LOWER SEMI-CONTINUITY FOR $\mathcal{A}$-QUASICONVEX FUNCTIONALS UNDER CONVEX RESTRICTIONS

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ABSTRACT. We show weak lower semi-continuity of functionals assuming the new notion of a “convexly constrained” $\mathcal{A}$-quasiconvex integrand. We assume $\mathcal{A}$-quasiconvexity only for functions with values in a set $K$ which is convex. Assuming this and sufficient integrability of the sequence we show that the functional is still (sequentially) weakly lower semi-continuous along weakly convergent “convexly constrained” $\mathcal{A}$-free sequences. In a motivating example, the integrand is $\det^{-1}$ and the convex constraint is positive semi-definiteness of a matrix field.

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1. INTRODUCTION

We will study (sequential) weak lower semi-continuity criteria for functionals of the form

$$u \mapsto \int_{\Omega} f(u(x)) \, dx$$

with respect to weakly converging sequences $u_n \rightharpoonup u$ in $L^p$. With appropriate growth bounds for $f$, for instance $c \leq f(v) \leq C(1 + |v|^p)$, we ask for which integrands $f$ we have this lower semi-continuity property.

If one considers all sequences in $L^p$ then we need $f$ to be convex. When we restrict the space of functions further then one can weaken the requirements on $f$. It was shown by Morrey [11] that, considering only sequences of gradients, it suffices to require $f$ to be quasiconvex, that is,

$$f(\zeta) \leq \frac{1}{|Q|} \int_{Q} f(\zeta + \nabla w(x)) \, dx$$

for all $\zeta$ and all smooth $w$ with compact support on a cube $Q$. One can view this as a Jensen-type inequality but for gradients only. On a cube, a function is a gradient if and only if its curl is zero and thus we obtain an equivalent condition by considering weakly converging sequences in $L^p$ that are curl-free, i.e. $\text{curl} u = 0$, and thus assume that $f$ satisfies

$$f(\zeta) \leq \frac{1}{|Q|} \int_{Q} f(\zeta + u(x)) \, dx$$

for all $\zeta$ and all smooth curl-free $u$ with compact support on a cube $Q$.

In order to study other physical systems we can generalise these methods to study $\mathcal{A}$-free systems with $\mathcal{A}$ a linear, homogeneous, differential operator with constant coefficients and, in most cases, constant rank. The $\mathcal{A}$-free approach was introduced in the work of Murat [12] and Tartar [20], [21] studying compensated compactness, and has applications in elasticity, plasticity, elasto-plasticity, electromagnetism etc. From the work of Dacorogna [2] it is known that this generalisation of quasiconvexity to $\mathcal{A}$-quasiconvexity is sufficient for lower semi-continuity and in the work of Fonseca and Müller [7] necessity is proved as well, if one assumes that $w$ is periodic over the cube.
We say that $\mathcal{L}$ is a potential operator for the $\mathcal{A}$-free system if $\mathcal{L}$ is a linear homogeneous differential operator with constant coefficients that maps into the kernel of $\mathcal{A}$ and such that for any smooth $u$ where $\mathcal{A}u = 0$ there exists a smooth potential $\varphi$ such that $\mathcal{L}\varphi = u$. We know, for instance, that the gradient is the potential operator for curl-free systems. With the recent work of Raïta [13] one can derive potentials in the general $\mathcal{A}$-free setting assuming $\mathcal{A}$ has constant rank. Using these potentials one establishes a new definition for $\mathcal{A}$-quasiconvexity of the form

$$f(\zeta) \leq \frac{1}{|\Omega|} \int_{\Omega} f(\zeta + \mathcal{L}\varphi(x)) \, dx$$

for all $\zeta$ and $\varphi \in C^\infty_c(\Omega)$ where $\Omega$ is an arbitrary domain with Lipschitz boundary. We will go into more detail later as we will use and develop the tools in [13] for our own situation.

Many physical models fall into the constant rank $\mathcal{A}$-free framework, but with a codomain restricted to a convex set. There are examples from compressible fluids studying the Euler, the Euler-Fourier, and relativistic Euler equations, models for rarefied gases (Boltzmann, discrete kinetic models, BGK), and other models for electromagnetic fields in a vacuum and the mass-momentum tensor in the Schrödinger equation. In these cases it makes sense to consider the problem of $\mathcal{A}$-quasiconvexity and (sequential) lower-semicontinuity only for physically relevant values inside such a convex set. Let $K \subset \mathbb{R}^N$ be a convex set with non-empty interior (i.e. of full dimension), then we define $K$-$\mathcal{A}$-quasiconvexity and $K$-$\mathcal{A}$-free sequences, in analogy to the standard definitions, by adding the constraint that $U \in K$ for almost every $x \in \Omega$. We will define these terms more rigorously later on. We can now state our main result of this paper.

**Theorem 1.1 (Main Theorem).** Let $p > d$, $f : \mathbb{R}^N \to [0, \infty)$ be continuous and satisfy the growth bound $f(z) \leq C(1 + |z|^r)$ for $C > 0$, $r < p$ and let $f$ be $K$-$\mathcal{A}$-quasiconvex. Let $(U_n) \in L^p(\Omega)$ be a sequence that is $K$-$\mathcal{A}$-free on $\Omega$ for all $n$ and converges weakly in $L^p$ to $U \in L^p(\Omega)$, then

$$\liminf_{n \to \infty} \int f(U_n) \, dx \geq \int f(U) \, dx.$$ 

In some cases the constant rank condition does not apply to $\mathcal{A}$, for example, the incompressible Euler equations. In this case to apply the theory of this paper one needs to derive a well-defined potential and for the incompressible Euler equations, this is done in [4]. Further, there are cases where the constraint does not define a convex set: For instance, imposing local non-interpenetration of matter in nonlinear elasticity, one assumes that $\det \nabla u > 0$ almost everywhere. This restriction models the assumption that under a deformation the matter will not change orientation or be compressed to a point, see [1] and [9] for more details.

