Comment on ”New apparatus design for high-precision measurement of G with atom interferometry”

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It is shown that even in the case of a negligibly small change in the gradient of the gravitational field of the mass source in the axial direction, the dependence of this gradient in the radial direction leads to a systematic error in Newton gravitational constant value. The magnitude of this error is calculated for two configurations of the field. For both configurations it was found that this error is larger than the inaccuracy predicted in the article by M. Jain et al [Eur. Phys. J. D 75, 197 (2021)]. In addition, we found the geometry of the source mass, for which this systematic error disappears.

Atom interference \[^{1}\] can be used to measure Newtonian gravitational constant \(G\) \[^{2}\]. The best measurement of this constant was performed in the experiment \[^{3}\]. It was stated that the measurement accuracy of 148 ppm was achieved. However, we have attempted \[^{4}\] to build an error budget for the conditions of the experiment \[^{3}\]. we have shown that due to uncertainties in the initial position of the centers of atomic clouds in the coordinate and velocity spaces, the accuracy of the measurement of \(G\) is not less than 275ppm. We also found that the nonlinear dependence of the phase of atomic interferometers (AI) on these uncertainties should have led to a systematic error of \(-199\)ppm in the measurement of \(G\).

An alternative measurement method is proposed in the article \[^{5}\]. If the gravitational field of the source mass has a strictly linear dependence on the coordinates along the atomic trajectory, then with a certain change in the effective wave vector of the second Raman pulse, the dependence of the phase AI on the initial coordinates and velocities disappears \[^{6}\]. This effect was observed in the articles \[^{5}\] \[^{7}\] in the articles \[^{5}\] \[^{8}\] \[^{9}\], the budget of errors of the alternative method was studied. Two configurations of the source mass were considered in the article \[^{6}\]. Based on the built error budget, the authors came to the conclusion that the accuracy of the measurement of \(G\) can be 13ppm and 5ppm for configurations I and II, respectively. In this commentary, we will calculate the contributions to the relative standard deviation (RSD) and systematic error (RSE) due, respectively, to uncertainties in the radial position of the center of the atomic cloud and the finite radial size of the atomic cloud. We will show that the RSEs will exceed the measurement accuracy predicted in \[^{6}\].

Near the center, the axial component of the gravitational field in the presence of a source mass is given by

\[
g_z(x) = g + \gamma(r) z, \tag{1a}
\]
\[
\gamma(r) = \Gamma_{33}^{(1)} (1 + \varepsilon r^2), \tag{1b}
\]
\[
\gamma = \Gamma_{33}^{(1)}, \tag{1c}
\]
\[
\varepsilon = \Gamma_{3311}^{(3)} / 2 \gamma, \tag{1d}
\]

where \(g\) is the gravitational field of the Earth, \(x = (x_\perp, z)\), \(x_\perp = (x, y)\) is the radial component of the vector \(x\), \(r = |x_\perp|\), \(\Gamma^{(i)}\) is the order \(i\) gravity-gradient tensor of the source mass gravitational field \[^{10}\]. Suppose that the basic idea of the alternative method \[^{5}\] \[^{8}\] \[^{9}\] \[^{11}\] is implemented, one can neglect completely nonlinear terms in the field dependence on the axial coordinate \(z\). Let’s also assume that one can neglect the recoil effect, that the atoms are launched in a strictly axial direction, their initial velocity \(v(0) = (0, 0, v_0)\), and that there is no radial component of the gravitational field. Then, when an atom interacts with 3 Roman pulses, the phase of the atomic interferometer is given by \[^{11}\]

\[
\phi = k_1 z(T_1) - 2k_2 z(T_2) + k_3 z(T_3), \tag{2}
\]

where \(k_m\) is the effective wave vector of the Raman pulse \(m\), \(T_m = T_1 + (m-1)T\) is the time delay between first pulse and the atoms launching moment, \(T\) is the interrogation time, \(z(t)\) is the axial component of the atomic trajectory. Considering the second term in Eq. \[^{11}\] as a small addition, one gets

\[
z(t) = z_0 + v_0 t + \frac{1}{2} g t^2 + \gamma(r) \left( \frac{1}{2} z_0 t^2 + \frac{1}{6} v_0 t^3 + \frac{1}{24} g t^4 \right). \tag{3}
\]

In the absence of radial components of the gravitational field and the initial velocity of the atom, \(r\) is the distance from the axis of the source mass to the position of the atom. For the choice of wave vectors \(k_1 = k_3 = k, k_2 = k + \Delta k_2\) proposed in \[^{6}\], one has \[^{12}\]

\[
\phi = kgT^2 + [z_0 + v_0 (T_1 + T)] [k\gamma(r)T^2 - 2\Delta k_2] + g \left[ k\gamma(r) \frac{6T_2^2T^2 + 12T_1T^3 + 7T^4}{12} - \Delta k_2 \left( T_1^2 + 2TT_1 + T^2 \right) \right]. \tag{4}
\]

One sees that at

\[
\gamma(r) = \frac{2\Delta k_2}{kT^2}, \tag{5}
\]

the AI phase does not depend on the atomic axial coordinates and velocity \((z_0, v_0)\) \[^{6}\], which makes it possible to measure the gradient of the gravitational field of the source mass \(\gamma\) and Newtonian gravitational constant \(G\) \[^{3}\] \[^{8}\] \[^{9}\]. The dependence of \(\gamma\) on the radial atomic
coordinates \( x_L \) leads to the RSE \( s \) and RSD \( \sigma \) of this measurement.

