T-EQUIVARIANT DISC POTENTIALS FOR TORIC CALABI-YAU MANIFOLDS

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Abstract. We formulate and compute the equivariant disc potentials of immersed SYZ fibers in toric Calabi-Yau manifolds, which are closely related to the open Gromov-Witten invariants of Aganagic-Vafa branes. The main tool is an equivariant version of the gluing method in [CHL18, HKL18].

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1. Introduction

Open Gromov-Witten invariants of Aganagic-Vafa Lagrangian branes in toric Calabi-Yau threefolds were first predicted by physicists [AV00, AV12, AKV02, BKMnP09]. The works of Katz-Liu [KL01] and Graber-Zaslow [GZ02] used $S^1$-equivariant localization to formulate and compute these invariants. There have been numerous recent developments [FL13, FLT12, FLZ16] in formulating and proving the physicists’ predictions using the localization technique. There are also vast conjectural generalizations of these invariants in relation with knot theory, see for instance [AENV14, TZ18].

On the other hand, in [CLL12, CLT13, CCLT16], the third-named author together with his collaborators used the SYZ $T$-duality approach [SYZ96] and wall-crossing [Aur07, GS06, GS10, GS11, KS00, KS06] to understand mirror symmetry for toric Calabi-Yau manifolds. Moreover, they proved that the generating functions of open Gromov-Witten invariants for a Lagrangian toric fiber can be computed from the inverse mirror map.

Let us illustrate by the typical example of a toric Calabi-Yau manifold $K_{P_{n-1}}$, the total space of the canonical line bundle of the projective space $P^{n-1}$. There is a $T^{n-1}$-action on $K_{P_{n-1}}$ whose symplectic quotients are identified with the complex plane. The Aganagic-Vafa Lagrangian brane $L^{AV}$ for $n = 3$ can be realized as a ray in the moment polytope (see Figure 1). In this dimension, the genus-zero open Gromov-Witten potential of $L^{AV}$ equals to the integral

$$\int \log(-z_1(z_2, q))dz_2$$

where $z_1(z_2, q)$ is obtained by solving the mirror curve equation

$$z_1 + z_2 + \frac{q}{z_1 z_2} + \exp(\phi_3(q)/3) = 0.$$
In the expression, $\phi_3(q)$ is the inverse mirror map on the Kähler parameter $q$, which has an explicit expression by solving the Picard-Fuchs differential equation. The statement was proved by [FL13] via localization.

In the SYZ approach, one has the Lagrangian torus fibration on a toric Calabi-Yau manifold due to [Gro01, Gol01]. In [CLL12], the mirror of this Lagrangian fibration was constructed via wall-crossing. Moreover, in [CCLT16], the generating function of open Gromov-Witten invariants for a Lagrangian toric fiber was computed and proved to be equal to the inverse mirror map. It states as follows for $K_{\mathbb{P}^2}$.

**Theorem 1.1 ([CLL12, LLW11, CCLT16, Lau15]).** The SYZ mirror of $K_{\mathbb{P}^2}$ equals to

$$u v = z_1 + \ldots + z_{n-1} + \frac{q}{z_1 \ldots z_{n-1}} + (1 + \delta(q))$$

where $(1 + \delta(q))$ is the generating function of one-pointed Gromov-Witten invariants of a moment-map fiber. Moreover, $(1 + \delta(q))$ equals to $\exp(\phi_n(q) / n)$ (where $\phi_n(q)$ is the inverse mirror map), which also equals to the Gross-Siebert normalized slab function.

We are interested in relating these two different approaches and their corresponding invariants. In this paper, we use the gluing method developed in [CHL18, HKL18] and the Morse model of equivariant Lagrangian Floer theory [SS10, HLS16a, HLS16b, DF17] formulated in the recent work [KLZ19] to attack this problem.

More specifically, we study immersed SYZ fibers that correspond to Aganagic-Vafa branes. They bound the same moduli of holomorphic discs. We compute the $S^1$-equivariant disc potential for the immersed SYZ fiber. In dimension three, the $S^1$-equivariant disc potential that we obtain equals to the derivative of the physicists’ generating function of genus-zero open Gromov-Witten invariants. Our approach works in all dimensions and computes the contribution of holomorphic polygons bounded by the immersed SYZ fibers which are not present in the original setting of Aganagic-Vafa branes.

The key step is to derive the gluing formula between an immersed SYZ fiber and a smooth Lagrangian torus in a toric Calabi-Yau manifold. The gluing formula involves the open Gromov-Witten invariants of a Lagrangian toric fiber. Here is the main theorem of the paper.
Theorem 1.2 (Theorem 3.18). Let $X$ be a toric Calabi-Yau $n$-fold and $L_0 \cong S^2 \times T^{n-2}$ an immersed SYZ fiber intersecting a codimension-two toric strata, where $S^2$ denotes the immersed sphere with a single nodal point. Recall that the SYZ mirror (for a given choice of a chamber and a basis of $\mathbb{Z}^n$) takes the form

$$\{(u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \mid uv = f(z_1, \ldots, z_{n-1})\}$$

where $f$ is a Laurent polynomial in the variables $z_1, \ldots, z_{n-1}$ (and also a series in Kähler parameters). Then the $S^1$-equivariant disc potential (with respect to the same choice of chamber and basis) takes the form

$$\lambda \cdot \log g(uv, z_2, \ldots, z_{n-1})$$

where $\lambda$ is the $S^1$-equivariant parameter, and $-z_1 = g(uv, z_2, \ldots, z_{n-1})$ is a solution to the mirror equation $uv = f(z_1, \ldots, z_{n-1})$.

$f(z_1, \ldots, z_{n-1})$ is the generating function of open Gromov-Witten invariants bounded by a Lagrangian torus fiber. In [CCLT16], it was proved that $f(z_1, \ldots, z_{n-1})$ (which is a priori a formal power series in the Kähler parameters) is convergent over $\mathbb{C}$, and hence it can be treated as a holomorphic function.

Notice that we consider the immersed SYZ fibers $S^2 \times S^1$ (when $\dim X = 3$) rather than the Aganagic-Vafa branes $\mathbb{R}^2 \times S^1$. The reason is the following: although they bound the same set of holomorphic discs, the Aganagic-Vafa branes are obstructed, while the immersed SYZ fibers are unobstructed due the existence of an anti-symplectic involution swapping the immersed sectors (see Section 3 for more details). Therefore it is easier to apply Lagrangian Floer theory to $S^2 \times S^1$ in order to extract an invariant by counting holomorphic discs.

More precisely, the disc potential is defined over the Novikov ring $\Lambda$,

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \mid A_i \geq 0 \text{ and increases to } +\infty \right\}.$$ 

The unitary elements of Each Kähler parameter $q^C$ of a curve class $C$ is substituted by $T^{\omega(C)}$, and $z_2$ for $i = 2, \ldots, n$ are replaced by $T^{A_i} z_2$ respectively, where $A_i > 0$ are symplectic areas of certain primitive discs depending on the position of $L_0$ in the codimension-two toric strata. Then the $T$-valuation zero term of $g$ is 1, and $\log g$ makes sense as a series (where $\log 1 = 0$).

Throughout the paper, we will also be using the following notations:

$$\Lambda_+ = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \mid A_i > 0 \text{ and increases to } +\infty \right\},$$

the maximal ideal of $\Lambda$, and

$$\Lambda_U = \mathbb{C}^\times \oplus \Lambda_+,$$

the valuation-zero subset of $\Lambda$, which forms a multiplicative group.

The organization of the paper is as follows. We review SYZ mirror symmetry for toric Calabi-Yau manifolds and the Morse model of equivariant Lagrangian Floer theory in Section 2. In Section 3, we study the gluing formulas between the immersed SYZ fiber and other Lagrangian tori, and, as a result, obtain the equivariant potential for the immersed fiber. Finally, in Section 4, we compute the equivariant potential for a certain immersed Lagrangian torus which plays an important role in the mirror construction of toric Calabi-Yau manifolds.
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2. A review on toric Calabi-Yau manifolds and Lagrangian Floer theory

In this section, we briefly review SYZ mirror symmetry for toric Calabi-Yau manifolds and the Morse model for equivariant Lagrangian Floer theory.

2.1 Toric Calabi-Yau manifolds. Let $N \cong \mathbb{Z}^n$ be a lattice of rank $n$ and $M = N^\vee$ the dual lattice. We fix a primitive vector $v \in M$ and take a closed lattice polytope $\Delta$ of dimension $n - 1$ contained in the affine hyperplane \( \{ v \in N_R \mid v(v) = 1 \} \). By choosing a lattice point $v \in \Delta$, we have a lattice polytope $\Delta - v$ in the hyperplane $v_R \subset N_R$. We choose a triangulation of $\Delta$ such that each maximal cell is a standard simplex. By taking a cone over this triangulation, we obtain a fan $\Sigma$ supported in $N_R$. The fan defines a toric Calabi-Yau manifold $X = X_{\Sigma}$, whose anti-canonical divisor $-K_X$ is linearly equivalent to zero. We denote by $w$ the toric holomorphic function corresponding to $v \in M$.

Let $v_1, \ldots, v_m$ be the lattice points in $\Delta$ corresponding to primitive generators of the one-dimensional cones of $\Sigma$. By relabeling them if necessary, we may assume that \( \{ v_1, \ldots, v_m \} \) is a basis of $N$ and generates a maximal cone $\sigma$ of $\Sigma$. For each $i \in \{ 1, \ldots, m \}$, $v_i$ can be uniquely written as $\sum_{\ell=1}^n a_{i,\ell} v_\ell$ for some $a_{i,\ell} \in \mathbb{Z}$. In particular, $a_{i,\ell} = \delta_{i\ell}$ for $i \in \{ 1, \ldots, n \}$.

We denote by $D_i$ the toric prime divisor corresponding to $v_i$. For each toric prime divisor $D_i$ and a Lagrangian toric fiber $L \cong (S^1)^n$ in a toric manifold $X$, one can associate the disc class $\beta_i \in \pi_2(X, L)$ represented by a holomorphic disc emanated from $D_i$ and bounded by $L$. Such a class will be called a basic disc class (associated to $D_i$), see [CO06]. It is well-known that $\pi_2(X, L)$ is generated by the basic disc classes, that is, $\pi_2(X, L) \cong \mathbb{Z} \langle \beta_1, \ldots, \beta_m \rangle$, and there is an exact sequence

\[
0 \to H_2(X; \mathbb{Z}) \to H_2(X, L; \mathbb{Z}) (\cong \pi_2(X, L)) \to H_1(L; \mathbb{Z}) (\cong \mathbb{N}) \to 0. \tag{2.1}
\]

For a disc class $\beta \in \pi_2(X, L)$, its Maslov index $\mu_L(\beta)$ is equal to $2 \sum_{i=1}^m D_i \cdot \beta$ as in [CO06, Aur07]. In particular, each basic disc class is of Maslov index two.

For $i = n + 1, \ldots, m$, consider the curve class $C_i$ in the kernel given by

\[
C_i := \beta_i - \sum_{\ell=1}^n a_{i,\ell} \beta_\ell. \tag{2.2}
\]

Then \( \{ C_{n+1}, \ldots, C_m \} \) is a basis of $H_2(X; \mathbb{Z})$ and generates the cone of effective curve classes $H_2^{\text{eff}}(X) \subset H_2(X; \mathbb{Z})$. The corresponding Kähler parameters are respectively denoted by $q^{C_{n+1}}, \ldots, q^{C_m}$. As $X$ is Calabi-Yau, $c_1(\alpha) := -K_X \cdot \alpha = \sum_{i=1}^m D_i \cdot \alpha = 0$ for all $\alpha \in H_2(X; \mathbb{Z})$.

Let $T^n$ denote the $n$-dimensional torus $(S^1)^n$ acting on $X$. We have an $(n - 1)$-dimensional subtorus $\mathbb{R}_+^1 / \mathbb{Z} \cong T^{n-1}$ acting trivially on $-K_X$. Let $\rho$ be the moment map of the $T^{n-1}$-action on $X$. It is simply given by the composition of the moment map of the $T^n$-action and the projection to $M_R / \mathbb{R} \cdot v \cong \mathbb{R}^{n-1}$. Take the symplectic reduction $X /_{\rho} T^{n-1}$ by fixing $a_1 \in M_R / \mathbb{R} \cdot v$. Since the holomorphic function $w$ is invariant under the $T^{n-1}$-action, it descends to a holomorphic function on the reduced space $X /_{a_1} T^{n-1}$, which
gives an isomorphism \( w : X \equiv a_T^{n-1} \sim \mathbb{C} \). Then each embedded loop in \( X \equiv a_T^{n-1} \) corresponds to a Lagrangian in \( X \) given by the preimage of the loop in the level set \( \rho^{-1}(a) \).

This method of constructing Lagrangians using symplectic reduction was introduced in [Gro01, Gol01].

If we take a circle centered at \( w = 0 \), the corresponding Lagrangian is a regular toric fiber \( L \). It bounds holomorphic discs of Maslov index two emanated from the toric prime divisors. These holomorphic discs have interesting sphere bubbling phenomenon and their classes in \( H_2(X, L; \mathbb{Z}) \) are of the form \( \beta_i^L + \alpha \), where \( \beta_i^L \) is the basic disc class corresponding to primitive generator \( v_i \) and \( \alpha \in H_2^\text{eff}(X) \) is an effective curve class.

Let \( n_1(\beta_i^L + \alpha) \) be the one point open Gromov-Witten invariant associated to the disc \( \beta_i^L + \alpha \). We note that \( n_1(\beta_i^L) = 1 \). In [CCLT16], the generating functions \( 1 + \delta_i(T) \) (known as slab functions for wall-crossing in the Gross-Siebert program [GS11]) defined by

\[
1 + \delta_i(T) = \sum_{\alpha \in H_2^\text{eff}(X)} n_1(\beta_i^L + \alpha) T^{\alpha},
\]

were proved to be coefficients of the inverse mirror maps which have explicit expressions. These will play an important role in this paper.

**Theorem 2.1** ([CLL12, Theorem 4.37]). The SYZ mirror of the toric Calabi-Yau manifold \( X \) is given by

\[
X^+ = \{(u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \mid uv = f(z_1, \ldots, z_{n-1})\},
\]

where \( f \) is the generation function of open Gromov-Witten invariants

\[
f(z_1, \ldots, z_{n-1}) = \sum_{i=1}^{m} T^{\nu_i} \exp(\delta_i(T)) (1 + \delta_i(T))^{2\nu'_i}, \quad \nu'_i = v_i - \nu_1.
\]

**Theorem 2.2.** [CCLT16, Theorem 1.4 restricted to the manifold case] Given a toric Calabi-Yau manifold \( X \) as above, for each \( i = 1, \ldots, m \), let

\[
g_i(\hat{q}) := \sum_{\alpha} \frac{(-1)^{(D_i \cdot \alpha)}(- (D_i \cdot \alpha) - 1)!}{\prod_{j \neq i} (D_j \cdot \alpha)!} \hat{q}^\alpha,
\]

in which the summation is over all effective curve classes \( \alpha \in H_2^\text{eff}(X) \) satisfying

\[-K_X \cdot \alpha = 0, D_i \cdot \alpha < 0 \text{ and } D_j \cdot \alpha \geq 0 \text{ for all } j \neq i.
\]

Then

\[
1 + \delta_i(q) = \exp g_i(\hat{q}(q))
\]

where \( \hat{q}(q) \) is the inverse of the mirror map

\[
\hat{q}^C_k(\hat{q}) := \hat{q}^C_k \cdot \exp \left( - \sum \left( C_k, D_i \right) g_i(\hat{q}) \right), \quad k = n + 1, \ldots, m.
\]

Here \( C_{n+1}, \ldots, C_m \) are the curve classes given by \( \alpha_T \).

### 2.2 Immersed Lagrangian Floer theory

Let \( (X, \omega) \) be a 2\( n \)-dimensional tame (compact, convex at infinity or geometrically bounded) symplectic manifold. Let \( L \) be a closed, connected, relatively spin, and immersed Lagrangian submanifold of \( X \) with clean self-intersections. The immersed Lagrangian \( L \) is then the image of the immersion \( i : L \rightarrow X \) with the self-intersection \( I \subset L \). As in Akaho-Joyce [AJ10], the inverse image of \( I \) under the immersion \( i \) is assumed to be the disjoint union \( I^- \bigcup I^+ \subset L \) of two components,
each of which is diffeomorphic to $\mathcal{I}$. Then the following fiber product consists of three components

$$\tilde{L} \times_L \tilde{L} = \bigsqcup_{j=-1,0,1} R_j,$$

where $R_0$ is the diagonal, and

$$R_1 = \{(p_-, p_+) \in \tilde{L} \times \tilde{L} \mid p_- \in \mathcal{I}^-, p_+ \in \mathcal{I}^+, \iota(p_-) = \iota(p_+)\},$$

$$R_{-1} = \{(p_-, p_+) \in \tilde{L} \times \tilde{L} \mid p_+ \in \mathcal{I}^-, p_- \in \mathcal{I}^+, \iota(p_-) = \iota(p_+)\}.$$

(2.7)

We have canonical isomorphisms $R_0 \cong \tilde{L}$ and $R_{-1} \cong R_1 \cong \mathcal{I}$. Also, we have the involution $\sigma : R_{-1} \bigsqcup R_1 \rightarrow R_{-1} \bigsqcup R_1$ swapping the two immersed sectors, i.e., $\sigma(p_-, p_+) = (p_+, p_-)$.

In [AJ10], Akaho and Joyce developed immersed Lagrangian Floer theory on the singular chain model. For an immersed Lagrangian $L$ with transverse self-intersection (which is a nodal point), they produced an $A_\infty$-algebra structure on a countably generated subcomplex $C^*(L; \Lambda)$ of the smooth singular cochain complex $S^*(\tilde{L} \times_L \tilde{L}; \Lambda)$, generalizing the work of Fukaya-Oh-Ohta-Ono [FOOO09b] in the embedded case. As in the following, their construction can be further generalized to the case where $\tilde{L}$ is an immersed Lagrangian with clean self-intersection described above, see [Sch16, CW15, Fuk18] for the development of Floer theory of clean intersections in different settings. We denote the resulting $A_\infty$-algebra by $(C^*(L; \Lambda), \tilde{m})$, $\tilde{m} = (\tilde{m}_k)_{k \geq 0}$.

To construct the structure maps for an $A_\infty$-algebra, we choose a compatible almost complex structure $J$ on $(X, \omega)$. For $\alpha : \{0, \ldots, k\} \rightarrow \{-1, 0, 1\}$, consider quintuples $(\Sigma, \tilde{z}, u, \tilde{u}, l)$ where

- $\Sigma$ is a prestable genus 0 bordered Riemann surface,
- $\tilde{z} = (z_0, \ldots, z_k)$ are distinct counter-clockwise ordered smooth points on $\partial \Sigma$,
- $u : (\Sigma, \partial \Sigma) \rightarrow (X, L)$ is a $J$-holomorphic map with $(\Sigma, \tilde{z}, u)$ stable,
- $\tilde{u} : S^1 \setminus \{\xi_i \mid i \in \mathbb{N} : \alpha(i) \neq 0\} \rightarrow \tilde{L}$ is a local lift of $u|_{\Sigma}$, i.e.,

$$\iota \circ \tilde{u} = u \circ \iota \text{ and } \left(\lim_{\theta \rightarrow 0^-} \tilde{u}(e^{i\theta} \xi_i), \lim_{\theta \rightarrow 0^+} \tilde{u}(e^{i\theta} \xi_i)\right) \in R_{\alpha(i)},$$

where $\alpha(i) \neq 0$ and $i := \sqrt{\iota}$.

- $l : S^1 \rightarrow \partial \Sigma$ is an orientation preserving continuous map (unique up to a reparametrization) characterized by that the inverse image of a smooth point is a point and the inverse image of a singular point consists of two points.

Let $[\Sigma, \tilde{z}, u, \tilde{u}, l]$ be the equivalence class of $(\Sigma, \tilde{z}, u, \tilde{u}, l)$ given by the automorphisms. For $\beta \in H_2(X; \mathbb{Z})$, we denote by $\mathcal{M}_{k+1}(\alpha, \beta)$ the moduli space of (equivalence classes of) such quintuples $[\Sigma, \tilde{z}, u, \tilde{u}, l]$ satisfying $u_*([\Sigma]) = \beta$. The moduli spaces come with the following evaluation maps $\text{ev}_i : \mathcal{M}_{k+1}(\alpha, \beta) \rightarrow \tilde{L} \times_L \tilde{L}$ defined by

$$\text{ev}_i([\Sigma, \tilde{z}, u, \tilde{u}, l]) = \begin{cases} \tilde{u}(z_i) \in R_0 & \alpha(i) = 0 \\ \left(\lim_{\theta \rightarrow 0^-} \tilde{u}(e^{i\theta} \xi_i), \lim_{\theta \rightarrow 0^+} \tilde{u}(e^{i\theta} \xi_i)\right) \in R_{\alpha(i)} & \alpha(i) \neq 0, \end{cases}$$

at the input marked points $i = 1, \ldots, k$, and

$$\text{ev}_0([\Sigma, \tilde{z}, u, \tilde{u}, l]) = \begin{cases} \tilde{u}(z_0) \in R_0 & \alpha(0) = 0 \\ \sigma \left(\lim_{\theta \rightarrow 0^-} \tilde{u}(e^{i\theta} \xi_0), \lim_{\theta \rightarrow 0^+} \tilde{u}(e^{i\theta} \xi_0)\right) \in R_{-\alpha(0)} & \alpha(0) \neq 0, \end{cases}$$

at the output marked point.

For the convenience of writing, we will call an element of $\mathcal{M}_{k+1}(\alpha, \beta)$ a stable polygon if $\alpha(i) \neq 0$ for some $i \in \{0, \ldots, k\}$. In this case, the corners of a polygon are the boundary
marked points $z_i$ with $\alpha(i) \neq 0$. If $\alpha(i) = 0$ for all $i$, we will simply refer to an element of $\mathcal{M}_{k+1}(\alpha, \beta)$ as a stable disc.

The Kuranishi structures (see [FOOO09b]) on $\mathcal{M}_{k+1}(\alpha, \beta)$ are taken to be weakly submersive, which means that the evaluation map $\text{ev} = (\text{ev}_1, \ldots, \text{ev}_k)$ from the Kuranishi neighborhood of each $[\Sigma, z, u, \bar{u}, l] \in \mathcal{M}_{k+1}(\alpha, \beta)$ to $\tilde{L} \times L \bar{L}$ is a submersion.

For $\vec{P} = (P_1, \ldots, P_k)$ where $P_1, \ldots, P_k \in C^*(L; \mathbb{Q})$, we denote by $\mathcal{M}_{k+1}(\alpha, \beta, \vec{P})$ the fiber product

$$\mathcal{M}_{k+1}(\alpha, \beta, \vec{P}) = \mathcal{M}_{k+1}(\alpha, \beta) \times_{(\tilde{L} \times L \bar{L})^k} \vec{P}$$

in the sense of Kuranishi structures. We write $\mathcal{M}_{k+1}(\alpha, \beta; \vec{P})^s = s^{-1}(0)$, where $s$ is a multi-valued section of the obstruction bundle $E$ transversal to the zero section, chosen in the construction of $(C^*(L; \Lambda), \tilde{m})$.

