Refined model for rectangular thin plates on two-parameter foundations using an iterative procedure

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Abstract. The main purpose of this study is to investigate the influence of soil heterogeneity on the mechanical responses of rectangular plates resting on elastic foundations and to determine reasonable physical model parameters. The bending problem of rectangular thin plates lying on Gibson elastic foundations is solved using an iterative procedure. First, plates and foundations are discussed as a whole system, based on the principle of minimum potential energy. Second, the governing equations and boundary conditions for plates are established. And the equation for the attenuation parameter in the mathematical model is also derived. Then an iterative procedure is used to iterate the two parameters simultaneously, which solves a key issue for the two-parameter foundation model. Third, a double Fourier series with supplementary functions is adopted, and the solution to the bending problem is obtained. The agreements between the numerical results and the literature results prove that the refined model is practical and achievable. The proposed method exhibits a certain universality in analyzing the interactions between rectangular thin plates and elastic foundations. Furthermore, Gibson soil has an influence on the structural responses of rectangular plates on elastic foundations.

1. Introduction
Beam, plate and shell models on elastic foundations are common structural components in engineering, and are being used for airport runways, highway pavements, dock floors, chemical containers, etc. Reasonable assumptions and accurate calculations of these models are of great significance. Owing to the difficulties in mathematics and mechanics, the analysis and study of the classic components has become a popular research topic. In particular, the foundation engineering is concealed, which has a significant influence on the safety of structures. The interactions between structures and foundations include the mechanical analysis of the structures and the selection of foundation models. The two-parameter foundation model can be classified into two types. The first one makes the foundation continuous in the transverse direction by adding transverse constraints between the independent springs in the Winkler foundation model. In the second type of model, some assumptions are introduced to simplify the displacement and stress distribution of a foundation on the basis of the elastic continuum model.

The existing theoretical systems of soil-structure interaction (SSI) are relatively perfect. Ozgan and Daloglu studied the influence of transverse shear strains on plates lying on elastic foundations using a modified Vlasov model [1]. And they applied the modified Vlasov foundation to the free vibration analysis of thick plates lying on elastic foundations [2]. The Galerkin solution to the problem of an irregular Kirchhoff plate resting on a Winkler foundation has been derived [3]. Dalgolu et al. [4] performed a dynamic analysis of plates using a Vlasov foundation model.
There are some aspects that need to be further analyzed and gradually improved in the research on SSI. For example, in the theory of elastic foundations, the medium is considered to be continuous, linear elastic, homogeneous and isotropic. However, the behavior of soil is not simply determined by a certain parameter, but by the coexisting physical parameters and their relationships. The complexity of the actual soil behavior leads to many rational foundation models. The Gibson-type foundation is more in line with the engineering practice. Eisenberger and Clastornik [5] presented and compared two methods for solving the eigenvalue problems of vibration and stability of a beam lying on a variable Winkler foundation. Dempsey and Li [6] studied the problem of a footing lying on a nonhomogeneous elastic half-space. Mayne and Poulos [7] presented a new solution for Gibson soil of a finite thickness.

It is difficult to determine the characteristic parameters of two-parameter elastic foundation model, which has hindered the popularization of the physical model. In view of this, Vallabhan and Das [8] further enhanced the Vlasov physical model for beams lying on elastic foundations by evaluating the third parameter. Arani and Zamani [9] performed a bending analysis of a carbon nanotube-reinforced beam lying on a modified Vlasov foundation.

Based on Gibson soil, the static bending problem of rectangular thin plates lying on refined Vlasov foundations is analyzed systematically in the current study. The influence of soil heterogeneity and the interactions between foundations and rectangular plates are thoroughly investigated. First, according to the principle of minimum potential energy, the governing equations for rectangular plates on Gibson-type two-parameter foundations are established. Then, a double Fourier series with supplementary terms is used to solve the problem. The equation that the attenuation parameter needs to satisfy is obtained by variation, and the two characteristic parameters are calculated by using the third parameter obtained through an iterative procedure. Finally the relatively accurate mechanical responses of rectangular plates on refined elastic foundations are obtained.

