Wiener integrals with respect to Yeh processes

Jae Gil Choi*
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803, USA
jgchoi@math.lsu.edu

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Abstract

We define Wiener integrals with respect to Yeh processes and study their properties. In particular, we obtain the martingale property of the associated stochastic processes and give a series expansion of Wiener integrals with respect to centered Yeh process. Moreover, we derive a representation of an Yeh process in terms of a random series.

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1 Introduction

The theory of stochastic integrals and stochastic differential equations was initiated and developed by K. Itô [2, 3]. There has been a tremendous amount of papers and books in the literature on the Itô theory. For an elementary introduction, see the recent book [5].

Let $B(t)$, $t \geq 0$, $\omega \in \Omega$, be a Brownian motion and let $[a, b] \subset [0, \infty)$ be a finite interval. Since with probability one the function $t \mapsto B(t)$ is nowhere differentiable, the integral $\int_{a}^{b} f(t) \, dB(t)$ can be defined pathwise by the ordinary calculus only for a very small class of deterministic functions $f(t)$. However, by using the special properties of a Brownian motion, we can define the Wiener integral $\int_{a}^{b} f(t) \, dB(t)$ for any deterministic function $f$ in $L^2[a, b]$. Moreover, the

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Wiener integral can be extended to the Itô stochastic integral $\int_{a}^{b} f(t) \, dB(t)$ for stochastic processes $f(t, \omega)$ satisfying certain conditions (see Chapters 4 and 5 in [3]).

In this paper we will extend the Wiener integral from a Brownian motion to a more general stochastic process defined in [6], which we call an Yeh process. An Yeh process on $[a, b]$ is a continuous additive stochastic process $X(t, \omega)$, $t \in [a, b]$, $\omega \in \Omega$, such that for any $a \leq s < t \leq b$,

$$X(t) - X(s) \sim N\left(\lambda(t) - \lambda(s), \rho(t) - \rho(s)\right),$$

where $N(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu$ and variance $\sigma^2$, $\lambda$ is a continuous real-valued function on $[a, b]$ and $\rho(t)$ is a continuous monotonically increasing real-valued function on $[a, b]$. Thus an Yeh process is determined by the functions $\lambda(t)$ and $\rho(t)$. We will further assume throughout this paper that $\lambda(t)$ is a function of bounded variation on $[a, b]$ and the measure defined by $\rho(t)$ is equivalent to the Lebesgue measure on $[a, b]$. These conditions are weaker than those in the paper [1]. In particular, the function $\lambda(t)$ in [1] is assumed to be absolutely continuous with $\lambda' \in L^2[a, b]$. Thus we can take $\lambda(t)$ to be the Cantor function in this paper, but not in [1].

Note that when $\lambda(t) = 0$ and $\rho(t) = t$, the Yeh process is a Brownian motion. On the other hand, we need to point out that a Brownian motion is stationary in time, while in general an Yeh process is not stationary in time and is subject to a shift $\lambda(t)$.

Suppose $X(t)$ is an Yeh process associated with functions $\lambda(t)$ and $\rho(t)$ on $[a, b]$. Then we have the following equalities:

$$E[X(t, \cdot)] = \lambda(t), \quad a \leq t \leq b, \quad (1.1)$$

$$E[X(s, \cdot)X(t, \cdot)] = \rho(s) + \lambda(s)\lambda(t), \quad a \leq s < t \leq b. \quad (1.2)$$

Next we define two Hilbert spaces needed in this paper. Let $L^2_{\rho}[a, b]$ be the Hilbert space of functions on $[a, b]$ given by

$$L^2_{\rho}[a, b] = \left\{ f : \int_{a}^{b} |f(t)|^2 \, d\rho(t) < \infty \right\}$$

equipped with the inner product defined by

$$\langle f, g \rangle_{\rho} = \int_{a}^{b} f(t)g(t) \, d\rho(t).$$

Note that by the assumption on $\rho(t)$, we have $L^2_{\rho}[a, b] = L^2[a, b]$ as sets and the norm $\| \cdot \|_{\rho}$ is equivalent to the $L^2[a, b]$-norm $\| \cdot \|_2$. Similarly, let

$$L^2_{\lambda,\rho}[a, b] = \left\{ f \in L^2_{\rho}[a, b] : \int_{a}^{b} |f(t)|^2 \, d|\lambda|(t) < \infty \right\},$$

