No-Regret Learning in Unknown Games with Correlated Payoffs

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Abstract

We consider the problem of learning to play a repeated multi-agent game with an unknown reward function. Single player online learning algorithms attain strong regret bounds when provided with full information feedback, which unfortunately is unavailable in many real-world scenarios. Bandit feedback alone, i.e., observing outcomes only for the selected action, yields substantially worse performance. In this paper, we consider a natural model where, besides a noisy measurement of the obtained reward, the player can also observe the opponents’ actions. This feedback model, together with a regularity assumption on the reward function, allows us to exploit the correlations among different game outcomes by means of Gaussian processes (GPs). We propose a novel confidence-bound based bandit algorithm GP-MW, which utilizes the GP model for the reward function and runs a multiplicative weight (MW) method. We obtain novel kernel-dependent regret bounds that are comparable to the known bounds in the full information setting, while substantially improving upon the existing bandit results. We experimentally demonstrate the effectiveness of GP-MW in random matrix games, as well as real-world problems of traffic routing and movie recommendation. In our experiments, GP-MW consistently outperforms several baselines, while its performance is often comparable to methods that have access to full information feedback.

1 Introduction

Many real-world problems, such as traffic routing [14], market prediction [10], and social network dynamics [21], involve multiple learning agents that interact and compete with each other. Such problems can be described as repeated games, in which the goal of every agent is to maximize her cumulative reward. In most cases, the underlying game is unknown to the agents, and the only way to learn about it is by repeatedly playing and observing the corresponding game outcomes.

The performance of an agent in a repeated game is often measured in terms of regret. For example, in traffic routing, the regret of an agent quantifies the reduction in travel time had the agent known the routes chosen by the other agents. No-regret algorithms for playing unknown repeated games exist, and their performance depends on the information available at every round. In the case of full information feedback, the agent observes the obtained reward, as well as the rewards of other non-played actions. While these algorithms attain strong regret guarantees, such full information feedback is often unrealistic in real-world applications. In traffic routing, for instance, agents only observe the incurred travel times and cannot observe the travel times for the routes not chosen.

In this paper, we address this challenge by considering a more realistic feedback model, where at every round of the game, the agent plays an action and observes the noisy reward outcome. In addition to this bandit feedback, the agent also observes the actions played by other agents. Under this feedback model and further regularity assumptions on the reward function, we present a novel
A no-regret algorithm for playing unknown repeated games. The proposed algorithm alleviates the need for full information feedback while still achieving comparable regret guarantees.

**Related Work.** In the full information setting, multiplicative-weights (MW) algorithms \[17\] such as HEDGE \[11\] attain optimal \(O(\sqrt{T \log K_i})\) regret, where \(K_i\) is the number of actions available to agent \(i\). In the case of convex action sets in \(\mathbb{R}^{d_i}\), and convex and Lipschitz rewards, online convex optimization algorithms attain optimal \(O(\sqrt{T})\) regret \[23\]. By only assuming Lipschitz rewards and bounded action sets, \(O(\sqrt{d_i T \log T})\) regret follows from \[18\], while in \[13\] the authors provide efficient gradient-based algorithms with ‘local’ regret guarantees. Full information feedback requires perfect knowledge of the game and is unrealistic in many applications. Our proposed algorithm overcomes this limitation while achieving comparable regret bounds.

In the more challenging bandit setting, existing algorithms have a substantially worse dependence on the size of the action set. For finite actions, EXP3 \[3\] and its variants ensure optimal \(O(\sqrt{TK_i \log K_i})\) regret. In the case of convex action sets, and convex and Lipschitz rewards, bandit algorithms attain \(O(\text{poly}(d_i) \sqrt{T})\) regret \[6\], while in the case of Lipschitz rewards \(O(T^{\frac{d_i+1}{2}} \log T)\) regret can be obtained \[22\]. In contrast, our algorithm works in the noisy bandit setting and requires the knowledge of the actions played by other agents. This allows us to, under some regularity assumptions, obtain substantially improved performance. In Table 1 we summarize the regret and feedback model of our algorithm together with the existing no-regret algorithms.

The previously mentioned online algorithms reduce the unknown repeated game to a single agent problem against an adversarial and adaptive environment that selects a different reward function at every time step \[7\]. A fact not exploited by these algorithms is that in a repeated game, the rewards obtained at different time steps are correlated through a static unknown reward function. In \[24\] the authors use this fact to show that, if every agent uses a regularized no-regret algorithm, their individual regret grows at a lower rate of \(O(T^{1/4})\), while the sum of their rewards grows only as \(O(1)\). In contrast to \[24\], we focus on the single-player viewpoint, and we do not make any assumption on opponents strategies instead, we show that by observing opponents’ actions, the agent can exploit the structure of the reward function to reduce her individual regret.