Now, we define an $A_{\infty}$-operation $\tilde{m}_k : C^*(L; \Lambda)^{\otimes k} \to C^*(L; \Lambda)$ for $k = 0, 1, 2, \ldots$. When $\beta$ is the constant disc class $\beta_0$ and $k = 0, 1$, we set

$$\begin{align*}
\tilde{m}_{0, \beta}(1) &= 0, \\
\tilde{m}_{1, \beta}(P) &= (-1)^n \partial P,
\end{align*}$$

where $\partial$ is the coboundary operator on $C^*(L; \Lambda)$. For $(k, \beta) \neq (1, \beta_0), (0, \beta_0)$, we define

$$\tilde{m}_{k, \beta}(P_1, \ldots, P_k) = (\text{ev}_0)\left( \mathcal{M}_{k+1}(\alpha, \beta; \vec{P})^s \right),$$

and

$$\tilde{m}_{k, \beta}(P_1, \ldots, P_k) = \sum_{\alpha} \tilde{m}_{k, \alpha}(P_1, \ldots, P_k).$$

Notice that $m_{k, \beta}(P_1, \ldots, P_k) = 0$ unless $P_i$ is a singular chain on $R_{\alpha(i)}$. The $A_{\infty}$-operation $\tilde{m}_k$ is given by

$$\tilde{m}_k(P_1, \ldots, P_k) = \sum_{\beta \in H^2_{\text{eff}}(X, L)} T^{\omega(\beta)} \tilde{m}_{k, \beta}(P_1, \ldots, P_k)$$

where $H^2_{\text{eff}}(X, L)$ is the cone of effective disc classes in $H_2(X, L; \mathbb{Z})$.

### 2.2.1 Anti-symplectic involutions

Let $\tau : X \to X$ be an anti-symplectic involution, i.e., $\tau^*\omega = -\omega$. We consider a $\tau$-invariant immersed Lagrangian $L$ such that the immersed locus $\tilde{L}$ is also $\tau$-invariant. Then $\tau|_L$ lifts to a diffeomorphism $\tilde{\tau} : \tilde{L} \to \tilde{L}$ satisfying $\tilde{\tau}^2 = \text{id}$, $\tilde{\tau}(\tilde{I}^-) = \tilde{I}^+$, and $\tilde{\tau}(\tilde{I}^+) = \tilde{I}^-$. Then $\tau$ induces the involution $\sigma : R_{-1} \coprod R_{1} \to R_{-1} \coprod R_{1}$ swapping the immersed sectors. Suppose the compatible almost complex structure $J$ is $\tau$-anti-invariant, i.e., $\tau^*J = -J$. Then $\tau$ induces an involution on the moduli spaces.

Let us fix a non-negative integer $k$, $\alpha : \{0, \ldots, k\} \to \{-1, 0, 1\}$, and $\beta \in H^2_{\text{eff}}(X, L)$. Let $[\Sigma, z, u, \bar{u}, l] \in \mathcal{M}_{k+1}(\alpha, \beta)$. We define $\hat{u} : (\tau, \partial \Sigma) \to (X, L)$ by

$$\hat{u}(z) := \tau \circ u(z).$$

Let $\check{\alpha} : \{0, \ldots, k\} \to \{-1, 0, 1\}$ be given by $\check{\alpha}(0) = \alpha(0)$ and $\check{\alpha}(i) = \alpha(k + 1 - i)$ for $i = 1, \ldots, k$. We put $\hat{z} = (\hat{z}_0, \ldots, \hat{z}_k) := (z_0, z_k, z_{k-1}, \ldots, z_1)$ and define $\hat{u} : S^1 \setminus \{\hat{z}_i := l^{-1}(\hat{z}_i) | \hat{z}_i \neq 0\} \to \hat{L}$ by

$$\hat{u}(z) := \tilde{\tau} \circ \hat{u}(z).$$

Then $\hat{u}$ satisfies

$$\iota \circ \hat{u} = \hat{u} \circ \iota$$

and

$$\left( \lim_{\theta \to 0^-} \hat{u}(e^{i\theta} \check{z}_i), \lim_{\theta \to 0^+} \hat{u}(e^{i\theta} \check{z}_i) \right) \in R_{\check{\alpha}(i)}, \quad \check{\alpha}(i) \neq 0.$$
Note that both the complex conjugation on the domain and the involution on \(X\) swap the immersed sectors and \(\hat{\alpha}\) is obtained from \(\alpha\) simply by relabeling the boundary marked points.

For \(\beta = [u]\), setting \(\hat{\beta} = [\hat{u}]\), the map \(\tau_*^{\text{main}} : \mathcal{M}_{k+1}(\alpha, \beta) \to \mathcal{M}_{k+1}(\hat{\alpha}, \hat{\beta})\) defined by
\[
[\Sigma, z, u, \hat{u}, l] \mapsto [\Sigma, \hat{z}, \hat{u}, \hat{u}, l]
\]
is a homeomorphism (between topological spaces) satisfying \(\tau_*^{\text{main}} \circ \tau_*^{\text{main}} = \text{id}\). We can then choose Kuranishi structures respecting the involution \(\tau\) as follows.

**Theorem 2.3** ([FOOO17, Theorem 4.11]). The map \(\tau_*^{\text{main}}\) is induced by an isomorphism of Kuranishi structures.

A choice of relative spin structure on \(L\) together with a choice a path in the Lagrangian Grassmannian of \(T_p X\), for each \(p \in \mathbb{Z}\), connecting \(d_1(T_{p-1} L)\) and \(d_1(T_{p+1} L)\), \(d_1(p_-) = d_1(p_+) = p\), determine orientations on the moduli spaces \(\mathcal{M}_{k+1}(\alpha, \beta)\) [AJ10, Section 5] (see also [FOOO09b, Chapter 8.8] for the choices of paths in the Bott-Morse setting). We will fix a choice of connecting paths for the following discussion.

Let \(\sigma \in H_2(X, L; \mathbb{Z}/2\mathbb{Z})\) be a (stable conjugacy class of) relative spin structure on \(L\). We write \(\mathcal{M}_{k+1}(\alpha, \beta)^\sigma\) to emphasize the Kuranishi structure \(\mathcal{M}_{k+1}(\alpha, \beta)\) is equipped with the orientation determined by \(\sigma\). Let \(\mathcal{M}_{k+1}^\text{clock}(\alpha, \beta)\) denote the moduli space with the boundary marked points respecting the clockwise order and put \(\tilde{z} = (\hat{z}_0, \ldots, \hat{z}_k)\). By [FOOO17, Theorem 4.10], the map
\[
\tau_* : \mathcal{M}_{k+1}(\alpha, \beta)^\sigma \to \mathcal{M}_{k+1}^\text{clock}(\alpha, \beta)^\sigma,
[\Sigma, z, u, \hat{u}, l] \mapsto [\Sigma, \hat{z}, \hat{u}, \hat{u}, l],
\]
is an orientation preserving isomorphism of Kuranishi structures if and only if \(\mu_L(\beta)/2 + k + 1\) is even.

Let \(P_1, \ldots, P_k \in C^*(R_{-1} \coprod R_1) \subset C^*(L; \Lambda)\) be singular chains on the immersed sectors. We have an isomorphism
\[
\tau_*^{\text{main}} : \mathcal{M}_{k+1}(\alpha, \beta; P_1, \ldots, P_k)^\sigma \to \mathcal{M}_{k+1}(\hat{\alpha}, \hat{\beta}; P_1, \ldots, P_k)^\sigma
\]
(2.9)
of Kuranishi structures, satisfying \(\tau_*^{\text{main}} \circ \tau_*^{\text{main}} = \text{id}\).

Let \(\mathcal{M}_{k+1}^\text{unordered}(\alpha, \beta; P_1, \ldots, P_k)\) be the moduli space with unordered boundary marked points. Note that the unordered moduli space contains both \(\mathcal{M}_{k+1}(\alpha, \beta; P_1, \ldots, P_k)\) and \(\mathcal{M}_{k+1}^\text{clock}(\alpha, \beta; P_1, \ldots, P_k)\) as connected components. Let \(\{i, i+1\} \subset \{1, \ldots, k\}\) and let \(\alpha^{i \to i+1} : \{0, \ldots, k\} \to \{-1, 0, 1\}\) be the map obtained from \(\alpha\) by swapping \(\alpha(i)\) and \(\alpha(i+1)\). By [FOOO17, Lemma 3.17], the action of changing the order of marked points induces an orientation preserving isomorphism
\[
\mathcal{M}_{k+1}^\text{unordered}(\alpha, \beta; P_1, \ldots, P_k) \xrightarrow{\sim} (-1)^{(\deg P_{i+1}-1)} \mathcal{M}_{k+1}^\text{unordered}(\alpha^{i \to i+1}, \beta; P_1, \ldots, P_{i+1}, P_i, \ldots, P_k).
\]
(2.10)
Combining (2.8) and (2.10), we derive the following theorem.

**Theorem 2.4.** The map (2.9) is orientation preserving (resp. reversing) if \(\epsilon\) is even (resp. odd) where
\[
\epsilon = \frac{\mu_L(\beta)}{2} + k + 1 + \sum_{1 \leq i < j \leq k} (\deg P_i + 1)(\deg P_j + 1).
\]
(2.11)
2.2.2 Pearl complexes. In order to produce pearl complexes, we choose a Morse function $f : \tilde{L} \times I, \tilde{L} \to \mathbb{R}$. We are particularly interested in the situation where $f$ has a unique maximum point $1^*$ on $\tilde{L}$. Let $C^*(f; \Lambda)$ be the cochain complex generated by the critical points of $f$. Namely,

$$C^*(f; \Lambda) = \bigoplus_{p \in \text{Crit}(f)} \Lambda \cdot p. \quad (2.12)$$

We begin by setting up some notations. We denote

- by $\mathcal{W}$ a (negative) pseudo gradient vector field of $f$ satisfying the Smale condition,
- by $\Phi_t$ the flow of $\mathcal{W}$,
- by $W^s(p)$ (resp. $W^u(p)$) the stable (resp. unstable) submanifold of $p \in \text{Crit}(f)$,
- by $\overline{W^s(p)}$ (resp. $\overline{W^u(p)}$) the natural compactification of $W^s(p)$ (resp. $W^u(p)$) to a smooth manifold with corners.

The $A_{\infty}$-operations on the complex $C^*(f; \Lambda)$ in (2.12) can be defined by counting configurations called *pearly trees* (see Figure 2 for an illustration). This was systematically developed by Biran and Cornea [BC07, BC09] under the monotonicity assumption on Lagrangians (a similar complex previously appeared in [Oh96]). Since we will not restrict ourselves to the setting of monotone Lagrangians, we shall derive the Morse model from the singular chain model using the homological method as in Fukaya-Oh-Ohta-Ono [FOOO09a].

The $A_{\infty}$-structure on $C^*(f; \Lambda)$ can be constructed by applying homological perturbation to the subcomplex $C^*_{(-1)}(L; \Lambda)$ quasi-isomorphic to $C^*(f; \Lambda)$ and generated by certain singular chains $\Delta_p$ representing $\overline{W^s(p)}$ for $p \in \text{Crit}(f)$. Here, the chain $\Delta_p$ is chosen so that the assignment $p \to \Delta_p$ is a chain map inducing an isomorphism on cohomology (see [KLZ19, Theorem 2.3]). We will refer to $\Delta_p$ as the the unstable chain of $p$, and implicitly identify $C^*(f; \Lambda)$ with $C^*_{(-1)}(L; \Lambda)$ from now on.

We now turn to the $A_{\infty}$-structure maps on $C^*(f; \Lambda)$ denoted by $m^* = (m^*_k)_{k \geq 0}$. In order to define the maps $m^*$, we recall *decorated planar rooted trees* as follows.

**Definition 2.5** ([FOOO09a]). A decorated planar rooted tree is a quintet $\Gamma = (T, \iota, v_0, V_{\text{tad}}, \eta)$ consisting of

- $T$ is a tree;
- $\iota : T \to D^2$ is an embedding into the unit disc;
- $v_0$ is the root vertex and $\iota(v_0) \in \partial D^2$;
- $V_{\text{tad}}$ is the set of interior vertices with valency 1;
- $\eta = (\eta_1, \eta_2) : V(\Gamma)_{\text{int}} \to \mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0}$,

where $V(\Gamma)$ is the set of vertices, $V(\Gamma)_{\text{ext}} = \iota^{-1}(\partial D^2)$ is the set of exterior vertices and $V(\Gamma)_{\text{int}} = V(\Gamma) \setminus V(\Gamma)_{\text{ext}}$ is the set of interior vertices. For $k \geq 0$, denote by $\Gamma_{k+1}$ the set of isotopy classes represented by $\Gamma = (T, \iota, v_0, \eta)$ with $|V(\Gamma)_{\text{ext}}| = k + 1$ and $\eta(v) > 0$ if the valency $\ell(v)$ of $v$ is 1 or 2. In other words, the elements of $\Gamma_{k+1}$ are stable. We will refer to the elements of $\Gamma_{k+1}$ as stable trees.

For each $\Gamma \in \Gamma_{k+1}$, we label the exterior vertices by $v_0, \ldots, v_k$ respecting the counterclockwise orientation, and orient the edges along the direction from the $k$ input vertices $v_1, \ldots, v_k$ towards the root vertex $v_0$. 

We define the map \( \Pi : C^*(L; \Lambda) \to C^*(f; \Lambda) \) by

\[
\Pi(P) = \begin{cases} 
\sum_{p \in \text{Crit}(f), \deg_p = n - \deg P} \sharp(P \cap W^s(p)) \cdot \Delta_p & \text{if the intersection } P \cap W^s(p) \text{ is transverse,} \\
0 & \text{if the intersection } P \cap W^s(p) \text{ is not transverse.}
\end{cases}
\]

For the map \( G : C^*(L; \Lambda) \to C^*(f; \Lambda) \), \( G(P) \) is defined by a singular chain representing the forward orbit of \( P \) satisfying

\[
\Pi(P) - P = \partial G(P) + G(\partial P). \tag{2.13}
\]

We refer to [KLZ19, Theorem 2.4] for the construction of \( G(P) \).

Denote by \( T^0 \in \Gamma_2 \) the unique tree with no interior vertices. For this tree \( \Gamma^0 \), we define

\[
m_{\Gamma^0} : C^*(f; \Lambda) \to C^*(f; \Lambda) \text{ by } m_{\Gamma^0} := \tilde{m}_{1, \beta_0},
\]

\[
f_{\Gamma^0} : C^*(f; \Lambda) \to C^*(L; \Lambda) \text{ by the inclusion.}
\]

For each \( k \geq 0 \), \( \Gamma_{k+1} \) contains a unique element that has a single interior vertex \( v \), which we denote by \( \Gamma_{k+1} \). Let \( \alpha_{k+1} \) denote the set of maps \( \alpha : \{0, \ldots, k\} \to \{-1, 0, 1\} \). We fix a labeling \( \{\beta_0, \beta_1, \ldots\} \) of elements of \( H^*_c(X, L) \) with \( \beta_0 \) the constant disc class and labeling \( \{\alpha_{k+1, 0}, \alpha_{k+1, 1}, \ldots\} \) of elements of \( \alpha_{k+1} \) with \( \alpha_{k+1, 0} \) the map \( \alpha_{k+1, 0}(i) = 0 \) for \( i = 1, \ldots, k \).

We define

\[
m_{\Gamma_{k+1}} := \Pi \circ \tilde{m}_{k, (\alpha_{(i)}(v), \eta_{(i)}(v)) \beta_{(v)}} \text{ and } f_{\Gamma_{k+1}} := G \circ \tilde{m}_{k, (\alpha_{(i)}(v), \eta_{(i)}(v)) \beta_{(v)}}.
\]

For a general rooted tree \( \Gamma \in \Gamma_{k+1} \), cut it at the vertex \( v \) closest to the root vertex \( v_0 \) so that \( \Gamma \) is decomposed into \( \Gamma^{(1)}, \ldots, \Gamma^{(\ell)} \) and an interval adjacent to \( v_0 \) in the counterclockwise order. The maps \( m_\Gamma : (C^*(f; \Lambda))^{\otimes k} \to C^*(f; \Lambda) \) and \( f_\Gamma : (C^*(f; \Lambda))^{\otimes k} \to C^*(L; \Lambda) \) are inductively defined by

\[
m_\Gamma := \Pi \circ \tilde{m}_{\ell, (\alpha_{(i)}(v), \eta_{(i)}(v)) \beta_{(v)}} \circ (f_{\Gamma^{(i)}} \otimes \ldots \otimes f_{\Gamma^{(i)}}) \tag{2.14}
\]

and

\[
f_\Gamma := G \circ \tilde{m}_{\ell, (\alpha_{(i)}(v), \eta_{(i)}(v)) \beta_{(v)}} \circ (f_{\Gamma^{(i)}} \otimes \ldots \otimes f_{\Gamma^{(i)}}). \tag{2.15}
\]

At last, we define the \( A^* \)-operations \( m_k^* : (C^*(f; \Lambda))^{\otimes k} \to C^*(f; \Lambda) \) by

\[
m_k^* := \sum_{\Gamma \in \Gamma_{k+1}} T^{\omega(\Gamma)} m_\Gamma,
\]

where \( \omega(\Gamma) = \sum_{v} \omega(\beta_{(v)}) \).

The constructed \( A^* \)-algebra \( (C^*(f; \Lambda), m^*) \) on the pearl complex does not have a strict unit in general. In [KLZ19], a unital \( A^* \)-algebra \( (CF^*(L; \Lambda), m) \) was constructed from the \( A^* \)-algebra \( (C^*(f; \Lambda), m^*) \) by applying the homotopy unit construction [FOOO09b, Chapter 7]. Below, we recall a few basic properties of \( (CF^*(L; \Lambda), m) \) which we use in this paper and refer the reader to [KLZ19] for the details.

- We have
  \[
  CF^*(L; \Lambda) = C^*(f; \Lambda) \oplus \Delta \cdot 1^\top \oplus \Delta \cdot 1^*, \tag{2.17}
  \]
  as graded modules. \( 1^\top \) and \( 1^* \) are generators in degree 0 and \(-1\), respectively.
- The restriction of \( m \) to \( C^*(f; \Lambda) \) agrees \( m^* \).
\[ \mathbf{1}^\triangledown \text{ is the strict unit, i.e., } \]
\[
m_2(\mathbf{1}^\triangledown, x) = (-1)^{\deg x}m_2(x, \mathbf{1}^\triangledown) = x,
\]
for \( x \in C^\cdot(f; \Lambda)^+ \), and
\[
m_k(\ldots, \mathbf{1}^\triangledown, \ldots) = 0
\]
for \( k \geq 2 \).

- Assuming the minimal Maslov index of \( L \) is nonnegative, we have
\[
m_1(1^\triangledown) = \mathbf{1}^\triangledown - (1 - h)\mathbf{1}^*, \quad h \in \Lambda_+.
\]

We remark that \( \mathbf{1}^* \) is a homotopy unit in the sense of [FOOO09b, Definition 3.3.2].

Let \((L_1, L_2)\) be a pair of closed, connected, relatively spin, and embedded Lagrangian submanifolds intersecting cleanly. Then \( L = L_1 \cup L_2 \) is an immersed Lagrangian with clean self-intersections with \( \bar{L} = L_1 \coprod L_2 \). We choose the splitting \( i^{-1}(\mathcal{I}) = \mathcal{I}^- \coprod \mathcal{I}^+ \) so that \( \mathcal{I}^- \subset L_1 \) and \( \mathcal{I}^+ \subset L_2 \). In this case, \( R_1 \) indicates a branch jump in \( L = L_1 \cup L_2 \) from \( L_1 \) to \( L_2 \) and \( R_{-1} \) indicates a branch jump from \( L_2 \) to \( L_1 \), see (2.7).

We can define a pearl complex \((CF^\bullet(L_1, L_2; \Lambda), m_{L_1, L_2}^1)\) for the Lagrangian intersection Floer theory of \((L_1, L_2)\) as follows:
\[ CF^\bullet(L_1, L_2; \Lambda) = \bigoplus_{p \in \text{Crit}(f|_{R_1})} \Lambda \cdot p \]
is the subcomplex of \( CF^\bullet(L; \Lambda) \) generated by critical points of \( f \) in \( R_1 \), and \( m_{L_1, L_2}^1 \) counts stable pearly trees \( \Gamma \in \Gamma_2 \) with input and output vertices in \( R_1 \).

\[ \text{Figure 2. A pearly tree} \]

2.2.3 Disc potentials of a Lagrangian. We next define the disc potential of \((CF^\bullet(L; \Lambda), m)\). For a unital \( A_\infty \)-algebra \((A, m)\) over \( \Lambda \) with the strict unit \( e_A \), the weak Maurer-Cartan equation for \( b \in A \) with \( b \equiv 0 \mod \Lambda_+ \cdot A \) is given by
\[ m_0^b(1) = m_0(1) + m_1(b) + m_2(b, b) + \ldots \in \Lambda \cdot e_A. \] (2.18)

Note that the requirement that \( b \equiv 0 \mod \Lambda_+ \cdot A \) ensures the convergence of \( m_0^b(1) \). We denote by
\[ \text{MC}(A) = \left\{ b \in A^{\text{odd}} \mid b \equiv 0 \mod \Lambda_+ \cdot A; m_0(1) \in \Lambda \cdot e_A \right\}, \]
the space of odd solution to the weak Maurer-Cartan equation. We say an \( A_\infty \)-algebra \( A \) is weakly unobstructed if \( \text{MC}(A) \) is nonempty, in which case we have \((m_1^b)^2 = 0\) for any \( b \in \text{MC}(A) \), thus defining a cohomology theory \( H^\bullet(A, m_1^b) \).
For any \( b \in \text{MC}(A) \), putting
\[
e^b := 1 + b + b \otimes b + \ldots,
\]
we define the deformation of the \( A_\infty \) structure \( m \) by \( b \)
\[
m^b_k(x_1, \ldots, x_k) = m(e^b, x_1, e^b, x_2, e^b, \ldots, e^b, x_k, e^b).
\]
(2.19)

In particular, \( m^b_0(1) = m(e^b) \).

The following lemma concerns the weakly unobstructedness of \((\text{CF}^*(L; \Lambda), m)\). This technique of finding weak bounding cochains in the presence of a homotopy unit was introduced in [FOOO09b, Chapter 7] and [CW15, Lemma 2.44].

