NONLOCAL TIME-POROUS MEDIUM EQUATION: WEAK SOLUTIONS AND FINITE SPEED OF PROPAGATION

JEAN-DANIEL DJIDA*
Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

JUAN J. NIETO AND IVÁN AREA
Departamento de Estatística, Análise Matemática e Optimización, Instituto de Matemáticas Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
Departamento de Matemática Aplicada II, E.E. Aeronáutica e do Espazo, Universidade de Vigo, Campus As Lagoas s/n, 32004 Vigo, Spain
(Communicated by Tomas Caraballo)

Abstract. We study a fractional time porous medium equation with fractional potential pressure. The initial data is assumed to be a bounded function with compact support and fast decay at infinity. We establish existence of weak solutions for which we determine whether the property of compact support is conserved in time depending on some parameters of the problem. Special attention is paid to the property of finite propagation for specific values of the parameters.

1. Introduction. In this paper we study the existence and finite speed of propagation of the following nonlocal time evolution equation

\[
\begin{align*}
\frac{\partial^s w(x,t)}{\partial t^s} &= \nabla \cdot (w^{m-1} \nabla p) + f, \quad p = (-\Delta)^{-s} w, \quad \text{for } x \in \mathbb{R}^N, \quad t > 0, \\
w(0,x) &= w_0(x) \quad \text{for } x \in \mathbb{R}^N,
\end{align*}
\]

for exponents \( m \geq 2, \quad 0 < s < 1, \quad 0 < s < 1 \) and both \( w \) and \( f \) are nonnegative. As for the time derivative we shall consider the extended Caputo or Marchaud fractional derivative \([38, 30, 3, 15]\), which is nonlocal and takes into account the past. The pressure \( p \) takes into consideration nonlocal effects through the inverse fractional laplacian operator \( (-\Delta)^{-s} \), that is the Riesz potential of order \( 2s \) \([47, 45]\), which depends linearly on the density function \( w \) according to the Darcy law. Our aim is to construct weak solutions and analyze finite speed of propagation for initial data \( w_0 : \mathbb{R}^N \to [0, \infty) \) and the right hand side \( f \) being bounded with compact support and having exponential decay at infinity.

2010 Mathematics Subject Classification. Primary: 35B65, 26A33; Secondary: 35K55.

Key words and phrases. Nonlinear fractional diffusion, fractional Laplacian, fractional derivatives, existence of weak solutions, finite speed of propagation, free boundary.

This work has been partially supported by the Agencia Estatal de Investigación (AEI) of Spain under grant MTM2016–75140-P, co-financed by the European Community fund FEDER, and Xunta de Galicia, grants GRC 2015–004 and R 2016/022.

* Corresponding author: jeandaniel.djida@usc.es.
The model (1) arises from the consideration of a continuum, which formally resembles the porous medium equation (PME) \[\partial_t w = \nabla \cdot (w^{m-1} \nabla w)\] when \(\gamma = 1, s = 0,\) and \(f = 0.\) If \(\gamma = 1\) and \(f = 0,\) we observe a dependence via the inverse fractional Laplacian operator, \(\partial_t w = \nabla \cdot (w^{m-1} \nabla p)\) with \(p = (-\Delta)^{-s} w,\) which accounts for nonlocal effects in the diffusive process \([44].\)

The problem for \(\gamma = 1, m = 2\) and \(f = 0\) was studied by Caffarelli and Vázquez starting with \([19, 18,\) followed later on by \([17, 22, 26, 27].\) In \([19]\) existence of weak solutions is proved for initial data \(w_0 \in L^1 \cap L^\infty\) via an approximation method that requires a suitable decay of the data.

Our model is the general case for the problem \(\partial_t^\gamma w = \nabla \cdot (w \nabla p) + f\) studied by the authors in \([4]\). The authors successfully adapted the technique of “true exaggerated supersolution” introduced in \([19, 18]\) to study the existence of weak solutions and the Hölder regularity solution.

We are specially interested in better understanding well-posedness and velocity of propagation when the fractional time derivative which takes into account the “memory” effect of the solution is involved. We point out that some materials have the so-called thermal memory \([35]\). On the other hand, when the permeability of the medium changes over time such as in porous medium equation, it might be interesting to use a fractional time derivative.

In our model (1) we have chosen the Caputo fractional derivative, first for its well established theory and moreover for its resemblance with the Marchaud derivative and fractional Laplacian \([2, 3]\). Indeed several different classes of fractional differential operators have been defined: from the classical Riemann–Liouville, Caputo, Marchaud, Weyl, and Grünwald–Letnikov formulae \([30, 38, 39, 42]\) to more recently developed models such as Caputo–Fabrizio \([21]\) or Atangana–Baleanu \([7]\), just to cite some of them. Each of these various nonlocal fractional models has properties and applications which the others do not, and they all present different problems and challenges in their analysis (see, e.g. \([5, 8, 9, 23, 24, 36, 41]\) and references therein).

It turns out that our problem (1) has quite interesting properties like, finite speed of propagation for \(m \geq 2\) which was left out in \([4]\). In \([33]\) the authors studied non-autonomous evolution equations with spatially variable exponents appearing in porous media.

Since the scope of the paper is based on the well-posedness and finite speed of propagation, we will not discuss about the asymptotic behaviour of the solution which is also of great importance \([29]\). This analysis is now under consideration. The paper combines several compactness techniques, based on the method introduced in \([19, 18]\). The main difficulties of the construction are: the nonlocal and nonlinear character of the equation, absence of comparison principle, as well as absence of explicit self-similar solutions when the fractional derivative is involved.

We will also discuss the notion of finite speed of propagation for the PME with nonlocal space and time operators. This phenomenon could take place when \(w_0 = 0;\) then the pressure \(p = 0\) and the equation becomes degenerate. This degeneracy result is the phenomenon of the finite speed of propagation. This of course is also related to the notion of free boundary. So one could show that under certain assumptions on the initial data, the free boundary \(\Gamma = \partial \{w(x, t) > 0\}\), is a smooth surface when \(0 < t < T,\) for some \(T > 0.\) If \(w_0\) is nonnegative, integrable and with compact support, then the Cauchy problem for the PME admits a unique solution on \((-\infty, T) \times \mathbb{R}^N\) which has constant mass. Furthermore, since the equation

\[\partial_t w = \nabla \cdot (w^{m-1} \nabla w)\]
becomes degenerate when $w_0 = 0$, the solution is not expected to be smooth. The physical interpretation of the equation points out that under ideal conditions, the free boundary should be a smooth surface and the pressure a smooth function up to the interface. In order to discuss these properties, we would like to mention that there are some previous works in that direction in the case of the local PME \cite{46, 19, 44, 6, 14} and also in the nonlocal versus where the potential pressure $p = (-\Delta)^{-s}$ is nonlocal and implies the fractional Laplacian \cite{44, 12}.

As one of the novelties of this work, we investigate the property of the speed of propagation for the fully nonlocal PME, where the operator in time involved is a fractional time derivative operator with a nonlocal kernel.

The main result on the existence of weak solution reads as follows.

**Theorem 1.1.** Let $0 \leq w_0(x) \leq Ae^{-a|x|}$ and $0 \leq f(t, x) \leq Ae^{-|x|}$ both satisfy the exponential bounds decay for some $a, A \geq 0$, and let us assume that $w_0 \in C^2$. Then, there exists a solution $w$ to (5) in $(0, \infty) \times \mathbb{R}^N$ with right hand side $f$ and $w(0, x) = w_0$.

In order to state and prove the result regarding the finite speed of propagation, we shall consider the method developed by Caffarelli and Vázquez in \cite{19}. A similar method has also been used in \cite{44} to establish existence of a class of weak solutions for which the properties of compact support is conserved. This was for the model with nonlocal PME with fractional potential pressure only.

We prove that for $m \in [2, 3)$, whenever the parameter $0 < \gamma < 1$ and $0 < s < \frac{1}{2}$, the solution becomes compactly supported for all $t \geq 0$ for a given compacted initial data $w_0$. Our quantitative estimate does not involve any control of the $L^1$ norm and the speed of propagation obtained is influenced by the time parameter $\gamma$. We would like to notice here that in the limit as $\gamma \uparrow 1$, we recover the speed of propagation known for the case of PME with long-range interaction and nonlocal potential pressure. Furthermore the solution is bounded for all the time and supported in the complement of the ball $B_r(t)$. This is summarized in the Theorem just below.

**Theorem 1.2.** Let $0 < \gamma < 1$ and $0 < s < \frac{1}{2}$. Assume $w$ is a bounded weak solution $0 \leq w \leq L$ to (5) with the nonlocal fractional potential pressure $p = (-\Delta)^{-s}w$. Assume that $w_0(x) = w(x, 0)$ has compact support. Then $w(\cdot, t)$ is compactly supported for all $t \geq 0$. In other words, if $w_0$ is below the parabola like function

$$\tilde{w}_0 = a(|x| - b)^2,$$

for some constant $a, b > 0$, with support in the ball $B_r(0)$ then there exists a large enough constant $C$, such that

$$w(x, t) \leq a(Ct - (|x| - b))^2,$$

with the finite speed of propagation

$$C(L, a) = C(1, 1)L \frac{2m-3+2s}{2s} a^{\frac{1-2s}{s}}.$$
Let us also assume that \( w_0 \) vanishes a.e. in a ball \( B_r(x_0) \subset B_R \). Then, there exists a time \( t_1 \) such that for every \( 0 < t < t_1 \) the solution \( w(t) \) vanishes at least in a smaller ball \( B_{r(t)}(x_0) \) with \( 0 < r(t) \leq R \), providing \( |x| \leq R \) and the function \( r(t) \) is monotone non-increasing. In these conditions we obtain an estimate for the free boundary point of the form
\[
|x(t)| \leq R + C_2 t^{\gamma/(2-2s)} ,
\]
if \( 0 < \gamma < 1 \) and \( 0 < s < 1/2 \).

The paper is organized as follows. In Section 2, we give some definitions and properties of the fractional derivative, fractional Laplacian and inverse fractional Laplacian. Next, we provide the weak formulation of the problem and recall some functional inequalities and some useful lemmas. In Section 3, we start with a regularized version of the problem (1). Later on, we provide an exponential tail control for \( m \geq 2 \). We end the section with the proof of existence and uniqueness of solutions stated in Theorem 1.1. Section 4 is devoted for the proof of finite speed of propagation and free boundary property solution for the problem (1) given by Theorem 1.2.

