EDGE IDEALS WITH ALMOST MAXIMAL FINITE INDEX AND THEIR POWERS

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Abstract. A graded ideal in \( \mathbb{K}[x_1, \ldots, x_n] \), \( \mathbb{K} \) a field, is said to have almost maximal finite index if all steps of its minimal free resolution are linear except for the last two steps. In this paper we classify the graphs whose edge ideals have this property. This in particular shows that for edge ideals, unlike the general case, the property of having almost maximal finite index does not depend on the characteristic of \( \mathbb{K} \). We also compute the non-linear Betti numbers of these ideals. Finally, we show that for the edge ideal \( I \) of a graph \( G \) with almost maximal finite index, the ideal \( I^s \) has a linear resolution for \( s \geq 2 \) if and only if the complementary graph \( \bar{G} \) does not contain induced cycles of length 4.

Introduction

In this paper, we consider the edge ideals whose minimal free resolution has relatively large number of linear steps. Let \( I \) be a graded ideal in the polynomial ring \( S = \mathbb{K}[x_1, \ldots, x_n] \), \( \mathbb{K} \) a field, generated by homogeneous polynomials of degree \( d \). The ideal is called \( r \)-steps linear, if \( I \) has a linear resolution up to the homological degree \( r \), that is the graded Betti numbers \( \beta_{i,i+j}(I) \) vanish for all \( i \leq r \) and all \( j > d \). The number

\[
\text{index}(I) = \inf\{r : I \text{ is not } r\text{-steps linear}\}
\]

is called the Green–Lazarsfeld index (or briefly index) of \( I \). A related invariant, called the \( N_{d,r} \)-property, was first considered by Green and Lazarsfeld in [16, 17]. In the paper [3] the index was introduced for the quotient ring \( S/I \), where \( I \) is generated by quadratics, to be the largest integer \( r \) such that the \( N_{2,r} \)-property holds. It is in general very hard to determine the value of the index. One reason is that this value, in general, depends on the characteristic of \( \mathbb{K} \). However, the index of quadratic monomial ideals is more studied in the literature taking advantage of some combinatorial methods. Indeed, since the index is preserved passing through polarization, one may reduce to the case of squarefree quadratic monomial ideals which can be viewed as the edge ideals of simple graphs, and the index of these ideals is proved to be characteristic independent, see [11, Theorem 2.1].

The main question regarding the study of the index of edge ideals is to classify the graphs with respect to the index of their edge ideals, in particular, it is more interesting to see when the index attains its largest or smallest value. In 1990, Fröberg [14] classified the graphs whose edge ideals have a linear resolution. A graded ideal \( I \) is said to have a linear resolution if \( \text{index}(I) = \infty \). In fact Fröberg showed that given a graph \( G \), its edge ideal \( I(G) \) has a linear resolution over all fields if and only if the complement \( \bar{G} \) of \( G \) is chordal, which means that all cycles in \( \bar{G} \) of length \( > 3 \) have a chord. In 2005, Eisenbud et al. gave a purely combinatorial description of the index of edge ideals in terms of the size of the smallest cycle(s) of length \( > 3 \) in the complementary graph, c.f. Theorem 1.1.
result shows that the index gets its smallest value 1 if and only if $G$ is gap-free, i.e. $G$ does not contain induced cycles of length 4. If the index of $I$ attains the largest finite value, we have $\text{index}(I) = \text{pd}(I)$, where $\text{pd}(I)$ denotes the projective dimension of $I$. In this case the ideal $I$ is said to have maximal finite index, see [2]. In [2, Theorem 4.1], it was shown that the edge ideal $I(G)$ has maximal finite index if and only if $G$ is an induced cycle of length $> 3$. In this paper, we proceed one more step and consider the edge ideals $I(G)$ with $\text{index}(I(G)) = \text{pd}(I(G)) - 1$. We call them edge ideals with almost maximal finite index. In Section 2 of this paper we precisely determine the simple graphs whose edge ideals have this property, see Theorem 2.6. These graphs are presented in Figures 1 to 4. In particular, it is deduced that the property of having almost maximal finite index is characteristic independent for edge ideals, although this is not the case for ideals generated in higher degrees, as discussed in the beginning of Section 2. It is also seen that the graded Betti numbers of these edge ideals do not depend on the characteristic of the base field.

We will compute the Betti numbers in the non-linear strands in Proposition 2.10. The main tool used throughout this section is Hochster's formula, Formula (1).

In the second half of the paper we study the index of powers of edge ideals with almost maximal finite index. Although, for arbitrary ideals, many properties such as depth, projective dimension or regularity stabilize for large powers (see e.g [1, 5, 6, 8, 9, 18, 20, 21, 22]), their initial behaviour is often quite mysterious. However, edge ideals behave more controllable from the beginning. In the study of the index of powers of edge ideals, one of the main results is due to Herzog, Hibi and Zheng [20, Theorem 3.2]. They showed that for a graph $G$, all powers of the edge ideal $I(G)$ have a linear resolution if and only if so does $I(G)$. On the other hand, it was shown in [2, Theorem 3.1] that all powers of $I(G)$ have index 1 if and only if $I(G)$ has also index 1. In the same paper it was proved that if $I(G)$ has maximal finite index $> 1$, then $I(G)^s$ has a linear resolution for all $s \geq 2$. This shows that chordality of the complement of $G$ is not a necessary condition on $G$ so that all high powers of its edge ideal have a linear resolution. Francisco, Há and Van Tuyl proved, in a personal communication, that being gap-free is a necessary condition for a graph $G$ in order that a power of its edge ideal has a linear resolution. However, Nevo and Peeva showed, by an example, that being gap-free alone is not a sufficient condition [25, Counterexample 1.10] so that all high powers of the edge ideal have a linear resolution. Later, Banerjee [1], and Erey [12] respectively proved that if a gap-free graph $G$ is also cricket-free or diamond-free, then the ideal $I(G)^s$ has a linear resolution for all $s \geq 2$. The definition of these concepts are recalled in Section 3.

Section 3 is devoted to answer the question whether the high powers of edge ideals with almost maximal finite index have a linear resolution. Not all graphs whose edge ideals have this property are cricket-free or diamond-free. However, using some formula for an upper bound of the regularity of powers of edge ideals offered in [1, Theorem 5.2], we give a positive answer to this question in case the graphs are gap-free, see Theorem 3.1.

Theorem 3.1 together with [2, Theorem 4.1] yield the following consequence which is a partial generalization of the result of Herzog et al. in [20, Theorem 3.2].

**Theorem 0.1.** Let $G$ be a simple gap-free graph and let $I \subset S$ be its edge ideal. Suppose $\text{pd}(I) - \text{index}(I) \leq 1$. Then $I^s$ has a linear resolution over all fields for any $s \geq 2$.

One may ask which is the largest integer $c$ such that Theorem 0.1 remains valid if one replaces $\text{pd}(I) - \text{index}(I) \leq 1$ by $\text{pd}(I) - \text{index}(I) \leq c$. Computation by Macaulay 2, [15], shows that in the example of Nevo and Peeva [25, Counterexample 1.10], $\text{index}(I) = 2$, and $\text{pd}(I) = 8$. Hence $c$ must be an integer with $1 \leq c \leq 5$. 
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1. Preliminaries

In this section we recall some concepts, definitions and results from Commutative Algebra and Combinatorics which will be used throughout the paper. Let \( S = \mathbb{K}[x_1, \ldots, x_n] \) be the polynomial ring over a field \( \mathbb{K} \) with \( n \) variables, and let \( M \) be a finitely generated graded \( S \)-module. Let the sequence
\[
0 \to F_0 \to F_1 \to F_2 \to \cdots \to F_d \to M \to 0
\]
be the minimal graded free resolution of \( M \), where for all \( i \geq 0 \) the modules \( F_i = \oplus_j S(-j)^{\beta_{i,j}^R(M)} \) are free \( S \)-modules of rank \( \beta_{i,j}^R(M) := \sum_j \beta_{i,j}^S(M) \). The numbers \( \beta_{i,j}^S(M) = \dim_{\mathbb{K}} \text{Tor}_j^R(M, \mathbb{K}) \) are called the \emph{graded Betti numbers} of \( M \) and \( \beta_i^2(M) \) is called the \( i \)-th \emph{Betti number} of \( M \). We write \( \beta_{i,j}(M) \) for \( \beta_{i,j}^S(M) \) when the field is fixed. The \emph{projective dimension} of \( M \), denoted by \( \text{pd}(M) \), is the largest \( i \) for which \( \beta_i(M) \neq 0 \). The \emph{Castelnuovo-Mumford regularity} of \( M \), \( \text{reg}(M) \), is defined to be
\[
\text{reg}(M) = \sup\{ j - i : \beta_{i,j}(M) \neq 0 \}.
\]

Let \( I \) be a graded ideal of \( S \) generated in a single degree \( d \). The \emph{Green–Lazarsfeld index} (briefly index) of \( I \), denoted by \( \text{index}(I) \), is defined to be
\[
\text{index}(I) = \inf\{ i : \beta_{i,j}(I) \neq 0, \text{ for some } j > i + d \}.
\]
Since \( \beta_{0,j}(I) = 0 \) for all \( j > d \), one always has \( \text{index}(I) \geq 1 \). The ideal \( I \) is said to have a \emph{\( d \)-linear resolution} if \( \text{index}(I) = \infty \). This means that for all \( i, \beta_i(I) = \beta_{i,i+d}(I) \), and this is the case if and only if \( \text{reg}(I) = \text{pd}(I) \). Otherwise \( \text{index}(I) \leq \text{pd}(I) \). In case \( I \) has the largest possible finite index, that is \( \text{index}(I) = \text{pd}(I) \), \( I \) is said to have \emph{maximal finite index}.

In Section 2 of this paper we deal with squarefree monomial ideals generated in degree 2. These ideals are the edge ideals of simple graphs. Recall that a \emph{simple} graph is a graph with no loops and no multiple edges, and given a graph \( G \) on the vertex set \( [n] := \{1, \ldots, n\} \), its edge ideal \( I(G) \subset S \) is an ideal generated by all quadratics \( x_ix_j \), where \( \{i, j\} \) is an edge in \( G \). We denote by \( E(G) \) the set of all edges of \( G \), and by \( V(G) \) the vertex set of \( G \). For a vertex \( v \in V(G) \), the neighbourhood \( N_G(v) \) of \( v \) in \( G \) is defined to be
\[
N_G(v) = \{ u \in V(G) : \{ u, v \} \in E(G) \}.
\]
The complement \( \bar{G} \) of \( G \) is a graph on \( V(G) \) whose edges are those pairs of \( V(G) \) which do not belong to \( E(G) \). The simplicial complex
\[
\Delta(G) = \{ F \subset V(G) : \text{ for all } \{i, j\} \subset F \text{ one has } \{i, j\} \in E(G) \}
\]
is called the \emph{flag complex} of \( G \). The \emph{independence complex} of \( G \) is the flag complex of \( G \). One can check that \( I(G) = I_{\Delta(\bar{G})} \), where \( I_{\Delta(\bar{G})} \) is the Stanley-Reisner ideal of \( \Delta(\bar{G}) \).

We assume that the reader is familiar with the definition and elementary properties of simplicial complexes. For more details consult with [19].

The main tool used widely in Section 2 for the computation of the graded Betti numbers is Hochster’s formula [19, Theorem 8.1.1]. Let \( \Delta \) be a simplicial complex on \( [n] \), and let
\( \tilde{C}(\Delta, K) \) be the augmented oriented chain complex of \( \Delta \) over a field \( K \) with the differentials
\[
\partial_i : \bigoplus_{F \in \Delta \atop \dim F = i} \mathbb{K} F \to \bigoplus_{G \in \Delta \atop \dim G = i-1} \mathbb{K} G,
\]
\[
\partial_i([v_0, \ldots, v_i]) = \sum_{0 \leq j \leq i} (-1)^j [v_0, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_i],
\]
where by \([v_0, v_1, \ldots, v_i]\) we mean the face \( \{v_0, v_1, \ldots, v_i\} \subseteq [n] \) of \( \Delta \) with \( v_0 < v_1 < \cdots < v_i \).

