Super Poincaré inequality for a dynamic model for the two-parameter Dirichlet process

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Abstract

We investigate the dynamic model for the two-parameter Dirichlet process introduced in [8]. To establish the super Poincaré inequality for the projection measure of the two-parameter Dirichlet process, the main difficulty is that the diffusion coefficients are degenerate. We have two methods to overcome the difficulty. One is the localization method in [17], the other one is the perturbation method in [16]. As a consequence, we obtain the super Poincaré inequality for the two-parameter Dirichlet process when the partition number of the projection measure is finite. Furthermore, if the partition number is infinite, the super Poincaré inequality doesn’t hold.

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1 Introduction

The two-parameter Dirichlet process is the natural generalization of the single-parameter Dirichlet process, which first appeared in the context of Bayesian statistics. And both two-parameter Dirichlet process and single-parameter Dirichlet process are pure atomic random measure.

For any $0 \leq \alpha < 1$ and $\theta > -\alpha$, let $\{U_k\}_{k \geq 1}$ be a sequence of independent random variables such that $U_k$ has $Beta(1 - \alpha, \theta + k\alpha)$ distribution. Set

$$V_{1}^{\alpha,\theta} = U_1, \quad V_{n}^{\alpha,\theta} = (1 - U_1) \cdots (1 - U_{n-1})U_n, \quad n \geq 2.$$

The distribution of $(V_{1}^{\alpha,\theta}, V_{2}^{\alpha,\theta}, \ldots)$ is called two-parameter GEM distribution, denoted by $GEM(\alpha, \theta)$. When $\alpha = 0$, $GEM(0, \theta)$ is the well know GEM distribution. Let $P(\alpha, \theta) =$ *

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(\rho_1, \rho_2, \cdots) denote \((V_1^{\alpha,\theta}, V_2^{\alpha,\theta}, \cdots)\) in descending order, the law of \(P(\alpha, \theta)\) is called two-parameter Poisson-Dirichlet distribution. When \(\alpha = 0\), \(P(0, \theta)\) is the Poisson-Dirichlet distribution, which was introduced by Kingman in \([10]\) to describe the distribution of gene frequencies in a large \(k\)-th most frequency locus. For \(S = \mathbb{N}\), and a sequence of independent identically distributed \(S\)-value random variables \(\xi_1, \xi_2, \cdots\) with common distribution \(\nu_0\) on \(S\), which are independent of \(P(\alpha, \theta)\), let

\[
\Theta_{\alpha,\theta,\nu_0} = \sum_{i=1}^{\infty} \rho_i \delta_{\xi_i},
\]

the distribution of \(\Theta_{\alpha,\theta,\nu_0}\) is called two-parameter Dirichlet process and denoted by \(\Pi_{\alpha,\theta,\nu_0}\). Both \(GEM(\alpha, \theta)\) and \(PD(\alpha, \theta)\) carry the information only on proportions while \(\Pi_{\alpha,\theta,\nu_0}\) contains information on both proportions and types or labels.

In the context of population genetics, the Dirichlet process appears as approximations to the equilibrium behavior of certain large populations evolving under the influence of mutation and random genetic drift. That is, the labeled infinite-many-neutral-alleles model (Fleming-Viot process with neutral parent independent mutation) is time-reversible with the Dirichlet process \(\Pi_{0,\theta,\nu_0}\). Based on \([8]\), when the range of parameters is \(\alpha = \frac{1}{2}, \theta > -\frac{1}{2}\), the Fleming-Viot process has a two-parameter analogue which is time-reversible with the two-parameter Dirichlet process \(\Pi_{\alpha,\theta,\nu_0}\).

We consider the bilinear form

\[
(1.1) \quad \begin{cases}
\mathcal{E}(F, G) = \frac{1}{2} \int_{\mathcal{P}_1(\mathbb{N})} \langle \nabla F(\mu), \nabla G(\mu) \rangle \mu \Pi_{\alpha,\theta,\nu_0}(d\mu), \ F, G \in \mathcal{F}; \\
\mathcal{F} = \{F(\mu) = f(\mu(1), \cdots, \mu(d)): f \in C^\infty(\mathbb{R}^d), d \geq 1\}.
\end{cases}
\]

According to \([8], \text{Theorem 2.1}\), the bilinear form is closable on \(L^2(\mathcal{P}_1(\mathbb{N}), \Pi_{\alpha,\theta,\nu_0})\) and its closure \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is a quasi-regular Dirichlet form. The diffusion process associated with \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is reversible with the stationary distribution \(\Pi_{\alpha,\theta,\nu_0}\). Denote by \((L, \mathcal{D}(L))\) the generator of \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) on \(L^2(\mathcal{P}_1(\mathbb{N}), \Pi_{\alpha,\theta,\nu_0})\). By \([8], \text{Theorem 2.2}\), \(\forall F \in \mathcal{F}_d\), where

\[
(1.2) \quad \mathcal{F}_d = \{F(\mu) = f(\mu(1), \cdots, \mu(d)): f \in C^\infty(\mathbb{R}^d)\}, d \geq 1,
\]

we have

\[
LF(\mu) = \frac{1}{2} \sum_{i,j=1}^{d} \mu(i)(\delta_{ij} - \mu(j))\partial_i f(\mu(1), \cdots, \mu(d))
\]

\[
+ \frac{1}{2} \sum_{i=1}^{d} \left( - \frac{1}{2} - \theta \mu(i) + \frac{1}{2} B_i(\mu) \right) \partial_i f(\mu(1), \cdots, \mu(d)),
\]

where

\[
B_i(\mu) = \lim_{d \to \infty} \frac{(d + 1)(\nu_0(i))^2}{\mu(i)} \sum_{i=1}^{d} \frac{(\nu_0(i))^2}{\mu(i)} + \frac{\nu_0(d+1)}{1 - \sum_{i=1}^{d} \mu(i)}
\]

exists in \(L^2(\mathcal{P}(\mathbb{N}), \Pi_{\alpha,\theta,\nu_0})\).
Below, we consider the projection case. For any \(d \geq 2\), we define

\[
\Delta^{(d)} := \{x \in [0, 1]^d : \sum_{1 \leq i \leq d} x_i \leq 1\}, \quad x_{d+1} = 1 - \sum_{1 \leq i \leq d} x_i.
\]

