Abstract—Despite the rich literature on scheduling algorithms for wireless networks, algorithms that can provide deadline guarantees on packet delivery for general traffic and interference models are very limited. In this paper, we study the problem of scheduling real-time traffic under a conflict-graph interference model with unreliable links due to channel fading. Packets that are not successfully delivered within their deadlines are of no value. We consider traffic (packet arrival and deadline) and fading (link reliability) processes that evolve as an unknown finite-state Markov chain. The performance metric is efficiency ratio which is the fraction of packets of each link which are delivered within their deadlines compared to that under the optimal (unknown) policy. We first show a conversion result that shows classical non-real-time scheduling algorithms can be ported to the real-time setting and yield a constant efficiency ratio, in particular, Max-Weight Scheduling (MWS) yields an efficiency ratio of 1/2. We then propose randomized algorithms that achieve efficiency ratios strictly higher than 1/2, by carefully randomizing over the maximal schedules. We further propose low-complexity and myopic distributed randomized algorithms, and characterize their efficiency ratio. Simulation results are presented that verify that randomized algorithms outperform classical algorithms such as MWS and GMS.

Index Terms—Scheduling, Real-Time Traffic, Markov Processes, Stability, Wireless Networks

I. INTRODUCTION

There has been vast research on scheduling algorithms in wireless networks which mostly focus on maximizing long-term throughput when packets have no strict delay constraints. Max-Weight Scheduling (MWS) policy has been shown to be throughput optimal in such settings, attaining any desired throughput vector in the feasible throughput region [1]. Further, greedy scheduling policies such as LQF [2], [3], or distributed policies such as CSMA [4]–[6] have been proposed that alleviate the computational complexity of MWS and achieve a certain fraction of the throughput region. However, in many emerging applications, such as Internet of Things (IoT), vehicular networks, and edge computing, delays and deadline guarantees on packet delivery play an important role [7]–[9], as packets that are not received within specific deadlines are of little or no value, and are typically discarded. This discontinuity in the packet value as a function of latency makes the problem significantly more challenging than traditional scheduling where packets do not have strict deadlines.

There is an increasing body of work attempting to address the above challenge, however they either assume a frame-based traffic model [10]–[14], relax the interference graph constraints [15], or use greedy scheduling approaches like LDF [16], [17]. In the frame-based traffic model, time is divided into frames, and packet arrivals and their deadlines during a frame are assumed to be known at the beginning of the frame, and deadlines are constrained by the frame’s length [10]–[14]. Under such assumptions, the optimal solution in each frame is a Max-Weight schedule. Note that unless the traffic is restricted to be synchronized across the users, such solutions are non-causal. The optimal scheduling policy (and the real-time throughput region) for general traffic patterns and interference graphs is unknown and very difficult to characterize. Largest-Deficit-First (LDF) is a causal policy which extends the well-studied Largest-Queue-First (LQF) from traditional scheduling to real-time scheduling. The performance of LDF has been studied in terms of efficiency ratio, which is the fraction of the real-time throughput region guaranteed by LDF. Under i.i.d. packet arrivals and deadlines, with no fading, LDF was shown to achieve an efficiency ratio of at least \( \frac{1}{\beta + 1} \) [16], where \( \beta \) is the interference degree of the network (which is the maximum number of links that can be scheduled simultaneously out of a link and its neighboring links).

Recently, the work [18] has shown that through randomization it is possible to design algorithms that can significantly improve the prior algorithms, in terms of both efficiency ratio and traffic assumptions. Specifically, [18] proposed two randomized scheduling policies: AMIX-ND for collocated networks with an efficiency ratio of at least \( \frac{1}{\beta + 1} \approx 0.63 \), and AMIX-MS for general interference graphs with an efficiency ratio of at least \( \frac{|I|}{2|I| - 1} > \frac{1}{2} \) (\(|I|\) is the number of maximal independent sets of the graph). However, the complexity of AMIX-MS can be prohibitive for implementation in large networks. Moreover, intrinsic wireless channel fading has not been considered in [18], and packet transmission over a link is assumed to be always reliable.

In this work, we consider an interference graph model of wireless network subject to fading, where packet transmissions over links are unreliable. We consider a joint traffic (packet arrival and deadline) and fading (link’s success probability) process that evolves as an unknown Markov chain over a finite state space. Can the existing traditional scheduling algorithms (which focus on long-term throughput with no deadline constraints) be used to provide guarantees for scheduling in this setting, without making frame-based traffic assumptions?
Application of Traditional Scheduling Algorithms to Real-Time Scheduling. Each non-empty link (i.e., with unexpired packets for transmission) is associated with a weight which is the product of its deficit counter and channel fading probability at that time, otherwise if the link is empty, the weight is zero. We show that any algorithm that provides a ψ-approximation to Max-Weight Schedule (MWS) under such weights, achieves an efficiency ratio of at least \( \frac{\psi}{\psi + 1} \) for real-time scheduling under any Markov traffic-fading process. As a consequence, MWS policy achieves an efficiency ratio of at least \( \frac{1}{2} \), and GMS (Greedy Maximal Scheduling) provides an efficiency ratio of at least \( \frac{1}{\sqrt{T}} \).

Randomized Scheduling of Real-Time Traffic Over Fading Channels. We extend [18] to show the power of randomization for scheduling real-traffic traffic over fading channels in general and collocated networks. By carefully randomizing over the maximal schedules, the algorithms can achieve an efficiency ratio of at least \( \frac{1}{1-e^{-\frac{1}{2}}} + 1 - q \) in any general graph under any unknown Markov traffic-fading process. In the special case of a collocated network with i.i.d. channel success probability \( q \), the algorithm can achieve an efficiency ratio of at least \( \left( \frac{1}{1-e^{-\frac{1}{2}}} + 1 - q \right)^{-1} \), which ranges from 0.5 to 0.63.

Low-Complexity and Distributed Randomized Algorithms. To address the high complexity of the randomized algorithm in general graphs, we propose a low-complexity covering-based randomization and a myopic distributed randomization. Given a coloring of the graph using \( \chi \) colors, we can achieve an efficiency ratio of at least \( \frac{1}{2^\chi - 1} \) by randomizing over \( \chi \) schedules. Moreover, we show that a myopic distributed randomization, which is simple and easily implementable, can achieve an efficiency ratio of \( \frac{1}{\Delta + 1} \) in any graph with maximum degree \( \Delta \).

B. Notations

Some of the basic notations used in this paper are as follows. Given \( y \in \mathbb{R}^n \), \( ||y|| = \sum_{i=1}^n |y_i| \). We use \( \text{int}(A) \) to denote the interior of set \( A \). \( [x]^+ = \max\{x, 0\} \). \( 1(E) \) is the indicator function of event \( E \). \( |A| \) is the cardinality of set \( A \). \( \mathbb{E}^Y[\cdot] \) is used to indicate that expectation is taken with respect to random variable \( Y \).

II. MODEL AND DEFINITIONS

Wireless Network and Interference Model. Consider a set of \( K \) links, denoted by the set \( K \). We assume time is divided into slots, and in every slot \( t \), each link \( l \in K \) can attempt to transmit at most one packet. To model interference between links, we use the standard interference graph \( G_I = (K, E_I) \): Each vertex of \( G_I \) is a link, and there is an edge \( (l_1, l_2) \in E_I \) if links \( l_1, l_2 \) interfere with each other. Hence, no two links that transmit packets can share an edge in \( G_I \). Let \( I \) be the set of all maximal independent sets of graph \( G_I \). Also let \( B(t) \subseteq K \) denote the set of nonempty links (i.e., links that have packets available to transmit) at time \( t \). We use \( M(t) \) to denote the set of links scheduled at time \( t \). By definition, \( M(t) \) is a valid schedule if links in \( M(t) \) are nonempty and form an independent set of \( G_I \), i.e.,

\[
M(t) \subseteq (B(t) \cap D), \text{ for some } D \in I. \tag{1}
\]

A schedule is said to be maximal if no nonempty link can be added to the schedule without violating the interference constraints. In this case, \( \subseteq \) in (1) holds with \( '=' \).

Fading Model. Transmission over a link is unreliable due to wireless channel fading. To capture the channel fading over link \( l \), we use an ON-OFF model where link \( l \) at time \( t \) is ON \((C_l(t) = 1)\) with probability \( q_l(t) \), otherwise it is OFF \((C_l(t) = 0)\). If scheduled, transmission over link \( l \) at time \( t \) is successful only if link \( l \) is ON. Let \( q(t) = (q_l(t), l \in K) \) be the vector of success probabilities of the links. We assume that at any time slot \( t \), the link’s success probability is known to the scheduler before making a decision. At the end of time slot \( t \), link’s transmitter receives a feedback from its receiver indicating whether transmission was successful or not. A special case of this model is when \( q_l(t) \in \{0, 1\}, \forall l \in K \), in which case the channel state \( C_l(t) \) is deterministically known to the scheduler, which requires periodic channel state estimation. Another special case is when \( C_l(t) \) is i.i.d. with some probability \( q_l \) which eliminates the need for periodic estimation. Similar models have been used in [15], [19].
Overall, we define \( I(t) = (I_l(t), l \in \mathcal{K}) \) to denote the successful packet transmissions over the links at time \( t \). Note that by definition, \( I_l(t) = \mathbb{1}(l \in M(t))C_l(t) \), where \( M(t) \) is the valid schedule at time \( t \), and \( C_l(t) \) is the channel state of link \( l \).