Hochster’s formula states that for the Stanley-Reisner ideal \( I := I_\Delta \subset S \) one has
\[
\beta_{i,j}(I) = \sum_{W \subseteq [n], |W| = j} \dim_{\mathbb{K}} \tilde{H}_{j-i-2}(\Delta_W; \mathbb{K}),
\]
where \( \Delta_W \) is the induced subcomplex of \( \Delta \) on \( W \) and \( \tilde{H}_{i}(\Delta_{W}; \mathbb{K}) \) is the \( i \)-th reduced homology of the complex \( \tilde{C}(\Delta_{W}, \mathbb{K}) \). We denote by \( \partial^W_i \) the differentials of the chain complex \( \tilde{C}(\Delta_{W}, \mathbb{K}) \).

Theorem 1.1 which is due to Eisenbud et al. \([11]\) provides a combinatorial method for determining the index of the edge ideal of a graph. To this end, one needs to consider the length of the minimal cycles of the complementary graph. A minimal cycle is an induced cycle of length \( > 3 \), and by an induced cycle we mean a cycle with no chord. The length of an induced cycle \( C \) is denoted by \( |C| \).

**Theorem 1.1** \([11, \text{Theorem 2.1}]\). Let \( I(G) \) be the edge ideal of a simple graph \( G \). Then
\[
\text{index}(I(G)) = \inf\{|C| : C \text{ is a minimal cycle in } \overline{G}\} - 3.
\]

2. **Edge ideals with almost maximal final index**

A graded ideal \( I \subset S \) is said to have almost maximal finite index over \( \mathbb{K} \) if \( \text{index}(I) = \text{pd}(I) - 1 \). Since, in general, \( \text{pd}(I) \) and \( \text{index}(I) \) depend on the characteristic of the base field, the property of having almost maximal finite index may also be characteristic dependent. For example, setting \( \Delta \) to be a triangulation of a real projective plane, the Stanley-Reisner ideal of \( \Delta \) is generated in degree 3 and it has almost maximal finite index over all fields of characteristic 2, while it has linear resolution over other fields (cf. \([4, \S 5.3]\)). However, as we will see in Corollary 2.7, in the case of quadratic monomial ideals, having almost maximal finite index is characteristic independent. Note that, although by Theorem 1.1, the index of an arbitrary edge ideal does not depend on the base field, its projective dimension may depend. M. Katzman presents a graph in \([23, \text{Section 4}]\) whose edge ideal has different projective dimensions over different fields.

In this section, we give a classification of the graphs whose edge ideals have almost maximal finite index. We will present this classification in Theorem 2.6, but before, we need some intermediate steps which give more insight about the complement of such graphs.

Unless otherwise stated, throughout this section, \( G \) is a simple graph on the vertex set \( [n] \) and \( \overline{G} \) is its complement, \( \Delta \) denotes the independence complex \( \Delta(\overline{G}) \), and \( \partial, \partial^W \) denote respectively the differentials of the augmented oriented chain complexes of \( \Delta(G), \Delta(\overline{G}) \) over a fixed field \( \mathbb{K} \). Moreover, for an integer \( k \), we show by \( \bar{k} \) the remainder of \( k \) modulo \( t + 3 \), i.e. \( \bar{k} \equiv k \pmod{t + 3} \), where \( t \geq 1 \) is an integer.

First, in order to avoid repetition of some arguments, we gather some facts which will be used frequently in the sequel in the following Observation. Meanwhile, we also fix some notation.
Notation and Observation 2.1.

Let \( G \) be a simple graph on the vertex set \([n]\) and let \( I := I(G) \subset S \) be its edge ideal.

(O-1) The graph \( G \) is connected if and only if its flag complex \( \Delta(G) \) is connected. On the other hand for an arbitrary simplicial complex \( \Gamma \) and any field \( \mathbb{K} \),
\[
\dim_{\mathbb{K}} \widetilde{H}_0(\Gamma; \mathbb{K}) = \text{Number of connected components of } \Gamma - 1,
\]
see [19, Problem 8.2]. Moreover, for any subset \( W \subseteq [n] \) one has \( \Delta(G_W) = \Delta(G)_W \), where \( G_W \) is the induced subgraph of \( G \) on the vertex set \( W \). It follows that \( G_W \) is connected if and only if \( \widetilde{H}_0(\Delta(G)_W; \mathbb{K}) = 0 \). Now if \( \beta_{i+2}(I) = 0 \) for some \( i \), then by Hochster’s formula \( \widetilde{H}_0(\Delta_W; \mathbb{K}) = 0 \) and hence \( G_W \) is connected for all \( W \subseteq [n] \) with \(|W| = i + 2\).

(O-2) Throughout, by \( P = u_1 - u_2 - \cdots - u_r \) in \( G \) we mean a path in \( G \) on \( r \) distinct vertices with the set of edges \( \bigcup_{1 \leq i \leq r-1} \{u_i, u_{i+1}\} \). If, in addition \( \{u_1, u_r\} \in E(G) \), then \( C = u_1 - u_2 - \cdots - u_r - u_1 \) is a cycle in \( G \). Then
\[
T(C) := (\sum_{1 \leq i \leq r-1} [u_i, u_{i+1}]) - [u_1, u_r] \in \ker \partial_1^\Delta(G), \tag{2}
\]
where \( \partial_1^\Delta(G) \) denotes the differentials of the chain complex of \( \Delta(G) \). It is shown in [8, Theorem 3.2] that \( \widetilde{H}_1(\Delta(G); \mathbb{K}) \neq 0 \) if and only if there exists a minimal cycle \( C \) in \( G \) such that \( T(C) \notin \text{Im } \partial_2^\Delta(G) \). Indeed, it is proved that \( \widetilde{H}_1(\Delta(G); \mathbb{K}) \) is minimally generated by the nonzero homology classes \( T(C) + \text{Im } \partial_2^\Delta(G) \), where \( C \) is a minimal cycle in \( G \).

If \( C \) is the base of a cone whose apex is the vertex \( u_{r+1} \), then
\[
T(C) = \partial_2^\Delta(G)((\sum_{1 \leq i \leq r-1} [u_{r+1}, u_i, u_{i+1}]) - [u_{r+1}, u_1, u_r])
\]
which implies that \( T(C) + \text{Im } \partial_2^\Delta(G) = 0 \).

(O-3) Now let \( D \) be an \( r \)-gonal dipyramid in \( G \); that is a subgraph of \( G \) with the vertex set \( V(D) = V(C) \cup \{a, b\} \) and \( E(D) = \bigcup_{1 \leq i \leq r} \{\{a, u_i\} \cup \{b, u_i\} \cup E(C) \} \) where \( C \) is a cycle as above which is called the waist of \( D \). Then
\[
T(D) := (\sum_{1 \leq i \leq r-1} [a, u_i, u_{i+1}]) - [b, u_i, u_{i+1}] - [a, u_1, u_r] + [b, u_1, u_r] \in \ker \partial_2^\Delta(G). \tag{3}
\]

(O-4) Suppose \( \text{index}(I) = t \). By Theorem 1.1, \( \widetilde{G} \) contains a minimal cycle \( C = u_1 - u_2 - \cdots - u_{t+3} - u_1 \) which has the smallest length among all minimal cycles of \( \widetilde{G} \).

(i) If \( \beta_{t+1,t+4}(I) = 0 \), then \( \widetilde{H}_1(\Delta_W; \mathbb{K}) = 0 \) for all \( W \subseteq [n] \) with \(|W| = t + 4\). Set \( W = \{u_{t+4}\} \cup V(C) \) for an arbitrary vertex \( u_{t+4} \in [n] \setminus V(C) \). Then \( C \) is a minimal cycle in \( G_W \) and \( T(C) \in \ker \partial_W^1 \) implies that \( T(C) \in \text{Im } \partial_W^2 \). It follows that each edge \( e \) of \( C \) is contained in a 2-face \( F_e \) of \( \Delta_W \). Since \( C \) is minimal, we must have \( F_e = e \cup \{u_{t+4}\} \) which means that \( u_{t+4} \) is adjacent to all vertices of \( C \) and hence \( G_W \) is a cone.

(ii) If \( \beta_{t+2,t+5}(I) = 0 \), then \( \widetilde{H}_1(\Delta_W; \mathbb{K}) = 0 \) for all \( W \subseteq [n] \) with \(|W| = t + 5\). Set \( W = \{u_{t+4}, u_{t+5}\} \cup V(C) \) for arbitrary vertices \( u_{t+4}, u_{t+5} \in [n] \setminus V(C) \). As in (i), \( T(C) = \partial_W^2(L) \) for some \( L \in \bigoplus_{F \in \Delta_W} \mathbb{K}F \), and hence each edge of \( C \) is contained in a 2-face of \( \Delta_W \). It follows that for each edge \( e \) of \( C \) either \( \{u_{t+4}\} \cup e \in \Delta_W \) or \( \{u_{t+5}\} \cup e \in \Delta_W \). If for all \( e \in E(C) \) one has \( \{u_{t+4}\} \cup e \in \Delta_W \), then \( \Delta_W \) contains a cone. Same holds if we replace \( u_{t+4} \) with \( u_{t+5} \). Suppose \( \{u_{t+4}\} \cup e, \{u_{t+5}\} \cup e' \notin \Delta_W \) for some \( e, e' \in E(C) \), which implies that \( u_{t+4}, u_{t+5} \) are not adjacent to all vertices
of $C$. Without loss of generality suppose \{u_{t+4}, u_1, u_2\} \notin \Delta_W$. It follows that \{u_{t+4} \cup \{u_1, u_2\} \in \Delta_W$. If \{u_1, u_2\} is the only edge $e$ of $C$ with \{u_{t+4} \cup \in \Delta_W$, then for all $e' \in E(C)$ with \$e' \neq \{u_1, u_2\}$ one has \{u_{t+5} \cup e' \in \Delta_W$. In particular, \{u_1, u_{t+3}, u_{t+5}\}, \{u_2, u_3, u_{t+5}\} \in \Delta_W$ which implies by the definition of $\Delta_W = \Delta(G_W)$ that \{u_{t+5}, u_1, u_2\} \in \Delta_W$, a contradiction. By our assumption there exists $3 \leq j \leq t + 3$ such that \{u_j, u_{t+4}\} \notin E(G)$. Let $a, b$ be respectively the biggest and the smallest integers with $2 \leq a < j < b \leq t + 3$ for which \{u_a, u_{t+4}\}, \{u_b, u_{t+4}\} \in E(G)$. If such $b$ does not exist we let $b = 1$ which implies that $a \neq 2$ because otherwise $u_{t+5}$ is adjacent to all vertices of $C$. Now $C' := u_{t+4} - u_a - u_{a+1} - \cdots - u_b - u_{t+4}$ is a minimal cycle of length $t + 6 - a$ if $b = 1$, and of length $b - a + 2$ if $b \leq t + 3$. Since $\text{index}(I) = t$ one has $t + 6 - a, b - a + 2 \geq t + 3$, and so $(a, b) \in \{(1, 3), (2, t + 3)\}$. In the both cases the vertex $u_{t+4}$ is adjacent to only three successive vertices of $C$. Without loss of generality we may assume that $u_1, u_2, u_3$ are these three vertices. Thus $u_{t+4}$ is adjacent to only two edges \{u_1, u_2\}, \{u_2, u_3\} of $C$ and hence \{u_1, u_2, u_3, u_{t+3}\} \subseteq N_G(u_{t+5})$. It follows that \{u_{t+5}, u_2\} \notin E(G)$ because $u_{t+5}$ is not adjacent to all vertices of $C$. Thus we get the minimal 4-cycle $C'' := u_{t+4} - u_a - u_{a+1} - \cdots - u_b - u_{t+4}$, a contradiction. We show that $u_{t+4}$ is a minimal cycle of length $t + 6 - a$ if $b = 1$, and of length $b - a + 2$ if $b \leq t + 3$. Since $\text{index}(I) = t$ one has $t + 6 - a, b - a + 2 \geq t + 3$, and so $(a, b) \in \{(1, 3), (2, t + 3)\}$. In the both cases the vertex $u_{t+4}$ is adjacent to only three successive vertices of $C$. Without loss of generality we may assume that $u_1, u_2, u_3$ are these three vertices. Thus $u_{t+4}$ is adjacent to only two edges \{u_1, u_2\}, \{u_2, u_3\} of $C$ and hence \{u_1, u_2, u_3, u_{t+3}\} \subseteq N_G(u_{t+5})$. It follows that \{u_{t+5}, u_2\} \notin E(G)$ because $u_{t+5}$ is not adjacent to all vertices of $C$. Thus we get the minimal 4-cycle $C'' := u_{t+4} - u_a - u_{a+1} - \cdots - u_b - u_{t+4}$, a contradiction.

