ALGORITHMIC CONSTRUCTIONS AND PRIMITIVE ELEMENTS IN THE FREE GROUP OF RANK 2

ADAM PIGGOTT

Abstract. The centrepiece of this paper is a normal form for primitive elements which facilitates the use of induction arguments to prove properties of primitive elements. The normal form arises from an elementary algorithm for constructing a primitive element \( p \) in \( F(x, y) \) with a given exponent sum pair \((X, Y)\), if such an element \( p \) exists. Several results concerning the primitive elements of \( F(x, y) \) are recast as applications of the algorithm and the normal form.

1. Introduction

Let \( F = F(x, y) \) denote the free group on two generators \( x \) and \( y \) and let \( F_{ab} = F_{ab}(x, y) \) denote the free abelian group on two generators \( x \) and \( y \). For an element \( w \in F \), the exponent sum pair is the ordered pair of integers \((X, Y)\) such that the exponent sum of \( x \) in \( w \) is \( X \) and the exponent sum of \( y \) in \( w \) is \( Y \). Clearly, conjugate elements of \( F \) have the same exponent sum pair. In the present paper, functions are written to act on the right. Denote some specific automorphisms of \( F \) as follows, where \( v \) is an element of \( F \):

\[
\begin{align*}
\alpha_x & : x \mapsto x^{-1} & \alpha_y & : x \mapsto x & \beta & : x \mapsto y & \iota_v & : x \mapsto v^{-1}xv \\
y & \mapsto y & y & \mapsto y^{-1} & y & \mapsto x & y & \mapsto v^{-1}yv
\end{align*}
\]

An automorphism \( \phi \in \text{Aut}(F) \) is said to be basic if there exists \( n \in \mathbb{N} \) such that either \( \phi \) is defined by \( x \mapsto xy^n \) and \( y \mapsto xy^{n+1} \) or \( \phi \) is defined by \( x \mapsto xy^{n+1} \) and \( y \mapsto xy^n \). Let \( \mathcal{B} \) denote the set of basic automorphisms. Let \( \Psi : F \to F \) be the map such that \( w\Psi = w^{-1}\alpha_x\alpha_y \) for each element \( w \in F \).

An element \( w \in F \) is said to be primitive if it is the image of \( x \) under some automorphism \( \theta_w \in \text{Aut}(F) \). Much is known about the structure of primitive elements of \( F \). For example, it was shown by Cohen, Metzler and Zimmermann[?] that, other than the conjugacy class containing \( x \) and the conjugacy class containing \( y \), each conjugacy
class of primitive elements in $F$ contains an element of the form
\[ xy^{m_1} xy^{m_2} \ldots xy^{m_s} \]  

(*)

where $s \geq 0$ and $m_i \in \{ n, n+1 \}$ for some $n \in \mathbb{N}$, or contains an element obtained from an element of the form (*) by application of some combination of $\alpha_x$, $\alpha_y$ and $\beta$; we shall refer to this fact as the (first) normal form property (for primitive elements). An element $p \in F$ is said to be a palindrome if $p \Psi = p$ (that is, "$p$ reads the same forwards and backwards"). It has recently been shown that each conjugacy class of primitive elements in $F$ contains an element $a$ such that either $xay^{-1}$ is a palindrome or $yax^{-1}$ is a palindrome[?, Theorem on p.613], and further that each primitive element in $F$ is the product of at most two palindromes[?, Lemma 1.6].

A theme of the present paper is the analysis of exponent sum pairs to inform about primitive elements. Such methods have been applied since the seminal work of Nielsen in the early 20th century.

It is observed in [?] that an elementary algorithm for determining whether or not a particular element $w \in F$ is primitive follows from the normal form property. The algorithm is modelled on the second of two proofs of the normal form theorem, and provides evidence of the fundamental role that basic automorphisms play in understanding primitive elements in $F$. Taking inspiration from Cohen, Metzler and Zimmermann’s insight, this paper records an algorithm which was developed from the algorithm in [?] and which solves the following problem:

**Problem 1.** For relatively prime integers $X, Y \in \mathbb{Z}$, write down a primitive element $p \in F$ with exponent sum pair $(X, Y)$.

The utility of the above result is framed by the following two well-known results.