Lemma 2.6 ([KLZ19, Lemma 2.7]). Let \( b \in \text{CF}^1(L; \Lambda_+) \). Suppose \( m^b_0(1) = W(b)1^* \) and the minimal Maslov index of \( L \) is nonnegative. Then there exists \( b^+ \in \text{CF}^1(L; \Lambda_+) \) such that \( m^b_0(1) = W^\vee(b)1^\vee \), i.e., \((\text{CF}^*(L; \Lambda), m)\) is weakly unobstructed. In particular, if the minimal Maslov index of \( L \) is at least two, then \( W^\vee(b) = W(b) \).

Definition 2.7. The deformation of \( L \) by \( b \) (or simply \((L, b)\)) is said to be unobstructed (resp. weakly unobstructed) if \( m^b_0(1) = 0 \) (resp. \( m^b_0(1) \in \Lambda \cdot 1 \)). In particular, if \( b = 0 \), we simply say \( L \) is unobstructed.

2.3 Equivariant Lagrangian Floer theory. Equivariant Lagrangian Floer theory has been substantially developed in the recent years [SS10, HLS16a, HLS16b, DF17]. In [KLZ19], a Morse model for equivariant Lagrangian Floer theory was constructed by counting pearly trees in the Borel construction. It was inspired by the family Morse theory of Hutchings [Hut08] and the pearl complex of Biran-Cornea [BC07, BC09]. It also incorporated the works of Fukaya-Oh-Ohta-Ono [FOOO09b, FOOO09a], allowing the theory to work in a very general setting. We give a brief summary of this equivariant Morse model in this section for the purpose of applications in this paper.

2.3.1 Equivariant Floer complexes. Let \((X, \omega)\) be a tame symplectic manifold equipped with a symplectic action of a compact Lie group \( G \). Let \( L \subset X \) be closed, connected, relatively spin, and \( G \)-invariant immersed Lagrangian submanifold with \( G \)-invariant clean self-intersections. We choose smooth finite dimensional approximations \( EG(N) \) and \( BG(N) \) for the universal bundle \( EG \) and classifying space \( BG \), namely, we have sequences of smooth embeddings
\[
EG(0) = G \hookrightarrow EG(1) \hookrightarrow EG(2) \hookrightarrow \ldots, \\
BG(0) = \text{pt} \hookrightarrow BG(1) \hookrightarrow BG(2) \hookrightarrow \ldots,
\]
(2.20)
such that
\[
EG = \varinjlim EG(N), \quad BG = \varprojlim BG(N).
\]

Let \( \mu^{(N)} : T^*EG(N) \to g^* \) be the moment map for the Hamiltonian \( G \)-action on \( T^*EG(N) \) lifted from the \( G \)-action on \( EG(N) \). Since \( G \) acts on \( T^*EG(N) \) freely, we have
\[
T^*EG(N) \# G := \mu_N^{-1}(0)/G \cong T^*BG(N),
\]
canonically, as symplectic manifolds. Let us set \( L(N) = L \times_G EG(N) \), \( \tilde{L}(N) = \tilde{L} \times_G EG(N) \) and \( X(N) = X \times_G \mu_N^{-1}(0) \). Then (2.20) induces a commutative diagram:
We choose Morse-Smale pairs \((\bar{L}, f)\). The symplectic structures on \(X(N)\) are chosen such that they are compatible with the inclusions. We note that \(X(N)\) and \(\bar{L}(N)\) are fiber bundles over \(T^*BG(N)\) and \(BG(N)\) with fibers \(X\) and \(\bar{L}\), respectively. Since the zero section \(BG(N)\) is an exact Lagrangian submanifold of \(T^*BG(N)\), we have an identification of the effective disc classes

\[
H^\text{eff}_2(X, L) = H^\text{eff}_2(X(N), L(N)).
\]

For simplicity of notations, we will denote \(L(N) = \bar{L}(N) \times_{L(N)} L\) and \(L = L(0)\). There is also a sequence:

\[
L \leftarrow L(1) \leftarrow L(2) \leftarrow \ldots
\]

We choose Morse-Smale pairs \((f_N, \varphi_N)\) on \(L(N)\) compatible with the inclusion \(L(N) \subset L(N + 1)\) and satisfying additional conditions stated in [KLZ19, Section 2.3]. For simplicity, we will assume \(f_N\) has a unique maximum point \(1\) in \(L \subset L(N)\). Let \((CF^\bullet(L(N); \Lambda), m^N)\) be the unital Morse model as in Section 2.3, and let \(m^N\) be the operations associated to stable trees in the definition of \(m^N\). We define the equivariant Floer complex \(CF^\bullet_G(L; \Lambda)\) by

\[
CF^\bullet_G(L; \Lambda) = \lim_{\rightarrow} CF^\bullet(L(N); \Lambda),
\]

where the arrows are inclusions of graded submodules.

The perturbations for the moduli spaces in the construction of \((CF^\bullet(L(N); \Lambda), m^N)\) can be chosen such that for fixed inputs \(p_1, \ldots, p_k \in CF^\bullet_G(L; \Lambda)\) and a stable tree \(\Gamma \in \Gamma_{k+1}\), \(m^N(p_1, \ldots, p_k)\) is independent of \(N\) for sufficiently large \(N\), and \(N\) depends on the degrees of the inputs and the Maslov indices of the decorations [KLZ19, Proposition 2.13]. (We orient the moduli spaces so that they are compatible with inclusions.) We can therefore define an \(A_{\infty}\)-algebra \(m^G = (m^G)_{k \geq 0}\) structure on \(CF^\bullet_G(L; \Lambda)\) by

\[
m^G_k(p_1, \ldots, p_k) = m^N_\Gamma(p_1, \ldots, p_k),
\]

for \(N\) sufficiently large (depending on the degrees of the inputs and the Maslov indices), and

\[
m^G_k = \sum_{\Gamma \in \Gamma_{k+1}} \mathcal{T}^{\omega(\Gamma)} m^G_\Gamma.
\]

We call the resulting \(A_{\infty}\)-algebra \((CF^\bullet_G(L; \Lambda), m^G)\) the Morse Model for \(G\)-equivariant Lagrangian Floer theory (\(G\)-equivariant Morse model) of \(L\).

Let \((L_1, L_2)\) be a pair of closed, connected, relatively spin, and embedded \(G\)-invariant Lagrangian submanifolds intersecting cleanly such that their intersection is also \(G\)-invariant. Then, \(L = L_1 \cup L_2\) is a immersed Lagrangian with \(G\)-invariant clean self-intersection and \(\bar{L} = L_1 \amalg L_2\). We choose the splitting \(\iota^{-1}(\mathbb{I}) = \mathbb{I}^- \amalg \mathbb{I}^+\) so that \(\mathbb{I}^- \subset L_1\) and \(\mathbb{I}^+ \subset L_2\).

Similar to the non-equivariant setting, we can define a pearl complex \((CF^\bullet_G(L_1, L_2; \Lambda); m^G_{\iota^{-1}(L_1, L_2)})\) for the \(G\)-equivariant Lagrangian intersection Floer theory as follows:

\[
CF^\bullet_G(L_1, L_2; \Lambda) \subset CF^\bullet_G(L; \Lambda)
\]
is the subcomplex generated by critical points on \((R_1)_G = R_1 \times_G E\) (suitably approximated in the sense of 2.3.1). The differential \(m_{1}^{G,(L_1,L_2)}\) counts stable pearly trees \(\Gamma \in \Gamma_2\) with input and output vertices in \((R_1)_G\).

2.3.2 Partial units. When \(G = T^n\) or \(G = U(n)\), there are well-known perfect Morse functions on \(BG\) such that their restrictions to \(BG(N)\) are also perfect Morse functions. In such cases, we can choose Morse functions \(f_N\) to be of the form

\[
f_N = \pi_N^*\phi_N + \phi_N,
\]

where \(\pi_N : L(N) \to BG(N)\) is the projection map, \(\phi_N\) is a perfect Morse function on \(BG(N)\), and \(\phi_N\) is a (generically) fiberwise Morse function over \(BG(N)\) such that \(\phi_N\) restricted to the fiber \(L = \pi_N^{-1}(\{\lambda\})\) over each critical point \(\lambda \in \text{Crit}(\phi_N)\) is a Morse function \(f = \phi_0\) on \(L\).

With such choices of Morse functions, we constructed an \(A_{\infty}\)-algebra \((CF^*_G(L;\Lambda)^\dagger, m^{G,\dagger})\) homotopy equivalent to \((CF^*_G(L;\Lambda), m^G)\). As a graded module, we have

\[
CF^*_G(L;\Lambda)^\dagger = CF^*(L;\Lambda) \otimes \Lambda H^*_G(pt;\Lambda),
\]

where

\[
CF^*(L;\Lambda) = C^*(f;\Lambda) \oplus \Lambda \cdot 1_L^\upnu \oplus \Lambda \cdot 1_L^\ast,
\]

and \(H^*_G(pt;\Lambda) = H^*(BG;\Lambda)\) is a polynomial ring generated in even degree elements.

For \(\lambda \in H^*_G(pt;\Lambda)\), we write

\[
\lambda^\ast = 1_L^\ast \otimes \lambda, \quad \lambda^\upnu = 1_L^\upnu \otimes \lambda, \quad \lambda^\dagger = 1_L^\dagger \otimes \lambda.
\]

In particular,

\[
1^\ast = 1_L^\ast \otimes 1, \quad 1^\upnu = 1_L^\upnu \otimes 1, \quad 1^\dagger = 1_L^\dagger \otimes 1.
\]

The \(A_{\infty}\)-structure \(m^{G,\dagger}\) has the following properties:

- The restriction of \(m^{G,\dagger}\) to \(CF^*_G(L;\Lambda)^\dagger\) agrees with \(m^G\).
- \(1^\dagger\) is the strict unit.
- The elements \(\lambda^\upnu\) are partial units, namely, they satisfy

\[
m^G_{2}(\lambda^\upnu, x \otimes y) = x \otimes \lambda \cup y = (-1)^{\deg x \otimes y} x \otimes y \cup \lambda = (-1)^{\deg x \otimes y} m^G_{2}(x \otimes y, \lambda^\upnu),
\]

for \(x \otimes y \in CF^*_G(L;\Lambda)^\dagger\), where \(\cup\) denotes the cup product on \(H^*_G(pt)\), and

\[
m^{G,\dagger}(\dot{\ldots}, \lambda^\upnu, \dot{\ldots}) = 0,
\]

for \(k \neq 0\).
- Assuming the minimal Maslov index of \(L\) is nonnegative, then

\[
m^{G,\dagger}_{1}(\lambda^\dagger) = \lambda^\upnu - (1 - h)\lambda^\ast, \quad h \in \Lambda_+.
\]

In particular,

\[
m^{G,\dagger}_{1}(1^\ast) = 1^\upnu - (1 - h)1^\ast.
\]

Let \(\lambda \cdot x \otimes y\) denote \(m^G_{2}(\lambda^\upnu, x \otimes y)\). It follows from the \(A_{\infty}\)-relations that

**Theorem 2.8 ([KLZ19, Theorem 2.14]).** Let \(X_1, \ldots, X_k \in CF^*_G(L;\Lambda)^\dagger\). Let \(\ell \in \{1, \ldots, k\}\) and write \(X_\ell = \sum x_i \otimes \lambda_j\). We have

\[
m^{G,\dagger}_{k}(X_1, \ldots, X_k) = (-1)^{\ell} \sum \lambda_j \cdot m^{G,\dagger}_{k}(X_1, \ldots, X_{\ell-1}, x_1 \otimes 1, X_{\ell+1}, \ldots, X_k).
\]

(2.24)
This shows \((\text{CF}_G^\bullet(L;\Lambda)^\dagger, \text{m}_G^\bullet)\) can be defined over the graded coefficient ring \(H_G^\bullet(\text{pt}; \Lambda)\), namely
\[
(\text{CF}_G^\bullet(L;\Lambda)^\dagger, \text{m}_G^\bullet) = (\text{CF}^\bullet(L; H_G^\bullet(\text{pt}; \Lambda)), \text{m}_G^\bullet).
\] (2.25)

\[\text{2.3.3 Equivariant Maurer-Cartan spaces.}\] We consider a modified version of weak Maurer-Cartan equation for \((\text{CF}_G^\bullet(L;\Lambda)^\dagger, \text{m}_G^\bullet)\). For \(b \in \text{CF}_G^\bullet(L;\Lambda_+)\text{t}^{\text{odd}}\), let \(m_k^{G,t,b}\) be the deformation of \(m_k^G\) by \(b\) as in (2.19). If
\[
m_0^{G,t,b}(1) \in H_G^\bullet(\text{pt}; \Lambda) \cdot 1^\vee,
\] then \((m_k^{G,t,b})^2 = 0\), which leads to a well-defined cohomology theory \(H^\bullet(\text{CF}_G^\bullet(L;\Lambda)^\dagger, \text{m}_G^\bullet)\). Therefore, we define the \textit{equivariant weak Maurer-Cartan space} of \(\text{CF}_G^\bullet(L;\Lambda)^\dagger\) to be
\[
\text{MC}_G \left( \text{CF}_G^\bullet(L;\Lambda)^\dagger \right) = \left\{ b \in \text{CF}_G^\bullet(L;\Lambda_+)\text{t}^{\text{odd}} \mid b \text{ satisfies (2.26)} \right\}.
\]
We say that \((\text{CF}_G^\bullet(L;\Lambda)^\dagger, \text{m}_G^\bullet)\) is weakly unobstructed if \(\text{MC}_G \left( \text{CF}_G^\bullet(L;\Lambda)^\dagger \right)\) is nonempty.

\[\text{Definition 2.9.}\] We say that the deformation of \((L,G)\) by \(b\) (or simply \(((L,G),b)\)), is unobstructed (resp. weakly unobstructed), if \(m_1^{G,t,b}(1) = 0\) (resp. \(m_1^{G,t,b}(1) \in \Lambda \cdot 1\)). In particular, if \(b = 0\), we simply say \((L,G)\) is unobstructed.

Similar to Lemma 2.6, we have

\[\text{Lemma 2.10 (}[\text{KLZ19}, \text{Lemma 2.16}]\). Let \(b \in \text{CF}_G^\dagger(L;\Lambda_+)\). Suppose the minimal Maslov index of \(L\) is nonnegative and \(m_0^{G,t,b}(1) = W(b)1^\star + \sum_{\deg \alpha = 2} \phi_\alpha(b)\alpha^\star\). Then there exists \(b^\dagger \in \text{CF}_G^\dagger(L;\Lambda_+)\) such that \(m_0^{G,t,b^\dagger}(1) = W^\circ(b)1^\vee + \sum_{\deg \alpha = 2} \phi_\alpha^\circ(b)\alpha^\vee\), i.e., \((\text{CF}_G^\bullet(L;\Lambda)^\dagger, \text{m}_G^\bullet)\) is weakly unobstructed. Moreover, if the minimal Maslov index of \(L\) is at least two, then \(W^\circ(b) = W(b)\) and \(\phi_\alpha^\circ(b) = \phi_\alpha(b)\).

\[\text{Corollary 2.11 (}[\text{KLZ19}, \text{Corollary 2.17}]\). In the setting of Lemma 2.10, if \(b \in \text{CF}_G^\dagger(L,\Lambda_+)\) and \((L,b)\) is weakly unobstructed, then \(((L,G),b)\) is weakly unobstructed in the sense of Definition 2.9.

We note that even if the minimal Maslov index of \(L\) is 2 and \((L,b)\) is (strictly) unobstructed, one can only expect \(((L,G),b)\) to be weakly unobstructed in general. This is due to the possibility of the constant disc class contributing to degree 2 equivariant parameters.

3. \textit{Equivariant disc potentials for immersed SYZ fibers}

In this section, we compute the equivariant disc potentials for immersed SYZ fibers in a toric Calabi-Yau manifold, which are homeomorphic to the product of an immersed 2-sphere and a torus. We begin by deriving a gluing formula between Maurer-Cartan spaces of smooth fibers and immersed fibers, making use of (quasi-)isomorphisms in Lagrangian Floer theory. The potential for the immersed fiber, then, follows by applying the gluing formula to that of a smooth fiber which is much easier to compute.

\[\text{3.1 Gluing formulas.}\] Let \(X = X_{\Sigma}\) be the toric Calabi-Yau manifold defined by a fan \(\Sigma\) as in Section 2.1. Let \(a_1 \in \text{M}_R/\mathbb{R} \cdot 1\) be an interior point of a codimension 2 face \(F\) of the polytope dual to \(\Sigma\). For each \(i = 0, 1, 2\), we obtain the Lagrangian submanifold \(L_i\) of \(X\) given by the preimage in \(\mu^{-1}_F(a_1)\) of the circle \(\ell_i := w(L_i)\) in the \(w\)-plane \(X/\!/_w T^{n-1} \cong \mathbb{C}\) depicted in Figure 3. Observe that \(L_1\) and \(L_2\) are Lagrangian tori, and \(L_0\) is an immersed Lagrangian homeomorphic to \(S^2 \times T^{n-2}\) where \(S^2\) denotes the immersed two-sphere with exactly one nodal self-intersection point.
Likewise, the subfamily $\tau$ (in a way that under the identification $M$ of Morse function $\epsilon$ number one around with respect to the holomorphic volume form oriented by the $\Omega$-gauge cycles for the flat $\Lambda$-connections on their trivial line bundles. In what follows, we fix parameters and gauge cycles for the flat $\Lambda$-connections, which enable us to compute the gluing formulas and the disc potentials explicitly later on.

We may assume that the face $F$ of codimension 2 containing $a_1$ is dual to the 2-cone $R^{\geq 0}.\{v_1, v_2\}$ without loss of generality. Then the vector $v'_1 := v_2 - v_1$ is perpendicular to $F$. We extend $v'_1$ to a basis $\{v'_1, v'_2, \ldots, v'_{n-1}\}$ of $v^\perp \subset \mathbb{N}$. For instance, one can take $v'_i = v_{i+1} - v_1$ for $i = 1, \ldots, n - 1$. We will refer to a choice of $\{v'_1, \ldots, v'_{n-1}\}$ as a framing for $L_0$ as the choice corresponds to a framing for an Aganagic-Vafa brane of a toric Calabi-Yau 3-fold (cf. [AV00]).

The restriction of $w : X^0 \to \mathbb{C}$ to $\{y_2, \ldots, y_n \neq 0\}$ is a trivial $(\mathbb{C}^\times)^{n-1}$-fibration with the base coordinate $w$ and the fiber coordinates $y_2, \ldots, y_n$. The map

$$(y_1, \ldots, y_n) \mapsto \left( \frac{w - \epsilon}{|w - \epsilon|}, \frac{y_2}{|y_2|}, \ldots, \frac{y_n}{|y_n|} \right)$$

trivializes the SYZ torus fibration in the chamber dual to $v_1$ (see [CLL12]). Also, the trivialization fixes identifications of $L_1$ and $L_2$ with the standard torus $T^n$ whose $S^1$-factors are in the directions of $v_1, v'_1, \ldots, v'_{n-1}$. We then have

$$\pi_1(L_i) \cong \pi_1(T^n) \cong \mathbb{Z} \cdot \{v_1, v'_1, \ldots, v'_{n-1}\} \quad (i = 1, 2),$$

$$\pi_1(L_0) \cong \mathbb{Z} \cdot \{v_1, v'_2, \ldots, v'_{n-1}\}.$$  \hspace{1cm} (3.1)

Now, we parametrize the space $\text{hom}(\pi_1(L_1), \Lambda_U)$ of flat connections (up to gauge) by $(z_1, \ldots, z_n)$, each of which is aligned in parallel with the order of basis elements in (3.1) where $z_j \in \Lambda_U$. The flat connection associated to $(z_1, \ldots, z_n)$ is denoted by $\nabla(z_1, \ldots, z_n)$. Similarly, one can associate the flat connection $\nabla(z'_1, \ldots, z'_n)$ for $L_2$ to $(z'_1, \ldots, z'_n) \in (\Lambda_U)^n$. Likewise, the subfamily $\{\nabla \in \text{hom}(\pi_1(L_0), \Lambda_U) \mid \nabla(\bar{v}_1) = 1\}$ of flat connections for $L_0$ is parametrized by $(z^{(0)}_2, \ldots, z^{(0)}_{n-1})$ and denoted by $\nabla(z^{(0)}_2, \ldots, z^{(0)}_{n-1})$.

We next fix the gauge cycles for the flat connections as follows. For the Lagrangian tori $L_1$ and $L_2$, it suffices to fix a union of cycles of codimension one in $T^n$. Let us fix a perfect Morse function $f^{S^1}$ on $S^1$. We choose a perfect Morse function $f^{L_i}$ on $L_i$, $i = 1, 2$ in such a way that under the identification $L_i \cong T^n$, $f^{L_i}$ is the sum of perfect Morse function $f^{S^1}$ on the $S^1$-factors in the directions of $v'_1, \ldots, v'_{n-1}$. We also fix a perfect Morse function on the $S^1$-factor of $L_1$ and $L_2$ in the $\bar{v}_1$-direction with positions of the critical points depicted in Figure 3 (the minimum points of $\ell_1$ and $\ell_2$ are denoted by $z_3$ and $z'_3$, respectively and the maximum points are denoted by $1^*$ in respective colors). The unstable chains of the degree one critical points of $f^{L_i}$ are hypertori dual to $v'_1, \ldots, v'_{n-1} \in \pi_1(T^n)$, co-oriented by the $S^1$-orbits in the respective directions. We then choose these hypertori dual to $v'_1, \ldots, v'_{n-1}$ to be gauge cycles. That is, the flat connections $\nabla(z_1, \ldots, z_n)$ (resp. $\nabla(z'_1, \ldots, z'_n)$)
are trivial away from the gauge hypertori, and have holonomy \( z_j \) (resp. \( z'_j \)) along a path positively crossing the gauge hypertorus dual to \( v_j \) in \( L_1 \) (resp. \( L_2 \)) once, for \( j = 1, \ldots, n - 1 \), and have holonomy \( z_n \) (resp. \( z'_n \)) along a path positively crossing the gauge hypertorus dual to \( v_1 \).

For the immersed Lagrangian \( L_0 \), consider a splitting \( L_0 \cong S^2 \times T^{n-2} \) where the \( T^{n-2} \)-factor is in the directions of \( v'_2, \ldots, v'_{n-1} \). Let \( \iota : \tilde{L}_0 = S^2 \times T^{n-2} \rightarrow X \) be the Lagrangian immersion such that \( \iota(\tilde{L}_0) = L_0 \). We denote

- by \( \{r\} \times T^{n-2} \) the clean self-intersection loci in \( L_0 \)
- by \( \{p\} \times T^{n-2} \) and \( \{q\} \times T^{n-2} \) the two disjoint connected components of the preimage of \( \{r\} \times T^{n-2} \) in \( \tilde{L}_0 \)

We then have three components of \( L_0 = \tilde{L}_0 \times_{L_0} \tilde{L}_0 \): the diagonal component \( R_0 \cong \tilde{L}_0 \) and the two non-diagonal components \( R_{-1} = \{(p,q)\} \times T^{n-2} \) and \( R_{1} = \{(q,p)\} \times T^{n-2} \).

