The generalized Levinger transformation

M. Adam and J. Maroulas

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Abstract

In this paper, we present new results relating the numerical range of a matrix $A$ with the generalized Levinger transformation $L(A, \alpha, \beta) = \alpha H_A + \beta S_A$, where $H_A$ and $S_A$ are, respectively, the Hermitian and skew-hermitian parts of $A$. Using these results, we then derive expressions for eigenvalues and eigenvectors of the perturbed matrix $A + L(E, \alpha, \beta)$, for a fixed matrix $E$ and $\alpha, \beta$ are real parameters.

Keywords: Numerical range, Generalized inverses, Perturbation theory, Eigenvalues-eigenvectors, Control and Systems Theory, Sensitivity.

AMS Subject Classifications: 15A60, 15A09, 47A55, 65F15, 93B60, 93B35.

1 Introduction

Let $\mathcal{M}_n(\mathbb{C})$ (or $\mathcal{M}_n(\mathbb{R})$) be the algebra of all $n \times n$ complex (real) matrices, and let $A \in \mathcal{M}_n(\mathbb{C})$. The numerical range of $A$, also known as the field of values [6], is the set

$NR[A] = \{ x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n \text{ with } x^*x = 1 \}$.

The numerical range $NR[A]$ is a compact and convex subset of $\mathbb{C}$, that contains the spectrum $\sigma(A)$ of $A$. If $\lambda \in \sigma(A) \cap \partial NR[A]$ with multiplicity $m$, then $\lambda$ is a normal eigenvalue, i.e., $A$ is unitarily similar to $\lambda I_m \oplus B$, with $\lambda \notin \sigma(B)$. Clearly, when $A$ is normal, then its eigenvalues are normal and $NR[A] = \text{Co}\{ \sigma(A) \}$, where $\text{Co}\{ \cdot \}$ denotes the convex hull of the set. For any $A \in \mathcal{M}_n(\mathbb{C})$, if we write $A = H_A + S_A$, where

$H_A = \frac{A + A^*}{2}$ and $S_A = \frac{A - A^*}{2}$,
are the Hermitian and the skew-hermitian parts of $A$ respectively, then

$$\text{Re} \ NR[A] = NR[H_A] \quad \text{and} \quad \text{i Im} \ NR[A] = NR[S_A].$$

Moreover, if $P$ is any $n \times m$ matrix with $n \geq m$ and $P^*P = I_m$, then

$$NR[P^*AP] \subseteq NR[A],$$

and the equality holds only for $m = n$.

Given a matrix $A \in M_n(\mathbb{C})$, we define "generalized Levinger transformation" of $A$ as the double parametrized family of matrices

$$L(A, \alpha, \beta) = \alpha H_A + \beta S_A = \frac{\alpha + \beta}{2} A + \frac{\alpha - \beta}{2} A^*, \quad \text{with} \quad \alpha, \beta \in \mathbb{R}. \quad (1)$$

In (1), for $\alpha = 1$ and $\beta = 2t - 1$, $t \in \mathbb{R}$, we have

$$L(A, 1, 2t - 1) = tA + (1 - t)A^*, \quad i.e., \quad L(A, 1, 2t - 1) \text{ is just the ordinary Levinger's transformation, which has been studied in } [5, 9, 10, 11].$$

Moreover, for $\alpha = 2t - 1$, $t \in \mathbb{R}$ and $\beta = 1$, we have

$$L(A, 2t - 1, 1) = tA + (t - 1)A^*. \quad (2)$$

The equation (2) is a different formulation of Levinger transformation, where in $L(A, 2t - 1, 1)$ the difference of coefficients $t$ and $t - 1$ of matrices is equal to unity (skew convex expression). Note, that

$$L(iA, 1, 2t - 1) = iL(A, 2t - 1, 1).$$

Clearly, for every $\alpha, \beta \in \mathbb{R}$, we have from (1)

$$H_{L(A, \alpha, \beta)} = \alpha H_A, \quad S_{L(A, \alpha, \beta)} = \beta S_A.$$

Hence,

$$\text{Re} \ NR[L(A, \alpha, \beta)] = NR[H_{L(A, \alpha, \beta)}] = \alpha NR[H_A] = \alpha \text{ Re} \ NR[A]$$

$$\text{i Im} \ NR[L(A, \alpha, \beta)] = NR[S_{L(A, \alpha, \beta)}] = i\beta \text{ Im} \ NR[A],$$

and consequently

$$NR[L(A, \alpha, \beta)] = \{ \alpha x + i\beta y : x, y \in \mathbb{R}, \quad \text{with} \quad x + iy \in NR[A] \}. \quad (3)$$
Moreover, the boundary of $\text{NR}[\mathcal{L}(A,\alpha,\beta)]$ is given by

$$\partial \text{NR}[\mathcal{L}(A,\alpha,\beta)] = \{ \alpha x + i\beta y : x, y \in \mathbb{R}, \text{ with } x + iy \in \partial \text{NR}[A] \}. \quad (4)$$

For $A \in \mathcal{M}_n(\mathbb{R})$, since $\text{NR}[A] = \text{NR}[A^T]$, $\text{NR}[\mathcal{L}(A,\alpha,\beta)]$ is symmetric with respect to the real axis and

$$\text{NR}[\mathcal{L}(A,\alpha,\beta)] = \text{NR}[\mathcal{L}(A^*,\alpha,-\beta)],$$

the domain of $\beta$ can be reduced to $[0, +\infty)$. Additionally, if $0 < \beta_1 < \beta_2$, then due to (3), (4) and the symmetry of the numerical ranges with respect to the real axis, we have (in some sense) a vertical dilation, i.e., for $z_1 \in \partial \text{NR}[\mathcal{L}(A,\alpha,\beta_1)]$, $z_2 \in \partial \text{NR}[\mathcal{L}(A,\alpha,\beta_2)]$, holds

$$|z_1| = \sqrt{\alpha^2(x^*H_Ax)^2 + \beta_1^2(x^*S_Ax)^2} < \sqrt{\alpha^2(x^*H_Ax)^2 + \beta_2^2(x^*S_Ax)^2} = |z_2|,$$

and consequently

$$\text{NR}[\mathcal{L}(A,\alpha,\beta_1)] \subset \text{NR}[\mathcal{L}(A,\alpha,\beta_2)].$$

**Example.** For $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & -6 \\ -1 & 3 & 5 \end{bmatrix}$, the numerical ranges of $A$ and $\mathcal{L}(A,\alpha,\beta)$ are illustrated in the following figures. On the right, due to (3), the vertical dilation is presented only when $\alpha$ is fixed (here $\alpha = 0.4$) and $\beta$ is altered ($0.5 \leq \beta \leq 1.2$), reminding that, this property holds only for the ordinary Levinger’s transformation. Otherwise, $\text{NR}[\mathcal{L}(A,\alpha,\beta)]$ is moved as it is shown on the left figure, for the values $\alpha = -0.9, \beta = 0.8$, $\alpha = 1.3, \beta = 0.6$ and $\alpha = 1.4, \beta = 1.3$. 
The paper is devoted to the study of the generalized Levinger transformation of a matrix. Specifically, we establish an interesting relationship between the numerical range of a matrix \( A \) and its generalized Levinger transformation. This relationship is then used to obtain results on the eigenvalue and eigenvector of the perturbed matrix of the form \( A + \mathcal{L}(E, \alpha, \beta) \), where \( E \) is fixed and \( \alpha \) and \( \beta \) are small parameters.

Our motivation for such study comes from the fact that a great deal of effort has been made in the literature to establish bounds on the eigenvalues of a perturbed matrix. For results on this topic, see [14] and the well-known books on linear and numerical linear algebra by Datta [4], Stewart and Sun [13], Kato [7], Lancaster and Tismenetsky [8], and Bhatia [2].