**Lemma 2.** If $(X, Y)$ is the exponent sum pair of a primitive element $w$ in $F$, then $X$ and $Y$ are relatively prime.

*Proof.* The element $w$ projects to $w_{ab} = x^X y^Y$ in $F_{ab}$. Since $w$ is primitive in $F$, $w_{ab}$ is primitive in $F_{ab}$ and the result follows from the well-known analogous result in $F_{ab}$. \qed

**Lemma 3** (Nielsen, see [?], pp. 166-169)). Each conjugacy class of primitive elements is determined uniquely by the corresponding exponent sum pair.

A proof of Lemma 3 is provided in §4.

Combining the solution to Problem 1 and Lemma 2 immediately yields the following.
Theorem 4. There exists a primitive element \( p \) in \( F \) with exponent sum pair \((X,Y)\) if and only if \( X \) and \( Y \) are relatively prime integers.

Combined with a simple observation and Lemma 3, the solution to Problem 1 suggests another type of normal form for primitive elements — one which, for each primitive \( p \in F \), describes an automorphism with the property that \( y \mapsto p \).

Theorem 5 (Second normal form for primitive elements). For each primitive element \( p \in F \) with exponent sum pair \((X,Y)\), there exist unique \( \epsilon, \gamma, \delta \in \{0,1\} \), a unique minimal length \( v \in F \) such that the following conditions hold:

1. if \(|X| + |Y| = 1\), then \( p = y^{\alpha_y \delta} \beta^\epsilon t_v \);
2. if \(|X| + |Y| = 2\), then \( p = y^{\phi_1 \alpha_x^2 \alpha_y \delta} t_v \) where \( \phi_1 \in B \) is the basic automorphism such that \( x \mapsto xy^2 \) and \( y \mapsto xy \);
3. if \(|X| + |Y| > 2\), then there exist exactly two sequences of basic automorphisms \( \phi_0, \phi_1, \ldots, \phi_s \in B \) such that

\[ p = y^{\phi_s \phi_{s-1} \ldots \phi_0 \alpha_x^2 \alpha_y \delta} t_v. \]

Further, the values \( \epsilon, \gamma, \delta, s \), the element \( v \in F \) and the basic automorphisms \( \phi_0, \phi_1, \ldots, \phi_s \in B \) may be found in time proportional to \( \log_2 |p| \), where \(|p|\) denotes the word-length of \( p \).

The second normal form confirms the importance of basic automorphisms and offers a useful new perspective on the primitive elements in \( F \). In particular, the second normal form facilitates the use of inductive arguments (inducting on \( s \)) when proving properties of primitive elements. Although such arguments are rarely elegant, they are simple to implement. For example, inductive arguments may be used to reprove the results from \([?]\) and \([?]\) mentioned above. The use of an inductive argument and the second normal form provides common ground between these results, which at first sight appear to be unrelated.

Let \( r \in F \). An algorithm for finding cyclically reduced primitive \( p \) such that \( r \) is contained in the normal closure of \( p \) follows immediately from Algorithm 8 and the following result.

Theorem 6. Let \( r \in F \) and let \((A,B)\) be the exponent sum pair of \( r \). If \((A,B) = (0,0)\), then \( r \in \langle p \rangle^F \) for each primitive element \( p \) in \( F \). If \((A,B) \neq (0,0)\), then \( r \in \langle p \rangle^F \) for primitive \( p \) in \( F \) if and only if the exponent sum pair of \( p \) is \( \pm \frac{1}{d}(A,B) \) for \( d \) the greatest common divisor of \( A \) and \( B \).

The structure of this paper is as follows: in §2 a solution to Problem 1 is described and the second normal form theorem proved; in §3 some
applications of the second normal form are detailed, including new proofs of those results in [?][?] described above; in §4 Theorem 6 is proved.

