WZ-PROOFS OF “DIVERGENT” RAMANUJAN-TYPE SERIES

JESÚS GUILLERA

Dedicated to Herb Wilf on his 80th birthday

Abstract. We prove some “divergent” Ramanujan-type series for $1/\pi$ and $1/\pi^2$ applying a Barnes-integrals strategy of the WZ-method.

1. Wilf-Zeilberger’s pairs

We recall that a function $A(n,k)$ is hypergeometric in its two variables if the quotients

$$\frac{A(n+1,k)}{A(n,k)} \text{ and } \frac{A(n,k+1)}{A(n,k)}$$

are rational functions in $n$ and $k$, respectively. Also, a pair of hypergeometric functions in its two variables, $F(n,k)$ and $G(n,k)$, is said to be a Wilf and Zeilberger (WZ) pair [13, Chapt. 7] if

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

In this case, H. S. Wilf and D. Zeilberger [17] have proved that there exists a rational function $C(n,k)$ such that

$$G(n,k) = C(n,k)F(n,k).$$

The rational function $C(n,k)$ is the so-called certificate of the pair $(F,G)$. To discover WZ-pairs, we use Zeilberger’s Maple package EKHAD [13, Appendix A]. If EKHAD certifies a function, we have found a WZ-pair! We will write the functions $F(n,k)$ and $G(n,k)$ using rising factorials, also called Pochhammer symbols, rather than the ordinary factorials. The rising factorial is defined by

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n \in \mathbb{Z}^+, \\ 1, & n = 0, \end{cases}$$

or more generally by $(x)_t = \Gamma(x+t)/\Gamma(x)$. For $t \in \mathbb{Z} - \mathbb{Z}^-$, this last definition coincide with (3). But it is more general because it is also defined for all complex $x$ and $t$ such that $x+t \in \mathbb{C} - (\mathbb{Z} - \mathbb{Z}^+)$.

Key words and phrases. Hypergeometric series; WZ-method; Ramanujan-type series for $1/\pi$ and $1/\pi^2$; Barnes Integrals.
2. A Barnes-integrals WZ strategy

If we sum \( \sum_{n=0}^{\infty} G(n, k) \) over all \( n \geq 0 \), we get

\[
\sum_{n=0}^{\infty} G(n, k) - \sum_{n=0}^{\infty} G(n, k+1) = -F(0, k) + \lim_{n\to\infty} F(n, k)
\]

whenever the series above are convergent and the limit is finite. D. Zeilberger was the first to apply the WZ-method to prove a Ramanujan-type series for \( 1/\pi \) \[4\]. Following his idea, in a series of papers \[5\], \[6\], \[9\], \[10\] and in the author’s thesis \[8\], we use WZ-pairs together with formula (4) to prove a total of eleven Ramanujan-type series for \( 1/\pi \) and four Ramanujan-like series for \( 1/\pi^2 \). However, while we discovered those pairs we also found some WZ-pairs corresponding to “divergent” Ramanujan-type series \[12\], like the following pair:

\[
F(n, k) = A(n, k) \frac{(-1)^n}{\Gamma(n+1)} \left(\frac{16}{9}\right)^n, \quad G(n, k) = B(n, k) \frac{(-1)^n}{\Gamma(n+1)} \left(\frac{16}{9}\right)^n,
\]

where

\[
A(n, k) = U(n, k) \frac{-n(n-2)}{3(n+2k+1)}, \quad B(n, k) = U(n, k)(5n+6k+1),
\]

and

\[
U(n, k) = \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4} + \frac{3k}{2}\right)_n \left(\frac{1}{4} + \frac{3k}{2}\right)_n \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{(1+k)_n(1+2k)_n (1/2)_k}.
\]

