MAXIMUM-LIKELIHOOD ALGORITHM FOR QUANTUM TOMOGRAPHY∗

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Optical homodyne tomography is discussed in the context of classical image processing. Analogies between these two fields are traced and used to formulate an iterative numerical algorithm for reconstructing the Wigner function from homodyne statistics.

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1 Introduction

Several years after the first demonstration [1], optical homodyne tomography has become a well established tool for measuring quantum statistical properties of optical radiation. What is particularly fascinating, this technique provides practical means to visualise the measured quantum state in the form of the Wigner function. This success is a result of combining a complete quantum mechanical measurement of field quadratures with a filtered back-projection algorithm used in medical imaging.

The purpose of this contribution is to trace some other analogies between quantum state reconstruction and classical image processing, with the motivation to develop novel numerical methods for quantum tomography. Our interest will be focused on imperfect detection [2, 3], which has deleterious effects on quantum interference phenomena exhibited by non-classical states [4]. As we will discuss in Sec. 2, such effects can be related in the phase space representation to image blurring. Restoration of blurred images is a well known problem in classical imaging, and a number of methods has been developed for this purpose. Specifically, we shall briefly describe in Sec. 3 the Richardson algorithm [5] (known also in statistics as the expectation-maximization algorithm [6]), which provides an effective iterative procedure to perform image deblurring.

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An interesting question is, whether classical deblurring methods can be transferred to quantum tomography. We discuss this, in the case of the Richardson algorithm, in Sec. 4. The answer is not straightforward: the Richardson algorithm assumes positive definiteness of the original undegraded image, and this condition is not satisfied by the quantum mechanical Wigner function. We will show that this difficulty can be overcome by expressing the Wigner function in terms of the phase space displaced photon distribution. This yields an iterative algorithm for reconstructing the Wigner function, which incorporates compensation for detection losses in a numerically stable way [7].

2 Imperfect detection and image blurring

Homodyne detection is a realization of the quantum mechanical measurement of field quadratures only in the idealized loss-free limit. In practice, a fraction of the signal field is always lost due to the mode-mismatch and the non-unit efficiency of photodiodes. The homodyne statistics collected using a realistic setup is described by the distribution

$$h(x; \theta) = \int_{-\infty}^{\infty} dx' \frac{1}{\sqrt{\pi(1 - \eta)}} \exp \left( -\frac{(x - \sqrt{\eta}x')^2}{1 - \eta} \right) \langle x'_0 | \hat{\rho} | x'_0 \rangle,$$

where $\theta$ is the local oscillator phase, $\eta$ characterizes the overall detector efficiency, and $\langle x'_0 | \hat{\rho} | x'_0 \rangle$ denote diagonal elements of the density matrix $\hat{\rho}$ in the quadrature basis.

Application of the back-projection transformation to $h(x; \theta)$ yields, instead of the Wigner function, a generalized phase space quasidistribution function $P(q, p; s)$ with the ordering parameter $s = -(1 - \eta)/\eta$. This function can be expressed as a convolution of the Wigner function $W(q, p)$ with a gaussian function:

$$P(q, p; -|s|) = \int dq' dp' \frac{1}{\pi |s|} \exp \left( -\frac{1}{|s|} [(q - q')^2 + (p - p')^2] \right) W(q', p').$$

Thus what we reconstruct from imperfect homodyne data, is a blurred version of the Wigner function. The question is, whether we can get rid of this blurring in numerical processing of experimental data.

A similar problem appears in the following classical context: suppose we observe an image using an imperfect apparatus (for example ill-matched glasses), which generates some blurring. Such blurring can be described by a so-called point spread function specifying the shape generated by a single point of the original image. The observed degraded image is consequently given by a convolution of the original image with the point spread function. Using this language, we can assign the following names to the terms of Eq. (2):

$$\frac{1}{\pi |s|} \exp \left( -\frac{1}{|s|} [(q - q')^2 + (p - p')^2] \right)W(q', p') \quad \text{original image}$$

$$P(q, p; -|s|) \quad \text{point spread function}$$

$$P(q, p; -|s|) \quad \text{degraded image}$$
The common problem now is the restoration of the original image from the degraded one, assuming that the point spread function is known.

