LARGE $N$ LIMIT OF THE $O(N)$ LINEAR SIGMA MODEL IN 3D

HAO SHEN, RONGCHAN ZHU, AND XIANGCHAN ZHU

Abstract. In this paper we study the large $N$ limit of the $O(N)$-invariant linear sigma model, which is a vector-valued generalization of the $\Phi^4$ quantum field theory, on the three dimensional torus. We study the problem via its stochastic quantization, which yields a coupled system of $N$ interacting SPDEs. We prove tightness of the invariant measures in the large $N$ limit. For large enough mass or small enough coupling constant, they converge to the (massive) Gaussian free field at a rate of order $1/\sqrt{N}$ with respect to the Wasserstein distance. We also obtain tightness results for certain $O(N)$ invariant observables. These generalize some of the results in [SSZZ22] from two dimensions to three dimensions. The proof leverages the method recently developed by [GH21] and combines many new techniques such as uniform in $N$ estimates on perturbative objects as well as the solutions.

Contents

1. Introduction 1
2. Stochastic terms and decomposition of the equation 6
3. Uniform in $N$ estimates 18
4. Convergence of measures and tightness of observables 32
Appendix A. Notations and Besov spaces 38
Appendix B. Extra estimates 42
Appendix C. Notation index 44
References 45

1. Introduction

In this paper, we continue the study initiated in [SSZZ22] on the application of singular SPDE methods to large $N$ problems in quantum field theory (QFT). Large $N$ problems in QFT generally refer to the study of asymptotic behaviors of QFT models as the dimensionality of the target space where the quantum fields take values tends to infinity. Physicists’ study of large $N$ problems in QFT originates from the seminal work by Wilson [Wil73] and Gross–Neveu [GN74], and soon flourished following the work by t’Hooft [t’H74] who applied it to gauge theories, which influenced many aspects of probability. We refer to [SSZZ22, Section 1.1] for a more thorough exposition of the background and motivation for large $N$ methods in QFT. A prototype model, which we consider in this paper, is the $O(N)$-invariant linear sigma model given by the (formal) measure

$$d\nu^N(\Phi) \overset{\text{def}}{=} \frac{1}{C_N} \exp \left( - \int_{\mathbb{T}^d} \sum_{j=1}^{N} |\nabla \Phi_j|^2 + m \sum_{j=1}^{N} \Phi_j^2 + \frac{\lambda}{2N} \left( \sum_{j=1}^{N} \Phi_j^2 \right)^2 dx \right) D\Phi$$  (1.1)

over $\mathbb{R}^N$ valued fields $\Phi = (\Phi_1, \Phi_2, ..., \Phi_N)$, where $\mathbb{T}^d$ is the $d$-dimensional torus and $C_N$ is a normalization constant (partition function). This is an $N$-component generalization of the $\Phi^4_d$ model, which
is symmetric under the action by the orthogonal group $O(N)$ on the field $\Phi$. In this paper we focus on $d = 3$.

Our main tool to investigate the large $N$ behavior of the above model (1.1) is the system of SPDEs arising as its stochastic quantization:

$$\mathcal{L}_i \Phi_i = -\frac{\lambda}{N} \sum_{j=1}^{N} \Phi_j^2 \Phi_i + \xi_i,$$

where $\mathcal{L}_i = \partial_i - \Delta + m$ with $m \geq 0$, and $i \in \{1, \ldots, N\}$. The collection $(\xi_i)_{i=1}^{N}$ consists of $N$ independent space-time white noises on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The utilization of the dynamic (1.2) to probe large $N$ properties of the QFT model (1.1) lead to connections with dynamical mean field theory, see [SSZZ22, Section 1.2] for more discussions on this connection. We also note that for finite $N$, the measure $\nu_N$ formally given by (1.1) can be constructed rigorously as the unique invariant measure of SPDEs (1.2) – see Lemma 4.2, and we henceforth denote by $\nu_N$ this well-defined measure.

In the 2D paper [SSZZ22], since the SPDE is less singular, a series of results was obtained. It is proved that the non-stationary dynamics converge to a mean field dynamic $\mathcal{L}_i \Psi_i = -\lambda \mathbb{E}[\Psi_i^2] \Psi_i + \xi_i$ (which needs to be interpreted via a suitable renormalization) as $N \to \infty$. Also, for large enough mass, the invariant measures converge to the (massive) Gaussian free field which is the unique invariant measure of the mean-field dynamic. Moreover, this paper obtained tightness in suitable Besov spaces of some $O(N)$ invariant observables as $N \to \infty$, and also proved exact correlation formulae for these observables.

In $d = 3$, the SPDE becomes much more singular. For $N = 1$, namely the dynamical $\Phi^3$ model, the construction of local solutions was achieved by [Hai14] using the theory of regularity structures and then [CC18] using paracontrolled distributions developed in [GIP15]; and global solutions were studied in [MW17a, GH19, AK17, MW20, GH21]. One should be able to extend these constructions and estimates to the vector-valued case for a finite and fixed $N > 1$.

The goal of this article is to study the asymptotic behavior as $N \to \infty$ of the invariant measures (1.1) as well as $O(N)$-invariant observables, in $d = 3$. Given the more singular nature in $d = 3$, it is not the purpose of this article to extend all the results in the 2D paper [SSZZ22] to 3D. Instead we only focus on large $N$ limit of the measure $\nu_N$ in (1.1) and tightness of certain observables in $d = 3$. Our proof is self-contained based on the techniques from mean field theory and recent progress of singular SPDEs, and does not rely on methods in constructive field theory such as cluster expansion and correlation inequalities. Our approach may bring a new perspective to study the large $N$ limit problem.

1.1. Main results. Our first result shows that assuming $m$ to be sufficiently large or $\lambda$ to be sufficiently small, the invariant measures converge to the (massive) Gaussian free field. Note that since the interaction term in the measure (1.1) has a sum besides the factor $1/N$, it is far from being obvious that the large $N$ limit of the measures is a Gaussian free field. This was only heuristically predicted by physicists (e.g. [Wil73]) at the level of perturbation theory: namely, after suitable renormalization, all the Feynman diagrams are of order $N^\alpha$ for some $\alpha < 0$ except for the diagrams corresponding to the Gaussian free field (see Section 1.3 below for more discussion).

To state the result, let $\nu \overset{\text{def}}{=} \mathcal{N}(0, \frac{1}{N}(m - \Delta)^{-1})$ be the (massive) Gaussian free field. Consider the projection onto the $i^{th}$ component of the field $\Phi$

$$\Pi_i : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d), \quad \Pi_i(\Phi) \overset{\text{def}}{=} \Phi_i.$$

Noting that $\nu_N$ is a measure on $\mathcal{S}'(\mathbb{T}^d)^N$, we define the marginal law $\nu_N^{i} \overset{\text{def}}{=} \nu_N \circ \Pi_i^{-1}$. Furthermore, consider

$$\Pi^{(k)} : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d)^k, \quad \Pi^{(k)}(\Phi) = (\Phi_i)_{1 \leq i \leq k}$$

(1.4)
and define the marginal law of the first $k$ components by $\nu^N_{i\leq k} \equiv \nu^N \circ (\Pi^{(k)})^{-1}$. We denote by $\mathbb{W}_{2,k}$ the Wasserstein distance, with the precise definition given in (4.5). We recall the definitions of Besov spaces $B^p_{q,\alpha}$, $C^\alpha$ and $H^\alpha$ in Section 1.6 and Appendix A.

**Theorem 1.1.** For any $(m, \lambda) \in (0, \infty) \times [0, \infty)$ and every $i \geq 1$, the sequence of probability measures $(\nu^{N,i})_{N \geq 1}$ is tight on $H^{-\frac{1}{2}-\kappa}$ for any $\kappa > 0$. Moreover, there exists a constant $c_0 > 0$ such that for all $(m, \lambda) \in [1, \infty) \times [0, \infty)$ satisfying $m \geq 1 + c_0 \lambda (1 + \lambda^{3/2})$ and every $k \geq 1$, there exists a constant $C_k > 0$ such that $\mathbb{W}_{2,k}(\nu^N, \nu^{N,k}) \leq C_k N^{-\frac{k}{2}}$.

Our next result is concerned with $O(N)$-invariant observables for the invariant measure. We refer to [SSZZ22, Section 1] for more discussion on the motivation of studying observables for QFT models with continuous symmetries. Here a natural quantity that is invariant under $O(N)$-rotations is the squared “length” of $\Phi$, that is suitably renormalized and scaled with respect to $N$:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N \Phi_i^2, \quad \Phi = (\Phi_i)_{1 \leq i \leq N} \sim \nu^N. \quad (1.5)$$

The precise definition is given by (4.10) in Section 4. We establish the large $N$ tightness of these observables as random fields in suitable Besov spaces.

**Theorem 1.2.** For $(m, \lambda) \in [1, \infty) \times [0, \infty)$ satisfying $m \geq 1 + c_0 \lambda (1 + \lambda^{3/2})$ as in Theorem 1.1, the sequence of random variables $\left(\frac{1}{\sqrt{N}} \sum_{i=1}^N \Phi_i^2, \right)_{N \geq 1}$ is tight on the Besov space $B_{1,1}^{-1-\kappa}$ for any $\kappa > 0$.

The proofs of Theorem 1.1 (split into Theorems 4.5 and 4.6) and Theorem 1.2 will be given in Section 4.

Let us briefly mention some of the earlier related results on large $N$ problems in QFT, and we refer to [SSZZ22, Sec. 1.1 and 1.2] for a more complete historical remarks and literature on the physical and mathematical results.

On a fixed lattice, Kupiainen [Kup80c] studied the $1/N$ expansion for the nonlinear sigma (classical Heisenberg) model, see also [FMR82], and more recently, Chatterjee and Jafarov [Cha19, CJ16] obtained the $1/N$ expansion for lattice gauge theories. The lattice cutoff allows one to avoid the ultraviolet or singularity problem, which is convenient especially when the continuum limit of the model is not constructed or may not exist. On the other hand, Lévy [L17] and Anshelevich–Sengupta [AS12] constructed the so called master field for the two-dimensional Yang–Mills model in continuum which is the large $N$ limit of Wilson loop observables. While the two-dimensional Yang–Mills field in continuum is very singular (and the random connection field was only constructed recently by Cheryev [Che19]), the model in $d = 2$ has a certain solvability property, so these large $N$ results are very difficult to be generalized to 3D continuum space, if possible, since the solvability would be lost. Back to the vector $\Phi^4$ or $O(N)$ linear sigma model in continuum, the first large $N$ result was achieved by Kupiainen [Kup80b] (see also the review [Kup80a]) in which $1/N$ expansion for the pressure (vacuum energy) of the model (1.1) in $d = 2$ was obtained using constructive field theory methods. See also [BR82]. Generalizing the results in [Kup80b] to 3D using constructive field theory methods would be very interesting but, as far as we understand, might be difficult.

To our best knowledge nothing was known in a very singular setting such as the model being considered here in $d = 3$, which is now analyzable thanks to the new SPDE methods developed more recently. See Section 1.4 for more discussion.

### 1.2. Methodology and difficulties.

To obtain our main results, we outline the main steps of our strategy. To prove tightness of $\nu^{N,i}$ and the observables we turn to uniform in $N$ moment estimates for the stationary solutions $\Phi$ to the stochastic quantization (1.2). Here $\Phi$ is only a distribution and (1.2) requires renormalization. Our general strategy is to apply SPDE techniques, especially energy method to derive uniform in $N$ estimates of $\Phi$. Due to the singularity of the noise $\xi$ and also $\Phi$, we
first decompose
\[ \Phi_i = Z_i + X_i + Y_i, \]  
(1.6)
to isolate parts of different regularities. Here \( Z_i \) is the stationary solution to the linear equation \( \mathcal{L} Z_i = \xi_i \) and is the most irregular part in \( \Phi_i \), and \( X_i \) involves the second irregular part \( \mathcal{Z}^Y \) (see Section 2.1 for the definition of \( \mathcal{Z}^Y \) and (2.6) for the definition of \( X_i \)).

With (1.6) at hand, \( Y_i \) becomes more regular and we prove uniform in \( N \) energy estimate for \( Y_i \), which solves an equation driven by \( X \) and renormalized powers of \( Z \) (see (2.25) below). As in the 2D case, the extra factor \( 1/N \) before the nonlinear terms makes the damping effect from \( Y_j^2Y_i \) weaker as \( N \) becomes large, so we cannot exploit the strong damping effect at the level of a fixed component of \( Y \). Instead we consider aggregate quantities, and ultimately we focus on the empirical average of the \( L^2 \)-norms. The advantage is that the dissipation term \( \frac{1}{N} \sum_{i=1}^{N} Y_i^2 \) can be used to control the nonlinear terms. We then show that the following averaged quantities
\[ \frac{1}{N} \sum_{j=1}^{N} \| Y_j(t) \|^2_{L^2} + \frac{1}{N} \sum_{j=1}^{N} \| Y_j \|_{L^2_{\nu}H^{1-2\kappa}} + \frac{\lambda}{N^2} \sum_{i=1}^{N} Y_i^2 \|_{L^2_{\nu}L^2} \]  
(1.7)
are controlled pathwise in terms of the averages of the renormalized powers of \( Z \) (see Theorem 3.4), which is enough to derive tightness of \( \nu^N,i \) in Theorem 1.1. Furthermore, we derive uniform in \( N \) bounds for the quantities in (1.7) multiplied by \( N \), i.e. the sum of \( L^2 \)-norm of \( Y_i \) (see Theorem 3.3).

We also remark that in order to establish uniform in \( N \) bounds, it would be a natural attempt to exploit the method in [MW17a] or [GH19] (which obtained global a priori estimates for dynamical \( \Phi^N,i \)). However, it is not clear how to exploit the aforementioned dissipation effect to deduce uniform in \( N \) bounds with \( L^p \) \( (p > 2) \) estimates developed in [MW17a]; also it seems difficult to apply the maximum principle used in [GH19, MW20] to the vector valued case. In this paper, we exploit the approach developed in [GH21] (which relies on interesting cancellations) to establish \( L^2 \) uniform bounds; due to the cancellation certain higher regularity estimate which would be rather technical is not required anymore for \( L^2 \) uniform estimates (see Section 2.4 for more details). On the other hand, compared to the dynamical \( \Phi^N,i \) model, the dissipation effect from the term \( \| \frac{1}{N} \sum_{i=1}^{N} Y_i^2 \|_{L^2} \) is weaker than the \( L^4 \) norm. We will perform further decomposition and choose suitable parameter to balance the competing contributions for the estimate of the cubic term (see Lemma 3.8 and discussion before it for more details.)

An important ingredient in our proof is a cancelation mechanism which emerges from Section 2.3 and arises from the following reasons. In our decomposition of the solution to (1.2), the leading order terms consist of \( (\text{polynomials of}) \) \( N \) independent Gaussian processes; so when calculating moments of certain sums of these leading order terms in a suitably chosen Hilbert space, many terms do not contribute, which allows us to gain “factors of \( 1/N \)”. See Lemma 2.5 (and also Lemmas 2.6-2.7) for these effects.

As a final step, as in the 2D case, we follow the idea in [GH21] and construct a jointly stationary process \( (\Phi, Z) \) whose components satisfy (1.2) and the linear equation (2.2) respectively. The law of \( Z \) is the Gaussian free field \( \nu \) and the law of \( (\Phi, Z) \) gives a coupling between the measures \( \nu^N,i \) and \( \nu \). We use this coupling to prove convergence of \( \nu^N,i \) to \( \nu \) by invoking the uniform in \( N \) estimates on the stationary processes. More precisely, we further use mean field limit techniques to center the stochastic objects \( (\text{i.e.} \ R^1_N, R^2_N \text{ and } Q^3_N \text{ in Theorem 3.3}) \) in our uniform estimates; their means will contribute to a mass term, which ultimately requires the assumptions on \( m, \lambda \). The part involving the centered stochastic objects could be absorbed by the dissipation terms from nonlinearity; and we will gain a factor of \( 1/N \) when computing variance of the centered stochastic objects (essentially a generalization of the elementary fact that the variance of the average of \( N \) mean-zero i.i.d. random variables is \( O(1/N) \)). Finally tightness of observables almost immediately follows from the uniform in \( N \) estimates in Theorem 4.6.
1.3. Background and heuristics. We review here how physicists (e.g. Wilson [Wil73]) predicted convergence of the model (1.1) (after inserting suitable renormalization constants in a way that corresponds to (2.1)) to GFF as $N \to \infty$, and why their heuristic argument was far from being rigorous. Since this is completely formal, we simply drop the renormalization and pretend that our fields can be evaluated at spatial points.

The prediction was based on viewing (1.1) as a perturbation of GFF, namely, to formally Taylor expand (1.1) in $\lambda$. As an example, one can calculate the two-point correlation $\mathbb{E}[\Phi_i(y_1)\Phi_j(y_2)]$ (for a fixed $i$) in this way: the zeroth order term in this Taylor expansion is obtained by simply taking $\lambda = 0$, which gives $\mathbb{E}[Z_i(y_1)Z_i(y_2)]$, where $Z_i \sim N(0, \frac{1}{2}(m - \Delta)^{-1})$. Each of the higher order terms in this Taylor expansion can be calculated as an expectation of a product of Gaussians using Wick theorem: for instance at order $\lambda^2$ one of the terms has the form

$$\frac{\lambda^2}{N^2} \int \mathbb{E}\left[\prod_{i_1,j_1=1}^{N} Z_{i_1}(x_1)^2 Z_{j_1}(x_1)^2 \prod_{i_2,j_2=1}^{N} Z_{i_2}(x_2)^2 Z_{j_2}(x_2)^2 \right] \, dx_1 \, dx_2.$$

One of the terms obtained from applying Wick theorem to the above expectation has the form

$$\frac{\lambda^2}{N^2} \sum_{i_1,j_1,i_2,j_2=1}^{N} \delta_{i_1,i_2} \delta_{j_1,j_2} \int \mathbb{E}[Z_{i_1}(y_1)Z_{i_1}(x_1)] \mathbb{E}[Z_{i_2}(y_2)Z_{i_2}(x_2)]$$

$$\times \mathbb{E}[Z_{i_2}(x_1)Z_{i_2}(x_2)] \mathbb{E}[Z_{j_2}(x_1)Z_{j_2}(x_2)] \, dx_1 \, dx_2,$$

where the Kronecker $\delta$’s come from the independence of $(Z_j)_{j=1}^N$. (This integral is convergent after introducing the renormalization constant $\delta_x$ as in (2.1).) Thanks to these Kronecker $\delta$’s, the summation has $N$ terms (rather than $N^4$ terms), so the above expression is of order $1/N$ and thus converge to 0 as $N \to \infty$. This argument is far from being rigorous because even if it could be shown that every order of the Taylor expansion (except for the zeroth order) converges to 0 as $N \to \infty$, and even if this could be done for higher order correlations, the above Taylor expansion in $\lambda$ does not converge.

This type of predictions can be rigorously proved now thanks to the recent development of singular SPDE techniques and the methods developed in this paper. On the other hand, as we will see, some of our calculations such as Lemma 2.5 have a similar flavor with the formal perturbative arguments discussed above.

1.4. Future directions. We mention a few possible future directions. The proof of tightness in Theorem 1.2 requires establishing uniform bounds on moment of the observables. It will be more interesting to prove exact formulae of correlations of the observables (as done in [SSZZZ22] in $d=2$) in the $N \to \infty$ limit, by first using integration by parts (i.e. Dyson–Schwinger equations) to find the leading contribution to the formula and then establishing uniform bounds which show that the remainder terms vanish. In 3D it would require more effort to interpret the Dyson–Schwinger equations and prove the bounds for the remainder, so we leave it to future work. We hope that the SPDE approach and uniform estimates derived in Section 3 can be used to characterize the tightness limit of $O(N)$ invariant observables, prove exact correlation formulas in the large $N$ limit or systematic $1/N$ expansions, for which we may need to combine with the methods from QFT, for instance, the dual field representation rigorously studied in [Kup80b].

Another interesting question is whether our convergence results hold for a larger range of $(m, \lambda)$, or even over the entire $(m, \lambda) \in \mathbb{R}_+^2$, at least on the torus. It would be interesting to generalize some of our results to infinite volume setting by introducing suitable weights. Let us also mention that proving or disproving phase transitions for $N$ large is an important question, see for instance [DCPSS17] for a different but related model and references therein, and we mention [CGW20] on phase transition of $\Phi^4_3$ using stochastic analysis methods.

1For $\Phi^4_2$, the expansion in $\lambda$ was proved to be divergent in [Jaf65], but proved to be asymptotic in [Dim74, SZZ21].
Finally, as discussed in Section 1.1, it would certainly be interesting to investigate large $N$ problems for more challenging models without lattice cutoff, such as Lie algebra valued [CCHS20, CCHS22] or manifold valued [BGHZ21, Hai16, RWZZ20, CWZZ21] models with large dimension of target space via stochastic quantization.

1.5. Structure of the paper. This paper is organized as follows. In Section 2, we set up the decomposition of our SPDE system, discuss renormalization, derive the energy balance identity, and prove uniform in $N$ bounds for stochastic terms. Section 3 is devoted to the proof of uniform in $N$ energy estimates for the solution.

Section 4 is concerned with the proof of Theorem 1.1 and Theorem 1.2. The convergence of invariant measures from $
u^{N,i}$ to the Gaussian free field $\nu$ is shown in Section 4.2. Section 4.3 is devoted to the study of the observables and the proof of Theorem 1.2. In Appendix A we introduce the basic notations and recall useful tools such as paraproducts and commutator estimates used throughout the paper. In Appendix B we give an extra estimate and finally we collect some notation in Appendix C.

1.6. Some notations. Throughout the paper, we use the notation $a \lesssim b$ if there exists a proportional constant $c > 0$ such that $a \leq cb$, and we write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. We say that $a \lesssim b$ uniformly in a parameter if the proportional constant $c$ does not depend on this parameter.

Given a Banach space $E$ with a norm $\| \cdot \|_E$ and fixing $T > 0$, we write $C_T E = C([0,T];E)$ for the space of continuous functions from $[0,T]$ to $E$, equipped with the supremum norm $\| f \|_{C_T E} = \sup_{t \in [0,T]} \| f(t) \|_E$. For $p \in [1,\infty]$ we write $L^p_T E = L^p([0,T];E)$ for the space of $L^p$-integrable functions from $[0,T]$ to $E$, equipped with the usual $L^p$-norm. Let $H$ be a separable Hilbert space with norm $\| \cdot \|_H$ and inner product $\langle \cdot, \cdot \rangle_H$. Given $p > 1$, $\alpha \in (0,1)$, let $W^\alpha_{T,p} H$ be the Sobolev space of all $f \in L^p_T H$ such that the following norm is finite

$$\| f \|^p_{W^\alpha_{T,p} H} \overset{\text{def}}{=} \int_0^T \| f(t) \|^p_H \mathrm{d} t + \int_0^T \int_0^T \frac{\| f(t) - f(s) \|^p_H}{|t-s|^{1+\alpha p}} \mathrm{d} t \mathrm{d} s.$$ 

Let $S'$ be the space of distributions on $\mathbb{T}^d$. We write $B^p_{\alpha,q}$ for Besov spaces on the torus with general indices $\alpha \in \mathbb{R}$, $p,q \in [1,\infty]$ and the Hölder–Besov space $C^{\alpha}$ is given by $C^{\alpha} = B^\alpha_{\infty,\infty}$. We will often write $\| \cdot \|_{C^\alpha}$ instead of $\| \cdot \|_{B^\alpha_{\infty,\infty}}$. For $\alpha \in \mathbb{R}$, set $H^{\alpha} = B^\alpha_{2,2}$. Set $\Lambda = (1-\Delta)^{\frac{1}{2}}$. We put the definition of Besov spaces and useful lemmas in Appendix A.

Acknowledgments. We would like to thank Scott Smith for the numerous and very helpful discussions on mean field limits and large $N$ problems of singular SPDEs. H.S. gratefully acknowledges financial support from NSF grants DMS-1712684 / 1909525, DMS-1954091 and CAREER DMS-2044415. R.Z. is grateful to the nancial supports of the NSFC (No. 11922103), X.Z. is grateful to the financial supports in part by National Key R&D Program of China (No. 2020YFA0712700) and the NSFC (No. 12288201, 12090014) and the support by key Lab of Random Complex Structures and Data Science, Youth Innovation Promotion Association (2020003), Chinese Academy of Science.

2. Stochastic terms and decomposition of the equation

Due to the singular noise, the solutions to (1.2) only belong to the negative order Besov-Hölder space and we need renormalization and decomposition to understand the nonlinearity. In this section, we first introduce the renormalizations and the relevant stochastic terms for the decomposition of equation (1.2). We then derive uniform in $N$ estimates for the stochastic terms in Section 2.2 and Section 2.3. In Section 2.4 we give the decomposition of equation (1.2), which gives the important cancellations for the nonlinearity.

We start with the renormalized version of equation (1.2). Let $\xi_{i,\varepsilon}$ be a space-time mollification of $\xi_i$ defined on $\mathbb{R} \times \mathbb{T}^3$. The formal equation (1.2) is then interpreted as the limit of the following

\[\int_0^T \mathbb{E} \left[ \int_{\mathbb{T}^3} \xi_{i,\varepsilon} \left( \langle \Delta \phi_{\varepsilon} + \xi_{i,\varepsilon} \rangle \right) \mathrm{d} x \right] \mathrm{d} t = \int_0^T \mathbb{E} \left[ \int_{\mathbb{T}^3} \phi_{\varepsilon} \left( \Phi_{\varepsilon} \right) \xi_{i,\varepsilon} \mathrm{d} x \right] \mathrm{d} t.\]
approximate equation

\[ \mathcal{L}\Phi_{t,\varepsilon} + \frac{\lambda}{N} \sum_{j=1}^{\lambda} \Phi_{j,\varepsilon}\Phi_{t,\varepsilon} + \left( - \frac{N + 2}{N} \lambda a_{\varepsilon} + \frac{3(N + 2)}{N^2} \lambda^2 \tilde{b}_{\varepsilon} \right) \Phi_{t,\varepsilon} = \xi_{t,\varepsilon}, \]

(2.1)

where \( a_{\varepsilon} \) and \( \tilde{b}_{\varepsilon} \) are renormalization constants given below. Let \( Z_t \) be a stationary solution to

\[ \mathcal{L} Z_t = \xi_t. \]

(2.2)

**Remark 2.1.** To briefly motivate the \( N \) dependent coefficients in the renormalization constants in (2.1), note that we can “recombine” the terms as

\[ \frac{\lambda}{N} \left( \Phi_{t,\varepsilon}^3 - 3a_{\varepsilon} \Phi_{t,\varepsilon} + \sum_{j \neq 1} \left( \Phi_{j,\varepsilon}^2 - a_{\varepsilon} \Phi_{j,\varepsilon} - a_{\varepsilon}\Phi_{t,\varepsilon} \right) \right) \]

and this is then consistent with our renormalization (2.3) below. The \( N \) dependence of the coefficient in front of \( \tilde{b}_{\varepsilon} \) will also be clear by similar consideration based on renormalization computation in paracONTrolled calculus in Section 2.1 below.

