A NEW COUPLED COMPLEX BOUNDARY METHOD (CCBM) FOR AN INVERSE OBSTACLE PROBLEM

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Abstract. In the present work we introduce and study a new method for solving the inverse obstacle problem. It consists in identifying a perfectly conducting inclusion $\omega$ contained in a larger bounded domain $\Omega$ via boundary measurements on $\partial \Omega$. In order to solve this problem, we use the coupled complex boundary method (CCBM), originally proposed in [16]. The new method transforms our inverse problem to a complex boundary problem with a complex Robin boundary condition coupling the Dirichlet and Neumann boundary data. Then, we optimize the shape cost function constructed by the imaginary part of the solution in the whole domain in order to determine the inclusion $\omega$. Thanks to the tools of shape optimization, we prove the existence of the shape derivative of the complex state with respect to the domain $\omega$. We characterize the gradient of the cost functional in order to make a numerical resolution. We then investigate the stability of the optimization problem and explain why this inverse problem is severely ill-posed by proving compactness of the Hessian of cost functional at the critical shape. Finally, some numerical results are presented and compared with classical methods.

1. Introduction. This paper deals with the problem of reconstructing an object living in a larger bounded domain from boundary measurements. A method to solve this inverse problem is the shape optimization method. To fix ideas, we consider in this work the following inverse problem: let $\Omega$ be a known smooth and bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Assume that $\omega$ is an unknown simply connected subdomain of $\Omega$ with regular boundary such that $\partial \omega \cap \partial \Omega = \emptyset$, we fix $\delta > 0$ and we denote by $\Omega_{\delta}$ the set of all open subsets $\omega$ strictly included in $\Omega$, with a $C^{2,1}$ boundary, such that $d(x, \partial \Omega) > \delta$ for all $x \in \omega$ and such that $\Omega \setminus \overline{\omega}$ is connected. Let us further assume that the material composing $\omega$ is perfectly conducting, contrary to the “background material” inside $\Omega \setminus \overline{\omega}$, which is assumed to have a constant conductivity $\sigma = 1$. For a given Dirichlet boundary measurement $f \in H^\frac{1}{2}(\partial \Omega)$ and a Neumann boundary data $g \in H^{-\frac{1}{2}}(\partial \Omega)$, the inclusion $\omega$ and the electrostatic

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potential $u$ solve the overdetermined problem:

$$\begin{cases}
-\Delta u &= 0 \text{ in } \Omega \setminus \overline{\omega} \\
u &= f \text{ on } \partial \Omega, \\
\partial_n u &= g \text{ on } \partial \Omega, \\
u &= 0 \text{ on } \partial \omega.
\end{cases}$$

(1)

where $\partial_n = \frac{\partial}{\partial n}$ stands for the outward normal derivative.

To be more precise, the inverse obstacle problem is reformulated as follows:

find $\omega \in \Omega_0$ and $u$ that satisfies the overdetermined system (1).

(2)

This problem is a particular case of the tomography problem [24, 19] and arises in many applications such as nondestructive testing of materials.

The fundamental question of the existence and uniqueness of the solution of the inverse obstacle problem from boundary measurements data has been investigated by several authors [5, 6, 9, 25]. We recall in the following theorem, the important identifiability result for this inverse problem which shows that the inclusion $\omega$ is unique.

**Theorem 1.** ([9])

The inclusion $\omega$ and the potential $u$ satisfying (1) are uniquely defined by the Cauchy data $(f, g) \neq (0, 0)$.

The classical way to solve this inverse problem, is to transform it into a shape optimization problem. Thus, Eppler and Harbrecht [19], Afraites and al. [1] considered two cost functions, the first is the least squares defined by:

$$J_{LS}(\omega) = \frac{1}{2} \int_{\partial \Omega} |u - f|^2 d\sigma,$$

(3)

where $u$ solves the Neumann problem:

$$\begin{cases}
-\Delta u &= 0 \text{ in } \Omega \setminus \overline{\omega}, \\
\partial_n u &= g \text{ on } \partial \Omega, \\
u &= 0 \text{ on } \partial \omega.
\end{cases}$$

(4)

The second is the Kohn-Vogelius cost function defined by

$$J_{KV}(\omega) = \frac{1}{2} \int_{\Omega \setminus \overline{\omega}} |\nabla (u_D - u_N)|^2 dx,$$

(5)

where $u_D$ is the solution of the Dirichlet problem:

$$\begin{cases}
-\Delta u &= 0 \text{ in } \Omega \setminus \overline{\omega}, \\
u &= f \text{ on } \partial \Omega, \\
u &= 0 \text{ on } \partial \omega.
\end{cases}$$

(6)

and $u_N$ is the solution of Neumann problem (4). The cost function $J_{KV}$ measures the gap of energy between the solutions of the Dirichlet and Neumann problems corresponding to the data given above. This function is positive and vanishes only if $u_D = u_N$ which is the case when the shape $\omega$ fits the exact inclusion. The authors studied those cost functionals using the tools of shape optimization via the shape derivative with respect to domain $\omega$ and characterize the shape gradient in order to make a numerical resolution. Then, the instability of the inverse problem is investigated by computing the shape Hessian at critical point.

