SOME CONJECTURES ON MODULAR REPRESENTATIONS OF AFFINE $\hat{\mathfrak{sl}}_2$ AND VIRASORO ALGEBRA

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Abstract. We conjecture an explicit bound on the prime characteristic of a field, under which the Weyl modules of affine $\hat{\mathfrak{sl}}_2$ and the minimal series modules of Virasoro algebra remain irreducible, and Goddard-Kent-Olive coset construction for $\hat{\mathfrak{sl}}_2$ is valid.

1. Introduction

This note contains some speculation on modular representations of infinite-dimensional Lie algebras. The modular representations of infinite-dimensional Lie algebras are little understood, and in particular Lusztig-type conjecture (cf. [6]) on irreducible characters in modular representation theory seems to be out of reach in the infinite-dimensional setting for now. We hope our explicit conjectures, though modest, might help to stimulate others to continue further in this challenging new direction.

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 2$. We speculate on an explicit bound for the prime characteristic of $\mathbb{F}$ such that the Weyl modules of the affine Lie algebra $\hat{\mathfrak{sl}}_2$ and the minimal series modules of Virasoro algebra remain irreducible over $\mathbb{F}$. One remarkable feature is that the two family of modules share the same bound on primes. This leads to another conjecture that the Goddard-Kent-Olive coset construction for $\hat{\mathfrak{sl}}_2$ is valid under the same bound on char $\mathbb{F}$.

There are two (a priori unrelated) works [2, 5] which motivated this note. Lai [5] constructed nontrivial homomorphisms between Weyl modules of $\hat{\mathfrak{sl}}_2$ at positive integral levels, and showed that the modular representations change dramatically once we go beyond level one and there was no obvious conjectural bound on the prime $p$ for which the Weyl modules remain irreducible. In [3, Table 2], a list of reducible Weyl modules of $\hat{\mathfrak{sl}}_2$ (together with the lowest level $\ell$ detected by the method therein relative to a given prime $p$) is given, in which one sees the prime $p$ can increase rather quickly relative to the level $\ell$.

Recall there are 3 minimal series modules of Virasoro algebra of central charge $\frac{1}{2}$ over $\mathbb{C}$, of highest weight $0, \frac{1}{2}, \frac{1}{16}$, respectively. In [2], Dong and Ren showed that the minimal series modules of central charge $\frac{1}{2}$ remain irreducible over $\mathbb{F}$, if char $F \neq 2, 7$. The prime 7 is bad since the values $\frac{1}{2}$ and $\frac{1}{16}$ coincide in $\mathbb{F}$ of characteristic 7.

Our main conjecture is that, under the assumption char $\mathbb{F} > 2\ell^2 + \ell - 3$ (for each $\ell \in \mathbb{Z}_{\geq 2}$), the Weyl modules of $\hat{\mathfrak{sl}}_2$ of level $\ell$ and the minimal series modules of Virasoro algebra of central charge $c_\ell$ remain irreducible, and the GKO coset construction is valid. This numerical bound $2\ell^2 + \ell - 3$ ensures the highest weights of the minimal series modules of Virasoro algebra of central charge $c_\ell$ given by (2.1) are distinct in $\mathbb{F}$ (the setting for [2] corresponds to $\ell = 2$, with $c_2 = \frac{1}{2}$ and $2\ell^2 + \ell - 3 = 7$). We then check this bound does not contradict with constraints coming from $\hat{\mathfrak{sl}}_2$ in [5].
2. The conjectures

2.1. On irreducibility of Virasoro minimal series. Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 2$. Recall that the Virasoro algebra is the Lie algebra over $\mathbb{F}$, $\text{Vir} = \mathbb{F}C \oplus \oplus_{n \in \mathbb{Z}} \mathbb{F}L_n$, subject to the commutation relations: $C$ is central, and

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{1}{2} \binom{m+1}{3} C, \quad (m, n \in \mathbb{Z}).$$

Set $\text{Vir}_+ = \bigoplus_{n=1}^{\infty} \mathbb{F}L_n, \text{Vir}_- = \bigoplus_{n=1}^{\infty} \mathbb{F}L_{-n}$. Given $c, h \in \mathbb{F}$, the Verma module $M_{c,h}$ over Vir is a free $U(\text{Vir}_-)$-module generated by 1, such that $\text{Vir}_+ \cdot 1 = 0, L_0 \cdot 1 = h1$ and $C \cdot 1 = c1$. Denote by $L_{c,h}$ the unique irreducible quotient Vir-module of $M_{c,h}$. The scalar $c$ is called the central charge of $L_{c,h}$.