### 2.1. Renormalization

We now introduce the renormalized terms. Let \( Z_{t,\varepsilon} \) be the stationary solution to \( \mathcal{L} Z_{t,\varepsilon} = \xi_{t,\varepsilon} \). For convenience, we assume that all the noises are mollified with a common bump function. In particular, \( Z_{t,\varepsilon} \) are i.i.d. mean zero Gaussian. The Wick products are defined as

\[ Z_{ij} = \begin{cases} \lim_{z \to 0} (Z_{i,z} - a_{\varepsilon}) & (i = j) \\ \lim_{z \to 0} Z_{i,z} Z_{j,z} & (i \neq j) \end{cases} \]

and this is then consistent with our renormalization (2.3) below. The \( N \) dependence of the coefficient in front of \( \tilde{b}_{\varepsilon} \) will also be clear by similar consideration based on renormalization computation in paracontrolled calculus in Section 2.1 below.

\[ \mathcal{L} \tilde{Z}_{ij,j,\varepsilon} = \tilde{Z}_{ij,j,\varepsilon}, \] i.e. 2

\[ \tilde{Z}_{ij,j,\varepsilon} = \int_{-\infty}^{t} P_{1-\varepsilon} Z_{ij,j,\varepsilon}(s) ds := \tilde{I} Z_{ij,j,\varepsilon}. \]

Set \( \tilde{Z}_{ij} = \lim_{\varepsilon \to 0} \tilde{Z}_{ij,j,\varepsilon} \), where the limit is in \( C_T C^{1-\kappa}_{\varepsilon} \) for \( \kappa > 0 \). Here in the “tree” type superscripts, a dot denotes a noise, a line denotes a heat kernel, and the subscripts specify the component indices of the noises showing in the trees. In particular we can write \( \tilde{Z}_{ij,j} \)

To introduce further stochastic objects, let \( G = m - \Delta \) and define

\[ \tilde{Z}_{ij,j,k,\varepsilon}^{\Psi} \triangleq G^{-1}(Z_{ij,j,\varepsilon}) \circ Z_{k,\varepsilon} - c_1 \hat{b}_{\varepsilon}, \quad \tilde{Z}_{ij,j,k,\varepsilon}^{\Psi} \triangleq \mathcal{I}(Z_{ij,j,\varepsilon}) \circ Z_{k,\varepsilon} - c_1 \hat{b}_{\varepsilon}(t), \]

\[ \tilde{Z}_{ij,j,k,\varepsilon}^{\Psi} \triangleq \mathcal{I}(Z_{ij,j,\varepsilon}) \circ Z_{k,\varepsilon} - c_2 \hat{b}_{\varepsilon}(t). \]

Here \( c_1 \) equals \( \frac{1}{2} \) if \( i = k \neq j = \ell \) or \( i = \ell \neq j = k \), equals 1 if \( i = k = j = \ell \), and is 0 otherwise; and \( c_2 \) equals 1 if \( j = k \neq i \), equals 3 if \( j = k = i \), and is 0 otherwise. Also, \( \hat{b}_{\varepsilon}(t) = \mathbf{E}[\mathcal{I}(Z_{ij,j,\varepsilon}) \circ Z_{k,\varepsilon}] \) and \( \hat{b}_{\varepsilon} = \mathbf{E}[\mathcal{I}(Z_{ij,j,\varepsilon}) \circ Z_{k,\varepsilon}] \) are renormalization constants, and one has \( |b_{\varepsilon} - \hat{b}_{\varepsilon}| \lesssim t^{-\gamma} \) for any \( \gamma > 0 \) uniformly in \( \varepsilon \). We denote collectively

\[ Z_{\varepsilon} = (Z_{i,\varepsilon}, Z_{ij,\varepsilon}, Z_{ij,j,\varepsilon}^{\Psi}, Z_{ij,k,\varepsilon}, Z_{ij,k,\varepsilon}^{\Psi}, Z_{ij,k,\varepsilon}^{\Psi}, Z_{ij,j,\varepsilon}^{\Psi}, Z_{ij,k,\varepsilon}^{\Psi}). \]

\[ ^{2}\text{Recall that } \mathcal{I} \text{ is defined similarly but with } f_{t,\varepsilon} \text{ replaced by } f_{0,\varepsilon}. \]
To state the next lemma, for $\kappa > 0$ we define the “homogeneities” $\alpha_\tau \in \mathbb{R}$ as in the following table

| $\tau$ | $Z_{i,\varepsilon}$ | $\mathcal{Z}_{ij,\varepsilon}^{\mathcal{V}}$ | $\mathcal{Z}_{ij,k,\varepsilon}^{\mathcal{V}}$ | $\mathcal{Z}_{ij,k,l,\varepsilon}^{\mathcal{V}}$ | $\mathcal{Z}_{ij,k,l,m,\varepsilon}^{\mathcal{V}}$ | $\mathcal{Z}_{ij,k,l,m,n,\varepsilon}^{\mathcal{V}}$ |
|--------|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\alpha_\tau$ | $-\frac{1}{2} - \kappa$ | $-1 - \kappa$ | $\frac{1}{2} - \kappa$ | $-\kappa$ | $-\kappa$ | $-\kappa$ |

**Lemma 2.2.** For every $\kappa, \sigma > 0$ and some $0 < \delta < 1/2$, there exist random distributions

$Z \overset{\text{def}}{=} (Z_i, Z_{ij}^{\mathcal{V}}, Z_{ij,j,k}^{\mathcal{V}}, Z_{ij,k,l}^{\mathcal{V}}, Z_{ij,k,l,m}^{\mathcal{V}}, Z_{ij,k,l,m,n}^{\mathcal{V}})$

(2.4)

such that if $\tau_\varepsilon$ is a component in $Z_{\varepsilon}$ and $\tau$ is the corresponding component in $Z$ then $\tau_\varepsilon \to \tau$ in $C_T C^{\alpha_\tau} \cap C_T^{3/2} C^{\alpha_\tau - \delta}$ a.s. as $\varepsilon \to 0$. Furthermore, for every $p > 1$

$$
\sup_{m \geq 1} \mathbb{E} \|\tau_\varepsilon\|^p_{C_T C^{\alpha_\tau}} + \sup_{m \geq 1} \mathbb{E} \|\tau_\varepsilon\|^p_{C_T^{3/2} C^{\alpha_\tau - \delta}} \lesssim 1, \\
\sup_{m \geq 1} \mathbb{E} \|\tau\|^p_{C_T C^{\alpha_\tau}} + \sup_{m \geq 1} \mathbb{E} \|\tau\|^p_{C_T^{3/2} C^{\alpha_\tau - \delta}} \lesssim 1,
$$

where the proportional constants in the inequalities are independent of $\varepsilon, i, j, N$.

We emphasize that Wick renormalization is used for the terms in (2.3) as in the 2D setting. But for the convergence of the higher order stochastic objects, we need to use the renormalization procedure in [Hai14, Section 10] or [CC18, MWX17]. We also refer the proof of Lemma 2.2 to the above references.

**Convention.** In the following if we do not need the precise powers of the renormalized terms in $Z$, we denote by $Q(Z)$ a generic polynomial in terms of the above norms of $\tau$ with $\mathbb{E} Q(Z)^q \lesssim 1$ for any $q \geq 1$. We note that $Q(Z)$ may depend on $N$ and summations from 1 to $N$, for instance, it could take the form $\sum_{i=1}^N \pi_i$, but the expectation of $Q(Z)$ is uniformly bounded and independent of $N$.

By Lemma 2.2 there exists a measurable $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for $\omega \in \Omega_0$

$$
\|\tau\|_{C_T C^{\alpha_\tau}} + \|\tau\|_{C_T^{3/2} C^{\alpha_\tau - \delta}} < \infty,
$$

and $\tau_\varepsilon \to \tau$ in $C_T C^{\alpha_\tau} \cap C_T^{3/2} C^{\alpha_\tau - \delta}$ as $\varepsilon \to 0$ for every $\tau \in Z$. We fix such $\omega \in \Omega_0$ in the following. Now we fix $\kappa$ small enough and $\delta = 1/2 - 2\kappa$ in the above estimate.

### 2.2. Additional stochastic terms.

Recall the Littlewood–Paley blocks $\Delta_j$ in Section A. Let

$$
\mathcal{U}_< \overset{\text{def}}{=} \sum_{j > L} \Delta_j, \quad \mathcal{U}_\leq \overset{\text{def}}{=} \sum_{j \leq L},
$$

(2.5)

for some constant $L > 0$ to be chosen below. Note that the limit $\Phi_i$ of $\Phi_{i,\varepsilon}$ has the same regularity as $Z_i$ which is $C_T C^{-\frac{1}{2} - \kappa}$, and it would be natural to decompose

$$
\Phi_i = Z_i - \frac{\lambda}{N} \sum_{j=1}^N \mathcal{Z}_{ijj}^{\mathcal{V}} + \zeta_i
$$

with $\zeta_i$ being a function of better regularity. Writing down the dynamics for $\zeta_i$, the most irregular terms would be the paraproduct $\mathcal{Z}_{ijj}^{\mathcal{V}} \sim (\Phi_k - Z_k)$, which indicates that $\zeta_i \in C^{1-\kappa}$ since $\mathcal{Z}_{ijj}^{\mathcal{V}} \in C^{-1-\kappa}$ as in Lemma 2.2, but then in an energy estimate for $\zeta_i$, the term $\langle \zeta_i, \mathcal{Z}_{ijj}^{\mathcal{V}} \rangle_\varepsilon (\Phi_k - Z_k)$ cannot be controlled.

To overcome this difficulty, as in [GH21] (for $N = 1$ case) we introduce one more stochastic object:

$$
X_i = -\frac{\lambda}{N} \sum_{j=1}^N \left( 2\mathcal{I}(X_j \sim \mathcal{U}_< \mathcal{Z}_{ijj}^{\mathcal{V}}) + \mathcal{I}(X_i \sim \mathcal{U}_< \mathcal{Z}_{ijj}^{\mathcal{V}}) + \mathcal{Z}_{ijj}^{\mathcal{V}} \right).
$$

(2.6)

In the Section 2.4 we will see that after a further decomposition, $Y_i = \Phi_i - Z_i - X_i$ satisfies a suitable energy inequality. For fixed $N$ and $L > 0$, fixed point argument and Lemma A.6 easily imply local well-posedness of (2.6) in $C_T C^{\frac{1}{2} - 2\kappa}$. Furthermore, by a suitable choice of $L$, we have the following
uniform in $N$ estimates, which imply global well-posedness. Note that the right-hand sides of the following bounds only involve stochastic terms appearing in Lemma 2.2.

**Lemma 2.3.** There exists $L = L(\lambda, N) > 0$ such that

\[
\frac{1}{N} \sum_{i=1}^{N} \| X_i \|^2_{C^\frac{1}{2} c^{-\infty}} \lesssim \frac{\lambda^2}{N^2} \sum_{i,j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}},
\]

and

\[
\| X_i \|^2_{C^\frac{1}{2} c^{-\infty}} \lesssim R^0_N + \frac{\lambda}{N} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}},
\]

\[
\| X_i \|^2_{C^\frac{1}{2} c^{-\infty}} \lesssim R^0_N + \frac{\lambda}{N} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} L_N + \frac{\lambda}{N} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}},
\]

uniformly in $N$, $\lambda$ and $m \geq 1$, where

\[
R^0_N \equiv \frac{\lambda^2}{N^2} \sum_{i,j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} + \frac{\lambda^2}{N} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}}.
\]

**Convention.** Below we will often have expressions of the form $2 A_i B_{ikl} + A_i B_{ikl}$. To streamline our notation, we will write this as $A_{ijkl} B_{ijkl}$, which stands for the same expression without the brackets multiplied by 2, plus the expression with the two indices in brackets swapped. For instance, in (2.6), we can write

\[
\mathcal{I}(X_{ij} < \mathcal{U}) \tilde{Z}_{ij} \| C_{T} c^{-3/2 - \infty} \lesssim 2^{-L/2} \| \tilde{Z}_{ij} \| C_{T} c^{-1 - \infty}.
\]

Note that when we prove inequalities later, often the factor 2 does not matter since those inequalities will have an implicit proportional constant anyway.

**Proof.** The proof is motivated by [GH21, Lemma 4.1]. By definitions of $\mathcal{U}$ and Besov spaces,

\[
\| \mathcal{U} \|_{C_{T} c^{-3/2 - \infty}} \lesssim 2^{-L/2} \| \tilde{Z}_{ij} \| C_{T} c^{-1 - \infty}.
\]

This combined with Schauder estimate Lemma A.4 and Lemma A.6 implies that

\[
\| X_i \|_{C^\frac{1}{2} c^{-\infty}} \lesssim 2^{-L/2} \frac{\lambda}{N} \sum_{j=1}^{N} \left( \| X_{ij} \|_{C^\frac{1}{2} c^{-\infty}} \right)^{\frac{1}{2}} \left( \frac{\lambda^2}{N^2} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} \right)^{\frac{1}{2}}
\]

\[
\lesssim 2^{-L/2} \left( \frac{\lambda}{N} \sum_{j=1}^{N} \| X_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} \right)^{\frac{1}{2}} \left( \frac{\lambda^2}{N^2} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} \right)^{\frac{1}{2}} + \frac{\lambda}{N} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}}.
\]

Taking square on both sides and summing over $i$ then dividing by $N$ we deduce

\[
\frac{1}{N} \sum_{i=1}^{N} \| X_i \|^2_{C^\frac{1}{2} c^{-\infty}} \lesssim 2^{-L} \left( \frac{\lambda}{N} \sum_{j=1}^{N} \| X_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} \right) \left( \frac{\lambda^2}{N^2} \sum_{i,j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} \right) + \frac{\lambda}{N} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}}.
\]

Choosing $L = L(\lambda, N)$ such that

\[
2^L = 2C^2 \left( \frac{\lambda^2}{N^2} \sum_{i,j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} \right) + 2C^2 \left( \frac{\lambda}{N} \sum_{j=1}^{N} \| \tilde{Z}_{ij} \|^2_{C^\frac{1}{2} c^{-\infty}} \right)^{\frac{1}{2}} + 1,
\]
with $C$ given as the maximum of proportional constant in (2.11) and (2.21) below, we easily deduce (2.7). Using (2.11) and (2.7), (2.8) follows. (2.9) follows from using Schauder estimate Lemma A.4 to bound $\|X_t\|_{C^T_c C^{-\frac{1}{2} - \kappa} L_\infty}$ in the analogous way as (2.11) and then applying (2.7) (2.8) and (2.12).

By Lemma 2.3 we have $X_i \in C_T C^{\frac{1}{2} - \kappa}$. We also introduce $X_{i,\varepsilon}$ defined by (2.6) with the elements in $Z$ replaced by $Z_{\varepsilon}$. It is easy to deduce that
\[ X_{i,\varepsilon} \to X_i \quad \text{in } C_T C^{\frac{1}{2} - \kappa} \quad \text{P-a.s., as } \varepsilon \to 0. \]

In our energy estimates below, we will have terms $X_j \circ Z_{ij}$, $X_i \circ Z_{ij}$ and $X_j \circ Z_i$, which are not classical well-defined. We will use the renormalized terms introduced in Section 2.1 to define them. We first consider $X_j \circ Z_{ij}$. For $i \neq j$, note that $2Z_{ij}^{\varepsilon}$ requires renormalization when $i = l$, and $Z_{ij}^{\varepsilon}$ does not. Hence, for $i \neq j$

\[ X_j \circ Z_{ij}^{\varepsilon} \overset{-def}{=} \lim_{\varepsilon \to 0} (X_{j,\varepsilon} \circ Z_{ij}^{\varepsilon} + \frac{\lambda \delta_{ij}}{N}(Z_{i,\varepsilon} + X_{i,\varepsilon})) \quad (2.13) \]

\[ = \frac{-\lambda}{N} \sum_{l=1}^{N} \lim_{\varepsilon \to 0} \left( 2 Z_{il,ij}^{\varepsilon} + \mathcal{I}(X_{l|\varepsilon}, \zeta_{ij}) \circ Z_{ij}^{\varepsilon} - \mathcal{I}(X_{l|\varepsilon} \times \mathcal{U}_{\leq}(Z_{ij}^{\varepsilon})) \circ Z_{ij}^{\varepsilon} - \frac{\delta_{ij}}{N} X_{i,\varepsilon} \right) \]

\[ = \frac{-\lambda}{N} \sum_{l=1}^{N} \left( 2 Z_{il,ij}^{\varepsilon} + X_{l|\varepsilon} Z_{ij}^{\varepsilon} + \tilde{C}(X_{l|\varepsilon}, Z_{ij}^{\varepsilon}) - \mathcal{I}(2X_{l|\varepsilon} \times \mathcal{U}_{\leq}(Z_{ij}^{\varepsilon})) \circ Z_{ij}^{\varepsilon} \right) + \frac{\lambda}{N} (\tilde{b} - b(t)) X_i \]

with
\[ \tilde{b} - b(t) = \lim_{\varepsilon \to 0} (\tilde{b}_{c} - b_{c}(t)), \quad |\tilde{b} - b(t)| \lesssim t^{-\gamma} \quad (2.14) \]

for $t > 0$ and $\gamma > 0$. Here $\tilde{C}$ is the commutator introduced in Lemma A.11. Using Lemma A.6, Lemma A.11 and Lemma 2.2 the limit in (2.13) is understood in $C((0,T]; C^{\frac{1}{2} - \kappa})$ P-a.s.. For $i = j$, we have similar decomposition as in (2.13) with $\delta_{ij}$ replaced by $3\delta_{ij}$, with limits also in $C((0,T]; C^{\frac{1}{2} - \kappa})$ P-a.s.. $2Z_{ij}^{\varepsilon}$ gives an extra $2\tilde{b}$, and $\tilde{Z}_{ij}^{\varepsilon}$ gives an extra $\tilde{b}$, in the case $i = j = l$.

Similarly for $i \neq j$ we define, again with limit in $C((0,T]; C^{\frac{1}{2} - \kappa})$ P-a.s.

\[ X_i \circ Z_{ij}^{\varepsilon} \overset{-def}{=} \lim_{\varepsilon \to 0} (X_{i,\varepsilon} \circ Z_{ij}^{\varepsilon} + \frac{\lambda \delta_{ij}}{N}(Z_{i,\varepsilon} + X_{i,\varepsilon})) \quad (2.15) \]

\[ = \frac{-\lambda}{N} \sum_{l=1}^{N} \left( 2 Z_{il,ij}^{\varepsilon} + X_{l|\varepsilon} Z_{ij}^{\varepsilon} + \tilde{C}(X_{l|\varepsilon}, Z_{ij}^{\varepsilon}) - \mathcal{I}(X_{l|\varepsilon} \times \mathcal{U}_{\leq}(Z_{ij}^{\varepsilon})) \circ Z_{ij}^{\varepsilon} \right) + \frac{\lambda}{N} (\tilde{b} - b(t)) X_i . \]

Finally we define, with the limit taken in $C_T C^{-\kappa}$ P-a.s.

\[ X_j \circ Z_i = \lim_{\varepsilon \to 0} (X_{j,\varepsilon} \circ Z_{i,\varepsilon}) = \frac{-\lambda}{N} \sum_{l=1}^{N} \left( 2 Z_{il,j}^{\varepsilon} + \mathcal{I}(X_{l|\varepsilon} \times \mathcal{U}_{\leq}(Z_{ij}^{\varepsilon} \circ Z_{i,\varepsilon})) \right) . \quad (2.16) \]

In the following lemma we derive uniform bounds for the above stochastic terms. Regarding $X_j Z_i$ we have $X_j Z_i = X_j \prec Z_i + X_j \circ Z_i + X_j \succ Z_i$.

**Lemma 2.4.** It holds that
\[ \frac{1}{N^2} \sum_{i,j=1}^{N} \| X_j Z_i \|_{C^T_c C^{\frac{1}{2} - \kappa} L_\infty}^2 \lesssim \lambda^2 (1 + \lambda^2) Q_N^{52}, \]
\[ \frac{1}{N} \sum_{j=1}^{N} \| X_j Z_j \|_{C^T_c C^{\frac{1}{2} - \kappa} L_\infty}^2 \lesssim \lambda^2 (1 + \lambda^2) Q_N^{53}, \]

where $Q_N^{52}, Q_N^{53}$ are given in the proof with $\mathbb{E} Q_N^{52} + \mathbb{E} Q_N^{53} \lesssim 1$ uniformly in $N$ and $\lambda$. 

Proof. By (2.16) and Lemmas A.4, A.6 we obtain
\[
\|X_j \circ Z_i\|_{C_{\tau}^{-1}} \lesssim \frac{\Lambda}{N} \sum_{i=1}^{N} \left[ \|Z_{ij}\|_{C_{\tau}^{-1}} + \|X_{ij}\|_{C_{\tau}^{2}} \right].
\] (2.17)
Moreover, by Lemma A.6 we have
\[
\|X_j \prec Z_i\|_{C_{\tau}^{-1}} + \|X_j \succ Z_i\|_{C_{\tau}^{-1}} \lesssim \|X_j\|_{C_{\tau}^{2}} + \|Z_i\|_{C_{\tau}^{2}}.
\]
By (2.7) we have
\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \|X_j Z_i\|_{C_{\tau}^{-1}}^2 \lesssim \lambda^2 (1 + \lambda^2) \left[ \frac{1}{N^3} \sum_{i,j,l=1}^{N} \|Z_{ij}\|_{C_{\tau}^{-1}}^2 \right]
\]
\[
+ \frac{1}{N^3} \sum_{i,j,l=1}^{N} \|Z_{ij} Z_{kl}\|_{C_{\tau}^{2}}^2 \left[ \frac{1}{N} \sum_{i,j,k,l=1}^{N} \|Z_{ij}\|_{C_{\tau}^{-1}}^2 \right] \lesssim \lambda^2 (1 + \lambda^2) Q_N^3,
\]
where the first line corresponds to \(X_j \prec Z_i + X_j \succ Z_i\), and the second and third line are estimates for \(X_j \circ Z_i\).
Moreover, we have the paraproduct decomposition \(X_j Z_j = X_j \prec Z_j + X_j \succ Z_j + X_j \circ Z_j\). Using Lemma A.6, we can bound the first two terms by \(\|X_j\|_{C_{\tau}^{2}} \|Z_j\|_{C_{\tau}^{-1}}\). By (2.8) we obtain
\[
\frac{1}{N} \sum_{j=1}^{N} \|X_j Z_j\|_{C_{\tau}^{-1}}^2 \lesssim \frac{\lambda^2}{N} \sum_{i=1}^{N} \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \|Z_{ij}\|_{C_{\tau}^{-1}}^2 \right) \lesssim \lambda^2 (1 + \lambda^2) Q_N^3,
\]
with
\[
Q_N^6 \defeq \frac{1}{N^2} \sum_{i,j=1}^{N} \|Z_{ij}\|_{C_{\tau}^{-1}}^2 \lesssim \lambda^2 (1 + \lambda^2) Q_N^3.
\]
Using (2.16) and (2.17) with \(i = j\) we deduce
\[
\frac{1}{N} \sum_{j=1}^{N} \|X_j Z_j\|_{C_{\tau}^{-1}}^2 \lesssim \lambda^2 (1 + \lambda^2) \left[ Q_N^6 \left( \frac{1}{N} \sum_{i=1}^{N} \|Z_{ij}\|_{C_{\tau}^{-1}}^2 \right) \right]
\]
\[
+ \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \|Z_{ij}\|_{C_{\tau}^{-1}}^2 \right) \left( \frac{1}{N^2} \sum_{l,k=1}^{N} \|Z_{kl}\|_{C_{\tau}^{-1}}^2 \right) \lesssim \lambda^2 (1 + \lambda^2) Q_N^3.
\]
The result follows by the above estimates. \(\square\)
2.3. Improved uniform in $N$ estimates for stochastic terms. In this section we derive uniform in $N$ estimates for the stochastic terms introduced in Section 2.2, which shows that one obtains “improved estimates” by gaining “factors of 1/N”.

Recall that for mean-zero independent random variables $U_1, \ldots, U_N$ taking values in a Hilbert space $H$, we have

$$E\left\| \sum_{i=1}^{N} U_i \right\|_H^2 = E \sum_{i=1}^{N} \left\| U_i \right\|_H^2. \quad (2.18)$$

This simple fact is important for us since the square of the sum on the LHS of (2.18) appears to have “$N^2$ terms” but under expectation it’s only a sum of $N$ terms, in a certain sense giving us a “factor of 1/N”. This motivates us to derive the following uniform in $N$ estimate in suitable Hilbert spaces. We first prove the following result for renormalization terms $\tilde{Z}_{ij}$ and $\tilde{Z}_{ij}^{\Psi}$.

**Lemma 2.5.** Set

$$Q_N^0 \overset{\text{def}}{=} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left\| \tilde{Z}_{ij}^{\Psi} \right\|^2_{L^2_T L^2}\text{,} \quad Q_N^1 \overset{\text{def}}{=} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left\| \tilde{Z}_{ij}^{\Psi} \right\|^2_{W_T^{\frac{1}{2}-2\kappa} L^2}\text{,}$$

$$Q_N^2 \overset{\text{def}}{=} \frac{1}{N^2} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} \left\| \tilde{Z}_{ij}^{\Psi} \right\|^2_{L^2_T L^2} \right)^2.$$

One has $E[Q_N^i]^q \lesssim 1$ for every $q \geq 1$ and $i = 0, 1, 2$, uniformly in $N$ and $m$.

