s-Numbers of compact embeddings of function spaces on quasi-bounded domains

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Abstract
We prove asymptotic formulas for the behavior of approximation, Gelfand, Kolmogorov and Weyl numbers of Sobolev embeddings between Besov and Triebel-Lizorkin spaces defined on quasi-bounded domains.

Keywords: approximation numbers, Gelfand numbers, Kolmogorov numbers, Weyl numbers, Sobolev embedding, quasi-bounded domain

1. Introduction

Today we have a good knowledge about asymptotic behavior of entropy and approximation numbers of compact embeddings between spaces of Sobolev-Besov-Hardy type. The results regarding asymptotic behavior of the entropy numbers and the approximation numbers of the embeddings of spaces defined on bounded domains are the oldest part of this theory. We owe M.S. Birman, M.Z. Solomjak, D. Edmunds, H. Triebel, A. Caetano and others the estimates in this respect. The Gelfand, the Kolmogorov and the Weyl numbers on bounded domains were studied by C. Lubitz, R. Linde, A. Caetano and J. Vybíral. But, there is a wider class of domains for which the embeddings of spaces of Besov-Sobolev type can be compact. These are so called quasi-bounded domains. The quasi-bounded domain may be unbounded, and can be even a domain of infinite Lebesgue measure. Recently H.-G. Leopold and L. Skrzypczak gave necessary and sufficient conditions for compactness of these embeddings and studied the asymptotic behavior of entropy numbers for function spaces defined on quasi-bounded domains. In this article we investigate asymptotic behavior of s-numbers of these embeddings. Our approach is based on wavelet decomposition of function spaces on domains introduced by H. Triebel in 2008; cf. [18]. In the case of the Weyl numbers our estimates complement the earlier results even for bounded domains.

The concept of quasi-bounded domains is not new. Function spaces on these domains were studied in the 60s of the last century by Clark and Hewgill; see [7, 8]. Overview of some of these results can be found in [1]. All these results assume

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Definition 1. In particular, each bounded domain is quasi-bounded.

\[ \Omega \text{ domain} \]

\[ \infty \text{ spaces} \]

We denote the continuous embeddings between quasi-Banach spaces by \( \hookrightarrow \) and also we use the common notations \( A^s_{p,q}(\mathbb{R}^d) \), \( A^s_{p,q}(\Omega) \) with \( A = B \) or \( A = F \), if it makes no difference.

Let \( \Omega \) be an open and nonempty set in \( \mathbb{R}^d \). Such set is called a domain. The domain \( \Omega \) in \( \mathbb{R}^d \) is called quasi-bounded if

\[
\lim_{x \in \Omega, |x| \to \infty} d(x, \partial \Omega) = 0.
\]

In particular, each bounded domain is quasi-bounded.

**2. Function spaces on quasi-bounded domain**

We assume that the reader is acquainted with the definition and basic properties of Besov spaces \( B^s_{p,q}(\mathbb{R}^d) \), \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \) and Triebel-Lizorkin spaces \( F^s_{p,q}(\mathbb{R}^d) \), \( 0 < p < \infty \), \( 0 < q \leq \infty \), \( s \in \mathbb{R} \). We denote the continuous embeddings between quasi-Banach spaces by \( \hookrightarrow \) and also we use the common notations \( A^s_{p,q}(\mathbb{R}^d) \), \( A^s_{p,q}(\Omega) \) with \( A = B \) or \( A = F \), if it makes no difference.

Let \( \Omega \) be an open and nonempty set in \( \mathbb{R}^d \). Such set is called a domain. The domain \( \Omega \) in \( \mathbb{R}^d \) is called quasi-bounded if

\[
\lim_{x \in \Omega, |x| \to \infty} d(x, \partial \Omega) = 0.
\]

In particular, each bounded domain is quasi-bounded.

**Definition 1.** Let \( \Omega \) be the domain in \( \mathbb{R}^d \), such that \( \Omega \neq \mathbb{R}^d \) and let \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \) with \( p < \infty \) for \( F \)-spaces.

(i) Let

\[
A^s_{p,q}(\Omega) = \{ f \in D'(\Omega) : f = g|_{\Omega} \text{ for some distribution } g \in A^s_{p,q}(\mathbb{R}^d) \},
\]

\[
\| f \|_{A^s_{p,q}(\Omega)} = \inf \| g \|_{A^s_{p,q}(\mathbb{R}^d)},
\]

where the infimum is taken over all \( g \in A^s_{p,q}(\mathbb{R}^d) \) with \( f = g|_{\Omega} \).

(ii) Let

\[
\tilde{A}^s_{p,q}(\Omega) = \{ f \in A^s_{p,q}(\mathbb{R}^d) : \text{supp } f \subset \Omega \},
\]

\[
\tilde{A}^s_{p,q}(\Omega) = \{ f \in D'(\Omega) : f = g|_{\Omega} \text{ for some distribution } g \in \tilde{A}^s_{p,q}(\Omega) \},
\]

\[
\| f \|_{\tilde{A}^s_{p,q}(\Omega)} = \inf \| g \|_{\tilde{A}^s_{p,q}(\mathbb{R}^d)},
\]

where the infimum is taken over all \( g \in \tilde{A}^s_{p,q}(\Omega) \) with \( f = g|_{\Omega} \).

(iii) We define

\[
\check{B}^s_{p,q}(\Omega) = \begin{cases} 
\check{B}^s_{p,q}(\Omega), & \text{if } 0 < p \leq \infty, 0 < q \leq \infty, s > \sigma_p, \\
B^s_{p,q}(\Omega), & \text{if } 1 < p < \infty, 1 \leq q \leq \infty, s = 0, \\
B^s_{p,q}(\Omega), & \text{if } 0 < p < \infty, 0 < q \leq \infty, s < 0,
\end{cases}
\]

and

\[
\check{F}^s_{p,q}(\Omega) = \begin{cases} 
\check{F}^s_{p,q}(\Omega), & \text{if } 0 < p \leq \infty, 0 < q \leq \infty, s > \sigma_{p,q}, \\
F^s_{p,q}(\Omega), & \text{if } 1 < p < \infty, 1 \leq q \leq \infty, s = 0, \\
F^s_{p,q}(\Omega), & \text{if } 0 < p < \infty, 0 < q \leq \infty, s < 0,
\end{cases}
\]

where

\[
\sigma_p = d\left(\frac{1}{p} - 1\right)_+, \quad \sigma_{p,q} = d\left(\frac{1}{\min(p,q)} - 1\right)_+, \quad 0 < p, q \leq \infty.
\]
In this paper we consider only the so called uniformly $E$-porous domain introduced by H. Triebel; cf. [18]. This assumption of the uniformly $E$-porous property allows us to use the wavelet characterization of these function spaces.

**Definition 2.**

(i) A close set $\Gamma \subset \mathbb{R}^d$ is said to be porous if there exists a number $0 < \eta < 1$ such that one finds for any ball $B(x,r) \subset \mathbb{R}^d$ centered at $x \in \Gamma$ and of radius $r$ with $0 < r < 1$, a ball $B(y,\eta r)$ with

$$B(y,\eta r) \subset B(x,r) \quad \text{and} \quad B(y,\eta r) \cap \Gamma = \emptyset.$$  

(ii) A close set $\Gamma \subset \mathbb{R}^d$ is said to be uniformly porous if it is porous and there is a locally finite positive Radon measure $\mu$ on $\mathbb{R}^d$ such that $\Gamma = \text{supp} \mu$ and for some constants $C, c > 0$

$$C \, h(r) \leq \mu(B(x,r)) \leq c \, h(r), \quad \text{for} \quad x \in \Gamma, 0 < r < 1,$$

where $h : [0, 1] \to \mathbb{R}$ is a continuous strictly increasing function with $h(0) = 0$ and $h(1) = 1$ (the constants $C, c$ are independent of $x$ and $r$).

**Definition 3.** Let $\Omega$ be an open set in $\mathbb{R}^d$ such that $\Omega \neq \mathbb{R}^d$ and $\Gamma = \partial \Omega$.

(i) The domain $\Omega$ is said to be $E$-porous if there is a number $0 < \eta < 1$ such that one finds for any ball $B(x,r) \subset \mathbb{R}^d$ centered at $x \in \Gamma$ and of radius $r$ with $0 < r < 1$, a ball $B(y,\eta r)$ with

$$B(y,\eta r) \subset B(x,r) \quad \text{and} \quad B(y,\eta r) \cap \Omega = \emptyset.$$  

(ii) The domain $\Omega$ is called uniformly $E$-porous if it is $E$-porous and $\Gamma = \partial \Omega$ is uniformly porous.

