INITIAL-BOUNDARY VALUE PROBLEM OF THE NAVIER-STOKES SYSTEM IN THE HALF SPACE

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ABSTRACT. In this paper, we study the initial-boundary value problem of the Navier-Stokes system in the half space. We prove the unique solvability of the weak solution on some short time interval $(0, T)$ with the velocity in $C^\alpha_\mathcal{F} (\mathbb{R}^n_+ \times (0, T))$, $0 < \alpha < 1$, when the initial given data is in $C^\alpha (\mathbb{R}^n_+)$ and the given boundary data is in $C^{\alpha, \frac{n}{2}} (\mathbb{R}^{n-1} \times (0, T))$. Our result generalizes the result in [30] considering nonhomogeneous Dirichlet boundary data.

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1. Introduction

Let $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n | x_n > 0 \}$, $n \geq 2$ and $0 < T < \infty$. In this paper, we consider the following initial-boundary value problem of the Navier-Stokes system in $\mathbb{R}^n_+ \times (0, T)$:

\[
\begin{align*}
    u_t - \Delta u + \nabla p &= -\text{div} (u \otimes u), & \text{div} u &= 0, & \text{in} \quad \mathbb{R}^n_+ \times (0, T), \\
    u|_{t=0} &= h, & u|_{x_n=0} &= g,
\end{align*}
\]

where $u = (u_1, \cdots, u_n)$ and $p$ are unknown velocity and the pressure, respectively, and $g = (g_1, \cdots, g_n)$, $h = (h_1, \cdots, h_n)$ are the given data.

In this paper, we show the unique solvability of the Navier-Stokes system (1.1) with initial and boundary data in anisotropic Besov spaces. The following states the main result of this paper.

**Theorem 1.1.** For $0 < \alpha < 1$ and $T > 0$, let $h \in C^\alpha (\mathbb{R}^n_+)$, $g \in C^{\alpha, \frac{n}{2}} (\mathbb{R}^{n-1} \times (0, T))$. We assume that

\[
g|_{t=0} = h|_{x_n=0}, \quad \text{div} h = 0, \quad R' g_n \in L^\infty (\mathbb{R}^{n-1} \times (0, T)), \quad R' h_n \in L^\infty (\mathbb{R}^n_+)
\]

where, $R' = (R'_1, \cdots, R'_{n-1})$ is the $n - 1$ dimensional Riesz operator. We also assume that there is $\tilde{h} \in C^\alpha (\mathbb{R}^n)$ an extension of $h$ to $\mathbb{R}^n$ satisfying that $\text{div} \tilde{h} = 0$, $R' h_n \in L^\infty (\mathbb{R}^n)$. Then, there is $T^*$ ($0 < T^* < T$) such that the Navier-Stokes system (1.1) has a weak solution $u \in C^{\alpha, \frac{n}{2}} (\mathbb{R}^n_+ \times (0, T^*))$ with appropriate distribution $p$. Moreover, $u$ is a unique in the class $C^{\alpha, \frac{n}{2}} (\mathbb{R}^n_+ \times (0, T^*))$.

There are abundant literature for the solvability of the Navier-Stokes system (1.1) when $g = 0$. When $h \in C^s (\mathbb{R}^n_+)$ for $s > 2$, V.A. Solonnikov [30] showed the local in time existence of the unique solution $u \in C^{s, \frac{n}{2}} (\mathbb{R}^n_+ \times (0, T))$. See also [10]. In [32], he also showed the local in time existence of the unique solution $u \in C (\mathbb{R}^n_+ \times (0, T))$ when $h \in C (\mathbb{R}^n_+)$. In [25], P. Maremonti showed the unique existence of classical solution of the Navier-Stokes system when the initial data
is nonconvergent at infinity. Theorem 1.1 generalizes the solvability result in \[30\] to a nonzero boundary data \( g \in C^{\alpha,\frac{3}{2}}(\mathbb{R}^{n-1} \times (0, \infty)) \) for \( 0 < \alpha < 1 \).

Navier-Stokes system in the half space has been studied mostly in \( p \)-frame work (Here \( p \)-frame work means function spaces such as \( L^p \)'s, \( W^{k,p} \)'s, \( 1 < p < \infty \), or their interpolation spaces, and \( \infty \)-frame work means such as \( L^{\infty} \)'s, \( W^{k}_{\infty} \)'s, or their interpolation spaces). See \[2, \ 7, \ 22, \ 30\] and references therein for the solvability of the Navier-Stokes system in the half space with homogeneous boundary data, that is, with \( g = 0 \). See \[1, \ 3, \ 4, \ 24, \ 41\] and references therein for the solvability of the Navier-Stokes system in the half space with the nonhomogeneous boundary data, that is, with \( g \neq 0 \).

There are many literatures for the study of the Navier-Stokes system in other domain such as whole space, a bounded domain, or exterior domain (with homogeneous or nonhomogeneous boundary data). If we mention papers using \( \infty \)-frame work, see \[16, \ 20, \ 23, \ 28, \ 2, \ 33\] and the references therein. If we mention papers using \( p \)-frame work, see \[2, \ 3, \ 4, \ 11, \ 12, \ 13, \ 15, \ 18, \ 19\] and the references therein.

Although the unsteady Navier-Stokes equations with low regular boundary data have been studied in several papers such as \[1, \ 3, \ 4, \ 11, \ 12, \ 13\], etc, we are interested in finding optimal regularity (in space-time) of the solution corresponding to the given data. As a first step we consider the Hölder continuous Dirichlet boundary data. In our forthcoming paper we would like to consider optimal regularity (in space-time) of the solution when low regular boundary data is given.

For the proof of Theorem 1.1, it is necessary to study the initial-boundary value problem of the Stokes system in \( \mathbb{R}^n_+ \times (0, T) \):

\[
\begin{align*}
  u_t - \Delta u + \nabla p &= f, & \text{div } u &= 0, & \text{in } \mathbb{R}^n_+ \times (0, T), \\
  u|_{t=0} &= h, & u|_{\partial \mathbb{R}^n_+} &= g.
\end{align*}
\]

(1.3)

The following states our result on the unique solvability of the Stokes system (1.3).

**Theorem 1.2.** For \( 0 < \alpha < 1 \) and \( T > 0 \), let \( h \in C^\alpha(\mathbb{R}^n_+) \), \( g \in C^{\alpha,\frac{3}{2}}(\mathbb{R}^{n-1} \times (0, T)) \) satisfy the same hypotheses as in Theorem 1.1. Moreover, we assume that \( f = \text{div } F \), where \( F \in C^{\alpha,\frac{3}{2}}(\mathbb{R}^n_+ \times (0, T)) \) with an extension \( \tilde{F} \in C^{\alpha,\frac{3}{2}}(\mathbb{R}^n \times (0, T)) \). Then, Stokes system (1.3) has a unique solution \( u \in C^{\alpha,\frac{3}{2}}(\mathbb{R}^n_+ \times (0, T)) \) with appropriate distribution \( p \) with

\[
\|u\|_{C^{\alpha,\frac{3}{2}}(\mathbb{R}^n_+ \times (0, T))} \leq C\left(\|h\|_{C^\alpha(\mathbb{R}^n_+)} + \max\{T^\frac{\alpha}{2}, T^\frac{\alpha}{2} + \frac{3}{2}\}\|F\|_{C^{\alpha,\frac{3}{2}}(\mathbb{R}^n_+ \times (0, T))} + \|g\|_{C^{\alpha,\frac{3}{2}}(\mathbb{R}^{n-1} \times (0, T))} + \|R^2 h_0\|_{L^\infty(\mathbb{R}^n)} + \|R^2 g_0\|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))}\right). \tag{1.4}
\]

When \( h \in C^\alpha(\mathbb{R}^3_+), f \in C^{\alpha-\frac{3}{2},\frac{3}{2}}(\mathbb{R}^3_+ \times (0, T)) \) and \( g \in C^{\alpha,\frac{3}{2}}(\mathbb{R}^2 \times (0, T)) \) for \( s > 2 \), V.A. Solonnikov \[31\] showed that there is a unique solution of the Stokes system (1.3) so that

\[
\|u\|_{C^{\alpha,\frac{3}{2}}(\mathbb{R}^3_+ \times (0, T))} \leq C\left(\|h\|_{C^\alpha(\mathbb{R}^3_+)} + \|f\|_{C^{\alpha-\frac{3}{2},\frac{3}{2}}(\mathbb{R}^3_+ \times (0, T))} + \|g\|_{C^{\alpha,\frac{3}{2}}(\mathbb{R}^2 \times (0, T))} + \|R^3(D_t g_0)\|_{L^\infty(\mathbb{R}^3, C^{\alpha,\frac{3}{2}}(0, T))}\right).
\]
Theorem 1.2 generalizes the result of [31] to $0 < s < 1$. Our result could be compared with the result in [27], where $V^{s,2}(\Omega \times (0, T))$, $0 \leq s \leq 2$, has been considered as a solution spaces in a bounded domain (see [27] for the definition of $V^{s,2}(\Omega \times (0, T))$).

