Final State Interactions in Kaon Decays Revisited

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March 25, 2022

Abstract

We examine effects of final state interactions (FSI) in kaon decays using a new scheme without the Omnès function, and show that while $\Delta I = 1/2$ rule is impossible to be explained in terms of FSI alone, $\epsilon'/\epsilon \sim 16.5 \times 10^{-4}$ is obtained through FSI, if the calculation is concerned only with the CP-violating parts. The $\pi\pi$ contribution to the $m_L - m_S$ mass difference is shown to be about 14%, if the sizes of the CP conserving amplitudes $|A_0|$ and $|A_2|$ are correctly given.

It seems that the established theory to explain simultaneously both of the $\Delta I = 1/2$ rule and the large $\epsilon'/\epsilon$ ratio in kaon decays has not yet been obtained [1, 2, 3]. In order to take a step toward resolving the issues much efforts have been made on the evaluation of hadronic matrix elements of quark operators [4, 5, 6]. Among them it is advocated that taking account of final state interactions (FSI) is crucial to enhance or suppress the decay amplitudes besides giving the strong phases [7, 8, 9, 10]. The final state interactions (FSI) in the kaon decays are usually described in terms of the Omnès function [11] as seen in Refs. [12, 7, 8, 9]. The FSI effect written in terms of the Omnès function is, however, involved with some ambiguities, inherent in dispersion integrals, coming from the dependence on a subtraction, a cutoff and a multiplicative arbitrary polynomial function [13, 14, 2]. We propose, therefore, a new scheme in this note, which makes the decay amplitude satisfy the final state interaction theorem without any dispersion relations.

In order to evaluate our scheme of FSI, we briefly see FSI in production processes of $S$-wave dipion states such as $\gamma + \gamma \rightarrow \pi\pi$, $\phi \rightarrow \pi\pi$, cascade decays of heavy quarkonia and others. Let us consider a production amplitude of an $S$-wave dipion state with a mass $\sqrt{s}$, called an $f$-state: It is composed of a direct production term of the $f$-state given by tree diagrams, which we call the Born term and denote by $B_f$, and an indirect production term, in which at first an $i$-state, denoted by $B_i$, is directly produced and then transformed into the final $f$-state through the $s$-channel loop and the $S$-wave scattering amplitude $T_{if}$. The loop integral should be properly regularized, and it may, then, depend on the renormalization scale parameter or a cutoff parameter. We can write the production amplitude, $F_f(s)$ as

$$F_f(s) = B_f(s) + \sum_i G_i(s) \cdot T_{if}(s),$$

where $G_i(s)$ is the loop integral composed of the meson propagators in the $i$-th channel, whose imaginary part (discontinuity along the physical cut) is given as

$$\text{Im} G_i(s) = -B_i(s) \rho_i(s) \theta(s - s_i),$$

where $\rho_i = p_i/(8\pi\sqrt{s})$ with $p_i(s_i)$ being the CM momentum (threshold energy squared) of the $i$-th channel. We can easily see that the above production amplitudes satisfy the final state interaction theorem,

$$\text{Im} F_f = -\sum_i F_i^* \rho_i \theta(s - s_i) T_{if}$$

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owing to Eq. (2), where we define the sign and normalization of $T_{ij}$ so as to satisfy $\text{Im}T_{ij} = -T_{ji} \rho_k T_{kj}$. We have found it very effective to use this scheme as shown in Refs. [15, 16, 17, 18, 19]. If the Born term contains two vertices, the loop diagram is a triangle, the third vertex of which is the starting vertex of $S$-wave $T_{ij}$ scattering. The triangle loop integrals are used in the two-photon collision processes [15], and the radiative $\phi$ meson decay [20, 21]. In the case that the Born term is made of a single vertex, $G(s)$ is a two-point loop integral. If the off-shell momentum dependence of the Born term is rewritten as the on-shell part and the remaining off-shell part having a form of $p^2 - m^2$, the off-shell factor $p^2 - m^2$ cancels one of the propagators, and the divergence is absorbed into counter terms in the Lagrangian with higher chiral order. The on-shell part is written as a factorized form of the Born term and the chiral loop integral such as $G_i(s) = B_i(s) \cdot J_i(s)$ as seen in ChPT and in weak ChPT. Here $J_i(s)$ is the renormalization scale dependent chiral loop integral [22], and $\text{Im}J_i(s) = -\rho_i(s)\theta(s - s_i)$. The scheme respects $s$-channel unitarity over the left-hand contributions, and this is shown to be rather valid in low energy dimeson scattering below 1 GeV or more [23, 24, 25, 26]. Vector or heavy mesons are allowed to be exchanged in the loop, but we discard them at all, because their contributions are not significant for low mass dimeson states. This scheme has some advantages over the scheme using the Omnès function; our scheme is applicable to inelastic multichannel reactions, it does not use dispersion integrals over the whole range of the physical cut, and then it is free of the ambiguities coming from the subtraction and from asymptotic behaviors of the integrands, though it depends on the logarithm of the renormalization scale $\nu$. Thus, probably this scheme is also useful for the $K \to \pi\pi$ decays.