We recall some notation that will be intensively used throughout the paper:

- \( \gamma \) denotes the order the extended Caputo derivative or Marchaud derivative.
- \( s \) denotes the order of the spatial fractional operator associated to the fractional Laplacian.
- \( a \) stands for the initial time for which our equation is defined.
- \( N \) refers to the space dimension.
- \( \Lambda_1, \Lambda_2 \) denotes the elliptic positive constants which gives the bound of the kernel of the fractional derivative.
- \( C \) refers to the time length of the discrete approximation.
- \( t, \tau \) denote time variables.
- \( B_R \) denotes the ball with radius \( R \).

2. Preliminary results. In this section we recall some previous results as well as we prove some new results that will be useful in our further analysis.

2.1. The fractional time derivative. Among the different fractional derivatives existing in the literature, in this paper we consider the extended Caputo or Marchaud derivative [20, 38]. The usual Caputo derivative for \( 0 < \gamma < 1 \) is defined by
\[
^C_a D^\gamma_t v(t) := \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-\tau)^{-\gamma} v'(\tau) d\tau .
\]
By using integration by parts and proceeding as defined in [30, 3, 15, 1, 16], defining \( v(t) \equiv v(a) \) for \( t < a \), the extended form or the Marchaud derivative is defined as
\[
\partial^\gamma_t v(t, \cdot) = \gamma \int_{-\infty}^t [v(t, \cdot) - v(\tau, \cdot)] \mathcal{K}(t, \tau) \, d\tau .
\] (2)

The kernel \( \mathcal{K} \) also satisfies the conditions
\[
\mathcal{K}(t, t-\tau) = \mathcal{K}(t+\tau, t) \quad \text{and} \quad \frac{\Lambda_1}{(t-\tau)^{1+\gamma}} \leq \mathcal{K}(t, \tau) \leq \frac{\Lambda_2}{(t-\tau)^{1+\gamma}} .
\] (3)

The formulation (2) is also known as the Marchaud derivative [42, 38, 10]. The reason of working with formulation (2) is that it allows one to easily utilize the nonlocal nature of the fractional time derivative for regularity purposes. This was
succefully accomplished for divergence problems in \([3]\) as well as for non-divergence problems in \([1, 2]\).

**Proposition 1.** Let \(g, h \in C^1(\mathbf{a}, T)\). Then

\[
\int_0^T u D_t^\gamma h + h D_t^\gamma u = \int_0^T u(t) h(t) \left[ \frac{1}{(T - t)^\gamma} + \frac{1}{(t - a)^\gamma} \right] dt \\
+ \gamma \int_0^T \int_a^t \frac{[u(t) - u(\tau)][h(t) - h(\tau)]}{(t - \tau)^{1+\gamma}} d\tau dt - \int_0^T u(t) h(a) + h(t) u(a) \frac{dt}{(t - a)^\gamma}.
\] (4)

We now give the exact formulation of our weak solutions. We say that \(w\) is a weak solution if for any \(\xi \in C_0^\infty(\mathbb{R}^N)\) we have

\[
\int_{\mathbb{R}^N} \int_{-\infty}^T \int_{-\infty}^t [w(t, x) - w(\tau, x)][\zeta(t, x) - \zeta(\tau, x)] \mathcal{K}(t, \tau, x) d\tau dt dx
\]

\[
+ \int_{\mathbb{R}^N} \int_{-\infty}^T \int_{-\infty}^{2-T} w(t, x) \xi(t, x) \mathcal{K}(t, \tau, x) d\tau dt dx - \int_{\mathbb{R}^N} \int_{-\infty}^T \int_{-\infty}^T w(t, x) D_t^\gamma \zeta(t, x) d\tau dx
\]

\[
+ \int_{-\infty}^T \int_{\mathbb{R}^N} \nabla \zeta(t, x) w(t, x) \nabla (-\Delta)^{-\gamma} w dx dt = \int_{\mathbb{R}^N} \int_{B_R} f(t, x) \zeta(t, x).\] (5)

We will also utilize a fractional Sobolev norm that arises from the fractional derivative.

**Lemma 2.1.** Let \(w\) be defined on \([a, T]\). We have for two constants \(c_1, c_2\) depending on \(\gamma, |T - a|\)

\[
\|w\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R}^N, a, T)} \leq c_1 \|w\|_{H^{\gamma/2}(\mathbb{R}^N, a, T)}^2 \leq c_2 \left( \gamma \int_a^T \int_a^t \frac{|w(t) - w(\tau)|^2}{|t - \tau|^{1+\gamma}} d\tau dt + \int_a^T \frac{w^2(t)}{(T - a)^\gamma} \right).
\]

We point out that if \(u = u_+ - u_-\) (positive and negative parts, respectively), then

\[
\int_a^T u_\pm(t) D_t^\gamma u_\pm(t) \geq 0.\] (6)

Next we recall the following lemma from \([3]\). For a given convex function \(F\) with \(F'' \geq \gamma, F' \geq 0, F(0) = 0\) and under the assumption that \(u \geq 0, u(0) = 0\), then there exists a constant \(c\) depending on \(\gamma\) and \(A\) such that

\[
\varepsilon \sum_{j \leq k} F(u(\varepsilon j)) D_t^\gamma u(\varepsilon j) \geq \varepsilon \sum_{j \leq k} \frac{F(u(\varepsilon j))}{(\varepsilon(j - i))^{1+\gamma}} + c_2 \int_{-\infty}^t \int_{-\infty}^t \frac{|u(\varepsilon j) - u(\varepsilon i)|^2}{(\varepsilon(j - i))^{1+\gamma}} d\tau dt.
\]

In the continuous version, if \(u\) is a limit of \(u_\varepsilon\), then it follows that

\[
\int_a^T F(u(t)) D_t^\gamma u(t) dt \geq \gamma \int_a^T \int_a^t \frac{F(u(t))}{(T - a)^\gamma} + c_2 \int_a^T \int_a^t \frac{|u(t) - u(\tau)|^2}{(t - \tau)^{1+\gamma}} d\tau dt.\] (7)

Next we recall the so-called Mainardi function which is a particular Wright function \([37]\)

\[
\mathcal{M}_\gamma(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n! \Gamma(-\gamma n + 1 - \gamma)} = \frac{1}{2i\pi} \int_G \lambda^{\gamma-1} e^{(\lambda-z)\gamma} d\lambda, \quad 0 < \gamma < 1,\] (8)
where \( G \) is a contour which starts and ends at \(-\infty\) and encircles the origin once clockwise. We also have the following relation between the Wright function and the Mittag-Leffler function:

\[
E\gamma(z) = \int_0^\infty \mathcal{M}\gamma(t) e^{zt} \, dt, \quad 0 < \gamma < 1. \tag{9}
\]

### 2.2. The fractional Laplacian and the inverse operator

In this part, we recall some definitions and basic notions for the functional setting of the problem related to the fractional Laplacian and its inverse. All the definitions, lemmas and theorems we recall here are basically borrowed from [47, 28, 44].

Let \( F \) denote the Fourier transform. For given \( s \in (0, 1) \) we consider the space

\[
H^s(\mathbb{R}^N) := \left\{ w : L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\mathcal{F}w(\xi)|^2 d\xi < +\infty \right\},
\]

with the norm

\[
\|w\|_{H^s(\mathbb{R}^N)} := \|w\|_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}w(\xi)|^2 d\xi.
\]

For functions \( w \in H^s(\mathbb{R}^N) \), the fractional Laplacian operator is defined by

\[
(-\Delta)^s w(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x) - w(y)}{|x - y|^{N+2s}} dy = F^{-1}(|\xi|^{2s} |\mathcal{F}w|),
\]

for \( x \in \mathbb{R}^N \), where \( C_{N,s} = \pi^{-s} \Gamma(N/2 + s) / \Gamma(-s) \). Then,

\[
\|w\|_{H^s(\mathbb{R}^N)} = \|w\|_{L^2(\mathbb{R}^N)} + C \|(-\Delta)^{s/2} w\|_{L^2(\mathbb{R}^N)}.
\]

The inverse operator \( K_s := (-\Delta)^{-s} \) coincides with the Riesz potential of order \( 2s \) and can be represented by convolution with the Riesz kernel \( K_s \) for \( N > 2s \):

\[
(-\Delta)^{-s} w = K_s \ast w, \quad K_s(x) = \frac{1}{c(N,s)} |x|^{-(N-2s)},
\]

where \( c(N,s) = \pi^{N/2 - 2} \Gamma(s) / \Gamma((N-2s)/2) \). When \( N = 1 \) and \( s \in [1/2, 1) \) we have to consider the composed operator \( V(-\Delta)^{-s} \) (see [11]).

Let \( \epsilon > 0 \) and \( w : \mathbb{R}^N \to \mathbb{R} \). The approximated fractional Laplacian operator is defined by

\[
\mathcal{K}_\epsilon^s[w](x) := C_{N,s} \int_{\mathbb{R}^N} \frac{w(x) - w(y)}{|x - y|^2 + \epsilon^2} dy. \tag{10}
\]

For any \( \epsilon > 0 \), \( \mathcal{K}_\epsilon^s \) is an integral operator with non-singular kernel and \( \mathcal{K}_\epsilon^s[w] \to (-\Delta)^s w \) pointwise in \( \mathbb{R}^N \) as \( \epsilon \to 0 \) for suitable functions \( w \).

Next we define the bilinear form

\[
\mathcal{E}_\epsilon(w, v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))(v(x) - v(y))}{(|x - y|^2 + \epsilon^2)^{N+2s/2}} \, dx dy \quad \text{for } w, v \in D(\mathcal{K}_\epsilon^s),
\]

and the quadratic form

\[
\mathcal{E}_\epsilon(w) := \mathcal{E}_\epsilon(w, w) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{(|x - y|^2 + \epsilon^2)^{N+2s/2}} \, dx dy.
\]

The bilinear form \( \mathcal{E}_\epsilon \) is well defined for functions in the space \( \dot{H}^s_\epsilon(\mathbb{R}^N) \), which is the closure of \( C_c^\infty(\mathbb{R}^N) \) with respect to the Gagliardo seminorm given by \( \overline{\mathcal{E}}_\epsilon \). We define

\[
H^s_\epsilon(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \mathcal{E}_\epsilon(u) < \infty \right\}. \tag{11}
\]

The space \( H^s_\epsilon(\mathbb{R}^N) \) is endowed with the standard norm.
The main idea is to consider a regularized version of (1) where all the terms that might cause blow up are approximated. We call the regularized solution of \( w \) the zero level set by putting \( \psi = 0 \).

