Abstract

We show that any $d$-colored set of points in general position in $\mathbb{R}^d$ can be partitioned into $n$ subsets with disjoint convex hulls such that the set of points and all color classes are partitioned as evenly as possible. This extends results by Holmsen, Kyncl & Valculescu (2017) and establishes a special case of their general conjecture. Our proof utilizes a result obtained independently by Soberón and by Karasev in 2010, on simultaneous equipartitions of $d$ continuous measures in $\mathbb{R}^d$ by $n$ convex regions. This gives a convex partition of $\mathbb{R}^d$ with the desired properties, except that points may lie on the boundaries of the regions. In order to resolve the ambiguous assignment of these points, we set up a network flow problem. The equipartition of the continuous measures gives a fractional flow. The existence of an integer flow then yields the desired partition of the point set.

1 Introduction

A (finite) set $X$ of points in $\mathbb{R}^d$ is in general position if every subset of size at most $d + 1$ is affinely independent. A partition $X = X_1 \sqcup \cdots \sqcup X_m$ of $X$ into $m$ disjoint subsets is an $m$-coloring of $X$. The sets $X_1, \ldots, X_m$ are called color classes and we say that the set $X$ is $m$-colored. A subset $Y \subseteq X$ containing points from at least $j$ distinct color classes is said to be $j$-colorful.

In this language, the classical partition result of Akiyama and Alon reads as follows.

**Theorem 1** (Akiyama–Alon [2]). Let $n, d$ be positive integers, and let $X$ be a $d$-colored set of points in general position in $\mathbb{R}^d$, with each color class containing $n$ points. Then there is a partition of $X$ into $n$ $d$-colorful sets of size $d$ whose convex hulls are pairwise disjoint.

*The authors are supported by DFG via the Collaborative Research Center TRR 109 “Discretization in Geometry and Dynamics.” PVMB is also supported by the grant ON 174008 of the Serbian Ministry of Education and Science. GMZ is also supported by DFG via the Berlin Mathematical School BMS.*
Akiyama and Alon gave a beautifully simple proof using a discrete version of the ham-sandwich theorem, which is a well known consequence of the Borsuk–Ulam theorem. The use of such topological methods created a lot of progress in solving discrete partitioning problems. In fact, many related partition results have both a continuous mass partition as well as a discrete colored version—often equivalent.

In this paper, we consider the following conjecture of Holmsen, Kynčl and Valculescu [5, Conjecture 3].

**Conjecture 2** (Holmsen–Kynčl–Valculescu, 2016). Let \( m, k, n, \) and \( d \) be positive integers, and let \( X \) be an \( m \)-colored set of \( kn \) points in general position in \( \mathbb{R}^d \). Suppose there is a partition of \( X \) into \( n \) \( d \)-colorful sets of size \( k \). Then there is also such a partition with the additional geometric property that the convex hulls of the \( n \) sets are pairwise disjoint.

Here, the assumption of the existence of a partition depends only on the number of color classes and their sizes, and it involves no geometry. It is obviously a necessary condition. In particular, it implies that \( m \geq d \) and \( k \geq d \).

**Theorem 1** answers the case when \( k = m = d \). The case \( m \geq k = d = 2 \) was settled by Aichholzer et al. [1] and by Kano, Suzuki and Uno [8]. Further developments on the planar case were made independently by Bespamyatnikh, Kirkpatrick and Snoeyink [3], Ito, Uehara and Yokoyama [6] as well as Sakai [11], who confirmed the conjecture for two colors \( (m = d = 2) \) when the sizes of the color classes are divisible by \( n \). Holmsen, Kynčl and Valculescu resolved the conjecture for the remaining cases in the plane, as well as for the case when \( k - 1 = m = d \geq 2 \), the latter by giving a particular discretization of the ham-sandwich theorem [5]. Their method is similar to the one used previously by Kano and Kynčl [7] to establish the case \( m - 1 = d = k \), who for the proof developed a generalization of the ham-sandwich theorem for \( d + 1 \) measures in \( \mathbb{R}^d \), which they called the hamburger theorem.

