1. Introduction

We consider the following mixed problem:

\[ u_{tt} = (\sigma(u_x))_x + \varepsilon u_{xx}, \quad x \in (0,1), \quad t > 0, \quad \varepsilon \in \mathbb{R} \]  
\[ u(0,t) = 0, \quad u(1,t) = P, \quad t > 0 \]  
\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,1) \]

where \( P > 0 \) and \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is an application with the following properties: 

- \( \sigma(0) = 0 \), \( \sigma \) is continuously differentiable on \( \mathbb{R} \), \( \sigma(\xi) > 0 \), \( \forall \xi > 0 \), \( \sigma(\xi) < 0 \), \( \forall \xi < 0 \); there exist \( \bar{\alpha}, \bar{\beta} \) such that \( 0 < \bar{\alpha} < \bar{\beta} < \beta \) and \( \sigma'(\xi) > 0 \), \( \forall \xi \in (-\infty, \bar{\alpha}) \cup (\bar{\beta}, \infty) \), \( \sigma'(\xi) < 0 \), \( \forall \xi \in (\bar{\alpha}, \bar{\beta}) \), and \( \bar{\alpha}, \bar{\beta} \), such that \( 0 < \bar{\alpha} < \bar{\beta} < \beta < \bar{\beta} \), such that \( \sigma(\bar{\alpha}) = \sigma(\bar{\beta}) \) and \( \sigma(\bar{\alpha}) = \sigma(\bar{\beta}) \).

In the paper [1] it is shown that for \( \varepsilon = 1 \) the mixed problem 1-3 has a unique solution on the appropriate function space. The uniphase and multiphase steady solutions are defined and it is studied the stability of these steady solutions. It is presented also a discretization in the spatial variable \( x \) of the equation (1.1) and some numerical simulations are presented. For \( \varepsilon = 0 \) the problem was studied in [2].

For \( \sigma(\xi) = a^2 \xi, \ a \in \mathbb{R} \) we obtain the wave equation studied in a lot of papers.

The aim of this paper is the study of uniphase steady solutions and partially the study of multiphase steady solutions of the system obtained via
a discretization of (1.1) in spatial variable $x$ and then a discretization in temporal variable $t$.

In the section 2 it is shown that the system obtained from the discretization of (1.1) in the spatial variable $x$ represents an equation Euler-Lagrange by rapport with a Lagrange function discrete continuous with damped term.

The system has a finite number of steady solutions (uniphase and multiphase).

In the section 3, using the linearized system of the system used in the section 2, we study the stability of uniphase steady solutions. We prove that such a solution is stable if $P \in (\alpha, \bar{\alpha})$ or $P \in (\beta, \bar{\beta})$ and $\varepsilon > 0$. If $P \in (\alpha, \beta)$ the uniphase steady solution is hyperbolic so it is unstable. We have determined a polycycle curve for this solution.

For $P \in (\alpha, \bar{\alpha})$ or $P \in (\beta, \bar{\beta})$, $\varepsilon = 0$, we find a curve $u^c(t)$, $t \in \mathbb{R}$ such that every of this components is periodic with the period

$$T_k = \frac{\pi}{n\sqrt{\tau}} \csc \frac{k\pi}{2n}$$

For the steady solution 2-phase we present stability conditions depending on $\varepsilon$, $\tau_1 = \sigma'(\alpha)$, $\rho = \sigma'(\beta)$, $\alpha \in (\alpha, \bar{\alpha})$, $\beta \in (\beta, \bar{\beta})$.

In the section 4 it is shown that the discretization of the equation (1.1) leads us to a difference system of equations which represents the equations Euler-Lagrange with damping for an associated discrete Lagrange function. The steady solutions, uniphase and multiphase, are presented and we prove that they are in finite number.

Also, in the section 4 we are studying the stability of the uniphase solutions, the conditions for a uniphase steady solution to be hyperbolic and the linear stable and unstable associated manifolds.

A similar study for the multiphase steady solutions will be done in a future paper.

### 2. Semidiscretization of the equation

$$u_{tt} = (\sigma(u_x))_x + \varepsilon u_{xxx}$$

For $\varepsilon = 0$, the equation (1.1) represents the equation Euler-Lagrange for the Lagrange function $L : J^2(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}, L(x, t, u, u_x, u_{tx}) = \frac{1}{2}u_x^2 - \sigma(u_x), \quad $
where $J^2(R^2, R)$ is the bundle of the jets of order 2 of the fibration $\pi : R^3 \to R^2$.

Be $x_k = kh_1$, $k = 1, 2, ..., n$ the division points of the interval $[0, 1]$, $h_1 = 1/n$ and $u_k(t) = u(kh_1, t)$, $k = 1, 2, ..., n$. For the boundary conditions we consider: $u_0 = u(0, t) = 0$, $u_n = u(1, t) = P$.