3 Image restoration

An analytical way to deconvolve Eq. (2) is to apply the Fourier transform, which maps a convolution onto a direct product. Dividing both the sides by the Fourier-transformed point spread function and evaluating the inverse Fourier transform thus yields an explicit expression for the original image. However, this procedure is very sensitive to statistical fluctuations and numerical truncation errors, which makes its practical application a very delicate matter. These problems have been noted also in the context of quantum tomography [9].

The numerical instability of the Fourier deconvolution has led to the development of techniques dedicated for image restoration. The basic observation is that statistical noise does not allow us to specify precisely the original image that was ’behind’ the blurred data. In principle, the measured degraded image could be generated by a variety of original images. However, comparing various original images we intuitively expect that some of them were more likely to generate the measured data than other ones. The maximum-likelihood methodology quantifies this intuition, and selects as an estimate the original image which maximizes the likelihood.

In order to discuss this idea in detail, we shall consider a discretized version of Eq. (2):

\[ p_\nu = \sum_n A_{\nu n} w_n, \]  

(3)

where \( w_n \) is the original image, \( A_{\nu n} \) is the point spread function, and \( p_\nu \) is the degraded image. Note that this formulation is more general compared to Eq. (2), because it allows the point spread function to be of different form for each ’element’ of the original image indexed with \( n \). The likelihood can be quantified using the function

\[ \mathcal{L} = \sum_\nu p_\nu \ln \left( \sum_n A_{\nu n} w_n \right) \]  

(4)

which has a rigorous derivation when the degraded image is observed as a histogram of events governed by Poissonian statistics. The likelihood function for quantum measurement has been discussed in Ref. [10], where its maximization has been proposed as a method for quantum state estimation.

In classical imaging, it is natural to assume that \( w_n \), as well as \( A_{\nu n} \) as a function of \( \nu \) for each \( n \), are positive definite distributions with sum equal to one. Under these assumptions, it is possible to find the maximum of the likelihood function \( \mathcal{L} \) via simple iteration:

\[ w_n^{(i+1)} = \sum_\nu p_\nu \frac{A_{\nu n} w_n^{(i)}}{\sum_m A_{\nu m} w_m^{(i)}}, \]  

(5)
which is the essence of the Richardson algorithm for image restoration [5]. A simple heuristic derivation of this algorithm can be found in Ref. [7].

Of course, the maximum-likelihood approach is not a magic wand solving unconditionally the problem of image restoration. With increasing blurring, the quality of the reconstructed image worsens, and the convergence of the iterative algorithm may be very slow. In many cases, however, it offers superior performance compared to the Fourier deconvolution technique.

4 Quantum tomography

An obvious difficulty with applying the iterative restoration algorithm to quantum tomography is that the object to be reconstructed in the quantum case, i.e. the Wigner function, is not positive definite. Nevertheless, there are some other quantum mechanical reconstruction problems, where positivity constraints appear in a natural way. An interesting and nontrivial example is determination of the photon number distribution from random phase homodyne statistics [11]. The relation between the phase-averaged homodyne statistics and the photon number distribution $\varrho_n$ is given by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \ h(x; \theta) = \sum_n A_n(x) \varrho_n,$$

where $A_n(x)$ describe contributions to the homodyne statistics generated by different Fock states $|n\rangle$. This formula, after discretization of $x$, is exactly of the form assumed in Eq. (3). Thus we arrive at the following formal analogy:

$$\varrho_n \quad \text{original image}$$

$$A_n(x) \quad \text{point spread function}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \ h(x; \theta) \quad \text{degraded image}$$

which allows us to apply directly the iterative reconstruction algorithm [12]. In this procedure, there is one $a$ priori parameter: it is the cut-off of the distribution $\varrho_n$ specifying the maximum number of photons.

