Unavoidable chromatic patterns in 2-colorings of the complete graph

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Funding information
Dirección General de Asuntos del Personal Académico, Universidad Nacional Autónoma de México, Grant/Award Numbers: PAPIIT IN111819, PAPIIT IN116519; Consejo Nacional de Ciencia y Tecnología, Grant/Award Number: 282280

Abstract
Given a graph $G$ on $k$ edges, we consider the following two extremal problems: provided $n$ is large enough, what is the minimum integer $\text{bal}(n, G)$, if it exists, such that any 2-coloring of the edges of a complete graph on $n$ vertices having more than $\text{bal}(n, G)$ edges in each color class, contains a balanced copy of $G$, that is, a copy of $G$ with exactly $\lceil k/2 \rceil$ edges in one of the colors? Graphs for which this is possible are called balanceable. The second problem deals with a similar question but we seek to guarantee copies of $G$ in every tone, that is, having exactly $r$ edges in, say, color red, for every $0 \leq r \leq k$. Graphs with this property are called omnitonal. We study these problems for different graph families, including paths, stars, and trees in general. When studying such extremal parameters, the question of its existence is obliged. In this line, two universal unavoidable patterns in 2-edge-colorings of the complete graph with sufficient representation in each of the colors emerge naturally and they are the key to characterizing balanceable as well as omnitonal graphs. For the two universal unavoidable patterns, which were already known to exist via a Ramsey-theoretic approach, we present here a Turán-type counterpart.

KEYWORDS
edge colorings, extremal graph theory, Ramsey theory, unavoidable patterns
INTRODUCTION

Our main interest in this paper is a certain kind of problems that lie in the junction of Ramsey theory, extremal graph theory, zero-sum Ramsey theory, and interpolation theorems in graph theory; general references to these topics are [1–3,6,20,23–25].

We consider 2-colorings of the set of edges $E(K_n)$ of the complete graph $K_n$. Given a graph $G$ with $e(G)$ edges, nonnegative integers $r$ and $b$ such that $r + b = e(G)$, and a 2-coloring $f : E(K_n) \to \{\text{red}, \text{blue}\}$, we say that $f$ induces an $(r, b)$-colored copy of $G$, if there is a copy of $G$ in $K_n$ such that $f$ assigns the color red to exactly $r$ edges and the color blue to exactly $b$ edges of that copy of $G$.

By Ramsey’s theorem we know that any 2-coloring of $E(K_n)$ (where $n$ is sufficiently large) induces either an $(e(G), 0)$-colored copy of $G$, or a $(0, e(G))$-colored copy of $G$. To force the existence of $G$ with other color patterns, we need, as a natural minimum requirement, not only to ensure a large $n$, but also a minimum amount of edges of each color. In this paper, we study which graphs are unavoidable under a prescribed color pattern in every 2-coloring of $E(K_n)$, whenever $n$ is sufficiently large and there are enough edges from each color. A similar approach has been studied in [5,12,17], where the emphasis is given on determining the minimum $n$ required to guarantee the existence of a given graph with a prescribed color pattern in every coloring of $K_n$ where each color appears in some positive fraction of the edges of $K_n$. In contrast, our approach has a Turán flavor in the sense that we focus our attention, on the one hand, on the maximum edge number that can have the smallest color class in a 2-coloring of $E(K_n)$ which is free of a copy of the given graph in the prescribed pattern, and, on the other hand, in characterizing the extremal colorings.

Observe that, in case $e(G) \equiv 0 \pmod{2}$, the study of the existence of a zero-sum copy of $G$ over $\mathbb{Z}$-weightings of $E(K_n)$, in particular over $\{-1, 1\}$-weightings of $E(K_n)$, carries along similarity to classical Ramsey theory by simply defining all red edges to have weight $-1$ and all blue edges to have weight 1. Thus, a zero-sum copy of $G$ translates into a copy of $G$ with equal number of red and blue edges, or equivalently to an $(e(G)/2, e(G)/2)$-colored copy of $G$. The study of the existence of such a balanced copy will be one of the purposes of this paper, thus further developing the line of research studied in [8,9,11]. The second problem we will focus on, and in which we will make use of interpolation techniques, deals with the existence of an $(r, b)$-colored copy of $G$ for every pair of nonnegative integers $r$ and $b$ such that $r + b = e(G)$. As in the previous case, this problem is related to the study of the existence of a zero-sum copy of $G$ over $\mathbb{Z}$-weightings of $E(K_n)$ with range $\{-p, q\}$, where $p$ and $q$ are positive integers with $\gcd(p, q) = 1$ and $e(G) \equiv 0 \pmod{p + q}$. These problems will lead us to the definition of two graph families which will be the center of this study: balanceable and omnitonal graphs.

1.1 Balanceable and omnitonal graphs

In the entire paper, we will deal with red–blue 2-edge-colorings of the complete graph, that is, mappings $f : E(K_n) \to \{\text{red}, \text{blue}\}$ or, equivalently, partitions $E(K_n) = E(R) \cup E(B)$ where we implicitly assume that $R$ and $B$ are the graphs induced by the red and the blue edges, respectively. Thus, $e(R)$ and $e(B)$ are the number of red and blue edges in a given 2-edge-coloring of $K_n$.

For a given graph $G$, we say that a 2-edge-coloring of $K_n$ contains a balanced copy of $G$, if we can find a $(e(G)/2, e(G)/2)$-colored copy of $G$ in case $e(G) \equiv 0 \pmod{2}$, and a
([\bar{e}(G)/2], [\bar{e}(G)/2])$-colored copy of $G$ or a $([\bar{e}(G)/2], [\bar{e}(G)/2])$-colored copy of $G$ in case $e(G) \equiv 1 \pmod{2}$.

**Definition 1.1.** For a given graph $G$ let $\text{bal}(n, G)$ be the minimum integer, if it exists, such that any 2-coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} > \text{bal}(n, G)$ contains a balanced copy of $G$. If $\text{bal}(n, G)$ exists for every sufficiently large $n$, we say that $G$ is balanceable. For a balanceable graph $G$, let $\text{Bal}(n, G)$ be the family of graphs with exactly $\text{bal}(n, G)$ edges such that a coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} = \text{bal}(n, G)$ contains no balanced copy of $G$ if and only if $R$ or $B$ is isomorphic to some $H \in \text{Bal}(n, G)$.

Omnitonal graphs will be those that appear in all possible tonal variations of red and blue in every 2-edge-coloring of the complete graph, as long as the latter is large enough. Clearly, omnitonal graphs are balanceable.

**Definition 1.2.** For a given graph $G$, let $\text{ot}(n, G)$ be the minimum integer, if it exists, such that any 2-coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} > \text{ot}(n, G)$ contains an $(r, b)$-colored copy of $G$ for any $r \geq 0$ and $b \geq 0$ such that $r + b = e(G)$. If $\text{ot}(n, G)$ exists for every sufficiently large $n$, we say that $G$ is omnitonal. For an omnitonal graph $G$, let $\text{Ot}(n, G)$ be the family of graphs with exactly $\text{ot}(n, G)$ edges such that a coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} = \text{ot}(n, G)$ contains no $(r, b)$-colored copy of $G$ for some pair $r, b \geq 0$ with $r + b = e(G)$ if and only if $R$ or $B$ is isomorphic to some $H \in \text{Ot}(n, G)$.

We are unaware of a systematic study along the lines suggested by the omnitonal graphs, which we start here. We shall be interested in finding balanceable and omnitonal graph families as well as in determining or finding good estimates for $\text{bal}(n, G)$ and/or $\text{ot}(n, G)$ and in characterizing the extremal colorings, that is, in finding $\text{Bal}(n, G)$ and/or $\text{Ot}(n, G)$.

The connection from balanceable graphs to the zero-sum analogue in $\{-1, 1\}$-weightings is already explained, establishing the bridge between the current paper and the results given in [8, 9, 11]. The connection of omnitonal graphs to the zero-sum problem with $\{-p, q\}$-weightings is explained below.

**Remark 1.3.** Let $p$ and $q$ be positive integers with $\gcd(p, q) = 1$ and let $G$ be an omnitonal graph with $e(G) \equiv 0 \pmod{p + q}$. Then, for large enough $n$, any coloring $f: E(K_n) \rightarrow \{-p, q\}$ with $\min\{|f^{-1}(-p)|, |f^{-1}(q)|\} > \text{ot}(n, G)$ contains a zero-sum copy $G^*$ of $G$ (i.e., a copy $G^*$ of $G$ where $\sum_{e \in E(G^*)} f(e) = 0$). To see this, define another coloring $g: E(K_n) \rightarrow \{\text{red}, \text{blue}\}$ where $g(e) = \text{red}$ iff $f(e) = -p$ and $g(e) = \text{blue}$ iff $f(e) = q$. Since $G$ is omnitonal, and $g$ is such that $\min\{e(R), e(B)\} > \text{ot}(n, G)$, we can find an $(r, b)$-colored copy $G^*$ of $G$ with $r = \frac{pe(G)}{p + q}$ and $b = \frac{qe(G)}{p + q}$. Then $G^*$ is a zero-sum copy of $G$ under coloring $f$:

$$\sum_{e \in E(G^*)} f(e) = -qr + pb = \frac{-qpe(G^*)}{p + q} + \frac{pqe(G^*)}{p + q} = \frac{(-qp + qp)e(G^*)}{p + q} = 0.$$
1.2 | Structure of the paper

After this introductory part where we established the general problem setting, the article is divided into two sections.

In Section 2 we give necessary and sufficient conditions for a graph to be balanceable and, respectively, omnitonal (Theorems 2.6 and 2.5). To this aim, we study two particular colorings, called the type-A and type-B colorings (to be defined at the beginning of Section 2) that arise naturally in the study of unavoidable chromatic patterns in 2-edge-colorings of the complete graph. To provide the structural characterizations of balanceable and omnitonal graphs, we use two important facts about the type-A and type-B colorings: first, that there are infinitely many values of \( n \) for which both patterns can be balanced (see Lemma 2.1); second, that precisely these colorings are proved to be the universal unavoidable patterns (see Theorem 2.3). The latter fact was shown, in the domain of Ramsey theory, by Cutler and Montágh in [12], where they explicitly require \( \epsilon \left( \frac{n}{2} \right) \) edges of each color to imply the existence of a type-A or a type-B coloring. In Section 2.1 we present a Turán-type version of Theorem 2.3, where without seeking sharp bounds, we prescind from the quadratic amount of edges of each color, and replace it with a subquadratic constraint. In particular, we also provide an alternative proof of the existence of this universal unavoidable patterns that rely on the classical Ramsey theorem and the Kővari–Sós–Turán theorem, with which the Ramsey–Turán nature of the problem is put into evidence.

