Linear stability of a supersonic boundary layer on a plate under conditions of vibrational excitation and of viscous stratification

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Abstract. An asymptotic theory of the neutral stability curve for a supersonic boundary layer of a vibrationally excited molecular gas on a plate is developed. The initial mathematical model of flow consists of equations of two-temperature viscous heat-conducting gas dynamics, which are used to derive a spectral problem for a linear system of eighth-order ordinary differential equations within the framework of the classical linear stability theory. Unified transformations of the system for all shear flows are performed in accordance with the classical Lin scheme. The problem is reduced to an algebraic secular equation with separation into the inviscid and viscous parts, which is solved numerically. It is shown that the thus-calculated neutral stability curves agree well with the previously obtained results of the direct numerical solution of the original spectral problem. In this case, the critical Reynolds number increases with excitation enhancement of the internal degrees of freedom of molecules, and the neutral stability curve is shifted toward the domain of higher wave numbers. This is also confirmed by means of solving an asymptotic equation for the critical Reynolds number.

1. Introduction
The classical investigations of the hydrodynamic stability theory in its early period (1940s–1950s) were exclusively based on asymptotical methods. The first meaningful results, mainly for the boundary layer on a flat plate, were summarized in [1, 2]. Further improvement of analytical approaches made it possible to find some general features in the field of hydrodynamic stability and transition to turbulence, and also to estimate the critical parameters of some flows important for practice, which were fairly close to actually observed values. At the same time, asymptotic methods have some constraints, first of all, on the flow parameter values, i.e., Mach and Reynolds numbers, phase velocities, and disturbance wavelengths. Moreover, activities based on asymptotic methods often involved a number of assumptions whose validity could not be verified at that time.

As computer hardware and software were developed, they provided a possibility of the direct numerical solution of problems of the linear stability theory for flows important in practice without constraints on flow parameters. The founder of the new research direction started in the 1960s was L.M. Mack [3]. These and subsequent numerical studies of other authors shifted the use of asymptotic approximations to the background. However, these new investigations also assisted in improvement of the classical analytical methods, particularly, in refining the
area of their applicability and avoiding unjustified assumptions. Therefore, verified analytical results are described in monographs on the hydrodynamic stability theory and are still used for comparisons with results of numerical and experimental studies.

New problems of hydrodynamic stability, e.g., for flows of chemically reacting, of optically active, and of thermally nonequilibrium molecular gases [4] stimulate generalization of the classical results [1, 2], first of all approximate estimates of the critical Reynolds numbers, for environments and flows that were not considered in earlier researches.

The present paper describes the asymptotic theory of the neutral stability curve for a supersonic boundary of a vibrationally excited molecular gas.

2. Statement of problem and basic equations
In the framework of the linear stability theory, the development of two-dimensional disturbances is considered for a plane semi-infinite plate flow of a vibrationally excited gas. The origin of the Cartesian coordinate system \((x, y)\) coincides with a leading edge of the plate, the \(x\) coordinate is oriented along the plate in the direction of the carried flow, the \(y\) coordinate is directed to the flow along the normal to the plate, respectively. The flow is described by a system of equations of the two-temperature gas dynamics [4].

The current distance \(x = L\) along the plate and parameters of the unperturbed flow outside the boundary layer, marked by the index "\(\infty\)" were chosen for nondimensionalization: the speed, \(U_\infty\), the density, \(\rho_\infty\), and the temperature, \(T_\infty\), the coefficients of the shear and bulk viscosities, \(\eta_\infty\) and \(\eta_b\), correspondingly, the thermal conductivity coefficient due to the energy transfer in translational and rotational degrees of freedom, \(\lambda_\infty = \lambda_{t}\\infty + \lambda_{r}\\infty\), the coefficient of thermal conductivity describing the diffusion transfer of the energy of vibrational quanta, \(\lambda_v\). For nondimensionalization of the pressure and time, the combined values of \(\rho_\infty U_\infty^2\) and \(L/U_\infty\), respectively were used. For the temperature dependence of the shear viscosity the Sutherland formula is used. The coefficients of thermal conductivity are analogous to [4] expressed in terms of the shear viscosity and heat capacities by means of semi-empirical Eucken relations. It is assumed that heat capacities are constant. It is also supposed the translational and rotational degrees of freedom of the molecules are in quasi-equilibrium and are determined by the equilibrium relations.

