A POSTERIORI ERROR ESTIMATES FOR THE STATIONARY NAVIER STOKES EQUATIONS WITH DIRAC MEASURES

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Abstract. In two dimensions, we propose and analyze an a posteriori error estimator for finite element approximations of the stationary Navier Stokes equations with singular sources on Lipschitz, but not necessarily convex, polygonal domains. Under a smallness assumption on the continuous and discrete solutions, we prove that the devised error estimator is reliable and locally efficient. We illustrate the theory with numerical examples.

Key words. A posteriori error estimates, Navier Stokes equations, Dirac measures, Muckenhoupt weights.

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1. Introduction. Let \( \Omega \subset \mathbb{R}^2 \) be an open and bounded domain with Lipschitz boundary \( \partial \Omega \). In this work we will be interested in the design and analysis of a posteriori error estimates for finite element approximations of the stationary Navier Stokes problem

\[
\begin{aligned}
-\Delta u + (u \cdot \nabla)u + \nabla p &= F\delta_z \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \delta_z \) corresponds to the Dirac delta supported at the interior point \( z \in \Omega \) and \( F \in \mathbb{R}^2 \). Here, \( u \) represents the velocity of the fluid, \( p \) the pressure, and \( F\delta_z \) is an externally applied force. Notice that, for simplicity, we have taken the viscosity to be equal to one.

Since the stationary Navier Stokes equations model the motion of a stationary, incompressible, Newtonian fluid, it is no surprise that their analysis and approximation, at least in energy-type spaces, is very well developed; see, for instance, [36, 22, 10, 34, 35] for an account of this theory.

On the other hand, there are situations where one wishes to allow this model to be driven by singular forces, like in (1). As a first example of this we mention [27], where the linear version of (1) is considered, and it is argued that it can be used to model the movement of active thin structures in a viscous fluid. A numerical scheme is proposed, but no complete analysis of this method is provided. Local error estimates, away from the support of the delta, were later provided in [9].

A second example comes from PDE–constrained optimization (optimal control). Reference [11] sets up a problem where the state is governed by the stationary Navier Stokes equations, but with a forcing (control) that is measure valued, like in (1). The motivation behind this is what the authors denote sparsity of the control, meaning that its support is small, even allowing it to have Lebesgue measure zero. The analysis of [11] assumes that the domain has \( C^2 \) boundary, and seeks for a solution to (1) in...
\( W^{1,q}_0(\Omega) \times L^q(\Omega)/\mathbb{R} \) with \( q \in [4/3, 2) \). In this setting a complete existence theory for the state is provided, and the optimization problem is analyzed. Necessary and sufficient optimality conditions are deduced. This work, however, is not concerned with approximation.

In this work we continue our program aimed at developing numerical methods for models of fluids under singular forces. The guiding principle that we follow is that by introducing a weight, and working in the ensuing weighted function spaces, we can allow for data that is singular, so that (1) fits our theory. We immediately must comment that the literature already presents an analysis of the stationary Navier Stokes equations on Muckenhoupt weighted spaces; see [32]. This paper however, requires the domain to be \( C^{1,1} \), which is not suitable for a finite element approximation. We, in contrast, assume only that the domain is Lipschitz. In [29] we developed existence and uniqueness for the Stokes problem over a reduced class of weighted spaces, see Definition 1 below. The numerical analysis of this linear model is presented in [17, 5], where a priori and a posteriori, respectively, error analyses are discussed. The non-linear case, that is (1) is considered in [30] where existence and uniqueness for small data, and in the same functional setting, is proved. In the setting of uniqueness, an a priori error analysis for a numerical scheme is also developed. This brings us to this work and its contributions. The solution to (1), because of the singular data, is not expected to be smooth, and thus adaptive methods must be developed to efficiently approximate it. Our goal here is to develop and analyze a reliable and efficient a posteriori error estimator, and show its performance when used in a standard adaptive procedure.

To achieve these goals we organize our presentation as follows. We set notation in section 2, where we also recall the definition of Muckenhoupt weights and introduce the weighted spaces we shall work with. In section 3, we introduce a suitable weak formulation for problem (3) in weighted spaces and review existence and uniqueness results for small data. Section 4 presents basic ingredients of finite element methods. Section 5 is one of the highlights of our work. In section 5.1 we introduce a Ritz projection of the residuals and prove, in section 5.2, that the energy norm of the error can be bounded in terms of the the energy norm of the Ritz projection. We thus propose, in section 5.3, an a posteriori error estimator for inf–sup stable finite element approximations of problem (1); the devised error estimator is proven to be locally efficient and globally reliable. We conclude, in Section 6, with a series of numerical experiments that illustrate our theory.

2. Notation and preliminaries. Let us set notation and describe the setting we shall operate with. Throughout this work \( \Omega \subset \mathbb{R}^2 \) is an open and bounded polygonal domain with Lipschitz boundary \( \partial \Omega \). Notice that we do not assume that \( \Omega \) is convex. If \( \mathcal{W} \) and \( \mathcal{Z} \) are Banach function spaces, we write \( \mathcal{W} \hookrightarrow \mathcal{Z} \) to denote that \( \mathcal{W} \) is continuously embedded in \( \mathcal{Z} \). We denote by \( \mathcal{W}' \) and \( \| \cdot \|_W \) the dual and the norm of \( \mathcal{W} \), respectively.

For \( E \subset \bar{\Omega} \) of finite Hausdorff \( i \)-dimension, \( i \in \{1, 2\} \), we denote its measure by \( |E| \). If \( E \) is such a set and \( f : E \to \mathbb{R} \) we denote its mean value by

\[
\bar{f}_E = \frac{1}{|E|} \int_E f.
\]

The relation \( a \lesssim b \) indicates that \( a \leq Cb \), with a constant \( C \) that depends neither on \( a \), \( b \) nor the discretization parameters. The value of \( C \) might change at each occurrence.
2.1. Weights. A notion which will be fundamental for further discussions is that of a weight. By a weight we mean a locally integrable, nonnegative function defined on \( \mathbb{R}^2 \). Of particular interest in our constructions will be weights that belong to the Muckenhoupt class \( A_2 \) [15], which consist of all weights \( \omega \) such that

\[
[w]_{A_2} := \sup_B \left( \frac{\int_B \omega}{\int_B \omega^{-1}} \right) < \infty,
\]

where the supremum is taken over all balls \( B \) in \( \mathbb{R}^2 \). For \( \omega \in A_2 \) the quantity \([w]_{A_2}\) is the Muckenhoupt characteristic of \( \omega \). We refer the reader to [15, 23, 28, 37] for basic facts about the class \( A_2 \). An \( A_2 \) weight which will be essential for our subsequent developments is the following. Let \( z \in \Omega \) and \( \alpha \in (−2, 2) \). Then

\[
d_z^\alpha(x) = |x − z|^\alpha \in A_2.
\]

An important property of the weight \( d_z^\alpha \) is that there is a neighborhood of \( \partial \Omega \) where \( d_z^\alpha \) is strictly positive, and continuous. This observation motivates us to define a restricted class of Muckenhoupt weights [21, Definition 2.5].

**Definition 1 (class \( A_2(\Omega) \)).** Let \( \Omega \subset \mathbb{R}^2 \) be a Lipschitz domain. We say that \( \omega \in A_2 \) belongs to \( A_2(\Omega) \) if there is an open set \( G \subset \Omega \), and \( \varepsilon, \omega > 0 \) such that:

\[
\{ x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \} \subset G, \quad \omega|_G \in C(\bar{G}), \quad \omega_1 \leq \omega(x) \quad \forall x \in \bar{G}.
\]

As we have mentioned in the introduction, what allows us to consider rough forcings in (1) is the use of weights and weighted spaces, as we will define below. We must note, however, that since we are not assuming our polygonal domain \( \Omega \) to be convex, the same considerations given in the counterexample of [14, page 2] show that we cannot work with general weights, and we cannot allow our forcings to have singularities near the boundary. This is the importance of the class \( A_2(\Omega) \) of Definition 1.

2.2. Weighted spaces. Let \( E \) be an arbitrary domain in \( \mathbb{R}^2 \) and \( \omega \in A_2 \). We define \( L^2(\omega, E) \) as the space of Lebesgue measurable functions in \( E \) such that

\[
\|v\|_{L^2(\omega, \Omega)} = \left( \int_E \omega|v|^2 \right)^{\frac{1}{2}} < \infty.
\]

We define the weighted Sobolev space \( H^1(\omega, E) \) as the set of functions \( v \in L^2(\omega, E) \) such that, for every multiindex \( \alpha \in \mathbb{N}_0^2 \) with \( |\alpha| \leq 1 \) we have that the distributional derivatives \( D^\alpha v \in L^2(\omega, E) \). We endow \( H^1(\omega, E) \) with the norm

\[
\|v\|_{H^1(\omega, E)} := \left( \|v\|^2_{L^2(\omega, E)} + \|\nabla v\|^2_{L^2(\omega, E)} \right)^{\frac{1}{2}}.
\]

We define \( H^1_0(\omega, E) \) as the closure of \( C_0^\infty (E) \) in \( H^1(\omega, E) \). We notice that, owing to a weighted Poincaré inequality [20], over \( H^1_0(\omega, E) \) the seminorm \( \|\nabla v\|_{L^2(\omega, E)} \) is equivalent to the norm defined in (4).