**Contributions.** We propose a novel no-regret bandit algorithm GP-MW for playing unknown repeated games. GP-MW combines the ideas of the multiplicative weights update method \[17\], with GP upper confidence bounds, a powerful tool used in GP bandit algorithms (e.g., \[23, 5\]). When a repeated games. GP-MW combines the ideas of the multiplicative weights update method \[17\], with GP upper confidence bounds, a powerful tool used in GP bandit algorithms (e.g., \[23, 5\]). When a

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### Table 1: Finite action set regret bounds that depend on the available feedback observed by player \(i\)

| Feedback                                      | HEDGE \[11\] | EXP3 \[3\] | GP-MW [this paper] |
|-----------------------------------------------|--------------|------------|--------------------|
| Regret                                        | \(O(\sqrt{T \log K_i})\) | \(O(\sqrt{TK_i \log K_i})\) | \(O(\sqrt{T \log K_i} + \gamma_T \sqrt{T})\) |

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1In fact, they are allowed to be adaptive and adversarial.
and the actions \(a^{-i}_t := (a_i^t, \ldots, a_{i-1}^t, a_{i+1}^t, \ldots, a_N^t)\) of all the other players. The goal of player \(i\) is to maximize the cumulative reward \(\sum_{t=1}^T r^i(a_i^t, a_{-i}^t)\). After \(T\) time steps, the regret of player \(i\) is defined as
\[
R^i(T) = \max_{a \in \mathcal{A}^i} \sum_{t=1}^T r^i(a, a_{-i}^t) - \sum_{t=1}^T r^i(a_i^t, a_{-i}^t),
\]
i.e., the maximum gain the player could have achieved by playing the single best fixed action in case the sequence of opponents’ actions \(\{a_{-i}^t\}_{t=1}^T\) and the reward function were known in hindsight. An algorithm is no-regret for player \(i\) if \(R^i(T)/T \to 0\) as \(T \to \infty\) for any sequence \(\{a_{-i}^t\}_{t=1}^T\).

First, we consider the case of a finite number of available actions \(K_i\), i.e., \(|\mathcal{A}^i| = K_i\). To achieve no-regret, the player should play mixed strategies \([11, 3]\), i.e., probability distributions \(w^i_t \in [0, 1]^{K_i}\), over \(\mathcal{A}^i\). With full-information feedback, at every time \(t\) player \(i\) observes the vector of rewards \(r_t^i = [r^i(a_i^t, a_{-i}^t)]_{a \in \mathcal{A}^i} \in \mathbb{R}^{K_i}\). With bandit feedback, only the reward \(r^i(a_i^t, a_{-i}^t)\) is observed by the player. Existing full-information and bandit algorithms \([11, 3]\), reduce the repeated game to a sequential decision making problem between player \(i\) and an adaptive environment that, at each time \(t\), selects a reward function \(r_t^i : \mathcal{A}^i \to [0, 1]\). In a repeated game, the reward that player \(i\) observes at time \(t\) is a static fixed function of \((a_i^t, a_{-i}^t)\), i.e., \(r_t^i(a_i^t, a_{-i}^t) = r^i(a_i^t, a_{-i}^t)\), and in many practical settings similar game outcomes lead to similar rewards (see, e.g., the traffic routing application in Section 4.2).

In contrast to existing approaches, we exploit such correlations by considering the feedback and reward function models described below.

**Feedback model.** We consider a noisy bandit feedback model where, at every time \(t\), player \(i\) observes a noisy measurement of the reward \(r_t^i = r^i(a_i^t, a_{-i}^t) + c_t^i\) where \(c_t^i\) is \(\sigma_i\)-sub-Gaussian, i.e., \(\mathbb{E}[\exp(c_t^i)] \leq \exp(\sigma_i^2 c_t^i/2)\) for all \(c \in \mathbb{R}\), with independence over time. The presence of noise is typical in real-world applications, since perfect measurements are unrealistic, e.g., measured travel times in traffic routing.

Besides the standard noisy bandit feedback, we assume player \(i\) also observes the played actions \(a_{-i}^t\) of all the other players. In some applications, the reward function \(r^i\) depends only indirectly on \(a_{-i}^t\) through some aggregative function \(\psi(a_{-i}^t)\). For example, in traffic routing \([14]\), \(\psi(a_{-i}^t)\) represents the total occupancy of the network’s edges, while in network games \([15]\), it represents the strategies of player \(i\)’s neighbours. In such cases, it is sufficient for the player to observe \(\psi(a_{-i}^t)\) instead of \(a_{-i}^t\).

