Volume growth and the topology of manifolds with nonnegative Ricci curvature

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Abstract
Let $M^n$ be a complete, open Riemannian manifold with $\text{Ric} \geq 0$. In 1994, Grigori Perelman showed that there exists a constant $\delta_n > 0$, depending only on the dimension of the manifold, such that if the volume growth satisfies $\alpha_M := \lim_{r \to \infty} \frac{\text{Vol}(B_p(r))}{\omega_n r^n} \geq 1 - \delta_n$, then $M^n$ is contractible. Here we employ the techniques of Perelman to find specific lower bounds for the volume growth, $\alpha(k, n)$, depending only on $k$ and $n$, which guarantee the individual $k$-homotopy group of $M^n$ is trivial.

1 Introduction

Let $M^n$ be an $n$-dimensional complete Riemannian manifold with nonnegative Ricci curvature. For a base point $p \in M^n$, denote by $B_p(r)$ the open geodesic ball in $M^n$ centered at $p$ and with radius $r$. Let $\text{Vol}(B_p(r))$ denote the volume of $B_p(r)$ and denote by $\omega_n$ the volume of the unit ball in Euclidean space. By the Bishop-Gromov Relative Volume Comparison Theorem, the function $r \to \text{Vol}(B_p(r))/\omega_n r^n$ is non-increasing and bounded above by 1.

Definition 1.1. Define $\alpha_M$, the volume growth of $M^n$, as

$$\alpha_M := \lim_{r \to \infty} \frac{\text{Vol}(B_p(r))}{\omega_n r^n}.$$ 

The manifold $M^n$ is said to have Euclidean (or large) volume growth when $\alpha_M > 0$.

The constant $\alpha_M$ is a global geometric invariant of $M^n$, i.e. it is independent of base point. Also, when $\alpha_M > 0$,

$$\text{Vol}(B_p(r)) \geq \alpha_M \omega_n r^n, \quad \text{for all } p \in M \text{ and for all } r > 0.$$ 

It follows from the Bishop-Gromov Volume Comparison Theorem that $\alpha_M = 1$ implies $M^n$ is isometric to $\mathbb{R}^n$.
In this paper, we study complete manifolds with $\text{Ric}_M \geq 0$ and $\alpha_M > 0$. Anderson [2] and Li [15] have independently shown that the order of $\pi_1(M^n)$ is bounded from above by $\frac{1}{\alpha_M}$. In particular, if $\alpha_M > \frac{1}{2}$, then $\pi_1(M^n) = 0$. Furthermore, Zhu [20] has shown that when $n = 3$, if $\alpha_M > 0$, then $M^3$ is contractible. It is interesting to note that this is not the case when $n = 4$ as Menguy [12] has constructed examples of 4-manifolds with large volume growth and infinite topological type based on an example by Perelman [16]. In 1994, Perelman [17] proved that there exists a small constant $\delta_n > 0$ which depends only on the dimension $n \geq 2$ of the manifold, such that if $\alpha_M \geq 1 - \delta_n$, then $M^n$ is contractible. It was later shown by Cheeger and Colding [6] that the conditions in Perelman’s theorem are enough to show that $M^n$ is $C^{1,\alpha}$ diffeomorphic to $\mathbb{R}^n$. In this paper, we follow the method of proof in Perelman’s theorem. Employing this method, we determine specific bounds on $\alpha_M$ which imply the individual $k$-th homotopy groups of the manifold are trivial.

We prove

**Theorem 1.2.** Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq 0$. If

$$\alpha_M > \alpha(k, n),$$

where $\alpha(k, n)$ are the constants given in Table 4, then $\pi_k(M^n) = 0$.

**Remark.** Table 4 contains values of $\alpha(k, n)$ for $1 \leq k \leq 3$ and $1 \leq n \leq 10$. In general, the value of $\alpha(k, n)$ is determined by Equation (169), where the function $h_{k,n}(x)$ and the values of $\delta_{k,n}$ are defined in Definition 3.2.

In section 1.1, we state general results from Riemannian geometry that will be required for the proof. The key ingredients are the excess estimate of Abresch-Gromoll, the Bishop-Gromov Volume Comparison Theorem, and a Maximal Volume Lemma of Perelman [Lemma 1.5].

In section 2, we apply the theory of almost equicontinuity from [18] to prove a general Homotopy Construction Theorem [Theorem 2.7] that will be needed when constructing the homotopies for Theorem 1.2.

In section 3, we prove Theorem 1.2 using a double induction argument for the general case. This argument follows Perelman’s except that we carefully determine the necessary constants to build each step. Perelman’s double induction argument is built from two lemmas each of
which depends on a parameter $k \in \mathbb{N}$. The Main Lemma$(k)$ [Lemma 3.4] says that given a constant $c > 1$ and an appropriate estimate on volume growth, any given continuous function $f : \mathbb{S}^k \to B_p(R)$ can be extended to a continuous function $g : \mathbb{D}^{k+1} \to B_p(cR)$ . This lemma is proven by defining intermediate functions $g_j$ on finer and finer nets in $\mathbb{D}^{k+1}$. To define $g_j$ on these nets one uses the Moving In Lemma, described below. To prove the limit $g(x) = \lim_{j \to \infty} g_j(x)$ exists and is continuous, we apply results from section 2.

The Moving In Lemma$(k)$ [Lemma 3.5] states that given a constant $d_0 > 0$ and a map $\phi : \mathbb{S}^k \to B_q(\rho)$ then with an appropriate bound on volume growth one can move $\phi$ inward obtaining a new map $\tilde{\phi} : \mathbb{S}^k \to B_q((1 - d_0)\rho)$. The new map $\tilde{\phi}$ is uniformly close to the map $\phi$ with respect to the radius $\rho$. The maps $\phi$ and $\tilde{\phi}$ are not necessarily homotopic; however, a homotopy is constructed by controlling precisely the uniform closeness of these maps on smaller and smaller scales. The Moving In Lemma$(k)$ and Main Lemma$(i)$, for $i = 0, \ldots, k - 1$, are used to produce finer and finer nets that then converge on the homotopy required for Main Lemma$(k)$. Moving In Lemma$(k)$ is proven by constructing the map $\tilde{\phi}$ inductively on successive $i$-skeleta of a triangulation of $\mathbb{S}^k$. The conclusion of Main Lemma$(i)$, for $i = 0, \ldots, k - 1$, is needed in the induction step of the proof of Moving In Lemma$(k)$.

The key place in the argument where the volume growth bound is introduced occurs in the proof of the Moving In Lemma; specifically, in producing a small, thin triangle in an advantageous location. However, due to the double inductive argument, and the fact that lower dimensional lemmas are applied on a variety of scales where the choice of $c$ and $d_0$ depend on $n$ and $k$, the actual estimate on the volume is produced using inductively defined functions $\beta(k, c, n)$ [Definition 3.3] and constants $C_{k,n}$ [Definition 3.1].

In the appendix, we complete our analysis of $\beta(k, c, n)$ to find the optimal bounds, $\alpha(k, n)$, over all constants $c > 1$. Through this analysis we are able to construct a table of values containing the optimal lower bounds for the volume growth, as stated in Theorem 1.2 which guarantee the $k$-th homotopy group is trivial. The bounds that we obtain are the best that can be achieved via Perelman’s method. A portion of this analysis was done using Mathematica 6. The code for these commands is available in [13].

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1.1 Background

Here we review two facts from the Riemannian geometry of manifolds with non-negative Ricci curvature. Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq 0$.

**Theorem 1.3.** [Abresch-Gromoll Excess Theorem]. Let $p, q \in M^n$ and let $\overline{pq}$ be a minimal geodesic connecting $p$ and $q$. For any $x \in M^n$, we define the excess function with respect to $p$ and $q$ as

$$e_{p,q}(x) = d(p, x) + d(q, x) - d(p, q).$$

Define $h(x) = d(x, \overline{pq})$ and set $s(x) = \min\{d(p, x), d(q, x)\}$. If $h(x) \leq s(x)/2$, then

$$e_{p,q}(x) \leq 8 \left( h(x) \right)^{1/n} \left( s(x) \right)^{1/n-1} \frac{h(x)}{s(x)}. $$

This excess estimate is due to Abresch-Gromoll [1] (c.f. [5]).

**Definition 1.4.** For constants $c > 1$, $\epsilon > 0$ and $n \in \mathbb{N}$, define

$$\gamma(c, \epsilon, n) = \left[ 1 + \left( \frac{c}{\epsilon} \right)^n \right]^{-1}. $$

**Lemma 1.5.** [Perelman’s Maximal Volume Lemma]. Let $p \in M^n$, $R > 0$, for any constants $c_1 > 1$ and $\epsilon > 0$, if $\alpha_M > 1 - \gamma(c_1, \epsilon, n)$, then for every $a \in B_p(R)$, there exist $q \in M^n \setminus B_p(c_1 R)$ such that $d(a, \overline{pq}) \leq \epsilon R$, where $\overline{pq}$ denotes a minimal geodesic connecting $p$ and $q$.

This fact was observed without proof by Perelman in [17]. Our statement and proof differ in that we utilize the global volume growth control on $\alpha_M$ rather than only a local volume bound in a neighborhood of $B_p(c_1 R)$. This global bound allows us to determine an expression for $\gamma$ not given in [17]. The proof of Perelman’s original statement follows from the proof of the Bishop-Gromov Volume Comparison Theorem and can be found in [21].

**Proof.** Let $c_2 > c_1 > 1$ be finite constants. Define $\Gamma \equiv \{ \hat{\sigma} \mid d(a, \hat{\sigma}) \leq \epsilon R \} \subset S_p^{n-1}(M^n) \subset T_p M^n$, where $\hat{\sigma}$ denotes a minimal geodesic in $M^n$ and $\sigma$ its velocity vector. Suppose that
for all $v \in \Gamma$, we have $\text{cut}(v) < c_1 R$. In what follows, we determine an upper bound on the
volume growth, $\alpha_M$, which would allow such a contradiction to occur. In turn, by requesting
the volume growth be bounded below by this upper bound, the lemma will follow.

By definition, we have

$$\text{Vol}(B_p(c_2 R)) = \int_{\Gamma} \int_0^{\min\{\text{cut}(v), c_2 R\}} A_M^n(t, v) dt dv$$

$$+ \int_{S^{n-1}\setminus \Gamma} \int_0^{\min\{\text{cut}(v), c_2 R\}} A_M^n(t, v) dt dv$$

$$\leq \text{Vol}(\Gamma) \int_0^{c_1 R} A^0(t) dt + \text{Vol}(S^{n-1}\setminus \Gamma) \int_0^{c_2 R} A^0(t) dt$$

$$= \text{Vol}(S^{n-1}) \int_0^{c_2 R} A^0(t) dt - \text{Vol}(\Gamma) \left( \left( \int_0^{c_2 R} - \int_0^{c_1 R} \right) A^0(t) dt \right)$$

$$= -\text{Vol}(\Gamma) \int_{c_1 R}^{c_2 R} A^0(t) dt + \text{Vol}(S^{n-1}) \int_0^{c_2 R} A^0(t) dt$$

$$= -\text{Vol}(\Gamma) \int_{c_1 R}^{c_2 R} A^0(t) dt + \text{Vol}(B^0(c_2 R)).$$

Here $A_M^n(t, v)$ denotes the volume element on $M^n$ and $A^0(t)$ denotes the volume element
on $\mathbb{R}^n$; that is, $A^0(t) = t^{n-1}$. From the assumption on the volume growth, we have that
$\text{Vol}(B_p(c_2 R)) \geq (1 - \gamma)\text{Vol}(B^0(c_2 R))$ and therefore

$$(1 - \gamma)\text{Vol}(B^0(c_2 R)) \leq -\text{Vol}(\Gamma) \int_{c_1 R}^{c_2 R} A^0(t) dt + \text{Vol}(B^0(c_2 R))$$  \hspace{1cm} (1)$$

$$\gamma\text{Vol}(B^0(c_2 R)) \geq \text{Vol}(\Gamma) \int_{c_1 R}^{c_2 R} A^0(t) dt$$  \hspace{1cm} (2)$$

$$\text{Vol}(\Gamma) \leq \frac{\gamma \text{Vol}(B^0(c_2 R))}{\int_{c_1 R}^{c_2 R} A^0(t) dt}.$$  \hspace{1cm} (3)$$

