General properties of the expansion methods of Lie algebras

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Abstract

The study of the relation between Lie algebras and groups, and especially the derivation of new algebras from them, is a problem of great interest in mathematics and physics, because finding a new Lie group from an already known one also means that a new physical theory can be obtained from a known one. One of the procedures that allow us to do so is called expansion of Lie algebras, and has been recently used in different physical applications—particularly in gauge theories of gravity. Here we report on further developments of this method, required to understand in a deeper way their consequences in physical theories. We have found theorems related to the preservation of some properties of the algebras under expansions that can be used as criteria and, more specifically, as necessary conditions to know if two arbitrary Lie algebras can be related by some expansion mechanism. Formal aspects, such as the Cartan decomposition of the expanded algebras, are also discussed. Finally, an instructive example that allows us to check explicitly all our theoretical results is also provided.

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1. Introduction

Global and local symmetries of a physical system play an essential role in modern theoretical physics and its physical applications. Apart from a well-known relation between symmetries and conserved charges via the Noether theorem, the knowledge of a symmetry group of a certain theory is deeply built-in in its theoretical description and may lead to restrictive ‘no-go’ theorems. Introduction of new algebras is, in that sense, required by possible solutions
to these theoretical problems in the hope that they could circumvent the ‘no-go’ theorems. A beautiful example is the introduction of Lie superalgebras that unify in a non-trivial way spacetime and internal symmetries of the microscopic world, not allowed in a purely bosonic context by the Coleman–Mandula theorem. In this way, the study of the relation between Lie algebras and groups, and especially the derivation of new algebras from them, is a problem of great interest in mathematics and physics, because finding a new Lie group from an already known one also means that a new physical theory can be obtained from a known one. This is particularly useful, for example, in gauge theories (like Yang–Mills and Chern–Simons (CS) theories) which have the symmetry group as a fundamental ingredient.

Thus, setting aside the trivial problem of finding whether a Lie algebra is a subalgebra of another one, there are, essentially, three different ways of relating and/or obtaining new algebras from those given. In fact, it was during the second half of the 20th century that certain mechanisms were developed to obtain non-trivial relations between different Lie groups and algebras. These mechanisms are known as contractions [1–4], deformations [5–7] and extensions [8], which all share the property of maintaining the dimension of the original group or algebra. As we are going to see now, this work is focused on a generalization of the contraction procedure called expansion that, starting from a given algebra, permits us to generate algebras of a higher dimension than the original one.

Expansions of Lie algebras are generalizations of the Weimar–Woods (WW) contraction method [3] and were introduced some years ago in [9–12]. While in a contraction a suitable rescaling of some generators of the Lie algebra is done, in the expansion method the starting point is to consider an algebra $G$, with the basis of generators $X_i$, as described by the Maurer–Cartan (MC) forms $\omega_i(\varphi)$ on the manifold $\mathcal{M}(\varphi)$ of its associated group $G$. As it is known, the local structure of the Lie group is encoded in the so-called MC equations:

$$d\omega^k = -\frac{1}{2} c_{ij}^k \omega^i \wedge \omega^j,$$

which are just an equivalent description to the one given in terms of Lie brackets among Lie algebra generators, $[X_i, X_j] = c_{ij}^k X_k$. When a rescaling by a parameter $\lambda$ is performed on some of the group coordinates $\varphi^i$, the forms $\omega^i(\varphi, \lambda)$ can be expanded as power series in $\lambda$. Inserting these expansions back in the original MC equations for $G$, one obtains the MC equations of a new finite-dimensional expanded Lie algebra. As shown in [10–12], this method reproduces the Inönü–Wigner (IW) contractions [2] and its generalizations in the sense of WW [3]. In these cases, the dimension of the algebra is preserved but, in general, this expansion method also permits to obtain higher dimensional Lie algebras.

A generalization of the above method is the $S$-expansion [13] that combines the structure constants of the algebra $G$ with the inner law of an Abelian semigroup $S$ in order to define the Lie bracket of a new $S$-expanded algebra. When certain conditions are met, the method permits the extraction of smaller algebras which are called resonant subalgebras and reduced algebras. This method reproduces the results of the first expansion method described above (that uses a parameter $\lambda$) for a particular choice of the semigroup: one of the semigroups is denoted as $S_{\lambda}^{(S)}$ and its definition is given in [13]. Since it reproduces the method above, it also reproduces all the generalized IW contractions, as is shown in [13] explicitly. This paper is dedicated to a further development of the $S$-expansion method and its application to the theory of Lie algebras.

5 By non-trivial relations we mean that these mechanisms allow us to obtain some Lie algebras starting from other algebras that have completely different properties. Also, the original algebra is not necessarily (though it could be in specific cases) contained as a subalgebra of the algebra obtained by these processes.

6 However, it is not clear if $S$-expansions exhaust all possible contractions since there exist contractions that are not realized by generalized IW-contractions [14–16]. Besides, there exist $S$-expansions which are not equivalent to contractions as was recently shown in [17].
Interesting physical applications of these methods, particularly of the $S$-expansion, appeared recently in the literature. One of the advantages used in those applications is that if we know the invariant tensors of a certain Lie algebra then the mechanism gives the invariant tensors for the expanded algebras, even if the last ones are not semisimple. This feature is especially useful in the construction of transgressions and CS gauge theories of (super)gravity [18–25] where the invariant tensors of the symmetry group under which the theory is invariant are fundamental ingredients of the theory. On the one hand, the problem of finding all the invariant tensors for a non-semisimple Lie algebra remains an open problem until now. But this is not only an important mathematical problem but also one with physical relevance, because given an algebra the choice of the invariant tensor fixes the classes of interactions that may occur between the different fields in the theory. In fact, the standard procedure to obtain an invariant tensor of range $r$ is to use the symmetrized (super)trace for the product of $r$ generators in some matrix representation of the algebra. However this procedure has some limitations in the case of non-semisimple Lie algebras. A concrete example is provided in [25] where an 11-dimensional gauge theory for the M algebra (the maximal supersymmetric extension of the Poincaré algebra in 11 dimensions) is constructed. It is shown that the supertrace is not a good choice for constructing the theory, because it has so many vanishing components that a transgression or a CS Lagrangian constructed from it depends only on the spin connection and it represents therefore a kind of exotic gravity. Therefore, with the supertrace as an invariant tensor, it is not possible to reproduce general relativity neither to include fermionic fields or fields associated with central charges. However, by using the $S$-expansion procedure it is possible to consider the M algebra as an expansion of the osp(32/1) algebra providing an invariant tensor, different from the supertrace, which leads to a theory with a richer structure. It is worth noting here that while the $S$-expansion method does not solve the problem of classifying all invariant tensors for non-semisimple algebras, it at least gives invariant tensors different from the supertrace that are useful for the construction of the mentioned gauge theories of gravity.

A dual formulation of the $S$-expansion procedure for the Lie algebra was also constructed in [26], which permits us to understand this procedure at the level of the Lagrangians. Another interesting application of the expansion methods consists in establishing a relation between General Relativity and CS gravity [27, 28]. As shown in [28], General Relativity in five-dimensional spacetime may emerge at a special critical point from a CS action. To achieve this result, both the Lie algebra (called the $B$ algebra) and the symmetric invariant tensor that defines the CS Lagrangian are constructed by means of the Lie algebra $S$-expansion method with the semigroup $S^3_{kE}$. It is also interesting to mention that a black hole [29] and a cosmological solution [30] have also been constructed for the CS theory constructed in terms of this expanded $B$ algebra.

On the other hand, in [31, 32] the $S$-expansion method was extended to the case of higher order Lie algebras [33]. These kinds of algebras, defined in terms of a multilinear antisymmetric product, are a particular case of the so-called strongly homotopy Lie algebras [34], which are in turn related to the structure of higher spin (HS) interactions [35, 36]. In [37], the $S$-expansion method was also extended to study the expansion of loop algebras, which are infinite dimensional and can be useful in applications related to HS theories as, for example, in the Vasiliev formulation thereof [38–42]. Finally, a procedure that permits us to construct Casimir operators for the expanded algebra was recently found in [43].

Therefore, as many physical applications appearing in the literature depend on the possibility of relating two given algebras by some contraction or expansion procedure, an exhaustive study of the general properties of this method is needed to deeply understand their consequences in physical theories. In fact, a feature that almost all those physical applications
have in common is that they depend on the following question: *given two symmetry algebras, can they be related by means of some contraction or expansion procedure?* Here we attempt to give the first steps to answer the above question. By studying the preserved properties of the algebras under expansions we find certain theorems that can be used as criteria and, more specifically, as necessary conditions to solve this problem. Then the relation might still exist or not. The second part of the process, which consists in finding the relation explicitly with the help of computer programs, is not presented here but mainly developed in [44, 45]. In order to check our theoretical results, examples are provided by performing all the possible expansions that can be made for $\mathfrak{sl}(2, \mathbb{R})$ with semigroups up to and including order 6.

This paper is organized as follows: in section 2, we present a brief technical description of the basic ingredients that we are going to use. In section 3, we study under which conditions properties like solvability, nilpotency, semisimplicity and compactness are preserved under $S$-expansions on each level of the procedure. Particularly, in section 3.4, we review some aspects of the Cartan decomposition of the expanded algebra when compactness is preserved. In section 4, we make an exhaustive study of the possible expansions that can be made for the algebra $\mathfrak{sl}(2, \mathbb{R})$ and check our theoretical results. In section 5, it is shown that a classification of semigroups in terms of the eigenvalues of certain matrices, that are related to intrinsic properties of the semigroup, is a useful tool to predict semisimplicity and compactness properties for the expanded algebras. Section 6 is a summary of the results and finally, in section 7, some comments about future applications are given.

### 2. Preliminaries

#### 2.1. The $S$-expansion procedure

In this section we briefly describe the general Abelian semigroup expansion procedure ($S$-expansion for short). We refer the interested reader to [13] for further details.

Consider a Lie algebra $\mathcal{G}$ and a finite Abelian semigroup $S = \{\lambda_a\}$. According to theorem 3.1 from [13], the direct product

$$G_S = S \otimes \mathcal{G}$$

is also a Lie algebra. The elements of this expanded algebra are denoted by

$$X_{(i,a)} = \lambda_a \otimes X_i$$

where the product is understood as a direct product of the matricial representations of the generators $X_i$ of $\mathcal{G}$ and the elements $\lambda_a$ of the semigroup $S$. The Lie product in $G_S$ is defined as

$$[T_{(i,a)}, T_{(j,b)}] = \lambda_a \cdot \lambda_b \otimes [T_i, T_j] = K_{\gamma}^{\alpha \beta} C_{(i,a)(j,b)}^{(k,\gamma)} T_k = C_{(i,a)(j,b)}^{(k,\gamma)} (\lambda_k \otimes T_k) = C_{(i,a)(j,b)}^{(k,\gamma)}$$

where $C_{ij}^{(k,\gamma)}$ are the structure constants of $\mathcal{G}$, and $K_{\gamma}^{\alpha \beta}$, which stores information about the multiplication law of the semigroup, is called the 2-selector. The set (1) with the composition law (3) is called an $S$-expanded Lie algebra.

Interestingly, there are cases where it is possible to systematically extract subalgebras from $S \otimes \mathcal{G}$ like, for example, if we start by decomposing $\mathcal{G}$ in a direct sum of subspaces, as in

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7 It is worth remarking here that we use the $S$-expansion method because it reproduces the other expansion methods for a particular choice of the semigroup (the semigroups $S_n^E$ mentioned before).

8 The structure of the semigroup is encoded in a quantity $K_{\gamma}^{\alpha \beta}$ which defines the composition law: $\lambda_a \cdot \lambda_b = K_{\gamma}^{\alpha \beta} \lambda_\gamma$. 

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\[ G = \bigoplus_{p \in I} V_p, \]

where \( I \) is a set of indices. The internal structure of \( G \) can be codified through the mapping \( i : I \otimes I \rightarrow 2^I \) according to

\[
[V_p, V_q] \subset \bigoplus_{r \in (p,q)} V_r. \tag{5}
\]

When the semigroup \( S \) can be decomposed in subsets \( S_p, S = \bigcup_{p \in I} S_p, \) such that they satisfy the resonant condition \(^{10}\)

\[
S_p \cdot S_q \subset \bigcap_{r \in (p,q)} S_r, \tag{6}
\]

then we have that

\[
G_{S,R} = \bigoplus_{p \in I} S_p \otimes V_p \tag{7}
\]

is a subalgebra of \( G_S \) which is called resonant subalgebra (see theorem 4.2 from \([13]\)). The commutation relations of this subalgebra are given by

\[
[T_{(i_p,\alpha_p)}, T_{(k_r,\gamma_r)}] = C_{(i_p,\alpha_p)(k_r,\gamma_r)}^{(k_r,\gamma_r)} T_{(k_r,\gamma_r)} = K_{\gamma_r}^{\alpha_p} C_{\alpha_p \gamma_r}^{\gamma_r} T_{(k_r,\gamma_r)} \tag{8}
\]

where a sum on the indices \( r \in I \) and \( k_r, \gamma_r \) on the respective subspaces \( V_r, S_r \) is assumed. The indices \( \alpha_p, \beta_q \) are fixed for each \( p, q \in I \). In this work, we write \( G_{S,R} \) to denote this resonant subalgebra of the expanded algebra \( G_S \).

An even smaller algebra can be obtained when there is a zero element in the semigroup, i.e., an element \( 0_S \in S \) such that, for all \( \lambda \in S \), \( 0_S \cdot \lambda = 0_S \). When this is the case, the whole \( 0_S \otimes G \) sector can be removed from the resonant subalgebra by imposing \( 0_S \otimes G = 0 \) (see definition 3.3 from \([13]\)). The resulting algebra continues to be a Lie algebra and here it will be denoted by \( G_{S,R}^{\text{red}} \).

