From $U(1)$ Maxwell Chern-Simons to Azbel-Hofstadter: Testing Magnetic Monopoles and Gravity to $\sim 10^{-15} \text{m}$?

P. Castelo Ferreira
Dep. de Mat. – I. S. T., Av. Rovisco Pais, 1049-001 Lisboa, Portugal
P. A. C. T. – University of Sussex, Falmer, Brighton BN1 9QJ, U.K.

It is built a map between an Abelian Topological Quantum Field Theory, $2+1$D compact $U(1)$ gauge Maxwell Chern-Simons Theory and the nonrelativistic quantum mechanics Azbel-Hofstadter model of Bloch electrons. The $U_q(sl_2)$ quantum group and the magnetic translations group of the Azbel-Hofstadter model correspond to discretized subgroups of $U(1)$ with linear gauge parameters. The magnetic monopole confining and condensate phases in the Topological Quantum Field Theory are identified with the extended (energy bands) and localized (gaps) phases of the Bloch electron. The magnetic monopole condensate is associated, at the nonrelativistic level, with gravitational white holes due to deformed classical gauge fields. These gravitational solutions render the existence of finite energy pure magnetic monopoles possible. This mechanism constitutes a dynamical symmetry breaking which regularizes the solutions on those localized phases allowing physical solutions of the Shr"odinger equation which are chains of electron filaments connecting several monopole-white holes. To test these results would be necessary a strong external magnetic field $B \sim 5 \text{T}$ at low temperature $T < 1 \text{K}$. To be accomplished, it would test the existence of magnetic monopoles and classical gravity to a scale of $\sim 10^{-15} \text{meters}$, the dimension of the monopole-white hole. A proper discussion of such experiment is out of the scope of this theoretical work.

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*current address: CENTRA - Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
†Electronic address: pcastelo@catastropha.org
I. DISCUSSION

A. Introduction and Results

The motivations to this work are three fold. First magnetic monopoles have not been observed experimentally. Although due to the quantization of electrical charge, they should exist. Secondly, testing the theoretical results obtained it would be necessary a strong magnetic field. In order to test the theoretical results obtained it would be necessary to test the existence of these configurations are in this work, and based in the results of [10], the conditions for the existence of these configurations are in principle be attained as in [10] where that symmetry is broken. On compact Maxwell Chern-Simons theories there exist vortexes (or magnetic monopoles) which may be in a Confined or Condensate phase. Given the map build in this work, the Confinement phase and the Condensate phase are related to the condensation and confinement of magnetic charge, meaning that magnetic charge can only be physically observed on the condensate phase. In the Confinement phase only magnetic dipoles are observed.

In Section IV the configurations on both phases are analysed. In the Confinement phase the gauge symmetry is present while in the Condensate phase the gauge symmetry is broken. This phase transition is shown to correspond to dynamical symmetry breaking. The symmetry breaking is achieved by considering a particular magnetic monopole solution of the gauge fields together with a nontrivial metric configuration which constitutes the white hole. One crucial consequence of this construction is that the association of the white hole to pure magnetic monopoles renders the energy of the configuration finite. In the Confinement phase, gravity plays no special role, we are in a quantum regime where the solutions of the Shr"odinger equation are the usual Landau levels in the presence of a magnetic field while in the Condensate phase we obtain electron

\[ U(1) \text{ Maxwell Chern-Simons Quantum Field Theory and the Azbel-Hofstadter } \]
ments attaching several magnetic monopole-white holes which we call in this work chains. We refer the reader to the work for $SU(2)$ BPS monopole chains.

B. Open Issues

The most important point to further explore seems to be the relation between the gravity and gauge sectors of the theory. The configurations presented in this work are given at ansatz level without a full theoretical justification. A semi-classical treatment is given in [4, 5]. However the solutions of these works do not possess similar characteristics to our ansatze, namely the charge and energy of the configurations are not finite. Also following [45, 46, 47] the author studied classical solutions for gauge Chern-Simons and a gravitational scalar field. However the results are not very promising. For the magnetic case [50], although horizons do exist, the usual Maxwell charges diverge. Only the Page charges are finite [51].

It is also important to stress that the usual $2+1$ gravity has no degrees of freedom. This problem may be solved by considering noncommutative gravity [52, 53, 54], but it would be interesting to study the solutions of the EOM corresponding to a full noncommutative metric. The main draw back is that, in this case, we are dealing with complex gravity.

The values of temperature ($< 1K$) and magnitude of the magnetic field (5T) to achieve a condensate phase (Localized phase), are based on the experimental results of [10] and on the usual definition of strong magnetic field. Nevertheless we did not study the precise values for the phase transition. Based on the Coulomb Gas approach, Kogan and Kovner [36] derived that the phase transition corresponds to the value of the CS coefficient $1/\pi$ (in natural units $\hbar = 1$). In the framework presented here it corresponds to a relatively weak magnetic field $B = 0.07435\ T$. The conditions for the validity of the Coulomb Gas approach may not be appropriated for a low matter density experiment in the laboratory but certainly it could be appropriate for high matter density regions of our universe.

Once we are talking about magnetic charges, it would not be polite, not to discuss electromagnetic duality (see [59] and references therein). Taking the usual quantization relation $eg = 2\pi\hbar\ N$, between pure magnetic ($g$) and electric charges ($e$) we obtain from equation (IV.15) that $Q_{mon} := B/4 = 2\pi\hbar\ N$. This simply implies that the system will react to values of the magnetic field when it raises (or decreases) over some threshold $B = 8\pi\hbar\ N$.

Also it remains to explain what are the dynamics of the mechanisms presented here, namely how the white holes emerge dynamical, is it a mechanism similar to the formation of black holes or is it a completely different mechanism?

Finally the relation with $U_q(sl_2)$ and area preserving diffeomorphisms suggests some relation with the Quantum Hall Effect. Note however that the solutions presented here are in the Gaps of the nonrelativistic quantum theory. They are not related to the Landau level solutions for the energy Bands. They correspond to Localized phases which (so far as the author knows) have only been detected recently on the experiments described and conducted by the authors of reference [10].