One potential use for these techniques would be in the area of convex integration. In [4] as in other examples one has a functional used to control the sequence of sub-solutions and a convex set of values which each sub-solution can take. In [4] lower semi-continuity is shown directly for the relevant functional.

### 1.1. Compensated Integrability

One motivational result inspiring this paper has been [14], where D. Serre recently introduced the theory of compensated integrability. For dimension $d \geq 2$ and the function $F(A) = \left(\det A\right)^{\frac{1}{d-1}}$, Serre studied the specific functional $\int_{\Omega} F(U(x)) \, dx$, where $\Omega$ is either a bounded convex domain or the torus, and $U : \Omega \to \mathbb{R}^{d \times d}$ a matrix field whose values are positive semi-definite. More specifically, the method applies to symmetric, divergence-free, positive semi-definite tensors (DPTs) defined as follows:

**Definition 1.2.** Let $\Omega$ be an open subset of $\mathbb{R}^d$, then a DPT is defined as a locally integrable, divergence-free, positive symmetric tensor $x \mapsto U(x)$, that is, $U \in L^1_{\text{loc}}(\Omega; \mathbb{R}^{d \times d})$ with the properties

...
that
\[ U(x) \in \text{Sym}^+_d := \{ A \in \mathbb{R}^{d \times d} : A' = A, \ A \geq 0 \} \]
almost everywhere and \( \text{div} U = 0 \) (row-wise) in the sense of distributions.

For this functional on the torus a specific result of Serre was the following.

**Theorem 1.3.** Let the DPT \( x \mapsto U(x) \) be periodic over \( \mathbb{T}^d \), with \( U \in L^1(\mathbb{T}^d) \). Then \( F(U) \in L^1(\mathbb{T}^d) \) and there holds
\[
\frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} F(U) \, dx \leq F \left( \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} U(x) \, dx \right).
\]

This can be used to obtain a gain of integrability and can be applied to compressible fluid models to improve the a-priori estimates. These methods have been used recently in many cases, see [16], [15], [17] and [18].

For this specific \( F \) and \( U \), further questions were raised. In the work of [6], the authors ask whether or not this estimate could be improved to work for general \( L^p \). Specifically, they ask the following.

**Problem 1.4.** Let \( U \in L^p(\mathbb{T}^d; \text{Sym}^+_d) \) and \( \text{div} U \in L^p(\mathbb{T}^d) \) with \( 1 < p < d \). Defining \( \frac{1}{p'} = \frac{1}{p} - \frac{1}{d} \), is it true that
\[ \det(U)^{\frac{1}{d}} \in L^{p'}(\mathbb{T}^d)? \]

In [6] it is shown that there are counterexamples disproving this statement and thus that there exist many \( U \) in \( L^p \) for \( 1 < p < \frac{d}{d-1} \) such that \( (\det U)^{\frac{1}{d-1}} \in L^1 \setminus L^{1+\epsilon} \), for all \( \epsilon > 0 \).

Another question, closely related to the results in this paper, is whether and for what \( U \in L^p \) this functional is weakly upper semi-continuous. In the recent work [5], after defining the space
\[ X_p := \{ A \in L^p(\mathbb{T}^d; \text{Sym}^+_d) : \text{div} A \in \mathcal{M}(\mathbb{T}^d, \mathbb{R}^d) \}, \]
the following is proved:

**Theorem 1.5.** Let \( p > \frac{d}{d-1} \) and \( \{ U_k \}_k \subset X_p \) be such that \( U_k \rightharpoonup U \) in \( X_p \). Then we have
\[
\limsup_k \int_{\mathbb{T}^d} (\det U_k(x))^{\frac{1}{d-1}} \, dx \leq \int_{\mathbb{T}^d} (\det U(x))^{\frac{1}{d-1}} \, dx.
\]

Further, for the case of \( p \leq \frac{d}{d-1} \), explicit counterexamples are provided. This completes the picture of upper semi-continuity for this specific functional and \( U \) a DPT.

We see that if we let \( \mathcal{A} = \text{div} \), then this is a form of \( \mathcal{A} \)-quasiconcavity but on a restricted set for \( U \), which also has to be symmetric and positive definite. Thus this is a specific case of our problem we have set in this paper. In such a case, we can generate a potential for symmetric divergence-free matrices where for the potential \( \Phi \) we have \( \| \Phi \|_{W^{r,p}} \leq C \| U \|_{L^p} \) for a suitable integer \( r \). Thus we can apply our general theory to the specific case above and obtain results (although by our current techniques we need to require \( p > d \), which is sharp for \( d = 2 \)).

In Section 2 we will introduce notation and definitions to that we can state our main result (Theorem 2.6). In section 3 we will introduce Young measures and show that our main theorem can be reduced to showing that there exists appropriate sequences of functions that generate homogeneous Young measures, similarly to the methods of Fonseca and Müller in [7]. In Section 4 we introduce potential operators and potentials and prove an ellipticity bound to gain Sobolev control for all the derivatives (Theorem 4.1). This may be considered of independent interest. Finally,
in Section 5 we prove the main result by generating the sequences of functions that generate the homogenous Young measures we need.

2. Lower semi-continuity

We want to show lower semi-continuity of the functional

\[ I(U) := \int_\Omega f(U(x)) \, dx \]

from \( \mathcal{A} \)-quasiconvexity of \( f \). Here \( \Omega \) is an open, bounded subset of \( \mathbb{R}^d \), with Lipschitz boundary (or more generally satisfying the cone condition, see [10]), \( U(x) : \Omega \to \mathbb{R}^N \) and \( f : \mathbb{R}^N \to \mathbb{R} \). By \( \mathcal{A} \) we mean a \( k \)-homogeneous, linear, differential operator \( \mathcal{A} : C^\infty(\Omega;\mathbb{R}^N) \to C^\infty(\Omega;\mathbb{R}^m) \),

\[ \mathcal{A} := \sum_{|\alpha|=k} A^\alpha \partial_\alpha, \]

for \( U : \mathbb{R}^d \to \mathbb{R}^N \) where \( A^\alpha \in \text{Lin}(\mathbb{R}^N,\mathbb{R}^m) \). Here, we sum over all \( d \)-dimensional multi-indices \( \alpha \) such that \( |\alpha| = k \). We define the associated Fourier symbol map

\[ A[\xi] := \sum_{|\alpha|=k} \xi^\alpha A^\alpha \]

for \( \xi \in \mathbb{R}^d \). Here \( \mathcal{A} \) is assumed to satisfy the constant rank property, i.e. there exists an \( r \in \mathbb{N} \) such that \( \text{Rank} \mathcal{A}[\xi] = r \) for all \( \xi \in \mathbb{R}^d \setminus \{0\} \).