In Section 3, we study and determine when possible, \( \text{bal}(n, G) \), \( \text{Bal}(n, G) \), \( \text{ot}(n, G) \), and \( \text{Ot}(n, G) \) when \( G \) belongs to different families of graphs. To determine if a graph \( G \) is omnitonal is a problem within the scope of interpolation theorems in graph theory. We dedicate Section 3.1 to the study of a class of graphs called amoebas (see Definition 3.2) developed here along the proof techniques used in interpolation theory, building upon ideas from [8,9]. In particular, we prove that every amoeba is balanceable (see Theorem 3.6) and that every bipartite amoeba \( G \) is omnitonal with \( \text{ot}(n, G) = \text{ex}(n, G) \) and \( \text{Ot}(n, G) = \text{Ex}(n, G) \), where \( \text{ex}(n, G) \) is the Turán number of \( G \) and \( \text{Ex}(n, G) \) the corresponding family of extremal graphs (see Theorem 3.5). Also, from the characterization of omnitonal graphs (Theorem 2.5), we can deduce that trees are omnitonal and, therefore, balanceable graphs (see Corollary 2.7). In Sections 3.2 and 3.3 we handle in detail the classes of paths and stars, determining all the parameters described above. Finally, in Section 3.4, we provide a linear (on \( n \)) upper bound for \( \text{ot}(n, T) \) where \( T \) is a tree; this bound yields naturally an upper bound for \( \text{bal}(n, T) \). We finish the paper by discussing, in Section 4, further variants of the concepts and presenting several open problems.

2 | UNIVERSAL UNAVOIDABLE PATTERNS AND CHARACTERIZATION OF BALANCEABLE AND OMNITONAL GRAPHS

Let \( t \) and \( m \) be integers with \( 1 \leq t < m \). A 2-edge-coloring of \( K_m \) is said to be of type-A \((t)\) if the edges of one of the colors induce a complete graph \( K_t \), and it is said to be of type-B \((t)\) if the edges of one of the colors induce a complete bipartite graph \( K_{t,m-t} \). If \( m = 2t \), we eliminate the parameter \( t \) and write for short type-A and type-B colorings.

When considering the problem of determining if a graph is balanceable or omnitonal, the study of type-A and type-B colorings arises naturally. For instance, to prove that a graph \( G \) is not omnitonal, it is enough to exhibit infinitely many values of \( n \) for which there is a particular
2-edge coloring of $K_n$ with the same number of red and blue edges without an $(r, b)$-colored copy of $G$ for some specific values of $r \geq 0$ and $b \geq 0$ such that $r + b = e(G)$. We will see below (in Theorems 2.6 and 2.5) that precisely the type-A and type-B colorings will play this role. In [8], the $n$’s for which there are balanced type-A or type-B colorings of $K_n$ were determined, which are in both cases an infinite number. The latter fact can be derived from Lemmas 3 and 4 in [8] and, since we will use it later on, we formulate it in the following lemma.

Lemma 2.1 (Caro et al [8]). For infinitely many positive integers $n$, we can choose $t = t(n)$ in a way that a type-A$(t)$ (resp., type-B$(t)$) coloring of $K_n$ is balanced.

Because of the fact that there are infinitely many values of $n$ for which the type-B$(t)$ pattern can be balanced, it is easy to conclude that omnitonal graphs are bipartite (since they, in particular, have to appear monochromatic in both colors).

Observation 2.2. Omnitonal graphs are bipartite.

We will use Lemma 2.1 to characterize the structure of omnitonal and of balanceable graphs. However, to this aim, we still need one more ingredient, namely, that the type-A and type-B colorings will be those proving to be the universal unavoidable patterns.

In the domain of Ramsey theory, it was conjectured by Bollobás (see [12]) and shown by Cutler and Montágh that, for sufficiently large $n$, every 2-edge-coloring of $K_n$ with a positive fraction of edges of each color contains a type-A or a type-B colored copy of $K_{2t}$.

Theorem 2.3 (Cutler and Montágh [12]). Let $0 < \varepsilon \leq \frac{1}{2}$ be a real number and $t \geq 1$ an integer. For large enough $n$, any 2-edge-coloring of $K_n$ with at least $\varepsilon \binom{n}{2}$ edges in each color contains a type-A or a type-B colored copy of $K_{2t}$.

The bound on the Ramsey parameter concerning this result was further improved by Fox and Sudakov in [17]. In both papers, the authors explicitly assume $\min\{e(R), e(B)\} = \varepsilon \binom{n}{2}$ for some $\varepsilon > 0$ and estimate an upper bound on the smallest $n$ for which this $\varepsilon$-balancing forces a type-A or a type-B colored copy of $K_{2t}$.

From Theorem 2.3 and Lemma 2.1, it is not hard to provide characterizations of omnitonal graphs and balanceable graphs (see Theorems 2.5 and 2.6), which will be presented in Section 2.2. Before that, we give in Theorem 2.4 a Turán-type version of Theorem 2.3, in which (without seeking sharp bounds) we prescind from the quadratic amount of edges of each color, and replace it with a subquadratic constraint. Aside from the Turán-type perspective that offers Theorem 2.4, together with the simplicity of its proof that relies on pure Ramsey–Turán combinatorial arguments, it also becomes relevant in our context because it implies that, in case of existence, $\text{bal}(n, G)$ and $\text{ot}(n, G)$ are always subquadratic as functions of $n$ (Theorems 2.5 and 2.6).

2.1 Turán-type approach for universal unavoidable patterns

Let $\mathcal{F}_i$ be the family of type-A and type-B colored $K_{2t}$’s. Define $\text{ex}_2(n, \mathcal{F}_i)$ as the maximum integer such that there is a 2-edge-coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} = \text{ex}_2(n, \mathcal{F}_i)$ and without a member of $\mathcal{F}_i$. 
For a given graph $G$, we denote by $R(G, G)$ the 2-color Ramsey number, that is, the minimum integer $R(G, G)$ such that, whenever $n \geq R(G, G)$, any coloring $E(K_n) = E(R) \cup E(B)$ contains either a blue or a red copy of $G$. For a given graph $G$, we denote by $\text{ex}(n, G)$ the Turán number for $G$, that is, the maximum number of edges in a graph with $n$ vertices containing no copy of $G$. The well-known Kővari–Sós–Turán theorem \cite{21} implies that, for the balanced complete bipartite graph $K_{t,t}$, \[
_{\text{ex}}(n, K_{t,t}) < \left(1 + \frac{1}{2}\right)\left(2^{t-1} + \frac{1}{2}(t-1)n\right). \tag{2} \]

Theorem 2.4. Let $t$ be a positive integer. For $n$ sufficiently large, there exists a positive integer $m = m(t)$ such that \[
_{\text{ex}}(n, \mathcal{F}_t) = O\left(n^2 - \frac{1}{m}\right). \]

Proof. Let $q \geq t$ be an integer such that \[(t - 1)^{1/t}(2q)^{2-1/t} + (t - 1)q + 1 \leq 2q^2, \tag{3} \]
and set $m = R(K_q, K_q)$. Now define \[\varphi(n,t) = \text{ex}(n, K_{m,m}) + m(m - 1) + m(n - 2m) + 1, \]
which, by (1), is clearly $O(n^{2 - \frac{1}{m}})$. Assume $n$ to be large enough such that we can take a coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} \geq \varphi(n,t)$. This is possible since $\varphi(n,t) = o(n^2)$. By definition, there is a monochromatic, say red, copy of $K_{m,m}$ in $K_n$. Let $X \cup Y$ be a vertex set partition of such a red copy of $K_{m,m}$, where $|X| = |Y| = m$ and all edges between $X$ and $Y$ are red. Consider now the complete graph $K_{n-2m}$ obtained from $K_n$ by removing the vertex set $X \cup Y$. Since we lose at most $2\left(\begin{array}{c}m \\ 2\end{array}\right) + 2m(n - 2m)$ blue edges, by the definition of $\varphi(n,t)$, there are at least $\text{ex}(n - 2m, K_{m,m}) + 1$ blue edges in $K_{n-2m}$. Hence, there is a blue copy of $K_{m,m}$ in $K_{n-2m}$. Let $Z \cup W$ be a vertex set partition of such a blue copy of $K_{m,m}$, where $|Z| = |W| = m$ and all edges between $Z$ and $W$ are blue. Observe that the red and the blue copies of $K_{m,m}$ that we obtain are vertex disjoint.

Now consider the 2-edge colored graph induced by $X \cup Y$. By the definition of $m$, we know that there are monochromatic copies of $K_q$ inside both $X$ and $Y$. If at least one of these monochromatic copies of $K_q$ is blue, then, since all edges between $X$ and $Y$ are red, we will have a copy of $K_{2q}$ which is either of type-$A$ or of type-$B$; and since $q \geq t$ we are
done in this case. Otherwise, we get two red monochromatic copies of $K_2$, one inside $X$ and the other inside $Y$, which indeed is a monochromatic, red, copy of $K_2$. Similarly, by looking at the 2-edge colored graph induced by $Z \cup W$, either we are done or we get a blue copy of $K_2$. Hence, we can assume that we have two vertex disjoint monochromatic copies of $K_2$, one red and one blue. Call $C$ the set of vertices of the red one, and $D$ the set of vertices of the blue one.

Finally, we consider the 2-edge colored complete bipartite graph, $K_{2q,2q}$, induced by $C \cup D$. Clearly, one of the colors, say red, has at least half of the edges. In other words, there are at least $\frac{1}{2}(2q)^2 = 2q^2$ red edges in $K_{2q,2q}$. By computing the upper bound (2) of the Zarankiewicz number $z(2q, t)$, we obtain the left-hand side of (3). Thus, by the definition of $q$, we gain a monochromatic copy of $K_t$ in $K_{2q,2q}$. That is, there are subsets $C' \subset C$ and $D' \subset D$, with $|C'| = |D'| = t$, such that all edges between $C'$ and $D'$ are red. Observe that the 2-edge colored complete graph $K_{2q}$ induced by $C' \cup D'$ is of type-$A$, which completes the proof.

Hence, we have shown that $\text{ex}_2(n, \mathcal{F}_t) \leq \varphi(n, t) = \mathcal{O}(n^{2-\frac{1}{q}})$. □

We would like to add here that, after the submission of this paper, some other results of this flavor have been announced. In particular, in a recent paper uploaded to ArXiv, Girão and Narayanan [19] proved that $\text{ex}(n, \mathcal{F}_t) = \mathcal{O}(n^{2-\frac{1}{q}})$ and, conditional on the Kővari–Sós–Turán conjecture that $\text{ex}(K_t, r) = \Omega(n^{2-1/\ell})$ for all $t$ [21], they showed that this bound is sharp up to the involved constants. Also, Theorem 2.4 has been already extended (and the constant factor was tightened, too) to an arbitrary number of colors in a more recent work by Bowen, Müyesser and the second and third authors of this paper [4]. Other interesting Ramsey–Turán results concerning families of 2-colored graphs contained in arbitrarily 2-colorings of graphs with at least an $\epsilon$-density on each of the colors have been obtained in [14,22].

