The prediction and observation of topological crystalline insulators (TCIs) in the SnTe material class has expanded the scope of topological matter and gained wide interest [1][5]. These TCIs possess topological surface states that are protected by mirror symmetry of the rocksalt crystal and became gapped under symmetry-breaking structural distortions [6][8]. These surface states are predicted to exhibit a plethora of novel phenomena ranging from large quantum anomalous Hall conductance [11][9] to strain-induced pseudo-Landau levels and superconductivity [10], which are currently under intensive study [11][12].

In this work, we study the effect of electron interactions in TCIs with mirror symmetry. Our work is motivated by both theoretical and experimental considerations. According to band theory of non-interacting electrons, these TCIs are classified by an integer topological invariant, the mirror Chern number [13]. However, recent theoretical works have shown that the classifications of interacting systems are different from the noninteracting limit in many symmetry classes and in various dimensions [14][15][40]. This raises the question: what is the classification of interacting TCIs with mirror symmetry? On the experimental side, there is a growing evidence that interactions have important effects in existing TCIs, as shown by the observation of spontaneous gap generation [6], surface structural transition [7] and anomalous band inversion [20]. Moreover, new TCI materials have been predicted in transition metal oxides [21][22] and heavy fermion compounds [23][24], where strong interactions are expected. In view of these developments, theory of interacting TCIs is strongly called for.

Our main results are summarized as follows. First, we find that interactions reduce the classification of TCIs from $Z$ in the noninteracting limit to $Z_4$ in two dimensions, and $Z_8$ in three dimensions. Second, we find a novel type of edge states with an interaction-driven spin gap in a class of two-dimensional (2D) TCIs predicted in SnTe films [25]. Lastly, we find a general relation between 3D TCIs protected by crystal symmetries and 2D topological phases protected by internal symmetries, which enables us to classify 3D interacting topological phases protected by spatial symmetry, for the first time.

**Interacting TCIs in two dimensions:** 2D TCIs considered in this work are symmetric under the reflection about the surface normal: $z \rightarrow -z$. The mirror operator $M$ is a product of the two-fold rotation $C_2$ and the inversion $P$. Importantly, due to the $C_2$ part, in spin-orbit-coupled systems, $M$ acts on electron’s spin in addition to its $z$-coordinate. For a single spin-1/2 particle, $M^2 = C_2^2 = 1$ (note that $P^2 = 1$) implies mirror eigenvalue can be either $+i$ or $-i$. In a noninteracting 2D insulator, single-particle Bloch states with $+i$ and $-i$ mirror eigenvalues cannot mix, thus forming two orthogonal subspaces. Therefore, one can define Chern number for each subspace, denoted by $n_{+i}$ and $n_{-i}$. The sum $N = n_{+i} + n_{-i}$ yields the well-known Chern number, a chiral topological index that does not require any symmetry. Interestingly, for non-chiral systems with $N = 0$, which are the subject of this work, the Chern number of a given mirror subspace, $n_M = n_{+i} = -n_{-i}$, can be a nonzero integer termed the mirror Chern number [13], which classifies 2D noninteracting TCIs with mirror symmetry. For example, (001) thin films of SnTe are predicted to have $|n_M| = 2$ [25].

From a theory point of view, for 2D systems, the above mirror symmetry does not affect the $x, y$ coordinate and in effect acts as a $Z_2$ internal symmetry. Therefore, 2D TCIs with mirror symmetry is synonymous to $U(1) \times Z_2$ symmetry-protected topological (SPT) phase, where the additional $U(1)$ symmetry comes from charge conservation. A generic, defining property of 2D SPT phases is the presence of one-dimensional (1D) edge states that are either gapless and preserve the symmetry, or gapped and spontaneously break the symmetry [26][27]. Therefore, classifying 2D interacting TCIs amounts to finding all possible types of edge states that satisfy the above criterion, with the symmetry being $U(1) \times Z_2$.

For noninteracting TCIs, a nonzero mirror Chern number $n_M$ implies the presence of $n = |n_M|$ pairs of counter-
propagating edge modes with opposite mirror eigenvalues. The low-energy Hamiltonian for the edge is given by

$$H_0 = \sum_a v_F \int dx (-i\psi^\dagger_{a,R} \partial_x \psi_{a,R} + i\psi^\dagger_{a,L} \partial_x \psi_{a,L}).$$  (1)

Here the fermion fields $\psi_{a,R/L}$ denote respectively the a-th right and left movers with $a = 1, ..., n$, which transform differently under mirror:

$$M\psi^\dagger_{a,R} M^{-1} = i\eta \psi^\dagger_{a,R}, \quad M\psi^\dagger_{a,L} M^{-1} = -i\eta \psi^\dagger_{a,L}.  \quad (2)$$

where $\eta = \text{sgn}(n_M)$. The difference in mirror eigenvalues forbids single-particle backscattering between left- and right-movers, and hence protects the gapless edge states of a noninteracting TCI, for any integer $n_M \neq 0$.

