Iterated Monodromy Groups

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Abstract

We associate a group $\text{IMG}(f)$ to every covering $f$ of a topological space $\mathcal{M}$ by its open subset. It is the quotient of the fundamental group $\pi_1(\mathcal{M})$ by the intersection of the kernels of its monodromy action for the iterates $f^n$. Every iterated monodromy group comes together with a naturally defined action on a rooted tree. We present an effective method to compute this action and show how the dynamics of $f$ is related to the group. In particular, the Julia set of $f$ can be reconstructed from $\text{IMG}(f)$ (from its action on the tree), if $f$ is expanding.

1 Introduction

The aim of this paper is to show a new connection between dynamical systems and algebra. A group, called *iterated monodromy group* is associated to every covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ of a topological space by its open subset. This group encodes the combinatorial information about the iterations of the map $f$. If the map $f$ is expanding (*hyperbolic*) then the iterated monodromy group (together with the associated *virtual endomorphism*) contains all the “essential” information about the dynamics of $f$: one can reconstruct from it the action of $f$ on its Julia set.

Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a $d$-fold covering map of an arcwise connected and locally arcwise connected topological space $\mathcal{M}$ by its arcwise connected open subset $\mathcal{M}_1$. By $f^n$ we denote the $n$th iteration of the mapping $f$. It is a covering of the space $\mathcal{M}$ by its open subset $\mathcal{M}_n$. Choose an arbitrary point $t \in \mathcal{M}$ and let $T_t$ be the formal disjoint union of the sets $f^{-n}(t)$ of preimages of $t$ under $f^n$. The set $T_t$ has a natural structure of a regular rooted tree with the root $t \in f^0(t)$ in which every vertex $z \in f^{-n}(t)$ is connected to the vertex $f(z) \in f^{-(n-1)}(t)$. The tree $T_t$ is called *preimage tree*.

The fundamental group $\pi_1(\mathcal{M}, t)$ naturally acts on each set $f^{-n}(t)$. It is easy to see that the obtained action of $\pi_1(\mathcal{M}, t)$ on $T_t$ is an action by automorphisms of the rooted tree. This action is called *iterated monodromy action* of

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\[ \pi_1(M) \]. It does not depend, up to a conjugacy of actions, on the choice of \( t \) (Proposition 3.2).

The iterated monodromy action is not faithful in general. Therefore, the following definition is introduced.

**Definition 1.1.** *Iterated monodromy group* \( \text{IMG}(f) \) of the covering \( f \) is the quotient of the fundamental group \( \pi_1(M, t) \) by the kernel of its iterated monodromy action on \( T_t \).

Iterated monodromy groups are discrete analogs of the following Galois groups, defined by R. Pink. Let \( f \in k[x] \) be a polynomial over a field \( k \). Denote by \( f^{on}(x) \) its \( n \)th iteration and define the polynomials \( F_n(x) = f^{on}(x) - t \) over the field \( k(t) \). Let \( \Omega_n \) be the splitting field of \( F_n \) over \( k(t) \) and let \( \Omega = \cup_{n \geq 1} \Omega_n \). We obtain the Galois group \( \text{IMG}(f) = \text{Aut}(\Omega/k(t)) \).

It is not hard to prove that if \( f \in \mathbb{C}[x] \) is a post-critically finite polynomial then the Galois group \( \text{IMG}(f) \) is the closure of the iterated monodromy group \( \text{IMG}(f) \) in the automorphism group of the rooted tree. Here \( \text{IMG}(f) \) is computed as the iterated monodromy group of the covering \( f : M_1 \rightarrow M \) for \( M = \mathbb{C} \setminus P, M_1 = \mathbb{C} \setminus f^{-1}(P) \), where \( P \) is the set of post-critical points of \( f \) (or any other finite set, for which \( f : M_1 \rightarrow M \) is a covering map).

We present in our paper a method to compute the action of the iterated monodromy group \( \text{IMG}(f) \) on the rooted tree.

Automorphisms of rooted trees are conveniently encoded using automata (see the survey [41]). Any regular tree is isomorphic to the tree \( X^* \) of finite words over an alphabet \( X \). We connect two words by an edge in the tree \( X^* \) if and only if they are of the form \( vx \) and \( vx' \), where \( v \in X^* \) and \( x \in X \). The root of the tree \( X^* \) is the empty word \( \emptyset \).

If \( g \) is an automorphism of the rooted tree \( X^* \) then for every \( x \in X \) there exists a uniquely defined automorphism \( g|_x \) such that

\[
(xv)^g = x^g v^g|_x
\]

for all \( v \in X^* \). This can be interpreted in terms of automata in the following way. The automorphism \( g \) is considered to be a state of an automaton, which when reading an input letter \( x \in X \) gives on output the letter \( x^g \) and then changes its state to \( g|_x \). Automata of this type are called sometimes transducers or sequential machines (see [9]).

Suppose now that we have a \( d \)-fold covering \( f : M_1 \rightarrow M \) of a topological space by its open subspace. Let \( t \in M \) be a basepoint, let \( X \) be an alphabet of cardinality \( d \). We choose some bijection \( \Lambda : X \rightarrow f^{-1}(X) \) and paths \( \ell_x \) in \( M \) connecting \( t \) to \( \Lambda(x) \).

The choice of the paths \( \ell_x \) defines an isomorphism of the rooted trees \( \Lambda : X^* \rightarrow T_t \), which can be used to encode the vertices of the tree \( T_t \) by words over the alphabet \( X \) (see Definition 3.2). We get in this way a standard action of \( \text{IMG}(f) \) on \( X^* \), conjugating the iterated monodromy action on \( T_t \) by \( \Lambda \) (i.e., identifying \( T_t \) with \( X^* \) by \( \Lambda \)).
The standard action is computed by the following recurrent formula (see Proposition 3.1):

\[(xv)^\gamma = y (v^\ell_x \gamma_x \ell_y^{-1}),\]

where \(\gamma_x\) is the \(f\)-preimage of the loop \(\gamma\), which starts at \(\Lambda(x)\) (and ends at \(\Lambda(y)\)).

This formula can be interpreted as a description of the automaton, whose action on \(X^*\) coincides with the standard action of the loop \(\gamma\).

It is therefore possible to apply the techniques developed for the study of groups generated by automata to the iterated monodromy groups. We review the main definitions and results about groups generated by automata in Subsection 3.3 and in Subsection 3.4 (where an algebraic description of such actions is introduced and studied). For more details on this topic, see the surveys [41, 37].

Theory of groups generated by automata is developing intensively in the last two decades (see the works [41, 10, 11, 37] and their bibliography). It was discovered, that many interesting groups can be easily defined and studied using their actions on trees.

The first example of a group of this sort was the Grigorchuk group, defined in [22]. It was constructed originally as a simply defined example of an infinite finitely generated torsion group (thus related to the General Burnside Problem). It was discovered later that it is a group of intermediate growth [20] (and thus answering on the Milnor’s question) and that it possesses many other interesting properties (see [10]). Later other interesting related examples of groups acting on rooted trees were constructed [33, 42].

One of the main properties of these examples is the fact that the restriction \(g|_x\) (see (1)) is asymptotically shorter than \(g\). Such groups are called contracting. The contraction property provides inductive proofs of most result about these groups. See for instance the original proof of the fact that the Grigorchuk group is periodic in [22]. We discuss the basic properties of contracting actions in Subsection 4.1.

It was also discovered that groups generated by automata have rich geometry. For example, in [43, 44] Schreier graphs of some of such groups were described and their spectra where computed. In particular, it became clear that the graphs of the action of a group on the levels of the tree may converge to some fractal space.

This observation was formalized later by the author in [26]. It was shown that if the group action is contracting, then a naturally defined limit space \(\mathcal{J}_G\) together with a continuous map \(s: \mathcal{J}_G \to \mathcal{J}_G\) is associated to it.

The space \(\mathcal{J}_G\) is defined as a quotient of the space \(X^{-\omega}\) of left-infinite sequences \(\ldots x_2 x_1\) over the alphabet \(X\) by the asymptotic equivalence relation. Two sequences \(\ldots x_2 x_1, \ldots y_2 y_1 \in X^{-\omega}\) are said to be asymptotically equivalent if there exists a sequence \(\{g_k\}_{k=1}^\infty\) taking a finite number of different values \(g_k \in G\) such that

\[(x_k \ldots x_1)^{g_k} = y_k \ldots y_1\]

for every \(k \geq 1\) (see Definition 4.3).
The asymptotic equivalence relation is described by a finite directed labeled graph (the *Moore diagram of the nucleus*): two sequences are equivalent if and only if they are read on a directed path of the graph (Proposition 4.2).

The shift $\sigma : \ldots x_2 x_1 \mapsto \ldots x_3 x_2$ preserves the asymptotic equivalence relation, hence it induces a continuous map $s : \mathcal{J}_G \to \mathcal{J}_G$ on the limit space $\mathcal{J}_G$. The obtained dynamical system $(\mathcal{J}_G, s)$ is called *limit dynamical system* of the contracting action (see Definition 4.4).

The main result of our paper is Theorem 4.6 showing that the limit dynamical system of the iterated monodromy group of an expanding self-covering $f : \mathcal{M}_1 \to \mathcal{M}$ is topologically conjugate to the dynamical system $(\mathcal{J}(f), f)$, where $\mathcal{J}(f)$ is the Julia set of $f$.

We illustrate in the last section the results of our paper on some examples. The first class of examples are expanding endomorphisms of Riemannian manifolds. They were studied before by M. Schub, J. Franks and M. Gromov. We show that a result of M. Schub and J. Franks, saying that an expanding endomorphism is uniquely determined by its action on the fundamental group, is a partial case of Theorem 4.6 (see Theorem 5.2 and Theorem 5.3). This is illustrated on some concrete examples like self-coverings of torus and Heisenberg group. Theorem 4.6 and the definition of the limit space provide an encoding of the manifolds by infinite sequences. These encodings are interesting numeration systems on nilpotent Lie groups (in particular on $\mathbb{R}^n$), which were studied by many authors (see [45, 34]).

Another very interesting class of examples of iterated monodromy groups comes from holomorphic dynamics. Every hyperbolic rational map (i.e., a map for which orbits of all critical points are converging to an attracting cycle) is expanding on a neighborhood of its Julia set and Theorem 4.6 can be applied.

Iterated monodromy groups appeared implicitly in the paper [23] of M. Lyubich and Y. Minsky, since they can be defined as the holonomy groups of the laminations, studied in [23].

The encoding of the Julia set by infinite sequences, given by Theorem 4.6 was also studied before. See, for example the papers of M. V. Yacobson [46, 34].

We compute several examples of iterated monodromy groups of post-critically finite rational functions. The “smooth” examples give “usual” groups: $\mathbb{Z}$ for $z^2$, infinite dihedral for $z^2 - 2$ and a $\mathbb{Z}/2\mathbb{Z}$-extension of $\mathbb{Z}^2$ (the group of the affine transformations $\pm x + a$, of $\mathbb{Z}^2$) for the Lattès examples.

On the other hand, the group $\text{IMG}(z^2 - 1)$ does not have a finite presentation by defining relations (though a simple recursive definition is known, see Theorem 5.4). It has exponential growth but no free non-abelian subgroups. Every its proper quotient is solvable.

The group $\text{IMG}(z^2 - 1)$ is the first example of an amenable group, which can not be constructed from groups of sub-exponential growth using the extensions and direct limits. Amenability of $\text{IMG}(z^2 - 1)$ was proved by L. Bartholdi and B. Virág [8] using self-similarity of the random walks on it.

It is interesting to mention that the group $\text{IMG}(z^2 - 1)$ was defined and studied for the first time by R. Grigorchuk and A. Žuk (see [17]) before iterated
monodromy groups where defined. They introduced this group as just an interesting example of a group generated by a three-state automaton. The fact that \( \text{IMG} \left( z^2 - 1 \right) \) cannot be obtained from the groups of sub-exponential growth is a result of [47].

Most results of this paper were announced in [37], where also some more examples of iterated monodromy groups were presented.

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2 Preliminary definitions

2.1 Rooted trees

A rooted tree \( T \) is a simplicial graph without cycles with a marked vertex \( v_\varnothing \) called the root. An isomorphism of two rooted trees is an isomorphism of the graphs, which preserves the roots.

Every rooted tree \( T \) can be defined by a sequence of sets and maps

\[
X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \cdots, \tag{2}
\]

where \( X_0 = \{v_\varnothing\} \) contains only the root. Here \( V = \cup_{n \geq 0} X_n \) is the set of vertices of \( T \) and every vertex \( v \in X_n \) is connected by an edge with the vertex \( f_n(v) \in X_{n-1} \).

The set \( X_n \) is called the \( n \)th level of the tree \( T \) and is uniquely defined as the set of the vertices which are on distance \( n \) from the root.

The rooted \( T \) tree is \( d \)-regular if every point \( v \in X_n \) has exactly \( d \) preimages under \( f_{n+1} \) for every \( n \).

Let \( X \) be a finite set (an alphabet). Denote by \( X^* \) the free monoid generated by \( X \), i.e., the set of all finite words of the form \( x_1 x_2 \ldots x_n \), where \( x_i \in X \), (including the empty word \( \varnothing \)). It has a natural structure of a tree, where every word \( v \in X^* \) is connected to the word \( vx \) for every \( x \in X \). In this way we get a rooted tree with the root in \( \varnothing \). It is the tree defined by the sequence

\[
X^0 \xleftarrow{f_1} X^1 \xleftarrow{f_2} X^2 \cdots,
\]

where \( f_n(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_{n-1} \) (and \( X^0 = \{\varnothing\} \) consists only of the empty word).

