(Non-)amenability of the Fourier algebra in the cb-multiplier norm

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Abstract

For a locally compact group $G$, let $A(G)$ denote its Fourier algebra, $M_{cb}(A(G))$ the completely bounded multipliers of $A(G)$, and $A_{M_{cb}}(G)$ the closure of $A(G)$ in $M_{cb}(A(G))$. We show that $A_{M_{cb}}(G)$ is not amenable if $G$ contains a copy of $\mathbb{F}_2$, the free group in two generators, as a closed subgroup.

Keywords: amenable Banach algebra; Fourier algebra; cb-multiplier; amenability constant; Connes-amenability constant; free group.

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Introduction

The Fourier algebra $A(G)$ of a locally compact group $G$ was introduced by P. Eymard in [Eym]; if $G$ is abelian with dual group $\hat{G}$, these algebras are isometrically isomorphic to $L^1(\hat{G})$, the group algebra of $\hat{G}$. In [Lep], H. Leptin characterized the amenable, locally compact groups in terms of their Fourier algebras: a locally compact group $G$ is amenable if and only if $A(G)$ has a bounded approximate identity (in fact, of bound one).

In [Joh 1], B. E. Johnson initiated the theory of amenable Banach algebras and proved that a locally compact group $G$ is amenable if and only if its group algebra $L^1(G)$ is amenable ([Joh 1, Theorem 2.5]). As all amenable Banach algebras have bounded approximate identities, this prompted the conjecture that $A(G)$ is amenable if and only if $G$ is amenable. Alas, in [Joh 3], Johnson showed that there are compact groups $G$—among them $SO(3)$—for which $A(G)$ fails to be amenable. In [F–R 1] (see also [Run 4]), B. E. Forrest and the author finally showed that $A(G)$ is amenable if and only if $G$ has an abelian subgroup of finite index.

It turns out that, if one wants to obtain a satisfactory amenability theory for Fourier algebras, one has to take their canonical operator space structure into account. Such a

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theory was initiated in [Rua] by Z.-J. Ruan under the name operator amenability. He was able to show: a locally compact group $G$ is amenable if and only if $A(G)$ is operator amenable ([Rua Theorem 2.5]).

The operator space structure of the Fourier algebra $A(G)$ of a locally compact group $G$, can be used to define another norm on it. A multiplier on $A(G)$ is a function $\phi: G \to \mathbb{C}$ such that $\phi A(G) \subset A(G)$. We consider the cb-multipliers on $A(G)$, i.e., those multipliers $\phi$ such that the map $A(G) \ni f \mapsto \phi f$ is completely bounded. They form a closed subalgebra of the completely bounded operator on $A(G)$, which we denote by $M_{cb}(A(G))$. As $A(G)$ embeds contractively into $M_{cb}(A(G))$, it inherits a new norm from $Mcb(A(G))$, the cb-multiplier norm. For amenable $G$, this norm equals the given norm (as a consequence of [Lep]). For non-amenable groups, however, the two norms are inequivalent. We denote the closure of $A(G)$ in $M_{cb}(A(G))$ by $A_{Mcb}(G)$. Unlike $A(G)$, the algebra, $A_{Mcb}(G)$ may have a bounded approximate identity for non-amenable $G$: this is the case, for instance, if $G = \mathbb{F}_2$, the free group in two generators ([HC–H]). Even more surprisingly, $A_{Mcb}(\mathbb{F}_2)$ is operator amenable as shown by B. E. Forrest, N. Spronk, and the author ([F–R–S]).

The result from [F–R–S] naturally begets the question if $A_{Mcb}(\mathbb{F}_2)$ is amenable in the sense of Johnson’s original definition. The methods used in [F–R–S] are not suited to produce an affirmative answer to this question. On the negative side, Forrest and the author were able to show that a locally compact group $G$ has to be abelian if $AM(A_{Mcb}(G)) < \frac{2}{\sqrt{3}}$, where $AM(A_{Mcb}(G))$ denotes the amenability constant of $A_{Mcb}(G)$ as introduced in [Joh 3]. (This estimate was recently improved by N. Juselius in [Jus].)

In the present paper, we prove that $A_{Mcb}(G)$ cannot be amenable if $G$ contains $\mathbb{F}_2$ as a closed subgroup. Of course, the interesting case is $G = \mathbb{F}_2$. Again, the amenability constant of $A_{Mcb}(G)$ plays a crucial rôle. More precisely, we show that, if $A_{Mcb}(\mathbb{F}_2) < \infty$, then $A_{Mcb}(F) \leq A_{Mcb}(\mathbb{F}_2)$ for every finite group $F$, which contradicts earlier work by Johnson ([Joh 3]).