Furthermore, we recall that
\[
H^s(\mathbb{R}^N) \subset H^s_{\epsilon_1}(\mathbb{R}^N) \subset H^s_{\epsilon_2}(\mathbb{R}^N) \quad \text{for} \quad 0 < \epsilon_1 < \epsilon_2.
\]

We refer to [44] for a precise discussion of these spaces in a more general framework.

Next we state the following generalized Stroock-Varopoulos inequality [44, Lemma 2.2].

**Lemma 2.2 (Generalized Stroock-Varopoulos Inequality for \( K^s_c \)).** Let us assume that \( w \in H^s_c(\mathbb{R}^N) \), and let \( \psi : \mathbb{R} \to \mathbb{R} \) such that \( \psi \in C^1(\mathbb{R}) \) and \( \psi' \geq 0 \). Then
\[
\int_{\mathbb{R}^N} \psi(w)K^s_c[w]dx \geq \int_{\mathbb{R}^N} \left( K^s_c \right)^\frac{1}{2} \left[ \psi(w) \right] dx,
\]
where \( \psi' = (\Psi')^2 \).

**3. Existence of weak solutions.** In order to construct existence of weak solutions of (1) for the general initial data \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and for a well behaved forcing term \( f \in L^\infty(Q) \) having exponential decay at infinity, we proceed by the regularization method, which will ensure compactness, together with an \( L^1 - L^\infty \) smoothing effect.

### 3.1. A regularized problem.

The main idea is to consider a regularized version of (1) where all the terms that might cause blow up are approximated. We call \( u \) the regularized solution of \( w \). We add a vanishing viscosity term \( \delta \Delta u \) to (1) that ensures good properties of regularity for solution; we eliminate the degeneracy at the zero level set by putting \( w^{m-1} \sim (w+\mu)^{m-1} \). Next we eliminate the singularity character of the fractional derivative by discrete approximation. Restricting our problem (1) in a bounded domain \( B_R \) with radius \( R \), the regularized approximated problem reads
\[
\begin{cases}
\frac{\partial}{\partial t} D^\gamma_{\nu} u = \delta \Delta u + \nabla \cdot (\mu(u) \nabla K^s_c u) + f & \text{in} \quad B_R \times (0, T), \\
u(x, 0) = \tilde{u}_0(x) & \text{in} \quad B_R, \\
u(x, t) = 0 & \text{in} \quad B_R \times (0, T),
\end{cases}
\]
depending on the parameters \( \delta, \mu, R > 0 \), and we define \( \mu(u) = (u+\mu)^{m-1} : \mathbb{R} \to \mathbb{R} \).

We say that \( u \) is a weak solution of (14) if
\[
\int_0^T \int_{B_R} \zeta D^\gamma_{\nu} u \, dx \, dt - \int_0^T \int_{B_R} \left( \delta \nabla u + (u+\mu)^{m-1} \nabla K^s_c u \right) \cdot \nabla \zeta \, dx \, dt
= \int_0^T \int_{B_R} f \zeta \, dx \, dt
\]
for smooth test functions \( \zeta \in C_0^\infty \) that vanish on the spatial boundary \( \partial B_R \) and for large \( t \).

### 3.2. Solution representation of the regularized problem.

Indeed, existence of smooth weak solutions of (14) is proved via mild solutions, i.e, \( u \) is the fixed point of (14). We look for the solution representation of the abstract fractional differential equation associated to (14). More precisely, we consider the fractional Cauchy problem for initial data \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and \( f \in L^\infty(Q) \) and having exponential decay at infinity as
\[
\begin{cases}
\frac{\partial}{\partial t} D^\gamma_{\nu} u(t) = Au(t) + \nabla \cdot \Psi(u)(t) + f(t), & t \in (0, T], \\
u(x, 0) = u_0(x),
\end{cases}
\]
where \( A u := \delta \Delta u \), \( (X, \| \cdot \|_X) \) is a Banach space, \( \gamma \in (0, 1) \), \( u_0 \in D(A) \) and \( \Psi(u) = (u + \mu)^{m-1} \nabla \mathcal{K}_\delta u \). The operator \( A : D(A) \subset X \to X \) is a densely defined linear operator which fulfills the following assumption [40]:

- (H1) the operator \( A \) is \( m \)-accretive in \( X \); this means maximal monotone.
- (H2) The operator \( A \) is the generator of a semigroup of contractions \( (Q(t))_{t \geq 0} \).

\[
\sup_{t \geq 0} \| Q(t) \|_{BX} \leq 1, \quad (17)
\]

where \( (B(X), \| \cdot \|_X) \) is a Banach space of all linear bounded operators on \( X \).

\( A \) is an \( m \)-accretive operator in \( L^2 \) and \( (A, D(A)) = (\delta \Delta, H^2(B_R) \cap H^1_0(B_R)) \), for \( \delta > 0 \) small enough. Hence \( A \) is a generator of contraction semigroup [40, 25].

**Remark 1.** It is worth to mentioned that for \( \gamma = 1 \) and \( f = 0 \) in (16), we fall in the classical case from which by Duhamel’s principle (see also [11, Proposition 5.2]), the solution representation is given by

\[
u(t) = e^{t \Delta} u_0 + \int_0^t \nabla e^{(t-\tau)\Delta} \cdot \Psi(u(\tau)) \, d\tau \quad \text{with} \quad \Psi(u) = (u + \mu)^{m-1} \nabla \mathcal{K}_\delta u, \quad (18)
\]

in \( (C[0, T], L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N) \) where \( e^{t \Delta} \) denotes the heat semigroup. The map

\[
T : u \mapsto e^{t \Delta} u_0 + \int_0^t \nabla e^{(t-s)\Delta} \cdot \Psi(u(s)) \, ds,
\]

with \( X = L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) has a fixed point by the Banach contraction principle as soon as \( T = T(||u_0||_X) > 0 \) is sufficiently small [11].

For \( f \neq 0 \), one can consider the classical elliptic problem

\[
gu - \delta \Delta u - \nabla \cdot \Psi(u) = f, \quad \text{on} \quad B_R \quad (19)
\]

with \( u \equiv 0 \) on \( \partial B_R \), \( g, f \geq 0 \) are smooth in \( L^\infty(Q) \). As it has been shown in [4, 31] for \( h \in C_0^{0, \beta} \) with \( h_1 \geq 0 \) we can apply Schauder’s estimate theory and by bootstrapping argument to conclude that

\[
\| u \|_{C^{1, \beta}} \leq C(m, \delta, s) \| h \|_{C_0^{0, \beta}} .
\]

This implies that the map \( T : h_1 \to u \) is a compact map. So the set \( \{h_1\} \) is a closed convex set. Hence we can apply the fixed point theorem [31] to conclude that there exists a solution to (19). Hence, the existence of the classical case for \( \gamma = 1 \) is recovered.

**Proposition 2.** Let \( u_0 \in X \) and \( f \in L^\infty(Q) \). Assume that (H1) and (H2) hold true. Let \( u \) be the solution representation to (16). Then

\[
u(t) = \mathcal{P}_1(t) u_0 + \int_0^t \nabla \cdot \left( \mathcal{P}_2(t-\tau) \Psi(u(\tau)) \right) \, d\tau + \int_0^t \mathcal{P}_2(t-\tau) f(\tau) \, d\tau, \quad (20)
\]

where

\[
\mathcal{P}_1(t) = \int_0^\infty \mathcal{M}_\gamma(\zeta) \mathcal{Q}(t^\gamma \zeta) \, d\zeta, \quad \text{and} \quad \mathcal{P}_2(t) = \gamma \int_0^\infty \zeta t^{\gamma-1} \mathcal{M}_\gamma(\zeta) \mathcal{Q}(t^\gamma \zeta) \, d\zeta. \quad (21)
\]

The proof of this proposition is provided in Appendix A.

The next step will be to show that the solution \( w \in C([0, T], X) \) can be seen as a fixed point of

\[
\mathcal{P} : w \mapsto \mathcal{P}_1(t) w_0 + \int_0^t \nabla \cdot \left( \mathcal{P}_2(t-\tau) \Psi(w(\tau)) \right) \, d\tau + \int_0^t \mathcal{P}_2(t-\tau) f(\tau) \, d\tau. \quad (22)
\]
which maps $\overline{B}(0, R)$ into itself, and is a contraction.

3.3. **Contraction.** Next we give the following lemma which state the contraction property for the map $\mathcal{P}$ defined in (22).

**Proposition 3.** For all $\gamma, s \in (0, 1)$, there exist a positive $T \in (0, 1)$ depending only on the initial data $u_0$, and a function $u \in C([0, T], X)$ such that (22) holds true. Moreover the operator $\mathcal{P}$ maps $C([0, T], X)$ into itself, and there exist $C > 0$ and $\kappa > 0$ such that for all $u, v \in \overline{B}(0, R) \subset C([0, T], X)$,

$$
\|\mathcal{P}(u) - \mathcal{P}(v)\|_{C([0,T],X)} \leq C_2(R)T^\kappa\|u - v\|_{C([0,T],X)},
$$

(23)

with $C_2(R)$ is a constant which depends on $\gamma, s, N, m, \delta$.

To prove this lemma, we adapted the proof of [11, Lemma 5.4].

**Proof.** From (22) we write the difference of $\mathcal{P}$'s as

$$
\mathcal{P}(u)(t) - \mathcal{P}(v)(t) = \mathcal{P}_1(t)(u_0 - v_0) + \int_0^t \nabla \cdot \mathcal{P}_2(t - \tau)(\Psi(u(\tau)) - \Psi(v(\tau)))d\tau
$$

$$
= \int_0^t \nabla \cdot \mathcal{P}_2(t - \tau)(\Psi(u(\tau)) - \Psi(v(\tau)))d\tau.
$$

We consider the estimate of $\|\nabla \mathcal{P}_2v\|_p$ (see for example [32, 34] were this type of estimate where successfully computed)

$$
\|\nabla \mathcal{P}_2v\|_p = C_1(\gamma, p, r)\tau^{-(1-\gamma)-\frac{mr}{2}(\frac{1}{r} - \frac{1}{2}) - \frac{3}{2}}\|v\|_p,
$$

(24)

for $1 \leq r \leq p \leq \infty$.