There are various literatures for the solvability of the Stokes system [12] with homogeneous boundary data, that is, with $g = 0$. See [17, 25, 26, 30, 32], and references therein. In particular, the following estimate is derived in [32]:

$$\|u\|_{L^\infty(\mathbb{R}_+^n \times (0, T))} \leq c \left( \|h\|_{L^\infty(\mathbb{R}_+^n)} + T^{\frac{1}{2}} \|F\|_{L^\infty(\mathbb{R}_+^n \times (0, T))} \right),$$

where $h \in C(\mathbb{R}_+^n)$ and $f = \text{div } F = (F_{kj})_{j,k=1}^n \in C(\mathbb{R}_+^n \times (0, T))$ with $\text{div } h = 0$, $h|_{x_n=0} = 0$, $(F_{n1}, \cdots, F_{nn})|_{x_n=0} = 0$. See also [17].

When $f = 0$ and $h = 0$, T.K. Chang and H.J. Choe [9] showed that

$$\|u\|_{L^\infty(\mathbb{R}_+^n \times (0, T))} \leq c \left( \|g\|_{L^\infty(\mathbb{R}_+^{n-1} \times (0, T))} + \|R^t g_n\|_{L^\infty(\mathbb{R}_+^{n-1} \times (0, T))} \right),$$

where $g \in L^\infty(\mathbb{R}_+^{n-1} \times (0, T))$, $R^t g_n \in L^\infty(\mathbb{R}_+^{n-1} \times (0, T))$, $g|_{t=0} = 0$. See also [34, 35].

We organized this paper as follows. In section 2 we introduce the notations and the function spaces such as anisotropic Besov spaces and the anisotropic Hölder spaces. In section 3 we consider Stokes system [12] with the homogeneous external force and homogeneous initial velocity, and give the proof of Theorem 3.1. In section 4, we complete the proof of Theorem 1.2 with the help of Theorem 3.1. In section 5, we give the proof of Theorem 1.2 by constructing approximate solutions.

2. Notations and Definitions

The points of spaces $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ are denoted by $x'$ and $x = (x', x_n)$, respectively. The multiple derivatives are denoted by $D^k_c D^m_l = \frac{\partial^{k+m} f}{\partial x'^{k} \partial x_n^m}$ for multi index $k$ and nonnegative integer $m$. For vector field $f = (f_1, \cdots, f_n)$ on $\mathbb{R}^n$, set $f' = (f_1, \cdots, f_{n-1})$ and $f = (f', f_n)$. Throughout this paper we denote by $c$ various generic constants.

For the Banach space $X$, $X'$ denotes the dual space of $X$. For the a $m$-dimensional smooth domain $\Omega$, $C_0^\infty(\Omega)$ stands for the collection of all complex-valued infinitely differentiable functions in $\mathbb{R}^m$ compactly supported in $\Omega$. Let $1 \leq p \leq \infty$ and $k$ be a nonnegative integer. The usual Sobolev spaces and homogeneous Sobolev spaces are denoted by $W^k_p(\Omega)$ and $\dot{W}^k_p(\Omega)$, respectively. Note that $W^0_p(\Omega) = \dot{W}^0_p(\Omega) = L^p(\Omega)$. Let $0 < \alpha < 1$. The usual Hölder spaces and the homogeneous Hölder spaces are denoted by $C^{k+\alpha}(\Omega)$ and $\dot{C}^{k+\alpha}(\Omega)$, respectively.

It is known that $C^{k+\alpha}(\Omega) = B^{k+\alpha}_\infty(\Omega)$ and $\dot{C}^{k+\alpha}(\Omega) = \dot{B}^{k+\alpha}_\infty(\Omega)$ with equivalent norms, where $B^s_p(\Omega)$ and $\dot{B}^s_p(\Omega)$ are the usual Besov spaces and the homogeneous Besov spaces, respectively. See [5, 96, 37, 35, 99, 40] for the definition of Besov spaces and their properties.
Let $\Omega$ be a domain in $m$-dimensional domain and $I$ be an open interval. Anisotropic Hölder spaces $C^{k+\alpha, k+\alpha} (\Omega \times I)$ and homogeneous anisotropic Hölder spaces $\dot{C}^{k+\alpha, k+\alpha} (\Omega \times I)$ are the set of functions on $\Omega \times I$ normed with

$$
\|f\|_{C^{k+\alpha, k+\alpha} (\Omega \times I)} := \sum_{|l|+2l_0 \leq k} \|D_t^l D_x^l f\|_{L^\infty (\Omega \times I)} + \sum_{|l|+2l_0 = k} [D_t^l D_x^l f]_{\alpha, \Omega \times I} < \infty
$$

and

$$
\|f\|_{\dot{C}^{k+\alpha, k+\alpha} (\Omega \times I)} := \sum_{|l|+2l_0 = k} [D_t^l D_x^l f]_{\alpha, \Omega \times I} < \infty,
$$

where

$$
[f]_{\alpha, \Omega \times I} := \sup_{t \in I} \sup_{x \neq y \in \Omega} \frac{|f(x, t) - f(y, t)|}{|x - y|^\alpha} + \sup_{x \in \Omega} \sup_{s \neq t \in I} \frac{|f(x, t) - f(x, s)|}{|t - s|^\alpha}.
$$

The properties of anisotropic Hölder spaces are the same as the properties of Hölder spaces. For example, $C^{k+\alpha, k+\alpha} (\Omega \times I) = B^{k+\alpha, k+\alpha}_\infty (\Omega \times I)$ and $\dot{C}^{k+\alpha, k+\alpha} (\Omega \times I) = \dot{B}^{k+\alpha, k+\alpha}_\infty (\Omega \times I)$ with equivalent norms, where $B^{k+\alpha, k+\alpha}_p (\Omega \times I)$ and $\dot{B}^{k+\alpha, k+\alpha}_p (\Omega \times I)$ are the anisotropic Besov spaces and the homogeneous anisotropic Besov spaces, respectively (For the definition of anisotropic Besov spaces and the homogeneous anisotropic Besov spaces, see [5, 38]). The properties of the anisotropic Besov spaces in $\Omega \times I$ are comparable with the properties of Besov spaces in $\Omega$, whose proof can be shown by the same arguments as in [5, 39, 40, 41].

**Definition 2.1** (Weak solution to the Stokes system). Suppose that $f = \text{div} \mathcal{F}, \mathcal{F} = \{F_j\}_{i,j=1}^n \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^n \times (0, T))$, $g \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^{n-1} \times (0, T))$ and $h \in C^{\alpha}(\mathbb{R}_+^n)$. Then a vector field $u \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^n \times (0, T))$ is called a weak solution of the Stokes system if the following conditions are satisfied:

1) $\int_0^T \int_{\mathbb{R}_+^n} \nabla u : \nabla \Phi dx dt = \int_0^T \int_{\mathbb{R}_+^n} u \cdot \Phi_t - \mathcal{F} : \nabla \Phi dx dt$

for each $\Phi \in C_0^\infty(\mathbb{R}_+^n \times (0, T))$ with $\text{div}_x \Phi = 0$,

2) $u(x, 0) = h(x)$ in $\mathbb{R}_+^n$ in trace sense.

3) $u(x', t) = g(x', t)$ in $\mathbb{R}_+^{n-1} \times (0, T)$ in trace sense.

**Definition 2.2** (Weak solution to the Navier-Stokes system). Suppose that $g \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^{n-1} \times (0, T))$ and $h \in C^{\alpha}(\mathbb{R}_+^n)$. Then a vector field $u \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^n \times (0, T))$ is called a weak solution of the Navier-Stokes system if the following conditions are satisfied:

1) $\nabla u \in L^\infty(K \times (\delta, T))$ for each $\delta > 0$ and for each compact subset $K$ of $\mathbb{R}_+^n$,

2) $\int_0^T \int_{\mathbb{R}_+^n} \nabla u : \nabla \Phi dx dt = \int_0^T \int_{\mathbb{R}_+^n} u \cdot (\Phi_t - (\Phi \cdot \nabla) u) dx dt$

for each $\Phi \in C_0^\infty(\mathbb{R}_+^n \times (0, T))$ with $\text{div}_x \Phi = 0$,

3) $u(x, 0) = h(x)$ in $\mathbb{R}_+^n$ in trace sense.