Even though the main result of the paper is entirely Banach algebraic, its object is defined in terms of operator space theory. It should therefore come as no surprise that operator spaces play a pivotal rôle in its proof. Our reference for background on operator spaces is [E–R].

1 Amenability and Connes-amenability constants

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then its dual $E^*$ becomes a Banach $\mathcal{A}$-bimodule as well via

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \quad \text{and} \quad \langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \quad (a \in \mathcal{A}, x \in E, \phi \in E^*)$$

We call Banach modules of this form dual.
Definition 1.1. Let \( \mathfrak{A} \) be a Banach algebra. Then \( \mathfrak{A} \) is called amenable if and only if, for every dual Banach \( \mathfrak{A} \)-bimodule \( E \), every bounded derivation \( D: \mathfrak{A} \to E \) is inner.

The reason for the choice of terminology is \cite{joh1, theoreme 2.5}: a locally compact group \( G \) is amenable if and only if its group algebra \( L^1(G) \) is amenable.

If \( \mathfrak{A} \) is a Banach algebra, then the projective tensor product \( \mathfrak{A} \otimes \gamma \mathfrak{A} \) becomes a Banach \( \mathfrak{A} \)-bimodule via

\[
a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya \quad (a, x, y \in \mathfrak{A})
\]

The multiplication map

\[
\Delta = \Delta_\mathfrak{A} : \mathfrak{A} \otimes \gamma \mathfrak{A} \to \mathfrak{A}, \quad a \otimes b \mapsto ab
\]

is then an \( \mathfrak{A} \)-bimodule homomorphism.

Definition 1.2. Let \( \mathfrak{A} \) be a Banach algebra. Then \( D \in (\mathfrak{A} \otimes \gamma \mathfrak{A})^{**} \) is called a virtual diagonal for \( \mathfrak{A} \) if

\[
a \cdot D = D \cdot a \quad \text{and} \quad a \cdot \Delta^{**} D = a \quad (a \in \mathfrak{A}).
\]

In \cite{joh2}, it was shown:

Theorem 1.3. The following are equivalent for a Banach algebra \( \mathfrak{A} \):

(i) \( \mathfrak{A} \) is amenable;

(ii) \( \mathfrak{A} \) has a virtual diagonal.

This characterization can be used to quantify the notion of amenability for Banach algebras, as was first done in \cite{joh3}:

Definition 1.4. Let \( \mathfrak{A} \) be a Banach algebra. Then

\[
AM(\mathfrak{A}) := \inf \{ \| D \| : D \text{ is a virtual diagonal for } \mathfrak{A} \}
\]

is the amenability constant of \( \mathfrak{A} \).

Remarks. 1. With the convention that \( \inf \emptyset = \infty \), we thus have that \( \mathfrak{A} \) is amenable if and only if \( AM(\mathfrak{A}) < \infty \).

2. It is clear the the infimum in Definition 1.4 is actually attained and thus a minimum.

Definition 1.5. A dual Banach algebra is a pair \( (\mathfrak{A}, \mathfrak{A}_*) \) of Banach spaces such that:
(a) \( \mathfrak{A} \) is a Banach algebra;
(b) \( \mathfrak{A} = (\mathfrak{A}_*)^* \);
(c) multiplication in \( \mathfrak{A} \) is separately \( \sigma(\mathfrak{A}, \mathfrak{A}_*) \) continuous.

Examples. 1. All von Neumann algebras are dual Banach algebras.

2. If \( G \) is a locally compact group, then its measure algebra \( M(G) \) is a dual Banach algebra with \( M(G)_* = C_0(G) \), the space of a continuous functions on \( G \) vanishing at infinity.

Remark. The somewhat pedantic formulation of Definition 1.5 is due to the fact that the predual space \( \mathfrak{A}_* \) need not be unique. It is unique if \( \mathfrak{A} \) is a von Neumann algebra, but, for instance, \( \ell^1(\mathbb{Z}) \) has a continuum of different preduals ([Daw et al.]). Often, there is a canonical dual, and we shall often speak simply of the weak* topology of a dual Banach algebra without explicitly mentioning the given predual.

For dual Banach algebra a weaker version of Definition 1.1, which takes the dual space structure into account, is more appropriate:

**Definition 1.6.** Let \( \mathfrak{A} \) be a dual Banach algebra, and let \( E \) be a dual Banach \( \mathfrak{A} \)-bimodule. We call \( E \) normal if, for each \( x \in E \), the maps

\[
\mathfrak{A} \to E, \quad a \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}
\]

are weak*-weak* continuous.

