SURFACES WITH INVOLUTION AND PRYM CONSTRUCTIONS

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ABSTRACT. An involution on a surface induces involutions on the cohomology, the Chow group and the Brauer group of the surface. We give a detailed study of those actions and show that the odd part of these groups can be used to describe the geometry of cubic fourfolds and conic bundles over \( \mathbb{P}^3 \).

1. INTRODUCTION

When a curve \( C \) admits an involution \( \sigma : C \to C \), its Jacobian \( J(C) \) splits into the even part and the odd part. When \( \sigma \) has no fixed point or exactly two fixed points, the odd part \( \text{Pr}(C, \sigma) = \text{Im}\{\sigma - 1 : J(C) \to J(C)\} \) admits a natural principal polarization. The resulting principally polarized abelian variety is defined to be the Prym variety of \( C \) with the involution \( \sigma \). See [11] for more details. Among many interesting applications, the theory was used to characterize the intermediate Jacobian of certain types of Fano threefolds and quadrics bundles (see, for example, [6, 12, 13, 3]). Together with the geometry of the theta divisor, such a description was used to study various rationality questions.

In this paper, we study the case where the curve is replaced by a surface. Throughout the paper, we will be working over the field \( \mathbb{C} \) of complex numbers. Let \( S \) be a smooth projective surface with \( H_1(S, \mathbb{Z}) = 0 \) and \( \sigma : S \to S \) an involution with only isolated fixed points. Then \( \sigma \) induces involutions on the cohomology \( H^2(S, \mathbb{Z}) \), the Chow group \( A_0(S) \) of 0-cycles with degree zero and the Brauer group \( \text{Br}(S) \).

**Definition 1.1** (see [13]). Let \( \Lambda \) be an abelian group together with an involution \( \sigma : \Lambda \to \Lambda \), then we define the Prym part of \( \Lambda \) to be \( \text{Pr}(\Lambda, \sigma) = \text{Im}\{\sigma - 1 : \Lambda \to \Lambda\} \).

With the above definition we can talk about the Prym part of the cohomology, the Chow group and the Brauer group of \( S \). For the cohomology group, we usually have a geometrically defined \( \sigma \)-invariant subgroup \( M \subset H^2(S, \mathbb{Z}) \) of algebraic classes. Hence \( \sigma \) induces an action on its orthogonal complement \( M^\perp \) and we take the Prym part of \( M^\perp \). This should be viewed as the Prym part of the “primitive cohomology”. Then this Prym construction can be used to describe the Hodge structure, the Chow group and the “second Brauer groups”, \( \text{Br}_2(X) \) (see Definition 3.11), of cubic fourfolds and conic bundles. To be more precise, if \( X \subset \mathbb{P}^5 \) is a smooth cubic fourfold, then we take \( S = S_l \) to be the surface of lines meeting a given general line \( l \subset X \). Then there is a natural involution \( \sigma \) on \( S \) induced by taking the residue of two intersecting lines. If \( f : X \to B = \mathbb{P}^3 \) is a standard conic
bundle, then we take $S$ to be the surface of lines in broken conics. We assume that
$S$ is smooth projective with $H_1(S, \mathbb{Z}) = 0$. Then there is again an involution $\sigma$
which is induced by switching the two lines in a broken conic. In the second case, we also assume
that the degree of the degeneration divisor is odd. We show that in both cases the Hodge
structure and the Chow group of $X$ can be recovered as the Prym part of the corresponding
structures of the surface $S$. The group $Br_2(X)$ can also be recovered in generic cases.
When $X$ is special, there might be a 2-torsion subgroup $K$ of $Br_2(X)$ which does not appear
in the Prym part of the Brauer group of $S$. In the case when $X$ is a cubic fourfold, this group $K$
is independent of the choice of the general line $l \subset X$. So far we do not have an example
where the group $K$ is nontrivial.

The idea of using the space of lines on a hypersurface to describe the Hodge
structure has been studied by many mathematicians. In [19], I. Shimada
considered the total space of lines on a general hypersurface and showed that the associated
cylinder homomorphism is an isomorphism in many cases. In [10], E. Izadi gave
a Prym construction for the Hodge structure of cubic hypersurfaces. Our result
on cubic fourfolds gives such a construction for the Chow group and the Brauer
group as well. Another advantage of our approach is that are potentially applica-
tions of Theorem 3.13 to other situations. See [18] for more general Prym-Tjurin
construction.

The plan of this article is as follows. In Section 2, we give a detailed study of
the action of $\sigma$ on $H^2(S, \mathbb{Z})$. The main results are summarized in Theorem 2.8 and
Theorem 2.9. It turns out that the structure of this action is quite sensitive to
whether $\sigma$ has fixed points or not. In section 3, we study the Prym part of the
cohomology, Chow group and Brauer group of $S$. Sections 4 and 5 are devoted
the application to the case of cubic fourfolds. Section 6 studies the case of conic
bundles over $\mathbb{P}^3$.

Notation and conventions.

(1) Let $\Lambda$ be an integral Hodge structure of weight $2k$. Then we use $Hdg(\Lambda) = 
\Lambda^{k,k} \cap \Lambda$ to denote the group of integral Hodge classes. If $\Lambda$ comes from geometry,
we also use $Alg(\Lambda) \subset Hdg(\Lambda)$ to denote the subgroup of algebraic classes. When
$\Lambda$ is polarized (which boils down to an integral symmetric bilinear form), we define
$\Lambda_{tr} \subset \Lambda$ to be the orthogonal complement of $Hdg^{2k}(\Lambda)$. We also use $T(\Lambda)$
to denote the quotient of $\Lambda$ by $Hdg(\Lambda)$. If $\Lambda = H^2(Y, \mathbb{Z})/tor$ comes from geometry,
we also use $Hdg^{2k}(Y)$, $Alg^{2k}(Y)$ and $T^{2k}(Y)$ to denote $Hdg(\Lambda)$, $Alg(\Lambda)$ and $T(\Lambda)$
respectively.

(2) In all sections except Section 5, we use $G = \{1, \sigma\}$ to denote the cyclic group
of order 2. For any abelian group $A$, we use $A_{+}$ to denote the $G$-module $A$
on which $\sigma$ acts as $+1$ and $A_{-}$ to denote the $G$-module $A$ on which $\sigma$ acts as $-1$.

(3) Let $V$ be a vector space of dimension $n$, and $1 \leq r_1 < \ldots < r_m < n$ a
sequence of increasing integers. We use $G(r_1, \ldots, r_m, V)$ to be the flag variety
parameterizing flags $V_1 \subset \ldots \subset V_m \subset V$ with $\dim V_i = r_i$. If $m = 1$ and $r_1 = 1$,
then we use $F(V)$ to denote $G(1, V)$. If we replace $V$ by a vector bundle $\mathcal{E}$, then
we define the relative flag varieties in a similar way.

(4) For any morphism $f : Y \to X$ between smooth varieties which is birational
onto image, we define the normal bundle $\mathcal{N}_{Y/X}$ of $Y$ in $X$ to be the cokernel of the
homomorphism $df : T_Y \to f^*T_X$. 

2. Surfaces with involution

In this section, we fix $S$ to be a smooth projective surface over $\mathbb{C}$. Assume that there is an involution $\sigma : S \rightarrow S$ with finitely many isolated fixed points $x_1, \ldots, x_r \in S$. This section is devoted to the study of the action of $\sigma$ on the cohomology group $H^*(S, \mathbb{Z})$.

Let $Y = S/\sigma$ be the quotient. Then $Y$ is a projective surface with finitely many ordinary double points $y_1, \ldots, y_r \in Y$. Let $\pi : S \rightarrow Y$ be the natural morphism of degree 2. We use $\tilde{S}$ to denote the blow up of $S$ at the points $x_1, \ldots, x_r$ with $E'_i$ being the corresponding exceptional curves. We use $\tilde{Y}$ to denote the minimal resolution of $Y$ with $E_i$ being the exceptional divisors. Note that the $E_i$’s are $(-2)$-curves on $\tilde{Y}$. Then $\sigma$ induces an involution, still denoted by $\sigma$, on $\tilde{S}$ with all points in the $E'_i$’s being fixed. One also easily sees that $\tilde{Y}$ is naturally the quotient of $\tilde{S}$ by the action of $\sigma$. If we use $\tilde{\pi} : \tilde{S} \rightarrow \tilde{Y}$ to denote the quotient map, then we have the following commutative diagram,

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{g} & S \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
Y & \xrightarrow{f} & Y
\end{array}
$$

where $f$ and $g$ are the blow-up’s.

On a complex analytic space $X$, we use $\mathcal{H}_p(X)$ to denote the sheaf associated to $U \mapsto H_p(X, X - U)$. Note that the stalk of $\mathcal{H}_p(X)$ at smooth point $x \in X$ is, by excision, equal to $H_p(B, \partial B)$ where $B \subset X$ is a small closed ball centered at $x$. Hence $\mathcal{H}_p(X)$, for $p \neq 2 \dim \mathbb{C} X$, is supported on the singular locus of $X$.

**Lemma 2.1.** Let $Y$ be the quotient of $S$ by $\sigma$ as above, then

(i) $\mathcal{H}_p(Y) = 0$, for $p \neq 2, 4$.

(ii) $\mathcal{H}_2(Y)$ is the skyscraper sheaf supported on $y_i$’s with $\mathcal{H}_2(Y)_y \cong \mathbb{Z}/2\mathbb{Z}$.

(iii) $\mathcal{H}_4(Y)$ is the constant sheave isomorphic to $\mathcal{H}_4(Y, \mathbb{Z}) \cong \mathbb{Z}$.

**Proof.** We know that $\mathcal{H}_p(Y)$ is zero at smooth points of $Y$ for $p \neq 4$. By assumption, we can embed $Y$ in some projective space. Let $B_i \subset Y$ be the intersection of $Y$ and a small closed ball (in the ambient projective space) centered at $y_i$. By construction, $B_i$ is contractible. By excision, we have isomorphism

$$\mathcal{H}_p(Y)_{y_i} \cong H_p(B_i, \partial B_i)$$

Consider the long exact sequence associated to the pair $(B_i, \partial B_i)$,

$$\cdots \xrightarrow{} H_p(\partial B_i) \xrightarrow{} H_p(B_i) \xrightarrow{} H_p(B_i, \partial B_i) \xrightarrow{} H_{p-1}(\partial B_i) \xrightarrow{} \cdots$$

where all cohomology groups have integral coefficients. Since $B_i$ is contractible, we have $H_p(B_i, \mathbb{Z}) = 0$ for $p \neq 0$ and $H_0(B_i, \mathbb{Z}) = \mathbb{Z}$. It follows that

$$H_p(B_i, \partial B_i) = H_{p-1}(\partial B_i, \mathbb{Z}), \quad p = 2, 3, 4$$

and we also have $H_p(B_i, \partial B_i) = 0$ for $p = 0, 1$. Let $\tilde{B}_i \subset \tilde{Y}$ be the inverse image of $B_i$. Then $\tilde{B}_i$ can be thought of as a tubular neighbourhood of $E_i \cong \mathbb{P}^1$ in $\tilde{Y}$. Then $\partial \tilde{B}_i = \partial \check{B}_i$ is homeomorphic to the unit circle bundle over $E_i$, sitting inside $\mathcal{N}_{E_i/\tilde{Y}} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ (equipped with some hermitian metric). Hence $\partial \tilde{B}_i$ can be
viewed as a circle bundle over \( \mathbb{P}^1 \) whose Euler class \( e_i = -2 \in H^2(E_i, \mathbb{Z}) \). Consider the associated Gysin sequence

\[
0 = H^1(E_i) \longrightarrow H^1(\partial B_i) \longrightarrow H^0(E_i) \cup e_i \longrightarrow H^2(\partial B_i) \longrightarrow 0
\]

where all cohomology groups have integral coefficients. One easily finds \( H^1(\partial B_i, \mathbb{Z}) = 0 \) and \( H^2(\partial B_i, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \). Note that \( \partial B_i \) is an orientable 3-dimensional compact manifold. By Poincaré duality, we get

\[
H_p(\partial B_i, \mathbb{Z}) = \begin{cases} 
0, & p = 2 \\
\mathbb{Z}, & p = 0, 3 \\
\mathbb{Z}/2\mathbb{Z}, & p = 1
\end{cases}
\]

Hence we have

\[
H_p(Y)_{y_i} = \begin{cases} 
0, & p = 0, 1, 3 \\
\mathbb{Z}/2\mathbb{Z}, & p = 2 \\
\mathbb{Z}, & p = 4
\end{cases}
\]

The lemma follows easily from this computation.

On a complex analytic space \( X \), we have Zeeman’s spectral sequence

\[
E^2_{p,q} = H^q(X, H_p(X)) \Longrightarrow H_{p-q}(X, \mathbb{Z})
\]

which measures the failure of Poincaré duality, see [22]. The differential \( d^r \) goes as follows

\[
d^r : E^r_{p,q} \longrightarrow E^r_{p+r-1,q+r}
\]

The convergence means that there is a filtration

\[
H_s(X) = F^0_s \supseteq F^1_s \supseteq \cdots \supseteq F^{n-s}_s \supseteq 0, \quad n = \dim_{\mathbb{R}}(X)
\]

such that \( E^r_{s+q,q} \cong F^q_s / F^{q+1}_s \).

We apply Zeeman’s spectral sequence to the singular surface \( Y \), and note that \( E^2_{p,q} = 0 \) unless \( p = 4 \) or \( (p,q) = (2,0) \). This means that \( E^3 = E^2 \) and we have the only nontrivial differential map

\[
d^3 : H^0(Y, H_2(Y)) \cong (\mathbb{Z}/2\mathbb{Z})^r \rightarrow H^3(Y, \mathbb{Z})
\]

We set

\[
(1) \quad M = \ker(d^3), \quad N = \text{im}(d^3).
\]

Then from the spectral sequence, we get the following

**Proposition 2.2.** Let \( Y \) be a projective surface with only isolated ordinary double points \( y_1, \ldots, y_r \in Y \) as above. Then the following statements are true.

(i) \( \cap[Y] : H^i(Y, \mathbb{Z}) \rightarrow H_{4-i}(Y, \mathbb{Z}) \) is an isomorphism for \( i = 0, 1, 4 \).

(ii) There is an exact sequence

\[
0 \longrightarrow H^2(Y) \longrightarrow H_2(Y) \longrightarrow H^0(Y, H_2(Y)) \xrightarrow{d^3} H^3(Y) \xrightarrow{\cap[Y]} H_1(Y) \longrightarrow 0
\]

where all (co)homology groups have integral coefficients.

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1Make references more precise.
Remark 2.3. If we split the above exact sequence in (ii), we get the following short exact sequences,

\[ 0 \to H^2(Y, \mathbb{Z}) \to \cap[Y] \to H_2(Y, \mathbb{Z}) \to M \to 0 \]
and

\[ 0 \to N \to H^3(Y, \mathbb{Z}) \to \cap[Y] \to H_1(Y, \mathbb{Z}) \to 0 \]

Consider the Leray spectral sequence associated to the resolution \( f : \tilde{Y} \to Y \), i.e.

\[ E_2^{p,q} = H^p(Y, R^q f_* \mathbb{Z}) \Rightarrow H^{p+q}(\tilde{Y}, \mathbb{Z}). \]

Note that

\[ R^q f_* \mathbb{Z} = \begin{cases} \mathbb{Z}, & q = 0; \\ 0, & q = 1; \\ \oplus_{r=1}^r H^2(E_r, \mathbb{Z})_y, & q = 2. \end{cases} \]

From this we get that

\[ f^* : H^i(Y, \mathbb{Z}) \to H^i(\tilde{Y}, \mathbb{Z}) \]

is an isomorphism for \( i = 0, 1, 4 \). There is also an exact sequence

\[ 0 \to H^2(Y) \to f^* H^2(\tilde{Y}) \to \oplus_{r=1}^r H^2(E_r) \to H^3(Y) \to f^* H^3(\tilde{Y}) \to 0 \]

where all cohomology groups have integral coefficients. This sequence is compatible with the one in (ii) of Proposition 2.2 in the following sense. Using Poincaré duality on \( \tilde{Y} \), we can define

\[ f_* : H^i(\tilde{Y}, \mathbb{Z}) \cong H_{4-i}(\tilde{Y}, \mathbb{Z}) \to H_{4-i}(Y, \mathbb{Z}). \]

Then \( f_* f^* = \cap[Y] : H^i(Y, \mathbb{Z}) \to H_{4-i}(Y, \mathbb{Z}) \). If we identify \( H^0(Y, H_2(Y)) \) with \( \oplus_{r=1}^r H^2(\partial B_r, \mathbb{Z}) \) as in the proof of Lemma 2.1 then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & H^2(Y) \\
\| & & \| \downarrow f_* \\
0 & \to & H^2(\tilde{Y})
\end{array}
\quad \begin{array}{ccc}
\oplus_{r=1}^r H^2(E_r) & \to & H^3(Y) \\
\downarrow \beta \quad & & \downarrow f^* \quad \\
\oplus_{r=1}^r H^2(\partial B_r) & \to & H^3(\tilde{Y})
\end{array} \quad \begin{array}{ccc}
0 & \to & H^3(Y) \\
\| & & \| \downarrow f_* \\
0 & \to & H_1(Y)
\end{array}
\]

where all (co)homology groups have integral coefficients. Here the homomorphism \( \beta \) is nothing but the pull back induced from the circle bundle structure \( \partial B_r \to E_r \).

The Gysin sequence tells us that \( \ker(\beta) \) is generated by the Euler classes \( e_i = -2 \in H^2(E_i, \mathbb{Z}) \). By diagram chasing, we easily get that

\[ f_* : H^2(\tilde{Y}, \mathbb{Z}) \to H_2(Y, \mathbb{Z}) \]

is surjective and

\[ f_* : H^3(\tilde{Y}, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \]

is an isomorphism. Then it follows that

\[ N \cong \text{im}(\alpha). \]
Before moving on, we recall some basic facts about group cohomology. Let $G$ be a finite group. The group cohomology functors $H^i(G, -)$ are defined to be the right derived functors of the functor

$$M \mapsto M^G = \{ x \in M : gx = x, \forall g \in G \}$$

on the category of $G$-modules. If $G = \{1, \sigma\}$ is the cyclic group of order 2, then the group cohomology can be written explicitly as

$$H^i(G, M) = \begin{cases} M^G, & i = 0; \\ \{ x \in M : \sigma x = -x \}, & \text{i is odd;} \\ \{ x \in M : \sigma x = x \}, & \text{i is even.} \end{cases}$$

where $M$ is a $G$-module. This implies that $H^i(G, M)$ is of 2-torsion for all positive $i$. For example, if $\alpha \in H^1(G, M)$, then $\alpha$ can be represented by $\tilde{\alpha} \in M$ such that $\sigma(\tilde{\alpha}) = -\tilde{\alpha}$. Then $2\tilde{\alpha} = \tilde{\alpha} - \sigma(\tilde{\alpha}) \in (\sigma - 1)M$. This means that $2\alpha = 0$. We state this as a lemma for future reference.

Lemma 2.4. Let $G$ be the cyclic group of order 2 and $M$ a $G$-module, then $H^i(G, M)$ is of 2-torsion for all $i > 0$. If $M$ is a torsion group with no 2-torsion elements, then $H^i(G, M) = 0$ for all $i > 0$.

The surface $S$ can be viewed as a space with $G$-action, where $G = \{1, \sigma\}$ is the cyclic group of order 2. On the category of $G$-sheaves on $S$, we have the $G$-invariant global section functor $\Gamma^G(\mathcal{F}) = \Gamma(S, \mathcal{F})^G$. The right derived functor $R^q(\Gamma^G) (\mathcal{F})$ is also denoted by $H^p(G; S, \mathcal{F})$. Then we have two natural spectral sequences attached to this situation,

\begin{equation}
\begin{align*}
\ iE_2^{p,q} &= H^p(\mathcal{Y}, R^q(\pi_*^G)\mathcal{F}) \Longrightarrow H^{p+q}(G; S, \mathcal{F}) \\
\ H_E_2^{p,q} &= H^p(G, H^q(S, \mathcal{F})) \Longrightarrow H^{p+q}(G; S, \mathcal{F})
\end{align*}
\end{equation}

See chapter V of [S] for more details. We apply the first spectral sequence to the case $\mathcal{F} = \mathbb{Z}$, and note that

$$R^q(\pi_*^G)\mathbb{Z} = \begin{cases} \mathbb{Z}, & q = 0; \\ \oplus_{i=1}^r H^q(G, \mathbb{Z}_{x_i}), & q > 0 \text{ even; } \\ 0, & \text{otherwise.} \end{cases}$$

It follows that all the differential homomorphisms

$$d_2^{p,q} : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$$

in the spectral sequence $\ iE_2$ are zero except for

$$d_2 = d_2^{2,0} : \oplus_{i=1}^r H^2(G, \mathbb{Z}_{x_i}) \rightarrow H^3(\mathcal{Y}, \mathbb{Z}).$$

Lemma 2.5. (i) There is a natural isomorphism $H^2(G, \mathbb{Z}_{x_i}) \cong H_2(\mathcal{Y})_{x_i}$, such that the differential map $d_2$ is identified with $d_3$ in Zeeman’s spectral sequence.

