Induced gravity in $\mathbb{Z}_N$ orientifold models

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Abstract

We consider non-compact $\mathbb{Z}_N$ orientifold models of type IIB superstring theory with four-dimensional gravity induced on a set of coincident D3-branes. For the models with $N = 3(2l - 1)$, $l \in \mathbb{N}$ and shift vector $v = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ the contribution to the one-loop renormalization of the Planck mass from the torus is shown to be $O(N)$ whereas the contributions from annulus, Möbius strip and Klein bottle are $O(1)$. One can therefore realize the Dvali-Gabadadze-Porrati idea that four-dimensional gravity is induced by quantum effects at the one-loop level by considering large $N$. The number of matter and gauge fields on the D3-brane does not grow with $N$.

1 Introduction

Today we know of several ways to realize four-dimensional gravity (see e.g. [1] for a review). First, we get a four-dimensional Einstein-Hilbert term from a D-dimensional one if we compactify $D - 4$ dimensions. Second, we can have a cosmological constant in the bulk and on a codimension one 3-brane. This is the Randall-Sundrum mechanism [2] and [3]. Third, we can have brane induced gravity, i.e. we can have a four-dimensional Einstein-Hilbert term that is induced on a 3-brane by the fields that live on the 3-brane. This has been put forward by Dvali, Gabadadze and Porrati ([4] – [6]). Obviously we can also combine these three ideas in a given model.

Both Randall-Sundrum and Dvali-Gabadadze-Porrati originally work in a five-dimensional bulk (i.e. with codimension one). Whereas the Randall-Sundrum setup leads to a four-dimensional behaviour of gravity at long distances and a five-dimensional behaviour of gravity at short distances (that

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is similar to compactification) in the case of the Dvali-Gabadadze-Porrati setup gravity is four-dimensional at short distances and five-dimensional at long distances. Therefore to be consistent with experiments and astronomical observations the cross-over scale has either to be astronomically large or one has to compactify the extra dimension [7]. In the later case Kaluza-Klein graviton emission is suppressed in the ultraviolet and therefore also the energy loss from brane to bulk. Constraints from experiments on the size of the compact dimension are therefore less stringent than for standard compactifications.

With a codimension greater than one the effects of a finite brane thickness can no longer be neglected ([8] – [12]), but the conclusions on the cross-over scale or on the need for compactification remain the same. On the other hand one can also merge the setups of Randall-Sundrum and of Dvali-Gabadadze-Porrati [13].

In a complete theory we need four-dimensional gravity and we will consider how this can be achieved in superstring theory. We may ask the question if there are string models with brane induced gravity where the one-loop induced four-dimensional Planck mass is large in string units. For heterotic string theory such corrections vanish for \( \mathcal{N} \geq 1 \) supersymmetry ([14] – [16]). For type II vacua such corrections can be non-vanishing for \( \mathcal{N} \leq 2 \) supersymmetry [14],[17]. In particular for a background of the type \( M_4 \times CY_3 \), where \( M_4 \) is four-dimensional Minkowski space and \( CY_3 \) is a Calabi-Yau, the one-loop correction is proportional to the Euler number ([17] – [19]).

In order to have models with gauge and matter fields on the D3-branes that come close to the standard model (and supersymmetric generalizations thereof) we will consider type I/orientifold vacua in this paper. Explicitly we will consider models that are non-compact (non-standard) orientifolds of type IIB superstring theory on symmetric \( \mathbb{Z}_N \) orbifolds and we will compute the induced gravity on a set of coincident D3-branes. The reasons are the following: As in this setup we have the gauge fields, matter fields and gravity localized on the D3-branes we do not need to compactify. Besides being interesting for its own, this has the advantage that we can have arbitrary \( N \) and may consider the large \( N \) limit and that we can have an arbitrary number of D3-branes (no untwisted tadpole cancellation condition).

Our orientifold models are similar to the ones discussed in [20] and [21] but more general as we consider models with arbitrary possibly large \( N \). Whereas in [20] and [21] the contributions to the Planck mass are only stated to exist and to be determined by the string scale we explicitly determine them by a string computation and show that they are large if \( N \) is large. This is a generalization of the computation of [22] and [19] that
considered compactification on K3. We can then compare the torus versus the annulus, Moebius strip and Klein bottle contributions.

Writing the one-loop renormalization of the four-dimensional Planck mass as

\[ \Delta L_{\text{eff}}^{1\text{-loop}} = \delta M^2_s \sqrt{-gR} \]  

we will show that the torus contribution is \( O(N) \) and that annulus, Moebius strip and Klein bottle contributions are \( O(1) \). Therefore, by considering large \( N \) the one-loop contribution can be arbitrary large. The number of gauge and matter fields on the D3-branes is not growing with \( N \) and we can in principle find models that are quite close to supersymmetric generalizations of the standard model.

There is also a different string theory realization of induced gravity presented in [8] that is based on the orientifold of K3 and two extra compactified dimensions as in [22]. There the contribution to the four-dimensional Planck mass comes only from the Kaluza-Klein tower from annulus, Moebius strip and Klein bottle as the torus does not contribute.

In section 2 we consider \( \mathbb{Z}_N \) orbifolds of type IIB and review the contribution to the Planck mass from the torus. In sections 3 we consider \( \mathbb{Z}_N \) orientifolds of type IIB and compute the contribution to the Planck mass from the annulus, the Moebius strip and the Klein bottle. In section 4 we give our conclusions. Some details of the computations are left to appendices.

2 \( \mathbb{Z}_N \) orbifolds

In order to have localized twisted sectors and therefore a localized Einstein term and in order to have a parameter \( N \) that we may e.g. assume to be large we consider \( \mathbb{Z}_N \) orbifolds of type IIB superstring theory in this section. In the compact case we compactify on \( M^4 \times T^6/\mathbb{Z}_N \), where \( M^4 \) is four-dimensional Minkowski space. In the non-compact case the background is \( M^4 \times \mathbb{R}^6/\mathbb{Z}_N \). We have \( \mathcal{N} = 2 \) supersymmetry in \( d = 4 \). In section 2.1 we will first review how the contribution of the torus to the one-loop renormalization of the Planck mass is determined by the Euler number or the second helicity supertrace. In section 2.2 we then show how the second helicity supertrace follows from the helicity generation partition function. In section 2.3 we analyse the large \( N \) limit. In 2.4 we compute the torus contribution from a two graviton amplitude in order to fix the vertex operator normalization that we will need in section 3.
2.1 The second helicity supertrace

The helicity supertraces are defined by (see [16])

\[ B_{2n} = \text{Tr} \left[ (-1)^{2\lambda} \lambda^{2n} \right] \quad n \in \mathbb{N}, \tag{2} \]

where the \( \lambda \)'s are the helicity eigenvalues. The contribution of the \( \mathcal{N} = 2 \) supergravity multiplet and of \( \mathcal{N} = 2 \) vector multiplets to the second helicity supertrace \( B_2 \) is 1, whereas the contribution of \( \mathcal{N} = 2 \) hyper multiplets is \(-1\). This gives

\[ B_2 = 1 + n_V - n_H, \tag{3} \]

where \( n_V \) and \( n_H \) count the number of vector and hyper multiplets. The \( \mathbb{Z}_N \) orbifolds we consider are singular limits of Calabi-Yau 3-folds with hodge numbers

\[ h^{1,1} = n_V, \quad h^{2,1} = n_H - 1, \tag{4} \]

where we have subtracted the universal hyper multiplet. The Euler number is

\[ \chi = 2 \left( h^{1,1} - h^{2,1} \right). \tag{5} \]

This gives

\[ B_2 = \frac{1}{2} \chi. \tag{6} \]

The only one-loop surface is the torus \( T \). In [17] it was shown (see also [19] and [23]) that this gives a one-loop renormalization of the Planck mass of

\[ \Delta L_{\text{1-loop}}^{\text{eff}} = \delta_T M_s^2 \sqrt{-g_R} = \frac{1}{12\pi} \chi M_s^2 \sqrt{-g_R} = \frac{1}{6\pi} B_2 M_s^2 \sqrt{-g_R}. \tag{7} \]

From the helicity generating partition function \( Z(v, \bar{v}) \) we get the second helicity supertrace as (see [16])

\[ B_2 = - \left( \frac{1}{2\pi i} \partial_v - \frac{1}{2\pi i} \partial_{\bar{v}} \right)^2 Z(v, \bar{v}) \bigg|_{v=\bar{v}=0}. \tag{8} \]

2.2 The helicity generating partition function

Let us define the complex bosons as \( Z^i = X^{2i+2} + iX^{2i+3}, \quad i = 1, 2, 3 \). The helicity generating torus partition function for the \( \mathbb{Z}_N \) orbifold of type IIB is (see also [18])

\[ Z(v, \bar{v}) = N_0(N) \int \frac{d^2 \tau}{\tau_2} Z_X^2(\tau) \sum_{h, g = 0}^{N-1} Z_1 \left[ ^hv_1 g v_1 \right](\tau) \]

\[ \times Z_2 \left[ ^hv_2 g v_2 \right](\tau) Z_3 \left[ ^hv_3 g v_3 \right](\tau) Z_3^+(\tau) Z_3^+(\tau)^*, \tag{9} \]

\(^1\)We have \( M_s^2 = 1/\alpha' \).
where

$$Z_X(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4}$$ (10)

$$Z_i[0][0](\tau) = \frac{\Gamma_{2,2}}{|\eta(\tau)|^4}$$ (11)

$$Z_i[hv_i, gv_i](\tau) = C(N)(hv_i, gv_i) \left| \frac{\eta(\tau)}{\theta[1/2 + hv_i, 1/2 + gv_i]}(0, \tau) \right|^2$$ for \((hv_i, gv_i) \neq (0, 0)\) (12)

$$Z^+_\psi(v, \tau) = \frac{\xi(v)}{2 \eta(\tau)^4} \sum_{\alpha, \beta=0}^1 (-1)^{\alpha+\beta+\alpha\beta} \theta[\alpha/2 + hv_1, \beta/2 + gv_1](0, \tau) \times \theta[\alpha/2 + hv_2, \beta/2 + gv_2](0, \tau) \theta[\alpha/2 + hv_3, \beta/2 + gv_3](0, \tau)$$ (13)

$$\xi(v) = \frac{\sin \pi v}{\pi} \frac{\partial_\theta[1/2, 1/2](u, \tau)_{u=0}}{\theta[1/2, 1/2](v, \tau)}$$ (14)

For the untwisted sector

$$\left| C(N)(0, gv_i) \right| = 4 (\sin(\pi gv_i))^2$$ (15)

and for the twisted sectors \((h \neq 0)\) \(\left| C(N)(hv_i, gv_i) \right|\) counts the fixed points multiplicity that is always 1 in the non-compact case. The torus partition function

$$Z_T = Z(0, 0)$$ (16)

is as expected modular invariant (use \(\xi(0) = 1\)). Using the Riemann identity we get

$$Z^+_\psi(v, \tau) = \frac{\xi(v)}{\eta(\tau)^4} \theta[1/2, 1/2](v/2, \tau) \theta[1/2 - hv_1, 1/2 - gv_1](v/2, \tau) \times \theta[1/2 - hv_2, 1/2 - gv_2](v/2, \tau) \theta[1/2 - hv_3, 1/2 - gv_3](v/2, \tau).$$ (17)

Let \(\mathcal{M}\) be the set of elements \(\{(h, g)\}\) that solve

$$hv_1 = gv_1 = 0 \mod 1$$ (18)

or $$hv_2 = gv_2 = 0 \mod 1$$ (19)

or $$hv_3 = gv_3 = 0 \mod 1.$$ (20)
Obviously \((0,0) \in \mathcal{M}\). From (8) we get the second helicity supertrace for the \(Z_N\) orbifold

\[
B_2 = \frac{\log 3}{2} N_0(N) \sum_{\substack{h, g = 0 \\ (h, g) \notin \mathcal{M}}} \left| C^{(N)}(hv_1, gv_1) C^{(N)}(hv_2, gv_2) C^{(N)}(hv_3, gv_3) \right|. \tag{21}
\]

The normalization

\[
N_0(N) = \frac{1}{N \log 3} \tag{22}
\]

is fixed by matching the massless spectrum that one gets from the operator approach with the one one derives from the helicity generating partition function. In appendix B we first show this for the example of the \(Z_3\) orbifold and then give the proof for prime \(N\). The proof for the case with general \(N \in \mathbb{N}\) is only sketched as it is straightforward and lengthy.

