Schwarzschild black holes in Starobinsky-Bel-Robinson gravity

Ruben Campos Delgado and Sergey V. Ketov

\textsuperscript{a}Bethe Center for Theoretical Physics, Physikalisches Institut der Universität Bonn, Nussallee 12, 53115 Bonn, Germany
\textsuperscript{b}Department of Physics, Tokyo Metropolitan University, 1-1 Minami-ohsawa, Hachioji-shi, Tokyo 192-0397, Japan
\textsuperscript{c}Interdisciplinary Research Laboratory, Tomsk State University, 36 Lenin Avenue, Tomsk 634050, Russia
\textsuperscript{d}Kavli Institute for the Physics and Mathematics of the Universe, The University of Tokyo Institutes for Advanced Study, Kashiwa 277-8583, Japan

Abstract

We study physical properties of a Schwarzschild black hole in the framework of the recently proposed Starobinsky-Bel-Robinson (SBR) modified theory of gravity, working perturbatively in the coupling constant. In particular, we compute the temperature, entropy, pressure and lifetime of a Schwarzschild black hole.

ruben.camposdelgado@gmail.com, ketov@tmu.ac.jp
1 Introduction

The Starobinsky model of modified $(R + R^2)$ gravity [1] is known as an excellent model of cosmological inflation, in very good agreement with WMAP/Planck/BICEP/KECK precision measurements of the cosmic microwave background radiation [2]. The Starobinsky model has no free parameters because its only parameter, given by the inflaton mass $m$, is determined by the COBE/WMAP normalization as $m \sim 10^{-5}M_{\text{Pl}}$ in terms of the reduced Planck mass $M_{\text{Pl}} = 1/\sqrt{8\pi G_N}$. The ultra-violet (UV) cutoff of the Starobinsky model is given by $M_{\text{Pl}}$ also, while the Starobinsky inflation belongs to the class of the single-large-field inflationary models that are UV-sensitive to quantum gravity corrections. It is therefore of importance to identify those corrections and study their applications in the high-curvature regimes relevant to the early universe and black holes.

An extension of the Starobinsky $(R + R^2)$ gravity in $D = 4$ spacetime dimensions by terms quartic in the curvature was proposed in Ref. [4]. This extension was motivated by the leading quantum gravity correction obtained by dimensional reduction of M-theory in $D = 11$ dimensions. The extra terms in the $D = 4$ gravitational low-energy effective action have a peculiar structure given by the square of the Bel-Robinson tensor [5, 6]. The new modified gravity was thus dubbed the Starobinsky-Bel-Robinson (SBR) gravity in Ref. [4]. The SBR action reads

$$S_{\text{SBR}}[g_{\mu\nu}] = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{6m^2} R^2 - \frac{\beta}{32M_{\text{Pl}}^6} (\mathcal{P}^2 - \mathcal{G}^2) \right],$$

where $\beta > 0$ is the new dimensionless coupling constant whose (unknown) value is supposed to be determined by compactification of M-theory, $\mathcal{G}$ and $\mathcal{P}$ are the Euler and Pontryagin topological densities in $D = 4$, respectively, coming from the BR-tensor squared,

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},$$

and

$$\mathcal{P} = \frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma}_{\alpha\beta} R^{\mu\nu\alpha\beta}.$$ 

In this letter we study how the physical properties of a Schwarzschild black hole get modified by the presence of the $\beta$-dependent terms in Eq. (1). Since the value of the $\beta$-parameter is expected to be small, $\beta \ll 1$, we work perturbatively at first order in $\beta$ only. Our paper is organized as follows. In Section 2 we compute the $\beta$-dependent correction to the Schwarzschild metric. This correction shifts the position of the event horizon. Having the displaced event horizon leads to the corresponding change in the black hole entropy obtained by integrating over the horizon area. In Section 3 we derive the $\beta$-corrections to the temperature, pressure and lifetime of a Schwarzschild black hole. Section 4 is our conclusion. We use natural units with $\hbar = c = k_B = 1$.

