PERFECT FLUID COSMOLOGICAL MODELS WITH
TIME-VARYING CONSTANTS

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Abstract

In this paper, we study in detail a perfect fluid cosmological model with time-varying “constants” using dimensional analysis and the symmetry method. We examine the case of variable “constants” in detail without considering the perfect fluid model as a limiting case of a model with a causal bulk viscous fluid as discussed in a recent paper. We obtain some new solutions through the Lie method and show that when matter creation is considered, these solutions are physically relevant.

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I. INTRODUCTION

Cosmological models with time-varying “constants” have been studied for quite some time ever since Dirac\(^1\) proposed a theory with a time-varying gravitational constant \(G\). Several works have investigated cosmological models with variable cosmological constant within a framework of dissipative thermodynamics\(^2\)\(-\)\(^{14}\). In a recent paper\(^15\), we have studied a causal bulk viscous cosmological model with time-varying constants. We arrived to the conclusion that our cosmological model tends to a perfect fluid one in the matter predominance era. The approach of our work was to study the symmetries of the model beginning with the dimensional analysis of the field equations. The method of Lie group allowed us to arrive to the conclusion that under the hypotheses considered there is only one solution for our model, the scaling one that was trivially obtained using dimensional analysis.

Since our viscous model tends to a perfect fluid one, the purpose of this work is to perform a detailed study of all the possible symmetries of a perfect fluid model with time varying constants showing that in this case it is possible to find more solutions in addition to the scaling one. In order to carry out this study, we begin in section 2 by outlining the equations that govern the model as well as the notation employed. In section 3, we review the scaling solution obtained in previous works highlighting the “assumed” hypotheses that we need to make in order to obtain a solution using dimensional analysis, these are: \(\text{div}(T^i_j) = 0\), conservation principle, and that the relation \(G/c^2\) remain constant for all value of \(t\) (cosmic time). We emphasize the special case \(\omega = 0\) i.e. the dust case as this is the scenario that describes our model in the matter predominance era (in agreement with our previous paper). We discuss some interesting relationships that arise with the similarity method.

In Section 4, we work towards finding other possible solutions to the field equations using the Lie group method. We start this section by rewriting the field equations in such a way that we can use the standard Lie procedure that allow us to find more symmetries. After outlining the equation and the constraint, we proceed to study some cases. The first one is the obtained previously using dimensional analysis since dimensional analysis is just a special class of symmetry. We would like to emphasize that the Lie method show us that one of the assumptions made with the dimensional method, \(G/c^2 = \text{const.}\), is at least correct from the mathematical point of view. This result allow us to validate completely the solution obtained through similarity. This solution connects perfectly with our previous work\(^15\).
In order to show that the other two solutions are physically relevant, we examine them in a matter creation scenario in section 5. We find that the horizon and entropy problems are solved when we consider matter creation and this leads to a set of physically relevant solutions. We note that no new constants or assumptions are required for this exercise. We conclude the paper by summarizing the results in section 6.

II. THE MODEL

We will use the field equations in the form:
\[ R_{ij} - \frac{1}{2} g_{ij} R = \frac{8 \pi G(t)}{c^4(t)} T_{ij} + \Lambda(t) g_{ij}, \]  

where the energy momentum tensor is:
\[ T_{ij} = (\rho + p) u_i u_j - pg_{ij}, \]

and \( p = \omega \rho \) in such a way that \( \omega \in [0, 1] \).

The cosmological equations are now (for a flat FRW Universe as the most recent observations suggest us\textsuperscript{16–19}):  
\[ 2H' + 3H^2 = -\frac{8 \pi G(t)}{c(t)^2} \rho + c(t)^2 \Lambda(t), \]
\[ 3H^2 = \frac{8 \pi G(t)}{c(t)^2} \rho + c(t)^2 \Lambda(t), \]

where \( H = (f'/f) \) is the Hubble function. Applying the covariance divergence to the second member of equation (1) we get:
\[ T^j_{ij} = \left( \frac{4c_j}{c} - \frac{G_j}{G} \right) T^j_i - c^4(t) \delta^j_i \Lambda_j - \frac{8 \pi G c_j}{c} \]

which simplifies to:
\[ \rho' + 3(\omega + 1) \rho H = \frac{N c^4}{8 \pi G} - \rho G' \frac{c}{G} + 4 \rho c', \]