**Traffic Model.** We assume a single-hop real-time traffic. We use \( a_l(t) \leq a_{\text{max}} \) to denote the number of packet arrivals at link \( l \) at time \( t \), where \( a_{\text{max}} < \infty \) is a constant. Each arriving packet has a deadline which indicates the maximum delay that the packet can tolerate before successful transmission. A packet with deadline \( d \) at time \( t \) has to be successfully transmitted before the end of time slot \( t + d - 1 \), otherwise it will be discarded. We define the traffic process \( \tau(t) = (\tau_{l,d}(t), d = 1, \ldots, d_{\text{max}}, l \in \mathcal{K}) \), where \( \tau_{l,d}(t) \) is the number of packets with deadline \( d \) arriving to link \( l \) at time \( t \), and \( d_{\text{max}} < \infty \).

**Traffic and Fading Process.** In general, we assume that the joint traffic and fading process \( z(t) = (q(t), \tau(t)) \) evolves as an “unknown” irreducible Markov chain over a finite state space \( \mathcal{Z} \). See Figure 1 for an example of a Markovian traffic-fading process.

Without loss of generality, we make the following assumption to make this Markov chain non-trivial: For every link \( l \), there are two states \( z^l, z'^l \in \mathcal{Z} \) such that \( z' \) has a packet arrival with deadline \( d \), and \( z'^l \) has \( q_l > 0 \), and there is a positive probability that \( z(t) \) can go from \( z' \) to \( z'^l \) in at most \( d \) time slots. This assumption simply states that it is possible to successfully transmit some packets of every link \( l \) within their deadlines. If a link does not satisfy this condition, we can simply remove it from the system. Note that \( z(t) \) is an irreducible finite-state Markov chain, hence it is positive recurrent [20], and time-average of any bounded function of \( z(t) \) is well defined, in particular the packet arrival rate for link \( l \):

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} a_l(s) =: \bar{\alpha}_l. \quad (2)
\]

**Buffer Dynamics.** The buffer of link \( l \) at time \( t \), denoted by \( \Psi_l(t) \), contains the existing packets at link \( l \) which have not expired yet and also the newly arrived packets at time \( t \). The remaining deadline of each packet in \( \Psi_l(t) \) decreases by one at every time slot, until the packet is successfully transmitted or reaches the deadline 0, which in either case the packet is removed from \( \Psi_l(t) \). We also define \( \Psi(t) = (\Psi_l(t); l \in \mathcal{K}). \)

**Delivery Requirement and Deficit.** As in [10]–[13], [16], we assume that there is a minimum delivery ratio requirement \( p_l \) (QoS requirement) for each link \( l \in \mathcal{K} \). This means we must successfully deliver at least \( p_l \) fraction of the incoming packets on each link \( l \) within their deadlines. Formally,

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} \frac{b_l(s)}{\sum_{s=1}^{t} a_l(s)} \geq p_l. \quad (3)
\]

We define a deficit \( w_l(t) \) which measures the number of successful packet transmissions owed to link \( l \) up to time \( t \) to fulfill its minimum delivery ratio. As in [13], [16], [18], the deficit evolves as

\[
w_l(t + 1) = \left[w_l(t) + \bar{\alpha}_l(t) - I_l(t)\right]^+, \quad (4)
\]

where \( \bar{\alpha}_l(t) \) indicates the amount of deficit increase due to packet arrivals. For each packet arrival, we should increase the deficit by \( p_l \) on average. For example, we can increase the deficit by exactly \( p_l \) for each packet arrival to link \( l \), or use a coin tossing process as in [13], [16], i.e., each packet arrival at link \( l \) increases the deficit by one with the probability \( p_l \), and zero otherwise. We refer to \( \bar{\alpha}_l(t) \) as the **deficit arrival process** for link \( l \). Note that it holds that

\[
\lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} \bar{\alpha}_l(s) = \bar{\pi}_l p_l := \lambda_l, \quad l \in \mathcal{K}. \quad (5)
\]

We refer to \( \lambda_l \) as the deficit arrival rate of link \( l \). Note that an arriving packet is always added to the link’s buffer, regardless of whether and how much deficit is added for that packet. Also note that in (4) each time a packet is transmitted successfully from link \( l \), i.e., \( I_l(t) = 1 \), the deficit is reduced by one. The dynamics in (4) define a deficit queueing system, with bounded increments/decrements, whose stability, e.g., in the sense

\[
\lim sup_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} \mathbb{E}[w_l(s)] < \infty, \quad (6)
\]

implies (3) holds [21]. Define the vector of deficits as \( w(t) = (w_l(t), l \in \mathcal{K}) \). The system state at time \( t \) is then defined as

\[
S(t) = (q(t), \tau(t), \Psi(t), w(t)). \quad (7)
\]

**Objective.** Define \( \mathcal{P}_C \) to be the set of all causal policies, i.e., policies that do not know the information of future arrivals, deadlines, and channel success probabilities in order to make scheduling decisions. For a given traffic-fading process \( z(t) \), with fixed \( \bar{\pi}_l \), defined in (2), we are interested in causal policies that can stabilize the deficit queues for the largest set of delivery rate vectors \( p = (p_l, l \in \mathcal{K}) \), or equivalently largest set of \( \lambda = (\lambda_l := \bar{\pi}_l p_l, l \in \mathcal{K}) \) possible. For a given traffic process, we say the rate vector \( \lambda = (\lambda_l, l \in \mathcal{K}) \) is supportable under some policy \( \mu \in \mathcal{P}_C \) if all the deficit queues remain stable for that policy. Then one can define the supportable real-time rate region of the policy \( \mu \) as

\[
\Lambda_\mu = \{ \lambda \geq 0 : \lambda \text{ is supportable by } \mu \}. \quad (8)
\]

The supportable real-time rate region under all the causal policies is defined as \( \Lambda = \bigcup_{\mu \in \mathcal{P}_C} \Lambda_\mu \). The overall performance of a policy \( \mu \) is evaluated by the efficiency ratio defined as

\[
\gamma_\mu^* = \sup \{ \gamma : \gamma \Lambda \subseteq \Lambda_\mu \}. \quad (9)
\]

For a casual policy \( \mu \), we aim to provide a universal lower bound on the efficiency ratio that holds for “all” Markovian traffic-fading processes, without knowing the Markov chain.

For clarity, we will use \( \mathbb{E}^{F}[\cdot] \) to denote expectation with respect to the random outcomes of the fading channel, and \( \mathbb{E}^{R}[\cdot] \) to denote expectation with respect to the random decisions of a randomized algorithm ALG, whenever applicable.

### III. Scheduling Algorithms and Main Results

Recall that \( \mathcal{I} \) is the set of all maximal independent sets of interference graph \( G_t \), and \( B(t) \) is the set of links that have packets to transmit at time \( t \). At any time \( t \), we define the set of maximal schedules \( \mathcal{M}(t) = \{ D \cap B(t), D \in \mathcal{I} \} \).

...
Define the gain of maximal schedule \( M \in \mathcal{M}(t) \) to be the total deficit of packets transmitted successfully at time \( t \), i.e., \( G_M(t) := \sum_{i \in M} C_i(t) w_i(t) \). We define the weight of maximal schedule \( M \) to be its expected gain, conditional on \( S(t) \), i.e.,

\[
W_M(t) := \mathbb{E}[G_M(t)|S(t)] = \sum_{i \in M} w_i(t) q_i(t). \tag{10}
\]

We use \( M^*(t) := \arg \max_{M \in \mathcal{M}(t)} W_M(t) \) to denote the Max-Weight Schedule (MWS) at time \( t \), and \( W^*(t) := W_{M^*(t)}(t) \) to denote its weight.

Given a Markov policy ALG, let \( \pi_M(t) \) be the probability that ALG selects maximal schedule \( M \) at time \( t \). Hence, the probability that link \( l \) is scheduled at time \( t \) is

\[
\phi_l(t) = \sum_{M \in \mathcal{M}(t)} \pi_M(t) \mathbb{I}(l \in M). \tag{11}
\]

Further, the expected gain of ALG is given by

\[
\mathbb{E}^{\mathbb{R},F}[G_{\text{ALG}}(t)|S(t)] = \sum_{M \in \mathcal{M}(t)} \pi_M(t) W_M(t) \tag{12}
\]

\[
= \sum_{i \in K} \phi_i(t) q_i(t) w_i(t). \tag{13}
\]

Without loss of generality, we consider natural policies that transmit the earliest-deadline packet from every selected link in the schedule. This is because, similar to [18], if a policy transmits a packet that is not the earliest-deadline, that packet can be replaced with the earliest-deadline packet of that link, and this only improves the state. Similarly, the optimal policy always selects a maximal schedule at any time.

### A. Converting Classical Non-Real-Time Algorithms for Real-Time Scheduling

The following theorem allows us to convert non-real-time scheduling policies to real-time scheduling policies, hence enabling the use of numerous policies from the literature of traditional non-real-time scheduling. More specifically, policies whose expected gain at any time \( t \) is \( \psi \)-fraction of the gain of MWS (i.e., \( W^*(t) \)) yield an efficiency ratio of at least \( \psi/(\psi + 1) \). The result is stated formally in Theorem 1.

