MODULAR UNITS AND CUSPIDAL DIVISOR CLASSES ON $X_0(n^2M)$ WITH $n|24$ AND $M$ SQUAREFREE

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ABSTRACT. For a positive integer $N$, let $\mathcal{C}(N)$ be the subgroup of $J_0(N)$ generated by the equivalence classes of cuspidal divisors of degree 0 and $\mathcal{C}(N)(\mathbb{Q}) := \mathcal{C}(N) \cap J_0(N)(\mathbb{Q})$ be its $\mathbb{Q}$-rational subgroup. Let also $\mathcal{C}_0(N)$ be the subgroup of $\mathcal{C}(N)(\mathbb{Q})$ generated by $\mathbb{Q}$-rational cuspidal divisors. We prove that when $N = n^2M$ for some integer $n$ dividing 24 and some squarefree integer $M$, the two groups $\mathcal{C}(N)(\mathbb{Q})$ and $\mathcal{C}_0(N)$ are equal. To achieve this, we show that all modular units on $X_0(N)$ on such $N$ are products of functions of the form $\eta(m\tau + k/h), mh^2|N$ and $k \in \mathbb{Z}$ and determine the necessary and sufficient conditions for products of such functions to be modular units on $X_0(N)$.

1. INTRODUCTION

Let $N$ be a positive integer. In this note, we are primarily concerned with modular units on the modular curve $X_0(N)$, i.e., modular functions on $X_0(N)$ whose divisors are supported on cusps, and the cuspidal subgroup of the Jacobian variety $J_0(N)$ of $X_0(N)$.

To describe relevant results in literature, we recall that a divisor $D \in \text{Div}(X_0(N))$ is said to be cuspidal if its support lies on cusps of $X_0(N)$. We let $\mathcal{C}(N)$ be the subgroup of $J_0(N)$ generated by the equivalence classes of cuspidal divisors of degree 0 on $X_0(N)$, and refer to it as the cuspidal subgroup of the Jacobian variety $J_0(N)$. By a well-known result of Manin and Drinfeld [10], $\mathcal{C}(N)$ is contained in the torsion subgroup $J_0(N)_{\text{tor}}$ of $J_0(N)$. Let also

$$\mathcal{C}(N)(\mathbb{Q}) := \mathcal{C}(N) \cap J_0(N)(\mathbb{Q})$$

and $\mathcal{C}_0(N)$ be the subgroup of $\mathcal{C}(N)$ generated by $\mathbb{Q}$-rational cuspidal divisors of degree 0 on $X_0(N)$. (Here we say a cuspidal divisor $D$ is $\mathbb{Q}$-rational if $\sigma(D) = D$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.) Since the study of $\mathcal{C}(N)$ is equivalent to the study of modular units, we introduce the following two groups:

$$\mathcal{U}(N) := \{\text{modular units on } X_0(N)\}/\mathbb{C}^\times$$

and

$$\mathcal{U}_0(N) := \{f \in \mathcal{U}(N) : \text{div } f \text{ is } \mathbb{Q}\text{-rational}\}/\mathbb{C}^\times.$$

Now we have the inclusions of three groups

$$\mathcal{C}_0(N) \subseteq \mathcal{C}(N)(\mathbb{Q}) \subseteq J_0(N)(\mathbb{Q})_{\text{tor}}.$$

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This work was motivated by a remark about the equality between $\mathcal{C}_0(N)$ and $\mathcal{C}(N)(\mathbb{Q})$ made by Hwajong Yoo in his talk given at Workshop on Eisenstein Ideals and Iwasawa Theory, Beijing, June 17–22, 2019. The second author would like to thank the organizers, Emmanuel Lecouturier in particular, for inviting him to this wonderful workshop. He enjoyed discussions with the participants of the workshop, including Yuan Ren, Ken Ribet, Takao Yamazaki, and Hwajong Yoo. The authors would also like to thank the anonymous referees for many valuable comments that greatly improve the exposition of the paper.
When the level $N$ is $2^r M$ for some odd squarefree integer $M$ and some nonnegative integer $r \leq 3$, every cusp of $X_0(N)$ is $\mathbb{Q}$-rational and hence $\mathcal{E}_\mathbb{Q}(N) = \mathcal{E}(N)(\mathbb{Q})$. However, as pointed out by Ken Ribet and other mathematicians, it is not clear a priori whether $\mathcal{E}_\mathbb{Q}(N)$ and $\mathcal{E}(N)(\mathbb{Q})$ are equal in general. It could happen that even though $D$ itself is not a $\mathbb{Q}$-rational cuspidal divisor, one still has $\sigma(D) \sim D$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ so that $D \in \mathcal{E}(N)(\mathbb{Q})$. For example, for $N = 25$, the cusps $a/5$, $a = 1, \ldots, 4$, are defined over $\mathbb{Q}((e^{2\pi i}/5))$ and they are Galois conjugates of each other, but since $X_0(25)$ has genus 0 and $J_0(25)$ is trivial, any cuspidal divisor class $(a/5) - (b/5)$ is a $\mathbb{Q}$-rational point of the (trivial) Jacobian. In fact, it took quite an effort in [23, Pages 1268–1273] to prove that in the case of $X_1(2p)$, $p$ a prime, two analogously defined groups are indeed equal.

For the second inclusion in (1), Ogg [14] conjectured and later Mazur [11] proved that in the case $N = p$ is a prime, one has $J_0(p)(\mathbb{Q})_{\text{tor}} = \mathcal{E}_\mathbb{Q}(p)$ and the group is cyclic of order $(p - 1)/(p - 1, 12)$ generated by the class of $(0) - (\infty)$. Since then, many mathematicians have tried to extend Mazur’s theorem to general cases. Here we list some known results in literature about $\mathcal{E}_\mathbb{Q}(N)$ and $J_0(N)(\mathbb{Q})_{\text{tor}}$.

(a) Lorenzini [9] showed that when $N = p^n$ is a prime power with $p \geq 5$ and $p \not\equiv 11 \pmod{12}$, one has
$$\mathcal{E}_\mathbb{Q}(p^n) \otimes \mathbb{Z}[1/2p] \simeq J_0(p^n)(\mathbb{Q})_{\text{tor}} \otimes \mathbb{Z}[1/2p].$$

(b) Assume that $N = p^n$ is a prime power with $p \geq 5$. Ling [8] computed the cardinality and the structure of $\mathcal{E}_\mathbb{Q}(N)$ and proved that
$$\mathcal{E}_\mathbb{Q}(p^n) \otimes \mathbb{Z}[1/6p] \simeq J_0(p^n)(\mathbb{Q})_{\text{tor}} \otimes \mathbb{Z}[1/6p].$$

One key property used in the proof is the fact that all modular units in $\mathcal{W}_\mathbb{Q}(N)$ are products of the Dedekind eta functions. (This follows from either [13, Theorem 1] or [7, Proposition 3.2.1].) Later on, Yamazaki and Yang [25] obtained a basis for $\mathcal{W}_\mathbb{Q}(p^n)$, $p \geq 5$, using Ling’s cuspidal class number formula.

(c) Assume that $N$ is squarefree. Takagi [21] also used the fact that all modular units on $X_0(N)$ are products of the Dedekind eta functions to compute the cuspidal class number and described the structure of $\mathcal{E}(N)(= \mathcal{E}_\mathbb{Q}(N))$. Note that the special case where $N$ is a product of two primes was treated earlier in [1].

