Abstract. The main result: for every sequence \( \{\omega_m\}_{m=1}^{\infty} \) of positive numbers (\( \omega_m > 0 \)) there exists an isometric embedding \( F : [0,1] \rightarrow L_1[0,1] \) which is nowhere differentiable, but for each \( t \in [0,1] \) the image \( F_t \) is infinitely differentiable on \([0,1]\) with bounds \( \max_{x \in [0,1]} |F_t^{(m)}(x)| \leq \omega_m \) and has an analytic extension to the complex plane which is an entire function.

Keywords: Differentiability of a Banach space valued map, isometric embedding, Lipschitz map

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1 Introduction

It is well-known that there exist nowhere differentiable Lipschitz maps \( F : [0,1] \rightarrow L_1[0,1] \) (in this paper we denote the function in \( L_1 \) corresponding to \( t \in [0,1] \) by \( F_t \)). Apparently the first and the simplest example of such a map was constructed by Clarkson \([7, \text{p. 405}]\) (earlier an example with the same property in \( \ell_\infty[0,1] \) was constructed by Bochner \([2]\)). Clarkson's example is \( F_t = 1_{[0,t]} \), where \( 1_A \) is the indicator function of \( A \), that is:

\[
1_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{otherwise.}
\end{cases}
\]

It is worth mentioning that the map constructed by Clarkson is not only a Lipschitz map, but is an isometric embedding.

So the analogue of the Lebesgue Theorem (see \([12, \text{p. 11}]\)) fails for Lipschitz maps from \([0,1]\) to \( L_1[0,1] \). Later the study initiated by Bochner and Clarkson developed into a direction in Banach space theory devoted to study Banach spaces for which such phenomena occur. Recall that one of the equivalent definitions of a Banach spaces with the Radon-Nikodým property is: A Banach space \( X \) has the Radon-Nikodým property if and only if every Lipschitz map \( F : \mathbb{R} \rightarrow X \) is differentiable almost everywhere. The corresponding part of the Banach space theory is reflected in many monographs, see \([1, \ldots]\).
Chapter 5], [3], [4], [8], [11, Chapter 2]. Radon-Nikodým property plays an important role in the theory of metric embeddings, see [5, 6, 10].

This paper is motivated by a question of Alessio Figalli [9] who asked if one may hope for a differentiability result for Lipschitz maps $F : [0, 1] \to L_1[0, 1]$ after adding some smoothness hypothesis on the functions $F_t = F_t(x)$ as functions of $x$ (for each fixed $t$). His main interest was the case where we suppose that $F_t(x)$ is 1-Lipschitz in $x$ for every $t$. Unfortunately the answer to this question is negative even under substantially stronger assumptions. The corresponding example is the main result of this paper. Namely, we prove the following theorem.

**Theorem 1.1.** Let $\{\omega_m\}_{m=1}^\infty$ be a sequence of positive numbers ($\omega_m > 0$). Then there exists an isometric embedding $F : [0, 1] \to L_1[0, 1]$ such that

(A) For each $t \in [0, 1]$ the function $F_t(x)$ is infinitely differentiable on $[0, 1]$ with bounds

$$\max_{x \in [0, 1]} \left| F_t^{(m)}(x) \right| \leq \omega_m$$

and has an analytic extension to the complex plane which is an entire function.

(B) The map $t \mapsto F_t$ does not have points of differentiability in $[0, 1]$.

**2 Proof of Theorem 1.1**

The idea of the proof can be described in the following way: we are going to join the function $F_0 = 0$ (identically 0 on $[0, 1]$) with the function $F_1 = 1$ (identically 1 on $[0, 1]$) with a 1-Lipschitz curve in $L_1[0, 1]$ consisting of functions satisfying the condition (A) in such a way that this curve “c-changes direction” at binary points of infinitely many “generations” with the same $c > 0$. The map $F : [0, 1] \to L_1[0, 1]$ which we construct is not only 1-Lipschitz, but is also an isometric embedding.