**Proof.** Since we will have several similar calculations in the sequel, we first demonstrate such calculation in the case $q = 1$. We have for $s = \frac{1}{2} - 2\kappa$

$$E\left\| \Lambda^{i} \tilde{Z}_{ij}^{\Psi} \right\|^2_{L^2_T L^2} = \frac{1}{N^2} \sum_{i, j, s=1}^{N} E\left\langle \Lambda^{i} \tilde{Z}_{ij}^{\Psi}, \Lambda^{j} \tilde{Z}_{ij}^{\Psi} \right\rangle_{L^2_T L^2},$$

where $\Lambda^i = (1 - \Delta)^{\frac{i}{2}}$ is introduced in Section A.1 and we used permutation invariance in law to drop the sum over $i$. We have 3 summation indices and a factor $1/N^2$. The contribution to the sum from the cases $j_1 = i$ or $j_2 = i$ or $j_1 = j_2$ is bounded by a constant in light of Lemma 2.2. If $i, j_1, j_2$ are all different, by independence and the fact that Wick products are mean zero, the terms are zero.

For general $q \geq 1$, by Gaussian hypercontractivity and the fact that $Q_N^i$ is a random variable in finite Wiener chaos, we have for $i = 0, 1, 2$

$$E[|Q_N^i|^q] \lesssim E[|Q_N^i|^2]^{q/2}.$$ 

So it suffices to consider $q = 2$. We write $E[|Q_N^0|^2]$ as

$$\frac{1}{N^4} \sum_{i, j, k, \ell=1}^{N} E\left\langle \Lambda^{i} \tilde{Z}_{1j1}, \Lambda^{j} \tilde{Z}_{1j1}, \Lambda^{k} \tilde{Z}_{ij2}, \Lambda^{\ell} \tilde{Z}_{ij2} \right\rangle_{L^2_T L^2} \left\langle \Lambda^{i} \tilde{Z}_{ij3j3}, \Lambda^{j} \tilde{Z}_{ij3j4} \right\rangle_{L^2_T L^2},$$

We have 6 indices $i_1, i_2, j_1, \ldots, j_4$ summing from 1 to $N$ and an overall factor $1/N^4$. Using again Lemma 2.2, we reduce the problem to the cases where five or six of the indices are distinct. However, in these two cases, at least one of $j_k$ is different from others. Then by independence the expectation is zero, so $E[|Q_N^0|^2] \lesssim 1$.

Since $W_T^{\frac{1}{2}-2\kappa} L^2$ is a Hilbert space with inner product given by

$$\langle f, g \rangle_{W_T^{\frac{1}{2}-2\kappa} L^2} = \langle f, g \rangle_{L^2_T L^2} + \int_0^T \int_0^T \frac{\langle f(t) - f(r), g(t) - g(r) \rangle_{L^2}}{|t - r|^{\frac{1}{2}-4\kappa}} dt dr,$$

we deduce $E[|Q_N^1|^2] \lesssim 1$ by similar arguments.
In the following we consider $E|Q_N^2|^2$. First note that $Q_N^2$ is bounded by

$$\frac{1}{N^2} \sum_{i,j=1}^N \left\| \sum_{l=1}^N \tilde{Z}_{l,j,i} \right\|_{L_T^2 H^{-1/2 - 2\kappa}}^2,$$

which implies that $E|Q_N^2|^2$ is bounded by

$$\frac{1}{N^6} \sum_{i,j=1}^N \sum_{k=1}^N E \left( \lambda^s \tilde{Z}_{l_1,i,j,k}^* \lambda^s \tilde{Z}_{l_2,i,j,k} \right) \left\| \tilde{Z}_{l_1,i,j,k} \right\|_{L_T^2 L^2} \left\| \tilde{Z}_{l_2,i,j,k} \right\|_{L_T^2 L^2},$$

for $s = -\frac{1}{2} - 2\kappa$. We have 8 indices $i, i_1, j_1, j_2, l_1, ..., l_4$ summing from 1 to $N$ and an overall factor $1/N^6$. Using again Lemma 2.2, we reduce the problem to the cases where seven or eight of the indices are different. However, in these two cases at least one of $l_k$ is different from others. Then by independence the expectation is zero, so $E(|Q_N^2|^2) \lesssim 1$. □

With the help of Lemma 2.5 we also have the following bounds for $X$, which states that summing $N$ terms of suitable Hilbert norms of $X_1$ actually “behaves like order 1”.

**Lemma 2.6.** The following bounds hold with proportional constants independent of $N$ and $\lambda$

\[ \sum_{i=1}^N \|X_i\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}^2 \lesssim \lambda^2 Q_N^0, \quad (2.19) \]

\[ \sum_{i=1}^N \|X_i\|_{W_T^{\frac{1}{2} - 3\kappa}}^2 \lesssim \lambda^2 Q(Z), \quad (2.20) \]

where $Q_N^0$ satisfies the moment bounds in Lemma 2.5, and $(E|Q(Z)|^q)^{1/q} \lesssim 1 + \lambda^2$ for any $q \geq 1$.

**Proof.** The start-point of the proof is similar as that of Lemma 2.3, but now it’s crucial that we estimate $X_1$ in a Hilbert space. Using the Schauder estimate in Lemma A.5 now (instead of Lemma A.4) and Lemma A.6 and (2.10) we have

\[
\|X_i\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}} \lesssim 2^{-L/2} \frac{\lambda}{N} \sum_{j=1}^N \left( \|X_j\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}} \|Z_{ij}^Y\|_{C_T \mathbf{C}^{1-\kappa}} \right) + \left\| \frac{\lambda}{N} \sum_{j=1}^N \tilde{Z}_{ij}^Y \right\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}},
\]

\[
\lesssim 2^{-L/2} \left( \sum_{j=1}^N \|X_j\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}^2 \right)^{1/2} \left( \frac{\lambda^2}{N^2} \sum_{j=1}^N \|Z_{ij}^Y\|_{C_T \mathbf{C}^{1-\kappa}}^2 \right)^{1/2},
\]

\[
+ 2^{-L/2} \|X_i\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}} \left( \frac{\lambda}{N} \sum_{j=1}^N \|Z_{ij}^Y\|_{C_T \mathbf{C}^{1-\kappa}} \right) + \left\| \frac{\lambda}{N} \sum_{j=1}^N \tilde{Z}_{ij}^Y \right\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}.
\]

Note that a key difference between the bound here and the proof of Lemma 2.3 is that by considering $\|X_i\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}$ instead of $\|X_i\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}$ and $\|Z_{ij}^Y\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}$, we gain a “factor of $1/N$” by Lemma 2.5 and the discussion before Lemma 2.5. Then taking square on both sides and summing over $i$ we obtain

\[ \sum_{i=1}^N \|X_i\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}^2 \lesssim 2^{-L} \left( \sum_{j=1}^N \|X_j\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}^2 \right)^{1/2} \left( \frac{\lambda^2}{N^2} \sum_{i,j=1}^N \|Z_{ij}^Y\|_{C_T \mathbf{C}^{1-\kappa}}^2 \right)^{1/2},
\]

\[ + 2^{-L} \left( \sum_{i=1}^N \|X_i\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}^2 \right) \left( \frac{\lambda}{N} \sum_{j=1}^N \|Z_{ij}^Y\|_{C_T \mathbf{C}^{1-\kappa}} \right)^2 + \left\| \frac{\lambda}{N} \sum_{j=1}^N \sum_{i=1}^N \tilde{Z}_{ij}^Y \right\|_{L_T^2 H^{\frac{1}{2} - 2\kappa}}^2.
\]

By the choice of $2^L$ in (2.12), (2.19) follows.
To show (2.20), again by Lemma A.5 and Lemma A.6 we have
\[ \|X_i\|_{W^{3n-2k}_n L^2}^2 \lesssim \frac{\lambda^2}{N^2} \left( \sum_{j=1}^N \|Z_{ij}\|_{L^2 H^{3n-2k}}^2 + \left( \frac{\lambda}{N} \sum_{j=1}^N \|X_j\|_{L^2_f H^{3n-2k}} \right)^2 \right), \]
which combined with (2.19) and Hölder’s inequality for the sum over j implies that
\[ \sum_{i=1}^N \|X_i\|_{W^{3n-2k}_n L^2}^2 \lesssim \lambda^2 Q_N^0 + \left( \sum_{j=1}^N \|X_j\|_{L^2_f H^{3n-2k}}^2 \right) \left( \frac{\lambda^2}{N^2} \sum_{i,j=1}^N \|Z_{ij}\|_{C^{1-k}}^2 \right)^2, \]
where \( Q_N^0, Q_N^1 \) are introduced in Lemma 2.5. Using (2.19) with Lemma 2.2 and Lemma 2.5, (2.20) follows.

In the following lemma we deduce the estimates in a suitable Hilbert space for the renormalized terms introduced in (2.13), (2.15) and (2.16) before Lemma 2.4.

**Lemma 2.7.** The following bounds hold with \( EQ(Z)^q \lesssim 1 + \lambda^q \) uniformly in \( N \) and \( \lambda \) for all \( q \geq 1 \)
\[ \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \| \bullet \|_{L^2_f H^{3n-2k}}^2 \right) \lesssim \lambda^2 Q(Z), \quad \bullet \in \{ X_j^2 Z_i, X_j^2 X_i Z_j \}, \]
\[ \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \| \bullet \|_{L^2_f H^{3n-2k}}^2 \right) \lesssim \lambda^2 Q(Z), \quad \bullet \in \{ X_i \circ Z_{ij}^\Upsilon, X_j \circ Z_{ij}^\Upsilon \}. \]

**Proof.** From the proof of Lemma 2.4 we have
\[ \frac{1}{N^2} \sum_{i,j=1}^N \| X_j \circ Z_i \|_{C^{1-k}}^2 \lesssim \lambda^2 (1 + \lambda^2) Q(Z), \] (2.22)
with \( Q(Z) \) satisfying \( EQ(Z)^q \lesssim 1 \) for any \( q \geq 1 \). Now we consider \( X_j^2 Z_i \) and use paraproduct to have the following decomposition
\[ X_j^2 \circ Z_i = 2(X_j \times X_j) \circ Z_i + (X_j \circ X_j) \circ Z_i \]
\[ = 2X_j(X_j \circ Z_i) + 2\tilde{C}(X_j, X_j, Z_i) + (X_j \circ X_j) \circ Z_i, \] (2.23)
where \( \tilde{C} \) is introduced in Lemma A.9. By Lemma A.6 and Lemma A.9 we have
\[ \| X_j^2 \circ Z_i \|_{L^2_f H^{3n-2k}} \lesssim \| X_j \|_{L^2_f H^{3n-2k}} \| X_j \circ Z_i \|_{C^{1-k}}, \]
\[ + \| X_j \|_{L^2_f H^{3n-2k}} \| X_j \|_{C^{1-k}} \| Z_i \|_{C^{1-k}}, \]
and
\[ \| X_j^2 \circ Z_i \|_{L^2_f H^{3n-2k}} \lesssim \| X_j \|_{L^2_f H^{3n-2k}} \| X_j \|_{C^{1-k}} \| Z_i \|_{C^{1-k}}. \]

Thus Hölder’s inequality implies that
\[ \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \| X_j^2 Z_i \|_{L^2_f H^{3n-2k}}^2 \right) \lesssim \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \| X_j \|_{L^2_f H^{3n-2k}}^2 \right) \left( \sum_{j=1}^N \| X_j \circ Z_i \|_{C^{1-k}}^2 \right), \]
\[ + \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \| X_j \|_{L^2_f H^{3n-2k}}^2 \right) \left( \sum_{j=1}^N \| X_j \|_{C^{1-k}}^2 \right) \| Z_i \|_{C^{1-k}}^2. \]
Then by (2.22), Lemma 2.3, Lemma 2.6 and Lemma 2.5 we obtain the bound for $X_j^2 Z_i$. The bound for $X_i X_j Z_j$ follows in the same way.

Moreover, by the decomposition for $X_j \circ Z_{ij}^\nu$ in (2.13) and Lemmas A.5, A.6 and A.12,

$$
\|X_j \circ Z_{ij}^\nu\|_{L^2_{ij} H^{-\frac{k}{2} - 2\kappa}} \lesssim \frac{\lambda}{N} \left( \sum_{l=1}^{N} \bar{Z}^\nu_{l,j,ij} \right)_{L^2_{ij} H^{-\frac{k}{2} - 2\kappa}} + \frac{\lambda}{N} \| (\bar{b} - b) X_i \|_{L^2_{ij} H^{-\frac{k}{2} - 2\kappa}}
$$

$$
\quad + \frac{\lambda}{N} \sum_{l=1}^{N} \left( \|X_i\|_{L^2_{ij} H^{\frac{k}{2} - 2\kappa}} \|Z_{ij}^\nu\|_{C_T C^{-\kappa}} \right)
$$

$$
\quad + \left( (1 + 2^{3\kappa L}) \|X_i\|_{L^2_{ij} H^{\frac{k}{2} - 2\kappa}} \|Z_{ij}^\nu\|_{C_T C^{-1 - \kappa}} + \|X_i\|_{W^{\frac{k}{2} - 3\kappa, 2}_2} \|Z_{ij}^\nu\|_{C_T C^{-1 - \kappa}} \right) \|Z_{ij}^\nu\|_{C_T C^{-1 - \kappa}}
$$

where we used (2.12) and

$$
\| \bar{P} \leq Z^\nu_{ij}\|_{C^{-1 + 2\kappa}} + \| \bar{P} \leq Z^\nu_{ij}\|_{C^{-1 + 2\kappa}} \lesssim 2^{3\kappa L} \|Z_{ij}^\nu\|_{C^{-1 - \kappa}} + \|Z_{ij}^\nu\|_{C^{-1 - \kappa}}.
$$

Moreover, using (2.14) we get

$$
\frac{1}{N^2} \sum_{i=1}^{N} \left( \frac{1}{N} \sum_{j=1}^{N} \| (\bar{b} - b) X_i \|_{L^2_{ij} H^{-\frac{k}{2} - 2\kappa}} \right)^2 \lesssim \frac{\lambda^2}{N^2} \sum_{i=1}^{N} \|X_i\|_{C_T C^\kappa C^{-\kappa}}^2.
$$

Thus by Lemma 2.6 and Lemma 2.3 we have

$$
\frac{1}{N^2} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} \|X_j \circ Z_{ij}^\nu\|_{L^2_{ij} H^{-\frac{k}{2} - 2\kappa}} \right)^2 \lesssim \lambda^2 Q^2_N + \lambda^4 Q(N),
$$

with $Q(N)$ satisfying $(E Q(Z)^q)^{1/q} \lesssim 1 + \lambda^2$ which combined with Lemma 2.5 implies the bound for $X_j \circ Z_{ij}^\nu$. The last bound regarding $X_i \circ Z_{ij}^\nu$ then follows in the same way.

\[ \square \]

2.4. Decomposition. In this section we consider the following equation

$$
\mathcal{L} \Phi_i = -\frac{\lambda}{N} \sum_{j=1}^{N} \Phi_i^2 \Phi_i + \xi_i, \quad \Phi_i(0) \in C^{-\frac{k}{2} - \kappa},
$$

(2.24)

for $\kappa > 0$. For fixed $N$ and $\omega \in \Omega_0$ by using regularity structure theory [Hai14] or paracontrolled distribution method [GIP15], we easily deduce the local well-posedness of (2.24) in $C_T C^{-\frac{k}{2} - \kappa}$ and $\Phi_i$ is the limit of $\Phi_{i,e}$, which is the unique solution to equation (2.1). Furthermore, by similar arguments as in [MW17a, GH19] global well-posedness also holds by uniform estimates, which may depend on $N$. In the following we concentrate on the uniform in $N$ bounds.

With the stochastic objects at hand, we have the following decompositions: $\Phi_i = Z_i + X_i + Y_i$ with $Y_i$ satisfying the following equation

$$
\mathcal{L} Y_i = -\frac{\lambda}{N} \sum_{j=1}^{N} \left( Y_i^2 Y_i + (X_i^2 + 2 X_i Y_j)(X_i + Y_i) + Y_i^2 X_i + (X_i + Y_j)^2 Z_i \right)
$$

$$
+ 2 (X_i + Y_j)(X_i + Y_j) Z_j + 2 X_i < Y_{ij} < Z_{ij} + 2 X_i < Y_{ij} < Z_{ij}
$$

$$
+ 2 Y_{ij} \langle Z_{ij} \rangle + Y_i < Z_{ij} + 2 (X_j + Y_j) \langle Z_{ij} + (X_i + Y_i) \rangle \langle Z_{ij} \rangle,
$$

(2.25)

$$
Y_i(0) = \Phi_i(0) - Z_i(0) - X_i(0).
$$

\[ \text{This decomposition follows [GH21] which uses different notation than } Z + X + Y \text{ here. Our choice of this notation is close to our 2D paper [SSZ22].} \]
Here the first line in (2.25) are the expansion of \((Y_j + X_j)^2(Y_i + X_i + Z_i)\); the terms containing \(Z_{ij}\) and \(Z_{jj}\) correspond to the remaining terms in the paraproduct expansion of \(2(Y_j + X_j)Z_jZ_i\) and \((Y_i + X_i + Z_i)Z_{ij}\), respectively.

Also, note that some products in (2.25) are understood via renormalization. Namely, \(X_j^2Z_i\), \(X_jZ_i\), \(X_jX_iZ_j\), \(X_j \circ Z_{ij}\) and \(X_i \circ Z_{jj}\) are understood using (2.13)-(2.16), (2.23) and Lemmas 2.4+2.7. Since \(Z_{ij}\) and \(Z_{jj}\) are not well-defined in the classical sense and we need to use the renormalization terms \(\tilde{Z}_{ij}^{Y}\) and \(\tilde{Z}_{jj}^{Y}\) to define these terms, which from the approximation level requires to subtract \(\frac{3\lambda(N+2)}{N^2}b_iY_{i,\epsilon}\) on the R.H.S of (2.25), where \(Y_{i,\epsilon} = \Phi_{i,\epsilon} - X_{i,\epsilon} - Z_{i,\epsilon}\).

In the following we establish \(L^2\)-energy estimate for \(Y_i\), and as explained in the introduction we follow the idea in [GH21] to use the duality between \(\prec\) and \(\circ\), i.e. Lemma A.10 to cancel \(Y_j \circ Z_{ij}\) and \(Y_i \circ Z_{jj}\) which would require paracontrolled ansatz and higher regularity estimate than \(H^1\).

Recall that \(\mathcal{D} = m - \Delta\) and we define

\[
\varphi_i \overset{\text{def}}{=} Y_i + \mathcal{D}^{-1}\sum_{j=1}^{N} (Y_{ij} \prec Z_{ij}^{Y}) \overset{\text{def}}{=} Y_i + \mathcal{D}^{-1}P_i, \tag{2.26}
\]

where in the last step we defined \(P_i\).

Now we turn to uniform in \(N\) bounds on (2.25) and note that \(Y_i\) depends on \(N\), but we omit this throughout. Similar as the 2d case in [SSZZ22], we do \(L^2\)-energy estimate of \(Y_i\) and take sum over \(i\). Using (2.26), we have the following decomposition.

**Lemma 2.8.** *(Energy balance)*

\[
\frac{1}{2} \sum_{i=1}^{N} \frac{d}{dt} \|Y_i\|_{L^2}^2 + m \sum_{i=1}^{N} \|\varphi_i\|_{L^2}^2 + \sum_{i=1}^{N} \|\nabla \varphi_i\|_{L^2}^2 + \frac{\lambda}{N} \sum_{i=1}^{N} \|Y_{i}^2\|_{L^2}^2 = \Theta + \Xi.
\]

Here

\[
\Theta = \sum_{i=1}^{N} \left( \mathcal{D}^{-1}P_i, P_i \right) - \frac{\lambda}{N} \sum_{i,j=1}^{N} \left( 2D(Y_i, Z_{ij}^Y, Y_j) + D(Y_i, Z_{jj}^Y, Y_i) \right),
\]

and

\[
\Xi = \frac{\lambda}{N} \sum_{i,j=1}^{N} \left( (X_j^2 + 2X_jY_j)(X_i + Y_i) + Y_j^2X_i + (X_j + Y_j)^2Z_i + 2(X_j + Y_j)(X_i + Y_i)Z_j + 2X_j \prec \mathcal{U}_{\epsilon} Z_{ij}^Y + X_i \prec \mathcal{U}_{\epsilon} Z_{jj}^Y + 2(X_j + Y_j) \prec Z_{ij}^Y + 2X_j \circ Z_{ij}^Y + (X_i + Y_i) \prec Z_{jj}^Y, Y_i \right).
\]

**Proof.** We will first focus on a formal derivation for the claimed identity. Then we will remark that although new renormalization appears to be necessary when we take inner product, in the end they cancel each other.

Taking inner product with \(Y_i\) in \(L^2\) on (2.25), we realize that the first term on the R.H.S. of (2.25) leads to the term \(-\frac{\lambda}{N} \|\sum_{i=1}^{N} Y_{i}^2\|_{L^2}^2\), and it is straightforward to check that the other terms all lead to \(\Xi\) except the following terms

\[
-\frac{\lambda}{N} \sum_{i,j=1}^{N} (2Y_j \prec Z_{ij}^Y + Y_i \prec Z_{jj}^Y, Y_i). \tag{2.27}
\]
We claim that we can write (2.27) plus \( \sum_{i=1}^{N} ((\Delta - m)Y_i, Y_i) \) as \( \Theta - m \sum_{i=1}^{N} \| \varphi_i \|^2_{L^2} - \sum_{i=1}^{N} \| \nabla \varphi_i \|^2_{L^2} \) (see Eq. (2.32)) and we will prove this claim for the rest of this proof.

Using (2.26) we write it as
\[
((\Delta - m)Y_i, Y_i) = ((\Delta - m)\varphi_i, \varphi_i) + 2(Y_i, P_i) + \langle \mathcal{D}^{-1} P_i, P_i \rangle.
\]  
(2.28)

We will realize below that \( 2(Y_i, P_i) \) cancels the irregular part (i.e. the paraproduct \( \preceq \) part) in (2.27). For the term in (2.27), by (2.26) we have
\[
-\frac{\lambda}{N} \sum_{i,j=1}^{N} \langle 2Y_j \preceq Z_{ij}^\mathcal{V} + Y_j \preceq Z_{jj}^\mathcal{V}, Y_i \rangle = -\langle Y_i, P_i \rangle - \frac{\lambda}{N} \sum_{i,j=1}^{N} (2Y_j \circ Z_{ij}^\mathcal{V} + Y_j \circ Z_{jj}^\mathcal{V}, Y_i).
\]  
(2.29)

where we note that the first term \( -\langle Y_i, P_i \rangle \) on the R.H.S. precisely cancels one \( \langle Y_i, P_i \rangle \) on the R.H.S. of (2.28). Using Lemma A.10, the other terms in (2.29) containing \( \circ \) can be written as
\[
\langle Y_i, 2Y_j \circ Z_{ij}^\mathcal{V} \rangle = 2\langle Y_i \times Z_{ij}^\mathcal{V}, Y_j \rangle + 2D(Y_i, Z_{ij}^\mathcal{V}, Y_j),
\]  
(2.30)
\[
\langle Y_i, Y_j \circ Z_{jj}^\mathcal{V} \rangle = \langle Y_i \times Z_{jj}^\mathcal{V}, Y_j \rangle + D(Y_i, Z_{jj}^\mathcal{V}, Y_j). 
\]  
(2.31)

where \( D(f,g,h) \) is the commutator introduced in Lemma A.10. The first terms on the R.H.S. of (2.30) and (2.31) cancel the other \( \langle Y_i, P_i \rangle \) from (2.28) when taking sum.

We sum all the terms in (2.28) and (2.29) w.r.t. \( i \) and obtain the following:
\[
\frac{\lambda}{N} \sum_{i,j=1}^{N} \left( \langle (\Delta - m)Y_i, Y_i \rangle - \frac{\lambda}{N} \sum_{i,j=1}^{N} (2Y_j \preceq Z_{ij}^\mathcal{V} + Y_j \preceq Z_{jj}^\mathcal{V}, Y_i) \right)
\]  
(2.32)

This completes the derivation of the claimed identity.

Finally, we remark that several terms in the above derivation should be understood in the renormalized sense. As we have noticed above, the term \( \sum_{i=1}^{N} 2(P_i, Y_i) \) cancels with the irregular part from (2.27), so no extra renormalization is needed for them.

For the last term in (2.28) we have
\[
\langle \mathcal{D}^{-1} P_i, P_i \rangle = \frac{\lambda}{N} \sum_{i,j=1}^{N} \left( \langle \mathcal{D}^{-1} P_i \circ Z_{ij}^\mathcal{V}, 2Y_j \rangle - 2D(Y_j, Z_{ij}^\mathcal{V}, \mathcal{D}^{-1} P_i) + \langle \mathcal{D}^{-1} P_i \circ Z_{jj}^\mathcal{V}, Y_i \rangle - D(Y_i, Z_{jj}^\mathcal{V}, \mathcal{D}^{-1} P_i) \right).
\]  
(2.33)

Using the definition of \( \mathcal{D}^{-1} P_i \) in (2.26) and the commutator \( C \) introduced in Lemma A.11 we can write the above term as
\[
\frac{\lambda}{N} \sum_{j=1}^{N} \left( \frac{\lambda}{N} \sum_{i=1}^{N} \left( \langle Z_{i\mu,jj}^{\mathcal{V}} Y_j \rceil + 4\langle C(Y_i, Z_{i\mu}^{\mathcal{V}}, Z_{jj}^{\mathcal{V}}, Y_j) \rangle + \langle Z_{i\mu,jj}^{\mathcal{V}} 2Y_j Y_i \rangle + 2\langle C(Y_i, Z_{i\mu}^{\mathcal{V}}, Z_{jj}^{\mathcal{V}}, Y_j) \rangle, Y_j \rangle + \langle Z_{i\mu,jj}^{\mathcal{V}} Y_i \rangle \right) \right)
\]  
(2.34)

Here – recall the renormalization of \( \hat{Z}_{ij}^{\mathcal{V}} \) in Section 2.1 – we need to subtract
\[
\frac{\lambda^2(N+8)}{N^2} \hat{b}_e(Y_{i,e}, Y_{i,e}) + \frac{2\lambda^2}{N^2} \sum_{j \neq i} \hat{b}_e(Y_{j,e}, Y_{j,e})
\]  
(2.35)
from the approximation level to go from \((2.33)\) to \((2.34)\). This precisely matches the renormalization from the SPDE \((2.1)\): indeed, summing \((2.35)\) over \(i\), we get \(\frac{3N+6}{N} \lambda \delta^i\) times \(\sum_i (Y_i, Y_{i,e})\).