(d) Again, assume that $N$ is squarefree. Ohta [15] showed that
$$\mathcal{E}(N) \otimes \mathbb{Z}[1/6] \simeq J_0(N)(\mathbb{Q})_{\text{tor}} \otimes \mathbb{Z}[1/6],$$

and in addition, if $3 \nmid N$, then
$$\mathcal{E}(N) \otimes \mathbb{Z}[1/2] \simeq J_0(N)(\mathbb{Q})_{\text{tor}} \otimes \mathbb{Z}[1/2].$$

(2)

In [27], Yoo showed that if $p$ is a prime greater than 3 such that either $p \not\equiv 1 \pmod{9}$ or $3(p-1)/3 \not\equiv 1 \pmod{p}$, then (2) also holds for $N = 3p$.

(e) Ren [16] proved that for any positive integer $N$,
$$J_0(N)(\mathbb{Q})_{\text{tor}} \otimes \mathbb{Z}[1/N'] \simeq 0,$$

where $N' = 6N \prod_{p|N}(p^2 - 1)$. That is, for a prime $p'$, the $p'$-primary part of $J_0(N)(\mathbb{Q})_{\text{tor}}$ is trivial unless $p'$ divides $N'$.

(f) In a very recent preprint [28], Yoo completely determined the structure of $\mathcal{E}_\mathbb{Q}(N)$ for all $N$.

Note that the cuspidal divisor subgroups of $J_1(N)$, the Jacobian of $X_1(N)$, have also been studied by many authors. See, for instance, [8] [18] [19] [20] [22] [23] [26] [29].
In this note, we will consider the case where \(N\) is of the form \(N = n^2M\) for some integer \(n\) dividing 24 and squarefree \(M\) (2\(M\) and 3\(M\) permitted). The primary reason for considering such levels is that modular units in these cases can still be expressed in terms of the Dedekind eta functions. The key observation is that if \(h|24\), then \(\eta(m\tau + k/h)\) is modular on \(\Gamma_0(h^2m)\) in the sense that
\[
\eta(m\gamma\tau + k/h) = \epsilon \sqrt{\frac{c\tau + d}{i}} \eta(m\tau + k/h)
\]
for all \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(h^2m)\) for some root of unity \(\epsilon\) depending on \(\gamma\) (see Lemma 5 below). Our approaches and results rely crucially on this observation and cannot be extended to the cases \(n \nmid 24\).

Throughout the remainder of the paper, we assume that \(N = n^2M\) with \(n|24\) and \(M\) squarefree. For a positive divisor \(m\) of \(N\), let \(h = h(m)\) be the largest integer such that \(mh^2|N\), and for an integer \(k\), we define
\[
\eta_{m,k}(\tau) := \eta(m\tau + k/h) = e^{2\pi i (m\tau + k/h)/24} \prod_{\ell=1}^{\infty} \left(1 - e^{2\pi i k/h} q^{m\ell}\right), \quad q = e^{2\pi i \tau}.
\]

Our first main result gives the necessary and sufficient conditions for a product of \(\eta_{m,k}\) to be a modular function on \(X_0(N)\). The conditions are reminiscent of a well-known criterion (see [7, Proposition 3.2.8]) for \(\prod_{d|N} \eta(d\tau)^{e_d}\) to be a modular function on \(X_0(N)\). Note that since \(\eta_{m,k}\) and \(\eta_{m,k+h}\) differ only by a root of unity, we may assume that \(k\) is in the range \(0 \leq k \leq h(m) - 1\).

**Theorem 1.** Let \(N = n^2M\) with \(n|24\) and \(M\) squarefree and let \(\eta_{m,k}\) and \(h(m)\) be defined as above. Then a product of the form
\[
\prod_{m|N} \prod_{k=0}^{h(m)-1} \eta_{m,k}^{e_{m,k}}, \quad e_{m,k} \in \mathbb{Z},
\]
is a modular function on \(X_0(N)\) if and only if the integers \(e_{m,k}\) satisfy the following conditions:

(a) \(\sum_{m,k} e_{m,k} = 0\),

(b) \(\sum_{m,k} e_{m,k} m \equiv 0 \mod 24\),

(c) \(\sum_{m,k} e_{m,k} N(h(m),k)^2 m^2 \equiv 0 \mod 24\),

(d) (i) In the case \(n = 3\) is odd,
\[
\sum_{m,k} e_{m,k} k \equiv 0 \mod 3
\]
and

(ii) In the case \(n\) is even,
\[
\prod_{m,k} m^{e_{m,k}}
\]
is the square of a rational number.

(iii) In the case \(n\) is even,
\[
\sum_{m,k} e_{m,k} \left(\frac{kn}{h(m)} + \frac{n}{2} \text{ord}_2(m)\right) \equiv 0 \mod n
\]
and the odd part of (4) is the square of a rational number, where \( \ord_2(m) \)

denotes the 2-adic valuation of \( m \).

Here the summation \( \sum_{m,k} \) and the product \( \prod_{m,k} \)

are understood to be over pairs \((m, k)\)
of integers with \( m \mid N \) and \( 0 \leq k \leq h(m) - 1 \).

**Remark 1.** Note that the first three conditions represent the requirements that the weight

is 0, and the orders of the function at the cusps \( \infty \) and 0 are integers, respectively. See Corollary 4 below.

Noticing that the number of such functions \( \eta_{m,k} \) exceeds the rank of \( \mathcal{M}(N) \), in the next theorem, we shall find a subset of such functions so that every modular unit is uniquely

expressed as a product of functions from this subset.

**Theorem 2.** Let \( N = n^2 M \) with \( n \mid 24 \) and \( M \) squarefree. Then every modular unit on

\( X_0(N) \) can be uniquely expressed as

\[
ce \prod_{m \mid N} \eta_{m,k}^{e_{m,k}}
\]

for some nonzero complex numbers \( e \) and integers \( e_{m,k} \) satisfying the conditions in Theorem 7 where \( \phi \) is Euler’s totient function.

**Remark 2.** The interested reader may use the following relations

(6)

\[
\eta(\tau + 1/2) = e^{2\pi i/48} q^{1/24} \prod_{n \text{ even}} (1 - q^n) \prod_{n \text{ odd}} (1 + q^n) \\
= e^{2\pi i/48} q^{1/24} \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{n=1}^{\infty} \frac{1 + q^n}{1 + q^{2n}} \\
= e^{2\pi i/48} q^{1/24} \prod_{n=1}^{\infty} (1 - q^{2n}) \prod_{n=1}^{\infty} \frac{1 - q^{2n})^2}{(1 - q^n)(1 - q^{4n})} \\
= e^{2\pi i/48} \frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)} \\
\eta(\tau + 1/3)\eta(\tau + 2/3) = e^{2\pi i/24} q^{1/12} \prod_{\text{3}}(1 - q^n)^2 \prod_{\text{3}}(1 - e^{2\pi i/3}) q^n(1 - q^{4\pi i/3}) q^n \\
= e^{2\pi i/24} q^{1/12} \prod_{n=1}^{\infty} (1 - q^{3n})^2 \prod_{n=1}^{\infty} \frac{1 - q^{3n}}{1 - q^n} \\
= e^{2\pi i/24} q^{1/12} \prod_{n=1}^{\infty} (1 - q^{3n}) \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^2}{(1 - q^{3n})(1 - q^n)} \\
= e^{2\pi i/24} \frac{\eta(3\tau)^4}{\eta(\tau)\eta(9\tau)}
\]

to check that the remaining \( \eta_{m,k} \) can all be expressed as a product of those in (3). For instance, we have