On the other hand, since $B_a(\epsilon R) \subset \text{Ann}_\Gamma(p; 0, c_1 R)$, it follows that
$\text{Vol}(B_a(\epsilon R)) \leq \text{Vol}(\Gamma) \int_0^{c_1 R} A^0(t) dt$. Hence

$$\text{Vol}(B_a(\epsilon R)) \leq \gamma \text{Vol}(B^0(c_2 R)) \frac{\int_0^{c_1 R} A^0(t) dt}{\int_{c_1 R}^{c_2 R} A^0(t) dt}.$$  \hspace{1cm} (4)$$

Furthermore, since $B_p(c_2 R) \subset B_a(R + c_2 R)$, we know that

$$\frac{\text{Vol}(B_p(c_2 R))}{\text{Vol}(B_a(\epsilon R))} \leq \frac{\text{Vol}(B_a(R + c_2 R))}{\text{Vol}(B_a(\epsilon R))} \leq \frac{(R + c_2 R)^n}{(\epsilon R)^n};$$
and therefore,

\[ \text{Vol}(B_a(\epsilon R)) \geq \text{Vol}(B_a(R + c_2 R)) \frac{(\epsilon R)^n}{(R + c_2 R)^n} \]  

(5)

\[ \geq \text{Vol}(B_p(R)) \frac{\epsilon^n}{(1 + c_2)^n} \]  

(6)

\[ \geq (1 - \gamma) \text{Vol}(B^0(c_2 R)) \frac{\epsilon^n}{(1 + c_2)^n}. \]  

(7)

Combining (6) and (7), we get

\[ (1 - \gamma) \text{Vol}(B^0(c_2 R)) \frac{\epsilon^n}{(1 + c_2)^n} \leq \gamma \text{Vol}(B^0(c_2 R)) \int_{c_1 R}^{c_1 R} A^0(t) dt \]  

(8)

\[ \left( \frac{\epsilon}{1 + c_2} \right)^n - \gamma \left( \frac{\epsilon}{1 + c_2} \right)^n \leq \frac{\epsilon^n}{c_2^n - c_1^n} \]  

(9)

\[ \left( \frac{\epsilon}{1 + c_2} \right)^n \leq \gamma \left[ \frac{\epsilon^n}{c_2^n - c_1^n} + \left( \frac{\epsilon}{1 + c_2} \right)^n \right]. \]  

(10)

By solving (10) for \( \gamma \), we can deduce a lower bound for \( \gamma \) dependent only on the constants \( c_2, c_1, \epsilon \) and \( n \). That is,

\[ \gamma \geq \left( \frac{\epsilon}{1 + c_2} \right)^n \frac{c_1^n}{c_2^n - c_1^n} + \left( \frac{\epsilon}{1 + c_2} \right)^n \]  

(11)

\[ = \left[ 1 + \frac{c_1^n}{c_2^n - c_1^n} \left( \frac{1 + c_2}{\epsilon} \right)^n \right]^{-1}. \]  

(12)

Note that, throughout the proof we required a restriction on the volume growth only within the larger ball \( B_p(c_2 R) \). Since \( \alpha_M \) is a global restriction on volume growth, it is possible to take \( c_2 \to \infty \) and thus refine the lower bound on \( \gamma \) determined above. Since

\[ \lim_{c_2 \to \infty} \left[ 1 + \frac{c_1^n}{c_2^n - c_1^n} \left( \frac{1 + c_2}{\epsilon} \right)^n \right]^{-1} = \left[ 1 + \frac{c_1^n}{\epsilon^n} \right]^{-1}, \]

the above lower bound on \( \gamma \) can be expressed more simply as

\[ \gamma \geq \left[ 1 + \frac{c_1^n}{\epsilon^n} \right]^{-1}. \]

Recall that this lower bound on \( \gamma \) provides the upper bound on \( \alpha_M = 1 - \gamma \) which leads to the contradiction of the Lemma. Thus, by requiring \( \alpha_M > 1 - \left[ 1 + \frac{c_1^n}{\epsilon^n} \right]^{-1} \), as originally
prescribed in the assumption, we have proven the Lemma.

Remark. Perelman’s Maximal Volume Lemma proves the existence of a geodesic in $M^n$ of length at least $c_1 R > 1$ that is within a fixed distance of a given point. Consider, for example, the case when $M^n = \mathbb{R}^n$. Given a point $a \in \mathbb{R}^n$, it is possible to find a geodesic of any length (in fact, there exists a ray) that is arbitrarily close to $a$. Indeed, letting $c_1 \to \infty$ in the expression for $\alpha_M$, while keeping $\epsilon$ and $n$ fixed, we find that $\alpha_M \to 1$. Similarly, letting $\epsilon \to 0$ (with $c_1, n$ fixed), forces $\alpha_M \to 1$ as well. Recall that by the Bishop-Gromov Volume Comparison Theorem, $\alpha_M = 1$ implies $M^n$ is isometric to $\mathbb{R}^n$.

Remark. Allowing the dimension of $M^n$ to increase while keeping $\epsilon$ and $c_1$ constant also pushes the lower bound on $\alpha_M$ closer to 1.

2 Almost Equicontinuity and the Construction of Homotopies

In this section, we prove a general method of constructing homotopies from sequences of increasingly refined nets. We begin by reviewing a definition and theorem from [18].

2.1 Background and Definitions

Definition 2.1. [18, Definition 2.5] A sequence of functions between compact metric spaces $f_i : X_i \to Y_i$, is said to be almost equicontinuous if there exists $\epsilon_i$ decreasing to 0 such that for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$d_{Y_i}(f_i(x_1), f_i(x_2)) < \epsilon + \epsilon_i, \quad \text{whenever} \quad d_{X_i}(x_1, x_2) < \delta_\epsilon. \tag{13}$$

Theorem 2.2. [18, Theorem 2.3] If $f_i : X_i \to Y_i$ is almost equicontinuous between complete length spaces $(X_i, x_i) \to (X, x)$ and $(Y_i, y_i) \to (Y, y)$ which converge in the Gromov-Hausdorff sense where $X$ and $Y$ are compact, then a subsequence of the $f_i$ converge to a continuous limit function $f : X \to Y$.

Let $X$ be a complete length space and let $K_j$ be a sequence of finite cell decompositions of $X$. Each such decomposition $K_j$ is composed of a collection of cells $\sigma_i$ so that, for each $j$, $X = \bigsqcup_{\sigma_i \in K_j} \sigma_i$. Each $K_{j+1}$ is a refinement of $K_j$. 
Definition 2.3. Let $K$ be a finite cell decomposition of a complete length space $X$. A map $\psi_K : X \to X$ which maps all the points in a cell $\sigma$ of $K$ to a single point $p \in \sigma$ is called a discrete decomposition map of $K$.

Lemma 2.4. Let $K_j$ be a sequence of finite cell decompositions of $X$ and $\{\psi_{K_j}\}$ a sequence of discrete decomposition maps of $K_j$. This sequence of maps is almost equicontinuous provided $\max\{\text{diam}(\sigma) | \sigma \in K_j\} \to 0$ as $j \to \infty$.

Proof. For each $j$, let $d_j = \max\{\text{diam}(\sigma) | \sigma \in K_j\}$. Pick $\epsilon > 0$ and suppose $x, y \in X$ such that $d(x, y) < \epsilon$. By the triangle inequality,

$$d(\psi_j(x), \psi_j(y)) \leq d(\psi_j(x), x) + d(x, y) + d(y, y) < \epsilon + 2d_j.$$ 

Each $K_{j+1}$ is a refinement of $K_j$ and so by assumption the sequence $d_j$ decreases to 0. Thus, the sequence $\{\psi_j\}$ is almost equicontinuous as claimed. 

Lemma 2.5. The composition of two almost equicontinuous sequences of maps is again almost equicontinuous; i.e. if $\{f_j\}$ and $\{g_j\}$ are two sequences of maps which are almost equicontinuous. Then $\{f_j \circ g_j\}$ is also almost equicontinuous.

Proof. Suppose $\{f_j\}$ and $\{g_j\}$ are two almost equicontinuous sequences of maps. Since $\{f_j\}$ is almost equicontinuous, given $\epsilon > 0$, there exists $\delta^f_\epsilon > 0$ and positive integer $K^f$ such that $d(f_j(x), f_j(y)) \leq \epsilon$ for all $j > K^f$, provided $d(x, y) < \delta^f_\epsilon$. Choose $\delta^{f\circ g}_\epsilon = \delta^f_\delta g$ and choose a positive integer $K = \max\{K^f, K^g\}$, where $\delta^g_\epsilon$ and $K^g$ are chosen so that when $d(a, b) < \delta^g_\epsilon$, we have $d(g_j(a), g_j(b)) < \delta^f_\epsilon$, for all $j > K^g$.

Therefore, if $d(a, b) < \delta^{f\circ g}_\epsilon$, then $d(g_j(a), g_j(b)) < \delta^f_\epsilon$, for all $j > K \geq K^g$ and thus, $d(f_j(g_j(a)), f_j(g_j(b))) < \epsilon$, for all $j > K \geq K^f$. Therefore, the sequence $\{f_j \circ g_j\}$ is almost equicontinuous.
2.2 Homotopy Construction Theorem

The following theorem is crucial in constructing the homotopies in the manifold setting. In the statement of the theorem and in what follows we often refer to the i-skeleton of a cell decomposition $K$. We define an i-skeleton here.

**Definition 2.6.** The i-skeleton of a k-dimensional cell decomposition $K$, denoted $\text{skel}_i(K)$ for $i = 0, 1, \ldots, k$, is defined as the collection of all i-dimensional cells contained in $K$.

Note that if $X = D^{k+1}$ then $S^k \subset D^{k+1}$ is contained in $\text{skel}_k(K)$ for any cell decomposition $K$ of $D^{k+1}$.

**Theorem 2.7. (Homotopy Construction Theorem).** Let $Y$ be a complete, locally compact metric space, $p \in Y$, $R > 0$ and $f : S^k \to B_p(R) \subset Y$ a continuous map. Given constants $c > 1$, $\omega \in (0, 1)$, and a sequence of finite cell decompositions $K_j$ of $D^{k+1}$ with maps $f_j : \text{skel}_k(K_j) \to Y$ satisfying the following three properties

(A) $K_{j+1}$ is a subdivision of $K_j$ and $f_{j+1} \equiv f_j$ on $K_j$ and $\max\{\text{diam}(\sigma) | \sigma \in K_j\} \to 0$,

(B) For each $(k+1)$-cell, $\sigma \in K_j$, there exists a point $p_{\sigma} \in B_p(cR) \subset Y$ and a constant $R_{\sigma} > 0$ such that $f_j(\partial \sigma) \subset B_{p_{\sigma}}(R_{\sigma})$;

and, if $\sigma' \subset \sigma$, where $\sigma' \in K_{j+1}$, $\sigma \in K_j$, then

$$B_{p_{\sigma'}}(cR_{\sigma'}) \subset B_{p_{\sigma}}(cR_{\sigma}), \quad \text{and} \quad R_{\sigma'} \leq \omega R_{\sigma}, \text{for} \ \omega \in (0, 1).$$

(C) $\text{skel}_k(K_0) = S^k = \partial D^{k+1}$, $p_{\sigma_0} = p$, and $R_{\sigma_0} = R$,

then the map $f$ can be continuously extended to a map $g : D^{k+1} \to B_p(cR) \subset Y$.

