Sharp results concerning disjoint cross-intersecting families

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Abstract

For an \( n \)-element set \( X \) let \( (X)_k \) be the collection of all its \( k \)-subsets. Two families of sets \( A \) and \( B \) are called cross-intersecting if \( A \cap B \neq \emptyset \) holds for all \( A \in A, B \in B \). Let \( f(n,k) \) denote the maximum of \( \min\{|A|,|B|\} \) where the maximum is taken over all pairs of disjoint, cross-intersecting families \( A, B \subset (\{n\}_k) \). Let \( c = \log_2 e \). We prove that \( f(n,k) = \left\lfloor \frac{1}{2} (n-1)_k \right\rfloor \) essentially iff \( n > ck^2 \) (cf. Theorem 1.4 for the exact statement). Let \( f^*(n,k) \) denote the same maximum under the additional restriction that the intersection of all members of both \( A \) and \( B \) are empty. For \( k \geq 5 \) and \( n \geq k^3 \) we show that \( f^*(n,k) = \left\lfloor \frac{1}{2} \left( \binom{n-1}{k-1} - \binom{n-2k}{k-1} \right) \right\rfloor + 1 \) and the restriction on \( n \) is essentially sharp (cf. Theorem 5.4).

1 Introduction

Let \( [n] = \{1,2,3,\ldots,n\} \) be the standard \( n \)-element set and let \( 2^{[n]} \) denote its power set. A subset \( \mathcal{F} \subset 2^{[n]} \) is called a family. For \( 0 \leq k \leq n \) let \( (\{n\}_k) = \{ F \subset [n] : |F| = k \} \). Subsets of \( (\{n\}_k) \) are called \( k \)-uniform. A family \( \mathcal{F} \) is called intersecting if \( F \cap F' \neq \emptyset \) for all \( F,F' \in \mathcal{F} \). Let us state one of the central results in extremal set theory

Erdős–Ko–Rado Theorem ([EKR]). Suppose that \( \mathcal{F} \subset 2^{[n]} \) is intersecting. Then (i) and (ii) hold.

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(i) $|\mathcal{F}| \leq 2^{n-1}$.

(ii) Assuming that $\mathcal{F}$ is $k$-uniform and $n \geq 2k$ one has

\begin{equation}
|\mathcal{F}| \leq \binom{n-1}{k-1}.
\end{equation}

We should mention that (i) is a trivial consequence of the fact that $F \in \mathcal{F}$ implies $([n] \setminus F) \notin \mathcal{F}$. There are many different proofs for (1.1). E.g., [D], [Ka2], [Py], [FF1], [HK], to mention a few.

**Definition 1.1.** If for some $x \in [n]$, $x \in F$ for all $F \in \mathcal{F}$ then $\mathcal{F}$ is called a star.

The full star $S_x = \{S \in \binom{[n]}{k} : x \in S\}$ shows that (1.1) is sharp.

**Hilton–Milner Theorem ([HM]).** Suppose that $n > 2k$, $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $\mathcal{F}$ is not a star. Then

\begin{equation}
|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.
\end{equation}

There are many known proofs for this important result as well. E.g. [FF2], [F87], [KZ], to mention a few.

Trying to prove results about intersecting families one arrives naturally at the following notion.

**Definition 1.2.** Two families $A, B$ are called cross-intersecting if $A \cap B \neq \emptyset$ for all $A \in A$, $B \in B$.

Noting that for $A = B$ the cross-intersecting property reduces to $A$ being intersecting, one can generalize (i) to:

\begin{equation}
|A| + |B| \leq 2^n
\end{equation}

whenever $A, B \subset 2^{[n]}$ are cross-intersecting. Although the bound $|A| + |B| \leq \binom{n}{k}$ is true, for $n > 2k$ it is not sufficient to derive (1.1). However, considering products does the job.
Pyber Theorem [Py]. Suppose that $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross-intersecting, $n \geq 2k$. Then

\[(1.3) \quad |\mathcal{A}| \cdot |\mathcal{B}| \leq \left(\frac{n-1}{k-1}\right)^2.\]

Let us mention [FK1] where a short proof of (1.1) is given.

The following natural question was first considered in [DF].