(7)

\[
\eta(\tau + 3/4) = \eta(\tau + 1/4 + 1/2) = \epsilon \frac{\eta(2\tau + 1/2)^3}{\eta(\tau + 1/4)\eta(4\tau)} \\
= \frac{\epsilon}{\eta(\tau + 1/4)\eta(4\tau)} \left( \frac{\eta(4\tau)^3}{\eta(2\tau)\eta(8\tau)} \right)^3 = \frac{\epsilon \eta(4\tau)^8}{\eta(\tau + 1/4)\eta(2\tau)^3\eta(8\tau)^3}.
\]
Here $\epsilon$ represents some root of unity and may not be the same at each occurrence.

As an application of our determination of modular units, in the next theorem, we prove that $\mathcal{C}_Q(N) = \mathcal{C}(N)(\mathbb{Q})$ for $N = n^2M$ with $n|24$ and $M$ squarefree.

**Theorem 3.** Assume that $N = n^2M$ with $n|24$ and $M$ squarefree. Then $\mathcal{C}_Q(N) = \mathcal{C}(N)(\mathbb{Q})$.

**Remark 3.** In fact, our main motivation for undertaking this research project is to seek for examples with $\mathcal{C}(N)(\mathbb{Q}) \neq \mathcal{C}_Q(N)$. Such an example will be a direct counterexample to the conjecture that $\mathcal{C}_Q(N) = \mathcal{J}_0(N)(\mathbb{Q})_{tor}$. However, after computing many examples and studying properties of modular units more thoroughly, we found that the equality actually holds for levels under consideration. In view of Theorem 3 it is perhaps reasonable to conjecture that the two inclusions in (1) are both equalities for all levels $N$.

2. **Modular Units on $X_0(N)$**

We remind the reader that the level $N$ is assumed to be $n^2M$ with $n|24$ and $M$ squarefree. We first recall the transformation formula for the Dedekind eta function.

**Lemma 4** ([24, Pages 125–127]). For $\gamma = \left( \begin{array} {cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z})$, the transformation formula for $\eta(\tau)$ is given by, for $c = 0$,

$$\eta(\tau + b) = e^{2\pi ib/24}\eta(\tau),$$

and, for $c \neq 0$,

$$\eta(\gamma\tau) = \epsilon(a, b, c, d)\sqrt{c\tau + d}i\eta(\tau)$$

with

$$\epsilon(a, b, c, d) = \begin{cases} 
    d/c & \text{if } c \text{ is odd}, \\
    c/d & \text{if } d \text{ is odd}, 
\end{cases}$$

where $\left( \frac{d}{c} \right)$ is the Jacobi symbol.

**Lemma 5.** Assume that $m|N$. Let $h = h(m)$ and for an integer $k$, let $\eta_{m,k}(\tau) := \eta(m\tau + k/h)$.

(a) For $\gamma = \left( \begin{array} {cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$, we have, when $c = 0$

$$\eta_{m,k}(\tau + b) = e^{2\pi ibm/24}\eta_{m,k}(\tau)$$

and when $c \neq 0$,

$$\eta_{m,k}(\gamma\tau) = \epsilon \left( a + \frac{kc}{hm}, k(d - a)h + bm - \frac{k^2c}{h^2m}d - \frac{kcd}{hm} \right) \sqrt{\frac{c\tau + d}{i}}\eta_{m,k}(\tau),$$

where $\epsilon$ is defined by (8).

(b) Let $a/c$ with $c|N$ be a cusp of $X_0(N)$. Write $(mha + kc)/hc$ in the reduced form $a'/c'$, $(a', c') = 1$. Then the order of $\eta_{m,k}(\tau)$ at $a/c$ is

$$\frac{cN}{24m(c')^2(c, N/c)}.$$
Note that when $k = 0$, we have $c' = c/(m, c)$ and the formula shows that the order of $\eta(m\tau)$ at $a/c$ is

$$\frac{N(m, c)^2}{24mc(c, N/c)},$$

agreeing with the formula given in [7, Proposition 3.2.8].

**Proof.** It is clear that $\eta_{m,k}(\tau+b) = e^{2\pi ibm/24}\eta_{m,k}(\tau)$. Let $\sigma = \left(\begin{smallmatrix} mh & k \\ 0 & h \end{smallmatrix}\right)$ so that $\eta_{m,k}(\tau) = \eta(\sigma \tau)$. Let

$$\gamma' = \sigma \gamma \sigma^{-1} = \left(\begin{array}{cc} a + kc/mh & k(d-a)/h + bm - k^2c/mh^2 \\ c/m & d - kc/mh \end{array}\right).$$

Since $h$ is a divisor of 24, we have $a \equiv d \mod h$ and $\gamma' \in SL(2, \mathbb{Z})$. Then by Lemma 4

$$\eta_{m,k}(\gamma \tau) = \eta(\sigma \gamma \tau) = \eta(\gamma' \sigma \tau)$$

$$= \epsilon \left(\begin{array}{cc} a + \frac{kc}{mh} & k(d-a)/h + bm - \frac{k^2c}{mh^2} \\ c/m & d - \frac{kc}{mh} \end{array}\right) \sqrt{\frac{c\tau + d}{i}} \eta_{m,k}(\tau).$$

We now prove Part (b).

Let $a/c$ with $c|N$ be a cusp of $X_0(N)$. Let $b, d, b', d'$ be integers such that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. We check that

$$m\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) + \frac{k}{h} = \left(\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right) \left(\frac{c(\epsilon \tau + d)}{m(c')^2} - \frac{d'}{c'}\right).$$

It follows that

$$\eta_{m,k}(\gamma \tau) = u \sqrt{\frac{c\tau + d}{i}} \eta \left(\frac{c(\epsilon \tau + d)}{m(c')^2} - \frac{d'}{c'}\right)$$

for some nonzero complex number $u$. Since a cusp of level $c$ on $X_0(N)$ has width $N/c(c, N/c)$, we find that the order of $\eta_{m,k}(\tau)$ at $a/c$ is

$$\frac{c^2}{24m(c')^2} \cdot \frac{N}{e(c, N/c)} = \frac{\epsilon N}{24m(c')^2(c, N/c)}.$$

This completes the proof of the lemma. □

**Corollary 6.** If the product in (5) is a modular function on $\Gamma_0(N)$, then the integers $e_{m,k}$ satisfy

(9) \hfill \sum_{m,k} e_{m,k} = 0,

(10) \hfill \sum_{m,k} me_{m,k} \equiv 0 \mod 24,

and

(11) \hfill \sum_{m,k} \frac{N(h(m), k)^2}{mh(m)^2} e_{m,k} \equiv 0 \mod 24.