**Theorem 1.** Consider any policy ALG such that, at every time \( t \), \( \mathbb{E}^{\mathbb{R},F}[G_{\text{ALG}}(t)|S(t)] \geq \psi W_M^*(t) \), whenever \( \|w(t)\| \geq W' \), for some finite \( W' \). Then

\[
\gamma^* \geq \frac{\psi}{\psi + 1}.
\]

Note that this conversion results in deadline-oblivious policies which can be preferred in cases where information about the deadlines of packets is either not accurate or not available. We remark that for certain policies the result of Theorem 1 is tight as seen in Corollary 1.1 below, whereas for other policies the bound can be considerably loose.

**Corollary 1.1.** MWS policy provides efficiency ratio \( \gamma_{\text{MWS}}^* = \frac{1}{2} \) for real-time scheduling under Markov traffic-fading processes.

**Proof.** Using Theorem 1 for MWS with \( \psi = 1 \), we directly obtain \( \gamma_{\text{MWS}}^* \geq 1/2 \). We can get the opposite inequality through an adversarial example. If we consider a simple network with two interfering links without fading, then MWS reduces to LDF, which has been shown to have \( \gamma^* \leq 1/2 \) for Markovian traffic with deterministic deficit admission [16], [18] (recall that efficiency ratio \( \gamma^* \) is defined as a universal bound for all traffic-fading processes for any given graph). □

Consider a Greedy Maximal Scheduling (GMS) policy defined as follows: Order the nonempty links \( B(t) \) in the decreasing order of the product \( w_i(t) q_i(t) \). Then construct a schedule recursively by including the nonempty link with the largest \( w_i(t) q_i(t) \), removing the interfering links, and repeating the same procedure over the remaining links.

The following corollary extends the result of [16] which was shown for i.i.d. traffic without fading for LDF.

**Corollary 1.2.** GMS provides an efficiency ratio \( \gamma_{\text{GMS}}^* \geq \frac{1}{\beta+1} \) for real-time scheduling under any Markov traffic-fading process, where \( \beta \) is the interference degree of the graph.

**Proof.** Suppose \( M = \{l_1, \ldots, l_n\} \) is the scheduled constructed by GMS, where \( l_i \) is the link selected by GMS in iteration \( i \). It is sufficient to show that \( \sum_{i \in M} w_i(t) q_i \geq \frac{1}{\beta} W_M^*(t) \), as in this case we can apply Theorem 1 with \( \psi = 1/\beta \) and consequently get \( \gamma^* \geq \frac{1}{\beta+1} \). Consider the first iteration where GMS selects \( l_1 \). The Max-Weight schedule \( M^* \) will either include link \( l_1 \), or at most \( \beta \) links from the interfering links of \( l_1 \), but since \( l_1 \) has the highest \( w_i(t) q_i \), it holds that \( w_{l_1}(t) q_{l_1} \geq \frac{1}{\beta} \sum_{i \in N(l_1)} w_i(t) q_i \), where \( N(l) \) denotes the neighborhood of link \( l \), including link \( l \) itself, in graph \( G_1 \). Then we remove all the links in \( N(l_1) \) and repeat the same argument for \( K \setminus N(l_1) \) and obtain \( w_{l_1}(t) q_{l_1} \geq \frac{1}{\beta} \sum_{i \in N(l_2) \setminus N(l_1)} \sum_{l \in M^*} w_i(t) q_i \). Repeating this argument, and summing the obtained inequalities over \( l_1, \ldots, l_n \), we obtain the required inequality. □

Note that many other results from traditional scheduling literature can be converted using Theorem 1. For example, the DISTGREEDY policy [22] would also obtain \( \gamma_{\text{DISTGREEDY}} \geq \frac{1}{\beta+1} \) but has a lower complexity than GMS.

### B. Randomized Scheduling Algorithms

In this section, we extend AMIX policies for collocated networks and general networks introduced in [18] to incorporate fading. We refer to the generalized policies as FAMIX (Fading-based Adaptive MIX). We describe these policies below.

1) **FAMIX-MS:** Randomized Scheduling in General Graphs:

Let \( R := |\mathcal{M}(t)| \) be the number of maximal schedules at time \( t \). We index and order the maximal schedules such that \( M^{(i)} \in \mathcal{M}(t) \) has the \( i \)-th largest weight (based on (10)), i.e.,

\[
W_{M^{(i)}}(t) \geq W_{M^{(i+1)}}(t) \cdots \geq W_{M^{(R)}}(t). \tag{14}
\]

Define the subharmonic average of weight of the first \( n \) maximal schedules, \( n \leq R \), to be

\[
C_n(t) = \frac{n - 1}{\sum_{i=1}^{n} (W_{M^{(i)}}(t))^{-1}}. \tag{15}
\]
Then select schedule $M^{(t)}$ with probability

$$\pi_{M^{(t)}}(t) = \begin{cases} 1 - \frac{C_i(t)}{W_{M^{(t)}}(t)}, & 1 \leq i \leq \bar{n} \\ 0, & \bar{n} < i \leq R \end{cases}$$

where $\bar{n}$ is the largest $n$ such that $\{\pi_{M^{(t)}}(t), 1 \leq i \leq R\}$ defines a valid probability distribution, i.e., $\pi_{M^{(t)}}(t) \geq 0$, and $\sum_{i=1}^{R} \pi_{M^{(t)}}(t) = 1$.

**Remark 1.** It has been shown in [18] that $\bar{n}$ in such a distribution can be found through a binary search. Essentially

**Theorem 2.** In a general interference graph $G_I$ with maximal independent sets $\mathcal{I}$, the efficiency ratio of FAMIX-MS is

$$\gamma_{\text{FAMIX-MS}}^* \geq \frac{|\mathcal{I}|}{2|\mathcal{I}| - 1} > \frac{1}{2}.$$  

**Remark 2.** Theorem 2 shows that with randomization we can do strictly better than MWS (Corollary 1.1). In particular, in the case of a complete bipartite interference graph FAMIX-MS (where there are two maximal independent sets) yields an efficiency ratio of at least 2/3 while MWS yields 1/2.

2) FAMIX-ND: Randomized Scheduling in Collocated Graphs: Extending AMIX-ND [18] to fading channels is more challenging. In particular, the derivation in [18] relied on two main ideas: (1) it is sufficient to consider only a restricted set of “non-dominated” links for transmission, and (2) there is an ordering among the non-dominated links such that given two non-dominated links, having packet in buffer from one of the links is always preferred to that from the other link. Finding such domination relationship for a general Markov fading process is difficult. Here, we describe an extension under a simplified fading process, where $q_l(t) = q_0 = q$, i.e., the links’ success probabilities are fixed and equal. We allow the channel state across links to be either independent or positively correlated, i.e., $\Pr[C_l(t) = 1|C_i(t) = 1] \geq q_0$. The above setting could be a reasonable approximation in collocated networks where links have similar reliabilities, and an active channel for one link implies a better condition for the overall shared wireless medium.

Let $e_l(t)$ denote the deadline of the earliest-deadline packet of link $l$ at time $t$. We say that link $l_1$ dominates link $l_2$ at time $t$ if $e_{l_1}(t) \leq e_{l_2}(t)$, $w_{l_1}(t) \geq w_{l_2}(t)$, i.e., link $l_1$ is more urgent and has a higher deficit. Based on this definition, the set of non-dominated links at any time can be found through a simple recursive procedure as in [18], i.e., add the largest-deficit nonempty link, remove all the links dominated by it from consideration and repeat the process for the remaining links. The following theorem describes FAMIX-ND and its efficiency ratio for a collocated network with a channel success probability $q$.

**Theorem 3.** Consider a collocated network, where $q_l = q$, $\forall l \in \mathcal{K}$, and channels of links are independent or positively correlated. Order and re-index the non-dominated links such that $w_1(t) \geq w_2(t) \geq ... \geq w_n(t)$.

Starting from $i = 1$, assign probability $\pi_i(t)$ to the $i$-th non-dominated link,

$$\pi_i(t) = \min \left\{ \frac{1}{q} \left(1 - \frac{w_{i+1}(t)}{w_i(t)}\right), 1 - \sum_{j<i} \pi_j(t) \right\}.$$  

FAMIX-ND selects the $i$-th non-dominated link with probability $\pi_i(t)$ and transmits its earliest-deadline packet. Then

$$\gamma_{\text{FAMIX-ND}}^* \geq \left(\frac{q}{1 - e^{-q}} + (1 - q)\right)^{-1} := h(q).$$

**Remark 3.** Note that due to channel uncertainty, FAMIX-ND boosts the probability of larger-weight links. Intuitively, as $q$ becomes small, deadlines of packets are “effectively” reduced, as each packet will need to be transmitted several times before success.

**Remark 4.** The lower bound $h(q)$ on efficiency ratio in (18) is a monotone function of $q$, which increases from 0.5 and to $\frac{e^q - 1}{e^q}$, as $q$ goes from 0 to 1. For $q = 1$, this recovers the result of [18] for non-fading channels, i.e., $\gamma_{\text{FAMIX-ND}}^* \geq \frac{1}{2}$. In the case of unequal $q_l(t) \in (q_{\min}, q_{\max})$, using (17) by replacing $q$ with $q_{\min}$, will give an efficiency ratio of at least $\left(\frac{q_{\max} - q_{\min} + 1}{1 - e^{-q_{\min}}} + 1 - q_{\max}\right)^{-1} := h(q_{\min}, q_{\max})$. Depending on $q_{\max}, q_{\min}, K$, we can choose either FAMIX-ND or FAMIX-MS and achieve $\gamma^* \geq \max\{h(q_{\min}, q_{\max}), \frac{1}{K}\}$.

