Navier-Stokes problems in half space with parameters
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Abstract
The existence, uniqueness and uniformly \( L^p \) estimates for solutions of the parameter dependent abstract Navier-Stokes problem on half space are derived. In application the existence, uniqueness and uniformly \( L^p \) estimates for solution of the Wentzell-Robin type mixed problem for Navier-Stokes equation is established.

Key Word: Stokes systems, Navier-Stokes equations, Differential equations with small parameters, Semigroups of operators, Boundary value problems, Differential-operator equations, Maximal \( L^p \) regularity

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1. Introduction
We will consider the initial boundary value problems (IBVP) for Navier-Stokes equation (NSE) with small parameter

\[
\frac{\partial u}{\partial t} - \triangle \varepsilon u + (u, \nabla) u + \nabla \varphi + Au = f(x,t), \quad \text{div } u = 0, \tag{1.1}
\]

\[
\sum_{i=0}^{\nu} \varepsilon_i \sigma_i \frac{\partial^2 u}{\partial x_n^2} (x',0,t) = 0, \quad \nu \in \{0,1\}, \tag{1.2}
\]

\[
u(x,0) = a(x), \quad x \in R^n_+, \quad t \in (0,T), \tag{1.3}
\]

where

\[
R^n_+ = \{x \in R^n, \quad x_n > 0, \quad x = (x',x_n), \quad x' = (x_1,x_2,...,x_{n-1})\},
\]

\[
\triangle \varepsilon u = \sum_{k=1}^{n} \varepsilon_k \frac{\partial^2 u}{\partial x_n^2}, \quad \sigma_i = \frac{1}{2} \left( i + \frac{1}{q} \right), \quad q \in (1,\infty),
\]

\[
\alpha_i \text{ are complex numbers, } \varepsilon = (\varepsilon_1,\varepsilon_2,...,\varepsilon_n), \quad \varepsilon_k \text{ are small positive parameters and } A \text{ is a linear operator in a Banach space } E. \text{ Here}
\]

\[
u = u_{\varepsilon} (x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t)), \quad u_k (x,t) = u_{\varepsilon_k} (x,t)
\]

and \( \varphi = \varphi(x,t) \) are represent the \( E \)-valued unknown velocity and pressure like functions, respectively; \( f = (f_1(x,t), f_2(x,t), ..., f_n(x,t)) \) and \( a \) represent a given \( E \)-valued external force and the initial velocity. In this work, we show the
uniform existence and uniqueness of the stronger local and global solution of the Navier-Stokes problem with small parameter (1.1) – (1.3). This problem is characterized by presence abstract operator $A$ and a small parameters $\varepsilon_k$ which corresponds to the inverse of Reynolds number $Re$ very large for the Navier-Stokes equations. The regularity properties of Navier-Stokes equations studied in e.g. [4 – 6] and [9 – 15]. Navier-Stokes equations with small viscosity when the boundary is either characteristic or non-characteristic have been well-studied see, e.g. in [9, 11, 21]. Moreover, regularity properties of differential operator equation (DOE) were investigated e.g. in [1, 2, 16-20, 23]. Here we consier Navier-Stokes operator equation in a Banach space $E$. Since the Banach space $E$ is arbitrary and $A$ is a possible linear operator, by chosing spaces $E$ and operators $A$ we can obtained existence, uniqueness and $L^p$ estimates of solutions for numerous class of Navier-Stokes type problems.

In this paper, firstly we prove that the Stokes problem

$$
\frac{\partial u}{\partial t} - \Delta_x u + Au + \nabla \varphi = f(x, t), \; \text{div } u = 0, \; x \in R^n_+, \; t \in (0, T),
$$

$$
\sum_{i=0}^{\nu} \varepsilon_n^{\alpha_i} \frac{\partial^i u}{\partial x_i^n}(x', 0, t) = 0, \; \nu \in \{0, 1\}, \; u(x, 0) = a(x)
$$

has a unique solution $(u, \nabla \varphi)$ for $f \in L^p(0, T; L^q(R^n_+; E)) = B(p, q), \; p, q \in (1, \infty)$ and the following uniform estimate holds

$$
\left\| \frac{\partial u}{\partial t} \right\|_{B(p, q)} + \sum_{k=1}^{n} \left\| \varepsilon_k \frac{\partial^2 u}{\partial x_k^n} \right\|_{B(p, q)} + \left\| Au \right\|_{B(p, q)} + \left\| \nabla \varphi \right\|_{B(p, q)} \leq C \left( \left\| f \right\|_{B(p, q)} + \left\| a \right\|_{B^{2-\frac{2}{p}}_{p, q}} \right)
$$

with $C = C(T, p, q)$ independent of $f$ and $\varepsilon$.

Then, by following Kato and Fujita [6, 10] method and using the above uniform coercive estimate for Stokes problem we derive a local a priori estimates for solutions of (1.1) – (1.3), i.e., we prove that for $\gamma < 1$ and $\delta > 0$ such that $\frac{\gamma}{\delta(\gamma - 1)} - \frac{1}{2} \leq \gamma, \; -\gamma < \delta < 1 - |\gamma|$, $a \in D\left(O_{\varepsilon_k}^\alpha\right)$ there is $T_* \in (0, T)$ independent of $\varepsilon_k \in (0, 1]$ such that $\left\| O_{\varepsilon_k}^{-\alpha} f(t) \right\|$ is continuous on $(0, T)$ and satisfies $\left\| O_{\varepsilon_k}^{-\alpha} f(t) \right\| = a(t^{\gamma - \alpha})$ as $t \to 0$, then there is a local solution of (1.1) – (1.3) such that $u \in C\left( (0, T_*); D\left(O_{\varepsilon_k}^\alpha\right) \right)$, $u(0) = a, \; u \in C\left( (0, T_*); D\left(O_{\varepsilon_k}^\alpha\right) \right)$ for some $T_* > 0$, $\left\| O_{\varepsilon_k}^\alpha u(t) \right\| = o(t^{\gamma - \alpha})$ as $t \to 0$ for all $\alpha$ with $\gamma < \alpha < 1 - \delta$ uniformly in $\varepsilon$. Moreover, the solution of (1.1) – (1.3) is unique if $u \in C\left( (0, T_*); D\left(O_{\varepsilon_k}^\alpha\right) \right)$, $\left\| O_{\varepsilon_k}^\alpha u(t) \right\| = o(t^{\gamma - \beta})$ as $t \to 0$ for some $\beta$ with $\beta > |\gamma|$ uniformly in $\varepsilon$. For sufficiently small date we show that, there is a global solution of the problem (1.1) – (1.3). Particularly, we prove that there is a $\delta > 0$ such that if $\|a\|_{L^\infty(R^n_+; E)} < \delta$, then there is a global solution $u_\varepsilon$ of (1.1) – (1.3) so that

$$
t^{(1-\frac{2}{p})/2} u_\varepsilon, \; t^{(1-\frac{2}{p})} \nabla u_\varepsilon \in C\left( (0, \infty); L^q(R^n_+; E) \right) \text{ for } n \leq q \leq \infty.$$
Moreover, the following uniform estimates hold
\[
\sup_{t,\varepsilon} \left\| t^{(1 - \frac{2}{q})/2} u_t \right\|_{L^q(R^n_+; E)} \leq C, \quad \sup_{t,\varepsilon} \left\| t^{(1 - \frac{2}{q})} \nabla u_x \right\|_{L^q(R^n_+; E)} \leq C, \quad k = 1, 2, \ldots, n
\]