In 2008 H. Triebel presented wavelet characterization of function spaces on $E$-porous domains; cf. [18, Chapter 2 and 3]. By Theorem 3.23 in [18] there exists an isomorphism between spaces $\bar{B}^s_{p,q}(\Omega)$ and $\ell_q(2^{(s-d/p)M_j})$, where the last space is defined below.

**Definition 4.** Let $0 < p \leq \infty$, $0 < q \leq \infty$, $\{\beta_j\}_{j=0}^\infty$ be a sequence of positive numbers and $\{M_j\}_{j=0}^\infty \in \mathbb{N}$, $j \in \mathbb{N}_0$, $\mathbb{N} = \mathbb{N} \cup \{\infty\}$. Then

$$\ell_q(\beta_j \ell_p^{M_j}) = \left\{ x : x = \{x_{j,l}\}_{j \in \mathbb{N}_0, l=1,...,M_j} \right\}$$

with

$$||x|\ell_q(\beta_j \ell_p^{M_j})|| = \left( \sum_{j=0}^\infty \beta_j^q \left( \sum_{l=1}^{M_j} |x_{j,l}|^p \right)^{q/p} \right)^{1/q} < \infty$$

with the obvious modifications if $p = \infty$ or $q = \infty$. 

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In [12] H.-G. Leopold and L. Skrzypczak introduced also the so-called box packing constant of a domain in $\mathbb{R}^d$.

Let $\Omega \subset \mathbb{R}^d$ be the domain such that $\Omega \neq \mathbb{R}^d$. We put

$$b_j(\Omega) = \sup \left\{ k : \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega, \quad Q_{j,m_\ell} \text{ being pairwise disjoint dyadic cubes of size } 2^{-j} \right\},$$

for $j = 0, 1, \ldots$ and we put

$$b(\Omega) = \sup \left\{ t \in \mathbb{R}_+ : \limsup_{j \to \infty} b_j(\Omega)2^{-jt} = \infty \right\}.$$

Moreover it is proved in [12], that for any nonempty open set $\Omega \subset \mathbb{R}^d$ we have $d \leq b(\Omega) \leq \infty$. If $\Omega$ is unbounded and is not quasi-bounded, then $b(\Omega) = \infty$. If Lebesgue measure $|\Omega|$ is finite, then $b(\Omega) = d$. In particular if the domain is bounded, then $b(\Omega) = d$. If the domain is quasi-bounded and its Lebesgue measure is infinite then $b(\Omega) \leq \infty$. However there are quasi-bounded domains such that $b(\Omega) = \infty$.

Furthermore in [12] it is shown that

$$M_j \sim b_j(\Omega). \quad (2)$$

The box packing constant helps to prove the conditions for the compactness of Sobolev embeddings.

**Theorem 5.** Let $\Omega$ be an uniformly $E$-porous quasi-bounded domain in $\mathbb{R}^d$ and let $\frac{1}{p} = \left( \frac{1}{p_2} - \frac{1}{p_1} \right)_+$, $b(\Omega) < \infty$. Then

$$\tilde{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \tilde{B}^{s_2}_{p_2,q_2}(\Omega) \quad (3)$$

is compact if

$$s_1 - s_2 - d\left( \frac{1}{p_1} - \frac{1}{p_2} \right) > \frac{b(\Omega)}{p^*}.$$

If the embedding (3) is compact and $\frac{1}{p} = 0$ then $s_1 - s_2 - d\left( \frac{1}{p_1} - \frac{1}{p_2} \right) > 0$.

If the embedding (3) is compact and $\frac{1}{p} > 0$ then $s_1 - s_2 - d\left( \frac{1}{p_1} - \frac{1}{p_2} \right) \geq \frac{b(\Omega)}{p^*}$.

Now we will formulate the key definitions, where we denote by $L(X,Y)$ the class of all linear continuous operators from $X$ into $Y$.

**Definition 6.** Let $X$ and $Y$ be quasi-Banach spaces and $T \in L(X,Y)$. 

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For \( k \in \mathbb{N} \), we define the \( k \)-th approximation number by
\[
a_k(T) := \inf \{ \| T - A \| : A \in L(X,Y), \ \text{rank}(A) < k \},
\]
where \( \text{rank}(A) \) denotes the dimension of the range \( A(X) = \{ A(x), x \in X \} \).

For \( k \in \mathbb{N} \), we define the \( k \)-th Gelfand number by
\[
c_k(T) := \inf \{ \| T J_{X,M} \| : M \subset X, \ \text{codim}(M) < k \},
\]
where \( J_{X,M} \) is the natural injection of \( M \) into \( X \) and \( M \) is a closed subspace of the quasi-Banach space \( X \).

For \( k \in \mathbb{N} \), we define the \( k \)-th Kolmogorov number by
\[
d_k(T) := \inf \{ \| Q_{N}^Y T \| : N \subset Y, \ \dim(N) < k \},
\]
where \( Q_{N}^Y \) is the natural surjection of \( Y \) onto the quotient space \( Y/N = \{ y + N : \| y + N \| = \inf_{z \in N} \| y + z \| \} \) and \( N \) is a closed subspace of the quasi-Banach space \( Y \).

For \( k \in \mathbb{N} \), we define the \( k \)-th Weyl number by
\[
x_k(T) := \sup \{ a_k(T S) : S \in L(\ell_2, X) \ with \ \| S \| \leq 1 \}.
\]

The approximation numbers, the Gelfand, the Kolmogorov and the Weyl numbers in the context of Banach spaces are examples of \( s \)-numbers, that were introduced by A. Pietsch; cf. [15]. In [6] we can find definition of \( s \)-numbers extended to the context of quasi-Banach spaces.

Now we recall the well known properties of the above \( s \)-numbers.

**Proposition 7.** Let \( s_k \in \{ a_k, c_k, d_k, x_k \} \), \( W, X, Y \) be quasi-Banach spaces and let \( Z \) be a \( t \)-Banach space, \( t \in (0,1] \). Then

1. \( \| T \| = s_1(T) \geq s_2(T) \geq s_3(T) \geq \ldots \geq 0 \) for all \( T \in L(X,Y) \),
2. (additivity)
\[
s_{n+k-1}(T_1 + T_2) \leq s_{k}(T_1) + s_{n}(T_2)
\]
for all \( T_1, T_2 \in L(X,Z) \) and \( n, k \in \mathbb{N} \),
3. (multiplicativity)
\[
s_{n+k-1}(T_1 T_2) \leq s_{k}(T_1)s_{n}(T_2)
\]
for all \( T_1 \in L(X,Y), T_2 \in L(W,X) \) and \( n, k \in \mathbb{N} \),
4. \( x_k(T) \leq c_k(T) \leq a_k(T), \ d_k(T) \leq a_k(T) \) for all \( T \in L(X,Y) \) and \( k \in \mathbb{N} \).

Estimates of the Weyl numbers of embeddings of finite dimensional \( \ell_p^N, N \in \mathbb{N} \), will be very useful for us therefore we recall the already known estimates. Lemma [8] and Lemma [10] can be found in Caetano [3]. Lemma [9] is taken from König [10, p. 186], see also in [4].
Lemma 8. Let $0 < p_1 \leq \max(2, p_2) \leq \infty$ and $1 \leq k \leq N/2$, $N \in \mathbb{N}$. Then

$$x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \sim \begin{cases} \sqrt{N}^{\frac{1}{p_1}}, & \text{if } 0 < p_1 \leq p_2 \leq 2, \\ 1, & \text{if } 2 \leq p_1 \leq p_2 \leq \infty, \\ \sqrt{N}^{\frac{1}{p_1}}, & \text{if } 0 < p_1 \leq 2 \leq p_2 \leq \infty, \\ N^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } 0 < p_2 \leq p_1 \leq 2. \end{cases} \quad (4)$$

Lemma 9. Let $2 \leq p_2 < p_1 \leq \infty$. Then there is a positive constant $C$ independent of $N$ and $k$ such that for all $k, N \in \mathbb{N}$

$$x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \leq C (N/k)^{(1/p_2 - 1/p_1)/(1 - 2/p_1)}. \quad (5)$$

Lemma 10. Let $0 < p_2 < 2 < p_1 \leq \infty$. Then there is a positive constant $C$ independent of $N$ and $k$ such that for all $k, N \in \mathbb{N}$

$$x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \geq C \begin{cases} N^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } k \leq N/2, \\ N^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } k \leq N^{1/2}, \end{cases} \quad (6)$$

Lemma 11. Let $2 < p_2 < p_1 \leq \infty$. Then there is a positive constant $C$ independent of $N$ and $k$ such that for all $k, N \in \mathbb{N}$

$$x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \geq C \quad \text{if } k \leq N/4. \quad (7)$$

Proof. Using the multiplicativity of the Weyl numbers, together with Lemma 8 and the hypothesis $k \leq N/4$, we can write

$$C_1 \leq x_{2k}(\text{id} : \ell^N_2 \to \ell^N_{p_2}) \leq x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_1}) \leq C_2 x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}),$$

which completes the proof. \hfill \Box

The following result was already proved in the case $1 \leq p_2 \leq 2 < p_1 \leq \infty$; see [10, p. 186].

