Cosmological Perturbations in Hořava-Lifshitz Gravity

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We study cosmological perturbations in Hořava-Lifshitz Gravity, a recently proposed potentially ultraviolet-complete quantum theory of gravity. We consider scalar metric fluctuations about a homogeneous and isotropic space-time. Starting from the most general metric, we work out the complete second order action for the perturbations. We then make use of the residual gauge invariance and of the constraint equations to reduce the number of dynamical degrees of freedom. At first glance, it appears that there is an extra scalar metric degree of freedom. However, introducing the Sasaki-Mukhanov variable, the combination of spatial metric fluctuation and matter inhomogeneity for which the action in General Relativity has canonical form, we find that this variable has the standard time derivative term in the second order action, and that the extra degree of freedom is non-dynamical. The limit $\lambda \to 1$ is well-behaved, unlike what is obtained when performing incomplete analyses of cosmological fluctuations. Thus, there is no strong coupling problem for Hořava-Lifshitz gravity when considering cosmological solutions. We also compute the spectrum of cosmological perturbations. If the potential in the action is taken to be of “detailed balance” form, we find a cancellation of the highest derivative terms in the action for the curvature fluctuations. As a consequence, the initial spectrum of perturbations will not be scale-invariant in a general spacetime background, in contrast to what happens when considering Hořava-Lifshitz matter leaving the gravitational sector unperturbed. However, if we break the detailed balance condition, then the initial spectrum of curvature fluctuations is indeed scale-invariant on ultraviolet scales. As an application, we consider fluctuations in an inflationary background and draw connections with the “trans-Planckian problem” for cosmological perturbations. In the special case in which the potential term in the action is of detailed balance form and in which $\lambda = 1$, the equation of motion for cosmological perturbations in the far UV takes the same form as in GR. However, in general the equation of motion is characterized by a modified dispersion relation.

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I. INTRODUCTION

Recently, Hořava proposed [1, 2] (see also [3]) a model for quantum gravity which is power-counting renormalizable and hence potentially ultra-violet (UV) complete. The model is based on a scaling symmetry which treats space and time differently. Hence, the model explicitly breaks Lorentz (and hence also general coordinate) invariance. The action chosen by Hořava is hence potentially ultra-violet (UV) complete. The model is based on a scaling symmetry which treats space and time differently.

Since it was proposed Hořava-Lifshitz (HL) gravity, as this model is now called, has attracted a lot of attention (for a complete list of references, the reader is referred in the recent paper [6]). We will only mention some papers relevant to our study. Initially, gravitational wave solutions in HL cosmology were studied [7]. The first papers on the early universe cosmology of HL gravity are [8, 9] where it was realized that the analogs of the Friedmann equations in HL gravity include a term which scales as dark radiation and contributes negatively to the energy density. Thus, it is possible in principle to obtain a nonsingular cosmological evolution with the Big Bang of Standard and Inflationary Cosmology replaced by a bounce. In this context, it becomes possible [10] to provide a realization of the “matter bounce” alternative [11] to cosmological inflation for explaining the origin of an almost scale-invariant spectrum of cosmological perturbations. As realized originally in [2] the different ultraviolet behavior of the theory might provide an alternative to cosmological inflation for solving the problems of Standard Cosmology such as the horizon and flatness problems [9]. The specific UV scaling of HL gravity could change the usual arguments for the origin of the scale-invariance of cosmological perturbations in inflationary cosmology, as pointed out in [8]. In fact, the dominant UV terms in the action may lead to the possibility of obtaining a scale-invariant spectrum of cosmological perturbations in the expanding phase of HL cosmology without inflation [12] (see also [13]). To substantiate these conclusions, however, a careful analysis of the theory of cosmological perturbations in HL gravity is required, and this is the topic of the current paper.

A possibly more important reason to study cosmological perturbations in HL gravity is basic consistency issues of HL gravity itself [3]. For a particular value of one of the parameters in the HL action, namely $\lambda = 1$, the theory has its IR fixed point the action of GR. HL gravity has the same dynamical degrees of freedom as General Relativity (GR), but it does not have the complete diffeomorphism invariance of GR. Spatial diffeomorphisms are still a symmetry, but space-dependent time reparameterizations are no longer allowed. Thus, one loses one out of the four gauge modes of GR, and hence one extra physical mode is expected survive. This fact was already pointed out in [14] and has been further discussed in [15, 16, 17]. This extra physical mode, if dynamical, would lead to severe problems for HL gravity, since no effects of extra gravitational degrees of freedom have been observed, and since there are stringent limits on the presence of such degrees of freedom. In [14, 15], cosmological fluctuations in the absence of matter were considered. A new physical scalar gravitational mode was found. In the limit $\lambda = 1$, this mode was claimed to be non-dynamical [14], although the constraint equation in both [14] and [16] showed a singularity in this limit. Perturbations in the presence of matter were very recently considered in [17], where it was claimed that the extra gravitational degree of freedom is physical and becomes strongly coupled in the limit $\lambda = 1$ in which GR is to be recovered.

These differing claims form our second motivation to perform a careful study of cosmological perturbations in the presence of matter in HL gravity. It is crucial to perturb about a dynamical Friedmann universe as opposed to perturbing about Minkowski space-time [4]. Our work builds on the paper [15] in which the groundwork for the present study was provided. We start with the general metric including scalar metric fluctuations. We use the spatial diffeomorphism invariance to choose a gauge in which the spatial metric is diagonal (we focus on the scalar metric fluctuations [5]). Making use of the perturbed constraint equations, we determine the action for the remaining degrees of freedom of the cosmological perturbations. At this stage there are indeed two apparent physical degrees of freedom present. After expressing the action in terms of the usual Sasaki-Mukhanov variable [20, 21], the variable in terms of which the action for cosmological perturbations has canonical kinetic term, we find that the extra degree of freedom for scalar metric fluctuations is non-dynamical. In particular, there is no strong coupling problem for the fluctuations: consistently including cosmological expansion regulates the divergence found in [17]. In the case of cosmological

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1 See also [3] for a recent study of the renormalizability issue.
2 See also [4] for a study of the IR limit.
3 One of the authors (RB) thanks G. Dvali and E. Witten for emphasizing this issue, and K. Zarembo for interesting discussions on this point.
4 Doing the latter is inconsistent with the background constraint equations since the presence of any matter will lead to a non-vanishing average energy density and hence to cosmological expansion.
5 See [13] for a review of the theory of cosmological perturbations and [19] for a shorter overview.
perturbations, the absence of new dynamical degrees of freedom holds for any value of \( \lambda \), whereas for gravitational waves our analysis only covers the case \( \lambda = 1 \), the most interesting case.

The action for cosmological perturbations derived in this paper allows us to study the spectrum of curvature fluctuations in HL gravity. If the potential term in the Hořava action is taken to be of “detailed balance” form, we find a cancellation of the leading UV terms in the action. Thus, unlike what happens in the case of HL matter on a fixed background, the initial spectrum of fluctuations is not scale-independent. Scale invariance of the fluctuations in the UV region is maintained if we add terms which break the detailed balance condition. More generally, we study ways of obtaining a scale-invariant spectrum during the course of cosmological evolution.

The outline of this paper is as follows: We begin with a brief review of HL gravity. In Section 3 we discuss cosmological perturbations and show that no strong coupling problem arises. In Section 4, we compute the power spectrum of cosmological perturbations and discuss applications to inflationary cosmology. The final section contains a discussion and conclusions.

II. SETUP

A. Brief review of Hořava-Lifshitz theory

The dynamical degrees of freedom in HL gravity are the usual metric degrees of freedom which appear in the ADM approach to canonical gravity, namely the spatial metric \( g_{ij} \), the lapse function \( N \) and the shift vector \( N_i \). In terms of these fields, the full space-time metric is

\[
ds^2 = -N^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right),
\]

where the indices of \( N_i \) are raised and lowered using the spatial metric \( g_{ij} \).