2. Mathematical model

2.1. Potential energy of the system

The total potential energy $U$ of the plate-foundation system is:

$$U = U_p + U_s + U_q$$  \hspace{1cm} (1)

where $U_p = \text{deformation potential energy of the plate}$, $U_s = \text{deformation potential energy of the foundation}$, $U_q = \text{external force potential energy}$. They can be calculated as follows:

$$U_p = \frac{D}{2} \iint_\Omega \left\{ (V^2w)^2 - 2(1-\mu) \left[ \frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2} - \left( \frac{\partial^2w}{\partial x\partial y} \right)^2 \right] \right\} \mathrm{d}x\mathrm{d}y$$ \hspace{1cm} (2)

where $D = \frac{Eh^3}{12(1-\mu)}$ is the bending stiffness, $\mu$ and $E$ represent the Poisson’s ratio and elastic modulus, $w$ denotes the deflection of the rectangular thin plate, and $\Omega$ represents the region of the rectangular thin plate.

$$U_q = -\iint_{\Omega_0} qw \mathrm{d}x\mathrm{d}y$$ \hspace{1cm} (3)

where $q$ is the external load and $\Omega_0$ denotes the area under the load.

$$U_s = \frac{1}{2} \iiint \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{xz} \gamma_{xz} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$ \hspace{1cm} (4)

The terms $(w_s, u_s, v_s)$ denote the deformations of the foundation in the $x$, $y$ and $z$ directions,
respectively.

\[ u_s(x, y, z) = 0, \quad v_s(x, y, z) = 0, \quad w_s(x, y, z) = w_s(x, y)\varphi(z) \]  \hspace{1cm} (5)

where \( w_s(x, y) \) is the displacement of the foundation surface and \( \varphi(z) \) denotes the attenuation function. On the contact surface between the rectangular thin plate and foundation, there are continuous displacement conditions.

The elasticity moduli at the top and bottom of the Gibson foundation are \( E_{s1} \) and \( E_{s2} \), respectively, and the following dimensionless parameter \( \eta \) is introduced.

\[ \eta = \frac{E_{s1}}{E_{s2}} \]  \hspace{1cm} (6)

As shown in Figures 1 and 2, the length and width of the rectangular thin plate are \( a \) and \( b \), and its thickness is \( h \). The depth of the foundation is \( H \). The modulus of elasticity \( E_s \) at depth \( z \) is

\[ E_s = E_{s2} \left[ \eta + (1 - \eta) \frac{z}{H} \right] \]  \hspace{1cm} (7)

It can be seen that the soft and hard conditions of foundation soil depend on the value of the dimensionless parameter \( \eta \).

Figure 1. Rectangular thin plate resting on the Gibson elastic foundation.

Figure 2. Rectangular thin plate and subdivision of the elastic foundation.
Therefore, the expression (4) for the potential energy of the soil can be rewritten as follows:

\[
U_s = \frac{1}{2} \iint_{t_s} \left\{ \frac{(1 - \mu_s)}{(1 + \mu_s)(1 - 2\mu_s)} \left( \frac{d\phi}{dz} \right)^2 w_s^2 + \frac{E_s}{2(1 + \mu_s)} \phi^2 \left[ \left( \frac{\partial w_s}{\partial x} \right)^2 + \left( \frac{\partial w_s}{\partial y} \right)^2 \right] \right\} \, dx \, dy \, dz
\]
\[= \frac{1}{2} \iint_{t_s} \left\{ kw_s^2 + G_p \left[ \left( \frac{\partial w_s}{\partial x} \right)^2 + \left( \frac{\partial w_s}{\partial y} \right)^2 \right] \right\} \, dx \, dy \]

(8)

where \( V_s \) represents the soil volume of the foundation and \( \Omega_s \) is the area of the foundation surface, including the area \( \Omega_s^+ \) under the rectangular plate and the area \( \Omega_s^- \) outside the rectangular plate. Attention should be paid to the deformation compatible condition for rectangular plates and foundations. The two characteristic parameters of the Gibson elastic foundation are as follows:

\[
k = \int_0^H \frac{E_s}{2(1 + \mu_s)} \left[ \left( \frac{\partial \phi}{\partial z} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] \, dz
\]
\[G_p = \int_0^H \frac{E_s}{2(1 + \mu_s)} \left[ \left( \frac{\partial \phi}{\partial z} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] \, dz
\]

