A new hyperelliptic solution class for the hyperbolic Ernst equation is obtained by transforming the regarding solution of the elliptic Ernst equation. Furthermore, a nontrivial way for obtaining general polarized colliding wave solutions from this hyperelliptic family of solutions is presented. The explicit form of the solutions for a Riemann surface of genus \( n = 1 \) is given. In addition, an explicit example in terms of a Khan-Penrose seed is provided, emphasizing the importance of the presented procedure for generating general polarized colliding plane-wave space times from space-times with a collinear polarization of the colliding waves.

PACS numbers: 02.30.Jr, 04.30.-w

I. INTRODUCTION

The search for new, inherently non-linear solutions of the hyperbolic Ernst equation, gains importance particularly in the context of colliding plane wave space-times. Here, the similarity between the elliptic and hyperbolic Ernst equations in certain coordinate frames can be employed for easily transforming already known solutions to new ones. Of course, the knowledge of the underlying symmetry algebra of the corresponding equations is of uttermost importance for the transformation procedure. In this paper the hyperelliptic solution class, that has been found by Meinel and Neugebauer ([5]) for the elliptic Ernst equation, will be transformed into a solution class of the hyperbolic Ernst equation. Furthermore, an outlook on utilizing this solution class for generating colliding plane wave space-times with non-collinear polarization of the colliding waves is provided.

II. THE HYPERBOLIC AND ELLIPTIC ERNST EQUATIONS

The elliptic Ernst equation has first been considered by Ernst ([2]) in the context of stationary axisymmetric space-times and can be written in polar coordinates \((\rho, \zeta)\) as follows:

\[
\left( \mathcal{R} \hat{Z} \right) \nabla^2 \hat{Z} = \left( \nabla \hat{Z} \right)^2,
\]

where \( \hat{Z} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \nabla = \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \zeta} \right) \) and \( \nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \zeta^2} \). Transforming (1) to complex coordinates \((z, \bar{z})\), given by

\[
z = \rho + i\zeta, \quad \bar{z} = \rho - i\zeta,
\]

yields the following form of the elliptic Ernst equation

\[
\left( \hat{Z} + \bar{Z} \right) \left[ 2\hat{Z}_{z\bar{z}} + \frac{\hat{Z}_z + \bar{Z}_{\bar{z}}}{z + \bar{z}} \right] = 4\hat{Z}_z \bar{Z}_{\bar{z}}
\]

In full analogy, the hyperbolic Ernst equation can be written in the following form

\[
(Z + \bar{Z}) \left[ 2Z_{fg} + \frac{Z_f + Z_g}{f + g} \right] = 4Z_f Z_g
\]
where \( Z : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) is a complex function of the two independent real coordinates \((f, g)\).

According to [7], the Ernst equation (4) admits the following point symmetries

\[
Z(f, g) \rightarrow Z(f + \beta, g - \beta), \tag{5}
\]
\[
Z(f, g) \rightarrow Z(e^\alpha f, e^\alpha g), \tag{6}
\]
\[
Z(f, g) \rightarrow Z(f, g) + \gamma, \tag{7}
\]
\[
Z(f, g) \rightarrow e^\delta Z(f, g), \tag{8}
\]

where \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \). Of course, the elliptic equation (3) exhibits the same symmetries.

III. THE HYPERELLIPTIC SOLUTION CLASS

The hyperelliptic solution class has been first considered by Meinel and Neugebauer (5). Accordingly, it has been shown that a solution class in terms of hyperelliptic integrals can be obtained by considering solutions of the stationary axisymmetric vacuum field equations associated with Jacobi’s inversion problem

\[
\hat{Z} = \exp \left( \sum_{m=1}^{n} \frac{\kappa(m) \, d\kappa}{\hat{W}} - u_n \right), \tag{9}
\]

where

\[
\hat{W} (\kappa; z, \bar{z}) = (\kappa + iz) (\kappa - i\bar{z}) \prod_{j=1}^{n} (\kappa - \kappa_j) (\kappa - \bar{\kappa}_j), \tag{10}
\]

and \( \kappa_j \) \((j = 1, \ldots, n)\) are arbitrary complex constants. Furthermore, the upper integration limits in (9) are functions of \( z \) and \( \bar{z} \), which have to be calculated by solving the following inversion problem

\[
\sum_{m=1}^{n} \int_{\kappa_m}^{\kappa(m)} \frac{\kappa^j d\kappa}{\hat{W}} = u_j, \quad j = 0, 1, 2, \ldots, n - 1, \tag{11}
\]

where the \( u_j \) are real functions of \( z \) and \( \bar{z} \), which solve the Euler-Poisson-Darboux (EPD) equation

\[
2 (z + \bar{z}) \partial_{zz} u_j + \partial_z u_j + \partial_{\bar{z}} u_j = 0. \tag{12}
\]

In addition, the \( u_j \) are required to satisfy the following recursive relations

\[
i \partial_z u_j = \frac{1}{2} u_{j-1} + z \partial_{\bar{z}} u_{j-1}, \quad j = 1, 2, \ldots, n, \tag{13}
\]
\[
- \partial_{\bar{z}} u_j = \frac{1}{2} u_{j-1} + \bar{z} \partial_z u_{j-1}, \quad j = 1, 2, \ldots, n. \tag{14}
\]

Note, that the EPD equation for \( u_{j-1} \) occurs as the integrability condition for the system (13)-(14).