For $\ell \in \mathbb{Z}_{\geq 2}$, set

$$(2.1) \quad c_\ell = 1 - \frac{6}{(\ell + 1)(\ell + 2)}.$$  

Note $c_2 = \frac{1}{2}, c_3 = \frac{7}{10}, c_4 = \frac{4}{5}$. For $1 \leq m \leq \ell, 1 \leq n \leq \ell + 1$, we let

$$(2.2) \quad h_{m,n} = h_{m,n;\ell} = \frac{(m(\ell + 2) - n(\ell + 1))^2 - 1}{4(\ell + 1)(\ell + 2)}.$$  

The scalars $c_\ell$ and $h_{m,n}$ in $\mathbb{F}$ are understood after canceling out common integer factors in the numerators and denominators, and so it is possible that these scalars are well defined even when char $\mathbb{F}$ divides $\ell + 1$ or $\ell + 2$. (For example, $c_2 = 1 - \frac{6}{10} = \frac{1}{2} \in \mathbb{F}$ makes sense in characteristic $p = 3$.) Note that $h_{m,n} = h_{\ell+1-m, \ell+2-n}$. To avoid such a double counting, it is well known that one can simply impose the constraint $n \leq m$. The Vir-modules $L_{c_\ell, h_{m,n}} (1 \leq n \leq m \leq \ell)$ are usually referred to as the (unitary) minimal series.

**Conjecture 1.** Let $\ell \in \mathbb{Z}_{\geq 2}$. Assume char $\mathbb{F} > 2\ell^2 + \ell - 3$.

1. The minimal series Vir-modules $L_{c_\ell, h_{m,n}}$ (for $1 \leq n \leq m \leq \ell$) over $\mathbb{F}$ are irreducible.
2. Let $L, L'$ be minimal series Vir-modules over $\mathbb{F}$ of the same central charge $c_\ell$. Then $\text{Ext}^i(L, L') = 0$.

One can further expect that the main theorem of [8] remain valid over $\mathbb{F}$ under the assumption that char $\mathbb{F} > 2\ell^2 + \ell - 3$; that is, the Virasor vertex algebra $L_{c_\ell,0}$ is rational and has the same fusion rule as in characteristic zero. (This is known to hold for $c_2 = \frac{1}{2}$ by the work of Dong-Ren.)

2.2. On irreducibility of Weyl modules for $\widehat{\mathfrak{sl}_2}$. For basics on modular representations of affine Kac-Moody algebra $\widehat{\mathfrak{g}}$, we refer to Mathieu [7]; also cf. [11, 4]. The level one Weyl modules of $\widehat{\mathfrak{g}}$ has been shown to be irreducible under the assumption that $p \geq h$ (the Coxeter number) by various authors (deConcini-Kac-Kazhdan, Chari-Jing, Brundan-Kleshchev) in different ways (though all these approaches are built on the fact that the level one Weyl modules afford an explicit combinatorial/vertex operator realization); cf. [4] for references.

Denote by $X^+_\ell$ the set of dominant integral weights of level $\ell \in \mathbb{Z}_{\geq 1}$. Note that $X^+_1 = \{\omega_0, \omega_1\}$ consists of 2 fundamental weights. Then $X^+_\ell = \{\lambda_{\ell,n} = (\ell-n)\omega_0 + n\omega_1 \mid 0 \leq n \leq \ell\}$. We can define the Weyl module $V(\lambda)$, for $\lambda \in X^+_\ell$ (for $\ell \in \mathbb{Z}_{> 0}$) of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ over $\mathbb{F}$ as in [7].
Conjecture 2. Let $\ell \in \mathbb{Z}_{>0}$. Assume $\text{char } F > 2\ell^2 + \ell - 3$.

1. The Weyl modules $V(\lambda)$ of $\widehat{\mathfrak{sl}}_2$ over $F$ are irreducible, for $\lambda \in X_\ell^+$. 
2. Let $\lambda, \mu \in X_\ell^+$. Then $\text{Ext}^1(V(\lambda), V(\mu)) = 0$.