**Proof.** Suppose we have such a sequence of finite cell decompositions $K_j$ of $D^{k+1}$ and continuous maps $f_j : \text{skel}_k(K_j) \to M$ satisfying (A), (B), and (C) above. For any $x \in D^{k+1}$, choose a sequence of $(k+1)$-cells $\sigma_j \in K_j$, such that $\sigma_{j+1} \subset \sigma_j$ and $x \in \text{clos}(\sigma_j)$ for all $j$. Therefore, each point $x \in D^{k+1}$ determines a sequence of $(k+1)$-cells ‘converging to’ $x$. Each of these cells determines a point, $p_{\sigma_j}$, and a radius, $R_{\sigma_j} > 0$, which we assume satisfy the properties outlined in (A), (B), and (C) above.
As in Perelman’s homotopy construction [17], define \( g \) by 
\[
g(x) = \lim_{j \to \infty} p_{\sigma_j}.
\]
If \( x \in \text{skel}_k(K_j) \) for some \( j \), set \( g(x) = f_j(x) \). If \( x \notin \text{skel}_k(K_j) \) for all \( j \), then for \( j, k > 0 \), property (B) implies 
\[
d(p_{\sigma_j}, p_{\sigma_{j+k}}) \leq cR_{\sigma_j} \leq c\omega^j R.
\]
Since \( \omega \in (0, 1) \), the sequence \( \{p_{\sigma_j}\} \) is a Cauchy sequence and thus converges. Hence, \( g(x) \) is well-defined.

Note that \( \partial D^{k+1} = S^k = \text{skel}_k(K_0) \) and so by the definition of \( g \), for any \( x \in \partial D^{k+1}, g(x) = f_0(x) = f(x) \). Thus, \( g|_{\partial D^{k+1}} = f \).

The continuity of \( g \) is not verified in [17]. Here we prove that \( g \) is continuous. Define a sequence of maps \( g_j : D^{k+1} \to Y \) by \( g_j(x) = p_{\sigma_j} \) for each \( j \).

**Claim.** The sequence of maps \( \{g_j\} \) is uniformly almost equicontinuous.

**Proof of Claim.** Define a sequence of intermediate maps \( \psi_{K_j} : D^{k+1} \to D^{k+1} \), where \( \psi_{K_j} \) is a discrete decomposition map for \( K_j \). Note that \( \text{Im}(\psi_{K_j}) \) is a discrete metric space. Define \( \overline{g_j} : \text{Im}(\psi_{K_j}) \to X \) in such a way that \( \overline{g_j} = g_j|_{\text{Im}(\psi_{K_j})} \).

By (A) we have that \( \max\{\text{diam}(\sigma) | \sigma \in K_j\} \to 0 \) as \( j \to \infty \). Therefore, the sequence of decomposition maps \( \psi_{K_j} \) is almost equicontinuous by Lemma 2.4.

The maps \( \overline{g_j} \) are discrete and thus the sequence \( \{\overline{g_j}\} \) is almost equicontinuous.

Since \( g_j = \overline{g_j} \circ \psi_{K_j} \), by Lemma 2.5 the sequence of maps \( \{g_j\} \) is also uniformly almost equicontinuous. This completes the proof of the Claim.

Finally, by Theorem 2.2 (see [18] for proof), the limiting map \( g \) is continuous. This completes the proof of Proposition 2.7.

\[\square\]

### 3 Double Induction Argument

In this section we use Perelman’s double induction argument outlined in section [1] to prove Theorem 1.2. We introduce a collection of constants which are defined inductively. We define them here as they are necessary for the induction statements.
**Definition 3.1.** For $k, n \in \mathbb{N}$ and $i = 0, 1, \ldots, k$, define constants $C_{k,n}(i)$ iteratively as follows:

$$C_{k,n}(i) = (16k)^{n-1} (1 + 10C_{k,n}(i-1))^n + 3 + 10C_{k,n}(i-1)),$$  
and $C_{k,n}(0) = 1$. We denote $C_{k,n} = C_{k,n}(k)$.

**Definition 3.2.** Define a function

$$h_{k,n}(x) = \left[1 - 10^{k+2}C_{k,n} \left(1 + \frac{x}{2k}\right)^k\right]^{-1}.$$  

This function $h_{k,n}$ has a vertical asymptote at $x = \delta_{k,n}$ for some small value $\delta_{k,n} > 0$, where $10^{k+2}C_{k,n}\delta_{k,n}\left(1 + \frac{\delta_{k,n}}{2k}\right)^k = 1$. Note that $h_{k,n} : (0, \delta_{k,n}) \to (1, \infty)$ is a smooth, one-to-one, onto, increasing function. Thus $h_{k,n}^{-1} : (1, \infty) \to (0, \delta_{k,n})$ is well-defined.

Toward proving Theorem 1.2, we need to build the homotopy as described earlier. This requires control on the volume growth of $M^n$. We now define the expression $\beta(k,c,n)$ which we will use to control the volume growth of $M^n$.

**Definition 3.3.** For constants, $c > 1$ and $k, n \in \mathbb{N}$, the value of $\beta(k,c,n)$ represents a minimum volume growth necessary to guarantee that any continuous map $f : S^k \to B_p(R)$ has a continuous extension $g : \mathbb{D}^{k+1} \to B_p(cR)$. Define

$$\beta(k,c,n) = \max\{1 - \gamma(c,h_{k,n}^{-1}(c),n); \beta(j,1 + \frac{h_{k,n}^{-1}(c)}{2k},n), j = 1, \ldots, k-1\},$$

where $\beta(0,c,n) = 0$ for any $c$ and $\beta(1,c,n) = 1 - \gamma(c,h_{k,n}^{-1}(c),n)$. Recall that $\gamma(c,d,n) = \left[1 + \frac{c^n}{d^n}\right]^{-1}$ [Definition 1.4] was used in proving Perelman’s Maximal Volume Lemma [Lemma 1.5].

### 3.1 Key Lemmas

In this section we state the Main Lemma and the Moving In Lemma. These are similar to the lemmas used in Perelman’s paper [17] except that we are controlling the constants carefully so as to be able to determine the best bounds for the volume growth later.
Lemma 3.4. [Main Lemma($k$)]. Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq 0$ and let $p \in M^n$ and $R > 0$. For any constant $c > 1$ and $k, n \in \mathbb{N}$, if

$$\alpha_M \geq \beta(k, c, n),$$

then any continuous map $f : \mathbb{S}^k \to B_p(R)$ can be continuously extended to a map $g : \mathbb{D}^{k+1} \to B_p(cR)$.

Lemma 3.5. [Moving In Lemma($k$)]. Let $M^n$ be a Riemannian manifold with $\text{Ric} \geq 0$. For any constant $d_0 \in (0, \delta_{k,n})$ and $k, n \in \mathbb{N}$ if

$$\alpha_M \geq \beta(k, h_{k,n}(d_0), n),$$

then given $q \in M^n$, $\rho > 0$, a continuous map $\phi : \mathbb{S}^k \to B_q(\rho)$ and a triangulation $T^k$ of $\mathbb{S}^k$ such that $\text{diam}(\phi(\Delta^k)) \leq d_0 \rho$ for all $\Delta^k \in T^k$, there exists a continuous map $\tilde{\phi} : \mathbb{S}^k \to B_q((1-d_0)\rho)$ such that

$$\text{diam}(\phi(\Delta^k) \cup \tilde{\phi}(\Delta^k)) \leq 10^{-k-1} \left(1 + \frac{d_0}{2^k}\right)^{-k} (1 - h_{k,n}(d_0)^{-1})\rho.$$

In the next two sections we prove these lemmas.

Before proceeding to the proofs, it is perhaps helpful to provide some insight to the main ideas behind the two lemmas above and how they are related to one another. The Moving In Lemma is, in some sense, the primary tool in constructing the homotopy. In fact, this lemma is precisely the point in the argument where the volume growth restriction is introduced. The new map $\tilde{\phi}$ constructed in the Moving In Lemma is not necessarily homotopic the original map $\phi$; however, we require their images to be ‘close’ in the manifold by controlling very carefully and uniformly the distance between the images of triangulations between the two maps. The proof is constructive and to construct a map with these properties requires large amount of volume growth in $M^n$. The Main Lemma provides a way of keeping track of the volume growth required to produce the homotopy. It’s requirement on the volume growth arises only in the fact that it’s proof requires an application of the Moving In Lemma in the same dimension.

The two lemmas are related to one another through the choice of the small constant $d_0$ in the
Moving In Lemma, the constant $c > 1$ in the Main Lemma and the double induction argument relating the two. For example, taking $d_0$ very small in the Moving In Lemma weakens the restriction on the volume growth there. However, the Main Lemma is proven by induction using the constant $c = 1 + d_0/2k$ in lower dimensions. Taking $c$ very close to 1 in the Main Lemma ultimately forces the volume growth to be very large, close to 1. Contrarily, taking a much larger $d_0 < 1$ in the Moving In Lemma immediately forces the volume growth to be close to 1. The difficulty in determining optimal bounds (via this method) for the volume growth as stated in Theorem 1.2 arises in finding the balance between these two competing lemmas and choosing the best constants $d_0$ and $c$.

In section 3.2 we prove Main Lemma$(k)$ assuming Moving In Lemma$(k)$ and Main Lemma$(j)$ for $j = 1,..,k - 1$. In section 3.3 we prove Moving In Lemma$(k)$ assuming Main Lemma$(i)$, $i = 0,..,k - 1$. In section 3.4 we apply these lemmas to prove Theorem 1.2. We begin by proving Main Lemma (0).

**Lemma 3.6. [Main Lemma(0)].** Let $X$ be a complete length space and let $p \in X$, $R > 0$. For any constant $c > 1$, any continuous map $f : S^0 \to B_p(R) \subset X$ can be continuously extended to a map $g : D^1 \to B_p(cR) \subset X$.

**Proof.** The image $f(S^0)$ consists of two points, $p_1, p_2 \in X$. Since $X$ is a complete length space, it is possible to find length minimizing geodesics $\sigma_i$ connecting $p_i$ to $p$, for $i = 1, 2$. Define $g$ so that $\text{Im}(g) = \sigma_1 \cup \sigma_2$ and $g(-1) = p_1$ and $g(1) = p_2$. Thus, $g$ is a continuous extension of the map $f$ and by construction $\text{Im}(g) \subset B_p(cR) \subset X$.

3.2 Proof of Main Lemma$(k)$

**Proof.** The proof is by induction on $k$. When $k = 0$, the result follows from Lemma 3.6. No assumption on volume growth is necessary. Assume now that Main Lemma$(i)$ holds for $i = 1,..,k - 1$: Given any constants $c_i > 1$, a continuous map $f : S^i \to B_p(R)$ has a continuous extension to a map $g : D^{i+1} \to B_p(c_i R)$ provided $\alpha_M \geq \beta(i, c_i, n)$. We will now show that the result is true for dimension $k$.

Let $f : S^k \to B_p(R) \subset M^n$ be a continuous map. Choose $c > 1$ and suppose $\alpha_M \geq \beta(k, c, n)$. Our goal now is to show that the map $f : S^k \to B_p(R)$ has a continuous extension. To do
this we will show that there exists a sequence of finite cell decompositions, $K_j$, of $\mathbb{D}^{k+1}$ and maps $f_j$ that satisfy the hypothesis of the Homotopy Construction Theorem [Theorem 2.7] and thus create the homotopy $g : \mathbb{D}^{k+1} \to B_p(cR)$.

For $j = 0$, define $K_0$ to be the cell decomposition consisting of a single cell (i.e. $K_0 \cong \mathbb{D}^{k+1}$) so $\text{ske}l_k(K_0) = S^k$. Recall that we use the notation $\text{ske}l_k(K_j)$ to denote the union of the boundaries of the cell decomposition of $K_j$ [Definition 2.6].

As in [17], inductively define $K_{j+1}$ given $K_j$ in the following way. For a $(k+1)$-cell, $\sigma \in K_j$, note that $\sigma$ is homeomorphic to a disk so it can be viewed in polar coordinates as $(S^k \times (0, 1]) \cup \{0\}$. Let $T^k_\sigma$ be a triangulation of $S^k$, where $S^k \cong \partial \sigma$ and $\text{diam}_\sigma(\Delta^k) < 1/k$ for all $\Delta^k \in T^k_\sigma$. Define $K_{j+1}$ so that $\sigma \cap \text{ske}l_k(K_{j+1}) = (S^k \times \{1\}) \cup (S^k \times \{1/2\}) \cup \left(\text{ske}l_{k-1}(T^k_\sigma) \times [1/2, 1]\right). \quad (21)$

This inductive construction of the $K_j$ provides us with a sequence of finite cell decompositions of $\mathbb{D}^{k+1}$. Note that with an appropriate selection of $S^k \times \{1/2\}$ this sequence of decompositions satisfies Condition A on cell decompositions as required by the Homotopy Construction Theorem [Theorem 2.7] because $\max\{\text{diam}(\sigma) | \sigma \in K_j\} \to 0$.