II. FROM 3D TOPOLOGICAL MASSIVE GAUGE THEORIES TO ELECTRONS ON MAGNETIC FIELDS

In this section is considered a compact $U(1)$ topological massive gauge theory, Abelian Maxwell Chern-Simons. This theory will have as a particular limit an effective description of some other system, electrons moving on perpendicular magnetic fields. The coupling of the topological term $k$ will be interpreted as an effective physical quantity, the external magnetic field.

A. TMGT, 3D Maxwell Chern-Simons

Let us start with a 3D Abelian Compact Maxwell Chern-Simons Gauge Theory,

$$S_{TMGT} = \frac{\mu_0}{V_3} \int_M d^3x \sqrt{-g} \left[ -\frac{1}{4\gamma^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{8\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + A_\mu J^\mu \right]$$ (II.1)

where $F=dA$ is the usual connection of the gauge field $A$ and a generic external 3D current $J^\mu$ for the gauge field.
field was introduced. The coupling constants are introduced on a nonstandard way in order to simplify the phenomenology of the nonrelativistic limit studied in this work. \( M \) is some Minkowski manifold allowing a \( 2 + 1 \) splitting and having a topology \( M = \Sigma \times R \), where \( \Sigma \) is a 2D compact Riemann surface (with \( \partial \Sigma = 0 \)) and \( R \) some finite interval which is going to play the role of time, the metric is considered to be \( 2 + 1 \) decomposable (see for example [16])

\[
ds^2 = -dt^2 + h_{ij}dx^i dx^j \tag{II.2}
\]

and therefore the factor containing the determinant of the metric reads \( \sqrt{-g} = \sqrt{h} \). The antisymmetric tensor \( \epsilon^{\mu\nu\lambda} \) is defined covariantly as

\[
\epsilon^{\mu\nu\lambda} = \frac{\hat{\epsilon}^{\mu\nu\lambda}}{\sqrt{h}}, \tag{II.3}
\]

where \( \hat{\epsilon}^{\mu\nu\lambda} \) is the usual numerical antisymmetric matrix with \( \epsilon^{012} = 1 \).

\[
H_{TMGT} = \int_{\Sigma} d^2 x \sqrt{h} \left[ -A_0 \left( \partial_i \pi^i + \frac{k}{8\pi} \epsilon^{ij} \partial_j A_j - J^0 \right) \right] + \mu_0 \frac{\alpha_T}{2 V_3} \int_{\Sigma} d^2 x \sqrt{h} \left[ \frac{1}{4\gamma^2} (\epsilon_{ij} F^{ij})^2 + c^2 \gamma^2 (\pi_i - \frac{k}{8\pi} \epsilon_i^k A_k) \left( \pi^i - \frac{k}{8\pi} \epsilon^{ij} A_j \right) - A_i J^i \right], \tag{II.4}
\]

where \( \pi^i \) are the canonical momenta conjugated to \( A_i \)

\[
\pi^i = -\frac{1}{c^2 \gamma^2} F^{0i} + \frac{k}{8\pi} \epsilon^{ij} A_j \tag{II.5}
\]

and the \( A_0 \) was taken to be a Lagrange multiplier imposing the gauss law

\[
\partial_i E^i + \frac{k}{4\pi} b = J_0 \tag{II.6}
\]

\( J_0 \) is some external charge distribution and the electric and magnetic fields are defined as

\[
E^i = \pi^i - \frac{k}{8\pi} \epsilon^{ij} A_j = -\frac{1}{c^2 \gamma^2} F^{0i}, \tag{II.7}
\]

\[
b = \frac{1}{2} \epsilon^{ij} F_{ij} = F_{12}. \tag{II.7}
\]

Note that the 2D antisymmetric tensor is actually induced from the \( 2 + 1 \) dimensional one as \( \epsilon^{ij} = \epsilon^{0ij} \).

\( V_3 \) stands for a unit 3D spatial volume and has units of length cube \((m^3)\), \( \alpha_T \) stands for an effective constant which relates the \( 2 + 1 \) physical system to the \( 3 + 1 \) physics. In simple terms is related to the thickness of the system on the perpendicular direction, say \( x^3 = z \), it has therefore units of length \((m)\). The physical meaning of \( \gamma \) and \( k \) will be addressed later on.

These theories are usually named Topological Massive Gauge Theories (TMGT) [11, 12], topological due to the topological character of the CS term (it is a topological invariant) and massive since the gauge boson \( A \) acquires a topological mass due to the coexistence of both Maxwell and CS terms \( M = k\gamma^2/4\pi \). This can be checked by the equations of motion for \( A \). Taking its curl and rewriting it in terms of the dual field strength \((*F')^\mu = \epsilon^{\mu\nu\lambda} F_{\nu\lambda}/2 \) we obtain \((\partial^2 - M^2)(*F')^\mu = 0\).

The Hamiltonian of such a system is easily computed to be (for further details see for example [16, 19, 20]),

\[
\text{Upon quantization the commutation relations are}
\]

\[
\left[ \pi^i(x'), A^j(x') \right] = -i \hbar \delta^{ij} \delta^{(2)}(x - x') \tag{II.8}
\]

and the commutation relations between the fields are computed to be

\[
\left[ E^i(x), E^j(x') \right] = -i \frac{\hbar}{\mu_0} \frac{k}{4\pi} \epsilon^{ij} \delta^{(2)}(x - x') ,
\]

\[
\left[ E^i(x), b(x') \right] = -i \frac{\hbar}{\mu_0} \epsilon^{ij} \partial_j \delta^{(2)}(x - x') ,
\]

where we assume the 2d delta functions to be normalized with a unit factor of area \([V_2] = m^2\).