(iii) If $\beta_{t+2, t+6}(I) = 0$, then $\bar{H}_2(\Delta(G_W); \mathbb{K}) = 0$ for all $W \subseteq [n]$ with $|W| = t + 6$. Suppose $G$ contains a dipyramid $D$ with the vertex set \{u_{t+4}, u_{t+5}\} \cup V(C)$ and the waist $C$ which is a minimal cycle of length $t + 3$. Set $W = \{u_{t+4}, u_{t+5}, u_{t+6}\} \cup V(C)$ for arbitrary vertex $u_{t+6} \in [n] \setminus (V(C) \cup \{u_{t+4}, u_{t+5}\})$. Then by (O-3) one has $T(D) \in \ker \partial^W_2$ and hence $T(D) \in \text{Im} \partial^W_2$. This implies that each 2-face of $D$ is contained in a 3-face of $\Delta_W$. Since $C$ is minimal, it follows that either \{u_{t+4}, u_{t+5}\} \in E(G)$ or \{u_{t+4}, u_{t+5}\} \cup V(C) \subseteq N_G(u_{t+6})$.

Example 2.2. Here we give 7 types of the graphs $G$ whose edge ideal $I := I(G)$ has almost maximal finite index over all fields. Indeed, we present the complementary graphs $G$ for which $\text{pd}(I) = \text{index}(I) + 1$. Since the smallest minimal cycles in the following graphs $G$ are of length $t + 3 \geq 4$, by Theorem 1.1, we have $\text{index}(I) = t$. We show that $\text{pd}(I) = t + 1$. Note that as it is also clear from Hochster’s formula, $\beta_{i,j}(I) = 0$ for all $j < i + 2$ and hence, in order to show that $\text{pd}(I) = t + 1$ it is enough to prove $\beta_{t+1, j}(I) \neq 0$ for some $j \geq t + 3$ and $\beta_{t+2, j}(I) = 0$ for all $t + 4 \leq j \leq n$. The argument below is independent of the choice of the base field.

(a) Let $G$ be either of the graphs $G_{(a)_1}, G_{(a)_2}, G_{(a)_3}$ shown in Figure 1 with $t \geq 1$. The two graphs $G_{(a)_1}, G_{(a)_2}$ have one minimal cycle $C = 1 - 2 - \cdots - (t + 3) - 1$, and the graph $G_{(a)_3}$, has two minimal cycles $C$ and $C' = 1 - (t + 4) - 3 - 4 - \cdots - (t + 3) - 1$. Setting $W = \{t + 4\}$, we have $T(C) \in \ker \partial^W_1$ by (O-2). Since $t > 0$, there are edges of $C$ in all three graphs which are not contained in a 2-face of $\Delta_W$. In particular, $T(C) \notin \text{Im} \partial^W_2$. Hence $\bar{H}_1(\Delta_W; \mathbb{K}) \neq 0$ which implies that $\beta_{t+1, t+4}(I) \neq 0$. Thus $\text{pd}(I) \geq t + 1$. If $\beta_{t+2, j}(I) \neq 0$ for some $j$, then there exists $W \subseteq [t + 4]$ with $|W| = j$ such that $\bar{H}_{|W|-t-4}(\Delta_W; \mathbb{K}) \neq 0$. Therefore $W = \{t + 4\}$. But $\bar{G}_W = G$ is connected meaning that $\Delta_W$ is connected, using
(O-1). Hence \( \tilde{H}_0(\Delta_W; \mathbb{K}) = 0 \) which implies that \( \beta_{t+2,t+4}(I) = 0 \). Therefore \( \text{pd}(I) = t + 1 \).

\[
\begin{align*}
G_{(a)_1} & & G_{(a)_2} & & G_{(a)_3} \\
\begin{array}{ccc}
t + 3 & t + 4 & 1 \\
2 & 3 & \vdots
\end{array} & & \begin{array}{ccc}
t + 3 & t + 4 & 1 \\
2 & 3 & \vdots
\end{array} & & \begin{array}{ccc}
t + 3 & t + 4 & 1 \\
2 & 3 & \vdots
\end{array}
\end{align*}
\]

**Figure 1.** The graphs \( G_{(a)_i} \)

(b) Let \( \bar{G} \) be the graph \( G_{(b)} \) in Figure 2, where \( t \geq 1 \) and \( \{i, t + 4\} \in E(\bar{G}) \) for all \( i \in [t + 3] \). Then \( \bar{G} \) has one minimal cycle \( C \) as in (a). Setting \( W = [t + 3] \cup \{t + 5\} \), we have \( T(C) \in \ker \partial_W^1 \setminus \text{Im} \partial_W^2 \). It follows that \( \beta_{t+1,t+4}(I) \neq 0 \). Therefore \( \text{pd}(I) \geq t + 1 \). For any \( W \subseteq [t + 5] \) with \( |W| = t + 4 \), \( \bar{G}_W \) is connected. So \( \beta_{t+2,t+4}(I) = 0 \). Suppose \( W = \{t + 5\} \). Although \( T(C) \in \ker \partial_W^1 \) one also has \( T(C) \in \text{Im} \partial_W^2 \), because \( C \) is the base of a cone with apex \( u_{t+4} \). Hence according to (O-2), \( \beta_{t+2,t+5}(I) = 0 \). It follows that \( \text{pd}(I) = t + 1 \).

\[
\begin{array}{ccc}
t + 3 & t + 4 & 1 \\
2 & 3 & \vdots
\end{array}
\]

**Figure 2.** The graph \( G_{(b)} \)

(c) Let \( \bar{G} \) be the graph \( G_{(c)} \) shown in Figure 3. This graph consists of 3 minimal cycles of length 4. So \( \text{index}(I) = 1 \). Moreover, \( \beta_{2,5}(I) \neq 0 \) because \( T(C) \in \ker \partial_1 \setminus \text{Im} \partial_2 \), for all minimal cycles \( C \) in \( \bar{G} \), and \( \beta_{3,5}(I) = 0 \) because \( \bar{G} \) is connected. Hence \( \text{pd}(I) = 2 \).

\[
\begin{array}{ccc}
4 & 5 & 3 \\
1 & 2 & \vdots
\end{array}
\]

**Figure 3.** The graph \( G_{(c)} \)

(d) Let \( \bar{G} \) be either of the graphs \( G_{(d)_1}, G_{(d)_2} \) in Figure 4. Both graphs have three minimal cycles of length 4. Since \( G_{(d)_1} \) is a dipyramid, by (O-3) one has \( \tilde{H}_2(\Delta(\bar{G}_{(d)_1}); \mathbb{K}) \neq 0 \)}
which implies that $\beta_{2,6}(I(G_{(d_1)})) \neq 0$. Although, $G_{(d_2)}$ is not a dipyramid, it contains the minimal cycle $C = 1 - 2 - 3 - 4 - 1$ which gives a nonzero homology class of $\overline{H}_1(\Delta(G_{(d_2)}); \mathbb{K}) \neq 0$, where $W = V(C) \cup \{5\}$. Hence $\beta_{2,5}(I(G_{(d_2)})) \neq 0$. Therefore $\text{pd}(I) \geq 2$ in both cases. To prove that $\text{pd}(I) = 2$ it is enough to show that $\beta_{3,5}(I) = \beta_{3,6}(I) = 0$.

Considering any subset $W$ of $[6]$ with $|W| = 5$, $\overline{G}_W$ and so $\Delta_W$ is connected in both cases. It follows that $\overline{H}_0(\Delta_W; \mathbb{K}) = 0$, and hence $\beta_{3,5}(I) = 0$. Now $\overline{H}_1(\Delta; \mathbb{K}) = 0$ because except for the cycle $C = 1 - 2 - 3 - 4 - 1$ in $G_{(d_2)}$, all other minimal cycles in $G_{(d_1)}, G_{(d_2)}$ are bases of some cones and for the cycle $C$, we have

$$T(C) = \partial_2^W([1, 2, 5] + [2, 3, 5] + [3, 4, 6] - [1, 4, 6] - [3, 5, 6] + [1, 5, 6]).$$

Consequently, $\beta_{3,6}(I) = 0$ and hence $\text{pd}(I) = 2$.

**Figure 4.** The graphs $G_{(d_i)}$

Next lemma gives more intuition about the length of minimal cycles in $\overline{G}$, when $I(G)$ has almost maximal finite index.

**Lemma 2.3.** Let $G$ be a simple graph on $[n]$ and let $I := I(G)$ have almost maximal finite index. Then any minimal cycle in $\overline{G}$ is of length $\text{index}(I) + 3$.

**Proof.** Let $\text{index}(I) = t$. Then $\text{pd}(I) = t + 1$ which means that $\beta_{i,j}(I) = 0$ for all $i > t + 1$ and all $j$. Using Theorem 1.1, there exists a minimal cycle $C$ in $\overline{G}$ of length $t + 3$ which has the smallest length among all the minimal cycles in $\overline{G}$. Let $C = u_1 - u_2 - \cdots - u_{t+3} - u_1$.

Suppose $C' \neq C$ is a minimal cycle in $\overline{G}$ with $C' = v_1 - v_2 - \cdots - v_l - v_1$. Setting $W = V(C')$ and $T(C')$ as defined in (2) one has $T(C') \in \ker \partial_1^W$, while $\text{Im} \partial_2^W = 0$. Hence $\overline{H}_1(\Delta_W; \mathbb{K}) \neq 0$. Hochster's formula implies that $\beta_{l-1,2j}(I) \neq 0$. Since $\beta_{i,j}(I) = 0$ for all $i > t + 1$, we have $l \leq t + 4$. We claim that $l < t + 4$. Since $t+3$ is the smallest length of a minimal cycle in $\overline{G}$, it follows that $l = t + 3$, as desired.

**Proof of the claim:** Suppose $l = t + 4$ and let $u \in [n] \setminus V(C')$. Note that such $u$ exists since otherwise $n = l$ and hence $V(C) \subset V(C')$ which implies that $C'$ is not minimal. Let $W = V(C') \cup \{u\}$. Since $\beta_{t+2,t+5}(I) = 0$, it follows that $\overline{H}_1(\Delta_W; \mathbb{K}) = 0$. Therefore, $T(C') \in \ker \partial_1^W$ implies that $T(C') \in \text{Im} \partial_2^W$. Hence, $\{u, v_i\} \in E(\overline{G})$ for all $1 \leq i \leq t + 4$. Since $u$ is an arbitrary element in $[n] \setminus V(C')$, it follows that $\{u_j, v_i\} \in E(\overline{G})$ for all $u_j \in V(C) \setminus V(C')$ and all $v_i$.

Suppose $V(C) \cap V(C') \neq \emptyset$, say $u_1 \in V(C')$. Then $\{u_j, u_1\} \in E(\overline{G})$ for all $u_j \in V(C) \setminus V(C')$. Since $C$ is minimal we conclude that $V(C) \setminus V(C') \subseteq \{u_2, u_{t+3}\}$. Therefore
all minimal cycles. Since \( t \geq 1 \), \( \{u_3, \ldots, u_{t+2}\} \neq \emptyset \). It follows that \( \{u_1, u_3\} \in E(G) \) which contradicts the minimality of \( C \). Thus \( V(C) \cap V(C') = \emptyset \). Consequently \( \{u_1, u_3\}, \{u_3, v_1\} \in E(G) \) for all \( 1 \leq i \leq t + 4 \). Let \( W = V(C') \cup \{u_1, u_3\} \). Then \( \Delta_W \) consists of an induced dipyramid \( D \). Thus \( T(D) \in \ker \partial^t_2 \), while \( \Im \partial^t_3 = 0 \), where \( T(D) \) is defined in (3). It follows that \( \widetilde{H}_2(\Delta_W; \mathbb{K}) \neq 0 \) and so \( \beta_{t+2,t+6}(I) \neq 0 \), a contradiction. Therefore \( l < t + 4 \).

In the next corollary we highlight some information obtained from Observation 2.1 about the vertices not belonging to a minimal cycle.

**Corollary 2.4.** Let \( G \) be a simple graph on \([n]\) and let \( I := I(G) \) have almost maximal finite index. Let \( C \) be a minimal cycle in \( G \). Then

(a) all vertices in \([n]\setminus V(C)\) are adjacent to some vertex in \( V(C) \).

(b) For any pair of vertices \( v, v' \in [n]\setminus V(C) \) whenever \(|N_G(v) \cap V(C)| \leq 2\), then \( V(C) \subseteq N_G(v') \).

(c) If \( \text{index}(I) = 1 \), then there are at most two vertices in \([n]\setminus V(C)\) which are not adjacent to all vertices of \( C \).

