An investigation of the Buchdahl inequality for spherically symmetric static shells

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Abstract. A classical result by Buchdahl [9] shows that a class of static spherically symmetric solutions of the Einstein equations obey the inequality $2M/R \leq \frac{8}{9}$, where $M$ is the total ADM mass and $R$ the area radius of the body. Buchdahl’s proof rests on the hypotheses that the energy density is non-increasing outwards and that the pressure is isotropic. In this work neither of Buchdahl’s hypotheses are assumed. We consider non-isotropic spherically symmetric shells, supported in $[R_0, R_1]$, $R_0 > 0$, of matter models for which the energy density $\rho \geq 0$, and the radial- and tangential pressures $p \geq 0$ and $q$, satisfy $p + q \leq \Omega \rho$, $\Omega \geq 1$. Note that this inequality holds with $\Omega = 3$ for any matter model which satisfies the dominant energy condition. We show a Buchdahl type inequality for shells which are thin; given an $\epsilon < 1/4$ there is a $\kappa > 0$ such that $2M/R_1 \leq 1 - \kappa$ when $R_1/R_0 \leq 1 + \epsilon$. It is also shown that for a sequence of solutions such that $R_1/R_0 \to 1$, the limit supremum of $2M/R_1$ of the sequence is bounded by $(\frac{2\Omega+1}{2})^2 - 1$ (which equals $8/9$ if $\Omega = 1$). We emphasize that no field equations for the matter are used for this result. However, in the second part we consider the Einstein-Vlasov system and use the matter field equation to construct a family of static solutions with the property that $R_1/R_0 \to 1$. We also show that for this sequence the value $8/9$ of $2M/R_1$ is attained in the limit (note that $\Omega = 1$ for Vlasov matter). The energy density of this sequence get more and more peaked, which should be contrasted to the solution for which $2M/R = 8/9$ in Buchdahl’s original work where $\rho$ is constant and $p$ blows up at $r = 0$. Clearly, this solution cannot satisfy the inequality $p + q \leq \Omega \rho$, in particular it violates the dominant energy condition.

1. Introduction
The metric of a static spherically symmetric spacetime takes the following form in Schwarzschild coordinates

$$ds^2 = -e^{2\mu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. Asymptotic flatness is expressed by the boundary conditions

$$\lim_{r \to \infty} \lambda(r) = \lim_{r \to \infty} \mu(r) = 0,$$

and a regular centre requires

$$\lambda(0) = 0.$$

The Einstein equations read

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho,$$  (1)
\[ e^{-2\lambda}(2\mu_r + 1) - 1 = 8\pi r^2 p, \quad (2) \]
\[ \mu_{rr} + (\mu_r - \lambda_r)(\mu_r + 1/r) = 4\pi q e^{2\lambda}, \quad (3) \]
Here \( \rho \) is the energy density, \( p \) the radial pressure and \( q \) is the tangential pressure. If the pressure is isotropic, i.e. \( 2p = q \), a solution will satisfy the well-known Tolman-Oppenheimer-Volkov equation for equilibrium
\[ p_r = -\mu_r(p + \rho). \quad (4) \]
In the case of non-isotropic pressure this equation generalizes to
\[ p_r = -\mu_r(p + \rho) - \frac{1}{r}(2p - q). \quad (5) \]
In 1959 Buchdahl [9] showed that static fluid spheres satisfy the inequality
\[ 2M/R \leq \frac{8}{9}, \]
where \( M \) is the ADM mass,
\[ M = \lim_{r \to \infty} m(t, r), \quad \text{where } m(t, r) = \int_0^r 4\pi \eta^2 \rho(\eta)d\eta, \]
and \( R \) is the outer boundary of the fluid sphere. The fluid spheres considered by Buchdahl have isotropic pressure and are in addition assumed to have an energy density which is non-increasing outwards. Note that the isotropy assumption implies that also the pressure is monotonic. This follows from (4) since \( \mu_r \geq 0 \), which is a consequence of (1) and (2). It is sometimes argued that the assumption of non-increasing energy density is natural in the sense that the fluid sphere is unstable otherwise [21]. However, at least for Vlasov matter this is certainly not the case. The existence of stable, spherically symmetric static shells of Vlasov matter (i.e. the matter is supported in \([R_0, R_1]\), \( R_0 > 0 \)) has been demonstrated numerically eg. in [4]. Moreover, these shells are in general non-isotropic. Static solutions of Vlasov matter can also be constructed which do not have the shell structure [18], i.e. \( R_0 = 0 \). Also for these solutions, the hypotheses of non-increasing energy density and isotropic pressure are in general violated. The method of proof by Buchdahl rests on the monotonicity of both \( \rho \) and \( p \), and a natural question is then if there is a Buchdahl type inequality also for static solutions where neither \( \rho \) nor \( p \) is monotonic? Quite surprisingly, numerical results in [5] support that \( 2M/R_1 < 8/9 \) for any static solution of the Einstein-Vlasov system and that there are solutions with \( 2M/R_1 \) arbitrary close to 8/9. (\( R_1 \) will always denote the outer boundary of the static solution.)