We thus conclude that although new renormalization appears to be necessary when we take inner product, in the end no extra renormalization other than the ones introduced at the level of the SPDE is actually needed. \(\square\)

### 3. Uniform in \(N\) estimates

In this section we prove uniform in \(N\) estimates based on Lemma 2.8. The main results are Theorems 3.3 and 3.4. The key step to prove these theorems is to bound \(\int_0^T (\Theta + \Xi) dt\) by

\[
\delta \left( \sum_{j=1}^N \| \nabla \varphi_j \|_{L^2_t L^2}^2 + \sum_{j=1}^N \| Y_j \|_{L^2_t H^{1-2\kappa}}^2 + \frac{\lambda}{N} \sum_{i=1}^N \| Y_i \|_{L^2_t L^2}^2 \right) + C_3 (1 + \lambda^{5\kappa}) \int_0^T \left( \sum_{j=1}^N \| Y_j \|_{L^2}^2 \right) \left( R_N^1 + R_N^2 + Q_N^1 \right) ds + \lambda (1 + \lambda) Q_N^1
\]

for a small constant \(\delta > 0\) with \(R_N^1, R_N^2, Q_N^1, Q_N^1\) introduced in Lemma 3.1, Propositions 3.5 and 3.6. \((3.1)\) will immediately follow from Propositions 3.5 and 3.6 in Section 3.2. The finite moments of \(R_N^1, R_N^2, Q_N^1, Q_N^1\) are bounded uniformly in \(N\), which follows from Lemma 2.2 and Lemma 3.7, \((3.12)\).

#### 3.1. Uniform estimates based on \((3.1)\)

We prove \((3.1)\) in Section 3.2. In this subsection we prove Theorem 3.3 and Theorem 3.4, assuming \((3.1)\). Before this we first prove the following two results. The first one is used to turn \(\| \varphi_i \|_{L^2}^2\) on the LHS of the identity in Lemma 2.8 to \(\| Y_i \|_{L^2}^2\). The second one gives uniform in \(N\) estimates of various norms of \(Y_i, \varphi_i, \mathcal{D}^{-1} P_i\) in terms of \((3.1)\) by using \((2.26)\).

**Lemma 3.1.** The following bound holds with \(C\) independent of \(\lambda\) and \(m \geq 1\)

\[
m \sum_{i=1}^N \| \varphi_i \|_{L^2}^2 \geq \left( \frac{m}{2} - C \lambda^{\frac{\kappa}{2}} (R_N^1) \right) \sum_{i=1}^N \| Y_i \|_{L^2}^2 + m \sum_{i=1}^N \| \mathcal{D}^{-1} P_i \|_{L^2}^2,
\]

with

\[
R_N^1 \overset{def}{=} \frac{1}{N^2} \sum_{i,j=1}^N \| Z_{ij} \|_{C_{1-\kappa}}^2 + \frac{1}{N} \sum_{j=1}^N \| Z_{jj} \|_{C_{1-\kappa}}^2 + 1.
\]

**Proof.** By definition of \(\varphi_i\) in \((2.26)\),

\[
m \sum_{i=1}^N \| \varphi_i \|_{L^2}^2 = m \sum_{i=1}^N \| Y_i \|_{L^2}^2 - 2m \sum_{i=1}^N (Y_i, \mathcal{D}^{-1} P_i) + m \sum_{i=1}^N \| \mathcal{D}^{-1} P_i \|_{L^2}^2.
\]

It remains to control the second term on the RHS. Note that for \(\kappa > 0\)

\[
\| m \mathcal{D}^{-1} f \|_{L^2}^2 = \sum_k \frac{m^2}{(m + |k|^2)^{1+\kappa}} | \hat{f}(k) |^2 \leq m^{1+\kappa} \sum_k \frac{1}{(m + |k|^2)^{1+\kappa}} | \hat{f}(k) |^2 \leq m^{1+\kappa} \| f \|_{H^{-1-\kappa}}^2.
\]

By \((3.3)\) and Lemma A.6 we have

\[
\sum_{i=1}^N | (Y_i, m \mathcal{D}^{-1} P_i) | \leq \sum_{i=1}^N \| Y_i \|_{L^2} \| m \mathcal{D}^{-1} P_i \|_{L^2} \leq m^{\frac{1+\kappa}{2}} \sum_{i=1}^N \| Y_i \|_{L^2} \left( \frac{\lambda}{N} \sum_{j=1}^N (Y_j, Z_{[j],[j]} \right) \|_{H^{-1-2\kappa}}.
\]
The following holds with Proof.

The second bound for also yields the first bound. The third bound for

\[ \| \varphi_i \|^2_{L^2} \leq \binom{N}{i,j=1} \left( \frac{\lambda^2}{N^2} \sum_{j=1}^N \| Z_{\varphi_j}^Y \|_{L^2}^2 \right) + \binom{\lambda N}{j=1} \sum_{j=1}^N \| Z_{\varphi_j}^Y \|_{C^{-\kappa}} \]

where we used Young’s inequality in the last step. Now the result follows. \( \square \)

The following estimates will be useful in the sequel. Recall \( R_N^1 \) from Lemma 3.1.

**Lemma 3.2.** The following holds with \( C \) independent of \( \lambda \) and \( m \geq 1 \)

\[ \sum_{i=1}^N \| Y_i \|^2_{H^{1-2\kappa}} \leq 2 \left( \sum_{i=1}^N \| \varphi_i \|^2_{H^1} \right) + C \lambda^2 \left( \sum_{i=1}^N \| Y_i \|^2_{L^2} \right) R_N^1, \] (3.4)

\[ \sum_{i=1}^N \| \mathcal{D}^{-1} P_i \|^2_{H^{1-2\kappa}} \leq C \lambda^2 \left( \sum_{i=1}^N \| Y_i \|^2_{L^2} \right) R_N^1, \] (3.5)

\[ \sum_{i=1}^N \| \varphi_i \|^2_{L^2} \leq \binom{N}{i,j=1} \left( \frac{\lambda^2}{N^2} \sum_{j=1}^N \| Z_{\varphi_j}^Y \|_{L^2}^2 \right) + C \left( \lambda^2 R_N^1 + 2 \right), \]

\[ \frac{1}{N} \sum_{i,j=1}^N \| \varphi_i \varphi_j \|^2_{L^2} \leq C \left( \sum_{i=1}^N \| Y_i \|^2_{L^2} \right) \left( \lambda^2 R_N^1 + 1 \right)^2. \]

**Proof.** Recalling the relation \( \varphi_i = Y_i + \mathcal{D}^{-1} P_i \) in the definition (2.26), we will see that we essentially only need to estimate \( \| \mathcal{D}^{-1} P_i \|_{H^{1-2\kappa}} \) for the first three inequalities in the lemma. Since \( \| \mathcal{D} f \|_{H^{\beta}} \approx \| f \|_{H^{\beta+2}} \) for \( \beta \in \mathbb{R} \), by Lemma A.6 we have

\[ \| \mathcal{D}^{-1} P_i \|_{H^{1-2\kappa}} \leq \binom{\lambda N}{j=1} \sum_{j=1}^N \left( \| Y_j \|_{L^2} \| Z_{\varphi_j}^Y \|_{C^{-\kappa}} + \| Y_i \|_{L^2} \| Z_{\varphi_j}^Y \|_{C^{-\kappa}} \right) \] (3.6)

\[ \leq C \left( \sum_{j=1}^N \| Y_j \|^2_{L^2} \right)^2 \left( \frac{\lambda^2}{N^2} \sum_{j=1}^N \| Z_{\varphi_j}^Y \|_{L^2}^2 \right) + C \| Y_i \|^2_{L^2} \left( \lambda^2 \sum_{j=1}^N \| Z_{\varphi_j}^Y \|_{C^{-\kappa}} \right). \]

The second bound for \( \mathcal{D}^{-1} P_i \) follows from taking square on both sides of (3.6) and summing over \( i \); this together with

\[ \| Y_i \|^2_{H^{1-2\kappa}} \leq \| \varphi_i \|_{H^1} + \| \mathcal{D}^{-1} P_i \|_{H^{1-2\kappa}} \]

also yields the first bound. The third bound for \( \varphi_i \) follows by \( \| \varphi_i \|_{L^2} \leq \| Y_i \|_{L^2} + \| \mathcal{D}^{-1} P_i \|_{H^{1-2\kappa}} \) and simply plugging in the above bound on \( \mathcal{D}^{-1} P_i \).

Moreover, by the bound (3.5) for \( \mathcal{D}^{-1} P_i \) we have

\[ \frac{1}{N} \sum_{i,j=1}^N \| \varphi_i \varphi_j \|^2_{L^2} = \frac{1}{N} \left( \sum_{i=1}^N \| \varphi_i \|^2_{L^2} \right)^2 \leq C \left( \sum_{i=1}^N \| Y_i \|^2_{L^2} \right)^2 + C \left( \sum_{i=1}^N \| \mathcal{D}^{-1} P_i \|^2_{L^2} \right)^2 \]

\[ \leq C \left( \sum_{i=1}^N \| Y_i \|^2_{L^2} \right)^2 \left( 1 + \lambda^2 R_N^1 \right)^2. \]

\( ^4 \)In the first bound, we keep the explicit constant 2 for the purpose of the proof of Theorem 3.3; in the third bound, we keep the explicit constant 2 in order to derive the condition for \( m \) and \( \lambda \) later.
where we used Sobolev embedding \( \| f \|_{L^4} \lesssim \| f \|_{H^{1-2\kappa}} \) for \( \kappa > 0 \) small enough and

\[
\left\| \sum_{i=1}^{N} Y_i^2 \right\|^2_{L^2} \geq \left( \sum_{i=1}^{N} \| Y_i \|^2_{L^2} \right)^2. \tag{3.7}
\]

\( \square \)

The main result of this section is given as follows, which will be used in Section 4.2.

**Theorem 3.3.** The following holds with constant \( C \) independent of \( N, \lambda \) and \( m \geq 1 \):

\[
\left( \sum_{j=1}^{N} \| Y_j(T) \|^2_{L^2} \right) + \frac{1}{2} \sum_{j=1}^{N} \| \nabla \varphi_j \|^2_{L^2} + m \sum_{j=1}^{N} \| Y_j \|^2_{L^2} + \frac{2\lambda}{N} \| \sum_{i=1}^{N} Y_i^2 \|^2_{L^2} + \frac{3}{8} \sum_{j=1}^{N} \| Y_j \|^2_{L^2, H^{1-2\kappa}}
\]

\[
\leq \sum_{j=1}^{N} \| Y_j(0) \|^2_{L^2} + \lambda(1 + \lambda) Q_N^4 + \sum_{j=1}^{N} \| Y_j \|^2_{L^2} + C \lambda(1 + \lambda^{55}) \int_0^T \left( \sum_{i=1}^{N} \| Y_i \|^2_{L^2} \right) (R_N^2 + R_N^3 + Q_N^3) ds.
\]

**Proof.** Integrating the energy equality in Lemma 2.8 over time, and using Lemma 3.1 and (3.1), \( Q_N^1 \geq (R_N^1)^{\frac{1}{\kappa}} \), we deduce

\[
\left( \sum_{j=1}^{N} \| Y_j(T) \|^2_{L^2} \right) + 2 \sum_{j=1}^{N} \| \nabla \varphi_j \|^2_{L^2} + m \sum_{j=1}^{N} \| Y_j \|^2_{L^2} + \frac{2\lambda}{N} \| \sum_{i=1}^{N} Y_i^2 \|^2_{L^2} + \frac{3}{8} \sum_{j=1}^{N} \| Y_j \|^2_{L^2, H^{1-2\kappa}}
\]

\[
\leq \delta \left( \sum_{j=1}^{N} \| \nabla \varphi_j \|^2_{L^2} + \sum_{j=1}^{N} \| Y_j \|^2_{L^2, H^{1-2\kappa}} + \frac{\lambda}{N} \| \sum_{i=1}^{N} Y_i^2 \|^2_{L^2} \right)
\]

\[
+ \left( \sum_{j=1}^{N} \| Y_j(0) \|^2_{L^2} \right) + C \delta \lambda(1 + \lambda^{55}) \int_0^T \left( \sum_{i=1}^{N} \| Y_i \|^2_{L^2} \right) (R_N^2 + R_N^3 + Q_N^3) ds
\]

\[
+ \lambda(1 + \lambda) Q_N^4,
\]

for some \( \delta > 0 \) small enough. Applying (3.4) to \( \sum_{j=1}^{N} \| Y_j \|^2_{L^2, H^{1-2\kappa}} \), the result follows. \( \square \)

Furthermore, using the dissipation effect from the term \( \frac{1}{N} \| \sum_{i=1}^{N} Y_i^2 \|^2_{L^2} \), the empirical averages of the \( L^2 \) norms of \( Y_i \) can be controlled pathwise in terms of the averages of the renormalized terms \( Q(\mathbb{Z}) \) with finite moment, as stated in the following theorem (which will be used in Sec 4.1).

**Theorem 3.4.** The following bound holds with \( EQ(\mathbb{Z}) \lesssim C(\lambda) \) where \( C(\lambda) \) is independent of \( N \)

\[
\sup_{t \in [0, T]} \frac{1}{N} \sum_{j=1}^{N} \| Y_j(t) \|^2_{L^2} + \frac{m}{N} \sum_{j=1}^{N} \| Y_j \|^2_{L^2} + \frac{\lambda}{2N} \sum_{j=1}^{N} \| \nabla \varphi_j \|^2_{L^2} + \frac{1}{8N} \sum_{j=1}^{N} \| Y_j \|^2_{L^2, H^{1-2\kappa}} \tag{3.8a}
\]

\[
+ \frac{\lambda}{2N^2} \sum_{i=1}^{N} \| Y_i^2 \|^2_{L^2} \lesssim Q(\mathbb{Z}) + \frac{2}{N} \sum_{i=1}^{N} \| Y_i(0) \|^2_{L^2}. \tag{3.8b}
\]
Proof. Similarly as in the proof of Theorem 3.3 we have

\[(3.8a) + \frac{\lambda}{N^2} \left\| \sum_{i=1}^{N} Y_i^2 \right\|_{L^2}^2 \leq \frac{2}{N} \sum_{j=1}^{N} \| Y_j(0) \|_{L^2}^2 + \lambda(1 + \lambda)Q_N^4 \]

\[+ C\lambda(1 + \lambda^{55}) \int_{0}^{T} \left( \frac{1}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) (R^3_N + R^2_N + Q^3_N) ds + \frac{1}{N} \sum_{j=1}^{N} \| Y_j \|_{L^2}^2 \]

where the last term is bounded by \( \frac{1}{N^2} \| \sum_{i=1}^{N} Y_i^2 \|_{L^2}^2 + C(\lambda) \). Using \( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \geq (\frac{1}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2)^2 \), Young’s inequality and \( E(Q_N^4)^2 \lesssim C(\lambda) \) from Lemma 3.7 below, \( E(R_N^3)^2 + E(R_N^2)^2 \lesssim 1 \) from Lemma 2.2, and moment bound (3.12) for \( Q_N^4 \) below, the result follows. \( \square \)

3.2. Proof of (3.1). We first consider the easier part \( \Theta \) defined in Lemma 2.8.

Proposition 3.5. It holds for \( \delta > 0 \) small that

\[ |\Theta| \leq \delta \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 + C\lambda(1 + \lambda^{4}) \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) R_N^4. \] (3.9)

Here \( C \) is independent of \( \lambda, N \) and \( m \geq 1 \), and for \( \theta = \frac{1+2\kappa}{2-4\kappa} \) we define

\[ R_N^2 \overset{\text{def}}{=} 1 + \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{\nu} \|_{C^{1-\kappa}}^2 \right) + \left( \frac{1}{N} \sum_{j=1}^{N} \| Z_{jj}^{\nu} \|_{C^{1-\kappa}}^2 \right) \]

\[+ \frac{1}{N^3} \sum_{i,j,l=1}^{N} \left( \| Z_{iiij}^{\nu} \|_{C^{1-\kappa}} \right)^2 + \frac{\lambda}{N} \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 + C(\lambda) \].

Proof. We estimate each term in \( \Theta \). By Young’s inequality we will use the first line in \( R_N^2 \) to control the renormalization terms evolving \( Z_{ij}^{\nu} \) below and use the second and the third line in \( R_N^2 \) to bound the renormalization terms containing \( Z_{ij,k}^{\nu} \).

**Step 1** (Estimates of \( D(Y_i, Z_{ij}^{\nu}, Y_j) \) and \( D(Y_i, Z_{ij,k}^{\nu}, Y_j) \)).

We first control the terms containing the commutator \( D \) in \( \Theta \). By Lemma A.10, Hölder’s inequality, and interpolation Lemma A.2, we have

\[ \left| \frac{\lambda}{N} \sum_{i,j=1}^{N} 2D(Y_i, Z_{ij}^{\nu}, Y_j) \right| \lesssim \frac{\lambda}{N} \sum_{i,j=1}^{N} \| Y_i \|_{H^{1-2\kappa}} \| Y_j \|_{C^{1-\kappa}} \| Z_{ij}^{\nu} \|_{H^{1-2\kappa}} \]

\[\leq \frac{\lambda}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{\nu} \|_{C^{1-\kappa}}^2 \left( \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 \right)^{\frac{1}{2}} \]

\[\leq \frac{\lambda}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{\nu} \|_{C^{1-\kappa}}^2 \left( \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 \right)^{\frac{1}{2}} \]

\[\leq \frac{\lambda}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{\nu} \|_{C^{1-\kappa}}^2 \left( \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 \right)^{\frac{1}{2}} \]

\[\leq \delta \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 + C(\lambda) \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{\nu} \|_{C^{1-\kappa}}^2 \right)^{\frac{1}{2}} \]

\[\leq \delta \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 + C(\lambda) \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{\nu} \|_{C^{1-\kappa}}^2 \right)^{\frac{1}{2}} \]
where \( \theta = \frac{1 - 2\kappa}{4\kappa} \in (\frac{1}{2}, \frac{3}{4}) \). Similarly we have

\[
\left| \frac{\lambda}{N} \sum_{i,j,l=1}^{N} D(Y_i, Z_{ij}^Y, Y_j) \right| \lesssim \frac{\lambda}{N} \sum_{i,j,l=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 \| Z_{ij}^Y \|_{C^{-1-\kappa}} \]

\[
\leq \delta \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}} + C_{\delta} \lambda^{\frac{1}{1-2\kappa}} \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) \left( \frac{1}{N} \sum_{j=1}^{N} \| Z_{ij}^Y \|_{C^{-1-\kappa}} \right)^{\frac{1}{1-2\kappa}}.
\]

Therefore the terms containing \( D \) in \( \Theta \) can be controlled by the right hand side of (3.9).

**Step 2** (Estimate of terms in \( (\mathcal{G}^{-1}P_i, P_i) \))

In the following we estimate each term in (2.34). We have three types of terms:

**I.** Terms without \( C \) or \( D \) such as \( \langle \hat{Z}_{ij}^Y, 4Y_i Y_j \rangle \),

**II.** Terms with \( C \), such as \( \langle C(Y_i, Z_{ij}^Y, Z_{ij}^Y), Y_j \rangle \),

**III.** Terms with \( D \), such as \( D(Y_i, Z_{ij}^Y, \mathcal{G}^{-1}P_i) \).

The terms having the same type can be estimated in the same way.

**I.** For the first type we use Lemma A.2 and Lemma A.7 to have

\[
\left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} \langle \hat{Z}_{ij}^Y, 4Y_i Y_j \rangle \right| \lesssim \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}} \| Y_i \|_{H^{2\kappa}} \| Y_j \|_{H^{2\kappa}}
\]

\[
\lesssim \frac{\lambda^2}{N^2} \sum_{i=1}^{N} \left( \frac{1}{N^2} \sum_{j,l=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}} \right)^2 \left( \sum_{j=1}^{N} \| Y_j \|_{H^{2\kappa}}^2 \right)
\]

\[
\leq \delta \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 + C_{\delta} \lambda^{\frac{2}{1-2\kappa}} \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) \left( \frac{1}{N} \sum_{j=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}}^2 \right)^{\frac{1}{1-2\kappa}},
\]

where \( \theta_1 = \frac{2\kappa}{1-2\kappa} \in (0, 1) \) and we used Lemma A.7 and Besov embedding \( H^{2\kappa} \subset B^\kappa_{2,1} \) to have

\[
\| Y_i Y_j \|_{B^\kappa_{2,1}} \lesssim \| Y_i \|_{B^\kappa_{2,1}} \| Y_j \|_{B^\kappa_{2,1}} \lesssim \| Y_i \|_{H^{2\kappa}} \| Y_j \|_{H^{2\kappa}},
\]

in the first inequality. By the exactly same arguments the same bounds hold for

\[
\left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} \langle \hat{Z}_{ij}^Y, 2Y_i Y_j \rangle \right|, \quad \left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} \langle \hat{Z}_{ij}^Y, 2Y_i Y_j \rangle \right|, \quad \left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} \langle \hat{Z}_{ij}^Y, Y_i^2 \rangle \right|
\]

with \( \frac{1}{N^3} \sum_{i,j,l=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}}^2 \) on the right hand side replaced by, respectively,

\[
\frac{1}{N^3} \sum_{i,j,l=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}}^2, \quad \frac{1}{N^2} \sum_{i,j,l=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}}^2, \quad \frac{1}{N^2} \sum_{j=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}}^2.
\]

By Young’s inequality the terms with \( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \) are all bounded by \( \lambda^2(1 + \lambda) \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) R_N^2 \).

**II.** By Lemma A.11, interpolation Lemma A.2 and Hölder inequality,

\[
\left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} \langle C(Y_i, Z_{ij}^Y, Z_{ij}^Y), Y_j \rangle \right|
\]

\[
\lesssim \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} \| Z_{ij}^Y \|_{C^{-1-\kappa}} \| Z_{ij}^Y \|_{C^{-1-\kappa}} \| Y_i \|_{H^{1-2\kappa}} \| Y_j \|_{H^{1-2\kappa}}
\]

\[
\leq \delta \sum_{i=1}^{N} \| Y_i \|_{H^{1-2\kappa}}^2 + C_{\delta} \lambda^{\frac{2}{1-2\kappa}} \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) \left( \frac{1}{N^3} \sum_{i,j,l=1}^{N} \| \hat{Z}_{ij}^Y \|_{C^{-\kappa}}^2 \right)^{\frac{1}{1-2\kappa}}.
\]
where we use similar argument as in (3.10) and by Young’s inequality the last term can be controlled by \( \lambda(1 + \lambda^4) \left( \sum_{i=1}^{N} \| Y_{i} \|_{L^2}^2 \right) R_{N}^2 \).

By the exactly same arguments the same bounds hold for

\[
\left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} 2(C(Y_l, Z_{il}^{Y}, Z_{ij}^{Y}), Y_j) \right|, \quad \left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} 2(C(Y_l, Z_{il}^{Y}, Z_{ij}^{Y}), Y_l) \right|
\]

and

\[
\left| \frac{\lambda^2}{N^2} \sum_{i,j,l=1}^{N} 2(C(Y_l, Z_{il}^{Y}, Z_{ij}^{Y}), Y_l) \right|
\]

with \( \| Z_{il}^{Y} \|_{C^{1,-\kappa}} \| Z_{ij}^{Y} \|_{C^{1,-\kappa}} \) on the right hand side replaced by, respectively,

\[
\| Z_{il}^{Y} \|_{C^{1,-\kappa}} \| Z_{ij}^{Y} \|_{C^{1,-\kappa}}, \quad \| Z_{il}^{Y} \|_{C^{1,-\kappa}} \| Z_{jj}^{Y} \|_{C^{1,-\kappa}}, \quad \| Z_{il}^{Y} \|_{C^{1,-\kappa}} \| Z_{jj}^{Y} \|_{C^{1,-\kappa}}.
\]

As above by Young’s inequality the terms containing \( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \) can be all controlled by \( \lambda(1 + \lambda^4) \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) R_{N}^2 \).

**III.** By Lemma A.10 and setting \( \theta_0 = \frac{4 \kappa}{1 - 2\kappa} \in (0, 1) \), we have

\[
\left| \frac{\lambda}{N^2} \sum_{i,j=1}^{N} -2D(Y_j, Z_{ij}^{Y}, \varphi^{-1} P_l) \right| \lesssim \frac{\lambda}{N} \sum_{i,j=1}^{N} \| Y_j \|_{H^{1+\kappa}} \| Z_{ij}^{Y} \|_{C^{1,-\kappa}} \| \varphi^{-1} P_l \|_{H^{1-2\kappa}}
\]

\[
\lesssim \lambda \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{Y} \|_{C^{1,-\kappa}} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \| Y_j \|_{H^{1+\kappa}} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \| \varphi^{-1} P_l \|_{H^{1-2\kappa}} \right)^{\frac{1}{2}}
\]

\[
\lesssim \lambda^2 \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{Y} \|_{C^{1,-\kappa}} \right) \left( \sum_{j=1}^{N} \| Y_j \|_{H^{1+\kappa}} \| Y_j \|_{L^2}^{2(1-\theta_0)} \right) + \delta \sum_{j=1}^{N} \| \varphi^{-1} P_j \|_{H^{1-2\kappa}}^2
\]

\[
\leq C_\delta \lambda^{1-\frac{4\kappa}{1-2\kappa}} \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^{Y} \|_{C^{1,-\kappa}} \right)^{\frac{1}{1-2\kappa}} + \delta \sum_{j=1}^{N} \left( \| Y_j \|_{H^{1-2\kappa}}^2 + \| \varphi^{-1} P_j \|_{H^{1-2\kappa}}^2 \right),
\]

where we used Young’s inequality and interpolation Lemma A.2 in the third inequality.

Similarly we have

\[
\left| \frac{\lambda}{N} \sum_{i,j=1}^{N} 2D(Y_l, Z_{ij}^{Y}, \varphi^{-1} P_l) \right| \lesssim \frac{\lambda}{N} \sum_{i,j=1}^{N} \| Y_l \|_{H^{1+\kappa}} \| Z_{ij}^{Y} \|_{C^{1,-\kappa}} \| \varphi^{-1} P_l \|_{H^{1-2\kappa}}
\]

\[
\leq C_\delta \lambda^{1-\frac{4\kappa}{1-2\kappa}} \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) \left( \frac{1}{N^2} \sum_{j=1}^{N} \| Z_{lj}^{Y} \|_{C^{1,-\kappa}} \right)^{\frac{1}{1-2\kappa}} + \delta \sum_{j=1}^{N} \left( \| Y_j \|_{H^{1-2\kappa}}^2 + \| \varphi^{-1} P_j \|_{H^{1-2\kappa}}^2 \right),
\]

which implies the result by the first two bounds in Lemma 3.2. \( \square \)

For \( \Xi \), since we will use Lemma 2.7 to bound the \( L^2 \) norm of the stochastic terms, we bound \( \| \Xi \|_{L^2} \).