Spaces of vector valued functions will be denoted by boldface, that is

\[
\mathbf{H}^1_0(\omega, E) = [H^1_0(\omega, E)]^2, \quad \|\nabla v\|_{\mathbf{L}^2(\omega, E)} := \left( \sum_{i=1}^2 \|\nabla v_i\|^2_{L^2(\omega, E)} \right)^{\frac{1}{2}}.
\]
where \( v = (v_1, v_2)^T \).

The following product spaces with the weight \( d^\alpha_z \) will be of particular importance. For \( \alpha \in (-2, 2) \), we define

\[
\mathcal{X}(E) = H^1_0(d^\alpha_z, E) \times L^2(d^\alpha_z, E)/\mathbb{R}, \quad \mathcal{Y}(E) = H^1_0(d^{\alpha-\alpha}_z, E) \times L^2(d^{\alpha-\alpha}_z, E)/\mathbb{R},
\]

which we endow with standard product space norms. When \( E = \Omega \), and in order to simplify the presentation of the material, we write \( \mathcal{X} = \mathcal{X}(\Omega) \) and \( \mathcal{Y} = \mathcal{Y}(\Omega) \).

### 3. The stationary Navier Stokes equations under singular forcing.

For \( \alpha \in (-2, 2) \), we define the bilinear forms

\[
a : H^1_0(d^\alpha_z, \Omega) \times H^1_0(d^{\alpha-\alpha}_z, \Omega) \to \mathbb{R}, \quad a(w, v) := \int_\Omega \nabla w : \nabla v,
\]

and

\[
b_\pm : H^1_0(d^\alpha_z, \Omega) \times L^2(d^{\pm\alpha}_z, \Omega) \to \mathbb{R}, \quad b_\pm(v, q) := -\int_\Omega q \text{div} v.
\]

We also define the trilinear form

\[
c : [H^1_0(d^\alpha_z, \Omega)]^2 \times H^1_0(d^{\alpha-\alpha}_z, \Omega) \to \mathbb{R}, \quad c(u, w; v) := -\int_\Omega u \otimes w : \nabla v.
\]

The results of [29] yield an inf–sup condition for the bilinear form \( a \) on weighted spaces, i.e., we have

\[
\inf_{0 \neq v \in H^1_0(d^\alpha_z, \Omega)} \sup_{0 \neq w \in H^1_0(d^{\alpha-\alpha}_z, \Omega)} \frac{a(v, w)}{\|\nabla v\|_{L^2(d^\alpha_z, \Omega)} \|\nabla w\|_{L^2(d^{\alpha-\alpha}_z, \Omega)}} = \inf_{0 \neq w \in H^1_0(d^{\alpha-\alpha}_z, \Omega)} \sup_{0 \neq v \in H^1_0(d^\alpha_z, \Omega)} \frac{a(v, w)}{\|\nabla v\|_{L^2(d^{\alpha-\alpha}_z, \Omega)} \|\nabla w\|_{L^2(d^\alpha_z, \Omega)}} > 0.
\]

On the other hand, since we are in two dimensions and \( d^\alpha_z \in A_2 \), [20, Theorem 1.3] shows that \( H^1_0(d^\alpha_z, \Omega) \hookrightarrow L^4(d^\alpha_z, \Omega) \). Thus, if we denote by \( C_{4 \to 2} \) the best embedding constant, we have that the convective term can be bound as follows:

\[
|c(u, w; v)| = \left| \int_\Omega u \otimes w : \nabla v \right| \leq \|u\|_{L^4(d^\alpha_z, \Omega)} \|w\|_{L^4(d^\alpha_z, \Omega)} \|\nabla v\|_{L^2(d^{\alpha-\alpha}_z, \Omega)} \leq C^{4 \to 2}_{4 \to 2} \|\nabla u\|_{L^4(d^{\alpha-\alpha}_z, \Omega)} \|\nabla w\|_{L^4(d^\alpha_z, \Omega)} \|\nabla v\|_{L^2(d^{\alpha-\alpha}_z, \Omega)}.
\]

#### 3.1. Weak formulation.

With definitions (6)–(8) at hand, we consider the following weak formulation for problem (1): Find \( (u, p) \in \mathcal{X} \) such that

\[
a(u, v) + b_-(v, p) + c(u, u; v) = \langle F \delta_z, v \rangle, \quad b_+(u, q) = 0,
\]

for all \((v, q) \in \mathcal{Y}\). Here and in what follows, \( \langle \cdot, \cdot \rangle \) denotes a duality pairing. The spaces used for such pairing shall be evident from the context. We must immediately comment that, in order to guarantee that \( \delta_z \in H^1_0(d^{\alpha-\alpha}_z, \Omega) \), and thus that \( \langle F \delta_z, v \rangle \) is well–defined for \( v \in H^1_0(d^{\alpha-\alpha}_z, \Omega) \), the parameter \( \alpha \) should be restricted to belong to the interval \((0, 2)\); see [26, Lemma 7.1.3] and [24, Remark 21.18].
3.2. Existence and uniqueness for small data. Let us define the mappings $S : \mathcal{X} \to \mathcal{Y}'$, $\mathcal{N}L : \mathcal{X} \to \mathcal{Y}'$, and $\mathcal{F} \in \mathcal{Y}'$ by

\[
\langle S(u, p), (v, q) \rangle = a(u, v) + b_-(v, p) + b_+(u, q),
\]
\[
\langle \mathcal{N}L(u, p), (v, q) \rangle = c(u, v),
\]
and $\langle \mathcal{F}, (v, q) \rangle = \langle \mathbf{F}\delta_z, v \rangle$, respectively. With this notation (11) can be equivalently written as the following operator equation in $\mathcal{Y}'$:

\[
S(u, p) + \mathcal{N}L(u, p) = \mathcal{F}.
\]

In what follows, by $\|S^{-1}\|$ we shall denote the $\mathcal{Y}' \to \mathcal{X}$ norm of $S^{-1}$. We recall that $C_{4 \to 2}$ denotes the best constant in the embedding $H_0^1(\mathbf{d}_z^0, \Omega) \hookrightarrow L^4(\mathbf{d}_z^0, \Omega)$. Assume that the forcing term $\mathbf{F}\delta_z$ is sufficiently small so that

\[
C_{4 \to 2}^2 \|S^{-1}\|^2 \|\mathbf{F}\delta_z\|_{H_0^1(\mathbf{d}_z^+, \Omega)'} < \frac{1}{6}.
\]

With this assumption at hand, we have existence and uniqueness for small data.

**Proposition 2** (existence and uniqueness). Let $\Omega$ be Lipschitz. Assume that the forcing term $\mathbf{F}\delta_z$ is sufficiently small so that (12) holds. If $\alpha \in (0, 2)$, then there is a unique solution of (11). Moreover, this solution satisfies the estimates

\[
\|\nabla u\|_{L^2(\mathbf{d}_z^+, \Omega)} \leq \frac{3}{2} \|S^{-1}\| \|\mathbf{F}\delta_z\|_{H_0^1(\mathbf{d}_z^+, \Omega)'}
\]

and

\[
\|p\|_{L^2(\mathbf{d}_z^+, \Omega)} \leq \|\nabla u\|_{L^2(\mathbf{d}_z^+, \Omega)} + \|\nabla u\|^2_{L^2(\mathbf{d}_z^+, \Omega)} + \|\mathbf{F}\delta_z\|_{H_0^1(\mathbf{d}_z^+, \Omega)'}
\]

where the hidden constant is independent of $u$, $p$, and $\mathbf{F}\delta_z$.

**Proof.** Existence, uniqueness, and the velocity estimate are the content of [30, Corollary 1].

To show the estimate on the pressure, we invoke the weighted inf–sup condition [16, Theorem 3.1], [31, Theorem 1], [17, Lemma 15]

\[
\|p\|_{L^2(\mathbf{d}_z^+, \Omega)} \leq \sup_{0 \neq \psi \in H_0^1(\mathbf{d}_z^+, \Omega)} \frac{b_-(\psi, p)}{\|\nabla \psi\|_{L^2(\mathbf{d}_z^+, \Omega)}} \quad \forall p \in L^2(\mathbf{d}_z^+, \Omega)/\mathbb{R}.
\]

The hidden constant depends only on $\Omega$ and $[\mathbf{d}_z^+]_A$. Using this estimate for $p$, the first equation in (11), and the estimate on the convective term of (10) yields the desired pressure estimate. \qed

4. Discretization. We now propose a finite element scheme to approximate the solution to (11). To accomplish this task, we first introduce some terminology and a few basic ingredients.

4.1. Triangulation. We denote by $\mathcal{T} = \{T\}$ a conforming partition of $\Omega$ into closed simplices $T$ with size $h_T = \text{diam}(T)$ and define $h_\mathcal{T} = \max_{T \in \mathcal{T}} h_T$. We denote by $\mathcal{T}$ the collection of conforming and shape regular meshes that are refinements of an initial mesh $\mathcal{T}_0$ [12, 19].

