SHOCK WAVES FOR RADIATIVE HYPERBOLIC–ELLIPTIC SYSTEMS

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ABSTRACT. The present paper deals with the following hyperbolic–elliptic coupled system, modelling dynamics of a gas in presence of radiation,

\[
\begin{align*}
  u_t + f(u)_x + Lq_x &= 0, \\
  -q_{xx} + Rq + G \cdot u_x &= 0,
\end{align*}
\]

where \( u \in \mathbb{R}^n \), \( q \in \mathbb{R} \) and \( R > 0, G, L \in \mathbb{R}^n \). The flux function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is smooth and such that \( \nabla f \) has \( n \) distinct real eigenvalues for any \( u \).

The problem of existence of admissible radiative shock wave is considered, i.e. existence of a solution of the form \( (u, q)(x, t) := (U, Q)(x - st) \), such that \( (U, Q)(\pm \infty) = (u_{\pm}, 0) \), and \( u_{\pm} \in \mathbb{R}^n \), \( s \in \mathbb{R} \) define a shock wave for the reduced hyperbolic system, obtained by formally putting \( L = 0 \).

It is proved that, if \( u_- \) is such that \( \nabla \lambda_k(u_-) \cdot r_k(u_-) \neq 0 \) (where \( \lambda_k \) denotes the \( k \)-th eigenvalue of \( \nabla f \) and \( r_k \) a corresponding right eigenvector) and

\[
(\ell_k(u_-) \cdot L) (G \cdot r_k(u_-)) > 0,
\]

then there exists a neighborhood \( U \) of \( u_- \) such that for any \( u_+ \in U \), \( s \in \mathbb{R} \) such that the triple \( (u_-, u_+; s) \) defines a shock wave for the reduced hyperbolic system, there exists a (unique up to shift) admissible radiative shock wave for the complete hyperbolic–elliptic system. The proof is based on reducing the system case to the scalar case, hence the problem of existence for the scalar case with general strictly convex fluxes is considered, generalizing existing results for the Burgers’ flux \( f(u) = u^2/2 \). Additionally, we are able to prove that the profile \((U, Q)\) gains smoothness when the size of the shock \(|u_+ - u_-|\) is small enough, as previously proved for the Burgers’ flux case.

Finally, the general case of nonconvex fluxes is also treated, showing similar results of existence and regularity for the profiles.

1. INTRODUCTION

The dynamics of a gas in presence of radiation, due to high–temperature effects, can be modeled by compressible Euler equations with an additional term in the flux of energy. Dealing with small perturbation of a fixed equilibrium state in one space dimension, this leads to consider an hyperbolic–elliptic coupled system of the form

\[
\begin{align*}
  u_t + f(u)_x + Lq_x &= 0, \\
  -q_{xx} + Rq + G \cdot u_x &= 0,
\end{align*}
\]

where \( x \in \mathbb{R} \), \( t > 0 \), \( u \in \mathbb{R}^n \), \( q \in \mathbb{R} \) and \( R > 0, G, L \in \mathbb{R}^n \) are constant vectors. The flux function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is assumed to be smooth and such that the reduced system

\[
u_t + f(u)_x = 0 \tag{1.1}
\]

is strictly hyperbolic, i.e. \( \nabla f(u) \) has \( n \) distinct real eigenvalues for any state \( u \) under consideration. For later use, we denote such eigenvalues with \( \lambda_1(u) < \cdots < \lambda_n(u) \) and with
\( \ell_1(u), \ldots, \ell_n(u), r_1(u), \ldots, r_n(u) \) the corresponding left and right eigenvectors normalized so that \( \ell_i(u) \cdot r_j(u) = \delta_{ij} \) for any \( i, j \).

The system (1.1) can be obtained from the complete gas dynamics equation with heat–flux radiative term by using a differential approximation of the integral equation for the radiative term. This approach has been proposed in the pioneering paper [4]. After that, equations (1.1) are also called Hamer system for radiating gas (see also [11, 12] for details on the derivation). Since the quantity \( q \) represents the radiative heat-flux term, considering it as a scalar quantity is physically meaningful (see [24] for a complete physical description of the phenomenon).

Under the hyperbolic rescaling \((\partial_t, \partial_x) \mapsto (\varepsilon \partial_t, \partial_x)\), the system (1.1) becomes

\[
\begin{align*}
  u_t + f(u)_x + Lq_x &= 0 \\
  -\varepsilon^2 q_{xx} + Rq + \varepsilon G \cdot u_x &= 0.
\end{align*}
\]

Eliminating the \( q \) variable,
\[
  u_t + f(u)_x = \varepsilon R^{-1} L \otimes G u_{xx} + \varepsilon^2 R^{-1} (u_t + f(u)_x)_{xx}.
\]

Hence the rescaled hyperbolic–elliptic system can be rewritten as a singular perturbation of a system of conservation laws. In particular, for \( \varepsilon \) sufficiently small, the system can be seen as a higher order correction of a viscous system of conservation laws with (degenerate) diffusion term given by the rank–one matrix \( \varepsilon R^{-1} L \otimes G \). In particular, this suggests that qualitative properties of solutions of (1.1) should resemble analogous properties of viscous system of conservation law as soon as: \( u \) varies along a direction not orthogonal to both vectors \( L \) and \( G \) (non degeneracy); variations of \( u \) are mainly in the small frequencies regime (small perturbations). Indeed, failing of the first condition would imply degeneration of the diffusion term \( \varepsilon R^{-1} L \otimes G u_{xx} \), and failing of the second would give to the higher order term \( \varepsilon^2 R^{-1} (u_t + f(u)_x)_{xx} \) a dominating rôle.

The above effects are present also in the scalar case, i.e. \( u \in \mathbb{R} \) (for a complete and introductory presentation see [23]). As noted in [19], in this situation, the \( 2 \times 2 \) system (1.1) enjoys many of the properties of scalar viscous conservation laws: \( L^1 \)–contraction, comparison principle, conservation of mass, constant solutions. Thanks to these properties, global existence and uniqueness of solutions can be proved (see [6] for data in \( BV \), [13] for data in \( L^1 \cap L^{\infty} \), [4] for the multidimensional case for data in \( L^1 \cap L^{\infty} \)). Nevertheless, regularization property does not hold: in [9] and in [16] (with a more detailed description) it is shown that there are initial data such that the corresponding solution to the Cauchy problems develops discontinuity in finite time.

The loss of regularity appears also when dealing with the problem

\[
\text{given } u_\pm, \text{ asymptotic states of an admissible shock wave solution to (1.1), does there exist a traveling wave solution to (1.1) with same speed of the shock and asymptotic states } (u_\pm, 0)?
\]

From now on, we refer to such a solution as a RADIATIVE SHOCK WAVE. In the scalar case and for \( f(s) = \frac{1}{2} s^2 \), in [10] it has proved that the answer is affirmative, but that the

\footnote{The analogous problem for the viscous regularization of a system of hyperbolic conservation laws is sometimes referred to as Gel’fand problem, and the first rigorous mathematical result has been proved in [3].}
profile of the radiative shock wave is discontinuous whenever the hyperbolic shock is large enough, i.e. \(|u_- - u_+|\) is large. The precise statement will be recalled later on. In the same article, stability and decay rate of perturbations are determined. The absence/presence of jumps in small/large radiative shock wave is again a manifestation of regularity properties for solution of (1.1): small transitions can be obtained through smooth solution, large transitions cannot.

For the scalar case, many other results are available in literature. For the sake of completeness, let us mention them, collected in two different groups:

- **Large-time behavior**: to prove stability, possibly with decay rate of the perturbations, of constant states [6, 21], shock profiles [10, 23], rarefaction waves [12], and to find asymptotic profiles for such perturbations [13, 2];
- **Weak solutions**: to find evolution/regularity of discontinuity curves [18], to determine relaxation limit under hyperbolic/parabolic rescaling [1, 13].

The theory for system is still at the very beginning and just few results are available. The first paper in this direction is [7], where global existence, asymptotic behavior and decay rate are proved for the solution to the Cauchy problem with initial data that are small perturbations of constant states. Generalizations have been given in [5] and [8], especially in the precise description of asymptotic profiles (diffusion waves). In particular, it has been proved that, for large time, the solutions to (1.1) are well–approximated by the solutions to the viscous system of conservation laws obtained from (1.3) when disregarding the \(O(\varepsilon^2)\) term. The singular limit counter part, i.e. \(\varepsilon \to 0^+\), has been dealt with in [11]. A different approach for analyzing the singular limit, based on the notion of positively invariant domain, has been considered in [22].

The present paper deals with the problem of proving existence of radiative shock waves in the case of general systems of the form (1.1). First of all let us recall the definition of shock wave and radiative shock wave, where, for the sake of clarity, we refer here to the genuinely nonlinear case.

**Definition 1.1.** A **SHOCK WAVE** of the hyperbolic system (1.2) is a weak solution of the form

\[
u(x, t) := u_- \chi_{(-\infty, x_0)}(x - st) + u_+ \chi_{(x_0, +\infty)}(x - st), \quad x_0 \in \mathbb{R}
\]

where \(u_\pm \in \mathbb{R}^n\), \(s \in \mathbb{R}\) satisfy, for some \(k \in \{1, \ldots, n\}\), the conditions \(\lambda_{k-1}(u_-) < s < \lambda_k(u_-)\) and \(\lambda_k(u_+) < s < \lambda_{k+1}(u_+)\), where \(\lambda_1(u) < \cdots < \lambda_n(u)\) denote the (real) eigenvalues of \(\nabla f(u)\) and \(\chi_I(x)\) is the characteristic function of the set \(I\).

In the general case, the entropy condition for Lax shocks, \(\lambda_{k-1}(u_-) < s < \lambda_k(u_-)\) and \(\lambda_k(u_+) < s < \lambda_{k+1}(u_+)\), should be replaced by the Liu–E condition (see the end of Section 5 for details).

**Definition 1.2.** A **RADIATIVE SHOCK WAVE** of the hyperbolic–elliptic system (1.1) is a weak solution \((u, q)(x, t) := (U, Q)(x - st)\) such that

\[
\lim_{\xi \to \pm \infty} (U, Q)(\xi) = (u_\pm, 0),
\]

where \(u_\pm \in \mathbb{R}^n\), \(s \in \mathbb{R}\) defines a shock wave for the reduced hyperbolic system (1.2).
The usual Rankine–Hugoniot condition
\[ f(u_+) - f(u_-) = s(u_+ - u_-), \] (1.4)
relating the states \( u_\pm \) and the speed of propagation \( s \), holds also for radiative shock waves. This is readily seen by integrating over all \( \mathbb{R} \) the first equation (1.1) and taking in account the asymptotic limits of the wave.

Being a weak solution, a radiative shock wave may be discontinuous at some \( \xi_0 \). Let \((U,Q)\) be piecewise \( C^1 \) in a neighborhood \((\xi_-,\xi_+)\) of \( \xi_0 \), that is
\[ U \in C^1((\xi_-,\xi_0]) \cap C^1([\xi_0,\xi_+)), \quad Q \in C^2((\xi_-,\xi_0]) \cap C^2([\xi_0,\xi_+)). \]

If the profile \( U(\xi_0-) \neq U(\xi_0+) \), then \( U' \) has a delta term concentrated at \( \xi_0 \) and, as a consequence of the second equation in (1.1), the same holds for \( Q'' \) and, therefore, \( Q \) is continuous at \( \xi_0 \). Hence, the first equation of (1.1) suggests to call the discontinuity at \( \xi_0 \) admissible if and only if the triple \((U(\xi_0\pm),s)\) is a shock wave for the reduced system (1.2).

Definition 1.3. An admissible radiative shock wave of the hyperbolic–elliptic system (1.1) is a radiative shock wave \((U,Q)\) if there exists a discrete set \( J \subset \mathbb{R} \) for which \( U \) is \( C^1 \) off \( 2J \), \( Q \) is continuous on \( \mathbb{R} \) and \( C^2 \) off \( J \) and the triple \((U(\xi_0\pm),s)\) is a shock wave for (1.2) for any \( \xi_0 \in J \).

Later on, it will be shown that the admissible radiative shock wave we deal with have at most one discontinuity point.

Now we are in position to state the main problem we deal with:

Problem. Given a triple \( u_\pm \in \mathbb{R}^n, s \in \mathbb{R} \), defining a shock wave for the reduced system (1.2), does there exist a corresponding admissible radiative shock wave for the hyperbolic–elliptic system (1.1)?

The analogous problem has been affirmatively solved for other regularizations of (1.2) (see [17] and [25] for the viscous and the relaxation approximation, respectively).

In the context of radiative gas model, this problem has been addressed for the scalar case [19, 10] and for specific systems [15].

The result in [19] concerns the existence of radiative shocks for convex fluxes \( f \). Elementary computations shows that the parameters \( \varepsilon \) and \( m \) in the cited paper are related with \( L,G,R \) by the relations \( \varepsilon = LG/R \) and \( m = \sqrt{R}/LG \). Hence the existence result in [19] can be stated as follows.

Theorem 1.4. [19] Assume \( f'' > 0 \). If
\[ \frac{R}{L^2G^2} \sup_{u \in [u_+,u_-]} \left\{ f''(u) \left( f(u_-) + s(u - u_-) - f(u) \right) \right\} \leq \frac{1}{4}, \] (1.5)
then there exist a continuous radiative shock.

\( ^2 \)Given \( J \subset \mathbb{R} \), we say that a function \( F : \mathbb{R} \to \mathbb{R} \) is \( C^k \) off \( J \) if for \( F \in C^k([a,b]) \) for any interval \([a,b] \) such that \( J \cap (a,b) \) is empty.
Condition (1.5) is satisfied in the case of small shocks, i.e. if $|u_+ - u_-|$ is small. Also, by a hyperbolic rescaling $\partial_x \mapsto L^{-1} \partial_x$, $\partial_t \mapsto L^{-1} \partial_t$, the system (1.1) becomes
\[
\begin{cases}
  u_t + f(u)_x + q_x = 0 \\
  -m^2 \nu^2 L^{-2} q_{xx} + q + \nu u_x = 0,
\end{cases}
\]
where $m^2 = R/L^2 G^2$ and $\nu = G L/R$. As $m \to 0$, formally, we get a scalar viscous conservation law. Hence condition (1.5) can be read as a measure of the strength of the viscosity term, needed for smoothness.