The partial derivatives $u_x$, $u_xx$, $(\sigma(u_x))_x$ will be approximate by:

$$u_x(kh_1, t) \sim \frac{1}{h_1} (u_k(t) - u_{k-1}(t))$$

$$u_{xx}(kh_1, t) \sim \frac{1}{h_1^2} (u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)) \tag{1}$$

$$(\sigma(u_x))_x(kh_1, t) \sim \frac{1}{h_1} \left[ \sigma \left( \frac{1}{h_1} (u_{k+1}(t) - u_k(t)) \right) - \sigma \left( \frac{1}{h_1} (u_k(t) - u_{k-1}(t)) \right) \right]$$

Let be the sequence $(u_k, \dot{u}_k) \in TR$, $k = 1, ..., n$ on the tangent bundle at $R$ and $L : TR \to R$ the Lagrange function defined by:

$$L(u_k, \dot{u}_k) = \frac{1}{2} \dot{u}_k^2 + w \left( \frac{1}{h_1} (u_k - u_{k-1}) \right) , \ k = 1, ..., n - 1 \tag{2}$$

where $w : R \to R$ with $w'(\xi) = \sigma(\xi)$. The action of $L$ is defined by:

$$\mathcal{A}(u, \dot{u}) = \sum_{k=1}^{n} L(u_k, \dot{u}_k) \tag{3}$$

In order to obtain the first variation of $\mathcal{A}$ we consider the sequence $(u_k(\eta), \dot{u}_k(\eta)) \in TR$, $k = 1, ..., n$ with $\eta \in (-a, a)$ and $u_k(0) = u_k$, $\dot{u}_k(0) = \dot{u}_k$. The action (2.3) on this sequence is:

$$\mathcal{A}(\eta) = \sum_{k=1}^{n-1} L(u_k(\eta), \dot{u}_k(\eta))$$

The first variation of (2.3) is given by

$$\frac{\partial \mathcal{A}(\eta)}{\partial \eta} \bigg|_{\eta=0} = 0$$

The first variation of (2.3) is:

$$\ddot{u}_k - \frac{1}{h_1} \sigma(\Delta u_k) + \frac{1}{h_1} \sigma(\Delta u_{k-1}) = 0 , \ k = 1, ..., n - 1 \tag{4}$$
where
\[ \Delta u_{k+1} = \frac{1}{h_1}(u_{k+1} - u_k), \quad k = 1, \ldots, n - 1 \]
and
\[ \Delta u_n = \frac{1}{h_1}(P - u_{n-1}) \]

The system (2.4) represent the semidiscretized system of the equation (1.1) for \( \varepsilon = 0 \).

The semidiscretized system of the equation (1.1) for \( \varepsilon \neq 0 \) is:
\[ \ddot{u}_k - \frac{1}{h_1}\sigma(\Delta u_{k+1}) + \frac{1}{h_1}\sigma(\Delta u_k) = \frac{\varepsilon}{h_1^2} (\dot{u}_{k+1} - 2\dot{u}_k + \dot{u}_{k-1}), \quad k = 1, \ldots, n - 1 \] (5)

and represents the equation Euler-Lagrange for the function \( L \) defined in (2.2) with the dispersion:
\[ \frac{\varepsilon}{h_1^2} (\dot{u}_{k+1} - 2\dot{u}_k + \dot{u}_{k-1}) \]

To the system (2.5) we associate the equivalent system on \( T^*\mathbb{R} \):
\[ \dot{u}_k = v_k, \quad k = 1, \ldots, n - 1 \]
\[ \dot{v}_k = \frac{1}{h_1} \left[ \sigma(\Delta u_{k+1}) - \sigma(\Delta u_k) \right] + \frac{\varepsilon}{h_1^2} (v_{k+1} - 2v_k + v_{k-1}) \] (6)

with the action:
\[ V(u, v) = \sum_{k=1}^{n-1} \left[ \frac{1}{2} v_k^2 + w(\Delta u_k) \right] + w(\Delta u_n) \] (7)

From (2.3) and (2.5) it follows that:
\[ \frac{dA(u, \dot{u})}{dt} = -\sum_{k=1}^{n} \left[ \frac{1}{h_1} (\dot{u}_{k+1} - \dot{u}_k) \right]^2 \] (8)

From (2.5) and (2.6) it follows that any steady solution of the system (2.5) satisfies the conditions:
\[ \sigma(\Delta u_k) = C, \quad k = 1, \ldots, n, \quad \sum_{k=1}^{n} \frac{1}{h_1} \Delta u_k = nP, \quad C \in \mathbb{R} \] (9)
Because the function $\sigma$ is not monotone we obtain two types of steady solutions:

**Definition 2.1.** A uniphase steady solution $\bar{u}$ for the system (2.5) is a steady solution with the property: $\bar{u}_k = kh_1 P$, $k = 1, ..., n - 1$.

**Definition 2.2.** A multiphase steady solution for the system (2.5) is a steady solution with the property: $u_{k+1} - u_k = h_1 f(k)$, $k = 1, ..., n - 1$.

We denote by $E$ the set of multiphase steady solutions and:

$$E^+ = \{ \bar{u} \in E : \sigma'(\Delta u_k) > 0, \ k = 1, ..., n \}$$

$$E^- = \{ \bar{u} \in E : \exists k \in \{1, ..., n\} : \sigma'(\Delta u_k) < 0 \}$$

In [1] it is shown that:

**Lemma 2.1.** For any steady solution $\bar{u} \in E^+$ there is $\alpha \in (\alpha, \bar{\alpha})$ and $\beta \in (\bar{\beta}, \beta)$, $C \in (\sigma, \bar{\sigma})$, such that:

1) $\sigma(\Delta \bar{u}_k) = C, \ k = 1, ..., n$
2) $\Delta(\bar{u}_k) = \alpha$ or $\Delta(\bar{u}_k) = \beta$, $k = 1, ..., n$
3) $k\alpha + (n - k)\beta = nP$

**Lemma 2.2.** If $P < \alpha$ or $P > \bar{\beta}$ then $E^+ = \emptyset$

**Lemma 2.3.** The system (2.5) has a finite number of steady solutions.
3. Properties of the steady solutions of the system (2.5)