**Proposition 3.6.** It holds for \( \delta > 0 \) small that

\[
\int_0^T |\Xi| ds \leq \delta \sum_{i=1}^{N} \| \nabla \varphi_i \|_{L^2}^2 + \delta \frac{\lambda}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 + \frac{\lambda}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^2\| \varphi^{-1} P_i \|_{H^{1-2\kappa}} + C\lambda(1 + \lambda^{5\kappa}) \int_0^T \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^2 \right) Q_N^2 ds + \lambda(1 + \lambda)Q_N^4.
\]
Here $C$ is independent of $\lambda, N$ and can be chosen uniform for $m \geq 1$ and

\[ Q_N^3 \overset{\text{def}}{=} R_N^4 + 1 + \left( \frac{1}{N} \sum_{j=1}^{N} \| X_j \|_{C_{\tau}^{\frac{d}{2} - \kappa}}^2 \right)^2 + \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \| X_j Z_i \|_{C_{\tau}^{\frac{d}{2} - \kappa}}^2 \right) \]

\[ + \left( \frac{1}{N} \sum_{j=1}^{N} \| X_j Z_i \|_{C_{\tau}^{\frac{d}{2} - \kappa}}^2 \right) + \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} + \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} \right), \]

with $\theta = \frac{3 + \kappa}{1 + 2\kappa}$ and

\[ Q_N^4 \overset{\text{def}}{=} \left( \sum_{i=1}^{N} \| X_i \|_{L_{\tau}^{2, H^\frac{d}{2} - \kappa}}^2 \right)^2 \left[ 1 + \frac{1}{N^2} \sum_{i,j=1}^{N} 2^{8\kappa L} \left( \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} + \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} \right) \right] \]

\[ + \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} + \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} \right) \]

\[ + \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \sum_{j=1}^{N} \| X_j Z_i \|_{L_{\tau}^{2, H^\frac{d}{2} - \kappa}}^2 \right) \right)^2 \]

\[ + \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \sum_{j=1}^{N} \| X_{ij} Z_j \|_{L_{\tau}^{2, H^\frac{d}{2} - \kappa}}^2 \right) \right)^2 \]

\[ + \left( \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \sum_{j=1}^{N} \| X_j \circ Z_{ij}^Y \|_{L_{\tau}^{2, H^\frac{d}{2} - \kappa}} + \| X_j \circ Z_{ij}^Y \|_{L_{\tau}^{2, H^\frac{d}{2} - \kappa}} \right) \right)^2 \]

(3.11)

where $L$ is chosen as in (2.12) and $R_N^4$ is defined in Lemma 3.8.

Note that with all the estimates obtained above we can easily deduce for any $q \geq 1$

\[ [E(Q_N^q)]^{1/q} \lesssim \lambda^2(1 + \lambda^6). \]  

(3.12)

Indeed, using Lemmas 2.6 and 2.7 we can bound all the terms in $Q_N^3$, that are $L_{\tau}^q$ norms and involving $X$. These bounds together with Lemma 2.2 and the definition of $L$ in (2.12) imply (3.12).

In the following lemma we use the renormalized terms in $Z$ to bound $Q_N^1$, which will be useful in Section 4 because, $Q_N^3$ involves $(X_i)_{i=1}^{N}$ which are not independent for different $i$ whereas for $Q_N^5$ below it will be easier to exploit independence.

**Lemma 3.7.** One has $Q_N^3 \lesssim (1 + \lambda^4)Q_N^5$ uniformly in $\lambda$ and $N$, with

\[ Q_N^5 \overset{\text{def}}{=} R_N^4 + \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} + \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}} \right) + \sum_{i=1}^{3} Q_i^N, \]

(3.13)

where $R_N^4$ is defined in Lemma 3.8, $Q_N^{51}$ is defined in the proof and $Q_N^{52}, Q_N^{53}$ are defined in the proof of Lemma 2.4.

**Proof.** In the proof we bound each term in $Q_N^3$ in terms of the renormalized terms in $Z$. Obviously nothing needs to be done for the term $R_N^4$ in $Q_N^4$. By (2.7) in Lemma 2.3

\[ \left( \frac{1}{N} \sum_{j=1}^{N} \| X_j \|_{C_{\tau}^{\frac{d}{2} - \kappa}}^2 \right)^2 \lesssim \frac{\lambda^4}{N^2} \sum_{i,j=1}^{N} \| Z_{ij}^Y \|_{C_{\tau}^{\frac{d}{2} - \kappa}}^2 \overset{\text{def}}{=} \lambda^4 Q_N^{51}. \]

Thus the result follows from Lemma 2.4. \(\square\)

Now we focus on the estimate of $\Xi$. First we consider the cubic term $(Y_i^2 Y_i, Z_i)$ in $\Xi$. Comparing to the dynamical $\Phi_3^1$ model, the dissipation from $\frac{1}{N} \sum_{i=1}^{N} Y_i^2 \| Y_i \|_{L_2}$ is weaker, which requires further decomposition and more delicate estimates. Also unlike 2D case, the best regularity for $Y_i$ is $H^{1-}$ ($H^1$ in 2D case). Here we decompose $Y_i$ as $\varphi_i$ (having better regularity) and $D^{-1} P_i$ (bound of which only need $L^2$-norm of $Y_i$, see Lemma 3.2). For the most complicated terms (see Step 3 in the following
proof) we also need to decompose $Z_i$ by localization operator and choosing $L$ to balance the competing contributions.

**Lemma 3.8.** It holds for $\delta > 0$ small that

$$
\left| \frac{\lambda}{N} \sum_{i,j=1}^{N} \langle Y_j Y_i, Z_i \rangle \right| \leq \delta \sum_{i=1}^{N} \| \varphi_i \|_{H^1}^2 + \delta \frac{\lambda}{N} \sum_{i=1}^{N} Y_i^2_{L^2} + \delta \sum_{i=1}^{N} \| Y_i \|_{H^{1-\kappa}}^2 \\
+ C_3 \lambda (1 + \lambda^{55}) \left( \sum_{j=1}^{N} \| Y_j \|_{L^2}^2 \right) R_N^4,
$$

(3.14)

where $C_3$ is independent of $\lambda, N$ and can be chosen uniform for $m \geq 1$ and

$$
R_N^4 = (R_N^1)^{25} \left( \frac{1}{N} \sum_{j=1}^{N} \| Z_j \|_{C^{-\frac{1}{2} - \kappa}}^2 + 1 \right)^9.
$$

**Proof.** We use (2.26) to get a decomposition for $Y_i$ and we have

$$
\frac{\lambda}{N} \sum_{i,j=1}^{N} \langle Y_j^2 Y_i, Z_i \rangle = \frac{\lambda}{N} \sum_{i,j=1}^{N} \left[ \langle \varphi_j^2 \varphi_i, Z_i \rangle + \langle [\mathcal{G}^{-1} P_j]^2 Y_i, Z_i \rangle - 2 \langle [\mathcal{G}^{-1} P_j] \varphi_j \varphi_i, Z_i \rangle \\
- \langle \varphi_j^2 [\mathcal{G}^{-1} P_j], Z_i \rangle + 2 \langle [\mathcal{G}^{-1} P_j] \varphi_j [\mathcal{G}^{-1} P_j], Z_i \rangle \right].
$$

In the following we show that each term can be bounded by the RHS of (3.14).

**Step 1** (Estimate of $\langle \varphi_j^2 \varphi_i, Z_i \rangle$.)

In this step we use Lemma A.8 with $s = \frac{1}{2} + \kappa$ to have

$$
\left| \frac{\lambda}{N} \sum_{i,j=1}^{N} \langle \varphi_j^2 \varphi_i, Z_i \rangle \right| = \left| \frac{\lambda}{N} \sum_{i,j=1}^{N} \langle \varphi_i^2 \varphi_j, Z_j \rangle \right|
$$

$$
\lesssim \frac{\lambda}{N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \| \nabla (\varphi_i^2 \varphi_j) \|_{L^1} \right)^{\frac{1}{\kappa}} \| \varphi_i^2 \varphi_j \|_{L^1}^{\frac{1}{\kappa}} + \| \varphi_i^2 \varphi_j \|_{L^1} \right) \| Z_j \|_{C^{-\frac{1}{2} - \kappa}} \right) \| Z_j \|_{C^{-\frac{1}{2} - \kappa}} 
$$

(3.15)

where we used Hölder’s inequality for the summation in $i$ in the last step. By Hölder’s inequality it holds that

$$
\sum_{i=1}^{N} \| \varphi_i^2 \varphi_j \|_{L^1} \leq \sum_{i=1}^{N} \| \varphi_i \|_{L^2} \| \varphi_j \|_{L^2} \leq \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2}^2 \right)^{\frac{1}{2}},
$$

(3.16)

and

$$
\sum_{i=1}^{N} \| \nabla (\varphi_i^2 \varphi_j) \|_{L^1} \lesssim \sum_{i=1}^{N} \| \varphi_i \varphi_j \nabla \varphi_i \|_{L^1} + \| \varphi_i^2 \nabla \varphi_j \|_{L^1} \\
\lesssim \left( \sum_{i=1}^{N} \| \varphi_i \varphi_j \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{H^1} \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{N} \| \varphi_i^2 \|_{L^2} \right)^{\frac{1}{2}} \| \nabla \varphi_j \|_{L^2}.
$$

(3.17)

Substituting (3.16)–(3.17) into (3.15) we obtain that (3.15) is bounded by

$$
\frac{\lambda}{N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \| \varphi_i \varphi_j \|_{L^2} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{H^1} \right)^{\frac{1}{2} - \kappa} \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2} \right)^{\frac{1}{2} - \kappa} \| Z_j \|_{C^{-\frac{1}{2} - \kappa}}
$$
\[ + \frac{\lambda}{N} \sum_{j=1}^{N} \left\| \sum_{i=1}^{N} \varphi_i^2 \right\|_{L^2}^{\frac{1}{4}+\kappa} \| \nabla \varphi_j \|_{L^2}^{\frac{1}{4}+\kappa} \left( \sum_{i=1}^{N} \| \varphi_i \varphi_j \|_{L^2}^{2} \right)^{\frac{1+2\kappa}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2}^{2} \right)^{\frac{1-2\kappa}{2}} \| Z_j \|_{C^{-\frac{1}{2}-\kappa}} \]
\[ + \frac{\lambda}{N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \| \varphi_i \varphi_j \|_{L^2}^{2} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2}^{2} \right)^{\frac{1}{2}} \| Z_j \|_{C^{-\frac{1}{2}-\kappa}}, \]

which by Hölder’s inequality for the summation in \( j \) is bounded by

\[ \lambda \left( \frac{1}{N} \sum_{i,j=1}^{N} \| \varphi_i \varphi_j \|_{L^2}^{2} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2}^{2} \right)^{\frac{1+2\kappa}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2}^{2} \right)^{\frac{1-2\kappa}{2}} \left( \frac{1}{N} \sum_{j=1}^{N} \| Z_j \|_{C^{-\frac{1}{2}-\kappa}}^{2} \right)^{\frac{1}{2}} \]
\[ + \lambda \left( \frac{1}{N} \sum_{i,j=1}^{N} \| \varphi_i \varphi_j \|_{L^2}^{2} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \| \varphi_i \|_{L^2}^{2} \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{j=1}^{N} \| Z_j \|_{C^{-\frac{1}{2}-\kappa}}^{2} \right)^{\frac{1}{2}} (1 + \lambda^2 R_N^4)^{3/2} \]
\[ \leq \frac{\delta \lambda}{N} \sum_{i=1}^{N} \| Y_i \|_{L^2}^{2} + C_\delta \lambda (1 + \lambda^2)^3 \left( \sum_{i=1}^{N} \| Y_i \|_{L^2}^{2} \right) \left( \frac{1}{N} \sum_{j=1}^{N} \| Z_j \|_{C^{-\frac{1}{2}-\kappa}}^{2} \right) (R_N^4)^3, \]

and the first term is bounded in the same way.

**Step 2** (Estimates of \( \langle [\mathcal{D}^{-1} P_j]^2 Y_i, Z_i \rangle \) and \( \langle [\mathcal{D}^{-1} P_j] \varphi_j [\mathcal{D}^{-1} P_i], Z_i \rangle \))

We use (A.2) and Lemma A.1 and Sobolev embedding \( H^\frac{2}{3} \subset L^4 \) to have

\[ \| fg \|_{B_{\frac{1}{2},1}^{\frac{2}{3}}} \lesssim \| f \|_{L^2} \| g \|_{H^{\frac{1}{2}}} + \| fg \|_{B_{\frac{1}{2},1}^{\frac{2}{3}}} \| h \|_{L^4} \lesssim \| f \|_{H^{1-2\kappa}} \| g \|_{H^{1-2\kappa}} \| h \|_{H^{\frac{1}{2}}}, \]

which combined with (ii) in Lemma A.7 gives

\[ \left| \frac{\lambda}{N} \sum_{i,j=1}^{N} \langle [\mathcal{D}^{-1} P_j]^2 Y_i, Z_i \rangle \right| \lesssim \frac{\lambda}{N} \sum_{i,j=1}^{N} \| [\mathcal{D}^{-1} P_j]^2 Y_i \|_{B_{\frac{1}{2},1}^{\frac{2}{3}}} \| Z_i \|_{C^{-\frac{1}{2}-\kappa}} \]
\[ \lesssim \frac{\lambda}{N} \sum_{i,j=1}^{N} \| [\mathcal{D}^{-1} P_j]^2 \|_{H^{1-2\kappa}} \| Y_i \|_{H^{\frac{1}{2}}} \| Z_i \|_{C^{-\frac{1}{2}-\kappa}} \]
\[ \lesssim \frac{\lambda}{\sqrt{N}} \left( \sum_{j=1}^{N} \| Y_j \|_{L^2}^{2} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \| Y_j \|_{H^{1-2\kappa}}^{2} \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{j=1}^{N} \| Z_j \|_{C^{-\frac{1}{2}-\kappa}}^{2} \right)^{\frac{1}{2}} \]
\[ \leq C_\delta \lambda^{\frac{1-2\kappa}{2}} (1 + R_N^4)^{\frac{1}{2}} \left( 1 + \frac{1}{N} \sum_{j=1}^{N} \| Z_j \|_{C^{-\frac{1}{2}-\kappa}}^{2} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N} \| Y_j \|_{L^2}^{2} \right)^{\frac{1}{2}} \]
\[ + \delta \lambda \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| Y_i \|_{L^2}^{2} \right)^{\frac{1}{2}} + \delta \left( \sum_{j=1}^{N} \| Y_j \|_{H^{1-2\kappa}}^{2} \right), \]

(3.19)
where \( \theta = \frac{3}{4(1-2\kappa)} \) and we used Lemma 3.2 to bound \( \| \mathcal{D}^{-1} P_j \|_{H^{1-2\kappa}}^2 \) in the third inequality and (3.7) in the last inequality. Similarly we use (3.18) and Lemma 3.2 to have

\[
\frac{\lambda}{N} \sum_{i,j=1}^N \langle (\mathcal{D}^{-1} P_j) \varphi_j, [\mathcal{D}^{-1} P_i], Z_i \rangle 
\leq \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i^2 \right\|_{L^2}^2 \lambda^2 R_N^4 \left( \sum_{j=1}^N \| \varphi_j \|_{H^1}^2 \right)^{\frac{3}{2}} \left( \sum_{j=1}^N \| \varphi_j \|_{L^4}^4 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{j=1}^N \| Z_j \|_{C^{1+3\kappa}}^2 \right)^{\frac{1}{2}}
\leq C_\delta \lambda^\frac{5}{16} (1 + \lambda^2 R_N^4)^{\frac{5}{8}} \left( \frac{1}{N} \sum_{j=1}^N \| Z_j \|_{C^{\frac{1}{2}+\delta}}^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^N \| Y_j \|_{L^2}^4 \right)^{\frac{1}{2}} + \delta \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i^2 \right\|_{L^2}^2 + \delta \left( \frac{1}{N} \sum_{j=1}^N \| \varphi_j \|_{H^1}^2 \right).
\tag{3.20}
\]

**Step 3 ( Estimates for \( \langle (\mathcal{D}^{-1} P_j) \varphi_j \varphi_i, Z_i \rangle \) and \( \langle (\mathcal{D}^{-1} P_i) \varphi_j^2, Z_i \rangle \)**

For the last two terms we use the localization operator \( \mathcal{W}_\sigma \) and \( \mathcal{W}_{\leq} \) introduced in (2.5) to separate \( Z_i = \mathcal{W}_\sigma Z_i + \mathcal{W}_{\leq} Z_i \) with \( L \) chosen below. By the definition of \( \mathcal{W}_\sigma \) and \( \mathcal{W}_{\leq} \) we know

\[
\| \mathcal{W}_\sigma Z_i \|_{C^{1+3\kappa}} \lesssim \| Z_i \|_{C^{\frac{1}{2}+\delta}} 2^{\left( -\frac{1}{2} + 4\kappa \right) L}, \quad \| \mathcal{W}_{\leq} Z_i \|_{L^\infty} \lesssim \| Z_i \|_{C^{\frac{1}{2}+\delta}} 2^{\left( \frac{1}{2} + 2\kappa \right) L},
\tag{3.21}
\]

Using (ii) in Lemma A.7 followed by (A.2) and Lemma A.1 and \( H^{1-2\kappa} \subset H^{\frac{3}{2}} \subset L^4 \), we find

\[
\frac{\lambda}{N} \sum_{i,j=1}^N \langle (\mathcal{D}^{-1} P_j) \varphi_j \varphi_i, \mathcal{W}_\sigma Z_i \rangle \lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \| \varphi_i \varphi_j \|_{H^{1-2\kappa}} \| \mathcal{D}^{-1} P_j \|_{H^{1-2\kappa}} \| \mathcal{W}_\sigma Z_i \|_{C^{1+3\kappa}}
\leq \lambda \left( \sum_{i=1}^N \| \varphi_i \|_{H^1}^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{i=1}^N \| \varphi_i \|_{H^\frac{3}{2}}^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{j=1}^N \| \mathcal{W}_\sigma Z_i \|_{C^{1+3\kappa}}^2 \| \mathcal{D}^{-1} P_j \|_{H^{1-2\kappa}}^2 \right)^{\frac{1}{2}}
\leq \lambda \left( \frac{1}{N} \sum_{i=1}^N \| \varphi_i \|_{H^1}^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{i=1}^N \| \varphi_i \|_{H^\frac{3}{2}}^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^N \| Z_j \|_{C^{\frac{1}{2}+\delta}}^2 \right)^{\frac{1}{2}} 2^{\left( -\frac{1}{2} + 4\kappa \right) L} (1 + \lambda^2 R_N^4)^{\frac{5}{8}},
\tag{3.22}
\]

where we used Lemma A.2, Lemma 3.2 and (3.21) in the last step. By Hölder’s inequality, (3.21) and Lemma 3.2 we obtain that

\[
\frac{\lambda}{N} \sum_{i,j=1}^N \langle (\mathcal{D}^{-1} P_j) \varphi_j \varphi_i, \mathcal{W}_{\leq} Z_i \rangle \leq \frac{\lambda}{N} \sum_{i,j=1}^N \| \mathcal{D}^{-1} P_j \|_{L^2} \| \varphi_j \varphi_i \|_{L^2} \| \mathcal{W}_{\leq} Z_i \|_{L^\infty}
\leq \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N \varphi_i^2 \right\|_{L^2}^2 \left( \sum_{j=1}^N \| \mathcal{D}^{-1} P_j \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{j=1}^N \| Z_j \|_{C^{\frac{1}{2}+\delta}}^2 \right)^{\frac{1}{2}} 2^{\left( \frac{1}{2} + 2\kappa \right) L}
\leq \lambda \left( \frac{1}{N} \sum_{i=1}^N \| \varphi_i \|_{H^1}^2 \right)^{\frac{1}{2}} \left( 1 + \lambda^2 R_N^4 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{j=1}^N \| Z_j \|_{C^{\frac{1}{2}+\delta}}^2 \right)^{\frac{1}{2}} \left( 1 + \lambda^2 R_N^4 \right)^{\frac{1}{2}} 2^{\left( \frac{1}{2} + 2\kappa \right) L}.
\tag{3.23}
\]

Now we choose \( L \) to balance the competing contributions:

\[
2^{L\left( -\frac{1}{2} + 4\kappa \right)} = \left( \sum_{j=1}^N \| Y_j \|_{L^2}^4 \right)^{\frac{1}{7}} \left( \sum_{i=1}^N \| \varphi_i \|_{H^1}^2 \right)^{-\frac{7}{16}}.
\]
This choice of $L$ leads to
\[
\left| \frac{\lambda}{N} \sum_{i,j=1}^{N} (\mathcal{P}_{ij}^{-1} \varphi_j \varphi_i, Z_i) \right| \leq \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i \right\|_{L^2} \left( \frac{1}{N} \sum_{j=1}^{N} \left\| Z_j \right\|_{C^k} \right)^{1/2} \\
\times \left[ \left( \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2} \right)^{2/5} \left( \sum_{j=1}^{N} \left\| \varphi_j \right\|_{\mathcal{H}^1} \right)^{7/5} (1 + \lambda^2 R_N^1)^{2/5} \right] \\
+ \left( \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2} \right)^{2/5} \left( \sum_{j=1}^{N} \left\| \varphi_j \right\|_{\mathcal{H}^1} \right)^{7/5} (1 + \lambda^2 R_N^1)^{2/5} \right] \\
\leq C_6 \lambda (1 + \lambda^{55}) (R_N^1)^2 \left( 1 + \frac{1}{N} \sum_{j=1}^{N} \left\| Z_j \right\|_{\mathcal{H}^1} \right)^9 \left( \sum_{j=1}^{N} \left\| Y_j \right\|_{L^2} \right)^{2} \\
+ \delta \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i \right\|_{L^2}^2 + \delta \left( \sum_{j=1}^{N} \left\| \varphi_j \right\|_{\mathcal{H}^1} \right)^2 \right),
\] (3.24)
where $a = \frac{1}{2} + \frac{28}{5}$ and the second line comes from (3.23) and the third line corresponds to (3.22).

Similar argument also implies that $\left| \frac{\lambda}{N} \sum_{i,j=1}^{N} (\mathcal{P}_{ij}^{-1} \varphi_j \varphi_i, Z_i) \right|$ can be bounded by the right hand side of (3.24). Combining the above estimates we obtain the result. \hfill \Box

**Proof of Proposition 3.6.** The desired estimate for $\frac{\lambda}{N} \sum_{i,j=1}^{N} (Y_i^2 Y_i, Z_i)$ has been deduced in Lemma 3.8. It remains to consider the other terms in $\Xi$. We write the remaining terms in $\Xi$ as

\[
I_i^N \overset{\text{def}}{=} - \frac{\lambda}{N} \sum_{i,j=1}^{N} \left\langle (X_j^2 + 2X_j Y_j)(X_i + Y_i) + Y_j^2 X_i, Y_i \right\rangle, \\
I_2^N \overset{\text{def}}{=} - \frac{\lambda}{N} \sum_{i,j=1}^{N} \left\langle (X_j^2 + 2X_j Y_j)Z_i + 2(X_j + Y_j)X_i Z_j + 2X_j Y_i Z_j, Y_i \right\rangle, \\
I_3^N \overset{\text{def}}{=} - \frac{\lambda}{N} \sum_{i,j=1}^{N} \left\langle X_{[j]} - X_{[j]} + (X + Y)_{[j]} - Z_{[j]} + X_{[j]} \circ Z_{[j]} + Y_{[j]} \right\rangle.
\]

**Step 1 (Estimate of $I_1^N$)**

By symmetry with respect to $i$ and $j$ we write $I_1^N$ as

\[
I_1^N = - \frac{\lambda}{N} \sum_{i,j=1}^{N} \left\langle X_j^2 X_i, Y_i \right\rangle - \frac{\lambda}{N} \sum_{i,j=1}^{N} \left\langle X_j^2 Y_i, Y_i \right\rangle - \frac{2\lambda}{N} \sum_{i,j=1}^{N} \left\langle X_j X_i Y_j, Y_i \right\rangle \\
- \frac{3\lambda}{N} \sum_{i,j=1}^{N} \left\langle X_j Y_j, Y_i^2 \right\rangle \overset{\text{def}}{=} 4 \sum_{k=1}^{4} I_{1k}^N,
\]

and we will prove that for every $k = 1, \ldots, 4$,

\[
\left\| I_{1k}^N \right\|_{L^1_k} \leq C_6 \lambda \left( \sum_{i=1}^{N} \left\| X_i \right\|^2_{\mathcal{H}^{1/2} - 2k} \right) + \delta \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i \right\|_{L^2}^2 \\
+ C_6 \lambda \left( \frac{1}{N} \sum_{j=1}^{N} \left\| Y_j \right\|^2_{C^k, \mathcal{H}^{1/2} - 2k} + 1 \right)^2 \left( \sum_{i=1}^{N} \left\| Y_i \right\|_{L^2}^{2} \right)^2.
\] (3.25)
We consider each term separately. For $I_{11}^N$, applying Hölder’s inequality we have
\[ \|X_j^2 X_i Y_i\|_{L^1_L} \lesssim \int_0^T \|X_j\|_{L^\infty}^2 \|X_i\|_{L^2} \|Y_i\|_{L^2} ds \lesssim \|X_j\|_{L^\infty}^2 \|X_i\|_{L^2} \|Y_i\|_{L^2}, \]
so by embedding $C^{\frac{2}{N}-\epsilon} \subset L^\infty$ we have
\[
\|I_{11}^N\|_{L^1_L} \lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \|X_j\|_{C_T C^{\frac{2}{N}-\epsilon}}^2 \|X_i\|_{L^2} \|Y_i\|_{L^2},
\]
which by Young’s inequality gives (3.25) for $I_{11}^N$. Applying Hölder’s inequality in a similar way as above we find
\[
\|I_{12}^N\|_{L^1_L} \lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \|X_j\|_{C_T C^{\frac{2}{N}-\epsilon}}^2 \|Y_i\|_{L^2},
\]
\[
\|I_{13}^N\|_{L^1_L} \lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \|X_j\|_{C_T C^{\frac{2}{N}-\epsilon}} \|X_i\|_{C_T C^{\frac{2}{N}-\epsilon}} \|Y_i\|_{L^2} \|Y_j\|_{L^2},
\]
which give (3.25) for $I_{12}^N$ and $I_{13}^N$. Moreover by Hölder’s inequality we have
\[
\|I_{14}^N\|_{L^1_L} \lesssim \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i^2 \right\|_{L^2 L^2} \left( \sum_{j=1}^N \|Y_j\|_{L^2}^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{j=1}^N \|X_j\|_{C_T C^{\frac{2}{N}-\epsilon}}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \delta \lambda \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i^2 \right\|_{L^2 L^2} + \lambda \left( \sum_{i=1}^N \|Y_i\|_{L^2} \right) \left( \frac{1}{N} \sum_{j=1}^N \|X_j\|_{C_T C^{\frac{2}{N}-\epsilon}}^2 \right),
\]
which implies (3.25).