Adapting the hyperelliptic class to the hyperboliv Ernst equation

Some requirements have to be met for transforming the solution (9) of the elliptic Ernst equation to a new solution of the hyperbolic Ernst equation. Note, that the proof in [5] of (9) constituting a solution of the elliptic Ernst equation relies essentially on three pillars: The functions \( u_j \) defined via (12)-(14) need to be real-valued, and the relations

\[
\sum_{m=1}^{n} \frac{\kappa(m) + iz}{\hat{W}(m)} \left( \kappa(m) \right)^{j-1} \partial_z \kappa(m) = 0, \quad j = 1, 2, \ldots, n - 1, \tag{15}
\]
\[
\sum_{m=1}^{n} \frac{\kappa(m) - i\bar{z}}{\hat{W}(m)} \left( \kappa(m) \right)^{j-1} \partial_{\bar{z}} \kappa(m) = 0, \quad j = 1, 2, \ldots, n - 1, \tag{16}
\]
where \( \hat{W}^{(m)} = \hat{W}(\kappa^{(m)}) \), and

\[
\partial_z \ln \hat{Z} = \sum_{m=1}^{n} \frac{\kappa^{(m)} + iz}{W^{(m)}} (\kappa^{(m)})^{n-1} \partial_z \kappa^{(m)},
\]

\[
\partial_{\bar{z}} \ln \hat{Z} = \sum_{m=1}^{n} \frac{\kappa^{(m)} - iz}{W^{(m)}} (\kappa^{(m)})^{n-1} \partial_{\bar{z}} \kappa^{(m)},
\]

are required to hold. Hence, we have to consider the class of transformations relating \((z, \bar{z})\) to \((f, g)\), which leave the Ernst equation \((3)\) and these three conditions invariant. It turns out that the following transformation meets all the requirements

\[
z \to i(\alpha f + \beta),
\]

\[
\bar{z} \to i(\alpha g - \beta),
\]

\[
Z(f, g) = \hat{Z}(i(\alpha f + \beta), i(\alpha g - \beta)),
\]

\[
W(\kappa; f, g) = \hat{W}(\kappa; i(\alpha f + \beta), i(\alpha g - \beta)),
\]

where \(\alpha \neq 0, \beta \in \mathbb{R}\).

Accordingly, a new solution class for the hyperbolic Ernst equation has been generated

\[
Z = \exp \left( \sum_{m=1}^{n} \int_{\kappa_m}^{\kappa} \kappa^{3} d\kappa \frac{d}{W} - u_n \right),
\]

where

\[
W(\kappa; f, g)^{2} = (\kappa - \alpha f - \beta)(\kappa + \alpha g - \beta) \prod_{j=1}^{n} (\kappa - \kappa_j)(\kappa - \bar{\kappa}_j),
\]

and \(\kappa_j (j = 1, \ldots, n)\) are arbitrary complex constants. In full analogy to the elliptic case, the upper integration limits in \((9)\) are functions of \(f\) and \(g\), which have to be calculated by solving the following inversion problem

\[
\sum_{m=1}^{n} \int_{\kappa_m}^{\kappa} \kappa^{3} d\kappa \frac{d}{W} = u_j, \quad j = 0, 1, 2, \ldots, n - 1
\]

where the \(u_j\) are real functions of \(f\) and \(g\), which solve the EPD equation\(^1\)

\[
2(f + g) \partial_g^2 u_j + \partial_f u_j + \partial_g u_j = 0.
\]

In addition, the \(u_j\) are required to satisfy the following recursive relations

\[
\partial_f u_j = -\frac{\alpha}{2} u_{j-1} + (\alpha f + \beta) \partial_f u_{j-1}, \quad j = 1, 2, \ldots, n,
\]

\[
\partial_g u_j = -\frac{\alpha}{2} u_{j-1} - (\alpha g - \beta) \partial_g u_{j-1}, \quad j = 1, 2, \ldots, n.
\]

Therefore, the \(u_j\) are certainly real-valued for \(\alpha, \beta \in \mathbb{R}\), if \(u_0\) is real-valued.

Consequently, the proof of the invariance of relations \((15)-(16)\) proceeds as follows. Differentiating \((25)\) with respect to \(f\) and \(g\), yields the important intermediate result:

\[
\sum_{m=1}^{n} \left( \frac{(\kappa^{(m)})^{2}}{W^{(m)}} \right) \partial_f \kappa^{(m)} + \frac{\alpha}{2} \int_{\kappa_m}^{\kappa} \frac{\kappa^{3} d\kappa}{(\kappa - \alpha f - \beta) W} \right) = \partial_f u_j, \quad j = 0, 1, 2, \ldots, n - 1,
\]

\[
\sum_{m=1}^{n} \left( \frac{(\kappa^{(m)})^{2}}{W^{(m)}} \right) \partial_g \kappa^{(m)} - \frac{\alpha}{2} \int_{\kappa_m}^{\kappa} \frac{\kappa^{3} d\kappa}{(\kappa + \alpha g - \beta) W} \right) = \partial_g u_j, \quad j = 0, 1, 2, \ldots, n - 1,
\]

\(^1\) The EPD equation is invariant under the transformation \((19)-(20)\), cf. \((6)\).
where $W^{(m)} = W(\kappa^{(m)})$ and the following identities have been used

\[
\frac{\partial f}{W} = \frac{\alpha}{2(\kappa - \alpha f - \beta)W},
\]

\[
\frac{\partial g}{W} = -\frac{\alpha}{2(\kappa + \alpha g - \beta)W}.
\]

Hence, the transformed relations (15)-(16) follow immediately

\[
\frac{\partial f}{W} u_j = \sum_{m=1}^{n} \left\{ \frac{(\kappa^{(m)})}{W^{(m)}} \partial f \kappa^{(m)} + \frac{\alpha}{2} \int_{\kappa_m}^{\kappa} \frac{\kappa^j d\kappa}{(\kappa - \alpha f - \beta)W} \right\}
\]

\[
= (\alpha f + \beta) \frac{\partial f}{W} u_{j-1} + \frac{\alpha}{2} u_{j-1}
\]

\[
= (\alpha f + \beta) \sum_{m=1}^{n} \left\{ \frac{(\kappa^{(m)})}{W^{(m)}} \partial f \kappa^{(m)} + \frac{\alpha}{2} \int_{\kappa_m}^{\kappa} \frac{\kappa^j d\kappa}{(\kappa - \alpha f - \beta)W} \right\}
\]

\[
+ \frac{\alpha}{2} \sum_{m=1}^{n} \int_{\kappa_m}^{\kappa} \frac{\kappa^j d\kappa}{W}
\]

And by equating both expressions one obtains

\[
\sum_{m=1}^{n} \frac{\kappa^{(m)} - \alpha f - \beta}{W^{(m)}} \left( \kappa^{(m)} \right)^{j-1} \partial f \kappa^{(m)} = 0, \quad j = 1, 2, \ldots, n - 1,
\]