### 2.3 Large \(N\) behaviour of \(Z_N\) orbifolds

For a non-compact \(Z_N\) orbifold the part of the spectrum coming from the untwisted sector is a sub-spectrum of the \(N = 8\) supergravity multiplet. Explicitly the \(N = 8\) supergravity multiplet

\[
\left( -2, -\frac{3}{2}, -1^{28}, -\frac{1}{2}, 0^{70}, \frac{1}{2}, 1^{28}, \frac{3}{2}, 2 \right) \tag{23}
\]

decomposes into the sum of the \(N = 2\) supergravity multiplet, 6 \(N = 2\) gravitino multiplets, 15 \(N = 2\) vector multiplets and 10 \(N = 2\) hyper multiplets. This means \(B_2^U \in [-9, 16]\).

The part of the second helicity supertrace coming from twisted sectors is \(B_2^T = n_V^T - n_H^T\). We have

\[
\begin{align*}
N \text{ even:} & \quad n_V^T = \frac{N - 2}{2} + 1 = \frac{N}{2}, \quad n_H^T = 0 \quad \text{(24)} \\
N \text{ odd:} & \quad n_V^T = \frac{N - 1}{2}, \quad n_H^T = 0. \quad \text{(25)}
\end{align*}
\]

For the total second helicity supertrace \(B_2 = B_2^U + B_2^T\) we therefore have the behavior

\[
B_2 \xrightarrow{N \to \infty} \frac{N}{2} + O(1) \tag{26}
\]

that gives using (4)

\[
\Delta \mathcal{L}_{1\text{-loop}}^{\text{eff}} \xrightarrow{N \to \infty} \frac{N + O(1)}{12\pi} M_s^2 \sqrt{-g} R. \tag{27}
\]
In the compact case the shift vector $v$ for a $\mathbb{Z}_N$ orbifold has to be such that the orbifold acts crystallographically. In the non-compact case this is obviously not necessary. We may consider families of models build on the $\mathbb{Z}_3$ or $\mathbb{Z}_7$ orbifolds/orientifolds. Let us define the $\mathbb{Z}_3$ family as $\mathbb{Z}_N$ orbifold/orientifold with $N = 3(2l - 1)$, $l \in \mathbb{N}$ and $v = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$. The $\mathbb{Z}_7$ family can similarly be defined as $\mathbb{Z}_N$ orbifold/orientifold with $N = 7(2l - 1)$, $l \in \mathbb{N}$ and $v = (\frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$ or $v = (\frac{1}{7}, \frac{4}{7}, -\frac{5}{7})$, but we will focus on the $\mathbb{Z}_3$ family.

We get

$$B_2 = 9 + (2l - 1)$$

(28)

that gives using (7)

$$\Delta \mathcal{L}_{1}\text{-loop}^{\text{eff}} = \frac{1}{6\pi} (9 + (2l - 1)) M_s^2 \sqrt{-g} R.$$  

(29)

2.4 The Planck mass from the two graviton amplitude

In this section we compute the torus contribution from a two graviton amplitude in order to fix the vertex operator normalization that we will need in section 3.

2.4.1 Matching amplitudes and effective actions

Let us define

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$  

(30)

$$\Gamma_{\nu\rho}^{\mu} = \frac{1}{2} g^{\mu\lambda} (\partial_{\rho} g_{\lambda\nu} - \partial_{\lambda} g_{\nu\rho} + \partial_{\nu} g_{\rho\lambda})$$  

(31)

$$R_{\nu\rho\sigma}^{\mu} = \partial_{\rho} \Gamma_{\nu\sigma}^{\mu} - \partial_{\sigma} \Gamma_{\nu\rho}^{\mu} + \Gamma_{\rho\lambda}^{\mu} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\rho\lambda}^{\mu} \Gamma_{\nu\sigma}^{\lambda}.$$  

(32)

With the graviton

$$h_{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \varepsilon_{\mu\nu},$$  

(33)

neglecting terms proportional to $p_1^2, p_2^2, p_1 \cdot p_2, p_1 \varepsilon_{1\mu}, p_2 \varepsilon_{2\rho}, \eta_{\mu\nu} \varepsilon_{1\mu}, \eta_{\rho\sigma} \varepsilon_{2\rho}$ (due to the fact that gravitons are massless and that the polarization tensors are physical) and keeping the momenta arbitrary (no momentum conservation) we find

$$\sqrt{-g} R\big|_{O(\kappa^2)} = -\kappa^2 8 \left( \eta_{\mu\rho} p_1 \varepsilon_{1\mu} p_2 + \eta_{\mu\sigma} p_1 \varepsilon_{1\mu} P_2 + \eta_{\nu\rho} p_1 \varepsilon_{1\nu} + \eta_{\nu\sigma} P_1 \varepsilon_{2\mu} + \eta_{\nu\sigma} P_1 \varepsilon_{2\nu} \right) \varepsilon_{1\mu} \varepsilon_{2\rho}.$$  

(34)

It is enough to consider only one tensor structure as it follows from covariance that the only term in the effective action that contributes in second order in momentum is $\sqrt{-g} R$. Let us write the (off-shell) two graviton amplitude as

$$A^{(2)} = -\frac{1}{4 C_m} \delta \left( \eta_{\mu\rho} P_1 \varepsilon_{1\mu} + \eta_{\mu\sigma} P_1 \varepsilon_{1\nu} + \eta_{\nu\rho} P_1 \varepsilon_{1\nu} + \eta_{\nu\sigma} P_1 \varepsilon_{2\mu} + \eta_{\nu\sigma} P_1 \varepsilon_{2\nu} \right) \varepsilon_{1\mu} \varepsilon_{2\rho}.$$  

(35)
where the momenta are measured in string units (i.e. they are dimensionless) and we have introduced a matching coefficient $C_m$ that we will determine and that accommodates the fact that we use vertex operators that are not normalized properly. Then the contribution to the effective action is precisely

$$\Delta L_{\text{eff}} = M_s^2 \delta \sqrt{-g} R,$$

(36)

where a factor of $\frac{1}{2}$ from Bose symmetry is taken into account.

### 2.4.2 The two graviton amplitude

The Einstein term is CP even and gets contributions from the even-even and the odd-odd spin structure two graviton amplitudes. Using the notation of appendix C the even-even spin structure two graviton amplitude is

$$A_{(e-e)}^{(2)} = \sum_{(\alpha,\beta)=0,1} \sum_{(\bar{\alpha},\bar{\beta})=0,1} \int_{\Gamma} d^2\tau \frac{d^2}{\tau_2} (-)^{\alpha+\beta+\alpha \beta} (-)^{\bar{\alpha}+\bar{\beta}+\bar{\alpha} \bar{\beta}} Z(\tau, \bar{\tau}, (\alpha, \beta), (\bar{\alpha}, \bar{\beta}))
\times \int d^2 z_1 \int d^2 z_2 \langle V^{(0,0)}(z_1, \bar{z}_1) V^{(0,0)}(z_2, \bar{z}_2) \rangle_{(\alpha, \beta), (\bar{\alpha}, \bar{\beta})},$$

(37)

with the graviton vertex operator in the $(0,0)$-ghost picture

$$V^{(0,0)}(z, \bar{z}) = -\frac{2g_s}{\alpha'} \epsilon_{\mu\nu} : \left( i\bar{\partial}X^\mu - \frac{\alpha'}{2} \psi^\mu p \cdot \psi \right) \left( i\bar{\partial}X^\nu + \frac{\alpha'}{2} \bar{\psi}^\nu p \cdot \bar{\psi} \right) e^{ip \cdot X} :.$$

(38)

The piece in second order in momentum vanishes due to (169) and (170) and there are no pinching contributions from $O(p^4)$. The other possible contribution comes from the odd-odd spin structure two graviton amplitude

$$A_{(o-o)}^{(2)} = \int_{\Gamma} d^2\tau \frac{d^2}{\tau_2} Z(\tau, \bar{\tau}, (1,1), (1,1)) \int d^2 z_1 \int d^2 z_2
\times \langle V^{(0,0)}(z_1, \bar{z}_1) V^{(-1,-1)}(z_2, \bar{z}_2) X^{pc}(z_{pc}, \bar{z}_{pc}) \rangle,$$

(39)

where the $(-1,-1)$-ghost picture vertex operator is

$$V^{(-1,-1)} = g_s \epsilon_{\mu\nu} : \psi^\mu \bar{\psi}^\nu e^{ip \cdot X} :$$

(40)

and the picture changing operator is

$$X^{pc} = \partial X^{\alpha} \psi_\alpha \bar{\partial} X^{\beta} \bar{\psi}_\beta.$$  

(41)

Actually to get the write tensor structure we have to consider a different distribution of the pictures. This is due to our choice of off-shell procedure
and can be avoided by considering amplitudes with more gravitons \(^2\). We start with

\[
A^{(2)}_{(o-o)} = \int \frac{d^2\tau}{\tau_2} Z(\tau, \bar{\tau}, (1, 1), (1, 1)) \int d^2 z_1 \int d^2 z_2 \\
\times \langle V(0,-1)(z_1, \bar{z}_1)V(-1,0)(z_2, \bar{z}_2)X^{\mu\nu}(z_{pc}, \bar{z}_{pc}) \rangle. \tag{42}
\]

The amplitude is independent on the position of the picture changing operator \(z_{pc}\). We have 4 fermion zero modes and the first non-vanishing correlator has 4 fermions

\[
\langle \psi^\mu(z_1)\psi^\nu(z_1)\psi^\rho(z_2)\psi^\sigma(z_{pc}) \rangle = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{\alpha'} g_1(z_1, z_2, z_{pc}, \tau) \\
\langle \tilde{\psi}^\mu(z_1)\tilde{\psi}^\nu(z_2)\tilde{\psi}^\rho(z_2)\tilde{\psi}^\sigma(z_{pc}) \rangle = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{\alpha'} g_2(z_1, z_2, z_{pc}, \tau)^* \tag{43}
\]