**Regularity assumption on rewards.** In this work, we assume the unknown reward function \(r^i : \mathcal{A} \to [0, 1]\) has a bounded norm in a reproducing kernel Hilbert space (RKHS) associated with a positive semi-definite kernel function \(k^i(\cdot, \cdot)\), that satisfies \(k^i(a, a') \leq 1\) for all \(a, a' \in \mathcal{A}\). The RKHS norm \(\|r^i\|_{k^i} = \sqrt{\langle r^i, r^i \rangle_{k^i}}\) measures the smoothness of \(r^i\) with respect to the kernel function \(k^i(\cdot, \cdot)\), while the kernel encodes the similarity between two different outcomes of the game \(a, a' \in \mathcal{A}\). Typical kernel choices are polynomial, Squared Exponential, and Matérn:
\[
k_{\text{poly}}(a, a') = \left(1 + \frac{a - a'}{l} \right)^n, \quad k_{\text{SE}}(a, a') = \exp \left(-\frac{s^2}{2l^2} \right), \quad k_{\text{Matérn}}(a, a') = \frac{2^{1-n} \Gamma(n)}{\Gamma \left(\frac{n}{2} \right)} \frac{s^{\frac{n}{2}}}{l} B_{\nu} \left(\frac{s^{\frac{n}{2}}}{l} \right),
\]
where \(s = \|a - a'\|_2\), \(B_{\nu}\) is the modified Bessel function, and \(l, n, \nu > 0\) are kernel hyperparameters \([20]\ Section 4\). This is a standard smoothness assumption used in kernelized bandits and Bayesian optimization (e.g., \([24, 9]\)). In our context it allows player \(i\) to use the observed history of play to learn about \(r^i\) and predict unseen game outcomes. Our results are not restricted to any specific kernel function, and depending on the application at hand, various kernels can be used to model different types of reward functions. Moreover, composite kernels (see e.g., \([16]\)) can be used to encode the differences in the structural dependence of \(r^i\) on \(a'\) and \(a^{-i}\).

It is well known that Gaussian Process models can be used to learn functions with bounded RKHS norm \([24, 9]\). A GP is a probability distribution over functions \(f(a) \sim \mathcal{GP}(\mu(a), k(a, a'))\), specified by its mean and covariance functions \(\mu(\cdot)\) and \(k(\cdot, \cdot)\), respectively. Given a history of measurements \(\{y_j\}_{j=1}^t\) at points \(\{a_j\}_{j=1}^t\) with \(y_j = f(a_j) + \epsilon_j\) and \(\epsilon_j \sim \mathcal{N}(0, \sigma^2)\), the posterior distribution under
We now introduce GP-MW, a novel no-regret bandit algorithm, which can be used by a generic player $d$

**Algorithm 1**

**Input:** Set of actions $A^i$, GP prior $(\mu_0, \sigma_0, k^i)$, parameters $\{\beta_t\}_{t \geq 1}, \eta$

1. Initialize: $w_t^i = \frac{1}{K_i}(1, \ldots, 1) \in \mathbb{R}^{K_i}$
2. for $t = 1, 2, \ldots, T$
   3. Sample action $a_t^i \sim w_t^i$
   4. Observe noisy reward $\tilde{r}_t^i$ and opponents’ actions $a_t^{-i}$:
      $$\tilde{r}_t^i = r^i(a_t^i, a_t^{-i}) + \epsilon_t^i$$
   5. Compute optimistic reward estimates $\hat{r}_t^i \in \mathbb{R}^{K_i}$:
      $$[\hat{r}_t^i]_a = \min \{1, UCB_t(a, a_t^{-i}) \} \quad \text{for every } a = 1, \ldots, K_i$$
6. Update mixed strategy:
      $${w}_{t+1}^i = \frac{[w_t^i]_a \exp(-\eta(1 - [\hat{r}_t^i]_a))}{\sum_{k=1}^{K_i} [w_t^i]_k \exp(-\eta(1 - [\hat{r}_t^i]_k))} \quad \text{for every } a = 1, \ldots, K_i$$
7. Update $\mu_t, \sigma_t$ according to (2)–(3) by appending $(a_t, \tilde{r}_t^i)$ to the history of play.
8. end for

**In Theorem 1,** we present a high-probability regret bound for GP-MW while all the proofs of this section can be found in the supplementary material. The obtained bound depends on the maximum information gain, a kernel-dependent quantity defined as:

$$\gamma_t := \max_{a_1, \ldots, a_t} \frac{1}{2} \log \det (I_t + \sigma^{-2}K_t).$$

It quantifies the maximal reduction in uncertainty about $r^i$ after observing outcomes $\{a_{j}^i\}_{j=1}^t$ and the corresponding noisy rewards. The result of [23] shows that this quantity is sublinear in $T$, e.g.,

$$\gamma_T = \mathcal{O}((\log T)^{d+1})$$

in the case of $k_{\text{SE}}$, and

$$\gamma_T = \mathcal{O}(T^{\frac{d^2 + d}{2}} \log T)$$

in the case of $k_{\text{Matern}}$, where $d$ is the total dimension of the outcomes $a \in A$, i.e., $d = \sum_{i=1}^{N} d_i$.
Theorem 1. Fix \( \delta \in (0, 1) \) and assume \( \epsilon_i^t \)'s are \( \sigma_i \)-sub-Gaussian with independence over time. For any \( r^i \) such that \( \| r^i \|_k \leq B \), if player \( i \) plays actions from \( A_i \), \( |A_i| = K_i \), according to GP-MW with \( \beta_i = B + \sqrt{2(\gamma_{i-1} + \log(2/\delta))} \) and \( \eta = \sqrt{8 \log K_i}/T \), then with probability at least \( 1 - \delta \),

\[
R^i(T) = O \left( \sqrt{T \log K_i} + \sqrt{T \log(2/\delta)} + B \sqrt{T \gamma_T} + \sqrt{T \gamma_T} (\gamma_T + \log(2/\delta)) \right).
\]

The proof of this theorem follows by the decomposition of the regret of GP-MW into the sum of two terms. The first term corresponds to the regret that player \( i \) incurs with respect to the sequence of computed upper confidence bounds. The second term is due to not knowing the true reward function \( r^i \). The proof of Theorem 1 then proceeds by bounding the first term using standard results from adversarial online learning [7], while the second term is upper bounded by using regret bounding techniques from GP optimization [23][4].