Denote \(p_i = \nu_0(i), 1 \leq i \leq d\), and \(p_{d+1} = 1 - \sum_{i=1}^d p_i\). Let

\[
\mu^{(d)}(dx) = \frac{\Gamma(\theta + \frac{d+1}{2}) \prod_{i=1}^{d+1} p_i}{\pi^{\frac{d}{2}} \Gamma(\theta + \frac{1}{2}) \left(\sum_{i=1}^{d+1} p_i^2\right)^{\theta + \frac{d+1}{2}}} dx = \rho(x) dx
\]

on the set \(\Delta^{(d)}\). Define the operator

\[
L^{(d)} f(x) = \frac{1}{2} \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j)(\partial_{ij} f)(x) + \frac{1}{2} \sum_{i=1}^{d} \left( -\frac{1}{2} - \theta x_i + \frac{(\theta + \frac{d+1}{2}) p_i^2}{\sum_{i=1}^{d+1} p_i^2} \right) \partial_i f(x),
\]

\(f \in C^2(\Delta^{(d)})\).

Dirichlet form

\[
\mathcal{E}^{(d)}(f, g) := \frac{1}{2} \mu^{(d)} \left( \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j) \partial_i f \partial_j g \right)
\]

with domain \(\mathcal{D}(\mathcal{E}^{(d)})\) being the closure of \(C^1(\Delta^{(d)})\). We consider the map

\[
\gamma_d : \mathcal{P}_1(\mathbb{N}) \to \Delta^{(d)},
\]

\[
\mu \to \gamma_d(\mu) = (\mu(1), \ldots, \mu(d)).
\]

By [2, Theorem 3.1], we have

\[
\Pi_{\alpha, \theta, \nu_0} \circ \gamma^{-1}_d = \mu^{(d)}.
\]

That is, \(\forall F, G \in \mathcal{F}_d\), we have

\[
\Pi_{\alpha, \theta, \nu_0}(h(F)) = \mu^{(d)}(h(f)), \forall h \in C^\infty(\mathbb{R}),
\]

\[
\int \frac{1}{2} (\nabla F(\mu), \nabla G(\mu)) \mu_{\alpha, \theta, \nu_0}(d\mu) = \frac{1}{2} \int_{\Delta^{(d)}} \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j)(\partial_i f)(x)(\partial_j g)(x) \mu^{(d)}(dx).
\]

This is a analogue of single-parameter Dirichlet process whose projection on finite partition of \(S\) is Dirichlet distribution.
1.1 Functional inequalities

In general, let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be a conservative symmetric Dirichlet form on \(L^2(\mu)\) for some probability space \((E, \mathcal{F}, \mu)\), let \((L, \mathcal{D}(L))\) be the associated Dirichlet operator, and let \(P_t := e^{tL} , t \geq 0\), be the Markov semigroup. The following is a brief summary from \cite{16} for the Poincaré inequality, log-Sobolev inequality, super Poincaré inequality, F-Sobolev inequality, see also \cite{11} and references within.

Firstly, we consider the Poincaré inequality
\[
\mu(f^2) \leq c\mathcal{E}(f, f) + \mu(|f|^2) , \quad f \in \mathcal{D}(\mathcal{E}).
\]
The Poincaré inequality is equivalent to the \(L^2\)-exponential convergence of \(P_t\):
\[
\|P_t f\|_{L^2(\mu)} \leq e^{-ct} \|f\|_{L^2(\mu)} , \quad t \geq 0 , \quad f \in L^2(\mu) , \mu(f) = 0.
\]
Let \(F \in C(0, \infty)\) be an increasing function such that \(\sup_{r \in [0,1]} |r F(r)| < \infty\) and \(F(\infty) := \lim_{r \to \infty} F(r) = \infty\). We say the F-Sobolev inequality holds if there exist two constants \(C_1 > 0, C_2 \geq 0\) such that
\[
\mu(f^2 F(f^2)) \leq C_1 \mathcal{E}(f, f) + C_2, \quad f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1.
\]
In particular, if \(F = \log\), we call \((1.6)\) the log-Sobolev inequality. The F-Sobolev inequality is equivalent to the super Poincaré inequality. Later, we say that \((\mathcal{E}, \mu)\) satisfies the super Poincaré inequality with rate function \(\beta : (0, \infty) \to (0, \infty)\), if
\[
\mu(f^2) \leq r \mathcal{E}(f, f) + \beta(r) \mu(|f|^2) , \quad r > 0 , \quad f \in \mathcal{D}(\mathcal{E}).
\]
This inequality is equivalent to the uniform integrability of \(P_t\), i.e. \(P_t\) has zero tail norm:
\[
\|P_t\|_{\text{tail}} := \lim_{R \to \infty} \sup_{\mu(f^2) \leq 1} \mu((P_t f)^2 1_{\{|P_t f| \geq R\}}) = 0 , \quad t > 0.
\]
When \(P_t\) has a heat kernel with respect to \(\mu\), it is also equivalent to the absence of the essential spectrum of \(L\) (i.e. the spectrum of \(L\) is purely discrete). The super Poincaré inequality generalizes the classical Sobolev/Nash type inequalities. For instance, when \(\text{gap}(L) > 0\), \((1.7)\) with \(\beta(r) = e^{c (1+r^{-1})}\) for some \(c > 0\) is equivalent to the log-Sobolev inequality for some constant \(C > 0\); while for a constant \(p > 0\), \((1.7)\) with \(\beta(r) = c(1+r^{-p})\) holds for some \(c > 0\) if and only if the Nash inequality
\[
\mu(f^2) \leq C\mathcal{E}(f, f)^{\frac{p}{p+1}} \mu(|f|)^{\frac{2}{p+1}} , \quad f \in \mathcal{D}(\mathcal{E}), \mu(f) = 0
\]
holds for some constant \(C > 0\), they are also equivalent to
\[
\|P_t - \mu\|_{L^1(\mu) \to L^{\infty}(\mu)} \leq \frac{c'}{(t \wedge 1)^p} e^{-\text{gap}(L)t} , \quad t > 0.
\]
The later implies the hypercontractivity of \(P_t\), and hence the log-Sobolev inequality for some constant \(C > 0\).