Using

\[
\varepsilon_\alpha^{\mu\nu} \varepsilon_\alpha^{\rho\sigma} = - \det \begin{pmatrix} \eta^{\mu\nu} & \eta^{\mu\rho} & \eta^{\mu\sigma} \\ \eta^{\nu\mu} & \eta^{\nu\rho} & \eta^{\nu\sigma} \\ \eta^{\rho\mu} & \eta^{\rho\nu} & \eta^{\rho\sigma} \end{pmatrix} \tag{44}
\]

and neglecting terms proportional to \(p_1^2, p_2^2, p_1\cdot p_2, p_1\varepsilon_1^{\mu\nu}, p_2\varepsilon_2^{\rho\sigma}, \eta_{\mu\nu} \varepsilon_1^{\mu\nu}, \eta_{\rho\sigma} \varepsilon_2^{\rho\sigma}\) (what leaves only one term from \(44\)) we find the piece in second order in momentum

\[
A^{(2)}_{(o-o)}|_{O(p^2)} = g_s^2 \int \frac{d^2\tau}{\tau_2} Z(\tau, \bar{\tau}, (1, 1), (1, 1)) \frac{1}{\alpha'\tau_2} h(\tau, \bar{\tau}) \\
\times \frac{\alpha'}{8} \left( \eta_{\mu\rho} p_{1\sigma} p_{2\nu} + \eta_{\mu\sigma} p_{1\rho} p_{2\nu} + \eta_{\nu\rho} p_{1\sigma} p_{2\mu} + \eta_{\nu\sigma} p_{1\rho} p_{2\mu} \right) \varepsilon_1^{\mu\nu} \varepsilon_2^{\rho\sigma}, \tag{45}
\]

where

\[
h(\tau, \bar{\tau}) = \int d^2 z_1 \int d^2 z_2 \ g_s^2 \langle \partial X(z_{pc}, \bar{z}_{pc}) \bar{\partial} X(z_{pc}, \bar{z}_{pc}) \rangle. \tag{46}
\]

From now on we measure positions and momenta in string units (\(\frac{1}{\alpha'}d^2 z \to d^2 z, \alpha' p_\mu p_\nu \to p_\mu p_\nu\)). Comparing with \(35\) we get

\[
\delta_T = -\frac{1}{2} g_s^2 C_m \int \frac{d^2\tau}{\tau_2} Z(\tau, \bar{\tau}, (1, 1), (1, 1)) h(\tau, \bar{\tau}). \tag{47}
\]

The odd-odd partition function (see \(39\)) is proportional to \(\left| \theta \left[ \frac{1}{2}, \frac{1}{2} \right] (0, \tau) \right|^2\) and therefore vanishes (see \(110\)) and \(h(\tau, \bar{\tau})\) is singular because of \(168\). Suitable regularization gives

\[
\left| \theta \left[ \frac{1}{2}, \frac{1}{2} \right] (0, \tau) \right|^2 h(\tau, \bar{\tau}) = C \cdot \left| \partial_v \theta \left[ \frac{1}{2}, \frac{1}{2} \right] (v, \tau) \right|_{v=0}^2 = C \cdot 4\pi^2 |\eta(\tau)|^6, \tag{48}
\]

\(^2\)The author thanks P. Vanhove for clarifying this point.
where $C$ is a real constant. With (21) we finally arrive at

$$\delta_T = -B_2 C \pi^2 g_s^2 C_m.$$  \hfill (49)

One the other hand we have (4), i.e.

$$\delta_T = \frac{B_2}{6\pi}.$$  \hfill (50)

This fixes the matching coefficient

$$C_m = -\frac{1}{6C \pi^3 g_s^2}.$$  \hfill (51)

3 Non-compact orientifolds with $\Omega J$ projection

In this section we consider D-branes in orientifold models because they are (as far as we know it today) among the best possibilities to get a setup in superstring theory that comes close to the standard model. We will work in the non-compact case because as the matter fields, gauge fields and gravity are localized on the D-branes we will not need to compactify. This will also have the advantage that we will have more models at our disposal. The $\mathbb{Z}_N$ orbifold action e.g. will no longer have to act crystallographically and the number of D-branes will not be fixed. We will consider a non-standard orientifold projection in order to have D3-branes. For simplicity we assume $N$ to be odd so that we only have D3-branes and we will assume that the D3-branes are coincident and on top of O3$^+$-planes. For the discussed orientifolds of $\mathbb{Z}_N$ orbifolds we have $\mathcal{N} = 1$ supersymmetry in $d = 4$. After we review the partition functions for annulus, Moebius strip and Klein bottle and find the tadpole conditions in section 3.1 we derive their contributions to the one-loop renormalization of the Planck mass in section 3.2. In section 3.3 we finally give some examples.

3.1 Tadpole conditions

Let $\Omega$ be the world sheet parity transformation and $J$ act on the transverse complex bosons $Z^i = X^{2i+2} + iX^{2i+3}$ as

$$J Z^i = -Z^i.$$  \hfill (52)

We consider the $\Omega J$ orientifold of the non-compact $\mathbb{Z}_N$ orbifold of type IIB superstring theory and we assume $N$ to be odd. Therefore we only have D3-branes (for $N$ even we would also have D7-branes). This model has been presented in [20] (see [21] for a review). The one-loop amplitudes are the torus $T$, the annulus $A$, the Moebius strip $M$ and the Klein bottle $K$. The torus contribution to the one-loop renormalization of the Planck mass
is one half of the corresponding orbifold result. As noted in [20] the annulus, Moebius and Klein bottle amplitudes for the $ΩJ$ orientifolds are the same as the ones for the standard $Ω$ orientifolds (that has only D9 branes if $N$ is odd) if one replaces the D9 Chan-Paton implementation matrices by the D3 ones. The $A$, $M$ and $K$ partition functions can e.g. be found in [24] (see also [25] and [26]) and we use the same convention as in [24] that we suppress the winding and momentum sums. Let us define $q = e^{2πiτ}$. For the annulus $τ = \frac{1}{2}$, the annulus partition function is given by

$$Z_A = \frac{1}{4N} \int_0^\infty \frac{dτ}{τ^2} \sum_{k=0}^{N-1} \Tr \left[ (1 + (-1)^F)θ^k q^L_0 \right],$$

and we find

$$Z_A = \frac{1}{4N} \int_0^\infty \frac{dτ}{τ^2} \sum_{k=0}^{N-1} \sum_{α,β=0,1} (-1)^{α+β+αβ} \frac{θ^{[α/2]} [β/2]}{η^2} \times \prod_{i=1}^3 \left( -2 \sin(πkτ_i) \right) \frac{θ^{[0]} [1/2 + kτ_i]}{θ^{[1/2]} [1/2 + kτ_i]} (\Tr γ_{k,3})^2$$

(54)

$$= \frac{(1 - 1)}{4N} \int_0^\infty \frac{dτ}{τ^2} \sum_{k=0}^{N-1} \frac{θ^{[0]} [1/2]}{η(\frac{1}{2}iτ)^3} \times \prod_{i=1}^3 \left( -2 \sin(πkτ_i) \right) \frac{θ^{[0]} [1/2 + kτ_i]}{θ^{[1/2]} [1/2 + kτ_i]} (\Tr γ_{k,3})^2.$$  

For the Moebius amplitude we have $τ = \frac{1}{2} + \frac{1}{2}iτ_2$. The Moebius partition function is given by

$$Z_M = \frac{1}{4N} \int_0^\infty \frac{dτ}{τ^2} \sum_{k=0}^{N-1} \Tr \left[ (1 + (-1)^F)Ωθ^k q^L_0 \right].$$

(56)

If we let everything depends on

$$q_{new} = q_{old}^2 = e^{4iπτ} = e^{-2πτ_2},$$

(57)
then we find

\[ Z_M = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \theta \left[ \frac{1/2}{0} \right] (0, i\tau_2) \theta \left[ 0 \right] (0, i\tau_2) \eta(i\tau_2)^3 \theta \left[ 0 \right] (0, i\tau_2) \]

\[ \times \prod_{i=1}^3 (-2 \sin(\pi k v_i)) \frac{\theta \left[ 0 \right] (0, i\tau_2) \theta \left[ 0 \right] (0, i\tau_2)}{\theta \left[ \frac{1/2}{1/2 + k v_i} \right] (0, i\tau_2) \theta \left[ \frac{1/2}{1/2 + k v_i} \right] (0, i\tau_2)} \text{Tr} \bar{\gamma}_{\Omega_{k,3}}^1 \bar{\gamma}_{\Omega_{k,3}}^T \]

(58)

For the Klein bottle we have \( \tau = 2i\tau_2 \). The Klein bottle partition function is given by

\[ Z_K = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{n,k=0}^{N-1} \text{Tr} \left[ (1 + (-1)^F) \Omega \theta^k q^{L_0(\theta^n)} q^{-L_0(\theta^n)} \right] . \]

(59)

\( \Omega \) exchanges \( \theta^n \) with \( \theta^{N-n} \). As we have chosen \( N \) to be odd only \( n = 0 \) does survive in the trace. We arrive at

\[ Z_K = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \alpha, \beta = 0, 1 \left( -1 \right)^{\alpha+\beta+\alpha \beta} \frac{\theta \left[ \frac{\alpha/2}{\beta/2} \right]}{\eta^3} \]

\[ \times \prod_{i=1}^3 (-2 \sin(2\pi k v_i)) \frac{\theta \left[ \frac{0}{1/2 + 2k v_i} \right]}{\theta \left[ \frac{0}{1/2 + 2k v_i} \right]} \]

(60)

\[ = \frac{1}{4N} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_{k=0}^{N-1} \theta \left[ 0 \right] (0, 2i\tau_2) \eta(2i\tau_2)^3 \]

\[ \times \prod_{i=1}^3 (-2 \sin(2\pi k v_i)) \frac{\theta \left[ \frac{0}{1/2 + 2k v_i} \right]}{\theta \left[ \frac{0}{1/2 + 2k v_i} \right]} (0, 2i\tau_2) \]

(61)

We show in appendix D that this leads to the tadpole conditions

\[ 0 = \frac{1}{4} \prod_{i=1}^3 (-2 \sin(\pi k v_i)) (\text{Tr} \bar{\gamma}_{k,3})^2 + 2 \prod_{i=1}^3 (-2 \sin(\pi k v_i)) \text{Tr} \bar{\gamma}_{\Omega,3}^1 \bar{\gamma}_{\Omega,3}^T \]

\[ + 4 \prod_{i=1}^3 (-2 \sin(2\pi k v_i)) \]

(62)
that are equivalent to

\[ 0 = \left( \text{Tr} \gamma_{2k,3} \mp 32 \prod_{i=1}^{3} \cos(\pi kv_i) \right)^2. \]  \hspace{1cm} (63)

As we are considering the non-compact case we have no untwisted tadpole cancellation condition and the number of D3-branes that we call \( n_3 \) is arbitrary. But we still have to impose the twisted tadpole cancellation conditions \((k = 1, \ldots, N-1)\). For \( \mathbb{Z}_N \) orientifolds we have \( \gamma_{k,3} = \gamma_{1,3}^k, k = 1, \ldots, N - 1, \) and \( \gamma_{1,3}^N = 1 \). Remember that \( \sum_{k=0}^{N-1} e^{2i\pi k/N} = 0. \)

### 3.2 Contribution of \( A, M \) and \( K \) to the renormalization of the Planck mass

We generalize the results of \([22]\) and \([19]\) that considered compactifications on K3 (and therefore of the \( \mathbb{Z}_2 \) orientifold) to general non-compact \( \mathbb{Z}_N \) orientifolds with \( N \) odd.