(9)  (10)

As shown in Figure 2, the range of the rectangular plate is \( x \in [0, a] \), \( y \in [0, b] \). The foundation outside the plate is divided into four boundary regions (2, 4, 6, 8) and four corner regions (1, 3, 5, 7). The deformation of the foundation soil is assumed to decay exponentially. Take the surface displacements of elastic foundations in subzones 1 and 2 and corner point \( (a, b) \) as examples. All the formulas satisfy the assumption that the deflection at infinity is zero.

\[
w_{s1} = w(a, b)e^{-a(x-a)}e^{-a(y-b)}
\]
\[w_{s2} = w(a, y)e^{-a(x-a)}
\]

(11)  (12)

where \( \alpha \) is the attenuation exponent of the deflection.

For subzone 1, the deformation potential energy of the soil is calculated as follows:

\[U_{s1} = G_p \frac{3}{8} w^2(a, b)
\]

(13)

For subzone 2, the deformation potential energy of the soil is calculated as follows:

\[U_{s2} = \frac{1}{2} G_p \left[ \alpha \int_0^b w_a^2 \, dy - \frac{1}{2\alpha} \int_0^b (w_a^2 \frac{\partial^2 w_a}{\partial y^2}) \, dy \right]
\]

(14)

In the same way, the expressions for the potential energy in other subregions of the foundation are obtained, and then the deformation potential energy of the soil-plate system is established by summation.

### 2.2. Governing equations and boundary conditions

The governing differential equations for the rectangular thin plate and foundation soil surface are derived by a complex variational deduction.

\[D \nabla^4 w - G_p \nabla^2 w + kw = q
\]
\[-G_p \nabla^2 w_i + kw_i = 0
\]

(15)  (16)

When \( 0 < z < H \), the coefficient terms of \( \delta \phi \) are sorted out, and the equation for the attenuation function \( \phi \) is obtained as follows:

\[-m \frac{d^2 \phi}{dz^2} + n\phi = 0
\]

(17)
The attenuation function satisfies the following conditions

$$\phi(0) = 1, \quad \phi(H) = 0$$

where

$$m = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_s (1-\mu_s) \frac{w^2}{(1+\mu_s)(1-2\mu_s)} \, dx \, dy$$

$$n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{E_s}{2(1+\mu_s)} (\nabla w)^2 \, dx \, dy$$

According to displacement continuity, the following equation can be derived.

$$\alpha^2 = k / G_p$$

According to differential equation (17) and boundary conditions (18), the attenuation function \( \phi \) and attenuation parameter \( \gamma \) can be solved as equations (22) and (23), respectively. These relationships will be used in the following iteration process.

$$\phi(z) = \frac{\sinh \gamma(1-z/H)}{\sinh \gamma}$$

$$\left(\frac{\gamma}{H}\right)^2 = \frac{n}{m}$$

From the arbitrariness of the variational items, the boundary conditions for the edge \( x = a \) and the corner point \( (a,b) \) are discussed as follows.

If the boundary condition is a free edge, the following two formulas can be obtained:

$$M_x \bigg|_{x=a} = 0, \quad V_y + G_p (\alpha w + \frac{\partial w}{\partial x} - \frac{1}{2\alpha} \frac{\partial^2 w}{\partial y^2}) \bigg|_{x=a} = 0$$

where \( M_x \) and \( V_y \) are the bending moment and shear force, respectively.

When the point \( (a,b) \) is a free corner, then the following formula can be obtained:

$$2(1-\mu)D \frac{\partial^2 w}{\partial x \partial y} \bigg|_{x=a,y=b} + \frac{3}{4} G_p w \bigg|_{x=a,y=b} = 0$$

In addition, other boundary conditions and corner conditions for a rectangular plate are similar to those mentioned above. After degradation, equations (8) to (23) are consistent with the governing equations and boundary conditions for the isotropic two-parameter foundation plate. That is, the equations and conditions do not change, but the two characteristic parameters of the model change.

2.3. Iteration process

In application of the two-parameter elastic foundation model, the value of the attenuation parameter must be estimated and is usually determined by experiments or experience. However, in this paper, the reaction coefficient and shear parameter of elastic foundations are not given in advance. That is, the attenuation parameter values are not estimated by experiments or experience in this study, but the relatively accurate values are obtained by an iterative process.

