1/2-BPS Wilson Loops and Vortices in ABJM Model

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Abstract

We explore the low-energy dynamics of 1/2-BPS heavy particles coupled to the ABJM model via the Higgsing of M2-branes, with focus on physical understanding of the recently discovered 1/2-BPS Wilson loop operators. The low-energy theory of 1/2-BPS heavy particles turns out to have the $U(N|N)$ supergauge symmetry, which explains the novel structure of the 1/2-BPS Wilson loop operator as a holonomy of a $U(N|N)$ superconnection. We show that the supersymmetric transformation of the Wilson loop operator can be identified as a fermionic supergauge transformation, which leads to their invariance under half of the supersymmetry. We also argue that 1/2-BPS Wilson loop operators appear as 1/2-BPS vortices with vorticity $1/k$. Such a vortex can be naturally interpreted as a membrane wrapping the $\mathbb{Z}_k$ cycle once, or type IIA fundamental string.
1 Introduction and Conclusion

It has been recently proposed by Aharony, Bergman, Jafferis and Maldacena that the \( \mathcal{N} = 6 \) superconformal Chern-Simons theory with gauge group \( U(N) \times U(N) \) describes the low-energy dynamics on \( N \) M2-branes at the tip of the orbifold space \( \mathbb{C}^4/\mathbb{Z}_k \) [1]. The gravity dual of the theory is either M-theory on \( \text{AdS}_4 \times S^7/\mathbb{Z}_k \) background, or type IIA string theory on \( \text{AdS}_4 \times \mathbb{CP}^3 \), depending on the range of the Chern-Simons level \( k \).

A set of physical observables in the conformal theories is spanned by correlation functions of gauge invariant operators. As important and interesting observables, the BPS Wilson loop operators have gotten great attention due to their clear identification as macroscopic strings in the gravity dual [2, 3]. In the \( \mathcal{N} = 6 \) Chern-Simons theory, the half BPS Wilson line operator corresponding to the most symmetric string configuration has remained unidentified for a while, despite the immediate initial discoveries of 1/6-BPS Wilson line operator [4–6] and 1/2-BPS vortices [7].

The very 1/2-BPS Wilson line operator of \( \mathcal{N} = 6 \) Chern-Simons theory has been constructed recently in [8]. It turns out that the operator has some interesting and novel features. More precisely, it takes the form as a holonomy of the superconnection in the super Lie group \( U(N|N) \) which is related to the \( U(N) \times U(N) \) ABJM model. The superconnection also involves certain constant spinors. Using the localization technique as in [9], one can even compute the vacuum expectation value of the 1/2 BPS Wilson line operator exactly [10–12].

In this work, we focus on how to understand the above peculiar structure of the 1/2 BPS Wilson line operator via a systematic and physical procedure, so-called Higgsing.

One possible interpretation of the Wilson line operator is an insertion of an external charged particle into the given system. The Wilson line operator appears as how the wave-function of the external particle evolves under its interactions with the given system. In order to study the BPS Wilson line operator, it is therefore essential how to introduce such very heavy particles to the system in a supersymmetric fashion. It is simply provided by the suitable Higgsing procedure.

We systematically explore the low-energy dynamics of 1/2-BPS very massive particles in the Coulomb phase of the M2-brane theory that describes a separation of a single M2-brane far away from the rest M2-branes placed near the orbifold singularity. In the infinite separation limit, these infinitely massive particles obviously provide the external source as the 1/2 BPS Wilson line operator. In this paper, we pay attention in particular how the fermion fields can affect the time-evolution of the external particles, which leads to a physical explanation of the super Lie algebraic structure of the Wilson line operator. Although the full ABJM model itself does not have the \( U(N|N) \) supergauge symmetry, the low-energy theory of these external 1/2-BPS particles coupled to ABJM model turns out to respect the supergauge symmetry.

We also show in addition that the supersymmetry transformation of 1/2-BPS Wilson line operator can be regarded as a gauge transformation of the superconnection.
with fermionic super Lie algebra elements. It immediately implies the SUSY invariance of loop operators under trace or supertrace depending on whether the fermionic field is anti-periodic or periodic. For the abelian ABJM model, the insertion of the 1/2-BPS Wilson line operator creates a fundamental 1/2-BPS vortex, which can be described as an M2-brane spike wrapping the $S^1$ fibre of $S^7/Z_k$. We also study several properties of the vortices in relation to the 1/2-BPS Wilson line operator.

Our analysis can be also applied to less supersymmetric cases, either Wilson line operators or theories. For examples, for 1/6-BPS Wilson lines, one can quickly show that all interactions of 1/6-BPS external particles that could deliver fermionic contributions to Wilson lines are averaged out in the infinite mass limit, due to their highly oscillatory behaviors. The absence of those interactions explains the usual expression for 1/6-BPS Wilson line operator. One can also show that there is 2/5-BPS Wilson line operators in $\mathcal{N} = 5$ Chern-Simons matter theories [13, 14], similar to 1/2-BPS Wilson line operators in the ABJM model after suitable changes in gauge group representation and additional reality conditions. The low-energy dynamics of corresponding external particles are again expected to have the $OSp(2N|2N)$ supergauge symmetry.

The super Lie algebraic structure of 1/2-BPS Wilson line operators however severely restricts their possible representations under the gauge group, since the fermionic components should transform as bi-fundamental representations of $U(N) \times U(N)$. It would be interesting how to circumvent this difficulty, or understand the obstruction in the membrane picture. The insertion of the Wilson-loop affects the system to break the scale symmetry, based on the physics of 1/2-BPS vortex in abelian theories. We expect that this could persist in the weak coupling or large $k$ limit. It would be also interesting to understand this physics in detail.

This paper is organized as follows. In Section 2, we briefly review the ABJM model and discuss the mass spectrum in the Higgsing procedure. The massive modes can be identified as macroscopic membranes, interpolating the separated M2-branes and wrapping the $S^1$ fibre of orbifold $C^4/Z_k$ once. We study in Section 3 the Higgsing procedure in more details to provide external particles to the ABJM model in a supersymmetric fashion. We discuss first how to read off the dynamical modes from the off-diagonal massive fields in the infinite mass limit, which parallels to what we perform in non-relativistic limit. We then present some delicate points in obtaining the low-energy theory of 1/2-BPS heavy particles, which originates from non-trivial interactions between the heavy particles and ABJM fields. In section 4, we show that low-energy theory of external 1/2-BPS particles coupled to the ABJM model preserves the $U(N|N)$ supergauge symmetry. It explains the physical origin of the novel structure of the 1/2-BPS Wilson line operators. We give an alternative proof on the invariance of 1/2-BPS Wilson line operator under the half of supersymmetry in relation to the supergauge transformation. We finally study in Section 5 the interesting relations between 1/2-BPS vortices of vorticity $1/k$ and 1/2-BPS Wilson line. We briefly discuss in Appendix the infinite mass limit in the free field theories, with care given to dynamical modes which can survive in the low-energy theory.
2 Preliminaries

2.1 Short review on ABJM model

Let us start with a short description on the ABJM model [1], believed to describe the dynamics of multiple M2-branes probing a orbifold $\mathbb{C}^4/\mathbb{Z}_k$. It is the $\mathcal{N} = 6$ supersymmetric Chern-Simons matter theory with the gauge group $G = U(N) \times U(N)$. The gauge fields are now denoted by $A_\mu$ and $\tilde{A}_\mu$ with the Chern-Simons levels $(k, -k)$. The matter fields are composed of four complex scalars $Z_\alpha$ ($\alpha = 1, 2, 3, 4$) and four three-dimensional spinors $\Psi^\alpha$, both of which transform under the gauge symmetry as bi-fundamental representations ($\mathbf{N}, \bar{\mathbf{N}}$). As well as the gauge symmetry, the present model also has additional global $SU(4)_R$ symmetry, under which the scalars $Z_\alpha$ furnish the representation $\mathbf{4}$ while the fermions $\Psi^\alpha$ furnish $\bar{\mathbf{4}}$. Both $Z_\alpha, \Psi^\alpha$ carry an abelian charge which identifies particles and anti-particles.

