Adiabatic Quantum Counting by Geometric Phase Estimation

Chi Zhang,1 Zhaohui Wei,2,3 and Anargyros Papageorgiou1,‡

1Department of Computer Science, Columbia University, New York, USA, 10027
2State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, Beijing, China, 100084
3Center for Quantum Technologies, National University of Singapore, Singapore, 117542

(Dated: August 21, 2009)

We design an adiabatic quantum algorithm for the counting problem, i.e., approximating the proportion, $\alpha$, of the marked items in a given database. As the quantum system undergoes a designed cyclic adiabatic evolution, it acquires a Berry phase $2\pi \alpha$. By estimating the Berry phase, we can approximate $\alpha$, and solve the problem. For an error bound $\epsilon$, the algorithm can solve the problem with cost of order $\left(\frac{1}{\epsilon}\right)^{3/2}$, which is not as good as the optimal algorithm in the quantum circuit model, but better than the classical random algorithm. Moreover, since the Berry phase is a purely geometric feature, the result may be robust to decoherence and resilient to certain noise.

PACS numbers: 03.67.Ac, 03.67.Lx

I. INTRODUCTION

Quantum algorithms can solve certain problems significantly faster than the known classical algorithms. However, one main obstacle to realize the faster quantum algorithms is the decoherence induced by the coupling environment. To overcome the obstacle, a novel quantum computation model, quantum adiabatic computation, is a promising candidate [1, 2, 3, 4, 5, 6, 7, 8]. The model is believed to enjoy inherent robustness against the impact of decoherence in [4, 5, 6]. Even though not all researchers share this view [7, 8], adiabatic quantum computation still attracts considerable attention. It has been proved to be polynomially equivalent to the quantum circuit model [1, 2]. For instance, in the adiabatic model searching an unordered database requires time of the same order of magnitude as Grover’s algorithm [3, 9], but few adiabatic algorithms are known that have performance similar to that of the corresponding algorithm in the quantum circuit model. In this paper, we show an adiabatic algorithm for the counting problem. The task of the counting problem is to approximate the proportion of marked items in an $N$-item database, which is denoted by $\alpha$. In classical computation, the counting problem needs $O(N)$ evaluations in the worst case setting, and $O((\frac{1}{\epsilon})^2)$ in the randomized setting, for error $\epsilon$. There exists a quantum algorithm solving the problem in $O(\frac{1}{\epsilon})$ evaluations [10]. Moreover, Nayak and Wu [11] showed that the quantum algorithm given in [10] is optimal in the quantum circuit model. The counting problem is also central for many continuous problems, such as high dimensional integration, path integration [12, 13, 14] and eigenvalue approximation [15].

Our algorithm for the counting problem is based on the Berry phase acquired in the adiabatic evolution. When a quantum system undergoes a cyclic adiabatic evolution, it acquires a geometric phase, which is known as the Berry phase [16]. In the algorithm, we design an adiabatic evolution such that the resulting Berry phases encode the solution of the problem. Then we can solve the problem by estimating the Berry phase after the adiabatic evolution. Since Berry phases are global phases and cannot be measured directly, we let two parts of a superposition undergo the same cyclic adiabatic evolution in different directions. After the adiabatic evolution, the dynamic phases of the two parts cancel out, and the Berry phases are $\pm 2\pi \alpha$. Then, by estimating the relative phase between the two parts of the superposition, we can estimate $\alpha$. In the static model, the algorithm has a runtime of order $(\frac{1}{\epsilon})^{3/2}$, which beats the optimal classical algorithm in the randomized setting. Usually, in adiabatic algorithms it is the final state that encodes the solution of the problem, while in our algorithm it is the Berry phase. Since the Berry phase is a purely geometric feature, i.e., it only depends on the path of evolution, and is independent of details of how the evolution is executed, the result is resilient to certain small errors.

We begin with some preliminaries about adiabatic algorithms and the Berry phase, which are helpful in presenting our main result. Consider a quantum system in a state $|\psi(t)\rangle$, $(0 \leq t \leq T)$, which evolves according to the Schrödinger
equation
\[ i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle, \]
where \( H(t) \) is the Hamiltonian of the system at time \( t \). If the system is initially in its ground state, and the Hamiltonian varies slowly enough, it will remain close to the ground state of \( H(t) \), at time \( t \). Let \( |E_0(t)\rangle \) be the ground state of the Hamiltonian, and \( E_0(t) \) be the corresponding eigenvalue. If \( H(T) = H(0) \), then \( |\psi(T)\rangle \) is close to \( |\psi(0)\rangle \), with the exception of a global phase. The phase can be divided into two parts: the dynamic phase
\[ \theta = - \int_0^T E_0(t) dt, \]
and the geometric Berry phase
\[ \gamma = i \int_0^T \langle E_0 | \frac{d}{dt} E_0 \rangle dt. \]

The Berry phase depends only on the path taken, not on how fast the path is traversed. Hence, if we design a cyclic path of Hamiltonians, the Berry phase is totally determined.