Determine or estimate $f(n) \overset{\text{def}}{=} \min\{|\mathcal{A}|, |\mathcal{B}|\}$ where $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross-intersecting and $\mathcal{A} \cap \mathcal{B} = \emptyset$. The following, rather surprising result was proved in [DF].

\[(1.4) \quad f(n) < \frac{3 - \sqrt{5}}{2} \cdot 2^n\]

and the constant $\frac{3 - \sqrt{5}}{2}$ is optimal.

The second author was the first to consider the corresponding function $f(n,k)$ for $k$-uniform families.

Conjecture 1.3 ([K17]). Suppose that $n > 2k \geq 4$. Let $f(n,k) = \max \min\{|\mathcal{A}|, |\mathcal{B}|\}$ where the maximum is over all disjoint and cross-intersecting families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$. Then

\[(1.5) \quad f(n,k) = \left\lfloor \frac{1}{2} \left(\frac{n-1}{k-1}\right) \right\rfloor.\]

A few months later Huang and independently the present authors disproved (1.5) for $n < k^2$ (cf. [H]). On the positive side Huang [H] proved (1.5) for $n > 2k^2$.

In the present paper we determine the range where (1.5) holds almost completely. Set $c = \log_2 e$ and note that $c < 3/2$.

Theorem 1.4. (i) If $n \geq ck^2 + (2-c)k$ then (1.5) is true.

(ii) If $n \leq ck^2 - 2ck + 1$ then (1.5) fails.

2 Tools of proofs

For a family $\mathcal{F}$ and a positive integer $l$ define the $l$-shadow $\sigma^{(l)}(\mathcal{F}) = \{G : |G| = l, \exists F \in \mathcal{F}, G \subset F\}$. In other words, $\sigma^{(l)}(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \binom{F}{l}$.
One of the most important results in extremal set theory is the Kruskal–Katona Theorem. For given positive integers \( m, k, l, k > l \) it determines the minimum of \( |\sigma(l)(F)| \) where \( F \) is \( k \)-uniform and \( |F| = m \).

It was Daykin’s proof [D] of the Erdős–Ko–Rado Theorem that established the connection between these two important theorems. To state this connection in the more general setting of pairs of cross-intersecting families let us define the lexicographic order, \(<_L\) on \( ([n]_t) \), \( 1 \leq t \leq n \). For \( G, H \in ([n]_t) \),

\[ G <_L H \iff \min\{i : i \in G \setminus H\} < \min\{j : j \in H \setminus G\}. \]

That is, \( \{1, 2, 77\} <_L \{1, 3, 4\} \).

For fixed \( t \) and \( m \), \( 1 \leq m < (n \atop t) \) let \( \mathcal{L}(n, t, m) \) denote the initial segment, the first \( m \) subsets of size \( t \) in the lexicographic order.

Let now \( a, b \) be positive integers, \( a + b < n \). Hilton [Hi] observed that \( A \subset ([n]_a) \) and \( B \subset ([n]_b) \) are cross-intersecting iff

\[ A \cap \sigma(a)(B^c) = \emptyset, \]

where \( B^c = \{[n] \setminus B : B \in \mathcal{B}\} \) is the family of complements. This permits the following equivalent formulation of the Kruskal–Katona Theorem.

**Kruskal–Katona Theorem** ([Ka1], [Kr]). Let \( a, b \) be positive integers, \( n > a + b \). Suppose that \( A \subset ([n]_a) \) and \( B \subset ([n]_b) \) are cross-intersecting. Then \( \mathcal{L}(n, a, |A|) \) and \( \mathcal{L}(n, b, |B|) \) are cross-intersecting too.

To get the reader familiar with this formulation let us show an easy consequence that we need later.

