L2 DECAY OF WEAK SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS IN GENERAL DOMAINS

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ABSTRACT

Let $u$ be a weak solution of the in-stationary Navier-Stokes equations in a completely general domain in $\mathbb{R}^3$. Firstly, we prove that the time decay rates of the weak solution $u$ in the $L^2$-norm like ones of the solutions for the homogeneous Stokes system taking the same initial value in which the decay exponent is less than $\frac{3}{4}$. Secondly, we show that under some additive conditions on the initial value, then $u$ coincides with the solution of the homogeneous Stokes system when time tends to infinity. Our proofs use the theory about the uniqueness arguments and time decay rates of strong solutions for the Navier-Stokes equations in the general domain when the initial value is small enough.

Keywords: Navier-Stokes equations; Decay; Weak solutions; Stokes equations; Uniqueness of solution.

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DÁNG ĐIỆU TIỆM CẬN CỦA NGHIỆM YẾU CHO HỆ PHƯƠNG TRÌNH NAVIER-STOKES TRONG MIỀN TỔNG QUÁT VỚI CHUẨN L2

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TÓM TẮT

Giả sử $u$ là một nghiệm yếu của hệ phương trình Navier-Stokes không dừng trong một miền tổng quát trong $\mathbb{R}^3$. Trước hết, chúng tôi chứng minh rằng tốc độ hội tụ theo thời gian của nghiệm yếu $u$ với chuẩn $L^2$ giống tốc độ hội tụ theo thời gian của nghiệm trong hệ Stokes thuần nhất với cùng giá trị ban đầu và số mũ hội tụ nhỏ hơn $\frac{3}{4}$. Thứ hai, chúng tôi chỉ ra rằng với một số điều kiện của giá trị ban đầu thì $u$ trùng với nghiệm của hệ Stokes thuần nhất khi thời gian dần tới vô cùng. Phân chứng minh các kết quả trong bài báo dựa trên lý thuyết về tính duy nhất và tốc độ hội tụ theo thời gian của nghiệm mạnh cho hệ phương trình Navier-Stokes trong miền tổng quát khi giá trị ban đầu đủ nhỏ.

Từ khóa: Hệ phương trình Navier-Stokes; Dáng điều tiệm cận; Nghiệm yếu; Hệ phương trình Stokes; Tính duy nhất nghiệm.

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1 Introduction and main result

We consider the in-stationary problem of the Navier-Stokes system
\[
\begin{cases}
    u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\
    \text{div } u = 0, \\
    u|_{\partial \Omega} = 0, \\
    u(0, x) = u_0,
\end{cases}
\tag{1}
\]
in a general domain $\Omega \subseteq \mathbb{R}^3$, i.e a non-empty connected open subset of $\mathbb{R}^3$, not necessarily bounded, with boundary $\partial \Omega$ and a time interval $[0, T), 0 < T \leq \infty$ and with the initial value $u_0$, where $u = (u_1, u_2, u_3), u \cdot \nabla u = \text{div}(u)u, uu = (u_1u_1, u_2u_2, u_3u_3)$, if $\text{div } u = 0$.

In this paper we discuss the behavior as $t \to \infty$ of weak solutions of the Navier-Stokes equations in space $L^2(\Omega)$, which goes to zero with explicit rates. The $L^2$-decay problem for the Navier-Stokes system was first posed by Leray [1] in $\mathbb{R}^3$. The first (affirmative) answer was given by Kato [2] in case $D = \mathbb{R}^3, n = 3, 4$, through his study of strong solutions in general spaces $L^p$, see also [3, 4, 5]. The idea of Schonbek was then applied by [6, 7] to the case where $D$ is a half-space of $\mathbb{R}^n, n \geq 2$ or an exterior domain of $\mathbb{R}^n, n \geq 3$. W. Borchers and T. Miyakawa [8] developed the method in [3, 6, 7] for the case of an arbitrary unbounded domain. They showed that if $\|e^{-tA}u_0\|_2 = O(t^{-\alpha})$ for some $\alpha \in (0, \frac{1}{2})$, then $\|u(t)\|_2 = O(t^{-\alpha})$. Our purpose in this paper is to improve and generalize the result of [8]. Firstly, we obtain the same result as that of them but under more general condition on $\alpha$, in which the condition $\alpha \in (0, \frac{1}{2})$ is replaced by $\alpha \in (0, \frac{1}{4})$. Secondly, we obtain the stronger result than theirs by assuming some additive conditions on the initial value.