### 2.2 | Characterization of balanceable and omnitonal graphs

In this section, we give a characterization of balanceable and omnitonal graphs that is derived from Theorem 2.4 and the fact stated in Lemma 2.1.

Given a partition $V(G) = X \cup Y$ of the vertex set of a graph $G$, we denote by $e(X, Y)$ the number of edges of $G$ with one end in $X$ and the other one in $Y$. Also, for a set $W \subseteq V(G)$, $G[W]$ stands for the subgraph of $G$ induced by the vertices in $W$ and $e(G[W])$ for its number of edges.

**Theorem 2.5.** A graph $G$ is omnitonal if and only if, for every integer $r$ with $0 \leq r \leq e(G)$, $G$ has both a partition $V(G) = X \cup Y$ and a set of vertices $W \subseteq V(G)$ such that

$$e(X, Y) = e(G[W]) = r.$$ 

Moreover, if $G$ is omnitonal, $\text{ot}(n, G) = \mathcal{O}(n^{2-\frac{1}{q}})$, where $m = m(G)$ depends only on $G$.

**Proof:** Suppose that $G$ is omnitonal. Let $n$ be large enough such that $\text{ot}(n, G)$ exists and chosen such that there is a balanced type-$A(t)$ coloring of $K_n$ for some $t = t(n)$, which is
possible by Lemma 2.1. Suppose, without loss of generality, that the graph induced by the red edges in such a coloring of $K_n$ is isomorphic to $K_t$. Since $G$ is omnitonal and $\text{ot}(n, G) \leq \left\lfloor \frac{n}{2} \right\rfloor = e(R) = e(B)$, there must be a copy of $G$ in $K_n$ with $r$ red edges for every $0 \leq r \leq e(G)$. This implies that there is a set $W \subseteq V(G)$ with $e(G[W]) = r$ for every $0 \leq r \leq e(G)$. Analogously, we take now an $n$ large enough such that $\text{ot}(n, G)$ exists and chosen such that there is a balanced type-$B(t)$ coloring of $K_n$ for some $t = t(n)$, which, again, is possible by Lemma 2.1. Suppose, without loss of generality, that the graph induced by the red edges in such a coloring of $K_n$ is isomorphic to $K_n - t$. Since $G$ is omnitonal and $\text{ot}(n, G) = (n - 1) - n$, there must be a copy of $G$ in $K_n$ with $r$ red edges for every $0 \leq r \leq e(G)$. It follows that there is a partition $V(G) = X \cup Y$ with $e(X, Y) = r$ for every $0 \leq r \leq e(G)$.

Conversely, suppose that $G$ has both a partition $V(G) = X \cup Y$ and a set of vertices $W \subseteq V(G)$ such that $e(X, Y) = e(G[W]) = r$ for every $0 \leq r \leq e(G)$. Let $E(K_n) = E(R) \cup E(B)$ be an edge coloring of $K_n$ with $\min\{e(R), e(B)\} \geq \text{ex}(n, \mathcal{F}_t)$, where $t = n(G)$. Hence, by Theorem 2.4, for $n$ sufficiently large, there is a type-$A$ or a type-$B$ colored copy of $K_{2t}$. If this copy is of type-$A$, then there are two possibilities: either we have one red $K_t$ and one blue $K_t$ and all edges in between are blue or the colors are reversed. In the first case, we can use a set $W$ with $e(G[W]) = r$ to find a copy of $G$ with $r$ red edges and $e(G) - r$ blue edges. In the second, we can use a set $W'$ with $e(G[W']) = e(G) - r$ to find a copy of $G$ with $r$ red edges and $e(G) - r$ blue edges. The case of having a type-$B$ colored copy of $K_{2t}$ is similar. $\square$

With similar arguments, we can obtain the following characterization of balanceable graphs.

**Theorem 2.6.** A graph $G$ is balanceable if and only if $G$ has both a partition $V(G) = X \cup Y$ and a set of vertices $W \subseteq V(G)$ such that

$$e(X, Y), e(G[W]) \in \left\{ \left\lfloor \frac{e(G)}{2} \right\rfloor, \left\lceil \frac{e(G)}{2} \right\rceil \right\}.$$

Moreover, if $G$ is balanceable, $\text{bal}(n, G) = O(n^{2-\frac{1}{m}})$, where $m = m(G)$ depends only on $G$. $\square$

By means of this characterization, a study of different graph families with respect to their balanceability is given in [13].

The next result is a consequence of Theorem 2.5.

**Corollary 2.7.** Every tree is omnitonal (and, therefore, balanceable).

**Proof.** Let $T$ be a tree. According to Theorem 2.5, we have to verify that, for every integer $r$ with $0 \leq r \leq e(T)$, $T$ has both a partition $V(T) = X \cup Y$ and a set of vertices $W \subseteq V(T)$ such that $e(X, Y) = e(G[W]) = r$. We proceed by induction on $e(T)$.

If $e(T) = 1$, then both conditions are clearly satisfied for $0 \leq r \leq 1$. Let $T$ be a tree with $e(T) = m$, and let $v \in V(T)$ be a leaf where $u$ is the only vertex of $T$ adjacent to $v$. By the induction hypothesis, the tree $T' = T - \{v\}$ satisfies that, for every $0 \leq r \leq m - 1 = e(T')$, there
are both a partition $V(T') = X' \cup Y'$ and a set of vertices $W \subseteq V(T')$ such that $e(X', Y') = e(T'[W]) = r$. Note that for every $0 \leq r \leq m - 1$ the subset $W \subseteq V(T') \subset V(T)$ satisfies $e(T[W]) = r$. Likewise, for every $0 \leq r \leq m - 1$ we can obtain a partition $V(T) = X \cup Y$ with $e(X, Y) = r$ by taking $X = X' \cup \{v\}$ and $Y = Y'$ if $u \in X'$, or $X = X'$ and $Y = Y' \cup \{v\}$ if $u \in Y'$. To show that there are both a partition $V(T) = X \cup Y$ and a set of vertices $W \subseteq V(T)$ such that $e(X, Y) = e(T[W]) = m = e(T)$ is trivial and the proof is concluded. □

It is not difficult to see that the disjoint union of two vertex disjoint omnitonal graphs is again an omnitonal graph. Hence, it follows directly from Corollary 2.7 that every forest is omnitonal.

3 | BALANCEABLE AND OMNITONAL GRAPH FAMILIES

In this section, we study families of graphs which are omnitonal and/or balanceable. To begin, we note that the case of complete graphs with an even number of edges has already been settled in [9]. Actually, the study of balanceable graphs (in disguise) started in three recent papers [8,9,11]. In [11], the authors introduced zero-sum weighting over $\mathbb{Z}$ and present several zero-sum theorems that fit to the framework of balanceable graphs as explained in the introduction. The other two [8,9] develop further the study on $\{-1, 1\}$-weightings on the set of positive integers $\{1, 2, ..., n\}$ or on the set of edges of $K_n$, forcing zero-sum copies of given structures (blocks of consecutive integers in the first case—which can also be seen as balanced subpaths in 2-colored paths—and copies of complete graphs in the second case). We restate here, in the language of red–blue coloring, instead of $\{-1, 1\}$-weighting, the main theorem from [9], which is a sort of role-model for the results in this section.

**Theorem 3.1** (Caro et al [9]).

(i) For any positive integer $m \geq 2$, $m \neq 4$, $m \equiv 0, 1 \pmod{4}$ the complete graph $K_m$ is not balanceable.

(ii) The complete graph $K_4$ is balanceable with

$$\text{bal}(n, K_4) = \begin{cases} n & \text{for } n \equiv 0 \pmod{4}, \\ n - 1 & \text{else}. \end{cases}$$

Moreover, $\text{Bal}(n, K_4) = \{H\}$ with $H = J \cup \bigcup_{i=1}^q C_4$, where $J \in \{\emptyset, K_1, K_2, P_2\}$, depending on the residue of $n \pmod{4}$, and $q = \lfloor \frac{n}{4} \rfloor$.

Next we describe a class of graphs which we call amoebas. Amoebas were born from the search of a family with nice interpolation properties that could fit into the balanceability and omnitonal study. Indeed, as we shall see below, amoebas are balanceable and provide a wide family of omnitonal graphs. Besides this, we think that amoebas are interesting by their own not only because they constitute a very rich and large family of graphs (including dense amoebas, as well as amoebas with large cliques and amoeba-trees with arbitrarily large maximum degree having an interesting Fibonacci-like structure) but also because of their underlying algebraic structure, see [7]. After this, we focus our attention to the family of trees,
particularly paths and stars (see Sections 3.4, 3.2, and 3.3, respectively). We shall mention here that other recent results in this flavor were obtained in [10].

3.1 Amoebas

Given a graph $G$ of order $n(G)$ embedded in a complete graph $K_n$, where $n$ is sufficiently large, we say that $H$, also embedded in $K_n$, is obtained from $G$ by an edge-replacement, if for some $e_1 \in E(G)$ and $e_2 \in E(K_n) \setminus E(G)$, $E(H) = (E(G) \setminus \{e_1\}) \cup \{e_2\}$. Isolated vertices will play no role here, so all graphs considered further on may be the ones induced by its corresponding edge set.

**Definition 3.2.** A graph $G$ is an amoeba if, for $n$ sufficiently large, and for any two copies $F$ and $H$ of $G$ in $K_n$, there is a chain $F = G_0, G_1, G_2, ..., G_t = H$ such that, for every $i \in \{1, ..., t\}$, $G_i \cong G$ and $G_i$ is obtained from $G_{i-1}$ by an edge-replacement.

For example, it is not hard to see that a path $P_k$ is an amoeba for every $k \geq 1$, while a cycle $C_k$ is not an amoeba for any $k \geq 3$.

The following is a basic interpolation lemma for amoebas.

**Lemma 3.3.** Let $G$ be an amoeba and consider a 2-coloring $E(K_n) = E(R) \cup E(B)$ where $n \geq n_0(G)$. Let $\alpha, \beta, \alpha', \beta'$ be integers such that $\alpha + \beta = \alpha' + \beta' = e(G)$ and $0 \leq \alpha \leq \alpha'$ and $0 \leq \beta' \leq \beta$. If there are both an $(\alpha, \beta)$- and an $(\alpha', \beta')$-colored copies of $G$, then, there is an $(r, b)$-colored copy of $G$ for all integers $r$ and $b$ such that $r + b = e(G)$, $\alpha \leq r \leq \alpha'$, and $\beta' \leq b \leq \beta$.

