Noncommutative Quantum Scattering in a Central Field

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Abstract

In this paper the problem of noncommutative elastic scattering in a central field is considered. General formulas for the differential cross-section for two cases are obtained. For the case of high energy of an incident wave it is shown that the differential cross-section coincides with that on the commutative space. For the case in which noncommutativity yields only a small correction to the central potential it is shown that the noncommutativity leads to the redistribution of particles along the azimuthal angle, although the whole cross-section coincides with the commutative case.

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1 Introduction

In recent years considerable attention was paid to the investigation of noncommutative spaces. The reasons for the emergence of this interest were the predictions of String theory \cite{1} in the low-energy limit which, along with the Brane-world scenario \cite{2}, led to the fact that the space-time could be noncommutative. Later intensive investigation of the field theory on noncommutative spaces was prompted by M-theory \cite{3} and the matrix formulation of the quantum Hall effect \cite{4}. Let us mention the fact that noncommutative spaces may arise as the gravitation quantum effect and may serve as a possible way for the regularization of quantum field theories \cite{5}.

Noncommutative spaces are characterized by the fact that their coordinate operators satisfy the equation

\begin{equation}
[x^i, x^j] = i\hbar \Theta^{ij},
\end{equation}

where $\Theta^{ij}$ is the constant parameter of noncommutativity. The parameter $\Theta^{ij}$ is real, antisymmetric and has the dimension $\text{(length)}^2$. In order to determine the physical system on the noncommutative space (according to \cite{6}), the Lagrangian of the corresponding commutative system is taken and all the usual derivatives in it are replaced with the (Weyl-Moyal) star product

\begin{equation}
(f \star g)(x) = e^{\frac{i}{2} \hbar \Theta^{ij} \partial_i x^j} f(x) g(x) |_{x = x'},
\end{equation}

where $f$ and $g$ are infinitely differentiable functions.

The noncommutative deformation of the Standard model \cite{7} was suggested for the study of the phenomenological consequences of noncommutativity.\textsuperscript{1} For these purposes quantum mechanics on noncommutative spaces was also extensively studied. The following steps were carried out: oscillator models on various noncommutative spaces were constructed, hydrogen-like atoms on these spaces were thoroughly considered. Stark, Zeeman and other effects were examined on these spaces. See details in \cite{9} and references therein.

At the same time very little attention was payed to the study of the quantum mechanical scattering problem on noncommutative spaces. There is only one paper \cite{10} where scattering on the noncommutative plane is investigated. It is obvious that the scattering problem is a key one because it may connect noncommutative effects with the reality on the experimental level. The consideration of the scattering problem is focal also for another reason. As it was shown in \cite{11}, in most "attraction" problems noncommutativity does not play any role.

\textsuperscript{1}For the renormalization of the energy-momentum tensor in noncommutative field theories, see \cite{8}).
The purpose of the present work is to investigate the scattering in the central field on the noncommutative space. In Section 2 the two-body problems are briefly considered. In Section 3.1 the scattering problem in the arbitrary central field for the case of high energies (the Born approximation) is investigated. In Section 3.2 we study the case when the potential cannot be considered small and another approximation has to be applied. In the Conclusion the main results are summarized.

2 The Two Body Problem

Let us consider a system of two quantum particles with respective masses and charges \((m_1, e_1)\) and \((m_2, e_2)\) on a noncommutative space. As it was shown in [11], the noncommutativity of a particle is proved to differ from its antiparticle by the sign. Consequently the Moyal product for this case can be written as

\[
f(\tilde{r}_1, \tilde{r}_2) \star_{Z_1, Z_2} g(\tilde{r}_1, \tilde{r}_2) = e^{\frac{\hbar}{2} \Theta^{ij}(Z_1, \frac{\partial}{\partial \tilde{r}^1_i}, \frac{\partial}{\partial \tilde{r}^2_i}) + \frac{\hbar}{2} \Theta^{ij}(Z_2, \frac{\partial}{\partial \tilde{r}^1_i}, \frac{\partial}{\partial \tilde{r}^2_i})}
\]

where \(Z_k\) are charge numbers \((e_k = Z_k e, k = 1, 2)\). This means that the commutation relations for \(\tilde{x}^i_k\) and momentum operators \(\hat{\tilde{p}}^i_k\) take the form

\[
[\tilde{x}^i_k, \tilde{x}^j_l] = i\hbar \Theta^{ij} Z_k \delta_{kl}, \quad [\tilde{x}^i_k, \hat{\tilde{p}}^j_l] = i\hbar \delta^{ij} \delta_{kl}.
\]