Let \( f^{S^2} : S^2 \rightarrow \mathbb{R} \) be a perfect Morse function such that the critical points of \( f^{S^2} \) are away from \( p \) and \( q \), and the two flow lines connecting \( p \) and \( q \) to the minimum point are disjoint. Using the splitting above, we choose a perfect Morse function \( f^{L_0} \) on \( L_0 \) of the form

\[
f^{L_0}|_{R_0} = f^{S^2} + f^{T^{n-2}} \quad \text{and} \quad f^{L_0}|_{R_{\pm 1}} = f^{T^{n-2}}
\]
where $f^{T^{n-2}}$ is the sum of $f^{S^1}$ on the $S^1$-factors of $T^{n-2}$. Also, recall that the gauge cycles for $\nabla^{(v'_{0}, \ldots, v'_{n-1})}$ consist of a union of $(n-2)$ different cycles of codimension one. The cycles are taken by the product of $S^2$ with hypertori in $T^{n-2}$ dual to $v'_{2}, \ldots, v'_{n-1} \in \pi_1(T^{n-2})$.

3.1.2 Unobstructedness of the Lagrangians (in $X^\ast$). Let $(CF^\ast(L_0), m^L)$ be a (unital) $\mathbb{A}_\ast$-algebra defined on the pearl complex associated to the Morse function $f|_{L_0}$ in (2.17). We choose the following perturbations for the moduli spaces $\mathcal{M}_{k+1}(\alpha, \beta, L_0)$ in order to simplify computations.

Note that the Hamiltonian $T^{n-2}$-action corresponding to the sublattice generated by $\{v'_{2}, \ldots, v'_{n-1}\}$ acts freely on the Lagrangian $L_i$ by rotating its $T^{n-2}$-factor for all $i = 0, 1, 2$. Moreover, the subtorus $T^{n-2}$ preserves the complex structure. Then this induces a free $T^{n-2}$-action on $\mathcal{M}_{k+1}(\alpha, \beta, L_0)$. We can therefore choose $T^{n-2}$-equivariant Kuranishi structures and perturbations for $\mathcal{M}_{k+1}(\alpha, \beta, L_0)$ as in [FO010] (see also [Fuk17]).

Let $\tau : X \rightarrow X$ be the anti-symplectic involution characterized by that it reverses the regular fibers of the moment map of the $T^{n}$-action. We note that the complex structure of $X$ is $\tau$-anti-invariant. We will orient the moduli spaces with a $\tau$-invariant relative spin structure By Theorem 2.3, the $T^{n-2}$-equivariant Kuranishi structures on $\mathcal{M}_{k+1}(\alpha, \beta; L_0)$ and $\mathcal{M}_{k+1}(\hat{\alpha}, \hat{\beta}; L_0)$ can be chosen to be isomorphic. The $T^{n-2}$-equivariant perturbations $s$ for $\mathcal{M}_{k+1}(\alpha, \beta; L_0)$ and $\hat{s}$ for $\mathcal{M}_{k+1}(\hat{\alpha}, \hat{\beta}; L_0)$ can then be chosen such that they satisfy $(\tau_*^{\text{main}})^s \hat{s} = s$.

The effective cone $H^{\text{eff}}_2(X^\ast, L_0)$ is generated by the Maslov index 0 disc classes. For any stable disc $u$ in a non-trivial class $\beta \in H^{\text{eff}}_2(X^\ast, L_0)$, we observe $u(\partial D^2) \subset \{r\} \times T^{n-2}$. This observation leads to the following simple consequences for the moduli spaces:

If $\alpha : \{0, \ldots, k\} \rightarrow \{-1, 0, 1\}$ and $\beta \in H^{\text{eff}}_2(X^\ast, L_0)$, then

- If $\alpha(i) = 0$ for all $i$, and $\beta \neq 0$, then $\mathcal{M}_{k+1}(\alpha, \beta; L_0)$ has two connected components corresponding to the lifts of the boundary of the holomorphic discs in class $\beta$ to $\{p\} \times T^{n-2}$ and $\{q\} \times T^{n-2}$.
- Suppose $i \in \{0, \ldots, k\}$ and $\alpha(i) \neq 0$, i.e., $z_i$ is a corner. Let $z_j$ be a corner adjacent to $z_i$, namely, $\alpha(j) \neq 0$, and $\alpha(k) = 0$ for all boundary marked points $z_k$ between $z_i$ and $z_j$. We then have $\mathcal{M}_{k+1}(\alpha, \beta; L_0) = \emptyset$ unless $\alpha(i)$ and $\alpha(j)$ have opposite signs. In particular, when $k = 0$, we have $\mathcal{M}_{1}(\alpha, \beta; L_0) = \emptyset$ unless $\alpha(0) = 0$, i.e., the output marked point lies in $R_0 = L_0$.

We denote by $1_{T^{n-2}}$ the maximum point of $f^{T^{n-2}}$, by $X_1, \ldots, X_{n-2}$ the degree 1 critical points, and by $X_{ij}$, $1 \leq i < j \leq n-2$ the degree 2 critical points. We also denote the maximum and minimum points of $f^{S^2}$ by $1_{S^2}$ and $a_{S^2}$, respectively.

We will abuse notations and denote the critical points of $f^{L_0}$ in the form of tensor products. In particular, we write $U = (p, q) \otimes 1_{T^{n-2}}$ and $V = (q, p) \otimes 1_{T^{n-2}}$ to denote the maximum points on $R_1$ and $R_1$ respectively. The holomorphic volume form $\Omega_X$ equips $CF^\ast(L_0)$ with a $\mathbb{Z}$-grading, under which the two critical points $U$ and $V$ are of degree 1, while the critical points $(p, q) \otimes X_i$ and $(q, p) \otimes X_i$ are of degree 2. Let $b = uU + vV$ for $u, v \in \Lambda^2$ with $\text{val}(u \cdot v) > 0$.

We define $L_0$, $L_1'$, $L_2$ to be the family of formal Lagrangian deformations (or the corresponding objects of the Fukaya category)

$$(L_0, \nabla^{(v'_{0}, \ldots, v'_{n-1})}), (L_1, \nabla^{(v_{1}, \ldots, v_{n})}) \text{ and } (L_2, \nabla^{(v_{1}', \ldots, v_{n}')}).$$
respectively. (The notation $L_i$ is reserved for another Lagrangian brane which will appear shortly.) We then denote by $(CF^*(L'_i), m^{L'_i})$ the formally deformed $A_{\infty}$-algebras on the pearl complexes associated to the Morse functions $f^{L_i}$. More precisely, the $A_{\infty}$-operations $m^{L_i}_k$ are defined by

$$m^{L'_i}_k = \sum_{\Gamma \in \Gamma_{k+1}} T^{\omega(\sum_\nu \beta_\nu)} \cdot Hol_{\Gamma_{(z_1, \ldots, z_k)}} \left( \sum_\nu \partial \beta_\nu \right) \cdot m_{L'_i}$$

$$m^{L_i}_k = \sum_{\Gamma \in \Gamma_{k+1}} T^{\omega(\sum_\nu \beta_\nu)} \cdot Hol_{\Gamma_{(z_1, \ldots, z_k)}} \left( \sum_\nu \partial \beta_\nu \right) \cdot m_{L_i}$$

$$m^{L_0}_k = \sum_{\Gamma \in \Gamma_{k+1}} T^{\omega(\sum_\nu \beta_\nu)} \cdot Hol_{\Gamma_{(z_1, \ldots, z_k)}} \left( \sum_\nu \partial \beta_\nu \right) \cdot m_{L_0,b}$$

where the summation $\sum_\nu$ is over the interior vertices of the stable tree $\Gamma$, $m^{L'_i}_k$ was defined in (2.14), and the deformation by $b$ for $m^{L_0,b}_k$ was defined in (2.19).

Figure 4. The domain $S^2 \times S^1$ of the Lagrangian immersion $L_0$. The immersed loci are shown as top and bottom circles in the figure, which are also boundaries of the holomorphic discs of Maslov index zero.

**Lemma 3.1.** $L_0$ is unobstructed.

**Proof.** Recall that $m^{L_0}_1(1)$ is the summation over operations indexed by stable trees $\Gamma \in \Gamma_{k+1}$, $k \geqslant 0$, with repeated inputs $b = uU + vV$. Since $U$ and $V$ are of degree 1 and $\mu_{L_0}(\beta) = 0$ for all $\beta \in H^2_{\text{eff}}(X^0, L_0)$, $m^{L_0}_1(1)$ is a linear combination of degree 2 critical points of $f^{L_0}$ of the form $(p, q) \otimes X_{ij}$, $(q, p) \otimes X_i$, $a_{\mathbb{S}^2} \otimes 1_{T^{n-2}}$, and $1_{\mathbb{S}^2} \otimes X_{ij}$. The coefficient of $1_{\mathbb{S}^2} \otimes X_{ij}$ is zero for an obvious reason: there are no flow lines from $p$ or $q$ to $1_{\mathbb{S}^2}$.

(1) Let us consider the coefficient of $(p, q) \otimes X_i$. Since the output of a stable tree with no input vertices must be in $R_0 = L_0$, it suffices to consider stable trees $\Gamma \in \Gamma_{k+1}$ with $k \geqslant 1$.

We first exclude the contribution of $\Gamma$ with at least one interior vertex. In this case, $\Gamma$ must have an interior vertex $v$ such that each input vertex of $v$ is an input vertex of $\Gamma$. Note that this includes vertices $v$ with no input vertices (tadpoles). We denote the output vertex of $v$ by $w$. Let $(\alpha_\nu, \beta_\nu)$ be the stable polygon decorating $v$. Since the unstable chains of $U$
and $V$ are both $T^{n-2}$-invariant, by choosing a generic small $T^{n-2}$-equivariant perturbation for the fiber product at $v$, the output singular chain $\Delta_v$ is then $T^{n-2}$-invariant.

If $a_v(0) = \pm 1$, i.e., the output marked point is a corner, then the virtual dimension of $\Delta_v$ is $n - 3$. Since $\Delta_v$ is $T^{n-2}$-invariant, it must be the zero chain. Now, suppose $a_v(0) = 0$, i.e., the output marked point is a smooth point. Then, $\Delta_v$ has virtual dimension $n - 2$. The evaluation image of $\Delta_v$ is a union of $T^{n-2}$-orbits $\bigcup_{i=1}^{d} \{p_i\} \times T^{n-2} \subset \dot{L}_0$, where $p_1, \ldots, p_d \in S^2 \setminus \{p, q\}$ are points situated near either $p$ or $q$. (We remark that the evaluation image at smooth output point can be perturbed outside of $\{p\} \times T^{n-2}$ and $\{q\} \times T^{n-2}$ due to the fact that the Kuranishi structures are weakly submersive.) Note that if $w$ is the root vertex, then $\Gamma$ does not contribute to $p \otimes X_i$ since its stable submanifold is contained in $R_{-1}$. We may therefore assume $w$ is an interior vertex and denote its decoration by $(\alpha_w, \beta_w)$.

For $\beta_w \neq \beta_0$, the evaluation image of $\mathcal{M}_{\ell(w)}(\alpha_w, \beta_w; \dot{L}_0)$ at the corresponding input marked point is contained in $\{p, q\} \times T^{n-2}$, which does not intersect with the flow lines from $\Delta_v$ since $f^{S^2}$ was chosen so that the two flow lines connecting $p$ and $q$ to $a_{S^2}$ are disjoint. For $\beta_w = \beta_0$, we have the following three cases:

(i) $w$ has an input vertex $v'$ such that $v'$ is an input vertex of $\Gamma$.
(ii) $w$ has an input vertex $v' \neq v$ such that each input vertex of $v'$ is an input vertex of $\Gamma$.
(iii) There is a subtree $\Gamma'$ of $\Gamma$ ending at $w$ which does not contain $v$. $\Gamma'$ has an interior vertex $v^+$ such that each input vertex of $v^+$ is an input vertex of $\Gamma$ and its output vertex is not $w$.

For the case (i), the fiber product at $w$ is empty since the flow lines from $U$ and $V$ are in the immersed sectors. For the case (ii), we again choose generic small $T^{n-2}$-equivariant perturbation for the fiber product at $v'$, and denote the output singular chain by $\Delta_{v'}$. The flow lines from $\Delta_v$ and $\Delta_{v'}$ do not intersect generically. Finally, for the case (iii), we repeat the arguments above for $\Gamma'$. It easy to see that such iterations terminate. This rules out the contribution of $\Gamma$ with at least one interior vertices.

Consequently, the only possible contributions to $(p, q) \otimes X_i$ are from Morse flow lines, and we know that flow lines from $U$ to $(p, q) \otimes X_i$ cancel pairwise. Hence the coefficient of $(p, q) \otimes X_i$ vanishes. The vanishing of the coefficient of $(q, p) \otimes X_i$ follows from the exact same argument.

(2) We next consider the coefficient of $a_{S^2} \otimes 1_{T^{n-2}}$. Since the Morse flow lines from $U$ and $V$ are contained in $R_{-1}$ and $R_1$, respectively, $\Gamma^0$ does not contribute to $a_{S^2} \otimes 1_{T^{n-2}}$. It suffices to consider stable trees with at least one interior vertex. Moreover, by the same argument as in (1), the only possible contributions are from stable trees $\Gamma_{k+1} \in \Gamma_{k+1}$, $k \geq 0$, with exactly one interior vertex $v$ (in which case its output vertex $w$ is the root vertex). If $v$ has an odd number of inputs, then its output marked point must be a corner, but $a_{S^2} \otimes 1_{T^{n-2}} \in \dot{L}_0$. Thus, we are left with the case of the stable trees $\Gamma_{2k+1} \in \Gamma_{2k+1}$. We note that since the boundary of a stable polygon bounded by $L_0$ is contained in the the immersed loci, the input corners $U$ and $V$ must appear in pairs.

For $k > 1$, since $U$ and $V$ are of degree 1, the map $\tau^\text{main}_* : \mathcal{M}_{2k+1}(\alpha_v, \beta_v; U, V, \ldots, U, V) \to \mathcal{M}_{2k+1}(\alpha_v, \beta_v; V, U, \ldots, V, U)$ is an orientation reversing isomorphism of Kuranishi structures by Theorem 2.4. Let $\hat{\Gamma}_{2k+1} \in \Gamma_{2k+1}$ be the tree obtained from $\Gamma_{2k+1}$ by changing the decoration at $v$ to $(\hat{\alpha}_v, \hat{\beta}_v)$. Then, by choosing $\tau^\text{main}_*$-invariant perturbations for
\[ \mathcal{M}_{2k+1}(a_v, \beta_v; U, V, \ldots, U, V) \text{ and } \mathcal{M}_{2k+1}(\alpha_v, \beta_v; U, V, \ldots, V, U), \] the contribution of \( \Gamma_{2k+1} \) and \( \hat{\Gamma}_{2k+1} \) to \( a_{S^2} \otimes 1_{T^{n-2}} \) cancel with each other.

For \( k = 0 \), we note that \( a_v : \{0\} \to \{-1, 0, 1\}, a_v(0) = 0 \) and \( \beta_v \neq 0 \). Then \( \mathcal{M}_1(a_v, \beta_v; L_0) \) has two connected components corresponding to the two possible lifts of the disc boundary to \( \{p\} \times T^{n-2} \) and \( \{q\} \times T^{n-2} \). By Theorem 2.4, \( \tau^\text{main}_*: \mathcal{M}_1(a_v, \beta_v; L_0) \to \mathcal{M}_1(a_v, \beta_v; L_0) \) is an orientation reversing isomorphism swapping the two components. By choosing \( \tau^\text{main}_*-\text{invariant perturbation for } \mathcal{M}_1(a_v, \beta_v; L_0) \), the contribution of \( \Gamma_{2k+1} \) to \( a_{S^2} \otimes 1_{T^{n-2}} \) vanishes.

The cancellation of contributions to \( a_{S^2} \otimes 1_{T^{n-2}} \) can be intuitively visualized as in Figure 5 if we make a transverse perturbation of the self-intersection loci.

![Figure 5. After perturbing the clean intersection loci, some of canceling pairs for \( m_0^b \) are intuitively visible.](image)

Finally, \( L_1' \) and \( L_2' \) are unobstructed as Lagrangian submanifolds of \( X^\circ \), since the underlying Lagrangians \( L_1 \) and \( L_2 \) do not bound any non-constant holomorphic discs in \( X^\circ \). Thus we have:

**Lemma 3.2.** \( L_i' \) is unobstructed for \( i = 1, 2 \).

### 3.1.3 Computation of the gluing formula.

For \( i < j \in \{0, 1, 2\} \), we denote the clean intersections of \( L_i \) with \( L_j \) by \( \{a_{i,j}^{\ell_i, \ell_j}, b_{i,j}^{\ell_i, \ell_j}\} \times T^{n-1} \), where \( a_{i,j}^{\ell_i, \ell_j} \), \( b_{i,j}^{\ell_i, \ell_j} \) are the two intersection points of the base circles \( \ell_i \) and \( \ell_j \) on the \( w \)-plane as depicted in Figure 3, and the \( T^{n-1} \)-factor lies along the directions of \( v'_1, \ldots, v'_{n-1} \). We choose a perfect Morse function \( f_{L_i, L_j} \) on \( \{a_{i,j}^{\ell_i, \ell_j}, b_{i,j}^{\ell_i, \ell_j}\} \times T^{n-1} \) such that its restriction to each connected component is the sum of \( f \) on the \( S^1 \)-factors.

In [CLL12], it was shown that the disc potentials of \( L_1 \) and \( L_2 \) are equal under the change of variables \( z_i' = z_i \) for \( i = 1, \ldots, n-1 \), and \( z_n' = z_n \cdot f(z_1, \ldots, z_{n-1}) \) where \( f \) is the generating function given in (2.4). In the following, we will deduce the same gluing formula by finding a (quasi-)isomorphism between \( L_i' \) and \( L_j' \) regarded as objects the Fukaya category.

Let \( (CF^*(L_1', L_2'), m_1^{L_1', L_2'}) \) and \( (CF^*(L_2', L_1'), m_1^{L_2', L_1'}) \) be the pearl complexes associated to the perfect Morse function \( f_{L_1, L_2} \), which are further deformed by the flat connections
\[ \nabla^{(z_1, \ldots, z_n)} \text{ and } \nabla^{(z'_1, \ldots, z'_n)}. \] We denote by
\[ a_{12} \otimes 1_{T^{n-1}} \in CF^0(L'_1, L'_2), \quad a_{21} \otimes 1_{T^{n-1}} \in CF^1(L'_2, L'_1) \]
\[ b_{12} \otimes 1_{T^{n-1}} \in CF^1(L'_1, L'_2), \quad b_{21} \otimes 1_{T^{n-1}} \in CF^0(L'_2, L'_1) \]
generators corresponding to the maximum points on \( \{a^{f_1,f_2}\} \times T^{n-1} \) and \( \{b^{f_1,f_2}\} \times T^{n-1} \), respectively. We will prove that \( a_{12} \otimes 1_{T^{n-1}} \) and \( b_{21} \otimes 1_{T^{n-1}} \) provide the desired isomorphisms, namely
\[ m_1^{L'_1,L'_2}(a_{12} \otimes 1_{T^{n-1}}) = 0, \]
\[ m_1^{L'_1,L'_2}(b_{21} \otimes 1_{T^{n-1}}) = 0, \]
\[ m_2^{L'_1,L'_2}(b_{21} \otimes 1_{T^{n-1}}, a_{12} \otimes 1_{T^{n-1}}) = c_1 \cdot 1_{L'_1} + m_1^{L'_1}(\gamma_1), \]
\[ m_2^{L'_1,L'_2}(a_{12} \otimes 1_{T^{n-1}}, b_{21} \otimes 1_{T^{n-1}}) = c_2 \cdot 1_{L'_2} + m_1^{L'_2}(\gamma_2), \]
for some nonzero \( c_1, c_2 \in \Lambda, \gamma_1 \in CF^*(L'_1), \) and \( \gamma_2 \in CF^*(L'_2). \) Notice that \( 1_{L'_1} \) and \( 1_{L'_2} \) in the above equations are the strict units of \( CF^*(L'_1) \) and \( CF^*(L'_2), \) respectively.

**Remark 3.3.** The relation (3.3) between Clifford type tori and Chekanov type tori was studied by Seidel in his lecture notes, see [Sei].

We may assume \( L_1 \) and \( L_2 \) are chosen such that the holomorphic strip classes \( \beta_0, \beta_1 \in \pi_2(X^\circ, L_1 \cup L_2) \) (see Corollary 3.7) have the same symplectic area. We then have the following.

**Theorem 3.4.** \( a_{12} \otimes 1_{T^{n-1}} \) is a quasi-isomorphism between \( L'_1 \) and \( L'_2 \) with an inverse \( b_{21} \otimes 1_{T^{n-1}} \) if and only if
\[ z'_i = z_i, \quad i = 1, \ldots, n-1 \]
\[ z'_n = z_n \cdot f(z_1, \ldots, z_{n-1}), \]
(3.4)

where \( f \) is given in (2.4).

The proof requires a few preliminary steps, analyzing relevant moduli spaces of holomorphic strips.

Let us first focus on \( m_1^{L'_1,L'_2}(a_{12} \otimes 1_{T^{n-1}}). \) A priori, the output of \( m_1^{L'_1,L'_2}(a_{12} \otimes 1_{T^{n-1}}) \) is a linear combination of \( b_{12} \otimes 1_{T^{n-1}} \) and \( a_{12} \otimes X_i, \) where \( X_1, \ldots, X_{n-1} \) are the degree 1 critical points of \( fT^{n-1}. \) Since there are no non-constant holomorphic polygon with both input and output corners in \( \{a^{f_1,f_2}\} \times T^{n-1} \), the coefficient of \( a_{12} \otimes X_i \) is given by the two Morse flow lines from \( a_{12} \otimes 1_{T^{n-1}} \) in \( \{a^{f_1,f_2}\} \times T^{n-1}. \) These flow lines contribute \( z_i - z'_i \) to the coefficient of \( a_{12} \otimes X_i. \)

Let \( \mathcal{M}_1^{L'_1,L'_2}(\beta; \{a^{f_1,f_2}\} \times T^{n-1}, \{b^{f_1,f_2}\} \times T^{n-1}) \) be the moduli space of stable holomorphic strips in class \( \beta \in \pi_2(X^\circ, L_1 \cup L_2) \) with input corner in \( \{a^{f_1,f_2}\} \times T^{n-1} \) and output corner in \( \{b^{f_1,f_2}\} \times T^{n-1}. \) The coefficient of \( b_{12} \otimes 1_{T^{n-1}} \) is given by the fiber products
\[ \mathcal{M}_2^{L'_1,L'_2}(\beta; \{a^{f_1,f_2}\} \times T^{n-1}, \{b^{f_1,f_2}\} \times T^{n-1}) \times L_1 \cap L_2 \{b_{12} \otimes 1_{T^{n-1}}, \}
\]
(3.5)

where \( \beta \) has Maslov index 1.