**Proof.** Under the hypothesis of the lemma, let $F$ be an $(\alpha, \beta)$-colored copy of $G$, and $H$ be an $(\alpha', \beta')$-colored copy of $G$ with $0 \leq \alpha \leq \alpha'$ and $0 \leq \beta' \leq \beta$. Since $G$ is an amoeba, and $n \geq n_0(G)$, we know there is a chain $F = G_0, G_1, G_2, ..., G_t = H$ such that, for every $i \in \{1, ..., t\}$, $G_i \cong G$ and $G_i$ is obtained from $G_{i-1}$ by an edge-replacement. Let $\eta = |R \cap E(G_i)|$ be the number of red edges in $G_i$, and $b_i = |B \cap E(G_i)|$ be the number of blue edges in $G_i$, so that, for every $i \in \{1, ..., t\}$, $G_i$ is an $(\eta_i, b_i)$-colored copy of $G$. Observe that an edge-replacement modifies the color pattern in at most one unit, that is, for every $i \in \{1, ..., t\}$, $|\eta_i - \eta_{i-1}| \leq 1$ as well as $|b_i - b_{i-1}| \leq 1$ and $\eta_i + b_i = e(G)$. Thus, if we start with an $(\eta_0, b_0)$-colored copy of $G$, and we end with an $(\eta_t, b_t)$-colored copy of $G$, we must cover all $(r, b)$-color patterns with $\alpha = \eta_0 \leq r \leq \eta_t = \alpha'$ and $\beta' \leq b \leq \eta_0 = \beta$. □

**Remark 3.4.** Since, by the Kővari–Sós–Turán theorem [21], $ex(n, G) = o(n^2)$ for any bipartite graph $G$, we have, for large enough $n$, $2(ex(n, G) + 1) \leq \binom{n}{2}$. This means that we can consider 2-colorings $E(K_n) = E(R) \cup E(B)$ with $min\{e(R), e(B)\} \geq ex(n, G) + 1$ if $n$ is sufficiently large.

Note that Lemma 3.3 implies that, for a given amoeba $G$ and a given 2-coloring $E(K_n) = E(R) \cup E(B)$, where $n \geq n_0(G)$, if we can find both a $(0, e(G))$-colored copy and an $(e(G), 0)$-colored copy of $G$, then the so colored $K_n$ will contain the graph $G$ in every possible $(r, b)$-color pattern for $r$ and $b$ with $r + b = e(G)$, $0 \leq r \leq e(G)$ and $0 \leq b \leq e(G)$. Therefore, by means of Lemma 3.3 and Remark 3.4, we can prove our next theorem.
Theorem 3.5. Let $G$ be an amoeba. Then $G$ is omnitonal if and only if $G$ is bipartite. Moreover, if $G$ is a bipartite amoeba, then, for every $n$ such that $\binom{n}{2} \geq 2(\text{ex}(n, G) + 1)$ and $n \geq n_0(G)$, $\text{ot}(n, G) = \text{ex}(n, G)$ and $\text{Ot}(n, G) = \text{Ex}(n, G)$.

Proof. The necessity follows directly from Observation 2.2. For the sufficiency, let $G$ be a bipartite amoeba. By Remark 3.4 we can consider, for sufficiently large $n$, 2-colorings of $E(K_n)$ with $n \geq n_0(G)$ and at least $\text{ex}(n, G) + 1$ edges of each color. Since any coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} \geq \text{ex}(n, G) + 1$ contains a $(0, e(G))$-colored copy of $G$ and an $(e(G), 0)$-colored copy of $G$, by Lemma 3.3, there is an $(r, b)$-colored copy of $G$ for all integers $r$ and $b$ such that $0 \leq r, b \leq e(G)$, and $r + b = e(G)$. Thus, $G$ is omnitonal and $\text{ot}(n, G) \leq \text{ex}(n, G)$. To see that $\text{ex}(n, G) \leq \text{ot}(n, G)$, notice that we can give a 2-coloring of $E(K_n)$ with $\min\{e(R), e(B)\} = \text{ex}(n, G)$ such that there are no $(e(G), 0)$-colored copies of $G$, and therefore $G$ cannot be omnitonal. Further, observe that the fact that $\text{ot}(n, G) = \text{ex}(n, G)$ implies that $\text{Ex}(n, G) \subseteq \text{Ot}(n, G)$. Suppose now there is a graph $H \in \text{Ot}(n, G) \setminus \text{Ex}(n, G)$ and let $E(K_n) = E(R) \cup E(B)$ be a coloring of the edges of $K_n$ such that $R \cong H$. Then $e(R) = \text{ex}(n, G)$ but, since $R \notin \text{Ex}(n, G)$, $R$ contains a subgraph isomorphic to $G$, that is, there is an $(e(G), 0)$-copy of $G$ contained in the colored $K_n$. Since $2(\text{ex}(n, G) + 1) \leq \binom{n}{2}$, clearly $e(B) \geq \text{ex}(n, G) + 1$ and there is also a $(0, e(G))$-copy of $G$ in $K_n$. Hence, by Lemma 3.3, there is an $(r, b)$-copy of $G$ for every pair of nonnegative integers $r, b$ with $r + b = e(G)$, a contradiction to the hypothesis that $R \cong H \in \text{Ot}(n, G)$. Therefore, $\text{Ot}(n, G) = \text{Ex}(n, G)$.

Since the balanceable property is not as restrictive as the omnitonal property, we will see that we can prescind from the bipartite condition to prove that every amoeba is balanceable. For the proof, we will make use of an old argument of Erdős which states that every graph $G$ has a bipartition $V(G) = X \cup Y$ such that $e(X, Y) \geq \lfloor e(G)/2 \rfloor$ (see Lemma 2.14 in [18]). Deleting edges if necessary, one can easily see that every graph $G$ contains a bipartite subgraph $H$ with $e(H) = \lfloor e(G)/2 \rfloor$.

Theorem 3.6. Every amoeba is balanceable.

Proof. Let $G$ be an amoeba. By the observation above, we may consider a bipartite subgraph $H$ of $G$ having exactly $e(H) = \lfloor e(G)/2 \rfloor$ edges. Let $E(K_n) = E(R) \cup E(B)$ be a 2-coloring with $\min\{e(R), e(B)\} \geq \text{ex}(n, H) + 1$, which is possible for $n$ large enough because of Remark 3.4. Hence, we know that $K_n$ contains a $(0, e(H))$-colored copy of $H$ and an $(e(H), 0)$-colored copy of $H$. Now we can complete those copies of $H$ into copies of $G$ in an arbitrary way to get an $(\alpha, \beta)$-colored copy of $G$, and an $(\alpha', \beta')$-colored copy of $G$, where $\lfloor e(G)/2 \rfloor \leq \beta$ and $\lfloor e(G)/2 \rfloor \leq \alpha'$. Since $\alpha + \beta = \alpha' + \beta' = e(G)$, we also have $\alpha \leq \lfloor e(G)/2 \rfloor$ and $\beta' \leq \lfloor e(G)/2 \rfloor$. Altogether we have $\alpha \leq \lfloor e(G)/2 \rfloor \leq \lfloor e(G)/2 \rfloor \leq \alpha'$ and $\beta' \leq \lfloor e(G)/2 \rfloor \leq \lfloor e(G)/2 \rfloor \leq \beta$. Hence, Lemma 3.3 implies that $K_n$ contains a $(\lfloor e(G)/2 \rfloor, \lfloor e(G)/2 \rfloor)$-copy and a $(\lfloor e(G)/2 \rfloor, \lfloor e(G)/2 \rfloor)$-copy of $G$.

\[
\text{ex}(n, G) \leq \text{ex}(n, H) + 1.
\]
3.2 | Paths

When a graph $G$ is a bipartite amoeba, Theorem 3.5 yields $ot(G) = ex(G)$ and $Ot(G) = Ex(G)$. Hence, in particular, we have

$$ot(n, P_k) = ex(n, P_k) \leq \left( \frac{k-1}{2} \right) n,$$

where the second inequality is well known [15] and the exact values for $ex(n, P_k)$ can be found in [16].

Hence we are left with the problem of determining $bal(n, P_k)$ and $Bal(n, P_k)$, which is done in the next theorem. Before continuing we need some notation. Let $p, q$, and $k$ be nonnegative integers. A graph $G$ is a $(p, q)$-split graph if there is a partition of the vertex set $V(G) = X \cup Y$ with $|X| = p$ and $|Y| = q$ such that $G[X] \cong K_p$ and $Y$ is an independent set. Furthermore, the split graph $G$ is called complete if $G[EXY] \cong K_{pq}$.

**Theorem 3.7.** Let $k \geq 2$ and $n$ be integers with $k$ even and such that $n \geq \frac{9}{32}k^2 + \frac{1}{4}k + 1$. Then

$$bal(n, P_{k+1}) = bal(n, P_k) = \begin{cases} 
\left( \frac{k-2}{4} \right)n - \frac{k^2}{32} + \frac{1}{8} & \text{for } k \equiv 2 \pmod{4}, \\
\left( \frac{k-4}{4} \right)n - \frac{k^2}{32} + \frac{k}{8} + 1 & \text{for } k \equiv 0 \pmod{4}, 
\end{cases}$$

and $Bal(n, P_{k+1}) = Bal(n, P_k)$ contains only one graph, namely, the complete $(\frac{k-2}{4}, n - \frac{k-2}{4})$-split graph, if $k \equiv 2 \pmod{4}$, and the complete $(\frac{k-4}{4}, n - \frac{k-4}{4})$-split graph plus one edge, if $k \equiv 0 \pmod{4}$.

**Proof.** Define $h(n, k)$ to be the right-hand side of Equation (5). First, we observe that the condition $min\{e(R), e(B)\} > h(n, k)$ is satisfiable, that is, we need to prove that $e(K_n) = \frac{n(n-1)}{2} \geq 2h(n, k) + 2$ holds true for all $n \geq \frac{9}{32}k^2 + \frac{1}{4}k + 1$. If $k = 2$, then $h(n, k) = 0$ and the condition is satisfied for every $n \geq 3$. Since $h(n, k) \leq \left( \frac{k-2}{4} \right)n - \frac{k^2}{32} + \frac{k}{8} + 1$ we have to verify, for $k \geq 4$, that

$$\left( \frac{k-2}{2} \right)n - \frac{k^2}{16} + \frac{k}{4} + 4 \leq \frac{n(n-1)}{2}. $$

Equivalently, $n^2 - n(k-1) + \frac{k^2}{8} - \frac{k}{2} - 8 \geq 0$, which is indeed the case for $n \geq \frac{9}{32}k^2 + \frac{1}{4}k + 1$ and $k \geq 4$.

Let $H$ be the complete $\left( \left[ \frac{k-2}{4} \right], n - \left[ \frac{k-2}{4} \right] \right)$-split graph, plus one edge if $k \equiv 0 \pmod{4}$. It is straightforward to check that $H$ has exactly $h(n, k)$ edges.