The remainder of the present paper is organized as follows. In section II, we provide the basic adiabatic evolution used in the algorithm, which can encode the solution to a Berry phase of a quantum system. Then we give the adiabatic algorithm for the counting problem. In section III, we will show the relationship between the accuracy of the algorithm and the time it used. In the Appendix, we show the detailed derivation of the difference of real relative phase and the Berry phase.

II. THE ADIABATIC ALGORITHM FOR THE COUNTING PROBLEM

In this section, we will show how to encode the solution to the Berry phases. We use a function \( f : \{0, \cdots, N-1\} \rightarrow \{0, 1\} \) to denote whether an item is marked, i.e., \( f(s) = 1 \), if the \( s \)-th item is marked; \( f(s) = 0 \), otherwise. For an error bound \( \epsilon \), the algorithm has \( m = \log(\frac{1}{\epsilon}) \) adiabatic evolutions. In each evolution, we use \( n + 1 \) qubits, where \( n = \log N \). The quantum state can be divided into two systems: the control system, which has 1 qubit, and the computing system, which has \( n \) qubits. The computing system is in an \( N \)-dimensional Hilbert space, whose basis states are denoted as \( |k\rangle \), where \( k = 0, \cdots, N-1 \). We use an equal superposition of all basis states as the initial state in the computing system
\[ |\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle, \]
and use \( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) as the initial state in the control system. Hence, the initial state of the whole system is
\[ |\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |\psi_0\rangle. \]

In each evolution, the initial Hamiltonian of the computing system is
\[ H_0 = I - |\psi_0\rangle \langle \psi_0|. \]

Typically, in quantum algorithms, the fundamental oracle used is
\[ |x\rangle |y\rangle \rightarrow |x\rangle |y \oplus f(x)\rangle. \]

In our algorithm, we modify the oracle to
\[ |x\rangle |y\rangle |z\rangle \rightarrow |x\rangle |y \oplus f(x)\rangle |z \oplus (y \cdot f(x))\rangle. \]

With a 2-qubit auxiliary state \( \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \), using the oracle on the initial state, and then discarding the auxiliary qubits, the operation of the oracle can be written as
\[ \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp(-i\frac{\pi}{2} f(k)) |k\rangle. \]
Let $\beta = 1 - \alpha$, $M = \alpha N$, and

\[
|\hat{0}\rangle = \frac{1}{\sqrt{N-M}} \sum_{s: f(s) = 0} |s\rangle,
|\hat{1}\rangle = \frac{1}{\sqrt{M}} \sum_{s: f(s) = 1} |s\rangle,
\]

the initial state can be rewritten as

\[
|\psi_0\rangle = \sqrt{\beta} |\hat{0}\rangle + \sqrt{\alpha} |\hat{1}\rangle,
\]

and the states after repeatedly using the oracle are

\[
|\psi_k\rangle = \sqrt{\beta} |\hat{0}\rangle + (-i)^k \sqrt{\alpha} |\hat{1}\rangle,
\]

for $k = 1, 2, 3$.

In the adiabatic algorithm, we define four Hamiltonian oracles as

\[
H_k = I - |\psi_k\rangle \langle \psi_k|,
\]

for $k = 0, 1, 2, 3$. Then consider a linear interpolation between the four oracles, that is

\[
H(t) = \sum_{k=0}^{3} s_k(t) H_k,
\]

where $\sum s_k(t) = 1$, for any $0 \leq t \leq T$.

In the $j$-th evolution, we choose

\[
s_0 = \frac{1}{2} (1 + \cos \theta_j),
 s_1 = \frac{1}{2} \sin \theta_j,
 s_2 = \frac{1}{2} (1 - \cos \theta_j),
 s_3 = -\frac{1}{2} \sin \theta_j,
\]

where $\theta_j$ is a function from $[0, T]$ to $[0, 2\pi]$, satisfying $\theta_j(0) = 0$ and $\theta_j(T) = 2^j \pi$. Since the Berry phase only depends on the path of the evolution of the Hamiltonian, the choice of the function $\theta(t)$ does not affect our result. Combining Eq. (13) and Eq. (15), the Hamiltonian of the $j$-th evolution is

\[
H(\theta_j) = I - \frac{1}{2} (1 + \cos \theta_j) |\psi_0\rangle \langle \psi_0| - \frac{1}{2} \sin \theta_j |\psi_1\rangle \langle \psi_1| - \frac{1}{2} (1 - \cos \theta_j) |\psi_2\rangle \langle \psi_2| + \frac{1}{2} \sin \theta_j |\psi_3\rangle \langle \psi_3|
\]

\[
= I - |\psi(\theta_j)\rangle \langle \psi(\theta_j)|,
\]

where

\[
|\psi(\theta_j)\rangle = \sqrt{\beta} |\hat{0}\rangle + e^{-i\theta_j} \sqrt{\alpha} |\hat{1}\rangle,
\]

and $\theta_j \in [0, 2\pi]$. Clearly, the ground state of $H(\theta_j)$ is $|\psi(\theta_j)\rangle$. We set the Hamiltonian of the whole system in the $j$-th adiabatic evolution as

\[
|0\rangle \langle 0| \otimes H(\theta_j) + |1\rangle \langle 1| \otimes H(-\theta_j).
\]