For \( K, A, M \) there is only one spin structure. The even spin structure two graviton amplitude is given by (see e.g. \([15]\))

\[
A^{(2)} = \sum_{(\alpha, \beta) = 0, 1}^{\text{even}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha\beta} Z(\tau, \bar{\tau}, (\alpha, \beta)) \times \int d^2z_1 \int d^2z_2 \langle V(0,0)(z_1, \bar{z}_1) V(0,0)(z_2, \bar{z}_2) \rangle_{(\alpha, \beta)}. \] \hspace{1cm} (64)

The piece in second order in momentum will give us the one-loop renormalization of the Planck mass. Neglecting terms proportional to \( p_1^2, p_2^2, p_1 \cdot p_2, p_1 \varepsilon_1^{\mu \nu}, p_2 \varepsilon_2^{\rho \sigma}, \eta_{\mu \nu} \varepsilon_1^{\mu \nu}, \eta_{\rho \sigma} \varepsilon_2^{\rho \sigma} \) we find

\[
A^{(2)}(\tau, \bar{\tau}, (\alpha, \beta)) \bigg|_{O(p^2)} = \int d^2z_1 \int d^2z_2 \langle V(0,0)(z_1, \bar{z}_1) V(0,0)(z_2, \bar{z}_2) \rangle_{(\alpha, \beta)} \bigg|_{O(p^2)}
= \sum_{\sigma = K, A, M} g_s^2 \int d^2z_1 \int d^2z_2 \varepsilon_1^{\mu \nu} \varepsilon_2^{\rho \sigma} p_2^\sigma p_1^\rho \eta^{\mu \rho} \left[ \langle \partial X \partial X \rangle \langle \bar{\psi} \bar{\psi} \rangle_{(\alpha, \beta)}^2 - \langle \bar{\partial} X \partial X \rangle \langle \bar{\psi} \bar{\psi} \rangle_{(\alpha, \beta)}^2 - \langle \bar{\partial} X \partial X \rangle \langle \bar{\psi} \bar{\psi} \rangle_{(\alpha, \beta)}^2 \right]. \] \hspace{1cm} (65)

From now on we again measure everything in string units. For \( K, A, M \) we have to act with the following involutions on the covering tori

\[
I_A = I_M = 1 - \bar{z}, \quad I_K = 1 - \bar{z} + \frac{\tau}{2}. \] \hspace{1cm} (66)
If on the covering torus we have \( \langle \psi(z)\psi(w) \rangle = P_F((\alpha, \beta); z, w) \) then by the method of images (see appendix of [22])

\[
\langle \psi(z)\psi(w) \rangle_{\sigma, (\alpha, \beta)} = P_F((\alpha, \beta); z, w) \quad (67)
\]

\[
\langle \psi(z)\psi(\bar{w}) \rangle_{\sigma, (\alpha, \beta)} = P_F((\alpha, \beta); z, L_\sigma(w)) \quad (68)
\]

\[
\langle \bar{\psi}(z)\bar{\psi}(\bar{w}) \rangle_{\sigma, (\alpha, \beta)} = \bar{P}_F((\bar{\alpha}, \bar{\beta}); \bar{z}, \bar{w}). \quad (69)
\]

On the other hand

\[
(\langle \psi(z)\psi(0) \rangle_{\mathcal{T}, (\alpha, \beta)})^2 = -\delta^2 \log \theta \left[ \frac{1}{2} \right] (z, \tau) + 4\pi i \partial_\tau \log \theta \left[ \frac{\alpha}{2} \right] (0, \tau) \quad (70)
\]

i.e. it can be written as a sum of a term that is independent of the spin structure (but dependent on the position \( z \) on the world-sheet) and therefore vanishes when summed over the spin structure (as the partition function vanishes due to supersymmetry) and a term independent of the position \( z \) on the world-sheet (but dependent on the spin structure) that can be taken outside the world-sheet integral. The surviving piece will be the same in \( \langle \bar{\psi}\psi \rangle^2, \langle \bar{\psi}\psi \rangle^2, \langle \bar{\psi}\psi \rangle^2, \langle \bar{\psi}\psi \rangle^2 \) so we replace it by \( \langle \psi\psi \rangle^2 \) everywhere. The remaining integral over the bosonic correlators gives (see again [22])

\[
\int d^2 z_1 \int d^2 z_2 \left[ \langle \partial X \partial X \rangle - \langle \bar{\partial} X \bar{\partial} X \rangle + \langle \bar{\partial} X \partial X \rangle - \langle \bar{\partial} X \bar{\partial} X \rangle \right] = \begin{cases} 
\pi \tau_2 / 8 & \text{for } \sigma = A, M \\
\pi \tau_2 / 2 & \text{for } \sigma = K 
\end{cases} \quad (71)
\]

Using (35) and (109) we find for the annulus

\[
\delta_A = -g_s^2 C_m \sum_{(\alpha, \beta)=0,1} \int_0^\infty \frac{d \tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha \beta} Z_A(\tau, \bar{\tau}, (\alpha, \beta)) \frac{\partial^2 \theta \left[ \frac{\alpha}{2} \right]}{\theta \left[ \frac{\beta}{2} \right]} \bigg|_{v=0}^{\pi \tau_2} \frac{\pi \tau_2}{8}.
\]

(72)

for the Moebius strip

\[
\delta_M = -g_s^2 C_m \sum_{(\alpha, \beta)=0,1} \int_0^\infty \frac{d \tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha \beta} Z_M(\tau, \bar{\tau}, (\alpha, \beta)) \frac{\partial^2 \theta \left[ \frac{\alpha}{2} \right]}{\theta \left[ \frac{\beta}{2} \right]} \bigg|_{v=0}^{\pi \tau_2} \frac{\pi \tau_2}{8}.
\]

(73)

and for the Klein bottle

\[
\delta_K = -g_s^2 C_m \sum_{(\alpha, \beta)=0,1} \int_0^\infty \frac{d \tau_2}{\tau_2^2} (-)^{\alpha+\beta+\alpha \beta} Z_K(\tau, \bar{\tau}, (\alpha, \beta)) \frac{\partial^2 \theta \left[ \frac{\alpha}{2} \right]}{\theta \left[ \frac{\beta}{2} \right]} \bigg|_{v=0}^{\pi \tau_2} \frac{\pi \tau_2}{2}.
\]

(74)
Using (110) to (112), the Riemann identity (113) and the partition functions \( \delta \times g \delta \times g \times \delta \times g \) we get for the annulus

\[
\delta_A = -g_s^2 C_m \frac{1}{2N} \partial_v^2 \int_0^{\infty} \frac{dt_2}{t_2^3} \sum_{k=1}^{N-1} \frac{\theta \left[ \frac{1}{2} \right]}{\eta(\frac{1}{2} i t_2)^3} \theta \left[ \frac{1}{2} \right] \left( \frac{v}{2}, \frac{1}{2} i t_2 \right) \times \left[ \prod_{i=1}^3 (-2 \sin(\pi k v_i)) \right] \frac{\theta \left[ \frac{1}{2} \right]}{\theta \left[ \frac{1}{2} \right]} \left( \frac{v}{2}, \frac{1}{2} i t_2 \right) \left( 0, \frac{1}{2} i t_2 \right) \right] \left( \operatorname{Tr} \gamma_{k,3} \right)^2 \frac{\tau_2}{8} \bigg|_{v=0}^{(75)}
\]

for the Moebius strip

\[
\delta_M = g_s^2 C_m \frac{1}{2N} \partial_v^2 \int_0^{\infty} \frac{dt_2}{t_2^3} \sum_{k=1}^{N-1} \frac{\theta \left[ \frac{1}{2} \right]}{\eta(i t_2)^3} \theta \left[ \frac{1}{2} \right] \left( \frac{v}{2}, i t_2 \right) \theta \left[ 0 \right] \left( 0, i t_2 \right) \times \left[ \prod_{i=1}^3 (-2 \sin(\pi k v_i)) \right] \frac{\theta \left[ \frac{1}{2} \right]}{\theta \left[ 1/2 \right]} \left( \frac{v}{2}, i t_2 \right) \left( 0, i t_2 \right) \theta \left[ 0 \right] \left( 0, i t_2 \right) \right] \left( \operatorname{Tr} \gamma_{k,3} \gamma_{k,3} \right) \frac{\pi \tau_2}{8} \bigg|_{v=0}^{(76)}
\]

and for the Klein bottle

\[
\delta_K = -g_s^2 C_m \frac{1}{2N} \partial_v^2 \int_0^{\infty} \frac{dt_2}{t_2^3} \sum_{k=1}^{N-1} \frac{\theta \left[ \frac{1}{2} \right]}{\eta(2 i t_2)^3} \times \left[ \prod_{i=1}^3 (-2 \sin(2 \pi k v_i)) \right] \frac{\theta \left[ \frac{1}{2} \right]}{\theta \left[ 1/2 \right]} \left( \frac{v}{2}, 2 i t_2 \right) \left( 0, 2 i t_2 \right) \right] \frac{\tau_2}{2} \bigg|_{v=0}^{(77)}
\]

We go to the transverse channel. For the annulus we have the transformations \( t = \frac{1}{2} t_2, l = \frac{1}{4} \)

\[
\delta_A = -g_s^2 C_m \frac{1}{2} \frac{1}{2N} \partial_v^2 \int_0^{\infty} \frac{dt}{d} \sum_{k=1}^{N-1} (-i e^{-\frac{\pi v^2}{4}}) \theta \left[ \frac{1}{2} \right] \theta \left[ \frac{1}{2} \right] \theta \left( \frac{i v l}{2}, il \right) \eta(i l)^3
\]

15
For the Moebius strip we have the transformations \( t = \frac{1}{2}, l = \frac{t}{2} \)

\[
\delta_M = g_s^2 C_m 2^2 \delta^2 \int_0^\infty dt \sum_{k=1}^{N-1} \frac{e^{-\frac{\pi l v^2}{4}} \theta \left[ 1/2 + kv, 1/2 \right] (iv, il) \left( \frac{\pi}{2}, il \right)}{\eta(2l)^3 \theta \left[ 0, 2il \right] (0, 2il)} \theta \left[ 1/2 + kv, 1/2 \right] (0, il) \theta \left[ 0, il \right] (0, il) \right]
\]

\[
\times \left. \begin{array}{c}
\text{Tr} \gamma^{-1}_{\Omega,3} \gamma^{T}_{\Omega,3} \frac{\pi}{16l} \bigg|_{v=0}
\end{array} \right.
\]

(79)

For the Klein bottle we have the transformations \( t = 2\tau_2, l = \frac{1}{t} \)

\[
\delta_K = -g_s^2 C_m 2^2 \delta^2 \int_0^\infty dt \sum_{k=1}^{N-1} \frac{e^{-\frac{\pi l v^2}{4}} \theta \left[ 1/2 + kv, 1/2 \right] (iv, il) \left( \frac{\pi}{2}, il \right)}{\eta(2l)^3 \theta \left[ 0, 2il \right] (0, 2il)} \theta \left[ 1/2 + kv, 1/2 \right] (0, il) \theta \left[ 0, il \right] (0, il) \right]
\]

\[
\times \left. \begin{array}{c}
\frac{\pi}{4l} \bigg|_{v=0}
\end{array} \right.
\]

(80)

Note that it is important here to take the derivatives with respect to \( v \) only after one goes to the transverse channel. Using (102) to (112) we arrive at

\[
\delta_M = 2\pi i g_s^3 C_m 2^2 \delta^2 \int_0^\infty dt \sum_{k=1}^{N-1} \left( il \right)^2 \left[ \prod_{j=1}^3 (-2 \sin(\pi kv)) \right] \left. \begin{array}{c}
\frac{\pi}{4l} \bigg|_{v=0}
\end{array} \right.
\]

\[
\times \left[ f(0, 0, 2il) + \sum_{i=1}^3 \left( i\pi + 2i\pi kv + f \left( \frac{1}{2} + kv, \frac{1}{2}, 2il \right) \right) + \sum_{i=1}^3 (2i\pi kv + f(kv, 0, 2il)) \right] \left. \begin{array}{c}
\text{Tr} \gamma^{-1}_{\Omega,3} \gamma^{T}_{\Omega,3} \frac{\pi}{16l} \bigg|_{v=0}
\end{array} \right.
\]

