Lepton mass and mixing in a simple extension of the Standard Model based on $T_7$ flavor symmetry

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A simple Standard Model Extension based on $T_7$ flavor symmetry which accommodates lepton mass and mixing with non-zero $\theta_{13}$ and CP violation phase is proposed. At the tree-level, the realistic lepton mass and mixing pattern is derived through the spontaneous symmetry breaking by just one vacuum expectation value ($v$) which is the same as in the Standard Model. Neutrinos get small masses from one $SU(2)_L$ doublet and two $SU(2)_L$ singlets in which one being in $\mathbf{1}$ and the two others in $\mathbf{3}$ and $\mathbf{3}^*$ under $T_7$, respectively. The model also gives a remarkable prediction of Dirac CP violation $\delta_{CP} = 172.598^\circ$ in both normal and inverted hierarchies which is still missing in the neutrino mixing matrix.

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I. INTRODUCTION

The discovery of neutrino mass is a great breakthrough for particle physics, and up to now, this is the unique evidence of New Physics. Neutrinos have tiny masses and this is

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probably related to the existence of a new mass scale in physics. Recently it has been shown that neutrinos can also play an important role in providing answer for the Baryon Asymmetry of Universe (BAU).

Theoretically, there exist various models describing the smallness of neutrino mass and large $\theta_{13}$ mixing\(^1\). Among the possible extensions of the Standard Model (SM), probably the simplest one is the neutrino minimal SM which has been studied in Refs. [2–6]. However, these extensions do not provide a natural explanation for large mass splitting between neutrinos and the lepton mixing was not explicitly explained [7].

There are five well-known patterns of lepton mixing [8], however, the Tri-bimaximal one proposed by Harrison-Perkins-Scott (HPS) [9–12]

\[
U_{\text{HPS}} = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix},
\]

seems to be the most popular and can be considered as a leading order approximation for the recent neutrino experimental data. Up to now, the absolute values of the entries of the lepton mixing matrix $U_{\text{PMNS}}$ have not yet been determined exactly, however, their scales are given in Ref. [13]

\[
|U_{\text{PMNS}}| = \begin{pmatrix}
0.801 \rightarrow 0.845 & 0.514 \rightarrow 0.580 & 0.137 \rightarrow 0.158 \\
0.225 \rightarrow 0.517 & 0.441 \rightarrow 0.699 & 0.614 \rightarrow 0.793 \\
0.246 \rightarrow 0.529 & 0.464 \rightarrow 0.713 & 0.590 \rightarrow 0.776
\end{pmatrix}.
\]

The range of experimental values of neutrino mass squared differences and leptonic mixing angles are given in Ref. [14] as below

\[
\sin^2 \theta_{12} = 0.304 \pm 0.014, \quad \sin^2 \theta_{13} = (2.19 \pm 0.12) \times 10^{-2}, \\
\Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5} \text{eV}^2, \\
\sin^2 \theta_{23} = 0.514^{+0.055}_{-0.056} \quad \text{(normal mass hierarchy)}, \\
\sin^2 \theta_{23} = 0.511 \pm 0.055 \quad \text{(inverted mass hierarchy)}, \\
\Delta m^2_{32} = (2.44 \pm 0.06) \times 10^{-3} \text{eV}^2, \quad \text{(normal mass hierarchy)}, \\
\Delta m^2_{32} = (2.49 \pm 0.06) \times 10^{-3} \text{eV}^2, \quad \text{(inverted mass hierarchy)}. \tag{3}
\]

\(^1\) The references for these models are mentioned in Ref. [1]
In fact, the models that successfully explain the experimental data are often mathematically
complicate. An ideal physical model should be mathematically quite simple but successfully
explains the experimental data and its physical parameters can be tested by the experiments
in near future. This desired model, up to now, has not yet been effective because each model
has its own advantages and disadvantages. To explain the specific neutrino mixings, it is
simple to use discrete symmetry such as $A_4, S_3, S_4$, etc. The use of non-abelian discrete
symmetries to construct the models describing the lepton masses and mixings is a new
method first proposed by E. Ma and G. Rajasekaran in 2001 [15]. In this treatment, there
are various models which have been proposed, see for example $A_4$ [15–33], $S_3$ [34–74], $S_4$
[75–103], $D_4$ [104–114], $T'$ [115–124], $T_7$ [125–129]. However, in all above mentioned papers,
the fermion masses and mixings generated from non-renormalizable interactions or at loop
levels but not at tree-level. The models involving only renormalizable interactions were
implemented in our previous works [131–143] in which the discrete symmetries have been
added to the 3-3-1 models. As we know the 3-3-1 model itself is an extension of the SM
where the gauge group $SU(2)_L$ is extended to $SU(3)_L$. In order to overcome such limitations,
we studied a neutrino mass model by adding the discrete symmetry $S_4$ to the SM which
accommodates the realistic lepton mass, mixing with non-zero $\theta_{13}$ and CP violation phase
at the tree-level with renormalizable interactions only [1].