In application we choose \( E = L_{p_1}(\Omega) \) and \( A \) to be differential operator with generalized Wentzell-Robin boundary condition defined by
\[
D(A) = \left\{ u \in W^{2, p}_1(0, 1), \ B_j u = A(u) + \sum_{i=0}^{1} \alpha_{ji} u^{(i)}(j), \ j = 0, 1 \right\},
\]
\[
Au = au^{(2)} + bu^{(1)} + cu,
\]
in (1.1) – (1.2), where \( \alpha_{ji} \) are complex numbers, \( a, b, c \) are complex-valued functions. Then, we obtain the following Wentzell-Robin type mixed problem for Navier-Stokes equation
\[
\frac{\partial u}{\partial t} - \triangle u + (u, \nabla) u + \nabla \varphi + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + cu = f(x, y, t), \quad (1.5)
\]
\[
\text{div}_x u = 0, \quad u = u(x, y, t), \quad x \in R^n_+,
\]
\[
\sum_{i=0}^{\nu} \varepsilon_{ii} \alpha_{i} \frac{\partial^2 u}{\partial x_i^2}(x', 0, y, t) = 0, \quad \nu \in \{0, 1\}, \quad x' \in R^{n-1}, \quad y \in (0, 1) \quad (1.6)
\]
\[
Au(x, j, t) + \sum_{i=0}^{1} \alpha_{ji} u^{(i)}(x, j, t) = 0, \quad u(x, 0) = a(x). \quad (1.7)
\]

Note that, the regularity properties of Wentzell-Robin type BVP for elliptic equations were studied e.g. in [7, 8] and the references therein. Here
\[
\hat{\Omega} = R^n_+ \times (0, 1), \quad p = (p_1, p).
\]
\( L_{p_1}(\hat{\Omega}) \) denotes the space of all \( p \)-summable complex-valued functions with mixed norm i.e., the space of all measurable functions \( f \) defined on \( \hat{\Omega} \), for which
\[
\|f\|_{L_{p_1}(\hat{\Omega})} = \left( \int_{R^n_+} \left( \int_0^1 |f(x, y)|^{p_1} \, dy \right)^{1/p_1} \, dx \right)^{1/p_1} < \infty.
\]

By using the above general abstract result, the existence, uniqueness and uniformly \( L_{p_1}(\hat{\Omega}) \) estimates for solution of the problem (1.5) – (1.7) is obtained.
Let $E$ be a Banach space and $L^p(Ω; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $Ω ⊂ R^n$ with the norm

$$\|f\|_{L^p} = \|f\|_{L^p(Ω; E)} = \left(\int_{Ω} \|f(x)\|^p_E \, dx\right)^{\frac{1}{p}}, 1 \leq p < ∞.$$  

The Banach space $E$ is called an UMD-space if the Hilbert operator $(Hf)(x) = \lim_{ε→0} \int_{|x−y|>ε} \frac{f(y)}{x−y} \, dy$ is bounded in $L^p(R, E), p ∈ (1, ∞)$ (see e.g. [2, § 4]). UMD spaces include e.g. $L^p, L^p$ spaces and Lorentz spaces $L_{pq}, p, q ∈ (1, ∞)$.

Let $E_1$ and $E_2$ be two Banach spaces. Let $B(E_1, E_2)$ denote the space of all bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

A linear operator $A$ is said to be $R$-positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\left\|(A + λI)^{-1}\right\|_{B(E)} ≤ M(1 + |λ|)^{-1}$ for any $λ ∈ (-∞,0]$ where $I$ is the identity operator in $E$ (see e.g. [22, §1.15.1]).

The positive operator $A$ is said to be $R$-positive in a Banach space $E$ if the set $L_A = \{ξ (A + ξ)^{-1} : ξ ∈ (-∞,0)\}$, is $R$-bounded (see [2, § 4]).

The operator $A(s)$ is said to be positive in $E$ uniformly with respect to parameter $s$ with bound $M > 0$ if $D(A(s))$ is independent on $s$, $D(A(s))$ is dense in $E$ and $\left\|(A(s) + λ)^{-1}\right\| ≤ \frac{M}{|1+λ|}$ for all $λ ∈ S_0, 0 ≤ ψ < π$, where $M$ does not depend on $s$ and $λ$.

Assume $E_0$ and $E$ are two Banach spaces and $E_0$ is continuously and densely embeds into $E$. Here $Ω$ is a measurable set in $R^n$ and $m$ is a positive integer. Let $W^{m,p}(Ω; E_0, E)$ denote the space of all functions $u ∈ L^p(Ω; E_0)$ that have the generalized derivatives $\frac{∂^m u}{∂x_k^m} ∈ L^p(Ω; E)$ with the norm

$$\|u\|_{W^{m,p}(Ω; E_0, E)} = \|u\|_{L^p(Ω; E_0)} + \sum_{k=1}^n \left\|\frac{∂^m u}{∂x_k^m}\right\|_{L^p(Ω; E)} < ∞.$$  

2. Regularity properties of solutions for DOEs with parameters

In this section, we consider the boundary value problem (BVP) for the elliptic DOE with small parameters in half-space. We will derive the maximal regularity properties of the following problem

$$- Δ_x u + (A + λ) u = f(x), \quad x ∈ R^n_+,$$  

$$\sum_{i=0}^{κ} \varepsilon_i \alpha_i \frac{∂^i u}{∂x_i^i}(x', 0, t) = 0,$$  

(2.1)
where $A$ is a linear operator in $E$, $\alpha_i$ are complex numbers, $\varepsilon_k$ are positive and $\lambda$ is a complex parameters and

$$\triangle_{\varepsilon} u = \sum_{k=1}^{n} \varepsilon_k \frac{\partial^2 u}{\partial x_k^2}, \quad \sigma_i = \frac{1}{2} \left( i + \frac{1}{q} \right), \nu \in \{0,1\}. $$

By virtue of [19, Theorem 2.2] we have

**Theorem 2.1.** Let $E$ be a UMD space space and $A$ is an $R$-positive operator in $E$. Assume $m$ is a nonnegative number, $q \in (1, \infty)$, $\alpha_{ij} \neq 0$, $0 < t_k \leq 1$, $k = 1, 2, ..., n$. Then for all $f \in W^{m,q}(R^n_+;E)$, $\lambda \in S_{\psi,\kappa}$ and sufficiently large $\kappa > 0$ problem (2.1) – (2.2) has a unique solution $u$ that belongs to $W^{2+m,q}(R^n_+;E(A);E)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n} \sum_{i=0}^{n} \varepsilon_k^{1/2} \frac{1}{\alpha_i} \left| \lambda \right|^{1-\frac{1}{q}} \frac{\partial^i u}{\partial x_k^{i}} \right\|_{L^q(R^n_+;E)} + \|Au\|_{L^q(R^n_+;E)} \leq C \|f\|_{W^{m,q}(R^n_+;E)} \quad \text{(2.3)}$$

with $C = C(q,A)$ independent of $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$, $\lambda$ and $f$.