2.2. History of finite semigroups programs

The number of finite non-isomorphic semigroups of order \( n \) are given in the following table:

| Order | \( Q = \) No. of semigroups |
|-------|---------------------------|
| 1     | 1                         |
| 2     | 4                         |
| 3     | 18                        |
| 4     | 126 [46]                  |
| 5     | 1 100 [47]                |
| 6     | 15 973 [48–50]            |
| 7     | 836 021 [51]              |
| 8     | 18 431 120 128 [52]      |
| 9     | 52 989 400 714 478 [53–55] |

All the semigroups of order 4 have been classified by Forsythe in \([46]\), of order 5 by Motzkin and Selfridge in \([47]\), of order 6 by Plemmons in \([48–50]\), of order 7 by Jürgensen and Wick in \([51]\), and of order 8 by Satoh, Yama and Tokizawa in \([52]\), and monoids and semigroups of order 9 by Distler and Kelsey in \([53, 54]\) and by Distler and Mitchell in \([55]\). Also, for semigroups of order 9 the result can be found in \([56]\).

\(^{9}\) Here \( 2^I \) stands for the set of all subsets of \( I \).

\(^{10}\) Here \( S_p \cdot S_q \) denotes the set of all the products of all elements from \( S_p \) with all elements from \( S_q \).
As shown in the table, the problem of enumerating all the non-isomorphic finite semigroups of a certain order is a non-trivial task. In fact, the number $Q$ of semigroups increases very quickly with the order of the semigroup.

In [59], a set of algorithms was given that allows us to make certain calculations with finite semigroups. The first program, gen.f, gives all the non-isomorphic semigroups of order $n$ for $n = 1, 2, \ldots, 8$. The input is the order, $n$, of the semigroups we want to obtain and the output is a list of all the non-isomorphic semigroups that exist at this order. In this work, the elements of the semigroup are labeled by $\lambda_\alpha$ with $\alpha = 1, \ldots, n$ and each semigroup will be denoted by $S_{\alpha(n)}^\prime$ where the supra-index $\alpha = 1, \ldots, Q$ identifies the specific semigroup of order $n$.

Note also that this history includes the classification of all Abelian semigroups (which are the ones used in the $S$-expansion mechanism introduced in the last section) as a particular case. In fact, the second program in [59], com.f, takes as an input one of the mentioned lists for a certain order, picks up just the symmetric tables and generates another list with all the Abelian semigroups. For example for $n = 2$ the elements are labeled by $[\lambda_1, \lambda_2]$ and the program com.f gives the following list of semigroups:

$$
\begin{array}{cccc}
S_1^1 & \lambda_1 & \lambda_2 & S_2^1 \\
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_1 \\
\end{array}
$$

Note that the semigroup $S_1^2$ is not given in the list (10) because it is not Abelian.

So in general the program com.f of [59] gives a list of tables of all the Abelian non-isomorphic semigroups of a certain order\(^{12}\) (up to order 8).

In section 4, we are going to use those lists to perform all possible expansions of the algebra $\mathfrak{sl}(2)$ that can be made (with semigroups up to the order 6) and check explicitly the theoretical properties we are going to find in the next section.

3. Properties preserved under the $S$-expansion procedure

3.1. Expansion of solvable and nilpotent Lie algebras

It is known that a solvable algebra $\mathcal{G}$ is one for which the sequence

$$
\mathcal{G}^{(0)} = \mathcal{G}, \quad \mathcal{G}^{(1)} = [\mathcal{G}^{(0)}, \mathcal{G}^{(0)}], \ldots, \quad \mathcal{G}^{(n)} = [\mathcal{G}^{(n-1)}, \mathcal{G}^{(n-1)}]
$$

terminates, i.e., such that $\mathcal{G}^{(n)} = 0$ for some $n$. On the other hand, a nilpotent algebra $\mathcal{G}$ is the one for which the sequence

$$
\mathcal{G}^{(0)} = \mathcal{G}, \quad \mathcal{G}^{(1)} = [\mathcal{G}^{(0)}, \mathcal{G}], \ldots, \quad \mathcal{G}^{(n)} = [\mathcal{G}^{(n-1)}, \mathcal{G}]
$$

terminates, i.e., for which $\mathcal{G}^{(n)} = 0$ for some $n$ (see [57] for details).

In order to study the expansions of solvable and nilpotent Lie algebras, it will be useful to have an expression of the solvability and nilpotency condition in terms of the structure constants. We make this explicitly for the solvable case.

Let $\{X_1\}$ be a basis of an algebra $\mathcal{G}$. Since $\mathcal{G} = \mathcal{G}^{(0)}$ is solvable, one can find, from the commutation relations

$$
[X_i, X_j] = C_{ij}^k X_k
$$

\(^{11}\) The order $n = 9$ is non-trivial and the algorithms in the mentioned reference fail. As mentioned in the table, this non-trivial problem was solved in 2009 by Andreas Distler, Tom Kelsey and James Mitchell. However, in this paper we are going to consider calculations with semigroups of at most order 6.

\(^{12}\) As the number of non-isomorphic semigroups increases very quickly with the order $n$ (see the table in equation (9)), the mentioned lists are very large for higher orders.
with \(i, j = 1, \ldots, \dim G\), that there exists at least one value of \(k\) such that \(C_{ij}^{(k)} \neq 0\). Let \(k^{(1)}\) represent the set of values such that \(C_{ij}^{(k)} \neq 0\). Then \(k^{(1)}\) runs over all values of the basis elements \(\{X_i\}\) except for those values for which
\[
C_{ij}^{(k^{(1)})} = 0.
\]
(13)

It is clear then that the set \(\{X_{k(0)}\}\) is smaller than \(\{X_k\}\) and \(\{X_{k(0)}\} \subset \{X_k\}\). Let us consider now \(G^{(2)} = \left\{ G^{(1)}(1), G^{(1)}(2) \right\}\) where \(G^{(1)} = \left\{ G^{(0)}, G^{(0)} \right\}\) = \(\{X_{k(0)}\}\) according to the notation established above. We have
\[
[X_{j(1)}, X_{j(1)}] = C_{j(1)j(1)}^{(2)} X_{k(2)}.
\]
(14)

and again as \(G\) is solvable the index \(k^{(2)}\) must run in a smaller subset with respect to that of \(k^{(1)}\). Therefore, if the algebra is solvable, there must exist some \(n\) for which \(G^{(n)} = 0\) and \(G^{(n-1)}\) is Abelian, i.e.,
\[
[X_{j(n-1)}, X_{j(n-1)}] = C_{j(n-1)j(n-1)}^{(n)} X_{k(n)} = 0.
\]
(15)

Then, for solvable algebras there exists some \(n\) such that the range of values of \(k^{(n)}\) is empty. Equivalently, the solvability of an algebra can be expressed in terms of its structure constants as the condition that there is some \(n\) for which \(C_{k(n-1)j(n-1)} = 0\). In a similar way, it can be shown that an algebra is nilpotent if the structure constants \(C_{k(n-1)j(n-1)}^{(n)}\) vanish for some \(n\).

**Theorem 1.** Let \(\{X_i\}\) be a basis of a solvable Lie algebra \(G = \{λ_α\}\) a finite Abelian semigroup and
\[
G_S = S \otimes G = \{λ_α \otimes X_i\} = \{X_{(i, α)}\}
\]
the \(S\)-expanded algebra, which satisfies
\[
[X_{(i, α)}, X_{(j, β)}] = C_{ij}^{(k, γ)} X_{(k, γ)} = C_{ij}^{(k, γ)} K_{k}^{(γ)} X_{(k, γ)}.
\]
(17)

Then the expanded algebra \(G_S = S \otimes G\) is solvable.

**Proof.** Let us consider the following sequence for the expanded algebra
\[
G_S^{(0)} = G_S, \quad G_S^{(1)} = [G_S^{(0)}, G_S^{(0)}], \quad \ldots, \quad G_S^{(n)} = [G_S^{(n-1)}, G_S^{(n-1)}].
\]
(18)

For \(G_S^{(n)}\) we have
\[
[X_{(i(n-1), α(n-1))}, X_{j(n-1), β(n-1)}] = C_{i(n-1)j(n-1)}^{(k, γ)} K_{k}^{(γ)} X_{(k, γ)}. \quad \quad \quad \quad \quad (19)
\]

So as \(G\) is solvable by hypothesis, then there exists some \(n\) for which the sequence (18) terminates, i.e., for which \(G_S^{(n)} = 0\) (or in terms of the structure constants, for which \(C_{i(n-1)j(n-1)}^{(k, γ)} = 0\).

The result that the expansion of a solvable Lie algebra is solvable too was also found in a different way in [17], but only at the first level of the expanded algebra, \(G_S = G \otimes S\). Here we prove that this result holds also for the resonant and the reduced algebra.

**Theorem 2.** The resonant subalgebra \(G_{S,R}\) (defined in (7)) of the expanded algebra \(G_S\) is always solvable if the original algebra \(G\) is solvable.

**Proof.** By the theory of classification of Lie algebras, any subalgebra of a solvable algebra must be solvable.

**Theorem 3.** Consider the expansion of a solvable Lie algebra \(G\) with a zero element. Then the reduced algebra \(G_{S,0}^{\text{red}}\) is always solvable.
Proof. According to the $S$-expansion procedure (of [13]) when the semigroup has a zero element $0_S$, $S = \{\lambda_n, 0_S\}$ the commutation relations of the expanded algebra $G_S$ are given by

\[
\begin{align*}
[X_{(i,a)}, X_{(i,b)}] &= C_{ij}^l K_{a\beta}^l X_{(l,\gamma)} + C_{ij}^l K_{a\beta}^l X_{(k,0)} \\
[X_{(i,0)}, X_{(j,b)}] &= C_{ij}^l X_{(l,0)} \\
[X_{(i,0)}, X_{(j,0)}] &= C_{ij}^l X_{(k,0)}
\end{align*}
\]

and the $0_S$-reduced algebra $G_S^{red}$ is given by

\[
[X_{(i,a)}, X_{(i,b)}] = C_{ij}^l K_{a\beta}^l X_{(l,\gamma)}.
\]

So if $G$ is solvable then there exists some $n$ for which $C_{(p-n+1,q-n+1)} = 0$. Therefore, for the same $n$

\[
C_{(p-n+1,q-n+1)} = C_{(p-n+1,q-n+1)}^{(k,\alpha)} K_{(p-n+1,q-n+1)}^{(k,\beta)} = 0
\]

and we can conclude that if $G$ is solvable, then the reduced algebra $G_S^{red}$ is solvable, too. □

In a similar way, it can be directly shown that if $G$ is nilpotent, then $G_S$, $G_{S,R}$ and $G_S^{red}$ are nilpotent too.

3.2. Expansion of semisimple and compact Lie algebras

An algebra $G$ is **semisimple** if its Killing–Cartan metric, defined in terms of the structure constants by

\[
g_{ij} = C_{ik}^l C_{jl}^k,
\]

is non-degenerate, i.e., if $\det(g_{ij}) \neq 0$. On the other hand, $g_{ij}$ is diagonalizable so if we denote by $(\mu_i)$ its spectra of eigenvalues then a semisimple Lie algebra $G$ is **compact** if and only if $\mu_i < 0$ (see [57] for details).

If we now perform the expansion of a semisimple algebra $G$, the Killing–Cartan metric of the expanded algebra $G_S$ will be given by

\[
g_{(i,a)(j,b)} = C_{ij}^{(l,\lambda)} K_{(p-n+1,q-n+1)}^{(l,\lambda)} = K_{ij}^{\alpha\beta} K_{\alpha\beta}^{\gamma\delta} C_{\gamma\delta}^{\gamma\delta}.
\]

Let us define the following matrices:

\[
g^E = (g_{(i,a)(j,b)}); \quad g^S = (g_{a\beta}) = (K_a^\alpha, K_{\alpha\beta}^\gamma); \quad g = (g_{ij}) = (C_{ik}^l C_{jl}^k).
\]

From equation (20) it follows that the first matrix is the Kronecker product of the last two:

\[
g^E = g^S \otimes g.
\]

Both $g^S$ and $g$ are diagonalizable. Let us denote by $\{\xi_a\}$ and $\{\mu_i\}$ their spectra of eigenvalues respectively. From the general theory of Kronecker products we know that

- the eigenvalues of $g^S$ are $\{\xi_a\}$ and $\{\mu_i\}$;
- $\det(g^E) = \det(g^S)^{\dim(G)} \det(g)^{|S|}$ where $|S|$ is the order of the semigroup.

Thus, if $G$ is semisimple ($\det(g) \neq 0$), $G_S$ is semisimple if and only if $\det(g^S) \neq 0$. If $G$ is compact ($\mu_i < 0$), $G_S$ is compact only if $\xi_a > 0$. Thus, the real form of $G_S$ strongly depends on the signs of $\xi_a$.
When considering a decomposition \( \mathcal{G} = \bigoplus_{\mu \in I} V_{\mu} \) and \( S = \bigcup_{\mu \in I} S_{\mu} \), where \( I \) is a set of indices, the Killing–Cartan metric of the expanded algebra is given by

\[
\mathbf{g}^E = \mathbf{g}^S \otimes \mathbf{g}
\]

\[
= \begin{pmatrix}
  g_{00} & g_{02} & \cdots \\
  g_{20} & g_{22} & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix} \otimes \begin{pmatrix}
  g_{00} & g_{01} & \cdots \\
  g_{10} & g_{11} & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix}
\]

(23)

where the sets \( S_{\mu} \) and subspaces \( V_{\mu} \) are mixed all together. If \( \mathcal{G} \) and \( S \) have a structure given by (5) and (6), then the Killing–Cartan metric of the resonant subalgebra \( \mathcal{G}_{S,R} \) of equation (7) is given by

\[
g^{E,R}_{(i,p),(j,q)} = C_{(i,p),(j,q)}^{(k,r)} C_{(j,q),(l,s)}^{(k,r)}
\]

and from its matrix form,

\[
\mathbf{g}^{E,R} = \begin{pmatrix}
  g_{E,R}^{(i,j)} \\
  g_{E,R}^{(j,i)} \\
  \vdots
\end{pmatrix}
\]

(24)

it can be seen that sets \( S_{\mu} \) and subspaces \( V_{\mu} \) are mixed in a resonant form.