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See e.g. Ref. [3] for a recent review.
2  \( \beta \)-correction to the Schwarzschild metric

In this section we derive and solve the equations of motion obtained by varying the action \( \Pi \) with respect to the metric. Our task is simplified by the fact that the Pontryagin density \( \mathcal{P} \) vanishes for a spherically symmetric metric. The equations of motion are

\[
J_{\mu \nu} := G_{\mu \nu} + \frac{1}{6m^2} H_{\mu \nu} - \frac{\beta}{32 M_{\text{Pl}}^6} K_{\mu \nu} = 0 ,
\]

where \( G_{\mu \nu} = \mathcal{R}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \mathcal{R} \) is the Einstein tensor,

\[
H_{\mu \nu} = 2 \mathcal{R}_{\mu \nu} \mathcal{R} - \frac{1}{2} g_{\mu \nu} \mathcal{R}^2 - 2 \nabla_\mu \nabla_\nu \mathcal{R} + 2 g_{\mu \nu} \Box \mathcal{R}
\]

and

\[
K_{\mu \nu} = \frac{1}{2} g_{\mu \nu} \mathcal{G}^2 - 2 \left[ 2 \mathcal{G} \mathcal{R} R_{\mu \nu} - 4 \mathcal{G} R^\rho_\mu R_{\nu \rho} 
+ 2 \mathcal{G} R_{\mu \rho \lambda} R_{\nu \rho \lambda} + 4 \mathcal{G} R^\rho_\mu R_{\nu \rho \lambda} + 2 g_{\mu \nu} R \Box \mathcal{G} - 2 R \nabla_\mu \nabla_\nu \mathcal{G}
- 4 R_{\mu \rho \lambda} \mathcal{G} + 4 \left( R_{\mu \rho} \nabla^\sigma \mathcal{G} + R_{\nu \rho} \nabla^\sigma \mathcal{G} \right)
- 4 g_{\mu \nu} R_{\rho \sigma} \nabla^\rho \nabla^\sigma \mathcal{G} + 4 R_{\mu \rho \sigma \lambda} \nabla^\rho \nabla^\sigma \mathcal{G} \right].
\]

We search for a solution in the form

\[
ds^2 = -a(r) dt^2 + \frac{1}{a(r)} dr^2 + r^2 d\Omega_2^2,
\]

where

\[
d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2 ,
\]

and with the function \( a(r) \) beginning with the Schwarzschild solution

\[
a(r) = 1 - \frac{2 G_N M}{r} - \beta f(r),
\]

in the presence of a perturbation \(- \beta f(r)\). We assume that \( \beta \ll 1 \) and linearize our equations with respect to \( \beta \).

Because of spherical symmetry, there are only four independent equations to consider. Plugging (9) into (4) we find, at first order in \( \beta \),

\[
J_{tt} = \frac{r - 2 G_N M}{r^{13}} \left[ 36 G_N^3 M^3 \left( \frac{M_{\text{Pl}}^6}{67 G_N M - 32 r} \right) f''(r)
+ r^7 \left( r^3 + \frac{4 G_N M}{m^2} - \frac{4 r}{3 m^2} \right) f(r) + r^8 \left( r^3 - \frac{2 G_N M}{m^2} + \frac{4 r}{3 m^2} \right) f'(r)
- 2 r^{10} f''(r) + \frac{r^{10}}{3 m^2} (11 G_N M - 6 r) f^{(3)}(r)
+ \frac{r^{11}}{3 m^2} (2 G_N M - r) f^{(4)}(r) \right] \beta + \mathcal{O}(\beta^2) = 0,
\]
\[ J_{rr} = \frac{1}{(2G_N M - r) r^{11}} \left[ \frac{36G_N^3 M^3}{M_{\text{Pl}}^3} (11G_N M - 4r) - r^7 \left( r^3 - \frac{4G_N M}{m^2} + \frac{8r}{3m^2} \right) f(r) + r^8 \left( r^3 - \frac{2G_N M}{m^2} + \frac{4r}{3m^2} \right) f'(r) + \frac{4r^9}{3m^2} (3G_N M - 2r) f''(r) + \frac{r^{10}}{3m^2} (3G_N M - 2r) f^{(3)}(r) \right] \beta + \mathcal{O}(\beta^2) = 0, \]

\[ J_{\theta\theta} = \frac{1}{2r^{10}} \left[ \frac{72G_N^3 M^3}{M_{\text{Pl}}^3} (41G_N M - 18r) + \frac{16r^7}{3m^2} (r - 3G_N M) f(r) - 2r^9 \left( r^2 + \frac{2}{3m^2} \right) f'(r) - r^9 \left( r^3 + \frac{4}{3m^2} (r - 6G_N M) \right) f''(r) + \frac{2r^{10}}{3m^2} (5r - 8G_N M) f^{(3)}(r) + \frac{2r^{11}}{3m^2} (r - 2G_N M) f^{(4)}(r) \right] \beta + \mathcal{O}(\beta^2) = 0, \]

\[ J_{\phi\phi} = \frac{\sin^2 \theta}{2r^{10}} \left[ \frac{72G_N^3 M^3}{M_{\text{Pl}}^3} (41G_N M - 18r) + \frac{16r^7}{3m^2} (r - 3G_N M) f(r) - 2r^9 \left( r^2 + \frac{2}{3m^2} \right) f'(r) - r^9 \left( r^3 + \frac{4}{3m^2} (r - 6G_N M) \right) f''(r) + \frac{2r^{10}}{3m^2} (5r - 8G_N M) f^{(3)}(r) + \frac{2r^{11}}{3m^2} (r - 2G_N M) f^{(4)}(r) \right] \beta + \mathcal{O}(\beta^2) = 0. \]