4) $u(x', 0) = g(x', t)$ in $\mathbb{R}_+^{n-1} \times (0, T)$ in trace sense.
3. Stokes system with homogeneous external force and initial velocity

Let us consider the following initial-boundary value problem of a nonstationary Stokes system in \( \mathbb{R}_+^n \times (0, T) \):

\[
\begin{align*}
    w_t - \Delta w + \nabla q &= 0, \quad \text{div } w = 0, \quad &\text{in } \mathbb{R}_+^n \times (0, T), \\
    w|_{t=0} &= 0, \quad w|_{x_n=0} = G.
\end{align*}
\]  

(3.1)

In [30], an explicit formula for \( w \) of the Stokes system (3.1) with boundary data \( G = (G', 0) \) is obtained by

\[
\begin{align*}
    w_i(x, t) &= \sum_{j=1}^{n-1} \int_0^t \int_{\mathbb{R}^{n-1}} K_{ij}(x' - y', x_n, t - s) G_j(y', s) dy' ds, \\
    q(x, t) &= \sum_{j=1}^{n-1} \int_0^t \int_{\mathbb{R}^{n-1}} \pi_j(x' - y', x_n, t - s) G_j(y', s) dy' ds.
\end{align*}
\]  

(3.2)

(3.3)

Here,

\[
\begin{align*}
    K_{ij}(x, t) &= -2 \delta_{ij} D_{x_n} \Gamma(x, t) + 4D_{x_j} \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(z, t) D_{x_i} N(x - z) dz, \\
    \pi_j(x, t) &= -2 \delta(t) D_{x_j} D_{x_n} N(x) + 4D_{x_j} D_{x_n}^2 A(x, t) + 4D_{t} D_{x_j} A(x, t), \\
    A(x, t) &= \int_{\mathbb{R}^{n-1}} \Gamma(z', 0, t) N(x' - z', x_n) dz',
\end{align*}
\]

where \( \Gamma \) and \( N \) are fundamental solutions of heat equation and Laplace equation in \( \mathbb{R}^n \), respectively, that is,

\[
\Gamma(x, t) = \begin{cases} 
\frac{c}{2\pi t^{n/2}} e^{-|x|^2/4t} & \text{if } t > 0, \\
0 & \text{if } t \leq 0,
\end{cases}
\]

and

\[
N(x) = \begin{cases} 
\frac{1}{\omega_n (2n-1)|x|^{n-1}} & \text{if } n \geq 3, \\
\frac{1}{2\pi} \ln |x| & \text{if } n = 2.
\end{cases}
\]

**Theorem 3.1.** Let \( 0 < \alpha < 1 \). Let \( G \in \dot{C}^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^{n-1} \times (0, \infty)) \) with \( G_n = 0 \). We also assume that \( G(t) = 0 \) for \( t \geq 0 \). Then, the function \( w \) defined by (3.2) is in \( \dot{C}^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^{n-1} \times (0, \infty)) \) and satisfies

\[
||w||_{\dot{C}^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^{n-1} \times (0, \infty))} \leq c ||G||_{\dot{C}^{\alpha, \frac{\alpha}{2}}(\mathbb{R}_+^{n-1} \times (0, \infty))}.
\]

**Proof.** According to the result of V.A. Solonnikov [31], if \( G = (G', 0) \in \dot{C}^{s, \frac{s}{2}}(\mathbb{R}^2 \times (0, T)) \) for \( s > 2 \), then \( w \) defined by (3.2) satisfies

\[
||w||_{\dot{C}^{s, \frac{s}{2}}(\mathbb{R}^2 \times (0, T))} \leq c ||G||_{\dot{C}^{s, \frac{s}{2}}(\mathbb{R}^2 \times (0, T))}, \quad s > 2.
\]

The argument in [31] can be applied for any \( n \geq 2 \) to obtain the same estimates as the above.

According to the result of T.K. Chang and H.J. Choe [9], if \( G = (G', 0) \in L^\infty(\mathbb{R}^{n-1} \times (0, T)) \), then \( w \) defined by (3.2) satisfies

\[
||w||_{L^\infty(\mathbb{R}^+_n \times (0, T))} \leq c ||G||_{L^\infty(\mathbb{R}^{n-1} \times (0, T))}.
\]

Interpolate the above two estimate, then we obtain the estimate in Theorem 3.1.
Remark 3.2. Let $G_{ij} = D_{x_j} \int_{0}^{\infty} \int_{\mathbb{R}^{n-1}} D_{x_i} \Gamma(z,t)D_{x_j}N(x-z)dz$. It is known that
\[
|D_{x_i}^{k} D_{x_j}^{m} D_{t}^{n} G_{ij}(x,t)| \leq c t^{-m+\frac{1}{2}(|x|^2 + t)^{\frac{n}{2}+\frac{1}{2}k}(x^2 + t)^{\frac{1}{2}k}}, \tag{3.4}
\]
where $1 \leq i \leq n$ and $1 \leq j \leq n-1$ (see [20]). Using the properties of Heat kernel $\Gamma$ and the estimates of $G_{ij}$, it is easy to see that
\[
x_n^{-\alpha_k - 1} t^{-\frac{1}{2}} |D_{x}^{k}w(x,t)| \leq c \|G\|_{C^\alpha([\mathbb{R}^{n-1} \times (0,T)])},
\]
Therefore, $w$ is smooth in $\mathbb{R}^n_+$ for each $t > 0$.

4. Stokes system with nonhomogeneous external force and initial velocity

In this section we consider the Stokes system (1.3) with $f = \text{div} \mathcal{F}, g, h$ satisfying the hypothesis of Theorem 1.2.

4.1. Formal decompositions. Let $\tilde{\mathcal{F}}$ be an extension of $\mathcal{F}$ to $\mathbb{R}^n \times (0,T)$ and let $\tilde{f} = \text{div} \tilde{\mathcal{F}}$. Let $\mathbb{P}$ be the Helmholtz projection operator on $\mathbb{R}^n$ defined by
\[
[\mathbb{P} \tilde{f}]_j(x,t) = \delta_{ij} \tilde{f}_j + \int_{\mathbb{R}^n} D_{x_j}D_{x_j}N(x-y)\tilde{f}_i(y,t)dy = \delta_{ij} \tilde{f}_j + R_i R_j \tilde{f}_i
\]
and define $\mathbb{Q}$ by
\[
\mathbb{Q} \tilde{f} = - \int_{\mathbb{R}^n} D_{x_j}N(x-y)\tilde{f}_i(y,t)dy.
\]
Then, we have
\[
\text{div} \mathbb{P} \tilde{f} = 0 \text{ in } \mathbb{R}^n \times (0,T) \quad \text{and} \quad \tilde{f} = \mathbb{P} \tilde{f} + D_{x} \mathbb{Q} \tilde{f}.
\]
Note that $[\mathbb{P} \tilde{f}]_j = D_{x_k}[\delta_{ij} \tilde{F}_{ki} + R_i R_j \tilde{F}_{ki}]$ for $\tilde{f} = \text{div} \tilde{\mathcal{F}}$. Define $V$ by
\[
V_j(x,t) = \int_{0}^{t} \int_{\mathbb{R}^n} D_{x_k} \Gamma(x-y, t-s)[\delta_{ij} \tilde{F}_{ki} + R_i R_j \tilde{F}_{ki}](y,s)dyds. \tag{4.1}
\]
Observe that $V$ satisfies the equations
\[
V_t - \Delta V = \mathbb{P} \tilde{f}, \quad \text{div} V = 0 \text{ in } \mathbb{R}^n \times (0,T), \quad V|_{t=0} = 0 \text{ on } \mathbb{R}^n.
\]
Let $\tilde{h}$ be an extension of $h$ satisfying that $\text{div} \tilde{h} = 0$ in $\mathbb{R}^n$. Define $v$ by
\[
v(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)\tilde{h}(y)dy. \tag{4.2}
\]
Observe that $v$ satisfies the equations
\[
v_t - \Delta v = 0, \quad \text{div} v = 0 \text{ in } \mathbb{R}^n \times (0,T), \quad v|_{t=0} = \tilde{h} \text{ on } \mathbb{R}^n.
\]
Define $\phi$ by
\[
\phi(x,t) = 2 \int_{\mathbb{R}^{n-1}} N(x'-y',x_n)(g_n(y',t) - v_n(y',0,t) - V_n(y',0,t))dy'. \tag{4.3}
\]
Observe that
\[
\Delta \phi = 0, \quad \nabla \phi|_{x_n=0} = (R'(g_n - v_n)|_{x_n=0} - V_n|_{x_n=0}), \quad g_n - v_n|_{x_n=0} - V_n|_{x_n=0}.
\]
Moreover, note that $\nabla \phi|_{t=0} = 0$ if $g_n|_{t=0} = h_n|_{x_n=0}$. Let

$$G = (G', 0), \text{ where } G' = g' - V'|_{x_n=0} - v'|_{x_n=0} - R'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}). \quad (4.4)$$

Note that $G'|_{t=0} = 0$ if $g|_{t=0} = h|_{x_n=0}$.

