations for \( Z \) and \( \tilde{Z} \). Since \( Z \) is bounded and \( E[Z] \) is continuous and bounded on \( [t, T] \), the Riemann sum under the sign of conditional expectation in (3.12) converges to the Lebesgue–Stieltjes integral in the expression (3.13); hence the convergence of the conditional expectations follows as well by the dominated convergence theorem. \( \square \)

Now we are able to calculate the first component on the right-hand side of (3.8). We define

\[
\tilde{m}(t) := \mathbb{E} \left[ \tilde{N}_t \right] \quad \text{if } t \geq 0,
\]

\[
\tilde{m}(t) := 0 \quad \text{if } t < 0,
\]

where \( \tilde{N} \) denotes the ground process of \( \left( \tilde{\tau}^j, \tilde{X}^j_j \right)_{j \in \mathbb{N}_0} \), i.e.

\[
\tilde{N}_t := \sum_{j=1}^{\infty} 1_{\{ \tilde{\tau}^j \leq t \}}, \quad t \geq 0.
\]

(3.15)

Note that \( \tilde{m} \) does not depend on \( i \) because of Assumption 1(2).
The following result also holds under different integrability and measurability conditions.

**Proposition 2.** Let \( Z := (Z_t)_{t \in [0, T]} \) be a continuous, bounded, and \( \mathbb{F} \)-adapted process and let \( Y \) be as in (3.6); then for any \( t \in [0, T] \),

\[
\mathbb{E}
\left[
1_{\{\tau_1^i > t\}} Y \big| \mathcal{H}_i^i \vee \mathcal{F}_t
\right] =
1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
Y \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right]
= 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
\sum_{j=1}^{\infty} 1_{\{\tau_j^i \leq t\}} X_j^i Z_j \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right] + 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
\sum_{j=2}^{\infty} 1_{\{\tau_j^i \leq t\}} X_j^i Z_j \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right],
\]

where \( \tilde{m} \) is defined in (3.14).

**Proof.** By applying (3.5) in Proposition 1 to \( Y \) as defined in (3.6), we get

\[
\mathbb{E}
\left[
1_{\{\tau_1^i > t\}} Y \big| \mathcal{H}_i^i \vee \mathcal{F}_t
\right] = 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
Y \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right]
= 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
\sum_{j=1}^{\infty} 1_{\{\tau_j^i \leq t\}} X_j^i Z_j \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right] + 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
\sum_{j=2}^{\infty} 1_{\{\tau_j^i \leq t\}} X_j^i Z_j \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right].
\]

For the first component of (3.16), it is sufficient to use (3.4) in Proposition 1 and an argument similar to Proposition 5.1.1 of [11], taking into account the independence condition in Assumption 1(3) and Lemma 2. We have hence

\[
1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
1_{\{\tau_1^i \leq T\}} X_1^i Z_{\tau_1^i} \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right] = 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
1_{\{t < \tau_1^i \leq T\}} X_1^i Z_{\tau_1^i} \big| \mathcal{H}_i^{i,1} \vee \mathcal{F}_t
\right]
= 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
1_{\{t < \tau_1^i \leq T\}} X_1^i Z_{\tau_1^i} \big| \mathcal{F}_t
\right]
= 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
\int_t^T E[X_1^i] Z_u \, d\mathbb{P}(\tau_1^i \leq u \big| \mathcal{F}_u) \big| \mathcal{F}_t
\right] = 1_{\{\tau_1^i > t\}}
\mathbb{E}
\left[
\int_t^T E[X_1^i] Z_u \, d\mathbb{P}(\tau_1^i > u \big| \mathcal{F}_u) \big| \mathcal{F}_t
\right].
\]

Now we focus on the second component of (3.16). We assume first that restricted to the interval \([t,T]\), \( Z \) is a bounded, stepwise, \( \mathbb{F} \)-predictable process, i.e.

\[
Z_u = \sum_{k=0}^{n} Z_{tk} 1_{\{t_k < u \leq t_{k+1}\}},
\]

(3.17)
for $t < u \leq T$, where $t_0 = t < \ldots < t_{n+1} = T$ and $Z_{t_k}$ is $\mathcal{F}_{t_k}$-measurable for all $k = 0, \ldots, n$. In this case, we have

\[
1\{\tau_i > t\} \mathbb{E}\left[ \sum_{j=2}^{\infty} 1\{\tau'_j \leq T\} \bar{X}_j^i Z_j^i \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right] = 1\{\tau_i > t\} \mathbb{E}\left[ \sum_{j=1}^{n} \sum_{k=0}^{\infty} 1\{t_k < \tau_i' + \bar{\tau}'_j \leq t_{k+1}\} \bar{X}_j^i Z_{t_k} \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right] \]

\[
= 1\{\tau_i > t\} \mathbb{E}\left[ \sum_{k=0}^{n} Z_{t_k} \mathbb{E}\left[ \sum_{j=1}^{\infty} 1\{t_k < \tau_i' + \bar{\tau}'_j \leq t_{k+1}\} \bar{X}_j^i \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right] \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right] \]

\[
= 1\{\tau_i > t\} \mathbb{E}\left[ \sum_{k=0}^{n} Z_{t_k} \mathbb{E}\left[ \sum_{j=1}^{\infty} 1\{t_k < \tau_i' + \bar{\tau}'_j \leq t_{k+1}\} \bar{X}_j^i \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right] \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right] \]

\[
= 1\{\tau_i > t\} \mathbb{E}\left[ \int_{t}^{T} Z_u \tilde{m}(u - \tau_i') \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right]. \tag{3.18} \]