The general form of the system (2.6) is:

\[ F_k(u_k, u_{k+1}, \dot{u}_{k-1}, \dot{u}_k, \ddot{u}_k) = 0, \ k = 1, 2, ..., n \] (1)

Let \( \bar{u} \) be a steady solution of the system (2.6). The linearized system in a neighborhood of \( \bar{u} \) is given by:

\[ \left. \frac{\partial F_k}{\partial \eta^j} \left( \bar{u} + \sum_{j=1}^{n} \eta^j w_j, \dot{\bar{u}} + \sum_{j=1}^{n} \eta^j \dot{w}_j, \ddot{\bar{u}} + \sum_{j=1}^{n} \eta^j \ddot{w}_j \right) \right|_{\eta^j} = 0 \] (2)

where \( \eta_j \in (-a, a), \ j = 1, ..., n \).

The linearized system associated to (2.5) is:

\[ \dot{w_j} - \frac{1}{h_1^2} \sigma'(\Delta \bar{u}_{j+1})(w_{j+1} - w_j) + \frac{1}{h_1^2} \sigma'(\Delta \bar{u}_j)(w_j - w_{j-1}) = \varepsilon \frac{1}{h_1^2} \lambda \left( \exp \left( i a_k j \right) \right), \ j = 1, ..., n - 1 \] (3)

A solution for the system (3.3) is:

\[ w_j = \exp \left( \lambda j \right) \exp \left( i a_k j \right), \ a_k = \frac{\pi k}{n}, \ k = 1, ..., n - 1, \ \lambda \in \mathbb{C} \] (4)

From (3.3) and (3.5) yields:

\[ \lambda^2 \exp i a_k - \frac{1}{h_1^2} \sigma'(\Delta \bar{u}_{k+1}) \exp i a_k \exp i a_{k-1} + \frac{1}{h_1^2} \sigma'(\Delta \bar{u}_k)(\exp i a_{k-1}) = \varepsilon \frac{1}{h_1^2} \lambda (\exp i a_{k-1})^2 \] (5)
**Theorem 3.1.** If \( P \in (\alpha, \bar{\alpha}) \) or \( P \in (\beta, \bar{\beta}) \) the uniphase steady solution \( \bar{u}_k = kh_1P, k = 1, \ldots, n - 1 \) of the system (2.5) is asymptotically stable if \( \varepsilon > 0 \) and unstable if \( \varepsilon < 0 \).

**Proof:** For the uniphase steady solution \( \bar{u} \), the equations (3.5) become:

\[
\lambda^2 \exp ia_k - \frac{1}{h_1^2} \tau (\exp ia_{k-1})^2 - \frac{\varepsilon}{h_1^2} \lambda (\exp ia_{k-1})^2 = 0, \ k = 1, \ldots, n - 1 \quad (6)
\]

where \( \tau = \sigma'(P) \). We denote:

\[
\mu_k = \frac{(\exp ia_{k-1})^2}{\exp ia_k} = -4\sin^2 \frac{\pi k}{2n}
\]

The equations (3.6) become:

\[
\lambda^2 - \varepsilon \mu_k n^2 \lambda - \mu_k n^2 \tau = 0, \ k = 1, \ldots, n - 1
\]

with the roots:

\[
\lambda = \frac{1}{2} \mu_k n^2 \left[ \varepsilon \pm \left( \varepsilon^2 + \frac{4\tau}{\mu_k n^2} \right)^{\frac{1}{2}} \right], \ k = 1, \ldots, n - 1 \quad (8)
\]

If \( \varepsilon > 0 \), because \( \mu_k < 0 \) yields that \( \Re \lambda_k < 0 \) so it follows that the steady uniphase solution \( \bar{u} \) is asymptotically stable.

If \( \varepsilon < 0 \), it follows that \( \Re \lambda_k < 0 \) so the steady uniphase solution is unstable.

**Theorem 3.2.** If \( P \in (\bar{\alpha}, \beta) \) the uniphase steady solution \( \bar{u} \) of the system (2.5) is hyperbolic so it is unstable.

**Proof:** For the uniphase steady solution \( \bar{u} \) in the equations (3.7) become:

\[
\lambda^2 - \varepsilon \mu_k n^2 \lambda + \mu_k n^2 \rho = 0, \ k = 1, \ldots, n - 1
\]

where \( \rho = -\sigma'(P) \); the solutions of these equations are:

\[
\lambda_k^+ = \frac{1}{2} \mu_k n^2 \left[ \varepsilon + \left( \varepsilon^2 - \frac{4\rho}{\mu_k n^2} \right)^{\frac{1}{2}} \right], \quad k = 1, \ldots, n - 1
\]

\[
\lambda_k^- = \frac{1}{2} \mu_k n^2 \left[ \varepsilon - \left( \varepsilon^2 - \frac{4\rho}{\mu_k n^2} \right)^{\frac{1}{2}} \right], \quad k = 1, \ldots, n - 1
\]
The real solutions $\lambda^+, \lambda^-$ have the property: $\lambda^+ \cdot \lambda^- = \mu_k n^2 \rho < 0$. It follows that they have opposite signs. The steady solution is unstable, it is hyperbolic.