2. A Solution to Problem 1

Let $\mathcal{E}$ denote the map from $F$ to the set of ordered pairs of integers, which maps $w \in F$ to the exponent sum pair of $w$. Let $M$ denote the map $\text{Aut}(F) \mapsto \text{GL}(2, \mathbb{Z})$ such that, for each automorphism $\theta \in \text{Aut}(F)$,

$$\theta M = \begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix},$$

where $(X_1, Y_1)$ is the exponent sum pair of $x\theta$ and $(X_2, Y_2)$ is the exponent sum pair of $y\theta$. Let $\text{GL}(2, \mathbb{Z})$ act on the set of ordered pairs of integers by matrix post-multiplication (where for this purpose an ordered pair of integers is regarded as a $1 \times 2$ matrix of integers). It is easily verified that $w\theta \mathcal{E} = (w \mathcal{E}).(\theta M)$ for each automorphism $\theta \in \text{Aut}(F)$ and each $w \in F$.

**Notation 7.** For integers $X, Y$ with $X \neq 0$, write $Y \mod X$ for the unique integer $r$ such that $0 \leq r < |X|$ and there exists $q \in \mathbb{Z}$ such that $Y = qX + r$.

**Algorithm 8.** Let $(X, Y)$ be an ordered pair of relatively prime integers such that $1 \leq X < Y$. Define $(X_0, Y_0) := (X, Y)$. Inductively, for $i = 0, 1, 2, \ldots$, proceed as follows:

- if $X_i = 1$ then terminate the inductive process;
- if $X_i \geq 2$ and $Y_i \mod X_i \leq X_i - Y_i \mod X_i$ then define
  $$X_{i+1} := Y_i \mod X_i$$
  $$Y_{i+1} := X_i - Y_i \mod X_i$$
  $$n_i := \max\{n \in \mathbb{N} \mid nX_i < Y_i\}$$
  $$\phi_i \in \mathcal{B} \text{ such that } x \mapsto xy^{n_i+1} \text{ and } y \mapsto xy^{n_i};$$
- if $X_i \geq 2$ and $Y_i \mod X_i > X_i - Y_i \mod X_i$ then define
  $$X_{i+1} := X_i - Y_i \mod X_i$$
  $$Y_{i+1} := Y_i \mod X_i$$
  $$n_i := \max\{n \in \mathbb{N} \mid nX_i < Y_i\}$$
  $$\phi_i \in \mathcal{B} \text{ such that } x \mapsto xy^{n_i} \text{ and } y \mapsto xy^{n_i+1}.$$
Proof. It suffices to prove the following two claims, the first of which confirms that the algorithm is well-defined and the second that it achieves its goal.

(A) for each integer \( i = 0, \ldots, s - 2 \), if \((X_i, Y_i)\) is an ordered pair of relatively prime integers then \((X_{i+1}, Y_{i+1})\) is also an ordered pair of relatively prime integers;

(B) for each integer \( i = s - 1, \ldots, 1 \), if \( p_{i+1} \) is a primitive element with exponent sum pair \((X_{i+1}, Y_{i+1})\), then \((p_{i+1})\phi_i\) is a primitive element with exponent sum pair \((X_i, Y_i)\).

Both claims are proved by inductive arguments. The inductive steps are shown below.

Let \( i \) be an integer such that \( 0 \leq i < s - 1 \) and assume that \((X_i, Y_i)\) is an ordered pair of relatively prime integers. Suppose that \( d \) is a positive integer such that \( d \) divides both \( X_{i+1} \) and \( Y_{i+1} \). By definition, \( d \) divides both \( Y_i \mod X_i \) and \( (X_i - Y_i \mod X_i) \). Since \( X_i = (X_i - Y_i \mod X_i) + Y_i \mod X_i \), it follows that \( d \) divides \( X_i \). Since \( Y_i = n_i X_i + Y_i \mod X_i \), it follows that \( d \) divides \( Y_i \). Now, \( d \) divides both \( X_i \) and \( Y_i \) and \( X_i, Y_i \) relatively prime implies that \( d = 1 \), hence \((X_{i+1}, Y_{i+1})\) is also an ordered pair of relatively prime integers.