In this paper, we construct a simple extension of the SM based on $T_7$ symmetry that
leads to lepton mass, mixing with non-zero $\theta_{13}$ and CP violation phase\(^2\). For this purpose,
two $SU(2)_L$ doublets and two $SU(2)_L$ singlets are introduced. The result follows without
perturbation and the number of scalars required to generate lepton masses are fewer than
those in Ref. [1].

The future content of this paper reads as follows. In Sec. III we present the fundamental
elements of the model and introduce necessary Higgs fields responsible for the lepton masses.
We summarize the results in the section III. Finally, the appendices A and B provide in detail
solutions for neutrino masses in the normal and the inverted hierarchies, respectively.

\(^2\) We note that $T_7$ symmetry has not been previously considered in this kind of the model with the mentioned
scenario. Furthermore, this model is different from our previous works [136, 138] because the 3-3-1 model
(based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$) itself is an extension of the SM.
II. LEPTON MASS AND MIXING

The lepton content of the model, under SU(2)$_L \otimes$ U(1)$_Y \otimes$ U(1)$_X \otimes$ T$_7$ symmetries, is given in Tab. I. The charged lepton masses arise from the couplings of \( \bar{\psi}_L l_{1R}, \bar{\psi}_L l_{2R} \) and \( \bar{\psi}_L l_{3R} \) to scalars, where \( \bar{\psi}_L l_i \) (\( i = 1, 2, 3 \)) transforms as 2 under SU(2)$_L$ and 3* under T$_7$. In order to generate masses for charged leptons, we need only one SU(2)$_L$ Higgs doublets (\( \phi \)) lying in 3 under T$_7$, as given in Tab. I.

The Yukawa interactions read

\[
-L = h_1(\bar{\psi}_L \phi)_1 l_{1R} + h_2(\bar{\psi}_L \phi)_2 l_{2R} + h_3(\bar{\psi}_L \phi)_3 l_{3R} + H.c
\]

\[
= h_1(\bar{\psi}_1 \phi_1 + \bar{\psi}_2 \phi_2 + \bar{\psi}_3 \phi_3) l_{1R}
+ h_2(\bar{\psi}_1 \phi_1 + \omega \bar{\psi}_2 \phi_2 + \omega^2 \bar{\psi}_3 \phi_3) l_{2R}
+ h_3(\bar{\psi}_1 \phi_1 + \omega \bar{\psi}_2 \phi_2 + \omega^2 \bar{\psi}_3 \phi_3) l_{3R} + H.c.
\]

(4)

In this work we impose only the breaking T$_7 \to Z_3$ in charged lepton sector, and this happens with the first alignment, i.e, \( \langle \phi \rangle = (\langle \phi_1 \rangle, \langle \phi_1 \rangle, \langle \phi_1 \rangle) \) under T$_7$, where

\[
\langle \phi_1 \rangle = (0 \quad v)^T.
\]

(5)

With the vacuum expectation value (VEV) of \( \phi_1 \) in Eq. (5), the mass Lagrangian for the charged leptons can be written in matrix form as

\[
-L_{\text{mass}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L}) M_l (l_{1R}, l_{2R}, l_{3R})^T + H.c,
\]

(6)

where

\[
M_l = \begin{pmatrix}
    h_1 v & h_2 v & h_3 v \\
    h_1 v & \omega^2 h_2 v & \omega h_3 v \\
    h_1 v & \omega h_2 v & \omega^2 h_3 v
\end{pmatrix}.
\]

(7)
The mass matrix $M_l$ in Eq. (7) is diagonalized:

$$U_L^\dagger M_l U_R = \begin{pmatrix}
\sqrt{3}h_1 v & 0 & 0 \\
0 & \sqrt{3}h_2 v & 0 \\
0 & 0 & \sqrt{3}h_3 v
\end{pmatrix} \equiv \begin{pmatrix}
m_e & 0 & 0 \\
0 & m_\mu & 0 \\
0 & 0 & m_\tau
\end{pmatrix}, \quad (8)$$

where

$$m_e = \sqrt{3}h_1 v, \quad m_\mu = \sqrt{3}h_2 v, \quad m_\tau = \sqrt{3}h_3 v, \quad (9)$$

and

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega^2 & \omega \\
1 & \omega & \omega^2
\end{pmatrix}, \quad U_R = 1. \quad (10)$$

The Yukawa couplings $h_{1,2,3}$ in charged lepton sector are defined:

$$h_1 = \frac{m_e}{\sqrt{3}v}, \quad h_2 = \frac{m_\mu}{\sqrt{3}v}, \quad h_3 = \frac{m_\tau}{\sqrt{3}v}. \quad (11)$$

The experimental values for masses of the charged leptons are given in [14]:

$$m_e \simeq 0.510998928 \text{ MeV}, \quad m_\mu = 105.6583715 \text{ MeV}, \quad m_\tau = 1776.86 \text{ MeV} \quad (12)$$

It follows that $h_1 \ll h_2 \ll h_3$. Furthermore, if we choose the VEV $v \sim 100 \text{ GeV}$ then

$$h_1 \sim 10^{-6}, \quad h_2 \sim 10^{-4}, \quad h_3 \sim 10^{-2}, \quad (13)$$

i.e, in the model under consideration, the hierarchy between the masses for charged-leptons can be achieved if there exists a hierarchy between Yukawa couplings $h_i (i = 1, 2, 3)$ in charged-lepton sector as given in Eq. (13). We note that the masses of charged leptons are self-separated by only one Higgs triplet $\phi$ (the same as in the SM), and this is a good feature of the $T_7$ group. We remind that the models with the other discrete symmetry groups need more than one Higgs scalar in the charged lepton sector.