Holmsen et al. emphasized the connection of the conjecture with a continuous analogue for the case \( m = d \), proved in the plane by Sakai [11] and extended to arbitrary dimension by Soberón [13] and independently by Karasev [9]. (A more general version, for functions that are not necessarily measures, was obtained soon after by Karasev, Hubard and Aronov [10] and by Blagojević and Ziegler [4].)

**Theorem 3** (Soberón–Karasev, 2010). Let \( n, d \) be positive integers, and let \( \mu_1, \ldots, \mu_d \) be absolutely continuous finite measures on \( \mathbb{R}^d \) with respect to the Lebesgue measure. Then there exists a partition of \( \mathbb{R}^d \) into \( n \) convex regions \( C_1, \ldots, C_n \) that simultaneously equipartitions all \( d \) measures, that is,

\[
\mu_i(C_j) = \frac{1}{n} \mu_i(\mathbb{R}^d)
\]

for all \( i \in \{1, \ldots, d\} \) and all \( j \in \{1, \ldots, n\} \).

Holmsen, Kynčl and Valculescu state:

“However, going from the continuous version to the discrete version seems to require, in many cases, a non-trivial approximation argument, and we do not see how the continuous results […] could be used to settle our Conjecture 3 for the case \( m = d \).”
Indeed, this is not straightforward. However, in this paper we show how it can be done: We confirm Conjecture 2 when \( m = d \), as a direct corollary of the following main result. For this we say that a partition of a finite set \( A \) into \( n \) parts is an equipartition if each of the parts contains \( \lceil \frac{|A|}{n} \rceil \) or \( \lfloor \frac{|A|}{n} \rfloor \) elements of \( A \).

**Theorem 4.** Let \( n, d \) be positive integers, and let \( X \) be a \( d \)-colored set of points in general position in \( \mathbb{R}^d \). Then there exists an equipartition of \( X \) into \( n \) subsets which simultaneously equipartition each of the color classes and whose convex hulls are pairwise disjoint.

To see that Theorem 4 implies Conjecture 2 for the case \( m = d \), observe that in this case the condition on \( X \) of admitting a partition into \( n \) pairwise disjoint \( d \)-colorful sets of size \( k \) implies that each color class has at least \( n \) elements. In an equipartition of a color class \( X_i \), each part contains at least \( \lfloor \frac{|X_i|}{n} \rfloor \geq 1 \) points. Thus, each part of \( X \) contains all \( d \) colors. With \( |X| = kn \) and an equipartition of \( X \), we get \( n \) sets of size \( k \) that each contain at least one point of each of the \( d \) colors.

### 2 Preliminaries

In order to discretize Theorem 3, we start by employing a classical idea (see Alon & Akiyama [2]): We replace the points in \( X \) with small enough closed balls and then define measures on these. The problem with applying the continuous result is that the boundaries of the regions may cut through some balls, see Figure 1 (left). We will assign every such “ambiguous” point to one of the regions intersected by the ball centered at the point.

The following lemma shows that, if the radius \( \varepsilon \) of the balls is small enough, we will always get a partition of \( X \) with disjoint convex hulls, no matter how we resolve the ambiguities. In Section 3 we will prove that we can resolve these ambiguities in such a way that we get an equipartition of the full point set \( X \) as well as of each of the color classes \( X_i \). Note that this does not guarantee the existence of a partition of \( \mathbb{R}^d \) into \( n \) convex regions such that each region would contain one of the disjoint convex hulls.

By general position, no \( \ell \) points of \( X \) with \( 1 \leq \ell \leq d \) lie on a common \((\ell - 1)\)-flat (affine subspace of dimension \( \ell - 1 \)). When we replace the points by balls, we make their radius small enough so that no \( \ell \) of these balls are intersected by any \((\ell - 1)\)-flat.