Now, we will show that any 2-coloring $E(K_n) = E(R) \cup E(B)$ with $min\{e(R), e(B)\} = h(n, k)$ where $R$ or $B$ is isomorphic to $H$ contains no balanced copy of $P_k$. Suppose without loss of generality that $R$ is the one isomorphic to $H$. Let $V(K_n) = V_1 \cup V_2$ be a partition such that all edges induced by $V_2$, minus one if $k \equiv 0$
(mod 4), are blue and all remaining edges are red. A balanced copy of $P_k$ or of $P_{k+1}$
must contain $\frac{k}{2}$ edges of each color; if $k \equiv 2 \pmod{4}$, then $|V_i| = \frac{k-2}{4}$, hence, the
maximal number of red edges that a path can contain is $2\left(\frac{k-2}{4}\right) = \frac{k}{2} - 1$; if $k \equiv 0 \pmod{4}$, then $|V_i| = \frac{k-4}{4}$ and we have and extra red edge in $V_2$. Hence, the maximal number
of red edges that a path can contain is $2\left(\frac{k-4}{4}\right) + 1 = \frac{k}{2} - 1$. Thus, no coloring where $R$
or $B$ is isomorphic to $H$ can have a balanced $P_k$ nor a balanced $P_{k+1}$.

Clearly, if $K_n$ contains a balanced $P_k$, then we can complete it to a balanced $P_{k+1}$,
yielding the inequality $\text{bal}(n, P_k) \geq \text{bal}(n, P_{k+1})$. Hence, so far we have proved that
$\text{bal}(n, P_k) \geq \text{bal}(n, P_{k+1}) \geq h(n, k) = e(H)$. To prove that $\text{bal}(n, P_k) \leq h(n, k)$ and that $\text{Bal}(n, P_{k+1}) = \text{Bal}(n, P_k) = \{H\}$, we will show by induction on $k$ that any coloring $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} \geq h(n, k)$ and such that $R$ and $B$ are not isomorphic to $H$ contains a balanced copy of $P_k$ (and, consequently, a balanced copy of $P_{k+1}$, too).

If $k = 2$, then $h(n, 2) = 0$ and $H$ is the complete $(0, n)$-split graph. It is evident that
every 2-coloring of $E(K_n)$, where $n \geq 3$, with at least one edge of each color contains a balanced $P_2$.

Let $k \geq 4$ and assume that the theorem is valid for $k - 2$. Let $n \geq \frac{9}{32} k^2 + \frac{1}{4} k + 1$, and
consider $E(K_n) = E(R) \cup E(B)$ with $\min\{e(R), e(B)\} \geq h(n, k)$ and such that $R$ and $B$
are not isomorphic to $H$. Since $h(n, k - 2) < h(n, k)$ for every $k \geq 4$ and every $n \geq 1$, then,
by the induction hypothesis, there exists a balanced $(k - 2)$-path, say $P = x_1 x_2 \cdots x_{k-1}$. Let $U = V(K_n) \setminus V(P)$. Since $n \geq \frac{9}{32} k^2 + \frac{1}{4} k + 1$, we have $|U| \geq 3$.
Suppose for contradiction that there is no balanced $k$-path. Next, we will analyze the
edges from $U$ to $\{x_i, x_{k-1}\}$ and the edges induced by vertices in $U$.

Claim 1. The edges in $(U, \{x_i, x_{k-1}\})$ are all of the same color.

Suppose there are two edges $x_i u, x_{k-1} v$ with $u, v \in U, u \neq v$, such that one is blue and one is red. Then $uPv$ is a balanced $k$-path, a contradiction. ⌋

In the following, we will assume, without loss of generality, that all edges from $U$ to $\{x_1, x_{k-1}\}$ are blue. Then the following claim easily follows.

Claim 2. All edges in $E(U)$ are blue.

Suppose there is a red $uv \in E(U)$. Then, due to Claim 1, $uvP$ is a balanced $k$-path, a contradiction. ⌋

Let $X = \{x_2, \ldots, x_{k-2}\}$. We will analyze the edges contained in $E(U, X)$. From Claims 1 and 2
we know that all red edges in $E(K_n)$ are incident to a vertex in $X$. Let $W \subseteq X$ be the set of
vertices that are incident to at least one red edge from $E(U, X)$. We will call a vertex $x_i \in X$ a
red vertex, if both $x_{i-1} x_i$ and $x_i x_{i+1}$ are red edges.

Claim 3. All vertices in $W$ are red vertices.

Suppose to the contrary that there is a vertex $x_i \in W$ such that $x_{i-1} x_i$ is a blue edge. Since
$x_i \in W$, there is a red edge $x_i u$ for some $u \in U$. Now take $v \in U \setminus \{u\}$ and note that
\(v_{x_{k-1}}x_{k-2} \cdots x_iux_i \cdots x_{i-1}\) is a balanced \(k\)-path, a contradiction. Hence, \(x_{i-1}x_i\) is a red edge. By a symmetric argument, we can conclude that \(x_ix_{i+1}\) must be also red. \(\Box\)

**Claim 4.** If \(x_i \in W\), then all edges in \(E(U, \{x_{i-1}, x_{i+1}\})\) are blue.

Suppose to the contrary that \(x_{i-1}u\) is a red edge for some \(u \in U\). By Claim 3 we know that \(x_{i-1}x_i\) is also a red edge. If \(ux_i\) is red, take \(v \in U\) such that \(v \neq u\) and note that \(vx_i\cdots x_{i-1}ux_i \cdots x_{k-1}\) is a balanced \(k\)-path, a contradiction. If \(ux_i\) is not red, then, since \(xWi \in \mathcal{W}\), we know that there is a vertex \(vU \in \mathcal{V}\) such that \(vux_i\) is a red edge. In this case note that \(ux_{i-1}uvx_{i+1} \cdots x_{k-1}\) is a balanced \(k\)-path, a contradiction. Hence, all edges from \(x_{i-1}\) to \(U\) are blue edges. By a symmetric argument, we can conclude that all edges from \(x_{i+1}\) to \(U\) must also be blue. \(\Box\)

**Claim 5.** \(|W| = \left\lfloor \frac{k-2}{4} \right\rfloor\) and all red edges of \(P\), with exception of one if \(k \equiv 0 \pmod{4}\), are incident with a vertex in \(W\).

By Claim 3, we know that \(W\) contains only red vertices, and, by Claim 4, we conclude that \(W\) contains no consecutive red vertices. Hence, since \(P\) contains exactly \(\frac{k-2}{2}\) red edges, \(|W| \leq \left\lfloor \frac{k-2}{4} \right\rfloor\). Now suppose for contradiction that \(|W| \leq \left\lfloor \frac{k-2}{4} \right\rfloor - 1\). Then, considering that all edges from \(W\) to \(U\) and, with exception of the \(\frac{k-2}{2}\) blue edges on \(P\), all edges induced by \(V(P)\) may be red edges, we have at most the following number of red edges:

\[
e(R) \leq \left(\left\lfloor \frac{k-2}{4} \right\rfloor - 1\right)(n - (k - 1)) + \frac{(k-1)(k-2)}{2} - \frac{k-2}{2}
= \left\lfloor \frac{k-2}{4} \right\rfloor n - (k-1)\left\lfloor \frac{k-2}{4} \right\rfloor - n + k - 1 + \frac{(k-2)^2}{2}.
\]

On the other hand, we know by hypothesis that

\[
\left\lfloor \frac{k-2}{4} \right\rfloor n - \frac{k^2}{32} \leq h(n, k) \leq e(R).
\]

Thus,

\[
\left\lfloor \frac{k-2}{4} \right\rfloor n - \frac{k^2}{32} \leq e(R) \leq \left\lfloor \frac{k-2}{4} \right\rfloor n - (k-1)\left\lfloor \frac{k-2}{4} \right\rfloor - n + k - 1 + \frac{(k-2)^2}{2}.
\]

This gives, together with the inequalities \(\left\lfloor \frac{k-2}{4} \right\rfloor \geq \frac{k-4}{4}\) that

\[
n \leq \frac{k^2}{32} - (k-1)\left\lfloor \frac{k-2}{4} \right\rfloor + k - 1 + \frac{(k-2)^2}{2}
\leq \frac{k^2}{32} - (k-1)(k-4)/4 + k - 1 + \frac{(k-2)^2}{2} = \frac{9}{32}k^2 + \frac{1}{4}k,
\]
a contradiction to the assumption that $n \geq \frac{9}{32} k^2 + \frac{3}{4} k + 1$.

Note that, to achieve $|W| = \left\lfloor \frac{k-2}{4} \right\rfloor$, all red edges from $P$, with exception of one if $k \equiv 0 \pmod{4}$, appear in pairs surrounding a vertex from $W$. Therefore, if $k \equiv 2 \pmod{4}$, then all red edges in $P$ are incident with a vertex in $W$ and, if $k \equiv 0 \pmod{4}$, then all red edges except one are incident with a vertex in $W$. ◇

Our purpose now is to prove that the remaining red edges induced by $V(P)$ are all incident with a vertex in $W$.

Claim 6. $x_ix_{k-1}$ is a blue edge.

Suppose that $x_{k-1}x_i$ is red. Then take a blue edge $x_ix_{i+1}$ in $P$ and two vertices $u, v \in U, u \neq v$. By Claim 3, we know that $ux_i$ and $vx_{i+1}$ are blue edges, then $vx_{i+1} \cdots x_{k-1}x_i \cdots x_iu$ is a balanced $k$-path, a contradiction. ◇

Claim 7. If $x_ix_j$ is a red edge for some $1 \leq i < j \leq k-1, j \neq i + 1$, then either $x_i$ or $x_j$ is in $W$.