(ii) $H^i(\mathcal{Y}, \mathbb{Z}) \cong H^i(G; S, \mathbb{Z})$ for $i = 0, 1$. 
(iii) We have the following short exact sequences
\[\begin{array}{ccccccc}
0 & \longrightarrow & H^4(Y, \mathbb{Z}) & \longrightarrow & H^4(G; S, \mathbb{Z}) & \oplus & H^4(G, \mathbb{Z}_x) & \longrightarrow & 0 \\
0 & \longrightarrow & H^2(Y, \mathbb{Z}) & \longrightarrow & H^2(G; S, \mathbb{Z}) & \longrightarrow & M & \longrightarrow & 0 \\
0 & \longrightarrow & N & \longrightarrow & H^3(Y, \mathbb{Z}) & \longrightarrow & H^3(G; S, \mathbb{Z}) & \longrightarrow & 0 \\
\end{array}\]

(iv) \[H^i(G; S, \mathbb{Z}) \cong H_{4-i}(Y, \mathbb{Z}) \text{ for } i = 0, 1, 2, 3.\]

Proof. (i) Let \(B_i\) be the intersection of \(Y\) and a small closed ball centered at \(y_i\) as before. Let \(B'_i \subset S\) be the inverse image of \(B_i\). Then \(B'_i\) is a small neighbourhood of \(x_i \in S\) and \(B'_i\) is homeomorphic to a 4-dimensional ball. The restriction of \(\pi\) to \(\partial B'_i\) gives a covering map, \(\pi_i : \partial B'_i = S^3 \rightarrow \partial B_i\), of degree 2. The spectral sequence \(_I\) for the covering map \(\pi_i\) gives a natural isomorphism
\[H^q(G; \partial B'_i, \mathbb{Z}) \cong H^q(\partial B_i, \mathbb{Z}), \quad \forall q \geq 0.\]

Apply the second spectral sequence \(_II\) to the covering map \(\pi_i\), we get
\[H^1(G; \partial B'_i, \mathbb{Z}) \cong H^1(G, \mathbb{Z}_x) \cong H^1(G, \mathbb{Z}_x),\]

Hence we have a natural isomorphism \(H^2(\partial B_i, \mathbb{Z}) \cong H^2(G, \mathbb{Z}_x)\). (ii) follows directly from the spectral sequence \(_I\). To prove (iii) and (iv), we only need to compare the spectral sequence \(_I\) with Zeeman’s spectral sequence. \(\square\)

From now on, we assume the following

Assumption 2.6. \(H_1(S, \mathbb{Z}) = 0.\)

One easily sees that the assumption above implies \(H^1(S, \mathbb{Z}) = 0\) by the universal coefficient theorem and \(H^3(S, \mathbb{Z}) = 0\) by the Poincaré duality. Consider the second spectral sequence \(_II\). Since \(H^q(S, \mathbb{Z}) = \mathbb{Z}, q = 0, 4,\) with the trivial action of \(G,\) we have
\[H^p(G, H^q(S, \mathbb{Z})) = \begin{cases} 
\mathbb{Z}, & p = 0; \\
\mathbb{Z}/2\mathbb{Z}, & p = 2k, k \geq 1; \\
0, & p = 2k - 1, k \geq 1,
\end{cases}\]

where \(q = 0, 4.\)

Lemma 2.7. Under the assumptions 2.6, the following statements are true.

(i) \(H^0(G; S, \mathbb{Z}) = \mathbb{Z}\) and \(H^1(G; S, \mathbb{Z}) = 0.\)

(ii) There is a short exact sequence
\[0 \longrightarrow H^2(G, H^0(S, \mathbb{Z})) = \mathbb{Z}/2\mathbb{Z} \longrightarrow H^2(G; S, \mathbb{Z}) \longrightarrow H^2(S, \mathbb{Z})^G \longrightarrow 0\]

(iii) If \(r \geq 1,\) then \(H^3(G; S, \mathbb{Z}) \cong H^1(G, H^2(S, \mathbb{Z})).\)

(iv) If \(r \geq 1,\) then there is a short exact sequence
\[0 \longrightarrow H^4(G, H^0(S, \mathbb{Z})) \longrightarrow H^4(G; S, \mathbb{Z})_{\text{tor}} \longrightarrow H^2(G, H^2(S, \mathbb{Z})) \longrightarrow 0\]

Proof. This is essentially a restatement of the spectral sequence \(_II\). (i) and (ii) are easy. For (iii) and (iv), we note that the spectral sequence gives an exact sequence
\[0 \longrightarrow H^3(G; S, \mathbb{Z}) \longrightarrow H^3(G, H^2(S, \mathbb{Z})) \overset{\partial}{\longrightarrow} H^4(G, H^0(S, \mathbb{Z})) \overset{\tau}{\longrightarrow} H^4(G; S, \mathbb{Z})\]

The composition of the natural map \(H^4(G; S, \mathbb{Z}) \rightarrow \oplus_{i=1}^r H^4(G, \mathbb{Z}_x)\) (see Lemma 2.5) with \(\tau\) is the natural map induced by the restriction homomorphism \(H^4(S, \mathbb{Z}) \rightarrow \)
\( \oplus \mathbb{Z}_{x_i} \). One sees easily that this composition is nonzero. It follows that \( \tau \) is nonzero and hence injective. Thus \( \rho = 0 \). Then (iii) and (iv) are easily deduced from the spectral sequence. \( \square \\
\)

**Theorem 2.8.** Let \( S \) be a smooth complex algebraic surface with an involution \( \sigma : S \to S \). Assume that \( \sigma \) has finitely many fixed points \( x_1, \ldots, x_r \in S \), \( r \geq 1 \). Let \( Y = S/\sigma \) be the quotient of \( S \) by the involution. If \( H_1(S, \mathbb{Z}) = 0 \), then the following are true.

(i) The homology groups of \( Y \) are given by
\[
H_0(Y, \mathbb{Z}) = \mathbb{Z}, \quad H_1(Y, \mathbb{Z}) = 0, \quad H_2(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus H^2(S, \mathbb{Z})^G
\]
\[
H_3(Y, \mathbb{Z}) = 0, \quad H_4(Y, \mathbb{Z}) = \mathbb{Z}.
\]

(ii) The cohomology groups of \( Y \) are given by
\[
H^0(Y, \mathbb{Z}) = \mathbb{Z}, \quad H^1(Y, \mathbb{Z}) = 0, \quad H^2(Y, \mathbb{Z}) \subset H^2(S, \mathbb{Z})^G
\]
\[
H^3(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \quad H^4(Y, \mathbb{Z}) = \mathbb{Z}.
\]

(iii) The failure of Poincaré duality on \( Y \) are measured by
\[
M = \ker \{ [Y] : H^2(Y, \mathbb{Z}) \to H_2(Y, \mathbb{Z}) \} \cong (\mathbb{Z}/2\mathbb{Z}) \cong 1
\]
\[
N = \ker \{ [Y] : H^2(Y, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \} \cong \mathbb{Z}/2\mathbb{Z}.
\]

(iv) We always have \( r \geq 2 \) and \( \ker \{ [Y] : H^2(Y, \mathbb{Z}) \to H^2(S, \mathbb{Z})^G \} \cong (\mathbb{Z}/2\mathbb{Z})^{r-2} \).

(v) \( H^1(G, H^3(S, \mathbb{Z})) = 0 \) and \( H^2(G, H^2(S, \mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z})^{r-2} \).

(vi) We have the following isomorphism as \( G \)-modules
\[
H^2(S, \mathbb{Z}) \cong \mathbb{Z}[G]^{r_0} \oplus \mathbb{Z}_{r-2}^{r-2}.
\]

where \( r_0 = \frac{h^2(S) - r + 2}{2} \).

**Proof.** First we note that the assumption \( H_1(S, \mathbb{Z}) = 0 \) implies that \( H^1(S, \mathbb{Z}) = 0 \), \( H^2(S, \mathbb{Z}) \) is torsion free (the universal coefficients) and \( H^3(S, \mathbb{Z}) = 0 \) (the Poincaré duality). Since the composition of the natural maps
\[
\varphi : H^2(G, H^0(S, \mathbb{Z})) \to H^2(G; S, \mathbb{Z}) \to \bigoplus_{i=1}^r H^2(G, \mathbb{Z}_{x_i})
\]
comes from the natural diagonal/restriction map \( H^0(S, \mathbb{Z}) = \mathbb{Z} \to \bigoplus_{i=1}^r \mathbb{Z}_{x_i} = \mathbb{Z}^r \), it follows that \( \varphi \) is injective if \( r \geq 1 \). Actually, under the isomorphisms \( H^2(G, H^0(S, \mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z} \) and \( \oplus H^2(G, \mathbb{Z}_{x_i}) \cong (\mathbb{Z}/2\mathbb{Z})^r \), the homomorphism \( \varphi \) is nothing but the diagonal map. Consider the diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\
\varphi & & \downarrow \\
& & H_2(Y, \mathbb{Z}) \\
\downarrow & & \downarrow \pi^* \\
& & H^2(Y, \mathbb{Z})
\end{array}
\]

One easily sees that the injectivity of \( \varphi \) implies that of \( \pi^* \). Hence \( H^2(Y, \mathbb{Z}) \) is torsion free.

Now we start to prove (i). By (iv) of Lemma 2.5 and (i) of Lemma 2.7, we get
\[
H^1(Y, \mathbb{Z}) = H_3(Y, \mathbb{Z}) = H^1(G; S, \mathbb{Z}) = 0
\]
Similarly, by (iv) of Lemma 2.5 and (ii) of Lemma 2.7 we get

\[ H_2(Y, Z) = H^2(G; S, Z) = H^2(S, Z)^G \oplus \mathbb{Z}/2\mathbb{Z} \]

We also note that (ii) of Lemma 2.5 together with (i) of Lemma 2.7 implies that \( H^1(Y, Z) = 0 \). We have already seen that \( H_2(Y, Z) \) is torsion free. Hence the universal coefficient theorem tells us that \( H_1(Y, Z) = 0 \).

The conclusions in (ii) follow from (i) by the universal coefficient theorem.

To prove (iii), we first note that \( H_1(Y, Z) = 0 \), together with the exact sequence in (ii) of Proposition 2.2, implies that \( N = H^3(Y, Z) = \mathbb{Z}/2\mathbb{Z} \). Since \( M \) and \( N \) fit into the following short exact sequence by (1).

\[
\begin{array}{c}
0 \rightarrow M \oplus \mathbb{H}(Y)_y = (\mathbb{Z}/2\mathbb{Z})^r \rightarrow N \rightarrow 0,
\end{array}
\]

we get \( M \cong (\mathbb{Z}/2\mathbb{Z})^{r-1} \).

To prove (iv), we consider the following diagram

\[
\begin{array}{c}
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi'} M \xrightarrow{(\mathbb{Z}/2\mathbb{Z})^{r-2}} 0 \\
\downarrow \downarrow \downarrow \\
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi^*} H^2(Y, Z) \xrightarrow{H^2(S, Z)^G} 0 \\
\downarrow \\
H^2(Y, Z) \xrightarrow{\pi^*} H^2(Y, Z)
\end{array}
\]

Note that \( H^2(Y, Z) \) is torsion free. This forces \( \varphi' \) to be injective. Hence \( M \neq 0 \) and \( r \geq 2 \). The last column shows that \( \text{coker}(\pi^*) = (\mathbb{Z}/2\mathbb{Z})^{r-2} \).

By (iii) of Lemma 2.7 we get

\[ H^1(G, H^2(S, Z)) = H^3(G; S, Z) = H_1(Y, Z) = 0. \]

This proves the first half of (v). Consider the following diagram

\[
\begin{array}{c}
0 \rightarrow H^4(Y, Z) \xrightarrow{H^4(S, Z)^G} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\oplus H^4(G, \mathbb{Z}_x)} 0 \\
\downarrow \downarrow \downarrow \\
0 \rightarrow H^4(Y, Z) \xrightarrow{H^4(G; S, Z)} \oplus H^4(G, \mathbb{Z}_x) \xrightarrow{\oplus H^4(G; S, Z)} 0 \\
\downarrow \\
H^4(G; S, Z) \xrightarrow{\oplus H^4(G; S, Z)} H^4(G; S, Z)
\end{array}
\]

We easily get \( H^4(G; S, Z)^{\text{tor}} \cong (\mathbb{Z}/2\mathbb{Z})^{r-1} \). By (iv) of Lemma 2.7, we get

\[ H^2(G, H^2(S, Z)) = \text{coker}(\mathbb{Z}/2\mathbb{Z} \rightarrow H^4(G; S, Z)^{\text{tor}}) = (\mathbb{Z}/2\mathbb{Z})^{r-2} \]

This proves the second half of (v).

The statement (vi) follows from Lemma 2.10 on structure of \( G \)-modules. \( \square \)

We will be mainly interested in the case when \( \sigma \) has fixed points, but to complete the picture, we state the corresponding results on the case where \( \sigma \) is fixed point free.
Theorem 2.9. Let $S$ be a smooth projective surface with $H_1(S, \mathbb{Z}) = 0$. Let $\sigma : S \to S$ be an involution that has no fixed point and $G = \{1, \sigma\}$. Let $\pi : S \to Y$ be the quotient morphism, which is étale of degree 2. Then the following are true.

(i) The homology groups of $Y$ are given by

$$H_0(Y, \mathbb{Z}) = \mathbb{Z}; \quad H_1(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}; \quad H_2(Y, \mathbb{Z}) = H_2(S, \mathbb{Z})^G \oplus \mathbb{Z}/2\mathbb{Z}; \quad H_3(Y, \mathbb{Z}) = 0; \quad H_4(Y, \mathbb{Z}) = \mathbb{Z}.$$

(ii) The cohomology groups of $Y$ are given by

$$H^0(Y, \mathbb{Z}) = \mathbb{Z}; \quad H^1(Y, \mathbb{Z}) = 0; \quad H^2(Y, \mathbb{Z}) = H^2(S, \mathbb{Z})^G \oplus \mathbb{Z}/2\mathbb{Z}; \quad H^3(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}; \quad H^4(Y, \mathbb{Z}) = \mathbb{Z}.$$

(iii) As a $G$-module, the group cohomology of $H^2(S, \mathbb{Z})$ is given by the following

$$H^1(G, H^2(S, \mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z})^2; \quad H^2(G, H^2(S, \mathbb{Z})) = 0.$$

(iv) We have the following isomorphism of $G$-modules

$$H^2(S, \mathbb{Z}) = \mathbb{Z}[G]^{r_0} \oplus \mathbb{Z}^2.$$ 

where $r_0 = \frac{H^2(S, \mathbb{Z})}{2}$. In particular, the second Betti number of $S$ is even.

Proof. The natural quotient morphism $\pi : S \to Y$ is étale. This implies that

$$R^q_{\pi_*}G\mathbb{Z} = \begin{cases} \mathbb{Z}, & q = 0; \\ 0, & q > 0. \end{cases}$$

Hence the first spectral sequence $E_2$ degenerates and we have

$$H^i(G; S, \mathbb{Z}) = H^i(Y, \mathbb{Z}), \forall i \geq 0.$$

By (i) of Lemma 2.7, we get $H^0(Y, \mathbb{Z}) = \mathbb{Z}$ and $H^1(Y, \mathbb{Z}) = 0$. By (ii) of Lemma 2.7 we get $H^2(Y, \mathbb{Z}) = H^2(S, \mathbb{Z})^G \oplus \mathbb{Z}/2\mathbb{Z}$. Then by the universal coefficient theorem for cohomology, we know that $H_1(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. This implies that $H^3(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ by Poincaré duality on $Y$. This finishes the proof of statement (ii). Then (i) follows automatically by Poincaré duality. The spectral sequence $E_2$ gives an exact sequence

$$0 \longrightarrow H^3(Y, \mathbb{Z}) \longrightarrow H^1(G, H^2(S, \mathbb{Z})) \longrightarrow H^4(G, H^0(S, \mathbb{Z})) \longrightarrow H^4(Y, \mathbb{Z}) = \mathbb{Z}.$$

Since $H^1(G, H^0(S, \mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, the last homomorphism $\alpha$ is zero. Then we can conclude that $H^1(G, H^2(S, \mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z})^2$ since this group is always 2-torsion by Lemma 2.7. The above long exact sequence goes further as

$$0 \longrightarrow H^2(G, H^2(S, \mathbb{Z})) \longrightarrow H^4(Y, \mathbb{Z}) \longrightarrow H^4(S, \mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Since $H^4(Y, \mathbb{Z})$ is torsion free, $H^2(G, H^2(S, \mathbb{Z}))$ has to be zero since it is always 2-torsion by Lemma 2.7. This proves (iii). The statement (iv) follows from the following lemma.

Lemma 2.10. Let $G$ be the cyclic group of order 2 with generator $\sigma$. Assume that $G$ acts on a finitely generated free abelian group $\Lambda$. Then we have

$$\Lambda = \mathbb{Z}[G]^{r_0} \oplus \mathbb{Z}_+^{r_+} \oplus \mathbb{Z}_-^{r_-}$$

as $G$-modules, where $\mathbb{Z}_-$ is the abelian group $\mathbb{Z}$ on which $\sigma$ acts as multiplication by $-1$ and $\mathbb{Z}_+$ is the trivial $G$-module. In particular, $\text{rk}(\Lambda) = 2r_0 + r_+ + r_-$, $H^1(G, \Lambda) \cong (\mathbb{Z}/2\mathbb{Z})^{r_-}$ and $H^2(G, \Lambda) = (\mathbb{Z}/2\mathbb{Z})^{r_+}$. 

Proof. Let $B = \ker(1 - \sigma) = \Lambda G \subset \Lambda$ and $P = \ker(1 + \sigma) \subset \Lambda$. Consider the exact sequence,

$$0 \rightarrow B \oplus P \rightarrow \Lambda \rightarrow Q \rightarrow 0$$

For each element $q \in Q$, let $\tilde{q} \in \Lambda$ be a lift of $q$. Since

$$2\tilde{q} = (1 + \sigma)\tilde{q} + (1 - \sigma)\tilde{q} \in B + P,$$

we see that $Q$ is of 2-torsion. If we write $b_0 = (1 + \sigma)\tilde{q} \in B$ and $p_0 = (1 - \sigma)\tilde{q} \in Q$, then we have $2\tilde{q} = b_0 + p_0$. We may choose $\tilde{q}$ such that both $b_0$ and $p_0$ are primitive.

Let $\varphi : \mathbb{Z}[G] \rightarrow \Lambda$ be the $G$-module homomorphism with $\varphi(1) = \tilde{q}$. We show that $\varphi$ is injective. If $\varphi(x + y\sigma) = 0$, for some $x, y \in \mathbb{Z}$, then by construction,

$$0 = (x + y\sigma)(2\tilde{q}) = (x + y\sigma)(b_0 + p_0) = (x + y)b_0 + (x - y)p_0.$$

It follows that $x + y = 0$ and $x - y = 0$ and hence $x = y = 0$. Let $\Lambda' = \text{coker}(\varphi)$. We next show that $\Lambda'$ is torsion free. We will prove this by contradiction. Note that $\Lambda'_\text{tor}$ is isomorphic to the saturation of $\varphi(\mathbb{Z}[G])$ modulo $\varphi(\mathbb{Z}[G])$. Assume that for some primitive element $x + y\sigma \in \mathbb{Z}[G]$ we have $p | \varphi(x + y\sigma)$, for some prime number $p$, then we can write

$$x + y\sigma = p\alpha$$

for some $\alpha \in \Lambda$. If we multiply both sides of (9) by $\sigma + 1$ and note that $b_0 = (\sigma + 1)\tilde{q}$, then we have $(x + y)b_0 = p\alpha$. Since $b_0$ is primitive, we see that $p | x + y$. If we multiply both side of (9) by $\sigma - 1$ instead, we get $p | x - y$. Hence we get $p | 2x$ and $p | 2y$. Since $x + y\sigma$ is primitive, we see that $x$ and $y$ are coprime. Hence $p = 2$. It follows that $x$ and $y$ have the same parity and hence both of them are odd. We write $x = 2x' + 1$ and $y = 2y' + 1$. Hence

$$\varphi(x + y\sigma) = 2\varphi(x' + y'\sigma) + \varphi(1 + \sigma) = 2\varphi(x' + y'\sigma) + b_0.$$

The assumption $2 | \varphi(x + y\sigma)$ implies that $2 | b_0$. This is a contradiction since we choose $b_0$ to be primitive. Thus $\Lambda'$ is torsion free.

Consider the following diagram

$$0 \rightarrow B' \oplus P' \rightarrow \Lambda' \rightarrow Q' \rightarrow 0$$

$$0 \rightarrow B \oplus P \rightarrow \Lambda \rightarrow Q \rightarrow 0$$

$$0 \rightarrow \mathbb{Z}b_0 \oplus \mathbb{Z}p_0 \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

The middle column implies that $H^1(G, \Lambda') = H^1(G, \Lambda)$ and $H^2(G, \Lambda') = H^2(G, \Lambda)$.