The Hamiltonian (in the absence of external sources) depends only on the coordinate combinations \( \pi^i - k\epsilon^{ij} A_j/8\pi \) such that

\[
H = H \left( \pi^1 - \frac{k}{8\pi} A_2, \pi^2 + \frac{k}{8\pi} A_1 \right). \tag{II.10}
\]

Being so, the most generic operator \( O \) which commutes with the Hamiltonian, must depend on the coordinate
combinations $\pi^i + k\epsilon^{ij}A_j/8\pi$ such that

$$\left[H\left(\pi^1 - \frac{k}{8\pi} A_2, \pi^2 + \frac{k}{8\pi} A_1\right), \right] = 0$$

This fact is simply proved by using the commutation relations (II.11).

The most usual and known example of such an operator is, of course, the gauge transformation operator

$$V = \exp \left\{ i \frac{k}{8\pi} A_2 \right\} = \exp \left\{ i \frac{\mu_0 V_3}{\hbar} \right\} \frac{1}{\sqrt{\hbar}} \Lambda(x) \left( \partial_i E^i + \frac{k}{4\pi} b - J_0 \right)$$

In this case the generator is simply a representation of $U(1)$. Note that, accordingly, the gauss law

$$\partial_i (\pi^i + k\epsilon^{ij}A_j/8\pi) = 0$$

We consider from now on that the measure in the 2-dimensional integral is defined covariantly. For the sake of compactness of the equations we do not write the area factor either, so the factor $\sqrt{\hbar} \alpha_T/V_3$ is implicit on all the 2d integration over $\Sigma$.

**B. Compact TMGT**

Being compact means that large gauge transformations are allowed. That is that the fields are defined up to close shifts around the holonomic cycles of the manifold $\Sigma$. The generators of this cycles are none other than the Wilson lines

$$W_{\alpha} = \exp \left\{ i \oint_{\alpha} A_i dx^i \right\}$$

for some non contractible close cycle $\alpha$. Also note that these objects are gauge invariant and constitute a basis for the topological wave functions of the theory as extensively studied [14, 15] (see also [17, 20] for more generic geometries).

Therefore $\Lambda$ is a compact parameter, say in the interval $[0, 2\pi]$ and the boundary conditions on the gauge fields need to be compatible with the identification

$$\Lambda \cong \Lambda + 2\pi$$

which must be invariant under a shift $\Lambda \rightarrow \Lambda + 2\pi$ due to the compactness of the gauge group. Then it imposes the group (dimensionless) charge quantization condition

$$q = s_1 + \frac{\mu_0}{\hbar} \frac{k}{8\pi} \int_{\Sigma} b = s_1 + \frac{\mu_0}{\hbar} \frac{k}{4} s_2$$

for some integer $s_1$ and $s_2$. Note that $s_1$ stands for the winding of the gauge group, i.e. how many times the gauge field winds on $\Sigma$ when a shift $\Lambda \rightarrow \Lambda + 2\pi$ is considered.

In the case of the compact gauge theory there are local operators which create magnetic vortexes creating a charge of flux $2\pi N$ for some integer $N$ [32, 33, 34, 35]. The physical Hilbert space of the compact theory is obtained by constraining the Hilbert space of the noncompact theory. This is done by demanding invariance of the physical states under the action of any combination (product) of those vortex operators. They are obtained integrating by parts $U$(II.12)

$$V(\Lambda) = \exp \left\{ i \frac{\mu_0}{\hbar} \int_{\Sigma} \partial_\lambda \left[ \pi^i + \frac{k}{8\pi} \epsilon^{ij} A_j - J_0 \right] \Lambda \right\}.$$  

Interpreting $\Lambda$ as an angle in the 2D plane and using the identity $\partial_\lambda (\pi^i + k\epsilon^{ij}A_j/8\pi) = -\epsilon_i^j \partial_j \ln |x|$ (from the the Cauchy-Riemann equations) one obtains

$$V(x) = \exp \left\{ -i \frac{\mu_0}{\hbar} \int_{\Sigma} \left[ E^i + \frac{k}{4\pi} \epsilon_i^j A_j \right] \right\} \times$$

$$\epsilon_i^k \partial_k \ln |x - x_0| - \Lambda(x - x_0) J_0$$

This operator creates a vortex that generates magnetic flux. Its commutator with the magnetic field is

$$[\hat{b}(x), V N(x)] = 2\pi N V N(x) \delta^{(2)}(x - x_0)$$

where the 2D identity $\partial^2 \ln |x| = 2\pi \delta(x)$ was used.

For compactness of the equations, in the following subsection, we use the notation $\Delta x_0 = x - x_0$.

**C. Vortex, Monopoles and Field Decomposition**

These operators can be also interpreted as generators of discontinuities or cuts from 0 to $\infty$ in the complex plane that generate a change on the charge (or tunnelling effect between states with the same energy and different charges). On other words they are associated with singular gauge transformations or monopole-instantons processes.
Note that two operators \( V(x) \) and \( V(x') \) do not commute, generally

\[
V(\Lambda, x_0) V(\Lambda', x'_0) = \exp \left\{ -\frac{i k}{4 \pi \hbar} \int_{\Sigma} \partial \Lambda_0 (\Delta x_0) \times \partial \Lambda'_0 (\Delta x'_0) \right\} V(x'_0) V(x_0)
\]

where the integral

\[
\int_{\Sigma} \int_{\Sigma} \varepsilon^{ij} \partial_i \Lambda_0 (\Delta x_0) \partial_j \Lambda'_0 (\Delta x'_0) \delta^{(2)} (\Delta x_0 - \Delta x'_0) = \int_{\Sigma} \varepsilon^{ij} \partial_i \Lambda_0 (\Delta x_0) \partial_j \Lambda'_0 (\Delta x_0)
\]

was considered and the explicit dependence on \( \Lambda \) restored.