The \( n \)th level of the tree \( X^* \) is the set \( X^n \) of the words of length \( n \). We denote the length of a word \( v \in X^* \) by \( |v| \), so that \( v \in X^n \) if and only if \( |v| = n \).

The tree \( X^* \) is \( d \)-regular for \( d = |X| \) and every \( d \)-regular tree is isomorphic to \( X^* \).

It is easy to see that automorphisms of rooted trees preserve the levels and that the following simple lemma holds.
Lemma 2.1. A bijection \( g : \bigcup_{n \geq 0} X_n \longrightarrow \bigcup_{n \geq 0} X_n \) is an automorphism of the rooted tree defined by the inverse sequence (2), if and only if \((X_n)^g = X_n\) for every \(n \geq 0\) and
\[(f_n(v))^g = f_n(v^g)\]
for every \(v \in X_n\).

\[\square\]

2.2 Partial self-coverings

We will use the standard terminology and facts about the covering maps (see, for example [14]).

Definition 2.1. Let \( \mathcal{M} \) be an arcwise connected and locally arcwise connected topological space. A \( d \)-fold partial self-covering map on the space \( \mathcal{M} \) is a \( d \)-fold covering map \( f : \mathcal{M}_1 \longrightarrow \mathcal{M} \), where \( \mathcal{M}_1 \) is an open arcwise connected subset of \( \mathcal{M} \).

Recall that \( f : \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \) is a \( d \)-fold covering map if it is surjective and every point \( x \in \mathcal{M}_2 \) has a neighborhood \( \mathcal{U} \) such that the preimage \( f^{-1}(\mathcal{U}) \) is a disjoint union of \( d \) subsets \( \mathcal{U}_i \) for which the restriction \( f : \mathcal{U}_i \longrightarrow \mathcal{U} \) is a homeomorphism.

Examples. 1. Self-covering is a covering \( f : \mathcal{M} \longrightarrow \mathcal{M} \) of a space by itself. As a simple example, consider the double self-covering \( x \mapsto 2x \) of the circle \( \mathbb{R}/\mathbb{Z} \) (equivalently, the map \( z \mapsto z^2 \) on the unit circle \( \{ z \in \mathbb{C} : |z| = 1 \} \)).

2. Branched coverings. Let \( \hat{\mathcal{M}} \) be a topological space. A map \( f : \hat{\mathcal{M}} \longrightarrow \hat{\mathcal{M}} \) is a branched covering if there exists a set \( R \subset \hat{\mathcal{M}} \) of branching points such that \( f \) is a local homeomorphism in every point \( x \in \hat{\mathcal{M}} \setminus R \). Then the set \( P = \bigcup_{k=0}^\infty f^k(R) \) is called the postcritical set. If the set \( M = \hat{\mathcal{M}} \setminus \overline{P} \) is arcwise connected and locally arcwise connected, then \( f : \mathcal{M}_1 \longrightarrow \mathcal{M} \) is a partial self-covering of the set \( \mathcal{M} \), where \( \mathcal{M}_1 = f^{-1}(M) \). Here \( \overline{P} \) denotes the closure of the set \( P \).

For example, the famous theorem of Thurston considers postcritically finite branched coverings \( f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}} \) of the complex sphere, i.e., the branched coverings for which the set \( P \) is finite (see [18]).

3. Rational functions. In particular, a rational function \( f \in \mathbb{C}(z) \) defines a branched covering of the complex sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \). The set of branching points of a rational function is the set of its critical values, i.e., the values of the function \( f \) in the critical points of the function. If the postcritical set \( P \) is small enough (for instance, if it is finite or has a finite number of accumulation points), then the function \( f \) is a partial self-covering of the set \( \mathcal{M} = \hat{\mathbb{C}} \setminus \overline{P} \).

4. Polynomial-like maps. Let \( U \) and \( V \) be open disks in \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \). A holomorphic map \( f : U \longrightarrow V \) is said to be proper, if an \( f \)-preimage of every compact subset \( K \subset V \) is a compact subset of \( U \).

The following notion was introduced in [35].

Definition 2.2. A polynomial-like map \( f : U \longrightarrow V \) is a proper map between open disks such that the closure of \( U \) is a compact subset of \( V \).
Let $\overline{P}$ be the closure of the set of post-critical points of the polynomial-like map and suppose that the sets $V \setminus \overline{P}$ and $f^{-1}(V \setminus \overline{P})$ are arcwise connected. Then the polynomial-like map $f$ is a partial self-covering of the set $\mathcal{M} = V \setminus \overline{P}$.

3 The definition

3.1 Iterated monodromy groups

Let us fix some $d$-fold partial self-covering $f$ of an arcwise connected and locally arcwise connected space $\mathcal{M}$. We have the following classical

**Lemma 3.1.** For every path $\gamma$ in $\mathcal{M}$ and every $f$-preimage $z$ of the beginning of $\gamma$ there exists a unique path $\gamma'$ in $\mathcal{M}_1$ beginning at $z$ and such that $f(\gamma') = \gamma$.

**Notation.** If $\gamma$ is a path in $\mathcal{M}$ and a point $z \in \mathcal{M}$ is such that $f^n(z)$ is the beginning of $\gamma$, then we denote by $f^{-n}(\gamma)[z]$ the preimage of $\gamma$ under $f^n$, which starts at the point $z$.

Let $t \in \mathcal{M}$ be an arbitrary point. We get an inverse sequence

$$
\{t\} \overset{f}{\leftarrow} f^{-1}(t) \overset{f}{\leftarrow} f^{-2}(t) \overset{f}{\leftarrow} f^{-3}(t) \overset{f}{\leftarrow} \ldots
$$

defining a rooted tree $T_t$ called the *preimages tree* of the point $t$. The tree $T_t$ is $d$-regular, since the map $f$ is a $d$-fold covering.

Let now $\gamma$ be a loop in $\mathcal{M}$ based at $t$, i.e., a path starting and ending at $t$. For every vertex $z \in f^{-n}(t)$ of the $n$th level of the preimage tree $T_t$ denote by $z^\gamma$ the end of the path $f^{-n}(\gamma)[z]$.

Then we obviously have $f^n(z^\gamma) = t$, so the element $z^\gamma$ also belongs to the $n$th level of $T_t$.

**Proposition 3.2.** The map $z \mapsto z^\gamma$ is an automorphism of the preimage tree, which depends only on the homotopy class of $\gamma$ in $\mathcal{M}$. In this way we get an action of the fundamental group $\pi_1(\mathcal{M}, t)$ on the tree $T_t$. Up to a conjugacy, the action does not depend on the choice of the basepoint.

We say that an action of a group $G_1$ on a set $M_1$ is conjugate to an action of a group $G_2$ on a set $M_2$ if there exists an isomorphism $\phi : G_1 \rightarrow G_2$ and a bijection $l : M_1 \rightarrow M_2$ such that

$$
l(x^g) = l(x)^{\phi(g)}
$$

for every $x \in M_1$ and $g \in G$.

**Proof.** The fact that the map $z \mapsto z^\gamma$ defines for every $n \geq 1$ an action of the fundamental group $\pi(\mathcal{M}, t)$ on the set $f^{-n}(t)$ is classical (see, for example, § 7 of [14]).

Let us prove that the map $g : z \mapsto z^\gamma$ is an automorphism of the tree $T_t$, using Lemma 2.1. The map $g$ defines a permutation of every level $f^{-n}(t)$ of the
tree $T_i$. Let $z \in f^{-n}(t)$ be an arbitrary point of the $n$th level. The point $z^\gamma$ is the end of the path $\gamma = f^{-n}(\gamma)[z]$. Then $f(\gamma')$ is equal to $f^{-1}(\gamma)[f(z)]$, thus the end of $f(\gamma')$ is equal to $f(z)^\gamma$. But it is also obviously equal to $f(z^\gamma)$. Therefore, 

$$f(z)^\gamma = f(z^\gamma)$$

for every $z \in f^{-n}(t)$ and Lemma 2.1 shows that the map $z \mapsto z^\gamma$ is an automorphism of the tree $T_i$.

Let $t'$ be another basepoint. Choose a path $\ell$ starting at $t$ and ending at $t'$. Let us define a map $l : T_i \rightarrow T_{t'}$, which maps every point $z \in f^{-n}(t)$ to the end of the path $f^{-n}(\ell)[z]$. Same considerations as above show that $l$ is an isomorphism of the rooted trees. The path $\ell$ defines also an isomorphism $\phi : \pi(M, t) \rightarrow \pi(M, t')$ of the fundamental groups by the formula $\phi(\gamma) = \ell^{-1}\gamma\ell$.

Let $\gamma$ be an arbitrary loop at $t$ and let $z \in f^{-n}(t)$ be an arbitrary vertex of the $n$th level of the tree $T_i$. Then $z^\gamma$ is the end of the path $f^{-n}(\gamma)[z]$. The path $f^{-n}(\ell)[z^\gamma]$ begins at $z^\gamma$ and ends in $l(z)$. The path $f^{-n}(\ell)[z]$ begins at $z$ and ends in $l(z)$.

Consequently, $(f^{-n}(\ell)[z])^{-1} \cdot f^{-n}(\gamma)[z] \cdot f^{-n}(\ell)[z^\gamma]$ is a path, starting at $l(z)$, ending at $l(z^\gamma)$ and equal to $f^{-n}(\ell^{-1}\gamma\ell) [l(z)]$. This implies that $l(z^\gamma) = l(z)^{\phi(\gamma)}$. \hfill \Box

**Definition 3.1.** The action of the fundamental group $\pi_1(M, t)$ on the preimage tree $T_i$, described in Proposition 3.2, is called *iterated monodromy action* for the map $f$. The quotient of $\pi_1(M, t)$ by the kernel of this action is called the *iterated monodromy group (i.m.g.)* of the map $f$, denoted $\text{IMG}(f)$.

Proposition 3.2 implies that the group $\text{IMG}(f)$ and its action on the tree do not depend on the choice of the point $t$.

### 3.2 Standard actions of iterated monodromy groups on $X^*$

If we want to compute the action of the iterated monodromy group on $T_i$, then we have to find some convenient way to encode the vertices of the tree $T_i$ as finite words over an alphabet $X$ of $d$ letters. We show here a class of naturally defined encodings which use the paths in the set $\mathcal{M}$. With respect to these encodings the iterated monodromy action will be self-similar, which will make the methods, developed for self-similar groups of automata, applicable to the iterated monodromy groups. We will discuss self-similar actions in general later.

Let us consider an alphabet $X$ with $d$ letters together with a bijection $\Lambda : X \rightarrow f^{-1}(t)$. For every $x \in X$ we choose a path $\ell_x$ in $\mathcal{M}$, starting at $t$ and ending at $\Lambda(x)$.

**Definition 3.2.** The isomorphism $\Lambda : X^* \rightarrow T_i$ is defined putting $\Lambda(\emptyset) = t$, and then inductively putting $\Lambda(xv)$ to be equal to the end of the path 

$$f^{-(n-1)}(\ell_x)[\Lambda(v)],$$

where $v \in X^{n-1}$ and $x \in X$.
Proposition 3.4. The following relation

Note that \( f^{n-1}(\Lambda(xv)) \) is the end of the path \( \ell_x \), so that \( f^n(z) = t \) and \( \Lambda(xv) \) belongs to the \( n \)th level of the preimage tree.

Proposition 3.3. The constructed map \( \Lambda : X^* \longrightarrow T_i \) is an isomorphism of the rooted trees.

Proof. It follows from the construction that the map \( \Lambda \) preserves the levels of the trees and is surjective on them. Let us prove by induction on \( n \) that the equality

\[
f(\Lambda(vx)) = \Lambda(v)
\]

holds for all \( v \in X^* \) and \( x \in X \). This will imply that the map \( \Lambda \) preserves the vertex adjacency and thus is an isomorphism.

The equality is true for \( n = 0 \). Suppose that it holds for \( n = k \). Let \( v \in X^k \) and \( x, y \in X \) be arbitrary. We are going to prove that \( f(\Lambda(yvx)) = \Lambda(yv) \). The end of the path \( \gamma_1 = f^{-1(n-1)}(\ell_y) [\Lambda(v)] \) is, by definition, \( \Lambda(yv) \). The end of the path \( \gamma_2 = f^{-n}(\ell_y) [\Lambda(vx)] \) is \( \Lambda(yvx) \). By assumption, \( f(\Lambda(vx)) = \Lambda(v) \), so we get \( f(\Lambda(vx)) = \gamma_1 \), thus we get their endpoints: \( f(\Lambda(yvx)) = \Lambda(yv) \). □ □

Definition 3.3. The standard action of the group \( \text{IMG}(f) \) (or of \( \pi_1(\mathcal{M}, t) \)) on the tree \( X^* \) is the action obtained from the action on the preimage tree \( T_i \) conjugating it by the isomorphism \( \Lambda : X^* \longrightarrow T_i \), i.e., the action

\[
v^g = \Lambda^{-1}(\Lambda(v)^g).
\]

The standard actions can be computed using the following recurrent formulæ.

Proposition 3.4. Let \( L = \{\ell_x\} \) be a collection of paths defining a standard action of \( \pi_1(\mathcal{M}, t) \) on \( X^* \). Then we have for \( \gamma \in \pi_1(\mathcal{M}, t), x \in X, v \in X^* \) the following relation

\[
(xv)^\gamma = y \left( v^{\ell_x \gamma_x \ell_y^{-1}} \right),
\]

where \( \gamma_x = f^{-1}(\gamma) [\Lambda(x)] \) and \( y = x^\gamma \).