In terminology of Section 2.2, (3.5) corresponds to the moduli spaces of Maslov index 2 stable polygons with one input critical point \( a_{12} \otimes 1_{T^{n-1}} \) (whose unstable chain is \( \{a^{f_1,f_2}\} \times T^{n-1} \)) and the output corner in \( \{b^{f_1,f_2}\} \times T^{n-1}. \) By degree reason, the output is a multiple of \( b_{12} \otimes 1_{T^{n-1}}. \) Since \( b_{12} \otimes 1_{T^{n-1}} \) is the maximum point on \( \{b^{f_1,f_2}\} \times T^{n-1}, \) any contributing stable polygon must intersect it at the output corner.
To make this more explicit, we first consider the example $X = \mathbb{C}^n$. Let $y_1, \ldots, y_n$ be the standard complex coordinates on $\mathbb{C}^n$. Then, $w = y_1 \ldots y_n$. The moment map of the $T^{n-1}$ action is given by

$$
\mu(y_1, \ldots, y_n) = (|y_2|^2 - |y_1|^2, \ldots, |y_n|^2 - |y_1|^2).
$$

For the moment, let us take $a_1 = (0, \ldots, 0)$ so that the Lagrangians $L_i$ for $i = 1, 2$ satisfy $|y_1| = \ldots = |y_n|$. Moreover, we take $\ell_2$ to be $|w| = 1$ and $\ell_1$ to be the image of $\mathbb{R} \cup \{\infty\}$ under the transformation $\frac{w-\bar{a}}{1-\bar{a}w}$ for some $a \in i \cdot (1,0)$ in the $w$-plane, where $i := \sqrt{-1}$.

**Lemma 3.5.** Let $X = \mathbb{C}^n$ and $L_1, L_2$ be given as above and let $\beta \in \pi_2(X^\circ, L_1 \cup L_2)$ be an holomorphic strip class of Maslov index 1. The moduli space $\mathcal{M}^{L_1, L_2}_2(\beta; \{a^i, \ell_i\} \times T^{n-1}, \{b^i, \ell_i\} \times T^{n-1})$ is regular and is isomorphic to $T^{n-1}$.

**Proof.** In the punctured $w$-plane $\mathbb{C}\setminus \{e\}$, $e \in i \cdot (1, \frac{a}{1})$, there are two holomorphic strips $u_L$ (which contains $w = 0$) and $u_R$ bounded by $\ell_1$ and $\ell_2$. The projection of a holomorphic strip $u$ in class $\beta$ to the $w$-plane must cover either $u_L$ or $u_R$. We denote the class of holomorphic strips covering $u_R$ by $\beta_R$. It is easy to see that the Lemma holds for $\beta_R$ by trivializing the $(\mathbb{C}^\times)^{n-1}$-fibration $w : X^\circ \to \mathbb{C}\setminus \{e\}$ away from $\{w = 0\}$. The strip classes covering $u_L$ can be identified with the Maslov index 1 disc classes bounded by a regular toric fiber of $\mathbb{C}^n$, and are hence regular (see [CO06]). Moreover, the holomorphic strips $u$ must intersects a toric Calabi-Yau manifold. The toric fiber of $\mathbb{C}^n$ acts freely).

Without loss of generality, we may assume $i = 1$. The image of $u$ is contained in the complement of $\{y_j = 0\}$ for $j = 2, \ldots, n$, which can be identified with $\mathbb{C} \times (\mathbb{C}^\times)^{n-1}$, equipped with coordinates $y_2, \ldots, y_n \in \mathbb{C}^\times$ and $w = y_1 \ldots y_n \in \mathbb{C}$. Then, $u$ can be written as a holomorphic map

$$
u(\zeta) = (w(\zeta), y_2(\zeta), \ldots, y_n(\zeta))$$

from the upper-half disc $\{\zeta \in \mathbb{C} \mid |\zeta| \leq 1, \text{Im} \zeta \geq 0\}$. We have $w(\zeta) = \frac{\zeta - a}{1 - \bar{a} \zeta}$ up to a reparametrization of the domain, which satisfies the boundary conditions $|w| = 1$ on the upper arc $|\zeta| = 1$ and $\{\frac{w-\bar{a}}{1-\bar{a}w} \mid w \in \mathbb{R} \cup \{\infty\}\}$ on the lower arc $[-1,1]$.

For the $y_j(\zeta)$-components of $u(\zeta)$, the boundary condition $|y_1| = \ldots = |y_n|$ implies $|w| = |y_j|^n$ for $j = 2, \ldots, n$. We consider the holomorphic map $\tilde{w}(\zeta) = \frac{\zeta - a}{1 - \bar{a} \zeta}$, which satisfies $|\tilde{w}| = |w| = 1$ on the upper arc $|\zeta| = 1$ and $|\tilde{w}| = |w|$ on the lower arc $\zeta \in [-1,1]$. Thus, it also satisfies $|\tilde{w}| = |y_j|^n$. Moreover, $\tilde{w}(\zeta) \neq 0$ on the upper half disc $\{\zeta \in \mathbb{C} \mid |\zeta| \leq 1, \text{Im} \zeta \geq 0\}$. By the maximum principle, $\tilde{w} = \rho y_i^n$ for some constant $\rho \in U(1)$. This determines $y_j = \tilde{w}^{1/n}$ up to a rotation by $\rho$. Thus, the moduli space is isomorphic to $T^{n-1}$ (on which $T^{n-1}$ acts freely).

In the proof above, we have chosen base loops $\ell_1$ and $\ell_2$, and a level $a_1$ for the simplicity of argument. It is easy to see that the lemma holds for other choices of a level $a_1$ and base loops $\ell_1$ and $\ell_2$ enclosing $\epsilon$ and intersecting transversely at two points.

Now, let $X$ be any toric Calabi-Yau manifold. The $T^{n}$-moment map is of the form

$$(\rho_1(|y_1|^2, \ldots, |y_n|^2), \ldots, \rho_n(|y_1|^2, \ldots, |y_n|^2))$$

for a toric chart $(y_1, \ldots, y_n)$, where $\rho$ is a Legendre transform determined by the toric Kähler metric [Gui94].
Lemma 3.6. For a toric Calabi-Yau manifold $X$ and $L_1,L_2$ given in the beginning of this section, let $\beta \in \pi_2(X^\circ,L_1 \cup L_2)$ be a holomorphic strip class of Maslov index 1. The moduli space $\mathcal{M}_2^{L_1,L_2}(\beta; \{a^{f_1,f_2}\} \times T^{n-1}, \{b^{f_1,f_2}\} \times T^{n-1})$ is regular and isomorphic to $T^{n-1}$.

Proof. Under the projection to the punctured $w$-plane, a holomorphic strip $u$ in class $\beta$ must cover one of the two strips $u_1$ (which contains $w = 0$) and $u_2$ bounded by the two base circles $\ell_1$ and $\ell_2$. Since it has Maslov index 1, it intersects at most one of the toric prime divisors in $\{w = 0\} \subset X^\circ$. If follows that $\text{Im} u$ is contained in a certain toric $\mathbb{C}^n$-chart. Note that holomorphic strips in a class $\beta$ are contained in the same toric chart, since they have the same the intersection numbers with the toric prime divisors. Moreover, by the maximum principle they are contained in a compact subset of $\mathbb{C}^n$, and thus the moduli space is compact by Gromov’s compactness theorem.

Both the standard symplectic form $\omega_{\text{std}}$ on $\mathbb{C}^n$ and the restriction of the symplectic form $\omega$ on $X$ to the toric $\mathbb{C}^n$ chart are $T^n$-invariant. By a $T^n$-equivariant Moser argument, we have a one-parameter family of $T^n$-equivariant symplectomorphic embeddings $\rho_t : (\mathbb{C}^n, \omega) \to (\mathbb{C}^n, \omega_{\text{std}})$ for $t \in [0,1]$ such that $\rho_0 = \text{Id}$ and $\rho_1^n \omega_{\text{std}} = \omega$. The isotopy $\rho_t$ can be realized as follows: take a one-parameter family of toric Kähler forms $\omega_t$ on $X$ whose corresponding moment map polytopes $P_t \subset \mathbb{R}_{\geq 0}^n$ interpolate between that of $\omega$ and $\mathbb{R}_{\geq 0}^n$. Using the symplectic toric coordinates, $\rho_t$ are simply given by the inclusions $P^o \to \mathbb{R}_{\geq 0}^n$, where $P^o$ is the part of the moment polytope that corresponds to the toric $\mathbb{C}^n$-chart.

Let $L_{i,t} = \rho_t^{-1}(L_i)$ be the Lagrangians in $(\mathbb{C}^n, \rho_t^n \omega_{\text{std}})$. We note that the standard complex structure remains invariant under this isotopy. The moduli space of holomorphic discs bounded by $L_{i,t}$ for $t \in [0,1]$ gives a cobordism of moduli spaces. Indeed, this is the simplest case of Fukaya’s trick [Fuk10]: since $\beta$ has the minimal Maslov index and all holomorphic discs in $\beta$ are supported in $\mathbb{C}^n$, disc and sphere bubbling do not occur, and therefore the cobordism has no extra boundary component.

The above proof gives a classification of the holomorphic strip classes:

Corollary 3.7. The holomorphic strip classes in $\pi_2(X^\circ,L_1 \cup L_2)$ of Maslov index 1 with input corner in $\{a^{f_1,f_2}\} \times T^{n-1}$ and output corner in $\{b^{f_1,f_2}\} \times T^{n-1}$ are given by $\beta_0$ which does not intersect $w = 0$, and $\beta_i$ for $i = 1, \ldots, m$, which intersects the toric divisor $D_i$ exactly once but does not intersect $D_j$ for $j \neq i$. This gives a one-to-one correspondence between $\beta_1, \ldots, \beta_m$ and the Maslov index 2 basic disc classes $\beta_1^1, \ldots, \beta_m^1 \in \pi_2(X,L)$ bounded by a regular toric fiber $L$ of $X$.

The correspondence above can be realized by smoothing the corners $\{a^{f_1,f_2}\} \times T^{n-1}$ and $\{b^{f_1,f_2}\} \times T^{n-1}$. The $T^{n-1}$-equivariant smoothing of $L_1 \cup L_2$ at these two corners (by using the symplectic reduction $X//T^{n-1}$) gives a union of Clifford torus and a Chekanov torus. Under this smoothing, the strips classes $\beta_i$ for $i = 1, \ldots, m$ correspond to discs classes bounded by the Clifford torus, while the strip class $\beta_0$ corresponds to the basic disc class bounded by the Chekanov torus.

Since $L_1$ and $L_2$ bound no non-constant holomorphic discs in $X^\circ$, a stable holomorphic strip class has no disc components. On the other hand, recall that $H_2(X; \mathbb{Z})$ is generated by primitive effective curve classes $\{C_{n+1}, \ldots, C_m\}$. We have rational curves in $X$ in classes $a \in \mathbb{Z}_{>0} \cdot \{C_{n+1}, \ldots, C_m\}$, and $c_1(a) = 0$. Moreover, all rational curves are contained in the divisor $D = \sum_{j=1}^m D_j$. This means we could have stable holomorphic strip classes of the form $\beta_i + a$, where $\beta_i$ is a holomorphic strip class in Corollary 3.7. We now classify all stable holomorphic strip classes of Maslov index 1.
Lemma 3.8. The holomorphic strip classes $\beta \in \pi_2(X^2, L_1 \cup L_2)$ of Maslov index 1 with input corner in \( \{a^{i_1,i_2}\} \times T^{n-1} \) and output corner in \( \{b^{i_1,i_2}\} \times T^{n-1} \) are of the form $\beta = \beta_i + \alpha$ for $i = 0, \ldots, m$, and $\alpha \in H_2(X; \mathbb{Z})$ is an effective curve class. In particular $\alpha = 0$ for $\beta_i = \beta_0$.

Proof. This follows from the S.E.S (2.1) and the correspondence between holomorphic strip class covering $u_L$ (resp. $u_R$) and the disc classes bounded by the Clifford (resp. Chekanov) torus.

Since $\mathcal{M}_2^{L_1,L_2}(\beta_i; \{a^{i_1,i_2}\} \times T^{n-1}, \{b^{i_1,i_2}\} \times T^{n-1}) \cong T^{n-1}$ and $ev_0 : \mathcal{M}_2^{L_1,L_2}(\beta_i; \{a^{i_1,i_2}\} \times T^{n-1}, \{b^{i_1,i_2}\} \times T^{n-1}) \rightarrow \{b^{i_1,i_2}\} \times T^{n-1}$ is $T^{n-1}$-equivariant for $i = 0, \ldots, m$, there is exactly one holomorphic strip in class $\beta_i$ passing through $\{b_{12}\} \otimes 1_{T^{n-1}}$ and hence $\mathcal{M}_2^{L_1,L_2}(a_{12} \otimes 1_{T^{n-1}})$ with moduli space of holomorphic strips bounded by a Clifford torus passing through a generic point. This allows us to use the result of [CCLT16] and relate the counts of the strips with the inverse mirror map.

Proposition 3.9. The moduli space

$$\mathcal{M}_2^{L_1,L_2}(\beta_i + \alpha; \{a^{i_1,i_2}\} \times T^{n-1}, \{b^{i_1,i_2}\} \times T^{n-1}) \times_{L_1 \cap L_2} \{b_{12} \otimes 1_{T^{n-1}}\}$$

for $i = 1, \ldots, m$, and $\alpha \in H_2(X; \mathbb{Z})$ is an effective curve class, is isomorphic to $\mathcal{M}_1^{L_1}(\beta_i + \alpha, L) \times L \{p\}$ as Kuranishi structures, for a Lagrangian toric fiber $L$ and a generic point $p \in L$.

Proof. Let $L$ be a regular toric fiber and $p \in L$ a generic point. It is well known that $\mathcal{M}_1(\beta_i^L + \alpha, L) \cong T^n$ and $ev_0 : \mathcal{M}_1(\beta_i^L, L) \rightarrow L$ is $T^n$-invariant (see [CLL12]). It follows that

$$\mathcal{M}_2^{L_1,L_2}(\beta_i; \{a^{i_1,i_2}\} \times T^{n-1}, \{b^{i_1,i_2}\} \times T^{n-1}) \times_{L_1 \cap L_2} \{b_{12} \otimes 1_{T^{n-1}}\} = \mathcal{M}_1(\beta_i^L, L) \times L \{p\} = \{pt\}.$$

The holomorphic strips in class $\beta_i$ intersect the toric prime divisor $D_l$ at a certain point $q$. For a rational curve in class $\alpha$, we can take a toric chart $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$ containing $q = (0, c_2, \ldots, c_n)$, $c_j \neq 0 \in \mathbb{C}$, $j = 2, \ldots, n$, and let $L = \{|z_1| = |c_1|, \ldots, |z_n| = |c_n|\} \subset \mathbb{C}^n$ for some $c_1 \neq 0 \in \mathbb{C}$. The choice of $p \in L$ is arbitrary. For instance, we can take $p = (c_1, \ldots, c_n)$. Then, the holomorphic disc $u$ in $\mathcal{M}_1(\beta_i^L, L)$ which passes through $p$ on the boundary and intersects $q$ is given by $u(\xi) = (c_1, c_2, \ldots, c_n)$.

Let $\mathcal{M}_1^{\text{sph}}(\alpha)$ denote the moduli space of 1-pointed genus 0 stable maps to $X$ in class $\alpha$. We have

$$\mathcal{M}_2^{L_1,L_2}(\beta_i + \alpha; \{a^{i_1,i_2}\} \times T^{n-1}, \{b^{i_1,i_2}\} \times T^{n-1}) \times_{L_1 \cap L_2} \{b_{12} \otimes 1_{T^{n-1}}\} \cong \left( \mathcal{M}_2^{L_1}(\beta_i; \{a^{i_1,i_2}\} \times T^{n-1}, \{b^{i_1,i_2}\} \times T^{n-1}) \times_{L_1 \cap L_2} \{b_{12} \otimes 1_{T^{n-1}}\} \right) \times_X \mathcal{M}_1^{\text{sph}}(\alpha),$$

and

$$\mathcal{M}_1(\beta_i^L + \alpha) \times L \{p\} \cong (\mathcal{M}_1(\beta_i^L) \times L \{p\}) \times_X \mathcal{M}_1^{\text{sph}}(\alpha),$$

where the subscript $(-1)$ means there is one interior marked point, and the fiber product over $X$ is taken using the interior evaluation maps. In both of the expression above, the first factor of the RHS is unobstructed (it is topologically a unit disc), and therefore the obstruction simply comes from $\mathcal{M}_1^{\text{sph}}(\alpha)$.


We are now ready to prove Theorem 3.4.

**Proof of Theorem 3.4.** \( m_{L_1, L_2}^i (a_{12} \otimes 1_{T^{n-1}}) \) is a linear combination of \( b_{12} \otimes 1_{T^{n-1}} \) and \( a_{12} \otimes X_i \), \( i = 1, \ldots, n - 1 \). The coefficient of \( a_{12} \otimes X_i \) is \( z_i - z_i' \) as per previous discussion. On the other hand, the coefficient of \( b_{12} \otimes 1_{T^{n-1}} \) is given by the fiber products \( \mathcal{M}_{L_1, L_2}^i (\beta_i + \alpha; \{ a_{i1}, a_{i2} \} \times T^{n-1}, \{ b_{i1}, b_{i2} \} \times T^{n-1}) \times L_1 \cap L_2 \{ b_{12} \otimes 1_{T^{n-1}} \} \), where \( \beta_i + \alpha \) is a stable holomorphic strip class in Lemma 3.6 and [CCL16]. For the strip class \( \beta_0 \), the fiber product is simply one point. Due to our choices of gauge cycles, \( \partial \beta_0 \) only intersects the gauge hypertorus dual to \( v_1 \) in \( L_2 \) (with intersection number 1). Thus, \( \beta_0 \) contributes the term \( -T^{\omega(\beta_0)} \).

Since \( L_1 \) and \( L_2 \) are chosen such that \( \beta_0 \) and \( \beta_1 \) have the same symplectic area, this equals to \( -T^{\omega(\beta_1)} \).

For \( \beta_i + \alpha \), \( i = 1, \ldots, m \), the number of holomorphic strips in the fiber product equals to the open Gromov-Witten invariant \( n_1 (\beta_i^L + \alpha) \) of a regular toric fiber \( L \) of \( X \) by Proposition 3.9. The boundary \( \partial \beta_i^L \) corresponds to the primitive generator \( u_i = (v_i - v_1) + v_1 \) and has holonomy \( z_{\beta_i} z_i^\beta \). The generating function of open Gromov-Witten invariants of \( L \) is given by

\[
\sum_{i=1}^m T^{\omega(\beta_i)} z_n z_i^\beta = \sum_{\alpha} n_1 (\beta_i^L + \alpha) T^{\omega(\alpha)}.
\]

Thus, the coefficient of \( b_{12} \otimes 1_{T^{n-1}} \) equals to

\[
T^{\omega(\beta_i)} (-1 + (z_i'^n)^{-1} z_n \sum_{i=1}^m T^{\omega(\beta_i - \beta_i)} z_i^{\beta_i^L} \sum_{\alpha} n_1 (\beta_i^L + \alpha) T^{\omega(\alpha)}) = T^{\omega(\beta_i)} (-1 + (z_i'^n)^{-1} z_n f(z_1, \ldots, z_{n-1})).
\]

Therefore, the cocycle condition \( m_{L_1, L_2}^i (a_{12} \otimes 1_{T^{n-1}}) = 0 \) implies the gluing formula (3.11). Here we have implicitly chosen \( L \) that \( \omega(\beta_i) = \omega(\beta_i^L) \) for \( i = 1, \ldots, m \). Conversely, the gluing formula (3.11) implies \( m_{L_1, L_2}^i (a_{12} \otimes 1_{T^{n-1}}) = 0 \), and similarly \( m_{L_1, L_2}^i (b_{21} \otimes 1_{T^{n-1}}) = 0 \).

\( m_{L_1, L_2, L_3}^i (b_{21} \otimes 1_{T^{n-1}}, a_{12} \otimes 1_{T^{n-1}}) \) and \( m_{L_1, L_2, L_3}^i (a_{12} \otimes 1_{T^{n-1}}, b_{21} \otimes 1_{T^{n-1}}) \) are given by the moduli spaces

\[
\mathcal{M}_{L_1, L_2, L_3}^i (\beta_i + \alpha; \{ a_{i1}, a_{i2} \} \times T^{n-1}, \{ b_{i1}, b_{i2} \} \times T^{n-1}) \times L_1 \{ 1_{L_1} \},
\]

and

\[
\mathcal{M}_{L_2, L_1, L_2}^i (\beta_i + \alpha; \{ b_{i1}, b_{i2} \} \times T^{n-1}) \times L_1 \{ 1_{L_2} \},
\]

where the target of the evaluation maps at the third marked points are \( L_1 \) and \( L_2 \), respectively. Since \( f_{L_1} \) and \( f_{L_2} \) were chosen such that the image of the maximum points \( 1_{L_1} \) and \( 1_{L_2} \) lie on the boundary of \( u_R \), the moduli spaces are empty except for \( \beta_0 \). In which case we get

\[
m_{L_1, L_2, L_3}^i (b_{21} \otimes 1_{T^{n-1}}, a_{12} \otimes 1_{T^{n-1}}) = T^{\omega(\beta_0)} 1_{L_1},
\]

and

\[
m_{L_1, L_2, L_3}^i (a_{12} \otimes 1_{T^{n-1}}, b_{21} \otimes 1_{T^{n-1}}) = T^{\omega(\beta_0)} 1_{L_2},
\]

where \( 1_{L_1} \) and \( 1_{L_2} \) are the maximum points on \( L_1 \) and \( L_2 \), respectively. This bears no effect on the gluing formula.

\[\square\]

**Remark 3.10.** Recall from Section 2.2.2 that the maximum points \( 1_{L_1} = 1_{L_1}^* \) and \( 1_{L_2} = 1_{L_2}^* \) are only homotopy units. Since \( L_1 \) and \( L_2 \) do not bound any non-constant holomorphic discs, we have

\[
m_{1_{L_1}}^i (1_{L_1}) = 1_{L_1} - 1_{L_1}^*, \quad i = 1, 2,
\]
where \( 1^*_{L_i} \) is the degree \(-1\) homotopy between \( 1^*_{L_i} \) and \( 1^*_{L_i} \). This means \( 1^*_{L_i} \) and \( 1^*_{L_i} \) are cohomologous to the strict units, and therefore the isomorphism equations (3.3) are indeed satisfied.

