JOINT ESTIMATION AND MODEL ORDER SELECTION FOR ONE DIMENSIONAL ARMA MODELS VIA CONVEX OPTIMIZATION: A NUCLEAR NORM PENALIZATION APPROACH

STÉPHANE CHRÉTIEN, TIANWEN WEI AND BASAD ALI HUSSAIN AL-SARRAY

Abstract. The problem of estimating ARMA models is computationally interesting due to the non-concavity of the log-likelihood function. Recent results were based on the convex minimization. Joint model selection using penalization by a convex norm, e.g. the nuclear norm of a certain matrix related to the state space formulation was extensively studied from a computational viewpoint. The goal of the present short note is to present a theoretical study of a nuclear norm penalization based variant of the method of [2, 3] under the assumption of a Gaussian noise process.

Keywords: ARMA models, Time series, Low rank model, Prediction, Nuclear norm penalization.

1. Introduction

The Auto-regressive with moving average (ARMA) model is central to the field of time serie analysis and has been studied since the early thirties in the field of econometrics [12]. ARMA time series are sequences of the form \((x_t)_{t \in \mathbb{N}}\) satisfying the following recursion

\[
x_t = \sum_{i=1}^{p} a_i x_{t-i} + \sum_{j=1}^{q} b_j e_{t-j} + e_t
\]

for all \(t \geq \max\{p, q\}\), and we focus on the case where \((e_t)_{t \in \mathbb{N}}\) is a sequence of zero mean independent identically distributed Gaussian random variables with variance denoted by \(\sigma^2_e\) for simplicity. As is well known [12], time series model are adequate for a wide range of phenomena in economics, engineering, social science, epidemiology, ecology, signal processing, etc. They can also be helpful as a building block in more complicated models such as GARCH models, which are particularly useful in financial time series analysis.

Two problems are to be addressed when studying ARMA time series:

1. estimate \(p\) and \(q\), the intrinsic orders of the model.
2. estimate \(a = (a_1, a_2, \ldots, a_p)\) and \(b = (b_1, b_2, \ldots, b_q)\).

In the case where \(q = 0\), the convention is to write (1.1) as:

\[
x_t = \sum_{i=1}^{p} a_i x_{t-1} + e_t
\]

and \(x_t\) to simply called an AR process. Estimation of \(a\) is often performed using the conditional likelihood approach, given \(x_0, \ldots, x_{p-1}\) yielding to the standard Yule-Walker equations. On the other hand, the model order selection problem is often performed using a penalized log-likelihood approach such as AIC, BIC, ..., may also use the plain likelihood. We refer the reader to the standard text of Brockwell and Davis for more details on these standard problems. Turning back to the full ARMA model, it is well known that the log-likelihood is not a concave function, and that multiple stationary points exist which can lead to severe bias when using local optimization routines for such as gradient or Newton-type methods for the joint estimation of \(a\) and \(b\). In Shumway and Stoffer [12] and iterative procedure resembling the EM algorithm is proposed, which seems more appropriate for the ARMA model than...
standard optimization algorithms. However, no convergence guarantee towards a global maximizer is provided. Concerning the model selection problem, penalties play a prominent role in modern statistical theory and practice, in particular since the recent successes of the LASSO in regression and its multiple generalization. The nuclear norm penalization has played an import for many problems in engineering, machine learning and statistics such as matrix completion, . . . Application of nuclear norm penalization to state space model estimation and model order selection using a moment-like estimator in a convex optimization framework is proposed in [6]. The approach of [6] is a remarkable contribution since convex model selection and state space estimation were combined for the first time in the problem of Time Series. However the approach of [6] is supported by no theoretical guarantee yet. Another approach for State Space model estimation was proposed in [2, 3] where good practical performances are reported and an asymptotic analysis is provided. This method as well as the unpenalized version of the method in [6] can be recast into the family of subspace methods; see [15].

Based on the evidence of the practical efficiency of subspace-type methods [15], our goal in the present note is to propose a theoretical study of a nuclear norm penalized version of the subspace method from [2] which incorporates the main ideas in [6].

2. The subspace method

2.1. Recall on the subspace approach. A real valued random discrete dynamical system \( (x_t)_{t \in \mathbb{N}} \) admits a State Space representation if there exists a discrete time process \( (s_t)_{t \in \mathbb{N}} \) such that

\[
\begin{align*}
    s_{t+1} &= As_t + Ke_t \\
    x_t &= Bs_t + e_t
\end{align*}
\]

where \( (e_t)_{t \in \mathbb{N}} \) is the noise, and \( A \in \mathbb{R}^{p \times p} \), \( B \in \mathbb{R}^{1 \times p} \), \( K \in \mathbb{R}^{p \times 1} \) are parameter matrices. It is well known that ARMA processes admit a State Space representation and vice versa [12].

2.2. Prediction. The problem of predicting \( x_{t+j} \) for \( j \geq 0 \) based on the knowledge of \( x_{t'}, t' < t \) and \( s_t \) can be solved easily following the approach by Bauer [2, 3]. For given initial values \( x_0, e_0 \), the State Space representation gives

\[
x_{t+h} = e_{t+h} + \sum_{j=1}^{h} BA^{j-1}K e_{t+h-j} + BA^h s_t
\]

On the other hand, the State Space representation implies that

\[
\begin{align*}
    s_t &= As_{t-1} + Ke_{t-1} \\
         &= As_{t-1} + K (x_{t-1} - Bs_{t-1}) \\
         &= (A - KB) s_{t-1} + K x_{t-1} \\
         &\quad \vdots
\end{align*}
\]

Thus, we obtain

\[
s_t = (A - KB)^j s_0 + \sum_{j=0}^{t-1} (A - KB)^j K x_{t-1-j}.
\]

In what follows, we will assume that we observe \( x_0, \ldots, x_T \) and that \( t > 0 \) is such that \( T - 2t + 1 > 0 \).