We recall some well-known function spaces, the definitions of weak and strong solutions to (1) and introduce some notations before describing the main results. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant $C$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. The expression $(\cdot, \cdot)_\Omega$ denotes the pairing of functions, vector fields, etc, on $\Omega$ and $(\cdot, \cdot)_{\Omega, T}$ means the corresponding pairing on $[0, T) \times \Omega$.

For $1 \leq q \leq \infty$ we use the well-known Lebesgue and Sobolev $L^q(\Omega), W^{k,p}(\Omega)$, with norms $\| \cdot \|_{L^q(\Omega)} = \| \cdot \|_q$ and $\| \cdot \|_{W^{k,p}(\Omega)} = \| \cdot \|_{k,p}$. Further, we use the Bochner spaces $L^s(0, T; L^p(\Omega))$, $1 \leq s, p \leq \infty$ with the norm $\| \cdot \|_{L^s(0, T; L^p(\Omega))} := \left( \int_0^T \| \|u_s\|_p \|_t \right)^{1/s}$.

To deal with solenoidal vector fields we introduce the spaces of divergence-free smooth compactly supported functions $C^\infty_0(\Omega) = \{ u \in C^\infty(\Omega), \text{div} (u) = 0 \}$, and the spaces $L^2_0(\Omega) = C^\infty_0(\Omega); W^{1,2}_0(\Omega) = C^\infty_0(\Omega); W^{1,2}_0(\Omega)$, and $W^{1,2}_0(\Omega) = C^\infty_0(\Omega)$.

Let $P : L^2(\Omega) \rightarrow L^2_0(\Omega)$ be the Helmholtz projection. Let the Stokes operator
\[ A = -\nabla \Delta : \mathcal{D}(A) \rightarrow L^2_0(\Omega) \]
with the domain of definition
\[ \mathcal{D}(A) = \{ u \in W^{1,2}_0(\Omega), \exists f \in L^2_0(\Omega) : \langle \nabla u, \nabla \varphi \rangle_\Omega = \langle f, \varphi \rangle_\Omega, \forall \varphi \in W^{1,2}_0(\Omega) \} \]
be defined as
\[ Au = -\nabla \Delta u = f, u \in \mathcal{D}(A). \]

As in [9], we define the fractional powers
\[ A^\alpha : \mathcal{D}(A^\alpha) \rightarrow L^2_\alpha(\Omega), -1 \leq \alpha \leq 1. \]

We have $\mathcal{D}(A) \subset \mathcal{D}(A^\alpha) \subset L^2_\alpha(\Omega)$ for $\alpha \in (0, 1]$. It is known that for any domain $\Omega \subseteq \mathbb{R}^3$ the operator $A$ is self-adjoint and generates a bounded analytic semigroup $e^{-tA}, t \geq 0$ on $L^2_\alpha(\Omega)$.

The following embedding properties play a basic role in the theory of the Navier-Stokes system
\[ \| A^{-\frac{\beta}{2}} P u \|_2 \leq C \| u \|_q, u \in L^2_\alpha(\Omega) \quad (2) \]
where $\frac{1}{2} \leq \beta < \frac{3}{2}, \frac{1}{q} = \frac{1}{2} + \beta$. Furthermore, we mention the Stokes semigroup estimates
\[ \| A^\alpha e^{-tA} u \|_2 \leq t^{-\alpha} \| u \|_2, \quad (3) \]
with $u \in L^2_\alpha(\Omega), 0 \leq \alpha \leq 1$. Now we recall the definitions of weak and strong solutions to (1).