We next derive the gluing formula between \( L_i' \), \( i = 1, 2 \), and the immersed Lagrangian brane \( L_0 \). Let \((CF^*(L'_1, L_0), m_1^{L'_1, L_0})\) and \((CF^*(L'_2, L_0), m_1^{L'_2, L_0})\) be the pearl complexes associated to the perfect Morse functions \( f^{L'_1, L_0} \) and \( f^{L'_2, L_0} \), respectively, formally deformed by the flat connections \( \nabla(z_{1, \ldots, n}) \), \( \nabla(z_{1, \ldots, n}') \), and \( \nabla(z^{(0), \ldots, n-1}) \), and the weak bounding cochain \( b = uU + vV \). We denote by

\[
a_{10} \otimes 1_{T_n-1} \in CF^0(L'_1, L_0), \quad b_{10} \otimes 1_{T_n-1} \in CF^1(L'_1, L_0), \\
a_{02} \otimes 1_{T_n-1} \in CF^0(L_0, L'_2), \quad b_{02} \otimes 1_{T_n-1} \in CF^1(L_0, L'_2)
\]

the generators corresponding to the maximum points on \( \{a^f_{0,0} \} \times T^0, \{b^f_{0,0} \} \times T^0, \{a^f_{0,0} \} \times T^0, \) \( \{b^f_{0,0} \} \times T^0 \), respectively.

There are two holomorphic strip classes \( \beta_{L_0, L_i}' \), \( \beta_{R_0, L_i}' \) in \( \pi_2(X^*, L_0 \cup L_i) \) of Maslov index 1 with the input corner in \( \{a^f_{0,0} \} \times T^0 \) and output corner in \( \{b^f_{0,0} \} \times T^0 \). They are labeled such that the image in the \( w \)-plane of a holomorphic strip in class \( \beta_{L_0, L_i}' \) contains 0. We assume the Lagrangians are chosen such that \( \beta_{L_0, L_i}' \) and \( \beta_{R_0, L_i}' \) have the same symplectic area \( A \). There are exactly one holomorphic strip in each of these strip classes passing through \( b_{10} \otimes 1_{T_n-1} \) at the output corner. The lift of boundaries of these holomorphic strips in \( \tilde{L}_0 \cong S^2 \times T^{n-2} \) are curve segments in the \( S^2 \)-factor and points in the \( T^{n-2} \)-factor.

In particular, the lower arc of the holomorphic strip in class \( \beta_{L_0, L_i}' \) is a curve segment connecting \( (p, 1_{T_n-2}) \) and \( (q, 1_{T_n-2}) \). The perfect Morse function \( f^{S^2} \) on \( S^2 \) is chosen such that the two flow lines connecting \( p \) and \( q \) to the minimum point \( a_{S^2} \) do not intersect with these curve segments. (See [HKL18, Section 3.3].)

The proof of the following gluing formula is similar to that of [HKL18, Theorem 3.7], except that here we need to take care of the contributions of non-constant holomorphic discs of Maslov index 0 bounded by \( L_0 \) using \( T^{n-2} \)-equivariant perturbations introduced in [FOOO10].

---

**Theorem 3.11.** There exists a series \( g(u, z^{(0), \ldots, n-1}) \) such that \( a_{10} \otimes 1_{T_n-1} \in CF^0(L'_1, L_0) \) and \( a_{02} \otimes 1_{T_n-1} \in CF^0(L_0, L'_2) \) are isomorphisms if and only if

\[
\begin{align*}
z_1 &= z'_1 = g(u, z^{(0), \ldots, n-1}) \\
z_i &= z'_i = z^{(0)} \quad \text{for } i = 2, \ldots, n-1 \\
z_n' &= u^{-1} \\
z'_n &= v.
\end{align*}
\]

Moreover, \( z = g(u, z^{(0), \ldots, n-1}) \) satisfies

\[
u v = f(z, z^{(0), \ldots, n-1}),
\]

with the same series \( f \) as in (2.4) (or in Theorem 3.4).

---

**Proof.** Let us first consider the pair \((L'_1, L_0)\). The output of \( m_1^{L'_1, L_0} (a_{10} \otimes 1_{T_n-1}) \) is a priori a linear combination of \( b_{10} \otimes 1_{T_n-1} \) and \( a_{10} \otimes x_i \) for \( i = 1, \ldots, n-1 \), where \( x_1, \ldots, x_{n-1} \) are the degree one critical points in \( \{a^f_{0,0} \} \times T^0 \). Let \( \Gamma_{k+1} \in \Gamma_{k+1} \) denote the stable trees with exactly \( k \) input vertices and one interior vertex \( v \). The only possible stable trees contributing to \( b_{10} \otimes 1_{T_n-1} \) are the following:

---
Since all domain discs in $M$ underlying Kuranishi structure of the moduli spaces in (3.7) is the fiber product $\beta = \beta^0_L + \beta$, where $\beta$ is a stable disc class of Maslov index 0 bounded by $L_0$.

Stable trees $\Gamma$ in $L_0$ with at least one interior vertex and inputs $U$ and $V$ attached to the interior vertex $\Gamma$ in (1) or $\Gamma_{k+1}$ in (2) via flow lines.

For case (1), there is exactly one holomorphic strip in class $\beta^0_L + \beta$ with input corner in $(a^0 \omega_L) \times T^{n-1}$ and output corner intersecting $b_{10} \otimes 1_{T^{n-1}}$, similar to the proof of Theorem 3.4. This contributes $-TA$ to $b_{10} \otimes 1_{T^{n-1}}$.

For case (2), the moduli spaces are of the form

$$ M_{2k+1}(\beta^0_L + \beta; a_{10} \otimes 1_{T^{n-1}}, U, V, \ldots, U, V, U). $$

Since all domain discs in $M_{2k+1}(\beta^0_L + \beta)$ are singular (for $\beta = 0$, we require $k \geq 2$), the underlying Kuranishi structure of the moduli spaces in (3.7) is the fiber product

$$ M_3(\beta^0_L; a_{10} \otimes 1_{T^{n-1}}, U) \times_{L_0} M_{2k}(\beta; U, V, \ldots, U, V, U). $$

We choose $T^{n-2}$-equivariant Kuranishi structures for both factors of (3.8) so that their fiber product is also $T^{n-2}$-equivariant. Let $E_1 \oplus E_2$ denote the obstruction bundles on (3.8). We choose $T^{n-2}$-equivariant multi-sections $s_1, s_2, (s_1, s_2)$, of $E_1, E_2$ and $E_1 \oplus E_2$ (transversal to the zero section), respectively. By compatibility of perturbations, we have $s_1 = s_1$ and $(s_2)|_{s_1(n)} = s_2$. Since $s_2^{-1}(0)$ is a $T^{n-2}$-invariant chain with expected dimension $n - 3$, it must be the zero chain. This means $(s_1, s_2)^{-1}(0) = \emptyset$, and the moduli spaces in (3.7) do not contribute unless $\beta = 0$ and $k = 1$. There is exactly one holomorphic strip in $M_3(\beta^0_L; a_{10} \otimes 1_{T^{n-1}}, U)$ with output corner passing through $b_{10} \otimes 1_{T^{n-1}}$. This contributes $z_2 uTA$ to $b_{10} \otimes 1_{T^{n-1}}$

Finally, for case (3), if a stable tree $\Gamma$ in $L_0$ has at least two interior vertices and inputs $U$ and $V$, then by choosing $T^{n-2}$-equivariant perturbations for the fiber products at its interior vertices whose inputs are $U$ and $V$, the moduli space of such configurations is empty by the same argument as in Lemma 3.1. If a stable tree $\Gamma$ has exactly one interior vertex, then by choosing $T^{n-2}$-equivariant perturbations for the fiber product at the interior vertex, the output are free $T^{n-2}$-orbits situated near either $p \times T^{n-2}$ or $q \times T^{n-2}$, which do not flow to the boundary of holomorphic strip in class $\beta^0_L$ or $\beta^0_L$ intersecting $b_{10} \otimes 1_{T^{n-1}}$ due to our choice of the Morse function $f^{S^2}$. Therefore, the coefficient of $b_{10} \otimes 1_{T^{n-1}}$ is $(u - z_2^{-1})TA$.

Let us now consider the coefficient of $a_{10} \otimes X_i$, $i = 1, \ldots, n - 1$. We have a pair of Morse flow lines from $a_{10} \otimes 1_{T^{n-1}}$ to $a_{10} \otimes X_i$. For $i = 2, \ldots, n - 1$, this gives $(z_i - z_i^{(0)}) \cdot a_{10} \otimes X_i$. For $i = 1$, one of the flow lines intersects the gauge hypertorus dual to $v'_1$ in $L_1$ and contributes $z_1$ to $a_{10} \otimes X_1$. While the other flow line does not intersect with any gauge cycles, it can however be attached by Maslov index 0 polygons bounded by $L_0$ via an isolated Morse flow line from $(p) \times T^{n-2}$ or $(q) \times T^{n-2}$ to $(a^{*L_2}) \times T^{n-1}$. Recall that the corners $U$ and $V$ must appear in pairs for a stable polygon bounded by $L_0$, since its boundary is contained in the the immersed loci. Thus, this contributes a series
each of the degree 2 critical points in $L$ Morse flow trees with repeated inputs way to compute since the moduli spaces involved are highly obstructed. However, Theorem 3.11 gives a contributions come from Morse theory. There are two flow lines from $X$ respectively. Let $L$

Proof. \[ \text{Lemma 3.13.} \]

Similarly, for the pair $(L_0, L'_2)$, we have

\[
m_1^{L_1, L_0, L'_2}(a_{02} \otimes 1_{T^{n-1}}) = \left( (z'_n)^{-1} \gamma - 1 \right) T^A \cdot b_{02} \otimes 1_{T^{n-1}} + \left( z'_1 - \hat{g}(uv, z'_2, \ldots, z'_{n-1}) \right) \cdot a_{02} \otimes X'_1 + \sum_{i=2}^{n-1} (z'_i - z'_i) \cdot a_{02} \otimes X'_i
\]

for some series $\hat{g}$, where $X'_1, \ldots, X'_{n-1}$ are the degree one critical points in $\{a_{0, L_2}^t \} \times T^{n-1}$, $a_{10} \otimes 1_{T^{n-1}}$ and $a_{02} \otimes 1_{T^{n-1}}$ are isomorphisms if and only if the coefficients in (3.9) and (3.10) are zero.

We have $m_2^{L_1, L_0, L'_2}(a_{02} \otimes 1_{T^{n-1}}, a_{10} \otimes 1_{T^{n-1}}) = T^A \cdot a_{12} \otimes 1_{T^{n-1}}$ contributed by a holomorphic section of symplectic area $\Delta$ over the triangle in the $w$-plane with corners $a_{10}^t, a_{02}^t, a_{12}^t$. Recall from Theorem 3.4 that $a_{12} \otimes 1_{T^{n-1}}$ is an isomorphism between $L_1$ and $L_2$ if only if $z'_i = z_i$ for $i = 1, \ldots, n-1$ and $z'_n = z_n \cdot f(z_1, \ldots, z_{n-1})$. Thus, we have $g = z_1 = z'_1 = \hat{g}$, and

\[
uv = z'_n z'^{-1} = f(z_1, \ldots, z_{n-1}) = f(g, z'_2, \ldots, z'_{n-1}).
\]

The series $g$ is the generating function of Maslov index 0 stable polygons bounded by $L_0$. The stable polygons can attach to an isolated Morse trajectory in $L_0$ flowing into $\{a_{0, L_2}^t \} \times T^{n-1} \subset L_1 \cap L_0$ and contribute to $a_{10} \otimes X_1$. It is difficult to compute $g$ directly since the moduli spaces involved are highly obstructed. However, Theorem 3.11 gives a way to compute $g$ by solving for $z_1$ from the equation $uv = f(z_1, \ldots, z_{n-1})$.

**Example 3.12.** Let $X = \mathbb{C}^3$. We have

\[
uv = z'_3 z'^{-1} = 1 + z_1 + z_2,
\]

and hence, $g(uv, z'_2) = z_1 = uv - z_2 - 1$.

**3.1.4 Another deformation families $L_1$ and $L_2$.** Let $X_1 \in \text{Crit}(f^{L_1})$ and $X'_1 \in \text{Crit}(f^{L_2})$ be the degree 1 critical points whose unstable chain is the hypertori dual to $v'_1$ in $L_1$ and $L_2$, respectively. Let $x_1, x'_1 \in \Lambda_+$. We denote the family of formal Lagrangian deformations $(L_1, \nabla^{(1, x_2, \ldots, x_n)}, x_1 X_1)$ and $(L_2, \nabla^{(1, x'_2, \ldots, x'_n)}, x'_1 X'_1)$ by $L_1$ and $L_2$, respectively. The reason for introducing these new deformation families is that they will fit more into our $S^1$-equivariant theory later.

**Lemma 3.13.** $L_1$ and $L_2$ are unobstructed.

**Proof.** Since $L_1$ and $L_2$ bound no non-constant holomorphic discs in $X^c$, the only possible contributions come from Morse theory. There are two flow lines from $X_1$ (resp. $X'_1$) to each of the degree 2 critical points in $L_1$ (resp. $L_2$) which cancel with each other. The Morse flow trees with repeated inputs $X_1$ (resp. $X'_1$) are empty for generic perturbations. \[ \square \]
In [KLZ19, Theorem 3.7], a particular type of perturbations was used for the computation of the toric superpotential so that the divisor axiom holds, and we have the change of variables $z = e^x$. Below, we choose analogous perturbations for the gluing formulas between the Lagrangian branes $L_0$, $L_1$, and $L_2$.

**Theorem 3.14.** $a_{12} \otimes 1_{T^n-1}$ is an isomorphism between $L_1$ and $L_2$ with an inverse $b_{21} \otimes 1_{T^n-1}$ if and only if

\[
e^{x_1} = e^{x_i}, \quad z_i' = z_i, \quad i = 2, \ldots, n - 1, \quad z_n' = z_n \cdot f(e^{x_1}, \ldots, z_{n-1}),
\]

where $f$ is given by (2.4).

**Proof.** The proof parallels that of Theorem 3.4, so we will simply highlight the parts that depend on such choices of perturbations.

$m_{1_1,1}^L(b_{12} \otimes 1_{T^n-1}, a_{12} \otimes X_i, i = 1, \ldots, n - 1)$ is a linear combination of $b_{12} \otimes 1_{T^n-1}$ and $a_{12} \otimes X_i$, $i = 1, \ldots, n - 1$. The coefficient of $a_{12} \otimes X_i$ for $i = 2, \ldots, n - 1$ is $z_i - z_i'$ as before. For $a_{12} \otimes X_1$, there are two Morse flow lines $\gamma_1$ and $\gamma_2$ from $a_{12} \otimes 1_{T^n-1}$. The flow line $\gamma_1$ which originally contributed $z_1$ in the setting of Theorem 3.4 can now intersect with flow lines from $k$ copies of $X_1$ in $L_1$ at a point and then follow to $a_{12} \otimes X_1$, forming Morse flow trees. The moduli spaces of such configurations are not transverse for $k > 1$, since the unstable chain of $X_1$ (which is the hypertorus dual in $L_1$) is not transverse with itself. Therefore, we have to choose transversal perturbations. For $k > 1$, let $X_1^{(1)}, \ldots, X_1^{(k)}$ be disjoint small perturbations of the hypertorus along the direction of $\gamma_1$ (and ordered along that direction). We choose the perturbation to be the average of all the cyclic permutations of $X_1^{(1)}, \ldots, X_1^{(k)}$, resulting in the count $1/k!$. These configurations together contribute $e^{x_1}$ to $a_{12} \otimes X_1$. Similarly, the Morse flow trees formed by the flow line $\gamma_2$ and the flow lines from copies of $X_1^{(1)}$ contribute $e^{x_1}$ to $a_{12} \otimes X_1$

For the coefficient of $b_{12} \otimes 1_{T^n-1}$, the strip class $\beta_0$ contributes $-T^{\omega(\beta_0)}$ as before. On the other hand, flow lines from $k$ copies of $X_1$ can now attach to the upper arc (in $L_1$) of stable strips in classes $\beta_i + \alpha$, $i = 1, \ldots, m$, forming pearly trees. Again, these fiber products are not transversal. The transversal perturbations we choose is the following. For the strip class $\beta_i + \alpha$ and $k > 1$, let $X_1^{(i,1)}, \ldots, X_1^{(i,k)}$ be disjoint small perturbations of the hypertorus along the direction of $\partial \beta_i$ (and ordered along that direction). We choose the perturbation of the fiber product to be the average of all the cyclic permutations of $X_1^{(i,1)}, \ldots, X_1^{(i,k)}$. By choosing such perturbations, we get $T^{\omega(\beta_1)} \cdot (\partial x_i) \cdot z_n f(e^{x_1}, \ldots, z_{n-1}) \cdot b_{12} \otimes 1_{T^n-1}$.

Finally, we can eliminate the possibility of Morse flow trees with repeated inputs $X_1$ attaching to the above configurations, since the moduli space of these flow trees are empty for generic perturbations.

By choosing similar perturbations as in the proof above, we also have the gluing formulas between $L_1$ and $L_0$.

**Theorem 3.15.** $a_{10} \otimes 1_{T^n-1} \in CF^0(L_1, L_0)$ and $a_{02} \otimes 1_{T^n-1} \in CF^0(L_0, L_2)$ are isomorphisms if and only if

\[
e^{x_1} = e^{x_i} = g(uv, z_2^{(0)}, \ldots, z_{n-1}^{(0)}), \quad z_i = z_i' = z_i^{(0)} \quad \text{for } i = 2, \ldots, n - 1, \quad z_n = u^{-1} \quad \text{and} \quad z_n' = v. \tag{3.12}
\]
Moreover, \( z = g(uv, z_2^{(0)}, \ldots, z_{n-1}^{(0)}) \) satisfies

\[
u v = f(z, z_2^{(0)}, \ldots, z_{n-1}^{(0)}),
\]

where the series \( f \) is same as in (2.4) and Theorem 3.4, and the series \( g \) as in Theorem 3.11.

### 3.2 \( S^1 \)-equivariant disc potential of \( L_0 \)

For the \( T^n \)-action on a compact toric semi-Fano manifold \( X \) with \( \text{dim}_C X = n \), the equivariant disc potential \( W_{T^n} \) of regular toric fibers in \( X \) was computed in [KLZ19]. It recovers the \( T^d \)-equivariant toric Landau-Ginzburg mirror of Givental [Giv98], namely,

\[
W_{T^n} = W(e^{x_1}, \ldots, e^{x_n}) + \sum_{i=1}^{n} x_i \lambda_i,
\]

Here, \( W \) is the Givental-Hori-Vafa superpotential [Giv98, HV00, LLY99], \( x_1, \ldots, x_n \) are parameters for the boundary deformations by hypertori, and \( \lambda_1, \ldots, \lambda_n \) are equivariant parameters generating \( H^*_T(\text{pt}; \Lambda) = \Lambda[\lambda_1, \ldots, \lambda_n] \). The \( S^1 \)-equivariant disc potential for immersed 2-sphere \( S^2 \) was also computed in [KLZ19] in the unobstructed setting using gluing formulas.

In this section, we study the \( S^1 \)-equivariant disc potential of the immersed SYZ fiber \( L_0 \cong S^2 \times T^{n-2} \) in a toric Calabi-Yau \( n \)-fold \( X \). The Lagrangians \( L_0, L_1 \) and \( L_2 \) and their pair-wise clean intersections are invariant under the \( T^{n-1} \)-action rotating along the directions of \( v'_1, \ldots, v'_{n-1} \). The sub \( S^1(\leq T^{n-1}) \)-action of our interest is the rotation along the \( v'_1 \)-direction. It acts on \( L_1 \) and \( L_2 \) by rotating the second \( S^1 \)-factor and acts on \( L_0 \) by rotating the \( S^2 \)-factor fixing the nodal point.

For \( i = 0, 1, 2 \), let \( (CF^*_S(L_i), m^{(L_i; S^1)}) \) the \( S^1 \)-equivariant Morse model such that

\[
CF^*_S(L_i) = CF^*(L_i; H^*_S(\text{pt}; \Lambda)),
\]

where \( CF^*(L_i; H^*_S(\text{pt}; \Lambda)) \) is generated by critical points of the Morse function \( f^L_i \) as explained in 2.3.2. Here, we suppress the superscript \( \dagger \) for simplicity of notations.

Let us make a decomposition \( L_i = S^1 \times T^{n-1} \) for \( i = 1, 2 \) so that \( S^1 \) acts freely on the \( S^1 \)-factor and trivially on the \( T^{n-1} \)-factor. Then

\[
(L_i)_{S^1} = (S^1)_{S^1} \times T^{n-1},
\]

and

\[
(L_0)_{S^1} = (S^2)_{S^1} \times T^{n-2},
\]

(Recall \( L_G = L \times C \text{EG} \)). We have

\[
\pi_1((L_i)_{S^1}) \cong \pi_1(T^{n-1}) = \mathbb{Z} \cdot \{v_1, v'_2, \ldots, v'_{n-1}\}, \quad i = 0, 1, 2.
\]

The flat \( \Lambda_U \)-connections \( \nabla^{(z_2^{(0)} \ldots z_{n-1}^{(0)})}, \nabla^{(1, z_2, \ldots, z_n)} \), and \( \nabla^{(1, \ast_{2}^{(0)} \ast_{n})} \) are \( S^1 \)-equivariant, and thus induce flat connections on \( (L_i)_{S^1} \) for \( i = 0, 1, 2 \).

Let \( X_1 \in CF^*_S(L_1), X'_1 \in CF^*_S(L_2), \) and \( U, V \in CF^*_S(L_0) \). We denote by \( (L_0, S^1), (L_1, S^1) \) and \( (L_2, S^1) \) the families of formal deformations of \( (L_0, S^1) \) by \( \nabla^{(z_2^{(0)} \ldots z_{n-1}^{(0)})} \) and \( uU + vV \) for \( u, v \in \Lambda^2 \) with \( \text{val}(u \cdot v) > 0 \), \( (L_1, S^1) \) by \( \nabla^{(1, z_2, \ldots, z_n)} \) and \( x_1X_1, \; x_1 \in \Lambda_+ \), and \( (L_2, S^1) \) by \( \nabla^{(1, \ast_{2}^{(0)} \ast_{n})} \) and \( x'_1X'_1, \; x'_1 \in \Lambda_+ \), respectively. By Corollary 2.11, Lemma 3.1 and Lemma
3.13, \((\mathbb{L}_i, S^1)\) is weakly unobstructed for \(i = 0, 1, 2\). This implies that

\[
m_0^{(\mathbb{L}_0, S^1)}(1) = W^{\mathbb{L}_0}_S(u, v, z_{2}^{(0)}, \ldots, z_{n}^{(0)}) \cdot 1_{\mathbb{L}_0},
\]

\[
m_0^{(\mathbb{L}_1, S^1)}(1) = W^{\mathbb{L}_1}_S(x_1, z_{2}, \ldots, z_{n}) \cdot 1_{\mathbb{L}_1},
\]

\[
m_0^{(\mathbb{L}_2, S^1)}(1) = W^{\mathbb{L}_2}_S(x_1', z_{2}', \ldots, z_{n}') \cdot 1_{\mathbb{L}_2}.
\]

for some series \(W^{\mathbb{L}_i}_S\) \((i = 0, 1, 2)\). We will refer to the functions \(W^{\mathbb{L}_i}_S\) as the \(S^1\)-equivariant disc potential of \(\mathbb{L}_i\).