(33)

and similarly

\[
\sum_{m=1}^{n} \frac{\kappa^{(m)} + \alpha g - \beta}{W^{(m)}} \left( \kappa^{(m)} \right)^{j-1} \partial g \kappa^{(m)} = 0, \quad j = 1, 2, \ldots, n - 1.
\]

(34)
Furthermore, considering the first derivative yields

$$\partial_f \ln Z = \sum_{m=1}^{n} \frac{(k^{(m)}(m))^{n} \partial_{f}k^{(m)}}{W(m)} \frac{\alpha^{(m)}}{2} \int_{\kappa^{(m)}}^{\kappa^{n} d\kappa} \frac{\kappa^{n} d\kappa}{(\kappa - \alpha f - \beta) W^{(m)}}$$

$$- \frac{\alpha}{2} \int_{\kappa^{(m)}}^{\kappa^{n} d\kappa} \frac{\kappa^{n} d\kappa}{(\kappa - \alpha f - \beta) W^{(m)}}$$

$$= \sum_{m=1}^{n} \frac{k^{(m)} - \alpha f - \beta}{W(m)} \left(\kappa^{(m)}\right)^{n-1} \partial_{f}k^{(m)}$$

$$= \frac{\alpha}{2} \int_{\kappa^{(m)}}^{\kappa^{n} d\kappa} \frac{\kappa^{n-1} d\kappa}{(\kappa - \alpha f - \beta) W^{(m)}} + \frac{(\alpha f + \beta) \kappa^{n-1}}{(\kappa - \alpha f - \beta) W^{(m)}} \right) d\kappa$$

$$= \sum_{m=1}^{n} \frac{k^{(m)} - \alpha f - \beta}{W(m)} \left(\kappa^{(m)}\right)^{n-1} \partial_{f}k^{(m)}, \quad (35)$$

and analogous

$$\partial_g \ln Z = \sum_{m=1}^{n} \frac{k^{(m)} + \alpha g - \beta}{W(m)} \left(\kappa^{(m)}\right)^{n-1} \partial_{g}k^{(m)}. \quad (36)$$

Thus, the relations \((15)-(16)\) and \((17)-(18)\) still hold, where \(i\bar{z}\) and \(-i\bar{z}\) have been replaced by \(-\alpha f - \beta\) and \(\alpha g - \beta\), respectively. Now, by following exactly the same steps of the proof provided in \([5]\), it can be shown that \((23)\) constitutes a solution of the hyperbolic Ernst equation \([4]\).

**IV. COLLIDING PLANE WAVE SOLUTIONS FROM THE HYPERELLPTIC SOLUTION CLASS**

Colliding plane wave space-times emerge as special solution class of the hyperbolic Ernst equation, satisfying a special type of boundary conditions on two null surfaces of the underlying manifold. It is possible to write these wave-conditions in the form of two simple limit-processes (cf. [3]), namely

$$\frac{1}{2} - f \left(1 + \frac{Z f \bar{Z} f}{(Z + \bar{Z})^2}\right) = \frac{1}{2} - f \left(1 + \frac{Z f \bar{Z} f}{(Z + \bar{Z})^2}\right)$$

$$\frac{1}{2} - g \left(1 + \frac{Z g \bar{Z} g}{(Z + \bar{Z})^2}\right) = \frac{1}{2} - g \left(1 + \frac{Z g \bar{Z} g}{(Z + \bar{Z})^2}\right)$$

$$\left(1 + \frac{Z f \bar{Z} f}{(Z + \bar{Z})^2}\right) = \left(1 + \frac{Z g \bar{Z} g}{(Z + \bar{Z})^2}\right)$$

For examining the wave conditions in dependence of the functions \(u_j\), the first derivatives of \(Z\) in terms of the derivatives of the \(u_j\) are needed. After some rather lengthy manipulations, that can be found in the appendix, \(\partial_f \ln Z\) and \(\partial_g \ln Z\) read

$$\partial_f \ln Z = A(f, g) \partial_f u_0(f, g) + B(f, g), \quad (39)$$

$$\partial_g \ln Z = C(f, g) \partial_g u_0(f, g) + D(f, g), \quad (40)$$
where

\[ A(f, g) = \sum_{k,i=1}^{n} \left( \kappa^{(k)} - \alpha f - \beta \right) \left( \kappa^{(k)} \right)^{n-1} F_{i,k} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) (\alpha f + \beta)^{i-1}, \tag{41} \]

\[ B(f, g) = \frac{\alpha}{2} \sum_{k,i=1}^{n} \left( \kappa^{(k)} - \alpha f - \beta \right) \left( \kappa^{(k)} \right)^{n-1} F_{i,k} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) \]

\[ \times \left\{ \sum_{l=1}^{i-1} (\alpha f + \beta)^{l-1} u_{i-l-1} - \sum_{m=1}^{n} \kappa^{(m)} \int_{\kappa_{m}}^{\kappa_{m+1}} \frac{\kappa^{l-1} d\kappa}{(\kappa - \alpha f - \beta) W} \right\}, \tag{42} \]

\[ C(f, g) = \sum_{k,i=1}^{n} \left( \kappa^{(k)} + \alpha g - \beta \right) \left( \kappa^{(k)} \right)^{n-1} F_{i,k} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) (\beta - \alpha g)^{i-1}, \tag{43} \]

\[ D(f, g) = -\frac{\alpha}{2} \sum_{k,i=1}^{n} \left( \kappa^{(k)} + \alpha g - \beta \right) \left( \kappa^{(k)} \right)^{n-1} F_{i,k} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) \]

\[ \times \left\{ \sum_{l=1}^{i-1} (\beta - \alpha g)^{l-1} u_{i-l-1} - \sum_{m=1}^{n} \kappa^{(m)} \int_{\kappa_{m}}^{\kappa_{m+1}} \frac{\kappa^{l-1} d\kappa}{(\kappa + \alpha g - \beta) W} \right\}, \tag{44} \]

and \( F_{i,k} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) \) is a rational function of the \( \kappa^{(j)}, j = 1, \ldots, n \).