(d) If \( \text{index}(I) > 1 \), then there is at most one vertex in \([n]\setminus V(C)\) which is not adjacent to all vertices of \( C \).

**Proof.** Let \( \text{index}(I) = t \). By assumption \( \text{pd}(I) = t + 1 \). By Lemma 2.3 all minimal cycles of \( G \) are of length \( t + 3 \). Let \( C = u_1 - u_2 - \cdots - u_{t+3} - u_1 \) be a minimal cycle of \( G \).

(a) Let \( u_{t+4} \in [n]\setminus V(C) \), and set \( W = V(C) \cup \{u_{t+4}\} \). Since \( \beta_{t+2,t+4}(I) = 0 \) we conclude that \( G_W \) is connected using (O-1). It follows that \( u_{t+4} \) is adjacent to some vertex of \( C \).

(b) If \(|[n]\setminus V(C)| \leq 1\), then there is nothing to prove. Suppose \( u_{t+4}, u_{t+5} \in [n]\setminus V(C) \).

Set \( W = V(C) \cup \{u_{t+4}, u_{t+5}\} \). Since \( \beta_{t+2,t+5}(I) = 0 \), (O-4)(ii) implies that for each edge \( e \) of \( C \) we either have \( e \cup \{u_{t+4}\} \in \Delta_W \) or \( e \cup \{u_{t+5}\} \in \Delta_W \). This in particular shows that if \( u_{t+4} \) is adjacent to at most 2 vertices of \( C \), then \( u_{t+5} \) is adjacent to all of them.

(c) Suppose \( u_5, u_6 \in [n]\setminus V(C) \) are not adjacent to all vertices of \( C \). The argument in (O-4)(ii) shows that \( \{u_1, u_2, u_3\} \subseteq N_G(u_5) \) but \( u_4 \notin N_G(u_5) \) and \( \{u_1, u_3, u_4\} \subseteq N_G(u_6) \) but \( u_2 \notin N_G(u_6) \). Now suppose \( u_7 \in [n]\setminus V(C) \) is not adjacent to all vertices of \( C \). By replacing \( u_4 \) with \( u_7 \) in (O-4)(ii) one sees that \( u_7 \) is not adjacent to \( u_2 \), and replacing \( u_5 \) with \( u_7 \) in the same argument shows that \( u_7 \) is adjacent to \( u_2 \), a contradiction.

(d) Suppose \( u_{t+4}, u_{t+5} \) are two vertices in \([n]\setminus V(C)\) which are not adjacent to all vertices of \( C \). The argument in (O-4)(ii) shows that \( t = 1 \), a contradiction.

The crucial point in the classification of the edge ideals with almost maximal finite index is to determine the number of the vertices of the graph with respect to the index of the ideal. In the following, we compute this number.

**Proposition 2.5.** Let \( G \) be a simple graph on \([n]\) with no isolated vertex and let \( I = I(G) \) have almost maximal finite index \( t \). Then \( G \) has either \( n = t + 4 \) or \( n = t + 5 \) vertices.

**Proof.** Since \( \text{index}(I) = t \) there is a minimal cycle \( C = u_1 - u_2 - \cdots - u_{t+3} - u_1 \) in \( G \). Moreover, \( G \neq C \), because otherwise \( \text{pd}(I) = \text{index}(I) \) by [2, Theorem 4.1]. Since \( C \) is a minimal cycle, \( G \neq C \) means that there exists \( v \in [n]\setminus V(C) \). Therefore \( n \geq t + 4 \).

Suppose on contrary that \( n > t + 5 \). So \( n - |V(C)| > 2 \).

Suppose first \( t > 1 \). It follows from Corollary 2.4(d) that there exist \( u_{t+4}, u_{t+5} \in [n]\setminus V(C) \) such that \( u_{t+4}, u_{t+5} \) are adjacent to all vertices of \( C \). Therefore \( C' := u_{t+4} - u_1 - u_{t+5} - u_3 - u_{t+4} \) is a 4-cycle. Since \( t > 1 \), \( C' \) is not minimal and hence \( \{u_{t+4}, u_{t+5}\} \in E(G) \).
Since $u_{t+4}, u_{t+5}$ are not isolated in $G$, there exist $v_1, v_2 \in [n] \setminus \{v_{t+4}, v_{t+5}\}$ such that $\{v_1, u_{t+4}\}, \{v_2, u_{t+5}\} \notin E(G)$. Note that there exists $1 \leq j \leq t+3$ such that $v_1$ is not adjacent to $u_j$, since otherwise setting $W = V(C) \cup \{u_{t+4}, v_1\}$, $\Delta_W$ is an induced dipyramid and hence $\beta_{t+2,t+6}(I) \neq 0$ by (O-3) which contradicts $pd(I) = t + 1$. Same holds for $v_2$. Now Corollary 2.4(d) implies that $v_1 = v_2$. Without loss of generality suppose $j = 1$.

By Corollary 2.4(a), $v_1$ is adjacent to some vertex of $C$. Suppose there are at least two vertices of $C$ adjacent to $v_1$ and suppose $1 < a < b \leq t + 3$ are respectively the smallest and biggest integers such that $u_a, u_b$ are adjacent to $v_1$. Then $C' = v_1 - u_b - u_{b+1} - \cdots - u_{t+3} - u_1 - u_2 - \cdots - u_a - v_1$ is a minimal cycle of length $t + 5 + a - b = t + 3$ by Lemma 2.3. It follows that $b = a + 2$ and consequently $u_{t+4} - u_a - v_1 - u_b - u_{t+4}$ is a minimal 4-cycle which contradicts $t > 1$. Therefore $v_1$ is adjacent to only one vertex $u_i$ of $C$. Setting $W = \{u_{t+4}, v_1\} \cup V(C) \setminus \{u_i\}$, $\Delta_W$ is not connected and so $\beta_{t+2,t+4}(I) \neq 0$, a contradiction. Therefore $n \leq t + 5$ when $t > 1$.

Now suppose $t = 1$. Since $n - |V(C)| > 2$ we have $n \geq 7$. By Corollary 2.4(c), at least one vertex, say $v_1$ in $[n] \setminus V(C)$ is adjacent to all vertices of $C$. Since $v_1$ is not isolated in $G$, there exists $v_2 \in [n] \setminus (V(C) \cup \{v_1\})$ such that $\{v_1, v_2\} \notin E(G)$. We claim that $v_2$ is not adjacent to some vertex of $C$.

Proof of the claim: Suppose on contrary that $v_2$ is adjacent to all vertices of $C$. Then we get an induced dipyramid $D$ on the vertex set $V(C) \cup \{v_1, v_2\}$. Now set $W = V(C) \cup \{v_1, v_2, v_3\}$ for some $v_3 \in [n] \setminus (V(C) \cup \{v_1, v_2\})$. Since $\beta_{3,7}(I) = 0$ we have $T(D) \in \operatorname{Im} \partial^W_1$, with $T(D)$ similar to the one in (3), which implies that each 2-face of $D$ is contained in a 3-face of $\Delta_W$ and hence $v_3$ is adjacent to all vertices of $V(C) \cup \{v_1, v_2\}$. As $v_3$ is not isolated in $G$, there exists $v_4 \in [n] \setminus (V(C) \cup \{v_1, v_2, v_3\})$ such that $\{v_3, v_4\} \notin E(G)$. Replacing $v_3$ with $v_4$ in the above argument we conclude that $v_4$ is also adjacent to all vertices of $V(C) \cup \{v_1, v_2\}$. Now set $W = \{v_1, v_2, v_3, v_4\} \cup V(C)$. Then

$$T = \left( \sum_{1 \leq i \leq 3} [v_1, v_3, u_i, u_{i+1}] - [v_1, v_4, u_i, u_{i+1}] - [v_2, v_3, u_i, u_{i+1}] + [v_2, v_4, u_i, u_{i+1}] \right)$$

$$- [v_1, v_3, u_1, u_4] + [v_1, v_4, u_1, u_4] + [v_2, v_3, u_1, u_4] - [v_2, v_4, u_1, u_4] \in \ker \partial^W_3$$

while $T \notin \operatorname{Im} \partial^W_1$, because $\Delta_W$ contains no 4-face. This implies that $\beta_{3,8}(I) \neq 0$ which is a contradiction. So the claim follows.

Without loss of generality suppose $\{v_2, v_4\} \notin E(G)$. Now consider $v'_3 \in [n] \setminus (V(C) \cup \{v_1, v_2\})$. We show that $v'_3$ is adjacent to all vertices of $C$. Otherwise, setting $W = \{v_2, v'_3\} \cup V(C)$ the same discussion as in (O-4)(ii) shows that $G_W$ is isomorphic to the graph $G_{(d)2}$ in Figure 4, where $\{v'_3, u_2\} \notin G_W$. Hence setting $W = \{v_1 v_2, v'_3\} \cup V(C)$,

$$T' = \left( \sum_{1 \leq i \leq 3} [v_1, u_i, u_{i+1}] - [v_1, u_1, u_4] - [v_2, u_2, u_3] - [v'_3, u_3, u_4] + [v'_3, u_1, u_4] - [v_2, v'_3, u_1] + [v_2, v'_3, u_3] \right) \in \ker \partial^W_2$$

while $T' \notin \operatorname{Im} \partial^W_1$ because $\{v_2, u_1, u_2\}$ is not contained in a 3-face of $\Delta_W$, and we get a contradiction. Thus $v'_3$ is adjacent to all vertices of $C$. It follows that a dipyramid $D$ with the vertex set $V(C) \cup \{v_1, v'_3\}$ lies in $\Delta_W$ and so $T(D) \in \ker \partial^W_2$ which implies that $T(D) \notin \operatorname{Im} \partial^W_1$. Thus each 2-face of $D$ is contained in a 3-face of $\Delta_W$. Since $v_2$ is not adjacent to $u_4$, we conclude that $\{v_1, v'_3\} \in E(G)$. Note that by $\beta_{3,5}(I) = 0$, setting $W = V(C) \cup \{v_2\}$, the vertex $v_2$ is adjacent to some vertex $u_i$ of $V(C)$, see Corollary 2.4. Now setting $W = \{v_1, v_2\} \cup V(C) \setminus \{u_i\}$, the same reason implies that $v_2$ is adjacent to some vertex $u_j$ in $V(C) \setminus \{u_i\}$. Finally, setting $W = \{v_1, v_2, v'_3\} \cup V(C) \setminus \{u_i, u_j\}$ indicates
that \( v_2 \) is adjacent to either three vertices \( u_i, u_j, u_k \) of \( C \) and to \( v'_3 \). We show that in the first case \( v_2 \) is also adjacent to the two vertices \( u_i, u_j \) of \( C \) and to \( v'_3 \). Suppose the first case happens. Since \( \{v_2, u_4\} \notin E(G) \), setting \( W = \{v_1, v_2, v'_3, u_2, u_4\} \) we have a minimal cycle \( C' := v_2 - v_2 - u_4 - u_1 - v_2 \) in \( \bar{G}_W \) with \( T(C') \in \ker \partial^W_2 \). Since \( \beta_{3,6}(I) = 0 \), any edge of \( C' \) must be contained in a 2-face of \( \Delta_W \) and since \( \{v_1, v_2\} \notin E(G) \) it follows that \( v_2 \) is adjacent to \( v'_3 \).

Now since \( v'_3 \) is not isolated in \( G \), there exists \( v'_4 \in [n] \setminus (V(C) \cup \{v_1, v_2, v'_3\}) \) with \( \{v'_3, v'_4\} \notin E(G) \). Replacing \( v'_3 \) with \( v'_4 \) in the above discussion, we see that \( v'_4 \) is adjacent to all vertices of \( C \). Setting \( W = V(C) \cup \{v'_2, v'_3, v'_4\} \), we have an induced dipyramid \( D \) on the vertex set \( V(C) \cup \{v'_3, v'_4\} \) with \( T(D) \in \ker \partial^W_2 \) and since \( \{v_2, u_4\} \notin G_W \) we have \( T(D) \notin \Im \partial^W_3 \) that is a contradiction with \( \beta_{3,7}(I) = 0 \). Thus \( n \leq t + 5 \) when \( t = 1 \).

Now we are ready to state the main result of this section which determines the graphs whose edge ideals have almost maximal finite index.

**Theorem 2.6.** Let \( G \) be a simple graph on \([n]\) with no isolated vertex and let \( I = I(G) \subset S \). Then \( I \) has almost maximal finite index if and only if \( G \) is isomorphic to one of the graphs given in Example 2.2.