Next, we define the continuous maps $f_j : \text{ske}l_k(K_j) \to M^n$. Begin by setting $f_0 \equiv f$. In this way, $f_0 : \text{ske}l_k(K_0) \to B_p(R) \subset M^n$ and the initializing hypothesis (C) of Theorem 2.7 is satisfied. We verify the rest of the hypothesis inductively.

Suppose $f_j$ satisfies hypotheses (A) and (B) of Theorem 2.7. It remains to define $f_{j+1}$ and check that hypotheses (A) and (B) hold for this $f_{j+1}$. We describe the process to define $f_{j+1}$ on the refinement of a single $(k+1)$-cell $\sigma \in K_j$. To define $f_{j+1}$ on all of $\text{ske}l_k(K_{j+1})$, repeat this process on each $(k+1)$-cell of $K_j$.

Given a $(k+1)$-cell $\sigma \in K_j$, by hypothesis (B), there exists a point $p_\sigma \in B_p(cR) \subset M^n$ and a constant $R_\sigma > 0$ such that $f_j(\partial \sigma) \subset B_{p_\sigma}(R_\sigma)$. As before, view $\sigma$ as $(S^k \times (0, 1]) \cup \{0\}$, and think of $f_j$ as a map $f_j : S^k \to B_{p_\sigma}(R_\sigma)$.

Define $f_{j+1} : \text{ske}l_k(K_{j+1}) \to M^n$ in three stages.

First we set

$$f_{j+1} \equiv f_j \quad \text{on } S^k \times \{1\}, \quad (22)$$
which is all that is required to satisfy hypothesis (A).

We claim that we can apply the Moving In Lemma\((k)\) to the map \(f_j\). Set \(d_0 = h_{k,n}^{-1}(c)\) and keep \(k, n\) as before. The volume growth assumption \((19)\) is satisfied since \(\alpha_M \geq \beta(k, c, n) = \beta(k, h_{k,n}(d_0), n)\).

Take \(q = p_\sigma, \rho = R_\sigma, \) and \(\phi = f_j\) and take a sufficiently fine triangulation, \(T^k_\sigma\), of \(S^k \cong \partial \sigma\) such that \(\text{diam}(f_j(\Delta^k)) \leq d_0 R_\sigma\) for all \(\Delta^k \in T^k_\sigma\). Applying the Moving In Lemma\((k)\) \[Lemma 3.5\], we obtain a map \(\tilde{f}_j : S^k \rightarrow B_{p_\sigma}((1 - d_0)R_\sigma)\). We set
\[
f_{j+1} = \tilde{f}_j \quad \text{on } S^k \times \{1/2\}.
\]

This completes the second stage of our construction of \(f_{j+1}\). Furthermore, by \((20)\),
\[
\text{diam}(f_j(\Delta^k) \cup \tilde{f}_j(\Delta^k)) \leq 10^{-k-1} \left(1 + \frac{d_0}{2k}\right)^{-k} (1 - (h_{k,n}(d_0))^{-1})R_\sigma \quad (24)
\]
\[
= 10^{-k-1} \left(1 + \frac{d_0}{2k}\right)^{-k} (1 - c^{-1})R_\sigma, \quad (25)
\]
for all \(\Delta^k \in T^k_\sigma\).

For the third stage and to complete the definition of \(f_{j+1}\) on \(\sigma \cap \text{skel}_k(K_{j+1})\), it remains to define \(f_{j+1}\) on \(\text{skel}_i(T^k_\sigma) \times [1/2, 1]\) for \(i = 0, 1, \ldots, k - 1\). Below we describe this procedure (inductively) for a single \(k\)-simplex \(\Delta^k\) of the triangulation \(T^k_\sigma\). Here we use the induction hypothesis and assume the Main Lemma\((j)\) is true for \(j = 1, \ldots, k - 1\). First, we apply Lemma \[Lemma 3.6\] to the 0-skeleton [note that Lemma \[Lemma 3.6\] is an analog of Main Lemma\((0)\)]. Then, we apply Main Lemma \[Lemma 3.4\] repeatedly starting with the 1-dimension skeleton and continuing to the \((k - 1)\)-dimension skeleton.

Let \(\Delta^0 \in T^k_\sigma\) be a 0-simplex. Consider the map \(f_{j+1,0}\) on \(S^0\) defined by \(f_{j+1,0}(-1) = f_{j+1}(\Delta^0 \times \{1\})\) and \(f_{j+1,0}(1) = f_{j+1}(\Delta^0 \times \{1/2\})\). On these components, the map \(f_{j+1,0}\) is obtained
from (22) and (23). We want to define \( f_{j+1} \) on \( \Delta^0 \times [1/2, 1] \). Note that,

\[
\begin{align*}
\text{diam}(\text{Im}(f_{j+1,0})) &= d(f_{j+1,0}(-1), f_{j+1,0}(1)) \\
&= \text{diam}(f_{j+1}(\Delta^0 \times \{1\}) \cup f_{j+1}(\Delta^0 \times \{1/2\})) \\
&= \text{diam}(f_{j}(\Delta^0) \cup \tilde{f}_{j}(\Delta^0)) \\
&\leq \text{diam}(f_{j}(\Delta^k) \cup \tilde{f}_{j}(\Delta^k)) \\
&\leq 10^{-k-1} \left( 1 + \frac{d_0}{2k} \right)^{-k} (1 - c^{-1})R_{\sigma}.
\end{align*}
\]

In this last line we have applied (25).

If we set

\[
R_{j+1,0} = 1/2 \cdot 10^{-k-1} \left( 1 + \frac{d_0}{2k} \right)^{-k} (1 - c^{-1})R_{\sigma},
\]

then, by our estimate on the diameter of its image, we have

\[
f_{j+1,0} : S^0 \to B_{p_{j+1,0}}(R_{j+1,0}),
\]

for some point \( p_{j+1,0} \in M^n \). We now apply Main Lemma(0) [Lemma 3.6] taking \( c = 1 + d_0/2k \), \( p = p_{j+1,0} \), \( R = R_{j+1,0} \) and \( f = f_{j+1,0} \). Clearly, the hypotheses of Main Lemma(0) are satisfied since \( \beta(0, c, n) = 0 \) and \( M^n \) is a complete Riemannian manifold. Therefore, there exists a continuous extension

\[
g_{j+1,1} : \mathbb{D}^1 \to B_{p_{j+1,0}} \left( \left( 1 + \frac{d_0}{2k} \right) R_{j+1,0} \right)
\]

and we use it to define \( f_{j+1} \) on \( \text{skel}_0(T_{\sigma}^k) \times [1/2, 1] \). Furthermore,

\[
\text{diam}(f_{j+1}(\Delta^0 \times [1/2, 1])) = \text{diam}(\text{Im}(g_{j+1,1}))
\]

\[
\leq 2 \cdot \left( 1 + \frac{d_0}{2k} \right) R_{j+1,0}
\]

\[
\leq 2 \cdot \left( 1 + \frac{d_0}{2k} \right) \cdot 1/2 \cdot \left( 10^{-k-1} \left( 1 + \frac{d_0}{2k} \right)^{-k} (1 - c^{-1})R_{\sigma} \right)
\]

\[
\leq 10^{-k-1} \left( 1 + \frac{d_0}{2k} \right)^{-k+1} (1 - c^{-1})R_{\sigma}.
\]
We will use induction on $i$ to define $f_{j+1}$ on $\Delta^i \times [1/2, 1]$, for $0 \leq i < k$. Assume we have defined $f_{j+1} = f_j$ on all simplices $\Delta^i \in T^k_\sigma$ and we have defined $f_{j+1}$ on all possible $\Delta^{i-1} \times [1/2, 1]$ so that

$$\text{diam}(f_{j+1}(\Delta^{i-1} \times [1/2, 1])) \leq 10^{i-1-k} \left(1 + \frac{d_0}{2^k}\right)^{i-k} (1 - c^{-1})R_\sigma. \quad (39)$$

Note that this holds for $i = 1$ by (38). Also, note that (25) implies

$$\text{diam}(f_{j+1}(\Delta^i \times \{1\}) \cup f_{j+1}(\Delta^i \times \{1/2\})) \quad (40)$$

$$= \text{diam}(f_j(\Delta^i) \cup \tilde{f}_j(\Delta^i)) \quad (41)$$

$$\leq \text{diam}(f_j(\Delta^k) \cup \tilde{f}_j(\Delta^k)) \quad (42)$$

$$\leq 10^{-k-1} \left(1 + \frac{d_0}{2^k}\right)^{-k} (1 - c^{-1})R_\sigma. \quad (43)$$

We now build a new map $f_{j+1,i+1}$ on $\Delta^i \times [1/2, 1]$. View $(\Delta^i \times \{1\}) \cup (\Delta^i \times \{1/2\}) \cup (\partial \Delta^i \times [1/2, 1])$ as $S^i$. Since $\partial \Delta^i \times [1/2, 1]$ is a collection of $\Delta^{i-1} \times [1/2, 1]$, we have a map

$$f_{j+1,i} : S^i \to B_{p_{j+1,i}}(R_{j+1,i}), \quad (44)$$

for some point $p_{j+1,i} \in M^n$ and where by (39) and (43) we have

$$2R_{j+1,i} = \text{diam}(f_{j+1}\mid_{\Delta^i \times \{1\}} \cup f_{j+1}\mid_{\Delta^i \times \{1/2\}}) + \text{diam}(f_{j+1}(\partial \Delta^i \times [1/2, 1])) \quad (45)$$

$$\leq \text{diam}(f_j(\Delta^i) \cup \tilde{f}_j(\Delta^i)) + \text{diam}(f_{j+1}(\Delta^{i-1} \times [1/2, 1])) \quad (46)$$

$$\leq 10^{-k-1} \left(1 + \frac{d_0}{2^k}\right)^{-k} (1 - c^{-1})R_\sigma + 10^{i-1-k} \left(1 + \frac{d_0}{2^k}\right)^{i-k} (1 - c^{-1})R_\sigma \quad (47)$$

$$\leq 10^{i-k} \left(1 + \frac{d_0}{2^k}\right)^{-k+i} (1 - c^{-1})R_\sigma. \quad (48)$$
Therefore,
\[ \text{diam}(\text{Im}(f_{j+1,i})) \leq 10^{i-k} \left(1 + \frac{d_0}{2k}\right)^{-k+i} (1 - c^{-1})R_\sigma. \] (52)

Apply Main Lemma(i) taking \( c = 1 + d_0/2k \) and \( k, n \) as before. This is allowed because the volume growth requirement for Main Lemma(i) is satisfied by (17) and because the volume growth satisfies
\[ \alpha_M \geq \beta(k,c,n) \geq \beta(i,1 + \frac{h^{-1}_{k,n}(c)}{2k},n) \]
(54)
\[ = \beta(i,1 + \frac{d_0}{2k},n). \] (55)

Therefore, there exists a continuous extension
\[ g_{j+1,i+1} : \mathbb{D}^{i+1} \to B_{p_{j+1,i}}((1 + d_0/2k)R_{j+1,i}) \] (56)
of the continuous map \( f_{j+1,i} \). This extension defines \( f_{j+1} \) on skel(\( T_k^\sigma \)) \( \times [1/2,1] \) and we have the bound
\[ \text{diam}(f_{j+1}(\Delta^i \times [1/2,1])) = \text{diam}(\text{Im}(g_{j+1,i+1})) \] (57)
\[ \leq 2 \cdot \left(1 + \frac{d_0}{2k}\right) \cdot R_{j+1,i} \] (58)
\[ = 2 \cdot \left(1 + \frac{d_0}{2k}\right) \cdot 1/2 \cdot \]
\[ 10^{i-k} \left(1 + \frac{d_0}{2k}\right)^{-k+i} (1 - c^{-1})R_\sigma \]
(60)
\[ = 10^{i-k} \left(1 + \frac{d_0}{2k}\right)^{-k+i+1} (1 - c^{-1})R_\sigma. \] (61)

Furthermore, we have the bound
\[ \text{diam}(f_{j+1}(\Delta^i \times [1/2,1])) \leq 10^{i-k} \left(1 + \frac{d_0}{2k}\right)^{i+1-k} (1 - c^{-1})R_\sigma, \] (62)
for all \( \Delta^i \subset \Delta^k, \ i = 0,1,..,k-1 \), which implies our induction hypothesis on \( i \). Thus, we have defined \( f_{j+1} \) on skel(\( T_k^\sigma \)) \( \times [1/2,1] \) for each \( i = 0,1,..k-1 \).
We now complete the proof by showing that the hypotheses (A) and (B) of the Homotopy Construction Theorem [Theorem 2.7] hold for the function \( f_{j+1} \).