Suppose for contradiction that neither $x_i$ nor $x_j$ belongs to $W$. We will prove the existence of a balanced $k$-path. Consider $P' = ux_{i+1} \cdots x_jx_i \cdots x_{k-1} \cdots x_{j+1}v$. Observe that

$$E(P') = (E(P) \setminus \{x_ix_{i+1}, x_jx_{j+1}\}) \cup \{ux_{i+1}, vx_{j+1}, x_ix_{k-1}, x_jx_i\},$$

where $x_ix_{k-1}$ is a blue edge, and $x_jx_i$ is a red edge. Thus, to show that $P'$ is a balanced $k$-path, it remains to see that $x_ix_{i+1}$ and $ux_{i+1}$ are edges of the same color as well as $x_jx_{j+1}$ and $vx_{j+1}$. If $x_ix_{i+1}$ and $x_jx_{j+1}$ are blue, then $ux_{i+1}$ and $vx_{j+1}$ are also blue (by Claim 3) so we are done. Suppose then, without loss of generality, that $x_ix_{i+1}$ is red. Since $x_i \notin W$, $x_ix_{i+1}$ must be blue, which implies that $x_jx_{j+1}$ is red (otherwise, by symmetric arguments we obtain that the $k$-path $P'' = vx_{j-1} \cdots x_{j}x_{j+1} \cdots x_{k-1,x_i} \cdots x_{i+1}u$ is balanced). Now notice that since $x_j \notin W$, $x_jx_{j+1}$ must be blue and the cardinality of $W$ forces that one of $x_{i+1}$ or $x_{j-1}$ belongs to $W$. If $x_{i+1} \in W$, we can choose $u$ such that $ux_{i+1}$ is a red edge, and we are done; if $x_{j-1} \in W$, we use the path $P''$ instead of $P'$ to find the balanced $k$-path. In all cases there is a balanced $k$-path which is a contradiction, and so either $x_i$ or $x_j$ is in $W$. ◇

To conclude the proof, we will count which is the maximum number of possible red edges in $K_n$. Since all edges induced by $W$, and all edges from a vertex in $W$ to a vertex in $V(K_n) \setminus W$, plus one if $k \equiv 0 \pmod{4}$, are the only ones being possibly red, we obtain

$$e(R) \leq \left\lfloor \frac{k-2}{4} \right\rfloor \left( n - \left\lfloor \frac{k-2}{4} \right\rfloor \right) + \frac{1}{2} \left\lfloor \frac{k-2}{4} \right\rfloor \left( \left\lfloor \frac{k-2}{4} \right\rfloor - 1 \right) + \epsilon,$$

(7)

where $\epsilon = 0$ if $k \equiv 2 \pmod{4}$, and $\epsilon = 1$ if $k \equiv 0 \pmod{4}$. Note that the right-hand side of (7) is exactly the number of edges of $H$, that is exactly $h(n,k)$ as shown at the beginning of the proof. Since we assume $\min \{e(R), e(B)\} \geq h(n,k)$, then $e(R) = h(n,k)$. Moreover, note that $R$ is forced to be isomorphic to $H$, which is a contradiction. So, a balanced $k$-path exists.
3.3 Stars

In this section, we determine \( n_{K^1_k} b(, ) \) and \( n_{K^1_k} b(, ) \), for \( k \geq 2 \) even, and \( \text{ot}(n, K^1_k) \) and \( \text{Ot}(n, G K^1_k) \), for arbitrary \( k \geq 1 \).

**Theorem 3.8.** Let \( k \) and \( n \) be integers with \( k \geq 2 \) even and such that \( n \geq \max \{3, \frac{k^2}{4} + 1\} \). Then

\[
\text{bal}(n, K^1_{k+1}) = \text{bal}(n, K^1_k) = \left( \frac{k-2}{2} \right) n - \frac{k^2}{8} + \frac{k}{4}, \tag{8}
\]

and \( \text{Bal}(n, K^1_{k+1}) = \text{Bal}(n, K^1_k) \) contains only one graph, namely, the complete \( \left( \frac{k-2}{2}, n - \frac{k-2}{2} \right) \)-split graph.

**Proof.** Define \( h(n, k) \) to be the right-hand side of Equation (8). First observe that the condition \( \min\{e(R), e(B)\} > h(n, k) \) is satisfiable, that is, we need to prove that \( e(K^1_n) = \frac{n(n-1)}{2} \geq 2h(n, k) + 2 \) holds true for all \( n \geq \max \{3, \frac{k^2}{4} + 1\} \). If \( k = 2 \), then \( h(n, k) = 0 \) and the condition is satisfied for every \( n \geq 3 = \max \{3, \frac{k^2}{4} + 1\} \). If \( k \geq 4 \), we have to verify that \( 2h(n, k) + 2 = n(k - 2) - \frac{k^2}{4} + \frac{k}{2} + 2 \leq \frac{n(n-1)}{2} \). Equivalently, \( n^2 - (2k - 3)n + \frac{k^2}{4} - k - 4 \geq 0 \), which is indeed the case for \( n \geq \frac{k^2}{4} + 1 \) and \( k \geq 4 \).

It is straightforward to check that \( H \) has exactly \( h(n, k) \) edges. Now, observe that any 2-coloring \( E(K^1_n) = E(R) \cup E(B) \) where \( R \) or \( B \) is isomorphic to \( H \) contains no balanced copy of \( K^1_k \) nor of \( K^1_{k+1} \). To see this, note that, for such a coloring, there are two types of vertices \( v \in V(K^1_n) \), the ones for which \( \{\deg_R(v), \deg_B(v)\} = \{0, n - 1\} \), and the ones for which \( \{\deg_R(v), \deg_B(v)\} = \{\frac{k}{2} - 1, n - \frac{k}{2}\} \). In any case, it is impossible to have a balanced \( K^1_k \).

Clearly, if \( K^1_n \) contains a balanced \( K^1_k \), then we can complete it to a balanced \( K^1_{k+1} \), yielding the inequality \( \text{bal}(n, K^1_k) \geq \text{bal}(n, K^1_{k+1}) \). Hence, so far we have proved that \( \text{bal}(n, K^1_k) \geq \text{bal}(n, K^1_{k+1}) \geq h(n, k) = e(H) \). To prove that \( \text{bal}(n, K^1_k) \leq h(n, k) \) and that \( \text{Bal}(n, K^1_k) = \text{Bal}(n, K^1_{k+1}) = \{H\} \) we will show that any coloring \( E(K^1_n) = E(R) \cup E(B) \) with \( \min\{e(R), e(B)\} \geq h(n, k) \) and such that \( R \) and \( B \) are not isomorphic to \( H \) contains a balanced copy of \( K^1_k \) (and, consequently, a balanced copy of \( K^1_{k+1} \), too). For this purpose, we define the following sets:

\[
V_R = \left\{ v \in V(K^1_n) \mid \deg_R(v) \geq \frac{k}{2} \right\} \quad \text{and} \quad V_B = \left\{ v \in V(K^1_n) \mid \deg_B(v) \geq \frac{k}{2} \right\}.
\]

Let \( E(K^1_n) = E(R) \cup E(B) \) be a coloring with \( \min\{e(R), e(B)\} \geq h(n, k) \) and such that \( R \) and \( B \) are not isomorphic to \( H \). If there is a vertex \( v \in V_R \cap V_B \), then we are done as there would be a balanced \( K^1_k \). So we may assume that \( V_R \cap V_B = \emptyset \). Note that since every vertex in \( K^1_n \) has degree \( n - 1 \geq k \), then \( V(K^1_n) = V_R \cup V_B \), hence...
\[ |V_R| + |V_B| = n. \] (9)

Assume without loss of generality that \(|V_R| \leq |V_B|\).

**Case 1.** Suppose \(|V_R| \leq \frac{k}{2} - 1\). Thus,

\[
2e(R) = \sum_{v \in V_R} \deg_R(v) + \sum_{v \in V_B} \deg_R(v) \leq |V_R|(n - 1) + |V_B|\left(\frac{k - 2}{2}\right)
\]

\[
\leq \left(\frac{k}{2} - 1\right)(n - 1) + \left(n - \frac{k}{2} + 1\right)\left(\frac{k - 2}{2}\right)
\]

\[
= 2\left(\frac{k - 2}{2}\right)n - \frac{k^2}{4} + \frac{k}{2} = 2h(n, k). \tag{10}
\]

Consequently, \(e(R) \leq h(n, k)\). By assumption we know that \(\min\{e(R), e(B)\} \geq h(n, k)\), so we must have \(e(R) = h(n, k)\). Looking back to the inequalities in (10), it must be that \(|V_R| = \left(\frac{k}{2} - 1\right)\) and \(R\) is isomorphic to \(H\), a contradiction to our assumption.

**Case 2.** Suppose now that \(|V_R| \geq \frac{k}{2}\). Denote by \(e'(R)\) the number of red edges between \(V_R\) and \(V_B\). Since a vertex \(v \in V_R\) satisfies \(\deg_B(v) < \frac{k}{2}\), then each vertex in \(V_R\) contributes to \(e'(R)\) with at least \(|V_R| - \frac{k}{2} + 1\) edges, thus

\[
e'(R) \geq |V_R|\left(|V_B| - \frac{k}{2} + 1\right) \geq \frac{k}{2}\left(|V_B| - \frac{k}{2} + 1\right). \tag{11}
\]

On the other hand, each vertex in \(V_B\) contributes to \(e'(R)\) with no more than \(\frac{k}{2} - 1\) edges, so that

\[
e'(R) \leq \left(\frac{k}{2} - 1\right)|V_B|. \tag{12}
\]

Now, from (11) and (12), we obtain

\[
\frac{k}{2}\left(|V_B| - \frac{k}{2} + 1\right) \leq \left(\frac{k}{2} - 1\right)|V_B|,
\]

from which, by means of (9) and the assumption that \(|V_R| \geq \frac{k}{2}\), it follows that

\[
-\frac{k^2}{4} + \frac{k}{2} \leq -|V_B| = |V_R| - n \leq \frac{k}{2} - n. \tag{13}
\]

This yields \(n \leq \frac{k^2}{4}\), a contradiction to the hypothesis. \(\Box\)
Theorem 3.9. Let $n$ and $k$ be positive integers such that $n \geq 4k$. Then

$$\text{ot}(n, K_{1,k}) = \begin{cases} \left\lfloor \frac{(k - 1)n}{2} \right\rfloor & \text{for } k \leq 3, \\ (k - 2)n - \frac{k^2}{2} + \frac{3}{2}k - 1 & \text{for } k \geq 4, \end{cases}$$

and $\text{Ot}(n, K_{1,k})$ is the family of graphs containing

1. the empty graph $K_n$, if $k = 1$;
2. a disjoint union of $\frac{n}{2}K_2$'s, when $n$ is even, and of $\frac{n-1}{2}K_2$'s and a $K_1$, when $n$ is odd, if $k = 2$;
3. a disjoint union of cycles, if $k = 3$;
4. a complete $(k - 2, n - k + 2)$-split graph, if $k \geq 4$.

Proof. Define $h(n, k)$ to be the right-hand side of Equation (14). First observe that the condition $\min\{e(R), e(B)\} > h(n, k)$ is satisfiable, that is, we need to prove that $e(K_n) = \frac{n(n-1)}{2} \geq 2h(n, k) + 2$ is satisfied for $n \geq 4k$. If $k \leq 3$, it is easy to check that $2h(n, k) + 2 \leq n(k - 1) + 2 \leq \frac{n(n-1)}{2}$. If $k \geq 4$, we have to verify that $2h(n, k) + 2 = 2n(k - 2) - k^2 + 3k \leq \frac{n(n-1)}{2}$, which is equivalent to $n^2 - (4k - 7)n + 2k^2 - 6k \geq 0$. This is indeed true for $n \geq 4k$. Next, observe that the colorings described in items 1–3 contains no $(r, b)$-colored copy of $K_{1,k}$ for some pair $(r, b) \in \{(0, k), (k, 0)\}$ (i.e., there is no blue or red copy of $K_{1,k}$). The coloring of item 4 does not contain a $K_{1,k}$ with $k - 1$ blue edges and one red edge or the other way around.

