Poincaré series of filtrations corresponding to ideals on surfaces *

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In [2], [9], [4], [5], there were considered and, in some cases, computed Poincaré series of two sorts of multi-index filtrations on the ring of germs of functions on a complex (normal) surface singularity \((S,0)\) (in particular on \((\mathbb{C}^2,0)\)). A filtration from the first class was defined by a curve (with several branches) on \((S,0)\). The other one (so called divisorial filtration) was defined by a set of components of the exceptional divisor of a modification of the surface singularity \((S,0)\). Here we define a filtration corresponding to an ideal or to a set of ideals in the ring \(\mathcal{O}_{S,0}\) of germs of functions on \((S,0)\) and compute the corresponding Poincaré series in some cases. For \(S = \mathbb{C}^2\) this notion unites the two classes of filtrations considered earlier.

The discussed notion of the filtration corresponding to an ideal was inspired by the notion of the zeta function of an ideal given in [14]. For our aim it is convenient to extend this notion to a finite set of ideals \(\{I_1, \ldots, I_r\}\), defining the “Alexander polynomial” of this set. This is a mixture of the notions introduced in [14] and [15].

Let \((V,0)\) be a germ of an analytic space with an isolated singular point at the origin and let \(I_1, \ldots, I_r\) be ideals in the ring \(\mathcal{O}_{V,0}\) of germs of functions on \((V,0)\). Let \(\pi : (X, D) \to (V,0)\) be a resolution of the singularity of \(V\) and also of the set of the ideals \(\{I_1, \ldots, I_r\}\). This means that:

1) \(X\) is a complex analytic manifold;

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2) $\pi$ is a proper analytic map which is an isomorphism outside of the union of the zero loci of the ideals $I_1, \ldots, I_r$;

3) the exceptional divisor $D = \pi^{-1}(0)$ is a normal crossing divisor on $X$;

4) for $i = 1, \ldots, r$, the lifting $I_i^* = \pi^* I_i$ of the ideal $I_i$ to the space $X$ of the modification is locally principal (and therefore $\pi$ is a principalization of the ideal $I_i$);

5) the union of the zero loci of the liftings of the ideals $I_1, \ldots, I_r$ to the space $X$ of the modification is a normal crossing divisor on $X$.

For $k = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 0}^r$, let $S_k$ be the set of points $x \in D$ such that, in a neighbourhood of $x$, the zero locus of the ideal $I_i^*$, $i = 1, \ldots, r$, is a smooth hypersurface with multiplicity $k_i$, i.e. $I_i^* = \langle g_{ix} \rangle$ where, in some local coordinates $z_1, \ldots, z_n$ on $\tilde{X}$ centered at the point $x$, one has $g_{ix}(z) = u_i(z)z_1^{k_i}$ with $u_i(0) \neq 0$.

**Definition:** The Alexander polynomial of the set of ideals $\{I_i\}$ is the rational function (or a power series) in the variables $t_1, \ldots, t_r$ given by the A’Campo type (\(\Pi\)) formula:

$$\Delta_{\{I_i\}}(t_1, \ldots, t_r) = \prod_{k \in \mathbb{Z}_{\geq 0}^r} (1 - t_k^{-1})^{-\chi(S_k)},$$

where $\chi(\cdot)$ is the Euler characteristic and $t_k := t_1^{k_1} \cdots t_r^{k_r}$.

**Remarks.** 1. For $r = 1$, this gives the notion of the zeta function $\zeta_I(t)$ of an ideal $I$: [14]. In [14, Theorem 4.2] one forgot to write that $\pi$ should be an isomorphism outside of the zero locus of the ideal $I$. This is the reason why we formulate the notion for germs $(V, 0)$ with isolated singularities. Another option is to demand that the singular locus of $V$ is contained in the zero locus of the ideal $I$.