The inductive step in the proof of Claim (B) is easily verified by calculation as follows. In the case that \( Y_i \mod X_i \leq X_i - Y_i \mod X_i \), then

\[
(p_{i+1})\phi_i = (p_{i+1})\phi_i \cdot \phi_i M
= (X_{i+1}, Y_{i+1}) \begin{pmatrix} 1 & n_i \\ 1 & n_i + 1 \end{pmatrix}
= (X_{i+1} + Y_{i+1}, n_i (X_{i+1} + Y_{i+1}) + Y_{i+1})
= (X_i, Y_i)
\]

In the case that \( Y_i \mod X_i > X_i - Y_i \mod X_i \), then

\[
(p_{i+1})\phi_i = (p_{i+1})\phi_i \cdot \phi_i M
= (X_{i+1}, Y_{i+1}) \begin{pmatrix} 1 & n_i + 1 \\ 1 & n_i \end{pmatrix}
= (X_{i+1} + Y_{i+1}, n_i (X_{i+1} + Y_{i+1}) + X_{i+1})
= (X_i, Y_i)
\]

\(\square\)
Algorithm 8 is easily extended to all relatively prime ordered pairs of integers \((X,Y)\), and hence a solution to Problem 4 by the following observations:

1. it follows from Lemma 2 that there is no primitive element in \(F\) with exponent sum pair \((0,0)\);
2. it follows from the properties of the automorphisms \(\alpha_x, \alpha_y, \beta \in \text{Aut}(F)\), that there exists a primitive element in \(F\) with exponent sum pair \((X,Y)\) if and only if there exists a primitive element in \(F\) with exponent sum pair \((\min\{|X|, |Y|\}, \max\{|X|, |Y|\})\).

**Example 9.** Find a primitive element with exponent sum pair \((34,-27)\).

Define \(X_0 := 27, Y_0 := 34\).

Since \(7 = Y_0 \mod X_0 < X_0 - Y_0 \mod X_0 = 20\), define \(X_1 := 7, Y_1 := 20, n_0 := 1\) and \(\phi_0 \in \text{Aut}(F)\) such that \(x \mapsto xy^2\) and \(y \mapsto xy\).

Since \(6 = Y_1 \mod X_1 > X_1 - Y_1 \mod X_1 = 1\), define \(X_2 := 1, Y_2 := 6, n_1 := 2\) and \(\phi_1 \in \text{Aut}(F)\) such that \(x \mapsto xy^2\) and \(y \mapsto xy^3\).

Then \((xy^6)\phi_1\phi_0 = (xy^2(xy^3)^6)\phi_0 = xy^2(xy^2(xy^3)^6)^6\) is a primitive element with exponent sum pair \((27,34)\), and

\[(xy^6)\phi_1\phi_0 \alpha_x \beta = y^{-1}x^2(y^{-1}x)^2(y^{-1}x)^3\]

is a primitive element with exponent sum pair \((34,-27)\).

To prove the second normal form theorem, it is convenient to use the following lemma.

**Lemma 10.** Let \((X,Y)\) be an ordered pair of relatively prime natural numbers such that \(1 \leq X < Y\). There exists a unique sequence of ordered pairs \((1,Y_s) = (X_s,Y_s), (X_{s-1},Y_{s-1}), \ldots (X_0,Y_0) = (X,Y)\) such that, for each \(i = s-1, \ldots, 1,0:\)

\[(X_{i+1},Y_{i+1}).(\phi_i M) = (X_i,Y_i),\]

for some basic automorphism \(\phi_i \in \mathcal{B}\).

**Proof.** It is easily verified that for each basic automorphism \(\phi \in \mathcal{B}\) and each ordered pair \((U,V)\), either \((U,V).(\phi M) = (U + V, n(U + V) + U)\) or \((U,V).(\phi M) = (U + V, n(U + V) + V)\). In either case, if \((X_{i+1},Y_{i+1}).(\phi_i M) = (X_i,Y_i)\), then \(X_{i+1} = \min\{Y_i \mod X_i, X_i - Y_i \mod X_i\}\) and \(Y_{i+1} = \max\{Y_i \mod X_i, X_i - Y_i \mod X_i\}\). Thus the sequence \((1,Y_s) = (X_s,Y_s), (X_{s-1},Y_{s-1}), \ldots (X_0,Y_0) = (X,Y)\) determined in Algorithm 8 is the unique sequence with the desired properties. \(\Box\)