2.3. Prediction with Hankel matrices. We will rewrite the prediction problem in terms of some Hankel matrices. For this purpose, define

\[
\begin{align*}
    \bar{A} &= A - KB, \quad \bar{A}_0 = [\bar{A}^1 s_0, \bar{A}^{t+1}_0, \ldots, \bar{A}^{T-t+1} s_0], \quad \bar{K} = [\bar{A}^{t-1} K, \cdots, \bar{A}^2 K, \bar{A} K, K],
\end{align*}
\]
Then, we have

\[
\begin{bmatrix}
x_t \\
\vdots \\
x_{2t-1}
\end{bmatrix}
= \mathcal{O}_t + \mathcal{N}
\begin{bmatrix}
e_t \\
\vdots \\
e_{t+h}
\end{bmatrix}
\]

and

\[
s_t = \mathcal{K}
\begin{bmatrix}
x_0 \\
\vdots \\
x_{t-1}
\end{bmatrix}
+ (A - KB)^t s_0.
\]

Combining (2.3) and (2.4), we thus obtain

\[
\begin{bmatrix}
x_t \\
\vdots \\
x_{2t-1}
\end{bmatrix}
= \mathcal{O} \mathcal{K}
\begin{bmatrix}
x_0 \\
\vdots \\
x_{t-1}
\end{bmatrix}
+ \mathcal{O} (A - KB)^t s_0
+ \mathcal{N}
\begin{bmatrix}
e_t \\
\vdots \\
e_{t+h}
\end{bmatrix}.
\]

Now, define

\[
X_{\text{past}} =
\begin{bmatrix}
x_0 & x_1 & \cdots & x_{T-t+1} \\
x_1 & x_2 & \cdots & x_{T-t+2} \\
\vdots & \vdots & \ddots & \vdots \\
x_{t-1} & x_t & \cdots & x_T
\end{bmatrix}
\quad \text{and} \quad
X_{\text{future}} =
\begin{bmatrix}
x_t & x_{t+1} & \cdots & x_{T-t+1} \\
x_{t+1} & x_{t+2} & \cdots & x_{T-t+2} \\
\vdots & \vdots & \ddots & \vdots \\
x_{2t-1} & x_{2t} & \cdots & x_T
\end{bmatrix}.
\]

Both matrices are Hankel matrices. The first one represents the past values and and second one the future values. Define also the noise matrix

\[
E =
\begin{bmatrix}
e_t & e_{t+1} & \cdots & e_{T-t+1} \\
e_{t+1} & e_{t+2} & \cdots & e_{T-t+2} \\
\vdots & \vdots & \ddots & \vdots \\
e_{2t-1} & e_{2t} & \cdots & e_T
\end{bmatrix}.
\]

All these Hankel matrices are related by the following equation

\[
X_{\text{future}} = \mathcal{O} \mathcal{K} X_{\text{past}} + \mathcal{O} \hat{A}_0 + \mathcal{N} E.
\]

### 3. The estimation problem

In the last section, we showed that the matrices \(A, B\) and \(K\) of the State Space model entered nicely into an equation allowing prediction of future values based on past values of the dynamical system. Our goal is now to use this equation to estimate the matrices \(A, B\) and \(C\). One interesting feature of our procedure is that the dimension \(p\) of the State Space model can be estimated jointly with the matrices themselves.

#### 3.1. Estimating \(\mathcal{O} \mathcal{K}\)

The matrix \(\mathcal{O} \mathcal{K}\) can be estimated using a least squares approach corresponding to solving

\[
\min_{\mathcal{L} \in \mathbb{R}^{r \times t}} \frac{1}{2} \| X_{\text{future}} - \mathcal{L} X_{\text{past}} \|_F^2.
\]

This procedure will make sense if the term \(\mathcal{O} \hat{A}_0\) is small. This can indeed be justified if \(t\) is large and if \(\|A\|\) is small. Let us call \(\hat{L}\) a solution of (3.5).
3.2. **Nuclear Norm penalized $\ell_1$-norm for low rank estimation.** An interesting property of the matrix $OK$ is that its rank is the State’s dimension $p$ when $A$ has full rank. Moreover, $OK$ has small rank compared to $t$ when $t$ is large compared to $p$. Therefore, one is tempted to penalize the least squares problem (3.5) with a low-rank promoting penalty.

One option is to try to solve

$$\min_{L \in \mathbb{R}^{t \times t}} \frac{1}{2} \|X_{\text{future}} - LX_{\text{past}}\|_F^2 + \lambda \operatorname{rank}(L) \quad (3.6)$$

The main drawback of this approach is that the rank function is non continuous and non convex. This renders the optimization problem intractable in practice. Fortunately, the rank function admits a well known convex surrogate, which is the nuclear norm, i.e. the sum of the singular values, denoted by $\|\cdot\|_\star$.

Thus, a nice convex relaxation of (3.6) is given by

$$\min_{L \in \mathbb{R}^{t \times t}} \frac{1}{2} \|X_{\text{future}} - LX_{\text{past}}\|_F^2 + \lambda \|L\|_\star. \quad (3.7)$$

It has been observed in practice that nuclear norm penalized least squares provide low rank solution for many interesting estimation problems [11].

4. **Main results**

The penalized least-squares problem (3.7) can be transformed into the following constrained problem

$$\min_{L \in \mathbb{R}^{t \times t}} \|L\|_\star \quad \text{subject to} \quad \|X_{\text{future}} - LX_{\text{past}}\|_F \leq \eta, \quad (4.8)$$

for some appropriate choice of $\eta$.

Let $\Sigma$ denote the covariance matrix of $[x_0, \ldots, x_{t-1}]^t$ and let $\Sigma^{\pm \frac{1}{2}}$ denote the square root of $\Sigma^{\pm 1}$. Then, Let $H$ be the random matrix whose components are given by

$$H_{s,r} = \sum_{s'=0}^{T-2t+1} \varepsilon_{s,s'} z_{s'+r}.$$ 

where $\varepsilon_{s,s'}$, $s = 0, \ldots, t - 1$ and $s' = 0, \ldots, T - 2t + 1$ are independent Rademacher random variables which are independent of $z_{s'}$, $s' = 0, \ldots, T - t$. Let $\Sigma^H$ denote the covariance matrix of $\text{vec}(H)$. Let $\mathcal{M}$ denote the operator defined by

$$\mathcal{M} = \text{Mat} \left( \Sigma^{-1/2} \text{vec}(\cdot) \right) \quad (4.9)$$

and let $\mathcal{M}^{-\star}$ denote the adjoint of the inverse of $\mathcal{M}$. The fact that $\mathcal{M}$ is invertible is easily obtained (see Section 6.3.3) and is seen from the fact that $\Sigma^H$ has all its eigenvalues equal to $T - 2t + 1$ according to Section 6.3.2. Let $S$ be the operator defined by

$$S(\cdot) \mapsto \mathcal{M}^{-\star}(\cdot) \Sigma^{-1/2}. \quad (4.10)$$

and let $T$ be the mapping

$$T(\cdot) \mapsto \frac{1}{\sqrt{t(T-2t+2)}} \mathcal{M}^{-1} \left( \cdot \Sigma^{\frac{1}{2}} \right).$$

Our main result is the following theorem.