Note that here we will allow \( N \equiv N(t, x) \), as the most general form in the ADM decomposition. Because if one assumes \( N \equiv N(t) \), then the Hamiltonian constraint for perturbations will be lost, and one can not recover GR in the IR limit and in the case \( \lambda = 1 \). For studies of cosmological perturbations assuming \( N = N(t) \), the reader is referred to [14, 15, 16].

The action of Horava-Lifshitz gravity contains a “kinetic” part and a “potential” part,

\[
S^g = S^g_K + S^g_V,
\]

with

\[
S^g_K = \frac{2}{\kappa^2} \int dtd^3x \sqrt{g}N \left( K_{ij} K^{ij} - \lambda K^2 \right),
\]

and

\[
K_{ij} = \frac{1}{2N} \left( g_{ij} - \nabla_i N_j - \nabla_j N_i \right),
\]

is the extrinsic curvature and \( K = g^{ij} K_{ij} \). Hořava chose the potential to be of the “detailed-balance” form \(^6\)

\[
S^g_V = \int dtd^3x \sqrt{g}N \left[ -\frac{\kappa^2}{2w^2} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2 w^2} \epsilon^{ijk} R_{il} \nabla_j R_k^l - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( 1 - \frac{4\lambda}{4} R^2 + \Lambda R - 3\Lambda^2 \right) \right],
\]

where \( C_{ij} \) is the Cotton tensor defined by

\[
C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}} \nabla_k \left( R^l_i - \frac{1}{4} R \delta^l_i \right). \tag{2.5}
\]

Note that in \( (2.3) \), \( \lambda \) is a dimensionless coupling of the theory and therefore runs as a function of energy. The extra terms in the potential contain two further constants \( w \) and \( \mu \). For our applications to cosmology, the terms involving \( w \) will not play a role. The action of GR is recovered in the IR limit if \( \lambda = 1 \).

The general structure of the action of scalar field matter in Hořava-Lifshitz gravity contains two parts: a quadratic kinetic term invariant under foliation-preserving diffeomorphisms and a potential term:

\[
S^\varphi = \int dtd^3x \sqrt{g}N \left[ \frac{1}{2N^2} \left( \dot{\varphi} - N^i \partial_i \varphi \right)^2 + F(\varphi, \partial_i \varphi, g_{ij}) \right], \tag{2.6}
\]

\(^6\) It is argued in [6, 17] that solutions of GR are often not recovered if the potential term is taken to be of detailed balance form, a problem already encountered in [23] in the context of spherically symmetric metrics. This problem is due to a strong coupling signature which does not arise if renormalizable terms which break the detailed balance condition are added to the potential. Going beyond the detailed balance form of the potential also allows IR solutions with a positive or vanishing cosmological constant [24, 25], whereas maintaining the detailed balance condition yields a negative cosmological constant.
with the “potential terms”

$$F(\varphi, \partial_i \varphi, g_{ij}) = -V(\varphi) + g_1 \xi_1 + g_{11} \xi_1^2 + g_{111} \xi_1^3 + g_2 \xi_2 + g_{12} \xi_1 \xi_2 + g_3 \xi_3,$$  

(2.7)

where the $\xi_i$ are invariants built out of spatial gradients of $\varphi$:

$$\xi_1 = \partial^i \varphi \partial_i \varphi,$$

(2.8)

$$\xi_2 = (\Delta \varphi)^2,$$

(2.9)

$$\xi_3 = (\Delta \varphi)(\Delta^2 \varphi).$$

(2.10)

Here and in what follows, we use $\Delta \equiv \partial^i \partial_i \equiv \partial^2 / a^2$ as a shorthand. Note that $g_1$ must be negative in order to obtain the standard form of the kinetic term in the IR.

### B. Background equations of motion

The equations of motion for $N$ and $N_i$ are the energy constraint and momentum constraints, respectively. They take the general form

$$0 = -\frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) - \frac{\kappa^2}{2w^2} C_{ij} C^{ij} + \frac{\kappa^2 \mu^2}{2w^2} e^{ijkl} R_{il} \nabla_j R^l_k - \frac{\kappa^2 \mu^2}{8 \kappa^2} \frac{R_{ij} R^{ij}}{2},$$

(2.11)

$$0 = \frac{4}{\kappa^2} \nabla_j \left( K^j_i - \lambda K \delta^j_i \right) - \frac{1}{N} \left( \dot{\varphi} - N^i \partial_i \varphi \right) \partial_j \varphi.$$

In this work, we focus on a spatially flat background. Thus the background values for the metric are

$$N = 1, \quad N_i = 0, \quad g_{ij} = a^2 \delta_{ij}, \quad \varphi_0 = \varphi_0(t),$$

(2.12)

where $a = a(t)$ is the traditional scale-factor. In this background, one has $C_{ij} = 0$ and $e^{ijkl} R_{il} \nabla_j R^l_k = 0$. Hence, at the background level, the energy constraint gives

$$0 = \frac{3 \kappa^2 \Lambda^2 \mu^2}{8(3\lambda - 1)} - V_0 - \frac{\dot{\varphi}_0^2}{2} + \frac{6(3\lambda - 1) H^2}{\kappa^2},$$

(2.13)

while the momentum constraint is trivially satisfied.

The space diagonal component of the generalized Einstein equation takes the form

$$\frac{2(3\lambda - 1)}{\kappa^2} \left( 2 \dot{H} + 3 H^2 \right) + \frac{3 \kappa^2 \Lambda^2 \mu^2}{8(3\lambda - 1)} + \frac{1}{2} \dot{\varphi}_0^2 - V_0 = 0,$$

(2.14)

where $H \equiv \dot{a} / a$ is the Hubble parameter and $V_0 \equiv V(\varphi_0)$.

Combining Eq. (2.14) and the energy constraint (2.13) equations, we find another useful equation:

$$\frac{4(3\lambda - 1) \dot{H}}{\kappa^2} + \dot{\varphi}_0^2 = 0.$$

(2.15)

The background equation of motion for the scalar field is as usual

$$\ddot{\varphi}_0 + 3H \dot{\varphi}_0 + V' = 0.$$

(2.16)

### III. PERTURBATION THEORY AND THE DYNAMICAL DEGREES OF FREEDOM

The scalar metric fluctuations about our background can be written as (see [18] for an overview of the theory of cosmological perturbations)

$$\delta g_{00} = -2\phi,$$

(3.1)

$$\delta g_{0i} = a^2 \partial_i \psi,$$

(3.2)

$$\delta g_{ij} = -2a^2 (\psi \delta_{ij} - \partial_i \partial_j E)$$

(3.3)

where $\phi, \psi, B$ and $E$ are functions of space and time. Matter is perturbed, as well. The scalar field perturbation is denoted by $\delta \varphi \equiv Q$. 
The theory is invariant under spatial diffeomorphisms
\[ x^i \rightarrow x^i + f^i(x^j, t) \]  
and under space-independent time reparametrizations. Compared to the situation in GR, one has lost the invariance under space-dependent changes in time.

We can use the spatial diffeomorphism invariance to choose the gauge \( E = 0 \), but one then no longer has the extra gauge freedom to choose spatially flat gauge (\( \psi = 0 \) in addition to \( E = 0 \)) or longitudinal gauge (\( B = 0 \) in addition to \( E = 0 \)). We are left with three variables, namely \( \phi, \psi \) and \( B \). The lapse and shift functions become
\[ N = 1 + \phi(t, x^i), \]
\[ N_i = \partial_i B(t, x^i), \] (3.5)
and that \( \phi, B \) and \( \psi \) are of the same order as the scalar field perturbation \( Q \equiv \delta \phi \). As in GR, in order to get the second order (and even the third order) action, we only need to expand \( N \) and \( N_i \) to first order. Higher order contributions of \( N \) and \( N_i \) can be eliminated making use of the equations of motion.

One should also note that in the \( E = 0 \) gauge, the spatial metric \( g_{ij} \) is conformal flat. In this gauge, making use of the local Weyl transformation, one has \( C_{ij} = 0 \) and \( \epsilon^{ijk} R_{il} \nabla_j R_k^l = 0 \). So the parameter \( \omega \) does not enter the cosmic perturbation theory.

