Separation and Estimation of Periodic/Aperiodic State

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Abstract

Periodicity and aperiodicity can exist in a state simultaneously and typically become quasi-periodicity and quasi-aperiodicity in a dynamically changing state. The quasi-periodic and quasi-aperiodic states existing in the periodic/aperiodic state mostly correspond to different phenomena and require different controls. For separation control of these states, this paper defines the periodic/aperiodic, quasi-periodic, and quasi-aperiodic states to construct a periodic/aperiodic separation filter that separates the periodic/aperiodic state into the quasi-periodic and quasi-aperiodic states. Based on these definitions, the linearity of periodic-pass and aperiodic-pass functions and the orthogonality of quasi-periodic and quasi-aperiodic state functions are proved. Subsequently, the periodic/aperiodic separation filter composed of periodic-pass and aperiodic-pass filters that realize the periodic-pass and aperiodic-pass functions is designed and integrated with a Kalman filter for estimation of the quasi-periodic and quasi-aperiodic states.

Key words: Periodic/Aperiodic separation filter, Lifting, Time delay, Comb filter, Kalman filter

1 Introduction

Periodicity and aperiodicity are typical trends of states and systems; in particular, control strategies with periodicity have been widely studied. Repetitive control proposed by [Inoue et al. (1981); Hara et al. (1988)] can realize precise periodic state control and has been developed to improve the precision [Bristow et al. (2006); Chen and Tomizuka (2014); Nagahara and Yamamoto (2016)]. Additionally, Muramatsu and Katsura (2018); Karttunen et al. (2014) studied the periodicity of a disturbance to estimate and compensate for a periodic disturbance, and Bittanti and Bolzern (1985); Bamieh and Pearson (1992) investigated controls for periodic systems. To address the periodicity, Bittanti and Colaneri (2000); Ebihara et al. (2010); Markovsky et al. (2014); Yang (2018) used a lifting technique to represent the periodic system as a time-invariant system.

Although many control strategies with periodicity have been proposed, a control strategy with both periodicity and aperiodicity is rare. Using the lifting technique, Muramatsu and Katsura (2019) proposed a periodic/aperiodic separation filter (PASF) that separates a periodic/aperiodic state into a quasi-periodic state and a quasi-aperiodic state; they used the PASF for separation control of the quasi-periodic and quasi-aperiodic states existing in the periodic/aperiodic state. The PASF is similar to a comb filter, which rejects a periodic signal [Sugiura et al. (2013); Liu and Declercq (2017); Aslan and Altintas (2018)], but differs from the comb filter in addressing the quasi-periodic and quasi-aperiodic signals. Nevertheless, the periodic/aperiodic separation control studies had three issues: the definitions of the periodic/aperiodic, quasi-periodic, and quasi-aperiodic states were qualitative; linearity and orthogonality were not proved; and variations of the PASF were not considered.

This paper addresses the aforementioned issues and makes the following contributions: it provides new definitions of the periodic/aperiodic, quasi-periodic, and quasi-aperiodic states; proves the linearity of periodic-pass and aperiodic-pass functions and orthogonality of quasi-periodic and quasi-aperiodic state functions; designs high-order infinite-impulse-response (IIR) and finite-impulse-response (FIR) realizations of the PASF; proposes a KF-PASF integrating the PASF and a Kalman filter (KF) [Kalman (1960); Auger et al. (2013)]; and proves the unbiased estimation and equivalent sum of periodic-error and aperiodic-error covari-
The discrete-time Fourier transform $F$ of $x_τ(k)$ is

$$F[x_τ(k)] := X_τ(ω) := \sum_{k=-\infty}^{\infty} x_τ(k)e^{-jωk}, \quad \omega \in \{ω \in \mathbb{R} \mid -\pi \leq ω \leq \pi\},$$

where $ω = \tilde{ω}IT \text{[rad/sample]}$, $\tilde{ω} \text{[rad/s]}$, and $T \text{[s]}$ denote the normalized angular frequency, angular frequency, and sampling time, respectively. Note that the sampling time of $t$ is $T$, and that of $k$ is $T$. Subsequently, this paper defines a set of a zero function $S_0$, a set of lifted state functions that have quasi-periodicity $S_p$, and a set of lifted state functions that have quasi-aperiodicity $S_κ$ as

$$S_0 := \{x_τ \in \mathbb{S}[\forall \omega, X_τ(\omega) = 0\}, \quad \text{(3a)}$$
$$S_p := \{x_τ \in \mathbb{S}[\exists \omega, X_τ(\omega) \neq 0 \land |\omega| \leq \rho\}, \quad \text{(3b)}$$
$$S_κ := \{x_τ \in \mathbb{S}[\exists \omega, X_τ(\omega) \neq 0 \land \rho < |\omega|\}, \quad \text{(3c)}$$
$$|\rho \in \{\rho \in \mathbb{R} | 0 \leq \rho \leq \pi\}, \quad \text{(3d)}$$

where the quasi-periodicity and quasi-aperiodicity are defined to be low-frequency waves and high-frequency waves of the lifted state function $x_τ$, respectively. This paper uses $\wedge, \lor, \Rightarrow$, and $\Leftrightarrow$ to denote the logical conjunction, disjunction, implication, and equivalence, respectively. The variable $\rho \text{[rad/sample]}$ denotes the normalized separation frequency, which is the boundary between the quasi-periodicity and quasi-aperiodicity. The separation frequency $\hat{\rho}$ [rad/s] is given by

$$\hat{\rho} = \frac{\rho}{IT}. \quad \text{(4)}$$

The whole set $S$ is the union of the three sets

$$S = S_0 \cup S_p \cup S_κ,$$

and $S_p$ and $S_κ$ do not contain $S_0$

$$S_0 \notin (S_p \cup S_κ).$$

According to the sets, a lifted quasi-periodic-state function $x_τp$ and lifted quasi-aperiodic-state function $x_τa$ are
defined by

\[ x_{\text{rp}} := x_r \text{ s.t. } x_r \in (S_0 \cup S_A)^c, \quad (5a) \]
\[ x_{\text{ra}} := x_r \text{ s.t. } x_r \in (S_0 \cup S_\Phi)^c, \quad (5b) \]

and the quasi-periodic state \( x_p(t) \) and quasi-aperiodic state \( x_a(t) \) are defined by

\[ x_p(t) := L^{-1}(x_{\text{rp}}(k)), \]
\[ x_a(t) := L^{-1}(x_{\text{ra}}(k)). \]

Note that the quasi-periodic state \( x_p(t) \) and quasi-aperiodic state \( x_a(t) \) are not low-frequency waves and high-frequency waves even though the lifted quasi-periodic state \( x_{\text{rp}}(k) \) and lifted quasi-aperiodic state \( x_{\text{ra}}(k) \) are low-frequency waves and high-frequency waves, respectively. Furthermore, this paper defines the lifted periodic/aperiodic-state function:

\[ x_{\text{rpa}} := x_r \text{ s.t. } x_r \in S_\Phi \cap S_A \]

and the periodic/aperiodic state \( x_{\text{pa}}(t) \):

\[ x_{\text{pa}}(t) := L^{-1}(x_{\text{rpa}}(k)), \]

which is the sum of the quasi-periodic and quasi-aperiodic states

\[ x_{\text{pa}}(t) = x_{\text{p}}(t) + x_{\text{a}}(t). \]

In summary, the state \( x(t) \) varies as

\[ x(t) = \begin{cases} 
   x_{\text{pa}}(t) & \text{if } x_r \in S_\Phi \cap S_A, \\
   x_{\text{p}}(t) & \text{if } x_r \in (S_0 \cup S_A)^c, \\
   x_{\text{a}}(t) & \text{if } x_r \in (S_0 \cup S_\Phi)^c, \\
   0 & \text{if } x_r \in S_0.
\end{cases} \quad (6) \]

Because the state function \( x \) has II lifted state functions \( x_0, \ldots, x_{\Pi-1} \), the domain \( Z \) of the state function \( x \) has II corresponding subsets: \( T_0, \ldots, T_{\Pi-1} \) defined by

\[ T_r := \{ t \in \mathbb{Z} | \forall k \in \mathbb{Z}, t = k\Pi + r \}, \]

which is illustrated in Fig. 1. In each domain \( T_r \), the state function \( x \) is periodic/aperiodic if \( x_r \in S_\Phi \cap S_A \), quasi-periodic if \( x_r \in (S_0 \cup S_A)^c \), quasi-aperiodic if \( x_r \in (S_0 \cup S_\Phi)^c \), or zero if \( x_r \in S_0 \).