Hypothesis (A) follows immediately from this construction since each \( K_{j+1} \) is a subdivision of the previous \( K_j \) and by definition \( f_{j+1} \equiv f_j \) on \( K_j \).

To check (B) holds, let \( \sigma' \in K_{j+1} \) and suppose \( \sigma' \equiv \Delta^k \times [1/2, 1] \) for some \( \Delta^k \in S^k \). Notice that

\[
\text{diam}(f_{j+1}(\partial\sigma')) \leq \text{diam}(f_{j+1}|_{\Delta^k \times \{1\} \cup f_{j+1}|_{\Delta^k \times \{1/2\}}) + \text{diam}(f_{j+1}(\partial\Delta^k \times [1/2, 1]))
\]

\[
\leq \text{diam}(f_j(\Delta^k) \cup \tilde{f}_j(\Delta^k)) + \text{diam}(f_{j+1}(\Delta^{k-1} \times [1/2, 1]))
\]

\[
\leq 10^{-k-1} \left(1 + \frac{d_0}{2k}\right)^{-k} (1 - c^{-1})R_\sigma + 10^{-1}(1 - c^{-1})R_\sigma,
\]

where the last line follows from (25) and (62) with \( i = k - 1 \).

Set

\[
R_{\sigma'} = 1/2 \cdot [10^{-k-1} \left(1 + \frac{d_0}{2k}\right)^{-k} (1 - c^{-1}) + 10^{-1}(1 - c^{-1})]R_\sigma.
\]

Then, by (68), there exists a point \( p_{\sigma'} \in M^n \) such that \( f_{j+1}(\partial\sigma') \subset B_{p_{\sigma'}}(R_{\sigma'}) \).

To verify \( B_{p_{\sigma'}}(cR_{\sigma'}) \subset B_{p_\sigma}(cR_\sigma) \), let \( x \in B_{p_\sigma}(cR_\sigma) \) and notice that for \( q \in f(\Delta^k \times \{1/2\}) \subset B_{p_\sigma}(cR_\sigma) \),

\[
d(x, p_\sigma) \leq d(x, q) + d(q, p_\sigma)
\]

\[
\leq 2 \cdot \frac{1}{2}(1 - c^{-1})cR_\sigma + (1 - d_0)R_\sigma
\]

\[
\leq (c - 1)R_\sigma + (1 - d_0)R_\sigma
\]

\[
< cR_\sigma.
\]

Therefore, \( B_{p_{\sigma'}}(cR_{\sigma'}) \subset B_{p_\sigma}(cR_\sigma) \).
Furthermore, since \( B_{p_{\sigma'}}(cR_{\sigma'}) \subset B_{p_{\sigma}}(cR_{\sigma}) \) for all nested sequences \( \sigma' \subset \sigma \), it follows that

\[
d(p_{\sigma'}, p) \leq d(p_{\sigma'}, p_{\sigma}) + \ldots + d(p_{\sigma}, p) \leq cR_{\sigma} - cR_{\sigma'} + \ldots + cR - cR_{\sigma} = cR - cR_{\sigma'} < cR.
\] (77)

Thus, \( p_{\sigma'} \in B_{p}(cR) \) as required.

Lastly, we have \( R_{\sigma'} \leq \omega R_{\sigma} \) for

\[
\omega = 1/2 \cdot \left[ 10^{-k-1} \left( 1 + \frac{d_0}{2k} \right)^{-k} (1 - c^{-1}) + 10^{-1} (1 - c^{-1}) \right]. \tag{78}
\]

Note that \( \omega \in (0, 1) \) because \( k \geq 1 \) and \( d_0 < 1 \).

Thus, we have constructed a sequence of maps \( f_j : \text{ske}l_k(K_j) \to M^n \) satisfying the hypotheses of the Homotopy Construction Theorem [Theorem 2.7]. Therefore, the map \( f \) can be continuously extended to a map

\( g : D^{k+1} \to B_{p}(cR) \subset M^n \). This completes the proof of Main Lemma(\( k \)). \( \square \)

### 3.3 Proof of Moving In Lemma(\( k \))

We now prove Moving In Lemma(\( k \)) assuming that Main Lemma(\( j \)) is true for \( j = 0, \ldots, k-1 \).

**Proof.** Recall that \( \alpha_M \geq \beta(k, h_{k,n}(d_0), n) \) and we are given \( q \in M^n, \rho > 0 \), a continuous map \( \phi : S^k \to B_q(\rho) \) and a triangulation \( T^k \) of \( S^k \) such that \( \text{diam}(\phi(\Delta^k)) \leq d_0 \rho \) for all \( \Delta^k \in T^k \).

We must show that there exists a continuous map \( \tilde{\phi} : S^k \to B_q((1-d_0)\rho) \) such that

\[
\text{diam}(\phi(\Delta^k) \cup \tilde{\phi}(\Delta^k)) \leq 10^{-k-1} \left( 1 + \frac{d_0}{2k} \right)^{-k} (1 - h_{k,n}(d_0)^{-1}) \rho. \tag{79}
\]

We will construct \( \tilde{\phi} \) inductively on \( \text{ske}l_i(T^k) \) for \( i = 0, \ldots, k \) in such a way that \( \tilde{\phi}(\Delta^i) \equiv \phi(\Delta^i) \).
if $\phi(\Delta^i) \subset B_q((1 - 2d_0)\rho)$; and, if $\phi \not\subset B_q((1 - 2d_0)\rho)$, then

$$\tilde{\phi}(\Delta^i) \subset B_q((1 - d_0(2 - i/k))\rho),$$  

$$\text{diam}(\phi(\Delta^i) \cup \tilde{\phi}(\Delta^i)) \leq 10d_i \rho, \quad (81)$$

for all $\Delta^i \subset T^k$, $i = 0, \ldots, k$. The constants $d_i > 0$ satisfy

$$d_0 + 10d_i \leq b_i(d_{i+1} - 3d_0 - 10d_i) \quad (82)$$

$$d_0 + 10d_i \leq b_i(c - 1 + d_0(2 - i/k)) \quad (83)$$

$$8b_i^{-1}(d_0 + 10d_i) \leq \frac{d_0}{2k} \quad (84)$$

$$10d_k \leq 10^{-k-1}(1 + d_0/2k)^{-k}(1 - h_{k,n}(d_0)^{-1}) \quad (85)$$

for some constants $b_i \in (0, 1/2]$. The existence of such constants $d_i$ and $b_i$ is proven in Lemma 4.1. Note that (81) and (85) together immediately imply (79). Thus, we need only define $\tilde{\phi}$ so that the above conditions are obeyed. To do so, we construct $\tilde{\phi}$ successively on the $i$-skeleta of $T_k$.

Begin with the case $i = 0$. Let $\Delta^0 \in \text{skel}_0(T^k)$ and assume $\phi(\Delta^0) \not\subset B_q((1 - 2d_0)\rho)$, else we are done. Let $\sigma_{\Delta^0}$ denote a length minimizing geodesic from $\phi(\Delta^0)$ to $q$ and define $\tilde{\phi}(\Delta^0) = \sigma_{\Delta^0}(1 - 2d_0)\rho)$. In this way, $\tilde{\phi}(\Delta^0) \in B_q((1 - 2d_0)\rho)$ and (80) is satisfied for $i = 0$.

Furthermore,

$$\text{diam}(\phi(\Delta^0) \cup \tilde{\phi}(\Delta^0)) = d(\phi(\Delta^0), \tilde{\phi}(\Delta^0))$$

$$= d(q, \phi(\Delta^0)) - d(q, \tilde{\phi}(\Delta^0))$$

$$\leq \rho - (1 - 2d_0)\rho = 2d_0\rho \leq 10d_0\rho. \quad (88)$$

Thus, (81) is also satisfied when $i = 0$.

Now assume that $\tilde{\phi}$ is defined on $\text{skel}_i(T^k)$ and that (81) and (85) for $0 \leq i \leq k - 1$. We now construct $\tilde{\phi}$ on $\text{skel}_{i+1}(T^k)$. Let $\Delta^{i+1} \subset \text{skel}_{i+1}(T^k)$. As before, suppose $\phi(\Delta^{i+1}) \not\subset B_q((1 - 2d_0)\rho)$, else we are done by simply setting $\tilde{\phi}(\Delta^{i+1}) \equiv \phi(\Delta^{i+1})$.

Next apply Perelman’s Maximal Volume Lemma [Lemma 1.5], taking $c_1 = h_{k,n}(d_0)$, $\epsilon = d_0$, taking
and \( p \equiv q, R \equiv \rho \). Since, by our hypothesis,

\[
\alpha_M \geq \beta(k, h_{k,n}(d_0), n)
\]

\[
= \max \left\{ 1 - \gamma(h_{k,n}(d_0), d_0, n); \beta \left( j, 1 + \frac{d_0}{2k}, n \right), j = 1, \ldots, k - 1 \right\}
\]

\[
\geq 1 - \gamma(h_{k,n}(d_0), d_0, n),
\]

there exists a point \( r_\Delta \in M^n \setminus B_q(h_{k,n}(d_0)\rho) \) such that \( d(\phi(\Delta^{i+1}), \overline{qr_\Delta}) \leq d_0\rho \). Recall, \( \overline{qr_\Delta} \) denotes a minimal geodesic connecting \( q \) and \( r_\Delta \). Let \( \sigma_\Delta \) be a length minimizing geodesic from \( q \) to \( r_\Delta \) and define a point \( q_\Delta = \sigma_\Delta((1 - d_{i+1})\rho) \). For any \( x \in \partial\Delta^{i+1} \), the triangle with vertices \( \tilde{\phi}(x), q_\Delta, \) and \( r_\Delta \) is small and thin. To verify this, we use the induction hypothesis that \( \tilde{\phi} \) has already been defined on \( \text{skel}_i(T^k) \), \( 0 \leq i \leq k - 1 \), and that the properties \( (\text{S}0), (\text{S}1) \) are satisfied in dimension \( i \).

Note that,

\[
d(\tilde{\phi}(x), \overline{q_\Delta r_\Delta}) \leq d(\phi(x), \overline{q_\Delta r_\Delta}) + d(\tilde{\phi}(x), \phi(x))
\]

\[
\leq d_0\rho + \text{diam}(\tilde{\phi}(\Delta^i) \cup \tilde{\phi}(\Delta^i))
\]

\[
\leq d_0\rho + 10d_i\rho.
\]

And

\[
d(\tilde{\phi}(x), q_\Delta) \geq d(q_\Delta, \phi(x)) - d(\phi(x), \tilde{\phi}(x))
\]

\[
\geq d(q, \phi(x)) - d(q_\Delta, q) - \text{diam}(\phi(\Delta^i) \cup \tilde{\phi}(\Delta^i))
\]

\[
\geq (1 - 2d_0)\rho - \text{diam}(\phi(\Delta^{i+1}) - (1 - d_{i+1})\rho - 10d_i\rho
\]

\[
\geq (1 - 2d_0)\rho - d_0\rho - (1 - d_{i+1})\rho - 10d_i\rho
\]

\[
\geq (d_{i+1} - 3d_0 - 10d_i)\rho
\]
And finally,

\[ d(\tilde{\phi}(x), r_\Delta) \geq d(r_\Delta, q) - d(q, \tilde{\phi}(x)) \geq C \rho - d(q, \tilde{\phi}(\Delta^i)) \geq c \rho - (1 - d_0(2 - i/k)) \rho \]

\[ = (c - 1 + d_0(2 - i/k)) \rho. \]

The inequalities (82) and (83) guarantee that the triangle with vertices \( \tilde{\phi}(x), q, r_\Delta \) is small and thin for some constants \( 0 < b_i \leq 1/2 \).