If $\dim_{\mathbb{Z}/2\mathbb{Z}} Q = 0$, then the lemma holds automatically. By induction on $\dim Q$, we may assume that $\Lambda' = \mathbb{Z}[G]^{r_0 - 1} \oplus \mathbb{Z}^{r_+} \oplus \mathbb{Z}^{r_-}$. Then it follows that

$$\text{Ext}_G^1(\Lambda', \mathbb{Z}[G]) = \text{Ext}_G^1(\mathbb{Z}^{r_+}, \mathbb{Z}[G])^{r_+} \oplus \text{Ext}_G^1(\mathbb{Z}^{r_-}, \mathbb{Z}[G])^{r_-} = 0.$$

This means that the column in the middle splits and hence

$$\Lambda = \mathbb{Z}[G]^{r_0} \oplus \mathbb{Z}^{r_+} \oplus \mathbb{Z}^{r_-}.$$

This proves the lemma. □
3. Prym Constructions

In this section, we fix $S$ to be a smooth projective complex surface with $H_1(S, \mathbb{Z}) = 0$. Assume that $\sigma : S \to S$ is an automorphism with $\sigma^2 = 1$. In this section, we will use $H^2(S)$ to denote $H^2(S, \mathbb{Z})$; we use $h^2(S)$ to denote the rank of $H^2(S, \mathbb{Z})$.

3.1. Hodge structures. Let $M \subset \text{Pic}(S)$ be a saturated subgroup that is also a $G$-submodule. Let $M^\perp \subset H^2(S, \mathbb{Z})$ be the orthogonal complement of $M$. Then $M^\perp$ is again a $G$-module.

**Definition 3.1.** The Prym part of $M^\perp$ is defined to be

$$\text{Pr}(M^\perp, \sigma) = (\sigma - 1)(M^\perp),$$

with the induced Hodge structure and an integral bilinear form $\langle x, y \rangle = \frac{1}{2}(x \cdot y)$, where $x \cdot y$ is the intersection pairing.

Note that the bilinear form $\langle x, y \rangle$ is integral. Indeed, we have $x = (\sigma - 1)x'$ and $y = (\sigma - 1)y'$ for some $x', y' \in M^\perp$. Then we get

$$\frac{1}{2}x \cdot y = \frac{1}{2}(\sigma x' \cdot \sigma y' + x' \cdot y' - \sigma x' \cdot y' - x' \cdot \sigma y') = x' \cdot y' - \sigma x' \cdot y' \in \mathbb{Z}.$$

**Proposition 3.2.** Assume that $H^1(G, M) = (\mathbb{Z}/2\mathbb{Z})^{a_1}$ and $H^2(G, M) = (\mathbb{Z}/2\mathbb{Z})^{a_2}$. If $\sigma$ is fixed point free and $a_1 = 0$, then we have

$$M^\perp \cong \mathbb{Z}[G]^{r_0} \oplus \mathbb{Z}^{a_2 + 2},$$

for some $r_0 \geq 0$. If $\sigma$ has $r > 0$ isolated fixed points and $a_2 = 0$, then

$$M^\perp \cong \mathbb{Z}[G]^{r_0} \oplus \mathbb{Z}^{a_1 + r - 2},$$

for some $r_0 \geq 0$.

**Proof.** For any $G$-module $N$, we can define $N^* = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$, which is again a $G$-module in a natural way. If $N$ is free as a $\mathbb{Z}$-module, then we have $N^* \cong N$ as $G$-modules. By Lemma [2.10] this is reduced to $(\mathbb{Z}_+)^* \cong \mathbb{Z}_+$, $(\mathbb{Z}_-)^* \cong \mathbb{Z}_-$ and $\mathbb{Z}[G]^* \cong \mathbb{Z}[G]$, which are easy to verify. We consider the following exact sequence:

$$0 \longrightarrow M \longrightarrow H^2(S, \mathbb{Z}) \longrightarrow (M^\perp)^* \longrightarrow 0.$$

If $\sigma$ is fixed point free, then Theorem [2.2] shows that $H^2(G, H^2(S)) = 0$ and $H^1(G, H^2(S)) = (\mathbb{Z}/2\mathbb{Z})^2$. If furthermore $a_1 = 0$, then from the above short exact sequence we get the following exact sequence

$$0 \longrightarrow H^1(G, H^2(S)) \longrightarrow H^1(G, (M^\perp)^*) \longrightarrow H^2(G, M) \longrightarrow 0$$

and $H^2(G, (M^\perp)^*) = 0$. Since $M^\perp \cong (M^\perp)^*$, we get

$$H^1(G, M^\perp) = (\mathbb{Z}/2\mathbb{Z})^{a_2 + 2}$$

and $H^2(G, M^\perp) = 0$.

Then the conclusion follows from Lemma [2.10].

If $\sigma$ has $r > 0$ fixed points, then Theorem [2.8] shows that $H^1(G, H^2(S)) = 0$ and $H^2(G, H^2(S)) = (\mathbb{Z}/2\mathbb{Z})^{r - 2}$. If furthermore $a_2 = 0$, then the associated long exact sequence becomes

$$0 \longrightarrow H^2(G, H^2(S)) \longrightarrow H^2(G, (M^\perp)^*) \longrightarrow H^3(G, M) \longrightarrow 0$$

and $H^1(G, (M^\perp)^*) = 0$. From this we get

$$H^1(G, M^\perp) = 0$$

and $H^2(G, M^\perp) = (\mathbb{Z}/2\mathbb{Z})^{a_1 + r - 2}$.
Then the conclusion follows again from Lemma 2.10.

**Corollary 3.3.** If $\sigma$ is fixed point free and $a_1 = 0$, then

$$\text{rk} \text{Pr}(M^\perp, \sigma) = \frac{h^2(S) - \text{rk} M + a_2}{2} + 1.$$  

If $\sigma$ has $r > 0$ fixed points and $a_2 = 0$, then

$$\text{Pr}(M^\perp, \sigma) = \{ \alpha \in M^\perp : \sigma(\alpha) = -\alpha \}$$

and

$$\text{rk} \text{Pr}(M^\perp, \sigma) = \frac{h^2(S) - \text{rk} M - a_1 - r}{2} + 1.$$  

**Proof.** If $\sigma$ is fixed point free and $a_1 = 0$, then $2s_0 + a_2 + 2 = h^2(S) - \text{rk} M$. Hence we get $s_0 = \frac{h^2(S) - \text{rk} M - a_2}{2} - 1$. By definition, we get

$$\text{rk} \text{Pr}(M^\perp, \sigma) = s_0 + a_2 + 2 = \frac{h^2(S) - \text{rk} M + a_2}{2} + 1.$$  

The remaining case is proved similarly.

We fix some notations. For any $G$-module $N$, we write

$$N^{\sigma=1} = N^G = \{ x \in N : \sigma(x) = x \} \text{ and } N^{\sigma=-1} = \{ x \in N : \sigma(x) = -x \}.$$  

Let $N$ be a free $\mathbb{Z}$-module with a symmetric bilinear form, then $\text{det}(N)$ is defined to be the determinant of a matrix representation of the bilinear form. For example, when $\Sigma$ is a smooth projective surface over $\mathbb{C}$, the cup-product (or intersection form) defines a symmetric bilinear form on $\Lambda = H^2(\Sigma, \mathbb{Z})/\text{tor}$. The Poincaré duality implies that $\text{det}(\Lambda) = \pm 1$.

**Proposition 3.4.** Let $M \subset \text{Pic}(S)$ be a saturated submodule as above such that the intersection form restricted to $M$ is nondegenerate and $2 \nmid \text{det}(M)$. Assume that $H^1(G, M) = (\mathbb{Z}/2\mathbb{Z})^{a_1}$ and $H^2(G, M) = (\mathbb{Z}/2\mathbb{Z})^{a_2}$. Let $M \hookrightarrow M^*$ be the natural induced inclusion with quotient denoted by $Q_M$. Let $q' = \text{det}(M^{\sigma=-1})$ and $q = \vert (Q_M)^{\sigma=-1} \vert$. If $\sigma$ is fixed point free and $a_1 = 0$, then

$$\text{det}(\text{Pr}(M^\perp, \sigma), \langle , \rangle) = \pm 2^{a+2} \frac{q^2}{q'}, \quad a = \frac{\text{rk} M + 3a_2}{2}.$$  

If $\sigma$ has $r > 0$ fixed points and $a_2 = 0$, then

$$\text{det}(\text{Pr}(M^\perp, \sigma), \langle , \rangle) = \pm 2^b \frac{q^2}{q'}, \quad b = \frac{\text{rk} M + a_1}{2}.$$  

**Proof.** We first prove the following

**Claim 1:** Let $r_0$ be the number of copies of $Z[G]$ in $H^2(S, \mathbb{Z})$, then $\text{det}(H^2(S)^{\sigma=-1}) = \pm 2^{r_0}$.

Consider the following exact sequence

$$0 \longrightarrow H^2(S)^{\sigma=1} \oplus H^2(S)^{\sigma=-1} \longrightarrow H^2(S) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{r_0} \longrightarrow 0.$$  

By Lemma 3.5, we get

$$\det(H^2(S)^{\sigma=1}) \det(H^2(S)^{\sigma=-1}) = \pm 2^{2r_0}. \tag{10}$$
Let $Y = S/G$ be the quotient of $S$ by the involution $\sigma$ and $\pi : S \to Y$ the quotient map. We use $H^2(Y)$ to denote $H^2(Y, \mathbb{Z})$ modulo torsion. If $\sigma$ is fixed point free, then $Y$ is smooth and there is an isomorphism $\pi^*H^2(Y) = H^2(S)^G$. Hence we have

$$\det H^2(S)^G = \pm 2^{k^2(Y)} = \pm 2^{r_0}.$$ 

By combining this with the identity above, we get

$$\det(H^2(S)^{\sigma=-1}) = \pm 2^{r_0}.$$ 

If $\sigma$ has $r > 0$ fixed points, the $Y$ has $r$ ordinary double points. Let $f : \tilde{Y} \to Y$ be the minimal resolution of $Y$. Let $E_i \subset \tilde{Y}$, $i = 1, \ldots, r$, be the exceptional divisors. As in Remark 2.3 we see that $f^*H^2(Y) \subset H^2(\tilde{Y})$ is simply the orthogonal complement of $\{E_1, \ldots, E_r\}$. Let $E \subset H^2(\tilde{Y})$ be the subgroup $\oplus_{i=1}^r \mathbb{Z}E_i$. From Remark 2.3, we see that the homomorphism $H^2(\tilde{Y}) \to \mathbb{Z}^r$, defined by intersecting with $E_i$’s, has cokernel $N = \mathbb{Z}/2\mathbb{Z}$. Note that $E^\perp = f^*H^2(Y)$, then the Leray spectral sequence in Remark 2.3 gives rise to the following exact sequence

$$0 \longrightarrow E^\perp \longrightarrow H^2(\tilde{Y}) \longrightarrow \oplus H^2(E_i, \mathbb{Z}) \longrightarrow N = \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

From this exact sequence, we can construct the following diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^2(\tilde{Y})_{\mathbb{Z}[E_{\perp}]} & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^r & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^2(\tilde{Y})_{E_{\perp}} & \longrightarrow & \bigoplus_{i=1}^r H^2(E_i, \mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & E & & E & & E
\end{array}
$$

It follows that $H^2(\tilde{Y})_{E_{\perp}} = (\mathbb{Z}/2\mathbb{Z})^{r-1}$, and hence we get the following exact sequence

$$0 \longrightarrow E^\perp \oplus E \longrightarrow H^2(\tilde{Y}) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{r-1} \longrightarrow 0$$

From this and Lemma 3.5 we deduce that $\det(E)\det(E^\perp) = \pm 2^{2r-2}$. Since $\det(E) = (-2)^r$, it follows that

$$\det(H^2(Y)) = \det(f^*H^2(Y)) = \det(E^\perp) = \pm 2^{r-2}.$$ 

Note that $\det(\pi^*H^2(Y)) = 2^{k^2(Y)} \det(H^2(F))$ since $\deg(\pi) = 2$ and also that

$$\operatorname{coker}\{\pi^*H^2(Y) \hookrightarrow H^2(S)^G\} = (\mathbb{Z}/2\mathbb{Z})^{r-2}.$$ 

which follows from (iv) of Theorem 2.8. By Lemma 3.5, we have

$$2^{k^2(Y)} \det(H^2(Y)) = \det(\pi^*H^2(Y)) = 2^{2r-4} \det(H^2(S)^G).$$ 

Note that $h^2(Y) = r_0 + r - 2$. Hence $\det(H^2(S)^G) = \pm 2^{r_0}$. Then it follows from [11] that $\det(H^2(S)^{\sigma=-1}) = \pm 2^{r_0}$. This proves Claim 1.

Now consider the following short exact sequence

$$0 \longrightarrow M \oplus M^\perp \longrightarrow H^2(S) \longrightarrow Q_M \longrightarrow 0.$$
By taking the anti-invariant part (meaning the part on which \( \sigma = -1 \)), we have a long exact sequence

\[
0 \longrightarrow M^{\sigma = -1} \oplus (M^\perp)^{\sigma = -1} \longrightarrow H^2(S)^{\sigma = -1} \longrightarrow (Q_M)^{\sigma = -1} \delta \longrightarrow H^2(G, M \oplus M^\perp) \longrightarrow \cdots
\]

Note that as a \( G \)-module, \( Q_M \) is canonically isomorphic to \( M^* / M \) where \( M \subset M^* \) is induced by the intersection form. The fact that \( 2 \nmid \det(M) \) implies that \( Q_M \) has no 2-torsion. Since \( H^2(G, M \oplus M^\perp) \) is always of 2-torsion by Lemma 2.4, we conclude that \( \delta = 0 \). By Lemma 3.5, we have

\[
\det(M^{\sigma = -1}) \det((M^\perp)^{\sigma = -1}) = q^2 \det(H^2(S)^{\sigma = -1}).
\]

or equivalently,

\[
\det((M^\perp)^{\sigma = -1}) = \frac{q^2}{q'} \det(H^2(S)^{\sigma = -1}).
\]

If \( \sigma \) is fixed point free, then the quotient of \( \text{Pr}(M^\perp, \sigma) \subset (M^\perp)^{\sigma = -1} \) is \((\mathbb{Z}/2\mathbb{Z})^{2a+2}\). Then we have

\[
\det(\text{Pr}(M^\perp, \sigma), (,)) = \frac{1}{2^s} \det(\text{Pr}(M^\perp, \sigma), (,) \cdot \cdot s = \text{rk Pr}(M^\perp, \sigma) \\
= 2^{2a-2s+4} \det(M^\perp)^{\sigma = -1} \\
= \pm 2^{a+2} q^2 \frac{q'}{q'},
\]

where \( a = \frac{\text{rk } M + 3a_2}{2} \). In the case when \( \sigma \) has at least one fixed point, we have

\[
\det(\text{Pr}(M^\perp, \sigma), (,)) = \frac{1}{2^s} \det(\text{Pr}(M^\perp, \sigma), (,) \cdot \cdot s = \text{rk Pr}(M^\perp, \sigma) \\
= 2^{-s} \det(M^\perp)^{\sigma = -1} \\
= \pm 2^{b} q^2 \frac{q'}{q'},
\]

where \( b = \frac{\text{rk } M + a}{2} \).

\[\text{Lemma 3.5.}\] Let \( N \) be a free \( \mathbb{Z} \)-module of finite rank which is equipped with an integral symmetric bilinear form and \( N' \subset N \) a submodule of same rank. Then we have \( \det(N') = |N/N'|^2 \det(N) \).

\[\text{Proof.}\] Let \( \mathbf{v} = (v_1, \ldots, v_n) \) be an integral basis of \( N \), where \( n = \text{rk } N \). Let \( \mathbf{v'} = (v'_1, \ldots, v'_n) \) be an integral basis of \( N' \). Then there is a matrix \( A \) such that \( \mathbf{v'} = \mathbf{v} A \) and \( |N/N'| = |\det(A)| \). Let \( T \) be the matrix representing the bilinear form of \( N \) under the basis \( \mathbf{v} \). Then the matrix representation of the bilinear form on \( N' \) with respect to \( \mathbf{v'} \) is \( T' = A^T T A \). Hence by definition, we have

\[
\det(N') = \det(T') = \det(A^T T A) = \det(A)^2 \det(T) = |N/N'|^2 \det(N).
\]

This proves the lemma.

\[\Box\]
3.2. Chow group and Brauer group. We use $A_0(S)$ to be the Chow group of zero cycles of degree 0. Then $\sigma$ also acts on $A_0(S)$. Let $\text{Br}(S)$ be the Brauer group of the surface $S$. Again $\sigma$ induces an involution on $\text{Br}(S)$.

**Definition 3.6.** The Prym part of $A_0(S)$ is

$$\text{Pr}(A_0(S), \sigma) = (\sigma - 1)A_0(S).$$

The Prym part of $\text{Br}(S)$ is

$$\text{Pr}(\text{Br}(S), \sigma) = (\sigma - 1)\text{Br}(S).$$

**Remark 3.7.** Under the assumption that $H^1(S, \mathbb{Z}) = 0$, By Roitman’s theorem [15], the group $A_0(S)$ is always torsion free and uniquely divisible. Then $\text{Pr}(A_0(S), \sigma)$ is also torsion free and uniquely divisible. This further implies that

$$\text{Pr}(A_0(S), \sigma) = A_0(S)^{\sigma = -1}.$$ 

However, the same identity for Prym part of Brauer group does not hold since $\text{Br}(S)$ is torsion.

Consider the long exact sequence associated to the exponential sequence on $S$, we have

$$0 \rightarrow \text{Hdg}^2(S) \rightarrow H^2(S, \mathbb{Z}) \rightarrow H^2(S, \mathcal{O}_S) \rightarrow H^2(S, \mathcal{O}_S^*) \rightarrow 0.$$ 

Since $\text{Br}(S) = H^2(S, \mathcal{O}_S^*)_{\text{tor}}$ (see [15, Cor. 2.2]), we get

$$\text{Br}(S) = T^2(S) \otimes \mathbb{Q}/\mathbb{Z} = \text{Hom}(H^2(S, \mathbb{Z})_{\text{tr}}, \mathbb{Q}/\mathbb{Z}).$$

Let $M \subset \text{Hdg}^2(S) = \text{Pic}(S)$ be saturated and nondegenerate as above. We still set $Q_M = H^2(S, \mathbb{Z})/(M \oplus M^\perp)$. Let $\text{Hdg}(Q_M)$ be the image of $\text{Hdg}^2(S)$ in $Q_M$. Set $T(Q_M) = Q_M/\text{Hdg}(Q_M)$. Then we have the following diagram:

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \text{Hdg}(Q_M) & Q_M & T(Q_M) & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \text{Hdg}^2(S) & H^2(S, \mathbb{Z}) & T^2(S) & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \text{Hdg}(M^\perp) \oplus M & M^\perp \oplus M & T(M^\perp) & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

If we take the $\sigma = -1$ parts and assume that $2 \nmid \det(M)$, then we get the following short exact sequence:

$$0 \rightarrow T(M^\perp)^{\sigma = -1} \rightarrow T^2(S)^{\sigma = -1} \rightarrow T(Q_M)^{\sigma = -1} \rightarrow 0.$$ 

The surjectivity on the right hand side follows from the fact that $T(Q_M)$ has no 2-torsion and that $H^2(G, T(M^\perp))$ is of 2-torsion (Lemma 2.4).
Lemma 3.8. Let $N$ be a $G$-module which is free of finite rank over $\mathbb{Z}$. Then
\[ N^{\sigma = -1} \otimes \mathbb{Q}/\mathbb{Z} = \text{Pr}(N \otimes \mathbb{Q}/\mathbb{Z}, \sigma) = (\sigma - 1)(N \otimes \mathbb{Q}/\mathbb{Z}). \]

Proof. By Lemma 2.10, we only need to check for $N = \mathbb{Z}_+$, $N = \mathbb{Z}_-$ and $N = \mathbb{Z}[G]$, which are easy to verify. \hfill \square

The above lemma leads to the following

Proposition 3.9. Let $M \subset \text{Pic}(S)$ be a $G$-module that is saturated and nondegenerate. Assume that $2 \nmid \det(M)$. Then there is a short exact sequence
\[ 0 \longrightarrow T(Q_M)^{\sigma = -1} \longrightarrow T(M^\perp)^{\sigma = -1} \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \text{Pr}(\text{Br}(S), \sigma) \longrightarrow 0. \]
We also have another short exact sequence
\[ 0 \longrightarrow K \longrightarrow T((M^\perp)^{\sigma = -1}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow T(M^\perp)^{\sigma = -1} \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \]
where $K = \ker\{H^1(G, H^2(S, \mathbb{Z})) \rightarrow H^1(G, T^2(S))\}$.