In order to incorporate the above scheme into the kaon decay amplitudes, we have to know the Born amplitudes, where the decay vertex is regarded as a point, and does not have any discontinuities along the physical cut, $s > 4m_K^2$. As $B_I$ we take the real amplitude for $K_0 \to (\pi\pi)_I$ given as [27, 28]

$$B_0 = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{X}{\sqrt{6}} \left[ -z_1 + 2z_2 + 3z_4 + 3z_6 \frac{Y^0}{X} \right],$$

$$B_2 = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{X}{\sqrt{3}} [z_1 + z_2] ,$$

where we take the large $N_c$ limit for $< Q_i >_I$ and

$$X = \sqrt{2} f_\pi (m_K^2 - m_\pi^2) Y^0 = -4\sqrt{2} (f_K - f_\pi) \left( \frac{m_K^2}{m_s + m_u} \right)^2.$$

For the Wilson coefficients $z_i$'s we take the values of the leading order (LO) approximation obtained in Ref. [29] at $\mu = 1$ GeV with $\Lambda_{\overline{MS}}^{(4)} = 435$ MeV as

$$(z_1, z_2, z_4, z_6) = (-0.901, 1.541, -0.016, -0.018),$$

which give

$$B_0 = \begin{cases} 9.33 \times 10^{-8} \text{GeV} & m_s + \bar{m} = 175 \text{MeV}, \\ 9.61 \times 10^{-8} \text{GeV} & m_s + \bar{m} = 150 \text{MeV}, \\ 10.33 \times 10^{-8} \text{GeV} & m_s + \bar{m} = 120 \text{MeV}, \end{cases}$$

$$B_2 = 1.94 \times 10^{-8} \text{GeV}. $$

The amplitudes of $O(G_F p^4)$ include one-loop contributions, which is of $O(N_c^{-1})$. The $s$-channel pion loop term is written

$$T_{ij}^{(2)}(s) \cdot J_{\pi\pi}(s) \cdot B_I(s)$$

evaluated at $s = m_K^2$. [32, 33], where $T_{ij}^{(2)}$ denotes the $O(p^2)$ amplitude for $S$-wave $\pi\pi \to \pi\pi$ scattering with the isospin $I$. This structure is the same as we encounter in the strong ChPT amplitudes, and implies that the off-shell momentum dependence of $X$ is the same as described above. Thus, we rewrite the decay amplitudes as

$$A_I e^{i\delta_I} = B_I [1 + J_{\pi\pi}(s) T_{ij}^\pi(s)]_{s=m_K^2}.$$
where we replace $T^{(2)}$ by the full $\pi\pi$ scattering amplitude $T_{ij}$, which can be calculated through a unitarized ChPT. Higher order corrections should be added to $B_I$, except for the $s$-channel loop terms, but $B_I$ is left unchanged here in order to know the pure FSI effect. This means that we include higher order terms specially in the $s$-channel in order for the amplitudes to satisfy exact unitarity on the elastic physical cut. The factor,

$$R_I(s) = 1 + J_{\pi\pi}(s)T_I(s) = |R_I(s)|e^{i\delta_I(s)},$$

(12)

gives the whole effect of FSI coming from the strong $\pi\pi$ interaction, which is shown in Fig. 1. Using the scattering amplitudes calculated in terms of a unitarized ChPT [26], we have

$$|R_0(m_K^2)| = 1.506 \text{ and } |R_2(m_K^2)| = 0.836,$$

(13)

$$\delta_0(m_K^2) = 41.5^\circ \text{ and } \delta_2(m_K^2) = -7.76^\circ,$$

(14)

where the renormalization scale parameter of the $\pi\pi$ interaction is taken to be $\nu = 1$ GeV. We note that recent ChPT with dispersion relations gives $\delta_0(m_K^2) - \delta_2(m_K^2) = 47.7^\circ \pm 1.5^\circ$ [41]. We do not take into account of $KK$ and $\eta\eta$ channels, because their effects are very small; for example, adding the $KK$ loop integral $J_{KK}(s)T_{KK}^\pi(s)$ gives only 0.05 at $s = m_K^2$. We also note that it is shown that the contributions from the $u$- and $t$-channel loop diagrams are small [13].