**Definition 1.7.** Let \( \mathfrak{A} \) be a dual Banach algebra. Then \( \mathfrak{A} \) is called Connes-amenable if, for every normal, dual Banach \( \mathfrak{A} \)-bimodule \( E \), every weak*-weak* continuous derivation \( D: \mathfrak{A} \to E \) is inner.

**Example.** Let \( G \) be a locally compact group. Then \( M(G) \) is Connes-amenable if and only if \( G \) is amenable ([Run 1]), but is amenable in the sense of Definition 1.1 if and only if \( G \) is discrete and amenable ([D–G–H]).

To obtain an analog of Theorem 1.3, we define:

**Definition 1.8.** Let \( \mathfrak{A} \) be a dual Banach algebra, and let \( E \) be a Banach \( \mathfrak{A} \)-bimodule. Then \( x \in E \) is called \( \sigma \)-weakly continuous if the maps

\[
\mathfrak{A} \to E, \quad a \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}
\]

are weak*-weakly continuous. We denote the collection of all weak*-weakly continuous elements of \( E \) by \( C^w_\sigma(E) \).
Remarks. 1. $C_w^r(E)$ is a closed submodule of $E$.

2. If $F$ is another Banach $\mathfrak{A}$-bimodule, and $\theta : E \to F$ is a bounded $\mathfrak{A}$-bimodule homomorphism, then $\theta(C_w^r(E)) \subset C_w^r(F)$.

3. $C_w^r(E)^*$ is a normal Banach $\mathfrak{A}$-bimodule.

Let $(\mathfrak{A}, \mathfrak{A}^*)$ be a dual Banach algebra. As $\mathfrak{A}_* \subset C_w^r(\mathfrak{A}^*)$, it follows that $\Delta^*\mathfrak{A}_* \subset C_w^r((\mathfrak{A} \otimes \gamma\mathfrak{A})^*)$. Consequently, $\Delta^{**}$ drops to an $\mathfrak{A}$-bimodule homomorphism $\Delta_{\sigma,w} : C_w^r((\mathfrak{A} \otimes \gamma\mathfrak{A})^*)^r \to \mathfrak{A}$.

We define:

**Definition 1.9.** Let $\mathfrak{A}$ be a dual Banach algebra. Then $D \in C_w^r((\mathfrak{A} \otimes \gamma\mathfrak{A})^*)^r$ is called a $C_w^r$-virtual diagonal for $\mathfrak{A}$ if

$$a \cdot D = D \cdot a \quad \text{and} \quad a\Delta_{\sigma,w}D = a \quad (a \in \mathfrak{A}).$$

The following is [Run 2, Theorem 4.8]:

**Theorem 1.10.** The following are equivalent for a dual Banach algebra $\mathfrak{A}$:

(i) $\mathfrak{A}$ is Connes-amenable;

(ii) there is a $C_w^r$-virtual diagonal for $\mathfrak{A}$.

In analogy with Definition 1.4, we define:

**Definition 1.11.** Let $\mathfrak{A}$ be a Banach algebra. Then

$$AM_{\sigma,w}(\mathfrak{A}) := \inf \{ \| D \| : D \text{ is a } C_w^r\text{-virtual diagonal for } \mathfrak{A} \}$$

is the Connes-amenability constant of $\mathfrak{A}$.

**Remark.** As in Definition 1.4, the infimum is actually attained.

We record:

**Theorem 1.12.** Let $\mathfrak{A}$ be a Banach algebra, let $\mathfrak{B}$ be a dual Banach algebra, and let $\theta : \mathfrak{A} \to \mathfrak{B}$ be a bounded algebra homomorphism with weak$^*$ dense range. Then the following are true:

(i) if $\mathfrak{A}$ is amenable, then $\mathfrak{B}$ is Connes-amenable such that

$$AM_{\sigma,w}(\mathfrak{B}) \leq \| \theta \|^2 AM(\mathfrak{A});$$

(ii) if $\mathfrak{A}$ is a Connes-amenable, dual Banach algebra, and $\theta$ is weak$^*$-weak$^*$ continuous, then $\mathfrak{B}$ is Connes-amenable such that

$$AM_{\sigma,w}(\mathfrak{B}) \leq \| \theta \|^2 AM_{\sigma,w}(\mathfrak{A}).$$
Proof. Suppose that \( A \) is amenable. Let \( D \in (\mathfrak{A} \otimes \mathfrak{A})^{**} \) be a virtual diagonal for \( \mathfrak{A} \). Then it is routinely checked that \( (\theta \otimes \theta)^{(1)}(D) \in C_\sigma^w((\mathfrak{B} \otimes \gamma \mathfrak{B})^*)^* \) is a virtual diagonal for \( \mathfrak{B} \). Clearly,

\[
\|((\theta \otimes \theta)^{(1)}(D))\| \leq \|\theta\|^2 \|D\| \leq \|\theta\|^2 \|\mathfrak{A}\|.
\]

holds. This proves (i).