### 2.2 Linearity and Orthogonality

Let \( f_{\text{rp}} : \mathbb{R} \to \mathbb{R} \) and \( f_{\text{ra}} : \mathbb{R} \to \mathbb{R} \) be the lifted periodic-pass function and lifted aperiodic-pass function, respectively

\[ f_{\text{rp}}(x_r(k)) := \begin{cases} 
   x_{\text{rp}}(k) & \text{if } x_r \in S_\Phi, \\
   0 & \text{if } x_r \in S_\Phi^c.
\end{cases} \]

Moreover, let \( f_p : \mathbb{R} \to \mathbb{R} \) and \( f_a : \mathbb{R} \to \mathbb{R} \) be the periodic-pass function and aperiodic-pass function, respectively

\[ f_p(x(t)) := \begin{cases} 
   x_{\text{p}}(t) & \text{if } x_r \in S_\Phi, \\
   0 & \text{if } x_r \in S_\Phi^c, \quad (7a) \\
   f_a(x(t)) := \begin{cases} 
   x_{\text{a}}(t) & \text{if } x_r \in S_A, \\
   0 & \text{if } x_r \in S_A^c, \quad (7b)
\end{cases}
\]

where

\[ f_{\text{rp}}(x(t)) = L^{-1}\left(f_{\text{rp}}\left(L(x(t))\right)\right), \]
\[ f_a(x(t)) = L^{-1}\left(f_a\left(L(x(t))\right)\right). \]

Fig. 2 shows the relationship of the definitions, lifting, and separation for the sets, states, and state functions with quasi-periodicity and quasi-aperiodicity.

Preliminarily, this paper presents Lemma 1. Then, for the states and functions, Theorems 1 and 2 demonstrate that the sets \( S_A^c \) and \( S_\Phi^c \) are closed under addition and multiplication. This implies that the quasi-periodicity and quasi-aperiodicity are retained or become zero after addition and multiplication. Furthermore, Theorem 3 demonstrates the linearity of the periodic-pass and aperiodic-pass functions. Theorem 4 and Proposition 1 demonstrate the orthogonality of the quasi-periodic and quasi-aperiodic states and interference between the periodic-pass and aperiodic-pass functions, respectively.

**Lemma 1**

\[ x_r \in S_A^c \iff X_r(\omega) = \sum_{k=-\infty}^{\infty} x_r(k) e^{-j\omega k} \quad \text{if } |\omega| \leq \rho, \]
\[ 0 \quad \text{if } \rho < |\omega| \]
\[ x_r \in S_\Phi^c \iff X_r(\omega) = \sum_{k=-\infty}^{\infty} x_r(k) e^{-j\omega k} \quad \text{if } |\omega| \leq \rho, \]
\[ 0 \quad \text{if } |\omega| > \rho. \]

**Proof.** (3b) and (3c) give

\[ x_r \in S_\Phi \iff X_r(\omega) \neq 0 \text{ if } |\omega| \leq \rho, \]
\[ x_r \in S_A \iff X_r(\omega) \neq 0 \text{ if } \rho < |\omega|, \]

and their contrapositives:

\[ x_r \in S_\Phi^c \iff X_r(\omega) = 0 \text{ if } |\omega| \leq \rho, \]
\[ x_r \in S_A^c \iff X_r(\omega) = 0 \text{ if } \rho < |\omega|. \]
Proof. Lemma 1 gives

\[ x_\tau \in S^c_\Delta \Leftrightarrow \mathcal{F}[x_\tau(k)] = \begin{cases} \sum_{k=-\infty}^{\infty} x_\tau(k)e^{-j\omega k} & \text{if } |\omega| \leq \rho, \\ 0 & \text{if } \rho < |\omega| \end{cases} \]

\[ x_\tau \in S^c_\Psi \Leftrightarrow \mathcal{F}[x_\tau(k)] = \begin{cases} 0 & \text{if } |\omega| \leq \rho, \\ \sum_{k=-\infty}^{\infty} x_\tau(k)e^{-j\omega k} & \text{if } \rho < |\omega| \end{cases} \]

\[ z_\tau \in S^c_\Delta \Leftrightarrow \mathcal{F}[z_\tau(k)] = \begin{cases} \sum_{k=-\infty}^{\infty} z_\tau(k)e^{-j\omega k} & \text{if } |\omega| \leq \rho, \\ 0 & \text{if } \rho < |\omega| \end{cases} \]

Using the linearity of the Fourier transform \( \mathcal{F}[x_\tau(k) + z_\tau(k)] = \mathcal{F}[x_\tau(k)] + \mathcal{F}[z_\tau(k)] \) and \( \mathcal{F}[x_\tau(k) + z_\tau(k)] \) are calculated as

\[ x_\tau, z_\tau \in S^c_\Delta \Rightarrow y_\tau \in S^c_\Delta, x_\tau, z_\tau \in S^c_\Psi \Rightarrow y_\tau \in S^c_\Psi, \]

\[ y(t) := x(t) + z(t), \ y_\tau(k) = x_\tau(k) + z_\tau(k). \]

The sum of the quasi-periodic (quasi-aperiodic) states or zero is quasi-periodic (quasi-aperiodic) or zero.

Using (2), it is proved that

\[ x_\tau \in S^c_\Delta \Leftrightarrow \mathcal{F}[x_\tau(k)] = \begin{cases} \sum_{k=-\infty}^{\infty} x_\tau(k)e^{-j\omega k} & \text{if } |\omega| \leq \rho, \\ 0 & \text{if } \rho < |\omega| \end{cases} \]

\[ x_\tau \in S^c_\Psi \Leftrightarrow \mathcal{F}[x_\tau(k)] = \begin{cases} 0 & \text{if } |\omega| \leq \rho, \\ \sum_{k=-\infty}^{\infty} x_\tau(k)e^{-j\omega k} & \text{if } \rho < |\omega| \end{cases} \]

\[ z_\tau \in S^c_\Delta \Leftrightarrow \mathcal{F}[z_\tau(k)] = \begin{cases} \sum_{k=-\infty}^{\infty} z_\tau(k)e^{-j\omega k} & \text{if } |\omega| \leq \rho, \\ 0 & \text{if } \rho < |\omega| \end{cases} \]
Therefore, the product of any value \( a \in \mathbb{R} \) and the quasi-periodic (quasi-aperiodic) state or zero is quasi-periodic (quasi-aperiodic) or zero.

\[
\begin{align*}
& a \in \mathbb{R} \rightarrow y_t \in \mathbb{S}_a^c, \\
& x_t, z_t \in \mathbb{S}_a^c \Rightarrow y_t \in \mathbb{S}_a^c, \\
& x_t, z_t \in \mathbb{S}_p^c \Rightarrow y_t \in \mathbb{S}_p^c.
\end{align*}
\]

Thus, the sum of the quasi-periodic (quasi-aperiodic) states or zero is quasi-periodic (quasi-aperiodic) or zero. This implies that the sum of the quasi-periodic (quasi-aperiodic) states is quasi-periodic (quasi-aperiodic) or zero, and it is trivial that the sum of the quasi-periodic (quasi-aperiodic) state and zero is quasi-periodic (quasi-aperiodic).