The ABJM Lagrangian takes the following forms

$$L = L_{\text{CS}} + L_{\text{kin}} + L_{\text{Yukawa}} + L_{\text{potential}}.$$  \hfill \text{(2.1)}

The Chern-Simons terms and matter kinetic terms are

$$L_{\text{CS}} + L_{\text{kin}} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho - \tilde{A}_\mu \partial_\nu \tilde{A}_\rho + \frac{2i}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right) - \text{Tr} \left( D_\mu Z^\alpha D^\mu Z_\alpha + i \bar{\Psi}_\alpha \gamma^\mu D_\mu \Psi^\alpha \right),$$  \hfill \text{(2.2)}

where $D_\mu Z_\alpha = \partial_\mu Z_\alpha - i A_\mu Z_\alpha + i Z_\alpha \tilde{A}_\mu$ and so on. The Yukawa-like interactions are

$$L_{\text{Yukawa}} = \frac{2\pi i}{k} \text{Tr} \left( \bar{Z}_\alpha Z_\beta \bar{\Psi}_\beta \Psi^\beta - Z_\alpha \bar{Z}_\beta \Psi^\beta \bar{\Psi}_\beta + 2Z_\alpha \bar{Z}_\beta \Psi^\beta \bar{\Psi}_\beta - 2\bar{Z}_\alpha Z_\beta \Psi^\beta \bar{\Psi}_\beta \right. \left. + \epsilon_{\alpha\beta\gamma\delta} \bar{Z}_\alpha Z_\beta \bar{Z}_\gamma Z_\delta \Psi^\beta \bar{\Psi}_\beta \right).$$  \hfill \text{(2.3)}

The sextic scalar interactions are summarized simply as

$$L_{\text{potential}} = -U = \frac{4\pi^2}{3k^2} \text{Tr} \left( 6Z_\alpha \bar{Z}_\beta Z_\gamma \bar{Z}_\delta Z_\epsilon - 4Z_\alpha \bar{Z}_\beta Z_\gamma \bar{Z}_\delta Z_\epsilon \right. \left. - Z_\alpha \bar{Z}_\beta Z_\gamma \bar{Z}_\delta Z_\epsilon - Z_\alpha \bar{Z}_\beta Z_\gamma \bar{Z}_\delta Z_\epsilon \right).$$  \hfill \text{(2.4)}

The positive definite bosonic potential $U$ can be expressed in terms of third order polynomials $W$ and its hermitian conjugate $\bar{W}$:

$$U = \frac{2}{3} \text{Tr} \left( W^\alpha_{\beta\gamma} W^\beta_{\alpha\gamma} \right) \geq 0$$  \hfill \text{(2.5)}

with

$$W^\alpha_{\beta\gamma} = -\frac{\pi}{k} \left( 2Z_\beta \bar{Z}_\alpha Z_\gamma + \delta^\alpha_\beta (Z_\gamma \bar{Z}_\rho Z_\gamma - Z_\rho \bar{Z}_\gamma Z_\gamma) \right) - (\beta \leftrightarrow \gamma),$$  \hfill \text{(2.6a)}

$$W^\beta_{\alpha\gamma} = \frac{\pi}{k} \left( 2\bar{Z}_\beta Z_\alpha Z_\gamma + \delta^\beta_\alpha (Z_\gamma Z_\rho \bar{Z}_\gamma - Z_\rho \bar{Z}_\gamma Z_\gamma) \right) - (\beta \leftrightarrow \gamma).$$  \hfill \text{(2.6b)}
The ABJM model is invariant under the $N = 6$ supersymmetry whose transformation rules are summarized as

\[
\begin{align*}
\delta Z_\alpha &= i\xi_{\alpha\beta}\Psi^\beta, & \delta\Psi^\alpha &= -\gamma^\mu\xi_{\alpha\beta} D_\mu Z_\beta + W^\alpha_{\beta\gamma} \xi_{\beta\gamma}, \\
\delta\bar{Z}_\alpha &= i\xi_{\alpha\beta}\bar{\Psi}^\beta, & \delta\bar{\Psi}^\alpha &= -\gamma^\mu\xi_{\alpha\beta} \bar{D}_\mu \bar{Z}_\beta + \bar{W}^\alpha_{\beta\gamma} \xi_{\beta\gamma}, \\
\delta A_\mu &= +\frac{2\pi}{k} (Z_\alpha \bar{\Psi}_\beta \gamma^\mu \xi_{\alpha\beta} + \Psi^\alpha \bar{Z}_\beta \gamma^\mu \xi_{\alpha\beta}), \\
\delta\tilde{A}_\mu &= -\frac{2\pi}{k} (\bar{\Psi}_\alpha Z^\beta \gamma^\mu \xi_{\alpha\beta} + \bar{Z}^\alpha \Psi^\beta \gamma^\mu \xi_{\alpha\beta}).
\end{align*}
\] (2.7)

Moreover, this M2-brane theory is also invariant under the parity operation accompanied by

\[
Z_\alpha, \Psi^\alpha, A_\mu, \tilde{A}_\mu \leftrightarrow \bar{Z}_\alpha, \bar{\Psi}^\alpha, \tilde{A}_\mu, A_\mu.
\]

Note that the supersymmetry transformation parameters $\xi_{\alpha\beta} = -\xi_{\beta\alpha}$ satisfy the reality condition

\[
\xi_{\alpha\beta} = (\xi_{\alpha\beta})^* = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \xi^{\gamma\delta},
\] (2.8)

with the convention $\epsilon_{1234} = \epsilon^{1234} = 1$. For later convenience, one summarizes the equation of motions for gauge fields

\[
\begin{align*}
\frac{k}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} - i (Z_\alpha D^\mu \bar{Z}_\alpha - D^\mu Z_\alpha \bar{Z}_\alpha) &= 0, \\
-\frac{k}{4\pi} \epsilon^{\mu\nu\rho} \tilde{F}_{\nu\rho} - i (\bar{Z}^\alpha D^\mu Z_\alpha - D^\mu \bar{Z}\bar{Z}_\alpha) &= 0.
\end{align*}
\] (2.9)

Let us now examine the vacuum moduli space of the present model at the classical level, i.e., solutions of $U(Z_\alpha, \bar{Z}^\beta) = 0$ up to gauge transformations. It leads to the equation for its minima

\[
Z_\alpha \bar{Z}^\beta Z_\gamma = Z_\gamma \bar{Z}^\beta Z_\alpha, \quad \bar{Z}^\alpha Z_\beta \bar{Z}^\gamma = \bar{Z}^\gamma Z_\beta \bar{Z}^\alpha.
\] (2.10)

This implies that the hermitian matrices $Z_\alpha \bar{Z}^\beta$ commute with each other, and similarly for $\bar{Z}^\alpha Z_\beta$. The vacuum solutions are thus given by diagonal $Z_\alpha$ up to gauge equivalences,

\[
Z_\alpha = \text{diag}(z_\alpha^1, z_\alpha^2, ..., z_\alpha^N).
\] (2.11)

On a generic point of the vacuum moduli space, the gauge group $G = U(N) \times U(N)$ is spontaneously broken down to $U(1)^N \subset U(N)_D$, diagonal part of $G$.

