A posteriori error bounds for discontinuous Galerkin methods for quasilinear parabolic problems

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Abstract We derive a posteriori error bounds for a quasilinear parabolic problem, which is approximated by the $hp$-version interior penalty discontinuous Galerkin method (IPDG). The error is measured in the energy norm. The theory is developed for the semidiscrete case for simplicity, allowing to focus on the challenges of a posteriori error control of IPDG space-discretizations of strictly monotone quasilinear parabolic problems. The a posteriori bounds are derived using the elliptic reconstruction framework, utilizing available a posteriori error bounds for the corresponding steady-state elliptic problem.

1 Introduction

Discontinuous Galerkin (DG) methods [2, 17, 1], have enjoyed substantial development in recent years. For parabolic problems DG methods are interesting due to their good local conservation properties as well as due to their block-diagonal mass matrices.

This work is concerned with the derivation of a posteriori error bounds for the space-discrete interior penalty discontinuous Galerkin method (IPDG) for quasilinear parabolic problems with strictly monotone non-linearities of Lipschitz growth.

A posteriori error bounds for $h$-version DG methods are derived in [12, 13, 3, 10] and for DG-in-space parabolic problems in [16, 6, 18, 5, 8, 7]. The contribution of this work is twofold:

- the derivation of a posteriori energy-norm error bounds for IPDG methods for quasilinear parabolic problems, and
• the resulting a posteriori bounds are are explicit with respect to the local elemental polynomial degree.

A key tool in our a posteriori error analysis is the elliptic reconstruction technique \([15, 14, 8]\). Roughly speaking, in the elliptic reconstruction framework the error is split into a parabolic and an elliptic part, respectively. In the interest of being explicit with respect to the dependence of the a posteriori error bounds in the elemental polynomial degree \(p\), we restrict the presentation to quadrilateral elements of tensor-product type (cf. Remark 2).

## 2 Model problem and the IPDG method

Let \(\Omega\) be a bounded open (curvilinear) polygonal domain with Lipschitz boundary \(\partial \Omega\) in \(\mathbb{R}^d\), \(d = 2, 3\). For \(\omega \subset \Omega\), we consider the standard spaces \(L^2(\omega)\) (whose norm is denoted by \(\|\cdot\|_\omega\) for brevity), \(H^1(\omega)\) and \(H^1_0(\omega)\), whose norm will be denoted by \(\|\cdot\|_1\), along with its dual \(H^{-1}(\Omega)\), with norm \(\|\cdot\|_{-1}\). For brevity, the standard inner product on \(L^2(\Omega)\) will be denoted by \(\langle \cdot, \cdot \rangle\) and the corresponding norm by \(\|\cdot\|\). We also define the spaces \(L^2(0, T, \omega)\), \(X = \{L^2(\omega), H^1_0(\omega)\}\) and \(L^\infty(0, T; L^2(\Omega))\), consisting of all measurable functions \(v : [0, T] \to X\), for which \(\|v\|_{L^2(0, T, X)} := \left(\int_0^T \|v(t)\|^2_X\right)^{1/2} < +\infty\) and \(\|v\|_{L^\infty(0, T; L^2(\Omega))} := \text{ess}\sup_{t \in [0, T]} \|v(t)\|\). (The differentials in the integrals with respect to \(t\) are suppressed for brevity throughout this work.)

We identify function \(v \in [0, T] \times \Omega \to \mathbb{R}\) with \(v : t \to X\) and we denote \(v(t), t \in [0, T]\), for \(v \in [0, T] \times \Omega \to \mathbb{R}\).

For \(t \in (0, T]\), we consider the problem of finding a function \(u\) satisfying

\[
\frac{d}{dt} u(t, x) - \nabla \cdot (a(t, x, |\nabla u(t, x)|) \nabla u(t, x)) = f(t, x) \quad \text{in} \quad (0, T] \times \Omega, \tag{1}
\]

where \(f \in L^\infty(0, T; L^2(\Omega))\) and a scalar uniformly continuous function, subject to initial condition \(u(0, x) = u_0(x)\) on \([0] \times \Omega\), for \(u_0 \in L^2(\Omega)\), and homogeneous Dirichlet boundary conditions on \([0, T] \times \partial \Omega\).