We cannot use formula (4) with this pair because the series is divergent and the limit is infinite, due to the factor \((-16/9)^n\). To deal with this kind of WZ-pairs we will proceed as follows: First we replace the factor \((-1)^n\) with \(\Gamma(n+1)\Gamma(-n)\). By doing it we again get a WZ-pair, because \((-1)^n\) and \(\Gamma(n+1)\Gamma(-n)\) transform formally in the same way under the substitution \( n \to n + 1 \); namely, the sign changes. To fix ideas, the modified version of the WZ-pair above is

\[
\tilde{F}(s, t) = A(s, t)\Gamma(-s) \left(\frac{16}{9}\right)^s, \quad \tilde{G}(s, t) = B(s, t)\Gamma(-s) \left(\frac{16}{9}\right)^s.
\]

Then, integrating from \( s = -i\infty \) to \( s = i\infty \) along a path \( \mathcal{P} \) (curved if necessary) which separates the poles of the form \( s = 0, 1, 2, \ldots \) from all the other poles, we obtain

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} B(s, t)\Gamma(-s)(-z)^s ds = \sum_{n=0}^{\infty} B(n, t) \frac{z^n}{n!}, \quad |z| < 1,
\]

where we have used the Barnes integral theorem, which is an application of Cauchy’s residues theorem using a contour which closes the path with a right side semicircle of center at the origin and infinite radius. The Barnes integral gives the analytic continuation of the series to \( z \in \mathbb{C} - [1, \infty) \). Integrating along the same path the
identity \( \widetilde{G}(s, t + 1) - \widetilde{G}(s, t) = \widetilde{F}(s + 1, t) - \widetilde{F}(s, t) \), we obtain

\[
\int_{-i\infty}^{i\infty} \widetilde{G}(s, t + 1)ds - \int_{-i\infty}^{i\infty} \widetilde{G}(s, t)ds = \int_{-i\infty}^{i\infty} \widetilde{F}(s + 1, t)ds - \int_{-i\infty}^{i\infty} \widetilde{F}(s, t)ds
\]

\[
= \int_{-i\infty}^{1+i\infty} \widetilde{F}(s, t)ds - \int_{-i\infty}^{i\infty} \widetilde{F}(s, t)ds = - \int_\mathcal{C} \widetilde{F}(s, t)ds,
\]

where \( \mathcal{C} \) is the contour limited by the path \( \mathcal{P} \), the same path but moved one unit to the right, and the lines \( y = -\infty \) and \( y = +\infty \). As the only pole inside this contour is at \( s = 0 \) and the residue at this point is zero, the last integral is zero and we have

\[
\int_{-i\infty}^{i\infty} \widetilde{G}(s, t)ds = \int_{-i\infty}^{i\infty} \widetilde{G}(s, t + 1)ds.
\]

This implies, by Weierstrass’s theorem [16], that

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \widetilde{G}(s, t)ds = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \widetilde{G}(s, t)ds = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \lim_{t \to \infty} \widetilde{G}(s, t)ds
\]

\[
= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{3}{\pi} \left( \frac{1}{2} \right)_s (\Gamma(-s))^{2s}ds = \frac{\sqrt{3}}{\pi},
\]

where the last equality holds because

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{1}{2} \right)_s (\Gamma(-s))^{2s}ds = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n z^n = \frac{1}{\sqrt{1-z}}, \quad |z| < 1, \]

implies that

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{1}{2} \right)_s (\Gamma(-s))^{2s}ds = \frac{1}{\sqrt{1-z}}, \quad z \in \mathbb{C} - [1, \infty).
\]

Hence, we have

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{1}{2} \right)_s \left( \frac{1}{4} + \frac{3t}{2} \right)_s \left( \frac{3}{4} + \frac{3t}{2} \right)_s \left( \frac{3}{4} + \frac{3t}{2} \right)_s \left( \frac{5}{4} + \frac{5t}{2} \right)_s (5s + 6t + 1)\Gamma(-s) \left( \frac{4}{3} \right) ds = \frac{\sqrt{3}}{\pi},
\]

or equivalently

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{1}{2} \right)_s \left( \frac{1}{4} + \frac{3t}{2} \right)_s \left( \frac{3}{4} + \frac{3t}{2} \right)_s (5s + 6t + 1)\Gamma(-s) \left( \frac{4}{3} \right) ds = \frac{\sqrt{3}}{\pi} \left( \frac{1}{1-t} \right)_t \left( \frac{1}{2} \right)_t.
\]