First, we set the initial value of the attenuation parameter as 1, and get the attenuation function according to equation (22). Second, we substitute this into equations (9) and (10) to get the values of the two characteristic parameters. Then, we solve for the deflection values of rectangular plates and foundation soil according to a Fourier series method. Third, we substitute the deflections into equations (19) and (20) to get the values of \( m \) and \( n \), and use equation (23) to calculate a new attenuation parameter. After the above process is completed, the new attenuation parameter value is obtained. We then take the average value of the new and old values as the new parameter value, repeat
the previous process, and iterate until the difference between the two continuous values is less than a specified small value, such as 0.001. Take the final attenuation parameter value to analyze the bending of rectangular plates on refined elastic foundations.

3. Numerical results

3.1. Methods

A Fourier series method can be used to obtain the exact solution to thin plate bending. By using a double Fourier series with supplementary terms as a trial function, simultaneous equations are solved to obtain the undetermined coefficients based on the governing equations and boundary conditions. To verify the theoretical model and method, a rectangular thin plate with four free edges on an elastic foundation is analyzed. Specifically, the form of biaxial symmetry such as physics and geometry is selected to simplify the calculation. The following deflection trial functions is selected for the rectangular thin plate with four free edges.

\[
\begin{align*}
    w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \\
    &+ \sum_{m=1}^{\infty} \left[ \frac{n^2 \pi^2 \mu A_n}{2b^2} x^2 - \frac{n^2 \pi^2 a \mu A_n}{2b^2} x + A_n \right] \sin \frac{n\pi y}{b} + \\
    &+ \sum_{m=1}^{\infty} \left[ \frac{m^2 \pi^2 \mu C_m}{2a^2} y^2 - \frac{m^2 \pi^2 b \mu C_m}{2a^2} y + C_m \right] \sin \frac{m\pi x}{a} + w_{00}
\end{align*}
\]

where \( b_{mn}, A_n, \) and \( C_m \) are the coefficients of the boundary deflection and boundary moment expansion terms, respectively, and \( w_{00} \) is the corner deflection of a rectangular thin plate.

The load can be expanded into the following Fourier series

\[
q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

where

\[
q_{mn} = \frac{4}{ab} \int_{0}^{a} \int_{0}^{b} q(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy
\]

The four groups of undetermined coefficients \( (b_{mn}, A_n, C_m, w_{00}) \) in the deflection expression can be determined according to the governing equations, boundary conditions and corner conditions. Because the number of unknowns is equal to the number of equations, the problem is solvable.

3.2. Numerical examples

We consider two examples. In the first example, the square plate on a elastic foundation subjected to a concentrated force and uniformly distributed load at the center is analyzed [10]. The refined Gibson foundation models are degraded to the traditional Vlasov foundation model. The following parameters are used in the calculation. The width of the thin plate \( a = 1 \) m, thickness of the plate \( h = 0.04 \) m, elastic modulus of the plate \( E = 1.7658 \times 10^{10} \) N/m², the Poisson’s ratio of the plate \( \mu = 0.167 \), elastic modulus of the foundation \( E_f = 8.6 \times 10^{6} \) N/m², the Poisson’s ratio of the foundation \( \mu_f = 0.2 \), depth of the foundation \( H = 1 \) m, a concentrated force \( P = 9.8 \times 10^{4} \) N, and a uniformly distributed load \( q = 9.8 \times 10^{3} \) N/m².
Table 1. Deflection of a rectangular plate on the isotropic two-parameter foundation in Example 1.

|             | $w/\text{m}$ | $y = a/2$ | $y = 5a/8$ | $y = 3a/4$ | $y = 7a/8$ |
|-------------|--------------|-----------|-----------|-----------|-----------|
| $x = a/2$   | 0.00267      | 0.00194   | 0.00108   | 0.00043   |           |
| $x = 5a/8$  | 0.00194      | 0.00157   | 0.00092   | 0.00037   |           |
| $x = 3a/4$  | 0.00108      | 0.00092   | 0.00058   | 0.00025   |           |
| $x = 7a/8$  | 0.00043      | 0.00037   | 0.00025   | 0.00010   | -0.0010   |

The calculation results for a rectangular thin plate on an isotropic two-parameter foundation obtained by a direct method are listed in the first row of Table 1, where $m = n = 50$. The exact solutions for a rectangular thin plate on the traditional two-parameter foundation in reference [10] are listed in the second row. The comparison between the two results shows that the theoretical derivation and calculation results are consistent.