\cite{13} established log-Sobolev inequality for the projection measure of the single-parameter Dirichlet process, and then proved that the Poincaré inequality for the single-parameter Dirichlet process holds but the super Poincaré inequality doesn’t hold. \cite{17} established the super Poincaré inequality for the projection measure of the single-parameter Dirichlet process. In this paper, we follow the line of thinking in \cite{13} and the main result is the following.
1.2 Main result

Let \( \nu_0 \) be the probability measure on type space \( S = \mathbb{N} \) in the definition of the two parameter Dirichlet process. Denote \( d = \sharp \{ i \in S, \nu_0(i) > 0 \} \).

**Theorem 1.1.** If \( d < \infty \), then there exist \( c > 0 \) such that the super Poincaré inequality

\[
\Pi_{\alpha, \theta, \nu_0}(F^2) \leq r\mathcal{E}(F, F) + cr^{-\frac{1}{2}(\theta + \frac{d}{2})(2d + 1) - 1}\Pi_{\alpha, \theta, \nu_0}(|F|^2), \quad r > 0, \quad F \in \mathcal{F}_d
\]

holds, where \( \mathcal{F}_d \) defined in (1.2).

**Theorem 1.2.** If \( d = \infty \), then the super Poincaré inequality doesn’t hold.

There is an question we haven’t finished: if \( d = \infty \), does the Poincaré inequality for \( \Pi_{\alpha, \theta, \nu_0} \) hold?

To establish the super Poincaré inequality for the measure-valued process, we firstly establish the super Poincaré inequality for the projection measure of \( \Pi_{\alpha, \theta, \nu_0} \) in Section 2, then prove Theorem 1.1 and Theorem 1.2 in Section 3.

2 Super Poincaré inequality for the projection measure of \( \Pi_{\alpha, \theta, \nu_0} \)

To establish the super Poincaré inequality for the projection measure of \( \Pi_{\alpha, \theta, \nu_0} \) with an explicit rate function \( \beta \), the main difficulty comes from the degeneracy of the diffusion coefficient on the boundary

\[
\partial \Delta^{(d)} = \left\{ x = (x_i)_{1 \leq i \leq d} \in \Delta^{(d)} : \min\{x_i : 1 \leq i \leq d + 1\} = 0 \right\}, \quad x_{d+1} := 1 - \sum_{i=1}^{d} x_i.
\]

We have two methods to establish the super Poincaré inequality. Firstly, from \cite{17}, we have known the super Poincaré inequality for another probability measure \( \tilde{\mu}^{(d)} \) already, then we get the local super Poincaré inequality for the measure \( \mu^{(d)} \). By \cite{17} Theorem 2.1, Lemma 3.4], we establish the super Poincaré inequality for \( \mu^{(d)} \). Secondly, from \cite{17}, we have known the super Poincaré inequality for another probability measure \( \tilde{\mu}^{(d)} \) already, by the perturbation result for the super Poincaré inequality \cite{16} Theorem 3.4.7], we establish the super Poincaré inequality for \( \mu^{(d)} \).

**Assumption (A):** Let \( (E, \mathcal{F}, \mu) \) be a separable complete probability space, and let \((\mathcal{E}, \mathcal{D}(\mathcal{E})) \) be a conservative symmetric local Dirichlet form on \( L^2(\mu) \) as the closure of

\[
\mathcal{E}(f, g) = \mu(\Gamma(f, g)), \quad f, g \in \mathcal{D}(\Gamma),
\]

where \( \Gamma : \mathcal{D}(\Gamma) \times \mathcal{D}(\Gamma) \rightarrow \mathcal{B}(E) \) is a positive definite symmetric bilinear mapping, \( \mathcal{B}(E) \) is the set of all \( \mu \)-a.e. finite measurable real functions on \( E \), \( \mathcal{D}(\Gamma) \) is a sub-algebra of \( \mathcal{B}(E) \), and \( \mathcal{D}_0(\Gamma) := \{ f \in \mathcal{D}(\Gamma) : f^2, \Gamma(f, f) \in L^1(\mu) \} \) such that

(a) \( \mathcal{D}_0(\Gamma) \) is dense in \( L^2(\mu) \).
(b) \( \mathcal{D}(\Gamma) \) is closed under combinations with \( \psi \in C([−\infty, \infty]) \) such that \( \psi \) is \( C^1 \) in \( \mathbb{R} \) and \( \psi' \) has compact support, and \( \Gamma(\psi \circ f, g) = \psi'(f)\Gamma(f, g) \) \( \mu \)-a.e. for \( f, g \in \mathcal{D}(\Gamma) \).