Theorem 3
The periodic-pass function \( f_p(x(t)) \) and aperiodic-pass function \( f_a(x(t)) \) are linear

\[
\begin{align*}
f_p(x(t) + z(t)) &= f_p(x(t)) + f_p(z(t)), \quad \forall x(t), \ z(t) \in \mathbb{R}, \\
f_a(x(t) + z(t)) &= f_a(x(t)) + f_a(z(t)), \quad \forall x(t), \ z(t) \in \mathbb{R}, \\
f_p(ax(t)) &= af_p(x(t)), \quad \forall x(t), \ z(t) \in \mathbb{R}, \\
f_a(ax(t)) &= af_a(x(t)), \quad \forall x(t), \ z(t) \in \mathbb{R}.
\end{align*}
\]

Proof. The sum of the states \( x(t) \) and \( z(t) \) can be expressed by (6) as

\[
x(t) + z(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t),
\]

\[
y_1(t) := \begin{cases} x_p(t) & \text{if } x_t \in \mathbb{S}_p^c, \\
0 & \text{if } x_t \in \mathbb{S}_a^c,
\end{cases} \quad y_2(t) := \begin{cases} x_a(t) & \text{if } x_t \in \mathbb{S}_a^c, \\
0 & \text{if } x_t \in \mathbb{S}_a^c.
\end{cases}
\]

\[
y_3(t) := \begin{cases} z_p(t) & \text{if } z_t \in \mathbb{S}_p^c, \\
0 & \text{if } z_t \in \mathbb{S}_a^c,
\end{cases} \quad y_4(t) := \begin{cases} z_a(t) & \text{if } z_t \in \mathbb{S}_a^c, \\
0 & \text{if } z_t \in \mathbb{S}_a^c.
\end{cases}
\]

Hence,

\[
y_{1t} \in \mathbb{S}_a^c, \; y_{2t} \in \mathbb{S}_p^c, \; y_{3t} \in \mathbb{S}_a^c, \; y_{4t} \in \mathbb{S}_p^c,
\]

and Theorem 1 gives

\[(L(y_1(t) + y_3(t)))_{k \in \mathbb{Z}} \in \mathbb{S}_a^c, \quad (L(y_2(t) + y_4(t)))_{k \in \mathbb{Z}} \in \mathbb{S}_p^c.
\]

Then, the periodic-pass and aperiodic-pass functions \( f_p \) and \( f_a \) output

\[
f_p(x(t) + z(t)) = y_1(t) + y_3(t),
\]

\[
f_a(x(t) + z(t)) = y_2(t) + y_4(t).
\]

Using

\[
f_p(x(t)) = y_1(t), \quad f_a(x(t)) = y_2(t),
\]

\[
f_p(z(t)) = y_3(t), \quad f_a(z(t)) = y_4(t),
\]

based on (7a) and (7b), the additivity is obtained as

\[
f_p(x(t) + z(t)) = f_p(x(t)) + f_p(z(t)), \tag{8a}
\]

\[
f_a(x(t) + z(t)) = f_a(x(t)) + f_a(z(t)). \tag{8b}
\]

Next, according to (6),

\[
ax(t) = ay_1(t) + ay_2(t).
\]
Theorem 2 gives
\[
(L(ax_1(t)))_{k \in \mathbb{Z}} \in \mathbb{S}_h^c, \quad (L(ax_2(t)))_{k \in \mathbb{Z}} \in \mathbb{S}_p^c;
\]
hence,
\[
f_p(ax(t)) = ay_1(t), \quad f_a(ax(t)) = ay_2(t).
\]
Then, using
\[
f_p(x(t)) = y_1(t), \quad f_a(x(t)) = y_2(t),
\]
based on (7), the homogeneity is obtained as
\[
f_p(ax(t)) = af_p(x(t)), 
\quad f_a(ax(t)) = af_a(x(t)). \tag{9a} \tag{9b}
\]
(8) and (9) prove that the periodic-pass function \(f_p(x(t))\) and aperiodic-pass function \(f_a(x(t))\) are linear.

\section*{Theorem 4}

The quasi-periodic-state function \(x_p\) and quasi-aperiodic-state function \(x_a\) are orthogonal to each other
\[
\sum_{t=-\infty}^{\infty} x_p(t)x_a(t) = 0. \tag{10}
\]

\section*{Proof.}

\[
\sum_{t=-\infty}^{\infty} x_p(t)x_a(t) = \sum_{\tau=0}^{\Pi-1} \sum_{k=-\infty}^{\infty} x_{\tau p}(k)x_{\tau a}(k),
\]
where
\[
\sum_{k=-\infty}^{\infty} x_{\tau p}(k)x_{\tau a}(k) = \frac{1}{2} \sum_{k=-\infty}^{\infty} (x_{\tau p}(k) + x_{\tau a}(k))^2 - \frac{1}{2} \sum_{k=-\infty}^{\infty} x_{\tau p}^2(k) - \frac{1}{2} \sum_{k=-\infty}^{\infty} x_{\tau a}^2(k). \tag{11}
\]
The Parseval’s theorem
\[
\sum_{k=-\infty}^{\infty} |x_{\tau}(k)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X_{\tau}(\omega)|^2 d\omega
\]
rewrites the terms in the discrete-time domain into
\[
\sum_{k=-\infty}^{\infty} (x_{\tau p}(k) + x_{\tau a}(k))^2
\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} (X_{\tau p}(\omega) + X_{\tau a}(\omega))^2 d\omega, \tag{12a}
\]
in the frequency domain. According to (5), the lifted quasi-periodic-state and quasi-aperiodic-state functions satisfy \(x_{\tau p} \in \mathbb{S}_h^c\) and \(x_{\tau a} \in \mathbb{S}_p^c\), respectively. Because of Lemma 1, \(x_{\tau p} \in \mathbb{S}_h^c\), and \(x_{\tau a} \in \mathbb{S}_p^c\), the Fourier transformed quasi-periodic and quasi-aperiodic states satisfy
\[
X_{\tau p}(\omega) = \begin{cases} X_{\tau p}(\omega) & \text{if } |\omega| \leq \rho \\ 0 & \text{if } \rho < |\omega| \end{cases},
\quad X_{\tau a}(\omega) = \begin{cases} 0 & \text{if } |\omega| \leq \rho \\ X_{\tau a}(\omega) & \text{if } \rho < |\omega| \end{cases}.
\]
They calculate (12) as follows:
\[
\sum_{k=-\infty}^{\infty} x_{\tau p}^2(k) + x_{\tau a}^2(k) = \frac{1}{2\pi} \int_{-\rho}^{\rho} X_{\tau p}^2(\omega) d\omega + \frac{1}{2\pi} \int_{-\rho}^{\rho} X_{\tau a}^2(\omega) d\omega,
\quad \sum_{k=-\infty}^{\infty} x_{\tau p}^2(k) + x_{\tau a}^2(k) = \frac{1}{2\pi} \int_{-\rho}^{\rho} X_{\tau p}^2(\omega) d\omega + \frac{1}{2\pi} \int_{-\rho}^{\rho} X_{\tau a}^2(\omega) d\omega.
\]
The terms and (11) result in the orthogonality of the lifted quasi-periodic-state function \(x_{\tau p}\) and lifted quasi-aperiodic-state function \(x_{\tau a}\)
\[
\sum_{k=-\infty}^{\infty} x_{\tau p}(k)x_{\tau a}(k) = 0.
\]
This orthogonality and (10) yield the orthogonality of the quasi-periodic-state function \(x_p\) and quasi-aperiodic-state function \(x_a\)
\[
\sum_{t=-\infty}^{\infty} x_p(t)x_a(t) = 0. \tag{12b} \tag{12c}
\]