Now let $E(K_n) = E(R) \cup E(B)$ be a 2-coloring with $\min\{e(R), e(B)\} \geq h(n, k)$ and such that $R$ and $B$ are not as in items 1–4 from the theorem. We will show that the so colored $K_n$ contains an $(r, b)$-colored copy of $K_{1,k}$ for every pair $r, b \geq 0$ with $r + b = k$.

With this purpose, we define the sets

$$R_r = \{v \in V(K_n) | \deg_R(v) \geq r\} \quad \text{and} \quad B_b = \{v \in V(K_n) | \deg_B(v) \geq b\},$$

for integers $b, r \geq 0$ such that $b + r = k$. If there is a vertex $x \in B_b \cap R_r$ for a pair $b, r \geq 0$ with $b + r = k$, then $x$ is the center of a star $K_{1,k}$ with $b$ blue edges and $r$ red edges. Hence, if $B_b \cap R_r \neq \emptyset$ for every pair $b, r \geq 0$ with $b + r = k$, then $K_n$ contains an $(r, b)$-colored copy of $K_{1,k}$ for every pair $r, b \geq 0$ with $r + b = k$ and we are done. So we may assume that there is a particular pair $b, r \geq 0$ with $b + r = k$ such that $B_b \cap R_r = \emptyset$. Assume for contradiction that there is no $(r, b)$-colored copy of $K_{1,k}$. Clearly, $V(K_n) \setminus (B_b \cup R_r) = \emptyset$, otherwise there would be a vertex of degree at most $b + r - 2 = k - 2$, which is not possible since every vertex in $K_n$ has degree $n - 1 \geq k - 1$. Hence, $V(K_n) = B_b \cup R_r$, where the union is disjoint. Observe that $B_0 = R_0 = V(K_n)$ and so, if $b = 0$ and $r = k$, we obtain $R_k = \emptyset$.

Case 1. Let $k = 1$. Then say $b = 0$ and $r = 1$, giving $B_0 = V(K_n)$ and $R_1 = \emptyset$, and thus $R$ is the empty graph, which is not possible by assumption.
Case 2. Let $k = 2$. Then $\{r, b\} = \{0, 2\}$ or $r = b = 1$. Say, in the first case, that $b = 0$ and $r = 2$. Then $B_0 = V(K_n)$ and $R_2 = \emptyset$. Then we have

$$2e(R) \leq \begin{cases} n - 1 & \text{if } n \text{ odd}, \\ n & \text{if } n \text{ even}, \end{cases} = 2h(n, 2).$$

Since by assumption $e(R) \geq h(n, k)$, we obtain equality in the above inequality chain. This is only possible if $R$ is a disjoint union of $\frac{n}{2}K_2$'s, when $n$ is even, and of $\frac{n - 1}{2}K_2$'s and a $K_1$, when $n$ is odd, which is not allowed by hypothesis. Hence, $b = r = 1$. Since $\min\{e(R), e(B)\} \geq h(n, k) = \left\lfloor \frac{n}{2} \right\rfloor$, $B_1, R_1 \neq \emptyset$. If $\deg_B(v) \geq 1$ for some $v \in R_1$, we have a $(1, 1)$-colored $K_{1,2}$. So we may assume that $\deg_B(v) = 0$ for all $v \in R_1$, implying that the edges between $B_1$ and $R_1$ are all red. Then it is easy to see that there is again a $(1, 1)$-colored $K_{1,2}$.

Case 3. Let $k = 3$. Then $\{r, b\} = \{0, 3\}$ or $\{r, b\} = \{1, 2\}$. Say, in the first case, that $b = 0$ and $r = 3$. Then $B_0 = V(K_n)$ and $R_3 = \emptyset$, and $2e(R) \leq 2n = 2h(n, 3)$. Since by assumption $e(R) \geq h(n, k)$, it follows that $e(R) = n$ and that all vertices have degree 2 in $R$, that is, $R$ is a union of cycles, which is not possible by assumption. Thus $\{r, b\} = \{1, 2\}$, so say that $b = 1$ and $r = 2$. Then $\deg_B(v) = 0$ for all $v \in R_2$, implying that all edges between $B_1$ and $R_2$ are red. Hence, it follows that all edges inside $R_2$ are red, too. Then $|B_1| \geq 2$, since otherwise there would not be any blue edges. Since $\deg_R(u) \leq 1$ for all $u \in B_1$, we infer that $|R_3| \leq 1$. If $R = \emptyset$, we cannot have more than $\left\lfloor \frac{n}{2} \right\rfloor$ red edges without having a $(2, 1)$-colored $K_{1,2}$. If $|R| = 1$, there are no red edges within vertices of $B_1$, and so there are at most $n - 1$ red edges. In both cases we obtain a contradiction to the hypothesis that $e(B) \geq h(n, 3) = n > 0$.

Case 4. Let $k \geq 4$ Observe that $B_0 = R_0 = V(K_n)$ and so, if $b = 0$ and $r = k$, we obtain $R_k = \emptyset$, leading to the contradiction $2e(R) \leq n(k - 1) < 2h(n, k)$. The case $b = k$ and $r = 0$ is analogous. Hence, we have $1 \leq r, b \leq k - 1$. If $B_b = \emptyset$, then we would have the same contradiction with

$$2e(B) \leq n(b - 1) \leq n(k - 1) < 2h(n, k). \tag{15}$$

The same happens if $R_r = \emptyset$. Hence, $B_b, R_r \neq \emptyset$ and, assuming without loss of generality that $|B_b| \leq |R_r|$, we have $1 \leq |B_b| \leq |R_r| \leq n - 1$. Now we distinguish two cases.

Subcase 4.1. Suppose that $|B_b| \leq b - 1$. Then we have

$$2e(B) = \sum_{v \in R_r} \deg_B(v) + \sum_{v \in B_b} \deg_B(v) \leq (n - |B_b|)(b - 1) + |B_b|(n - 1)$$

$$= |B_b|(n - b) + n(b - 1) \leq (b - 1)(n - b) + n(b - 1)$$

$$= -b^2 + (2n + 1)b - 2n. \tag{16}$$

Define the function $g(b) = -b^2 + (2n + 1)b - 2n$ and observe that $g'(b) = -2b + 2n + 1 > 0$ for $b \in [1, k - 1]$. Hence, the maximum of the function $g(b)$ on the
domain $[1, k - 1]$ is attained when $b = k - 1$, and thus

$$2e(B) \leq -b^2 + (2n + 1)b - 2n$$

$$\leq -(k - 1)^2 + (2n + 1)(k - 1) - 2n$$

$$= 2nk - 4n - k^2 + 3k - 2 = 2h(n, k).$$

(17)

Since, by assumption, $e(B) \geq h(n, k)$, we obtain equality all along the inequality chains (16) and (17). This gives us that $b = k - 1$, $r = 1$, $|B_b| = |B_{k-1}| = b - 1 = k - 2$, and that each $u \in R_1$ and $v \in B_{k-1}$ have $\deg_B(u) = b - 1 = k - 2$ and $\deg_B(v) = n - 1$. Hence, $B$ is a complete $(k - 2, n - k + 2)$-split graph, a contradiction to our assumptions.

Subcase 4.2. Suppose that $|B_b| \geq b$. Considering $e_B(R_r, B_b)$, the number of red edges with one vertex in $R_r$ and one in $B_b$, we have

$$|B_b|(|B_b| - b + 1) \leq |R_r|(|B_b| - b + 1) \leq e_R(R_r, B_b) \leq |B_b|(r - 1).$$

(18)

In particular, it follows that $|B_b| - b + 1 \leq r - 1$ which is the same as $|B_b| \leq r + b - 2 = k - 2$. Hence, we have $1 \leq b \leq |B_b| \leq k - 2$. Moreover, counting the blue edges and using that $\deg_B(v) \leq b - 1$ for every $v \in R_r$ and (18), we obtain

$$2e(B) = \sum_{v \in R_r} \deg_B(v) + \sum_{v \in B_b} \deg_B(v)$$

$$\leq |R_r|(b - 1) + |B_b|(n - 1) - e_R(R_r, B_b)$$

$$\leq |R_r|(b - 1) + |B_b|(n - 1) - |R_r||B_b| - b + 1$$

$$= |R_r|(-|B_b| + 2b - 2) + |B_b|(n - 1)$$

$$= (n - |B_b|)(-|B_b| + 2b - 2) + |B_b|(n - 1)$$

$$= |B_b|^2 - 2|B_b|b + |B_b| + 2nb - 2n.$$
 Altogether we have shown that \( ot(n, K_{1,k}) = h(n, k) \) and that \( Ot(n, K_{1,k}) \) is the family of graphs described in items 1–4. □

### 3.4 Trees

By Corollary 2.7, we know that trees are omnital. Moreover, Theorem 3.5 yields that a tree \( T \) which is an amoeba satisfies \( ot(n, T) = ex(n, T) \), but we also know that not every tree is an amoeba (like, stars with at least three leaves). However, we will prove that \( ot(n, T) \) is linear on \( n \) for every tree \( T \). More precisely, we will show that more than \( (k-1)n \) edges from each color are enough to guarantee the existence of every tree on \( k \) edges in all different tonal variations.

In 1962, Erdős and Sós conjectured that the trivial lower bound \( ex(n, T) \geq n\left(\frac{k-1}{2}\right) \) is tight (see [18]). A proof of this conjecture for sufficiently large \( k \) was announced years ago by Ajtai, Komlós, Simonovits, and Szemerédi, but the proof remains unpublished. For our purpose, we will use the following weaker statement, which is folklore (see, e.g., [18]). Denote by \( T_k \) the class of trees on \( k \) edges.

**Remark 3.10.** Let \( k \) be a positive integer and let \( T \in T_k \). Then, \( ex(T, n) < (k-1)n \).

Before stating the theorem, we need one more definition. In a tree that is not a star, there are at least two vertices such that all but one of its neighbors are leaves. The star induced by such a vertex together with its neighbor-leaves is called an end-star. Hence, for every tree \( T \) different from a star, there is an end-star vertex \( v \) with \( \deg(v) \leq \frac{e(T) + 1}{2} \).

**Theorem 3.11.** Let \( n \) and \( k \) be positive integers such that \( n \geq 4k \). Then, for every \( T \in T_k \), \( ot(n, T) \leq (k-1)n \).

**Proof.** First observe that the condition \( \min\{e(R), e(B)\} \geq (k-1)n \) is satisfiable. This is clearly so, since \( e(K_n) = \frac{n(n-1)}{2} \geq 2(k-1)n \) is satisfied for \( n \geq 4k \).

