Investigation of trajectory planning of a travel route while turning at an intersection

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Abstract
This paper presents theoretical consideration of a travel route while turning at an intersection. The curvature distribution of a travel route is an important physical quantity that generates a traveling track. The formulation and characteristics of a transition curve at connection points for a non-interpolation curve, a single clothoid curve, and a multiple clothoid curve are investigated. The physical variable derived from a curvature distribution function is demonstrated, and the features of the three types of transition curves are explained. The integration value of a curvature distribution function represents an azimuthal angle. Therefore, the curvature distribution curve is the velocity of the azimuthal angle. Then, the first-order differential value of the curvature distribution curve is the acceleration of the azimuthal angle, and the second-order differential value of the curvature distribution curve is the jerk of the azimuthal angle. The theoretical equation of a traveling path is derived from the curvature distribution curve of the three types of transition curves, and their characteristics are formulated. Theoretical expressions of a speed profile of a real running vehicle at a traffic intersection are also proposed. The validity of the multiple clothoid curve is discussed analytically as compared with the other two cases where it is interpolated by the non-interpolation curve or the single clothoid curve. It was also found that the influence of the multiple clothoid curve on vehicle movement and ride comfort was superior to those of the non-interpolation and single clothoid curve. Some results are presented in the form of parametric plots.

Keywords: Optimum transition curve, Multiple clothoid curve, Curvature distribution, Theoretical formulation

1. Introduction

The curve of a highway and main trunk roads, etc., has an inlet straight-line part, an inlet clothoid-curve part, a circular arc part, an outlet clothoid-curve part, and an outlet straight-line part. If the road is constituted in this way, vehicles can run a curve smoothly, and it becomes possible to increase passenger ride comfort (McConnell, 1957). However, an intersection of a city area consists of only an inlet straight-line part, a circular arc part, and an outlet straight-line part, and does not have an inlet clothoid-curve part or an outlet clothoid-curve part in many cases. In such a case, the curvature of a running track becomes discontinuous at the connection point between an inlet straight-line part and a circle part, and the connection point between a circle part and an outlet straight-line part. Therefore, vehicles cannot run smoothly (Malcolm, 1979), (Lima et al., 2015).

The vibration of vehicles is amplified, especially in vehicles (Ino et al., 2015) carrying driving support equipment that performs the steering operation in place of the driver and supports the driver’s steering operation. This will become unstable, and passenger ride comfort will decrease (Yamakado et al., 2013). The discontinuity of the curvature at a connection point makes highly precise control of the electric equipment mounted on vehicle very difficult. The authors developed a three-dimensional vehicle-passenger system in a previous paper (Yamamoto et al., 2017), and built an optimal running lane with a transition curve when vehicles turned at an intersection. The paper showed the influence of the curvature distribution on a vehicle and its occupants’ dynamic behavior. The multiple clothoid curve was devised as a new transition curve in which the curvature changes smoothly at the connection point. Then, the validity of the model
and system were shown. Furthermore, an application was filed for a related industrial patent (Yamamoto, 2016). However, as far as the authors know, with regard to investigations of the theoretical considerations of a running track on which a vehicle turns at an intersection (Ran et al., 2010), (Shibuya et al., 2012), (Yamagishi et al., 2003), there is no reported example.

In this research, theoretical considerations are examined with regard to an optimal running lane in which a multiple clothoid curve (Yamamoto et al., 2017) is applied as a transition curve at the connection point. This point is located between a straight line and a circle, a circular arc and a circular arc, and a straight line and a straight line. Furthermore, a theoretical equation of the running speed distribution of vehicles in an actual intersection is proposed.

2. Theory

2.1 Curvature distribution

Figure 1 shows part of a running track of vehicles (Eliou and Kaliabetsos, 2014), where $x$ and $y$ are coordinates, $O$ is the center of curvature, $R$ is the curvature radius, $S$ is the starting point, $A$ and $B$ are points on a route, $\theta$ is the course angle, and $dx$ and $dy$ are small segments.

![Running track diagram](image)

Fig. 1 Running track, where $x$ and $y$ are coordinates, $O$ center of curvature, $R$ curvature radius, $S$ starting point, $E$ ending point, $A$ and $B$ points on a route, and $\theta$ course angle. Then, $dx$ and $dy$ are small segments.

The course angle at point A is set to $\theta$. The small segment from point A to B is $ds$. The course angle of point B can be displayed as $\theta + d\theta$.

Moreover, $\angle AOB = \theta - (\theta + d\theta) = -d\theta$, and the course angle at point A is defined as the following equation:

$$\frac{dy}{dx} = \tan \theta$$

(1)

where $x$ and $y$ are the coordinate system, and $R$ is the curvature radius.

The length between points AB is defined as the following equation:

$$ds = R(-d\theta) = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \approx dx$$

(2)

The curvature can be introduced from Equation (2) as the following equation:

$$\frac{1}{R} = \frac{-1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \frac{d\theta}{dx}$$

(3)

The curvature distribution is proportional to the value that differentiates course angle $\theta$ with respect to $x$ from Equation (3). Differentiating both sides of Equation (1) with respect to $x$ gives the following equation:

$$\frac{d^2y}{dx^2} = (1 + \tan^2 \theta) \frac{d\theta}{dx}$$

(4)

The following equation can be introduced from Equations (3) and (4):
Moreover, Equation (5) can be simplified as follows:

\[
\frac{1}{R} = \frac{d^2 y}{dx^2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}^{3/2}
\]  

(5)

By substituting \( x = vt \) in Equation (6), Equation (6) can be expressed as the following equation:

\[
\frac{1}{R} = \frac{d^2 y}{dx^2} \left(\frac{1}{v}\right)^2 \frac{d^2 y}{dt^2} < 1
\]  

(6)

Therefore, the curvature distribution is proportional to the second-order differential value \( y \) with respect to \( t \). That is, when a point mass moves at a constant speed along an arbitrary curvature distribution, the lateral acceleration \( d^2 y/dt^2 \) of a point mass is proportional to the curvature distribution \( 1/R \).