**Definition 1.1.** (See [9].) Let $u_0 \in L^2_\alpha(\Omega)$.

1. A vector field
\[ u \in L^\infty(0, T; L^2_\alpha(\Omega)) \cap L^2_{\text{loc}}([0, T); W^{1,2}_0(\Omega)) \quad (4) \]
is called a weak solution in the sense of Leray-Hopf of the Navier-Stokes system (1) with the initial value \( u(0, x) = u_0 \) if the relation

\[
- \langle u, w \rangle_{Ω,T} + \langle \nabla u, \nabla w \rangle_{Ω,T} - \langle uu, \nabla w \rangle_{Ω,T} = \langle u_0, w \rangle_{Ω}
\]

is satisfied for all test functions \( w \in C_0^∞(0, T); C_0^∞(Ω) \), and additionally the energy inequality

\[
\frac{1}{2} \|u(t)\|^2_2 + \int_0^t \|\nabla u(\tau)\|^2_2 d\tau \leq \frac{1}{2} \|u_0\|^2_2 + \int_0^t \langle uu, \nabla w \rangle_{Ω,\tau} d\tau
\]

for all \( t \in (0, T) \). A weak solution is called a strong solution of the Navier-Stokes equation (1) if additionally local Serrin’s condition

\[
u \in L^q_{loc}([0, T]; L^2(Ω))
\]

is satisfied with \( \frac{2}{3} < s < \infty, 3 < q < \infty \) where \( \frac{2}{s} + \frac{3}{q} \leq 1 \).

As is well known, in the case the domain \( Ω \) is bounded, it is not difficult to prove the existence of a weak solution \( u \) as in Definition 1.1 which additionally satisfies the strong energy inequality

\[
\frac{1}{2} \|u(t)\|^2_2 + \int_0^t \|\nabla u(\tau)\|^2_2 d\tau \leq \frac{1}{2} \|u_0\|^2_2 + \int_0^t \langle uu, \nabla w \rangle_{Ω,\tau} d\tau
\]

for almost all \( t' \in [0, T) \) and all \( t \in [t', T) \), see [9], p. 340. For further results in this context for unbounded domains we refer to [10].

Now we can state our main results.

**Theorem 1.1.** Let \( Ω \subseteq \mathbb{R}^3 \) be a general domain, \( u_0 \in L^2(Ω) \) and \( u \) is a weak solution of the Navier-Stokes system (1) satisfying strong energy inequality (8). Then

(a) If \( \|e^{-tA}u_0\|_2 = O(t^{−α}) \) for some \( 0 < α < \frac{3}{4} \), then \( \|u(t)\|_2 = O(t^{−α}) \) as \( t \to ∞ \).

(b) If \( \|e^{-tA}u_0\|_2 = o(t^{−α}) \) for some \( 0 < α < \frac{3}{4} \), then \( \|u(t)\|_2 = o(t^{−α}) \) as \( t \to ∞ \).

**Theorem 1.2.** Let \( Ω \subseteq \mathbb{R}^3 \) be a general domain, \( u_0 \in L^2(Ω) \) and \( u \) is a weak solution of the Navier-Stokes system (1) satisfying strong energy inequality (8). If \( u_0 \in L^q(Ω) \cap L^2(Ω), 1 < q < 2 \), then

\[
\|u(t)\|_2 = o\left(t^{−\frac{1}{2} \left(\frac{2}{q}−\frac{1}{2}\right)}\right) \text{ as } t \to ∞.
\]

**Theorem 1.3.** Let \( Ω \subseteq \mathbb{R}^3 \) be a general domain, \( u_0 \in L^2(Ω) \) and \( u \) is a weak solution of the Navier-Stokes system (1) satisfying strong energy inequality (8). If there exist positive constants \( t_0, C_1, C_2 \) such that

\[
C_1 t^{−α_1} \leq \|e^{−tA}u_0\|_2 \leq C_2 t^{−α_2} \text{ for } t ≥ t_0,
\]

where \( α_1, α_2 \) are constants satisfying

\[
0 ≤ α_2 < \frac{1}{2} \text{ and } α_2 ≤ α_1 < α_2 + \frac{1}{4},
\]

then \( u \) coincides with the solution of the homogeneous Stokes system with the initial value \( u_0 \) when time tends to infinity in the sense that

\[
\lim_{t \to ∞} \frac{\|u(t) − e^{−tA}u_0\|_2}{\|u(t)\|_2} = 0. \quad (9)
\]