**Theorem 4.1.** Let $\xi$ be any positive real number. Assume that $\eta$ is such that

$$\|O \hat{A} s_0 + NE\| \geq \eta \quad (4.10)$$

with probability less than or equal to $e^{-\nu^2/2}$ for some $\nu > 0$. Then, with probability greater than or equal to $1 - e^{-\nu^2/2}$,

$$\|OK - \hat{L}\|_F \leq \frac{2\eta}{\Lambda} \quad (4.11)$$
where
\[
\lambda \geq \xi \sqrt{t(T-2t+1)} \left(1 - \frac{4\xi}{\sqrt{\pi}} \left(\frac{e}{2}\right)^{\frac{1}{2}} \sigma_{\max} \left(\Sigma^{1/2}\right) \sqrt{t}\right)
\]
\[-2\sqrt{2} \sqrt{\frac{t}{T-2t+1}} + \sigma_{\max}(\Sigma) \sqrt{(2cT + 1 + c\sqrt{t}) \sqrt{t} \frac{\text{rank}(\Sigma)}{c \sqrt{\sigma_{\min}(\Sigma)}} + 2t - \nu \xi}.
\]

In the remainder of this section, we introduce the results, notations and tools for proving this theorem. The proof is given in Section 4.6.

4.1. Some notations. For all \(s = 0, \ldots, t-1\) and \(s' = 0, \ldots, T-2t+1\), let \(\mathcal{A}_{s,s'}\) denote the operator defined by
\[
\mathcal{A}_{s,s'}(L) = \sum_{r=0}^{t-1} L_{s,r} x_{s'+r}
\]
and let \(\mathcal{A}\) denote the operator
\[
L \mapsto (\mathcal{A}_{s,s'}(L))_{s=1, \ldots, t, s'=0, \ldots, T-2t+1}.
\]
The descent cone of the nuclear norm at \(\mathcal{O}\mathcal{K}\), denoted by \(\mathcal{D}(\|\cdot\|_*, \mathcal{O}\mathcal{K})\), is defined by
\[
\mathcal{D}(\|\cdot\|_*, \mathcal{O}\mathcal{K}) = \bigcup_{\tau > 0} \left\{ D \in \mathbb{R}^{t \times t} \mid \|H + \tau D\|_* \leq \|H\|_* \right\}.
\]

4.2. A deterministic inequality. The following result will be the key of our analysis.

**Theorem 4.2.** [14] Assume that
\[
\|\mathcal{O}\hat{A}s_0 + \mathcal{N}E\| \leq \eta.
\]
Let \(\hat{L}\) denote any solution of \(\hat{A}\). Then,
\[
\|\mathcal{O}\mathcal{K} - \hat{L}\|_F \leq \frac{2\eta}{\lambda_{\min}(\mathcal{A}, \mathcal{D}(\|\cdot\|_*, \mathcal{O}\mathcal{K}))},
\]
where
\[
\lambda_{\min}(\mathcal{A}, \mathcal{D}(\|\cdot\|_*, \mathcal{O}\mathcal{K})) = \min_{D \in \mathcal{D}(\|\cdot\|_*, \mathcal{O}\mathcal{K})} \|\mathcal{A}(D)\|_F.
\]

4.3. A lower bound on \(\lambda_{\min}(\mathcal{A}, \mathcal{D}(\|\cdot\|_*, \mathcal{O}\mathcal{K}))\). We will closely follow the approach of Tropp based on Mendelson’s bound. For this purpose, we will need the definition of the Gaussian mean width \(w_G(\mathcal{X})\) of a set \(\mathcal{X} \in \mathbb{R}^d\)
\[
w_G(\mathcal{X}) = \mathbb{E} \left[ \sup_{x \in \mathcal{X}} \langle G, x \rangle \right],
\]
where the expectation is taken with respect to the Gaussian random vector \(G\) taking values in \(\mathbb{R}^d\). The statistical dimension of \(\mathcal{X}\) (see e.g. [1]) is Let us also denote by \(Q_\xi\) the quantity
\[
Q_\xi(D) = \frac{1}{t} \sum_{s=0}^{t-1} \mathbb{P} \left( \sum_{r=0}^{t-1} D_{s,r} z_{s'+r} \geq \xi \right),
\]
which, as one might easily check, does not depend on \(s'\). Recall that \(\Sigma\) is the covariance matrix of \(x_0, \ldots, x_{t-1}\) and that \(\Sigma^{\pm \frac{1}{2}}\) denotes the square root of \(\Sigma^{\pm 1}\). Thus,
\[
\begin{bmatrix}
z_0 \\
\vdots \\
z_{t-1}
\end{bmatrix} = \Sigma^{-\frac{1}{2}}
\begin{bmatrix}
x_0 \\
\vdots \\
x_{t-1}
\end{bmatrix}
\]
follows the standard Gaussian distribution \(\mathcal{N}(0, I)\). Let \(\hat{D} = D\Sigma^{-\frac{1}{2}}\). We now state Tropp’s result.
Lemma 4.3. Define

\[ K = \frac{1}{\sqrt{t(T-2t+2)}} M^{-1} \left( \Sigma^{-1/2} D(\| \cdot \|_*, \mathcal{O}K) \right). \]

We have

\[ \lambda_{\text{min}}(A, D(\| \cdot \|_*, \mathcal{O}K)) \geq \xi \sqrt{t(T-2t-2)} \inf_{\| \tilde{D} \|_{\Sigma}^{1/2}} Q_{2\xi}(\tilde{D}) - \frac{2}{\sigma_{\text{min}}(\mathcal{S})} w_G(K) - \nu \xi \]

with probability greater than or equal to 1 - exp(-\nu^2/2).

Proof. See Section 6.7. \qed

4.4. A lower bound on \( \inf_{\| \tilde{D} \|_{\Sigma}^{1/2}} Q_{2\xi}(\tilde{D}) \). Since

\[ Z = \sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \]

follows the law \( \mathcal{N}(0, \sum_{r=0}^{t-1} \tilde{D}_{s,r}^2) \), using Lemma 6.2 from the Appendix, we get

\[ \mathbb{P} \left( Z^2 \leq \left( \sum_{r=0}^{t-1} \tilde{D}_{s,r}^2 \right) \sqrt{u} \right) \leq \frac{2}{\sqrt{\pi}} \left( \frac{\epsilon}{2} \right)^{1/4} \frac{\xi}{\sqrt{\sum_{r=0}^{t-1} \tilde{D}_{s,r}^2}}. \]