We denote by $\mathcal{I}$ the set of internal one dimensional interelement boundaries $S$ of $\mathcal{T}$. For $S \in \mathcal{I}$, we indicate by $h_S$ the diameter of $S$. If $T \in \mathcal{T}$, we define $\mathcal{I}_T$ as the subset of $\mathcal{I}$ that contains the sides of $T$. For $S \in \mathcal{I}$, we set $\mathcal{N}_S = \{T^+, T^-\}$, where $T^+, T^- \in \mathcal{T}$ are such that $S = T^+ \cap T^-$. For $T \in \mathcal{T}$, we define the following stars or patches associated with the element $T$

\[
N_T := \{T' \in \mathcal{I} : \mathcal{I}_T \cap \mathcal{I}_{T'} \neq \emptyset\}, \quad S_T := \{T' \in \mathcal{T} : T \cap T' \neq \emptyset\}.
\]
4.2. Finite element spaces. Given a mesh \( \mathcal{T} \in \mathbb{T} \), we denote by \( \mathbf{V}(\mathcal{T}) \) and \( \mathcal{P}(\mathcal{T}) \) the finite element spaces that approximate the velocity field and the pressure, respectively, constructed over \( \mathcal{T} \). The following elections are popular.

(a) The mini element. This pair is studied, for instance, in [8], [19, Section 4.2.4], and it is defined by

\[
\begin{align*}
(15) & \quad \mathbf{V}(\mathcal{T}) = \{ v_\mathcal{T} \in C(\bar{\Omega}) : \forall T \in \mathcal{T}, v_\mathcal{T}|_T \in [P_1(T) \oplus \mathbb{B}(T)]^2 \} \cap H_0^1(\Omega), \\
(16) & \quad \mathcal{P}(\mathcal{T}) = \{ q_\mathcal{T} \in L^2(\Omega) / \mathbb{R} \cap C(\bar{\Omega}) : \forall T \in \mathcal{T}, q_\mathcal{T}|_T \in P_1(T) \},
\end{align*}
\]

where \( \mathbb{B}(T) \) denotes the space spanned by a local bubble function.

(b) The Taylor–Hood pair. The lowest order Taylor–Hood element [25], [38], [19, Section 4.2.5] is defined by

\[
\begin{align*}
(17) & \quad \mathbf{V}(\mathcal{T}) = \{ v_\mathcal{T} \in C(\bar{\Omega}) : \forall T \in \mathcal{T}, v_\mathcal{T}|_T \in P_2(T)^2 \} \cap H_0^1(\Omega), \\
(18) & \quad \mathcal{P}(\mathcal{T}) = \{ q_\mathcal{T} \in L^2(\Omega) / \mathbb{R} \cap C(\bar{\Omega}) : \forall T \in \mathcal{T}, q_\mathcal{T}|_T \in P_1(T) \}.
\end{align*}
\]

It is important to observe that, if \( \omega \in A_2 \), we have, for the elections given by (15)–(18),

\[
\mathbf{V}(\mathcal{T}) \subset W_0^{1,\infty}(\Omega) \subset H_0^1(\omega, \Omega), \quad \mathcal{P}(\mathcal{T}) \subset L^\infty(\Omega) / \mathbb{R} \subset L^2(\omega, \Omega) / \mathbb{R}.
\]

In addition, these spaces are compatible, in the sense that they satisfy weighted versions of the classical LBB condition [19, 22]. Namely, there exists a positive constant \( \beta > 0 \), which is independent of \( \mathcal{T} \) and for which we have, [17, Theorems 6.2 and 6.4]

\[
\beta \| q_\mathcal{T} \|_{L^2(d_\mathcal{T}^+ \omega, \Omega)} \leq \sup_{0 \neq v_\mathcal{T} \in \mathbf{V}(\mathcal{T})} \frac{b_+(v_\mathcal{T}, q_\mathcal{T})}{\| \nabla v_\mathcal{T} \|_{L^2(d_\mathcal{T}^+ \omega, \Omega)}} \quad \forall q_\mathcal{T} \in \mathcal{P}(\mathcal{T}).
\]

4.3. Finite element approximation. We now define a finite element approximation of problem (11) as follows: Find \( (u_\mathcal{T}, p_\mathcal{T}) \in \mathbf{V}(\mathcal{T}) \times \mathcal{P}(\mathcal{T}) \) such that

\[
(20) \quad a(u_\mathcal{T}, v_\mathcal{T}) + b_-(v_\mathcal{T}, p_\mathcal{T}) + c(u_\mathcal{T}, u_\mathcal{T}; v_\mathcal{T}) = F \cdot v_\mathcal{T}(z), \quad b_+(u_\mathcal{T}, q_\mathcal{T}) = 0,
\]

for all \( v_\mathcal{T} \in \mathbf{V}(\mathcal{T}) \) and \( q_\mathcal{T} \in \mathcal{P}(\mathcal{T}) \).

Denote by \( S_\mathcal{T} \) the discrete version of \( S \). Since the pairs \((\mathbf{V}(\mathcal{T}), \mathcal{P}(\mathcal{T}))\) satisfy all the assumptions of [17], we have that the Stokes projection onto \((\mathbf{V}(\mathcal{T}), \mathcal{P}(\mathcal{T}))\) is stable in \( H_0^1(d_\mathcal{T}^+ \Omega) \times L^2(d_\mathcal{T}^+ \Omega) / \mathbb{R} \). As a consequence, \( S_\mathcal{T} \) is a bounded linear operator whose inverse \( S_\mathcal{T}^{-1} \) is bounded uniformly over all \( \mathcal{T} \in \mathbb{T} \); see [17, Theorem 4.1]. Assume that the forcing term \( F \delta_\mathcal{T} \) is sufficiently small so that (12) holds with \( S_\mathcal{T} \) replaced with \( S_\mathcal{T} \). Then there is a unique \( (u_\mathcal{T}, p_\mathcal{T}) \in \mathbf{V}(\mathcal{T}) \times \mathcal{P}(\mathcal{T}) \) that solves (20). Moreover, we have

\[
(21) \quad \| \nabla u_\mathcal{T} \|_{L^2(d_\mathcal{T}^+ \omega, \Omega)} \leq \frac{3}{2} \| S_\mathcal{T}^{-1} \| \| F \delta_\mathcal{T} \|_{H_0^1(d_\mathcal{T}^+ \omega, \Omega)'},
\]

with a pressure estimate similar to that of Proposition 2.

4.4. A quasi–interpolation operator. In order to derive reliability properties for the proposed a posteriori error estimator, it is useful to have at hand a suitable quasi–interpolation operator with optimal approximation properties [39]. We consider the operator \( \Pi_\mathcal{T} : L^1(\Omega) \to \mathbf{V}(\mathcal{T}) \) analyzed in [28]. The construction of \( \Pi_\mathcal{T} \) is inspired in the ideas developed by Clément [13], Scott and Zhang [33], and Durán...
and Lombardi [18]: it is built on local averages over stars and thus well-defined for functions in $L^1(\Omega)$; it also exhibits optimal approximation properties. In what follows, we shall make use of the following estimates of the local interpolation error [5, 28].

To present them, we first define, for $T \in \mathcal{T}$,

$$(22) \quad D_T := \max_{x \in T} |x - z|.$$ 

**Proposition 3** (stability and interpolation estimates). Let $\alpha \in (-2, 2)$, and $T \in \mathcal{T}$. Then, for every $v \in H^1(d_s^z, S_T)$, we have the local stability bound

$$(23) \quad \| \nabla \Pi_T v \|_{L^2(d_s^z, S_T)} \lesssim \| v \|_{H^1(d_s^z, S_T)}$$

and the interpolation error estimate

$$(24) \quad \| v - \Pi_T v \|_{L^2(d_s^z, S_T)} \lesssim h_T D_T^2 \| \nabla v \|_{L^2(d_s^z, S_T)}.$$ 

In addition, if $\alpha \in (0, 2)$, then we have that

$$(25) \quad \| v - \Pi_T v \|_{L^2(T)} \lesssim h_T D_T^2 \| \nabla v \|_{L^2(d_s^z, S_T)},$$

The hidden constants, in the previous inequalities, are independent of $v$, the cell $T$, and the mesh $\mathcal{T}$.

**Proposition 4** (trace interpolation error). Let $\alpha \in (0, 2)$, $T \in \mathcal{T}$, $S \subset \mathcal{T}$, and $v \in H^1(d_s^z, S_T)$. Then we have the following interpolation error estimate for the trace

$$(26) \quad \| v - \Pi_T v \|_{L^2(S)} \lesssim h_T D_T^2 \| \nabla v \|_{L^2(d_s^z, S_T)},$$

where the hidden constant is independent of $v$, $T$, and the mesh $\mathcal{T}$.

5. **A posteriori error estimates.** In this section, we analyze a posterior error estimates for the finite element approximation (20) of problem (11). To begin with such an analysis, we define the velocity and pressure errors $(e_u, e_p)$ by

$$(27) \quad e_u := u - u_\mathcal{T} \in H_0^1(d_s^z, \Omega), \quad e_p := p - p_\mathcal{T} \in L^2(d_s^z, \Omega)/\mathbb{R},$$

respectively.

5.1. **Ritz projection.** In order to perform a reliability analysis for the proposed a posteriori error estimator, inspired by the developments of [3], we introduce a Ritz projection $(\Phi, \psi)$ of the residuals. The pair $(\Phi, \psi)$ is defined as the solution to the following problem: Find $(\Phi, \psi) \in \mathcal{X}$ such that

$$(28) \quad a(\Phi, v) = a(e_u, v) + b_-(v, e_p) + c(u, e_u; v) + c(e_u, u_\mathcal{T}; v),$$

$$(\psi, q)_{L^2(\Omega)} = b_+(e_u, q),$$

for all $(v, q) \in \mathcal{Y}$.

The following results yields the well-posedness of problem (28).

