DYNAMICS OF PAIRWISE MOTIONS

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ABSTRACT

We propose a simple closed-form expression relating \( v_{12}(r) \)—the mean relative velocity of pairs of galaxies at fixed separation \( r \)—to the two-point correlation function of mass density fluctuations, \( \xi(r) \). Our Ansatz is an interpolation between the perturbative and stable clustering expressions for \( v_{12} \). We compare our analytic model for \( v_{12}(r) \) with \( N \)-body simulations and find excellent agreement in the entire dynamical range probed by the simulations \((0.1 \leq \xi \leq 1000)\). Our results can be used to estimate the cosmological density parameter, \( \Omega \), directly from redshift-distance surveys.

Subject headings: cosmology: observations — cosmology: theory — galaxies: kinematics and dynamics

1. AN ANALYTICAL MODEL FOR \( v_{12}(r) \)

Most dynamical estimates of the cosmological density parameter, \( \Omega \), use the gravitational effect of departures from a strictly homogeneous distribution of mass. One such dynamical estimator can be constructed by using an equation expressing the conservation of particle pairs in a self-gravitating gas. This equation was derived by Davis & Peebles (1977) from the BBGKY theory (see also Peebles 1980, hereafter LSS). Let us consider a pair of particles at a comoving separation vector \( x \) and comoving time \( t \), moving with a mean (pair-weighted) relative velocity \( v_{12}(x,t)/x \). Its magnitude is related to the two-point correlation function of density fluctuations, \( \xi(x,t) \), by the pair conservation equation (LSS),

\[
\frac{a}{3[1 + \xi(x,a)]} \frac{\partial \xi(x,a)}{\partial a} = - \frac{v_{12}(x,a)}{Hr},
\]

where \( a(t) \) is the expansion factor, \( r = ax \) is the proper separation, \( H(a) \) is the Hubble parameter, and \( \xi(x,a) \) is the two-point correlation function, averaged over a ball of comoving radius \( x \); \( \xi(x,a) = 3x^{-3} \int \xi(y,a) y^2 d y \). At the present cosmological time, \( a = 1 \), \( x = r \), and \( H = 100 \) \( h \) \( \text{km \ s}^{-1} \text{Mpc}^{-1} \), where \( h \) is the conventional dimensionless parameterization for the Hubble constant. There are two well-known approximate solutions of equation (1). They are the small separation limit, where \( \xi(x) \gg 1 \) (stable clustering regime), and the large separation limit, where \( |\xi| \ll 1 \) (linear regime). The stable clustering solution (LSS, § 71) \( v_{12}(x,a) = -Hr \), as expected for virialized clusters on sufficiently small scales. The linear solution is given by the first terms in the perturbative expansions for \( v_{12} \) and \( \xi \), which for the correlation is given by \( \xi = \xi^{(1)} + \xi^{(2)} + \ldots \), with \( \xi^{(1)} = O[\xi^{(1)}]^{2} \), etc. The growing mode of the linear solution is \( \xi^{(1)}(x,a) = \xi^{*}(x,a) D(a)^2 \), where \( D(a) \) is the usual linear growth factor (LSS, § 11; we neglect the decaying mode). The general technique for deriving \( \xi^{(2)} \) and higher order terms for initially Gaussian density fluctuations in an Einstein–de Sitter universe was introduced by Juszkiewicz (1981), Juszkiewicz, Sonoda, & Barrow (1984), Vishniac (1983), and Fry (1984); their results were recently generalized to a wider class of cosmological models, including those considered below (Juszkiewicz, Bouchet, & Colombi 1993; Bouchet et al. 1995). These calculations show that the second-order term in the expansion for \( \xi \) is well approximated by \( \xi^{(2)}(x,a) \approx D^4(a) \xi^{*}(x,a) \), where \( \xi^{*} \) is a function of \( x \) alone. Substituting \( \xi = D^2 \xi^{*} + D^2 \xi^{*} \) into equation (1) and solving for \( v_{12} \), we get

\[
v_{12} = -\frac{2}{3} H r [\xi^{(1)} - \xi^{(1)} + 2 \xi^{(1)} + O[\xi^{(1)}]],
\]