This paper is divided in two parts. The first part contains geometric properties of numerical range of \( \mathcal{L}(A, \alpha, \beta) \). Also bounds are given for real and imaginary parts of eigenvalues of \( \mathcal{L}(A, \alpha, \beta) \). This provides us a framework to study variation of the spectrum of \( \mathcal{L}(A, \alpha, \beta) \). In the second part, we use the Levinger transformation for a fixed matrix \( E \) as a perturbation matrix, whose activity on a matrix \( A \) depends only on the parameters \( \alpha \) and \( \beta \). First, we formulate a necessary and sufficient condition for a normal matrix to remain normal, under a perturbation by a symmetric and rank one matrix. Next, we present an approximation of a perturbed eigenpair of a diagonalizable matrix \( A \) using two parameters, which generalizes a known result in [13, p. 183], where the eigenvector of perturbed eigenvalue is not mentioned and even the perturbed eigenvector is investigated in [4, p. 431], without giving further details for the perturbation of the corresponding eigenvalue. Further we simplify these formulae using the notion of generalized inverse extending the corresponding result in [5]. As an application, we present a sufficient condition such that the perturbed eigenpair is first order approximation of the corresponding simple eigenpair of initial matrix \( A \), and we give two numerical examples to illustrate our results.

2 Geometric properties

**Proposition 1 a.** Let \( A \in \mathcal{M}_n(\mathbb{C}) \). The image of a line segment \( \epsilon_A \in NR[A] \) by the Levinger transformation, is a line segment \( \epsilon_{\mathcal{L}} \in NR[\mathcal{L}(A, \alpha, \beta)] \).

**b.** If \( A \in \mathcal{M}_n(\mathbb{R}) \) and \( NR[A] \) is an ellipse, then \( NR[\mathcal{L}(A, \alpha, \beta)] \), for \( \alpha \neq 0 \), is also an ellipse.

**Proof.** a. For any \( x_1 + iy_1, x_2 + iy_2 \in NR[A] \), and \( t \in [0, 1] \), we observe that

\[
\alpha[(1-t)x_1 + tx_2] + \beta[(1-t)i y_1 + ti y_2] = (1 - t)(\alpha x_1 + i \beta y_1) + t(\alpha x_2 + i \beta y_2),
\]

where \( \alpha x_1 + i \beta y_1 \) and \( \alpha x_2 + i \beta y_2 \) lie in the convex set \( NR[\mathcal{L}(A, \alpha, \beta)] \). Hence, the proof of a follows readily.
b. Consider that \( NR[A] = \{ x + iy : \frac{x^2}{c^2} + \frac{y^2}{k^2} \leq 1, \text{ with } c > k > 0 \} \). For \( \alpha \neq 0 \) and \( \beta > 0 \), changing the variables \( x = \alpha^{-1}X, \ y = \beta^{-1}Y \), by (3) we have that the boundary \( \partial NR[\mathcal{L}(A, \alpha, \beta)] \) is the ellipse \( \frac{X^2}{\alpha^2c^2} + \frac{Y^2}{\beta^2k^2} = 1 \). The foci are on the real axis, when \( \alpha \in \mathbb{R}\setminus[-\frac{\beta k}{c}, \frac{\beta k}{c}] \), otherwise, they lie on the imaginary axis. \( \square \)

**Proposition 2** Let \( A \in \mathcal{M}_n(\mathbb{C}), \ B \in \mathcal{M}_m(\mathbb{C}) \), and let \( NR[B] \) be a polygon circumscribed to \( NR[A] \). Then the geometric relationship of numerical ranges \( NR[\mathcal{L}(A, \alpha, \beta)] \) and \( NR[\mathcal{L}(B, \alpha, \beta)] \) remains the same, for \( \alpha, \beta \in \mathbb{R} - \{0\} \).

**Proof.** By Proposition 1a and 1b, \( NR[\mathcal{L}(B, \alpha, \beta)] \) is a convex polygon. Moreover, it is easy to see that there do not exist other common boundary points of \( NR[\mathcal{L}(A, \alpha, \beta)] \) and \( NR[\mathcal{L}(B, \alpha, \beta)] \), except those, which correspond to \( z \in \partial NR[A] \cap \partial NR[B] \). \( \square \)

**Proposition 3** Let \( A \in \mathcal{M}_n(\mathbb{C}) \), and \( \alpha, \beta \in \mathbb{R} - \{0\} \).

a. If \( A \) is a normal matrix, then \( \partial NR[\mathcal{L}(A, \alpha, \beta)] \) is a \( k \)-polygon, as \( \partial NR[A] \).

b. \( NR[\mathcal{L}(A, \alpha, \beta)] \cap \mathbb{R} = \mathbb{R} \cap NR[\alpha A] \).

**Proof.** a. The proof of this part follows readily by Proposition 1a and from the observation that \( A \) is unitarily similar to \( \text{diag}\{ x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n \} \) if and only if \( \mathcal{L}(A, \alpha, \beta) \) is unitarily similar to \( \text{diag}\{ \alpha x_1 + i\beta y_1, \alpha x_2 + i\beta y_2, \ldots, \alpha x_n + i\beta y_n \} \).

b. Clearly, if \( \alpha x^* Ax \in NR[\alpha A] \cap \mathbb{R} \), then \( x^* \mathcal{L}(A, \alpha, \beta)x \) \( = \frac{\alpha + \beta}{2} x^* Ax + \frac{\alpha - \beta}{2} x^* A^* x = \alpha x^* Ax \), concluding that \( NR[\alpha A] \cap \mathbb{R} \subset NR[\mathcal{L}(A, \alpha, \beta)] \cap \mathbb{R} \). Moreover, for \( \alpha, \beta \in \mathbb{R}\setminus\{0\} \), because

\[
\alpha A = \frac{\alpha + \beta}{2 \beta} \mathcal{L}(A, \alpha, \beta) - \frac{\alpha - \beta}{2 \beta} \mathcal{L}^*(A, \alpha, \beta),
\]

in an analogous way, we obtain \( NR[\mathcal{L}(A, \alpha, \beta)] \cap \mathbb{R} \subset NR[\alpha A] \cap \mathbb{R} \). \( \square \)

**Remark**

The eigenvalue \( \lambda \in \sigma(A) \) is normal if and only if \( \lambda_L \) is a normal eigenvalue of \( \mathcal{L}(A, \alpha, \beta) \).

In fact, from (1) and the relationship \( U^* A U = \lambda I_m \oplus B \), where \( \lambda \notin \sigma(B) \), it follows that \( U^* \mathcal{L}(A, \alpha, \beta) U = \lambda L_I m \oplus \mathcal{L}(B, \alpha, \beta) \).

In the following proposition we present a compression of \( NR[\mathcal{L}(A, \alpha, \beta)] \), when \( A \) is a normal matrix, based on our results in [1].

**Proposition 4** Let \( A \in \mathcal{M}_n(\mathbb{C}) \) be a normal matrix and let the polygon \( \langle \lambda(A)_1, \lambda(A)_2, \ldots, \lambda(A)_k \rangle \) be the numerical range of \( A \). If \( x_j \) is a corresponding eigenvector of \( \lambda(A)_j \), \( j = 1, 2, \ldots, k \),
and \( v = \sum_{j=1}^{k} v_j x_j \) is a unit vector, denoting by \( E = \text{span}\{ v \} \) and \( E_W^\perp \) the orthogonal complement of \( E \) with respect of \( W = \text{span}\{ x_1, x_2, \ldots, x_k \} \), then
\[
NR[P^* \mathcal{L}(A, \alpha, \beta)P] = NR[\mathcal{L}(P^* AP, \alpha, \beta)] < \lambda_{(L)_1}, \lambda_{(L)_2}, \ldots, \lambda_{(L)_k} >,
\]
where \( P = [w_1 \ w_2 \ \ldots \ w_{k-1}] \), and \( w_1, w_2, \ldots, w_{k-1} \) is an orthonormal basis of \( E_W^\perp \). Moreover, \( \partial NR[P^* \mathcal{L}(A, \alpha, \beta)P] \) is tangential to the edges of the polygon \( < \lambda_{(L)_1}, \lambda_{(L)_2}, \ldots, \lambda_{(L)_k} > \) at the points
\[
\mu_{(L)_\tau} = \alpha \text{Re} \mu_{(A)_\tau} + i \beta \text{Im} \mu_{(A)_\tau}, \quad (\tau = 1, \ldots, k)
\]
where
\[
\mu_{(A)_\tau} = \frac{|\nu_{\tau+1}|^2 \lambda_{(A)_\tau} + |\nu_{\tau}|^2 \lambda_{(A)_{\tau+1}}}{|\nu_{\tau+1}|^2 + |\nu_{\tau}|^2} \quad (\tau = 1, \ldots, k-1), \quad \mu_{(A)_k} = \frac{|\nu_1|^2 \lambda_{(A)_k} + |\nu_{k-1}|^2 \lambda_{(A)_1}}{|\nu_1|^2 + |\nu_{k-1}|^2}.
\]