Then the Berry phases of the computing system after the evolution are $\gamma_j$ and $-\gamma_j$, where

\[
\gamma_j = i \int_0^{2^j \pi} \langle \psi | \frac{d}{d\theta} \psi \rangle d\theta = \int_0^{2^j \pi} \alpha d\theta = 2^j \pi \alpha.
\]

Since the dynamic phase is the same in both parts, the final state is

\[
|\psi_j(T)\rangle = \frac{1}{\sqrt{2}} (e^{i\gamma_j} |0\rangle + e^{-i\gamma_j} |1\rangle) \otimes |\psi_0\rangle.
\]
Hence, at the end of the evolution, the relative phase of the first qubit is \( \Gamma_j = 2\gamma_j = 2\pi(2^j \alpha) \). In this way, we successfully encode the solution to the relative phases of a set of quantum states.

In the algorithm, we do not use the phase estimation procedure in [17] to estimate \( \alpha \) from the Berry phases, in order to avoid unnecessary entanglement. We use Kitaev’s equivalent procedure instead [18]. As described above, in the \( j \)-th adiabatic evolution of the algorithm, we prepare a quantum state whose relative phase is \( 2\pi(2^j \alpha) \), for \( j = 1, \cdots, m \). A measurement for the first qubit in \( |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) and \( |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \) basis gives the result \( |+\rangle \) with probability

\[
p = \cos^2(\gamma) = \cos^2(2^j \pi \alpha).
\]

Hence, if we apply the process several times, we can approximate the probability. More precisely, let \( q = r/R \) be the ratio between the number of \( r \) of results \( |+\rangle \) and the number \( R \) of measurements. Then Chernoff’s bound

\[
Prob(|p - q| \geq \delta) \leq 2e^{-2\delta^2 R}
\]

shows that for a fixed \( \delta \), the error is smaller than \( \epsilon \) for only \( O(\log(1/\epsilon)) \) number of measurements. In the application, we only need an error that is smaller than \( \pi/8 \). Then, we obtain an estimation of \( 2^j \alpha \), modulo 1, with error 1/16. Let

\[
\alpha_j = \sum_{j=1}^{\infty} 2^{-j} \alpha_j,
\]

for \( \alpha_j \in \{0, 1\} \), and \( \alpha_1 = 0 \) since \( \alpha < 1/2 \). We also use \( \alpha_1 \cdots \alpha_p \) to denote the binary fraction \( \sum_{j=1}^{p} 2^{-j} \alpha_j \). For \( j = 1, \cdots, m \), we replace the known approximate value of \( 2^j \alpha \) by \( \eta_j \), the closest number from the set \( \{0, 1/8, 2/8, \cdots, 7/8\} \). Hence, we have

\[
|2^j \alpha - \eta_j|_1 < 1/16 + 1/16 = 1/8.
\]

Since if \( |y - 2\alpha|_1 < \delta < 1/2 \), then \( |y_0 - \alpha|_1 < \delta/2 \) or \( |y'_1 - \alpha|_1 < \delta/2 \), where \( y_0', y_1' \) are the solutions to the equation \( 2y' \equiv y (\text{mod} 1) \), we can start from \( 2^m \alpha \) and increase the precision in the following way: Set \( \eta_m = \alpha_m \alpha_{m+1} \cdots \alpha_{m+2} = \eta_m \) and proceed by iteration:

\[
\alpha_j = \begin{cases} 0 & \text{if } |\alpha_{j+1} \alpha_{j+2} - \eta_j|_1 < 1/4, \\ 1 & \text{if } |\alpha_{j+1} \alpha_{j+2} - \eta_j|_1 < 1/4, \end{cases}
\]

for \( j = m - 1, \cdots, 1 \). By a simple induction, \( \alpha_1 \alpha_2 \cdots \alpha_m \) can estimate \( \alpha \) with error less than \( 2^{-m} = \epsilon \).

### III. Running Time of the Adiabatic Algorithm

In this section, we consider the accuracy of the evolutions and the running time of the algorithm. It is easy to see that under the Hamiltonian given in Eq. (16) the actual state in the computing system, \( |\varphi(\theta)\rangle \), always stays in the subspace spanned by \( \{|0\rangle, |1\rangle\} \). Then \( H(\theta) \) can be rewritten as

\[
H(\theta) = \begin{pmatrix} \alpha & -\sqrt{\alpha \beta}e^{i\theta} \\ -\sqrt{\alpha \beta}e^{-i\theta} & -\alpha \end{pmatrix}
\]

in the subspace. Assume \( \omega = \frac{4\pi}{\beta} \) is constant, and \( \omega \ll 1 \). Let \( |\varphi(t)\rangle \) be the \( t \)-time state in the system which is initially in \( |\psi_0\rangle \), and evolving under \( H(\omega t) \). By solving the Schrodinger equation (16), we attain