The neutrino masses arise from the couplings of $\bar{\psi}_L \nu_R$ and $\bar{\nu}_R^c \nu_R$ to scalars, where $\bar{\psi}_L \nu_R$ transforms as 2 under SU(2)$_L$ and $1 \oplus 1' \oplus 1'' \oplus 3 \oplus 3^*$ under $T_7$; $\bar{\nu}_R^c \nu_R$ transform as 1 under

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3 In the SM, the Higgs VEV $v$ is 246 GeV, fixed by the W boson mass and the gauge coupling $m_W^2 = \frac{g^2 v^2}{4} v_{\text{weak}}$. In the model under consideration $M_W^2 \simeq \frac{3}{2} g^2 v^2$. Therefore, we can identify $v_{\text{weak}}^2 = 6 v^2 = (246 \text{ GeV})^2$. It follows $v \simeq 100 \text{ GeV}$. 


SU(2)\textsubscript{L} and \(\mathbf{3} \oplus \mathbf{3}^* \oplus \mathbf{3}^*\) under \(T_7\). Note that \(\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}\) has two invariants and \(\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}^*\) has one invariant under \(T_7\). In order to generate mass for neutrinos, we additionally introduce one \(SU(2)\textsubscript{L}\) doublet (\(\varphi\)) and two \(SU(2)\textsubscript{L}\) singlets (\(\chi, \zeta\)), respectively, put in \(\mathbf{1}, \mathbf{3}\) and \(\mathbf{3}^*\) under \(T_7\) as given in Tab. II. We note that the \(U(1)\textsubscript{X}\) symmetry forbids the Yukawa terms of the form \((\bar{\psi}_L \bar{\phi})\mathbf{3}_\mu R\) and yield the expected results in neutrino sector, and this is interesting feature of \(X\)-symmetry. It is also interesting to note that \(\varphi\) contributes to the Dirac mass matrix, \(\chi\) and \(\zeta\) contribute to the Majorana mass matrix of the right-handed neutrinos. In fact, there exist no one-dimensional representation in \(\mathbf{3} \otimes \mathbf{3}\) under \(T_7\). Hence, \(\zeta\) put in \(\mathbf{3}^*\) will be responsible for a realistic neutrino spectrum without any perturbation and soft breaking in both lepton and neutrino sectors. This feature is different from the one in Ref. [130].

It needs to note that \(\varphi\) contributes to the Dirac mass matrix in the neutrino sector, \(\chi\) and \(\zeta\) contribute to the Majorana mass matrix of the right-handed neutrinos. The interesting feature of \(X\)-symmetry is to prevents the unwanted interaction of the form \((\bar{\psi}_L \bar{\phi})\mathbf{3}_\mu R\) and provides the expected results in the neutrino sector.

In this work we impose that the breaking \(T_7 \rightarrow \{\text{identity}\}\) must be taken place, i.e., \(T_7\) is completely broken in neutrino sector. This can be achieved within each case below.

1. A new \(SU(2)\textsubscript{L}\) singlet \(\chi\) lies in \(\mathbf{3}\) under \(T_7\) with the VEV is given by \(\langle \chi \rangle = (0, \langle \chi_2 \rangle, 0)^T\) under \(T_7\), where

\[
\langle \chi_2 \rangle = v_\chi. \tag{14}
\]

2. Another singlet \(\zeta\) lies in \(\mathbf{3}^*\) under \(T_7\) with the VEV is given by \(\langle \zeta \rangle = (\langle \zeta_1 \rangle, \langle \zeta_2 \rangle, \langle \zeta_3 \rangle)^T\) under \(T_7\), i.e. \(\langle \zeta_1 \rangle \neq \langle \zeta_2 \rangle \neq \langle \zeta_3 \rangle \neq 0\), where

\[
\langle \zeta_i \rangle = u_i \quad (i = 1, 2, 3). \tag{15}
\]

The neutrino Yukawa interactions are given by

\[
-L_\nu = x(\bar{\psi}_L \bar{\phi})\mathbf{3}\nu_R + \frac{y}{2}(\bar{\nu}_R \chi)\mathbf{3}\nu_R + \frac{z}{2}(\bar{\nu}_R \zeta)\mathbf{3}\nu_R + H.c
\]

\[
= x(\bar{\psi}_1 L \bar{\phi}_1 \nu_R + \bar{\psi}_2 L \bar{\phi}_2 \nu_R + \bar{\psi}_3 L \bar{\phi}_3 \nu_R)
\]

\[
+ \frac{y}{2}[(\bar{\nu}_R \chi_3 + \bar{\nu}_R \chi_2)\nu_1 R + (\bar{\nu}_R \chi_1 + \bar{\nu}_R \chi_3)\nu_2 R + (\bar{\nu}_R \chi_2 + \bar{\nu}_R \chi_1)\nu_3 R]
\]