Note that $L^2_{\lambda,\rho}[a, b]$ is a subset of $L^2_{\rho}[a, b]$ and the norm $\| \cdot \|_{\lambda,\rho}$ is equivalent to the $L^2_{\lambda,\rho}[a, b]$-norm $\| \cdot \|_{\lambda,\rho}$.
where $|\lambda|$ is the total variation function of $\lambda$. Then $L^2_{\lambda,\rho}[a,b]$ is a Hilbert space with the inner product defined by

$$\langle f, g \rangle_{\lambda,\rho} = \int_a^b f(t)g(t) \, d[\rho(t) + |\lambda|(t)].$$

It is easy to see that $\|f\|_{\lambda,\rho} = 0$ if and only if $f = 0$ for $m_\rho$-a.e. and $f = 0$ for $m_{|\lambda|}$-a.e. where $m_\rho$ and $m_{|\lambda|}$ are Lebesgue-Stieltjes measures induced by $\rho$ and $|\lambda|$, respectively.

## 2 Wiener integral with respect to an Yeh process

Let $S[a,b]$ be the set of all step functions on $[a,b]$,

$$f = \sum_{i=1}^n c_i 1_{[t_{i-1}, t_i)},$$

(2.1)

where $a = t_0 < t_1 < \cdots < t_n = b$ and $c_i \in \mathbb{R}$. Obviously, $S[a,b]$ is a dense subspace of $L^2_{\lambda,\rho}[a,b]$.

For a step function $f(t)$ represented by Equation (2.1), we define the Wiener integral $I(f)$ with respect to an Yeh process $X(t)$ to be the random variable

$$I(f)(\omega) = \sum_{i=1}^n c_i (X(t_i, \omega) - X(t_{i-1}, \omega)), \quad \omega \in \Omega.$$

It is easy to check that $I(f)$ is well-defined, namely, $I(f)$ is independent of the representation of $f$ in Equation (2.1). Moreover, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for any $\alpha, \beta \in \mathbb{R}$ and $f, g \in S[a,b]$.

Using Equations (1.1) and (1.2), and the same ideas as in the proof of Lemma 2.3.1 in [5], we have the following theorem.

**Theorem 2.1.** For $f, g \in S[a,b]$, the following hold:

1. $E[I(f)] = \int_a^b f(t) \, d\lambda(t)$,
2. $E[I(f)I(g)] = \int_a^b f(t)g(t) \, d\rho(t) + \int_a^b f(t) \, d\lambda(t) \int_a^b g(t) \, d\lambda(t)$,
3. $E[(I(f))^2] = \int_a^b f(t)^2 \, d\rho(t) + \left( \int_a^b f(t) \, d\lambda(t) \right)^2$,
4. $I(f)$ has normal distribution $N(\int_a^b f(t) \, d\lambda(t), \int_a^b f(t)^2 \, d\rho(t))$. 

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Next we extend the Wiener integral $I(f)$ from $S[a, b]$ to $L^2_{\lambda, \rho}[a, b]$. Let $f \in L^2_{\lambda, \rho}[a, b]$. By the denseness of $S[a, b]$ in $L^2_{\lambda, \rho}[a, b]$, there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in $L^2_{\lambda, \rho}[a, b]$ such that $\lim_{n \to \infty} \|f - f_n\|_{\lambda, \rho} = 0$. Then by the linearity of the mapping $I$ and assertion (3) of Theorem 2.1, we have
\[
\|I(f_m) - I(f_n)\|_{2}^2 \\
\leq \int_a^b |f_n(t) - f_m(t)|^2 d\rho(t) + \left( \int_a^b |f_n(t) - f_m(t)| d|\lambda|(t) \right)^2 \\
\leq \int_a^b |f_n(t) - f_m(t)|^2 d\rho(t) + \left( \int_a^b d|\lambda|(t) \right) \left( \int_a^b |f_n(t) - f_m(t)|^2 d|\lambda|(t) \right) \\
\leq (1 + |\lambda|(b) - |\lambda|(a))\|f_n - f_m\|_{\lambda, \rho}^2.
\]
Hence $\{I(f_n)\}$ is a Cauchy sequence in $L^2(\Omega)$ and so it converges in $L^2(\Omega)$. Define
\[
I(f) = \lim_{n \to \infty} I(f_n), \quad \text{in } L^2(\Omega). \tag{2.2}
\]
It is easy to check that $I(f)$ is independent of the choice of the sequence $\{f_n\}_{n \in \mathbb{N}}$. Thus we can make the following definition.