\[
|\varphi(t)\rangle = e^{-i\frac{\omega}{2}(Ae^{i\omega_1 t} + Be^{i\omega_2 t})}|0\rangle + e^{-i\frac{\omega}{2}(Ce^{-i\omega_1 t} + De^{-i\omega_2 t})}|1\rangle,
\]

where

\[
\omega_{1,2} = \frac{\omega \pm \sqrt{(1 - \omega)^2 + 4\omega^2}}{2},
\]
From the assumption $\omega$, the relative phase will be the argument of $\langle H \rangle$. Hence, we have
\[
A = \frac{(1 - \omega)^2 - \alpha(1 - 3\omega) + (\beta - \omega)E}{(1 - \omega)^2 + 4\omega + (\beta - \alpha - \omega)E} \sqrt{\beta},
\]
\[
B = \frac{\alpha(1 + \omega) - \alpha E}{(1 - \omega)^2 + 4\omega + (\beta - \alpha - \omega)E} \sqrt{\beta},
\]
\[
C = \frac{(1 + \omega)^2 - \beta(1 + 3\omega) - (\alpha + \omega)E}{(1 - \omega)^2 + 4\omega + (\beta - \alpha - \omega)E} \sqrt{\alpha},
\]
\[
D = \frac{\beta(1 - \omega) + \beta E}{(1 - \omega)^2 + 4\omega + (\beta - \alpha - \omega)E} \sqrt{\alpha},
\]
(29)
where $E = \sqrt{(1 - \omega)^2 + 4\omega}$, see the Appendix. On the other hand, denote the quantum state evolving under the Hamiltonian $H(-\theta) = H(-\omega t)$ by $|\varphi'(t)\rangle$, which can be obtained from $|\varphi\rangle$ by exchanging all $\omega$ by $-\omega$.

Then, the final state of the $j$-th evolution is
\[
|\psi_j(T)\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\varphi(T)\rangle + |1\rangle|\varphi'(T)\rangle),
\]
(30)
where $T = 2^j \pi/\omega$, which is the time of the $j$-th evolution, for $j = 1, \ldots, m$. As indicated before, we will measure the relative phase of the first qubit, and use it as an approximation of $2\pi(2^j\alpha)$. Let $|\varphi_\perp\rangle$ be a state in the span space of $|0\rangle$ and $|1\rangle$, which is orthogonal to $|\varphi\rangle$, from Eq.(30),
\[
|\psi_j(T)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \langle\varphi(T)|\varphi'(T)\rangle|1\rangle)|\varphi\rangle + \frac{1}{\sqrt{2}}(|\varphi_\perp(T)|\varphi'(T)\rangle|1\rangle)|\varphi_\perp(T)\rangle.
\]
(31)
Hence, with probability
\[
p_s = \frac{1 + |\langle\varphi(T)|\varphi'(T)\rangle|^2}{2},
\]
(32)
the relative phase will be the argument of $\langle\varphi(T)|\varphi'(T)\rangle$. It can be checked that
\[
\langle\varphi(T)|\varphi'(T)\rangle = (AA' + DD')e^{i\mu_1} + (BB' + CC')e^{-i\mu_1} + (AB' + C'D)e^{i\mu_2} + (A'B + C'D)e^{-i\mu_2},
\]
(33)
where $A', B', C', D'$ are derived from $A, B, C, D$ by exchanging $\omega$ by $-\omega$, and
\[
\mu_1 = \frac{1}{2}(\sqrt{(1 - \omega)^2 + 4\omega} - \sqrt{(1 + \omega)^2 - 4\omega})T,
\]
\[
\mu_2 = \frac{1}{2}(\sqrt{(1 - \omega)^2 + 4\omega} + \sqrt{(1 + \omega)^2 - 4\omega})T
\]
(34)
From the assumption $\omega \ll 1$, we have
\[
AA' + DD' = 1 - 3\alpha\beta\omega^2 + O(\omega^3),
\]
\[
BB' + CC' = -\alpha\beta\omega^2 + O(\omega^3),
\]
\[
AB' + C'D = 2\alpha\beta\omega^2 + O(\omega^3),
\]
\[
A'B + C'D = 2\alpha\beta\omega^2 + O(\omega^3).
\]
(35)
Hence,
\[
p_s = \frac{1 + |\langle\varphi(T)|\varphi'(T)\rangle|^2}{2} \geq 1 - 8\alpha\beta\omega^2,
\]
(36)
and
\[
|\text{arg}(\langle\varphi(T)|\varphi'(T)\rangle) - \mu_1| \leq 8\alpha\beta\omega^2.
\]
(37)
Moreover,
\[
\mu_1 = 2\alpha\omega T + 2\alpha\beta(\beta - \alpha)\omega^3 T + O(\omega^5T)
\]
\[
= 2\pi(2^j\alpha)(1 + \beta(\beta - \alpha)\omega^2 + O(\omega^3)),
\]
(38)
in the \( j \)-th evolution. Hence, the difference between the expected relative phase and \( 2\pi(2^j\alpha) \) is

\[
\Delta \leq 2\pi(1 - p_s) + |\arg((\varphi_+(T)|\varphi_-(T))) - \mu_1| + |\mu_1 - 2\pi(2^j\alpha)| = O(2^j\omega^2). \tag{39}
\]