\[
+ \frac{z}{2}(\bar{\nu}_1 R \zeta_2 \nu_1 R + \bar{\nu}_2 R \zeta_3 \nu_2 R + \bar{\nu}_3 R \zeta_1 \nu_3 R) + H.c. \tag{16}
\]
The neutrino mass Lagrangian are given as

\[
- \mathcal{L}^\text{mass}_\nu = x v (\bar{\nu}_1 L \nu_1 R + \bar{\nu}_2 L \nu_2 R + \bar{\nu}_3 L \nu_3 R) \\
+ \frac{y}{2} (v_1 \bar{\nu}_3 R \nu_1 L + v_2 \bar{\nu}_3 R \nu_1 L + v_3 \bar{\nu}_1 R \nu_2 R) \\
+ \frac{z}{2} (u_2 \bar{\nu}_1 R \nu_1 R + u_3 \bar{\nu}_2 R \nu_2 R + u_1 \bar{\nu}_3 R \nu_3 R) + H.c.
\]

(17)

We can rewrite in the matrix form

\[
- \mathcal{L}^\text{mass}_\nu = \frac{1}{2} \chi_c L M_\nu \chi_c L + H.c., \quad \chi_c \equiv \begin{pmatrix} \nu_1 c_l \\ \nu_2 c_l \\ \nu_3 c_l \end{pmatrix}, \quad M_\nu \equiv \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix},
\]

(18)

\[
\nu_c = (\nu_1 c_L \nu_2 c_L \nu_3 c_L)^T, \quad \nu_R = (\nu_1 R \nu_2 R \nu_3 R)^T
\]

where the Dirac neutrino mass matrix \((M_D)\) and the right-handed Majorana neutrino mass matrix \((M_R)\) are given by

\[
M_D = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad M_R = \begin{pmatrix} N_2 & 0 & b \\ 0 & N_3 & 0 \\ b & 0 & N_1 \end{pmatrix},
\]

(19)

with

\[
a = v_\phi x, \quad b = v_\chi y, \quad N_i = u_i z \quad (i = 1, 2, 3).
\]

(20)

The seesaw mechanism generates small masses for neutrinos is given by

\[
M_{\text{eff}} = -M_D M_R^{-1} M_D^T = \begin{pmatrix} A_1 & 0 & B \\ 0 & A_3 & 0 \\ B & 0 & A_2 \end{pmatrix},
\]

(21)

where

\[
A_1 = \frac{a^2 N_1}{b^2 - N_1 N_2}, \quad A_2 = \frac{a^2 N_2}{b^2 - N_1 N_2}, \quad A_3 = \frac{-a^2}{N_3}, \quad B = \frac{a^2 b}{N_1 N_2 - b^2}.
\]

(22)

The matrix \(M_{\text{eff}}\) in Eq. (21) has three exact eigenvalues given by

\[
m_1 = \frac{1}{2} \left( A_1 + A_2 - \sqrt{(A_1 - A_2)^2 + 4B^2} \right), \quad m_2 = A_3,
\]

\[
m_3 = \frac{1}{2} \left( A_1 + A_2 + \sqrt{(A_1 - A_2)^2 + 4B^2} \right).
\]

(23)
and the corresponding eigenstates are

$$U_\nu = \begin{pmatrix} \frac{K}{\sqrt{K^2+1}} & 0 & \frac{1}{\sqrt{K^2+1}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{K^2+1}} & 0 & -\frac{K}{\sqrt{K^2+1}} \end{pmatrix},$$

(24)

where

$$K = A_1 - A_2 - \sqrt{(A_1 - A_2)^2 + 4B^2} \frac{2B}{2B},$$

(25)

and \(A_{1,2}, B\) are given in Eq. (22).

The lepton mixing matrix is then expressed as

$$U = U_L^\dagger U_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1+K}{\sqrt{K^2+1}} & 1 & \frac{1-K}{\sqrt{K^2+1}} \\ \frac{K+\omega^2}{\sqrt{K^2+1}} & \omega & \frac{1-K\omega}{\sqrt{K^2+1}} \\ \frac{K+\omega}{\sqrt{K^2+1}} & \omega^2 & \frac{1-K\omega}{\sqrt{K^2+1}} \end{pmatrix},$$

(26)

where \(K\) is defined in Eq. (25). Comparing the lepton mixing matrix in Eq. (26) and the standard parametrization \(^4\) in Ref. [14] yields:

$$s_{13}e^{-i\delta} = \frac{1}{\sqrt{3}} \frac{1-K}{\sqrt{K^2+1}},$$

(27)

$$t_{12}^2 = \frac{\sqrt{K^2+1}}{1+K},$$

(28)

$$t_{23}^2 = \frac{1-K\omega^2}{1-K\omega}.$$

(29)