For the proof of (ii), we first claim that \( (\theta \otimes \theta)^{(1)}(C_w^\sigma((B \otimes \gamma B)^*)) \subset C_w^\sigma((\mathfrak{A} \otimes \gamma \mathfrak{A})^*) \). This is not a simple consequence of the remarks following Definition 1.8 because we are dealing with modules over different algebras. Let \( \Phi \in C_w^\sigma((\mathfrak{B} \otimes \gamma \mathfrak{B})^*) \), and define

\[
L_\Phi : B \rightarrow (\mathfrak{B} \otimes \gamma \mathfrak{B})^*, \quad b \mapsto \Phi \cdot b
\]

and

\[
R_\Phi : B \rightarrow (\mathfrak{B} \otimes \gamma \mathfrak{B})^*, \quad b \mapsto b \cdot \Phi
\]

These are weak*-weakly continuous, which means that their adjoints attain their values in \( B_* \). As \( \theta \) is weak*-weak* continuous, there is a unique \( \theta_* : B_* \rightarrow \mathfrak{A}_* \) such that \( (\theta_*)^* = \theta \). It is routinely checked that

\[
\mathfrak{A} \rightarrow (\mathfrak{A} \otimes \gamma \mathfrak{A})^*, \quad a \mapsto \begin{cases} a \cdot (\theta \otimes \theta)^*(\Phi) \\ (\theta \otimes \theta)^*(\Phi) \cdot a \end{cases}
\]

are the adjoints of \( \theta_* \circ L_\Phi^* \) and \( \theta_* \circ R_\Phi^* \), respectively. It follows that \( (\theta \otimes \theta)^* (\Phi) \in C_w^\sigma((\mathfrak{A} \otimes \gamma \mathfrak{A})^*) \).

To wrap up the proof, let \( D \in C_\sigma^w((\mathfrak{A} \otimes \gamma \mathfrak{A})^*) \) be a virtual diagonal for \( \mathfrak{A} \). Then it is routinely checked that \( \left((\theta \otimes \theta)^*|_{C_\sigma^w((\mathfrak{A} \otimes \gamma \mathfrak{A})^*)}\right)^*(D) \) is a virtual diagonal for \( \mathfrak{B} \). The norm estimates follow as in the proof of (i).

\[\square\]

2 The Fourier algebra in the cb-multiplier norm

Let \( G \) be a locally compact group. The Fourier algebra \( A(G) \) of \( G \) was defined by P. Eymard in [Eym]. Recently, a comprehensive monograph by E. Kaniuth and A. T.-M. Lau has appeared ([K–L]).

A multiplier of \( A(G) \) is a function \( \phi : G \rightarrow \mathbb{C} \) such that \( \phi A(G) \subset A(G) \). A straightforward application of the closed graph theorem yields that, if \( \phi : G \rightarrow \mathbb{C} \) is a multiplier of \( A(G) \), then the operator

\[
M_\phi : A(G) \rightarrow A(G), \quad f \mapsto \phi f
\]

is bounded. As \( A(G) \) is an operator space in a natural manner ([E–R Section 16.2]), it makes sense to define:
Definition 2.1. Let $G$ be a locally compact group. Then a multiplier $\phi: G \to \mathbb{C}$ is called completely bounded if $M_\phi: A(G) \to A(G)$ is completely bounded. We denote the collection of completely bounded multipliers—short: cb-multipliers—of $A(G)$ by $M_{cb}(A(G))$. We set

$$\|\phi\|_{M_{cb}} := \|M_\phi\|_{cb} \quad (\phi \in M_{cb}(A(G))).$$

Here, $\|\cdot\|_{cb}$ stands for the completely bounded norm ([E–R]).

Remarks. 1. It is clear that $(M_{cb}(A(G)), \|\cdot\|_{M_{cb}})$ is a (completely contractive; see [E–R, Section 16.1]) Banach algebra.