Corollary 3.10. There is a short exact sequence
\[ 0 \longrightarrow K \oplus T(Q_M)^{\sigma = -1} \longrightarrow T((M^\perp)^{\sigma = -1}) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \text{Pr}(\text{Br}(S), \sigma) \longrightarrow 0. \]

Proof of Proposition 3.9. By Lemma 3.8, we have
\[ T^2(S)^{\sigma = -1} \otimes \mathbb{Q}/\mathbb{Z} = \text{Pr}(\text{Br}(S), \sigma). \]

We tensor sequence (12) with $\mathbb{Q}/\mathbb{Z}$ and get
\[ 0 \longrightarrow \text{Tor}_1(T(Q_M)^{\sigma = -1}, \mathbb{Q}/\mathbb{Z}) \longrightarrow T(M^\perp)^{\sigma = -1} \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \text{Pr}(\text{Br}(S), \sigma) \longrightarrow 0. \]

Since $\text{Tor}_1(T(Q_M)^{\sigma = -1}, \mathbb{Q}/\mathbb{Z}) \cong T(Q_M)^{\sigma = -1}$, the first exact sequence follows. Consider the short exact sequence
\[ 0 \longrightarrow \text{Hdg}(M^\perp) \longrightarrow M^\perp \longrightarrow T(M^\perp) \longrightarrow 0. \]
By taking the $\sigma = -1$ part, we get
\[ 0 \longrightarrow \text{Hdg}(M^\perp)^{\sigma = -1} \longrightarrow (M^\perp)^{\sigma = -1} \longrightarrow T(M^\perp)^{\sigma = -1} \longrightarrow K' \longrightarrow 0, \]
where $K' = \ker\{H^2(G, \text{Hdg}(M^\perp)) \rightarrow H^2(G, M^\perp)\}$. In the diagram (11), we note that $Q_M$ (and hence $\text{Hdg}(Q_M)$ and $T(Q_M)$) is free of 2-torsion. This implies $H^i(G, Q_M) = 0$ (and hence $H^i(G, \text{Hdg}(Q_M)) = 0$ and $H^i(G, T(Q_M)) = 0$) for all $i > 0$. The left and middle columns of (11) show that
\[ H^2(G, M \oplus \text{Hdg}(M^\perp)) \cong H^2(G, \text{Hdg}^2(S)) \]
and
\[ H^2(G, M^\perp \oplus M) \cong H^2(G, H^2(S, \mathbb{Z})). \]
Hence we have
\[ K' = \ker\{H^2(G, \text{Hdg}(M^\perp) \oplus M) \rightarrow H^2(G, M^\perp \oplus M)\} \]
\[ = \ker\{H^2(G, \text{Alg}^2(S)) \rightarrow H^2(G, H^2(S, \mathbb{Z}))\} \]
\[ = \text{coker}\{H^1(G, H^2(S, \mathbb{Z})) \rightarrow H^1(G, T^2(S))\} \]
\[ = K. \]
This gives the following short exact sequence

\[ 0 \to T((M^\perp)^{\sigma=1}) \to T(M^\perp)^{\sigma=1} \to K \to 0. \]

By tensoring with \( \mathbb{Q}/\mathbb{Z} \), we get the second exact sequence. \( \square \)

### 3.3. Applications to Fourfolds

On a variety \( Y \), the Deligne complex, \( Z(k)_D \), is the complex

\[ Z(k) \to O_Y \to \Omega^1_Y \to \cdots \to \Omega^{k-1}_Y \]

where \( Z(k) = (2\pi i)^k \mathbb{Z} \) is in degree 0. The Deligne cohomology groups are defined to be the hypercohomology of the Deligne complex. To be more precise,

\[ H^j_D(Y, Z(k)) = H^j(Y, Z(k)_D). \]

For small values of \( j \) and \( k \), these groups are easy to understand. For example, the exponential sequence shows that \( Z^1_D \) is quasi-isomorphic to \( O^*_Y \). Hence we know that

\[ H^2_D(Y, Z^1_D) \cong \text{Pic}(Y) \] and \( H^3_D(Y, Z^1_D) \cong \text{Br}(Y) \).

**Definition 3.11.** Let \( Y \) be a smooth projective variety. We define

\[ \text{Br}_2(Y) = H^5_D(Y, Z(2)) \]

**Remark 3.12.** If \( H^i(Y, O_Y) = 0 \), for all \( i > 0 \), then the group \( \text{Br}_2(Y) \) is quite similar to the Brauer group of a surface. If we further assume that \( H^5(Y, Z) = 0 \), then the long exact sequence associated to

\[ 0 \to H^1_d(Y) \to H^1(Y, Z) \to H^1_\text{tr}(Y, Z) \to 0 \]

Here we use the fact that \( H^1(Y, Z) \cong H^1(Y, \{Z(2) \to O_Y\}) \), which is a consequence of the vanishing of \( H^i(Y, O_Y) \), \( i > 0 \). The above exact sequence implies that

\[ \text{Br}_2(Y) = H^5_d(Y, Z(2)) \]

Let \( X \) be a smooth projective fourfold with \( H^i(X, O_X) = 0 \), \( \forall i > 0 \). Assume that \( S \) parameterizes a family of curves on \( X \) given by

\[ \begin{array}{ccc} C & \xrightarrow{q} & X \\ p \downarrow & & \downarrow \phi \\ S & & \end{array} \]

Let \( \Phi = p_\ast q^\ast \) be the Abel-Jacobi homomorphism defined on the cohomology groups, the Chow groups and the Brauer groups. To be more precise we have

\[ \Phi : H^1(X, Z) \to H^2(S, Z), \]

\[ \Phi : CH^1(X) \to CH^0(S), \]

\[ \Phi : \text{Br}_2(X) \to \text{Br}(S). \]

The cylinder homomorphism \( \Psi = q_\ast p^\ast \) is defined similarly. Let \( N \) be the saturation of the subgroup \( \text{Sym}^2(H^2(X, Z)) \subset H^4(X, Z) \) of cohomology classes coming from \( H^2(X, Z) \). We define the primitive cohomology of \( X \) to be

\[ H^4(X, Z)_{\text{prim}} = N^\perp \subset H^4(X, Z). \]
Let $A_1(X) = \text{CH}_1(X)_{\text{hom}}$ be the Chow group of homologically trivial 1-cycles on $X$. In our case, $A_1(X)$ consists all elements of $\alpha \in \text{CH}_1(X)$ such that the intersection number $\alpha \cdot D$ is zero for all divisors $D$ on $X$. Let $M \subset \text{Pic}(S)$ be a $G$-submodule such that

$$\Phi(N) \subset M \text{ and } \Psi(M) \subset N.$$  

Then the homomorphism $\Phi$ induces

$$\Phi : H^4(X, \mathbb{Z})_{\text{prim}} \to M^\perp,$$

$$\Phi : A_1(X) \to A_0(S),$$

where $A_0(S)$ is the Chow group of 0-cycles of degree zero. Similarly, we have

$$\Psi : M^\perp \to H^4(X, \mathbb{Z})_{\text{prim}},$$

$$\Psi : A_0(S) \to A_1(X).$$

**Theorem 3.13.** Let $S$ be a smooth projective surface with $H_1(S, \mathbb{Z}) = 0$ and $\sigma : S \to S$ an involution with isolated fixed points. Let $X$ be a 4-dimensional smooth projective variety with $H^i(X, \mathcal{O}_X) = 0$, for all $i > 0$ and $H^5(X, \mathbb{Z}) = 0$. Assume that $S$ parameterizes a family, $p : \mathcal{C} \to S$, of curves on $X$ with $q : \mathcal{C} \to X$ being the natural map. Set $\Phi = p_*q^*$ and $\Psi = q_*p^*$ as above, which are defined on the cohomology groups, the Chow groups and the Brauer group. Let $N \subset H^4(X, \mathbb{Z})$ be the saturation of the image of $\text{Sym}^2(H^2(X, \mathbb{Z}))$. Let $M \subset \text{Pic}(S)$ be a saturated and nondegenerate $G$-submodule such that $\Phi(N) \subset M$ and $\Psi(M) \subset N$.

(i) Assume that the following three conditions hold,

(a). $\Psi \circ \Phi = -2 : H^4(X, \mathbb{Z})_{\text{prim}} \to H^4(X, \mathbb{Z})_{\text{prim}}$;

(b). $\Phi \circ \Psi = \sigma - 1 : M^\perp \to M^\perp$;

(c). $H^1(G, M^\perp) = 0$.

Then $\Phi$ induces an isomorphism

$$\Phi : H^4(X, \mathbb{Z})_{\text{prim}} \to \text{Pr}(M^\perp, \sigma)(-1),$$

which respects the bilinear forms of the two sides, namely

$$\langle \Phi(x), \Phi(y) \rangle = -(x \cdot y)_X, \quad \forall x, y \in H^4(X, \mathbb{Z})_{\text{prim}}.$$  

The condition (c) above is fulfilled if $\sigma$ has fixed points and $H^1(G, M) = 0$.

(ii) Assume that the following two conditions hold,

(a). $\Psi \circ \Phi = -2 : A_1(X) \to A_1(X)$;

(b). $\Phi \circ \Psi = \sigma - 1 : A_0(S) \to A_0(S)$.

Then there is a canonical isomorphism

$$A_1(X) \cong \text{Pr}(A_0(S), \sigma) \oplus (2\text{-torsion}),$$

such that $\Phi : A_1(X) \to \text{Pr}(A_0(S), \sigma)$ is simply the projection.

(iii) Let the assumptions be the same as in (i). If $2 \nmid \det(M) \det(N)$, then for “Brauer” groups, we have a short exact sequence

$$0 \to K \to \text{Br}_2(X) \xrightarrow{\Phi} \text{Pr}(\text{Br}(S), \sigma) \to 0,$$

where $K = \text{coker}(H^1(G, H^2(S, \mathbb{Z})) \to H^1(G, T^2(S)))$.  

Proof. We note that the assumption $H^6(X, \mathbb{Z}) = 0$ implies that $H^3(X, \mathbb{Z}) = 0$ by the Poincaré duality and hence $H^4(X, \mathbb{Z})$ is torsion-free by the universal coefficient theorem for cohomology. To prove (i), we first show that $\Phi : H^4(X, \mathbb{Z})_{\text{prim}} \to M^\perp$ is injective with $\text{Im}(\Phi) \subset \text{Pr}(M^\perp, \sigma)$. For any element $\alpha \in H^4(X, \mathbb{Z})_{\text{prim}}$, set $a = \Phi(\alpha) \in M^\perp$. If $a = 0$, then

$$-2\alpha = \Psi(\Phi(\alpha)) = \Psi(a) = 0.$$  

This implies $\alpha = 0$ since $H^4(X, \mathbb{Z})$ is torsion free. This shows that $\Phi$ is injective. We also have

$$(\sigma - 1)a = \Phi \circ \Psi(\Phi(\alpha)) = \Phi(-2\alpha) = -2a,$$

which implies $\sigma(a) = -a$, namely $a \in (M^\perp)^{\sigma=-1}$. Since $H^1(G, M^\perp) = 0$, we know that $a \in \text{Pr}(M^\perp, \sigma)$. We next show that $\text{Im}(\Phi) = \text{Pr}(M^\perp, \sigma)$. To do this we only need to show that $\Psi : M^\perp \to H^4(X, \mathbb{Z})_{\text{prim}}$ is surjective. Indeed for any $\alpha \in H^4(X, \mathbb{Z})_{\text{prim}}$, we know that $a = \Phi(\alpha)$ can be written as $a = (\sigma - 1)a_0$, for some $a_0 \in M^\perp$. Let $a' = \Psi(a_0)$, then

$$\Psi(\Phi(a')) = \Phi \circ \Psi(a_0) = (\sigma - 1)a_0 = a = \Phi(\alpha).$$

Since $\Phi$ is injective, this forces $\alpha = \alpha' = \Psi(a_0)$. Hence $\Psi$ is surjective. We still need to check the compatibility of the bilinear forms. Let $x, y \in H^4(X, \mathbb{Z})_{\text{prim}}$, then

$$\langle \Phi(x), \Phi(y) \rangle = \frac{1}{2}(\Phi(x) \cdot \Phi(y))_S = \frac{1}{2}(x \cdot \Psi\Phi(y))_X = -(x \cdot y)_X.$$

Let $\beta \in A_1(X)$ be arbitrary and set $\Phi(\beta) = \Phi(\beta) \in A_0(S)$. Then, in the same way as for cohomology, we can show that $\sigma(\beta) = -\beta$. If $\beta = 0$, then $\Psi(\beta) = 0$. Namely $\beta$ is 2-torsion. Conversely, if $\beta$ is 2-torsion, then $\Phi(\beta) = 0$ since $A_0(S)$ is torsion free by a theorem of Roitman \cite{15}. This implies that

$$\ker(\Phi) = A_1(X)[2] = \{ \beta \in A_1(X) : 2\beta = 0 \}.$$ 

Since $A_0(S)$ is uniquely divisible, the element $b_0 = -\frac{1}{2}b \in A_0(S)$ is well-defined and $\sigma(b_0) = -b_0$. Let $\beta' = \Psi(b_0) \in A_1(X)$. Then

$$\Phi(\beta') = (\sigma - 1)b_0 = -2b_0 = b = \Phi(\beta).$$

This implies that $\beta - \beta' \in \ker(\Phi)$ and hence $\Psi$ is surjective modulo 2-torsion. Thus we conclude that $\Phi : A_1(X) \to \text{Pr}(A_0(S), \sigma)$ is surjective with kernel equal to $A_0(S)[2]$. The image of $\Psi$ gives the natural splitting. This proves (ii).

We start to prove (iii). Consider the following short exact sequence

$$0 \rightarrow N \oplus N^\perp \rightarrow H^1(X, \mathbb{Z}) \rightarrow Q_N \rightarrow 0.$$

Note that all classes in $N$ are hodge classes and hence $T(N \oplus N^\perp) = T(N^\perp)$. Then we can deduce the following exact sequence from the one above,

$$0 \rightarrow T(N^\perp) \rightarrow T^4(X) \rightarrow T(Q_N) \rightarrow 0,$$

where $T(Q_N)$ is the quotient of $Q_N$ be the image of Hodge classes. By tensoring with $\mathbb{Q}/\mathbb{Z}$ and note that $\text{Tor}_1(T(Q_N), \mathbb{Q}/\mathbb{Z}) \cong T(Q_N)$ and $\text{Br}_2(X) = T^4(X) \otimes \mathbb{Q}/\mathbb{Z}$, we get a short exact sequence relating $\text{Br}_2(X)$ to the group $T(H^4(X, \mathbb{Z})_{\text{prim}})$, namely

$$0 \rightarrow T(Q_N) \rightarrow T(H^4(X, \mathbb{Z})_{\text{prim}}) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow \text{Br}_2(X) \rightarrow 0.$$ 

Here $Q_N = N^* / N = H^4(X, \mathbb{Z}) / (N \oplus H^4(X, \mathbb{Z})_{\text{prim}})$. Let $\text{Hdg}(Q_N)$ be the image of $\text{Hdg}^4(X)$ in $Q_N$, then $T(Q_N) = Q_N / \text{Hdg}(Q_N)$. Note that by assumption, $Q_N$
has no 2-torsion and hence neither does \(T(Q_N)\). The above short exact sequence together with the one obtained in Corollary 3.10 we get the following diagram

\[
\begin{array}{cccccccccc}
0 & \to & K \oplus T(Q_M)^{\sigma=1} & \to & T((M^1)^{\sigma=1}) \otimes \mathbb{Q}/\mathbb{Z} & \to & \mathrm{Pr}(\mathcal{B}(S), \sigma) & \to & 0 \\
0 & \phi & \to & T(Q_N) & \iso & T(H^4(X,\mathbb{Z}_{\text{prim}})) \otimes \mathbb{Q}/\mathbb{Z} & \phi & \to & Br_2(X) & \to & 0 \\
\end{array}
\]

where the middle column is isomorphism by (i). The assumptions imply that \(\Psi \circ \Phi = -2 : Br_2(X) \to Br_2(X)\). This implies \(\ker(\Phi)\) is of 2-torsion. The snake lemma shows that \(\ker(\Phi)\) is isomorphic to \(\mathrm{coker}(\phi)\). Since \(T(Q_M)\) and \(T(Q_N)\) are all free of 2-torsion, we conclude that \(\phi : T(Q_N) \to T(Q_M)^{\sigma=1}\) is isomorphism and \(\ker(\Phi) \cong K\).

4. Cubic Fourfolds

In this section, we fix \(X \subseteq \mathbb{P}_\mathbb{C}^5\) to be a smooth cubic fourfold. Let \(F = F(X)\) be the variety of lines on \(X\). One naturally embeds \(F\) into the Grassmannian \(G(2,6)\). It is known that \(F\) is smooth of dimension 4, see [6, 1]. Let \(P = P(X)\) be the total space of lines on \(X\). We have the following diagram

\[
P \xrightarrow{q} X \xleftarrow{p} F
\]

A. Beauville and R. Donagi showed in [5] that \(F\) is actually irreducible holomorphic symplectic, i.e. \(F\) is simply connected and \(H^0(F, \Omega_F^2)\) is generated by an everywhere nondegenerate holomorphic 2-form \(\omega\). Furthermore, they proved that the Abel-Jacobi homomorphism

\[\Phi = p_*q^* : H^4(X,\mathbb{Z})(1) \to H^2(F,\mathbb{Z})\]

is an isomorphism of Hodge structures. The primitive cohomology of \(X\), denoted by \(H^4(X,\mathbb{Z})_{\text{prim}}\), is defined to be all elements orthogonal to \(h^2\) under the intersection pairing, where \(h\) is the class of a hyperplane section on \(X\). Let \(b(-,-)\) be the Beauville-Bogomolov bilinear form on \(H^2(F,\mathbb{Z})\); see [4]. The primitive cohomology of \(F\), denoted \(H^2(F,\mathbb{Z})_{\text{prim}}\), is defined to be all elements orthogonal to \(\Phi(h^2)\) under the bilinear form \(b\). Then it is further proved in [5] that \(\Phi\) induces an isomorphism

\[\Phi : H^4(X,\mathbb{Z})_{\text{prim}} \to H^2(F,\mathbb{Z})_{\text{prim}}\]

and the intersection form on \(X\) and the Beauville-Bogomolov form on \(F\) are related by

\[b(\Phi(x), \Phi(y)) = -(x \cdot y)_X\]

for \(x, y \in H^4(X,\mathbb{Z})_{\text{prim}}\).

**Definition 4.1.** Let \(M\) be a free abelian group of finite rank with an integral symmetric bilinear form \(b_1 : M \times M \to \mathbb{Z}\). Such an \(M\) will be called a lattice. We define the kernel of the bilinear form \(b_1\) to be

\[\ker(b_1) = \{x \in M : b_1(x,y) = 0, \ \forall y \in M\}\]

\(M\) will be called non-degenerate if \(\ker(b_1) = 0\). Let \(x \in M \setminus \ker(b_1)\). We define the scale, \(s_{b_1}(x)\), of \(x\) to be the largest integer that divides \(b_1(m,x)\) for all \(m \in M\). The
The modification (++) of $(M, b_1)$ by $x$ is a new lattice whose underlying abelian group is still $M$ while the new bilinear form, $b_2 = m_{s_1}^+(b_1)$, is given by

$$b_2(\alpha, \beta) = b_1(\alpha, \beta) + \frac{1}{(s_{b_1}(x))^2} b_1(x, \alpha)b_1(x, \beta).$$

Similarly, the (−)-modification of $(M, b_1)$ by $x$ is $(M, b'_2 = m_{s_1}^-(b_1))$ where the new bilinear form is given by

$$b'_2(\alpha, \beta) = b_1(\alpha, \beta) - \frac{1}{(s_{b_1}(x))^2} b_1(x, \alpha)b_1(x, \beta).$$

If $x \in \ker(b_1)$, by convention, we define the scale of $x$ in $(M, b_1)$ to be zero.

**Lemma 4.2.** Let $(M, b_1)$ be a lattice and $x \in M \setminus \ker(b_1)$. Let $(M, b_2)$ the (++)-modification of $(M, b_1)$ by $x$ and $(M, b'_2)$ be the (−)-modification. Then the following are true

(i) The modification by $x$ is the same as the modification by $ax$ for all nonzero integer $a$.

(ii) Let $s$ be the scale of $x$ in $(M, b_1)$ and $b_1(x, x) = sc_0$ with $c_0 \in \mathbb{Z}$. Then the scale of $x$ in $(M, b_2)$ is $s + c_0$ and the scale of $x$ in $(M, b'_2)$ is $s - c_0$.

(iii) If $s + c_0 \neq 0$, then the (−)-modification of $(M, b_2)$ by $x$ is $(M, b_1)$; if $s - c_0 \neq 0$ then the (++)-modification of $(M, b'_2)$ by $x$ is $(M, b_1)$.