**Proof.** The “if” implication follows from Example 2.2. We prove the converse. Suppose \( \text{index}(I) = t \). Then there is a minimal cycle \( C := u_1 - u_2 - \cdots - u_{t+3} \) in \( G \). Moreover, by Proposition 2.5 there exists \( u_{t+4} \in [n] \setminus V(C) \) which by Corollary 2.4(a) is adjacent to some vertex \( u_i \) in \( V(C) \). Without loss of generality we may assume that \( i = 1 \). By Proposition 2.5 we have \( n - (t + 3) \leq 2 \). We consider two cases:

**Case (i):** Suppose \( [n] \setminus V(C) = \{u_{t+4}\} \) and let \( 1 \leq l \leq t + 3 \) be the largest integer such that \( \{u_l, u_{t+4}\} \notin E(G) \).

- If \( l = 1 \), then \( G = G(a)_1 \) in Figure 1.
- If \( l = 2 \), then \( G = G(a)_2 \) in Figure 1.
- If \( 3 \leq l < t+3 \), then there is a minimal cycle \( C' = u_1 - u_{t+4} - u_l - u_{t+1} - \cdots - u_{t+3} - u_1 \) of length \( t + 6 - l \). By Lemma 2.3, \( t + 6 - l = t + 3 \) which implies \( l = 3 \). If \( \{u_2, u_{t+4}\} \notin E(G) \), then we will have a minimal 4-cycle on the vertex set \( \{u_1, u_2, u_3, u_{t+4}\} \). It follows from Lemma 2.3 that \( |C| = 4 \). Hence, \( G \) is isomorphic to \( G(c) \) in Figure 3. If \( \{u_2, u_{t+4}\} \in E(G) \), then \( G = G(a)_3 \) in Figure 1.

- If \( l = t + 3 \), then since \( G \) does not have isolated vertices, there exists \( 1 < j < t + 3 \), such that \( \{u_{t+4}, u_j\} \notin E(G) \). Let \( k, k' \) with \( 1 \leq k < j < k' \leq t + 3 \) be respectively the largest index and the smallest index such that \( \{u_k, u_{t+4}\}, \{u_{k'}, u_{t+4}\} \notin E(G) \). It follows that \( C'' = u_{t+4} - u_k - u_{k+1} - \cdots - u_{k'} - u_{t+4} \) is a minimal cycle and hence \( |C''| = k' - k + 2 = t + 3 \). Therefore we have either \((k, k') = (1, t + 2)\) or \((k, k') = (2, t + 3)\).

In both cases \( G \) is isomorphic to \( G(a)_3 \).

**Case (ii):** Suppose \( [n] \setminus V(C) = \{u_{t+4}, u_{t+5}\} \). By Corollary 2.4(a) both \( u_{t+4}, u_{t+5} \) are adjacent to at least one vertex of \( C \).

- Suppose \( u_{t+4} \) is adjacent to at most 2 vertices of \( C \) one of which is \( u_1 \). Then \( u_{t+5} \) is adjacent to all vertices of \( C \) by Corollary 2.4(b). Since \( \Delta_W \) is connected for \( W = [n] \setminus \{u_1\} \), we conclude that \( u_{t+4} \) is adjacent to some vertex in \( [n] \setminus \{u_1\} \) and since \( u_{t+5} \) is not isolated in \( G \), \( u_{t+5} \) is not adjacent to \( u_{t+4} \) and hence \( u_{t+4} \) is adjacent to some \( u_j \in V(C) \) with \( j \neq 1 \). We show that either \( j = 2 \) or else \( j = t + 3 \). Otherwise there is a minimal cycle \( C'' := u_{t+4} - u_1 - u_2 - \cdots - u_j - u_{t+4} \) of length \( j + 1 = t + 3 \), by Lemma 2.3. Thus \( j = t + 2 \) which implies that \( C''' := u_{t+4} - u_{t+2} - u_{t+3} - u_1 - u_{t+4} \) is a minimal 4-cycle and hence \( t = 1 \). But \( T(C''') \in \ker \partial_1 \setminus \partial_2 \), where \( C''' := u_5 - u_3 - u_4 - u_1 - u_5 \) is a minimal 4-cycle.
in \( \bar{G} \). It follows that \( \beta_{3,6}(I) \neq 0 \), a contradiction. Thus either \( j = 2 \) or else \( j = t + 3 \) and hence \( G \) is isomorphic to \( G_{(b)} \) in Figure 2. Same holds if we interchange \( u_{t+4} \) and \( u_{t+5} \) in the above argument.

- Now suppose \( u_{t+4}, u_{t+5} \) are adjacent to at least 3 vertices of \( C \). If \( u_{t+4} \) and \( u_{t+5} \) are not adjacent to some vertices of \( C \), then as seen in the argument of (O-4)(ii), the graph \( G \) is isomorphic to \( G_{(d)_2} \) in Figure 4.

Now consider the case that at least one of the vertices \( u_{t+4}, u_{t+5} \), say \( u_{t+5} \), is adjacent to all vertices of \( C \). The argument below also works if we interchange \( u_{t+4}, u_{t+5} \).

Suppose \( u_{t+4} \) is adjacent to (at least) three vertices \( u_1, u_k, u_j \) of \( C \) with \( 1 < k < j \leq t+3 \). Since \( u_{t+5} \) is not isolated in \( G \), we have \( \{u_{t+4}, u_{t+5}\} \notin E(\bar{G}) \). If \( (k, j) \neq (2, t + 3) \), then we get the minimal 4-cycle \( C' = u_{t+4} - u_1 - u_{t+5} - u_t - u_{t+4} \), where \( l = k \) if \( k \neq 2 \), and else \( l = j \), and hence \( t = 1 \). If \( (k, j) = (2, t + 3) \), then we get the minimal 4-cycle \( C'' = u_{t+4} - u_2 - u_{t+5} - u_{t+3} - u_{t+4} \) and so \( t = 1 \) also in this case. From \( t = 1 \) we conclude that \( u_1, u_k, u_j \) are successive vertices in \( C \). Without loss of generality we may assume that \( (k, j) = (2, 3) \). Assume first that \( \{u_5, u_4\} \notin E(\bar{G}) \). Since \( u_6 \) is adjacent to all vertices of \( C \) and it is not isolated in \( G \), we have \( \{u_5, u_6\} \notin E(\bar{G}) \). Hence \( G \) is again isomorphic to \( G_{(d)_2} \). Finally, assume \( \{u_5, u_4\} \in E(\bar{G}) \), which implies that \( \{u_5, u_6\} \notin E(\bar{G}) \) and hence \( G \) is isomorphic to \( G_{(d)_1} \) in Figure 4. This completes the proof.

All the arguments so far in this section were characteristic independent; consequently

**Corollary 2.7.** The property of having almost maximal finite index for edge ideals is independent of the characteristic of the base field. In other words, given a simple graph \( G \), its edge ideal \( I(G) \) has almost maximal finite index over some field if and only if it has this property over all fields.

**Corollary 2.8.** Let \( G \) be a simple graph on \( [n] \) with no isolated vertex and let \( I = I(G) \subset S \) have almost maximal finite index. Then over all fields

\[
\text{pd}(I) = \begin{cases} 
  n - 3 & \text{if } \bar{G} = G_{(c)} \text{ or } G_{(a)_i}, \ i = 1, 2, 3, \\
  n - 4 & \text{if } \bar{G} = G_{(b)} \text{ or } G_{(d)_i}, \ i = 1, 2.
\end{cases}
\]

In particular, \( 3 \leq \text{depth}(I) \leq 4 \).

**Proof.** Let \( \text{index}(I) = t \). By Theorem 2.6, \( \bar{G} \in \{G_{(a)_i}, G_{(b)}, G_{(c)}, G_{(d)_j}, 1 \leq i \leq 3, 1 \leq j \leq 2\} \). It follows that

\[
n = \begin{cases} 
  t + 4 & \text{if } \bar{G} = G_{(c)}, G_{(a)_i}, \ i = 1, 2, 3, \\
  t + 5 & \text{if } \bar{G} = G_{(b)}, G_{(d)_i}, \ i = 1, 2.
\end{cases}
\]

Since \( \text{pd}(I) = t + 1 \), the assertion follows from the Auslander-Buchsbaum formula.

In the rest of this section we study the last graded Betti numbers of edge ideals with almost maximal finite index. We first see in the following lemma that the graded Betti numbers of the edge ideals with this property are independent of the characteristic of the base field. The proof takes a great benefit of Katzman’s paper [23].

**Lemma 2.9.** Let \( I \subset S \) be the edge ideal of a simple graph with almost maximal finite index. Then the Betti numbers of \( I \) are characteristic independent.

**Proof.** [23, Theorem 4.1] states that the Betti numbers of the edge ideals of the graphs with at most 10 vertices are independent of the characteristic of the base field. It follows that the graded Betti numbers of \( I = I(G) \) with \( \text{index}(I) = t \) are characteristic independent when \( \bar{G} \in \{G_{(c)}, G_{(d)_i}, \ i = 1, 2\} \).
By [23, Corollary 1.6, Lemma 3.2(b)], if $G$ has a vertex $v$ of degree 1 or at least $|V(G)| - 4$, then the Betti numbers of $I(G)$ are characteristic independent if and only if the Betti numbers of $I(G - \{v\})$ are characteristic independent. Here $G - \{v\}$ is the induced subgraph of $G$ on $V(G) \setminus \{v\}$. Since the vertex $t + 4$ is of degree one in $G_{(b)}$, it follows that the Betti numbers of $I(G_{(b)})$ are characteristic independent if and only if so do the Betti numbers of $I(G_{(a)2})$. For $G \in \{G_{(a)} : 1 \leq i \leq 3\}$, since $t + 4$ is adjacent to at least $t$ vertices of $G$ and since $|V(G)| = t + 4$, it is enough to show that the Betti numbers of $I(G - \{t + 4\})$ are characteristic independent. But $G - \{t + 4\}$ is the complement of a minimal cycle of length $t + 3$. Note that by Hochster’s formula, all the linear Betti numbers $\beta_{i,i+2}(I)$ are obtained from computing the dimension of $H_0(\Delta(\overline{G})_W; \mathbb{K})$ with $W \subseteq V(G)$ and $|W| = i + 2$, and this dimension equals the number of connected components of $\overline{G}_W$ minus one. Therefore these Betti numbers do not depend on the characteristic of the base field, see also [23, Corollary 1.2(b)]. Moreover, as seen in [2, Proposition 4.3], the edge ideal of the complement of a minimal cycle has one nonzero non-linear Betti number $\beta_{t,t+3}(I) = 1$ over all fields. Therefore the Betti numbers of $I(G - \{t+4\})$ are characteristic independent, as desired.

For the edge ideals with linear resolution all non-linear Betti numbers are zero. For the edge ideals with maximal finite index $t$, it is seen in [11, 2] that there is only one nonzero non-linear Betti number $\beta_{t,t+3}(I) = 1$ over all fields. In the case of ideals with almost maximal finite index with index$(I) = t$, the non-linear Betti numbers appear in the last two homological degrees of the minimal free resolution. By the arguments that we had so far, it is easy to compute the $(t + 1)$-th graded Betti numbers and also $t$-th non-linear Betti numbers, where $I$ is the edge ideal with almost maximal finite index. Nevertheless, in the cases $G = G_{(c)}$ and $\bar{G} = G_{(d)}$, for $i = 1, 2$ one can see the whole Betti table, using Macaulay 2, [15]. Note that since all the graphs in Example 2.2 have at most $t + 5$ vertices, where index$(I(G)) = t$, and since the edge ideals are generated in degree 2, by Hochster’s formula it is enough to consider $\beta_{i,j}(I(G))$ for $i + 2 \leq j \leq t + 5$.