where in the second-to-last equality we use the independence between the marked point process $\left(\bar{\tau}_j^i, \bar{X}_j^i\right)_{j \in \mathbb{N}_0}$ and the $\sigma$-algebra $\mathcal{H}_t^{i,1} \lor \mathcal{F}_t$ in Assumption 1. This shows that for any bounded, stepwise, $\mathbb{F}$-predictable process $Z$, we have

\[
1\{\tau_i > t\} \mathbb{E}\left[ \sum_{j=2}^{\infty} 1\{\tau'_j \leq T\} \bar{X}_j^i Z_j^i \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right] = 1\{\tau_i > t\} \mathbb{E}\left[ \int_{t}^{T} Z_u \tilde{m}(u - \tau_i') \mid \mathcal{H}_t^{i,1} \lor \mathcal{F}_t \right]. \]

A continuous bounded process $Z$ can be approximated by a sequence of bounded, stepwise, and $\mathbb{F}$-predictable processes; i.e. there is a sequence $Z^n$ of the form (3.17) such that

\[
Z^n \longrightarrow Z \quad \text{and} \quad |Z^n| \leq M, \tag{3.19} \]

with $M > 0$. Since $\tilde{m}$ is right-continuous and monotone, the Lebesgue–Stieltjes integral

\[
\int_{t}^{T} Z_u \tilde{m}(u - \tau_i') \tag{3.19} \]

is well defined. It holds by the Lebesgue theorem that

\[
\int_{t}^{T} Z_u^n \tilde{m}(u - \tau_i') \longrightarrow \int_{t}^{T} Z_u \tilde{m}(u - \tau_i'). \]

Furthermore,

\[
\left| \int_{t}^{T} Z_u^n \tilde{m}(u - \tau_i') \right| \leq M \left| \int_{t}^{T} \tilde{m}(u - \tau_i') \right| = M \left| \tilde{m}(T - \tau_i') - \tilde{m}(t - \tau_i') \right|. \tag{3.20} \]
The right-hand side of (3.20) is uniformly bounded by (3.14) and (2.4). By again applying the Lebesgue theorem, we obtain also the convergence of the conditional expectations

\[ 1\{\tau_i > t\} \mathbb{E}\left[ \int_t^T Z_u d\tilde{m}(u - \tau_i) \bigg| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right] \rightarrow \mathbb{E}\left[ \int_t^T Z_u d\tilde{m}(u - \tau_i) \bigg| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right]. \]

We note that \( \tilde{m}(u) = 0 \) for \( u < 0 \); hence,

\[ 1\{\tau_i > t\} \mathbb{E}\left[ \sum_{j=2}^{\infty} 1\{\tau_j \leq T\} X_i^j Z_{\tau_j}^i \bigg| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right] = 1\{\tau_i > t\} \mathbb{E}\left[ \int_t^T Z_u d\tilde{m}(u - \tau_i) \bigg| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right] \]

\[ = \mathbb{E}\left[ 1\{t < \tau_i \leq T\} \int_t^T Z_u d\tilde{m}(u - \tau_i) \bigg| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right]. \]

By again applying (3.4) in Proposition 1 to the above expression, we get

\[ 1\{\tau_i > t\} \mathbb{E}\left[ \sum_{j=2}^{\infty} 1\{\tau_j \leq T\} X_i^j Z_{\tau_j}^i \bigg| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right] = 1\{\tau_i > t\} \mathbb{E}\left[ 1\{t < \tau_i \leq T\} \int_t^T Z_u d\tilde{m}(u - \tau_i) \bigg| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right]. \]

Let \( \tilde{Z}_s := \int_t^T Z_u d\tilde{m}(u - s), \ s \in [0, T] \). We note that \( \tilde{m} \) is right-continuous and monotone. On one hand, for fixed \( s \in [0, T] \), the function \( d_s(u) := \tilde{m}(u - s), \ u \in [0, T] \), is also right-continuous and monotone and defines the cumulative distribution function of a finite positive measure, in view of (2.4). On the other hand, for fixed \( u \in [0, T] \), the function \( \tilde{m}(u - s), \ s \in [0, T] \), is left-continuous in \( s \); i.e. for every series \( s_n \not\to s \), we have the pointwise convergence

\[ \lim_{s_n \not\to s} d_{s_n}(u) = d_s(u) \quad \text{for all } u \in [0, T] \]

of the cumulative distribution functions, equivalent to convergence in distribution or weak convergence in measure. Note that a series of positive finite measures \( (\nu_n)_{n \in \mathbb{N}} \) converges weakly to a positive finite measure \( \nu \) if for all bounded continuous functions \( f \) the following holds:

\[ \int f d\nu_n \rightarrow \int f d\nu. \]

This yields the convergence

\[ \tilde{Z}_{s_n} \rightarrow \tilde{Z}_s, \quad \text{P.a.s.}; \]
that is, \( \bar{Z}_s := \int_s^T Z_u \bar{d}\tilde{m}(u-s), \ s \in [0, T] \), is left-continuous. Furthermore, it is also bounded. Now we apply Lemma 6 and obtain

\[
\mathbb{E}\left[1_{\{t_i^r > t\}} \int_t^T Z_u \bar{d}\tilde{m}(u - t_i^r) \bigg| \mathcal{F}_t\right] = \mathbb{E}\left[1_{\{t_i^r > t\}} \bar{Z}_{t_i^r} \bigg| \mathcal{F}_t\right]
\]

\[
= \mathbb{E}\left[1_{\{t_i^r > t\}} \int_t^T Z_u \bar{d}\tilde{m}(u) \bigg| \mathcal{F}_t\right] = \mathbb{E}\left[1_{\{t_i^r > t\}} \int_t^T (\int_t^T Z_u \bar{d}\tilde{m}(v-u)) \bigg| \mathcal{F}_t\right]
\]

