On an exact hydrodynamic solution for the elliptic flow

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Looking for the underlying hydrodynamic mechanisms determining the elliptic flow we show that for an expanding relativistic perfect fluid the transverse flow may derive from a solvable hydrodynamic potential, if the entropy is transversally conserved and the corresponding expansion “quasi-stationary”, that is mainly governed by the temperature cooling. Exact solutions for the velocity flow coefficients $v_2$ and the temperature dependence of the spatial and momentum anisotropy are obtained and shown to be in agreement with the elliptic flow features of heavy-ion collisions.

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I. INTRODUCTION

The hydrodynamic description of the formation and development of a quark-gluon plasma (QGP) in high-energy heavy-ion collisions has met a considerable success \cite{1}. In particular, the hydrodynamic features seem to be, at least partly, required in order to take into account the second Fourier coefficient $v_2$ of the transverse flow of particles, the so-called elliptic flow. One writes \cite{2,3} for the azimuthal multiplicity distribution

$$\frac{dN}{d\phi} = \frac{N}{2\pi} \left(1 + 2v_2 \cos(2\phi) + \cdots\right), \quad (1)$$

discarding for simplicity other Fourier coefficients non-relevant here. The experimentally observed values of $v_2$, which are due to the anisotropy of the initial state collisions at nonzero impact parameter, are sizeable enough to require important collective effects of particle production. These are better reproduced by hydrodynamical properties of the flow in some early stage of the quark-gluon plasma formation.

The theoretical estimates of the elliptic flow are obtained from numerical studies based on various versions of the hydrodynamic models. Indeed, a full study requires not only to deal with the solution of the relativistic hydrodynamic equations but also with the definition of appropriate initial conditions and a model for the mutation of the QGP pieces of fluid into particles. The numerical studies (cf. \cite{1}) reveal that the QGP as a fluid is “almost perfect” since its viscosity is remarkably weak, even if the model dependence may account for some variation on the quantitative estimates. This observation has a considerable theoretical impact, since it points to a strongly coupled plasma, guiding a large part of theoretical interest towards strongly coupled gauge field theories.

Our goal in the present work is to try and identify by explicit analytic solutions the basic hydrodynamic mechanisms at work in the elliptic flow. For this sake, we have to simplify (or even idealize) the description of the QGP formation in a heavy-ion collision while keeping the main physical ingredients. Among other simplifications that we will discuss now, our study will assume the QGP to be a perfect fluid without viscosity. We will also restrict our analysis to the transverse flow in the central rapidity region where the hydrodynamic description is better suited.

The main characteristic feature of the hydrodynamic description of QGP formation in heavy-ion collisions appears to be a nontrivial combination of: a) the large longitudinal momentum and energy boosts provided to the created medium by the initial state, b) the (presumably fast) equilibration of the energy density and all three pressure components due to local thermalization required by hydrodynamics. As a matter of fact, the first stage of the hydrodynamical description of particle production in high energy collisions is mainly governed by the longitudinal flow, that is the expansion of the relativistic fluid in (1+1) dimensions. Already the pioneering papers of the hydrodynamic approach \cite{4,5} based their analysis on this property.

In the mean time, the 4-dimensional hydrodynamical feature of the system is kept with the thermodynamic relations which, through the local temperature $T$ and the Equation of State (EoS), lead to

$$\frac{T}{T_0} \sim \left(\frac{\tau}{\tau_0}\right)^{\frac{1}{\gamma}_c}; \quad \tau \equiv \sqrt{x_0^2 + x_3^2}; \quad \gamma_c^2 = \frac{dp}{de} = \frac{sdT}{Td} \quad (2)$$

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where \( \tau \) is the proper-time, \( c_s \) the speed-of-sound, which we will assumed to be constant in the following, and \( e, p \) the energy and (isotropic) pressure density, respectively. Our remaining notations are \( s \) for the entropy density, \( x_{\mu=(0,\cdots,3)} \) for the space-time coordinates \( (x_0 \equiv t) \), and \( u_{\mu=(0,\cdots,3)} \) for the fluid 4-velocity (with lower indices) in the Minkowski metric \( \eta^{\mu\nu} \), with signature \( (1,-1,-1,-1) \) satisfying the normalization condition

\[
u_u u^\nu = u_0^2 - u_3^2 - u_\perp^2 = 1; \quad u_0^2 = u_0^2 + u_\perp^2 . \tag{3}\]

Our idea for studying the transverse motion of the fluid is that it is also driven by the longitudinal evolution, but slowly enough to be considered as “quasi-stationary”, that is in such a way that its time evolution is essentially related to the temperature cooling. Indeed, the seed of transverse momenta is indirect and there should be, at least during some first stage of the hydrodynamic evolution, no strong back-reaction on the longitudinal motion. The “quasi-stationarity” hypothesis will allow for an exact solution for the elliptic flow and in general for the hydrodynamic regime in the transverse plane. To be concrete we state the following conjectured properties of the hydrodynamic flow:

- **a)** **Transversally isentropic.** Since the overall entropy should be conserved, we will conjecture that the transverse flow is itself (approximately) isentropic, i.e. we write the following equation

\[
[\partial_{x_1}(su_1) + \partial_{x_2}(su_2)]_{\text{transverse}} = 0 . \tag{4}
\]

- **b)** **Quasi-stationary:** Since the time dependence of the transverse entropy distribution is absent from (4), we will close the equations for the transverse flow by assuming that its hydrodynamic evolution is smooth enough to be driven only by the temperature change. We thus consider the equation and solutions of a temperature-dependent stationary flow, with a source emitting a fluid at a given temperature (and thus transverse speed, see further).