The two body problem Hamiltonian can be written as

\[
\hat{H} = \frac{\hat{\tilde{p}}^2_1}{2m_1} + \frac{\hat{\tilde{p}}^2_2}{2m_2} + V(|\tilde{r}_1 - \tilde{r}_2|).
\]

For a further separation of variables let us pass to the ”center mass” system. For this purpose we introduce ”relative” coordinates

\[
\tilde{r} = \tilde{r}_1 - \tilde{r}_2
\]

and ”center of mass” coordinates

\[
\tilde{R} = \frac{m_1 \tilde{r}_1 + m_2 \tilde{r}_2}{m_1 + m_2}.
\]

The appropriate momenta have the form

\[
\hat{\tilde{p}} = \frac{m_2 \hat{\tilde{p}}_1 - m_1 \hat{\tilde{p}}_2}{m_1 + m_2}
\]

and

\[
\hat{\tilde{K}} = \hat{\tilde{p}}_1 + \hat{\tilde{p}}_2
\]

respectively. The corresponding Hamiltonian reads

\[
\hat{H} = \frac{\hat{\tilde{p}}^2}{2\mu} + \frac{\hat{\tilde{K}}^2}{2M} + V(|\tilde{r}|)
\]

and the commutation relations are

\[
[\tilde{x}^i, \tilde{x}^j] = i\hbar \Theta^{ij}(Z_1 + Z_2), \quad [\tilde{x}^i, \hat{\tilde{x}}^j] = i\hbar \Theta^{ij} \frac{m_1 Z_1 - m_2 Z_2}{m_1 + m_2},
\]

\[
[\tilde{X}^i, \tilde{X}^j] = i\hbar \Theta^{ij} \frac{m_1^2 Z_1 + m_2^2 Z_2}{(m_1 + m_2)^2}, \quad [\tilde{x}^i, \hat{\tilde{p}}_j] = [\tilde{X}^i, \hat{\tilde{K}}_j] = i\hbar \delta^{ij},
\]

where \(\mu\) and \(M\) are the ”reduced” and ”total” masses.
One can also pass to the variables in which the commutation relations have a canonical form. These new variables depend on the previous ones in following way:

\[
\mathbf{r} = \hat{\mathbf{r}} - \frac{1}{2}(Z_1 + Z_2)\Theta \times \hat{\mathbf{p}} - \frac{1}{2} \frac{m_1 Z_1 - m_2 Z_2}{m_1 + m_2} \Theta \times \hat{\mathbf{K}},
\]

\[
\mathbf{R} = \hat{\mathbf{R}} - \frac{1}{2} \frac{m_1 Z_1 - m_2 Z_2}{m_1 + m_2} \Theta \times \hat{\mathbf{p}} - \frac{1}{2} \frac{m_1^2 Z_1 + m_2^2 Z_2}{(m_1 + m_2)^2} \Theta \times \hat{\mathbf{K}},
\]

\[
\hat{p} = \hat{\mathbf{p}}, \quad \hat{\mathbf{K}} = \hat{\mathbf{K}},
\]

where \( \Theta_k = \varepsilon_{kij} \Theta^{ij} \). Using them one can write down the Hamiltonian as follows:

\[
\hat{H} = \frac{\hat{\mathbf{p}}^2}{2\mu} + \frac{\hat{\mathbf{K}}^2}{2M} + V(|\mathbf{r}| + \frac{1}{2}(Z_1 + Z_2)\Theta \times \hat{\mathbf{p}} + \frac{1}{2} \frac{m_1 Z_1 - m_2 Z_2}{m_1 + m_2} \Theta \times \hat{\mathbf{K}})|.
\]