2.2 Transition curve at connection points

Figure 2 depicts the running track to point D through B and C from point A. Between A and B is a straight-line part, between B and C is a quarter of the circumference of a circle of radius \( R \), and between C and D is a straight-line part. The length of line segment AB is \( L_1 \), BC is \( L_2 + L_3 \), and CD is \( L_4 \). A circular arc is represented by 1/4 circles. However, the theory shown in this study can be applied in different central angles of circular arcs. The effect of the central angle of a circular arc on dynamic behavior of vehicle-passenger was shown in previous paper (Yamamoto, et al., 2017).

Figure 3 displays the curvature distribution of the running track as shown in Fig. 2. The curvature of radius \( R \) is referred to as \( 1/R = 2D_6 \). At the point that is connected with a straight line and a circular arc at points B and C, the curvature changes discontinuously from 0 to \( 2D_6 \). As for a machine product that has a precision mechanism and high efficiency, control will become difficult owing to the discontinuity of the curvature when vehicles pass points B and C.

Furthermore, useless energy is consumed, and automobile ride comfort decreases. As shown in Fig. 3, the case where the curvature changes perpendicularly and discontinuously at connection points B and C is called non-interpolation in this report.
In order to reduce the discontinuity of curvature at points B and C, the curvature distribution shown in Fig. 4 is devised so that it may be changed linearly from 0 to $2D_d$ at points. At this time, the function at point B is set to $Y_1$, and the function at point C is set to $Y_2$. $Y_1$ and $Y_2$ are defined as follows:

$$Y_1 = \frac{D_d}{A_y} X + D_d - \frac{D_d}{A_y} L_1, \quad L_1-A_y \leq X \leq L_1 + A_y$$

$$Y_2 = -\frac{D_d}{A_y} X + \frac{D_d}{A_y} (L_1 + L_2 + L_3 + A_y), \quad L_1 + L_2 + L_3 - A_y \leq X \leq L_1 + L_2 + L_3 + A_y$$

where $D_d = 1/(2R)$, $A_y = K(L_2 + L_3)$, $R = 2(L_2 + L_3)/\pi$. In addition, $K$ is a parameter representing the slope of a linear function. As shown in Fig. 4, the case where the curvature is interpolated by a straight line at points B and C is called a single clothoid curve in this report. Dubins proved the Theorem that average curvature always less than or equal to $R^{-1}$ (Dubins, 1957). Bertolazzi and Frego proposed an algorithm for the solution the problem of Hermite $G^1$ interpolation with a clothoid curve (Bertolazzi and Frego, 2013). These reports are closely related to a single clothoid.

Then, $A_y^* = K'(L_2 + L_3)$.

The first- and second-order differential values of Equation (10) are represented as Equations (11) and (12), respectively:

$$\frac{dy}{dx} = D_d \sec h^2\left(\frac{2}{A_y} X\right) \cdot \frac{2}{A_y}$$

$$\frac{d^2y}{dx^2} = -2\left(\sec h \left(\frac{2}{A_y} X\right) \right)^2 \cdot \tanh \left(\frac{2}{A_y} X\right) \cdot \left(\frac{2}{A_y}\right)^2 \cdot D_d$$

Figure 5 shows a function in which the curvature may still be smoother and may change continuously at points B and C. As shown in the figure, the case where the curvature is smooth and changes continuously at points B and C is called a multiple clothoid curve in this report. It can be indicated by a function as the following equation. In Fig. 5, $A_y^*$ is a parameter that expresses a tangential gradient of a function at points B and C. The tangential gradient of a function at points B and C becomes small, according to $A_y^*$ becomes large. $X_l$ is the length of the distance.

$$Y = D_d + D_d \tanh\left(\frac{2}{A_y^*} X\right)$$

$$X = -(X_1 - (L_1 + L_2 + L_3))$$

where $X = X_1 - L_1$ for $0 \leq X_1 < L_1 + L_2$, and $X = -(X_1 - (L_1 + L_2 + L_3))$ for $L_1 + L_2 < X_1 < L_1 + L_2 + L_3 + L_4$.

Then, $A_y^* = K'(L_2 + L_3)$.
As a numerical computation, the transition curves at connection points for the three types of curvature distribution curves are shown in the case of \( R = 50.0 \text{ m} \), \( L_1 = 50.0 \text{ m} \), \( L_2 = 39.25 \text{ m} \), \( L_3 = 39.25 \text{ m} \), \( L_4 = 50.0 \text{ m} \), \( K = 0.16 \), and \( K^* = 0.3 \).

![Fig. 6 Comparison of three types of transition curves at connection points.](image)

Figure 6 shows three different curvature distribution curves for non-interpolation, a single clothoid curve, and a multiple clothoid curve. The horizontal axis of the figure indicates the distance, and the vertical axis indicates the curvature. In the case of the non-interpolation, the curvature changes suddenly from 0 to 0.02 at the connection point. In the case of a single clothoid curve, it changes linearly. On the other hand, the curvature for a multiple clothoid curve is smoothly and continuously changed at the connection point.

![Fig. 7 Curvature and first- and second-order differential values of multiple clothoid function.](image)

Figure 7 simultaneously shows the curvature distribution function and the first- and second-order differential values of the curvature distribution for a multiple clothoid curve. In the figure, the horizontal axis indicates the distance, the right vertical axis indicates the curvature distribution, and the left vertical axis indicates the first- and second-order differential values of the curvature distribution function simultaneously. The characteristic relation of the first- and second-order differential values of the curvature distribution and the curvature distribution function can be clearly seen. The first-order differential value of the curvature distribution function takes a peak value when the curvature distribution changes smoothly. The second-order differential value of the curvature distribution function is displaced from a positive peak to a negative peak when the curvature increases smoothly. Furthermore, when the curvature distribution decreases smoothly, the second-order differential value of the curvature distribution function is changed from a negative peak to a positive peak.