**Lemma 5.** Let \( P \subseteq \mathbb{R}^d \) be a finite set of points in general position, and let \( \varepsilon > 0 \) be chosen such that no \( \ell \) closed balls \( B_{\varepsilon}(x) \) of radius \( \varepsilon \) centered at points \( x \) from \( P \) with \( 1 \leq \ell \leq d \) can be intersected by a common \((\ell - 1)\)-flat. Suppose we are given an affine hyperplane \( H \subseteq \mathbb{R}^d \) and a partition of \( P = P^+ \sqcup P^- \) satisfying

\[
P^+ \subseteq \{ x \in P : B_{\varepsilon}(x) \cap H^+ \neq \emptyset \} \quad \text{and} \quad P^- \subseteq \{ x \in P : B_{\varepsilon}(x) \cap H^- \neq \emptyset \},
\]

where \( H^+ \) and \( H^- \) are the open half-spaces determined by \( H \). Then

\[
\text{conv } P^+ \cap \text{conv } P^- = \emptyset.
\]

**Proof.** The proof is based on the perturbation argument from [2, Proof of Lemma 2], which we make more explicit.
We perform a reverse induction on the number \( \ell \) of \( \varepsilon \)-balls intersected by \( H \), starting with the maximum possible value \( \ell = d \) and proceeding downwards. For the induction basis, when \( \ell = d \), we choose points \( q_1, \ldots, q_d \) in the intersection of the \( d \) balls with \( H \). We move them straight to the ball centers at constant speed, and we let the hyperplane \( H \) through the points follow along. By our assumptions, \( H \) is always uniquely defined throughout the motion, and it will not intersect any other ball at any time. When the points \( q_i \) arrive, \( H \) goes through \( d \) points of \( P \). We now perturb each point \( q_i \) perpendicular to \( H \) in the appropriate direction, so that the corresponding point of \( P \) will be on the desired side, \( H^+ \) or \( H^- \). The position of the remaining \( n - d \) points with respect to \( H \) is unchanged throughout this process, and it was correct from the beginning since \( H \) did not intersect their balls. Thus, we have a hyperplane strictly separating the sets \( P^+ \) and \( P^- \). Consequently, \( \text{conv} \: P^+ \cap \text{conv} \: P^- = \emptyset \).

Let us now consider the case that \( H \) intersects \( \ell < d \) balls. As above, we choose points \( q_1, \ldots, q_\ell \) from \( H \) inside these balls, and we move them towards the ball centers. The motion of \( H \) is no longer uniquely defined, but since the moving points are never in degenerate position, they span an \((\ell - 1)\)-flat \( L \) that moves continuously, and we can continuously move the hyperplane \( H \) while containing \( L \). If, during this motion, \( H \) intersects an additional ball, we have increased \( \ell \) and we proceed by induction. Otherwise, we arrive at a position where \( H \) goes through \( \ell \) points of \( P \), and we perform the perturbation as above.

In order to assign boundary points to regions we will set up a flow network with a fractional flow; from this we obtain an integer flow, which in turn will determine the assignment.

In a directed graph \( D = (V, A) \) with a set of vertices \( V \) and a set of arcs \( A \), a flow is a function \( f : A \rightarrow \mathbb{R} \) that assigns a real number to each arc. The excess of the flow \( f : A \rightarrow \mathbb{R} \) at the vertex \( v \) of the graph \( V \) is the difference between the inflow and the outflow:

\[
\text{excess}(f, v) := \sum_{(u,v) \in A} f(u, v) - \sum_{(v,w) \in A} f(v, w)
\]

To obtain an integer flow from a fractional one, we will use the following statement.

**Proposition 6.** Let \( D = (V, A) \) be a directed graph, and let \( F_-, F_+ : A \rightarrow \mathbb{Z} \), and \( E_-, E_+ : V \rightarrow \mathbb{Z} \) be integer-valued functions on the arcs and on the vertices, respectively. If there is some flow \( f : A \rightarrow \mathbb{R} \) on \( D \) such that

\[
F_-(a) \leq f(a) \leq F_+(a) \quad \text{for } a \in A \quad \text{and} \quad E_-(v) \leq \text{excess}(f, v) \leq E_+(v) \quad \text{for } v \in V,
\]

then there is also an integer flow \( f' : A \rightarrow \mathbb{Z} \) that satisfies the same bounds.

**Proof.** This is a variation of the well-known integrality results on network flows. Classical flow networks involve only a single vertex with negative excess (source) and a single vertex with positive excess (sink), conserving the flow at all other vertices. The network we consider has several sources and sinks. Additionally, the excesses at these vertices are not fixed but allowed to vary within bounds, and we have lower as well as upper capacities on the arcs.