We assume that the non-linearity \(a\) in equation (1) is of strongly monotone type with Lipschitz growth so that there exist positive constants \(\alpha\) and \(\overline{\alpha}\) such that the following inequalities hold:

\[
|a(t, x, |y|)y - a(t, x, |z|)z| \leq \overline{\alpha}|y - z| \tag{2}
\]

\[
(a(t, x, |y|)y - a(t, x, |z|)z) \cdot (y - z) \geq \alpha|y - z|^2, \tag{3}
\]

for all vectors \(y, z \in \mathbb{R}^d\), and all \((t, x) \in [0, T] \times \Omega\).

Let \(\mathcal{T}\) be a shape-regular subdivision of \(\Omega\) into disjoint closed quadrilateral elements \(\kappa \in \mathcal{T}\). We assume that \(\kappa \in \mathcal{T}\) are constructed via \(C^\infty\)-diffeomorphisms with non-singular Jacobian \(F_\kappa : (-1, 1)^d \to \kappa\), so as to ensure \(\overline{\Omega} = \bigcup_{\kappa \in \mathcal{T}} \overline{\kappa}\).
For \( p \in \mathbb{N} \), \( Q_p(\kappa) \) is the set of all tensor-product polynomials on \((-1,1)^d\) of degree \( p \) in each variable and let

\[ S^p := \{ v \in L^2(\Omega) : v|_{\Gamma_e} \in Q_p(\kappa), \kappa \in T \}, \]

be the (discontinuous) \textit{finite element space}. Let \( \Gamma \) be the union of all \((d-1)\)-dimensional element faces \( e \) associated with the subdivision \( T \) (including the boundary). Let also \( \Gamma_{\text{int}} := \Gamma \setminus \partial \Omega \), so that \( \Gamma = \partial \Omega \cup \Gamma_{\text{int}} \).

Let \( \kappa^+, \kappa^- \) be two (generic) elements sharing a face \( e := \kappa^+ \cap \kappa^- \subset \Gamma_{\text{int}} \) with respective outward normal unit vectors \( n^+ \) and \( n^- \) on \( e \). For \( q : \Omega \to \mathbb{R} \) and \( \phi : \Omega \to \mathbb{R}^d \), let \( q^\pm := q|_{\partial \kappa^{\pm}} \) and \( \phi^\pm := \phi|_{\partial \kappa^{\pm}} \), and set

\[
\{ q \}_e := \frac{1}{2}(q^+ + q^-), \quad \{ \phi \}_e := \frac{1}{2}(\phi^+ + \phi^-), \\
[q]_e := q^+ n^+ - q^- n^-, \quad [\phi]_e := \phi^+ \cdot n^+ - \phi^- \cdot n^-;
\]

if \( e \subset \partial \kappa \cap \partial \Omega \), we set \( \{ \phi \}_e := \phi^+ \) and \( [q]_e := q^+ n^+ \). Finally, we introduce the \textit{meshsize} \( h : \Omega \to \mathbb{R} \), defined by \( h(x) = \text{diam} \kappa \), if \( x \in \kappa \setminus \partial \kappa \) and \( h(x) = \{ h \} \), if \( x \in \Gamma \).