We assume that \text{div}(T^j_i) = 0 which leads to the two following equations:
\[ \rho' + 3(\omega + 1) \rho H = 0, \]
\[ -\frac{N c^4}{8 \pi G} - \rho G' \frac{c}{G} + 4 \rho c' = 0. \]
Hence the field equations are:

\[ 2H' + 3H^2 = -\frac{8\pi G}{c^2} p + \Lambda c^2, \]  
\[ 3H^2 = \frac{8\pi G}{c^2} \rho + \Lambda c^2, \]  
\[ \rho' + 3 (\omega + 1) \rho H = 0, \]  
\[ -\frac{\Lambda c^4}{8\pi \rho G} - \frac{G'}{G} + 4\frac{c'}{c} = 0. \]  

III. REVIEW OF THE SCALING SOLUTION

As have been pointed out by Carr and Coley\textsuperscript{20}, the existence of self-similar solutions (Barenblatt and Zeldovich\textsuperscript{21}) is related to conservation laws and to the invariance of the problem with respect to the group of similarity transformations of quantities with independent dimensions. This can be characterized within general relativity by the existence of a homotetic vector field and for this reason one must distinguish between geometrical and physical self-similarity. Geometrical similarity is a property of the spacetime metric, whereas physical similarity is a property of the matter fields (our case). In the case of perfect fluid solutions admitting a homotetic vector, geometrical self-similarity implies physical self-similarity.

As we show in this section as well as in previous works, the assumption of self-similarity reduces the mathematical complexity of the governing differential equations. This makes such solutions easier to study mathematically. Indeed self-similarity in the broadest Lie sense refers to an invariance which allows such a reduction.

Perfect fluid space-times admitting a homotetic vector within general relativity have been studied by Eardley\textsuperscript{22}. In such space-times, all physical transformations occur according to their respective dimensions, in such a way that geometric and physical self-similarity coincide. It is said that these space-times admit a transitive similarity group and space-times admitting a non-trivial similarity group are called self-similar. Our model i.e. a flat FRW model with a perfect fluid stress-energy tensor has this property and as already have been pointed out by Wainwright\textsuperscript{23}, this model has a power law solution.

Under the action of a similarity group, each physical quantity \( \phi \) transforms according to its dimension \( q \) under the scale transformation. For space-times with a transitive similarity group, dimensionless quantities are therefore spacetime constants. This implies that the
ratio of the pressure of the energy density is constant so that the only possible equation of state is the usual one in cosmology i.e. \( p = \omega \rho \), where \( \omega \) is a constant. In the same way, the existence of homotetic vector implies the existence of conserved quantities.

In this section we would like to review the solution obtained through Dimensional Analysis\(^{24-25}\). Our starting point is the condition \( \text{div} (T^i_j) = 0 \) which allows us to obtain one of the dimensional constants that we need in order to apply the method of dimensional analysis. Therefore, from eq. \((11)\)

\[
\rho' + 3 (\rho + p) H = 0,
\]

we obtain the following relation between the energy density and the scale factor as well as the constant of integration that we shall need for our subsequent calculations:

\[
\rho = A_\omega f^{-3(\omega+1)},
\]

where \( A_\omega \) is the integration constant that depends on the equation of state that we want to consider i.e. constant \( \omega \), \([A_\omega] = L^{3(\omega+1)-1}MT^{-2}\). We consider a second dimensional constant by considering the relation \( G/c^2 = B \), where, \( B \) is the constant. This is a hypothesis which is necessary in order to apply dimensional analysis. In this next section, we will see that this condition on \( G \) and \( c \) is mathematically correct. Our purpose here is to show that no more hypotheses are necessary to solve the differential equations that govern the model (see \(^{26-28}\) for standard text-book on Dimensional Analysis and \(^{29}\) for applications to Cosmology).