As the last step, we note that for \( u < s, \int_s^T Z_u \bar{d}\tilde{m}(u-s) = \int_s^T Z_u \bar{d}\tilde{m}(u-s) \) since \( \tilde{m}(u-s) = 0 \). This concludes the proof. \( \square \)

**Remark 4.** The proof of Proposition 2 relies on Assumption 1. Another sufficient condition would be the continuity of \( \tilde{m} \), as in the case of a compound Poisson process or a Cox process with continuous intensity process and integrable marks. Indeed, since \( \tilde{m}(u) = 0 \) for \( u < 0 \),

\[
\mathbb{E}\left[1_{\{t_i^r > t\}} \int_t^T Z_u \bar{d}\tilde{m}(u - t_i^r) \bigg| \mathcal{F}_t\right] = \mathbb{E}\left[1_{\{t_i^r > t\}} \int_t^T \bar{d}\tilde{m}(u - t_i^r) \bigg| \mathcal{F}_t\right]
\]

and the right-hand side is uniformly bounded if \( \tilde{m} \) is continuous.

The following proposition gives a representation of the second component on the right-hand side of (3.8).

**Proposition 3.** Under the same assumptions as those of Proposition 2, if for each \( i = 1, \ldots, n \), the process

\[
\left( \sum_{j=1}^{N_t} \tilde{X}_j^i \right)_{t \in [0, T]},
\]

where \( \tilde{N} \) is defined in (3.15), is of independent increments with respect to its natural filtration \( \tilde{\mathbb{H}}^i \), then for \( t \in [0, T] \) and \( Y \) as in (3.6), it holds that

\[
\mathbb{E}\left[1_{\{t_i \leq t\}} Y \bigg| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T Z_u \bar{d}\tilde{m}(u) \bigg| \mathcal{H}^{i,1}_\infty \vee \tilde{\mathcal{H}}_t \vee \mathcal{F}_t\right]_{x = t_i^r},
\]

for \( i = 1, \ldots, n \).

**Proof.** It follows from Lemma 4 that

\[
\mathbb{E}\left[1_{\{t_i \leq t\}} Y \bigg| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbb{E}\left[1_{\{t_i \leq t\}} Y \bigg| \mathcal{H}^{i,1}_\infty \vee \mathcal{H}^{i,>1}_r \vee \mathcal{F}_t\right].
\]
As in the proof of Proposition 2, we assume first $Z$ of the form (3.17). Similar calculations lead to

$$
\mathbb{E} \left[ \mathbf{1}_{\{\tau_1^i \leq t\}} Y \mid \mathcal{H}^{i,1}_\infty \vee \mathcal{H}^{i,>1}_t \right]
$$

$$
= \mathbf{1}_{\{\tau_1^i \leq t\}} \mathbb{E} \left[ \sum_{j=1}^{\infty} \mathbf{1}_{\{t \leq \tau_1^i + j \leq t\}} \tilde{X}^i_j Z_{t_j} \mid \mathcal{H}^{i,1}_\infty \vee \mathcal{H}^{i,>1}_t \right]
$$

$$
= \mathbf{1}_{\{\tau_1^i \leq t\}} \mathbb{E} \left[ \sum_{i=0}^{n} \sum_{j=1}^{\infty} \mathbf{1}_{\{t \leq \tau_1^i + j \leq t\}} \tilde{X}^i_j Z_{t_j} \mid \mathcal{H}^{i,1}_\infty \vee \mathcal{H}^{i,>1}_t \right]_{x=\tau_1^i}
$$

where the last step follows from the definitions of the filtrations. Using the tower property, the independence between the marked point process $(\tilde{\tau}_j^i, \tilde{X}^i_j)_{j \in \mathbb{N}_0}$ and $\mathcal{F}_\infty \vee \mathcal{H}^{i,1}_\infty$ (see Assumption 1), and the independence of increments of the process

$$
\left( \sum_{j=1}^{N_t} \tilde{X}^i_j \right)_{t \in [0,T]},
$$

we get furthermore

$$
\mathbb{E} \left[ \mathbf{1}_{\{\tau_1^i \leq t\}} Y \mid \mathcal{H}^{i,1}_\infty \vee \mathcal{H}^{i,>1}_t \right]
$$

$$
= \mathbf{1}_{\{\tau_1^i \leq t\}} \mathbb{E} \left[ \sum_{i=0}^{n} \tilde{Z}_i \left( \mathbb{E} \left[ \mathbf{1}_{\{\tilde{\tau}_i^i \leq t_i - 1\}} \tilde{X}^i_{\tilde{\tau}_i^i} \mid \mathcal{H}^{i,1}_\infty \vee \mathcal{H}^{i,>1}_{t_i} \right] \right) \right]_{x=\tau_1^i}
$$

$$
= \mathbf{1}_{\{\tau_1^i \leq t\}} \mathbb{E} \left[ \sum_{i=0}^{n} \tilde{Z}_i \left( \tilde{m}(t_i - 1) - \tilde{m}(t_i - x) + \tilde{X}^i_{\tilde{\tau}_i^i} \right) \right]_{x=\tau_1^i}
$$

$$
= \mathbf{1}_{\{\tau_1^i \leq t\}} \mathbb{E} \left[ \tilde{Z}_i (\tilde{m}(t_i - 1) - \tilde{m}(t_i - x)) \right]_{x=\tau_1^i}.
$$