C. Low-Complexity and Distributed Randomized Variants

The general algorithm FAMIX-MS in Section III-B potentially randomizes over all the maximal schedules. This can be computationally expensive in large networks that may have many maximal schedules. In this section, we design variants that only need to consider a subset of the maximal schedules, or are distributed.

1) Covering-Based Randomized Algorithms: We propose two variants that only need to consider a subset of the maximal schedules. Proposition 1 below states a sufficient condition under which randomization over a subset of the maximal schedules can provide a related approximation on the efficiency ratio.

**Proposition 1.** Consider policy ALG = FAMIX-MS $|\mathcal{M}_0(t)|$ which, at any time $t$, selects a schedule from a subset $\mathcal{M}_0(t) \subseteq \mathcal{M}(t)$, according to probabilities of FAMIX-MS computed for $\mathcal{M}_0(t)$. Suppose that for every $M \in \mathcal{M}(t) \setminus \mathcal{M}_0(t)$,

$$\psi(t) \sum_{l \in M} w_l(t) q_l(t) \phi_l(t) \leq \max_{M' \in \mathcal{M}_0(t)} \sum_{l \in M'} q_l(t) w_l(t) \phi_l(t) + (1 - \psi(t)) \mathbb{E}_{F}[G_{\text{ALG}}(t)|S(t)],$$

for some $\psi(t) \in (0, 1)$, where $\phi_l(t)$ was defined in (11) and $\phi_l(t) = 1 - \phi_l(t)$. Then

$$\gamma_{\text{ALG}}^* \geq \min_{t \geq 0} \left\{ \psi(t) \frac{|\mathcal{M}_0(t)|}{2|\mathcal{M}_0(t)| - 1} \right\}.$$
This condition in Proposition 1 has parameter $\psi(t)$. If the condition is satisfied for higher $\psi(t)$ the resulting bound is improved.

Next, by focusing on a special case in which Condition (19) holds, we can design provably efficient policies by considering a small set of schedules such that any other maximal schedule can be covered by them.

**Lemma 4.** Suppose every $M \in \mathcal{M}(t) \setminus \mathcal{M}_0(t)$ is covered by at most $\zeta$ maximal schedules from $\mathcal{M}_0(t)$, i.e.,

$$M \subseteq \bigcup_{M' \in S_M} M', \text{ for some } S_M \subseteq \mathcal{M}_0(t) : |S_M| \leq \zeta. \tag{21}$$

Then Condition (19) holds with fixed $\psi(t) = \frac{1}{\zeta}$.

**Proof.** Using the covering definition (21), we have

$$\sum_{t \in M} q(t)w(t)\hat{\phi}(t) \leq \sum_{M' \in S_M} \sum_{t \in M'} q(t)w(t)\hat{\phi}(t) \leq \zeta \max_{M' \in \mathcal{M}_0(t)} \sum_{t \in M'} q(t)w(t)\hat{\phi}(t), \tag{22}$$

and hence condition (19) trivially holds for $\psi = \frac{1}{\zeta}$. \qed

In general, $\mathcal{M}_0(t)$ can be adaptive and constructed based on the link deficits and fading probabilities. Here, we apply Proposition 1 and Lemma 4 for a constant $\psi(t) = \psi$ and a family $\mathcal{M}_0(t)$ induced by fixed subset of the independent sets $\mathcal{I}_0 \subseteq \mathcal{I}$, i.e., $\mathcal{M}_0(t) = \{D \cap \mathcal{B}(t), D \in \mathcal{I}_0\}$. With minor abuse of notation, we refer to such algorithms as FAMIX-MS|\mathcal{I}_0|.

Below, we present two such covering-based algorithms.

**Corollary 4.1** (Coloring-based Randomization). Consider a coloring of graph $G_1$ with $\chi$ colors, which partitions the vertices of $G_1$ into $\chi$ independent sets $\{D_1', ..., D_{\chi}'\}$. Extend these independent sets arbitrarily so they are maximal $\{D_1, ..., D_{\chi}\} := \mathcal{I}_0$. Then $\gamma_{\text{FAMIX-MS}}^{\chi} \geq \frac{1}{2\chi-1}.$

**Proof.** Any maximal schedule in $\mathcal{M}_1 \setminus \mathcal{M}_0$ can be covered by at most $\chi$ maximal schedules in $\mathcal{M}_0$, thus $\psi = \frac{1}{\chi}$ by Lemma 4. Further, $\frac{|\mathcal{M}_0(t)|}{2|\mathcal{M}_0(t)| - 1} \geq \frac{\chi}{2\chi-1}$. Thus applying Proposition 1, we get the stated efficiency ratio. \qed

**Remark 5.** In general, finding an efficient coloring might be computationally demanding, but it needs to be done only once for a given $G_1$. There are many interesting families of graphs for which coloring can be solved efficiently. For example, for a (not necessarily complete) bipartite graph (e.g. a tree) where $\chi = 2$, we obtain $\gamma^* \geq \frac{1}{2}$. This performs much better than LDF whose efficiency ratio is in a tree with maximum degree $\beta$ is $\gamma^* \geq \frac{2}{\beta^{\beta-1}}$ in the case without fading (which is a special case of our setting). Another family that admits an efficient coloring are planar graphs where, by the four-color theorem, always have a 4-coloring which can be found in polynomial time [23]. Further, we remark that the independent sets $D_i'$ could be extended adaptively at every time $t$, e.g., using GMS.

**Corollary 4.2** (Partition-based Randomization). Partition the vertices of graph $G_1$ into two sets $K_1$ and $K_2$, each with at most $\lceil K/2 \rceil$ vertices. Consider the set of maximal independent sets of each induced subgraph, denoted by $\mathcal{I}_1', \mathcal{I}_2'$. Then extend these independent sets arbitrarily so that they form maximal independent sets $\mathcal{I}_1, \mathcal{I}_2$ for the whole graph $G_1$. Then

$$\gamma^*_{\text{FAMIX-MS}|\mathcal{I}_1 \cup \mathcal{I}_2|} \geq \frac{1}{2} \frac{|\mathcal{I}_1| + |\mathcal{I}_2|}{|\mathcal{I}_1| + |\mathcal{I}_2| - 1} > \frac{1}{4}. \tag{23}$$

**Proof.** By construction, every maximal independent set in $G_1$ can be covered by at most 2 maximal independent sets in $\mathcal{I}_1 \cup \mathcal{I}_2$, and hence we have $\psi = 1/2$. Following similar arguments as in Corollary 4.1, we obtain the result. \qed

**Remark 6.** By Corollary 4.2, we only need to randomize over at most $|\mathcal{I}_1| + |\mathcal{I}_2|$ maximal schedules at any time which can be much smaller than $|\mathcal{I}|$ in large graphs. The corollary can be generalized to more than 2 partitions to allow a trade-off between the guaranteed efficiency ratio and complexity.

2) **Myopic Distributed Randomized Algorithm:** We present a simple distributed algorithm that has constant complexity. Assume each slot is divided in two parts, a control part of duration $T_C$, and a packet transmission part with duration normalized to 1. At the beginning of the control phase, every non-empty link $l \in \mathcal{B}(t)$ starts a timer $T_l \sim \text{Exp}(\nu_l)$, where $\text{Exp}(\nu)$ denotes an exponential distribution with rate $\nu$. Once the timer of a link $l$ runs down to zero, it broadcasts an announcement informing its neighbors that it will participate in data transmission, unless it has heard an earlier announcement from its neighboring links, or the control phase ends.

Given any $\delta \in (0, 1)$, let $T_C = \max_{\nu_l} \left\{ \frac{\log(\delta)}{\nu_l} \right\}$. The next corollary states the efficiency ratio for the uniform timer rates.

**Corollary 4.3.** Consider the myopic randomized algorithm where every link $l$ has the same timer value $\nu_l = \nu$. If the maximum degree of $G_1$ is $\Delta$, then

$$\gamma^*_{\text{myopic}} \geq \frac{1}{\Delta - \frac{1}{1 - \delta} + 1}. \tag{23}$$

Note that theoretically we can scale up the timer rate $\nu$, so that the control phase $T_C$ becomes very small.

**Remark 7.** We note that direct application of Theorem 1 would yield an efficiency ratio $\approx \frac{1}{\Delta}$ as the myopic algorithm obtains $\frac{1}{\Delta + 1}$ approximation of the MWS (To see this note that every link has probability $1/(\Delta + 1)$ of getting service. Thus all the links of the MWS are included with this probability). Therefore Corollary 4.3 also serves as an example in which more careful analysis can improve the bound of a direct conversion.

IV. **Analysis Techniques and Proofs**

We provide an overview of the techniques in our proofs.