**Remark 2.1.** We have asked for these properties on \( \mathcal{A} \) so that we can find a potential operator \( \mathcal{L} \) for \( \mathcal{A} \) such that \( ||\Phi||_{W^{k,p}} \leq C ||U||_{L^p} \) for \( \mathcal{L} \Phi = U \). However, if there is a well-defined potential for \( \mathcal{A} \) and if this bound holds, then the results will still follow without the constant rank assumption. This pertains, for example, to the incompressible Euler equations and the potential defined in [4].

We will have extra convex restrictions on the set of values our functions can take. The usual form of \( \mathcal{A} \)-quasiconvexity as stated in [7] reads as follows:

**Definition 2.2.** A function \( f : \mathbb{R}^N \to \mathbb{R} \) is said to be \( \mathcal{A} \)-quasiconvex if

\[ f(\zeta) \leq \frac{1}{|1_d|} \int_{1_d} f(\zeta + w(x)) \, dx \]

for all \( \zeta \in \mathbb{R}^N \) and for all periodic \( w \in C^\infty(\mathbb{T}^d;\mathbb{R}^N) \) such that \( \mathcal{A}w = 0 \) and \( \int_{1_d} w(x) \, dx = 0 \).

Let \( K \) be a (not necessarily bounded) convex subset of \( \mathbb{R}^N \) with non-empty interior (i.e. a convex set of full dimension, so to speak). We can add \( K \) into the definitions of \( \mathcal{A} \)-quasi convexity to obtain a convexly restricted form. For this, we assume there exists a potential operator for \( \mathcal{A} \), i.e. a linear homogeneous differential operator \( \mathcal{L} \) of order \( l \) with \( \text{im} \mathcal{L} = \text{ker} \mathcal{A} \).

**Definition 2.3.** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \). Then \( f \) is \( K \)-\( \mathcal{A} \)-quasi convex if

\[ f(\zeta) \leq \frac{1}{|\Omega|} \int_\Omega f(\zeta + U(x)) \, dx \]

for all \( U \in C^\infty_c(\Omega) \), \( \zeta + U(x) \in K \) for almost every \( x \) and \( \mathcal{A}U = 0 \), or, equivalently, if

\[ f(\zeta) \leq \frac{1}{|\Omega|} \int_\Omega f(\zeta + \mathcal{L}\Phi(x)) \, dx \]

for all \( \Phi \in C^\infty_c(\Omega) \) where \( \zeta + \mathcal{L}\Phi(x) \in K \) for almost every \( x \).

**Remark 2.4.** As in [7] Remark 3.3 (ii), if \( f \) is upper semi-continuous and locally bounded above and \( f(V) \leq C(1 + |V|^p) \) for all \( V \in K \) for some \( C > 0 \), then one can replace \( C^\infty \) by \( L^p \) in (2.1).
We see that this is the same definition with potentials as in [13] except we have asked for the restriction to the set $K$.

**Definition 2.5.** We will define a function $U \in L^1_{\text{loc}}(\Omega)$ to be $K$-$\mathcal{A}$-free if $U(x) \in K$ for almost every $x$ and $U$ is weakly $\mathcal{A}$-free, that is, $\mathcal{A}U = 0$ in the sense of distributions.

Here we want to prove lower semi-continuity of an $f$ that is $K$-$\mathcal{A}$-quasiconvex for weakly converging sequences of functions $U_n$ which are $K$-$\mathcal{A}$-free.

**Theorem 2.6 (Main Theorem).** Let $p > d$, $f : \mathbb{R}^N \rightarrow [0, \infty)$ be continuous and satisfy the growth bound $f(z) \leq C(1 + |z|^r)$ for $C > 0$, $r < p$ and let $f$ be $K$-$\mathcal{A}$-quasiconvex. Let $(U_n) \in L^p(\Omega)$ be a sequence of $K$-$\mathcal{A}$-free maps on $\Omega$ that converges weakly in $L^p$ to $U \in L^p(\Omega)$, then

$$\liminf_{n \to \infty} \int_{\Omega} f(U_n) \, dx \geq \int_{\Omega} f(U) \, dx.$$

**Remark 2.7.** Consider an integrand with explicit dependence on the domain and the potential, for instance, $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^N \rightarrow \mathbb{R}$. Under our previous assumptions on the last variable, and the standard assumptions in [7] for the other variables, one can even treat such integrands of the form $f(x, u(x), U(x))$. Further, we can just assume that the $f$ is bounded from below and not necessarily by 0.

3. **Young Measures**

We will be using Young measures (parametrised probability measures) $\nu_x$ for $x \in \Omega$. These are useful objects that hold more information about the limit of weakly converging sequences of functions and a natural tool when discussing lower semicontinuity. We will state a version of the fundamental theorem of Young measures which demonstrates this link. Below is a modified version of the theorem by Fonseca and Müller in [7], see also [13].