One can rephrase Conjecture 2 as that the category of rational representations of level $\ell$ of the Kac-Moody group (associated to $\mathfrak{sl}_2$) over $F$ is semisimple if $\text{char } F > 2\ell^2 + \ell - 3$.

One can further hope that under the assumption that $\text{char } F > 2\ell^2 + \ell - 3$ the affine vertex algebra $V(\omega_0)$ is rational and has the same fusion rule as over the complex field $\mathbb{C}$ (which was computed by I. Frenkel and Y. Zhu).

2.3. Modular GKO coset construction. Let $\ell \geq 2$. Recall the Goddard-Kent-Olive $(\widehat{\mathfrak{sl}}_2|_{\ell-1} \oplus \widehat{\mathfrak{sl}}_2|_{\ell})$-coset construction over $\mathbb{C}$ [1] refers to the following tensor product decomposition into a direct sum of multiplicity-free irreducible $(\widehat{\mathfrak{sl}}_2|_{\ell}, \text{Vir})$-modules:

\[
V(\lambda_{\ell-1};n) \otimes V(\omega_\ell) = \bigoplus_{0 \leq j \leq n} V(\lambda_{\ell,j}) \otimes L_{\ell,\ell,\lambda_{\ell-1,n},j+1} \bigoplus_{n+1 \leq j \leq \ell} V(\lambda_{\ell,j}) \otimes L_{\ell,\ell,j-\ell-1}\]

for all $0 \leq n \leq \ell - 1$ and $\epsilon \in \{0, 1\}$.

Conjecture 3 (Modular GKO conjecture). The multiplicity-free decomposition (2.3) into a direct sum of irreducible $(\widehat{\mathfrak{sl}}_2|_{\ell}, \text{Vir})$-module is valid over $F$, if $\text{char } F > 2\ell^2 + \ell - 3$.

Remark 4. The following “partial semisimple tensor product” statement is a consequence of Conjectures 2 and 3. Assume $\text{char } F > 2\ell^2 + \ell - 3$. For any positive integers $\ell_1, \ell_2$ such that $\ell_1 + \ell_2 \leq \ell$ and any $\lambda \in X_{\ell_1}^+, \mu \in X_{\ell_2}^+$, $V(\lambda) \otimes V(\mu)$ is a semisimple $\widehat{\mathfrak{sl}}_2$-module.

3. EVIDENCE AND GENERALIZATIONS

3.1. Supporting evidence for Conjecture 1. Recall $h_{m,n}$ from (2.2). A prime $p$ is called $\text{Vir}|_{\ell}$-good if the scalars $h_{m,n}$ $(1 \leq n \leq m \leq \ell)$ are pairwise distinct (and hence there are $\ell(\ell + 1)/2$ distinct values of such $h_{m,n}$); otherwise, a prime $p$ is called $\text{Vir}|_{\ell}$-bad.

Proposition 5. Every prime greater than $2\ell^2 + \ell - 3$ are $\text{Vir}|_{\ell}$-good. Equivalently, every $\text{Vir}|_{\ell}$-bad prime does not exceed $2\ell^2 + \ell - 3$.

Note $2\ell^2 + \ell - 3 = (2\ell + 3)(\ell - 1)$ is a prime only when $\ell = 2$.

Proof. Let $1 \leq m, m' \leq \ell, 1 \leq n, n' \leq \ell + 1$. Denote the numerator of $h_{m,n}$ by

\[
\bar{h}_{m,n} = (m(\ell + 2) - n(\ell + 1))^2 - 1,
\]

and denote

\[
D_{m,m',n,n'}^{\ell,\pm} = (m \pm m')(\ell + 2) - (n \pm n')(\ell + 1).
\]

Then

\[
\bar{h}_{m,n} - \bar{h}_{m',n'} = D_{m,m',n,n'}^{\ell,+}D_{m,m',n,n'}^{\ell,-}.
\]

Hence $\bar{h}_{m,n} = \bar{h}_{m',n'}$ if and only if $(i)$ $D_{m,m',n,n'}^{\ell,+=0}$ or $(ii)$ $D_{m,m',n,n'}^{\ell,-=0}$. Let us consider (i) and (ii) as equations over $\mathbb{Z}$ for now. Assume (i) holds. Since $\{\ell + 1, \ell + 2\}$ are relatively prime, we have $(\ell + 2) | (n + n')$ and $(\ell + 1) | (m + m')$, which further imply that $n + n' = \ell + 2$
and $m + m' = \ell + 1$, respectively (recall $m, m' \leq \ell$, and $n, n' \leq \ell + 1$). Similarly, (ii) holds imply $m = m'$ and $n = n'$.