The second normal form is proved by collating some of the results obtained above.
Proof of the second normal form theorem. By the properties of $\alpha_x$, $\alpha_y$, $\beta$ and the set of inner automorphisms, it suffices to consider cyclically reduced primitive elements $p \in F$ with exponent sum pairs $(X, Y)$ such that $0 \leq X \leq Y$. In the case that $|X| + |Y| = 1$, Lemma 2 implies that $X = 0$, $Y = 1$ and $p$ is a cyclic permutation of $(y)\alpha_x^0\alpha_y^0\beta_1^0$. In the case that $|X| + |Y| = 1$, then $X = 1$ and $Y = 1$ and the result is clear. In the case that $X = 1$ but $|X| + |Y| > 2$, define $\phi_0 \in \mathcal{B}$ such that $y \mapsto xy^Y$ and either $x \mapsto xy^{Y-1}$ or $x \mapsto xy^{Y+1}$, then $p$ is a cyclic permutation of $(y)\phi_0\alpha_x^0\alpha_y^0\beta_1^0$. In the case that $X \geq 2$, Lemma 2 implies that $X < Y$. It follows from the algorithm and Lemma 10 that there is a unique sequence of exponent sum pairs $(1, Y_s) = (X_s, Y_s), (X_{s-1}, Y_{s-1}), \ldots (X_0, Y_0) = (X, Y)$ such that, for each $i = s - 1, \ldots, 1, 0, (X_{i+1}, Y_{i+1}),(\phi_i, M) = (X_i, Y_i)$, for some basic automorphism $\phi_i \in \mathcal{B}$. If $Y_s = 1$, then $\phi_{s-1}$ is such that $x \mapsto xy^2$ and $y \mapsto xy$, but $\phi_{s-2}$ may be defined such that $x \mapsto xy^n$ and $y \mapsto xy^{n+1}$ or $x \mapsto xy^n$ and $y \mapsto xy^{n+1};$ it is clear that the remaining basic automorphisms are uniquely determined by the sequence of exponent sum pairs. If $Y_s > 1$, then $\phi_{s-1}$ may be defined such that $y \mapsto xy^Y$ and $x \mapsto xy^{Y+1}$ or $x \mapsto xy^{Y-1};$ it is clear that the remaining basic automorphisms are uniquely determined by the sequence of exponent sum pairs.

3. Some Applications of the Second Normal Form

In this section, some applications of the second normal form and are described. It is convenient to first record the following lemma, the proof of which is trivial.

Lemma 11. Let $\phi$ be a basic automorphism. If $w \in F$ is a palindrome in which only positive exponents appear, then $(w)\phi = xv$ for some palindrome $v \in F$ in which only positive exponents appear.

It is now possible to reprove the result from [2] mentioned in the introduction, using an induction technique based on the second normal form.

Theorem 12 (Shpilrain, Bardakov and Tolstykh [2]). Each primitive element $p \in F$ is either a palindrome, or is the product of two palindromes.

Proof. Let $\epsilon, \gamma, \delta, v, s$, and $\phi_i$ (for $i = 0, 1, \ldots s$) be as in the statement of Corollary 4. If $s = -1$, then $p \in \{x^{\pm 1}, y^{\pm 1}\}$ and $p$ is a palindrome. If $s = 0$, then $p \in \{x^{\pm 1}y^z \mid z \neq 0\} \cup \{x^{\pm 1}y \mid z \neq 0\}$ and $p$ is the product of two palindromes. Assume the result holds for each primitive element where $s = k$, for some $k \geq 0$. Consider the case that $s = k + 1$. By the inductive hypothesis, $(y)\phi_s \ldots \phi_1$ is a either a palindrome or a
product of two palindromes. In the former case, Lemma \[\text{[14]}\] informs that \( (y)\phi_\ldots\phi_0 = xv \) for some palindrome \( v \in F \); in the latter case, say \((y)\phi_\ldots\phi_1 = v_1v_2 \) for palindromes \( v_1, v_2 \in F \), Lemma \[\text{[14]}\] informs that \((y)\phi_\ldots\phi_0 = xv_3xv_4 = (xv_3x)v_4 \) for some palindromes \( v_3, v_4 \in F \). Hence in either case, \((y)\phi_\ldots\phi_0 \) is the product of two palindromes (and possibly also a palindrome itself). It is clear that application of \( \alpha_x, \alpha_y, \beta \) and inner automorphisms preserve the property of being a palindrome or being a product of two palindromes, hence \( p \) has the required property. \[\square\]

The author is grateful to Peter Nickolas for pointing out the following corollary to Theorem \[\text{[12]}\].