(iv) Assume that $s \pm c_0 \neq 0$, then the orthogonal complement $x^\perp$ of $x$ is independent of the bilinear forms $b_1$, $b_2$ or $b'_2$.

(v) Assume that $M$ carries a Hodge structure of weight $2k$ and $x \in M^{k,k}$. If $b_1$ respects the Hodge structure in the sense that

$$b_1(M^{i,j}, M^{i',j'}) = 0, \text{ if } i \neq j',$$

then both $b_2$ and $b'_2$ respect the Hodge structure of $M$.

**Proof.** For any $\alpha \in M$, we have

$$b_2(x, \alpha) = b_1(x, \alpha) + \frac{1}{s^2} b_1(x, x)b_1(x, \alpha) = (s + c_0) \frac{b_1(x, \alpha)}{s}.$$

When $\alpha$ runs through $M$, $\frac{b_1(x, \alpha)}{s}$ runs through every integer. Hence the scale of $x$ in $(M, b_2)$ is $s + c_0$. This proves the first part of (ii). The second part is proved in a same way. The remaining statements follow quite easily from the definitions. □

From now on, we will identify $H^4(X, \mathbb{Z})$ with $H^2(F, \mathbb{Z})$ as free abelian groups with Hodge structures, which will be denoted $\Lambda$. Let $\lambda_0 \in \Lambda$ be the element that corresponds to $h^2 \in H^2(X, \mathbb{Z})$, where $h$ is the class of a hyperplane section on $X$. On $F$, the element $\lambda_0$ is nothing but the polarization coming from the Plücker embedding of the Grassmannian $G(2,6)$. Let $b_0 : \Lambda \times \Lambda \to \mathbb{Z}$ be the intersection form on $H^4(X, \mathbb{Z})$ and $b : \Lambda \times \Lambda \to \mathbb{Z}$ be the Beauville-Bogomolov form on $H^2(F, \mathbb{Z})$.

**Proposition 4.3.** (i) The bilinear form $b$ can be written as $b = -m_{\lambda_0}^-(b_0)$, i.e.

$$b(x, y) = b_0(x, \lambda_0)b_0(y, \lambda_0) - b_0(x, y), \quad \forall x, y \in \Lambda.$$

(ii) The bilinear form $b_0$ can be written as $b_0 = -m_{\lambda_0}^+(b)$, i.e.

$$b_0(x, y) = \frac{1}{4} b(x, \lambda_0)b(y, \lambda_0) - b(x, y), \quad \forall x, y \in \Lambda.$$
Remark 4.4. Let $\Lambda$ be the normal bundle of $l \subset S$ polarization on $N$. We will use $\lambda$ computed in [2, Cor. 1.7]. The quotient of $Y$ is a scale of $\lambda$. We know that $b(x,y) = -b_0(x,y)$ for $x,y \in H^4(X,\mathbb{Z})_{prim}$. By Proposition 6 of [5], we know that $\lambda_0$ is orthogonal to $H^4(X,\mathbb{Z})_{prim}$ under $b$; then $b$ is uniquely determined by its restriction to $\lambda_0$ and $H^4(X,\mathbb{Z})_{prim}$. One checks that the bilinear form given by the formula in (i) coincides with $b$ after restricting to $\lambda_0$ and $H^4(X,\mathbb{Z})_{prim}$. This forces (i) to be true. To prove (ii), we only need to check that the scale of $\lambda_0$ in $(\Lambda, b)$ is equal to 2. This can be seen easily by considering the special case when $X$ is a Pfaffian cubic fourfold. In this case $F = S[2]$ where $S$ is a K3-surface of degree 14. Under the canonical orthogonal decomposition $H^2(F,\mathbb{Z}) \cong H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta$, we have $\lambda_0 = 2l - 5\delta$ with $l$ being the class of the degree 14 polarization on $S$. Here $\delta \in \operatorname{Pic}(F)$ is the half of the boundary divisor with $b(\delta, \delta) = -2$; see [4]. From this, we computes that $b(\lambda_0, \Lambda) = 2\mathbb{Z}$ and hence the scale of $\lambda_0$ in $(\Lambda, b)$ is 2. \hfill $\square$

**Remark 4.4**. Let $\Lambda_0 = (\lambda_0)^+ \subset \Lambda$, which is independent of the bilinear form. If $x \in \Lambda_0$, then $b(x,y) = -b_0(x,y)$, for all $y \in \Lambda$.

Let $l \subset X$ be a general line on $X$. Then we have the following splitting type of the normal bundle of $l$ in $X$, see [R Proposition 6.19],

$$\mathcal{N}_{l/X} \cong \mathcal{O}^2 \oplus \mathcal{O}(1).$$

We will use $\mathcal{N}_{l/X}^+$ to denote the positive sub-bundle $\mathcal{O}(1) \subset \mathcal{N}_{l/X}$. Let $S_l \subset F$ be the space of all lines meeting $l$. It is known that $S_l$ is a smooth surface, see [21]. On $S_l$, there is a natural involution $\sigma : S_l \to S_l$. If $[l'] \in S_l$ is a point different from $[l]$, then $\sigma([l'])$ is the residue line of $l \cup l'$. The line $l$ determines a unique linear $P_l = \mathbb{P}^2 \subset \mathbb{P}^5$ which is the span of $l$ and the positive subbundle $\mathcal{N}_{l/X}^+$. The intersection of $P_l$ and $X$ is given by

$$P_l : X = 2l + l_0.$$

Then $\sigma([l]) = [l_0]$. The involution $\sigma$ has 16 isolated fixed points. This number is computed in [2, Cor. 1.7]. The quotient of $Y_l = S_l/\sigma$ is a quintic surface in $\mathbb{P}^3$ with 16 ordinary double points. To apply the results from the previous sections, we need the following lemma whose proof will be given later.

**Lemma 4.5.** The surface $S_l$ satisfies $H_1(S_l,\mathbb{Z}) = 0$.

Let $p_l : \mathcal{G}_l \to S_l$ be the total space of lines meeting $l$ and $q_l : \mathcal{G}_l \to X$ be the natural morphism. Let

$$\Phi_l = (p_l)_*(q_l)^* : H^4(X,\mathbb{Z}) \to H^2(S_l,\mathbb{Z})$$

be the associated Abel-Jacobi homomorphism and

$$\Psi_l = (q_l)_*(p_l)^* : H^2(S_l,\mathbb{Z}) \to H^4(X,\mathbb{Z})$$

be the associated cylinder homomorphism. Similarly we can define the Abel-Jacobi and cylinder homomorphisms for the Chow groups. By abuse of notations, we will still use $\Phi_l$ and $\Psi_l$ to denote them, i.e.

$$\Phi_l = (p_l)_*(q_l)^* : \operatorname{CH}_i(X) \to \operatorname{CH}_{i-1}(S_l),$$

and

$$\Psi_l = (q_l)_*(p_l)^* : \operatorname{CH}_i(S_l) \to \operatorname{CH}_{i+1}(X).$$
Note that the action $\sigma$ induces an involution on $H^2(S_t, \mathbb{Z})$ and also on $CH^1(S_t)$ via pull-back. We will still use $\sigma$ to denote this action. On $S_t$, we have two natural divisor classes $g$ and $g'$. The class $g$ is the restriction of $\lambda_0$. The class $g'$ is defined as follows. For any point $x \in l$, all lines passing through $x$ form a curve $C_x \subset S_t$. We define $g'$ to be the class of the curve $C_x$.

**Definition 4.6.** The primitive cohomology of $S_t$ is defined to be

$$H^2(S_t, \mathbb{Z})_{\text{prim}} = \{ \alpha \in H^2(S_t, \mathbb{Z}) : \alpha \cup g = 0, \alpha \cup g' = 0 \}$$

Note that on the group $B_2(X)$, we have

$$\Phi_t : (p_1)_* (q_1)^* : B_2(X) \rightarrow H^3_2(S_t, \mathbb{Z}(1))_{\text{tor}} = Br(S_t).$$

Now we state the main theorem of this section.

**Theorem 4.7.** Let notations be as above, then the following are true.

(i) The action $\sigma$ preserves primitive classes on $S_t$, i.e.

$$\sigma : H^2(S_t, \mathbb{Z})_{\text{prim}} \rightarrow H^2(S_t, \mathbb{Z})_{\text{prim}}$$

(ii) The Abel-Jacobi homomorphism induces an isomorphism

$$\Phi_t : H^4(X, \mathbb{Z})_{\text{prim}} \rightarrow \Pr(H^2(S_t, \mathbb{Z})_{\text{prim}}, \sigma)(-1)$$

of Hodge structures that respects the bilinear forms. This means that for any $\alpha, \beta \in H^4(X, \mathbb{Z})_{\text{prim}}$, we have

$$\langle \Phi_t(\alpha), \Phi_t(\beta) \rangle = -\langle \alpha \cdot \beta \rangle_X.$$ 

(iii) For the Chow groups, let $A_1(X)$ be the Chow group of 1-cycles on $X$ of degree 0. Let $A_0(S_t)$ be the Chow group of 0-cycles on $S_t$ of degree 0. Then the Abel-Jacobi homomorphism also induces an isomorphism

$$\Phi_t : A_1(X) \rightarrow \Pr(A_0(S_t), \sigma)$$

of abelian groups. In particular, $A_1(X)$ is uniquely divisible.

(iv) For the "Brauer groups", we have that $\Phi_t$ fits into an exact sequence

$$0 \rightarrow K \rightarrow B_2(X) \xrightarrow{\Phi_t} \Pr(Br(S_t), \sigma) \rightarrow 0,$$

where $K \cong H^1(G, T^2(S_t))$.

(v) The group $K \subset Br_2(X)$ is independent of the choice of the general line $l \subset X$.

**Proof.** The main idea of the proof follows that of [18, Theorem 3.5]. We give a sketch here, for more details we refer to [18]. Let $\{l_t \subset X : t \in T \}$ be a 1-dimensional family of lines on $X$ with $l_0 = l$ for some closed point $0 \in T$. Let $I \subset F \times F$ be the incidence correspondence. Let

$$I_t = I_{[S_t \times S_t]} \subset CH^2(S_t \times S_t)$$

be the restriction of the incidence correspondence. Let

$$I_0 = \lim_{\rightarrow 0} I_t \subset CH^2(S_t \times S_t).$$

By definition, $I_0$ induces the homomorphism $\Phi_t \circ \Psi_t$ on the cohomology, the Chow groups and the Brauer group. To give a description of $I_0$, we denote $v = \frac{d}{dt}|_{t=0} \in H^0(l, \mathcal{N}^1_{l/X})$. Assume that the incidence lines of the pair $(l_t, l)$, i.e. lines meeting both $l$ and $l_t$ (see [17] for a more precise definition, where we use the terminology of secant line instead), specialize to $E_1, \ldots, E_5 \in S_t$. For each $[l'] \in S_t - [l]$, we can canonically associate a linear $P^3 \subset P^5$ passing through $x = l' \cap l$ in the following
way. If \( l' \neq E_i, \forall i = 1, \ldots, 5 \), then we take the linear span of \( l', l \) and \( v_x \) to be the \( \mathbb{P}^3 \) associated to \( l' \); if \( l' = E_i \) for some \( i \), then we take the corresponding \( \mathbb{P}^3 \) to be the linear span of \( l, v_x \) and the \( \mathcal{O}(1) \)-direction \( \mathcal{N}_{E_i/X}^+ \) of \( \mathcal{N}_{E_i/X}^- \). This linear \( \mathbb{P}^3 \) will be denoted by \( \Pi_\nu \). There are 6 lines, denoted \( \{l, l', L_1, L_2, L_3, L_4\} \), on \( X \) that pass through \( x \) and lie on \( \Pi_\nu \). Then \( I_0 \) is generically defined by

\[
\rho : [l'] \mapsto \sigma([l']) + [L_1] + [L_2] + [L_3] + [L_4].
\]

This means that \( I_0 \) is represented by the closure \( \bar{\Gamma} \) of \( \nu \). Hence we get the following key identity

\[

I_0 = \bar{\Gamma} - \Delta_{S_i} + \Gamma_v.
\]

Let \( \tilde{S}_1 \) be the blow up of \( S_1 \) at the point \([l]\). Then we get a fibration structure \( \pi : \tilde{S}_1 \to l \), which extends the map \( [l'] \mapsto x = l' \cap l \). The exceptional divisor of the blow-up is mapped isomorphically onto \( l \). Hence we can identify the exceptional divisor with \( l \). For \( x \in l \), we can define a linear \( \mathbb{P}^3 \) to be the linear span of \( l, v_x \) and the \( \mathcal{O}(1) \)-direction \( \mathcal{N}_{l/X}^+ \) of \( \mathcal{N}_{l/X}^- \). There are again 6 lines, denoted \( 2l, L_{x,1}, \ldots, L_{x,4} \), on \( X \) passing through \( x \) that lie in the linear \( \mathbb{P}^3 \). The rule

\[
x \mapsto \sigma([l]) + [L_{x,1}] + [L_{x,2}] + [L_{x,3}] + [L_{x,4}]
\]

extends \( \rho \) to a 1-to-5 multiple valued map \( \hat{\rho} \) from \( \tilde{S}_1 \) to \( S_1 \). When \( x \) runs through \( l \), the points \( \{L_{x,1}, \ldots, L_{x,4}\} \) traces out a divisor \( D \subset S_1 \). Hence we have

\[
\hat{\Gamma}_\rho = \{(l_1), [l_2] : [l_1] \neq [l], [l_2] \in \rho([l_1]) \} \cup \{[l]\} \times D \subset S_1 \times S_1.
\]

**Claim 1:** The divisor class of \( D \) is equal to \( \sigma(C_x) \). The class \( g \) can be written as \( g = 2C_x + \sigma(C_x) \). [Also see Lemme 2 in §3 of [21].]

This can be seen as follows. By definition we have \( D|C_x| + 2[l] = K_{C_x} \) since we know that \( C_x \) is a (2,3)-complete intersection in \( \mathbb{P}(T_{X,x}) = \mathbb{P}^3 \). We can embed \( C_x \subset \mathbb{P}^1 \times \mathbb{P}^1 \). We see that \( \sigma(C_x) \cap C_x \) consists of points on \( C_x \) which are on the same horizontal or vertical rulings as \([l]\). This implies that \( \sigma(C_x)|C_x| + 2[l] \) is the restriction of the \((1,1)\) class on \( \mathbb{P}^1 \times \mathbb{P}^1 \). Hence by the adjunction formula we have \( \sigma(C_x)|C_x| + 2[l] = K_{C_x} \). This gives \( D - \sigma(C_x)|C_x| = 0 \) for all general \( x \in C_x \). This forces \( D \) to be the same as \( \sigma(C_x) \). The class \( \Psi_l(C_x) \subset \text{CH}_2(X) \) is represented by the surface swept by all lines passing through \( x \). It is known that is class is 2\( h^2 \), where \( h \) is the hyperplane, see [18 Lemma 3.26]. Then it follows that

\[
2g = \Phi_l \circ \Psi_l(C_x) = \Phi_l \circ (\hat{\Gamma}_\rho) \circ C_x = \sigma(C_x) + 4C_x + D.
\]

Hence we get \( g = 2C_x + \sigma(C_x) \).

**Claim 2:** The primitive lattice \( H^2(S_1, \mathbb{Z})_\text{prim} \) is simply the orthogonal complement of \( M = ZC_x \oplus ZC_x^\perp \), where \( C_x^\perp = \sigma(C_x) \). As a \( G = Z[\sigma] \)-module, we have

\[
H^1(G, H^2(S_1, \mathbb{Z})_\text{prim}) = 0.
\]

This follows from Claim 1 and Proposition [32.2]. Here we note that \( M \) is saturated in \( H^2(S, \mathbb{Z}) \) and \( M \cong \mathbb{Z}[G] \). This follows from the fact that \( \det(M) = -15 \) which
On the primitive cohomology, the homomorphism \( \gamma \) is divisible. It follows that in the isomorphism of (ii) of Theorem 3.13, there is no 2-torsion part. By Proposition 3.9, we have

\[
0 \longrightarrow T(Q_{M_1})^\sigma_2 \cong 1 \rightarrow T(Q_{M_2})^\sigma_2 \rightarrow T(P_2) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow \text{Pr}(Br(S_2), \sigma_2) \rightarrow 0,
\]

where the vertical arrow in the middle is an isomorphism. Hence \( \gamma' \) is surjective.

The composition of \( \Phi : Br_2(X) \rightarrow \text{Pr}(Br(S_1), \sigma_1) \) and \( \gamma' \) is equal to \( \Phi' \). It follows that \( \ker(\gamma') \) is of 2-torsion. Since the groups in the first column are of 3-torsion, we know that \( \ker(\gamma') \) can only have 3-torsion. This forces \( \gamma' \) to be injective and hence an isomorphism. Then we have

\[
K_1 = \ker\{ \Phi : Br_2(X) \rightarrow \text{Pr}(Br(S_1), \sigma_1) \} = \ker\{ \gamma' \circ \Phi : Br_2(X) \rightarrow \text{Pr}(Br(S_2), \sigma_2) \} = \ker\{ \Phi' : Br_2(X) \rightarrow \text{Pr}(Br(S_2), \sigma_2) \} = K_2.
\]

This finishes the proof of the Theorem. \( \square \)

**Corollary 4.8.** For the transcendental lattices, the Abel-Jacobi map

\[
\Phi : H^2(X, \mathbb{Z})_{\text{tr}} \rightarrow H^2(S_1, \mathbb{Z})_{\text{tr}}^\sigma = -1
\]
defines a morphism $X$ equality. Thus the second inclusion, which is the same as $\Phi_l$ the quotient of the last term by the first has the same size as that of the second. Let Proposition 4.9. Let $X$ be a smooth cubic fourfold and $l \subset X$ a general line. Let $Y_l$ be the quotient of $S_l$ be the involution $\sigma$. If $\text{rk Pic}(Y_l) = 1$ and $\text{rk Pic}(S_l) \leq 3$, then $K = 0$, i.e.

$$
\Phi_l : \text{Br}_2(X) \to \text{Pr}((\text{Br})(S_l), \sigma)
$$

is an isomorphism.

Proof. We have the natural identifications

$$
\text{Br}_2(X) = \text{Hom}(H^4(X, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}), \quad \text{Pr}(\text{Br}(S_l), \sigma) = \text{Hom}(H^2(S_l, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})^{\sigma - 1}.
$$

The Abel-Jacobi homomorphism $\Phi_l : \text{Br}_2(X) \to \text{Pr}(\text{Br}(S_l), \sigma)$ is induced by

$$
\Psi_l : H^2(S_l, \mathbb{Z})_{\text{tr}} \to H^4(X, \mathbb{Z})_{\text{tr}}.
$$

By applying $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$, we see that $K$ is isomorphic to the cokernel of $\Psi_l$ on the transcendental lattice. Note that $T^2(S_l)$ and $H^2(S_l, \mathbb{Z})_{\text{tr}}$ are isomorphic as $G$-modules. By definition, We have

$$
K = H^1(G, T^2(S_l))
$$

$$
\cong H^1(G, H^2(S_l, \mathbb{Z})_{\text{tr}})
$$

$$
= H^2(S_l, \mathbb{Z})_{\text{tr}}^{\sigma - 1}/(\sigma - 1)H^2(S_l, \mathbb{Z})_{\text{tr}}
$$

$$
= H^2(S_l, \mathbb{Z})_{\text{tr}}^{\sigma - 1}/\Phi_l\Psi_l(H^2(S_l, \mathbb{Z})_{\text{tr}})
$$

as abstract groups. This shows that in the sequence of inclusions

$$
\Psi_l(H^2(S_l, \mathbb{Z})_{\text{tr}}) \hookrightarrow H^4(X, \mathbb{Z})_{\text{tr}} \xrightarrow{\Phi_l} H^2(S_l, \mathbb{Z})_{\text{tr}}^{\sigma - 1},
$$

the quotient of the last term by the first has the same size as that of the second by the first. Thus the second inclusion, which is the same as $\Phi_l$, is actually an equality.

It is not clear yet whether $K \neq 0$ actually happens. The following proposition shows that one expects $K = 0$ to happen generically.

**Proposition 4.9.** Let $X$ be a smooth cubic fourfold and $l \subset X$ a general line. Let $Y_l$ be the quotient of $S_l$ be the involution $\sigma$. If $\text{rk Pic}(Y_l) = 1$ and $\text{rk Pic}(S_l) \leq 3$, then $K = 0$, i.e.

$$
\Phi_l : \text{Br}_2(X) \to \text{Pr}(\text{Br}(S_l), \sigma)
$$

is an isomorphism.