Finally, substituting \( t = 0 \), we see that

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{1}{2} \right)_s \left( \frac{1}{4} + \frac{3t}{2} \right)_s \left( \frac{3}{4} + \frac{3t}{2} \right)_s (5s + 1)\Gamma(-s) \left( \frac{4}{3} \right) ds = \frac{\sqrt{3}}{\pi}.
\]

It is very convenient to write the Barnes integral in hypergeometric notation. By the definition of hypergeometric series, we see that for \( -1 \leq z < 1 \), we have

\[
\sum_{n=0}^{\infty} \frac{(1/2)_n (s)_n (1-s)_n z^n}{(1)_n^3} = {}_3F_2 \left( \frac{1}{4}, s, 1-s \mid z \right).
\]
and
\[ \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (s)_n (1-s)_n}{(1)_n^3} n z^n = \frac{1}{2} s(1-s) z \, _3F_2 \left( \frac{3}{2}, \frac{1+s}{2}, \frac{2-s}{2} \mid z \right), \]
where the notation on the right side stands for the analytic continuation of the series on the left. Hence, we can write (8) in the form
\[ _3F_2 \left( \frac{3}{2}, \frac{1}{4}, \frac{3}{4} \mid -\frac{16}{9} \right) - \frac{5}{6} _3F_2 \left( \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \mid -\frac{16}{9} \right) = \frac{\sqrt{3}}{\pi}. \]

If, instead of integrating to the right side, we integrate (8) along a contour which closes the path \( P \) with a semicircle of center \( s = 0 \) taken to the left side with an infinite radius, then we have poles at \( s = \pm 3/4 \) for \( n = 0, 1, 2, \ldots \), and we obtain
\[ \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n (\frac{3}{4})_n} (10n + 3)(-1)^n \left( \frac{3}{4} \right)^{2n} = \frac{\sqrt{2}}{8 \pi^2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n (\frac{3}{4})_n} (20n + 1)(-1)^n \left( \frac{3}{4} \right)^{2n} - \frac{3\sqrt{2}}{16 \pi^2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n (\frac{3}{4})_n} (20n + 11)(-1)^n \left( \frac{3}{4} \right)^{2n} = 1. \]
which is an identity relating three convergent series.

3. OTHER EXAMPLES

In a similar way we can prove other identities of the same kind, for example,
\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(\frac{1}{2} + t)_s}{(1 + t)^3} \frac{1}{(1 + 2t)^s} (10s^2 + 6s + 1 + 14st + 4t^2 + 4t) \Gamma(-s) 2^{2s} ds = \frac{4}{\pi^2} \left( \frac{1}{2} \right)_t^4, \]
\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(\frac{1}{2})_s}{(1)_s (1 + 2t)^s} (3s^2 + 2t + 1) \Gamma(-s) 2^{3s} ds = \frac{1}{\pi} \left( \frac{1}{2} \right)_t, \]
and
\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(\frac{1}{2})_s}{(1)_s (1 + 2t)^s} \frac{1}{(1 + t)^s} \left( \frac{3}{2} + t \right)_s (\frac{3}{2} + t)_s \left( \frac{5}{2} + t \right)_s \left( \frac{5}{2} + t \right)_s \times \frac{(15s + 4)(2s + 1) + t(33s + 16)}{2s + t + 1} \Gamma(-s) 2^{2s} ds = \frac{3\sqrt{3}}{\pi} \frac{1}{2^{2s} \left( \frac{1}{2} \right)_t \left( \frac{2}{4} \right)_t}. \]