By a binary point in $[0, 1]$ we mean a point representable by a fraction $\frac{i}{2^n}$, where $n$ is a nonnegative integer and $i$ is an integer satisfying $0 \leq i \leq 2^n$. The canonical form of a positive binary point is $\frac{i}{2^n}$, where $i$ is odd. The generation of a binary point is the number $n$ in its canonical form, and we define that 0 is of generation 0.

We say that $F$ “c-changes direction” at a binary point $i/2^n$ of generation $n \geq 1$ if

$$\| F_{(i-1)/2^n} + F_{(i+1)/2^n} - 2F_i/2^n \|_{L_1} \geq c2^{-n}. \quad (1)$$

Using a version of a well-known argument we show that if the “c-change of direction” condition is satisfied at infinitely many generations of binary points, then the map does not have points of differentiability, see Lemma 2.3.

The main content of the construction is to show that the “c-change of direction” condition at sufficiently rare generations of binary points is compatible with $F_t$ being infinitely continuously differentiable functions with suitable bounds on the restrictions of derivatives to $[0, 1]$, and having analytic extensions to the complex plane. First we describe how to find the functions $F_t$ for all binary $t$. These functions will be chosen to be polynomials. Their construction is done generation by generation. It is easy to see from the construction below that the map $t \mapsto F_t$ is monotone in the following sense:

$$0 \leq s \leq t \leq 1 \implies 0 \leq F_s(x) \leq F_t(x) \leq 1 \quad \text{for all } x \in [0, 1]. \quad (2)$$

For this reason the notion of the “region bounded by the graphs of $F_{\frac{1}{2^n}}$ and $F_{\frac{1}{2^n}}$” which we use below is well-defined.
The property that $t \mapsto F_t$ is an isometric embedding is achieved by choosing the image of the binary point $\frac{i}{2^n}$ (where $1 \leq i \leq 2^n - 1$ and $i$ is odd) of generation $n$ to be mapped onto a function $F_{\frac{i}{2^n}}$ whose graph divides the region bounded by the graphs of $F_{\frac{i-1}{2^n}}$ and $F_{\frac{i+1}{2^n}}$ (observe that $\frac{i-1}{2^n}$ and $\frac{i+1}{2^n}$ belong to previous generations) into two parts of equal area. It is clear that the continuous extension of this map to non-binary points will be an isometric embedding of $[0, 1]$ into $L_1[0, 1]$.

The condition that functions $F_t$ satisfy (A) is achieved in the following way: For several (usually very many) consecutive generations of binary points we define $F_{\frac{i}{2^n}}$ (where $\frac{i}{2^n}$ is of generation $n$) by

$$F_{\frac{i}{2^n}} = \frac{1}{2} \left( F_{\frac{i-1}{2^n}} + F_{\frac{i+1}{2^n}} \right). \quad (3)$$

Note that since $(i-1)$ and $(i+1)$ are even, the functions on the right hand side are defined earlier. We show that if we use the definition (3) for sufficiently many generations of binary points, then for the immediately following generation we can define $F_{\frac{i+1}{2^n}}$ in such a way that it satisfies both the "c-change of direction" condition and the conditions which lead to bounds for the derivatives and the existence of an analytic extension.

2.1 Plan for estimating the derivatives and showing the analyticity

The bounds on the derivatives of $F_t$ and analyticity are obtained as follows. For each point $t \in [0, 1]$ we find a sequence $\{p(r, t)\}_{r=1}^{\infty}$ of binary points converging to $t$ such that the following conditions hold:

1. The series

$$F_{p(1,t)} + \sum_{r=1}^{\infty} \left( F_{p(r+1,t)} - F_{p(r,t)} \right) \quad (4)$$

converges uniformly on each bounded disc in the complex plane (this statement has a natural meaning - recall that $F_{\frac{i}{2^n}}$ are polynomials). It is clear that restrictions of the terms of this series to $[0, 1]$ form a series which converges to $F_t$, and so this condition together with the fact that $F_{\frac{i}{2^n}}$ are polynomials, implies that $F_t$ has an analytic extension to $\mathbb{C}$.