In the case \(K\) being real numbers, Eqs. (27) and (29) imply \(\theta_{23} = 45^o\) and \(\delta = 0\). As we know, the recent experimental data imply \(\delta \neq 0\). To overcome this, we will consider \(K\) as a complex variable. Substituting \(\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\) into Eqs. (27), (28) and (29) we obtain:

$$s_{13} = \frac{1}{\sqrt{3}} \frac{[(k_1 - 1)^2 + k_2^2]^{1/2}}{\alpha^{1/4}},$$

(30)

$$t_{12}^2 = \frac{\alpha^{1/2}}{(1+k_1)^2 + k_2^2},$$

(31)

$$t_{23}^2 = 1 - \frac{2\sqrt{3}k_2}{1+k_1+k_1^2+k_2^2+\sqrt{3}k_2},$$

(32)

\(^4\) In fact, the Majorana phases do not contribute to neutrino oscillations so they will be ignored for the rest of this work.
where

\[ \alpha = (1 + k_1^2 - k_2^2)^2 + 4k_1^2k_2^2, \] (33)

and \( k_1 \) and \( k_2 \) being the real and imaginary parts of \( K \), respectively.

On the other hand, from Eq. (27), we get:

\[ e^{-i\delta} = \frac{1}{s_{13}\sqrt{3}} \frac{1 - K}{\sqrt{K^2 + 1}} \equiv \cos \delta - i \sin \delta, \] (34)

with

\[ \cos \delta = \left( 1 + 2k_1 - k_1^2 - k_2^2 - \sqrt{\alpha} \right) \beta, \]

\[ \sin \delta = \left\{ k_2^2 - 1 + k_1(1 - k_1 + k_1^2 + k_2^2) + (1 - k_1)\sqrt{\alpha} \right\} \beta, \] (35)

where

\[ \beta = \frac{\alpha^{1/4}}{\sqrt{2}} \sqrt{-1 - k_1^2 + k_2^2 + \sqrt{\alpha}} \left[ (1 - k_1^2 + k_2^2)\sqrt{\alpha} + k_1^4 + (k_2^2 - 1)^2 + 2k_2^2(1 + k_2^2) \right], \] (36)

which is satisfying the relation \( \sin^2 \delta + \cos^2 \delta = 1 \) with all \( k_1, k_2 \).

The neutrino mass spectrum can be the normal hierarchy (\( |m_1| \simeq |m_2| < |m_3| \)), the inverted hierarchy (\( |m_3| < |m_1| \simeq |m_2| \)) or nearly degenerate (\( |m_1| \simeq |m_2| \simeq |m_3| \)). The mass ordering of neutrino depends on the sign of \( \Delta m_{23}^2 \) which is currently unknown. However, some tight upper limits on the total neutrino mass \( \sum m_\nu \) have given by the recent studies. For example, the total mass of three degenerate neutrinos was given by Planck satellite mission \[144\], \( \sum m_\nu < 0.72 \text{ eV} \) (95\% CL) by using Planck TT-lowP data, and \( \sum m_\nu < 0.49 \text{ eV} \) (95\% CL) by using Planck TT,TE,EE+lowP data. While the improved constraints are given by adding the baryon acoustic oscillation (BAO) measurements \[145\], i.e., \( \sum m_\nu < 0.21 \text{ eV} \) (95\% CL) and \( \sum m_\nu < 0.17 \text{ eV} \) (95\% CL), respectively. Another upper limit was given in Ref. \[146\], \( \sum m_\nu < 0.113 \text{ eV} \) (95\% CL).

As will see, in the model under consideration, the two possible signs of \( \Delta m_{23}^2 \) correspond to two types of neutrino mass spectrum as well as the values of the atmospheric neutrino mixing angle \( \theta_{23} \) can be provided.

Combining Eq. (30) with the experimental values of \( \theta_{13} \) given in Ref. \[14\] as shown in Eq. (3), we have a solution as follow:\(^5\):

\[ k_2 = -\frac{1}{2} \sqrt{(8.03468 - 4k_1)k_1 - 4.03468 + 2\sqrt{0.069663 + (0.139025k_1 - 0.139326)k_1}}. \] (37)

\(^5\) There exist four mathematical solutions, however, these solutions differ only by the sign of \( m_{1,2,3} \) which has no effect on the neutrino oscillation experiments.
Next, from Eqs. (37) and (32) with the experimental values of $\theta_{23}$ in Eq. (3), we get two solutions:

$$k_1 = 0.690532, \quad k_2 = -0.035032, \quad K = 0.690532 - 0.035032i,$$

(38)

and the lepton mixing matrix in (26) then takes the form

$$|U| \simeq \begin{pmatrix} 0.803441 & 0.57735 & 0.147986 \\ 0.437621 & 0.57735 & 0.709451 \\ 0.405089 & 0.57735 & 0.689859 \end{pmatrix},$$

(39)

which is consistent with constraint in Eq. (2). Now, substituting $k_{1,2}$ from (38) in to (31) yields $t_{12}^2 = 0.516381$ (or $t_{12} = 0.718597$), i.e, $\theta_{12} \simeq 35.7^\circ$. It follows $\cos \delta = -0.991667$, $\sin \delta = 0.128827$, i.e, $\delta \simeq 172.598^\circ$.

