Abstract. The basic concepts of the formulation of Maxwellian electromagnetism in the absence of a Minkowski scalar product on spacetime are summarized, with particular emphasis on the way that the electromagnetic constitutive law on the space of bivectors over spacetime supplants the role of the Minkowski scalar product on spacetime itself. The complex geometry of the space of bivectors is also summarized, with the intent of showing how an isomorphic copy of the Lorentz group appears in that context. The use of complex 3-spinors to represent electromagnetic fields is then discussed, as well as the expansion of scope that the more general complex projective geometry of the space of bivectors suggests.

Keywords: Pre-metric electromagnetism, complex structures, complex 3-spinors, representations of the Clifford algebra of Minkowski space, complex projective geometry

1. Introduction

The role of spinors in electromagnetism has been well established since they were first introduced in non-relativistic form by Pauli, and later, in relativistic form, by Dirac. In the first case, the necessity came from the Stern-Gerlach experiment, which showed that the electron has two distinct spin states. In the second case, the necessity came from the purely mathematical desire to eliminate the physically puzzling negative energy solutions to the Klein-Gordon equation, which was the most logical relativistic formulation of the Schrödinger equation for quantum wavefunctions. Ironically, the resulting relativistic Dirac wave equation did not actually eliminate the negative energy solutions, which represented the positron wavefunctions in a historical era in which such particles had yet to be discovered.

We denote four-dimensional Minkowski space by \( \mathbb{M}_4 = (\mathbb{R}^4, \eta) \), in which we have chosen the signature convention \( \eta = \text{diag}(+1, -1, -1, -1) \) for specificity. The wavefunctions \( \psi: \mathbb{M}_4 \to \mathbb{C}^4 \) that obey the Dirac equation:

\[
(i\gamma^\mu \partial_\mu + m)\psi = 0
\]

are then associated with moving charge distributions that define the sources of electromagnetic fields.

In this equation, the four Dirac matrices \( \gamma^\mu \), \( \mu = 0, 1, 2, 3 \) are associated with the four corresponding legs of an oriented time-oriented orthonormal frame \( e_\mu \), \( \mu = 0, 1, 2, 3 \) on Minkowski space, or more precisely – its reciprocal coframe \( \theta^\mu \). These matrices generate a matrix representation of \( Cl(\mathbb{R}^4, \eta) \), the Clifford algebra of Minkowski space; the space \( \mathbb{C}^4 \) on which they act is the space of field values for the fields on \( \mathbb{M}_4 \) that one calls Dirac spinors. Since the group of units in \( Cl(\mathbb{R}^4, \eta) \) is isomorphic to \( SL(2; \mathbb{C}) \), in which the proper orthochronous Lorentz
group can be represented as a subgroup, the Dirac spinors carry a representation of the proper orthochronous Lorentz group.

If $\psi$ represents the wavefunction for a moving electron/positron then one can associate it with a conserved current 1-form $J$ whose components relative to $\theta^\mu$ are:

$$J_\mu = \pm ie\bar{\psi}\gamma_\mu\psi,$$

in which we have lowered the index of by means of the Minkowski scalar product.

One always has one – admittedly trivial – Clifford algebra that is associated with Minkowski space by ignoring its scalar product and giving $\mathbb{R}^4$ the completely degenerate bilinear form that associates all pairs of vectors with zero. The Clifford algebra that is associated with this orthogonal structure is simply the exterior algebra of $\mathbb{R}^4$, which clearly relates to the formulation of Maxwellian electromagnetism.

The introduction the Minkowski scalar product into the definition of the Clifford algebra associated with the laws of electromagnetism is intimately related to the introduction of the Minkowski scalar product into the formulation of the field equations for electromagnetism. Because both aforementioned Clifford algebras are defined over the same vector space there is an obvious linear isomorphism between the two, namely, the identity map; it is not, of course, an algebra isomorphism.

The 2-forms, which include the Minkowski field strength 2-form for an electromagnetic field, are contained in the even subalgebra of either Clifford algebra. The even subalgebra of $\text{Cl}(\mathbb{R}^4; \eta)$ is isomorphic to the Clifford algebra $\text{Cl}(\mathbb{R}^3, \delta)$ of Euclidian $\mathbb{R}^3$, which is why the electromagnetic 2-form $F$ is often represented in terms of the Pauli matrices $\sigma_i$, $i = 1, 2, 3$, which are usually regarded as non-relativistic objects.