2. The derivatives of $\{F_{p(r,t)}\}_{r=1}^{\infty}$ satisfy

$$|F_{(m)p(1,t)}(x)| = 0 \quad \forall x \in [0, 1], \quad (5)$$

$$|F_{(m)p(r+1,t)}(x) - F_{(m)p(r,t)}(x)| \leq \frac{\omega_m}{2^r} \quad \forall x \in [0, 1]. \quad (6)$$

As is well-known, (5) and (6) together with the uniform convergence of (4) imply that $|F_t^{(m)}(x)| \leq \omega_m$ for all $m \in \mathbb{N}$ and $x \in [0, 1]$. (In more detail, we repeatedly use the statement: if the series $\sum u'$ of derivatives of a uniformly convergent series $\sum u$ is uniformly convergent, then $\sum u'$ converges to the derivative of the $(\sum u)'$.)

A similar argument in a different context was used by Bernstein in Constructive Function Theory, see [13, Theorem 13.20 on p. 119].
2.2 Construction of an example

We start by letting \( F_0 = 0 \) (identically 0 on \([0, 1]\)) and \( F_1 = 1 \) (identically 1 on \([0, 1]\)). After that we define polynomials (with real coefficients) \( F_t \) for all other binary points \( t \) in \([0, 1]\) generation by generation. As will be easy to see from our construction, the map \( t \mapsto F_t \) is monotone in the sense of (2).

We define \( F_{\pm t} \) at a point \( \frac{1}{2n} \) of generation \( n \) so that its graph bisects the region between the graphs of \( F_{\pm 1} \) and \( F_{\pm 1} \) into two regions of equal area. This will ensure the isometry condition.

Specifically, for every \( \frac{1}{2n} \) of generation \( n \) we employ a polynomial \( \eta_{\pm t} : [0, 1] \to [0, 1] \) to define

\[
F_{\pm t} = \eta_{\pm t} F_{\pm \frac{1}{2}} + \left( 1 - \eta_{\pm t} \right) F_{\pm \frac{1}{2}},
\]

so that the graph of the restriction of \( F_{\pm t} \) to \([0, 1]\) splits the region between the graphs of the restrictions of \( F_{\pm 1} \) and \( F_{\pm 1} \) to \([0, 1]\) into two regions of equal area. For many generations of binaries, these polynomials are constant: \( \eta_{\pm t}(x) = \frac{1}{2} \). In this case (7) is just (3). However, with this choice of \( \eta \) there is no change of direction at \( \frac{1}{2n} \) since in this case

\[
||F_{(i-1)/2^n} + F_{(i+1)/2^n} - 2F_{i/2^n}||_{L_1} = 0.
\]

To introduce the “c-change of direction” (with \( c = 1 \)) at an infinite sequence of generations \( \{n_r\}_{r=1}^\infty \) we construct functions \( \eta \) with the help of the following Lemma.

**Lemma 2.1.** Let \( G \) be a polynomial of degree \( d \geq 0 \) with \( G(x) \geq 0 \) for all \( x \in [0, 1] \). Then there exists a polynomial \( U \) of degree \((d + 1)\) with \( U(x) \in [0, 1] \) for all \( x \in [0, 1] \) such that the degree \((2d + 1)\) polynomials \( \tilde{G} := UG \) and \( \tilde{G} := (1 - U)G \) satisfy

\[
||\tilde{G}||_{L_1} = ||\tilde{G}||_{L_1} = \frac{1}{2}||G||_{L_1}, \quad \text{and} \quad ||\tilde{G} - \tilde{G}||_{L_1} = \frac{1}{2}||G||_{L_1}.
\]