From the representation theory of Lie algebras (see appendix A) it turns out that if \( \mathcal{G}_S \) and \( \mathcal{G}_{S,R} \) are semisimple then \( \mathcal{G}_S \) is completely reducible with respect to the action of \( \mathcal{G}_{S,R} \) and therefore

\[
\frac{\mathcal{G}_S}{\mathcal{G}_{S,R}} \cong \mathcal{G}_S / \mathcal{G}_{S,R}.
\]

(25)

Then it is possible to obtain the Killing–Cartan metric for \( \mathcal{G}_{S,R} \) in terms of the one for \( \mathcal{G}_S \)

\[
g^{E,R}_{(i,j)} = \alpha g^{E}_{(i,j)}
\]

(26)

where

\[
\alpha = \left(1 + \frac{d_{\mathcal{G}_S/\mathcal{G}_{S,R}} g_{\mathcal{G}_S/\mathcal{G}_{S,R}}}{d_{\mathcal{G}_{S,R}}}ight)^{-1}
\]

(27)

is a positive number, defined in appendix A, characterizing \( \mathcal{G}_S / \mathcal{G}_{S,R} \) and \( \mathcal{G}_{S,R} \). On the other hand, if \( \mathcal{G}_S \) and \( \mathcal{G}_{S,R} \) are not semisimple, then the general expression for the Killing–Cartan form will be given just by equation (24).

An interesting case, which often appears in physical applications, is to take \( I = \{0, 1\} \), i.e., \( \mathcal{G} = V_0 \oplus V_1 \) with \( \mathcal{G} \) having a subspace structure given by \( ^{13} \)

\[
[V_0, V_0] \subset V_0
\]

\[
[V_0, V_1] \subset V_1
\]

\[
[V_1, V_1] \subset V_0.
\]

(28)

If the semigroup has a resonant decomposition \( S = S_0 \cup S_1 \) satisfying

\[
S_0 \times S_0 \subset S_0
\]

\[
S_0 \times S_1 \subset S_1
\]

\[
S_1 \times S_1 \subset S_0,
\]

(29)

\[
^{13} \text{As an example we will perform, in section 4, all the possible expansions of an algebra that satisfy this particular kind of decomposition.}
\]
with $S_0 = \{\lambda_{a_0}\}$ and $S_1 = \{\lambda_{a_1}\}$, then the resonant subalgebra (7) takes the form

$$G_{S,R} = (S_0 \otimes V_0) \oplus (S_1 \otimes V_1)$$

with a Killing–Cartan metric given by

$$g^{E,R} = \begin{pmatrix} g^{E,R} & 0 \\ 0 & g^{E,R} \end{pmatrix}$$

and where

$$g^{E,R}_{(i_0,a_0),(j_0,b_0)} = K_{(i_0,a_0),(j_0,b_0)}^0 K_{(i_0,a_0),(j_0,b_0)}^{i_0} C_{i_0}^{j_0} + K_{(i_0,a_0),(j_0,b_0)}^{i_0} K_{(i_0,a_0),(j_0,b_0)}^{j_0} C_{i_0}^{j_0} C_{i_0}^{j_0}$$

Moreover, if $G_S$ and $G_{S,R}$ are semisimple, then we can use (27) to write

$$g^{E,R}_{(i_0,a_0),(j_0,b_0)} = \alpha g^{E,R}_{(i_0,a_0),(j_0,b_0)} = \alpha g^{E,R}_{(i_0,a_0),(j_0,b_0)}$$

or equivalently

$$g^{E,R} = \alpha \begin{pmatrix} g (S_0) \otimes g (V_0) \\ 0 \\ g (S_1) \otimes g (V_1) \end{pmatrix}$$

where

$$g^S (S_0) = (g^S_{a_0,b_0}) = (K_{a_0,b_0}^0 K_{a_0,b_0}^{i_0} C_{i_0}^{j_0} + K_{a_0,b_0}^{i_0} K_{a_0,b_0}^{j_0} C_{i_0}^{j_0})$$

$$g^S (S_1) = (g^S_{a_1,b_1}) = (K_{a_1,b_1}^0 K_{a_1,b_1}^{i_0} C_{i_0}^{j_0} + K_{a_1,b_1}^{i_0} K_{a_1,b_1}^{j_0} C_{i_0}^{j_0})$$

$$g (V_0) = (g_{a_0,b_0}) = (C_{i_0}^{j_0} C_{i_0}^{j_0} + C_{i_0}^{j_0} C_{i_0}^{j_0})$$

$$g (V_1) = (g_{a_1,b_1}) = (C_{i_1}^{j_1} C_{i_1}^{j_1} + C_{i_1}^{j_1} C_{i_1}^{j_1})$$

From the fact that $\alpha$ is a positive number, the eigenvalues of $g^{E,R}$, $(\alpha \mu_{i_0} \xi_{a_0}, \alpha \mu_{i_0} \xi_{a_0})$, allow us to analyze the compactness of $G_{S,R}$ in terms of the eigenvalues of the matrices $g^S (S_0)$ and $g^S (S_1)$, i.e., in terms of $\xi_{a_0}$ and $\xi_{a_1}$.

Finally, if the semigroup has a zero element, a reduction can be performed and the result is similar:

$$g^{E,R,\text{red}} = \alpha \begin{pmatrix} g^{\text{red}} (S_0) \otimes g (V_0) \\ 0 \\ g^{\text{red}} (S_1) \otimes g (V_1) \end{pmatrix}$$

where $g^{\text{red}} (S_0)$ and $g^{\text{red}} (S_1)$ are calculated by using (33) and (34) but considering that their indices do not take a value on the zero element $0_3$ and considering that $K_{a_0 b_0}^{i_0} = 0$ if one of the indices $\alpha, \beta, \gamma$ is evaluated on the zero element.

Here we consider the case when a reduction of this specific resonant subalgebra is performed. Note however that the $0_3$-reduction procedure is a step that can be performed independently from the extraction of the resonant subalgebra. Explicit examples of this fact are given in section 4, figure 1.
3.3. Expansion of a general Lie algebra

By the Levi–Malcev theorem (see [57] for details), an arbitrary Lie algebra $\mathcal{G}$ always has a decomposition

$$\mathcal{G} = N \oplus S,$$

where $N$ is the radical (maximal solvable ideal) and $S$ is semisimple.

In this section, the semigroup used to perform the expansions will be denoted by the symbol $\mathcal{S}$, because in this section we are using $\mathcal{S}$ to denote the semisimple part of the Levi–Malcev decomposition of the arbitrary algebra (38).

3.3.1. The expanded algebra.

Let us call $X_{iN}$ and $X_{iS}$ the generators of $N$ and $S$, respectively. And let $\mathcal{S} = \{\lambda_{\alpha}\}$ be a finite Abelian semigroup. Then the expanded algebra of $\mathcal{G}$ (38) is given by

$$\mathcal{G}_S = \{X_{(iN,\alpha)}, X_{(iS,\beta)}\}$$

and we have the following commutation relations:

$$[X_{(iN,\alpha)}, X_{(jN,\beta)}] = C_{(iN,\alpha)(jN,\beta)} X_{(kN,\gamma)}$$
$$[X_{(iN,\alpha)}, X_{(jS,\beta)}] = C_{(iN,\alpha)(jS,\beta)} X_{(kN,\gamma)}$$
$$[X_{(iS,\alpha)}, X_{(jS,\beta)}] = C_{(iS,\alpha)(jS,\beta)} X_{(kS,\gamma)}$$

where we have used the fact that $\mathcal{G}$ is a semi-direct sum, (38). So the expanded algebra has also a semi-direct structure

$$\mathcal{G}_S = E_N \oplus E_S,$$

where $E_N = \{X_{(iN,\alpha)}\}$ is an ideal (which is also nilpotent being an expansion of a nilpotent algebra) and $E_S = \{X_{(iS,\alpha)}\}$ is a subalgebra. The important issue to mention here is that the subalgebra $E_S = \{X_{(iS,\alpha)}\}$ is not necessarily semisimple, because it is the expansion of a semisimple algebra $S$ which, as we found in the last section, is arbitrary, i.e., semisimplicity can be broken. Thus, it must have again a Levi–Malcev decomposition

$$E_S = N' \oplus S_{\exp}$$

where $N' = \{X_{(iS,\alpha)}\}$ and $S_{\exp} = \{X_{(iS,\alpha)}\}$ and

$$(iS, \alpha) = \{(iS, \alpha)' \oplus (iS, \alpha)_S\}$$

expresses the fact that the expansion of the semisimple part is arbitrary and will have a Levi–Malcev decomposition. Then, the expanded algebra is also arbitrary and must have a Levi–Malcev decomposition:

$$\mathcal{G}_S = E_N \oplus (N' \oplus S_{\exp})$$

and

$$= (E_N \oplus N') \oplus S_{\exp}$$

$$= N_{\exp} \oplus S_{\exp},$$

where $N_{\exp} = E_N \oplus N'$ is the radical of the expanded algebra $\mathcal{G}_S$. In fact, the last statement can be easily proved by showing that $N_{\exp}$ is the maximal solvable ideal, which is made in appendix B. In this way we have the following:

**Theorem 4.** If $\mathcal{G}$ is an arbitrary Lie algebra with a Levi–Malcev decomposition $\mathcal{G} = N \oplus S = \{X_{iN}\} \oplus \{X_{iS}\}$, then the expanded algebra is also arbitrary and has a Levi–Malcev decomposition given by

$$\mathcal{G}_S = N_{\exp} \oplus S_{\exp}$$
where $N_{\text{exp}} = E_N \cup N'$ is the radical of $G_S$, $E_S = N' \cup S_{\text{exp}}$ is the expansion of $S$, $N'$ is the radical of $E_S$, $S_{\text{exp}}$ is the semisimple part of $E_S$ and $G_S$ with

\[
E_N = \{X_{(i_s,a)}\} \\
E_S = \{X_{(i_s,a')}\} \\
N' = \{X_{(i_s,a)\gamma}\} \\
S_{\text{exp}} = \{X_{(i_s,a)\gamma'}\}.
\]

The resonant subalgebra $G_{S,R}$ of this arbitrary Lie algebra $G_S$ must also have a Levi–Malcev decomposition and its specific form can be studied in each case with a similar procedure.

### 3.3.2. The $0_S$-reduced algebra

Let us consider the case when the semigroup has a zero element, so $S = \{\lambda\Gamma\} = \{\lambda\} \cup \{0_S\}$. The reduced algebra is given by

\[
[\{X_{(i_s,a)}, X_{(j_s,b)}\}] = C^i_{j_s \alpha} K^\gamma_{\alpha \beta} X_{(k_s,\gamma)}
\]

where the index $\alpha, \beta, \ldots$ runs over all the non-zero elements of the semigroup.

In the last section, we obtained that for the algebra $G = N \cup S = \{X_{i_s}\} \cup \{X_{i_s}\}$, the complete expanded algebra is given by

\[
G_S = E_N \cup N' \cup S_{\text{exp}}
\]

where $E_N = \{X_{(i_s,\Gamma)}\}$, $N' = \{X_{(i_s,\Gamma)\gamma}\}$ and $S_{\text{exp}} = \{X_{(i_s,\Gamma)\gamma'}\}$. Then the reduced algebra is then given by

\[
G_S^{\text{red}} = \{X_{(i_s,a)}X_{(j_s,a)\gamma'}, X_{(i_s,a)\gamma}\}
\]

with commutation relations

\[
[\{X_{(i_s,a)}, X_{(j_s,b)}\}] = C^i_{(i_s,a),(j_s,b)} X_{(k_s,\gamma)} \\
[\{X_{(i_s,a)}, X_{(j_s,b)\gamma'}\}] = C^i_{(i_s,a),(j_s,b)\gamma'} X_{(k_s,\gamma)} \\
[\{X_{(i_s,a)}, X_{(j_s,b)\gamma}\}] = C^i_{(i_s,a),(j_s,b)\gamma} X_{(k_s,\gamma)} \\
[\{X_{(i_s,a)\gamma'}, X_{(j_s,b)\gamma'}\}] = C^i_{(i_s,a)\gamma'(j_s,b)\gamma'} X_{(k_s,\gamma)\gamma'} \\
[\{X_{(i_s,a)\gamma'}, X_{(j_s,b)\gamma}\}] = C^i_{(i_s,a)\gamma'(j_s,b)\gamma} X_{(k_s,\gamma)\gamma'} \\
[\{X_{(i_s,a)\gamma}, X_{(j_s,b)\gamma}\}] = C^i_{(i_s,a)\gamma(j_s,b)\gamma} X_{(k_s,\gamma)\gamma}
\]

so that

\[
E_{\text{red}}^N = \{X_{(i_s,a)}\} \text{ is an ideal of } G_S^{\text{red}} \\
N'_{\text{red}} = \{X_{(i_s,a)\gamma'}\} \text{ is a subalgebra of } G_S^{\text{red}} \text{ and an ideal of } E_S' = N'_{\text{red}} \cup S_{\text{exp}}^{\text{red}} \\
S_{\text{exp}}^{\text{red}} = \{X_{(i_s,a)\gamma}\} \text{ is a subalgebra of } G_S^{\text{red}},
\]

therefore, the semidirect structure of $G_S^{\text{red}}$ is confirmed,

\[
G_S^{\text{red}} = E_{\text{red}}^N \cup N_{\text{red}}^{\text{exp}} \cup S_{\text{exp}}^{\text{red}}.
\]

Now it only remains to prove that $S_{\text{exp}}^{\text{red}}$ is semisimple. We can write

\[
S_{\text{exp}}^{\text{red}} = \{X_{(i_s,\Gamma)\gamma}\} = \{X_{(i_s,a)}, X_{(i_s,0)\gamma}\}.
\]

Since $S_{\text{exp}}^{(0)} = \{X_{(i_s,0)\gamma}\}$ is an ideal of the semisimple Lie algebra $S_{\text{exp}}$, by a corollary of the structure theorem for semisimple Lie algebras, $S_{\text{exp}}^{(0)}$ has to be semisimple itself. In other words, if we express $S_{\text{exp}}$ as a direct sum of commuting simple Lie algebras:

\[
S_{\text{exp}} = \bigoplus_{k=0}^l S_{\text{exp},k}, \quad [S_{\text{exp},k}, S_{\text{exp},k'}] = 0 \quad \text{for } k \neq k',
\]
then $S^{(0)}_{\exp}$ coincides with the direct sum of a subset of these simple algebras:

$$S^{(0)}_{\exp} = \bigoplus_{k \in I} S^{\exp}_{k}$$

with $I$ a subset of $\{0, 1, \ldots, l\}$. Thus $S^{\exp}_{\text{red}} = S^{\exp} \ominus S^{(0)}_{\exp}$ is semisimple. Then we have that

$$S^{\text{red}}_{\exp} = N^{\text{red}}_{\exp} \ominus S^{\text{red}}_{\exp}$$

is the Levi–Malcev decomposition for the reduced algebra where $N^{\text{red}}_{\exp} = E^{\text{red}}_{N} \ominus N^{\text{red}}$ is the radical and $S^{\text{red}}_{\exp} = S^{\exp} \ominus S^{(0)}_{\exp}$ the semisimple part.