### A. Constraints

At first order, the energy constraint gives
\[ 0 = 2(1 - 3\lambda) (\kappa^2 \dot{\phi}^2 + 12(1 - 3\lambda) H^2) \phi + 8(3\lambda - 1)^2 H \Delta B + 2 \kappa^2 (3\lambda - 1) \left( \dot{\phi} \dot{Q} + V' Q \right) + \kappa^4 \mu^2 \Lambda \frac{\partial^2 \psi}{a^2} + 24(3\lambda - 1)^2 H \ddot{\psi} \]  
and the momentum constraint gives
\[ 0 = (3\lambda - 1) H \phi + (\lambda - 1) \Delta B + (3\lambda - 1) \dot{\psi} - \frac{\kappa^2}{4} \dot{\phi} Q. \]  
(3.7)

From (3.6) and (3.7) we can solve for \( \phi \) and \( B \) explicitly to get
\[ \phi = \frac{1}{16H^2(1-3\lambda)^2 + 2\kappa^2(\lambda - 1)(3\lambda - 1)\dot{\phi}^2} \left\{ \Delta \psi \kappa^4 (-1 + \lambda) \Lambda \mu^2 - 16H(1 - 3\lambda)^2 \psi \right. \]
\[ + 2 \kappa^2 (-1 + \lambda)(-1 + 3\lambda) \ddot{\phi} 0 + 2Q\kappa^2 (-1 + 3\lambda) \left( H(-1 + 3\lambda) \dot{\phi} 0 + (-1 + \lambda)V' \right) \}
\[ \Delta B = \frac{1}{32H^2(3\lambda - 1) + 4\kappa^2(\lambda - 1)\dot{\phi}^2} \left\{ -2H \Delta \psi \kappa^4 \Lambda \mu^2 - 4\kappa^2 (-1 + 3\lambda) \dot{\phi} 0 \left( H \ddot{Q} + \psi \dot{\phi} 0 \right) \right. \]
\[ + Q\kappa^2 \left( 12H^2 (1 - 3\lambda) \dot{\phi} 0 + \kappa^2 \dot{\phi}^3 0 + 4H(1 - 3\lambda)V' \right) \}. \]  
(3.8)

Note that these constraint equations allow us to solve for \( \phi \) and \( B \) without any singularities, even in the case \( \lambda = 1 \). If one (incorrectly) had expanded about Minkowski space-time instead of about an expanding universe, one would have obtained a singularity in the constraint equation (see e.g. Eq. (68) of [17]). Note also that \( \phi \) and \( B \) remain perturbatively small. We learn the important lesson that the expansion of space which is inevitable in the presence of matter removes the potential strong coupling problem for cosmological perturbations. This is an important consistency check for the cosmology of HL gravity.

### B. Second-order action

At this stage, we have two independent degrees of freedom for scalar metric fluctuations (instead of only one as would be the case in GR), namely \( \psi \) and \( Q \). In order to set up the linear theory of cosmological perturbations, we need to find the second order action for the fluctuations.

After inserting the perturbed metric and perturbed matter into the action for gravity and matter, expanding to second order in the fluctuations, and make use of the constraints (3.8), we obtain
\[
S_2[\psi, Q] = \int dt d^3 x \left[ a_0^3 \left\{ c_\psi \ddot{\psi}^2 + f_\psi \dot{\psi} \ddot{Q} + g_1 \dot{Q} \Delta Q + g_2 (\Delta Q)^2 + g_3 (\Delta Q) (\Delta^2 Q) + m_\psi \psi^2 \right. \right. \\
+ c_\psi \psi^2 + f_\psi \psi \dot{\psi} + h_\psi \dot{\psi} \Delta \psi + \omega_\psi \psi \Delta \psi + d_\psi (\Delta \psi)^2 + m_\psi \psi^2 \\
\left. \left. + c_\psi \psi \ddot{\psi} + f_\psi \psi \dot{\psi} + h_\psi \dot{\psi} \Delta \psi + \omega_\psi \psi \Delta \psi + d_\psi (\Delta \psi)^2 + m_\psi \psi^2 \right\} \right]. 
\]  
(3.9)
It appears that there are two dynamical degrees of freedom, \( \psi \) and \( Q \) respectively. However, it will be shown that this is an illusion (see the next subsection for a detailed discussion on the issue of dynamical degrees of freedom in Ho\'rava theory). In fact, there is only one dynamical degree of freedom, the same as in GR. The easiest way to see this is to look at the combination of all of the kinetic terms in the above action, i.e. the \( Q^2 \), \( \psi^2 \) and \( Q\psi \) terms (see Appendix A.1), and to realize that they can be brought into a "perfect square" form

\[
c_f \dot{Q}^2 + c_\psi \dot{\psi}^2 + c_{\psi \phi} \dot{\psi} \dot{\phi} \propto \left( \dot{\psi} + \frac{H}{\dot{\phi} Q} \right)^2.
\]

This fact implies that there is indeed only one dynamical degrees of freedom in our system. This degree of freedom is precisely the Sasaki-Mukhanov combination of matter and metric fluctuations, the variable in terms of which the action for cosmological perturbations in GR has canonical form.

The Sasaki-Mukhanov variable \( \zeta \)

\[
- \zeta \equiv \psi + \frac{H}{\dot{\phi} Q} \ ,
\]

is the gauge-invariant curvature perturbation on uniform-density hypersurfaces. From (3.10), we can express \( Q \) in terms of \( \psi \) and \( \zeta \),

\[
Q = -\frac{\dot{\phi} Q}{H} (\zeta + \psi) ,
\]

\[
\dot{Q} = -\frac{\dot{\phi} Q}{H^2} (\zeta + \psi) - \frac{\dot{\phi}}{H} (\dot{\psi} + \dot{\phi}) - \frac{\dot{\phi} Q}{H} (\dot{\zeta} + \dot{\psi}) ,
\]

where the various coefficients can be found in Appendix A.2.

It is important to note that there is no \( \psi^2 \) term in (3.12). Moreover, the coefficient of the kinetic term for \( \dot{\zeta} \) is \( c_\zeta \propto \dot{\phi}_0^2 \), and thus, \( \dot{\zeta} \) also becomes non-dynamical in the absence of matter field \( \varphi \). In the presence of matter field \( \varphi \), there is only one dynamical degree of freedom in our system, which we can identify as \( \zeta \).

Eq. (3.12) is rather complicated. However, we can further simplify the expression. First we note that all the coefficients in (3.12) are evaluated in terms of the background fields, and thus are functions of time only. Then, for a general function \( F(t) \) and spacetime field \( \phi \), we have the following convenient relation

\[
\int dt d^3x \alpha^3 F(t) \phi \phi \sim \int dt d^3x \frac{d}{dt} \left( -\frac{\alpha^3}{2} F(t) \right) \phi \phi = \int dt d^3x \alpha^3 \left[ -\frac{1}{2} \left( \dot{F} + 3HF \right) \right] \phi \phi ,
\]

where "\( \sim \)" denotes up to total derivative terms. Similarly, we have (noticing that \( \Delta \equiv \partial^2 / a^2 \))

\[
\int dt d^3x \alpha^3 F(t) \phi \Delta \phi \sim \int dt d^3x \alpha^3 \left[ -\frac{1}{2} \left( \dot{F} + H F \right) \right] \phi \Delta \phi .
\]