where \( f = d \ln D/d \ln a \approx Q^{(1)} \) (which is a good approximation when \( \Lambda = 0 \) or \( \Omega + \Omega_{c} = 1; \Omega_{c} = \Lambda/3H^2 \)). The bars above the \( \xi^{*} \)’s denote the averaging over a ball of comoving radius \( x \). Now, let us suppose that the logarithmic slope of the correlation function, \( \gamma(x) = -\ln \xi^{(1)}(x,a)/a \ln x \), is a constant in the range \( 0 \leq \gamma < 2 \). Then \( \xi^{(2)} \) is related to \( \xi^{(1)} \) by a simple closed-form expression,

\[
\xi^{(2)}(x,a) = \alpha(\gamma)[\xi^{(1)}(x,a)]^2,
\]

where \( \alpha \) is a constant (Lokas et al. 1996), and the function \( \alpha(\gamma) \) can be expressed in terms of Euler’s \( \Gamma \) functions (Scoccimarro & Frieman 1996). In the range \( 0 \leq \gamma < 2 \), this expression is well approximated by a fitting formula:

\[
\alpha \approx 1.84 - 1.1 \gamma - 8.4(\gamma/2)^{10}.
\]

For \( \gamma \geq 2 \), perturbation theory diverges, but \( N \)-body simulations suggest that the parabolic relation (eq. [3]) remains valid, provided the \( \alpha(\gamma) \) dependence for \( \gamma \gtrsim 1.5 \) is derived from the so-called universal scaling relation (USH)—an empirically fitted formula for the nonlinear power spectrum, extracted from \( \gamma \gtrsim 1 \) scale-free and cold dark matter (CDM) simulations (Jain, Mo, & White 1995). Since the USH approach fails when \( \gamma < 1 \), while the perturbation theory (PT) fails for \( \gamma \geq 2 \), we propose trading accuracy for an extended range of validity, interpolating the expression \( \alpha(\gamma) \) between the PT formula for \( \gamma < 1 \) and the USH result for \( \gamma > 1 \) (Scoccimarro & Frieman 1996, Fig. 20), and replacing equation (4) with

\[
\alpha \approx 1.2 - 0.6 \gamma.
\]

To extend the validity of our perturbative solution (eq. [2]) into

\[
\dot{\alpha} \approx 1.2 - 0.6 \gamma.
\]
In the nonlinear regime, we propose the following Ansatz:

\[ v_{12}(x, a) \approx -\frac{2}{3} Hr f \tilde{\xi}(x, a) [1 + \alpha \tilde{\xi}(x, a)]. \]  

(6)

Here \( \tilde{\xi}(x, a) = \xi(x, a)/[1 + \xi(x, a)] \). The above expression agrees with the perturbative expansion (eq. [2]) when \( \xi \to 0 \); it also approximates the stable clustering limit when \( x \to 0 \). Indeed, when \( \xi \) is large, \(-v_{12}/Hr \approx (2/3)f(\Omega)(1 + \alpha)\), which is of order unity for all reasonable values of \( \Omega \) and \( \alpha \).

2. N-BODY SIMULATIONS

In this section, we compare our analytic expression for \( v_{12}(x, a) \) with the results from high-resolution AP'M simulations of 256³ dark matter particles, kindly provided to us by the Virgo collaboration (Jenkins et al. 1998). We consider four members of the CDM family: an open model (OCDM), a zero curvature low-\( \Omega \) model (ΛCDM), and two models with an Einstein-de Sitter metric—the “standard CDM” model (SCDM) and its modified version (τCDM). Following Jenkins et al., the values assigned to parameters \( (h, \Omega, \Omega_m, \sigma_8) \) are \((0.7, 0.3, 0, 0.85)\) for OCDM, \((0.7, 0.3, 0.7, 0.9)\) for ΛCDM, and \((0.5, 1, 0, 0.6)\) for the SCDM and τCDM, which has extra large-scale power (added in an ad hoc manner; described by Jenkins et al. 1998). Here and below, \( \sigma_8 \) is the rms dark matter density contrast in a ball of radius 8 h⁻¹ Mpc.

In order to measure the correlation function from the N-body experiments, we find all neighbors at separations \( \leq 40 \) h⁻¹ Mpc for a random subset of particles in each realization. Using logarithmically spaced bins, we then count the pairs of a certain separation. These counts provide an estimate of \( \xi(r) \). The pairwise velocity \( v_{12}(r) \) is obtained by averaging the expression \( (v_1 - v_2) \cdot r/r \) over all pairs of particles at fixed separation \( r = |r_1 - r_2| \). Here \( v_1 \) and \( v_2 \) are the peculiar velocity and position of the 1th particle, respectively (\( A = 1, 2 \)).