Lemma 12. Let $0 < p_2 < 2 < p_1 \leq \infty$. Then there is a positive constant $C$ independent of $N$ and $k$ such that for all $k, N \in \mathbb{N}$

$$x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \leq CN^{\frac{1}{2} - \frac{1}{p_1}}. \quad (8)$$

Proof. Using the multiplicativity of the Weyl numbers, together with Lemma 8 and the hypothesis $k \leq N/2$, we can write

$$x_{2k}(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \leq x_k(\text{id} : \ell^N_2 \to \ell^N_{p_2}) x_k(\text{id} : \ell^N_{p_1} \to \ell^N_2) \leq CN^{\frac{1}{2} - \frac{1}{p_1}} (N/k)^{(1/2 - 1/p_1)/(1 - 2/p_1)} \leq CN^{\frac{1}{2} - \frac{1}{p_1}},$$

which finishes the proof, by the monotonicity of the Weyl numbers. \hfill \Box
The following lemma for Gelfand numbers will be useful for the upper estimates of Weyl numbers.

**Lemma 13.** If \( 1 \leq k \leq N < \infty \) and \( 0 < p_2 \leq p_1 \leq \infty \), then

\[
c_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) = (N - k + 1)^{\frac{1}{p_2} - \frac{1}{p_1}}.
\]

The proof of this lemma follows literally [16, Section 11.11.4], see also [17]. Indeed the original proof is used only to deal with the Banach setting. However, the same proof works also in the quasi-Banach setting \( 0 < p_2 \leq p_1 \leq \infty \).

Based on the definitions of Weyl and Gelfand numbers, it is known that for any linear continuous operator \( T \) between two complex quasi-Banach spaces, \( x_k(T) \leq c_k(T) \). Then in terms of Lemma 9, Lemma 12 and Lemma 13, we have the following two propositions.

**Proposition 14.** Let \( 0 < p_2 \leq 2 < p_1 \leq \infty \). Then there is a positive constant \( C \) independent of \( N \) and \( k \) such that for all \( k, N \in \mathbb{N} \)

\[
x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \leq C \begin{cases} N^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } 1 \leq k \leq N^{\frac{1}{p_1}}, \\ (N^{\frac{1}{p_2} - \frac{1}{p_1}})^{\frac{1}{(1/p_2 - 1/p_1)/(1 - 2/p_1)}}, & \text{if } N^{\frac{1}{p_1}} \leq k \leq N. \end{cases}
\]

(9)

**Proposition 15.** Let \( 2 \leq p_2 < p_1 \leq \infty \). Then there is a positive constant \( C \) independent of \( N \) and \( k \) such that for all \( k, N \in \mathbb{N} \)

\[
x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) \leq C \begin{cases} N^{\frac{1}{p_2} - \frac{1}{p_1}}, & \text{if } 1 \leq k \leq N^{\frac{1}{p_1}}, \\ (N^{\frac{1}{p_2} - \frac{1}{p_1}})^{\frac{1}{(1/p_2 - 1/p_1)/(1 - 2/p_1)}}, & \text{if } N^{\frac{1}{p_1}} \leq k \leq N. \end{cases}
\]

(10)

Of course when \( k > N \) then

\[ x_k(\text{id} : \ell^N_{p_1} \to \ell^N_{p_2}) = 0. \]

3. The embeddings of sequence spaces

In this part we consider the embeddings of sequence spaces from Definition [4] that will play an important role in this paper. In [11], we find the following theorem.

**Theorem 16.** Let \( 0 < p_1, p_2 \leq \infty \), \( 0 < q_1, q_2 \leq \infty \), \( \{M_j\}_j=0^\infty \) be an arbitrary sequence of natural numbers and \( \{\beta_j\}_j=0^\infty \) be an arbitrary weight sequence. The embedding

\[ \text{id} : \ell^N_{q_1}(\beta_j \ell^M_{p_1}) \to \ell^N_{q_2}(\ell^{M_j}_{p_2}) \]

is compact if and only if

\[ \{\beta_j^{-1} M_j^{1/(p_2 - 1/p_1)}\}_j=0^\infty \in \ell^{q^*}_{q^*} \quad \text{if } q^* < \infty \]

or

\[ \lim_{j \to \infty} (\beta_j^{-1} M_j^{1/(p_2 - 1/p_1)}) = 0 \quad \text{if } q^* = \infty, \]
where

\[
\frac{1}{q^*} = \left( \frac{1}{q_2} - \frac{1}{q_1} \right)^+ \quad \text{i.e.} \quad q^* = \begin{cases} \infty, & \text{if } 0 < q_1 \leq q_2 \leq \infty, \\ (q_1 q_2)/(q_1 - q_2), & \text{if } 0 < q_2 < q_1 < \infty, \\ q_2, & \text{if } q_1 = \infty. \end{cases}
\]

We need also the following lemma which will be used in the proof of the next theorem.

**Lemma 17.** Let \( \{\beta_j^{-1} M_j^{(1/p_2-1/p_1)_+}\}_{j=0}^{\infty} \in \ell_{q^*} \text{ and} \)

\[
\text{id}_j x = (\delta_{j,k} x_{k,l})_{k \in N_0, l=1, \ldots, M_k} = \begin{cases} 0, & \text{if } k \neq j, \\ x_{j,l}, & \text{if } k = j \text{ and } l = 1, \ldots, M_j. \end{cases}
\]

Then

\[
\left\| \left( \text{id} - \sum_{j=0}^{N} \text{id}_j \right) x \right\|_{\ell_{q_2}(\ell_{p_2}^{M_j})} \leq \left\| \left\{ \beta_j^{-1} M_j^{(1/p_2-1/p_1)_+}\right\}_{j=N+1}^{\infty} \right\|_{\ell_{q^*}} \left\| x \right\|_{\ell_{q_1}(\beta_j p_{M_j})}. \]

The proof can be found in [11].

Now we consider the asymptotic behavior of the Weyl, the Gelfand numbers and the approximation numbers for these embeddings. We start with the estimates of the Weyl numbers.

**Theorem 18.** Let \( 0 < p_1, p_2 \leq \infty, 0 < q_1, q_2 \leq \infty \text{ and } 0 < b < \infty, \delta > b(p_2 - \frac{1}{p_1})_+ \). Suppose \( \lambda = \left( \frac{1}{p_2} - \frac{1}{p_1}\right)/(1 - \frac{2}{p_1}) \), \( M_j \sim 2^{j\delta} \) and that the operator \( \text{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}^{M_j}) \rightarrow \ell_{q_2}(\ell_{p_2}^{M_j}) \) is compact.

(i) If \( 0 < p_1 \leq \max(2, p_2) \leq \infty \text{ or } 0 < p_2 \leq 2 < p_1 \leq \infty \), then

\[
x_k(\text{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}^{M_j}) \rightarrow \ell_{q_2}(\ell_{p_2}^{M_j})) \sim k^{-\beta},
\]

where

\[
\beta = \begin{cases} \frac{\delta}{\delta} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 0 < p_1, p_2 \leq 2, \\ \frac{\delta}{\delta} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 2 \leq p_1 \leq p_2 \leq \infty, \\ \frac{\delta}{\delta} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 0 < p_2 \leq 2 < p_1 \leq \infty \text{ and } \delta < \frac{b}{p_2}, \\ \frac{\delta}{\delta} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 0 < p_2 \leq 2 < p_1 \leq \infty \text{ and } \delta > \frac{b}{p_2}. \end{cases}
\]

(ii) If \( 2 < p_2 < p_1 \leq \infty \text{ and } \delta > b\lambda \), then

\[
x_k(\text{id} : \ell_{q_1}(2^{j\delta} \ell_{p_1}^{M_j}) \rightarrow \ell_{q_2}(\ell_{p_2}^{M_j})) \sim k^{-\frac{\delta}{\delta}}.
\]