Consider the operator $Q_{\varepsilon}$ generated by problem (2.1) – (2.2), i.e.,

$$D(Q_{\varepsilon}) = W^{2,q}(R^n_+;L_{1\varepsilon}) = \{ u \in W^{2,q}(R^n_+) , L_{1\varepsilon}u = 0 \},$$

$$Q_{\varepsilon}u = -\triangle_{\varepsilon} u + Au.$$

From Theorem 2.1 we obtain the following

**Result 2.1.** Suppose the conditions of Theorem 2.1 are satisfied. For $\lambda \in S_{\psi,\kappa}$ there is a resolvent $(Q_{\varepsilon} + \lambda)^{-1}$ of the operator $Q_{\varepsilon}$ satisfying the following uniform estimate

$$\sum_{k=1}^{n} \sum_{i=0}^{n} \varepsilon_k^{1/2} \frac{1}{\alpha_i} \left| \lambda \right|^{1-\frac{1}{q}} \frac{\partial^i u}{\partial x_k^{i}} \right\|_{B(L^q(R^n_+;E))} \leq C.$$

It is clear that the solution of the problem (2.1) – (2.2) depend on parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, i.e. $u = u_{\varepsilon}(x)$. In view of the Theorem 2.1, we derive the properties of the solutions (2.1) – (2.2). Particularly, by resoning as [19, Theorem 2.2] we show the following:

**Corollary 2.1.** Let all conditions of the Theorem 2.1. hold. Then, the solution of (2.1) – (2.2) satisfies the following uniform estimate

$$\sum_{k=1}^{n} \sum_{i=0}^{n} \varepsilon_k^{1/2} \frac{1}{\alpha_i} \left| \lambda \right|^{1-\frac{1}{q}} \frac{\partial^i u}{\partial x_k^{i}} \right\|_{L^q(R^n_+;E)} \leq \frac{C}{\left| \lambda \right|} \|(Q_{\varepsilon} + \lambda) u\|_{L^q(R^n_+;E)}.$$

From Theorem 2.1 we obtain the following

**Result 2.2.** For $\lambda \in S_{\psi,\kappa}$ there is a resolvent $(Q_{\varepsilon} + \lambda)^{-1}$ of the operator $Q_{\varepsilon}$ satisfying the following uniform estimate

$$\sum_{k=1}^{n} \sum_{i=0}^{n} \varepsilon_k^{1/2} \frac{1}{\alpha_i} \left| \lambda \right|^{1-\frac{1}{q}} \frac{\partial^i u}{\partial x_k^{i}} \right\|_{B(L^q(R^n_+;E))} \leq C. \quad \text{(2.9)}$$
3. Initial-boundary value problems for Stokes system with small parameters

Consider the following BVP for the stationary Stokes equation with parameter
\[ -\triangle u + Au + \nabla \varphi + \lambda u = f(x), \quad \text{div} \, u = 0, \quad x \in R^n_+, \quad \text{(3.1)} \]

\[ L_{1\varepsilon}u = \sum_{i=0}^{\nu} \varepsilon_n^\alpha_i \frac{\partial^i u}{\partial x^i_n} (x', 0, t) = 0, \quad \nu \in \{0, 1\}. \quad \text{(3.2)} \]

The function \( u \in W^{2,q}_s (R^n_+; E(A), E, L_{1\varepsilon}) = \{ u \in W^{2,q}_s (R^n_+; E(A), E), L_{1\varepsilon}u = 0, \quad \text{div} \, u = 0 \} \)
satisfying the equation (3.1) a.e. on \( R^n_+ \) is called the stronger solution of the problem (3.1) – (3.2).

Let \( W^{s,q}_s (R^n_+; E) \), \( 0 < s < \infty \) be the \( E \)-valued Sobolev space of order \( s \) such that \( W^{s,0}(R^n_+; E) = L^q(R^n_+; E) \). For \( q \in (1, \infty) \) let \( X_q = L^q(R^n_+; E) \) denote the closure of \( C^{0,s}_0(R^n_+; E) \) in \( L^p(R^n_+; E) \), where
\[ C^{0,s}_0(R^n_+; E) = \{ u \in C^\infty_0(R^n_+; E), \quad \text{div} \, u = 0 \}. \]

By virtue of [19], vector field \( u \in L^q(R^n_+; E) \) has a Helmholtz decomposition, i.e. all \( u \in L^q(R^n_+; E) \) can be uniquely decomposed as \( u = u_0 + \nabla \varphi \) with \( u_0 \in L^q(R^n_+; E) \), \( u_0 = P_q u \) where \( P_q = P \) is a projection operator from \( L^q(R^n_+; E) \) to \( L^q(R^n_+; E) \) and \( \varphi \in L^q_{loc}(R^n_+; E) \), \( \nabla \varphi \in L^q(R^n_+; E) \) so that
\[ \| \nabla \varphi \|_q \leq C \| u \|_q, \quad \| \varphi \|_{L^q(G \cap B)} \leq C \| u \|_q \]
with \( C \) independent of \( u \), where \( B \) is an open ball in \( R^n \) and \( \| u \|_p \) denotes the norm of \( u \) in \( L^q(R^n_+; E) \).

Then the problem (3.1) – (3.2) can be reduced to the following BVP
\[ -P \triangle \varepsilon u + PAu + \lambda u = f(x), \quad x \in R^n_+, \quad \text{(3.3)} \]

\[ L_{1\varepsilon}u = \sum_{i=0}^{\nu} \varepsilon_n^\alpha_i \frac{\partial^i u}{\partial x^i_n} (x', 0) = 0, \quad \nu \in \{0, 1\}. \quad \text{(3.4)} \]

Consider the parameter dependent Stokes operator \( O_{\varepsilon} = O_{\varepsilon,q} \) generated by problem (3.3) – (3.4), i.e.,
\[ D(O_{\varepsilon}) = W^{2,q}_s (R^n_+; E(A), E, L_{1\varepsilon}), \quad O_{\varepsilon}u = -P \triangle \varepsilon u + PAu. \]

From the Rezult 2.2 we get that the operator \( O_{\varepsilon} \) is positive and generates a bounded holomorphic semigroup \( S_{\varepsilon}(t) = \exp(-O_{\varepsilon}t) \) for \( t > 0 \).

In a similar way as in [6] we show
Proposition 3.1. The following estimate holds
\[ \| O_\varepsilon^\alpha \| \leq C t^{-\alpha}, \]
uniformly in \( \varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \) for \( \alpha \geq 0 \) and \( t > 0 \).