**Theorem 5** (Ritz projection). Problem (28) has a unique solution in $\mathcal{X}$. In addition, this solution satisfies the estimate

$$(29) \quad \| \nabla \Phi \|_{L^2(d_s^z, \Omega)} + \| \psi \|_{L^2(d_s^z, \Omega)} \lesssim \| \nabla e_u \|_{L^2(d_s^z, \Omega)} + \| e_p \|_{L^2(d_s^z, \Omega)} + \| \nabla u_\mathcal{T} \|_{L^2(d_s^z, \Omega)} + \| \nabla u_\mathcal{T} \|_{L^2(d_s^z, \Omega)} + \| \nabla u_\mathcal{T} \|_{L^2(d_s^z, \Omega)},$$

where the hidden constant is independent of $(\Phi, \psi)$, $(u, p)$, and $(u_\mathcal{T}, p_\mathcal{T})$. 

Proof. Define
\[ \mathcal{G} : H^1_0(d_z^{-\alpha}, \Omega) \to \mathbb{R}, \quad \mathbf{v} \mapsto a(\mathbf{e}_u, \mathbf{v}) + b_-(\mathbf{v}, \mathbf{e}_p) + c(\mathbf{u}, \mathbf{e}_u; \mathbf{v}) + c(\mathbf{e}_u, \mathbf{u}_\mathcal{F}; \mathbf{v}). \]

Notice that \( \mathcal{G} \) is linear. To prove that \( \mathcal{G} \in H^1_0(d_z^{-\alpha}, \Omega)' \), we observe that
\[
|\mathcal{G}(\mathbf{v})| \leq (\|\nabla \mathbf{e}_u\|_{L^2(d_z^\alpha, \Omega)} + \|\mathbf{e}_p\|_{L^2(d_z^\alpha, \Omega)} + \|\mathbf{u}\|_{L^4(d_z^\alpha, \Omega)} \|\mathbf{e}_u\|_{L^4(d_z^\alpha, \Omega)} \\
+ \|\mathbf{u}\|_{L^4(d_z^\alpha, \Omega)} \|\nabla \mathbf{v}\|_{L^2(d_z^{-\alpha}, \Omega)}).
\]

This, combined with the Sobolev embedding \( H^1_0(d_z^\alpha, \Omega) \to L^4(d_z^\alpha, \Omega) \), allows us to conclude.

Since \( d_z^\alpha \in A_2(\Omega) \) and \( \mathcal{G} \in H^1_0(d_z^{-\alpha}, \Omega)' \), we can thus invoke the results of [29] to conclude the existence and uniqueness of \( \Phi \in H^1_0(d_z^\alpha, \Omega) \) together with the bound
\[
(30) \quad \|\nabla \Phi\|_{L^2(d_z^{-\alpha}, \Omega)} \leq \|\nabla \mathbf{e}_u\|_{L^2(d_z^\alpha, \Omega)} + \|\mathbf{e}_p\|_{L^2(d_z^\alpha, \Omega)} + \|\nabla \mathbf{u}\|_{L^2(d_z^\alpha, \Omega)} + \|\nabla \mathbf{u}_\mathcal{F}\|_{L^2(d_z^\alpha, \Omega)}.
\]

On the other hand, since \( \mathbf{e}_u \in H^1_0(d_z^\alpha, \Omega) \), \( b_+(\mathbf{e}_u, \cdot) \) defines a linear and bounded functional in \( L^2(d_z^{-\alpha}, \Omega)/\mathbb{R} \). This immediately yields the existence and uniqueness of \( \psi \in L^2(d_z^\alpha, \Omega)/\mathbb{R} \) together with the estimate
\[
\|\psi\|_{L^2(d_z^\alpha, \Omega)} \leq \|\div \mathbf{e}_u\|_{L^2(d_z^\alpha, \Omega)}.
\]

A collection of the derived estimates yields (29). This concludes the proof. \( \square \)

5.2. An upper bound for the error. With the results of Theorem 5 at hand, we observe that the pair \( (\mathbf{e}_u, \mathbf{e}_p) \) can be seen as the solution to the following Stokes problem: Find \( (\mathbf{e}_u, \mathbf{e}_p) \in X \) such that
\[
(31) \quad a(\mathbf{e}_u, \mathbf{v}) + b_-(\mathbf{v}, \mathbf{e}_p) = \mathcal{G}(\mathbf{v}), \quad b_+(\mathbf{e}_u, \mathbf{q}) = (\psi, \mathbf{q})_{L^2(\Omega)}
\]
for all \( (\mathbf{v}, \mathbf{q}) \in \mathcal{Y} \), where
\[ \mathcal{G} : H^1_0(d_z^{-\alpha}, \Omega) \to \mathbb{R}, \quad \mathbf{v} \mapsto a(\mathbf{\Phi}, \mathbf{v}) - c(\mathbf{u}, \mathbf{e}_u; \mathbf{v}) - c(\mathbf{e}_u, \mathbf{u}_\mathcal{F}; \mathbf{v}). \]

It is clear that \( \mathcal{G} \) is linear in \( H^1_0(d_z^{-\alpha}, \Omega) \). In fact, \( \mathcal{G} \in H^1_0(d_z^{-\alpha}, \Omega)' \), since
\[
(32) \quad \|\mathcal{G}\|_{H^1_0(d_z^{-\alpha}, \Omega)'} \leq \|\nabla \mathbf{\Phi}\|_{L^2(d_z^\alpha, \Omega)} + C_{4 \to 2}^2 \|\nabla \mathbf{e}_u\|_{L^2(d_z^\alpha, \Omega)} + \|\nabla \mathbf{u}_\mathcal{F}\|_{L^2(d_z^\alpha, \Omega)}.
\]

With the aid of this identification, we now prove that the energy norm of the error can be bounded in terms of the the energy norm of the Ritz projection, which in turn will allow us to provide computable upper bounds for the error. To do so, we must assume that the forcing term \( \mathbf{F}\delta_z \) is sufficiently small so that
\[
(33) \quad 1 - \|\mathcal{S}^{-1}\| C_{4 \to 2}^2 \left[ \|\nabla \mathbf{u}\|_{L^2(d_z^\alpha, \Omega)} + \|\nabla \mathbf{u}_\mathcal{F}\|_{L^2(d_z^\alpha, \Omega)} \right] \geq \lambda > 0.
\]

Corollary 6. (Upper bound for the error). Assume that the forcing term \( \mathbf{F}\delta_z \) is sufficiently small so that (33) holds. We then have that
\[
(34) \quad \|\nabla \mathbf{e}_u\|_{L^2(d_z^\alpha, \Omega)} + \|\mathbf{e}_p\|_{L^2(d_z^\alpha, \Omega)} \lesssim \|\nabla \mathbf{\Phi}\|_{L^2(d_z^\alpha, \Omega)} + \|\psi\|_{L^2(d_z^\alpha, \Omega)},
\]
where the hidden constant is independent of \( (\mathbf{u}, \mathbf{p}), (\mathbf{u}_\mathcal{F}, \mathbf{p}_\mathcal{F}) \), and \( (\Phi, \psi) \).
Proof. Since \( d^0 \in A_2(\Omega) \) and \( \mathfrak{g} \in H^1_0(\Omega) \), we can apply [29, Theorem 17] to conclude that

\[
\begin{align*}
\|\nabla e_u\|_{L^2(d^0,\Omega)} + \|e_p\|_{L^2(d^0,\Omega)} &\leq \|S^{-1}\| \left( \|\nabla \Phi\|_{L^2(d^0,\Omega)} + \|\psi\|_{L^2(d^0,\Omega)} \right) \\
&+ C_{\ref{thm:aposteriori}} \|\nabla e_u\|_{L^2(d^0,\Omega)} \left( \|\nabla u\|_{L^2(d^0,\Omega)} + \|\nabla u\|_{L^2(d^0,\Omega)} \right),
\end{align*}
\]

where we have also used estimate (32). The smallness assumption (33) allows us to absorb the last term in this estimate on the left hand side and obtain (34). This concludes the proof. \( \square \)

5.3. A residual-type error estimator. In this section, we propose an a posteriori error estimator for the finite element approximation (20) of problem (11).