Consider the IPDG semi-linear form \( B(\cdot, \cdot) : S^p \times S^p \to \mathbb{R} \), introduced in [9] for the solution of the corresponding steady-state problem, defined by

\[
B(w, v) := \sum_{\kappa \in T} \int_{\kappa} \alpha(w) \cdot \nabla v \, dx + \int_{\Gamma} (\theta(a(t, x, h^{-1}[w])) \nabla v) \cdot [w] \] 
\[
\quad - \{ \alpha(w) \} \cdot [v] + \sigma[w] \cdot [v] \, ds,
\]

where \( \alpha(w) := a(t, \cdot, ||\nabla w||) \nabla w, w \in H^1(\Omega) + S^p \), for \( \theta \in \{-1,0,1\} \), with the function \( \sigma : \Gamma \to \mathbb{R}_+ \) defined piecewise by \( \sigma|_e := C_\sigma p^2/(h|_e) \), for some sufficient large constant \( C_\sigma > 0 \). The corresponding energy norm \( \| \cdot \| \) is defined \( \| w \| := \left( \sum_{\kappa \in T} \| \nabla w \|_2^2 + \int_{\Gamma} \sigma[w]^2 \, ds \right)^{1/2} \), for \( w \in H^1(\Omega) + S^p \). The (spatially semidiscrete) \textit{interior penalty discontinuous Galerkin method} (IPDG) for the initial/boundary value model problem reads:

find \( U : (0,T] \to S^p \) such that \( \langle U_t, V \rangle + B(U, V) = \langle f, V \rangle \quad \forall t \in (0,T], V \in S^p \).

3 A posteriori error bounds

For \( w \in H^1(\Omega) + S^p \), and \( T > 0 \), we define the norm \( ||| w |||_{L^2(0,T;H^1(\Omega))} \) := \( \left( \int_0^T \| w \|^2 \right)^{1/2}, t > 0 \). We shall derive a posteriori bounds for the error \( ||| u - U |||_{L^2(0,T;H^1(\Omega))} \).
Definition 1 (elliptic reconstruction). Let \( U \) be the (semi-discrete) solution to the problem (6) and fix \( t \in [0, T] \). We define the elliptic reconstruction \( w \equiv w(t) \in H^1_0(\Omega) \) of \( U \) to be the solution to the elliptic problem

\[
\langle \alpha(w), \nabla v \rangle = \langle g, v \rangle \quad \forall v \in H^1_0(\Omega),
\]

where \( g \equiv g(t) \) is given by \( g := -AU + f - \Pi f \), with \( \Pi : L^2(\Omega) \to S^p \) is the orthogonal \( L^2 \)-projection operator onto \( S^p \) and \( A \equiv A(t) : S^p \to S^p \) is the discrete operator defined by

\[
f(Z) = B(Z, V) \quad \forall V \in S^p.
\]

The construction of \( w \) and that of \( AZ \) are both well defined in view of the elliptic problem’s unique solvability and the Riesz representation, respectively.

Remark 1. The key property of the construction in Definition 1 is that \( U \) is the IPDG solution of an elliptic problem with analytical solution \( w \). Namely, for each fixed \( t \in [0, T] \) it satisfies

\[
\text{find } U \in S^p \text{ such that } B(U, V) = \langle g, V \rangle \quad \forall V \in S^p.
\]

We can now decompose the error as follows:

\[
U - u = \rho - \epsilon, \quad \text{with } \rho := w - u, \quad \text{and } \epsilon := w - U,
\]

where \( w \equiv w(t) \) denotes the elliptic reconstruction of \( U \equiv U(t), t \in [0, T] \).

Lemma 1 (differential error relation). Let \( u, w, U, e, \rho, \epsilon \) as above. Then, for all \( v \in H^1_0(\Omega) \), we have

\[
\langle e_t, v \rangle + \langle \alpha(w) - \alpha(u), \nabla v \rangle = 0.
\]

Proof. We have

\[
\langle e_t, v \rangle + \langle \alpha(w) - \alpha(u), \nabla v \rangle = \langle U_t, v \rangle + \langle \alpha(w), \nabla v \rangle - \langle f, v \rangle
\]

\[
= \langle U_t, v \rangle + \langle g, v \rangle - \langle f, v \rangle = \langle U_t, v \rangle + \langle -AU, v \rangle - \langle \Pi f, v \rangle
\]

\[
= \langle U_t, \Pi v \rangle + \langle -AU, \Pi v \rangle - \langle f, \Pi v \rangle = 0,
\]

using (1), (7) and the properties of the \( L^2 \)-projection, respectively. \( \square \)

We consider further the decomposition of \( U \) into conforming and non-conforming (discontinuous) parts \( U = U^c + U^d \), where \( U^c \in S^p \cap H^1_0(\Omega) \) and \( U^d := U - U^c \in S^p \). Note that there are many ways of performing this decomposition (e.g., by projecting \( U \) onto the conforming space) whereof the specific nature remains at our disposal until further.