For differential technique, one uses two ALs. Let’s assume that the atomic clouds in these interferometers are identical. Then, averaging (5) over the atomic cloud, one finds

\[
s = -\varepsilon \left( a_x^2 + a_y^2 \right), \tag{6}
\]

where \((a_x, a_y) = \left[ f \, d(x_L \cdot x, y^2) \cdot f(x_L) \right]^{1/2} \) and \( f(x_L) \) are the radii of the atomic cloud and the radial atomic distribution function.

Uncertainties in the position of cloud centers with standard deviations \((\sigma_x, \sigma_y)\) lead to uncertainty \( \gamma \) with RSD \( \sigma_\gamma \). The usual formula for RSD

\[
\sigma_\gamma = \gamma^{-1} \left[ (\partial \gamma/\partial x)_{x_L=0}^2 \sigma_x^2 + (\partial \gamma/\partial y)_{x_L=0}^2 \sigma_y^2 \right]^{1/2} \tag{7}
\]

leads to a zero result, because due to the symmetry of the source mass, its center is an extreme point, \( (\partial \gamma/\partial x)_{x_L=0} = 0 \). In the article [4], we obtained the formula (16a) for RSD, which one can use at extreme points. From it, one will get that

\[
\sigma_\gamma = \gamma^{-1/2} \left[ (\partial^2 \gamma/\partial x^2)_{x_L=0} \sigma_x^4 + (\partial^2 \gamma/\partial y^2)_{x_L=0} \sigma_y^4 \right]^{1/2} \tag{8}
\]

Here we assumed that the cummulants do not contribute, \( \kappa (x) = \kappa (y) \). Quadratic terms in the dependence of signal variations on the uncertainties of atomic variables lead to an additional RSE \( s' \), for which from the Eq. (16b) in [4] one gets

\[
s' = -\varepsilon \left( \sigma_x^2 + \sigma_y^2 \right). \tag{9}
\]

We applied the results (6, 8, 9) for the source mass configurations proposed in [4]. For the configuration I, using the formula for the axial component of the gravitational field of the cylinder, derived in [4], one obtains \( \varepsilon = -1.45 \text{m}^{-2} \). From the Table 2 in the Ref. [9] one can conclude that the radius of the atomic cloud \( a = \sqrt{a_x^2 + a_y^2} \) is in the range of \( 2.6 \text{mm} < a < 3.4 \text{mm} \). Then for RSE (9) one obtains

\[
9.8 \text{ppm} < s < 17 \text{ppm}. \tag{10}
\]

This RSE may be greater than the measurement accuracy of 13ppm predicted in [4]. From the same Table, one can conclude that the uncertainties in the position of the atomic cloud \( \sigma_x = \sigma_y = 1 \text{mm} \). Then \( \sigma_\gamma = s' = 2.9 \text{ppm} \).

For the configuration II, one finds that

\[
\varepsilon = \frac{3}{4} \left[ \frac{y}{(h^2/4 + y)^{5/2}} \right]_{y=R_2^2}^{y=R_1^2} \sqrt{\frac{1}{(h^2/4 + y)}}, \tag{11}
\]

which, again, is more than the accuracy of measuring \( G \) 5ppm predicted in [9]. At the same time, RSD (3) and RSE (9) are equal to \( \sigma_\gamma = s' = 0.75 \text{ppm} \).

We propose in this comment to choose other sizes of the source mass at which the parameter (11) disappears. The numerical solution of the equation \( \varepsilon = 0 \) is shown in the figure [4]. So, for example, with the same internal and external radius of a hollow cylinder, \( \{R_1, R_2\} = \{0.15 \text{m}, 0.52 \text{m}\} \), it is sufficient to reduce the height of the cylinder to the value \( h = 0.701 \text{m} \) to turn off the systematic and statistical measurement errors caused by the finite radial size of the atomic cloud.

Conclusion. The finite size of the atomic cloud leads to a systematic error that is greater than the accuracy of measuring Newtonian gravitational constant \( G \), predicted in the article Ref. [9].

\[
\text{FIG. 1: Contour plots of the solution of the equation } \varepsilon = 0 \text{ in the plane of the height and the outer radius of the hollow cylinder } \{h, R_2\} \text{ at the different values of the inner radius } R_1
\]
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[10] In the Eq. (1b) we neglected the gravity-gradient tensor of the Earth’s field. It can be excluded, when the double differential technique [2] is used. We also took into account that the third order tensors $\Gamma^{(3)}_{3311}$ and $\Gamma^{(3)}_{3322}$ coincide for both source mass configurations proposed in [2].

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