In [10], it has been considered the scalar case with the choice of a Burgers’ like flux function $f(u) = \frac{1}{2} u^2$ and, without loss of generality, $L = R = G = 1$.

**Theorem 1.5.** [10] Let $u_\pm$ be such that $u_+ < u_-$. Then there exists a (unique up to shift) admissible radiative shock wave $(U, Q)$.

The authors can also determine the smoothness of the profiles $(U, Q)$ as the size $|u_+ - u_-|$ varies, showing that regularity improves as the amplitude of the shock decreases:
- if $|u_+ - u_-| > \sqrt{2}$, then $U \in C^0(\mathbb{R} \setminus \{\xi_0\})$ for some $\xi_0 \in \mathbb{R}$ and $Q$ is Lipschitz continuous;
- if $|u_+ - u_-| < 2\sqrt{n}/(n + 1)$ for some $n \in \mathbb{N}$, then $U \in C^n(\mathbb{R})$ and $Q \in C^{n+1}(\mathbb{R})$.

In the case of systems (i.e. $u \in \mathbb{R}^n$), the literature on the subject restricts to the very recent article [15]. There, the authors consider the problem of existence of smooth radiative shocks for a specific model describing the gases not in thermodynamical equilibrium with radiations. In their case, the coupled elliptic equation is nonlinear with respect to the temperature.

In the present article, we deal with general systems (1.1) and consider admissible radiative shock waves, hence possibly discontinuous. We are able to prove that, for small amplitude shock waves of (1.2), there always exists a radiative shock profiles.

**Theorem 1.6.** Let $u_- \in \mathbb{R}^n$ be such that the $k$–th characteristic field of (1.2) is genuinely nonlinear at $u_-$, that is $\nabla \lambda_k(u_-) \cdot r_k(u_-) \neq 0$. Assume that
\[
(\ell_k(u_-) \cdot L) (G \cdot r_k(u_-)) > 0. \tag{1.6}
\]
Then there exists a sufficiently small neighborhood $U$ of $u_-$ such that for any $u_+ \in U$, $s \in \mathbb{R}$ such that the triple $(u_-, u_+; s)$ defines a shock wave for (1.2), there exists a (unique up to shift) admissible radiative shock wave for (1.7).

The proof is essentially divided in two steps. The first one is to reduce the system case to the scalar case. This (surprising!) possibility is essentially due to the fact that the diffusion matrix $L \otimes G$ is rank–one. The second step is to generalize the existing proof for Burgers’ like flux to general strictly convex fluxes and to show the existence of an heteroclinic orbit connecting the asymptotic states. The analysis is complicated by the fact that the differential equation for the profile is not in normal form and that discontinuity of $U$ can arise. In dealing with this problem, it turns to be useful to work with the integrated variable $z$ such that $q = -z_x$, hence to deal with the equations
\[
\begin{cases}
  u_t + f(u)_x = L z_{xx} \\
  -z_{xx} + R z = G \cdot u,
\end{cases} \tag{1.7}
\]
instead of the original system (1.1).

Remark 1.7. The smallness assumption on the shock \((u_-, u_+; s)\) is needed only in the first step of the proof, that is in the reduction procedure from system to scalar case, which is carried out in terms of Implicit Function Theorem. Since we are able to prove the existence of (possibly discontinuous) radiative shocks for general strictly convex scalar models and general large admissible shocks, our existence result includes in principle the case of discontinuous radiative shocks for (1.1).

By applying a similar strategy to the one used in [10], we can also get analogous result on smoothness of the profile \((U, Q)\).

**Theorem 1.8.** There exists a sequence \(\{\varepsilon_n\}\), \(\varepsilon_n \to 0\) as \(n \to \infty\), such that the profile \(u\) is \(C^{n+1}\) whenever \(|u_+ - u_-| < \varepsilon_n\).

For completeness, once the convex case has been treated in full generality, we consider the case of nonconvex fluxes, again showing the existence and regularity of admissible radiative shock profiles. The number of possible jumps of the profiles and the construction of the profiles themselves are strictly related with the number of inflection points of the flux function (see Section 4).

The paper is organized as follows. Section 2 is devoted to prove the existence result in the scalar case for strictly convex flux functions \(f\). Gaining of regularity of the profile as the amplitude decrease is considered in Section 3 again in the strictly convex case. The scalar case with a general (nonconvex) flux function \(f\) is treated in Section 4. Finally, in Section 5 we show how to reduce the problem from the system case to the scalar one.

## 2. Existence of the Profile in the Convex Case

We start our analysis with the case of a scalar conservation law, with convex flux, coupled with the linear elliptic equation describing the radiating effects. Specifically, we discuss the existence of a travelling wave profile for the \(2 \times 2\) hyperbolic–elliptic system

\[
\begin{align*}
    u_t + \tilde{f}(u)_x + Lq_x &= 0 \\
    -q_{xx} + Rq + Gu_x &= 0,
\end{align*}
\]

where \(x \in \mathbb{R}, \ t > 0, \ u\) and \(q\) are scalar functions, the flux function \(\tilde{f}\) is strictly convex, \(R > 0\) and \(L, G\) are constants such that \(LG > 0\). As reported in the Introduction, previous results on the same problem are contained in [19] (general convex fluxes, smooth radiative shocks) and in [10] (Burgers’ flux, general radiative shocks).

Applying the rescaling

\[
\partial_t \mapsto LG \partial_t, \quad \partial_x \mapsto \sqrt{R} \partial_x, \quad q \mapsto Gq/\sqrt{R},
\]

and setting \(f(s) := \sqrt{R} \tilde{f}(s)/LG\), we get the (adimensionalized) version

\[
\begin{align*}
    u_t + f(u)_x + qx &= 0 \\
    -q_{xx} + q + ux &= 0,
\end{align*}
\]

where \(f\) is a strictly convex function.
We stress that, thanks to the discussion of Section 5, the results of the present section will give the existence of a $k$–travelling wave solutions for system (1.1) with genuinely nonlinear $k$ characteristic field. The study of the general scalar case is left to Section 4 and the results for system (1.1) without GNL assumptions is again a consequence of the reduction arguments of Section 5.

Remark 2.1. The existence of the profile does not need the strict convexity of the flux function $f$. Indeed, our proof applies also to the case of a function $f$ such that $f(u) - cu$ is monotone on two intervals $(u_+, u_-)$ and $(u_-, u_+)$. Let us consider a solution to (2.1) of the form $(u,q) = (u(x-st), q(x-st))$ such that $u(\pm\infty) = u_\pm$, $s = f(u_+) - f(u_-)$, $u_+ < u_-$. We introduce the variable $z$ as the opposite of the antiderivative of $q$, that is $z_x = -q$, with $z(\pm\infty) = z_\pm = u_\pm$. Hence, after an integration of the second equation, system (2.1) rewrites as

$$\begin{align*}
  u_x + f(u)x - z_{xx} = 0 \\
  -z_{xx} + z - u = 0.
\end{align*}$$

Thus, the equations for the profiles $u(x-st)$ and $z(x-st)$ take the form

$$\begin{align*}
  -su' + f'(u)' - z'' = 0 \\
  -z'' + z - u = 0,
\end{align*}$$

that is, after integration of the first equation

$$\begin{align*}
  -s(u - u_\pm) + f(u) - f(u_\pm) = z' \\
  u = z - z''.
\end{align*}$$

At this point, we rewrite (2.2) as the following second order equation for $z$

$$z' = F(z - z''; s), \quad (2.3)$$

where $F(u; s) = f(u) - f(u_\pm) - s(u - u_\pm)$. We shall prove the existence of a profile for $z$ solution of (2.3) between the states $z_- = u_-$ and $z_+ = u_+$, $z_- > z_+$, which will give the existence of our profile for $u = z - z''$. To this end, let us note that, thanks to the strict convexity of $f$, the function $F(\cdot; s)$ is strictly decreasing in an interval $[z_+, z_-]$, and strictly increasing in $[z_-, z_+]$, where $F(z_\pm; s) = 0$ and $F(z_\pm; s) = -m < 0$. Hence, $F(\cdot; s)$ is invertible in the aforementioned intervals and we denote with $h_\pm$ the corresponding inverse functions. In the next proposition, we analyze the behavior of the two ordinary differential equations $z'' = z - h_\pm(z')$ in the corresponding intervals of existence.

**Proposition 2.2.** Let us denote with $z_+ = z_+(x)$ the (unique up to a shift) maximal solution of

$$z'' = z - h_+(z'), \quad (2.4)$$

with $z_+(+\infty) = z_+$ and $z'_+(+\infty) = 0$. Then $z_+$ is monotone decreasing, $z'_+$ is monotone increasing and moreover $z_+$ is not globally defined, that is, there exists a point, assumed
to be 0 (thanks to translation invariance), such that
\[ z_+(0) - z_+''(0) = z_* , \quad z_+'(0) = -m. \]  
\[ (2.5) \]

Proof. We rewrite (2.4) as a system of first order equations as follows
\[ X' = H_+(X), \]  
\[ (2.6) \]
where
\[ X = \begin{pmatrix} z \\ z' \end{pmatrix}, \quad H_+(X) = \begin{pmatrix} z' \\ z - h_+(z') \end{pmatrix}. \]
Then
\[ \nabla H_+ \left( \begin{array}{c} z_+ \\ 0 \end{array} \right) = \begin{pmatrix} 0 \\ 1 - \frac{1}{F'(z_+; s)} \end{pmatrix} \]
and \((z_+, 0)\) is a saddle point. We are interested in the stable manifold at this point and, since we want \(z\) to be decreasing, let us follow the trajectory which exits from \((z_+, 0)\) in the lower half plane of the states space. Then we claim that \(z\) is monotone decreasing and \(z'\) is monotone increasing. Indeed, if by contradiction \(z\) attains a local maximum, say at \(x = x_0\), then \(z'(x_0) = 0\) and \(0 \leq z'' = z - h_+(0) = z - z_+\) at \(x = x_0\), which is impossible. Thus \(z\) is monotone decreasing and \(z' < 0\). Now, if by contradiction \(z'\) attains a local minimum at \(x = x_0\), then the trajectory \(z' = \varphi(z)\) in the \((z, z')\) plane must attain a local minimum at \(x = x_0\), that is \(\varphi'(z) = 0\) and \(\varphi''(z) \geq 0\). Thus, at \(x = x_0\),
\[ 0 = \varphi'(z) = \frac{z''}{z'} = \frac{z - h_+(z')}{z'} \]
and
\[ \varphi''(z) = \frac{d}{dz} \left( \frac{z - h_+(z')}{z'} \right) = \frac{(z' - h_+(z')z'')z' - (z - h_+(z'))z''}{(z')^3} = \frac{1}{z'} < 0, \]
which is impossible. Hence \(z'\) is monotone increasing and clearly \(h_+(z') \in [z_+, z_*]\). Therefore \(z'' = z + O(1)\), which implies the solution does not blow up in finite time.

Thus, we are left with the two following possibilities:

(i) the solution is defined for any \(x \in \mathbb{R}\);

(ii) the solution reach the boundary of the domain of definition of the differential equation in a finite point, which can be assumed to be 0 up to a space translation, namely, \((2.5)\) holds.

Hence, the proof is complete if we can exclude the case (i). To this end, let us assume that (i) holds. Thus in particular \(z'\) is bounded for any \(x \in \mathbb{R}\), because \(z' \in (-m, 0)\). Since \(z\) is monotone decreasing, then \(z \to l \in (z_+, +\infty)\) as \(x \to -\infty\). If \(l < +\infty\), since \(z'\) is increasing, then \(z' \to 0\) as \(x \to -\infty\) and therefore \(z'' = z - h_+(z') \to l - h_+(0) = l - z_+ = 0\), which is impossible. On the other hand, let us assume \(l = +\infty\). Since \(z'\) is globally bounded, \(z\) decreases at most linearly, which is incompatible with \(z'' = z + O(1)\), because the latter implies an exponential rate. Therefore, the solution cannot be defined for any \(x \in \mathbb{R}\) and the proof is complete. \(\square\)
With the same kind of arguments, it is possible to analyze the behavior of the maximal solution $z_-(x)$ of the equation $z'' = z - h_-(z')$. Hence, the following proposition holds.

**Proposition 2.3.** Let us denote with $z_-(x)$ the (unique, up to a space shift) maximal solution of
\[ z'' = z - h_-(z'), \]  \hspace{1cm} (2.7)
with $z_-(\infty) = z_-$ and $z''_-(\infty) = 0$. Then $z_-$ and $z'_-$ are monotone decreasing and moreover $z_-$ is not globally defined, that is, there exists a point, which we can assume to be 0 (thanks to translation invariance), such that
\[ z_-(0) - z''_-(0) = z_*, \quad z'_-(0) = -m. \]

With the aid of Propositions 2.2 and 2.3, we shall build a $C^1$ trajectory joining $z_-$ and $z_+$ by finding a point of intersection for the orbits of the maximal solutions $z_-(x)$ and $z_+(x)$ in the state space $(z, z')$. The monotonicity of $z_+ (x)$ and $z'_+ (x)$ shall guarantee such an intersection is unique and therefore the resulting $C^1$ trajectory from $z_-$ to $z_+$ is unique, up to a space translation. The existence of the aforementioned intersection is a straightforward consequence of the following lemma.