The initial system of equations was linearized on a stationary self-similar boundary layer solution in the absence of vibrational excitation [5]. In deriving the linearized equations for disturbances the instantaneous values of the hydrodynamic variables were represented in the form

\[
\begin{align*}
  u_x &= U_s + u_x^*, & u_y &= u_y^*, & \rho &= \rho_s + \rho^*, & T &= T_s + T^*, & T_v &= T_s + T_v^*, & p &= p_s + p^*, \\
  \eta(T) &= \eta(T_s) + \eta(T_s) T^* = \eta_s + \eta^*, & \lambda(T) &= \lambda(T_s) + \lambda(T_s) T^* = \lambda_s + \lambda^*, \\
  \lambda_v(T) &= \lambda_v(T_s) + \lambda_v(T_s) T^* = \lambda_s + \lambda_v^*, \quad (1)
\end{align*}
\]

The temperature perturbations of the transport coefficients were taken into account

\[
\eta_{Ts} = \left. \frac{d\eta}{dT} \right|_{T=T_s}, \quad \lambda_{Ts} = \left. \frac{d\lambda}{dT} \right|_{T=T_s}, \quad \lambda_{v,Ts} = \left. \frac{d\lambda_v}{dT} \right|_{T=T_s}.
\]

Here the subscript \(s\) denotes the values of the hydrodynamic variables related to the stationary flow, and the prime denotes the disturbances of these variables.

Substituting (1) into the initial system of equations of two-temperature gas dynamics and linearizing it relative to disturbances of hydrodynamic variables up to the values of the first order
of smallness leads to a system of equations for disturbances [4]. The stability of disturbances periodic over the longitudinal coordinate $x$ in the form of traveling plane waves

$$q^*(x, y, t) = q_0(y) \exp \left[ i \alpha (x - ct) \right], \quad q^* = (u^*, u_y^*, \rho^*, T^*, T_v^*, p^*), \quad q_0(y) = (u, \alpha v, \rho, \theta, \theta_v, p)$$

was considered. Here $\alpha$ is the wavenumber along the periodic variable $x$, $c = c_r + ic_i$ is the complex phase velocity, $i$ is the imaginary unit.

The system of equations for the perturbation amplitudes $q_0$ has the form

$$D \rho + \alpha \rho_s' v + \alpha \rho_s \sigma = 0,$$

$$\frac{\eta_s}{Re} \Delta u - \rho_s D u - \alpha \rho_s v U_s' - i \alpha \varepsilon + \frac{\eta T_s T'}{Re} \left( u' + i \alpha^2 v \right) + \frac{(\eta T_s U_s')'}{Re} \theta + \frac{\eta T_s U_s'}{Re} \theta' = 0,$$

$$\frac{\alpha \eta_s}{Re} \Delta v - \alpha \rho_s D v - \varepsilon' + \frac{\alpha \eta T_s T'}{Re} \left( v' - i u \right) + \frac{\alpha \eta T_s U_s'}{Re} \theta = 0,$$

$$\frac{\eta_s \gamma}{Re Pr} \Delta \theta - \rho_s D \theta - \alpha \rho_s v T_s' - \alpha (\gamma - 1) \sigma + \frac{2 \gamma (\gamma - 1) \eta_s M^2}{Re} \left( u' + i \alpha^2 v \right) U_s' +$$