2 Proof of main theorems

Let us construct a weak solution of the following integral equation

\[
u(t) = e^{−tA}u_0 − \int_0^t A^\frac{1}{2} e^{−(t−τ)A} \frac{1}{2} P(u \cdot \nabla u) dτ. \quad (10)
\]

We know that

\[
u \in L^∞(0, T; L^2(Ω)) \cap L^2_{loc}([0, T); W^{1,2}(Ω))
\]

is a weak solution of the Navier-Stokes system (1) iff \( u \) satisfies the integral equation (10), see [9]. In order to prove the main theorems, we need the following lemmas.

**Lemma 2.1.** Let \( γ, θ \in \mathbb{R} \) and \( t > 0 \), then

(a) If \( θ < 1 \), then

\[
\int_0^t (t−τ)^{−γ} τ^θ dτ = K_1 t^{1−γ−θ}
\]

where \( K_1 = \int_0^1 (1−τ)^{−γ} τ^θ dτ < ∞ \).

(b) If \( γ < 1 \), then

\[
\int_0^t (t−τ)^{−γ} τ^θ dτ = K_2 t^{1−γ−θ}
\]

where \( K_2 = \int_0^1 (1−τ)^{−γ} τ^θ dτ < ∞ \).
The proof of this lemma is elementary and may be omitted.

**Lemma 2.2.** Let \( u \in L^2(\Omega) \) and \( \nabla u \in L^2(\Omega) \). Then
\[
\left\| e^{-tA}P(u \cdot \nabla u) \right\|_2 \lesssim t^{\frac{\beta}{2}} \left\| u \right\|_2^{\beta - 1} \left\| \nabla u \right\|_2^{\frac{\beta}{2} - \beta}
\]
where \( \beta \) is positive constant such that \( \frac{1}{2} \leq \beta < \frac{3}{2} \).

**Proof.** Applying inequalities (6), (3), Holder inequality, interpolation inequality, and Lemma 2.1, we obtain
\[
\left\| e^{-tA}P(u \cdot \nabla u) \right\|_2 = \left\| A^{\frac{3}{2}} e^{-tA} A^{-\frac{3}{2}} P(u \cdot \nabla u) \right\|_2 \leq t^{-\frac{\beta}{2}} \left\| A^{\frac{3}{2}} P(u \cdot \nabla u) \right\|_2 \leq t^{-\frac{\beta}{2}} \left\| u \cdot \nabla u \right\|_q \lesssim t^{-\frac{\beta}{2}} \left\| u \right\|_2 \left\| \nabla u \right\|_2 \lesssim t^{-\frac{\beta}{2}} \left\| u \right\|_2^{\beta - \frac{1}{2}} \left\| \nabla u \right\|_2^{\frac{3}{2} - \beta} \lesssim t^{-\frac{\beta}{2}} \left\| u \right\|_2^{\beta - \frac{1}{2}} \left\| \nabla u \right\|_2^{\frac{3}{2} - \beta}.
\]
The proof of Lemma 2.2 is complete.

**Lemma 2.3.** There exists a positive constant \( \delta \) such that if \( u_0 \in \mathbb{D}(A^\frac{3}{2}) \) and \( \left\| A^{\frac{3}{2}} u_0 \right\|_2 \lesssim \delta \), then the Navier-Stokes system (1) has a strong solution with the initial value \( u_0 \) satisfying \( \left\| \nabla u(t) \right\|_2 \lesssim t^{-\frac{\beta}{2}} \) for all \( t \geq 0 \).