Therefore, if we take into account the standard dimensional procedure, we find that the set of governing parameters are \( M = M \{ A_\omega, B, t \} \), which bring us to obtain the next relations:

\[

g \propto A_\omega^{\frac{2}{\gamma+1}} B^{\frac{2}{\gamma+1}+1} t^{\frac{2(1-\gamma)}{\gamma+1}},
g \propto A_\omega^{\frac{1}{\gamma+1}} B^{\frac{1}{\gamma+1}} t^{\frac{(1-\gamma)}{\gamma+1}},
\]

\[
\rho \propto B^{-1} t^{-2},
f \propto A_\omega^{\frac{1}{\gamma+1}} B^{\frac{1}{\gamma+1}} t^{\frac{2}{\gamma+1}},
k_B \theta \propto A_\omega^{\frac{3}{\gamma+1}} B^{\frac{3}{\gamma+1}-1} t^{\frac{2}{\gamma+1}},
\]

\[
\Lambda \propto A_\omega^{\frac{2}{\gamma+1}} B^{\frac{2}{\gamma+1}} t^{\frac{4}{\gamma+1}},
\]

\[
q = \frac{\gamma - 1}{2},
\]

where \( \gamma = 3(\omega + 1) - 1 \), and \( q \) is the deceleration parameter.

From the set of equations \((15)\), we see that \( \frac{G}{c^2} = B \) (trivially), \( f = ct \) (the horizon problem is missing in this model), \( \Lambda \propto \frac{1}{c^2 t^2} \propto f^{-2} \) for all value of \( \gamma \) i.e. \( \forall \omega \).
We would like to point out that the same results have been obtained by Midy and Pettit in a more general context\textsuperscript{30}.

As we have indicated in the introduction, in a recent paper\textsuperscript{15} we have studied a full causal bulk viscous cosmological model with time varying constants. The main conclusions in that paper was that under the “assumed” hypotheses, i.e. $\text{div}(T^i_j) = 0$, conservation principle, and $\Pi \propto \rho$, the bulk viscous pressure behaves as the energy density and we obtained the following results:

1. there is only one solution for the field equations (scaling solution),

2. the bulk viscous pressure behaves as an adiabatic matter mechanism (solving the entropy problem),

3. the “constants” $G, c$ and $\Lambda$ are decreasing functions on time (solving the horizon problem) and

4. in the matter predominance era, i.e. $\omega = 0$, we cannot consider the bulk viscosity and therefore our viscous fluid tends to a perfect fluid one (see \textsuperscript{15} for details).

Due to these conclusions, in the present paper, we have studied a perfect fluid model with variable “constant” emphasizing the case of matter predominance era or a “dust” solution; in other words, considering $\omega = 0$ in the equation of state. In this case, we obtain the following solutions:

\begin{equation}
G \propto t^{-2/3}, \quad c \propto t^{-1/3}, \quad \rho \propto t^{-2}
\end{equation}

\begin{equation}
f \propto t^{2/3}, \quad \theta \propto t^0, \quad \Lambda \propto t^{-4/3}, \quad q = 1/2
\end{equation}

while in the viscous model, the solutions are:

\begin{equation}
G \propto A_{\omega, j, k} t^{-2} k^{\frac{3+\alpha}{3(\alpha+1)}} t^{-\frac{3+\alpha}{6(\alpha+1)}},
\end{equation}

\begin{equation}
c \propto A_{\omega, j, k} t^{-1} k^{\frac{1}{3(\alpha+1)}},
\end{equation}

\begin{equation}
\rho \propto k^{-b-1} t^{b-1} \propto \Pi,
\end{equation}

\begin{equation}
f \propto A_{\omega, j, k} t^{-1} k^{-\frac{1}{3(\alpha+1)}},
\end{equation}

\begin{equation}
\Lambda \propto A_{\omega, j, k} t^{-\frac{2}{3(\alpha+1)}},
\end{equation}

\begin{equation}
\omega = 0
\end{equation}
where, $\alpha = 3(\omega + 1 + \kappa) - 1$, $b = \gamma - 1$, (see 15 for details). If we consider $\gamma = 1/2$, $\omega = 0$ and $\kappa = 0$ (viscous pressure vanishes and therefore $k_{\gamma}$ vanishes too) then we obtain the same results as in equation (16), but, as we have indicated previously this approach is inconsistent when $\omega = 0$ and for this reason we have studied the perfect fluid case.

As we can see in the perfect fluid case, “constants” $G, c$ and $\Lambda$ are decreasing functions on time, but in this case decrease slower than in the radiation predominance era, while $\rho$ and $f$ behave as in the FRW model.

To obtain these solutions we needed two dimensional constants, viz., $A_{\omega}$ and $B$. In the next section, we shall show that constant $B$ is a reasonable hypothesis since with the Lie group method such condition holds as a result in the scaling solution.