(iii) If \( 2 < p_2 < p_1 \leq \infty \text{ and } \delta < b\lambda \), then

\[
ck^{-\frac{\delta}{\delta}} \leq x_k(\text{id} : \ell_{q_1}(2\ell_{p_1}^{M_j}) \rightarrow \ell_{q_2}(\ell_{p_2}^{M_j})) \leq CK^{-\frac{\delta}{\delta}}(\frac{1}{p_1} - \frac{1}{p_2}).
\]
Proof. Step 1. Estimation from below. We consider the commutative diagram

\[
\begin{array}{c}
\ell_{M_j}^{M_j} \\
\downarrow \id^{(j)} \\
\ell_{M_j}^{M_j} \\
\downarrow T_j \\
\ell_{q_1}^{M_j}
\end{array}
\quad
\begin{array}{c}
\ell_{q_1}^{M_j} \\
\downarrow \id \\
\ell_{q_2}^{M_j}
\end{array}
\]

where \(\id^{(j)}\) denotes the embedding from \(\ell_{M_j}^{M_j}\) in \(\ell_{p_2}^{M_j}\). The operator \(T_j\) is a projection, such that

\[
T_j x = \{\delta_{j,k} x_k\}_{k=1}^{M_j},
\]

and \(S_j\) is the natural embedding which maps \(\{x_l\}_{l=1}^{M_j}\) to the \(j\)-th block in \(\ell_{q_1}(2^j \ell_{p_1}^{M_j})\),

\[
S_j(\{x_l\}_{l=1}^{M_j}) = (\hat{x}_{v,l}),
\]

where

\[
\hat{x}_{v,l} = \begin{cases} 
0, & \text{if } v \neq j, \\
 x_l, & \text{if } v = j \text{ and } 1 \leq l \leq M_j.
\end{cases}
\]

Then

\[
||S_j : \ell_{p_1}^{M_j} \to \ell_{q_1}(2^j \ell_{p_1}^{M_j})|| = 2^j, \quad ||T_j : \ell_{q_2}^{M_j} \to \ell_{p_2}^{M_j}|| = 1
\]

and

\[
\id^{(j)} = T_j \circ \id \circ S_j.
\]

Consequently

\[
x_k(\id^{(j)} : \ell_{p_1}^{M_j} \to \ell_{p_2}^{M_j}) \leq 2^j x_k(\id : \ell_{q_1}(2^j \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j})).
\]

Let \(0 < p_1 \leq \max(2, p_2) \leq \infty\). By (4) with \(k = [M_j/2]\), we obtain

\[
x_k(\id) \geq C \begin{cases} 
k^{-((\delta/b+1)/p_1-1/p_2)}, & \text{if } 0 < p_1, p_2 \leq 2, \\
k^{-\delta/b}, & \text{if } 2 \leq p_1 \leq p_2, \quad 0 \leq \delta \leq \infty, \\
k^{-((\delta/b+1)/p_1-1/2)}, & \text{if } 0 < p_1 \leq 2 \leq p_2 \leq \infty.
\end{cases}
\]

Let \(0 < p_2 \leq 2 < p_1 \leq \infty\).

If \(\delta < \frac{b}{p_2}\) then by (5) with \(k = [M_j/2]\), we obtain

\[
C k^{-(p_1/2)(\delta/b+1)/(p_1-1/p_2)} \leq C 2^{-j\delta} 2^{(1/p_2-1/p_1)} \leq C 2^{-j\delta} M_j^{1/p_2-1/p_1} \leq x_k(\id).
\]

If \(\delta > \frac{b}{p_2}\) then the formula (3) with \(k = [M_j/2]\), yields

\[
C k^{-(\delta/b+1/2-1/p_2)} \leq C 2^{-j\delta} M_j^{1/p_2-1/2} \leq x_k(\id).
\]

Let \(2 < p_2 < p_1 \leq \infty\). By (4) with \(k = [M_j/4]\), we obtain

\[
C k^{-\delta/b} \leq C 2^{-j\delta} \leq x_k(\id).
\]
Step 2. Estimation from above. Let $\text{id}_j$ be defined as in Lemma 17. Then we have

$$\left\| (\text{id} - \sum_{j=0}^{N} \text{id}_j)x |\ell_q(2^{j\delta}q^M_{p_2}) \right\| \leq E_N \left\| x |\ell_q(2^{j\delta}q^M_{p_1}) \right\|,$$

where

$$E_N = \left\| \left\{ 2^{-j\delta} M_j^{(1/p_2-1/p_1)} \right\}_{j=N+1}^{\infty} \right\|.$$ 

Let $\rho = \min(1, p_2, q_2)$, then $l_{q_2}(q^M_{p_2})$ is a $\rho$-Banach space. Therefore from the properties of the Weyl numbers, we get

$$x^p_k (\text{id} : \ell_{q_1}(2^{j\delta}q^M_{p_1}) \rightarrow \ell_{q_2}(q^M_{p_2})) \leq E^p_N + \sum_{j=0}^{L} x^p_{k_j} (\text{id}_j) + \sum_{j=L+1}^{N} x^p_{k_j} (\text{id}_j), \quad (11)$$

where

$$k = \sum_{j=0}^{N} k_j - (N + 1). \quad (12)$$

We will choose $N$ later.

Substep 2.1. We consider the commutative diagram

$$\ell_{q_1}(2^{j\delta}q^M_{p_1}) \xrightarrow{T_j} \ell^M_{p_1}$$

$$\downarrow \text{id}_j \quad \downarrow \text{id}^{(j)}$$

$$\ell_{q_2}(q^M_{p_2}) \xrightarrow{S_j} \ell^M_{p_2},$$

where $T_j$ and $S_j$ are defined similarly to $T_j$ and $S_j$ in Step 1. Now

$$\|T_j : \ell_{q_1}(2^{j\delta}q^M_{p_1}) \rightarrow \ell^M_{p_1}\| = 2^{-j\delta} \quad \text{and} \quad \|S_j : \ell^M_{p_1} \rightarrow \ell_{q_2}(q^M_{p_2})\| = 1.$$ 

Therefore $\text{id}_j = \text{S}_j \circ \text{id}^{(j)} \circ T_j$ and

$$x_{k_j} (\text{id}_j) \leq 2^{-j\delta} x_{k_j} (\text{id}^{(j)} : \ell^M_{p_1} \rightarrow \ell^M_{p_2}). \quad (13)$$

Consequently we get from (11)

$$x^p_k (\text{id} : \ell_{q_1}(2^{j\delta}q^M_{p_1}) \rightarrow \ell_{q_2}(q^M_{p_2})) \leq E^p_N + \sum_{j=0}^{L} 2^{-j\delta p} x^p_{k_j} (\text{id}^{(j)} (\text{id}_j) + \sum_{j=L+1}^{N} 2^{-j\delta p} x^p_{k_j} (\text{id}^{(j)}),$$

with arbitrary $L, N$ and $k_j$ satisfying (12).

Substep 2.2. Now we estimate the sum $\sum_{j=0}^{L} x^p_{k_j} (\text{id}_j).$

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Let \( 0 < p_1 \leq \max(2, p_2) \leq \infty \). First we show that there exists a constant \( C > 0 \) independent of \( N \) and \( k \), such that

\[
x_k(\text{id} : ℓ^N_{p_1} \to ℓ^N_{p_2}) \leq C \begin{cases} 
  k^{1/p_2 - 1/p_1}, & \text{if } 0 \leq p_1 \leq p_2 \leq 2, \\
  1, & \text{if } 2 \leq p_1 \leq p_2 \leq \infty, \\
  N^{1/p_2 - 1/p_1}, & \text{if } 0 < p_1 \leq 2 \leq p_2 \leq \infty, \\
  1, & \text{if } 2 \leq p_1 \leq p_2 \leq \infty,
\end{cases}
\]

(14)

for \( k \leq N \). We consider the commutative diagram

\[
\begin{array}{ccc}
ℓ^N_{p_1} & \xrightarrow{S} & ℓ^2N_{p_1} \\
\downarrow \text{id} & & \downarrow \text{id} \\
ℓ^N_{p_2} & \xleftarrow{T} & ℓ^2N_{p_2},
\end{array}
\]

where

\[
S(\lambda_1, \ldots, \lambda_N) = (\lambda_1, \ldots, \lambda_N, 0, \ldots, 0) \quad \text{and} \quad T(\lambda_1, \ldots, \lambda_2N) = (\lambda_1, \ldots, \lambda_N).
\]

Both norms \(||S||\) and \(||T||\) are equal to 1. Therefore \( x_k(\text{id}) \leq x_k(\text{Id}) \), and then by (13) we get (14).