**Corollary 2.1.** Let \( n > 2k > 0 \) and suppose that \( G \subset ([n-1]_k) \) and \( \mathcal{H} \subset ([n-1]_k) \) are cross-intersecting and \( |G| > (n-1) \atop (k-1) - (n-k) \). Then

\[ |\mathcal{H}| \leq k - 1. \]

**Proof.** The first \( (n-1-k) \) subsets \( G \in ([n-1]_k) \) are \( \mathcal{L}(n - 1, k - 1, (n-1-k) \atop (k-1)) = \{G \in ([n-1]_k) : G \cap [k - 1] \neq \emptyset\} \). The next \( (k - 1) \)-set is \([k, 2k - 2]\). The only \( k \)-subsets intersecting all of them are \([k - 1] \cup \{j\}, k \leq j \leq 2k - 2\). \( \square \)

The proof of Theorem 1.4 (i) is based on the following result extending the Hilton–Milner Theorem to two families.
Mörs Theorem ([M]). Suppose that \( A, B \subset \binom{[n]}{k} \) are cross-intersecting, \( n > 2k > 0 \) and neither \( A \) nor \( B \) is a star. Then

\[
\min\{|A|, |B|\} \leq \binom{n - 1}{k - 1} - \binom{n - 2k - 1}{k - 1} + 1.
\]

Let us note that for \( k = 1 \) all the problems considered so far are trivial. For \( k = 2 \), and neither \( A \) nor \( B \) being a star, either \( A = B = \binom{T}{2} \) for a 3-element set \( T \), or one of \( A \) and \( B \) consists of two pairwise disjoint 2-sets and the other is a subset of the four 2-sets intersecting both. Therefore from now on we are going to assume \( k \geq 3 \).

In a recent work [FK2] we extended the Mörs Theorem using the notion of diversity (cf. the definition and statement in Section 4). This result appears to be essential in establishing the exact value of \( f^*(n, k) := \min\{|A|, |B| : A, B \subset \binom{[n]}{k}, A \cap B = \emptyset, A \text{ and } B \text{ are cross-intersecting, and neither of them is a star}\} \).

Proposition 2.2. For \( n > 2k \) one has

\[
f^*(n, k) \geq \left\lfloor \frac{1}{2} \left( \binom{n - 1}{k - 1} - \binom{n - 2k - 1}{k - 1} \right) \right\rfloor + 1.
\]

3 The proof of Theorem 1.4 and Proposition 2.2

Let \( A, B \subset \binom{[n]}{k} \) be disjoint and cross-intersecting. If one of them (say \( A \)) is a star then there are two cases. For definiteness suppose \( 1 \in A \) for all \( A \in A \). Should \( 1 \in B \) hold for all \( B \in B \) then \( |A \cup B| \leq \binom{n - 1}{k - 1} \) implies (1.5).

If there is some \( B_0 \in B \) with \( 1 \notin B_0 \) then the cross-intersecting property implies

\[
|A| \leq \binom{n - 1}{k - 1} - \binom{n - k - 1}{k - 1}.
\]

On the other hand, if neither of \( A \) and \( B \) is a star, we may apply the Mörs Theorem and get the bound (2.2). Consequently, if

\[
\binom{n - 1}{k - 1} - \binom{n - k - 1}{k - 1} + 1 \leq \frac{1}{2} \binom{n - 1}{k - 1}
\]
then \((1.3)\) holds. Equivalently

\[
\binom{n-k-1}{k-1} \geq \frac{1}{2} \binom{n-1}{k-1} + 1.
\]

Recall the inequality \(1 + x < e^x\) valid for all \(x\). Suppose that \(n \geq c(k^2 - k) + 2k - 1\). Then

\[
\binom{n-1}{k-1}/\binom{n-k-1}{k-1} = \prod_{1 \leq i \leq k-1} 1 + \frac{k}{n-k-i} < \left(1 + \frac{k}{ck(k-1)}\right)^{k-1} < e^{\frac{1}{2}} = 2.
\]

This proves \((3.2)\) for the corresponding range thereby establishing Theorem \(1.4\) (i).