IV. LIE METHOD

As we have seen earlier, the $\pi$ – monomia is the main object in dimensional analysis. It may be defined as a product of quantities which are invariant under changes of fundamental units. $\pi$ – monomia are dimensionless quantities, their dimensions are equal to unity. Dimensional analysis has the structure of a Lie group31. The $\pi$ – monomia are invariant under the action of the similarity group. On the other hand, we must mention that the similarity group is only a special class of the mother group of all symmetries that can be obtained using the Lie method. For this reason, when one uses dimensional analysis, only one of the possible solutions to the problem is obtained.

As we have been able to find a solution through dimensional analysis, it is possible that there are other symmetries of the model, since dimensional analysis is a reminiscent of scaling symmetries, which obviously are not the most general form of symmetries. Hence, we shall study the model through the method of Lie group symmetries, showing that under the assumed hypotheses there are other solutions of the field equations. In this section we shall show how the lie method allows us to obtain different solutions for the field equations. In particular we seek the forms of $G$ and $c$ for which our field equations admit symmetries i.e. are integrable (see 32–37).

An alternative use of the Lie groups have been performed by M. Szydlowki et. al. 38–39 where they study the Friedman equations in order to find the correct equation of state following pionerr works of Collins40.
In order to use the Lie method, we rewrite the field equations as follows. From (9) – (10), we obtain
\[ 2\frac{f''}{f} - 2\left(\frac{f'}{f}\right)^2 = -\frac{8\pi G}{c^2} (p + \rho), \] (22)
and therefore
\[ 2 (H)' = -\frac{8\pi G}{c^2} (p + \rho). \] (23)

From equation (11), we can obtain
\[ H = -\frac{\rho'}{3((\omega + 1)\rho)}, \] (24)
therefore
\[ \left(\frac{\rho'}{\rho}\right)' = 12\pi (\omega + 1)^2 \frac{G}{c^2} \rho. \] (25)
Taking \(12\pi (\omega + 1)^2 = A\) and then expanding, we obtain
\[ \rho'' = \frac{\rho'^2}{\rho} + A\frac{G}{c^2} \rho^2. \] (26)

Now, we apply the standard Lie procedure to this equation. A vector field \(X = \xi(t, \rho) \partial_t + \eta(t, \rho) \partial_\rho\), (27)
is a symmetry of (26) iff
\[-\xi f_t - \eta f_\rho + \eta_t + (2\eta_\rho - \xi_{tt}) \rho' + (\eta_{\rho \rho} - 2\xi_{t \rho}) \rho'^2 - \xi_{\rho \rho} \rho'^3 + ... \]
\[ ... + (\eta_\rho - 2\xi_t - 3\rho' \xi_\rho) f - [\eta_t + (\eta_\rho - \xi_t) \rho' - \rho'^2 \xi_\rho] f_\rho = 0. \] (28)
By expanding and separating (28) with respect to powers of \(\rho'\), we obtain the overdetermined system:
\[ \xi_{\rho \rho} + \rho^{-1} \xi_\rho = 0, \] (29)
\[ \eta_{\rho \rho} - 2\xi_{t \rho} + \rho^{-2} \eta - \rho^{-1} \eta_\rho = 0, \] (30)
\[ 2\eta_\rho - \xi_{tt} - 3A\frac{G}{c^2} \rho^2 \xi_\rho - 2\rho^{-1} \eta_t = 0, \] (31)
\[ \eta_t - A\left(\frac{G'}{c^2} - 2G\frac{c'}{c^3}\right) \rho^2 \xi - 2\eta A\frac{G}{c^2} \rho + (\eta_\rho - 2\xi_t) A\frac{G}{c^2} \rho^2 = 0. \] (32)
Solving (29-32), we find that
\[ \xi(t, \rho) = -2et + a, \eta(t, \rho) = (bt + d) \rho, \] (33)
subject to the constrain
\[
\frac{G'}{G} = 2\frac{c'}{c} + \frac{bt + d - 4e}{2et - a},
\]
with \(a, b, e,\) and \(d\) as constants. In order to solve (34), we consider the following cases.

**A. Case I: \(b = 0\) and \(d - 4e = 0\)**

In this case, the solution (34) reduces to
\[
\frac{G'}{G} = 2\frac{c'}{c} \implies \frac{G}{c^2} = B = \text{const.}
\]
which means that “constants” \(G\) and \(c\) vary but in such a way that the relation \(\frac{G}{c^2}\) remains constant.