Proof. From Result 2.2 we obtain that the operator \( O_\varepsilon \) is uniformly positive in \( L_q (R^n; E) \), i.e. for \( \lambda \in (0, \infty) \) the following uniform estimate holds
\[ \left\| (O_\varepsilon + \lambda)^{-1} \right\| \leq M |\lambda|^{-1}, \]
where the constant \( M \) is independent of \( \lambda \) and \( \varepsilon \). Then, by using Danford integral and operator calculus as in [6] we obtain the assertion.

From [19] we obtain the following result

Theorem 3.1. Let \( E \) be a a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). For any \( 0 \leq \alpha \leq 1 \) the domain \( D(O_\varepsilon^\alpha) \) is the complex interpolation space \([X^q, D(O_\varepsilon)]_{\alpha}\).

4. Existence and Uniqueness for Navier-Stokes equation with parameters

In this section, we study the Navier-Stokes problem (1.1) – 1.3 in \( X^q \). The problem (1.1) – (1.3) can be expressed as
\[ \frac{du}{dt} + O_\varepsilon u = Fu + Pf, \quad u(0) = 0, \quad t > 0, \quad Fu = -P(u, \nabla)u. \] (4.1)
We consider this equation in integral form
\[ u(t) = S_\varepsilon(t) a + \int_0^t S_\varepsilon(t - s) [Fu(s) + Pf(s)] ds, \quad t > 0. \] (4.2)

For the proving the main result we need the following lemma which is obtained from [4, Theorem 2].

Lemma 4.1. Let \( E \) be a a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). For any \( 0 \leq \alpha \leq 1 \) the domain \( D(O_\varepsilon^\alpha) \) is the complex interpolation space \([X^q, D(O_\varepsilon)]_{\alpha}\).
Lemma 4.2. Let $E$ be a a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. For each $k = 1, 2, \ldots, n$ the operator $u \to \Omega^{-\frac{\varepsilon}{2}} P \left( \frac{\partial}{\partial x_k} \right) u$ extends uniquely to a uniformly bounded linear operator from $L^q \left( R^n_+ ; E \right)$ to $X_q$.

Proof. Since $\Omega \varepsilon$ is a positive operator, it has a fractional powers $\Omega^{\alpha} \varepsilon$. From Lemma 4.1 it follows that the domain $D (\Omega^{\alpha} \varepsilon)$ is continuously embedded in $X_q \cap H^{2\alpha} (R^n_+ ; E (A) ,E)$ for any $\alpha > 0$. Then by using the duality argument and due to uniform positivity of $\Omega^{1/2} \varepsilon$ we obtain the following uniformly in $\varepsilon$ estimate holds
\[
\left\| \Omega^{-\frac{\varepsilon}{2}} P \left( \frac{\partial}{\partial x_k} \right) u \right\|_{L^q (R^n_+ ; E)} \leq C \left\| u \right\|_{X_q}.
\] (4.3)

By reasoning as in [3] we obtain the following

Lemma 4.3. Let $E$ be a a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $0 \leq \delta < \frac{1}{2} + \frac{n}{2q} \left( 1 - \frac{1}{q} \right)$. Then the following estimate holds
\[
\left\| \Omega^{-\delta} P (u, \nabla) v \right\|_q \leq M \left\| \Omega^\theta u \right\|_q \left\| \Omega^\sigma u \right\|_q
\]
uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ with constant $M = M (\delta, \theta, q, \sigma)$ provided that $\theta > 0$, $\sigma > 0$, $\sigma + \delta > \frac{1}{2}$ and
\[
\theta + \sigma + \delta > \frac{n}{2q} + \frac{1}{2}.
\]

Proof. Assume that $0 < \nu < \frac{n}{2} \left( 1 - \frac{1}{q} \right)$. Since $D (\Omega^\nu)$ is continuously embedded in $X_q \cap H^{2\alpha} (R^n_+ ; E (A) ,E)$ and $L^q (R^n_+ ; E) \cap X_q'$ is the same as $X_s'$, by Sobolev imbedding theorem we obtain that the operators
\[
\Omega^{-\nu} : X_q' \to D (\Omega^\nu) \to X_s'
\]
is bounded, where
\[
\frac{1}{s'} = 1 - \frac{1}{q} = \frac{1}{q'} + \frac{1}{q} = 1.
\]

By duality argument then, we get that the operator $u \to \Omega^{-\nu} u$ is bounded from $X_s$ to $X_q$, where
\[
\frac{1}{s} = 1 - \frac{1}{q} = \frac{1}{q'} + \frac{2\nu}{n}.
\]

Consider first the case $\delta > \frac{1}{2}$. Since $P (u, \nabla) v$ is bilinear in $u, v$, it suffices to prove the estimate on a dense subspace. Therefore assume that $u$ and $v$ are smooth. Since $\text{div} u = 0$, we get
\[
(u, \nabla) v = \sum_{k=1}^n \frac{\partial}{\partial x_k} (u_k v).
\]
Taking \( \nu = \delta - \frac{1}{2} \) and using the uniform boundedness of \( O_{\varepsilon,q}^{\nu} \) from \( X_s \) to \( X_q \) and Lemma 4.2 for all \( \varepsilon > 0 \) we obtain

\[
\| O_{\varepsilon}^{-\delta} P (u, \nabla) v \|_q = \left\| \varepsilon_k O_{\varepsilon,q}^{\frac{\delta}{2} - \nu} \sum_{k=1}^{n} P \frac{\partial}{\partial x_k} (u_k v) \right\|_q \leq \| u \| \| v \|_s.
\]

By assumption we can take \( r \) and \( \eta \) such that

\[
\frac{1}{r} \geq \frac{1}{q} - \frac{2\theta}{n}, \quad \frac{1}{\eta} \geq \frac{1}{q} - \frac{2\sigma}{n}, \quad \frac{1}{r} + \frac{1}{\eta} = \frac{1}{s}, \quad r > 1, \quad \eta < \infty.
\]

Since \( D \left( O_{\varepsilon,q}^{\gamma} \right) \) is continuously embedded in \( X_q \cap H^{2n}_q \left( R^n; E (A) , E \right) \), then by Sobolev imbedding we get

\[
\| u \| \| v \|_s \leq \| u \|_r \| v \|_\eta \leq M \| O_{\varepsilon,q}^\theta u \|_r \| O_{\varepsilon,q}^{\sigma} v \|_\eta,
\]

i.e., we have the required result for \( \delta > \frac{1}{2} \). In particular, we get

\[
\| O_{\varepsilon}^{-\frac{\delta}{2}} P (u, \nabla) v \|_q \leq M \| O_{\varepsilon,q}^\theta u \|_r \| O_{\varepsilon,q}^{\sigma} v \|_\eta, \quad \theta + \beta \geq \frac{n}{2q}, \quad \beta > 0.
\]