2. The Alexander polynomial defined this way is not, generally speaking, a polynomial. It is really a polynomial in the case $V = \mathbb{C}^2$ and $I_i = \langle f_i \rangle$, where $f_i = 0$ are equations of the irreducible components $C_i$ of a (reduced) plane curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$: $C = \bigcup_{i=1}^r C_i$, $r > 1$. In this case $\Delta_{\{I_i\}}(t_1, \ldots, t_r)$ coincides with the classical Alexander polynomial in several variables of the algebraic link $C \cap S^3_\varepsilon \subset S^3_\varepsilon$ ($S^3_\varepsilon$ is the sphere of small radius $\varepsilon$ centered at the origin in $\mathbb{C}^2$). We prefer to keep the name in the general case.

Now let $(S, 0)$ be a germ of a normal surface singularity. For a function germ $f \in \mathcal{O}_{S,0}$ and a divisor $\gamma$ on $(S, 0)$, there is defined the intersection
number \((\gamma \circ f) \in \mathbb{Z} \cup \{\infty\}\). If \(\gamma\) is a Cartier divisor, \(\gamma = \{g = 0\}\), \(g \in \mathcal{O}_{S,0}\), we shall write \((g \circ f)\) instead of \((\gamma \circ f)\). Let \(I\) be an ideal in the ring \(\mathcal{O}_{S,0}\).

**Definition:** The filtration corresponding to the ideal \(I\) is the filtration
\[
\mathcal{O}_{S,0} = J_I(0) \supset J_I(1) \supset J_I(2) \supset \ldots
\]
by the ideals \(J_I(v)\) defined by
\[
J_I(v) = \{g \in \mathcal{O}_{S,0} : \forall f \in I, (g \circ f) \geq v\}.
\]

This filtration is defined by the order function \(v_I : \mathcal{O}_{S,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}\): \(v_I(g) := \min\{(g \circ f) : f \in I\}\). (A function \(v : \mathcal{O}_{S,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}\) is called an order function if \(v(\lambda g) = v(g)\) for \(\lambda \neq 0\) and \(v(g_1 + g_2) \geq \min\{v(g_1), v(g_2)\}\).)

Now let \(\{I_1, \ldots, I_r\}\) be a set of ideals in the ring \(\mathcal{O}_{S,0}\).

**Definition:** The multi-index filtration corresponding to the set of ideals \(\{I_1, \ldots, I_r\}\) is the filtration by the ideals
\[
J_{\{I_i\}}(\underline{v}) = \bigcap_{i=1}^{r} J_{I_i}(v_i)
\]
\((\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r)\).

This filtration is defined by the set \(\{v_{I_i}\}\) of the order functions corresponding to the ideals \(I_i\).

**Examples.** 1. Let \((C, 0) \subset (S, 0)\) be a germ of a (reduced) curve on the surface \(S\), let \(C = \bigcup_{i=1}^{r} C_i\) be the decomposition of the curve \(C\) into irreducible components, let \(I_i\) be the ideal of the curve \(C_i\). One has the multi-index filtration corresponding to the set of ideals \(\{I_1, \ldots, I_r\}\). For \(S = \mathbb{C}^2\) or if all the components \(C_i\) of the curve \(C\) are Cartier divisors on \((S, 0)\) (this takes place, e.g., for any curve on the rational double point of type \(E_8\)), this is the filtration corresponding to the curve \(C\) considered in [2] and [5]. Otherwise this is not, generally speaking, the case.