**Proof.** Assume first that \( ||G||_{L_1} = 1 \), i.e. \( G \) is the density of a probability distribution on \([0, 1]\). Then the cumulative distribution function \( U(x) := \int_0^x G(t) \, dt \) is a polynomial of degree \((d + 1)\) that maps \([0, 1]\) onto \([0, 1]\). Since both \( U \) and \( G \) are nonnegative on \([0, 1]\) we have

\[
||\tilde{G}||_{L_1} = \int_0^1 U(x)G(x) \, dx = \frac{1}{2} \int_0^1 d \frac{dx}{dx} U^2(x) \, dx = \frac{1}{2} U^2(1) - \frac{1}{2} U^2(0) = \frac{1}{2}.
\]

Similarly,

\[
||\tilde{G}||_{L_1} = \int_0^1 (1 - U(x))G(x) \, dx = \int_0^1 G(x) \, dx - \int_0^1 U(x)G(x) \, dx = \frac{1}{2}.
\]

Note that since \( G \) is a polynomial, there exists a unique \( m \in (0, 1) \) such that

\[
U(m) = \int_0^m G(x) \, dx = \int_0^1 G(x) \, dx = \frac{1}{2}.
\]
Then
\[
||\tilde{G} - \bar{G}||_{L_1} = \int_0^1 |1 - 2\mathcal{U}(x)|G(x)\ dx = \int_0^m (1 - 2\mathcal{U}(x))G(x)\ dx
\]
\[
+ \int_m^1 (-1 + 2\mathcal{U}(x))G(x)\ dx = \int_0^m G(x)\ dx - \int_0^m \frac{d}{dx}\mathcal{U}^2(x)\ dx
\]
\[
- \int_m^1 G(x)\ dx + \int^1 m \frac{d}{dx}\mathcal{U}^2(x)\ dx = \frac{1}{2} - \frac{1}{4} - \frac{1}{2} + 1 - \frac{1}{4} = \frac{1}{2}.
\]

In the case where \(||G||_{L_1} > 0\) is not necessarily one, we apply the previous construction to the probability density \(\frac{G}{||G||_{L_1}}\) on \([0,1]\).

\[\square\]

At certain, rather rare sequence \(\{n_r\}_{r=1}^\infty\) of generations, instead of defining \(F_{\frac{1}{2^n}}\) using (3) we do the following. Let \(i\) be odd, \(1 \leq i \leq 2^{n_r} - 1\). Define
\[
G := F_{\frac{i+1}{2^n}} - F_{\frac{i-1}{2^n}}.
\]
Then \(||G||_{L_1} = \frac{2}{2^n}\). Since \(\frac{i-1}{2^n}\) and \(\frac{i+1}{2^n}\) are binary of generations strictly smaller than \(n_r\), \(F_{\frac{i+1}{2^n}}\) and \(F_{\frac{i-1}{2^n}}\) are already defined polynomials, and the monotonicity condition (2) guarantees that \(G\) is nonnegative on \([0,1]\). We apply Lemma 2.1 to \(G\) to obtain \(\mathcal{U}\) and we use \(\eta_{\frac{1}{2^n}} := \mathcal{U}\) in the bisection formula (7) to get
\[
F_{\frac{i}{2^n}} = \mathcal{U}F_{\frac{i-1}{2^n}} + (1 - \mathcal{U}) F_{\frac{i+1}{2^n}}.
\]
We make the following observation.

**Observation 2.2.** Note that
\[
||F_{\frac{i-1}{2^n}} + F_{\frac{i+1}{2^n}} - 2F_{\frac{i}{2^n}}||_{L_1} = ||\mathcal{U}G - (1 - \mathcal{U})G||_{L_1} = \frac{1}{2}||G||_{L_1} = 2^{-n_r},
\]
i.e. at \(t = i/2^n\) there is a change of direction (cf (1)) in \(F\) with \(c = 1\).