This yields that for any bounded, stepwise, $\mathbb{F}$-predictable process $Z$, we have

$$
\mathbb{E} \left[ \mathbf{1}_{\{\tau_1^i \leq t\}} Y \mid \mathcal{H}^{i}_t \vee \mathcal{F}_t \right] = \mathbf{1}_{\{\tau_1^i \leq t\}} \mathbb{E} \left[ \int_t^T Z_u \tilde{m}(u - x) \mid \mathcal{H}^{i,1}_\infty \vee \mathcal{F}_t \right]_{x=\tau_1^i}.
$$

If $Z$ is continuous, bounded, and $\mathbb{F}$-adapted, then $Z$ can be approximated by a sequence of bounded, stepwise, and $\mathbb{F}$-predictable processes. This together with the fact that $\tilde{m}$ is
right-continuous and monotone guarantees that the Riemann sum in (3.21) under the sign of conditional expectation converges to the Lebesgue–Stieltjes integral, using the same arguments as Proposition 2.

We summarize the results in the following theorem, which gives an explicit representation of $\mathbb{G}$-conditional expectation with respect to the first reporting time $\tau^i_1$. Note that, like most of our results, the conclusion also holds under alternative integrability and measurability conditions.

**Theorem 5.** Let $Z := (Z_t)_{t \in [0,T]}$ be a continuous, bounded, and $\mathbb{F}$-adapted process, and let $Y$ be of the form (3.6). If the process $\left(\tilde{N}_t \sum_{j=1}^{\tilde{X}_j^i} \right)_{t \in [0,T]}$ has independent increments and $\tilde{m}$ is as defined in (3.14), then

$$
\mathbb{E}[Y \mid \mathcal{G}_t] = \mathbf{1}_{\{\tau^i_1 \leq t\}} \mathbb{E}\left[ \int_t^T Z_u d\tilde{m}(u - x) \mid \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,1} \vee \mathcal{F}_t \right] |_{x = \tau^i_1} + \mathbf{1}_{\{\tau^i_1 > t\}} \mathbb{E}\left[ \int_t^T \left( E[X]^i_u Z_u + \int_u^T Z_v d\tilde{m}(v - u) \right) d\mathbb{P}(\tau^i_1 \leq u \mid \mathcal{F}_u) \right] \mathcal{F}_t,
$$

for $i = 1, \ldots, n$, where

$$
\mathbb{P}(\tau^i_1 \leq t \mid \mathcal{F}_t) = \int_0^t \left( \alpha_0 e^{-\int_0^u \mu_t dv} \mu_u + \int_u^T g(u - v) e^{-\int_0^u \mu_t ds} \mu_s dv \right) du,
$$

with $\alpha_0$ and $g$ defined in (2.6).

**Proof:** It is enough to combine Corollary 2, Lemma 5, Corollary 4, Proposition 2, and Proposition 3.

Compared to Theorem 4, Theorem 5 is more explicit and has the advantage that the representation is expressed as function of $\mu$, the distribution of $\theta^i$, and the distribution of $\left(\tilde{\tau}^i_1, \tilde{X}^i_1\right)_{j \in \mathbb{N}_0}$. This result will be useful for the concrete reserving problem in a hybrid market in Section 5.

### 4. Comparison with the compensator approach

In this section, we compare our framework with the compensator approach for non-life insurance in the existing literature. Within this section, the filtration $\mathbb{H}$ denotes the natural filtration of a marked point process $(\tau_n, X_n)_{n \in \mathbb{N}_0}$, with marked cumulative process $N$, and $\mathbb{G}$ is a generic enlargement of $\mathbb{H}$. We set $\mathcal{H} := \mathcal{H}_\infty$ and $\mathcal{G} := \mathcal{G}_\infty$.

In most of the current literature, e.g. [3], [32], [31], and [35], the study of non-life insurance contracts is based on modeling the $\mathbb{G}$-compensator of $N$, since the $\mathbb{G}$-compensator is involved in the pricing formula and in the calculation of the hedging strategy. In the reduced-form framework for life insurance, the direct modeling approach and the compensator approach coincide; see e.g. [11]. However, the compensator approach presents several difficulties in a non-life insurance setting with nontrivial dependence between the filtrations.
Definition 1. The $\mathbb{G}$-mark-predictable $\sigma$-algebra on the product space $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega$ is the $\sigma$-algebra generated by sets of the form $(s, t] \times B \times A$ where $0 < s < t$, $B \in \mathcal{B}(\mathbb{R}_+)$, and $A \in \mathcal{G}_t$.

Definition 2. The $\mathbb{G}$-compensator of a marked point process $(\tau_n, X_n)_{n \in \mathbb{N}_0}$ is any $\mathbb{G}$-mark-predictable, cumulative process $\Lambda(t, B, \omega)$ such that $(\Lambda(t, B))_{t \geq 0}$ with $\Lambda(t, B)(\cdot, \cdot) := \Lambda(t, B, \cdot)$ is the $\mathbb{G}$-compensator of the point process $(N(t, B))_{t \geq 0}$. We use the notation $(\Lambda_t)_{t \geq 0}$, $\Lambda_t := \Lambda(t, \mathbb{R}_+)$, to denote the $\mathbb{G}$-compensator of the ground process $(N_t)_{t \geq 0}$.

