The Gromov width of symplectic toric manifolds associated with graphs

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1. Symplectic toric manifold

2. Nestohedron

3. Computation of Gromov width
Symplectic toric manifold

Nestohedron

Computation of Gromov width

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Hamiltonian torus action

- $(M^{2n}, \omega)$: symplectic manifold of dimension $2n$
- $T^k \cong (\mathbb{R}/\mathbb{Z})^k$: $k$-dimensional torus acting on $M$
- $\mathfrak{t} \cong \mathbb{R}^k$: Lie algebra of $T^k$ with dual Lie algebra $\mathfrak{t}^*$
- $X$: fundamental vector field on $M$ for given $X \in \mathfrak{t}$

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**Definition**

$T^k$-action on $(M, \omega)$ is called **Hamiltonian** if for each $X \in \mathfrak{t}$ there exists $\mu: M \to \mathfrak{t}^*$ such that

$$\omega(-, X) = d\langle \mu, X \rangle.$$

Such $\mu$ is called a **moment map**.
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Such \(\mu\) is called a **moment map**.

Fix basis \(X_1, \ldots, X_k\) of \(\mathfrak{t}_\mathbb{Z}\). This identifies \((\mathfrak{t}^*, \mathfrak{t}_\mathbb{Z}^*)\) with \((\mathbb{R}^n, \mathbb{Z}^n)\).

Any quantity should be measured with respect to the lattice.
Examples of moment maps

1. $T^2$ acts on $(\mathbb{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$ by $(t_1, t_2) \cdot (z_1, z_2) := (e^{2\pi i t_1} z_1, e^{2\pi i t_2} z_2)$.

$$
\mu(z_1, z_2) = (\pi |z_1|^2, \pi |z_2|^2).
$$

2. $\sigma$: volume form on $S^2$ so that $\int_{S^2} \sigma = 1$. $T^2$ acts on $(S^2 \times S^2, a\sigma + b\sigma)$ diagonally.
Delzant theorem

A symplectic manifold \((M^{2n}, \omega)\) equipped with an effective Hamiltonian \(T^n\)-action is called toric.

In toric case, the image of a moment map \(\mu : M \to \mathbb{R}^n\) is a polytope satisfying the following conditions:

1. There are \(n\) edges at each vertex \(p\).
2. The edges at \(p\) have the form \(p + tu_i\) for some basis vectors \(u_1, \ldots, u_n \in \mathbb{Z}^n\).

Such polytopes are called Delzant polytopes.

Theorem (Delzant)

There is a 1-1 correspondence

\[
\{\text{closed symplectic toric manifolds}\} \leftrightarrow \{\text{Delzant polytopes}\}.
\]

Want: recover geometric data from Delzant polytopes
**Theorem (Gromov nonsqueezing)**

\[ B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2} \text{ symplectically if and only if } r \leq R. \]

\[ \pi r^2 \]

\[ B^4(r) \]

\[ \pi R^2 \]

\[ B^2(R) \times \mathbb{R}^2 \]

(Caution: this picture is far from a proof.)

**Definition (Gromov width)**

\[ w_G(M^{2n}, \omega) := \sup \{ \pi r^2 \mid B^{2n}(r) \hookrightarrow (M, \omega) \text{ symplectically} \}. \]
These two are different as symplectic toric manifolds, but they are isomorphic as symplectic manifolds.

- Delzant polytope = symplectic structure + torus action
  \[\implies\] symplectic data should be independent of the action.

- In these examples, the Gromov width is the height.
A building set $\mathcal{B}$ on $[n+1]$ is a collection of subsets in $[n+1]$ satisfying the following conditions:

1. Each singleton is an element of $\mathcal{B}$.
2. $I, J \in \mathcal{B}$ with $I \cap J \neq \emptyset \implies I \cup J \in \mathcal{B}$.

For $I \in \mathcal{B}$, let $\Delta_I := \text{conv}\{e_i | i \in I\}$.

The nestohedron $P_\mathcal{B}$ is defined to be the Minkowski sum

$$P_\mathcal{B} := \sum_{I \in \mathcal{B}} \Delta_I.$$ 

Example ($n = 2, \mathcal{B} = \{1, 2, 3, 12, 13, 23, 123\}$)

$$P_\mathcal{B} = e_1 + e_2 + e_3$$

$$+ \text{conv}\{e_1, e_2\} + \text{conv}\{e_1, e_3\} + \text{conv}\{e_2, e_3\}$$

$$+ \text{conv}\{e_1, e_2, e_3\}.$$
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Properties

- $P_B \subset \{ \sum_{i=1}^{n+1} x_i = |B| \}$.
- $P_B$ is a Delzant polytope.
- If $[n+1] \in B$, then $P_B$ is $n$-dimensional.
  ([n + 1] \notin B, then $P_B$ is a product of smaller nestohedra.)
- $I \subset [n + 1]$ defines a facet of $P_B$:
  \[
  F_I \subset \left\{ \sum_{i \in I} x_i = |B|_I \right\}, \text{ where } B |_I = \{ J \in B \mid J \subset I \}.
  \]
Building set from a simple graph

- \( G \): simple graph with vertex set \([n + 1]\).
- \( \mathcal{B}(G) := \{ I \subset [n + 1] \mid \text{the subgraph } G|_I \text{ is connected} \} \).