### 2.3 Physical variable introduced by the curvature distribution function

The curvature is expressed as \( 1/R = -d^2y/dx^2 \) from Equation (6). Integrating both sides of the Equation (6) with respect to \( x \), the integration of the curvature yields \( \int (1/R) dx = -dy/dx + C \). However, \( C \) is a constant of integration. Therefore, the integration value of the curvature shows the gradient \( dy/dx \), i.e., it is the course angle of arbitrary time (the angle between a tangential line of curve AB and the \( x \) axis).

Figure 8 shows the running track, which consists of a straight line, a circular arc, and a straight line. Points B and C show a connection point. The transition curves at a connection point for non-interpolation, a single clothoid curve, and a
multiple clothoid curve are analyzed. The vehicle travels the running route, which starts at point A, and reaches point D through points B and C. The length of the straight-line part AB is 50 m, and the length of the circular-arc part BC is $25\pi$ m. The straight-line part CD is 50 m, and radius $R$ is 50 m. Moreover, X and Y show plane coordinates.

![Diagram of the running lane consisting of inlet straight line, circular arc, and outlet straight line.](image)

Fig. 8 Running lane consisting of inlet straight line, circular arc, and outlet straight line, where AB = 50 m, BC = $25\pi$ m, CD = 50 m, and $R$ = 50 m.

### 2.3.1 Case of non-interpolation

Three cases are of interest and will be considered in turn. In this case, the transition curve at a connection point is a non-interpolation, theoretical formulation, and can be written as follows:

$$Y = C_1$$

$$Y_d = \int YdX = S_1 \cdot X + S_2$$

(13)

where $Y$ represents the curvature distribution curve, and $Y_d$ is the integral value of $Y$ that is the angular displacement. $S_1$ and $S_2$ are constants of the integral, which must be determined from the boundary conditions and represented as the following equations:

$$C_1 = \begin{cases} 
0 & (0 \leq X < L_1) \\
2D_d & (L_1 \leq X < L_1 + L_2 + L_3) \\
0 & (L_1 + L_2 + L_3 \leq X < L_1 + L_2 + L_3 + L_4) 
\end{cases}$$

(14)

$$S_1 = \begin{cases} 
0 & (0 \leq X < L_1) \\
2D_d & (L_1 \leq X < L_1 + L_2 + L_3) \\
0 & (L_1 + L_2 + L_3 \leq X < L_1 + L_2 + L_3 + L_4) 
\end{cases}$$

(15)

$$S_2 = \begin{cases} 
0 & (0 \leq X < L_1) \\
-2D_d L_4 & (L_1 \leq X < L_1 + L_2 + L_3) \\
\pi/2 & (L_1 + L_2 + L_3 \leq X < L_1 + L_2 + L_3 + L_4) 
\end{cases}$$

(16)

### 2.3.2 Case of single clothoid curve

In this instance, the transition curve at a connection point is a single clothoid curve. Its theoretical formulation can be written as the following equations:

$$Y = C_1X + C_2$$

$$Y_d = \int YdX = S_1 \cdot X^2 + S_2 \cdot X + S_3$$

(17)

where $C_1$ and $C_2$ shown in Equation (17) are coefficients, and $S_1$, $S_2$, and $S_3$ are constants of integration. We obtain these values using the following equations. The coefficients $C_1$ and $C_2$ are defined as Equations (18) and (19).

The constants of integration $S_1$, $S_2$, and $S_3$ are defined as Equations (20) to (22).

$$C_1 = \begin{cases} 
0 & (0 \leq X < L_1 - A_0) \\
\frac{D_d}{A_0} & (L_1 - A_0 \leq X < L_1 + A_0) \\
0 & (L_1 + A_0 \leq X < L_1 + L_2 + L_3 - A_0) \\
-\frac{D_d}{A_0} & (L_1 + L_2 + L_3 - A_0 \leq X < L_1 + L_2 + L_3 + A_0) \\
0 & (L_1 + L_2 + L_3 + A_0 \leq X < L_1 + L_2 + L_3 + L_4) 
\end{cases}$$

(18)

$$C_2 = \begin{cases} 
0 & (0 \leq X < L_1 - A_0) \\
\frac{D_d}{A_0} & (L_1 - A_0 \leq X < L_1 + A_0) \\
0 & (L_1 + A_0 \leq X < L_1 + L_2 + L_3 - A_0) \\
-\frac{D_d}{A_0} & (L_1 + L_2 + L_3 - A_0 \leq X < L_1 + L_2 + L_3 + A_0) \\
0 & (L_1 + L_2 + L_3 + A_0 \leq X < L_1 + L_2 + L_3 + L_4) 
\end{cases}$$

(19)
\[ C_2 = \begin{cases} 0 & (0 \leq X < L_1 - A) \\ D_2 / A_2 \cdot (A_2 - L_2) & (L_1 - A \leq X < L_2 + A) \\ D_2 / A_2 \cdot (L_2 + L_3 + A_2) & (L_2 + L_3 - A_2 \leq X < L_2 + L_3 + A_2) \\ 0 & (L_2 + L_3 + A_2 \leq X < L_2 + L_3 + L_4) \end{cases} \] (19)

\[ S_1 = \begin{cases} 1/2 \cdot D_2 / A_2 & (L_1 - A \leq X < L_1 + A) \\ 0 & (L_1 + A_2 \leq X < L_1 + L_2 + L_3 - A_2) \\ -1/2 \cdot D_2 / A_2 & (L_1 + L_2 + L_3 - A_2 \leq X < L_1 + L_2 + L_3 + A_2) \\ 0 & (L_1 + L_2 + L_3 + A_2 \leq X < L_1 + L_2 + L_3 + L_4) \end{cases} \] (20)