**Lemma 2.4.** Let us denote with $z_+(x)$ and $z_-(x)$ the maximal solutions of (2.4) and (2.7) respectively. Then
\[ z_+(0) \geq z_* \geq z_-(0). \]  \hspace{1cm} (2.8)

**Proof.** We prove only the inequality on the left of (2.8), the right one to be proved similarly. Let us denote with $\bar{y}$ the heteroclinic orbit of
\[
\begin{cases}
y' = F(y; s) \\
y(\pm \infty) = z_\pm.
\end{cases}
\]

Our aim is to compare this solution with the solution $z_+(x)$ in the phase space. To this end, as before we denote with $\varphi = \varphi(z)$ the function whose graph is the trajectory of $(z_+(x), z'_+(x))$ in the $(z, z')$ plane and with $\psi = \psi(z) = F(z; s)$ the function whose graph is the trajectory of $(\bar{y}(x), \bar{y}'(x))$ in the $(z, z')$ plane. Then
\[ \psi'(z_+) = F'(z_+; s) < 0 \]
and $\varphi'(z_+) = \lambda_1$ is the negative eigenvalue of $\nabla H_+(z_+, 0)$, with $H_+$ defined in the proof of Proposition 2.2, namely, the negative root of
\[ P(\lambda) = \lambda^2 + \frac{\lambda}{F'(z_+; s)} - 1. \]
Since $P(F'(z_+; s)) = (F'(z_+; s))^2 > 0$, then $F'(z_+; s) < \lambda_1$ and therefore the trajectory $\psi(z)$ leaves the $z$-axis in $z_+$ below the trajectory $\varphi(z)$.

If $\varphi(z)$ intersect $\psi(z)$ in a point of the $(z, z')$ plane, then in that point we have $\varphi(z) = \psi(z)$ and $\varphi'(z) \leq \psi'(z)$. Hence, there exist a point $\bar{x}$ such that $\bar{y}(\bar{x}) = z_+(\bar{x})$ and $\bar{y}'(\bar{x}) = z'_+(\bar{x}) < 0$. Moreover, since
\[ \varphi'(z) = \frac{z''_+}{z'_+}. \]
and

\[ \psi'(z) = \frac{d}{dz} \frac{\bar{y}''}{\bar{y}'}, \]

in that point we also have \( \bar{y}'' \leq z''_+ = z_+ - h_+(z'_+) = \bar{y} - h_+(\bar{y}') = 0 \). On the other hand, if \( 0 \geq \bar{y}'' = \psi'(\bar{y}; s)\bar{y}' = F'(-\bar{y}; s)\bar{y}'F(-\bar{y}; s), \) that is \( F'(-\bar{y}; s) \geq 0 \) because \( F(-\bar{y}; s) < 0 \), namely \( \bar{y} = z_+ \geq z_* \) in that point.

Conversely, let us assume that \( \varphi(z) \) remains above \( \psi(z) \). From Proposition 2.2 we know that \( z'_+(0) = -m \), which is the minimum of \( \psi(z) \), attained at \( z_* \). Therefore we must have \( z_+(0) = z_* \) and the proof is complete. \( \square \)

With the aid of Lemma 2.4, we are able to prove the main result of this section, which is contained in the following theorem.

**Theorem 2.5.** There exists a (unique up to space translations) \( C^1 \) profile \( z \) with \( z(\pm \infty) = z_\pm = u_\pm \) such that the function \( z(x - st) \) is solution of (2.2), where \( s \) is given by the Rankine–Hugoniot condition. The solution \( z \) is \( C^2 \) away from a single point, where \( z'' \) has at most a jump discontinuity.

Moreover, there exists an (unique up to space translations) profile \( u \) with \( u(\pm \infty) = u_\pm \) such that the function \( u(x - st) \) is solution of (2.3), where \( s \) is given by the Rankine–Hugoniot condition. This profile is continuous away from a single point, where it has at most a jump discontinuity which verifies the Rankine–Hugoniot and the admissibility conditions of the scalar conservation law \( u_t + f(u)_x = 0 \).

**Proof.** Let us observe that Lemma 2.4 implies that there exists a point in the \((z, z')\) plane where the graphs of \( z_+ \) and \( z_- \) intersects, namely \( z_- = z_+ \) and \( z'_- = z'_+ \) in that point. Moreover, due to the monotonicity of these graphs, which comes from the proofs of Proposition 2.2 and Proposition 2.3, this intersection is indeed unique. Hence with appropriate space translations, we can find a point \( \bar{x} \) such that \( z_-(\bar{x}) = z_+(\bar{x}) = \bar{z} \) and \( z'_-(\bar{x}) = z'_+(\bar{x}) = \bar{z} \). Set

\[ z(x) := \begin{cases} z_-(x) & x \leq \bar{x} \\ z_+(x) & x \geq \bar{x}. \end{cases} \]

Thus, this profile is the unique (up to space translations) \( C^1 \) solution of (2.2) and it verifies \( z(\pm \infty) = z_\pm \). Moreover, \( z(x) \) is \( C^2 \) in the intervals \((-\infty, \bar{x}] \) and \([\bar{x}, +\infty) \) and finally \( z''(\bar{x} - 0) = z''(\bar{x}) = z_-(\bar{x}) - h_-(z'_-(\bar{x})) = \bar{z} - h_-(\bar{z}) \) and \( z''(\bar{x} + 0) = z''(\bar{x}) = z_+(\bar{x}) - h_+(z'_+(\bar{x})) = \bar{z} - h_+(\bar{z}). \)

The regularity of \( u = u(x - st) \) is a direct consequence of the first part of the theorem and of the relation \( u = z - z'' \). Moreover, in the case of a discontinuity in \( u \), namely \( u(\bar{x} - 0) \nmid u(\bar{x} + 0), (u(\bar{x} - 0), u(\bar{x} + 0); s) \) verifies the Rankine–Hugoniot condition for the strictly convex conservation law \( u_t + f(u)_x = 0 \). Indeed, \( u(\bar{x} - 0) = h_-(\bar{z}), u(\bar{x} + 0) = h_+(\bar{z}) \) and a direct calculation shows

\[ \frac{f(h_+(\bar{z})) - f(h_-(\bar{z}))}{h_+(\bar{z}) - h_-(\bar{z})} = s. \]

Finally, \( u(\bar{x} - 0) = h_-(\bar{z}) > h_+(\bar{z}) = u(\bar{x} + 0) \), that is, this shock is admissible and the proof is complete. \( \square \)
Remark 2.6. The above theorem contains an uniqueness result, among the class of radiative shocks given in Definition 1.2. An uniqueness result in a wider class of solutions is contained in Theorem 4.3. We postpone this result because it is valid for general flux functions, disregarding convexity properties.

Remark 2.7. From the last lines of the above proof, it is clear that the profile $u$ can have a jump discontinuity in a point, as it was already proved in [10] for sufficiently large shocks in the case of the Burgers’ equation. Moreover, in that paper it is proved also that, below an explicit threshold, the profile is continuous and it smooths out as the strength of the shock decreases. In our case, the profile in $u$ is continuous if $h'(\bar{z}) = h'_{\pm}(\bar{z}) = z_*$, that is $-m = \bar{z} = z'_{\pm}(\bar{x})$, $z_{\pm}(\bar{x}) = z_*$ and $z''_{\pm}(\bar{x}) = 0$. This property and the further regularity of the profile for sufficiently small shocks is proved in Section 3 below.

3. Regularity of the profile in the convex case

In Section 2 we proved the existence of the travelling wave $(u, q) = (u(x-st), q(x-st))$,

$$
u(\pm\infty) = u_{\pm}, \quad s = \frac{f(u_{\pm}) - f(u_{-})}{u_{+} - u_{-}},$$

for (2.1), when the flux $f(u)$ is smooth and strictly convex; in the present section we focus our attention to its regularity.

Let us start by observing that this issue is related to the smallness of the shock we are dealing with [10]. Here we shall prove that it is possible to recast such a property in terms of smallness of the term $-q_{xx}$ in (2.1), namely when this system increases its diffusive nature, being closer to its parabolic equilibrium. This will be made in terms of a diffusive scaling with respect to the shock strength $\varepsilon = |u_{+} - u_{-}|$ and, following the ideas of [10], by analyzing the system for $(u, v = u')$. In that way, we will obtain the same kind of results of [10] for the special case $f(u) = \frac{1}{2}u^2$. Let us finally observe that similar phenomena arise in the discussions of Section 4 to prove the existence and regularity of travelling wave profiles of (4.1) with flux functions $f$ with change of convexity. In that case, the small parameter $\varepsilon$ is already present in the model and it is not connected with the smallness of the shock. However, existence and regularity of the profile will require once again the smallness of that parameter, namely, as before, when the hyperbolic–elliptic model is close enough to its diffusive underlying dynamic.

From (2.1) and after integration with respect to $\xi$ of the first line, the equations for the profiles $u$ and $q$ are given by

$$
\begin{align*}
-s(u - u_{\pm}) + f(u) - f(u_{\pm}) + q &= 0 \\
u' &= q'' - q,
\end{align*}
$$

because $q(\pm\infty) = 0$. Thus, using $q = -[f(u) - f(u_{\pm}) - s(u - u_{\pm})] = -F(u; s)$ in (2.1), we end up with

$$
u' = -F(u; s)'' + F(u; s) = -(f'(u) - s)u'' - f''(u)(u')^2 + F(u; s).$$
Hence, we obtain the following system in the state space \((u, v = u')\)
\[
\begin{align*}
u' &= v \\
(f'(u) - s)v' &= -f''(u)v^2 - v + F(u; s).
\end{align*}
\tag{3.2}
\]

It is worth to observe that system (3.2) is singular where \(f'(u) - s = 0\) and, thanks to the strict convexity of \(f\), the latter occurs at a unique \(\bar{u} \in (u_+, u_-)\). At this stage, driven by \([13]\), we scale both the independent and the dependent variable as follows:
\[
\tilde{u} = \frac{u - u_+}{\varepsilon} \quad \tilde{v} = \frac{v}{\varepsilon^2} \quad \tilde{\xi} = \varepsilon \xi.
\]
Then, dropping the tildas, system (3.2) becomes
\[
\begin{align*}
u' &= v \\
f_{\varepsilon}'(u)v' &= \frac{1}{\varepsilon^2} \left(-\varepsilon^2 f''_{\varepsilon}(u)v^2 - v + f_{\varepsilon}(u)\right),
\end{align*}
\tag{3.3}
\]
where \(f_{\varepsilon}(u) := \frac{1}{\varepsilon^2} F(u_+ + \varepsilon u; s)\) and \(u \in [0, 1]\). Once again, system (3.3) is singular at \(u = \bar{u}^\varepsilon\), for a unique \(\bar{u}^\varepsilon \in (0, 1)\). Finally, as in \([10]\), we remove the singularity of (3.3) by using once again a new independent variable \(\eta\) defined by
\[
\xi = \int_{\eta}^{\infty} f_{\varepsilon}'(u(\zeta))d\zeta.
\]
The resulting system reads as follows
\[
\begin{align*}
u' &= f_{\varepsilon}'(u)v \\
v' &= \frac{1}{\varepsilon^2} \left(-\varepsilon^2 f''_{\varepsilon}(u)v^2 - v + f_{\varepsilon}(u)\right).
\end{align*}
\tag{3.4}
\]
We notice that (3.4) admits \((0, 0)\) and \((1, 0)\) as equilibrium points, corresponding to the two equilibrium points \((u_\pm, 0)\) of the original system.

**Remark 3.1.** (i) For any \(n \geq 2\), if the original flux function \(f\) is \(C^n\), then \(f_{\varepsilon} \to \frac{1}{2}f''(u_+)u(u-1)\) in \(C^n([0, 1])\), as \(\varepsilon \downarrow 0\).

(ii) The last change of independent variable gives a reparametrization of the orbit of (3.3) in the two regions \([0, \bar{u}^\varepsilon)\) and \((\bar{u}^\varepsilon, 1]\). Let us first observe that \(f_{\varepsilon}'(u) < 0\) for \(0 \leq u < \bar{u}^\varepsilon\) and \(f_{\varepsilon}'(u) > 0\) for \(\bar{u}^\varepsilon < u \leq 1\). Hence, we obtain continuity of the orbit \(u\) of (3.3), provided we prove the two previous reparametrized orbits verify \(u(-\infty) = 0\), \(u(+\infty) = \bar{u}^\varepsilon\), and \(u(+\infty) = \bar{u}^\varepsilon\), \(u(-\infty) = 1\) respectively. We shall prove the former property, the latter being similar.

(iii) As for \([10]\), further regularity for the profiles solutions of (3.4) implies further regularity of the original profile, thanks to their exponential decay toward the asymptotic states at \(\pm \infty\).

For the sake of clarity, we state first the result concerning the continuity and the \(C^1\) regularity for the profile \(u\) solution of (3.4) and then the one for the \(C^2\) regularity. The general case, including the former ones, is then stated in the last Proposition. Here below, we shall assume \(f\) to be smooth.
Proposition 3.2. There exist two values \( \bar{\varepsilon}, \varepsilon_0 > 0 \) such that, for \( \varepsilon < \min \{ \bar{\varepsilon}, \varepsilon_0 \} \), the orbits of (3.4) which pass through the equilibrium points \((0, 0)\) and \((1, 0)\) meet at the equilibrium point \( (\bar{u}^\varepsilon, \bar{v}^\varepsilon_2) \), where

\[
\bar{v}^\varepsilon_2 = \frac{-1 + \sqrt{1 + 4\varepsilon^2 f''(\bar{u}^\varepsilon) f(\bar{u}^\varepsilon)}}{2\varepsilon^2 f''(\bar{u}^\varepsilon)}.
\]  

(3.5)

In particular, the orbit \( u \) is \( C^1 \).