For the estimation of \( 2^j\alpha \) modulo 1 with error less than \( 1/16 \), it is enough to make \( \Delta < \frac{2\pi}{32} \), and then estimate the relative phase to its expected value within error \( 2\pi/32 \). To satisfy the first condition, in the \( j \)-th evolution we set

\[
\omega = \omega_j = O\left(\frac{1}{\sqrt{2^j}}\right) \tag{40}
\]

in Eq. \ref{eq:39} When estimating the relative phase, we can boost the success probability using repetitions. In our case, \( O(m - j) \) repetitions yield overall success probability greater than \( 1/2 \). Since the time of the \( j \)-th evolution is \( \frac{2\pi}{\omega_j} \), the total running time is

\[
T_{\text{total}} = \sum_{j=1}^{m} O\left(\frac{2^j}{\omega_j}(m - j)\right) = O((2^m)^{3/2}) = O\left(\frac{1}{\epsilon}^{3/2}\right). \tag{41}
\]

Therefore, the adiabatic algorithm has an \( O\left(\frac{1}{\epsilon}^{3/2}\right) \) running time. As we know the optimal quantum algorithm has a running time \( O\left(\frac{1}{\epsilon}\right) \), our adiabatic algorithm is not as good as the quantum algorithm given in the circuit model. However, it is better than the best classical algorithm, which needs \( O\left(\frac{1}{\epsilon}^2\right) \) running time.

### IV. CONCLUSION

In conclusion, we have proposed an adiabatic algorithm for the quantum counting problem. The key idea of the algorithm is to construct Berry phases which equal \( 2\pi(2^j\alpha) \), for \( j = 1, \cdots, m \), where \( \alpha \) is the proportion of marked items in the database, \( m = \log\left(\frac{1}{\epsilon}\right) \). The algorithm has a running time of \( O\left(\frac{1}{\epsilon}^{3/2}\right) \), which beats the classical random algorithm.

There are some special characteristics of our algorithm. Firstly, the solution of the problem is encoded in the phase difference of the final state, rather than the ground state of the final Hamiltonian. The final information is achieved by measuring relative phase, rather than the usual quantum measurement. Next, different from usual adiabatic algorithms, such as in \[3\], we use more than one Hamiltonian as oracles to construct the evolution path. The idea of designing adiabatic algorithms based on geometric Berry phases could also be applied to solve other problems, such as searching an unordered database \[21\].

We are grateful to Joseph F. Traub, Henryk Woźniakowski, Columbia University, Luming Duan, Yongjian Han, University of Michigan, and Mingsheng Ying, Tsinghua University, for their very helpful discussions and comments.

### V. APPENDIX

In the Appendix, we will show the details of how to derive the quantum state under the Hamiltonian \( H(\theta) = H(\omega t) \). The quantum system is initially in a state \( |\psi\rangle = (\sqrt{\beta}, \sqrt{\alpha}) \), then evolves under the Hamiltonian

\[
H(\theta) = \begin{pmatrix}
\alpha & -\sqrt{\alpha\beta}e^{-i\theta} \\
-\sqrt{\alpha\beta}e^{i\theta} & \beta
\end{pmatrix}. \tag{42}
\]

Let \( |\varphi(t)\rangle \) be the \( t \)-time quantum state in the system. Then the Schrödinger equation \( (\Box) \) turns out to be,

\[
i\frac{d}{dt}|\varphi\rangle = H|\varphi\rangle = \begin{pmatrix}
\alpha & -\sqrt{\alpha\beta}e^{i\omega t} \\
-\sqrt{\alpha\beta}e^{-i\omega t} & \beta
\end{pmatrix}|\varphi\rangle. \tag{43}
\]

Let \( |\varphi\rangle = \begin{pmatrix} x \\ y \end{pmatrix} \), then

\[
i\frac{d}{dt}x = \alpha x - \sqrt{\alpha\beta}e^{i\omega t} y, \tag{44}
\]

\[
i\frac{d}{dt}y = -\sqrt{\alpha\beta}e^{-i\omega t} x + \beta y. \tag{45}
\]
From Eq (43),

\[ y = \sqrt{\frac{\alpha}{\beta}} e^{-i\omega t} x - i \frac{1}{\sqrt{\alpha \beta}} e^{-i\omega t} \frac{dx}{dt}, \]

\[ \frac{dy}{dt} = \frac{1}{\sqrt{\alpha \beta}} e^{-i\omega t} \left[ -i\alpha \omega x + (\alpha - \omega) \frac{dx}{dt} - i \frac{d^2 x}{dt^2} \right]. \tag{46} \]