The linear manifolds associated to the steady solution of hyperbolic type are:

$$
\tilde{u}_k^h(t) = h_1 k P + \exp(t \lambda^+_k) \sum_{j=1}^{n-1} \eta_j \exp(ia_k j), \ t \in \mathbb{R}, \ k = 1, ..., n - 1 \quad (11)
$$

$$
\tilde{u}_k^s(t) = h_1 k P + \exp(t \lambda^-_k) \sum_{j=1}^{n-1} \eta_j \exp(ia_k j), \ t \in \mathbb{R}, \ k = 1, ..., n - 1 \quad (12)
$$

$\tilde{u}_k^u$ and $\tilde{u}_k^s$ have the properties:

$$
\lim_{t \to -\infty} \tilde{u}_k^h(t) = \bar{u}_k, \ \lim_{t \to \infty} \tilde{u}_k^s(t) = \bar{u}_k, \ k = 1, ..., n - 1
$$

**Theorem 3.3.** The curves $u_{kk+1}^h, k = 1, 2, ..., n, \ t \in \mathbb{R}$ given by:

$$
u_{kk+1}^h = \frac{\tilde{u}_k^h(t)}{\bar{u}_k^h(t)} \bar{u}_{k+1} + \frac{\tilde{u}_k^u(t)}{\bar{u}_k^u(t)} \bar{u}_k, \ t \in \mathbb{R} \quad (13)
$$

are policycles for the uniphase steady solution, that is

$$
\lim_{t \to \infty} u_{kk+1}^h(t) = \bar{u}_k
$$

**Proof:** From (3.12) and (3.13) we have:

$$
\lim_{t \to \infty} u_{kk+1}^h(t) = \frac{\bar{u}_k}{\bar{u}_{k+1}} \bar{u}_{k+1} + \bar{u}_k \lim_{t \to \infty} \frac{\tilde{u}_k^u(t)}{\bar{u}_k^u(t)}
$$

From (3.11) yields:

$$
\lim_{t \to \infty} \frac{\tilde{u}_k^u(t)}{\bar{u}_k^u(t)} = \lim_{t \to \infty} \frac{h_1(k + 1)P + \exp(t \lambda^+_k) \sum_{j=1}^{n} \eta_j \exp(ia_{k+1} j)}{h_1(k)P + \exp(t \lambda^-_k) \sum_{j=1}^{n} \eta_j \exp(ia_{k} j)} = \frac{\lambda^+_{k+1}}{\lambda^+_k} \lim_{t \to \infty} \exp(t(\lambda^+_k - \lambda^+_{k+1}))) \frac{\sum_{j=1}^{n} \eta_j \exp(ia_{k+1} j)}{\sum_{j=1}^{n} \eta_j \exp(ia_{k} j)}
$$
From (3.10) we have $\lambda^+_{k+1} < \lambda^+_{k}$ and $\lambda^-_{k+1} < \lambda^-_{k}$. It follows that

$$\lim_{t \to \infty} \frac{\tilde{u}^h_{k+1}(t)}{\tilde{u}^h_k(t)} = 0$$

Thus we obtain:

$$\lim_{t \to \infty} u^h_{k+1}(t) = \bar{u}_k$$

Using (3.11) and (3.13) it follows:

$$\lim_{t \to -\infty} u^h_{kk+1}(t) = \bar{u}_{k+1} + \bar{u}_{k+1} \lim_{t \to -\infty} \frac{\tilde{u}^s_k(t)}{\tilde{u}^s_{k+1}(t)}$$

From (3.12) yields:

$$\lim_{t \to -\infty} \frac{\tilde{u}^s_k(t)}{\tilde{u}^s_{k+1}(t)} = \frac{\lambda^-_k}{\lambda^-_{k+1}} \lim_{t \to -\infty} \exp(t(\lambda^-_k - \lambda^-_{k+1})).$$

$$\sum_{j=1}^{n} \eta_j \exp(ia_k j) \cdot \frac{\sum_{j=1}^{n} \eta_j \exp(ia_{k+1} j)}{\sum_{j=1}^{n} \eta_j \exp(ia_k j)} = 0$$

Thus we obtain:

$$\lim_{t \to -\infty} u^h_{kk+1} = \bar{u}_{k+1}$$

The curves (3.13) are a polycycle considering:

$$u_{n+1} = u_1$$

**Theorem 3.4.** If $P \in (\alpha, \bar{\alpha})$ or $P \in (\beta, \bar{\beta})$ and $\varepsilon = 0$ the curves:

$$u^c_k(t) = \bar{u}_k + \exp(\lambda^c t) \sum_{j=1}^{n} \eta_j \exp(ia_k j), \ k = 1, ..., n, \ t \in \mathbb{R}$$

are periodic, with the period $T_k = \frac{\pi}{n \sqrt{\tau}} \sec \frac{k\pi}{2n}, \ k = 1, ..., n$, where $\lambda^c = \pm 2in\sqrt{\tau} \sin \frac{k\pi}{2n}$. 

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Proof: For \( \varepsilon = 0 \) the equation (3.7) is:
\[
\lambda^2 - \mu_k n^2 \tau = 0, \quad k = 1, \ldots, n-1
\]
with the roots:
\[
\lambda^c = \pm 2n \sqrt{\tau} \sin \frac{k\pi}{2n}
\]

The uniphase steady solution \( \bar{u}_k = kh_1 P \) is center for the system (2.5) with \( \varepsilon = 0 \). From the condition \( u_k^c(t) = u_{k+1}^c(t + T_k), \quad k = 1, \ldots, n-1 \), it follows that \( \exp (\lambda^c (t + T_k)) = \exp (\lambda^c (t)) \). Hence:
\[
2n \sqrt{\tau} \sin \frac{k\pi}{2n} T_k = 2\pi
\]