Now using the inequality (24), for $\langle p, r \rangle = (1, 1)$ and $\langle p, r \rangle = (\infty, q)$, respectively, we have that

$$
\|\mathcal{P}(u) - \mathcal{P}(v)\|_{C([0,T], L^1(\mathbb{R}^N))} \leq C_0(R)C_1(\gamma, p, r)\|u - v\|_{C([0,T], X)} \int_0^T t^{-1+\frac{1}{2}}dt
$$

$$
\leq C_1(\gamma, p, N, r, C_0(R))T^{-\kappa}\|u - v\|_{C([0,T], X)},
$$

(25)

and

$$
\|\mathcal{P}(u) - \mathcal{P}(v)\|_{C([0,T], L^\infty(\mathbb{R}^N))} \leq C_0(R)C_1(\gamma, p, r)\|u - v\|_{C([0,T], X)} \int_0^T t^{-1+\frac{1}{2}(1-\frac{N}{q^\#})}dt
$$

$$
\leq C_3(\gamma, p, N, r, C_0(R))T^{-\kappa}\left(1-\frac{N}{q^\#}\right)\|u - v\|_{C([0,T], X)}.
$$

(26)

Finally, combining (22), (25) and (26), we obtain the desired estimate (23) for $s, \gamma \in (0, 1)$, with $\kappa = \frac{1}{2} - \frac{\gamma N}{2q^\#}$, now with a new constant $C_2(R) = C(\gamma, s, N, q^\#, R, G'(2R))$, where $q^# > N$ have been chosen to ensure $\frac{\gamma N}{2q^#} < \frac{\gamma}{2}$.

**Corollary 2.** For $v = 0$ in the estimate (23) we have that

$$
\|\mathcal{P}(u)\|_{C([0,T], X)} \leq \|u_0\|_X + RC_2(R)T^\kappa.
$$

(27)

Therefore one can choose $R = 2\|u_0\|_X$ and $T > 0$ such that $C_2(R)T^\kappa \leq \frac{1}{2}$ in order to ensure that $\mathcal{P}$ maps $\overline{B}(0, R)$ into itself, hence a contraction.
Now we use the existence of solutions to (19) to obtain –via recursion arguments– solutions to the discretized problem

\[ \mathcal{D}_\varepsilon^\gamma u - \delta \Delta u - \nabla \cdot \left( (u + \mu)^{m-1} \nabla K_s \right) = f \quad \text{on } [0, T] \times B_R, \]

(28)

with \( u(0, x) = u_0(x) \) an initially defined smooth function with compact support, \( \varepsilon = T/k \) for some \( k \in \mathbb{N} \). We will eventually let \( k \to \infty \), so that \( \varepsilon \to 0 \).

We first rewrite the problem (16) in the form

\[ \mathcal{D}_\varepsilon^\gamma Y(t) = cY(t) + h(t), \]

(29)

with \( h(t) \) a nonlinear term. Hence the solution (16) takes the form

\[ Y(t) = Y(0)E_\alpha(ct^\alpha) + \gamma \int_0^t (t - s)^{\alpha-1}E_{\gamma,\gamma}(c(t-s)^\alpha)h(s) \, ds, \]

where \( E_\gamma \) and \( E_{\gamma,\gamma} \) are respectively the Mittag-Leffler and the parametric Mittag-Leffler function of order \( \gamma \).

We will utilize in the next Lemma one specific instance of (29). We define \( Y_1(t) \) to be the solution to (29) with \( Y(0) = \sup u(0, x), c = 0, h = 2\Lambda f \).

**Lemma 3.1.** Let \( u \) be a solution to (28). Then, there exists \( \varepsilon_0 \) depending only on \( T, \gamma, ||f||_{L^\infty} \) such that if \( \varepsilon \leq \varepsilon_0 \), then

\[ u(\varepsilon j) \leq Y_1(\varepsilon j) \]

**Proof.** We have from Proposition 3 that \( Y_1(t) \) is an increasing and bounded function. This means that we can write

\[ \mathcal{D}_\varepsilon^\gamma Y_1(\varepsilon j) \geq \Lambda^{-1} \mathcal{D}_\varepsilon^\gamma Y_1(\varepsilon j) \geq \frac{2}{3\Lambda} \mathcal{D}_\varepsilon^\gamma Y_1(\varepsilon j) = \frac{4}{3} f. \]

Next we use \( (u(t, x) - Y_1(t)+)_+ \) as a test function. Since \( (u - Y_1)+_+(0) = 0 \) it follows from the weak formulation that

\[ \varepsilon \sum_{j < k} \int_{B_R} f(\varepsilon j, x) (u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \, dx \]

\[ = \varepsilon \sum_{j < k} \int_{B_R} (u - Y_1)_+ \mathcal{D}_\varepsilon^\gamma [(u - Y_1)_+ - (u - Y_1)_- + Y_1] \, dx \]

\[ + \delta \varepsilon \sum_{j < k} \int_{B_R} \nabla u \nabla (u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \, dx \]

\[ + \varepsilon \sum_{j < k} \int_{B_R} \left( (u + \mu)^{m-1} \nabla (-\Delta)^{-1}K_{1-s} \right) \varepsilon (u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \, dx \]

Then for \( \varepsilon \) small enough and \( j > 0 \), we have the following estimate

\[ \varepsilon \sum_{j \leq k} \int_{B_R} f(\varepsilon j, x) (u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \]

\[ \geq \int_{B_R} \varepsilon \sum_{j \leq k} \frac{4}{3} (u - Y_1)_+ f + \delta \varepsilon \sum_{j \leq k} \chi_{\{u > Y_1\}} \|\nabla u\|^2 \, dx \]

\[ + \left( (u + \mu)^{m-1} \chi_{\{u > Y_1\}} \nabla u \nabla (-\Delta)^{-1}K_{1-s} \right) (u(\varepsilon j, x) - Y_1(\varepsilon j))_+ \, dx \]

\[ \geq \frac{4}{3} \varepsilon \sum_{j \leq k} \int_{B_R} f(\varepsilon j, x) (u(\varepsilon j, x) - Y_1(\varepsilon j))_+. \]
Thus \((u - Y_1)_+ \equiv 0\).

3.4. Exponential tail control. In order to prove existence and finite speed of propagation properties, the estimation of certain rate of decay of our solutions as \(|x| \to \infty\) is necessary. This method is known as a comparison method with suitable family of barrier functions, that in [19] received the name of “true supersolutions”.

**Lemma 3.2.** Let \(0 < \gamma < 1\), \(0 < s < 1/2\), \(m \geq 2\) and let \(u\) be the solution to problem (14). We assume that \(u\) is bounded \(0 \leq u(x, t) \leq L\) and that \(u_0\) and \(f\) lay below a function of the form

\[
u_0(x) = Ae^{-a|x|}, \quad f = Ae^{-|x|}, \quad A, a > 0.\]

If \(A\) is large enough, then there exists a positive constant \(C\) depending on \(N, \gamma, s, a, L, A, \|u(0, x)\|_{L^\infty}, \|f\|_{L^\infty}\), such that for any \(T > 0\) we will have the comparison

\[
u(x, t) \leq AY_2(\varepsilon j)e^{-|x|},
\]

for all \(x \in \mathbb{R}^N\) and all \(0 < t \leq T\) providing that \(t = \varepsilon j\).

**Proof.** We basically adapt the technique of tail control [19] by constructing some kind of “exaggerated supersolutions” \(Y_2\) solution of (29), with the properties \(Y_2(0) = 2, h = 0,\) and \(c = CA^{-1}\).

- **Reduction.** By scaling we may put \(a = L = 1\). This is done by considering instead of \(u\), the function \(\hat{u}\) defined as

\[
u(x, t) = L\hat{u}(ax, bt), \quad \hat{b}^1 = L^{m-1}a^{2-2s},
\]

which satisfies the equation

\[\partial_t^\alpha \hat{u} = \delta_1 \Delta \hat{u} + \nabla_x (d_e \nabla \hat{u}) + \delta_2 \hat{f},\]

with \(\delta_1 = a^{2s}d/e^{m-1}\) and \(\delta_2 = a^{2s-2}/L^m\).

One should notice that \(\hat{u}(x, 0) \leq A_1e^{-|x|}\) with \(A_1 = A/L\). The corresponding bound for \(\hat{u}(x, t)\) will be \(\hat{u}(x, t) \leq A/L Y_2(\varepsilon j)e^{-|x|}\) with a new constant \(C_1\) embedded in \(Y_2(\varepsilon j)\) as \(C_1 = C/b = C(L^{(m-1)/\gamma}a^{(2-2s)/\gamma})^{-1}\).

- **Contact analysis.** We assume that \(0 \leq u(x, 0) \leq 1\) and also that

\[
u(x, 0) \leq Y_2(0, x)e^{-r} \leq A\nu^{-r}, \quad r = |x| > 0,
\]

where \(A > 0\) is a constant that will be chosen below, say larger than 2.

Next, we consider a radially symmetric candidate for the upper barrier function of the form

\[\hat{u}(x, \varepsilon j) = AY_2(\varepsilon j)e^{-r}.\]

The constant \(C\) embedded in \(Y_2(\varepsilon j)\), will be determined in terms of \(A\) to satisfy a true supersolution condition which is obtained by contradiction at the first point \((x_c, t_c)\) of possible contact of \(u\) and \(\hat{u}\).

Since \(u\) is smooth and continuous, and satisfies a “true supersolution”, then

\[\nu(x, \varepsilon j) \leq \hat{u}(x, \varepsilon j) = LY_2(\varepsilon j)e^{-r} \leq AY_2(\varepsilon j)e^{-r}\]

for some \(L > A\), which could be determined in order to \(LY_2(\varepsilon j)e^{-r}\) to satisfy a “true supersolution” condition. We lower \(L \geq A\) until it touches \(u\) for the first time at the contact point \(u(x_c, t_c) \in Q\).

Note that if there exists a contact point, it cannot happen at the boundary \(\partial B_R\) since \(u = 0\). Also since \(u\) is smooth, the contact cannot happen at a point \((\varepsilon j, 0)\). Furthermore this cannot happen at the initial time due to the fact that \(LY_2(0, x) \geq 2A \geq 2u(0, x)\).
The equation satisfied by \( u \) can be written in the form
\[
\mathcal{C} \partial^2_t u = \delta u + (m - 1) (u + \mu)^{-1} \nabla u \cdot \nabla p + (u + \mu)^{-1} \Delta p + f, \quad p = \mathcal{K}^\gamma_p[u]. \tag{32}
\]
We will obtain necessary conditions in order that (32) holds at the contact point \((x_c, t_c)\). Then, we prove that there exists a suitable choice of parameters \( C, A, \mu \) such that the contact cannot hold.