Let

\[
k_j = \lfloor M_j 2^{(L-j)\epsilon} \rfloor \quad \text{for} \quad j = 0, 1, \ldots, L,
\]

with \( 0 < \epsilon < b \). Then

\[
\sum_{j=0}^{L} k_j \leq \sum_{j=0}^{L} \lfloor e^{2^{j+2}(L-j)\epsilon} \rfloor \leq \lfloor e^{2Lb} \rfloor \sum_{j=0}^{L} 2^{(j-2)(b-\epsilon)} \leq \lfloor e^{2Lb} \rfloor.
\]

(15)

We put

\[
t = \begin{cases} 
  1/p_2 - 1/p_1, & \text{if } 0 < p_1, p_2 \leq 2, \\
  0, & \text{if } 2 \leq p_1 \leq p_2 \leq \infty, \\
  1/2 - 1/p_1, & \text{if } 0 < p_1 \leq 2 \leq p_2 \leq \infty.
\end{cases}
\]

(16)

So by (13) and (14) we get

\[
x_{k_j}(\text{id}_j) \leq C(2^{-j^2} M_j^2 2^{(L-j)\epsilon} t),
\]

(17)

if \( k_j \leq M_j \) and

\[
x_{k_j}(\text{id}_j) = 0,
\]

if \( k_j > M_j \). Moreover if \( j < L - \frac{1}{\epsilon} \), then \( k_j = \lfloor M_j 2^{(L-j)\epsilon} \rfloor > M_j \) and then \( x_{k_j}(\text{id}_j) = 0 \). Therefore by (17) we have:
for \( t < 0 \)
\[
\sum_{j=0}^{L} x_{k_j}(id_j) = \sum_{j=[L-1/c]}^{L} x_{k_j}(id_j) \\
\leq c(2^{-L \delta} 2^{L b t})^\rho \sum_{j=[L-1/c]}^{L} (2(L-j) \delta 2^{-(L-j)b t} 2^{(L-j) e t})^\rho \\
\leq c 2^{-L b p (\delta / b - t)} \sum_{j=[L-1/c]}^{L} 2^{\delta / e 2^{-b t / e} 2^{(L-j) e t}} \leq C 2^{-L b p (\delta / b - t)}; \quad (18)
\]
for \( t = 0 \)
\[
\sum_{j=0}^{L} x_{k_j}(id_j) = \sum_{j=[L-1/c]}^{L} x_{k_j}(id_j) \\
\leq c 2^{-L \rho \delta} \sum_{j=[L-1/c]}^{L} 2^{\delta / e} \leq c 2^{-L \rho \delta} \frac{1}{\epsilon} 2^{\delta / e} \leq C 2^{-L \rho \delta}; \quad (19)
\]
for \( t > 0 \)
\[
\sum_{j=0}^{L} x_{k_j}(id_j) = \sum_{j=[L-1/c]}^{L} x_{k_j}(id_j) \\
\leq c 2^{-L b p (\delta / b - t)} \sum_{j=[L-1/c]}^{L} (2^{\delta / e} 2^{-(L-j) b t})^\rho \leq C 2^{-L b p (\delta / b - t)}; \quad (20)
\]
where \( \lceil x \rceil = \inf \{k \in \mathbb{N} : k \geq x\} \) and constant \( C \) is dependent on \( \epsilon, \delta, t \) but not on \( L \).

Let \( 0 < p_2 \leq 2 < p_1 \leq \infty \).

If \( \frac{\delta}{b} < \frac{1}{p_2} \) then we take
\[
k_j = \lceil M 2^{2/p_1} 2^{(L-j) e} \rceil \quad \text{for} \quad j = 0, 1, \ldots, L,
\]
with \( \epsilon > 0 \). Since \( \frac{\delta}{b} < \frac{1}{p_2} \) we can choose \( \epsilon > 0 \), such that \( \frac{\delta}{b} + \frac{1}{p_1} - \frac{1}{p_2} < \frac{\epsilon}{p_1} \). So we have
\[
\sum_{j=0}^{L} k_j \leq \sum_{j=0}^{L} \lceil 2^{(2/p_1)} 2^{(L-j) e} \rceil \leq \lceil 2^{L b (2/p_1)} \rceil \sum_{j=0}^{L} 2^{b (j-L) (2/p_1 - \epsilon / b)} \leq \lceil 2^{L b (2/p_1)} \rceil.
\]
(21)

Now by (13) and (9) we get
\[
x_{k_j}(id_j) \leq C 2^{-j \delta} M_j^{1/p_2 - \epsilon / b} 2^{-(L-j)(\epsilon/2)} \\
\leq C 2^{-L \delta} 2^{L b (1/p_2 - 1/p_1)} 2^{-(L-j)(\delta/2)}, \quad (22)
\]

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since $M_j^{2/p_1} \leq k_j$. Hence
\[\sum_{j=0}^{L} x_{k_j}^\rho (\text{id}_j) \leq c 2^{-Lb_0(\delta/b+1/p_1-1/p_2)} \sum_{j=0}^{L} 2^{b_0(L-j)(\delta/b+1/p_1-1/p_2-\epsilon/(2b))} \leq C 2^{-Lb_0(\delta/b+1/p_1-1/p_2)},\] (23)
with constant $C$ independent of $L$.

If $\frac{\delta}{b} > \frac{1}{p_2}$ then we take
\[k_j = \lfloor M_j 2^{(L-j)\epsilon} \rfloor \quad \text{for} \quad j = 0, 1, \ldots, L,\] (24)
with a fixed $\epsilon$, $0 < \epsilon < b$. Hence
\[\sum_{j=0}^{L} k_j \leq \sum_{j=0}^{L} \lfloor c 2^{j b_0 [2^{(L-j)\epsilon}]} \rfloor \leq \lfloor c 2^{L b_0} \rfloor \sum_{j=0}^{L} 2^{(j-L)(b-\epsilon)} \leq c 2^{L b_0}.\] (25)

If $j < L - \frac{1}{\epsilon}$ then $k_j > M_j$. Therefore from (13) and (9) we get
\[\sum_{j=0}^{L} x_{k_j}^\rho (\text{id}_j) \leq \sum_{j=0}^{L} \lfloor c 2^{j b_0 [2^{(L-j)\epsilon}]} \rfloor \leq c 2^{L b_0} \sum_{j=0}^{L} 2^{(L-j)\delta 2^{-(L-j)b(1/p_2-1/2)} 2^{-(L-j)(\epsilon/2)}}.\]

Here
\[\sum_{j=0}^{L} 2^{(L-j)\delta 2^{-(L-j)b(1/p_2-1/2)} 2^{-(L-j)(\epsilon/2)}} \leq 2^{\delta b_0/\epsilon} \sum_{j=0}^{L} 2^{-(L-j)b(1/p_2-1/2)} 2^{-(L-j)(\epsilon/2)} \leq C.\]

So
\[\sum_{j=0}^{L} x_{k_j}^\rho (\text{id}_j) \leq C 2^{-Lb_0(\delta/b+1/2-1/p_2)}.\] (26)

Let $2 < p_2 < p_1 \leq \infty$.

If $\delta < b\lambda$ then we take
\[k_j = \lfloor M_j^{2/p_1} 2^{(L-j)\epsilon} \rfloor \quad \text{for} \quad j = 0, 1, \ldots, L,\]
with $\epsilon > 0$. Since $\delta < b\lambda$ we can choose $\epsilon > 0$, such that $\frac{1}{\lambda} (\frac{\delta}{b} + \frac{1}{p_1} - \frac{1}{p_2}) < \frac{\epsilon}{b} < \frac{2}{p_2}$.

We should mention that the equality $\frac{1}{\lambda} (\frac{\delta}{b} + \frac{1}{p_1} - \frac{1}{p_2}) = \frac{2}{p_1}$ holds true if and only if $\delta = b\lambda$. Then
\[\sum_{j=0}^{L} k_j \leq \lfloor c 2^{L b(2/p_1)} \rfloor.\] (27)
By means of (10) and (13), we have

\[ x_{k_j}(\text{id}_j) \leq C2^{-j\delta} M_j \lambda^{(1-2/p_1)} 2^{-(L-j)\epsilon_\lambda} \leq C2^{-Lb(\frac{L}{2}+\frac{L}{p_1}-\frac{L}{p_2})} \lambda^{(1-2/p_1)} 2^{b(L-j)(\frac{L}{p_1}+\frac{L}{p_2}-\frac{L}{p_2}-\frac{L}{p_2})}. \]

Therefore,

\[ \sum_{j=0}^{L} x_{k_j}^p(\text{id}_j) \leq C2^{-Lbp(\frac{L}{p_1}+\frac{L}{p_2})}, \quad (28) \]

with constant \( C \) independent of \( L \).