We note from the above statements that the minimization problems (3) and (5) use the Neumann data $g$ and the Dirichlet data $f$ sequentially. In this paper,
we propose a new coupled complex boundary method (CCBM) that uses both $g$ and $f$ data in a single PDE. The idea of the CCBM is to couple the Neumann data and Dirichlet data in a Robin boundary condition in such a way that the Neumann data and Dirichlet data are the real part and imaginary part of the Robin boundary condition, respectively. As a result, the data needed to fit defined on the boundary $\partial \Omega$ are transferred to the volume problem defined on $\Omega \setminus \overline{\omega}$. The coupled complex boundary method (CCBM) was first proposed by Cheng et al in ([16]) for solving an inverse source problem, Rongfang et al in ([21]) applied it to an inverse conductivity problem with one measurement and recently, this method is applied to parameter identification in elliptic problems by Cheng et al in ([31]). To the best of our knowledge, in the literature, this is the first time that the idea of the coupled complex boundary condition has been explored for solving inverse obstacle problem.

The outline of this paper is organized as follows. In section 2, we present the new coupled complex boundary method appropriate to our inverse obstacle problem and its reformulation at the shape optimization by introducing the Least Squares fitting for the imaginary part of the complex PDE’s solution. In section 3 we give a rigorous proof of the existence of the material derivative of the state and we deduce its shape derivative. In section 4 we establish the shape gradient calculus and the second order shape derivative of the cost function at a critical shape. Moreover, based on the latter results, in section 5 we investigate the stability of our identification problem by proving the compactness of the shape Hessian of the cost functional at the critical shape. Finally, in the last section, we give an algorithm based on the gradient method and we solve an elliptic problem in order to find the steepest descend direction in the space of $H^1$ velocity vector field that satisfies certain boundary conditions. Then, we present detailed numerical results and extensive comparative experiments for nonparametric shapes.

2. A novel coupled complex boundary method. We first introduce some notations for function spaces and assumptions on the data [17]. For a set $G$ (e.g., $\Omega$, $\omega$, $\Omega \setminus \overline{\omega}$), we denote by $W^{m,s}(G)$ the standard real Sobolev space with the norm $\|\cdot\|_{m,s,G}$. Let $W^{0,s}(G) := L^s(G)$. In particular, $H^m(G)$ represents $W^{m,2}(G)$ with the corresponding inner product $\langle \cdot, \cdot \rangle_{m,G}$ and norm $\|\cdot\|_{m,G}$. Also, let $H^m(G)$ be the complex version of $H^m(G)$ with the inner product $\langle \cdot, \cdot \rangle_{m,G}$ and norm $\|\cdot\|_{m,G}$. Denote $V = H^1_\phi(\Omega)$, $\mathcal{V} = H^1_\phi(\Omega)$ where $H^1_\phi(\Omega) = \{ u \in H^1(\Omega), u = 0 \text{ on } \partial \omega \}$, $Q = L^2(\Omega)$, $\mathcal{Q} = L^2(\Omega)$.

We consider a complex PDE:

$$
\begin{cases}
-\Delta u = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
\partial_n u + iu = g + if & \text{on } \partial \Omega, \\
u = 0 & \text{on } \partial \omega.
\end{cases}
$$

(7)

where $i = \sqrt{-1}$ is the imaginary unit. Assume that $u = u_1 + iu_2$ is a weak solution of (7). Then real-valued functions $u_1, u_2$ satisfy:

$$
\begin{cases}
-\Delta u_1 = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
\partial_n u_1 - u_2 = g & \text{on } \partial \Omega, \\
u_1 = 0 & \text{on } \partial \omega.
\end{cases}
$$

(8)
If \( u_2 = 0 \) in \( \Omega \setminus \overline{\omega} \), then \( u_2 = \partial_n u_2 = 0 \) on \( \partial \Omega \) and \( u_2 = 0 \) on \( \partial \omega \). As a result, from PDE (9) and (8), \((u_1, \omega)\) is a solution of the original problem (1). Conversely, if \((u, \omega)\) is a solution of the original problem (1), then immediately, it satisfies (7).

We conclude from the above discussion that the original inverse shape problem is equivalent to the following problem.

**Problem 1.** Find \( \omega \in \Omega_\delta \) such that

\[
\begin{array}{rcl}
u_2 &=& 0 \text{ in } \Omega \setminus \overline{\omega}, \\
-\Delta u &=& 0 \quad \text{in } \Omega \setminus \overline{\omega}, \\
\partial_n u + iu &=& f \quad \text{on } \partial \Omega, \\
u_2 &=& 0 \quad \text{on } \partial \omega.
\end{array}
\]

(10)

Before to solve the inverse problem, we first show a well-posedness result of the complex PDE (10). For this, let \( u, v \in V \) such that :

\[
a(u, v) = \int_{\Omega \setminus \overline{\omega}} \nabla u \cdot \nabla \bar{v} \, dx + i \int_{\partial \Omega} \bar{u} \bar{v} \sigma \, d\sigma, \quad l(v) = \int_{\partial \Omega} g \bar{v} \sigma + i \int_{\partial \Omega} f \bar{v} \sigma.
\]

Then the weak formulation of the complex PDE (10) is given by :

Find \( u \in V \), such that \( a(u, v) = l(v), \quad \forall v \in V \). (11)

The existence and uniqueness of the weak formulation problem is given by the following proposition.