In the first part of this presentation we will consider static shells of any matter model (which has static solutions) for which \( \rho \) and \( p \) are non-negative and
\[ p + q \leq \Omega \rho, \quad \Omega \geq 1. \quad (6) \]

Remark: Any matter model with \( \rho \) and \( p \) non-negative, which satisfies the dominant energy condition is admitted with \( \Omega = 3 \). Note that the condition that \( \rho \geq 0 \) can be replaced by requiring that the weak energy condition is satisfied. In the case of Vlasov matter \( \rho, p \) and \( q \) are all non-negative and (6), with strict inequality, is satisfied with \( \Omega = 1 \).

We consider shells which are thin in the sense that \( R_1/R_0 \leq 1 + \epsilon \), and we show that for any \( \epsilon < 1/4 \) there is a \( \kappa > 0 \) such that \( 2M/R_1 \leq 1 - \kappa \). We also show that for a sequence of static shells supported in \([R_0^j, R_1^j]\) such that \( R_1^j/R_0^j \to 1 \), it holds that
\[ \limsup_{j \to \infty} \frac{2M^j}{R_1^j} \leq \frac{((2\Omega + 1)^2 - 1)/(2\Omega + 1)^2}{}, \quad (7) \]
where $M^j$ is the ADM mass of each solution in the sequence. In section 2 these two results are discussed in more detail.

Let us now turn to the second topic of this presentation, i.e., the Einstein-Vlasov system. Vlasov matter is described within the framework of kinetic theory. The fundamental object is the distribution function $f$ which is defined on phase-space, and models a collection of particles. The particles are assumed to interact only via the gravitational field created by the particles themselves and not via direct collisions between them. For an introduction to kinetic theory in general relativity and the Einstein-Vlasov system in particular we refer to [3] and [19]. The static spherically symmetric Einstein-Vlasov system is given by (1)-(3) together with the static Vlasov equation

$$\frac{w}{\varepsilon} \partial_r f - \left( \mu_r \varepsilon - \frac{L}{r^3 \varepsilon} \right) \partial_w f = 0,$$

(8)

where

$$\varepsilon = \varepsilon(r, w, L) = \sqrt{1 + w^2 + L/r^2}.$$

The matter quantities are then given in terms of $f$ through

$$\rho(r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \varepsilon f(r, w, L) \, dLdw,$$

$$p(r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \frac{w^2}{\varepsilon} f(r, w, L) \, dLdw,$$

$$q(r) = \frac{\pi}{r^4} \int_{-\infty}^{\infty} \frac{L}{\varepsilon} f(r, w, L) \, dLdw.$$

The variables $w$ and $L$ can be thought of as the momentum in the radial direction and the square of the angular momentum respectively. Let

$$E = e^{\mu} \varepsilon,$$

the ansatz

$$f(r, w, L) = \Phi(E, L),$$

(9)

then satisfies (8), and the the matter quantities become functionals of the metric coefficient $\mu$. Substituting these into the Einstein equations one obtains a single first order equation for $\mu$ (the metric function $\lambda$ is given by $e^{-2\lambda(t,r)} = 1 - 2m(r)/r$.) Analyzing this equation constitutes an efficient way to construct static solutions with finite ADM mass and finite extension, cf. [18], [17]. It should be pointed out that spherically symmetric static solutions which do not have the form (9) globally exist, cf. [20], which contrasts the Newtonian case where all spherically symmetric static solutions have this form, cf. [6]. As a matter of fact, the solutions we construct in Theorem 3 below are good candidates for solutions which are not globally given by (9).