If \( \delta > b\lambda \) then we take

\[ k_j = \lceil M_j 2^{(L-j)\epsilon} \rceil \quad \text{for} \quad j = 0, 1, \ldots, L, \]

with fixed \( \epsilon, \quad 0 < \epsilon < b \). Then \( \delta > \lambda\epsilon \) and

\[ \sum_{j=0}^{L} k_j \leq \lceil c2^Lb \rceil. \quad (29) \]

If \( j < L - \frac{1}{b} \), then \( k_j > M_j \). In view of (13) and (13), we have

\[ \sum_{j=0}^{L} x_{k_j}^p(\text{id}_j) = \sum_{j=[L-1/\epsilon]}^{L} x_{k_j}^p(\text{id}_j) \leq C1 \sum_{j=[L-1/\epsilon]}^{L} (2^{-j\delta} 2^{-(L-j)\lambda\epsilon})^p \]

\[ \leq C1 2^{-L\delta\rho} \sum_{j=[L-1/\epsilon]}^{L} (2^{(L-j)(\delta-\lambda\epsilon)})^p \leq C2 2^{-L\delta\rho} 2^{(\delta-\lambda\epsilon)p/\epsilon} \leq C3 2^{-L\delta\rho}. \quad (30) \]

**Substep 2.3.** Now we estimate the sum \( \sum_{j=L+1}^{N} x_{k_j}^p(\text{id}_j) \) and choose \( N \) such that \( E_N \) is small enough.

Consider the first case \( 0 < p_1 \leq \max(2, p_2) \leq \infty \). We show that

\[ \sum_{j=L+1}^{N} x_{k_j}^p(\text{id}_j) \leq C2^{-Lbp(\delta/b-t)}, \quad (31) \]

where \( t \) is defined in (10). Let

\[ k_j = \max \{ \lceil M_L(j-L)^{-2} \rceil, 1 \} \quad \text{for} \quad j = L + 1, \ldots, N. \]

Then

\[ \sum_{j=L+1}^{N} k_j \leq cM_L + (N-L). \quad (32) \]

If \( t \leq 0 \) then \( k_j^p \leq (M_L(j-L)^{-2})^t \). Therefore from (13) and (14) we get

\[ \sum_{j=L+1}^{N} x_{k_j}^p(\text{id}_j) \leq C(2^{-L\delta_2 Lb(t)})^\rho \sum_{j=L+1}^{N} (2^{(L-j)^\delta}(j-L)^{-2t})^\rho \leq C2^{-Lbp(\delta/b-t)}. \quad (33) \]
If \( t > 0 \) and since \( \delta > b\left(\frac{1}{p_2} - \frac{1}{p_1}\right) = \frac{1}{t} \) then from (13) and (14) we obtain

\[
\sum_{j=L+1}^{N} x_{k_j}^\rho (\text{id}_j) \leq C(2^{-L\delta}2^{Lb(t)})^\rho \sum_{j=L+1}^{N} (2(L-j)^\delta 2^{-(L-j)b})^\rho \leq C2^{-Lb(\delta/b-t)},
\]

where \( C \) is independent of \( L \) and \( N \).

Now we can choose \( N \) such that \( E_N \leq c2^{-L\delta}M^L \). This is possible because \( \{E_N\}_{N=1}^\infty \) is a decreasing sequence such that \( \lim_{N \to \infty} E_N = 0 \). The formulas (12), (15) and (32) imply that \( k \leq [cM_L] \). So by (11), (18), (19), (20), (21) and (34) we have \( x_{[cM_L]}(\text{id}) \leq C2^{-Lb(\delta/b-t)} \). Finally

\[
x_k(\text{id}) \leq Ck^{-(\delta/b-t)}, \quad \text{for all } k \in \mathbb{N}.
\]

Consider the next case \( 0 < p_2 \leq 2 < p_1 \leq \infty \).

If \( \frac{\delta}{b} < \frac{1}{p_2} \) then we take

\[
k_j = \max \left\{ \left[ \frac{M^2}{p_1^2} (j - L)^{-2} \right], 1 \right\} \quad \text{for } j = L + 1, \ldots, N.
\]

Hence

\[
\sum_{j=L+1}^{N} k_j \leq cM^2 + (N - L).
\]

Now by (12), (13) and since \( \delta > b\left(\frac{1}{p_2} - \frac{1}{p_1}\right) \) we have

\[
\sum_{j=L+1}^{N} x_{k_j}^\rho (\text{id}_j) \leq C(2^{-L\delta}2^{Lb(1/p_2-1/p_1)})^\rho \sum_{j=L+1}^{N} (2(L-j)^\delta 2^{(L-j)b(1/p_1-1/p_2)})^\rho \leq C2^{-Lb(\delta/b+1/p_1-1/p_2)} \leq c2^{-Lb(\delta/b+1/p_1-1/p_2)}.
\]

As in the previous case we can choose \( N \) such that \( E_N \leq c2^{-L\delta}M^{L/p_2-1/p_1} \).

The formulas (12), (21) and (35) imply \( k \leq [cM^2/p_1] \). So by (11), (23) and (36) we have \( x_{[cM^2/p_1]}(\text{id}) \leq C2^{-Lb(\delta/b+1/p_1-1/p_2)} \). Finally

\[
x_k(\text{id}) \leq Ck^{-(p_1/2)(\delta/b+1/p_1-1/p_2)}, \quad \text{for all } k \in \mathbb{N}.
\]

But if we assume \( \frac{\delta}{b} > \frac{1}{p_2} \), then taking

\[
k_j = \max \left\{ \left[ M_L(j - L)^{-2} \right], 1 \right\} \quad \text{for } j = L + 1, \ldots, N,
\]

we get

\[
\sum_{j=L+1}^{N} k_j \leq cM^L + (N - L).
\]
We notice that if $1 \leq k_j \leq M_j^{2/p_1}$ then $M_j^{1/p_2-1/p_1} \leq M_j^{1/p_2} k_j^{-1/2}$. So by (9) and (13) we have

$$\sum_{j=L+1}^{N} x_{k_j}^p (id_j) \leq \sum_{j=L+1}^{N} c(2^{-j \delta 2^{(b_2)/p_2} 2^{-(b_L)/2} (j - L)})^p$$

$$\leq c2^{-Lb p(\delta/b - 1/p_2 + 1/2)} \sum_{j=L+1}^{N} (2^{b(L-j) \delta b - 1/p_2}) (j - L))^p$$

$$\leq C2^{-Lb p(\delta/b - 1/p_2 + 1/2)}.$$

(38)

Now we choose $N$ such that $E_N \leq c2^{-L\delta M_L^{1/p_2-1/2}}$. From (12), (25), (37) we have $k \leq [cM_L]$. So by (11), (26), (38) we get $x_{[cM_L]}(id) \leq C2^{-Lb p(\delta/b - 1/p_2 + 1/2)}$. Then

$$x_k(id) \leq C k^{-p_1}$$

for all $k \in \mathbb{N}$.

Consider the last case $2 < p_2 < p_1 \leq \infty$.

If $\delta < b \lambda$ then we take

$$k_j = \max\{[M_j^{2/p_1} (j - L)^{-2}], 1\} \text{ for } j = L + 1, \ldots, N.$$ 

Hence, $k_j \leq M_j^{2/p_1} \leq M_j^{2/p_1}$ and

$$\sum_{j=L+1}^{N} k_j \leq cM_L^{2/p_1} + (N - L).$$

(39)

Based on (11), (13) and the previous assumption $\delta > b(1 - 1/p_1)^+$, we have

$$\sum_{j=L+1}^{N} x_{k_j}^p (id_j) \leq C \sum_{j=L+1}^{N} (2^{-j \delta 2^{b(1/p_1 - 1/p_2)}})^p \leq C2^{-Lb p(\delta/b + 1/p_1 - 1/p_2)}.$$ 

(40)

We choose $N$ such that $E_N \leq c2^{-L\delta M_L^{1/p_2 - 1/p_1}}$. The formulas (27) and (39) imply that $k = \sum_{j=0}^{N} k_j - (N + 1) \leq [cM_L^{2/p_1}]$. Finally, by (11), (28) and (40), we have

$$x_k(id) \leq C k^{-p_1} \left(\frac{\delta}{1 + \frac{1}{p_1}}\right)^p,$$

for all $k \in \mathbb{N}$.