Define, for \( \alpha \in (0,2) \) and \( T \in \mathcal{T} \), the element indicator

\[
\begin{align*}
\mathcal{E}_\alpha^2(u_T,p_T; T) &:= h_T^2 \|\Delta u_T - (u_T \cdot \nabla) u_T - \text{div} u_T - \nabla p_T\|_{L^2(T)}^2 \\
&+ \|\text{div} u_T\|_{L^2(d^0; T)}^2 + h_T^2 \|\nabla u_T - p_T I\|_{L^2(T \setminus \partial T)}^2 + h_T^2 |\{ z \cap T \}|,
\end{align*}
\]

where \((u_T,p_T)\) denotes the solution to the discrete problem (20), \( I \in \mathbb{R}^{d \times d} \) denotes the identity matrix, and, for a set \( E \), by \( |\{ z \} \cap E| \) we mean its cardinality. Thus \( |\{ z \} \cap T| \) equals one if \( z \in T \) and zero otherwise. Here we must recall that we consider our elements \( T \) to be closed sets. For a discrete tensor valued function \( W_T \), we denote by \([W_T \cdot \nu]\) the jump or interelement residual, which is defined, on the internal side \( S \in \mathcal{T} \) shared by the distinct elements \( T^+, T^- \in \mathcal{N}_S \), by

\[
[W_T \cdot \nu] = W_T^+ \cdot \nu^+ + W_T^- \cdot \nu^-.
\]

Here \( \nu^+, \nu^- \) are unit normals on \( S \) pointing towards \( T^+, T^- \), respectively. The error estimator is thus defined as

\[
\mathcal{E}_\alpha(u_T,p_T; \mathcal{T}) := \left( \sum_{T \in \mathcal{T}} \mathcal{E}_\alpha^2(u_T,p_T; T) \right)^{\frac{1}{2}}.
\]

5.4. Reliability. We present the following global reliability result.

**Theorem 7** (global reliability). Let \((u,p) \in X\) be the solution to (11) and the pair \((u_T,p_T) \in (V(T),P(T))\) be its finite element approximation defined as the solution to (20). Assume that the forcing term \( F_0 \) is sufficiently small so that (33) holds. If \( \alpha \in (0,2) \), then

\[
\|\nabla e_u\|_{L^2(d^0,\Omega)} + \|e_p\|_{L^2(d^0,\Omega)} \leq \mathcal{E}_\alpha(u_T,p_T; \mathcal{T}),
\]

where the hidden constant is independent of the continuous and discrete solutions, the size of the elements in the mesh \( \mathcal{T} \), and \#\( \mathcal{T} \).

Proof. We proceed in three steps.

**Step 1.** Using the first equation of (28) and (11) we obtain that

\[
a(\Phi,v) = \langle F_0, v \rangle - \sum_{T \in \mathcal{T}} \int_T (\nabla u_T : \nabla v - u_T \otimes u_T : \nabla v - p_T \text{div} v),
\]

where \( a(\Phi,v) \) is the bilinear form associated with the variational formulation.
for every \( v \in H^0_0(d_x^{-\alpha}, \Omega) \). Integrating by parts we arrive at the identity

\[
(40) \quad a(\Phi, v) = (F\delta_z, v) + \sum_{S \in \mathcal{T}} \int_S \left[ (\nabla u_{v} - p_{v} I) \cdot \nu \right] \cdot v \\
+ \sum_{T \in \mathcal{T}} \int_T (\Delta u_{v} - (u_{v} \cdot \nabla) u_{v} - \text{div} u_{v} u_{v} - \nabla p_{v}) \cdot v.
\]

On the other hand, the first equation of problem (20) can be rewritten as

\[
(F\delta_z, v_{v}) - a(u_{v}, v_{v}) - b_{\nu}(v_{v}, p_{v}) - c(u_{v}, u_{v}; v_{v}) = 0 \quad \forall v_{v} \in V(\mathcal{T}).
\]

Set \( v_{v} = \Pi_{\mathcal{T}} v \) in the previous expression, apply, again, an integration by parts formula and invoke (39) to arrive at the identity

\[
(41) \quad a(\Phi, v) = (F\delta_z, v - \Pi_{\mathcal{T}} v) + \sum_{S \in \mathcal{T}} \int_S \left[ (\nabla u_{v} - p_{v} I) \cdot \nu \right] (v - \Pi_{\mathcal{T}} v) \\
+ \sum_{T \in \mathcal{T}} \int_T (\Delta u_{v} - (u_{v} \cdot \nabla) u_{v} - \text{div} u_{v} u_{v} - \nabla p_{v}) \cdot (v - \Pi_{\mathcal{T}} v) = I + II + III.
\]

Notice that, to derive the previous expression, we have used that \( \int_S [u_{v} \otimes u_{v} \cdot \nu] \cdot (v - \Pi_{\mathcal{T}} v) = 0 \), which follows from the fact our finite element velocity space consists of continuous functions.

We now control each term separately. To control the term \( I \), we first invoke the local bound of [1, Theorem 4.7] for \( \delta_z \) and then the interpolation error estimate (24) and the stability bound (23) to arrive at

\[
(42) \quad I \lesssim |F| \left( \frac{h_T^{\frac{n-1}{2}}}{h_T} \| v - \Pi_{\mathcal{T}} v \|_{L^2(d_x^{-\alpha}, T)} + h_T^{\frac{n}{2}} \| \nabla (v - \Pi_{\mathcal{T}} v) \|_{L^2(d_x^{-\alpha}, T)} \right) \\
\lesssim |F| h_T^{\frac{n}{2}} \| \nabla v \|_{L^2(d_x^{-\alpha}, \mathcal{T})}.
\]

The control of \( II \) follows from the trace interpolation error estimate (26),

\[
(43) \quad II \lesssim \sum_{S \in \mathcal{T}} \left\| [[(\nabla u_{v} - p_{v} I) \cdot \nu] \|_{L^2(S)} \| v - \Pi_{\mathcal{T}} v \|_{L^2(S)} \right. \\
\lesssim \sum_{S \in \mathcal{T}} h_T^{-D_T} \left\| [[(\nabla u_{v} - p_{v} I) \cdot \nu] \|_{L^2(S)} \| \nabla v \|_{L^2(d_x^{-\alpha}, \mathcal{T})}. \right.
\]

We finally bound \( III \) using the error estimate (25):

\[
(44) \quad III \lesssim \sum_{T \in \mathcal{T}} h_T D_T^{\frac{n}{2}} \| \Delta u_{v} - (u_{v} \cdot \nabla) u_{v} - \text{div} u_{v} u_{v} - \nabla p_{v} \|_{L^2(T)} \| \nabla v \|_{L^2(d_x^{-\alpha}, \mathcal{T})}.
\]

We now apply the inf–sup condition (9), the identity (41) and the estimates obtained for the terms \( I, II, \) and \( III \) to arrive at

\[
\| \nabla \Phi \|_{L^2(d_x^{-\alpha}, \Omega)}^2 \lesssim \left( \sup_{0 \neq v \in H^1_0(d_x^{-\alpha}, \Omega)} \frac{a(\Phi, v)}{\| v \|_{H^1_0(d_x^{-\alpha}, \Omega)}} \right)^2 \\
\lesssim \sum_{T \in \mathcal{T}} \left( h_T D_T^{\frac{n}{2}} \left\| [[(\nabla u_{v} - p_{v} I) \cdot \nu] \right\|_{L^2(\partial T \setminus \partial \Omega)}^2 \\
+ h_T^2 D_T^{\frac{n}{2}} \| \Delta u_{v} - (u_{v} \cdot \nabla) u_{v} - \text{div} u_{v} u_{v} - \nabla p_{v} \|_{L^2(T)}^2 \right) + h_T^{\frac{n}{2}} |F| \#(\{z\} \cap T).
\]
Notice that, to obtain the last estimate, we have also used the finite overlapping property of stars.

**Step 2.** Notice that since \( \psi \in L^2(d_x^\alpha, \Omega) \), then \( \tilde{q} = d_x^\alpha \psi \in L^2(d_x^{-\alpha}, \Omega) \). Define now \( q = \tilde{q} + c \), where \( c \in \mathbb{R} \) is chosen so that \( q \in L^2(d_x^{-\alpha}, \Omega)/\mathbb{R} \). This particular choice of test function for the second equation of (28) yields

\[
(45) \quad \| \psi \|^2_{L^2(d_x^\alpha, \Omega)} = b_+(e_{\alpha}, d_x^\alpha \psi) = -b_+(u, d_x^\alpha \psi) \leq \| \text{div} u \|_{L^2(d_x^\alpha, \Omega)} \| \psi \|_{L^2(d_x^\alpha, \Omega)}.
\]

Consequently, \( \| \psi \|^2_{L^2(d_x^\alpha, \Omega)} \leq \| \text{div} u \|_{L^2(d_x^\alpha, \Omega)} \).

**Step 3.** In light of the smallness assumption (33), we may apply the estimate (34) of Corollary 6 to write

\[
\| \nabla e_u \|_{L^2(d_x^\alpha, \Omega)} \leq \| e_\alpha \|_{L^2(d_x^\alpha, \Omega)} \lesssim \left( \| \nabla \Phi \|_{L^2(d_x^\alpha, \Omega)} + \| \psi \|_{L^2(d_x^\alpha, \Omega)} \right).
\]

The desired estimate (38) thus follows from the estimates derived in steps 1 and 2. This concludes the proof. \( \Box \)

**5.5. Local efficiency analysis.** To derive efficiency properties for the local indicator \( e_{\alpha}(u, p; T) \) we utilize standard residual estimation techniques but on the basis of suitable bubble functions, whose construction we owe to [1, Section 5.2].

Given \( T \in \mathcal{T} \), we introduce an element bubble function \( \varphi_T \) that satisfies \( 0 \leq \varphi_T \leq 1 \),

\[
(46) \quad \varphi_T(z) = 0, \quad |T| \lesssim \int_T \varphi_T, \quad \| \nabla \varphi_T \|_{L^\infty(R_T)} \lesssim h_T^{-1},
\]

and there exists a simplex \( T^* \subset T \) such that \( R_T := \text{supp}(\varphi_T) \subset T^* \). Notice that, since \( \varphi_T \) satisfies (46), we have that

\[
(47) \quad \| \theta \|_{L^2(R_T)} \lesssim \left\| \frac{1}{h_T} \theta \right\|_{L^2(R_T)} \quad \forall \theta \in \mathcal{P}_T(R_T).
\]

Given \( S \in \mathcal{T} \), we introduce an edge bubble function \( \varphi_S \) that satisfies \( 0 \leq \varphi_S \leq 1 \),

\[
(48) \quad \varphi_S(z) = 0, \quad |S| \lesssim \int_S \varphi_S, \quad \| \nabla \varphi_S \|_{L^\infty(R_S)} \lesssim h_S^{-1},
\]

and \( R_S := \text{supp}(\varphi_S) \) is such that, if \( N_S = \{T, T'\} \), there are simplices \( T_s \subset T \) and \( T'_s \subset T' \) such that \( R_S \subset T_s \cup T'_s \subset T \cup T' \).