### 2.3 Details

First we use Lemma 2.1 to split \(G_1 := F_1 - F_0\) and denote the obtained function by \(\mathcal{U}_1\) (it is clear that \(\mathcal{U}_1(x) = x\), so this step is somewhat trivial, we discuss it in detail for uniformity as it is quite similar to all further steps). Denote by \(D_R\) the centered at 0 disc of radius \(R\) in \(\mathbb{C}\). Since \(\mathcal{U}_1\) is a polynomial, there exists \(k_1 \in \mathbb{N}\) such that
\[
|\mathcal{U}_1^{(m)}(x)| \leq 2^{k_1-1}\omega_m \quad \text{and} \quad |(1 - \mathcal{U}_1)^{(m)}(x)| \leq 2^{k_1-1}\omega_m \quad \forall m \in \mathbb{N} \forall x \in [0,1],
\]
and
\[
\max\{|\mathcal{U}_1(z)|, |(1 - \mathcal{U}_1)(z)|\} \leq 2^{k_1-1} \quad \forall z \in D_1.
\]

We then use (3) to define \(F_i\) for the first \(k_1\) generations of binary points. Note that as \(F_0 = 0\) and \(F_1 = 1\), we get for all \(1 \leq n \leq k_1\), \(1 \leq i \leq 2^n\) that
\[
F_{\frac{i}{2^n}}(x) = \frac{i}{2^n} \quad \text{for all} \quad x \in [0,1],
\]
therefore
\[
F_{\frac{i}{2^n}}(x) - F_{\frac{i+1}{2^n}}(x) = \frac{1}{2^n} \quad \text{for all} \quad 1 \leq i \leq 2^n \quad \text{and} \quad x \in [0,1].
\]
For generation $n_1 := (k_1 + 1)$ we define $\eta_{\frac{1}{2^{n_1}}} := U_1$ and

$$F_{\frac{1}{2^{n_1}}} = \eta_{\frac{1}{2^{n_1}}} F_{\frac{1}{2^{n_1}}} + \left(1 - \eta_{\frac{1}{2^{n_1}}} \right) F_{\frac{1}{2^{n_1}}}$$

for every odd $1 \leq i \leq 2^{n_1} - 1$.

Since

$$F_{\frac{1}{2^{n_1}}} - F_{\frac{1}{2^{n_1}}} = \left(1 - \eta_{\frac{1}{2^{n_1}}} \right) \left( F_{\frac{1}{2^{n_1}}} - F_{\frac{1}{2^{n_1}}} \right)$$

and

$$F_{\frac{1}{2^{n_1}}} - F_{\frac{1}{2^{n_1}}} = \eta_{\frac{1}{2^{n_1}}} \left( F_{\frac{1}{2^{n_1}}} - F_{\frac{1}{2^{n_1}}} \right),$$

we obtain from (9) that for all $m \in \mathbb{N}$

$$|F_{\frac{1}{2^{n_1}}}(x) - F_{\frac{1}{2^{n_1}}}(x)| \leq 2^{k_1 - 1} \omega_m \frac{1}{2^{k_1}} = \frac{\omega_m}{2} \text{ for every } 1 \leq i \leq 2^{n_1} \text{ and } x \in [0, 1], \quad (12)$$

and from (10) that

$$|F_{\frac{1}{2^{n_1}}}(z) - F_{\frac{1}{2^{n_1}}}(z)| \leq 2^{k_1 - 1} \frac{1}{2^{k_1}} = \frac{1}{2} \text{ for every } z \in D_1. \quad (13)$$

In the next stage of the construction we use Lemma 2.1 to split all of the $2^{k_1 + 1} = 2^{n_1}$ functions $G_{\frac{1}{2^{n_1}}} := F_{\frac{1}{2^{n_1}}} - F_{\frac{1}{2^{n_1}}}$ with $1 \leq i \leq 2^{n_1}$. We denote the obtained $2^{n_1}$ functions by $U_{0,i}$, $1 \leq i \leq 2^{n_1}$.