Let us now prove that for \(n \leq c(k-1)^2 + 1\),

\[
\binom{n-1}{k-1} - \binom{n-k}{k-1} > \frac{1}{2} \binom{n-1}{k-1}.
\]

Equivalently,

\[
\binom{n-k}{k-1} < \frac{1}{2} \binom{n-1}{k-1}.
\]

We derive \((3.4)\) from the following chain of equalities and inequalities:

\[
\frac{\binom{n-k}{k-1}}{\binom{n-1}{k-1}} < \left(\frac{n-k}{n-1}\right)^{k-1} = \left(1 - \frac{k-1}{n-1}\right)^{k-1} \leq \left(1 - \frac{k-1}{c(k-1)^2}\right)^{k-1}
\]

\[
= \left(1 - \frac{1}{c(k-1)}\right)^{k-1} < e^{-\frac{1}{2}} = \frac{1}{2}.
\]

To use \((3.3)\), let us define two families \(\mathcal{F}\) and \(\mathcal{G}\).

\[
\mathcal{F} = \left\{F \in \binom{[n]}{k} : F \cap [k] = \{1\} \text{ or } \{2, 3, \ldots, k\}\right\},
\]

\[
\mathcal{G} = \left\{G \in \binom{[n]}{k} : 1 \in G, \ G \cap \{2, 3, \ldots, k\} \neq \emptyset\right\}.
\]
Then
\[(3.5)\quad |F| + |G| = \binom{n-1}{k-1} + n - k\]
and the families \(F, G\) are cross-intersecting while \(G\) is intersecting. In view of \((3.3)\), \(|G| > \frac{1}{2}\binom{n-1}{k-1}\).

If \(|F| > \frac{1}{2}\binom{n-1}{k-1}\) then we are done.

Suppose \(|F| \leq \frac{1}{2}\binom{n-1}{k-1}\) and let \(G_0 \subset G\) be an arbitrary subfamily satisfying \(|G_0| = \left\lfloor \frac{1}{2}\binom{n-1}{k-1} \right\rfloor + 1 - |F|\). Set \(A = F \cup G_0, B = G \setminus G_0\). By definition \(|A| = \left\lfloor \frac{1}{2}\binom{n-1}{k-1} \right\rfloor + 1\) and \((3.5)\) implies \(|B| > \frac{1}{2}\binom{n-1}{k-1}\) as well. This completes the proof of Theorem 1.4. □

Let us turn to the proof of Proposition 2.2.

Let \(P, Q \in \binom{[2n]}{k}\) satisfy \(P \cap Q = \{2\}\). Define
\[A = \left\{ A \in \binom{[n]}{k} : 1 \in A, A \cap Q \neq \emptyset \right\} \cup \{P\},\]
\[B = \left\{ B \in \binom{[n]}{k} : 1 \in B, B \cap P \neq \emptyset \right\} \cup \{Q\}.
\]

It is easy to see that \(A\) and \(B\) are cross-intersecting and neither of them is a star. However, they are not disjoint.

Noting \(|A| = |B|\) and \(|A \cup B| = \binom{n-1}{k-1} - \binom{n-2k+1}{k-1} + 2\), one can remove equitably the members of \(A \cap B\) from exactly one of the two families to obtain \(A_0 \subset A, B_0 \subset B\),
\[|A_0| = \left\lfloor \frac{1}{2} \left( \binom{n-1}{k-1} - \binom{n-2k}{k-1} \right) \right\rfloor + 1,\]
\[|B_0| = \left\lceil \frac{1}{2} \left( \binom{n-1}{k-1} - \binom{n-2k}{k-1} \right) \right\rceil + 1.\]

Obviously, \(A_0\) and \(B_0\) are cross-intersecting and for \(k \geq 2\) neither of them is a star. This concludes the proof of \((2.3)\). □

4 In the grey zone

In the previous section we proved Theorem 1.4. To be more exact, we proved that
\[f(n, k) = \left\lfloor \frac{1}{2} \binom{n-1}{k-1} \right\rfloor \quad \text{if} \quad \frac{1}{2} \binom{n-1}{k-1} \geq \binom{n-1}{k-1} - \binom{n-k}{k-1} \]
and also
\[ f(n, k) > \frac{1}{2} \binom{n-1}{k-1} \quad \text{if} \quad \frac{1}{2} \binom{n-1}{k-1} \leq \binom{n-1}{k-1} - \binom{n-k}{k-1}. \]

Let us try and say something about the “grey zone”, about the narrow range where
\[ \binom{n-1}{k-1} - \binom{n-k}{k-1} \leq \frac{1}{2} \binom{n-1}{k-1} < \binom{n-1}{k-1} - \binom{n-k}{k-1}, \]
or equivalently
\[ \binom{n-k-1}{k-1} < \frac{1}{2} \binom{n-1}{k-1} < \binom{n-k}{k-1}. \]

Let us prove that even if (1.5) fails it is almost true.