---

1This is where the assumption $h|24$ is required. For general $h$, $\eta_{m,k}| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ will equal to $\epsilon \eta_{m,k'}$ for some root of unity $\epsilon$, where $k'$ is an integer satisfying $ak' \equiv dk \mod h$. 

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Proof. In order for the product to be a modular function on $\Gamma_0(N)$, it is necessary that its weight is 0 and its orders at $\infty$ and 0 are integers. The condition that the weight is 0 translates to (9). Also, the order of $\eta_{m,k}$ at $\infty$ is $m/24$. Hence the condition that the order at $\infty$ is an integer translates to (10). Finally, the order of $\eta_{m,k}$ at 0 is determined by Part (b) of Lemma 5 (with $a = 0$, $c = 1$, $a' = k/(h(m), k)$, and $c' = h(m)/(h(m), k)$). We find that it is

$$\frac{N}{24m(h(m)/(h(m), k))^2}.$$  

This explains the condition (11). \qed

Lemma 7. Assume that

$$f(\tau) = \prod_{m|N} \prod_{k=0}^{\gamma(m)-1} \eta_{m,k}^{\gamma_{m,k}}$$

is a product satisfying the three conditions in Corollary 6. Let $G$ be the subgroup of $\Gamma_0(N)$ generated by $\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)$ and $\left(\begin{array}{cc} 1 & 0 \\ N & 1 \end{array}\right)$. Then for $\gamma \in \Gamma_0(N)$, the value of the root of unity $\mu$ in $f(\gamma \tau) = \mu f(\tau)$ depends only on the right coset $G\gamma$ of $\gamma$ in $\Gamma_0(N)$.

Proof. Let $\sigma = (\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array})$ and $\sigma' = (\begin{array}{cc} 1 & 0 \\ N & 1 \end{array})$. The condition in (10) clearly implies that $f(\sigma'\gamma \tau) = f(\gamma \tau)$ for all $\gamma \in \Gamma_0(N)$. Likewise, since $\sigma'$ is a generator of the isotropy subgroup of the cusp 0, the condition (11) implies that

$$f(\sigma'\gamma \tau) = f(\gamma \tau)$$

for all $\gamma \in \Gamma_0(N)$. (More concretely, we may set $g(\tau) = f(-1/N\tau)$ and verify that $f(\sigma'\tau) = f(\tau)$ holds if and only if $g(\tau - 1) = g(\tau)$ holds. Then notice that the latter follows from (11).) This proves the lemma. \qed

Lemma 8. Let $G$ be the subgroup of $\Gamma_0(N)$ generated by $\sigma = (\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array})$ and $\sigma' = (\begin{array}{cc} 1 & 0 \\ N & 1 \end{array})$. Then every right coset in $G\backslash \Gamma_0(N)$ contains an element $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$ such that $24N|c$.

Proof. Assume that $\gamma = (\begin{array}{cc} a & b \\ c & d \end{array}) \in \Gamma_0(N)$. Let $s = (a, 6)$ and $t = 6/s$. Since $(a, c) = 1$, we have $(c, s) = 1$ and hence $(a + tc, 6) = 1$. Let $c' = c/N$ and $r$ be an integer such that $r(a + tc) + c' \equiv 24$. Now

$$\left(\begin{array}{cc} 1 & 0 \\ rN & 1 \end{array}\right) \left(\begin{array}{cc} 1 & t \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \left(\begin{array}{cc} a + tc \\ d + rN(b + td) \end{array}\right).$$

By our choice of $r$, the $(2, 1)$-entry of the last matrix is divisible by $24N$. This proves the lemma. \qed

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let

$$f(\tau) = \prod_{m|N} \prod_{k=0}^{\gamma(m)-1} \eta_{m,k}^{\gamma_{m,k}}.$$  

By Corollary 6, in order for $f$ to be a modular function on $\Gamma_0(N)$, it is necessary that $f$ satisfies the three conditions (9), (10), and (11), which we assume from now on.

\[\text{This proof is suggested by one of the referees.}\]
Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. By Lemmas 7 and 8 we may assume that $24N|c$. When $c = 0$, i.e., when $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we have
\[ \eta_{m,k}(\tau + b) = e^{2\pi ibm/24}\eta_{m,k}(\tau) \]
and hence $f(\tau + b) = f(\tau)$ by Condition (10) that $\sum_{m,k} me_{m,k} \equiv 0 \mod 24$.

When $c \neq 0$, we apply Lemma 5 and obtain
\[ \eta_{m,k}(\gamma\tau) = \epsilon \left( a + \frac{kc}{hm}, bm + \frac{k(d-a)}{h} - \frac{k^2c}{h^2m}, \frac{c}{m}, d - \frac{kc}{hm} \right) \sqrt{\frac{c\tau + d}{i}} \eta_{m,k}(\tau), \]
where $\epsilon$ is given by (8). Since $24|N$, we have
\[ 24 \mid \frac{k}{hm}, \frac{k^2c}{h^2m}, \frac{c}{m} \]
for all $m|N$ and all $k$. Hence,
\[ \epsilon \left( a + \frac{kc}{hm}, bm + \frac{k(d-a)}{h} - \frac{k^2c}{h^2m}, \frac{c}{m}, d - \frac{kc}{hm} \right) = \left( \frac{c/m}{d-\frac{kc}{hm}} \right) e^{2\pi iS/24}, \]
where
\[ S = d \left( bm + \frac{k(d-a)}{h} + 3 \right). \]
Now $h$ is relatively prime to $d - \frac{kc}{hm}$ since $h^2m|c$ and $(d, N) = 1$. Therefore,
\[ \left( \frac{c/m}{d-\frac{kc}{hm}} \right) = \left( \frac{c/h^2m}{d-\frac{kc}{hm}} \right) = \left( \frac{c/h^2m}{d} \right) = \left( \frac{cm}{d} \right). \]
It follows that, by (9), $f(\gamma\tau) = \mu_1\mu_2f(\tau)$, where
\[ \mu_1 = \prod_{m,k} \left( \frac{m}{d} \right)^{e_{m,k}}, \quad \mu_2 = \exp \left\{ \frac{2\pi i}{24} \sum_{m,k} e_{m,k} \left( bdm + \frac{kd(d-a)}{h} \right) \right\}. \]
Since $f$ is assumed to satisfy (10), we have
\[ \sum_{m,k} e_{m,k} bdm \equiv 0 \mod 24. \]
Also, because $24|c$, we have $24|(d-a)$, say, $d-a = 24d'$. We deduce that
\[ \mu_2 = \exp \left\{ \frac{2\pi i}{n} \sum_{m,k} e_{m,k} \frac{kn}{h} \right\}. \]
Since the value of $\mu_1$ can only be $\pm 1$ and that of $\mu_2$ is an $n$th root of unity, when $n = 3$, we need to have $\mu_1 = \mu_2 = 1$. By varying $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we conclude that in the case $n = 3$, $f(\gamma\tau) = f(\tau)$ holds for all $\gamma \in \Gamma_0(N)$ if and only if $\prod_{m,k} me_{m,k}$ is the square of a rational number and
\[ \sum_{m,k} e_{m,k} \frac{kn}{h} \equiv 0 \mod 3, \]
which is the equivalent to $\sum_{m,k} e_{m,k} k \equiv 0 \mod 3$.