**Frame Construction.** A key step in the analysis of our scheduling algorithms is a frame construction similar to the one in [18], but based on the joint traffic-fading process. The definition of frame is as follows

**Definition 1** (Frames and Cycles). Starting from an initial traffic and fading state tuple $(\tau(0), q(0)) = z \in \mathcal{Z}$, let $t_i$
denote the i-th return time of traffic-fading Markov chain \( z(t) \) to \( z_i \), \( i=1,\ldots \). By convention, define \( t_0=0 \). The i-th cycle \( C_i \) is defined from the beginning of time slot \( t_{i-1}+1 \) until the end of time slot \( t_i \), with cycle length \( C_i = t_i - t_{i-1} \). Given a fixed \( k \in \mathbb{N} \), we define the i-th frame \( F^{(k)} \) as \( k \) consecutive cycles \( C_{(i-1)k+1}, \ldots, C_{ik} \), i.e., from the beginning of slot \( t_{(i-1)k+1} \) until the end of slot \( t_{ik} \). The length of the i-th frame is denoted by \( F^{(k)} = \sum_{j=1}^{ik} C_{ij} \). Define \( \mathcal{J}(F^{(k)}) \) to be the space of all possible \( (\tau(t),q(t)) \) patterns during a frame \( F^{(k)} \). Note that these patterns start after \( z \) and end with \( z \).

By the strong Markov property and the positive recurrence of traffic-fading Markov chain \( z(t) \), frame lengths \( F^{(k)}_i \) are i.i.d with mean \( \mathbb{E}[F^{(k)}_i] = k \mathbb{E}[C] \), where \( \mathbb{E}[C] \) is the mean cycle length which is a bounded constant \([20]\). In fact, since state space \( \mathcal{Z} \) is finite, all the moments of \( C \) and \( F^{(k)}_i \) are finite. We choose a fixed \( k \), and, when the context is clear, drop the dependence on \( k \) in the notation.

Define the class of non-causal \( F \)-framed policies \( \mathcal{P}_{NC}(F) \) to be the policies that, at the beginning of each frame \( F_i \), have complete information about the traffic-fading pattern in that frame, but have a restriction that they drop the packets that are still in the buffer at the end of the frame. Note that the number of such packets is at most \( \max_{i} \max_{n} K \), which is negligible compared to the average number of packets in the frame, \( \pi_1 \mathbb{E}[C] = \pi_1 k \mathbb{E}[C] \), as \( k \to \infty \). Define the rate region

\[
\Lambda_{NC}(F) = \bigcup_{\mu \in \mathcal{P}_{NC}(F)} \Lambda_{\mu}.
\]

(24)

Given a policy \( \mu \in \mathcal{P}_{NC}(F) \), the time-average real-time service rate \( I_l(t) \) of link \( l \) is well defined. By the renewal reward theorem (e.g. \([24]\), Theorem 5.10), and boundedness of \( \mathbb{E}[F] \),

\[
\lim_{t \to \infty} \frac{\sum_{s=1}^{t} I_l(s)}{t} = \frac{\mathbb{E} \left[ \sum_{t \in F} I_l(t) \right]}{\mathbb{E}[F]} = \bar{I}_l.
\]

(25)

Similarly for the deficit arrival rate \( \lambda_l \), defined in (5),

\[
\frac{\mathbb{E} \left[ \sum_{t \in F} F_l(t) \right]}{\mathbb{E}[F]} = \lambda_l, \quad l \in \mathcal{K}.
\]

(26)

In Definition 1, each frame consists of \( k \) cycles. Using similar arguments as in \([16], [18]\), it is easy to see that

\[
\lim_{k \to \infty} \inf_{\mu \in \mathcal{P}_{NC}(F)} \Lambda_{\mu} = \Lambda_{NC}(F) \supseteq \int(\Lambda),
\]

Hence, if we prove that for a causal policy ALG, there exists a constant \( \rho \) and a large \( k_0 \), such that for all \( k \geq k_0 \),

\[
\rho \int(\Lambda_{NC}(F^{(k)})) \subseteq \Lambda_{ALG},
\]

(27)

then it follows that \( \Lambda_{ALG} \supseteq \rho \int(\Lambda) \). For our algorithms, we find a \( \rho \) such that (27) holds for any traffic-fading process under our model. Then it follows that \( \gamma_{ALG} \geq \rho \).

**Lyapunov Argument.** To prove (27), we rely on comparing the expected gain of ALG with that of the non-causal policy that maximizes the expected gain over the frame (max-gain policy). The following proposition, which is similar to that in \([18]\), will be used to prove the main results. We omit its proof, as it is similar to the proof in \([18]\) with minor modifications to account for channel uncertainty.

**Proposition 2.** Consider a frame \( F = F^{(k)} \), for a fixed \( k \) based on the returns of the traffic-fading process \( z(t) \) to a state \( z \). Define the norm of initial deficits at the beginning of a frame \( \|w(t_0)\| = \sum_{l \in K} w_l(t_0) \). Suppose for a causal policy ALG, given any \( \epsilon > 0 \), there is a \( W' \) such that when \( \|w(t_0)\| > W' \),

\[
\frac{\mathbb{E} \left[ \sum_{t \in F} G_{ALG}(t) | S(t_0) \right]}{\mathbb{E} \left[ \sum_{t \in F} G_{\mu^*}(t) | S(t_0) \right]} \geq \rho - \epsilon,
\]

(28)

where \( S(t_0) = (q(t_0), \tau(t_0), \Psi(t_0), w(t_0)) \), and \( \mu^* \) is the non-causal policy that maximizes the gain over the frame. Then for any \( \lambda \in \rho \int(\Lambda_{NC}(F)) \), the deficit queues are bounded in the sense of (6).

**Amortized Gain Analysis.** To use Proposition 2, we need to analyze the achievable gain of ALG and the non-causal policy \( \mu^* \) over a frame. Since comparing the gains of the two policies directly is difficult, we adapt an amortized analysis technique from \([18]\), initially extended from \([25]-[28]\). The general idea is as follows. Let \((q(t), \tau(t), \Psi(t), w(t))\) be the state under our algorithm at time \( t \in F \), and \((\Psi^*(t), q(t), \tau(t), w^*(t))\) be the state under the optimal policy \( \mu^* \). The traffic-fading process \( z(t) = (q(t), \tau(t)) \) is identical for both algorithms as it is independent of the actions of the scheduling policy. We change the state of \( \mu^* \) (by modifying its buffers and deficits) to make it identical to \((q(t), \tau(t), \Psi(t), w(t))\), but also give \( \mu^* \) an additional gain that ensures the change is advantageous for \( \mu^* \) considering the rest of the frame. Let \( G_{\mu^*}(t) \) denote the amortized gain of \( \mu^* \) at time \( t \) with any compensated gain, which has the property that

\[
\mathbb{E} \left[ \sum_{t \in F} G_{\mu^*}(t) | J, S(t_0) \right] \geq \mathbb{E} \left[ \sum_{t \in F} G_{\mu^*}(t) | J, S(t_0) \right] + \mathbb{E}_F 
\]

(29) given any traffic-fading pattern \( J \in \mathcal{J}(F) \) and initial frame state \( S(t_0) \). Then, the following proposition will be useful in bounding the gain and thus the efficiency ratio of our policies.

**Proposition 3.** Consider a Markov policy ALG that for any traffic-fading pattern \( J \in \mathcal{J}(F) \), at any time \( t \in F \), satisfies

\[
\rho \mathbb{E} \left[ G_{\mu^*}(t) | J, S(t) \right] \leq \mathbb{E} \left[ G_{ALG}(t) | J, S(t) \right] + \mathbb{E}_F
\]

(30)

for some \( \mathbb{E}_F \) which is a measurable function of the frame length \( F \), with \( \mathbb{E}[F \mathbb{E}_F] < \infty \). Then \( \gamma_{ALG} \geq \rho \).

**Proof.** First note that \( \mathbb{E} \left[ G_{ALG}(t) | J, S(t) \right] = \mathbb{E} \left[ G_{ALG}(t) | J, S(t), S(t_0) \right] \) by the Markov property of ALG. Further \( \mathbb{E} \left[ G_{\mu^*}(t) | J, S(t) \right] = \mathbb{E} \left[ G_{\mu^*}(t) | J, S(t), S(t_0) \right] \) since the amortized gain of the max-gain policy does not depend on the past state given the current state and future traffic-fading pattern (note that this amortized gain might depend on policy ALG, which itself is Markov). Hence, taking expectation of both sides of (30), conditional on \( J, S(t_0) \), and using the law of iterated expectations, we get

\[
\rho \mathbb{E} \left[ G_{\mu^*}(t) | J, S(t_0) \right] \leq \mathbb{E} \left[ G_{ALG}(t) | J, S(t_0) \right] + \mathbb{E} \left[ \mathbb{E}_F | J, S(t_0) \right]
\]
Summing over the frame, and using the fact that
\[ E[E_F | J, S(t_0)] = E_F \] since \( F \) is determined by \( J \), we have
\[
\rho E \left[ \sum_{t \in F} G_{\mu_t}^* (t) | J, S(t_0) \right] \leq E \left[ \sum_{t \in F} G_{ALG}(t) | J, S(t_0) \right] + F E_F.
\]
Using the definition (29), and taking expectations over the randomness of the traffic-fading pattern, we obtain
\[
\rho E \left[ \sum_{t \in F} G_{\mu_t}^* (t) | S(t_0) \right] \leq E \left[ \sum_{t \in F} G_{ALG}(t) | S(t_0) \right] + E [F E_F].
\]
Note that by assumption, \( E [F E_F] < \infty \). If we show that
\[
\lim_{\| w(t_0) \| \to \infty} E \left[ \sum_{t \in F} G_{\mu_t}^* (t) | S(t_0) \right] = \infty,
\]

and using Proposition 2, we obtain that \( \gamma_{\mu} \geq \rho \).