Proof. We start by studying the “new” equilibrium points of (3.4) introduced by the last change of variable, besides the aforementioned points \((0, 0)\) and \((1, 0)\). Hence we must satisfy the relations

\[
\begin{cases}
  f'_\varepsilon(u) = 0 \\
  \varepsilon^2 f''(u) v^2 + v - f(\varepsilon(u)) = 0,
\end{cases}
\]

that is \( u = \bar{u}^\varepsilon \) and

\[
\bar{v}^\varepsilon_{1,2} = \frac{-1 + \sqrt{1 - 4\varepsilon^2 f''(\bar{u}^\varepsilon) |f(\varepsilon(u))|}}{2\varepsilon^2 f''(\bar{u}^\varepsilon)},
\]

provided \( \varepsilon \leq \varepsilon_0 \), thanks to Remark 3.1–(i). Moreover, using again Remark 3.1–(i), there exists an \( \bar{\varepsilon} > 0 \) such that for any \( \varepsilon \leq \bar{\varepsilon} \) the whole curves

\[
v^\varepsilon_{1,2}(u) = \frac{-1 + \sqrt{1 - 4\varepsilon^2 f''(u) |f(\varepsilon(u))|}}{2\varepsilon^2 f''(u)}
\]

(3.6)

with \( u \in [0, 1] \) are real. We shall now study the nature of these points to prove the first assertion of the theorem. The Jacobian associated to system (3.4) is given by

\[
J(u, v) = \begin{pmatrix}
  f''(u) v & f'(u) \\
  -f''(u) + \frac{1}{\varepsilon^2} f'(u) & -2f''(u) v - \frac{1}{\varepsilon^2}
\end{pmatrix}
\]

Evaluating \( J(0, 0) \), \( J(1, 0) \) and \( J(\bar{u}^\varepsilon, \bar{v}^\varepsilon_2) \) we conclude \((0, 0)\) and \((1, 0)\) are saddle points, while \((\bar{u}^\varepsilon, \bar{v}^\varepsilon_2)\) is a sink. Moreover, the curve \( v = v^\varepsilon_2(u) \) given in (3.6) passes through the point \((0, 0)\) and

\[
\frac{dv^\varepsilon_2(u)}{du}
\]

at \( u = 0 \) is \( f'(0) \).

Let us consider the orbit exiting from \((0, 0)\) at \(-\infty\), namely the one on the unstable manifold of that (saddle) point, in the halfplane \( v < 0 \). Its tangent vector is then given by the eigenvector related to the positive eigenvalue \( \lambda_+ \) of \( J(0, 0) \), that is

\[
\begin{pmatrix}
  1 \\
  \lambda_+ f'(0)
\end{pmatrix}
\]

A direct calculation shows

\[
0 > \frac{\lambda_+}{f'(0)} > f'(0),
\]

namely the trajectory leaves the equilibrium point above the curve \( v = v^\varepsilon_2(u) \). Hence, a straightforward analysis of the dynamical system (3.4) implies it must reach the equilibrium point \((\bar{u}^\varepsilon, \bar{v}^\varepsilon_2)\) at \(+\infty\). The same result holds for the orbit exiting from \((1, 0)\).
and therefore the first claim of the theorem is proved. Finally, since \( v \) is continuous and bounded everywhere, \( u \in C^1 \) and the proof is complete.

**Proposition 3.3.** There exists a value \( 0 < \varepsilon_1 < \varepsilon_0 \) such that for \( \varepsilon < \min\{\varepsilon, \varepsilon_1\} \), the orbit \( u \) of (3.4) is \( C^2 \).

**Proof.** We evaluate the Jacobian \( J(u, v) \) at the equilibrium point \((\bar{u}^\varepsilon, \bar{v}^\varepsilon)\) to obtain

\[
J(\bar{u}^\varepsilon, \bar{v}^\varepsilon) = \begin{pmatrix}
    f''_\varepsilon(\bar{u}^\varepsilon)\bar{v}_2^\varepsilon & 0 \\
    -f''_\varepsilon(\bar{u}^\varepsilon)(\bar{v}_2^\varepsilon)^2 & -2f''_\varepsilon(\bar{u}^\varepsilon)\bar{v}_2^\varepsilon - \frac{1}{\varepsilon^2}
\end{pmatrix}
\]

which admits the following (negative) eigenvalues

\[
\lambda_1^\varepsilon = f''_\varepsilon(\bar{u}^\varepsilon)\bar{v}_2^\varepsilon = -1 + \frac{\sqrt{1 - 4\varepsilon^2 f''_\varepsilon(\bar{u}^\varepsilon)\varepsilon^2}}{2}\]
\[
\lambda_2^\varepsilon = -2f''_\varepsilon(\bar{u}^\varepsilon)\bar{v}_2^\varepsilon - \frac{1}{\varepsilon^2} = -\frac{\sqrt{1 - 4\varepsilon^2 f''_\varepsilon(\bar{u}^\varepsilon)\varepsilon^2}}{\varepsilon^2}
\]

For \( \varepsilon = \varepsilon_0 \), \( \lambda_2^\varepsilon = 0 \) and \( \lambda_1^\varepsilon = -\frac{1}{2\varepsilon} < 0 \), while \( \lambda_2^\varepsilon \to -\infty \) and \( \lambda_1^\varepsilon \to -f''_\varepsilon(\bar{u}^\varepsilon)\varepsilon^2 < 0 \) as \( \varepsilon \downarrow 0 \). Thus, there exists an \( \varepsilon_1 < \varepsilon_0 \) such that for \( \varepsilon < \varepsilon_1 \), \( \lambda_2^\varepsilon < \lambda_1^\varepsilon < 0 \). Therefore, for any \( \varepsilon < \varepsilon_1 \), the orbit in \((\bar{u}^\varepsilon, \bar{v}^\varepsilon)\) is tangent to the eigenvector related to \( \lambda_1^\varepsilon \), that is

\[
\begin{pmatrix}
    f''_\varepsilon(\bar{u}^\varepsilon)(\bar{v}_2^\varepsilon)^2 \\
    -3f''_\varepsilon(\bar{u}^\varepsilon)(\bar{v}_2^\varepsilon)^2 - \frac{1}{\varepsilon^2}
\end{pmatrix} = \frac{1}{\lambda_2^\varepsilon - \lambda_1^\varepsilon}
\]

instead of the eigenvector related to \( \lambda_2^\varepsilon \), namely \((0, 1)^t\).

Finally,

\[
\frac{dv}{du}\bigg|_{u=\bar{u}^\varepsilon} = \frac{f''_\varepsilon(\bar{u}^\varepsilon)(\bar{v}_2^\varepsilon)^2}{\lambda_2^\varepsilon - \lambda_1^\varepsilon} = \bar{w}_1^\varepsilon \in \mathbb{R},
\]

which implies the \( C^1 \) regularity of the \( v \)-component of the orbit of (3.4), that is, the \( C^2 \) regularity of the \( w \)-component of that orbit.

To prove further regularity of the profile \((u, v)\) solution of (3.4), we follow again the ideas of [10]. For \( n \geq 0 \), let us consider by induction the new variable

\[
w_n = \frac{1}{(u - \bar{u}^\varepsilon)^n} \left( v - \sum_{j=0}^{n} \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^j \right),
\]

that is

\[
v = \sum_{j=0}^{n} \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^j + w_n (u - \bar{u}^\varepsilon)^n := h_n(u, w_n).
\]

Let us note that, with the notation introduced above, we choose \( \bar{w}_0^\varepsilon = \bar{v}_2^\varepsilon \) and \( \bar{w}_1^\varepsilon \) given by (3.7). Hence, for \( n = 1 \), (3.8) reduces to

\[
w_1 = \frac{v - \bar{v}_2^\varepsilon}{u - \bar{u}^\varepsilon} - \bar{w}_1^\varepsilon,
\]

that is, \( \bar{w}_1^\varepsilon \) is the value of the first derivative of the curve \( v = v(u) \) for \( u = \bar{u}^\varepsilon \) and therefore the rest point \((\bar{u}^\varepsilon, \bar{v}_2^\varepsilon)\) becomes \((\bar{u}^\varepsilon, 0)\) in the new pair of variables \((u, w_1)\). However, for
Proof. A direct calculation shows

\[ \bar{w}_j^\varepsilon = \frac{1}{j!} \frac{d^j v}{du^j} \bigg|_{u=\bar{u}^\varepsilon}, \quad \text{for any } j \geq 2 \]  

(3.10)
as a consequence. The system for \((u, w_n)\) is given by

\[
\begin{cases}
    u' = f'_\varepsilon(u) h_n(u, w_n) \\
    w'_n = \frac{\theta_n(u, w_n)}{\varepsilon^2 (u - \bar{u}^\varepsilon) n} 
\end{cases}
\]

(3.11)
where

\[
\theta_n(u, w_n) = -\varepsilon^2 f''_\varepsilon(u) h_n(u, w_n)^2 - h_n(u, w_n) + f_\varepsilon(u) - \varepsilon^2 \left( \sum_{j=1}^{n} j \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^{j-1} + n w_n (u - \bar{u}^\varepsilon)^{n-1} \right) f'_\varepsilon(u) h_n(u, w_n). 
\]

We are ready now to prove the main result of this section, namely, Proposition 3.4 (already proved for \(n = 0, 1\) in Propositions 3.2 and 3.3).

**Proposition 3.4.** For any \(n \geq 0\), there exists a decreasing sequence of positive values \(\{\varepsilon_n\}_{n \geq 0}\), such that, for any \(\varepsilon < \min\{\bar{\varepsilon}, \varepsilon_n\}\), \(v\) is a \(C^n\) function of \(u\) and admits the expansion

\[
v = \sum_{j=0}^{n} \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^j + o((u - \bar{u}^\varepsilon)^n), \quad (3.12)
\]
for \(u \to \bar{u}^\varepsilon\).

**Proof.** A direct calculation shows

\[
\Theta_n(u, w_n) = F_n(u) + G_n(u) w_n + H_n(u) w_n^2,
\]
where

\[
F_n(u) = -\varepsilon^2 f''_\varepsilon(u) \left( \sum_{j=0}^{n} \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^j \right)^2 - \sum_{j=0}^{n} \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^j + f_\varepsilon(u)
- \varepsilon^2 f'_\varepsilon(u) \left( \sum_{j=1}^{n} j \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^{j-1} \right) \left( \sum_{l=0}^{n} \bar{w}_l^\varepsilon (u - \bar{u}^\varepsilon)^{l} \right);
\]
\[
G_n(u) = -2\varepsilon^2 f''_\varepsilon(u) \left( \sum_{j=0}^{n} \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^{j+n} \right) - (u - \bar{u}^\varepsilon)^n
- \varepsilon^2 f'_\varepsilon(u) \left( \sum_{j=1}^{n} (j + n) \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^{j+n-1} \right) - \varepsilon^2 f'_\varepsilon(u) \bar{w}_0^\varepsilon (u - \bar{u}^\varepsilon)^{n-1},
\]
\[
H_n(u) = \varepsilon^2 f''_\varepsilon(u) \left( \sum_{j=0}^{n} \bar{w}_j^\varepsilon (u - \bar{u}^\varepsilon)^{j+n} \right) - \varepsilon^2 f'_\varepsilon(u) \bar{w}_0^\varepsilon (u - \bar{u}^\varepsilon)^{n-1}.
\]
\[ H_n(u) = -\varepsilon^2 \left( f''_\varepsilon(u)(u - \bar{u}^\varepsilon)^{2n} + n f'(u)(u - \bar{u}^\varepsilon)^{2n-1} \right). \]

In addition, for sufficiently smooth fluxes \( f \), since \( f'_\varepsilon(\bar{u}^\varepsilon) = 0 \), we obtain, for \( u \to \bar{u}^\varepsilon \),

\[ G_n(u) = O((u - \bar{u}^\varepsilon)^n), \quad H_n(u) = O((u - \bar{u}^\varepsilon)^{2n}). \]

Moreover, the expression of \( F_n(u) \) implies \( F_n(u) = F_{n-1}(u) + O((u - \bar{u}^\varepsilon)^n) \) and therefore the coefficients of the Taylor approximation of \( F_n(u) \) about \( \bar{u}^\varepsilon \) do not depend on \( n \), that is

\[ F_n(u) = \sum_{j=0}^{\infty} c_j (u - \bar{u}^\varepsilon)^j + O((u - \bar{u}^\varepsilon)^{n+1}), \]

with \( c_j \) independent from \( n \), for any \( j \). We write down these coefficients more explicitly as follows

\[ c_j = \alpha_j(\bar{w}_0^\varepsilon, \ldots, \bar{w}_{j-1}^\varepsilon) + \beta_j(\bar{w}_0^\varepsilon)\bar{w}_j^\varepsilon, \]

where

\[ \beta_j(\bar{w}_0^\varepsilon) = -\left( \varepsilon^2(2 + j)f''_\varepsilon(\bar{u}^\varepsilon)\bar{w}_0^\varepsilon + 1 \right). \]

Now, using again Remark 3.1 (i), there exists a decreasing sequence \( \{\varepsilon_j\}_{j \geq 0} \) such that

\[ \beta_j(\bar{w}_0^\varepsilon) < 0 \quad (3.13) \]

for any \( \varepsilon < \varepsilon_j \), and therefore we choose \( \bar{w}_j^\varepsilon \), for \( \varepsilon < \varepsilon_j \) and for any \( j \), such that

\[ c_j = 0 \iff F_n^{(j)}(\bar{u}^\varepsilon) = 0. \quad (3.14) \]

It is worth to observe that, if we characterize the above argument, and in particular the values \( \varepsilon_j \) and \( 3.14 \), to the cases \( n = 0 \) and \( n = 1 \), we recover the previous choices \( \bar{w}_0^\varepsilon = \bar{v}_2^\varepsilon \) and \( \bar{w}_1^\varepsilon \) given by (3.7).