**Proposition 1.** For given \( \omega \in \Omega_\delta \), \( f \in H^{\frac{1}{2}}(\partial \Omega) \) and \( g \in H^{-\frac{1}{2}}(\partial \Omega) \), the problem (11) admits a unique solution \( u \in V \) which depends continuously on \( f \) and \( g \). Moreover,

\[
\|u\|_{1, \Omega \setminus \overline{\omega}} \leq c \left( \|f\|_{H^{\frac{1}{2}}(\partial \Omega)} + \|g\|_{H^{-\frac{1}{2}}(\partial \Omega)} \right)
\]

(12)

**Proof.** We use the complex version of Lax-Milgram Lemma ([17], p.376) and we show that the form \( a(., .) \) is continuous and elliptic on \( V \), and the linear form \( l(.) \) is continuous on \( V \). Then, the problem (11) has a unique solution \( u \in V \). Furthermore, we have (12). For more details (see [16], proposition 2.2).

To solve the inverse shape **Problem 1**, we transform it into the following shape optimization functional :

\[
J(\omega) = \frac{1}{2} \|u_2\|_{L^2(\Omega \setminus \overline{\omega})}^2
\]

(13)

and we introduce the following minimization problem:

**Problem 2.** Find \( \omega^* \in \Omega_\delta \) in admissible shape, such that :

\[
\omega^* = \arg\min_{\omega \in \Omega_\delta} J(\omega).
\]

(14)

**Remark 1.** The new method allows us to define the cost function \( J \) in the whole domain \( \Omega \setminus \overline{\omega} \) which brings advantages of robustness in the reconstruction such as the Kohn-Vogelius cost function \( J_{KV} \) compared to the Least Squares fitting \( J_{LS} \) which is defined only on the boundary \( \partial \Omega \) (see [1, 2, 4]). Compared to the Kohn-Vogelius method, the latter requires two problems to be solved at each iteration, however the new method (CCBM), needs a single complex problem to be solved.
In order to compute the shape derivative of the cost function (13) with respect to the shape \( \omega \), we need to compute the shape derivative of the state (7). We use the classical shape calculus developed by Murat-Simon [27]. For details concerning the differentiation with respect to the domain( see[30, 28, 29] and the book [23]). We can also use the techniques developed in [13, 14, 15].

3. Shape derivatives. In this section, we present some preliminary notions concerning the shape derivative as velocity field deformation, material derivative and shape derivative. Then, we give a rigorous proof of the existence of the material derivative of state and we deduce its shape derivative.

3.1. Elements of shape calculus. Let us first fix the notations. We set \( n_{\partial \omega} \) the unit normal vector to \( \partial \omega \) pointing into \( \Omega \setminus \partial \omega \). Let \( V \) denote a smooth vector field with compact support in \( \Omega \), we denote \( U \) the space of admissible deformations \( V \) and we set \( V_{n} := \langle V, n_{\partial \omega} \rangle \) its normal component. In the sequel, the tangential gradient will be denoted by \( \nabla_{\tau} \). We set \( T_{t}(x) = x + tV(x) \), this transformation is as smooth as \( V \) and for small \( t \) is invertible. We also set \( J_{t}(x) = \det DT_{t}(x) \) and \( A_{t}(x) = DT_{t}^{-1}(x)^{T}DT_{t}^{-1}(x)J_{t}(x) \).

We recall some classical facts in shape optimization: \( A_{t}(x) \) is symmetric positive and for \( t < t_{0} \) one has \( y^{T}A_{t}(x)y \geq ||y||^{2}/2 \), moreover, \( A \) is a smooth application (it depends only on \( V \)) with \( A_{0}(x) = I \), \( J_{0}(x) = 1 \) and

\[
\frac{d}{dt}J_{t}(x)|_{t=0} = \text{div}(V), \quad A = \frac{d}{dt}A_{t}(x)|_{t=0} = \text{div}(V)I - (DV^{T} + DV).
\]

We denote \( u_{t} \) the solution of (7) with inclusion \( \omega_{t} = T_{t}(\omega) \) and \( u^{t} = u_{t} \circ T_{t} \). The function \( u^{t} \) is defined on the fixed domain \( \Omega \) and its material derivative (or Lagrangian derivative) is defined by

\[
\dot{u} := \lim_{t \to 0} \frac{u^{t} - u}{t}, \quad \forall x \in \Omega.
\]

The shape derivative ( or Eulerian derivative) is defined by :

\[
u' := \dot{u} - \nabla u \cdot V.
\]

3.2. Shape derivatives of the state functions. Theorem 2. The state \( u \) has a material derivative \( \dot{u} \in \mathcal{V} \) that solves

\[
\forall v \in \mathcal{V}(\Omega \setminus \partial) \left( \nabla \dot{u}, \nabla \bar{v} \right)_{\Omega \setminus \partial} = -\left( A\nabla u, \nabla \bar{v} \right)_{\Omega \setminus \partial} \quad (15)
\]

The state \( u \) is shape differentiable and its shape derivative \( u' \) satisfies

\[
\begin{align*}
-\Delta u' &= 0 \quad \text{in } \Omega \setminus \partial, \\
\partial_{n}u' + iu' &= 0 \quad \text{on } \partial \Omega, \\
u' &= -V_{n}\partial_{n}u \quad \text{on } \partial \omega.
\end{align*} \quad (16)
\]

Before proving this result, we have to make some comments. If we assume that the state \( u \) is shape differentiable, we formally obtain (16) by differentiation of the boundary conditions. We find it useful to explain how both existence of the derivative and (16) can be obtained by the classical methods of shape optimization. Hence we give a complete proof of Theorem 2.
Proof. Theorem 2: We decompose the proof into four classical parts: first, we formulate the problem in the fixing domain, we then prove its weak convergence to the material derivative, then strong convergence and we deduce the shape derivative of the state.