Thus, setting

\[ u = \frac{\xi^4}{\left( \sum_{r=0}^{t-1} \tilde{D}_{s,r}^2 \right)^2}, \]

we obtain

\[ \mathbb{P} \left( \left| \sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right| \geq \xi \right) \geq 1 - \frac{2}{\sqrt{\pi}} \left( \frac{\epsilon}{2} \right)^{1/4} \frac{\xi}{\sqrt{\sum_{r=0}^{t-1} \tilde{D}_{s,r}^2}}. \]

This finally gives

\[ Q_{2\xi}(\tilde{D}) \geq 1 - \frac{4\xi}{\sqrt{\pi}} \left( \frac{\epsilon}{2} \right)^{1/4} \frac{1}{t} \sum_{s=0}^{t-1} \frac{1}{\sqrt{\sum_{r=0}^{t-1} \tilde{D}_{s,r}^2}}. \]

Let us now compute a lower bound to the infimum of this quantity over the set of \( \tilde{D} \) satisfying \( \| \tilde{D} \|_{\Sigma}^{1/2} = 1 \). For this purpose, first note that

\[ \inf_{\| \tilde{D} \|_{\Sigma}^{1/2}=1} Q_{2\xi}(\tilde{D}) \geq 1 - \sup_{\| \tilde{D} \|_{\epsilon} \geq \sigma_{\text{max}}(\Sigma^{1/2})^{-1}} \frac{4\xi}{\sqrt{\pi}} \left( \frac{\epsilon}{2} \right)^{1/4} \frac{1}{t} \sum_{s=0}^{t-1} \frac{1}{\sqrt{\sum_{r=0}^{t-1} \tilde{D}_{s,r}^2}}. \]

On the other hand, simple manipulations of the optimality conditions using symmetry prove that

\[ \sup_{\| A \|_{\epsilon} \leq 1} \frac{1}{t} \sum_{s=0}^{t-1} \frac{1}{\sqrt{\sum_{r=0}^{t-1} A_{s,r}^2}} = \sqrt{t}. \]

Therefore,

\[ (4.18) \quad \inf_{\| \tilde{D} \|_{\Sigma}^{1/2}=1} Q_{2\xi}(\tilde{D}) \geq 1 - \frac{4\xi}{\sqrt{\pi}} \left( \frac{\epsilon}{2} \right)^{1/4} \sigma_{\text{max}}(\Sigma^{1/2}) \sqrt{t}. \]

4.5. The Gaussian mean width of \( K \). The Gaussian mean width of a set \( \mathcal{X} \) and its statistical dimension are related by

\[ (4.19) \quad w_G(\mathcal{X})^2 \leq \delta(\mathcal{X}) \leq w_G(\mathcal{X})^2 + 1. \]

See [11 Proposition 10.2] for a proof. In this subsection, we estimate the Gaussian mean width of \( K \) using its statistical dimension.
4.5.1. The descent cone $\mathcal{D}(\| \cdot \|, \mathcal{O}K)$. The descent cone of the nuclear norm satisfies [14, Eq. (4.1)] which we recall now

\begin{equation}
\mathcal{D}(\| \cdot \|, \mathcal{O}K)^{\circ} = \text{cone} (\partial\| \cdot \|_{\ast}(\mathcal{O}K)).
\end{equation}

4.5.2. Computation of $K^{\circ}$. Using Proposition 4.2 in [14], we obtain

\begin{equation}
\sup_{\left\| \tilde{\mathcal{D}}^{\ast} \right\|_{F} = 1} \left< \tilde{\mathcal{D}}^{\ast}, \tilde{\mathcal{H}} \right> \leq \text{dist}(\tilde{\mathcal{H}}, K^{\circ}).
\end{equation}

We now have to compute the polar cone of $K$. We have

\[
K^{\circ} = \{ \Delta \mid \langle \Delta, D \rangle \leq 0 \quad \forall D \in K \} = \left\{ \Delta \mid \left< \frac{1}{\sqrt{t(T-2t+2)}} M^{-1} \left( \Delta \Sigma^{\frac{1}{2}} \right), D \right> \leq 0 \quad \forall D \in \mathcal{D}(\| \cdot \|, \mathcal{O}K) \right\}.
\]

Recall that $T$ is the mapping $\Delta \mapsto \frac{1}{\sqrt{t(T-2t+2)}} M^{-1} \left( \Delta \Sigma^{\frac{1}{2}} \right)$. Then, we obtain that

\[
K^{\circ} = T^{-1}(\mathcal{D}(\| \cdot \|, \mathcal{O}K)^{\circ}).
\]

4.5.3. An upper bound on the statistical dimension of $K$. Let us write the singular value decomposition of $\mathcal{O}K$ as

\[
\mathcal{O}K = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \text{diag}(\sigma_{\mathcal{O}K}) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{\prime}
\]

where $\sigma_{\mathcal{O}K}$ is the vector of the singular values of $\mathcal{O}K$. Moreover, the subdifferential of the Schatten norm is given by

\[
\partial\| \cdot \|_{\ast}(\mathcal{O}K) = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \left\{ \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} \mid \|Y\| \leq 1 \right\} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{\prime}.
\]

Therefore, using (4.21), we obtain that

\[
\mathbb{E}\left[ \left( \sup_{\left\| \tilde{\mathcal{D}}^{\ast} \right\|_{F} = 1, \tilde{\mathcal{D}}^{\ast} \in K} \left< \tilde{\mathcal{D}}^{\ast}, \tilde{\mathcal{H}} \right> \right)^{2} \right] \leq \mathbb{E}\left[ \min_{\tau > 0, \|Y\| \leq 1} \|T^{-1}\left( \tau \begin{bmatrix} U_1 V_1^{\prime} & 0 \\ 0 & U_2 Y V_2^{\prime} \end{bmatrix} \right) - \tilde{\mathcal{H}} \|_{F}^{2} \right].
\]