Thus, by using the above relations and performing many integrations by parts, we find that the second-order action (3.12) can be recast into a rather convenient form:

\[
S_2[\zeta, \psi] \equiv \int dt d^3x \alpha^3 \left\{ c_\zeta \dot{\zeta}^2 + \zeta \Gamma_4(\Delta) \zeta + \psi \Gamma_1(\Delta) \psi + \left( \Gamma_2(\Delta) \zeta + \Gamma_3(\Delta) \dot{\zeta} \right) \psi \right\} ,
\]

where we have defined

\[
\Gamma_1(\Delta) \equiv -\frac{1}{2} \left( h_{\psi} + H h_{\psi} \right) \Delta + \dot{d}_{\psi} \Delta^2 + d_{\psi} \Delta^2 + \omega_{\psi} \Delta + m_{\psi} - \frac{1}{2} \left( \dot{f}_{\psi} + 3HF \right) ,
\]

\[
\Gamma_2(\Delta) \equiv \left( \omega_{\psi \phi} + \omega_{\psi \psi} \right) \Delta + d_{\psi \phi} \Delta^2 + \dot{d}_{\psi \phi} \Delta^2 + \left( m_{\psi \phi} - \dot{f}_{\psi \phi} - 3HF \dot{\psi}_0 \right) ,
\]

\[
\Gamma_3(\Delta) \equiv h_{\psi \phi} \Delta ,
\]

\[
\Gamma_4(\Delta) \equiv \omega_{\psi} \Delta + d_{\psi} \Delta^2 + \dot{d}_{\psi} \Delta^2 + m_{\psi} - \frac{1}{2} \left( \dot{f}_{\psi} + 3HF \right) ,
\]

(3.14)
for simplicity (in $\Gamma_3$, we have used the fact $f_{\zeta\psi} = \tilde{f}_{\zeta\psi}$, see Appendix A2 for details). Note that since $\Delta \equiv \partial^2/a^2$ and since several of the coefficients in (3.12) are time-dependent, the $\Gamma$’s are in general time-dependent. The above results should be understood in Fourier space, where we identify $\Delta \equiv -k^2/a^2$.

The important point is that now $\psi$ has no time-derivatives and acts as a new constraint. The equation of motion for $\psi$ is

$$2 \Gamma_1(\Delta) \psi + \Gamma_2(\Delta) \zeta + \Gamma_3(\Delta) \dot{\zeta} = 0,$$

(3.15)

from which we can solve $\psi$ explicitly to get

$$\psi = -\frac{\Gamma_2(\Delta) \zeta + \Gamma_3(\Delta) \dot{\zeta}}{2 \Gamma_1(\Delta)}.$$  

(3.16)

After plugging (3.16) into (3.13), and after some straightforward calculations, we obtain an effective quadratic action for a single variable $\zeta$,

$$S_2[\zeta] = \int dt d^3x a^3 \left\{ \left( c_{\zeta} - \frac{\Gamma_3}{4\Gamma_1} \right) \dot{\zeta}^2 + \left[ \Gamma_4 - \frac{\Gamma_2^2}{4\Gamma_1} + \frac{1}{4a^3} \frac{d}{dt} \left( a^3 \frac{\partial f_{\zeta\psi}}{\partial \psi} \right) \right] \zeta^2 \right\}.$$  

(3.17)

C. Subtleties of the dynamical degrees of freedom

The important lesson to be drawn from the previous subsection is that the extra degree of freedom which appears in HL gravity due to the loss of space-dependent time reparametrizations as a symmetry of the theory is non-dynamical in cosmology. Hence, the strong coupling problem discussed recently in [17] is not present in linear cosmological perturbation theory. Let us discuss this result, a result which holds for any value of $\lambda$.

The symmetry group of Hořava-Lifshitz gravity is that of “foliation-preserving” diffeomorphisms, which is smaller than the full diffeomorphism group of GR. Thus, one may naively expect that with smaller symmetry, we are left with less gauge artifacts and more physical modes. In particular, it was argued that there was an additional scalar dynamical degree of freedom in Hořava theory, arising from the gravity sector itself. This sounds very different from the situation in GR, where gravity has only two physical degrees from freedom — the two polarization states of gravitational waves. According to the standard treatment of cosmological perturbation theory in GR, the quantum perturbations of the scalar sector are expected to be generated by scalar matter fields. In this case, if there is no matter, all perturbation modes are gauge artifacts. Thus, if there is indeed one additional scalar dynamical degree of freedom from gravity itself in Hořava theory, our traditional picture of cosmological perturbation would not apply. In fact, the extra dynamical degree of freedom would lead to serious problems for the theory, as stressed in [17]. However, in this work, a detailed study of the perturbation theory shows that there is no additional dynamical scalar degree of freedom at all, when perturbations in the presence of matter are found by consistently expanding around the FRW metric and not around the flat Minkowski one.

Furthermore, another point is that, in Hořava’s original formulation of the theory, the lapse function $N$ was restricted to being a function of time only $N = N(t)$, where the corresponding Hamiltonian becomes non-local. In this work, we relax this restriction. By assuming $N = N(t, \kappa)$ and starting from the most general expansion of the action, we see that the apparent additional degree of freedom is non-dynamical.

Is there a new dynamical degree of freedom in the absence of matter? Let us take the limit of our equations when $H$ and the energy density of matter tend to zero. Since the coefficient of the kinetic term $\dot{\zeta}^2$ is proportional to the background scalar-field value $\varphi_0$ (see Appendix A2), $c_{\zeta} \propto \varphi_0^2$. Thus if there is no matter field, all perturbation modes become non-dynamical, which is exactly the case in GR. However, the limit we are discussing is singular unless $\lambda = 1$ because if $\lambda \neq 1$ one of the coefficients in the action (the $d_{\zeta}$ coefficient) blows up.

Thus, our work also shows that in the case $\lambda = 1$, the case in which the action of HL gravity reduces to that of GR in the IR limit, there are no extra gravitational degrees of freedom in the vacuum. This result is in agreement with the conclusions in [14], where a scalar degree of freedom in the gravity sector was identified for $\lambda \neq 1$, and where it was shown that this mode becomes non-dynamical when $\lambda = 1$. Our conclusions also agree with the recent analysis of [6] 7. The extra scalar gravitational degree of freedom in the vacuum sector of the theory was also discussed in [16]. That paper shows that there is a singularity for $\lambda = 1$ which is agreement with our conclusion that for this choice of $\lambda$ there are no new gravitational degrees of freedom.

IV. COSMOLOGICAL PERTURBATIONS AND THE POWER SPECTRUM

In [12] (see also [13]) it was pointed out that a scalar field with Hořava-Lifshitz form (2.6) will obtain a scale-invariant spectrum of cosmological fluctuations, and it was then argued that a scale-invariant spectrum of curvature fluctuations may similarly result.

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7 The absence of an extra dynamical degree of freedom in the vacuum sector of HL gravity is also observed in [22].
independent of the equation of state of the background. This result follows from the fact that the action contains terms with six space derivatives. These terms dominate in the UV and yield a scale-invariant spectrum.

In cosmology we are interested in the power spectrum of the induced curvature fluctuations. The formalism we established in the previous section now allows us to calculate this spectrum. We will show that for the gravitational potential of detailed balance form, the terms with six spatial derivatives cancel in the action for cosmological fluctuations. This happens independently of the detailed form of the coefficients of the higher derivative terms in the matter action. The reason for this result is that the terms in the gravitational action with six spatial derivatives (the terms involving the constant $w$) do not enter the action for cosmological perturbations. If we add terms to the action consistent with power counting renormalizability which break the detailed balance condition, terms with six spatial derivatives in the action for cosmological perturbations will survive.

If we keep the detailed balance condition on the potential, then the leading terms in the UV contain four spatial derivatives. Thus, the initial power spectrum of curvature fluctuations is not scale-invariant. Thus, scale-invariance of the late time power spectrum will only arise for specific background evolutions. e.g. for inflationary expansion (as discussed later in this section) or for a matter bounce background. If we drop the detailed balance condition, then an initially scale-invariant curvature power spectrum results.