**Proof.** See [11].

**Lemma 2.4.** Let \( u \) be a weak solution of the Navier-Stokes system (1) with the initial value \( u_0 \in L^2(\Omega) \). Then there exists the positive value \( t_0 \) large enough such that \( \left\| \nabla u(t) \right\|_2 \lesssim t^{-\frac{\beta}{2}} \) for all \( t \geq t_0 \).

**Proof.** Applying Holder inequality, we have
\[
\left\| A^{\frac{3}{2}} u \right\|_2^2 = \int_0^\infty \lambda^{\frac{3}{2}} d\|E_\lambda u\|_2^2 \leq (\int_0^\infty \lambda d\|E_\lambda u\|_2^2)^{\frac{1}{2}} (\int_0^\infty d\|E_\lambda u\|_2^2)^{\frac{1}{2}} \lesssim \left\| A^{\frac{3}{2}} u \right\|_2 \left\| u \right\|_2 (11)
\]
Consider the weak solution of the Navier-Stokes system (1) satisfying the energy inequality
\[
\frac{1}{2} \| u(t) \|_2^2 + \int_0^t \left\| \nabla u(\tau) \right\|_2^2 d\tau \lesssim \frac{1}{2} \| u(t_0) \|_2^2 (12)
\]
for all \( t \in [0, t_0) \) with \( N \) is a null set. Let \( \delta \) be a positive constant in Lemma 2.3. Since (11) and (12), it follows that there exists the large enough \( t_0 \in [0, \infty) \) such that \( \left\| u(t_0) \right\|_{\mathcal{D}(A^\frac{3}{2})} \lesssim \delta \).

Combining Lemma 2.3, inequality (12), and Serrin’s uniqueness criterion [9, 12], we obtain
\[
\left\| \nabla u(t) \right\|_2^2 \lesssim t^{-\frac{\beta}{2}} \text{ for all } t \geq t_0.
\]
The proof of Lemma 2.4 is complete.

**Proof of Theorem 1.1**

(a) Consider the weak solution of the Navier-Stokes system (1), then \( u \) holds the integral equation
\[
u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} P(u \cdot \nabla u) ds. \quad (13)
\]
From Lemma 2.2, we have
\[
\left\| u(t) \right\|_2 \lesssim \left\| e^{-tA}u_0 \right\|_2 + \int_0^t (t-s)^{-\frac{\beta}{2}} \left\| u(s) \right\|_2^{\beta - \frac{1}{2}} \left\| \nabla u(s) \right\|_2^{\frac{3}{2} - \beta} ds
\]
for all \( \frac{1}{2} \leq \beta < \frac{3}{2} \). We divide the above integral into two different parts as follow
\[
I = \int_0^t (t-s)^{-\frac{\beta}{2}} \left\| u(s) \right\|_2^{\beta - \frac{1}{2}} \left\| \nabla u(s) \right\|_2^{\frac{3}{2} - \beta} ds
\]
\[
+ \int_0^t (t-s)^{-\frac{\beta}{2}} \left\| u(s) \right\|_2^{\beta - \frac{1}{2}} \left\| \nabla u(s) \right\|_2^{\frac{3}{2} - \beta} ds = I_1 + I_2.
\]
We consider the following three cases:
\[
0 \leq \alpha \leq \frac{1}{4}, \frac{1}{4} \leq \alpha < \frac{1}{2}, \text{ and } \frac{1}{2} \leq \alpha < \frac{3}{4}.
\]
Case 1: $0 \leq \alpha \leq \frac{1}{4}$.

Applying the energy inequality and Hölder inequality, we obtain

$$I_1 \lesssim \|u_0\|_2 \int_{\frac{1}{2}}^t \int_0^s \|\nabla u(s)\|_2^\alpha ds$$

$$\lesssim \|u_0\|_2 \int_{\frac{1}{2}}^t \left( \int_0^s \|\nabla u(s)\|_2^2 \right)^{\frac{2-\alpha}{2}} ds$$

$$\lesssim \|u_0\|_2 \int_{\frac{1}{2}}^t \left( \int_0^s \|\nabla u(s)\|_2^2 \right)^{\frac{2-\alpha}{2}} ds = O(t^{-\frac{\alpha}{2}}).$$