The solution obtained through Dimensional Analysis needs to make this relations as hypothesis in order to obtain a complete solution for the field equations. This case shows us that such hypothesis is correct (at least has mathematical sense).

The knowledge of one symmetry \(X\) might suggest the form of a particular solution as an invariant of the operator \(X\) i.e. the solution of
\[
\frac{dt}{\xi(t, \rho)} = \frac{d\rho}{\eta(t, \rho)},
\]
this particular solution is known as an invariant solution (generalization of similarity solution), therefore the energy density is obtained as
\[
\frac{dt}{-2et + a} = \frac{d\rho}{4e\rho} \implies \rho = \frac{1}{(2et - a)^2},
\]
for simplicity we adopt
\[
\rho = \rho_0 t^{-2},
\]
Once we have obtained \(\rho\), we can obtain \(f\) (the scale factor) from
\[
\rho = A_\omega f^{-3(\omega + 1)} \implies f = (A_\omega t)^{\frac{2}{\omega + 1}},
\]
in this way we find \(H\) and from eq. (10), we obtain the behaviour of \(\Lambda\) as:
\[
c^2\Lambda = 3H^2 - \frac{8\pi G}{c^2} \rho,
\]
and therefore,
\[
\Lambda = (3\beta^2 - 8\pi B\rho_0) \frac{1}{c^2t^2} = \frac{l}{c^2t^2}.
\]
If we replace all these results into eq. (12), then we obtain the exact behaviour for \( c \), i.e.,

\[
- \left( \frac{1}{t} + \frac{c'}{c} \right) \lambda = \frac{c'}{c},
\]

where \( \lambda = \frac{t}{\pi \mu \rho_0} \), with \( \lambda \in \mathbb{R}^+ \), i.e. is a positive real number and thus,

\[
c = c_0 t^{-\alpha},
\]

with \( \alpha = \left( \frac{\lambda}{1 + \lambda} \right) \).

Hence, in this case we have found that (see fig.1):

\[
G = G_0 t^{-2\alpha}, \quad c = c_0 t^{-\alpha}, \quad \Lambda = \Lambda_0 t^{-2(1-\alpha)}, \quad f = (A \omega t) \frac{2}{(d+1)}, \quad \rho = \rho_0 t^{-2}.
\]

This is the solution that we have obtained with dimensional analysis in the previous section and we shall show this is only solution compatible with our previous solution\textsuperscript{15} obtained in the framework of a bulk viscous fluid (full causal theory).

\[\text{FIG. 1: We see the behaviour of } G, c \text{ and } \Lambda \text{ for the first class of solutions for different values of } \alpha : \alpha = 0.5 \text{ (solid curve), } \alpha = 0.9 \text{ (dotted curve), } \alpha = 0.3 \text{ (matter era)(dashed curve). In all cases the constants are decreasing functions.}\]

\[\text{B. Case II, } b = a = 0\]

In this case, we find that

\[
\frac{G}{c^2} = \tilde{B} t^\zeta,
\]

where \( \zeta = \delta - 2 \) and \( \delta = \frac{d}{2e} \). On following the same procedure as above, we find that

\[
\frac{dt}{\xi} = \frac{d\rho}{\eta} \implies \rho = \rho_0 t^{-\delta},
\]
we must impose the condition \( \text{sign}(d) = \text{sign}(e) \), i.e., \( \delta \in \mathbb{R}^+ \), in order that the solution has some physical meaning that the energy density is a decreasing function of time \( t \). It is observed that if \( d = 4e \) then we obtain same solution that the obtained one in the case I. The scale factor is found to be

\[
f = K_f t^{\delta/(\omega+1)},
\]

where \( K_f \) is an integration constant, and therefore, the Hubble parameter is:

\[
H = \frac{\delta}{3(\omega + 1) t},
\]

which is similar to the scale factor obtained in case I. To obtain the behaviour of the “constants” \( G, c \) and \( \Lambda \), we follow the same steps as in case I, i.e., from

\[
c^2 \Lambda = 3H^2 - \frac{8\pi G}{c^2} \rho,
\]

we obtain the behaviour of \( \Lambda \) being:

\[
\Lambda = \frac{l}{c^2 t^2},
\]

where, \( l = (K_1 - K_2) \), \( K_1 = \frac{\delta^2}{3(\omega+1)^2} \) and \( K_2 = 8\pi \rho_0 \tilde{B} \) i.e., \( l \in \mathbb{R}^+ \). Therefore,

\[
\Lambda' = -\frac{2l}{c^4 t^2} \left( \frac{c'}{c} + \frac{1}{t} \right)
\]