At this point, we have determined all coefficients \( \bar{w}_j^\varepsilon \), \( j \geq 0 \) in (3.14) and we are left to the proof of the regularity of \( v \) as a function of \( u \) close to \( u = \bar{u}^\varepsilon \). Assume by induction the result holds for \( n \) and introduce the new variable \( w_n \) defined in (3.8), with \( \bar{w}_j : j \leq n \) verifying (3.14). Therefore, \( w_n \to 0 \) as \( u \to \bar{u}^\varepsilon \) by the induction hypothesis and, thanks to (3.14), \( (\bar{w}, 0) \) is a rest point for (3.11), which correspond to the original rest point \( (\bar{u}^\varepsilon, \bar{v}_2^\varepsilon) \) for (3.4). Since \( h_n(\bar{w}, w_n) \equiv 0 \) for any \( w_n \) and in view of the structure of \( \theta_n(u, w_n) \) showed before, the Jacobian of (3.11) evaluated at this rest point is given by

\[ \begin{pmatrix} f''_\varepsilon(\bar{u}^\varepsilon) \bar{w}_0^\varepsilon & 0 \\ \lim_{u \to \bar{u}^\varepsilon} \frac{F'_n(u)(u - \bar{u}^\varepsilon) - n F_n(u)}{\varepsilon^2(u - \bar{u}^\varepsilon)^{n+1}} & \lim_{u \to \bar{u}^\varepsilon} \frac{G_n(u)}{\varepsilon^2(u - \bar{u}^\varepsilon)^n} \end{pmatrix}. \quad (3.15) \]

Since we know (3.14) for \( j \leq n \), then

\[ \lim_{u \to \bar{u}^\varepsilon} \frac{F'_n(u)(u - \bar{u}^\varepsilon) - n F_n(u)}{\varepsilon^2(u - \bar{u}^\varepsilon)^{n+1}} = s_n \in \mathbb{R}. \]

Moreover

\[ \lim_{u \to \bar{u}^\varepsilon} \frac{G_n(u)}{\varepsilon^2(u - \bar{u}^\varepsilon)^n} = -\left( (2 + n)f''_\varepsilon(\bar{u}^\varepsilon)\bar{w}_0^\varepsilon + \frac{1}{\varepsilon^2} \right) = \lambda_2^\varepsilon - n \lambda_1^\varepsilon. \]
Therefore, the eigenvalues of (3.15) are given by
\[ f''_\varepsilon(\bar{u}_\varepsilon)|_0 = \lambda^1_1 \text{ and } \lambda^2_2 - n\lambda^1_1 \]
with eigenvectors
\[ \left( \frac{1}{s_n}, \frac{1}{(n+1)\lambda^2_2 - \lambda^1_2} \right) \text{ and } \left( 0, -1 \right) \).
Thus, the result for \( n+1 \) is obtained proceeding as in Proposition 3.3, by imposing
\[ \lambda^2_2 - n\lambda^1_1 > \lambda^1_1 > \lambda^2_2, \]
that is, condition (3.13) for \( j = n+1 \). Finally, from the uniqueness of the Taylor expansion of \( v \) about \( \bar{u}_\varepsilon \), relations (3.10) is also verified. \( \square \)

*Remark 3.5.* In the case of a Burgers’ flow, that is, the case studied in [10], our scaling procedure will lead to a modified quadratic flux independent from \( \varepsilon \), namely
\[ \frac{1}{2}u(u - 1) \] (see also [13] for further comments on the behavior of the flux with respect to this scaling).

Therefore, for that particular case, the thresholds \( \varepsilon_j \) can be explicitly calculated from (3.13), \( \bar{\varepsilon} = \varepsilon_0 \) and our results coincide with the ones established in [10].

### 4. The non convex case

Now, let us consider existence and regularity of travelling wave profiles when the flux function may change its convexity. As already pointed out in Section 3, in that case, we shall prove these results if the behavior of (2.1) is close enough to its underlying diffusive equilibrium, that is, when \(-q_{xx}\) is sufficiently small. Hence, let us consider the 2×2 system
\[
\begin{align*}
\begin{cases}
  u_t + f(u)x + q_x &= 0 \\
  -\varepsilon q_{xx} + q + u_x &= 0,
\end{cases}
\end{align*}
\] (4.1)
with \( 0 < \varepsilon \ll 1 \). Let \((u_-, u_+; s)\) be an admissible shock for the inviscid conservation law \( u_t + f(u)x = 0 \), namely, it satisfies the Rankine–Hugoniot condition
\[ s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \]
and the the strict Oleinik condition
\[ \frac{f(u) - f(u_-)}{u - u_-} > \frac{f(u_+) - f(u_-)}{u_+ - u_-} > \frac{f(u_+) - f(u)}{u_+ - u}, \] (4.2)
for any \( u \) between \( u_- \) and \( u_+ \). As before, let us introduce a new variable \( z \) as follows
\[ -z_x := q \] so that the equations for \( u(x - st) \) and \( z(x - st) \) are given by
\[
\begin{align*}
\begin{cases}
  F(u; s) &= z' \\
  -\varepsilon z'' + z &= u,
\end{cases}
\end{align*}
\] (4.3)
where, as usual, \( F(u; s) = f(u) - f(u_+) - s(u - u_+) \) and \( z(\pm\infty) = z_\pm = u_\pm \). Without loss of generality, assume \( u_- > u_+ \). Then (4.2) becomes
\[ F(u; s) < 0, \text{ for any } u \in (u_+, u_-) \] (4.4)
and in addition it implies the Lax condition
\[ f'(u_+) \leq s \leq f'(u_-). \]
We start by considering the non-degenerate situation, that is
\[
\begin{align*}
  f'(u_+) &< s < f'(u_-), \\
  f''(u) &\neq 0 \text{ for any } u \in (u_+, u_-) \text{ with } f'(u) = 0.
\end{align*}
\] (4.5)

The result in the general case will be proved by an approximation procedure at the end of the section [20]. We shall construct a profile for the \( z \)-component by solving the equation

\[
z' = F(z - \varepsilon z''; s)
\] (4.6)

and then \( u \) will be given by \( u = z - \varepsilon z'' \).

Since \( f \) is not strictly convex, in order to invert the function \( F(\cdot; s) \), we decompose \( [z_+, z_-] \) in the disjoint (up to vertices) union of subintervals \( I_1, \ldots, I_{2n} \), \( n > 1 \), where \( F(\cdot; s) \) is monotone. More precisely, thanks to (1.4) and (1.5), \( F(\cdot; s) \) is decreasing in \( I_{2k-1} \) and increasing in \( I_{2k} \), \( k = 1, \ldots, n \). In the spirit of Propositions 2.2 and 2.3, we shall construct maximal solutions \( z_0, z_k', z_k^r \), \( k = 1, \ldots, n - 1 \) and \( z_n \) that correspond respectively to \( I_1, I_{2k}, I_{2k+1}, k = 1, \ldots, n - 1 \) and \( I_{2n} \) and then match the corresponding graphs in the phase plane to obtain a global \( C^1 \) profile as in Theorem 2.5 In addition, this solution will be \( C^2 \) in the points \( \{z_{2k}^*\} = I_{2k} \cap I_{2k+1}, k = 1, \ldots, n - 1 \), where \( F(\cdot; s) \) attains a local maximum, while the second derivative will have a jump discontinuity in the points \( \{z_{2k-1}^*\} = I_{2k-1} \cap I_{2k}, k = 1, \ldots, n \), where \( F(\cdot; s) \) attains a local minimum \( \{z_k^*, k = 1, \ldots, 2n \} \) are the zeros of \( F''(\cdot; s) \) in \( [z_+, z_-] \), ordered from the left to the right). In this way, the resulting profile for \( u \) will be regular, except for \( n \) points, where it has at most a jump discontinuity, which is an inviscid shock satisfying (4.4). Clearly, the the case \( n = 1 \) correspond to the strict convex case treated before. Moreover, the construction of the (unique, up to a space translation) maximal solutions \( z_+ \) and \( z_- \) of Propositions 2.2 and 2.3 can be repeated under assumptions (1.4) and (1.5) to obtain the (unique, up to a space translation) profiles \( z_0 \) and \( z_n \) corresponding to the intervals \( [z_+, z_0^*] \) and \( [z_{2n-1}^*, z_-] \). Taking into account also the results of Lemma 2.4, we know that such maximal solutions verify, up to a space translation, the following properties:

(i) \( z_0 : [0, +\infty) \to (z_+, z_0(0)) \), \( z_0 \) monotone decreasing, \( z_0' \) monotone increasing and
\( z_0(+\infty) = z_+; \)
(ii) \( z_n : (-\infty, 0] \to [z_n(0), z_-], z_n \) and \( z_n' \) monotone decreasing and \( z_n(-\infty) = z_-; \)
(iii) \( z_0(0) \geq z_1^* \) and \( z_n(0) \leq z_{2n-1}^* \).

It is worth observing that the existence of the above profiles with the aforementioned properties does not depend on the value of \( \varepsilon > 0 \). Hence, we are left with the construction of the intermediate maximal solutions \( z_{k,l} \) and \( z_{k,r} \), \( k = 1, \ldots, n - 1 \), solutions respectively of

\[
\varepsilon z'' = z - h_{2k}(z')
\] (4.7)

and

\[
\varepsilon z'' = z - h_{2k+1}(z'),
\] (4.8)

where \( h_i \) denotes the inverse of \( F(\cdot; s) \) on \( I_i, i = 1, \ldots, 2n \).
Proposition 4.1. Let us assume conditions (4.4) and (4.5) hold. Then, for any \( k = 1, \ldots, n - 1 \), there exists a (unique up to space translations) maximal solution \( z_{k,l} \) of (4.7) and \( z_{k,r} \) of (4.8), with initial data
\[
\begin{aligned}
z_{k,l}(0) &= z_{k,r}(0) = z_{2k}^*

z_{k,l}'(0) &= z_{k,r}'(0) = F(z_{2k}^*; s).
\end{aligned}
\]
Moreover
\[
\begin{aligned}
z_{k,l}''(0) = z_{k,r}''(0) = 0
\end{aligned}
\]
and
\[
\begin{aligned}
&\text{(i) } z_{k,l}, z_{k,l}' \text{ are monotone decreasing and } z_{k,l} \text{ is not globally defined, that is there exists a point } \xi_{k,l} > 0 \text{ such that}
\end{aligned}
\]
\[
\begin{aligned}
z_{k,l}(\xi_{k,l}) - \varepsilon z_{k,l}''(\xi_{k,l}) = z_{2k-1}^*, \quad z_{k,l}'(\xi_{k,l}) = F(z_{2k-1}^*; s);
\end{aligned}
\]
\[
\begin{aligned}
&\text{(ii) } z_{k,r} \text{ is monotone decreasing, } z_{k,r}' \text{ monotone increasing and } z_{k,r} \text{ is not globally defined, that is there exists a point } \xi_{k,r} < 0 \text{ such that}
\end{aligned}
\]
\[
\begin{aligned}
z_{k,r}(\xi_{k,r}) - \varepsilon z_{k,r}''(\xi_{k,r}) = z_{2k+1}^*, \quad z_{k,r}'(\xi_{k,r}) = F(z_{2k+1}^*; s);
\end{aligned}
\]
\[
\begin{aligned}
&\text{(iii) there exists an } \varepsilon_k \text{ such that, for any } \varepsilon < \varepsilon_k,
\end{aligned}
\]
\[
\begin{aligned}
z_{k,l}(\xi_k) \leq z_{2k-1}^*, \quad z_{k,r}(\xi_k) \geq z_{2k+1}^*.
\end{aligned}
\]

Proof. Let us start by justifying the choice of initial data in (4.9). Since we want to joint \( z_{k,l} \) and \( z_{k,r} \) in 0 to obtain a smooth (say, \( C^2 \)) profile, we are forced to choose (4.9). Moreover, differentiating (4.6) we get
\[
\begin{aligned}
z_{k,l}'' = (z - \varepsilon z'')'F'(z - \varepsilon z''; s).
\end{aligned}
\]
In the sequel, we shall also need \( z_{k,l}'''(0) \) and \( z_{k,r}'''(0) \) finite and therefore, assuming that requirement, form \( 4.12 \) we obtain \( 4.10 \), which implies \( 4.9 \), in view of \( 4.7 \) and \( 4.8 \).

Since \( F(z_{2k}^*; s) < 0 \), we can extend continuously the function \( h_{2k} \) at the right of \( z_{2k}^* \) to obtain the existence of a (not necessary unique) \( C^2 \) maximal solution of \( 4.7 \), which coincides with \( z_{k,l}(\xi) \), for any \( \xi > 0 \), as long as \( z_{k,l}(\xi) \leq z_{2k}^* \). Hence, property (i) is proved as before: \( h_{2k} \) is the inverse of \( F(\cdot; s) \) on an interval where \( F(\cdot; s) \) is increasing and therefore it corresponds to the case studied in Proposition 2.3. A symmetric argument will lead to the situation of Proposition 2.2 which gives (ii). The remaining part of the result will be proved only for \( z_{k,l} \), the ones for \( z_{k,r} \) being similar and thus we drop the subscript \( k,l \).