**Theorem 3.1 (Fundamental Theorem on Young Measures).** Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite measure and let $\{z_n\}$ be a sequence of measurable functions $\Omega \rightarrow \mathbb{R}^N$ such that the sequence does not lose mass at infinity, i.e., $\lim_{M \to \infty} \sup_n |\{z_n \geq M\}| = 0$. Then there exist a subsequence (not relabelled) and a weakly*-measurable map$^1$ $\nu : \Omega \rightarrow \mathcal{P}(\mathbb{R}^N)$ such that the following hold:

1. let $D \subset \mathbb{R}^N$ be a compact subset then $\sup \nu_x \subset D$ for almost every $x \in \Omega$ if and only if $\text{dist}(z_n, D) \to 0$ in measure;
2. if $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a normal integrand (Borel measurable in the first and lower semicontinuous in the second variable), bounded from below, then

$$\liminf_{n \to \infty} \int_{\Omega} f(x, z_n(x)) \, dx \geq \int_{\Omega} f(x) \, dx \quad \text{where} \quad f(x) := \langle \nu_x, f(x, \cdot) \rangle = \int_{\mathbb{R}^N} f(x, y) \, d\nu_x(y);$$

3. if $f$ is Carathéodory ($f$ and $-f$ are normal integrands) and bounded from below, then

$$\lim_{n \to \infty} \int_{\Omega} f(x, z_n(x)) \, dx = \int_{\Omega} f(x) \, dx < \infty$$

if and only if $\{f(z_n(\cdot))\}$ is equi-integrable. In this case $f(\cdot, z_n(\cdot)) \rightharpoonup f$ in $L^1(\Omega)$.

Here $\nu_x$ is the Young measure generated by the sequence $\{z_n\}$. The Young measure is homogeneous if there is a Radon measure $\nu \in \mathcal{M}(\mathbb{R}^N)$ such that $\nu_x = \nu$ for almost every $x \in \Omega$.

$^1$This means that $x \mapsto \int_{\mathbb{R}^N} f(z) \, d\nu_x(z)$ is measurable for every bounded continuous $f : \mathbb{R}^N \rightarrow \mathbb{R}$. 
The growth bound $f(z) \leq C(1 + |z|^r)$ for $C > 0$, $r < p$ is assumed so that we can prove equi-integrability of $\{f(z_n)\}$. If $\{z_n\}$ is bounded in $L^p$, $f$ is continuous and we assume the growth bound then $f(z_n) \to \tilde{f}$ in $L^\infty$. If $\{z_n\}$ is equi-integrable, then letting $f = \text{id}$ we see that $z_n \to \tilde{z}$ in $L^1(\Omega)$ where $\tilde{z}(x) := \langle v_x, \text{id} \rangle$.

**Theorem 3.2.** Let $p > d$ and $\{U_n\}$ a weakly converging sequence in $L^p$ that is $K$-$A$-free and generates the Young measure $\nu = \{v_x\}_{x \in \Omega}$. Then for almost every $a \in \Omega$ there exists a $p$-equi-integrable sequence $\bar{U}_n : T^d \to \mathbb{R}^N$ that is $K$-$A$-free, has mean zero, and generates the homogeneous Young measure $\nu_a$.

**Lemma 3.3.** To prove Theorem 2.6 it is enough to show Theorem 3.2.

**Proof.** Let $U_n$ be a sequence that is $K$-$A$-free, converges weakly in $L^p$ to $U$, and generates the Young measure $\nu$. Let $a \in \Omega$ be such that the conclusion of Theorem 3.2 holds at $a$. From Theorem 3.2 there exists a $p$-equi-integrable weakly converging sequence $\bar{U}_n$ that is $K$-$A$-free and generates the homogeneous Young measure $\nu_a$. Thus we see that

\[
\langle v_a, f \rangle = \frac{1}{|T^d|} \int_{T^d} \langle v_a, f \rangle \, dx = \lim_{n \to \infty} \frac{1}{|T^d|} \int_{T^d} f(U_n) \, dx \\
\geq \lim_{n \to \infty} f(U(a)) = f(\langle v_a, \text{id} \rangle).
\]

Here we used the homogeneity of the Young measure for the first equality, the properties of $f$ to see that $\{f(U_n)\}$ is equi-integrable and Theorem 3.1 part (3) for the second equality, and finally definition 2.3 and the remark thereafter for the middle inequality. Thus, we see that $\langle v_a, f \rangle \geq f(\langle v_a, \text{id} \rangle)$ for almost every $x$ and so

\[
\liminf_{n \to \infty} \int_{\Omega} f(U_n) \, dx = \int_{\Omega} \langle v_x, f \rangle \, dx \geq \int_{\Omega} f(\langle v_x, \text{id} \rangle) \, dx = \int_{\Omega} f(U) \, dx,
\]

where again we used Theorem 3.1 part (3) for the first and last equalities. 

Were it not for the pointwise convex constraint, we could use the theory of Fonseca and Müller [7] to prove Theorem 3.2. However, in order to generate the appropriate Young measure, they use a projection onto $A$-free elements which is formulated using Fourier techniques. This could destroy the convex constraint on the sequence and thus invalidate Definition 2.3. Instead of using a (nonlocal) projection operator, we will focus on potential operators (which are local).

4. POTENTIALS AND SOBOLEV BOUNDS

We will define a sequence of functions that need to be smoothly cut off, and if this is naively done, it will invalidate the $A$-free condition. In order to keep the sequence $A$-free, we will perform the cut-off at the level of the potential functions and then apply the potential operator to ensure we remain $A$-free. We will focus on potential operators as they act locally, and thus we can perform estimates uniformly to keep the convex constraint.

From the work of Raita [13] for any linear, homogeneous differential operator $A$ of constant rank and constant coefficients, there exists another such operator $\mathcal{L}$ of order, say, $l$, such that $\ker A[\xi] = \text{im} \mathcal{L}[\xi]$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Thus, on the torus, for any smooth function $U$ where $AU = 0$, we can find a smooth potential $\Phi$ such that $\mathcal{L}\Phi = U$.