Theorem 1 can be made more explicit by substituting bounds on \( \gamma_T \). For instance, in the case of the squared exponential kernel the regret bound becomes

\[
R^i(T) = O \left( ((\log K_i)^{1/2} + (\log T)^{d+1}) \sqrt{T} \right).
\]

In comparison to the standard multi-armed bandit regret bound \( O(\sqrt{d_i T \log T}) \) and \( O((d_i + 1)^{d+1} \log T) \) in the full information [18] and bandit setting [22], respectively. By using a discretization argument, we obtain a high probability regret bound for GP-MW.

Corollary 1. Let \( \delta \in (0, 1) \) and \( \epsilon_i^t \) be \( \sigma_i \)-sub-Gaussian with independence over time. Assume \( \| r^i \|_k \leq B \), \( A_i \subset [0, b]^{d_i} \), and \( r^i \) is \( L \)-Lipschitz in its first argument, and consider the discretization \( [A_i]_T \) with \( [A_i]_T = \left( Lb \sqrt{d_i} \right)^{d_i} \) such that \( \| a - [a]_T \|_1 \leq \sqrt{d_i T}/L \) for every \( a \in A_i \), where \([a]_T\) is the closest point to \( a \) in \([A_i]_T\). If player \( i \) plays actions from \([A_i]_T\) according to GP-MW with \( \beta_i = B + \sqrt{2(\gamma_{i-1} + \log(2/\delta))} \) and \( \eta = \sqrt{8d_i \log(Lb \sqrt{d_i})}/T \), then with probability at least \( 1 - \delta \),

\[
R^i(T) = O \left( \sqrt{d_i T \log(Lb \sqrt{d_i} T)} + \sqrt{T \log(2/\delta)} + B \sqrt{T \gamma_T} + \sqrt{T \gamma_T} (\gamma_T + \log(2/\delta)) \right).
\]

By substituting bounds on \( \gamma_T \), our bound becomes \( R^i(T) = O(T^{1/2} \text{polylog}(T)) \) in the case of the SE kernel (for fixed \( d \)). Such a bound has a strictly better dependence on \( T \) than the existing bandit bound \( O(T^{d_i+1}/\log T) \) from [22]. Similarly to [22][18], the algorithm resulting from Corollary 1 is not efficient in high dimensional settings, as its computational complexity is exponential in \( d_i \).

4 Experiments

In this section, we consider random matrix games and a traffic routing model and compare GP-MW with the existing algorithms for playing repeated games. Then, we show an application of GP-MW to robust BO and compare it with existing baselines on a movie recommendation problem.

4.1 Repeated random matrix games

We consider a repeated matrix game between two players with actions \( A_1 = A_2 = \{0, 1, \ldots, K - 1\} \) and payoff matrices \( A^i \in \mathbb{R}^{K \times K}, i = 1, 2 \). At every time step, each player \( i \) receives a payoff \( r^i(a_i^t, a^t_j) = [A^i]_{a_i^t, a^t_j} \), where \([A^i]_{i,j}\) indicates the \((i, j)\)-th entry of matrix \( A^i \). We select \( K = 30 \) and generate 10 random matrices with \( r^1 = r^2 \sim GP(0, k(\cdot, \cdot)) \), where \( k = k_{SE} \) with \( l = 6 \). We set the noise to \( \epsilon_i^t \sim \mathcal{N}(0, 1) \), and use \( T = 200 \). For every game, we distinguish between two settings: Against random opponent. In this setting, player-2 plays actions uniformly at random from \( A^2 \) at every round \( t \), while player-1 plays according to a no-regret algorithm. In Figure 1a, we compare the
We consider a repeated game, where agents choose routes using either of the following algorithms:

- GP-MW vs. E

The Sioux-Falls road network [14, 1], a standard benchmark model in the transportation literature, as well as travel time functions for each edge, are taken from [14, 1]. A more detailed description is provided in Appendix C.

Our experimental setup is provided in Appendix C.

The network is a directed graph with 24 nodes and 76 edges ($|E|$). To do so, agent $i$ can choose among $K_i = 5$ possible routes consisting of network edges $E(i) \subseteq E$. A route chosen by agent $i$ corresponds to action $a^i \in \mathbb{R}^{|E(i)|}$ with $[a^i]_e = u^i$ in case $e$ belongs to the route and $[a^i]_e = 0$ otherwise. The goal of each agent $i$ is to minimize the travel time weighted by the number of units $u^i$. The travel time of an agent is unknown and depends on the total occupancy of the traversed edges within the chosen route. Hence, the travel time increases when more agents use the same edges. The number of units $u^i$ and depends on the total occupancy of the traversed edges within the chosen route.