**Proof.** By the equation \( (1) \) we have \( \mathcal{L}(P^* AP, \alpha, \beta) = P^* \mathcal{L}(A, \alpha, \beta)P \), and it is evident the equality of the numerical ranges. Moreover, \( \mathcal{L}(A, \alpha, \beta) \) is normal, and it is known in \( [1] \) that, \( \partial NR[P^* \mathcal{L}(A, \alpha, \beta)P] \) tangents to the edges of the polygon \( < \lambda_{(L)_1}, \lambda_{(L)_2}, \ldots, \lambda_{(L)_k} > \) at the points
\[
\mu_{(L)_\tau} = \frac{|\nu_{\tau+1}|^2 \lambda_{(L)_\tau} + |\nu_{\tau}|^2 \lambda_{(L)_{\tau+1}}}{|\nu_{\tau+1}|^2 + |\nu_{\tau}|^2} \quad (\tau = 1, \ldots, k-1); \quad \mu_{(L)_k} = \frac{|\nu_1|^2 \lambda_{(L)_k} + |\nu_{k-1}|^2 \lambda_{(L)_1}}{|\nu_1|^2 + |\nu_{k-1}|^2}.
\]
Since, the eigenvalues of \( A \) and \( \mathcal{L}(A, \alpha, \beta) \) are related by \( \lambda_{(L)_\tau} = \alpha \text{Re} \lambda_{(A)_\tau} + i \beta \text{Im} \lambda_{(A)_\tau} \), the equation \( (5) \) is verified. \( \square \)

In the following applying the results of Rojo and Soto in \( [12] \) to \( \mathcal{L}(A, \alpha, \beta) \), one obtains bounds for the real and the imaginary parts of the eigenvalues of \( \mathcal{L}(A, \alpha, \beta) \).

**Theorem 1** Let the matrix \( A \in \mathcal{M}_n(\mathbb{R}) \) and \( \lambda_j \in \sigma(\mathcal{L}(A, \alpha, \beta)) \). Then for each \( \lambda_j \) we have
\[
|\text{Re} \lambda_j - \frac{\alpha \text{tr}(H_A)}{n}| \leq |\alpha| \left[ \frac{n-1}{n} \left( \frac{\| S_A H_A - H_A S_A \|^2_F}{3 \left( \alpha^2 \| H_A \|^2_F + \beta^2 \| S_A \|^2_F \right)} - \frac{\| \text{tr}(H_A) \|^2}{n} \right) \right]^{1/2}, \quad (6)
\]
and
\[
|\text{Im} \lambda_j| \leq |\beta| \left[ \frac{n-1}{n} \left( \frac{\| S_A \|^2_F - \frac{\alpha^2 \| S_A H_A - H_A S_A \|^2_F}{3 \left( \alpha^2 \| H_A \|^2_F + \beta^2 \| S_A \|^2_F \right)} \right) \right]^{1/2}, \quad (7)
\]
where \( \| \cdot \|_F \) denotes the Frobenius norm.
Proof. Observe that \( \text{tr}(\mathcal{L}(A, \alpha, \beta)) = \text{tr}(\alpha H_A + \beta S_A) = \alpha \text{tr}(H_A) \), and
\[
\|\mathcal{L}(A, \alpha, \beta)\mathcal{L}^T(A, \alpha, \beta) - \mathcal{L}^T(A, \alpha, \beta)\mathcal{L}(A, \alpha, \beta)\|_F \\
= \|(\alpha H_A + \beta S_A)(\alpha H_A^T + \beta S_A^T) - (\alpha H_A^T + \beta S_A^T)(\alpha H_A + \beta S_A)\|_F \\
= 2|\alpha\beta||S_A H_A - H_A S_A|_F.
\]
(8)

Since \( \text{tr}(H_A S_A^T) = -\text{tr}(H_A S_A) = -\text{tr} \left( \frac{(A + A^T)(A - A^T)}{4} \right) = -\frac{1}{4} (\text{tr}(A^2) - \text{tr}[(A^T)^2]) = 0 \)
we have
\[
\|\mathcal{L}(A, \alpha, \beta)\|_F^2 = \text{tr}[\mathcal{L}(A, \alpha, \beta)\mathcal{L}^T(A, \alpha, \beta)] = \text{tr}[(\alpha H_A + \beta S_A)(\alpha H_A^T + \beta S_A^T)] \\
= \alpha^2\|H_A\|_F^2 + \beta^2\|S_A\|_F^2.
\]
(9)

and
\[
\text{tr}[\mathcal{L}^2(A, \alpha, \beta)] = \text{tr}[(\alpha H_A + \beta S_A)(\alpha H_A + \beta S_A)] = \alpha^2\|H_A\|_F^2 - \beta^2\|S_A\|_F^2,
\]
(10)
and consequently by (9) and (10)
\[
\|\mathcal{L}(A, \alpha, \beta)\|_F^2 + \text{tr}(\mathcal{L}^2(A, \alpha, \beta)) = 2\alpha^2\|H_A\|_F^2, \\
\]
(11)
\[
\|\mathcal{L}(A, \alpha, \beta)\|_F^2 - \text{tr}(\mathcal{L}^2(A, \alpha, \beta)) = 2\beta^2\|S_A\|_F^2.
\]
Therefore, if we substitute (8), (9), and (11) in the relationships of Theorem 7 in [12], we have
\[
\left| \text{Re}\lambda_j - \frac{\text{tr}(\mathcal{L}(A, \alpha, \beta))}{n} \right| \leq \sqrt{\frac{n - 1}{n} \left( \frac{\|\mathcal{L}(A, \alpha, \beta)\|_F^2 + \text{tr}(\mathcal{L}^2(A, \alpha, \beta))}{2} - \frac{\nu(\mathcal{L}(A, \alpha, \beta))^2}{12\|\mathcal{L}(A, \alpha, \beta)\|_F^2} - \frac{[\text{tr}(\mathcal{L}(A, \alpha, \beta))]^2}{n} \right)}
\]
and
\[
\left| \text{Im}\lambda_j \right| \leq \sqrt{\frac{n - 1}{2n} \left( \frac{\|\mathcal{L}(A, \alpha, \beta)\|_F^2 - \text{tr}(\mathcal{L}^2(A, \alpha, \beta)) - \frac{\nu(\mathcal{L}(A, \alpha, \beta))^2}{6\|\mathcal{L}(A, \alpha, \beta)\|_F^2}}{2} \right)},
\]
where \( \nu(\mathcal{L}(A, \alpha, \beta)) = \|\mathcal{L}(A, \alpha, \beta)\mathcal{L}^T(A, \alpha, \beta) - \mathcal{L}^T(A, \alpha, \beta)\mathcal{L}(A, \alpha, \beta)\|_F \), thus we obtain the bounds for the real and the imaginary part for each eigenvalue of \( \mathcal{L}(A, \alpha, \beta) \) in (6) and (7). \( \square \)
3 Application to perturbation theory

The question "how close is a matrix $M$ to being normal", it is known that it is evaluated by the normality distance $\|AA^T - A^TA\|_p$ of $p$ norm. Since various matrix norms are equivalent, using the Frobenius norm we have:

**Proposition 5** Let $N \in \mathcal{M}_n(\mathbb{R})$ be a normal matrix and for a nonzero vector $x \in \mathbb{R}^n$, let $E = xx^T$. If $x$ is not eigenvector of $N$ corresponding to a real eigenvalue, the matrix $M = N + E$ is normal if and only if $N$ is symmetric.