We continue our study with a multiphase steady solution \( \bar{u} \in E^+ \). To be more specific we will consider only 2-phase steady solutions. From Lemma 2.1. it follows that for \( \alpha \in (\bar{\alpha}, \bar{\alpha}), \beta \in (\bar{\beta}, \bar{\beta}) \), with \( I(\alpha) = \{ i \in \{1, 2, \ldots, n\}, \Delta \bar{u}_i = \alpha \} \), \( I(\beta) = \{ i \in \{1, 2, \ldots, n\}, \Delta \bar{u}_i = \beta \} \) we have
\[
I(\alpha) \cup I(\beta) = \{1, \ldots, n\}
\]
\[
\text{card } I(\alpha) \cdot \alpha + (n - \text{card } I(\beta)) \beta = nP
\]

Let consider the following sets:
\[
J(\alpha) = \{ i \in I(\alpha): i + 1 \in I(\alpha) \}
\]
\[
J(\beta) = \{ i \in I(\beta): i + 1 \in I(\beta) \}
\]

and
\[
K(\alpha, \beta) = \{ i \in I(\alpha), i + 1 \in I(\beta) \}
\]

From (3.5) we obtain:
\[
\lambda^2 + 4\varepsilon n^2 \sin^2 \frac{\alpha_p}{2} \lambda + 4n^2 \sin^2 \frac{\alpha_p}{2} \tau = 0, \quad p \in J(\alpha) \tag{15}
\]
\[
\lambda^2 + 4\varepsilon n^2 \lambda \sin^2 \frac{\alpha_p}{2} + 4n^2 \sin^2 \frac{\alpha_p}{2} \tau = 0, \quad p \in J(\beta) \tag{16}
\]
\[
\lambda^2 + 4\varepsilon n^2 \lambda \sin^2 \frac{\alpha_p}{2} - n^2 [\rho (\exp i\alpha_{p-1}) + \tau (\exp (-i\alpha_p) - 1)] = 0, \quad p \in K(\alpha, \beta) \tag{17}
\]
where $\tau = \sigma'(\alpha)$, $\rho = \sigma'(\beta)$, $\rho > \tau$.

**Lemma 3.1.**

a). The equations (3.15) have complex roots with negative real part if and only if:

$$\varepsilon > 0 \text{ and } \varepsilon < \frac{\sqrt{\tau}}{n\sin \frac{\alpha}{2}}, \ r = \max J(\alpha)$$

The equations (3.15) have real negative roots if and only if:

$$\varepsilon > 0 \text{ and } \varepsilon > \frac{\sqrt{\rho}}{n\sin \frac{\beta}{2}}, \ q = \min J(\alpha)$$

b). The equations (3.16) have complex roots with negative real part if and only if:

$$\varepsilon > 0 \text{ and } \varepsilon < \frac{\sqrt{\rho}}{n\sin \frac{\beta}{2}}, \ r = \max J(\beta)$$

The equations (3.16) have real negative roots if and only if:

$$\varepsilon > 0 \text{ and } \varepsilon > \frac{\sqrt{\rho}}{n\sin \frac{\beta}{2}}, \ q = \min J(\beta)$$

c). The equations (3.17) have complex roots with negative real part if and only if:

$$\varepsilon > 0 \frac{\sqrt{\rho + \tau - \sqrt{\Delta}}}{n\sin \frac{\alpha}{2}} < \varepsilon < \frac{\sqrt{\rho + \tau + \sqrt{\Delta}}}{n\sin \frac{\alpha}{2}}$$

where

$$q = \min K(\alpha, \beta), \ r = \max K(\alpha, \beta)$$

**Proof:** From the equations (3.15) and (3.16) it follows directly a) and b).

The equations (3.17) may be written in the following form:

$$\lambda^2 + 4\varepsilon n^2 \sin^2 \frac{\alpha p}{2} - \lambda + 2n^2(\rho + \tau)\sin^2 \frac{\alpha p}{2} - in^2(\rho - \tau)\sin \alpha p = 0, \ p \in K(\alpha, \beta)$$
The necessary and sufficient condition for these equations to admit complex roots with negative real part is:

\[
2n^2(\rho + \tau)\sin^2 \frac{a_p}{2} > \frac{16\varepsilon n^4 \sin^4 a_p}{8} + \frac{2n^4(\rho - \tau)^2 \sin^2 a_p}{16\varepsilon n^4 \sin^4 a_p}
\] (24)

and

\[4\varepsilon n^2 \sin^2 \frac{a_p}{2} > 0, \quad 2n^2(\rho + \tau)\sin^2 \frac{a_p}{2} > 0\]

**Theorem 3.5.** The 2-phase steady solution is stable if one of the following conditions holds:

a). \[
\sqrt{\frac{\rho + \tau - \sqrt{\Delta}}{n\sin \frac{a_q}{2}}} < \varepsilon < \frac{\sqrt{\tau}}{n}, \quad q = \min K(\alpha, \beta)
\]

b). \[
\sqrt{\frac{\rho + \tau - \sqrt{\Delta}}{n\sin \frac{a_r}{2}}} < \varepsilon < \frac{\sqrt{\rho + \tau + \sqrt{\Delta}}}{n\sin \frac{a_r}{2}}, \quad r = \max K(\alpha, \beta)
\]

c). \[
\max \left( \frac{\rho}{n\sin \frac{a_{q_2}}{2}}, \frac{\sqrt{\rho + \tau - \sqrt{\Delta}}}{n\sin \frac{a_{q_3}}{2}} \right) < \varepsilon < \min \left( \frac{\sqrt{\tau}}{n\sin \frac{a_{r_1}}{2}}, \frac{\sqrt{\rho + \tau + \sqrt{\Delta}}}{n\sin \frac{a_{r_3}}{2}} \right)
\]