Theorem 14.2.IV(a) of [15] shows that given a marked point process $(\tau_n, X_n)_{n \in \mathbb{N}_0}$ with finite first moment measure, its $\mathbb{G}$-compensator $\Lambda$ always exists and is $(l \otimes \mathbb{P})$-almost everywhere unique, where $l$ denotes the Lebesgue measure on $\mathbb{R}_+$. In particular, for all $(t, B, \omega) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega$, the following relation holds:

$$
\Lambda(t, B, \omega) = \int_0^t \kappa(B|s, \omega) \Lambda(ds, \omega),
$$

(4.1)

where $\kappa(B|s, \omega)$, $B \in \mathcal{B}(\mathbb{R}_+)$, $s \geq 0$, $\omega \in \Omega$, is the unique predictable kernel such that for all $A \in \mathcal{G}_t$, $0 < s < t$, $B \in \mathcal{B}(\mathbb{R}_+)$,

$$
\int_A \int_s^t N(u, B)(\omega) du \mathbb{P}(d\omega) = \int_A \int_s^t \kappa(B|u, \omega) N_u(\omega) du \mathbb{P}(d\omega).
$$

However, under general conditions it is not always true that given a $\mathbb{G}$-mark-predictable and cumulative process $\Lambda$, there exists a marked point process $(\tau_n, X_n)_{n \in \mathbb{N}_0}$ with $\mathbb{G}$-compensator $\Lambda$. The problem is first mentioned in [21], where the case with $\mathbb{G} = \mathbb{H}$ is solved. An extension of the existence theorem to the case of $\mathbb{G} = \mathbb{F} \otimes \mathbb{H}$, i.e. when the filtrations $\mathbb{F}$ and $\mathbb{H}$ are independent, is provided in [17]. Furthermore while the law of $N$ is uniquely determined by the $\mathbb{H}$-compensator, this is not true for the $\mathbb{G}$-compensator. (See the discussion in [21] and in [20, Section 4.8].) Consequently, the literature with the compensator approach is mostly limited to the cases of $\mathbb{G} = \mathbb{H}$ (see e.g. [32], [31]) or $\mathbb{G} = \mathbb{F} \otimes \mathbb{H}$ (see e.g. [3]).

In the following we provide a sufficient condition in the general case of $\mathbb{G} = \mathbb{F} \lor \mathbb{H}$, such that the law of $N$ is uniquely determined by $\Lambda$. Similarly to e.g. [32] and [31], we assume that the $\mathbb{G}$-compensator of $(\tau_n, X_n)_{n \in \mathbb{N}_0}$ has the following form:

$$
\Lambda(t, B) = \int_0^t \int_B \lambda_s \eta_s(dx) ds \quad \text{for all } t \geq 0, \ B \in \mathcal{B}(\mathbb{R}_+),
$$

(4.2)

where $\lambda := (\lambda_t)_{t \geq 0}$ is a $\mathbb{G}$-progressively measurable process and the mapping $\eta$ given by

$$
\eta : \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega \rightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))
$$

$$(t, B, \omega) \mapsto \eta_t(B)(\omega)
$$

is such that for every $t \geq 0$, $\omega \in \Omega$, $\eta(t, \cdot, \cdot, \omega)$ is a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, and for every $B \in \mathcal{B}(\mathbb{R}_+)$, $(\eta_t(B))_{t \geq 0}$ is a $\mathbb{G}$-progressively measurable process. Clearly, we have

$$
\Lambda_t = \int_0^t \lambda_s ds \quad \text{for all } t \geq 0.
$$

In particular, we can choose a predictable version of both $\lambda$ and $\eta$; see Section 14.3 of [15] for details. The processes $\lambda$ and $\eta$ can be interpreted respectively as jump intensity and jump
size intensity. We recall that a marked point process \((\tau_n, X_n)_{n \in \mathbb{N}_0}\) has independent marks if the marks \((X_n)_{n \in \mathbb{N}_0}\) are mutually independent given \(N\).

**Proposition 4.** The law of a simple marked point process \((\tau_n, X_n)_{n \in \mathbb{N}_0}\) on \((\Omega, \mathcal{H})\) with finite first moment measure and independent marks, and of the form (4.2), is uniquely determined by \(\lambda\) and \(\eta\). If furthermore \(\lambda\) is \(\mathbb{H}\)-measurable, then the law of \(N\) on \((\Omega, \mathcal{G})\) is also uniquely defined.

**Proof.** By Proposition 6.4.IV(a) of [15], the law of a marked point process with independent marks is uniquely determined by the kernel \(\kappa\) and the distribution of \(N\). According to the relations (4.1) and (4.2), the kernel \(\kappa\) is given by

\[
\kappa(B(t, \omega)) = \eta(t)(B)(\omega), \quad (t, B, \omega) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega.
\]

Corollary 4.8.5 of [20] and Theorem 14.2.IV(c) of [15] show that, if \(N\) is simple and of the form (4.2), the process \((E[\lambda_t | \mathcal{H}_t])_{t \geq 0}\) uniquely determines the distribution of \(N\) on \((\Omega, \mathcal{H})\). If in addition \(\lambda\) is \(\mathbb{H}\)-adapted, then by Theorem 4.8.1 of [20], the distribution of \(N\) on \((\Omega, \mathcal{G})\) is also uniquely determined.

Nevertheless, Proposition 4 requires the jump intensity process \(\lambda\) to be \(\mathbb{H}\)-adapted in order to have \(N\) uniquely defined in law, which is an unnatural condition in our context.