**Convention** As a final comment, let us summarize our convention. A natural choice for gamma matrices $\gamma^\mu$ is in the Majorana representation:

\[
\gamma^0 = i\tau^2, \quad \gamma^1 = \tau^1, \quad \gamma^2 = \tau^3, \quad \gamma^{012} = 1_2.
\] (2.12)
We define constant two-component spinors with definite helicity $u_{\pm}$ as
\[ u_{\pm} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \end{pmatrix}, \quad \bar{u}_{\pm} \equiv \frac{1}{\sqrt{2}}(1, \mp i), \tag{2.13} \]
which obviously satisfy the following relations
\[ i\gamma^0 u_{\pm} = \pm u_{\pm}, \quad \bar{u}_{\pm} u_{\mp} = 1, \quad \bar{u}_+ u_- = 0, \tag{2.14} \]
and so on.

One can express the supersymmetry parameters $\xi^{\alpha\beta}$ and Dirac spinors in the helicity basis like below
\[ \xi^{\alpha\beta} = u_+^{T} \xi_{\alpha\beta}^{-} + u_-^{T} \xi_{\alpha\beta}^{+}, \quad \Psi^{\alpha} = u_+^{T} \Psi_{\alpha}^{-} + u_-^{T} \Psi_{\alpha}^{+}, \tag{2.15} \]
where $\xi_{\alpha\beta\pm} = [\bar{u}_{\pm} \xi_{\alpha\beta}] = u_+^{T} \xi_{\alpha\beta}, \Psi_{\pm}^{\alpha} = [\bar{u}_{\pm} \Psi^{\alpha}] = u_+^{T} \Psi^{\alpha}.$

### 2.2 Higgsing and massive particles

We now in turn discuss massive modes in the Higgsing procedure which separates a single M2-brane apart from the rest $N$ M2-branes at the orbifold singularity. For concreteness, we first present the mass spectrum at the generic point on the Coulomb branch of $U(2) \times U(2)$ ABJM model. The mass formula for massive modes, applicable particularly in the Higgsing, will be presented in order.

The generic vacuum of $U(2) \times U(2)$ ABJM model can be described as
\[ \langle Z_\alpha \rangle = \text{diag}(u_\alpha, v_\alpha), \tag{2.16} \]
where $u_\alpha$ and $v_\alpha$ denote the positions of two M2-branes in the orbifold space $\mathbb{C}^4/\mathbb{Z}_k$. When $u_\alpha \neq v_\alpha$, the linear fluctuation analysis tells us that the mass spectrum can be summarized as
\[
\left\{ \begin{array}{c}
\text{massless multiplet} & : 16 \text{ scalar bosons} + 16 \text{ fermions} \\
\text{massive multiplet} & : 12 \text{ scalar bosons} + 16 \text{ fermions} + 4 \text{ vector bosons} 
\end{array} \right. \tag{2.17}
\]

The massive modes are made of a pair of $1/2$ BPS massive vector multiplets of opposite parity as we will see. The above massive modes arise from the off-diagonal elements of matter fields, as usual. One can show that the perturbative mass $\mu$ of the massive multiplet takes the following form [15]
\[ \mu = \frac{2\pi}{|k|} \sqrt{(u_\alpha \bar{u}^{\alpha} + v_\alpha \bar{v}^{\alpha})^2 - 4|u_\alpha \bar{v}^{\alpha}|^2}. \tag{2.18} \]

For the Higgsing, we take the particular position of a single M2-brane far away from the origin such that $v_\alpha = (0,0,0,v)$ and for any $\alpha, |u_\alpha| \ll v$. Then, the mass formula (2.18) becomes
\[ \mu = \frac{2\pi}{k} \left(|u_1|^2 + |u_2|^2 + |u_3|^2 - |u_4|^2\right) + \frac{2\pi}{k} |v|^2. \tag{2.19} \]
The quadratic dependence on the position parameters can be understood as follows: the massive modes arise from an M2-brane which interpolates two separated M2-branes and also wraps the 11-dimensional circle of size proportional to \( \frac{2\pi}{k} \times \text{distance} \) [15].

The signs in (2.19) imply that the distance between two M2-branes increases along \( z_1, z_2, z_3 \) directions while decreases along \( z_4 \) direction, where \( z_\alpha \in \mathbb{C}^4/\mathbb{Z}_k \).

The presence of the W bosons and other massive particles can be treated as source terms for scalars with some sign difference. The end result becomes

\[
\mathcal{L}_{\text{scalar}} = -\frac{2\pi}{k} \left( |u_1|^2 + |u_2|^2 + |u_3|^2 - |u_4|^2 \right) \delta^2(z - z_p),
\]

where \( z_p \) denote the position of the source. The energy contribution to the \( u_1, u_2, u_3 \) is increasing and that to \( u_4 \) is decreasing. Indeed the insertion of 1/2 BPS Wilson line to an abelian theory leads to such scalar field source as we will see later in Section 5.

Let us finally consider the Higgsing for \( N \) M2-branes, i.e., put \( N - 1 \) M2-branes near the orbifold singularity and another away from the tip so that the vacuum expectation value becomes

\[
\langle Z_\alpha \rangle = \text{diag}(u_\alpha, u_\alpha, \cdots u_\alpha, v_{\delta_4}) \ . \quad (2.21)
\]

One can show that the mass formula again takes the form (2.19). In order to study the 1/2 BPS Wilson line, we are interested in the low-energy dynamics of those massive particles interacting with the ABJM model living on \( N - 1 \) M2-branes near the tip. In the next section, it will be discussed in details.

## 3 Low-energy Dynamics of Heavy Particles in M2 Theory

More precisely, let us start with \( U(N) \times U(N) \) ABJM model and separate a single M2-brane far away from the rests sitting at the origin of \( \mathbb{C}^4/\mathbb{Z}_k \) by giving some expectation values to complex scalar fields \( Z_\alpha \). For a 1/2-BPS Wilson line operator, the suitable choice of vacuum expectation value turns out to be

\[
\langle Z_\hat{\alpha} \rangle = 0 \ , \quad \langle Z_4 \rangle = \text{diag}(0, 0, \cdots, v) \ , \quad (\hat{\alpha} = 1, 2, 3) \quad (3.1)
\]

in order to preserve \( SU(3) \subset SU(4)_R \). The gauge group \( U(N) \times U(N) \) is obviously broken down to \( U(N - 1) \times U(N - 1) \).

As will be presented in order, there are massive super-multiplets arising from the standard Higgs mechanism at the above particular point (3.1) on the Coulomb branch. Those massive modes are coming from the off-diagonal modes

\[
\left\{(A_\mu)_{mN}, (Z_{\hat{\alpha}})_{mN}, (\Psi^\alpha)_{mN}\right\}, \quad \left\{(\tilde{A}_\mu)_{Nm}, (Z_{\hat{\alpha}})_{Nm}, (\Psi^\alpha)_{Nm}\right\}
\]

(3.2)

together with their complex conjugates. Here \( m = 1, 2, \cdots, N - 1 \). They transform as \( (N - 1, 1), (1, N - 1) \), and so on under the unbroken gauge symmetry \( U(N - 1) \times U(N - 1) \). The mass of massive modes is given by

\[
m = \frac{2\pi}{k} v^2 .
\]

(3.3)
Note that the off-diagonal modes \((Z_4)_{mN}, (Z_4)_{Nm}\) are massless, which can be understood as Goldstone bosons eaten by massive vector bosons.