In the case $n$ is even, we need $\mu_1 = \mu_2 = 1$ or $\mu_1 = \mu_2 = -1$. As $d$ is relatively prime to $n$ and there are $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ such that $24|c$ and $d' = (d-a)/24$ is also relatively prime to $n$, we find that the sum $\sum_{m,k} e_{m,k} kn/h$ must be a multiple of $n/2$, say,
\[ \sum_{m,k} e_{m,k} \frac{kn}{h} = \frac{k'n}{2}, \]
for some \( k' \in \mathbb{Z} \). Then
\[
\mu_2 = (-1)^d k'.
\]

Now since \( 32 | c \), we have
\[
(-1)^d = (-1)^{(d-a)/8} = (-1)^{(d^2-1)/8} = \left( \frac{2}{d} \right).
\]

It follows that
\[
\mu_1 \mu_2 = \left( \frac{2}{d} \right)^k \prod_{m,k} \left( \frac{m}{d} \right)^{e_{m,k}}.
\]

By varying \( \left( \frac{a}{c} \right) \in \Gamma_0(N) \), we see that \( f \) is a modular function on \( \Gamma_0(N) \) if and only if
\[
2^k \prod_{m,k} m^{e_{m,k}}
\]
is the square of a rational number, in addition to the three conditions \((\text{II}), (\text{III}), \) and \( (\text{IV}) \), or equivalently, the odd part of \( \prod_{m,k} m^{e_{m,k}} \) is the square of a rational number and
\[
\sum_{m,k} e_{m,k} \left( \frac{kn}{h} + \frac{n}{2} \text{ord}_2(m) \right) \equiv 0 \mod n.
\]

This completes the proof of Theorem \( \text{I} \) \( \square \)

We next prove Theorem \( \text{II} \).

**Proof of Theorem \( \text{II} \)** Theorem \( \text{II} \) will follow once we prove the following four claims.

(a) The number of pairs \( (m, k) \) with \( m | N \) and \( 0 \leq k \leq \phi(h(m)) - 1 \) is equal to the number of cusps of \( X_0(N) \).

(b) There are no multiplicative relations among \( \eta_{m,k} \) with \( m | N \) and \( 0 \leq k \leq \phi(h(m)) - 1 \).

(c) Let \( \mathcal{U}_0 \) be the subgroup of \( \mathcal{U} (N) \) formed by products \( \prod_{m | N} \prod_{k=0}^{\phi(h(m)) - 1} \eta_{m,k} \) satisfying the conditions in Theorem \( \text{I} \) Then \( \mathcal{U}_0 \) is of finite index in \( \mathcal{U} (N) \), which implies that if \( g \in \mathcal{U} (N) \), then there exists a positive integer \( \ell \) such that \( cg^\ell \) is in \( \mathcal{U}_0 \) for some non-zero complex number \( c \).

(d) If \( g \in \mathcal{U} (N) \) and \( \ell \) is a positive integer such that \( cg^\ell \) is in \( \mathcal{U}_0 \) for some \( c \in \mathbb{C}^\times \), then \( c'g \in \mathcal{U}_0 \) for some \( c' \in \mathbb{C}^\times \).

To prove Claim (a), we first observe that for a given divisor \( h_0 \) of \( n \), a divisor \( m \) of \( N \) satisfies \( h(m) = h_0 \) if and only if \( m | N/h_0^2 \) and \( N/mh_0^2 \) is squarefree. Let \( \mu \) be the Mobius function so that \( \mu^2 \) is the characteristic function of squarefree integers. Then the number of pairs \( (m, k) \) with \( m | N \) and \( 0 \leq k \leq \phi(h(m)) - 1 \) is
\[
\sum_{h | n} \phi(h) \sum_{m | N/h^2} \mu(m)^2.
\]

On the other hand, the number of cusps of \( X_0(N) \) is
\[
\sum_{m | N} \phi(m/N/m) = \sum_{h | n} \phi(h) \sum_{m | N/h^2, (m', N/m'h^2) = 1} 1 = \sum_{h | n} \phi(h) 2^{\omega(N/h^2)},
\]
where for a positive integer \( k \), \( \omega(k) \) denotes the number of prime factors of \( k \). Now we check that both functions \( k \mapsto \sum_{m | k} \mu(m)^2 \) and \( k \mapsto 2^{\omega(k)} \) are multiplicative and agree
on prime powers. Therefore, we have
\[ \sum_{m \mid N/h^2} \mu(m)^2 = 2^{\omega(N/h^2)}. \]
This proves Claim (a).

We next prove Claim (b). Assume that \( e_{m,k} \) are integers such that
\[ \prod_{m \mid N} \prod_{k=0}^{\phi(h(m)) - 1} \eta_{e_{m,k}} \]
is a constant function. Considering the second term in its Fourier expansion, we find that
\[ \sum_{k=0}^{\phi(n) - 1} e_{1,k} \zeta_n^k = 0, \quad \zeta_n = e^{2\pi i / n}. \]
Recall that \( 1, \ldots, \zeta_n^{\phi(n) - 1} \) form a basis of \( \mathbb{Q}(\zeta_n) \) over \( \mathbb{Q} \) (see, for instance, [12, Theorem 6.4]). Hence \( e_{1,k} = 0 \) for all \( k = 0, \ldots, \phi(n) - 1 \). Similarly, by considering the Fourier coefficients of \( g^m \) for the next divisor \( m \) of \( N \), we find that \( e_{m,k} = 0 \) for all \( k = 0, \ldots, \phi(h(m)) - 1 \) for the next divisor \( m \) of \( N \). Continuing in this way, we find that \( e_{m,k} = 0 \) for all \( (m,k) \).

For Claim (c), we observe that \( \mathcal{W}_0 \) contains at least those products having \( 24 \mid e_{m,k} \) for all \( e_{m,k} \) and \( \sum e_{m,k} = 0 \). It follows that, by Claim (b), the rank of \( \mathcal{W}_0 \) is at least
\[ \# \{(m,k) : m \mid N, 0 \leq k \leq \phi(h(m)) - 1 \} - 1, \]
which, by Claim (a), is equal to the number of cusps of \( X_0(N) \) minus 1. Therefore, \( \mathcal{W}_0 \) and \( \mathcal{W}(N) \) have the same rank. We now prove Claim (d).