$$\gamma c v \rho_s \tau (\theta_v - \theta) + \frac{2 \gamma \eta T_s T_s'}{Re Pr} \theta' + \left[ \frac{\gamma (\eta T_s T_s')}{Re Pr} + \frac{\gamma (\gamma - 1) \eta T_s M^2 U_s'^2}{Re} \right] \theta = 0.$$  (2)

$$\frac{20}{33} \frac{\gamma c v \eta_s}{Re Pr} \Delta \theta_v - \gamma c v \rho_s D \theta_v - \alpha c v \rho_s v T_s' - \gamma c v \rho_s \tau \theta_v = \frac{20}{33} \frac{\gamma c v (\eta T_s T_s')'}{Re Pr} \theta +$$

$$\frac{20}{33} \gamma c v \eta T_s T_s' \theta' + \frac{20}{33} \gamma c v \eta T_s T_s' \theta_v = 0,$$

$$\gamma M^2 \rho_s = \rho_s \theta + \rho T_s.$$}

Here we use the following notations

$$D = i \alpha (U_s - c), \quad \sigma = v' + i u, \quad \varepsilon = p - \frac{\alpha \eta_s}{Re} \left( \alpha_1 + \frac{1}{3} \right) \sigma, \quad \Delta = \frac{d^2}{dy^2} - \alpha^2,$$

and the primes in functions denote differentiation with respect to the variable $y$. The system (2) together with the homogeneous boundary conditions defines the spectral problem on the spectrum of disturbances whose eigenvalues are the complex-valued phase disturbance velocities $c = c_r + ic_i$. In system (2) $\alpha_1 = \eta_{\infty}/\eta_\infty$ is the ratio of the bulk viscosity to the shear viscosity, $\gamma = c_p/c_v$ is the ratio of specific heats, $c_v = c_{v,t} + c_{v,r}$ and $c_p = c_v + R$ are the specific heats at constant volume and pressure correspondingly, which are presented as the sum of specific heats induced by translational and rotational motion of molecules, $\gamma_v = c_{v,v}/(c_{v,t} + c_{v,r})$ is the parameter characterizing the degree of non-equilibrium of the vibrational mode, $c_{v,v}$ is the specific heat at constant volume corresponding to the relaxing vibrational mode, $\tau$ is the characteristic time of relaxation of the vibrational mode, $R$ is the gas constant, $Re = \rho_\infty U_\infty L/\eta_\infty$ and $M = U_\infty/\sqrt{\gamma R T_\infty}$ are the Reynolds and Mach numbers, respectively, $Pr = \eta_\infty c_v/\lambda_\infty$ is the Prandtl number.

Since in the subsequent calculations in the system (2) profiles of hydrodynamic variables depending on the self-similar Dorodnitsyn–Howard variable [6] were used, the derivatives with respect to the $y$ were replaced by the derivatives with respect to $\xi$ in accordance with relation

$$\frac{d}{dy} = \frac{\rho_s}{2} \frac{d}{d\xi}.$$
The asymptotic solutions of the system (2) for the large Reynolds numbers Re are constructed in the form of a perturbation series

\[ q(y) = q_0(y) + \frac{1}{\text{Re}} q_1(y) + \ldots \]

In the zero approximation one obtains a system of equations for inviscid perturbations. In the work [5] has been shown that this system is reduced to the second order linear equation for pressure perturbation. Thus the zero approximation allows us to find only two the linearly independent solutions. The remaining six solutions one has to determine by direct consideration a full system. In this case, only decreasing solutions at Re \( \rightarrow \infty \) should be used to derive the secular equation.