**Proof.** For the symmetric matrix $E = xx^T$, clearly $E^2 = \|x\|^2E$, and the normality distance of $M$ is equal to

$$\|MM^T - M^TM\|_F = \|(N+ E)(N^T + E) - (N^T + E)(N + E)\|_F$$

where $R = NE - EN$. Since

$$tr(R^2) = 2 \left[ tr(NEE) - tr(EN^2E) \right] = 2 \left[ (x^TNx)^2 - \|x\|^2(x^TN^2x) \right]$$

$$tr(RR^T) = 2 \left[ \|x\|^2tr(NEEN^T) - tr(ENEN^T) \right] = 2 \left[ \|x\|^2(x^TN^2N) - (x^TNx)^2 \right]$$

and $tr \left[ (R^T)^2 \right] = tr(R^2)$, we have:

$$\|R + R^T\|_F^2 = tr \left[ (R + R^T)^2 \right] = tr(R^2) + tr \left[ (R^T)^2 \right] + 2tr(RR^T) = 4\|x\|^2 x^T(N^TN - N^2)x.$$  

Hence, the matrix $M$ is normal if and only if $N^TN = N^2$. This equation is equivalent to

$$DD = D^2,$$  

(12)

where $D$ is diagonal and unitary similar to $N$, i.e., $N = UDU^*$. Thus, by (12), $D$ is real and $N$ is symmetric, since it is unitary similar to a real diagonal matrix. \qed

**Corollary 1** Let $N = \text{diag}(N_1, N_2, \ldots, N_\tau) \in \mathcal{M}_{n}(\mathbb{R})$ be a normal matrix and for a nonzero vector $x = \left[ x_1^T \ldots x_\tau^T \right] \in \mathbb{R}^n$, with all $x_j \neq 0$, let $E = xx^T$. If $x_j$ is not eigenvector of $N_j$, $(j = 1, \ldots, \tau)$, corresponding to a real eigenvalue, the matrix $M = N + E$ is normal if and only if $N$ is symmetric.

It is worth notice that, the result in Proposition 5 is combined by the special form of $E$, but it is interesting to look at more general perturbations, investigating how main properties of $N$ are influenced. For this, we consider $E \in \mathcal{M}_{n}(\mathbb{R})$, since

$$\|MM^T - M^TM\|_F = \|2H_{[N,E^T]} + EE^T - E^TE\|_F,$$
where \([N, E^T] = NE^T - E^TN\), we conclude that, the normality distance of \(M\) is related to the normality distance of \(E\). Hence, an outlet is to investigate if some properties of perturbed normal matrices remain.

Let the matrix \(A \in \mathcal{M}_n(\mathbb{C})\) be diagonalizable (keeping the property of \(N\)) and \(E \in \mathcal{M}_n(\mathbb{R})\) be fixed, without giving any attention to \(\|E\|\). Consider the matrix

\[
M_{\alpha,\beta} = A + \mathcal{L}(E, \alpha, \beta) = A + \alpha H_E + \beta S_E,
\]

where \(\alpha, \beta \in \mathbb{R}\) are small enough varying parameters. Clearly in (13), \(M_{\alpha,\beta}\) is continuous differentiable and the Hermitian and the skew-hermitian parts of \(E\) influence independently the matrix \(A\). Especially, when \(A\) is normal, \(H_E\) and \(S_E\) alter \(H_A\) and \(S_A\) separately.

Denote by \(\lambda_{\alpha,\beta}\) an eigenvalue of \(M_{\alpha,\beta}\) in (13) and by \(v_{\alpha,\beta}\) and \(\omega_{\alpha,\beta}\), the corresponding right and left eigenvectors, i.e., \((M_{\alpha,\beta} - \lambda_{\alpha,\beta}I)v_{\alpha,\beta} = 0, \ \omega_{\alpha,\beta}^*(M_{\alpha,\beta} - \lambda_{\alpha,\beta}I) = 0\). Since the coefficients of characteristic polynomial \(\det(I - M_{\alpha,\beta})\) are polynomials of two variables \(\alpha, \beta\) and \(\lambda_{\alpha,\beta}\) is continuous function of these coefficients, for \(\alpha = \beta = 0\) the perturbed eigenvalue \(\lambda_{\alpha,\beta}\) is equal to a semisimple eigenvalue \(\lambda_i\) of \(A\), and the eigenvectors are: \(v_{\alpha,\beta} = v_i, \ \omega_{\alpha,\beta} = \omega_i\), where \(v_i\) and \(\omega_i\) are the right and left eigenvectors of \(\lambda_i\) for the matrix \(A\). We remind the readers that an eigenvalue is called *semisimple*, when it is a simple root of the minimal polynomial of matrix. Moreover, \(\lambda_{\alpha,\beta}\) and \(v_{\alpha,\beta}, \omega_{\alpha,\beta}\) are continuous functions of \(\alpha, \beta\) and partial differentiable, but might have rather singularities on total differentiability [7] p. 116. For further details we refer to [7] and [8] Ch. 11. We will now give a result on the sensitivity of eigenvalues and eigenvectors of perturbed matrix \(M_{\alpha,\beta}\) in (13) in the neighborhood of \(\lambda_i\) in relation with the remaining eigenvalues and eigenvectors.

**Theorem 2** Let the matrix \(A \in \mathcal{M}_n(\mathbb{C})\) be diagonalizable and let \(v_j\) and \(\omega_j\) be the right and left eigenvectors of \(A\) corresponding to \(\lambda_j \in \sigma(A)\). If the eigenpair \((\lambda_{\alpha,\beta}, v_{\alpha,\beta})\) has continuous second order partial derivatives in the neighborhood of \(\lambda_i\) and \(v_i\), then:

\[
\lambda_{\alpha,\beta} = \lambda_i + \frac{\omega_i^* \mathcal{L}(E, \alpha, \beta) v_i}{s_i} \sum_{k \neq i} (\omega_i^* \mathcal{L}(E, \alpha, \beta) v_k)(\omega_k^* \mathcal{L}(E, \alpha, \beta) v_i)\left(\frac{1}{\lambda_i - \lambda_k}\right) s_i s_k + O\left(\alpha^2, \beta^3\right)
\]

\[
v_{\alpha,\beta} = v_i + \sum_{k \neq i} \frac{v_k \omega_k^* \mathcal{L}(E, \alpha, \beta) v_i}{(\lambda_i - \lambda_k)} s_k + \sum_{j \neq i} \sum_{k \neq i} \frac{(\omega_j^* \mathcal{L}(E, \alpha, \beta) v_k)(\omega_k^* \mathcal{L}(E, \alpha, \beta) v_i)}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)} s_k s_j v_j
\]

\[
- \sum_{j \neq i} \frac{(\omega_j^* \mathcal{L}(E, \alpha, \beta) v_i)(\omega_i^* \mathcal{L}(E, \alpha, \beta) v_i)}{(\lambda_i - \lambda_j)^2} s_i s_j v_j + O\left(\alpha^3, \beta^3\right),
\]