Here the following form of $\Phi$ will be used

$$\Phi(E, L) = (E_0 - E)^k \left( L - L_0 \right)^l,$$

(10)

where $l \geq 1/2$, $k \geq 0$, $L_0 > 0$, $E_0 > 0$, and $x_+ := \max\{x, 0\}$. In the Newtonian case with $l = L_0 = 0$, this ansatz leads to steady states with a polytropic equation of state. Note that when $L_0 > 0$ there will be no matter in the region

$$r < \sqrt{\frac{L_0}{(E_0 e^{\mu(0)})^2 - 1}},$$

(11)
since there necessarily $E > E_0$ and $f$ vanishes. The existence of solutions supported in $[R_0, R_1]$, $R_0 > 0$, with finite ADM mass has been given in [17], and we shall call such configurations static shells of Vlasov matter. We will always take $L_0 > 0$, and thus only consider shells. The case $L_0 = 0$ is left to a future study. However, we stress that a numerical investigation of this situation is contained in [5].

Numerical evidence is presented in [5] that the following hold true:

i) For any solution of the static Einstein-Vlasov system

$$\Gamma := \sup_r \frac{2m(r)}{r} < \frac{8}{9}. \tag{12}$$

ii) The inequality is sharp in the sense that there is a sequence of steady states such that $\Gamma = 2M/R = 8/9$ in the limit.

In the case of shells the sequence which realizes $\Gamma = 8/9$ is obtained numerically in [5] by constructing solutions where the inner boundary of the shells tend to zero. The outer boundary of these shells also tend to zero, and in [5] numerical support is obtained for the following claim:

iii) There is a sequence of static shells supported in $[R^j_0, R^j_1]$, such that $R^j_0 \to 0$, and $R^j_1/R^j_0 \to 1$, as $j \to \infty$.

Our main results concern issues (ii) and (iii). Issue (i) is clearly a very interesting open problem, and we believe that the results presented here are important for the understanding of (i).

Let us now consider issue (iii) more closely. Note that inequality (7) assumes the existence of a family of shell solutions with the property that $R^j_1/R^j_0 \to 1$ as $j \to \infty$. Our main result on the Einstein-Vlasov system is indeed to show that such a family of static solutions exist. We also improve statement (7), by proving that for Vlasov matter the limit of $2M/R^j_1$ as $j \to \infty$ equals $8/9$ (recall that $\Omega = 1$ for Vlasov matter), i.e., we prove that (ii) holds. These results are described in more detail in section 3 below.

Next we discuss some previous results on the Buchdahl inequality in cases where at least one of Buchdahl’s original hypotheses are relaxed.

Bondi [8] has investigated (not rigorously) if isotropic solutions, without the assumption of non-increasing energy density, obey a Buchdahl type inequality. He considers models for which $\rho \geq 0$, $\rho \geq p$, or $\rho \geq 3p$, and gets 0.97, 0.86 and 0.70 respectively as upper bounds of $2M/R_1$. The isotropic condition is crucial though, since our results show that the second bound is violated by non-isotropic steady states of Vlasov matter, cf. Theorem 4 below and [5]. Note that $\rho \geq p$ always holds for Vlasov matter.

Remark: We point out that steady states of Vlasov matter which are isotropic and which have non-increasing energy density do exist but the analysis of such states is identical to Buchdahl’s original analysis.

General non-isotropic solutions have previously been studied in [7] where it is shown that $2M/R_1 < 1$, and in [15] static shells where the density is concentrated at a single radius and the source in the Einstein equations is distributional, were found to satisfy $2M/R_1 \leq 24/25$ (which corresponds to $\Omega = 2$, cf. our discussion in section 2). In [16] the general non-isotropic case is considered but instead they require that there is a constant $B$ such that

$$\frac{(\rho - 3m/r^3) + 2(pt - p)}{p + m/r^3} \leq B.$$
Under this assumption they obtain a bound on $\Gamma$ which for large $B$ can be written in the form

$$\Gamma \leq 1 - 2/(2 + B)^2.$$ 

As $B \to \infty$ this bound degenerates to the estimate $\Gamma \leq 1$. This is interesting in view of (i) which indicates that $\Gamma$ is *always* less than $8/9$, and in view of the results presented here that the energy densities along a family of steady states which makes $\Gamma$ approach $8/9$ get more and more peaked so that $B \to \infty$ for this family.