We consider a repeated game, where agents choose routes using either of the following algorithms:

- **HEDGE.** To run HEDGE, each agent has to observe the travel time incurred had she chosen any different route. This requires knowing the exact travel time functions. Although these assumptions are unrealistic, we use HEDGE as an idealized benchmark.

- **Exp3.P.** In the case of Exp3.P, agents only need to observe their incurred travel time. This corresponds to the standard bandit feedback.

- **GP-MW.** Let $\psi(a_i^{-1}) \in \mathbb{R}^{|E(i)|}$ be the total occupancy (by other agents) of edges $E(i)$ at time $t$. To run GP-MW, agent $i$ needs to observe a noisy measurement of the travel time as well as the corresponding $\psi(a_i^{-1})$.

- **Q-BRI (Q-learning Better Replies with Inertia algorithm [8]).** This algorithm requires the same feedback as GP-MW and is proven to asymptotically converge to a Nash equilibrium (as the considered game is a potential game [19]). We use the same set of algorithm parameters as in [8].

For every agent $i$ to run GP-MW, we use a composite kernel $k^i$ such that for every $a_1, a_2 \in \mathcal{A}$, $k^i((a_1, a_1^{-1}), (a_2, a_2^{-1})) = k^1_i(a_1, a_2) \cdot k^2_i(a_1 + \psi(a_1^{-1}), a_2 + \psi(a_2^{-1}))$, where $k^1_i$ is a linear kernel and $k^2_i$ is a polynomial kernel of degree $n \in \{2, 4, 6\}$.

First, we consider a random subset of 100 agents that we refer to as learning agents. These agents choose actions (routes) according to the aforementioned no-regret algorithms for $T = 100$ game cycles.
rounds. The remaining non-learning agents simply choose the shortest route, ignoring the presence of the other agents. In Figure 2 (top plots), we compare the average regret (expressed in hours) of the learning agents when they use the different no-regret algorithms. We also show the associated average congestion in the network (see (13) in Appendix C for a formal definition). When playing according to GP-MW, agents incur significantly smaller regret and the overall congestion is reduced in comparison to EXP3.P and Q-BRI.

In our second experiment, we consider the same setup as before, but we vary the number of learning agents. In Figure 2 (bottom plots), we show the final (when $T = 100$) average regret and congestion as a function of the number of learning agents. We observe that GP-MW systematically leads to a smaller regret and reduced congestion in comparison to EXP3.P and Q-BRI. Moreover, as the number of learning agents increases, both HEDGE and GP-MW reduce the congestion in the network, while this is not the case with EXP3.P or Q-BRI (due to a slower convergence).

4.3 GP-MW and robust Bayesian Optimization

In this section, we apply GP-MW to a novel robust Bayesian Optimization (BO) setting, similar to the one considered in [4]. The goal is to optimize an unknown function $f$ (under the same regularity assumptions as in Section 2) from a sequence of queries and corresponding noisy observations. Very often, the actual queried points may differ from the selected ones due to various input perturbations, or the function may depend on external parameters that cannot be controlled (see [4] for examples).

This scenario can be modelled via a two player repeated game, where a player is competing against an adversary. The unknown reward function is given by $f : \mathcal{X} \times \Delta \rightarrow \mathbb{R}$. At every round $t$ of the game, the player selects a point $x_t \in \mathcal{X}$, and the adversary chooses $\delta_t \in \Delta$. The player then observes the parameter $\delta_t$ and a noisy estimate of the reward: $f(x_t, \delta_t) + \epsilon_t$. After $T$ time steps, the player incurs the regret

$$R(T) = \max_{x \in \mathcal{X}} \sum_{t=1}^{T} f(x, \delta_t) - \sum_{t=1}^{T} f(x_t, \delta_t).$$

Note that both the regret definition and feedback model are the same as in Section 2.
We have proposed GP-MW, a no-regret bandit algorithm for playing unknown repeated games. In addition to the standard bandit feedback, the algorithm requires observing the actions of other players after every round of the game. By exploiting the correlation among different game outcomes, it computes upper confidence bounds on the rewards and uses them to simulate unavailable full information feedback. Our algorithm attains high probability regret bounds that can substantially improve upon the existing bandit regret bounds. In our experiments, we have demonstrated the effectiveness of GP-MW on synthetic games, and real-world problems of traffic routing and movie recommendation.

5 Conclusions

We have proposed GP-MW, a no-regret bandit algorithm for playing unknown repeated games. In addition to the standard bandit feedback, the algorithm requires observing the actions of other players after every round of the game. By exploiting the correlation among different game outcomes, it computes upper confidence bounds on the rewards and uses them to simulate unavailable full information feedback. Our algorithm attains high probability regret bounds that can substantially improve upon the existing bandit regret bounds. In our experiments, we have demonstrated the effectiveness of GP-MW on synthetic games, and real-world problems of traffic routing and movie recommendation.
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A Proof of Theorem 1

We make use of the following well-known confidence lemma.