Since (39) and (40) are linear in \( \partial_f u_0 \) and \( \partial_g u_0 \), it is convenient to restrict the range of possible \( u_0 \) to those, satisfying

\[ 0 < \lim_{(f,g) \to \left( \frac{1}{2}, \frac{1}{2} \right) \frac{1}{2}} \left( \frac{1}{2} - g \right) |\partial_g u_0|^2 = \gamma_2 < \infty, \tag{45} \]

\[ 0 < \lim_{(f,g) \to \left( \frac{1}{2}, \frac{1}{2} \right) 1/2} \left( \frac{1}{2} - f \right) |\partial_f u_0|^2 = \gamma_1 < \infty. \tag{46} \]

Thus \( u_0 \) can be interpreted as a wave solution in its very own right. Furthermore, the wave-solution corresponding to \( u_0 \) is collinear, since \( u_0 \) is a real valued function, satisfying the EPD equation.

However, we also have to examine the structure of (41)-(44) for determining, whether \( Z \) is satisfying (37) and (37). First of all, equation (39) is considered together with condition (46).

Because \( u_0 \) satisfies

\[ 0 < \lim_{(f,g) \to (1/2,1/2)} \left( \frac{1}{2} - f \right) |\partial_f u_0|^2 = \gamma_1 < \infty, \tag{47} \]

it follows that \( \partial_f u_0 \) is locally of the form

\[ \partial_f u_0 \left( f, \frac{1}{2} \right) = a + b \frac{f}{\sqrt{1 - 2f}} + \mathcal{O}(X) \quad \text{close to } (f, g) = \left( \frac{1}{2}, \frac{1}{2} \right), \]

where \( a, b \in \mathbb{C} \) and \( X \) is representative for higher order terms in \( f \). Hence, \( u_0 \) assumes the local form

\[ u_0 \left( f, \frac{1}{2} \right) = a f - b \frac{f}{\sqrt{1 - 2f}} + c + \mathcal{O}(X) \quad \text{close to } (f, g) = \left( \frac{1}{2}, \frac{1}{2} \right), \]

where \( c \in \mathbb{C} \). In addition, all the \( u_j \), derived from \( u_0 \) via relation (27), are at least of the same order as \( u_0 \) close to \( (f, g) = \left( \frac{1}{2}, \frac{1}{2} \right) \), since they are obtained by integrating (27)

\[ u_j (f, g) = \frac{1}{2} \int_{f_0}^{f} \left[ \alpha u_{j-1} \left( \tilde{f}, g \right) + 2 \left( \alpha \tilde{f} + \beta \right) \partial_{\tilde{f}} u_{j-1} \left( \tilde{f}, g \right) \right] d\tilde{f} + v_j (g), \tag{48} \]

with some arbitrary constant \( f_0 \) and a function \( v_j (g) \) that needs to be determined by plugging \( u_j (f, g) \) into the EPD equation. As a result, the term \( \frac{\alpha}{2} \sum_{i=1}^{i-1} (\alpha f + \beta)^{i-1} u_{i-1-1} \) in (42) does not diverge for \( (f, g) \to \left( \frac{1}{2}, \frac{1}{2} \right) \) and the
The next step is to exclude possible roots or poles of the term \( (\kappa^{(k)} - \alpha f - \beta) \), by noting that it is always possible to ensure \( 0 < |\kappa^{(k)} - \alpha f - \beta| |\kappa^{(k)}|^{n-1} < \infty \) for \((f, g) \to \left( \frac{1}{2}, \frac{1}{2} \right) \) by continuously changing the constants \( \kappa_k \), \( \alpha \) and \( \beta \). This follows since \( \kappa^{(k)} \) is a nontrivial hyperelliptic function of \( u_k \) and thus of \((f, g)\), which has a discrete set of poles and roots containing no limit points. Therefore the location of poles and roots can be shifted by a continuous variation of \( \kappa_k \), such that there are no problems when \((f, g) \to \left( \frac{1}{2}, \frac{1}{2} \right) \). The possibility for \( \sum_{m=1}^{n} \int_{\kappa_m}^{(m)} \frac{\kappa^{(m)} d\kappa}{(\kappa - \alpha f - \beta)\kappa} \) to become singular is ruled out by the same argument.

Finally, \( F_{i,k} (\kappa^{(1)}, \ldots, \kappa^{(n)}) \) has no singular points, if all \( \kappa^{(i)} \) are assumed to be pairwise disjoint for each point \((f, g)\) in the domain of consideration (consult the appendix for the detailed structure of \( F_{i,k} (\kappa^{(1)}, \ldots, \kappa^{(n)}) \)). Accordingly, condition (37) becomes

\[
\frac{1}{4} \leq \lim_{(f,g) \to \left( \frac{1}{2}, \frac{1}{2} \right)} \left[ \left( \frac{1}{2} - f \right) \frac{Z_f \bar{Z}_f}{(Z + \bar{Z})^2} \right] \\
= \lim_{(f,g) \to \left( \frac{1}{2}, \frac{1}{2} \right)} \left[ \left( \frac{1}{2} - f \right) \frac{Z \bar{Z} \left( A \partial_f u_0 + B \right) \left( \bar{A} \partial_{\bar{f}} u_0 + \bar{B} \right)}{(Z + \bar{Z})^2} \right] \\
= \left| A \left( \frac{1}{2}, \frac{1}{2} \right) \right|^2 \left| Z \left( \frac{1}{2}, \frac{1}{2} \right) \right|^2 \gamma_1 \frac{1}{2} \leq \frac{1}{4}.
\]  

where (47) has been used. The requirement (49) can be satisfied by replacing \( u_0 \to \delta u_0 \) with \( \delta \in \mathbb{R} \), if \( 0 < \left| A \left( \frac{1}{2}, \frac{1}{2} \right) \right| < \infty \) and \( 0 < \left| \text{Re} \left( \frac{1}{2}, \frac{1}{2} \right) \right| < \infty \), since the EPD equation is invariant under a rescaling of \( u_0 \) with an arbitrary real parameter. Furthermore, the term \( Z \left( \frac{1}{2}, \frac{1}{2} \right) \) cannot cause any trouble, because neither \( \sum_{m=1}^{n} \int_{\kappa_m}^{(m)} \frac{\kappa^{(m)} d\kappa}{(\kappa - \alpha f - \beta)\kappa} \) nor \( u_n \) are permitted to become singular for \((f, g) \to \left( \frac{1}{2}, \frac{1}{2} \right) \) for an adequate choice of the constants \( \kappa_j \). Similarly, the case \( A \left( \frac{1}{2}, \frac{1}{2} \right) = 0 \) has already been excluded. Therefore, the first condition (37) can always be satisfied.