**Proposition 8** (estimates for bubble functions). Let \( T \in \mathcal{T} \) and \( \varphi_T \) be the bubble function that satisfies (46). If \( \alpha \in (0, 2) \), then

\[
(49) \quad h_T \| \nabla(\theta \varphi_T) \|_{L^2(d_x^{-\alpha}, T)} \lesssim D_T^{-\frac{\alpha}{2}} \| \theta \|_{L^2(T)} \quad \forall \theta \in \mathcal{P}_T(T).
\]

Let \( S \in \mathcal{T} \) and \( \varphi_S \) be the bubble function that satisfies (48). If \( \alpha \in (0, 2) \), then

\[
(50) \quad h_T^{-\frac{1}{2}} \| \nabla(\theta \varphi_S) \|_{L^2(d_x^{-\alpha}, N_S)} \lesssim D_T^{-\frac{\alpha}{2}} \| \theta \|_{L^2(S)} \quad \forall \theta \in \mathcal{P}_S(S),
\]

where \( \theta \) is extended to the elements that comprise \( N_S \) as a constant along the direction normal to \( S \).

**Proof.** See [1, Lemma 5.2]. \( \Box \)

Having constructed these local bubble functions, the local efficiency can be shown following more or less standard arguments.
THEOREM 9 (local efficiency). Let \((u, p) \in X\) be the solution to problem (11) and \((u_\mathcal{T}, p_\mathcal{T}) \in V(\mathcal{T}) \times \mathcal{P}(\mathcal{T})\) its finite element approximation given as the solution to (20). Assume that the forcing term \(F \delta_x\) is sufficiently small so that (33) holds. If \(\alpha \in (0, 2)\), then

\[
\delta_\alpha^2(u_{\mathcal{T}}, p_{\mathcal{T}}; T) \lesssim \|\nabla e_u\|_{L^2(d^2; \mathbb{N}_T)} + \|e_p\|_{L^2(d^2; \mathbb{N}_T)},
\]

where the hidden constant is independent of the continuous and discrete solutions, the size of the elements in the mesh \(\mathcal{T}\), and \#\(\mathcal{T}\).

**Proof.** We estimate each contribution in (35) separately, so the proof has several steps.

Step 1. For \(T \in \mathcal{T}\) we bound the bulk term \(h_T^2 D_T^2 \|\Delta u_{\mathcal{T}} - (u_{\mathcal{T}} \cdot \nabla) u_{\mathcal{T}} - \text{div} u_{\mathcal{T}} u_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|_{L^2(T)}^2\). To shorten notation, we define the functions

\[
X_T := (\Delta u_{\mathcal{T}} - (u_{\mathcal{T}} \cdot \nabla) u_{\mathcal{T}} - \text{div} u_{\mathcal{T}} u_{\mathcal{T}} - \nabla p_{\mathcal{T}})|_T, \quad Y_T := \varphi_T X_T.
\]

Since \(\varphi(z) = 0\), we immediately conclude that \(Y_T(z) = \varphi_T(z) X_T(z) = 0\). We utilize the definitions of \(X_T\) and \(Y_T\) and invoke (47) to conclude that

\[
\|\Delta u_{\mathcal{T}} - (u_{\mathcal{T}} \cdot \nabla) u_{\mathcal{T}} - \text{div} u_{\mathcal{T}} u_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|_{L^2(T)}^2 \lesssim \int_{R_T} |X_T|^2 \varphi_T = \int_T X_T \cdot Y_T.
\]

Set, in identity (40), the test function \(v = Y_T\), and use that \(Y_T(z) = 0\), and that \(Y_T|_S = 0\) for \(S \in \mathcal{T}\) to arrive at

\[
\int_T X_T \cdot Y_T = a(\Phi, Y_T).
\]

Since \(\text{supp } Y_T \subset T\), a local version of estimate (29) then implies that

\[
a(\Phi, Y_T) \lesssim \left(\|\nabla e_u\|_{L^2(d^2; T)} + \|e_p\|_{L^2(d^2; T)} + \|\nabla u\|_{L^2(d^2; T)} \|\nabla e_u\|_{L^2(d^2; T)}
\]

\[
+ \|\nabla e_u\|_{L^2(d^2; T)} \|\nabla u\|_{L^2(d^2; T)} \|\nabla Y_T\|_{L^2(d^2; T)} \right) \|\nabla Y_T\|_{L^2(d^2; T)}.
\]

Next, we utilize the smallness assumption (33), which implies the estimate

\[
\|\nabla u\|_{L^2(d^2; \Omega)} + \|\nabla u_{\mathcal{T}}\|_{L^2(d^2; \Omega)} \lesssim \frac{1 - \lambda}{\|S^{-1}\|_{C^2_4}^2},
\]

to obtain that

\[
(53) \quad a(\Phi, Y_T) \lesssim \left(\|\nabla e_u\|_{L^2(d^2; T)} + \|e_p\|_{L^2(d^2; T)} \right) \|\nabla Y_T\|_{L^2(d^2; T)}.
\]

We replace this estimate into (52) to derive

\[
(54) \quad \|X_T\|_{L^2(T)}^2 \lesssim \left(\|\nabla e_u\|_{L^2(d^2; T)} + \|e_p\|_{L^2(d^2; T)} \right) \|\nabla Y_T\|_{L^2(d^2; T)}.
\]

We recall that \(Y_T := \varphi_T X_T\) and invoke estimate (49) to arrive at

\[
\|\nabla Y_T\|_{L^2(d^2; T)} \lesssim h_T^{-1} D_T^{-1/2} \|X_T\|_{L^2(T)}.
\]
The previous two estimates yield the desired bound on the first term

\begin{equation}
(55) \quad h_T^2 D_T^2 \|\Delta u_T - (u_T \cdot \nabla) u_T - \nabla p_T \|_{L^2(T)}^2 \lesssim \|\nabla e_u\|_{L^2(d^T_{\text{d}^T_0}, T)}^2 + \|e_p\|_{L^2(d^T_{\text{d}^T_0}, T)}^2.
\end{equation}

Step 2. In this step we control the jump term \(h_T D_T^2 \|((\nabla u_T - p_T I) \cdot \nu)\|_{L^2(S)}^2\). Let \(T \in \mathcal{T}\) and \(S \in \mathcal{A}_T\). Define \(A_S = \varphi_S \|((\nabla u_T - p_T I) \cdot \nu)\|_{L^2(S)}^2\). Let \(T \in \mathcal{T}\) and since every \(T^\ast \in \mathcal{T}_*\), \(T^\ast \subset \cup\{T' : T' \in \mathcal{N}_S\}\), similar arguments to the ones that led to (53) allow us to obtain

\begin{equation}
\int_S \|((\nabla u_T - p_T I) \cdot \nu)\|_{L^2(S)}^2 A_S \leq \sum_{T^\ast \in \mathcal{N}_S} \|\nabla \Phi\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} \|\nabla A_S\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} + \sum_{T^\ast \in \mathcal{N}_S} \|X_{T^\ast'}\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} \|A_S\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} \lesssim \sum_{T^\ast \in \mathcal{N}_S} \|\nabla e_u\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} + \|e_p\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} \|\nabla A_S\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} + \sum_{T^\ast \in \mathcal{N}_S} \|X_{T^\ast'}\|_{L^2(T^\ast')} \|A_S\|_{L^2(T^\ast')}.
\end{equation}

By shape regularity we have that

\[ \|A_S\|_{L^2(T^\ast')} \approx |T^\ast'|^{\frac{d}{2}} |S|^{-\frac{d}{2}} \|A_S\|_{L^2(S)} \approx h_T^{-\frac{d}{2}} \|A_S\|_{L^2(S)}. \]

This, estimate (50), and the bound on \(X_{T^\ast'}\) derived in (55) yield

\begin{equation}
\int_S \|((\nabla u_T - p_T I) \cdot \nu)\|_{L^2(S)}^2 A_S \lesssim \sum_{T^\ast \in \mathcal{N}_S} \left( \|\nabla e_u\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} + \|e_p\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} \right) h_T^{-\frac{d}{2}} D_T^2 \|A_S\|_{L^2(S)}.
\end{equation}

We replace the previous estimate in (56) to arrive at

\begin{equation}
(57) \quad h_T D_T^2 \|((\nabla u_T - p_T I) \cdot \nu)\|_{L^2(S)}^2 \lesssim \sum_{T^\ast \in \mathcal{N}_S} \left( \|\nabla e_u\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} + \|e_p\|_{L^2(d^T_{\text{d}^T_0}, T^\ast')} \right),
\end{equation}

and since every \(T \in \mathcal{T}\) belongs to at most two \(\mathcal{N}_S\) for \(S \in \mathcal{T}\), we can conclude.