Proof. Note that the path \( \ell_x \gamma_x \ell_y^{-1} \) is obviously a loop based at \( t \).

Let \( |v| = n \) and \( \Lambda(yu) = (\Lambda(xv))^\gamma \), where \( y \in X \) and \( u \in X^n \). Denote

\[
\ell_x^v = f^{-n}(\ell_x) [\Lambda(v)], \quad \ell_y^u = f^{-n}(\ell_y) [\Lambda(u)],
\]

and let

\[
\gamma_{xv} = f^{-(n+1)}(\gamma) [\Lambda(xv)].
\]

Then the end of \( \ell_x^v \) is equal to \( xv \) and the end of \( \ell_y^u \) is equal to \( yu \). The end of \( \gamma_{xv} \) is also equal to \( yu \), thus we get a path \( \ell_x^v \gamma_{xv} \ell_y^u \)^{-1} from \( \Lambda(v) \) to \( \Lambda(u) \). Its image under \( f^n \) is the path \( \ell_x \gamma_x \ell_y^{-1} \), thus \( u = v^{\ell_x \gamma_x \ell_y^{-1}} \). □ □
3.3 The tree $X^*$ and automata

We recall here some basic facts about automorphisms of the rooted tree $X^*$, Mealy automata (transducers) and self-similar groups. More details can be found in [10, 41, 11, 16, 37].

Lemma 2.1 states that a map $g : X^* \rightarrow X^*$ is an automorphism if and only if it preserves the length of the words and for every $x_1x_2\ldots x_n \in X^n$ there exists $y_n \in X$ such that $(x_1x_2\ldots x_n)^g = (x_1x_2\ldots x_{n-1})^gy_n$. It follows that for every $v \in X^*$ and $u \in X^n$ there exists $w \in X^n$ such that $(vu)^g = v^gw$. Moreover, Lemma 2.1 implies that the map $u \mapsto w$ is again an automorphism of the rooted tree $X^*$. We denote this automorphism by $g|_v$ and call it restriction of $g$ at $v$.

The following properties of restrictions hold:

\[(vu)^g = v^g u^g v \quad (5)\]
\[g|_v = (g|_u)|_v \quad (6)\]
\[(g_1g_2)|_v = (g_1|_v)(g_2|_{v^g_1}) \quad (7)\]
\[(g^{-1})|_v = (g|_{v^g}^{-1})^{-1} \quad (8)\]

The set $x_1X^*$ of all the words starting with a fixed letter $x_1 \in X$ is a subtree of $X^*$ defined by the inverse sequence

\[
\{x_1\} \overset{f_2}{\leftarrow} x_1X_1 \overset{f_3}{\leftarrow} x_1X_2 \ldots
\]

Every automorphism $g$ of the rooted tree $X^*$ acts on every subtree $xX^*$ by the automorphism $g|_x$ and then permutes the subtrees $xX^*$ by the permutation induced by $g$ on the set $X^1 \subset X^*$. This leads to an interpretation of the automorphisms of rooted trees in terms of automata.

**Definition 3.4.** An automaton over the alphabet $X$ is a triple $\langle Q, \lambda, \pi \rangle$, where

1. $Q$ is a set (the set of the internal states);
2. $\lambda : Q \times X \rightarrow X$ is a map, called the output function;
3. $\pi : Q \times X \rightarrow Q$ is a map, called the transition function.

An automaton is finite if the set $Q$ is finite.

It is convenient to define automata by their Moore diagrams.

**Definition 3.5.** A Moore diagram of an automaton $A = \langle Q, \lambda, \pi \rangle$ is a labeled directed graph with the set of vertices $Q$ and the set of arrows $Q \times X$, where $(q, x)$ is an arrow starting at $q$, ending at $\pi(q, x)$, and labeled by the pair $(x, \lambda(q, x)) \in X \times X$.

On Figure an example of a Moore diagram is shown.

We interpret an automaton $A = \langle Q, \lambda, \pi \rangle$ as a machine, which being in a state $q \in Q$ and reading on the input tape a letter $x$ goes to the state $\pi(q, x)$ and
prints on the output tape the letter $\lambda(q, x)$. It processes the words in this way, so that the automaton $A$ with the initial state $q$ defines a map $A_q : X^* \rightarrow X^*$ by inductive formula

$$A_q(\emptyset) = \emptyset, \quad A_q(xv) = \lambda(q, x)A_{\pi(q, x)}(v),$$

where $x \in X$ and $v \in X^*$ are arbitrary.

It follows from Lemma 2.1 that if the map $A_q$ is bijective, then it is an automorphism of the tree $X^*$.

On the other hand it implies that if $g$ is an automorphism of the tree $X^*$, then it is defined by the automaton $A = (Q, \lambda, \pi)$ with the initial state $g = g|_{\emptyset}$, where $Q = \{g|_v : v \in X^*\}$ is the set of all possible restrictions of $g$ and the maps $\pi$ and $\lambda$ are defined by

$$\pi(g, x) = g|_x, \quad \lambda(g, x) = x^g.$$

An automorphism $g$ of the tree $X^*$ is said to be \textit{finite state} if it is defined by a finite automaton, i.e., if the set $\{g|_v : v \in X^*\}$ is finite. The set $\mathcal{F}A$ of all finite state automorphisms is a countable subgroup of the automorphism group $\mathcal{A}$ of the rooted tree $X^*$. The group $\mathcal{F}A$ is called the \textit{group of finite automata}.

An important class of automorphism groups of the tree $X^*$ is the class of \textit{self-similar}, or \textit{state-closed} groups.

\textbf{Definition 3.6.} An action of a group $G$ on the tree $X^*$ is said to be \textit{self-similar} if for every $g \in G$ and $x \in X$ there exists $h \in G$ and $y \in X$ such that

$$(xw)^g = y\left(w^h\right)$$

for every $w \in X^*$.

An automorphism group $G$ of the tree $X^*$ is self-similar if and only if for all $g \in G$ and $x \in X$ the restriction $g|_x$ belongs to $G$ (from this originates the often used term “state-closed group”).

Self-similar groups can be also defined as the groups, generated by automata, in the sense of the following definition (used for the first time in [31]).

\textbf{Definition 3.7.} Let $A = (Q, \lambda, \pi)$ be an automaton, such that the transformation $A_q$ is invertible for every $q \in G$. Then the automorphism group of the tree $X^*$ generated by the transformations $A_q$, $q \in Q$ is called the \textit{group, generated by the automaton} $A$ and is denoted $\langle A \rangle$.

Another interpretation of self-similar actions uses the following notion of a permutational wreath product.

\textbf{Definition 3.8.} Let $H$ be a group acting by permutations on a set $X$ and let $G$ be an arbitrary group. Then the \textit{permutational wreath product} $G \wr H$ is the semi-direct product $G^X \rtimes H$, where $H$ acts on the direct power $G^X$ by the respective permutations of the direct factors.
The elements of a permutational wreath product \( G \wr H \) are written in the form \( g \cdot h \), where \( h \in H \) is a permutation of \( X \) and \( g \in G^X \) is a function from \( X \) to \( G \). If we fix some indexing \( X = \{x_1, x_2, \ldots, x_d\} \) of the set \( X \), then the elements \( g \in G^X \) can be written as tuples \( g = (g_1, g_2, \ldots, g_d) \), where \( g_i = g(x_i) \).

For every self-similar automorphism group \( G \) of the tree \( X^* \) we get a naturally defined homomorphism \( \psi: G \to G \wr S(X) \), where \( S(X) \) is the symmetric permutation group of the alphabet \( X \). This homomorphism is defined by the formula \( \psi(g) = \tilde{g} \cdot \alpha_g \), where \( \alpha_g \in S(X) \) is the restriction of the automorphism \( g \) onto the set \( X = X^1 \subset X^* \), and \( \tilde{g} \in G^X \) is defined as \( \tilde{g}(x) = g|_x \), or, in other words

\[
\psi(g) = (g|_{x_1}, g|_{x_2}, \ldots, g|_{x_d}) \alpha_g.
\]

The homomorphism \( \psi \) is called the permutational recursion, associated to the self-similar action. Note that the permutational recursion is injective. In particular, we have an isomorphism \( A \cong A \wr S(X) \) for the automorphism group \( A \) of the tree \( X^* \). We will usually identify the elements \( g \in A \) with their images in \( A \wr S(X) \) under the permutational recursion \( \psi \).

In the other direction, if we have a homomorphism \( \psi: G \to G \wr S(X) \), then it defines a self-similar action of the group \( G \) on the tree \( X^* \) by the recurrent formula

\[
(x^g \cdot v)^g = x^{\alpha_g \cdot \tilde{g}(x)}, \quad \emptyset^g = \emptyset,
\]

where \( \alpha_g \in S(X) \) is a permutation and \( \tilde{g} \in G^X \) is a function \( X \to G \) such that \( \psi(g) = \tilde{g} \cdot \alpha_g \). Note, that this self-similar action is not faithful in general.

### 3.4 Virtual endomorphisms

Here we recall some facts about virtual endomorphisms of groups. For more details see the papers [40, 26, 18].

Every partial self-covering \( f \) of a topological space induces a virtual endomorphism \( \phi_f \) of the fundamental group of the space. The dynamics of the self-covering is very closely related to the dynamics of \( \phi_f \).

**Definition 3.9.** A virtual homomorphism \( \phi: G_1 \to G_2 \) is a homomorphism \( \phi: \text{Dom} \phi \to G_2 \), where \( \text{Dom} \phi \leq G_1 \) is a subgroup of finite index, called the domain of the virtual homomorphism. A virtual endomorphism of a group \( G \) is a virtual homomorphism \( \phi: G \to G \).

It is not hard to prove that a composition of two virtual homomorphisms is again a virtual endomorphism.

**Example 1.** Let us take a group \( G \) with a faithful self-similar action on the tree \( X^* \) (see Definition 3.6). Suppose that the action is transitive on the first level \( X^1 \). Then for any \( x_0 \in X \) we have the associated virtual endomorphism of the action, defined as

\[
\phi_{x_0}(g) = g|_{x_0}
\]
where Dom $\phi_{x_0}$ is equal to the stabilizer $G_{x_0}$ of the element $x_0$. The subgroup $G_{x_0}$ has index $|X|$ in $G$. The fact that the map $\phi_{x_0}$ is a virtual endomorphism follows from (7).

**Example 2.** Let $f : M_1 \to M$ be a $d$-fold partial self-covering map of the space $M$ and let $\ell_x$ be a path, connecting the basepoint $t$ with one of its $f$-preimages $x$. The set $G_1 \subset \pi_1(M, t)$ of all loops $\gamma$ such that the path $f^{-1}(\gamma)[x]$ is again a loop, is a subgroup of index $d$ in $\pi_1(M, t)$, which is isomorphic to the fundamental group $\pi_1(M_1, x)$. Let $\phi_f : \pi_1(M, t) \to \pi_1(M, t)$ be the virtual endomorphism, equal to the composition of the homomorphisms

$$\pi_1(M, t) > G_1 \rightarrow \pi_1(M_1, x) \rightarrow \pi_1(M, x) \rightarrow \pi_1(M, t),$$

where

(i) $G_1 \rightarrow \pi_1(M_1, x)$ is the isomorphism, which carries a loop to its $f$-preimage,

(ii) $\pi_1(M_1, x) \rightarrow \pi_1(M, x)$ is the homomorphism induced by the inclusion $M_1 \subseteq M$,

(iii) $\pi_1(M, x) \rightarrow \pi_1(M, t)$ is the isomorphism defined by the path $\ell_x$, i.e., the map $\gamma \mapsto \ell_x \gamma \ell_x^{-1}$.

We say that the virtual endomorphism $\phi_f$ of the fundamental group $\pi_1(M, t)$ is **induced** by the map $f$ and the path $\ell_x$.

**Definition 3.10.** Two virtual endomorphisms $\phi_1, \phi_2$ of a group $G$ are said to be **conjugate** if there exist $g, h \in G$ such that $\text{Dom } \phi_1 = g^{-1} \cdot \text{Dom } \phi_2 \cdot g$ and $\phi_2(x) = h^{-1} \phi_1(g^{-1} x g) h$ for every $x \in \text{Dom } \phi_2$.

In Example 1, if we take two different letters $x, y \in X$ and define the respective associated virtual endomorphisms $\phi_x$ and $\phi_y$, then they will be conjugate, since there will exist an element $h \in G$ such that $x^h = y$, and then

$$(yv)^{h^{-1}gh} = y \left(v^{h^{-1}gh} \cdot h_x v \cdot h_x^{-1}\right)$$

for every $g \in \text{Dom } \phi_x$ and every $v \in X^*$. Thus,

$$\phi_x(g) = h_x \phi_y(h^{-1}gh) h_x^{-1},$$

and the virtual endomorphisms $\phi_x$ and $\phi_y$ are conjugate.

For the case of Example 2 we have the following proposition.

**Proposition 3.5.** If the virtual endomorphisms $\phi_1$ and $\phi_2$ are induced by the map $f$ and the paths $\ell_1$ and $\ell_2$ respectively, then they are conjugate.
Proof. Let \( x_1 \) be the end of the path \( \ell_1 \) and \( x_2 \) be the end of the path \( \ell_2 \). Both points \( x_1 \) and \( x_2 \) belong to \( f^{-1}(t) \). There exists a path \( \rho \) starting at \( x_2 \) and ending at \( x_1 \). Its image \( h = f(\rho) \) is a loop at \( t \). Let \( \gamma \) be an arbitrary element of \( \text{Dom} \phi_2 \), i.e., such a loop at \( t \), that its preimage \( \gamma x_2 = f^{-1}(\gamma)[x_2] \) is a loop at \( x_2 \).