Since CDM-like models are not scale-free, equation (5) does not apply. In principle, we should therefore calculate \( \alpha(x) = \tilde{\xi}^{\prime\prime}(x, a)/[\tilde{\xi}^{\prime\prime}(x, a)]^{2} \) for each considered power spectrum and each separation \( x \), using standard perturbative techniques (Łokas et al. 1996; Scoccimarro & Frieman 1996). However, as we will show below, these calculations can be significantly simplified by finding an effective slope, \( \gamma_{\text{eff}} \), that provides a “best-fit” \( \alpha \) and \( v_{12} \) when substituted in equations (5) and (6). The precise value of \( \alpha \) is unimportant in the stable clustering regime as well as in the linear regime, when the term quadratic in \( \tilde{\xi} \) is subdominant. Hence, the precision in \( \alpha(x) \) matters only at the boundary between the linear and nonlinear regimes, say, at \( \xi(x, t) = 1 \). One of the possible definitions of the effective slope is therefore given by

\[ \gamma_{\text{eff}} = -(d \ln \xi/d \ln x)|_{x=1}. \]  

(7)

One can also choose \( \gamma_{\text{eff}} \) as follows. In Figure 1, we plot logarithmic slopes of \( \tilde{\xi}^{\prime\prime}(x) \) and \( \tilde{\xi}(x) \) (the latter is measured from the simulations). Both curves agree at large separations as they should, apart from small differences arising from noise in the measurement (we use only a finite number of bins and pairs to measure \( \xi \), and from finite box-size and cosmic-variance effects (Jenkins et al. 1998). However, there is a well-defined scale (marked with an asterisk) at which the nonlinear slope turns away from the linear theory prediction, marking the onset of the nonlinear regime.
of the nonlinear regime. We take $\gamma_{\text{eff}}$ to be the logarithmic slope of $\xi^{(1)}$ at that scale. The resulting slopes are $\gamma_{\text{eff}} = 1.67$ (SCDM), 1.46 (ΛCDM), 1.49 (OCDM), and 1.40 (τCDM). The alternative definition (eq. [7]) gives, respectively, $\gamma_{\text{eff}} = 1.67$, 1.47, 1.45, and 1.28. The advantage of the former definition is that it is more closely related to observations because it uses $\gamma$ along with components. The linear correlation function is easy to calculate under the controlled conditions of an N-body experiment, but it cannot be easily determined from observations.

In Figure 2, we test our Ansatz (eq. [6]) against N-body measurements. For comparison, we also plot three other approximations for $v_{12}(r)$, which were considered earlier in the literature:

$$v_{12} \approx -\frac{2}{3} H r f \xi^{(1)}, \quad (8a)$$

$$v_{12} \approx -\frac{2}{3} H r f \xi, \quad (8b)$$

$$v_{12} \approx -\frac{2}{3} H r f \bar{\xi}. \quad (8c)$$

Here equations (8a) and (8b) are two variants of linear theory predictions, and equation (8c) is an improvement over linear theory, suggested by LSS (§ 71). Figure 2 shows that the deviations from linear theory are small at large separations, as they should. The range of validity of the Peebles equation (8c) is already considerably wider than that of linear theory. However, our new Ansatz provides by far the best approximation. In fact, it covers the entire dynamical range!

The scale at which the linear approximation becomes acceptable depends on the amplitude of fluctuations; it increases with increasing $\gamma$. For example, for a power-law correlation function, $\xi(r) = \sigma_{8}^{2} F(\gamma)r^{-\gamma}$, where $F(\gamma) = (16 \ h^{-1} \ Mpc)^{\gamma} \times (4 - \gamma)(6 - \gamma)/24$, and equation (8b) can be rewritten as

$$v_{12}(r) \approx -\frac{2}{3} \sigma_{8}^{2} \Omega^{1/6} H F(\gamma) r^{1-\gamma} \approx -605 \sigma_{8}^{2} \Omega^{0.6} \ km \ s^{-1}, \quad (9)$$

where the expression after the last "≈" sign assumes $\gamma = 1.75$ and $r = 10 \ h^{-1} \ Mpc$. The relative error in the latter expression, introduced by linear theory, can be calculated from equations (5) and (6). It depends on $\sigma_{8}$ only; for $\sigma_{8} = 1$ and 0.6, linear theory overestimates $|v_{12}|$ by 24% and 10%, respectively.