Also they have the property

\[
V(\Lambda, x_0) V(\Lambda', x'_0) = \exp \left\{ -\frac{i k}{2 \pi \hbar} \int_{\Sigma} \partial \Lambda_0 (\Delta x_0) \times \partial \Lambda'_0 (\Delta x'_0) \right\} V(\Lambda + \Lambda', x_0).
\]

So this operators indeed form a group, in this case is simply \( U(1) \) as we already know.

For future use consider the decomposition of the spatial part of the gauge field \( A_1 \) into its longitudinal \( a_1 = \{ a_{1i} \} \) and transverse \( a_2 = \{ a_{2i} \} \) components

\[
A = a_1 + a_2
\]

such that by definition

\[
\begin{align*}
\nabla \cdot a_1 &\neq 0 \\
\nabla \times a_1 &= 0 \\
\n\nabla \cdot a_2 &= 0 \\
\n\nabla \times a_2 &\neq 0
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
\nabla \cdot A = \nabla \cdot a_1 \\
\n\nabla \times A = \nabla \times a_2
\end{cases}
\]

With this decomposition the magnetic field \( b \) depends uniquely on the field \( a_2 \). From the perspective of non-trivial magnetic configurations this fact can also be interpreted as \( a_2 \) containing the singular part of the gauge field and \( a_1 \) the regular part. Here singular means that it includes the discontinuities (singularities) that generate the magnetic charge.

Note that once we introduce an external source, it will reduce the symmetries of the theory. Nevertheless, depending on the specific configurations it will maintain some subgroup of the full gauge group. We will use this fact in the following section. A geometrical interpretation is that a local charge insertion (or more generally a current insertion) may be interpreted as a vertex insertion on the manifold \( \Sigma \), this will clearly change the topology of the manifold, in particular its holonomy group. Note that by each insertion we create a new non contractible loop around that insertion. Due to the massiveness of the photon, its perturbative effect decays exponentially although the topology has effectively being modified. This matter will reduce the gauge symmetry group of the theory since physically it is creating an effective potential.

In the case of a \( x y \) periodic lattice the \( U(1) \) gauge symmetry will be reduced to some discrete subgroup of \( U(1) \) as it will be argued below.

### D. Electrons on a Magnetic Field

Consider a particular longitudinal gauge field \( a_1 = \mu \), such that

\[
A_i = \frac{4 \pi \varepsilon_i}{\mu_0} \mu_0
\]

\[
1 = |\nabla \times a| \quad (\text{II.25})
\]

\[
0 = \nabla \cdot a.
\]

The second condition is simply a choice of normalisation for the field \( a \) and will be shown to be compatible with both the Landau and symmetric gauge. As argued above the magnetic field \( b \) is null for longitudinal fields

\[
b = \frac{1}{2} \varepsilon_{ij} F^{ij} = F_{12} = \partial_i a^i = 0 \quad (\text{II.26})
\]

Rewrite now (II.5) as the simplified Hamiltonian

\[
H_a = \frac{\hbar c^2 \varepsilon_i}{2} \left( \pi + \frac{k}{2 a} \right)^2 + V \quad (\text{II.27})
\]

with the effective potential

\[
V = \frac{4 \pi}{\mu_0} \left( a_i j^i + \frac{1}{e^2} \mu_0 \rho^0 \right), \quad (\text{II.28})
\]

where \( j^i = \epsilon_j^i J^j \) and \( \rho^0 \) are some effective current and charge distributions.

Making the identifications

\[
k = \frac{2 e B}{\mu_0},
\]

\[
\gamma = \frac{1}{e^2 m_e},
\]

where \( e \) and \( m_e \) stand for the physical charge and mass of the electron and \( \mu_0 \) is the magnetic constant.

The Hamiltonian is easily rewritten as

\[
H_B = \frac{1}{2m_e} \left( p + e B a \right)^2 + V. \quad (\text{II.30})
\]
It can be recognised as the Hamiltonian for electrons on a potential under an external perpendicular uniform magnetic field $B$. The momenta $\mathbf{p}$ stands for the nonrelativistic momenta and the original Chern-Simons coefficient $k$ is, in this way proportional to the external magnetic field. This means that the external magnetic field is related with the topological character of the Quantum Field Theory.

The coupling constants on the original TQFT were introduced in a not so standard way to achieve the correct phenomenology of the model which will be addressed in the last section of the paper. Their units on SI are

$$[k] = A^2 \text{ m}^{-1} \text{ s},$$

$$[\gamma^2] = A^{-2} \text{ m}^{-1},$$

With this convention the gauge field has units of length $[a] = \text{m}$.

The external magnetic field $B$ and the internal one $b$ should not be confused. The first one is encoded in the Chern-Simons coefficient and is imposed externally to the system. The second one is due to non trivial configurations of the transverse gauge fields. Also it is important to stress that the $b$ field is dimensionless, its dimension-full counterpart can be taken from the expression

$$\gamma = \frac{B}{4\pi b},$$

for the charges on the theory such that

$$\tilde{b} = \frac{B}{4\pi b}.$$  \hspace{1cm} (II.31)

The $2 + 1D$ magnetic field $\tilde{b}$ spreads over the thickness of the system on the third spatial dimension, in here is assumed that effectively it spreads uniformly over the distance $\alpha_T$ and is null away from the planar system where $\alpha_T$ stands for the thickness of the system on the third spatial direction (perpendicular to the $x \times y$ plane)

The vortex operator in this effective limit is defined as

$$V_\lambda = \exp \left\{ \frac{i}{\hbar} \partial_\lambda \left( \mathbf{p}^i + e B a^i \right) \right\}$$

up to a multiplicative constant due to the potential which we can include in the normalization of the operator.

Concerning the nonrelativistic quantum problem, the symmetric and Landau gauge correspond to

$$a_s = \left( \begin{array}{c} \frac{y}{2} \ y \end{array} \right),$$

$$a_L = (0, x)$$

respectively and both are related by a gauge transformation $\lambda' = xy/2$, such that $a_s = a_L + d\lambda'$.