\[q_2 = \min J(\beta), \quad q_3 = \min K(\alpha, \beta)\]

\[r_1 = \max J(\alpha), \quad r_3 = \max K(\alpha, \beta)\]

d). \[
\max \left( \frac{\sqrt{\tau}}{n\sin \frac{a_{q_1}}{2}}, \frac{\sqrt{\rho + \tau - \sqrt{\Delta}}}{n\sin \frac{a_{q_3}}{2}} \right) < \varepsilon < \min \left( \frac{\rho}{n\sin \frac{a_{r_2}}{2}}, \frac{\sqrt{\rho + \tau + \sqrt{\Delta}}}{n\sin \frac{a_{r_3}}{2}} \right)
\]

\[q_1 = \min J(\alpha), \quad q_3 = \min K(\alpha, \beta)\]
\[ r_2 = \max J(\beta), \ r_3 = \max K(\alpha, \beta) \]

The proof is a direct consequence of the Lemma 3.1.

**Theorem 3.6.** If \( \varepsilon = 0 \) the curves

\[
u_c^p(t) = \bar{u}_p + \exp(\lambda_p^c t) \sum_{k=1}^{n} \eta_k \exp(ia_p k), \ p \in J(\alpha)
\]

\[
u_c^p(t) = \bar{u}_p + \exp(\lambda_p^c t) \sum_{k=1}^{n} \eta_k \exp(ia_p k), \ p \in J(\beta)
\]

where \( \lambda_p^c, \ p \in J(\alpha) \) respectively \( \lambda_p^c, \ p \in J(\beta) \) are roots of the equations:

\[
\lambda^2 + 4n^2 \sin^2 \frac{a_p}{2} \tau = 0, \ p \in J(\alpha)
\]

\[
\lambda^2 + 4n^2 \sin^2 \frac{a_p}{2} \rho = 0, \ p \in J(\beta)
\]

are periodic with the period

\[
T_p = \frac{\pi}{n \sqrt{\tau} \csc p\pi/2n}, \ p \in J(\alpha)
\]

respectively

\[
T_p = \frac{\pi}{n \sqrt{\rho} \csc p\pi/2n}, \ p \in J(\beta)
\]

The proof follows from the above theorem.

**Remark.** For \( \varepsilon < 0 \) the equations (3.15) and (3.16) have positive roots and it follows that the 2-phase steady solution is unstable.

For \( \varepsilon < 0 \) or \( \varepsilon \) not satisfying one of the above conditions a), b), c), the roots of the equations (4.10), (4.11), (4.12) may have positive or negative real part in the case they are complex, or they are positive or negative real numbers. In this case the steady 2-phase solution is unstable; it is hyperbolic saddle. The associated linear manifolds may be described completely in the same manner we have done for the uniphase steady solutions.

**4. The discretization of the equation**

\[ u_{tt} = \left( \sigma(u_x) \right)_x + \varepsilon u_{xxt} \]
Let \( x_k = kh_1, \ k = 1, \ldots, n \) the division points of the interval \([0, 1]\) with \( h_1 = 1/n \) and \( t_p = ph_2, \ p = 1, \ldots, m \) the division points of the interval \([0, 1]\), \( h_2 = 1/n \).

We’ll approximate the derivates \( \dot{u}_k(t), \ddot{u}_k(t) \) by \( \frac{1}{h_2}[u_k^p - u_k^{p-1}] \) and \( \frac{1}{h_2^2}[u_k^{p+1} - 2u_k^p + u_k^{p-1}] \), where \( u_k^p = u(kh_1, ph_2) \).

We call the discrete system associated to the equation (1.1) the following system:

\[
\frac{1}{h_2^2}[u_k^{p+1} - 2u_k^p + u_k^{p-1}] - \frac{1}{h_1}\sigma \left( \frac{1}{h_1}(u_{k+1}^p - u_k^p) - \frac{1}{h_1}(u_k^p - u_{k-1}^p) \right) = \\
\frac{\varepsilon}{h_1^2 h_2}[u_{k+1}^p - u_k^p - 2(u_k^{p+1} - u_k^p) + u_{k-1}^{p+1} - u_{k-1}^p]
\]

\( k = 1, \ldots, n - 1, \ p = 1, \ldots, m - 1 \) \quad (1)

The corresponding boundary conditions are:

\[ u_0^p = 0, \ u_n^p = P > 0, \ p = 1, \ldots, m \] \quad (2)

The system (4.1) represent the discrete Euler-Lagrange equations for the discrete Lagrange function:

\[
L(u_{k-1}^p, u_k^p, u_{k+1}^p) = \frac{1}{2h_2^2}(u_{k+1}^p - u_k^p)^2 + w\left(\frac{1}{h_1}(u_k^p - u_{k-1}^p)\right)
\] \quad (3)

where \( w : \mathbb{R} \to \mathbb{R} \), \( w'(\xi) = \sigma(\xi) \), and the dispersive term is:

\[
\frac{\varepsilon}{h_1^2 h_2}[u_{k+1}^p - u_k^p - 2(u_k^{p+1} - u_k^p) + u_{k-1}^{p+1} - u_{k-1}^p]
\]