We also use the shorthand notation \( e^c := U^c - u \) and \( e^c := w - U^c \); note that \( e^c = \rho - \epsilon^c, e = e^c + U^d \) and that \( e^c \in H^1_0(\Omega) \).

Theorem 1 (abstract a posteriori energy-error estimate). With \( u, U, U^d, e, \) and \( \epsilon \) as defined above, the following error estimate is satisfied:
\[
\|e\|_{L^2(0,T,H^1(\Omega))} \leq C_1 \|e\|_{L^2(0,T,H^1(\Omega))} + g^{-\frac{1}{2}} \left( \|u_0 - U(0)\| + \|U^d(0)\| \right) + C_4 \|U^d\|_{L^2(0,T,H^{-1}(\Omega))} + C_2 \|U^d\|_{L^2(0,T,H^{-1}(\Omega))},
\]

with \(C_1 := 1 + \sqrt{2\pi a^{-1}}\) and \(C_2 := \sqrt{2a^{-1}}\).

**Proof.** Set \(v = e^\circ\) in (11), to deduce

\[
\langle e^\circ_t, e^\circ \rangle + \langle \alpha(U^\circ) - \alpha(u), \nabla e^\circ \rangle = -\langle U^d_t, e^\circ \rangle + \langle \alpha(U^\circ) - \alpha(w), \nabla e^\circ \rangle. \tag{14}
\]

Conditions (3) and (2) imply, respectively,

\[
\|\alpha(U^\circ) - \alpha(u), \nabla e^\circ \| \geq g \|\nabla e^\circ\|^2, \quad \text{and} \quad \|\alpha(U^\circ) - \alpha(w), \nabla e^\circ \| \leq \|\nabla e^\circ\| \|\nabla e^\circ\|,
\]

and the duality pairing \((H^{-1}, H^1_b)\) gives \(\|U^d_t, e^\circ\| \leq \|U^d_t\|_{-1} \|\nabla e^\circ\|\). Using the last 3 relations on (14), we deduce

\[
\langle e^\circ_t, e^\circ \rangle + g \|\nabla e^\circ\|^2 \leq \left( \|U^d_t\|_{-1} + \pi \|\nabla e^\circ\| \right) \|\nabla e^\circ\|, \tag{15}
\]

which, in turn, implies

\[
\langle e^\circ_t, e^\circ \rangle + \frac{g}{2} \|\nabla e^\circ\|^2 \leq \frac{1}{2g} \left( \|U^d_t\|_{-1} + \pi \|\nabla e^\circ\| \right)^2. \tag{16}
\]

Integrating (16) with respect to \(t\) between 0 and \(T\), yields

\[
\|e^\circ(t)\|^2 + g \int_0^T \|\nabla e^\circ\|^2 \leq \|e^\circ(0)\|^2 + \frac{1}{2g} \int_0^T \left( \|U^d_t\|_{-1} + \pi \|\nabla e^\circ\| \right)^2, \tag{17}
\]

or

\[
\left( \int_0^T \|\nabla e^\circ\|^2 \right)^{\frac{1}{2}} \leq g^{-\frac{1}{2}} \|e^\circ(0)\| + \frac{1}{2g} \int_0^T \left( \|U^d_t\|_{-1} + \pi \|\nabla e^\circ\| \right)^2 \leq g^{-\frac{1}{2}} \|e^\circ(0)\| + \frac{1}{2g} \left( \|U^d\|_{L^2(0,T,H^{-1}(\Omega))} \right)
\]