Substituting Eq (46) into Eq (45),

\[ \frac{d^2 x}{dt^2} + i(1 - \omega) \frac{dx}{dt} + \alpha \omega x = 0. \tag{47} \]

Similarly, we can derive

\[ \frac{d^2 y}{dt^2} + i(1 + \omega) \frac{dy}{dt} - \beta \omega y = 0. \tag{48} \]

The roots of Eq (47) and Eq (48) are \(-\frac{i}{2} \pm i\omega_{1,2}\) separately, where

\[ \omega_{1,2} = \frac{\omega \pm \sqrt{(1 - \omega)^2 + 4\alpha \omega}}{2}. \tag{49} \]

Then

\[ x = e^{-i\frac{\theta}{2} t} (A e^{i\omega_{1} t} + B e^{i\omega_{2} t}), \]

\[ y = e^{-i\frac{\theta}{2} t} (C e^{-i\omega_{1} t} + D e^{-i\omega_{2} t}). \tag{50} \]

From the initial state \(|\psi\rangle = \sqrt{\beta} |\hat{0}\rangle + \sqrt{\alpha} |\hat{1}\rangle\), we have

\[ A + B = \sqrt{\beta}; C + D = \sqrt{\alpha}. \tag{51} \]

Substituting Eq (50) into Eq (44) it can be derived that

\[ \frac{D}{A} = \frac{-B}{C} = \lambda = \frac{2\sqrt{\alpha \beta}}{(\beta - \alpha - \omega) + \sqrt{(1 - \omega)^2 + 4\alpha \omega}}. \tag{52} \]

Then, we attain that

\[ A = \lambda \sqrt{\alpha} + \sqrt{\beta} = \frac{(1 - \omega)^2 - \alpha(1 - 3\omega) + (\beta - \omega)E}{1 + \lambda^2}, \]

\[ B = \lambda \sqrt{\beta} - \sqrt{\alpha} = \frac{\alpha(1 + \omega) - \alpha E}{1 + \lambda^2}, \]

\[ C = \sqrt{\alpha} - \lambda \sqrt{\beta} = \frac{(1 + \omega)^2 - \beta(1 + 3\omega) - (\alpha + \omega)E}{1 + \lambda^2}, \]

\[ D = \lambda \sqrt{\alpha} + \sqrt{\beta} = \frac{\beta(1 - \omega) + \beta E}{1 + \lambda^2}, \tag{53} \]

where \(E = \sqrt{(1 - \omega)^2 + 4\alpha \omega}\). On the other hand, denote the quantum state evolving under the Hamiltonian \(H(-\theta) = H(-\omega t)\) by \(|\varphi'(t)\rangle\), and it can be derived from \(|\varphi\rangle\) by exchanging all \(\omega\) by \(-\omega\).

Then, the final state of the \(j\)-th evolution is

\[ |\psi_j(T)\rangle = \frac{1}{\sqrt{2}} (|0\rangle |\varphi(T)\rangle + |1\rangle |\varphi'(T)\rangle), \tag{54} \]

where \(T = 2^j \pi/\omega\), for \(j = 1, \ldots, m\). Let \(|\varphi_{\perp}\rangle\) be a state in the span space of \(|\hat{0}\rangle\) and \(|\hat{1}\rangle\), which is orthogonal to \(|\varphi\rangle\), from Eq (53),

\[ |\psi_j(T)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + \langle \varphi(T)|\varphi'(T)|1\rangle)|\varphi\rangle + \frac{1}{\sqrt{2}} (\langle \varphi_{\perp}(T)|\varphi'(T)|1\rangle)|\varphi_{\perp}(T)\rangle). \tag{55} \]
Hence, with probability
\[ p_s = \frac{1 + |\langle \varphi(T) | \varphi'(T) \rangle|^2}{2}, \]  
(56)
the relative phase will be the argument of $\langle \varphi(T) | \varphi'(T) \rangle$. Since
\[ \langle \varphi(T) | \varphi'(T) \rangle = (AA' + DD')e^{i\mu_1} + (BB' + CC')e^{-i\mu_1} + (AB' + C'D)e^{i\mu_2} + (A'B + CD')e^{-i\mu_2}, \]  
(57)
where $A', B', C', D'$ are derived from $A, B, C, D$ by exchanging $\omega$ by $-\omega$, and
\[ \mu_1 = \frac{1}{2}(\sqrt{(1 - \omega)^2 + 4\alpha\omega} - \sqrt{(1 + \omega)^2 - 4\alpha\omega})T, \] \[ \mu_2 = \frac{1}{2}(\sqrt{(1 - \omega)^2 + 4\alpha\omega} + \sqrt{(1 + \omega)^2 - 4\alpha\omega})T. \]  
(58)
From the assumption $\omega \ll 1$, the relative phase can be estimated by $\langle \varphi(T) | \varphi'(T) \rangle$, from $AA' + DD'$, $BB' + CC'$, $AB' + C'D$ and $A'B + CD'$.