The discussion of the second condition is in principle similar, when assuming that the first condition has not already been fixed. However, the situation turns out to be highly nontrivial, since both conditions need to be satisfied simultaneously. This can be seen as follows.

Plugging (40) into the condition (38) leads to the requirement

\[
\frac{1}{4} \leq \lim_{(f,g) \to \left( \frac{1}{2}, \frac{1}{2} \right)} \left[ \left( \frac{1}{2} - g \right) \frac{Z_g \bar{Z}_g}{(Z + \bar{Z})^2} \right] \\
= \lim_{(f,g) \to \left( \frac{1}{2}, \frac{1}{2} \right)} \left[ \left( \frac{1}{2} - g \right) \frac{Z \bar{Z} \left( C \partial_g u_0 + D \right) \left( \bar{C} \partial_{\bar{g}} u_0 + \bar{D} \right)}{(Z + \bar{Z})^2} \right] \\
= \left| C \left( \frac{1}{2}, \frac{1}{2} \right) \right|^2 \left| Z \left( \frac{1}{2}, \frac{1}{2} \right) \right|^2 \gamma_2 \frac{1}{2} \leq \frac{1}{4},
\]  

which appears to be problematic due to the obvious lack of symmetry between \( \partial_f \ln Z \) and \( \partial_g \ln Z \), leading to different values of \( A \left( \frac{1}{2}, \frac{1}{2} \right) \) and \( C \left( \frac{1}{2}, \frac{1}{2} \right) \). Rescaling \( u_0 \) a second time to satisfy condition (50) would also alter condition (49).

In all, it is not possible to give a general statement about the compliance of \( Z \) with the conditions (50) and (49) for arbitrary functions \( u_j \), since the hyperelliptic functions do not admit an explicit evaluation of \( A(f, g), B(f, g), C(f, g), \) and \( D(f, g) \) at \( f = \frac{1}{2} \) and \( g = \frac{1}{2} \). In terms of degrees of freedom, there are \( 2n + 3 \) real parameters from the \( n \) complex constants \( \kappa_i \), \( \alpha \), and \( \beta \) plus a possible rescaling of \( u_0 \), which can be adjusted in order to satisfy the two conditions (50) and (49). However, each specific case has to be considered separately, since the explicit dependencies of \( A(f, g), B(f, g), C(f, g), \) and \( D(f, g) \) on those parameters are unknown. An expansion of the regarding hyperelliptic functions to a power series close to \( f = \frac{1}{2} \) and \( g = \frac{1}{2} \) might give a deeper insight on their local dependency on the \( \kappa_i \) \((i = 1, 2, \ldots , n)\), \( \alpha \), \( \beta \) and \( u_0 \), such that a suitable choice of these parameters might become easier and a general statement can be formulated.
Note, the relaxed wave conditions (cf. [3])

\[
0 \leq \lim_{(f,g) \to (\frac{1}{2}, \frac{1}{2})} \left[ \left( \frac{1}{2} - g \right) \frac{Z_g Z_g}{(Z + Z)^2} \right] = k_2 \frac{1}{2} < \frac{1}{2}, \quad (51)
\]

\[
0 \leq \lim_{(f,g) \to (\frac{1}{2}, \frac{1}{2})} \left[ \left( \frac{1}{2} - f \right) \frac{Z_f Z_f}{(Z + Z)^2} \right] = k_1 \frac{1}{2} < \frac{1}{2}, \quad (52)
\]

can always be met by rescaling \( u_0 \) with some constant \( 0 < \delta < 1 \). This possibility will be ignored here, because of the unclear physical implications emerging from impulsive matter tensor components at the junctions \( f = \frac{1}{2} \) and \( g = \frac{1}{2} \), which can occur in this case.

One might ask, what possibilities remain for \( Z \) to satisfy the junction conditions \([37] + [38]\), if \( u_0 \) does not fulfill \([47]\). Possible singularities of the right order may occur in the functions \( A(f,g) \) and \( B(f,g) \) (cf. \([41] + [42]\)). However, the junction conditions cannot be met, if both functions only exhibit singularities of integer order, since the term \( \frac{Z_f Z_f}{(Z + Z)^2} \)

would then have a singularity of higher order than 1 and accordingly \( \lim_{(f,g) \to (\frac{1}{2}, \frac{1}{2})} \left[ \left( \frac{1}{2} - f \right) \frac{Z_f Z_f}{(Z + Z)^2} \right] \) would not be finite. However, no general statement can be made for arbitrary functions \( u_j \).

All in all, the procedure presented here corresponds to a particular simple way for generating arbitrary polarized solutions from collinear ones, which cannot be reduced to a simple coordinate transformation. Moreover, the solutions generated by this method are highly non-trivial due to the occurrence of hyperelliptic integrals. Meinel and Neugebauer have mentioned \([5]\), that it is generally possible to solve the inversion problem \([11]\) by means of hyperelliptic functions. Still, the analysis and interpretation of solution in terms of these special functions remain a hard challenge and their physical relevance might be doubtful.