If $\delta > b \lambda$ then we take

$$k_j = \max\{[M_L (j - L)^{-2}], 1\} \text{ for } j = L + 1, \ldots, N,$$

and we get

$$\sum_{j=L+1}^{N} k_j \leq cM_L + (N + L).$$

(41)
By (5) and (13), we have

\[
\sum_{j=0}^{N} x_{j}^{\rho}(id) \leq C \sum_{j=0}^{N} (2^{-j\delta}2^{j\lambda}2^{-Lj\lambda}(j-L)^{2\lambda})^{\rho} 
\leq C2^{-Lp\delta} \sum_{j=0}^{N} 2^{\rho(L-j)(\delta-b\lambda)}(j-L)^{2p\lambda} \leq C2^{-Lp\delta}.
\]

(42)

We choose \(N\) such that \(E_{N} \leq c2^{-L\delta}\). The formulas (29) and (41) yield that \(k = \sum_{j=0}^{N} k_{j} - (N + 1) \leq [cM_{L}]\). Then, by (11), (30) and (42), we have

\[
x_{k}(id) \leq Ck^{-\frac{\delta}{\lambda}}, \quad \text{for all } k \in \mathbb{N}.
\]

\[\square\]

**Remark 1.** Still, there are minor gaps left open in the estimates for the case \(\delta < b\lambda\). It should be mentioned that, for the bounds of the asymptotic order, the inequality \(\frac{\delta}{\lambda} = \frac{p\delta}{p\beta} + \frac{\delta}{\beta} = \frac{p\delta}{p\lambda} + \frac{\delta}{\lambda}\) holds if and only if \(\delta = b\lambda\). Furthermore, the problem becomes much more complicated if \(\delta = b\lambda\). How do the Weyl numbers behave in such limiting situation? Different from those previous cases, the corresponding behaviour herein may depend on the parameters \(q_{1}\) and \(q_{2}\) to a certain extent.

The asymptotic behavior of the approximation, the Gelfand and the Kolmogorov numbers can be found in [19], Theorem 3.5, 4.12 and 4.6. Since a bit different notation is used there we recall here the results using our symbols. Some partial results were proved earlier in [3].

**Theorem 19.** Let \(0 < p_{1}, p_{2} \leq \infty\), \(0 < q_{1}, q_{2} \leq \infty\) and \(0 < b < \infty\), \(\delta > b(\frac{1}{p_{2}} - \frac{1}{p_{1}})_{+}\). Suppose \(M_{j} \sim 2^{jb}\), \(\frac{1}{p} = \frac{1}{\min(p_{1}, p_{2})}\) and that the operator

\[
\alpha_{k}(id) : \ell_{q_{1}}(2^{jb}M_{j}^{p_{1}}) \rightarrow \ell_{q_{2}}(2^{jb}M_{j}^{p_{2}}) \quad \text{is compact. Then}
\]

\[a_{k}(id) : \ell_{q_{1}}(2^{jb}M_{j}^{p_{1}}) \rightarrow \ell_{q_{2}}(2^{jb}M_{j}^{p_{2}}) \sim k^{-\beta},
\]

where

\[
\beta = \begin{cases} 
\frac{\delta}{\lambda}, & \text{if } 0 < p_{1} \leq p_{2} \leq 2 \leq p_{1} \leq p_{2} \leq \infty, \\
\frac{\delta}{\lambda} + \frac{1}{\beta}, & \text{if } 0 < p_{1} < 2 < p_{2} < \infty \text{ and } \delta < \frac{\delta}{\lambda} \text{ or } 1 < p_{1} < 2 < p_{2} = \infty \text{ and } \delta \leq \frac{\delta}{\lambda}, \\
\frac{\delta}{\lambda} + \frac{1}{\beta} - \frac{1}{p_{2}}, & \text{if } 0 < p_{1} < 2 < p_{2} \leq \infty \text{ and } \delta > \frac{\delta}{\lambda}, \\
\frac{\delta}{\lambda} - \frac{1}{p_{2}}, & \text{if } 0 < p_{2} \leq p_{1} \leq \infty, \\
\frac{\delta}{\lambda} + \frac{1}{\beta} - \frac{1}{p_{2}}, & \text{if } 0 < p_{1} \leq 1 < p_{2} = \infty.
\end{cases}
\]

(43)

**Theorem 20.** Let \(0 < p_{1}, p_{2} \leq \infty\), \(0 < q_{1} q_{2} \leq \infty\) and \(0 < b < \infty\), \(\delta > b(\frac{1}{p_{2}} - \frac{1}{p_{1}})_{+}\). Suppose \(M_{j} \sim 2^{jb}\), \(\theta = (\frac{1}{p_{1}} - \frac{1}{p_{2}})/(\frac{1}{q_{1}} - \frac{1}{q_{2}})\) and that the operator

\[
\alpha_{k}(id) : \ell_{q_{1}}(2^{jb}M_{j}^{p_{1}}) \rightarrow \ell_{q_{2}}(2^{jb}M_{j}^{p_{2}}) \quad \text{is compact. Then}
\]
where

\[ \beta = \begin{cases} \frac{\delta}{q}, & \text{if } 2 \leq p_1 < p_2 \leq \infty, \\ \frac{\delta}{q} + \frac{1}{p_2} - \frac{1}{p_1}, & \text{if } 0 < p_2 \leq p_1 \leq \infty \text{ or } 0 < p_1 < p_2 \leq 2 \text{ and } \frac{\delta}{p_2} > \frac{1}{p_2}, \\ \frac{\delta}{q} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 0 < p_1 < 2 < \infty \text{ and } \frac{\delta}{p_2} < \frac{1}{p_1}, \\ \frac{\delta}{q} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 0 < p_1 < 2 < \infty \text{ and } \frac{\delta}{p_2} < \frac{1}{p_1}. \end{cases} \]

**Theorem 21.** Let \( 0 < p_1, p_2 \leq \infty, 0 < q_1, q_2 \leq \infty \) and \( 0 < b < \infty, \delta > b(\frac{1}{p_2} - \frac{1}{p_1})_+ \). Suppose \( M_j \sim 2^{j\beta}, \theta' = (\frac{1}{p_1} - \frac{1}{p_2})/(\frac{1}{q} - \frac{1}{p_2}) \) and that the operator \( \text{id} : \ell_{q_1}(2^{j\beta} \ell^{M_j}_{p_1}) \rightarrow \ell_{q_2}(\ell^{M_j}_{p_2}) \) is compact. Then

\[ d_k(\text{id} : \ell_{q_1}(2^{j\beta} \ell^{M_j}_{p_1}) \rightarrow \ell_{q_2}(\ell^{M_j}_{p_2})) \sim k^{-\beta}, \]

where

\[ \beta = \begin{cases} \frac{\delta}{q}, & \text{if } 0 < p_1 \leq p_2 \leq 2, \\ \frac{\delta}{q} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{\delta}{p_2} > \frac{1}{p_2}, \\ \frac{\delta}{q} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{\delta}{p_2} < \frac{1}{p_2} \text{ or } 2 < p_1 \leq p_2 \leq \infty \text{ and } \frac{\delta}{p_2} < \frac{1}{p_2}, \\ \frac{\delta}{q} + \frac{1}{p_1} - \frac{1}{p_2}, & \text{if } 2 < p_1 \leq p_2 \leq \infty \text{ and } \frac{\delta}{p_2} > \frac{1}{p_2} \text{ or } 0 < p_2 \leq p_1 \leq \infty. \end{cases} \]

4. The embeddings of function spaces on quasi-bounded domains

In this part, we use the estimates from the previous section to obtain the asymptotic behavior of the \( s \)-numbers of the following embeddings

\[ \tilde{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \tilde{A}^{s_2}_{p_2,q_2}(\Omega), \]

defined on the quasi-bounded domains. Theorem 3.23 in [18] guarantees that

\[ s_k(\tilde{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \tilde{A}^{s_2}_{p_2,q_2}(\Omega)) \sim s_k(\ell_{q_1}(2^{j\beta} \ell^{M_j}_{p_1}) \hookrightarrow \ell_{q_2}(\ell^{M_j}_{p_2})), \]

for any \( s \)-numbers, if \( \Omega \) is the uniformly \( E \)-porous domain described in Definition [3] If domain \( \Omega \) is quasi-bounded and the assumptions of Theorem [5] are satisfied then the embedding is compact. Moreover if

\[ 0 < \lim \inf_{j \to \infty} b_j(\Omega) 2^{-j b(\Omega)} \leq \lim \sup_{j \to \infty} b_j(\Omega) 2^{-j b(\Omega)} < \infty, \]  \hspace{1cm} (44)

then using (2) we get

\[ M_j \sim 2^{b(\Omega)}. \]
Using all the above remarks and the well known elementary embeddings

\[ B_{p,q_1}^s(\mathbb{R}^d) \hookrightarrow F_{p,q}^s(\mathbb{R}^d) \hookrightarrow B_{p,q_2}^s(\mathbb{R}^d), \quad \text{if } q_1 \leq \min(p,q), \quad q_2 \geq \max(p,q), \]

we get the following theorem.