Since the above 4 terms share the same denominator, we can first calculate the denominator $F$, then calculate their numerators separately. The denominator is
\[ F = [(1 - \omega)^2 + 4\alpha\omega + (1 - 2\alpha - \omega)\sqrt{(1 - \omega)^2 + 4\alpha\omega}] [(1 + \omega)^2 - 4\alpha\omega + (1 - 2\alpha + \omega)\sqrt{(1 + \omega)^2 - 4\alpha\omega}] \] \[ = [(1 - \omega)^2 + 4\alpha\omega] [(1 - 2\alpha - \omega)\sqrt{(1 - \omega)^2 + 4\alpha\omega}] [(1 + \omega)^2 - 4\alpha\omega] \] \[ + [(1 - \omega)^2 + 4\alpha\omega] [(1 + 2\alpha + \omega) + (1 - 2\alpha - \omega)\omega + 2\alpha\beta\omega^2] \] \[ + [(1 - 2\alpha - \omega)^2 - (1 - 2\alpha)\omega + 2\alpha\beta\omega^2] [(1 + 1 - 2\alpha)\omega + 2\alpha\beta\omega^2] + O(\omega^3) \] \[ = 1 - 2(1 - 8\alpha\beta)\omega^2 + 2(1 - 2\alpha)[1 - 2(1 - 5\alpha\beta)\omega^2] + (1 - 2\alpha)^2 - \omega^2 - (1 - 2\alpha)^2(1 - 8\alpha\beta)\omega^2 + O(\omega^3) \] \[ = 4\beta^2(1 - 2\beta(1 - 4\alpha)\omega^2) + O(\omega^3). \]  
(59)
Then
\[ (AA' + DD')F \] \[ = \beta[(1 - \omega)^2 - \alpha(1 - 3\omega) + (1 - \alpha - \omega)\sqrt{(1 - \omega)^2 + 4\alpha\omega}] [(1 + \omega)^2 - \alpha(1 + 3\omega) + (1 - \alpha + \omega)\sqrt{(1 + \omega)^2 - 4\alpha\omega}] \] \[ + \alpha\beta^2[(1 - \omega) + \sqrt{(1 - \omega)^2 + 4\alpha\omega}] [(1 + \omega) + \sqrt{(1 + \omega)^2 - 4\alpha\omega}] \] \[ = \beta[(1 - \omega)^2 - 2\alpha(1 - 5\omega^2) + \alpha^2(1 - 9\omega^2) + \alpha\beta(1 - \omega^2)] + \beta[\beta + \beta(2\beta - 1)\omega + (2\alpha - 1)\omega^2][(1 - 2\alpha)\omega + 2\alpha\beta\omega^2] \] \[ + \beta[\beta - \beta(2\beta - 1)\omega + (2\alpha - 1)\omega^2][(1 + 1 - 2\alpha)\omega + 2\alpha\beta\omega^2] + \beta(1 - \omega^2)(2\alpha - 1)\omega^2][(1 + 1 - 2\alpha)\omega + 2\alpha\beta\omega^2] + O(\omega^3) \] \[ = 4\beta^2(1 - 2\beta(1 - 5\alpha)\omega^2) + O(\omega^3). \]  
(60)
Hence,
\[ AA' + DD' = \frac{4\beta^2(1 - \beta(2 - 5\alpha)\omega^2) + O(\omega^3)}{4\beta^2(1 - 2\beta(1 - 4\alpha)\omega^2) + O(\omega^3)} = 1 - 3\alpha\beta\omega^2 + O(\omega^3). \]  
(61)
\[ (BB' + CC')F \] \[ = \alpha^2\beta(1 + \omega - \sqrt{(1 - \omega)^2 + 4\alpha\omega})(1 - \omega - \sqrt{(1 + \omega)^2 - 4\alpha\omega}) \] \[ + \alpha[(1 + \omega)^2 - \beta(1 + 3\omega) - (\alpha + \omega)\sqrt{(1 - \omega)^2 + 4\alpha\omega}] [(1 + \omega)^2 - \beta(1 - 3\omega) - (\alpha - \omega)\sqrt{(1 + \omega)^2 - 4\alpha\omega}] \] \[ = \alpha[\alpha + (9\alpha\beta + \beta^2 - 2)\omega^2] + [\alpha + (1 - 2\alpha)\alpha\omega + (1 - 2\alpha)\omega^2][(1 - 1 - 2\alpha)\omega + 2\alpha\beta\omega^2] \] \[ + [\alpha - (1 - 2\alpha)\alpha\omega + (1 - 2\alpha)\omega^2][(1 + 1 - 2\alpha)\omega + 2\alpha\beta\omega^2] + \alpha(\alpha - \omega^2)(1 - 1 - 2\alpha)\omega + 2\alpha\beta\omega^2] + O(\omega^3) \] \[ = -4\alpha\beta^3 + O(\omega^3). \]  
(62)
Hence,
\[ BB' + CC' = \frac{-4\alpha\beta^3 + O(\omega^3)}{4\beta^2(1 - 2\beta(1 - 4\alpha)\omega^2) + O(\omega^3)} = -\alpha\beta\omega^2 + O(\omega^3). \]  
(63)
\[(AB' + C'D)F\]
\[= \alpha\beta[(1 - \omega)^2 - \alpha(1 - 3\omega) + (\beta - \omega)\sqrt{(1 - \omega)^2 + 4\alpha\omega}] \cdot (1 - \omega - \sqrt{{(1 + \omega)^2 - 4\alpha\omega})}
\]
\[+ \alpha\beta[(1 - \omega)^2 - \beta(1 - 3\omega) + (\alpha - \omega)\sqrt{(1 + \omega)^2 - 4\alpha\omega}] \cdot (1 - \omega + \sqrt{{(1 - \omega)^2 + 4\alpha\omega})}
\]
\[= \alpha\beta(1 - \omega)(1 - \omega + 2\omega^2) + \alpha\beta(1 - (3 - \beta)x + 2\omega^2)(1 - (1 - 2\alpha)x + 2\alpha\beta^2)
\]
\[- \alpha\beta(1 + (3 - 2\beta)x + 2\omega^2)(1 + (1 - 2\alpha)x + 2\alpha\beta^2)
\]
\[- (1 - 2\omega)(1 - (1 - 2\alpha)x + 2\omega^2) \cdot (1 + (1 - 2\alpha)x + 2\alpha\beta^2) + O(\omega^3)
\]
\[= 8\alpha\beta^3\omega^3 + O(\omega^3).
\]
Hence,
\[AB' + C'D = \frac{8\alpha\beta^3\omega^3 + O(\omega^3)}{4\beta^2(1 - 2\beta(1 - 4\alpha)x)^2 + O(\omega^3)} = 2\alpha\beta^2 + O(\omega^3).
\]
Since \[A'B + CD' = (AB' + C'D)',\]
\[A'B + CD' = 2\alpha\beta^2 + O(\omega^3).
\]
Hence,
\[
\langle \varphi(T)|\varphi'(T)\rangle = (1 - 3\alpha\beta^2)e^{i\mu_1} - \alpha\beta^2e^{-i\mu_1}
\]
\[+ 4\alpha\beta^2 \cos(\mu_2) + O(\omega^3).
\]
So
\[p_n = \frac{1 + |\langle \varphi(T)|\varphi'(T)\rangle|^2}{2} \geq 1 - 8\alpha\beta^2,
\]
and
\[|\arg(\langle \varphi(T)|\varphi'(T)\rangle) - \mu_1| \leq 8\alpha\beta^2.
\]
\[T = O((\frac{1}{e})^{3/2})
\]