We eventually take the limit \(v \to \infty\) so that massive off-diagonal modes behave like external charged particles in \(U(N-1) \times U(N-1)\) ABJM model. In order to study \(1/2\)-BPS Wilson line operators, we are interested in the Lagrangian \(\hat{L}\) that governs the low-energy dynamics of such external charged particles interacting with \(U(N-1) \times U(N-1)\) ABJM fields. One can obtain such a Lagrangian via expanding the \(U(N) \times U(N)\) ABJM Lagrangian in the limit \(v \to \infty\)
\[
\mathcal{L}_{\text{ABJM}}^{U(N)} \to \mathcal{L}_{\text{ABJM}}^{U(N-1)} + \hat{\mathcal{L}}(\text{heavy modes, light modes}) + O(1/v) .
\] It needs however some elaborations and careful analysis for suitable expansion, which will be presented below.

### 3.1 Non-relativistic modes

In the limit \(v \to \infty\), massive modes can be treated as non-relativistic particles, due to the fact that one can barely create a particle/anti-particle at rest from each massive fields such as \((Z_4^\alpha)_{mN}\). The analysis to obtain the Lagrangian \(\mathcal{L}_{\text{massive}}\) is therefore inevitably similar to that performed in the non-relativistic limit of mass-deformed ABJM model [16, 17]. The focus is however different as we are interested in infinite mass limit where the spatial gradient terms become irrelevant. In the Appendix, we give a brief review on the infinite mass limit of massive particles with various helicity in the free field theory.

There can be many possible non-relativistic system obtained from a single relativistic system, depending on what kinds of particles/anti-particles we want to keep in the non-relativistic limit. Likewise, we will end up with several different heavy particle systems depending on our choice. We now discuss how to choose particle or anti-particle modes for off-diagonal massive fields in the infinite mass limit, compatible with particular gauge choice and preserved \(\mathcal{N} = 3\) supersymmetry or the 1/2 of the original supersymmetry.

In our discussions below, we choose the unitary gauge where all Goldstone bosons \((Z_4)_{mN}, (Z_4)_{Nm}\) with their complex conjugates are turned off. To maintain the unitary gauge, one has to demand the following supersymmetry transformation
\[
\delta Z_4 = \xi_{4\dot{\alpha}+} \Psi^-_\dot{\alpha} - \xi_{4\dot{\alpha}-} \Psi^\alpha_+ \quad (3.5)
\]
\(\xi_{4\dot{\alpha}+}, \xi_{4\dot{\alpha}-}\) represent the helicity. Here \(\xi \pm\) represents the helicity. Inspired by the macroscopic string or the vortex description for \(1/2\) BPS Wilson line [] (which will be also presented in Section 5), let us keep only half of the supersymmetry as follow
\[
\xi_{4\dot{\alpha}+} = (\xi_{4\dot{\alpha}}^-)^* , \quad \xi_{\dot{\alpha}+} = \epsilon_{\dot{\alpha}\dot{\beta}\gamma} \xi_{4\dot{\gamma}+} . \quad (3.6)
\]
It therefore implies that negative helicity modes for \(\Psi^\alpha\) should be turned off
\[
(\Psi^-_\dot{\alpha})_{mN} = 0 , \quad (\Psi^-_\dot{\alpha})_{Nm} = 0 , \quad (3.7)
\]
and their complex conjugates. Solving the free field equation for the above off-diagonal components, the suitable choice of non-relativistic modes turns out to be

\[
(\Psi_\hat{\alpha}^\dagger)_{nN} = u_+ \bar{\psi}_{\hat{\alpha}}^n(x)e^{-imt}, \quad (\Psi_\hat{\alpha})_{Nn} = u_+ \bar{\psi}_{\hat{\alpha}}^n(x)e^{+imt},
\]

and similar for their complex conjugates, as shown in the Appendix. For later convenience, we present the explicit dependence of the amplitudes on the space-time coordinates.

The non-relativistic modes for other massive fields can be fully determined by requiring that they are combined to generate \( \mathcal{N} = 3 \) vector multiplets, and together by solving their free field equations. As shown in the Appendix, the right choice for the non-relativistic modes are therefore given by

\[
(\vec{A})_{nN} = \sqrt{\frac{\pi}{k}} \vec{E}_- w_{+n}(x)e^{-imt}, \quad (\check{Z}_\hat{\alpha})_{Nn} = \frac{1}{\sqrt{2m}} \bar{\phi}_{\hat{\alpha}n}(x)e^{-imt},
\]

\[
(\check{\Psi}^\dagger)_{nN} = u_- \bar{\psi}_{\hat{\alpha}}^n(x)e^{-imt}, \quad (\check{\Psi}^4)_{nN} = u_- \bar{\psi}_{4}^n(x)e^{-imt},
\]

and

\[
(\vec{A})_{Nn} = \sqrt{\frac{\pi}{k}} \vec{E}_- \tilde{w}_{+n}(x)e^{+imt}, \quad (\check{Z}_\hat{\alpha})_{nN} = \frac{1}{\sqrt{2m}} \tilde{\phi}_{\hat{\alpha}n}(x)e^{+imt},
\]

\[
(\check{\Psi}^\dagger)_{Nn} = u_- \bar{\psi}_{\hat{\alpha}}^n(x)e^{+imt}, \quad (\check{\Psi}^4)_{Nn} = u_- \bar{\psi}_{4}^n(x)e^{+imt},
\]

where \( \vec{E}_\pm = (1, \pm i) \) denote the polarization vectors with definite helicity \( \pm 1 \). Here all normalization factors are determined by canonical kinetic terms for heavy particles.

As promised, the non-relativistic modes \( w_+, \psi_\hat{\alpha}, \phi_{\hat{\alpha}} \) and \( \psi_4 \) which transform as \( (\mathcal{N} - 1, 1) \) under the unbroken gauge symmetry \( U(N - 1) \times U(N - 1) \) are combined to generate the \( \mathcal{N} = 3 \) vector multiplet

|                  | \( w_+ \) | \( \psi_\hat{\alpha}^\dagger \) | \( \phi_{\hat{\alpha}} \) | \( \psi_4 \) |
|------------------|------------|-------------------------------|-----------------|--------------|
| helicity         | +1         | +1/2                          | 0               | -1/2         |
| degeneracy       | 1          | 3                             | 3               | 1            |

(3.11)

Similarly, the non-relativistic modes which furnish \( (1, \mathcal{N} - 1) \) representation are combined to generate another \( \mathcal{N} = 3 \) vector multiplet

|                  | \( \tilde{w}_- \) | \( \tilde{\psi}_{\hat{\alpha}} \) | \( \tilde{\phi}_{\hat{\alpha}}^\dagger \) | \( \tilde{\psi}_4 \) |
|------------------|------------------|-------------------------------|-----------------|--------------|
| helicity         | -1               | -1/2                          | 0               | +1/2         |
| degeneracy       | 1                 | 3                             | 3               | 1            |

(3.12)

### 3.2 Low-energy dynamics of 1/2-BPS particles

We present in this section the detailed steps to obtain the low-energy Lagrangian \( \hat{\mathcal{L}} \) for the non-relativistic modes. Basically all we need to do is to insert (3.9, 3.10) into the \( U(N) \times U(N) \) ABJM model and expand it to read off the leading terms. The
non-trivial interactions between heavy 1/2-BPS particles with \( U(N - 1) \times U(N - 1) \) ABJM model however leads to several delicate points we should take into account. In particular some \( 1/v \)-corrections to the non-relativistic modes (3.9, 3.10), obtained in the free field theory limit, can arise.

We hereafter use abusing notations for massless \( U(N - 1) \times U(N - 1) \) ABJM fields. They will be denoted by \( Z_\alpha, \Psi^\alpha, A_\mu \) and \( \bar{A}_\mu \) just like original \( U(N) \times U(N) \) ABJM fields unless any confusion arises.