**Theorem 22.** Let \( \Omega \) be the quasi-bounded domain satisfying \([44]\) and uniformly E-porous in \( \mathbb{R}^d \) with \( \Omega \neq \mathbb{R}^d \). Let \( 0 < p_1, p_2 \leq \infty, 0 < q_1, q_2 \leq \infty, b = b(\Omega) < \infty, \delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0, \lambda = \frac{1}{p_2} - \frac{1}{p_1}, \theta = \frac{1}{p_1} - \frac{1}{p_2}, \) \( \theta' = \frac{1}{p_1} - \frac{1}{p_2} \) and \( \frac{1}{p} = \frac{1}{\min(p_1,p_2)}. \) Then

(i)

\[ x_k(\bar{A}_{p_1,q_1}^s(\Omega)) \hookrightarrow \bar{A}_{p_2,q_2}^s(\Omega) \sim k^{-\gamma}, \]

where

\[ \gamma = \begin{cases} \frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_1, p_2 \leq 2, \\ \frac{s_1 - s_2}{b} - \frac{d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 2 \leq p_1 \leq p_2 \leq \infty \text{ or } 2 \leq p_1 < p_2 \leq \infty \text{ and } \delta > b\lambda, \\ \frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_1 \leq 2 < p_2 \leq \infty, \\ \frac{p_1}{2} \left( \frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}) \right), & \text{if } 0 < p_2 \leq 2 < p_1 \leq \infty \text{ and } \frac{\delta}{b} < \frac{1}{p_1}, \\ \frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_2 \leq 2 < p_1 \leq \infty \text{ and } \frac{\delta}{b} > \frac{1}{p_2}. \end{cases} \]

(ii) If \( 2 < p_2 < p_1 \leq \infty \) and \( \delta < b\lambda \), then

\[ c_k^{-\frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2})} \leq x_k(\bar{A}_{p_1,q_1}^s(\Omega)) \hookrightarrow \bar{A}_{p_2,q_2}^s(\Omega) \]

\[ \leq C k^{-\frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2})}. \]

(iii)

\[ a_k(\bar{A}_{p_1,q_1}^s(\Omega)) \hookrightarrow \bar{A}_{p_2,q_2}^s(\Omega) \sim k^{-\gamma}, \]

where

\[ \gamma = \begin{cases} \frac{s_1 - s_2}{b} - \frac{d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_1 \leq p_2 \leq 2 \text{ or } 2 \leq p_1 \leq p_2 \leq \infty, \\ \frac{\delta}{2} \left( \frac{s_1 - s_2}{b} - \frac{d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}) \right), & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{\delta}{b} < \frac{1}{p_2}, \\ \frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{\delta}{b} > \frac{1}{p_2}, \\ \frac{1}{2} \left( \frac{s_1 - s_2}{b} - \frac{d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}) \right), & \text{if } 0 < p_1 \leq 2 < p_2 = \infty, \\ \frac{s_1 - s_2}{b} - \frac{d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_2 \leq p_1 \leq \infty, \\ \frac{s_1 - s_2}{b} + \frac{b - d}{\bar{b}}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_1 \leq 1 < p_2 = \infty. \end{cases} \]

(iv)

\[ c_k(\bar{A}_{p_1,q_1}^s(\Omega)) \hookrightarrow \bar{A}_{p_2,q_2}^s(\Omega) \sim k^{-\gamma}, \]
and Corollary 23. Let $\Omega$ be uniformly $\alpha$-porous in $\mathbb{R}^d$ of finite Lebesgue measure and $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2})_+ > 0$. Let $\lambda$, $\theta$, $\theta'$ and $\frac{1}{p}$ denote the same as in Theorem 22. Then

(i) $x_k(A_{p_1,q_1}^{s_1} \hookrightarrow A_{p_2,q_2}^{s_2} (\Omega)) \sim k^{-\gamma}$,

where

$$
\gamma = \begin{cases} 
\frac{s_1-s_2}{b} - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if } 2 \leq p_1, p_2 \leq \infty, \\
\frac{s_1-s_2}{b} + \frac{b}{p_2}(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if } 0 < p_2 \leq p_1 \leq \infty \text{ or } 0 < p_1 < p_2 \leq 2 \text{ and } \frac{b}{p} > \frac{\theta}{p_1}, \\
\frac{s_1-s_2}{b} - \frac{d}{b}\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if } 1 < p_1 \leq p_2 \leq 2 \text{ and } \frac{b}{p} < \frac{\theta}{p_1} \text{ or } 1 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{b}{p} < \frac{\theta}{p_2}, \\
\frac{s_1-s_2}{b} + \frac{b-d}{b}\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{b}{p} > \frac{\theta}{p_1}, \\
\frac{s_1-s_2}{b} + \frac{b-d}{b}\left(\frac{1}{p_1} - \frac{1}{p_2}\right), & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{b}{p} > \frac{\theta}{p_2}.
\end{cases}
$$

The above formulas simplify if $\Omega$ is a domain of finite measure.

(ii) If $2 < p_2 < p_1 \leq \infty$ and $\delta < d\lambda$, then

$$
ck^{-\frac{s_1-s_2}{d} + \frac{1}{p_1} - \frac{1}{p_2}} \leq x_k(A_{p_1,q_1}^{s_1} \hookrightarrow A_{p_2,q_2}^{s_2} (\Omega)) \leq Ck^{-\frac{s_1-s_2}{d} + \frac{1}{p_1} - \frac{1}{p_2}}.
$$

(iii) $a_k(A_{p_1,q_1}^{s_1} \hookrightarrow A_{p_2,q_2}^{s_2} (\Omega)) \sim k^{-\gamma}$,
where
\[ \gamma = \begin{cases} \frac{s_1 - s_2}{d} - \frac{1}{p_1} + \frac{1}{p_2}, & \text{if } 0 < p_1 \leq p_2 \leq 2 \text{ or } 0 < p_2 \leq p_1 \leq 2 \\ \frac{p_2}{2} \left( \frac{2}{d} - \frac{1}{p_1} + \frac{1}{p_2} \right), & \text{if } 0 < p_1 < 2 < p_2 < \infty \text{ and } \frac{d}{p_2} < \frac{1}{p_1} \\ \frac{1}{2} - \frac{1}{p_1} + \frac{1}{p_2}, & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{d}{p_2} > \frac{1}{p_1} \\ \frac{1}{2} - \frac{1}{p_1}, & \text{if } 0 < p_2 \leq p_1 \leq \infty \\ \frac{1}{2} - \frac{1}{p_1}, & \text{if } 0 < p_2 \leq 1 < p_1 = \infty. \end{cases} \]

(iv)
\[ c_k(\tilde{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \tilde{A}^{s_2}_{p_2,q_2}(\Omega)) \sim k^{-\gamma}, \]

where
\[ \gamma = \begin{cases} \frac{s_1 - s_2}{d} - \frac{1}{p_1} + \frac{1}{p_2}, & \text{if } 2 \leq p_1 < p_2 \leq \infty, \\ \frac{p_1}{2} \left( \frac{2}{d} - \frac{1}{p_1} + \frac{1}{p_2} \right), & \text{if } 0 < p_2 \leq p_1 \leq \infty \text{ or } 0 < p_1 < p_2 \leq 2 \text{ and } \frac{d}{p_2} > \frac{d}{p_1} \\ \frac{1}{2} - \frac{1}{p_2}, & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{d}{p_2} > \frac{1}{p_1}. \end{cases} \]

(v)
\[ d_k(\tilde{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \tilde{A}^{s_2}_{p_2,q_2}(\Omega)) \sim k^{-\gamma}, \]

where
\[ \gamma = \begin{cases} \frac{s_1 - s_2}{d} - \frac{1}{p_1} + \frac{1}{p_2}, & \text{if } 0 < p_1 \leq p_2 \leq 2, \\ \frac{p_2}{2} \left( \frac{2}{d} - \frac{1}{p_1} + \frac{1}{p_2} \right), & \text{if } 0 < p_1 < 2 < p_2 \leq \infty \text{ and } \frac{d}{p_2} > \frac{d}{p_1} \text{ or } 2 < p_1 \leq p_2 \leq \infty \text{ and } \frac{d}{p_2} > \frac{d}{p_1} \\ \frac{1}{2} - \frac{1}{p_1}, & \text{if } 2 < p_1 \leq p_2 \leq \infty \text{ and } \frac{d}{p_2} > \frac{1}{p_1} \text{ or } 0 < p_2 \leq p_1 \leq 2. \end{cases} \]

Remark 2. Estimates of the approximation, the Gelfand, the Kolmogorov and the Weyl numbers given in the last corollary coincide with the previous results for these numbers in the case of bounded Lipschitz domains; cf. [11, Section 3.3.4], [3, 15, Theorem 3.5, 4.12 and 4.6]. To the best of our knowledge the estimates of the Weyl numbers were not complete to the very end; cf. A. Caetano [3, 6]. The above corollary complements the previous estimates.

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