**Proposition 4.1.** Suppose that (4.1) hold for the pair \((n, k), n > 2k.\) Then
\[ f(n, k) \leq \frac{1}{2} \left( \binom{n-1}{k-1} + n - k - 1 \right). \]

**Proof.** Let again \(A, B \subset \binom{[n]}{k}\) be disjoint and cross-intersecting, moreover, \(\min\{|A|, |B|\} \geq \frac{1}{2} \binom{n-1}{k-1}.\) If both are stars then (1.5) holds. Suppose now that \(A\) is a star where \(n \in A\) for all \(A \in A.\) In view of (4.1) we may apply Corollary 2.1 with \(G = A(n) = \{A \setminus \{n\} : A \in A\}, H = B(\bar{n}) = \{B \in B : n \notin B\}.\) This gives \(|H| \leq k - 1.\) Except for \(H,\) all members of \(A \cup B\) contain \(n.\) Thus we infer
\[ |A \cup B| \leq \binom{n-1}{k-1} + k - 1. \]

Since \(A\) and \(B\) are disjoint, (4.3) follows.

The case that remains is when neither \(A\) nor \(B\) is a star. To deal with this case we need the notion of diversity and the extension of Mörs Theorem.

**Definition 4.2.** For a family \(\mathcal{F} \in 2^{[n]}\) define its diversity \(\gamma(\mathcal{F})\) by \(\gamma(\mathcal{F}) = \min\{|\mathcal{F}(i)| : i \in [n]|.\)

It should be clear that \(\mathcal{F}\) is a star iff \(\gamma(\mathcal{F}) = 0.\)
Theorem 4.3 ([FK2]). Suppose that $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross-intersecting, $n > 2k$, $k > 3$. Let $u$ be an integer, $3 \leq u \leq k$ and suppose that

$$\min\{|A|, |B|\} \geq \binom{n-1}{k-1} - \binom{n-u-1}{k-1} + \binom{n-u-1}{n-k-1}.$$  

Then either equality holds in (4.4) for both $\mathcal{A}$ and $\mathcal{B}$ or

$$\max\{\gamma(\mathcal{A}), \gamma(\mathcal{B})\} < \binom{n-u-1}{n-k-1},$$

moreover both families share the same (unique) element of maximum degree.

Let us return to the proof of Proposition 4.1. We apply Theorem 4.3 with $u = k - 1$. If

$$\min\{|A|, |B|\} \leq \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k}{n-k-1},$$

then via (4.4) we derive

$$\min\{|A|, |B|\} \leq \frac{1}{2} \binom{n-1}{k-1} + n - k - 1$$

follows.

In the opposite case from (4.4) we derive

$$\max\{\gamma(\mathcal{A}), \gamma(\mathcal{B})\} \leq n - k - 1.$$ 

By symmetry we may assume

$$\max\{|\mathcal{A}(\bar{n})|, |\mathcal{B}(\bar{n})|\} \leq n - k - 1.$$ 

By disjointness of $\mathcal{A}$ and $\mathcal{B}$ we infer

$$|\mathcal{A}(n)| + |\mathcal{B}(n)| \leq \binom{n-1}{k-1}.$$ 

Thus

$$\frac{|\mathcal{A}| + |\mathcal{B}|}{2} \leq \frac{1}{2} \binom{n-1}{k-1} + n - k - 1$$

follows. □
5 Determining \( f^*(n, k) \) for \( n \geq k(k+5) \), \( k \geq 5 \)

Throughout this section let \( k \geq 5 \), \( n \geq k(k+5) \) and let \( \mathcal{A}, \mathcal{B} \subset \binom{[n]}{k} \) be disjoint, cross-intersecting. Moreover we assume that neither of \( \mathcal{A} \) and \( \mathcal{B} \) is a star.

Since we are trying to determine \( f^*(n, k) \), in view of Proposition 2.2 we may assume that

\[
\min\{|\mathcal{A}|, |\mathcal{B}|\} > \frac{1}{2} \left( \binom{n-1}{k-1} - \binom{n-2k}{k-1} \right).
\]

First we show that the above conditions on \( n \) and \( k \) guarantee that Theorem 4.3 can be applied with some \( 3 \leq u \leq k \).