Similarly we obtain

\[
\| P (u, \nabla) v \|_q \leq C \| u \|_r \| v \|_\eta \leq C \| O_{\varepsilon,q}^\theta u \|_r \| O_{\varepsilon,q}^{\beta + \frac{1}{2}} v \|_\eta
\]

for \( \frac{1}{r} + \frac{1}{\eta} = \frac{1}{q} \) and \( \delta = 0 \). The above two estimates show that the map \( v \to P (u, \nabla) v \) is a uniform bounded operator from \( D \left( O_{\varepsilon,q}^\theta \right) \) to \( D \left( O_{\varepsilon,q}^{\beta} \right) \) and from \( D \left( O_{\varepsilon,q}^{\beta + \frac{1}{2}} \right) \) to \( X_q \). By using the Lemma 4.1 and the interpolation theory for \( 0 \leq \delta \leq \frac{1}{2} \) we obtain

\[
\| P (u, \nabla) v \|_q \leq C \| O_{\varepsilon,q}^\theta u \|_r \| O_{\varepsilon,q}^{\sigma} v \|_\eta.
\]

By using Lemma 4.3 and iteration argument, by reasoning as in Fujita and Kato [6] we obtain the following

**Theorem 4.1.** Let \( E \) be a a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). Let \( \gamma < 1 \) be a real number and \( \delta \geq 0 \) such that

\[
\frac{n}{2q} - \frac{1}{2} \leq \gamma, \quad -\gamma < \delta < 1 - |\gamma|.
\]

Suppose that \( a \in D \left( O_{\varepsilon}^{\theta} \right) \), and that \( \| O_{\varepsilon}^{-\delta} P f (t) \|_q \) is continuous on \( (0, T) \) and satisfies

\[
\| O_{\varepsilon}^{-\delta} P f (t) \| = o \left( t^{\gamma + \delta - 1} \right) \text{ as } t \to 0.
\]

Then there is \( T_\ast \in (0, T) \) independent of \( \varepsilon \) and local solution of (4.1) such that
where we suppose \( \gamma \leq \varepsilon \) uniformly with respect to parameters \( \theta, \sigma, \delta \).

Moreover, the solution of (4.3) and (4.4) is unique if \( u \in C ([0, T_*]; D (O^\varepsilon)) \), \( u (0) = a, \)
and \( u \in C ([0, T_*]; D (O^\varepsilon)) \) for some \( T_* > 0 \),

\[
\| O^\varepsilon u (t) \| = O (t^{\gamma - \alpha}) \quad \text{as} \quad t \to 0 \quad \text{for all} \quad \alpha \quad \text{with} \quad \gamma < \alpha < 1 - \delta \quad \text{uniformly in} \quad \varepsilon.
\]

Proof. We introduce the following iteration scheme

\[
u_0 (t) = S^\varepsilon (t) a + \int_0^t S^\varepsilon (t - s) P f (s) \, ds,
\]

\[
u_{m+1} (t) = u_0 (t) + \int_0^t S^\varepsilon (t - s) F u_m (s) \, ds, \quad m \geq 0.
\]

By estimating the term \( u_0 (t) \) in (4.3) and by using the Lemma 4.3 for \( \gamma < \alpha < 1 - \delta \) we get

\[
\| O^\varepsilon u_0 (t) \| \leq \| O^\varepsilon S^\varepsilon (t) a \| + \int_0^t \| O^{\alpha + \delta} S^\varepsilon (t - s) \| \| O^{-\delta} P f (s) \| \, ds \leq
\]

\[
\| O^\varepsilon S^\varepsilon (t) a \| + C_{\alpha + \delta} \int_0^t (t - s) | s |^{- (\alpha + \delta)} \| O^{-\delta} P f (s) \| \, ds \leq M_\alpha t^{\gamma - \alpha}
\]

uniformly with respect to parameters \( \varepsilon_1, \varepsilon_2, ..., \varepsilon_n \) with

\[
M_\alpha = \sup_{0 < t \leq T, \varepsilon_1 > 0} t^{\alpha - \gamma} \| O^{\alpha + \delta} S^\varepsilon (t) a \| + C_{\alpha + \delta} NB (1 - \delta - \alpha, \gamma + \alpha),
\]

where \( N = \sup_{0 < t \leq T} t^{1 - \gamma - \delta} \| O^{-\delta} f (t) \| \) and \( B (a, b) \) is the beta function. Here we suppose \( \gamma + \delta > 0 \). By induction assume that \( u_m (t) \) satisfies the following

\[
\| O^\varepsilon u_m (t) \| \leq M_\alpha t^{\gamma - \alpha}, \quad \gamma < \alpha < 1 - \delta.
\]

We shall estimate \( O^\varepsilon u_{m+1} (t) \) by using (5.2). To estimate the term \( \| O^{-\delta} F u_m (s) \| \) we suppose

\[
\theta + \sigma + \delta = 1 + \gamma, \quad \gamma < \theta < 1 - \delta, \quad \gamma < \sigma < 1 - \delta,
\]

\[
\theta > 0, \quad \sigma > 0, \quad \delta + \sigma > \frac{1}{2},
\]

so that the numbers \( \theta, \sigma, \delta \) satisfy the assumptions of Lemma 4.3. Using Lemma 4.3 and (4.4), we get

\[
\| O^{-\delta} F u_m (s) \| \leq CM_\theta M_\sigma s^{\gamma + \delta - 1}.
\]

Therefore, we obtain

\[
\| O^\varepsilon u_m (t) \| \leq M_\alpha t^{\gamma - \alpha} + M_{\alpha + \delta} \int_0^t (t - s) | s |^{- (\alpha + \delta)} \| O^{-\delta} F u_m (s) \| \, ds
\]
\[
\leq M_{\alpha + 1}t^{-\alpha}
\]
with
\[
M_{\alpha + 1} = M_\alpha + M_{\alpha + \delta}MB(1 - \delta - \alpha, \gamma + \delta)M_{\theta m}M_{\sigma m}.
\]

We get the uniform estimate. So, the remaining part of proof is obtained the same as in [3, Theorem 2.3].

By reasoning as in [6] we obtain

**Lemma 4.4.** Let the parameter dependent operator \( A_\varepsilon \) be uniform positive in a Banach space \( E \) and \( \alpha \) be a positive number with \( 0 < \alpha < 1 \). Then, the following uniform inequality holds

\[
\|A^\alpha_\varepsilon (e^{-A_\varepsilon t} - I) u\|_E \leq \frac{t^\alpha}{\alpha} \|A^\alpha_\varepsilon u\|_E
\]

for all \( u \in E \).

**Proposition 4.1.** Let \( E \) be a space satisfying a multiplier condition, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). Let \( u \) be the solution given by Theorem 4.1. Then \( O^\varepsilon_\alpha u \) for \( \gamma < \alpha < 1 - \delta \) is uniform Hölder continuous on every interval \([\eta, T_*]\), \( 0 < \eta < T_* \) for all parameters \( \varepsilon_k > 0 \).

**Proof.** It suffices to prove the Hölder continuity of \( O^\varepsilon_\alpha v \), where

\[
v(t) = \int_0^t S_\varepsilon(t-s) \left[ Fu(s) + Pf(s) \right] ds.
\]

Using the Lemma 4.4 we get the uniform estimate

\[
\left\| (e^{-\hbar O^\varepsilon_\alpha} - I) O^{-\alpha} \right\|_{B(E)} \leq \frac{\hbar^\alpha}{\alpha}, \ h > 0.
\]

Then as a similar way as in [3, Proposition 2.4] we obtain the assertion.