In order to use the construction in [13] for our method, we will need to prove the extra property that $\|\Phi\|_{W^{l,p}} \leq C\|U\|_{L^p}$ for a suitable choice of $\Phi$. To do this we will need to find another condition on $\Phi$ that does not interfere with the property $\mathcal{L}\Phi = U$, yet creates an elliptic system that will give us the above bound. We see that as $\mathcal{L}$ is another linear homogeneous differential operator with
constant rank and constant coefficients, we can apply the theory of Raita again to find a linear homogeneous differential operator with constant rank and constant coefficients $G$ such that

$$\ker \mathcal{L} = \text{im } G$$

and so this will characterise the kernel of $\mathcal{L}$. Now, any $\Phi$ can be split into two parts $\Phi = \Psi + G h$, where $\Psi$ is orthogonal to the kernel of $\mathcal{L}$. If we impose the condition $G h = 0$, this will force $\Phi = \Psi$ and thus $\Phi$ will be orthogonal to the kernel of $\mathcal{L}$. Further, by construction, $G$ is self-adjoint and so for all $h \in C_c^\infty$ we have

$$0 = \langle Gh, \Phi \rangle = \langle h, \mathcal{G} \Phi \rangle.$$

Thus, having imposed $G h = 0$, we obtain the condition $\mathcal{G} \Phi = 0$. This suggests that $\mathcal{G} \Phi = 0$ would be a suitable condition to select only the $\Phi$ that has no component in the kernel of $\mathcal{L}$. Indeed we will show that this is enough to control the full derivative of $\Phi$ by $U$ in $L^p$ for $1 < p < \infty$.

Further, as $\mathcal{L}$ and $\mathcal{G}$ are self-adjoint, the domain and target spaces of the respective Fourier symbols are the same, say $W$, while the target space of the symbol of $\mathcal{A}$ is called $V$; see the following diagram:

$$\ldots W \xrightarrow{\mathcal{G}}_{\Phi, \mathcal{G} h = \Phi} W \xrightarrow{\mathcal{L}}_{U, \mathcal{L} \Phi = U} \mathcal{A} W \xrightarrow{\mathcal{A} U} V$$

An explicit example (for $\mathcal{A} = \text{div}$ in two dimensions) is given in Appendix A.1.

**Theorem 4.1.** Let $1 < p < \infty$ and let $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{G}$ be linear, homogeneous, differential operators, with constant rank and constant coefficients and assume that

$$\ker \mathcal{A} [\xi] = \text{im } \mathcal{L} [\xi] \quad \text{and} \quad \ker \mathcal{L} [\xi] = \text{im } \mathcal{G} [\xi]$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Then there exists a constant $C$ such that for all $U \in L^p(\mathbb{T}^d)$ such that $\mathcal{A} U = 0$ and $\int_{\mathbb{T}^d} U \, dx = 0$, there exists a $\Phi \in W^{l, p}(\mathbb{T}^d)$ (where $l = 2k$ and $k$ is the homogeneity of $\mathcal{A}$) such that

$$\mathcal{L} \Phi = U$$
$$\mathcal{G} \Phi = 0,$$

and such that $\|\Phi\|_{W^{l, p}} \leq C \|U\|_{L^p}$.

**Remark 4.2.** For the case of DPTs, it is possible to construct the potential operator “by hand” via successive application of Poincaré’s Lemma.

When $U \in C_c^\infty$, we observe from Theorem 1 in [13] that from $\mathcal{A}$ we have the existence of $\mathcal{L}$ and $\mathcal{G}$ and further from Lemma 5 we see that $\Phi \in C_c^\infty(\mathbb{T}^d)$ exists. Here we just have to prove that adding the extra condition $\mathcal{G} \Phi = 0$ gives the bound $\|\Phi\|_{W^{l, p}} \leq C \|U\|_{L^p}$, and the extension to $L^p$ then follows immediately from an easy approximation.

Before we begin the proof, let us fix some notation. For a matrix $M \in \mathbb{R}^{n \times m}$ we define its pseudo-inverse $M^\dagger \in \mathbb{R}^{m \times n}$ as the unique matrix determined by the relations

$$MM^\dagger M = M, \quad M^\dagger MM^\dagger = M^\dagger, \quad \left(M^\dagger M\right)^* = M^\dagger M, \quad (MM^\dagger)^* = MM^\dagger,$$

where $M^*$ is the adjoint of $M$. Further, from [3], for a matrix $M \in \mathbb{R}^{n \times m}$ denote by $p(\lambda) := (-1)^n(a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n)$ as the characteristic polynomial of $MM^*$, where $a_0 = 1$. Define $r = \max\{ j \in \mathbb{N} : a_j > 0 \}$. Then, if $r = 0$, we have that $M^\dagger = 0$; else

$$M^\dagger = -a_r^{-1} M^\dagger [a_0 (MM^*)^{r-1} + a_1 (MM^*)^{r-2} + \cdots + a_{r-1} \text{Id}_{n \times n}].$$
\textbf{Proof}. We can apply the theory in [13] to obtain the following formulas for $\mathcal{L}[\xi]$ and $\mathcal{G}[\xi]$:

$$
\mathcal{L}[\xi] = a_0^r(\xi) [\text{Id} - A^t[\xi]A[\xi]],
$$

where $a_0^r(\xi)$ is the $r^{th}$ term of the characteristic polynomial of $A[\xi]A^*[\xi]$, and similarly

$$
\mathcal{G}[\xi] = a_1^r(\xi) [\text{Id} - L^t[\xi]L[\xi]],
$$

where $a_1^r(\xi)$ is the $r^{th}$ term of the characteristic polynomial of $L[\xi]L^*[\xi]$. We want to represent $\mathcal{G}[\xi]$ in terms of $A[\xi]$. Firstly, we will expand $L^*[\xi]$ and so obtain

$$
L^*[\xi] = -(a_1^r(\xi)L^*[\xi])[d_0(\xi)(L[\xi]L^*[\xi])]r^{-1} + a_1^r(\xi)(L[\xi]L^*[\xi])^{r-2} + \cdots + a_{r-1}^r(\xi)\text{Id} [L[\xi]],
$$

and so $a_1^r(\xi)L^*[\xi]L[\xi]$ becomes

$$
L^*[\xi]a_1^r(\xi)(L[\xi]L^*[\xi])r^{-1} + a_1^r(\xi)(L[\xi]L^*[\xi])^{r-2} + \cdots + a_{r-1}^r(\xi)\text{Id} [L[\xi]] = -[d_0^r(\xi)(L^*[\xi][L[\xi]])r + a_1^r(\xi)(L^*[\xi][L[\xi]])^{r-1} + \cdots + a_{r-1}^r(\xi)\text{Id} [L[\xi][L[\xi]]]].
$$