For (4.1) it follows that the steady solutions \( \tilde{u} = \tilde{u}_k \) satisfy:

\[
\sigma\left(\frac{1}{h_1}(\tilde{u}_{k+1}^p - \tilde{u}_k^p)\right) = \sigma\left(\frac{1}{h_1}(\tilde{u}_k^p - \tilde{u}_{k-1}^p)\right), \ k = 1, \ldots, n - 1
\] \quad (4)

From (4.2) we obtain:

\[
\sum_{k=1}^{n} \frac{1}{h_1}(\tilde{u}_k^p - \tilde{u}_{k-1}^p) = \frac{1}{h_1}P
\]
Thus a steady solution $\bar{u}$ satisfies the following:

$$\bar{u}_k^p - \bar{u}_{k-1}^p = P - \bar{u}_{n-1}^p, \quad \sigma\left(\frac{1}{h_1}(\bar{u}_k^p - \bar{u}_{k-1}^p)\right) = C, \quad k = 1, ..., n, \quad C \in \mathbb{R} \quad (5)$$

**Definition 4.1.** A steady solution $\bar{u} = (\bar{u}_k^p)$ with $\bar{u}_k^p = kh_1P, \quad k = 1, ..., n, \quad p = 1, ..., m$ is called uniphase steady solution.

A steady solution $\bar{u} = (\bar{u}_k^p)$ with the property $\bar{u}_k^p - \bar{u}_{k-1}^p = h_1f(k), \quad k = 1, ..., n - 1$ is called multiphase steady solution.

Let $E = \{\bar{u} = (\bar{u}_k^p)\}$ the set of multiphase steady solution. We consider the following sets:

$$E^+ = \{\bar{u} = (\bar{u}_k^p) : \sigma\left(\frac{1}{h_1}(\bar{u}_k^p - \bar{u}_{k-1}^p)\right) > 0, \quad k = 1, ..., n - 1\}$$

$$E^- = \{\bar{u} = (\bar{u}_k^p) : \sigma\left(\frac{1}{h_1}(\bar{u}_k^p - \bar{u}_{k-1}^p)\right) > 0, \quad k = 1, ..., n - 1\}$$

**Theorem 4.1.** The system (4.1) has a finite number of multiphase steady solutions.

Let $\bar{u} = (\bar{u}_k^p)$ a steady solution of (4.1) and $\bar{u}_k^p = \bar{u}_k^p + \sum_{l=1}^{n} \eta_l w_l^p(k), \quad k = 1, ..., n - 1$ the components of a vector with $\eta_l \in (-a, a)$.

The linearized system associated to the system (4.1) in a neighboorhood of the steady solution $\bar{u}$ is:

$$h_2^2(w_l^{p+1} - 2w_l^p + w_l^{p-1}) - h_2^2h_1^2 \left[ \sigma\left(\frac{1}{h_1}(\bar{u}_{k+1}^p - \bar{u}_k^p)\right)(w_{l+1}^p - w_l^p) - \sigma\left(\frac{1}{h_1}(\bar{u}_k^p - \bar{u}_{k-1}^p)\right)(w_l^p - w_{l-1}^p) \right] =$$

$$= \varepsilon h_2[w_l^{p+1} + w_l^{p-1} - 2(w_l^{p+1} - w_l^p) + w_{l-1}^{p+1} - w_{l-1}^p] \quad (6)$$

A solution of (4.6) has the form:

$$w_l^p = \lambda^p exp(iakl), \quad a_k = \frac{4\pi}{n}, \quad k = 1, ..., n, \quad \lambda \in \mathbb{C} \quad (7)$$

Replacing in (4.6) we obtain:
\[ h_1^2(\lambda^2 - 2\lambda + 1)\exp(ia_k) - h_2^2h_1^2(\exp2ia_k - \exp ia_k) - \\
- s_k\lambda(\exp ia_k - 1) - \varepsilon h_2(\lambda^2 - \lambda)(\exp ia_k - 1)^2 = 0 \quad (8) \]

where
\[ s_k = \sigma' \left( \frac{1}{h_1} (\bar{u}_k^p - \bar{u}_{k-1}^p) \right) \]

From (4.8) it follows:
\[ \lambda^2[h_1^2 - \varepsilon h_2(\exp ia_k - 1)^2] - \lambda[2h_1^2\exp ia_k + h_2^2h_1(s_{k+1}\exp ia_k)(\exp ia_k - 1) - \\
- s_k(\exp ia_k - 1) - h_2\varepsilon(\exp ia_k - 1)^2] + h_1^2\exp ia_k = 0 \quad (9) \]

For a uniphase steady solution \( \bar{u} = (\bar{u}_k^p) \), \( \bar{u}_k^p = kh_1P \) and \( \tau = \sigma'(P) > 0 \), the equations (4.9) become:
\[ \lambda^2(h_1^2 - \varepsilon h_2\mu_k) - \lambda(2h_1^2 - h_2^2h_1\tau\mu_k - \varepsilon h_2\mu_k) + h_1^2 = 0 \quad (10) \]

where
\[ \mu_k = \frac{(\exp(ia_k - 1))^2}{\exp (ia_k - 1)} = -4\sin^2\frac{k\pi}{2n} \]

We suppose for the rest of this section that \( h_1, h_2 \in (0, 1) \) and \( \varepsilon \neq \frac{h_1}{h_2\mu_k} \).