**Step 2** (Estimate of $I_{2}^N$) By symmetry with respect $i$ and $j$ we write $I_{2}^N$ as
\[
I_{2}^N = -\frac{\lambda}{N} \sum_{i,j=1}^N \langle X_j^2 Z_i, Y_i \rangle - \frac{4\lambda}{N} \sum_{i,j=1}^N \langle X_j Y_j Z_i, Y_i \rangle - \frac{2\lambda}{N} \sum_{i,j=1}^N \langle X_j X_i Z_j, Y_i \rangle
\]
\[
- \frac{2\lambda}{N} \sum_{i,j=1}^N \langle X_j Y_i Z_j, Y_i \rangle \overset{\text{def}}{=} \sum_{i=1}^N I_{2i}^N.
\]
In the following we show that the $L^1_T$-norm of each term can be bounded by
\[
\delta \left( \sum_{i=1}^N \|Y_i\|_{L^2_H} \right) + C_\delta \lambda^2 Q_N^{41}
\]
\[
+ C_\delta \lambda^2 \left( \sum_{j=1}^N \|Y_j\|_{L^2} \right)^2 \left[ \left( \frac{1}{N^2} \sum_{i,j=1}^N \|X_j Z_i\|_{C_T C^{\frac{2}{N}-\epsilon}}^2 \right) + \left( \frac{1}{N} \sum_{j=1}^N \|X_j Z_j\|_{C_T C^{\frac{2}{N}-\epsilon}}^2 \right) \right],
\]
with $Q_N^{\frac{1}{2}}$ given by the third line in the definition of $Q_N^i$ in (3.11). Using (ii) in Lemma A.7 followed by Hölder’s inequality for the summation over $i$ we have
\[
\|I_{21}^N\|_{L^4_N} \lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \|X_j^2 Z_i\|_{L^{\frac{3}{2},H^{-\frac{1}{2}}} \to L^{2,H^{1-2\kappa}}} \|Y_i\|_{L^{2,H^{1-2\kappa}}}
\]
\[
\lesssim \lambda \left( \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \|X_j^2 Z_i\|_{L^{\frac{3}{2},H^{-\frac{1}{2}}} \to L^{2,H^{1-2\kappa}}})^2 \left( \sum_{i=1}^N \|Y_i\|_{L^{2,H^{1-2\kappa}}} \right)^\frac{1}{2},
\]
which by Young’s inequality gives the desired bound for $\|I_{21}^N\|_{L^4_N}$. Similarly,
\[
\|I_{23}^N\|_{L^4_N} \lesssim \lambda \left( \frac{1}{N^2} \sum_{i,j=1}^N \|X_j Z_i\|_{L^{\frac{3}{2},C^{\frac{1}{2} - \kappa}}}^2 \left( \sum_{i,j=1}^N \|Y_i Y_j\|_{L^{2,H^{1-2\kappa}}} \left( \sum_{i=1}^N \|Y_i\|_{L^{2,H^{1-2\kappa}}} \right)^\frac{1}{2},
\]
which implies the bound for $\|I_{23}^N\|_{L^4_N}$. Using (ii) in Lemma A.7 followed by Hölder’s inequality for the summation over $i,j$ and (A.2), we deduce
\[
\|I_{22}^N\|_{L^4_N} \lesssim \lambda \left( \frac{1}{N^2} \sum_{i,j=1}^N \|X_j Z_i\|_{L^{\frac{3}{2},C^{\frac{1}{2} - \kappa}}}^2 \left( \sum_{i,j=1}^N \|Y_i Y_j\|_{L^{2,H^{1-2\kappa}}} \left( \sum_{i=1}^N \|Y_i\|_{L^{2,H^{1-2\kappa}}} \right)^\frac{1}{2},
\]
which by Young’s inequality gives the required bound for $\|I_{22}^N\|_{L^4_N}$. Similarly we have
\[
\|I_{34}^N\|_{L^4_N} \lesssim \lambda \left( \frac{1}{N} \sum_{j=1}^N \|X_j Z_i\|_{L^{3/2, C^{-\frac{1}{2} - \kappa}}}^2 \left( \sum_{j=1}^N \|Y_j^2\|_{L^{2,H^{1-2\kappa}}} \left( \sum_{i=1}^N \|Y_i\|_{L^{2,H^{1-2\kappa}}} \right)^\frac{1}{2},
\]
which implies the bound for $I_{34}^N$.

**Step 3** (Estimate of $I_3^N$) We write $I_3^N = \sum_{i=1}^N I_{3i}^N$ with
\[
I_{31}^N \overset{\text{def}}{=} -\frac{\lambda}{N} \sum_{i,j=1}^N \langle X_{[i]} \prec \mathcal{W}_N \mathcal{Z}_{[i],j}, Y_i \rangle, \quad I_{32}^N \overset{\text{def}}{=} -\frac{\lambda}{N} \sum_{i,j=1}^N \langle X_{[i]} \succ \mathcal{Z}^N_{[i],j}, Y_i \rangle,
\]
\[
I_{33}^N \overset{\text{def}}{=} -\frac{\lambda}{N} \sum_{i,j=1}^N \langle X_{[i]} \circ \mathcal{Z}^N_{[i],j}, Y_i \rangle, \quad I_{34}^N \overset{\text{def}}{=} -\frac{\lambda}{N} \sum_{i,j=1}^N \langle Y_{[j]} \succ \mathcal{Z}^N_{[j],i}, Y_i \rangle.
\]

In the following we bound the $L^4_N$-norm of each term by
\[
\delta \left( \sum_{i=1}^N \|Y_i\|_{L^{2,H^{1-2\kappa}}}^2 \right) + \lambda^2 Q_N^{\frac{1}{2}}
\]
\[
+ \lambda \theta \left( \sum_{j=1}^N \|Y_j\|_{L^{2,H^{1-2\kappa}}}^2 \right) \left( \frac{1}{N^2} \sum_{i,j=1}^N \left( \|\mathcal{Z}^N_{ij}\|_{C_T^{1+\kappa}} + \|\mathcal{Z}^N_{ji}\|_{C_T^{1+\kappa}} \right) \right)
\]
with $\theta = \frac{1 + \kappa}{1 - 2\kappa}$. Recall that a product with the $[i],[j]$ notation expands into two terms. By definition of $\mathcal{W}_N$ we have
\[
\|\mathcal{W}_N \mathcal{Z}^N_{ij}\|_{C_T^{1+\kappa}} \lesssim 2^{4 \kappa L} \|\mathcal{Z}^N_{ij}\|_{C_T^{1+\kappa}},
\]
\[
\|\mathcal{Z}^N_{ij}\|_{C_T^{1+\kappa}} \lesssim 2^{4 \kappa L} \|\mathcal{Z}^N_{ij}\|_{C_T^{1-\kappa}},
\]
with $L$ given in (2.12), which combined with Lemma A.6 implies that for the first term in $I_{31}^N$,

$$
\int_0^T \left| \frac{\lambda}{N} \sum_{i,j=1}^N \langle X_j \prec Y_i \rangle \right| ds 
\lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \| X_j \|_{L^2 H^{\frac{1}{2} - 2\kappa}} \| Y_i \|_{L^2 H^{1 - 2\kappa}} 
\lesssim 2^{4\kappa} \lambda \left( \frac{1}{N^2} \sum_{i,j=1}^N \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}} \right)^{\frac{\theta}{2}} \left( \sum_{i,j=1}^N \| X_j \|_{L^2 H^{\frac{1}{2} - 2\kappa}} \| Y_i \|_{L^2 H^{1 - 2\kappa}} \right)^{\frac{\theta}{2}}.
$$

Similarly, for the second term in $I_{31}^N$,

$$
\int_0^T \left| \frac{\lambda}{N} \sum_{i,j=1}^N \langle X_i \prec Y_j \rangle \right| ds 
\lesssim 2^{4\kappa} \lambda \left( \frac{1}{N} \sum_{i,j=1}^N \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}} \right)^{\frac{\theta}{2}} \left( \sum_{i=1}^N \| X_i \|_{L^2 H^{\frac{1}{2} - 2\kappa}} \| Y_i \|_{L^2 H^{1 - 2\kappa}} \right)^{\frac{\theta}{2}},
$$

which by Young’s inequality shows that $\| I_{31}^N \|_{L^1_T}$ is bounded by the first line of $Q_N^1$ in (3.11) multiplied by $\lambda^2$ and $\delta \sum_{i=1}^N \| Y_i \|_{L^2 H^{1 - 2\kappa}}^2$. Moreover, by Lemma A.6 and Lemma A.7 we have

$$
\| I_{32}^N \|_{L^1_T} \lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \| X[j] \|_{L^2 H^{\frac{1}{2} - 2\kappa}} \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}} \| Y[i] \|_{L^2 H^{1 - 2\kappa}} 
\lesssim \lambda \left[ \left( \frac{1}{N} \sum_{i,j=1}^N \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}}^2 \right)^{\frac{\theta}{2}} + \left( \frac{1}{N} \sum_{j=1}^N \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}}^2 \right)^{\frac{\theta}{2}} \right]
\times \left( \sum_{i=1}^N \| X[i] \|_{L^2 H^{\frac{1}{2} - 2\kappa}} \right)^{\frac{\theta}{2}} \left( \sum_{i=1}^N \| Y[i] \|_{L^2 H^{1 - 2\kappa}}^2 \right)^{\frac{\theta}{2}},
$$

which by Young’s inequality shows that $\| I_{32}^N \|_{L^1_T}$ is bounded by the second line of $Q_N^1$ in (3.11) multiplied by $\lambda^2$ and $\delta \sum_{i=1}^N \| Y[i] \|_{L^2 H^{1 - 2\kappa}}^2$. Also we have

$$
\| I_{33}^N \|_{L^1_T} \lesssim \frac{\lambda}{N} \sum_{i=1}^N \| X[i] \| \circ \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{L^2 H^{\frac{1}{2} - 2\kappa}} \| Y[i] \|_{L^2 H^{1 - 2\kappa}} 
\lesssim \lambda \left[ \left( \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \| X[i] \| \circ \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{L^2 H^{\frac{1}{2} - 2\kappa}} \right)^2 \right)^{\frac{\theta}{2}} \left( \sum_{i=1}^N \| Y[i] \|_{L^2 H^{1 - 2\kappa}}^2 \right)^{\frac{\theta}{2}},
$$

which by Young’s inequality shows that $\| I_{33}^N \|_{L^1_T}$ is bounded by the last line of $Q_N^1$ in (3.11) multiplied by $\lambda^2$ and $\delta \sum_{i=1}^N \| Y[i] \|_{L^2 H^{1 - 2\kappa}}^2$.

Furthermore, using Lemma A.6 and Lemma A.7 (ii),

$$
\| I_{34}^N \|_{L^1_T} \lesssim \frac{\lambda}{N} \sum_{i,j=1}^N \| Y[i] \|_{L^2 H^{\frac{1}{2} + \kappa}} \| Y[i] \|_{L^2 H^{\frac{1}{2} + \kappa}} \left( \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}} + \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}} \right)
\lesssim \lambda \left( \frac{1}{N^2} \sum_{i,j=1}^N \left( \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}} + \| \mathcal{Z}^{\mathcal{Y}_{ij}} \|_{C_{\mathcal{T}} C^{-1 - \kappa}} \right) \right)^{\frac{\theta}{2}} \left( \sum_{i=1}^N \| Y[i] \|_{L^2 H^{1 - 2\kappa}}^2 \right)^{\frac{\theta}{2}} \left( \sum_{i=1}^N \| Y[i] \|_{L^2}^2 \right)^{1-\theta},
$$
with \( \theta = \frac{4 + \kappa}{1 - 2\kappa} \), where we used Hölder inequality for the sum over \( i \) and \( j \). Thus the required bound for \( \| I_{34}^N \|_{L^1_\theta} \) follows by Young’s inequality.

Proposition 3.6 now follows from (3.25), (3.26) and (3.27) and Lemma 3.8. \( \square \)

4. Convergence of Measures and Tightness of Observables

Now we return to the measure \( \nu^N \) in (1.1) which is invariant under (1.2). In fact, by standard argument (c.f. [HM18b]) the solutions \( (\Phi_i)_{1 \leq i \leq N} \) to (2.24) form a Markov process on \((C^{-\frac{1}{2} - \kappa})^N\) which, by strong Feller property in [HM18b] and irreducibility in [HS22], will turn out to admit a unique invariant measure.

One goal in this section is to show that for sufficiently large mass or small \( \lambda \), in the limit \( N \to \infty \), the marginals of \( \nu^N \) are simply products of the Gaussian measure \( \nu \), which is the unique invariant measure to the linear dynamics (2.2). Furthermore, the convergence rate is given in Theorem 4.6, which can be seen as a very non elementary version of the Projective Central Limit Theorem. The idea in this section follows similarly as in our previous work [SSZZ22, Sections 5-6.1]. Since the 3D case is far more complicated than its counterpart in 2D, our work heavily relies on the computations from Section 2.2 to Section 3 for the remainder \( X \) and \( Y \), but we leverage these estimates with consequences of stationarity.

**Remark 4.1.** Using lattice approximations (see e.g. [GH21], [HM18a, ZZ18]) one can show that the measure \( d\nu^N(\Phi) \) indeed has the form (1.1), with suitable renormalization that is consistent with (2.1), i.e. formally \(^5\) it is

\[
\exp \left( -\int \sum_{j=1}^N |\nabla \Phi_j|^2 + \left( m - \frac{N+2}{N} \lambda a_x + \frac{3(N+2)}{N^2} \lambda^2 \gamma \right) \sum_{j=1}^N \Phi_j^2 + \frac{\lambda}{2N} \left( \sum_{j=1}^N \Phi_j^2 \right)^2 dx \right) d\Phi
\]
divided by a normalization constant.

4.1. Tightness of the measure. It will be convenient to have a stationary coupling of the linear and non-linear dynamics (2.2) and (2.24), which is stated in the following lemma and the proof follows essentially the same as in [SSZZ22, Lemma 5.7].

**Lemma 4.2.** For \((m, \lambda) \in (0, \infty) \times [0, \infty)\), there exists a unique invariant measure \( \nu^N \) on \((C^{-\frac{1}{2} - \kappa})^N\) to (2.24). Furthermore, there exists a stationary process \( (\Phi_i^N, Z_i)_{1 \leq i \leq N} \) such that the components \( \Phi_i^N, Z_i \) are stationary solutions to (2.24) and (2.2) respectively and

\[
\mathbb{E}\| \Phi_i^N(0) - Z_i(0) \|^2_{L^2} \lesssim 1.
\]

Here the implicit constant may depend on \( \lambda, m \) and \( N \).

**Proof.** Let \( \Phi_i \) and \( \tilde{Z}_i \) be solutions to (2.24) and (2.2) with general initial conditions, respectively. We also recall that \( Z_i \) is the stationary solution to (2.2). By the general results of [HM18b], \( (\Phi_i, \tilde{Z}_i)_{1 \leq i \leq N} \) is a Markov process on \((C^{-\frac{1}{2} - \kappa})^{2N}\), and we denote by \((P_t^N)_{t \geq 0}\) the associated Markov semigroup. To derive the desired structural properties about the limiting measure, we will follow the Krylov-Bogoliubov construction with a specific choice of initial condition that allows to exploit the uniform estimate from Appendix B. Namely, we denote by \( \Phi_i \) the solution to (2.24) starting from \( Z_i(0) \) where \( Z_i \) is the stationary solution to (2.2), so that the process \( \Phi_i - \tilde{Z}_i \) starts from the origin. In this case \( \tilde{Z}_i \) is the same as the stationary solution \( Z_i \). By Lemma B.2, for every \( T \geq 1 \) and \( \kappa > 0 \)

\[
\int_0^T \mathbb{E}\left( \frac{1}{N} \sum_{i=1}^N \| (\Phi_i - Z_i)(t) \|_{L^2}^2 \right) dt + \int_0^T \mathbb{E}\left( \frac{1}{N} \sum_{i=1}^N \| (\Phi_i - Z_i)(t) \|^2_{C^{\alpha}} \right) dt \lesssim T, \quad (4.1)
\]

\(^5\)Namely, the integral, \( \nabla \) and \( d\Phi \) should be approximated by lattice summation, finite difference and Lebesgue measure.
where the implicit constant is independent of $T$. By (4.1) together with Lemma 2.2 we have

$$\int_0^T \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^N \|\Phi_i(t)\|_C^{1+\frac{2\kappa}{\epsilon}}\right) dt + \int_0^T \mathbb{E}\left(\frac{1}{N} \sum_{i=1}^N \|Z_i(t)\|_C^{1+\frac{2\kappa}{\epsilon}}\right) dt \lesssim T.$$

Defining $R^N_t := \frac{1}{t} \int_0^t P^N dr$, the above estimates and the compactness of the embedding $C^{-\frac{1+\kappa}{\epsilon}} \hookrightarrow C^{-\frac{1}{2} - \kappa}$ imply the induced laws of $\{R^N_t\}_{t \geq 0}$ started from $(Z_i(0), Z_i(0))$ are tight on $(C^{-\frac{1}{2} - \kappa})^{2N}$. Furthermore, by continuity w.r.t. initial data, it is easy to check that $(P^N_t)_{t \geq 0}$ is Feller on $(C^{-\frac{1}{2} - \kappa})^{2N}$. By Krylov-Bogoliubov existence theorem (see [DPZ96, Corollary 3.1.2]), these laws converge weakly in $(C^{-\frac{1}{2} - \kappa})^{2N}$ along a subsequence $t_k \to \infty$ to an invariant measure $\pi_N$ for $(P^N_t)_{t \geq 0}$. The desired stationary process $(\Phi^N_i, Z_i)_{1 \leq i \leq N}$ is defined to be the unique solution to (2.24) and (2.2) obtained by sampling the initial datum $(\phi_i, z_i)_i$ from $\pi_N$. By (4.1) we deduce

$$\mathbb{E}^\pi_N \|\Phi_i(0) - Z_i(0)\|_{L^2}^2 = \mathbb{E}^\pi_N \sup_f \left[\frac{1}{T} \int_0^T \langle\Phi_i(t) - Z_i(t), f\rangle dt\right]^2 \lesssim \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}\|\Phi_i(t) - Z_i(t)\|_{L^2}^2 dt \lesssim 1,$$

where $\mathbb{E}^\pi$ denotes the expectation w.r.t. $\pi_N$ and $\sup_f$ is over smooth functions $f$ with $\|f\|_{L^2} \leq 1$.

Finally, we project onto the first component and consider the map $\Pi_1 : \mathcal{S}(\mathbb{T}^3)^{2N} \to \mathcal{S}(\mathbb{T}^3)^N$ defined through $\Pi_1(\Phi, Z) = \Phi$. Letting $\nu_N = \pi_N \circ \Pi_1^{-1}$ yields an invariant measure to (2.24), and uniqueness follows from the general results of [HM18b] and [HS22].

By lattice approximation it is possible to prove that the field $\Phi$ under $\nu_N$ is $O(N)$ invariant, translation invariant and satisfies reflection positivity (see e.g. [GH21] for $\Phi^\epsilon_d$). As we do not prove convergence of lattice approximations for sigma models in this paper, we give a proof of $O(N)$ and translation invariances using ergodicity of the dynamics.

**Proposition 4.3.** The field $\Phi$ under $\nu_N$ is $O(N)$ invariant and translation invariant in law.

**Proof.** The equation (2.1) is $O(N)$ invariant and translation invariant, namely, for any $A \in O(N)$ and solution $\Phi$, $A\Phi$ satisfies the same equation with driven noise $A\xi \overset{law}{=} \xi$ and initial data $A\Phi(0)$, and similar property holds for translations of $\Phi$. By uniqueness of the solutions to (2.1), the solution $\Phi(\cdot)$ starting from zero satisfy $O(N)$ invariance and translation invariance. Hence, the limit $\Phi(t)$ of $\Phi(\cdot)$ as $\epsilon \to 0$ is $O(N)$ and translation invariant. As $\nu_N$ is the unique invariant measure to (1.2), the law of $\Phi(t)$ converges to $\nu_N$ as $t \to \infty$. Hence, the field $\Phi$ under $\nu_N$ is $O(N)$ and translation invariant. \qed

**Remark 4.4.** Lemma 4.2 also gives a dynamical / SPDE construction of the measure $\nu_N$.

Next we study tightness of the marginal laws of $\nu_N$ over $\mathcal{S}(\mathbb{T}^3)^N$. Recall that $\nu_{N,i} \overset{law}{=} \nu_N \circ \Pi_i^{-1}$ is the marginal law of the $i^{th}$ component and $\nu^N_k \overset{law}{=} \nu_N \circ (\Pi(k))^{-1}$ is the marginal law of the first $k$ components, with $\Pi_i$ and $\Pi(k)$ defined in (1.3) and (1.4), respectively.

**Theorem 4.5.** For $(m, \lambda) \in (0, \infty) \times [0, \infty)$, $\{\nu_{N,i}\}_{N \geq 1}$ form a tight set of probability measures on $H^{-\frac{1}{2} - \kappa}$ for $\kappa > 0$.

**Proof.** Let $(\tilde{\Phi}_i, Z_i)_{1 \leq i \leq N}$ be as in Lemma 4.2. By the compact embedding of $H^{\frac{1}{2} - 2\kappa} \hookrightarrow H^{-\frac{1}{2} - \kappa}$ for $\kappa > 0$ small enough and the stationarity of $\Phi$, it suffices to show that the second moment of
\[ \| \Phi_t(0) - Z_t(0) \|_{H^\frac{1}{2} - 2\alpha} \] is bounded uniformly in \( N \). To this end, let \( Y_i = \Phi_t - Z_t - X_i \) with \( X_i \) defined in (2.6). It holds that

\[
E \| \Phi_t(0) - Z_t(0) \|_{H^\frac{1}{2} - 2\alpha}^2 = \frac{1}{t} \int_0^t E \| \Phi_s(0) - Z_t(0) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds \\
= \frac{1}{t} \int_0^t E \| X_i(s) + Y_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds \\
\leq \frac{2}{t} \int_0^t E \| X_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds + \frac{2}{t} \int_0^t E \| Y_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds.
\] (4.2)

For the first term, by (2.19) of Lemma 2.6, we have

\[
\frac{1}{t} \int_0^t E \| X_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds = \frac{1}{tN} \int_0^t E \sum_{i=1}^N \| X_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds \lesssim \frac{1}{tN} \mathbb{E} Q_N^0 \lesssim \frac{C_t}{tN},
\]

with \( Q_N^0 \) defined in Lemma 2.5 and \( \mathbb{E} Q_N^0 \lesssim C_t \). For the second term, by \( O(N) \) symmetry in law

\[
\frac{1}{t} \int_0^t E \| Y_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds = \frac{1}{tN} \int_0^t E \left( \sum_{i=1}^N \| Y_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \right) \, ds.
\] (4.4)

Using Theorem 3.4\(^6\) we deduce that

\[
\frac{1}{8N} \int_0^t E \sum_{i=1}^N \| Y_i(s) \|_{H^\frac{1}{2} - 2\alpha}^2 \, ds + \frac{m}{N} \int_0^t E \sum_{i=1}^N \| Y_i(s) \|_{L^2}^2 \, ds + \frac{\lambda}{2N^2} \int_0^t E \sum_{i=1}^N \| Y_i^2 \|_{L^2}^2 \, ds \\
\leq 4E\| \Phi_t(0) - Z_t(0) \|_{L^2}^2 + \frac{4N}{N} \sum_{i=1}^N E \| X_i(0) \|_{L^2}^2 + C.
\]

By definition of \( X_i \) in (2.6) (and definition of \( \mathcal{I} \) in Section A.2) we have \( X_i(0) = -\frac{N}{N} \sum_{j=1}^N \tilde{Z}_i^{\psi_j}(0) \), which gives

\[
\frac{1}{N} \sum_{i=1}^N E \| X_i(0) \|_{L^2}^2 \lesssim \frac{\lambda^2}{N^2} \sum_{i,j=1}^N E \| \tilde{Z}_i^{\psi_j}(0) \|_{L^2}^2 \lesssim 1.
\]

Then we conclude that

\[
E \| \Phi_t(0) - Z_t(0) \|_{H^\frac{1}{2} - 2\alpha}^2 \leq C_t + \frac{64}{t} E \| \Phi_t(0) - Z_t(0) \|_{L^2}^2.
\]

Lemma 4.2 implies that \( E \| \Phi_t(0) - Z_t(0) \|_{L^2} \) is finite. Choosing \( t = 128 \) and using \( \| \cdot \|_{L^2} \leq \| \cdot \|_{H^\frac{1}{2} - 2\alpha} \), we find \( E \| \Phi_t(0) - Z_t(0) \|_{H^\frac{1}{2} - 2\alpha} \leq C_t \). The result then follows by the bound for \( Z_t(0) \) from Lemma 2.2. \( \Box \)

4.2. Convergence of measures. In the following we prove convergence of the measures to the unique invariant measure (which requires \( m \) large enough or \( \lambda \) small) using the estimate in Theorem 3.3.

Define the \((H^{-\frac{1}{2} - \alpha})^k\)-Wasserstein distance for the measures on \( (H^{-\frac{1}{2} - \alpha})^k \)

\[
\mathbb{W}_{2,k}(\nu_1, \nu_2) := \inf_{\pi \in \mathcal{C}(\nu_1, \nu_2)} \left( \int \| \varphi - \psi \|_{(H^{-\frac{1}{2} - \alpha})^k}^2 \pi(d\varphi, d\psi) \right)^{1/2}, \tag{4.5}
\]

where \( \mathcal{C}(\nu_1, \nu_2) \) denotes the set of all couplings of \( \nu_1, \nu_2 \) satisfying \( \int \| \varphi \|_{(H^{-\frac{1}{2} - \alpha})^k}^2 \nu_i(d\varphi) < \infty \) for \( i = 1, 2 \). Recall that \( \nu = \mathcal{N}(0, \frac{1}{2}(m - \Delta)^{-1}) \).