To that end, consider the link \( l_1 = \arg \max_{i \in J} w_i(t_0) \). Since \( \mu^* \) maximizes the expected gain over the frame, its expected gain cannot be lower than a policy that simply schedules link \( l_1 \) whenever it has available packets. Choose \( F \equiv F^k \) with any \( k \geq 2 \). Then by the non-trivial traffic-fading Markov chain assumption (Section II), there is a traffic-fading pattern \( J' \) of length \( F^k \) with some nonzero probability of occurring \( \Pr (J') > 0 \), in which a packet arrives for link \( l_1 \) at some time \( t_1 \in F \), and at a later time \( t_2 \in F \) before the packet expires, we have \( q_i(t_2) > 0 \), i.e., \( l_1 \in B(t_2) \) and \( E[C_{l_1}(t_2)] = q_i(t_2) > 0 \). Then trivially
\[
\lim_{\| w(t_0) \| \to \infty} E \left[ \sum_{t \in F} G_{\mu_t}^* (t) | S(t_0) \right] = \infty.
\]

The following lemma describes a generic amortized gain computation for general networks that allows us to modify the state of the max-gain policy during a frame to match the state of the considered policy ALG. Recall from (11) that \( \phi_i(t) \) is the probability that link \( l \) is scheduled under ALG, and \( \pi_M(t) \) is the probability that schedule \( M \) is selected. We drop their dependence on time to simplify the notation.

**Lemma 5.** For any pattern \( J \in \mathcal{J}(F) \) in a frame \( F \), given a Markov policy ALG, the amortized gain of the max-gain policy \( \mu^* \) (w.r.t. ALG) if it selects maximal schedule \( M \in M \) at time \( t \) in the frame, is given by
\[
E^{RF} \left[ \hat{G}_{\mu_t}^* (t) | J, S(t) \right] = W_M(t) + \sum_{l \in M} w_l(t) \phi_l(q_i(t) + E_m,
\]
\[
\leq W_M(t) (1 - \pi_M) + E^{RF} [G_{ALG}(t) | S(t)] + E_m,
\]
where \( E_m := K (F + a_{max} d_{max}) \).

**Proof.** The main idea is similar to the one in [18], but we have to account for fading. Suppose \( \mu^* \) attempts to transmit the earliest-deadline packets of links from schedule \( M \) whereas ALG attempts the earliest-deadline packets from schedule \( M' \). We need to modify the state of \( \mu^* \), i.e., the buffers and deficits, so it is identical with the state of ALG. To achieve this, we allow \( \mu^* \) to additionally transmit the packets of links successfully transmitted by ALG but not \( \mu^* \), i.e, \( S_1 = (M' \setminus M(t)) \cap \{ l \in K : C_l(t) = 1 \} \). Transmitting such packets for \( \mu^* \) might not be advantageous for its total gain as the weight of these packets can increase by \( a_{max} d_{max} \) before they could be transmitted. Thus giving an additional reward \( K a_{max} d_{max} \) to \( \mu^* \) guarantees that the modification is advantageous. Further, we insert the packets transmitted successfully by \( \mu^* \) but not ALG, i.e., \( S_2 = (M(t) \setminus M'(t)) \cap \{ l \in K : C_l(t) = 1 \} \), back to its buffers (which is advantageous for \( \mu^* \)). Further to make the deficits identical, we increase the deficit counters of \( \mu^* \) for links in \( S_2 \), which is advantageous for the total-gain within the frame. Additionally, we decrease the deficit counters of \( \mu^* \) for links in \( S_1 \). This change might not be advantageous for the total gain of \( \mu^* \), thus we give it extra reward for every possible subsequent transmission over links, which is at most \( K F \). Hence the total additional compensation \( E_m = K (F + a_{max} d_{max}) \).

Following the above argument, the expected amortized gain of \( \mu^* \), when it selects \( M \), is bounded as
\[
E^{RF} \left[ \hat{G}_{\mu_t}^* (t) | J, S(t) \right] = \sum_{l \in M} q_i(t) w_l(t) + \sum_{M' \in M \setminus \{ M \}} \pi_{M'}(t) \sum_{l \in M' \setminus M} q_i(t) w_l(t) + E_m
\]
\[
= W_M(t) + \sum_{l \in M} w_l(t) \phi_l(q_i(t)) + E_m \tag{31}
\]
where (a) follows from definitions of \( \phi_l \) and \( W_M(t) \). The inequality in the lemma’s statement follows by noting that
\[
(31) \leq W_M(t) + \sum_{M' \in M \setminus \{ M \}} W_{M'}(t) \pi_{M'} + E_m,
\]
since \( \sum_{l \in M' \setminus M} q_i(t) w_l(t) \leq W_{M'}(t) \), and by using (12). \( \square \)

**A. Proof of Theorem 1: Conversion Result**

In the rest of proofs, for notational compactness, we define
\[
E_{t,J} [.] := E [S(t), J] \text{ and } E_t [.] := E [S(t)].
\]
Now consider a pattern \( J \in \mathcal{J}(F) \). When \( \| w(t) \| \geq W^* \), the amortized gain of \( \mu^* \), if it selects schedule \( M \), is bounded as
\[
E^{RF}_{t,J} \left[ G_{\mu_t}^* (t) \right] \leq W_M(t) (1 - \pi_M) + E^{RF} [G_{ALG}(t)] + E_m
\]
\[
\leq W_M(t) + E^{RF} [G_{ALG}(t)] + E_m
\]
\[
= E^{RF} [G_{ALG}(t)] (\frac{W_M(t)}{E_t^{RF} [G_{ALG}(t)]} + 1) + E_m
\]
\[
\leq E^{RF} [G_{ALG}(1/\psi + 1) + E_m,
\]
where in (a) we used Lemma 5, and in (b) we used the main assumption that ALG obtains \( \psi \) fraction of the maximum.
weight schedule. This inequality does not depend on the particular choice $M$. Similarly in the case that $\|w(t)\| \leq W'$, it can be seen from (13) and Lemma 5 that $\mathbb{E}_{t,J}^{\text{RF}}[\hat{G}_{\mu}^{(M)}(t)] \leq 2W' + \mathcal{E}_m$. Consequently in either case, $\mathbb{E}_{t,J}^{\text{RF}}[\hat{G}_{\mu}^{(M)}(t)] \leq 2W' + \mathcal{E}_m + \mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)](1/\psi + 1)$. Applying Proposition 3 with $\rho = (1/\psi + 1)^{-1}$, we obtain the result.

B. Analysis of FAMIX-MS: Proof of Theorem 2.

Using Lemma 5, and probabilities of FAMIX-MS indicated in (16), in both cases of $i \leq \bar{n}$ and $i > \bar{n}$, the amortized gain of $\mu^*$ can be bounded as

$$\mathbb{E}_{t,J}^{\text{RF}}[\hat{G}_{\mu^*}^{(M)}(t)] \leq W_{M(i)}(t)(1 - \pi_i) + \mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)] + \mathcal{E}_m$$

$$\leq C_n(t) + \mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)] + \mathcal{E}_m.$$

Thus regardless of the schedule selected by $\mu^*$, we have

$$\mathbb{E}_{t,J}^{\text{RF}}[\hat{G}_{\mu}^{(M)}(t)] \leq \mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)]$$

Note that $|\mathcal{I}| \geq |\mathcal{M}| \geq \bar{n}$, hence it suffices to show that

$$\frac{C_n(t)}{\mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)]} \leq \frac{\bar{n} - 1}{\bar{n}}$$

(32), as then by Proposition 3 it will follow that $\gamma_{\text{FAMIX-MS}} \geq \frac{|\mathcal{I}|}{|\mathcal{I}|-1} + 1 = \frac{1}{2\bar{n}-1}$. To show (32), note that

$$\frac{C_n(t)}{\mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)]} = \frac{\bar{n}C_n(t) - (\bar{n} - 1)\mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)]}{\bar{n}\mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)]} + \frac{\bar{n} - 1}{\bar{n}}$$

and $\bar{n}C_n(t) - (\bar{n} - 1)\mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)] \leq 0$. This inequality holds because by (12) and (16), it follows that $\mathbb{E}_{t,J}^{\text{RF}}[\mathcal{G}_{\text{FAMIX-MS}}(t)] = \sum_{j=1}^{\bar{n}} W_{M(j)}(t) - \bar{n}C_n(t)$ and hence the inequality becomes

$$C_n(t)\bar{n}^2 \leq (\bar{n} - 1)\sum_{j=1}^{\bar{n}} W_{M(j)}(t),$$

which holds by (15) and the arithmetic mean-harmonic mean inequality applied to $\{W_{M(i)}(t), i = 1, \ldots, \bar{n}\}$.

C. Analysis of FAMIX-ND: Proof of Theorem 3.

We first state the following Lemma that allows us to focus on policies that transmit from non-dominated links.