First step. Let \( u_t \) denote the solution of (10) with inclusion \( \omega_t = T_t(\omega) \), we have:

\[
\forall v \in \mathcal{V}, \quad \int_{\Omega \setminus \omega_t} \nabla u_t \cdot \nabla \bar{v} + i \int_{\partial \Omega} u_t \bar{v} = \int_{\partial \Omega} g \bar{v} d\sigma + i \int_{\partial \Omega} f d\sigma.
\]

Then, the transported \( u^t = u_t \circ T_t \) solves the variational equation:

\[
\forall v \in \mathcal{V}, \quad \int_{\Omega \setminus \omega} A_t(x) \nabla u^t(x) \cdot \nabla \bar{v}(x) + i \int_{\partial \Omega} u_t \bar{v} J_t = \int_{\partial \Omega} g \bar{v} d\sigma + i \int_{\partial \Omega} f d\sigma,
\]

since we have on the boundary \( \partial \Omega \), \( u^t = u, \ J_t = 1 \), then:

\[
\forall v \in \mathcal{V}, \quad \int_{\Omega \setminus \omega} A_t(x) \nabla u^t \cdot \nabla \bar{v}(x) + i \int_{\partial \Omega} u^t \bar{v} J_t = \int_{\partial \Omega} g \bar{v} d\sigma + i \int_{\partial \Omega} f d\sigma, \quad (17)
\]

Second step. Subtracting the variational equation of the original problem (11) and the transported one (17), we obtain:

\[
\forall v \in \mathcal{V}, \quad \left( \frac{\nabla u^t - \nabla u}{t} \right) = \left( \frac{I - A_t}{t} \nabla u \right) \quad \text{in} \quad \Omega \setminus \omega. \quad (18)
\]

Taking \( \frac{(u^t - u)}{t} \) as test function, we get by the properties of \( A_t \):

\[
\alpha \left\| \frac{\nabla u^t - \nabla u}{t} \right\|^2 \leq \left\| \frac{A_t - I}{t} \right\| \left\| \nabla u \right\| \left\| \frac{\nabla u^t - \nabla u}{t} \right\| \quad \text{in} \quad \Omega \setminus \omega,
\]

we deduce that:

\[
\left\| \frac{\nabla u^t - \nabla u}{t} \right\| \leq C \left\| \frac{A_t - I}{t} \right\| \left\| \nabla u \right\|,
\]

where \( C \) is a positive constant. Therefore \( \frac{(u^t - u)}{t} \) is bounded in \( \mathcal{V} \). Hence the sequence is weakly convergent in \( \mathcal{V} \); indeed, setting \( v = \frac{(u^t - u)}{t} \) in (18), we get

\[
(A_t \nabla v, \nabla v) = \langle \frac{I - A_t}{t} \nabla u^t, \nabla v \rangle = \langle (A_t - I) \nabla v, \nabla v \rangle + \langle \frac{I - A_t}{t} \nabla u^t, \nabla v \rangle = E_{1,t} + E_{2,t},
\]

where

\[
E_{1,t} = \langle (A_t - I) \nabla v, \nabla v \rangle \quad \text{and} \quad E_{2,t} = \left( \frac{I - A_t}{t} \right) \nabla u^t, \nabla v \right\).
\]
The weak convergence of \((u' - u)/t\) leads after straightforward calculations to
\[ E_{1,t} \rightarrow 0 \text{ and } E_{2,t} \rightarrow -\left( \mathcal{A}\nabla u, \nabla \dot{u} \right) \text{ when } t \rightarrow 0. \]

By (15), we conclude that \(E_{2,t} \rightarrow \langle \nabla \dot{u}, \nabla \dot{u} \rangle\). This shows that \(\nabla v\) converges strongly to \(\nabla \dot{u}\) in \(\mathbf{V}\). The fact that \(\|\nabla u\|_Q\) is a norm equivalent to the norm \(\mathbf{V}\), we deduce the strong convergence of \(v\) to \(\dot{u}\) in \(\mathbf{V}\).

**Fourth Step.** We deduce the equations satisfied by the shape derivative \(u' = \dot{u} - V \nabla u\). Set \(b = \langle V \nabla u, \nabla v \rangle + \langle V \nabla v, \nabla u \rangle - \langle \nabla u, \nabla v \rangle V\). We will use the classical identity
\[ -\nabla u \cdot A \nabla v = \text{div}(b) - \langle V \nabla u \rangle \Delta v - \langle V \nabla v \rangle \Delta u. \tag{21} \]

From the identity satisfied by \(\dot{u}\), we have
\[ \int_{\Omega \setminus \overline{\omega}} \nabla \dot{u} \cdot \nabla v = \int_{\Omega \setminus \overline{\omega}} \text{div}(b) - \int_{\Omega \setminus \overline{\omega}} \langle V \nabla u \rangle \Delta v - \int_{\Omega \setminus \overline{\omega}} \langle V \nabla v \rangle \Delta u \tag{22} \]
from the divergence theorem and after an integration by parts and using \(\Delta u = 0\) in \(\Omega \setminus \overline{\omega}\), we get
\[ \int_{\Omega \setminus \overline{\omega}} \nabla \dot{u} \cdot \nabla v = -\int_{\partial \omega} \langle V \nabla u \rangle \partial_n u + \int_{\partial \omega} \langle \nabla u, \nabla v \rangle V_n + \int_{\Omega \setminus \overline{\omega}} \langle V \nabla u \rangle \nabla v \]
Since \(u = v = 0\) on \(\partial \omega\), then \(\nabla v = \nabla u = 0\).