where \(s_\ell = \omega_\ell^* v_\ell\).
Proof. The partial derivatives of the equation \((M_{\alpha,\beta} - \lambda_{\alpha,\beta}I)v_{\alpha,\beta} = 0\), with respect to \(\alpha, \beta\), are
\[
\left(HE - \frac{\partial \lambda_{\alpha,\beta}}{\partial \alpha} I\right)v_{\alpha,\beta} + (M_{\alpha,\beta} - \lambda_{\alpha,\beta}I) \frac{\partial v_{\alpha,\beta}}{\partial \alpha} = 0
\]
\[
\left(SE - \frac{\partial \lambda_{\alpha,\beta}}{\partial \beta} I\right)v_{\alpha,\beta} + (M_{\alpha,\beta} - \lambda_{\alpha,\beta}I) \frac{\partial v_{\alpha,\beta}}{\partial \beta} = 0
\]
Multiplying these by \(\omega^*_{\alpha,\beta}\), since \(\omega^*_{\alpha,\beta}M_{\alpha,\beta} = \lambda_{\alpha,\beta}\omega^*_{\alpha,\beta}\), we have
\[
\omega^*_{\alpha,\beta}\left(HE - \frac{\partial \lambda_{\alpha,\beta}}{\partial \alpha} I\right)v_{\alpha,\beta} = 0, \quad \omega^*_{\alpha,\beta}\left(SE - \frac{\partial \lambda_{\alpha,\beta}}{\partial \beta} I\right)v_{\alpha,\beta} = 0.
\]
For \((\alpha, \beta) \to (0,0)\) the expressions given above
\[
\frac{\partial \lambda_{(\alpha,\beta)=0}}{\partial \alpha} = \lim_{(\alpha, \beta) \to (0,0)} \frac{\partial \lambda_{\alpha,\beta}}{\partial \alpha} = \omega^*_i H_E v_i, \quad \frac{\partial \lambda_{(\alpha,\beta)=0}}{\partial \beta} = \lim_{(\alpha, \beta) \to (0,0)} \frac{\partial \lambda_{\alpha,\beta}}{\partial \beta} = \omega^*_i S_E v_i,
\]
and then, the first differential \(d\lambda_{\alpha,\beta}\) is equal to
\[
d\lambda_{\alpha,\beta} = \alpha \frac{\partial \lambda_{(\alpha,\beta)=0}}{\partial \alpha} + \beta \frac{\partial \lambda_{(\alpha,\beta)=0}}{\partial \beta} = \alpha \frac{\omega^*_i H_E v_i}{\omega^*_i v_i} + \beta \frac{\omega^*_i S_E v_i}{\omega^*_i v_i} = \omega^*_i \mathcal{L}(E, \alpha, \beta) v_i.
\]
Moreover, the first equality in (16) for \((\alpha, \beta) \to (0,0)\) gives
\[
\left(HE - \frac{\partial \lambda_{(\alpha,\beta)=0}}{\partial \alpha} I\right) v_i + (A - \lambda_i I) \frac{\partial v_{(\alpha,\beta)=0}}{\partial \alpha} = 0.
\]
Since \(A\) is diagonalizable, we can write \(\frac{\partial v_{(\alpha,\beta)=0}}{\partial \alpha} = \sum_{k=1}^n c_k v_k\), and so the last equality can be written as
\[
\left(HE - \frac{\partial \lambda_{(\alpha,\beta)=0}}{\partial \alpha} I\right) v_i + \sum_{k \neq i} c_k (\lambda_k - \lambda_i) v_k = 0.
\]
Furthermore, multiplying the above equality by the left eigenvector \(\omega_k\) of \(A\), and using the orthogonality of \(\omega_k\) and \(v_i\) \((k \neq i)\), we have
\[
c_k = \frac{\omega^*_k H_E v_i}{(\lambda_i - \lambda_k) \omega^*_k v_k}, \quad \text{for} \quad k \neq i,
\]
and consequently,
\[
\frac{\partial v_{(\alpha,\beta)=0}}{\partial \alpha} = \sum_{k \neq i} \frac{\omega^*_k H_E v_i}{(\lambda_i - \lambda_k) \omega^*_k v_k} v_k.
\]
Similarly, by the second equality in (16), we obtain \((\lambda_i - \lambda_k)\omega_k^* S_E v_i\), and thus

\[
\frac{\partial v_{(\alpha, \beta)}=0}{\partial \beta} = \sum_{k \neq i} \frac{\omega_k^* S_E v_i}{(\lambda_i - \lambda_k)\omega_k^* v_k} v_k.
\]  

(19)

Hence, the differential \(dv_{\alpha, \beta}\) can be computed as

\[
dv_{\alpha, \beta} = \alpha \frac{\partial v_{(\alpha, \beta)=0}}{\partial \alpha} + \beta \frac{\partial v_{(\alpha, \beta)=0}}{\partial \beta} = \alpha \sum_{k \neq i} \frac{\omega_k^* H_E v_i}{(\lambda_i - \lambda_k)\omega_k^* v_k} v_k + \beta \sum_{k \neq i} \frac{\omega_k^* S_E v_i}{(\lambda_i - \lambda_k)\omega_k^* v_k} v_k = \sum_{k \neq i} v_k \omega_k^* L(E, \alpha, \beta) v_i.
\]

Now, the partial derivatives of the equations in (16) with respect to \(\alpha, \beta\), are

\[
2 \left( H_E - \frac{\partial \lambda_{\alpha, \beta}}{\partial \alpha} I \right) \frac{\partial v_{\alpha, \beta}}{\partial \alpha} + (M_{\alpha, \beta} - \lambda_{\alpha, \beta} I) \frac{\partial^2 v_{\alpha, \beta}}{\partial \alpha^2} - \frac{\partial^2 \lambda_{\alpha, \beta}}{\partial \alpha^2} v_{\alpha, \beta} = 0
\]

\[
2 \left( S_E - \frac{\partial \lambda_{\alpha, \beta}}{\partial \beta} I \right) \frac{\partial v_{\alpha, \beta}}{\partial \beta} + (M_{\alpha, \beta} - \lambda_{\alpha, \beta} I) \frac{\partial^2 v_{\alpha, \beta}}{\partial \beta^2} - \frac{\partial^2 \lambda_{\alpha, \beta}}{\partial \beta^2} v_{\alpha, \beta} = 0
\]

\[
(M_{\alpha, \beta} - \lambda_{\alpha, \beta} I) \frac{\partial^2 v_{\alpha, \beta}}{\partial \alpha \partial \beta} + \left( H_E - \frac{\partial \lambda_{\alpha, \beta}}{\partial \alpha} I \right) \frac{\partial v_{\alpha, \beta}}{\partial \alpha} + \left( S_E - \frac{\partial \lambda_{\alpha, \beta}}{\partial \beta} I \right) \frac{\partial v_{\alpha, \beta}}{\partial \beta} - \frac{\partial^2 \lambda_{\alpha, \beta}}{\partial \alpha \partial \beta} v_{\alpha, \beta} = 0.
\]

Multiplying these expressions by \(\omega_{\alpha, \beta}^*\) and substituting \(\frac{\partial v_{(\alpha, \beta)=0}}{\partial \alpha}, \frac{\partial v_{(\alpha, \beta)=0}}{\partial \beta}\) from (18) and (19), for \((\alpha, \beta) \to (0,0)\), and noting that \(\omega_{\alpha, \beta}^* v_k = 0\), we obtain

\[
\frac{\partial^2 \lambda_{\alpha, \beta}=0}{\partial \alpha^2} = \frac{2}{\omega_{\alpha}^* v_i} \left( \omega_{\alpha}^* H_E - \omega_{\alpha}^* \frac{\partial \lambda_{\alpha, \beta}=0}{\partial \alpha} \right) \frac{\partial v_{(\alpha, \beta)=0}}{\partial \alpha} = \frac{2}{\omega_{\alpha}^* v_i} \sum_{k \neq i} \frac{(\omega_k^* H_E v_i)(\omega_k^* H_E v_k)}{(\lambda_i - \lambda_k)\omega_k^* v_k}
\]

\[
\frac{\partial^2 \lambda_{\alpha, \beta}=0}{\partial \beta^2} = \frac{2}{\omega_{\beta}^* v_i} \left( \omega_{\beta}^* S_E - \omega_{\beta}^* \frac{\partial \lambda_{\alpha, \beta}=0}{\partial \beta} \right) \frac{\partial v_{(\alpha, \beta)=0}}{\partial \beta} = \frac{2}{\omega_{\beta}^* v_i} \sum_{k \neq i} \frac{(\omega_k^* S_E v_i)(\omega_k^* S_E v_k)}{(\lambda_i - \lambda_k)\omega_k^* v_k}
\]

\[
\frac{\partial^2 \lambda_{\alpha, \beta}=0}{\partial \alpha \partial \beta} = \frac{1}{\omega_{\alpha}^* v_i} \left( \omega_{\alpha}^* H_E \frac{\partial v_{(\alpha, \beta)=0}}{\partial \beta} + \omega_{\beta}^* S_E \frac{\partial v_{(\alpha, \beta)=0}}{\partial \alpha} \right) = \frac{1}{\omega_{\alpha}^* v_i} \sum_{k \neq i} \frac{(\omega_k^* H_E v_i)(\omega_k^* S_E v_i) + (\omega_k^* H_E v_i)(\omega_k^* S_E v_k)}{(\lambda_i - \lambda_k)\omega_k^* v_k}
\]