This system of differential equations can be reduced to a single equation after eliminating the 3rd and 4th order derivatives. Indeed, solving \( J_{\theta\theta} \) or \( J_{\phi\phi} \) for \( f^{(4)}(r) \), \( J_{rr} \) for \( f^{(3)}(r) \) and plugging the resulting expressions into \( J_{\mu\nu} \), we obtain the equation

\[ \frac{72G_N^3 M^3}{M_{\text{Pl}}^3} (291G_N^2 M^2 - 343G_N M r + 96r^2) - 2r^{11} f(r) + 2r^{11} (r - 3G_N M) f'(r) + r^{12} (2r - 3G_N M) f''(r) = 0, \]

whose solution is

\[ f(r) = \frac{4G_N^3 M^3}{5M_{\text{Pl}}^3 r^{10}} (97G_N M - 54r) = \frac{2048\pi^3 G_N^6 M^3}{5r^{10}} (97G_N M - 54r). \]

As a result, we find the function \( a(r) \) in the form

\[ a(r) = 1 - \frac{r_S}{r} + \beta \frac{128\pi^3}{5} \left( \frac{G_N r_S}{r^3} \right)^3 \left( 108 - 97 \frac{r_S}{r} \right), \]

where we have introduced the (unperturbed) Schwarzschild radius \( r_S = 2G_N M \).
3 \hspace{1em} \beta\text{-corrections to the physical properties of a Schwarzschild black hole}

Let us now study how the solution (16) affects the physical properties of a Schwarzschild black hole, namely, temperature, entropy, pressure and lifetime. The horizon radius is determined by a solution of $a(r) = 0$. At first order in $\beta$ we find

$$r_H = 2G_N M - \beta \frac{44\pi^3}{5G_N^2 M^3} + \mathcal{O}(\beta^2).$$

Therefore, the Hawking temperature is given by

$$T_H = \frac{1}{4\pi \left. \frac{da(r)}{dr} \right|_{r=r_H}} = \frac{1}{8\pi G_N M} + \beta \frac{\pi^2}{G_N^4 M^7} + \mathcal{O}(\beta^2).$$

According to the Wald formula [7], the entropy is

$$S = -2\pi \int_{r=r_H} d\Sigma \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}},$$

where $d\Sigma = r^2 \sin \theta d\theta d\phi$, $\epsilon_{\mu\nu}$ is the Levi-Civita tensor in two dimensions and $\mathcal{L}$ is the Lagrangian density of the theory. In our case we have

$$\mathcal{L} = \frac{1}{16\pi G_N} \left( R + \frac{1}{6m^2} R^2 + 16\pi^3 G_N^3 \beta^2 \right).$$

The antisymmetry of $\epsilon_{\mu\nu}$ reduces the expression of the entropy to

$$S = -8\pi A \left. \frac{\partial \mathcal{L}}{\partial R_{trtr}} \right|_{r=r_H},$$

where $A = 4\pi r_H^2$ is the area enclosed by the event horizon. The partial derivative of the Lagrangian with respect to the Riemann tensor is computed by using the formulae [8]

$$\frac{\partial R}{\partial R_{\mu\nu\rho\sigma}} = \frac{1}{2} \left( g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right),$$

$$\frac{\partial R_{\alpha\beta\gamma\delta}}{\partial R_{\mu\nu\rho\sigma}} = g^{\mu\rho} R^{\nu\sigma} - g^{\nu\rho} R^{\mu\sigma},$$

$$\frac{\partial R_{\alpha\beta\gamma\delta}}{\partial R_{\mu\nu\rho\sigma}} = 2R^{\mu\nu\rho\sigma}.$$  

We find

$$\frac{\partial \mathcal{L}}{\partial R_{trtr}} = \frac{1}{16\pi G_N} \left\{ -\frac{1}{2} - \frac{R}{6m^2} + 16\pi^3 G_N^3 \beta \left[ 32g^{rr} R^{tt} R_{\alpha\beta} R^{\alpha\beta} - 8 \left( g^{rr} R^{tt} R^2 - R_{\alpha\beta} R^{\alpha\beta} R \right) - 2R^3 - 8 \left( g^{rr} R^{tt} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + 2R^{rtrtr} R_{\alpha\beta} R^{\alpha\beta} \right) + 2 \left( 2R^{rtrr} R^2 - RR_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \right) + 4R^{rtrr} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \right] \right\}.$$
The final expression for the entropy is given by

\[
S = \frac{A}{4G_N} + \beta \left( \frac{304\pi^4}{5G_N^3 M^4} + \frac{624\pi^4}{5m^2G_N^3 M^6} \right) + O(\beta^2).
\] (24)