**Theorem 4.2.** Let \( E \) be a a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). Assume \( Pf : [0, T_\ast] \rightarrow X_q \) is Hölder continuous on each subinterval \([\eta, T_*]\). Then, the solution of (4.2) given by Theorem 4.1 satisfies equation (4.1) for all parameters \( \varepsilon_k > 0 \). Moreover, \( u \in D(O^\varepsilon_\alpha) \) for \( t \in (0, T_\ast] \).

**Proof.** It suffices to show Hölder continuity of \( Fu(t) \) on each interval \([\eta, T_*]\). It is clear to see that \( u(\eta) \in X_q \) and

\[
u(t) = S_\varepsilon(t)u(\eta) + \int_0^t S_\varepsilon(t-s) \left[ Fu(s) + Pf(s) \right] ds, \ t \in [\eta, T_*].
\]

Since \( Pf \) is continuous on \([\eta, T_*]\) we get

\[
\|Pf(t)\| = o(t - \eta)^{-\alpha}, \ t \rightarrow \eta, \ \alpha > 0.
\]
The uniqueness of \( u(t) \), ensured by Theorem 4.1, implies the following estimates
\[
C ([\eta, T_*]; D (O^\nu_\varepsilon)) \cap C ((\eta, T_*]; D (O^\nu_\varepsilon)),
\]
\[
O^\nu_\varepsilon \| u(t) \| = o (t - \eta)^{\nu - \alpha}, \quad t \to \eta, \nu < \alpha < 1
\]
uniformly in \( \varepsilon_k \), where \( \nu = \max \{ \gamma, 0 \} \). So, by Proposition 5.1, \( O^\nu_\varepsilon u(t) \) is continuous on every subinterval \([\eta, T_*]\). Since we can choose \( \theta, \sigma \) so that
\[
\theta + \sigma = 1 + \nu, \quad \nu < \theta < 1, \quad \max \left\{ \gamma, \frac{1}{2} \right\} < \sigma < 1.
\]

Lemma 4.2 implies that \( Fu(t) \) is Hölder continuous on every interval \([\eta, T_*]\).

5. Regularity properties

The purposes of this section is to show that the solutions of the equation (1.1) are smooth if the data are smooth. For simplicity, we assume \( Pf = 0 \). The proof when \( Pf \neq 0 \) is the same. Consider first all of the Stokes problem (3.3) \( - \) (3.4).

By reasoning as in [6, Lemma 2.14] we obtain

**Lemma 5.1.** Let \( E \) be a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). Let \( f \in C^\mu ([0, T]; X_\mu) \), for some \( \mu \in (0, 1) \). Then for every \( \eta \in (0, \mu) \) we have
\[
v(t) = \int_0^t S_\varepsilon (t - s) f (s) \, ds \in C^\eta ([0, T]; D (O_{\varepsilon})) \cap C^{1+\eta} ((0, T]; X_\mu).
\]

In a similar way as Lemma 3.3, 3.6,3.7 in [3] we obtain, respectively:

**Lemma 5.2.** Let \( E \) be a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). For \( u, v \in W^{m,q} (R^a_+; E (A), E) \), \( q \in (1, \infty) \) the following hold:
1. \( Pu \in W^{m,q} (R^a_+; E (A), E) \cap X_q \) and \( \| Pu \|_{W^{m,q} (R^a_+; E)} \leq C_{m,q} \| u \|_{W^{m,q} (R^a_+; E)} \);
2. for \( m > \frac{n}{q} \) there exists a constant \( C_{m,q} \) such that
\[
\| P (u, \nabla) v \|_{W^{m,q} (R^a_+; E)} \leq C_{m,q} \| u \|_{W^{m,q} (R^a_+; E)} \| v \|_{W^{m+1,q} (R^a_+; E)};
\]
3. when \( q > n \) we have
\[
\| P (u, \nabla) v \|_{L^q (R^a_+; E)} \leq C_q \| u \|_{W^{1,q} (R^a_+; E)} \| v \|_{W^{1,q} (R^a_+; E)}.
\]

**Lemma 5.3.** Let \( E \) be a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). Let \( u = u_\varepsilon (t) \) be solution of (4.2) for \( Pf = 0 \), then \( u \in C^\mu ((0, T]; D (O_\varepsilon)) \) and \( \frac{du}{dt} \in C^\mu ((0, T]; X_\mu) \) for \( \mu \in (0, \frac{1}{2}) \). Moreover,
\[
Fu \in C^\mu ((0, T]; W^{1,q} (R^a_+; E (A), E)).
\]
Lemma 5.4. Let $E$ be a a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $u = u_\varepsilon(t)$ be solution of (4.2) for $Pf = 0$, then $u \in C^\mu \left(0, T \right); D \left(\mathcal{O}_x^{\frac{1}{2}} \right)$ for $\mu \in (0, \frac{1}{2})$.

Now by reasoning as in [3, Proposition 3.5 ] we can state the following

Proposition 5.1. Let $E$ be a a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1, \infty)$ and $0 < \varepsilon_k \leq 1$. Let $E$ be Banach algebra, $q > n$ and $a \in X_q$. Suppose that the solution $u = u_\varepsilon(t)$ of (4.2) for $Pf = 0$ given by Theorem 4.1 exists on $[0, T]$. Then $u \in C^\infty \left(R^n \times [0, T]; E \right)$.

Proof. The solution $u = u_\varepsilon(t)$ of (4.2) for $Pf = 0$ given by Theorem 4.1 is expressed as

$$u(t) = S_\varepsilon(t) a + \int_0^t S_\varepsilon(t - s) Fu(s) ds, \; t > 0, \quad (5.1)$$

where $Fu = -P(\partial, \nabla) u$. From (5.1) we get

$$O_x^{\frac{1}{2}} u(t) = S_\varepsilon(t - \eta) O_x^{\frac{1}{2}} u(\eta) + \int_\eta^t O_x S_\varepsilon(t - s) O_x^{\frac{1}{2}} Fu(s) ds, \; t > 0 =$$

$$S_\varepsilon(t - \eta) O_x^{\frac{1}{2}} u(\eta) + v(t), \; v(t) = v_\varepsilon(t) = \int_\eta^t O_x S_\varepsilon(t - s) O_x^{\frac{1}{2}} Fu(s) ds.$$

Since $S_\varepsilon(t - \eta) O_x^{\frac{1}{2}} u(\eta) \in C^\infty ((\delta, T]; X_q)$ and $0 < \eta < T$, we will examining only $v(t)$. Integrating by parts, we obtain

$$v(t) = \int_\eta^t \frac{d}{ds} S_\varepsilon(t - s) O_x^{\frac{1}{2}} Fu(s) ds = \varepsilon O_x^{\frac{1}{2}} Fu(t) -$$

$$S_\varepsilon(t - \eta) O_x^{\frac{1}{2}} Fu(\delta) - \int_\eta^t S_\varepsilon(t - s) O_x^{\frac{1}{2}} \frac{d}{ds} (Fu)(s) ds.$$