Claim 5.1.

\[
\min\{|A|, |B|\} > \binom{n-1}{k-1} - \binom{n-4}{k-1} - \binom{n-4}{n-k-1}.
\]

Proof. One can rewrite the RHS as

\[
\binom{n-2}{k-2} + \binom{n-3}{k-2} + \binom{n-4}{k-3} + \binom{n-4}{n-2} + 2\binom{n-3}{k-2} < 3\binom{n-2}{k-2}.
\]

In view of (5.1) it is sufficient to show that

\[
\binom{n-2}{k-2} + \binom{n-3}{k-2} + \cdots + \binom{n-10}{k-2} > 6\binom{n-2}{k-2}.
\]

Noting that \( \frac{(n-i-1)}{k-2} \) \( \left( \frac{(n-i)}{k-2} \right) = 1 - \frac{k-2}{n-i} \geq 1 - \frac{k-2}{n-8} \) for \( 2 \leq i \leq 8 \), using \( n \geq k^2 + 5k > (k-2)(k+7) + 8 \) we infer that the above ratio is always more than

\[
1 - \frac{k-2}{(k-2)(k+7)} = 1 - \frac{1}{k+7} \geq \frac{11}{12}.
\]

Now (5.3) follows from \( 1 + \frac{11}{12} + \left( \frac{11}{12} \right)^2 + \cdots + \left( \frac{11}{12} \right)^7 = 12 \cdot (1 - \left( \frac{11}{12} \right)^8) \) and \( \left( \frac{11}{12} \right)^8 < \frac{1}{2} \).

Now let \( u \) be the maximal integer, \( 3 \leq u \leq k \) such that

\[
\min\{|\mathcal{A}|, |\mathcal{B}|\} > \binom{n-1}{k-1} - \binom{n-u-1}{k-1} + \binom{n-u-1}{n-k-1}.
\]
In view of Theorem 4.3 we may assume that
\[
\max\{|A(\bar{1})|, |B(\bar{1})|\} < \binom{n-u-1}{n-k-1}.
\]
Since neither $A$ nor $B$ is a star, $u < k$ follows.

Our next goal is to show that for $A \in A(\bar{1})$ and $B \in B(\bar{1})$ necessarily
\begin{equation}
(5.5) \quad |A \cap B| = 1.
\end{equation}

Suppose for contradiction that $|A \cap B| \geq 2$. Set $D = A \cup B$. Note that $|D| = 2k - |A \cap B| \leq 2k - 2$.

The cross-intersecting property implies that every $C \in (A(1) \cup B(1))$ satisfies $D \cap C \neq \emptyset$. Consequently,
\begin{equation}
(5.6) \quad |A(1) \cup B(1)| \leq \binom{n-1}{k-1} - \binom{n-2k+1}{k-1} = \binom{n-1}{k-1} - \binom{n-2k}{k-1} - \binom{n-2k}{k-2}.
\end{equation}

By (5.4) we have
\[
|A(\bar{1})| + |B(\bar{1})| < \binom{n-4}{k-4}.
\]

Comparing with (5.1) we infer
\begin{equation}
(5.7) \quad 2 \binom{n-4}{k-4} > \binom{n-2k}{k-2}.
\end{equation}

However, simple computation shows that this inequality fails for $n \geq k(k+5)$.

Indeed,
\begin{align*}
\frac{n-2k}{k-2} \frac{k-2}{n-4} & = \frac{(n-2k)(n-2k-1)}{(k-2)(k-3)} \prod_{2 \leq i \leq k-3} \left(1 - \frac{2(k-1)}{n-2-i}\right). \\
\end{align*}

For $k \geq 5$ and $n \geq k(k+5)$, $\frac{n-2k-1}{k-3} > \frac{n-2k}{k-2} > k + 3 \geq 8$. Also $\frac{2(k-1)}{n-(k-1)} < \frac{2}{k+5}$ and $(1 - \frac{2}{k+5})^{k-2} > e^{-2}$. These show that
\[
\binom{n-2k}{k-2} > \binom{n-4}{k-4} \left(\frac{8}{e}\right)^2,
\]
contradicting (5.7). Now (5.5) is proved.
Claim 5.2. If $A, A' \in \mathcal{A}(\bar{1})$ and $B, B' \in \mathcal{B}(\bar{1})$ then $A \cup B = A' \cup B'$.