(c) \( \Gamma(fg, h) = g\Gamma(f, h) + f\Gamma(g, h) \) \( \mu \)-a.e. for \( f, g, h \in \mathcal{D}(\Gamma) \).

Let \( \phi \in \mathcal{D}(\Gamma) \) be an unbounded nonnegative function and let

\[
h(s) := \sup_{\phi \leq s} \Gamma(\phi, \phi), s \geq s_0 > 0.
\]

(2.1)

\[
D_s := \{ \phi \leq s \}, \quad s \geq s_0 > 0,
\]

where \( \sup_{\emptyset} = 0 \) by convention.

(2.2) \( 0 < \lambda(s) := \inf \{ \mathcal{E}(f, f) : \mu(f^2) = 1, f|_{D_s} = 0 \} \uparrow \infty \) as \( s \uparrow \infty \).

To verify Assumption \((A)\), we take

\[
\mathcal{D}(\Gamma) = \left\{ f \in C(\Delta^{(d)}; [−\infty, \infty]) : f \text{ is finite and } C^1 \text{ in } \Delta^{(d)} \setminus \{ x_i = 0, \ 1 \leq i \leq d + 1 \} \right\},
\]

and let

\[
\Gamma(f, g)(x) = 1_{\{x_k > 0, \ 1 \leq k \leq d\}} \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j)(\partial_i f)(x)(\partial_j g)(x), \ f, g \in \mathcal{D}(\Gamma).
\]

Obviously, conditions (a)-(c) in Assumption \((A)\) hold.

We take

(2.3)

\[
\phi(x) = \sum_{1 \leq i \leq d+1} x_i^{-2}, \quad x = (x_i)_{1 \leq i \leq d} \in \Delta^{(d)}.
\]

Then for \( s \geq s_0 > 0 \),

(2.4)

\[
D_s := \{ \phi \leq s \} = \{ x \in \Delta^{(d)} : \sum_{1 \leq i \leq d+1} x_i^{-2} \leq s \}.
\]

\[
h(s) := \sup_{\phi \leq s} \Gamma(\phi, \phi)
\]

\[
= \sup_{\phi \leq s} \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j)(\partial_i \phi)(x)(\partial_j \phi)(x)
\]

\[
= \sup_{\phi \leq s} \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j) \left( \frac{-2}{x_i^3} + \frac{2}{x_{d+1}^3} \right) \left( \frac{-2}{x_j^3} + \frac{2}{x_{d+1}^3} \right)
\]

\[
\leq c_3 s^{\frac{3}{2}}.
\]

In the following subsection, we estimate \( \lambda(s) \).
2.1 Estimate on $\lambda(s)$

Let $\lambda(s) = \inf \{ \mathcal{E}^{(d)}(f, f) : f \in C^1(\Delta^{(d)}), \mu^{(d)}(f^2) = 1, f|_{D_s} = 0 \}$. We will adopt the following Cheeger inequality to estimate $\lambda(s)$. Let

$$\partial D_s = \{ x \in \Delta^{(d)} : \exists 1 \leq i \leq d + 1, x_i = b_i s^{-\frac{1}{d}} \}, \quad s \geq s_0 > 0, b_i > 0.$$  

**Lemma 2.1.** If there exists a function $f \in C^2(\Delta^{(N)} \setminus \{ x_i = 0, \ 1 \leq i \leq d + 1 \})$ such that

$$|\sigma \nabla f(x)| := \left( \sum_{i=1}^{d} \left( \sum_{j=1}^{d} x_i(\delta_{ij} - x_j)(\partial_j f(x)) \right)^2 \right)^{\frac{1}{2}} \leq a_1, \quad |L^{(d)} f(x)| \geq a_2, \quad x \in D_s^c$$

holds for some constants $a_1, a_2 > 0$, and that

$$\lim_{r \to \infty} \int_{x \in \partial D_r} \left[ \sum_{i=1}^{d} \left( \sum_{j=1}^{d} x_i(\delta_{ij} - x_j)(\partial_j f(x)) \right)^2 \right]^{\frac{1}{2}} \rho(x)dA = 0,$$

then

$$\lambda(s) \geq \frac{a_2^2}{4a_1}.$$

**Proof.** By (2.5), we assume that $L^{(d)} f|_{D_s^c} \geq a_2$, otherwise simply use $-f$ replacing $f$. Let $\sigma(x) = \{ x_i(\delta_{ij} - x_j) \}_{1 \leq i, j \leq d}$. For any nonnegative $g \in C^1(\Delta^{(d)})$ with $g|_{D_s} = 0$, we have $g|_{\partial D_s} = 0$. So that by integration by parts formula,

$$\mu^{(d)}(g) \leq \mu^{(d)}(gL^{(d)} f) = \lim_{r \to \infty} \int_{D_r \setminus D_s} (\rho gL^{(d)} f)(x)dx$$

$$\leq -\mu^{(d)}(\langle \sigma \nabla g, \sigma \nabla f \rangle) + \| g \|_\infty \limsup_{r \to \infty} \int_{\partial D_r} \left[ \sum_{i=1}^{d} \left( \sum_{j=1}^{d} x_i(\delta_{ij} - x_j)(\partial_j f(x)) \right)^2 \right]^{\frac{1}{2}} \rho(x)dA,$$

where $A$ is the area measure on $\partial D_r$ induced by the Lebesgue measure. Combining this with (2.5), (2.6) and (2.7), we obtain

$$a_2 \mu^{(d)}(g) \leq |\mu^{(d)}(\langle \sigma \nabla g, \sigma \nabla f \rangle)| \leq \sqrt{a_1} \mu^{(d)}(|\sigma \nabla g|).$$