3.4. The Cartan decomposition under the $S$-expansion

As a brief explanation of what the Cartan decomposition is, let us cite some theorems (theorems 6.3 and 7.1) and definitions from [58]:

**Theorem.** Every semisimple Lie algebra over $\mathbb{C}$ has a compact real form, $G_{k}$.

In fact if $\{H_{\alpha}, E_{\alpha}, E_{-\alpha}\}$ is the Cartan–Weyl basis, then the compact real form $G_{k}$ is given by

$$G_{k} = \sum \mathbb{R} (iH_{\alpha}) + \sum \mathbb{R} (E_{\alpha} - E_{-\alpha}) + \sum \mathbb{R} (i (E_{\alpha} + E_{-\alpha})) .$$ (39)

The proof is given in [58].

**Theorem.** Let $G_{0}$ be a real semisimple Lie algebra, $G$ its complexification and $U$ any compact real form of $G$. Let $\sigma$ and $\tau$ be conjugations of $G$ with respect to $G_{0}$ and $U$, respectively. Then there exists an automorphism $\phi$ of $G$ such that the compact real form $\phi (U)$ is invariant under $\sigma$.

So having these results, it is possible to give (see [58]) the definition of a Cartan decomposition of a semisimple algebra:

**Definition.** Let $G_{0}$ be a semisimple real Lie algebra, $G$ its complexification and $\sigma$ a conjugation of $G$ with respect to $G_{0}$. Then

$$G_{0} = T_{0} + P_{0} ,$$ (40)

where $T_{0}$ is a subalgebra, is called a Cartan decomposition if there exists a compact real form, $G_{k}$, of $G$ such that

$$\sigma (G_{k}) \subset G_{k} \text{ and } T_{0} = G_{0} \cap G_{k}$$

$$P_{0} = G_{0} \cap (iG_{k}) .$$ (41)

The first two cited theorems imply that each real semisimple Lie algebra $G_{0}$ has a Cartan decomposition. It can also be demonstrated that $T_{0}$ is the maximal compactly embedded subalgebra of $G_{0}$. From now on, we are going to use the symbol ‘0’ to characterize structures related to real and semisimple Lie algebras.$^{15}$

Note also that the compact real form $G_{k}$ can be constructed in terms of $T_{0}$ and $P_{0}$ as

$$G_{k} = T_{0} + iP_{0} .$$ (42)

$^{15}$The only exception to this convention will be when we consider an algebra $G$ and a semigroup $S$ having a decomposition:

$$G = V_{0} + V_{1}$$

$$S = S_{0} \cup S_{1}$$
which satisfies
\[
[T_0, T_0] \subset T_0 \\
[T_0, iP_0] \subset iP_0 \\
iP_0, iP_0 \subset T_0.
\]

3.4.1. The expanded semisimple algebra. Let \( G_0 \) be a real semisimple Lie algebra, \( G \) its complexification and \( G_k \) its compact real form. If we perform an expansion of the compact real form \( G_k \) with a semigroup that preserves the compactness, then the expansion with this semigroup of \( G_0 \) will also preserve the semisimplicity. This is true because the first condition implies that the eigenvalues \( \xi_\alpha \) are all positive, so that \( \det g \neq 0 \). Therefore, it must have a Cartan decomposition, whose explicit form is given as follows:

**Theorem 5.** Let \( G_0 = T_0 + P_0 \) be the Cartan decomposition of the real semisimple algebra \( G_0 \), \( G \) its complexification and \( G_k \) its compact real form. Let \( \{X_0^m \}_{m=1}^{\dim \mathcal{G}_0}, \{X_0^m \}_{m=1}^{\dim \mathcal{T}_0}, \{X_0^m \}_{m=1}^{\dim \mathcal{P}_0} \) and \( \{X_1^m \}_{m=1}^{\dim \mathcal{G}_k}, \{X_1^m \}_{m=1}^{\dim \mathcal{T}_0}, \{X_1^m \}_{m=1}^{\dim \mathcal{P}_0} \) be the bases of \( G_0, T_0, P_0 \) and \( G_k \), respectively. If the semigroup used in the expansion preserves the compactness (\( \xi_\alpha \) are all positive) of the compact real form \( G_k \), i.e., if
\[
G_0, S = \mathcal{S}_0 \otimes T_0 \subset T_0 \\
\mathcal{P}_0, S = \mathcal{S}_0 \otimes P_0 \subset P_0
\]
is compact, then
\[
G_0, S = (T_0, S) + (P_0, S),
\]
where
\[
\mathcal{T}_0, S = \{X_0^{(0,0)}\} \\
\mathcal{P}_0, S = \{X_0^{(1,0)}\}
\]
is the Cartan decomposition of the expanded algebra \( G_0, S \).

**Proof.** See appendix C.

\[\square\]

3.4.2. The semisimple resonant subalgebra. The Cartan decomposition of a real semisimple Lie algebra
\[
G_0 = T_0 + P_0
\]
has the subspace structure (29) with \( V_0 = \mathcal{T}_0 \) and \( V_1 = \mathcal{P}_0 \), i.e.,
\[
[T_0, T_0] \subset T_0 \\
[T_0, P_0] \subset P_0 \\
P_0, P_0 \subset T_0
\]
which is a particular case of (5). So if we perform a special expansion by choosing a semigroup having a decomposition \( S = S_0 \cup S_1 \) satisfying the resonant condition (30), then the resonant subalgebra of \( G_{0,S,R} \) is given by
\[
G_{0,S,R} = (S_0 \otimes T_0) + (S_1 \otimes P_0)
\]
\[
= \mathcal{T}_{0,S,R} + \mathcal{P}_{0,S,R}
\]
where
\[
\mathcal{T}_{0,S,R} = \{X_0^{(0,0)}\} \\
\mathcal{P}_{0,S,R} = \{X_0^{(1,0)}\}
\]
Then we have the following:

**Theorem 6.** If the semigroup $S$ used in the expansion is such that all the eigenvalues $\xi_\alpha$ of the matrix $g^\alpha$ are positive then the resonant subalgebra $G_{0,S,R}$ (which is obviously semisimple because $\det g^\alpha \neq 0$) has a Cartan decomposition given precisely by (50)–(52).

**Proof.** See appendix C. □

These results can be presented in a more general form by relaxing the assumption that $S$ preserves compactness, i.e., $\xi_\alpha > 0$. In fact, one can consider the broader class of semigroups preserving semisimplicity, i.e., $\det g^\alpha \neq 0$. Nevertheless, under this condition the explicit form of the Cartan decomposition is not necessarily given by equations (50)–(52) (equations (45)–(47)). The reason is that the expression for $G_{0,S,R}$ given in equation (C.12) (for $G_{0,S}$ given in equation (44)) is not necessarily compact; compact generators may become non-compact and vice versa depending on the signature of $g^\alpha$. It is definitely true that to demand $\det g^\alpha \neq 0$, it is enough for the expanded algebra to admit a Cartan decomposition because under this condition $G_{0,S,R}$ is a real semisimple Lie algebra. To find it explicitly, one would have to diagonalize $g^{E,D}$ to find the new compact and non-compact generators, which would no longer be expressible as tensor products of elements of $S$ and $G$, but rather as combination of tensor products. For example, the diagonalization of $g^\alpha$ is given by

$$g^{E,D} = \tilde{O}^\alpha g^\alpha \tilde{O}$$

where $\tilde{O}^{(\alpha',i')} = \tilde{O}_\alpha^{i'}$ is an orthogonal matrix; $\tilde{O}_\alpha^{i'}$ and $\tilde{O}_i^{i'}$ are the orthogonal matrices diagonalizing $g^\alpha$ and $g$, respectively. Then the new basis of generators

$$X^{(\alpha',i')} = X^{(\alpha,i)} \tilde{O}_{(\alpha',i')}^{i'}$$

splits into two subsets, corresponding to negative and positive eigenvalues, generating the subspaces $T_{0,S}$ and $P_{0,S}$, so that the Cartan decomposition will be given by $G_{0,S} = T_{0,S} + P_{0,S}$. A similar procedure can be carried out in the case of the resonant subalgebra.

**3.4.3. The semisimple reduction of the resonant subalgebra.** The same results obtained above can be extended when the reduction of the resonant subalgebra, $G_{0,S,R}^{red}$, is compact. In this case we have

$$G_{0,S,R}^{red} = (S_0^{red} \otimes T_0) + (S_1^{red} \otimes iP_0)$$

where $S_0^{red} = S_0 - \{0\}$ and $S_1^{red} = S_1 - \{0\}$. The Cartan decomposition of $G_{0,S,R}^{red}$ is given by

$$G_{0,S,R}^{red} = (S_0^{red} \otimes T_0) + (S_1^{red} \otimes P_0) = T_{0,S,R}^{red} + P_{0,S,R}^{red}$$

with

$$T_{0,S,R}^{red} = G_{0,S,R}^{red} \cap G_{0,S,R}^{red} = \{X^{(\alpha,i)}_{(0,0)}\}$$

$$P_{0,S,R}^{red} = G_{0,S,R}^{red} \cap (G_{0,S,R}^{red} \otimes P_0) = \{X^{(\alpha,i)}_{(0,1)}\}$$

where $\alpha_0^{red}$ and $\alpha_1^{red}$ are indices running on $S_0^{red}$ and $S_1^{red}$, respectively.

**4. Expansions of $sl(2, \mathbb{R})$, an instructive example**

Consider the algebra $G_0 = sl(2, \mathbb{R})$ and its complex form $G = sl(2, \mathbb{C})$. As known, $G_k = su(2)$ is the compact real form of $sl(2, \mathbb{C})$ so the Cartan decomposition of $sl(2, \mathbb{R})$,
\[ G_0 = \mathfrak{sl}(2, \mathbb{R}) = T_0 + P_0, \]  
\[ T_0 = \{ -i\sigma_2 \} \]  
\[ P_0 = \{ \sigma_1, \sigma_3 \} \]  
is such that

\[ T_0 = \mathfrak{sl}(2, \mathbb{R}) \cap \mathfrak{su}(2) \]  
\[ P_0 = \mathfrak{sl}(2, \mathbb{R}) \cap i(\mathfrak{su}(2)). \]

This can be checked directly by considering that the basis of \( SU(2) \) is given by

\[ G_k = \mathfrak{su}(2) = \{ i\sigma_2, i\sigma_1, i\sigma_3 \} = T_0 + iP_0 \]

where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the usual Pauli matrices. It is also directly checked that \( \mathfrak{sl}(2, \mathbb{R}) \) has the subspace structure (29) with \( V_0 = T_0 \) and \( V_1 = P_0 \) and that the Killing–Cartan metric for \( \mathfrak{sl}(2, \mathbb{R}) \) is given by

\[ g = \begin{pmatrix} g(V_0) & 0 \\ 0 & g(V_1) \end{pmatrix} = \text{diag} (-8, 8) \]  
\[ g(V_0) = -8; \quad g(V_1) = \text{diag} (8, 8). \]

According to our notation:

\[ X_1 = -i\sigma_2, \quad X_2 = \sigma_1 \text{ and } X_3 = \sigma_3, \]

so the elements of the expanded algebra will have the form

\[ X_{(i,a)} = \lambda_a \otimes X_i \]  

as in (2), with \( i = 1, 2, 3 \) and \( a \) an index in a certain semigroup \( S \).

In what follows we study the expansions of \( \mathfrak{sl}(2, \mathbb{R}) \) with the standard semigroups introduced in section 2.2. The results obtained in section 3 will also be checked with this example.

4.1. Classification of the different kinds of expansions

We are going to study the properties of all the possible expansions of \( \mathfrak{sl}(2, \mathbb{R}) \) that can be made by using semigroups of order \( n = 1, 2, \ldots, 6 \). These expansions can be classified as one of the following kinds:

- expansions with all the Abelian semigroups;
- expansions with all the Abelian semigroups with a zero element;
- expansions with all the Abelian semigroups with a resonant decomposition of the form (30);
- expansions with all the Abelian semigroups with a zero element and simultaneously with a resonant decomposition of the form (30).\(^{16}\)

To perform all these possible expansions we use the following algorithm.

First we identify all the semigroups of a certain order \( n \) (we have limited this up to order 6) satisfying the conditions enumerated above. For example, for \( n = 3 \) the results are given in figure 1.

We identified the different semigroups \( S_{(3)} \) of order 3 with a label ‘a’ for reasons of space in figure 1 to name the different semigroups. The horizontal axis represents the set of semigroups used in some specific expansion while the vertical axis represents the different kinds of expansions that can be performed.

\(^{16}\) It is interesting to note that a semigroup can have more than one resonant decomposition leading then to different expanded algebras.
In the first level, we list all the Abelian semigroups that allow us to perform a general expansion $S \otimes G$. The Abelian semigroups are: $S^{1}_{(3)}$, $S^{2}_{(3)}$, $S^{3}_{(3)}$, $S^{6}_{(3)}$, $S^{7}_{(3)}$, $S^{9}_{(3)}$, $S^{10}_{(3)}$, $S^{12}_{(3)}$, $S^{15}_{(3)}$, $S^{16}_{(3)}$, $S^{17}_{(3)}$, $S^{18}_{(3)}$.