In the second example, the physical parameters, geometric parameters, and load conditions of the rectangular thin plate are the same as those in Example 1. However, the parameters of Gibson two-parameter elastic foundation are as follows: the elastic modulus at the bottom of the Gibson soil $E_{s2} = 4.0 \times 10^7 \text{N/m}^2$, the Poisson’s ratio of the foundation $\mu_s = 0.3$, and the depth of the foundation soil $H = 2\text{m}$.

When the characteristic parameters of the Gibson elastic foundation are selected as $\eta = 0.5, 1, \text{and } 2$, respectively, the results are listed in Table 2. $w$ in the table refers to the deflection at the center of the rectangular plate. In this example, $m = n = 25$ is selected to reduce the machine time and ensure the convergence of the results. Figures 3 and 4 depict the deflection and bending moment of the rectangular thin plate on the Gibson two-parameter foundation solved iteratively, respectively. The shape and trend of these figures are correct.

Table 2. Model parameters and deformation of three different Gibson two-parameter foundations in Example 2.

| Parameters | $\eta$ | $\gamma$ | $k \left(10^6 \text{N/m}^3\right)$ | $G_p \left(10^6 \text{N/m}\right)$ | $w (\text{mm})$ |
|------------|-------|----------|------------------------------------|----------------------------------|----------------|
|            | 0.5   | 0.89105  | 1.52803                            | 1.04514                          | 2.92           |
|            | 1     | 0.90559  | 2.12703                            | 1.68410                          | 2.39           |
|            | 2     | 0.97593  | 3.36041                            | 2.92681                          | 1.81           |
Figure 3. Deflection of a rectangular plate on a Gibson elastic foundation.

Figure 4. Bending moment of a rectangular plate on a Gibson elastic foundation.

4. Iterative analysis

The attenuation parameter represents the variation of the deflection of foundation soil along the z axis. In Table 3, $N$ denotes the number of iterations of the programs. $\gamma_a$, $\gamma_b$, and $\gamma_c$ are the attenuation parameters of the rectangular thin plate lying on Gibson two-parameter foundations in Example 2 corresponding to $\eta = 0.5, 1, \text{ and } 2$, respectively.

| $\gamma_a$ | 1.00 | 0.94494 | 0.91725 | 0.90333 | 0.89633 | 0.89282 | 0.89105 | 0.89105 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $\gamma_b$ | 1.00 | 0.95203 | 0.92805 | 0.91606 | 0.91007 | 0.90708 | 0.90559 | 0.90559 |
| $\gamma_c$ | 1.00 | 0.98739 | 0.98293 | 0.97762 | 0.97593 | 0.97593 | — | — |

Table 3. Attenuation parameters of Gibson two-parameter foundations.
5. Conclusion
In this study, with the heterogeneity of Gibson soil taken into account, the bending problem of rectangular thin plates on a refined Vlasov foundation is analyzed and calculated using the iterative procedure. The following conclusions can be drawn from the study.

(1) The foundation is divided into nine regions, and the deflection variation law of each region is reasonably assumed. The governing differential equations can be derived according to the principle of minimum potential energy.

(2) A double Fourier series with supplementary terms is selected as the deflection trial function, and the simultaneous equations are obtained by comparing the coefficients. The unknown coefficients are solved by using simultaneous equations, and then the deformation and internal force of rectangular plates on refined elastic foundations are obtained.

(3) With regard to foundation soil, considering it as Gibson foundation is more in line with the engineering practice than just considering it as isotropic soil. If the refined foundation is degenerated to a homogeneous and isotropic medium, the mathematical and physical formulas in this paper can be degraded to the classical two-parameter foundation plate.

(4) It is found that for the characteristic parameters, better results can be obtained by using the iterative procedure. The subgrade coefficient and shear stiffness can be obtained, which solves the problem of not being able to determine the characteristic parameters of two-parameter elastic foundations.

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