Let us end this section with a brief discussion on the possible role of the Buchdahl inequality for the time dependent problem with Vlasov matter. The cosmic censorship conjecture is fundamental in classical general relativity and to a large extent an open problem. In the case of gravitational collapse the only rigorous result is by Christodoulou who has obtained a proof in the case of the spherically symmetric Einstein-Scalar Field system [11]. One key result for this proof is contained in [10], cf. also [12], and states roughly that if there is a sufficient amount of matter within a bounded region then necessarily a trapped surface will form in the evolution. If a trapped surface forms, then Dafermos [13] has shown under some restrictions on the matter model, that cosmic censorship holds. In particular, Dafermos and Rendall [14] have proved that spherically symmetric Vlasov matter satisfies these restrictions. Thus cosmic censorship holds for the spherically symmetric Einstein-Vlasov system if there is a trapped surface in spacetime. Now, assume that a Buchdahl inequality holds in general for this system, then in view of the result by Christodoulou mentioned above, it is natural to believe that if $2m/r$ exceeds the value given by such an inequality, then a trapped surface must form.

2. Main results for matter models with $p + q \leq \rho$

The formulation of our first result is quite technical but a general discussion follows after the statement.

**Theorem 1** Let $a > 0$, $K \geq 5/4 + 3\Omega/4$, and let

$$\kappa = \frac{1}{(2\Omega + 1 + \frac{2(\Omega + 1)^2}{2l+1}(K + a))(2l+1)^2}.$$ 

Consider a solution to the Einstein equations such that $p + q \leq \Omega \rho$, with support in $[R_0, R_1]$, and such that $R_1/R_0 \leq (1 + \epsilon)$, where

$$\epsilon < \min \left\{ \frac{1 - (2\Omega + 3)\kappa - (\Omega + 1)(1 - \kappa)e^{-4K/9}}{2(\Omega + 1)}, a\sqrt{\kappa} \right\}.$$ 

Then

$$2M/R_1 \leq 1 - \kappa,$$

and

$$\sup_{r} \frac{2m}{r} \leq 1 - \kappa/2, \text{ if } \epsilon \leq \kappa/2.$$ 

The condition on $K$ is to ensure that the first term in the expression for $\epsilon$ is positive and can be sharpened. The numbers $\kappa$ and $\epsilon$ are related in the sense that a larger $\epsilon$ can be chosen if $\kappa$ is taken smaller and vice versa. In view of the next theorem, which gives a much improved value of $\kappa$ when $\epsilon$ is made arbitrary small, it seems more interesting to make $\epsilon$ large even if $\kappa$ then becomes smaller, since it gives us an estimate on the possible thickness of the shells that our method allows in order to admit a Buchdahl inequality at all. We see immediately from the formula above that our method cannot handle $\epsilon \geq 1/4$, but by fixing $K$ large and then taking $a$ sufficiently large we can have $\epsilon$ arbitrary close to $1/4$. A choice which favours a large value of
$\epsilon$ rather than a large $\kappa$, in the case when $\Omega = 1$, is $K = 6$ and $\alpha = 9$, which implies that when $\epsilon < 1/5$ it holds that $2M/R_1 \leq 1 - 1/43^2$.

Next we consider a sequence of shells supported in $[R_0^j, R_1^j]$, such that $R_1^j/R_0^j \to 1$. We find that, as the support gets thinner, the value

$$\frac{(2\Omega + 1)^2 - 1}{(2\Omega + 1)^2}$$

of $2M/R_1$ cannot be exceeded. In particular, if $\Omega = 1$, then the same bound $8/9$ as in Buchdahl's original work is obtained, and if $\Omega = 2$, then the bound is $24/25$ which agrees with the value found in [15]. The latter agreement is not surprising since the case considered in [15] is an infinitely thin shell with radial pressure zero which satisfies the dominant energy condition. In our terminology this means $\Omega = 2$. The former result is more surprising since the steady state that realizes $8/9$ in Buchdahl's original work is the well-known interior solution with constant energy density and where the pressure blows up at $r = 0$ (and hence does not satisfy (6) for any $\Omega$) which is completely different from our situation where the energy density gets more and more peaked at the radius $R_1$.