\[ S_2 = \begin{cases} 0 & (0 \leq X < L_1 - A) \\ -D_2 / A_2 \cdot (L_1 - A_2) & (L_1 - A_2 \leq X < L_1 + A_2) \\ 2D_2 / A_2 \cdot (L_1 + L_2 + L_3 - A_2) & (L_1 + L_2 + L_3 - A_2 \leq X < L_1 + L_2 + L_3 + A_2) \\ 0 & (L_1 + L_2 + L_3 + A_2 \leq X < L_1 + L_2 + L_3 + L_4) \end{cases} \] (21)

\[ S_3 = \begin{cases} 0 & (0 \leq X < L_1 - A) \\ 1/2 \cdot D_2 / A_2 \cdot (L_4 - A_4)^2 & (L_4 - A_4 \leq X < L_4 + A_4) \\ \pi/4 - 2D_2 / (L_4 + L_2) & (L_4 + A_4 \leq X < L_4 + L_2 + L_3 - A_4) \\ \pi/2 & (L_4 + L_2 + L_3 + A_4 \leq X < L_4 + L_2 + L_3 + L_4) \end{cases} \] (22)

\[ C^* = \frac{A^*}{L_2} K (L_2 + L_3) \]

Integrating Equation (23) with respect to \( x \) gives Equation (24):

\[ Y = E_2 X + \frac{\log \cosh \frac{2}{A^*} X}{E_3} \] (24)

Here, constants of integration \( E_1, E_2, \) and \( E_3 \) are obtained by integrating Equation (24), and are expressed as the following equations:

\[ E_1 = \begin{cases} D_2 & (0 \leq X_1 < L_1 + L_2) \\ -D_2 & (L_1 + L_2 \leq X_1 < L_1 + L_2 + L_3 + L_4) \end{cases} \] (25)

\[ E_2 = \begin{cases} A^* / 2 \cdot D_2 & (0 \leq X_1 < L_1 + L_2) \\ -A^* / 2 \cdot D_2 & (L_1 + L_2 \leq X_1 < L_1 + L_2 + L_3 + L_4) \end{cases} \] (26)

\[ E_3 = \begin{cases} D_2 \cdot L_1 - A^* / 2 \cdot D_2 \cdot \log \cosh \frac{2}{A^*} (-L_1) & (0 \leq X_1 < L_1 + L_2) \\ \pi/2 - D_2 \cdot L_4 + A^* / 2 \cdot D_2 \cdot \log \cosh \frac{2}{A^*} (-L_4) & (L_4 + L_2 \leq X_1 < L_4 + L_2 + L_3 + L_4) \end{cases} \] (27)

2.3.3 Case of multiple clothoid curve

For this case, the transition curve at a connection point is the multiple clothoid curve, whose theoretical formulation can be written as follows:

\[ Y = D_2 + D_2 \tanh \left( \frac{2}{A^*} X \right) \] (23)

where \( X_i \) is the distance length,

\[ X = X_1 - L_1 \quad \text{for} \quad 0 \leq X_1 < L_1 + L_2, \quad \text{and} \quad X = -(X_1 - (L_4 + L_2 + L_3)) \quad \text{for} \quad L_4 + L_2 \leq X_1 < L_4 + L_2 + L_3 + L_4 \]

Then, \( A^* = K (L_3 + L_2) \).

2.3.4 Comparison of three kinds of transition curves

Figure 9 displays the running track. The turning radius is \( R = 50 \) m. The relations and length of a circular-arc part and a straight-line part are \( AB = 50 \) m, \( BC = \pi R / 2 \), and \( AB = CD \). Points B and C show a connection point. We compare the three kinds of characteristics of the transition curves at connection points for non-interpolation, the single clothoid curve, and the multiple clothoid curve.
Figure 10 shows the angle distribution when a vehicle runs at a speed of 1.0 m/s on the running track shown in Fig. 9. Figure 10 (a) shows the course angle distribution in the case of non-interpolation, Figure 10 (b) exhibits the course angle distribution in the case of a single clothoid curve, and Figure 10 (c) illustrates the course angle distribution in the case of a multiple clothoid curve. In Fig. 10 (a), the vehicle passes the connection point at time 50.1 s and 128.5 s, and the course angle changes suddenly. The course angle distribution is changed smoothly in Fig. 10 (b) and 10(c).

Figure 11 shows the curvature distribution curve of the running track. Figure 11 (a) shows the curvature distribution curve for non-interpolation and for the case of a single clothoid curve in Figure 11 (b). Figure 11 (c) illustrates the curvature distribution curve in the case of a multiple clothoid curve. In Fig. 11 (a), the vehicle passes the connection point at time 50.1 s and 128.5 s, and an abrupt change occurs in the curvature distribution curve. In Fig. 11 (b), the curvature distribution changes suddenly at time 40.1 s, 60.5 s, 119.0 s, and 138.3 s. The curvature distribution shown in Fig. 11 (c) changes smoothly. Figure 10 shows the course angle distribution that integrates the curvature distribution function shown in Fig. 11. Moreover, Figure 11 shows the angular velocity that differentiates the course angle distribution function shown in Fig. 10, and refers to the course angle speed.

Figure 12 shows the course angle acceleration. Figure 12 (a) shows the angular acceleration for non-interpolation and for the single clothoid curve shown in Fig. 12 (b). Figure 12 (c) illustrates the course angle acceleration for a multiple
clothoid curve. In Figure 12 (a), the vehicle passes the connection point at time 50.1 s and 128.5 s, and the peak value of the course angle acceleration occurs there. In Fig. 12 (b), abrupt vertical changes of the course angle acceleration occur at time 40.1 s, 60.5 s, 119.0 s, and 138.3 s. The angular acceleration shown in Fig. 12 (c) changes smoothly. Figure 12 shows the course angle acceleration that differentiates the curvature distribution curve shown in Fig. 11 with respect to time.

![](image)

Fig. 12 Comparison of angular acceleration between three types of transition curve.