3. Asymptotic of inviscid and viscous of solutions. Secular equation

Two linearly independent inviscid solutions of equation for pressure perturbation

\[ p'' - a(\psi) p' - b(\psi) p = 0, \quad a(\psi) = 2W'/W, \quad b(\psi) = 4\chi\alpha^2 T_s, \quad \psi = \xi - \xi_c. \quad (3) \]

Here

\[ \chi = T_s - m^2 M^2 W^2, \quad W = U_s - c, \quad m^2 = m_i^2 + i m_i, \]

\[ m_i^2 = \frac{R_1 (1 + \gamma v) + \Delta^2}{R_1^2 + \Delta^2}, \quad m_i = \frac{- \gamma v (\gamma - 1) \Delta}{\gamma R_1^2 + \Delta}, \quad R_1 = 1 + (\gamma v/\gamma), \quad \Delta = \alpha \tau (y - c). \]

For the neutral perturbations considered below, we have \( c = c_r \). As the first Rayleigh condition \( c_r \in [0, 1] \) [4] has also to be satisfied for neutral perturbations, then \( \xi = \xi_c \) is a regular singular point of equation (3). Its solutions in the neighborhood of the singular point are found by the Frobenius method [6, 7]. In what follows the subscript “c” is used to indicate the values of the variables at the point \( \xi_c \), where the phase velocity is equal to the free-stream velocity (in the critical layer): \( U_s(\xi_c) = c_r \). Equation (3) was solved numerically. The algorithm of its solution can be found in the work [6]. The remaining inviscid solutions are expressed via the pressure perturbation as

\[ u_{\text{inv}} = -\frac{T_s}{W} \left( p + \frac{p'U'_s}{\alpha^2 W} \right), \quad v_{\text{inv}} = -\frac{ip' T_s}{\alpha^2 W}, \]

\[ \theta_{\text{inv}} = T_s \left[ \frac{(\gamma - 1) (1 + i \alpha \tau W) p M^2}{1 + (\gamma v/\gamma) + i \alpha \tau W} + \frac{p'U'_s}{\alpha^2 W} \right]. \]

To find viscous solutions the system (2) with using some simplifications was reduced to form which analytical to Dunn–Lin viscous system [6]:

\[ v' + iu - i (U_s - c) \frac{\theta}{T_s} = 0, \quad u'' - i \alpha \text{Re} (U_s - c) \frac{u'}{\eta_s T_s} = 0, \quad (5) \]

\[ \theta'' - i \alpha \text{PrRe} (U_s - c) \frac{\theta}{\eta_s T_s} + \frac{\gamma v \text{PrRe}}{\gamma \tau \eta_s T_s} (\theta_v - \theta) = 0, \quad (6) \]

\[ \theta_v'' - \frac{33}{20} i \alpha \text{PrRe} (U_s - c) \frac{\theta_v}{\gamma \eta_s T_s} - \frac{33 \text{PrRe}}{20 \gamma \eta_s T_s} (\theta_v - \theta) = 0. \quad (7) \]

For diatomic gases one can approximately suppose that \( 33/(20 \gamma) \approx 1 \). It allows one by summation and subtraction equations (6), (7) to introduce equations for auxiliary functions \( \theta_+ = \theta + (\gamma v \theta_v/\gamma), \theta_- = \theta - \theta_v \). Then temperatures are correspondingly expressed by formulae

\[ \theta = \frac{\gamma_+ \theta_+}{\gamma + \gamma v}, \quad \theta_v = \frac{(\theta_+ + \theta_-)}{\gamma + \gamma v}. \]
The momentum equation for $u$ and equations for functions $\theta_+$, $\theta_-$ were reduced to the Airy equations. Its solutions were presented through the generalized Airy functions of first and second orders $A_k(z, \rho)$ [8]. As a result, solutions for temperatures were found. This allowed one to splinter the last equation from system (5)–(7) and to transit by such a way to a sixth order system. The reduced system has two trivial solutions $u = v = \theta = 0$ and $u = 1$, $v = -iy$, $\theta = \theta_v = 0$. Since here we consider only decreasing (bounded) solutions as $Re \to \infty$, the only zero trivial solution satisfying this condition will be replaced by the inviscid solution (4):

$$V_1(y) = [u_{inv}(y), v_{inv}(y), \theta_{inv}(y)].$$

(8)