According to the excess estimate of Abresch-Gromoll [Theorem 1.3], we have that for any \( x \in \partial \Delta^{i+1} \), with \( i = 0, 1, \ldots, k - 1 \),

\[ e_{q_\Delta, r_\Delta}(\tilde{\phi}(x)) = d(\tilde{\phi}(x), q_\Delta) + d(\tilde{\phi}(x), r_\Delta) - d(q_\Delta, r_\Delta) \leq 8 \left( \frac{d(\tilde{\phi}(x), q_\Delta)}{\min\{d(\tilde{\phi}(x), q_\Delta), d(\tilde{\phi}(x), r_\Delta)\}} \right)^{1/n} d(\tilde{\phi}(x), q_\Delta) \]

\[ \leq 8b_i^{1/n} (d_0 + 10d_i) \rho. \]

Also, by the triangle inequality,

\[ d(q, q_\Delta) + d(q_\Delta, r_\Delta) = d(q, r_\Delta) \leq d(q, \tilde{\phi}(x)) + d(\tilde{\phi}(x), r_\Delta). \]

Adding \( \text{(106)} \) and \( \text{(107)} \), we get

\[ d(\tilde{\phi}(x), q_\Delta) \leq 8b_i^{1/n} (d_0 + 10d_i) \rho + d(q_\Delta, r_\Delta) - d(\tilde{\phi}(x), r_\Delta) \leq 8b_i^{1/n} (d_0 + 10d_i) \rho + d(q, \tilde{\phi}(x)) + d(\tilde{\phi}(x), r_\Delta) - d(q, q_\Delta) - d(\tilde{\phi}(x), r_\Delta) \leq 8b_i^{1/n} (d_0 + 10d_i) \rho + (1 - d_0(2 - i/k)) \rho - (1 - d_{i+1} \rho) \]

\[ = \left( 8b_i^{1/n} (d_0 + 10d_i) + d_{i+1} - d_0(2 - i/k) \right) \rho. \]
It then follows from (84) that, for all \( x \in \partial \Delta^{i+1} \),
\[
d(\hat{\phi}(x), q) \leq \left( \frac{d_0}{2k} + d_{i+1}d_0(2 - i/k) \right) \rho = \left( d_{i+1} - d_0(2 - \frac{2i + 1}{2k}) \right) \rho.
\] (113)

Now apply the Main Lemma [Lemma 3.4] in dimension \( i \) taking
\[
p = q_\Delta, \quad R = \left( d_{i+1} - d_0 \left( 2 - \frac{2i + 1}{2k} \right) \right) \rho, \quad c = 1 + d_0/2k;
\] (114) \hspace{1cm} (115) \hspace{1cm} (116)
and letting \( f = \hat{\phi} \). Since
\[
\alpha_M \geq \beta(k, h_{k,n}(d_0), n) = \max \left\{ 1 - \gamma(h_{k,n}(d_0), d_0, n); \beta \left( j, 1 + \frac{d_0}{2k}, n \right); j = 1, \ldots, k-1 \right\} \geq \beta(i, 1 + \frac{d_0}{2k}, n)
\] (117) \hspace{1cm} (118) \hspace{1cm} (119)
by our hypothesis, there exists a continuous extension of \( \hat{\phi} \) from \( \partial \Delta^{i+1} \) to \( \Delta^{i+1} \). Furthermore,
\[
d(\hat{\phi}(\Delta^{i+1}), q_\Delta) \leq (1 + d_0/2k) \left( d_{i+1} - d_0(2 - \frac{2i + 1}{2k}) \right) \rho \leq \left( d_{i+1} - d_0(2 - \frac{i + 1}{k}) \right) \rho,
\] (120) \hspace{1cm} (121)
promid \( d_i < 1 \), which is guaranteed by the fact that the \( d_i \)'s are increasing in \( i \) and, by (85), \( d_k < 1 \). Therefore, by the triangle inequality,
\[
d(\hat{\phi}(\Delta^{i+1}), q) \leq d(\hat{\phi}(\Delta^{i+1}), q_\Delta) + d(q_\Delta, q) \leq \left( d_{i+1} - d_0(2 - \frac{i + 1}{k}) \right) \rho + (1 - d_{i+1}) \rho \leq \left( 1 - d_0(2 - \frac{i + 1}{k}) \right) \rho.
\] (122) \hspace{1cm} (123) \hspace{1cm} (124)
Thus, (80) is satisfied for \( i + 1 \) for any choice of \( d_i, b_i \) satisfying the inequalities (82), (83) and (84).
Furthermore,

\[
\text{diam}(\phi(\Delta^{i+1} \cup \tilde{\phi}(\Delta^{i+1}))) \leq \text{diam}(\phi(\partial\Delta^{i+1} \cup \tilde{\phi}(\partial\Delta^{i+1}))) + \\
\text{diam}(\phi(\Delta^{i+1})) + \text{diam}(\tilde{\phi}(\Delta^{i+1}))
\]

(125)

\[
\leq 10d_i \rho + d_0 \rho + 2 \left( d_{i+1} - d_0 \left( 2 - \frac{i+1}{k} \right) \rho \right)
\]

(126)

\[
= \left( 2d_{i+1} + d_0 \left( -3 + \frac{2(i+1)}{k} \right) + 10d_i \right) \rho
\]

(127)

\[
\leq (2d_{i+1} + 10d_i - d_0) \rho.
\]

(128)

The inequality (82) and the fact that \(0 < b_i \leq 1/2\) imply that

\[
\text{diam}(\phi(\Delta^{i+1} \cup \tilde{\phi}(\Delta^{i+1}))) \leq 10d_{i+1} \rho.
\]

(130)

So, (81) are satisfied for dimension \(i+1\). Therefore, \(\tilde{\phi}\) has been defined so that (80) and (81) are satisfied for \(i = 0, \ldots, k\). When \(i = k\), (80) implies

\[
\tilde{\phi}(\Delta^k) \subset B_q((1 - d_0) \rho), \quad \forall \Delta^k \in T^k.
\]

(131)

Thus, we have constructed the map \(\tilde{\phi} : S^k \to B_q((1 - d_0) \rho)\); and furthermore,

\[
\text{diam}(\phi(\Delta^k) \cup \tilde{\phi}(\Delta^k)) \leq 10d_k \rho
\]

(132)

\[
\leq 10^{-k-1} \left( 1 + \frac{d_0}{2k} \right)^{-k} \left( 1 - h_{k,n}(d_0)^{-1} \right),
\]

(133)

where the last inequality follows from (81). \(\square\)

\section*{3.4 Proof of the Main Theorem}

In this section we prove Theorem 1.2 using Main Lemma\(k\).

As a direct consequence of Main Lemma\(k\) [Lemma 3.4], we have

\textbf{Proposition 3.7.} Let \(M^n\) be a complete Riemannian manifold with \(\text{Ric} \geq 0\). For \(k \in \mathbb{N}\), there exists a constant \(\delta_k(n) > 0\) such that if \(\alpha_M \geq 1 - \delta_k(n)\), then \(\pi_k(M^n) = 0\).

\textbf{Proof.} Choose some \(c > 1\) and set \(\delta_k(n) = 1 - \beta(k, c, n)\). The conclusion then follows from
Lemma 3.4

Thus, we recover Perelman’s result [17]:

Lemma 3.8. [17, Theorem 2]. Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq 0$. There exists a constant $\delta_n > 0$ such that if $\alpha_M \geq 1 - \delta_n$, then $M^n$ is contractible.

Proof. Choose some $c > 1$ and set $\delta_n = 1 - \max_{k=1,\ldots,n} \beta(k,c,n)$. Then Lemma 3.4 implies $\pi_k(M^n) = 0$ for all positive values $k$. Hence, $M^n$ is contractible by the Whitehead Theorem [19].

Remark. In the appendix we use the expression for $\beta(k,c,n)$ from Definition 3.3 to find the ‘best’ value (depending only on $k$ and $n$) of $\alpha_M$ which guarantees that $\pi_k(M^n) = 0$. This is the lower bound for $\alpha_M$ as stated in Theorem 1.2.

We now prove Theorem 1.2.

Proof. Let

$$\alpha(k,n) = \inf_{c \in (1,\infty)} \beta(k,c,n).$$

By assumption, $\alpha_M > \alpha(k,n)$ and thus there exists $c_0 > 1$ such that $\alpha_M \geq \beta(k,c_0,n)$. The result follows by applying Main Lemma$(k)$ with $c = c_0$. In the appendix we compute values of $\alpha(k,n)$.

4 Appendix

The constants $C_{k,n}$ explicitly determine the function $h_{k,n}(x)$ defined in section 3. In this appendix, we show that the constants $C_{k,n}$ as defined are optimal and use the definition of $h_{k,n}$ to compute explicit values for $\alpha(k,n)$ as stated in Theorem 1.2.

4.1 Optimal Constants

Recall Definition 3.1 of $C_{k,n}(i)$:

$$C_{k,n}(i) = (16k)^{n-1}(1 + 10C_{k,n}(i-1))^n + 3 + 10C_{k,n}(i-1), \quad i \geq 1 \quad (134)$$
and $C_{k,n}(0) = 1$. Denote $C_{k,n} = C_{k,n}(k)$.

The constants $C_{k,n}$ grow large very quickly. Preliminary values for $C_{k,n}$ where $1 \leq k \leq 3$ and $1 \leq n \leq 8$ are listed in Table 1.

Table 1: Table of $C_{k,n}$ values for $1 \leq k \leq 3$, $1 \leq n \leq 10$

|   | $k = 1$ | $k = 2$ | $k = 3$ |
|---|--------|--------|--------|
| $n = 1$ | 24     | -      | -      |
| $n = 2$ | 384    | $1.89 \times 10^8$ | -      |
| $n = 3$ | 6144   | $1.52 \times 10^{17}$ | $1.36 \times 10^{63}$ |
| $n = 4$ | 98304  | $1.25 \times 10^{29}$ | $1.00 \times 10^{133}$ |
| $n = 5$ | $1.57 \times 10^6$ | $1.06 \times 10^{44}$ | $9.53 \times 10^{248}$ |
| $n = 6$ | $2.51 \times 10^7$ | $9.15 \times 10^{61}$ | $1.43 \times 10^{118}$ |
| $n = 7$ | $4.03 \times 10^8$ | $8.10 \times 10^{82}$ | $4.12 \times 10^{550}$ |
| $n = 8$ | $6.44 \times 10^9$ | $7.35 \times 10^{106}$ | $2.80 \times 10^{596}$ |
| $n = 9$ | $1.03 \times 10^{11}$ | $6.82 \times 10^{134}$ | $5.50 \times 10^{1345}$ |
| $n = 10$ | $1.65 \times 10^{12}$ | $6.49 \times 10^{163}$ | $3.81 \times 10^{1528}$ |

**Lemma 4.1.** If $d_i = C_{k,n}(i)d_0$ and $b_i = [16k(1 + 10C_{k,n}(i))]^{-(n-1)}$, then

$$d_0 + 10d_i = b_i(d_{i+1} - 3d_0 - 10d_i),$$

(135)

and

$$8b_i^{n-1}(d_0 + 10d_i) = \frac{d_0}{2k},$$

(136)

for $i = 0, 1, \ldots, k$. Furthermore, (83) and (85) hold as well.