[1] E. Farhi, S. Gutmann, Phys. Rev. A 57, 2403 (1998)
[2] D. Aharonov, W. V. Dam, J. Kempe, Z. Landau, S. Lloyd, O. Regev, Proceedings 45th FOCS, 42-51 (2004)
[3] J. Roland, N. J. Cerf, Phys. Rev. A, 65, 042308 (2002)
[4] A. M. Childs, E. Farhi, J. Preskill, Phys. Rev. A 65, 012322 (2001)
[5] J. Roland, N. J. Cerf, Phys. Rev. A 71, 032330 (2005)
[6] D. A. Lidar, Phys. Rev. Lett. 100, 160506 (2008)
[7] M. S. Sarandy, D. A. Lidar, Phys. Rev. Lett. 95, 250503 (2005)
[8] S. Ashhab, J. R. Johansson, F. Nori, Phys. Rev. A. 74, 052330 (2006)
[9] L. K. Grover, Phys. Rev. Lett. 79, 325 (1997)
[10] G. Brassard, P. Hoyer, M. Mosca, A. Tapp, Contemporary Mathematics, vol. 305, Am. Math. Soc., 53-74 (2002); Also quant-ph/0005055 (2000)
[11] A. Nayak, F. Wu, Proceedings 31st STOC 384-393 (1999)
[12] S. Heinrich, J. Complexity, 18, 1-50 (2002)
[13] J. F. Traub, H. Woźniakowski, Quantum Information Processing 1(5), 365-388 (2002)
[14] S. Heinrich, E. Novak, Optimal Summation and Integration by Deterministic, Randomized, and Quantum Algorithms, 4th International Conference on Monte Carlo and Quasi-Monte Carlo Methods, Hong Kong, (2000)
[15] A. Papageorgiou, H. Woźniakowski, Quantum Information Processing 4(2), 87-127 (2005)
[16] M. V. Berry, Proc. R. Soc. Lond., A 392, 45-57 (1984)
[17] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information (2000)
[18] A. Y. Kitaev, A. H. Shen, M. N. Vyalyi, Classical and Quantum Computation (2002)
[19] L. I. Schiff, Quantum Mechanics (McGraw-Hill, New York, 1955)
[20] E. Farhi, J. Goldstone, S. Gutmann, quant-ph/0208135 (2002)
[21] Z. Wei, C. Zhang, C. OH, to be appear