Let \( g \) be a modular unit on \( X_0(N) \). Without loss of generality, we may assume that the leading coefficient of \( g \) is 1. Since \( g \) is naturally also a modular unit on \( X(N) \), by [4, 5], \( g \) is a product of Siegel functions and, in the case \( N \) is even, also functions of the forms \( q^{-d/48} \prod_n (1 + q^{d(n+1/2)}) \) and \( q^{d/24} \prod_n (1 + q^{dn}). \) (We refer the reader to [6] for the definition of Siegel functions.) Hence all its Fourier coefficients are algebraic integers. Also, by Claim (c), there exists a positive integer \( \ell \) such that \( g^\ell \in \mathcal{W}_0 \) up to a scalar, say,
\[ g^\ell = e \prod_{m \mid N} \prod_{k=0}^{\phi(h(m)) - 1} \eta_{e_{m,k}}. \]

Now for convenience, for a Puiseux series \( f \) in \( g \), we let
\[ S(f) = \frac{\text{second nonzero term of } f}{\text{leading term of } f}. \]
Comparing the \( \ell \)th roots of the two sides of (12), we find that
\[ S(g) = -\left( \sum_{k=0}^{\phi(n) - 1} \frac{e_{1,k}}{\ell^k} \zeta_n^{k} \right) q, \quad \zeta_n = e^{2\pi i / n}. \]
Since \( S(g) \) is an algebraic integer and \( 1, \zeta_n, \ldots, \zeta_n^{\phi(n) - 1} \) form an integral basis for the ring of integers in \( \mathbb{Q}(\zeta_n) \), we must have \( e_{1,k}/\ell \in \mathbb{Z} \) for all \( k \). By the same token, by considering \( S(g \prod_k \eta_{e_{1,k}/\ell^k}) \), we deduce that \( e_{m,k}/\ell \in \mathbb{Z} \) for all \( k \) for the next divisor \( m \) of \( N \). Continuing in this way, we conclude that \( g \in \mathcal{W}_0 \) up to a scalar. This completes the proof of Theorem.
\[ \square \]
3. THE TWO GROUPS $\mathcal{U}(N)(\mathbb{Q})$ AND $\mathcal{U}_0(N)$

We will prove Theorem 3 in this section. Let $D$ be a cuspidal divisor of degree 0 on $X_0(N)$ such that $D^\sigma \sim D$ for all $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$. (Here $\sim$ denotes the linear equivalence between divisors.) Our goal is to construct a $\mathbb{Q}$-rational cuspidal divisor $D'$ such that $D \sim D'$.

Assume that $[D]$ has order $r$ in $J_0(N)$ and $f$ is a modular unit such that $\text{div } f = rD$. By Theorem 2 we have

$$f = \prod_{m|N} \prod_{k=0}^{\phi(h(m))-1} \eta_{m,k}^{e_{m,k}}$$

for some integers $e_{m,k}$ satisfying the four conditions in Theorem 1. We first describe how $\text{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$ acts on $\text{div } f$.

**Lemma 9.** For an integer $\ell$ with $(\ell, N) = 1$, let $\sigma_{\ell}$ be the element of $\text{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$ that maps $e^{2\pi i/n}$ to $e^{2\pi i/n\ell}$. We have

$$(\text{div } \eta_{m,k})^{\sigma_{\ell}} = \text{div } \eta_{m,\ell k}.$$  

**Proof.** Let $\ell'$ be an integer such that $\ell\ell' \equiv 1 \mod N$. We first remark that because $n|24$, we have $\ell \equiv \ell' \mod n$. Thus, $\text{div } \eta_{m,k} = \text{div } \eta_{m,\ell k}$ since the two functions differ only by a root of unity. We will prove the lemma in the form

$$(\text{div } \eta_{m,k})^{\sigma_{\ell}} = \text{div } \eta_{m,\ell' k}.$$  

Let $a/c$ with $c|N$ be a cusp of $X_0(N)$. Recall that the action of $\sigma_{\ell}$ maps the cusp $a/c$ to the cusp $a/\ell'c$ (see, for instance, [17, Theorem 1.3.1]). This cusp $a/\ell'c$ is equivalent to $\ell' a/c$ (see, for example, [2, Proposition 2.2.3]). Thus, to prove the lemma, it suffices to show that the order of $\eta_{m,k}$ at $a/c$ is equal to that of $\eta_{m,\ell'k}$ at $\ell' a/c$. Now by Lemma 5 the former is

$$cN$$

while the latter is

$$cN$$

where $c'$ and $c''$ are the denominators in the reduced forms of $(mha+kc)/hc$ and $(mh\ell' a + \ell' kc)/hc$, respectively. Since $\ell'$ is relatively prime to $hc$, we have $c' = c''$. Then the lemma follows.

In view of the lemma, we naturally define

$$f^{\sigma_{\ell}} := \prod_{m|N} \prod_{k=0}^{\phi(h(m))-1} \eta_{m,\ell k}^{e_{m,k}}$$

for $\ell$ with $(\ell, N) = 1$ so that

$$\text{div } f^{\sigma_{\ell}} = (\text{div } f)^{\sigma_{\ell}} = rD^{\sigma_{\ell}}.$$  

Our strategy of proving the theorem is as follows.

(a) We first show (case by case) that $r|e_{m,k}$ for all $(m, k)$ with $0 < k < \phi(h(m))$. This is achieved by using the assumption that $D^\sigma \sim D$, $\sigma \in \text{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$, which implies that $f^\sigma/ f$ is the $r$th power of some modular unit on $X_0(N)$. 

Lemma 10. Assume that \( m, h, \) and \( k \) are positive integers such that \( h^2m|N \) and \( (k, h) = 1 \). Then the orders of the functions \( s(\tau) \) defined below at the cusps \( \infty \) and 0 are both integers.

(a) Assume that \( 3|h \). Set \( h' = h/3 \) and let
\[
s(\tau) = \frac{\eta(m\tau + k/h)\eta(3m\tau + k/h')^4}{\eta(m\tau + k/h')^4\eta(9m\tau + k/h')}.\]

(b) Assume that \( 4|h \). Set \( h' = h/4 \) and let
\[
s(\tau) = \frac{\eta(m\tau + k/h)\eta(m\tau + k/h')\eta(4m\tau + k/h')^3\eta(16m\tau + k/h')}{\eta(2m\tau + k/h')^3\eta(8m\tau + k/h')^3}.\]

Proof. The order at \( \infty \) is clearly 0. By Lemma 5 the order of the function in Part (a) at 0 is
\[
\frac{1}{24} \left( \frac{N}{mh^2} + \frac{4N}{3m(h')^2} - \frac{4N}{m(h')^2} - \frac{N}{9m(h')^2} \right) = \frac{N}{24mh^2}(1 + 12 - 36 - 1) = -\frac{N}{mh^2},
\]
while that of the function in Part (b) is
\[
\frac{1}{24} \left( \frac{N}{mh^2} + \frac{N}{m(h')^2} + \frac{3N}{4m(h')^2} + \frac{N}{16m(h')^2} - \frac{3N}{2m(h')^2} - \frac{3N}{8m(h')^2} \right)
= \frac{N}{24mh^2}(1 + 16 + 12 + 1 - 24 - 6) = 0.
\]
The orders are indeed integers. \( \square \)

We now describe our construction of \( g \) case by case. For convenience, all equalities among modular units stated below hold only up to nonzero scalars. Note that the cases \( n = 1 \) and \( n = 2 \) are trivial since every cusp is \( \mathbb{Q} \)-rational in these cases.
3.1. Case \( n = 3 \). Let \( D, f, \) and \( \eta_{m,k} \) be given as above and \( \sigma = \sigma_{-1} \) be the nontrivial element in \( \text{Gal}(\mathbb{Q}(e^{2\pi i/3})/\mathbb{Q}) \). As explained in the description of our strategy, we know that \( f^\sigma/f \) is the \( r \)th power of a modular unit. Now we have

\[
f^\sigma/f = \prod_{m \mid M} \left( \frac{\eta(m\tau - 1/3)}{\eta(m\tau + 1/3)} \right)^{e_{m,1}^r}.
\]

(Note that \( k = 1 \) occurs only when \( m \mid M \).) Using (6), we may write it as

\[
f^\sigma/f = \prod_{m \mid M} \left( \frac{\eta(3m\tau)}{\eta(3m\tau + 1)} \right)^{e_{m,1}^r}.
\]

Since this is the \( r \)th power of some modular unit, by the uniqueness of product expression described in Theorem[2] we must have \( r \mid e_{m,k} \) for all \( (m,k) \) with \( k = 1 \). (Alternatively, we may follow the argument for Claim (c) in the proof of Theorem[2] to show that \( r \mid e_{m,1} \) for all \( m \).) Set \( e_{m,1}' = e_{m,1}/r \). Note that since \( (f^\sigma/f)^{1/r} \) is a modular unit, we have

\[
\sum_{m \mid M} e_{m,1}' \equiv 0 \mod 3,
\]

by Theorem[2].