We now study the above edge states under interactions, and use the result to classify 2D interacting TCIs. Following the method of Ref. [30, 31], it is convenient to use the result to classify 2D interacting TCIs.

For an arbitrary phase shift. By definition, interactions in 2D edge states are gapped by sufficiently strong interactions that preserve the $U(1) \times Z_2$ symmetry. The condition for charge conservation requires

$$q_L = L^T t = 0.  \quad (7)$$

The transformation law of the field $\Phi_L$ under mirror symmetry follows from that of the fermion field $[2]$:

$$M\Phi_L M^{-1} = \Phi_L + \eta \frac{\pi}{2} L^T m,  \quad (8)$$

where $m$ is a 2n-component vector defined as $m = (1_n, -1_n)^T$. Therefore the condition for mirror symmetry is

$$L^T m \equiv 0 \mod 4.  \quad (9)$$

To diagnose SPT phases, we consider the scenario that edge states are gapped by sufficiently strong interactions that preserve the $U(1) \times Z_2$ symmetry. The 2n edge modes can be completely gapped by adding to the Chern-Simons theory $n$ cosine terms $(\cos(\Phi_L(x)))$, which pin $n$ different fields $\Phi_L$, specified by a set of linearly independent, integer-valued vectors $L_a, a = 1, ..., n$. To ensure that these fields can simultaneously have classical values, the commutator $[0]_b$ between any two of them must vanish, which requires

$$L^T a K L_b = 0,  \quad (10)$$

for any indices $a, b = 1, ..., n$. A set of such vectors $\{L_a\}$ will be referred to as a set of gapping vectors.

A SPT phase and a trivial phase (such as an atomic insulator) are distinguished by the symmetry property of the gapped edge states. While a trivial phase permits a gapped and symmetry-preserving edge, edge states of a SPT phase, if gapped, must spontaneously break the protecting symmetry. As shown by Levin and Stern [32, 33], spontaneous symmetry breaking may (but not necessarily) occur when a linear combination of gapping vectors $\sum c_i L_i$ for the coefficients $\{c_i\}$ with no common divisors is nonprimitive, i.e.,

$$\sum_i c_i L_i = cL  \quad (11)$$

and the integer $c$ is larger than 1. In this case, the set of pinned fields $\{\Phi_L\}$, which themselves are symmetry-preserving, also freezes the field $\Phi_L$. The latter may or may not break the original symmetry of the system, which needs to be checked case by case. Conversely, if $\sum_i c_i L_i$ is primitive for any coefficients with no common divisors, spontaneous symmetry breaking is guaranteed to be absent.

To diagnose 2D interacting TCIs, our strategy is to first find appropriate interactions that satisfy (a) the condition for gapping the original noninteracting edge states; (b) the $U(1)$ charge conservation symmetry; (c) the $Z_2$
mirror symmetry, and then check whether (d) the resulting gapped edge is symmetry-preserving.

We now show explicitly that edge states with \( n = 4k \) pairs of counter-propagating modes can be gapped and satisfy all the conditions (a)-(d). For example, this can be achieved by the following interaction

\[
V = \sum_a v_a \cos(\Phi_{L_a}(x))
\]  

with a set of gapping vectors \( L_a \) defined by:

\[
L_1 = (1,-1;0,0;1,-1,0,0),
L_2 = (0,1,0,0;1,0,-1,0,0),
\]

and

\[
L_n = (0,1,0,0;1,-1,0,0),
\]

One can check straightforwardly that these gapping vectors satisfy the conditions (a), (b) and (d). As for the condition (c), we find \( L_i^T m = 0 \) for \( i = 1, \ldots, n-2 \) and \( L_{n-1}^T m = L_n^T m = n \), which for \( n = 4k \) satisfies the mirror symmetry condition. Therefore, the above interaction leads to a gapped and symmetry-preserving edge. This immediately implies that by turning on interactions, a noninteracting 2D TCI with mirror Chern number \( n_M = 4k \) is equivalent, i.e., can be adiabatically connected, to a trivial phase.

Next we show, case by case, that the gapped edges of TCIs with \( n_M = \pm 1 \) and \( 2 \) all spontaneously break the mirror symmetry. For \( n_M = \pm 1 \), edge states consist of a single pair of counter-propagating modes. A two-particle backscattering term \( \psi_L^\dagger \partial_x \psi_R^\dagger \psi_R \partial_x \psi_L + \text{h.c.} \) is symmetry allowed, which has a bosonized form \( \cos(2\Phi_{L}) \). Therefore, the above interaction leads to a gapped and symmetry-preserving edge. This immediately implies that by turning on interactions, a noninteracting 2D TCI with mirror Chern number \( n_M = \pm 1 \) is equivalent, i.e., can be adiabatically connected, to a trivial phase.