Now, it was first observed by Cartan \[1\], and later expanded upon by Kottler \[2\] and Van Dantzig \[3\], that the only place where spacetime metric appears in Maxwell equations is in the Hodge duality isomorphisms:

$$* : \Lambda^*(M) \to \Lambda^{4-*}(M), \alpha \mapsto *\alpha. \quad (1, 3)$$

Kottler and Van Dantzig then succeeded in re-formulating Maxwell’s equations without the introduction of the usual Lorentzian pseudo-metric, but by substituting an electromagnetic constitutive law as the agent of this new formulation. (For the sake of brevity, in the sequel, we shall refer to the resulting theory of “pre-metric electromagnetism” by the acronym PMEM.)

One also must confront the purely physical consideration that the appearance and nature of the spacetime pseudo-metric of relativity theory is intimately linked with the propagation of electromagnetic waves, even though – paradoxically – the metric structure of spacetime seems to be ultimately more fundamental to the appearance of gravitational forces, not electromagnetic ones. Hence, there is reason to suspect that the much weaker gravitational force is, in some sense, subordinate to the much stronger electromagnetic one. In particular, one can define the Lorentzian structure as something that appears by way of the principal symbol of the d’Alembertian operator, and can be derived from the electromagnetic constitutive law by the use of the Fresnel analysis of waves in anisotropic media \[4\], suitably adapted to four-dimensional methods.

One gets more fundamental isomorphisms between $\Lambda_* (\mathbb{R}^4)$ and $\Lambda^* (\mathbb{R}^4)$ from Poincaré duality, which is defined by the use of a choice of unit volume element. This
suggests that projective geometry might be more appropriate for electromagnetism. Another hint in this direction comes from the consideration of the symmetries of the pre-metric Maxwell equations, in the sense of the symmetries of their space of solutions. In work done by the author [5], it was found that although in the absence of deeper analysis there seems to be a choice of four possible symmetry groups for PMEM, nevertheless, the one that seems to most directly extend the conformal Lorentz symmetry that was established by Bateman and Cunningham is the group SL(5; \( \mathbb{R} \)), which represents the group of projective transformations of \( \mathbb{RP}^4 \).

The conformal Lorentz group is associated with the introduction of light cones into the tangent spaces of the spacetime manifold. Indeed, physically, the measurement of distances in spacetime is facilitated by the introduction of electromagnetic waves. However "most" bivectors (2-forms, resp.) are not wave-like, so the use of a structure – namely, the Lorentzian structure \( g \) that is associated with the wave solutions restricts the generality of Maxwell’s equations.

In the next two sections, we shall summarize the basic notions of PMEM, as they relate to the issue of spinors and Clifford algebras. We then summarize some relevant concepts that pertain to the geometry of \( \mathbb{R}^4 \) when one introduces a complex structure on the six-dimensional vector space of bivectors (2-forms, resp.). In the central section of the article, we address the way that one introduces Clifford algebras into PMEM. Finally, we discuss some of the issues that are associated with expanding from Clifford algebras to more projective sorts of algebras.

2. Pre-metric Maxwell equations [4, 5, 6]

We assume that our spacetime manifold \( M \) is four-dimensional, orientable, and given a specific choice \( \varepsilon \in \Lambda^4(M) \) of unit-volume element on \( T(M) \). One can then define a unit-volume element \( \epsilon \in \Lambda_4(M) \) on \( T^*(M) \) by choosing the unique 4-vector field \( \epsilon \) such that \( \varepsilon(\epsilon) = 1 \). For a natural frame field \( \partial_{\mu} = \partial / \partial x^\mu \) that is defined by a local coordinate chart \( (U, x^\mu) \) on an open subset \( U \) in \( M \), and its reciprocal local co-frame field \( dx^\mu \), these two volume elements take the local form:

\[
\varepsilon = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \frac{1}{4!} \varepsilon_{\kappa\lambda\mu\nu} dx^{\kappa} \wedge dx^{\lambda} \wedge dx^{\mu} \wedge dx^{\nu}, \tag{2, 4}
\]

\[
\epsilon = \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3 = \frac{1}{4!} \varepsilon^{\kappa\lambda\mu\nu} \partial_\kappa \wedge \partial_\lambda \wedge \partial_\mu \wedge \partial_\nu. \tag{2, 5}
\]