Therefore, for any $g \in C^1(\Delta^{(d)})$ with $g|_{D_s} = 0$,

$$\mu^{(d)}(g^2) \leq \frac{\sqrt{a_1}}{a_2} \mu^{(d)}(|\sigma \nabla g|^2) \leq \frac{2\sqrt{a_1}}{a_2} \sqrt{\mu^{(d)}(g^2) \mu^{(d)}(|\sigma \nabla g|^2)}.$$

Noting that $\mu^{(d)}(|\sigma \nabla g|^2) = \mathcal{E}^{(d)}(g, g)$, we arrive at

$$\mu^{(d)}(g^2) \leq \frac{4a_1}{a_2^2} \mathcal{E}^{(d)}(g, g), \quad g \in C^1_b(\Delta^{(d)}), g|_{D_s} = 0,$$

which finishes the proof. □
Lemma 2.2. For the operator \( L^{(d)} \), there exist constants \( s_0, c_6 > 0 \) such that
\[
\lambda(s) \geq c_6 s^{\frac{7}{8}}, \quad s \geq s_0 > 0.
\]

Proof. Take
\[
f(x) = \sum_{1 \leq i \leq d} x_i^d, \quad x \in \Delta^{(d)}.
\]

Then \( \forall x \in D^c_s \),
\[
(2.8) \quad |\sigma \nabla f(x)| := \sqrt{\sum_{i=1}^{d} \left( \sum_{j=1}^{d} x_i (\delta_{ij} - x_j) (\partial_j f)(x) \right)^2} \leq \left[ \sum_{i=1}^{d} \left( \frac{1}{4} x_i^d - \frac{1}{4} x_i \sum_{j=1}^{d} x_j^d \right) \right]^{\frac{1}{2}} \leq c_4 s^{-\frac{1}{8}}.
\]

\[
|L^{(d)} f(x)| = \left| \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j) (\partial_{ij} f)(x) \right| \geq c_5 s^{\frac{3}{8}}
\]

Denote
\[
\partial D_{r,j} = \{ x | x_j = b_j r^{-\frac{1}{2}} \}
\]

\[
\int_{\partial D_r} \frac{\prod_{i=1}^{d+1} x_i^{-\frac{3}{2}}}{\left( \sum_{i=1}^{d+1} p_i^2 x_i \right)^{\theta + d + \frac{1}{2}}} \rho(x) dA
\]

\[
= \sum_{j=1}^{d+1} \int_{\partial D_{r,j}} \frac{\prod_{i \neq j, 1 \leq i \leq d+1} x_i^{-\frac{3}{2}}}{\left( \sum_{i \neq j, 1 \leq i \leq d+1} p_i^2 x_i \right)^{\theta + d + \frac{1}{2}}} dA
\]

\[
= \sum_{i=1}^{d} (1 - \sum_{j \neq i} b_j r^{-\frac{1}{2}})^{\theta + d - \frac{1}{2}} \int_{\Delta^{(d-1)}} \frac{\prod_{i=1}^{d-1} x_i^{-\frac{3}{2}}}{\left( \sum_{1 \leq i \leq d-1} p_i^2 x_i \right)^{\theta + d + \frac{1}{2}}} dx
\]

is bounded, so
\[
\limsup_{r \to \infty} \int_{\partial D_r} \sum_{i=1}^{d} \left( \sum_{j=1}^{d} x_i (\delta_{ij} - x_j) (\partial_j f)(x) \right)^2 \rho(x) dA = 0.
\]

We derive from Lemma 2.1 that
\[
\lambda(s) \geq \frac{c_5^2 s^{\frac{7}{8}}}{4c_4 s^{\frac{3}{8}}} = c_6 s^{\frac{7}{8}}, \quad s \geq s_0 > 0.
\]
2.2 Localization method

**Theorem 2.3.** Let $\mu^{(d)}$ defined as [1.3], then the super Poincaré inequality
\begin{equation}
\mu^{(d)}(f^2) \leq r\mathcal{E}^{(d)}(f, f) + \beta(r)\mu^{(d)}(|f|^2), \quad r > 0, f \in \mathcal{D}(\mathcal{E}^{(d)})
\end{equation}
holds with $\beta(r) = c_{13}(1 + r^{-1})^{(2\theta+d+\theta+\frac{d}{2}-1)+\frac{4}{d}}, \ c_{13} > 0$.

**Proof.** We know
\[
\mu^{(d)} = \frac{\Gamma(\theta + \frac{d+1}{2}) \prod_{i=1}^{d+1} p_i \prod_{i=1}^{d+1} x_i^{-\frac{d}{2}}}{\pi^\frac{d}{2} \Gamma(\theta + \frac{1}{2})(\sum_{i=1}^{d+1} \frac{p_i^2}{x_i})^{\theta + \frac{d+1}{2}}} dx.
\]
Denote
\[
\tilde{\mu}^{(d)} = \frac{\Gamma(|\alpha_1|)}{\prod_{1 \leq i \leq d+1} \Gamma(\alpha_i)} (1 - |x_1|)^{\alpha_{d+1}-1} \prod_{1 \leq i \leq d} x_i^{\alpha_i-1} dx.
\]
We set
\[
\tilde{\mu}^{(d)} = \mu^{(d)}(e^W),
\]
so
\[
e^W = \frac{\pi^\frac{d}{2} \frac{\Gamma(\theta + \frac{1}{2}) \Gamma(|\alpha_1|) \prod_{i=1}^{d+1} p_i \prod_{1 \leq i \leq d+1} \Gamma(\alpha_i)}{\Gamma(\theta + \frac{d+1}{2}) \prod_{i=1}^{d+1} p_i \prod_{1 \leq i \leq d+1} \Gamma(\alpha_i)}} x_i^{\alpha_i+\frac{1}{2}} dx.
\]
When $\alpha_i \geq \theta + \frac{d}{2}, 1 \leq i \leq d+1$, there are constants $C_1$ and $C_2$ such that
\[
C_1s^{-\sum_{i=1}^{d+1}(\alpha_i-(\theta+\frac{d}{2}))} \leq e^W \leq C_2.
\]
So the local super Poincaré inequality becomes
\[
\mu^{(d)}(f^2) \leq C_3 r s^{-\sum_{i=1}^{d+1}(\alpha_i-(\theta+\frac{d}{2}))} \mathcal{E}^{(d)}(f, f) + C_4 \beta_s(r) s^{-\sum_{i=1}^{d+1}(\alpha_i-(\theta+\frac{d}{2}))} \mu^{(d)}(|f|^2),
\]
where $\beta_s(v) = c_{14}(1 + v^{-1})^{p'}$. When $\alpha_i = \theta + \frac{d}{2}, 1 \leq i \leq d + 1$, we get the smallest $p' = \lceil (2\theta + d + \theta + \frac{d}{2} - 1) \rceil$. By [17] Theorem 2.1, without the condition that $h(s) < \infty$, we know the super Poincaré inequality holds with
\[
\beta(r) = c_{13}(1 + r^{-1})^{(p'+\frac{4}{d})}.
\]
\[\Box\]