**Scalar parts** Let us first blindly expand the bosonic potential to quadratic order in the off-diagonal massive components which could survive in the infinite mass limit

\[
U^{U(N)} = U^{U(N-1)} + m^2 \tilde{\Phi}^\alpha \left( 1 + \frac{2}{v^2} \Omega^{\beta}_\gamma Z_\beta \bar{Z}_\gamma \right) \Phi_\alpha + m^2 \tilde{\Phi}^\alpha \left( 1 + \frac{2}{v^4} \Omega^{\beta}_\gamma \bar{Z}_\beta Z_\gamma \right) \bar{\Phi}_\alpha \\
+ \frac{m^2}{v} \left( \tilde{\Phi}_4 \bar{Z}^\alpha \Phi_\alpha + \tilde{\Phi}^\alpha Z_\beta \bar{Z}_\gamma \Phi_\alpha + \tilde{\Phi}_4 \bar{Z}^\alpha \Phi_4 + \tilde{\Phi}_4 \bar{Z}_\beta Z_\gamma \Phi_\alpha \right) \\
- \frac{m^2}{v^4} \left( \tilde{\Phi}^\alpha Z_\beta \bar{Z}^\beta \Phi_\beta - \tilde{\Phi}^\beta Z_\gamma \bar{Z}_\ gamma \Phi_\gamma + \tilde{\Phi}_4 Z_\beta \bar{Z}^\beta \Phi_4 + \tilde{\Phi}_4 Z_\gamma \bar{Z}_\ gamma \Phi_\gamma \right), \tag{3.13}
\]

where \( \Omega^{\beta}_\gamma = \text{diag}(1, 1, 1, -1) \) and \( [\Phi_\alpha]_n = (Z_\alpha)_{nN}, [\bar{\Phi}^\alpha]_n = (\bar{Z}^\alpha)_{nN} \). The unitary gauge is not imposed yet for a reason clarified below. Non-trivial interactions in the last two lines of (3.13) lead to subleading corrections to the mass-eigenstates for scalar fields. It implies that the non-relativistic modes (3.9, 3.10) get slightly modified by

\[
\left( \delta_\beta^\alpha - \frac{1}{2v^2} Z_\beta \bar{Z}^\alpha \right) \Phi_\beta + \frac{1}{v} Z_\alpha \bar{\Phi}_4 = \frac{1}{\sqrt{2m}} \varphi_\alpha e^{-imt}, \\
\left( \delta_\beta^\alpha - \frac{1}{2v^2} \bar{Z}^\alpha Z_\beta \right) \bar{\Phi}^\beta + \frac{1}{v} \bar{Z}^\alpha \Phi_4 = \frac{1}{\sqrt{2m}} \bar{\varphi}_\alpha e^{-imt}, \tag{3.14}
\]

and the unitary gauge \( [Z_4]_{nN} = [\bar{Z}_4]_{Nn} = 0 \) is also rotated by

\[
0 = [Z_4]_{nN} \equiv \left( 1 - \frac{1}{2v^2} Z_\beta \bar{Z}^\alpha \right) \Phi_4 - \frac{1}{v} Z_\alpha \bar{\Phi}_4, \\
0 = [\bar{Z}_4]_{Nn} \equiv \left( 1 - \frac{1}{2v^2} \bar{Z}^\alpha Z_\beta \right) \bar{\Phi}_4 - \frac{1}{v} \bar{Z}^\alpha \Phi_4. \tag{3.15}
\]

The above redefinition of the scalar fields can be understood as an infinitesimal \( SU(4)_R \) rotation with the parameter \( \bar{Z}^\alpha / v \). We therefore end up with canonical kinetic terms and interactions for external scalar particles

\[
\mathcal{L}_{\text{scalar}} = i \bar{\varphi}^\alpha D_0 \varphi_\alpha + i \bar{\varphi}_\alpha D_0 \bar{\varphi}^\alpha - \frac{2\pi}{k} \left[ \bar{\varphi}^\alpha \left( \Omega^{\beta}_\gamma Z_\beta \bar{Z}_\gamma \right) \varphi_\alpha - \varphi_\alpha \left( \Omega^{\gamma}_\beta \bar{Z}_\beta Z_\gamma \right) \bar{\varphi}^\alpha \right]. \tag{3.16}
\]

Note that the interactions between massless ABJM scalars and heavy scalars are in a perfect matching with the mass formula (2.19).
Vector parts
Expanding the scalar kinetic terms gives us the interactions between $W$-bosons and massless scalar fields

$$
\mathcal{L}_{\text{vector}}^\text{int} = - |Z^\alpha \vec{W}|^2 - |Z^\alpha \vec{W}|^2 - v^2 |\vec{W} - \frac{Z^4}{v^2} \vec{W}|^2 - v^2 |\vec{W} - \frac{Z^4}{v^2} \vec{W}|^2 + \mathcal{O}(1/v) , (3.17)
$$

where $(\bar{A})_{mN} = \bar{W}_m$ and $(\bar{\bar{A}})_{mN} = \bar{\bar{W}}_m$. They imply again that mass-eigenstates for vector bosons $(3.9, 3.10)$ also get shifted by

$$
\begin{align*}
(1 + \frac{Z^4}{2v^2}) \vec{W} - \frac{Z^4}{v} \vec{W} &= \sqrt{\frac{\pi}{k}} \vec{E}_+ w_+ e^{-imt} , \\
(1 + \frac{Z^4}{2v^2}) \bar{\bar{W}} - \frac{Z^4}{v} \bar{\bar{W}} &= \sqrt{\frac{\pi}{k}} \vec{E}_+ \bar{\bar{w}}_- e^{-imt} .
\end{align*}
$$

This transformation is indeed an infinitesimal Lorentz transformation on the field space $\vec{W}$, leaving the Chern-Simons kinetic terms intact

$$
\mathcal{L}_{\text{CS}}^{U(N)} = \mathcal{L}_{\text{CS}}^{U(N-1)} + i\bar{w}_- D_0 w_+ + i\bar{\bar{w}}_+ D_0 \bar{\bar{w}}_- + m(\bar{w}_- w_+ + \bar{\bar{w}}_+ \bar{\bar{w}}_-) . (3.19)
$$

The Lagrangian for massive vector boson can therefore takes the following form

$$
\begin{align*}
\mathcal{L}_{\text{vector}} = i\bar{w}_- D_0 w_+ + i\bar{\bar{w}}_+ D_0 \bar{\bar{w}}_- - \frac{2\pi}{k} \left[ \bar{w}_- (\Omega^\beta_\alpha Z^\beta) w_+ + \bar{\bar{w}}_+ (\Omega^\beta_\alpha \bar{Z}^\alpha Z^\beta) \bar{\bar{w}}_- \right].
\end{align*}
$$

Fermion parts
In order to find out proper mass-eigenstates for fermion fields, it needs much careful analysis to expand the Yukawa interactions, which turns out to be little tricky. One can show that non-trivial interactions again lead to the sub-leading corrections to the mass-eigenstates as below

$$
\begin{align*}
\left[ (\tilde{\delta}^\gamma_\alpha - \frac{1}{2v^2}(Z^\gamma_\alpha \tilde{\delta}^\gamma_\beta - Z^{\gamma\beta}) \tilde{\delta}_\alpha^\beta) \right] \psi^\beta + \frac{1}{v} \bar{\psi}^\beta \bar{Z}^\beta \psi^\gamma \right]_{nN} = u_+ \psi^\alpha_{+n}(x) e^{-imt} , & (3.21) \\
\left[ (\delta^\alpha_\alpha - \frac{1}{2v^2}(\bar{Z}^\alpha \bar{Z}^\gamma \delta^\gamma_\alpha - \bar{Z}^{\alpha\beta} \bar{Z}^\beta) \right] \bar{\psi}^\beta + \frac{1}{v} \bar{\psi}^\beta \bar{Z}^\beta \bar{\psi}^\gamma \right]_{nN} = u_- \psi^\alpha_{-n}(x) e^{-imt} .
\end{align*}
$$