Denote
\[
B_\ell = \left\{ \left| D_{m,n,m',n'}^{\ell,\ell+1} \right| \middle| 1 \leq m, m' \leq \ell, 1 \leq n, n' \leq \ell + 1 \right\}.
\]
(A set defined using $|D_{m,n,m',n'}^{\ell,\ell+1}|$ instead is equivalent to $B_\ell$ thanks to $\tilde{h}_{m,n} = \tilde{h}_{\ell+1-m,\ell+2-n}$.) Note the maximal value in $B_\ell$ is achieved at $(\ell + 1)(\ell + 2) - (1 + 1)(\ell + 1) = 2\ell^2 + \ell - 3$; so under the assumption $\text{char } \mathbb{F} > 2\ell^2 + \ell - 3$, all values of $\tilde{h}_{m,n}$ for $1 \leq n \leq m \leq \ell$ are distinct.

Conjecture 1 is known to hold when $c_2 = \frac{1}{2}$ (i.e., $\ell = 2$) \cite{2}. A basic observation of \cite{2} can be rephrased that the $\text{Vir}_{c_\ell}$-bad primes are $\{2, 7\}$. Note that the $\text{Vir}_{c_\ell}$-good primes $p = 3, 5$ are not detected by Proposition 5; see Remark 9 below for an explanation of these missing primes.

3.2. Supporting evidence for Conjecture 2

Note [5, Table 2] provides a list of reducible Weyl modules (detected by the approach therein) of lowest levels $\ell$ at a given prime $p$. Table 1 below is a somewhat novel look at the data provided in [5, Table 2], and it indicates the maximal known prime $p$ for which there exists a reducible Weyl module at level $\ell \geq 2$.

**Table 1. Maximal known primes $p$ for reducible Weyl modules at level $\ell$**

| $\ell$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|
| $p$   | 3 | 13 | 11 | 23 | 37 | 47 | 53 |
| $2\ell^2 + \ell - 3$ | 7 | 18 | 33 | 52 | 75 | 102 | 133 |

We note that $p < 2\ell^2 + \ell - 3$ for all $\ell$ in the table, and so Conjecture 2 is consistent with the results of [5].

**Remark 6.** Lai and the author have formulated a (conjectural) linkage principle; see [3, Conjecture 6.1]. If one can show that the weights in $X_\ell^+$ are minimal in the Bruhat order in each linkage class assuming $\text{char } \mathbb{F} > 2\ell^2 + \ell - 3$, then Conjecture 2 would follow (modulo the linkage principle conjecture).

3.3. Supporting evidence for Conjecture 3

The evidence from Table 1 (based on [3, Table 2]) for affine algebra $\widehat{\mathfrak{sl}}_2$ is remarkably consistent with the constraints from Proposition 5 for Virasoro algebra. Conjecture 3 offers a reasonable and conceptual way of explaining such a coincidence, and it helps to relate Conjecture 2 and Conjecture 1.

3.4. More precise bound on char $\mathbb{F}$ for Virasoro algebra.

One could make a conjecture (which strengthens Conjecture 1) that whenever char $\mathbb{F}$ is a $\text{Vir}_{c_\ell}$-good prime the minimal series $L_{c_\ell,h_{m,n}}$ are irreducible. One drawback of this stronger conjectural bound of char $\mathbb{F}$ is that a precise description of the set of $\text{Vir}_{c_\ell}$-bad primes is difficult for general $\ell \geq 2$, in contrast to the explicit though coarser bound in Proposition 5. We will provide some partial answer below. The $\text{Vir}_{c_\ell}$-bad primes for small values of $\ell$ can be computed by hand.
Example 7. The \( \text{Vir}_{c_2} \)-bad primes are \{2, 7\} (i.e., the primes \( \leq 2\ell^2 + \ell - 3 = 7 \) except 3, 5). The \( \text{Vir}_{c_3} \)-bad primes are \{2, 3, 7, 9, 13, 17\} (i.e., primes \( \leq 2\ell^2 + \ell - 3 = 18 \) except 5, 11). The \( \text{Vir}_{c_4} \)-bad primes are \{all primes \( \leq 2\ell^2 + \ell - 3 = 33\}\}\{5, 19, 29, 31\}, The \( \text{Vir}_{c_5} \)-bad primes are \{all primes \( \leq 2\ell^2 + \ell - 3 = 52\}\}\{7, 29, 41, 43, 47\}. The \( \text{Vir}_{c_6} \)-bad primes are \{all primes \( \leq 2\ell^2 + \ell - 3 = 75\}\}\{7, 41, 71, 73\}.