2. Let \(\pi : (X, D) \to (S, 0)\) be a proper modification of the space \((S, 0)\) which is an isomorphism outside of the origin, with \(X\) smooth and \(D = \pi^{-1}(0)\) a normal crossing divisor on \(X\). Let \(D = \bigcup_{\sigma \in \Gamma} E_{\sigma}\) be the representation of the exceptional divisor \(D\) as the union of its irreducible components. For \(\sigma \in \Gamma\), i.e. for a component \(E_{\sigma}\) of the exceptional divisor \(D\), let \(\tilde{L}\) be a germ of a smooth irreducible curve on \(X\) intersecting \(E_{\sigma}\) transversally at a smooth point (i.e. not at an intersection point with other components of the exceptional...
divisor $D$), let $L = \pi(\tilde{L})$ be a curve on $(S, 0)$, let $I_L \subset O_{S,0}$ be the ideal of the curve $(L, 0)$, and let $I_\sigma$ be the ideal generated by all the ideals $I_L$ of the described type. For $r$ chosen components $E_1, \ldots, E_r$ of the exceptional divisor $D$, i.e. for $\{1, \ldots, r\} \subset \Gamma$, this way one gets a multi-index filtration corresponding to a set $\{E_1, \ldots, E_r\}$ of components of the exceptional divisor $D$. Again, if $S = \mathbb{C}^2$ or if all curves $L$ described above are Cartier divisors on $(S, 0)$, this filtration coincides with the divisorial filtration studied in [9] and [4]. Otherwise this is not the case.

The Poincaré series of the one-index filtration $\{J_I(v)\}$ is the series

$$P_I(t) = \sum_{v=0}^{\infty} \dim \left( J_I(v)/J_I(v+1) \right) t^v .$$

The notion of the Poincaré series of a multi-index filtration can be found in [8] and [2]. For computations it is convenient to write the definition in terms of an integral with respect to the Euler characteristic. Let $\mathbb{P}O_{S,0}$ be the projectivization of the ring (vector space) $O_{S,0}$, and let $\nu: \mathbb{P}O_{S,0} \to (\mathbb{Z}_{\geq 0} \cup \{\infty\})^r$ be the function $\nu(g) = (v_{I_1}(g), \ldots, v_{I_r}(g))$. One can consider the monomial $t^{\nu(g)}$ as a function on the projectivization $\mathbb{P}O_{S,0}$ with values in the group $\mathbb{Z}[[t_1, \ldots, t_r]]$ of power series in the variables $t_1, \ldots, t_r$ (with the sum as the group operation). The notion of integration with respect to the Euler characteristic over the space $\mathbb{P}O_{S,0}$ can be found, e.g., in [3]. One has

$$P_{\{I_i\}}(t_1, \ldots, t_r) = \int_{\mathbb{P}O_{S,0}} t^{\nu(g)} d\chi . \quad (1)$$

Let $\{I_1, \ldots, I_r\}$ be a set of ideals in $O_{\mathbb{C}^2,0}$. Let $\pi: (X, D) \to (\mathbb{C}^2, 0)$ be as above. Let $D = \bigcup_{\sigma \in \Gamma} E_\sigma$, where $E_\sigma$ are irreducible components of the exceptional divisor $D$ (each $E_\sigma$ is isomorphic to the complex projective line $\mathbb{CP}^1$). For $\sigma \in \Gamma$, i.e. for a component $E_\sigma$ of the exceptional divisor $D$, let $E^\circ_{\sigma}$ be the “smooth part” of $E_\sigma$ in the union of zero loci of the ideals $\{I_i^\circ\}$, i.e. $E_\sigma$ minus intersection points with all other components of the union. Let $k_{\sigma i}$ be multiplicity of the component $E_\sigma$ in the zero divisor of the ideal $I_i$ and let $k_\sigma := (k_{\sigma 1}, \ldots, k_{\sigma r}) \in \mathbb{Z}^r$.

Theorem 1.

$$P_{\{I_i\}}(t_1, \ldots, t_r) = \prod_{\sigma \in \Gamma} (1 - t^{k_\sigma})^{-\chi(E^\circ_{\sigma})} . \quad (2)$$

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This statement generalizes those from [2] and [9] for the ideals described in Examples 1 and 2 (for \( S = \mathbb{C}^2 \)). One can see that the right-hand side of the equation (2) is equal to the Alexander polynomial \( \Delta_{\{I_i\}}(t_1, \ldots, t_r) \) of the set of ideals \( \{I_i\} \) and thus in this case the Poincaré series coincides with the Alexander polynomial. For \( I_i = \langle f_i \rangle \) (see Example 1 above), this is a statement from [2].