We have:

\[
\phi_2(\gamma) = \ell_2 \gamma x_2 \ell_2^{-1} \\
\phi_1(h^{-1} \gamma h) = \ell_1 \rho^{-1} \gamma x_2 \rho \ell_1^{-1}.
\]

Therefore,

\[
\phi_2(\gamma) = (\ell_2 \ell_1^{-1} \rho) \phi_2(\gamma) (\ell_1 \rho^{-1} \ell_2^{-1}) = g^{-1} \phi_1(\gamma) g,
\]

where \( g = \ell_1 \rho^{-1} \ell_2^{-1} \) is a loop at \( t \).

Let us show how to reconstruct the self-similar action from the associated virtual endomorphism.

Recall, that if \( H \) is a subgroup of a group \( G \), then a right coset transversal for \( H \) is a set \( T \subset G \) such that \( G \) is a disjoint union of the sets \( Hg, g \in T \).

**Definition 3.11.** Let \( \phi \) be a virtual endomorphism of a group \( G \), let \( T \) be a right coset transversal \( \{r_x\}_{x \in X} \) for the subgroup \( \text{Dom} \phi \) and let \( C \) be a sequence \( \{h_x\}_{x \in X} \) of elements of the group \( G \). An action defined by the triple \( (\phi, T, C) \) is the action of the group \( G \) by automorphisms of the rooted tree \( X^* \) defined recurrently by the formula

\[
(xw)^g = y u h_{x}^{-1} \phi(r_x g r_y^{-1}) h_y,
\]

where \( x \in X, w \in X^*, g \in G \) and the element \( y \in X \) is defined by the condition \( r_x g r_y^{-1} \in \text{Dom} \phi \).

An action defined by the pair \( (\phi, T) \) is the action, defined by the triple \( (\phi, T, C_0) \) where \( C_0 = \{1, 1, \ldots 1\} \).

Note, that if \( \phi \) is onto, then the action, defined by the triple \( (\phi, T, C) \) is defined by the pair \( (\phi, T') \), where \( T' = \{r'_x = g_x^{-1} r_x\}_{x \in X} \), for \( g_x \) such that \( \phi(g_x) = h_x \).

**Proposition 3.6.** Suppose that \( \phi \) is a virtual endomorphism of a group \( G \), \( T = \{r_x\}_{x \in X} \) is a right coset transversal for the subgroup \( \text{Dom} \phi \) and \( C = \{h_x\}_{x \in X} \) is a sequence of elements of \( G \). Then \( \phi \) gives a well defined action of the group \( G \) on \( X^* \).

**Proof.** Formula \( (14) \) gives a well defined action of \( G \) on the first level \( X^1 \) of the tree \( X^* \), conjugate to the natural action of \( G \) on the right cosets \( \text{Dom} \phi \cdot h \). Let \( g_1, g_2 \in G \) be two arbitrary elements. Suppose that \( r_x g_1 r_y^{-1} \) and \( r_y g_2 r_z^{-1} \) belong to \( \text{Dom} \phi \). Then \( r_x g_1 g_2 r_z^{-1} \) also belongs to \( \text{Dom} \phi \) and

\[
h_x^{-1} \phi(r_x g_1 r_y^{-1}) h_z = h_x^{-1} \phi(r_x g_1 r_y^{-1}) h_y \cdot h_y^{-1} \phi(r_y g_2 r_z^{-1}) h_z.
\]

It follows now by induction on \( n \) that formula \( (14) \) gives a well defined action of the group \( G \) on the \( n \)-th level \( X^n \) of the tree \( X^* \).
Proposition 3.7. Every faithful self-similar action is defined by the associated virtual endomorphism $\phi_{x_0}$, a coset transversal $T = \{r_x\}_{x \in X}$ such that $(x_0)r_x = x$ and the sequence $C = \{h_x = r_x|x_0\}_{x \in X}$.

Proof. Take any $x \in X$, $g \in G$ and suppose that $y \in X$ and $h \in G$ are such that $(xw)y = ywh$ for all $w \in X^*$. Such $y$ and $h$ are uniquely defined, namely $y = g(x)$ and $h = g|_x$. Then:

$$x_0^{r_xgr_y^{-1}} = x_0$$

and thus $r_xgr_y^{-1} \in \text{Dom} \phi_{x_0}$. We have

$$\phi_{x_0}(r_xgr_y^{-1}) = (r_xgr_y^{-1})|_{x_0}.$$  

But then

$$(xw)y = (xw)^{-1}r_xgr_y^{-1}r_y = y \left( w^{r_x^{-1}}\cdot \phi_{x_0}(r_xgr_y^{-1}) \cdot h_y \right).$$

We have used $\Box$ and $\Box$.

Thus we see that $\Box$ is true for the triple $(\phi_{x_0}, T, C)$ and consequently, the action of $G$ is defined by it. $\Box$ $\Box$

If the virtual endomorphism $\phi$ is induced by a partial self-covering, then the set of actions defined by the triples $(\phi, T, C)$ coincides with the set of standard actions, as Propositions 3.8 and 3.9 show.

Proposition 3.8. Let $L = \{\ell_x\}_{x \in X}$ and $\Lambda : X \rightarrow f^{-1}(t)$ be a collection of paths and a bijection defining a standard action. Take the virtual endomorphism $\phi_f$, induced by $f$ and by a path $\ell$, connecting $t$ with $z_0 \in f^{-1}(t)$. For every $x \in X$, choose a path $\rho_x$ in $M_1$, starting at $z_0$ and ending at $\Lambda(x)$. Denote

$$r_x = f(\rho_x), \quad h_x = \ell \rho_x \ell^{-1}.$$  

Then the standard action is defined by the triple $(\phi_f, \{r_x\}_{x \in X}, \{h_x\}_{x \in X})$.

Proof. The set $\{r_x\}_{x \in X}$ is a right coset transversal for the domain of $\phi_f$, since $z_0^{r_x} = \Lambda(x)$ for every $x$. Let $\gamma$ be an arbitrary loop at $t$. The path $\gamma_x = f^{-1}(\gamma) [\Lambda(x)]$ ends in $\Lambda(y)$ for $y = x\gamma$. The element $r_xgr_y^{-1}$ belongs to $\text{Dom} \phi_f$, since it is the loop $f(\rho_x \gamma_x \rho_y)$. The element $\phi_f(r_xgr_y^{-1})$ is then the loop $\ell \rho_x \gamma_x \rho_y \ell^{-1}$. Consequently

$$h_x^{-1} \phi_f(r_xgr_y^{-1})h_y = (\ell \rho_x \ell^{-1})^{-1} \ell \rho_x \gamma_x \rho_y \ell^{-1} (\ell \rho_x \ell^{-1}) = \ell \gamma_x \ell_y^{-1},$$

and Proposition 3.4 ends the proof. $\Box$ $\Box$

Proposition 3.9. Let $\phi$ be the virtual endomorphism of $\pi(M, t)$, defined by the partial self-covering $f$. Then for every right coset transversal $T = \{r_x\}_{x \in X}$ and sequence $C = \{h_x\}_{x \in X}$ there exists a collection of paths $L = \{\ell_x\}_{x \in X}$ and a bijection $\Lambda : X \rightarrow f^{-1}(t)$ such that the respective standard action is the action defined by the triple $(\phi, T, C)$.  

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Proof. Let \( \ell \) be the path from \( t \) to \( z_0 \in f^{-1}(t) \), inducing together with \( f \) the endomorphism \( \phi \). For every \( x \in X \) put \( \rho_x = f^{-1}(r_x)|z_0] \). Define \( \Lambda(x) \) to be the end of \( \rho_x \). It follows from the fact that \( r_x \) is a right coset transversal of \( \text{Dom} \phi \), that the defined map \( \Lambda : X \rightarrow f^{-1}(t) \) is a bijection.

Take \( \ell x = h^{-1}_x \rho_x \). Then \( h_x = \ell \rho_x \ell^{-1}_x \), \( r_x = f(\rho_x) \) and Proposition 3.8 shows that the standard action defined by the collection \( L = \{ \ell_x \} \in X \) and the bijection \( \Lambda : X \rightarrow f^{-1}(t) \) coincides with the action defined by the triple \((\phi,T,C)\).

Finally, let us mention a description of the kernel of a self-similar action (see [40] and [49]).

**Proposition 3.10.** The kernel of the action defined by a triple \((\phi,T,C)\) is equal to 
\[ \mathcal{C}(\phi) = \bigcap_{n \geq 1} \bigcap_{g \in G} g^{-1} \cdot \text{Dom} \phi^n \cdot g. \]

In particular, the iterated monodromy group \( \text{IMG}(f) \) is the quotient of the fundamental group \( \pi_1(M) \) by \( \mathcal{C}(\phi_f) \), where \( \phi_f \) is the virtual endomorphism induced by the self-covering.

Proof. Let \( N \) be the kernel of the action defined by the triple. If \( g \in N \), then for every \( x \in X \) we have \( r_x gr^{-1}_x \in \text{Dom} \phi \) and \( h_x^{-1} \phi(r_x gr^{-1}_x) h_x \in N \). In particular, if we take \( x = x_0 \) such that \( r_{x_0} \in \text{Dom} \phi \), then we get that \( g \in \text{Dom} \phi \), since \( r_{x_0} gr^{-1}_{x_0} \in \text{Dom} \phi \). Besides, we get that
\[ h_{x_0}^{-1} \phi(r_{x_0} gr^{-1}_{x_0}) h_{x_0} \in N, \]
hence
\[ \phi(g) = (\phi(r_{x_0})^{-1} h_{x_0}) \cdot \phi(g) \cdot (\phi(r_{x_0})^{-1} h_{x_0})^{-1} \in N. \]

We have proved that \( g \in N \) implies \( g \in \text{Dom} \phi \) and \( \phi(g) \in N \). From this, by induction, we get that \( \phi^n(g) \in N \) and therefore, \( \mathcal{C}(\phi) \geq N \).

On the other hand, if \( g \in \mathcal{C}(\phi) \), then \( r_x gr^{-1}_x \in \text{Dom} \phi \) for every \( x \in X \) and
\[ h_x^{-1} \phi(r_x gr^{-1}_x) h_x \in \mathcal{C}(\phi) \]. It follows by induction on \( n \) that the action of \( g \) on \( X^n \) is trivial. Hence \( \mathcal{C}(\phi) \leq N. \)

4 Contracting groups and expanding maps

4.1 Contracting self-similar actions and their limit spaces

We recall here some definitions and results of the paper [20].

**Definition 4.1.** A self-similar action of a group \( G \) on the set \( X^* \) is said to be **contracting** if there exists a finite subset \( \mathcal{N} \subset G \) such that for every \( g \in G \) there exists \( n \in \mathbb{N} \) such that for every \( v \in X^* \), \(|v| > n \) the restriction \( g|_v \) belongs to \( \mathcal{N} \).
It is proved in [26], that the property of an action to be contracting depends only on the associated virtual endomorphism \( \phi \).

The minimal set \( \mathcal{N} \) satisfying the conditions of Definition 4.1 is called the nucleus of the action. It follows from definition that if \( h \in \mathcal{N} \) and \( x \in X \) then \( h|x \) belongs to \( \mathcal{N} \), so we consider \( \mathcal{N} \) as an automaton.

Contraction of actions of finitely generated groups can be defined using the contraction of the length of the group elements under the action of the virtual endomorphism.

**Definition 4.2.** Let \( \phi \) be a virtual endomorphism of a finitely generated group \( G \). Denote by \( |g| \) the length of an element \( g \in G \) with respect to a fixed finite generating set \( S = S^{-1} \), i.e., the minimal length of a representation of \( g \) as a product of the elements of \( S \).

Then the number
\[
\rho = \lim_{n \to \infty} \sqrt{n \limsup_{|g| \to \infty} \frac{\phi^n(g)}{|g|}}
\]
is called the contraction coefficient of the virtual endomorphism \( \phi \).

The following proposition is proved in [26].

**Proposition 4.1.** The contraction coefficient of a virtual endomorphism of a finitely generated group is finite and does not depend on the choice of the generating set.

A level-transitive self-similar action of a finitely generated group is contracting if and only if the contraction coefficient of the associated virtual endomorphism is less than one.

Denote by \( X^{-\omega} \) the set of all infinite to the left sequences of the form \( \ldots x_2 x_1 \) with the topology of the direct product of discrete sets \( X^{-\omega} = \cdots \times X \times X \).

**Definition 4.3.** Consider a self-similar action of a group \( G \) on the set \( X^* \). Two sequences \( \ldots x_2 x_1 \) and \( \ldots y_2 y_1 \) are said to be asymptotically equivalent with respect to the action of \( G \) if there exists a bounded sequence \( \{g_k\}_{k \geq 1} \) of elements of the group \( G \), such that
\[
(x_k x_{k-1} \ldots x_1)^{g_k} = y_k y_{k-1} \ldots y_1
\]
for all \( k \geq 1 \).

Here a sequence \( \{g_k\}_{k \geq 1} \) is called bounded if it takes only a finite number of different values.

In other words, two sequences \( \ldots x_2 x_1 \) and \( \ldots y_2 y_1 \) are asymptotically equivalent if and only if the words \( x_k \ldots x_1 \) and \( y_k \ldots y_1 \) stay on a uniformly bounded distance from each other with respect to the action of \( G \).

It is easy to see that the asymptotic equivalence is an equivalence relation.
Proposition 4.2. Two sequences \(x_2 x_1\) and \(y_2 y_1\) are asymptotically equivalent with respect to a contracting action if and only if there exists a directed path \(e e_1\) in the Moore diagram of the nucleus, for which the arrow \(e_i\) is labeled by \((x_i, y_i)\).