**Lemma 1** (Confidence Lemma). Let $\mathcal{H}_k$ be a RKHS with underlying kernel function $k$. Consider an unknown function $f : \mathcal{A} \rightarrow \mathbb{R}$ in $\mathcal{H}_k$ such that $\|f\|_k \leq B$, and the sampling model $y_t = f(a_t) + \epsilon_t$, where $\epsilon_t$ is $\sigma$-sub-Gaussian with dependence between times. By setting

$$
\beta_t = B + \sqrt{2(\gamma_{t-1} + \log(1/\delta))}
$$

the following holds with probability at least $1 - \delta$:

$$
|\mu_t(a) - f(a)| \leq \beta_t \sigma_{t-1}(a), \quad \forall a \in \mathcal{A}, \quad \forall t \geq 1,
$$

where $\mu_{t-1}(\cdot)$ and $\sigma_{t-1}(\cdot)$ are given in [23, Lemma 5.4].

Lemma 1 follows directly from [23, Theorem 3.11 and Remark 3.13] as well as the definition of the maximum information gain $\gamma_{t-1}$.

We can now prove Theorem 1. Recall the definition of regret

$$
R^i(T) = \max_{a \in \mathcal{A}} \sum_{t=1}^{T} r_i(a_t) - \sum_{t=1}^{T} r_i(a_t, a_t^-) .
$$

Defining $\bar{a} = \arg\max_{a \in \mathcal{A}} \sum_{t=1}^{T} r_i(a_t, a_t^-)$, $R^i(T)$ can be rewritten as

$$
R^i(T) = \sum_{t=1}^{T} r_i(\bar{a}^i, a_t^-) - \sum_{t=1}^{T} r_i(a_t^i, a_t^-) .
$$

By Lemma 1 and since rewards are in $[0, 1]$, with probability $1 - \frac{\delta}{2}$ the true unknown reward function can be upper and lower bounded as:

$$
UCB_t(a) - 2\beta_t \sigma_{t-1}(a) \leq r_i(a) \leq \min\{1, UCB_t(a)\}, \quad \forall a \in \mathcal{A}^1 \times \cdots \times \mathcal{A}^N, \quad \forall t \geq 1, \quad (7)
$$

with $UCB_t$ defined in [4] and $\beta_t$ chosen according to Theorem 1. Thus, $UCB_t(a) - 2\beta_t \sigma_{t-1}(a)$ is a lower confidence bound of $r_i(a)$.

Hence,

$$
R^i(T) \leq \sum_{t=1}^{T} \min\{1, UCB_t(\bar{a}^i, a_t^-)\} - \sum_{t=1}^{T} [UCB_t(a_t^i, a_t^-) - 2\beta_t \sigma_{t-1}(a_t^i, a_t^-)]
$$

$$
\leq \sum_{t=1}^{T} \min\{1, UCB_t(\bar{a}^i, a_t^-)\} - \sum_{t=1}^{T} \min\{1, UCB_t(a_t^i, a_t^-)\} + 2\beta T \sum_{t=1}^{T} \sigma_{t-1}(a_t^i, a_t^-) ,
$$

where the first inequality follows by (7) and the second one since $\beta_t$ is increasing in $t$.

Moreover, by [23, Lemma 5.4] and the choice $\beta_T = B + \sqrt{2(\gamma_T + \log(2/\delta))}$, we have

$$
2\beta_T \sum_{t=1}^{T} \sigma_{t-1}(a_t^i, a_t^-) = O \left( B\sqrt{T \gamma_T} + \sqrt{T \gamma_T (\gamma_T + \log(2/\delta))} \right) .
$$

Next, we show that with probability $1 - \frac{\delta}{2}$,

$$
\sum_{t=1}^{T} \min\{1, UCB_t(\bar{a}^i, a_t^-)\} - \sum_{t=1}^{T} \min\{1, UCB_t(a_t^i, a_t^-)\}
$$

$$
= O \left( \sqrt{T \log K} + \sqrt{T \log(2/\delta)} \right) . \quad (8)
$$
As in the proof of Theorem 1, let

\[ A = \min \{ \text{UCB}^i(\cdot), a^{-i}_t \} \]

Therefore, by \cite[Corollary 4.2]{7}, with probability \( 1 - \delta / 2 \),

To show (8), define the function

\[ \hat{r}_t(\bar{a}) = \min_{a \in A^i} \max_{a' \in A^i} \sum_{t=1}^{T} r^i(a', a^{-i}_t) \]

The statement of the theorem then follows by standard probability arguments:

\[ P[E_1 \cap E_2] = P[E_1] + P[E_2] - P[E_1 \cup E_2] \geq \left( 1 - \frac{\delta}{2} \right) + \left( 1 - \frac{\delta}{2} \right) - 1 = 1 - \delta, \]

where \( E_1 \) and \( E_2 \) are the events (7) and (5), respectively.