Such networks can be reduced to the classical situation by standard transformations; We sketch these transformations. (See [12, Corollary 11.2i] for an alternative approach via Hoffman’s circulation theorem.)
• First, any arc \((u, v)\) with a positive lower bound \(F_-(u, v)\) and upper bound \(F_+(u, v)\) is replaced by a conventional arc with nonnegative flow and upper bound \(F_+(u, v) - F_-(u, v)\). To compensate this offsetting of the flow, the excesses have to be adapted. We subtract \(F_-(u, v)\) from the excess bounds \(E_-(v)\) and \(E_+(v)\) at \(v\) and add it to the excess bounds \(E_-(u)\) and \(E_+(u)\) at \(u\). In the network that we will use, there are no arcs with negative bounds. Such an arc \((u, v)\) could be treated by introducing the reverse arc \((v, u)\).

• To deal with the variation of the excess, we create an additional “balancing sink” \(B\). We fix the excess of each vertex at its lower bound. Thus, a vertex \(v\) with excess bounds \(E_-(v)\) and \(E_+(v)\) (as modified in the previous step) is declared to be a sink with demand \(E_-(v)\) if \(E_-(v) \geq 0\), or a source with supply \(-E_-(v)\) if \(E_-(v) < 0\). The excessive inflow at \(v\) is then absorbed by an arc \((v, B)\) of capacity \(E_+(v) - E_-(v)\). We fix the demand of \(B\) to make the overall sum of demands equal to the overall supply.

• Finally, we add a new super-source \(S\) with an arc to every source \(v\), of capacity \(F_+(S, v)\) equal to the supply at \(v\), and a new super-sink \(T\), with an arc from every sink \(v\) to \(T\), of capacity \(F_+(v, T)\) equal to the demand at \(v\).

Flows in the original graph \(D\) correspond, in the transformed network, to classical flows from \(S\) to \(T\) that saturate all edges out of \(S\). Integrality is preserved throughout the transformation.

3 Proof of the main result

Proof of Theorem\(^4\). Let \(n\) and \(d\) be positive integers, and let \(X\) be a \(d\)-colored set of points in general position in \(\mathbb{R}^d\). Using the tools presented in Section\(^2\), we now prove our claim that we can partition \(X\) into \(n\) sets of size \(\lfloor \frac{|X|}{n} \rfloor\) or \(\lceil \frac{|X|}{n} \rceil\) that have pairwise disjoint convex hulls and simultaneously equipartition the color classes. The proof is done in several steps.

1. From points to measures.

   We replace each point \(x \in X\) by a ball \(B_\varepsilon(x)\) centered at \(x\), with \(\varepsilon > 0\) a real number small enough such that no \(\ell\)-flat with \(\ell < d\) intersects more than \(\ell + 1\) balls. With each ball centered at a point in \(X\), we associate a uniformly distributed measure of unit total mass. For each \(i \in \{1, \ldots, d\}\) and for every measurable subset \(A \subseteq \mathbb{R}^d\), let \(\mu_i(A)\) be the total measure of balls centered at points in \(X_i\) that is captured by \(A\). Clearly, \(\mu_1, \ldots, \mu_d\) are absolutely continuous finite measures on \(\mathbb{R}^d\) with \(\mu_i(\mathbb{R}^d) = |X_i|\). According to Theorem\(^3\), there exists a partition of \(\mathbb{R}^d\) into \(n\) convex regions \(C_1, \ldots, C_n\) which equipartitions the measures, that is,

   \[ \mu_i(C_j) = \frac{|X_i|}{n} \]

   for all \(i \in \{1, \ldots, d\}\) and all \(j \in \{1, \ldots, n\}\).