Linearly independent solutions for disturbances of transversal velocity are obtained by integration of the momentum equation for $u$ for two alternatives: 1) $\theta = 0$, $u \neq 0$ and 2) $u = 0$, $\theta \neq 0$. Its also were expressed through the generalized Airy functions of first and second orders. As a result, it was shown that the linearly independent decreasing (bounded) solutions as $Re \to \infty$ of simplified viscous system have the form:

$$V_2(y) = [u_{vis}(y), v_{vis1}(y), 0], \quad V_3(y) = [0, v_{vis2}(y), \theta_{vis}].$$

(9)

For derivation of a secular equation it is necessary to require that a linear combination of independent solutions (8), (9) should satisfy the homogeneous boundary conditions

$$c_1 V_1(0) + c_2 V_2(0) + c_3 V_3(0) = 0.$$  

(10)

The homogeneous system (10) has nontrivial solutions $(c_1, c_2, c_3)$ if its determinant equals to zero:

$$\Delta = \begin{vmatrix} u_{inv}(0) & u_{vis}(0) & 0 \\ v_{inv}(0) & v_{vis1}(0) & v_{vis2}(0) \\ \theta_{inv}(0) & 0 & \theta_{vis}(0) \end{vmatrix} = 0.$$

This equality represents the original form of the secular equation.

Use of the Airy functions allows one to represent the viscous part of secular equation through the tabulated Tietjens function $F(z)$ and the auxiliary function $G(z, \rho)$, which are usually applied in the traditional asymptotic theory of stability [1, 6, 8]. The secular equation obtained by such a way has the form:

$$\Pi(0) = \frac{p'(0)}{\alpha^2 p(0)} = -\frac{c \Omega [F(z) + NG(z, \rho)]}{\Omega U_s(0)F(z) - i},$$

$$N = \frac{(\gamma - 1) M^2 c^2}{T_s(0)} \left[ \frac{1 - i\alpha c\tau}{1 + (\gamma_\nu/\gamma) - i\alpha c\tau} \right], \quad \Omega = \frac{3}{2} \sqrt{\frac{\eta_s T_s(0)}{c^3}} \int_0^{y_e} \sqrt{\frac{U_s - c}{\eta_s T_s}} dt.$$

(11)

4. **Numerical calculations of neutral curves**

In the calculations of the spectra viscous perturbation the Reynolds number $Re$ determined from the current length $L$, was replaced by the number, $Re_3 = \sqrt{Re}$, which is determined from the local of boundary layer thickness. The secular equation (11) has a characteristic structure, which coincides with analogical equations [2, 6]. Left inviscid part of (11) depends on the phase velocity $c$ and wave number $\alpha$. At the same time a right viscous part of (11) is expressed through tabulated functions of variable $z$ also depended on the phase velocity $c$. Therefore, points on neutral stability curves $Re_3(\alpha, \gamma_\nu, M)$ are calculated on plane $(Re_3, \alpha)$ in the next sequence. The value of the phase velocity $c$ was set for fixed values of the Mach number $M$ and the degree of vibrational non-equilibrium $\gamma_\nu$ in interval $c = [0, 1]$ with step $\Delta c = 10^{-4}$. The integral in right-hand side of (11) was calculated by the Simpson formula. The real and imaginary parts of
Figure 1. Neutral stability curves $\text{Re} \delta(\alpha)$ for modes I and II. The solid and dashed curves show the results of calculations by spectral problem (2) and the secular equation (11) of the asymptotic theory, correspondingly. $K_1$ and $K'_1$ are the critical points of the mode I and $K_2$ and $K'_2$ are the critical points of the mode II.

right-hand side of (11) are depended on the variable $z$. The arrays of their values was calculated for $z = [0, 10]$ using tables [6, 9]. The table step $\Delta z = 0.04$ was picked out.