**Corollary 13.** Let \( p \) be a primitive element in \( F \). One of the following two statements holds:

1. \( p = z^{-1}wz \) for some \( z \in F \) and some palindrome \( w \in F \);
2. \( p = z^{-1}awz \) for some \( z \in F \), some \( a \in \{x, x^{-1}, y, y^{-1}\} \) and some palindrome \( w \in F \).

**Proof.** Let \( p \) be a primitive element in \( F \). If \( p \) is a palindrome, there is nothing to prove, so we may assume that \( p \) is not a palindrome. By the Theorem, there exist palindromes \( w_1, w_2 \in F \) such that \( p = w_1w_2 \). Consider first the case that \( w_1 \) has even length, say \( w_1 = v(v\Psi) \) for some \( v \in F \). Then \( v((v\Psi)w_2v)v^{-1} = v(v\Psi)w_2 = p \); hence Case \[\text{[1]}\] holds with \( z = v^{-1} \). Next, consider the case that \( w_1 \) has odd length, say \( w_1 = va(v\Psi) \) for some \( v \in F \) and some \( a \in \{x, x^{-1}, y, y^{-1}\} \). Then \( v(a(v\Psi)w_2v)v^{-1} = va(v\Psi)w_2 = p \); hence Case \[\text{[2]}\] holds with \( z = v^{-1} \). \[\square\]

We may use a similar strategy to reprove the result from \[\text{[?]}\] mentioned in the introduction.

**Lemma 14** (Helling \[\text{[?]}\]). For each primitive element \( p \in F \), there exists a palindrome \( v \in F \) and an element \( z \in F \) such that either \( p = zy^{-1}vxz^{-1} \) or \( p = zx^{-1}vyz^{-1} \).

**Proof.** Define a subset \( Q \) of \( F \) as follows:

\[
Q := \{ w \in F \mid \exists z \in F \text{ and a palindrome } v \in F \text{ such that } w = zy^{-1}vxz^{-1} \text{ or } w = zx^{-1}vyz^{-1} \}\]

It is clear from the definition that \( Q \) is closed under the action of inner automorphisms.
Let \( p \) be such that \( p = zy^{-1}vxz^{-1} \) for some \( z \in F \) and for some palindrome \( v \in F \). Then

\[
\begin{align*}
(p)\alpha_x &= (z\alpha_x)y^{-1}(v\alpha_x)x^{-1}(z^{-1}\alpha_x) \\
&= (z\alpha_x)(xx^{-1})y^{-1}(v\alpha_x)(y^{-1}y)x^{-1}(z^{-1}\alpha_x) \\
&= ((z\alpha_x)x)x^{-1}(y^{-1}(v\alpha_x)y^{-1})y(x^{-1}(z^{-1}\alpha_x)) \\
&= z'x^{-1}uy(z')^{-1},
\end{align*}
\]

where \( u = y^{-1}(v\alpha_x)y^{-1} \) is a palindrome and \( z' = (z\alpha_x)x \);

\[
\begin{align*}
(p)\alpha_y &= (z\alpha_y)y(v\alpha_y)x(z^{-1}\alpha_y) \\
&= (z\alpha_y)y(x^{-1}x)(v\alpha_y)x(yy^{-1})(z^{-1}\alpha_y) \\
&= ((z\alpha_y)y)x^{-1}(x(v\alpha_y)x)y(y^{-1}(z^{-1}\alpha_y)) \\
&= z'x^{-1}uy(z')^{-1}
\end{align*}
\]

where \( u = x(v\alpha_y)x \) is a palindrome and \( z' = (z\alpha_y)y \); and

\[
\begin{align*}
(p)\beta &= (z\beta)x^{-1}(v\beta)y(z^{-1}\beta) \\
&= z'x^{-1}uy(z')^{-1}
\end{align*}
\]

where \( u = v\beta \) is a palindrome and \( z' = z\beta \). A similar treatment shows that if \( p \) is such that \( p = zx^{-1}vyz^{-1} \), then \( p\alpha_x, p\alpha_y \) and \( p\beta \) are of the form \( z'x^{-1}uy(z')^{-1} \) or \( z'y^{-1}ux(z')^{-1} \), for some palindrome \( u \in F \) and for some \( z' \in F \). Hence \( Q \) is also closed under the action of \( \alpha_x, \alpha_y \), and \( \beta \).