\section*{Proposition 1}

The output of the state \(x(t)\) by the periodic-pass function \(f_p\) and aperiodic-pass function \(f_a\) is zero
\[
f_a(f_p(x(t))) = f_p(f_a(x(t))) = 0.
\]
Fig. 3. Realization flow of the PASF from the periodic-pass and aperiodic-pass functions to the periodic-pass and aperiodic-pass filters.

**Proof.** According to (7a) and (7b),

\[ f_p(x(t)) \in S_h^c, \ f_a(x(t)) \in S_p^c, \]

and

\[ f_a(f_p(x(t))) = f_p(f_a(x(t))) = 0. \]

3 Realization of Periodic/Aperiodic Separation Filter

3.1 Realization Framework

Based on the definitions of the quasi-periodic and quasi-aperiodic states, this paper constructs causal linear time-invariant periodic-pass and aperiodic-pass filters that are an approximate realization of the periodic-pass function \( f_p \) in (7a) and aperiodic-pass function \( f_a \) in (7b). The PASF, which comprises the periodic-pass and aperiodic-pass filters, provides separated quasi-periodic state \( \tilde{x}_p(t) \) and quasi-aperiodic state \( \tilde{x}_a(t) \) from the periodic/aperiodic state \( x_{pa}(t) \). As shown in Fig. 3, the PASF is realized by lifting the periodic-pass and aperiodic-pass functions, realizing the ideal low-pass and high-pass filters as the lifted periodic-pass and aperiodic-pass filters, and inverse lifting the lifted filters into the periodic-pass and aperiodic-pass filters. For the realization, this paper introduces two \( Z \)-transforms with \( z \) and \( Z \), which are related as Proposition 2.

**Proposition 2**

The \( z \)-transform and inverse \( z \)-transform for the state \( x(t) \):

\[ \hat{x}(t) := X(z^{-1}) := \sum_{t=-\infty}^{\infty} x(t)z^{-t}, \]

\[ \hat{x}^{-1}[X(z^{-1})] := \frac{1}{2\pi j} \int_C X(z^{-1})z^{t-1}dz = x(t), \]

and the \( Z \)-transform and inverse \( Z \)-transform for the lifted state \( x_r(k) \):

\[ \mathcal{Z}[x_r(k)] := X_r(Z^{-1}) := \sum_{k=-\infty}^{\infty} x_r(k)Z^{-k}, \]

\[ \mathcal{Z}^{-1}[X_r(Z^{-1})] := \frac{1}{2\pi j} \int_C X_r(Z^{-1})Z^{k-1}dZ = x_r(k), \]

are related as

\[ L^{-1}(\mathcal{Z}^{-1}[Z^{-1}X_r(Z^{-1})]) = z^{-1}[\hat{x}^{-1}[X(z^{-1})]] = x(t - \Pi). \]

**Proof.** The inverse \( Z \)-transform of \( Z^{-1}X_r(Z^{-1}) \) is

\[ \mathcal{Z}^{-1}[Z^{-1}X_r(Z^{-1})] = x_r(k - 1), \]

which is inverse lifted by (1) as

\[ L^{-1}(x_r(k - 1)) = x((k - 1)\Pi + \tau) = x(t - \Pi). \]

Additionally, the inverse \( z \)-transform of \( z^{-1}X(z^{-1}) \) is

\[ z^{-1}[\hat{x}^{-1}[X(z^{-1})]] = x(t - \Pi). \]

In this paper, the lifted periodic-pass and aperiodic-pass functions are realized by causal linear filters as lifted periodic-pass and aperiodic-pass filters

\[ \hat{x}(t) = -\sum_{i=1}^{N} a_i \hat{x}(t - i) + \sum_{i=0}^{N} b_i x_{pa}(k - i), \]

\[ \hat{x}_a(t) = -\sum_{i=1}^{N} c_i \hat{x}_a(t - i) + \sum_{i=0}^{N} d_i x_{pa}(k - i), \]

\[ a_i, \ b_i, \ c_i, \ d_i \in \mathbb{R}, \ N \in \mathbb{Z}_{>0}, \]

respectively. Then, the filters are \( Z \)-transformed with respect to \( k \) into

\[ \hat{X}(Z^{-1}) = F(Z^{-1})X_{pa}(Z^{-1}), \]

\[ \hat{X}_a(Z^{-1}) = F_a(Z^{-1})X_{pa}(Z^{-1}), \]
The lifted periodic-pass and aperiodic-pass filters can be utilized to construct IIR periodic-pass and aperiodic-pass filters. According to Proposition 2, the inverse Z-transform and inverse lifting function $L^{-1}$ derive the periodic-pass and aperiodic-pass filters:

$$F_{\text{p}}(Z^{-1}) := \frac{b_0 + b_1 Z^{-1} + \ldots + b_N Z^{-N}}{1 + a_1 Z^{-1} + \ldots + a_N Z^{-N}},$$

$$F_{\text{a}}(Z^{-1}) := \frac{d_0 + d_1 Z^{-1} + \ldots + d_N Z^{-N}}{1 + c_1 Z^{-1} + \ldots + c_N Z^{-N}}.$$

The coefficients $a_i$ and $b_i$ are determined to construct $F_{\text{p}}(Z^{-1})$ to be a low-pass filter; $c_i$ and $d_i$ are determined to construct $F_{\text{a}}(Z^{-1})$ to be a high-pass filter. The limitation of the realization is that the filters cannot ideally realize the periodic-pass and aperiodic-pass functions owing to the impossibility of realizing ideal and causal low-pass and high-pass filters. This realization error results in the interference of the periodic and aperiodic states between the separated quasi-periodic and quasi-aperiodic states.

$$\tilde{x}_p(t) = -\sum_{i=1}^{N} a_i \tilde{x}_p(t - i\Pi) + \sum_{i=0}^{N} b_i x_{pa}(t - i\Pi),$$

$$\tilde{x}_a(t) = -\sum_{i=1}^{N} c_i \tilde{x}_a(t - i\Pi) + \sum_{i=0}^{N} d_i x_{pa}(t - i\Pi),$$

which are the PASF. Additionally, the $z$-transformed periodic-pass filter $F_p(z^{-1})$ and aperiodic-pass filter $F_a(z^{-1})$ are

$$\tilde{X}_p(z^{-1}) = F_p(z^{-1}) X_{pa}(z^{-1}),$$

$$\tilde{X}_a(z^{-1}) = F_a(z^{-1}) X_{pa}(z^{-1}),$$

$$F_p(z^{-1}) = \frac{b_0 + b_1 z^{-\Pi} + \ldots + b_N z^{-N\Pi}}{1 + a_1 z^{-\Pi} + \ldots + a_N z^{-N\Pi}},$$

$$F_a(z^{-1}) = \frac{d_0 + d_1 z^{-\Pi} + \ldots + d_N z^{-N\Pi}}{1 + c_1 z^{-\Pi} + \ldots + c_N z^{-N\Pi}}.$$

The limitation of the realization is that the filters cannot ideally realize the periodic-pass and aperiodic-pass functions owing to the impossibility of realizing ideal and causal low-pass and high-pass filters. This realization error results in the interference of the periodic and aperiodic states between the separated quasi-periodic and quasi-aperiodic states.