By contrast, the approach proposed in Section 2 allows us to take into account a dependence structure between the filtrations \(\mathbb{H}\) and \(\mathbb{G}\) by directly modeling the \(\mathbb{F}\)-adapted intensity process \(\mu\). Furthermore, this allows us to obtain analytical results for valuation formulas as shown in Section 3.

**5. Pricing and hedging of non-life insurance liability cash flows in a hybrid market**

In this section, we address the issue of pricing and hedging non-life insurance liability cash flows by applying the results of Section 3 under the benchmarked risk-minimization approach. We assume that \(P\) is the real-world measure and consider a general structure for a hybrid insurance and financial market. We fix a time horizon \(T\) with \(0 < T < \infty\), and denote the inflation index process by \(I := (I_t)_{t \in [0, T]}\), which represents the percentage increments of the Consumer Price Index and follows a nonnegative \((P, \mathbb{F})\)-semimartingale. We distinguish real price value, i.e. inflation-adjusted, from nominal price value, which can be converted to the real value at any time \(t \in [0, T]\) if divided by the inflation index \(I_t\). If not otherwise specified, all prices are expressed as nominal values.

We consider \(d\) liquidly traded primary assets on the financial market described by the price process vector \(S := (S^1_t, \ldots, S^d_t)_{t \in [0, T]}\), which follows a real-valued \((P, \mathbb{F})\)-semimartingale. We assume that there is a publicly accessible index, based on the intensity process \(\mu\) and modeled by the process \(L := (L_t)_{t \in [0, T]}\) with

\[
L_t := e^{-\Gamma t}, \quad t \in [0, T];
\]

see e.g. [13]. This index reflects the underlying systematic risk factors related to the insurance portfolio, such as mortality risk, weather risk, car accident risk, etc. We distinguish three kinds of primary assets as elements of the vector \(S\):

1. Financial assets, such as the zero-coupon bond, call and put options, futures, etc.
2. Inflation-linked derivatives, such as inflation-linked zero-coupon bonds (also called zero-coupon Treasury Inflation-Protected Securities (TIPS)), which pay off \(I_T\) equivalent to 1 real unit at time \(T\); inflation-linked call and put options; etc.
3. Macro-risk-factor-linked derivatives based on the index $L$, such as longevity bonds which pay off $L_T$ at time $T$, weather-index-based derivatives, etc. Note that apart from longevity bonds, other macro-index-based derivatives are still not common in the market.

We denote by $L(S, P, \mathbb{G})$ the space of $\mathbb{R}^d$-valued $\mathbb{G}$-predictable $S$-integrable processes. Following the definitions in [6], we define the portfolio or value process $S^\delta := (S_t^\delta)_{t \in [0,T]}$ associated to a trading strategy $\delta := (\delta_t)_{t \in [0,T]}$ in $L(S, P, \mathbb{G})$ as the following càdlàg optional process:

$$S^\delta_t = \delta^\top_t S_t = \sum_{i=1}^d \delta^i_t S^i_t, \quad t \in [0, T].$$

It is called self-financing if

$$S^\delta_t = S^\delta_0 + \int_0^t \delta^\top_u dS_u = \sum_{i=1}^d \int_0^t \delta^i_u dS^i_u, \quad t \in [0, T].$$

We introduce the following set:

$$V^+_x = \{ S^\delta \text{ self-financing} : \delta \in L(S, P, \mathbb{G}), \; S^\delta_0 = x > 0, \; S^\delta > 0 \}.$$ 

**Definition 3.** A benchmark or numéraire portfolio $S^* := (S^*_t)_{t \in [0,T]}$ is an element of $V^+_1$ such that

$$\frac{S^*_s}{S^*_t} \geq \mathbb{E} \left[ \frac{S^\delta_t}{S^\delta_s} \bigg| \mathcal{G}_s \right], \quad s, t \in [0, T], \quad t \geq s.$$ 

We follow the approach of [34] and work under the following assumption.

**Assumption 6.** There exists a benchmark portfolio $S^*$. 

In [19], it is shown that Assumption 6 is weaker than assuming the existence of an equivalent martingale measure. As discussed in [4], this weak no-arbitrage assumption is more suitable for modeling a hybrid market as in our case and allows us to work directly under the real-world measure $P$. Extensive background on the benchmark approach and its application can be found in [34]. Note that, as discussed in [34], the benchmark portfolio $S^*$ can be identified with a sufficiently diversified portfolio such as the MSCI World Index.

Given a generic random variable or process $X$, we denote by $\hat{X} := X/S^*$ the benchmarked value of $X$. The following lemma is proved in [5].

**Lemma 7.** If the vector process of primary assets $S$ is continuous, then the benchmarked vector process $\hat{S} := S/S^*$ is a $(\mathbb{F}, \mathbb{P})$-local martingale.

For the sake of simplicity, we assume the following conditions, which are similar to the ones in [10].

**Assumption 7.** The inflation index process $I = (I_t)_{t \in [0,T]}$ and the vector process of primary assets $S$ are continuous. The benchmark portfolio $S^* := (S^*_t)_{t \in [0,T]}$ is continuous and $\mathbb{F}$-adapted, and the benchmarked value process $\hat{S} := S/S^*$ is an $(\mathbb{F}, \mathbb{P})$-local martingale. An inflation-linked zero-coupon bond (or TIPS) is a primary asset, i.e. an element of the vector $S$. 
The payment stream in real units of the insurance company towards policyholders is modeled by a nonnegative \((P, \mathbb{G})\)-semimartingale \(D := (D_t)_{t \in [0, T]}\). We denote by \(A := (A_t)_{t \in [0, T]}\) the nominal benchmarked cumulative payment, namely
\[
A_t := \int_0^t \frac{I_u}{S^*_u} \, dD_u, \quad t \in [0, T].
\]

**Definition 4.** We call the following formula the real-world pricing formula associated to \(A\):
\[
V_t := \frac{S^*_t}{I_t} \mathbb{E} \left[ A_T - A_t \mid \mathcal{G}_t \right] = \frac{S^*_t}{I_t} \mathbb{E} \left[ \int_t^T \frac{I_u}{S^*_u} \, dD_u \mid \mathcal{G}_t \right],
\]
for \(t \in [0, T]\).