A first key point of departure from the conventional formulation of Maxwell’s equations is the fact the divergence operator on \( \Lambda_*(M) \), viz., \( \delta \equiv \#^{-1}d\# \), can be defined by the Poincaré duality isomorphism:

\[
\# : \Lambda_4(M) \rightarrow \Lambda^{4-*}(M), \ a \mapsto i_a \varepsilon = a^{\mu...\nu} \varepsilon_{\kappa\lambda\mu\nu}. \tag{2, 6}
\]

which comes from the volume element, not Hodge duality, which requires a metric. Indeed, this is an important subtlety concerning the divergence operator in general, which is often presented as something that requires the introduction of a metric. However, one must recall that divergenceless vector fields are the infinitesimal generators of local volume-preserving diffeomorphisms, which indicates the fundamental relationship between the divergence operator and the volume element.

A second point of departure is that the role of an explicitly specified electromagnetic constitutive law is given more prominence than in most treatments of Maxwell’s equations using exterior differential forms.
In general, an electromagnetic constitutive law takes the form of an invertible fiber-preserving map:

$$\chi : \Lambda^2(M) \rightarrow \Lambda^2(M), \quad F \mapsto h = \chi(F),$$

(2, 7)

that is a diffeomorphism of the fibers in the nonlinear case and a linear isomorphism:

$$h^{\mu\nu} = \frac{1}{2} F_{\kappa\lambda} \chi^{\kappa\lambda\mu\nu}$$

(2, 8)
in the linear case. Furthermore, one expects this bundle map to cover the identity, i.e., to take a fiber of $\Lambda^2(M)$ at a given point to a fiber of $\Lambda^2(M)$ at the same point. The bivector field $h$ that corresponds to a given 2-form $F$ is referred to as its electromagnetic excitation bivector field.

If $F$ is the usual Minkowski 2-form of electromagnetic field strengths, $d$ is the exterior derivative operator on $\Lambda^*(M)$, and $J$ is the vector field of electric current (i.e., the source of the electromagnetic field) then the pre-metric Maxwell equations take the form:

$$dF = 0, \quad \delta h = J, \quad h = \chi(F).$$

(2, 9)

In local form, these are:

$$\partial_{\lambda} F_{\mu\nu} + \partial_{\mu} F_{\nu\lambda} + \partial_{\nu} F_{\lambda\mu} = 0, \quad \partial_{\mu} h^{\mu\nu} = J^{\nu}, \quad h^{\kappa\lambda} = \frac{1}{2} \chi^{\kappa\lambda\mu\nu} F_{\mu\nu}.$$ 

(2, 10)

### 3. Electromagnetic Constitutive Laws \[4, 6, 7\]

Clearly, the pre-metric Maxwell equations closely resemble the usual Maxwell equations, except that the role of the electromagnetic constitutive law has supplanted that of the Lorentzian pseudo-metric. Hence, we now briefly discuss both physical and mathematical aspects of postulating an electromagnetic constitutive law as a fundamental object.

In classical vacuum (Maxwellian) electromagnetism, $h$ is linear on the fibers, and if the Minkowski 2-form takes the local form:

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = E_i dx^0 \wedge dx^i + \frac{1}{2} \varepsilon^{ijk} B^j dx^j \wedge dx^k$$

(3, 11)

then $\chi$ takes the homogeneous isotropic form:

$$\chi(F) = D^i \partial_0 \wedge \partial_i + \frac{1}{2} \varepsilon^{ijk} H_j \partial_j \wedge \partial_k = \varepsilon_0 \delta^{ij} E_j \partial_0 \wedge \partial_i + \frac{1}{2\mu_0} \varepsilon^{ijk} B_i \partial_j \wedge \partial_k.$$ 

(3, 12)
in which the indices $i, j, k$ range over the spatial values 1, 2, 3.

The constant $\varepsilon_0$ is referred to as the electric permittivity (or dielectric constant) of the vacuum and $\mu_0$ is its magnetic permeability.