In the second level, we find all the Abelian semigroups that contain at least one resonant decomposition so that a resonant subalgebra can be extracted from the expanded one. The Abelian semigroups are: $S^{1}_{(3)}$, $S^{2}_{(3)}$, $S^{3}_{(3)}$, $S^{6}_{(3)}$, $S^{12}_{(3)}$, $S^{15}_{(3)}$, $S^{16}_{(3)}$, $S^{17}_{(3)}$.

In the third level, we see all the Abelian semigroups that contain a zero element so that a reduced algebra can be extracted from the expanded one. Here are the semigroups: $S^{1}_{(3)}$, $S^{2}_{(3)}$, $S^{4}_{(3)}$, $S^{6}_{(3)}$, $S^{9}_{(3)}$, $S^{10}_{(3)}$, $S^{12}_{(3)}$.

In the fourth level, we find all the Abelian semigroups that contain at least one resonant decomposition and also a zero element. So a reduced algebra can be obtained from the resonant subalgebra. The semigroups that allow us to do that are: $S^{1}_{(3)}$, $S^{2}_{(3)}$, $S^{3}_{(3)}$, $S^{6}_{(3)}$, $S^{12}_{(3)}$.

Then, for all these expansions, we identify the semigroups that preserve the semisimplicity of the original algebra. In figure 1, those semigroups are labeled with a gray number.

As we know, the $S$-expansion method [13] contains as a particular case the expansions by means of a parameter [10] (and at the same time this method includes all types of contractions that are known in the literature). In fact, this kind of expansion is recovered when one specific semigroup [17], in the fourth level of figure 1, is used. So, it is evident that the use of semigroups leads to more general kinds of expansions, i.e., to expansions that are not only on the fourth level of figure 1, but also in the other three lower levels. Besides, in each of them there are cases where semisimplicity can or cannot be preserved.

4.2. General properties of the expansions with $n = 3, \ldots, 6$

For higher orders $n \geq 4$ it is not possible to show a graphic like that given in figure 1. Instead, we give a table listing the number of semigroups that leads to the different kinds of expansions

17 In the case of expansions with semigroups of order 3, this happens for the semigroup $S^{6}_{(3)}$ given by

$$S^{6}_{(3)} = \begin{pmatrix}
\lambda_{1} & \lambda_{2} & \lambda_{3} \\
\lambda_{1} & \lambda_{1} & \lambda_{1} \\
\lambda_{2} & \lambda_{1} & \lambda_{2} \\
\lambda_{3} & \lambda_{1} & \lambda_{3}
\end{pmatrix}$$

which is isomorphic to the semigroup $S^{6}_{(n)}$ with $n = 1$ that is used in the original article [13] where $S$-expansions were introduced.
we have mentioned in the previous section. We also give in each case the number of semigroups preserving semisimplicity:

| order | 3   | 4   | 5   | 6   |
|-------|-----|-----|-----|-----|
| expanded $S \otimes G$ preserving semisimplicity | #12 | #58 | #325 | #2,143 |
| expanded and reduced preserving semisimplicity | #5  | #16 | #51  | #201  |
| resonant subalgebra preserving semisimplicity | #8  | #39 | #226 | #1,538 |
| reduction of resonant subalg. preserving semisimplicity | #3  | #9  | #34  | #135  |

To construct this table, we have written a set of computer programs (a Java library) that takes as an input all the Abelian semigroups given by the second program in [59], con.f, and then permits to classify those with zero elements and also to study all the resonant decompositions they have. A deeper description of those programs will be submitted soon (see [45]).

In the different rows we see the various kinds of expansions that can be done (the expanded algebra, the resonant subalgebra, the reduced algebra and the reduction of the resonant subalgebra) for each order of the semigroups. In each case the number of semigroups with which the expansions can be performed is specified. The number of semigroups preserving semisimplicity is also given. However, it may happen that a certain semigroup has more than one resonance, which then leads to different expanded algebras. In what follows we summarize this information:

**For the order $n = 3$.**
- There are eight semigroups of order 3 with at least one resonant decomposition,
- with nine different resonant decompositions (so this is the number of different kinds of expansions that can be made for $n = 3$)
- and one expansion that gives a semisimple Lie algebra with one of its resonances.

**For the order $n = 4$.**
- There are 48 semigroups of order 4 with at least one resonant decomposition,
- with 124 different resonant decompositions (so this is the number of different kinds of expansions that can be made for $n = 4$)
- and four expansions that give a semisimple Lie algebra with one of its resonances.

**For the order $n = 5$.**
- There are 299 semigroups of order 5 with at least one resonant decomposition,
- with 1653 different resonant decompositions (so this is the number of different kinds of expansions that can be made for $n = 5$)
and seven expansions that give a semisimple Lie algebra with one of its resonances.

For the order \( n = 6 \),

- There are 2059 semigroups of order 6 with at least one resonant decomposition,
- with 25,512 different resonant decompositions (so this is the number of different kinds of expansions that can be made for \( n = 6 \))
- and 23 expansions that give a semisimple Lie algebra with one of its resonances.

In general, all these expansions with \( n = 3, 4, 5, 6 \) share the following property:

**Remark 7.** Consider a semigroup of order \( n = 3, 4, 5, 6 \) having more than one resonance. Then if it preserves semisimplicity, this happens just for one of its resonances. There is no semigroup preserving semisimplicity with more than one of its resonances.

In this way it is explicitly verified that starting from a semisimple algebra the expanded algebras are not necessarily semisimple, as was suggested in section 3. In fact the major part of the expansions do not preserve semisimplicity.

### 5. Semigroups preserving semisimplicity

We have seen that semisimplicity and compactness are preserved only in expansions performed with semigroups satisfying certain conditions related to the signature (i.e., number of positive, negative and zero eigenvalues) of the symmetric matrices \( g^S, g^S(S_0), g^S(S_1), g^{red}(S_0) \) and \( g^{red}(S_0) \) defined in section 3.2. It is therefore relevant to classify the semigroups also in relation to that property, which is done here for semigroups of order 3, 4 and 5 preserving semisimplicity.

Using the standard classification and notation of the semigroups given in section 2.2 we have found that the semigroups \( S^{\text{red}}_{(n)} = \{\lambda_1, \ldots, \lambda_n\} \) of order \( n = 3, 4, 5 \) with a zero element and a resonant decomposition (30) that preserve semisimplicity are: \( S^{12}_{(3)}, S^{88}, S^{770}, S^{779}, S^{922}, S^{908}, S^{900}, S^{991} \). In what follows, we list their main properties and explicit expansions of \( \mathfrak{sl}(2, \mathbb{R}) \) will be performed with those semigroups.

The semigroup \( S^{12}_{(3)} = \{\lambda_1, \lambda_2, \lambda_3\} \) is defined by

\[
\begin{array}{ccc}
\lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_2 & \lambda_3 \\
\lambda_3 & \lambda_1 & \lambda_2 \\
\end{array}
\]

with the resonant decomposition

\( S_0 = \{\lambda_1, \lambda_2\}, S_1 = \{\lambda_1, \lambda_3\} \).

The eigenvalues of the matrices \( g^S, g^S(S_0), g^S(S_1), g^{red}(S_0), g^{red}(S_0) \) for this semigroup are all positive so, besides semisimplicity, compactness properties are also preserved. However, an expansion with this semigroup is trivial in the sense that the reduction of the resonant subalgebra is equal to the original algebra. Something similar happens for the semigroup \( S^{88} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \) defined by

\[
\begin{array}{cccc}
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\lambda_3 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_2 & \lambda_4 & \lambda_4 \\
\end{array}
\]

with the resonant decomposition

\( S_0 = \{\lambda_1, \lambda_2, \lambda_3\}, S_1 = \{\lambda_1, \lambda_2, \lambda_4\} \).

It also preserves compactness properties, but two consecutive reductions\(^{18} \) of the resonant subalgebra lead finally to the original algebra, so it is not interesting either.

---

\(^{18} \) In general, more than one reduction can be performed when a semigroup (here \( S^{88}_{(4)} \) with a certain zero element \( \lambda_3 \) contains a sub-semigroup (the set \( \{\lambda_2, \lambda_3, \lambda_4\} \) having its own zero element (in this case \( \lambda_2 \)).
More interesting cases can be found among the six semigroups of order 5 that preserve semisimplicity of the reduction of the resonant subalgebra (see the table in equation (63)). Those semigroups are

\[
\begin{array}{c|cccccccc}
S_5^{\text{770}} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_5 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_5 & \lambda_1 & \lambda_1 \\
\lambda_3 & \lambda_1 & \lambda_3 & \lambda_1 & \lambda_2 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_4 & \lambda_1 & \lambda_3 & \lambda_2 \\
\lambda_5 & \lambda_1 & \lambda_5 & \lambda_1 & \lambda_4 & \lambda_3 \\
\end{array}
\hspace{1cm}
\begin{array}{c|cccccccc}
S_5^{\text{779}} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_1 & \lambda_2 & \lambda_5 \\
\lambda_3 & \lambda_1 & \lambda_3 & \lambda_1 & \lambda_3 & \lambda_5 \\
\lambda_4 & \lambda_1 & \lambda_4 & \lambda_1 & \lambda_4 & \lambda_5 \\
\lambda_5 & \lambda_1 & \lambda_5 & \lambda_1 & \lambda_5 & \lambda_5 \\
\end{array}
\hspace{1cm}
\begin{array}{c|cccccccc}
S_5^{\text{368}} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_5 \\
\lambda_3 & \lambda_1 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_5 \\
\lambda_4 & \lambda_1 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_5 \\
\lambda_5 & \lambda_1 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 \\
\end{array}
\hspace{1cm}
\begin{array}{c|cccccccc}
S_5^{\text{690}} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_4 \\
\lambda_3 & \lambda_1 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\lambda_5 & \lambda_1 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 \\
\end{array}
\hspace{1cm}
\begin{array}{c|cccccccc}
S_5^{\text{691}} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\hline
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\
\lambda_2 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_4 \\
\lambda_3 & \lambda_1 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_4 \\
\lambda_4 & \lambda_1 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\
\lambda_5 & \lambda_1 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 \\
\end{array}
\]

with resonant decomposition: with resonant decomposition:

\[
S_0 = \{\lambda_1, \lambda_2, \lambda_3\} \\
S_0 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \\
S_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}
\]

\[
S_0 = \{\lambda_1, \lambda_2, \lambda_3\} \\
S_0 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_3\} \\
S_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}
\]

\[
S_0 = \{\lambda_1, \lambda_2, \lambda_3\} \\
S_0 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_3\} \\
S_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}
\]

\[
S_0 = \{\lambda_1, \lambda_2, \lambda_3\} \\
S_0 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_3\} \\
S_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}
\]

The semigroups \(S_5^{\text{770}}\) and \(S_5^{\text{368}}\) also lead to trivial expansions: three reductions can be performed on the resonant subalgebra leading finally to the original algebra\(^{19}\). However, more interesting structures are obtained with the other semigroups.

5.1. The semigroup \(S_5^{\text{770}}\)

Expanding \(\mathfrak{g}(2, \mathbb{R})\) with the semigroup \(S_5^{\text{770}}\), the reduction of the resonant subalgebra is given by

\[
G_{0,3,5}^{\text{red}} = (S_0^{\text{red}} \otimes T_0) + (S_1^{\text{red}} \otimes P_0)
\]

\[
= \{X_{(1,2)}, X_{(1,3)}\} + \{X_{(2,4)}, X_{(2,5)}, X_{(3,4)}, X_{(3,5)}\}
\]

\[
= \{\lambda_2 \otimes (-i\sigma_2), (\lambda_3 \otimes -i\sigma_2)\} + \{\lambda_4 \otimes \sigma_1, \lambda_5 \otimes \sigma_1, \lambda_4 \otimes \sigma_3, \lambda_5 \otimes \sigma_3\}
\]

\(^{19}\) See the structure of their multiplication tables and remember the comment made about semigroups containing sub-semigroups with its own zero elements. In some sense these semigroups can be considered as an enlargement of \(S_5^{\text{[1]}}\) by new zero elements.
where we have used the notation given in (61) and (62). The generators were renamed as
\begin{align}
Y_1 &= X_{1(2)} \\
Y_2 &= X_{1(3)} \\
Y_3 &= X_{1(4)} \\
Y_4 &= X_{2(5)} \\
Y_5 &= X_{3(4)} \\
Y_6 &= X_{3(5)}.
\end{align}

Using equation (3), the commutation relations of \(\mathfrak{s}(2, \mathbb{R})\) and the multiplication table of \(S_{70}(S)\) (given in (64)), we obtain the following commutation relations for \(G_{0, S, R}^{\text{red}}\):
\begin{align}
[Y_1, Y_2] &= 0 \\
[Y_1, Y_3] &= 0 \\
[Y_1, Y_4] &= 2Y_6 \\
[Y_1, Y_5] &= 2Y_7 \\
[Y_1, Y_6] &= 2Y_3 \\
[Y_2, Y_3] &= 2Y_5 \\
[Y_2, Y_4] &= 2Y_6 \\
[Y_2, Y_5] &= 0 \\
[Y_2, Y_6] &= 0 \\
[Y_3, Y_4] &= 2Y_6 \\
[Y_3, Y_5] &= 0 \\
[Y_3, Y_6] &= 2Y_4 \\
[Y_4, Y_5] &= 0 \\
[Y_4, Y_6] &= 2Y_1.
\end{align}

It is direct to see that this algebra is just \(\mathfrak{s}(2, \mathbb{R}) \oplus \mathfrak{s}(2, \mathbb{R})\), which is obviously semisimple. The two commuting \(\mathfrak{s}(2, \mathbb{R})\) algebras are respectively generated by \([Y_1, Y_2, Y_3]\) and \([Y_2, Y_3, Y_4]\). The Killing–Cartan metric, calculated with those structure constants, is given by
\begin{equation}
g_{E, R, \text{red}} = \text{diag}(-8, -8, 8, 8, 8, 8). \tag{70}
\end{equation}