(21)

Therefore, the second differential \(d^2 \lambda_{\alpha, \beta}\) is equal to
\[ d^2 \lambda_{\alpha\beta} = \alpha^2 \frac{\partial^2 \lambda_{(\alpha\beta)=0}}{\partial \alpha^2} + 2\alpha\beta \frac{\partial^2 \lambda_{(\alpha\beta)=0}}{\partial \alpha \partial \beta} + \beta^2 \frac{\partial^2 \lambda_{(\alpha\beta)=0}}{\partial \beta^2} \]

\[ = 2\alpha^2 \sum_{k \neq i} \frac{(\omega_k^* H_E v_i)(\omega_k^* H_E v_k)}{(\lambda_i - \lambda_k) \omega_k^* v_k} + 2\alpha \beta \sum_{k \neq i} \frac{(\omega_k^* H_E v_i)(\omega_k^* S_E v_i)(\omega_k^* S_k v_k)}{(\lambda_i - \lambda_k) \omega_k^* v_k} \]

\[ + \frac{2\beta^2}{\omega_i^* v_i} \sum_{k \neq i} \frac{(\omega_k^* S_E v_i)(\omega_k^* S_E v_k)}{(\lambda_i - \lambda_k) \omega_k^* v_k} \]

\[ = 2\alpha \sum_{k \neq i} \frac{(\omega_k^* H_k v_k)(\omega_k^* \mathcal{L}(E,\alpha,\beta) v_i)}{(\lambda_i - \lambda_k) \omega_k^* v_k} + 2\beta \sum_{k \neq i} \frac{(\omega_k^* S_E v_k)(\omega_k^* \mathcal{L}(E,\alpha,\beta) v_i)}{(\lambda_i - \lambda_k) \omega_k^* v_k} \]

\[ = 2 \frac{\alpha}{\omega_i^* v_i} \sum_{k \neq i} \frac{(\omega_k^* \mathcal{L}(E,\alpha,\beta) v_k)(\omega_k^* \mathcal{L}(E,\alpha,\beta) v_i)}{(\lambda_i - \lambda_k) \omega_k^* v_k}, \]

and by

\[ \lambda_{\alpha\beta} = \lambda_i + d\lambda_{\alpha\beta} + \frac{1}{2} d^2 \lambda_{\alpha\beta} + \mathcal{O}(\alpha^3, \beta^2) \]

we receive (14), whereas we have declared \( s_i = \omega_i^* v_i \).

Multiplying the first of (20) by \( \omega_i^* \), due to \( \omega_j^* v_i = 0 \) \( (j \neq i) \), for \( (\alpha, \beta) \rightarrow (0, 0) \), we obtain

\[ (\lambda_i - \lambda_j) \omega_j^* \frac{\partial^2 v_{(\alpha\beta)=0}}{\partial \alpha^2} = 2 \left( \omega_j^* H_E - \omega_j^* \frac{\partial \lambda_{(\alpha\beta)=0}}{\partial \alpha} \right) \frac{\partial v_{(\alpha\beta)=0}}{\partial \alpha}. \]

Substituting the formulae of \( \frac{\partial \lambda_{(\alpha\beta)=0}}{\partial \alpha}, \frac{\partial v_{(\alpha\beta)=0}}{\partial \alpha} \) from (17) and (18), since \( \omega_j^* v_k = 0 \) \( (j \neq k) \),

we take

\[ \omega_j^* \frac{\partial^2 v_{(\alpha\beta)=0}}{\partial \alpha^2} = 2 \left( \sum_{k \neq i} \frac{(\omega_k^* H_E v_i)(\omega_k^* H_E v_k)}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_j) \omega_k^* v_k} \right) - 2 \frac{(\omega_j^* H_E v_i)(\omega_j^* H_E v_i)}{\omega_i^* v_i (\lambda_i - \lambda_j)^2} ; \quad j \neq i \]

and then

\[ \frac{\partial^2 v_{(\alpha\beta)=0}}{\partial \alpha^2} = 2 \sum_{j \neq i} \left( \sum_{k \neq i} \frac{(\omega_k^* H_E v_i)(\omega_k^* H_E v_k)}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)(\omega_k^* v_k)(\omega_j^* v_j)} \right) v_j - 2 \sum_{j \neq i} \frac{(\omega_j^* H_E v_i)(\omega_j^* H_E v_i)}{(\lambda_i - \lambda_j)^2(\omega_i^* v_i)(\omega_j^* v_j)} v_j. \quad (22) \]

Similarly, the last two expressions of (20) lead to

\[ \frac{\partial^2 v_{(\alpha\beta)=0}}{\partial \beta^2} = 2 \sum_{j \neq i} \left( \sum_{k \neq i} \frac{(\omega_k^* S_E v_i)(\omega_k^* S_E v_k)}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)(\omega_k^* v_k)(\omega_j^* v_j)} \right) v_j - 2 \sum_{j \neq i} \frac{(\omega_j^* S_E v_i)(\omega_j^* S_E v_i)}{(\lambda_i - \lambda_j)^2(\omega_i^* v_i)(\omega_j^* v_j)} v_j \]

\[ \frac{\partial^2 v_{(\alpha\beta)=0}}{\partial \alpha \partial \beta} = \sum_{j \neq i} \left( \sum_{k \neq i} \frac{(\omega_k^* S_E v_i)(\omega_k^* H_E v_k) + (\omega_k^* H_E v_i)(\omega_k^* S_E v_k)}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)(\omega_k^* v_k)(\omega_j^* v_j)} \right) v_j. \quad (23) \]
Furthermore, no results on the perturbation of the eigenpairs of the partial derivatives depend on the invertibility of a matrix and the eigenvectors earlier by Chu in \[3\]. Chu has follow different methodology considering that $\upsilon_d$ and by $\upsilon_E = \upsilon_\alpha, \beta = \upsilon_i$.

In the following we present a lemma, which will contribute in the approximation formulae (17) and (21) of $\alpha, \beta = \alpha, \beta$. The simplified presentation of partial differential formulae (17) and (21) of $\lambda_\alpha, \beta$ and (18), (19), (22) and (23) of $\upsilon_\alpha, \beta$ for $\alpha = \beta = 0$ are independent results on those, which were obtained earlier by Chu in \[3\]. Chu has follow different methodology considering that $\lambda_i$ is simple eigenvalue and an additional normalized condition that $\omega_i^* v_i = 1$, and even Chu’s formulations of the partial derivatives depend on the invertibility of a matrix and the eigenvectors $v_i$, $\omega_i$.

Furthermore, no results on the perturbation of the eigenpairs $\lambda_\alpha, \beta$, and $\upsilon_\alpha, \beta$ are given in \[3\].

In the following we present a lemma, which will contribute in the approximation formulae (14) and (15).