**An example: The hyperelliptic class for \( n = 1 \)**

For \( n = 1 \), the regarding hyperelliptic integrals reduce to elliptic integrals, that can be expressed in terms of elliptic functions. The details of the calculation can be found in the appendix, the result is an inherently non-linear solution \( Z \) of the hyperbolic Ernst equation \([4]\):

\[
Z(f,g) = \exp \left\{ \pm \frac{2i}{\kappa_1 - \kappa} \Pi \left[ k(f,g), \text{am} (\tilde{u}_0 (f,g), m(f,g)) \right] \right\}
\]

\[
\pm \kappa_1 u_0 (f,g) - u_1 (f,g), \quad (53)
\]

where \( \Pi (\cdot, \cdot, \cdot) \) denotes the incomplete elliptic integral of the third kind, \( \text{am}(\cdot, \cdot) \) is the Jacobian amplitude function (cf. [4]) and \( u_1 (f,g) \) follows from \([48]\):

\[
u_1 (f,g) = \frac{1}{2} \int_0^f \left[ \alpha u_0 \left( \tilde{f}, g \right) + 2 \left( \alpha \cdot \beta \right) \partial \tilde{f} u_0 \left( \tilde{f}, g \right) \right] d \tilde{f} + v_1 (g),
\]

and the following abbreviations have been used

\[
\tilde{u}_0 = \mp \sqrt{\kappa_1 - \alpha f - \beta} \left( \tilde{k}_1 + \alpha g - \beta \right) \frac{u_0}{2}, \quad (54)
\]

\[
m(f,g) = \frac{(\kappa_1 - \alpha f - \beta) (\kappa_1 + \alpha g - \beta)}{(\kappa_1 - \alpha f - \beta) (\tilde{k}_1 + \alpha g - \beta)}, \quad (55)
\]

\[
k (f,g) = \frac{(\kappa_1 + \alpha g - \beta)}{(\tilde{k}_1 + \alpha g - \beta)}. \quad (56)
\]

**An elliptic solution from a Khan-Penrose seed**

For demonstrating the practical relevance of the methods discussed so far, a special solution for the \( n = 1 \) case is presented, where the well-known Khan Penrose solution (cf. [4]) is taken as a seed solution \( u_0 \):
\[ u_0 = 2\gamma \arctanh \left( \sqrt{\frac{1}{2} - f \sqrt{\frac{1}{2} + g}} - \sqrt{\frac{1}{2} + f \sqrt{\frac{1}{2} - g}} \right) \]  

where \( \gamma \) is used for fitting the generated solution to the junction conditions.

After trying some different values for \( \alpha, \beta, \) and \( \kappa_1 \), the following choice turns out to be compatible with the wave conditions

\[ \alpha = 1, \quad \beta = 0, \quad \kappa_1 = \frac{i}{4}, \]  

yielding

\[ u_1 (f, g) = \frac{\gamma}{2} \left\{ \sqrt{1 + 2f} \sqrt{1 - 2g} - \sqrt{1 + 2g} \sqrt{1 - 2f} - 2(f - g) \arctanh \left[ \frac{1}{2} \left( \sqrt{1 + 2f} \sqrt{1 - 2g} + \sqrt{1 + 2g} \sqrt{1 - 2f} \right) \right] \right\} \]  

which is a valid solution of the EPD equation that satisfies (27)-(28).

Hence, the following solution has been generated

\[ Z(f, g) = \exp \left\{ \frac{i}{\sqrt{(f - \frac{i}{4})(g - \frac{i}{4})}} \Pi[k(f, g), \text{am}(-\tilde{u}_0(f, g), m(f, g)), m(f, g)] + \frac{i}{4} u_0(f, g) - u_1(f, g) \right\}, \]  

where

\[ \tilde{u}_0 = \sqrt{\left( \frac{i}{4} - f \right) \left( g - \frac{i}{4} \right) u_0}, \]  

and all signs have been chosen appropriately, such that the underlying square-roots are taken consistently.

Furthermore, choosing \( \gamma = 2 \) gives the following limit values

\[ \lim_{(f, g) \to (\frac{1}{2}, \frac{1}{2})} \left[ \sqrt{\frac{1}{2} - g} \frac{Z_f}{Z} \right] = 1 + \frac{i}{2}, \]

\[ \lim_{(f, g) \to (\frac{1}{2}, \frac{1}{2})} \left[ \sqrt{\frac{1}{2} - f} \frac{Z_g}{Z} \right] = -1 + \frac{i}{2}, \]

leading to

\[ \lim_{(f, g) \to (\frac{1}{2}, \frac{1}{2})} \left[ \frac{1}{2} - g \right] \frac{Z_g Z_f}{(Z + \bar{Z})^2} = \frac{5}{16}, \]  

\[ \lim_{(f, g) \to (\frac{1}{2}, \frac{1}{2})} \left[ \frac{1}{2} - f \right] \frac{Z_f Z_g}{(Z + \bar{Z})^2} = \frac{5}{16}. \]

Thus, (60) satisfies the junction conditions (37)-(38). As a result, a potential \( Z \) describing a new colliding plane wave space-time has been created. Figures 1-2 show the real and imaginary parts of \( Z \) in the domain \( \Omega_{f,g} = \{(f, g) \in \mathbb{R}^2, \ f < \frac{1}{2}, \ g < \frac{1}{2}, \ f + g > 0\} \).
As a main result, a new solution class for the hyperbolic Ernst equation has been obtained. Furthermore, this class offers a way for creating non-parallel polarized wave solutions from collinear wave solutions. Consequently, it has been shown that solutions of the EPD equation that satisfy the wave conditions can be used as seed solutions for generating general polarized waves related to a hyperelliptic class of solutions of the Ernst equation. In this context, a specific example of an elliptic solution generated from the Khan-Penrose solution has been provided. However, it has not been possible to formulate a general argument, delivering an exact mathematical statement about the compliance of solutions obtained from arbitrary seeds with the wave conditions emerging in relation to plane-wave collisions. A series expansion of the regarding hyperelliptic functions with respect to the external parameters could be one possibility for obtaining the desired statement. Obviously, this problem leaves space for further research. Moreover, analytical features of the new hyperelliptic solution class can be examined by considering more specific examples, of which a possible wave analogue for the stationary rotating disk of dust (see [5]) might be of particular importance.