Proof. Set $D = A \cup B$, $D' = A' \cup B'$ and suppose indirectly $D \neq D'$. As we pointed out before, the cross-intersecting property implies both $C \cap D \neq \emptyset$ and $C \cap D' \neq \emptyset$ for all $C \in \mathcal{A}(1) \cup \mathcal{B}(1)$. It is easy to see that the total number of $(k-1)$-subsets $\tilde{C} \subset [2n]$, intersecting both $D$ and $D'$ is largest if $|D \cap D'| = 2k - 2$.

In that case the number is

$$\left(\begin{array}{c} n-1 \\ k-1 \end{array}\right) - \left(\begin{array}{c} n-2k+1 \\ k-1 \end{array}\right) + \left(\begin{array}{c} n-2k-1 \\ k-3 \end{array}\right)$$

$$= \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right) - \left(\begin{array}{c} n-2k \\ k-1 \end{array}\right) - \left(\begin{array}{c} n-2k \\ k-2 \end{array}\right) - \left(\begin{array}{c} n-2k-1 \\ k-3 \end{array}\right).$$

It is only a slight difference with respect to (5.6) and we can get a contradiction in exactly the same way. □

Claim 5.3. $\min\{|\mathcal{A}(\bar{1})|, |\mathcal{B}(\bar{1})|\} = 1$.

Proof. Suppose the contrary. WLOG let $[2, k+1] \in \mathcal{A}(\bar{1})$, $[k+1, 2k] \in \mathcal{B}(\bar{1})$. Choose some different $A \in \mathcal{A}(\bar{1})$ and $B \in \mathcal{B}(\bar{1})$. By Claim 5.2, $A \cup B = [2, 2k]$. By (5.5), $A \cap B = \{j\}$ for some $j \in [2, 2k]$.

In the case $j = k + 1$, $A \neq [2, k+1]$ implies $|A \cap [k+1, 2k]| \geq 2$, contradicting (5.5). Suppose by symmetry $2 \leq j \leq k$. Now (5.5) implies $[2, k+1] \cap B = \{j\}$ whence $B = \{j\} \cup [k+2, 2k]$. From $A \cap B = \{j\}$, $A = [2, k+1]$ follows which is in contradiction with our choice $A \neq [2, k+1]$. □

By symmetry, let us suppose that $|\mathcal{A}(\bar{1})| = 1$. WLOG let $[2, k+1]$ be the unique member of $\mathcal{A}(\bar{1})$ and let $[k+1, 2k]$ be one of the members of $\mathcal{B}(\bar{1})$. Using Claim 5.2 and (5.5) we infer that

$$\mathcal{B}(\bar{1}) \subset \{\{j\} \cup [k+2, 2k] : 2 \leq j \leq k+1\},$$

in particular

$$|\mathcal{B}(\bar{1})| \leq k.\quad(5.9)$$

Together with

$$|\mathcal{A}(1) \cup \mathcal{B}(1)| \leq \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right) - \left(\begin{array}{c} n-2k \\ k-1 \end{array}\right)$$
this implies

\[ |A| + |B| \leq \binom{n - 1}{k - 1} - \binom{n - 2k}{k - 1} + 1 + |\mathcal{B}(\overline{1})| \]
\[ \leq \binom{n - 1}{k - 1} - \binom{n - 2k}{k - 1} + 1 + k. \]

In the case \(|\mathcal{B}(\overline{1})| = 1\) we infer

\[ \min \{|A|, |B|\} \leq \left\lfloor \frac{1}{2} \left( \binom{n - 1}{k - 1} - \binom{n - 2k}{k - 1} \right) \right\rfloor + 1 \]

in accordance with (2.3).

Let us show that for \(n \geq k^3\) the RHS of the last displayed inequality is the value of \(f^*(n, k)\).