**Definition 2.2.** Let $f \in L^2_{\lambda, \rho}[a, b]$. The limit $I(f)$ defined by Equation (2.2) is called the Wiener integral of $f$ with respect to the Yeh process $X(t)$. The Wiener integral $I(f)$ will be denoted by
\[
I(f)(\omega) = \left( \int_a^b f(t) dX(t) \right)(\omega), \quad \text{for a.s. } \omega \in \Omega.
\]

**Theorem 2.3.** The Wiener integral $I(\cdot)$ is a linear mapping from $L^2_{\lambda, \rho}[a, b]$ into $L^2(\Omega)$. Moreover, the assertions (1), (2), (3), and (4) in Theorem 2.1 hold for any $f, g \in L^2_{\lambda, \rho}[a, b]$.

In particular, for any $f, g \in L^2_{\lambda, \rho}[a, b]$, we have the following equality which will be used later.
\[
E[I(f)I(g)] = \int_a^b f(t)g(t) d\rho(t) + \int_a^b f(t) d\lambda(t) \int_a^b g(t) d\lambda(t). \tag{2.3}
\]

**Corollary 2.4.** Let $f, g \in L^2_{\lambda, \rho}[a, b]$. Then $\langle f, g \rangle_{\rho} = 0$ if and only if the Gaussian random variables $I(f)$ and $I(g)$ are independent.

The next theorem relates the Wiener integral of a function $f$ of bounded variation to the pathwise Riemann-Stieltjes integral of $f$. Using the same ideas as in the proof of Theorem 2.3.7 in [5], we have the following theorem.

**Theorem 2.5.** Let $f$ be a function of bounded variation on $[a, b]$. Then
\[
I(f)(\omega) = (RS) \int_a^b f(t) dX(t, \omega), \quad \text{a.s. } \omega \in \Omega,
\]
where the right hand side is a Riemann-Stieltjes integral for each sample path of $X(t)$.
3 Properties of Wiener integrals

It is well known that a Brownian motion $B(t)$ is a martingale with respect to the filtration $\{F^B_t : t \geq 0\}$ given by $F^B_t = \sigma\{B(s) : a \leq s \leq t\}$. Moreover, for any $f \in L^2[a, b]$, the stochastic process

$$M(t) = \int_a^t f(s) dB(s), \quad t \in [a, b],$$

is also a martingale with respect to $\{F^B_t\}$. However, an Yeh process $X(t)$ determined by $\lambda$ and $\rho$ may not be a martingale with respect to the filtration $F_t = \sigma\{X(s) : a \leq s \leq t\}, a \leq t \leq b$. In fact, for any $a \leq s \leq t \leq b$, we have

$$E[X(t)|F_s] = E[X(t) - X(s)] + X(s) = \lambda(t) - \lambda(s) + X(s).$$

Hence if $\lambda$ is an increasing function on $[a, b]$, then $X(t)$ is a submartingale with respect to $\{F_t\}$. But if $\lambda$ is a decreasing function on $[a, b]$, then $X(t)$ is a supermartingale with respect to $\{F_t\}$.

**Theorem 3.1.** Suppose the mean function $\lambda$ of an Yeh process $X(t), a \leq t \leq b$, is increasing on $[a, b]$ and let $f \in L^2_{\lambda, \rho}[a, b]$ be a nonnegative function. Then the stochastic process

$$M(t) = \int_a^t f(s) dX(s), \quad a \leq t \leq b, \quad (3.1)$$

is a submartingale with respect to the filtration $\{F_t : a \leq t \leq b\}$ defined by $F_t = \sigma\{X(s) : a \leq s \leq t\}, a \leq t \leq b$.