We considering the effective potential to be periodic with unit cell $\mathbf{d} = (dx, dy)$. The previous local $U(1)$ gauge transformations operators $V_\lambda$ are therefore identified with lattice translations operators. For the case of the symmetric gauge, with $\lambda = \mathbf{d} \cdot \mathbf{r}$, they correspond to magnetic translation operators while for the Landau gauge they correspond to the generators of $U_q(sl_2)$ as will be computed in the next subsection. In this sense both the Magnetic Translation Group and $U_q(sl_2)$ correspond to distinct discrete subgroups of $U(1)$ with linear gauge parameter and are related by a gauge transformation $\lambda' = xy/2$.

III. THE AZBEL-HOFSTADTER MODEL

The Azbel-Hofstadter is a model that describes Bloch electrons, i.e. electrons on a 2D periodic potential under a strong perpendicular magnetic field. It is a model on the lattice and the energy eigenvalues are computed by demanding the wave functions solutions periodicity to be compatible with the lattice (see \cite{23, 24} for exact asymptotic solutions). In this way one gets the band/gap structure (depending on the magnetic flux per unit cell) of the theory which constitutes a fractal (Cantor set) known as the Hofstadter butterfly (see \cite{29} for a recent review). A key point of this model is that the periodicity of wave functions is only possible for rational magnetic flux per unit cell. The physics for irrational values of the magnetic flux can nevertheless be studied by considering a succession of rational values which converge to that irrational value. Even in this irrational limit the structure of the spectrum is maintained although the number of bands and gaps become infinite.

A. Periodic Potential and the Magnetic Translations Group

Take now a periodic potential represented by a 2D lattice of unit cell with sides $dx$ and $dy$ (i.e. a planar rectangular crystal) of size $L_x$ and $L_y$ with periodic boundary conditions on both directions $x$ and $y$. This means our space becomes a square torus of area $A_{T^2} = L_x L_y$. For other boundary conditions one would get other topologies.

In the presence of a periodic potential the spatial translation symmetry is broken down to a discrete version, which spans the equipotential points on both directions of the potential lattice. When a Magnetic field is present
the theory is no longer symmetric under the usual trans-
lations, but it still is under the magnetic translations. On
the presence of a periodic potential and for a particular
gauge choice (Landau gauge), the magnetic translations
are reduced to only two generators that constitute the
$U_q(s_{l2})$ quantum group.

To see it explicitly let us build the magnetic genera-
tors $[22]$ such that $R_d$ stands for some vector of the potential
lattice. Note that these operators indeed commute with the
Hamiltonian

$$[T(R_d), H_B] = 0 \quad (III.2)$$

since the potential value is the same at points related by
the shift $R_d$ and the commutation relation $[\pi - e B a, \pi +
e B a] = 0$ holds. Note that in the absence of external
magnetic field these operators become simply the trans-
lation operators $T(R_d) = \exp\{-iR_d, \pi / h\}$ as argued before.

Furthermore only two of such generators are necessary
to span each of the equipotential spaces. They are

$$T_x = T(R_{dx}) \quad (III.3)$$
$$T_y = T(R_{dy})$$

They indeed work as local gauge operators $V_a$ given
by $[23]$ for the symmetric gauge with parameters $\lambda_x =
d_x x$ and $\lambda_y = d_y y$.

Acting on wave functions we get

$$T_x \Psi(x, y) = \exp\left\{ e B \frac{h d_x}{h} \right\} \Psi(x + d_x, y)$$
$$T_y \Psi(x, y) = \exp\left\{ e B \frac{h d_y}{h} \right\} \Psi(x, y + d_y) \quad (III.4)$$
in accordance to $[21]$.

Also one gets

$$T_x T_y = \exp\left\{ -i e B \frac{h d_x}{h} d_y \right\} T_y T_x \quad (III.5)$$
in accordance with $[21]$.

To complete the definition of the group consider

$$T_x T_y = \exp\left\{ -i e B \frac{h d_x}{2h} d_y \right\} T_x T_y \quad (III.6)$$

So we just defined the group of magnetic translations
and become apparent that it is described correctly in
terms of discretized local $U(1)$ gauge operators of a com-
act Abelian Maxwell Chern-Simons Massive Gauge the-
ory. Moreover in the Topological Quantum Field Theory
they are vortex operators. Here, in the Azbel-Hofstadter
model, they are interpreted as the generators of the quan-
tum group that generalises translations on the presence
of an external magnetic field. This group can easily be
interpreted as the possible interpolations (tunnelling ef-
fec) between non-distinguishable field configurations of
the theory or equivalent vacua, also commonly called on
the 3D Field Theory framework monopole-instantons.

B. Harper Equation from Landau Gauge

For the Landau gauge $a_L = (0, x)$ the previous group
generators acting on wave functions are

$$T_x \Psi(x, y) = \exp\{i k_x \} \Psi(x + d_x, y) \quad (III.7)$$
$$T_y \Psi(x, y) = \exp\{i k_y \} \exp \{i \pi \} \Psi(x, y)$$

where $\omega$ is the flux per unit cell over $h$

$$\omega = \frac{e B}{h} d_x d_y \quad (III.8)$$

In this gauge the group has a one-dimensional represen-
tation for each Bloch wave vector $(k_x, k_y)$. There is
no shift on the $y$ direction because in the Landau
gauge the canonical momentum $p_y$ is null (see the original
works $[21, 22, 23, 24]$ for further details).