- **Estimates on \( u \) and \( p \) at the first contact point.**

  For \( 0 < \gamma < 1, 0 < s < 1/2 \), at the first contact point \((x_c, t_c)\) we have the estimates
  \[
  \partial_t u = -AY_2(t_c)e^{-r c}, \quad \Delta u \leq AY_2(t_c)e^{-r c}, \quad \partial^2_t u(t_c) \geq \frac{2}{3} \mathcal{C} \partial^2_t Y_2(t_c).
  \]

  From the assumption that our solution \( u \) is bounded by \( 0 \leq u \leq 1 \), then
  \[
  u(x_c, t_c) = AY_2(t_c)e^{-r c} \leq Y_1(T)K \leq 1. \tag{33}
  \]

  Moreover, from [19] we have the following upper bounds for the pressure term at the contact point for \( 0 < s < 1/2 \):
  \[
  \Delta p(x_c, t_c) \leq K_1 \leq Y_1(T)K_1, \quad (-\partial_t p)(x_c, t_c) \leq K_2 \leq Y_1(T)K_2. \tag{34}
  \]

  Now assume that there exists a first contact for \( t > 0 \), at a space and time of contact point \((x_c, t_c)\). At the contact point \((x_c, t_c)\) with \( r_c = |x_c| \), equation (32) implies that
  \[
  \frac{2}{3} CY_2(t_c)e^{-r c} = \frac{\Lambda}{3} L_0^\gamma Y_2(t_c)e^{-r c} \leq \partial^2_t Y_2(t_c)e^{-r c} \leq \delta Y_2(t_c)e^{-r c} + (m - 1) (u(x_c, t_c) + \mu)^{m-2} (-Y_2(t_c)e^{-r c})(\partial_t p) + (u(x_c, t_c) + \mu)^{m-1} \Delta p + f(t_c, r_c). \tag{35}
  \]

  From (34) with \( K = \max\{K_1 Y_1(T), K_2 Y_1(T)\} \), we obtain from the previous inequality, after simplification by \( LY_2(t_c)e^{-r c} \),
  \[
  \frac{2}{3} C \leq \delta + (m - 1) (u(x_c, t_c) + \mu)^{m-2} K \quad \text{with} \quad (u(x_c, t_c) + \mu)^{m-2} \left(1 + \frac{\mu}{LY_2(t_c)}e^{r c}\right) K \quad \text{and} \quad f(t_c, r_c)e^{r c}.
  \]

  Thus,
  \[
  C \leq 2\delta + 2K (u(x_c, t_c) + \mu)^{m-2} \left(m + \frac{\mu e^{r c}}{LY_2(t_c)}\right) + 2 + f(t_c, r_c)e^{r c}.
  \]

  Moreover, by (33) we have that \( \mu < u(x_c, t_c) + \mu < 1 + \mu \) and \( f(t_c, r_c) \leq Le^{-r c} \). Since \( m \geq 2 \), then
  \[
  C \leq 2\delta + 2K (1 + \mu)^{m-2} (m + \mu) + 2.
  \]

  Choosing \( \delta, \mu \) small enough, we get from the above inequality
  \[
  C \leq 2mK^{m-2} + 2.
  \]

  If we choose now \( C > mK^{m-2} + 1 \), we obtain a contradiction since \( C \) depends on \( N, \|u_0\|_{L^\infty} \) and \( \|f\|_{L^\infty} \). \[ \square \]
3.5. **Some Sobolev estimates.** In what follows, we perform some estimations on the solution of the problem (28). Due to the fact that the tail of the fractional time derivative naturally introduces a right hand side, we cannot choose the natural test function ln(u) as a test function, because this introduces integrability issues since u can evaluate zero. To overcome this difficulty, as shown in [4], we shall make use of a test function of the form

\[ F(t) = \frac{1}{\sigma + 1}(t + \mu)^\sigma - \mu^\sigma t, \]

where \( 0 < \sigma < 1 \), which satisfies the hypothesis of Lemma 3.2. For \( u \) being a solution of (28) and satisfying the condition \( |u| \leq Ke^{-|x|} \) for some large \( K \). The use of the extension of \( u(x, t) = u(0, x) \) for \( j < 0 \), makes that \( u \) is the solution to (28) with nonnegative right hand side

\[ \delta \Delta (u(0, x)) + \nabla \cdot K^\varepsilon_\sigma [u(0, x)], \quad \text{for } j \leq 0. \]

We fix cut-off function \( \zeta(t) \) with \( \zeta(t) \geq K \) for \( t \leq -2 \) and \( \zeta(t) = 0 \) for \( t \geq -1 \) and define

\[ u = \zeta + (u - \zeta)_+ - (u - \zeta)_- =: u_\zeta^+ - u_\zeta^- + \zeta. \]

Following the strategy used in [4], we take our test function as \( \varepsilon F'([u(t, x) - \zeta(t)]) \).

On the other hand \( u \) is weak solution of (28) and satisfies

\[
\varepsilon \sum_{j \leq k} \int_{B_R} \zeta \partial_t^j u(\varepsilon j) dx + \varepsilon \delta \sum_{j \leq k} \int_{B_R} \nabla \zeta \cdot \nabla u(\varepsilon j) dx \\
+ \varepsilon \sum_{j \leq k} \int_{B_R} \nabla \zeta (u(\varepsilon j) + \mu)^{m-1} \nabla K^\varepsilon_\sigma[u] dx = \varepsilon \sum_{j \leq k} f(\varepsilon j, x) \zeta dx. \quad (36)
\]

Next we proceed with the a priori estimate of each terms of (36).

**Estimates of the nonlocal operator in time**

We have defined \( \tilde{u} = u(t) \) for \( \varepsilon j - 1 < t \leq \varepsilon j \). From [4, Lemma 4.3] there exist two constants \( c \) and \( c_1 \) depending on \( \gamma, T, \Lambda \) such that for \( \varepsilon < 1 \), the following relation holds true

\[
\varepsilon \sum_{j \leq k} F'(u^\varepsilon_\zeta(j)) \partial_t^j u(\varepsilon j) \geq c \int_{-\infty}^{T} \int_{-\infty}^{t} [\tilde{u}^\varepsilon_\zeta(t) - \tilde{u}^\varepsilon_\zeta(\tau)]^2 (t - \tau)^{-\gamma - 1} d\tau d\tau \\
+ c \int_{-\infty}^{T} F(u^\varepsilon_\zeta(t))(T - t)^{-\gamma} dt - c_1 \int_{-\infty}^{T} F'(\tilde{u}^\varepsilon_\zeta(t)) \zeta^\varepsilon_\sigma(t) dt.
\]

So we have that

\[
\varepsilon \sum_{j \leq k} F'(u^\varepsilon_\zeta(j)) \partial_t^j u(\varepsilon j) \approx \|u\|_{W^{\gamma/2,2}[0,T]}. \]

**Estimates of the local and nonlocal spatial terms.** For the local spatial term we have

\[
\delta \varepsilon \sum_{j \leq k} \int_{B_R} \nabla F'(u^\varepsilon_\zeta(\varepsilon j, x)) \nabla u dx \\
= \delta \varepsilon \sigma \sum_{j \leq k} \int_{B_R} \left( u^\varepsilon_\zeta(\varepsilon j, x) + \mu \right)^{\sigma - 1} \nabla u^\varepsilon_\zeta(\varepsilon j, x) \nabla u dx \\
\geq \delta \sigma \int_{0}^{T} \int_{B_R} (\tilde{u} + \mu)^{\sigma - 1} |\nabla \tilde{u}|^2 dx dt. \quad (37)
\]
Next we provide the estimate for the nonlocal spatial term

\[ ε \sum_{j \leq k} \int_{B_R} \nabla \zeta (u + \mu)^{m-1} \nabla K_1^\epsilon[u] dx \]

\[ = ε \sum_{j \leq k} \int_{B_R} \nabla F'(u^+_{\xi}) (u + \mu)^{m-1} \nabla (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx \]

\[ = ε \sum_{j \leq k} \int_{B_R} \nabla F'(u^+_{\xi}) \left( u^+_{\xi} + \zeta + \mu \right)^{m-1} \nabla (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx. \]

But since \( \nabla F'(u^+_{\xi}(\varepsilon, j, x)) = \sigma \left( u^+_{\xi} + \mu \right)^{\sigma-1} \nabla u^+_{\xi}, \) then we have

\[ ε \sum_{j \leq k} \int_{B_R} \nabla \zeta (u + \mu)^{m-1} \nabla K_1^\epsilon[u] dx \]

\[ = ε \sum_{j \leq k} \int_{B_R} \nabla F'(u^+_{\xi}) (u + \mu)^{m-1} \nabla (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx \]

\[ \geq ε \sum_{j \leq k} \int_{B_R} \nabla u^+_{\xi} \left( u^+_{\xi} + \mu \right)^{m+\sigma-2} \nabla (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx \]

\[ \geq \frac{ε}{m + \sigma - 1} \sum_{j \leq k} \int_{B_R} \nabla \left( u^+_{\xi} + \mu \right)^{m+\sigma-1} \nabla (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx \]

\[ \geq \frac{ε}{m + \sigma - 1} \int_{-2}^{T} \mathcal{E}_{\xi} \left( \left( u^+_{\xi} + \mu \right)^{m+\sigma-1}, u \right) dt. \]

The last line involving the bilinear form is well defined. Indeed, if we set

\( \nabla Z(u) = \nabla \left( u^+_{\xi} + \mu \right)^{m+\sigma-1}, \)

then

\[ ε \sum_{j \leq k} \int_{B_R} \nabla Z(u) \nabla (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx = ε \sum_{j \leq k} \int_{B_R} Z(u) (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx \]

\[ = ε \sum_{j \leq k} \int_{B_R} Z(u) K_1^\epsilon_{s-x}[u] dx. \]

Now using the generalized Stroock-Varopoulos inequality (13) with \( Z' = (V')^2 \) and \( \nabla Z(u) = \nabla F' \left( u^+_{\xi}(\varepsilon, j, x) \right) (u + \mu)^{m-1}, \) we get that

\[ ε \sum_{j \leq k} \int_{B_R} \nabla F'(u^+_{\xi}) (u + \mu)^{m-1} \nabla (-\Delta)^{-1} K_1^\epsilon_{s-x}[u] dx \]

\[ \geq ε \sum_{j \leq k} \int_{B_R} \left| V(u) \left( K_1^\epsilon_{s-x}[u] \right)^{\frac{1}{2}} \right|^2 dx, \]

where

\[ Z(z) = \sigma \int_{0}^{z} (y + \mu)^{\sigma+m-2} dy = \frac{\sigma}{\sigma + m - 1} \left[ z + \mu \right]^\sigma, \]
and

\[ V(z) = \int_0^z [Z'(y)]^{1/2} dy. \]

As a consequence \( V(u) \in L^2 ((-\infty, T); H^{1-s} (\mathbb{R}^N)) \).