From Lemma 2.4 and Lemma 2.1(b), we have

$$I_2 \lesssim \|u_0\|_2 \int_{\frac{1}{2}}^t (t-s)^{-\frac{\alpha}{2}} s^{-\frac{1}{2}} \left( s^{-\frac{\alpha}{2}} \right) ds$$

$$= O(t^{-\frac{\alpha}{2}}) \text{ for } t \geq 2t_0$$

where $t_0$ is the constant in Lemma 2.4. It follows that

$$\|u(t)\|_2 \lesssim \|e^{-tA}u_0\|_2 + I \leq O(t^{-\alpha}) + O(t^{-\frac{\alpha}{2}}) = O(t^{-\alpha}) \text{ as } t \to \infty.$$

Case 2: $\frac{1}{4} \leq \alpha < \frac{1}{2}$.

Applying the above inequality for $\alpha = -\frac{1}{4}$ and Hölder inequality, we obtain

$$I_1 \lesssim t^{-\frac{\alpha}{2}} \int_{\frac{1}{2}}^t (s^{-\frac{1}{2}})^{\alpha-\frac{1}{2}} \|\nabla u(s)\|_2^\alpha ds$$

$$\lesssim t^{-\frac{\alpha}{2}} \int_{\frac{1}{2}}^t s^{-\frac{1}{2} \alpha} \left( \int_0^s \|\nabla u(s)\|_2^2 \right)^{\frac{2-\alpha}{2}} ds$$

$$\lesssim t^{-\frac{\alpha}{2}} \left( t^{-\frac{\alpha}{2}} \right)^{\frac{2-\alpha}{2}} = O(t^{-\frac{\alpha}{2}}).$$

On the other hand, from Lemma 2.4 and Lemma 2.1(b), we have

$$I_2 \lesssim \int_{\frac{1}{2}}^t (t-s)^{-\frac{\alpha}{2}} (s^{-\frac{1}{2}})^{\alpha-\frac{1}{2}} (s^{-\frac{1}{2}})^{\frac{3}{2}-\beta} ds$$

$$\lesssim \int_{\frac{1}{2}}^t (t-s)^{-\frac{\alpha}{2}} s^{-\frac{1}{2} \alpha} s^{-\frac{1}{2} \alpha} s^{-\frac{1}{2} \alpha} ds$$

$$= O(t^{-\frac{\alpha}{2}-\frac{1}{2}}) \text{ for } t \geq 2t_0.$$

So, we have

$$\|u(t)\|_2 \lesssim \|e^{-tA}u_0\|_2 + I$$

$$\leq O(t^{-\alpha}) + O(t^{-\frac{\alpha}{2}}) \text{ for } t \geq 2t_0.$$
Proof. The proof is derived directly from the proof of Case 3 of Theorem 1.1.

Proof of Theorem 1.2

Theorem 1.2 is an immediate consequence of Theorem 1.1(b) and the following lemma.

Lemma 2.5. Let \( u_0 \in L^2_q(\Omega) \). Then
(a) \( \| e^{-tA}u_0 \|_2 \to 0 \) as \( t \to \infty \).
(b) If \( u_0 \in L^2_q(\Omega) \cap L^q(\Omega) \) for some \( 1 < q \leq 2 \), then
\[
\| e^{-tA}u_0 \|_2 = O\left(t^{-\frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)}\right) \text{ as } t \to \infty. \quad (15)
\]

Proof. (a) See Lemma 1.5.1 in [9], p. 204. (b) Applying inequality (3), we obtain
\[
\| e^{-tA}u_0 \|_2 = \| e^{-\frac{tA}{2}} e^{-\frac{tA}{2}} u_0 \|_2 \\
= \| A^{\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{q}\right) e^{-\frac{tA}{2}} A^{-\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{q}\right) u_0 \|_2 \\
\leq t^{-\frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)} \| e^{-\frac{tA}{2}} A^{-\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{q}\right) u_0 \|_2. \quad (16)
\]
On the other hand, using inequality (2), we get
\[
A^{-\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{q}\right) u_0 \in L^2_q(\Omega) \quad (17)
\]
Property 15 is deduced from Lemma 2.5(a), (16), and (17).