Since \(L_1\) does not bound any non-constant holomorphic discs in \(X\), the following is an immediate consequence of [KLZ19, Lemma 3.4].

**Proposition 3.16.** The \(S^1\)-equivariant disc potential \(W^{\mathbb{L}_1}_S\) of \(\mathbb{L}_1\) is given by

\[
W^{\mathbb{L}_1}_S = \lambda_1 x_1,
\]

where \(\lambda_1\) is the equivariant parameter of the \(S^1\)-action, i.e., \(H^\bullet_S(pt; \Lambda) = \Lambda[\lambda_1]\).

Let \((C^\bullet_S(L_1, \mathbb{L}_0), m^{(\mathbb{L}_1, S^1)}, (\mathbb{L}_0, S^1))\) and \((C^\bullet_S(L_0, \mathbb{L}_2), m^{(\mathbb{L}_0, S^1), (\mathbb{L}_2, S^1)})\) be the \(S^1\)-equivariant Morse models with

\[
(C^\bullet_S(L_1, \mathbb{L}_0) = CF^\bullet((\mathbb{L}_1, \mathbb{L}_0); H^\bullet_S(pt; \Lambda))
\]

and

\[
(C^\bullet_S(L_0, \mathbb{L}_1) = CF^\bullet((\mathbb{L}_0, \mathbb{L}_2); H^\bullet_S(pt; \Lambda)).
\]

The gluing formulas between \((\mathbb{L}_0, S^1)\), \((\mathbb{L}_1, S^1)\) and \((\mathbb{L}_2, S^1)\) remain the same as before.

**Proposition 3.17.** \(a_{10} \otimes 1_{T^{n-1}} \in CF^0_S(L_1, \mathbb{L}_0)\) and \(a_{02} \otimes 1_{T^{n-1}} \in CF^0_S(L_0, \mathbb{L}_2)\) are isomorphisms if and only if

\[
e^{x_1} = e^{x_1} = g(uv, z_{2}^{(0)}, \ldots, z_{n}^{(0)})
\]

\[
z_i = z_i' = z_i^{(0)} \text{ for } i = 2, \ldots, n - 1
\]

\[
z_n = u^{-1}
\]

\[
z_n' = v.
\]

Moreover, \(z = g(uv, z_{2}^{(0)}, \ldots, z_{n}^{(0)})\) satisfies

\[
u v = f(z, z_{2}^{(0)}, \ldots, z_{n-1}^{(0)}),
\]

where the series \(f\) is the same as in (2.4) (or in Theorem 3.4), and hence, \(g\) equals the one in Theorem 3.11.

**Proof.** Since the equivariant parameter \(\lambda_1\) has degree 2, it does not appear in the isomorphism equations (3.3). Thus, we have

\[
m_1^{(\mathbb{L}_1, S^1), (\mathbb{L}_0, S^1)}(a_{10} \otimes 1_{T^{n-1}}) = m_1^{(\mathbb{L}_1, \mathbb{L}_0)}(a_{10} \otimes 1_{T^{n-1}}),
\]

\[
m_1^{(\mathbb{L}_0, S^1), (\mathbb{L}_1, S^1)}(a_{02} \otimes 1_{T^{n-1}}) = m_1^{(\mathbb{L}_0, \mathbb{L}_1)}(a_{02} \otimes 1_{T^{n-1}}),
\]

\[
m_2^{(\mathbb{L}_1, S^1), (\mathbb{L}_0, S^1)}(a_{02} \otimes 1_{T^{n-1}}, a_{10} \otimes 1_{T^{n-1}}) = m_2^{(\mathbb{L}_1, \mathbb{L}_0)}(a_{02} \otimes 1_{T^{n-1}}, a_{10} \otimes 1_{T^{n-1}}).
\]

The statement then follows from Theorem 3.11 and Theorem 3.15.

Since the disc potentials are compatible with the gluing formulas, we can compute \(W^{\mathbb{L}_0}_S\) in terms of \(g\) making use of the propositions above.
Theorem 3.18. The $S^1$-equivariant disc potential $W_{S^1}^{L_0}$ of $L_0$ is given by

$$W_{S^1}^{L_0} = \lambda_1 \log g(uv, z_0, \ldots, z_{n-1}).$$  \hspace{1cm} (3.14)

Remark 3.19. The expression $\log g(uv, z_0, \ldots, z_{n-1})$ depends on choices of a framing $\{v'_1, \ldots, v'_{n-1}\}$.

In dimension three, by setting $u = v = 0$, we obtain the equivariant term $\log g(0, z^0_2)$. Integration of this term is the disc potential of the Aganagic-Vafa brane [AKV02, GZ02]. Our formulation has the advantage that it works in all dimensions.

3.3 Examples. Before we finish the section, we provide explicit numerical computations for a few interesting toric Calabi-Yau manifolds in various dimensions.

3.3.1 $X = K_{P^2}$. In this case, the primitive generators of the fan $\Sigma$ are

$$v_1 = (0,0,1), \quad v_2 = (1,0,1), \quad v_3 = (0,1,1), \quad v_4 = (-1,-1,1).$$

Let $L_0 \cong S^2 \times S^1$ be an immersed Lagrangian at the ‘inner branch’ dual to the cone $\mathbb{R}_{\geq 0}(v_1, v_2)$, which bounds a primitive holomorphic disc $\beta$ with area $A$, as depicted in Figure 6. The area of the curve class is denoted by $\tau$. We fix the compact chamber dual to $v_1$ and the framing $\{v'_i = v_{i+1} - v_1 \mid i = 1, 2\}$ to compute the $S^1$-equivariant disc potential of $L_0$. Note that different choices of a chamber and a framing give different Morse functions on $L_0$, and the disc potentials will change accordingly. The choice of chamber matters for $(uv)$ but not the discs with no corners.

The gluing formula between $L_0$ and a regular toric fiber $L_1$ reads

$$uv = (1 + \delta(T^T)) + \exp x_1 + T^A z_2 + T^{\tau-A} \exp(-x_1) \cdot z_2^{-1}$$

where

$$\delta(T^T) = -2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + O(q^6), \quad q = T^T,$$

is the generating function of open Gromov-Witten invariants bounded by a regular toric fiber of $K_{P^2}$ computed in [CLL12].

However, notice that the left hand side lies in $\Lambda_+$. The right hand side has leading term 2 and hence does not lie in $\Lambda_+$. Hence the gluing formula has empty solution. It means that the formally deformed Lagrangian $L_1$ is not isomorphic to the formally deformed immersed Lagrangian $L_0$.

To remedy this, we take a non-trivial spin structure of the tori $L_1$ and $L_2$ in the $v'_1$-direction. This systematically introduces extra signs to the orientations of moduli spaces. Formally it gives the change of coordinates $x_1 \mapsto x_1 + i\pi$. Then the gluing formula becomes

$$uv = (1 + \delta(T^T)) - \exp(x_1) + T^A z_2 - T^{\tau-A} \exp(-x_1) \cdot z_2^{-1}.$$ 

Both sides are now in $\Lambda_+$. We can then solve $\exp(x_1)$ in terms of $uv$ and $z = T^A z_2$, which gives the series $g$ in Theorem 3.18.
Figure 6. Immersed Lagrangian branes in $K_{\mathbb{P}^2}$

A direct calculation gives that the coefficient of $\lambda_1$ equals to $a_0 + a_1uv + a_2 \cdot (uv)^2 + O((uv)^3)$ where

$$a_0 = \left(z - \frac{z^2}{2} + \frac{z^3}{3} + O(z^4)\right) + \left(-z^{-1} - z + 2z^2 - 3z^3 + O(z^4)\right)q$$

$$+ \left(-\frac{3}{2}z^{-2} + 2z^{-1} + 5z - \frac{27}{2}z^2 + 27z^3 + O(z^4)\right)q^2 + O(q^3)$$

$$a_1 = (-1 + z - z^2 + z^3 + O(z^4)) + (-2z^{-1} + 4 - 8z + 14z^2 - 22z^3 + O(z^4))q$$

$$+ (-6z^{-2} + 18z^{-1} - 41 + 92z - 189z^2 + 356z^3 + O(z^4))q^2 + O(q^3)$$

$$a_2 = \left(-\frac{1}{2} + z - \frac{3}{2}z^2 + 2z^3 + O(z^4)\right) + \left(-3z^{-1} + 10 - 24z + 48z^2 - 85z^3 + O(z^4)\right)q$$

$$+ \left(-15z^{-2} + 66z^{-1} - 196 + 489z - 1080z^2 + 2170z^3 + O(z^4)\right)q^2 + O(q^3).$$

This differs by $z \mapsto -z$ from physicists’ convention, which results from different ways of parametrizing the flat connections. As noted in [AKV02], all the $z^j$-coefficients of the leading order term $(q^0)$ in $a_0$ multiplied by $j$ are integers. This reflects that the automorphism group of a holomorphic disc in class $j\beta$ ($\beta$ is the primitive disc class) is of order $j$.

On the other hand, we note that all the $z^j$-coefficients in $\ell \cdot a_{\ell}$ are integers in the examples we have computed.

We write the series as $\sum -a_{j\ell}(-z)^j(-q)^k(\nu \nu)^\ell$ and record the coefficients in the following table.
3.3.2 Outer branes in $K_{p3}$. We may also consider an outer brane in $K_{p3}$, corresponding to the codimension-two strata dual the cone $\mathbb{R}_{\geq 0}(v_2,v_3)$. We fix the chamber dual to the generator $v_3$ and fix the framing $\{v_2 - v_3, v_1 - v_3\}$ (corresponding to the holonomy variables $\tilde{z}_1$ and $\tilde{z}_2$ respectively). The gluing formula becomes

$$uv = (1 + \delta) T^4 \tilde{z}_2 - \exp(\tilde{x}_1) + 1 - T^{r+3A} \exp(-\tilde{x}_1) \tilde{z}_2^3,$$

where $A$ is the area of the primitive holomorphic disc bounded by the outer brane.

Putting $z = T^4 \tilde{z}_2$ and writing the series as $\sum -a_{jk\ell}(z)^j(-q)^k(uv)^\ell$, record the coefficients in the following table.

| ord$(uv) = 0$ | ord$(uv) = 1$ | ord$(uv) = 2$ |
|---------------|---------------|---------------|
| ord$(z)$     | ord$(q)$     | ord$(z)$     | ord$(q)$     | ord$(z)$     | ord$(q)$     |
| 0 1 2 3      | 0 1 2 3      | 0 1/2 0 0    | 0 1/2 0 0    | 0 1/2 0 0    | 0 1/2 0 0    |
| 1 1 2 5 32   | 1 1 2 5 32   | 1 1 2 5 32   | 1 1 2 5 32   | 1 1 2 5 32   | 1 1 2 5 32   |
| 2 1/2 2 7 42 | 2 1 4 14 84  | 2 3/2 6 21 126 | 2 3/2 6 21 126 | 2 3/2 6 21 126 | 2 3/2 6 21 126 |
| 3 1/3 3 9 164/3 | 3 1 8 27 164 | 3 2 15 54 328 | 3 2 15 54 328 | 3 2 15 54 328 | 3 2 15 54 328 |
| 4 1/4 4 15 80 | 4 1 14 56 310 | 4 5/2 32 134 760 | 4 5/2 32 134 760 | 4 5/2 32 134 760 | 4 5/2 32 134 760 |

Note that the first column (for ord$(q) = 0$) are the same for both branes. This reflects the fact that the holomorphic polygons (without sphere bubbling) are the same.

3.3.3 $X = K_{p3}$. Our method also works for higher dimensions. For instance, the toric Calabi-Yau 4-fold $X = K_{p3}$ is associated with the fan $\Sigma$ which has generators

$$v_1 = (0,0,0,1), v_2 = (1,0,0,1), v_3 = (0,1,0,1), v_4 = (0,0,1,1), v_5 = (-1,-1,-1,1).$$

Let $L_0 = S^2 \times T^2$ be an inner brane at the codimension-two toric stratum corresponding to the cone $\mathbb{R}_{\geq 0}(v_1,v_2)$ (a toric divisor of $\mathbb{P}^3$). We fix the chamber dual to $v_1$ and the basis $\{v_2 - v_1, v_3 - v_1, v_4 - v_1\}$.

The SYZ mirror of $K_{p3}$ is given by

$$uv = (1 + \delta(T^4)) + z_1 + z_2 + z_3 + T^{r}z_1^{-1}z_2^{-1}z_3^{-1}$$

where $1 + \delta(T^4)$ is the generating function counting stable discs bounded by a Lagrangian toric fiber. By [CLT13, CCLT16], it can be computed from the mirror map as follows. The mirror map for $K_{p3}$ is given by $q = Q e^{i(T^4)}$, where $q = T^4$ is the Kähler parameter for
the primitive curve class in $K_{\mathbb{P}^3}$, $Q$ is the mirror complex parameter and $f(Q)$ is the hypergeometric series

$$f(Q) = \sum_{k=1}^{\infty} \frac{(4k)!}{k!(k!)^4} Q^k.$$ 

Taking the inverse, we get the inverse mirror map

$$Q(q) = q - 24q^2 - 396q^3 - 39104q^4 - 4356750q^5 - O(q^6).$$

Then the generating function of open Gromov-Witten invariants of a Lagrangian toric fiber is given by

$$1 + \delta(q) = \exp(\frac{f(Q(q))}{4}) = 1 + 6q + 189q^2 + 14366q^3 + 1518750q^4 + O(q^5).$$

(see also [CLLT17, Theorem 1.1]).

Now we use the above method to deduce from this the $S^1$-equivariant disc potential for the immersed $L_0$. Namely, the gluing formula between a smooth torus and $L_0$ is given by substituting $z_1 = -e^{x_1}$ (and replacing $z_2, z_3$ by $T^{A_2} z_2, T^{A_3} z_3$ respectively) in the above equation for the SYZ mirror. Then we solve $x_1$ in terms of $z_2, z_3, u, v$, and substitute into $x_1 \lambda_1$ (which is the $S^1$-equivariant disc potential for $L_1$). The following table shows the leading coefficients $a_{jk\mu}$ of the generating function $\sum -a_{jk\mu} z_1^j z_2^k q^\mu$ for $\mu = 0$ (corresponding to stable discs with no corners).

| ord($q$) = 0 | ord($q$) = 1 |
|--------------|--------------|
| ord($z_1$) | ord($z_2$) | ord($z_1$) | ord($z_2$) |
| 0 | 1 | 2 | 3 | -1 | 0 | 1 | 2 |
| 1 | 1 | 1 | 1 | -1 | 1 | 2 | 3 | 4 |
| 2 | 1/2 | 1 | 3/2 | 2 | 0 | 2 | 6 | 12 | 20 |
| 3 | 1/3 | 1 | 2 | 10/3 | 1 | 3 | 12 | 30 | 60 |

| ord($q$) = 2 |
|--------------|
| ord($z_1$) | ord($z_2$) |
| -2 | 3/2 | 6 | 15 | 30 | 105/2 |
| -1 | 6 | 36 | 108 | 246 | 480 |
| 0 | 15 | 108 | 387 | 1020 | 2250 |
| 1 | 30 | 246 | 1020 | 3060 | 7560 |
| 2 | 105/2 | 480 | 2250 | 7560 | 20685 |

3.3.4 Local Calabi-Yau surfaces. We next consider the local Calabi-Yau surface $X_{(d)}$ of type $\tilde{A}_{d-1}$. The surface can be realized as the $\mathbb{Z}$-quotient of the following infinite-type toric Calabi-Yau surface. Such a toric construction was found by Mumford [Mum72, AMRT10] and its mirror symmetry was studied by Gross-Siebert [GS16, ABC+09].

Let $N = \mathbb{Z}^2$. For $i \in \mathbb{Z}$, define the cone

$$\sigma_i = \mathbb{R}_{\geq 0} \cdot \langle (i, 1), (i + 1, 1) \rangle \subset N_{\mathbb{R}}.$$ 

The fan $\Sigma_0 = \bigcup_{i \in \mathbb{Z}} \sigma_i \subset \mathbb{R}^2$ is defined as the infinite collection of these cones (and their boundary cones). The corresponding toric surface $X = X_{\Sigma_0}$ is Calabi–Yau since all the primitive generators $(i, 1) \in N$ have second coordinates being 1.

The fan $\Sigma_0$ has an obvious symmetry of the infinite cyclic group $\mathbb{Z}$ given by $k \cdot (a, b) = (a + k, b)$ for $k \in \mathbb{Z}$ and $(a, b) \in N$. We can take an open neighborhood $X^o$ of the toric divisors which is invariant under the $\mathbb{Z}$-action, and take the quotient $X_{(d)} = X^o / (d \cdot \mathbb{Z})$. 

The SYZ mirror of $X(d) = X^o/(d \cdot \mathbb{Z})$ was constructed in [KL19]. For simplicity, we examine the case $d = 1$, only. The SYZ mirror is given by

$$uv = \prod_{i=1}^{\infty}(1 + q^i z^{-1}) \prod_{j=0}^{\infty}(1 + q^j z).$$

The right hand side can be rewritten as

$$\prod_{k=1}^{\infty} \frac{1}{1 - q^k} \cdot \sum_{\ell = -\infty}^{\infty} q^{-\frac{(\ell-1)}{2}} z^\ell = \frac{e^{\frac{\pi i}{2}}}{\eta(\tau)} \cdot \vartheta \left( \frac{\zeta - \tau \cdot \bar{\tau}}{2i} \right)$$

where $q := e^{2\pi i \tau}$, $z := e^{2\pi i \zeta}$, $\eta$ is the Dedekind eta function, and $\vartheta$ is the Jacobi theta function.

Now, we consider the equivariant disc potential of the immersed 2-sphere $S^2$. We have fixed the chamber dual to $(0,1)$ and the framing $v_1 = (1,0)$ for the equivariant Morse model of $S^2$. The above mirror equation can be understood as the relation between the immersed variables $u,v$ of the immersed sphere $S^2$ and the formal deformations $x$ of the torus $T^2$, where $z = -e^x$. The $S^1$-equivariant disc potential of $T^2$ is given by $x \lambda$, and that of $S^2$ can be found by solving $x$ in the above equation.

We solve the equation order-by-order in $q = T^t$ where $t$ is the area of the primitive holomorphic sphere class. Namely, the equation is rewritten as

$$(1 - uv - e^x) + (uv - e^{-x} + e^{2x})q + O(q^2) = 0.$$ 

Then we put $x = \sum_{k \geq 0} h_k(uv)q^k$ and solve $h_k$ order by order. For instance, the leading term is $h_0 = \log(1 - uv) = -\sum_{i \geq 0} \frac{(uv)^i}{i}$. Write the generating function as $-\sum a_{k\ell}(uv)^\ell q^k$ and the coefficients are shown in the following table.

| ord(uv) | 0 | 1 | 2 | 3 | 4 | 5 |
|---------|---|---|---|---|---|---|
| ord(q) |   |   |   |   |   |   |
| 0       | 0 | 0 | 0 | 0 | 0 | 0 |
| 1       | 1 | 2 | 5 | 10| 20| 36|
| 2       | 1/2| 2 | 7 | 20| 105/2| 126|
| 3       | 1/3| 3 | 18| 245/3| 315| 1071|
| 4       | 1/4| 4 | 33| 192| 1815/2| 3696|
| 5       | 1/5| 5 | 55| 410| 2415| 60252/5|

The first column records the counts of constant polygons and it is the same as the result for $X = \mathbb{C}^2$ in [KLZ19, Theorem 4.6] (namely they are coefficients of the series $-\log(1 - uv)$). The constant polygons are merely local and are not affected by the presence of the holomorphic $A_1$-fiber. We also note that the coefficients multiplied with the corresponding orders of $(uv)$ are integers.

### 3.3.5 The total space of a family of Abelian surfaces.

This is the three-dimensional version of the last example. Consider the fan $\Sigma$ consisting of the maximal cones

$$\langle (i,j,1), (i+1,j,1), (i,j+1,1), (i+1,j+1,1) \rangle, \quad i, j \in \mathbb{Z}.$$ 

It admits an action by $\mathbb{Z}^2$: $(k,\ell) \cdot (a,b,c) = (a+k, b+\ell, c)$ on $N$. A crepant resolution is obtained by refining each maximal cone (which is a cone over a square) into 2 simplices, and the refinement is taken to be $\mathbb{Z}^2$-invariant. Then we take a toric neighborhood $X^o_\Sigma$ of
the toric divisors where $\mathbb{Z}^2$ acts freely. This is a toric Calabi-Yau manifold, which is also the total space of a family of Abelian surfaces.

The SYZ mirror was constructed in [KL19, Theorem 5.6]. It is given by

$$ uv = \Delta(\Omega) \cdot \sum_{(j,k) \in \mathbb{Z}^2} \left[ \frac{j}{2} \right] q_{1j} q_{2k} \cdot z_1^{j} z_2^{k} = \Delta(\Omega) \cdot \Theta_2 \left[ \frac{0}{2} \left( \frac{\tau}{2}, -\rho \right) \right] (\xi_1, \xi_2; \Omega). $$

Here $\Omega := \begin{bmatrix} T & \sigma \\ \sigma & \rho \end{bmatrix}$. There are three Kähler parameters $q_\tau = e^{2\pi i \tau} = q_1 q_\sigma$, $q_\rho = e^{2\pi i \rho} = q_2 q_\sigma$, $q_\sigma = e^{2\pi i \sigma}$. $z_i$ is the genus 2 Riemann theta function, and

$$ \Delta(\Omega) = \exp \left( \sum_{j \geq 2} \frac{(-1)^j}{j} \sum_{(l_1, l_2) \in (\mathbb{Z}^2)_{-1}} \exp \left( \sum_{k=1}^{j} \pi i k \cdot \Omega \cdot \mathbf{l}_k^T \right) \right). \quad (3.15) $$

As before, we choose a Lagrangian immersion $L_0 \cong S^2 \times S^1$ which bounds a primitive holomorphic disc with area $A$ whose boundary has holonomy parametrized by $z_2$. We have fixed the chamber dual to $(0,0,1)$ and the framing $v'_1 = (1,0,0), v'_2 = (0,1,0)$ for the $S^1$-equivariant Morse model of $L_0$. Then the above mirror equation can be understood as the gluing formula between $L_0$ and a embedded Lagrangian torus, with $z_2$ replaced by $T^A z_2$, and $z_1 = -e^{\tau_1}$. The equivariant disc potential is given by $x_1 \lambda_1$, where $x_1$ is solved from the above equation.