Next we recall from [4, Proposition 10.1], that if \( u_\sigma^+(x) - u_\sigma^+(y) \geq 0 \), then

\[ (u_\sigma^+ + \mu)^{m+\sigma-1}(x) - (u_\sigma^+ + \mu)^{m+\sigma-1}(y) \geq (u_\sigma^+(x) - u_\sigma^+(y))^{m+\sigma-1}. \]

Hence

\[ c \sum_{j \leq k} \int_{B_R} \nabla F' \left( u_\sigma^+ (\varepsilon_j, x) \right) (u + \mu)^{m-1} \nabla p \, dx \]

\[ \geq \frac{\sigma c}{m + \sigma + 1} \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_{1-s}(x,y) |\tilde{u}(x) - \tilde{u}(y)|^{m+\sigma} \, dx \, dy \, dt. \quad (38) \]

**Estimate of the whole problem.** Combining these three previous estimates with the right hand side term \( f \) we have for a certain constant \( C \) depending on \( \gamma, s, \Lambda, \sigma, K, T, N \)

\[ C \int_{-2}^T \int_{B_R} \left[ (u_\sigma^+ + \mu)^{\sigma} - \mu^{\sigma} \right] \, dx \, dt \leq C \int_{-2}^T \int_{B_R} \left[ (K e^{-|x|} + \mu)^{\sigma} - \mu^{\sigma} \right] \, dx \, dt \]

\[ \leq C \int_{-2}^T \int_{B_R} K^{\sigma} e^{-\sigma|x|} \, dx \, dt \leq C(\gamma, s, \Lambda, \sigma, K, T, N). \]

(39)

From [4, Proposition 10.2],

\[ (u_\sigma^+ + \mu)^{\sigma} - \mu^{\sigma} \leq 2^{\sigma} (u_\sigma^+)^{\sigma}. \]

So

\[ C \int_{-2}^T \int_{B_R} \left[ (u_\sigma^+ + \mu)^{\sigma} - \mu^{\sigma} \right] \, dx \, dt = \int_{-2}^T \int_{B_R} \left[ (K e^{-|x|} + \mu)^{\sigma} - \mu^{\sigma} \right] \, dx \, dt \]

\[ \leq 2^{\sigma} C \int_{-2}^T \int_{B_R} K^{\sigma} e^{-\sigma|x|} \, dx \, dt \leq C. \]

We then obtain an estimate which remain uniform as \( \sigma, \mu \to 0 \)

\[ \delta \sigma \int_0^T \int_{B_R} (\tilde{u} + \mu)^{\sigma-1} |\nabla \tilde{u}|^2 \, dx \, dt + \int_0^T \left\| \tilde{u} \right\|_{W^{(2-\sigma)/\sigma}}^{m+\sigma} \, dt \]

\[ + \int_{B_R} \left\| \tilde{u} \right\|_{L^{\gamma/2}((0,T))}^2 \, dx \leq C, \quad (40) \]
with the constant $C$ depending only on the exponential decay of $f, u_0$, and on $\gamma, s, N, T$, but not on $\delta, R$.

Note that the existence of a weak solution of problem (1) is done by passing to the limit step-by-step in the approximating problems. With the estimate (40) we take $u = \lim_{\mu, \epsilon \to 0}$, in order to obtain

$$\partial_t^2 u - \delta \Delta u - \nabla \cdot \left( u^{m-1} \nabla (-\Delta)^{-s} u \right) = f \quad [0, T] \times B_R. \tag{41}$$

Now the next step will be to provide a compactness result. We state the two following Lemmas from which we refer to [4] for similar proofs with slightly differences.

**Lemma 3.3.** Assume for any $v \in \mathcal{F}$,

$$\int_0^T \|v(t, x)\|_{W^{m+\sigma}}^{m+\gamma} + \int \|v(t, x)\|^2_{W^{m+\sigma}} \leq C. \tag{42}$$

Then $\mathcal{F}$ is totally bounded in $L^p([0, T] \times B_R)$ for $m \geq 2$ and for $1 \leq p \leq 2$.

The second Lemma gives the guarantee that $\nabla (-\Delta)^{-s} u \in L^p$ as $\delta \to 0, R \to \infty$.

**Lemma 3.4.** Let $u$ be a solution to (41) with right hand side $f$ and $u_0$ both satisfying the exponential bound (30). Then

$$\int_0^T \|(-\Delta)^{-s} u(t, \cdot)\|_{W^{m+\sigma}}^{m+\gamma} dt \leq C$$

with the constant $C$ depending only on the exponential bounds in (30), $N, \sigma, T, m$.

**Corollary 3.** Let $u_k$ be a sequence of solutions to (41) with $R \to \infty$ and $\delta \to 0$. For fixed $\rho > 0$, there exist a subsequence and limit with

$$u_k \to u_0 \in L^p(B_{\rho}) \text{ for } 1 \leq p \leq 2 \text{ and } u_k \to u_0 \in L^{m+\sigma} \left(0, T; W^{(2-2s)/(m+\sigma), m+\sigma} \right).$$

Furthermore, for any compactly supported $\zeta$

$$\varepsilon \sum_{j \leq k} \int_{\mathbb{R}^N} \left[ (\zeta(x, \varepsilon) \nabla \cdot u_0 (\varepsilon) + u_0^{m-1} \nabla \zeta \nabla (-\Delta)^{-s} u_0 \right] = \varepsilon \sum_{j \leq k} \int_{\mathbb{R}^N} f \zeta \ dx. \tag{43}$$

**Proof.** The proof of this corollary follows from (40) and Lemma 3.3. On the other hand, from Lemma 3.4, and for $\sigma$ small enough depending on $s$, then

$$\frac{2 - 2s}{m + \sigma} + 2s > 1, \quad 0 < \sigma < 1,$$

we have that

$$\nabla (-\Delta)^{-1} K_{1-s}[u_k] \to \nabla (-\Delta)^{-s} u_0 \in L^{m+\sigma} \left(0, T; W^{(2-2s)/(m+\sigma), m+\sigma} \right).$$

Hence as in [4]

$$\nabla (-\Delta)^{-s} u_k \to \nabla (-\Delta)^{-s} u_0 \in L^{m+\sigma} \left((0, T) \times \mathbb{R}^N \right), \tag{44}$$

which shows that $u_0$ is a solution. \qed

Now we give the proof of Theorem 1.1
Proof of Theorem 1.1. The proof of the theorem is based on passing to the limit of the regularized solution obtained in Section 3.1 and it follows also the original technique presented in [4]. We first assume \( f, u_0 \) smooth and satisfying the exponential bounds (30). Next consider solutions \( u_\varepsilon \) to (43) over a finite interval \((0, T)\). The first limit is done as \( \varepsilon \to 0 \) and it is based on the compactness criteria of type Simon-Aubin-Lions [43] so that there exist a subsequence and a limit \( u_\varepsilon \to u_0 \) with the weak convergence as in (44) and strong convergence over compact sets for \( 1 \leq p \leq 2 \) just as in Lemma 3.3 and in [4]. We now consider a sequence of solution \( \{w_j\} \) with \( \{f_j\}, \{(w_0)_j\} \in C^\infty \) with \( f_j \to f \) and \( (w_0)_j \to w_0 \) in weak sense to \( L^\infty \). Then, there exists a limit solution \( w \) with right hand side \( f \) satisfying (30). From Lemma 3.1 we can let \( T \to \infty \), which ends the proof.

4. Finite speed of propagation and free Boundary property. Proof of Theorem 1.2.

4.1. Finite speed of propagation properties. We start by recalling the definition of finite speed of propagation [46].

Definition 4.1. We say that finite propagation holds for a certain class of solutions \( w \) of an evolution equation if

(a) for given times \( 0 \leq t_1, t_2 \leq T \), the support of the solution at time \( t_2 \) is included in a neighbourhood of radius \( g(|t_2 - t_1|) \) of the support of \( w(t_1) \), where \( g \) is a continuous function \( \mathbb{R}_+ \to \mathbb{R}_+ \) with \( g(0+) = 0 \),

(b) the function \( g \) is independent of the solution under consideration, —we call it uniform finite propagation,

(c) \( g(t) \leq Ct \) for some constant \( C > 0 \) —we say that propagation has finite speed.

We now prove Theorem 1.2, by using a method similar to the exponential tail control section and the original idea introduced in [19], (also used for further models [44, 12]) but with some technical adaptation to our model.

Proof of Theorem 1.2. We assume a nonnegative solution \( w \) of (1) having bounded initial data \( w(x, 0) = w_0(x) \leq Ae^{-|x|} \leq L \). We also assume that \( w_0 \) is below the parabola like function \( \bar{w}_0(x) = a(|x| - b)^2, a, b > 0 \). The support of \( \bar{w}_0 \) is the ball of radius \( b \) and the graphs of \( w_0 \) and \( \bar{w}_0 \) are strictly separated in that ball. If we take as a comparison function \( \bar{w}(x, t) = a(Ct - (|x| - b))^2 \), the goal is to argue the fact that the first point in time and space where the function \( w(x, t) \) touches the parabola \( \bar{w} \) from below happens for \( x \neq \infty \) and for \( t > 0 \). As in the exponential tail control section 3.4, we called \((x_c, t_c)\) the contact point for which

\[
w(x_c, t_c) = \bar{w}(x_c, t_c) = a(Ct_c - (|x_c| - b))^2.
\]

Similarly as shown in [44, Lemma 7.2] the contact can not happen at the vanishing point \( |x_f(t_c)| = (b + Ct_c) \) of the barrier or the boundary of the support of the parabola \( \bar{w} \) at time \( t_c \).

Let us assume that \( w \) is of class \( C^2 \). We consider two cases.