### 3.2 IIR Realization

Consider IIR periodic-pass and aperiodic-pass filters. The lifted periodic-pass and aperiodic-pass filters can utilize $N$th-order IIR low-pass and high-pass filters as

$$F_{\text{p}}(Z^{-1}) = \left(\frac{\rho}{\tilde{s} + \rho}\right)^N, \quad F_{\text{a}}(Z^{-1}) = \left(\frac{\tilde{s}}{\tilde{s} + \rho}\right)^N,$$

where $\tilde{s}$ is the $Z$-transformed approximate representation of the Laplace operator by the bilinear transform with $Z$:

$$\tilde{s} := \frac{2}{\Pi T} \frac{1 - Z^{-1}}{1 + Z^{-1}}.$$

Fig. 4 shows the Bode plots of the $N$th-order IIR periodic-pass filter $F_p(z^{-1})$ and aperiodic-pass filter $F_a(z^{-1})$, where the increase in the order deepens the band-stop characteristics. Moreover, Fig. 5 shows the filters with variations in the separation frequency $\rho$ in addition to the variations in the order $N$, where the increase in the separation frequency $\rho$ extends the band-pass bandwidth of the periodic-pass filter $F_p(z^{-1})$. The
3.3 FIR Realization

Consider FIR periodic-pass and aperiodic-pass filters. In contrast to the IIR filters, the FIR filters set $a_i$ and $c_i$ of the denominator polynomials to zero; hence, the FIR filters are inherently stable. This study designed three equi-ripple FIR low-pass and high-pass filters using the Parks-McClellan algorithm for the lifted periodic-pass filter $F_{p}(z^{-1})$ and the lifted aperiodic-pass filter $F_{a}(z^{-1})$, where MATLAB function firceqrip() was used to calculate the coefficients $b_i$ and $d_i$ of the 20th, 30th, and 50th FIR low-pass and high-pass filters. Fig. 6 shows the Bode plots of the Nth-order FIR periodic-pass filter $F_{p}(z^{-1})$ and aperiodic-pass filter $F_{a}(z^{-1})$, where the slope increases as the order increases. The filter gain has a steeper slope with a higher order than those of the IIR filters.

3.4 Complementary Realization

The complementary realization designs the aperiodic-pass filter to be a complementary filter of the periodic-pass filter as

$$ F_a(z^{-1}) = 1 - F_p(z^{-1}). $$

The first-order IIR periodic-pass and aperiodic-pass filters based on (13a) and (13b) are the complementary filters as

$$ F_p(z^{-1}) = \frac{\tilde{\rho}IT(1 + z^{-\Pi})}{2(1 - z^{-\Pi}) + \rho IT(1 + z^{-\Pi})}, $$

$$ F_a(z^{-1}) = \frac{2(1 - z^{-\Pi})}{2(1 - z^{-\Pi}) + \rho IT(1 + z^{-\Pi})}. $$

The complementary realization can improve the phase lag of the FIR aperiodic-pass function. Compared to the phase of the FIR aperiodic-pass filters in Fig. 6, that of the complementary FIR aperiodic-pass filters is zero at the band-pass frequencies, as shown in Fig. 7.

4 Periodic/Aperiodic Separation Filter with Kalman Filter

4.1 Preliminaries

Consider an observable linear time-invariant system:

$$ x_{pa}(t + 1) = Ax_{pa}(t) + Bu(t) + v(t), $$

$$ y(t) = Cx_{pa}(t) + w(t), $$

$$ E[v(t)] = 0, \quad E[w(t)] = 0, $$

$$ Q := E[w(t)w^T(t)], \quad R := E[w(t)w^T(t)], $$

$$ x_{pa}(t), \quad v(t), \quad u(t) \in \mathbb{R}^n, \quad y(t), \quad w(t) \in \mathbb{R}^m, $$

$$ A, Q, R \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times p}, \quad C \in \mathbb{R}^{m \times n}, $$

the Kalman filter:

$$ \dot{x}_{pa}(t|t-1) = A\hat{x}_{pa}(t-1|t-1) + Bu(t-1), $$

$$ P(t|t-1) = AP(t-1|t-1)A^T + Q, $$

$$ g(t) = P(t|t-1)C^T[CP(t|t-1)C^T + R]^{-1}, $$

$$ \hat{x}_{pa}(t|t) = \hat{x}_{pa}(t|t-1) + g(t)[y(t) - C\hat{x}_{pa}(t|t-1)], $$

where $\hat{x}_{pa}(t|t-1)$ is the state estimate of the aperiodic pass filter at time $t$, and $\hat{x}_{pa}(t|t)$ is its updated estimate at time $t$. The matrices $A$, $B$, $C$, $Q$, and $R$ represent the state transition, input, output, process noise, and measurement noise covariance matrices, respectively. The Kalman gain $g(t)$ is used to weight the difference between the actual measurement and the prediction made by the model. The time update equation updates the state estimate and uncertainty at time $t$ using the previous state estimate and the process noise. The measurement update equation corrects the state estimate and uncertainty at time $t$ using the measurement at time $t$. The Kalman filter is widely used in control systems, navigation, and signal processing to estimate the state of a system accurately.
\[ P(t|t) = (I - g(t)C)P(t|t-1), \]
\[ \hat{x}_{pa}(t) \in \mathbb{R}^n, \quad P(t|t) \in \mathbb{R}^n, \quad g(t) \in \mathbb{R}^{n \times m}, \]
and the PASF:
\[ \dot{x}_p(t) = \dot{\theta}_p(t) - \Pi + S_p \dot{x}_{pa}(t), \]
\[ \dot{x}_a(t) = \dot{\theta}_a(t) - \Pi + S_a \dot{x}_{pa}(t), \]
\[ \dot{\theta}_p(t) = \dot{\theta}_p(t) - \Pi + (G_p \dot{x}_p(t) - H_p \dot{x}_{pa}(t)), \]
\[ \dot{\theta}_a(t) = \dot{\theta}_a(t) - \Pi + (G_a \dot{x}_a(t) - H_a \dot{x}_{pa}(t)), \]
where
\[ \dot{\theta}_p(t) = \theta_p(t), \quad \dot{\theta}_a(t) = \theta_a(t), \quad \theta_p(t), \theta_a(t) \in \mathbb{R}^n, \]

Assumption 1

The PASF in (16) satisfies (7a) and (7b) as
\[ \dot{x}_p(t) = x_p(t), \quad \dot{x}_a(t) = x_a(t), \quad x_p(t), \quad x_a(t) \in \mathbb{R}^n, \]
\[ \text{which gives} \]
\[ \dot{\theta}_p(t) = \theta_p(t), \quad \dot{\theta}_a(t) = \theta_a(t), \quad \theta_p(t), \theta_a(t) \in \mathbb{R}^n, \]
\[ \text{where} \]
\[ \theta_p(t - \Pi) := \sum_{i=1}^{N} G_{pi} x_p(t - i\Pi) + H_{pi} x_{pa}(t - i\Pi), \]
\[ \theta_a(t - \Pi) := \sum_{i=1}^{N} G_{ai} x_a(t - i\Pi) + H_{ai} x_{pa}(t - i\Pi). \]