Step 3. We now bound the residual term associated with the incompressibility constraint. Since \(\text{div} u = 0\), for any \(T \in \mathcal{T}\), we immediately arrive at

\begin{equation}
\|\text{div} u_T\|_{L^2(d^T_{\text{d}^T_0}, T)} = \|\text{div} e_u\|_{L^2(d^T_{\text{d}^T_0}, T)} \lesssim \|\nabla e_u\|_{L^2(d^T_{\text{d}^T_0}, T)}.
\end{equation}
We now use the estimates (59) that yield the existence of a smooth function \( \eta \) such that

\[
\eta(z) = 1, \quad \| \eta \|_{L^\infty(\Omega)} = 1, \quad \| \nabla \eta \|_{L^\infty(\Omega)} \lesssim h_T^{-1}, \quad \text{supp}(\eta) \subset \mathcal{N}_T.
\]

Define \( \mathbf{v}_\eta := \mathbf{F}_\eta \in H_0^1(d_z^{-\alpha}, \Omega) \). Since \((\mathbf{u}, p)\) and \((\Phi, \psi)\) solve (11) and (28), respectively, we obtain

\[
\| \mathbf{F} \|^2 = \langle \mathbf{F} \delta_z, \mathbf{v}_\eta \rangle = a(\mathbf{u}, \mathbf{v}_\eta) + b_+ (\mathbf{v}_\eta, p) + c(\mathbf{u}, \mathbf{v}_\eta) \\
= a(\Phi, \mathbf{v}_\eta) + a(\mathbf{u}_\mathcal{F}, \mathbf{v}_\eta) + b_+ (\mathbf{v}_\eta, p_\mathcal{F}) + c(\mathbf{u}_\mathcal{F}, \mathbf{u}_\mathcal{F}; \mathbf{v}_\eta).
\]

Since \text{supp}(\eta) \subset \mathcal{N}_T, we apply similar arguments to the ones that led to (53), integration by parts, and basic estimates to arrive at

\[
\| \mathbf{F} \|^2 \lesssim \left( \| \nabla \mathbf{e}_\mathcal{F} \|_{L^2(d_z^{-\alpha}, \mathcal{N}_T)} + \| \mathbf{F} \|_{L^2(d_z^{-\alpha}, \mathcal{N}_T)} \right) \| \nabla \mathbf{v}_\eta \|_{L^2(d_z^{-\alpha}, \mathcal{N}_T)} \\
+ \sum_{T' \in \mathcal{F}: T' \subset \mathcal{N}_T} \| \Delta \mathbf{u}_\mathcal{F} - (\mathbf{u}_\mathcal{F} \cdot \nabla) \mathbf{u}_\mathcal{F} - \text{div} \mathbf{u}_\mathcal{F} \mathbf{u}_\mathcal{F} - \nabla p_\mathcal{F} \|_{L^2(T')} \| \mathbf{v}_\eta \|_{L^2(T')},
\]

We now use the estimates

\[
\| \eta \|_{L^2(S)} \lesssim h_T^d, \quad \| \eta \|_{L^2(\mathcal{N}_T)} \lesssim h_T^d, \quad \| \nabla \eta \|_{L^2(d_z^{-\alpha}, \mathcal{N}_T)} \lesssim h_T^{d-2-\alpha},
\]

and the fact that, since \( z \in T \), we have \( h_T \approx D_T \), to assert the bound

\[
\| \mathbf{F} \|^2 \lesssim h_T^{4d-2-\alpha} | \mathbf{F} | \left( \| \nabla \mathbf{e}_\mathcal{F} \|_{L^2(d_z^{-\alpha}, \mathcal{N}_T)} + \| \mathbf{F} \|_{L^2(d_z^{-\alpha}, \mathcal{N}_T)} \right)^{\frac{d}{d-2+\alpha}} \\
+ h_T^{4d-2-\alpha} | \mathbf{F} | \left( \sum_{T' \in \mathcal{F}: T' \subset \mathcal{N}_T} h_T D_T^{\frac{d}{d-2+\alpha}} \| \Delta \mathbf{u}_\mathcal{F} - (\mathbf{u}_\mathcal{F} \cdot \nabla) \mathbf{u}_\mathcal{F} - \text{div} \mathbf{u}_\mathcal{F} \mathbf{u}_\mathcal{F} - \nabla p_\mathcal{F} \|_{L^2(T')} \\
+ \sum_{T' \in \mathcal{F}: T' \subset \mathcal{N}_T} \sum_{S \in \mathcal{F}^{T'}: S \nabla \partial \mathcal{N}_T} D_T^{\frac{d}{d-2+\alpha}} h_T^{\frac{d}{d-2+\alpha}} \| \left( \nabla \mathbf{u}_\mathcal{F} - p_\mathcal{F} \mathbf{1} \right) \cdot \nu \|_{L^2(S)} \right)
\]

Invoke (55) and (57) to conclude.

**Step 5.** Collect the estimates derived in the previous steps to arrive at the desired local efficiency estimate (51).

**6. Numerical results.** In this section we present a series of numerical examples that illustrate the performance of the devised error estimator \( E_\alpha(\mathbf{u}_\mathcal{F}, p_\mathcal{F}; \mathcal{F}) \) defined in (37). In some of these examples, we go beyond the presented theory and perform numerical experiments where we violate the assumption of homogeneous Dirichlet boundary conditions. The examples have been carried out with the help of a code that we implemented using C++. All matrices have been assembled exactly and global linear systems were solved using the multifrontal massively parallel sparse direct solver (MUMPS) [6, 7]. The right hand sides and terms involving the weight, and the approximation errors, are computed by a quadrature formula which is exact for polynomials of degree nineteen (19).
For a given partition $\mathcal{T}$, we solve the discrete problem (20) with the discrete spaces (17)–(18). This setting will be referred to as Taylor–Hood approximation. To obtain the solution of (20) we use a fixed–point strategy, which is described in Algorithm 2. In this algorithm, the initial guess is obtained as a discrete approximation of the solution to a Stokes problem with singular sources [5] and tol $= 10^{-8}$. Once the discrete solution $(u_\mathcal{T}, p_\mathcal{T})$ is obtained, we compute, for $T \in \mathcal{T}$, the a posteriori error indicators $\mathcal{E}_\alpha(u_\mathcal{T}, p_\mathcal{T}; T)$, given in (35), to drive the adaptive mesh refinement procedure described in Algorithm 1. Every mesh $\mathcal{T}$ is adaptively refined by marking for refinement the elements $T \in \mathcal{T}$ that are such that the step 3 in Algorithm 1 holds. A sequence of adaptively refined meshes is thus generated from the initial meshes shown in Figure 1.

We define the total number of degrees of freedom as $\text{Ndof} := \dim V(\mathcal{T}) + \dim P(\mathcal{T})$. We recall that the discrete spaces $V(\mathcal{T})$ and $P(\mathcal{T})$ are as in (17) and (18), respectively.

**Algorithm 1 Adaptive Algorithm**

Input: Initial mesh $\mathcal{T}_0$, interior point $z \in \Omega$, and $\alpha \in (0, 2)$;

1: Solve the discrete problem (20) by using Algorithm 2;

2: For each $T \in \mathcal{T}$ compute the local error indicators $\mathcal{E}_\alpha(u_\mathcal{T}, p_\mathcal{T}; T)$ given in (35);

3: Mark an element $T \in \mathcal{T}$ for refinement if

$$\mathcal{E}_\alpha(u_\mathcal{T}, p_\mathcal{T}; T) > \frac{1}{2} \max_{T' \in \mathcal{T}} \mathcal{E}_\alpha(u_\mathcal{T}, p_\mathcal{T}; T');$$

4: From step 3, construct a new mesh, using a longest edge bisection algorithm. Set $i \leftarrow i + 1$, and go to step 1.

**Algorithm 2 Fixed-Point Algorithm**

Input: Initial guess $(u_0^i, p_0^i) \in V(\mathcal{T}) \times P(\mathcal{T})$ and tol. Set $i = 1$;

1: Find $(u_i^j, p_i^j) \in V(\mathcal{T}) \times P(\mathcal{T})$ such that

$$a(u_i^j, v_\mathcal{T}) + b_-(v_\mathcal{T}, p_i^j) + c(u_i^j - u_i^{i-1}, v_\mathcal{T}; v_\mathcal{T}) = F \cdot v_\mathcal{T}(z), \quad b_+(u_i^j, q_\mathcal{T}) = 0,$$

for all $(v_\mathcal{T}, q_\mathcal{T}) \in V(\mathcal{T}) \times P(\mathcal{T})$;

2: If $|(u_i^j, p_i^j) - (u_i^{i-1}, p_i^{i-1})| >$ tol, set $i \leftarrow i + 1$, and go to step 1. Otherwise, return $(u_\mathcal{T}, p_\mathcal{T}) = (u_i^j, p_i^j)$.

6.1. **Convex and non–convex domains with homogeneous boundary conditions.** We first explore the performance of the devised a posteriori error estimator in problems with homogeneous boundary conditions on convex and non-convex domains $\Omega$. 
We present the experimental rates of convergence for the error estimator $\epsilon_{\alpha}(u, p; \mathcal{F})$ considering $\alpha \in \{0.25, 0.5, 1.0, 1.25, 1.5, 1.75\}$ (left) and the mesh obtained after 15 adaptive refinements with $\alpha = 1.5$ (right). The mesh contains 204 elements and 113 vertices.