Combining (25) and the values of $K$ in (38), we obtain

$$A_1 = A_2 - (0.753905 + 0.108377i)B.$$  

(40)

**A. Normal spectrum ($\Delta m_{23}^2 > 0$)**

Substituting $A_1$ from (40) into (23) and combining with the two experimental constraints on squared mass differences of neutrinos for the normal spectrum as shown in (3), i.e, $\Delta m_{21}^2 = 7.53 \times 10^{-5}$ eV$^2$, $\Delta m_{32}^2 = 2.44 \times 10^{-3}$ eV$^2$, we get the analytical expressions of $A_2, B, m_{1,2,3}$ (in [eV]) given in Appendix [A]

By using the upper limits on neutrino mass [144–146] we can restrict $A_3 \leq 0.72$ eV. However, in the normal spectrum case in (A), $A_3 \in [0.0087, 0.03]$ eV or $A_3 \in [-0.03, -0.0087]$ eV are good regions of $A_3$ that can reach the realistic neutrino mass hierarchies. With $m_2 \in [0.0087, 0.03]$ eV, $m_{1,2,3}$ as functions of $A_3 = m_2$ are plotted in Fig. [A]. This figure shows that there exist allowed regions of the parameter $A_3$ where either normal or quasi-degenerate neutrino masses spectrum is achieved. The quasi-degenerate mass hierarchy$^8$ is obtained when $A_3 \in [0.03 \text{ eV}, +\infty)$ or $A_3 \in (-\infty, -0.03 \text{ eV})$ ($|A_3|$ increases but must be

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$^6$ Here we only consider one case because another value has no effect on the neutrino oscillation experiments.

$^7$ $\theta_{12} \simeq 35.7^\circ$ obtained from the model is an acceptable prediction.

$^8$ There is no clear limits between neutrino mass hierarchies by the recent experimental results on neutrino oscillations.
small enough because of the scale of $m_{1,2,3}$). The normal mass hierarchy will be obtained if $A_3 \in [0.0087, 0.03]$ eV or $A_3 \in [-0.03, -0.0087]$ eV. The total neutrino masses in the model under consideration $\sum_{i=1}^{3} m_i$ and $\sum_{i=1}^{3} |m_i|$ with $m_2 \in [0.0087, 0.05]$ eV is depicted in Fig. 2.

![Fig. 1: $m_{1,2,3}$ as functions of $A_3$ in the normal spectrum with $A_3 \in (-0.03, -0.0087)$ eV (left) and $A_3 \in (0.0087, 0.03)$ eV (right).](image1)

![Fig. 2: The sum $\sum_{i=1}^{3} m_i$ as a function of $A_3$ with $A_3 \in (0.0087, 0.03)$ eV in the normal spectrum.](image2)

It is easily to obtain the effective mass $\langle m_{ee} \rangle$ governing neutrinoless double beta decay

$$\langle m_{ee} \rangle = |\sum_{i=1}^{3} U_{ei}^2 m_i|, \quad m_\beta = \left\{ \sum_{i=1}^{3} |U_{ei}|^2 m_i^2 \right\}^{1/2}$$

by combining the expressions (26), (38), (A1), (A2) and (A3), the values of $m_{ee}, m_\beta$ and $m_{\text{light}}$ are plotted in Fig. 3 together with $m_1$ with $A_3 \in (0.0087, 0.03)$ eV. We also note that in the normal spectrum, $m_1 \approx m_2 < m_3$ so $m_{\text{light}} = m_1$ given in (A2) is the lightest neutrino mass.

To get explicit values of the model parameters, we assume $A_3 \equiv m_2 = 10^{-2}$ eV, which is safely small. Then the other neutrino masses and the other parameters are explicitly given in Tab. [II]
FIG. 3: $m_{ee}$, $m_\beta$ and $m_{light}$ as functions of $A_3$ with $A_3 \in (0.00867, 0.05)$ eV in the normal spectrum.

TABLE II: The model parameters in the case $A_3 = 10^{-2}$ eV in the normal spectrum

| Parameters [eV] | The derived values                  |
|-----------------|------------------------------------|
| $A_1$           | $0.0357232 + 0.00100894i$          |
| $A_2$           | $0.0196452 - 0.00100894i$          |
| $B$             | $-0.0212715 + 0.000381301i$       |
| $m_1$           | $0.00496991$                       |
| $m_3$           | $0.0503984$                        |
| $\sum m_I^I$   | $0.0653683$                        |
| $\langle m_{ee}^I \rangle$ | $0.00761271$              |
| $m_\beta^I$     | $0.0171627$                        |

Now, comparing Eqs. (22) and derived values in Tab. II we get the relations:

\[
N_1 = -(142.621 + 4.02808i)a^2, \quad N_2 = (-78.4315 + 4.02808i)a^2,
N_3 = -100a^2, \quad b = (-84.9243 + 1.52231i)a^2.
\]  
(41)

or

\[
|N_1|/|b| = 1.67979, \quad |N_2|/|b| = 0.924614, \quad |N_3|/|b| = 1.17733,
\]
(42)

\[
|N_1/a^2| = 142.678, \quad |N_2/a^2| = 78.5348, \quad |N_3/a^2| = 100, \quad |b/a^2| = 84.938,
\]
(43)

i.e., $N_1$, $N_2$, $N_3$ and $b$ have the same order of magnitude, and approximately two orders of magnitude of $a^2$. 