Thus, we obtain the values of the PC-conserving amplitudes except for the strong phase $\exp[i\delta_0]$ and $\exp[i\delta_2]$ as

$$A_0 = (14.06, 14.48, 15.56) \times 10^{-8} \text{ GeV}, \text{ and } A_2 = 1.62 \times 10^{-8} \text{ GeV},$$

(15)

where three values of $A_0$ correspond to the values of $m_s + m$ at 175 MeV, 150 MeV and 120 MeV, respectively. The size of $A_0$ is merely about a half of the experimental value $33.3 \times 10^{-8}$ GeV, while the size of $A_2$ is rather close to the experimental value $A_2 = 1.50 \times 10^{-8}$ GeV. The ratio $\omega^{-1} = A_0/A_2$ is

$$\omega^{-1} = 8.68, \ 8.94, \ 9.60$$

(16)

depending on the value of $m_s + m$, respectively, which is smaller than a half of the observed value 22.1. Since the enhancement factor by FSI would be almost the maximum, it is insufficient to explain the $\Delta I = 1/2$ rule in terms of the FSI effects alone and then we have to search for higher order contributions or more fundamental mechanism to enlarge $B_2$ about twice.

The values of the enhancement of FSI are very similar to those obtained in Ref. [9], where the once-subtracted Omnés function is used with the parameter set, the subtraction point $s_0 = 0$ and the cutoff energy squared $\bar{s} = (1.6 \text{ GeV})^2$ of the dispersion integral. The dispersion integrals seem to be sensitive to these parameters, however.

Next, we consider the FSI effect on the $\epsilon'/\epsilon$ value. The ratio $\epsilon'/\epsilon$ is given as

$$\frac{\epsilon'}{\epsilon} = e^{i\phi} \frac{\omega}{\sqrt{2|\epsilon|}} \left[ \frac{\text{Im}A_2}{\text{Re}A_2} - \frac{\text{Im}A_0}{\text{Re}A_0} \right],$$

(17)
where $\Phi = \pi/2 - \delta_0 + \delta_2 - \theta_e \approx 0$, where $A_I e^{i\delta_I}$ denotes here the full amplitude involving the CP-violating part. Usually, the CP conserving amplitudes, $\text{Re}A_2$ and $\text{Re}A_0$, their ratio $\omega$ and $|\epsilon|$ are set to their experimental values, and the theoretical calculation is focused on the evaluation of the CP-violating amplitudes, $\text{Im}A_I$'s. The ratio is estimated through the formula \[ \frac{\epsilon'}{\epsilon} \approx 13 \text{Im} \lambda_I \left[ B_6^{(1/2)} (1 - \Omega_{IB}) - 0.4 B_8^{(3/2)} \right] \] (18)

at $m_s(m_c) = 130$ MeV, $m_t(m_t) = 165$ GeV, $\Lambda^{(4)}_{\text{MS}} = 340$ MeV and $\text{Im}\lambda_I = 1.33 \times 10^{-4}$. Taking both parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$ to be 1 at the large $N_c$ limit and $\Omega_{IB} = 0.25$ in the lowest order estimate \[33\], we have $\epsilon'/\epsilon \approx 6 \times 10^{-4}$.

The FSI effects modify these values to $B_6^{(1/2)}|R_0| = 1.506$, $B_8^{(3/2)}|R_2| = 0.836$ and $B_6^{(1/2)}\Omega_{IB}|R_2| = 0.209$, since the last term is proportional to $\text{Im}A_2$ \[3\], and then the value of the square brackets in Eq.(18) is 0.963, that is larger by a factor of 2.75 than the value at the large $N_c$ limit. Thus, we have

\[ \frac{\epsilon'}{\epsilon} \approx 16.5 \times 10^{-4}. \] (19)

This enhanced ratio is almost the same as the values obtained in Refs. \[7, 8, 9, 10, 34\]. If higher order corrections except for the $s$-channel loops are taken into the calculation of $B_6$ and $B_8$, we would have a larger value of $\epsilon'/\epsilon$.

If we do not distinguish the physics underlying the CP-violating part from that of the CP-conserving part, both numerator and denominator of the first and of the second term in the square brackets of Eq.(17) are multiplied by the same factor, $|R_2|$ and $|R_0|$, respectively, and then the effect of FSI does not work at all. The parameter $\epsilon$ is also independent of FSI, but $\omega$ is suppressed. In this case it is also impossible to understand the large size of the direct CP violation in terms of the FSI effects.