Setting $u = v = 0$, we obtain the generating function of stable discs with no corners. This gives an enumerative meaning of the zero locus of the Riemann theta function (on $(z_1, z_2) \in (\mathbb{C}^\times)^2$) as a section over the $z_2$-axis. The leading-order terms are given as follows (where $w = T^A z_2$):

\[
\left( \left( \frac{3q_1^2 q_2^2}{2} - 2q_1 q_2^2 + q_1^2 q_2 \right) + 4q_1^3 q_2^4 q_4 + \left( \frac{3q_1^3 q_2^4}{2} - q_1^2 q_2 \right) \right) w^-2 \\
+ \left( (-q_1 + q_1 q_2 - 2q_1^2 q_2 + 4q_1^3 q_2^2) + (q_1 + q_2 - 2q_1 q_2 + 4q_1^2 q_2 + 9q_1 q_2^2) q_4 + (q_1^2 + 2q_1 q_2 - 2q_1^2 q_2 - 9q_1 q_2^2 + 12q_2^2 q_2^2) q_4^2 \right) w^{-1} \\
+ \left( (6q_1^2 q_2^2 - 4q_1 q_2 + 2q_1^2 q_2 - q_1 + 1) + (15q_1^2 q_2 + 33q_1 q_2 - 11q_1 q_2 + q_1 - 3q_2^2 + 3q_2 - 1) q_4 \\
+ (60q_1^3 q_2^4 - 62q_1 q_2 + q_1 - 62q_1 q_2^2 + 23q_1 q_2 + q_1 + 9q_2 - 3q_2^2) q^2 \right) w \\
+ \left( \left( 2q_1 q_2^3 + 12q_1 q_2 - 2q_1 q_2^3 + \frac{3q_2^2}{2} + 2q_2 - \frac{1}{2} \right) + (-24q_1 q_2 + 9q_1 q_2 + 16q_1 q_2 + 12q_2 - 4q_2) q_4 \\
+ \left( -\frac{257q_1 q_2}{2} + 78q_1 q_2 - \frac{q_1^2}{2} + 26q_1 q_2 - 20q_1 q_2 - \frac{3q_2}{2} + \frac{1}{2} \right) \right) w.
\]

4. **Equivariant disc potentials of immersed Lagrangian tori**

So far, we have mainly focused on the immersed Lagrangian $S^2 \times T^{n-2}$, whose Maurer-Cartan deformation space fills the codimension-two toric strata of a toric Calabi-Yau manifold. In this section, we consider an immersed Lagrangian torus that copes with lower dimensional toric strata. The construction of a Landau-Ginzburg mirror associated to such an immersed torus, using the method in [CHL17], has been introduced in the survey article [Lau]. The immersed torus and its application to HMS have also been addressed in recent talks by Abouzaid.

The immersed torus $L$ is constructed by symplectic reduction. Recall from Section 2 that we have the reduction $X \sslASH a_1 T^{n-1} := \rho^{-1}(a_1) / T^{n-1} \cong C$ where $\rho$ denotes the $T^{n-1}$-moment map. Previously, we take $a_1$ from the image of a strictly codimension-two toric
stratum so that a path passing through $0 \in C$ gives rise to a Lagrangian immersion in $X$. In this section, we do not make such a restriction and allow $a_1$ to lie in the image of any toric stratum.

Let us take the immersed curve $C$ in the $w$-plane shown in Figure 7a. This immersed circle was first introduced by Seidel [Sei11] for proving homological mirror symmetry of genus-two curves. Here the punctured plane $C - \{0, 1\}$ is identified with $\mathbb{P}^1 - \{0, 1, \infty\}$.

The curve $C$ has three self-intersection points $U, V, H$, each of which corresponds to two immersed generators (in the Floer complex of $C$). By abuse of notations, they are denoted by $U, V, H$ (with odd degree) and $\hat{U}, \hat{V}, \hat{H}$ (with even degree) respectively.

Since $C$ does not pass through $0 \in C$ (that would create additional singularity), its preimage $\mathcal{L}$ in $\rho^{-1}\{a_1\} \subset X$ is a Lagrangian immersion in $X$ from an $n$-dimensional torus, where the fiber over each point of $C$ is a torus $T^{n-1}$. Thus $\mathcal{L}$ is an immersed torus which has three clean self-intersections isomorphic to $T^{n-1}$ that lie over the three self-intersection points $U, V, H$ of $C$. Let us denote the immersion by $i : \hat{\mathcal{L}} \to \mathcal{L}$ where $\hat{\mathcal{L}} = T^n$ is an $n$-dimensional torus.

In order to express $\mathcal{L}$ as the product of the immersed curve $C$ with a fiber $T^{n-1}$ precisely (this splitting is not canonical), we take a trivialization of the holomorphic fibration $w : X \to C$ as follows. All the fibers of $w$ are $(\mathbb{C}^\times)^{n-1}$ except $w^{-1}(0)$, which is the union of toric prime divisors. By relabeling $v_i$ if necessary, we may assume that the toric stratum that $a_1$ is located is adjacent to the $v_1$-facet. We fix a basis $v'_i \in \mathbb{N}$ for $i = 1, \ldots, \rho^{-1}\{a_1\}$ and some $v_1, \ldots, v_{n-1} \in \mathbb{N}$. This gives a set of coordinate functions (where $v$ corresponds to $w$), which gives a biholomorphism between the complement of all the toric divisors corresponding to $v_i$ for $i \neq 1$ and $C \times (\mathbb{C}^\times)^{n-1}$. The projections to the factors $C$ and $(\mathbb{C}^\times)^{n-1}$ give the splitting $\mathcal{L} \cong C \times T^{n-1}$. This also fixes a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\pi_1(\hat{\mathcal{L}})$.

The Floer complex of $\mathcal{L}$ admits the following description. Observe that $\hat{\mathcal{L}} \times \hat{\mathcal{L}} \cong \hat{\mathcal{L}}$ consists of seven connected components, as the self intersection loci of $\mathcal{L}$ has three components. One of them is simply $\hat{\mathcal{L}}$, which is responsible for non-immersed generators, which are represented by critical points of Morse functions on $\hat{\mathcal{L}}$. The remaining six components are copies of $T^{n-1}$, and they give rise to the generators of the form $G \otimes X_I$ for $G = U, V, H, \hat{U}, \hat{V}, \hat{H}$ and $I \subset \{1, \ldots, n - 1\}$, which are represented by critical points in the corresponding $T^{n-1}$-components of $\hat{\mathcal{L}} \times \hat{\mathcal{L}}$. We will specify the choice of relevant Morse
functions shortly. For simplicity, we will often write $G$ to denote $G \otimes 1_{T^n}$ when there is no danger of confusion.

We have the holomorphic volume form

$$\Omega = \frac{dw}{(w - 1)} \wedge d \log z_1 \wedge \ldots \wedge d \log z_{n-1}$$

on the divisor complement $X^\circ = X - \{ w = 1 \}$, where $z_1, \ldots, z_{n-1}$ are the local toric coordinates corresponding to $\{ n \nu_1, \ldots, \nu_{n-1} \}$. It induces the one-form $d \log (w - 1)$ on the reduced space $X //_{a_1} T^{n-1} \cong \mathbb{C}$.

**Lemma 4.1.** $\mathcal{L}$ is graded with respect to $\Omega$. The generators $U, V, H$ have degrees $1, 1, -1$ respectively, and the generators $\bar{U}, \bar{V}, \bar{H}$ have degrees $0, 0, 2$. Here $1_{T^{n-1}}$ denotes the maximum point of the Morse function on $T^{n-1}$-component.

To be more precise, the degree of $G \otimes X_I$ for $G = U, V, H, \bar{U}, \bar{V}, \bar{H}$ and $I \subset \{1, \ldots, n-1\}$ is obtained by adding the degree of $X_I$ to that of $G$ given in this lemma.

**Proof.** By the reduction, it suffices to check that the curve $C$ is graded with respect to $-i d \log (w - 1)$. The preimage of $C$ under $w = e^{iy} + 1$ is a union of ‘figure-8’ as shown in Figure 8a. There is a well-defined phase function for the one-form $dy$ on (the normalization of) all the figure-8 components, which is invariant under translation by $2\pi$. Thus $C$ is graded. The degrees of the immersed generators can be directly computed from the phase function. □

![Figure 8. The gradings on C by two different holomorphic volume forms.](image)

For the purpose of computing Maslov indices, we also consider another grading for $\mathcal{L}$ given as follows. We take the meromorphic volume form

$$\tilde{\Omega} = \frac{-2i \, dw \wedge d \log z_1 \wedge \ldots \wedge d \log z_{n-1}}{(w + i)(w - i)}$$

on restricted on $X^\circ$, which corresponds to $d \log \frac{w + i}{w - i}$ in the reduced space. It has the pole divisors $w = \pm i$. Similar to the above lemma, we can directly check the following. See Figure 8b.

**Lemma 4.2.** $\mathcal{L}$ is graded with respect to $\tilde{\Omega}$. The generators $U, V, H$ all have degree one, and the generators $\bar{U}, \bar{V}, \bar{H}$ all have degree zero.

We extend the Maslov index formula [CO06, Aur07] to the immersed situation.

**Lemma 4.3.** Let $L$ be an immersed Lagrangian graded by a meromorphic nowhere zero $n$-form $\Omega$. For a holomorphic polygon in class $\beta$ bounded by $L$ with corners at degree one immersed generators of $L$, its Maslov index equals to $2\beta \cdot D \geq 0$ where $D$ is the pole divisor of $\Omega$. 
Proof. We trivialize $TX$ pulled back over the domain polygon $\Delta$, and we have a Lagrangian sub-bundle $TL$ over the boundary edges of the polygon (which can be understood as a disc with boundary punctures). It is extended to be a Lagrangian sub-bundle over $\partial \Delta$ by taking positive-definite paths at the corners. Since the corners have degree one with respect to (the pull-back of) $\Omega$, we have a well-defined real-valued phase function for the Lagrangian sub-bundle. Thus the phase change with respect to $\Omega$ equals to zero with respect to $\Omega$.

The pull-back of $\Omega$ has poles at $a_i \in \Delta$. Then \( \left( \prod_{i} \frac{z-a_i}{1-\bar{a}_i \bar{z}} \right) \Omega \) becomes a nowhere zero holomorphic section of $\bigwedge^n T^*X|_{\Delta}$. Each factor $\frac{z-a_i}{1-\bar{a}_i \bar{z}}$ results in adding $2\pi$ for the phase change. Thus the total phase change equals to $2\pi k$ where $k$ is the number of poles. Thus the Maslov index equals to Maslov index equals to $2\beta \cdot D$. \( \square \)

For the Fukaya category of $X^\circ$, the objects are Lagrangians graded by $\Omega$. The grading of $\mathcal{L}$ by $\bar{\Omega}$ is auxiliary. We apply the above lemma to $\mathcal{L}$ using the grading by $\bar{\Omega}$. Thus the Maslov index of a holomorphic polygon with corners at $U,V,H$ equals to two times the intersection number with the pole divisor $\{w = i\} \cup \{w = -i\}$. In particular $L_T$ does not bound any holomorphic polygon with corners at $U,V,H$ which has negative Maslov index.

We take a perfect Morse function on each component of $\tilde{\mathcal{L}} \times_\mathcal{L} \tilde{\mathcal{L}}$. First we take a perfect Morse function on $\tilde{\mathcal{L}} \cong S^1 \times T^{n-1}$. We take a perfect Morse function on the normalization $S^1$ of $C$, whose maximum and minimal points lie in the upper and lower parts of $C$ respectively, see Figure 7. Let us denote the maximum and minimal points by $p_{\text{max}}$ and $p_{\text{min}}$ respectively. We take a perfect Morse function on the $T^{n-1}$-factor, whose unstable hypertori of the degree-one critical points are dual to the $S^1$-orbits of $\nu_i'$. The sum of these two functions gives the desired perfect Morse function.

We also take such a perfect Morse function on the $T^{n-1}$-components of $\tilde{\mathcal{L}} \times_\mathcal{L} \tilde{\mathcal{L}}$. They are identified with the clean intersections of $\mathcal{L}$. The perfect Morse function is taken such that the unstable hypertori of the degree-one critical points are dual to the $S^1$-orbits of $\nu_i'$ for $i = 1, \ldots, n-1$.

As a result, the Morse complex of $\tilde{\mathcal{L}} \times_\mathcal{L} \tilde{\mathcal{L}}$ has the following generators. The component $\tilde{\mathcal{L}} \cong T^n$ has the generators $X_I$ for $I \subset \{0, \ldots, n-1\}$ (where $X_\emptyset = 1_{\tilde{\mathcal{L}}}$). We also have the immersed generators $G \otimes X_I$ for $G = U,V,H,\tilde{U},\tilde{V},\tilde{H}$ and $I \subset \{1, \ldots, n-1\}$. They are critical points in the corresponding $T^{n-1}$-components of $\tilde{\mathcal{L}} \times_\mathcal{L} \tilde{\mathcal{L}}$. (Recall that we have written $G$ for $G \otimes 1_{T^{n-1}}$ by abuse of notation so far.)

Let us equip $\mathcal{L}$ with flat $\Lambda_U$-connections. We do it in a ‘minimal way’. Namely, we only consider connections which are trivial along $\gamma_1$, because such a deformation direction is already covered by the formal deformations of the immersed generators $U,V,H$. We denote by $z_i$ the holonomy variables associated to the loops $\gamma_i$. We consider the formally deformed immersed Lagrangian $(\mathcal{L}, uU + vV + hH, \nabla(z_1, \ldots, z_{n-1}))$. We will prove that these are weak bounding cochains. Notice that $\deg u = \deg v = 0$ and $\deg h = 2$ with respect to $\Omega$, whereas, with respect to $\bar{\Omega}$, $\deg u = \deg v = \deg h = 0$. (This ensures $b = uU + vV + hH$ always has degree one.)

The only non-constant holomorphic polygons bounded by $C$ of Maslov index two are the two triangles with corners $U,V,H$ (shown in Figure 7b), or the two one-gons with the corner $H$ (shown in Figure 7c). Using this, we classify holomorphic polygons bounded by $\mathcal{L}$ in what follows.
Lemma 4.4. Any non-constant holomorphic polygon bounded by $L$ must project to a non-constant holomorphic polygon bounded by $C$ under $w$.

Proof. If $w$ is constant, the holomorphic polygon is contained in the corresponding fiber of $w$, which is $(\mathbb{C}^\times)^{n-1}$. $L$ intersects this fiber at $T^{n-1}$ which is isotopic to the standard torus (the product of unit circles) in $(\mathbb{C}^\times)^{n-1}$. By the maximal principle, such a torus does not bound any non-constant holomorphic disc. □

Since $C$ does not bound any non-constant Maslov-zero holomorphic disc, by the above lemma, there is no non-constant Maslov-zero holomorphic disc bounded by $\hat{L}$.

The classification of holomorphic disc classes bounded by $L$ is similar to Lemma 3.6 and Corollary 3.7. Moreover Lemma 3.8 and Proposition 3.9 also have their counterparts for $L$. The proofs are parallel to the corresponding ones for $L_0 \cong S^2 \times T^{n-2}$, and hence omitted.

Proposition 4.5.  
(i) The only holomorphic polygon classes of Maslov index two bounded by $L$ are $\beta_i^\pm$ for $i = 0, \ldots, m$, where $\beta_0^\pm$ (or $\beta_0^\pm$) is the class which never intersects $w = 0$ and projects to the triangle passing through $p_{\max}$ in $C$ (or passing through $p_{\min}$ respectively); $\beta_i^+$ (or $\beta_i^-$) is the class which intersects the toric divisor $D_i$ once but not the other $D_j$ for $j \neq i$, and projects to the one-gon passing through $p_{\max}$ in $C$ (or passing through $p_{\min}$ respectively).

(ii) The stable polygon classes of Maslov index two bounded by $L$ are $\beta_i^\pm + \alpha$ for $i = 0, \ldots, m$ and $\alpha$ is an effective curve class. For $i = 0, \alpha = 0$.

(iii) The moduli space
$$\mathcal{M}_2(L; \beta_i + \alpha) \times_{ev_0} \{p_{\max}\}$$
for $i = 1, \ldots, m$ is isomorphic to $\mathcal{M}_1(\beta_i^\pm + \alpha) \times_{ev} \{p\}$ for a certain Lagrangian toric fiber $L$ and a certain point $p \in L$.

(iv) The moduli space
$$\mathcal{M}_2(H, V, U; \beta_0) \times_{ev_0} \{p_{\max}\}$$
is simply a point.

We equip $L$ with a non-trivial spin structure, which is represented by the $T^{n-1}$-fiber of the point $p_{\max} \in C$. By abuse of notation, we denote the three generators which has base degree 1 and fiber degree 0 by the same letters $U, V, H$ corresponding to the three immersed points of $C$. Take the formal deformations $b = uU + vV + hH$ for $u, v, h \in \mathbb{C}$. We also have a flat $\mathbb{C}^\times$ connection in the fiber $T^{n-1}$-direction over $C$; its holonomy is given by $(z_1, \ldots, z_{n-1}) \in (\mathbb{C}^\times)^{n-1}$.

Using cancellation in pairs of holomorphic polygons due to symmetry along the dotted line shown in Figure 7a, we prove the following statement.

Proposition 4.6. $(L, b, \nabla^z)$ is weakly unobstructed.

Proof. This is similar to the proof of weakly unobstructedness for the Seidel Lagrangian in [CHL17]. The anti-symplectic involution identifies the moduli spaces $\mathcal{M}_3(U, V, H; \beta_i^\pm + \alpha)$ with $\mathcal{M}_3(H, V, U; \beta_i^\pm + \alpha)$, and $\mathcal{M}_1(H; \beta_i^\pm + \alpha)$ with $\mathcal{M}_1(H; \beta_i^\pm + \alpha)$ for $i = 1, \ldots, m$. Since the holomorphic polygons in $\beta_i^\pm$ for $i = 0, \ldots, m$ pass through the spin cycle $\{p_{\max}\} \times T^{n-1}$, while the holomorphic polygons in $\beta_i^\pm$ do not, the pairs of moduli spaces have opposite signs. Thus their contributions to $\hat{U}, \hat{V}, \hat{H}$ cancel and equal to zero. □

In particular, the superpotential associated to $(L, b, \nabla^z)$ is well-defined. Moreover, we can compute it explicitly.
Theorem 4.7. The superpotential of \((L, b, \nabla^z)\) is

\[ W = -u \nu h + h f(z) \]

defined on \(((u, v, h), z) \in \mathbb{C}^3 \times (\mathbb{C}^\times)^{n-1}, \text{ where } f \text{ is given in (2.4).} \) Its critical locus is

\[ \bar{X} = \left\{ ((u, v), z) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \mid uv = f(z) \right\}. \]

Proof. Since the smooth fibers are conics which topologically do not bound any non-constant discs, the image of a Maslov-two disc must be either one of the regions shown in Figure 7. For the region with corners \(u, v, h\), there is no singular conic fiber and hence there is only one holomorphic polygon over it passing through a generic marked point (corresponding to the constant section). This gives the term \(-u \nu h\). For the region with one corner \(h\), by Riemann mapping theorem the polygons over it are one-to-one corresponding to those bounded by a toric fiber. They contribute \(h \cdot f(z)\) to \(W\). \(\square\)

We fix \(a_1 \in M_{\mathbb{R}}/V^1_{\mathbb{R}}, \) which is the level of the symplectic reduction \(X \parallel _{a_1} T^{n-1} \cong \mathbb{C}, \) by requiring that \(a_1\) lies in image of strictly codimension-two toric strata of \(X\) under the moment map \(\rho. \) This guarantees that we have the immersed Lagrangian \(S^2 \times T^{n-2}\) on this level, and enables us to compare with \(L.\)

To distinguish the variables between \(L\) and \(S^2 \times T^{n-2}\), we denote the degree-one immersed generators of \(L\) by \(U_0, V_0, H\) and that of \(S^2 \times T^{n-2}\) by \(U, V.\) \(L\) and \(S^2 \times T^{n-2}\) intersect cleanly at two tori \(T^{n-1}.\) We fix a basis of generators of \(H^1(T^{n-2})\) (the factor contained in \(S^2 \times T^{n-2}\)) and extend it to \(H^1(T^{n-1})\) (the clean intersections). Denote the corresponding holonomy variables for \(L\) by \(z_1^{(0)}, \ldots, z_{n-1}^{(0)}\) and those for \(S^2 \times T^{n-2}\) by \(z_2^{(1)}, \ldots, z_{n-1}^{(1)}\).

![Figure 9. The strips in computing the embedding of \(S^2 \times T^{n-2}\) to \(L_1^2.\)](image)

Theorem 4.8. There exists a non-trivial morphism \(a\) from \(\mathcal{L}_0 := (S^2 \times T^{n-2}, uU + vV, \nabla z^{(1)})\) to \(\mathcal{L} = (\mathcal{L}, u_0U + v_0V + hH, \nabla z^{(0)})\) for \(u_0 = u, v_0 = v, h = 0, z_i^{(0)} = z_i^{(1)}\) for \(i = 2, \ldots, n - 1,\) and \(z_1^{(0)} = g(uv, z_2^{(1)}, \ldots, z_{n-1}^{(1)})\), where \(g\) is the same as that in Theorem 3.18. Moreover \(a\) has a one-sided inverse \(\beta,\) namely \(m_2^{\mathcal{L}, \mathcal{L}_0}(\beta, \alpha) = 1_{\mathcal{L}_0}.

Remark 4.9. We have different choices of the one-sided inverse \(\beta.\) Note that \(m_2^{\mathcal{L}, \mathcal{L}_0}(\alpha, \beta)\) has a non-zero output to \(\bar{U}_0.\) Thus \(\beta\) cannot be a two-sided inverse. This is natural since the Maurer-Cartan deformation space of \(\mathcal{L}\) is bigger than that of \(\mathcal{L}_0.\)
On the other hand, one can check that if neither $u_0$ nor $v_0$ vanishes, $\bar{U}_0$ determines an idempotent, and we speculate that the corresponding object in the split-closed Fukaya category is isomorphic to $L_0$ (under the coordinate change above).

**Remark 4.10.** While the mirror chart of the immersed torus $L$ already covers the mirror chart of $S^2 \times S^1$, the study of $S^2 \times S^1$ is still interesting since it passes through the discriminant locus and has an analogous role of the Aganagic-Vafa brane [AV00] as they bound the same set of Maslov index zero holomorphic discs.

We can also consider $S^1$-equivariant theory of $L$. It is similar to the previous sections and so we will be brief.

**Theorem 4.11.** The equivariant disc potential for $L = (L, u_0 U + v_0 V + h H, \nabla z(0))$ equals to

$$W^{L}_{S^1} = -u v h + h \cdot f(\exp x_1^{(0)}, \ldots, \exp x_{n-1}^{(0)}) + \sum_{i=1}^{n-1} x_i^{(0)} \lambda_i.$$ 

It restricts to the equivariant disc potential of $L_0$ via the embedding in Theorem 4.8.

Thus we obtain the LG model $(\Lambda^{n+2}_+ \times \text{Spec}(\Lambda_+ [\lambda]), W^{L}_{S^1})$, which can be understood as an equivariant mirror for the toric Calabi-Yau manifold $X$.

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