The arrays of left side of (11) for a fixed value of the phase velocity $c$ were calculated for $\alpha = [1, 3]$ with a step $\Delta \alpha = 10^{-4}$. Calculated arrays of right and left sides of (11) were compared up to obtaining coincide within the accuracy $10^{-8}$ if such a state could be achieved for a given value of $c$. Then calculation was repeated for next value of the phase velocity $c$. As a result, we constructed arrays of values of wave numbers $\alpha_k$, phase velocities $c_k$, and of variable $z_k$, which correspond to points on neutral curve. Using formula

$$\text{Re} \delta_k(\alpha_k) = \frac{2}{3} \sqrt{\frac{z_k^3}{\alpha_k I_k}}, \quad I_k = \int_0^{y_k} \sqrt{\frac{U_{s,k} - c_k}{\eta_{s,k} T_{s,k}}} dt,$$

values of the Reynolds number $\text{Re} \delta_k$ on neutral curve were calculated. The neutral curves of modes I and II were determined analogical the work [3].

Figure 1 shows the results of calculations of neutral stability curves for perfect ($\gamma_v = 0$) and of vibrationally excited ($\gamma_v = 0.667$) gases at the Mach number $M = 2.2$ and 4.5. It is seen, that asymptotic curves are in satisfactory agreement with numerical calculation of origin spectral problem (2). In particular, dissipative effect of the vibrational relaxation is reflected clearly.
Table 1. Numerical values $\text{Re}_{\delta,\text{cr}}$ and $\alpha_{\text{cr}}$ for modes I and II.

| $\gamma_v$ | $\text{Re}_{\delta,\text{cr}}^s$ | $\alpha_{\text{cr}}^s$ | $\text{Re}_{\delta,\text{cr}}^a$ | $\alpha_{\text{cr}}^a$ | $\text{Re}_{\text{cr}}^s$ | $\alpha_{\text{cr}}^s$ | $\text{Re}_{\text{cr}}^a$ | $\alpha_{\text{cr}}^a$ |
|------------|-----------------|----------------|-----------------|----------------|-----------------|----------------|----------------|----------------|
| M = 2.2    | M = 4.5         | Mode I          | Mode II         |
| 0          | 281             | 0.0580          | 240             | 0.0626         | 462             | 0.0650         | 396             | 0.0653         |
| 0.667      | 317             | 0.0635          | 271             | 0.0685         | 521             | 0.0708         | 446             | 0.0712         |
| 0.667      | —               | —               | —               | —              | 221             | 0.2001         | 191             | 0.2015         |
| 0.667      | —               | —               | —               | —              | 250             | 0.2060         | 214             | 0.2072         |

The transition to the asymptotic theory extends slightly an instability area and reduces the critical Reynolds numbers.

Table 1 allow one to compare data of numerical calculations of spectral problem (2) and asymptotic theory with respect to critical values of the Reynolds numbers $\text{Re}_{\delta,\text{cr}}$ and wave numbers $\alpha_{\text{cr}}$. It is seen that values $\text{Re}_{\delta,\text{cr}}^a$ are approximately 14 ÷ 15 percent less than corresponding values $\text{Re}_{\delta,\text{cr}}^s$. At the same time, the critical Reynolds number increases with excitation enhancement of the internal degrees of freedom of molecules, and the neutral stability curve is shifted toward the domain of higher wave numbers.

5. Conclusion

An asymptotic theory of the neutral stability curve for a supersonic boundary layer of a vibrationally excited molecular gas on a plate is developed. It is shown that the thus-calculated neutral stability curves agree well with the previously obtained results of the direct numerical solution of the original spectral problem. In this case, the critical Reynolds number increases with excitation enhancement of the internal degrees of freedom of molecules, and the neutral stability curve is shifted toward the domain of higher wave numbers. This is also confirmed by means of solving an asymptotic equation for the critical Reynolds number.

These calculations once again confirm existence in a molecular gas of a dissipative effect, which is caused by relaxation of vibrational modes.

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