For \( n = 2 \), by an exhaustive enumeration, we find two sets of gapping vectors that satisfy the conditions (a)-(c):

\[
\begin{align*}
L_1 &= (1,1;1,1)^T, \\
L_2 &= (1,-1;1,-1)^T,
\end{align*}
\]

and

\[
\begin{align*}
L_1 &= (1,1;1,1)^T, \\
L_2 &= (1,-1;1,-1)^T.
\end{align*}
\]

In the first set, \( L_1 \) specifies a umklapp term that backscatters fermions of the two flavors \( a = 1, 2 \), and \( L_2 \) a backscattering term that flips the flavor of both left- and right-movers. In terms of the fermion fields, these terms are given by

\[
\begin{align*}
V_1 &= v_1(\psi_{1R}^\dagger \psi_{2R}^\dagger \psi_{2L} \psi_{1L} + \text{h.c.}), \\
V_2 &= v_2(\psi_{1R}^\dagger \psi_{2L} \psi_{1L} \psi_{2R} + \text{h.c.}).
\end{align*}
\]

Both terms conserve the number of fermions in each flavor and mutually commute. It is convenient to introduce boson fields for each flavor: \( \varphi_a = (\phi_{a,R} + \phi_{a,L})/2 \) and \( \theta_a = (\phi_{a,R} - \phi_{a,L})/2 \), so that \( n_a = \partial_x \theta_a \) is the density of flavor \( a \). In bosonized forms, the above interaction terms are given by:

\[
\begin{align*}
V_1 &\sim v_1 \cos(2\theta_1 + 2\theta_2), \\
V_2 &\sim v_2 \cos(2\theta_1 - 2\theta_2).
\end{align*}
\]

When \( v_1 \) and \( v_2 \) are sufficiently large, the fields \( \theta_1 \) and \( \theta_2 \) are both pinned and the edge becomes gapped. However, this gap opening is accompanied by nonzero expectation values of single-particle backscattering within each flavor: \( \langle e^{i2\theta_1} \rangle \sim \langle \psi_{1R}^\dagger \psi_{1L} \rangle \neq 0 \) and \( \langle e^{i2\theta_2} \rangle \sim \langle \psi_{2R}^\dagger \psi_{2L} \rangle \neq 0 \), hence breaking the mirror symmetry spontaneously.

For the second set of gapping vectors, the same umklapp term as in the first set is used, but the backscattering term is slightly different: \( \tilde{V}_2 = \tilde{v}_2(\psi_{1R}^\dagger \psi_{1L}^\dagger \psi_{2L} \psi_{2R} + \text{h.c.}) \). Nonetheless, a redefinition of the flavor index of the left-movers \( \psi_{1L} \to \psi_{1L}^\dagger \psi_{2L} \) brings \( \tilde{V}_2 \) back to the previous backscattering term \( V_2 \). Moreover, this redefinition does not change any symmetry property of fermion fields. Therefore the case of gapping vectors \( \{L_1, L_2\} \) is completely equivalent to the previous case of \( \{L_1, \tilde{L}_2\} \).

The above analysis shows that the gapped edge states with \( n = 2 \) pairs of counter-propagating modes necessarily break the mirror symmetry. This proves that TCIs with \( n_M = \pm 2 \) (as predicted for SnTe films) remain topologically nontrivial in the presence of interactions. At generic filling, only the backscattering is allowed by momentum conservation. Our renormalization group analysis shows that for repulsive Luttinger interaction \( K < 1 \) there appears a novel spin gap phase while the charge sector remains gapless and fluctuates. Boundaries and impurities affect the charge mode by pinning a fluctuating charge density wave, which can be detected by STM measurement.

Given the additive topology of SPT phases, the results of \( n_M = \pm 1, 2 \) combined with that of \( n_M = 4k \) suffice to show that 2D TCIs with \( n_M \neq 4k \) remain topologically distinct from a trivial phase and from each other in the presence of interactions. Therefore, we conclude that interactions reduce the classification of 2D TCIs from \( Z \) in the noninteracting limit to \( Z_2 \).