Finally, we get :
\[ \int_{\Omega \setminus \overline{\omega}} \nabla (\dot{u} - V \nabla u) \cdot \nabla v \, dx = 0 \]
Hence, the shape derivative \(u' = \dot{u} - V \nabla u\) is solution of
\[ \int_{\Omega \setminus \overline{\omega}} \nabla u' \cdot \nabla v = 0 \]
and Green’s formula enables us to get
\[ -\int_{\Omega \setminus \overline{\omega}} \Delta u' v + \int_{\partial(\Omega \setminus \overline{\omega})} v \partial_n u' = 0. \]
We take \(v \in D(\Omega \setminus \overline{\omega})\), this shows that \(u'\) satisfies \(\Delta u' = 0\) in \(\Omega \setminus \overline{\omega}\) with the condition
\(\partial_n u' + iu' = 0\) on \(\partial \Omega\).

Since \(\dot{u} = 0\) on \(\partial \omega\), we get
\[ u' = -V_n \partial_n u \text{ on } \partial \omega. \tag{23} \]
This ends the proof of the shape differentiability of \(u\).

4. **Shape derivative of the cost function.** In this section, we characterize the shape derivatives of the cost functional (13) then, we give the second order shape derivative at critical shape in order to study the stability of our process in the next section.
4.1. **First order shape derivative.** The shape cost function is given by:

**Proposition 2.** For $V \in \mathcal{U}$, we have

$$DJ(\omega)[V]) = \int_{\partial \omega} \left( \partial_n w_1 \partial_n u_2 - \partial_n w_2 \partial_n u_1 \right) V_n$$

where $u$ is the solution of the state (7) and $w$ is the solution of the following adjoint problem

$$\begin{cases}
-\Delta w = u_2 & \text{in } \Omega \setminus \overline{\omega}, \\
\partial_n w - iw = 0 & \text{on } \partial \Omega, \\
w = 0 & \text{on } \partial \omega.
\end{cases}$$

(24)

**Proof.** The state function is differentiable and the compact support of the objective is strictly separated from the moving boundary $\partial \omega$. Hence, $J$ has also a shape derivative that is obtained by the chain rule while using local derivatives:

$$DJ(\omega)[V]) = \int_{\Omega \setminus \overline{\omega}} u_2 u'_2 + \int_{\partial \Omega} \text{div}(|u_2|^2 V)$$

$$= \int_{\Omega \setminus \overline{\omega}} u_2 u'_2 + \int_{\partial \Omega} |u_2|^2 V_n + \int_{\partial \omega} |u_2|^2 V_n$$

(25)

because $V = 0$ on $\partial \Omega$ and $u_2 = 0$ on $\partial \omega$.

We multiple the solution of the derivative of $u'$ by $w$ solution of the adjoint state and after applying integration by parts, we obtain:

$$\int_{\Omega \setminus \overline{\omega}} \nabla u' \nabla \overline{w} + i \int_{\partial \Omega} u' \overline{w} = 0.$$  

(26)

In the same way, we apply the Green formula for adjoint state and with $u'$ as a test function, we get:

$$\int_{\Omega \setminus \overline{\omega}} -\Delta \overline{w} u' = \int_{\Omega \setminus \overline{\omega}} \nabla \overline{w} \nabla u' - \int_{\partial \Omega} \partial_n w \overline{u}' - \int_{\partial \omega} \partial_n w u'.$$

(27)

(27) implies that

$$\int_{\Omega \setminus \overline{\omega}} -\Delta \overline{w} u' = \int_{\Omega \setminus \overline{\omega}} \nabla \overline{w} \nabla u' + i \int_{\partial \Omega} \overline{w} u' - \int_{\partial \omega} \partial_n \overline{w} u'.$$

(28)

Finally, since we have $-\Delta w = u_2$ in $\Omega \setminus \overline{\omega}$, $u' = -\partial_n u V_n$ on $\partial \omega$ and using (26) we obtain:

$$\int_{\Omega \setminus \overline{\omega}} u_2 u'_2 = \int_{\partial \omega} \partial_n \overline{w} \partial_n u V_n$$

which gives:

$$\int_{\Omega \setminus \overline{\omega}} u_2 u'_2 = \int_{\partial \omega} (\partial_n w_1 \partial_n u_2 - \partial_n w_2 \partial_n u_1) V_n,$$

hence, we get the desired expression. \qed
4.2. Second order shape derivative of the cost function. Let us consider $\omega^* \in \Omega_5$ solution of the inverse problem (1). In order to study the stability of the optimization problem (14) at $\omega^*$, we want to compute the second order shape derivative of $J$, i.e. the shape Hessian. First of all, notice that we prove in exactly the same way as we proved the existence of the shape derivative $u'$ that the adjoint state $w$ is differentiable with respect to the shape $\omega \in \Omega_5$ and we denote by $w'$ its shape derivative. Then, we have the following result :