If we substitute all this results into the next equation

\[
\frac{\Lambda' c^4}{8\pi G\rho} + \frac{G'}{G} - 4\frac{c'}{c} = 0,
\]

we obtain an ODE for \( c \), i.e.,

\[
\frac{c'}{c} (\lambda - 2) = -(\lambda - 2 + \delta) \frac{1}{t}
\]

where, \( \lambda = \left( -\frac{l}{4\pi \rho_0 \tilde{B}} \right), \lambda \in \mathbb{R}^- \), which leads to

\[
c = c_0 t^{-\alpha}
\]

with \( \alpha = \left( 1 + \frac{\delta}{\lambda - 2} \right) \) such that \( \alpha \in [0, 1) \). In this way we can find the rest of quantities:

\[
G = G_0 t^{-2(\alpha+1)+\delta}, \quad \Lambda = \Lambda_0 t^{-2(1-\alpha)},
\]

note that \( \alpha < 1 \). The case \( \alpha = 1 \iff \delta = 0 \) is forbidden and \( \alpha = 0 \) brings us to the limiting case of the \( G, \Lambda \) variable cosmologies\(^{41}\).

We notice that this solution is very similar to the case I but in this case all the parameters are perturbed by \( \delta \) and more important is the result, \( \frac{G}{\rho} = \tilde{B} t^\kappa \) (see figs. 2, 3 and 4).
FIG. 2: Time variation of $c(t)$ for the second class of solutions for different values of $\alpha: \alpha = 0.5$ (solid curve), $\alpha = 0.9$ (dotted curve), $\alpha = 0.1$ (dashed curve) and $\alpha = 0.000001$ (long dashed curve), the last solution describes the case $c(t) = \text{const.}$

C. Case III, $b = e = 0$

Following the same procedure as above, we find in this case that such restrictions imply $\xi(t, \rho) = a$, $\eta(t, \rho) = d\rho$ and therefore:

$$\frac{G'}{G} = 2 \frac{c'}{c} - \frac{d}{a}, \quad (56)$$

which brings us to:

$$\frac{G}{c^2} = K \exp(-\alpha t), \quad (57)$$

where $\frac{d}{a} = \alpha$ and note that $[K] = [B]$ i.e has the same dimensional equation,

$$\frac{dt}{\xi(t, \rho)} = \frac{d\rho}{\eta(t, \rho)} \Rightarrow \frac{dt}{a} = \frac{d\rho}{d\rho} \Rightarrow \rho = \rho_0 \exp(\alpha t), \quad (58)$$

this expression only has sense if $\alpha \in \mathbb{R}^-$, note that $[\alpha] = T^{-1}$.

The scale factor $f$ satisfies the relationship:

$$\rho = A_\omega f^{-3(\omega+1)} \Rightarrow f = K_f \exp(\alpha t)^{\frac{1}{3(\omega+1)}}, \quad (59)$$

that is to say, it is a growing function without singularity. In this way, we find that

$$H = -\frac{\alpha}{3(\omega + 1)} = \text{conts.} \quad H > 0. \quad (60)$$
FIG. 3: The variation of the gravitational “constant” $G(t)$, for different values of $\alpha$ and $\delta$: $\alpha = 0.5$ and $\delta = 1$ (solid curve), $\alpha = 0.9$ and $\delta = 1$ (dotted curve), $\alpha = 0.1$ and $\delta = 1$ (dashed curve) and $\alpha = 0.000001$ and $\delta = 5$ (long dashed curve), the last curve describes a growing solution.

The cosmological “constant” is obtained as

$$c^2 \Lambda = \frac{\alpha^2}{3(\omega + 1)^2} - 8\pi K \rho_0 \implies c^2 \Lambda = l,$$

note that $[l] = T^{-2}$, if we replace all these results into eq. (12) then we shall obtain the exact behaviour for $c$, i.e.

$$\left(\frac{l}{8\pi K \rho_0} + 2\right) \frac{c'}{c} = \alpha,$$

and hence,

$$c = K \exp(c_0 t),$$

where $c_0 = \frac{\alpha}{(\frac{l}{8\pi K \rho_0} + 2)}$ with $c_0 \in \mathbb{R}^-$ since $\alpha \in \mathbb{R}^-$, that is, $c$ is a decreasing function on time $t$.