2. A directed graph of incidences.

   In order to apply Lemma\(^5\), we show the existence of an assignment of the points in \(X\) to the \(n\) regions \(C_1, \ldots, C_n\) such that for each point \(x\) assigned to a region \(C_j\), \(B_\varepsilon(x)\) intersects \(C_j\), while in total \(\lfloor \frac{|X|}{n} \rfloor\) or \(\lceil \frac{|X|}{n} \rceil\) points are assigned to each region, with \(\lfloor \frac{|X_i|}{n} \rfloor\) or \(\lceil \frac{|X_i|}{n} \rceil\) of color class \(i\) for every \(i \in \{1, \ldots, d\}\). Such an assignment may be modeled as an integer flow from the
Figure 1: A configuration of 11 points/small balls with $d = 2$ colors in $d = 2$ dimensions, partitioned into $n = 3$ regions, and the corresponding directed graph $D$ with some upper and lower bounds on the flow and its excess indicated as intervals.

points in $X$ to the regions in the partition, where each $x \in X$ has an outflow of 1 and each region has an inflow of $\lceil |X|/n \rceil$ or $\lfloor |X|/n \rfloor$, the number of points assigned to it. To guarantee an equipartition of the color classes, we add a middle layer of vertices, one for each color and region, and set the constraints on these vertices and arcs accordingly.

We define the directed graph $D = (V, A)$ with $V = X \sqcup Y \sqcup Z$, where

$$Y = \{ y^i_j : 1 \leq i \leq d, \ 1 \leq j \leq n \}$$

contains a vertex $y^i_j$ for each color $i$ and each region $C_j$, and the set $Z = \{C_1, \ldots, C_n\}$ contains a vertex for each region. We have arcs from a point $x \in X$ to those vertices in $Y$ corresponding to the color of $x$ and the regions incident to the ball $B_\varepsilon(x)$ centered at $x$, as well as arcs from the vertices in $Y$ to their respective region in $Z$. More precisely, the set of arcs is

$$A := \{ (x, y^i_j) : 1 \leq i \leq d, \ 1 \leq j \leq n, \ x \in X, \ B_\varepsilon(x) \cap C_j \neq 0 \}$$

$$\cup \ \{ (y^i_j, C_j) : 1 \leq i \leq d, \ 1 \leq j \leq n \}.$$

For the vertices of $D$, we define lower bounds $E_- : V \rightarrow Z$ and upper bounds $E_+ : V \rightarrow Z$ on the excess as follows:

$$E_- (x) := -1 \quad E_+ (x) := -1$$

$$E_- (y) := 0 \quad E_+ (y) := 0$$

$$E_- (C_j) := \lfloor |X|/n \rfloor \quad E_+ (C_j) := \lceil |X|/n \rceil$$

6
For the arcs of $D$, we define lower bounds $b^- : A \to \mathbb{Z}$ and upper bounds $b^+ : A \to \mathbb{Z}$ as follows:

$$b^-(x, y) := 0 \quad b^+(x, y) := 1$$

$$b^-(y^j_i, C_j) := \left\lfloor \frac{|X_i|}{n} \right\rfloor \quad b^+(y^j_i, C_j) := \left\lceil \frac{|X_i|}{n} \right\rceil$$

In all five cases, the lower bounds don’t exceed the upper bounds.

(3) A fractional flow.

We now construct a fractional flow $f : A \to \mathbb{R}$ by setting

$$f(x, y^j_i) := \mu_i(B_{\varepsilon}(x) \cap C_j) \quad \text{and} \quad f(y^j_i, C_j) := \frac{|X_i|}{n}.$$ 

The lower and upper constraints on the arcs are trivially satisfied,

$$b^-(a) \leq f(a) \leq b^+(a) \quad \text{for all} \quad a \in A.$$ 

With $\mu_i(B_{\varepsilon}(x)) = 1$ for all $x \in X_i$, we get

$$E^-(x) = -1 = \text{excess}(f, x) = -\sum_{j=1}^n \mu_i(B_{\varepsilon}(x) \cap C_j) = -1 = E^+(x).$$

With $\mu_i(C_j) = \frac{|X_i|}{n} = f(y^j_i, C_j)$ for a vertex $y^j_i \in Y$, the values yield

$$E^-(y^j_i) = 0 = \text{excess}(f, y^j_i) = \sum_{x \in X} \mu_i(B_{\varepsilon}(x) \cap C_j) - f(y^j_i, C_j) = 0 = E^+(y^j_i).$$

Lastly, for a $C_j \in Z$ we get

$$E^-(C_j) = \left\lfloor \frac{|X|}{n} \right\rfloor \leq \text{excess}(f, C_j) = \sum_{i=1}^d f(y^j_i, C_j) = \frac{|X|}{n} \leq \left\lceil \frac{|X|}{n} \right\rceil = E^+(C_j),$$

and consequently $E^-(v) \leq \text{excess}(f, v) \leq E^+(v)$ for all $v \in V$.