Hence, in the regime when the longitudinal expansion is dominant, the transverse motion is conjectured to be smoothly driven by the overall local temperature of the fluid, which provides the 4-dimensional feature\(^1\) of the system through the thermodynamic relations \((2)\).

The plan of the paper is the following: in the next section II, using the hydrodynamic potential for a stationary flow \([5]\), we derive the analytic equation obeyed by the azimuthal distribution of entropy and thus the elliptic flow. Then, in section III, we find the exact solutions for the transverse flow. In section IV we apply our solution showing that the obtained elliptic flow retain good qualitative features observed in reality or in realistic numerical hydrodynamic model studies. A discussion of the hypotheses and our conclusions and outlook form the final section V.

**II. AZIMUTHAL ENTROPY DISTRIBUTION**

As we shall see now, the conditions **a)** and **b)** lead to nontrivial properties of the fluid and to analytic solutions for the elliptic flow. It is obtained from an hydrodynamical (KK) potential \([7]\) derived by Khalatnikov and Kamenshchik for a stationary transverse isentropic flow. The “quasi-stationary” hypothesis allows us to extend its applicability to a slow transverse motion of the fluid and to find the general exact solution of the elliptic flow. In fact, the existence of a hydrodynamical potential obeying a linear equation is known since long \([8, 9]\) for the longitudinal evolution. Recently \([10]\), it was possible to express interesting analytic solutions for the entropy distribution \(dS/dy\) where the (hydrodynamic) rapidity is defined by \( y = \frac{1}{2} \log(u_0 + u_3)/\log(u_0 - u_3) \). We shall follow the same method as in \([10]\) for the transverse flow case, and find the general solution of the KK potential in order to obtain the azimuthal distribution of the entropy \(dS/d\phi\) giving access to the elliptic flow.

However, one crucial difference of the transverse w.r.t. the longitudinal case is the velocity-temperature relation between \( u_0 \), the time component of the velocity (and thus also \( u_\perp = \sqrt{u_0^2 - 1} \), the modulus of the transverse one) to the local temperature. This comes from the relativistic Bernoulli relation, verified by a stationary fluid \([11]\), namely

\[
Tu_0 = T \sqrt{1 + u_\perp^2} = T_0 . \tag{5}
\]

\(^1\) In that respect, our picture is different from a purely transverse hydrodynamic flow \([6]\).
In the same conditions, the whole evolution from some initial temperature to the final freeze-out one is constrained to be in the supersonic regime \[11\], as we will verify later through our equations. The condition writes

$$v_\perp \equiv \frac{u_\perp}{u_0} = \frac{u_\perp}{\sqrt{1 + u_\perp^2}} = \left\{ 1 - \left( \frac{T}{T_0} \right)^2 \right\}^{1/2} \geq c_s.$$ \hspace{1cm} (6)

Hence, the isentropic transverse evolution starts at a given temperature \(T_I\) such that

$$T_I \leq T_s \equiv T_0 \sqrt{1 - c_s^2},$$ \hspace{1cm} (7)

and the velocity increases when the temperature decreases from \(T_I\), reaching eventually ultra-relativistic values before hadronization. For convenience, we will from now on introduce the variable

$$l = \frac{1}{2} \log \left( 1 - \left( \frac{T}{T_0} \right)^2 \right) = \frac{1}{2} \log \left[ \frac{u_\perp^2}{1 + u_\perp^2} \right] = \log v_\perp.$$

The derivation of the KK potential comes briefly as follows \[7\]: Together with the transverse entropy conservation \[4\], the equations for the transverse flow close with the projection to the transverse plane of the energy-momentum conservation relation \(\partial_\nu T_{\mu\nu} = 0\), again by neglecting the time derivatives w.r.t. the transverse gradients. After nontrivial transforms, presented in appendix \[A\], one obtains the system of equations

$$\partial_x_1(su_1) + \partial_x_2(su_2) = 0$$

$$\partial_x_1(Tu_2) - \partial_x_2(Tu_1) = 0.$$ \hspace{1cm} (9)