Proof. Let $X_l$ be the blow-up of $X$ along the line $l$. Then the projection from $l$ defines a morphism $X_l \to \mathbb{P}^3$ which realizes $X_l$ as a conic bundle over $\mathbb{P}^3$. The surface $S_l$ parameterizes lines in the singular fibers and $Y_l \subset \mathbb{P}^3$ is the discriminant divisor. By assumption, Pic($Y_l$) is generated by the hyperplane class $\lambda$. It is known that $\pi^*\lambda = C_x + C_y^\sigma$, where $\pi : S_l \to Y_l$ is the natural double cover; see LeMme 2 in [21] [3]. If the Picard rank of $S_l$ is 2, then $\text{Alg}^2(S_l) = M$ and $M^\perp = H^2(S_l, \mathbb{Z})_{\text{tr}}$. In this case we already know that $K \cong H^1(G, M^\perp) = 0$ by Proposition 3.2.

Assume that the Picard rank of $S_l$ is 3. Then $\text{Pic}(S_l) = \mathbb{Z}C_x \oplus \mathbb{Z}C_y^\sigma \oplus \mathbb{Z}a$ for some class $a$. Since $\sigma(a) + a = \pi_*\pi^*a$ is a multiple of $\pi^*\lambda$, we have

$$
a + \sigma(a) = n(C_x + C_y^\sigma), \quad n \in \mathbb{Z}.
$$

We replace $a$ by $a - nC_x$ and assume that $\sigma(a) = -a$. Hence $\text{Pic}(S_l) \cong \mathbb{Z}[G] \oplus \mathbb{Z}$ as a $G$-module. In particular, $H^4(G, \text{Pic}(S_l)) = \mathbb{Z}/2\mathbb{Z}$ for odd $i$. The intersection
form on the Picard group, with respect to the basis chosen above, is given by

\[
A = \begin{pmatrix}
1 & 4 & m \\
4 & 1 & -m \\
m & -m & d
\end{pmatrix}
\]

where \( m = a \cdot C_x \) and \( d = (a)^2 \). One computes that \( \det(A) = -5(3d + 2m^2) \). Let \( \{f, f^r, \alpha\} \) be the dual basis of \( \text{Pic}(S_l)^r \) and \( \{\bar{f}, f^r, \bar{\alpha}\} \) its image in \( \text{Pic}(S_l)^r / \text{Pic}(S_l) \). Let \( \iota : \text{Pic}(S_l) \to \text{Pic}(S_l)^r \) be the natural inclusion induced by intersection pairing. Then we have

\[
\begin{align*}
\iota(C_x) &= f + 4f^r + m\alpha \\
\iota(C_x^r) &= f + f^r - m\alpha \\
\iota(a) &= mf - mf^r + d\alpha
\end{align*}
\]

It follows that

\[
5(f + f^r) = \iota(c_x + c_x^r), \quad (2m^2 + 3d)\alpha = \iota(m(C_x - C_x^r) + 3a).
\]

Then \( \bar{f} + f^r \) has order 5 and \( (3d + 2m^2)\bar{\alpha} = 0 \). If \( 3 \nmid m \), then \( \bar{\alpha} \) is of order \( N = |3d + 2m^2| \) and we have

\[
\text{Pic}(S_l)^r / \text{Pic}(S_l) \cong (\mathbb{Z}/5\mathbb{Z})_+ \oplus (\mathbb{Z}/N\mathbb{Z})_+.
\]

If \( m = 3m_0 \), then \( \bar{x} \) is of order \( N' = |d + 6m_0^2| \) and

\[
\text{Pic}(S_l)^r / \text{Pic}(S_l) \cong (\mathbb{Z}/5\mathbb{Z})_+ \oplus (\mathbb{Z}/N'\mathbb{Z})_+ \oplus (\mathbb{Z}/3\mathbb{Z})_+.
\]

In either case, we know that the 2-primary part \( T_2 \) of \( \{\text{Pic}(S_l)^r / \text{Pic}(S_l)\}^r = 1 \) is either isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) (if \( N \) is even) or zero (if \( N \) is odd; but we will see that this does not happen). The exact sequence

\[
0 \longrightarrow \text{Pic}(S_l) \oplus H^2(S_l, \mathbb{Z})_{\text{tr}} \longrightarrow H^2(S_l, \mathbb{Z}) \longrightarrow \text{Pic}(S_l)^r / \text{Pic}(S_l) \longrightarrow 0.
\]

gives an exact sequence

\[
\cdots \longrightarrow T_2 \longrightarrow H^1(G, \text{Pic}(S_l)) \oplus H^1(G, H^2(S_l, \mathbb{Z})_{\text{tr}}) \longrightarrow H^1(G, H^2(S_l, \mathbb{Z})) = 0.
\]

Since \( H^1(G, \text{Pic}(S_l)) = \mathbb{Z}/2\mathbb{Z} \), we get \( T_2 = \mathbb{Z}/2\mathbb{Z} \) and \( H^1(G, H^2(S_l, \mathbb{Z})_{\text{tr}}) = 0 \). This implies that \( K = 0 \). \( \square \)

5. **Proof of Lemma 4.5**

Let \( V \) be a complex vector space with \( \dim V = 6 \) and \( G \in \text{Sym}^3 V^\ast \) such that \( G = 0 \) defines a smooth cubic fourfold \( X \subset \mathbb{P}(V) \). Let \( \{e_0, e_1, \ldots, e_5\} \) be a basis of \( V \) and \( \{X_0, X_1, \ldots, X_5\} \) the dual basis of \( V^\ast \). Let \( G(r, V) \) be the Grassmannian parameterizing \( r \)-dimensional subspaces of \( V \). On \( G(r, V) \), there is the canonical rank \( r \) subbundle \( \mathcal{V}_r \) of the trivial bundle \( V \otimes \mathcal{O}_{G(r,V)} \). More generally, we will use \( G(r_1, r_2, V) \) to denote the flag variety parameterizing \( V_1 \subset V_2 \subset V \) with \( r_1 = \dim V_1 < r_2 = \dim V_2 \). In the particular case of \( r_1 = 1 \) and \( r_2 = 2 \), we have the following diagram

\[
\begin{array}{ccc}
G(1,2,V) & \overset{f}{\longrightarrow} & G(1,V) = \mathbb{P}(V) \\
\downarrow g & & \\
G(2,V)
\end{array}
\]
On $G(1,2,V)$, we have the natural inclusions
\[ f^*\mathcal{V}_1 \subset g^*\mathcal{V}_2 \subset V. \]
By definition, $\text{Sym}^3 V^*$ is a quotient of $V^* \otimes V^* \otimes V^*$. However, we can naturally identify $\text{Sym}^3 V^*$ with the symmetric tensors in $(V^*)^\otimes 3$. More precisely, the inclusion $\text{Sym}^3 V^* \subset (V^*)^\otimes 3$ is given by
\[ \alpha_1 \alpha_2 \alpha_3 \mapsto \frac{1}{6} \left( \sum \alpha_i \otimes \alpha_j \otimes \alpha_k \right), \]
where $(i,j,k)$ runs through all permutations of $\{1,2,3\}$. Hence $G \in \text{Sym}^3 V^*$ maps to an element $\tilde{G}$ in $(V^*)^\otimes 3$ and we write
\[ \tilde{G} = \sum a_{ijk} X_i \otimes X_j \otimes X_k. \]
Let $G'$ be the image of $\tilde{G}$ under the natural map $V^* \otimes V^* \otimes V^* \rightarrow \text{Sym}^2 (V^*) \otimes V^*$.
If we identify $\text{Sym}^2 V^* \otimes V^*$ with $\text{Hom}(V,\text{Sym}^2 V^*)$, then $G'$ gives an element $G_1 \in \text{Hom}(V,\text{Sym}^2 V^*)$.
If we identify $\text{Sym}^2 V^* \otimes V^*$ with $\text{Hom}(\text{Sym}^2 V,V^*)$, then $G'$ gives an element $G_2 \in \text{Hom}(\text{Sym}^2 V,V^*)$.
We can also write
\[ G_1 = \frac{1}{3} \sum_{i=0}^5 X_i \otimes \frac{\partial G}{\partial X_i}, \quad G_2 = \frac{1}{3} \sum_{i=0}^5 \frac{\partial G}{\partial X_i} \otimes X_i. \]
The canonical element $1 \in \text{Hom}(V,\text{Sym}^2 V^*) = V \otimes V^* = \text{Hom}(\mathcal{O}_{\mathbb{P}(V)}, V \otimes \mathcal{O}_{\mathbb{P}(V)}(1))$ gives a homomorphism
\[ e : \mathcal{O}_{\mathbb{P}(V)} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)}(1), \quad 1 \mapsto \sum e_i \otimes X_i. \]
It is well known that this fits into the following Euler exact sequence
\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}(V)} \rightarrow e \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \rightarrow T_{\mathbb{P}(V)} \rightarrow 0. \]
There is a natural identification
\[ \text{Hom}(V,\text{Sym}^2 V^*) = \text{Hom}(V \otimes \mathcal{O}_{\mathbb{P}(V)}, \mathcal{O}_{\mathbb{P}(V)}(2)). \]
Hence $G_1$ induces a homomorphism
\[ G_1 : V \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{O}_{\mathbb{P}(V)}(2). \]
Similarly, $G_2$ also induces a homomorphism
\[ G_2 : \text{Sym}^2 V \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{O}_{\mathbb{P}(V)}(1). \]
It is easy to check that the composition
\[ \mathcal{O}_{\mathbb{P}(V)} \overset{e}{\longrightarrow} V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \overset{G_1 \otimes 1}{\longrightarrow} \mathcal{O}_{\mathbb{P}(V)}(3) \]
is simply given by the element $G \in \text{Sym}^3 V^*$. Hence on the cubic fourfold $X$, the above composition is 0 and hence $G_1 \otimes 1$ factors through $T_{\mathbb{P}(V)}|_X$. Let $\rho : T_{\mathbb{P}(V)}|_X \rightarrow$
\( O_X(3) \) be the resulting homomorphism. Then all these homomorphisms fit into the following diagram

\[
\begin{array}{cccc}
T_X & \xrightarrow{\rho} & T_{\mathbb{P}(V)}|_X & \xrightarrow{\rho} & O_X(3) \\
\uparrow & & \uparrow & & \\
\mathcal{F} & \xrightarrow{V \otimes O_X(1)} & G_3 \otimes 1 & \xrightarrow{G_3 \otimes 1} & O_X(3) \\
\downarrow & & \downarrow & & \\
O_X & \xrightarrow{\varepsilon} & O_X & \xrightarrow{\pi} & O_X
\end{array}
\]

where all the horizontal and vertical sequences are exact. We identify \( f : f^{-1}X \to X \) with \( \pi : \mathbb{P}(T_{\mathbb{P}(V)}|_X) \to X \), and let \( \xi \) be the relative \( O(1) \)-class. Then by the identity

\[
\text{Hom}(T_{\mathbb{P}(V)}|_X, O_X(3)) = H^0(X, O_X(3) \otimes \pi_* O(\xi)) = H^0(\mathbb{P}(T_{\mathbb{P}(V)}|_X), O(\xi) \otimes \pi^* O_X(3)),
\]

the homomorphism \( \rho \) defines an element \( \rho_1 \in H^0(\mathbb{P}(T_{\mathbb{P}(V)}|_X), O(\xi) \otimes \pi^* O_X(3)) \). The vanishing of \( \rho_1 \) defines \( \mathbb{P}(T_X) \subset \mathbb{P}(T_{\mathbb{P}(V)}|_X) \). This can already be seen from the above diagram. However, we give another proof which gives more information.

Given a point \( v = [V_1 \subset V_2 \subset V] \in G(1, 2, V) \), we take a basis \( \{v_1, v_2\} \) of \( V_2 \) such that \( V_1 = \mathbb{C}v_1 \). Let \( t \in \mathbb{A}^1 \) be an affine coordinate, then

\[
G(v_1 + tv_2) = G(v_1) + t \sum_{i=0}^{5} \frac{\partial G}{\partial X_i}(v_1)X_i(v_2) + t^2 \sum_{i=0}^{5} X_i(v_1) \frac{\partial G}{\partial X_i}(v_2) + t^3 G(v_2).
\]

If \( v \in f^{-1}X \), then \( G(v_1) = 0 \). Hence the line \( l_{V_2} \), defined by \( V_2 \subset V \), is tangent to \( X \) at the point \( \pi(v) \in X \) if and only if

\[
\sum_{i=0}^{5} \frac{\partial G}{\partial X_i}(v_1)X_i(v_2) = 0.
\]

Note that the truth of the above equality is independent of the choice of the basis \( \{v_1, v_2\} \). Tracing the definition of \( \rho_1 \), we see that equation \( \text{(15)} \) is the same as \( \rho_1 = 0 \). This can be carried out explicitly as follows. The composition

\[
\pi^*(\mathcal{V}_1|_X \otimes \mathcal{V}_1|_X) \otimes (g^* \mathcal{V}_2)|_{f^{-1}X} \xrightarrow{\pi^*} V \otimes V \otimes V \xrightarrow{G} \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}(V)}|_X)}
\]

factors through \( \pi^*(\mathcal{V}_1|_X \otimes \mathcal{V}_1|_X) \otimes (g^* \mathcal{V}_2/f^* \mathcal{V}_1)|_{f^{-1}X} \). The resulting homomorphism is exactly

\[
\sum_{i=0}^{5} \frac{\partial G}{\partial X_i} \otimes X_i : \pi^*(\mathcal{V}_1|_X \otimes \mathcal{V}_1|_X) \otimes (g^* \mathcal{V}_2/f^* \mathcal{V}_1)|_{f^{-1}X} \to \mathcal{O}.
\]

Note that \( \mathcal{V}_1 = \mathcal{O}_{\mathbb{P}(V)}(-1) \), \( g^* \mathcal{V}_2/f^* \mathcal{V}_1 = \mathcal{O}_f(-1) \) where \( \mathcal{O}_f(1) \) is the relative \( O(1) \) bundle of

\[
f : G(1, 2, V) = \mathbb{P}(V/V_3) \to \mathbb{P}(V)
\]

Since \( T_{\mathbb{P}(V)} = (V/V_1) \otimes \mathbb{V}_1^* \), we get

\[
\mathcal{O}_f(1) = \mathcal{O}(\xi) \otimes f^* \mathcal{O}_{\mathbb{P}(V)}(1).
\]
Hence we have
\[
\sum_{i=0}^{5} \frac{\partial G}{\partial X_i} \otimes X_i \in \text{Hom}(\pi^*(\mathcal{V}_1|_X \otimes \mathcal{V}_1|_X) \otimes (g^*\mathcal{V}_2/f^*\mathcal{V}_1)|_{f^{-1}X}, \mathcal{O})
\]
\[
= \text{Hom}(\pi^*\mathcal{O}_X(-2) \otimes (\mathcal{O}(-\xi) \otimes \pi^*\mathcal{O}(-1)), \mathcal{O})
\]
\[
= H^0(\mathbb{P}(T_{\mathbb{P}|_X} \otimes \mathcal{O}(\xi) \otimes \pi^*\mathcal{O}_X(3)),
\]
and this element is exactly $\rho_1$. Assume that $\rho_1 = 0$, then equation (15) tells us that $l_{\mathcal{V}_2}$ intersects $X$ with multiplicity at least 3 at the point $[v_1] \in X$ if and only if
\[
\sum_{i=0}^{5} X_i(v_1) \frac{\partial G}{\partial X_i}(v_2) = 0.
\]
Proceed in a similar way, we see that the above condition is the same as a vanishing of some element
\[
\rho_2 \in H^0(\mathbb{P}(T_X), \mathcal{O}(2\xi) \otimes \pi^*\mathcal{O}_X(3)).
\]
If we further assume $\rho_2 = 0$, then $l_{\mathcal{V}_2}$ is contained in $X$ if and only if some element
\[
\rho_3 \in H^0(\mathbb{P}(T_X), \mathcal{O}(3\xi) \otimes \pi^*\mathcal{O}_X(3))
\]
vanes. These facts can summarized in the following

**Proposition 5.1.** Let $X \subset \mathbb{P}(V)$ be a smooth cubic fourfold defined by $G = 0$ as above. Let $F$ be the variety of lines on $X$ with $p : P \to F$ being the total family of lines and $q : P \to X$ the natural morphism. Then there is a natural closed immersion $P \subset \mathbb{P}(T_X)$ and $P$ is a $(2\xi + 3h, 3\xi + 3h)$-complete intersection, where $\xi$ is the relative $\mathcal{O}(1)$ class of $\pi : \mathbb{P}(T_X) \to X$ and $h = \pi^*\mathcal{h}$ is the pull back of the hyperplane class.

Let $l \subset X$ be a general line and $S_l$ the surface parameterizing all lines meeting $l$. The line $l$ itself defines a closed point $[l] \in S_l$. It is known that $S_l$ is smooth. Let $S_l'$ be the blow-up of $S_l$ at the point $[l]$. Then there is a morphism $\pi_l : S_l' \to l$, $[l'] \mapsto l' \cap l$. Then $\tilde{S_l}$ is naturally identified with $q^{-1}(l)$.

**Corollary 5.2.** There is a natural closed immersion $\tilde{S_l} \subset \mathbb{P}(T_X|_l)$ such that $\tilde{S_l}$ is a $(2\xi + 3f, 3\xi + 3f)$-complete intersection and $\pi_l = \pi|_{\tilde{S_l}}$, where $f = \pi^*[pt]$.

To show that $H_1(S_l, \mathbb{Z}) = 0$, it suffices to show that $H_1(S^c, \mathbb{Z}) = 0$, where $S^c = S_l - [l]$. The above corollary allows us to prove Lemma 4.5 using some generalized version of the Lefschetz Hyperplane Theorem for quasiprojective varieties, see §2.2 of [7].

**Proof of Lemma 4.5.** Let $Z = \mathbb{P}(T_X|_l)$. Since $l \subset X$ is general, we have $T_X|_l = \mathcal{O}_l(2) \oplus \mathcal{O}_l(1) \oplus \mathcal{O}_l^2$. This canonically gives
\[
Z_1 = \mathbb{P}(\mathcal{O}_l(2)) \subset Z_2 = \mathbb{P}(\mathcal{O}_l(2) \oplus \mathcal{O}_l(1)) \subset Z.
\]

Easy computation shows
\[
H^0(Z, \mathcal{O}(2\xi + 3f)) = H^0(l, \pi_*(\mathcal{O}(2\xi + 3f)))
\]
\[
= H^0(l, \text{Sym}^2(\Omega_X|_l) \otimes \mathcal{O}(3))
\]
\[
= H^0(l, \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1)^3 \oplus \mathcal{O}(2)^2 \oplus \mathcal{O}(3)^3)
\]
\[
= \mathbb{C}^{25}.
\]
This means $|2\xi + 3f| \cong \mathbb{P}^{24}$.

**Claim 1:** The base locus of $|2\xi + 3f|$ is $Z_1$ and this complete linear system induces an immersion $\varphi : Z - Z_1 = U \to \mathbb{P}^{24}$.

First note that $Z_1 \cong \mathbb{P}^1$ is the section of $\pi : Z \to l$ that corresponds to $\Omega^1_{|l}| \to \mathcal{O}_l(-2)$. This means that $\xi|_{Z_1} = \mathcal{O}(-2)$. Hence $\mathcal{O}(2\xi + 3f)|_{Z_1} \cong \mathcal{O}_l(1)$. Hence $Z_1$ is contained in the base locus of $|2\xi + 3f|$. Let $z_1, z_2 \in Z - Z_1$ be two arbitrary points. The natural quotient homomorphism

$$\Omega^1_{\mathcal{X}}|_l \to \mathcal{O}_l^2$$

defines a morphism $\text{pr}_2 : Z - Z_2 \to \mathbb{P}(\mathcal{O}_l^2) = l \times \mathbb{P}^1$. Let $p_2 : l \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to the second factor. Assume that $\pi(z_1) \neq \pi(z_2)$ and either $z_1, z_2 \notin Z_2$, $p_2 \circ \text{pr}_2(z_1) \neq p_2 \circ \text{pr}_2(z_2)$ or one of the $z_i$'s is in $Z_2$, then we can find a section $C = \mathbb{P}^1 \subset Z$ that corresponds to $\Omega^1_{\mathcal{X}}|_l \to \mathcal{O}(1)$, such that $C$ passes through both $z_1$ and $z_2$. For such a section, we have $\mathcal{O}(2\xi + 3f)|_C \cong \mathcal{O}(5)$. One checks that the image of the restriction map

$$H^0(Z, \mathcal{O}(2\xi + 3f)) \to H^0(C, \mathcal{O}_C(5))$$

defines a closed immersion of $C$. If $z_1, z_2 \notin Z_2$, $\pi(z_1) \neq \pi(z_2)$ and $p_2 \circ \text{pr}_2(z_1) = p_2 \circ \text{pr}_2(z_2)$, then we can find a section $C$ that corresponds to $\Omega^1_{\mathcal{X}}|_l \to \mathcal{O}$ such that $C$ passes through both $z_1$ and $z_2$. In this case $\mathcal{O}(2\xi + 3f)|_C \cong \mathcal{O}(3)$ and the restriction map

$$H^0(Z, \mathcal{O}(2\xi + 3f)) \to H^0(C, \mathcal{O}_C(3))$$

is surjective. If $z_1, z_2 \in Z_2$, then we can take a section $C$ such that $\xi|_C \cong \mathcal{O}(-1)$. In this case, we again have that

$$H^0(Z, \mathcal{O}(2\xi + 3f)) \to H^0(C, \mathcal{O}_C(1))$$

is surjective. If $z_1, z_2$ are in the same fiber, we simply take $C$ to be the line the fiber connecting them. In this case, if $C$ does not meet $Z_1$, then the restriction map

$$H^0(Z, \mathcal{O}(2\xi + 3f)) \to H^0(C, \mathcal{O}_C(2))$$

is surjective. If $C$ meets $Z_1$ then the image of the above map consists of all sections vanishing at $Z_1 \cap C$. In summary, there is always a rational curve $C$ passing through $z_1, z_2$ such that the linear system $|2\xi + 3f|$, restricted to $C$, defines an immersion of $C - Z_1$. This show that the linear system separates points of $Z - Z_1$. By a similar argument, one shows that for any point $z \in Z - Z_1$ and a tangent vector $v \in T_{Z,z}$, there is a curves $C$ such that $z \in C$ and $C$ is tangent to $v$ and furthermore, the linear system defines an immersion of $C - Z_1$. This shows that the linear system separates tangent vectors on $Z - Z_1$.