In both of the expressions for $F$ and $\chi(F)$ we have implicitly used the Euclidian spatial metric, whose components in the chosen frame are $\delta_{ij}$, and its inverse $\delta^{ij}$, to raise and lower the index of $B_i$ and $H^j$, respectively. Hence, one should carefully note that the expression for $\chi(F)$ is not actually invariant under Lorentz transformations of the local frame field $\partial_\mu$, but only under Euclidian rotation of its spatial members. What makes this intriguing is that the speed of propagation for electromagnetic waves in vacuo is derived from $\varepsilon_0$ and $\mu_0$:

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}.$$ 

(3, 13)
Hence, a fundamental assumption of special relativity – viz., that \( c \) is a constant that is independent of the choice of Lorentz frame – seems to resolve at the pre-metric level to the statement that the constants from which \( c \) is constructed can change with the choice of Lorentz frame.

One can eliminate the homogeneity restriction and allow \( \varepsilon \) and \( \mu \) to vary with position, which is essentially what one does in linear optics. In such a case, it is usually not the speed of propagation in the medium that one considers, but its index of refraction:

\[
 n(x) = \frac{c_0}{c(x)} = \sqrt{\frac{\varepsilon_0 \mu_0}{\varepsilon(x) \mu(x)}}. \tag{3, 14}
\]

Furthermore, one can drop the isotropy restriction, in which case, \( \varepsilon_0 \) and \( \mu_0 \) are replaced by \( 3 \times 3 \) matrices whose components are functions of position. This situation relates to the propagation of electromagnetic waves in crystal media, in which the speed of propagation can vary with direction, as well as position.

The case of a nonlinear \( \chi \) not only has immediate application to nonlinear optics, but also a possible application to effective QED, perhaps in the effective models for vacuum polarization, such as the Born-Infeld model. However, our immediate concern is with how one introduces Clifford algebras into the pre-metric formalism, so we shall concentrate only on the linear case.

Returning to the linear case, \( \chi \) also defines a non-degenerate bilinear form on \( \Lambda^2(M) \):

\[
 \chi(F, G) = G(\chi(F)) = \chi_{IJ} F^I G^J = \frac{1}{2} \varepsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda} G_{\mu\nu}. \tag{3, 15}
\]

This bilinear form admits a decomposition that is irreducible under the action of \( GL(6; \mathbb{R}) \) on it by congruence (viz., \( \chi \mapsto A \chi A^T \)):

\[
 \chi = (1) \chi + (2) \chi + (3) \chi, \tag{3, 16}
\]

in which:

\[
 (1) \chi = \chi - (2) \chi - (3) \chi \tag{3, 17}
\]

is called the principal part. It is symmetric and “traceless,” in the sense that it does not contain a contribution that is proportional to the volume element \( \varepsilon \).

The tensor field:

\[
 (2) \chi = \frac{1}{2} (\chi - \chi^T) \tag{3, 18}
\]

is the anti-symmetric skewon part of \( \chi \). It is associated with established physical phenomena, such as the Faraday effect, and natural optical activity \([7, 8, 9, 10]\), so it is not a purely abstract generalization to include it in \( \chi \).

The tensor field:

\[
 (3) \chi = \chi_0 (E_I, E_I) \varepsilon \tag{3, 19}
\]

is the axion part of \( \chi \), which proportional to volume element. It does not affect the propagation of electromagnetic waves, but Lindell and Sivola \([11]\) have suggested that it might still play a role in some electromagnetic media.

In the language of projective geometry, the case of a general \( \chi \) defines a correlation on the fibers of \( \Lambda^2(M) \) (more precisely, their projectivizations), namely, a linear isomorphism from each fiber of \( \Lambda^2(M) \) to its dual vector space, which is a fiber of \( \Lambda^2(M) \). A symmetric correlation is called a polarity and defines a quadratic form. An anti-symmetric correlation defines a null polarity, much like a symplectic form on an even-dimensional vector space. One can always polarize a correlation into a symmetric and an anti-symmetric part, although the individual parts do not have to both be non-degenerate.
The manner by which $\chi$ gives rise to a Lorentzian metric on $T(M)$ follows from adding certain restricting assumptions on $\chi$. Essentially, one looks for an “exterior square root” $\chi = "g \wedge g."$ since, in the case of the Hodge * isomorphism, the role of $\chi$ is played by the map whose tensor components are:

$$\frac{1}{2}(g_{\kappa\lambda}g_{\mu\nu} - g_{\kappa\mu}g_{\lambda\nu})$$

(3, 20)

Physically, the absence of birefringence is often a necessary and sufficient restricting assumption for the reduction to take place. Birefringence is an optical phenomenon that takes the form of the speed of light in a medium − hence, the index and angle of refraction − depending on the polarization direction of the light wave [8].