(81)
\[
\delta_A = -2\pi ig_s^2 C_m \frac{1}{2^2} \frac{1}{2N} \int_0^\infty dl \sum_{k=1}^{N-1} \left( i\pi \right) \left( \prod_{j=1}^3 (-2 \sin(\pi kv_j)) \right) \prod_{j=1}^3 (-2 \sin(\pi kv_j)) \right] \\
\times \sum_{i=1}^3 \left( i\pi + 2i\pi kv_i + f \left( \frac{1}{2} + kv_i, \frac{1}{2}, il \right) \right) \left( \text{Tr} \gamma_{k,3} \right)^2 \frac{\pi}{4l} 
\]

\[
\delta_K = -2\pi ig_s^2 C_m 2^2 \frac{1}{2N} \int_0^\infty dl \sum_{k=1}^{N-1} \left( i\pi \right) \left( \prod_{j=1}^3 (-2 \sin(2\pi kv_j)) \right) \prod_{j=1}^3 (-2 \sin(2\pi kv_j)) \right] \\
\times \sum_{i=1}^3 \left( i\pi + 4i\pi kv_i + f \left( \frac{1}{2} + 2kv_i, \frac{1}{2}, il \right) \right) \frac{\pi}{4l} 
\]

The tadpole cancellation condition (82) guarantees that the contribution of terms in the sum proportional to a constant (see the \(i\pi\)) in \(\delta = \delta_A + \delta_M + \delta_K\) vanishes. On the other hand \(\sum_{i=1}^3 v_i = 0\) because we consider supersymmetric models. All remaining terms in \(\delta\) are proportional to some \(f(a, b, il)\). These functions (for \(a \in (-1/2, 1/2)\)) fall off rapidly as \(l \to \infty\). Therefore \(\delta\) is free of ultraviolet divergences due to tadpole cancellation. We are left with

\[
\delta = \frac{i\pi^2}{16N} g_s^2 C_m \int_0^\infty dl \sum_{k=1}^{N-1} \left\{ \prod_{j=1}^3 (-2 \sin(\pi kv_j)) \right\} \left[ f \left( \frac{1}{2} + kv_i, \frac{1}{2}, il \right) \right] \left( \text{Tr} \gamma_{k,3} \right)^2 \\
- 2 \left[ \prod_{j=1}^3 (-2 \sin(\pi kv_j)) \right] \left[ f(0, 0, 2il) + \sum_{i=1}^3 f \left( \frac{1}{2} + kv_i, \frac{1}{2}, 2il \right) \right] \\
+ \sum_{i=1}^3 f \left( kv_i, 0, 2il \right) \text{Tr} \gamma_{0,3}^{-1} \gamma_{0,3} \gamma_{0,3} \\
+ 2 \left[ \prod_{j=1}^3 (-2 \sin(2\pi kv_j)) \right] \sum_{i=1}^3 f \left( \frac{1}{2} + 2kv_i, \frac{1}{2}, il \right) \right\}, 
\]

which may be computed explicitly. Though the integral is free of ultraviolet divergences we may still have some infrared divergences that can only be handled by considering the Wilsonian couplings. \(C_m\) is given by (51).

### 3.3 Examples

We first consider the \(\mathbb{Z}_3\) orientifold as we will conclude from it the results for more general \(\mathbb{Z}_N\) orientifolds. For the \(\mathbb{Z}_3\) orientifold (already presented in [20]) with \(v = \left( \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right)\) we get \(\text{Tr} \gamma_{k,3} = -4, k=1,2\). We define \(M = \frac{n_3 + 4}{3} \in \mathbb{N}\) and find as solution to the tadpole conditions

\[
\gamma_{1,3} = \text{diag}(\alpha (\text{M-times}), \alpha^{-1} (\text{M-times}), 1 ((\text{M-4})\text{-times})), 
\]

17
where $\alpha = e^{2i\pi/3}$. The open sector gauge and matter fields are best determined using the shift vector as in \cite{23}. We find the gauge group $U(M) \times SO(M-4)$. With $M = 5$, i.e. $n_3 = 11$, one gets e.g. a 3 generation model with gauge group $SU(5)$ (but with a Higgs content that cannot be used to break to the standard model gauge group) see \cite{21}. To compute $\delta_{A+M+K}$ we start with (84) and use the properties (104) to (106). It is a non-trivial check to see that all factors of $i\pi$ that would again lead to ultraviolet divergences indeed cancel after one shifts to values of $a$ in $(-1/2, 1/2)$. We get

$$\delta = \frac{i\pi^2}{16 \cdot 3} g_s^2 C_m \int_0^\infty dl l 4 \cdot 3 \cdot \sqrt{3} \cdot 12 [-f(1/6, 1/2, il) - f(1/6, 1/2, 2il) + f(1/3, 0, 2il)].$$

(86)

Using the computer program Mathematica and after subtracting the infrared divergences we get

$$\delta_{\text{finite}}^{A+M+K} = - \frac{1}{16} \sqrt{3\pi^3} g_s^2 C_m \left[ 5 \text{PolyGamma} \left( 1, \frac{1}{3} \right) - 5 \text{PolyGamma} \left( 1, \frac{2}{3} \right) \\
+ \text{PolyGamma} \left( 1, \frac{1}{6} \right) - \text{PolyGamma} \left( 1, \frac{5}{6} \right) \right] \approx -236g_s^2 C_m.$$

(87)

where $\text{PolyGamma}(1, z)$ is the first derivative of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$.

For the $\mathbb{Z}_3$ family, i.e. the $\mathbb{Z}_N$ orientifolds with $N = 3(2l - 1)$, $l \in \mathbb{N}$ and $v = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ defined in section 2.3 we get the same tadpole conditions and therefore the same implementation matrices and gauge group as for the non-compact $\mathbb{Z}_3$ orientifold. Notice that the number of D3-branes $n_3$ and therefore the gauge group is independent of $N$. From (84) it follows that for the $\mathbb{Z}_3$ family $\delta_{A+M+K}$ is the same for all $l \in \mathbb{N}$ and given by the $\mathbb{Z}_3$ result. This is because the factor of $(2l-1)$ in the number of twisted sectors is cancelled by the the factor $(2l-1)$ in the normalization of the partition functions. Notice that the $(2l-2)$ twisted sectors that are of the form $k = 3m$, $m \in \mathbb{N}$ do not contribute. The torus contribution is given by one half of (29). Naively we can estimate that for large $N$ the torus contribution dominates over the contributions from $A$, $M$ and $K$ because

$$\delta_T \propto \frac{1}{N} \sum_{k,g=0}^{N-1} = O(N)$$

(88)

$$\delta_{A+M+K} \propto \frac{1}{N} \sum_{k=0}^{N-1} = O(1).$$

(89)

We have checked this for the $\mathbb{Z}_3$ family.
4 Conclusions

We have considered D-branes in orientifold models because they are (as far as we know it today) among the best possibilities to get a setup in superstring theory that comes close to the standard model. We focused on the non-compact case because in these models the matter fields, gauge fields and gravity are localized on the D-branes and we do not need to compactify. The issue was to show that the one-loop correction of the Planck mass can be arbitrary large in string units. It is therefore possible to accommodate the measured four-dimensional Planck mass as a one-loop effect and to have a string scale far below the Planck scale.

To be more precise we have constructed non-compact orientifolds of $\mathbb{Z}_N$ orbifolds of type IIB with induced gravity on coincident D3-branes that are on top of O3$_+$-planes. That we consider orientifolds of $\mathbb{Z}_N$ orbifolds is because they have localized twisted sectors and therefore localized gravity. As we consider the non-compact case the orbifold need not act crystallographically. That we assumed $N$ to be odd and the D3-branes to be coincident and on top of O3$_+$-planes was just for simplicity.

As an example we have shown for the $\mathbb{Z}_3$ family, i.e. the $\mathbb{Z}_N$ orientifolds with $N = 3(2l - 1), \ l \in \mathbb{N}$ and shift vector $v = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$, that the contribution to the one-loop renormalization of the Planck mass from the torus is $O(N)$ and from annulus, Moebius strip and Klein bottle is $O(1)$. The idea that four-dimensional gravity may be induced by quantum corrections at the one-loop level can therefore be realized by considering sufficiently large $N$. On the other hand the number of D3-branes and therefore of matter and gauge fields on the D3-branes are independent of $N$. If $n_3$ is the number of coincident D3-branes and $M = \frac{n_3 + 4}{3} \in \mathbb{N}$ then the gauge group is $U(M) \times SO(M - 4)$. With $M = 5$, i.e. $n_3 = 11$, one gets a 3 generation model with gauge group $SU(5)$. This is as close we get to the standard model in this paper.

Obviously the models presented in this paper are only toy models of orientifold realizations of the standard model and there is plenty of room for generalization. The aim will be to construct more realistic brane induced gravity models that come closer to (supersymmetric generalizations) of the standard model by considering e.g. more general D-brane configurations, Scherk-Schwarz directions or Wilson lines. One will have to check the higher loop corrections to the Planck mass and also the renormalization of higher derivative terms as e.g. the $R^2$-terms.
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A Theta functions

We use the definitions of \[27\]
\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right](z, \tau) = \sum_{n=-\infty}^{\infty} \exp \left[ i\pi(n + a)^2 \tau + 2i\pi(n + a)(z + b) \right] \]
\[90\]
\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \]
\[91\]

This gives
\[
\theta \left[ \begin{array}{c} -a \\ -b \end{array} \right](z, \tau) = \theta \left[ \begin{array}{c} a \\ b \end{array} \right](-z, \tau) \]
\[92\]
\[
\theta \left[ \begin{array}{c} a + 1 \\ b \end{array} \right](z, \tau) = \theta \left[ \begin{array}{c} a \\ b \end{array} \right](z, \tau) \]
\[93\]
\[
\theta \left[ \begin{array}{c} a \\ b + 1 \end{array} \right](z, \tau) = e^{2i\pi a} \theta \left[ \begin{array}{c} a \\ b \end{array} \right](z, \tau). \]
\[94\]

We have the modular transformations
\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right](z, \tau + 1) = e^{-i\pi(a^2 + a)} \theta \left[ \begin{array}{c} a \\ \frac{1}{2} + a + b \end{array} \right](z, \tau) \]
\[95\]
\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] \left( \frac{z}{\tau}, -\frac{1}{\tau} \right) = (-i\tau)^{1/2} \exp \left[ 2\pi iab + \frac{i\pi z^2}{\tau} \right] \theta \left[ \begin{array}{c} b \\ -a \end{array} \right](z, \tau) \]
\[96\]
\[
\eta(\tau + 1) = e^{i\pi \tau^2} \eta(\tau) \]
\[97\]
\[
\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2} \eta(\tau), \]
\[98\]

where the second property is shown using Poisson resummation
\[
\sum_{n=-\infty}^{\infty} \exp \left[ -\pi an^2 + 2\pi ibn \right] = a^{-1/2} \sum_{m=-\infty}^{\infty} \exp \left[ -\frac{\pi(m - b)^2}{a} \right]. \]
\[99\]

The theta-functions have the product representation
\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right](u, \tau) = e^{2i\pi ab} q^{2a^2} z^a \prod_{m=1}^{\infty} (1 - q^m) \left( 1 + e^{2i\pi b} q^{m - \frac{1}{2} - a} \right) \left( 1 + e^{-2i\pi b} q^{m - \frac{1}{2} - a} \right), \]
\[100\]
where $q = e^{2i\pi \tau}$, $z = e^{2i\pi v}$. In particular we have

$$\theta \left[ \frac{a}{b} \right] (0, \tau) = e^{2\pi i a b} q^{\frac{a^2}{24}} \prod_{m=1}^{\infty} \left[ 1 + e^{-2\pi i b q (m - \frac{1}{2} - a)} \right] \left[ 1 + e^{2\pi i b q (m - \frac{1}{2} + a)} \right].$$