For \( m \mid M \), define

\[
\tilde{\eta}_{m,1}(\tau) = \frac{\eta(m\tau)^4}{\eta(3m\tau)^4},
\]

and set

\[
g(\tau) = \prod_{m \mid M} \left( \frac{\eta_{m,1}}{\eta_{m,1}} \right)^{e_{m,1}'}.
\]

By Lemma[10] the order of \( g \) at \( \infty \) and \( 0 \) are integers and hence \( g \) satisfies Conditions (a), (b), and (c) in Theorem[1]. Also, by (14). Condition (d) is fulfilled. Hence \( g \) is a modular unit on \( X_0(N) \), and \( D' = D - \text{div} \ g \) is a \( \mathbb{Q} \)-rational cuspidal divisor equivalent to \( D \).

3.2. Case \( n = 6 \). Let \( \sigma \) be the nontrivial element in \( \text{Gal}(\mathbb{Q}(e^{2\pi i/3})/\mathbb{Q}) \). As in the case \( n = 3 \), we can show that \( r \mid e_{m,1} \) for all \( m \mid N \) with \( \phi(h(m)) \neq 1 \) (i.e., \( h(m) = 3 \) or \( h(m) = 6 \)). Set \( e_{m,1}' = e_{m,1}/r \) for such \( m \). The fact that \( (f^\sigma/f)^{1/r} \) is a modular unit implies that

\[
\prod_{m \mid N, h(m)=3,6} \left( \frac{\eta_{m,1}}{\eta_{m,1}} \right)^{e_{m,1}'}
\]

is a modular unit, which in turn shows that

\[
-2 \sum_{m \mid N, h(m)=6} e_{m,1}' + 2 \sum_{m \mid N, h(m)=3} e_{m,1}' \equiv 0 \mod 6,
\]

by Condition (d) of Theorem[1].

For \( m \mid N \) with \( h(m) = 6 \), define

\[
\tilde{\eta}_{m,1}(\tau) = \frac{\eta(m\tau + 1/2)^4}{\eta(3m\tau + 1)^4},
\]

and for \( m \mid N \) with \( h(m) = 3 \), define

\[
\tilde{\eta}_{m,1}(\tau) = \frac{\eta(m\tau)^4}{\eta(3m\tau)^4}.
\]
Note that by (6), \( \eta(m\tau + 1/2) \) can be written as a product of \( \eta(d\tau) \), \( d | N \). Set
\[
g = \prod_{m | M, h(m) = 3, 6} \left( \frac{\eta_{m, 1}}{\bar{\eta}_{m, 1}} \right)^{e'_{m, 1}} \]

By Lemma 5, \( g \) satisfies Conditions (a), (b), and (c) of Theorem 1. Also, the left-hand side of (5) for the function \( g \) is
\[
-2 \sum_{m | N : h(m) = 6} e'_{m, 1} + 2 \sum_{m | N : h(m) = 3} e'_{m, 1},
\]
which by (15), is congruent to 0 modulo 6. Hence Condition (d) is also satisfied, and \( g \) is a modular unit. This proves the theorem for the case \( n = 6 \).

3.3. Case \( n = 4 \). Let \( \sigma \) be the nontrivial element of \( \text{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \). Again, we omit the proof of \( r | e_{m, 1} \) for all \( m | M \). (Note that \( h(m) = 4 \) if and only if \( m | M \).) Set \( e'_{m, 1} = e_{m, 1}/r \) for those \( m \). The fact that
\[
\prod_{m | M} \left( \frac{\eta(m\tau - 1/4)}{\eta(m\tau + 1/4)} \right)^{e'_{m, 1}} = \left( \frac{f\tau}{f} \right)^{1/r}
\]
is a modular unit implies that
\[
\sum_{m | M} e'_{m, 1} \equiv 0 \mod 2,
\]
by Condition (d) in Theorem 1.
Define
\[
\bar{\eta}_{m, 1}(\tau) = \frac{\eta(2m\tau)^3\eta(8m\tau)^3}{\eta(m\tau)^3\eta(4m\tau)^3\eta(16m\tau)},
\]
and
\[
g = \prod_{m | M} \left( \frac{\eta_{m, 1}}{\bar{\eta}_{m, 1}} \right)^{e'_{m, 1}}.
\]
By Lemma 5, \( g \) satisfies Conditions (a), (b), (c) in Theorem 1. Moreover, if
\[
\sum_{m | M} e'_{m, 1} \equiv 0 \mod 4,
\]
then Condition (d) is also fulfilled (note that for an individual \( \eta_{m, 1}/\bar{\eta}_{m, 1} \), the sum of the left-hand side of (5) is congruent to 1 modulo 4 and hence \( g \) is a modular unit on \( X_0(N) \). If
\[
\sum_{m | M} e'_{m, 1} \equiv 2 \mod 4
\]
instead, we replace \( g \) by
\[
g = \frac{\eta(\tau)^2\eta(4\tau)^7}{\eta(2\tau)^7\eta(8\tau)^2} \prod_{m | M} \left( \frac{\eta_{m, 1}}{\bar{\eta}_{m, 1}} \right)^{e'_{m, 1}},
\]
which is a modular unit under the assumption \( \sum_{m | M} e'_{m, 1} \equiv 2 \mod 4 \). Either way, we find that \( D' = D - \div g \) is a \( \mathbb{Q} \)-rational cuspidal divisor linearly equivalent to \( D \).
3.4. **Case** \( n = 8 \). For \( a \in \{ \pm 1, \pm 3 \} \), let \( \sigma_a \) be the element of \( G = \text{Gal}(\mathbb{Q}(e^{2\pi i/8})/\mathbb{Q}) \) that maps \( e^{2\pi i/8} \) to \( e^{2\pi ia/8} \). We have

\[
Q_\sigma f = \prod_{m|N,h(m)=8} \prod_{k=1}^{3} \left( \frac{\eta_{m,k}}{\eta_{m,k}} \right)^{e_{m,k}} \times \prod_{m|N,h(m)=4} \left( \frac{\eta_{m,1}}{\eta_{m,1}} \right)^{e_{m,1}}
\]

for \( a \in \{ \pm 1, \pm 3 \} \). As \( Q_\sigma f / f \) is the \( r \)th power of some modular unit, by considering \( a = 3 \) and using (9), we find that \( r|e_{m,1} \) for \( m \) with \( h(m) = 8 \) and \( r|e_{m,1} \) for \( m \) with \( h(m) = 4 \). By considering \( a = -3 \) instead, we conclude also that \( r|e_{m,1}, e_{m,3} \) for \( m \) with \( h(m) = 8 \). Set \( e'_{m,k} = e_{m,k}/r \) for those \( (m, k) \). The fact that