In this case, we have found

$$c = K \exp(c_0 t),$$

$$G = G_0 \exp((-\alpha + 2c_0) t),$$

$$\Lambda = l \exp(c_0 t)^{-2},$$

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FIG. 4: Time variation of $\Lambda(t)$ for the second class of solutions for different values of $\alpha$: $\alpha = 0.5$ (solid curve), $\alpha = 0.9$ (dotted curve), $\alpha = 0.1$ (dashed curve) and $\alpha = 0.000001$ (long dashed curve). In all cases, $\Lambda(t)$ is a decreasing function.

therefore the solutions for this case are (see fig. 3):

\begin{align}
G &= G_0 \exp((-\alpha + 2c_0) t), c = K \exp(c_0 t), \Lambda = l \exp(c_0 t)^{-2}, \\
\rho &= \rho_0 \exp(\alpha t), f = K_f \exp(\alpha t)^{\frac{-1}{3(\omega + 1)}},
\end{align}

(67) (68)

FIG. 5: We see the behaviour of constants for the third class of solutions for different values of $c_0$: $c_0 = -0.5$ (solid curve), $c_0 = -0.3$ (dotted curve), $c_0 = -0.01$ (dashed curve) and $\alpha = -0.1$. In all cases, $\Lambda(t)$ is a growing function.
As we can see, equation (34) allows us to obtain more cases, however, we examine their physical validity by considering matter creation in our model in the next section. The behavior of this class of solutions for different values of \( c_0 \) is shown in figure 5.

V. MATTER CREATION

As we have seen in the previous section, solutions IVB and IVC suggest us a new scenario since these solutions may describe the early universe in a very different way than our previous solution\(^{15} \). For this reason, in this section, we study briefly the important case in which adiabatic matter creation\(^{42-45} \) can be taken into account, in order to get rid of the entropy problem. The matter creation theory is based on an interpretation of the matter energy-stress tensor in open thermodynamic systems, which leads to the modification of the adiabatic energy conservation law and as a result including the irreversible matter creation. The matter creation corresponds to an irreversible energy flow from the gravitational field to the constituents of the particles created and this involves the addition of a creation pressure \( p_c \) in the matter energy-momentum tensor which we discuss below.

The field equations that now govern our model are as follows:

\[
2H' + 3H^2 = - \frac{8\pi G(t)}{c^2(t)}(p + p_c) + c^2(t)\Lambda(t),
\]

\[
3H^2 = \frac{8\pi G(t)}{c^2(t)} \rho + c^2(t)\Lambda(t),
\]

\[
n' + 3nH = \psi,
\]

and taking into account our general assumption i.e.

\[
T^j_{ij} \delta_{ij}^j - \left( \frac{4c_j}{c} - \frac{G_j}{G} \right) T^j_i + \frac{c^4(t)\delta^j_i\Lambda_j}{8\pi G} = 0,
\]

with \( T^j_{ij} = 0 \), we obtain the two equations

\[
\rho' + 3(\rho + p + p_c)H = 0,
\]

and

\[
\frac{\Lambda'c^4}{8\pi G\rho} + \frac{G'}{G} - 4\frac{c'}{c} = 0,
\]

where \( n \) is the particle number density, \( \psi \) is the function that measures the matter creation, \( H = f'/f \) represents the Hubble parameter (\( f \) is the scale factor that appears in the metric),
$p$ is the thermostatic pressure, $\rho$ is energy density and $p_c$ is the pressure that generates the matter creation.

The creation pressure $p_c$ depends on the function $\psi$. For adiabatic matter creation this pressure takes the following form:

$$p_c = -\left[\frac{\rho + p}{3nH^2}\right]. \quad (75)$$

The state equation that we next use is the well-known expression

$$p = \omega \rho, \quad (76)$$

where $\omega = \text{const.}$ and $\omega \in (-1, 1]$. We assume that the matter creation function follows the law:

$$\psi = 3\beta nH, \quad (77)$$

where $\beta$ is a dimensionless constant (if $\beta = 0$ then there is no matter creation since $\psi = 0$).