**Proof.** First we show that $E|M(t)| < \infty$ for all $t \in [a, b]$ in order to take conditional expectation of $M(t)$. Apply Equation (2.3) with $f = g$ to get

$$E[|M(t)|^2] = \int_a^t f(s)^2 d\rho(s) + \left( \int_a^t f(s) d\lambda(s) \right)^2 \leq \int_a^b f(s)^2 d\rho(s) + \left( \int_a^b f(s) d\lambda(s) \right)^2.$$

Hence $E|M(t)| \leq \{E[|M(t)|^2]\}^{1/2} < \infty$. Next we need to show that

$$E[M(t)|F_s] \geq M(s), \quad \text{almost surely,} \quad (3.2)$$

for any $a \leq s \leq t \leq b$. Note that for any $s < t$,

$$M(t) = M(s) + \int_s^t f(u) dX(u).$$
Hence we have

\[ E[M(t)|\mathcal{F}_s] = M(s) + E\left[ \int_s^t f(u) \, dX(u) \bigg| \mathcal{F}_s \right]. \]

Thus in order to prove Equation (3.2), it suffices to show that for any \( s \leq t \),

\[ E\left[ \int_s^t f(u) \, dX(u) \bigg| \mathcal{F}_s \right] \geq 0. \tag{3.3} \]

First suppose \( f \) is a nonnegative step function represented by

\[ f = \sum_{i=1}^n c_i 1_{[t_{i-1}, t_i)}, \]

where \( t_0 = s \) and \( t_n = t \). In this case, we have

\[ \int_s^t f(u) \, dX(u) = \sum_{i=1}^n c_i (X(t_i) - X(t_{i-1})), \quad c_i \geq 0. \]

But \( X(t_i) - X(t_{i-1}), i = 1, \ldots, n \), are all independent of the \( \sigma \)-field \( \mathcal{F}_s \). Hence

\[
E\left[ \int_s^t f(u) \, dX(u) \bigg| \mathcal{F}_s \right] = \sum_{i=1}^n c_i E[X(t_i) - X(t_{i-1})|\mathcal{F}_s] \\
= \sum_{i=1}^n c_i E[X(t_i) - X(t_{i-1})] \\
= \sum_{i=1}^n c_i (\lambda(t_i) - \lambda(t_{i-1})).
\]

Thus Equation (3.3) holds for any nonnegative step function \( f \).

Next suppose \( f \in L^2_{\lambda,\rho}[a, b] \) and \( f \geq 0 \). Choose a sequence \( \{f_n\}_{n=1}^\infty \) of nonnegative step functions converging to \( f \) in \( L^2_{\lambda,\rho}[a, b] \) monotonically. Then by the conditional Jensen’s inequality, we have the inequality

\[ |E[X|\mathcal{F}]|^2 \leq E[X^2|\mathcal{F}], \]

which implies that

\[
\left| E\left[ \int_s^t (f_n(u) - f(u)) \, dX(u) \bigg| \mathcal{F}_s \right]\right|^2 \leq E\left[ \left( \int_s^t (f_n(u) - f(u)) \, dX(u) \right)^2 \bigg| \mathcal{F}_s \right].
\]
Moreover, we use the property $E[E[X|\mathcal{F}]]=E[X]$ of conditional expectation and then apply Equation (2.3) with $f=g$ to get

$$E\left[ E\left[ \int_s^t (f_n(u) - f(u)) \, dX(u) \big| \mathcal{F}_s \right]^2 \right]$$

$$\leq E\left[ E\left[ \left( \int_s^t (f_n(u) - f(u)) \, dX(u) \right)^2 \big| \mathcal{F}_s \right] \right]$$

$$= \int_s^t (f_n(u) - f(u))^2 \, d\rho(u) + \left( \int_s^t (f_n(u) - f(u)) \, d\lambda(u) \right)^2$$

$$\leq \int_a^b (f_n(u) - f(u))^2 \, d\rho(u) + \left( \int_a^b |f_n(u) - f(u)| \, |d\lambda|(u) \right)^2$$

$$\leq (1 + |\lambda|(b) - |\lambda|(a)) \|f_n - f_m\|_{\lambda, \rho}^2$$

$$\to 0,$$

as $n \to \infty$. This shows that the sequence $E[\int_s^t f_n(u) \, dX(u)|\mathcal{F}_s]$, $n \geq 1$, of random variables converges to $E[\int_s^t f(u) \, dX(u)|\mathcal{F}_s]$ in $L^2(\Omega)$. Note that the convergence of a sequence in $L^2(\Omega)$ implies convergence in probability, which implies the existence of a subsequence converging almost surely. Thus by choosing a subsequence, if necessary, we conclude that the following equality holds with probability one,

$$\lim_{n \to \infty} E\left[ \int_s^t f_n(u) \, dX(u) \big| \mathcal{F}_s \right] = E\left[ \int_s^t f(u) \, dX(u) \big| \mathcal{F}_s \right]. \quad (3.4)$$