Proof. The proof essentially repeats the arguments from [3], [9]. One uses the representation of the Poincaré series as an integral with respect to the Euler characteristic: see (1). There is a map from the projectivization \( \mathbb{P}O_{\mathbb{C}^2,0} \) of the ring \( O_{\mathbb{C}^2,0} \) onto the space of effective divisors on \( \hat{D} = \bigcup_{\sigma \in \Gamma} \hat{E}_\sigma \): to a function germ \( f \) on \( (\mathbb{C}^2,0) \) one associates the intersection of the strict transform of the curve \( \{f = 0\} \) with the exceptional divisor \( D \). (For any chosen \( N \) this map is defined for all function germs \( f \) with \( v_{I_i}(f) \leq N \) if one makes sufficiently many additional blowing-ups at intersection points of the components of the union of zero loci of the ideals \( I_i^* \) on \( X \). For all additional components \( E_\sigma \) of the exceptional divisor \( D \), their smooth parts \( \hat{E}_\sigma \) are isomorphic to the complex projective line \( \mathbb{CP}^1 \) without two points. Therefore their Euler characteristic are equal to zero and they do not contribute to the right-hand side of the equation (2).) Proposition 2 from [3] implies that the preimage of a point with respect to this map is a complex affine space and thus has the Euler characteristic equal to 1. The Fubini formula implies that the Poincaré series \( P_{\{I_i\}}(t_1, \ldots, t_r) \) is equal to the integral with respect to the Euler characteristic of the monomial \( t^\nu \) over the space of effective divisors on \( \hat{D} \). Here \( \nu \) is an additive function on the space of effective divisors on \( \hat{D} \) (with values in \( \mathbb{Z}_{\geq 0} \)) equal to \( k_\sigma \) for a point from the component \( \hat{E}_\sigma \).

The space of effective divisors on \( \hat{D} \) is the direct product of the spaces of effective divisors on the components \( \hat{E}_\sigma, \sigma \in \Gamma \). Each of the latter ones is the disjoint union of the symmetric powers \( S^\ell \hat{E}_\sigma \) of the component \( \hat{E}_\sigma \). Therefore

\[
P_{\{I_i\}}(t_1, \ldots, t_r) = \prod_{\sigma \in \Gamma} \left( \sum_{\ell=0}^\infty \chi(S^\ell \hat{E}_\sigma) \cdot t^{\ell k_\sigma} \right).
\]

Now equation (2) follows from the formula

\[
\sum_{\ell=0}^\infty \chi(S^\ell X) t^\ell = (1 - t)^{-\chi(X)}.
\]
Example 3. By the definition, for ideals $I_1, \ldots, I_r \subset \mathcal{O}_{\mathbb{C}^2,0}$, their Alexander polynomial coincides with the Alexander polynomial of their integral closures $\overline{I}_1, \ldots, \overline{I}_r$. An integrally closed ideal $I = \overline{I}$ of finite length (i.e. $\dim \mathcal{O}_{\mathbb{C}^2,0}/I < \infty$) has a representation of the form $I = \prod_{\sigma} I_{\sigma}$ where $I_{\sigma}$ is the ideal corresponding to a divisor $E_{\sigma}$ of a resolution of the ideal $I$ (see [16]). Let $s = \# \Gamma$ be the number of components of the exceptional divisor $D$ of the resolution. One has $n_{\sigma} = \sum_{\delta \in \Gamma} m_{\sigma \delta} k_\delta$, where $k_\delta$ is the multiplicity of the ideal $I$ on the component $E_{\delta}$ of the exceptional divisor $D$, $(m_{\sigma \delta})$ is the inverse matrix of minus the intersection matrix $(E_{\sigma} \circ E_{\delta})$ of the components $E_{\sigma}$ on $X$. Therefore one has the following equation for the order function $v_I: \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$:

$$v_I = \sum_{\sigma} n_{\sigma} v_{\sigma} = \sum_{\sigma, \delta} m_{\sigma \delta} k_\delta v_{\sigma},$$

where $v_{\sigma}$ is the order function corresponding to the component $E_{\sigma}$ (Example 2). One has the following equation

$$P_I(t) = P_{\{I_{\sigma}\}}(t^{n_{1,\sigma}}, \ldots, t^{n_{s,\sigma}})$$

where $P_{\{I_{\sigma}\}}(t_1, \ldots, t_s)$ is the Poincaré series of the divisorial valuations $v_1, \ldots, v_s$ (see [9]). Moreover, if $I_i = \prod_{\sigma} I_{\sigma}^{n_i}$, $i = 1, \ldots, r$, one has

$$P_{\{I_i\}}(t_1, \ldots, t_r) = P_{\{I_{\sigma}\}}(\prod_i t_i^{n_i^1}, \ldots, \prod_i t_i^{n_i^s}).$$

In a similar way one can prove versions of the main statements from [4] and [7] for ideals in the ring of functions on a rational surface singularity or on their universal abelian covers. Let $(S, 0)$ be a rational surface singularity. The link $S \cap S^3_\varepsilon$ of the singularity $(S, 0)$ is a rational homology sphere and its first homology group $H = H_1(S \setminus \{0\})$ is isomorphic to the cokernel $\mathbb{Z}^\Gamma/\text{Im } j$ of the map $j: \mathbb{Z}^\Gamma \to \mathbb{Z}^\Gamma$ defined by the intersection matrix $(E_{\sigma} \circ E_{\sigma'})$ (the order of the group $H$ is equal to the determinant $d$ of minus the intersection matrix $-(E_{\sigma} \circ E_{\sigma'})$). For $\sigma \in \Gamma$, let $h_\sigma$ be the element of the group $H$ represented by the loop in the manifold $X \setminus D \simeq S \setminus \{0\}$ going around the component $E_{\sigma}$ in the positive direction. The group $H$ is generated by the elements $h_\sigma$ for all $\sigma \in \Gamma$.

Let $p: (\tilde{S}, 0) \to (S; 0)$ be the universal abelian cover of the surface singularity $S$ (see e.g. [12], [13], [14]). The group $H$ acts on $(\tilde{S}, 0)$ and the restriction
$p|_{\tilde{S}\setminus\{0\}}$ of the map $p$ to the complement of the origin is a (usual, nonramified) covering $\tilde{S}\setminus\{0\} \to S\setminus\{0\}$ with the structure group $H$. One can lift the map $p$ to a (ramified) covering $p' : (\tilde{X}, \tilde{D}) \to (X, D)$ where $\tilde{X}$ is a normal surface (generally speaking, not smooth) and $\tilde{X}\setminus\tilde{D} \cong \tilde{S}\setminus\{0\}$:

\[
\begin{array}{ccc}
(\tilde{X}, \tilde{D}) & \xrightarrow{\tilde{\pi}} & (\tilde{S}, 0) \\
p' & \downarrow & p \\
(X, D) & \xrightarrow{\pi} & (S, 0)
\end{array}
\]

(one can define $\tilde{X}$ as the normalization of the fibre product $X \times_S \tilde{S}$ of the varieties $X$ and $\tilde{S}$ over $S$).