Proof. Let \(\{g_k\}_{k \geq 1}\) be a bounded sequence of group elements such that

\[(x_k \ldots x_1)^{g_k} = y_k \ldots y_1.\]

There exists a number \(n_0\) such that \(g_k|x_k \ldots x_{k-n_0}\) belongs to the nucleus for all \(n \geq n_0\). Let \(A_k\) be the set of all elements of the nucleus of the form \(g_k+n|x_{k+n} \ldots x_{k+1}\).

It follows from the definitions that for every \(a_k \in A_k\) we have

\[(x_k \ldots x_1)^{a_k} = y_k \ldots y_1\]

and \(a_k|x_k \in A_{k-1}\).

All the sets \(A_k\) are finite and non-empty. Thus, by a standard argument, there exists a sequence \(a_k \in A_k\) such that \(a_k|x_k = a_{k-1}\) and \(a_k(x_k) = y_k\). This sequence is the necessary path in the Moore diagram.

Definition 4.4. The quotient of the topological space \(X^{-\omega}\) by the asymptotic equivalence relation is called the limit space of the action, denoted \(\mathcal{J}_G\). The limit dynamical system is the dynamical system \((\mathcal{J}_G, s)\), where the map \(s : \mathcal{J}_G \rightarrow \mathcal{J}_G\) is induced by the shift

\[\sigma : \ldots x_2 x_1 \mapsto \ldots x_3 x_2\]

on \(X^{-\omega}\).

The asymptotic equivalence is a shift-invariant, so that the map \(s\) are well defined.

It is proved in [26] that the limit space \(\mathcal{J}_G\) of a contracting action is metrizable and finite dimensional. If the group \(G\) is finitely generated and level-transitive, then the limit space is connected.

The following is also proved in [26].

Proposition 4.3. The dynamical system \((\mathcal{J}_G, s)\) is uniquely determined, up to topological conjugacy, by the group \(G\) and the conjugacy class of the virtual endomorphism \(\phi\).

We have the following criterion for a metric space to be homeomorphic to the limit space \(\mathcal{J}_G\).

Another aspect of the limit space is that it can be represented as a limit of the graphs of the action of \(G\) on the levels \(X^n\) of the tree \(X^*\). This can be formalized (see [26]), but we will use this fact here only as an illustration (see Figure 6).
4.2 Iterated monodromy groups of expanding maps

The space $\mathcal{M}$ in this subsection is a differentiable manifold and the partial self-covering $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ is smooth. If $\mu$ is a Riemannian metric on a manifold, then by $d_\mu(x, y)$ we denote the distance between the points $x$ and $y$ (i.e., the greatest lower bound of the lengths of the piecewise smooth paths connecting them).

**Definition 4.5.** A map $f : \mathcal{M}_1 \rightarrow \mathcal{M}$, where $\mathcal{M}_1$ is an open subset of $\mathcal{M}$, is expanding if there exist a Riemannian metric $\mu$ on $\mathcal{M}$ and numbers $c > 0, k > 1$ such that $\|Df^n v\|_\mu \geq c \cdot k^n \|v\|_\mu$ for every $n \in \mathbb{N}$ and every tangent vector $v \in T_0 f^{-n}(\mathcal{M})$.

If the map $f$ is expanding and surjective, then it is a partial self-covering of the space $\mathcal{M}$ and every piecewise smooth path of length $l$ is mapped by $f^n$ onto a smooth path of length $\geq c \cdot k^n l$.

**Definition 4.6.** The Julia set of an expanding map $f$, denoted $\mathcal{J}(f)$, is the set of the accumulation points of the set $\bigcup_{n=0}^\infty f^{-n}(z_0)$, where $z_0 \in \mathcal{M}$ is arbitrary.

If $f$ is an expanding map, then we say that the set $\mathcal{M}$ is complete on the Julia set if every Cauchy subsequence of the set $\bigcup_{n=0}^\infty f^{-n}(z_0)$ converges in $\mathcal{M}$.

**Lemma 4.4.** If $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ is an expanding map, then the Julia set $\mathcal{J}(f)$ does not depend on the choice of the initial point $z_0$ and $f(\mathcal{J}(f)) = \mathcal{J}(f)$, $f^{-1}(\mathcal{J}(f)) = \mathcal{J}(f)$.

**Proof.** Let $z_1$ be another point in $\mathcal{M}$ and let $\gamma$ be a piecewise smooth path, connecting $z_0$ with $z_1$. Then for every $x \in f^{-n}(z_0)$ the end $y$ of the path $\gamma' = f^{-n}(\gamma)[x]$ belongs to $f^{-n}(z_1)$. The length of $\gamma'$ is not greater than $c^{-1}k^{-n}(\text{length}(\gamma))$. Therefore $d_\mu(x, y) \leq c^{-1}k^{-n}d_\mu(z_0, z_1)$, what implies that the set of accumulation points of $\bigcup_{n=0}^\infty f^{-n}(z_1)$ is equal to the set of the accumulation points of $\bigcup_{n=0}^\infty f^{-n}(z_0)$, thus the Julia set does not depend on the choice of $z_0$.

The set $\bigcup_{n=0}^\infty f^{-n}(z_0)$ is mapped by $f$ onto the set $\bigcup_{n=0}^\infty f^{-n}(f(z_0))$, so that $f(\bigcup_{n=0}^\infty f^{-n}(z_0)) = \bigcup_{n=0}^\infty f^{-n}(z_0) \cup \{f(z_0)\}$, thus the Julia set is $f$-invariant.

We have also $f^{-1}(\bigcup_{n=0}^\infty f^{-n}(z_0)) \cup \{z_0\} = \bigcup_{n=0}^\infty f^{-n}(z_0)$, so that $f^{-1}(\mathcal{J}(f)) = \mathcal{J}(f)$. \hfill $\Box$

**Definition 4.7.** We say that the fundamental group of a Riemannian manifold $\mathcal{U}$ has finite balls if for every two points $x, y \in \mathcal{U}$ and every $R > 0$ there exists only a finite number of homotopy classes of paths of length $\leq R$, connecting $x$ with $y$ in $\mathcal{U}$.

The following lemma can be used to prove that the fundamental group of a manifold has finite balls.

**Lemma 4.5.** Let $\mathcal{U}$ be an arcwise connected Riemannian manifold and let $\hat{\mathcal{U}}$ be its completion. If the inclusion $\mathcal{U} \subset \hat{\mathcal{U}}$ induces an isomorphism of the fundamental groups $\pi_1(\mathcal{U})$ and $\pi_1(\hat{\mathcal{U}})$, then the fundamental group of $\mathcal{U}$ has finite balls.

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Proof. The completion \( \hat{\mathcal{U}} \) is a length space, i.e., the distance between two its points is equal to the infimum of the lengths of rectifiable paths, connecting them. Let \( \hat{\mathcal{U}} \) be the universal covering of \( \mathcal{U} \). It is also a length space and its completion is the universal covering \( \tilde{\mathcal{U}} \) of \( \hat{\mathcal{U}} \). Then by Hopf-Rinow Theorem (see \([36]\) and \([24]\) p. 35), every bounded closed subset of \( \tilde{\mathcal{U}} \) is compact. Let \( x_0 \) be a preimage of the point \( x \in \mathcal{U} \) in \( \tilde{\mathcal{U}} \). Every path of length not greater than \( R \) connecting \( x \) with \( y \in \mathcal{U} \) can be lifted to a path of the same length, connecting \( x_0 \) with a preimage \( y_i \) of \( y \). The homotopy class of the path is uniquely defined by the end \( y_i \) of the preimage. The set of all possible values of \( y_i \) is bounded and closed. Thus it is compact, i.e., finite.

\[ \square \]

Theorem 4.6. Let \( f : \mathcal{M}_1 \rightarrow \mathcal{M} \) be an expanding partial self-covering map on \( \mathcal{M} \). Suppose that \( \mathcal{M} \) is complete on the Julia set \( \mathcal{J}_\mathcal{M}(f) \) and that the Julia set has an open arcwise connected neighborhood whose fundamental group has finite balls. Then every standard action of \( G = \text{IMG}(f) \) on \( X^* \) is contracting and the limit dynamical system \( (\mathcal{J}_G, s) \) is topologically conjugated to the dynamical system \( (\mathcal{J}(f), f) \).

Proof. Let \( L = \{ \ell_x \}_{x \in \mathcal{X}} \) be a set of paths defining a standard action of \( \text{IMG}(f) \) on \( X^* \). We may assume that the paths \( \ell_x \) are piecewise smooth. Let \( \ell \) be the maximal length of the paths from \( L \).

The Julia set \( \mathcal{J}(f) \) is equal to the set of the accumulation points of

\[ \cup_{k=0}^{\infty} f^{-k}(t) = T_1 = \Lambda(X^*). \]

Let us define for all \( v, u \in X^* \) a piecewise smooth path \( \ell(v; uv) \) starting at \( \Lambda(v) \) and ending at \( \Lambda(uv) \) by the conditions:

1. \( \ell(v; v) \) is the trivial path at \( \Lambda(v) \);
2. \( \ell(v; xv) = f^{-|v|}(\ell_x)[\Lambda(v)] \);
3. \( \ell(w; uv) = \ell(w; vw)\ell(vw; uvw) \).

It easily follows from the definition of \( \Lambda \) that the paths \( \ell(v; uv) \) are well defined. We have also that the length of the path \( \ell(v; uv) \) is not greater than \( c^{-1}k^{-|v|} \cdot (l + k^{-1}l + k^{-2}l + \ldots + k^{-|v|+1}l) \).

Let \( \ldots x_2x_1 \in X^{-\omega} \). Each path from the sequence of the paths

\[ \{ \ell(x_n \ldots x_2x_1; x_{n} \ldots x_{n+1}x_{n} \ldots x_2x_1) \}_{m \geq n}, \]

is a continuation of the previous one. In the limit we get a path denoted \( \ell(x_n \ldots x_2x_1; \ldots x_2x_1) \) of length not greater than \( c^{-1}(k^{-n} + k^{-n-1} + \ldots) = \frac{1}{1-k} \).

Every \( \ldots x_2x_1 \in X^{-\omega} \) defines a sequence \( z_n = \Lambda(x_n x_{n-1} \ldots x_1), n = 0, 1, \ldots \) of points of the preimage tree \( T_1 \), i.e., a sequence of elements of the set \( \cup_{n=0}^{\infty} f^{-n}(t) \).

\[ \square \]
The length of the path $\ell (x_{n-1} \ldots x_1; x_n x_{n-1} \ldots x_1)$, connecting $z_{n-1}$ with $z_n$ is not greater than $c^{-1}k^{-n+1}l$. This implies that the sequence $z_n$ is a Cauchy sequence in the metric space $(M, d_\mu)$. Let us denote its limit by $\Lambda(x_{n-1} \ldots x_2 x_1)$. Then the path $\ell (x_{n-1} \ldots x_2 x_1; x_2 x_1)$ starts at $\Lambda(x_n \ldots x_1)$ and ends at $\Lambda(\ldots x_2 x_1)$.

We have the following obvious properties of the map $\Lambda : X^\omega \rightarrow J(f)$:

$$d_\mu(\Lambda(v), \Lambda(x_2 x_1)) \leq c^{-1} \left( \frac{1}{k} \right) \left( \frac{1}{1 - k^{-1}} \right) \sum_{i=0}^{\infty} k^{-|v|+i} = \frac{c^{-1}l}{1 - k^{-1}} \quad (10)$$

$$d_\mu(\Lambda(x_2 x_1), \Lambda(y_2 y_1)) \leq k^{-\varphi} \frac{2lc^{-1}}{1 - k^{-1}}, \quad (11)$$

for all $v \in X^\omega$ and $y \in X^\omega$.

It follows from (11) that the map $\Lambda : X^\omega \rightarrow J(f)$ is continuous. Inequality (10) implies that $\Lambda$ is onto. Consequently, $J(f)$ is compact as an image of the compact space $X^\omega$.

Since $f(\Lambda(x_n x_{n-1} \ldots x_1)) = \Lambda(x_n x_{n-1} \ldots x_2)$ for every $n$, we have

$$f(\Lambda(x_2 x_1)) = \Lambda(x_3 x_2). \quad (12)$$

The spaces $X^\omega$ and $J(f)$ are compact, thus the map $\Lambda$ is a quotient map (see [6] Theorem 9 on p. 114). So it is sufficient to prove that the group $IMG(f)$ is contracting and that two points $\ldots x_2 x_1 \ldots y_2 y_1 \in X^\omega$ are asymptotically equivalent if and only if their $\Lambda$-images are equal.

Let us prove that the group $IMG(f)$ is contracting. Let $U$ be an arcwise connected open neighborhood of $J(f)$ such that the fundamental group of $U$ has finite balls. Consider an arbitrary infinite word $\ldots x_2 x_1 \in X^\omega$. The sequence $\{\Lambda(x_n x_{n-1} \ldots x_1)\}_{n=1,2,\ldots}$ converges to the point

$$\Lambda(\ldots x_2 x_1) \in J(f) \subseteq U.$$

It follows from (11) that there exists $n_0 \in \mathbb{N}$, such that the path

$$\ell (x_{n_0} x_{n_0-1} \ldots x_1; w x_{n_0} x_{n_0-1} \ldots x_1)$$

is inside the set $U$ for every $w \in X^\omega$.