The theory is in this way mapped into a one dimen-
sional model by writing the Hamiltonian as

$$H = T_x + T_x^{-1} + \lambda \left( T_y + T_y^{-1} \right) \quad (III.9)$$

where $\lambda = d_y / d_x$. The resulting discretized shrödinger
equation is the well know second order Harper equa-
tion $[21]$

$$\psi_{n+1} + \psi_{n-1} + 2 \lambda \cos(2 \pi (\omega n + \phi_0)) \psi_n = E \psi_n \quad (III.10)$$

where $k_x = 0$, $\phi_0 = k_y$ and $E$ is the energy eigenvalue.
Under the map $x_n = \psi_{n-1} / \psi_n$ the first order Harper
map is obtained $[22]$

$$x_{n+1} = \frac{1}{x_n - E + 2 \lambda \cos(2 \pi \phi_n)} \quad (III.11)$$
$$\phi_{n+1} = \omega n + \phi_0$$

The Lyapunov exponent corresponding to the $x$ dy-
namics is given by

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} y_n \quad (III.12)$$
$$y_n = \log \frac{\partial x_{n+1}}{\partial x_n} = \log x_{n+1}^2$$
Aubry and André proved that the Lyapunov exponent $\gamma = -2\gamma$, being $\gamma$ the localisation length and that $\gamma \leq 0$ always. For $\gamma = 0$, the phase is Extended and for $\gamma < 0$ the phase is Localized.

C. Phases of Azbel-Hofstadter model: Bands and Gaps

Numerically the two phases of these maps where studied to some extend in [8, 25, 26].

Relating them with the original Azbel-Hofstadter one checks that the Extended phases corresponds to the original energy bands and the localized phases to the gaps.

In [8] were studied the attractors in the Localized phases in the $x \times \phi$ plane. It was suggested a classification of the many localized phases of this map by integers $\eta$. The magnetic flux $\omega^* = (\sqrt{\eta} - 1)/2$ is taken to be the limit of the succession $\omega_n = F_{n-1}/F_n$ where $F_0 = F_1 = 1$ and the relation $F_n = F_{n-1} + F_{n-2}$ define the Fibonacci succession.

The full phase diagram for that irrational magnetic flux $\omega = (\sqrt{\eta} - 1)/2$ is shown in figure 1.

The integers labels are the winding of the attractors on the $x \times \phi$ torus. This torus is obtained by considering the identification on $\phi : 0 \cong 1$ and on $x : -\infty \cong \infty$ on the plane $x \times \phi$. Figure 2 is an example of such attractors for $\eta = 11$ and $\eta = 10$.

Note that each winding corresponds to $x$ going trough $x = \pm\infty$ and crossing, necessarily, the line $x = 0$. The first case means simply that the wave function is going trough zero and does not constitute a problem, but the second one is translated in terms of the original wave function into the existence of divergences for the null values of $\phi$. These are essential singularities. Therefore the conclusion is that the solutions of the wave functions do exist on the Localized phases (gaps) but are not physical since they are not normalizable.

Then the question which remains to answer is if the localized phases and all the structure described in [8] does correspond to any physics at all or if it is just a mathematical curiosity.

IV. PHYSICS ON THE GAPS

As explained so far one manage to go from a Topological Quantum Field Theory to a quantum mechanics model of nonrelativistic electrons on magnetic fields and then mapped it into a simple dynamical system described by an one-dimensional map.

At the dynamical system level it was concluded that there are two kind of phases, one Extended and the other one Localized. While on the Extended phase the map have $2D$ attractors and the iterations spread in the $x \times \phi$ plane according to some probability distributions, on the Localized phases there are stable one-dimensional attractors.

At the nonrelativistic quantum level these results must be interpreted in other terms. The Azbel-Hofstadter model is precisely based on finding the solutions of the Shrödinger equation which preserve the magnetic translations symmetry. Therefore the previous Extended phases correspond to the energy bands of the quantum system while the Localized ones to the gaps of the theory. So, although numerical solutions for the Schrödinger equation were found, they are not physical since are not square integrable due to essential divergences. Something must be missing in our picture.

A. Symmetry Breaking and Phase Transitions

Let us go back to the TQFT framework and re-examine the previous results. The magnetic translations are now the gauge transformations, the Extended phases correspond to a regime where the gauge symmetry is present while the Localized ones to a regime where the gauge symmetry is somehow broken.

The issue here is symmetry breaking and the theory being renormalizable or not. In the broken regime the theory is nonrenormalizable which translates to our nonrelativistic quantum limit as the wave functions being non normalizable. So the problem to solve is a very old one: symmetry breaking maintaining the theory renormalizable!

As widely known the symmetry breaking must however be nonexplicitly in order that the broken theory is still renormalizable. This issue has been well studied and is in the origin of the Higgs-Kibble mechanism of spontaneous symmetry breaking. There the symmetry is broken by choosing a particular point of the several degenerated vacua. For the setup presented here, the natural way out, is to consider specific field configurations allowed on the theory which break the gauge invariance. Again this is an old subject, we have some degenerate vacua and we are choosing some particular configuration which minimises the energy of the system.
constituting then a classical background. These are often called monopoles \[39, 40, 41, 42, 43\] and constitute what is know as dynamical symmetry breaking. By allowing such configurations the gauge symmetry is broken in exchange of field configurations but the theory is still renormalizable. This is also the same mechanism which in M/string-theory compactifications break the supersymmetry due to the choice of some particular vacua manifold (classical solution of the gravity theory).

Note that in 2+1D the dynamical Symmetry breaking is also associated to a confined-condensate phase transition of the theory as will be discussed to some extension later on.

B. Monopoles and White Holes

Consider now the results computed on the work of Stichel \[5\] (see also its references and in particular \[3, 4\]) where a classical deformation of the gauge field yields both a monopole-like configuration and a deformation of the metric which constitutes a gravitational white hole. Note that although the framework of that paper is different it translates to the framework presented here if in the decomposition \[11, 24\] \[\mathbf{A} = \mathbf{a}_1 + \mathbf{a}_2\] the second field \[\mathbf{a}_2\] is considered at classical level to be a deformation of the first one \[\mathbf{a}_1\].