**Theorem 2** Assume that $\{(\rho^j, p^j, q^j, \mu^j)\}_{j=1}^\infty$ is a sequence of solutions to the static Einstein-matter system such that $p^j + q^j \leq \Omega \rho^j$, with support in $[R_0^j, R_1^j]$, and such that

$$\lim_{j \to \infty} \frac{R_1^j}{R_0^j} = 1.$$ 

Then

$$\limsup_{j \to \infty} \frac{2M^j}{R_1^j} \leq \frac{(2\Omega + 1)^2 - 1}{(2\Omega + 1)^2}.$$ 

We again point out that the hypothesis (13) holds for a class of static shell solutions of the Einstein-Vlasov system, cf. Theorem 3 below, and we emphasize that the statement (14) is improved in that case in the sense that $8/9$ is attained in the limit, cf. Theorem 4.

The proofs of Theorem 1 and Theorem 2 are found in [1].

3. Main results for Vlasov matter

In view of (11) it is clear that the region where $f$ necessarily vanishes can be made arbitrary small if the values of $E_0$ and $\mu(0)$ are such that $E_0 e^{-\mu(0)}$ is large. That this is always possible can be seen as follows. Set $E_0 = 1$, and construct a solution by specifying an arbitrary non-positive value $\mu(0)$, in particular $e^{-\mu(0)}$ can be made as large as we wish. The metric function $\mu$ is then obtained by integrating from the centre using equation (2),

$$\mu(r) = \mu(0) + \int_0^r \left( \frac{m}{r^2} + 4\pi \eta p \right) e^{2\lambda} d\eta.$$ 

This implies that the boundary condition at $\infty$ of $\mu$ will be violated in general. However, by letting $\hat{E}_0 = e^{\mu(\infty)}$, and $\hat{\mu}(r) := \mu(r) - \mu(\infty)$, then $\hat{\mu}$, and the distribution function $f$ associated with $\hat{\mu}$ and $\hat{E}_0$, will solve the Einstein-Vlasov system and satisfy the boundary condition at infinity. Hence we can take $E_0 = 1$ and obtain arbitrary small values of $R_0$ by taking $-\mu(0)$ sufficiently large.

Let us define

$$R_0 := \sqrt{\frac{L_0}{e^{-2\mu(0)} - 1}}.$$ 

It now holds that $f(r, \cdot, \cdot) > 0$, when $r$ is sufficiently close to but larger than $R_0$, cf. [2]. Hence, given any number $R_0 > 0$ we can construct a solution having inner radius of support equal to $R_0$. The following result proves issue (iii).
Theorem 3 Consider a shell solution with a sufficiently small inner radius of support $R_0$, and let $q := l + k + 3/2$. The distribution function $f$ then vanishes somewhere within the interval

$$[R_0 + B_0 R_0^{(q+3)/(q+1)}, (1 - B_1 R_0^{2/(q+1)})^{-1} (R_0 + B_2 R_0^{(q+2)/(q+1)})],$$

where $B_0, B_1$ and $B_2$ are positive constants. The solution can thus be joined with a Schwarzschild solution at the point where $f$ vanishes and a static shell is obtained with support within $[R_0, R_1]$, where

$$R_1 = \frac{R_0 + B_2 R_0^{(q+2)/(q+1)}}{1 - B_1 R_0^{2/(q+1)}},$$

so that $R_1/R_0 \to 1$ as $R_0 \to 0$.

This result is interesting in its own right since it gives a detailed description of the support of a class of shell solutions of the Einstein-Vlasov system. Moreover, the solutions constructed in Theorem 3 can be used to obtain a sequence of shells of Vlasov matter with the property that $2M/R = 8/9$ in the limit.

Theorem 4 Let $(f^j, \mu^j)$ be a sequence of shell solutions with support in $[R_0^j, R_1^j]$, and such that $R_1^j/R_0^j \to 1$ and $R_0^j \to 0$, as $j \to \infty$, and let $M^j$ be the corresponding ADM mass of $(f^j, \mu^j)$. Then

$$\lim_{j \to \infty} \frac{2M^j}{R_1^j} = \frac{8}{9}.$$  \hfill (15)

The proofs of Theorem 3 and Theorem 4 are found in [2].