**Lemma 6.** Given $J \in \mathcal{J}(\mathcal{F})$, let $\mu^*$ be the maximum-gain policy that transmits only from non-dominated links at any time (we refer to $\mu^*$ as max-gain ND-policy) and the maximum-gain policy $\mu^*$ that can transmit any packet, then

$$\mathbb{E}[\sum_{t \in \mathcal{F}} \mathcal{G}_{\mu^*}(t)S(t_0), J] \geq \mathbb{E}[\sum_{t \in \mathcal{F}} \mathcal{G}_{\mu^*}(t)S(t_0), J] - (a_{\text{max}} + 1)F^2.$$

**Proof.** First we consider the case where deficits do not vary over time regardless of arrivals or transmissions. We argue that in this case transmitting from a non-dominated link has always higher total expected gain from any state considering the remaining frame for any pattern $J$. Let $V(t, \Psi, J)$ denote the maximum expected gain over the remaining frame for the given pattern $J$, if at the current time slot $t \in \mathcal{F}$ with buffers $\Psi$, link $l$ is scheduled. Further, define $V(t, \Psi, J) = \max \{V_l(t, \Psi, J)\}$. Finally define $V_{ND}(t, S, J)$ to be the maximum expected gain for the remaining frame over policies that only schedule non-dominated links. Consider the earliest-deadline packets $p_u, p_v$ of links $u, v$, with $u$ dominating $v$. We argue that transmitting $p_u$ yields a higher expected gain than that of $p_v$. If $p_u$ is scheduled we have:

$$V_u(t, \Psi, J) = q(w_u + V(t + 1, \Psi(p_u), J)) + (1 - q)V(t + 1, \Psi', J),$$

where $\Psi'$ are the updated buffers due to regular buffer dynamics but ignoring the scheduled packet by our policy, and $\Psi^{(p_u)}$ indicates buffer $\Psi'$ without packet $p$. Similar expression can be written if $p_v$ is scheduled. Hence, to show $V_u(t, \Psi, J) \geq V_v(t, \Psi, J)$, equivalently we can show

$$w_u + V(t + 1, \Psi(p_u), J) \geq w_v + V(t + 1, \Psi(p_v), J).$$

(33)

Now note that by the domination definitions, the deadline of packet $v$ is at least as long as that of $u$, hence the optimal policy $\mu$ for $V(t + 1, \Psi(p_u), J)$ that attempt packet $u$ can be used to construct a policy $\mu'$ for $V(t + 1, \Psi(p_v), J)$ that attempts $v$ instead of $u$ whenever $\mu$ would have scheduled $p_u$. This is possible since $p_v$ does not expire before $p_u$. Now consider a coupling for the channel outcomes in $\mu$ and $\mu'$ such that channel $C_{\mu}^{(t)}(u) \mu$ under $\mu$ is identical to $C_{\mu'}^{(t)}(t)$ under $\mu'$. This does not affect the marginal success probability of links as both links $u, v$ have the same probability of success $q$. Further each success when transmitting $v$ under $\mu'$ results in reward $w_v$, instead of $w_u$ under $\mu$, therefore the gain of policy $\mu'$ is stochastically larger than that of policy $\mu$, and hence in the expectation.

$$V(t + 1, \Psi(p_v), J) \geq \langle w_v - w_u \rangle + V(t + 1, \Psi(p_v), J)$$

Thus in the fixed-deficit case we have shown that $V_{ND}(t, \Psi, J) \geq V(t, \Psi, J)$.

Now consider the case with time-varying deficits, and let $\bar{V}(t, \Psi, J)$ denote the corresponding quantities. In this notation $\bar{V}(t_0, \Psi(t_0), J) = \mathbb{E}[\sum_{t \in \mathcal{F}} \mathcal{G}_{\mu^*}(t)J, S(t_0)]$. If pattern $J$ has length $F$, then (a): $\bar{V}(t, \Psi, J) \geq \bar{V}(t, \Psi, J) - a_{\text{max}}F^2$ since any transmission under any policy in the varying-deficit case will yield a gain of at most $a_{\text{max}}F^2$ higher than the fixed-deficit case, since the deficit cannot increase more than $a_{\text{max}}F$ during a frame, and we can have at most $F$ such transmissions. Similarly we obtain (b): $V_{ND}(t, \Psi, J) + F^2 \geq V_{ND}(t, \Psi, J)$, as in the time-varying-deficit case every transmission can result in at most $F$ gain less than the fixed-deficit case, and we can have at most $F$ such transmissions. By (a) and (b), we obtain

$$\bar{V}_{ND}(t, \Psi, J) \geq \bar{V}(t, \Psi, J) - (a_{max} + 1)F^2.$$

**Proof of Theorem 3.** In view of Lemma 6, we perform an amortized gain analysis for FAMIX-ND in comparison with ND-policies. Suppose FAMIX-ND decides to schedule the earliest-deadline packet $p_f = (w_f, e_f)$ of link $f$ and ND-policy $\mu^*$ transmits a packet $p_z = (w_z, e_z)$ from a different
non-dominated link $z$ ($z \neq f$). The state of $\hat{\mu}^*$ and FAMIX-ND will be different in the following four cases,

1) $C_f(t) = 1, C_z(t) = 0$: In this case, we make the buffers of the two identical by allowing $\hat{\mu}^*$ to also transmit $P_f$, and give extra reward $a_{\text{max}}d_{\text{max}}$. Further we reduce the deficit in $\hat{\mu}^*$ for link $f$ by 1 and we give reward to $\hat{\mu}^*$ similarly as in the proof of Lemma 5.

2) $C_f(t) = 0, C_z(t) = 1$: We replace $P_z$ in the buffer of $\hat{\mu}^*$ and increase the deficit of link $z$ in $\hat{\mu}^*$ which are advantageous for $\hat{\mu}^*$.

3) $e_f \leq e_z, w_f \leq w_z$, and $C_f(t) = 1, C_z(t) = 1$: We replace $P_f$ from link $f$ with $P_z$ in link $z$ in buffers of $\hat{\mu}^*$. Packet $P_z$ has higher deadline and higher weight at time $t$. Since both packets will expire in at most $d_{\text{max}}$ slots, the deficit of $f$ can only increase by at most $d_{\text{max}}a_{\text{max}}$ before $P_f$ expires, whereas the deficit of $z$ can decrease by at most $d_{\text{max}}$. Therefore giving $\hat{\mu}^*$ additional compensation of $(1 + a_{\text{max}})d_{\text{max}}$ will guarantee that the modification is advantageous. Further, we decrease the deficit of link $f$ by one $(w_f - 1 + \hat{\mu}^*)$ and we increase the deficit of link $z$ by one $(w_z + 1 + \hat{\mu}^*)$. To compensate for the decrease in deficit of link $f$ we give $\hat{\mu}^*$ extra gain of $F$. Hence, the total compensation is bounded by $F + (a_{\text{max}} + 1)d_{\text{max}}$.

4) $e_z \leq e_f, w_z \leq w_f$ and $C_f(t) = 1, C_z(t) = 0$: In this case, we allow $\hat{\mu}^*$ to additionally transmit packet $P_f$ at time $t$ and give extra reward $a_{\text{max}}d_{\text{max}}$, and inject a copy of packet $P_z$ to the buffer of link $z$. This makes the buffers identical, but results in the decrease of deficit of link $f$ by one. To guarantee the change is advantageous for $\hat{\mu}^*$ considering the rest of the frame, we give it extra reward $F$.

In all cases, the additional compensation is bounded by $E_0 = F + (a_{\text{max}} + 1)d_{\text{max}}$. Let $q_{ij} = \Pr(C_i(t) = 1, C_j(t) = 1) \geq q^2$, or equivalently $q_{ij} = q^2 + \epsilon_{ij}$ for some $\epsilon_{ij} \geq 0$, by the positive correlation assumption, and hence $\Pr(C_i(t) = 1, C_j(t) = 0) = q - q_{ij} = q(1 - q) - \epsilon_{ij}$. Then,

$$E_{t,J} \left[ \hat{\gamma}_{t,J}^{(i)}(t) \right] \leq qw_i + \sum_j \pi_j(q - q_{ij})w_j + \sum_{j < i} \pi_j \pi_{ij}w_j + \sum_{j > i} \pi_j \pi_{ij}w_j + E_0$$

$$= qw_i + (1 - q) \sum_j \pi_jqw_j + \sum_{j < i} \pi_j \pi_{ij}w_j + E_0$$

Now notice that for $\pi_i \leq \frac{1}{q}(1 - \frac{w_{i+1}}{w_i})$ the right-hand-side of the above inequality is maximized for $i = 1$. This can be verified by computing the difference of the gains for choices $i, i + 1$. Hence, we have

$$E_{t,J} \left[ \hat{\gamma}_{t,J}^{(i)}(t) \right] \leq \max_i E_{t,J} \left[ \hat{\gamma}_{t,J}^{(i)}(t) \right] = E_{t,J} \hat{\gamma}_{t,J}^{(i)}(t) = qw_i + (1 - q)E_{t} \left[ \hat{\gamma}_{t,J}^{(i)}(t) \right] + E_0.$$