Thus, we get

\[
\mathbb{E}\left[ \left( \sup_{\left\| \tilde{\mathcal{D}}^{\ast} \right\|_{F} = 1, \tilde{\mathcal{D}}^{\ast} \in K} \left< \tilde{\mathcal{D}}^{\ast}, \tilde{\mathcal{H}} \right> \right)^{2} \right] \leq \mathbb{E}\left[ \min_{\tau > 0, \|Y\| \leq 1} \|T^{-1}\|^{2} \left( \tau^{2}\|U_1 V_1^{\prime}\|_{F}^{2} + \|U_2 Y V_2^{\prime} - T_{2,2}(\tilde{\mathcal{H}})\|_{F}^{2} \right)
\right.
\]

\[
+ \|T_{1,2}(\tilde{\mathcal{H}})\|_{F}^{2} + \|T_{2,1}(\tilde{\mathcal{H}})\|_{F}^{2}\right)\]

where

\[
T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},
\]
and the dimension of $T_{11}$ is $\text{rank}(\mathcal{O}K) \times \text{rank}(\mathcal{O}K)$ and the dimension of $\mathcal{T}_{j, j'}$ for all other combinations of $j$ and $j'$ is easily deduced from the dimension of $\mathcal{T}$. which gives, after taking $\tau = \|U_2 \mathcal{T}_{2,2}(\mathcal{H}) V_2\|$, 

$$
\mathbb{E} \left[ \left( \sup_{\tilde{D}^* \in K, \tilde{F} \in F} \langle \tilde{D}^*, \mathcal{H} \rangle \right)^2 \right] \leq \sigma_{\min}(\mathcal{T})^{-1} \left( \mathbb{E} [\tau^2] \text{ rank}(\mathcal{O}K) \right. 
\left. + \left( \sigma_{\max}(\mathcal{T}_{1,2})^2 + \sigma_{\max}(\mathcal{T}_{2,1})^2 \right) \mathbb{E} \left[ \| \mathcal{H} \|_F^2 \right] \right).
$$

Note that 

$$
\tau \leq \|\mathcal{T}_{2,2}\| \|\mathcal{H}\|.
$$

By Gordon’s theorem [16, Theorem 10.2], $\mathbb{E}[\|\mathcal{H}\|] \leq 2 \sqrt{t}$. Moreover, by Lemma 6.1 in the Appendix, 

$$
\mathbb{E}[\|\mathcal{H}\|^2] \leq \frac{2}{c} (2 ct + 1) + 2 \sqrt{t}.
$$

On the other hand, $\mathbb{E}[\|\mathcal{H}\|_F^2] = 2t$. Therefore, we obtain that 

$$
\delta(K) = \mathbb{E} \left[ \left( \sup_{\tilde{D}^* \in K, \tilde{F} \in F} \langle \tilde{D}^*, \mathcal{H} \rangle \right)^2 \right] 
\leq 2 \sigma_{\min}(\mathcal{T})^{-1} \left( \frac{1}{c} \|\mathcal{T}_{2,2}\|^2 \left( (2 ct + 1) + c \sqrt{t} \right) \text{ rank}(\mathcal{O}K) \right. 
\left. + 2 \left( \sigma_{\max}(\mathcal{T}_{1,2})^2 + \sigma_{\max}(\mathcal{T}_{2,1})^2 \right) t \right).
$$

Using (4.19), we obtain that 

$$
(4.22) \quad w_G(K) \leq 
\sqrt{\frac{2 \sigma_{\min}(\mathcal{T})^{-1} \left( \frac{1}{c} \|\mathcal{T}_{2,2}\|^2 \left( (2 ct + 1) + c \sqrt{t} \right) \text{ rank}(\mathcal{O}K) + 2 \left( \sigma_{\max}(\mathcal{T}_{1,2})^2 + \sigma_{\max}(\mathcal{T}_{2,1})^2 \right) t \right) \left[ \left( \frac{(c/2)^{1/2}}{\sqrt{T}} \right) - \frac{2 \sqrt{2}}{\sigma_{\min}(\mathcal{S})} \right] \left( \frac{(1/c) \|\mathcal{T}_{2,2}\|^2 \left( (2 ct + 1) + c \sqrt{t} \right) \text{ rank}(\mathcal{O}K)}{\sigma_{\min}(\mathcal{T})} + \left( \sigma_{\max}(\mathcal{T}_{1,2})^2 + \sigma_{\max}(\mathcal{T}_{2,1})^2 \right) t \right) - \nu \xi \right)}.
$$

4.6. Proof of Theorem 4.4. Combining Lemma 4.3 with (4.18) and (4.22), we obtain that 

$$
\lambda_{\min}(\mathcal{A}, \mathcal{D}(\| \cdot \|, \mathcal{O}K)) \geq t \sqrt{T - 2t - 2} - \frac{4 \xi^2}{\sqrt{T}} \left( \frac{c}{2} \right)^{1/2} \sigma_{\min}(\Sigma^{1/2}) 
- \frac{2 \sqrt{2}}{\sigma_{\min}(\mathcal{S})} \left[ \left( \frac{(1/c) \|\mathcal{T}_{2,2}\|^2 \left( (2 ct + 1) + c \sqrt{t} \right) \text{ rank}(\mathcal{O}K)}{\sigma_{\min}(\mathcal{T})} + \left( \sigma_{\max}(\mathcal{T}_{1,2})^2 + \sigma_{\max}(\mathcal{T}_{2,1})^2 \right) t \right) - \nu \xi \right).
$$

Using that 

$$
\|\mathcal{T}_{2,2}\|^2 \leq \|T\|^2,
$$

and 

$$
\sigma_{\max}(\mathcal{T}_{1,2})^2 + \sigma_{\max}(\mathcal{T}_{2,1})^2 \leq 2 \|T\|^2,
$$

and combining this last inequality with Theorem 4.2, we obtain the following proposition. 

**Proposition 4.4.** Let $\xi$ be any positive real number. Assume that $\eta$ is such that 

$$
(4.23) \quad \|\mathcal{O}A s_0 + \mathcal{N}E\| \geq \eta
$$
with probability less than or equal to $e^{-\nu^2/2}$ for some $\nu > 0$. Then, with probability greater than or equal to $1 - e^{-\nu^2/2}$,

$$\|\mathcal{O}\mathcal{K} - \hat{\mathcal{L}}\|_F \leq \frac{2\eta}{\Lambda},$$

where

$$\Lambda \geq \xi \sqrt{t(T - 2t + 1)} \left( 1 - \frac{4\xi}{\sqrt{\pi}} \left( \frac{c}{2} \right)^{\frac{1}{3}} \sigma_{\max} \left( \Sigma^{1/2} \right) \sqrt{t} \right)$$

$$- \frac{2\sqrt{2}\|T\|}{\sigma_{\min}(\mathcal{S})} \sqrt{\left( 2ct + 1 + c\sqrt{t} \right) \frac{\text{rang}(\mathcal{O}\mathcal{K})}{c\sigma_{\min}(T)} + 2t - \nu\xi}.$$

Combining this result with the bounds from Section 6.3.3, the proof is completed.

5. Conclusion

The goal of the present note is to show that the performance of nuclear norm penalized subspace-type methods can be studied theoretically. We concentrated on a special approach due to Bauer [2]. Our approach can easily be extended to the case of the method promoted in [3]. Our next objective for future research is to address the case of more general noise sequences such as in [9].