Then, using the “hodograph” \[8, 9, 10, 11\] inversion of variables \((x_1, x_2) \rightarrow (l, \varphi)\) and combining Eqs.\(4\) and \(9\), one arrives at the formulae expressing the kinematic (now dynamical) variables \((x_1, x_2)\) in terms of the hydrodynamic variables through a suitably defined KK potential function \(\chi(\varphi, l)\), namely

$$x_\perp(\varphi, l) = \frac{e^{-l}}{T_0} \sqrt{\left( \frac{\partial \chi}{\partial l} \right)^2 + \left( \frac{\partial \chi}{\partial \varphi} \right)^2}$$

$$\alpha(\varphi, l) = \varphi + \text{arctan} \left( \frac{\partial \chi}{\partial \varphi} \left( \frac{\partial \chi}{\partial l} \right)^{-1} \right),$$ \hspace{1cm} (10)

where we have parameterized

$$u_1 = u_\perp \cos \varphi \hspace{1cm} u_2 = u_\perp \sin \varphi$$

$$x_1 = x_\perp \cos \alpha \hspace{1cm} x_2 = x_\perp \sin \alpha.$$ \hspace{1cm} (11)

The KK potential function \(\chi(\varphi, l)\) is solution of a linear equation obtained by closing the hydrodynamic equations system using the EoS

$$\left( 1 - e^{2l}/c_s^2 \right) \frac{\partial^2 \chi}{\partial l^2} + \left( 1 - e^{2l}/c_s^2 \right) \partial^2 \chi/\partial \varphi^2 + \left( 1 - 1/c_s^2 \right) e^{2l} \partial \chi/\partial l = 0.$$ \hspace{1cm} (12)

Note the zero coefficient at \(e^l = c_s\) which signals the supersonic bound \[6, 7\] at \(T = T_s\). In fact the system expands in the vacuum for \(T \leq T_s\), while it is compressed when \(T > T_s\), cf. \[4\]. Hence the physical solutions are restricted to the supersonic range \(T \leq T_I \leq T_s\).

The KK potential and its equation have been reproduced from \[7\]. The calculation of the entropy distribution is now parallel to the one \[10\] (see \[12\] for an early version) used in the \((1+1)\) dimensional case. Considering an entropy flux normal to the tangential line element \((dx_1, dx_2)\), one has to compute

$$dS = su_2 dx_1 - su_1 dx_2.$$ \hspace{1cm} (13)

Using the formulae \[10\] for the expression of the line element in terms of the potential, one has

$$-\frac{e^{-l}}{T_0} \frac{\partial \chi}{\partial l} = x_1 \cos \varphi + x_2 \sin \varphi = x_\perp \cos(\alpha - \varphi)$$

$$-\frac{e^{-l}}{T_0} \frac{\partial \chi}{\partial \varphi} = x_2 \cos \varphi - x_1 \sin \varphi = x_\perp \sin(\alpha - \varphi).$$ \hspace{1cm} (14)
which, by differentiation with respect to \(l\) and \(\phi\), gives

\[
dS = \frac{sT_0}{T} \left\{ \left[ \frac{\partial^2 \chi}{\partial l \partial \phi} - \frac{\partial \chi}{\partial \phi} \right] dl + \left[ \frac{\partial^2 \chi}{\partial \phi^2} + \frac{\partial \chi}{\partial l} \right] d\phi \right\},
\]

where we used the relation \(u_\perp e^{-l} \equiv u_0 = T_0 / T\).

At fixed temperature (and thus fixed \(l\)), which is the case considered further on, one gets the azimuthal entropy distribution

\[
\frac{dS}{d\phi} = \frac{sT_0}{T} \left[ \frac{\partial^2 \chi(\phi, l)}{\partial \phi^2} + \frac{\partial \chi(\phi, l)}{\partial l} \right] = \frac{sT}{T_0 (1 - e^{2l}/c^2)} \left[ \frac{\partial \chi(\phi, l)}{\partial l} - \frac{\partial^2 \chi(\phi, l)}{\partial l^2} \right],
\]

where the second expression comes from the KK potential equation \([12]\). Note again the singular coefficient at \(T_s\), corresponding to the lower bound of temperature.

### III. EXACT SOLUTION OF THE TRANSVERSE FLOW

In our idealized hydrodynamic framework, without hadronization, one relates the entropy distribution to multiplicity, \(dS/S \sim dN/N\). Hence, the elliptic flow is defined by the azimuthal entropy distribution \([16]\) through a Fourier expansion similar to \([1]\), namely

\[
\frac{dS}{d\phi} = \frac{S}{2\pi} \{ 1 + 2v_2 \cos(2\phi) + \cdots \},
\]

and thus

\[
v_2 = \frac{\int d\phi \cos(2\phi) \frac{dS}{d\phi}(\phi)}{\int d\phi \frac{dS}{d\phi}(\phi)}. \tag{18}
\]

The eccentricity can be obtained as a function of temperature (or \(l\)) through Eqs.\([14]\) in terms of the KK potential \(\chi(\phi, l)\) as:

\[
\varepsilon \equiv \frac{(x_2^2 - x_1^2)}{(x_2^2 + x_1^2)} = \frac{\int d\phi \left( x_2^2(\phi, l) - x_1^2(\phi, l) \right)}{\int d\phi \left( x_2^2(\phi, l) + x_1^2(\phi, l) \right)} \frac{\int d\phi \left\{ \cos 2\phi \left[ \left( \frac{\partial \chi}{\partial \phi} \right)^2 - \left( \frac{\partial \chi}{\partial l} \right)^2 \right] + 2 \sin 2\phi \frac{\partial \chi}{\partial \phi} \frac{\partial^2 \chi}{\partial \phi \partial l} \right\}}{\int d\phi \left( \left( \frac{\partial \chi}{\partial \phi} \right)^2 + \left( \frac{\partial \chi}{\partial l} \right)^2 \right)} . \tag{19}
\]

Note the characteristic feature of the hodograph method: a geometrical parameter, here \(\varepsilon\), is expressed in terms of dynamical ones, here the temperature. Once the solution is found, one has to invert these relations in order to restore the hierarchy between “cause” and “effect”.