Reconstruction of the photon distribution may seem to be quite distant from the starting point of our considerations, which was deblurring of the Wigner function. However, let us recall that the Wigner function can be represented as an alternating sum of the photon distribution $\varrho_n(q, p)$ corresponding to the phase space displaced state $\hat{D}^\dagger(q, p) \hat{\rho} \hat{D}(q, p)$:

$$W(q, p) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \varrho_n(q, p)$$

Obviously, we can apply this formula to evaluate the Wigner function at $q = p = 0$. What would be of interest, is the generalization of the maximum-likelihood algorithm to determination of an arbitrarily displaced photon distribution $\varrho_n(q, p)$. This would yield a numerically stable procedure for reconstructing the Wigner function from homodyne statistics, even in the case of the non-unit detection efficiency.
Maximum-likelihood algorithm for quantum tomography

Fig. 1. Reconstruction of the Schrödinger cat state \(|\Psi\rangle \propto |2\rangle - |-2\rangle\) from Monte Carlo simulated homodyne experiment. The homodyne data consisted of $10^5$ events generated for each of 64 phases spaced uniformly between 0 and $\pi$. At each point of the grid, the displaced photon statistics was obtained from $10^4$ iterations, starting from a flat distribution for $0 \leq n \leq 39$. In the simulations, the homodyne variable $x$ has been discretized into 16000 bins over the range $-8 \leq x \leq 8$.

Surprisingly, this generalization is quite straightforward. The basic observation is that the displacement transformation has a simple effect on the homodyne statistics, shifting it by $\sqrt{\eta}(q \cos \theta + p \sin \theta)$ for a given local oscillator phase $\theta$. Consequently, we have the relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta h(x + \sqrt{\eta}q \cos \theta + \sqrt{\eta}p \sin \theta; \theta) = \sum_n A_n(x) \rho_n(q, p),$$

which can be readily implemented in the iterative algorithm. Thus, we have arrived at the following two-step algorithm for quantum tomography: for a given phase space point $(q, p)$, construct the phase-averaged histogram according to the right-hand side of Eq. (8), and iteratively reconstruct $\rho_n(q, p)$. Then, calculate the value of the Wigner function according to Eq. (7). In Fig. 1 we present Monte Carlo simulated reconstruction of the Wigner function for a Schrödinger cat state detected using a homodyne setup with the efficiency $\eta = 90\%$. 
The standard filtered back-projection algorithm used in quantum tomography is based on the inverse Radon transform, whose integral kernel is singular. Therefore, a regularization scheme is necessary in processing experimental data. This aspect has a counterpart in the maximum-likelihood algorithm. In this approach, we have the cut-off for the photon distribution which can be regarded as a regularization parameter. Its proper choice is an important matter. Setting it too small perturbs the reconstructed photon distribution. On the other hand, the larger number of $\varrho_n$s, the slower iterations converge. The expected shape of the photon distribution can be quite easily predicted, if we roughly know the region of the phase space where the Wigner function is localized. For this purpose it is useful to recall the semiclassical picture of projections on Fock states as rings in the phase space characterized by the radius $\sqrt{2n}$. The photon distribution is nonzero over the range of $n$ for which the corresponding rings overlap with the localization region for the Wigner function.

Truncation of the photon distribution can be introduced as a regularization scheme also in the standard linear reconstruction approach. In such a scheme, the Wigner function would be evaluated from a finite part of the photon distribution reconstructed using the pattern function technique. However, properties of the reconstructed photon distribution make this method very sensitive to statistical noise. This can be straightforwardly seen in the most regular case of $\eta = 1$. For large $n$, the error of $\varrho_n$ tends to a fixed nonzero value [13], and moreover deviations of consecutive $\varrho_n$s are strongly anticorrelated. Consequently, the alternating sum defined in Eq. (7) accumulates the statistical uncertainty of the photon distribution [14].

Let us also note that in principle we could apply the restoration algorithm to homodyne histograms described by Eq. (1), in order to obtain deblurred quadrature distributions $\langle x_\theta | \hat{\varrho} | x_\theta \rangle$. In this case statistical fluctuations would play a much more significant role. The advantage of using Eq. (8) is that we use the whole available sample of experimental data to determine a relatively small number of parameters $\varrho_n(q,p)$.

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