Let us now prove
\[
\begin{aligned}
|z'''(0)| < +\infty.
\end{aligned}
\]
To this end, let us come back to the original variables \( u \) and \( q \). From \( 4.11 \), and after integration with respect to \( \xi \) of the first line, the equations for the profiles \( u \) and \( q \) are given by
\[
\begin{aligned}
&\begin{cases}
F(u; s) = -q \\
u' = \varepsilon q'' - q,
\end{cases}
\end{aligned}
\]
because \( q(\pm \infty) = 0 \). Thus, the profile \( u \), regular away from 0 and continuous there, verifies

\[
  u' = -\varepsilon F''(u; s) + F(u; s) = -\varepsilon F'(u; s)u'' - \varepsilon F''(u; s)(u')^2 + F(u; s).
\]

Hence, we obtain the following system in the state space \((u, v = u')\)

\[
\begin{aligned}
  u' &= v \\
  \varepsilon F'(u; s)v' &= -\varepsilon F''(u; s)v^2 - v + F(u; s).
\end{aligned}
\]

(4.14)

At this point, as in [10], we shall introduce a new independent variable \( \eta \) given by

\[
  \xi = \int_{\eta}^{\xi} F'(u(\zeta); s)d\zeta,
\]

which will move the singularity attained for \( \xi = 0 \) to \( \eta = \infty \), thanks to exponential decay of \( F'(u(\eta); s) \) toward zero, as \( \eta \to \infty \) (the same kind of procedure used for regularity results in Section 3). Hence, (4.14) becomes

\[
\begin{aligned}
  u' &= F'(u; s)v \\
  v' &= \frac{1}{\varepsilon} \left( -\varepsilon F''(u; s)v^2 - v + F(u; s) \right).
\end{aligned}
\]

(4.15)

The singular point \( u = z^s_{2k} \) for (4.14) corresponds to a pair of equilibrium points \((z^s_{2k}, v^*_{1,2})\) for (4.15), with

\[
v^*_{1,2} = \frac{-1 \pm \sqrt{1 + 4\varepsilon|F''(z^s_{2k}; s)F(z^s_{2k}; s)|}}{2\varepsilon F''(z^s_{2k}; s)}.
\]

Moreover, from (i) we know that

\[
v = u' = z' - \varepsilon z''' = \frac{z''}{\varepsilon F'(z - \varepsilon z'''; s)} < 0.
\]

Hence the orbit lies in the lower halfplane \( v < 0 \). Moreover, \( u \to z^s_{2k} \) as \( \eta \to -\infty \), in view of the requirements 4.13 in the original variables and a straightforward analysis of the dynamical system (4.15) ensures the \((z^s_{2k}, v^*_1)\) is a saddle point (see also Section 3 for further details on a very close dynamical system). Thus there exists an unique orbit for which \( u(-\infty) = z^s_{2k} \), namely the unique orbit ending at the saddle point. In particular, the value \( v \) converges toward the finite value \( v^*_1 < 0 \), as \( \eta \to -\infty \). Hence, coming back to the original independent variable \( \xi \), \( u' \) is bounded in 0, that is (4.13). Moreover, as already pointed out before, that property implies \( z''(0) = 0 \).

We are now left with the proof of (iii), namely \( z(\xi) \leq z^s_{2k-1} \). As for Lemma 2.4 we shall prove that property by comparing in the phase space \((z, z')\), the graph \( \varphi(z) \) of the trajectory of \( z(\xi) \) with the one \( \psi(z) = F(z; s) \) of \( z' = F(z; s) \) close to the point \((z^s_{2k}, F(z^s_{2k}; s))\). Since \( \psi'(z^s_{2k}) = F'(z^s_{2k}; s) = 0 \) and

\[
\varphi'(z^s_{2k}) = \frac{z''}{z'} \bigg|_{\xi=0} = 0,
\]

we get
we need to evaluate the second derivative at this point. Clearly, \( \psi''(z_{2k}^*) = F''(z_{2k}^*; s) \), while

\[
\varphi''(z_{2k}^*) = \frac{d}{d\xi} \left( \frac{z''(\xi)}{z'} \right) \bigg|_{\xi=0} = \frac{z''(0)z'(0) - z''(0)^2}{(z'(0))^3} = \frac{z''(0)}{F(z_{2k}^*; s)^2}.
\]

Moreover, from the above calculations, we know that

\[
z''(0) = \frac{z'(0) - u'(0)}{\varepsilon} = \frac{F(z_{2k}^*; s) - v_1^*}{\varepsilon}.
\]

Therefore, \( z''(0) \) is the biggest solution of the equation

\[
Q(\lambda) := (\varepsilon\lambda - F(z_{2k}^*; s))^2 - \frac{\lambda}{F''(z_{2k}^*; s)} = 0.
\]

Since

\[
Q(F(z_{2k}^*; s)^2F''(z_{2k}^*; s)) = \varepsilon F(z_{2k}^*; s)^3F''(z_{2k}^*; s)(\varepsilon F(z_{2k}^*; s)F''(z_{2k}^*; s) - 2),
\]

and \( F(z_{2k}^*; s)F''(z_{2k}^*; s) > 0 \), there exists an \( \varepsilon_0 > 0 \) such that, for any \( \varepsilon < \varepsilon_0 \), \( Q(F(z_{2k}^*; s)^2F''(z_{2k}^*; s)) < 0 \), that is

\[
F(z_{2k}^*; s)^2F''(z_{2k}^*; s) < z''(0),
\]

which implies

\[
0 > \varphi''(z_{2k}^*) > \psi''(z_{2k}^*).
\]

Therefore, the graph \( \varphi(z) \) leaves the point \( z_{2k}^* \) above \( \psi(z) \), while going to the left. Hence, we can argue as in Lemma 2.4 to conclude \( z(\tilde{\xi}) \leq z_{2k-1}^* \) and the proof is complete. \( \square \)

In view of the above result, as in Theorem 2.5, we can glue together the profiles \( z_0, z_{k_1}, z_{k_1}, k_1, \ldots, n-1, z_n \) to obtain the desired radiating profile, joining \( u_- = z_- \) and \( u_+ = z_+ \).

**Proposition 4.2.** Under conditions (4.4) and (4.7), there exists an \( \bar{\varepsilon} > 0 \) such that, for any \( \varepsilon < \bar{\varepsilon} \), there exists a (unique up to space translations) \( C^1 \) profile \( z \) with \( z(\pm \infty) = z_{\pm} \) and a speed \( s \) such that the function \( z(x - st) \) is solution of (4.6). This solution is \( C^2 \) away from the \( n \) points \( z_{2k-1}^*, k = 1, \ldots, n \), where \( z'' \) has at most a jump discontinuity.

Moreover, there exists a (unique up to space translations) profile \( u \) with \( u(\pm \infty) = u_{\pm} \) and a speed \( s \) (given by the Rankine–Hugoniot condition) such that the function \( u(x - st) \) is solution of (4.6). This profile is continuous away from the \( n \) points \( z_{2k-1}^*, k = 1, \ldots, n \), where it has at most a jump discontinuity. At these points, the Rankine–Hugoniot and the admissibility conditions of the scalar conservation law \( u_t + f(u)_x = 0 \) are satisfied.

**Proof.** First of all, from Proposition 4.1 we can define \( C^2 \) profiles \( z_1, \ldots, z_{n-1} \), gluing together the profiles \( z_{k,l}, z_{k,r} \), after an appropriate space translation. Hence, we end up with \( n + 1 \) profiles \( z_0, \ldots, z_n \), all of them decreasing.

Moreover, for \( \varepsilon < \bar{\varepsilon} = \min\{\varepsilon_k\}_{k=1,\ldots,n-1} \), there exist points in the \((z, z')\) plane where the graphs of two consecutive \( z_k \) can be glued together in that point, in order to give a \( C^1 \) solution of (4.6). Moreover, due to the monotonicity of these graphs, again consequence
of Proposition 4.4, this intersection is indeed unique. Hence with appropriate space translations, we can find a points $\bar{\xi}_k$, $k = 1, \ldots, n - 1$, such that $z_k(\bar{\xi}_k) = z_{k+1}(\bar{\xi}_k) = \bar{z}_k$ and $z_k'(\bar{\xi}_k) = z_{k+1}'(\bar{\xi}_k) = \bar{z}_k$, $k = 1, \ldots, n - 1$. Now we define

$$z(\xi) = \begin{cases} 
  z_n(\xi) & \xi \in (-\infty, \bar{\xi}_{n-1}] \\
  z_k(\xi), & \xi \in [\bar{\xi}_k, \bar{\xi}_{k-1}), \\
  z_0(\xi) & \xi \in [\bar{\xi}_0, +\infty).
\end{cases}$$

Thus, this profile defines a $C^1$ solution of (4.6) and it verifies $z(\pm \infty) = z_\pm$. Moreover, $z(x)$ is $C^2$ away from the points $\bar{\xi}_k$, where it verifies $\varepsilon z''(\bar{\xi}_k - 0) = \varepsilon z''(\bar{\xi}_k) = \bar{z}_k - h_{2k+2}(\bar{\xi}_k)$ and $\varepsilon z''(\bar{\xi}_k + 0) = \varepsilon z''(\bar{\xi}_k) = \bar{z}_k - h_{2k+1}(\bar{\xi}_k)$.

The regularity of $u = u(x - st)$ is a direct consequence of the first part of the theorem and of the relation $u = z - \varepsilon z''$. Moreover, in the case of a discontinuity in $u$, namely $u(\bar{\xi}_k - 0) \neq u(\bar{\xi}_k + 0)$, $(u(\bar{\xi}_k - 0), u(\bar{\xi}_k + 0); s)$ verifies the Rankine–Hugoniot condition for the strictly convex conservation law $u_t + f(u)_x = 0$. Indeed, $u(\bar{\xi}_k - 0) = h_{2k+2}(\bar{\xi}_k)$, $u(\bar{\xi}_k + 0) = h_{2k+2}(\bar{\xi}_k)$ and a direct calculation shows

$$\frac{f(h_{2k+1}(\bar{\xi}_k)) - f(h_{2k+2}(\bar{\xi}_k))}{h_{2k+2}(\bar{\xi}_k) - h_{2k+1}(\bar{\xi}_k)} = s.$$ 

In addition, $u(\bar{\xi}_k - 0) = h_{2k+2}(\bar{\xi}_k) \in I_{2k+2}$, $u(\bar{\xi}_k + 0) = h_{2k+1}(\bar{\xi}_k) \in I_{2k+1}$ and $F(u(\bar{\xi}_k - 0); s) = F(u(\bar{\xi}_k + 0); s) = \bar{z}_k$. Since $F(; s)$ has a local minimum and no local maximum in $(u(\bar{\xi}_k + 0), u(\bar{\xi}_k - 0))$, we conclude $F(u; s) < F(u(\bar{\xi}_k \pm 0); s)$, for any $u \in (u(\bar{\xi}_k + 0), u(\bar{\xi}_k - 0))$, that is the Oleinik condition for the inviscid shock $(u(\bar{\xi}_k - 0), u(\bar{\xi}_k + 0); s)$.

Thanks to the above theorem, we know there exist radiative shocks, if Oleinik condition (4.4) and the non degeneracy (4.5) are satisfied. Hence, we can pass now to the proof of qualitative properties of that solutions. In particular, for a profile $u$ that belongs to $BV$, we shall prove monotonicity and uniqueness in the class modulo $L^1$. It is worth to observe that these properties do not require assumption (4.5), because they are based on the contraction properties of the model under consideration. Actually, they will be used in the existence result of Theorem 4.4 in the case of general smooth flux functions which may violate (4.5).

**Theorem 4.3.** Let $(u, q)$ be a radiative shock solution of (4.7). If $u \in BV$, then $u$ is monotone. Moreover, given two $BV$ radiative shocks $(u, q_1)$ and $(v, q_2)$, such that $u - v \in L^1$, then $u$ and $v$ are equal up to a space translation.

**Proof.** As pointed out before by several authors, [7] [9] [10] [21] [23], it is convenient to rewrite the $2 \times 2$ system (4.1) as follows:

$$u_t + f(u)_x = -\frac{1}{\varepsilon} (u - K^\varepsilon * u), \quad (4.16)$$

where the convolution kernel $K^\varepsilon$ is given by

$$K^\varepsilon(x) = \frac{1}{2\sqrt{\varepsilon}} e^{-\frac{|x|}{\sqrt{\varepsilon}}}.$$ 

Let $u$ and $v$ be radiative profiles of class $BV$, associated with a shock triplet $(u_-, u_+; s)$. Assume that $v - u$ is integrable, a fact that is certainly true if $u$ is in $BV$ and $v$ is a shift.
\[ \tau_h u \] of \( u \). Finally, let us denote \( w := v - u \). We integrate the entropy inequality

\[ |w|_t + \left((f(v) - f(u))\text{sgn}(w)\right)_x \leq \frac{1}{\varepsilon} (K^\varepsilon * w - w)\text{sgn}(w) \]

and obtain, with an argument à la Kruzhkov,

\[ \varepsilon \frac{d}{dt} \|w\|_1 \leq \int_{\mathbb{R}} (K^\varepsilon w)\text{sgn}(w) \, dx - \|w\|_1. \]

Since \( w \) is a function of \( x - st \), the norm \( \|w\|_1 \) is constant and the left-hand side above is zero. Therefore, we have

\[ \|w\|_1 \leq \int_{\mathbb{R}} (K^\varepsilon w)\text{sgn}(w) \, dx. \]

Since \( K^\varepsilon \) has positive values and unit integral, we deduce that \( w \) has a constant sign.

Let \( h \) be a real number. Since \( \tau_h v := v(\cdot + h) \) is also a profile and \( \tau_h v - v \in L^1(\mathbb{R}) \), the above fact may be applied to the pair \((\tau_h v, u)\). We obtain that for every real number \( h \), \( \tau_h v - u \) has a constant sign.

Let us focus on the case \( v = u \). Since the integral of \( \tau_h u - u \) equals \( h(u_+ - u_-) \), we see that the sign of \( \tau_h u - u \) is that of \( h(u_+ - u_-) \). In other words, \( u \) is monotonous.

Going back to the general case, the integral of \( \tau_h v - u \) equals \( h(u_+ - u_-) \) plus a constant (the integral of \( v - u \)). Hence there exists a number \( h \) for which this integral equals zero. But since \( \tau_h v - u \) has a constant sign, this means that \( \tau_h v - u \equiv 0 \) almost everywhere. \( \square \)

Now we are ready to remove the non degenerate assumption \((4.5)\) in the existence of a radiative shock. We perform this task by means of an approximating procedure already used in [20] for discrete shock profiles for conservation laws.

**Theorem 4.4.** Under conditions \((4.4)\), there exists an \( \varepsilon > 0 \) such that, for any \( \varepsilon < \varepsilon \), there exists a (unique up to space translations) \( C^1 \) profile \( z \) with \( z(\pm\infty) = z_\pm \) such that the function \( z(x - st) \) is a solution of \((4.6)\), where the speed \( s \) is given by the Rankine–Hugoniot condition. This solution is \( C^2 \) away from the \( n \) points \( z^{*}_{2k-1} \), \( k = 1, \ldots, n \), where \( z'' \) has at most a jump discontinuity.