Since $G_{\frac{1}{2^{n_1}}}$ and $U_{2,i}$ are polynomials, we can pick $k_2 \in \mathbb{N}$, such that for all $i \in \{1, \ldots, 2^{n_1}\}$

$$\max\{|(U_{2,i}G_{\frac{1}{2^{n_1}}})(m)(x)|, |((1 - U_{2,i})G_{\frac{1}{2^{n_1}}})(m)(x)|\} \leq 2^{k_2 - 2} \omega_m \forall m \in \mathbb{N} \forall x \in [0, 1], \quad (14)$$

and

$$\max\{|(U_{2,i}G_{\frac{1}{2^{n_1}}})(z)|, |((1 - U_{2,i})G_{\frac{1}{2^{n_1}}})(z)|\} \leq 2^{k_2 - 2} \forall z \in D_2. \quad (15)$$

Then we use (3) to define $F_i$ for the next $k_2$ generations of binary points. Let $n_2 := n_1 + k_2 + 1$. We obtain from (14) that for all $m \in \mathbb{N}$

$$|F_{\frac{1}{2^{n_2}}}(x) - F_{\frac{1}{2^{n_2}}}(x)| \leq 2^{k_2 - 2} \omega_m \frac{1}{2^{k_2}} = \frac{\omega_m}{2^2} \text{ for every } 1 \leq i \leq 2^{n_2} \text{ and } x \in [0, 1], \quad (16)$$

and from (15) that

$$|F_{\frac{1}{2^{n_2}}}(z) - F_{\frac{1}{2^{n_2}}}(z)| \leq 2^{k_2 - 2} \frac{1}{2^{k_2}} = \frac{1}{2^2} \text{ for every } z \in D_2. \quad (17)$$

Assuming the construction was done up to “c-changing direction” generation $n_{r-1} := k_1 + \cdots + k_{r-1} + r - 1$, we define $k_r$ as follows. We use Lemma 2.1 to split

$$G_{\frac{1}{2^{n_{r-1}}}} := F_{\frac{1}{2^{n_{r-1}}}} - F_{\frac{1}{2^{n_{r-1}}}}, \quad 1 \leq i \leq 2^{n_{r-1}},$$

denoting the obtained functions $U_{r,i}$ and finding $k_r \geq 1$ such that for all $1 \leq i \leq 2^{n_{r-1}}$,

$$\max\left\{|(U_{r,i}G_{\frac{1}{2^{n_{r-1}}}})(m)(x)|, |((1 - U_{r,i})G_{\frac{1}{2^{n_{r-1}}}})(m)(x)|\right\} \leq 2^{k_{r-1}} \omega_m \forall m \in \mathbb{N} \forall x \in [0, 1], \quad (18)$$
and
\[ \max \left\{ \left| (U_{r,i}G_{\frac{r}{2^r} - 1})(z) \right|, \left| ((1 - U_{r,i})G_{\frac{r}{2^r} - 1})(z) \right| \right\} \leq 2^{k_r - r} \quad \forall z \in D_r. \] (19)

Then we use (3) to define \( F_t \) for the next \( k_r \) generations of binary points. Let \( n_r := n_{r-1} + k_r + 1 \). We obtain from (18) that for all \( m \in \mathbb{N} \)
\[ |F_{\frac{r}{2^r}}^{(m)}(x) - F_{\frac{r}{2^r}}^{(m)}(x)| \leq 2^{k_r - r} \omega_m \frac{1}{2^{k_r}} = \frac{\omega_m}{2^r} \quad \text{for every } 1 \leq i \leq 2^n \text{ and } x \in [0, 1], \] (20)
and from (19) that
\[ |F_{\frac{r}{2^r}}(z) - F_{\frac{r}{2^r}}(z)| \leq 2^{k_r - r} \frac{1}{2^r} = \frac{1}{2^r} \quad \text{for every } z \in D_r. \] (21)

We continue in an obvious way and define \( F_t \) for all binary \( t \in [0, 1] \). As is mentioned, this defines \( F_t \in L_1[0, 1] \) for all \( t \in [0, 1] \).