noting that \(\|e^\circ\| = \|\nabla e^\circ\|\). Using the bounds \(\|e^\circ\| \leq \|e\| + \|U^d\|\), \(\|e^\circ(0)\| \leq \|e(0)\| + \|U^d(0)\|\) on (17) and the resulting bound on the triangle inequality

\[
\|e\|_{L^2(0,T,H^1(\Omega))} \leq \|e^\circ\|_{L^2(0,T,H^1(\Omega))} + \|U^d\|_{L^2(0,T,H^1(\Omega))},
\]

yields the result. \(\Box\)

For the above result to yield a formally a posteriori bound, we need to estimate \(\|e\|_{L^2(0,T,H^1(\Omega))}\) further. In particular, in view of Remark 1, we require an a posteriori error bound for the IPDG method for the corresponding elliptic quasilinear problem (9). Such a result is available in [11], an instance of which and is presented next.
Theorem 2 ([11]). Let \( w \in H^1_0(\Omega) \) be the elliptic reconstruction defined in (7) and let \( W \in S^p \) be the solution of (9). Then, for \( C_\sigma > 1 \) sufficiently large the bound

\[
\|w - W\|^2 \leq \mathcal{E}(W, g, S^p) := C_{\text{est}} \sum_{\kappa \in \mathcal{T}} \left( \eta^2_{\kappa} + \mathcal{O}(g, W) \right),
\]

holds, with

\[
\eta^2_{\kappa} = \frac{h^2_{\kappa}}{p^2} \|\tilde{\Pi}(g + \nabla \cdot \alpha(W))\|_{\kappa}^2 + \frac{h_{\kappa}}{p} \|\tilde{\Pi}_F[\alpha(W)]\|_{\partial\kappa \setminus \partial\Omega}^2 + C_\sigma^2 \frac{p^3}{h_{\kappa}} \|W\|_{\partial\kappa}^2,
\]

and

\[
\mathcal{O}(g, w_{\text{DG}}) = \sum_{\kappa \in \mathcal{T}} \left( \frac{h^2_{\kappa}}{p^2} \|\mathbb{1} - \tilde{\Pi}(g + \alpha(W))\|_{\kappa}^2 + \frac{h_{\kappa}}{p} \|\mathbb{1} - \tilde{\Pi}_F[\alpha(W)]\|_{\partial\kappa \setminus \partial\Omega}^2 \right).
\]

where \( \mathbb{1} \) denotes a generic identity operator, \( \tilde{\Pi} \) denotes the \( L^2 \)-projection operator onto \( S^{p-1} \), \( \tilde{\Pi}_F \) is defined piecewise by \( \tilde{\Pi}_F v|_e := \pi^{p-1}_e v \), for all elemental faces \( e \subset \Gamma \), \( v \in L^2(\Omega) \), where \( \pi^{p-1}_e : L^2(\Omega) \to \mathcal{P}_{p-1}(e) \) denotes the \( L^2 \)-projection operator of the trace on the face \( e \) of a function in \( S^{p-1} \) (with \( \mathcal{P}_{p-1}(e) \), for \( e \subset \bar{\kappa} \) the space of mapped univariate polynomials of degree at most \( p-1 \) on \( e \)), and \( C_{\text{est}} > 0 \) is independent of \( C_\sigma, \theta, h \), and \( p \).

Also, it is possible to further estimate the terms involving \( U^d \), to avoid computing \( U^d \) explicitly. This is done (with, crucially, explicit dependence on \( p \)) using the following result based on [4, Lemma 3.2].

Lemma 2. Suppose \( \mathcal{T} \) does not contain any hanging nodes. Then, for any \( v \in S^p \) and any multi-index \( \gamma \), with \( |\gamma| = 0, 1 \), there exists a function \( v^\gamma \in S^p \cap H^1_0(\Omega) \) such that

\[
\sum_{\kappa \in \mathcal{T}} \|D^\gamma (v - v^\gamma)\|_{\kappa}^2 \leq C_3 \|v\|_{\mathcal{T}, 2},
\]

with \( C_3 > 0 \) depending on the maximal angle of \( \mathcal{T} \) only.