• Case where \( a = L = 1 \):

At the first contact point \((x_c, t_c)\), since \( w \leq 1 \) we have

\[
w(x_c, t_c) = h^2, \quad \partial_t w(x_c, t_c) = -2h, \quad \Delta w(x_c, t_c) \leq 2N,
\]
where we defined \( h = b + C t_c - |x_c| \), to be the distance from which \( x_c \) lies from \( |x_f(t_c)| \).

Furthermore,

\[
\partial_t^\alpha w(x_c, t_c) = \frac{1}{\Gamma(1 - \gamma)} \int_0^{t_c} (t_c - \tau)^{-\gamma} u'(x_c, \tau) d\tau
\]

\[
= \frac{2aC}{\Gamma(2 - \gamma)} \left( (-|x_c| + b)t_c^{1-\gamma} + \frac{C}{(2 - \gamma)} t_c^{2-\gamma} \right)
\]

\[
= \frac{2aC}{\Gamma(2 - \gamma)} \left( \frac{C}{(2 - \gamma)} t_c - (|x_c| - b) \right) t_c^{1-\gamma} \leq 2c_1 C |C t_c - (|x_c| - b)| t_c^{1-\gamma}
\]

\[
\partial_t^\alpha w(x_c, t_c) \geq 2Ch t_c^{-\gamma}.
\]

For \( p = K_s(w) \) and using the following equation

\[
\partial_t^\alpha w = (m - 1)w^{m-2}\nabla w \cdot \nabla p + w^{m-1}\Delta p + f,
\]

we get the inequality

\[
2Ch t_c^{-\gamma} \leq 2(m - 1)h^{2m-3} \left( -\partial_t p + \frac{h}{2} \Delta p + Ae^{-r_c} \right), \tag{45}
\]

where \( \partial_t p \) and \( \Delta p \) are the values of \( \nabla p \) and \( \Delta p \) at the point \( (x_c, t_c) \) as in section 3.4.

We also used the fact that the function \( f(x_c, t_c) \) is controlled by \( Ae^{-r_c} \).

So in order to get a contradiction, we use the estimate for the value of \( \partial_t p \) and \( \Delta p \) already obtained in [19] which reads as

\[
-\partial_t p \leq K_1 + K_2 h^{1+2s} + K_3 h \quad \text{and} \quad \Delta p \leq K_4.
\]

We set \( K = \max\{K_1, K_2, K_3, K_4\} \), we can rewrite the previous inequality as

\[
-\partial_t p \leq (1 + h^{1+2s} + h) K \quad \text{and} \quad \Delta p \leq K. \tag{46}
\]

Therefore, combining the inequalities (45) and (46) we get

\[
2Ch t_c^{-\gamma} \leq 2(m - 1)h^{2m-3} K \left( 1 + \frac{3}{2} h + h^{1+2s} \right) + Ae^{-r_c} \tag{47}
\]

\[
C \leq (m - 1)h^{2m-4} \left( K \left( 1 + \frac{3}{2} h + h^{1+2s} \right) + \frac{A}{2h} \right) t_c^{-\gamma}, \tag{48}
\]

which is impossible for \( C \) large and independent of the distance \( h \) of the boundary of the support of the parabola \( \tilde{w} \) at time \( t_c \), since \( m > 2 \) and \( |h| \leq 1 \). Hence for the minimal constant

\[
C = C(\gamma, s, N) = \min \left\{ (m - 1)h^{2m-4} \left( K \left( 1 + \frac{3}{2} h + h^{1+2s} \right) + \frac{A}{2h} \right) t_c^{-\gamma} \right\},
\]

it cannot be a contact point with \( h \neq 0 \), proving that \( m > 2 \). For \( m < 2 \), we do not obtain a contradiction in the estimate (47), since the term \( Kh^{2m-4} \) can be very large for small values of \( |h| \).

- **Case where** \( a \neq 1 \) and \( L \neq 1 \):

In this part, the goal is to show that the equation is invariant under the scaling. For this purpose we use the scaling property that if \( w \) is solution to (1), then

\[
\tilde{w}(x, t) = Aw(Bx, Tt), \tag{49}
\]

is also solution to (1) if \( T^\gamma = A^{m-1}B^{2-2s} \), for \( A, B, T > 0 \).

We search the parameters \( A, B, T \) for which \( \tilde{w} \) defined by (49) satisfies

\[
0 \leq \tilde{w}(x, t) \leq L, \quad \tilde{w}(x, 0) \leq \tilde{a}(|x| - \tilde{b})^2.
\]
In fact at the initial point, the function satisfies \( Aw(Bx,0) \leq A(B|x| - b)^2 \), then \( w(x,t) \leq \tilde{w}(x,t) = (Ct - (|x| - b))^2 \) for all \( t > 0 \). From this observation we have that
\[
Aw(x,0) \leq A(B|x| - b)^2 \leq AB^2 \left( |x| - \frac{b}{B} \right)^2 \leq \tilde{a} \left( |x| - \tilde{b} \right)^2,
\]
where \( \tilde{a} = AB^2 \), \( \tilde{b} = \frac{b}{B} \), \( A = L \). Using the relation between \( A, B \) and \( T \), we get that \( B = (\tilde{a}/L)^{\frac{1}{2}} \), and \( T = L^{\frac{m-2s}{\gamma}} \tilde{a}^{-\frac{1}{\gamma}} \).

Then \( \tilde{w}(x,t) \) is below the upper barrier \( \hat{w} = \tilde{a} \left( \hat{C}t - (|x| - \hat{b}) \right)^2 \), where the new speed is given by
\[
\hat{C} = C(1,1)T/B = C(1,1)L^{\frac{m-2s}{\gamma}} \tilde{a}^{-\frac{1}{\gamma}}.
\]

**Remark 2.** The previous result about the finite speed propagation shows the dependency of the parameter \( \gamma \). This speed increases in the limit when \( \gamma \to 0 \), for \( 0 < s < 1/2 \). However we recover the result of speed of propagation for PME with potential pressure \([44]\) in the limit when \( \gamma \nearrow 1 \) as
\[
\hat{C} = C(1,1)L^{m-\frac{2}{\gamma} + s} a^{-s+\frac{1}{\gamma}}.
\]

4.2. **Topological boundary of the support of the solution: growth estimate of the support.** As a consequence of the results of finite speed of propagation, we assert the existence of free boundary points by showing that the growth of the support is bounded in a finite time which was stated in Corollary 1, proved below.

**Proof of Corollary 1.** We follow the idea of the proof to the one given in \([46]\). From section 3.5, the support of \( w(\cdot; t) \) is bounded and is contained in a ball of radius \( r(t) \). For a given time \( t_1 \), we look for a barrier as in the previous section with positive constant \( a \). For this purpose as in \([19]\) we choose \( L = ar_1^2 \) and we use the formula of the parabolic barrier so that \( b = r(t_1) + r_1 + \varepsilon \), for \( \varepsilon \in (0,1) \). In the limit as \( \varepsilon \to 0 \) and using the speed estimate in Theorem 1.2 we get the inequality
\[
r(t) - r(t_1) - r_1 \leq \hat{C}(t_1 - t) = CL^{\frac{2m-3+2s}{\gamma}}(t_1) a^{\frac{1+2s}{\gamma}}(t_1 - t_1).
\]
Using the \( L^\infty \) bound \( L(t_1) \leq L(0) = L \) and the fact that \( L = ar_1^2 \), we get
\[
r(t) - r(t_1) - r_1 \leq CL^{\frac{2m-3+2s}{\gamma}} L^{\frac{1-2s}{\gamma}} r_1^{\frac{2s-1}{\gamma}}(t_1 - t) \leq r(t) - r(t_1) + r_1 + CL^{\frac{m-1}{\gamma}} r_1^{\frac{2s-1}{\gamma}}(t_1 - t_1).
\]
By setting \( t_1 = 0 \) and estimating the right-hand side in \( r_1 \), it yields
\[
|x(t)| \leq R + C_2 t^{\gamma/(2-2s)},
\]
which implies the desired result.

**Remark 3.** One should notice that our estimates do not give the exact vanishing set, but only a lower bound \( R(t) \) for the radius of the ball where the solution \( u \) vanishes. Furthermore in the limit \( \gamma \nearrow 1 \) and \( s \to 1/2 \) we can also show that we get the linear growth. In a similar way, for \( s \to 0 \) and \( \gamma \nearrow 1 \) we get the standard \( t^{1/2} \) growth of the PME.
Appendix A.

Proof. Using the fact that the Laplace transform of $c^0 D^\gamma_t u$ is given by
\[ L\{c^0 D^\gamma_t u\}(p) = p^\gamma \tilde{u}(p) - p^{\gamma-1} u_0, \]
we can write the Laplace transform of the fractional differential equation (16) as
\[ p^\gamma \tilde{u}(p) = p^{\gamma-1} u_0 + A \tilde{u}(p) + \nabla \cdot \tilde{\Psi}(p) + \tilde{f}(p), \]
\[ \tilde{u}(p) = (p^\gamma I - A)^{-1} p^{\gamma-1} u_0 + \nabla \cdot (p^\gamma I - A)^{-1} \tilde{\Psi}(p) + (p^\gamma I - A)^{-1} \tilde{f}(p), \]
(50)
where $I$ is the identity operator.

Next we compute each of the terms $(p^\gamma I - A)^{-1}$ and $(p^\gamma I - A)^{-1} p^{\gamma-1}$.