But $E[\int_s^t f_n(u) \, dX(u)|\mathcal{F}_s] \geq 0$ since we have already shown that Equation (3.3) holds for nonnegative step functions. Hence by Equation (3.4),

$$E\left[ \int_s^t f(u) \, dX(u) \big| \mathcal{F}_s \right] \geq 0,$$

which shows that the inequality in Equation (3.3) holds for any nonnegative function $f$ in $L^2_{\lambda, \rho}[a, b].$ 

From the proof of the above theorem, we get the following assertion under various conditions on the mean function $\lambda(t)$ and the integrand $f(t)$:

1. If the mean function $\lambda(t)$ of an Yeh process $X(t)$ is increasing on $[a, b]$ and $f \in L^2_{\lambda, \rho}[a, b]$ is nonpositive, then the stochastic process $M(t)$ given by Equation (3.1) is a supermartingale.

2. If the mean function $\lambda(t)$ of an Yeh process $X(t)$ is decreasing on $[a, b]$ and $f \in L^2_{\lambda, \rho}[a, b]$ is nonnegative, then the stochastic process $M(t)$ given by Equation (3.1) is a supermartingale.
If the mean function $\lambda(t)$ of an Yeh process $X(t)$ is decreasing on $[a, b]$ and $f \in L_\lambda^2[a, b]$ is nonpositive, then the stochastic process $M(t)$ given by Equation (3.1) is a submartingale.

In Theorem 3.1 and the above assertions (1), (2), and (3), the condition on the positivity or negativity of the integrand $f$ is necessary. For example, consider the case $\lambda(t) = t$ on $[0, 1]$. Let $f$ be the following step function

$$f(t) = \begin{cases} 1/2, & \text{if } 0 \leq t < 1/3; \\ -1/2, & \text{if } 1/3 \leq t < 2/3; \\ 2, & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

Then we have

$$E[M(1/2)|F_{1/4}] = M(1/4) - 1/24 < M(1/4),$$
$$E[M(3/4)|F_{1/4}] = M(1/4) + 1/24 > M(1/4).$$

Thus the stochastic process $M(t)$ in Equation (3.1) given by the above function $f(t)$ is neither a submartingale nor a supermartingale.

4 Random series expansion of Wiener integrals

Let $X(t)$ be an Yeh process with mean function $\lambda(t)$ and variance function $\rho(t)$. The centered Yeh process $\tilde{X}(t)$ is defined by

$$\tilde{X}(t) = X(t) - \lambda(t), \quad a \leq t \leq b.$$ 

Thus $\tilde{X}(t)$ is an Yeh process with mean function 0 and variance function $\rho(t)$. We will use $\tilde{I}(f)$ to denote the Wiener integral of $f \in L_\rho^2[a, b]$ with respect to $\tilde{X}(t)$. Obviously, we have the equality

$$\tilde{I}(f) = I(f) - \int_a^b f(t) \, d\lambda(t), \quad f \in L_\lambda^2[a, b].$$

Moreover, by Theorem 2.3, $\tilde{I}(f)$ is a Gaussian random variable and

$$E[\tilde{I}(f)] = 0, \quad E[\tilde{I}(f)\tilde{I}(g)] = \langle f, g \rangle_\rho.$$ 

Therefore, $\tilde{I}(f)$ and $\tilde{I}(g)$ are independent if and only if $\langle f, g \rangle_\rho = 0.$
Let \( \{\phi_n\}_{n=1}^{\infty} \) be an orthonormal basis for the Hilbert space \( L^2_\rho[a, b] \). Each \( f \in L^2_\rho[a, b] \) has the following expansion

\[
f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle_\rho \phi_n. \tag{4.1}
\]

Moreover, we have the Parseval identity \( \|f\|_\rho^2 = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle_\rho^2 \).