**Interacting TCIs in three dimensions:** We now turn to TCIs in three dimensions, protected by a single mirror symmetry, say \( x \to -x \). Within band theory, one can define the mirror Chern number \( n_M \) on the 2D plane.
\( k_z = 0 \) in \( k \)-space, which is invariant under this reflection. The integer \( n_M \) thus classifies 3D noninteracting TCIs \([1, 35–37]\). The hallmark surface states are present on crystal surfaces symmetric under mirror, and consist of \( n = |n_M| \) Dirac cones:

\[
H = \sum_{a=1}^{n} v_F \int dxdy \psi_a^\dagger(-i\partial_x s_y + i\partial_y s_z)\psi_a
\]  

where \( \psi_a^\dagger = (\psi_a^\dagger, \psi_a^\dagger) \) is a two-component fermion field. Reflection acts on both electron’s coordinate and spin as follows:

\[
M\psi_a^\dagger(x, y)M^{-1} = is_x\psi_a^\dagger(-x, y).
\]

The mirror symmetry forbids any Dirac mass term \( s_z\psi_b \), and thus protects these \( n \) copies of gapless Dirac fermions in the absence of interactions.

What is the fate of the above surface states under strong interactions? Can they be gapped without breaking the charge conservation and mirror symmetry? Answer to these questions is the key to classifying interacting TCIs in three dimensions. We will prove below that interacting TCI surface states with \( n \) Dirac cones, for \( n \neq 8k \), cannot be in a gapped and symmetric phase that has no intrinsic topological order; and they can for \( n = 8k \).

To show this, we consider the following scenario: the mirror symmetry is broken externally on the surface in the region \( x < 0 \) and \( x > 0 \) in exactly opposite directions, and a spatially-varying and flavor-dependent Dirac mass term is generated,

\[
V = \sum_{a=1}^{n} \int dxdy m_a(x)\psi_a^\dagger(x, y)s_z\psi_a(x, y),
\]

\[
m_a(x) = m_a\text{sgn}(x).
\]

Importantly, in the above setup (see Fig. 1b), the surface as a whole described by the Hamiltonian \( H + V \) remains invariant under the mirror \( x \to -x \).

As is well-known, the mass domain wall of a 2D noninteracting Dirac fermion hosts a 1D chiral fermion mode, whose directionality depends on the sign of Dirac mass at a given side, say \( x > 0 \). Consequently, the domain wall at \( x = 0 \) in our setup (20) hosts \( n \) flavors of chiral fermion modes, of which \( n_+ \) modes move in the \( +y \) direction and \( n_- \) modes move in the \( -y \) direction. Here \( n_+ \) \((n_-)\) is the number of Dirac fermions with \( m_a > 0 \) \((m_a < 0)\). Importantly, these counter-propagating modes are different eigenstates of the \( s_x \) and hence have opposite mirror eigenvalues, which will play a crucial role below.

We now use the domain wall setup to study whether interactions can turn \( n \) flavors of Dirac fermions into in a gapped, symmetric and non-fractionalized phase, i.e., a completely trivial phase. Suppose such a completely trivial surface exists, it must be adiabatically connectable to \( n \) flavors of massive Dirac fermions when one further breaks the mirror symmetry explicitly. With this in mind, we now introduce a sandwich setup shown in Fig. 1b by making the region \( |x| < L/2 \) the completely trivial phase, which is symmetric under \( x \to -x \); making the region \( x > L/2 \) a gapped phase by explicitly introducing a set of Dirac masses \( m_a \) \((a = 1, ..., n)\); and making the region \( x < -L/2 \) the mirror image of the region \( x > L/2 \), with opposite Dirac masses. We choose \( L \) to be much larger than the correlation length in \( x \), and make the surface Hamiltonian vary slowly with \( x \) across the interface. Under this condition, the completely trivial phase adiabatically evolves into the massive Dirac fermion phase as a function of \( x \), and the gap does not close at the interface. In this sandwich setup, the surface is everywhere gapped and as a whole preserves the mirror symmetry.

On the other hand, the sandwich setup is topologically equivalent to the domain wall setup introduced earlier \([35]\). Without interactions, the domain wall has been shown to host gapless 1D fermions, whereas the sandwich setup is gapped under the assumption that a completely trivial surface exists. This apparent contradiction can only be resolved in the following way: whenever the 1D domain wall fermions cannot be gapped and symmetric under any interactions, the completely trivial surface does not exist. Thus, we have found a relation between the 2D interacting surface states of TCIs and the 1D interacting fermions on the domain wall.