Further observe that

$$
\mathcal{L}[\xi]L^*[\xi] = a_0^r(\xi)[\text{Id} - A^t[\xi]A[\xi]](a_0^r(\xi)[\text{Id} - A^t[\xi]A[\xi]])^r = (a_0^r(\xi))^2(\xi)[\text{Id} - A^t[\xi]A[\xi]]
$$

and thus

$$
\mathcal{G}[\xi] = a_1^r(\xi)\text{Id} + [d_0^r(\xi)(a_0^r)^2(\xi)[\text{Id} - A^t[\xi]A[\xi]] + a_1^r(\xi)(a_0^r)^2(\xi)[\text{Id} - A^t[\xi]A[\xi]] + \cdots + a_{r-1}^r(\xi)(a_0^r)^2(\xi)[\text{Id} - A^t[\xi]A[\xi]]] = a_1^r(\xi)\text{Id} + [d_0^r(\xi)(a_0^r)^2(\xi) + a_1^r(\xi)(a_0^r)^2(\xi)] [\text{Id} - A^t[\xi]A[\xi]].
$$

Consider the system

$$
\mathcal{L}[\xi]\Phi(\xi) = \dot{U}(\xi) \quad \mathcal{G}[\xi]\dot{\Phi}(\xi) = 0,
$$

then $\mathcal{G}[\xi]\dot{\Phi}(\xi) = 0$ implies that

$$
-a_1^r(\xi)\dot{\Phi}(\xi) = [d_0^r(\xi)(a_0^r)^2r-1(\xi) + a_1^r(\xi)(a_0^r)^2r-3(\xi) + \cdots + a_{r-1}^r(\xi)(a_0^r)]\dot{U}(\xi)
$$

and thus

$$
\dot{\Phi}(\xi) = -\frac{[d_0^r(\xi)(a_0^r)^2r-1(\xi) + a_1^r(\xi)(a_0^r)^2r-3(\xi) + \cdots + a_{r-1}^r(\xi)(a_0^r)]}{a_1^r(\xi)} \dot{U}(\xi).
$$

From this and using (4.1) we see that

$$
\dot{\Phi}(\xi) = -\frac{C}{a_1^r(\xi)} \dot{U}(\xi),
$$

Finally, note that $a_0^r(\xi)$ is the last term in the characteristic polynomial of $A[\xi]A^*[\xi]$, which is a symmetric matrix, and $A$ is homogeneous and has constant rank. Therefore, $a_0^r$ is real valued and has order $2k$ in $\xi$. Thus (4.2) simplifies to

$$
\dot{\Phi}(\xi) = -\frac{C}{(\xi)^{2k}} \dot{U}(\xi).
$$
and so we have a relation with an elliptic symbol and can apply Calderón-Zygmund theory (see [8] and also [19]) to the singular integral kernel generated from this symbol. Thus using techniques in [10], (assuming a Lipschitz domain, here a torus), we obtain
\[ \|\Phi\|_{W^{2k,p}} \leq C\|U\|_{L^p}. \]
We can use density of \( C^\infty \) in \( L^p \) and \( W^{2k,p} \) with the bound above to see that the estimate extends to \( U \in L^p \).

\[ \square \]

5. Proof of Main Theorem

Having developed all the tools needed, we can now prove the main theorem.

Proof of Theorem 2.6. Firstly, we can modify the sequence so that \( U_n \) is bounded away uniformly from \( \partial K \). Indeed, take \( Y \in \text{Int}(K) \) (which is non-empty by assumption) and define
\[ V_n := (1 - \frac{1}{n})(U_n - Y) + Y. \]
This has the effect of shrinking the codomain of \( U_n \) from \( K \) to a subset of \( K_{\delta_n} \) where
\[ K_{\delta_n} := \{ x \in K : \text{dist}(x, \partial K) \geq \delta_n \} \]
for some \( \delta_n > 0 \) where \( \delta_n \to 0 \) as \( n \to \infty \). We see that \( AV_n = (1 - \frac{1}{n})A U_n = 0 \) for all \( n, V_n \in L^p \) and \( V_n \to U \) in \( L^p \). Further, by using [7] Proposition 2.4 we see that \( V_n \) will generate the same Young measure as \( U_n \). We denote \( U_n \) as this \( V_n \) for the rest of the proof by slight abuse of notation, and assume henceforth that \( U_n \in K \) and \( \text{dist}(U_n, \partial K) \geq \delta > 0 \).

Without loss of generality we can consider \( \mathbb{T}^d \) as the domain, as the construction is entirely local. In the construction we will zoom in and cut off smoothly to zero. Thus, when we let \( R \) be small enough such that the cut-off function has support inside \( \Omega \) and so we can embed \( \Omega \) inside \( \mathbb{T}^d \) and extend by zero to make the functions periodic over \( \mathbb{T}^d \).