If we informally take the Wiener integral with respect to \( \tilde{X}(t) \) in both sides of Equation (4.1), then we would get

\[
\int_a^b f(t) d\tilde{X}(t) = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle_\rho \int_a^b \phi_n(t) d\tilde{X}(t). \tag{4.2}
\]

We claim that this equality is indeed true in the \( L^2(\Omega) \) sense. To prove this claim, use Equation (2.3) to show that

\[
E \left[ \left( \tilde{I}(f) - \sum_{n=1}^{N} \langle f, \phi_n \rangle_\rho \tilde{I}(\phi_n) \right)^2 \right]
\]

\[
= E \left[ \left( \tilde{I}(f - \sum_{n=1}^{N} \langle f, \phi_n \rangle_\rho \phi_n) \right)^2 \right]
\]

\[
= \left\| f - \sum_{n=1}^{N} \langle f, \phi_n \rangle_\rho \phi_n \right\|_\rho^2 
\]

\[
\to 0,
\]

as \( N \to 0 \). Hence the random series in Equation (4.2) converges in \( L^2(\Omega) \) to the random variable in the left-hand side of Equation (4.2). But the \( L^2(\Omega) \) convergence implies convergence in probability. On the other hand, note that the random variables \( \tilde{I}(\phi_n) \), \( n \geq 1 \), are independent. Hence we can apply the Lévy equivalence theorem (page 173 [4]) to conclude that the random series in Equation (4.2) converges almost surely. Thus we have proved the next theorem for the random series expansion of Wiener integral with respect to the centered Yeh process \( \tilde{X}(t) = X(t) - \lambda(t) \).

**Theorem 4.1.** Let \( \{\phi_n\}_{n=1}^{\infty} \) be an orthonormal basis for \( L^2_\rho[a, b] \). Then for each \( f \in L^2_\rho[a, b] \), the Wiener integral of \( f \) with respect to \( \tilde{X}(t) \) has the following random series expansion,

\[
\int_a^b f(t) d\tilde{X}(t) = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle_\rho \int_a^b \phi_n(t) d\tilde{X}(t), \tag{4.3}
\]

where the right hand side converges in \( L^2(\Omega) \) and almost surely.
It follows from Equation (4.3) that we also have the equality for Wiener integral with respect to the Yeh process $X(t)$,

$$
\int_a^b f(t) dX(t) = \int_a^b f(t) d\lambda(t) + \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \rho \int_a^b \phi_n(t) d\tilde{X}(t).
$$

In particular, take the function $f = 1_{[a,t]}$. Then we have the random series representations of $\tilde{X}(t)$ and $X(t)$ by:

$$
\tilde{X}(t) = \sum_{n=1}^{\infty} \left( \int_a^t \phi_n(s) d\rho(s) \right) \left( \int_a^b \phi_n(s) d\tilde{X}(s) \right),
$$

$$
X(t) = \lambda(t) + \sum_{n=1}^{\infty} \left( \int_a^t \phi_n(s) d\rho(s) \right) \left( \int_a^b \phi_n(s) d\tilde{X}(s) \right).
$$

Note that the sequence $\tilde{I}(\phi_n) = \int_a^b \phi_n(s) d\tilde{X}(s)$, $n \geq 1$, is an independent sequence of standard normal random variables. Thus, given a function $\rho(t)$ satisfying the conditions in Section I, we can consider the random series

$$
\tilde{X}(t) = \sum_{n=1}^{\infty} \left( \int_a^t \phi_n(s) d\rho(s) \right) \xi_n,
$$

where $\{\phi_n : n \geq 1\}$ is an orthonormal basis for $L^{2}_\rho[a,b]$ and $\{\xi_n : n \geq 1\}$ is an independent sequence of standard Gaussian random variables. It can be checked that this random series indeed converges in $L^2(\Omega)$ and almost surely and that the stochastic process $\tilde{X}(t)$ is an Yeh process with mean function 0 and variance function $\rho(t)$. In addition, if we are also given a function $\lambda(t)$ satisfying the conditions in Section I, then the following random series

$$
X(t) = \lambda(t) + \sum_{n=1}^{\infty} \left( \int_a^t \phi_n(s) d\rho(s) \right) \xi_n,
$$

is an Yeh process with mean function $\lambda(t)$ and variance function $\rho(t)$.

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