In this way we cover the space $X^\omega$ by cylindrical sets $X^\omega x_{n_0} \ldots x_1$, so that for any $w \in X^\omega x_{n_0} \ldots x_1 \cup X^\omega x_{n_0} \ldots x_1$ and $v \in X^\omega x_{n_0} \ldots x_1$ the path $\ell (v; w)$ belongs to $U$. The space $X^\omega$ is compact, so we can choose a finite sub-cover $\{X^\omega v_i\}_{i=1,\ldots,m}$, where $v_i \in X^\omega$. Denote the set $\{v_i\}_{i=1,\ldots,m}$ by $V$.

For a given $R > 0$ define $K(R)$ to be the set of the elements of $IMG(f)$ defined by the loops of the form $\ell (\mathcal{O}; v) \gamma \ell (\mathcal{O}; u)^{-1}$, where $v, u \in V$ and $\gamma$ is a path in $U$ of length not greater than $R$. Then the set $K(R)$ is finite for every $R$.

Let $\gamma$ be a loop at $t$, defining an element $g \in IMG(f)$. It follows from Proposition 3.4 that

$$g|_v = \ell (\mathcal{O}; v) \gamma \ell (\mathcal{O}; u)^{-1},$$

where
where \( \gamma_v = f^{-|v|}(\gamma) [\Lambda(v)] \) and \( u = v\). There exist, for \( v \) long enough, words \( v', u' \in \forall \) such that \( v = w_1v' \) and \( u = w_2u' \). Then

\[
\ell(\emptyset; v)\gamma_v\ell(\emptyset; u)^{-1} = \ell(\emptyset; v')\ell(v'; w_1v')\gamma_v\ell(u'; w_1u')^{-1} \ell(\emptyset; u')^{-1}.
\]

The middle part \( \ell(v'; w_1v')\gamma_v\ell(u'; w_1u')^{-1} \) of the path is inside the set \( \mathcal{U} \), if \( v \) is long enough (since then \( \gamma_v \) is short), and its length is not greater than

\[
c^{-1}k^{-|v'|}(l + k^{-1}l + \cdots + k^{-|w_1|+1}l) + c^{-1}k^{-|v|}\text{length}(\gamma) + c^{-1}k^{-|u'|}(l + k^{-1}l + \cdots + k^{-|w_2|+1}l) < R_1 + c^{-1}k^{-|v|}\text{length}(\gamma),
\]

where \( R_1 = \max_{v' \in \forall} \frac{k^{-|v|}k^{-1}}{1-k^{-1}} \).

So for all \( v \in X^* \) long enough the restriction \( g|_v \) belongs to the set \( K(R_1+1) \), which is finite. Therefore, the action of \( \text{IMG}(f) \) on \( X^* \) is contracting.

Suppose that the points \( x_2x_1, \ldots x_2y_1 \in X^{-w} \) are asymptotically equivalent. Then there exists a bounded sequence \( \{\gamma_n\} \) of loops at \( t \) such that \( (x_nx_{n-1} \ldots x_1)^n = y_ny_{n-1} \ldots y_1 \) for every \( n \geq 1 \). Denote by \( m \) the maximal length of the paths \( \gamma_n \). Let

\[
\gamma_n' = f^{-n}(\gamma_n) [\Lambda(x_nx_{n-1} \ldots x_1)].
\]

Then the end of \( \gamma_n' \) is \( \Lambda(y_ny_{n-1} \ldots y_1) \) and its length will be not greater than \( c^{-1}k^{-n}m \). Therefore

\[
d(\Lambda(x_nx_{n-1} \ldots x_1), \Lambda(y_ny_{n-1} \ldots y_1)) \leq c^{-1}k^{-n}m,
\]

so

\[
\Lambda(x_nx_{n-1} \ldots x) = \lim_{n \to \infty} \Lambda(x_nx_{n-1} \ldots x_1) = \Lambda(y_ny_{n-1} \ldots y_1) = \Lambda(x_nx_{n-1} \ldots x_1).
\]

Suppose now that \( \Lambda(x_nx_{n-1} \ldots x_1) \) is \( \Lambda(y_ny_{n-1} \ldots y_1) \). It follows from (12) that

\[
\Lambda(x_nx_{n+1} \ldots x_{n+1}) = \Lambda(y_ny_{n+1} \ldots y_{n+1})
\]

for every \( n \).

Then the path

\[
\gamma_n' = \ell(x_n \ldots x_1; \ldots x_nx_{n+1}y_1 \ldots y_ny_{n+1})^{-1}
\]

begins at \( \Lambda(x_nx_{n-1} \ldots x_1) \) and ends in \( \Lambda(y_ny_{n-1} \ldots y_1) \). Its image under \( f^n \) is the loop \( \gamma_n = \ell(\emptyset; \ldots x_2x_1)\ell(\emptyset; y_2y_1)^{-1} \).

The path \( \gamma_n \) is equal to

\[
\ell(\emptyset; x_nx_{n-1}x_1) \ell(x_nx_{n-1} \ldots x_2x_1) \ell(y_ny_{n-1}x_2y_2y_1)^{-1} \ell(\emptyset; y_ny_{n-1} \ldots y_1)^{-1}.
\]
for some \(x_{n_1}x_{n_1-1} \ldots x_1; y_{n_2}y_{n_2-1} \ldots y_1 \in V\). The middle part
\[
\ell(x_{n_1}x_{n_1-1} \ldots x_1; x_2x_1)\ell(y_{n_2}y_{n_2-1} \ldots y_1; y_2y_1)^{-1}
\]
is inside the set \(U\) and its length is not greater than \(c^{-1}(k^{-1}n + k^{-2}) = R_2\), so the path \(\gamma_n\) belongs to a finite set \(K(R_2)\). Then \((x_nx_{n-1} \ldots x_1)^{\gamma_n} = y_ny_{n-1} \ldots y_1\), and the words \(\ldots x_2x_1, \ldots y_2y_1\) are asymptotically equivalent.

So the map \(\Lambda: X^{-\omega} \rightarrow J_M(f)\) induces a homeomorphism of \(J_G\) with \(J(f)\). It follows from (12) that this homeomorphism conjugates the shifts with the map \(f\).

\[\square\]

5 Examples and applications

5.1 Self-coverings and expanding endomorphisms of manifolds

Here we consider the case when \(f\) is defined on the whole space \(M\). Then the respective virtual endomorphism \(\phi_f\) of the fundamental group \(\pi_1(M)\) is an isomorphism from a subgroup of finite index \(\text{Dom} \phi_f < \pi_1(M)\) to \(\pi_1(M)\). Its inverse is an injective endomorphism \(f_\#: \pi_1(M) \rightarrow \pi_1(M)\) induced by the map \(f\) (more pedantically, \(f_\#\) is a map from \(\pi_1(M, x)\) to \(\pi_1(M, t)\), where \(x\) is a preimage of \(t\), but we identify \(\pi_1(M, x)\) with \(\pi_1(M, t)\) using a path \(\ell_x\) connecting \(t\) to \(x\).

The kernel of the iterated monodromy action of the fundamental group \(\pi_1(M)\) is equal, by Proposition 3.10, to the subgroup
\[
N_f = \bigcap_{k \geq 1} \bigcap_{g \in G} g^{-1} \cdot f_\#^k(\pi_1(M)) \cdot g.
\]
The iterated monodromy group \(\text{IMG}(f)\) is isomorphic then to the quotient \(\pi_1(M)/N_f\).

The following properties of expanding endomorphisms of Riemannian manifolds where proved by M. Shub and J. Franks [19].

**Theorem 5.1 (M. Shub, J. Franks).** Suppose that the map \(f: M \rightarrow M\) on a compact Riemannian manifold \(M\) is expanding. Then the following is true.

1. The map \(f\) has a fixed point.
2. The universal covering space of \(M\) is diffeomorphic to \(\mathbb{R}^n\).
3. The periodic points of \(f\) are dense in \(M\).
4. There exists a dense orbit of \(f\) (i.e., the dynamical system \((M, f)\) is topologically transitive).
5. The fundamental group \(\pi_1(M)\) is a torsion free group of polynomial growth.
6. \[ \bigcap_{k \geq 1} f_k^\#(\pi_1(M)) = \{1\}. \]

Theorem 4.6 and 5.1 imply Theorem 5.2.

Let \( f : M \rightarrow M \) be an expanding map on a compact manifold \( M \). Then the iterated monodromy group \( G = \text{IMG}(f) \) is isomorphic to the fundamental group \( \pi_1(M) \). Every its standard self-similar action on the tree \( X^* \) is contracting and the limit dynamical system \((\mathcal{J}_G, \mathbf{s})\) is topologically conjugate with the system \((M, f)\).

Proof. The only thing we need to prove is that the set \( \bigcup_{k \geq 1} f^{-1}(x) \) is dense in \( M \), i.e., that \( \mathcal{J}(f) = M \). But this follows easily from the topological transitivity of \((M, f)\). Let \( x_0 \in M \) be a point with a dense \( f \)-orbit. Then for every \( y \in M \) and \( \epsilon > 0 \) there exist \( m \) and \( n > m \) such that \( d(f^n(x_0), x) < \epsilon \) and \( d(f^m(x_0), y) < \epsilon \), but then
\[
d(x', y) \leq d(x', f^m(x)) + d(f^m(x), y) < c^{-1} \cdot k^{m-n} \epsilon + \epsilon < (c^{-1} + 1) \epsilon
\]
for some \( x' \in f^{-(n-m)}(x) \). Here \( k > 1 \) and \( c > 0 \) are the constants from the definition of an expanding map. \( \square \)

Theorem 5.2 and Proposition 5.9 imply the following (see [19] Theorems 4 and 5).

**Theorem 5.3 (M. Shub).** The expanding map \( f : M \rightarrow M \) is uniquely determined, up to topological conjugacy, by the action of \( f^\# \) on its fundamental group \( \pi_1(M) \).

M. Gromov, using his theorem on groups of polynomial growth, has proved a conjecture of M. Shub (see [1] and [15]), which describes all possible expanding endomorphisms of Riemannian manifolds.

Let \( L \) be a connected and simply connected nilpotent Lie group and let \( \text{Aff}(L) \) be the group generated by the left translations and automorphisms of the group \( L \). Chose a subgroup \( G < \text{Aff}(L) \) acting freely and discretely on \( L \). Suppose that the quotient \( M = L/G \) is compact. Then it is a manifold. If an expanding endomorphism \( \hat{f} \) of the Lie group \( L \) conjugates \( G \) to its subgroup, then \( \hat{f} \) induces an expanding map \( f : M \rightarrow M \). Such map is called expanding endomorphism of the infranil-manifold \( M \).

Note that an endomorphism of a Lie group is expanding if and only if its derivative at 1 is an expanding linear map.

**Theorem 5.4 (M. Gromov).** Every expanding map of a compact manifold is topologically conjugate to an expanding endomorphism of an infranil-manifold.

Explicit self-similar actions on the tree \( X^* \) are interesting from computational and dynamical points of view. They produce faithful actions of the groups by finite-automatic automorphisms of the tree \( X^* \) and correspond to generalized numeration systems on the group.
Let us compute the standard iterated monodromy action of the element \( \tau \). This induces a two-fold self-covering of the circle. The fundamental group of the circle is generated by the loop \( \{0\} \) of the tree. The adding machine.

Let \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) be the circle. The map \( f : x \mapsto 2x \) induces a two-fold self-covering of the circle \( \mathbb{T} \). The fundamental group of the circle is generated by the loop \( \tau \) equal to the image of the segment \([0, 1]\) in \( \mathbb{R}/\mathbb{Z} \). Let us compute the standard iterated monodromy action of the element \( \tau \) on the tree \( \{0, 1\}^\omega \).

Choose the base-point \( t \) equal to 0. It has two preimages: itself and \( 1/2 \). So we can take \( \ell_0 \) equal to the trivial path in the point 0 and \( \ell_1 \) equal to the image of the segment \([0, 1/2]\).

The path \( \tau \) has two preimages. One is \( \tau_0 = \ell_1 = [0, 1/2] \), another is \( \tau_1 = [1/2, 1] \). Therefore \( \tau \) acts on the first level of the tree \( X^\ast \) by the transposition. So using (4) we get
\[
(0w)^\tau = 1w, \quad (1w)^\tau = 0w^\tau,
\]
since the path \( \ell_0 \tau_0 \ell_1^{-1} \) is trivial and \( \ell_1 \tau_1 \ell_0^{-1} \) is equal to \( \tau \).

The recurrent definition of the transformation \( \tau \) coincides with the rules of adding 1 to a binary number. More precisely, \( (a_0 \ldots a_n)^\tau = (b_0 \ldots b_n) \) is equivalent to the equality
\[
(a_0 + a_1 \cdot 2 + \cdots + a_n \cdot 2^n) + 1 = (b_0 + b_1 \cdot 2 + \cdots + b_n \cdot 2^n)
\]
modulo \( 2^{n+1} \).

If we identify the infinite words \( w = a_0 a_1 \ldots \in \{0, 1\}^\omega \) with the dyadic integers \( \Phi(w) = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \cdots \), then \( \Phi(w^\tau) = \Phi(w) + 1 \).

The transformation \( \tau \) is called the adding machine. It is an important example of a minimal dynamical system (see [26], for example).

The map \( f \) is obviously expanding and the described action of the group \( \mathbb{Z} \) on \( X^\ast \) is contracting, since the respective virtual endomorphism \( \phi_f \) is the map \( x \mapsto x/2 \). The nucleus of the action is the set \( \{-1, 0, 1\} \). The Moore diagram of the nucleus is shown on Figure 4.