Figure 13 shows a course angle jerk. Figure 13 (a) shows the course angle jerk for non-interpolation and for the case of the single clothoid curve shown in Fig. 13 (b). Figure 13 (c) illustrates the course angle jerk for the case of a multiple clothoid curve. In Fig. 13 (a), the vehicle passes the connection point at time 50.1 s and 128.5 s, and the peak value of the course angle jerk occurs. In Fig. 13 (b), the course angle jerk changes suddenly at time 40.1 s, 60.5 s, 119.0 s, and 138.3 s. Figure 13 (c) shows the jerk changing smoothly. Figure 13 shows the distribution curve that differentiates the second order of the curvature distribution curve shown in Fig. 11, and shows the course angle jerk. Although the peak value for the multiple clothoid curve was about 0.0001 rad/s$^3$, the peak value for the single clothoid curve was about $10^4$ times the peak value for the multiple clothoid curve, and the peak value was 1.0 rad/s$^3$. Furthermore, the peak value of the non-interpolation was 20,000 rad/s$^3$, and which was $2 \times 10^6$ times the value for the single clothoid curve. The Jerk is derivative of acceleration with respect to time. Therefore, the value of Jerk is largely depend on a time step. In this study, the time step is 0.0025 s. The multiple clothoid curve eliminates non differentiable points and the Jerk can be well suppressed.

### 2.4 Running track

Figure 14 depicts a running track in which vehicles pass points A and B from the starting point S (Eliou, Kaliabetsos, 2014). $\alpha$ is the course angle, and $\beta$ is the central angle of the turning radius. The relation between $\alpha$ and $\beta$ is presented as follows:

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_1 + \beta_2, \quad \alpha_3 = \beta_1 + \beta_2 + \beta_3, \quad \ldots \quad \alpha_n = \beta_1 + \beta_2 + \ldots + \beta_n$$ (28)

A small segment $\Delta l$ can be denoted by the following equation:

$$\Delta l = \beta_i R_i$$ (29)

On the other hand, $\beta_1$ is obtained from the relation between $\Delta l$ and $\beta_1$ as the following equation:
Substituting $\beta_1, \beta_2, \beta_3, \ldots \beta_n$ into Eq. (28), one obtains the following equation:

$$\alpha_n = 2\sin^{-1} \frac{\Delta l}{2R_1} + 2\sin^{-1} \frac{\Delta l}{2R_2} + \cdots + 2\sin^{-1} \frac{\Delta l}{2R_n}.$$  \hfill (31)

Therefore, the displacement in the $x$ and $y$ directions is introduced as the following equation:

$$\begin{align*}
x &= \Delta l \sum_{i=1}^{n} \cos \alpha_i \\
y &= \Delta l \sum_{i=1}^{n} \sin \alpha_i
\end{align*}.$$  \hfill (32)

The line shape of curve line SE is expressed in terms of vehicle speed $V$ and course angle $\alpha$ as the following equation:

$$\begin{align*}
x &= \int V \cos \alpha \cdot dt \\
y &= \int V \sin \alpha \cdot dt
\end{align*}.$$  \hfill (33)

Figure 15 displays a running track that is interpolated by three transition curves at a connection point. Once the curvature distribution curve is determined, the running track is introduced from Equation (33). Figure 15 shows the running track for the three types of transition curve at a connection point. Figure 15 (b) shows a detailed drawing from 20 to 50 m for the $x$ coordinate. It should be noted that the distribution curve $y$ is shifted upward in order of the non-interpolation, the single clothoid curve, and the multiple clothoid curve. The turning radius becomes large as the distribution curve moves upward.
Figure 16 shows the vehicle azimuthal angle as interpolated by the three transition curves at a connection point. The course angle can be introduced by integrating the curvature distribution function with respect to distance $x$. Figure 16 (a) exhibits the three types of transition curves at a connection point. Figure 16 (b) is a detailed drawing of a vehicle azimuthal angle traveling route from 20 to 40 m. The distribution curve is shifted upward in order of the non-interpolation, the single clothoid curve, and the multiple clothoid curve. The course angle is smoothly displaced as the distribution curve moves upward. At a connection point (31.4 m), an abrupt edge can be seen in the case of non-interpolation in the course angle distribution.

3. Speed profile at an intersection

It seems that the running speeds of vehicles that turn to the right or left at an intersection have specific characteristics. Wolfemann et al. tried to express the speed profile in terms of a polynomial degree-three function from analysis results of real running-vehicle action data when the vehicles turned to the right (Wolfemann et al., 2011). Watabe et al. analyzed the influence of road structures and traffic employment on a right-turning vehicle’s action (Watabe et al., 2014). The running speed distribution was expressed in terms of a third-order polynomial function curve, and took into consideration a running track that consisted of a single clothoid curve, a circular arc, and a single clothoid curve.

Figure 17 explains the speed distribution of the right turn of a free run at an intersection. The horizontal axis shows the time, and the vertical axis shows the speed. The vehicle runs into an intersection at time $t_1$ at inflow velocity $v_1$, slows down to minimum speed $v_2$ at time $t_2$, then accelerates gradually and flows out at time $t_3$ at outflow speed $v_3$. Here, it is supposed that the speed at the time of the inflow and outflow is constant, and it should be noted that the acceleration is zero at the time of inflow and outflow. Moreover, since the changes in the speed distribution at the time of the slowdown at the side of the inflow and at the time of acceleration at the side of the outflow are not symmetrical, the speed profile is divided and displayed as two types for the inflow and outflow sides bordering at time $t_2$ and recorded at minimum speed.
3.1 Speed profile in the case of a polynomial function