- Computation of $(p^\gamma I - A)^{-1}$: Observe that for any $g \in X$, we have that
\[ (p^\gamma I - A)^{-1} g = \int_0^\infty e^{-p^\gamma \theta} Q(\theta) g \, d\theta. \]
Since $\|Q(t)\| \leq K$, it follows that the integral above is absolutely convergent for all $p > 0$. Hence
\[ (p^\gamma I - A)^{-1} g = \int_0^\infty e^{-p^\gamma \theta} \rho_\gamma(\theta) g \, d\theta, \]
where $\rho_\gamma$ is the one-side stable probability density, whose Laplace transform is given by \[40\]
\[ \int_0^\infty \rho_\gamma(t) e^{-p^\gamma t} \, dt = e^{-p^\gamma}, \]
(51)
and which satisfies
\[ \gamma M_\gamma(\zeta) = \zeta^{-1-1/\gamma} \rho_\gamma(\zeta^{-1/\gamma}), \]
(52)
which is in terms of the Mainardi function defined in (8). By using (52) it yields
\[ (p^\gamma I - A)^{-1} g = \int_0^\infty e^{-p^\gamma t} \left( \int_0^\infty \theta^{-1/\gamma} \rho_\gamma(t\theta^{-1/\gamma}) \, d\theta \right) \, dt. \]
Hence
\[ (p^\gamma I - A)^{-1} g = \mathbb{P}_2(p) g, \]
with
\[ \mathbb{P}_2(t) = \int_0^\infty \gamma \zeta t^{\gamma-1} M_\gamma(\zeta) Q(t^\gamma \zeta) \, d\zeta. \]

- Computation of $p^{\gamma-1} (p^\gamma I - A)^{-1}$: Similarly as in the previous computation,
\[ p^{\gamma-1} (p^\gamma I - A)^{-1} g = p^{\gamma-1} \int_0^\infty e^{-p^{\gamma-1} \theta} Q(\theta) g \, d\theta. \]
From (51) we have
\[- \int_0^\infty e^{-pt} \left( \int_0^\infty \rho_\gamma(\theta) Q(t^\gamma \theta^{-\gamma}) g \, d\theta \right) \, dt = \int_0^\infty e^{-pt} \left( \int_0^\infty \frac{1}{\gamma} \zeta^{-1/\gamma} \rho_\gamma(\zeta^{-1/\gamma}) Q(t^\gamma \zeta) g \, d\zeta \right) \, dt \]

Hence we get that
\[p^{\gamma^{-1}} (p^\gamma I - A)^{-1} g = \widehat{\mathcal{P}}_1(p)g,\]
with
\[\mathcal{P}_1(t) = \int_0^\infty \mathcal{M}_\gamma(\zeta) Q(t^\gamma \zeta) \, d\zeta.\]
Thus replacing \(p^{\gamma^{-1}} (p^\gamma I - A)^{-1} g\) and \((p^\gamma I - A)^{-1}\) in (50) we get
\[\tilde{u}(p) = \widehat{\mathcal{P}}_1(p)u_0 + \nabla \cdot \left( \widehat{\mathcal{P}}_2(p) \ast \tilde{\Psi}(p) + \widehat{\mathcal{P}}_2(p) \ast \tilde{f}(p) \right).\]

Then consequently applying the inverse Laplace transform we get
\[u(t) = \mathcal{P}_1(t)u_0 + \int_0^t \nabla \cdot (\mathcal{P}_2(t - \tau)\Psi(u(\tau))) \, d\tau + \int_0^t \mathcal{P}_2(t - \tau)f(\tau) \, d\tau.\]

Acknowledgments. The authors are grateful to the reviewers for their helpful comments and remarks that improved a preliminary version of the manuscript.

REFERENCES

[1] M. Allen, Hölder regularity for nondivergence nonlocal parabolic equations, *Calc. Var. Partial Differential Equations*, 57 (2018), Art. 110, 29 pp, [arXiv:1610.10073](https://arxiv.org/abs/1610.10073)

[2] M. Allen, A nondivergence parabolic problem with a fractional time derivative, *Differential Integral Equations*, 31 (2018), 215–230.

[3] M. Allen, L. Caffarelli and A. Vasseur, A parabolic problem with a fractional time derivative, *Arch. Ration. Mech. Anal.*, 221 (2016), 603–630.

[4] M. Allen, L. Caffarelli and A. Vasseur, Porous medium flow with both a fractional potential pressure and fractional time derivative, *Chin. Ann. Math. Ser. B*, 38 (2017), 45–82.

[5] I. Area, J. Losada and J. J. Nieto, A note on the fractional logistic equation, *Phys. A*, 444 (2016), 182–187.

[6] D. G. Aronson and J. Serrin, Local behavior of solutions of quasilinear parabolic equations, *Arch. Ration. Mech. Anal.*, 25 (1967), 81–122.

[7] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Thermal Science*, 20 (2016), 763–769.

[8] D. Baleanu and A. Fernandez, A generalisation of the Malgrange-Ehrenpreis theorem to find fundamental solutions to fractional PDEs, *Electronic Journal of Qualitative Theory of Differential Equations*, 2017 (2017), 1–12.

[9] D. Baleanu and A. Fernandez, On some new properties of fractional derivatives with Mittag-Leffler kernel, *Communications in Nonlinear Science and Numerical Simulation*, 59 (2018), 444–462.

[10] A. Bernardis, F. J. Martín-Reyes, P. R. Stinga and J. L. Torrea, Maximum principles, extension problem and inversion for nonlocal one-sided equations, *J. Differential Equations*, 260 (2016), 6333–6362.

[11] P. Biler, C. Imbert and G. Karch, The nonlocal porous medium equation: Barenblatt profiles and other weak solutions, *Archive for Rational Mechanics and Analysis*, 215 (2015), 497–529.

[12] P. Biler, G. Karch and R. Monneau, Nonlinear diffusion of dislocation density and self-similar solutions, *Commun. Math. Phys.*, 294 (2010), 145–168.
[13] M. Bonforte, A. Figalli and X. Ros-Oton, Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains, *Comm. Pure Appl. Math.*, **70** (2017), 1472–1508.

[14] M. Bonforte and J. L. Vázquez, Quantitative local and global a priori estimates for fractional nonlinear diffusion equations, *Adv. Math.*, **250** (2014), 242–284.

[15] C. Bucur, Some nonlocal operators and effects due to nonlocality, preprint, arXiv:1705.00953

[16] C. Bucur and F. Ferrari, An extension problem for the fractional derivative defined by Marchaud, *Fract. Calc. Appl. Anal.*, **19** (2016), 867–887.

[17] L. Caffarelli, F. Soria and J. L. Vázquez, Regularity of solutions of the fractional porous medium flow, *Journal Europ. Math. Society.*, **15** (2013), 1701–1746.

[18] L. Caffarelli and J. L. Vázquez, Asymptotic behaviour of a porous medium equation with fractional diffusion, *Discrete Contin. Dyn. Syst.*, **29** (2011), 1378–1409.

[19] L. Caffarelli and J. L. Vázquez, Nonlinear porous medium flow with fractional potential pressure, *Arch. Ration. Mech. Anal.*, **202** (2011), 537–565.

[20] M. Caputo, Diffusion of fluids in porous media with memory, *Geothermics*, **28** (1999), 113–130.

[21] M. Caputo and M. Fabrizio, A new Definition of Fractional Derivative without Singular Kernel, *Progr Fract Differ Appl*, **1** (2015), 73–85.

[22] J. A. Carrillo, Y. Huang, M. C. Santos and J. L. Vázquez, Exponential convergence towards stationary states for the 1D porous medium equation with fractional pressure, *J. Differ. Equations*, **258** (2015), 736–763.

[23] J. D. Djida, A. Atangana and I. Area, Numerical computation of a fractional derivative with non-local and non-singular kernel, *Math. Model. Nat. Phenom.*, **12** (2017), 4–13.

[24] J. D. Djida, J. J. Nieto and I. Area, Parabolic problem with fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel, *Discrete Continuous Dyn. Syst. Ser. S* (to appear), 2018.

[25] R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Masson, Paris, 1987.

[26] A. de Pablo, F. Quirós, A. Rodríguez and J. L. Vázquez, A fractional porous medium equation, *Adv. Math.*, **226** (2011), 1378–1409.

[27] A. de Pablo, F. Quirós, A. Rodríguez and J. L. Vázquez, A general fractional porous medium equation, *Comm. Pure Appl. Math.*, **65** (2012), 1242–1284.

[28] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573.

[29] S. Dipierro, E. Valdinoci and V. Vespri, Decay estimates for evolutionary equations with fractional time diffusion, preprint, arXiv:1707.08278v1.

[30] F. Ferrari, Weyl and Marchaud derivatives: A forgotten history, *Mathematics*, **6** (2018), p6.

[31] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations Of Second Order*, Berlin: Springer, reprint of the 1998 ed. edition, 2001.

[32] J. Kemppainen, J. Siljander and R. Zacher, Representation of solutions and large-time behavior for fully nonlocal diffusion equations, *Journal of Differential Equations*, **263** (2017), 149–201.

[33] P. E. Kloeden and J. Simsen, Pullback attractors for non-autonomous evolution equations with spatially variable exponents, *Communications on Pure and Applied Analysis*, **13** (2014), 2543–2557.

[34] A. N. Kochubei, Fractional-order diffusion, *Differ. Equations*, **26** (1990), 485–492.

[35] L. Liu, T. Caraballo and P. E. Kloeden, Long time behavior of stochastic parabolic problems with white noise in materials with thermal memory, *Revista Matemática Complutense*, **30** (2018), 687–717.

[36] F. Mainardi, Yu. Luchko and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fract. Calc. Appl. Anal.*, **4** (2001), 153–192.

[37] F. Mainardi and G. Pagnini, *The Wright functions as solutions of the time-fractional diffusion equation*, *Applied Mathematics and Computation*, **141** (2003), 51–62.

[38] A. Marchaud, *Sur Les Dérivées et Sur les Différences des Fonctions De Variables Réelles*, PhD thesis, Faculté des Sciences de Paris, 1927.

[39] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[40] G. M. Mophou and G. M. N’Guérékata, On a class of fractional differential equations in a Sobolev space, *Applicable Analysis*, **91** (2012), 15–34.
[41] M. K. Saad, A. Atangana and D. Baleanu, New fractional derivatives with non-singular kernel applied to the Burgers equation, Chaos: An Interdisciplinary Journal of Nonlinear Science, 28 (2018), 063109, 6 pp.
[42] S. Samko, A. A. Kilbas and O. Marichev, Fractional Integrals and Derivatives, Taylor & Francis, 1993.
[43] J. Simon, Compact sets in the space $L^p(0,t;b)$, Annali di Matematica Pura ed Applicata, 146 (1987), 65–96.
[44] D. Stan, F. del Teso and J. L. Vázquez, Finite and infinite speed of propagation for porous medium equations with fractional pressure, Comptes Rendus Acad. Sci., 352 (2014), 123–128.
[45] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.
[46] J. L. Vázquez, The Porous Medium Equation, Oxford University Press, 2007.
[47] J. L. Vázquez, Nonlinear diffusion with fractional laplacian operators, Nonlinear Partial Differential Equations, 271–298, Abel Symp., 7, Springer, Heidelberg, 2012. Available online at http://www.umm.es/personal_pdi/ciencias/jvazquez/JLVABEL-2010.pdf.

Received March 2018; revised September 2018.
E-mail address: jeandaniel.djida@usc.es
E-mail address: juanjose.nieto.roig@usc.es
E-mail address: area@uvigo.es