Finally, let $(w, q)$ be defined by (3.2) and (3.3) with $G$ defined by (4.4). Then, $u = w + \nabla \phi + v + V$ and $p = q - \phi_t + \mathbb{B} \dot{f}$ satisfies formally the nonstationary Stokes system (3.3).

### 4.2 Preliminary Estimates

The subsequent propositions are the basic tool for the estimate of $v, V, \nabla \phi$ and $w$ introduced in the previous section. See Appendix A, Appendix B, Appendix C and Appendix D for the proof of the Proposition 4.1, Proposition 4.2, Proposition 4.3 and Proposition 4.4 respectively.

**Proposition 4.1.** Let $0 < \alpha$. For $f \in B^\alpha_{\infty}(\mathbb{R}^n)$, define $u(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)f(y)dy$. Then $u \in B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))$ with

$$\|u\|_{B^\alpha_{\infty, \infty}(\mathbb{R}^n \times (0, \infty))^{'}} \leq c\|f\|_{B^\alpha_{\infty}(\mathbb{R}^n)}, \quad \|u\|_{L^\infty(\mathbb{R}^n \times (0, \infty))} \leq c\|f\|_{L^\infty(\mathbb{R}^n)}.$$

Moreover, $u$ is smooth in $\mathbb{R}^n \times (0, T)$ with

$$\sup_{x \in \mathbb{R}^n, t \in (0, \infty)} t^{m+\frac{1}{4}}|D_x^m D_t^k u(x, t)| \leq c\|f\|_{L^\infty(\mathbb{R}^n)}.$$

**Proposition 4.2.** Let $0 < \alpha$. Let $f \in B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))$. Define $u(x, t) = \int_0^t \int_{\mathbb{R}^n} D_x \Gamma(x - y, t - s)f(y, s)dyds$. Then $u \in B^{\alpha + 1 + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))$ with

$$\|u\|_{B^{\alpha + 1 + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))} \leq c\|f\|_{B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))},$$

$$\|u\|_{B^{\alpha + 1}_{\infty, \infty}(\mathbb{R}^n \times (0, T))} \leq cT^{\frac{1}{2}}\|f\|_{B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))},$$

$$\|u\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq cT^{\frac{1}{2}}\|f\|_{L^\infty((0, T); BMO(\mathbb{R}^n))}.$$

Here $BMO(\mathbb{R}^n)$ denotes the usual BMO space, which is the dual space of Hardy space $H^1(\mathbb{R}^n)$.

**Proposition 4.3.** Let $\alpha \in \mathbb{R}$. Then

$$\|Rf\|_{B^\alpha_{\infty}(\mathbb{R}^n)} \leq c\|f\|_{B^\alpha_{\infty}(\mathbb{R}^n)}, \quad \|Rf\|_{BMO(\mathbb{R}^n)} \leq c\|f\|_{BMO(\mathbb{R}^n)}, \quad \|Rf\|_{H^1(\mathbb{R}^n)} \leq c\|f\|_{H^1(\mathbb{R}^n)}.$$

Moreover, if $0 < \alpha$, then

$$\|Rf\|_{B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))} \leq c\|f\|_{B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))}.$$

**Proposition 4.4.** Let $0 < \alpha$. Define $Pf(x, t) = \int_{\mathbb{R}^{n-1}} \frac{x_n - y_n}{(|x'|^2 + |y'|^2)^{\frac{n-1}{2}}} f(y', t)dy'$. Then

$$\|Pf(t)\|_{B^\alpha_{\infty}(\mathbb{R}^n)} \leq c\|f(t)\|_{B^\alpha_{\infty}(\mathbb{R}^{n-1})},$$

$$\|Pf u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c\|f(t)\|_{L^\infty(\mathbb{R}^{n-1})},$$

$$\|Pf\|_{B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^n \times (0, T))} \leq c\|f\|_{B^{\alpha + \frac{1}{2}}_{\infty, \infty}(\mathbb{R}^{n-1} \times (0, T))}.$$

Moreover, $u(t)$ is smooth in $\mathbb{R}^n$ with

$$\sup_{x \in \mathbb{R}^n} x^k |D_x^k u(x, t)| \leq c \|f(t)\|_{L^\infty(\mathbb{R}^{n-1})}.$$  

4.3. Proof of Theorem 1.2

Choose $\tilde{h} \in \dot{C}^\alpha(\mathbb{R}^n)$ and $\tilde{F} \in \dot{C}^\alpha(\mathbb{R}^n \times (0, T))$ which are the extension of $h$ and $F$, respectively. Let $V$, $v$ and $\phi$ be the corresponding vector fields defined by Proposition 4.1, 4.2, and 4.3, respectively, and let $w$ be defined by (3.2) with $G$ as (4.4).

- At this step, we will show that $u = v + V + \nabla \phi + w \in \dot{C}^\alpha(\mathbb{R}^n \times (0, T))$.

From Proposition 4.1 and the property of the extension $\tilde{h}$

$$\|v\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))} \leq c \|h\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))}. \quad (4.5)$$

From Proposition 4.2, Proposition 4.3 and the property of the extension $\tilde{f}$

$$\|V\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))} \leq c T^{\frac{2}{2\alpha}} \|\delta_{ij} \tilde{F}_{ik} + R_i R_j \tilde{F}_{ik}\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))} \leq c T^{\frac{2}{2\alpha}} \|\tilde{f}\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))}. \quad (4.6)$$

According to the well known trace theorem, $V, v \in \dot{C}^\alpha(\mathbb{R}^n \times (0, T))$ imply $V|_{x_n=0}, v|_{x_n=0} \in \dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))$ with

$$\|V|_{x_n=0}\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \leq c \|V\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))},$$

$$\|v|_{x_n=0}\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \leq c \|v\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))}. \quad (4.7)$$

Again, according to Proposition 4.3, $g_n, V|_{x_n=0}, v|_{x_n=0} \in \dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))$ imply $R'g_n, R'(v|_{x_n=0}), R'(v|_{x_n=0}) \in \dot{C}^\alpha(\mathbb{R}^n \times (0, T))$ with

$$\|R'g_n\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \leq c \|g_n\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))},$$

$$\|R'(v|_{x_n=0})\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \leq c \|v|_{x_n=0}\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))},$$

$$\|R'(V|_{x_n=0})\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \leq c \|V|_{x_n=0}\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))}. \quad (4.8)$$

Observe that

$$D_x \phi = 2 \int_{\mathbb{R}^{n-1}} D_x N(x' - y', x_n) \left( g_n(y', t) - v_n(y', 0, t) - V_n(y', 0, t) \right) dy', $$

$$D_{x'} \phi = 2 \int_{\mathbb{R}^{n-1}} D_{x'} N(x' - y', x_n) \left( R'g_n(y', t) - R'v_n(y', 0, t) - R'V_n(y', 0, t) \right) dy'. $$

According to the Proposition 4.3 and Proposition 4.3 (4.7) and (4.8) imply $\nabla \phi \in \dot{C}^\alpha(\mathbb{R}^n \times (0, T))$ with

$$\|\nabla \phi\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))} \leq c \left( \|g_n\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} + \|v_n\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} + \|V_n\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \right) \quad (4.9)$$

(4.7) and (4.8) also imply $G \in \dot{C}^\alpha(\mathbb{R}^n \times (0, T))$ with

$$\|G\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \leq c \left( \|v\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} + \|V\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} + \|g\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))} \right). \quad (4.10)$$

Applying Theorem 3.1, we have

$$\|u\|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))} \leq c \|G\|_{\dot{C}^\alpha(\mathbb{R}^{n-1} \times (0, T))}, \quad (4.11)$$
Combining (4.5), (4.6), (4.9) and (4.11) together with (4.10), we conclude that \( u = v + V + \nabla \phi + w \in \mathring{C}^{\infty, 2}(\mathbb{R}^n_+ \times (0, T)) \) with the inequality

\[
\|u\|_{C^{\infty, 2}(\mathbb{R}^n_+ \times (0, T))} \leq c \left( \|h\|_{C^0(\mathbb{R}^n_+)} + T^{\frac{3}{2}} \|\mathcal{F}\|_{C^{\infty, 2}(\mathbb{R}^n_+ \times (0, T))} + \|g\|_{C^{\infty, 2}(\mathbb{R}^{n-1} \times (0, T))} \right). \tag{4.12}
\]

- At this step, we will show that \( u = v + V + \nabla \phi + w \in L^\infty(\mathbb{R}^n_+ \times (0, T)) \).