4. Geometry of bivectors [12]

One might say that the basic theme of PMEM is that one must make a shift of emphasis from considering the geometry of $M$ by way of a metric $g$ on $T(M)$ to considering the geometry $M$ by way of the various structures that one defines on $\Lambda^2(M)$. Hence, we now summarize some of those geometric structures. Furthermore, we restrict our scope to the manifold $\mathbb{R}^4$, which amounts to considering a single tangent space to a more nonlinear manifold.

The volume element $\varepsilon$ defines a real scalar product of signature type $(3, 3)$ on $\Lambda_2(\mathbb{R}^4)$:

$$< F, G > \equiv \varepsilon(F \wedge G).$$

(4, 21)

The isotropic bivectors of this scalar product − i.e., $< F, F > = 0$ − define the Klein hypersurface in $\Lambda_2(\mathbb{R}^4)$. A bivector $F$ is isotropic iff it is decomposable, where $F$ decomposable iff $F = a \wedge b$ for some $a, b \in \mathbb{R}^4$.

When we express $F$ as $E^i E_i + B^i \ast E_i$, we find that:

$$< F, F > = E^2 - B^2.$$  

(4, 22)

We define the isomorphism $\kappa: \Lambda_2(\mathbb{R}^4) \to \Lambda_2(\mathbb{R}^4)$, $\kappa = \# \chi$, which we assume to be proportional to a complex structure $\ast$ on $\Lambda_2(\mathbb{R}^4)$:

$$\kappa^2 = -\lambda^2 I, \ast \equiv \lambda^{-1} \kappa.$$  

(4, 23)

Hence, by definition, $\ast^2 = -I$, which is also a property of the Hodge $\ast$ when it acts on 2-forms on a four-dimensional Lorentz manifold. However, in the present case, we have not defined the isomorphism $\Lambda_1(\mathbb{R}^4) g \to \Lambda_3(\mathbb{R}^4)$ that Hodge duality defines. Interestingly, the remaining isomorphisms $\Lambda_0(\mathbb{R}^4) \to \Lambda_4(\mathbb{R}^4)$ and $\Lambda_0(\mathbb{R}^4) \to \Lambda_4(\mathbb{R}^4)$ can still be defined in the absence of a metric. One assigns the real number $a$ to the 4-vector $a \epsilon$, and the 4-vector $a \epsilon$ to the real number $-a$, respectively. Hence, we have defined Hodge duality only on the even subalgebra of $\Lambda_4(\mathbb{R}^4)$, namely:

$$\Lambda_4(\mathbb{R}^4) = \Lambda_0(\mathbb{R}^4) \oplus \Lambda_2(\mathbb{R}^4) \oplus \Lambda_4(\mathbb{R}^4) = \mathbb{R} \oplus \Lambda_2(\mathbb{R}^4) \oplus \mathbb{C}.$$  

(4, 24)

In order to complete the isomorphism of $\Lambda_2(\mathbb{R}^4)$ with $\mathbb{C}^3$, we must give it a complex scalar multiplication, which we define by:

$$(\alpha + i\beta)F \equiv \alpha F + \beta \ast F$$  

(4, 25)

A (non-canonical) $\mathbb{C}$-linear isomorphism from $\Lambda_2(\mathbb{R}^4)$ to $\mathbb{C}^3$ is then defined by a choice of complex 3-frame on $\Lambda_2(\mathbb{R}^4)$. That is, if:

$$F = E^i E_i + B^i \ast E_i = (E^i + iB^i)E_i,$$  

(4, 26)
as above, and the complex 3-frame $E_i$ on $\Lambda_2(\mathbb{R}^4)$ corresponds to the canonical
3-frame on $\mathbb{C}^3$ then the vector in $\mathbb{C}^3$ that corresponds to $F$ has the complex
components $E^i + iB^i$.