2.3 Perturbation method

Below is the perturbation theorem which is similar to [16] Theorem 3.4.7.

**Theorem 2.4.** Under Assumption (A), let $W$ is bounded on $\{\phi \leq r\}$ for any $r > 0$. Let $S(W) \in \mathcal{B}$ be such that for any nonnegative $f \in \mathcal{D}(\Gamma)$ with $\text{supp} f \subset \{\phi \leq r\}$ for some $r > 0$, one has
\[
\int \Gamma(f, W) d\mu \geq - \int f S(W) d\mu \in \mathbb{R}.
\]
Put
\[ \varphi(r) = \sup \{ e^W : \phi \leq r \}, \]
\[ \psi(r) = \frac{1}{4} \sup \{ \Gamma(W, W) + 2S(W) : \phi \leq r \}. \]

If there exist \( c_1, p > 0 \) such that
\[ (2.11) \quad \mu(f^2e^W) \leq r\mu(e^W\Gamma(f, f)) + c_1(1 + r^{-1})p\mu(e^W|f|^2), \]
then \( (1.7) \) holds with
\[ \beta(r) = \frac{\sqrt[10]{\varphi(s)}}{2(1+\varphi(s))^{2}} \cdot \left( 1 + \frac{2(1+\varphi(s))^{2}h(3s) + \psi(3s)}{r(1 - \frac{2(1+\varphi(s))^{2}h(2s)}{\lambda(2s)s^{2}})} \right)^{\frac{p}{2}} \cdot \varphi(3s), \]
where \( s = c_2r^{-\frac{8}{7}}. \)

Proof. Let \( S(W) \in \mathcal{B} \) be such that for any nonnegative \( f \in \mathcal{D}(\Gamma) \) with \( \text{supp}f \subset \{ \phi \leq r \} \) for some \( r > 0 \), one has
\[ \int \Gamma(f, W)d\mu \geq -\int fS(W)d\mu \in \mathbb{R}. \]

Put
\[ \varphi(r) = \sup \{ e^W : \phi \leq r \}, \]
\[ \psi(r) = \frac{1}{4} \sup \{ \Gamma(W, W) + 2S(W) : \phi \leq r \}. \]

It suffices to consider \( f \in \mathcal{D}_0(\Gamma) \). For any \( s \geq s_0 \) and small \( \varepsilon \in (0, 1) \), let \( \varphi_i \in C^1([0, \infty]) \) with \( 0 \leq \varphi_i \leq 1, |\varphi_i'(s)| \leq (1 + \varepsilon)s^{-1}, i = 1, 2 \) such that
\[ \varphi_1|_{[0,s]} = 0, \quad \varphi_1|_{[s,\infty]} = 1; \quad \varphi_2|_{[0,2s]} = 1, \quad \varphi_2|_{[2s,\infty]} = 0. \]

Let \( f_i = f \cdot \varphi_i \circ \phi, 1 \leq i \leq 2 \). Then \( f^2 \leq f_1^2 + f_2^2 \) and by conditions (b) and (c),
\[ \Gamma(f_1, f) \leq 2\Gamma(f, f) + 2(1+\varepsilon)^2f^2s^{-2}h(2s) \leq 2\Gamma(f, f) + \frac{2(1+\varepsilon)^2h(2s)}{s^2}f^2, \]
\[ \Gamma(f_2, f) \leq 2\Gamma(f, f) + \frac{2(1+\varepsilon)^2h(3s)}{s^2}f^2. \]

In particular, \( f_1, f_2 \in \mathcal{D}_0(\Gamma) \subset \mathcal{D}(\mathcal{E}). \)

\[ (2.12) \quad \mu(f_i^2) \leq \frac{1}{\lambda(s)}\mu\left( 2\Gamma(f, f) + \frac{2(1+\varepsilon)^2h(2s)}{s^2}f^2 \right) \]
Thus, we put it and (2.12) together, then

\[ \mu(f_2^2) \leq r \mu \left( \Gamma(f_2, f_2) - \frac{1}{2} \Gamma((f_2)^2, W) + \frac{1}{4} (f_2)^2 \Gamma(W, W) \right) + c_1 \left( 1 + r^{-1} \right)^p \varphi(3s) \mu(|f|)^2 \]

\[ \leq 2r \mu(\Gamma(f, f)) + (1 + \varepsilon)^2 f^2 s^{-2} h(3s) + \frac{r}{4} \mu((f_2)^2 \{ \Gamma(W, W) + 2S(W) \}) \]

\[ + c_1 \left( 1 + r^{-1} \right)^p \varphi(3s) \mu(|f|)^2 \]

\[ \leq 2r \mu(\Gamma(f, f)) + r [2(1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s)] \mu(f^2) + c_1 \left( 1 + r^{-1} \right)^p \varphi(3s) \mu(|f|)^2. \]