**Proposition 2.10.** Let $G$ be a graph and let $I := I(G)$ have almost maximal finite index $t$. Then over all fields, $\beta_{t,t+4}(I) = \beta_{t,t+5}(I) = 0$ and

$$
\beta_{t,t+3}(I) = \begin{cases} 
1 & \text{if } \bar{G} = G_{(a)1} \text{ or } G_{(a)2} \text{ or } G_{(b)}, \\
2 & \text{if } \bar{G} = G_{(a)3}, \\
3 & \text{otherwise,}
\end{cases}
$$

$$
\beta_{t+1,t+3}(I) = \begin{cases} 
1 & \text{if } \bar{G} = G_{(a)1} \text{ or } G_{(b)}, \\
0 & \text{otherwise,}
\end{cases}
$$

$$
\beta_{t+1,t+4}(I) = \begin{cases} 
2 & \text{if } \bar{G} = G_{(c)} \text{ or } G_{(d)2}, \\
0 & \text{if } \bar{G} = G_{(d)1}, \\
1 & \text{otherwise,}
\end{cases}
$$

$$
\beta_{t+1,t+5}(I) = \begin{cases} 
1 & \text{if } \bar{G} = G_{(d)1}, \\
0 & \text{otherwise.}
\end{cases}
$$

In particular,

$$
\text{reg}(I) = \begin{cases} 
4 & \text{if } \bar{G} = G_{(d)1}, \\
3 & \text{otherwise.}
\end{cases}
$$
Proof. All the equalities are straightforward consequences of the use of Hochster’s formula and Observation 2.1. However, the Betti number $\beta_{t, t+3}(I)$ can be also deduced from [13, Theorem 4.6]. It is worth to emphasize that although $\widetilde{H}_1(\Delta, K)$ is spanned by the set of $0 \neq T(C) + \text{Im} \partial_2$ for all minimal cycles $C$ of $G$, this set may not be a basis. In case $G = G_{(c)}$, the graph $\overline{G}$ has three minimal cycles $C$ of length 4 with $T(C) \notin \text{Im} \partial_2$, but for the cycle $C = 1 - 2 - 3 - 4 - 1$, $T(C)$ is a linear combination of $T(C'), T(C'')$, where $C', C''$ are the two other cycles of $G_{(c)}$. Hence $\dim_K \widetilde{H}_1(\Delta(G_{(c)}), K) = 2$. In the case of $G_{(a)\delta}$, we have $0 \neq T(C) + \text{Im} \partial_2$, where $C$ is the minimal cycle on $[t + 3]$, but $T(C)$ is a linear combination of $T(C'), T(C''), T(C''')$, where $C'$ is the minimal cycle on $[t + 4] \setminus \{2\}$, and $C'', C'''$ are the two triangles in $G_{(a)\delta}$ and hence $\dim_K \widetilde{H}_1(\Delta(G_{(a)\delta}), K) = 1$. \[\square\]

3. Powers of edge ideals with large Index

Due to a result of Herzog, Hibi and Zheng, [20, Theorem 1.2], if the edge ideal $I := I(G)$ has a linear resolution, that is $\text{index}(I) = \infty$, then all of its powers have a linear resolution as well. In case $I$ has maximal finite index $t > 1$, then by [2, Corollary 4.4] the ideal $I^s$ has a linear resolution for all $s \geq 2$. Note that in general for any edge ideal $I$ with $\text{index}(I) = 1$, one has $\text{index}(I^s) = 1$ for all $s \geq 2$, see Remark 1 below. In this section we investigate when the higher powers of the edge ideal $I$ with almost maximal finite index have a linear resolution. The main result of this section is the following:

Theorem 3.1. Let $G$ be a simple graph with no isolated vertex whose edge ideal $I(G) \subset S$ has almost maximal finite index. Then $I(G)^s$ has a linear resolution for $s \geq 2$ if and only if $G$ is gap-free.

Recall that a gap in a graph $G$ is an induced subgraph on 4 vertices and a pair of edges with no vertices in common which are not linked by a third edge; see the graph $G_1$ in Figure 5. The graph $G$ is called gap-free if it does not admit a gap; equivalently if $G$ does not contain an induced 4-cycle. This property plays an important role in the study of the resolution of powers of edge ideals; for example

Proposition 3.2. (Francisco-Há-Van Tuyl; unpublished, see [25, Proposition 1.8] and [2, Theorem 3.1]) Let $G$ be a simple graph. If $I(G)^s$ has a linear resolution for some $s \geq 1$, then $G$ is gap-free.

On the other hand,

Remark 1. A more precise statement about the gap-free graphs is given in [2, Theorem 3.1] which says that for a graph $G$ the following are equivalent:

(a) $G$ admits a gap;
(b) $\text{index}(I(G)^s) = 1$ for all $s \geq 1$;
(c) there exists $s \geq 1$ with $\text{index}(I(G)^s) = 1$.

If $G$ is the graph whose complement belongs to the set $\{G_{(b)}, G_{(a), i = 1 \leq i \leq 3, t = 1} \cup G_{(c)}, G_{(d), i = 1, 2}\}$, then $G$ has a gap. So by the above equivalence $\text{index}(I(G)^s) = 1$ for all $s \geq 1$ in this case.

To prove Theorem 3.1, we need some intermediate steps. We first review some known results which will be used in the proof of this theorem.

Theorem 3.3. [1, Theorem 5.2] Let $G$ be a simple graph and let $I := I(G)$ be its edge ideal. Let $\mathcal{G}(I^t) = \{m_1, \ldots, m_r\}$. Then for all $s \geq 1$

$$\text{reg}(I^{s+1}) \leq \max\{\text{reg}(I^s), \text{reg}(I^{s+1} : m_k) + 2s \text{ for } 1 \leq k \leq r\},$$
where \((I^{s+1} : m_k)\) is the colon ideal \(\{f \in S : fm_k \in I^{s+1}\}\).

As a consequence of this theorem, Banerjee showed in [1, Theorem 6.17] that for any gap-free and cricket-free graph \(G\), the ideal \(I(G)^s\) has a linear resolution for all \(s \geq 2\). A cricket is a graph isomorphic to the graph \(G_2\) in Figure 5, and a graph \(G\) is called cricket-free if \(G\) contains no cricket as an induced subgraph.

Another class of graphs which produce edge ideals whose higher powers have linear resolution was given by Erey. She proved in [12] that \(I(G)^s\) has a linear resolution for all \(s \geq 2\) if \(G\) is both gap-free and diamond-free. A diamond is a graph isomorphic to the graph \(G_3\) in Figure 5, and a diamond-free graph is a graph with no diamond as its induced subgraph.

\[
\begin{array}{ccc}
G_1 & G_2 & G_3 \\
\end{array}
\]

**Figure 5.** \(G_1\) a gap, \(G_2\) a cricket, \(G_3\) a diamond

**Remark 2.** Clearly, the graphs \(G_{(a)\bar{i}}\), \(i = 1, 2\), are cricket-free and hence the statement of Theorem 3.1 holds in these two cases using [1, Theorem 6.17]. Note that these graphs are gap-free for \(t \geq 2\).

On the other hand, the graphs \(G_{(a):i}\) and \(G_{(b):i}\) contain crickets for large enough \(t\). Indeed, if \(t \geq 3\), then the induced subgraph of \(G_{(a):3}\) on the vertex set \(\{1, 2, 3, 5, t+4\}\), and if \(t \geq 2\), then the induced subgraph of \(G_{(b):i}\) on \(\{3, 4, 5, t+4, t+5\}\) are isomorphic to a cricket. These graphs \(G_{(a):3}\) and \(G_{(b):i}\) are not diamond-free in general as well, because for \(t \geq 3\), the induced subgraphs on the vertex sets \(\{2, 4, 6, t+4\}\) and \(\{3, 5, 6, t+5\}\) form respectively diamonds in \(G_{(a):3}\) and \(G_{(b):i}\). Therefore, when \(G \in \{G_{(a):3}, G_{(b):i}\}\) and \(t\) is large enough, one can not take benefit of the results of Banerjee or Erey to deduce Theorem 3.1.

It is shown in [1, Section 6] that for the edge ideal \(I\) of a simple graph \(G\) and the minimal generator \(m_k\) of \(I^s\), \(s \geq 1\), the ideal \((I^{s+1} : m_k)\) is a quadratic monomial ideal whose polarization coincides with the edge ideal of a simple graph with the construction explained in Lemma 3.4. For the details about the polarization technique, the reader may consult with [19].

**Lemma 3.4.** [1, Lemma 6.11] Let \(G\) be a simple graph with the edge ideal \(I := I(G)\), and let \(m_k = x_{e_1} \cdots x_{e_s}\) be a minimal generator of \(I^s\), where \(e_1, \ldots, e_s\) are some edges of \(G\), and \(x_e = x_ix_j\) for \(e = \{i, j\}\). Then the polarization \((I^{s+1} : m_k)^{pol}\) of the ideal \((I^{s+1} : m_k)\) is the edge ideal of a new graph \(G_{e_1 \cdots e_s}\) with the following structure:

1. \(V(G) \subseteq V(G_{e_1 \cdots e_s})\), \(E(G) \subseteq E(G_{e_1 \cdots e_s})\).
2. Any two vertices \(u, v\), \(u \neq v\), of \(G\) that are even-connected with respect to \(e_1 \cdots e_s\) are connected by an edge in \(G_{e_1 \cdots e_s}\).
3. For every vertex \(u\) which is even-connected to itself with respect to \(e_1 \cdots e_s\) there is a new vertex \(u' \notin V(G)\) which is connected to \(u\) in \(G_{e_1 \cdots e_s}\) by an edge and not connected to any other vertex (so \(\{u, u'\}\) is a whisker in \(G_{e_1 \cdots e_s}\)).
In [1], two vertices \( u \) and \( v \) of a graph \( G \) (\( u \) may be same as \( v \)) are said to be even-connected with respect to an \( s \)-fold product \( e_1 \cdots e_s \) in \( G \) if there is a path \( P = p_0 - p_1 - \cdots - p_{2k+1}, k \geq 1 \), in \( G \) such that:

1. \( p_0 = u, p_{2k+1} = v. \)
2. For all \( 0 \leq l \leq k - 1 \), \( \{p_{2l+1}, p_{2l+2}\} = e_i \) for some \( 1 \leq i \leq s. \)
3. For all \( i \), \( \{|t: 0 \leq l \leq k - 1, \{p_{2l+1}, p_{2l+2}\} = e_i\| \leq |\{j: 1 \leq j \leq s, e_j = e_i\|\}. \)
4. For all \( 0 \leq r \leq 2k, \{p_r, p_{r+1}\} \) is an edge in \( G. \)

Although Theorem 3.3 is an efficient tool to prove Theorem 3.1, it is not enough. Indeed, one can check by Macaulay 2, [15], that the ideal \( (I(G_b))^2 : x_{t+4} x_{t+5} \) has regularity 3 for different choices of \( t \) and hence one can not rely only on Theorem 3.3 to prove Theorem 3.1 in this case. So we need some other tools:

**Lemma 3.5.** [10, Lemma 2.10] Let \( I \subset S \) be a monomial ideal, and let \( x \) be a variable appearing in some generator of \( I \). Then

\[
\text{reg}(I) \leq \max\{\text{reg}((I : x)) + 1, \text{reg}(I + (x))\}.
\]

Moreover, if \( I \) is squarefree, then \( \text{reg}(I) \) is equal to one of these terms.

In the next result we move a step forward to compute the regularity of the ideal \( (I(G_b))^2 : x_{t+5} \) by showing that it has linear quotients. The proof is a bit long yet easy to follow. Recall that a graded ideal \( I \) is said to have linear quotients if there exists a homogeneous generating set of \( I \), say \( \{f_1, \ldots, f_m\} \), such that the colon ideal \( (f_1, \ldots, f_{i-1}) : f_i \) is generated by variables for all \( i > 1 \). By [19, Theorem 8.2.1] equigenerated ideals with linear quotients have a linear resolution.

For a monomial \( m \in S \) we denote by \( \text{supp}(m) \) the set of all variables dividing \( m \) and denote by \( \text{deg}_m x_i \) the largest integer \( r \) such that \( x_i^r \) divides \( m \).

**Proposition 3.6.** Let \( I \subset S \) be the edge ideal of the graph \( G = G_b \), with \( n = t + 5 \) vertices, \( t \geq 1 \). Then the ideal \( (I^2 : x_{t+5}) \) has linear quotients.