As shown in equation (37) of section 3.2, the semisimplicity and compactness properties of the expanded algebra can be directly predicted from intrinsic properties of the semigroup used in the expansion. In fact, the matrices \(g_{\text{red}}^{(S_0)}\) and \(g_{\text{red}}^{(S_1)}\) can be directly calculated from the multiplication table of the semigroup \(S_{70}(S)\).
\begin{equation}
g_{\text{red}}^{(S_0)} = g_{\text{red}}^{(S_1)} = \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}. \tag{71}
\end{equation}

From equation (37) and since the eigenvalues \(\xi_{a_0}\) and \(\xi_{a_1}\) of \(g_{\text{red}}^{(S_0)}\) and \(g_{\text{red}}^{(S_1)}\) are all positive, we conclude that the expansion of whatever algebra, with a decomposition \(\mathcal{G} = V_0 \oplus V_1\) and having the subspace structure (29), will preserve semisimplicity and compactness properties. In particular, if \(\mathcal{G} = \mathfrak{s}(2, \mathbb{R})\) (with \(V_0 = \mathcal{T}_0\) and \(V_1 = \mathcal{P}_0\) then from (37), (60) and (70) we obtain \(\alpha = 1/2\). Besides,
\begin{equation}
G_{0, S, R}^{\text{red}} = \mathcal{T}_{0, S, R}^{\text{red}} + \mathcal{P}_{0, S, R}^{\text{red}} \tag{72}
\end{equation}

where
\begin{align}
\mathcal{T}_{0, S, R}^{\text{red}} &= \mathcal{T}_0 \oplus S_0^{\text{red}} \\
\mathcal{P}_{0, S, R}^{\text{red}} &= \mathcal{P}_0 \oplus S_1^{\text{red}} \tag{73}
\end{align}
is the Cartan decomposition for the algebra \(G_{0, S, R}^{\text{red}}\), while the real compact form of \(G_{S, R}^{\text{red}}\) (the complex form of \(G_{0, S, R}^{\text{red}}\)) is given by
\begin{equation}
G_{S, R}^{\text{red}} = \left(S_0^{\text{red}} \otimes T_0\right) + \left(S_1^{\text{red}} \otimes \mathbb{R}P_0\right) = \{\lambda_2 \otimes i\sigma_2, \lambda_3 \otimes i\sigma_2\} + \{\lambda_4 \otimes i\sigma_1, \lambda_5 \otimes i\sigma_1, \lambda_4 \otimes i\sigma_3, \lambda_5 \otimes i\sigma_3\}. \tag{74}
\end{equation}

Finally, relations (55) can also be directly checked,
\begin{align}
\mathcal{T}_{0, S, R}^{\text{red}} \cap G_{S, R}^{\text{red}} &= \{\lambda_2 \otimes (-i\sigma_2), \lambda_3 \otimes (-i\sigma_2)\} \\
\mathcal{P}_{0, S, R}^{\text{red}} \cap G_{S, R}^{\text{red}} &= \{\lambda_4 \otimes \sigma_1, \lambda_5 \otimes \sigma_1, \lambda_4 \otimes \sigma_3, \lambda_5 \otimes \sigma_3\}. \tag{75}
\end{align}
5.2. The semigroup $S^{068}_{(5)}$

For this semigroup the matrices $g^{\text{red}}(S_0), g^{\text{red}}(S_1)$ are given by
\[
g^{\text{red}}(S_0) = g^{\text{red}}(S_1) = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.
\]

Since their eigenvalues are all positive we conclude that semisimplicity and compactness properties should be preserved. Considering $G = sl(2, \mathbb{R})$, equation (37) takes the form
\[
g^{E,R,\text{red}} = \alpha \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \otimes \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 \end{pmatrix} \otimes \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix},
\]
from where we can predict that the expanded algebra will have two compact generators and four non-compact generators. In what follows, we are going to show that the expanded algebra is $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$.

Expanding with the semigroup $S^{068}_{(5)}$, we have
\[
G^{\text{red}}_{0,8,R} = (y_0^{\text{red}} \otimes T_0) + (s_1^{\text{red}} \otimes P_0)
= \{X_{(2,1)}, X_{(3,2)}\} + \{X_{(2,4)}, X_{(3,5)}, X_{(3,6)}\}
= \{\lambda_2 \otimes (-i\sigma_2), \lambda_3 \otimes (-i\sigma_2)\} + \{\lambda_4 \otimes \sigma_1, \lambda_5 \otimes \sigma_1, \lambda_4 \otimes \sigma_3, \lambda_5 \otimes \sigma_3\}.
\]

Note that the structure is very similar to the algebra (67). What makes the difference between those algebras is that the elements of the semigroups obey different tables of multiplication. Let us also define $Y_1, Y_2, \ldots, Y_6$ as in equation (68). Using equation (3), the commutation relations of $sl(2, \mathbb{R})$ and the multiplication table of $S^{068}_{(5)}$ (given in (65)), we obtain the following commutation relations for $G^{\text{red}}_{0,8,R}$:
\[
\begin{align*}
[Y_1, Y_2] &= -2Y_5 \\
[Y_1, Y_3] &= -2Y_5 \\
[Y_1, Y_4] &= 2Y_3 \\
[Y_1, Y_5] &= 2Y_3 \\
[Y_2, Y_3] &= 2Y_5 \\
[Y_2, Y_4] &= 2Y_5 \\
[Y_2, Y_5] &= -2Y_5 \\
[Y_2, Y_6] &= -2Y_5 \\
[Y_3, Y_4] &= 2Y_6
\end{align*}
\]

The Killing–Cartan metric for this algebra is given by
\[
g^{E,R,\text{red}} = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 16 & 0 & 0 \\ 0 & 0 & 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 8 & 16 \end{pmatrix},
\]

which, in fact, possesses two negative and four positive eigenvalues corresponding to two compact and four non-compact generators. The expanded algebra is $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ and the two commuting $sl(2, \mathbb{R})$ subalgebras are generated respectively by
\[
\begin{align*}
sl(2, \mathbb{R})_1 &= \{Y_2 - Y_1, Y_3 - Y_4, Y_5 - Y_6\} \\
sl(2, \mathbb{R})_2 &= \{Y_1, Y_3, Y_5\}.
\end{align*}
\]

Furthermore, from (60), (77) and (80) it can be directly obtained that $\alpha = 1/2$ and relations similar to (72) and (73) hold for the Cartan decomposition of this algebra.

\[20\text{In this case, the identification of the algebra with an } sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \text{ directly follows from the inspection of the Cartan–Killing matrix (its being non-singular and having just two negative eigenvalues).}
Finally, the real compact form of $G_{5,R}^{\text{red}}$ (the complex form of $G_{0,5,R}^{\text{red}}$) is given by

$$G_{5,R}^{\text{red}} = (S_0^{\text{red}} \otimes T_0) + (S_1^{\text{red}} \otimes P_0)$$

$$= \{ \lambda_2 \otimes i\sigma_2, \lambda_3 \otimes i\sigma_2 \} + [\lambda_4 \otimes i\sigma_1, \lambda_5 \otimes i\sigma_1, \lambda_4 \otimes i\sigma_3, \lambda_5 \otimes i\sigma_3]. \quad (82)$$

It is also easy to check the relations (55):

$$T_{5,R}^{\text{red}} = G_{5,R}^{\text{red}} \cap G_{4,3,R}^{\text{red}} = \{ \lambda_2 \otimes (-i\sigma_2), \lambda_3 \otimes (-i\sigma_2) \}$$

$$T_{0,5,R}^{\text{red}} = G_{0,5,R}^{\text{red}} \cap \left( G_{5,R}^{\text{red}} \right) = \{ \lambda_4 \otimes \sigma_1, \lambda_5 \otimes \sigma_1, \lambda_4 \otimes \sigma_3, \lambda_5 \otimes \sigma_3 \}. \quad (83)$$

5.3. The semigroup $S_{(5)}^{0\text{red}}$

For this semigroup the matrices $g^{\text{red}}(S_0)$, $g^{\text{red}}(S_1)$ are given by

$$g^{\text{red}}(S_0) = g^{\text{red}}(S_1) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}. \quad (84)$$

Again, since their eigenvalues are all positive we conclude that semisimplicity and compactness properties will be preserved. Taking $G = \mathfrak{sl}(2, \mathbb{R})$ we can predict, using equation (37), that the expanded algebra will have two compact generators and four non-compact generators and it will be then $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. Let us see this explicitly.

Expanding with the semigroup $G_{(5)}^{\text{red}}$, we have

$$G_{0,5,R}^{\text{red}} = (S_0^{\text{red}} \otimes T_0) + (S_1^{\text{red}} \otimes P_0)$$

$$= \{ X_{(1,2)}, X_{(1,5)} \} + [X_{(2,3)}, X_{(2,4)}, X_{(3,3)}, X_{(3,4)}],$$

$$= \{ \lambda_2 \otimes (-i\sigma_2), \lambda_3 \otimes (-i\sigma_2) \} + [\lambda_3 \otimes \sigma_1, \lambda_4 \otimes \sigma_1, \lambda_3 \otimes \sigma_3, \lambda_4 \otimes \sigma_3]. \quad (85)$$

Renaming the generators as

$$Y_1 = X_{(1,2)}, \quad Y_2 = X_{(1,5)}, \quad Y_3 = X_{(2,3)}, \quad Y_4 = X_{(2,4)}$$

and using (3), the commutation relations of $\mathfrak{sl}(2, \mathbb{R})$ and the multiplication table of $S_{(5)}^{0\text{red}}$ (given in (66)), we obtain

$$[Y_1, Y_1] = -2Y_3 \quad [Y_2, Y_3] = 2Y_4$$

$$[Y_1, Y_2] = -2Y_6 \quad [Y_2, Y_6] = 2Y_5$$

$$[Y_1, Y_3] = 2Y_5 \quad [Y_3, Y_5] = 2Y_1$$

$$[Y_1, Y_4] = 2Y_4 \quad [Y_4, Y_6] = 2Y_2$$

$$[Y_2, Y_5] = -2Y_6 \quad [Y_5, Y_5] = 2Y_2$$

$$[Y_2, Y_4] = -2Y_3 \quad [Y_4, Y_6] = 2Y_1 \quad (87)$$

whose Killing–Cartan metric is given by

$$g^{E,R,\text{red}} = \text{diag} (-16, -16, 16, 16, 16, 16) \quad (88)$$

which again has two negative and four positive eigenvalues. Then the expanded algebra is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, the two commuting $\mathfrak{sl}(2, \mathbb{R})$ algebras being generated respectively by

$$\mathfrak{sl}(2, \mathbb{R})_1 = \left\{ \frac{1}{2} (Y_1 - Y_2), \frac{1}{2} (Y_1 - Y_4), \frac{1}{2} (Y_3 - Y_6) \right\}$$

$$\mathfrak{sl}(2, \mathbb{R})_2 = \left\{ \frac{1}{2} (Y_1 + Y_2), \frac{1}{2} (Y_5 + Y_4), \frac{1}{2} (Y_5 + Y_6) \right\}. \quad (89)$$

Besides, from (37), (60) and (88) it is also obtained that $\alpha = 1/2$ and similar relations to (72) and (73) hold for the Cartan decomposition of this algebra.
Finally, the real compact form of $G^{\text{red}}_{S, R}$ is given by
\[
G^{\text{red}}_{S, R} = (S_{0}^{\text{red}} \otimes T_{0}) + (S_{1}^{\text{red}} \otimes i P_{0})
\]
\[
= \{ \lambda_{2} \otimes i \sigma_{2}, \lambda_{3} \otimes i \sigma_{2} \} + \{ \lambda_{3} \otimes i \sigma_{1}, \lambda_{4} \otimes i \sigma_{1}, \lambda_{3} \otimes i \sigma_{3}, \lambda_{4} \otimes i \sigma_{3} \},
\]
so we can check again the relations (55):

\[
T_{0, S, R} = G_{0, S, R}^{\text{red}} \cap G_{S, R}^{\text{red}} = \{ \lambda_{2} \otimes (-i \sigma_{2}), \lambda_{5} \otimes (-i \sigma_{2}) \}
\]
\[
P_{0, S, R} = G_{0, S, R}^{\text{red}} \cap i G_{S, R}^{\text{red}} = \{ \lambda_{3} \otimes \sigma_{1}, \lambda_{4} \otimes \sigma_{1}, \lambda_{3} \otimes \sigma_{3}, \lambda_{4} \otimes \sigma_{3} \}.
\]

5.4. The semigroup $S_{(5)}^{991}$

This case is different from the ones above. Indeed, the matrices $g^{\text{red}} (S_{0}), g^{\text{red}} (S_{1})$ are given by
\[
g^{\text{red}} (S_{0}) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} ; \quad g^{\text{red}} (S_{1}) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}
\]
so $g^{\text{red}} (S_{0})$ has again positive eigenvalues, while $g^{\text{red}} (S_{1})$ have a positive and negative one. Then in this case the structure of the expanded algebra is different from the cases above and indeed, taking $G = \mathfrak{sl}(2, \mathbb{R})$, the Killing–Cartan form of the expanded algebra is now
\[
g_{E,R, \text{red}} = \alpha \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \otimes (-8) \quad 0 \\ 0 & 4 \end{pmatrix}
\]
\[
= \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \otimes (8 \ 0 \\ 0 \ 8)
\]
which has four negative and two positive eigenvalues (taking into account that $\alpha$ is a positive number), so while semisimplicity will again be preserved, compactness properties will not. Let us see this explicitly.