Let \( I \) be a countable dense subset of \( L^1(\mathbb{T}^d) \) and similarly, \( C \) be a countable dense subset of \( C(\mathbb{R}^N) \). We know that if \( U_n \in L^p \) generates the Young measure \( v_n \), then for any \( g \in C \), \( g(U_n) \rightharpoonup \langle v_n, g \rangle \) in \( L^p(\Omega) \). Let \( \Omega_0 \) be the set of all Lebesgue points \( a \in \mathbb{T}^d \) of \( U \) which are at the same time Lebesgue points for the functions
\[ x \mapsto \int_{\mathbb{R}^N} |\xi|^p \, dv_n(\xi) \]
and \( x \mapsto \langle v_n, g \rangle \) for \( g \in C \), so that in particular
\[ \lim_{R \to 0} \int_{\mathbb{T}^d} |\langle v_n + Rx, g \rangle - \langle v_n, g \rangle| \, dx = 0. \]

Let \( \Phi_n \) be the potential for the sequence \( U_n - U \), thus, \( \mathcal{L}\Phi_n(x) = U_n(x) - U(x) \). Identifying the torus \( \mathbb{T}^d \) with a cube \( Q \), let \( \chi_j \) be a sequence of positive smooth cut-off functions in \( C_0^\infty(Q) \) converging monotonically up to 1. Then for a fixed \( a \in \Omega_0 \) we define a new sequence \( U_{j,R,n} (R > 0, j, n \in N) \) by
\[ U_{j,R,n}(x) := \mathcal{L}[\chi_j(x) R^{-1} \Phi_n(a + Rx)] + U(a + Rx) \]
for \( x \in \mathbb{T}^d \).

We have assumed that \( U_n - U \rightharpoonup 0 \) in \( L^p \), so by linearity of \( \mathcal{L} \) and the compact embedding \( W^{1,p} \hookrightarrow C^{0,\varepsilon} \) for some \( \varepsilon > 0 \) we obtain uniform convergence of \( \partial^\beta \Phi_n \to 0 \) for \( |\beta| \leq l - 1 \).

We clearly see that, as \( \mathcal{L} \) maps to the kernel of \( A \),
\[ A(U_{j,R,n}) = A(\mathcal{L}[\chi_j(x) R^{-1} \Phi_n(a + Rx)] + U(a + Rx)) = 0 \]
for all \( j, R, n \).
We can expand the expression to see that

\[
\begin{align*}
U_{j,R,n}(x) &= \mathcal{L}[\chi_j(x)R^{-l}\Phi_n(a+Rx)] + U(a+Rx) \\
&= \chi_j(x) \left( U_n(a+Rx) - U(a+Rx) \right) + U(a+Rx) \\
&\quad + \sum_{|\alpha|+|\beta|=l,|\beta|>1} R^{[\alpha]} \Phi_n(a+Rx)L(\alpha,\beta) \partial_\beta \chi_j(x).
\end{align*}
\]

One notices that the \( R^{-l} \) was added so that it scales with the \( R \) gained from each differentiation, as the operator \( \mathcal{L} \) is homogeneous of degree \( l \). We will use the assumption that \( p > d \) so that we obtain uniform convergence of the \( \partial_\beta \Phi_n \to 0 \) to control the remainder terms, even with the potential problem of \( R^{[\alpha]} \).

If we fix \( j \) then we see that \( U_{j,R,n} - U(a) \to 0 \) in \( \text{L}^p \) as \( n \to \infty \) and \( R \to 0 \). For the first term, by weak convergence, \( U_n(a+Rx) \to U(a+Rx) \). For the term \( U(a+Rx) - U(a) \) we have assumed that \( a \) is a Lebesgue point so this term will converge to zero. For the remainder terms we see that \( \frac{\partial_s \Phi_n(a+Rx)}{R^{[\alpha]}} \) converges to 0 uniformly with \( n \).

We will now show that the sequence \( U_{j,R,n} \in \text{L}^p(\mathbb{T}^d;\mathbb{R}^d) \) will generate the homogeneous Young measure \( \nu_a \). For all \( \psi \in \mathcal{I} \) and \( g \in \mathcal{C} \),

\[
\begin{align*}
\lim_{j \to \infty} \lim_{R \to 0} \lim_{n \to \infty} \int_{\mathbb{T}^d} \psi(x)g(U_{j,R,n}(x)) \, dx \\
&= \lim_{j \to \infty} \lim_{R \to 0} \lim_{n \to \infty} \int_{\mathbb{T}^d} \psi(x)g \left( \chi_j(x) \left( U_n(a+Rx) - U(a+Rx) \right) + U(a+Rx) \\
&\quad + \sum_{|\alpha|+|\beta|=l} R^{[\alpha]} \partial_\alpha \Phi_n(a+Rx)L(\alpha,\beta) \partial_\beta \chi_j(x) \right) \, dx \\
&= \lim_{j \to \infty} \int_{\mathbb{T}^d} \psi(x) \int_{\mathbb{T}^d} g \left( \chi_j(x) \left( \xi - \langle \text{id}, \nu_{a+Rx} \rangle \right) + \langle \text{id}, \nu_{a+Rx} \rangle \right) \, d\nu_{a+Rx}(\xi) \\
&\quad + \langle \text{id}, \nu_{a+Rx} \rangle \, d\nu_{a+Rx}(\xi) \\
&= \langle \nu_a, g \rangle \int_{\mathbb{T}^d} \psi(x) \, dx.
\end{align*}
\]

Further, we can show that \( \{ |U_{j,R,n}|^p \} \) is equi-integrable,

\[
\begin{align*}
\lim_{j \to \infty} \lim_{R \to 0} \lim_{n \to \infty} \int_{\mathbb{T}^d} |U_{j,R,n}(x)|^p \, dx \\
&= \lim_{j \to \infty} \lim_{R \to 0} \lim_{n \to \infty} \int_{\mathbb{T}^d} \left| \chi_j(x) \left( U_n(a+Rx) - U(a+Rx) \right) + U(a+Rx) \\
&\quad + \sum_{|\alpha|+|\beta|=l} R^{[\alpha]} \partial_\alpha \Phi_n(a+Rx)L(\alpha,\beta) \partial_\beta \chi_j(x) \right|^p \, dx \\
&= \lim_{j \to \infty} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left| \chi_j(x) \left( \xi - \langle \text{id}, \nu_{a+Rx} \rangle \right) + \langle \text{id}, \nu_{a+Rx} \rangle \right|^p \, d\nu_{a+Rx}(\xi) \, dx \\
&\quad + \langle \text{id}, \nu_{a+Rx} \rangle \, d\nu_{a+Rx}(\xi) \\
&= \int_{\mathbb{R}^d} |\xi|^p \, d\nu_{a}(\xi)
\end{align*}
\]