We put it and (2.12) together, then

\[ \mu(f^2) \leq \mu(f_1^2) + \mu(f_2^2) \]

\[ \leq \frac{1}{\lambda(s)} \mu \left( 2 \Gamma(f, f) + \frac{2(1 + \varepsilon)^2 h(2s)}{s^2} f^2 \right) + 2r \mu(\Gamma(f, f)) + r [(1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s)] \mu(f^2) \]

\[ + c_1 \left( 1 + r^{-1} \right)^p \varphi(3s) \mu(|f|)^2 \]

\[ \leq \left( \frac{2}{\lambda(s)} + 2r \right) \mu(\Gamma(f, f)) + \left\{ \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2} + r \left[ 2(1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s) \right] \right\} \mu(f^2) \]

\[ + c_1 \left( 1 + r^{-1} \right)^p \varphi(3s) \mu(|f|)^2. \]

We set

\[ \varepsilon = \frac{2r + \frac{2}{\lambda(s)}}{1 - \left\{ \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2} + r \left[ 2(1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s) \right] \right\}}, \]

we choose

\[ \lambda(s) = r^{-1}, \]

so

\[ r = \frac{\varepsilon \left( 1 - \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2} \right)}{4 + \varepsilon \left[ (1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s) \right]}. \]

Thus,

\[ \beta(\varepsilon) = \frac{c_1}{1 - \left\{ \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2} + \varepsilon \left[ (1 - \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2}) \right] \left[ (1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s) \right] \right\}} \cdot \left( 1 + \frac{4 + \varepsilon \left[ (1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s) \right]}{\varepsilon \left( 1 - \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2} \right)} \right)^p \varphi(3s), \]

where \( s = c_2 \varepsilon^{\frac{1}{d}}. \)

\[ \square \]

**Theorem 2.5.** Let \( \mu^{(d)} \) defined as (1.3), then the super Poincaré inequality

(2.13) \[ \mu^{(d)}(f^2) \leq r \varepsilon^{\varphi^{(d)}(f, f)} + \beta(r) \mu^{(d)}(|f|)^2, \quad r > 0, f \in \mathcal{D}(\varphi^{(d)}) \]

holds, where \( \beta(r) = c_{12} (1 + r^{-1})^{\frac{1}{2}((\theta + \frac{d}{2})(2d + 1) - 1)} \).
Proof. We know
\[ \mu^{(d)} = \frac{\Gamma(\theta + \frac{d+1}{2}) \prod_{i=1}^{d+1} p_i}{\pi^{\frac{d}{2}} \Gamma(\theta + \frac{1}{2}) (\sum_{i=1}^{d+1} \frac{p_i}{x_i})^{\theta + \frac{d+1}{2}}} dx. \]
Denote
\[ \tilde{\mu}^{(d)} = \frac{\Gamma(|\alpha|_1)}{\prod_{1 \leq i \leq d+1} \Gamma(\alpha_i)} (1 - |x|_1)^{\alpha_d+1-1} \prod_{1 \leq i \leq d} x_i^{\alpha_i-1} dx. \]
We set
\[ \tilde{\mu}^{(d)} = \mu^{(d)}(e^W), \]
so
\[ e^W = \frac{\pi^{\frac{d}{2}} \Gamma(\theta + \frac{1}{2}) \Gamma(|\alpha|_1)(\sum_{i=1}^{d+1} \frac{p_i}{x_i})^{\theta + \frac{d+1}{2}}}{\Gamma(\theta + \frac{d+1}{2}) \prod_{i=1}^{d+1} p_i \prod_{1 \leq i \leq d+1} \Gamma(\alpha_i)} \prod_{1 \leq i \leq d+1} x_i^{\alpha_i+\frac{1}{2}} dx. \]
Let \( S(W) \in B \) be such that for any nonnegative \( f \in D(\Gamma) \) with \( \text{supp} f \subset \{ \phi \leq r \} \) for some \( r > 0 \), one has
\[ \int \Gamma(f, W) d\mu \geq - \int f S(W) d\mu \in \mathbb{R}. \]
Put
\[ \varphi(r) = \sup \{ e^W : \phi \leq r \}, \]
\[ \psi(r) = \frac{1}{4} \sup \{ \Gamma(W, W) + 2S(W) : \phi \leq r \}. \]
From Theorem 1.1 in [17], the super Poincaré inequality holds for \( \tilde{\mu}^{(d)} \),
\[ \tilde{\mu}^{(d)}(f^2 e^W) \leq r \tilde{\mu}^{(d)}(e^W \Gamma(f, f)) + c_1 (1 + r^{-1}) \tilde{\mu}^{(d)}(e^W |f|), \]
where \( p = \sum_{i=1}^{d+1} (2\alpha_i) + (\alpha_{d+1} - 1)^+, c_1 > 0 \). Then from Theorem 2.4, we know
\[ \beta(\varepsilon) = \frac{c_1}{1 - \left\{ \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2} + \frac{\varepsilon(1 - \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2})((1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s))}{2 + \varepsilon((1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s))} \right\}} \cdot \left( 1 + \frac{2 + \varepsilon((1 + \varepsilon)^2 s^{-2} h(3s) + \psi(3s))}{\varepsilon(1 - \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2})} \right)^p \varphi(3s). \]
When \( s \geq s_0 \), we have
\[ h(s) := \sup_{\phi \leq s} \Gamma(\phi, \phi) \leq c_7 s^{\frac{7}{2}}, \]
\[ \lambda(s) \geq c_6 s^{\frac{7}{2}}, \]
\[ \frac{2(1 + \varepsilon)^2 h(2s)}{\lambda(2s)s^2} \leq c_8 s^{\frac{3}{2}}, \]
\[ \psi(3s) \leq c_9 s^{-1}. \]
So
\[
\frac{\varepsilon(1 - \frac{2(1+\varepsilon)^2h(2\varepsilon)}{\lambda(2\varepsilon)s^2})[(1 + \varepsilon)^2 s^{-2}h(3s) + \psi(3s)]}{2 + \varepsilon[(1 + \varepsilon)^2 s^{-2}h(3s) + \psi(3s)]} \leq c_{10}s^{-1},
\]
\[
\frac{2 + \varepsilon[(1 + \varepsilon)^2 s^{-2}h(3s) + \psi(3s)]}{\varepsilon(1 - \frac{2(1+\varepsilon)^2h(2\varepsilon)}{\lambda(2\varepsilon)s^2})} \leq c_{11}\varepsilon^{-1}.
\]