2. In [dC–H], it was observed that $M_{cb}(A(G))$ is a dual Banach space in a canonical way for every locally compact group $G$. The resulting weak* topology coincides on norm bounded subsets of $M_{cb}(A(G))$ with the relative topology induced by $\sigma(L^\infty(G), L^1(G))$; by the Kreĭn–Šmulian Theorem ([D–S, Theorem V.6.4]), this property uniquely identifies this particular predual. It is straightforward to see that this duality turns $M_{cb}(A(G))$ into a dual Banach algebra.

3. In [Spr], N. Spronk provided an alternative characterization of this canonical predual. We sketch it here because we will make use of it below. Following [E–R], we denote the uncompleted Haagerup tensor product of operator spaces by $\otimes_h$ whereas we use $\otimes$ for its completion. The multiplication operator $\Delta : L^1(G) \otimes_h L^1(G) \to L^1(G)$ is surjective, but unbounded for infinite $G$. Let $\ker \Delta$ be the closure of $\ker \Delta$ in $L^1(G) \otimes_h L^1(G)$. Then

$$Q(G) := L^1(G) \otimes_h L^1(G)/\overline{\ker \Delta}$$

is a predual of $M_{cb}(A(G))$ (see [Spr] for the details). As the resulting weak* topology on $M_{cb}(A(G))$ coincides with $\sigma(L^\infty(G), L^1(G))|_{M_{cb}(A(G))}$ on norm bounded subsets of $M_{cb}(A(G))$, it follows that $Q(G)$ is the same space as the predual described in [dC–H].

Let $G$ be a locally compact group. As $A(G)$ is a completely contractive Banach algebra, it is clear that $A(G)$ embeds (completely) contractively into $M_{cb}(A(G))$. It follows from Leptin’s Theorem ([Lep]) that $\|\cdot\|_{M_{cb}}$ and the given norm on $A(G)$ are identical for amenable $G$. Surprisingly, there is a converse to this assertion: if $\|\cdot\|_{M_{cb}}$ and the given norm are equivalent on $A(G)$, then $G$ is amenable: this was proven by Z.-J. Ruan building on ideas previously developed by V. Losert ([Los]), but apparently never published (see the remarks at the end of [Spr]).

We thus define:

Definition 2.2. Let $G$ be a locally compact group. Then $A_{M_{cb}}(G)$ is defined to be the norm closure of $A(G)$ in $M_{cb}(A(G))$. 


We record:

**Lemma 2.3.** Let $G$ be a locally compact group, and let $H$ be a closed subgroup of $G$. Then $f|_H \in \mathcal{A}_{M_{cb}}(H)$ for each $f \in \mathcal{A}_{M_{cb}}(H)$, and the restriction map

$$\mathcal{A}_{M_{cb}}(G) \to \mathcal{A}_{M_{cb}}(H), \quad f \mapsto f|_H$$

is a quotient map.

*Proof.* This is [For, Theorem 1]. \hfill \Box

**Lemma 2.4.** Let $G$ be a locally compact group, and let $H$ be an open subgroup of $G$. Then

$$\{ f \in \mathcal{A}_{M_{cb}}(G) : \text{supp } f \subset H \} \to \mathcal{A}_{M_{cb}}(H), \quad f \mapsto f|_H$$

is an isometry.

*Proof.* This is [F–R 2, Proposition 2.1]. \hfill \Box

The following definition is from [C–H] (it has nothing to do with the notion of weak amenability in Banach algebra theory introduced in [B–C–D]):

**Definition 2.5.** Let $G$ be a locally compact group. Then $G$ is called weakly amenable if $\mathcal{A}_{M_{cb}}(G)$ has a bounded approximate identity.

**Example.** Leptin’s Theorem immediately yields that amenable locally compact groups are weakly amenable, but $\mathbb{F}_2$ is weakly amenable as well ([dC–H]).

We conclude this section with:

**Proposition 2.6.** Let $G$ be a locally compact group. Then:

(i) if $\mathcal{A}_{M_{cb}}(G)$ is amenable, then $\mathcal{M}_{cb}(A(G))$ is Connes-amenable, and

$$\text{AM}_{\sigma,w}^{\mathcal{M}_{cb}}(\mathcal{M}_{cb}(A(G))) \leq \text{AM}(\mathcal{A}_{M_{cb}}(G))$$

holds;

(ii) if $G$ is discrete and weakly amenable and $\mathcal{M}_{cb}(A(G))$ is Connes-amenable, then $\mathcal{A}_{M_{cb}}(G)$ is amenable.