All together, the complexity of the obtained solution class opens a wide field for further research also in pure mathematics and of course for colliding plane-wave space-times.
Computing the first derivatives

In this section, the reduction of \( \frac{\partial f^{(m)}}{W^{(m)}} \) into terms of the \( u_j \) and corresponding derivatives is performed. For this purpose, (29) is written in the form

\[
\sum_{m=1}^{n} \frac{\partial f^{(m)}}{W^{(m)}} \left( \kappa^{(m)} \right)^j = \partial_f u_j - \frac{\alpha}{2} \sum_{m=1}^{n} \int_{\kappa_m} \frac{\kappa^j d\kappa}{(\kappa - \alpha f - \beta) W}, \quad j = 0, 1, 2, \ldots, n - 1,
\]

\[
v^T A = w,
\]

where the vectors \( v, w \in \mathbb{C}^n \) and the matrix \( A \in \mathbb{C}^{n \times n} \) have been introduced according to

\[
(v)_i = \frac{\partial f^{(i)}}{W^{(i)}}, \quad i = 1, 2, \ldots, n,
\]

\[
(w)_j = \partial_f u_{j-1} - \frac{\alpha}{2} \sum_{m=1}^{n} \int_{\kappa_m} \frac{\kappa^{-1} d\kappa}{(\kappa - \alpha f - \beta) W}, \quad j = 1, 2, \ldots, n,
\]

\[
(A)_{ij} = \left( \kappa^{(i)} \right)^{-1}.
\]

Hence, \( A \) is a Vandermonde matrix

\[
A = \begin{pmatrix}
1 & \kappa^{(1)} & \ldots & (\kappa^{(1)})^{n-1} \\
1 & \kappa^{(2)} & \ldots & (\kappa^{(2)})^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \kappa^{(n)} & \ldots & (\kappa^{(n)})^{n-1}
\end{pmatrix},
\]

that can be inverted, if

\[
det A = \prod_{1 \leq i < j \leq n} \left( \kappa^{(j)} - \kappa^{(i)} \right) \neq 0.
\]

Therefore, the \( \kappa^{(i)} \) are assumed to be pairwise disjoint for each point \((f, g)\) in the domain of consideration. The inverse of \( A \) reads

\[
(A^{-1})_{ij} = (-1)^j \left\{ \sum_{1 \leq m_1 < \ldots < m_{n-1+1} \leq n \atop m_1 \ldots m_{n-1+1} \neq j} \prod_{1 \leq m \leq n \atop m \neq j} \left( \kappa^{(m)} - \kappa^{(j)} \right) \right\},
\]

and accordingly \( \frac{\partial f^{(j)}}{W^{(j)}} \) can be written as

\[
\frac{\partial f^{(j)}}{W^{(j)}} = \sum_{i=1}^{n} (w)_i (A^{-1})_{ij}
\]

\[
= \sum_{i=1}^{n} F_{ij} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) \left\{ \partial_f u_{i-1} - \frac{\alpha}{2} \sum_{m=1}^{n} \int_{\kappa_m} \frac{\kappa^{i-1} d\kappa}{(\kappa - \alpha f - \beta) W} \right\},
\]
where $F_{ij}(\kappa^{(1)}, \ldots, \kappa^{(n)}) = (A^{-1})_{ij}$ is a rational function of $\kappa^{(1)}, \ldots, \kappa^{(n)}$. Similarly, the derivative with respect to $g$ leads to

$$\frac{\partial_g \kappa^{(j)}}{W^{(j)}} = \sum_{i=1}^{n} F_{ij}(\kappa^{(1)}, \ldots, \kappa^{(n)}) \left\{ \partial_g u_{i-1} + \frac{\alpha}{2} \sum_{m=1}^{n} \int_{\kappa_{m}}^{\kappa^{(m)}} \frac{\kappa^{i-1} d\kappa}{(\kappa + \alpha g - \beta) W} \right\}. \quad (72)$$

As a result, the derivatives $\partial_f \ln Z$ and $\partial_g \ln Z$ can be written in terms of the $u_j$, when plugging (71) and (72) into (35) and (36)

$$\partial_f \ln Z = \sum_{k,i=1}^{n} \left( \kappa^{(k)} - \alpha f - \beta \right) (\kappa^{(k)})^{n-1} F_{ik}(\kappa^{(1)}, \ldots, \kappa^{(n)})$$

$$\times \left\{ \partial_f u_{i-1} - \frac{\alpha}{2} \sum_{m=1}^{n} \int_{\kappa_{m}}^{\kappa^{(m)}} \frac{\kappa^{i-1} d\kappa}{(\kappa - \alpha f - \beta) W} \right\}, \quad (73)$$

$$\partial_g \ln Z = \sum_{k,i=1}^{n} \left( \kappa^{(k)} + \alpha g - \beta \right) (\kappa^{(k)})^{n-1} F_{ik}(\kappa^{(1)}, \ldots, \kappa^{(n)})$$

$$\times \left\{ \partial_g u_{i-1} + \frac{\alpha}{2} \sum_{m=1}^{n} \int_{\kappa_{m}}^{\kappa^{(m)}} \frac{\kappa^{i-1} d\kappa}{(\kappa + \alpha g - \beta) W} \right\}. \quad (74)$$

The next step is to resolve the recursive relation (13) in order to shrink down the term $\partial_f u_{i-1}$ to its basic ingredients

$$\partial_f u_j = \frac{\alpha}{2} u_{j-1} + (\alpha f + \beta) \partial_f u_{j-1}$$

$$= \frac{\alpha}{2} (u_{j-1} + (\alpha f + \beta) u_{j-2}) + (\alpha f + \beta)^2 \partial_f u_{j-2}$$

$$= \frac{\alpha}{2} \left( u_{j-1} + (\alpha f + \beta) u_{j-2} + (\alpha f + \beta)^2 u_{j-3} \right) + (\alpha f + \beta)^3 \partial_f u_{j-3}$$

$$\vdots$$

$$= \frac{\alpha}{2} \sum_{k=1}^{j} (\alpha f + \beta)^{k-1} u_{j-k} + (\alpha f + \beta)^j \partial_f u_0.$$
A similar relation holds for $\partial_u g$. Consequently, (73) and (74) become

\[
\partial_f \ln Z = \sum_{k,i=1}^{n} \left( \kappa^{(k)} - \alpha f - \beta \right) \left( \kappa^{(k)} \right)^{n-1} F_{ik} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) 
\times \left\{ \frac{\alpha}{2} \sum_{l=1}^{i-1} (\alpha f + \beta) u_{i-l-1} + (\alpha f + \beta)^{i-1} \partial_f u_0 
- \frac{\alpha}{2} \sum_{m=1}^{n} \kappa^{(m)} \int \frac{\kappa^{i-1} d\kappa}{(\kappa - \alpha f - \beta) W} \right\},
\]