By Proposition 1.3, we have

\[
\|V\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq c T^{\frac{3}{2}} \|\delta_i \tilde{F}_{ik} + R_i R_j \tilde{F}_{ik}\|_{L^\infty(0, T; BMO(\mathbb{R}^n))} \leq c T^{\frac{3}{2}} \|\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+ \times (0, T))}, \tag{4.13}
\]

and by Proposition 4.4 we have

\[
\|v\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq c \|h\|_{L^\infty(\mathbb{R}^n_+)} \tag{4.14}
\]

To show that \( \nabla \phi, w \) are in \( L^\infty(\mathbb{R}^n_+ \times (0, T)) \), it is necessary that \( v_n|_{x_n=0}, V_n|_{x_n=0}, R'(v_n|_{x_n=0}), R'(V_n|_{x_n=0}) \) are in \( L^\infty(\mathbb{R}^n_+ \times (0, T)) \). Observe that from Proposition 4.4

\[
V(x', 0, t) = \int_0^t \int_{\mathbb{R}^n} D_x \Gamma(x' - y', y_n, t - s) \left[ \delta_{ij} \tilde{F}_{ik} + R_i R_j \tilde{F}_{ik} \right](y, s) dy ds 
\leq c \int_0^t \|D_x \Gamma(x' - \cdot, \cdot; t - s)\|_{H^1(\mathbb{R}^n)} \|\delta_{ij} \tilde{F}_{ik} + R_i R_j \tilde{F}_{ik}\|_{BMO(\mathbb{R}^n)} ds,
\]

\[
v(x', 0, t) = \int_{\mathbb{R}^n} \Gamma(x' - y', y_n, t) \tilde{h}(y) dy \leq c \|\Gamma(x' - \cdot, \cdot; t)\|_{L^1(\mathbb{R}^n)} \|\tilde{h}\|_{L^\infty(\mathbb{R}^n)}.
\]

Hence \( V|_{x_n=0}, v|_{x_n=0} \in L^\infty(\mathbb{R}^{n-1} \times (0, T)) \) with

\[
\|V|_{x_n=0}\|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} \leq c T^{\frac{3}{2}} \|\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+ \times (0, T))}, \quad \|v|_{x_n=0}\|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} \leq c \|h\|_{L^\infty(\mathbb{R}^n_+)}. \tag{4.15}
\]

The estimates of \( R'(v_n|_{x_n=0}), R'(V_n|_{x_n=0}) \) in \( L^\infty(\mathbb{R}^n_+ \times (0, T)) \) are rather delicate. Note that \( \mathbb{P}_N \tilde{f}_n = \Delta N \ast \mathbb{P}_N \tilde{f}_n = \sum_{l \neq n} D^2_{x_l} N \ast \mathbb{P}_N \tilde{f}_n - \sum_{l \neq n} D_{x_l} D_{x_l} \mathbb{P}_N \tilde{f}_l, \) since \( \text{div} \mathbb{P}_N \tilde{f} = 0 \). Hence, we have

\[
V_n(x', 0, t) = \sum_{l \neq n} \int_0^t \int_{\mathbb{R}^n} D_{x_l} \Gamma(x' - y', y_n, t - s) D_{y_l} D_{y_l} \left[ N \ast (\delta_{ln} + R_l R_n) \tilde{F}_{kl} \right](y, s) dy ds 
+ \sum_{l \neq n} \int_0^t \int_{\mathbb{R}^n} D_{y_l} \Gamma(x' - y', y_n, t - s) D_{y_l} D_{y_l} \left[ N \ast (\delta_{il} + R_l R_l) \tilde{F}_{kl} \right] dy ds
= \sum_{l \neq n} \int_0^t \int_{\mathbb{R}^n} D_{x_l} \Gamma(x' - y', y_n, t - s) R_l R_k (\delta_{ln} + R_l R_n) \tilde{F}_{kl} \right](y, s) dy ds 
+ \sum_{l \neq n} \int_0^t \int_{\mathbb{R}^n} D_{y_l} \Gamma(x' - y', y_n, t - s) R_l R_k (\delta_{il} + R_l R_l) \tilde{F}_{kl} \right](y, s) dy ds.
\]

Using the above representation, \( R'(V_n|_{x_n=0}) \) have the following representation:

\[
R'(V_n(t)|_{x_n=0})
\]
\[
= \sum_{l \neq n} \int_0^t \int_0^\infty k(y_n, t - s) \left( \int_{\mathbb{R}^{n-1}} R' D_{x_l} K(x' - y', t - s) R_l R_k (\delta_{ln} + R_l R_n) \tilde{F}_{kl}(y', y_n, s) dy' \right) dy_n ds 
+ \sum_{l \neq n} \int_0^t \int_0^\infty k(y_n, t - s) \left( \int_{\mathbb{R}^{n-1}} R' D_{y_l} K(x' - y', t - s) R_n R_k (\delta_{il} + R_l R_l) \tilde{F}_{kl}(y', y_n, s) dy' \right) dy_n ds.
\]
Here $K(x', t) = K_t(x') = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x'|^2}{4t}}$, $k(x_n, t) = k_t(x_n) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x_n^2}{4t}}$. Note that

\[
\int_{\mathbb{R}^{n-1}} R' D_{x'} K(x' - y', t - s) R_t R_k (\delta_{ij} + R_i R_j) \tilde{F}(\cdot, y_n, s) dy'
\leq c \| D_{x'} K(x' - y', t - s) \|_{\dot{B}_{1, \infty}^{-\alpha}(\mathbb{R}^{n-1})} \| R_t R_k (\delta_{ij} + R_i R_j) \tilde{F}(\cdot, y_n, s) \|_{\dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R}^{n-1})}
\leq c \| K(x' - y', t - s) \|_{\dot{B}_{1, \infty}^{-\alpha}(\mathbb{R}^{n-1})} \sup_{y_n} \| R_t R_k (\delta_{ij} + R_i R_j) \tilde{F}(\cdot, y_n, s) \|_{\dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R}^{n-1})}
\leq c (t - s)^{-\frac{1}{2} + \frac{\alpha}{4}} \| R_t R_k (\delta_{ij} + R_i R_j) \tilde{F}(\cdot, s) \|_{\dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R})} \leq c (t - s)^{-\frac{1}{2} + \frac{\alpha}{4}} \| \tilde{F}(\cdot, s) \|_{\dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R})}.
\]

Here we use the fact that $R : \dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R}^n) \rightarrow \dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R}^n)$ is bounded operator for $\alpha \in \mathbb{R}$, $L^\infty(\mathbb{R}; \dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R}^{n-1})) \cap L^\infty(\mathbb{R}^{n-1}; \dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R})) = \dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R}^n)$ for $\alpha > 0$, and $\dot{B}_{\infty, \infty}^{\alpha}(\mathbb{R}^n) = \dot{C}^\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$. Hence, we have

\[
\| R'(V_n(t)|_{x_n=0}) \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} \leq c T^{\frac{1}{2} + \frac{\alpha}{4}} \| \tilde{F} \|_{\dot{C}^\alpha(\mathbb{R}^n \times (0, T))}. \quad (4.16)
\]

Direct computation shows the identity

\[
R'(v_n(t)|_{x_n=0}) = \int_{\mathbb{R}^n} \Gamma(y, t) (R' \hat{h}_n)(x' - y', y_n) dy.
\]

Hence, if $R' \hat{h}_n \in L^\infty(\mathbb{R}^n)$, then

\[
\| R'(v_n(t)|_{x_n=0}) \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} \leq c \| R' \hat{h}_n \|_{L^\infty(\mathbb{R}^n)}. \quad (4.17)
\]

By Proposition 3.2, 4.16 and 4.17 imply $\nabla \phi \in L^\infty(\mathbb{R}_+^n \times (0, T))$ with

\[
\| D_{x_n} \phi(t) \|_{L^\infty(\mathbb{R}_+^n)} \leq c \left( \| g_n \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} + T^{\frac{1}{2}} \| \mathcal{F} \|_{L^\infty(\mathbb{R}_+^n \times (0, T))} + \| \tilde{h} \|_{L^\infty(\mathbb{R}_+^n)} \right), \quad (4.18)
\]

\[
\| D_{x'} \phi(t) \|_{L^\infty(\mathbb{R}_+^n)} \leq c \left( \| R' g_n(t) \|_{L^\infty(\mathbb{R}^{n-1})} + T^{\frac{1}{2} + \frac{\alpha}{4}} \| \mathcal{F} \|_{\dot{C}^\alpha(\mathbb{R}_+^n \times (0, T))} + \| R' \hat{h}_n \|_{L^\infty(\mathbb{R}_+^n)} \right). \quad (4.19)
\]