Substituting $A_1$ in (10) into (23) and combining with the experimental constraints on squared mass differences of neutrinos for the inverted spectrum as shown in (3), i.e, $\Delta m^2_{21} = 7.53 \times 10^{-5}$ eV$^2$, $\Delta m^2_{32} = -2.49 \times 10^{-3}$ eV$^2$, we get a solution (in [eV]) given in Appendix B.

In the inverted spectrum, with the solution in (B), $A_3 \in (0.055, 0.085)$ eV or $A_3 \in [-0.085, -0.055]$ eV are good regions of $A_3$ that can reach the inverted neutrino mass hierarchies. The absolute values $|m_{1,2,3}|$ as functions of $A_3 = m_2$ are plotted in Fig. 4 in which $A_3 \in [0.055, 0.085]$ eV. This figure shows that the quasi-degenerate mass hierarchy is obtained when $A_3 \in [0.085 \text{ eV, } +\infty)$ or $A_3 \in (-\infty, -0.085 \text{ eV})$. The inverted mass hierarchy will be obtained if $|A_3| \in [0.055, 0.085]$ eV. The total neutrino masses $\sum^3_{i=1} m^I_i$ and $\sum^3_{i=1} |m^I_i|$ with $A_3 \in [0.055, 0.085]$ eV is depicted in Fig.5.

![Graph](image-url)

**FIG. 4:** $|m_{1,2,3}|$ as functions of $A_3$ in the inverted spectrum with $A_3 \in (-0.085, -0.055)$ eV (left) and $A_3 \in (0.055, 0.085)$ eV (right).

In the inverted spectrum, the effective mass $\langle m^I_{\nu}\rangle$ governing neutrinoless double beta decay $\langle m^I_{\nu}\rangle$ and $m^I_{\beta}$ together with $m_3$ are plotted in Fig.6 with $A_3 \in [0.055, 0.085]$ eV by combining the expressions (26), (38), (B1), (B2) and (B3). In this case $m^I_{\nu\beta\text{light}} = m_3$ given in Eq. (A3) is the lightest neutrino mass.

To get explicit values of the model parameters, we assume $A_3 \equiv m_2 = 6 \times 10^{-2}$ eV. The other neutrino masses and the other parameters are explicitly given in Tab. III.
Comparing Eqs. (22) and derived values in Tab. III yields:

\[ N_1 = (21.0986 - 0.292525i)a^2, \quad N_2 = (25.7602 + 0.292525i)a^2, \]
\[ N_3 = -16.6667a^2, \quad b = (-6.16732 + 0.110552i)a^2. \]  

or

\[ |N_1|/|b| = 3.42082, \quad |N_2|/|b| = 4.17648, \quad |N_3|/|b| = 2.70198, \]  
\[ |N_1/a^2| = 21.1006, \quad |N_2/a^2| = 25.7618, \quad |N_3/a^2| = 16.6667, \quad |b/a^2| = 6.16831, \]  

i.e., \( N_1, N_2, N_3 \) and \( b \) have the same order of magnitude, and approximately one orders of magnitude of \( a^2 \).
TABLE III: The model parameters in the case $A_3 = 6 \times 10^{-2}$ eV in the inverted spectrum

| Parameters [eV] | The derived values |
|----------------|--------------------|
| $A_1$          | $-0.0417327 + 0.000578609i$ |
| $A_2$          | $-0.0509532 - 0.000578609i$ |
| $B$            | $-0.0121988 + 0.00021867i$ |
| $m_1$          | $-0.0593692$ |
| $m_3$          | $-0.0333167$ |
| $\sum m_i^l$  | $0.0326858$ |
| $\langle m_{ee}^l \rangle$ | $0.0190284$ |
| $m_\beta$      | $0.08723$ |

III. CONCLUSIONS

We have proposed a simple Standard Model extension based on $T_7$ flavor symmetry which accommodates lepton mass, mixing with non-zero $\theta_{13}$ and CP violation phase. The spontaneous symmetry breaking in the model is imposed to obtain the realistic lepton mass and mixing pattern at the tree-level with renormalizable interactions. In difference from other discrete groups, the $T_7$ flavor group requires only one VEV ($\langle \phi_1 \rangle = v$) which, the same as in the SM, is enough for production of the charged lepton masses. The neutrinos get small masses from one $SU(2)_L$ doublet and two $SU(2)_L$ singlets in which one being in $\underline{1}$ and the two others in $\underline{2}$ and $\underline{3}^*$ under $T_7$, respectively. The model also gives a remarkable prediction of Dirac CP violation $\delta_{CP} = 172.598^\circ$ in both normal and inverted spectrum which is still missing in the neutrino mixing matrix.