In terms of modified mass-eigenstates for fermions, the Yukawa coupling can be expanded as

$$
\begin{align*}
\mathcal{L}_{\text{Yukawa}}^{U(N)} &= \mathcal{L}_{\text{Yukawa}}^{U(N-1)} - m(\bar{\psi}_\alpha \psi^\alpha + \bar{\psi}^\alpha \psi^\alpha) - \frac{2\pi}{k} \left[ \bar{\psi}_\alpha (\Omega^\alpha_\beta Z^\beta) \psi^\alpha + \bar{\bar{\psi}}^\alpha (\Omega^\beta_\alpha \bar{Z}^\alpha Z^\beta) \psi^\beta \right] \\
&\quad + \sqrt{\frac{4\pi}{k}} \left[ \bar{\psi}^\alpha \Psi^\alpha_{\alpha -} + \bar{\psi}_\alpha \Psi^\alpha_{+} + \bar{\psi}_\alpha \Psi^\alpha_{\alpha +} + \bar{\bar{\psi}}^\alpha \bar{\psi}^\alpha_{\alpha -} + \bar{\bar{\psi}}^\alpha \bar{\psi}^\alpha_{+} \right] + \mathcal{O}(\frac{1}{\sqrt{m}}) . (3.22)
\end{align*}
$$

For clarity, we hereafter sometimes drop the helicity indices unless it does not make any confusion. The fermion kinetic terms can be expanded as below

$$
\begin{align*}
\mathcal{L}_{\text{kin}}^{U(N)} &= \mathcal{L}_{\text{kin}}^{U(N-1)} + m(\bar{\psi}_\alpha \psi^\alpha + \bar{\psi}^\alpha \psi^\alpha) + i\bar{\psi}_\alpha D_0 \psi^\alpha + i\bar{\psi}^\alpha D_0 \bar{\psi}^\alpha \\
&\quad + \sqrt{\frac{4\pi}{k}} \left[ \bar{\psi}_\alpha \Psi^\alpha_{\alpha -} + \bar{\psi}^\alpha \Psi^\alpha_{+} + \bar{\bar{\psi}}^\alpha \bar{\psi}^\alpha_{\alpha +} + \bar{\psi}_\alpha \Psi^\alpha_{\alpha +} + \bar{\bar{\psi}}^\alpha \bar{\psi}^\alpha_{+} \right] + \mathcal{O}(\frac{1}{\sqrt{m}}) .
\end{align*}
$$
As a consequence, the Lagrangian for heavy fermions becomes

\[
\mathcal{L}_{\text{fermion}} = i\bar{\psi}_\alpha D_0 \psi^\alpha + i\bar{\psi}^\alpha D_0 \bar{\psi}_\alpha - \frac{2\pi}{k} \left[ \bar{\psi}_\alpha \left( \Omega^\alpha_{\beta} Z^\beta \right) \psi^\alpha + \bar{\psi}^\alpha \left( \Omega^\beta_{\alpha} \bar{Z}^\beta \right) \bar{\psi}_\alpha \right]
\]

+ \sqrt{\frac{4\pi}{k}} \left[ \bar{\omega}_+ \bar{\Psi}_4 - \psi_4^+ \bar{\Psi}_4 - \psi_4^+ \bar{\omega}_- + \bar{\psi}_4 \bar{\Psi}_4^+ \psi_4^+ \bar{\Psi}_4^+ \psi_4^+ \bar{\omega}_- + \bar{\omega}_- \Psi_4^4 \bar{\psi}_4^+ \psi_4^+ \bar{\psi}_4^+ \psi_4^+ \bar{\omega}_- \right].

(3.23)

Summary Collecting the results (3.16), (3.20) and (3.23), the low-energy dynamics of external 1/2-BPS particles interacting with ABJM fields are governed by

\[
\mathcal{L} = i\bar{\omega} D_0 \omega + i\bar{\omega}_+ \tilde{D}_0 \bar{\omega}_- + i\bar{\psi} D_0 \psi + i\bar{\psi}^\alpha \tilde{D}_0 \bar{\psi}_\alpha + i\bar{\psi}^\alpha \tilde{D}_0 \bar{\psi}_\alpha + i\bar{\psi}^\alpha \tilde{D}_0 \bar{\psi}_\alpha + i\bar{\psi}^\alpha \tilde{D}_0 \bar{\psi}_\alpha
\]

+ \sqrt{\frac{4\pi}{k}} \left[ \bar{\omega}_+ \bar{\Psi}_4 - \psi_4^+ \bar{\Psi}_4 - \psi_4^+ \bar{\omega}_- + \bar{\psi}_4 \bar{\Psi}_4^+ \psi_4^+ \bar{\Psi}_4^+ \psi_4^+ \bar{\omega}_- \right].

(3.24)

where the covariant derivatives are defined as

\[
D_0 = \partial_0 - iA_0, \quad A_0 = A_0 - \frac{2\pi}{k} \Omega^\alpha_{\beta} Z^\beta, \quad \tilde{D}_0 = \partial_0 - i\tilde{A}_0, \quad \tilde{A}_0 = \tilde{A}_0 - \frac{2\pi}{k} \Omega^\beta_{\alpha} \bar{Z}^\alpha Z_\beta.
\]

(3.25)

4 Half BPS Wilson Line in M2-Theory

It is now ready to discuss the half BPS Wilson line with focus on physical origin of the superconnection. Let us begin by managing the low-energy Lagrangian for heavy particles (3.24) into an appealing expression

\[
\mathcal{L} = \text{Tr} \left[ i\bar{\Psi}^\alpha \tilde{D}_0 \bar{\Psi}_\alpha \right],
\]

(4.1)

where \( \Psi_\alpha \) are supermatrices defined as

\[
\Psi^\alpha = \begin{pmatrix} \psi^\alpha \\ \bar{\psi}_\alpha \end{pmatrix}, \quad \Psi^4 = \begin{pmatrix} \omega^+ \\ \bar{\omega}_- \end{pmatrix},
\]

(4.2)

and \( \tilde{D}_0 \) represent a super-covariant derivative with an \( U(N|N) \) superconnection \( \tilde{A}_0 \)

\[
\tilde{D}_0 = \partial_0 - i\tilde{A}_0, \quad \tilde{A}_0 = \begin{pmatrix} \tilde{A}_0 \\ \sqrt{\frac{4\pi}{k}} \bar{\Psi}_4^4 \tilde{A}_0 \end{pmatrix}.
\]

(4.3)
It implies that the low-energy dynamics of 1/2-BPS massive particles respects the supergauge symmetry $U(\mathcal{N}|\mathcal{N})$, provided that the matter fields $\Psi_\alpha$ and superconnection $\hat{A}_0$ transforms as

$$
\Psi \rightarrow U^\dagger \Psi_\alpha \ , \ \hat{A}_0 \rightarrow U^\dagger \hat{A}_0 U + i U^\dagger \partial_0 U \ , \ \ U = e^{-i\Lambda} \in U(\mathcal{N}|\mathcal{N}) \ .
$$

(4.4)

As mentioned repeatedly, one can understand the 1/2-BPS Wilson line evolve under the interactions to ABJM model. Since the equations of motions for massive particles are

$$
\hat{D}_0 \Psi_\alpha = 0 \ ,
$$

(4.5)

the time-evolution factor of the wavefunctions, or 1/2-BPS Wilson line is given by

$$
W(t) = \mathcal{P} \text{exp} \left[ i \int_0^t d\tau \ \hat{A}_0 \right] ,
$$

(4.6)

which exactly matches with the result of [8]. The supergauge symmetry of (4.1) explains the physical origin of the form of the 1/2-BPS Wilson line as the holonomy of $U(\mathcal{N}|\mathcal{N})$ superconnection. Free constant spinor parameters $\eta_\alpha$ of [8] can be also understood as helicity projections of massive 1/2-BPS particles in the infinite mass limit.