Proof of Theorem 1.3

Proof. Applying Corollary 2.1 for \( \gamma = \alpha_2, \theta = \frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8} \), there exists a positive constant \( M_1 \) such that
\[
\| u(t) - e^{-tA}u_0 \|_2 \leq M_1 t^{-\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8}\right)} \quad (18)
\]
It follows from the above inequality that
\[
\| u(t) \|_2 \geq \| u(t) \|_2 - \| u(t) - e^{-tA}u_0 \|_2 \\
\geq C_1 t^{-\alpha_1} - M_1 t^{-\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8}\right)} \\
\geq \left(C_1 - M_1 t^{-\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8}\right)}\right) t^{-\alpha_1} \\
\geq \frac{C_1}{2} t^{-\alpha_1} \text{ for } t \geq t_1,
\]
where
\[
t_1 = \max\left\{ t_0, \left(\frac{2M_1}{C_1}\right)^{\frac{8}{\alpha_2 - \alpha_1}}\right\}.
\]
From the above two estimates, we obtain that
\[
\frac{\| u(t) - e^{-tA}u_0 \|_2}{\| u(t) \|_2} \leq \frac{M_1 t^{-\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8}\right)}}{C_1 t^{-\alpha_1}} = \frac{2M_1}{C_1} t^{-\left(\frac{\alpha_1 - \alpha_2}{2} + \frac{1}{8}\right)} \to 0 \text{ as } t \to \infty.
\]
The proof of Theorem is complete.

References

[1] J. Leray, "Sur le mouvement d’un liquide visqueux emplissant l’espace," Acta Math., vol. 63, pp. 193-248, 1934.

[2] T. Kato, "Strong \( L^p \) solutions of the Navier-Stokes equation in \( \mathbb{R}^m \), with applications to weak solutions," Math. Z., vol. 187, pp. 471-480, 1984.

[3] R. Kajikiya, T. Miyakawa, "On \( L^2 \) decay of weak solutions of the Navier-Stokes equations in \( \mathbb{R}^n \), Math. Z., vol. 192, pp. 135-148, 1986.

[4] M. E. Schonbek, "Large time behaviour of solutions to the Navier-Stokes equations," Commun. in Partial Diff. Eq., vol. 11, pp. 733-763, 1986.

[5] M. Wiegner, "Decay results for weak solutions of the Navier-Stokes equations in \( \mathbb{R}^n \), J. London Math. Soc., vol. 35, pp. 303-313, 1987.

[6] W. Borchers, T. Miyakawa, "\( L^2 \)-Decay for the Navier-Stokes flows in halfspaces," Math. Ann., vol. 282, pp. 139-155, 1988.

[7] W. Borchers, T. Miyakawa, "Algebraic \( L^2 \) - decay for Navier-Stokes flows in exterior domains," Acta Math., vol. 165, pp. 189-227, 1990.

[8] W. Borchers, T. Miyakawa, "\( L^2 \)-Decay for Navier-Stokes flows in unbounded domains with application to Exterior Stationary Flows," Arch. Rational Mech. Anal., vol. 118, pp. 273-295, 1992.
[9]. H. Sohr, *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2001.

[10]. R. Farwig, H. Kozono, and H. Sohr, "An $L^q$-approach to Stokes and Navier–Stokes equations in general domains," *Acta Math.*, vol. 195, pp. 21-53, 2005.

[11]. H. Kozono, T. Ogawa, "Global strong solution and its decay properties for the Navier-Stokes equations in three dimensional domains with non-compact boundaries," *Mathematische Zeitschrift*, vol. 216, pp. 1-30, 1994.

[12]. J. Serrin, *The initial value problem for the Navier-Stokes equations*, Univ. Wisconsin Press, Nonlinear problems, Ed. R. E. Langer, 1963.