If Assumption 1 holds, the expectation of the product of the periodic-estimation and aperiodic-estimation errors is zero based on Theorem 4
\[ E[e_p(t|t)e_a^T(t|t)] = \sum_{t=-\infty}^{\infty} e_p(t|t)e_a^T(t|t) = 0. \]

4.2 Algorithm, Unbiased Estimation, and Equivalent Sum of Covariances

The Kalman filter in (15) and the PASF in (16) are integrated into the KF-PASF in Algorithm 1, where the predicted quasi-periodic state \( \dot{x}_p(t - 1) \), predicted quasi-aperiodic state \( \dot{x}_a(t - 1) \), and updated quasi-periodic state \( \dot{x}_p(t) \), and updated quasi-aperiodic state \( \dot{x}_a(t) \) are obtained as
\[ \dot{x}_p(t-1) = \dot{\theta}_p(t - \Pi|t - \Pi) + S_p \dot{x}_{pa}(t-1), \]
\[ \dot{x}_a(t-1) = \dot{\theta}_a(t - \Pi|t - \Pi) + S_a \dot{x}_{pa}(t-1), \]
\[ \dot{x}_p(t) = \dot{\theta}_p(t - \Pi|t - \Pi) + S_p \dot{x}_{pa}(t), \]
\[ \dot{x}_a(t) = \dot{\theta}_a(t - \Pi|t - \Pi) + S_a \dot{x}_{pa}(t). \]
\[ \text{where} \]
\[ \dot{\theta}_p(t - \Pi|t - \Pi) := \sum_{i=1}^{N} [G_{pi} \dot{x}_p(t - i\Pi|t - i\Pi) + H_{pi} \dot{x}_{pa}(t - i\Pi|t - i\Pi)], \]
\[ \dot{\theta}_a(t - \Pi|t - \Pi) := \sum_{i=1}^{N} [G_{ai} \dot{x}_a(t - i\Pi|t - i\Pi) + H_{ai} \dot{x}_{pa}(t - i\Pi|t - i\Pi)]. \]

The estimation errors of \( \dot{\theta}_p(t - \Pi|t - \Pi) \) and \( \dot{\theta}_a(t - \Pi|t - \Pi) \) are respectively defined as
\[ e_p(t - \Pi|t - \Pi) := \theta_p(t - \Pi) - \dot{\theta}_p(t - \Pi|t - \Pi), \]
\[ e_a(t - \Pi|t - \Pi) := \theta_a(t - \Pi) - \dot{\theta}_a(t - \Pi|t - \Pi). \]
Theorem 5

Assume Assumption 1 holds. The estimation error $e(t|t)$, periodic-estimation error $e_p(t|t)$, and aperiodic-estimation error $e_a(t|t)$ of the KF-PASF in Algorithm 1 are unbiased if the initial errors are unbiased.

$$E[e(-n|−n)] = E[e_p(-n|−n)] = E[e_a(-n|−n)] = 0,$$

$$n = 0, 1, \ldots, N_1 − 1,$$

$$\Rightarrow \forall t \in Z > 0, E[e(t|t)] = E[e_p(t|t)] = E[e_a(t|t)] = 0.$$

Proof. The dynamics of the expectation of the estimation error are

$$E[e(t|t)] = (I - g(t)C)AE[e(t−1|t−1)],$$

which can be derived from (14), (15), and (17). The PASF for the periodic/aperiodic state in (16), the PASF for the estimated periodic/aperiodic state in (19), and Assumption 1 give the periodic-estimation error dynamics and aperiodic-estimation error dynamics as

$$e_p(t|t) = e_p(t − \Pi|t − \Pi) + S_pe(t|t),$$

$$e_a(t|t) = e_a(t − \Pi|t − \Pi) + S_ae(t|t),$$

$$e_p(t|t) + e_a(t|t) = e(t|t) = e_p(t − \Pi|t − \Pi) + S_pe(t|t) + S_ae(t|t).$$

The expectation of the periodic-estimation error $E[e_p(t|t)]$ and that of the aperiodic-estimation error $E[e_a(t|t)]$ become zero as

$$E[e_p(t − \Pi|t − \Pi)] = E[e_a(t − \Pi|t − \Pi)] = 0$$

$$\Rightarrow E[e(t|t)] = E[e_p(t|t)] = E[e_a(t|t)] = 0.$$

Using $E[e(t−1|t−1)] = 0 \Rightarrow E[e(t|t)] = 0$ based on (20).

Additionally, the expectations of $e_p(t − \Pi|t − \Pi)$ and $e_a(t − \Pi|t − \Pi)$ become zero as

$$E[e(t−1|t−1)] = E[e_p(t−1|t−1) − e(t−1|t−1)] = E[e_a(t−1|t−1) − e(t−1|t−1)] = 0$$

$$\Rightarrow E[e_p(t − \Pi|t − \Pi)] = E[e_a(t − \Pi|t − \Pi)] = 0.$$

These yield

$$E[e(t−1|t−1)] = E[e_p(t−1|t−1)] = E[e_a(t−1|t−1)] = 0$$

$$\Rightarrow E[e_p(t|t)] = E[e_a(t|t)] = 0,$$

which and $E[e(t−1|t−1)] = 0 \Rightarrow E[e(t|t)] = 0$ derive

$$E[e(-n|−n)] = E[e_p(-n|−n)] = E[e_a(-n|−n)] = 0$$

$$\Rightarrow \forall t \in Z > 0, E[e(t|t)] = E[e_p(t|t)] = E[e_a(t|t)] = 0.$$

Thus, the estimation of Algorithm 1 is unbiased as

$$E[e(-n|−n)] = E[e_p(-n|−n)] = E[e_a(-n|−n)] = 0$$

$$\Rightarrow \forall t \in Z > 0, E[e(t|t)] = E[e_p(t|t)] = E[e_a(t|t)] = 0.$$
trix $P_a(t|t)$

$$P(t|t) = P_p(t|t) + P_a(t|t), \quad P_p(t|t), P_a(t|t) \in \mathbb{R}^{n \times n}$$

$$P_p(t|t) := E[e_p(t|t)e_p^T(t|t)], \quad P_a(t|t) := E[e_a(t|t)e_a^T(t|t)].$$

**Proof.**

$$P(t|t) = E[e(t|t)e^T(t|t)],$$

$$= E[(e_p(t|t) + e_a(t|t))(e_p(t|t) + e_a(t|t))^T].$$

According to (18),

$$P(t|t) = E[e_p(t|t)e_p^T(t|t)] + E[e_a(t|t)e_a^T(t|t)],$$

$$= P_p(t|t) + P_a(t|t).$$

The KF-PASF minimizes the sum of the periodic-error and aperiodic-error covariances $P_p(t|t) + P_a(t|t)$ by minimizing the error covariance $P(t|t)$ with the algorithm of the Kalman filter.