**Theorem 4.1.** The uniphase steady solution for the system (4.1) is asymptotically stable if and only if the following condition hold:
\[ \tau > \frac{h_1}{\frac{h_2^2}{2}\sin^2\frac{\pi}{2n}}, \; \varepsilon \in \left( -\frac{h_1^2}{2h_2\sin^2\frac{\pi}{2n}} + \frac{h_1h_2}{2} \tau, \infty \right) \quad (11) \]

**Proof:** The equations (4.10) have the roots in modulus less than 1 if and only if (4.11) hold.

A necessary and sufficient condition for the \( k \)-th equation in (4.10) to have the modulus of its roots less than one is:
\[ \frac{h_1^2}{h_1^2 - \varepsilon h_2\mu_k} - 1 < 0 \]
\[ -1 - \frac{h_1^2 - \varepsilon h_2 \mu_k}{h_2^2} < \frac{|2h_1^2 + h_2^2 h_1 \tau \mu_k - \varepsilon h_2 \mu_k|}{h_2^2} < 1 + \frac{h_1^2 - \varepsilon h_2 \mu_k}{h_1^2} \]  \hspace{1cm} (12)

The inequalities (4.12) hold if and only if

\[ \tau > -\frac{4h_1}{h_2^2 \mu_k}, \; \varepsilon \in \left( -\infty, -\frac{2h_1^2}{\mu_k h_2} + \frac{h_1 h_2}{2} \tau, \infty \right) \]  \hspace{1cm} (13)

But \( \mu_k > \mu_{k+1} \) so it follows that (4.13) hold for any \( k = 1, \ldots, n \) if and only if (4.11) hold.

**Theorem 4.2.** The uniphase steady solution \( \bar{u} = (\bar{u}_k^\alpha) \) of the system (4.1) is unstable if one of the following conditions hold:

a). \( \tau < -\frac{h_1}{h_1^2 \sin^2 \frac{\pi}{2n}}, \; \varepsilon \in \left( -\infty, -\frac{2h_1^2}{4h_2 \sin^2 \frac{\pi}{2n}} + \frac{h_1 h_2}{2} \tau \right) \)

b). \( \varepsilon \in \left( -\infty, -\frac{h_1^2}{2h_2 \sin^2 \frac{\pi}{2n}} - h_1 h_2 \tau \right) \)

c). \( \tau > \frac{h_1}{h_1^2 \sin^2 \frac{\pi}{2n}}, \; \varepsilon \in \left( -\frac{h_1^2}{4h_2 \sin^2 \frac{\pi}{2n}}, -\frac{h_1^2}{4h_2 \sin^2 \frac{\pi}{2n}} + h_1 h_2 \tau \right) \)

**Proof:** a). The equations (4.10) have the roots in modulus greater than one only if a). hold. A necessary and sufficient condition for the \( k \)-th equation from (4.9) to have roots in modulus greater than one is:

\[ \frac{h_1^2 - \varepsilon h_2 \mu_k}{h_2^2} - 1 < 0 \]

\[ -1 - \frac{h_1^2 - \varepsilon h_2 \mu_k}{h_1^2} < \frac{|2h_1^2 + h_2^2 h_1 \tau \mu_k - \varepsilon h_2 \mu_k|}{h_1^2} < \frac{h_1^2 - \varepsilon h_2 \mu_k}{h_1^2} + 1 \]  \hspace{1cm} (14)

From (4.14) we obtain

\[ \tau < -\frac{4h_1}{h_2^2 \mu_k}, \; \varepsilon \in \left( -\infty, \frac{2h_1^2}{h_2 \mu_k} + \frac{h_1 h_2}{2} \tau \right) \]  \hspace{1cm} (15)
b), c) can be provided in an analogous manner.

**Theorem 4.3.** The uniphase steady solution \( \bar{u} = (\bar{u}_p^k) \) for the system (4.1) with \( \varepsilon = 0 \) is unstable if

\[
\tau < \frac{h_1}{h_2 \sin^2 \frac{\pi}{2n}}
\]

**Proof:** For \( \varepsilon = 0 \), the equation (4.10) become:

\[
\lambda^2 - \lambda (2 + \frac{h_2^2}{h_1^2} \tau \mu_k) + 1 = 0, \; k = 1, \ldots, n
\]

(16)

From (4.16) we have \( \lambda_1 \lambda_2 = 1 \) and \( \lambda_1 + \lambda_2 = 2 + \frac{h_2^2}{h_1^2} \tau \mu_k \). A necessary and sufficient condition for \( k \)-th equation from (4.16) to have one root greater than one and the other less than one is:

\[
\tau < \frac{h_1}{h_2 \sin^2 \frac{\pi}{2n}}
\]

**Proposition 4.1.** Let \( u = \bar{u}_k^p \) a uniphase steady solution for the system (4.1) which satisfies one of the condition b). or c). from the theorem 4.2. and \( \lambda_k^+, \lambda_k^- \) the roots of the \( k \)-th equation (4.9) with \( |\lambda_k^+| > 1 \) and \( |\lambda_k^-| < 1 \). The linear stable manifold is given by:

\[
\bar{u}_k^{sp} = kh_1 P + (\lambda_k^-)^p \sum_{l=1}^{n} \eta_l \exp (ia_k l)
\]

\( p = 1, \ldots, m, \; k = 1, \ldots, n - 1 \)

The linear unstable manifold is given by:

\[
\bar{u}_k^{up} = kh_1 P + (\lambda_k^+)^p \sum_{l=1}^{n} \eta_l \exp (ia_k l)
\]

\( p = 1, \ldots, m, \; k = 1, \ldots, n - 1 \)

In these conditions it follows that:

\[
\lim_{p \to \infty} \bar{u}_k^{sp} = kh_1 P \quad \text{and} \quad \lim_{p \to -\infty} \bar{u}_k^{up} = kh_1 P
\]

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