The Moore diagram and Proposition 4.2 show that two sequences are asymptotically equivalent if and only if they are equal or are of the form \( \ldots 110v, \ldots 001v \), where \( v \in X^\ast \) is arbitrary, or of the form \( \ldots 11, \ldots 00 \). But this is exactly the usual identification of the real binary numbers, so the sequences \( \ldots x_2x_1 \ldots y_2y_1 \in X^\sim \) are asymptotically equivalent if and only if
\[
x_1/2 + x_2/4 + \cdots + x_n/2^n + \cdots = y_1/2 + y_2/4 + \cdots + y_n/2^n + \cdots \pmod{1}.
\]

It follows that the limit space is the circle \( \mathbb{R}/\mathbb{Z} \) with the shift map \( s(x) = 2x \pmod{1} \). We have returned back to the original self-covering, what agrees with Theorem 4.6.
Figure 1: The nucleus of the adding machine action

We see that the standard action of the iterated monodromy group on the limit space defines an encoding of the space $\mathcal{M} = \mathbb{R}/\mathbb{Z}$ by sequences of digits, which coincides with the binary numeration system on $\mathbb{R}$.

The torus. The adding machine example can be generalized to $n$ dimensions. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the $n$-dimensional torus. Let $A$ be an $n \times n$-matrix with integral entries and with determinant equal to $d > 1$. Then the linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ induces an $d$-fold self-covering of the torus $\mathbb{T}^n$.

The fundamental group of the torus $\mathbb{T}^n$ is the free Abelian group $\mathbb{Z}^n$. The iterated monodromy action is faithful if and only if the group $N_A = \bigcap_{k \geq 1} A^k(\mathbb{Z}^n)$ is trivial. This is equivalent to the condition that the no eigenvalue of $A^{-1}$ is an algebraic integer (see [49] Proposition 4.1 and [45] Proposition 10.1).

The corresponding standard iterated monodromy actions of $\mathbb{Z}^n$ on rooted trees can be interpreted as numeration systems on $\mathbb{Z}^n$. Consider a standard action of the group $\mathbb{Z}^n$ on $X^*$. Then it follows from Proposition 3.8 that there exists a coset transversal $\{r_0, r_1, \ldots, r_{d-1}\}$ for the subgroup $A(\mathbb{Z}^n)$ such that $(xw)^a = yw^{A^{-1}(a + r_x - r_y)}$, for all $x \in X$, $a \in \mathbb{Z}^n$ and $w \in X^*$, where $\{0, 1, 2, \ldots, d-1\} = X$ and $y \in X$ is such that $a + r_x - r_y \in A(\mathbb{Z}^n)$. We do not need the elements $h_x$, since $\phi_f$ in our case is onto (see the remark after Definition 3.11).

Consequently, $(x_0x_1 \ldots x_m)^a = y_0y_1 \ldots y_m$ is equivalent to the condition that the elements

$$r_{x_0} + A (r_{x_1}) + A^2 (r_{x_2}) + \cdots + A^m (r_{x_m}) + a$$

and

$$r_{y_0} + A (r_{y_1}) + A^2 (r_{y_2}) + \cdots + A^m (r_{y_m})$$

are equal modulo $A^{n+1}(\mathbb{Z}^n)$. So the self-similar action of $\mathbb{Z}^n$ on the tree corresponds to an “$A$-adic” numeration system on $\mathbb{Z}^n$.

If the matrix $A$ is expanding, i.e., if all its eigenvalues have absolute value greater than 1, then the series

$$\sum_{n=1}^{\infty} A^{-n}(r_{x_n})$$

is equivalent to the condition that

$$r_{x_0} + A (r_{x_1}) + A^2 (r_{x_2}) + \cdots + A^m (r_{x_m}) + a$$

and

$$r_{y_0} + A (r_{y_1}) + A^2 (r_{y_2}) + \cdots + A^m (r_{y_m})$$

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are equal modulo $A^{n+1}(\mathbb{Z}^n)$. So the self-similar action of $\mathbb{Z}^n$ on the tree corresponds to an “$A$-adic” numeration system on $\mathbb{Z}^n$.
is convergent in $\mathbb{R}^n$ for every $\ldots x_2 x_1 \in X^{-\omega}$. In this way the $A$-adic numeration system on $\mathbb{Z}^n$ extends to an $A$-adic numeration system on $\mathbb{R}^n$.

It is proved in [26], that in the case when $A$ is expanding, the limit space of the iterated monodromy action is the torus $\mathbb{R}^n/\mathbb{Z}^n$ and that the quotient map $X^{-\omega} \to \mathbb{R}^n/\mathbb{Z}^n$ comes from this $A$-adic numeration system on $\mathbb{R}^n$ in a similar way like for the case of the adding machine.

The set $T$ of all possible sums of the series (13) is called the digit tile defined by the $A$-adic numeration system. The $\mathbb{Z}^n$-translations of the set $T$ cover the space $\mathbb{R}^n$. In general the translates can overlap, but often they form a tiling of the space $\mathbb{R}^n$ (see a criterion in [26]).

The linear map $A$ maps every tile of such a tiling to a union of $d$ tiles. The tilings with this properties are called self-affine (self-replicating tilings, rep-tilings, digit-tilings).

For every $k \geq 1$, the images of the sets $A^{-k}T + r$, $r \in \mathbb{Z}^n$ in the torus $\mathbb{R}^n/\mathbb{Z}^n$ form a Markov partition of the dynamical system $(\mathbb{R}^n/\mathbb{Z}^n, A)$. They are equal to the images of the cylindrical sets $X^{-\omega}v, v \in X^k$ with respect to the presentation of the limit space $\mathbb{R}^n/\mathbb{Z}^n$ as a quotient of $X^{-\omega}$ by the asymptotic equivalence relation.

The set $T$ often have a fractal boundary. One of most known examples is the “dragon curve”, shown on Figure 2. It corresponds to the case $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ and the coset transversal $\{(0,0), (1,0)\}$. The respective numeration system on $\mathbb{R}^2$ can be interpreted as a numeration system on $\mathbb{C}$ with the base $(-1+i)$ and the digits 0, 1. See its discussion in [32].

See the survey [4] and bibliography in it for properties of the digit tiles and their applications. The question, when different $A$-adic expansions define the
same point in \( \mathbb{R}^2 \) was studied in [17].

**Heisenberg group.** This example of a self-covering is from [19]. Let \( L \) be the group of lower triangular matrices
\[
\begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
c & b & 1
\end{pmatrix}
\]
with \( a, b, c \in \mathbb{R} \) and let \( G \) be its subgroup of matrices with \( a, b, c \in \mathbb{Z} \). Then for all \( p, q \in \mathbb{Z} \), the map
\[
f_# : \begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
c & b & 1
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 0 & 0 \\
p \cdot a & 1 & 0 \\
(pq) \cdot c \cdot q & b & 1
\end{pmatrix}
\]
is an automorphism of the group \( L \), which maps \( G \) to a subgroup of index \( p^2q^2 \). The quotient \( L/G \) is a three-dimensional nil-manifold, and the map \( f_# \) induces its expanding \( p^2q^2 \)-fold self-covering.

Let us consider, for instance, the case \( p = q = 2 \). Then by Proposition 3.9, one of the standard self-similar actions of the iterated monodromy group \( G \) is defined by the pair \( (\phi, T) \), where \( \phi \) is the virtual endomorphism
\[
\begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
c & b & 1
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 0 & 0 \\
a/2 & 1 & 0 \\
c/4 & b/2 & 1
\end{pmatrix}
\]
with the domain \( \begin{pmatrix}2\mathbb{Z} & 1 & 0 \end{pmatrix} \) and \( T \) is the coset transversal \( \left\{ \begin{pmatrix}1 & 0 & 0 \\
a & 1 & 0 \\
c & b & 1 \end{pmatrix} : a, b \in \{0, 1\}, c \in \{0, 1, 2, 3\} \right\} \).

In the same way, as in the case of abelian groups, the images of cylindrical sets \( X^{-v} \) are tiles of self-replicating tilings of the Lie group \( L \). See the paper [29] for a treatment of self-replicating tilings of groups. See also a discussion of self-similar actions of nilpotent and solvable groups in [49].

### 5.2 Rational functions on \( \mathbb{C} \)

#### 5.2.1 Iterated monodromy group as a Galois group.

*Iterated monodromy group* of a rational function \( f \) is by definition the iterated monodromy group of the partial self-covering \( f : M_1 \rightarrow M \) of the the set \( M = \hat{\mathbb{C}} \setminus \overline{P} \), where \( \overline{P} \) is the closure of the post-critical set of \( f \) and \( M_1 = f^{-1}(M) \).

The following construction is due to R. Pink (private communication).

Let \( f \in \mathbb{C}[z] \) be a polynomial over \( \mathbb{C} \). For every \( n \geq 1 \) define a polynomial \( F_n(z) = f^n(z) - t \in \mathbb{C}(t)[z] \) over the field \( \mathbb{C}(t) \) of rational functions, where \( f^n(z) \) denotes the \( n \)th iteration of \( f \). Let \( \Omega_n \) be the splitting field of \( F_n \). It is easy to see that \( \Omega_n \subset \Omega_{n+1} \). It is a classical fact, that the Galois group \( \text{Aut}(\Omega_n/\mathbb{C}(t)) \) is isomorphic to the monodromy group of the branched covering \( f^n : \mathbb{C} \rightarrow \mathbb{C} \) (see [3] Theorem 8.12), i.e., to the group of permutations of the set \( f^{-n}(z_0) \) induced by the action of the fundamental group \( \pi_1(\mathbb{C} \setminus P_n, z_0) \), where \( P_n \) is the set of branching points of \( f^n \) and \( z_0 \notin P_n \) is an arbitrary point.

This implies the following interpretation of the iterated monodromy group \( \text{IMG}(f) \).

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Proposition 5.5. Let $f \in \mathbb{C}[z]$ be a post-critically finite polynomial. Then the closure of the iterated monodromy group $\text{IMG}(f)$ in the automorphism group of the preimage tree is isomorphic to the Galois group $\text{Aut}(\Omega/\mathbb{C}(t))$, where $\Omega = \bigcup_{n \geq 1} \Omega_n$.

5.2.2 Hyperbolic maps.

The Julia set $J(f)$ of a rational function $f \in \mathbb{C}(z)$ is the set of points $z \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that the set of functions $\{f^n : n \in \mathbb{N}\}$ is not normal on any neighborhood of $z$ (see [28]). Since the set $\bigcup_{n \geq 0} f^{-n}(z)$ is dense in $J(f)$ for every $z \in J(f)$ (see [28], or [27]), our definition of the Julia set of an expanding map agrees with the notion of the Julia set of a rational function.

Definition 5.1. A rational function $f \in \mathbb{C}(z)$ is said to be hyperbolic if it is expanding on a neighborhood of its Julia set.

We have the following criterion, originally due to Fatou (see [7], see also a proof in [28], Theorem 19.1).

Theorem 5.6. A rational function $f \in \mathbb{C}(z)$ is hyperbolic if and only if the closure of the postcritical set $\mathcal{P}$ does not intersect the Julia set $J(f)$, or equivalently, if and only if the orbit of every critical point converges to an attracting cycle.

So, if a rational function is hyperbolic, then the post-critical set has a finite number of accumulation points, which are all outside the Julia set and the set $\mathcal{M} = \hat{\mathbb{C}} \setminus \mathcal{P}$ is arcwise connected. Moreover, the rational function is then expanding on the set $\mathcal{M}$ (see the proof of Theorem 19.1 in [28]).

It is also easy to see that the Julia set has a neighborhood whose fundamental group has finite balls. One can take, for instance, the complement to the union of small closed disks around the points of the attracting cycles (there exists only a finite number of them) and around the post-critical points, which do not belong to the already chosen disks.

Consequently, Theorem 5.6 implies

Theorem 5.7. Let $f \in \mathbb{C}(z)$ be a hyperbolic rational function. Then every standard self-similar action of the iterated monodromy group $\text{IMG}(f)$ is contracting and the limit dynamical system $(\mathcal{J}_{\text{IMG}(f)}, s)$ is topologically conjugate to the dynamical system $(J(f), f)$. In particular, the limit space $\mathcal{J}_{\text{IMG}(f)}$ is homeomorphic to the Julia set of $f$.

Theorem 5.7 provides a finite-to-one encoding of the points of the Julia set $J(f)$ by infinite sequences over the alphabet $X$, which semi-conjugates the polynomial $f$ to the Bernoulli shift. This and similar encodings where constructed in [46, 34, 13], see also [27] p. 81–82.

Let us illustrate Theorem 5.7 and computation of the iterated monodromy groups (using Proposition 3.4) by some examples of polynomial mappings of degree 2. All iterated monodromy groups of these polynomials will act on
the binary tree $X^* = \{0, 1\}^*$. We will use here the permutational recursion, described at the end of Subsection 2.1. Namely:

$$g = (g_0, g_1) \iff (0w)^g = 0w^{g_0} \text{ and } (1w)^g = 1w^{g_1} \text{ for every } w \in X^*, $$

$$g = (g_0, g_1)\sigma \iff (0w)^g = 1w^{g_0} \text{ and } (1w)^g = 0w^{g_1} \text{ for every } w \in X^*. $$

Here $\sigma$ denotes the “switch” $(0w)^\sigma = 1w$, $(1w)^\sigma = 0w$. For example, the adding machine transformation $\tau$ is defined in this notation by the recurrent formula $\tau = (1, \tau)\sigma$.