**Proposition 3.** (Characterization of the shape Hessian at a critical shape) For $V \in U$, we have :

$$D^2 J(\omega^*)[V, V] = \int_{\partial \omega^*} (\partial_n w' \partial_n u_1) V_n,$$

where $w'$ is solution of the following problem :

$$\begin{cases}
-\Delta w' = u'_2 & \text{in } \Omega \setminus \overline{\omega}, \\
\partial_n w' - iw' = 0 & \text{on } \partial \Omega, \\
w' = 0 & \text{on } \partial \omega^*,
\end{cases}$$

(30)

**Proof.** Applying classical results to differentiate the volume integral with respect to the shape, we get :

$$D^2 J(\omega^*)[V, V] = \int_{\Omega \setminus \overline{\omega}} (u'_2)^2 + \int_{\Omega \setminus \overline{\omega}} u_2 u'' + \int_{\Omega \setminus \overline{\omega}} \text{div}(u_2 u'_2 V).$$

(31)

At critical shape of $J$, we have $u_2 = 0$ then (31) becomes :

$$D^2 J(\omega^*)[V, V] = \int_{\Omega \setminus \overline{\omega}} (u'_2)^2.$$

(32)

We then characterize the shape derivative of the adjoint state $w'$ in the same way that we characterized $u'$, i.e.

$$\begin{cases}
-\Delta w' = u'_2 & \text{in } \Omega \setminus \overline{\omega}, \\
\partial_n w' - iw' = 0 & \text{on } \partial \Omega, \\
w' = -\partial_n w V_n & \text{on } \partial \omega,
\end{cases}$$

(30)

At critical shape $w^*$, we have $u_2 = 0$ in $\Omega \setminus \overline{\omega}$ thereafter the adjoint state $w = 0$ in $\Omega \setminus \overline{\omega}$ which give $\partial_n w = 0$ on $\partial \omega^*$ and we obtain (30).

To obtain the expression of (29), we multiply the solution of (30) by $u'$ and we use the boundary conditions, we obtain :

$$-\int_{\Omega \setminus \overline{\omega}} \Delta w' \bar{u}' = \int_{\Omega \setminus \overline{\omega}} \nabla w' \nabla \bar{u}' - i \int_{\partial \Omega} w' \bar{u}' + \int_{\partial \omega} \partial_n w' \partial_n \bar{u} V_n$$

(33)

in the same way, we have :

$$0 = -\int_{\Omega \setminus \overline{\omega}} \Delta u' \bar{w}' = \int_{\Omega \setminus \overline{\omega}} \nabla u' \nabla \bar{w}' + i \int_{\partial \Omega} u' \bar{w}'$$

(34)

and (33) give :

$$\int_{\Omega \setminus \overline{\omega}} u'_2 \bar{w}' = -\int_{\Omega \setminus \overline{\omega}} \Delta w' \bar{u}' = \int_{\partial \omega^*} \partial_n \bar{w}' \partial_n u V_n.$$

(35)

Since at the critical shape $\partial_n u_2 = 0$ which give :

$$\int_{\Omega \setminus \overline{\omega}} u'_2 u'_2 = \int_{\partial \omega} \partial_n w'_2 \partial_n u_1 V_n.$$

(36)

We obtain the claimed expression. □
5. Instability of the problem. Let us investigate the properties of stability of our cost function. Thus, we assume that there exists an admissible inclusion \( \omega^* \) such that \( J(\omega^*) = 0 \). It realizes the absolute minimum of the criterion \( J \). This is satisfied by solution of the inverse problem. To prove this instability result of the inverse problem \( (1) \), we adapt the method already used in \([8, 10, 11, 12]\). For this, we use a local regularity argument in order to prove the compactness of the Riesz operator corresponding to the shape Hessian at a solution \( \omega^* \in \Omega_3 \) of the inverse problem. An alternative proof could be to use the potential layers as what is done in \([2, 3]\).

At a critical shape, we have the (32):

\[
D^2J(\omega^*)[V, V] = \int_{\Omega_\sigma} (u'_2)^2.
\] (37)

Moreover, if \( V_n \neq 0 \), then \( D^2J(\omega^*)(V, V) > 0 \) holds. Nevertheless, (37) does not means that the minimization problem is well posed. In fact, the following theorem explains the instability of standard minimization algorithms.

**Proposition 4.** (Compactness at a critical shape) If \( \omega^* \) is the critical shape of \( J \), then the Riesz operator associated to the quadratic shape Hessian

\[
D^2J(\omega^*) : H^{1/2}(\partial \omega^*) \to H^{-1/2}(\partial \omega^*)
\]

is compact.

**Proof.** The idea of the proof is to write the shape Hessian as a composition of linear continuous operators whose one is compact (the compactness being obtained using the compactness of the imbedding between two Sobolev spaces).

We write the formula of the characterization of the shape Hessian given in Proposition 3 by :

\[
D^2J(\omega^*)[V, V] = \int_{\partial \omega^*} \partial_n w'_2 \partial_n u_1 V_n = \langle \partial_n u_1 V_n, \partial_n w'_2 \rangle,
\]

where \( \langle ., . \rangle \) is the product of duality \( H^{1/2}(\partial \omega^*) \times H^{-1/2}(\partial \omega^*) \).