Moreover, there exist a (unique up to space translations) profile \( u \) with \( u(\pm\infty) = u_\pm \) and a speed \( s \) such that the function \( u(x - st) \) is solution of \((4.3)\), where the speed \( s \) is given by the Rankine–Hugoniot condition. This profile is continuous away from the \( n \) points \( z^{*}_{2k-1} \), \( k = 1, \ldots, n \), where \( u \) has at most a jump discontinuity which verifies the Rankine–Hugoniot and the admissibility conditions of the scalar conservation law \( u_t + f(u)_x = 0 \).

**Proof.** We start by approximating a \( C^2 \) flux function \( f \) satisfying \((1.1)\) with a sequence of smooth functions \( f^n \) satisfying both \((1.3)\) and \((1.5)\). Then, given an inviscid shock \((u_-, u_+; s) \), \( u_- > u_+ \), Proposition \(1.2\) gives the existence of radiative shocks \((u^n, z^n)\) for any \( n \) and for any \( \varepsilon < \varepsilon \) such that \( u^n(\pm\infty) = z^n(\pm\infty) = u_\pm \). This value \( \varepsilon \) is independent from \( n \), because \( f^n \) and \( (f^n)'' \) remain bounded, (see the constraints that gives the values \( \varepsilon_k \) in Proposition \(1.1\) and hence \( \varepsilon \) in Proposition \(1.2\)). We fix these profiles with the condition

\[ \frac{1}{2}(u_- + u_+) \in [u^n(0+), u^n(0-)]. \]

Since \( u^n \) is decreasing,

\[ \|\tau_h u^n - u^n\|_{L^1} = |h(u_- - u_+)|, \]
which implies that $u^n$ is equicontinuous in $L^1$. Moreover, $u_+ \leq u^n \leq u_-$, that is, $u^n$ is equibounded in $L^\infty$ and hence in $L^1_{loc}$. We recall that the solution $z^n$ of $-\varepsilon (z^n)'' + z^n = u^n$ with $z^n(\pm \infty) = u_\pm$ is given by

$$z^n(\xi) = \frac{1}{2\sqrt{\varepsilon}} \int_{-\infty}^{+\infty} e^{-\frac{|\zeta|}{\sqrt{\varepsilon}}} u^n(\xi - \zeta) d\zeta$$

and it satisfies the same estimates. Thus, passing if necessary to subsequences, we have $u^n \rightarrow u$ and $z^n \rightarrow z$ in $L^1_{loc}$ and bounded almost everywhere. Clearly, $u$ is decreasing and therefore

$$u_+ \leq u(+\infty) \leq \frac{1}{2}(u_- + u_+) \leq u(-\infty) \leq u_-.$$

Then, we pass to the limit in the profile equations

$$\begin{cases}
F^n(u^n; s) = (z^n)', \\
-\varepsilon (z^n)'' + z^n = u^n,
\end{cases} \quad (4.17)$$

where $F^n(u; s) = f^n(u) - f^n(u_\pm) - s(u - u_\pm)$ to conclude that $(u, z)$ defines a radiative profile for the inviscid shock $(u(-\infty), u(+\infty); s)$. Hence we are left to the proof of $u(\pm \infty) = u_\pm$. Integrating $(4.17)_1$ and passing into the limit a.e. we get

$$z(\xi) - z(\zeta) = \int_\zeta^\xi F(u(x); s) dx.$$ 

Since $z(\pm \infty) = u(\pm \infty) \in \mathbb{R}$ and $u \in [u_+, u_-]$, from the above relation and from $(4.4)$ we conclude

$$0 > \int_{-\infty}^{+\infty} F(u(x); s) dx > -\infty.$$ 

Therefore in particular $F(u(\pm \infty); s) = 0$ and, using $(4.4)$ and

$$u(+\infty) \in \left[u_+, \frac{1}{2}(u_- + u_+)\right], \quad u(-\infty) \in \left[\frac{1}{2}(u_- + u_+)u_-\right],$$

we conclude $u(\pm \infty) = u_\pm$. \hfill \Box

**Remark 4.5.** Proceeding as in Section 3, it is possible to prove the radiative shocks of $(4.4)$ increase their regularity as $\varepsilon \downarrow 0$ also in this general non convex case. Indeed, across the values $z_{2k-1}^*, k = 1, \ldots, n$, where $F(\cdot; s)$ has local minima, we are in the same situation of the one discussed in the convex case and the regularity increases as $\varepsilon \downarrow 0$. On the contrary, across the values $z_{2k}^*, k = 1, \ldots, n$, where $F(\cdot; s)$ has local maxima, we can repeat the arguments of Proposition 3.4 and we obtain that the trajectory in the saddle point corresponding to the value $u = z_{2k}^*$ is never tangent to the eigenvector $(0, 1)^t$ of the negative eigenvalue and therefore the profile is regular.

5. **Reduction from the system case to the scalar case**

Let us consider the following strictly hyperbolic–elliptic coupled system

$$\begin{cases}
\displaystyle u_t + f(u)_x + Lq_x = 0, \\
\displaystyle -q_{xx} + Rq + G \cdot u_x = 0, 
\end{cases} \quad (5.1)$$
where \( x \in \mathbb{R}, t > 0, u \in \mathbb{R}^n, q \) is scalar and \( R > 0, G, L \in \mathbb{R}^n \) are constants. In this section we shall prove that the existence of travelling wave solutions of (5.1), for sufficiently small shocks, reduces to the study of a scalar model, and therefore we shall obtain their existence (and regularity) as corollary of the previous sections.

According to Definition 5.2 let us consider a radiative shock wave solutions for (5.1), namely a solution of the form \( (u, q) = (u(x - st), q(x - st)) \), with \( s \) verifying the Rankine–Hugoniot condition

\[
  u(\pm \infty) = u_\pm, \quad s(u_+ - u_-) = f(u_+) - f(u_-).
\]

As usual, we denote with \( \lambda_1(u) < \cdots < \lambda_n(u) \) the \( n \) real eigenvalues of the matrix \( \nabla f(u) \).

Neglecting the higher order term \(-q_{xx} \) in (5.1) (see, for instance [7, 13]), we end up with the reduced system

\[
  u_t + f(u)_x = R^{-1}G \cdot u_{xx}L = R^{-1}L \otimes Gu_{xx}
\]

and the \( k \)th field has a diffusive dynamics near \( u_\pm \), provided

\[
  \ell_k(u_\pm) \cdot L \otimes Gr_k(u_\pm) = G \cdot r_k(u_\pm) \ell_k(u_\pm) \cdot L > 0,
\]

where, as usual, \( r_k(u) \) and \( \ell_k(u) \) denote the \( k \)th right and left eigenvector of \( \nabla f(u) \), normalized such that \( \ell_i(u) \cdot r_j(u) = \delta_{ij} \).

Once again, we introduce the variable \( z \) as the opposite of the antiderivative of \( q \), that is \(-z_x := q \) and therefore \( z(\pm \infty) = z_\pm = G \cdot u_\pm \). Hence, we proceed as in the previous sections to conclude the the system for \( u = u(x - st) \) and \( z = z(x - st) \) is given by

\[
  \begin{cases}
    Lz' = f(u) - f(u_+) - s(u - u_-) \\
    Rz - z'' = G \cdot u.
  \end{cases}
\]

Given the vector \( L \), let \( P : \mathbb{R}^n \to \mathbb{R}^n \) and \( Q : \mathbb{R}^n \to \mathbb{R} \) be linear applications such that \( \ker P = \text{span} \{L\} \), \( QL = 1 \).

Then system (5.3) becomes

\[
  \begin{cases}
    P (f(u) - f(u_+) - s(u - u_-)) = 0 \\
    z' = Q (f(u) - f(u_+) - s(u - u_-)) \\
    G \cdot u = Rz - z''.
  \end{cases}
\]

Hence, the existence of our profile reduces to a scalar case, provided the constraints (5.1), together with condition (5.2), ensure

\[
  Q (f(u) - f(u_+) - s(u - u_-)) = \tilde{F}(G \cdot u; s),
\]

in a neighborhood of \( (u_\pm; \lambda_k(u_\pm)) \), that is for \( |u_+ - u_-| \) sufficiently small. This property is proved in the next lemma.

**Lemma 5.1.** Let \( f \) be \( C^1 \) and assume condition (5.2) holds. Then, there exists a neighborhood \( \mathcal{U} \times I \) of \( (u_\pm; \lambda_k(u_\pm)) \) such that for any \((u; s) \in \mathcal{U} \times I,\)

\[
  P (f(u) - f(u_+) - s(u - u_-)) = 0 \iff u = \Phi(G \cdot u; s),
\]

with \( \Phi \in C^1(\mathbb{R} \times \mathbb{R}; \mathcal{U}) \), not depending on the choice of \( P \) such that \( \ker P = \text{span} \{L\} \).
Proof. Assume, without loss of generality, \((u_\pm; \lambda_k(u_\pm)) = 0\) and \(\|G\| = 1\). Let \(V_1, \ldots, V_{n-1}\) be a basis for \(G^\perp\), so that \(\{V_1, \ldots, V_{n-1}, G\}\) is a basis for \(\mathbb{R}^n\). For \(u = c_1V_1 + \cdots + c_{n-1}V_{n-1} + \alpha G\), let us denote with \(\phi(c_1, \ldots, c_{n-1}, \alpha; s)\) the function \(P (f(u) - f(u_\pm) - s(u - u_\pm))\). Then \(\phi(0, \ldots, 0; 0) = 0\). Moreover, for \(j \in \{1, \ldots, n-1\}\),

\[
\frac{\partial \phi}{\partial c_j}(0) = P \nabla f(0) V_j.
\]

Then the conclusion follows from the Implicit Function Theorem, provided the vectors \(P \nabla f(0)V_1, \ldots, P \nabla f(0)V_{n-1}, j = 1, \ldots, n-1\) are linearly independent, because, in that case, the relation \(\phi(c_1, \ldots, c_{j-1}, \alpha; s) = 0\) can be locally written as a function of \((\alpha = G \cdot U; s)\).

Let \(a_1, \ldots, a_{n-1} \in \mathbb{R}\) be such that

\[
a_1 P \nabla f(0)V_1 + \cdots + a_{n-1} P \nabla f(0)V_{n-1} = 0.
\]

Then

\[
P \nabla f(0) (a_1 V_1 + \cdots + a_{n-1} V_{n-1}) = 0
\]

and, by definition of \(P\), there exist \(\beta \in \mathbb{R}\) such that

\[
\nabla f(0) (a_1 V_1 + \cdots + a_{n-1} V_{n-1}) = \beta L.
\]

Applying \(\ell_k(0)\) at the left hand side, we have

\[
0 = \ell_k(0) \nabla f(0) (a_1 V_1 + \cdots + a_{n-1} V_{n-1}) = \beta \ell_k(0) \cdot L.
\]

Hence, condition (5.2) implies \(\beta = 0\), namely \(a_1 V_1 + \cdots + a_{n-1} V_{n-1} \in \ker \nabla f(0)\), which means

\[
a_1 V_1 + \cdots + a_{n-1} V_{n-1} = \gamma r_k(0),
\]

for some \(\gamma \in \mathbb{R}\). Since \(V_1, \ldots, V_{n-1}\) is a basis for \(G^\perp\), using again (5.2) we conclude \(\gamma = 0\) and therefore \(a_1 = \cdots = a_{n-1} = 0\). This means that \(P \nabla f(0)V_j, j = 1, \ldots, n-1\) are linearly independent and Implicit Function Theorem can be applied.

Finally, let us note that since \(\ker P = \text{span}\{L\}\), \(P (f(u) - f(u_\pm) - s(u - u_\pm)) = 0\) is equivalent to \(f(u) - f(u_\pm) - s(u - u_\pm) \in \text{span}\{L\}\), hence the function \(\Phi\) does not depend on the specific choice of \(P\).

The proof is complete. \(\Box\)

At this point, we assume the \(k\)th field is genuinely nonlinear, that is

\[
\nabla \lambda_k(u) \cdot r_k(u) \neq 0
\]

and let \((u_-, u_+; s)\) be a \(k\)-Lax radiating shock for 5.1, that is

\[
\lambda_k(u_+) < s < \lambda_k(u_-), \quad \lambda_{k-1}(u_-) < s < \lambda_{k+1}(u_+).
\]

We shall prove that in this framework, in the reduced scalar dynamics yielding the existence of our profile, that is (5.3), the flux function \(F(\cdot; s)\) does not change convexity. On the other hand, in the general case and assuming the states verify the Liu E–condition, the reduction will lead to the non convex model treated in Section 4.
Proposition 5.2. Let us assume condition (5.2) and (5.6) hold. Then the function $\hat{F}(\cdot; s) : \mathbb{R} \to \mathbb{R}$ defined in (5.5) is either strictly convex or strictly concave in a neighborhood of $z_\pm = G \cdot u_\pm$.

Proof. Since $(u_-, u_+; s)$ form a $k$–Lax shock for (5.1), it suffices to prove

$$\frac{d^2}{dw^2} \hat{F}(G \cdot u_\pm; s) \neq 0,$$

for sufficiently small shocks.