For integers \( a, b \) with \( a \leq b \), denote by \([a, b]\) the interval of integers between \( a \) and \( b \). Set

\[
B_\ell(a) = [\ell^2 + \ell + a(\ell + 2), \ell^2 + 2\ell - 1 + a(\ell + 1)], \quad \text{for } 0 \leq a \leq \ell - 1.
\]

Note \( k < k' \) for all \( k \in B_\ell(a), k' \in B_\ell(a') \) whenever \( a < a' \).

Proposition 8. The set \( B_\ell \) (3.2) is given by

\[
B_\ell = [1, \ell^2 + \ell - 2] \cup B_\ell(0) \cup B_\ell(1) \cup \cdots \cup B_\ell(\ell - 1).
\]

Proof. Recall \( B_\ell \) is defined using \(|D(m, n; m', n')|\); cf. (3.1). We first list the values of \((n+n')(\ell+1)\), for \( 1 \leq n, n' \leq \ell + 1 \), in an increasing row (there are \(2\ell + 1\) entries), and list the values of \((m+m')(\ell+2)\), for \( 1 \leq m, m' \leq \ell \), in an increasing column (there are \(2\ell - 1\) entries). By taking the absolute value of the difference of row and column entries, one produces a \((2\ell - 1) \times (2\ell + 1)\) matrix \( A \) whose entries are given by \(|D(m, n; m', n')| = |(m+m')(\ell+2) - (n+n')(\ell+1)|\). One observes that the matrix is symmetric under rotation by 180 degrees so we only need to consider the \((2\ell - 1) \times (\ell + 1)\) submatrix, denoted by \( D \), which consists of the \((\ell + 1)\) columns of \( A \). By listing the entries at the \( r \)th diagonals of \( D \) in the following order: \( r = 1, 0, 2, -1, 3, -2, \ldots, 2 - \ell, \ell \), one obtains exactly the interval \([1, \ell^2 + \ell - 2]\). (This is not so miraculous by noting the following: the values in each diagonal form a natural sequence, the last column of \( D \) is symmetric by flipping.) Now we are left with the lower \( \ell \) diagonals of \( D \), whose values are given by the \( \ell \) intervals \( B_\ell(a) \), for \( 0 \leq a \leq \ell - 1 \).

The easiest way for a reader to convince herself/himself of the above proof is to work out an example for one particular \( \ell \). For example let \( \ell = 5 \). Following the recipe in the proof above, we obtain the initial row and column vectors to be \((12, 18, 24, 30, 36, 42, \ldots, 72)\) and \((14, 21, 28, 35, 42, 49, 56, 63, 70)^\ell \). This leads to the following matrix

\[
D = \begin{bmatrix}
2 & 4 & 10 & 16 & 22 & 28 \\
9 & 3 & 3 & 9 & 15 & 21 \\
16 & 10 & 4 & 2 & 8 & 14 \\
23 & 17 & 11 & 5 & 1 & 7 \\
30 & 24 & 18 & 12 & 6 & 0 \\
37 & 31 & 25 & 19 & 13 & 7 \\
44 & 38 & 32 & 26 & 20 & 14 \\
51 & 45 & 39 & 33 & 27 & 21 \\
58 & 52 & 46 & 40 & 34 & 28 \\
\end{bmatrix}
\]

Then one sees clearly that the above recipe leads to the statement in the proposition. \( \Box \)