(75)

\[
\partial_g \ln Z = \sum_{k,i=1}^{n} \left( \kappa^{(k)} + \alpha g - \beta \right) \left( \kappa^{(k)} \right)^{n-1} F_{ik} \left( \kappa^{(1)}, \ldots, \kappa^{(n)} \right) 
\times \left\{ -\frac{\alpha}{2} \sum_{l=1}^{i-1} (\beta - \alpha g) u_{i-l-1} + (\beta - \alpha g)^{i-1} \partial_g u_0 
+ \frac{\alpha}{2} \sum_{m=1}^{n} \kappa^{(m)} \int \frac{\kappa^{i-1} d\kappa}{(\kappa + \alpha g - \beta) W} \right\},
\]

(76)

which shows that $\partial_f \ln Z$ and $\partial_g \ln Z$ are linear functions of $\partial_f u_0$ and $\partial_g u_0$, respectively.

**The elliptic case**

The hyperelliptic solution (23) reduces to the following form for $n = 1$

\[
Z = \exp \left( \int_{\kappa_1}^{\kappa^{(1)}} \frac{\kappa d\kappa}{W(\kappa)} - u_1 \right),
\]

(77)

where

\[
W(\kappa; f, g) = (\kappa - \alpha f - \beta) (\kappa + \alpha g - \beta) (\kappa - \kappa_1) (\kappa - \kappa_1),
\]

(78)

and $\kappa^{(1)}$ is the solution of the following inversion problem

\[
\int_{\kappa_1}^{\kappa^{(1)}} \frac{d\kappa}{W(\kappa)} = u_0.
\]

(79)

Writing $W^2$ a bit more general

\[
W^2(\kappa) = (\kappa - e_1) (\kappa - e_2) (\kappa - e_3) (\kappa - e_4),
\]

(80)

yields the following primitive for the integral (79)

\[
\int \frac{d\kappa}{W(\kappa)} = \pm \frac{2}{\sqrt{(e_1 - e_4) (e_3 - e_2)}} F \left[ \arcsin \sqrt{\Lambda(\kappa)}, m \right],
\]

(81)

where $F(\cdot, \cdot)$ is the incomplete elliptic integral of the first kind (see [1]) and

\[
\Lambda(\kappa) = \frac{(e_2 - e_3) (e_4 - k)}{(e_4 - e_3) (e_2 - k)},
\]

(82)

\[
m = \frac{(e_2 - e_1) (e_4 - e_3)}{(e_4 - e_1) (e_2 - e_3)}.
\]

(83)
The sign ambiguity in (81) is a reminder that some square-roots have been taken, which should have been treated with more care (especially when dealing with specific values of $\kappa_1$). The same holds for the following sign ambiguities. Choosing $e_4 = \kappa_1$, $e_2 = \bar{\kappa}_1$, $e_1 = \alpha f + \beta$ and $e_3 = \beta - \alpha g$ leads to

$$u_0 = \int_{\kappa_1}^{\kappa} \frac{dk'}{W(k')} = \pm \frac{2}{\sqrt{(e_1 - e_4)(e_3 - e_2)}} F \left[ \arcsin \sqrt{\Lambda(k)}, m \right],$$

(84)

because $F[0, m] = 0$ and therefore

$$\arcsin \sqrt{\Lambda(k)} = \text{am}(\tilde{u}_0, m),$$

(85)

$$\sqrt{\Lambda(k)} = \text{sn}(\tilde{u}_0, m),$$

(86)

where $\text{am}(\cdot, \cdot)$ denotes the amplitude function and $\text{sn}(\cdot, \cdot)$ is the Jacobian elliptic sine amplitude (see [1]) and the following abbreviation has been introduced

$$\tilde{u}_0 = \pm \sqrt{(e_1 - e_4)(e_3 - e_2)} \frac{u_0}{2}.$$  

(87)

The primitive for the integral occurring in (77) is

$$\int_{\kappa_1}^{\kappa} \frac{kdk}{W(k)} = \pm \frac{2}{\sqrt{(e_1 - e_4)(e_3 - e_2)}} \left\{ e_2 F \left[ \arcsin \sqrt{\Lambda(k)}, m \right] + (e_2 - e_4) \Pi \left[ k, \arcsin \sqrt{\Lambda(k)}, m \right] \right\},$$

(88)

where $\Pi(\cdot, \cdot, \cdot)$ denotes the incomplete elliptic integral of the third kind (cf. [1]) and

$$k = \frac{(e_4 - e_3)}{(e_2 - e_3)}.$$  

(89)

Hence, the regarding definite integral reads

$$\int_{\kappa_1}^{\kappa} \frac{kdk}{W(k)} = \left\{ e_2 u_0 \pm \frac{2(e_2 - e_4)}{\sqrt{(e_1 - e_4)(e_3 - e_2)}} \Pi \left[ k, \text{am}(\tilde{u}_0, m), m \right] \right\},$$

(90)

since $\Pi[k, 0, m] = 0$.

Putting everything together, the resulting Ernst potential $Z$ reads:

$$Z(f, g) = \exp \left\{ \pm \frac{2i(\bar{\kappa}_1 - \kappa_1)}{\sqrt{(\alpha f + \beta - \kappa_1)(\alpha g - \beta + \bar{\kappa}_1)}} \Pi \left[ k(f, g), \text{am}(\tilde{u}_0(f, g), m(f, g)), m(f, g) \right] \right\} \pm \bar{\kappa}_1 u_0(f, g) - u_1(f, g).$$

(91)

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, 1970.
[2] F.J. Ernst. *Physical Review*, 167:1175–1177, 1968.
[3] J.B. Griffiths. *Colliding plane waves in general relativity*. Clarendon Press, 1991.
[4] K.A. Khan and R. Penrose. *Nature*, 229:185–186, 1971.
[5] R. Meinel and G. Neugebauer. *Physics Letters A*, 210(3):160–162, 1996.
[6] Willard Miller, Jr. *SIAM Journal on Mathematical Analysis*, 4(2):314–328, 1973.
[7] S. Moeckel. *arXiv preprint arXiv:1312.0672*, 2013.