Note that the wavefunctions evolves by mixing the particles in different $\mathcal{N} = 3$ vector multiplets. It strongly implies that the Wilson line could be invariant under the supersymmetry parameters complement to (3.6)

$$
\xi_{\hat{\alpha}4}^+, \quad \xi_{\hat{\alpha}\hat{\beta}}^- ,
$$

(4.7)

which also mixes particles in different $\mathcal{N} = 3$ vector multiplets. It indeed turns out to be the case. Under the supersymmetry transformation with (4.7), one can show that the superconnection $\hat{A}_0$ transforms as

$$
\delta_{\text{SUSY}} \hat{A}_0 = \partial_0 \Lambda - i \{ \hat{A}_0 , \Lambda \}
$$

(4.8)

with

$$
\Lambda = \sqrt{\frac{4\pi}{k}} \begin{pmatrix}
0 & i Z_{\hat{\alpha}} \xi_{\hat{\alpha}4}^+

-iZ_{\hat{\alpha}} \xi_{\hat{\alpha}4}^- & 0
\end{pmatrix} .
$$

(4.9)

It is nothing but a specific supergauge transformation with parameter $\Lambda \in u(\mathcal{N}|\mathcal{N})$.

The Wilson line operator

$$
W(t_f, t_i) = \mathcal{P} \text{exp} \left[ i \int_{t_i}^{t_f} d\tau \ \hat{A}_0 \right] ,
$$

(4.10)

would transform covariantly under the supersymmetry

$$
W(t_f, t_i) \rightarrow U(t_f)^\dagger W(t_f, t_i) U(t_i) .
$$

(4.11)
For a closed loop, there are two possible periodic boundary conditions on the fermion fields $\Psi_4$ in the superconnection $\hat{A}$. The supersymmetric transformation (4.8) in turn decides the boundary condition on $\xi_{\hat{A}_4}$. For the periodic boundary condition, the 1/2-BPS Wilson loop involves supertrace,

$$W_{\text{periodic}} = \text{STr} \exp \left[ \oint d\tau \hat{A}_0 \right]$$  \hspace{1cm} (4.12)

while, for the anti-periodic one, taking the ordinary trace gives us the 1/2-BPS Wilson loop operator

$$W_{\text{anti-periodic}} = \text{Tr} \exp \left[ \oint d\tau \hat{A}_0 \right].$$  \hspace{1cm} (4.13)

In [8], it has been argued that the proper boundary condition for the circular Wilson loop in Euclidean $\mathbb{R}^3$ is the anti-periodic boundary condition, i.e., the ordinary trace leads to the supersymmetric Wilson loop.

5 1/2-BPS Vortices and External Particles

In the previous section we showed how the Wilson line operator arises when the external particles interact with the ABJM model. Let us now in turn think of a different aspect of the Wilson line operator. In particular, we are interested in how some classical pictures can be affected when the Wilson line operator is introduced in the path integral formulations of low-energy theory on M2-branes. Similar to our analysis below has been studied in [7] for different purposes.

Let us start with the $U(1) \times U(1)$ ABJM model whose bosonic Lagrangian takes the following form

$$\mathcal{L}^{(1)}_{\text{abelian}} = -D_{\mu} \tilde{Z}^{\alpha} D^\mu Z_\alpha + \frac{k}{2\pi} \epsilon^{\mu\nu\rho} b_{\mu} \partial_\nu c_\rho$$

$$= - \left| (\partial_{\mu} - ib_{\mu}) Z_\alpha \right|^2 + \frac{1}{4\pi} \epsilon^{\mu\nu\rho} (kb_{\mu} - \partial_{\mu} \sigma) f_{\nu\rho},$$  \hspace{1cm} (5.1)

where $b_{\mu} = A_{\mu} - \tilde{A}_{\mu}$ and $c_{\mu} = (A_{\mu} + \tilde{A}_{\mu})/2$. For the last equality, one introduces an auxiliary two-form field $f_{\mu\nu}$. The invisibility of magnetic monopoles of $2\pi f_{12}$-flux demands the scalar field $\sigma$ to have $2\pi$ periodicity $\sigma \sim \sigma + 2\pi$. Integrating over $f_{\mu\nu}$, one can rewrite (5.1) into the almost free Lagrangian

$$\mathcal{L}^{(1)}_{\text{abelian}} = - \left| \partial_{\mu} \tilde{Z}_\alpha \right|^2,$$

$$\tilde{Z}_\alpha = e^{i\sigma/k} Z_\alpha,$$  \hspace{1cm} (5.2)

with the residual gauge symmetry $\tilde{Z}_\alpha \sim e^{2\pi n/k} \tilde{Z}_\alpha$, which leads to the moduli space of a single M2 brane

$$\mathcal{M} = \mathbb{C}^4/\mathbb{Z}_k.$$  \hspace{1cm} (5.3)
1/2-BPS vortex  We first study a classical 1/2-BPS bosonic object in the $U(1) \times U(1)$ ABJM model for simplicity. Looking at the SUSY variation rules for fermions

$$\delta \psi^\alpha_+ = -i \xi^{\alpha \beta}_+ D_0 Z_\beta + i \xi^{\alpha \beta}_+ D_+ Z_\beta ,$$

$$\delta \psi^\alpha_- = +i \xi^{\alpha \beta}_- D_0 Z_\beta - i \xi^{\alpha \beta}_+ D_- Z_\beta ,$$

one can show that vortex solutions satisfying the following equations

$$Z_1 = Z_2 = Z_3 = 0 , \quad D_0 Z_4 = 0 , \quad D_+ Z_4 = 0 , \quad D_- Z_4 \neq 0 ,$$

preserves half of the supersymmetry along $\hat{\xi}_4^0, \hat{\xi}_4^\beta$. Here $D_\pm = D_1 \pm i D_2$ and $A_\pm = A_1 \pm i A_2$. The field equations for gauge bosons are given by

$$\frac{k}{2\pi} F_{i0} + D_i (Z_4 \bar{Z}^4) = 0 , \quad F_{12} = 0 ,$$

$$\frac{k}{2\pi} \tilde{F}_{i0} + D_i (\bar{Z}^4 Z_4) = 0 , \quad \tilde{F}_{12} = 0 .$$

For static solutions, they can be solved by

$$A_0 = \tilde{A}_0 = -\frac{2\pi}{k} |Z_4|^2 .$$

In terms of the gauge invariant variable $\hat{Z}_4$, the 1/2-BPS vortex equation $\partial_+ \hat{Z}_4 = 0$ implies that the vortex configuration can be described as a holomorphic function. Due to the residual gauge symmetry $Z_k$ ($\hat{Z}_4 \sim e^{2\pi i/k} \hat{Z}_4$), the 1/2-BPS elementary vortex becomes

$$\hat{Z}_4 = \frac{p_v}{(z - z_0)^{1/k}} ,$$

where $z_0$ denotes the position of the source. Note that the dimensionful parameter $p_v$ indicates the complicated internal structure of the solution\#1.