Since any choice of 3-frame, such as $E_i$, defines a direct sum decomposition
of $\Lambda_2(\mathbb{R}^4)$ into essentially a “real” subspace and an “imaginary” subspace, this
decomposition is clearly not unique; in fact, it closely analogous to a 3+1 decom-
position of $\mathbb{R}^4$, although we refer the curious to a more detailed treatment [12] by
the author.

Note further that we can extend the $\mathbb{C}$-linear isomorphism of $\Lambda_2(\mathbb{R}^4)$ with
$\mathbb{C}^3$ to a $\mathbb{C}$-linear isomorphism of the even subalgebra of $\Lambda_2(\mathbb{R}^4)$ with $\mathbb{C}^4$, by
simply regarding the $\mathbb{R} \oplus \mathbb{R} \epsilon$ part as corresponding to $\mathbb{C}$ by the isomorphism
$a + * b \mapsto a + ib$.

Since we now regard $*$ as the fundamental object on $\Lambda_2(\mathbb{R}^4)$, we should con-
sider the subgroup of $GL(6; \mathbb{R})$ that preserves $*$. One finds that this subgroup
is isomorphic to $GL(3; \mathbb{C})$. Its elements can be expressed as either $3 \times 3$ com-
plex invertible matrices or as $3+3$-partitioned $6 \times 6$ real invertible matrices by the
association:

$$A + iB \leftrightarrow \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad (4, 27)$$
in which we are representing the matrix of $*$ by:

$$* = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \quad (4, 28)$$

Although, as noted above, there is good reason to keep the isomorphism $\chi$ more
general, nevertheless, when $\chi$ is symmetric, the $*$-isomorphism defines another real
scalar product on $\Lambda_2(\mathbb{R}^4)$:

$$(\mathbf{F}, \mathbf{G}) = \langle \mathbf{F}, \mathbf{G} \rangle.$$

Relative to the frame that we have been habitually using we have:

$$(\mathbf{F}, \mathbf{F}) = 2E \cdot B. \quad (4, 30)$$

Combining both of the aforementioned real scalar products defines a Euclidian
structure on $\Lambda_2(\mathbb{R}^4)$ by way of

$$\langle \mathbf{F}, \mathbf{G} \rangle = \langle \mathbf{F}, \mathbf{G} \rangle + i < \mathbf{F}, \mathbf{G} > . \quad (4, 31)$$

The subgroup of $GL(3; \mathbb{C})$ that preserves this scalar product is isomorphic
to $O(3; \mathbb{C})$. The introduction of a unit-volume element on $\Lambda_2(\mathbb{R}^4)$, which is
straightforward, then defines a reduction to $SO(3; \mathbb{C})$, which is isomorphic to
$SO_0(3, 1)$. This isomorphism really embodies the essence of the reduction from
the geometry of $\Lambda_2(\mathbb{R}^4)$ given the $*$ isomorphism, which is essentially the geometry
of $\mathbb{C}^3$, to the geometry of $\mathbb{R}^4$ with the Lorentz scalar product.

5. Bivector Clifford algebras

In order to define a Clifford algebra – whether real or complex – one must have
a vector space with an orthogonal structure – i.e., a scalar product $t$ defined on it. The vector space in question at the moment is $\Lambda_2(\mathbb{R}^4)$, which we have just
observed can be regarded as having six real dimensions or three complex dimensions;
we have also defined a scalar product in either case.
When $\Lambda_2(\mathbb{R}^4)$ is regarded as a real vector space, one can use the real scalar product $\langle \ldots, \ldots \rangle$, whose orthogonal group has $SO(3, 3)$ as its identity component. This 64-dimensional real Clifford algebra has been dealt with by Harne tt [13]. It is representable by the matrix algebra $\text{End}(\mathbb{R}^4 \oplus \mathbb{R}^4^*)$. These matrices generalize the Dirac matrices that one uses to represent the Clifford algebra of Minkowski space. However, when one gives $\Lambda_2(\mathbb{R}^4)$ a complex structure, the role of $SO(3, 3)$ does not seem as physically meaningful as that of $SO(3, \mathbb{C})$. Hence, we shall not elaborate on this example at the moment.