**Theorem 5.4.** For \(n \geq k^3\)

\[ (5.10) \quad f^*(n, k) = \left\lfloor \frac{1}{2} \left( \binom{n - 1}{k - 1} - \binom{n - 2k}{k - 1} \right) \right\rfloor + 1. \]

**Proof.** In view of (5.10), all we have to show is \(|\mathcal{B}(\overline{1})| = 1\). Suppose for contradiction \(|\mathcal{B}(\overline{1})| \geq 2\). WLOG \([k + 1, 2k]\) and \(\{k\} \cup [k + 2, 2k]\) belong to \(|\mathcal{B}(\overline{1})|\). Then for \(A \in \mathcal{A}(1)\) either

- \(A \cap [k + 2, 2k] \neq \emptyset\) or
- \(A \cap [k + 2, 2k] = \emptyset\) but \(\{k, k + 1\} \subset A\).

The number of \((k - 1)\)-sets \(A \subset [2, n]\) satisfying these conditions is \(\binom{n - 1}{k - 1} - \binom{n - k}{k - 1} + \binom{n - k - 2}{k - 3}\). In view of \(|\mathcal{A}(\overline{1})| = 1\), to conclude the proof it is sufficient to show

\[ (5.11) \quad \binom{n - 1}{k - 1} - \binom{n - k}{k - 1} + \binom{n - k - 2}{k - 3} < \frac{1}{2} \left( \binom{n - 1}{k - 1} - \binom{n - 2k}{k - 1} \right) - \frac{1}{2}. \]

Equivalently,

\[ (5.12) \quad \binom{n - 1}{k - 1} + \binom{n - k - 2}{k - 2} + \cdots + \binom{n - 2k}{k - 2} < \binom{n - 2k}{k - 2} + \cdots + \binom{n - 2k}{k - 2} - 2 \left( \binom{n - k - 2}{k - 3} + 1 \right). \]
The last term in brackets is of smaller order of magnitude. E.g., for \( n - k > 4(k^2 - 4) \) it is smaller than \( \frac{1}{2(k+2)}(n-k-1) \).

We break up \( \binom{n-k-1}{k-2} \) into \( k+1 \) equal parts and use \( \frac{1}{k+1}\binom{n-k-1}{k-2} > \frac{1}{k+1}\binom{n-k-j}{k-2} \) for \( j \geq 2 \). One of these terms we use to compensate for the last term in (5.12). Consequently, instead of (5.12) it suffices to show that

\[
\binom{n-2}{k-2} + \cdots + \binom{n-k}{k-2} < \frac{k+2}{k+1}\binom{n-k-2}{k-2} + \cdots + \frac{k+2}{k+1}\binom{n-2k}{k-2}.
\]

This inequality follows once we show

(5.13) \( \frac{n-k-j}{k-2}/\binom{n-j}{k-2} > \frac{k+1}{k+2} \) for \( 2 \leq j \leq k \).

Let us expand the LHS of (5.13) and use the Bernoulli inequality

\[
\frac{n-k-j}{k-2}/\binom{n-j}{k-2} = \prod_{0 \leq i \leq k-3} \left(1 - \frac{1}{k} \frac{k}{n-j-i}\right) > \left(1 - \frac{k}{n-2k}\right)^{k-2} > 1 - \frac{k(k-2)}{n-2k}.
\]

Now \( n \geq k^3 \) implies \( n - 2k \geq k^3 - 2k > (k+2)k(k-2) \), i.e., the RHS is at least \( 1 - \frac{1}{k+2} = \frac{k+1}{k+2} \) completing the proof.

Let us mention that our argument was essentially sharp, that is for \( n < (1 - \varepsilon)k^3 \) the inequality (5.11) would fail completely. That is the difference of the two sides would be much more than \( k \). Consequently, imitating the proof of Proposition 2.2 we can show the following.

### Proposition 5.5.

For any \( \varepsilon > 0 \) and \( k > k_0(\varepsilon) \) in the range \( k^2 + 5k < n < (1 - \varepsilon)k^3 \) one has

\[
f^*(n,k) = \left\lfloor \frac{1}{2} \left( \frac{n-1}{k-1} - \frac{n-2k}{k-1} + k + 1 \right) \right\rfloor.
\]

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