We proceed to study the latter 1D problem. Can \( n_+ \) and \( n_- \) fermions moving in opposite directions be gapped by interactions? For any odd integer \( n = n_+ + n_- \), \( n_+ \neq n_- \) leads to a net chirality, which prevents a full gap. For an even integer \( n = 2k \), we need to find out whether the \( n_+ = n_- = k \) pairs of 1D counter-propagating fermions can be gapped without breaking the mirror symmetry. Here we make the key observa-
tion: since these 1D domain wall fermions are localized at \( x = 0 \), mirror symmetry \( x \to -x \) acts as an internal \( \mathbb{Z}_2 \) symmetry within this Hilbert space. As we pointed our earlier, counter-propagating fermions have opposite mirror eigenvalues. Therefore, mirror acts on these domain wall fermions in exactly the same way as on the edge states of a 2D TCI studied in the first part of this paper, see Eq. (2). As shown there, the 1D domain wall with \( k \neq 4l \) pairs of fermions cannot be gapped and symmetric.

The above analysis of 1D interacting domain wall fermions implies that 3D TCIs with mirror Chern number \( n \neq 8k \) cannot have a completely trivial surface and hence remain topologically nontrivial after turning on interactions. In contrast, for \( n = 8k \), domain walls of massive Dirac fermions with mass term \( m = m(1_{4k} - 1_{4k}) \) can be gapped by interactions. One can then envision quantum disordering the long-range Ising order leads to a disordered phase with \( \langle m \rangle = 0 \) and hence restores the mirror symmetry, thereby making a completely trivial surface. An explicit microscopic construction of such a surface for \( n = 8k \) will be presented elsewhere [39].

Putting everything together, we conclude that interactions reduce the classification of 3D TCIs with mirror symmetry from \( Z \) to \( Z_8 \).

In addition to reducing the classification of noninteracting TCIs, interactions may also enable new TCI phases that do not exist in free fermion systems, as recently found in other symmetry classes [40, 41]. We leave this interesting problem of interaction-enabled TCIs with mirror symmetry for future study.

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Supplemental Material

Tomonaga-Luttinger liquid description

We have proven in the main text the $Z_4$ classification of two dimensional (2D) topological crystalline insulators (TCIs) by the Chern-Simons formulation. In the supplemental material, we will make a connection between the Chern-Simons description and the Tomonaga-Luttinger liquid description, and will give a physical picture of SPT phases. In the Tomonaga-Luttinger theory, the fermion field $\psi^\dagger_{a,R/L}$ is

$$\psi^\dagger_{a,R} = \frac{1}{\sqrt{2\pi\alpha}} e^{i\phi_{a,R}} = \frac{1}{\sqrt{2\pi\alpha}} e^{i(\varphi_a + \theta_a)}, \quad (21)$$

$$\psi^\dagger_{a,L} = \frac{1}{\sqrt{2\pi\alpha}} e^{i\phi_{a,L}} = \frac{1}{\sqrt{2\pi\alpha}} e^{i(\varphi_a - \theta_a)}. \quad (22)$$

$\alpha$ is an infinitesimal convergence factor. We omit the Klein factor here. The fields for right and left movers $\phi_{a,R/L}$ are expressed by $\phi_i$ in the Chern-Simons theory as

$$\phi_{a,R} = \varphi_a, \quad \phi_{a,L} = -\varphi_{a+a}. \quad (23)$$

The two fields $\varphi_a(x)$ and $\theta_a(x)$ satisfy the commutation relation

$$[\theta_a(x), \varphi_n(x')] = i\frac{\pi}{2} \text{sgn}(x-x'), \quad (24)$$

and transform under the mirror symmetry as

$$M\varphi_a(x)M^{-1} = \varphi_a(x), \quad M\theta_a(x)M^{-1} = \theta_a(x) + \frac{\pi}{2} \quad (25)$$

to satisfy the mirror symmetry. The electron density for the right and left movers are given by $n_{a,R/L} = \pm \partial_x \varphi_{a,R/L}/(2\pi)$ and thus the total electron density of $a$-th pair is $n_a = n_{a,R} + n_{a,L} = \partial_x \theta_a/\pi$. With the definition above, the bosonized Hamiltonian without gap-opening scatterings is

$$H^0 = \sum_{a=1}^n H^0_a(v_a, K_a) \quad (26)$$

with

$$H^0_a(v_a, K_a) = \frac{v_a}{2\pi} \int dx \left[ K_a(\partial_x \varphi_a)^2 + \frac{1}{K_a}(\partial_x \theta_a)^2 \right]. \quad (27)$$

The forward scattering terms, $g_2$ and $g_4$, are included through the Luttinger parameter $K_a$ and the renormalized velocity $v_a$, defined as

$$K_a = \sqrt{\frac{1 + (g_4,a - g_2,a)/(2\pi v_{F,a})}{1 + (g_4,a + g_2,a)/(2\pi v_{F,a})}}, \quad (28)$$

$$v_a = v_{F,a} \sqrt{\left(1 + \frac{g_4,a}{2\pi v_{F,a}}\right)^2 - \left(\frac{g_2,a}{2\pi v_{F,a}}\right)^2} \quad (29)$$