To complete the proof it suffices to show that \( (y)\phi_s \ldots \phi_0 \in Q \) for each list of basic automorphisms \( \phi_0, \phi_1, \ldots, \phi_s \). Let \( s \geq 0 \) and let \( \phi_0, \phi_1, \ldots, \phi_s \) be a list of basic automorphisms. It is clear that \( (y)\phi_s \in Q \). Assume that, for some \( i \) such that \( 0 < i \leq s \), \( (y)\phi_s \ldots \phi_i \in Q \). Suppose first that \( (y)\phi_s \ldots \phi_i = zx^{-1}vyz^{-1} \) for some palindrome \( v \in F \) and for some \( z \in F \) and \( \phi_{i-1} \) is defined by \( x \mapsto xy^{n_i+1} \) and \( y \mapsto xy^{n_i} \). Then

\[
\begin{align*}
(y)\phi_s \ldots \phi_i\phi_{i-1}(y) &= (zx^{-1}vyz^{-1})\phi_{i-1} \\
&= (z\phi_{i-1})y^{-(n_i+1)}x^{-1}uxy^{n_i}(z^{-1}\phi_{i-1}) \\
&= ((z\phi_{i-1})y^{-n_i})y^{-1}ux(y^{n_i}(z^{-1}\phi_{i-1})) \\
&= z'y^{-1}ux(z')^{-1},
\end{align*}
\]
where $z' = (z\phi_{i-1})y^{-n_i}$. Now suppose that $\phi_{i-1}$ is defined by $x \mapsto xy^{n_i}$ and $y \mapsto xy^{n_i+1}$. Then
\[
(y)\phi_s \cdots \phi_i \phi_{i-1} = (zx^{-1}vyz^{-1})\phi_{i-1}
\]
\[
= (z\phi_{i-1})y^{-n_i}x^{-1}xuxy^{n_i+1}(z^{-1}\phi_{i-1})
\]
(for some palindrome $u \in F$, by Lemma 11)
\[
= ((z\phi_{i-1})y^{-n_i})x^{-1}(xux)y(y^{n_i}(z^{-1}\phi_{i-1}))
\]
\[
= z'x^{-1}(xux)y(z')^{-1},
\]
where $xux$ is a palindrome and $z' = (z\phi_{i-1})y^{-n_i}$. The case that
\[
(y)\phi_s \cdots \phi_i = zy^{-1}vzx^{-1}
\]
for some palindrome $v \in F$ and for some $z \in F$, is verified similarly. □

4. THE NORMAL CLOSURE OF A PRIMITIVE ELEMENT

Let $\{p, q\}$ be a basis for $F$. For each element $r \in F$, let $r_{p,q}$ denote
the unique reduced word in $\{p \pm 1, q \pm 1\}$ such that $r_{p,q}$ is equal to $r$ in $F$.
The normal closure of $p$ in $F$, denoted $\langle\langle p\rangle\rangle_F$, is defined to be
\[
\langle\langle p\rangle\rangle_F := \{w \in F | w = \prod_{i=1}^{s} f_i^{-1} p^\epsilon f_i, \text{ for some } s \geq 0, \epsilon_j \in \pm 1, f_j \in F\}.
\]

Lemma 15. For each $r \in F$, $r \in \langle\langle p\rangle\rangle_F$ if and only if the exponent sum of $q$ in $r_{p,q}$ is zero.

Proof. Suppose that the exponent sum of $q$ in $r_{p,q}$ is zero. Then
\[
r_{p,q} = p^{\alpha_1} q^{\beta_1} \cdots p^{\alpha_s} q^{\beta_s},
\]
for some $s \geq 0$, $\alpha_j$ non-zero (except perhaps $\alpha_1$), $\beta_j$ non-zero (except perhaps $\beta_s$) such that $\sum_{j=1}^{s} \beta_j = 0$. Insertion of trivial words yields
\[
r_{p,q} = (q^{-(\beta_1 + \cdots + \beta_s)} p^{\alpha_1} q^{\beta_1 + \cdots + \beta_s}) \cdots (q^{-(\beta_{s-1} + \beta_s)} p^{s-1} q^{\beta_{s-1} + \beta_s}) (q^{-\beta_s} p^{\alpha_s} q^{\beta_s}),
\]
and $r \in \langle\langle p\rangle\rangle_F$. The opposite direction of implication follows easily from the definition of $\langle\langle p\rangle\rangle_F$. □

As an aside to ensure that the present paper is self-contained, Lemma 15 may be used to prove Nielsen’s result, Lemma 3.