(4.16) and (4.17) also imply $G \in L^\infty(\mathbb{R}^{n-1} \times (0, T))$ with

\[
\| G \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} \leq c \left( \| g \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} + \| \tilde{h} \|_{L^\infty(\mathbb{R}_+^n)} + T^{\frac{1}{2}} \| \mathcal{F} \|_{L^\infty(\mathbb{R}_+^n \times (0, T))} + T^{\frac{1}{2} + \frac{\alpha}{4}} \| \mathcal{F} \|_{\dot{C}^\alpha(\mathbb{R}_+^n \times (0, T))} + \| R' g_n(t) \|_{L^\infty(\mathbb{R}^{n-1})} + \| R' \hat{h}_n \|_{L^\infty(\mathbb{R}_+^n)} \right). \quad (4.20)
\]

Note that $G_n = 0$, $G|_{t=0} = 0$. According to the result of (39),

\[
\| u \|_{L^\infty(\mathbb{R}_+^n \times (0, T))} \leq c \| G \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))}. \quad (4.21)
\]

Combining (4.13), (4.14), (4.18), (4.19) and (4.21) together with (4.20), we conclude that $u = v + V + \nabla \phi + w \in L^\infty(\mathbb{R}_+^n \times (0, T))$ with the inequality

\[
\| u \|_{L^\infty(\mathbb{R}_+^n \times (0, T))} \leq c \left( \| h \|_{L^\infty(\mathbb{R}_+^n)} + T^{\frac{1}{2}} \| \mathcal{F} \|_{L^\infty(\mathbb{R}_+^n \times (0, T))} + T^{\frac{1}{2} + \frac{\alpha}{4}} \| \mathcal{F} \|_{\dot{C}^\alpha(\mathbb{R}_+^n \times (0, T))} + \| g \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} + \| R' \hat{h}_n \|_{L^\infty(\mathbb{R}_+^n)} + \| R' g_n \|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} \right). \quad (4.22)
\]

Combining (4.12) and (4.22), we obtain the estimates (1.4) in Theorem 1.2. The uniqueness follows from (1.4).
Remark 4.5. Recall that formally, $u = w + \nabla \phi + v + V$, $p = q - \phi_t + \mathbb{Q}f$ satisfies the Stokes system \[\text{(1.3)}\] formally, and $q$ can be written by $q(x,t) = q_0(x,t) + D_t q_1(x,t)$ (see \[\text{(20)}\] for the details), where $q_0(t), q_1(t)$ are harmonic function in $x$ variable, but their differentiability in $t$ variable is equal to the differentiability of $g$ in $t$ variable. Therefore, for $g \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0,T))$, $0 < \alpha < 1$, $q$ is not a function but a distribution in terms of $t$ variables. From this reason, our solution $u \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n_+ \times (0,T))$ satisfies weak formulation of the Stokes system \[\text{(1.3)}\], but cannot satisfy the Stokes system \[\text{(1.3)}\] in classical sense.

5. Proof of Theorem \[\text{(1.1)}\]

Since
\[|(uv)(x,t) - (uv)(y,t)| \leq |v(x,t)||u(x,t) - u(y,t)| + |u(y,t)||v(x,t) - v(y,t)|\]
and
\[|(uv)(x,t) - (uv)(x,s)| \leq |v(x,t)||u(x,t) - u(x,s)| + |u(x,s)||v(x,t) - v(x,s)|,\]
the following bilinear estimate can be obtained.

Lemma 5.1. Let $u, v \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])$. Then
\[\|uv\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])} \leq c\|u\|_{L^\infty(\mathbb{R}^n \times [0,T])}\|v\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])} + c\|v\|_{L^\infty(\mathbb{R}^n \times [0,T])}\|u\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times [0,T])}.

5.1. Approximate solutions. Let $(u^1, p^1)$ be the solution of the system
\[u_t^1 - \Delta u^1 + \nabla p^1 = 0, \quad \text{div } u = 0, \quad \text{in } \mathbb{R}^n_+ \times (0,T),\]
\[u^1|_{t=0} = h, \quad u^1|_{x_n=0} = g.\]

Let $m \geq 1$. After obtaining $(u^1, p^1), \ldots, (u^m, p^m)$ construct $(u^{m+1}, p^{m+1})$ which satisfies the system
\[u_t^{m+1} - \Delta u^{m+1} + \nabla p^{m+1} = f^m, \quad \text{div } u^{m+1} = 0, \quad \text{in } \mathbb{R}^n_+ \times (0,T),\]
\[u^{m+1}|_{t=0} = h, \quad u^{m+1}|_{x_n=0} = g,\]
where $f^m = -\text{div}(u^m \otimes u^m)$.

5.2. Uniform boundeness. Let $T \leq 1$. By the result of Theorem \[\text{(1.2)}\] we have
\[\|u^{m+1}\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n_+ \times (0,T))} \leq c\left(\|h\|_{C^{\alpha}(\mathbb{R}^n_+)} + \|g\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0,T))} + \|R'g_n\|_{L^\infty(\mathbb{R}^n \times (0,T))} + \|R' h_n\|_{L^\infty(\mathbb{R}^n)} + \max\{T^\frac{\alpha}{2}, T^{\frac{\alpha}{2} + \frac{\alpha}{2}}\}\|u^m \otimes u^m\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n_+ \times (0,T))}\right).

By the bilinear estimate in Lemma \[\text{5.1}\] we have
\[\|(u^m \otimes u^m)\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n_+ \times (0,T))} \leq c\|u^m\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))}\|u^m\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n_+ \times (0,T))}.

Therefore, we have
\[\|u^{m+1}\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n_+ \times (0,T))} \leq c_1\left(\|h\|_{C^{\alpha}(\mathbb{R}^n_+)} + \|g\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0,T))} + \|R'g_n\|_{L^\infty(\mathbb{R}^n \times (0,T))} + \|R' h_n\|_{L^\infty(\mathbb{R}^n)} + \max\{T^\frac{\alpha}{2}, T^{\frac{\alpha}{2} + \frac{\alpha}{2}}\}\|u^m\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))}\|u^m\|_{C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n_+ \times (0,T))}\right).
\[\text{(5.1)}\]
Now send \( m \) section. In this section, we will show that existence, regularity and uniqueness.

Set
\[
M_0 = \|h\|_{C^\alpha(\mathbb{R}^n)} + \|g\|_{C^\alpha(\mathbb{R}^{n-1} \times (0, T))} + \|R'g_n\|_{L^\infty(\mathbb{R}^{n-1} \times (0, T))} + \|R'h_n\|_{L^\infty(\mathbb{R}^n)}.
\]

Choose \( M > 2c_1M_0 \). Then (5.1) implies that
\[
\|u^1\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \leq c_1M_0 < M,
\]
and under the condition that \( \|u^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \leq M \), we have
\[
\|u^{m+1}\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \leq c_1M_0 + c_1 \max\{T^{\frac{3}{2}}, T^{\frac{3}{2} + \frac{1}{2}}\}M^2.
\]

Choose \( 0 < T \leq \frac{1}{(2c_1M)^2} \), together with the condition \( T \leq 1 \). Then by mathematical induction argument we can conclude that
\[
\|u^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \leq M \text{ for all } m = 1, 2, \cdots.
\]

5.3. Uniform Convergence. Let \( U^m = u^{m+1} - u^m \) and \( P^m = p^{m+1} - p^m \). Then \( U^m \) satisfies the system
\[
U_t^m - \Delta U^m + \nabla P^m = -\nabla \text{div}(u^m \otimes U^m - 1 + U^m \otimes u^m), \quad \text{div} U^m = 0, \quad \text{in } \mathbb{R}^n_+ \times (0, T),
\]
\[
U^m|_{t=0} = 0, \quad U^m|_{x_n=0} = 0.
\]

By the result of Theorem 1.2 we have
\[
\|U^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \leq c \max\{T^{\frac{3}{2}}, T^{\frac{3}{2} + \frac{1}{2}}\}\|U^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} + \|u^m - u^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \leq c_2 \max\{T^{\frac{3}{2}}, T^{\frac{3}{2} + \frac{1}{2}}\}\|U^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} + \|u^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \frac{1}{\sqrt{2}} M^2.
\]

Choose \( 0 < T \leq \frac{1}{(2c_1M)^2} \), together with the condition \( T \leq 1 \). Then, the above estimate leads to the
\[
\|U^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))} \leq \frac{1}{2} \|U^m\|_{C^\alpha(\mathbb{R}^n \times (0, T))}.
\]

implies the infinite series \( \sum_{k=1}^\infty U^k \) converges in \( C^\alpha_\ast(\mathbb{R}^n_+ \times (0, T)) \). Observe that \( u^m = u^1 + \sum_{k=1}^m U^k, m = 2, 3, \cdots \). Hence \( u^m \) converges to \( u^1 + \sum_{k=1}^\infty U^k \) in \( C^\alpha_\ast(\mathbb{R}^n_+ \times (0, T)) \). Set \( u := u^1 + \sum_{k=1}^\infty U^k \).