We discuss last the $\pi\pi$ contributions to the $K_L - K_S$ mass difference $\Delta m = m_L - m_S$. It is said that the short distance component dominates the mass difference through the effective $\Delta S = 2$ Hamiltonian \[29, 30\]. According to them typically 70% of $\Delta m$ can be described by the short distance contribution, and the remaining of $\sim 30\%$ in $\Delta m$ is attributed to the long distance component. Our interest is in the estimate of the $\pi\pi$ contribution to the mass difference. Pennington calculated this by using the once-subtracted dispersion relation with the experimental $\pi\pi$ S-wave phase shift, and obtained $\Delta m/\Gamma_S = 0.22 \pm 0.03 \ [30\]$, where $\Gamma_S$ is the total $K_S$ decay width and the experimental value is $7.36 \times 10^{-12}$ GeV.

We apply our scheme to the self-energy with $\Delta S = 2$, and consider the following analytic function except for the physical cut;

\[ \Sigma(s) = -2 \sum_{I=0,2} B_I(s) \left[ J_{\pi\pi}(s) + J_{\pi\pi}(s)T_I(s)J_{\pi\pi}(s) \right] B_I(s), \] (20)

which gives

\[ \Sigma(m_K^2) = 2m_K \Delta m + i m_K \Gamma_S. \] (21)

Indeed, we have

\[ \text{Im}\Sigma(m_K^2) = \rho_{\pi\pi}[2A_0^2 + 2A_2^2]|_{s=m_K^2} = m_K \Gamma_S, \] (22)

where $A_I$ denotes the CP-conserving amplitude defined in Eq.(11). We see that the energy dependence of $\text{Im}\Sigma(s)$ is almost the same below 1 GeV as the one by Pennington, where $m_K^2$ in $X$ of $B_I$ is replaced by $s$. The numerical value of $\Sigma(m_K^2)$ gives the ratio of $\Delta m_{\pi\pi}$ to $\Gamma_S$ as

\[ \frac{\text{Re}\Sigma}{2 \text{Im}\Sigma} = \frac{\Delta m_{\pi\pi}}{\Gamma_S} = 0.072 \sim 0.074 \] (23)

for $m_s + m_u = 120$ and 175 MeV, respectively. We note that if we modify the sizes of $A_0$ and $A_2$ so as to reproduce the experimental ones, the above ratio does not change, but we have $\Delta m_{\pi\pi}/\Delta m_{\text{exp}} = 0.14$, which should be compared with the value obtained by Pennington, $\Delta m_{\pi\pi}/\Delta m_{\text{exp}} = 0.46$. 


Figure 2: Energy dependence of $\text{Im}\Sigma(\sqrt{s})/\text{Im}\Sigma(m_K)$. The solid (dotted, dashed) line is for $m_s + m_u = 120 (150, 175)$ MeV.

It should be noticed that the enhancement by $R_0$ near the kaon mass is not due to the $\sigma$ resonance, but due to the whole rescattering dynamics without the pre-existing $\sigma$ pole \cite{37}. The effects expected by introducing the $\sigma$ pole through the linear $\sigma$ model \cite{38} or the nonlinear $\sigma$ model \cite{28} can be replaced by the FSI effects without the degrees of freedom of the genuine $\sigma$ resonance. This replacement resolves a problem between the final state phase $\exp[i\delta_0]$ and the width ignored in the $\sigma$ propagator \cite{38}, and does the difficulty that the $\sigma$ pole dominance may give a too large contribution to the $K_L - K_S$ mass difference \cite{39}.

The scheme for the FSI effects explained in this note is useful in the sense that it is less worried by the ambiguities concerning with the dispersion integrals. We have shown in this note that FSI alone is impossible to reproduce the expected size of $A_0$, but it naturally gives a large $\epsilon'/\epsilon$ ratio, if we have a sufficient size of $|A_0|$. It turns out that the two-pion contribution to the $m_L - m_S$ mass difference is about 14\% under the condition that the size of $A_0$ is sufficient. Thus, the crucial point for the $K_S \rightarrow \pi\pi$ decay is how to obtain $B_0 \sim 22 \times 10^{-8}$ GeV.

Acknowledgments

The author thanks K. Funakubo, Saga University, and K. Terasaki, Institute for Fundamental Physics, Kyoto University, for valuable comments. He also thanks the Department of Physics, Saga University for the hospitality extended to him.

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