### A. Equation of motion

The action for a gravitational potential satisfying the detailed balance condition has the general structure:

$$S_2[\zeta] = \int dt d^3x \, a^3 \left( \gamma \zeta^2 - \Omega \zeta^2 \right),$$  

(4.1)

with

$$\gamma \equiv \left( c_\zeta - \frac{\Gamma_2}{4\Gamma_1} \right) ,$$  

$$-\Omega \equiv \Gamma_4 - \frac{\Gamma_2}{4\Gamma_1} + \frac{1}{4a^3} \frac{d}{dt} \left( \frac{a^3 \Gamma_2 \Gamma_3}{\Gamma_1} \right) ,$$  

(4.2)

where the $\Gamma$’s are defined in (3.14). In order to write the action (4.1) in canonical form, we introduce the new variable

$$u \equiv a \sqrt{\gamma} \zeta .$$  

(4.3)

After changing to conformal time (which is defined by $dt = ad\eta$) we have

$$S_2[\zeta] = \int d\eta d^3x \left\{ u'^2 + \left( \mathcal{H} + \frac{\gamma'}{2\gamma} \right)^2 + \left( \mathcal{H} + \frac{\gamma'}{2\gamma} \right)' - \frac{a^2 \Omega}{\gamma} u^2 \right\} ,$$  

(4.4)

where $\mathcal{H} \equiv a'/a$, and a prime indicates the derivative with respect to conformal time. Note that the above result should be understood in momentum space, that is we must make the replacement $\Gamma_i(\Delta) \rightarrow \Gamma_i(-k^2/a^2)$.

The classical equation of motion for the canonically-normalized variable $u$ is simply

$$u'' + \omega^2(\eta, k) \, u_k = 0 ,$$  

(4.5)

with

$$\omega^2(\eta, k) \equiv \frac{a^2 \Omega}{\gamma} - \left( \mathcal{H} + \frac{\gamma'}{2\gamma} \right)^2 - \left( \mathcal{H} + \frac{\gamma'}{2\gamma} \right)' ,$$  

(4.6)

where $\gamma$ and $\Omega$ are defined in (4.2) and the variously introduced parameters can be found in Appendix A 2. We emphasize that in deriving the equation of motion (4.5)-(4.6), no approximation (beyond the restriction to linear perturbation theory) has been made and that thus the equation is exact. One can use (4.5)-(4.6) as the starting point of a detailed investigation of the spectrum of scalar metric perturbations in Hořava-Lifshitz theory.

In the following, we shall first investigate the UV limit of the above equations in a general background, and then study the evolution in an inflationary background.

---

8 The possibility of obtaining a scale-invariant spectrum of metric perturbations in pure HL gravity was discussed in [26].

9 As already mentioned in [12], the scale-invariance of an initial matter entropy field spectrum can induce scale-invariance of the curvature fluctuations via the “curvaton” mechanism [27].
B. UV Limit of Perturbation Theory

To study the perturbations in the UV limit, i.e., for \( k \to \infty \), we consider the terms with highest power of \( \Delta \) in the equation of motion for the fluctuations. First, we note that the leading terms in the \( \Gamma \)’s are

\[
\Gamma_1 \sim \tilde{d}_\phi \Delta^3, \quad \Gamma_2 \sim \tilde{d}_\xi \Delta^3, \quad \Gamma_3 \sim h_\Delta, \quad \Gamma_4 \sim \tilde{c}_\zeta \Delta^3, \quad \gamma \sim c_\zeta, \quad \Omega \sim \left(-\tilde{d}_\zeta + \frac{d^2 \psi}{4d^2} \right) \Delta^3 ,
\]  

(4.7)

where the \( \simeq \) sign means that we are writing down only the leading term in the UV limit (but with the exact value of its coefficient).

Inserting the coefficients (see Appendix A 2):

\[
\tilde{d}_\zeta \equiv \frac{g_{\Delta \psi}^2}{H^2}, \quad \tilde{d}_\psi \equiv \frac{2g_{\Delta \psi}^2}{H^2}, \quad \tilde{d}_\phi \equiv \frac{g_{\Delta \phi}^2}{H^2},
\]

(4.8)

we find that

\[
\Omega \simeq 0.
\]

(4.9)

In other words, the \( k^6 \) term in \( \Omega \), which could be naively expected to arise in Hořava-Lifshitz theory based on the terms in the Lagrangian, vanishes. exactly. The leading order contribution in powers of \( k \) starts with a \( k^4 \) term. As shown in \[12\], it is the \( k^6 \) term which can naturally produce a scale invariant spectrum, while the \( k^4 \) term cannot. Thus, for a potential satisfying the detailed balance condition it is not true that in Hořava-Lifshitz theory a scale invariant spectrum can be produced in any background.

The above cancellation of the \( k^6 \) term is not an accident. One can prove that even for more general high order derivative terms in the scalar field Lagrangian, the above cancellation happens. To show that, we take the high order derivative part of the Lagrangian to be \[^{10} \]

\[
\sum_{m=1, n>m} g_{mn} \Delta^m \varphi \Delta^n \varphi.
\]

(4.10)

The corresponding piece of the matter Lagrangian quadratic in \( Q \) is

\[
\sum_{m=1, n>m} g_{mn} \Delta^m Q \Delta^n Q \rightarrow \sum_{m=1, n>m} \frac{g_{mn} \Delta^2}{H^2} \Delta^m+\Delta^n (\zeta+\psi)^2.
\]

(4.11)

Note that here \( \psi \) is not a dynamical field, and needs to be solved for by minimizing the action. In the \( k \to \infty \) limit, there are no other \( \Delta^3 \) or higher spatial derivative terms, because the highest derivative in the gravity sector is \( \Delta^2 \). Thus, Eq. (4.11) corresponds to the dominant term in the action. Minimizing the action, the solution is

\[
\psi \simeq -\zeta.
\]

(4.12)

Thus, the \( \Delta^3 \) or higher spatial derivative terms in the scalar field sector are canceled and thus do not contribute to cosmic perturbations.

If one assumes that there are terms such as \( \varphi \Delta^3 \varphi \) instead of \( \Delta \varphi \Delta^2 \varphi \) in the scalar field Lagrangian, then the above proof is no longer valid. It is then possible to generate scale invariant perturbations for a general background in this case. However, such a scalar field Lagrangian breaks the shift symmetry in the gradient term, and is thus not conventionally used in the literature.

At this stage, it is also important to recall that \( k^6 \) terms in the action for cosmological perturbations will survive if one adds terms to the gravitational action which do not conform to the detailed balance condition.

C. Perturbation Theory in an Inflationary Background

Now we turn to the perturbation equation of motion \[4.15\). Obviously, \[4.15\] reduces to the standard case in IR. Thus, in this work, in order to investigate the possible differences of cosmological perturbation in Hořava theory from those in General Relativity, we focus on the UV region. For simplicity, we consider an inflationary background as usual, where the Hubble parameter is approximately constant.

As discussed in the previous subsection, in the UV limit, the \( \sim k^6 \) term exactly cancels out. The equation of motion takes the form

\[
u''(c^2 k^2 + \frac{\varphi}{a^2} \frac{k^4}{a^2} - \frac{a''}{a} + M^2 a^2) = 0.
\]

(4.13)

[^{10}]: See also \[23\] for a study of scalar field Lifshitz actions.
We neglect slow-roll corrections to the scale factor evolution. The expressions for both $c_s^2$ and $M^2$ can be found in Appendix A.2. We should keep in mind that (4.13) only describes the behavior of perturbation in UV limit, that is, the left-hand-side of (4.13) should be compensated with terms of order $O(1/k^2)$, $O(1/k^4)$, · · · etc., which we neglect in the following analysis.

It is interesting to point out a connection with the “trans-Planckian problem” for fluctuations in inflationary cosmology. In [29], it was argued that the predictions of inflationary cosmology for the spectrum of cosmological perturbations are sensitive to hidden assumptions about physics on trans-Planckian scales. To demonstrate this point, fluctuations obeying a modified dispersion relation was studied in [30]. However, in [30], the modified dispersion relation was not derived from any action principle but simply postulated. We have shown here that HL gravity yields a specific modified dispersion relation for scalar metric fluctuations. We will return to this point in a followup paper [31].

Note that in the case $\lambda = 1$, the stability condition on the solutions in the UV requires that the coefficient $g_1$ in the matter Lagrangian be negative, the sign we expect (since it is the $g_1$ term which dominates in the IR and must give the standard kinetic term for the scalar field Lagrangian).