Then, one introduces two operators \( T \) and \( M \) (spaces related to \( \partial \omega^* \)) by :

\[
T : H^{1/2}(\partial \omega^*) \to H^{1/2}(\partial \omega^*) \quad \text{and} \quad M : H^{1/2}(\partial \omega^*) \to H^{-1/2}(\partial \omega^*)
\]

\[
V \mapsto \partial_n u_1 V_n \quad \text{and} \quad V \mapsto \partial_n w'_2
\]

So that Hessian is :

\[
D^2J(\omega^*)(V, V) = \langle T(V), M(V) \rangle.
\]

The operator \( T \) is clearly linear continuous as a product of smooth functions \([26]\) but the operator \( M \) is compact. For the latter, according to the characterization of \( w' \), we decompose \( M \) by \( M = M_2oM_1 \) with :

\[
M_1 : H^{1/2}(\partial \omega^*) \to H^1(\Omega) \quad \text{and} \quad M_2 : H^1(\Omega) \to H^{1/2}(\partial \omega^*)
\]

\[
V \mapsto u'_2 \quad \text{and} \quad \psi \mapsto \partial_n \Phi_2
\]

where \( u'_2 \) is solution of (16) and \( \Phi = \Phi_1 + i\Phi_2 \) solution of :

\[
\begin{align*}
-\Delta \Phi &= \psi \text{ in } \Omega \setminus \overline{\omega^*}, \\
\Phi - i\Phi &= 0 \text{ in } \partial \Omega, \\
\Phi &= 0 \text{ in } \partial \omega^*,
\end{align*}
\] (38)
$M_1$ is linear continuous. Now, we can decompose $M_2$ as $M_2 = M_{2,3} \circ M_{2,2} \circ M_{2,1}$ with:

\[
M_{2,1} : H^1(\Omega) \to H^3(\Omega_\delta) \\
\psi \mapsto \Phi_2
\]

\[
M_{2,2} : H^3(\Omega_\delta) \to H^{3/2}(\partial \omega^*) \\
\Phi_2 \mapsto \partial_n \Phi_2
\]

\[
M_{2,3} : H^{3/2}(\partial \omega^*) \to H^{-1/2}(\partial \omega^*) \\
\partial_n \Phi_2 \mapsto \partial_n \Phi_2
\]

The operators $M_{2,1}$ and $M_{2,2}$ are then linear continuous and the operator $M_{2,3}$ is the compact embedding of $H^{3/2}(\partial \omega^*)$ into $H^{-1/2}(\partial \omega^*)$. Note that the regularity $H^3(\Omega)$ is due to a local regularity argument (see [8, 10, 11, 12]). The solution of problem (38) is globally $H^1(\Omega)$, but locally $H^3(\Omega_\delta)$. Hence, we obtain the compactness result.

**Remark 2.** The instability study of our shape optimization problem is very important and prove that this inverse geometric problem is unstable in following sense: the functional $J$ is degenerate for the highly oscillating perturbations (frequencies) (see [2, 12], Numerical Section). For this, the choice of low frequency in measurements data is a regularization way of the cost functional considered.

6. **Algorithm and numerical results.** In this section, we present some numerical simulations in order to confirm and complete our previous theoretical results. The optimization method used for the numerical simulations is here the classical gradient algorithm for which we give more details in the next subsection.

6.1. **Algorithm.** The shape derivative of the cost function $J$ along a deformation field $V$ can be expressed as:

\[
DJ(\omega)[V]) = \int_{\partial \omega} RV_n d\sigma
\]

where $R = \left( \partial_n w_1 \partial_n u_2 - \partial_n w_2 \partial_n u_1 \right)$ where $u$ is solution of the direct problem (10) and $w$ is solution of the adjoint problem (24). The deformation field $V$ is chosen to provide a descent direction of the cost function $J(\omega)$, thus $V = -Rn$ on $\partial \omega$ is a descent direction. In addition, it is well known that the shape gradient is defined on the boundary of the moving shape ([18]) using this approach, the direction of descent must be defined only on $\partial \omega$. However, if the boundary measurements $(f, g)$ is not sufficiently smooth, the surface expression of the shape gradient may not exist or the direction of descent $V$ may be have a poor regularity. Therefore, it is interesting to compute a direction of descent $V$ on $\Omega$ from the volumetric expression of the shape gradient. Which requires solving another additional variational problem. Let $V$ be the Riesz representative of $-DJ(\omega)$, i.e (see [7], [20]):

\[
<V, \phi >_{H^1(\Omega \setminus \Gamma)} = -DJ(\omega)[\phi] = -< Rn, \phi >_{L^2(\partial \omega)}, \forall \phi \in D^2,
\]

(39)

where

\[
D = \left\{ \phi \in H^1(\Omega), \phi = 0 \text{ in } \partial \Omega \right\},
\]

and $<, >$ is the inner product on $D^2$ defined by:

\[
<V, \phi >_{H^1(\Omega \setminus \Gamma)} = \int_{\Omega \setminus \Gamma} \nabla V : \nabla \psi + V \cdot \psi.
\]
The equation (39) is the weak formulation for the following system:

\[
\begin{align*}
-\Delta V + V &= 0 \quad \text{in } \Omega \setminus \overline{\omega}, \\
V &= 0 \quad \text{on } \partial \Omega, \\
\partial_n V &= -Rn \quad \text{on } \partial \omega.
\end{align*}
\]  