\(^6\)Since we do not need to choose \( m \) and \( \lambda \) in the proof, we can use Theorem 3.4 for \( m > 0, \lambda > 0 \) with the proportional constant depending on \( m, \lambda \).
Theorem 4.6. Let \((m, \lambda) \in [1, \infty) \times [0, \infty)\). There exists \(c_0 > 0\) such that for \(m \geq 1 + \lambda(1 + \lambda^{59})c_0\)
\[
\mathbb{W}_{2,k}(\nu^N_k, \nu^{\otimes k}) \leq C_k N^{-\frac{1}{4}}.
\] (4.6)

Proof. By Lemma 4.2 we may construct a stationary coupling \((\tilde{\Phi}_i, Z_i)\) of \(\nu_N\) and \(\nu\) whose components satisfy (2.24) and (2.2), respectively. The stationarity of the joint law of \((\tilde{\Phi}_i, Z_i)\) implies that also \(\Phi_i - Z_i\) is stationary. We now claim that
\[
\mathbf{E}\|\tilde{\Phi}_i(0) - Z_i(0)\|^2_{H^{\frac{1}{2} - 2\kappa}} \leq CN^{-1},
\] (4.7)
which implies (4.6) by definition of the Wasserstein metric and the embedding \(H^{\frac{1}{2} - 2\kappa} \hookrightarrow H^{-\frac{1}{2} - \kappa}\).

Recall the inequality (4.2), for which the RHS is bounded by (4.3)+(4.4).

Using Theorem 3.3 and Lemma 3.7 (with \(R^1_N, R^2_N, Q^5_N\) introduced in Lemma 3.1, Proposition 3.5 and Lemma 3.7 respectively), we deduce that
\[
\frac{1}{8} \int_0^t \mathbf{E}\left(\sum_{i=1}^N \|Y_i(s)\|_{H^{\frac{1}{2} - 2\kappa}}^2\right)ds + (m - 1) \int_0^t \mathbf{E}\left(\sum_{i=1}^N \|Y_i(s)\|_{L^2}^2\right)ds + \frac{\lambda}{N} \mathbf{E}\left(\sum_{i=1}^N \|Y_i(s)\|_{L^2}^2\right)ds
\leq \mathbf{E}\left(\sum_{i=1}^N \|Y_i(0)\|_{L^2}^2\right) + C \int_0^t \mathbf{E}\left[\left(\sum_{i=1}^N \|Y_i(s)\|_{L^2}^2\right)\left(\lambda(1 + \lambda^{59})(R^1_N + R^2_N + Q^5_N)\right)\right]ds + C(t, \lambda)
\leq 2N\mathbf{E}\|\tilde{\Phi}_i(0) - Z_i(0)\|^2_{L^2} + 2 \sum_{i=1}^N \mathbf{E}\|X_i(0)\|^2_{L^2} + C(t, \lambda)
+ C \int_0^t \mathbf{E}\left(\sum_{i=1}^N \|Y_i(s)\|_{L^2}^2\right)\mathbf{E}\left(\lambda(1 + \lambda^{59})(R^1_N + R^2_N + Q^5_N)\right)ds
+ C \int_0^t \mathbf{E}\left[\left(\sum_{i=1}^N \|Y_i(s)\|_{L^2}^2\right)\lambda(1 + \lambda^{59})\left|R^1_N + R^2_N + Q^5_N - \mathbf{E}[R^1_N + R^2_N + Q^5_N]\right|\right]ds.
\] (4.8)

Here \(C\) is independent of \(\lambda, m\) and \(N\). By definition of \(X_i\) in (2.6) and similar cancelation as in the proof of Lemma 2.5 we have
\[
\sum_{i=1}^N \mathbf{E}\|X_i(0)\|^2_{L^2} = \frac{\lambda^2}{N^2} \sum_{i=1}^N \mathbf{E}\left\|\sum_{j=1}^N \mathbf{E}[\mathbf{Y}_{ijj}(0)]\right\|^2_{L^2} \lesssim \frac{1}{N^2} \sum_{i,j=1}^N \mathbf{E}\|\mathbf{Y}_{ijj}(0)\|^2_{L^2} \lesssim 1.
\]
Note that by Lemma 2.2 we can easily deduce for \(T > 0\)
\[
\mathbf{E}R^1_N + \mathbf{E}R^2_N + \mathbf{E}Q^5_N \lesssim 1.
\]
Here we recall the definition of \(Q^5_N\) depend on \(T\) and \(\mathbf{E}Q^5_N\) is increasing w.r.t. \(T\). The last line of (4.8) is controlled by
\[
\frac{\lambda}{2N} \int_0^t \mathbf{E}\left[\left(\sum_{i=1}^N \|Y_i(s)\|_{L^2}^2\right)\right]ds
+ C(\lambda)N \sum_{k=1}^t \int_0^t \mathbf{E}\left[(R^k_N - \mathbf{E}[R^k_N])\right]ds + C(\lambda)N \int_0^t \mathbf{E}\left[(Q^5_N - \mathbf{E}[Q^5_N])\right]ds.
\]
We claim that the last two terms here – although apparently have a large factor \(N\) – are actually bounded by constant independent of \(N\).

Indeed \(R^1_N - \mathbf{E}R^1_N, R^2_N - \mathbf{E}R^2_N\) and \(Q^5_N - \mathbf{E}Q^5_N\) are all summations of terms of the form
\[
\frac{1}{N^l} \sum_{i_1, \ldots, i_l=1}^N M_{i_1, \ldots, i_l},
\]
for different choices of \( l \) (namely \( l = 1, 2 \) for \( R^3_N - ER^3_N \), \( l = 1, 2, 3 \) for \( R^2_N - ER^2_N \), and \( l = 2, 3, 4, 5 \) for \( Q^5_N - EQ^5_N \)), where each \( M_{i_1, \ldots, i_l} \) is mean-zero, has bounded second moment, and they satisfy the independence assumption in Lemma 4.7 because they only involve the stochastic objects in (2.4) constructed from the independent noises \( (\xi_i)_{i=1}^N \). So by Lemma 4.7 below the claim is proved.

Recall that \( C \) is independent of \( \lambda, m \) and \( N \). Choosing

\[
m > C_\lambda(1 + \lambda^{9})(E[R^3_N + R^2_N] + E[Q^5_N]) + 1,
\]

with \( T = 64 \) in \( Q^5_N \) which is uniformly bounded w.r.t. \( m \) by Lemma 2.2, we have for \( t \leq 64 \)

\[
\frac{1}{8} \int_0^t E \sum_{i=1}^N \|Y_i(s)\|_{H^{\frac{1}{2}} - 2 \kappa}^2 \, ds + \int_0^t \frac{\lambda}{N} E \left\| \sum_{i=1}^N Y_i^2(s) \right\|_{L^2}^2 \, ds \leq C(t, \lambda) + 2NE\|\hat{\Phi}_t(0) - Z_t(0)\|_{L^2}^2.
\]

Combining (4.2) and (4.9) we conclude that for \( t \leq 64 \)

\[
E\|\hat{\Phi}_t(0) - Z_t(0)\|_{H^{\frac{1}{2}} - 2 \kappa}^2 \leq \frac{C(t, \lambda)}{N} + \frac{32}{t} E\|\hat{\Phi}_t(0) - Z_t(0)\|_{L^2}^2.
\]

By Lemma 4.2 we have \( E\|\hat{\Phi}_t(0) - Z_t(0)\|_{L^2}^2 \) finite. Choosing \( t = 64 \) and using \( \| \cdot \|_{L^2} \leq \| \cdot \|_{H^{\frac{1}{2}} - 2 \kappa} \) we obtain the claim (4.7).

**Lemma 4.7.** Let \( l \) be a fixed positive integer and \( (M_{i_1, \ldots, i_l} : i_1, \ldots, i_l \in \{1, \ldots, N\}) \) be a collection of mean-zero random variables such that \( E[M_{i_1, \ldots, i_l}] \leq 1 \) uniformly in \( N \) for any \( i_1, \ldots, i_l \in \{1, \ldots, N\} \), and assume that \( M_{i_1, \ldots, i_l} \) and \( M_{j_1, \ldots, j_l} \) are independent when the \( 2l \) indices \( i_1, \ldots, i_l, j_1, \ldots, j_l \) are all different. Then we have

\[
E \left[ \left( \frac{1}{N^l} \sum_{i_1, \ldots, i_l=1}^N M_{i_1, \ldots, i_l} \right)^2 \right] \leq C/N,
\]

where \( C \) only depends on \( l \) and is independent of \( N \).

**Proof.** Writing the LHS as

\[
\frac{1}{N^{2l}} \sum_{i_1, \ldots, i_l=1}^N \sum_{j_1, \ldots, j_l=1}^N E[M_{i_1, \ldots, i_l} M_{j_1, \ldots, j_l}],
\]

we see that the expectation is zero when \( i_1, \ldots, i_l, j_1, \ldots, j_l \) are all different by the mean-zero and the independence assumptions. When these indices are not all different, the number of summands is \( N^{2l} \frac{N!}{(N-2l)!} \leq C N^{2l-1} \) where \( C \) only depends on \( l \) but independent of \( N \), and each summand is bounded by our moment bound assumption, so we obtain the claimed bound.

### 4.3. Observables

In this section we write the stationary solutions constructed in Lemma 4.2 as \((\Phi_i, Z_i)\). In the following we study the observables (1.5). They are defined as follows. By Lemma 4.2 we decompose \( \Phi_i = X_i + Y_i + Z_i \) with \( (\Phi_i, Z_i) \) stationary and \( X_i \) introduced in (2.6). With this we define

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^N \Phi_{i}^2 = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( X_i^2 + Y_i^2 + 2X_i Y_i + 2X_i Z_i + 2Y_i Z_i \right).
\]

By (4.9) and (4.7) we also have

**Lemma 4.8.** Let \( (m, \lambda) \in [1, \infty) \times [0, \infty) \). For \( m \geq c_0 \lambda(1 + \lambda^{59}) + 1 \) with \( c_0 \) as in Theorem 4.6. It holds that

\[
\int_0^t E \sum_{i=1}^N \|Y_i(s)\|_{H^{\frac{1}{2}} - 2 \kappa}^2 \, ds + \int_0^t \frac{1}{N} E \left\| \sum_{i=1}^N Y_i^2(s) \right\|_{L^2}^2 \, ds \leq 1,
\]

where the proportional constant may depend on \( \lambda \) and is independent of \( N \).
Proof of Theorem 1.2. By stationarity we have
\[ \mathbf{E} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} : \Phi_i^2 : \right\|_{B_{1,1}^{-1-2\kappa}} = \frac{1}{T} \mathbf{E} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} : \Phi_i^2 : \right\|_{L^1 T B_{1,1}^{-1-2\kappa}}. \]

In the following we use the above equality to derive
\[ \mathbf{E} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} : \Phi_i^2 : \right\|_{B_{1,1}^{-1-2\kappa}} \lesssim 1, \quad (4.11) \]
which implies the desired result by the compact embedding \( B_{1,1}^{-1-2\kappa} \subset B_{1,1}^{-1-3\kappa} \).

We consider \( L^2 \) \( B_{1,1}^{-1-2\kappa} \)-norm of each term in (4.10). Using Lemma A.7 and (2.19) we find
\[ \mathbf{E} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i^2 \right\|_{L^2 T B_{1,1}^{-1-2\kappa}} \lesssim \mathbf{E} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| X_i \|_{L^2 H^{1/2-2\kappa}}^2 \lesssim \frac{1}{\sqrt{N}}. \]
Similarly, by Lemma A.7 and Lemma 4.8 and Lemma 2.6 we have
\[ \mathbf{E} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i^2 \right\|_{L^2 T B_{1,1}^{-1-3\kappa}} \lesssim \mathbf{E} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| Y_i \|_{L^2 H^{1/2-2\kappa}}^2 \lesssim \frac{1}{\sqrt{N}}. \]

Furthermore, by independence and similar argument as in the proof of Lemma 2.5 we deduce
\[ \mathbf{E} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Z_i^2 \right\|_{L^2 T B_{1,1}^{-1-\kappa}} \lesssim \mathbf{E} \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{L^2 T B_{1,1}^{-1-\kappa}}^2 \lesssim 1. \]
Using Lemma A.7 and Lemma 4.8 we obtain
\[ \mathbf{E} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i Z_i \right\|_{L^2 T B_{1,1}^{-1-\kappa}} \lesssim \mathbf{E} \frac{1}{\sqrt{N}} \left( \sum_{i=1}^{N} \| Y_i \|_{L^2 H^{1/2-2\kappa}}^2 \right)^{1/2} \left( \sum_{i=1}^{N} \| Z_i \|_{L^2 H^{1/2-2\kappa}}^2 \right)^{1/2} \lesssim 1. \]
It only remains to consider \( X_i Z_i \). By Lemma A.6 we have
\[ \mathbf{E} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| X_i < Z_i \| \lesssim \mathbf{E} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| X_i \| \| Z_i \| \lesssim \mathbf{E} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( \sum_{i=1}^{N} \| X_i \|_{L^2 H^{1/2-2\kappa}}^2 \right)^{1/2} \left( \sum_{i=1}^{N} \| Z_i \|_{L^2 H^{1/2-2\kappa}}^2 \right)^{1/2} \lesssim 1. \]
Here we use Lemma 2.6 in the last inequality. For \( X_i \circ Z_i \) we recall (2.16) that
\[ X_i \circ Z_i = -\frac{\lambda}{N} \sum_{j=1}^{N} \left[ Z^\mathcal{V}_{ij,i} + I(2X_j < \mathcal{X} Z^\mathcal{V}_{ij} + X_i < \mathcal{X} Z^\mathcal{V}_{ij} \circ Z_i) \right], \]
which by Lemma A.6 implies that
\[ \mathbf{E} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| X_i \circ Z_i \|_{L^2 T B_{1,1}^{-2\kappa}} \lesssim \mathbf{E} \frac{1}{N^{3/2}} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} \| Z^\mathcal{V}_{ij,i} \|_{L^2 T B_{1,1}^{-2\kappa}}^2 \right)^{1/2} + \mathbf{E} \frac{1}{N^{3/2}} \sum_{i=1}^{N} \| X_j \|_{L^2 H^{1/2-2\kappa}} \| Z^\mathcal{V}_{ij} \|_{C^{1-\kappa}} \| Z_i \|_{C^{1-\kappa}} \lesssim 1. \]
\[ + \mathbb{E} \frac{1}{N^{3/2}} \sum_{i,j=1}^{N} \| X_i \|_{L^2_{\infty} H^{1/2} - 2\epsilon} \| \mathcal{Z}_{ij}^\nu \|_{C^\gamma C^{-1-\epsilon}} \| Z_i \|_{C^\gamma C^{-1/2}} \ldots := \sum_{i=1}^{3} J_i^N. \]

Regarding \( J_2^N \) we can bound each summand by
\[ \frac{1}{N} \| X_j \|_{L^2_{\infty} H^{-1/2} - 2\epsilon} \| Z_i \|_{C^\gamma C^{-1/2}} \ldots \]
and after taking expectation and summation it is equal to
\[ \mathbb{E} \left( \sum_{j=1}^{N} \| X_j \|_{L^2_{\infty} H^{-1/2} - 2\epsilon}^2 \right) \left( \frac{1}{N} \sum_{i=1}^{N} \| Z_i \|_{C^\gamma C^{-1/2}}^2 \right) + \left( \frac{1}{N^2} \mathbb{E} \sum_{i,j=1}^{N} \| \mathcal{Z}_{ij}^\nu \|_{C^\gamma C^{-1-\epsilon}}^2 \right). \]
Applying Cauchy–Schwarz to the product in the first term followed by (2.19) and Lemma 2.2 and Lemma 2.6 we have \( J_2^N \lesssim 1 \). For the term \( J_3^N \), each summand is bounded by
\[ \frac{1}{N^2} \mathbb{E} \sum_{i,j=1}^{N} \| \mathcal{Z}_{ij}^\nu \|_{C^\gamma C^{-1-\epsilon}}^2 \]
and then \( J_3^N \lesssim 1 \) follows in the same way.

The first term \( J_1^N \) is bounded by
\[ \mathbb{E} \left( \frac{1}{N^2} \sum_{j=1}^{N} \| \mathcal{Z}_{ij,1} \|_{L^2_{\infty} B_{2,2}^{-1/2}}^2 \right) \lesssim T \left( \mathbb{E} \left( \frac{1}{N} \sum_{j=1}^{N} \| \mathcal{Z}_{ij,1} \|_{L^2_{B_{2,2}^{-1/2}}}^2 \right) \right)^{1/2}. \]
The quantity in the parenthesis is equal to
\[ \mathbb{E} \frac{1}{N} \sum_{j_1,j_2=1}^{N} \left( \Lambda^{-\epsilon} \mathcal{Z}_{ij,1} \Lambda^{-\epsilon} \mathcal{Z}_{ij,2} \right). \]
For the case \( j_1 \neq j_2 \), the expectation is zero. We thus conclude that \( J_1^N \lesssim 1 \). Combining all the calculations above (4.11) follows. \( \square \)

**Remark 4.9.** We believe that nontrivial formulae for the correlations of the observables in the large \( N \) limit can be calculated explicitly as in 2D case (see [SSZZ22, Section 6]). However, the solutions to the SPDEs (1.2) and the related field \( \nu^N \) are much more singular compared to its counterpart in 2D case. The uniform in \( N \) estimates in 3D, which are the main part in Sections 2-3, are much more complicated than the 2D case. Moreover, as \( \Phi^3 \) is not a random distribution in 3D, it would require more effort to interpret the new terms when applying Dyson–Schwinger equation. In the 2D case \( L^p \) energy estimates are enough to derive the convergence of the correlations of the observable, but in 3D case we need further decomposition and uniform in \( N \) Schauder estimate to bound the cubic term. These requires more complicated estimates than that in Sections 2-3, so we leave it to the future studies.

### Appendix A. Notations and Besov spaces

**A.1. Besov spaces.** We denote by \((\Delta_j)_{j \geq -1}\) the Littlewood–Paley blocks for a dyadic partition of unity. The Besov spaces \( B_{p,q}^\alpha \) on the torus with \( \alpha \in \mathbb{R}, p, q \in [1, \infty] \) are defined as the completion of \( C^\infty \) with respect to the norm
\[ \| u \|_{B_{p,q}^\alpha} := \left( \sum_{j \geq -1} 2^{jq\alpha} \| \Delta_j u \|_{L^p}^q \right)^{1/q}. \]

The following embedding results will be frequently used.

**Lemma A.1.** Let \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( 1 \leq q_1 \leq q_2 \leq \infty \), and let \( \alpha \in \mathbb{R} \). Then \( B_{p_1,q_1}^\alpha \subset B_{p_2,q_2}^{\alpha-d(1/p_1-1/p_2)} \). Here \( \subset \) means continuous and dense embedding. (cf. [GIP15, Lemma A.2])
We also recall the following interpolation lemma.

**Lemma A.2.** Suppose that \( \theta \in (0, 1) \), \( \alpha_1, \alpha_2 \in \mathbb{R} \). Then for \( \alpha = \theta \alpha_1 + (1 - \theta)\alpha_2 \)
\[
\| f \|_{H^\alpha} \lesssim \| f \|_{H^{\alpha_1}}^{1 - \theta} \| f \|_{H^{\alpha_2}}^\theta.
\]
(cf. [Tri78, Theorem 4.3.1])

### A.2. Smoothing effect of heat flow

We recall the following smoothing effect of the heat flow \( P_t = e^{t(D - \Delta)} \), \( m > 0 \) (e.g. [GIP15, Lemma A.7], [MW17a, Proposition A.13]).

**Lemma A.3.** Let \( u \in B^\alpha_{p,q} \) for some \( \alpha \in \mathbb{R}, p, q \in [1, \infty] \). Then for every \( \delta \geq 0 \), \( t \in [0, T] \)
\[
\| P_t u \|_{B^\alpha_{p,q}} \lesssim t^{-\delta/2} \| u \|_{B^\alpha_{p,q}},
\]
where the proportionality constants are uniform for \( m \geq 1 \). If \( 0 \leq \beta - \alpha \leq 2 \), then
\[
\| (I - P_t) u \|_{B^\alpha_{p,q}} \lesssim t^{\beta/2} \| u \|_{B^\beta_{p,q}},
\]
where the proportionality constants are uniform for \( m \geq 1 \).

We also define \( (I f)(t, x) := (\mathcal{L}^{-1} f)(t, x) = \int_0^t P_{t-s} f ds \).

**Lemma A.4.** ([GIP15, Lemma A.9], [ZZZ20, Lemma 2.8, Lemma 2.9]) Let \( \alpha \in \mathbb{R} \). Then the following bounds hold uniformly over \( 0 \leq t \leq T \)
\[
\| I f(t) \|_{C^{2+\alpha}} \lesssim \| f \|_{L^\infty T} C^\alpha.
\]
(A.1)

If \( 0 \leq 2 + \alpha < 2 \) then
\[
\| I f(t) \|_{C^{(2+\alpha)/2} L^\infty} \lesssim \| f \|_{L^\infty T} C^\alpha,
\]
where the proportionality constants are uniform for \( m \geq 1 \).

**Lemma A.5.** For \( \beta \in \mathbb{R} \) we have
\[
\| I f \|_{L^2 T H^\beta} \lesssim \| f \|_{L^2 T H^{2+2\alpha}} \quad \text{and} \quad \| I f \|_{W^{1+2\beta, 2} T L^2} \lesssim \| f \|_{L^2 T H^{2+2\alpha}}
\]
for \( 0 \leq \kappa < 1/20 \), where the proportionality constants are uniform for \( m \geq 1 \).

**Proof.** Let \( \{e_k(x)\} = \{2^{-\frac{2}{\beta}} e^{i\pi k x}, k \in \mathbb{Z}^3\} \) on \( T^3 \).
\[
\| I f \|_{L^2 T H^\beta}^2 = \int_0^T \sum_k (|k|^2 + m)^\beta \left| \int_0^t P_{t-s} f ds, e_k \right|^2 dt
\]
\[
= \int_0^T \sum_k (|k|^2 + m)^\beta \left| \int_0^t e^{-(t-s)(|k|^2 + m)} \langle f, e_k \rangle ds \right|^2 dt
\]
\[
\lesssim \int_0^T \sum_k (|k|^2 + m)^{\beta - 1} \int_0^t e^{-(t-s)(|k|^2 + m)} \langle f, e_k \rangle dt ds
\]
\[
= \sum_k (|k|^2 + m)^{\beta - 1} \int_0^T \int_s^T e^{-(t-s)(|k|^2 + m)} dt \langle f, e_k \rangle ds \lesssim \| f \|_{L^2 T H^{2+2\alpha}}^2,
\]
where we use Hölder’s inequality in the first inequality. Thus the first result follows.

By the definition of \( W^{1+2\beta, 2} T L^2 \) we have
\[
\| I f \|_{W^{1+2\beta, 2} T L^2}^2 = \int_0^T \| I f(s) \|_{L^2}^2 ds + \int_0^T \int_0^T \frac{\| I f(t) - I f(s) \|_{L^2}^2}{|t - s|^{1+2(\beta-3\kappa)}} ds dt = I_1 + I_2.
\]
can be easily controlled by the first result and we consider the $I_2$ part. Using the smoothing effect of the heat kernel given in Lemma A.3 we have

$$I_2 \lesssim \int_0^T \int_0^t \| \mathcal{I} f(s) \|^2_{L^2 H^{\frac{4}{3} - 2\kappa}} ds dt + \int_0^T \int_0^t \left( \int_0^s (t - r)^{-\frac{4}{3} - \kappa} \| f(r) \|_{H^{\frac{4}{3} + 2\kappa}} dr \right)^2 ds dt$$

$$\lesssim \int_0^T \int_0^t \frac{\| \mathcal{I} f(s) \|^2_{L^2 H^{\frac{4}{3} - 2\kappa}} ds dt + \int_0^T \int_0^t \int_0^s (t - r)^{-\frac{4}{3} - \kappa} \| f(r) \|^2_{H^{\frac{4}{3} + 2\kappa}} dr ds dt}{|t - s|^{1 + 4\kappa}}$$

$$\lesssim \| \mathcal{I} f \|^2_{L^2 H^{\frac{4}{3} - 2\kappa}} + \| f \|^2_{L^2 H^{\frac{4}{3} - 2\kappa}}$$

where we used the fact that the second term on the right is controlled by

$$\int_0^T \int_0^t \int_0^s (t - r)^{-\frac{4}{3} - \kappa} \| f(r) \|^2_{H^{\frac{4}{3} + 2\kappa}} ds dr ds dt$$

$$\lesssim \int_0^T \int_0^T (t - r)^{-1 + \kappa} \| f(r) \|^2_{H^{\frac{4}{3} + 2\kappa}} ds dr$$

in the last step. Thus the second result follows from the first result. \qed

A.3. Paraproducts and commutators. We recall the following paraproduct introduced by Bony (see [Bon81]). In general, the product $fg$ of two distributions $f \in C^\omega, g \in C^\beta$ is well defined if and only if $\alpha + \beta > 0$. In terms of Littlewood-Paley blocks, the product $fg$ of two distributions $f$ and $g$ can be formally decomposed as

$$fg = f < g + f \circ g + f \succ g,$$

with

$$f < g = g \succ f = \sum_{j \geq -1} \sum_{i < j - 1} \Delta_i f \Delta_j g, \quad f \circ g = \sum_{|i - j| \leq 1} \Delta_i f \Delta_j g.$$

We also denote

$$\succdef \succ + \circ, \quad \succapproxdef \succ + \circ.$$

The following results on paraproduct on Besov space is from [Bon81] (see also [GIP15, Lemma 2.1], [MW17a, Proposition A.7]).