As it is seen, the Hamiltonian does not depend on \( \mathbf{R} \) and consequently the momenta \( \hat{\mathbf{K}} \) are preserved. Shifting the origin of the coordinate system and neglecting the constant kinetic energy of the center of mass, one can bring the Hamiltonian to the form

\[
\hat{H} = \frac{\hat{\mathbf{p}}^2}{2\mu} + V(|\mathbf{r} + \frac{1}{2}(Z_1 + Z_2)\Theta \times \hat{\mathbf{p}}|).
\]

Taking into consideration the fact that \( \Theta \) is small and keeping in the Hamiltonian only the first order terms on \( \Theta \), we get

\[
\hat{H} = \frac{\hat{\mathbf{p}}^2}{2\mu} + V(\mathbf{r}) - \frac{Z_1 + Z_2}{2r} \frac{dV(\mathbf{r})}{dr} \Theta \cdot \hat{\mathbf{L}},
\]

where \( r = \sqrt{\mathbf{r}} \) and \( \hat{\mathbf{L}} = \mathbf{r} \times \hat{\mathbf{p}} \) is the angular momentum operator.

It has to be mentioned that in this case, besides the energy, we have two conserved quantities - i.e. the angular momentum magnitude and its projection in the \( \Theta \) direction. It can be easily seen by calculating the time derivative of the angular momentum operator:

\[
\frac{d\hat{\mathbf{L}}}{dt} = [\hat{\mathbf{L}}, \hat{H}] = -\frac{Z_1 + Z_2}{2r} \frac{dV(\mathbf{r})}{dr} \Theta \times \hat{\mathbf{L}}.
\]

### 3 Noncommutative Quantum Scattering

#### 3.1 Born Approximation

Let us consider elastic quantum scattering on noncommutative three dimensional space. Our purpose is to compute the differential cross-section. The Schrödinger equation according to the Hamiltonian has the form

\[
-\frac{\hbar^2}{2\mu} \Delta \Psi(\mathbf{r}) + U(\mathbf{r}) \Psi(\mathbf{r}) = E \Psi(\mathbf{r}),
\]

where we introduced the notation

\[
U(\mathbf{r}) = V(\mathbf{r}) - \frac{Z_1 + Z_2}{2r} \frac{dV(\mathbf{r})}{dr} \Theta \cdot \hat{\mathbf{L}}.
\]

The exact solution of equation, which describes scattering, can be obtained also from the following integral equation:

\[
\psi_+^a(\mathbf{r}) = e^{ik_\alpha \mathbf{r}} - \frac{\mu}{2\pi\hbar^2} \int \frac{\exp(\pm ik|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} U(\mathbf{r}') \psi_+^a(\mathbf{r}') d^3 \mathbf{r}',
\]

(3.3)
where \( {\mathbf{k}}_a \) is the wave vector of an incident plane wave and \( k \) is its magnitude. At large distances the solution of equation (3.3) corresponding to the scattering has the form

\[
\Psi_a^{(+)}(r) \approx \phi_a(r) + \frac{f_a^{(+)}(\theta, \varphi)}{r} e^{i k r},
\]

where the first term is an incoming plane wave and the second one is an outgoing spherical wave. Here

\[
f_a^{(+)}(\theta_b, \varphi_b) = - \frac{\mu}{2 \pi \hbar} \langle \phi_b | U | \Psi_a^{(+)} \rangle
\]

denotes a scattering amplitude and \( \phi_b \) is an outgoing plane wave. Differently from the case of the central field it depends also on the azimuthal angle. The differential cross-section \( d\sigma \) is expressed in terms of the scattering amplitude as follows:

\[
d\sigma = f_a^{(+)}(\theta_b, \varphi_b) \sin \theta_b \, d\theta \, d\varphi.
\]

The formula (3.5) is exact. In order to find the scattering amplitude we have to apply certain assumptions. In this Section we will assume that the energy \( E \) of the relative motion is large enough, i.e. \( V(r) \ll E \) (as it was mentioned the two-dimensional analog was investigated in [10]). Consequently, \( \Psi_a^{(+)} \) is very little different from the incoming plane wave \( \phi_a \) and in (3.5) we can substitute it by the latter one. This leads us to the first Born approximation

\[
f_a^{(+)}(\theta_b, \varphi_b) = - \frac{\mu}{2 \pi \hbar} \langle \phi_b | U | \phi_a \rangle.
\]