There the original theory is of a nonrelativistic point particles coupled to gauge fields which are deformed in
order to allow time-dependent area preserving diffeomorphisms. Here the starting point is a quantum field theory which has as a particular nonrelativistic quantum mechanical limit, classical electrons travelling on a perpendicular magnetic field. Some non-trivial field configurations on the original quantum field theory will translate, in the classical limit, to a deformation of the gauge field and a deformation of the metric which constitutes a geometric bag or monopole. It is this deformation that interests us in the scope of this work. Also note that as explained before, at the TQFT level, any magnetic field depends only on the field $A_{\alpha}$, where $\lambda$ is a regular function and $\phi$ is a spatial singular function

$$\phi(x) = \arctan \frac{x}{y}.$$  \hfill (IV.2)

Take only one singular point at $(x, y) = (0, 0)$ and consider a time independent hedgehog solution for the gauge field $A$

$$A_x = y \left(1 - \sqrt{1 - \frac{\theta}{r^2}}\right) \frac{2N}{\theta},$$

$$A_y = -x \left(1 - \sqrt{1 - \frac{\theta}{r^2}}\right) \frac{2N}{\theta},$$ \hfill (IV.3)

$$A_0 = \text{const}$$

which constitutes a monopole configuration, and the corresponding metric solution

$$h^{xx} = \frac{1}{\sqrt{2}} \left(\frac{1}{1 - \frac{\theta}{r^2}} - \frac{\tilde{\theta}(2 - \frac{\theta}{r^2}) x^2}{r^2 - \theta / r^2}\right),$$

$$h^{xy} = -\frac{1}{\sqrt{2}} \frac{\tilde{\theta}(2 - \frac{\theta}{r^2}) xy}{r^2 - \theta / r^2},$$ \hfill (IV.4)

$$h^{yy} = \frac{1}{\sqrt{2}} \left(1 - \frac{\theta}{r^2} - \frac{\tilde{\theta}(2 - \frac{\theta}{r^2}) y^2}{r^2 - \theta / r^2}\right),$$

where $r^2 = x^2 + y^2$ stands for the planar radius and $N$ is an integer with units of length square $[N] = m^2$ that parameterises the monopole charge.

The parameter of these configurations is given by

$$\tilde{\theta} = e^2 \alpha_n^4 \frac{k}{4\pi c} = \frac{e^2}{2\pi} \left(\frac{\mu_0}{m_e}\right)^4 B \frac{c\mu_0}{\epsilon},$$ \hfill (IV.5)

with units of length square $[\tilde{\theta}] = m^2$. It is therefore proportional to the external perpendicular magnetic field.

The electric field is null since $A_0 = \text{const}$ and the magnetic field of the configuration is computed to be

$$b(r) = \frac{2N}{\theta} \left(2 + \frac{\tilde{\theta} - 2r^2}{r^2 \sqrt{1 - \frac{\theta}{r^2}}}\right).$$ \hfill (IV.6)

The determinant of the inverse metric is

$$h^{-1}(r) = \frac{r^2}{r^2 - \theta},$$ \hfill (IV.7)

and as expected the metric is asymptotically flat

$$\lim_{r \to \infty} h(r) = 1.$$ \hfill (IV.8)

The gravitational configuration corresponds in this nonrelativistic quantum mechanical limit to a white hole with radius

$$r_0^2 = \frac{e^2}{2\pi} \left(\frac{\mu_0}{m_e}\right)^4 B \frac{c\mu_0}{\epsilon}$$ \hfill (IV.9)

which reflects any incident wave function as described in detail in the last section of the text.

By request of the referee we proceed to analyse the properties of the monopole-white hole and compute the asymptotic solutions of the Schrödinger equation.

In planar polar coordinates the metric is given by

$$ds^2 = \frac{r^4}{\sqrt{2(r^2 - \theta)}} d\phi.$$ \hfill (IV.10)
Although the curvature is null everywhere, $R = 0$ as expected in two dimensional gravity, the contraction of two Ricci Tensors is

$$R_{ij} R^{ij} = -\frac{(r^2 - 2 \theta)^4}{\bar r^4 (r^2 - \theta)^4}. \quad \text{(IV.11)}$$

The vanishing $\theta$ limit is well defined, $\lim_{\theta \to 0} R_{ij} R^{ij} = 1/r^4$ corresponding to the flat metric, as it should since $\theta$ is the deformation parameter. Clearly $r = r_0 = \sqrt{\theta}$ is singularity. We will next show that it also constitutes a white-hole quantum numbers. This issue is not going to be studied here, see for instance \cite{65} (and references therein). In the scope of this work, let us simply compute some relevant quantities of such a particle, the energy, charge and mass density.

In the limit $r = \sqrt{\theta}$ the wave function vanishes independently of the angular momentum eigenvalue $l$. Note also that the asymptotic solution does not depend on the energy of the solution. Then we proved that effectively the electron wave function does not penetrate the core of the monopole-white hole.

One can now interpret the white hole, a gravitational deformation induced by the gauge fields, as being a physical particle, the monopole or a geometric bag. A full description would also need to take into account the white hole quantum numbers. This issue is not going to be studied here, see for instance \cite{65} (and references therein). In the scope of this work, let us simply compute some relevant quantities of such a particle, the energy, charge and mass density.

The energy of the configuration is finite as required and is just the integral of the magnetic field squared

$$E_{mon} = \frac{\pi \mu_0}{\gamma^2 V_3} \int dr \ h^{-1}(r) \ r \ b(r)^2$$

$$= N \frac{\pi \mu_0}{\gamma^2 V_3} (3 - \ln 2)$$

The physically measurable charge of the magnetic monopole is

$$Q_m = \frac{B}{4} \int dr \ r \ b(r) = \frac{B}{4} N \quad \text{(IV.15)}$$

as expected. For $N = 1$ we have the fundamental charge allowed on the theory.
Considering that the monopole is spherical (in 3D terms), the constant $\alpha_T$ is of the same order of magnitude of the white hole’s radius. Therefore we obtain

$$\rho_{\text{mon}} = 1.195 \times 10^{17} \frac{1}{B}.$$  \hspace{1cm} \text{(IV.16)}

For strong values of the magnetic field (say of order $B = 25 \ T^2$ - see equation [IV.21] in the end of the article) we get a value for the density of order $10^{16} \ Kgm^{-3}$. This is somewhere between the density of the electron ($10^{13}$) and nuclear matter ($10^{18}$ - considering $r_p \sim 1 \ fm$).