(101)

For the derivatives we get

$$\frac{\partial_v \theta \left[ \frac{a}{b} \right] (v, \tau)}{\theta \left[ \frac{a}{b} \right] (v, \tau)} \bigg|_{v=0} = 2i\pi a + f(a, b; \tau),$$

(102)

where

$$f(a, b; \tau) = 2\pi i \sum_{m=1}^{\infty} \left[ e^{2\pi i b q (m - \frac{1}{2} + a)} - e^{-2\pi i b q (m - \frac{1}{2} - a)} \right].$$

(103)

From the behaviour of the theta functions follows

$$f(-a, -b; \tau) = -f(a, b; \tau)$$

(104)

$$f(a + 1, b; \tau) = -2i\pi + f(a, b; \tau)$$

(105)

$$f(a, b + 1; \tau) = f(a, b; \tau)$$

(106)

$$f(a, b; \tau + 1) = f(a, 1/2 + a + b; \tau)$$

(107)

$$f(a, b; -1/\tau) = 2i\pi(b - a) + f(b, -a; \tau).$$

(108)

The theta functions are solutions of the heat equation

$$\frac{\partial^2}{\partial z^2} \theta \left[ \frac{a}{b} \right] (z, \tau) = 4\pi i \frac{\partial}{\partial \tau} \theta \left[ \frac{a}{b} \right] (z, \tau)$$

(109)

moreover

$$\theta \left[ \frac{1/2}{1/2} \right] (0, \tau) = 0$$

(110)

$$\frac{\partial_v \theta \left[ \frac{1/2}{1/2} \right] (v, \tau)}{\theta \left[ \frac{1/2}{1/2} \right] (v, \tau)} \bigg|_{v=0} = -2\pi \eta(\tau)^3$$

(111)

$$\frac{\partial^2_v}{\partial v^2} \theta \left[ \frac{1/2}{1/2} \right] (v, \tau) \bigg|_{v=0} = 0.$$

(112)

For $\sum_{i=1}^{4} h_i = \sum_{i=1}^{4} g_i = 0$ we have the Riemann identity

$$\frac{1}{2} \sum_{\alpha,\beta=0}^{1} (-)^{\alpha + \beta + \alpha \beta} \prod_{i=1}^{4} \theta \left[ \frac{\alpha/2 + h_i}{\beta/2 + g_i} \right] (v_i) = -\prod_{i=1}^{4} \theta \left[ \frac{1/2 - h_i}{1/2 - g_i} \right] (v'_i)$$

(113)

$$v'_1 = \frac{1}{2}(v_1 + v_2 + v_3 + v_4), \quad v'_2 = \frac{1}{2}(v_1 - v_2 + v_3 + v_4)$$

(114)

$$v'_3 = \frac{1}{2}(v_1 + v_2 - v_3 + v_4), \quad v'_4 = \frac{1}{2}(v_1 + v_2 + v_3 - v_4).$$

(115)
B The normalization of the partition function

B.1 The example of the $\mathbb{Z}_3$ orbifold

Let us first consider the compact case.

B.1.1 The massless spectrum

Let $T^6 = T^2 \times T^2 \times T^2$ and let us define $\alpha = e^{\frac{2\pi i}{3}}$. The $\mathbb{Z}_3$ orbifold acts on the tori as $Z^i \rightarrow e^{2\pi iv_i}Z^i$, where $v = (v_1, v_2, v_3) = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$, i.e. we have the reflection

$$r : Z^1 \rightarrow \alpha Z^1, Z^2 \rightarrow \alpha Z^2, Z^3 \rightarrow \alpha^{-2} Z^3. \quad (116)$$

For the orbifold to act crystallographically the torus moduli of the three spacetime tori have to be $\tau_i = \alpha^{1/2} R_i$. For each torus we have 3 fixed points at $n\alpha^{1/4}R_i/3$, $n = 0, 1, 2$, giving a total of $3^3 = 27$ fixed points.

The untwisted massless spectrum

The zero point energy of a complex boson with twist $\theta$ is

$$f(\theta) = \frac{1 - 3(1 - 2\theta)^2}{24} \quad (117)$$

and the negative of this for a complex fermion. The shift is zero for the 4 complex bosons (the transverse and the three compact) so the zero point energy is in the NS sector

$$4f(0) - 4f(1/2) = -\frac{1}{2} \quad (118)$$

and the first exited states are massless. In the R sector instead we have the zero point energy

$$4f(0) - 4f(0) = 0 \quad (119)$$

and the massless states are the degenerate ground states.

Let us separate the lefthanded part of the massless states according to their eigenvalue under $\alpha$:

$$\alpha^0 : \psi_{-1/2}^i |0\rangle_{NS}, \left| \frac{1}{2}, \frac{1}{2} \right\rangle_R, \left| -\frac{1}{2}, \frac{1}{2} \right\rangle_R \quad (120)$$

$$\alpha^1 : \psi_{-1/2}^i |0\rangle_{NS}, \left| \frac{1}{2}, \frac{3}{2} \right\rangle_R \quad (121)$$

$$\alpha^2 : \psi_{-1/2}^i |0\rangle_{NS}, \left| -\frac{1}{2}, \frac{3}{2} \right\rangle_R \quad (122)$$
where

\[
\begin{align*}
|1\rangle_R &= \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rightangle_R \\
|\bar{1}\rangle_R &= \left| -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rightangle_R \\
|3\rangle_R &= \left\{ \left| \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \rightangle_R, \left| -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rightangle_R, \left| -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \rightangle_R \right\} \\
|\bar{3}\rangle_R &= \left\{ \left| -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rightangle_R, \left| \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \rightangle_R, \left| \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rightangle_R \right\}.
\end{align*}
\]

The untwisted massless states that are invariant under the orbifold action come from $\alpha^0\alpha^0$, $\alpha^1\alpha^2$ and $\alpha^2\alpha^1$. We find 44 bosonic states and 44 fermionic states that give the following $\mathcal{N} = 2$ multiplets

\[
\left[ \left( -2, \frac{3}{2}, -1 \right) + \left( 1, \frac{3}{2}, 2 \right) \right] + \left[ \left( -1, -\frac{1}{2}, 0 \right) + \left( 0, \frac{1}{2}, 1 \right) \right]^9
+ \left[ \left( -\frac{1}{2}, 0^2, \frac{1}{2} \right) + \left( -\frac{1}{2}, 0^2, \frac{1}{2} \right) \right],
\]

where the superscripts are not powers but give the number of fields of given helicity.

The twisted massless spectrum

The massless spectrum is 27 copies of the massless spectrum at one of the fixed points. Let us first consider the states twisted by $r$. The transverse complex bosons has shift 0 and the three compact complex bosons have shift $1/3$. In the NS sector we get using (117) the zero point energy

\[
f(0) - f(1/2) + 3f(1/3) - 3f(1/6) = 0
\]

and in the R sector

\[
f(0) - f(0) + 3f(1/3) - 3f(1/3) = 0
\]

so in both cases the massless states are ground states. Let us first consider the R case. There are 2 fermion zero modes coming from the transverse complex fermion leading to 2 possible states $|\pm \frac{1}{2}\rangle_{h_R}$. The GSO projection then only leaves only one state $|\frac{1}{2}\rangle_{h_R}$. In the NS case there is a unique ground state $|0\rangle_{h_{NS}}$. We find two bosonic states and two fermionic states. The states twisted by $r^2$ give the antiparticles of these. The total massless spectrum from the twisted states is in terms of $\mathcal{N} = 2$ multiplets

\[
\left[ \left( -1, -\frac{1}{2}, 0 \right) + \left( 0, \frac{1}{2}, 1 \right) \right]^{27}.
\]
B.1.2 The helicity generating partition function

For type IIB on $M^4 \times T^6 / \mathbb{Z}_3$ with $T^6 = T^2 \times T^2 \times T^2$ we find

$$Z(v, \bar{v}) = N_0 \int_{\mathcal{X}} \frac{d^2 \tau}{\tau_2^2} Z^2_X(\tau) \sum_{h,g=0}^{2} \frac{Z_1}{h/3 \ g/3} (\tau) Z_2 \frac{h/3}{g/3} (\tau) Z_3 \frac{-2h/3}{-2g/3} (\tau)$$

$$\times Z^+_{\psi}(v, \tau) Z^+(\bar{v}, \tau)^*,$$

(131)

where

$$Z^2_X(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4}$$

(132)

$$Z_i \frac{[q]}{0} (\tau) = \frac{\Gamma_{2,2}}{|\eta(\tau)|^4} \quad (\text{2, 2 lattice sum})$$

(133)

$$Z_i \frac{p}{q} (\tau) = -3 \left( \frac{\eta(\tau)}{\theta} \right)^2 \left[ \frac{1/2 + p}{1/2 + q} \right] (0, \tau) \quad \text{for } (p, q) \neq (0, 0)$$

(134)

$$Z^+_{\psi}(v, \tau) = \frac{\xi(v)}{2} \frac{1}{\eta(\tau)^4} \sum_{\alpha, \beta=0}^{1} (-)^{\alpha + \beta + \alpha\beta} \theta \left[ \frac{\alpha/2 + h/3}{\beta/2} \right] (v, \tau) \theta \left[ \frac{\alpha/2 + h/3}{\beta/2 + g/3} \right] (0, \tau)$$

$$\times \theta \left[ \frac{\alpha/2 + h/3}{\beta/2 - 2g/3} \right] (0, \tau) \theta \left[ \frac{\alpha/2 - 2h/3}{\beta/2 - 2g/3} \right] (0, \tau).$$

(135)

Using the Riemann identity we get

$$Z^+_{\psi}(v, \tau) = \frac{\xi(v)}{\eta(\tau)^4} \theta \left[ \frac{1/2}{1/2} \right] \left( v/2, \tau \right) \left( \theta \left[ \frac{1/2 - h/3}{1/2 - g/3} \right] \left( v/2, \tau \right) \right)^2 \left( \theta \left[ \frac{1/2 + h/3}{1/2 + g/3} \right] \left( v/2, \tau \right) \right).$$

(136)

We find the contribution to the helicity generating partition function from the massless modes from the limit $\tau_2 \to \infty$. The twisted states come from $h = 1, 2$ and the untwisted from $h = 0$. Using

$$\Gamma_{2,2} |\text{massless}| = 1, \quad \xi(v) \xrightarrow{\tau_2 \to \infty} 1, \quad \bar{\xi}(\bar{v}) \xrightarrow{\tau_2 \to \infty} 1$$

(137)

$$\frac{1}{|\eta(\tau)|^6} \theta \left[ \frac{1/2}{1/2} \right] \left( v/2, \tau \right) \bar{\theta} \left[ \frac{1/2}{1/2} \right] \left( \bar{v}/2, \tau \right) = 4 \sin \frac{\pi v}{2} \sin \frac{\pi \bar{v}}{2} (1 + O(q \bar{q}))$$

(138)

$$\int_{\mathcal{X}} \frac{d^2 \tau}{\tau_2^2} = \log 3$$

(139)

and

$$\left| \frac{\theta}{\tau_2 \to \infty} \left[ \frac{1/2}{1/6} \right] (v/2, \tau) \frac{\theta}{(0, \tau)} \right| \left| \frac{1}{\sqrt{3}} e^{i \pi v/2} + e^{-i \pi/3} e^{-i \pi v/2} \right|$$