The speed distribution approximated by the polynomial of a degree-three function is shown in Equation (34a) (Wolfemann et al., 2011). The speed profile $v_i$ at an intersection, the acceleration distribution $\alpha_i$, and the jerk $J_i$ are defined by the following equations:

$$v_i = C_{1i}t_i^3 + C_{2i}t_i^2 + C_{3i}t_i + C_{4i}$$

$$\alpha_i = 3C_{1i}t_i^2 + 2C_{2i}t_i + C_{3i}$$

$$J_i = 6C_{1i}t_i + 2C_{2i}$$

and $i = (1,2,3)$, $k = (in, out)$, $k = (in, out) = (1,3)$

The coefficients $C_{1i}$ shown in Equation (34) are defined as following equations from the boundary conditions:

$$C_{1in} = \frac{1}{(t_1 - t_3)^3} \left[ 2(v_1 - v_3) - (\alpha_1 - \alpha_3)(t_1 - t_3) \right]$$

$$C_{2in} = \frac{1}{2(t_1 - t_3)} \left( \alpha_1 - \alpha_3 - 3C_{1in}(t_1^2 - t_3^2) \right)$$

$$C_{3in} = \frac{1}{2} \left[ (\alpha_1 + \alpha_3) - 3C_{1in}(t_1^2 + t_3^2) - 2C_{2in}(t_1 + t_3) \right]$$

$$C_{4in} = \frac{1}{2} \left[ (v_1 + v_3) - C_{1in}(t_1^3 + t_3^3) - C_{2in}(t_1^2 + t_3^2) - C_{3in}(t_1 + t_3) \right]$$

$$C_{1out} = -\frac{1}{(t_2 - t_3)^3} \left[ 2(v_2 - v_3) - (\alpha_2 - \alpha_3)(t_2 - t_3) \right]$$

$$C_{2out} = \frac{1}{2(t_2 - t_3)} \left( \alpha_2 - \alpha_3 - 3C_{1out}(t_2^2 - t_3^2) \right)$$

$$C_{3out} = \frac{1}{2} \left[ (\alpha_2 + \alpha_3) - 3C_{1out}(t_2^2 + t_3^2) - 2C_{2out}(t_2 + t_3) \right]$$

$$C_{4out} = \frac{1}{2} \left[ (v_2 + v_3) - C_{1out}(t_2^3 + t_3^3) - C_{2out}(t_2^2 + t_3^2) - C_{3out}(t_2 + t_3) \right]$$

Since the speed distribution at the inflow side differs from that at the outflow side, two kinds of speed curves are distinguished by the value of $k$. As a boundary condition, the acceleration at the position of the inflow, outflow, and the minimum is assumed to be 0, and the inflow and outflow speeds are given. Coefficients $C_{1i}$ and the minimum speed $v_{\min}$ are needed to specify the speed distribution. Then, since these two parameters depend on intersectional geometric structures and running-vehicle characteristics, the vehicle’s speed distribution can be expressed.

where $k = (in, out)$: distinction between the inflow and an outflow, $i = (in, min, out)$: distinction of the boundary positioning conditions. Furthermore, since this speed distribution is related to the running track, the distance from the starting point to the position recorded at the minimum speed is curvilinear and is defined in a model. Thus, it becomes possible to reproduce and evaluate the right-turn movement at an intersection on a time axis. As a numerical computation, the case of $v_1 = 13.0$ m/s, $v_2 = 5.0$ m/s, $v_3 = 13.0$ m/s, $t_1 = 0.0$ s, $t_2 = 12.0$ s, and $t_3 = 24.0$ s is shown. (Wolfemann et al., 2011)

Figure 18 shows the time history of the displacement, in which the speed profile is expressed by a polynomial function. The horizontal axis of the figure indicates the time, and the vertical axis indicates the total running distance. As a feature of a distribution curve, the point of inflection exists at time 12 s. Since the slowdown region exists from 0 to 12 s, the curve projects upward. After passing through the point of inflection, since the acceleration region exists from 12 to 24 s, the curve projects downward.

Figure 19 shows the time history of the speed distribution. The horizontal axis of the figure shows the time, and the vertical axis shows the speed. The inflow and outflow speeds are the same. The minimum speed exists at time 12 s. The inflow speed, minimum speed, and outflow speed are a case of regularity.

Figure 20 exhibits the distribution of acceleration. The horizontal axis of the figure shows the time, and the vertical axis shows the acceleration. Since the speed is regular at the inflow, the minimum, and the outflow, the acceleration is 0 at time 0, 12, and 24 s. Going into an intersection, the value of acceleration has a negative value in the deceleration region and a positive value in the acceleration region. The maximum value is 1.00 m/s².

Figure 21 illustrates the time history of a jerk. The horizontal axis of the figure shows the time, and the vertical axis
shows the jerk. The jerk increases uniformly at a time from 0 to the time recorded at the minimum speed, and the maximum jerk is 0.333 m/s$^3$ at time 12 s. After that, the jerk decreased uniformly. When the speed distribution is defined by the polynomial of the third function, the distribution of the jerk changes suddenly at the time when the minimum speed is recorded.

Vehicles carrying a control apparatus that supports automatic operation cannot control the apparatus under conditions such as those characteristic of a jerk. It should be noted that those characteristic of a jerk induce the reduction of automobile ride comfort.

\[ v(t) = v_1 + \frac{1}{2} (v_1 - v_2) \left( \frac{2\pi}{t_1 - t_2} - 1 \right) \]  

\[ \frac{dv}{dt} = \frac{1}{2} (v_1 - v_2) (-\sin \frac{2\pi}{t_1 - t_2} \cdot \frac{2\pi}{t_1 - t_2}) \]  

\[ \frac{d^2v}{dt^2} = \frac{1}{2} (v_1 - v_2) (-1) \cdot \frac{2\pi}{t_1 - t_2} \cdot 2\cos \frac{2\pi}{t_1 - t_2} \]  

When the speed profile is symmetrical at time recorded the minimum speed, the following equation is defined as the inflow and outflow speed distribution. Moreover, the acceleration and jerk are calculated by the following equations:

\[ v(t) = v_2 + \frac{1}{2} (v_3 - v_2) \left( 1 - \cos \frac{\pi(t - t_2)}{t_3 - t_2} \right) \]  

Fig. 18 Distance profile.  
Fig. 19 Speed profile expressed by polynomial of degree 3.  
Fig. 20 Acceleration profile.  
Fig. 21 Jerk profile.