Proof of Lemma 3. Let $z \in F$ be a primitive element with exponent sum pair $(0, 1)$. By Lemma 15, $z \in \langle\langle y\rangle\rangle_F$ and $y \in \langle\langle z\rangle\rangle_F$. That is,
\[
z = \prod_{i=1}^{s} w_i^{-1} y^{\epsilon_i} w_i, \text{ and } y = \prod_{j=1}^{t} v_j^{-1} z^{\delta_j} v_j,
\]
for some $s, t \in \mathbb{N}$, some $w_i, v_j \in F$ and some $\epsilon_i, \delta_j \in \{\pm 1\}$ such that
\[ \sum_{i=1}^{s} \epsilon_i = 1 \quad \text{and} \quad \sum_{j=1}^{t} \delta_j = 1. \]
Substitution yields
\[ z = \prod_{i=1}^{s} w_i^{-1}(\prod_{j=1}^{t} v_j^{-1}z^{\delta_j}v_j)w_i \]
\[ = w_1^{-1}v_1^{-1}z^{\delta_1}v_1 \ldots v_t^{-1}z^{\delta_t}v_t w_1 \ldots w_s^{-1}v_1^{-1}z^{\delta_1}v_1 \ldots v_t^{-1}z^{\delta_t}v_t w_s. \]
It follows that $v_jv_{j+1}^{-1} = 1$ for each $j = 1, \ldots, t - 1$ and $w_iw_{i+1}^{-1} = 1$
for each $i = 1, \ldots, s - 1$, hence we may assume that $s = 1$ (and $t = 1$)
and $z$ is conjugate to $y$.

More generally, let $p$ and $q$ be primitive elements with the same
exponent sum pair. Since $p$ is primitive, there exists an automorphism
$\theta \in \text{Aut}(F)$ such that $p \mapsto y$. It follows that the exponent sum pair of
$q\theta$ is $(0, 1)$, hence $q\theta$ is conjugate to $p\theta$ and $q$ is conjugate to $p$. $\square$

Let $(X, Y)$ denote the exponent sum pair of $p$ and let $(U, V)$ denote
the exponent sum pair of $q$.

**Lemma 16.** It holds that $XV - YU = 1$.

**Proof.** Since $\{p, q\}$ is a basis for $F$, $\{x^Xy^Y, x^Uy^V\}$ is a basis for $F_{ab}$. The
result then follows from the well-known analogous result for $F_{ab}$. $\square$

**Corollary 17.** For each $r \in F$, $r \in \langle p \rangle^F$ if and only if the exponent
sum pair of $r$ is $(kX, kY)$ for some integer $k$.

**Proof.** Let $r \in F$, let $P$ denote the exponent sum of $p$ in $r_{p,q}$ and let
$Q$ denote the exponent sum of $q$ in $r_{p,q}$. Note that the exponent sum
pair of $r$ is given by $(PX + QU, PY + QV)$. It follows easily from the
definition of $\langle p \rangle^F$, that $r \in \langle p \rangle^F$ implies the exponent sum pair of
$r$ is $(PX, PY)$. Suppose that the exponent sum pair of $r$ is $(kX, kY)$
for some integer $k$. If $Q$ is non-zero, then $(PX + QU, PY + QV) =
(kX, kY)$ implies that $U = \frac{(k-P)X}{Q}$ and $V = \frac{(k-P)Y}{Q}$ and $XV - YU = 0$,
contradicting Lemma 15 hence $Q = 0$ and Lemma 15 implies that $r \in \langle p \rangle^F$.

$\square$

Theorem 6 follows immediately from Corollary 17.

School of Mathematics and Applied Statistics, University of Wol-
longong, NSW 2522, Australia