To show (9), define the function

\[ f_t^i(a) = \min_{a' \in A^i} \sum_{t=1}^{T} r^i(a', a^{-i}_t) \]

Hence, by (11),

\[ \max_{a \in A^i} \sum_{t=1}^{T} f_t^i(a) \leq \max_{a \in A^i} \sum_{t=1}^{T} f_t^i(a) - \sum_{t=1}^{T} f_t^i(a^i_t). \quad (9) \]

Observe that the right hand side of (9) is precisely the regret which player \( i \) incurs in an adversarial online learning problem with reward functions \( f_t^i(\cdot) \in [0, 1] \). The actions \( a^i_t \), moreover, are exactly chosen by the HEDGE \cite{11} algorithm which receives the full information feedback \( r_t = [f_t^i(a_1), \ldots, f_t^i(a_K)] \). Note that the original version of HEDGE works with losses instead of rewards, but the same happens in GP-MW since the mixed strategies are updated with \( 1 - \hat{r}_t \).

Therefore, by \cite[Corollary 4.2]{7}, with probability \( 1 - \delta / 2 \),

\[ \max_{a \in A^i} \sum_{t=1}^{T} f_t^i(a) - \sum_{t=1}^{T} f_t^i(a^i_t) = \mathcal{O} \left( \sqrt{T \log K_i + T \log \frac{2}{\delta}} \right) . \]

Note that according to \cite[Remark 4.3]{7}, the functions \( f_t^i(\cdot) \) can be chosen by an adaptive adversary depending on past actions \( a^i_1, \ldots, a^i_{t-1} \), but not on the current action \( a^i_t \). This applies to our setting since \( f_t^i \) depends only on \( a^i_1, \ldots, a^i_{t-1} \) and not on \( a^i_t \).

\section*{B Proof of Corollary 1}

A function \( f : \mathcal{X} \rightarrow \mathbb{R} \) is Lipschitz continuous with constant \( L \) (or \( L \)-Lipschitz) if

\[ |f(x) - f(x')| \leq L \|x - x'\|_1 \quad \forall x, x' \in \mathcal{X}. \]

Define \( A^{-i} = A^1 \times \cdots \times A^{i-1} \times \{a^i_1, \ldots, a^i_{T-1}\} \times \mathcal{A}^N \). The fact that \( r^i \) is \( L \)-Lipschitz in its first argument implies that

\[ |r^i(a, a^{-i}) - r^i(a', a^{-i})| \leq L \|a - a'\|_1 \quad \forall a, a' \in \mathcal{A}, \forall a^{-i} \in \mathcal{A}^{-i} . \quad (10) \]

Moreover, recall the discrete set \( [\mathcal{A}^i]_T \) with \( \| [\mathcal{A}^i]_T \|_1 = (Lb \sqrt{d_i T})^d \), such that \( \|a - [a]_T\|_1 \leq bd_i / Lb \sqrt{d_i T} = \sqrt{d_i T} / L \forall a \in \mathcal{A}^i \), where \([a]_T \) is the closest point to \( a \) in \([\mathcal{A}^i]_T \). An example of such a set can be obtained for instance by a uniform grid of points in \([0, b]^d \).

As in the proof of Theorem 1, let \( \bar{a} = \arg \max_{a \in A^i} \sum_{t=1}^{T} r^i(a, a^{-i}_t) \). Moreover, let \([\bar{a}]_T \) be the closest point to \( \bar{a} \) in \([\mathcal{A}^i]_T \). We have:

\[ R^i(T) = \sum_{t=1}^{T} r^i(\bar{a}, a^{-i}_t) - \sum_{t=1}^{T} r^i(a^i_t, a^{-i}_t) \]

\[ = \sum_{t=1}^{T} r^i(\bar{a}, a^{-i}_t) - \sum_{t=1}^{T} r^i([\bar{a}]_T, a^{-i}_t) + \sum_{t=1}^{T} r^i([\bar{a}]_T, a^{-i}_t) - \sum_{t=1}^{T} r^i(a^i_t, a^{-i}_t) . \]

We prove the corollary by bounding \( R^i_1(T) \) and \( R^i_2(T) \) separately.

By the Lipschitz property (10) of \( r^i \), and by construction of \([\mathcal{A}^i]_T \), we have that

\[ |r^i(\bar{a}, a^{-i}_t) - r^i([\bar{a}]_T, a^{-i}_t)| \leq L \|\bar{a} - [\bar{a}]_T\|_1 \leq L \sqrt{d_i T} / L = \sqrt{d_i T} , \quad \forall a^{-i} \in \mathcal{A}^{-i} . \quad (11) \]