(4) Back to geometry.

From this fractional flow, Proposition[6] produces an integer flow on $D$ that satisfies the constraints given by functions $b^-$, $b^+$, and $E^-$, $E^+$. This in turn gives an assignment of points into sets of size $\left\lfloor \frac{|X|}{n} \right\rfloor$ and $\left\lceil \frac{|X|}{n} \right\rceil$, equipartitioning $X$. The middle layer of $D$ ensures that each of the sets contains $\left\lfloor \frac{|X|}{n} \right\rfloor$ or $\left\lceil \frac{|X|}{n} \right\rceil$ points from the color class $X_i$, resulting in a simultaneous equipartition of $X$ and all $d$ color classes.

We now want that, for any two regions $C_j$ and $C_k$, the sets of points $P^+$ assigned to $C_j$ and $P^-$ assigned to $C_k$ have disjoint convex hulls. For each point $x$ assigned to a region, $B_{\varepsilon}(x)$ intersects that region, by the definition of the arc set $A$. We may therefore apply Lemma[5] to the set $P = P^+ \cup P^-$ and conclude that the convex hulls of $P^+$ and $P^-$ are disjoint.

Acknowledgements

We are grateful to the four DCG referees for many useful comments and suggestions.
References

[1] O. Aichholzer, S. Cabello, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, F. Hurtado, and D. R. Wood. Edge-removal and non-crossing configurations in geometric graphs. *Discrete Math. Theor. Comput. Sci. (DMTCS)*, 12:75–86, 2010.

[2] J. Akiyama and N. Alon. Disjoint simplices and geometric hypergraphs. In *Combinatorial Mathematics: Proc. of the Third International Conference, New York 1985*, volume 555 of *Annals of the New York Academy of Sciences*, pages 1–3, 1989.

[3] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink. Generalizing ham sandwich cuts to equitable subdivisions. *Discrete Comput. Geom.*, 24:605–622, 2000.

[4] P. V. M. Blagojević and G. M. Ziegler. Convex equipartitions via Equivariant Obstruction Theory. *Israel Journal of Mathematics*, 200:49–77, 2014.

[5] A. F. Holmsen, J. Kynčl, and C. Valculescu. Near equipartitions of colored point sets. *Comput. Geom. Theory Appl.*, 65:35–42, 2017.

[6] H. Ito, H. Uehara, and M. Yokoyama. 2-dimension ham sandwich theorem for partitioning into three convex pieces. In *Discrete and Computational Geometry: Japanese Conference, JDCG’98 Tokyo, Japan, December 9–12, 1998. Revised Papers*, pages 129–157. Springer-Verlag, 2000.

[7] M. Kano and J. Kynčl. The hamburger theorem. *Comput. Geom. Theory Appl.*, 68:167–173, 2018.

[8] M. Kano, K. Suzuki, and M. Uno. Properly colored geometric matchings and 3-trees without crossings on multicolored points in the plane. In *Discrete and Computational Geometry and Graphs*, volume 8845 of *Lecture Notes in Comput. Sci.*, pages 96–111. Springer-Verlag, 2014.

[9] R. Karasev. Equipartition of several measures. Preprint, June 2013, 11 pages, [arXiv:1011.4762v7](http://arxiv.org/abs/1011.4762v7).

[10] R. Karasev, A. Hubard, and B. Aronov. Convex equipartitions: The spicy chicken theorem. *Geometriae Dedicata*, 170:263–279, 2014.

[11] T. Sakai. Balanced convex partitions of measures in $\mathbb{R}^2$. *Graphs and Combinatorics*, 18:169–192, 2002.

[12] A. Schrijver. *Combinatorial Optimization—Polyhedra and Efficiency. Vol. A: Paths, Flows, Matchings*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003.

[13] P. Soberón. Balanced convex partitions of measures in $\mathbb{R}^d$. *Mathematika*, 58:71–76, 2012.