(140)
that follows from the product representation of the theta functions we find the massless contribution of the twisted sector

\[
Z^T(v, \bar{v})|_{\text{massless}} = 3(\log 3) N_0 (-3)^3 \frac{4}{2} \left| \sin \frac{\pi v}{2} \right|^2 e^{i\pi v/6}^6 \\
+ 3(\log 3) N_0 (-3)^3 \frac{4}{2} \left| \sin \frac{\pi v}{2} \right|^2 e^{-i\pi v/6}^6 \quad (144)
\]

and the massless contribution of the untwisted sector

\[
Z^U(v, \bar{v})|_{\text{massless}} = (\log 3) N_0 256 \left| \sin \frac{\pi v}{2} \right|^{12} \\
- (\log 3) N_0 4 \left| \sin \frac{\pi v}{2} \right|^{12} e^{i\pi v/2} e^{-i\pi v/2}^6 \\
- (\log 3) N_0 4 \left| \sin \frac{\pi v}{2} \right|^{12} e^{-i\pi v/3} e^{i\pi v/2} e^{-i\pi v/2}^6 . \quad (145)
\]

If we write a function \( f(v, \bar{v}) \) as

\[
f(v, \bar{v}) = \left( \sum_{\lambda_R} \tilde{c}_{\lambda_R} e^{2i\pi v\lambda_R} \right) \left( \sum_{\lambda_L} c_{\lambda_L} e^{-2i\pi \bar{v}\lambda_L} \right) \quad (146)
\]

with coefficients \( \tilde{c}_{\lambda_R}, c_{\lambda_L} \) then the contribution to the fixed helicity \( \lambda_{\text{tot}} = \lambda_R + \lambda_L \) is

\[
\sum_{\lambda_R} \tilde{c}_{\lambda_R} c_{(\lambda_{\text{tot}}-\lambda_R)} e^{2i\pi v\lambda_R} e^{-2i\pi \bar{v}(\lambda_{\text{tot}}-\lambda_R)} \bigg|_{v=\bar{v}=0} = \sum_{\lambda_R} \tilde{c}_{\lambda_R} c_{(\lambda_{\text{tot}}-\lambda_R)} . \quad (147)
\]

The helicity content of \( 4 \left| \sin \frac{\pi v}{2} \right|^2 e^{i\pi v/6}^6 \) is

\[
\begin{array}{c|ccc}
\lambda_{\text{tot}} & 0 & 1/2 & 1 \\
\text{value} & -1 & 2 & -1 \\
\end{array}
\]

The helicity content of \( 4 \left| \sin \frac{\pi v}{2} \right|^2 e^{-i\pi v/6}^6 \) is

\[
\begin{array}{c|ccc}
\lambda_{\text{tot}} & 0 & -1/2 & -1 \\
\text{value} & -1 & 2 & -1 \\
\end{array}
\]
Comparing with the twisted spectrum (130) fixes the normalization to be
\[ N_0 = \frac{1}{3 \log 3}. \] (148)

The helicity content of
\[ 4 \left| \sin \frac{\pi v}{2} \right|^2 e^{i\pi v/2} + e^{-i\pi/3}e^{-i\pi v/2} \] and of
\[ 4 \left| \sin \frac{\pi v}{2} \right|^2 e^{-i\pi/3}e^{i\pi v/2} + e^{-i\pi v/2} \] both give
\[
\begin{array}{cccc}
\lambda_{\text{tot}} & 0 & \pm 1/2 & \pm 1 \\
\text{value} & 2 & 2 & -1 \\
\end{array}
\]

The helicity content of
\[ 256 \left| \sin \frac{\pi v}{2} \right|^8 \] is
\[
\begin{array}{cccc}
\lambda_{\text{tot}} & 0 & \pm 1/2 & \pm 1 \\
\text{value} & 70 & -56 & 28 \\
\end{array}
\]
We see that with the normalization (148) we indeed reproduce the untwisted spectrum (127).

Let us also compute the second helicity supertrace
\[ B_2 = -\left( \frac{1}{2\pi i} \partial_v - \frac{1}{2\pi i} \partial_b \right)^2 Z(v, \bar{v}) \bigg|_{v=\bar{v}=0}. \] (149)

Using
\[ \partial_v^{\left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right]} (\nu, \tau) \bigg|_{\nu=0} = -2\pi \eta(\tau)^3, \quad \frac{\partial}{\partial \nu} = \frac{1}{2} \frac{\partial}{\partial \tau}, \] (150)
we find from (131)
\[ B_2 = 36. \] (151)

The massless contribution from (144) and (145) gives the same
\[ B_2|_{\text{massless}} = 27 + 9 = 36. \] (152)

The \( Z_3 \) orbifold is the singular limit of the Eguchi-Hanson space \( EH_3 \) that is a Calabi-Yau 3-fold with \( h^{1,1} = 36 \) and \( h^{2,1} = 0 \). We get form (7)
\[ \Delta L_{\text{1-loop}}^{\text{eff}} = \frac{6}{\pi} M_s^2 \sqrt{-gR}. \] (153)

**B.1.3 The non-compact case**

For the \( Z_3 \) orbifold we have in the non-compact case from the 27 fixed points of the compact case just the origin left. Instead of \( C = -3 \) in (131) we have \( C(p = 0) = -3 \) and \( C(p \neq 0) = -1 \). For the second helicity supertrace we get
\[ B_2 = B_2|_{\text{massless}} = 1 + 9 = 10 \] (154)
that gives using (7)
\[ \Delta L_{\text{eff}}^{\text{1-loop}} = \frac{5}{3\pi} M_s^2 \sqrt{-gR}. \] (155)
B.2 General \(N\)

We consider the non-compact case and fix the normalization from the twisted sectors. The massless spectrum is given by \(22\) or \(23\), where the sectors \(h\) and \(N - h\) for \(h \neq \frac{N}{2}\) together give one vector multiplet as does the sector \(h = \frac{N}{2}\) if \(N\) is even.

We will proof that the normalization of the partition function is given by \(23\) in the case that \(N\) is prime. The proof in the general case \(N \in \mathbb{N}\) relies on the fact that every natural number can uniquely be written as a product of prime numbers and is quite lengthy as there are sectors with \(hv_i \in \mathbb{Z}\) and one has more cases to consider. However, this generalization is straightforward. We will also assume that \(v_i \notin \mathbb{Z}\), \(i = 1, 2, 3\). The \(v_i\) are of the form \(v_i = \frac{k_i}{N}\) with \(k_i \in \mathbb{Z}\). As \(h = 1, \ldots, N - 1\) and \(N\) is prime it follows that \(hv_i \notin \mathbb{Z}\) for all \(h\).

From the partition function \(9\) we get the massless contribution of the twisted sectors from the limit \(\tau_2 \to \infty\). Using (137) to (139) and \(C(N) = -1\) we arrive at

\[
Z^T(v, \bar{v})\big|_{\text{massless}} = -N_0(\log 3) 4 \left| \sin \frac{\pi v}{2} \right|^2 \sum_{g=0}^{N-1} \sum_{h=1}^{N-1} \prod_{r_2 \to \infty} \left| \frac{\theta [1/2 + hv_i]}{1/2 + gv_i} (v/2, \tau) \right|^2.
\]

We write \(hv_i = [hv_i] + r(hv_i)\) with integer part \([hv_i]\) and rest \(r(hv_i) \in (0, 1)\).

From the product representation of the theta functions (100) we get

\[
\prod_{i=1}^{3} \prod_{g=0}^{N-1} \prod_{h=1}^{N-1} \left| \frac{\theta [1/2 + hv_i]}{1/2 + gv_i} (v/2, \tau) \right|^2 = \prod_{i=1}^{3} \prod_{g=0}^{N-1} \prod_{h=1}^{N-1} e^{i \pi v (\frac{3}{2} + r(hv_i))}.
\]

Due to supersymmetry we have \(v_3 = -v_1 - v_2\). On the other hand \([-x] = -[x] + 1\) for any \(x\) and \([x_1 + x_2] = \begin{cases} [x_1] + [x_2] & \text{for } r(x_1) + r(x_2) < 1 \\ [x_1] + [x_2] + 1 & \text{for } r(x_1) + r(x_2) > 1 \end{cases}\) for any \(x_1\) and \(x_2\). If \(r(hv_1) + r(hv_2) < 1 (\geq 1)\) then \(r((N-h)v_1)+r((N-h)v_2) = 1 - r(hv_1) + 1 - r(hv_2) > 1 (\leq 1)\). We are left with

\[
Z^T(v, \bar{v})\big|_{\text{massless}} = -N_0(\log 3) 4 \left| \sin \frac{\pi v}{2} \right|^2 \sum_{h=1}^{N-1} \left| e^{i \pi v (\frac{3}{2} + r(hv_i))} \right|^2.
\]

(158)
and from the helicity content of the functions after equation (147) follows the normalization (22) if one matches on the spectrum (25).

## C Some details of the two graviton amplitude

The part of the general n-point one-loop amplitude coming from even-even spin structures is given by (see [15])

\[
\mathcal{A}_{n}^{(e,e)} = \sum_{(\alpha,\beta) = 0,1} \sum_{(\bar{\alpha},\bar{\beta}) = 0,1} \int \frac{d^2 \tau}{\tau_2} (-)^{\alpha+\beta+\bar{\alpha}+\bar{\beta}} Z(\tau, \bar{\tau}, (\alpha, \beta), (\bar{\alpha}, \bar{\beta})) \\
\times \int \prod_{i=1}^{n-1} d^2 z_i \prod_{i=1}^{n} V^{(0,0)}(z_i, \bar{z}_i)_{(\alpha,\beta), (\bar{\alpha}, \bar{\beta})}
\]

(159)
as all vertex operators can be chosen in the (0,0)-ghost picture (see e.g. [28]), \(\Gamma\) is the fundamental region that is

\[
\Gamma = \{ \tau | \text{Im} \tau > 0, |\text{Re} \tau| \leq \frac{1}{2}, |\tau| \geq 1 \}
\]

(160)for the torus and \(\tau_2 \in [0, \infty)\) for \(\mathcal{K}, \mathcal{A}, \mathcal{M}\) and the \(z_i\) are integrated over the strip

\[
\Gamma_{\tau} = \{ z_i | |\text{Re} z_i| \leq \frac{1}{2}, 0 \leq |\text{Im} z_i| \leq |\text{Im} \tau| \}.
\]

(161)For the torus we can set \(z_n = \tau\) due to the conformal symmetry. The partition function is vanishing by supersymmetry so we need at least two fermion contractions (from the vertex operators) to get a non-vanishing result. The graviton vertex operator in the (0,0)-ghost picture is

\[
V^{(0,0)}(z, \bar{z}) = -\frac{2g_s}{\alpha'} \varepsilon_{\mu \nu} : \left( i\partial X^\mu - \frac{\alpha'}{2} \bar{\psi}^\mu p \cdot \psi \right) \left( i\bar{\partial} X^\nu + \frac{\alpha'}{2} \bar{\psi}^\nu \tilde{p} \cdot \tilde{\psi} \right) e^{i p \cdot X} :.
\]

(162)The bosonic Green function on the torus is

\[
\langle X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') \rangle = -\frac{\alpha'}{2} \eta^{\mu \nu} \log |\chi(z - z', \tau)|^2,
\]

(163)where

\[
\chi(z_{ij}, \tau) = 2\pi \exp \left[ -\pi \frac{\text{Im} z_{ij}^2}{|\text{Im} \tau|} \right] \theta \left[ \frac{1/2}{1/2} \right](z_{ij}, \tau) \theta' \left[ \frac{1/2}{1/2} \right](0, \tau).
\]