In the two last examples the hypothesis of Weierstrass theorem fail and hence we cannot apply it, but we obtain the sum using Meurman’s periodic version of Carlson’s theorem [2] p. 39 which asserts that if \( H(z) \) is a periodic entire function of period 1 and there is a real number \( c < 2\pi \) such that \( H(z) = O(\exp(c|Im(z)|)) \) for all \( z \in \mathbb{C} \), then \( H(z) \) is constant [11 Appendix] and [11] Thm. 2.3. In the second
and third examples we determine the constants $1/\pi$ and $3\sqrt{3}/\pi$ taking $t = 1/2$ and $t = -1/3$ respectively. Substituting $t = 0$ in the above examples, we obtain respectively

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(1/2)^5}{(1/2)^4} (10s^2 + 6s + 1)\Gamma(-s)2^{2s}ds = \frac{4}{\pi^2}, \]

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(1/3)^3}{(1/2)^2} (3s + 1)\Gamma(-s)2^{3s}ds = \frac{1}{\pi}, \]

and

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(1/2)^5 (1/3)(1/2)}{(1/2)^2} (15s + 4)\Gamma(-s)2^{2s}ds = \frac{3\sqrt{3}}{\pi}. \]

Using hypergeometric notation, we can write (9), (10) and (11) respectively in the following forms:

\[ 5F_4 \left( \begin{array}{cccc} 1/2, 3/2, 1/2, 1/2, 1/2 \\ 1, 1, 1, 1, 1 \end{array} \middle| -4 \right) - \frac{3}{4} \cdot 5F_4 \left( \begin{array}{cccc} 3/2, 3/2, 3/2, 3/2, 3/2 \\ 2, 2, 2, 2, 2 \end{array} \middle| -4 \right) = \frac{4}{\pi^2}, \]

\[ 3F_2 \left( \begin{array}{cc} 1/2, 1/2, 1/2 \\ 1, 1, 1 \end{array} \middle| -8 \right) - 3F_2 \left( \begin{array}{cc} 3/2, 3/2, 3/2 \\ 2, 2, 2 \end{array} \middle| -8 \right) = \frac{1}{\pi}, \]

and

\[ 4F_2 \left( \begin{array}{cc} 1/2, 1/2, 2/3 \\ 1, 1, 1 \end{array} \middle| -4 \right) - \frac{20}{3} \cdot 4F_2 \left( \begin{array}{cc} 3/2, 4/3, 5/3 \\ 2, 2, 2 \end{array} \middle| -4 \right) = \frac{3\sqrt{3}}{\pi}. \]

Related applications of the WZ-method for Barnes-type integrals are for example in [3, Sect. 5.2] and [14].

4. THE DUAL OF A “DIVERGENT” RAMANUJAN-TYPE SERIES

The WZ duality technique [13, Ch. 7] allows to transform pairs which lead to divergences into pairs which lead to convergent series. To get the dual \( \hat{G}(n,k) \) of \( G(-n, -k) \), we make the following changes:

\[ (a)_{-n} \rightarrow \frac{(-1)^n}{(1-a)_n}, \quad (1)_{-n} \rightarrow \frac{n(-1)^n}{(1)_n}, \quad (a)_{-k} \rightarrow \frac{(-1)^k}{(1-a)_k}, \quad (1)_{-k} \rightarrow \frac{k(-1)^k}{(1)_k}. \]
Example 1. The package EKHAD certifies the pair

\[
F(n, k) = U(n, k) \frac{2n^2}{2n + k}, \quad G(n, k) = U(n, k) \frac{6n^2 + 2n + k + 4nk}{2n + k},
\]

where

\[
U(n, k) = \frac{\left(\frac{1}{2}\right)_n (1 + \frac{k}{2})_n \left(\frac{1}{2} + \frac{k}{2}\right)_n \left(\frac{1}{2}\right)_k}{(1)_n^2 (1 + k)_n^2} 4^n = \frac{(2n)!^2 (2n + k)! (2k)!}{n!^2 (n + k)!^2 16^n 4^k}.
\]