\( P_{\mathcal{B}(G)} \) is called a graph associahedron.

\[ G = \begin{array}{ccc}
1 & \rightarrow & \bullet \\
\downarrow & & \\
2 & \rightarrow & 3
\end{array} \quad \mathcal{B}(G) = \{1, 2, 3, 12, 13, 23, 123\}. \]

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Main theorem

Theorem

Let $G$ be a simple graph with the vertex set $[n+1]$. The Gromov width of the symplectic toric manifold for $P_{B(G)}$ is

$$\min \{k_i > 1 \mid i = 1, \ldots, n+1 \} - 1,$$

where $k_i$ is the number of connected induced subgraphs of $G$ containing $i$.

$$G = \begin{array}{c}
1 \\
2 \\
3
\end{array}$$

$k_1 = k_2 = k_3 = 4$. $\implies w_G = 3$.

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$k_1 = k_3 = 3$, $k_2 = 4$. $\implies w_G = 2$. 
For $I \in \mathcal{B}$, $F_I$ has a parallel facet if and only if $[n + 1] \setminus I \in \mathcal{B}$.

When $\mathcal{B}$ is obtained from a graph,

1. Such $I \in \mathcal{B}$ always exists.
2. Minimal distance between such facets are attained when $I$ or its complement is a singleton.
3. $k_i - 1$ is the distance between $F_{\{i\}}$ and $F_{[n+1]\setminus\{i\}}$. 
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Distance between parallel facets bounds the Gromov width.
Stabilized embedding

- Gromov nonsqueezing: $w_G(S^2(\pi r^2) \times \mathbb{R}^2) = \pi r^2$.
- On the other hand, $w_G(\Sigma_g(\pi r^2) \times \mathbb{R}^2) = \infty$ for $g \geq 1$.

Let $M_1$, $M_2$ be symplectic toric manifolds. Is it true that

$$w_G(M_1 \times M_2) = \min \{w_G(M_1), w_G(M_2)\}?$$
Stabilized embedding

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For $M_G$ constructed from graph $G$, $w_G(M_G \times \mathbb{R}^2) = w_G(M_G)$.

**Corollary**

Let $H$ be a subgraph of $G$. Suppose $k = |G| - |H| > 0$. Then

$$M_G \times \mathbb{R}^{2m} \hookrightarrow M_H \times \mathbb{R}^{2k+2m}$$

*can never be symplectic for any $m \geq 0$.*

Is there a topological obstruction to this embedding?
Idea of proof

Theorem

Let $G$ be a simple graph with the vertex set $[n + 1]$. The Gromov width of the symplectic toric manifold for $P_{B(G)}$ is

$$
\lambda := \min \{ k_i > 1 \mid i = 1, \ldots, n + 1 \} - 1,
$$

where $k_i$ is the number of connected induced subgraphs of $G$ containing $i$.

(Lower bound) To show $w_G \geq \lambda$,

1. Use global action-angle coordinates given by moment map.
2. Find some shape (corresponding to a ball) of “size” $\lambda$ inside $P$.

(Upper bound) To show $w_G \leq \lambda$,

1. Find $J$-holomorphic sphere with symplectic area $\leq \lambda$.
2. Use McDuff–Tolman computation on Seidel representation to find suitable nonvanishing Gromov–Witten invariant.
3. Use semifree circle action with codimension 2 extrema.
Lower bound

Let $P$ be the moment map image of a symplectic toric manifold.

**Theorem (Mandini–Pabiniak, Latschev–McDuff–Schlenk)**

$\bigtriangleup^n(\rho) \subset P \quad \implies \quad w_G \geq \rho.$

Find $L_1, \ldots, L_n$ satisfying

- $L_1 \cap \cdots \cap L_n$ is a point.
- Primitive vectors parallel to $L_i$ form a basis for $\mathbb{Z}^n$.
- $L_i$ has affine length $\rho$.

$\bigtriangleup^n(\rho) := \text{conv}(L_1, \ldots, L_n)$.
Lower bound

Let $P$ be the moment map image of a symplectic toric manifold.

**Theorem (Mandini–Pabiniak, Latschev–McDuff–Schlenk)**

$\diamondsuit^n(\rho) \subset P \implies w_G \geq \rho$.

- $w_G \geq 3$.
- $p = (2, 2)$ if 3rd coordinate is ignored.

Find $L_1, \ldots, L_n$ satisfying

- $L_1 \cap \cdots \cap L_n$ is a point.
- Primitive vectors parallel to $L_i$ form a basis for $\mathbb{Z}^n$.
- $L_i$ has affine length $\rho$.