$n = 1$ case

A two-particle backscattering is allowed at half-filling $k_F = \pi/2$. To be precise, the points where the process occurs are split by the lattice constant $a$ and it is given by

$$V = g \int dx (e^{-4ik_F x} \psi^\dagger_{R}(x) \psi^\dagger_{R}(x+a) \times \psi_{L}(x+a)\psi_{L}(x) + \text{h.c.}). \quad (30)$$

The bosonized form of the process is

$$V = \frac{2g}{(2\pi\alpha)^2} \int dx \cos(2\phi_R - 2\phi_L)$$

$$= \frac{2g}{(2\pi\alpha)^2} \int dx \cos(2\Phi_L(x)). \quad (31)$$

with $L = (1; -1)^T$. This process is relevant for $K < 1/2$ due to the renormalization group (RG) analysis. When it is relevant, the cosine term pins the variable $2\Phi_L = 4\theta$ to have a expectation value. Its value depends on the sign of $g$: $\theta = 0$ or $\pi/2$ for $g < 0$, and $\theta = \pi/4$ or $3\pi/4$ for $g > 0$, noting that $\theta$ has a periodicity of $\pi$. Pinning of $\theta$ opens a gap. However recalling the transformation law Eq. (25), the pinning of the field $\theta$ spontaneously breaks the mirror symmetry. It happens because the gapping vector $L = (1; -1)^T$ violates the mirror symmetry.

$n = 2$ case

For $n = 2$, by an exhaustive enumeration, we find only two sets of gapping vectors satisfying the conditions (a)-(c), defined by

$$L_1 = (1,1; -1,1)^T,$$

$$L_2 = (1,-1; -1,1)^T, \quad (32)$$

and

$$\hat{L}_1 = (1,1; -1,1)^T,$$

$$\hat{L}_2 = (1,-1; -1,1)^T. \quad (33)$$

While these gapping vectors satisfy the mirror symmetry condition (c), they violate the primitive condition (d). An alternative choice $\frac{1}{2}(L_1 \pm L_2)$ or $\frac{1}{2}(L_1 \pm \hat{L}_2)$ is primitive, but violates the mirror symmetry. Therefore, we conclude that $n = 2$ edge states cannot be gapped without symmetry breaking.