(164)The fermionic Green function on the torus is

\[
\langle \psi^\mu(z) \psi^\nu(z') \rangle_{(\alpha, \beta)} = \eta^{\mu \nu} \theta \left[ \frac{\alpha/2}{\beta/2} \right] (z - z', \tau) \theta' \left[ \frac{1/2}{1/2} \right](0, \tau) \theta \left[ \frac{\alpha/2}{\beta/2} \right] (0, \tau).
\]

(165)
For the bosonic correlation functions we use as a starting point

\[
\langle \exp \left[ \sum_{i=1}^{N} \left( ik_i \cdot X(z_i, \bar{z}_i) + J^\mu \partial_{z_i} X_\mu(z_i, \bar{z}_i) + \bar{J}^\mu \partial_{\bar{z}_i} X^\mu(z_i, \bar{z}_i) \right) \right] \rangle = \\
= \exp \left[ \frac{1}{2} \sum_{i \neq j} \left( ik_{i\mu} + J_\mu(z_i) \partial_{z_i} + \bar{J}_\mu(\bar{z}_i) \partial_{\bar{z}_i} \right) \left( ik_{j\nu} + J_\nu(z_j) \partial_{z_j} + \bar{J}_\nu(\bar{z}_j) \partial_{\bar{z}_j} \right) \right] \\
\times \langle X^\mu(z_i, \bar{z}_i) X^\nu(z_j, \bar{z}_j) \rangle. \tag{166}
\]

Making functional derivatives with respect to the currents \( J(z) \) and \( \bar{J}(\bar{z}) \) and finally setting them to zero, we can compute the expectation value of any vertex operator that is a polynomial in derivatives of \( X \) times the exponential \( e^{ik \cdot X} \). We define

\[
G(z, \tau) = -\frac{1}{2} \log |\chi(z, \tau)| \tag{167}
\]

and find

\[
\partial_z G(z, \tau) = -\frac{1}{4} \sum_{k,m}' \frac{1}{k \tau - m} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\
\partial_{\bar{z}} G(z, \tau) = \frac{1}{4} \sum_{k,m}' \frac{1}{k \tau - m} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\
\partial_z \partial_z G(z, \tau) = \frac{\pi}{4 \text{Im} \tau} \sum_{k,m}' \frac{m - k \tau}{m - k \tau} \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\
\partial_z \partial_{\bar{z}} G(z, \tau) = \frac{\pi}{4 \text{Im} \tau} \sum_{k,m}' \exp \left[ 2\pi i k \left( \text{Re} z - \text{Re} \frac{\text{Im} z}{\text{Im} \tau} \right) \right] \exp \left[ 2\pi i m \frac{\text{Im} z}{\text{Im} \tau} \right] \\
-\frac{\pi}{4} \left( \delta(\text{Re} z) \delta(\text{Im} z) - \frac{1}{\text{Im} \tau} \right), \quad (168)
\]

where \( \sum_{k,m}' \) means that \((k, m) = (0, 0)\) is not in the sum. This gives

\[
\text{Im} \tau \int_0^{\frac{1}{2}} \text{d} \text{Im} z \int_0^{-\frac{1}{2}} \text{d} \text{Re} z \partial_z \partial_z G(z, \tau) = 0 \tag{169}
\]
\[
\text{Im} \tau \int_0^{\frac{1}{2}} \text{d} \text{Im} z \int_0^{-\frac{1}{2}} \text{d} \text{Re} z \partial_z \partial_{\bar{z}} G(z, \tau) = 0. \tag{170}
\]
D Deriving the tadpole conditions

First we find the transverse channel expressions for the amplitudes. For the annulus and Klein bottle this is achieved by the standard S-transformation as they depend on $\frac{1}{2} i \tau_2$ and $2 i \tau_2$ respectively. For the Moebius amplitude the functions do not depend on the standard $\frac{1}{2} + \frac{1}{2} i \tau_2$ that would lead to a $P$-transformation (with $P = ST^2ST$) but on $i \tau_2$ (because they depend on $q_{\text{new}} = q_{\text{old}}$ see [27]) so we again need a $S$-transformation to go to the transverse channel. The $S$-transformation of the theta functions is given by (90).

For the annulus we have the transformations $t = \frac{1}{2} \tau_2, l = \frac{1}{t}$

\[
Z_A = \frac{1}{2^2} (1 - 1) \frac{N-1}{4N} \int_0^\infty \frac{d\tau}{\eta} \sum_{k=0}^{N-1} \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, il) \theta \left[ \begin{array}{c} 1/2 + kv_i \\ 0 \end{array} \right] (0, il) \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, il) (\text{Tr} \gamma_k, 3)^2.
\]

(171)

For the Moebius strip we have the transformations $t = \frac{1}{\tau_2}, l = \frac{t}{2}$

\[
Z_M = -2 \frac{1}{2^2} (1 - 1) \frac{N-1}{4N} \int_0^\infty \frac{d\tau}{\eta} \sum_{k=0}^{N-1} \theta \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right] (0, 2il) \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, 2il) \theta \left[ \begin{array}{c} 1/2 + kv_i \\ 0 \end{array} \right] (0, 2il)
\]

\[
\times \prod_{i=1}^{3} (-2 \sin(\pi kv_i)) \left( 0, 2il \right) \theta \left[ \begin{array}{c} 1/2 + kv_i \\ 0 \end{array} \right] \gamma_{k, 3} T \gamma_{k, 3} (0, 2il)
\]

(172)

For the Klein bottle we have the transformations $t = 2 \tau_2, l = \frac{1}{t}$

\[
Z_K = 2^2 (1 - 1) \frac{N-1}{4N} \int_0^\infty \frac{d\tau}{\eta} \sum_{k=0}^{N-1} \theta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, il) \theta \left[ \begin{array}{c} 1/2 + 2kv_i \\ 0 \end{array} \right] (0, il) \theta \left[ \begin{array}{c} 1/2 + 2kv_i \\ 0 \end{array} \right] (0, il)
\]

(173)
The ultraviolet contribution comes from $\tau_2 \to 0$ or $l \to \infty$. We have

$$\lim_{l \to \infty} \frac{\theta \left[ \begin{array}{c} a \\ b_1 \end{array} \right] (0, il)}{\theta \left[ \begin{array}{c} a \\ b_2 \end{array} \right] (0, il)} = e^{2\pi i a(b_1-b_2)}. \tag{174}$$

In the sum $Z_A + Z_M + Z_K$ the NS and R contributions are both separately free of ultraviolet divergences, i.e. of tadpoles, under the condition that

$$0 = 1/8 \prod_{i=1}^{3} (-2 \sin(\pi k v_i))^2 \mp 2 \prod_{i=1}^{3} (-2 \sin(\pi k v_i)) \text{Tr} \left( \gamma_{\Omega^k,3}^{-1} \gamma_{\Omega^k,3}^T \right)$$

$$+ 4 \prod_{i=1}^{3} (-2 \sin(2\pi k v_i)). \tag{175}$$

We have $\sin(2\pi k v_i) = 2 \sin(\pi k v_i) \cos(\pi k v_i)$ and we can choose $\text{Tr} \left( \gamma_{\Omega^k,3}^{-1} \gamma_{\Omega^k,3}^T \right) = \text{Tr} \gamma_{2k,3}^2$ so (175) is equivalent to

$$0 = (\text{Tr} \gamma_{k,3})^2 \mp 8 \text{Tr} \gamma_{2k,3} + 16 \prod_{i=1}^{3} 2 \cos(\pi k v_i). \tag{176}$$

This has to be a perfect square and it is if

$$(\text{Tr} \gamma_{k,3})^2 = \frac{(\text{Tr} \gamma_{2k,3})^2}{\prod_{i=1}^{3} 2 \cos(\pi k v_i)} \tag{177}$$

then (176) is equivalent to

$$0 = \left( \text{Tr} \gamma_{2k,3} \mp 32 \prod_{i=1}^{3} \cos(\pi k v_i) \right)^2. \tag{178}$$

References

[1] E. Kiritsis, arXiv:hep-th/0310001.

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [arXiv:hep-ph/9905221].

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690 [arXiv:hep-th/9906064].

[4] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B484 (2000) 112 [arXiv:hep-th/0002190].
[5] G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B485 (2000) 208 [arXiv:hep-th/0005016].

[6] G. Dvali and G. Gabadadze, Phys. Rev. D63 (2001) 065007 [arXiv:hep-th/0008054].

[7] G. Dvali, G. Gabadadze, M. Kolanovic and F. Nitti, Phys. Rev. D64 (2001) 084004 [arXiv:hep-ph/0102216].

[8] E. Kiritsis, N. Tetradis and T. Tomaras, JHEP 0108 (2001) 012 [arXiv:hep-th/0106050].

[9] C. Middleton and G. Siopsis, [arXiv:hep-th/0210033].

[10] U. Ellwanger, [arXiv:hep-th/0304057].

[11] M. Kolanovic, Phys. Rev. D67 (2003) 106002 [arXiv:hep-th/0301116].

[12] M. Kolanovic, M. Porrati and J.-W. Rombouts, Phys. Rev. D68 (2003) 064018, [arXiv:hep-th/0304148].

[13] E. Kiritsis, N. Tetradis and T. Tomaras, JHEP 0203 (2002) 019 [arXiv:hep-th/0202037].

[14] E. Kiritsis and C. Kounnas, Nucl. Phys. B442 (1995) 472 [arXiv:hep-th/9501020].

[15] K. Förger, B.A. Ovrut, S.J. Theisen and D. Waldram, Phys. Lett. B388 (1996) 512 [arXiv:hep-th/9605145].

[16] E. Kiritsis, Leuven Univ. Press (1998) 315 p, (Leuven notes in mathematical and theoretical physics. B9), [arXiv:hep-th/9709062].

[17] I. Antoniadis, S. Ferrara, R. Minasian and K.S. Narain, Nucl. Phys. B507 (1997) 571 [arXiv:hep-th/9707013].

[18] E. Kohlprath, JHEP 0210 (2002) 026 [arXiv:hep-th/0207023].

[19] I. Antoniadis, R. Minasian and P. Vanhove, Nucl. Phys. B648 (2003) 69 [arXiv:hep-th/0209030].

[20] Z. Kakushadze, Nucl. Phys. B529 (1998) 157 [arXiv:hep-th/9803214].

[21] Z. Kakushadze, JHEP 0110 (2001) 031 [arXiv:hep-th/0109054].

[22] I. Antoniadis, C. Bachas, C. Fabre, H. Partouche and T.R. Taylor, Nucl. Phys. B489 (1997) 160 [arXiv:hep-th/9608012].

[23] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, Class. Quant. Grav. 20 (2003) 5079 [arXiv:hep-th/0307268].

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[24] G. Aldazabal, A. Font, L.E. Ibanez und G. Violero, Nucl. Phys. B536 (1998) 29 [arXiv:hep-th/9804026].

[25] E.C. Gimon und J. Polchinski, Phys. Rev. D54 (1996) 1667 [arXiv:hep-th/9601038].

[26] C. Angelatonj und A. Sagnotti, Phys. Rept. 371 (2002) 1, Erratum-ibid. 376 (2003) 339 [arXiv:hep-th/0204089].

[27] J. Polchinski, String Theory Volumes I and II, Cambridge University Press, 1998.

[28] W. Lerche, B.E.W. Nilsson and A.N. Schellekens, Nucl. Phys. B289 (1987) 609.