We can now proceed by looking for the general solution resulting from the KK potential solution of \([12]\). For this sake, it is convenient to expand the potential

\[
\chi(\phi, l) = \beta_0(l) + \sum_{p=1}^{\infty} \beta_p(l) \cos(2p \phi) \tag{20}
\]

in Fourier coefficients \(\beta_p(l)\) which verify the equations

\[
(e^{2l} - 1) \beta_p''(l) + e^{2l}(c_s^{-2} - 1)\beta_p'(l) - 4p^2 (c_s^{-2} e^{2l} - 1) \beta_p(l) = 0 , \tag{21}
\]

where primes denote derivatives with respect to \(l\).

As it is well known, the solution is in general a suitable combination, with constant coefficients, of two independent solutions of the second-order equations \([20]\). Using standard textbooks, one finds for \(p \neq 0\)

\[
\beta_p(l) = c_p^{(1)} \beta_p^{(1)} + c_p^{(2)} \beta_p^{(2)} \\
\equiv c_p^{(1)} (-)^{p+1} e^{2pl} 2 F_1 \left( p+\frac{1}{2} (c_s^{-2} - 1) - \sqrt{(c_s^{-2} - 1)^2/16 + c_s^{-2} p^2} , p+\frac{1}{2} (c_s^{-2} - 1) + \sqrt{(c_s^{-2} - 1)^2/16 + c_s^{-2} p^2} , e^{2l} \right) + c_p^{(2)} G_{2,2}^{2,0} \left( e^{2l} \left| \begin{array}{c} 5-c_s^{-2} \\ -p \end{array} \right| \frac{(c_s^{-2} - 1)^2}{4} + c_s^{-2} p^2 , 5-c_s^{-2} \\ \frac{c_s^{-2} - 1}{4} \right) , \tag{22}
\]

where

\[
\frac{dS}{d\phi} = \frac{sT_0}{T} \left[ \frac{\partial^2 \chi(\phi, l)}{\partial \phi^2} + \frac{\partial \chi(\phi, l)}{\partial l} \right] = \frac{sT}{T_0 (1 - e^{2l}/c^2)} \left[ \frac{\partial \chi(\phi, l)}{\partial l} - \frac{\partial^2 \chi(\phi, l)}{\partial l^2} \right],
\]

where the second expression comes from the KK potential equation \([12]\). Note again the singular coefficient at \(T_s\), corresponding to the lower bound of temperature.
while we single out the first component \( c_0 \beta_0(l) \), whose derivative simply writes

\[
\beta'_0(l) = (1 - e^{2l})^{1-\epsilon/2}.
\]  

(23)

In (22), \( _2F_1(a, b; c, z) \) is the usual hypergeometric function while \( G^{2,0}_{2,2}(e^{2l} | a, b \mid c, d) \) denotes a Meijer function \[13\].

The function \( \beta_p^{(1)} \) (resp. \( \beta_p^{(2)} \)) is the regular (resp. irregular) solution\(^2\) at \( e^l = 0 \) of (21). Hence, any arbitrary combination of these two independent functions is a solution of (21). The boundary conditions will define the specific linear combinations to be chosen for the general solution which may be obtained from the Green function of the problem. For illustration, in the special case \( p = 0 \), one writes

\[
\beta'_0(l) = \int_{-\infty}^{+\infty} \Theta(-l) \left( 1 - e^{2(l-l)} \right)^{1-\epsilon/2} F_0(l) \, dl,
\]  

(24)

where \( F_0(l) \) describes a distribution of sources in temperature convoluted with the Green function, which in this case is just \( \Theta(-l) \beta'_0(l) \) from (23). A straightforward but more tedious expression can be written for all values of \( p \) but is skipped here for brevity. A specific realization for the sake of our physical problem will be discussed in the application section.

Inserting the general solution (20, 22) for the KK potential in Eq. (16) for the azimuthal entropy distribution, one finds

\[
\frac{dS}{d\phi}(\varphi) = \frac{8T_b}{T} \left\{ \beta'_0(l) + \sum_{p=1}^\infty \cos(2p\varphi) \left[ (2p)^2 \beta_p(l) - \beta'_p(l) \right] \right\},
\]

\[
v_2(T) = \frac{4\beta_1(l) - \beta'_1(l)}{2\beta'_0(l)} \equiv \rho \left[ \frac{4\beta_1^{(1)}(l) - \beta'_1^{(1)}(l)}{2\beta'_0(l)} + \lambda \frac{4\beta_1^{(2)}(l) - \beta'_1^{(2)}(l)}{2\beta'_0(l)} \right],
\]  