**Claim 2:** A general element $Y \in |2\xi + 3f|$ is smooth.

By Bertini's theorem, $Y$ can only have singularities along $Z_1 \subset Y$. But we know that $S_1$ is a smooth $(2\xi + 3f, 3\xi + 3f)$-complete intersection and $Z_1 \subset S_1$. This forces $Y$ to be smooth along $Z_1$ for the $Y$ appearing in the complete intersection. Hence a general $Y$ is smooth along $Z_1$.

**Claim 3:** For a general element $Y \in |2\xi + 3f|$, we have $H_1(Y, Z) = 0$. 


Let $\tilde{Z} = Bl_{Z_1}(Z)$ be the blow up of $Z$ along $Z_1$. Then we have a diagram

$$
\begin{array}{ccc}
E_1 & \longrightarrow & \tilde{Z} \\
\downarrow \sigma_1 & & \downarrow \sigma \\
Z_1 & \longrightarrow & Z
\end{array}
$$

The normal bundle of $Z_1$ in $Z$ can be described as follows

$$\mathcal{N}_{Z_1/Z} = T_{Z_1}|_{Z_1} = \mathcal{O}_{Z_1}(\xi) \otimes (\pi^*T_X|/\mathcal{O}(-\xi))|_{Z_1} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2)^2.$$ 

This implies that $E_1 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)^2)$. Let $\xi_1$ be the relative $\mathcal{O}(1)$ class of $E_1 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)^2) \to Z_1 = \mathbb{P}^1$ and $f_1$ the class of a fiber. Then we have

$$\sigma^*(2\xi + 3f) - E_1|_{E_1} = \xi_1 - f_1.$$

There is a canonical section $\tilde{Z}_1 = \mathbb{P}(\mathcal{O}(-1)) \subset E_1 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)^2)$ of $\sigma_1$. Since $\xi_1|_{\tilde{Z}_1} = \mathcal{O}(1)$, we get $|\xi_1 - f_1|_{\tilde{Z}_1} = 0$. Using the above argument of restricting $\sigma^*(2\xi + 3f) - E_1$ to rational curves, we see that the complete linear system $|\sigma^*(2\xi + 3f) - E_1|$ defines a morphism

$$\varphi : \tilde{Z} \to \mathbb{P}^24,$$

such that $\varphi$ is an immersion on $\tilde{Z} - \tilde{Z}_1$ and contracts $\tilde{Z}_1$ to a singular point. As a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$, the variety $Z$ has trivial $H_1$, i.e. $H_1(Z, Z) = 0$. Since the center of the blow up $\sigma$ is a rational curve, we have $H_1(\tilde{Z}, Z) = 0$. Hence we also have

$$H_1(U_1, Z) = H_1(\tilde{Z}, Z) = 0,$$

where $U_1 = \tilde{Z} - \tilde{Z}_1$. Let $Y$ be the blow up of $Y$ along $Z_1$, then $Y \subset U_1$ is a hyperplane section and

$$H_1(Y, Z) = H_1(U_1, Z) = 0.$$

By the generalized Lefschetz hyperplane theorem (§2.2 of [1]), we get

$$H_1(Y, Z) = H_1(U_1, Z) = 0.$$

This proves Claim 3.

The combination of Claim 2 and Claim 3 implies $H_1(Y, Z) = 0$, where $Y = Y - Z_1$. We note that $\mathcal{O}_{Z_1}(3\xi + 3f) = \mathcal{O}(-3)$ and $\mathcal{O}_{Z_1}(3\xi + 3f) = \mathcal{O}_{Z_1}(3Z_1)$. The base locus of $|3\xi + 3f|$ is $Z_1$ and it defines a morphism $\psi : U = Z - Z_1 \to \mathbb{P}^37$ such that $Z_2 - Z_1$ is contracted to a point. Furthermore, $\psi|_{Z - Z_2}$ is an immersion. Let $Y' \in |3\xi + 3f|$ be general, then $\Sigma^0 = Y \cap Y'$ is a smooth surface that is homeomorphic to $S^0$. Furthermore, $\Sigma^0$ is a hyperplane section on $Y^0$. Again, by the Lefschetz hyperplane theorem for quasiprojective varieties, we have $H_1(\Sigma^0, Z) = 0$. This implies $H_1(S^0, Z) = 0$, which finishes the proof. \hfill $\square$

6. Conic bundles

In this section we study rationally connected fourfolds which admit a conic bundle structure over $\mathbb{P}^3$. To be more precise, let $X$ a smooth projective variety of dimension 4. Let $f : X \to B = \mathbb{P}^3$ be a flat dominant morphism. Assume that $X_b \cong \mathbb{P}^1$ for a general point $b \in B$. Such $X$ will be called a conic bundle over $B$. Let $\Delta \subset B$ be the degeneration divisor. It consists of points $b \in B$ such that $X_b$ is either a broken conic or a double line. Let $S$ be the surface parameterizing lines in the degenerate fibers, then there is a natural degree 2 morphism $\pi : S \to \Delta$. 
This induces an involution \( \sigma : S \to S \). For simplicity, we make the following assumptions.

**Assumption 6.1.** (1) The surface \( S \) is smooth and irreducible with \( H_1(S, \mathbb{Z}) = 0 \); (2) The involution \( \sigma \) has at most finitely many isolated fixed points; (3) The degeneration divisor \( \Delta \) is of odd degree.

**Remark 6.2.** Actually we know that \( \sigma \) has at least one fixed point. Otherwise, \( \pi : S \to \Delta \) is an étale double cover. But the Lefschetz Hyperplane Theorem for the fundamental group implies that \( \Delta \) is simply connected and does not have étale double covers.

By construction, we have the total family of lines in degenerate fibers given by the following diagram

\[
\begin{array}{c}
\mathcal{C} \\
p \\
\downarrow \\
S
\end{array} \quad \begin{array}{c}
\xrightarrow{q} X \\
\xrightarrow{\Phi = p \circ q^*}
\end{array}
\]

Let \( \Phi = p \circ q^* \) be the Abel-Jacobi homomorphism and \( \Psi = q \circ p^* \) be the cylinder homomorphism as in Section 3.3.

To understand the geometry of \( X \) and \( S \) better, we note that in our case, we can always find a rank 3 vector bundle \( \mathcal{E} \) on \( B \) such that \( i : X \hookrightarrow \mathbb{P}(\mathcal{E}) \), see [10]. Let \( f_1 : \mathbb{P}(\mathcal{E}) \to B \) be the natural projection with \( f = f_1 \circ i \). Let \( \xi \in \text{Pic}(\mathbb{P}(\mathcal{E})) \) be the Chern class of the relative \( O(1) \)-bundle. Then the divisor class of \( X \) on \( \mathbb{P}(\mathcal{E}) \) is equal to \( 2\xi + f_1^*c_1(\mathcal{L}) \) for some line bundle \( \mathcal{L} \) on \( B \). Hence \( X \) determines, up to a scalar, a global section \( s \in H^0(\mathbb{P}(\mathcal{E}), f_1^*\mathcal{L} \otimes O_{\mathbb{P}(\mathcal{E})}(2)) \) and \( X \) is the vanishing locus of \( s \). The Grassmannian \( G(2, \mathcal{E}) \) can be viewed as the parameter space of lines on \( \mathbb{P}(\mathcal{E}) \) and \( S \) naturally embeds into \( G(2, \mathcal{E}) \). Put these together, we get the following diagram

\[
\begin{array}{c}
\mathcal{C} \\
p \\
\downarrow \\
S
\end{array} \quad \begin{array}{c}
\xrightarrow{j} G(1, 2, \mathcal{E}) \\
\xrightarrow{h_1} \mathbb{P}(\mathcal{E}) \\
\xrightarrow{f_1} B
\end{array}
\]

On the Grassmannian \( G(2, \mathcal{E}) \), there is the natural rank 2 sub-bundle \( \mathcal{V}_2 \) of \( f_2^*\mathcal{E} \). Let \( \xi_1 = h_1^*\xi \), then we have

\[
H^0(\mathbb{P}(\mathcal{E}), f_1^*\mathcal{L} \otimes O(2\xi)) \subset H^0(G(1, 2, \mathcal{E}), h^*f_1^*\mathcal{L} \otimes O(2\xi_1)) \subset H^0(G(2, \mathcal{E}), g_*(g^*f_2^*\mathcal{L} \otimes O(2\xi_1))) \subset H^0(G(2, \mathcal{E}), f_2^*\mathcal{L} \otimes \text{Sym}^2(\mathcal{V}_2^*))
\]

Hence the element \( s \) determines a section \( s' \in H^0(G(2, \mathcal{E}), f_2^*\mathcal{L} \otimes \text{Sym}^2(\mathcal{V}_2^*)) \). Then by definition, \( S \) is simply the vanishing locus of \( s' \). This shows that

\[
[S] = c_3(f_2^*\mathcal{L} \otimes \text{Sym}^2(\mathcal{V}_2^*))
\]

holds in \( H^2(G(2, \mathcal{E}), \mathbb{Z}) \). Let \( M \subset H^2(S, \mathbb{Z}) \) be the saturation of the image of the restriction homomorphism \( H^2(G(2, \mathcal{E}), \mathbb{Z}) \to H^2(S, \mathbb{Z}) \).

**Lemma 6.3.** The subgroup \( M \) is actually a \( G \)-module and \( H^1(G, M^\perp) = 0 \).
Proof: First we note that $\text{H}^2(G(2, \mathcal{E}), \mathbb{Z})$ is generated by $a = c_1(\mathcal{V}_2)$ and $f_2^* h$, where $h$ is the class of a hyperplane on $B$. It is easy to see that $\sigma|_S = h|_S$. Let $\mathcal{V}_2' = j^* \mathcal{V}_2$. Let $\Delta^0$ be the smooth locus of $\Delta$ and $S^0 = \pi^{-1}(\Delta^0)$. By sending a point $t \in S^0$ to the singular point of $X_b$ where $b = \pi(t)$, we have a section $\tau : S^0 \to \mathcal{E}$. This section descends to a section $\tau_0 : \Delta^0 \to \mathbb{P}(\mathcal{E})$. This means that we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\tau} & \mathbb{P}(\mathcal{E}) \\
\downarrow \tau_0 & & \downarrow \pi \\
S^0 & \xrightarrow{\pi} & \Delta^0 
\end{array}
\]

The closure $\hat{\Delta}$ of $\tau(S^0)$ in $\mathcal{E}$ is the blow-up of $S$ at the fixed points of $\sigma$; the closure $\hat{\Delta}$ of $\tau_0(\Delta^0)$ in $\mathbb{P}(\mathcal{E})$ is the minimal resolution of $\Delta$. On the surface $S^0$, we have the following short exact sequence

\[
0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{V}_2' \oplus \sigma^* \mathcal{V}_2' \longrightarrow \mathcal{E}|_S \longrightarrow 0
\]

The projection $\mathcal{M} \to \mathcal{V}_2'$ to the first factor defines the section $\tau$. This means that $\tau^* \mathcal{O}_{\mathcal{E}}(-1) = \mathcal{M}$, where $\mathcal{O}_{\mathcal{E}}(-1)$ is the tautological line bundle on $\mathcal{E}$. Note that $c_1(\mathcal{O}_{\mathcal{E}}(-1)) = -\xi|_\mathcal{E}$ and $\tau^*(\xi|_\mathcal{E}) = \pi^* \tau_0^* \xi$. Hence we get

\[
c_1(\mathcal{V}_2') + \sigma^* c_1(\mathcal{V}_2') = c_1(\mathcal{E}|_S) - \tau^*(\xi|_\mathcal{E}) = c_1(\mathcal{E}|_S) - \pi^* \tau_0^* \xi.
\]

If we can show that $\tau_0^* \xi$ is a class that comes from $B = \mathbb{P}^3$, then we know that $\sigma^* c_1(\mathcal{V}_2')$ is again an element in $M$, which shows that $M$ is a $G$-module. Hence it remains to study the class $\tau_0^* \xi$. Note that the element

\[
s \in \text{H}^0(\mathbb{P}(\mathcal{E}), f_1^* \mathcal{L} \otimes \mathcal{O}(2\xi))
\]

\[
= \text{H}^0(B, \text{Sym}^2(\mathcal{E}^*) \otimes \mathcal{L})
\]

\[
\subset \text{H}^0(B, \mathcal{E}^* \otimes \mathcal{E}^* \otimes \mathcal{L})
\]

\[
= \text{Hom}(\mathcal{E}, \mathcal{E}^* \otimes \mathcal{L})
\]

determines a homomorphism

\[
s_0 : \mathcal{E} \to \mathcal{E}^* \otimes \mathcal{L}.
\]

Now consider the following composition

\[
\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \longrightarrow f_1^* \mathcal{E} \longrightarrow f_1^* s_0 \longrightarrow f_1^* (\mathcal{E}^* \otimes \mathcal{L}),
\]

which defines an element $\theta \in \text{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}(1) \otimes f_1^* (\mathcal{E}^* \otimes \mathcal{L}))$. The surface $\tau_0(\Delta^0)$ is defined by the vanishing of $\theta$. On $\mathbb{P}(\mathcal{E})$, by twisting the Euler short exact sequence, we have the following short exact sequence

\[
0 \longrightarrow \mathcal{O}(-1) \longrightarrow f_1^* \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,
\]

where $\mathcal{F} = \mathcal{T}_{\mathbb{P}(\mathcal{E})/B} \otimes \mathcal{O}(-1)$ is locally free of rank 2. Hence on the surface $\tau_0(\Delta^0)$, we have another short exact sequence

\[
0 \longrightarrow \mathcal{F} \longrightarrow f_1^* (\mathcal{E}^* \otimes \mathcal{L}) \longrightarrow \mathcal{L}_1 \longrightarrow 0,
\]

where the first injective homomorphism is induced by $f_1^* s_0$ and $\mathcal{L}_1$ is an invertible sheaf on $\Delta^0$. By dualizing the sequence (19), we get

\[
0 \longrightarrow \mathcal{L}_1^{-1} \otimes f_1^* \mathcal{L} \longrightarrow f_1^* \mathcal{E} \longrightarrow \mathcal{F}^* \otimes f_1^* \mathcal{L} \longrightarrow 0, \quad \text{on } \tau_0(\Delta^0).
\]
Since $s$ is symmetric, we know that the above sequence is essentially the same as \([\mathcal{S}]\). This implies that

$$\mathcal{S} \cong \mathcal{S}^* \otimes f_1^* \mathcal{L}, \quad \text{on } \tau_0(\Delta_0).$$

By taking the first Chern classes of both sides, we get $c_1(\mathcal{S}) = f_1^* c_1(\mathcal{L})$. Together with sequence \([\mathcal{S}]\), we see that $\xi|_{\tau_0(\Delta_0)}$ is a class that comes from $B$. Hence $\tau_0^* \xi$ comes from $B$. If we trace the quantities above, we get $\tau_0^* \xi = (c_1(\mathcal{L}) - c_1(\mathcal{E}))|_{\Delta_0}$. From this identity, we can get

$$c_1(V_2') + \sigma c_1(V_2') = (2c_1(\mathcal{E}) - c_1(\mathcal{L}))(\mod S).$$

We next show that $H^1(G, M) = H^1(G, M^\perp) = 0$. To do this, we only need to show $2 \not| \det(M)$. Indeed, the condition that $2 \not| \det(M)$ implies that the torsion module $Q_M$ has no 2-torsion and hence $H^1(G, Q_M) = 0$ for all $i > 0$; see Lemma \([28]\). Then the short exact sequence

$$0 \longrightarrow M \oplus M^\perp \longrightarrow H^2(S, \mathbb{Z}) \longrightarrow Q_M \longrightarrow 0$$

gives $H^1(G, M \oplus M^\perp) \cong H^1(G, H^2(S, \mathbb{Z}))$ for all $i > 0$. Since $\sigma$ has fixed points, by Theorem \([28]\) we have $H^1(G, H^2(S, \mathbb{Z})) = 0$.

It remains to show that $2 \not| \det(M)$. To do this, we only need to show that $2 \not| \det(M_0)$ for some submodule $M_0 \subset M$ of full rank. This is because we have $\det(M_0) = \det(M)|M/M_0|^2$. In our case, we take $M_0$ to be the $\mathbb{Z}$-linear span of $a$ and $\sigma(a)$, where $a = c_1(V_2')$. We take $a_1 = a$ and $a_2 = a + \sigma(a)$. We have already seen that $a_2 = 2c_1(\mathcal{E})|_S - c_1(\mathcal{L})|_S$. Note that

$$\beta_1|_S \cup \beta_2|_S = 2(\Delta \cup \beta_1 \cup \beta_2)$$

is even, for all $\beta_1, \beta_2 \in H^2(B, \mathbb{Z})$. It follows that $(a_2)^2$ is even. Let $d_0 = (a_1)^2(a_2)^2 - (a_1 + a_2)^2$. If we can show that $a_1 \cup a_2$ is odd, then $2 \not| d_0$ and hence $d_0 \neq 0$. This would imply that $M_0$ is of rank 2 and $\det(M_0) = d_0$ is odd. There is a natural identification $G(2, \mathcal{E}) \cong G(1, \mathcal{E}^*)$. Let $\eta$ be the relative $\mathcal{O}(1)$ class of $G(1, \mathcal{E}^*)$, then we have the following short exact sequence

$$0 \longrightarrow \mathcal{V}_2 \longrightarrow f_2^* \mathcal{E} \longrightarrow \mathcal{O}(\eta) \longrightarrow 0.$$

From this we deduce that

$$c_1(\mathcal{V}_2) = f_2^* c_1(\mathcal{E}) - \eta, \quad c_2(\mathcal{V}_2) = f_2^* c_2(\mathcal{E}) - f_2^* c_1(\mathcal{E}) \eta + \eta^2.$$

Then the class of $S$ in $H^6(G(2, \mathcal{E}), \mathbb{Z})$ is given by

$[S] = c_3(f_2^* \mathcal{L} \otimes \text{Sym}^2 \mathcal{V}_2^\ast)$

$$= c_3(\text{Sym}^2 \mathcal{V}_2^\ast) + c_2(\text{Sym}^2 \mathcal{V}_2^\ast)f_2^* c_1(\mathcal{L}) + c_1(\text{Sym}^2 \mathcal{V}_2^\ast)f_2^* c_1(\mathcal{L})^2 + f_2^* c_1(\mathcal{L})^3$$

$$= 4c_1(\mathcal{V}_2) c_2(\mathcal{V}_2^\ast) + (2c_1(\mathcal{V}_2) + 4c(\mathcal{V}_2^\ast))f_2^* c_1(\mathcal{L})$$

$$+ 3c_1(\mathcal{V}_2^\ast)f_2^* c_1(\mathcal{L})^2 + f_2^* c_1(\mathcal{L})^3$$

$$\equiv 3(\eta - f_2^* c_1(\mathcal{E}^*))f_2^* c_1(\mathcal{L})^2 + f_2^* c_1(\mathcal{L})^3, \quad \text{mod } 2.$$
Hence we have
\[
\begin{align*}
\alpha_1 \cup \alpha_2 &= \{S\} \cup (f_2^* c_1(\mathcal{E}) - \eta) \cup f_2^*(2c_1(\mathcal{E}) - c_1(\mathcal{L})) \\
&\equiv \{S\} \cup \eta \cup f_2^* c_1(\mathcal{L}), \quad \text{mod 2} \\
&\equiv 3(\eta - f_2^* c_1(\mathcal{E}))f_2^* c_1(\mathcal{L})^3 \cup \eta, \quad \text{mod 2} \\
&\equiv \eta^2 \cup f_2^* c_1(\mathcal{L})^3, \quad \text{mod 2} \\
&\equiv c_1(\mathcal{L})^3, \quad \text{mod 2}
\end{align*}
\]

Note that the degenerate locus $\Delta \subset B$ is defined by the vanishing of
\[\wedge^3 s_0: \det(\mathcal{E}) \longrightarrow \det(\mathcal{E}^* \otimes \mathcal{L}) = \det \mathcal{E}^* \otimes \mathcal{L}^3\]
Hence we see that the class of $\Delta$ is given by $[\Delta] = -2c_1(\mathcal{E}) + 3c_1(\mathcal{L})$. The fact that $\deg(\Delta)$ is odd implies that $c_1(\mathcal{L})$ is odd. Hence we conclude that $\alpha_1 \cup \alpha_2$ is odd. □

**Corollary 6.4.** The module $M$ is isomorphic to $\mathbb{Z}[G]$ as a $G$-module.

**Proof.** The above proof shows that $H^1(G, M) = 0$, which implies that as $G$-modules we have either $M \cong \mathbb{Z}[G]$ or $M \cong (\mathbb{Z}/2)^2$. But the fact $d_0 \neq 0$ implies that $\sigma(a) = a$. Hence $M$ is not isomorphic to $(\mathbb{Z}/2)^2$. □

As in Section 4, we use $N \subset H^4(X, \mathbb{Z})$ to denote the saturation of Sym$^2 H^2(X, \mathbb{Z})$. The following lemma assures that $\Phi(N^\perp) \subset M^\perp$ and $\Psi(M^\perp) \subset N^\perp$. Note that $N^\perp = H^4(X, \mathbb{Z})_{\text{prim}}$ by definition.