**Proof.** The proofs of (135) and (136) are by induction in $i$. When $i = 0$ the conclusion holds. Assume the conclusion holds for $i < k$. It remains to verify the conclusion for $i + 1$. Note that

$$b_{i+1}(d_{i+2} - 3d_0 - 10d_{i+1})$$

$$= [16k(1 + 10C_{k,n}(i + 1))]^{n-1}(C_{k,n}(i + 2)d_0 - 3d_0 - 10C_{k,n}(i + 1)d_0)$$

$$= (1 + 10C_{k,n}(i + 1))d_0 = d_0 + 10d_{i+1}.$$
Similarly, for the second equation we get

\[ 8b_{i+1}(d_0 + 10d_{i+1}) = 8[16k(1 + 10C_{k,n}(i + 1))]^{-1}(1 + 10C_{k,n}(i + 1))d_0 = \frac{d_0}{2k}. \]

To verify that (85) holds, note that \( d_k = C_{k,n}(k)d_0 = C_{k,n}d_0 \) and, by the definition of \( h_{k,n}(d_0) \) [Definition 3.2], we have exactly (85).

Lastly, both (85) and (135) imply (83). Note that, setting \( h_{k,n}(d_0) = c \), (85) implies

\[ d_k \leq 10^{-k-2}(1 + d_0/2k)^{-k}(1 - c^{-1}) \quad (137) \]
\[ = 10^{-k-2}(1 + d_0/2k)^{-k}1/c(c - 1) \quad (138) \]
\[ \leq c - 1, \quad (139) \]

where the last inequality follows because \( 10^{-k-2} < 1, (1 + d_0/2k)^{-k} < 1, \) and \( 1/c < 1. \)

Therefore, since \( 1 \leq i < k, \)

\[ b_i(c - 1 + d_0(2 - i/k)) \geq b_i(d_k + d_0(2 - i/k)) \quad (140) \]
\[ \geq b_i(d_k + d_0) \quad (141) \]
\[ \geq b_i \cdot d_k \quad (142) \]
\[ \geq b_i \cdot d_{i+1} \quad (143) \]
\[ \geq b_i(d_{i+1} - 3d_0 - 10d_i) \quad (144) \]
\[ = d_0 + 10d_i, \quad (145) \]

where the last equality follows from (135). Thus, (83) holds and this completes the proof. \( \square \)

**Remark.** So we see that the constants \( C_{k,n}(i) \) suffice for the proof of Theorem 1.2. Next we show that these constants provide the optimal choice.

**Lemma 4.2.** If (82) and (84) hold for all \( i \geq 0, \) then

\[ d_i \geq C_{k,n}(i) \]
and
\[ b_i \leq \frac{1}{[16k(1 + 10C_{k,n}(i))]^{n-1}}. \]

**Proof.** The proof is by induction on \( i \). When \( i = 0 \) the conclusion holds. From (82) and assuming the conclusion holds for \( i \), we have

\begin{align*}
  d_{i+1} &\geq \frac{1}{b_i}(d_0 + 10d_i) + 3d_0 + 10d_i \\
           &\geq [(16k)^{n-1}(1 + 10C_{k,n}(i))^n + 3 + 10C_{k,n}(i)]d_0 \\
           &= C_{k,n}(i + 1)d_0.
\end{align*}

Using this lower bound for \( d_{i+1} \) and (84) we get

\begin{align*}
  b_{i+1} &\leq \left( \frac{d_0}{2k d_0 + 10d_{i+1}} \right)^{n-1} \\
          &= \left( \frac{1}{16k(1 + 10C_{k,n}(i + 1))} \right)^{n-1}.
\end{align*}

This completes the proof.

4.2 Computing \( \alpha(k, n) \) values

The term \( \beta(k, c, n) \) denotes the minimal volume growth necessary to guarantee that any continuous map \( f : S^k \to B_p(R) \) has a continuous extension \( g : D^{k+1} \to B_p(cR) \) (see Definition 3.3). Recall that the expression for \( \beta(k, c, n) \) is iteratively defined.

By definition,

\[ \beta(k, c, n) = \max \left\{ 1 - \gamma(c, h_{k,n}^{-1}(c), n) ; \beta \left( j, 1 + \frac{h_{k,n}^{-1}(c)}{2k}, n \right), j = 1, \ldots, k - 1 \right\}. \]

Ultimately we are not concerned with the location of the homotopy map. Thus we have a certain amount of freedom when choosing which \( c \) value to take. To determine the optimal bound on volume growth guaranteeing \( \pi_k(M^n) = 0 \), it is necessary to choose the ‘best’ value of \( c \) for \( \beta(k, c, n) \); that is, the \( c \) which makes \( \beta(k, c, n) \) the smallest. Set \( \alpha(k, n) = \inf_{c > 1} \beta(k, c, n) \).
In order to compute explicit values for $\alpha(k, n)$, we must successively simplify the components of $\beta(k, c, n)$. Ultimately, because of its iterative definition, it is possible to express $\beta(k, c, n)$ as the maximum of a collection of $\gamma$ terms. Using the definition of $\gamma(c, e, n)$, we can then compute specific values for $\alpha(k, n)$. Here we describe in detail the method to compute $\alpha(k, n)$ and compile a table of these values for $k = 1, 2, 3$ and $n = 1, ..., 10$.

To begin, we have

$$\beta(1, c, n) = 1 - \gamma\left(c, h_{1,n}^{-1}(c), n\right). \quad (153)$$

By definition, when $k = 2$

$$\beta(2, c, n) = \max \left\{ 1 - \gamma\left(c, h_{2,n}^{-1}(c), n\right), \right.$$  
$$\left. \beta\left(1, 1 + \frac{h_{2,n}^{-1}(c)}{4}, n\right) \right\}. \quad (154) \quad (155)$$

To evaluate this expression for $\beta(2, c, n)$, simplify the $\beta\left(1, 1 + \frac{h_{2,n}^{-1}(c)}{4}, n\right)$ term by setting $c = 1 + \frac{h_{2,n}^{-1}(c)}{4}$ and applying (153). Therefore,

$$\beta(2, c, n) = \max \left\{ 1 - \gamma\left(c, h_{2,n}^{-1}(c), n\right), \right.$$  
$$\left. 1 - \gamma\left(1 + \frac{h_{2,n}^{-1}(c)}{4}, h_{1,n}^{-1}\left(1 + \frac{h_{2,n}^{-1}(c)}{4}\right), n\right) \right\}. \quad (156) \quad (157)$$
Similarly, to evaluate $\beta(3, c, n)$ we have, by definition,

$$
\beta(3, c, n) = \max \left\{ 1 - \gamma \left( c, h_{3,n}^{-1}(c), n \right) ;  \right.
\beta \left( j, 1 + \frac{h_{3,n}^{-1}(c)}{6}, n \right), j = 1, 2 \right\} \tag{158}
$$

$$
= \max \left\{ 1 - \gamma \left( c, h_{3,n}^{-1}(c), n \right) ,  \beta \left( 1, 1 + \frac{h_{3,n}^{-1}(c)}{6}, n \right), \right. \tag{159}
\beta \left( 2, 1 + \frac{h_{3,n}^{-1}(c)}{6}, n \right) \right\}. \tag{160}
$$

Substituting $\beta \left( 1, 1 + \frac{h_{3,n}^{-1}(c)}{6}, n \right)$ with the expression obtained by setting $c = 1 + \frac{h_{3,n}^{-1}(c)}{6}$ and evaluating (153) yields

$$
\beta(3, c, n) = \max \left\{ 1 - \gamma \left( c, h_{3,n}^{-1}(c), n \right) ,  \right. \tag{163}
1 - \gamma \left( 1 + \frac{h_{3,n}^{-1}(c)}{6}, h_{1,n}^{-1} \left( 1 + \frac{h_{3,n}^{-1}(c)}{6} \right), n \right), \tag{164}
\beta \left( 2, 1 + \frac{h_{3,n}^{-1}(c)}{6}, n \right) \right\}. \tag{165}
$$

Finally, apply (156) with $c = 1 + \frac{h_{3,n}^{-1}(c)}{6}$ to simplify the remaining $\beta \left( 2, 1 + \frac{h_{3,n}^{-1}(c)}{6}, n \right)$ term. We get
\[
\beta(3, c, n) = \max \left\{ 1 - \gamma \left( c, h_{3,n}^{-1}(c), n \right), \\
1 - \gamma \left( 1 + \frac{h_{3,n}^{-1}(c)}{6}, h_{1,n}^{-1} \left( 1 + \frac{h_{3,n}^{-1}(c)}{6} \right), n \right), \\
1 - \gamma \left( 1 + \frac{h_{3,n}^{-1}(c)}{6}, h_{2,n}^{-1} \left( 1 + \frac{h_{3,n}^{-1}(c)}{6} \right), n \right), \\
1 - \gamma \left( 1 + \frac{h_{2,n}^{-1} \left( 1 + \frac{h_{3,n}^{-1}(c)}{6} \right)}{4}, h_{1,n}^{-1} \left( 1 + \frac{h_{2,n}^{-1} \left( 1 + \frac{h_{3,n}^{-1}(c)}{6} \right)}{4} \right), n \right) \right\}.
\]

Because of the successive nesting, when completely expanded, the expression \( \beta(k, c, n) \) can be written as the maximum of \( 2^{k-1} \) terms of the form \( 1 - \gamma(.,.,n) \). However, given the nature of the functions \( h_{k,n}(x) \) and the behavior of \( \gamma(c, h_{k,n}^{-1}(c), n) \) when \( 1 < c < 2 \), the maximum of this collection of \( 1 - \gamma \) terms is determined by the the maximum of the leading \( 1 - \gamma(c, h_{k,n}^{-1}(c), n) \) term and the final \( 1 - \gamma \) term containing the most iterations. That is to say, for all \( k \) and \( n \),

\[
\beta(k, c, n) = \max \left\{ 1 - \gamma \left( c, h_{k,n}^{-1}(c), n \right); \beta \left( j, 1 + \frac{h_{k,n}^{-1}(c)}{2k}, n \right), j = 1, \ldots, k - 1 \right\} \\
= \max \left\{ 1 - \gamma \left( c, h_{k,n}^{-1}(c), n \right), \\
1 - \gamma \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{h_{k,n}^{-1}(c)}{2k} \right)}{2(k-1)}, h_{1,n}^{-1} \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{h_{k,n}^{-1}(c)}{2k} \right)}{2(k-1)} \right), n \right) \right\};
\]

which in turn can be written as
\[ \beta(k, c, n) = \max \left\{ 1 - \gamma \left( c, h_{k,n}^{-1}(c), n \right), 
1 - \gamma \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( \frac{1 + h_{k,n}^{-1}(c)}{2k} \right)}{2(k-1)}, h_{1,n}^{-1} \left( 1 + \ldots \frac{h_{1,n}^{-1} \left( \frac{1 + h_{k,n}^{-1}(c)}{2k} \right)}{2(k-1)} \right), n \right) \right\} 
= 1 - \min \left\{ \gamma \left( c, h_{k,n}^{-1}(c), n \right), 
\gamma \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( \frac{1 + h_{k,n}^{-1}(c)}{2k} \right)}{2(k-1)}, h_{1,n}^{-1} \left( 1 + \ldots \frac{h_{1,n}^{-1} \left( \frac{1 + h_{k,n}^{-1}(c)}{2k} \right)}{2(k-1)} \right), n \right) \right\}. \]

Recall that the constants \( \delta_{k,n} \) [Definition 3.2] represent the location of the vertical asymptote \( x = \delta_{k,n} \) of the function \( h_{k,n}(x) \). Therefore, the function \( h_{k,n}^{-1} \) is bounded above by the constant \( \delta_{k,n} \); that is, \( h_{k,n}^{-1}(c) < \delta_{k,n} \) for all \( c > 1 \). In Table 2 we list values of \( \delta_{k,n} \) for \( 1 \leq k \leq 3 \) and \( 1 \leq n \leq 10 \).