(25)

where we denote \( c^{(1)}_{\epsilon_0} \equiv \rho \), and \( c^{(2)}_{\epsilon_1}\equiv \lambda \), and

\[
\epsilon(T) = \frac{2\beta'_0(l) \left[ 2\beta_1(l) + \beta'_1(l) \right]}{2\beta'^2_0(l) + 4\beta^2_1(l) + \beta'^2_1(l)} \equiv \rho \left\{ \frac{2\beta'_0(l)}{2\beta'^2_0(l) + \rho^2} \left[ \frac{2 \left( \beta^{(1)}_1 + \lambda \beta^{(2)}_1 \right) + \beta^{(1)}_1' + \lambda \beta^{(2)}_1'}{4 \left( \beta^{(1)}_1 + \lambda \beta^{(2)}_1 \right)^2 + \left( \beta^{(1)}_1 + \lambda \beta^{(2)}_1 \right)'^2} \right] \right\}.
\]

(27)

Thanks to the analytic solutions we obtained, all the expressions contain an explicit dependence in temperature. One should only specify which temperature is physically relevant, \( e.g., T_f \) for the initial spatial eccentricity and some freeze-out temperature \( T_f \) for the observed \( v_2 \). Using our formulae, one may discuss the dynamical hydrodynamical process through the temperature dependence of both the spatial and momentum average anisotropy of the lump of quark-gluon plasma. For this sake, we note an interesting parameter-independent relation between the spatial eccentricity at any temperature \( T_I \) and \( v_2 \) at any temperature \( T_f \), namely

\[
\frac{v_2(T_f)}{\epsilon(T_f)} = \frac{\beta_1(T_f) \beta'(T_f)}{\beta'_1(T_f) \beta'(T_f)} \times \frac{1 - \beta'_1(T_f)}{1 + \beta'_1(T_f)} \times \left( \frac{1}{\sqrt{1 - 2\epsilon^2(T_f)}} + \frac{\beta_1^{(2)}(T_f)}{4 \beta_1^{(1)}(T_f)} \right)^{-1}.
\]  

(28)

\(^2\) For \( p = 0 \), Eq. (21) is only first-order for \( \beta'_0(l) \) and thus introduces only one arbitrary coefficient \( c_0 \).
A general physical comment is in order about the parameters $\lambda$ and $\rho$ defining the relevant solutions in the “elliptic approximation”. Using a source of given temperature, the parameter $\lambda$, which corresponds to the relative strength of the two independent solutions of the second-order differential equation (21), will specify the initial condition of the elliptic flow. The parameter $\rho$, which is geometrical in nature since it gives the relative strength of the elliptic harmonic in (20), will be related to the initial centrality of the reaction. For general initial conditions, the more general Green function formalism, cf. (24), has to be used.

IV. EXACT ELLIPTIC FLOW: APPLICATIONS

Taking into account the linear equations for the potential and entropy distributions, cf. (12,15), the determination of the elliptic flow boils down to defining properly the boundary conditions, i.e. the sources of the hydrodynamic expansion, which are given functions of temperature and azimuth. In the following we will assume that the source is simply given by a delta-function at the initial temperature $T_I$ of the process and a given initial eccentricity profile. We fix it by the condition that $v_2(T_I) = 0$ while $\varepsilon(T_I)$ is maximal. Note that the solution satisfies the constraint $T_I \lesssim T_s$, i.e. the fluid is always supersonic.

In the “elliptic approximation” for which the Fourier expansion of the potential (20) is limited to the two first orders, the observables $v_2$ (25) and $\varepsilon$ (27) depend only on two relevant parameters, namely $\rho = c_1^{(1)}/c_0$, obtained from the Fourier expansion (20) and $\lambda = c_1^{(2)}/c_1^{(1)}$, that is the coefficient ratio between the regular and irregular solutions (22) of the potential equations (12,21).

Determination of $\lambda$. From the previous discussion, $\lambda$ is chosen in such a way that $v_2(T_I) = 0$, where $\varepsilon(T_I)$ is maximal. As an illustration of the discussion, the temperature dependence of both $v_2$ and $\varepsilon$ that we obtain with our definition of the initial condition, is displayed in Fig.1 for a given value of the geometrical anisotropy parameter $\rho = 0.8$. The value of $T_I$ is lower but nearby the speed-of-sound lower limit of temperature $T_s$.

![Fig. 1](image)

FIG. 1: (Color online) Compared temperature dependence for the momentum $v_2(T)$ and the spatial $\varepsilon(T)$ anisotropies. The curves correspond to the initial temperature source at $T = T_I$ (see text). The dashed line is for the supersonic lower bound $T_s$. The geometrical anisotropy parameter (see text) is $\rho = 0.8$ and the speed of sound is the reference one $c_s = 1/\sqrt{3}$.