As a final remark note that the procedure described in [4] takes in account the gauss law constraint and therefore gauge fixes the theory. The point is that on a theory containing gravity both the degrees of freedom of gravity and the gauge theory will be constrained together such that for a particular nontrivial field configuration the gauge symmetry is broken (or gauge fixed if one prefers to say so) in exchange of a nontrivial gravitational background. Also it must be stressed that the metric [IV.4] is not the one presented on [3], there the ansatz corresponds to an anion (with both electric and magnetic charge) and renders the determinant of the metric to be always 1. The metric presented here has determinant 1 only asymptotically rendering finite energy pure magnetic monopole-white hole configurations. In terms of the quantum field theory these issues must be translated into the existence of a gravitational sector of the theory. This is not going to be studied here.

C. Confinement and Condensation of Monopoles

Let us analyse the results of the previous subsection at the level of the TQFT. In the early works on Topological Massive Gauge Theories (see for instance [13]) the monopole charge was imposed to average to zero over the entire manifold in order to preserve gauge invariance of the path integral. It was studied in [36] that $2 + 1D$ compact Maxwell Chern-Simons has two distinct phases. In the confined phase the magnetic charge is confined such that the monopoles are paired. The total charge is null for each pair and the monopole charge is in this way screened. This is the phase in which gauge symmetry is unbroken, there are only trivial configurations of magnetic field. In the condensate phase the magnetic charge condensates and the monopoles are not paired any longer. The total magnetic charge has now a non null value $\rho_{\text{mon}}$. See figure [3] for a suggestive picture of both phases. The above arguments strongly suggest that the existence of monopole condensation on the TQFT is intrinsically linked to the deformation of the classical gauge field and metric in the classical limit.

Furthermore it was argued in [37] that in the continuum limit of the Abelian theory the vortexes are not present due to being logarithmically UV divergent and therefore the monopole effect would be negligible anyway. The mechanisms which regularize the theory seem to be two folded. Both considering a periodic current in the same manner that the periodic potential is considered in the original Azbel-Hofstadter model such that the theory is always defined on a lattice and taking in account gravitational effects in order to correctly regularize the theory. Given the framework of the Azbel-Hofstadter model the theory is defined on a lattice due to the periodicity of the external current (the effective periodic potential). This is not enough since the wave functions are not normalizable and one has to account also for gravitational configurations.

D. Divergences Removed on the Gaps, Chains of Filaments and Monopoles

The punch line is that, if we allow configurations where monopoles have a non null net charge together with a gravitational deformation, they will preserve the finitude of the wave function. Moreover the value of the net charge must still be a multiple of the fundamental magnetic charge due to the fact that the full effect is still due to a set of such particles. The monopole number would be related to the number of essential (nonremovable) singularities of the bare wave function which are present in the pure gauge theory. That number is simply given by the winding number of the dynamical system on the $x \times \phi$ plane, that is the classification index of the phase diagrams on the last section!

Furthermore if one takes the wave function to be defined as a function of the phase $\phi$ one knows that the divergences are located on the points where the attractors are null. These will be the points where the white holes must stand in order the theory to be well defined and the wave functions divergences absent.

Translating it in more formal terms note that the phase $\phi_n = \phi_0 + \omega n$ parameterizes a discretized line. For irrational values of $\omega$ we can define a continuum phase $\phi$ which parameterises the attractors on the $x \times \phi$ plane

$$x(\phi) = \psi(\phi)/\psi(\phi + \delta \phi)$$  \hspace{1cm} \text{(IV.17)}

\[\text{[End of Document]}\]
FIG. 3: Monopole confinement and condensate.

FIG. 4: Chain containing 3 monopole-white holes of fundamental charge $Q = k/4$ and electron filaments $e$.

$(x$ should not be confused with the spatial coordinate - see figure 4). For a given string of length $L_I$ the phase is simply a normalized parameter

$$\phi = \frac{l}{L_I}, \quad l \in [0, L_I] \quad \text{(IV.18)}$$

Now one can consider wave functions of the form

$$\Psi(l) \sim e^{\int f(l) \, dl} \quad \text{(IV.19)}$$

such that the function $f(l)$ defines the linear attractor on the localized phases

$$x(l) = \frac{\psi(l)}{\psi(l + \delta l)} = e^{F(l) - F(l + \delta l)} \quad \text{(IV.20)}$$

where $F(l)$ stands for the value of the primitive of $f$ evaluated at $l$.

Once one considers the theory with monopole-white holes the wave function is physically well defined and one obtains a chain constitute by electron filaments attached to monopoles. The position of the monopole-white holes on the chain will be located at the divergences of the bare wave functions and its number will correspond to the before mention $\eta$ (the label of the phase). See figure 4 for a picture of such a chain.

E. Possible Phenomenology

A experimental analysis is out of the scope of this work and the expertise of the author. However a brief phenomenological analysis is carried out for the sake of completeness.

To have an order of magnitude of the physical effects consider some strong magnetic field $B$, meaning that

$$\mu B \gg 1 \quad \text{(IV.21)}$$

where $\mu$ is the electron mobility. Taking it to be of order $\mu \sim 10 m^2 V^{-1} s^{-1}$ one requires a magnetic field of order $B \sim 5 T$. Then the radius of the monopole-white hole is of order

$$r_0 = \sqrt{\frac{e^2}{2\pi} \left( \frac{\mu_0}{m_e} \right)^4 \frac{B}{\mu \mu_0}} \sim 10^{-15} m \quad \text{(IV.22)}$$

This results indicate that in principle it is possible to test the theoretical results presented in this work with present day technology.

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