Proof. [4, Lemma 3.2] implies that for every \( \kappa \in \mathcal{T} \) there exists an Oswald-type operator \( I_{O\kappa} : S^p \to S^p \cap H^1_0(\Omega) \), such that

\[
\|v - I_{O\kappa} v\|_{\kappa}^2 \leq C \sum_{e \in \mathcal{F}(\kappa)} \frac{h_{\kappa}}{p^2} \|v\|_{e, 2}^2,
\]

for all \( v \in S^p \), with \( \mathcal{F}(\kappa) := \{ e \in \Gamma : e \cap \bar{\kappa} \neq \emptyset \} \). Summing over all the elements \( \kappa \in \mathcal{T} \), and observing that the maximal angle and the lack of hanging nodes gives an upper bound on the cardinality of \( \mathcal{F}(\kappa) \) for all \( \kappa \in \mathcal{T} \), we deduce that
\[ \sum_{\kappa \in T} \| v - I_{\text{Os}} v \|_{\kappa}^2 \leq C \sum_{e \subset T} \frac{h_e}{p^2} \| v \|_{e}^2, \tag{21} \]

which shows (19) for $|\gamma| = 0$. To show (19) for $|\gamma| = 1$, we observe that $(v - I_{\text{Os}} v) \in S^p$; thus, the standard inverse estimate yields:

\[ \sum_{\kappa \in T} \| \nabla (v - I_{\text{Os}} v) \|_{\kappa}^2 \leq C \sum_{e \subset T} \frac{p^4}{h_e^2} \| v - I_{\text{Os}} v \|_{\kappa}^2 \leq C \sum_{e \subset T} \frac{p^2}{h_e^2} \| [v] \|_{e}^2, \tag{22} \]

using the shape regularity of $T$. Setting $v^c = I_{\text{Os}} v$, the result follows. $\square$

**Remark 2.** The assumptions of Lemma 2 pose the following restrictions on the finite element space $S^p$: the use of quadrilateral elements (as the tensor-product nature of the local elemental bases is of crucial importance here), the exclusion of hanging nodes and the uniformity of the polynomial degree. If explicit knowledge of the polynomial degree $p$ in the a posteriori bounds presented in this work is not required, then these restrictions are not needed in view of [12, Lemma 4.1], i.e., triangular elements containing hanging nodes can be employed.

Combining the results of Theorems 1 and 2, together with the approximation properties described in Lemma 2, we obtain an a posteriori error bound in the energy norm for the semi-discrete problem (6).

**Theorem 3 (energy-norm a posteriori bound).** With the notation of Theorem 1 and the assumptions of Lemma 2, the following error bound holds:

\[
\| e \|_{L^2(0,T,H^1(\Omega))} \leq C_1 \int_0^T E^2(U, g, S^p) + a^{-\frac{1}{2}} \| u_0 - U(0) \| \\
+ a^{-\frac{1}{2}} C_3 \| (\frac{h}{p})^{\frac{1}{2}} [U(0)] \|_{H^1(\Omega)}^2 + C_4 \| \sqrt{\sigma} [U] \|_{L^2(0,T;L^2(\Gamma))} \\
+ C_5 \| (\frac{h}{p^2})^{\frac{1}{2}} [U_t] \|_{L^2(0,T;L^2(\Gamma))},
\]  

with $C_4 := C_1 \sqrt{C_3/C}$, $C_5 := C_2 C_{PF}$ and $C_{PF} > 0$ (the Poincaré–Friedrichs constant), such that $\| v \|_{-1} \leq C_{PF} \| v \|$, for all $v \in L^2(\Omega)$.

**Proof.** Combining the results from Theorems 1 and 2, together with the approximation properties described in Lemma 2, the result follows. $\square$

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