### 3.2 Speed profile displayed by trigonometric function (inflow and outflow velocity are same)

When the speed distribution is determined by a third-order polynomial function, the jerk at the time of minimum speed changes suddenly. In order to improve the sudden changes of the jerk, a speed profile distribution curve expressed by trigonometric functions is proposed. The following equation is defined as the inflow speed distribution. Moreover, the acceleration and jerk are determined by the following equations:

\[ v(t) = v_1 + \frac{1}{2} (v_1 - v_2) \left( \cos \frac{2\pi}{t_1 - t_2} - 1 \right) \]  

\[ \frac{dv}{dt} = \frac{1}{2} (v_1 - v_2) (-\sin \frac{2\pi}{t_1 - t_2} \cdot \frac{2\pi}{t_1 - t_2}) \]  

\[ \frac{d^2v}{dt^2} = \frac{1}{2} (v_1 - v_2) (-1) \cdot \frac{2\pi}{t_1 - t_2} \cdot 2\cos \frac{2\pi}{t_1 - t_2} \]  

When the speed profile is symmetrical at time recorded the minimum speed, the following equation is defined as the inflow and outflow speed distribution. Moreover, the acceleration and jerk are calculated by the following equations:

\[ v(t) = v_2 + \frac{1}{2} (v_3 - v_2) \left( 1 - \cos \frac{\pi(t - t_2)}{t_3 - t_2} \right) \]
As numerical computations, the case of \(v_1 = 13.0\) m/s, \(v_2 = 5.0\) m/s, \(v_3 = 13.0\) m/s, \(t_1 = 0.0\) s, \(t_2 = 12.0\) s, and \(t_3 = 24.0\) s is shown.

Figure 22 shows the time history of the displacement when the speed distribution is determined by trigonometric functions. The horizontal axis of the figure shows the time, and the vertical axis shows the total mileage. As a feature of the distribution curve, the point of inflection exists at time 12 s, and since a slowdown region exists at time 0 to 12 s, the curve projects upward. After passing through the point of inflection, since the acceleration region exists from 12 to 24 s, the curve projects downward.

As numerical computations, the case of \(v_1 = 13.0\) m/s, \(v_2 = 5.0\) m/s, \(v_3 = 13.0\) m/s, \(t_1 = 0.0\) s, \(t_2 = 12.0\) s, and \(t_3 = 24.0\) s is shown.

Figure 22 shows the time history of the displacement when the speed distribution is determined by trigonometric functions. The horizontal axis of the figure shows the time, and the vertical axis shows the total mileage. As a feature of the distribution curve, the point of inflection exists at time 12 s, and since a slowdown region exists at time 0 to 12 s, the curve projects upward. After passing through the point of inflection, since the acceleration region exists from 12 to 24 s, the curve projects downward.

Figure 23 shows the time history of the speed distribution. The horizontal axis of the figure shows the time, and the vertical axis shows the speed. The inflow and outflow speeds are the same. The minimum speed exists at 12 s. The inflow speed, minimum speed, and outflow speed are a case of regularity.

Figure 24 illustrates the distribution of acceleration. The horizontal axis of the figure shows the time, and the vertical axis shows the acceleration. Since the speed is regular at the inflow, minimum, and outflow, the acceleration is 0 at time 0, 12, and 24 s. When the vehicle runs in an intersection, the value of acceleration has a negative value in the deceleration region and a positive value in the acceleration region. The maximum value of acceleration is 1.05 m/s².

Figure 25 illustrates the time history of a jerk. The horizontal axis of the figure shows the time, and the vertical axis shows the jerk. The jerk increases uniformly from time 0 to the time during which the minimum speed is recorded, and the maximum jerk is 0.274 m/s³ at time 12 s. After that, the jerk decreases uniformly. When the speed distribution is defined by trigonometric functions, the maximum jerk of the distribution is 0.274 m/s³ at the time during which the minimum speed is recorded, but it changes smoothly. The maximum jerk is smaller than that of the velocity distribution function represented by the third polynomial.
3.3 Speed profile displayed by trigonometric functions (when inflow and outflow velocity differ)

When the inflow and the outflow velocity distributions differ at time recorded the minimum speed, the following equation is defined as the inflow speed distribution. Moreover, the first-degree differentiation and the second-degree differentiation are as follows:

\[
\nu(t) = \nu_1 + \frac{1}{2}(\nu_1 - \nu_2)\left\{ \cos \frac{\pi}{t_2 - t_1} - 1 \right\} \\
\frac{dv}{dt} = \frac{1}{2}(\nu_1 - \nu_2)(-\sin \frac{\pi}{t_2 - t_1}) \frac{\pi}{t_2 - t_1} \\
\frac{d^2v}{dt^2} = \frac{1}{2}(\nu_1 - \nu_2)(-1)\left(\frac{\pi}{t_2 - t_1}\right)^2 \cos \frac{\pi}{t_2 - t_1}
\]

The outflow speed profile, acceleration, and jerk are defined as the following equations:

\[
\nu(t) = \nu_2 + \frac{1}{2}(\nu_2 - \nu_3)\left\{ 1 - \cos \frac{\pi(t - t_2)}{t_3 - t_2} \right\} \\
\frac{dv}{dt} = \frac{1}{2}(\nu_2 - \nu_3)\sin \frac{\pi(t - t_2)}{t_3 - t_2} \frac{\pi}{t_3 - t_2} \\
\frac{d^2v}{dt^2} = \frac{1}{2}(\nu_2 - \nu_3)\left( \cos \left[ \frac{\pi}{t_3 - t_2} \right] \right)^2 + C
\]

where, \( C = -\frac{1}{2}(\nu_1 - \nu_2)\left( \cos \left[ \frac{\pi}{t_2 - t_1} \right] \right)^2 - \frac{1}{2}(\nu_3 - \nu_2)\left( \frac{\pi}{t_3 - t_2} \right)^2 \)

The velocity profile is smoothly connected at a time \( t_2 \). Moreover, \( dv/dt=0 \) is satisfied at times \( t_1, t_2, \) and \( t_3 \). When trigonometric functions that have different amplitude or periodic time intervals will arise an error for the value of \( d^2v/dt^2 \) shown in Equation (40c) and (39c) at connecting point \( t_2 \). Assuming Equation (40c) equals Equation (39c) at time \( t_2 \), a supplementary term is added in Equation (40c).