In the following, we will consider the two sets in the Tomonaga-Luttinger description, and will show the corresponding microscopic origins of those scattering processes. At the beginning, we assume the two equivalent
edge modes by setting \( v_1 = v_2 = v \) and \( K_1 = K_2 = K \).
When two pairs of edge modes exist, two types of forward scatterings connecting two copies are allowed:
\[
V'_2 = g'_2 \int dx (\psi_1^R \psi_1^R \psi_1^L \psi_1^L + \psi_1^L \psi_1^L \psi_2^R \psi_2^R), \tag{34}
\]
\[
V'_4 = g_4 \int dx (\psi_1^R \psi_1^R \psi_2^R \psi_2^R + \psi_1^L \psi_1^L \psi_2^R \psi_2^L). \tag{35}
\]
Bosonizing the two processes \( V'_2 \) and \( V'_4 \), we obtain
\[
H = \frac{1}{2\pi} \int dx \left[ M_ϕ (\partial_x ϕ^2) + (\partial_x \bar{ϕ})^2 M_θ (\partial_x \bar{θ}) \right], \tag{36}
\]
where \( \bar{ϕ} = (ϕ_1, ϕ_2)^T, \ bar{θ} = (θ_1, θ_2)^T \), and the matrices \( M_ϕ \) and \( M_θ \) are given by
\[
M_ϕ = \begin{pmatrix} \frac{vK}{(g'_4 - g'_2)/2\pi} & \frac{(g'_4 - g'_2)/2\pi}{vK} \\ \frac{(g'_4 + g'_2)/2\pi}{vK} & \frac{vK}{(g'_4 + g'_2)/2\pi} \end{pmatrix} \tag{37}
\]
\[
M_θ = \begin{pmatrix} \frac{vK}{(g'_4 - g'_2)/2\pi} & \frac{(g'_4 - g'_2)/2\pi}{vK} \\ \frac{(g'_4 + g'_2)/2\pi}{vK} & \frac{vK}{(g'_4 + g'_2)/2\pi} \end{pmatrix} \tag{38}
\]
The matrices \( M_ϕ \) and \( M_θ \) can be diagonalized simultaneously to obtain
\[
H = \frac{v_+}{2\pi} \int dx \left[ \left( K_+ (\partial_x ϕ^2) + \frac{1}{K_+} (\partial_x ϕ_+)^2 \right) \right]
\]
\[
+ \frac{v_-}{2\pi} \int dx \left[ \left( K_- (\partial_x ϕ^2) + \frac{1}{K_-} (\partial_x ϕ_-)^2 \right) \right] \tag{39}
\]
with the new Luttinger parameter
\[
K_± = \sqrt{\frac{vK ± (g'_4 - g'_2)/2\pi}{vK ± (g'_4 + g'_2)/2\pi}} \tag{40}
\]
and the renormalized velocity
\[
v_± = \sqrt{\left( \frac{vK ± (g'_4 - g'_2)/2\pi}{vK ± (g'_4 + g'_2)/2\pi} \right)} \tag{41}
\]
The fields \( ϕ_± \) and \( θ_± \) are defined by
\[
ϕ_± = \frac{1}{\sqrt{2}} (ϕ_1 ± ϕ_2), \quad θ_± = \frac{1}{\sqrt{2}} (θ_1 ± θ_2). \tag{42}
\]
First we consider the scattering processes denoted by \( L_1 \) and \( L_2 \). The two scattering processes are written as
\[
V_1 = g_u \int dx (e^{-ik_F x} \psi_1^R \psi_2^R \psi_2^L \psi_1^L + h.c.), \tag{43}
\]
\[
V_2 = g_b \int dx (\psi_1^R \psi_2^R \psi_1^L \psi_2^L + h.c.). \tag{44}
\]
\( V_1 \) is an umklapp process and occurs at \( k_F = π/2 \). \( V_2 \) is a backscattering and is allowed at generic filling. Their bosonized forms are
\[
V_1 = \frac{2g_u}{(2\piα)^2} \int dx \cos(2θ_1 - 2θ_2), \tag{45}
\]
\[
V_2 = \frac{2g_b}{(2\piα)^2} \int dx \cos(2θ_1 + 2θ_2), \tag{46}
\]
or by using \( θ_± \)
\[
V_1 = \frac{2g_u}{(2\piα)^2} \int dx \cos(2\sqrt{2}θ_+), \tag{47}
\]
\[
V_2 = \frac{2g_b}{(2\piα)^2} \int dx \cos(2\sqrt{2}θ_-). \tag{48}
\]
The RG analysis shows that \( V_1 \) is relevant for \( K_+ < 1 \) and \( V_2 \) for \( K_- < 1 \). When a scattering process is relevant, it pins the field \( θ_± \) and generates a gap. The pinning of \( θ_± \) leads to the mass \( Δ_± \), estimated as \( Δ_± \approx (v_±/α)(g_u)^1/(2-2K_±) \) and \( Δ_± \approx (v_±/α)(g_b)^1/(2-2K_-) \).
This situation resembles the charge-spin separation of conventional spinful 1D systems. The fields \( ϕ_± \) and \( θ_± \) correspond to the charge degrees, and \( ϕ_- \) and \( θ_- \) to the spin degrees. The charge sector is gapped by the umklapp process and the spin sector by the backscattering process.

The mirror symmetry restricts the simultaneous gap opening of \( Δ_+ \) and \( Δ_- \) because the pinning of \( θ_± \) means the pinning of \( θ_1,2 \). Since \( θ_1 \) and \( θ_2 \) have a periodicity of \( π \), \( θ_1 + θ_2 \) is pinned at either 0 or \( π \) (mod 2π) for \( g_u < 0 \), and either \( π/2 \) or \( 3π/2 \) (mod 2π) for \( g_b > 0 \). Similar consideration applies for the backscattering process, which pins \( θ_1 - θ_2 \) and its value depends on the sign of \( g_b \). Therefore \( θ_1,2 \) have expectation values of either 0, \( π/4 \), \( π/2 \), or \( 3π/4 \), depending on the signs of \( g_u \) and \( g_b \), and hence the mirror symmetry is spontaneously broken. It also leads to non-zero expectation values of single-particle backscattering \( \langle e^{ik_1} \rangle \sim \langle ψ_1^R ψ_1^L \rangle \neq 0 \) and \( \langle e^{ik_2} \rangle \sim \langle ψ_2^R ψ_2^L \rangle \neq 0 \).

Next we consider \( L_1 \) and \( L_2 \). \( L_2 \) corresponds to
\[
\bar{V}_2 = \bar{g}_b \int dx (\psi_1^R ψ_1^R \psi_2^L ψ_2^R + h.c.), \tag{49}
\]
and its bosonized form is
\[
\bar{V}_2 = \frac{2\bar{g}_b}{(2\piα)^2} \int dx \cos(2\varphi_1 - 2\varphi_2), \tag{50}
\]
\( \bar{V}_2 \) is equivalent to \( V_2 \) by the redefinition \( ψ_1^R \rightarrow ψ_1^R \) and \( ψ_2^L \rightarrow ψ_2^L \). For the redefinition, the velocities of the two modes should be the same. When the two velocities are different, \( V_2 \) and \( \bar{V}_2 \) read different scattering processes (Fig. C).