(40)

We consider the following gradient descent algorithm to solve our shape optimization problem:

**Algorithm 1** Gradient algorithm for shape optimization

1. Choose an initial shape \( \omega_0 \), set \( k = 0 \).
2. Solve the state problem (10).
3. Solve the adjoint problem (24).
4. Compute the descent direction \( V_k \) by using (40).
5. Update the current boundary \( \partial \omega_k \) by \( V_k \) to obtain \( \partial \omega_{k+1} \), i.e., set \( \partial \omega_{k+1} := \{ x + t_k V_k(x) : x \in \partial \omega_k \} \), for some sufficiently small scalar \( t_k > 0 \).
6. While \( \| DJ(\omega_k) V_k \| \geq \epsilon, k = k + 1 \) and go back to step 2 and repeat.

**Remark 3.** A regularized Newton method could be used (as in [19, 2] for example) in order to perform the numerical procedure using shape Hessian informations. However, we do not use the optimization method of order two because the expression of the shape Hessian (31) needed to compute the second derivative of state (as in [3] for example) and the additional regularity is required. The order two analysis was introduce in this work in order to describe the ill-posedness of the inverse obstacle problem.

6.2. Numerical results. The numerical simulations presented are made in dimension two using the finite elements library Freefem++ (see [22]). The exterior boundary \( \partial \Omega \) is assumed to be unit circle. We construct the synthetic data on \( \partial \Omega \), by fixing the shape \( \omega \) and choosing the Neumann boundary condition \( q(t) t \in [0, 2\pi] \), then we solve the problem (4) using a \( P_2 \) finite elements discretization and extract the measurement \( f = u \) on \( \partial \Omega \). For the tests with noisy data, we perturbed the Dirichlet data \( f \) by a Gaussian noise with a fixed amplitude. Furthermore, in an effort to avoid committing an inverse crime, the number of discretizations for obtaining the synthetic data was chosen to be different from the number of discretizations within the inverse solver (\( P_2 \) for data construction and \( P_1 \) to solve direct and adjoint problem). Then, we use the function \( \text{movemesh} \) of Freefem++ in order to change the shape of the objects at each step (step 4 in the previous algorithm) \( \text{movemesh} \) applies a globally diffeomorphism to the mesh and the function \( \text{adaptmesh} \) to refine and avoid degeneracy of the triangles in the meshes (see the tutorial of Freefem++ [22] for the use of these classical functions).

The implementation also requires a stopping criteria. In our numerical experiments, it was difficult to find criteria which works well on both synthetic and measured test data. Therefore, in our case, the iterative process was interactively monitored and stopped when the change in the reconstruction obtained from an iteration step was no longer noticeable.

6.2.1. Results without noise. We use initially \( N_{\text{ext}} := 100 \) discretization points for the exterior boundary and \( N_{\text{int}} := 60 \) points for the interior boundary and we try to identify the following shapes, starting from the simple shape to the complex one. We precise that in all tests, the exterior boundary is represented by the black line,
the initial shape by green line, the exact shape to identify by blue line and the reconstructed shape by the dotted red line.

**Simple configuration:** We start by the circular shape. We present in the figure 1, the reconstruction result by our method and the evaluation of the cost functional according to the number of iterations and the evaluation of associated gradients. We notice that the proposed method gives appropriate results with a very good convergence.

**Medium configuration:** In the figures 2 and 3, we present the shape reconstruction for different configurations and more complex than a circle. The results obtained are significant and well identified.

**Complex configuration:** In the figure 4 and 5, we even observe results in the case of a more complex shapes and our method is more robust.
6.2.2. Results with noise. In this subsection, we present the results obtained from noisy data as follows:

\[ \tilde{f}(x) = f(x)(1 + \tau \xi) \]

where \( \xi \) is a uniformly distributed random variable in \([-1, 1]\) and \( \tau \) dictates the level of noise.
6.3. **Comparison between the classical method and proposed method.** In order to compare the CCBM Method with some classical method, we consider the first example presented in the figure (1) and we applied the Least Squares objective defined in (3). We trace the evolution of the cost function and the gradient for both method (Least Square method and CCBM Method) in figure (11). We observe
that our method (CCBM) gives better results than the least squares one. The cost function and the gradient converge very quickly in our method than in the least squares method.

7. Conclusion. The inverse obstacle problem is studied by the new complex coupled boundary method (CCBM) using both Dirichlet and Neumann boundary conditions. The inverse problem is reformulated as a shape optimization problem for the Least Squares fitting of the imaginary part of the complex PDE’s solution in the domain $\Omega \setminus \varnothing$ where the unknown is the a nonparametric shape. In order to compute the gradient of the cost functional, we use the shape derivative techniques. We show in a rigorous way the existence of the shape derivative of the state and we characterize the shape gradient of the cost function in order the make a numerical resolution based on the descent method. To study the inverse problem, the second derivative of the cost function at the critical shape is investigated. The latest study allows us to establish the compactness of the Riesz operator corresponding to the shape Hessian and the ill-posedness of the associated identification problem. Numerical results are displayed to show that the proposed method is feasible and effective.
A NEW CCBM FOR AN INVERSE OBSTACLE PROBLEM

Figure 11. The comparison between the evolution of the cost function and the gradient with respect the iteration number

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