5 Examples

This section shows estimation, comparison, and control examples. The examples shown in Sections 5.1, 5.2, and 5.4 used the system:

$$x_{pa}(t+1) = Ax_{pa}(t) + Bu(t) + v(t),$$

$$y(t) = Cx_{pa}(t) + w(t),$$

$$x_{pa}(t) = [x_1(t) \ x_2(t) \ x_3(t)]^T, \quad B = [0 \ 0 \ 1]^T, \quad C = [1 \ 0 \ 0], \quad v \sim \mathcal{N}(0,10^{-5}), \quad w \sim \mathcal{N}(0,0.5).$$

The PASF used by the examples was the first-order IIR $(N=1)$, second-order IIR $(N=2)$, third-order IIR $(N=3)$, or 50th-order FIR periodic-pass and aperiodic-pass filters. The parameters of the IIR filters were as follows

$$T = 1 \text{ ms}, \quad \Pi = 1000,$$

$$P(0|0) = 0, \quad Q = \text{diag}(0,0,10^{-8}), \quad R = 0.25,$$

$$G_{p1} = G_{a1} = -\frac{\rho (\Pi T - 2)}{\rho \Pi + 2}, \quad H_{p1} = \frac{\rho (\Pi T - 2)}{\rho \Pi + 2}, \quad H_{a1} = -\frac{2}{\rho \Pi + 2},$$

$$S_p = \frac{\rho (\Pi T - 2)}{\rho \Pi + 2}, \quad S_a = \frac{2}{\rho \Pi + 2},$$

$$N = 1 :$$

$$G_{p1} = G_{a1} = -\frac{\rho (\Pi T - 2)}{\rho \Pi + 2}, \quad H_{p1} = \frac{\rho (\Pi T - 2)}{\rho \Pi + 2}, \quad H_{a1} = -\frac{2}{\rho \Pi + 2},$$

$$S_p = \frac{\rho (\Pi T - 2)}{\rho \Pi + 2}, \quad S_a = \frac{2}{\rho \Pi + 2},$$

$$N = 2 :$$

$$G_{p1} = G_{a1} = -\frac{2 \rho (\Pi T - 2)}{\rho (\Pi T + 2)}, \quad H_{p2} = \frac{\rho (\Pi T - 2)}{\rho (\Pi T + 2)},$$

$$H_{a1} = -\frac{\rho (\Pi T - 2)}{\rho (\Pi T + 2)}, \quad H_{a2} = -\frac{2}{\rho (\Pi T + 2)},$$

$$S_p = \frac{\rho (\Pi T - 2)}{\rho (\Pi T + 2)}, \quad S_a = \frac{2}{\rho (\Pi T + 2)},$$

$$N = 3 :$$

$$G_{p1} = G_{a1} = -\frac{3 \rho (\Pi T - 2)}{\rho (\Pi T + 2)}, \quad G_{p2} = G_{a2} = -\frac{3 \rho (\Pi T - 2)^2}{\rho (\Pi T + 2)^2},$$

$$G_{p3} = G_{a3} = -\frac{3 \rho (\Pi T - 2)^3}{\rho (\Pi T + 2)^3},$$

$$H_{p1} = \frac{3 \rho (\Pi T)^3}{\rho (\Pi T + 2)^3}, \quad H_{p2} = \frac{3 \rho (\Pi T)^2}{\rho (\Pi T + 2)^3}, \quad H_{p3} = \frac{\rho (\Pi T)^3}{\rho (\Pi T + 2)^3},$$

$$H_{a1} = -\frac{24 \rho (\Pi T)^3}{\rho (\Pi T + 2)^3}, \quad H_{a2} = -\frac{8 \rho (\Pi T)^2}{\rho (\Pi T + 2)^3}, \quad H_{a3} = -\frac{\rho (\Pi T)^3}{\rho (\Pi T + 2)^3},$$

$$S_p = \frac{\rho (\Pi T)^2}{\rho (\Pi T + 2)^3}, \quad S_a = \frac{8 \rho (\Pi T + 2)^3}{\rho (\Pi T + 2)^3}.$$

The 50th-order FIR filter was designed using the MATLAB function firceqrip().

5.1 Estimation of Quasi-Periodic and Quasi-Aperiodic States with Separation Frequency Change

In this example, the KF-PASF in Algorithm 1, using the first-order IIR periodic-pass and aperiodic-pass filters, estimated the quasi-periodic and quasi-aperiodic states. The example employed the matrix, signals, and separation frequency

$$A = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ -2500 & -100 & 0 \end{bmatrix}, \quad u(t) = 2500(u_1(t) + u_2(t)),$$

$$\begin{cases} 10 \sum_{i=1}^{10} (-1)^i \sin((2i - 1)2\pi T) & \text{if } T t < 10 \text{ s} \\ -i & \text{if } 10 \text{ s} \leq T t < 40 \text{ s} \end{cases}$$

$$\begin{cases} \sum_{i=1}^{10} i^20.01 \sin(2i\pi T) & \text{if } 40 \text{ s} \leq T t < 80 \text{ s} \\ \sum_{i=1}^{10} \frac{1}{(2i - 1)} \sin((2i - 1)2\pi T) & \text{otherwise} \end{cases}$$

$$\begin{cases} 1 & \text{if } 25 \text{ s} \leq T t \leq 25.3 \text{ s} \\ \sqrt{70} & \text{if } 70 \text{ s} \leq T t \leq 70.3 \text{ s} \\ \sqrt{110} & \text{if } 110 \text{ s} \leq T t \leq 110.3 \text{ s} \\ 0 & \text{otherwise} \end{cases}$$

$$\rho = \begin{cases} 10 \text{ rad/s} & \text{if } T t < 40 \text{ s} \lor 80 \text{ s} \leq T t \leq 100 \text{ s} \\ 0.2 \text{ rad/s} & \text{if } 80 \text{ s} \leq T t \leq 80 \text{ s} \\ 0.01 \text{ rad/s} & \text{otherwise} \end{cases}$$

Fig. 8 shows the estimation result for $\dot{x}_{1p}(t|t)$ and $\dot{x}_{1a}(t|t)$, where the KF-PASF estimated the three different quasi-periodic states with the three separation frequencies. The large and small separation frequencies provided fast and slow convergences of the updated quasi-periodic and quasi-aperiodic states, respectively. Meanwhile, the large separation frequency provided the states that were much affected by $u_1(t)$ even while $u_2(t)$ became zero from one, and the small separation frequency provided the rigid separation that was not much affected by $u_2(t)$ while $u_2(t) = 0$. 

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5.2 Comparison of Realizations

This example compared the KF-PASFs based on the different realizations of the three IIR filters and an FIR filter with the matrix, signals, and the separation frequency

\[ A = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ -2500 & -100 & 0 \end{bmatrix} \], \quad u(t) = 2500(u_1(t) + u_2(t)), \]

\[ u_1(t) = 1 + \sum_{i=1}^{10} i^2 0.01 \sin(i 2 \pi T t), \]

\[ u_2(t) = \begin{cases} 2 & \text{if } 15 < t \leq 15.3 \\ 0 & \text{otherwise} \end{cases}, \quad \rho = 0.01 \text{ rad/s}. \]

Their initial updated states \( \hat{x}_{pa}(-n| -n), \hat{x}_p(-n| -n), \) and \( \hat{x}_a(-n| -n) \) were given beforehand.

Fig. 9 shows the estimation results for \( \hat{x}_{1p}(t|t) \) and \( \hat{x}_{1a}(t|t) \), where the output and actual state are depicted in (a) and the estimated states are depicted in (b)–(e). According to the estimated states, there was no significant difference among the four filters; however, the updated quasi-aperiodic states given by the IIR filters were slightly oscillatory in contrast to the state given by the FIR filter. Furthermore, (f) shows the four interferences that were the separated quasi-aperiodic states from the updated quasi-periodic states \( \hat{x}_{1p}(t|t) \). The increase in the order of the IIR filter reduced the interference, whereas the interference of the 50th-order FIR filter was larger than that of the third-order IIR filter.