**Lemma 1** Let the matrix $A \in M_n(\mathbb{C})$ be diagonalizable and $Y_i$, $W_i$ be matrices whose columns $v_i$ and rows $\omega_i^*$ respectively are the corresponding right and left eigenvectors of $A$, for $\lambda_i \in \sigma(A)$. A generalized inverse of $(A - \lambda_i I)^\mu$, $\mu \in \mathbb{N}$, is defined by

\[
[(A - \lambda_i I)^\mu]^+ = \sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i)^\mu s_k}; \quad s_k = \omega_k^* v_k. \quad (24)
\]
Proof. It is evident that \((A - \lambda_i I)^{\frac{v_k \omega_k^*}{\lambda_k - \lambda_i} \mu} = v_k \omega_k^*\), and then
\[
(A - \lambda_i I)^\mu \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu} = (A - \lambda_i I)^{\mu-1} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu}
\]
\[
= (A - \lambda_i I)^{\mu-1} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu-1} = \cdots = v_k \omega_k^*.
\]

Since \(\sum \frac{v_k \omega_k^*}{s_k} = I\), \(\omega_k^* A = \lambda_i \omega_k^*\) and for \(k \neq i\), \(\omega_k^* v_i = 0\), we have:
\[
(A - \lambda_i I)^\mu \left(\sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu s_k}\right) (A - \lambda_i I)^\mu = \left(\sum_{k \neq i} \frac{v_k \omega_k^*}{s_k}\right) (A - \lambda_i I)^\mu = (I - Y_i W_i) (A - \lambda_i I)^\mu
\]
\[
= (A - \lambda_i I)^\mu - Y_i W_i (A - \lambda_i I)^\mu = (A - \lambda_i I)^\mu - Y_i (W_i A - \lambda_i W_i) (A - \lambda_i I)^{\mu-1} = (A - \lambda_i I)^\mu
\]
and
\[
\left(\sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu s_k}\right) (A - \lambda_i I)^\mu \left(\sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu s_k}\right) = \left(\sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu s_k}\right) (I - Y_i W_i) = \sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) \mu s_k}.
\]

\(\square\)

In Lemma 1, if \(A\) is normal, then \(v_k = \omega_k\), and \([(A - \lambda_i I)^\mu]^+\) is Hermitian. In this case, we confirm that \([(A - \lambda_i I)^\mu]^+\) in (24) is the Moore-Penrose inverse of \((A - \lambda_i I)^\mu\).

Combining Equations (24), (14), and (15), Theorem 2 leads to a generalization of a corresponding result for simple eigenvalues of a Hermitian matrix, which was presented by M. Fiedler in [5].

**Theorem 3** Let the matrix \(A \in \mathcal{M}_n(\mathbb{C})\) be diagonalizable and \(\lambda_i\) be a semisimple eigenvalue of \(A\) with \(v_i, \omega_i\) corresponding right and left eigenvectors. If the assumptions for the equations (14) and (15) hold, then the following expressions for \(\lambda_{\alpha, \beta}\) and \(v_{\alpha, \beta}\) hold:
\[
\lambda_{\alpha, \beta} = \lambda_i + \frac{1}{s_i} \omega_i^* \mathcal{L}(E, \alpha, \beta) v_i - \frac{1}{s_i} \omega_i^* \mathcal{L}(E, \alpha, \beta) (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i + O(\alpha^3, \beta^3), \quad (25)
\]
\[
v_{\alpha, \beta} = v_i - (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i + [(A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta)]^2 v_i
\]
\[
- \frac{1}{s_i} [(A - \lambda_i I)^2]^+ \mathcal{L}(E, \alpha, \beta) v_i \omega_i^* \mathcal{L}(E, \alpha, \beta) v_i + O(\alpha^3, \beta^3). \quad (26)
\]
Proof. From (14) and (24) with $\mu = 1$, we immediately have

$$\lambda_{\alpha,\beta} = \lambda_i + \frac{1}{s_i} \omega_i^* \mathcal{L}(E, \alpha, \beta) v_i - \frac{1}{s_i} \omega_i^* \mathcal{L}(E, \alpha, \beta) \left( \sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) s_k} \right) \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{O}(\alpha^3, \beta^3),$$

proving (25). Also, from (15) and (24) with $\mu = 1, 2$, we have

$$v_{\alpha,\beta} = v_i - \left( \sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) s_k} \right) \mathcal{L}(E, \alpha, \beta) v_i + \sum_{k \neq i} \left( \sum_{j \neq i} \frac{v_j \omega_j^*}{(\lambda_j - \lambda_i) s_j} \mathcal{L}(E, \alpha, \beta) \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) s_k} \mathcal{L}(E, \alpha, \beta) v_i \right)$$

$$- \left( \sum_{j \neq i} \frac{v_j \omega_j^*}{(\lambda_j - \lambda_i) s_j} \right) \mathcal{L}(E, \alpha, \beta) \frac{v_i \omega_i^*}{s_i} \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{O}(\alpha^3, \beta^3)$$

$$= v_i - (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i + (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) \left( \sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) s_k} \mathcal{L}(E, \alpha, \beta) v_i \right)$$

$$- \left( \sum_{j \neq i} \frac{v_j \omega_j^*}{(\lambda_j - \lambda_i) s_j} \right) \mathcal{L}(E, \alpha, \beta) \frac{v_i \omega_i^*}{s_i} \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{O}(\alpha^3, \beta^3)$$

$$= v_i - (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i + [(A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta)]^2 v_i$$

$$- [(A - \lambda_i I)^2]^+ \mathcal{L}(E, \alpha, \beta) \frac{v_i \omega_i^*}{s_i} \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{O}(\alpha^3, \beta^3),$$

proving (26). \qed

In (25) and (26), if we consider the first-order approximation, then we simply have

$$\tilde{\lambda}_{\alpha,\beta} = \lambda_i + \frac{1}{s_i} \omega_i^* \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{O}(\alpha^2, \beta^2),$$

and

$$\tilde{v}_{\alpha,\beta} = v_i - (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{O}(\alpha^2, \beta^2),$$

(27)

for a simple eigenvalue $\lambda_i$. In these cases,

$$M_{\alpha,\beta} \tilde{v}_{\alpha,\beta} - \tilde{\lambda}_{\alpha,\beta} \tilde{v}_{\alpha,\beta} = -(A - \lambda_i I)(A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{L}(E, \alpha, \beta) v_i - \frac{1}{s_i} [\omega_i^* \mathcal{L}(E, \alpha, \beta) v_i] v_i$$

$$+ \left[ \frac{1}{s_i} (\omega_i^* \mathcal{L}(E, \alpha, \beta) v_i) I - \mathcal{L}(E, \alpha, \beta) \right] (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i$$

$$= - \left[ (A - \lambda_i I)(A - \lambda_i I)^+ + \frac{v_i \omega_i^*}{s_i} \right] \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{L}(E, \alpha, \beta) v_i$$

$$+ \left[ \frac{1}{s_i} (\omega_i^* \mathcal{L}(E, \alpha, \beta) v_i) I - \mathcal{L}(E, \alpha, \beta) \right] (A - \lambda_i I)^+ \mathcal{L}(E, \alpha, \beta) v_i + \mathcal{O}(\alpha^2, \beta^2).$$
Since,
\[(A - \lambda_i I)(A - \lambda_i I)^+ + \frac{v_i \omega_i^*}{s_i} = (A - \lambda_i I) \sum_{k \neq i} \frac{v_k \omega_k^*}{(\lambda_k - \lambda_i) s_k} + \frac{v_i \omega_i^*}{s_i} = \sum_k \frac{v_k \omega_k^*}{s_k} = I,\]
we have
\[M_{\alpha,\beta} \bar{v}_{\alpha,\beta} - \tilde{\lambda}_{\alpha,\beta} \bar{v}_{\alpha,\beta} = \left[ \frac{1}{s_i} \omega_i^* L(E, \alpha, \beta) v_i - L(E, \alpha, \beta) \right] (A - \lambda_i I)^+ L(E, \alpha, \beta) v_i + O(\alpha^2, \beta^2). (28)\]

**Proposition 6** Let \(\lambda_i\) be a simple eigenvalue of diagonalizable matrix \(A \in \mathcal{M}_n(\mathbb{C})\) with right and left eigenvectors \(v_i\) and \(\omega_i\). If there exist \(\alpha, \beta\) such that \(L(E, \alpha, \beta) v_i \in \ker(A - \lambda_i I)^+\), then \(\tilde{\lambda}_{\alpha,\beta}\) and \(\bar{v}_{\alpha,\beta}\) in (27) is an approximation of an eigenpair of \(M_{\alpha,\beta} = A + L(E, \alpha, \beta)\).