Let $\bar{n}$ be the number of links with positive probability in (17). For the gain of FAMIX-ND, we have

$$E_{t} \left[ \hat{\gamma}_{t,J}^{(i)}(t) \right] = \sum_{i=1}^{\bar{n}} q\pi_i w_i$$

$$= \sum_{i=1}^{\bar{n}-1} (w_i - w_{i+1}) + q(1 - \sum_{i=1}^{\bar{n}-1} \pi_i)w_\bar{n}$$

$$= w_1 + w_\bar{n}(-1 + q - \sum_{i=1}^{\bar{n}-1} \pi_i)$$

$$= w_1 \left( 1 - (1 - q + q \sum_{i=1}^{\bar{n}-1} \pi_i \prod_{i=1}^{\bar{n}-1} (1 - \pi_iq) \right)$$

$$\geq w_1 \left( 1 - \left( \frac{n - q}{n} \right) \frac{\bar{n}}{w_1} \right) \geq w_1(1 - e^{-q}),$$

where in (a), by the definition of $\pi_i$ in (17), we used: $w_\bar{n} = w_1 \prod_{i=1}^{\bar{n}-1} (1 - \pi_iq)$. In (b) we used the geometric-arithmetic inequality. Using the above relation and (34) we get

$$E_{\bar{n},J} \left[ \hat{\gamma}_{t,J}^{(i)}(t) \right] \leq E_{\bar{n}} \left[ \gamma_{t,J}^{(i)}(t) \right] \left( \frac{q}{1 - e^{-q}} + (1 - q) \right) + E_0$$

Using similar arguments as in the proof of Proposition 3, we can sum over the entire frame to obtain

$$E \left[ \sum_{t \in F} \hat{\gamma}_{t,J}^{(i)}(t) \right] J, S(t_0) \right) \leq E \left[ \sum_{t \in F} \gamma_{t,J}^{(i)}(t) \right] (\frac{q}{1 - e^{-q}} + (1 - q)) + F E_0.$$
E. Proof of Corollary 4.3: Myopic Distributed Algorithm

The amortized gain can be written as

\[ E_{t,J}\left[ G_{\text{ALG}}^{(M)}(t) \right] = \sum_{l \in M} q_l(t)w_l(t)\hat{\phi}_l + E\left[ G_{\text{ALG}}(t) \right] + \mathcal{E}_m \]

where in (a), we used (13).

Let \( A_l = \bigcap_{t \in N(t)} \{ T_l < T_c \} \) be the event that the timer of \( l \) expires earlier than that of its neighbors. We can obtain a bound on the probability of link getting scheduled as follows:

\[
\phi_l \geq \Pr [A_l \cap \{ T_l < T_c \}] = \Pr [A_l] \Pr [T_l < T_c | A_l]
\]

\[
\geq \frac{\nu_l}{\nu_l + \sum_{z \in N(z)} \nu_z} (1 - e^{-\nu_l + \sum_{z \in N(z)} \nu_z} T_c)
\]

\[
\geq \frac{\nu_l}{\nu_l + \sum_{z \in N(z)} \nu_z} (1 - e^{-\nu_l T_c})
\]

\[
\geq \frac{\nu_l}{\nu_l + \sum_{z \in N(z)} \nu_z} (1 - \delta),
\]

(38)

where (a) follows from the fact that \( T_l | A_l \sim Exp(\nu_l + \sum_{z \in N(z)} \nu_z) \) (easy to verify), and (b) from the choice of \( T_c \).

Hence, using (38), in the case that \( \nu_l = \nu \), we get \( \phi_l \geq \frac{1 - \delta}{1 + \Delta} \).

\[
\sum_{l \in M(t)} q_l(t)w_l(t)(1 - \phi_l) \leq \frac{\sum_{l \in M(t)} q_l(t)w_l(t)(1 - \frac{1 - \delta}{1 + \Delta})}{\sum_{l \in M(t)} q_l(t)w_l(t) \frac{1 - \delta}{1 + \Delta}} \leq \frac{\sum_{l \in M(t)} q_l(t)w_l(t)(1 - \frac{1 - \delta}{1 + \Delta})}{\sum_{l \in M(t)} q_l(t)w_l(t) \frac{1 - \delta}{1 + \Delta}} \leq \frac{\Delta - \delta}{1 - \delta},
\]

where the first inequality is due to the fact that the ratio is a decreasing function of \( \phi_l \), thus using (37).

\[
E_{t,J}\left[ G_{\text{ALG}}^{(M)}(t) \right] \leq E_{t,J}\left[ G_{\text{ALG}}(t) \right] \left( \frac{\Delta - \delta}{1 - \delta} + 1 \right) + \mathcal{E}_m,
\]

where \( \delta \) can be made arbitrarily small. Using Proposition 3, we obtain the final result.

V. CONCLUSION

We considered scheduling of real-time traffic over fading channels, where traffic (arrival and deadline) and links’ reliability evolves as an unknown finite-state Markov chain. We provided a conversion result that shows classical non-real-time scheduling algorithms like MWS and GMS can be ported to this setting and characterized their efficiency ratio. We then extended the randomized algorithms from [18] to fading channels. We further proposed low-complexity and myopic distributed randomized algorithms and characterized their efficiency ratio. Investigating more efficient low-complexity and distributed randomized algorithms could be an interesting future research.
REFERENCES

[1] L. Tassiulas and A. Ephremides, “Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks,” in 29th IEEE Conference on Decision and Control, 1990, pp. 2130–2132.

[2] C. Joo, X. Lin, and N. B. Shroff, “Understanding the capacity region of the greedy maximal scheduling algorithm in multihop wireless networks,” IEEE/ACM Transactions on Networking (TON), vol. 17, no. 4, pp. 1132–1145, 2009.

[3] A. Dimakis and J. Walrand, “Sufficient conditions for stability of longest-queue-first scheduling: Second-order properties using fluid limits,” Advances in Applied probability, vol. 38, no. 2, pp. 505–521, 2006.

[4] J. Ghaderi and R. Srikant, “On the design of efficient CSMA algorithms for wireless networks,” in 49th IEEE Conference on Decision and Control (CDC). IEEE, 2010, pp. 954–959.

[5] J. Ni, B. Tan, and R. Srikant, “Q-CSMA: Queue-length-based CSMA/CA algorithms for achieving maximum throughput and low delay in wireless networks,” IEEE/ACM Transactions on Networking (TON), vol. 13, no. 3, pp. 825–836, 2012.

[6] D. Shah and J. Shin, “Delay optimal queue-based CSMA,” in ACM SIGMETRICS Performance Evaluation Review, vol. 38, no. 1. ACM, 2010, pp. 373–374.

[7] C. Lu, A. Saifullah, B. Li, M. Sha, H. Gonzalez, D. Gunatilaka, C. Wu, L. Nie, and Y. Chen, “Real-time wireless sensor-actuator networks for industrial cyber-physical systems,” Proceedings of the IEEE, vol. 104, no. 5, pp. 1013–1024, 2015.

[8] J. Song, S. Han, A. Mok, D. Chen, M. Lucas, M. Nixon, and W. Pratt, “WirelessHart: Applying wireless technology in real-time industrial process control,” in 2008 IEEE Real-Time and Embedded Technology and Applications Symposium. IEEE, 2008, pp. 377–386.

[9] J. Gubbi, R. Buyya, S. Marusic, and M. Palaniswami, “Internet of things (iot): A vision, architectural elements, and future directions,” Future generation computer systems, vol. 29, no. 7, pp. 1645–1660, 2013.

[10] I. Hou, V. Borkar, and P. R. Kumar, “A theory of QoS for wireless,” in Proc. IEEE International Conference on Computer Communications (INFOCOM), Rio de Janeiro, Brazil, April 2009.

[11] I. Hou and P. R. Kumar, “Admission control and scheduling for QoS guarantees for variable-bit-rate applications on wireless channels,” in Proc. ACM international symposium on Mobile ad hoc networking and computing (MOBIHOC), New Orleans, Louisiana, May 2009.

[12] “Scheduling heterogeneous real-time traffic over fading wireless channels,” in Proc. IEEE International Conference on Computer Communications (INFOCOM), San Diego, California, March 2010.

[13] J. Jaramillo and R. Srikant, “Optimal scheduling for fair resource allocation in ad hoc networks with elastic and inelastic traffic,” in Proc. IEEE International Conference on Computer Communications (INFOCOM), San Diego, California, March 2010.

[14] B. Li and A. Eryilmaz, “Optimal distributed scheduling under time-varying conditions: A fast-csma algorithm with applications,” IEEE Transactions on Wireless Communications, vol. 12, no. 7, pp. 3278–3288, 2013.

[15] R. Singh and P. Kumar, “Throughput optimal decentralized scheduling of multihop networks with end-to-end deadline constraints: Unreliable links,” IEEE Transactions on Automatic Control, vol. 64, no. 1, pp. 127–142, 2018.

[16] X. Kang, W. Wang, J. J. Jaramillo, and L. Ying, “On the performance of largest-deficit-first for scheduling real-time traffic in wireless networks,” IEEE/ACM Transactions on Networking, vol. 24, no. 1, pp. 72–84, 2014.

[17] X. Kang, I.-H. Hou, L. Ying et al., “On the capacity requirement of largest-deficit-first for scheduling real-time traffic in wireless networks,” in Proceedings of the 15th ACM International Symposium on Mobile Ad Hoc Networking and Computing. ACM, 2015, pp. 217–226.

[18] C. Tsankikidis and J. Ghaderi, “On the power of randomization for scheduling real-time traffic in wireless networks,” arXiv preprint arXiv:2001.05146, 2020.

[19] I.-H. Hou, “Scheduling heterogeneous real-time traffic over fading wireless channels,” IEEE/ACM Transactions on Networking, vol. 22, no. 5, pp. 1631–1644, 2013.

[20] E. B. Dynkin, Theory of Markov processes. Courier Corporation, 2012.

[21] M. E. Neely, “Queue stability and probability 1 convergence via lyapunov optimization,” arXiv preprint arXiv:1008.5519, 2010.