*Proof.* For (i), note that a bounded approximate identity of $\mathcal{A}_{M_{cb}}(G)$ converges to the constant function $1 \in \mathcal{M}_{cb}(A(G))$ in the canonical weak* topology of $\mathcal{M}_{cb}(A(G))$. As $\mathcal{A}_{M_{cb}}(G)$ is an ideal of $\mathcal{M}_{cb}(A(G))$, this means that $\mathcal{A}_{M_{cb}}(A(G))$ is weak* dense in $\mathcal{M}_{cb}(A(G))$. The claim then follows from Theorem 1.12(i).

(ii) is implicitly proven in [F–R–S]: it is fairly straightforward to modify the proof of [F–R–S, Theorem 2.7] accordingly. \hfill \Box

**Remark.** Note that Proposition 2.6(ii) makes no assertion about how $\text{AM}(\mathcal{A}_{M_{cb}}(G))$ and $\text{AM}_{\sigma,w}^{\mathcal{M}_{cb}}(\mathcal{M}_{cb}(A(G)))$ are related.
3 Non-amenability of $A_{Mcb}(\mathbb{F}_2)$

Given a set $S$, we denote the free group generated by $S$ by $F_S$; we write $F_\infty$ instead of $F_N$ and $F_n$ instead of $F_{\{1,\ldots,n\}}$ for $n \in \mathbb{N}$.

It turns out that with regards to the question for which locally compact $G$ group, the Banach algebra $A_{Mcb}(G)$ is amenable, it makes no difference which free group in two or more generators we study:

**Proposition 3.1.** The following are equivalent:

(i) $A_{Mcb}(\mathbb{F}_2) < \infty$;

(ii) $\text{AM}(A_{Mcb}(F_S)) = \text{AM}(A_{Mcb}(\mathbb{F}_2)) < \infty$ for every set $S$ with $|S| \geq 2$.

**Proof.** Of course, only (i) $\implies$ (ii) needs proof.

Suppose that $\text{AM}(A_{Mcb}(\mathbb{F}_2)) < \infty$. Let $\{a, b\}$ a set of generators of $\mathbb{F}_2$. Then the subgroup of $\mathbb{F}_2$ generated by the set $S = \{a^{-n}ba^n : n \in \mathbb{N}\}$ is well known to be $F_S \cong F_\infty$ ([F-T–P, p. 2]). From Lemma 2.3 and (an inspection of the proof of) [Run 3, Proposition 2.3.1], we conclude that

$$\text{AM}(A_{Mcb}(F_S)) \leq \text{AM}(A_{Mcb}(\mathbb{F}_2)).$$

As $\mathbb{F}_2 \subset F_n \subset F_\infty$ canonically for each $n \in \mathbb{N}$, the same argument yields that

$$\text{AM}(A_{Mcb}(\mathbb{F}_2)) \leq \text{AM}(A_{Mcb}(\mathbb{F}_n)) \leq \text{AM}(A_{Mcb}(F_\infty)) \quad (n \in \mathbb{N}).$$

It follows that $\text{AM}(A_{Mcb}(F_S)) = \text{AM}(A_{Mcb}(\mathbb{F}_2))$ for each finite $S$ with $|S| \geq 2$.

Let $S$ with $|S| \geq 2$ be arbitrary, and let $\mathcal{F}(S)$ denote the collection of all finite subsets of $S$. For each $F \in \mathcal{F}(S)$, the subgroup of $\mathbb{F}_S$ generated by $F$ is $\mathbb{F}_F$. We define

$$\mathfrak{A}_F := \{f \in A_{Mcb}(F_S) : \text{supp} f \subset \mathbb{F}_F\}.$$ 

By Lemma 2.3 we have $\mathfrak{A}_F \cong A_{Mcb}(\mathbb{F}_F)$ isometrically, so that

$$\text{AM}(\mathfrak{A}_F) = \text{AM}(A_{Mcb}(\mathbb{F}_F)) = \text{AM}(A_{Mcb}(\mathbb{F}_2)).$$

Now, $\{\mathfrak{A}_F : F \in \mathcal{F}(S)\}$ is a directed family of closed subalgebras of $A_{Mcb}(\mathbb{F}_S)$ such that

$$A_{Mcb}(\mathbb{F}_S) = \bigcup \{\mathfrak{A}_F : F \in \mathcal{F}(S)\}.$$ 

It follows from [Run 3, Proposition 2.3.17] that $\text{AM}(A_{Mcb}(F_S)) \leq \text{AM}(A_{Mcb}(\mathbb{F}_2)).$ 

Let $G$ be a locally compact, and let $N$ be a closed normal subgroup of $G$ such that $G/N$ is compact. Then $A(G/N) = M_{cb}(A(G/N))$ embeds canonically and (completely) isometrically is $M_{cb}(A(G))$ by the functorial properties of algebras of completely bounded multipliers ([Spr Corollary 6.3(iv)]).