3. A PRACTICAL APPLICATION: AN ESTIMATOR OF $\Omega$

Equation (9) shows how measurements of $v_{12}(r)$ at sufficiently large separations could be used to estimate $\sigma_{8}^{2} \Omega^{0.6}$. Measurements on intermediate scales (the mildly nonlinear regime) can then be used to remove the degeneracy and to estimate $\Omega$ and $\sigma_{8}$ separately because the $\alpha \xi$ term in equation (6) is $\Omega^{0.6}$.
independent. Ferreira et al. (1999) have recently shown how this can be done using redshift-distance surveys, like the Mark III catalogue (Willick et al. 1997). The only problem that will need extra attention here is the possibility of the existence of velocity bias—systematic differences between the relative motions of pairs of dark matter particles and pairs of galaxies: \( v_{12}(r) \neq v_{12}(r) \); here and below, the index \( g \) stands for “galaxies.” Velocity bias should be distinguished from clustering bias, which can be characterized by the parameter \( b^2(r, t) = \xi_4(r, t)/\xi(r, t) \). In the linear bias model, \( b \) is a constant for a given morphological type of galaxies, and at large separations, the model predicts \( v_{123}(r) = b v_{12}(r) \) (Fisher et al. 1994). Hence, for pairs of galaxies at large separations, \( v_{12g} \propto b \Omega^{1/4} \), and \( v_{12g}(r) \) should depend on galaxy morphology. This effect is absent in Mark III data: spiral and elliptical subsamples give similar estimates of \( v_{12g} \) (Juszkiewicz et al. 1999). Although recent numerical simulations, which account for dissipative processes, do show clustering bias, there is no velocity bias: \( v_{12}(r) = v_{12}(r) \) (Kauffmann et al. 1999; see also Fry 1996, however, cf. Narayan, Berlind, & Weinberg 1998). Hence, the velocity bias may in the end turn out to be the proverbial red herring. We will address this issue in more detail in a separate paper; here we will instead keep our focus on understanding the dynamics of pairwise motions of dark matter particles and on providing an analytical insight into \( N \)-body simulations.

4. Conclusions

The main results of this Letter are in equations (5) and (6), relating the mean relative velocity of pairs of dark matter particles, \( v_{12}(r) \), to an integral of the two-point correlation function of mass density fluctuations. Our analytic expressions reproduce the results of numerical simulations on all scales, from the strongly nonlinear to the linear regime. These results can be used to estimate \( \Omega \) from redshift-distance surveys, like Mark III (Ferreira et al. 1999; Juszkiewicz et al. 1999). In the absence of velocity bias on large scales, \( v_{12}(r) \) scale with \( \Omega \), and \( \sigma_8 \) as \( \Omega^{1/4} \sigma_8 \). This property makes \( v_{12}(r) \) complementary to other estimators: the POTENT method (Sigad et al. 1998) and the cluster abundances (Bahcall & Fan 1998) are sensitive to \( \beta \equiv \Omega^{1/4} \sigma_8 \); the supernovae (Perlmutter et al. 1998) distances measure \( q_0 \equiv (\Omega/2) - \Omega \); finally, the position of acoustic peaks in the cosmic microwave background (CMB) power spectrum (Doroshkevich, Zeldovich, & Sunyaev 1978) is sensitive to \( \Omega + \Omega_\Lambda \). The advantage of our estimator over the CMB peaks method is model independence; our method’s advantage over POTENT is simplicity (Ferreira et al. 1999). Our results can also be used to study the nature of biasing and to test the gravitational instability theory. Indeed, one can use the galaxy correlation function, \( \xi(r) \), estimated from galaxy redshift or angular surveys to predict \( v_{12}(r) \), which can then be compared with a \( v_{12}(r) \) estimated from redshift-distance surveys (Gaztäna & Juszkiewicz 1999). Finally, equation (6) can be used to truncate the BBGKY hierarchy and obtain a closed \( 1 + 1 \) partial differential equation for the dynamical evolution of \( \xi \) (Caldwell et al. 1999).

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