Moreover, since $u(s) \in D(O_x)$ for all $\varepsilon > 0$, $0 < s \leq T$, we have

$$(Fu)(s) = - \sum_{k=1}^n P \left( \frac{\partial}{\partial x_k} \right) [u_k(s) u(s)],$$

where $u(s) = (u_1(s), u_2(s), ..., u_n(s))$, $u_k = u_{\varepsilon_k}$. Hence, by Lemma 4.1 we get the following uniform estimate

$$\left\| O_x^{\frac{1}{2}} \frac{d}{ds} Fu \right\|_{X_q} = \left\| \sum_{k=1}^n O_x^{\frac{1}{2}} P \left( \frac{\partial}{\partial x_k} \right) \left[ du_{\varepsilon_1} u_k du_{\varepsilon_1} du_k \right] \right\|_{X_q}$$

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\[ \leq C \|u_t\|_{L^\infty(R^d;E)} \left\| \frac{du_t}{ds} \right\|_{X_q} \leq C \left\| O_x^{\frac{1}{2}} u_t \right\|_{X_q} \left\| \frac{du_t}{ds} \right\|_{X_q}. \]

This estimates together with Lemma 5.3 shows that
\[ O_x^{\frac{1}{2}} \frac{d}{ds} F u \in C^\mu ((0, T]; X_q). \]

Lemma 5.1 and Lemma 5.2 now imply that
\[ \frac{dv}{dt} \in C^\mu ((0, T]; X_q). \]

Since \( D \left( O_x^{\frac{1}{2}} \right) \subset W^{1,q} (R^n; E (A), E) \), Corollary 5.1, Lemmas 5.3, 5.4 and the identity \( u(t) = O_x^{\frac{1}{2}} (F u - \frac{dv}{dt}) \) imply
\[ u \in C^\mu ((0, T]; W^{1,q} (R^n; E (A), E)). \]

Then the proof will be completed as in \[3, Proposition 3.5\] by using the induction.

Now we can state the main result of this section

**Theorem 5.1.** Let \( E \) be a a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \). Let \( E \) be Banach algebra and \( a \in X_q \). Suppose that the solution \( u = u(x, t) \) (of 4.2) for \( PF = 0 \) given by Theorem 4.1 exists on \([0, T]\). Then \( u \in C^\infty (R^n \times [0, T]; E) \).

**Proof.** For \( q > n \) the assertion is obtained from the Proposition 5.1. Let us show that the assertion is valid for \( 1 < q \leq n \). Indeed, the solution \( u = u(x, t) \) of (5.2) for \( PF = 0 \) given by Theorem 4.1 satisfies the equation (5.1) on every subinterval \([\eta, T]\), \( 0 < \eta < T \). Theorem 4.2 shows that \( u(\eta) \in D (O_x) \). Since \( 0 < \frac{\gamma}{n} - \frac{1}{2} \leq \gamma < 1 \), we have \( D (O_x) \subset X_n \) so that \( D (O_x) \subset X_s \) for some \( s > n \). By (4.2) this means that we may assume \( q > n \) and \( a \in X_q \).

### 6. Existence of global solutions

In this section, we prove the existence and estimate of global solution of the problem (1.1) - (1.3). The proofs of these theorems are based on the theory of holomorphic semigroups and fractional powers of generators. We assume for simplicity that \( f = 0 \), although it is not difficult to include nonzero \( f \) under appropriate conditions. The main result is the following

**Theorem 6.1.** Let \( E \) be a UMD space, \( A \) an \( R \)-positive operator in \( E \), \( q \in (1, \infty) \) and \( 0 < \varepsilon_k \leq 1 \) and \( a \in L^q (R^n) \). There is a \( T > 0 \) and a unique solution \( u = u(x) \) of (1.1) - (1.3) so that \( t^{(1-\frac{q}{2})} u \in C \left( [0, T]; L^q \left( R^n; E \right) \right) \) for \( n \leq q \leq \infty \) and \( t^{(1-\frac{q}{2})} \nabla u \in C \left( [0, T]; L^q \left( R^n; E \right) \right) \) for \( n \leq q < \infty \). Moreover, the following estimates hold
\[ \sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{q}{2})} u \right\|_{L^q} \leq C, \sup_{t \in [0, T], \varepsilon_k > 0} \left\| t^{(1-\frac{q}{2})} \nabla u \right\|_{L^q} \leq C. \]
Proof. The solution $u = u_\varepsilon(t)$ of (4.2) for $Pf = 0$ given by Theorem 4.1 is expressed as
\[ u(t) = u_0(t) + G_\varepsilon u(t), \]  
where,
\[ u_0(t) = S_\varepsilon(t)a, \quad G_\varepsilon u(t) = \int_0^t S_\varepsilon(t-s)Fu(s)\,ds, \quad t > 0. \]

By applying the generalized Minkovskii inequality and by Proposition 3.1 we can see that
\[ \|S_\varepsilon(t)u\|_{L^p} \leq C_\varepsilon n^2 \left(1 + \frac{1}{p} \right) \kappa \|u\|_{L^p}, \quad k = 1, 2, \ldots, n. \]

By using the above estimate we get
\[ \|S_\varepsilon(t)u\|_{L^q} \leq C_\varepsilon n^2 \left(\frac{2}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) t^{-\frac{\alpha}{\gamma} - \frac{1}{q}}, \quad 1 < p < q < \infty. \]

Moreover, by using (6.1), (6.2) and by applying the Hölder inequality, we get
\[ \|F(u, v)\|_{L^r} \leq C \|u\|_{L^r} \|\nabla v\|_{L^s}, \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{s}. \]

Then in view of (6.1)-(6.4) we obtain the following uniform estimate
\[ \|G_\varepsilon u\|_{L^{m/\gamma}} \leq C \int_0^t (t-s)^{-\alpha + \beta - \gamma/2} \|u(s)\|_{m/\alpha} \|\nabla u(s)\|_{m/\beta} \,ds, \]  
\[ \|\nabla G_\varepsilon u\|_{L^{m/\gamma}} \leq C \int_0^t (t-s)^{-1 + \alpha + \beta - \gamma/2} \|u(s)\|_{m/\alpha} \|\nabla u(s)\|_{m/\beta} \,ds, \]

where
\[ \alpha, \beta, \gamma > 0, \quad \gamma \leq \alpha + \beta < n. \]

Then solving the equation (6.1) by successive approximation, starting with $u_0 = S_\varepsilon(t)a$ we get
\[ u_{k+1} = u_0 + G_\varepsilon u_k, \quad u_k = u_{k\varepsilon}(t), \quad k = 0, 1, 2, \ldots, \]