6. Appendix: Technical intermediate results

In this section, we gather some technical results used in the proof of Theorem 1.4

6.1. Proof of Lemma 4.3

6.1.1. First step. We have

$$\lambda_{\min}(\mathcal{A}, \mathcal{D}(\|\cdot\|_s, \mathcal{O}\mathcal{K})) = \min_{\|D\|_F = 1, D \in \mathcal{D}(\|\cdot\|_s, \mathcal{O}\mathcal{K})} \|\mathcal{A}(D)\|_F$$

$$= \min_{\|D\|_F = 1, D \in \mathcal{D}(\|\cdot\|_s, \mathcal{O}\mathcal{K})} \left( \sum_{s=0}^{t-1} D_{s,r}x_{s' + r} \right)^2 \frac{1}{2}$$

Recall that $\Sigma$ is the covariance matrix of $[x_0, \ldots, x_{t-1}]^t$ and that $\Sigma^{\frac{1}{2}}$ denotes the square root of $\Sigma^{-1}$. Thus,

$$\begin{bmatrix} z_0 \\ \vdots \\ z_{t-1} \end{bmatrix} := \Sigma^{-\frac{1}{2}} \begin{bmatrix} x_0 \\ \vdots \\ x_{t-1} \end{bmatrix}$$

follows the standard Gaussian distribution $\mathcal{N}(0, I)$. Recall also that $\hat{D} = D\Sigma^{\frac{1}{2}}$. Then, we have

$$\lambda_{\min}(\mathcal{A}, \mathcal{D}(\|\cdot\|_s, \mathcal{O}\mathcal{K})) = \min_{\|D\|_F = 1, \hat{D} \in \mathcal{D}(\|\cdot\|_s, \mathcal{O}\mathcal{K})} \left( \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \sum_{r=0}^{t-1} \hat{D}_{s,r}z_{s' + r} \right)^2 \frac{1}{2} \geq \frac{1}{t(T - 2t + 1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \sum_{r=0}^{t-1} \hat{D}_{s,r}z_{s' + r}$$

which gives, by Markov’s inequality

$$\left( \frac{1}{t(T - 2t + 1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \sum_{r=0}^{t-1} \hat{D}_{s,r}z_{s' + r} \right)^{\frac{1}{2}} \geq \frac{\xi}{t(T - 2t + 1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \sum_{r=0}^{t-1} \hat{D}_{s,r}z_{s' + r} \geq \xi.$$
Thus, we obtain
\[
\left(\frac{1}{t(T-2t+1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \left(\sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right)^2\right)^{\frac{1}{2}} \geq \xi Q_{2\xi}(\tilde{D}) - \frac{\xi}{\Omega} \frac{1}{t(T-2t+1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \left(\sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right)^2 \geq \xi \left\{ \sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \geq \xi \right\}.
\]

6.1.2. Second step. Let
\[
f(z_0, \ldots, z_{T-t}) = \sup_{||\Delta^{1/2}||_{F=1}} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \left(\sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right)^2 \geq \xi Q_{2\xi}(\tilde{D}) - \frac{\xi}{\Omega} \frac{1}{t(T-2t+1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \left(\sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right)^2 \geq \xi \left\{ \sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \geq \xi \right\}.
\]

We will now use the bounded difference inequality to control this quantity. For this purpose, notice that
\[
|f(\zeta_0, \ldots, \zeta_{t-1}) - f(\zeta_0, \ldots, \zeta_t, \ldots, \zeta_{t-1})| \leq 2t(T-2t+2),
\]
for all \((\zeta_0, \ldots, \zeta_s, \ldots, \zeta_{T-t})\) in \(\mathbb{R}^T\) and \(\zeta_s \in \mathbb{R}\). Thus,
\[
f(z_0, \ldots, z_T) - \mathbb{E}[f(z_0, \ldots, z_{T-t})] \leq \nu \sqrt{t(T-2t+2)},
\]
with probability \(1 - e^{-\nu^2/2}\) for all \(\nu \in \mathbb{R}_+\). Now, the expected supremum can be bounded in the same manner as in [14, Equation 5.6].
\[
\mathbb{E}[f(z_0, \ldots, z_{T-t})] \leq \frac{2}{\xi} \mathbb{E}\left[ \sup_{||\Delta^{1/2}||_{F=1}} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \varepsilon_{s,s'} \sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right],
\]
where \(\varepsilon_{s,r}, s = 0, \ldots, t-1\) and \(r = 0, \ldots, t-1\) are independent Rademacher random variables which are independent of \(z_{s'}, s' = 0, \ldots, T-t\). Therefore, we obtain
\[
\inf_{||\Delta^{1/2}||_{F=1}} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \left(\sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right)^2 \geq \xi Q_{2\xi}(\tilde{D}) - \frac{\xi}{\Omega} \frac{1}{t(T-2t+1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \left(\sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right)^2 \geq \xi \left\{ \sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \geq \xi \right\}.
\]
which gives
\[
\inf_{||\Delta^{1/2}||_{F=1}} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \left(\sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \right)^2 \geq \xi \sqrt{t(T-2t+2)} Q_{2\xi}(\tilde{D}) - \frac{\xi}{\Omega} \frac{1}{t(T-2t+1)} \sum_{s=0}^{t-1} \sum_{s'=0}^{T-2t+1} \varepsilon_{s,s'} \sum_{r=0}^{t-1} \tilde{D}_{s,r} z_{s'+r} \geq -\nu \xi.
\]
Let us denote by \( W \) the quantity
\[
W = \mathbb{E} \left[ \sup_{\tilde{D} \in D(\|\cdot\|, \mathcal{O} K) \Sigma^{1/2}} \frac{1}{\sqrt{t(T-2t+2)}} \sum_{s=0}^{T-2t+1} \sum_{s'=0}^{t-1} \varepsilon_{s,s'} \sum_{r=0}^{t-1} \tilde{D}_{s,r} \varepsilon_{s'+r} \right].
\]