which is bounded as we have assumed that \( a \) is a Lebesgue point. We can now apply Theorem 3.1 part (3) with the function \( h(x) := |x|^p \) to see that \( \{ h(U_{j,R,n}) \} \) is equi-integrable.
We finally need to show that \( U_{j,R,n}(x) \in K \) for almost every \( x \). We see that
\[
\left| U_{j,R,n}(x) - \left[ \chi_j(x) U_n(a + Rx) + (1 - \chi_j(x)) U(a + Rx) \right] \right| \\
\leq \left| \sum_{|\alpha| + |\beta| = l} R^{[\alpha]} \Phi_n(a + Rx) L^{(\alpha,\beta)} \partial_\beta \chi_j(x) \right|.
\]
The right hand side converges to zero uniformly (when the limit is taken in the order \( n \to \infty \), then \( R \to 0 \), then \( j \to \infty \)), and the second two terms on the left hand side form a convex combination of functions with values in \( K \) and distance at least \( \delta \) from \( \partial K \). This ensures that \( U_{j,R,n}(x) \in K \) almost everywhere for sufficiently large \( n \).

We can now use a diagonalisation procedure (choosing \( R = R(n) \) and \( j = j(n) \) appropriately) to find a subsequence \( \tilde{U}_n \) (relabelled with index \( n \)) that contains all the properties shown above and we are done. \( \square \)

**Appendix A. Potential for Divergence**

Here we give an explicit example of a potential operator for the divergence of a two-dimensional vector field (though this also works in higher dimensions) in order to better illustrate the techniques from [13].

**Example A.1.** Let \( A = \text{div} \) in two dimensions. Then \( A[\xi] = \xi^T \) and \( A^*[\xi] = \bar{\xi} \) and so
\[
A[\xi] A^*[\xi] = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix}.
\]
We want to calculate the characteristic polynomial of this matrix so we must look at
\[
\det[A[\xi] A^*[\xi] - \lambda \text{Id}] = \det \begin{pmatrix} \xi_1 - \lambda & \xi_2 \\ \xi_3 & \xi_4 - \lambda \end{pmatrix} = \lambda^2 - |\xi|^2 \lambda = (a_0^0) \lambda^2 + (a_1^0) \lambda.
\]
From this we see that \( r^\alpha = 1 \) and
\[
A^*[\xi] = \frac{1}{|\xi|^2} \xi [a_0^0 (A[\xi] A^*[\xi])^0] = \frac{\bar{\xi}}{|\xi|^2} \text{Id}
\]
and thus
\[
\mathcal{L}[\xi] = |\xi| \left[ |\text{Id} - \frac{\bar{\xi}}{|\xi|^2} \text{Id} \xi^T \right] = (|\xi|^2 \text{Id} - \xi \otimes \xi).
\]
Similarly, we can apply the same method to \( \mathcal{L} \). We see that
\[
\det[\mathcal{L} \mathcal{L}^* - \lambda \text{Id}] = -(\lambda^2 + |\xi|^4 \lambda) = a_0^0 \lambda^2 + a_1^0 \lambda
\]
and so \( r^\alpha = 1 \) and we obtain that
\[
\mathcal{L}^*[\xi] = -\frac{1}{|\xi|^4} \mathcal{L}^*[\xi](-1) = \frac{1}{|\xi|^4} (|\xi|^2 \text{Id} - \xi \otimes \xi)
\]
and thus
\[
\mathcal{G}[\xi] = |\xi|^4 \left[ |\text{Id} - \frac{1}{|\xi|^4} (|\xi|^2 \text{Id} - \xi \otimes \xi)(|\xi|^2 \text{Id} - \xi \otimes \xi) \right] = 2|\xi|^2 \xi \otimes \xi - (\xi \otimes \xi)^2.
\]
We shall consider the system
\[
\mathcal{L} \Phi = U
\]
\[
\mathcal{G} \Phi = 0
\]
and see that
\[
\mathcal{L}[\xi] \hat{\Phi}(\xi) = (|\xi|^2 \text{Id} - \xi \otimes \xi) \hat{\Phi}(\xi) = \hat{U}(\xi)
\]
\[
\mathcal{G}[\xi] \hat{\Phi}(\xi) = (2|\xi|^2 \xi \otimes (\xi \otimes \xi)) \hat{\Phi}(\xi) = 0
\]
We can manipulate \(\mathcal{G}[\xi] \hat{\Phi}[\xi]\) and see that
\[
\mathcal{G}[\xi] \hat{\Phi}(\xi) = ((\xi \otimes \xi)|(\xi^2 \text{Id} + |\xi|^2 \text{Id} - (\xi \otimes \xi)) \hat{\Phi}(\xi) = 0
\]
and thus
\[
(\xi \otimes \xi)|\xi|^2 \text{Id} \hat{\Phi}(\xi) = -(\xi \otimes \xi)|(\xi^2 \text{Id} - (\xi \otimes \xi)) \hat{\Phi}(\xi) = (\xi \otimes \xi)(-\mathcal{L}[\xi]) \hat{\Phi}[\xi] = -(\xi \otimes \xi) \hat{U}(\xi).
\]
We can rearrange this and obtain that
\[
\hat{\Phi}(\xi) = -|\xi|^{-2} \hat{U}(\xi).
\]
We thus have an elliptic operator relating \(U\) and \(\Phi\) and can apply Calderón-Zygmund theory to the singular integral kernel generated from this symbol. Assuming a Lipschitz domain, we obtain for \(1 < p < \infty\) that \(\|\Phi\|_{W^{2,p}} \leq C\|U\|_{L^p}\) as we needed.

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