| \( n \)  | \( k = 1 \)      | \( k = 2 \)      | \( k = 3 \)      |
|-------|-----------------|-----------------|-----------------|
| 1     | 4.17 \times 10^{-3} | -               | -               |
| 2     | 2.60 \times 10^{-6} | 5.29 \times 10^{-13} | -               |
| 3     | 1.63 \times 10^{-7} | 6.58 \times 10^{-22} | 7.34 \times 10^{-66} |
| 4     | 1.02 \times 10^{-8} | 7.98 \times 10^{-34} | 9.96 \times 10^{-139} |
| 5     | 6.36 \times 10^{-10} | 9.45 \times 10^{-49} | 1.05 \times 10^{-254} |
| 6     | 3.97 \times 10^{-11} | 1.09 \times 10^{-66} | 7.01 \times 10^{-424} |
| 7     | 2.48 \times 10^{-12} | 1.23 \times 10^{-87} | 2.43 \times 10^{-656} |
| 8     | 1.55 \times 10^{-13} | 1.36 \times 10^{-111} | 3.57 \times 10^{-962} |
| 9     | 9.70 \times 10^{-15} | 1.47 \times 10^{-138} | 1.81 \times 10^{-1351} |
| 10    | 6.06 \times 10^{-16} | 1.54 \times 10^{-168} | 2.62 \times 10^{-1834} |

Fixing \( k \) and \( n \), the function \( \gamma(c, h_{k,n}^{-1}(c), n) \) is increasing as a function of \( c \) when \( 1 < c < 2 \). Further, we have that for all \( c > 1 \)

\[ h_{k,n}^{-1}(c) < \delta_{k,n} \quad (166) \]
\[ h_{k,n}^{-1}(c)/2k < \delta_{k,n}/2k \quad (167) \]
\[ 1 + h_{k,n}^{-1}(c)/2k < 1 + \delta_{k,n}/2k << 2. \quad (168) \]
Define $\epsilon_{k,n}$ as

$$
\epsilon_{k,n} = \lim_{c \to \infty} \gamma \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{h_{k,n}(c)}{2k} \right)}{2(k-1)}, h_{1,n}^{-1} \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{h_{k,n}(c)}{2k} \right)}{2(k-1)} \right), n \right)
$$

$$
= \gamma \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{\delta_{k,n}}{2k} \right)}{2(k-1)}, h_{1,n}^{-1} \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{\delta_{k,n}}{2k} \right)}{2(k-1)} \right), n \right)
$$

$$
= \left[ 1 + \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{\delta_{k,n}}{2k} \right)}{2(k-1)} \right) \left( 1 + \ldots \frac{h_{k-1,n}^{-1} \left( 1 + \frac{\delta_{k,n}}{2k} \right)}{2(k-1)} \right) \right]^{-1}
$$

With this simplification, it is possible to explicitly compute the values of $\epsilon_{k,n}$. Table 3 below lists values of $\epsilon_{k,n}$ for $k = 1, 2, 3$ and $n = 1, \ldots, 10$. These values were computing using Mathematica 6.0 and the source code for these computations as well as additional exposition is available in [14].

Table 3: Table of $\epsilon_{k,n}$ values for $1 \leq k \leq 3, 1 \leq n \leq 10$

| $k$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n = 9$ | $n = 10$ |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $k = 1$ | $1.04 \times 10^{-3}$ | - | $6.74 \times 10^{-25}$ | $4.18 \times 10^{-35}$ | $1.01 \times 10^{-49}$ | $9.61 \times 10^{-61}$ | $3.56 \times 10^{-86}$ | $5.14 \times 10^{-108}$ | $2.90 \times 10^{-132}$ | $6.41 \times 10^{-159}$ |
| $k = 2$ | - | $1.89 \times 10^{-37}$ | $1.92 \times 10^{-86}$ | $1.70 \times 10^{-107}$ | $7.64 \times 10^{-290}$ | $1.64 \times 10^{-362}$ | $1.55 \times 10^{-604}$ | $6.06 \times 10^{-995}$ | $9.08 \times 10^{-1373}$ | $4.87 \times 10^{-1837}$ |
| $k = 3$ | - | - | $3.52 \times 10^{-284}$ | $1.29 \times 10^{-722}$ | $1.25 \times 10^{-1363}$ | $4.16 \times 10^{-3096}$ | $2.75 \times 10^{-5289}$ | $9.42 \times 10^{-8693}$ | $1.94 \times 10^{-13536}$ | $1.24 \times 10^{-20180}$ |
The value \( \alpha(k, n) \), as described in Theorem 1.2, represents the optimal lower bound for the volume growth guaranteeing \( \pi_k(M^n) = 0 \). We can then set \( \alpha(k, n) = 1 - \epsilon_{k,n} \). Table 4 contains the values of \( \alpha(k, n) \) for \( k = 1, 2, 3 \) and \( n = 1, \ldots, 10 \).

Table 4: Table of \( \alpha(k, n) \) values for \( 1 \leq k \leq 3, 1 \leq n \leq 10 \)

|    | \( k = 1 \)             | \( k = 2 \)             | \( k = 3 \)             |
|----|-------------------------|-------------------------|-------------------------|
| \( n = 1 \) | \( 1 - 1.04 \times 10^{-3} \) | -                       | -                       |
| \( n = 2 \) | \( 1 - 4.24 \times 10^{-13} \) | \( 1 - 1.89 \times 10^{-37} \) | -                       |
| \( n = 3 \) | \( 1 - 6.74 \times 10^{-23} \) | \( 1 - 1.92 \times 10^{-86} \) | \( 1 - 3.52 \times 10^{-284} \) |
| \( n = 4 \) | \( 1 - 4.18 \times 10^{-49} \) | \( 1 - 1.70 \times 10^{-16} \) | \( 1 - 1.29 \times 10^{-122} \) |
| \( n = 5 \) | \( 1 - 1.01 \times 10^{-10} \) | \( 1 - 7.64 \times 10^{-29} \) | \( 1 - 1.25 \times 10^{-1563} \) |
| \( n = 6 \) | \( 1 - 9.61 \times 10^{-67} \) | \( 1 - 1.64 \times 10^{-462} \) | \( 1 - 4.16 \times 10^{-3006} \) |
| \( n = 7 \) | \( 1 - 3.56 \times 10^{-86} \) | \( 1 - 1.55 \times 10^{-694} \) | \( 1 - 2.75 \times 10^{-5289} \) |
| \( n = 8 \) | \( 1 - 5.14 \times 10^{-108} \) | \( 1 - 6.06 \times 10^{-993} \) | \( 1 - 9.42 \times 10^{-8036} \) |
| \( n = 9 \) | \( 1 - 2.90 \times 10^{-132} \) | \( 1 - 9.08 \times 10^{-1373} \) | \( 1 - 1.94 \times 10^{-13536} \) |
| \( n = 10 \) | \( 1 - 6.41 \times 10^{-159} \) | \( 1 - 4.87 \times 10^{-1837} \) | \( 1 - 1.24 \times 10^{-20180} \) |

In general, \( \alpha(1, n) = 1 - \left(1 + \frac{2}{h_{1,n}(2)}\right)^{-1} \); and for \( k > 1 \), we have

\[
\alpha(k, n) = 1 - \epsilon_{k,n} \quad \ldots \quad (169)
\]

\[
\alpha(k, n) = 1 - \left[1 + \left(\frac{1}{h_{k-1,n}(2)} + \frac{h_{k-1,n}(1+\frac{n\epsilon_{k,n}}{2})}{2(k-1)}\right)^{n-1}\right]^\frac{1}{n-1} \quad \ldots \quad (170)
\]

These are the bounds that can be achieved via Perelman’s method \[17\].

Combining this information with previous results of Anderson \[2\], Li \[15\], Cohn-Vossen \[9\] and Zhu \[20\] we can refine the table above.
Table 5: Table of revised $\alpha(k, n)$ values for $1 \leq k \leq 3, 1 \leq n \leq 10$

|     | $k = 1$ | $k = 2$ | $k = 3$ |
|-----|---------|---------|---------|
| $n = 1$ | -       | -       | -       |
| $n = 2$ | 0       | 0       | -       |
| $n = 3$ | 0       | 0       | 0       |
| $n = 4$ | $1/2$   | $1 - 1.70 \times 10^{-167}$ | $1 - 1.29 \times 10^{-722}$ |
| $n = 5$ | $1/2$   | $1 - 7.64 \times 10^{-290}$ | $1 - 1.25 \times 10^{-1563}$ |
| $n = 6$ | $1/2$   | $1 - 1.64 \times 10^{-462}$ | $1 - 4.16 \times 10^{-3086}$ |
| $n = 7$ | $1/2$   | $1 - 1.55 \times 10^{-694}$ | $1 - 2.75 \times 10^{-5289}$ |
| $n = 8$ | $1/2$   | $1 - 6.06 \times 10^{-995}$ | $1 - 9.42 \times 10^{-8693}$ |
| $n = 9$ | $1/2$   | $1 - 9.08 \times 10^{-1373}$ | $1 - 1.94 \times 10^{-13546}$ |
| $n = 10$ | $1/2$   | $1 - 4.87 \times 10^{-1837}$ | $1 - 1.24 \times 10^{-20180}$ |

References

[1] U. Abresch, D. Gromoll, On complete manifold with nonnegative Ricci curvature, J. Amer. Math. Soc. 3 (1990) 355-374.

[2] M. Anderson, On the topology of complete manifold of nonnegative Ricci curvature, Topology 3 (1990) 41-55.

[3] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, American Mathematical Society, Providence, RI, 2001.

[4] R. L. Bishop, R. J. Crittenden, Geometry on Manifolds, Academic Press, New York, 1964.

[5] J. Cheeger, Critical points of distance functions and applications to geometry. Geometric topology: recent developments (Montecatini Terme, 1990), 1–38, Lecture Notes in Math., 1504, Springer, Berlin, 1991. 53C23 (53-02)

[6] J. Cheeger, T. H. Colding, On the structure of spaces with Ricci curvature bounded below I, J. Diff. Geom. 46 (1997) 406-480.

[7] J. Cheeger, T. H. Colding, On the structure of spaces with Ricci curvature bounded below II, J. Diff. Geom. 54 (2000) 13-35.
[8] J. Cheeger, T. H. Colding, On the structure of spaces with Ricci curvature bounded below III, *J. Diff. Geom.* 54 (2000) 37-74.

[9] S. Cohn-Vossen, Totalkrümmung und geodätische Linien auf einfach zusammenhängenden offenen vollständigen Flächenstücken, *Recueil Mathématique de Moscou* 43 (1936) 139-163.

[10] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of the Laplace operator*, Invent. Math. 87 (1987) 517-547.

[11] M. Gromov, J. Lafontaine, and P. Pansu, Structures métriques pour les variétés riemanniennes, Cédic, Fernand Nathan, Paris (1981).

[12] Menguy, X. *Noncollapsing examples with positive Ricci curvature and infinite topological type*, Geom. Funct. Anal. 10 (2000), 600–627.

[13] X. Menguy, Ph. D. thesis, Courant Institute, New York University (2000).

[14] M. Munn, Ph. D. thesis, City University of New York, Graduate Center (2008).

[15] P. Li, Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature, Ann. Math. 124, (1986) 1-21.

[16] G. Perelman, Construction of manifolds of positive Ricci curvature with big volume and large Betti numbers, *Comparison Geometry (Berkeley, CA, 1993-94)* 157-163.

[17] G. Perelman, Manifolds of positive Ricci curvature with almost maximal volume, *J. Amer. Math. Soc.* 7, (1994) 299-305.

[18] C. Sormani, Friedmann Cosmology and Almost Isotropy, *Geom. Geom. Func. Anal* 14 (2004) 853-912.

[19] G. Whitehead, Elements of homotopy theory, *Springer-Verlag*, New York, (1978).

[20] S. Zhu, A finiteness theorem for Ricci curvature in dimension three, *J. Diff. Geom.* 37 (1993) 711-727.
[21] S. Zhu, The comparison geometry of Ricci curvature. In *Comparison Geometry (Berkeley, CA, 1993-1994)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 221-262. Cambridge Univ. Press, Cambridge, 1997.

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