**Proof.** Set \( J := (I^2 : x_{t+5}) \). We first determine the minimal generating set \( G(J) \) of \( J \). Note that \( E(G) = E(\bar{C}) \cup \{x_{i+5} x_i: 3 \leq i \leq t + 4\} \), where \( C = 1 - 2 - \cdots - (t + 3) - 1 \) is the unique induced cycle of \( G_b \) of length \( > 3 \). By [19, Proposition 1.2.2], the ideal \( J \) is generated by monomials \( x_i x_{i'} / \text{gcd}(x_i, x_{i'}, x_{t+5}) \), where \( e, e' \in E(G) \) are not necessarily distinct. It follows that \( J \) is generated by elements of the set

\[
\{x_i x_{i'} : e, e' \in E(\bar{C})\} \cup \{x_i x_e : e \in E(G), 3 \leq i \leq t + 4\}.
\]

Since for all \( e \in E(G) \) we have \( e \cap \{3, 4, \ldots, t + 4\} \neq \emptyset \) the ideal \( (\{x_i x_{i'} : e, e' \in E(\bar{C})\}) \) is generated by elements of the set \( \{x_i x_e : e \in E(G), 3 \leq i \leq t + 4\} \) and hence

\[
J = (\{x_i x_e : e \in E(G), 3 \leq i \leq t + 4\}.
\]

We order the edges of \( E(\bar{C}) \) as follows:

- for \( e = \{i, j\} \) with \( 1 \leq i < j \leq t + 3 \) and \( e' = \{i', j'\} \) with \( 1 \leq i' < j' \leq t + 3 \), we let \( e < e' \) if either \( i < i' \) or \( i = i' \) with \( j < j' \).

For \( e \in E(\bar{C}) \), let \( G_e = \{x_i x_e : 3 \leq i \leq t + 4, x_i x_e \notin G_{e'} \text{ for } e' < e\}. \)
Moreover, let
\[
\mathcal{G}_1 = \{ x_ix_{i+1}x_{t+5} : 3 \leq i \leq t+2 \} \cup \{ x_ix_jx_{t+5} : 3 \leq i < j - 1 \leq t + 2 \} \\
\cup \{ x_ix_{t+4}x_{t+5} : 3 \leq i \leq t + 3 \}, \\
\mathcal{G}_2 = \{ x_i^2x_{t+5} : 3 \leq i \leq t + 4 \}.
\]

The above sets \( \mathcal{G}_e, \mathcal{G}_1, \mathcal{G}_2 \) are pairwise disjoint and
\[
\mathcal{G}(J) = ( \bigcup_{e \in E(\bar{C})} \mathcal{G}_e ) \cup \mathcal{G}_1 \cup \mathcal{G}_2.
\]

Now we put an order \(<\) on the elements of \( \mathcal{G}(J) \). For \( m, m' \in \mathcal{G}(J) \) we let \( m < m' \) in the following cases:

- either \( m, m' \in \mathcal{G}_e \) for some \( e \in E(\bar{C}) \), or \( m, m' \in \mathcal{G}_1 \), or \( m, m' \in \mathcal{G}_2 \), and in all these cases \( m <_{\text{lex}} m' \) induced by \( x_1 < x_2 < \cdots < x_{t+5} \);
- \( m \in \mathcal{G}_e \) and \( m' \in \mathcal{G}_e' \) for some \( e, e' \in E(\bar{C}) \) with \( e < e' \);
- \( m \in \mathcal{G}_e \) for some \( e \in E(\bar{C}) \) and \( m' \in \mathcal{G}_1 \cup \mathcal{G}_2 \);
- \( m \in \mathcal{G}_1, m' \in \mathcal{G}_2 \).

Suppose \( \mathcal{G}(J) = \{ m_1, \ldots, m_r \} \) with \( m_1 < \cdots < m_r \). We show that for any \( m_l \in \mathcal{G}(J) \) with \( l > 1 \), the ideal \( (\{ m_1, \ldots, m_{l-1} \} : m_l) \) is generated by some variables. Set \( J_l := (m_1, \ldots, m_{l-1}) \). By [19, Proposition 1.2.2], the ideal \( (J_l : m_l) \) is generated by the elements of the set \( \{ m_s / \gcd(m_s, m_l) : 1 \leq s \leq l - 1 \} \). Let \( m_{s,l} := m_s / \gcd(m_s, m_l) \). We consider three cases:

(i) \( m_l \in \mathcal{G}_e \) for some \( e \in E(\bar{C}) \);  
(ii) \( m_l \in \mathcal{G}_1 \);  
(iii) \( m_l \in \mathcal{G}_2 \).

In each case we suppose on contrary that there exists \( m_s \) such that \( m_{s,l} \) is a minimal generator of \( (J_l : m_l) \) of degree \( > 1 \) and finally we get a contradiction.

First consider case (i). We have \( e = \{ i, j \}, m_l = x_i x_j x_k \) for some \( 1 \leq i < j \leq t + 3 \) with \( j - i > 1 \), and some \( 3 \leq k \leq t + 4 \). Note that \( j \geq 3 \). Moreover, by the construction of \( \mathcal{G}_e \), if \( e_1 = \{ i, k \} \in E(\bar{C}) \) (resp. \( e_2 = \{ j, k \} \in E(\bar{C}) \)), then \( e_1 \geq e \) (resp. \( e_2 \geq e \)). Note that

(a) for all \( 1 \leq j'' < j \) with \( j'' \neq i, j - 1, i + 1 \) we have \( e'' = \{ i, j'' \} \in E(\bar{C}) \) and \( e'' < e \). It follows that \( x_{e''} x_j \in J_l \) and thus \( x_{e''} \in (J_l : m_l) \).

(b) For all \( 1 < i'' < i \) with \( i'' \neq j - 1, j, j + 1 \) we have \( e'' = \{ i'', j \} \in E(\bar{C}) \), and \( i \geq 3 \) if such \( i'' \) exists. Hence \( x_{i''} x_i \in J_l \) and so \( x_{i''} \in (J_l : m_l) \). But \( i < j \) implies that \( i'' \neq j - 1, j, j + 1 \) for all \( 1 < i'' < i \). Hence \( x_{i''} \in (J_l : m_l) \) for all \( 1 < i'' < i \).

(c) For all \( 3 \leq k'' < k \) we have \( x_{e''} x_{k''} \in J_l \) and thus \( x_{k''} \in (J_l : m_l) \).

Since \( s < l \), we have \( m_s = x_{e''} x_{k''} \in \mathcal{G}_{e'} \) for some \( e' \in E(\bar{C}) \) with \( e' \leq e \) and some \( 3 \leq k' \leq t + 4 \). Since \( \deg m_{s,l} > 1 \) we have \( e' < e \). If \( e' = \{ i', j' \} \) with \( j' < j \), then \( m_{s,l} = x_{j'} x_{k'} \). Since \( x_{j'} x_{k'} \) is a minimal generator one deduces from (a) that \( j' \notin \{ i, i - 1, i + 1 \} \) which is impossible because \( e' \in E(\bar{C}) \). So we have \( e' = \{ i', j' \} \) with \( i' < i \).

If \( j' > 1 \), then by (b), \( x_{i'} \notin \supp(m_{s,l}) \). Thus \( m_{s,l} = x_{j'} x_{k'} \), where \( k' > k \) by (c), and also \( i' \notin \{ j, k \} \). Since \( 1 < i' < i < j \), we conclude that \( i' = k \) and hence \( k < i < j \). Therefore \( k \notin \{ j, j - 1, j + 1 \} \) and \( e'' = \{ k, j \} \in E(\bar{C}) \) with \( e'' < e' \). But \( 1 < i' < i \) implies that \( i > 3 \) and hence \( m_l = x_i x_{e''} \in J_l \) which is impossible. Thus \( i' = 1 \).

Since \( i' < i \), \( x_1 \) does not divide \( m_l \) and it follows that \( x_1 \notin (J_l : m_l) \) because \( m_{s,l} \) is a minimal generator. If \( i \neq 2 \), then \( e'' = \{ 1, i \} \in E(\bar{C}) \) and hence \( x_{e''} x_k \in J_l \) which implies that \( x_1 \in (J_l : m_l) \) which is not true. Therefore \( i = 2 \).
If \( j \neq t + 3 \), then \( e'' := \{1, j\} \in E(\overline{C}) \), so \( x_{e''} x_k \in J_l \) which implies that \( x_1 \in (J_l : m_t) \) which is not true. Therefore \( j = t + 3 \). This in particular implies that \( k \in \{3, t + 3, t + 4\} \) since otherwise setting \( e'' := \{2, k\} \) one has \( m_t = x_{t+3} x_{e''} \in J_l \) because \( e'' < e \).

If \( j'' \neq 3 \), then by (a), \( x_{j''} \in (J_l : m_t) \) and hence \( x_{j''} \notin \text{supp}(m_{s,l}) \) which means that \( j'' \in \{2, t + 3, k\} \), and hence \( j'' = k \) which is a contradiction because \( k \in \{3, t + 3, t + 4\} \). Therefore \( j'' = 3 \).

Finally, if \( k = 3 \), then \( x_{e''} x_{t+3} \in J_l \) which implies that \( x_1 \in (J_l : m_t) \) which is not the case. Hence \( k \in \{t + 3, t + 4\} \). It follows that \( x_{e''} x_3 \in J_l \) and hence \( x_3 \in (J_l : m_t) \) which implies that \( m_{s,l} = x_1 x_3 \). Therefore \( x_3 \in \text{supp}(m_t) = \{x_2, x_{t+3}, x_k\} \), a contradiction. So we are done in case (i).

Next consider case (ii). We have \( m_s \in \bigcup_{e \in E(\overline{C})} G_e \cup G_1 \) and either \( m_t = x_1 x_{i+1} x_{t+5} \) for some \( 3 \leq i \leq t + 2 \) or \( m_t = x_i x_j x_{t+5} \) for some \( 3 \leq i < j - 1 \leq t + 2 \) or \( m_t = x_i x_{t+4} x_{t+5} \) for some \( 3 \leq i \leq t + 3 \). Since \( \text{deg}_{m_{s,l}} x_{t+5} \leq 1 \) we have \( x_{t+5} \notin \text{supp}(m_{s,l}) \).

First let \( m_t = x_1 x_{i+1} x_{t+5} \) with \( 3 \leq i \leq t + 2 \). Note that

(a) we have \( e := \{i, j\} \in E(\overline{C}) \) for all \( j \in \{t + 3\} \setminus \{i, i - 1, i + 1\} \). Thus \( x_{e} x_{i+1} \in J_l \) implies that \( x_j \in (J_l : m_t) \).

(b) We have \( e := \{i + 1, j\} \in E(\overline{C}) \) with \( j \in \{t + 3\} \setminus \{i, i + 1, i + 2\} \). Hence \( x_{e} x_i \in J_l \) and so \( x_j \in (J_l : m_t) \).

Using (a), (b), one has \( \text{supp}(m_{s,l}) \subseteq \{x_1, x_i+1, x_{t+4}\} \). Thus \( m_s \notin G_e \) for all \( e \in E(\overline{C}) \). Therefore \( m_s \in G_1 \) and hence \( m_s \leq_{lex} m_t \). This implies that \( m_s = x_1 x_{t+5} \). Therefore \( m_{s,l} = x_1 \) which contradicts the assumption that \( \text{deg}_{m_{s,l}} > 1 \).

Next let \( m_t = x_i x_j x_{t+5} \) for some \( 3 \leq i < j - 1 \leq t + 2 \). Since \( \{i, j\} \in E(\overline{C}) \), one has \( x_i x_j x_k \in J_l \) for all \( 3 \leq k \leq t + 4 \). Hence \( x_k \in (J_l : m_t) \). It follows that \( \text{supp}(m_{s,l}) \subseteq \{x_1, x_2\} \). Since \( \text{deg}_{m_{s,l}} > 1 \) and \( x_1^2, x_2^2 \) divide none of the generators of \( J \), we conclude that \( x_1 x_2 \) divides \( m_s \). But there is no generator of \( J \) divided by \( x_1 x_2 \), a contradiction.

Now let \( m_t = x_i x_{t+3} x_{t+5} \) for some \( 3 \leq i \leq t + 3 \).

(a) For all \( j \in \{t + 3\} \setminus \{i, i - 1, i + 1\} \), we have \( e = \{i, j\} \in E(\overline{C}) \). Hence \( x_{e} x_{t+4} \in J_l \) and so \( x_j \in (J_l : m_t) \).

(b) If \( i \neq t + 3 \), we have \( m = x_i x_{i+1} x_{t+5} < m_t \) and hence \( x_{i+1} \in (J_l : m_t) \).

(c) If \( i = t + 3 \), we have \( m = x_i x_{t+3} x_{t+5} < m_t \) and hence \( x_{i-1} \in (J_l : m_t) \).

(d) Since \( \text{deg}_{m_{s,l}} x_{t+5} \leq 1 \), \( \text{deg}_{m_s} x_{t+4} \leq 1 \), we have \( x_{t+4}, x_{t+5} \notin \text{supp}(m_{s,l}) \).