Expanding explicitly with the semigroup $S_{(5)}^{968}$, we have
\[
G^{\text{red}}_{R, S, R} = (S_{0}^{\text{red}} \otimes T_{0}) + (S_{1}^{\text{red}} \otimes P_{0})
\]
\[
= \{ X_{(1,2)}, X_{(1,3)} \} + \{ X_{(2,4)}, X_{(2,5)}, X_{(3,4)}, X_{(3,5)} \}
\]
\[
= \{ \lambda_{2} \otimes (-i \sigma_{2}), \lambda_{3} \otimes (-i \sigma_{2}) \} + \{ \lambda_{3} \otimes \sigma_{1}, \lambda_{5} \otimes \sigma_{1}, \lambda_{4} \otimes \sigma_{3}, \lambda_{5} \otimes \sigma_{3} \}.
\]
Again the structure is very similar to the algebra (67) and (78), but obviously the algebra is completely different since the multiplication rules of the corresponding semigroups are different. Let us define
\[
Y_{1} = X_{(1,2)}, \quad Y_{4} = X_{(2,5)}
\]
\[
Y_{2} = X_{(1,3)}, \quad Y_{5} = X_{(3,4)}
\]
\[
Y_{3} = X_{(2,4)}, \quad Y_{6} = X_{(3,5)}.
\]
Using equation (3), the commutation relations of $\mathfrak{sl}(2, \mathbb{R})$ and the multiplication table of $S_{(5)}^{991}$ (given in (66)), we obtain the following algebra:
\[
[Y_{1}, Y_{5}] = -2Y_{5} \quad [Y_{2}, Y_{5}] = 2Y_{4}
\]
\[
[Y_{1}, Y_{6}] = -2Y_{6} \quad [Y_{2}, Y_{6}] = 2Y_{4}
\]
\[
[V_{1}, Y_{5}] = 2Y_{3} \quad [V_{2}, Y_{5}] = 2Y_{2}
\]
\[
[V_{1}, Y_{6}] = 2Y_{4} \quad [V_{2}, Y_{6}] = 2Y_{1}
\]
\[
[V_{2}, Y_{6}] = -2Y_{6} \quad [V_{4}, Y_{5}] = 2Y_{1}
\]
\[
[V_{2}, Y_{4}] = -2Y_{4} \quad [V_{4}, Y_{6}] = 2Y_{2}
\]
whose Killing–Cartan metric is given by

$$g^{E,R,\text{red}} = \begin{pmatrix}
-16 & 0 & 0 & 0 & 0 \\
0 & -16 & 0 & 0 & 0 \\
0 & 0 & 16 & 0 & 0 \\
0 & 0 & 0 & 0 & 16 \\
0 & 0 & 0 & 0 & 16
\end{pmatrix}$$

(97)

with eigenvalues $(-16, -16, -16, -16, 16)$ corresponding to four compact generators $\{Y_1, Y_2, Y_3 - Y_4, Y_5 - Y_6\}$ and two non-compact ones $\{Y_3 + Y_4, Y_5 + Y_6\}$. The expanded algebra is in this case $\mathfrak{su}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, the two commuting algebras being generated respectively by

$$\mathfrak{su}(2, \mathbb{R}) = \left\{ \frac{1}{2} (Y_1 - Y_2), \frac{1}{2} (Y_3 - Y_4), \frac{1}{2} (Y_5 - Y_6) \right\}$$

$$\mathfrak{sl}(2, \mathbb{R}) = \left\{ \frac{1}{2} (Y_1 + Y_2), \frac{1}{2} (Y_3 + Y_4), \frac{1}{2} (Y_5 + Y_6) \right\}.$$ 

(98)

Besides, from (60), (93) and (97) it is obtained again that $\alpha = 1/2$.

6. Summary

It was found that, under the action of the $S$-expansion procedure, some properties of the Lie algebras (like commutativity, solvability and nilpotency) are preserved, while others (like semisimplicity and compactness) are not in general. This was analyzed for all kinds of expansions that can be performed: for the direct product $G_S = S \otimes G$, for the resonant subalgebra $G_{S,R}$ and for the reduced algebras $G_{S,\text{red}}$ and $G_{S,R,\text{red}}$. The same analysis was done for the expansion of a general Lie algebra $G$ on its Levi–Malcev decomposition, $G = N \uplus S$.

These results are summarized in the following table:

| Properties preserved by the action of the $S$-expansion process | Original $G$ | Expanded $G_S$ | Resonant $G_{S,R}$ | Reduced $G_{S,\text{red}}$ |
|---------------------------------------------------------------|--------------|-----------------|-------------------|------------------------|
| Abelian                                                       | Abelian      | Abelian         | Abelian           |
| Solvable                                                      | Solvable     | Solvable        | Solvable          |
| Nilpotent                                                     | Nilpotent    | Nilpotent       | Nilpotent         |
| Compact                                                       | Arbitrary    | Arbitrary       | Arbitrary         |
| $G = S$                                                       | Arbitrary    | Arbitrary       | Arbitrary         |
| $G = N \uplus S$                                              | Arbitrary    | Arbitrary       | Arbitrary         |

Figure 2 illustrates the general scheme of the classification theory of Lie algebras and the arrows represent the action of the expansion method on this classification. Gray arrows are used when the expansion method maps algebras of one set on to the same set, i.e., they preserve some specific property. On the other hand, black arrows indicate that the expansion methods can map algebras of one specific set on to the same set and also can lead us outside the set, to an algebra that will have a Levi–Malcev decomposition.

21 Remember that the existence of the resonant subalgebra and of the reduced algebra are independent facts. As shown in figure 1, there are semigroups with resonant decomposition and with no zero element and vice versa. Also it can happen that both exist simultaneously.
Figure 2. Action of the expansion methods on the scheme of the theory of the classification of Lie algebras.

The Cartan decomposition of the expanded algebra was also obtained when the expansion preserves compactness. To check all these theoretical results, an example is provided by studying all the possible expansions of the semisimple algebra $\mathfrak{sl}(2, \mathbb{R})$ with semigroups of order up to 6. Finally, it was shown that a classification of semigroups in terms of the eigenvalues of its associated matrices $g^S$, $g^{S_0}(S_0)$, $g^{S_1}(S_1)$, $g^{\text{red}}(S_0)$, $g^{\text{red}}(S_0)$ is a useful tool to predict the semisimplicity and compactness properties of the expanded algebras.

Therefore, a procedure to answer the question: given two Lie algebras, can they be related by some expansion procedure? has mainly two steps.

(a) First we apply the theorems we have found in section 3. They are necessary conditions for the existence of the relation. If the relation is forbidden by some of those theorems, then the recipe finishes here.

(b) If the relation is not forbidden, then the relation can still exist or not. Then we have to study if one of the semigroups existing on each order $n = 2, \ldots, 6$ (those mentioned in section 2.2) with an appropriate resonant decomposition and maybe with a zero element can connect those algebras.

In this paper, we have mainly developed the step (a). On the other hand, part (b) can be performed by hand (as for example in [44]) but in more complicated cases it is necessary to use computer programs. A report with a set of computer programs (some of them used here to study expansions of $\mathfrak{sl}(2, \mathbb{R})$), more specifically a Java library, to study all the possible relations by means of an $S$-expansion between two arbitrary Lie algebras is a work in progress [45]. This library will give resonant subalgebras and reduced algebras for expansions performed with any semigroup and will also contain algorithms to classify the expansions through comparing, by means of isomorphisms or anti-isomorphisms, with those semigroups enumerated and characterized in section 2.2.

7. Comments and possible developments

The results used in this paper can be extended to study what happens to the Gauss and Iwasawa decompositions under the expansion procedure. Those decompositions play a crucial role in
the theory of representation of Lie algebras, so this study could be very useful for obtaining a general picture of the possible relations between arbitrary algebras, mainly in those used in physics.

As shown in the table of section 4.2, the fraction of semigroups preserving semisimplicity is very small (on each level of the expansion procedure) and expansions performed with those semigroups are, in general, direct sums of the original ones. Therefore, those expansions can be less interesting from the point of view of obtaining fundamentally new objects. However, there is an interesting observation that can be made here:

For expansions with semigroup of order \( n = 3, 4, 5 \), if semisimplicity is preserved, then the expanded algebra always contains the original algebra as a subalgebra.

This made us conjecture that there is no expansion procedure (at least an expansion procedure that just uses semigroups with the conditions considered here) that permits us to relate the simple Lie algebras (the classical Lie algebras \( A_n, B_n, C_n, D_n \) and the special ones) to each other. A study to prove this conjecture and some new physical applications with these results (particularly in gauge CS theories of gravity) is a work in progress (see [45]).

Apart from this fact, for physical applications it seems to be more interesting to use those semigroups that broke semisimplicity in the sense that they permit us to obtain fundamentally new objects. This is the case of two examples already mentioned in the introduction: (a) the non-semisimple M superalgebra, which is obtained as an expansion of the semisimple \( \text{osp}(32/1) \) superalgebra and (b) the non-semisimple \( \mathfrak{B} \) algebra obtained as an expansion of the AdS algebra. In both cases the original algebra is not a subalgebra of the expanded one, and their structure is richer than the expanded algebras obtained with semigroups preserving semisimplicity. In [61], we propose a method to obtain all the semigroups, that in the same way for the \( \mathfrak{B} \) algebra, permits us to obtain standard general relativity as a special limit of a CS theory in five dimensions.

On the other hand, as mentioned in [17], it would be interesting to know whether \( S \)-expansions fit the classification of solvable Lie algebras of a fixed dimension by means of \( S \)-expansions of simple (semisimple) Lie algebras of the same dimension. The theoretical results proposed in this work could be useful in solving that problem.

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Appendix A. Reducible representations

Let us consider a semisimple Lie algebra \( \mathcal{G} \) with a decomposition \( \mathcal{G} = \mathcal{H} + \mathcal{K} \) where \( \mathcal{H} \) is a subalgebra. Then \( \mathcal{G}/\mathcal{H} \) is a representation of \( \mathcal{H} \). Besides, if \( \mathcal{H} \) is semisimple, then \( \mathcal{G} \) is a completely reducible representation with respect to the adjoint action of \( \mathcal{H} \) and therefore

\[
[\mathcal{H}, \mathcal{K}] \subset \mathcal{K}.
\] (A.1)
If we use the indices \(i, j\) for \(H\) and \(a, b\) for \(K\) then we have that the Killing–Cartan metric of \(G\) restricted to \(H\) is given by

\[
 g_{ij}^G = C_{ik}^b c_{kjl}^b + C_{ib}^a c_{ajl}^a,
\]

(A.2)

where (A.1) was used, while the Killing–Cartan metric for \(H\) is given by

\[
 g_{ij}^H = C_{ik}^b c_{kjl}^b.
\]

(A.3)

Considering the adjoint action in \(K\) of a generator in \(H\),

\[
 C_{ub}^a c_{jb} = \text{Tr} (T^K_i T^K_j) = \alpha^K g_{ij}^H,
\]

where the last equality comes from the fact that for whatever matrix representation of the generators of \(H\), \(T^K\), the symmetric invariant form constructed as \(\text{Tr}(T^K_i T^K_j)\) must be proportional to \(g_{ij}^H\). The reason is that \(g_{ij}^H\) is the only symmetric 2-index invariant form that exist for a semisimple Lie algebra \(H\). The constant \(\alpha^K\) depends on the dimensions \(d_H\) and \(d_K\) of \(H\) and \(K\) and is given by

\[
 \alpha^K = \frac{d_K c_K}{d_H},
\]

(A.4)

where \(c_K\) is the positive eigenvalue of the second Casimir operator, relative to the representation \(K\) of \(H\) (see [60]: Gilmore, example 13 page 276). Therefore it is direct to see that

\[
 g_{ij}^G = \left(1 + \frac{d_K c_K}{d_H}\right) g_{ij}^H.
\]

Appendix B. Expansion of a general Lie algebra

Here we give the proof of theorem 4 of section 3.3 by showing that \(N_{\text{exp}} = E_N \uplus N'\) is the radical of the expanded algebra \(G_S\). Let us prove first that \(N_{\text{exp}}\) is an ideal:

\[
 [N_{\text{exp}}, N_{\text{exp}}] = [E_N \uplus N', E_N \uplus N']
\]

\[
 \subset [E_N, E_N] + [E_N, N'] + [N', N']
\]

\[
 \subset E_N + E_N + N' = N_{\text{exp}}
\]

\[
 [N_{\text{exp}}, S_{\text{exp}}] = [N_{\text{exp}}, S_{\text{exp}}] = [E_N \uplus N', S_{\text{exp}}]
\]

\[
 = [E_N, S_{\text{exp}}] + [N', S_{\text{exp}}]
\]

\[
 \subset E_N + N' = N_{\text{exp}}
\]

so \(N_{\text{exp}}\) is an ideal because

\[
 [N_{\text{exp}}, N_{\text{exp}}] \subset N_{\text{exp}}
\]

\[
 [N_{\text{exp}}, S_{\text{exp}}] \subset N_{\text{exp}}.
\]

Now, let us prove that \(N_{\text{exp}}\) is solvable. We see that

\[
 N_{\text{exp}} = E_N \uplus N' = \{X_{(i,s,a)}, X_{(i,s,a)}'\}
\]

where \(X_{(i,s,a)} \in E_N\) and \(X_{(i,s,a)'} \in N'\). We have to study the behavior of \(N^{(n)}_{\text{exp}}\) defined by

\[
 N^{(n)}_{\text{exp}} = [N^{(n-1)}_{\text{exp}}, N^{(n-1)}_{\text{exp}}]
\]

where the set

\[
 N^{(n)}_{\text{exp}} = \{X_{(i,s,a)^{(n)}}, X_{(i,s,a)'}^{(n)}\}
\]

28
has the following commutation relations:

\[
\begin{align*}
[X_{(j_{\gamma}^{(1)}, \alpha_{(\gamma)}^{(1)})}, X_{(j_{\beta}^{(1)}, \beta_{(\gamma)}^{(1)})}] &= C_{(j_{\gamma}^{(1)}, \beta_{(\gamma)}^{(1)})}^{(0)} K_{\alpha_{(\gamma)}^{(1)})(j_{\gamma}^{(1)})}^{(0)} X_{(j_{\beta}^{(1)}, \beta_{(\gamma)}^{(1)})} \\
\end{align*}
\]

(B.1)

\[
\begin{align*}
[X_{(j_{\gamma}^{(1)}, \alpha_{(\gamma)}^{(1)})}, X_{(j_{\beta}^{(1)}, \beta_{(\gamma)}^{(1)})}] &= C_{(j_{\gamma}^{(1)}, \beta_{(\gamma)}^{(1)})}^{(0)} K_{\alpha_{(\gamma)}^{(1)})(j_{\gamma}^{(1)})}^{(0)} X_{(j_{\beta}^{(1)}, \beta_{(\gamma)}^{(1)})} \\
\end{align*}
\]