Determination of $\rho$. The determination of the geometrical parameter $\rho$, the first anisotropy coefficient of the potential $\chi$, see (20), is governed by the centrality. In Fig.2 we display $\varepsilon(\rho)$ which shows a quasi-linear behavior. This is in good agreement with the observed feature of the experimentally reconstructed eccentricity with an observed proportionality relation with the centrality $c \sim N_{\text{part}}/N_{\text{max}}$, where $N_{\text{part}}$ is the number of participant nucleons. Indeed, one expects a simple relation between $\rho$ and $c$, up to a rescaling $\rho/\rho_{\text{max}} \sim 1-N_{\text{part}}/N_{\text{max}}$. With such a choice and using Eq. (25),...
FIG. 2: \( \varepsilon \) as a function of the geometrical anisotropy parameter \( \rho \). The observed dependence qualitatively reproduces simulations of \( \varepsilon \) as a function of centrality \( c \sim N_{\text{part}}/N_{\text{max}} \) (see text). In this figure we used \( c_s^2 = 1/3 \) and \( \rho_{\text{max}} \sim 0.8 \).

one finds a simple proportionality rule of \( v_2 \) with centrality, namely

\[
v_2 = \rho_{\text{max}} (1 - c) \left[ \frac{4\beta_1^{(1)}(l) - \beta_1^{(1)}(l)}{2\beta_0^{(l)}} + \lambda \frac{4\beta_1^{(2)}(l) - \beta_1^{(2)}(l)}{2\beta_0^{(l)}} \right],
\]

which is also expected from hydrodynamical simulations \[\text{[14]}\]. Note that, in this framework, \( \rho_{\text{max}} \) is indeed independent of the evolving temperature ratio \( T/T_0 \), but it may depend on the initial conditions such as the type of heavy-ion reaction and the initial c.o.m. energy (or \( T_0 \)). The \( T/T_0 \) dependence, given in \[\text{[29]}\] by the function within brackets, is uniquely defined from \[\text{[26]}\]. In our calculations, the linearity of the formula \[\text{[25]}\] for \( v_2 \) in terms of the normalized second Fourier coefficient \( \rho \) is a direct consequence of the equation \[\text{[12]}\] for the KK potential which, together with the azimuthal entropy distribution, is diagonalized by the Fourier expansion \[\text{[20]}\]. It is clear that the formulation of the initial eccentricity profile depends on the initial conditions, and we take the curve in Fig. 2 as an example.

In order to restore the time variable through its dependence on the temperature, we shall make use of a convenient rescaling of the temperature equivalent to the expansion time, similar to the one proposed in \[\text{[15]}\], where the ratio \( v_2(\tau - \tau_0)/\varepsilon(\tau_0) \) with initial time \( \tau_0 \) is displayed for different values of impact-parameter and various values of the speed-of-sound \( c_s \). One makes the rescaling substitution

\[
\tau \rightarrow \frac{c_s}{\bar{R}} (\tau - \tau_0) ; \quad \frac{1}{\bar{R}} = \sqrt{\frac{1}{\langle x_1^2 \rangle} + \frac{1}{\langle x_2^2 \rangle}}.
\]

where \( \bar{R} \) gives an appropriate average estimate of the expanding size of the plasma. In our temperature-dependent scheme, we define an analogous rescaling using the thermodynamical relation \[\text{[2]}\] by choosing a “rescaled time” variable defined in terms of the temperature as

\[
\theta \equiv \frac{c_s}{\bar{R}} \left\{ \left( \frac{T_0}{T} \right)^{c_s^2 - 2} - \left( \frac{T_0}{T} \right)^{c_s^2} \right\}.
\]

In Fig. 3 we show the theoretical results for \( v_2/\varepsilon \) as a function on the rescaled variable \( \theta \) for the solution we considered. Let us comment both sides of the figure. In the upper graph, the figure displays the dependence on centrality via
FIG. 3: (Color online) $v_2/\varepsilon$ as a function of the “rescaled time” $\theta$. Up: dependence on the centrality via the $\rho$-parameter at fixed $c_s = 1/\sqrt{3}$. Down: dependence on the EoS via $c_s^2 = (1/3, 1/5, 1/7, 1/10)$ at fixed eccentricity $\varepsilon = 0.8$. For this sake, one is led to choose, respectively, $\rho = (0.8, 1, 2, 4)$, by tuning the values of $\rho_{\text{max}}$. The “reduced time” is defined by Eq. (31).

$\rho = \rho_{\text{max}}(1 - c)$. The value of $\rho_{\text{max}}$ has been chosen fixing $v_2(\tau - \tau_0)/\varepsilon(\tau_0)$ to match with some realistic value (see, e.g. [1, 14]). One observes the general trend of the $\theta$ evolution as a function of increasing centrality (or decreasing $\rho$). This trend, which has been empirically observed in hydrodynamic simulations [15] is here explained by the nonlinear $\varepsilon$-dependent correction to $v_2/\varepsilon$ (see (28)). It is easy to realize that the remnant $\varepsilon$-dependence in (28) is such that it increases for increasing $\varepsilon$ (the denominator is smaller) and thus it decreases with centrality. Hence our scheme reproduces, at least qualitatively, a trend as a function of impact-parameter observed in hydrodynamical simulations.