Then, we have
\[
W = \mathbb{E} \left[ \sup_{\|\tilde{D}\|^{1/2}_F = 1, \tilde{D} \in D(\|\cdot\|, \mathcal{O} K) \Sigma^{1/2}} \frac{1}{\sqrt{t(T-2t+2)}} \langle \tilde{D}, H \rangle \right],
\]
where we recall that \( H \) is the random matrix whose components are given by
\[
H_{s,r} = \sum_{s'=0}^{T-2t+1} \varepsilon_{s,s'} \varepsilon_{s'_{s'+r}}
\]
and \( \Sigma^H \) denotes the covariance matrix of \( \text{vec}(H) \). Let \( \tilde{H} = \mathcal{M}(H) \) where \( \mathcal{M} \) denotes the operator defined by
\[
\mathcal{M}(\cdot) = \text{Mat} \left( \Sigma^{-1/2} \text{vec}(\cdot) \right).
\]
Then \( \tilde{H} \) is a Gaussian matrix with i.i.d. components with law \( \mathcal{N}(0, 1) \). Using the invertibility of \( \mathcal{M} \) proved in Section 6.3.3 we get
\[
W = \mathbb{E} \left[ \sup_{\|\tilde{D}\|^{1/2}_F = 1, \tilde{D} \in K \left[ \tilde{D}, \tilde{H} \right]} \frac{1}{\sqrt{t(T-2t+2)}} \langle \tilde{D}, \tilde{H} \rangle \right],
\]
where
\[
K = \frac{1}{\sqrt{t(T-2t+2)}} \mathcal{M}^{-\ast} \left( D(\|\cdot\|, \mathcal{O} K) \Sigma^{1/2} \right),
\]
where we recall that \( \mathcal{M}^{-\ast} \) is the adjoint of the inverse of \( \mathcal{M} \). Moreover, we have
\[
\sup_{\|\tilde{D}\|^{1/2}_F = 1, \tilde{D} \in K} \langle \tilde{D}, \tilde{H} \rangle \leq \sigma_{\text{min}}(S) \sup_{\|\tilde{D}\|^{1/2}_F = 1, \tilde{D} \in K} \langle \tilde{D}, \tilde{H} \rangle
\]
where \( \sigma_{\text{min}}(S) \) is the smallest singular value of the operator \( S \) defined by
\[
S(\cdot) \mapsto \mathcal{M}^{-\ast}(\cdot) \Sigma^{-1/2}.
\]
Thus,
\[
W \leq \frac{w_G(K)}{\sigma_{\text{min}}(S)}
\]
and the proof is completed.

6.2. Control of \( \mathbb{E} \left[ \|\tilde{H}\|^2 \right] \).

**Lemma 6.1.** We have
\[
\mathbb{E} \left[ \|\tilde{H}\|^2 \right] \leq \left( 1 + \frac{1}{2ct} \right) \mathbb{E} \left[ \|\tilde{H}\|^2 \right] + \mathbb{E} \left[ \|\tilde{H}\| \right].
\]

**Proof.** By Gaussian concentration [13 Proposition 4] and the fact that the spectral (operator) norm is 1-Lipschitz, we obtain that for all \( u > 0 \),
\[
P \left( \|\tilde{H}\| \geq \mathbb{E} \left[ \|\tilde{H}\| \right] + u \right) \leq e^{-cu^2}
\]
for some absolute positive constant \( c \). Taking \( u = \delta \mathbb{E} \left[ \| \tilde{H} \| \right] \), we obtain that
\[
\Pr \left( \| \tilde{H} \| \geq (1 + \delta) \mathbb{E} \left[ \| \tilde{H} \| \right] \right) \leq e^{-4c\delta^2 t}.
\]

Thus,
\[
\mathbb{E} \left[ \| \tilde{H} \|^2 \right] = \int_0^{+\infty} \mathbb{P} \left( \| \tilde{H} \|^2 \geq s \right) \, ds
\]
\[
= \int_0^{\mathbb{E}[\|\tilde{H}\|^2]} \mathbb{P} \left( \| \tilde{H} \|^2 \geq s \right) \, ds + \int_{\mathbb{E}[\|\tilde{H}\|^2]}^{+\infty} \mathbb{P} \left( \| \tilde{H} \|^2 \geq s \right) \, ds
\]
\[
= \mathbb{E} \left[ \| \tilde{H} \|^2 \right] + \int_{\mathbb{E}[\|\tilde{H}\|^2]}^{+\infty} \mathbb{P} \left( \| \tilde{H} \| \geq \sqrt{s} \right) \, ds
\]
\[
\leq \mathbb{E} \left[ \| \tilde{H} \|^2 \right] + \int_{\mathbb{E}[\|\tilde{H}\|^2]}^{+\infty} \exp \left( -4c \left( \frac{\sqrt{s} - \mathbb{E}[\|\tilde{H}\|]}{\mathbb{E}[\|\tilde{H}\|]} \right)^2 t \right) \, ds
\]

and making the change of variable \( r = (\sqrt{s} - \mathbb{E}[\|\tilde{H}\|])^2 \), we obtain
\[
\mathbb{E} \left[ \| \tilde{H} \|^2 \right] = \int_0^{+\infty} \mathbb{P} \left( \| \tilde{H} \|^2 \geq s \right) \, ds
\]
\[
\leq \mathbb{E} \left[ \| \tilde{H} \|^2 \right] + 2 \int_0^{+\infty} \exp \left( -4c \left( \frac{\mathbb{E}[\|\tilde{H}\|^2]}{\mathbb{E}[\|\tilde{H}\|]} \right)^2 r \right) \, dr
\]
\[
\leq \mathbb{E} \left[ \| \tilde{H} \|^2 \right] + 2 \int_0^{\mathbb{E}[\|\tilde{H}\|^2]} \frac{1}{\sqrt{r}} \, dr.
\]

Thus, we obtain
\[
\mathbb{E} \left[ \| \tilde{H} \|^2 \right] \leq \mathbb{E} \left[ \| \tilde{H} \|^2 \right] + \mathbb{E}[\|\tilde{H}\|] - \frac{\mathbb{E}[\|\tilde{H}\|^2]}{2c} \left[ \exp \left( -4c \left( \frac{\mathbb{E}[\|\tilde{H}\|^2]}{\mathbb{E}[\|\tilde{H}\|]} \right)^2 r \right) \right]_0^{+\infty}
\]
\[
\leq \left( 1 + \frac{1}{2c} \right) \mathbb{E} \left[ \| \tilde{H} \|^2 \right] + \mathbb{E}[\|\tilde{H}\|].
\]

This completes the proof. \( \square \)

6.3. Some properties of \( \Sigma, \Sigma^H, M, S \) and \( T \).

6.3.1. The spectrum of \( \Sigma \). The spectrum of \( \Sigma \) can be studied using the methods of Grenander and Szego [5]. In [5], the classical results are extended to the case of generalized fractional processes. It was shown in particular by Grenander and Szego in [5, Chapter 5] that if \( 2\pi m \leq \lambda \leq 2\pi M \) for any eigenvalue \( \lambda \) of \( \Sigma \), where \( m \) and \( M \) are the essential infimum and supremum of the spectral density function \( f \) of the process. For ARMA processes, this function is just
\[
f(\nu) = \frac{\sigma^2}{2\pi} \left| \frac{\theta(e^{i\nu})}{\phi(e^{i\nu})} \right|^2
\]
where
\[
\phi(z) = 1 - a_1z - \cdots - a_pz^p \text{ and } \theta(z) = 1 + b_1z + \cdots + b_qz^q.
\]