5.4. Existence, regularity and uniqueness. Let \( u \) be the same one constructed by the previous section. In this section, we will show that \( u \) satisfies weak formulation of Navier-Stokes system, that is, \( u \) is a weak solution of Navier-Stokes system with appropriate distribution \( p \).

Let \( \Phi \in C_0^\infty(\mathbb{R}^n_+ \times (0, T)) \) with \( \text{div} \Phi = 0 \). Observe that
\[
- \int_0^T \int_{\mathbb{R}^n_+} u^{m+1} \cdot \Delta \Phi dx dt = \int_0^T \int_{\mathbb{R}^n_+} u^m \cdot (\Phi_t + (u^m \cdot \nabla)) \Phi dx dt.
\]

Now send \( m \) to the infinity, then, since \( u^m \rightharpoonup u \) in \( C^\alpha_\ast(\mathbb{R}^n_+ \times (0, T)) \), we have
\[
\int_0^T \int_{\mathbb{R}^n_+} u \cdot \Delta \Phi dx dt = \int_0^T \int_{\mathbb{R}^n_+} u \cdot (\Phi_t + (u \cdot \nabla)) \Phi dx dt.
\]
Note that $u$ can be decomposed by $u = v + V + \nabla \phi + w$, where

$$v(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t) \hat{h}(y) dy,$$

$$V_j(x, t) = \int_0^t \int_{\mathbb{R}^n} D_{x_k} \Gamma(x - y, t - s)[\delta_{ij} \hat{u}_k \hat{u}_i + R_i R_j \hat{u}_k \hat{u}_j](y, s) dy ds,$$

$$\phi(x, t) = 2 \int_{\mathbb{R}^n} N(x' - y', x_n) \left(g_n(y', t) - v_n(y', 0, t) - V_n(y', 0, t)\right) dy'$$

$$w = \sum_{j=1}^{n-1} \int_0^t \int_{\mathbb{R}^{n-1}} K_{ij}(x' - y', x_n, t - s) G_j(y', s) dy' ds,$$

for $G = (g' - V'|_{x_n=0} - v'|_{x_n=0} - R'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}, 0)$. By Proposition 4.1, $v$ is infinitely differentiable in $(x, t) \in \mathbb{R}^n_+ \times (0, T)$, by Proposition 4.2 $V \in C^{\alpha+1}((\mathbb{R}^n_+ \times (0, T))$, by Proposition 4.3 $\nabla \phi(t)$ is infinitely differentiable in $x \in \mathbb{R}^n_+$ for each $t > 0$, and by Remark 3.2, $w(t)$ is infinitely differentiable in $x \in \mathbb{R}^n_+$ for each $t > 0$, concluding that $\nabla u \in L^\infty(K \times (\delta, T))$ for each $\delta > 0$ and for each compact subset $K$ of $\mathbb{R}^n_+$. Therefore, $(u, v)$ can be rewritten by

$$\int_0^T \int_{\mathbb{R}^n_+} \nabla u : \nabla \Phi dx dt = \int_0^T \int_{\mathbb{R}^n_+} u \cdot (\Phi_t - (\Phi \cdot \nabla)u) dx dt.$$

This leads to the conclusion that $u$ is a weak solution of the Navier-Stokes system (1.1).

Let $v \in C^{\alpha+\frac{1}{p}}((\mathbb{R}^n_+ \times (0, T))$ be another solution of Navier-Stokes system (1.1) with pressure $q$. Then $u - v$ satisfies the system

$$(u - v)_t - \Delta(u - v) + \nabla(p - q) = -\text{div}(u \otimes (u - v) + (u - v) \otimes v)$$

$$\text{div} (u - v) = 0, \text{ in } \mathbb{R}^n_+ \times (0, T),$$

$$(u - v)|_{t=0} = 0, \ (u - v)|_{x_n=0} = 0.$$

Applying Theorem 1.2 to the above Stokes system for $u - v$ and then applying Lemma 5.1

$$\|u - v\|_{C^{\alpha+\frac{1}{p}}((\mathbb{R}^n_+ \times (0, T_1))} \leq c \max\left\{ T_1^{\frac{1}{2}}, T_1^{\frac{1}{2} + \frac{1}{p}} \right\} \|u \otimes (u - v) + (u - v) \otimes v\|_{C^{\alpha+\frac{1}{p}}((\mathbb{R}^n_+ \times (0, T_1))}$$

$$\leq c_3 \max\left\{ T_1^{\frac{1}{2}}, T_1^{\frac{1}{2} + \frac{1}{p}} \right\} \left( \|u\|_{L^\infty((\mathbb{R}^n_+ \times (0, T_1))} + \|v\|_{L^\infty((\mathbb{R}^n_+ \times (0, T_1))} \right) \|u - v\|_{C^{\alpha+\frac{1}{p}}((\mathbb{R}^n_+ \times (0, T_1))}.$$ 

If we take $T_1 \leq \frac{1}{c_3 (\|u\|_{L^\infty((\mathbb{R}^n_+ \times (0, T))} + \|v\|_{L^\infty((\mathbb{R}^n_+ \times (0, T))} + 1)}$ together with $T_1 \leq 1$, then the above inequality leads to the conclusion that

$$\|u - v\|_{C^{\alpha+\frac{1}{p}}((\mathbb{R}^n_+ \times (0, T_1))} = 0$$

that is, $u \equiv v$ in $\mathbb{R}^n_+ \times (0, T_1)$.

By the same argument, we can show that

$$\|u - v\|_{C^{\alpha+\frac{1}{p}}((\mathbb{R}^n_+ \times (T_1, 2T_1))} = 0$$

that is, $u \equiv v$ in $\mathbb{R}^n_+ \times (T_1, 2T_1)$.

After iterating this procedure finite times, we obtain the conclusion that $u = v$ in $\mathbb{R}^n_+ \times (0, T)$. 
Appendix A. Proof of Proposition 4.1

By Young’s theorem, we have
\[ \|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c\|\Gamma_t\|_{L^1}\|f\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{L^\infty(\mathbb{R}^n)}, \]
and this gives the estimate
\[ \|u\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} \leq c\|f\|_{L^\infty(\mathbb{R}^n)} \tag{1.1} \]

Since \( \Delta_t u = \Delta_x u \), and \( D_x^2 u = \Gamma_t \ast D_x^2 f \), again, by Young’s theorem we have
\[ \|D_t u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c\|D_x^2 u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c\|\Gamma_t\|_{L^1}\|D_x^2 f\|_{L^\infty(\mathbb{R}^n)} \leq c\|D_x^2 f\|_{L^\infty(\mathbb{R}^n)}, \]
and this gives the estimate
\[ \|u\|_{W^{2,1}_\alpha(\mathbb{R}^n \times (0,\infty))} \leq c\|f\|_{W^{2,1}_\alpha(\mathbb{R}^n)} \tag{1.2} \]

According to the real interpolation theory,
\[ (L^\infty(\mathbb{R}^n), \dot{W}^{2,1}_\alpha(\mathbb{R}^n))_{\frac{2}{\alpha}} = \dot{B}^{\alpha}_{\infty}(\mathbb{R}^n), \]
and
\[ (L^\infty(\mathbb{R}^n \times (0,\infty)), \dot{W}^{2,1}_\alpha(\mathbb{R}^n \times (0,\infty)))_{\frac{2}{\alpha}} = \dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times (0,\infty)) \]
for \( 0 < \alpha < 2 \). Apply real interpolation theory to (1.1) and (1.2), then we have the estimate
\[ \|u\|_{\dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times (0,\infty))} \leq c\|f\|_{\dot{B}^{\alpha}_{\infty}(\mathbb{R}^n)}. \]

The argument can be extended to any \( \alpha > 0 \).

The last estimate concerning smoothness comes easily from the properties of the heat kernel.

Appendix B. Proof of Proposition 4.2

Let us derive the first estimate of the proposition. By properties of heat kernel, \( \Gamma \ast_{x,t} f \in \dot{B}^{\alpha+2\frac{\alpha-1}{\alpha}}_{\infty}(\mathbb{R}^n \times \mathbb{R}) \) with
\[ \|\Gamma \ast_{x,t} f\|_{\dot{B}^{\alpha+2\frac{\alpha-1}{\alpha}}_{\infty}(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times \mathbb{R})}, \]
where \( \ast_{x,t} \) means convolution in \((x,t)\) variables.