\[
\prod_{m|N,h(m)=8} \prod_{k=1}^{3} \left( \frac{\eta_{m,k}}{\eta_{m,k}} \right)^{e'_{m,k}} \times \prod_{m|N,h(m)=4} \left( \frac{\eta_{m,1}}{\eta_{m,1}} \right)^{e'_{m,1}} = \left( \frac{f^{\sigma-1}}{f} \right)^{1/r}
\]

is a modular unit implies that

\[
(16) \sum_{m|N,h(m)=8} (6e'_{m,1} + 4e'_{m,2} + 2e'_{m,3}) + 4 \sum_{m|N,h(m)=4} e'_{m,1} \equiv 0 \mod 8,
\]

by Condition (d) of Theorem[1] Define \( \eta_{m,k} \) by

\[
\eta_{m,k} = \frac{\eta(2m\tau + 1/2)^3\eta(8m\tau + 1/2)^3}{\eta(m\tau + 1/2)^3\eta(4m\tau + 1/2)^3\eta(16m\tau + 1/2)}
\]

for \((m, k)\) with \( h(m) = 8 \) and \( k = 1, 3 \), and by

\[
\eta_{m,0} = \frac{\eta(2m\tau)^3\eta(8m\tau)^3}{\eta(m\tau)^3\eta(4m\tau)^3\eta(16m\tau)}
\]

for \((m, k)\) with \((h(m), k) = (4, 1)\) or \((h(m), k) = (8, 2)\). Then set

\[
g_0 = \prod_{m|N,h(m)=8} \prod_{k=1}^{3} \left( \frac{\eta_{m,k}}{\eta_{m,k}} \right)^{e'_{m,k}} \times \prod_{m|N,h(m)=4} \left( \frac{\eta_{m,1}}{\eta_{m,1}} \right)^{e'_{m,1}}
\]

By Lemma[10] this function \( g_0 \) satisfies Conditions (a), (b), and (c) in Theorem[1]. Furthermore, the left-hand side of (15) for \( g_0 \) is congruent to

\[
\sum_{m|N,h(m)=8} (5e'_{m,1} + 6e'_{m,2} + 7e'_{m,3}) + 2 \sum_{m|N,h(m)=4} e'_{m,1}
\]

modulo 8. By (16), this sum is a multiple of 4. We set

\[
g = \begin{cases} g_0, & \text{if the sum is divisible by } 8, \\ g_0\eta(\tau)^2\eta(4\tau)^3/\eta(2\tau)^7\eta(8\tau)^2, & \text{if the sum is congruent to } 4 \text{ modulo } 8. \end{cases}
\]

Then this is a modular unit on \( X_0(N) \), as we are required to construct.

3.5. **Case** \( n = 12 \) or \( n = 24 \). The idea of proof is similar to previous cases, so we will only sketch the argument. Let \( n = 12 \) or \( n = 24 \). By using the property that \( Q_\sigma f / f \) is the \( r \)th power of a modular unit for all \( \sigma \in \text{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q}) \), we can show \( r|e_{m,1} \) for all \((m, k)\) with \( k \neq 0 \). Set \( e'_{m,k} = e_{m,k}/r \) for those \((m, k)\). Then the fact that

\[
\left( \frac{f^{\sigma-1}}{f} \right)^{1/r} = \prod_{m|N} \prod_{k=1}^{\phi(h(m))-1} \left( \frac{\eta_{m,k}}{\eta_{m,k}} \right)^{e'_{m,k}}
\]
is a modular unit implies that
\begin{equation}
2 \sum_{h \mid n} \frac{n}{h} \sum_{(m,k) : h(m) = h, k \neq 0} k e'_{m,k} \equiv 0 \mod n
\end{equation}
by Condition (d) of Theorem 1. Now Lemma 10 provides two procedures to find a function \( \iota \) such that \( \iota \) is a product of \( \eta(d m \tau + k/(h/3)) \) or a product of \( \eta(d m \tau + k/(h/4)) \), depending on whether \( h \) is divisible by 3 or 4, and \( \eta_{m,k}/\iota \) has weight 0, and its order at the cusps \( \infty \) and 0 are integers. Using these two procedures or the combination of the two procedures, we can construct \( \bar{\eta}_{m,k} \) that is a product of \( \eta(d \tau) \), \( d \mid N \), or a product of \( \eta(d \tau + 1/2), d \mid N/4 \), in the case \( h(m) = 6, 8, 24 \) such that it has weight 0 and its order at \( \infty \) and 0 are integers. For instance, for \( \eta_{1,1} = \eta(\tau + 1/24) \), we apply Part (a) of Lemma 10 and find that the function \( \iota \) can be chosen to be
\[
\frac{\eta(\tau + 1/24)\eta(3\tau + 1/8)^4}{\eta(\tau + 1/8)^4\eta(9\tau + 1/8)^3}.
\]
Then for each \( \eta(d \tau + 1/8), d \mid 9 \), we apply Part (b) of the same lemma and find that
\[
\frac{\eta(d \tau + 1/8)\eta(d \tau + 1/2)\eta(4d \tau + 1/2)^3\eta(16d \tau + 1/2)}{\eta(2d \tau + 1/2)^3\eta(8d \tau + 1/2)^3}
\]
has weight 0 and integer orders at \( \infty \) and 0. From these two procedures, we obtain a function \( \bar{\eta}_1 \) of the form \( \prod_{d \mid 144} \eta(d \tau + 1/2) \) such that \( \eta_{1,1}/\bar{\eta}_1 \) satisfies Conditions (a), (b), and (c) of Theorem 1.

Let \( \bar{\eta}_{m,k} \) be the eta-products constructed above and consider
\[
g_0 = \prod_{m \mid N} \phi(h(m))^{-1} \prod_{k=1}^{\phi(h(m))} \left( \frac{\eta_{m,k}}{\bar{\eta}_{m,k}} \right)^{e'_{m,k}}.
\]
By construction, \( g_0 \) satisfies Conditions (a), (b), and (c) in Theorem 1. Furthermore, the left-hand side of (5) for \( g_0 \) is
\[
\sum_{h \mid n} \frac{n}{h} \sum_{(m,k) : h(m) = h, k \neq 0} k e'_{m,k} - \frac{n}{2} \sum_{h=6,8,24} \sum_{(m,k) : h(m) = h, k \neq 0} e'_{m,k}
\]
modulo \( n \). (Note that the second sum is coming from \( \bar{\eta}_{m,k} \) that are products of the form \( \eta(d m \tau + 1/2) \) in the case \( h(m) = 6, 8, 24 \).) By (17), the sum above is congruent to \( 0 \mod n/2 \). Set
\[
g = \begin{cases} 
g_0, & \text{if the sum is divisible by } n, 
g_0 \eta(\tau)^2 \eta(4\tau)^7 / \eta(2\tau)^7 \eta(8\tau)^2, & \text{if the sum is congruent to } n/2 \mod n. 
\end{cases}
\]
Then this function \( g \) is a modular unit on \( X_0(N) \). The rest of proof is the same as before and is omitted.

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