Hence, by \cite{11},

\[ R^i_1(T) \leq T \sqrt{d_i T} = \sqrt{d_i T} . \]
To bound $R_2^i(T)$, note that $R_2^i(T) \leq \arg \max_{a \in [A]^T} \sum_{t=1}^T r^i(a, a_{t-1})$. Moreover, note that actions $a_t$ are chosen by running GP-MW on the discretized domain $[A]^T$ with $K_i = [A]^T = (Lb\sqrt{d_i}T)^d_i$. Hence, according to Theorem 1 it must hold that with probability at least $1 - \delta$,

$$R_2^i(T) \leq \mathcal{O} \left( \sqrt{T \log K_i} + \sqrt{T \log(2/\delta)} + B \sqrt{T \gamma_T} + \sqrt{T \gamma_T (\gamma_T + \log(2/\delta))} \right).$$

The final bound then follows by substituting $K_i = (Lb\sqrt{d_i}T)^d_i$ in the bound above and noting that $R_1^i(T)$ is dominated by $R_2^i(T)$.

\begin{center}
\textbf{C Repeated traffic routing - Experimental setup}
\end{center}

In this section we give a detailed explanation of our traffic routing experiment of Section 4.2.

We consider the Sioux-Falls road network [14,1], a directed graph with 24 nodes and 76 edges $e \in E$. We use the demand data from [14,1]. Such data indicate the units of flow to be sent from each node (origin) to any other node (destination) in the network. Each of those origin-destination pairs is represented by an agent, for a total of $N = 528$ agents. The goal of each agent $i$ is to send $u_i$ units of demand to destination, while minimizing the total travel time. The time to reach destination, however, depends on the total occupancy of the edges the agent chooses to traverse and hence on the routes chosen by all the other agents.

Each edge $e$ has a travel time $t_e(x)$ which is a function of the total number of units $x$ traversing $e$. Intuitively, we expect such travel time to increase with $x$. According to [14,1], we select $t_e$ to be the Bureau of Public Roads (BPR) function

$$t_e(x) = c_e \left( 1 + 0.15 \left( \frac{x}{C_e} \right)^4 \right),$$

where $c_e$ and $C_e$ are free-flow time and capacity of edge $e$, respectively. Values for $c_e$ and $C_e$ are taken from [1].

Each agent $i$ can choose among $K_i = 5$ routes, and we assume that she cannot split her demand over different routes. Hence, the action space $\mathcal{A}$ represents the 5 shortest routes that agent $i$ could possibly traverse. Each route in $\mathcal{A}$ corresponds to a vector $a^i \in \mathbb{R}^{K_i} \subseteq \mathcal{A}$ such that $[a^i]_e = u_i$ if edge $e$ belongs to the given route, and $[a^i]_e = 0$ otherwise. Moreover, we let $\psi(a^{-i}) = \in \mathbb{R}^{E(i)}$ be the total occupancy by the other agents on such edges, i.e., $[\psi(a^{-i})]_e = \sum_{j \neq i} [a^j]_e$ for every $e \in E(i)$. The travel time of agent $i$ can thus be written as

$$t^i(a^i, a^{-i}) = \sum_{e \in E(i)} [a^i]_e t_e([a^i]_e + [\psi(a^{-i})]_e), \quad (12)$$

i.e., the sum of the travel times on the selected edges, weighted by $u^i$. Hence, we let the reward function of agent $i$ be $r^i(a^i, a^{-i}) = -t^i(a^i, a^{-i})$.

Note that agents don’t know the actual $t_e$’s functions, hence their reward function is unknown. This does not limit the bandit Exp3.P algorithm, where agents only need to observe their experienced travel times. However, it makes the full information feedback HEDGE algorithm unrealistic. Nevertheless, we used HEDGE in our experiments as an idealized benchmark.

To run GP-MW, agent $i$ observes the experienced travel time as well as the vector of occupancies $\psi(a^{-i})$. This allows GP-MW to exploit the correlations in the unknown reward function by choosing a suitable kernel. For every agent $i$, we chose a composite kernel $k^i$ such that for every $a_1, a_2 \in \mathcal{A}$, $k^i((a_1, a_1^{-i}), (a_2, a_2^{-i})) = k_1^i(a_1, a_2) \cdot k_2^i(a_1^{-i} + \psi(a_2^{-i}), a_2^{-i} + \psi(a_2^{-i}))$, with $k_1^i$ and $k_2^i$ being linear and polynomial kernels, respectively. This reflects the different dependences that $r^i$ has on $a^i$ and $a^{-i}$. In fact, for fixed total occupancy in each edge, we expect $r^i$ to be linear in $a^i$, being the travel time an additive quantity (see (12)). On the other hand, given a specific route chosen, $r^i$ grows polynomially with the total occupancy on such route (see (12)). Kernels hyperparameters are optimized via maximum-likelihood over 200 random outcomes.

To scale their rewards in $[0,1]$ agents need to know upper bounds on their travel times. Such bounds are estimated by $10/000$ random outcomes and fed to the agents. Moreover, standard deviations of
measurement noises are chosen 0.1% of such upper bounds. Finally, to evaluate a given outcome \( a_t \) of the game, we compute the congestion on a given edge \( e \) via the expression:

\[
0.15 \cdot \left( \sum_{j=1}^{N} |a_j|^4 / C_e \right).
\]  

(13)

The average congestion in the network is obtained by averaging the quantity above over all the edges \( e \in E \).