If \( f|_{t=0} = 0 \), then there is \( \tilde{f} \in \dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times \mathbb{R}) \) with \( \text{supp } \tilde{f} \subset \mathbb{R}^n \times (0,2T) \) and \( \|\tilde{f}\|_{\dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times (0,T))}. \) Note that \( u(x,t) = \int_{0}^{T} \Gamma \ast_{x,t} \tilde{f} \) for \( t > 0 \). Since, \( \Gamma \ast_{x,t} \tilde{f} \in \dot{B}^{\alpha+2\frac{\alpha-1}{\alpha}}_{\infty}(\mathbb{R}^n \times \mathbb{R}) \) with
\[ \|\Gamma \ast_{x,t} \tilde{f}\|_{\dot{B}^{\alpha+2\frac{\alpha-1}{\alpha}}_{\infty}(\mathbb{R}^n \times \mathbb{R})} \leq c\|\tilde{f}\|_{\dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times (0,T))}, \]
we have
\[ \|u\|_{\dot{B}^{\alpha+1\frac{\alpha-1}{\alpha}}_{\infty}(\mathbb{R}^n \times (0,T))} \leq c\|\tilde{f}\|_{\dot{B}^{\alpha}_{\infty}(\mathbb{R}^n \times (0,T))}. \tag{2.1} \]
If \( f|_{t=0} \neq 0 \), then let \( F(s) = f(s) - \Gamma_s * (f|_{t=0}) \) and \( U = \int_0^t \int_{\mathbb{R}^n} D_y \Gamma_{t-s} * F(s) dy ds \). Then

\[
\|U\|_{B^{\alpha+1, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))} \leq c \|F\|_{B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))}.
\]

Note that

\[
\int_0^t \int_{\mathbb{R}^n} D_y \Gamma_{t-s} * (\Gamma_s * (f|_{t=0})) ds = t \int_{\mathbb{R}^n} D_y \Gamma_t * (f|_{t=0}) dy, \quad \text{and} \quad \int_{\mathbb{R}^n} \Gamma_t * (f|_{t=0}) dy.
\]

By the same reasoning as for the proof of Proposition 4.1, we can show that

\[
\|t \int_{\mathbb{R}^n} D_y \Gamma_t * (f|_{t=0}) dy\|_{B^{\alpha+1, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))} \leq c \|f|_{t=0}\|_{B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))} 
\]

Combining the above two estimates we conclude that

\[
\|u\|_{B^{\alpha+1, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))} \leq \|U\|_{B^{\alpha+1, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))} + \|t \int_{\mathbb{R}^n} D_y \Gamma_t * (f|_{t=0}) dy\|_{B^{\alpha+1, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))} \leq c \|f\|_{B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))}.
\]

(2.2)

It is well known that \( D_x \Gamma(t) \in H^1(\mathbb{R}^n) \), where \( H^1(\mathbb{R}^n) \) denotes Hardy space. Since \( \|D_x \Gamma(t)\|_{H^1(\mathbb{R}^n)} \leq c t^{-\frac{1}{2}} \), we have

\[
\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c \int_0^t \int_{\mathbb{R}^n} |D_y \Gamma(-y, t-s)||f(s)||_{BMO(\mathbb{R}^n)} ds \leq c T^{\frac{1}{2}} \|f\|_{L^\infty((0, T); BMO(\mathbb{R}^n))}.
\]

(2.3)

This gives the third estimate of the proposition.

Finally, we will derive the second estimate of the proposition. Since \( D_x^2 u = \int_0^t D_y \Gamma_{t-s} * D_y^2 f(s) ds \), by Young’s Theorem we have

\[
\|D_x^2 u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c \int_0^t \|D_y \Gamma(-y, t-s)\|_{L^1(\mathbb{R}^n)} \|D_y^2 f(s)\|_{L^\infty(\mathbb{R}^n)} ds \leq c T^{\frac{1}{2}} \|D_y^2 f\|_{L^\infty(\mathbb{R}^n \times (0, T))}.
\]

Since \( D_t u = \int_0^t D_y \Gamma(t-s) * x_D f(s) ds \),

\[
\|D_t u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c \int_0^t \|D_y \Gamma(-y, t-s)\|_{L^1(\mathbb{R}^n)} \|D_t f(s)\|_{L^\infty(\mathbb{R}^n)} ds \leq c T^{\frac{1}{2}} \|D_t f\|_{L^\infty(\mathbb{R}^n \times (0, T))}.
\]

This gives the estimate

\[
\|u\|_{W^{2, 1}_2(\mathbb{R}^n \times (0, T))} \leq c T^{\frac{1}{2}} \|f\|_{W^{2, 1}_2(\mathbb{R}^n \times (0, T))}.
\]

(2.4)

Apply real interpolation theory to (2.3) and (2.4), we have the estimate

\[
\|u\|_{B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))} \leq c T^{\frac{1}{2}} \|f\|_{B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times (0, T))}, \quad 0 < \alpha < 2.
\]

The argument can be extended to any \( \alpha > 0 \).

**Appendix C. Proof of Proposition 4.3**

The first estimates in Proposition 4.3 are well known properties of the singular integral operator (see [16], [28] and [36]). Hence we have only to prove the second estimates. By the similar argument as in [16], it holds

\[
\|Rf\|_{B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times \mathbb{R})} \leq c \|f\|_{B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times \mathbb{R})}, \quad \alpha \in \mathbb{R}, \quad \text{for any } f \in B^{\alpha, \frac{\alpha}{2} + \frac{1}{2}}_\infty (\mathbb{R}^n \times \mathbb{R}).
If \( f \in \dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T)) \) with \( f|_{t=0} = 0 \), then there is \( \tilde{f} \in \dot{B}^\alpha_{\infty} (\mathbb{R}^n \times \mathbb{R}) \) extension of \( f \) with
\[
\| \tilde{f} \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times \mathbb{R})} \leq c \| f \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))},
\]
hence
\[
\| Rf \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \leq c \| f \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \quad \alpha \in \mathbb{R}, \quad \text{for any} \quad f \in \dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T)) \quad \text{with} \quad f|_{t=0} = 0.
\]
Now let us consider \( f \in \dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T)) \) with \( f|_{t=0} \neq 0 \). Let \( F = f - \Gamma_t \ast (f|_{t=0}) \), then \( F|_{t=0} = 0 \). Hence
\[
\| RF \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \leq c \| F \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \quad \alpha \in \mathbb{R}.
\]
Note that \( R \left( \Gamma_t \ast (f|_{t=0}) \right) = \Gamma_t \ast (R(f|_{t=0})) \), and
\[
\| R \left( \Gamma_t \ast (f|_{t=0}) \right) \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \leq c \| R(f|_{t=0}) \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \leq c \| f \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \leq c \| f \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))}.
\]
Here the first inequality and the last inequality hold for \( 0 < \alpha \). Therefore we conclude that
\[
\| Rf \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \leq c \| f \|_{\dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T))} \quad \text{for any} \quad f \in \dot{B}^\alpha_{\infty} (\mathbb{R}^n \times (0, T)), \quad \alpha > 0.
\]

**APPENDIX D. PROOF OF PROPOSITION 4.4**

The first estimate of Proposition 4.4 is well known property of Poisson operator (see [36]). Hence we have only to prove the second two estimates. By the first estimate
\[
\| Pf \|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq c \| f \|_{L^\infty(\mathbb{R}^n \times (0, T))} \quad \text{(4.5)}
\]
Since \( D^2_{x^t} Pf(t) = P(D^2_{x^t} f) \), \( D^2_{x^t} Pf = -\Delta x^t Pf \) and \( D_t Pf = P(D_t f) \), we have
\[
\| D^2_{x^t} Pf \|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq \| D^2_{x^t} Pf \|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq c \| D^2_{x^t} f \|_{L^\infty(\mathbb{R}^n \times (0, T))},
\]
\[
\| D_t Pf \|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq c \| D_t f \|_{L^\infty(\mathbb{R}^n \times (0, T))}.
\]
This gives the estimate
\[
\| Pf \|_{\dot{W}^{2,1}(\mathbb{R}^n \times (0, T))} \leq c \| f \|_{\dot{W}^{2,1}(\mathbb{R}^n \times (0, T))} \quad \text{(4.6)}
\]
Apply real interpolation theory to (4.5) and (4.6), we have the estimate
\[
\| Pf \|_{\dot{B}^\alpha_{\infty}(\mathbb{R}^n \times (0, T))} \leq c \| f \|_{\dot{B}^\alpha_{\infty}(\mathbb{R}^n \times (0, T))}, \quad 0 < \alpha < 2.
\]
The argument can be extended to any \( \alpha > 0 \).

The last estimate concerning smoothness comes easily from the properties of the Poisson kernel.

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