$\diamondsuit^n(\rho) := \text{conv}(L_1, \ldots, L_n)$. 
Finding ♦ inside $P$

- $G$: connected simple graph with the vertex set $[n + 1]$.
- $B$: building set constructed from $G$.
- $k_i$: number of connected induced subgraphs of $G$ containing $i$.

Assume $k_{n+1}$ is minimal among $k_i$.

Regard $P_B$ as a subset in $\mathbb{R}^n$ by forgetting last coordinate.

Goal: find $\diamond n(k_{n+1} - 1)$ in $P_B$

Take $L_i = \{(a, \ldots, a, x, a, \ldots, a) \mid 1 \leq x \leq k_{n+1}\}$, where

$$a := \frac{|B| - k_{n+1} - 1}{n - 1}.$$ 

Checking $L_i \subset P_B$ and $1 \leq a \leq k_{n+1}$ reduces to the following.

Lemma

$$n \cdot k_i \geq |B| - 1 \quad \text{for any } i = 1, \ldots, n + 1.$$
Gromov–Witten invariants

(Upper bound) To show $w_G \leq \lambda$,

1. Find $J$-holomorphic sphere with symplectic area $\leq \lambda$.
2. Use Seidel representation to find nonzero GW-invariant.
3. Use semifree circle action with codimension 2 extrema.

The (genus zero) Gromov–Witten invariant

$$\text{GW}_{A,k}^M(\alpha_1, \ldots, \alpha_k) \in \mathbb{Q}$$

counts number of $J$-holomorphic spheres in class $A \in H_2(M, \mathbb{Z})$, passing through cycles $\alpha_i \in H^*(M)$.

**Theorem (Gromov)**

If $\text{GW}_{A,k}^M([pt], \alpha_2, \ldots, \alpha_k) \neq 0$ for some $A \in H_2(M, \mathbb{Z})$, $\alpha_i \in H^*(M)$, then the Gromov width of $(M, \omega)$ is at most $\omega(A) > 0$. 
Seidel representation

Goal: find $A$, $\alpha_i$ such that $GW_{A,k}^{M}([pt], \ldots, \alpha_k) \neq 0$, $\omega(A) = \lambda$.

- The Seidel morphism is a group homomorphism
  \[ S : \pi_1(\text{Ham}(M, \omega)) \rightarrow (QH^0(M; \Lambda)^\times, \ast). \]

- $QH^\bullet(M; \Lambda) = H^\bullet(M) \otimes \Lambda$ with quantum product $\ast$.
- $\Lambda = \{ \sum a_i q^{\mu_i} t^{\kappa_i} \mid \deg q = 2, \deg t = 0, \text{some condition} \}$.
- $a \ast b = \sum_{A \in H_2(M, \mathbb{Z})} (a \ast b)_A \otimes q^{c_1(A)} t^{\omega(A)}$, where for all $c$,
  \[ \int_M (a \ast b)_A \cup c = GW_{A,3}^{M}(a, b, c). \]

We obtain information on GW invariants by studying $S$. 
Upper bound

- $u \in \pi_1(\text{Ham}(M, \omega))$ is represented by Hamiltonian $S^1$-action. 
  \[ \implies -u \text{ is represented by the opposite } S^1\text{-action.} \]
- $S : \pi_1(\text{Ham}(M, \omega)) \to QH^0(M; \Lambda)^\times$ is a homomorphism.

$S(u) \ast S(-u) = S(u + (-u)) = 1$.

Therefore, at least one term on the LHS survives.

- McDuff–Tolman developed a way to compute $S(u)$. 
  In general $S(u)$ has infinitely many terms.
  \[ \implies \text{It is hard to see which term will survive on the LHS.} \]
- If $u$ is a semifree action whose maximum has codimension 2, some unwanted terms in $S(u)$ vanish.

We can find $A$, $\alpha_i$ such that $GW_{A,k}^M([pt], \ldots, \alpha_k) \neq 0$, $\omega(A) = \lambda$.

\[ \implies w_G \leq \lambda \text{ by Gromov's theorem.} \]
Let \((M, \omega)\) be a symplectic toric manifold whose moment polytope is \(P \subset \mathbb{R}^n\). Suppose that there exists a primitive vector \(u \in \mathbb{Z}^n\) satisfying the following two conditions.

1. \(\langle u, \eta \rangle \in \{0, \pm 1\}\) for any primitive \(\eta\) parallel to an edge of \(P\).
2. \(P\) has supporting hyperplanes of the form \(\{x \in \mathbb{R}^n \mid \langle x, u \rangle \leq \lambda\}\) and \(\{x \in \mathbb{R}^n \mid \langle x, u \rangle \geq \mu\}\).

Then the Gromov width of \((M, \omega)\) is at most \(\lambda - \mu\).

For general nestohedra,

- (1) is true but (2) is not.
- Even when (2) is true, the minimal distance might not be obtained from a singleton, so the formula will be different.