As it is seen from formulae (3.2) and (3.7) the scattering amplitude consists of two parts. The first one, which depends only on ”scattering” angle, is an amplitude in the central field on commutative space

\[
f_{\text{com}}(\theta) = - \frac{\mu}{2 \pi \hbar} \langle \phi_b | V(r) | \phi_a \rangle,
\]

whereas the second one is the addition due to noncommutativity

\[
f_{\text{noncom}}(\theta, \varphi) = \frac{(Z_1 + Z_2) \mu}{4 \pi \hbar^2} \langle \phi_b | \frac{dV(r)}{dr} \Theta \cdot \mathbf{L} | \phi_a \rangle.
\]

In order to simplify formulae (8) and (9) let us introduce spherical coordinates, directing the z axis along an incident plane wave. Applying the plane wave expansion in spherical harmonics presented below

\[
\phi = e^{i k r} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr) Y_{l,m}^* \begin{pmatrix} \frac{k}{r} \\ \frac{\mathbf{r}}{r} \end{pmatrix},
\]

where \( j_l(kr) \) is a Bessel spherical function, \( l \) and \( m \) are orbital and azimuthal quantum numbers respectively, integrating over the angles and considering the following equations:

\[
\hat{L}_z Y_{l,0} = 0, \quad \hat{L}_\pm Y_{l,0} = \hbar \sqrt{l(l+1)} Y_{l, \pm 1}
\]

and sums

\[
\sum_{l=0}^{\infty} (2l+1) j_l^2(kr) P_l(\cos \theta) = j_0(q r),
\]

\[
\sum_{l=1}^{\infty} (2l+1) j_l^2(kr) P_l^1(\cos \theta) = kr \cos \frac{\theta}{2} j_1(q r),
\]

where \( q = |{\mathbf{k}}_a - {\mathbf{k}}_b| = 2k \sin \frac{\theta}{2} \), \( P_l \) and \( P_l^1 \) are usual and adjoined Legendre polynomials, we obtain
\[ f_{\text{com}} = -\frac{2\mu}{\hbar^2} \int_0^\infty V(r)j_0(qr) r^2 dr, \quad (3.13) \]

\[ f_{\text{noncom}} = -i(\Theta \cdot \mathbf{n}) \frac{2\mu(Z_1 + Z_2)}{\hbar} k^2 \frac{\sin \theta}{q} \int_0^\infty \frac{dV(r)}{dr} j_1(qr) r^2 dr. \quad (3.14) \]

Here \( n \) is a unit vector which is normal to the scattering plane and is determined by the equation

\[ \mathbf{k}_a \times \mathbf{k}_b = n k^2 \sin \theta. \quad (3.15) \]

Its components can be expressed by the azimuthal angle in the following way:

\[ n = -\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j}, \quad (3.16) \]

where \( \mathbf{i} \) and \( \mathbf{j} \) are cartesian basis vectors. Using the equation \( x^2 j_0(x) = \frac{d}{dx}(x^2 j_1(x)) \) and integrating by parts, one can write

\[ f = \frac{2\mu}{\hbar^2 q} (1 - i(Z_1 + Z_2) \hbar k^2 \sin \theta(\Theta \cdot \mathbf{n})) \int_0^\infty \frac{dV(r)}{dr} j_1(qr) r^2 dr. \quad (3.17) \]

It can be easily seen from the equation above that the noncommutativity addition in the differential cross-section is of the second order in \( \Theta \). Taking into consideration that \( \Theta \) is small enough one can say that, in the Born approximation, we have no corrections due to noncommutativity.

### 3.2 Disordered-Wave Born Approximation

The Born approximation is used only when the potential is weak enough to give very rapid convergence. In this Section we investigate the case when the potential \( V(r) \) is not small, and for this we need an alternative approach. The disordered-wave Born approximation (DWBA) can serve as such, since our potential \( U(r) \) decomposes naturally into two parts: \( U(r) = V(r) + W(r) \), where \( W(r) = -\frac{Z_1 + Z_2}{2r} \frac{dV(r)}{dr} \). The first term is the primary potential and the second one is a small addition. It should be mentioned that this division is especially useful if the scattering wave function under the action of a primary part is obtained exactly. For example, as a primary potential one can take the exactly soluble Coulomb potential.