6.3.2. The spectrum of \( \Sigma^H \). Recall that \( H \) is the random matrix whose components are given by
\[
H_{s,t} = \sum_{s' = 0}^{T-2t+1} \varepsilon_{s,s'} z_{s'+t}
\]
where \( \varepsilon_{s,s'}, s = 0, \ldots, t-1 \) and \( s' = 0, \ldots, T-2t+1 \) are independent Rademacher random variables which are independent of \( z_{s'}, s' = 0, \ldots, T-t \).
Using matrix representation, we have

\[
H = \varepsilon z
= \begin{pmatrix}
  \varepsilon_{0,1} & \varepsilon_{0,2} & \cdots & \varepsilon_{0,T-2t+1} \\
  \varepsilon_{1,1} & \varepsilon_{1,2} & \cdots & \varepsilon_{1,T-2t+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \varepsilon_{t-1,1} & \varepsilon_{t-1,2} & \cdots & \varepsilon_{t-1,T-2t+1}
\end{pmatrix}
\begin{pmatrix}
  z_0 & z_1 & \cdots & z_{t-1} \\
  z_1 & z_2 & \cdots & z_t \\
  \vdots & \vdots & \ddots & \vdots \\
  z_{T-2t+1} & z_{T-2t+2} & \cdots & z_{T-1}
\end{pmatrix}.
\]

Let \( Z_p \) be the \((p + 1)\)-th column of \( z \). Then

\[
\text{vec}(H) = \begin{pmatrix}
  \varepsilon Z_0 \\
  \varepsilon Z_1 \\
  \vdots \\
  \varepsilon Z_{t-1}
\end{pmatrix}.
\]

The \((p, q)\)-th block of \( \Sigma^H \) is given by

\[
\Sigma^H_{[p, q]} = \mathbb{E}[\varepsilon E[Z_pZ_q^t] \varepsilon^t],
\]

where for \( p < q \)

\[
\mathbb{E}[Z_pZ_q^t] = \begin{pmatrix}
  0 & 0 \\
  I_{T - 2t + 1 - (q - p)} & 0
\end{pmatrix}.
\]

Here, \( I_{T - 2t + 1 - (q - p)} \) denotes the identity matrix of dimension \( T - 2t + 1 - (q - p) \).

Partitioning \( \varepsilon \) appropriately as

\[
\varepsilon = \begin{pmatrix}
  \varepsilon_{[1,1]} & \varepsilon_{[1,2]} \\
  \varepsilon_{[2,1]} & \varepsilon_{[2,2]}
\end{pmatrix},
\]

we deduce that

\[
\Sigma^H_{[p, q]} = E \begin{pmatrix}
  \varepsilon_{[1,1]} & \varepsilon_{[1,2]} \\
  \varepsilon_{[2,1]} & \varepsilon_{[2,2]}
\end{pmatrix} \begin{pmatrix}
  0 & 0 \\
  I_{T - 2t + 1 - (q - p)} & 0
\end{pmatrix} \begin{pmatrix}
  \varepsilon_{[1,1]} & \varepsilon_{[1,2]} \\
  \varepsilon_{[2,1]} & \varepsilon_{[2,2]}
\end{pmatrix}
\]

\[
= E \begin{pmatrix}
  \varepsilon_{[1,2]} \varepsilon_{[1,1]} & \varepsilon_{[1,2]} \varepsilon_{[2,1]} \\
  \varepsilon_{[2,2]} \varepsilon_{[1,1]} & \varepsilon_{[2,2]} \varepsilon_{[2,1]}
\end{pmatrix}
\]

\[
= 0
\]

for \( p < q \). Similarly, we can show that \( \Sigma^H_{[p, q]} = 0 \) for \( p > q \). As for \( p = q \), we have \( E[Z_pZ_p^t] = I_{T - 2t + 1} \).

Thus

\[
\Sigma^H_{[p, p]} = E[\varepsilon \varepsilon^t] = (T - 2t + 1) I_t
\]

It is then follows that

\[
\Sigma^H = (T - 2t + 1) I_t (T - 2t + 1).
\]

### 6.3.3. Consequences for \( \mathcal{M}, \mathcal{S} \) and \( \mathcal{T} \)

Recall that \( \mathcal{M} \) denotes the operator defined by

\[
(6.28) \quad \mathcal{M} = \text{Mat} \left( \Sigma^{H-1/2} \text{vec} (\cdot) \right)
\]

and \( \mathcal{M}^{-*} \) denotes the adjoint of the inverse of \( \mathcal{M} \). Using the results of Section 6.3.2, we obtain that

\[
(6.29) \quad \mathcal{M} = \frac{1}{\sqrt{T - 2t + 1}} \text{Id.}
\]

and

\[
(6.30) \quad \mathcal{M}^{-*} = \sqrt{T - 2t + 1} \text{Id.}
\]

Using these results, we obtain that \( \mathcal{S} \) is the operator defined by

\[
S(\cdot) \mapsto \frac{1}{\sqrt{T - 2t + 1}} \cdot \Sigma^{-1/2}.
\]

and \( \mathcal{T} \) is the mapping

\[
T(\cdot) \mapsto \frac{1}{\sqrt{t}} \cdot \Sigma^+.\]
We thus have the following results on $\mathcal{T}$.

$$\|\mathcal{T}\| \leq \frac{\sqrt{\sigma_{\text{max}}(\Sigma)}}{\sqrt{t}}$$

and

$$\sigma_{\text{min}}(\mathcal{T}) \geq \frac{\sqrt{\sigma_{\text{min}}(\Sigma)}}{\sqrt{t}}.$$ 

We also obtain that

$$\sigma_{\text{min}}(\mathcal{S}) \geq \frac{\sqrt{\sigma_{\text{min}}(\Sigma)}}{\sqrt{T - 2t + 1}}.$$ 

6.4. Some properties of the $\chi^2$ distribution. We recall the following useful bounds for the $\chi^2(\nu)$ distribution of degree of freedom $\nu$.

**Lemma 6.2.** [4, Lemma B.1] The following bounds hold:

$$\mathbb{P}\left(\chi(\nu) \geq \sqrt{\nu + \sqrt{2t}}\right) \leq \exp(-t)$$

$$\mathbb{P}\left(\chi(\nu) \leq \sqrt{u\nu}\right) \leq \frac{2}{\sqrt{\pi \nu}} \left(\frac{u e}{2}\right)^{\frac{\nu}{2}}.$$

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