Since $F(u; s) := f(u) - f(u_\pm) - s(u - u_\pm) = 0$, there hold $P F(u; s) = 0$ and $Q F(u; s) = \hat{F}(G \cdot u; s)$ for any $u \in \mathcal{U}$. Hence

$$F(u; s) = \hat{F}(G \cdot u; s) L.$$

Differentiating with respect to $u$, one has

$$\nabla f(u) - s I = \frac{d}{dw} \hat{F}(G \cdot u; s) L \otimes G$$

Applying $\ell_k(u)$ and $r_k(u)$ respectively to the left and to the right of (5.7), we get

$$\lambda_k(u) - s = \frac{d}{dw} \hat{F}(G \cdot u; s) \ell_k(u) \cdot L \otimes G r_k(w)$$

Choosing $u = u_\pm$ and assuming $|u_+ - u_-|$ small enough, the main assumption (5.2) implies

$$\left. \frac{d}{dw} \hat{F}(G \cdot u; s) \right|_{u=u_\pm} = o(1) \quad \text{as} \quad |u_- - u_+| \to 0. \quad (5.9)$$

Differentiating (5.8) with respect to $u$ in the direction of $r_k(u)$, we obtain

$$\nabla \lambda_k(u) \cdot r_k(u) = \frac{d}{dw} \hat{F}(G \cdot u; s) \nabla (\ell_k(u) \cdot L \otimes G r_k(u)) \cdot r_k(u)$$

$$+ \frac{d^2}{dw^2} \hat{F}(G \cdot u; s) (\ell_k(u) \cdot L) (G \cdot r_k(u))^2.$$

Evaluating this relation at $u = u_\pm$ and taking in account (5.9), we obtain

$$\frac{d^2}{dw^2} \hat{F}(G \cdot u_\pm; s) = \frac{\nabla \lambda_k(u_\pm) \cdot r_k(u_\pm)}{(\ell_k(u) \cdot L) (G \cdot r_k(u))^2} + o(1) \quad \text{as} \quad |u_- - u_+| \to 1,$$

The conclusion follows from GNL condition (5.6) and assumption (5.2). \qed

The previous results show that the existence of a $k$–radiative shock for (5.1) reduces to the study of a scalar model of the form

$$\begin{cases}
w_t + \hat{f}(w)x + q_x = 0 \\
-q_{xx} + R q + w_x = 0,
\end{cases}$$

provided (5.2) is satisfied for the original shock $(u_-, u_+; s)$. In addition, if (5.6) is also verified, we can assume, without loss of generality, that the flux in (5.10) is strictly convex (see Proposition 5.2). In this framework, the reduction of (5.1) to (5.10) is given by (5.5) and Lemma 5.1, namely

$$w = G \cdot u, \quad \hat{F}(G \cdot u; s) = Q(f(u) - f(u_\pm) - s(u - u_\pm)).$$
with \( Q \) (and \( P \) below) as before and, adding if necessary a linear function to the flux,

\[
\hat{F}(w; s) = \hat{f}(w) - \hat{f}(w_{\pm}) - s(w - w_{\pm}).
\]

For the sake of clarity, we start by considering the GNL case and we postpone the general case at the end of the section.

If the flux in (5.10) is strictly convex, Theorem 2.5 guarantees the existence of a radiative shock for that model, with at most a jump discontinuity, which is indeed an admissible shock for the inviscid related conservation law. To conclude with the results stated in Theorem 1.6, we shall analyze that discontinuity in the corresponding radiative shock of the original vectorial case (5.1), showing it forms an admissible radiative shock for that system.

**Proof of Theorem 1.6.** Let \((u_-, u_+; s)\) be an admissible \( k \)-shock for (5.1) and, let us consider \( w_\pm = G \cdot u_\pm \). Let \( U, W = \Phi(U; s) \) be the neighborhoods of \( u_\pm \) and \( w_\pm \) given by Lemma 5.1 and Proposition 5.2, and such that

\[
\ell_k(u) \cdot L \otimes Gr_k(u) > 0
\]

(5.11)

for any \( u \in U \). We start by proving that \((w_-, w_+; s)\) is an admissible shock for the reduced scalar conservation law (with strictly convex flux)

\[
w_t + \hat{f}(w)_x = 0.
\]

(5.12)

Indeed, from

\[
f(u_+) - f(u_-) - s(u_+ - u_-) = 0,
\]

we obtain in particular

\[
\hat{F}(w_+; s) = Q(f(u_+) - f(u_-) - s(u_+ - u_-)) = 0,
\]

that is, the Rankine–Hugoniot conditions of (5.12) for the shock \((w_-, w_+; s)\).

Proceeding as in the proof of Proposition 5.2, we obtain (5.8), that is

\[
\lambda_k(u) - s = (\hat{f}'(G \cdot u) - s)\ell_k(u) \cdot L \otimes Gr_k(u).
\]

(5.13)

Therefore the sign of \( \hat{f}'(G \cdot u) - s \) is given by the sign of \( \lambda_k(u) - s \) for any \( u \in U \), in view of (5.11). Thus, using (5.13) for \( u = u_\pm \) and taking into account the admissibility condition of \( u_\pm \), \( \lambda_k(u_+) < s < \lambda_k(u_-) \), we obtain

\[
\hat{f}'(w_-) > s > \hat{f}'(w_+),
\]

that is, the discontinuity \((w_-, w_+; s)\) is admissible for (5.12).

At this point, let \( w \) be the (unique up to space shifts) radiative profile of the reduced model (5.10) given by Theorem 2.5. Then, the above results guarantee the existence of the (unique up to space shift) radiative profile \((u, q)\) for (5.1), if \(|u_+ - u_-|\) is sufficiently small. Therefore we only have to prove that, if \( w \) is discontinuous, the corresponding discontinuity in \( u \) defines an admissible \( k \)-shock wave for the inviscid hyperbolic system of conservation laws \( u_t + f(u)_x = 0 \). We shall perform this task as before, connecting the properties of the shock for the reduced system with the ones of the shock of the original systems.
Denoting with \((w_l, w_r; s)\) the discontinuity of the radiating shock for the scalar reduced model, we have \(w_{l,r} \in W\) and therefore there exist unique \(u_{l,r} \in U\) solutions of
\[
\begin{align*}
G \cdot u &= w_{l,r} \\
F(u) - f(u_s) - s(u - u_s) &= 0,
\end{align*}
\] (5.14)
which are defined through the function \(\Phi\) constructed in Lemma 5.1.

As before, let us start by showing the Rankine–Hugoniot condition for \((u_l, u_r; s)\), namely
\[f(u_l) - f(u_r) = s(u_l - u_r).\]

Clearly, the above relation is equivalent to
\[F(u_l; s) = F(u_r; s),\]
(5.15)
In order to prove (5.15), we only have to prove
\[QF(u_l; s) = QF(u_r; s),\]
(5.16)
because \(PF(u_l; s) = 0 = PF(u_r; s)\) is given by (5.14). Moreover, \(QF(u; s) = \hat{F}(G \cdot u = w; s)\) for any \(u \in U\), namely, for \(u = \Phi(w; s)\) and therefore (5.16) is precisely the Rankine–Hugoniot condition
\[\hat{f}(w_l) - \hat{f}(w_r) = s(w_l - w_r)\]
for the reduced scalar model (5.10).

We turn now to the proof of the Lax conditions for the \(k\)-shock \((u_l, u_r; s)\) of \(u_t + f(u)_x = 0\), namely
\[
\begin{align*}
\lambda_k(u_r) &< s < \lambda_k(u_l), \\
\lambda_k-1(u_l) &< s < \lambda_k+1(u_r).
\end{align*}
\] (5.17) (5.18)
Since we are dealing with weak shocks and the system is assumed to be strictly hyperbolic, (5.18) follows from (5.17). Finally, using once again (5.8), this time for \(u = u_{l,r}\), and taking into account the admissibility condition of \(w_{l,r} = G \cdot u_{l,r}\), that is
\[\hat{f}'(w_l) > s > \hat{f}'(w_r),\]
we obtain (5.17).

Proof of Theorem 1.8. The proof is an immediate consequence of Proposition 3.4 and Theorem 1.6.

Let us pass now to the general case, namely when condition (5.6) is violated. As we have shown in Section 4, the existence of radiative shock without convexity assumptions is guaranteed only when the radiative effect is sufficiently dissipative. Hence, we shall consider the following system
\[
\begin{align*}
u_t + f(u)_x + Lq_x &= 0 \\
-\varepsilon q_{xx} + Rq + G \cdot u_x &= 0,
\end{align*}
\] (5.19)
with $0 < \varepsilon \ll 1$. It is worth to observe that the smallness in $\varepsilon$ is needed only for the existence of the profile for the reduced scalar model

$$
\begin{cases}
w_t + \hat{f}(w)x + q_x = 0 \\
-\varepsilon q_{xx} + Rq + w_x = 0
\end{cases}
$$

(5.20)

and it does not play any role in the connections between the admissibility conditions for the jumps of that model and the ones of (5.19).

**Theorem 5.3.** Let $(u_-, u_+; s)$ be an admissible $k$-shock for (1.2) and assume (5.2) holds. If $|u_- - u_+|$ and $\varepsilon$ are sufficiently small, then there exists a (unique up to shift) admissible radiative shock wave $(u = u(x-\xi), q = q(x-\xi))$ of (5.19) such that $(u(\pm \infty), q(\pm \infty)) = (u_\pm, 0)$.

**Proof.** The proof of this theorem follows the same lines of the one of Theorem 4.4. In particular, the existence of a radiative shock for (5.19) comes from Theorem 4.4 and the reduction to (5.20). Clearly, the analysis of Rankine–Hugoniot conditions is made as before, because it is independent from convexity assumptions. Therefore we are left to the proof that, given the neighborhoods $U$ and $W$ as before, a shock $(u_l, u_r; s), u_l, u_r \in U$, verifies the Liu E–condition for (1.2) if and only if the corresponding shock $(w_l, w_r; s), w_l, w_r \in W$, verifies the Oleinik condition for

$$
w_t + \hat{f}(w)x = 0.
$$

(5.21)

Let $(u_-, u_+; s)$ be a $k$-shock of (5.19) which verifies the (strict) Liu E–condition: denoting with $u_k(\tau)$ the $k$–th shock curve, $u_- = u_k(0), u_+ = u_k(\bar{\tau})$, then

$$
s = s_k(\bar{\tau}) - s_k(\tau)
$$

(5.22)

for any $\tau$ between 0 and $\bar{\tau}$. As before, consider $w_\pm = G \cdot u_\pm$ and assume $w_- > w_+$. By definition of shock curve, we have

$$
0 = f(u_k(\tau)) - f(u_-) - s_k(\tau)(u_k(\tau) - u_-)
$$

$$
= F(u_k(\tau); s) + (s - s_k(\tau))(u_k(\tau) - u_-)
$$

$$
= \hat{F}(G \cdot u_k(\tau); s)L + (s - s_k(\tau))(u_k(\tau) - u_-)
$$

which implies

$$
\hat{F}(G \cdot u_k(\tau); s)L = (s_k(\bar{\tau}) - s)(u_k(\bar{\tau}) - u_-).
$$

(5.23)

Since for small shocks, that is $|\tau|$ small, $u_k(\tau) - u_- = \tau r_k(u_-) + o(\tau)$, form (5.23) we conclude

$$
\hat{F}(G \cdot u_k(\tau); s)L = (\tau r_k(u_-) + o(\tau))(s_k(\bar{\tau}) - s)
$$

and, multiplying that relation on the left for $\ell_k(u_-)$

$$
\hat{F}(G \cdot u_k(\tau); s)\ell_k(u_-) \cdot L = (\tau + o(\tau))(s_k(\bar{\tau}) - s).
$$

(5.24)

Moreover, we differentiate $w_k(\tau) := G \cdot u_k(\tau)$ with respect to $\tau$ to conclude $\hat{w}_k(\tau) = G \cdot \hat{u}_k(\tau) = G \cdot r_k(u_-) + o(\tau)$, and, since we are dealing with small shocks and $G \cdot r_k(u_-) \neq 0$ for (5.21), it follows that $w_k(\tau)$ is decreasing for $\tau$ between 0 and $\bar{\tau}$. Hence, if $G \cdot r_k(u_-) > 0$ (resp. $G \cdot r_k(u_-) < 0$), then $\bar{\tau} < 0$ (resp. $\bar{\tau} > 0$) and $\ell_k(u_-) \cdot L > 0$ (resp. $\ell_k(u_-) \cdot L < 0$).
using again (5.24). Therefore, for \(|\tau|\) small and between 0 and \(\bar{\tau}\), \(\tau + o(\tau) < 0\) (resp. \(\tau + o(\tau) > 0\)) if \(\ell_k(u_-) \cdot L > 0\) (resp. \(\ell_k(u_-) \cdot L < 0\)) and therefore from (5.22) and (5.24) we conclude \(\hat{F}(G \cdot u_k(\tau); s) < 0\) for any \(\tau\) between 0 and \(\bar{\tau}\), that is, the strict Oleinik condition for the shock \((w_-, w_+; s)\) of (5.21).

Let us now consider a radiative profile for (5.20) which has a discontinuity \((w_l, w_r; s)\) admissible for (5.21). As in the proof of Theorem 1.6 for the GNL case, there exist unique \(u_{l,r}\) in the neighborhood under consideration such that the discontinuity \((u_l, u_r; s)\) verifies the Rankine–Hugoniot conditions of (1.2) and in particular it belongs to a \(k\)–shock curve \(u_k(\tau)\) with \(u_k(0) = u_l, u_k(\tau_r) = u_r\). The relation

\[
f(u_k(\tau)) - f(u_l) - s_k(\tau)(u_k(\tau) - u_l) = 0
\]

implies this time

\[
(\hat{F}(G \cdot u_k(\tau); s) - \hat{F}(G \cdot u_{l,r}(\tau); s))L = (s_k(\tau) - s)(u_k(\tau) - u_-),
\]

which becomes for \(|\tau|\) small

\[
(\hat{F}(G \cdot u_k(\tau); s) - \hat{F}(G \cdot u_{l,r}(\tau); s))\ell_k(u_l) \cdot L = (s_k(\tau) - s)(\tau + o(\tau)).
\]

Moreover, using (5.11) this time for \(u_l\), we argue as before to obtain the (strict) Liu E–condition

\[
s = s_k(\tau_r) < s_k(\tau)
\]

for any \(\tau\) between 0 and \(\tau_r\) from the (strict) Oleinik condition

\[
\hat{F}(G \cdot u_k(\tau); s) < \hat{F}(G \cdot u_{l,r}(\tau); s)
\]

for any \(w_k(\tau) = G \cdot u_k(\tau) \in (w_r, w_l)\) and the proof is complete. \(\square\)

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