**Lemma 6.5.** The modules $M$ and $N$ are related by $\Psi(M) \subset N$ and $\Phi(N) \subset M$.

**Proof.** The inclusion $\Phi(N) \subset M$ is quite straight forward. First we note that
\[N \otimes \mathbb{Q} = \text{Im}\{H^4(P(\mathcal{E}), \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q})\}.
\]
Namely the classes in $N$ come from $P(\mathcal{E})$. We view $G(2, \mathcal{E})$ as the space of lines in the fibers of $P(\mathcal{E}) \rightarrow B$. By considering the total space of such lines, we get an Abel-Jacobi homomorphism
\[\tilde{\Phi}: H^4(P(\mathcal{E}), \mathbb{Q}) \longrightarrow H^2(G(2, \mathcal{E}), \mathbb{Q})
\]
such that $\Phi(\alpha|_X) = \tilde{\Phi}(\alpha)|_S$. Hence it follows that $\Phi(N)$ comes from classes on $G(2, \mathcal{E})$, i.e. $\Phi(N) \subset M$.

For the inclusion $\Psi(M) \subset N$, the only non-trivial part is $\Psi(c_1(V_2)|_S) \in N$. On the total space $\mathcal{E}$, we have the Euler exact sequence
\[0 \longrightarrow \mathcal{O}_\mathcal{E} \longrightarrow p^*(j^*V_2) \otimes \mathcal{O}_\mathcal{E}(\xi) \longrightarrow T_{\mathcal{E}/S} \longrightarrow 0,
\]
By taking Chern classes, we get
\[p^*c_1(j^*V_2) = c_1(T_{\mathcal{E}/S}) - 2\xi|_\mathcal{E}.
\]
By the projection formula, we have
\[q_*(\xi|_\mathcal{E}) = \xi|_X \cdot q_*\mathcal{E} = \xi|_X \cdot f^*\Delta \in N.
\]
We also recall that $\Psi(c_1(V_2)|_S) = q_*p^*c_1(j^*V_2)$. This implies that $\Psi(j^*c_1(V_2)) \in N$ is equivalent to $q_*c_1(T_{\mathcal{E}/S}) \in N$.

The surface $S$ is naturally a double cover of $\Delta$ and we get $K_S = p^*K_\Delta = (d - 4)\pi^*(h|_\Delta)$, where $d = \deg(\Delta)$ and $h$ is the class of a hyperplane of $\mathbb{P}^3$. Hence we get
\[ c_1(K_{\mathcal{E}}) = c_1(T_{\mathcal{E}/S}) + p^* \pi^*(d - 4)[h_\Delta], \]

from which we conclude that \( q_* c_1(T_{\mathcal{E}/S}) \in N \) is equivalent to \( q_* K_{\mathcal{E}} \in N \). By the relation

\[ c_1(N_{\mathcal{E}/X}) - K_{\mathcal{E}} = -q^* K_X, \]

we see that it suffices to show \( q_* c_1(N_{\mathcal{E}/X}) \in N \). The following fact is standard. If we have two rational curves \( C_1 \) and \( C_2 \) on a surface \( Y \) such that they meet transversally in a point \( y \), then \( N_{C_i/Y}|C_i \cong N_{C_1/Y}(y) \), where \( C = C_1 \cup C_2 \). If we globalize this and apply to the family \( \mathcal{E}/S \), we get \( N_{\mathcal{E}/X} \cong p^* \pi^* \mathcal{O}_\Delta(d h) \otimes \mathcal{O}_\Delta(-\tau(S)) \). Thus we only need to show \( q_*[\tau(S)] \in N \). Note that \( q_*[\tau(S)] = 2[\Delta] \), where \( \Delta \) is the locus of singular points in the fibers of \( f : X \to B \). So every thing is reduced to showing \([\Delta]^* \in N \).

In the proof of the previous lemma (right above the sequence (18)), we showed that \( \Delta \) is defined by the vanishing of \( \theta \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}(1) \otimes f_1^*(\delta^* \otimes \mathcal{L})) \). We can view \( \theta \) as a homomorphism

\[ \theta : f_1^* \delta^* \otimes \mathcal{O}(-\xi) \to \mathcal{L}. \]

let \( \eta' : \mathcal{O}(-2\xi) = \mathcal{O}(-\xi) \otimes \mathcal{O}(-\xi) \to f_1^* \delta^* \otimes \mathcal{O}(-\xi) \) be the natural homomorphism induced by \( \mathcal{O}(-\xi) \to f_1^* \delta^* \). Then \( X \) is defined by the vanishing of \( \theta \circ \eta' \). Let \( \mathcal{F} \) be the quotient of \( \mathcal{O}(-\xi) \to f_1^* \delta^* \). Then on \( X \), since \( \theta \circ \eta' = 0 \), we have an induced homomorphism

\[ \theta' : \mathcal{F} \otimes \mathcal{O}_X(-\xi) \to f^* \mathcal{L}, \]

whose vanishing defines the class \([\Delta] \) on \( X \). Hence

\[ [\Delta] = c_2(\mathcal{F}^* \otimes \mathcal{O}(\xi) \otimes f_1^* \mathcal{L})|_X \]

is a class coming from \( \mathbb{P}(\mathcal{E}) \) and hence is in \( N \). \( \square \)

**Lemma 6.6.** The identity \( \Psi \circ \Phi = -2 \) holds as a homomorphism \( H^4(X, \mathbb{Z})_{prim} \to H^4(X, \mathbb{Z})_{prim} \) and also as a homomorphism \( \Lambda_1(X) \to \Lambda_1(X) \).

**Proof.** The idea of the proof is similar to that of [18, §2]. Let \( f_0 : \Gamma \to X \) be either a continuous map from a real 4-dimensional topological manifold to \( X \) or a morphism from an algebraic curve to \( X \). For each point \( t \in \Gamma \), the lines in \( \mathbb{P}(\mathcal{E}) \) passing through \( x = f_0(t) \) is parameterized by \( \mathbb{P}(T_{\mathbb{P}(\mathcal{E})/B,x}) \). This shows that here is a map/morphism \( j_0 : \Sigma = \mathbb{P}(\mathcal{F}_0) \to G(2, \mathcal{E}) \), where \( \mathcal{F}_0 = f_0^* T_{\mathbb{P}(\mathcal{E})/B} \). This gives rise to the following diagram

\[ \begin{array}{c}
\xymatrix{
D_1 \cup D_2 \ar[r]^{j_2} \ar[d] & X' \ar[r]^{h_2} \ar[d]^{j_1} & X \ar[d]^h \\
P \ar[r]^{j_0} & G(1,2,\mathcal{E}) \ar[r]^{h_1} & \mathbb{P}(\mathcal{E}) \\
\Sigma \ar[r]^{g} & G(2,\mathcal{E}) \ar[r]^{f_2} & B
} \end{array} \]

where all the squares are fiber products. Each line on \( \mathbb{P}(\mathcal{E}) \) meets \( X \) in two points; \( D_2 \) corresponds to the point on \( f_0(\Gamma) \) and \( D_1 \) corresponds to the remaining point. Let \( \varphi_i : D_i \to X, i = 1,2 \), be the natural maps. To make the exposition simpler, all the following identities will be understood to be modulo homological equivalence when \( \Gamma \) is a topological cycle and modulo rational equivalence when \( \Gamma \) is an algebraic
cycle. First we note that \( \varphi_2 \) contracts \( D_2 \) onto a variety of smaller dimension and hence \( \varphi_2_*, [D_2] = 0 \). It follows that
\[
\varphi_1_*, [D_1] = (h_2 \circ j_2)_*(|D_1| + |D_2|) = i^*(h_1 \circ j_1)_*, [P]
\]
is a class coming from \( \mathbb{P}(\mathcal{E}) \). We also easily see that \( (\varphi_1)_*, [D_1] = f^* f_* [\Gamma] \), where \( f = f_1 \circ i : X \to B \), and
\[
(\varphi_1)_*, (\xi|_{D_1}) = (\varphi_1)_*, [D_1] : \xi = (\pi^* \pi_* [D_1]) \cdot \xi.
\]
Let \( \rho_i : D_i \to \Sigma, i = 1, 2 \), be the natural maps, then \( \rho_2 \) is an isomorphism while \( \rho_1 \) is isomorphism way from points \( t \in \Sigma \) such that the corresponding line \( L_i \) is contained in \( X \). If \( L_i \) is contained in \( X \), then \( \rho_1^{-1}(t) = L_i \). Let \( E \subseteq D_1 \) be the exceptional loci of the map \( \rho_1 \). Since the restriction of \( \xi|_{D_1} + |E| \) to each \( E_i \) is trivial (see Claim 1 in the proof of [2], Theorem 2.2), we conclude that
\[
\xi|_{D_1} + E = \rho_1^* b,
\]
for some cycle class \( b \) from \( \Sigma \). It follows that \( b = (\rho_1)_*(\xi|_{D_1}) \). As classes on \( \Sigma \), we have
\[
(\rho_1)_*, (\xi|_{D_1}) + (\rho_2)_*, (\xi|_{D_2}) = g_* j_i^* h_i^* ([X] \cdot \xi)
\]
\[
= g'_* (2g^2 + c_1(\mathcal{L}) \cdot \xi)
\]
\[
= 2g'_* (\xi|_\Gamma)^2 + c_1(\mathcal{L})|_\Sigma.
\]
Note that \( (\rho_2)_*, (\xi|_{D_2}) = \nu^*(\xi|_\Gamma) \), where \( \nu : \Sigma = \mathbb{P}(\mathcal{F}_0) \to \Gamma \) is natural map. On \( P = \mathbb{P}(\mathcal{V}_2|_\Sigma) \), we have the following identity
\[
(\xi|_\Gamma)^2 + g'^* c_1(\mathcal{V}_2|_\Sigma)(\xi|_\Gamma) + g'^* c_2(\mathcal{V}_2|_\Sigma) = 0,
\]
which implies that \( g'_*(\xi|_\Gamma)^2 = -c_1(\mathcal{V}_2|_\Sigma) \). On \( \Sigma \), we have the following diagram
\[
\begin{array}{ccc}
Q_\lambda & \longrightarrow & Q_\lambda \\
\uparrow & & \uparrow \\
\mathcal{O}_\Sigma & \longrightarrow & \mathcal{O}_\Gamma(\xi) \\
\uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{O}_\Gamma(\xi) \\
\end{array}
\]
where \( \lambda \) is the class of the relative \( \mathcal{O}(1) \)-bundle of \( \Sigma = \mathbb{P}(\mathcal{F}_0) \to \Gamma \). The last row gives
\[
c_1(\mathcal{V}_2|_\Sigma) + 2\nu^*(\xi|_\Gamma) + \lambda = 0.
\]
Combine the above identities, we get
\[
b = (\rho_1)_*, (\xi|_{D_1})
\]
\[
= 2g'_* (\xi|_\Gamma)^2 + c_1(\mathcal{L})|_\Sigma - (\rho_2)_*, (\xi|_{D_2})
\]
\[
= -2c_1(\mathcal{V}_2|_\Sigma) + c_1(\mathcal{L})|_\Sigma - \nu^*(\xi|_\Gamma)
\]
\[
= 2\lambda + 3\nu^*(\xi|_\Gamma) + c_1(\mathcal{L})|_\Sigma.
\]
Thus we get other and get be the total space of lines with the natural projections of the image of $j$ Let $j$ associated double cover $\pi$ homomorphism and Theorem 6.8.

The intersection $\varphi_1 \circ \rho^*_1 b - \varphi_1\langle \xi|D_1 \rangle$

$= (\varphi_1 \circ \rho^*_1 b + \varphi_2 \circ \rho^*_2 b) - \varphi_2 \circ \rho^*_2 b - \varphi_1\langle \xi|D_1 \rangle$

$= i^* \hat{h}_1, j_1 \ast g^* b - \varphi_2 \circ \rho^*_2 b - \varphi_1\langle \xi|D_1 \rangle$

$= -\varphi_2 \circ \rho^*_2 b, \mod \text{ classes from } \mathbb{P}(\mathcal{E})$

$= -2\gamma.$

In other words, $\Psi \circ \Phi(\gamma) + 2\gamma$ is always a class from $\mathbb{P}(\mathcal{E})$. By linearity, this is true for any cycle class $\gamma$ in $H^4(X, \mathbb{Z})$ or $A_1(X)$. Note that $H^2(\mathbb{P}(\mathcal{E}), \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})$ and $H^4(\mathbb{P}(\mathcal{E}), \mathbb{Q}) \rightarrow N \otimes \mathbb{Q}$ are all isomorphisms. Let $\gamma \in H^4(X, \mathbb{Z})_{\text{prim}}$, then $\Psi \circ \Phi(\gamma) + 2\gamma$ is again in $H^4(X, \mathbb{Z})_{\text{prim}}$, this forces it to be zero. Hence $\Psi \circ \Phi(\gamma) = -2\gamma$. Similarly, we have $\Psi \circ \Phi(\gamma) = -\gamma$ for all $\gamma \in A_1(X)$. 

**Lemma 6.7.** The identity $\Phi \circ \Psi = \sigma - 1$ holds as an endomorphism of $M^\perp$ and also as an endomorphism of $A_0(S)$.

**Proof.** The total space $\mathcal{E}$ can be viewed as an element in $CH_0(S \times X)$. Let $\mathcal{E}^t \subset X \times S$ be the transpose of $\mathcal{E}$. Then $\Phi \circ \Psi$ is induced by $p_{13*}(p_{12*}^* \mathcal{E} \cdot p_{23*}^* \mathcal{E}^t) \in CH_2(S \times S)$, where $p_{ij}$ are the projections from $S \times X \times X$ to corresponding factors. The cycle $p_{12*}^* \mathcal{E}$ is represented by $C_1 = \{([l], x, [l']) \in S \times X \times S : x \in l\}$ and $p_{23*}^* \mathcal{E}^t$ is represented by $C_2 = \{([l], x, [l']) \in S \times X \times S : x \in l'\}.$

Note that $C_1 \cap C_2 = \Gamma_1 \cup \Gamma_2$, where

$\Gamma_1 = \{([l], x, \sigma([l])) : x \subset l \cup \sigma([l])\}, \quad \Gamma_2 = \{([l], x, [l]) : x \in l\}$.

The intersection $C_1 \cdot C_2 = \gamma_1 + \gamma_2$ where $\gamma_1 = \lceil \Gamma_1 \rceil$ and $\gamma_2 \in CH_2(\Gamma_2)$. Hence $p_{13*}(C_1 \cdot C_2) = \Gamma_\sigma + a\Delta_S,$

for some $a \in \mathbb{Z}$, where $\Gamma_\sigma$ is the graph of $\sigma$. Pick a general point $[l] \in S$, then we have $\Phi \circ \Psi([l]) = \sigma([l]) + a[l]$. We use the fact that $\Phi$ and $\Psi$ are transpose to each other and get

$1 + a = [S] \cdot \Phi([l]) = \Psi([S]) \cdot l = f^* [\Delta] \cdot l = l \cdot [\Delta] \cdot f_* l = [\Delta] \cdot 0 = 0.$

Thus we get $a = -1$. Hence we have $\Phi \circ \Psi = \sigma - 1$. 

By the above lemmas and Theorem 6.13, we easily get the following theorem.

**Theorem 6.8.** Let $f : X \subset \mathbb{P}(\mathcal{E}) \rightarrow B = \mathbb{P}^3$ be a standard conic bundle with the associated double cover $\pi : S \rightarrow \Delta$ that satisfies Assumption 6.7. Let $\mathcal{E} \subset X \times X$ be the total space of lines with the natural projections $p : \mathcal{E} \rightarrow S$ and $q : \mathcal{E} \rightarrow X$.

Let $j : S \hookrightarrow G(2, \mathcal{E})$ be the natural inclusion and $M \subset H^2(S, \mathbb{Z})$ be the saturation of the image of $j^* : H^2(G(2, \mathcal{E}), \mathbb{Z}) \rightarrow H^2(S, \mathbb{Z})$. Let $\Phi = p_* q^*$ be the Abel-Jacobi homomorphism and $\Psi = q_* p^*$ be the cylinder homomorphism as above. Then the following are true.

(i) The Abel-Jacobi homomorphism induces an isomorphism

$\Phi : H^4(X, \mathbb{Z})_{\text{prim}} \rightarrow \text{Pr}(M^\perp, \sigma)(-1),\Psi$
which is compatible with the bilinear forms, namely

\[ \langle \Phi(x), \Phi(y) \rangle = -(x \cdot y)_X, \quad \forall x, y \in H^4(X, \mathbb{Z})_{\text{prim}}. \]

(ii) There is a canonical isomorphism

\[ \Lambda_1(X) \cong \text{Pr}(\Lambda_0(S), \sigma) \oplus (2\text{-torsion}), \]

such that \( \Phi : \Lambda_1(X) \to \Lambda_0(S) \) is identified with the projection to the first factor.

(iii) There is a short exact sequence

\[ 0 \longrightarrow K \longrightarrow \text{Br}_2(X) \xrightarrow{\Phi} \text{Br}(S) \longrightarrow 0, \]

where \( K = H^1(G, T^2(S)) \).

Proof. To apply Theorem \ref{thm:main} in (iii), we only need to show that \( 2 \nmid \det(N) \). Since \( \det(N) = \pm \det(N^+) \) and \( N^+ = H^4(X, \mathbb{Z})_{\text{prim}} \cong \text{Pr}(M^+, \sigma) \), we only need to show that \( \det(P) \) is not divisible by 2. By Proposition \ref{prop:2-torsion} we have \( \det(P) = \pm 2q^2 \), where \( q = |Q_M|^{\sigma=-1} \) and \( q' = \det(M^\sigma=1) \). By assumption \( q \) is not divisible by 2. Hence we only need to show that 2 divides \( q' \). Since \( M \cong \mathbb{Z}[G] \), we can find \( x \in M \) such that \( M^{\sigma=-1} = \mathbb{Z}(\sigma(x) - x) \). Hence \( q' = (\sigma(x) - x)^2 = 2(x^2 - x \cdot \sigma(x)) \), which is divisible by 2.

Remark 6.9. (a) It is quite likely that the assumption \( H_1(S, \mathbb{Z}) = 0 \) holds automatically. If \( \text{Sym}^2(V_2^*) \otimes f_2^* \mathcal{L} \) were an ample vector bundle on \( G(2, \mathcal{S}) \), then \( H_1(S, \mathbb{Z}) = 0 \) by a theorem of Sommese \cite{Sommese} since \( S \) is the vanishing of a section of an ample vector bundle. Unfortunately, in our case the vector bundle \( \text{Sym}^2(V_2^*) \otimes f_2^* \mathcal{L} \) is never ample. However, we still expect that \( H_1(S, \mathbb{Z}) = 0 \) holds true.

(b) If \( X \) is very general such that \( \text{Pic}(S) = M \), then \( K = 0 \) and \( \Phi : \text{Br}_2(X) \to \text{Pr}(\text{Br}(S), \sigma) \) is an isomorphism. It would be very interesting to see if the group \( K \) is always trivial or not.

(c) The case of cubic fourfolds can be viewed as a special case of the conic bundle case as follows. Let \( X \subset \mathbb{P}^3 \) be a smooth cubic fourfold and \( l \subset X \) a general line. Let \( X_l \) be the blow-up of \( X \) along \( l \). Then the projection from the line \( l \) defines a conic bundle structure \( \pi : X_l \to \mathbb{P}^3 \). The surface parameterizes lines in broken conics is simply \( S_l \) and the degeneration divisor \( \Delta \subset \mathbb{P}^3 \) is a quintic surface with 16 nodes. Then the above theorem applies to this situation.

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