Also, in the lower graph of Fig. 3 we display the speed-of-sound dependence of $v_2(\theta)/\varepsilon$. Thanks to the “time” rescaling (30), it is possible to superimpose the different curves, provided an adequate choice of $\rho$ ensures the constant initial $\varepsilon(T)$. As also seen numerically, the analytical dependence over $c_s$ of our resulting formula (28) gives a decreasing value of the ratio $v_2(\theta)/\varepsilon(T)$ with decreasing speed-of-sound. Here also one finds the observed behavior [15]. However, this hierarchy is obtained at rather larger $\theta$ than observed in [15]. We will comment on that feature in the next section.

V. CONCLUSION, DISCUSSION AND OUTLOOK

Let us briefly summarize our results: using the conjecture of a “quasi-stationary” and transversally isentropic hydrodynamic regime governing the transverse flow, and for a given EoS, we arrive at a closed system of hydrodynamic equations which can be solved by a hodograph transform $x_{1,2} \rightarrow T, \varphi$. Thanks to the potential method [7] we can formulate the general solution and give explicit analytic expressions for the hydrodynamic features of the transverse flow. In an application to a source with given temperature and constant effective speed-of-sound $c_s$, we are able to give a complete analytic solution. The applications to the determination of the features of the elliptic flow are in good qualitative agreement with the observed (or numerical) characteristics: the temperature dependence of the spatial and velocity anisotropy (see Fig 1), the linear behavior of $v_2$ with centrality (cf. Eq. (29)) for a realistic initial eccentricity (see Fig 2) and the centrality and speed-of-sound dependence of the ratio $v_2/\varepsilon$ (see Fig 3).

Now, possessing an analytic solution, it is possible to go back to the initial assumptions and discuss their range of validity. In other terms we may address the question to which approximation can we consider our closed system of transverse equations to be a good approximation of the full hydrodynamic equations. To quantify this approximation
a meaningful comparison is to give estimates of two quantities which are relevant for the discussion of the two hypotheses, a) of a “transversally isentropic” and b) a “quasi-stationary transverse” flow.

In order to test our conjecture a) and looking to (4), we are led to consider the following ratio of entropy flow gradients

\[
\frac{\partial T_x(su)}{\partial (su_0)} \sim \frac{\partial T_x(su)}{\partial (su_0)} \frac{\partial_T x(T)}{\partial_T \tau(T)} \equiv \frac{1}{V} \frac{\partial T_x(su)}{\partial_T \tau(T)},
\]

where \( V \equiv \frac{\partial_T x(T)}{\partial_T \tau(T)} \) is the average, temperature-dependent, expansion rate. Indeed, this ratio governs the effect of the time-gradient compared with a typical transverse one. Note, however that the overall transverse entropy gradient is zero, by virtue of (4).

In (32), we have replaced the kinematical variables by their temperature-dependent averages defined by our solution. The approximation range of a “transversally isentropic” is thus related to the value of the (analytically known) expression (32) to be larger than 1 in a significant range of “reduced time”. In the upper graph of Fig. 4 one sees that the transverse over longitudinal entropy gradient becomes indeed significantly larger than 1 for high enough speed of sound. For small speed of sound this requires longer “reduced time”. This could explain the features of Fig. 3, low, with a “retarded” ordering w.r.t. \[15\].

In order to test the consistency of the “quasi-stationary” approach b), in the lower graph of Fig. 4 we present the expansion rate itself \( V \equiv \frac{\partial_T x(T)}{\partial_T \tau(T)} \), where the functions \( \tau(T), x(T) \) are analytically obtained from their definition within our temperature-dependent scheme, namely from (2) and (10) respectively. Note that this rate is also appearing in the denominator of (32), which shows that the two hypotheses of a “transversally isentropic” and “quasi-stationary transverse” flow are indeed connected, since a slow motion gives rise to a high transverse over time typical entropy gradient. From Fig. 4 we see that both the transversally isentropic and quasi-stationary hypotheses are consistent at not too short “reduced times” and not too small speed-of-sound. Thus, these hypotheses give a qualitative analytic understanding of the transverse flow. Our qualitative picture seems consistent. However, the time gradient is not negligible w.r.t. the transverse derivative, indicating, at least within the initial conditions we choose, that a quantitative agreement could be more difficult to be obtained.

\[\text{FIG. 4: (Color online) Comparison of entropy and kinematic gradients. The analytically known quantities } \frac{\partial_T x(T)}{\partial_T \tau(T)} \text{ and } V \equiv \frac{\partial_T x(T)}{\partial_T \tau(T)} \text{ (see text) are plotted as a function of the reduced time } \theta.\]
Another topic is the range of validity of our approximation in transverse space. Indeed, due to (6), the quasi-stationary approach is only valid in the supersonic dilatation regime, which requires a large enough transverse velocity $v_{\perp} \geq c_s$. This could limit the range of validity of the hydrodynamical flow which has been observed only at small transverse momentum. It could also compromise the dominance of the longitudinal Bjorken flow determining the thermodynamical relations (2). We think that this limitation, which should be taken into account for a quantitative study difficult to perform analytically, will not endanger the qualitative but explicit solution we found. The study of the implications for the transverse momentum dependence of the elliptic flow deserves per se a study which goes beyond the scope of the present work, where no mass relation between fluid velocity and transverse momentum has been introduced.