The classical relation \( TdS = dM \) receives a correction also. We find

\[
TdS = dM + \beta \left( \frac{8\pi^3}{G_N^3 M^6} - \frac{152\pi^3}{5G_N^3 M^6} - \frac{468\pi^3}{5m^2G_N^3 M^8} \right) dM.
\] (25)

The physical interpretation of this result is that the Bel-Robinson (quartic) term generates a pressure for the black hole. The first law of thermodynamics is then given by

\[
TdS - PdV = dM + \beta \left( \frac{8\pi^3}{G_N^3 M^6} - \frac{152\pi^3}{5G_N^3 M^6} - \frac{468\pi^3}{5m^2G_N^3 M^8} \right) dM,
\] (26)

where \( P \) is the pressure of the black hole and \( V \) is its volume, \( V = 4/3\pi r_H^3 \). We can then identify \( TdS = dM \) and obtain

\[
P = \beta \left( \frac{7\pi^2}{10G_N^3 M^8} + \frac{117\pi^2}{40m^2G_N^3 M^{10}} \right) + O(\beta^2).
\] (27)

A non-vanishing pressure for a Schwarzschild black hole (it has no pressure classically) is the common feature of quantum gravity [9].

The last physical quantity we want to consider is the mass loss rate due to the Hawking radiation. If the black hole mass is sufficiently large, then the temperature is low. Accordingly, all the mass lost in the Hawking process is due to the emission of massless particles. The mass loss rate is given by the Stefan-Boltzmann law [10]

\[
\frac{dM}{dt} = -\xi T^4 \left( \frac{21}{8} \sigma_\nu + \sigma_\gamma + \sigma_g \right),
\] (28)

where \( \xi = \pi^2/60 \) is the Stefan-Boltzmann constant and \( \sigma_\nu, \sigma_\gamma, \sigma_g \) are the thermally averaged cross sections of the black hole for neutrinos, photons and gravitons, respectively. We neglect the neutrino masses in what follows, and introduce the geometrical optics cross section \( \rho_0 \) for the Schwarzschild black hole. Defining also

\[
\rho = \left( \frac{21}{8} \sigma_\nu + \sigma_\gamma + \sigma_g \right) \rho_0^{-1},
\] (29)

we rewrite (28) as follows:

\[
\frac{dM}{dt} = -\xi T^4 \rho \sigma_0.
\] (30)

The cross sections for neutrinos, photons and gravitons can be estimated as [10]

\[
\sigma_\nu \approx 0.66852\sigma_0, \quad \sigma_\gamma \approx 0.24044\sigma_0, \quad \sigma_g \approx 0.02748\sigma_0.
\] (31)
while most of the power is emitted in neutrinos. The value of $\rho$ is then $\rho \approx 2.0228$.

Classically, we have $\sigma_0 = 27\pi G_N^2 M^2$. We now derive the $\beta$-dependent correction to the classical result. The emitted particles move along null geodesics governed by the equation

$$\left( \frac{dr}{d\lambda} \right)^2 = E^2 - a(r) \frac{J^2}{r^2},$$

where $\lambda$ is the affine parameter and $E, J$ are the energy and angular momentum, respectively. For the emitted particles to reach the infinity rather than falling back into the black hole horizon, one has to require $(dr/d\lambda)^2 \geq 0$, i.e.

$$\frac{1}{l^2} = \frac{E^2}{J^2} \geq \frac{a(r)}{r^2}$$

for any $r \geq r_H$. The inequality is saturated by the maximal value of $a(r)/r^2$. The solution of $\partial_r (a(r)/r^2) = 0$ corresponds to the critical, unstable orbit $r_c$, and the impact factor can be computed as $l_c = r_c/\sqrt{a(r_c)}$ [11,12]. The geometrical optics cross section of the black hole is then given by

$$\sigma_0 = \pi l_c^2 = \pi \frac{r_c^2}{a(r_c)}.$$
is increased also, whereas the black hole pressure $[27]$ becomes positive. As regards the black hole lifetime $[37]$, we find two additional contributions with the opposite signs, where the negative contribution given by the 2nd term apparently dominates but the last term may become important for black holes with small masses. However, our approximation is expected to break down for very small black hole masses because in that case the even higher orders (in the spacetime curvature) have to be taken into account in the effective quantum gravity action.

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