(B.2)

\[
\begin{align*}
[X_{(j_{\gamma}^{(1)}, \alpha_{(\gamma)}^{(1)})}, X_{(j_{\beta}^{(1)}, \beta_{(\gamma)}^{(1)})}] &= C_{(j_{\gamma}^{(1)}, \beta_{(\gamma)}^{(1)})}^{(0)} K_{\alpha_{(\gamma)}^{(1)})(j_{\gamma}^{(1)})}^{(0)} X_{(j_{\beta}^{(1)}, \beta_{(\gamma)}^{(1)})} \\
\end{align*}
\]

(B.3)

For \(N_{\exp}^{(1)}\) we have

\[
N_{\exp}^{(1)} = [N_{\exp}^{(0)}, N_{\exp}^{(0)}] = [E_N, E_N] + [E_N, N'] + [N', N']
\]

\[
\subset E_N^{(1)} + N'(1) + E_N = [E_N^{(1)}, N'(1), E_N^{(0)}]
\]

where \(N_{\exp}^{(0)} = N_{\exp}, E_N^{(1)} = [E_N, E_N], N'(1) = [N', N']\) and we have used the fact that \([E_N, N'] \subset E_N\) as can be seen in equation (B.2). Then for \(N_{\exp}^{(2)}\) we have

\[
N_{\exp}^{(2)} = [N_{\exp}^{(1)}, N_{\exp}^{(1)}] = [E_N^{(1)} + N'(1) + E_N^{(0)}, E_N^{(1)} + N'(1) + E_N^{(0)}]
\]

\[
\subset [E_N^{(2)}, N_{\exp}^{(1)}, E_N^{(0,1)}]
\]

where \(E_N^{(2)} = [E_N^{(1)}, E_N^{(1)}], N_{\exp}^{(2)} = [N_{\exp}^{(1)}, N_{\exp}^{(1)}]\) and \(E_N^{(0,1)} = [E_N^{(0)}, N'(1)]\). Thus, as \(N'\) is solvable, there exists some \(m\) such that

\[
N_{\exp}^{(m)} = [E_N^{(m)}, E_N^{(m-1)}, E_N^{(m-2, m-1)}, \ldots, E_N^{(0)}] \subset E_N.
\]

On the other hand, as \(E_N\) is solvable, there exists some \(n\) such that \(E_N^{(n)} = 0\). Therefore, \(N_{\exp}\) is solvable too, because there exists a value \(p = m + n\) such that

\[
N_{\exp}^{(p)} = N_{\exp}^{(m+n)} = (N_{\exp}^{(m)})^{(n)} = E_N^{(n)} = 0.
\]

Finally, suppose \(N_{\exp} = E_N \oplus N'\) is not maximal. This means that there exists a generator \(X \in S_{\exp}\) such that \(N'_{\exp} = N_{\exp} \oplus \{X\}\) is a solvable ideal of \(G_S\). Define \(N'' = N' \oplus \{X\} \subset E_S\). We have that

\[
N'' = N'_{\exp} \cap E_S,
\]

(B.4)

so that \(N''\) is a solvable Lie subalgebra of \(E_S\). Moreover, if \(S' = S_{\exp} \oplus \{X\}\),

\[
G_S = N'_{\exp} \oplus S',
\]

(B.5)

so that, \(N'_{\exp}\) being an ideal by assumption, \([S', N'_{\exp}] \subset N'_{\exp}\). Restricting both sides to \(E_S\), we then have

\[
[S', N'_{\exp}] \subset N''\]

(B.6)

that is \(N'' = N' \oplus \{X\}\) is a solvable ideal of \(E_S\), which cannot be since \(N'\) is maximal in \(E_S\).

**Appendix C. Cartan decomposition under the S-expansion**

**C.1. The expanded semisimple algebra**

Here is given the proof of theorem 5 of section 3.4, i.e. that (44)–(47) is the Cartan decomposition of \(G_{0,S}\) when \(G_{2,S}\) is compact. This is done by providing a conjugation \(s_\sigma\) of \(G\) with respect to \(G_{0,S}\) such that

\[
s_\sigma(G_{2,S}) \subset G_{2,S}
\]

(C.1)
and then by showing the following relations:

\[ T_{0,S} = \mathcal{G}_{0,S} \cap \mathcal{G}_{k,S} \]  \hspace{1cm} (C.2)

\[ \mathcal{P}_{0,S} = \mathcal{G}_{0,S} \cap (i\mathcal{G}_{k,S}). \]  \hspace{1cm} (C.3)

Let us find how the explicit form of \( \sigma_S \) is found. Consider the elements \( A = d^i X_i \in \mathcal{G} \) and \( B = b^{(i,\alpha)} X_{i,\alpha} \in \mathcal{G}_S \) and let \( \{X_i\} = \{X_{i}^{(m)}, iX_{i}^{(m)}\}_{m=0}^{\dim \mathcal{G}_0} \) and \( \{X_{i,\alpha}\} = \{X_{i,\alpha}^{(m)}, iX_{i,\alpha}^{(m)}\}_{m=0}^{\dim \mathcal{G}_0} \) be respectively the bases of \( \mathcal{G} \) and \( \mathcal{G}_S \). Then, we can write

\[ A = d^i X_i = d^{(m)} X_i^{(m)} + i\tilde{d}^{(m)} X_i^{(m)} \]

\[ B = b^{(i,\alpha)} X_{i,\alpha} = b^{(m,\alpha)} X_{i}^{(m,\alpha)} + i\tilde{b}^{(m,\alpha)} X_{i}^{(m,\alpha)} \]  \hspace{1cm} (C.4)

where \( d^{(m)}, \tilde{d}^{(m)}, b^{(m,\alpha)} \) and \( \tilde{b}^{(m,\alpha)} \) are real constants. Let us also define a mapping \( \sigma_S : \mathcal{G}_S \to \mathcal{G}_S \) such that

\[ \sigma_S (\alpha B_1 + \beta B_2) = \bar{\alpha} \sigma_S (B_1) + \bar{\beta} \sigma_S (B_2), \forall B_1, B_2 \in \mathcal{G}_S, \forall \alpha, \beta \in \mathbb{C} \]

\[ \sigma_S (\alpha \otimes A) = \bar{\alpha} \otimes \sigma (A), \forall \alpha \in \mathbb{C}, \forall A \in \mathcal{G} \]  \hspace{1cm} (C.5)

where \( \sigma \) is the conjugation of \( \mathcal{G} \) with respect to \( \mathcal{G}_0 \) and where \( \bar{\alpha} \) denotes the complex conjugate of \( \alpha \). Then the mapping \( \sigma_S \) of an arbitrary element \( B \in \mathcal{G}_S \) can be expressed in terms of \( \sigma \) as follows:

\[ \sigma_S (B) = \sigma_S (b^{(i,\alpha)} X_{i,\alpha}) = \lambda_{\alpha} \otimes \sigma (b^{(m,\alpha)} X_{i}^{(m,\alpha)}) \]

\[ = \lambda_{\alpha} \otimes \sigma (b^{(m,\alpha)} X_{i}^{(m,\alpha)} + i\tilde{b}^{(m,\alpha)} X_{i}^{(m,\alpha)}) \]  \hspace{1cm} (C.6)

Then it is straightforward to show that \( \sigma_S \) is a conjugation of \( \mathcal{G}_S \) with respect to \( \mathcal{G}_0, \) i.e., that also satisfies

\[ \sigma_S [B_1, B_2] = [\sigma_S (B_1), \sigma_S (B_2)] \forall B_1, B_2 \in \mathcal{G}_S \]  \hspace{1cm} (C.1)

Now let us prove (C.1), i.e., \( \mathcal{G}_{k,S} \) is invariant under the conjugation (C.6). In fact, consider the action of \( \sigma_S \) on an arbitrary element \( K = k^{(i,\alpha)} X_{i}^{(i,\alpha)} \in \mathcal{G}_{k,S} : \)

\[ \sigma_S (k^{(i,\alpha)} X_{i}^{(i,\alpha)}) = k^{(i,\alpha)} \sigma_S (\lambda_{\alpha} \otimes X_{i}^{(i,\alpha)}) = k^{(i,\alpha)} \lambda_{\alpha} \otimes \sigma (X_{i}^{(i,\alpha)}) \]

\[ = k^{(i,\alpha)} \lambda_{\alpha} \otimes X_{i}^{(i,\alpha)} = k^{(i,\alpha)} \lambda_{\alpha} \otimes X_{i}^{(i,\alpha)} \]

\[ = k^{(i,\alpha)} X_{i}^{(i,\alpha)} \in \mathcal{G}_{k,S}, \]

where we have used (C.5) and \( \sigma (\mathcal{G}_k) \subset \mathcal{G}_k \). In this way, (C.1) is satisfied.

Now proving (C.2) is easy because

\[ T_{0,S} = \{X_{i}^{(m,\alpha)}\} = \{\lambda_{\alpha} \otimes X_{i}^{(m,\alpha)}\} \]

where by hypothesis \( X_{i}^{(m)} \in \mathcal{G}_0 \cap \mathcal{G}_k \). Then \( X_{i}^{(m,\alpha)} \) is in \( \mathcal{G}_{0,S} \) and also in \( \mathcal{G}_{k,S} \), i.e., \( X_{i}^{(m,\alpha)} \in \mathcal{G}_{0,S} \cap \mathcal{G}_{k,S} \); therefore, (C.2) is true.

On the other hand, to prove (C.3) we just have to note that

\[ \mathcal{P}_{0,S} = \{X_{i}^{(m,\alpha)}\} \in \{\lambda_{\alpha} \otimes X_{i}^{(m,\alpha)}\} \]

where \( X_{i}^{(m)} \in \mathcal{G}_0 \cap (i\mathcal{G}_k) \). Then \( X_{i}^{(m,\alpha)} \) is in \( \mathcal{G}_{0,S} \) and also in \( i\mathcal{G}_{k,S} \), i.e., \( X_{i}^{(m,\alpha)} \in \mathcal{G}_{0,S} \cap (i\mathcal{G}_{k,S}) \) so we have also proven (C.3).
C.2. The semisimple resonant subalgebra

Here we give the proof of theorem 6 of section 3.4. We have to find a compact real form, $G_{k,S,R}$, of $G_{k,R}$ (the complex form of $G_{0,S,R}$) satisfying the conditions (41), that in this case reads as

$$\sigma_{S,R} (G_{k,S,R}) \subset G_{k,S,R} \quad \text{and}$$

$$T_{0,S,R} = G_{0,S,R} \cap \overline{G_{k,S,R}}$$

$$P_{0,S,R} = G_{0,S,R} \cap (\overline{G_{k,S,R}})$$

where $\sigma_{S,R}$ is a conjugation in $G_{k,S,R}$ with respect to $G_{0,S,R}$.

As we saw before, the expansion of the compact algebra $G_k$, $G_k = S \otimes G_k$, (C.10) with

$$G_k = T_0 + iP_0$$

is compact when $\xi_k > 0$. Besides, $G_k$ satisfies the resonant condition, as can be seen in (42) and (43), so

$$G_{k,S,R} = (S_0 \otimes T_0) + (S_1 \otimes iP_0)$$

is the resonant subalgebra of $G_{k,S}$ and it is compact because it is a subalgebra of a compact Lie algebra, $G_{k,S}$.

Let us prove now that (C.7)–(C.9) are satisfied. Considering that

$$\{X_{\alpha} \}_{\alpha = 1}^{\dim \mathfrak{g}_0} \quad \{X_{\alpha}^{(m)} \}_{\alpha = 1}^{\dim T_0} \quad \{X_{\alpha}^{(m)} \}_{\alpha = 1}^{\dim P_0}$$

are respectively bases of $G$, $T_0$ and $P_0$ we have that an arbitrary element in $G = T_0 + P_0$ can be written as follows:

$$A = a^0 X_i = a^{(m)} X_{\alpha} + i\tilde{a}^{(m)} X_{\alpha}$$

$$= a^{(m)} X_{\alpha}^{(m)} + i\tilde{a}^{(m)} X_{\alpha}^{(m)} \quad \text{with} \quad r = 0, 1,$$

where a sum on $r = 0, 1$ is also assumed. Then an arbitrary element on $G_{k,S,R}$ can be written as

$$B = \sum_{r=0,1} (b^{(m)} X_{\alpha}^{(m)} + i\tilde{t}^{(m)} X_{\alpha}^{(m)}),$$

so the conjugation $\sigma_S$, defined before, acting on this element gives

$$\sigma_S (B) = \sum_{r=0,1} \lambda_{\alpha} \otimes \sigma (b^{(m)} X_{\alpha}^{(m)} + i\tilde{t}^{(m)} X_{\alpha}^{(m)}), \quad \forall B \in G_{k,S,R}$$

for $r = 0, 1$. Now let us consider an arbitrary element $K \in G_{k,S,R}$, i.e.,

$$K = k^{(i)}(\text{a}_0)X_{\alpha_{\text{a}_0}}^{(i)} + \tilde{k}^{(i)}(\text{a}_1)X_{\alpha_{\text{a}_1}}^{(i)}$$

where $i^{(k)}$ and $i^{(k)}$ are indices living on $G_{k,T_0} = G_k \cap T_0$ and $G_k \cap P_0$, respectively. Then,

$$\sigma_S (K) = \tilde{k}^{(i)}(\text{a}_0) \sigma (X_{\alpha_{\text{a}_0}}^{(i)}) + \tilde{k}^{(i)}(\text{a}_1) \sigma (X_{\alpha_{\text{a}_1}}^{(i)})$$

and as

$$\lambda_{\text{a}_0} \otimes \sigma_S (X_{\alpha_{\text{a}_0}}^{(i)}) \subset S_0 \otimes T_0$$

$$\lambda_{\text{a}_1} \otimes \sigma_S (X_{\alpha_{\text{a}_1}}^{(i)}) \subset S_1 \otimes (iP_0)$$

we have that

$$\sigma_S (K) \subset (S_0 \otimes T_0) + (S_1 \otimes (iP_0)) = G_{k,S,R},$$

so (C.7) is proved to be true. In the same way is it possible to show (C.8) and (C.9).
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