We take a closer look at the case where $G/N$ is finite.
Lemma 3.2. Let $G$ be a discrete group, let $N$ be a normal subgroup of $G$ with $n := |G : N| < \infty$, let $x_1, \ldots, x_n \in G$ be coset representatives of $G / N$, and let

$$r : M_{cb}(A(G)) \to \mathbb{C}^\{x_1, \ldots, x_n\}, \quad f \mapsto f|_{\{x_1, \ldots, x_n\}}$$

with $\mathbb{C}^\{x_1, \ldots, x_n\}$ being equipped with the quotient norm induced by $r$. Then

$$r \circ j : A_{M_{cb}}(G / N) \to \mathbb{C}^\{x_1, \ldots, x_n\}$$

is an isometric isomorphism of Banach algebras where $j : A_{M_{cb}}(G / N) \to M_{cb}(A(G))$ is the canonical isometric embedding.

Proof. Let $\iota : \ell^1(\{x_1, \ldots, x_n\}) \to \ell^1(G)$ be the canonical embedding, and let $q : \ell^1(G) \to \ell^1(G / N)$ be the canonical quotient map, so that $q \circ \iota : \ell^1(\{1, \ldots, x_n\}) \to \ell^1(G / N)$ is an isometric isomorphism. When all the spaces involved are equipped with their canonical, i.e., maximal, operator space structure, $\iota$, $q$, and $q \circ \iota$ become complete isometries and quotient maps, respectively. Due to the peculiar functorial properties of the Haagerup tensor product ([E.R. Proposition 9.2.5]), we obtain a sequence

$$\{0\} \to \ell^1(\{x_1, \ldots, x_n\}) \otimes_h \ell^1(\{x_1, \ldots, x_n\}) \xrightarrow{\iota \otimes q} \ell^1(G) \otimes_h \ell^1(G) \xrightarrow{q \otimes q} \ell^1(G / N) \otimes \ell^1(G / N) \to \{0\}$$

of operator spaces with $\iota \otimes q$ and $(q \circ \iota) \otimes (q \circ \iota)$ being completely isometric and $q \otimes q$ a complete quotient map.

Let $\Delta_{\ell^1(G)} : \ell^1(G) \otimes \ell^1(G) \to \ell^1(G)$ and $\Delta_{\ell^1(G / N)} : \ell^1(G / N) \otimes \ell^1(G / N) \to \ell^1(G / N)$ denote the respective multiplication operators (note that we are looking at uncompleted tensor products). As

$$\ker \Delta_{\ell^1(G)} = \text{span}\{f \otimes g - \delta_e \otimes f \ast g : f, g \in \ell^1(G)\}$$

and

$$\ker \Delta_{\ell^1(G / N)} = \text{span}\{f \otimes g - \delta_e \otimes f \ast g : f, g \in \ell^1(G / N)\},$$

we see immediately that $(q \otimes q)(\ker \Delta_{\ell^1(G)}) = \ker \Delta_{\ell^1(G / N)}$. As $\dim \ell^1(G / N) < \infty$, it is clear that $(q \otimes q)(\overline{\ker \Delta_{\ell^1(G)}}) = \ker \Delta_{\ell^1(G / N)}$ as well where the closure of $\ker \Delta_{\ell^1(G)}$ is taken in $\ell^1(G) \otimes_h \ell^1(G)$. With $X := (\ell^1(\{1, \ldots, x_n\}) \otimes_h \ell^1(\{1, \ldots, x_n\})) / \ker \Delta_{\ell^1(G)}$, we thus obtain a sequence of quotient operator spaces

$$\{0\} \to Q(\{x_1, \ldots, x_n\}) := (\ell^1(\{1, \ldots, x_n\}) \otimes_h \ell^1(\{1, \ldots, x_n\})) / X \xrightarrow{\tilde{\iota}} Q(G) \xrightarrow{\tilde{q}} Q(G / N) \to \{0\}$$