As an application of the framework we have developed in this paper, let us consider the evolution of fluctuations in an inflationary background with $a(\eta) = -1/(H\eta)$, where $\eta$ is the conformal time. The equation of motion (4.13) has been extensively investigated. On super-Hubble scales the solution is

$$u_k(\eta) \sim a(\eta)$$

which corresponds to constant curvature fluctuation $\zeta$. In the special case $\lambda = 1$, the solution $u_k(\eta)$ will be oscillating on sub-Hubble scales.

More generally, the equation can be solved analytically to get

$$u_k(\eta) = \frac{1}{\sqrt{C}} e^{-z/2} \sqrt{\eta - c_s^2 k^2 \eta^2} U(\alpha, \nu + 1, z),$$

where $U(\alpha, \nu + 1, z)$ is a confluent hypergeometric function of the first kind with

$$\alpha \equiv \frac{1}{2}(\nu + 1) - \frac{i c_s^2}{4H \Xi},$$

$$\nu \equiv \sqrt{\frac{9}{4} - \frac{M^2}{H^2}},$$

$$z \equiv -iH \Xi k^2 \eta^2,$$

and the overall normalization constant is

$$C \equiv 2\nu e^{-\frac{\pi}{2}(7\nu+1)} \left( - \frac{c_s^2}{H \Xi} \right)^{\nu} \Gamma(\nu) \Gamma(-\nu) \left[ \frac{1}{\Gamma(\alpha - \nu) \Gamma(\alpha)} - \frac{e^{5\pi\nu}}{\Gamma(\alpha) \Gamma(\alpha - \nu)} \right],$$

where $\Xi = \frac{H^2(3\lambda - 1)^2 + \kappa^4 \Lambda^2 \mu^2}{2H^2\kappa^2(1 - \lambda)(3\lambda - 1)}$.

$11$ We neglect slow-roll corrections to the scale factor evolution.

$12$ Equation of motion with the same structure of (4.13) has also been analyzed in [33], in the investigation the statistical anisotropy and large-scale CMB anomalies.
where “∗” denotes a complex conjugate. Here the $k$-independent normalization constant $C$ is chosen so that $u_k(\eta)$ is normalized in the usual way (unit Wronskian):

$$u_k(\eta)u_k^*(\eta) - u_k^*(\eta)u_k(\eta) = i\,,$$  

which is the condition for canonical quantization.

The evolution of a perturbation mode is shown in Fig[1]. It can be seen that the behavior of perturbation in Hořava theory is very similar to that in GR. In particular, after the wavelength exits the sound horizon, the perturbation modes are frozen. On super-horizon scales, since

$$u_k(\eta) \xrightarrow[\eta \to 0]{} \Gamma(\nu)\sqrt{-\eta} \left(\frac{c_s}{iHk\eta}\right)^\nu,$$  

the dimensionless power spectrum $P_\zeta$ of the curvature fluctuation $\zeta$ can easily be calculated:

$$P_\zeta(k) \equiv \frac{k^3}{2\pi^2}|\zeta(k)|^2 = \frac{k^3}{2\pi^2} \frac{u_k(\eta)}{a(\eta)}$$

$$\approx \left(\frac{H}{2\pi}\right)^2 \frac{2\Gamma^2(\nu)}{c_s^2c_\zeta(C|\Gamma(\alpha)|^2)} \left(-c_s^2k\eta\right)^{2\nu},$$

where we have used the fact that in UV limit, $\gamma \approx c_\zeta$. The dimensionless power spectrum $P_\zeta(k)$ is shown in Fig[2] (up to an overall factor). The figure also shows for comparison the power spectrum obtained with the standard GR mode function

$$u_k(\eta) = \sqrt{\frac{\pi}{2}} e^{i(\nu + \frac{1}{2})\eta} \sqrt{-\eta}H_\nu^{(1)}(-c_s^2k\eta),$$

which describes the fluctuation of a massive light scalar field in de Sitter space-time. We observe that the perturbations are suppressed in Hořava-Lifshitz gravity compared with those in GR. This verifies the argument that renormalizability of gravity generally reduces the amplitude of perturbations [32]. It would be also interesting to see whether this suppression of perturbations could stabilize the inflationary background in the UV limit and prohibit eternal inflation.

We also would like to mention that we have not proved whether $\zeta$ is a conserved quantity beyond the slow roll approximation here. Although in the solution, we can see $\zeta$ is no more than slowly varying on super Hubble scales. In one Hubble time, the variation of $\zeta$ is at most as large as the slow roll parameters. It is still possible for $\zeta$ to receive $O(1)$ corrections during about 60 e-folds of inflation. Another possible contribution to $\zeta$ on super Hubble scales is the UV-IR transition. As the UV-IR transition happens at $O(1)$ e-folds, it may also yield a $O(1)$ contribution to $\zeta$.

Applications of the equation of motion for cosmological fluctuations in HL gravity to non-inflationary backgrounds will be left to a followup paper [31].

### D. IR Limit

Now we turn to the IR limit behavior of perturbations as an important consistency check of perturbation theory in Hořava gravity. In the IR limit ($k \to 0$), (4.3) takes the form

$$u_k'' + \left(\frac{c_s^2k^2}{a^2} + \frac{\alpha''}{a} + \tilde{M}^2a^2\right)u_k = 0,$$  

where the expressions for $\tilde{c}_s^2$ and $\tilde{M}^2$ can be found in Appendix A[4]. There are additional terms on the left-hand-side of the equation which are of the order $O(k^4)$, $O(k^6)$, etc. which we can neglect in IR limit. Then, Eq. (4.25) has the same form as the corresponding equation in standard perturbation theory in GR.

As we know, in the IR limit, Hořava theory reduces to GR with the following parameters:

$$c = \frac{\kappa^2\mu}{4\sqrt{\frac{\Lambda}{1 - 3\lambda}}}, \quad 16\pi G = \frac{\kappa^4\mu}{8\sqrt{\frac{\Lambda}{1 - 3\lambda}}}, \quad \Lambda_{\text{GR}} = \frac{3\kappa^4\mu^2\Lambda^2}{32(1 - 3\lambda)}.$$

Thus, setting the speed of light $c$ in the IR to $c = 1$ corresponds to choosing $\Lambda = \frac{16(1 - 3\lambda)}{1 - 3\lambda}$ in Hořava theory. With this value of $\Lambda$, from A[39] it is easy to show that

$$\tilde{c}_s^2 \equiv -2g_1,$$  

(4.27)
which is what we expected. Thus, the equation of motion for perturbations in Hořava-Lifshitz theory in the IR limit indeed reduces to the same form as in GR. However, the effective mass $\tilde{M}$ and the parameter $\gamma$ still remains different from GR. One can check that when one further takes $\lambda = 1$, we have $\gamma = \frac{\dot{\phi}^2}{2H^2}$, and $\tilde{M}$ is suppressed by slow roll parameters. So in the $g = -1/2$ case in the IR limit, the perturbation theory completely returns to GR up to the leading order of slow roll approximation. This is a consistency check for our calculation.

To summarize, we have found several cases in which the perturbation equation takes the same form as that in GR: (1) In the UV limit with $\lambda = 1$. In this case, the equation of motion has the same form with that in GR. However, the coefficients are different. (2) In the IR limit with arbitrary $\lambda$. In this case, the equation of motion is the same with that in GR up to leading order in slow roll approximation. (3) In the IR limit with $\lambda = 1$. In this case, the equation of motion should completely reduced to GR.

V. CONCLUSIONS AND DISCUSSION

In this paper we have studied the theory of linearized cosmological perturbations in Hořava-Lifshitz (HL) gravity. In this study, it is important to expand about a dynamical background since the presence of matter implies that the average energy density does not vanish. We have found that the extra degree of freedom which could be expected to arise because of the reduced symmetry of HL gravity is in fact not dynamical. This conclusion holds for any value of $\lambda$. Taking the flat space-time limit of our analysis, we can also show the absence of any new dynamical scalar metric degree of freedom in the absence of matter. Our limiting procedure, however, only works in the special case $\lambda = 1$, the case in which the HL gravitational action flows to the action of General Relativity (GR) in the infrared. We thus do not see any evidence of the "strong coupling problem" mentioned in [17].