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Appendix A: Neutrino masses in the normal spectrum

\[
A_2 = 8.44732 \times 10^{-9} \sqrt{\Gamma} + (8.13873 \times 10^{-6} + 2.91875 \times 10^{-7}i) \sqrt{\gamma} \sqrt{\Gamma} \\
- (0.188476 + 0.0270942i) \sqrt{\gamma'} - 2\sqrt{\gamma} + 7.14452 \times 10^{-6} A_3^2 \sqrt{\Gamma},
\]
\[
B = -0.5 \sqrt{\gamma'} - 2\sqrt{\gamma},
\]
\[
m_1 = -0.5 \sqrt{0.0023647 + 2A_3^2 - (2.27831 + 0.081706i)\sqrt{\gamma}} \\
+ (0.188476 + 0.0270943i) \sqrt{\gamma'} - 2\sqrt{\gamma} + 8.44732 \times 10^{-9} \sqrt{\Gamma} \\
+ (8.13873 \times 10^{-6} + 2.91875 \times 10^{-7}i) \sqrt{\gamma} \sqrt{\Gamma} \\
- (0.188476 + 0.0270942i) \sqrt{\gamma'} - 2\sqrt{\gamma} + 7.14452 A_3^2 \sqrt{\Gamma},
\]
\[
m_2 = A_3,\tag{A1}
\]
\[
m_3 = 0.5 \sqrt{0.0023647 + 2A_3^2 - (2.27831 + 0.081706i)\sqrt{\gamma}} \\
+ (0.188476 + 0.0270943i) \sqrt{\gamma'} - 2\sqrt{\gamma} + 8.44732 \times 10^{-9} \sqrt{\Gamma} \\
+ (8.13873 \times 10^{-6} + 2.91875 \times 10^{-7}i) \sqrt{\gamma} \sqrt{\Gamma} \\
- (0.188476 + 0.0270942i) \sqrt{\gamma'} - 2\sqrt{\gamma} + 7.14452 \times 10^{-6} A_3^2 \sqrt{\Gamma}. \tag{A2}
\]

where

\[
\gamma = (-1.4104 \times 10^{-7} + 1.01291 \times 10^{-8}i) + (0.00181524 - 0.000130366i)A_3^2 \\
+ (0.76764 - 0.0551299i)A_3^4, \tag{A4}
\]
\[
\gamma' = (0.00207317 - 0.0000743489i) + (1.75343 - 0.0628823i)A_3^2, \tag{A5}
\]
\[
\Gamma = (7.32235 \times 10^{12} + 0.000183105i) + (6.19305 \times 10^{15} - 0.0625i)A_3 \\
- (7.05485 \times 10^{15} + 2.53004 \times 10^{14}i)\sqrt{\gamma}. \tag{A6}
\]
Appendix B: Neutrino masses in the inverted spectrum

\[ A_2 = 9.5457 \times 10^{-9} \sqrt{\Gamma_1} - (8.4778 \times 10^{-6} + 3.04035 \times 10^{-7}i) \sqrt{\gamma_1 \Gamma_1} \]
\[ - (0.188476 + 0.0270942i) \sqrt{\gamma'_1 - 2\sqrt{\gamma_1} - 7.44217 \times 10^{-6} A_3^2 \sqrt{\Gamma_1}}, \]
\[ B = -0.5 \sqrt{\gamma'_1 - 2\sqrt{\gamma_1}}, \]
\[ m_1 = -0.5 \sqrt{(-0.0025653 + 2.71051 \times 10^{-20}i) + 2A_3^2 - (2.27831 + 0.081706i) \sqrt{\gamma_1}} \]
\[ + (0.188476 + 0.0270942i) \sqrt{\gamma'_1 - 2\sqrt{\gamma_1} - (8.4778 \times 10^{-6} + 3.04035 \times 10^{-7}i) \sqrt{\gamma_1 \Gamma_1}} \]
\[ + (9.5457 \times 10^{-9} - 7.44217 \times 10^{-6} A_3^2) \sqrt{\Gamma_1}, \]
\[ m_2 = A_3, \]
\[ m_3 = 0.5 \sqrt{(-0.0025653 + 2.71051 \times 10^{-20}i) + 2A_3^2 - (2.27831 + 0.081706i) \sqrt{\gamma_1}} \]
\[ + (0.188476 + 0.0270942i) \sqrt{\gamma'_1 - 2\sqrt{\gamma_1} - (8.4778 \times 10^{-6} + 3.04035 \times 10^{-7}i) \sqrt{\gamma_1 \Gamma_1}} \]
\[ + (9.5457 \times 10^{-9} - 7.44217 \times 10^{-6} A_3^2) \sqrt{\Gamma_1}, \]

where

\[ \gamma_1 = (1.4393 \times 10^{-7} - 1.03367 \times 10^{-8}i) - (0.00196923 - 0.000141425i)A_3^2 \]
\[ + (0.76764 - 0.0551299i)A_3^4, \]
\[ \gamma'_1 = (-0.00224904 + 0.000080656i) + (1.75343 - 0.0628823i)A_3^2, \]
\[ \Gamma_1 = -7.94351 \times 10^{12} + (6.19305 \times 10^{15} - 0.0625i)A_3^2 \]
\[ - (7.05485 \times 10^{15} + 2.53004 \times 10^{14}i) \sqrt{\gamma_1}. \]

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