First by reasoning as in [22, Theorem 1] and by using (6.3)-(6.5) we show by induction that $u_k = u_{k\varepsilon}$ exists, moreover,
\[ t^{(1-\delta)/2}u_{k\varepsilon} \in C \left(\left[0, \infty \right); L^{n/\delta}(R^n_+; E)\right), \quad t^{1/2}\nabla u_{k\varepsilon} \in C \left(\left[0, \infty \right); L^\gamma (R^n_+; E)\right)\]
and for $\delta \in (0,1)$ the following uniform estimates hold
\[
\sup_{t, \varepsilon k} \left\| t^{(1-\delta)/2} u_{\varepsilon k} \right\|_{L^n} \leq M_k, \quad \sup_{t, \varepsilon k} \left\| t^{1/2} \nabla u_{\varepsilon k} \right\|_{L^n} \leq M_k'.
\] (6.8)

By applying (6.3)-(6.5) for $q = n$ and $p = \frac{q}{n}$ we have
\[
M_0 = M_0' = C \|a\|_{L^q(R^n_+;E)},
\] (6.9)
where $C$ is a positive constant. From (6.5) and (6.7) for $n \leq p < \infty$ we obtain
\[
\left\| u_{\varepsilon k+1} \right\|_{L^p} \leq \left\| u_{\varepsilon 0} \right\|_{L^p} \leq \frac{C M_k M_k'}{t} \int_0^t (t-s)^{-\frac{1}{2}\left(1+\frac{n}{q}\right)} s^{-\frac{1}{2}(1-\delta)} ds \leq M t^{-\frac{1}{2}(1-n/q)}.\]

It follows that $u_{\varepsilon k}(t)$ converges to a limit function $u_\varepsilon$ uniformly with respect to $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, moreover, $u_\varepsilon \in C \left( [0,T); L^n(R^n_+;E) \right)$ for $p = n$ and $u_\varepsilon$ satisfies (6.1) for $n < p < \infty$.

**Theorem 6.2.** Let $E$ be a a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. There is a $\mu > 0$ such that if $\|a\|_{L^q(R^n_+;E)} < \mu$, then there is a global solution $u_\varepsilon$ of the problem (1.1) – (1.3), so that $t^{(1-\delta)/2} u_\varepsilon \in C \left( [0,\infty); L^q(R^n_+;E) \right)$ for $n \leq q \leq \infty$, $t^{(1-\delta)/2} \nabla u_\varepsilon \in C \left( [0,\infty); L^q(R^n_+;E) \right)$ for $n \leq q < \infty$. Moreover, the following uniform estimates hold
\[
\sup_{t, \varepsilon k} \left\| t^{(1-n)/q} u_\varepsilon \right\|_{L^q} \leq C, \quad \sup_{t, \varepsilon k} \left\| t^{(1-\delta)/q} \nabla u_\varepsilon \right\|_{L^q} \leq C.
\] (6.10)

**Proof.** It is clear to see from proof of Theorem 6.1 that $M_k$ and $M_k'$ are bounded by a constant $M$ if $M_0 \leq \lambda$. By (7.9) this is true if $\|a\|_{L^q(R^n_+;E)}$ is sufficiently small. In this case, as in [10] we prove that the sequences $t^{(1-\delta)/2} u_{\varepsilon k}$, $t^{1/2} \nabla u_{\varepsilon k}$ are bounded on $(0,\infty)$ uniformly in $k$ and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ i.e.,
\[
\sup_{t, \varepsilon k} \left\| t^{(1-\delta)/2} u_{\varepsilon k} \right\|_{L^n} \leq M_1, \quad \sup_{t, \varepsilon k} \left\| t^{1/2} \nabla u_{\varepsilon k} \right\|_{L^n} \leq M_2.
\] (6.11)

Then (6.11) is obtained from (6.10).

**Remark 6.1.** Let $E$ be a a UMD space, $A$ an $R$-positive operator in $E$, $q \in (1,\infty)$ and $0 < \varepsilon_k \leq 1$. Theorem 6.2 shows that all $L^p$ norms of $u_\varepsilon(t)$ decay as $t \to \infty$ for $p > q$ uniformly in $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$.

For $p = q$ we obtain the following result

**Theorem 6.3.** Let all conditions of Theorem 6.2 hold. Then $\|u_\varepsilon(t)\|_p \to 0$ uniformly in $\varepsilon$ as $t \to \infty$. More precisely, we have
\[
\|u_\varepsilon(t) - u_{0\varepsilon}(t)\|_p = O \left( t^{-\frac{\delta}{2}} \right) \text{ as } t \to \infty,
\]
where, $u_{0\varepsilon}(t) = S_\varepsilon(t)a$ and $\delta < \min \left\{ 1, n - \frac{n}{q}, \frac{n}{q} - 1 \right\}$. 

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7. The Wentzell-Robin type mixed problem for Novier-Stokes equations

Consider the problem (1.5) − (1.7). Here, $W^{2,p}(\tilde{\Omega})$ denotes the Sobolev space with corresponding mixed norm.

The main aim of this section is to prove the following result:

**Theorem 7.1.** Let $a \in W^{1,\infty}(0,1)$, $a(x) \geq \delta > 0$, $b, c \in L^{\infty}(0,1)$. Suppose the condition 7.1 hold. Let $\gamma < 1$ be a real number and $\delta \geq 0$ such that

$$\frac{n}{2q} - \frac{1}{2} \leq \gamma, \quad -\gamma < \delta < 1 - |\gamma|.$$ 

Suppose $a \in D(O^2_{\epsilon})$ such that $\|O_{\epsilon}^{-\delta} Pf(t)\|$ is continuous on $(0,T)$ and satisfies

$$\|O_{\epsilon}^{-\delta} Pf(t)\| = o(t^{\gamma+\delta-1})$$ 

as $t \to 0$.

Then there is $T_\epsilon \in (0,T)$ independent of $\epsilon$ and local solution of $(4.1)$ such that

$$u \in C([0,T_*]; Y_{2,p}^\infty),$$

where $T_0$ is a maximal time interval that is appropriately small relative to $M$.

Moreover, if

$$\sup_{t \in [0, T_0]} (\|u\|_{Y_{2,p}} + \|u\|_{X_{\infty}} + \|u_t\|_{Y_{2,p}} + \|u_{tt}\|_{X_{\infty}}) < \infty$$

then $T_0 = \infty$.

**Proof.** Let $E = L^{p_1}(0,1)$. It is known [2] that $L^{p_1}(0,1)$ is an UMD space for $p_1 \in (1,\infty)$. Consider the operator $A$ defined by

$$D(A) = W^{2,p_1}(\Omega; B_j u = 0), \quad Au = a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + cu.$$ 

Therefore, the problem (1.7) − (1.8) can be rewritten in the form of (1.1) − (1.3), where $u(x) = u(x,.)$ is a function with values in $E = L^{p_1}(0,1)$, $f(x) = f(x,.)$ are functions with values in $E = L^{p_1}(0,1)$, From [7, 8] we get that the operator $A$ generates analytic semigroup in $L^{p_1}(0,1)$. Moreover, we obtain that the operator $A$ is $R$-positive in $L^{p_1}$. Then from Theorem 4.1 we obtain the assertion.

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