Let \( i \neq t + 3 \). Then by (a), (b), (d) we have \( \text{supp}(m_{s,l}) \subseteq \{x_{i-1}, x_i\} \). Thus \( \text{supp}(m_s) \subseteq \{x_{i-1}, x_i, x_{t+4}, x_{t+5}\} \). This implies that \( x_{e} \) does not divide \( m_s \) for all choices of \( e \in E(\overline{C}) \) and hence \( m_s \notin G_e \). Consequently, \( m_s \in G_1 \) with \( m_s \leq_{lex} m_t \) and thus \( m_s \in \{x_{i-1}, x_i, x_{t+4}, x_{t+5}\} \). In either of the cases \( \text{deg}_{m_{s,l}} = 1 \), a contradiction.

Let \( i = t + 3 \). Then by (a), (c), (d) we have \( \text{supp}(m_{s,l}) \subseteq \{x_1, x_{t+3}\} \). It follows that \( \text{supp}(m_s) \subseteq \{x_1, x_{t+3}, x_{t+4}, x_{t+5}\} \). Hence \( x_{e} \) does not divide \( m_s \) for any choice of \( e \in E(\overline{C}) \) which implies that \( m_s \in G_1 \). Thus \( x_1 \notin \text{supp}(m_s) \). But for any \( m \in G(J) \) with \( \text{supp}(m) \subseteq \{x_{t+3}, x_{t+4}, x_{t+5}\} \) one has \( m \geq m_t \). So \( \text{supp}(m_s) \notin \{x_{t+3}, x_{t+4}, x_{t+5}\} \), a contradiction.

Finally consider case (iii). Then \( m_t = x_i^2 x_{t+5} \) for some \( 3 \leq i \leq t + 4 \).

(a) If \( i \neq t + 4 \) we have \( e = \{i, j\} \in E(\overline{C}) \) for \( j \in \{t + 3\} \setminus \{i, i - 1, i + 1\} \). It follows that \( x_{e} x_j \in J_l \) and hence \( x_j \in (J_l : m_t) \).

(b) If \( i = t + 4 \) we have \( x_i x_{t+4} x_{t+5} \in G_1 \) for all \( 3 \leq j \leq t + 3 \); thus \( x_j \in (J_l : m_t) \).

(c) If \( i \neq t + 4 \), \( x_i x_{i+1} x_{t+5}, x_i x_{t+4} x_{t+5} \in G_1 \); so \( x_{i+1}, x_{t+4} \in (J_l : m_t) \).

(d) If \( i \neq 3 \), \( x_{i-1} x_i x_{t+5} \in G_1 \); so \( x_{i-1} \in (J_l : m_t) \).
(e) Since $\deg_{m_s} x_{t+5} \leq 1$ and $\deg_{m_s} x_i \leq 2$, we have $x_{t+5}, x_i \notin \text{supp}(m_s,t)$.

If $i \neq t+3, t+4$, then $i+1 = i + 1$ and by (a), (c), (e), $\text{supp}(m_s,t) \subseteq \{x_{i-1}\}$. Since $\deg m_s,t > 1$ we have $m_s = x_{i-1}^j x_j$ for some $j \in \{i, t+5\}$. Since $x_{i-1}^j x_j$ is not a generator of $J$, we have $m_s = x_{i-1}x_{t+5}$ which is a generator of $J$ only if $i \neq 3$. Now (d) implies that $m_s,t$ is not a minimal generator of $(J_t : m_t)$, a contradiction.

If $i = t+3$, then by (a), (c), (d), (e), $\text{supp}(m_s,t) \subseteq \{x_1\}$. It follows that $m_s = x_1^j x_j$ for some $j \in \{t+3, t+5\}$ because $\deg m_s,t > 1$. But there is no such $m_s$ in $J$.

If $i = t+4$, then by (b), (e), $\text{supp}(m_s,t) \subseteq \{x_1, x_2\}$. As discussed in the case $m_t = x_i x_j x_{t+5}$, we have $m_s \notin \bigcup_{e \in E(G)} G_e \cup G_1$. But it does not belong to $G_2$ either, a contradiction.

We showed that for any choice of $m_s < m_t$, it is impossible to have $m_s,t \in G((J_t : m_t))$ with $\deg m_s,t > 1$. Hence $(J_t : m_t)$ is generated by variables, as desired. \hfill \Box

Now we use Proposition 3.6 to show that $I(G(b))^2$ has a linear resolution when $G(b)$ does not have an induced 4-cycle.

**Theorem 3.7.** Let $G$ be a graph on $n \geq 7$ vertices with no isolated vertex and let $G(b)$ be its complement. Let $I := I(G)$ be the edge ideal of $G$. Then $I^2$ has a linear resolution.

**Proof.** By construction, $n = t+5$. We apply Lemma 3.5 for $I^2$ and $x := x_{t+5}$ to prove the assertion. To do this, we first compute $\text{reg}(I^2 + (x_{t+5}))$. Setting $C = 1 - 2 \cdots - (t+3) - 1$, we have

$$I^2 + (x_{t+5}) = (I(C) + (x_{t+5})(x_3, \ldots, x_{t+4}))^2 + (x_{t+5}) = I(C)^2 + (x_{t+5}).$$

Since $x_{t+5}$ does not appear in the support of the generators of $I(C)^2$, we have

$$\text{reg}(I^2 + (x_{t+5})) = \text{reg}(I(C)^2 + (x_{t+5})) = \text{reg}(I(C)^2).$$

It is proved in [2, Corollary 4.4] that $I(C)^k$ has a linear resolution for $k > 1$ when $|C| > 4$, which is the case here because $t + 3 > 4$. Thus $\text{reg}(I^2 + (x_{t+5})) = 4$. On the other hand $(I^2 : x_{t+5})$ has linear quotients by Proposition 3.6, and it is seen in the proof of this proposition that $(I^2 : x_{t+5})$ is generated in degree 3. Therefore, $(I^2 : x_{t+5})$ has a 3-linear resolution, see [19, Theorem 8.2.1]. It follows that $\text{reg}((I^2 : x_{t+5})) = 3$. Now using Lemma 3.5 we have $\text{reg}(I^2) \leq 4$. Since $I^2$ is generated in degree 4 we conclude that $I^2$ has a linear resolution. \hfill \Box

Now we have all the required tools to prove the main theorem of this section.

**Proof of Theorem 3.1.** By Proposition 3.2 it suffices to show that if $G$ is gap-free and if $I(G)$ has almost maximal finite index, then $I(G)^s$ has a linear resolution for all $s \geq 2$.

According to Remarks 1 and 2, it remains to prove that for all $s \geq 2$ the ideal $I(G)^s$ has a linear resolution, where $G \in \{G(a)_3, G(b)_3\}$ and $t > 1$. In Theorem 3.7 we saw that $I(G(b))^2$ has a linear resolution. Now we use Theorem 3.3 to prove the assertion in general. To this end we first show that for any

(i) $s \geq 1$ when $\bar{G} = G(a)_3$,

(ii) $s \geq 2$ when $\bar{G} = G(b)_3$

and any $s$-fold product $e_1 \cdots e_s$ of the edges in $G$, the graph $\bar{G}_{e_1 \cdots e_s}$ is chordal, where $G_{e_1 \cdots e_s}$ is a simple graph explained in Lemma 3.4 with the edge ideal $I(G_{e_1 \cdots e_s}) = (I^{s+1} : x_{e_1} \cdots x_{e_s})_{\text{pol}}$.

Since by [1, Lemma 6.15], any induced cycle of $\bar{G}_{e_1 \cdots e_s}$ is an induced cycle of $\bar{G}$, we conclude that if $\bar{G}_{e_1 \cdots e_s}$ contains an induced cycle $C$ of length $> 3$, then in case $\bar{G} = G(a)_3$...
we have $C \in \{C_1, C_2\}$, where $C_1 = 1 - 2 - \cdots -(t + 3) - 1$ and $C_2 = 1 - (t + 4) - 3 - 4 - \cdots -(t + 3) - 1$, and in case $\bar{G} = G(b)$, we have $C = C_1$. Thus, in order to prove that $\bar{G}_{e_1 \cdots e_s}$ is chordal, we need to show that

(i) $C_1, C_2$ are not induced cycles in $\bar{G}_{e_1 \cdots e_s}$, where $\bar{G} = G(a)_3$ and $s \geq 1$;

(ii) $C_1$ is not an induced cycle in $\bar{G}_{e_1 \cdots e_s}$, where $\bar{G} = G(b)$ and $s \geq 2$.

We claim that there exist $k, l \in V(\bar{G}_{e_1 \cdots e_s})$ such that $\{k, l\} \in E(\bar{G}_{e_1 \cdots e_s}) \cap E(C_r)$, $r = 1, 2$. It follows that $C_r$ is not a subgraph of $\bar{G}_{e_1 \cdots e_s}$, and hence (i), (ii) hold.

**Proof of the claim:** Let $e_1 = \{i, j\}$ with $i < j$. First let $\bar{G} = G(a)_3$ and $s \geq 1$. We choose $\{k, l\} \in E(C_1)$ as follows:

(a) If $e_1 = \{4, t + 4\}$, then let $k = 1$ and $l = t + 3$;

(b) if $e_1 = \{1, t + 2\}$, then let $k = 3$ and $l = 2$;

(c) if $e_1 = \{2, t + 3\}$, then let $k = 4$ and $l = 3$;

(d) otherwise, let $k = i - 2$ and $l = i - 1$.

Since $C_2$ is obtained from $C_1$ by replacing 2 with $t + 4$, in order to find $\{k, l\} \in E(C_2)$, we choose $\{k, l\} \in E(C_2)$ as suggested in (a) – (d) with an extra condition that if $\{k, l\}$ is obtained from (b), (d) and it contains 2, then we replace 2 with $t + 4$ in this pair.

Now let $\bar{G} = G(b)$ and $s \geq 2$. Then there exists $e_i, 1 \leq i \leq s$, such that $e_i \notin \{t + 4, t + 5\}$. We may assume that $i = 1$. Since $\{t + 4, t + 5\}$ is the only edge in $G$ containing $t + 4$, we have $t + 4 \notin e_1$. We choose $\{k, l\} \in E(C_1)$ as given in (b) – (d) and $(a')$ below:

(a') if $e_1 = \{3, t + 5\}$, then let $k = 1$ and $l = t + 3$.

By the above choices of $k, l$, although $\{k, l\} \notin E(G)$, we have $\{k, i\}, \{j, l\} \in E(G)$. It follows that $k - i - j - l$ is a path in $G$ and hence, by definition, $k$ and $l$ are even-connected with respect to $e_1 \cdots e_s$. Therefore $\{k, l\} \in E(\bar{G}_{e_1 \cdots e_s})$. This completes the proof of the claim.

Now since $\bar{G}_{e_1 \cdots e_s}$ is chordal for $s \geq 1$ when $G = \bar{G}_{(a)_3}$, and for $s \geq 2$ when $G = \bar{G}_{(b)}$, by [14, Theorem 1], $I(\bar{G}_{e_1 \cdots e_s})$ has a 2-linear resolution for $s \geq 1$ when $G = \bar{G}_{(a)_3}$, and for $s \geq 2$ when $G = \bar{G}_{(b)}$. It follows that

$$\text{reg}(I^{s+1} : x_{e_1} \cdots x_{e_s}) = \text{reg}(I^{s+1} : x_{e_1} \cdots x_{e_s})^{pol} = \text{reg}(I(G_{e_1 \cdots e_s})) = 2 \quad \text{for} \quad \left\{ \begin{array}{ll} s \geq 1 & \text{if } G = \bar{G}_{(a)_3}, \\ s \geq 2 & \text{if } G = \bar{G}_{(b)}, \end{array} \right.$$ 

and for any choice of the edges $e_1, \ldots, e_s$ of $G$. The first equality follows from [19, Corollary 1.6.3]. Note that the above result does not hold for $G = \bar{G}_{(b)}$ with $s = 1$ as we have $\text{reg}(I(G)^2 : x_{e_1}) \neq 2$ for $e = \{t + 4, t + 5\}$ and large $t$.

Suppose $G = \bar{G}_{(a)_3}$. By Proposition 2.10 we have $\text{reg}(I) = 3$. Theorem 3.3 implies that $\text{reg}(I^2) \leq 4$. Since $I^2$ is generated in degree 4 we conclude that $\text{reg}(I^2) = 4$. In case $G = \bar{G}_{(b)}$, although we can not use Theorem 3.3 to compute the regularity of $I^2$, we know directly from Theorem 3.7 that $\text{reg}(I^2) = 4$.

Now induction on $s > 1$ and using Theorem 3.3 together with the fact that $\text{reg}(I^s) \geq 2s$ yield the assertion. 

□

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