Let $R(H)$ be the ring of (virtual) representations of the group $H$. For $\sigma \in \Gamma$, i.e. for a component $E_\sigma$ of the exceptional divisor $D$, let $\alpha_\sigma$ be the one-dimensional representation $H \to GL(1, \mathbb{C})$ of the group $H$ defined by $\alpha_\sigma(h) = \exp(-2\pi \sqrt{-1} m_\sigma h)$ (here the minus sign reflects the fact that the action of an element $h \in H$ on the ring $\mathcal{O}_{S,0}$ is defined by $(h \cdot f)(x) = f(h^{-1}(x))$). Let $I_i = p^*I_i$ ($i = 1, \ldots, r$) be the liftings of the ideals $I_i$ to the universal abelian cover $\tilde{S}$. The corresponding multi-index filtration on the ring $\mathcal{O}_{\tilde{S},0}$ of germs of functions on the abelian cover $\tilde{S}$ is an $H$-invariant one.

A notion of the equivariant Poincaré series of such multi-index filtration was defined in [6]. Similar to [7] one obtains the following result.

**Theorem 2.**

\[
P^H(t_1, \ldots, t_r) = \prod_{\sigma \in \Gamma} (1 - \alpha_\sigma t_1^{d_\sigma})^{-\chi(E_\sigma)}. \tag{3}
\]

The sum of monomials of this series with the trivial representation in the coefficients (i.e. those for which $\sum_\sigma m_{\delta\sigma}v_\sigma$ is an integer for any $\delta \in \Gamma$) with the change of variables $t_i^d \rightarrow t_i$ is the Poincaré series of the filtration on $\mathcal{O}_{S,0}$ corresponding to the set of ideals $\{I_i\}$ (cf. [4], [3]).

**Remark.** Integrally closed ideals in the ring $\mathcal{O}_{S,0}$ of germs of functions on a rational surface singularity $(S, 0)$ have a description somewhat similar to that in Example 3 for $S = \mathbb{C}^2$. Let $I = \mathcal{T}$ be an integrally closed ideal in $\mathcal{O}_{S,0}$ and let $\pi : (X, 0) \rightarrow (S, 0)$ be a resolution of it. For $\sigma \in \Gamma$, i.e. for a component $E_\sigma$ of the exceptional divisor $D$, let $\mathcal{L}$ be a germ of a smooth irreducible curve on $X$ intersecting $E_\sigma$ transversally at a smooth point, i.e. at a point of $\tilde{E}_\sigma$, and let $L = \pi(\mathcal{L})$. There exists the minimal natural number $d_\sigma$ such that $d_\sigma L$ is a Cartier divisor on $(S, 0)$: $d_\sigma L = (g_L)$ for $g_L \in \mathcal{O}_{S,0}$. (The number $d_\sigma$ is the minimal natural number such that $d_\sigma m_{\delta\sigma}$ are integers for all $\delta \in \Gamma$ and is equal to the order of the element $h_\sigma$ in the
group $H = H_1(S \setminus \{0\})$. Let $I'_\sigma \subset O_{S,0}$ be the ideal generated by all germs $g_L$ of the described type.

The integrally closed ideal $I$ has a unique representation of the form $I = \prod \sigma I^r_\sigma$ for non negative rational numbers $r_\sigma$ such that $\sum (E_\sigma \circ E_\delta) r_\sigma$ are integers for all $\delta \in \Gamma$ (see [10]). (One writes that $I_1 = I_{1/2}$ ($I_1$ and $I_2$ are integrally closed ideals in the ring $O_{S,0}$) if and only if $I^1 = I_2$. If such an ideal $I_1$ exists, it is defined in a unique way.) As it was mentioned, the divisorial filtration on the ring $O_{S,0}$ corresponding to the component $E_\sigma$ is not, generally speaking, the filtration corresponding to an ideal. However, for the corresponding order functions one has $v_{I'_\sigma} = d_\sigma v_\sigma$. Therefore the Poincaré series of the filtration corresponding to the ideal $I'_\sigma$ is obtained from the one of the divisorial filtration corresponding to the component $E_\sigma$ by substituting the variable $t$ by $t^{d_\sigma}$. This explains a relation between the Poincaré series of the filtration in the ring $O_{S,0}$ corresponding to a set of ideals and that of the divisorial filtration like in Example 3.

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