**The adding machine as $\text{IMG} (z^2)$.** The polynomial $z^2$ defines on the circle $\{z \in \mathbb{C} : |z| = 1\}$ a self-covering, conjugate to the self-covering $x \mapsto 2x$ of the circle $\mathbb{R}/\mathbb{Z}$. The post-critical set of $z^2$ is $\{0, \infty\}$, so the fundamental group of the space $\mathcal{M} = \mathbb{C} \setminus P$ is generated by a loop around the circle, and the computation of the standard action of the iterated monodromy group of $z^2$ repeats the computation of the iterated monodromy group of the self-covering $x \mapsto 2x$ of the circle $\mathbb{R}/\mathbb{Z}$.

The polynomial $z^2$ is clearly hyperbolic (both critical points $0, \infty$ are attracting fixed points). As we have already seen, the limit space of the adding machine action is the circle, what agrees with the fact that the Julia set of the polynomial $z^2$ is the circle $\{z : |z| = 1\}$.

**Computation of $\text{IMG} (z^2 - 1)$.** The critical points of the polynomial $z^2 - 1$ are $\infty$ and $0$. The infinity is a fixed point, and the orbit of $0$ is $0 \mapsto -1 \mapsto 0$, so the post-critical set is $P = \{0, -1, \infty\}$. The cycle $\{0, -1\}$ is attracting, since $0$ is a critical point (so it is even super-attracting).

Choose a basepoint $t = \frac{1 - \sqrt{5}}{2}$ (denoted by a star on Figure 3). It has two preimages: itself, and $-t$. Let $\ell_0$ be the trivial path at $t$ and let $\ell_1$ be the path, connecting $t$ and $-t$ as on the lower part of Figure 3. Let $a$ and $b$ be the elements of $\text{IMG} (z^2 - 1)$, defined by the loops in positive direction around $-1$ and $0$ respectively, shown on the upper part of the figure.

![Figure 3: Computation of the group $\text{IMG} (z^2 - 1)$](image)

The preimages of the loops $a$ and $b$ are shown on the lower part of Figure 3.
We have
\[ a = (b, 1)\sigma, \quad b = (a, 1). \]

Thus, the group \(\mathbf{IMG}(z^2 - 1)\) is generated by the automaton with the Moore diagram shown on Figure 4.

![Moore diagram](image)

**Figure 4:** The automaton generating the group \(\mathbf{IMG}(z^2 - 1)\).

In the papers [47, 50] the following properties of the group \(\mathbf{IMG}(z^2 - 1)\) are proved.

**Theorem 5.8 (R. Grigorchuk, A. Žuk).** The group \(\mathbf{IMG}(z^2 - 1)\)

1. is torsion free;
2. has exponential growth (actually, the semigroup generated by \(a\) and \(b\) is free);
3. is just non-solvable, i.e., every its proper quotient is solvable;
4. has solvable word and conjugacy problems;
5. has no free non-abelian subgroups of rank 2.

Theorem 5.8 is proved using the methods developed during the study of groups acting on rooted trees, in particular branch groups (see [10]) and of the just non-solvable group of A. Brunner, S. Sidki and A. Vieira in [42]. The properties of the group \(\mathbf{IMG}(z^2 - 1)\) are very similar to the properties of the group from [42], which is also a subgroup of the pro-finite completion of \(\mathbf{IMG}(z^2 - 1)\).

The following theorem is a result of L. Bartholdi.

**Theorem 5.9.** The group \(\mathbf{IMG}(z^2 - 1)\) has the following presentation by defining relations:

\[
\mathbf{IMG}(z^2 - 1) = \langle a, b \mid [a^{2k}, b^{2k}], [b^{2k}, a^{2k+1}], a^{2k+1}, k \geq 0 \rangle.
\]
Here \([x, y] = x^{-1}y^{-1}xy\).

The presentation of the group \(\text{IMG}(z^2 - 1)\) is similar to the presentation of the Grigorchuk group, which was constructed by I. Lysionok in [30] and generalized for many other contracting groups by L. Bartholdi [21].

Another interesting property of the group \(\text{IMG}(z^2 - 1)\) is its amenability, proved by B. Virág and L. Bartholdi (see [8]), using self-similar random walks. It was proved before in [50] that the group \(\text{IMG}(z^2 - 1)\) does not belong to the class of sub-exponentially amenable groups, i.e., can not be constructed from groups of sub-exponential growth using the group-theoretic operations, preserving amenability (passing to subgroups, quotients, extensions and direct limits). The group \(\text{IMG}(z^2 - 1)\) is the first example of an amenable group of this sort.

Figure 5 shows the Schreier graphs of action of the group \(\text{IMG}(z^2 - 1)\) on the levels of the tree \(X^*\). The shape of these Schreier graphs was described by L. Bartholdi (see [37]). The Julia set of the polynomial \(z^2 - 1\) is shown on Figure 6.

**Figure 5:** The Schreier graphs \(\Gamma_n(\text{IMG}(z^2 - 1), \{a, b\})\) for \(3 \leq n \leq 6\).

### 5.2.3 Sub-hyperbolic maps

The set of hyperbolic functions is a subset of a more general class of sub-hyperbolic functions. A function is sub-hyperbolic (see [28]) if it is expanding respectively to some orbifold metric on a neighborhood of the Julia set. An analog of Theorem 5.6 is the following criterion (see [28]).

**Theorem 5.10.** A rational function is sub-hyperbolic if and only if every orbit of a critical point is either finite, or converges to an attracting cycle.

In particular, every post-critically finite function (i.e., a function for which the post-critical set \(P\) is finite) is sub-hyperbolic.
Theorem \[\text{5.7}\] holds also for the sub-hyperbolic functions. The proof of Theorem \[\text{4.6}\] (with small modifications) is also valid for the case of orbifold coverings, defined by sub-hyperbolic functions. More details will appear in a subsequent paper, where iterated monodromy groups of orbifold coverings will be studied.

Let us consider some examples

**The dihedral group as \(\text{IMG} \,(z^2 - 2)\).**  The orbit of the finite critical point is \(0 \mapsto -2 \mapsto 2 \mapsto 2\). The post-critical set is \((-2, 2, \infty)\). Take, for instance, the base point \(t = 0\) and connect it with the preimages \(\pm \sqrt{2}\) by straight segment. The fundamental group of the space \(\mathcal{M} = \mathbb{C} \setminus \{-2, 2\}\) is generated by a small loop \(a\) around \(-2\), which is connected to \(t\) by a straight segment, and by a small loop \(b\) around 2, which is connected to \(t\) by a straight segment (both loops go around the points in the positive direction). Computation of the standard action shows that the respective generators of the iterated monodromy group \(\text{IMG} \,(z^2 - 2)\) are defined by the recursion

\[
a = (1, 1)\sigma, \quad b = (b, a).
\]

It follows from the formula that \(a^2 = 1\) and \(b^2 = (b^2, a^2) = (b^2, 1) = 1\). So, the elements \(a\) and \(b\) are both of order 2. They generate the infinite dihedral group \(\mathbb{D}_\infty\) (see \[\text{41}\] and \[\text{47}\]). The Schreier graph of this group on the level \(X^n\) is a chain of edges of length \(2^n - 1\). It follows that the limit space is homeomorphic to the real segment. This agrees with the fact that the Julia set of the polynomial \(z^2 - 2\) is the segment \([-2, 2]\).

This example can be generalized to the Chebyshev polynomials \(T_n(z) = \cos(n \arccos z)\), which all have the iterated monodromy group isomorphic to
The Chebyshev polynomials are the only polynomials with the Julia set $[-1, 1]$ (see [38]).

The sphere and example of Lattès. If every critical point $z_0$ of a rational function $f$ is pre-periodic (i.e., if $f^m(z_0) = f^k(z_0)$ for some $m \neq k$, but $f^n(z_0) \neq z_0$ for all $n \geq 1$), then the Julia set of $f$ is the whole sphere $\hat{\mathbb{C}}$ (see, for example Theorem 1.24 in [27] or Theorem 9.4.4 in [38]).

This is the case, for example, for the rational functions of S. Lattès [39]. Let $\Gamma$ be a lattice in $\mathbb{C}$, and let $\alpha$ be such that $\alpha \Gamma \subset \Gamma$. The Weierstrass elliptic function $\wp$ for the lattice $\Gamma$ is defined as

$$
\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left( \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right).
$$

It induces a two-fold branched covering of the sphere $\hat{\mathbb{C}}$ by the torus $\mathbb{C}/\Gamma$. The map $z \mapsto \alpha z$ defines an $|\alpha|^2$-covering of the torus $\mathbb{C}/\Gamma$ and respectively, an $|\alpha|^2$-fold branched covering of the sphere $\hat{\mathbb{C}}$, defined by a rational function $f$ of degree $|\alpha|^2$ such that $\wp(\alpha z) = f(\wp(z))$.

For example, for $\alpha = 2$ the function $f$ is

$$
f(z) = \frac{z^4 + \frac{g_2}{4}z^2 + 2g_3z + g_3^2}{4z^3 - g_2z - g_3},
$$

(see [38] p. 74), where $g_2 = 60s_4$ and $g_3 = 140s_6$ for

$$
s_m = \sum_{\omega \in \Gamma, \omega \neq 0} \omega^{-m}.
$$

A pair $(g_2, g_3)$ is realized by a lattice $\Gamma$ if and only if $g_3^2 - 27g_2^2 \neq 0$ (see [5], p. 39). In particular, there exists a lattice $\Gamma$ such that $g_3 = 0$ and $g_2 = 4$, so that

$$
f(z) = \frac{(z^2 + 1)^2}{4z(z^2 - 1)}, \quad (14)
$$

For the case of the lattice $\Gamma = \mathbb{Z}[i]$ we have $g_3 = 0$, thus $f(z) = \frac{(z^2 + g_2/4)^2}{4z(z^2 - g_2/4)}$, which is also conjugate to (14) (the conjugating map is $t(z) = \frac{z}{\sqrt[4]{g_2}}$).

**Proposition 5.11.** Let $\Gamma$ be a lattice in $\mathbb{C}$ and let $\alpha \in \mathbb{C}$ be such that $\alpha \Gamma \subset \Gamma$ and $|\alpha| \neq 1$. Let a rational function $f \in \mathbb{C}(z)$ be such that $\wp(\alpha z) = f(\wp(z))$.

Then the iterated monodromy group $\text{IMG}(f)$ is isomorphic to the group of affine transformations $(-1)^k z + \omega$, where $k \in \mathbb{Z}, \omega \in \Gamma$. The associated virtual endomorphism is the map $(-1)^k z + \omega \mapsto (-1)^k z + \alpha^{-1} \omega$. 

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Proof. The Weierstrass function $\wp$ defines a two-fold branched covering $\wp_0 : \mathbb{C}/\Gamma \rightarrow \hat{\mathbb{C}}$. The transformation $\alpha : z \mapsto \alpha z$ defines a $|\alpha|^2$-fold self-covering $f_0$ of the torus $\mathbb{C}/\Gamma$. The iterated monodromy group $\text{IMG}(f_0)$ is the group $\Gamma$ with the virtual endomorphism $\wp_0 : \omega \mapsto \alpha^{-1}\omega$.

Take the basepoint $t \in \hat{\mathbb{C}}$ not equal to a ramification point of the function $\wp_0$. Then the point $t$ has two preimages $z_0$ and $-z_0 \in \mathbb{C}/\Gamma$ under $\wp_0$. The $n$th level of the preimage tree $T_n$ of the point $z_0$ with respect to the map $f_0$ is the image of $\alpha^{-n}(\Gamma + z_0)$ in the quotient $\mathbb{C}/\Gamma$. The image of $\alpha^{-n}(\Gamma - z_0)$ in $\mathbb{C}/\Gamma$ is the $n$th level of the preimage tree of the point $-z_0$. Since $f^n(\wp(z)) = \wp(\alpha^n z)$, the $n$th level of the preimage tree $T_n$ of the point $t$ respectively to the map $f$ is $\wp_0(\alpha^{-n}(\Gamma + z_0)) = \wp_0(\alpha^{-n}(\Gamma - z_0))$.

Let $\gamma$ be a loop based at $t$. Its preimages under $\wp_0$ are either a loop $\gamma_0$ based at $z_0$ and the symmetrical loop $-\gamma_0$ based at $-z_0$ or a path $\gamma_0$ from $z_0$ to $-z_0$ and the symmetrical path $-\gamma_0$ from $-z_0$ to $z_0$. In the first case every lift of the path $\gamma_0$ to the universal cover $\mathbb{C}$ of the torus $\mathbb{C}/\Gamma$ connects the point $z_0 + \omega$, $\omega \in \Gamma$ to the point $z_0 + \omega + a$ for some fixed $a \in \Gamma$. Similarly, $-\gamma_0$ will connect the point $-z_0 - \omega - a$ to the point $-z_0 - \omega$. In the second case every lift of the path $\gamma_0$ to the universal covering $C$ of $\mathbb{C}/\Gamma$ connects the point $z_0 + \omega$, $\omega \in \Gamma$ to the point $-z_0 + \omega + a$ for some fixed $a \in \Gamma$ and every lift of the path $-\gamma_0$ connects the point $-z_0 - \omega - a$ with the point $-z_0 - \omega$.

Consequently, in the first case $\gamma$ acts on the $n$th level of the preimage tree $T_n$ by the transformation

$$\wp_0(\alpha^{-n}(z_0 + \omega)) \mapsto \wp_0(\alpha^{-n}(z_0 + \omega + a))$$

and in the second case by the transformation

$$\wp_0(\alpha^{-n}(z_0 + \omega)) \mapsto \wp_0(\alpha^{-n}(-z_0 + \omega + a)) = \wp_0(\alpha^{-n}(-z_0 - \omega - a)).$$

This implies immediately the statement of the proposition.

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