5.3 Comparison of PASF with Comb Filters

This example compared the third-order IIR PASF with three conventional comb filters. The input periodic/aperiodic signal \( x_{pa} \) was

\[ x_{pa}(t) = x_p(t) + x_a(t), \quad \nu \sim \mathcal{N}(0, 0.01), \]

\[ x_p(t) = \begin{cases} \sin(4 \pi T t) & \text{if } (t \mod 500) < 250 \\ 0 & \text{otherwise} \end{cases}, \]

\[ x_a(t) = \begin{cases} 0.5 & \text{if } 5.125 s < T t \leq 5.135 s \\ \nu & \text{if } 10 s < T t \leq 12 s \\ 0 & \text{otherwise} \end{cases}, \]

and the separation frequency \( \rho \) of the third-order PASF was

\[ \rho = \begin{cases} 1000 \text{ rad/s} & \text{if } T t < 4 \text{ s} \\ 0.001 \text{ rad/s} & \text{otherwise} \end{cases}. \]

The comparison used the comb filter presented by Sugiuara et al. (2013)

\[ C_{\text{comb}}(z^{-1}) := 1 - \frac{(1 - g)(1 - b)}{2} \frac{1 + z^{-\Pi}}{1 - bz^{-\Pi}}, \]
which was designed as $C_{\text{comb}1}(z^{-1})$ with $b = 0$ and $g = 0$ and $C_{\text{comb}2}(z^{-1})$ with $b = 0.5$ and $g = 0$

$$
C_{\text{comb}1}(z^{-1}) := \frac{1 - z^{-2}}{2}, \quad C_{\text{comb}2}(z^{-1}) := \frac{3}{2} - \frac{z^{-2}}{2}.
$$

They are equivalent to the feedforward and feedback comb filters of Liu and Declercq (2017), respectively. Additionally, the comparison employed the comb filter presented by Aslan and Altintas (2018)

$$
C_{\text{comb}3}(z^{-1}) := \frac{\beta(1 - z^{-2})}{1 - \alpha z^{-2}},
$$

$$
\alpha := \frac{1 - \gamma}{1 + \gamma}, \quad \beta := \frac{1}{1 + \gamma}, \quad \gamma := \frac{\sqrt{1 - |G_{cb}|^2} \tan \frac{\pi}{2Q}}{2},
$$

where $|G_{cb}| = 0.708$ and

$$
Q = \begin{cases}
1.717 & \text{if } Tt < 4 \text{ s} \\
1591 & \text{otherwise}
\end{cases}.
$$

The comb filters are filters that eliminate harmonics; hence, this example regarded the outputs of the comb filters as quasi-aperiodic signals for the comparison. Furthermore, the difference between the input periodic/aperiodic signal and output was accordingly regarded as a quasi-periodic signal. The periodic-pass and aperiodic-pass filters of the third-order IIR PASF were $F_{p}(z^{-1})$ and $F_{a}(z^{-1})$ with $N = 3$ of (13), respectively. All initial states of the PASF and comb filters were set to zero.

Fig. 10 shows the comparative results. Fig. 10(a) depicts the input periodic/aperiodic signal $x_{\text{pa}}$, and the separated signals are depicted in (b)–(e). According to (b), (c), and (e), the third-order IIR PASF realized a more rigid separation than the comb filters $C_{\text{comb}1}(z^{-1})$ and $C_{\text{comb}2}(z^{-1})$. According to (d) and (e), the separation results of the third-order IIR PASF and the comb filter $C_{\text{comb}3}(z^{-1})$ were similar, but the interference, which was the separated quasi-aperiodic signal from the separated quasi-periodic signal, of the PASF was smaller than that of the comb filter, as shown in (f).

5.4 Periodic/Aperiodic Separation Control Based on KF-PASF

In this control example, the KF-PASF, using the first-order IIR periodic-pass and aperiodic-pass filters, was applied to the periodic/aperiodic separation control, which is the control of the quasi-periodic and quasi-aperiodic states. This control example employed

$$
A = \begin{bmatrix}
1 & T \\
0 & 1 \\
0 & 0
\end{bmatrix}, \quad \dot{\rho} = \begin{cases}
10 \text{ rad/s} & \text{if } Tt < 20 \text{ s} \\
0.01 \text{ rad/s} & \text{otherwise}
\end{cases},
$$

$$
x_{\text{p}}^{\text{cmd}}(t) := 2 + \sum_{i=1}^{10} \frac{1}{2i-1} \sin((2i-1)2\pi Tt),
$$

$$
x_{\text{a}}^{\text{cmd}}(t) := \begin{cases}
\sin(\pi(Tt - 25)) & \text{if } 25 < Tt \leq 26 \text{ s} \\
0 & \text{otherwise}
\end{cases},
$$

$$
u(t) = \begin{cases}
u_{p}(t) + u_{a}(t) & \text{if } Tt < 5 \text{ s} \\
u_{p}(t) := 900(x_{\text{p}}^{\text{cmd}}(t) - \hat{x}_{1p}(t|t)) + 60(\hat{x}_{1p}^{\text{cmd}}(t) - \hat{x}_{2p}(t|t)),
\end{cases}
$$

$$
u_{a}(t) := 2500(x_{\text{a}}^{\text{cmd}}(t) - \hat{x}_{1a}(t|t)) + 100(\hat{x}_{1a}^{\text{cmd}}(t) - \hat{x}_{2a}(t|t)).
$$

Fig. 11 shows the control results for $\hat{x}_{1p}(t|t)$ and $\hat{x}_{1a}(t|t)$, where the quasi-periodic and quasi-aperiodic states converged at their commands under the noise with different feedback gains for the states.

6 Conclusion

This paper defined periodic/aperiodic state composed of orthogonal quasi-periodic and quasi-aperiodic states,
which further defined linear periodic-pass and aperiodic-pass functions. It was demonstrated that the sum of the quasi-periodic (quasi-aperiodic) states or zero is quasi-periodic (quasi-aperiodic) or zero. Similarly, the product of any value and the quasi-periodic state (quasi-aperiodic) or zero is quasi-periodic (quasi-aperiodic) or zero. Moreover, based on the definitions, the functions were realized as causal linear periodic-pass and aperiodic-pass filters, which represent the PASF. The realized high-order IIR and FIR periodic-pass and aperiodic-pass filters enhanced the slope of the band-stop characteristics and reduced the interference between the separated quasi-periodic and quasi-aperiodic states. Additionally, the complementary realization eliminated the phase lag of the aperiodic-pass filter at the band-pass frequencies. Lastly, the KF-PASF that integrates the PASF and Kalman filter achieved the unbiased estimation of the quasi-periodic and quasi-aperiodic states with the minimum sum of the periodic-error and aperiodic-error covariances.

A limitation of this study is that the realization error between the periodic-pass and aperiodic-pass functions and the causal periodic-pass and aperiodic-pass filters is inevitable. Hence, in practical use, the quasi-periodic and quasi-aperiodic states are only almost separated and Assumption 1 is only almost satisfied. Nevertheless, this imperfect realization is similar to the imperfect realization of low-pass and high-pass filters; therefore, the PASF is sufficiently practical as well. Furthermore, the new definitions, linearity and orthogonality, high-order IIR and FIR realization of the PASF, and KF-PASF are expected to be the basis of future separation control studies on the quasi-periodic and quasi-aperiodic states.

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