### 6.1.1. Convex domain.

We set $\Omega = (0,1)^2$, $z = (0.5,0.5)^T$, and $F = (1,1)^T$. We explore the performance of $\epsilon_{\alpha}$ when driving the adaptive procedure of Algorithm 1. We also investigate the effect of varying the exponent $\alpha$ in the Muckenhoupt weight. To accomplish this task, we consider $\alpha \in \{0.25, 0.75, 1.0, 1.25, 1.5, 1.75\}$.

In Figure 2 we present the experimental rates of convergence for the error estimator $\epsilon_{\alpha}$. We observe that optimal experimental rates of convergence are attained for all the values of the parameter $\alpha$ that we considered. We also observe that most of the refinement is concentrated around the singular source point.

### 6.1.2. Non-convex domain.

We set $\Omega = (-1,1)^2 \setminus [0,1) \times [-1,0)$, an L-shaped domain, $z = (0.5,0.5)^T$, and $F = (1,1)^T$. We again consider different values of the exponent $\alpha$ of the Muckenhoupt weight $d_z^\alpha$ defined in (3). We consider $\alpha \in \{0.25, 0.75, 1.0, 1.25, 1.5, 1.75\}$.

In Figure 3 we present the experimental rates of convergence for the error estimator $\epsilon_{\alpha}$ and the mesh obtained after 30 adaptive refinements with $\alpha = 1.5$. We observe that, for all the values of the parameter $\alpha$ that we have considered, optimal experimental rates of convergence are attained. We also observe that most of the refinement is concentrated around the singular source point and the reentrant corner.

### 6.2. A series of Dirac sources.

We now go beyond the presented theory and include a series of Dirac delta sources on the right-hand side of the momentum equation. To be precise, we will replace the momentum equation in (1) by

$$
- \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \sum_{z \in \mathcal{Z}} \mathbf{F}_z \delta_z \text{ in } \Omega,
$$

where $\mathcal{Z} \subset \Omega$ denotes a finite set with cardinality $\# \mathcal{Z}$ which is such that $1 < \# \mathcal{Z}$ and $\{\mathbf{F}_z\}_{z \in \mathcal{Z}} \subset \mathbb{R}^2$. We introduce the weight [4, Section 5]

$$
\rho(x) = \begin{cases} 
\frac{d_z}{d_{z'}}, & \exists z \in \mathcal{Z} : |x - z| < \frac{d_z}{d_{z'}}, \\
1, & |x - z| \geq \frac{d_z}{d_{z'}}, \forall z \in \mathcal{Z},
\end{cases}
$$

where $d_z = \min \{\text{dist}(\mathcal{Z}, \partial \Omega), \min \{|z - z'| : z, z' \in \mathcal{Z}, z \neq z'|\}\}$. This weight belongs to the Muckenhoupt class $A_2$ [2] and to the restricted class $A_2(\Omega)$. With the weight
we present the experimental rates of convergence for the erro-

$D_{\alpha}(u,\rho;\mathcal{T})$ considering $\alpha \in \{0.25,0.5,1.0,1.25,1.5,1.75\}$ (left) and the mesh obtained after 30 adaptive refinements with $\alpha = 1.5$ (right). The mesh contains 1208 elements and 639 vertices.

$\rho$ at hand, we modify the definition (5) of the spaces $\mathcal{X}$ and $\mathcal{Y}$ as follows:

$$\mathcal{X} = H^1_0(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}, \quad \mathcal{Y} = H^1_0(\rho^{-1}, \Omega) \times L^2(\rho^{-1}, \Omega)/\mathbb{R}.$$  

Define

$$DTz := \min_{z \in Z} \left\{ \max_{x \in \mathcal{T}} |x - z| \right\}.$$  

We propose the following error estimator when the Taylor–Hood scheme is considered:

$$\mathcal{D}_{\alpha}(u,\rho;\mathcal{T}) := \left( \sum_{T \in \mathcal{T}} \mathcal{D}^2_{\alpha}(u,\rho;T) \right)^{\frac{1}{2}},$$

where the local indicators are such that

$$\mathcal{D}_{\alpha}(u,\rho;T) := \left( h_T^2 DTz \| \Delta u - (u \cdot \nabla) u - \text{div} u \|_{L^2(T)}^2 \right)^{\frac{1}{2}} + \| \text{div} u \|_{L^2(\rho;\mathcal{T})} + h_T D_{\alpha}T \| [(\nabla u \cdot \rho \mathbb{I}) \cdot \nu] \|_{L^2(\partial T \setminus \partial \Omega)} + \sum_{z \in Z \cap T} h_f T |F_z|^2 \right)^{\frac{1}{2}}.$$  

**6.2.1. Convex domain with four Delta sources.** We set $\Omega = (0,1)^2$ and $Z = \{(0.25,0.25), (0.25,0.75), (0.75,0.25), (0.75,0.75)\}.$

We consider $F_z = (1,1)^T$ for all $z \in Z$ and fix the exponent of the Muckenhoupt weight $p = 1.5.$

In Figure 4 we present the experimental rates of convergence for the error estimator $\mathcal{D}_{\alpha}$ and the mesh obtained after 30 adaptive refinements with $\alpha = 1.5.$ It can be observed that the devised a posteriori error estimator exhibits an optimal experimental rate of convergence. It can also be observed that most of the refinement is concentrated around the singular source points. In Figure 5, we present the finite element approximations of $|u|_{\mathcal{T}}$ and $p_{\mathcal{T}}$ over the mesh that is obtained after 30 iterations of our adaptive loop with $\alpha = 1.5.$
6.3. A convex domain with nonhomogeneous boundary conditions. We now explore the performance of our devised a posteriori error estimator by considering a problem with nonhomogeneous boundary conditions; a framework that does not fit in our analysis.

6.3.1. A rectangular domain. We set $\Omega = (0, 4) \times (0, 1)$, $z = (0.5, 0.5)^T$ and $F = (10, 10)^T$. The boundary conditions are illustrated in the left panel of Figure 6. We prescribe the parabolic Dirichlet inflow condition $u_D = (y(1 - y), 0)^T$ on $\{0\} \times [0, 1]$ and $u_D = 0$ on $[0, 4] \times \{0\} \cup \{1\}$ and the homogeneous Neumann condition $(\nabla u_D - pI)\nu = 0$ on $\{4\} \times [0, 1]$. Here, $\nu$ denotes the unit normal on $\partial \Omega$ pointing outwards. We recall that $I$ denotes the identity matrix in $\mathbb{R}^{2 \times 2}$. 

Fig. 4. A series of Dirac sources: Experimental rate of convergence for the error estimator $D_{1.5}(u_T, p_T; T)$ considering $\alpha = 1.5$ (left) and the mesh obtained after 30 adaptive refinements (right). The mesh contains 1324 elements and 693 vertices.

Fig. 5. A series of Dirac sources: Finite element approximations $|u_T|$ (left) and $p_T$ (right) over the mesh obtained after 30 adaptive refinements with $\alpha = 1.5$. The mesh contains 1324 elements and 693 vertices.
A rectangular domain: Boundary conditions for problem (1) on \( \Omega = (0, 4) \times (0, 1) \). We prescribe the parabolic Dirichlet inflow condition \( \mathbf{u}_D = (y(1 - y), 0)^T \) on \( \{ 0 \} \times [0, 1] \) and \( \mathbf{u}_D = 0 \) on \( [0, 4] \times \{0\} \cup \{1\} \) and the homogeneous Neumann condition \( (\nabla \mathbf{u}_D - pI)\mathbf{\nu} = 0 \) on \( \{4\} \times [0, 1] \) (left). Experimental rate of convergence for the error estimator \( E_{\alpha}(\mathbf{u}_T, p_T; \mathcal{T}) \) with \( \alpha = 1.5 \) (right).

In the right panel of Figure 6 we observe an optimal decay rate for the devised error estimator \( E_{1.5}(\mathbf{u}_T, p_T; \mathcal{T}) \). Finally, Figure 7 shows the streamlines and velocity field for this problem.

6.3.2 A rectangular domain with and obstacle. We set \( \Omega = (0, 10) \times (0, 1) \setminus [2, 4] \times [0.4, 0.6] \) and \( \mathbf{F}_z = (10, 10)^T \) for \( z \in \mathcal{Z} \), where \( \mathcal{Z} = \{ z_i \}_{i=1}^4 \), with

\[
\begin{align*}
z_1 &= (1.011635, 0.198805)^T, \quad z_2 = (1.011635, 0.801195)^T, \\
z_3 &= (5.354725, 0.200869)^T, \quad z_4 = (5.264444, 0.719518)^T.\end{align*}
\]
The boundary conditions are illustrated in the top panel of Figure 8. We prescribe the parabolic Dirichlet inflow condition \( u_D = (y(1-y), 0)^T \) on \( \{0\} \times [0, 1] \) and \( u_D = 0 \) on \([0,10] \times \{0\} \cup \{1\}\) and the homogeneous Neumann condition \( (\nabla u_D - p I) \nu = 0 \) on \( \{0\} \times [0, 1] \). Here, \( \nu \) denotes the unit normal on \( \partial \Omega \) pointing outwards. We recall again that \( I \) denotes the identity matrix in \( \mathbb{R}^{2 \times 2} \).

The bottom panel of Figure 8 shows the experimental rate of convergence for the error estimator \( D_\alpha(u_T, p_T; T) \) with \( \alpha = 1.5 \). The estimator exhibits an optimal rate of decay. Finally, Figure 9 shows the streamlines, magnitude and vector plot of the velocity field and the mesh obtained after 100 iterations of our adaptive loop.

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