Here \(V_t\) in (5.2) is expressed as a real value, i.e. an inflation-adjusted value. According to the benchmark approach of [34], a portfolio’s process is fair if its benchmarked value process is a \(P\)-martingale. The real-world pricing formula (5.2) then provides the fair portfolio of minimal price among all replicating self-financing portfolios for a given benchmarked claim \(\hat{H}\), if \(\hat{H}\) is hedgeable. In the case of incomplete market models, it corresponds to the benchmarked risk-minimizing price for the payment process \(A\) at time \(t\), if \(A\) is square-integrable, i.e.
\[
\sup_{t \in [0, T]} \mathbb{E} \left[ A^2_t \right] < \infty.
\]

For hedging purposes, we now use the relation between the benchmark approach and risk minimization, illustrated in [34] and [5] for a single payoff and in the appendix of [10] for the case of dividend payments. In particular, Theorem A.7 of [10] shows that if
\[
A_T = \mathbb{E} [A_T] + \int_0^T \left( \delta^A_u \right) \top \, d\hat{S}_u + L^A_T, \quad P\text{-a.s.},
\]
is the Galtchouk–Kunita–Watanabe decomposition of \(A_T\), where \(\int_0^T \delta^A_u \, d\hat{S}_u\) is \(P\)-strongly orthogonal to \(L^A\), then \(\delta^A\) is the unique benchmarked risk-minimizing strategy for \(A\), i.e. the trading strategy which minimizes the expected quadratic risk as in Definition A.4 of [10]. Furthermore, the associated benchmarked cumulative cost process \(C^\delta_t\) is given by
\[
C^\delta_t = \mathbb{E} [A_T] + L^A_t, \quad t \in [0, T],
\]
and the benchmarked value process \(\hat{S}^\delta_t\) is given by the discounted value of the real-world pricing formula (5.2),
\[
\hat{S}^\delta_t = \mathbb{E} \left[ A_T - A_t \mid \mathcal{G}_t \right] = \frac{I_t}{S^*_t} V_t, \quad t \in [0, T].
\]

Note that the decomposition (5.3) shows the orthogonality between the perfectly hedgeable part \(\int_0^T \left( \delta^A_u \right) \top \, d\hat{S}_u\) of \(A_T\) and the totally unhedgeable part \(\mathbb{E} [A_T] + L^A_T\), covered by the benchmarked cumulative cost process \(C\).

### 5.1. Pricing and hedging non-life insurance claims

In the setting outlined above, we now apply the results of Section 3 to compute the real-world pricing formula for non-life insurance claims, under the interpretation of Section 2.2.
The cumulative payment at time $t$ for $i$th policy, expressed as a real value, is given by

$$\sum_{j=1}^{\infty} 1_{\{\tau_{ij} \leq t\}} X_{ij} = \sum_{j=1}^{N_t^i} X_{ij}.$$  

The nominal benchmarked cumulative payment process $A := (A_t)_{t \in [0,T]}$ is hence

$$A_t := \int_0^t \frac{I_s}{S_s^*} dD_s = \sum_{i=1}^{n} \sum_{j=1}^{N_t^i} \frac{I_{\tau_{ij}}}{S_{\tau_{ij}}^*} X_{ij}, \quad t \in [0, T]. \tag{5.4}$$

Following [1], the claim reserving problem in the context of non-life insurance can be formulated as the estimation of $A$. In particular, we are interested in estimating not only the overall reserve $E[A_T]$, but also the conditional reserve $E[A_T - A_t \mid \mathcal{G}_t]$ at a given time $t$, i.e. a predictor of the remaining nominal payment $A_T - A_t$, given the information up to time $t$. Unlike in the life insurance case, the risk related to non-life insurance policies is related not only to the accident itself, but also to the first reporting delay (this is the case of incurred but not reported (IBNR) claims), and to the time and the size of developments after the first reporting. We now focus on pricing and hedging the remaining nominal payment $A_T - A_t$, for $t \in [0, T]$. We assume that the process $I_s/S_s^*$ is $\mathbb{F}$-conditionally independent of $\tau_{ij}^i$, for all $i = 1, \ldots, n$, and that the cumulative payments related to the marked point processes $(\tilde{\tau}_{ij}^i, \tilde{X}_{ij}^i)_{j \in \mathbb{N}_0}$, $i = 1, \ldots, n$,

$$\tilde{N}_t^i \sum_{j=1}^{\tilde{X}_{ij}^i}, \quad t \in [0, T], \quad i = 1, \ldots, n,$$

are independent and identically distributed compound Poisson processes, i.e. the $\tilde{N}_t^i$ are mutually independent Poisson processes with parameter $\lambda$, and the $\tilde{X}_{ij}^i$ are independent and identically distributed integrable nonnegative random variables independent of $\tilde{N}_t^i$ with expectation $E[\tilde{X}_{ij}^i] = m$. In this case, we have

$$\tilde{m}(t) = \lambda mt, \quad t \in [0, T],$$

where $\tilde{m}$ is defined in (3.14).