Note that \( B_\ell(\ell - 1) = \{ 2(\ell^2 + \ell - 1) \} \) consists of the largest integer in \( B_\ell \); moreover \( 2\ell^2 + \ell - 3 \in B_\ell(\ell - 2) \) is the largest integer in \( B_\ell' := B_\ell \backslash B_\ell(\ell - 1) \), or the second largest integer in \( B_\ell \). By definition of \( B_\ell \) and Proposition 3 we have

\[
(3.3) \quad \{ \text{Vir}_{c_\ell} \text{-bad primes} \} \subseteq [1, 2\ell^2 + \ell - 3] \cap B_\ell.
\]
One may regard (3.3) as a sharper form of the description of the bound in Proposition 5 thanks to the concrete description of the set $B_\ell$ in Proposition 8.

**Remark 9.** Note $(\ell + 1)^2 \notin B_\ell$ and $(\ell + 2)^2 \notin B_\ell$. Recall the denominator for $h_{m,n}$ is $4(\ell + 1)(\ell + 2)$. It follows that if either $\ell + 1$ or $\ell + 2$ happens to be a prime it must be a $\text{Vir}|_{c_\ell}$-good prime. In the above examples for small $\ell$, this prime happens to be the smallest $\text{Vir}|_{c_\ell}$-good prime, and the second $\text{Vir}|_{c_\ell}$-good prime happens to be $\ell^2 + \ell - 1$ (where $\ell^2 + \ell - 1$ happens to be a prime).

Introduce $G_\ell = [1, 2\ell^2 + \ell - 3]\backslash B_\ell$. If follows by definition that the set of $\text{Vir}|_{c_\ell}$-good primes $\leq 2\ell^2 + \ell - 3$ is contained in $G_\ell$ (recall all primes $> 2\ell^2 + \ell - 3$ are $\text{Vir}|_{c_\ell}$-good by Proposition 5). Set

$$G_\ell(a) = [\ell^2 + \ell - 1 + a(\ell + 1), \ell^2 + \ell - 1 + a(\ell + 2)], \quad \text{for } 0 \leq a \leq \ell - 1.$$ 

Note $k < k'$ for all $k \in G_\ell(a), k' \in G_\ell(a')$ whenever $a < a'$. Then one derives by definition and Proposition 8 that

$$G_\ell = G_\ell(0) \bigcup G_\ell(1) \bigcup \cdots \bigcup G_\ell(\ell - 1).$$

### 3.5. Discussions.

We indicate below several possible generalizations, provided the conjectures of this paper are valid.

One can ask the same question in Conjecture 1 for non-unitary minimal series of Virasoro algebra. An analysis analogous to Proposition 5 should allow one to find a conjectural bound on the characteristic of the field $\mathbb{F}$ over which the non-unitary minimal series of Virasoro algebra remain irreducible.

It would be interesting to generalize the conjectures of this paper to affine algebras of higher ranks and W-algebras (in particular for $\widehat{\mathfrak{sl}}_n$ and the corresponding $W$-algebra $W_n$). One could try to find a conjectural bound on $\text{char } \mathbb{F}$ under which the Weyl modules are irreducible, by analyzing the highest weights of the unitary minimal series of the W-algebras. One can see by a similar method as in Proposition 5 this bound should be some quadratic polynomial on the level and the rank. One can also derive some more evidence from [5, Theorem 5.10] on $\text{char } \mathbb{F}$ for reducible Weyl modules of affine Lie algebras of higher rank (say, $\widehat{\mathfrak{sl}}_3$). Then one would check if the bound arising from W-algebra minimal series is compatible with the bound from affine Lie algebras.

Instead of Virasoro algebra, one can consider the super-Virasoro algebra, also known as Neveu-Schwarz (and Ramond) algebras, and its unitary minimal series. A similar analysis can lead to a conjectural bound on $\text{char } \mathbb{F}$ under which the minimal series of the super-Virasoro algebra are irreducible. In the same way that affine Lie algebra $\widehat{\mathfrak{sl}}_2$ is related to Virasoro algebra via the GKO construction, the Neveu-Schwarz algebra is related to the affine Lie superalgebra $\widehat{\mathfrak{osp}}_{1|2}$. So we can give a conjectural bound on $\text{char } \mathbb{F}$ (relative to the levels) under which the Weyl modules of $\widehat{\mathfrak{osp}}_{1|2}$ are irreducible.

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