Our next goal is to show that the condition (A) holds following the plan described in Section 2.1.

So for each point \( t \in [0, 1] \) we need to find a sequence \( \{p(r,t)\}_{r=1}^{\infty} \) of binary points converging to \( t \) and satisfying the conditions in Section 2.1. We do this as follows.

We let \( p(1,t) \) to be the closest to \( t \) binary number which corresponds to a constant function (this can be non-unique only if \( t \) is binary of generation \( n_1 \), the first “c-changing direction” generation). Here and later we resolve ties arbitrarily.

We let \( p(2,t) \) be the closest to \( t \) binary number of generation at most \( n_2 - 1 \) (this can be non-unique only if \( t \) is binary of generation \( n_2 \)).

So on, we let \( p(r,t) \) be the closest to \( t \) binary number of generation at most \( n_r - 1 \). In this way we define \( p(r,t) \) for all \( r \in \mathbb{N} \).

It suffices to show that for all \( r \geq 1 \)
\[ |F_{\frac{r}{2^r}}^{(m)}(x) - F_{\frac{r}{2^r}}^{(m)}(x)| \leq \frac{\omega_m}{2^r} \]
for all \( m \in \mathbb{N} \) and all \( x \in [0, 1] \), and
\[ |F_{\frac{r}{2^r}}(z) - F_{\frac{r}{2^r}}(z)| \leq \frac{1}{2^r} \quad \forall z \in D_r. \]

These conditions follow from (20) and (21), respectively, since \( F_{\frac{r}{2^r}}^{(m)} - F_{\frac{r}{2^r}}(x) \) is a multiple of \( F_{\frac{r}{2^r}}^{(m)} - F_{\frac{r}{2^r}}^{(m)} \) with coefficient in \([0, 1]\). Therefore the condition (A) is satisfied.

### 2.4 Nondifferentiability

**Lemma 2.3.** Suppose that a map \( F : [0, 1] \to L_1[0, 1] \) is such that there is \( c > 0 \) and an infinite sequence \( \{n_r\}_{r=1}^{\infty} \) of positive integers such that the condition (1) is satisfied for each \( n = n_r \) and each odd natural number \( i \leq 2^n \). Then the map \( F \) does not have points of differentiability.

This argument is essentially known, see [1, Theorem 5.21, (iii) \( \Rightarrow \) (i)]. For convenience of the reader we present it.
Proof. Assume that \( t_0 \in [0, 1] \) is a point of differentiability of \( F_t \) (with respect to \( t \)) and let \( D \in L_1[0, 1] \) be the corresponding derivative. Then
\[
\| F_{t_0+h} - F_{t_0} - hD \|_{L_1} = o(h) \quad \text{as} \quad |h| \downarrow 0 \quad \text{and} \quad t_0 + h \in [0, 1]. \tag{22}
\]
Since for every \( n \) fixed
\[
\bigcup_{i \text{ odd}} \left[ \frac{i-1}{2^n}, \frac{i+1}{2^n} \right] = [0, 1],
\]
there exists a sequence of binary numbers in canonical form \( i_n/2^n \) such that
\[
t_0 \in \left( \frac{i_n - 1}{2^n}, \frac{i_n + 1}{2^n} \right] \quad \text{for every} \quad n.
\]
Condition (22) implies
\[
2^n \| F_{i_n/2^n} - F_{t_0} - \left( \frac{i_n \pm 1}{2^n} - t_0 \right) D \|_{L_1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
and
\[
2^n \| F_{i_n/2^n} - F_{t_0} - \left( \frac{i_n}{2^n} - t_0 \right) D \|_{L_1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
By the triangle inequality we obtain
\[
2^n \| F_{i_n/2^n} + F_{i_n/2^n} - 2F_{i_n/2^n} \|_{L_1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
However, this contradicts the fact that \( F \) “c-changes direction” at every \( \frac{i_n}{2^n} \).

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