In view of the above assumptions, all conditions in Theorem 5 are satisfied in the case of $Y = A_T - A_t$, for $t \in [0, T]$. Let $R_t$ be the number of reported claims at time $t$, i.e.

$$R_t := \sum_{i=1}^{n} 1_{\{\tau_{i} \leq t\}}, \quad t \in [0, T].$$

The real-world pricing formula (5.2) together with Corollary 2, Theorem 5, and Assumption 7 yields

$$V_t \frac{I_t}{S_t^*} = \mathbb{E} [A_T - A_t \mid \mathcal{G}_t] = \mathbb{E} \left[ \sum_{i=1}^{n} \sum_{j=1}^{N_t^i} \frac{I_{\tau_{ij}}}{S_{\tau_{ij}}^*} X_{ij} \mid \mathcal{G}_t \right]$$

$$= \sum_{i=1}^{n} \mathbb{E} \left[ \sum_{j=1}^{N_t^i} \frac{I_{\tau_{ij}}}{S_{\tau_{ij}}^*} X_{ij} \mid \mathcal{F}_t \vee \mathcal{H}_t^i \right]$$
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where the conditional probability function \( \mathbb{P}(\tau_i \leq t | \mathcal{F}_t) \) is given in (3.11), i.e.

\[
\mathbb{P}(\tau_i \leq t | \mathcal{F}_t) = \int_0^t \left( \alpha_0 e^{-\int_0^s \mu_v dv} \mu_s + \int_s^t g(s-u)e^{-\int_0^u \mu_v dv} \mu_u du \right) ds.
\]

The first component on the left-hand side of (5.5),

\[
\lambda m R_i (T - t) \frac{I_t}{S^*_t}, \quad (5.6)
\]
corresponds to already reported claims. We observe that the valuation of this part does not involve the updating information after the first reporting. The second component on the right-hand side of (5.5),

\[
(n - R_i) \frac{1}{S^*_t} \mathbb{E} \left[ f_t^T \left( E[X^*_1] \frac{I_t}{S^*_t} + \lambda m \int_t^T \frac{I_s}{S^*_s} dv \right) \right] \mathbb{P}(\tau_i \leq u | \mathcal{F}_u) | \mathcal{F}_t, \quad (5.7)
\]

which can be further explicitly computed, corresponds to unreported claims and includes both IBNR claims and claims not yet incurred. The standard literature on non-life insurance is mainly focused on IBNR claims. However, for the pricing problem it is more appropriate to consider the entire expression (5.5). As already mentioned at the beginning of this section, this price equals the benchmarked risk-minimizing price, if we assume square-integrability of the claim.

We now calculate the associated benchmarked risk-minimizing strategy. For additional details, we refer to [38]. We mainly focus on the second component (5.7) related to not-yet-reported claims, since hedging strategy is additive with respect to claims, and the first component (5.6), related to already reported claims, is perfectly hedgeable by trading inflation-linked zero-coupon bonds. By the same arguments as in Proposition 4.11 in [2] and Section 4.1 of [10], the benchmarked risk-minimizing strategy \( \delta \) associated to not-yet-reported claims is given by

\[
\delta_t = (n - R_i) \left( e^{-\int_0^t \mu_u du} + \int_0^t \tilde{G}(t-u)e^{-\int_0^u \mu_v dv} \mu_u du \right)^{-1} \phi_t, \quad t \in [0, T],
\]
where $\phi_t$ is the benchmarked risk-minimizing strategy at $t$ related to

$$
U_t := \mathbb{E} \left[ \int_0^T \left( \mathbb{E}[X^t_1] \frac{I}{S^u} + \lambda m \int_u^T \frac{I}{S^v} \, dv \right) \, d\mathbb{P} \left( \tau^t_1 \leq u \mid \mathcal{F}_u \right) \right], \quad t \in [0, T].
$$

More precisely, the vector process $\phi := (\phi_t)_{t \in [0, T]}$ follows from the Galtchouk–Kunita–Watanabe decomposition of $(U_t)_{t \in [0, T]}$,

$$
U_t = \mathbb{E} \left[ \int_0^T \left( \mathbb{E}[X^t_1] \frac{I}{S^u} + \lambda m \int_u^T \frac{I}{S^v} \, dv \right) \, d\mathbb{P} \left( \tau^t_1 \leq u \mid \mathcal{F}_u \right) \right] + \int_0^t \phi_u^\top \, d\tilde{S}_u + L^U_t.
$$

Note that the benchmarked risk-minimizing strategy covers only the perfectly hedgeable part of the liability cash flow $A$. It is, however, the best possible hedging strategy in the benchmarked risk-minimizing sense. Indeed, the totally unhedgeable part, which is orthogonal to the hedgeable part, represents the basis hedging risk.

The form of $V$ in (5.5) suggests the design of derivatives which can be used to hedge risks in this market model. In particular, since $U$ involves only the stochastic processes $S^*, I$, and $\mu$, it is sufficient to introduce three kinds of instruments, namely pure financial assets, inflation-linked derivatives, and macro-risk-factor-linked derivatives, to hedge risks derived from $S^*$, $I$, and $\mu$ respectively.

6. Conclusion

In this paper, we introduce a general framework for modeling an insurance liability cash flow in continuous time by extending the reduced-form setting. This framework allows us to consider a nontrivial dependence between the reference information flow and the internal insurance information flow. In this setting, we compute explicit valuation and hedging formulas, which can be used for pricing non-life insurance products under the benchmark approach.

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