We proceed now by induction on \( k \). A tree \( T \) with \( k = 1, 2, \) or 3 edges is either a star or a path, and it follows from (4) and (14) that \( ot(n, T) \leq \left(\frac{k-1}{2}\right)n \leq (k-1)n \), so that the statement holds true in these cases. For a fix \( k \geq 3 \), assume that the statement is true for every tree with less than \( k \) edges. Let \( T \) be a tree with \( k \) edges, let \( n \geq 4k \), and consider a 2-coloring \( E(K_n) = E(R) \cup E(B) \) such that

\[
\min\{e(B), e(R)\} > (k-1)n. \tag{19}
\]

Note that, by Remark 3.10, we get both a \((0, k)\)-colored copy, and a \((k, 0)\)-colored copy of \( T \). Thus, we only need to prove that, for every \( 1 \leq r \leq k-1 \), there is an \((r, k-r)\)-colored copy of \( T \) under the given coloring. Also note that, if \( T \) is a star we are done by
Theorem 3.9. Then we assume that $T$ is not a star. By the discussion before the statement of the theorem, there is an end-star vertex $v \in V(T)$ such that $\text{deg}_T(v) = t$ and

$$2 \leq t \leq \frac{k + 1}{2}.$$  

(20)

Denote by $w$ the only neighbor of $v$ which is not a leaf, and let $T'$ be the tree on $k - t$ edges obtained by deleting $v$ and all its leaf-neighbors from $T$. By the induction hypothesis, there is a copy of $T'$ in every tone. For each $0 \leq s \leq k - t$, let $T'_s$ be an $(s, k - t - s)$-copy of $T'$. Let $W_s = V(K_n) \setminus V(T'_s)$ and let $B^*_s$ and $R^*_s$ be the graphs induced by the blue and, respectively, the red edges in $W_s$. Observe that $|W_s| = n - (k - t + 1) \geq 4k - (k - t + 1) = 3k + t - 1 > 4t$. Moreover,

$$\min\{e(B^*_s), e(R^*_s)\} > (k - 1)n - \left(\frac{n(T'_s)}{2}\right) - n(T')\left(n - n(T'_s)\right)$$

$$= (k - 1)n - \frac{1}{2}(k - t + 1)(k - t) - (k - t + 1)(n - (k - t + 1))$$

$$= (t - 2)(n - k + t - 1) + \frac{1}{2}(k - t + 1)(k + t - 2).$$

Since $k \geq 2t - 1$ by (1), we get $\frac{1}{2}(k - t + 1)(k + t - 2) \geq \frac{1}{2}t(3t - 3) > 0$ for $t \geq 2$, and so we can conclude that

$$\min\{e(B^*_s), e(R^*_s)\} > (t - 2)(n - k + t - 1).$$

Hence, by Theorem 3.9, there is a copy of $K_{1,t}$ in $W_s$ in every tone. In particular, there is a $(1, t - 1)$ and a $(t - 1, 1)$ copy of $K_{1,t}$. Now we will show that, for every $1 \leq r \leq k - 1$, there is an $(r, k - r)$-colored copy of $T$ in $K_n$. To this aim, we distinguish two cases.

**Case 1.** $r \geq t - 1$. Set $s = r - t + 1$ and consider $T'_s$, the $(s, k - t - s)$-colored copy of $T'$ in $W_s$. Let $w^*$ be the copy of $w$ in $T'_s$. By the discussion above, there is a $(t - 1, 1)$-colored copy of $K_{1,t}$ in $W_s$ with, say, central vertex $v^*$ and leaves $x_1, x_2, \ldots, x_t$, and such that $v^*x_i$ is red for $1 \leq i \leq t - 1$, and $v^*x_t$ is blue. To find the desired $(r, k - r)$ copy of $T$, proceed as follows:

- If $w^*v^*$ is red, take the set of vertices $V(T'_s) \cup \{v^*, x_2, \ldots, x_t\}$ and edge set $E(T'_s) \cup \{w^*v^*, v^*x_2, \ldots, v^*x_t\}$.

- If $w^*v^*$ is blue, take the set of vertices $V(T'_s) \cup \{v^*, x_1, \ldots, x_{t-1}\}$ and edge set $E(T'_s) \cup \{w^*v^*, v^*x_1, \ldots, v^*x_{t-1}\}$.

In both cases we obtain a copy of $T$ with $s + t - 1 = (r - t + 1) + t - 1 = r$ red edges and $k - t - s + 1 = k - t - (r - t + 1) + 1 = k - r$ blue edges.

**Case 2.** $r \leq t - 2$. Set $s = r - 1$ and consider $T'_s$, the $(s, k - t - s)$-colored copy of $T'$ in $W_s$. Let $w^*$ be the copy of $w$ in $T'_s$. In this case, we will use a $(1, t - 1)$-colored copy of $K_{1,t}$
contained in $W_s$ with, say, central vertex $v^*$ and leaves $x_1, x_2, ..., x_t$, and such that $vx_i$ is blue for $1 \leq i \leq t - 1$, and $v^*x_1$ is red. To find the desired $(r, k - r)$ copy of $T$, we do the following:

- If $w^*v^*$ is red, take the set of vertices $V\left(T_s'\right) \cup \{v^*, x_1, ..., x_{t-1}\}$ and edge set $E\left(T_s'\right) \cup \{w^*v^*, v^*x_1, ..., v^*x_{t-1}\}$.

- If $w^*v$ is blue, take the set of vertices $V\left(T_s\right) \cup \{v^*, x_2, ..., x_t\}$ and edge set $E\left(T_s\right) \cup \{w^*v^*, v^*x_2, ..., v^*x_t\}$.

In both cases we obtain a copy of $T$ with $s + 1 = (r - 1) + 1 = r$ red edges and $(k - t - s) + t - 1 = (k - t - (r - 1)) + t - 1 = k - r$ blue edges.

\[\square\]

The bound from Theorem 3.11 yields a better bound for $\text{bal}(n, T)$ than the one we get by means of a similar approach as in the proof of this theorem but making use of Theorem 3.8 instead. The problem with this approach is that, on the induction step, too many edges from one of the colors may be lost, so that the $(k - 1)$-factor cannot be improved. Hence, to find a better bound on $\text{bal}(n, T)$ another method will be needed.

**Corollary 3.12.** Let $n$ and $k$ be positive integers such that $n \geq 4k$. Then, for every $T \in T_k$, $\text{bal}(n, T) \leq (k - 1)n$.

## 4 | Discussion, Open Problems, and Potential Research Lines

In this section, we discuss some of the problems and variations that one can consider.

To find more families of graphs which are omnitonal and/or balanceable is one of our main interests. The recognition problem is clearly in NP, and it seems plausible that it is NP-complete, but this has yet to be settled.

**Problem 1.** Are the problems of determining if a graph $G$ is omnitonal or balanceable NP-complete?

The bound $\text{ot}(n, T) < (k - 1)n$ given in Theorem 3.11 for trees on $k$ edges is quite good if we compare it with the fact that $\text{ot}(n, K_{1,k}) \approx (k - 2)n$ for $k \geq 4$ (see Theorem 3.9). However, the problem of finding a good upper bound on $\text{bal}(n, T)$ remains open.

Analyzing the proofs of Theorems 2.5, 2.6, and 3.11, it lets us suspect that the diameter could play an important role in obtaining more precise bounds for these parameters. We pose the following problems.

**Problem 2.** Let $T$ be a tree on $k$ edges.

(i) Give better estimates for $\text{bal}(n, T)$.

(ii) Estimate $\text{bal}(n, T)$ and $\text{ot}(n, T)$ assuming $T$ has fixed diameter $d$. 
In the context of complete graphs, we know that $K_4$ is the only balanceable complete graph with an even number of edges [8]. However, for the odd case, it is not difficult to see that $K_3, K_7, K_{11},$ and $K_{14}$ are balanceable as they can be found balanced in both unavoidable patterns (type-$A$ and type-$B$), and if we look further, some larger graphs work, too, like, for example, $K_{52}$. So this problem, which relies clearly on a number-theoretical setting, together with the determination of the balancing number as well as the extremal colorings could be interesting to settle.

**Problem 3.** For $m \equiv 2, 3 \pmod{4}$, characterize the values of $m$ for which $K_m$ is balanceable. Determine, in such cases, $\text{bal}(n, K_m)$ and $\text{Bal}(n, K_m)$.

For an omnitonal graph $G$, the inequality $\text{ot}(n, G) \geq \text{ex}(n, G)$ clearly holds. Besides bipartite amoebas which are precisely the omnitonal amoebas (see Theorem 3.5), we do not know about the existence of other graph types where the omnitonal number equals the Turán number, a question that would be interesting to investigate.

**Problem 4.** Characterize the family of omnitonal graphs $G$ with $\text{ot}(n, G) = \text{ex}(n, G)$.

In this study, we have fully concentrated on the balanceable and the omnitonal problems, but one can also consider the problem of whether a graph $G$ is $r$-tonal, meaning that, as soon as there are sufficiently many edges from each color and $n$ is large enough, $G$ appears either $(r, e(G) - r)$-colored or $(e(G) - r, r)$-colored in every coloring of $K_n$. This problem can be also modified to the more restrictive version of whether the graph appears $(r, e(G) - r)$-colored, or even to the version of whether it appears in both fashions, $(r, e(G) - r)$ and $(e(G) - r, r)$-colored. For all these versions it should not be difficult to give results analogous to Theorems 2.5 and 2.6.

Another natural direction to consider is to have more than two colors. The notion of balanceable graphs, strongly balanceable graphs, and omnitonal graphs can be easily modified to the case where $c \geq 3$ colors are used. A generalization of Theorem 2.3 to $c \geq 2$ colors was already done in [5], while the Turán version for multiple colors has been given in [4]. However, when considering omnitonal graphs, it is easy to show that, for $c \geq 3$ colors, they must be disconnected, in sharp contrast to the case of $c = 2$.

**ACKNOWLEDGMENTS**

We thank Alp Müyesser for a discussion on part of this paper, pointing to us References [12,17], and to his manuscript [5] which also contains a sketch of a similar proof of Theorem 2.4 concerning $\epsilon$-balanced colorings. We also thank the anonymous referees, whose comments helped improving the presentation of this paper. Finally, we would like to thank BIRS-CMO for hosting the workshop Zero-Sum Ramsey Theory: Graphs, Sequences and More 19w5132, in which the authors of this paper were organizers and participants, and where many fruitful discussions arose that contributed to a better understanding of these topics. The second author was partially supported by PAPIIT IN111819 and CONACyT project 282280. The third author was partially supported by PAPIIT IN116519 and CONACyT project 282280.

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How to cite this article: Caro Y, Hansberg A, Montejano A. Unavoidable chromatic patterns in 2-colorings of the complete graph. J Graph Theory. 2021;97:123–147. https://doi.org/10.1002/jgt.22645