We cannot use this WZ-pair to obtain a Ramanujan-like evaluation because, as \(z > 1\), the corresponding series and also the corresponding Barnes integral are both divergent. However, we will see how to use it to evaluate a related convergent series. What we will do is to apply the WZ duality technique. Thus, if we take the dual of \(G(-n, -k)\) and replace \(k\) with \(k - 1\), we obtain

\[
\hat{G}(n, k) = \frac{1}{U(n, k)} \frac{2(2k - 1)(2n + k)}{n^2 (n + k)^2 (n + k - 1)^2} (6n^2 - 6n + 1 - k + 4nk),
\]

and EHKAD finds its companion

\[
\hat{F}(n, k) = \frac{1}{U(n, k)} \frac{-2(2n + k)(2n + k - 1)(2n - 1)^2}{n^2 (n + k)^2 (n + k - 1)^2}.
\]

Applying Zeilberger’s formula

\[
\sum_{n=j}^{\infty} (\hat{F}(n+1, n) + \hat{G}(n, n)) = \sum_{n=j}^{\infty} \hat{G}(n, j)
\]

with \(j = 1\), we obtain

\[
\sum_{n=1}^{\infty} \left(\frac{16}{27}\right)^n \frac{(1)_n^3}{(\frac{1}{2})_n^3 (\frac{1}{3})_n^3} \frac{11n - 3}{n^3} = \sum_{n=1}^{\infty} \frac{(1)_n^3}{4^n (\frac{1}{2})_n^3} \frac{3n - 1}{n^3}.
\]

The series in \((13)\) are dual to Ramanujan-type “divergent” series, and in [7, p. 221] we proved that the series on the right side is equal to \(\pi^2/2\). Hence

\[
\sum_{n=1}^{\infty} \left(\frac{16}{27}\right)^n \frac{(1)_n^3}{(\frac{1}{2})_n^3 (\frac{1}{3})_n^3} \frac{11n - 3}{n^3} = 8\pi^2.
\]

Formula \((14)\), as well as other similar formulas, was conjectured in [15, Conj 1.4] by Zhi-Wei Sun.

Example 2. The package EKHAD certifies the pair

\[
F(n, k) = U(n, k) \frac{64n^3}{(2k + 1)(2n - 2k + 1)}, \quad G(n, k) = U(n, k) \frac{(2n + 1)^2 (11n + 3) - 12k(2n^2 + 3nk + n + k)}{(2n + 1)^2},
\]
where

\[ U(n, k) = \frac{(\frac{1}{2} - k)_n (\frac{1}{2} + k)_n^2 (\frac{1}{3})_n (\frac{1}{3})_n (\frac{3}{2})_n}{(1)_n (\frac{3}{2})_n^3} \left( \frac{27}{16} \right)^n. \]

Taking the dual \( \hat{G}(n, k) \) of \( G(-n, -k) \), replacing \( n \) with \( n + x \) and applying Zeilberger’s theorem

\[
\sum_{n=0}^{\infty} \hat{G}(n + x, 0) = \lim_{k \to \infty} \sum_{n=0}^{\infty} \hat{G}(n + x, k) + \sum_{k=0}^{\infty} \hat{F}(x, k),
\]

where \( \hat{F}(n, k) \) is the companion of \( \hat{G}(n, k) \), we obtain

\[
\sum_{n=0}^{\infty} \frac{(1 + x)_n^3}{(\frac{1}{2} + x)_n (\frac{1}{3} + x)_n (\frac{3}{2} + x)_n} \left( \frac{16}{27} \right)^n \frac{11(n + x) - 3}{(n + x)^3} = \frac{6(3x - 1)(3x - 2)}{x^3(2x - 1)} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (\frac{3}{2} - x)_k}{(\frac{1}{2} + x)_k^2}.
\]

Taking \( x = 1 \) we again obtain (14).

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E-mail address: jguillera@gmail.com

Av. Cesáreo Alierta, 31 esc. izda 4º–A, Zaragoza (Spain)