with $\tilde{\iota}$ and $\tilde{q}$ denoting the maps induced by $\iota \otimes \iota$ and $q \otimes q$, respectively. Clearly, $\tilde{q}$ is a complete quotient map, $\tilde{\iota}$ is an injective complete contraction, and $\tilde{q} \circ \tilde{\iota}$ is a bijective
complete contraction. A simple dimension counting argument reveals that \((q \circ \iota) \otimes (q \circ \iota)\) maps \(X\) onto \(\ker \Delta_{\ell(G/N)}\). As both \(\mathcal{Q}(\{1, \ldots, x_n\})\) and \(\mathcal{Q}(G/N)\) are equipped with their respective quotient operator space structures, and since \((q \circ \iota) \otimes (q \circ \iota)\) is a complete isometry, it follows that \(\tilde{q} \circ \tilde{\iota}\) is also a complete isometry. Consequently, \(\tilde{\iota}\) must be a complete isometry, too.

As \(r = (\tilde{\iota})^*\) and \(j = (\tilde{q})^*\), the claim follows. \(\square\)

**Corollary 3.3.** Let \(G\) be discrete group, let \(N\) be a normal subgroup of \(G\) with finite index. Then

\[
AM(A(G/N)) \leq AM_{\sigma,w}(M_{cb}(A(G)))
\]

holds.

**Proof.** Suppose without loss of generality that \(AM_{\sigma,w}(M_{cb}(A(G))) < \infty\).

The map \(r\) in Lemma 3.2 is weak*-weak* continuous. Thus Theorem 1.12(ii) yields

\[
AM\left(C^{\{x_1, \ldots, x_n\}}\right) = AM_{\sigma,w}\left(C^{\{x_1, \ldots, x_n\}}\right) \leq AM_{\sigma,w}(M_{cb}(A(G))).
\]

As \(C^{\{x_1, \ldots, x_n\}} \cong A(G/N)\) isometrically by Lemma 3.2, this yields the claim. \(\square\)

We are one lemma away from our main result:

**Lemma 3.4.** Let \(C > 0\). Then there is a finite group \(F\) such that \(AM(F) > C\).

**Proof.** By [Joh 3, Corollary 4.2], we have

\[
AM(A(G \times H)) = AM(G) AM(H)
\]

for all finite groups \(G\) and \(H\), and by [Joh 3, Proposition 4.3], \(AM(G) \geq \frac{3}{2}\) holds for all finite, non-abelian groups \(G\). For finite, non-abelian \(G\) and sufficiently large \(n \in \mathbb{N}\), we thus have \(AM(A(G^n)) > C\). \(\square\)

**Theorem 3.5.** Let \(G\) be a locally compact group containing a closed subgroup isomorphic to \(F_2\). Then \(A_{M_{cb}}(G)\) is not amenable.

**Proof.** By Lemma 2.3, we can suppose that \(G = F_2\).

Assume that \(A_{M_{cb}}(F_2) < \infty\), and use Lemma 3.4 to obtain a finite group \(F\) such that

\[
AM(A(F)) > AM(A_{M_{cb}}(F_2)).
\]

By the universal property of free groups, there is a normal subgroup \(N\) of \(F\) with \(F/N \cong F\). By Proposition 2.6(i) and Corollary 3.3, this means that

\[
AM(A(F)) = AM(A(F/N)) \leq AM_{\sigma,w}(M_{cb}(A(F))) \leq AM(A_{M_{cb}}(F)) = AM(A_{M_{cb}}(F_2)),
\]

where the last equality holds by Proposition 3.1. This contradicts the choice of \(F\). \(\square\)
Remarks. 1. Completely bounded multipliers rely on operator space theory for their very definition, and the proof of Lemma 3.2 relies on operator space theory, in particular on properties of the Haagerup tensor product. In fact, an inspection of the results of this paper show that—with the obvious modifications—they all extend to the operator space category—with one exception: Lemma 3.4. If one defines the operator amenability constant $AM_{cb}(A)$ of a completely contractive Banach algebra $A$ in the obvious way, then an inspection of the proof of [Rua, Theorem 2.5] shows that $AM_{cb}(A(G)) = 1$ for all amenable, locally compact groups $G$.

2. Instead of $M_{cb}(A(G))$, one can consider the multiplier algebra $M(A(G))$ consisting of all multipliers of $A(G)$, equipped with the multiplier norm $\| \cdot \|_M$ given by $\| \phi \|_M := \| M\phi \|$ for $\phi \in M(A(G))$. The closure of $A(G)$ in $M(A(G))$ is denoted by $A_M(G)$. The proof of Theorem 3.5 leaves the question of whether $A_M(F_2)$ is amenable wide open—mainly because for the study of $M(A(F_2))$ and $A_M(F_2)$, operator space methods are not available.

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