The generalized principle of conservation $T^i_{ij} = 0$, for the stress-energy tensor (73) leads us to:

$$\rho' + 3(\omega + 1)(1 - \beta)\rho H = 0. \quad (78)$$

Therefore, the new set of field equations are:

$$2H' + 3H^2 = -\frac{8\pi G(t)}{c^2(t)}(p + p_c) + c^2(t)\Lambda(t), \quad (79)$$

$$3H^2 = \frac{8\pi G(t)}{c^2(t)}\rho + c^2(t)\Lambda(t), \quad (80)$$

$$\rho' + 3(\omega + 1)(1 - \beta)\rho H = 0, \quad (81)$$

$$\frac{N^i e^i}{8\pi G\rho} + \frac{G'}{G} - 4\frac{c'}{c} = 0. \quad (82)$$

Now, using the same procedure that was followed in section 4, we find the new equation on which we can apply the Lie method. We rewrite the field equations as follows: from (79) – (80) we obtain

$$2f'' - 2\left(\frac{f'}{f}\right)^2 = -\frac{8\pi G}{c^2} (p + p_c + \rho), \quad (83)$$

and therefore

$$2(H')' = -\frac{8\pi G}{c^2} (p + p_c + \rho). \quad (84)$$

From (81), we can obtain

$$H = -\frac{\rho'}{3(\omega + 1)(1 - \beta)\rho}. \quad (85)$$
therefore
\[
\left( \frac{\rho'}{\rho} \right)' = 12\pi (\omega + 1)^2 (1 - \beta)^2 \frac{G}{c^2} \rho,
\]
(86)
making \( 12\pi (\omega + 1)^2 (1 - \beta)^2 = \tilde{A} \) and expanding, we obtain
\[
\rho'' = \frac{\rho'^2}{\rho} + \frac{\tilde{A}G}{c^2} \rho^2.
\]
(87)
Therefore, we obtain the same equation that we obtained in section 4, namely equation (26) except the constant \( \tilde{A} \). Hence, within this framework the entropy and horizon problems are solved for our model.

VI. CONCLUSIONS.

In this paper we have studied the behaviours of time-varying “constants” \( G, c \) and \( \Lambda \) in a perfect fluid model. We began reviewing the scaling solution obtained through dimensional analysis. He have shown that this solution connects with our previous solution\(^{15} \) where we studied the behaviour of the “constants” \( G, c \) and \( \Lambda \) in a full causal bulk viscous model arriving to the conclusion that this model tends to a perfect fluid one when we impose the condition \( \omega = 0 \) in the equation of state (dust solution).

To obtain this solution, we imposed the assumption, \( \text{div}(T^i_j) = 0 \), from which we obtained the dimensional constant \( A_\omega \) that relates \( \rho \propto f^{-3(\omega+1)} \) and the relationship \( G/c^2 = \text{const.} = B \) remaining constants for all value of \( t \), i.e. \( G \) and \( c \) vary but in such a way that \( G/c^2 \) remain constant. With these two hypothesis, we have obtained the scaling solution that connects perfectly with one obtained in our previous paper\(^{15} \), i.e. with the bulk viscous model in the matter-dominated era.

In this context, the solution obtained through dimensional analysis show us that the “constants” \( G, c \) and \( \Lambda \) are decreasing functions of time, but in this case decrease slowly than in the radiation predominance era, while \( \rho \) and \( f \) behave as in the FRW model solving the horizon problem.

Since we have been able to found a solution through similarity, i.e. through dimensional analysis, it is possible that there are other symmetries of the model, since dimensional analysis is a reminiscent of scaling symmetries, which obviously are not the most general form of symmetries. Therefore, we studied the model through the method of Lie group
symmetries, showing that under the assumed hypotheses, there are other solutions of the field equations.

The first solution obtained is the already obtained one through similarity, but, in this case we have showed the condition $G/c^2$ arises as a result and not as an ad-hoc condition. We also have studied two other cases which can be considered as physically relevant solutions since $f$ is a growing function on time and $\rho$ is a decreasing function on time. They could describe very early cosmological solutions (inflationary ones) but in this context we cannot solve the entropy problem. We have considered matter creation and and using a recasted set of equations, shown that when matter creation is taken into account, the horizon and entropy problems are solved for the two solutions.

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