Starting from the most general metric including scalar cosmological perturbations, we have worked out the quadratic action for cosmological perturbations. It turns out that the distinguished dynamical variable for fluctuations is the usual Sasaki-Mukhanov

FIG. 1: Evolution of the mode function with the scale factor $a$. The red curve corresponds to the mode function in Hořava theory, the black curve shows the standard mode function in GR. The dashed vertical line denotes the sound horizon. The parameters were chosen to be $c_s = 1$, $k = H = 10$, $M = 0$, $\Xi = 0.1$.

FIG. 2: Power spectrum on super-Hubble scales. The red line denotes the spectrum of perturbations in Hořava theory, and the black line denotes the standard GR result. The parameters were chosen to be $c_s = 1$, $\Xi = 0.1$, $\nu = 1.52$, $H = 10$. The spectra are evaluated at the conformal time $\eta = -0.001$. 
variable. In terms of this variable, the kinetic terms in the action has canonical form. The second metric variable enters the action without kinetic term and is hence not a dynamical degree of freedom.

Based on the action for cosmological perturbations, we can compute the spectrum of these fluctuations. Rather surprisingly it turns out that the terms in the equation of motion for these fluctuations containing six spatial derivatives vanish if the potential term in the HL gravitational action is of detailed balance form. Thus, we conclude that the spectrum of cosmological perturbations is not scale-invariant in HL cosmology (with potential satisfying the detailed balance form), unlike the spectrum of spectator matter field fluctuations. However, the fact that spectator HL matter fields acquire a scale-invariant spectrum will likely make it possible to use the curvaton mechanism [27] to induce scale-invariant fluctuations in HL cosmology independent of the expansion rate of space.

If the gravitational action is not of detailed balance form (and this seems to be the preferred case [5, 17] if the IR limit of the theory is really to reproduce Einstein gravity), then the \( k^6 \) terms in the action for cosmological fluctuations will persist, making it possible to have a scale-invariant initial spectrum of adiabatic cosmological perturbations along the lines suggested by [12].

We would like to warn the reader that Hořava-Lifshitz gravity faces many challenges before it can be declared as a viable candidate theory for quantum gravity (see [17, 34] for some potential problems). We have only addressed one of these problems - the strong coupling problem for additional fluctuation modes, a problem which has been considered fatal for the theory - and shown that in fact does not arise. Other potential problems remain to be resolved. If they can be successfully resolved, then it becomes of great interest to explore applications of the equations of cosmological perturbations which we have derived here to non-inflationary backgrounds. Work on this issue is in progress [31].

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**APPENDIX A: VARIOUS COEFFICIENTS**

1. **Coefficients in (3.3)**

\[
\begin{align*}
    c_\varphi &\equiv \frac{4H^2(-1 + 3\lambda)}{8H^2(-1 + 3\lambda) + \kappa^2(-1 + \lambda)\dot{\varphi}_0^2}, \\
    c_\psi &\equiv \frac{4(-1 + 3\lambda)\dot{\varphi}_0^2}{8H^2(-1 + 3\lambda) + \kappa^2(-1 + \lambda)\dot{\varphi}_0^2}, \\
    c_{\psi\varphi} &\equiv \frac{8H(-1 + 3\lambda)\dot{\varphi}_0}{8H^2(-1 + 3\lambda) + \kappa^2(-1 + \lambda)\dot{\varphi}_0^2}, \\
    f_\varphi &\equiv \frac{\kappa^2\dot{\varphi}_0 ((H - 3\lambda)\dot{\varphi}_0 - (-1 + \lambda)V')}{{8H^2(-1 + 3\lambda) + \kappa^2(-1 + \lambda)\dot{\varphi}_0^2}}, \\
    f_\psi &\equiv \frac{12H(1 - 3\lambda)}{\kappa^2}, \\
    f_{\psi\varphi} &\equiv -3\dot{\varphi}_0^2, \\
    f_{\psi\psi} &\equiv \frac{(-1 + 3\lambda) ( - \kappa^2\dot{\varphi}_0^2 + 8HV'' )}{{8H^2(-1 + 3\lambda) + \kappa^2(-1 + \lambda)\dot{\varphi}_0^2}}, \\
    h_\psi &\equiv \frac{4H\kappa^2\Lambda\mu^2}{{8H^2(-1 + 3\lambda) + \kappa^2(-1 + \lambda)\dot{\varphi}_0^2}}.
\end{align*}
\]
\[ h_{\psi \varphi} = -\frac{\kappa^4 (-1 + \lambda) \Lambda \mu^2 \dot{\varphi}_0}{16H^2 (1 - 3\lambda)^2 + 2\kappa^2 (-1 + \lambda) (-1 + 3\lambda) \dot{\varphi}_0^2} , \]  
(A9)

\[ \omega_{\psi} \equiv \frac{\kappa^2 \Lambda \mu^2}{-4 + 12\lambda} , \]  
(A10)

\[ \omega_{\varphi \psi} \equiv -\frac{\kappa^4 \Lambda \mu^2 (H(-1 + 3\lambda) \dot{\varphi}_0 + (-1 + \lambda) V')}{2(-1 + 3\lambda) (8H^2 (-1 + 3\lambda) + \kappa^2 (-1 + \lambda) \dot{\varphi}_0^2)} , \]  
(A11)

\[ d_{\psi} \equiv -\frac{\kappa^2 (-1 + \lambda) \mu^2 (16H^2 (1 - 3\lambda)^2 + \kappa^4 \Lambda^2 \mu^2 + 2\kappa^2 (1 - 4\lambda + 3\lambda^2) \dot{\varphi}_0^2)}{8(1 - 3\lambda)^2 (8H^2 (-1 + 3\lambda) + \kappa^2 (-1 + \lambda) \dot{\varphi}_0^2)} , \]  
(A12)

\[ m_{\varphi} \equiv \frac{\kappa^4 \dot{\varphi}_0^4 + 8H \kappa^2 (1 - 3\lambda) \dot{\varphi}_0 V' - 4\kappa^2 (-1 + \lambda) (V')^2 + 32H^2 (1 - 3\lambda) V'' - 4\kappa^2 \dot{\varphi}_0^2 (3H^2 (-1 + 3\lambda) + (-1 + \lambda) V'')}{8 (8H^2 (-1 + 3\lambda) + \kappa^2 (-1 + \lambda) \dot{\varphi}_0^2)} , \]  
(A13)

\[ m_{\psi} = \frac{3 \left( 12H^2 (1 - 3\lambda) + \kappa^2 \dot{\varphi}_0^2 \right)}{2\kappa^2} , \]  
(A14)

\[ m_{\psi \varphi} = 3V' . \]  
(A15)