**Lemma A.6.** Let $\beta \in \mathbb{R}, p, p_1, p_2, q \in [1, \infty]$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then we have

$$\| f < g \|_{B^\beta_{p,q}} \lesssim \| f \|_{L^{p_1}} \| g \|_{B^\beta_{p_2,q}}$$

$$\| f < g \|_{B^{\alpha+\beta}_{p,q}} \lesssim \| f \|_{B^\alpha_{p_1,q}} \| g \|_{B^\beta_{p_2,q}} \quad \text{(for } \alpha < 0)$$

$$\| f \circ g \|_{B^{\alpha+\beta}_{p,q}} \lesssim \| f \|_{B^\alpha_{p_1,q}} \| g \|_{B^\beta_{p_2,q}} \quad \text{(for } \alpha + \beta > 0).$$

Furthermore, we have the following multiplicative inequality.

**Lemma A.7.** (i) Let $\alpha, \beta \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The bilinear map $(u, v) \mapsto uv$ extends to a continuous map from $B^\alpha_{p_1,q} \times B^\beta_{p_2,q}$ to $B^{\alpha+\beta}_{p,q}$ if $\alpha + \beta > 0$. (cf. [MW17b, Corollary 3.21])

(ii) (Duality.) Let $\alpha \in (0, 1), p, q \in [1, \infty]$, $p'$ and $q'$ be their conjugate exponents, respectively. Then the mapping $(u, v) \mapsto \int uv dx$ extends to a continuous bilinear form on $B^\alpha_{p,q} \times B^{-\alpha}_{p',q'}$, and one has $\| (u, v) \| \lesssim \| u \|_{B^\omega_{p,q}} \| v \|_{B^{-\omega}_{p',q'}}$ (cf. [MW17b, Proposition 3.23]).

(iii) (Fractional Leibniz estimate) Let $s \geq 0, p, p_1, p_2, p_3, p_4, q \in [1, \infty]$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then it holds that [CGW20, Proposition A.2]

$$\| uv \|_{B^s_{p,q}} \lesssim \| u \|_{B^s_{p_1,q}} \| v \|_{L^{p_2}} + \| u \|_{L^{p_3}} \| v \|_{B^s_{p_4,q}}.$$  \hspace{1cm} (A.2)
By duality and [MW17b, Proposition 3.25] we easily deduce the following (cf. [SSZZ22, Lemma 2.5]):

Lemma A.8. For \( s \in (0,1) \)
\[
\begin{align*}
|\langle g, f \rangle| & \lesssim \left( \|\nabla g\|_{L^p} \|g\|_{L^q}^{1-s} + \|g\|_{L^1} \right) \|f\|_{C^{-s}}.
\end{align*}
\]

We also recall the following commutator estimate ([GIP15, Lemma 2.4], [MW17a, Proposition A.9]).

Lemma A.9. Let \( \alpha, \beta, \gamma \in \mathbb{R} \) such that \( \alpha + \beta + \gamma > 0 \) and \( \beta + \gamma < 0 \), \( p, p_1, p_2 \in [1,\infty] \), \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then there exist a trilinear bounded operator \( \tilde{C}(f,g,h) : B_{p_1,\infty}^\alpha \times C^\beta \times B_{p_2,\infty}^\gamma \rightarrow B_{p,\infty}^{\alpha+\beta+\gamma} \) satisfying
\[
\|\tilde{C}(f,g,h)\|_{B_{p,\infty}^{\alpha+\beta+\gamma}} \lesssim \|f\|_{B_{p_1,\infty}^\alpha} \|g\|_{C^\beta} \|h\|_{B_{p_2,\infty}^\gamma}
\]
and for smooth functions \( f,g,h \)
\[
\tilde{C}(f,g,h) = (f \prec g) \circ h - f(g \circ h).
\]

The following lemmas are from [GH21, Lemma A.13, Lemma A.14]. In [GH21] these lemmas are stated in the discrete setting, but the same arguments lead to the following statements in the continuum setting.

Lemma A.10. Let \( \alpha, \beta, \gamma \in \mathbb{R} \) such that \( \alpha + \beta + \gamma > 0 \) and \( \beta + \gamma < 0 \). Then there exist a trilinear bounded operator \( D(f,g,h) : H^\alpha \times C^\beta \times H^{\gamma} \rightarrow \mathbb{R} \) satisfying
\[
|D(f,g,h)| \lesssim \|f\|_{H^\alpha} \|g\|_{C^\beta} \|h\|_{H^{\gamma}}
\]
and for smooth functions \( f,g,h \)
\[
D(f,g,h) = (f \circ g \circ h) - (f \prec g, h).
\]

Lemma A.11. Let \( \alpha, \beta, \gamma \in \mathbb{R} \) such that \( \alpha \in (0,1) \), \( \alpha + \beta + \gamma + 2 > 0 \) and \( \beta + \gamma + 2 < 0 \). Then there exist trilinear bounded operators
\[
\begin{align*}
C(f,g,h) : H^\alpha \times C^\beta \times C^{\gamma+\delta} & \rightarrow H^{\beta+\gamma+2}, \\
\tilde{C}(f,g,h) : C^\alpha \times C^\beta \times C^{\gamma+\delta} & \rightarrow C^{\beta+\gamma+2},
\end{align*}
\]
satisfying for \( \delta > 0 \)
\[
\begin{align*}
\|C(f,g,h)\|_{H^{\beta+\gamma+2}} & \lesssim \|f\|_{H^\alpha} \|g\|_{C^\beta} \|h\|_{C^{\gamma+\delta}}, \\
\|\tilde{C}(f,g,h)\|_{C^\alpha \times C^\beta \times C^{\gamma+\delta}} & \lesssim (\|f\|_{C^\alpha} + \|f\|_{C^\beta} + \|f\|_{C^{\gamma+\delta}}) \|g\|_{C^\alpha} \|h\|_{C^{\beta+\gamma+2}}
\end{align*}
\]
and for smooth functions \( f,g,h \)
\[
\begin{align*}
C(f,g,h) & = ((m - \Delta)^{-1}(f \prec g)) \circ h - f(h \circ (m - \Delta)^{-1} g), \\
\tilde{C}(f,g,h) & = \mathcal{I}(f \prec g) \circ h - f(\mathcal{I}g) \circ h \quad \text{(with } \mathcal{I} = \mathcal{L}^{-1}).
\end{align*}
\]

We also prove the following estimate for commutators:

Lemma A.12. For \( \tilde{C} \) as in Lemma A.11, \( T > 0 \), \( f \in L^2_T H^{\frac{1}{2}-2\kappa} \cap W^{\frac{1}{2}-3\kappa,2}_T L^2 \), and \( g, h \in C_T C^{1-\kappa} \) with \( 0 < \kappa < 1/22 \), it holds that
\[
\|C(f,g,h)\|_{L^2_T H^{\frac{1}{2}-2\kappa}} \lesssim \|f\|_{L^2_T H^{\frac{1}{2}-2\kappa}} + \|f\|_{W^{\frac{1}{2}-2\kappa,2}_T L^2} \|g\|_{C_T C^{1-\kappa}} \|h\|_{C_T C^{1-\kappa}}.
\]
Here the proportionality constants are uniform for \( m \geq 1 \) and may depend on \( T \).

Proof. We write \( C(f,g,h)(t) \) as
\[
\begin{align*}
\int_0^t P_{t-s} (f \prec g_s) ds \circ h_t + \int_0^t (P_{t-s}[f_t \prec g_s] - f_t \prec P_{t-s} g_s) ds \circ h_t \\
+ \left((f \prec \mathcal{I} g) \circ h - f(\mathcal{I} g \circ h)\right)(t) & \overset{\text{def}}{=} I_1(t) + I_2(t) + I_3(t),
\end{align*}
\]
with \( \delta_{st}f = f_s - f_t \). By Lemmas A.9 and A.4 we have
\[
\|I_3\|_{L^2_t L^2_x}^2 \lesssim \|h\|_{C^{1-\kappa}_T}^2 \|g\|_{C^{1-\kappa}_T}^2 \|f\|_{L^2_y}^2 \lesssim \|h\|_{C^{1-\kappa}_T}^2 \|g\|_{C^{1-\kappa}_T}^2 \|f\|_{L^2_y}^2.
\]
Using Lemma A.6, Lemma A.3 followed by Hölder’s inequality we find
\[
\|I_1\|_{L^2_t L^2_x}^2 \lesssim \|h\|_{C^{1-\kappa}_T}^2 \|g\|_{C^{1-\kappa}_T}^2 \int_0^T \left[ \int_0^t \|\delta_{st}f\|_{L^2}^2 \|t - s\|^{1+2\kappa} \text{ds} \right] \text{dt}
\]
\[
\lesssim \|h\|_{C^{1-\kappa}_T}^2 \|g\|_{C^{1-\kappa}_T}^2 \int_0^T \left[ \int_0^t \|\delta_{st}f\|_{L^2}^2 \|t - s\|^{1+5\kappa} \text{ds} \right] \text{dt}
\]
\[
\lesssim \|f\|_{W^{1,3}_{\kappa,2} L^2}^2 \|h\|_{C^{1-\kappa}_T}^2 \|g\|_{C^{1-\kappa}_T}^2 \|f\|_{L^2_y}^2.
\]
Moreover, Lemma A.6 and [MW17a, Proposition A.16] imply that
\[
\|I_2\|_{L^2_t L^2_x}^2 \lesssim \|h\|_{C^{1-\kappa}_T}^2 \|g\|_{C^{1-\kappa}_T}^2 \int_0^t \left[ \int_0^t \|\delta_{st}f\|_{L^2}^2 \|t - s\|^{1-2\kappa} \text{ds} \right] \text{dt}
\]
\[
\lesssim \|f\|_{L^2_y}^2 \|h\|_{C^{1-\kappa}_T}^2 \|g\|_{C^{1-\kappa}_T}^2 \|f\|_{L^2_y}^2.
\]
where we use Hölder’s inequality in the last step. \( \square \)

**Appendix B. Extra estimates**

By similar argument as in Section 3 we deduce

**Lemma B.1. (Energy balance)**

\[
\frac{1}{2N} \sum_{i=1}^N \frac{d}{dt} \|Y_i\|_{L^2}^2 + \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N Y_{ij}^2 \lesssim R_N, \quad \text{for } R_N \text{ given in the proof. Here the implicit constant may depend on } \lambda \text{ and } m.
\]

**Proof.** Using Lemma 2.8 we find
\[
\frac{1}{2N} \sum_{i=1}^N \frac{d}{dt} \|Y_i\|_{L^2}^2 + \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N Y_{ij}^2 + \frac{m}{N} \sum_{i=1}^N \|\varphi_i\|_{L^2}^2 + \frac{1}{N} \sum_{i=1}^N \|\nabla \varphi_i\|_{L^2}^2 \lesssim \frac{1}{N} \Theta + \frac{1}{N} \Xi.
\]

In Proposition 3.5 and Lemma 3.8 we already deduce the required bound for \( \frac{1}{N} \Theta \) and the cubic terms in \( \frac{1}{N} \Xi \). In the following we consider the remaining terms in \( \frac{1}{N} \Xi \). In the proof of Proposition 3.6 we give estimate for \( \|\Xi\|_{L^1} \). Following the same argument and estimate at a fixed time, we find
\[
\frac{1}{N} \Theta + \frac{1}{N} \Xi \lesssim \delta \frac{1}{N} \sum_{i=1}^N \|Y_i\|_{H^{1-2\kappa}}^2 + \frac{1}{N} \sum_{i=1}^N \|Y_i\|_{L^2}^2 (Q^3_N + R^2_N) + \frac{Q^4_N}{N}
\]
\[
\lesssim \delta \frac{1}{N} \sum_{i=1}^N \|Y_i\|_{H^{1-2\kappa}}^2 + \delta \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N Y_{ij}^2 + (Q^3_N + R^2_N)^2 + \frac{1}{N} \tilde{Q}^N_N,
\]

with
\[
\tilde{Q}^N_N \overset{\text{def}}{=} R^2_N + 1 + \left( \frac{1}{N} \sum_{j=1}^N \|X_j\|_{C^{1-2\kappa}}^2 \right)^2 + \left( \frac{1}{N^2} \sum_{i,j=1}^N \|X_j Z_i\|_{C^{1-2\kappa}}^2 \right)^2.
\]
\[
\theta = \frac{1}{2\pi} \quad \text{and}
\]
\[
\tilde{Q}_N^1 \overset{\text{def}}{=} \left( \sum_{i=1}^N \|X_i\|^2_{H^{\frac{1}{2}-2\epsilon}} \right) \left[ 1 + \frac{1}{N^2} \sum_{i,j=1}^N 2^{8\alpha L} \left( \|Z_{ij}^Y\|_{C^{1-\epsilon}} + \|Z_{ij}^Y\|_{C^{1-\epsilon}} \right) \right] \\
+ \left( \frac{1}{N^2} \sum_{i,j=1}^N \left( \sum_{i=1}^N \|X_j Z_i\|_{H^{-\frac{1}{2}-2\epsilon}} \right)^2 \right) + \left( \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \|X_i Z_j\|_{H^{-\frac{1}{2}-2\epsilon}} \right)^2 \right) \\
+ \left( \frac{1}{N^2} \sum_{i=1}^N \left( \sum_{j=1}^N \|X_i \circ Z_{ij}^Y\|_{H^{-\frac{1}{2}-2\epsilon}} + \|X_j \circ Z_{ij}^Y\|_{H^{-\frac{1}{2}-2\epsilon}} \right)^2 \right) + \frac{\gamma}{N^2} \sum_{i=1}^N \|X_i\|^2_{L^2},
\]
for \(\gamma \gt 0\). Using the first inequality in Lemma 3.2 the result holds with \(R_N\) given by \((\tilde{Q}_N^3 + R_N^2 + R_N^3)^2 + \frac{1}{N} \tilde{Q}_N^1 + 1\). \(\square\)

We denote by \(\Phi_i\) the solution to (2.24) starting from the stationary solution \(Z_i(0)\), so that the process \(\Phi_i - Z_i\) starts from the origin as in the proof of Lemma 4.2. By using Lemma B.1 and [TW18, Lemma 3.8], we obtain the following result.

**Lemma B.2.** For every \(T \gt 1\) and \(\kappa \gt 0\), the following holds uniformly in \(T\)

\[
\int_0^T \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N \|(\Phi_i - Z_i)(t)\|^2_{L^2} \right] dt \lesssim T,
\]
\[
\int_0^T \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^N \|(\Phi_i - Z_i)(t)\|^2_{C^{\frac{1}{2}-\epsilon}} \right] dt \lesssim T.
\]

**Proof.** On every interval \([s,s+2]\), \(s \gt 1\) we decompose \(\Phi_i - Z_i = X_i^s + Y_i^s\) with \(X_i^s\) satisfies

\[
X_i^s(t) = -\frac{\lambda}{N} \int_s^t \sum_{j=1}^N P_{t-s}(X_j^s - \mathcal{U}_s X_j^Y + X_i^s - \mathcal{U}_s Z_{ij}^Y) dr + \frac{\lambda}{N} \sum_{j=1}^N \tilde{Z}_{ijj}^Y(t).
\]

and \(Y_i^s\) satisfies (2.25) with \(X_i\) replaced by \(X_i^s\) and the initial condition \(Y_i^s(s) = \Phi_i(s) - Z_i(s) + \frac{\lambda}{N} \sum_{j=1}^N \tilde{Z}_{ijj}^Y(s)\). By similar argument as in Lemma 2.3 we deduce

\[
\mathbb{E} \left( \frac{1}{N} \sum_{i=1}^N \|X_i^s(t)\|^2_{C([s,s+2];C^{\frac{1}{2}-\epsilon})} \right) \lesssim 1,
\]
with the proportional constant independent of \(s, t\) and initial condition. We find Lemma B.1 also holds with \(Y_i, X_i\), and \(X_i\) replaced by \(Y_i^s, X_i^s\), respectively. For every \(\tau \in \mathbb{Z}\) in Lemma 2.2 we know

\[
\sup_{s \geq 0} \mathbb{E} \left[ \sup_{s \leq t \leq s+2} \tau \right] \lesssim 1,
\]
which implies that

\[
\sup_{s \geq 1} \mathbb{E} \sup_{s \leq t \leq s+2} R_N^s(t) \lesssim 1,
\]
for $R_N^s$ given as $R_N$ in the proof of Lemma B.1 with $X_i$ replaced by $X_i^r$. Now by [TW18, Lemma 3.8] and (3.7) we deduce for $1 \leq s \leq t \leq s + 1$
\begin{align*}
E \left( \frac{1}{N} \sum_{i=1}^{N} \|Y_i^s(t)\|_{L^2}^2 \right) \lesssim (t - s)^{-1/2},
\end{align*}
with the proportinal constant independent of $s$ and initial condition, which by taking integration over $t$ for (B.1) implies that for $s \geq 1$
\begin{align*}
\int_{s + \frac{1}{2}}^{s + 2} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|Y_i^s(t)\|_{L^2}^2 \right) dt \lesssim 1,
\end{align*}
(B.2)
with the proportinal constant independent of $s$. Then we know for $T \geq 1$
\begin{align*}
\int_{0}^{T} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|(\Phi_i - Z_i)(t)\|_{L^2}^2 \right) dt \\
\lesssim \sum_{s=1}^{T-1} \int_{s + \frac{1}{2}}^{s + 2} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|X_i^s(t)\|_{L^2}^2 \right) dt + \sum_{s=1}^{T-1} \int_{s + \frac{1}{2}}^{s + \frac{3}{2}} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|Y_i^s(t)\|_{L^2}^2 \right) dt \\
+ \int_{0}^{\frac{T}{2}} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|X_i^s(t)\|_{L^2}^2 \right) dt + \int_{0}^{\frac{T}{2}} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|Y_i^s(t)\|_{L^2}^2 \right) dt \lesssim T,
\end{align*}
where we used Theorem 3.4 for the bound of the integral from 0 to $\frac{T}{2}$. Similarly we obtain
\begin{align*}
\int_{0}^{T} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \|(\Phi_i - Z_i)(t)\|_{L^2}^2 \right) dt \lesssim T
\end{align*}
by using (B.2) and Besov embedding Lemma A.1. Since $\kappa$ is arbitrary, the second result follows. ~$\square$

**Appendix C. Notation index**

We collect some frequently used notations of this paper, with their meanings and the pages where they first occur. Remark that all the objects named with $Q$ or $R$ are explicit combinations of norms of $Z$ (see (2.4)), except that $Q^3_N$ and $Q^{1,1}_N$ also depend on $X$ (defined in (2.6)).

| Symbol | Place introduced | Page |
|--------|-----------------|------|
| $\varphi_i$, $P_i$ | Eq. (2.26) | 16 |
| $Q^0_N$, $Q^1_N$, $Q^2_N$ | Lemma 2.5 | 12 |
| $Q^3_N$, $Q^4_N$ | Proposition 3.6 | 23 |
| $Q^3_N$, $Q^3_N$ | Lemma 3.7 | 24 |
| $Q^2_N$, $Q^3_N$ | Lemma 2.4 | 10 |
| $Q^1_N$ | Lemma 2.4 | 11 |
| $R^0_N$ | Lemma 2.3 | 9 |
| $R^1_N$ | Lemma 3.1 | 18 |
| $R^2_N$ | Proposition 3.5 | 21 |
| $R^3_N$ | Lemma 3.8 | 25 |
| $X_i$ | Eq. (2.6) | 8 |
References

[AK17] S. Albeverio and S. Kusuoka. The invariant measure and the flow associated to the $\varphi^4_3$-quantum field model. arXiv:1711.07108, 2017.

[AS12] M. Anshelevich and A. N. Sengupta. Quantum free Yang-Mills on the plane. J. Geom. Phys., 62(2):330–343, 2012.

[BGHZ21] Y. Bruned, F. Gabriel, M. Hairer, and L. Zambotti. Geometric stochastic heat equations. J. Amer. Math. Soc., 35(1):1–80, 2021.

[Bon81] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. École Norm. Sup. (4), 14(2):209–246, 1981.

[BR82] C. Billionnet and P. Renouard. Analytic interpolation and Borel summability of the $(\hat{\Phi}^4_3)_2$ models. I. Finite volume approximation. Comm. Math. Phys., 84(2):257–295, 1982.

[CC18] R. Catellier and K. Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. Ann. Probab., 46(5):2621–2679, 2018.

[CCHS22] A. Chandra, I. Chevyrev, M. Hairer, and H. Shen. Stochastic quantisation of Yang-Mills-Higgs in 3D. arXiv preprint arXiv:2006.15933, 2020.

[CPS22] A. Chandra, I. Chevyrev, M. Hairer, and H. Shen. Stochastic quantisation of Yang-Mills-Higgs in 3D. arXiv preprint arXiv:2201.03487, 2022.

[CGW20] A. Chandra, T. S. Gunaratnam, and H. Weber. Phase transitions for $\varphi^4_3$. arXiv preprint arXiv:2006.15933, 2020.

[Cha19] S. Chatterjee. Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large $N$ limit. Comm. Math. Phys., 366(1):203–268, 2019.

[Che19] I. Chevyrev. Yang-Mills measure on the two-dimensional torus as a random distribution. Comm. Math. Phys., 372(3):1027–1058, 2019.

[CJ16] S. Chatterjee and J. Jafarov. The $1/N$ expansion for SO($N$) lattice gauge theory at strong coupling. arXiv preprint arXiv:1604.04777, 2016.

[CWZZ21] X. Chen, B. Wu, R. Zhu, and X. Zhu. Stochastic heat equations for infinite strings with values in a manifold. Trans. Amer. Math. Soc., 374(1):407–452, 2021.

[DCPSS17] H. Duminil-Copin, R. Peled, W. Samotij, and Y. Spinka. Exponential decay of loop lengths in the loop equation. Ann. Probab., 46(3):1176–1263, 2018.

[DCPSS18] H. Duminil-Copin, R. Peled, W. Samotij, and Y. Spinka. Exponential decay of loop lengths in the loop equation. Ann. Probab., 46(3):1176–1263, 2018.

[Dim74] J. Dimock. Asymptotic perturbation expansion in the $P(\varphi)^2$ quantum field theory. Comm. Math. Phys., 35:347–356, 1974.

[DPZ96] G. Da Prato and J. Zabczyk. Ergodicity for infinite-dimensional systems, volume 229 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996.

[FMR82] J. Fröhlich, A. Mardia, and V. Rivasseau. Borel summability of the $1/n$ expansion for a quantum field model. Commun. Math. Phys., 84(2):257–295, 1982.

[GH21] M. Gubinelli and M. Hofmanová. Global solutions to elliptic and parabolic $\Phi^4_3$ models in Euclidean space. Comm. Math. Phys., 368(3):1201–1266, 2019.

[GIK15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. Forum Math. Pi, 3:e6, 2015.

[GN74] D. J. Gross and A. Neveu. Dynamical symmetry breaking in asymptotically free field theories. Physical Review D, 10(10):3235, 1974.

[HA14] M. Hairer. A theory of regularity structures. Invent. Math., 198(2):269–504, 2014.

[HA16] M. Hairer. The motion of a random string. arXiv:1605.02192, 2016.

[HM18a] M. Hairer and K. Matetski. Discretisations of rough stochastic PDEs. Ann. Probab., 46(3):1651–1709, 2018.

[HM18b] M. Hairer and J. Mattingly. The strong Feller property for singular stochastic PDEs. Ann. Inst. Henri Poincaré Probab. Stat., 54(3):1314–1340, 2018.

[HSS22] M. Hairer and P. Schönbauer. The support of singular stochastic PDEs. Forum Math. Pi, 10:No. e1, 127, 2022.

[Ja65] A. Jaffe. Divergence of perturbation theory for bosons. Communications in Mathematical Physics, 1(2):127–149, 1965.

[Kup80a] A. Kupiainen. $1/n$ expansion—some rigorous results. In Mathematical problems in theoretical physics (Proc. Internat. Conf. Math. Phys., Lausanne, 1979), volume 116 of Lecture Notes in Phys., pages 208–210. Springer, Berlin-New York, 1980.

[Kup80b] A. J. Kupiainen. $1/n$ expansion for a quantum field model. Comm. Math. Phys., 74(3):199–222, 1980.

[Kup80c] A. J. Kupiainen. On the $1/n$ expansion. Comm. Math. Phys., 73(3):273–294, 1980.

[Lé65] T. Lévy. The master field on the plane. Astérisque, (388):ix+201, 2017.

[MW17a] J.-C. Mourrat and H. Weber. The dynamic $\Phi^4_3$ model comes down from infinity. Comm. Math. Phys., 356(3):673–753, 2017.
[MW17b] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic $\Phi^4$ model in the plane. *Ann. Probab.*, 45(4):2398–2476, 2017.

[MW20] A. Moinat and H. Weber. Space-time localisation for the dynamic $\varphi^4_3$ model. *Communications on Pure and Applied Mathematics*, 73(12):2519–2555, 2020.

[MWX17] J.-C. Mourrat, H. Weber, and W. Xu. Construction of $\Phi^4_3$ diagrams for pedestrians. In *From particle systems to partial differential equations*, volume 209 of *Springer Proc. Math. Stat.*, pages 1–46. Springer, Cham, 2017.

[RWZZ20] M. Röckner, B. Wu, R. Zhu, and X. Zhu. Stochastic heat equations with values in a manifold via Dirichlet forms. *SIAM J. Math. Anal.*, 52(3):2237–2274, 2020.

[SSZZ22] H. Shen, S. A. Smith, R. Zhu, and X. Zhu. Large $N$ limit of the $O(N)$ linear sigma model via stochastic quantization. *Ann. Probab.*, 50(1):131–202, 2022.

[SZZ21] H. Shen, R. Zhu, and X. Zhu. An SPDE approach to perturbation theory of $\varphi^4_3$: asymptoticity and short distance behavior. *arXiv preprint arXiv:2108.11312*, 2021.

[t’H74] G. ’t Hooft. A planar diagram theory for strong interactions. *Nuclear Physics. B*, 72(3):461–473, 1974.

[Tri78] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[TW18] P. Tsatsoulis and H. Weber. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1204–1249, 2018.

[Wil73] K. G. Wilson. Quantum field-theory models in less than 4 dimensions. *Physical Review D*, 7(10):2911, 1973.

[ZZ18] R. Zhu and X. Zhu. Lattice approximation to the dynamical $\Phi^4_3$ model. *Ann. Probab.*, 46(1):397–455, 2018.

[ZZZ20] X. Zhang, R. Zhu, and X. Zhu. Singular HJB equations with applications to KPZ on the real line. *arXiv preprint arXiv:2007.06783*, 2020.

(H. Shen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN - MADISON, USA

Email address: pkushenhao@gmail.com

(R. Zhu) DEPARTMENT OF MATHEMATICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING 100081, CHINA

Email address: zhurongchan@126.com

(X. Zhu) ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

Email address: zhuxiangchan@126.com