As an outlook, it will be interesting to develop the study of hydrodynamical mechanisms generating the elliptic flow by the investigation of other phenomenological aspects, such as the abovementioned $p_{\perp}$ dependence, the effect of a weak viscosity, etc.... In order to reach more quantitative features, it will be useful to refine the definition of the initial conditions. In fact, it could be worthwhile to typically defining a priori the dependence of $\varepsilon$ as a function of $\rho$ or centrality, and finding the corresponding initial conditions$^3$ by inverting e.g. (24). In particular, to examine whether they could identify more definitely the hydrodynamical mechanisms. On a more theoretical ground, the existence of rather simple mechanisms may facilitate the search for a relation to the fundamental gauge field theory, and in particular the gauge/gravity dual approach of the elliptic flow.

All in all, our results suggest an analytic approach to the transverse motion of the fluid, which can clarify the behavior of the elliptic flow obtained from the data or numerical simulations. This is related to an approximate “quasi-stationary” and “transversally isentropic” property, of the transverse flow for which the time dependence of the system comes mainly through the temperature.

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APPENDIX A: QUASI-STATIONARY TRANSVERSE FLOW OF A PERFECT FLUID

In this section we derive the basic equations determining the quasi-stationary transverse flow of a perfect fluid. The energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = (e + p)u^\mu u^\nu - p\eta^{\mu\nu},$$

(A1)

where $e$ is the energy density, $p$ is the pressure and $u^\mu$ ($\mu = \{0, 1, 2, 3\}$) is the 4-velocity in the Minkowski metric $\eta^{\mu\nu}$, with signature $(1, -1, -1, -1)$. It obeys the equations

$$\partial_\mu T^\mu_\nu = 0 \Rightarrow u_\nu \partial_\mu [(e + p)u^\mu] + (e + p)u^\mu \partial_\mu u_\nu - \partial_\nu p = 0,$$

(A2)

with $u_\mu u^\mu = 1$, and thus

$$u_\nu \partial_\mu u^\nu = 0.$$  

(A3)

From now on and for simplicity, $\partial_\mu$ denotes $\partial_{x^\mu}$. Multiplying the equations of motion (A2) by $u_\nu$, i.e projecting them on the direction of the 4-velocity, and using (A3) we acquire:

$$\partial_\mu [(e + p)u^\mu] - u^\mu \partial_\mu p = 0.$$  

(A4)

Finally, re-inserting (A4) into (A2) we obtain:

$$(e + p)u^\mu \partial_\mu u_\nu = -u_\nu u^\mu \partial_\mu p + \partial_\nu p.$$  

(A5)

$^3$ We thank C. Gombaud for this suggestion.
Relation (A5) holds for a general perfect fluid. For a stationary flow it gives rise to Bernoulli equation (11), namely
\[ T u_0 = T_0. \] (A6)

Let us now focus on the stationary transverse flow, namely setting \( u_3 = 0 \), which is the regime of interest of the present work. In this case the equation of motion (A5) for \( u_3 \) gives \( \partial_3 p = 0 \) and thus the various quantities do not depend on the longitudinal coordinate \( x_3 \). Therefore, \( e, p \) and the velocities are functions of \( x_1, x_2 \) only. The equations of motion (A5) boil down to:
\[
\begin{align*}
\partial_1 \left( e + p \right) u_1^2 + \partial_2 \left( e + p \right) u_2^2 & = 0 \\
\partial_1 \left( e + p \right) u_1 u_2 + \partial_2 \left( e + p \right) u_2 u_2 & = 0 \\
\partial_1 \left[ (e + p) u_1 u_0 \right] + \partial_2 \left[ (e + p) u_2 u_0 \right] & = 0,
\end{align*}
\] (A7)
with \( u_0^2 = 1 + u_1^2 + u_2^2 \). Equations (A7) can be expressed in terms of the temperature and the entropy density. Considering vanishing chemical potential we have:
\[ p + e = T s; \quad de = T ds; \quad dp = sdT. \] (A8)

Using relations (A8), equations (A7) become:
\[
\begin{align*}
\partial_1 \left( (T s) u_1^2 \right) + s \partial_1 T + \partial_2 \left( (T s) u_2 u_2 \right) & = 0 \\
\partial_1 \left( (T s) u_1 u_2 \right) + \partial_2 \left[ (T s) u_2^2 \right] + \partial_2 T & = 0 \\
\partial_1 \left[ (T s) u_1 u_0 \right] + \partial_2 \left[ (T s) u_2 u_0 \right] & = 0.
\end{align*}
\] (A9) (A10) (A11)

Now, the Bernoulli equation (A6) allows to transform (A11) to the entropy conservation in the transverse plane:
\[ \partial_1 (s u_1) + \partial_2 (s u_2) = 0. \] (A12)

Finally, equations (A9) and (A10), with the use of (A3), (A6), (A11), give rise to the same equation:
\[ \partial_1 (T u_2) = \partial_2 (T u_1). \] (A13)
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