Let \( \chi^{(+)}_a \) be a scattering wave function of an unperturbed potential (all notations are in accordance with the notations of the previous Section), then the exact expression of a scattering amplitude has the following form:

\[ f = -\frac{\mu}{2\pi \hbar^2} \langle \phi_b | V(r) | \chi^{(+)}_a \rangle - \frac{\mu}{2\pi \hbar^2} \langle \chi^{(-)}_b | W(r) | \Psi^{(+)}_a \rangle, \quad (3.18) \]

where \( \Psi^{(+)}_a \) is a scattering wave function of a perturbed Hamiltonian. The first term is the scattering amplitude in the absence of a perturbing potential and the second one is a correction due to \( W(r) \). Supposing \( W(r) \) to be small enough, one can change \( \Psi^{(+)}_a \) for \( \chi^{(+)}_a \) in the (3.18). Then, for the above-mentioned correction we have

\[ \langle \chi^{(-)}_b | W(r) | \chi^{(+)}_a \rangle = \int \chi^{(-)}_b(r) W(r) \chi^{(+)}_a(r) d^3r, \quad (3.19) \]

with the first-order exactness by \( W(r) \). We can simplify this formula by passing to spherical coordinates as it was done in the previous Section. The only difference is that in the latter case we must use the expansion of \( \chi^{(+)}_a \) on spherical harmonics instead of (3.10). This expansion has the following form:

\[ \chi^{(\pm)}_k = \frac{4\pi}{k^l} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \ell \epsilon^{\pm \ell m} F_l(k; r) Y^{*}_{l,m} \left( \frac{k}{r} \right) Y_{l,m} \left( \frac{r}{r} \right), \quad (3.20) \]
where $F_l(k; r)$ is a regular solution (corresponding to scattering) of an unperturbed radial Schrödinger equation. The coefficient $\eta_l$ is a phase shift of the $l$-wave. Substituting (3.20) into (3.19) we get

$$\langle \chi_b^{(-)} | \frac{1}{2r} \frac{dV(r)}{dr} \Theta \cdot \hat{L} | \chi_a^{(+)} \rangle = i(\Theta \cdot n)\frac{2\pi \hbar}{k^2} \sum_{l=1}^{\infty} (2l + 1) P_l^1(\cos \theta) e^{2i\eta_l} \int_0^{\infty} \frac{1}{r} \frac{dV(r)}{dr} |F_l(k; r)|^2 dr. \quad (3.21)$$

As seen from the above formula, the noncommutative correction permits to consider the scattering of each partial wave separately. This correction is not present for the $s$-wave. As distinguished from the case of the Born approximation, in this case we have a first-order correction in $\theta$. The dependence of this correction on the azimuthal angle is the same for all partial waves. It leads to such a redistribution of scattering particles by the angle $\varphi$, that integrating over this angle results in the disappearance of this correction, i.e. to the first order in $\theta$ we will have the same scattering amplitude for the complete scattering cone, as in the case of the unperturbed potential.

4 Conclusion

In this paper the problem of elastic noncommutative scattering in an arbitrary central field was considered. Two cases were investigated: firstly, the case when the whole potential can be considered small and secondly, the one when the noncommutativity is considered as a small addition to the main potential. In the first case it was established that, to the first order in the noncommutativity parameter, there are no additions to the formula of the "usual" scattering in an arbitrary central field. In the second case we came to the following results. It is estimated that the scattering of each partial wave as in the usual case of central scattering (commutative) can be considered separately. The scattering is independent from the azimuthal quantum number and the dependence on the azimuthal angle is the same for all types of waves. The $s$-wave differential cross-section is like in the case when the noncommutativity is absent. Differently with respect to the case with low potential, there is a first-order addition to the differential cross-section. The characteristic feature of this correction is that the scattering in the full scattering cone is like in the case when noncommutativity is absent, although it leads to the redistribution of particles along the azimuthal angle.

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