**Corollary 2** Let \(\lambda_i\) be a simple eigenvalue of normal matrix \(A \in \mathcal{M}_n(\mathbb{C})\) with eigenvector \(v_i\). If there exist \(\alpha, \beta\) such that \(L(E, \alpha, \beta) v_i = 0\), then \(\tilde{\lambda}_{\alpha,\beta}\) and \(\bar{v}_{\alpha,\beta}\) in (27) is an approximation of an eigenpair of \(M_{\alpha,\beta} = A + L(E, \alpha, \beta)\).

**Proof.** It is well-known that
\[\ker(A - \lambda_i I)^+ = \ker(A - \lambda_i I)^T.\]
Also, since \(A\) is normal, \(\omega_i = v_i\). Thus by (28), it is implied that
\[(A - \lambda_i I)^+ L(E, \alpha, \beta) v_i = (A - \lambda_i I)^+ \bar{v}_i = [v_i^* (A - \lambda_i I)]^T = 0.\]

In this case, (27) is simplified to
\[\tilde{\lambda}_{\alpha,\beta} = \lambda_i + \frac{v_i^* v_i}{s_i} + O(\alpha^2, \beta^2), \quad \bar{v}_{\alpha,\beta} = v_i + O(\alpha^2, \beta^2) \quad (29)\]
or, \(\tilde{\lambda}_{\alpha,\beta} = (\lambda_i + 1) + O(\alpha^2, \beta^2)\), \(\bar{v}_{\alpha,\beta} = v_i + O(\alpha^2, \beta^2)\) for real symmetric matrix \(A\).

**Example 1** Let \(A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\). Then \(\sigma(A) = \{\lambda_1 = 1, \lambda_{2,3} = 2\}\), and the corresponding right and left eigenvectors are given by:
\[
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]
Let the perturbation matrix $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. We then obtain

$$L(E, \alpha, \beta) = \begin{bmatrix} \alpha & 0 & \beta \\ 0 & \alpha & 0 \\ -\beta & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_{\alpha,\beta} = \begin{bmatrix} 1 + \alpha & 0 & 1 + \beta \\ 0 & 2 + \alpha & 0 \\ -\beta & 0 & 2 \end{bmatrix}.$$ 

Then, $\sigma(M_{\alpha,\beta}) = \{\lambda_{1,\alpha,\beta} = \frac{3+\alpha-\sqrt{(\alpha-1)^2-4\beta(\beta+1)}}{2}, \lambda_{2,\alpha,\beta} = \frac{3+\alpha+\sqrt{(\alpha-1)^2-4\beta(\beta+1)}}{2}, \lambda_{3,\alpha,\beta} = 2 + \alpha\}$, and the corresponding right eigenvectors are:

$$\begin{bmatrix} 1 + \beta & 0 & \lambda_{1,\alpha,\beta} - 1 - \alpha \end{bmatrix}^T, \quad \begin{bmatrix} 1 + \beta & 0 & \lambda_{2,\alpha,\beta} - 1 - \alpha \end{bmatrix}^T, \quad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T.$$

Clearly, the eigenvalues and the corresponding eigenvectors of $M_{\alpha,\beta}$ are real functions of two variables, with continuous partial derivatives for all permissible values of $\alpha, \beta$.

For $\alpha = 0.1, \beta = 0.01$, we have $\sigma(M_{\alpha,\beta}) = \{1.1114, 1.9886, 2.1\}$, and the corresponding unit eigenvectors are given by

$$\begin{bmatrix} 0.9999 & 0 & 0.0113 \\ 0.7508 & 0 & 0.6606 \end{bmatrix}^T, \quad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T.$$

Moreover by (24),

$$(A - I)^+ = [(A - I)^2]^+ = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Hence, a first-order approximations of the eigenvalue and unit eigenvector of $M_{\alpha,\beta}$ by (27) are:

$$\tilde{\lambda}_1 = 1 + 0.11 = 1.11, \quad \tilde{v}_1^T = \begin{bmatrix} 1 & 0 & 0.0099 \end{bmatrix}^T.$$

Also, by (25), (26), the corresponding second-order approximations are equal to

$$\tilde{\lambda}_1 = 1.1112, \quad \tilde{v}_1^T = \begin{bmatrix} 0.9999 & 0 & 0.0111 \end{bmatrix}^T.$$

By (24), $$(A - 2I)^+ = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [(A - 2I)^2]^+ = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$ and substituting these expressions in (25) and (26), using the eigenvectors $v_2, \omega_2$, the second-order approximations of the eigenpair of $M_{\alpha,\beta}$ are

$$\tilde{\lambda}_2 = 1.9888, \quad \tilde{v}_2^T = \begin{bmatrix} 0.7505 & 0 & 0.6609 \end{bmatrix}^T.$$
Similarly, using the eigenvectors \( v_3, \omega_3 \), the second-order approximations of the third eigenpair of \( M_{\alpha,\beta} \) are

\[
\tilde{\lambda}_3 = 2.1, \quad \tilde{v}_3^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T.
\]

Notice that, we obtain all eigenpair of \( M_{\alpha,\beta} \) with preciseness \( 10^{-3} \).

**Example 2** Let the matrix \( A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & -1 \\ 0 & -1 & 2 \end{bmatrix} \), with \( \sigma(A) = \{ \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 7 \} \), and by Corollary 2 and (29) the eigenpair of \( M_{\alpha,\beta} \) is

\[
\tilde{\lambda}_3 = \lambda_1 + \frac{v_1^T v_1}{s_1} = 2 + 1 = 3, \quad \tilde{v}_3^T = v_1^T.
\]

Moreover, \( \mathcal{L}(E, 0.04, 0.08) v_2 \neq v_2 \), and by (24), \( (A - I)\tilde{T} = \begin{bmatrix} 0.222 & -0.0556 & -0.3889 \\ -0.0556 & 0.1389 & -0.0278 \\ -0.3889 & -0.0278 & 0.8056 \end{bmatrix} \), then (25) and (26) lead to the second-order approximations

\[
\tilde{\lambda}_2 = 1.5890, \quad \tilde{v}_2^T = \begin{bmatrix} 0.7031 & 0.4409 & 0.5579 \end{bmatrix}^T,
\]

in contrast to the first-order approximations of the eigenpair of \( M_{\alpha,\beta} \), which are equal to

\[
\tilde{\lambda}_2 = 1.5933, \quad \tilde{v}_2^T = \begin{bmatrix} 0.6257 & 0.4242 & 0.6546 \end{bmatrix}^T.
\]
Moreover, \( \mathcal{L}(E, 0.04, 0.08)v_3 \neq v_3 \), and by (24), \((A−7I)^+ = \begin{bmatrix} -0.1511 & 0.0556 & 0.0244 \\ -0.0556 & -0.0278 & -0.0278 \\ 0.0244 & -0.0278 & -0.1878 \end{bmatrix} \),

\[
[(A−7I)^2]^+ = \begin{bmatrix} 0.0265 & 0.0093 & -0.0067 \\ 0.0093 & 0.0046 & 0.0046 \\ -0.0067 & 0.0046 & 0.0366 \end{bmatrix}.
\]

Also, by (25) and (26), the second-order approximations give

\[
\tilde{\lambda}_3 = 7.0910, \quad \tilde{v}_3^T = \begin{bmatrix} 0.3878 & -0.9049 & 0.1754 \end{bmatrix}^T,
\]

but the first-order approximations are not satisfactory enough

\[
\tilde{\lambda}_3 = 7.0867, \quad \tilde{v}_3^T = \begin{bmatrix} 0.3851 & -0.9056 & 0.1779 \end{bmatrix}^T.
\]

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