When \( \alpha_i \geq \theta + \frac{d}{2} \), \( \varphi(3s) \) is bounded. Thus,
\[
\beta(r) = c_{12}(1 + r^{-1})^{\frac{1}{2}}(\sum_{i=1}^{d} 1^{\nu(2\alpha_i) + (\alpha_{d+1} - 1)^{+}}).
\]

When \( \alpha_i = \theta + \frac{d}{2}, 1 \leq i \leq d + 1 \), we have
\[
r^{-\left(\sum_{i=1}^{d} 1^{\nu(2\alpha_i) + (\alpha_{d+1} - 1)^{+}}\right)} = r^{-(2\theta + d + \theta + \frac{d}{2} - 1)},
\]
so we get
\[
\mu^{(d)}(f^2) \leq r\mathscr{E}^{(d)}(f, f) + c_{12}r^{-(\theta + \frac{d}{2})(2d+1)-1}\mu^{(d)}(|f|)^2, \ f \in C^1(\Delta^{(d)}).
\]

3 Proof of Theorem 1.1 and 1.2

3.1 Proof of Theorem 1.1

Proof. For \( d \geq 2 \), \( \forall F \in \mathcal{F}_d \), which defined in (1.2). As
\[
(2\theta + d)d + \theta + \frac{d}{2} - 1 + \frac{3}{7} \geq (2\theta + d)d + \theta + \frac{d}{2} - 1,
\]
by Theorem 2.3 and Theorem 2.5, we have the super Poincaré inequality for \( \mu^{(d)} \) and \( \mathscr{E}^{(d)} \), i.e.
\[
\mu^{(d)}(f^2) \leq r\mathscr{E}^{(d)}(f, f) + cr^{-(2\theta + d)d + \theta + \frac{d}{2} - 1 - \frac{3}{7}}\mu^{(d)}(|f|)^2, \ r > 0, f \in \mathcal{D}(\mathscr{E}^{(d)}).
\]

From (1.4), (1.5), we have the super Poincaré inequality for \( \Pi_{\alpha,\theta,\nu_0} \) and \( \mathcal{E} \), i.e.
\[
\Pi_{\alpha,\theta,\nu_0}(F^2) \leq r\mathcal{E}(F, F) + cr^{-(2\theta + d)d + \theta + \frac{d}{2} - 1} \Pi_{\alpha,\theta,\nu_0}(|F|)^2, \ r > 0, F \in \mathcal{F}_d.
\]

3.2 Proof of Theorem 1.2

Proof. If \( d = \#\{i \in S, \nu_0(i) > 0\} = \infty \), we follow the proof in [13] to prove the validity of super Poincaré inequality for \( \Pi_{\alpha,\theta,\nu_0} \) and \( \mathcal{E} \). As defined before \( p_i = \nu_0(i), i \geq 1 \), and \( \lim_{i \to \infty} p_i = 0 \). Let
\[
F_n(\mu) = \left(\frac{1}{p_n(1 + \theta p_n)}\right)^{\frac{1}{2}} \mu(n).
\]
\[ \forall c > 0, \]
\[ \int_{\{ F_n^2 \geq c \}} F_n^2 d\Pi_{\alpha, \theta, \nu} = \frac{1}{p_n (1 + \theta p_n)} \int_{\{ \mu(n) \geq \sqrt{c p_n (1 + \theta p_n)} \}} \mu(n)^2 \Pi_{\alpha, \theta, \nu}(d\mu) \]
\[ = \frac{1}{p_n (1 + \theta p_n)} \int_1^{1 / \sqrt{c p_n (1 + \theta p_n)}} p_n (1 - p_n) \Gamma(\theta + \frac{3}{2}) t^{\frac{1}{2}} (1 - t)^{-\frac{3}{2}} dt \]
\[ \frac{n \rightarrow \infty}{\Gamma(\theta + \frac{3}{2}) \pi \Gamma(\theta + \frac{1}{2})} \int_0^1 t^{\frac{1}{2}} (1 - t)^{\theta} dt \]
\[ = \frac{\Gamma(\theta + \frac{3}{2}) \Gamma(\frac{1}{2}) \Gamma(\theta)}{\pi (\Gamma(\theta + \frac{1}{2}))^2}. \]

which implies that \( \{ F_n^2 \}_{n \geq 1} \) is not uniformly integrable. So the F-Sobolev inequality doesn’t hold then the super Poincaré inequality doesn’t hold.

\[ \square \]

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