Now we extend the analysis to the case where the two velocities are different \( v_1 \neq v_2 \) as well as \( K_1 \neq K_2 \). In this case, the charge and spin degrees are no longer separated. Here we concentrate on \( V_2 \). In the RG analysis, the scattering process \( V_2 \) is relevant for \( δ > 0 \), where \( δ \) is a scaling dimension of \( \bar{g}_b \), i.e., the coupling constant \( \bar{g}_b \) transforms into \( λ^δ \bar{g}_b \) under the scaling \( (x, r) \rightarrow λr \). The scaling dimension \( δ \) is given by \( δ = 2 + δ_{\cos \theta} \)}
The scaling dimension \( \delta \) being a scaling dimension of \( \cos(2\varphi_1 - 2\varphi_2) \). Following Ref. [43], \( \delta \) is calculated from the correlator

\[
K(r) = \langle \cos[2\varphi_1(r) - 2\varphi_2(r)] \cos[2\varphi_1(0) - 2\varphi_2(0)] \rangle
\]

as

\[
K(\lambda r) = \lambda^\delta K(r), \quad (51)
\]

If we assume an infinitely long system at zero temperature, the Euclidean action after integrating \( \theta \) fields is

\[
S_\varphi = \frac{1}{2} \int \frac{d\omega}{2\pi} \int \frac{dq}{2\pi} \bar{\varphi}(-q, -\omega)^T L(q, \omega) \varphi(q, \omega), \quad (52)
\]

where a \( 2 \times 2 \) matrix \( L(q, \omega) \) is defined as

\[
L(q, \omega) = \frac{1}{\pi} (q^2 M_\varphi + \omega^2 M_\theta^{-1}). \quad (53)
\]

Then the correlator \( K(r) \) will be

\[
K(r) = \frac{1}{2} e^{4I(r)} \quad (54)
\]

with

\[
I(r) = \int \frac{d\omega}{2\pi} \int \frac{dq}{2\pi} (e^{iqx - i\omega r} - 1)e^{-\alpha |q|} (L_{11}^{-1} + L_{22}^{-1} - L_{12}^{-1} - L_{21}^{-1}). \quad (55)
\]

To perform the integrations over \( q \) and \( \omega \), we differentiate \( I(r) \) with respect to \( x \), and then impose the boundary condition \( I(0) = 0 \). Following this procedure, we obtain

\[
I(r) = \frac{B - A\eta_1^2}{4\eta_1(\eta_2^2 - \eta_1^2)} \left[ \log \left( \frac{\alpha}{\alpha - i\pi + \eta_1 \tau} \right) + \log \left( \frac{\alpha}{\alpha + i\pi + \eta_1 \tau} \right) \right] - \frac{B - A\eta_2^2}{4\eta_2(\eta_2^2 - \eta_1^2)} \left[ \log \left( \frac{\alpha}{\alpha - i\pi + \eta_2 \tau} \right) + \log \left( \frac{\alpha}{\alpha + i\pi + \eta_2 \tau} \right) \right]. \quad (56)
\]

where

\[
A = \frac{v_1}{K_1} + \frac{v_2}{K_2} + \frac{1}{\pi} (g_1' + g_2'), \quad (57)
\]

\[
B = (\det M_\theta) \left[ v_1 K_1 + v_2 K_2 - \frac{1}{\pi} (g_1' - g_2') \right], \quad (58)
\]

\[
\eta_{1,2}^2 = \zeta \mp \sqrt{\zeta^2 - (\det M_\phi)(\det M_\theta)}, \quad (59)
\]

\[
\zeta = \frac{v_1^2 + v_2^2}{2} + \frac{1}{(2\pi)^2} (g_1'^2 - g_2'^2). \quad (60)
\]

Note that \( \eta_{1,2} \) can be regarded as renormalized velocities. The scaling dimension \( \delta_{\text{cos}} \) becomes

\[
\delta_{\text{cos}} = -\frac{A\eta_1\eta_2 + B}{\eta_1\eta_2(\eta_1 + \eta_2)}, \quad (61)
\]

and \( \bar{V}_2 \) is relevant when \( \delta > 0 \), i.e.,

\[
\frac{A\eta_1\eta_2 + B}{\eta_1\eta_2(\eta_1 + \eta_2)} < 2. \quad (62)
\]

[42] For bosonization, see for example T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2003).

[43] A.V. Moroz, K.V. Samokhin, and C.H.W. Barnes, Phys. Rev. B 62, 16900 (2000).