2. Coefficients in (3.12)

\[ c_{\zeta} \equiv \frac{4(-1 + 3\lambda) \dot{\varphi}_0^2}{8H^2 (-1 + 3\lambda) + \kappa^2 (-1 + \lambda) \dot{\varphi}_0^2} , \]  
(A16)

\[ f_{\zeta} = \frac{\dot{\varphi}_0 (3H \dot{\varphi}_0 + V')}{H^2} , \]  
(A17)

\[ f_{\psi} = \frac{12H (1 - 3\lambda)}{\kappa^2} + \frac{3 \dot{\varphi}_0^2}{H} - \frac{\dot{\varphi}_0 V'}{H^2} , \]  
(A18)

\[ f_{\zeta \psi} = \frac{\dot{\varphi}_0 V'}{H^2} , \]  
(A19)

\[ \tilde{f}_{\zeta \psi} = \frac{\dot{\varphi}_0 V'}{H^2} , \]  
(A20)

\[ h_{\psi} = \frac{-\kappa^2 \Lambda \mu^2}{2H - 6H\lambda} , \]  
(A21)

\[ h_{\zeta \psi} = \frac{\kappa^4 (-1 + \lambda) \Lambda \mu^2 \dot{\varphi}_0^2}{2H (-1 + 3\lambda) (8H^2 (-1 + 3\lambda) + \kappa^2 (-1 + \lambda) \dot{\varphi}_0^2)} , \]  
(A22)

\[ d_{\zeta} = \frac{g_{\zeta} \dot{\varphi}_0^2}{H^2} , \]  
(A23)

\[ \dot{d}_{\zeta} = \frac{g_{\zeta} \dot{\varphi}_0^2}{H^2} , \]  
(A24)
\[ d\psi \equiv \frac{g_2 \dot{\varphi}_0^2}{H^2}, \]  
\[ d_\zeta \equiv \frac{g_2 \dot{\varphi}_0^2}{H^2}, \]  
\[ \ddot{d}_\zeta \equiv \frac{2g_2 \dot{\varphi}_0^2}{H^2}, \]  
\[ \omega_\psi \equiv \frac{2H^2 \kappa^2(-1 + 3\lambda) \Lambda \mu^2 + (\kappa^4 \Lambda \mu^2 - 8(1 - 3\lambda)^2 \Lambda \mu^2) \varphi_0^2}{8(H - 3H\lambda)^2}, \]  
\[ \omega_\zeta \equiv -\frac{g_1 \dot{\varphi}_0^2}{H^2}, \]  
\[ \omega_\psi \zeta \equiv -\frac{g_1 \dot{\varphi}_0^2}{H^2}, \]  
\[ m_\zeta \equiv \frac{(-3H \kappa^2 \beta^4 + 24H^2(-1 + 3\lambda) \varphi_0 V' - 2\kappa^2 \beta^3 V' + 4H(-1 + 3\lambda) (V')^2 + 4H(-1 + 3\lambda) \varphi_0^2 (9H^2 - V''))}{8H^3(-1 + 3\lambda)}, \]  
\[ m_\psi \equiv \frac{1}{8H^3 \kappa^2(-1 + 3\lambda)} \left\{ 3H \kappa^4 \beta^4 + 24H^2 \kappa^2(-1 + 3\lambda) \varphi_0 V' - 2\kappa^4 \beta^3 V' - 4H(-1 + 3\lambda) \left( 36H^4(-1 + 3\lambda) + \kappa^2 (V')^2 \right) - 4H \kappa^2(-1 + 3\lambda) \varphi_0^2 (6H^2 + V'') \right\}, \]  
\[ m_\psi \zeta \equiv \frac{(-\kappa^2 \beta^3 V' + 2H(-1 + 3\lambda) (V')^2 + 2H(-1 + 3\lambda) \varphi_0^2 V'')}{2H^3(-1 + 3\lambda)}. \]  

3. \( c_s^2 \) and \( M^2 \) in (4.13)

The exact form for the “effective speed of sound” \( c_s^2 \) is given by

\[ c_s^2 \equiv \frac{1}{2048H^2(1 - 3\lambda)^6 g_3 \dot{\varphi}_0^6} \times \left\{ H^2 \kappa^4(-1 + \lambda)^2 \mu^4 \left( 16H^2(1 - 3\lambda)^2 + \kappa^4 \Lambda \mu^2 \right)^2 + 2\dot{\varphi}_0^2 \left[ 16H^4 \kappa^6 \left( 1 - 4\lambda + 3\lambda^2 \right)^3 \mu^4 + H^2 \kappa^{10}(-1 + \lambda)^3(-1 + 3\lambda) \Lambda \mu^2 \right. \right. \]  
\[ \left. + 32(-1 + 3\lambda)^3 g_3 \left( 16H^4 \kappa^2(-1 + 3\lambda)^2 \Lambda \mu^2 + (\kappa^4 \Lambda \mu^2 - 8(1 - 3\lambda)^2 \Lambda \mu^2) \varphi_0^2 \left( 8H^2(-1 + 3\lambda) + \kappa^2(-1 + \lambda) \varphi_0^2 \right) \right) \right\}. \]  

In the case \( \lambda = 1 \),

\[ c_s^2 = \frac{64H^4 \kappa^2 \mu^2 + 16H^2 \left( \kappa^4 \mu^2 - 32g_1 \right) \varphi_0^2}{256H^2 \dot{\varphi}_0^2}, \]  

(A37)
which reduces further to $\epsilon_s^2 = -2g_1$ if we set $\Lambda = 0$, as expected.

The effective mass-square term $M^2$ is given by

$$M^2 = \frac{1}{256c^4c^4} \left\{ \frac{h}{\zeta} \left( \frac{d^2}{\zeta^2} - 4d_{\zeta} \frac{d}{\zeta} \right) - 4c_{\zeta}h^4 \left( \frac{d}{\zeta} \left( -d_{\zeta} \frac{d}{\zeta} + 2d_{\zeta} \frac{d}{\zeta} \right) + d_{\zeta} \left( 3d_{\zeta}^2 - 8d_{\zeta} \frac{d}{\zeta} \right) \right) \\
+ 16c_{\zeta}^2 \left[ 4d_{\zeta} \frac{d}{\zeta} \left( -d_{\zeta} \frac{d}{\zeta} + d_{\zeta} \frac{d}{\zeta} \right) + d_{\zeta}^2 \left( 3d_{\zeta}^2 - 4d_{\zeta} \frac{d}{\zeta} \right) \right] \\
+ \frac{d}{\zeta} \left( Hh \left( \frac{d}{\zeta}^2 - 2d_{\zeta} \frac{d}{\zeta} \right) + d_{\zeta} \left( d_{\zeta}^2 - Hh \frac{d}{\zeta} \frac{d}{\zeta} \right) + 2d_{\zeta} \left( H \frac{d}{\zeta} \frac{d}{\zeta} + d_{\zeta} \left( d_{\zeta}^2 + \frac{d}{\zeta} \left( -3H \frac{d}{\zeta} + 2 \left( \omega \frac{d}{\zeta} + \overline{\omega} \frac{d}{\zeta} \right) \right) \right) \right) \\
- 32c_{\zeta}^2 \left[ 2d_{\zeta}^2 \frac{d}{\zeta} - 4d_{\zeta} \frac{d}{\zeta} e_{\zeta} - 2d_{\zeta} \left( m_{\zeta} \frac{d}{\zeta} + 3H f_{\zeta} \frac{d}{\zeta} - 2m_{\zeta} \frac{d}{\zeta} - 3H \frac{d}{\zeta} f_{\zeta} \right) \right] \\
+ 2d_{\zeta} \left( Hh \frac{d}{\zeta} \frac{d}{\zeta} + d_{\zeta} \left( -3H \frac{d}{\zeta} + 2 \left( \omega \frac{d}{\zeta} + \overline{\omega} \frac{d}{\zeta} \right) \right) \right) \right\} . \tag{A38}
$$

4. $\epsilon_s^2$ and $M^2$ in (4.25)

$$\epsilon_s^2 = -\frac{1}{2c_{\zeta} (3H f_{\zeta} - 2m_{\zeta})} \times \left\{ Hh \left( m_{\zeta} - 3H f_{\zeta} \right)^2 \right. \\
+ (3H f_{\zeta} - 2m_{\zeta}) \left[ -6HF_{\zeta} \omega_{\zeta} + 4m_{\zeta} \omega_{\zeta} + Hh \left( m_{\zeta} - 3H f_{\zeta} \right) - 2 \left( m_{\zeta} - 3H f_{\zeta} \right) \left( \omega_{\zeta} + \overline{\omega}_{\zeta} \right) \right] \right\} , \tag{A39}
$$

and

$$M^2 = -\frac{9H^2 f_{\zeta} f_{\zeta} + 6H f_{\zeta} m_{\zeta} - 2m_{\zeta} - 6H f_{\zeta} m_{\zeta} - 6H m_{\zeta} f_{\zeta} - 9h^2 f_{\zeta}^2}{6H c_{\zeta} f_{\zeta} - 4c_{\zeta} m_{\zeta}} , \tag{A40}
$$

with parameters are given in Appendix [A2].

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