By contrast, when $\Lambda_2(\mathbb{R}^4)$ is regarded as a complex vector space, we can use the complex orthogonal structure $\langle \ldots, \ldots \rangle_\mathbb{C}$, whose orthogonal group has the identity component $SO(3, \mathbb{C})$. The resulting complex eight- (real sixteen-) dimensional Clifford algebra is, by definition, $\mathcal{C}(\mathbb{R}^3, \delta)$, which is $\mathbb{R}$-isomorphic to $\mathcal{C}(\mathbb{R}^4, \eta)$ — i.e., the Clifford algebra of Minkowski space — and $\mathbb{C}$-isomorphic to the matrix algebra $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. Consequently, these matrices are representable by the usual Dirac matrices.

One notes that a key difference between using the complex Clifford algebra $\mathcal{C}(\mathbb{C}^3, \delta)$ and the $\mathbb{R}$-isomorphic real Clifford algebra $\mathcal{C}(\mathbb{R}^4, \eta)$ is that the most natural spinor fields that are associated with the real form take their values in $\mathbb{C}^4$, but the $\mathbb{C}$-isomorphism of $\Lambda_2(\mathbb{R}^4)$ with $\mathbb{C}^3$ suggests that the most natural spinor fields, at least from the standpoint of the electromagnetic field strength 2-form, are fields that take their values in $\mathbb{C}^3$; these fields are sometimes referred to as “three-component spinors [14, 15],” and seem to be fundamental in the complex formulation of general relativity. Like the Dirac spinors, they still carry a representation of the proper orthochronous Lorentz group, by way of its isomorphism with either an $SO(3; \mathbb{C})$ subgroup of $GL(3; \mathbb{C})$ or an $SL(2; \mathbb{C})$ subgroup.

Although, as just observed, the representation of electromagnetic field 2-forms by 3-spinor fields is immediate when one chooses a complex 3-frame for $\Lambda_2(\mathbb{R}^4)$, nevertheless, the use of 4-spinors is more related to the representation of the wave function for the source current $J$. In particular, the two representations of the Lorentz group have different weights (i.e., spins). Hence, one must still determine the manner by which one represents $J$ in terms of things derivable from complex 3-spinors.

Since Dirac 4-spinors are really Cartesian products of 2-spinors, corresponding to the isomorphism of $\mathcal{C}(\mathbb{R}^3, \eta)$ with $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, one might investigate the representation of 4-spinors as tensor products of 2-spinors; although the space of such tensor products is also four-complex-dimensional, nevertheless, the physical interpretation of such a construction would be that they are bound states of more elementary things. Furthermore, as we saw above, we have a $\mathbb{C}$-linear isomorphism of $\mathbb{C}^4$ with the even subalgebra of $\Lambda_2(\mathbb{R}^4)$. However, this suggests a natural decomposition of $\mathbb{C}^4$ into $\mathbb{C} \oplus \mathbb{C}^3$, whereas the more established decompositions of Dirac spinors are into pairs of 2-spinors, which suggests $\mathbb{C}^2 \oplus \mathbb{C}^2$.

Finally, along the same lines, one notes that if one defines the complex conjugation operator on $\Lambda_2(\mathbb{R}^4)$ that corresponds to the complex conjugation operator on $\mathbb{C}^3$ by way of the chosen isomorphism, then the complex orthogonal structure also defines a Hermitian structure, i.e., a Hermitian inner product, by way of:

$$\langle \mathbf{F}, \mathbf{G} \rangle = \langle \mathbf{F}, \bar{\mathbf{G}} \rangle_{\mathbb{R}}.$$  \hspace{1cm} (5, 32)

Consequently, one can reduce the $SO(3; \mathbb{C})$ subgroup of $SL(3; \mathbb{C})$ to $SU(3)$. Although this group represents the color gauge symmetry of the strong interaction
and the components of a complex 3-spinor field can represent the $u$, $d$, and $s$ quarks, clearly, the details of how this mathematical coincidence relates to physical theory needs to be developed further. One hint is that the quadratic form that is associated with the Hermitian structure takes the 1+3 form:

$$ (\mathbf{F}, \mathbf{F}) = E^2 + B^2, $$

which is proportional to the electromagnetic field Hamiltonian. Hence, unitary transformations of $\mathbb{C}^3$, given the Hermitian structure just defined, would preserve the field Hamiltonian. Whether this leads to a more direct route to the unification of the theories of the electromagnetic and strong interactions is a worthy point to ponder.