As numerical computations, the case of \( \nu_1 = 13.0 \) m/s, \( \nu_2 = 5.0 \) m/s, \( \nu_3 = 10.0 \) m/s, \( t_1 = 0.0 \) s, \( t_2 = 12.0 \) s, and \( t_3 = 24.0 \) s is shown. Figure 28 shows the acceleration distribution when the speed distribution is expressed by trigonometric functions. Figure 29 explains the distribution of the jerk when the speed distribution is expressed by trigonometric functions.

Figure 26 shows the time history of the displacement when the speed distribution is expressed by trigonometric functions. The horizontal axis of the figure shows the time, and the vertical axis shows the total mileage. As a feature of the distribution curve, the point of inflection exists at time 12 s, and since a slowdown region exists at a time from 0 to 12 s, the curve projects upward. After passing through the point of inflection, since the acceleration region exists from 12 to 24 s, the curve projects downward.

Figure 27 displays the time history of the speed distribution. The horizontal axis of the figure shows the time, and the vertical axis shows the speed. The inflow and outflow speeds are different. The minimum speed exists at time 12 s. The inflow speed, minimum speed, and outflow speed are a case of regularity.

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Fig. 26 Distance profile.  Fig. 27 Speed profile expressed by trigonometric function.
Figure 28 illustrates the distribution of acceleration. The horizontal axis of the figure shows the time, and the vertical axis shows the acceleration. Since the inflow speed, minimum speed, and outflow speed are constant, the acceleration is 0 at time 0, 12 and 24 s. Going into a crossing, the acceleration is a negative value in the slowdown region and a positive value in the acceleration region. The minimum and maximum value of acceleration are -1.05 m/s² and 0.654 m/s².

Figure 29 indicates the time history of a jerk. The horizontal axis of the figure shows the time, and the vertical axis shows the jerk. The jerk increases uniformly at time 0 to the time during which the minimum speed was recorded, and the maximum jerk is 0.274 m/s³ at time 12 s. After that, the jerk decreases uniformly. When the speed distribution is defined by trigonometric functions, the maximum jerk of the distribution is 0.274 m/s³ at the time during which the minimum speed was recorded, but it changes smoothly. The maximum jerk is smaller than that of the velocity distribution function represented by the third polynomial.

![Fig. 28 Acceleration profile.](image1)

![Fig. 29 Jerk profile.](image2)

![Fig. 30 Velocity profile of three expressions.](image3)

Figure 30 explains the three types of speed distribution shown above. The horizontal axis of the figure shows the time, and the vertical axis shows the speed. The polynomial in the figure shows the speed distribution, which displays the speed distribution curve by the third-order polynomial function. Trigonometric function 1 displays the speed distribution by trigonometric functions, and indicates the speed distribution when the inflow and outflow velocities are the same. Trigonometric function 2 shows the speed distribution when the speed distribution is displayed by trigonometric functions and the inflow speed differs from the outflow velocity. Since the speed distribution curves differ although the time from invasion to escape is the same, the total distance lengths of the three types of speed profile are different.

The peak value of the acceleration was 1.00 m/s², and the peak value of the jerk was 0.333 m/s³ for the speed distribution expressed by a polynomial function. For the speed distribution expressed by trigonometric function 1, the peak value of the acceleration was 1.05 m/s², and the peak value of the jerk was 0.274 m/s³. Moreover, as for the peak value of the acceleration, for the speed profile expressed by trigonometric function 2, the peak value of the acceleration was 1.05 m/s², and the peak value of the jerk was 0.274 m/s³. In the speed distribution expressed by trigonometric functions, the peak value of the acceleration and the jerk were smaller than those of the speed profile expressed by polynomials.
4. Conclusion

A theoretical investigation of a running track was shown in the turn of an intersection. A theoretical analysis was performed for cases of non-interpolation, a single clothoid curve, and a multiple clothoid curve as a transition curve at a connection point. The following results were obtained.

The transverse direction acceleration of an object that moves in accordance with a course was proportional to the curvature distribution of the running track. The integration of the curvature distribution function is the course angle, and the curvature distribution is the course angle speed. The first- and second-order differential values of the curvature distribution function are the course angle acceleration and the course angle jerk, respectively.

The physical quantity introduced from the curvature distribution function was shown theoretically as a transition curve for the cases of non-interpolation, a single clothoid curve, and a multiple clothoid curve. A theoretical equation for the speed distribution of vehicles that turn at an intersection was proposed. A distribution curve in which the acceleration and the jerk smoothly change was defined.

As a future subject, when a vehicle turns right or left with the application of the running speed distribution at an intersection, as proposed in this research, the optimal running track and running speed distribution are needed from a running-speed and other run environments. Moreover, research on an optimal parameter that determines the speed distribution curve for vehicles during automatic operation is required. As various running tracks in an intersection, the optimal run way transition curve in three-dimensional space also needs to be developed.

The result of the theoretical analysis about the multiple clothoid curve, which is the new transition curve shown in this paper, is useful in automotive engineering and other areas. We think that the theoretical results are expected to be applied in a wide range of fields, including theoretical research on the optimal paths of robots for NC (Numerical control) control in welding, painting, and adhesives applications; theoretical consideration of the orbital designs of vessels and airplanes; and theoretical research for the orbital design of machine elements, the tool loci of machine tools, the expression of outline forms of structures, furthermore, design of railway track for highspeed railways (Hodas, 2014), and so on.

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