6. Role of projective geometry in PMEM

So far, we have considered how the principal part of the linear electromagnetic constitutive tensor $\chi$ defines an orthogonal structure on the space of bivectors over $\mathbb{R}^4$. In light of the fact that apparently the existence of a non-vanishing skewon contribution to the electromagnetic constitutive law is not a physical triviality, apparently a complete analysis of pre-metric electromagnetism must confront the role of that constitutive law as a correlation, not merely a metric. Hence, one conjectures that the most appropriate geometrical context for pre-metric electromagnetism might be projective geometry – indeed, complex projective geometry.

One suggestive result in this direction was derived by the author in an analysis of the symmetries of pre-metric Maxwell system [5]. Just as the Maxwell equations in metric form admitted the conformal Lorentz group as a symmetry group that acts on their space of solutions, the pre-metric Maxwell equations seem to give a more ambiguous result: that the symmetry group is either the affine group for $\mathbb{R}^4$, the group of projective transformations of $\mathbb{RP}^4$, the group of diffeomorphisms of $\mathbb{R}^4$ whose volume varies as $t^3$, or the full diffeomorphism group. One’s intuition is that it is the group of projective transformations that is the most physically meaningful expansion from the conformal Lorentz group that Bateman and Cunningham established. However, the example of the expansion of a spherical wavefront shows that the transformations whose infinitesimal generators have constant divergence might also have a certain physical appeal.

As for the suggestion that it is complex projective geometry that we are concerned with, note that a duality plane in $\Lambda_2(\mathbb{R}^4)$ – viz., a space spanned by some bivector $\mathbf{F}$ and $*\mathbf{F}$ – corresponds with a complex line in $\mathbb{C}^3$ under a $\mathbb{C}$-isomorphism of $\Lambda_2(\mathbb{R}^4)$ with $\mathbb{C}^3$. Hence, the space of duality planes in $\Lambda_2(\mathbb{R}^4)$ is projectively equivalent to $\mathbb{CP}^2$. Physically, a duality plane in $\Lambda_2(\mathbb{R}^4)$ is also related to the polarization plane in $\mathbb{R}^4$ that is defined by $\mathbf{F}$ when it is isotropic.

The matter of what “geometric algebra” might represent projective geometry is possibly resolved when one remembers that Grassmann’s intent in defining what is now called the exterior – or Grassmann – algebra of a vector space was precisely that of representing the incidence relations between linear subspaces in terms of algebraic operators on elements of a vector space. Of course, the more physically fundamental question is: what happens when one restricts oneself to projective transformations that preserve a given correlation, i.e., linear electromagnetic constitutive law $\chi$? This suggests an expansion of the Lorentz group to a more complicated linear algebraic group whose algebraic structure would depend upon the properties of $\chi$. 
This expansion of the scope of the spinors in electromagnetism from Dirac spinors to projective spinors begs some further questions:

Insofar as conventional spinors are wavefunctions, what do spinor fields represent in the absence of the basis for defining waves, viz., light cones? Of course, the moving charge distributions that generate electromagnetic fields are always massive, so one might still be able to define the wavefunctions of massive matter, even if one could not define photons.

As pointed out in the introduction, Dirac spinors carry a representation of the proper orthochronous Lorentz group $SO_0(1,3)$, by way of its universal covering group $SL(2; \mathbb{C})$. Which group might one expect projective spinors to carry a representation of: $SL(5; \mathbb{R})$, which defines the projective transformations of $RP^4$, or $SL(3; \mathbb{C})$, which defines the projective transformations of $CP^2$? From the preceding discussions, one suspects that it is the complex projective group that plays the fundamental role, but one should recall that the essence of the projective relativity [16] that grew out of the Kaluza-Klein program was the embedding of the spacetime manifold in $\mathbb{R}^5$ in a projective manner. Hence, the real projective group might also be of interest.

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