The elementary vortex describes how the world-sheet of single M2-brane is deformed by the external point source. The configuration (5.8) implies that the M2-brane is pulled by the source to the spatial infinity, and wraps the $S^1$ fibre of the orbifold $\mathbb{C}^4/Z_k$ once. The source can be therefore identified as an infinitely long type IIA fundamental string, qualitatively similar to heavy particles of the Wilson line.

This elementary vortex has infinite energy due to its singularity at the point $z_0$. Although the gauge fields should be pure gauge, they can carry the non-zero flux at the source: suppose that

$$A_- = -\frac{i \alpha}{z - z_0} , \quad \tilde{A}_- = -\frac{i \beta}{z - z_0} ,$$

which describe point magnetic fluxes $2\pi \alpha$ and $2\pi \beta$ at the source $z_0$. The BPS equations for vortex solutions becomes

$$D_+ Z_4 = (\partial_+ + \frac{\alpha - \beta}{z - z_0}) Z_4 = 0 , \quad D_- Z_4 = (\partial_- + \frac{\alpha - \beta}{z - z_0}) Z_4 = 0 .$$

\#1There is also a 1/2-BPS funnel solution $\hat{Z}_4 = c_f z^{1/k}$ which has different boundary condition.
In the abelian Higgs system, the vorticity and magnetic flux are tightly correlated when we require the finite energy configurations so that the gauge invariant scalar has net-zero vorticity. Our vortex is somewhat different as the scalar $Z_4$ vanishes at the spatial infinity. All we need to require is that the gauge invariant $\hat{Z}_4$ carries $1/k$ vorticity. The natural choice for the 1/2-BPS external source then generates two types of vortices as

$$Z_4 = \frac{c}{|z-z_0|^{1/k}}, \quad A_- = -\frac{i}{k(z-z_0)} (A - \text{type vortex}),$$
$$Z_4 = \frac{\tilde{c}}{|z-z_0|^{1/k}}, \quad \tilde{A}_- = +\frac{i}{k(z-z_0)} (\tilde{A} - \text{type vortex}).$$ (5.11)

**Dual description of ABJM model**  In order to study another aspect of the 1/2-BPS vortex, let us present a dual description of the ABJM model with use of the vector-scalar duality, so-called Mukhi-Papageorgakis map [18].

Integrating over the gauge field $b_\mu$ in (5.1), one can obtain

$$C_{\mu\nu} \equiv \partial_\mu c_\nu - \partial_\nu c_\mu = 2\epsilon_{\mu\nu\rho} \sum_\alpha \left| Z_\alpha \right|^2 \left( \partial^\rho \text{arg} Z_\alpha - b^\rho \right).$$ (5.12)

Since we are now interested in the case where all scalars except $Z_4$ are turned off, $b_\mu$ can be expressed by

$$b_\mu = \frac{1}{\phi} \left[ \partial_\mu \text{arg} Z_4 - \frac{1}{4} \epsilon_{\mu\nu\rho} C^{\nu\rho} \right], \quad \phi = \frac{2\pi}{k} |Z_4|^2.$$ (5.13)

In terms of $C_{\mu\nu}$ and $\phi$, the Lagrangian (5.1) can be rewritten as

$$\mathcal{L}^{(2)}_{\text{abelian}} = -\frac{k}{8\pi \phi} (\partial_\mu \phi)^2 - \frac{k}{16\pi \phi} C_{\mu\nu} C^{\mu\nu}.$$ (5.14)

It is noteworthy here that $\phi$ and $c_\mu$ are in fact combined to form $N = 2$ vector multiplet in three dimensions. For more systematic analysis in a manifestly supersymmetric fashion, it is referred to [19].

**1/2-BPS vortex revisited**  In terms of the $\phi$ and $c_\mu$ variables, the 1/2 BPS vortices can be characterized by the solution

$$\phi = -c_0 = \frac{2\pi}{k} \frac{p_\nu^2}{|z|^{2/k}}$$ (5.15)

with fluxes at the source. One can roughly understand that it is the BPS solution from the complete squares of energy density

$$\mathcal{E} = \frac{k}{8\pi \phi} \left( (\partial_i \phi)^2 + C_{i0}^2 \right) = \frac{k}{8\pi \phi} (\partial_i \phi + C_{i0})^2 - \partial_i \left( \frac{k}{4\pi} C_{i0} \right),$$ (5.16)
up to the Gauss law. Here we ignored the source term. In order to generate the 1/2-BPS vortex (5.15), one should add to the Lagrangian (5.14) the point source terms as

\[ \mathcal{L}_{\text{source}} = (\phi + c_0) \delta^2(z) + \cdots, \quad (5.17) \]

where we do not specify the source terms for the flux yet.

Let us consider the insertion of the A-type 1/2-BPS Wilson-line to the abelian theory, ignoring fermionic contributions. We introduce the source terms below

\[ \mathcal{L}_{\text{source}}^{A\text{-type}} = \left( A_0 - \frac{2\pi}{k}(|Z_\alpha|^2 - |Z_4|^2) \right) \delta^2(z) \]

\[ = \left( c_0 + \phi \right) \delta^2(z) + \frac{1}{2}b_0\delta^2(z) - \frac{2\pi}{k} |Z_\alpha|^2 \delta^2(z) \quad (5.18) \]

Both the scalar field equation and the energetic consideration imply that \( Z_\alpha = 0 \) to prevent the energy increase. Here \( A_0 = c_0 + b_0/2 \). One can also show that the second terms of (5.18) is necessary to generate the \( f_{12} \) flux at the source \( f_{12} = \pi \delta^2(x)/k \). The above source terms can therefore be identified as the source terms (5.17) for 1/2-BPS vortices. This analysis confirms that the insertion of 1/2 BPS Wilson line parallels to the insertion of 1/2 BPS vortices.

The multi-vortex solutions are given by

\[ \hat{Z}_4 \equiv f(z) = \frac{P_v}{\prod_p (z - z_p)^{1/k}}. \quad (5.19) \]

where the parameter \( P_v \) is dimensionful in general. Note that the multi-vortex solution is multiplicative, not additive. When two vortices overlap, the factor therefore becomes \( 1/z^{2/k} \) so that the winding over the \( S^1 \) fibre is doubled. In the large \( k \) limit, one can show both vortex and funnel solutions become the logarithmic solutions similar to the reaction of a D2-brane under the external source.

Only for \( k = 2 \), a single vortex solution can be scale invariant due to the fact that the coefficient \( P_v \) become dimensionless. For even \( k \), \( k/2 \) vortices on the top of each other is scale invariant. Further properties of the scale invariant solution has been investigated in [7].

Ultimately we are interested in the insertion of non-abelian Wilson line operators with their quantum nature. It would be very interesting to see how such structure parameters would survive and manifest. At least from the string calculation of particle-antiparticle attractive force in finite distance, done in [4], the attractive energy is given by

\[ E \sim -\frac{1}{L} \sqrt{\frac{N}{k}} \quad (5.20) \]

which is blind to the structures of vortices. It is also falling off faster than the fall off \( 1/L^{2/k} \), naively expected for vortex/anti-vortex attraction.
Acknowledgement

We would like to thank Nakwoo Kim, Sangmin Lee, and Piljin Yi for valuable discussions. We also thank Jun-Bao Wu for collaboration at an early stage of this work. This work (KML) is supported in part by the Center for Quantum Spacetime of Sogang University with grant number R11-2005-021 and the Korea National Research Foundation (No.2006-0093850, No. 2009-0084601).
Appendix

A Infinite Mass Limit in Free Field Theories

The infinite mass limit \( v \to \infty \) somehow parallels with the standard non-relativistic limit \( c \to \infty \), due to the fact that factors \( (\vec{p}/mc) \) and \( (E/mc^2) \) become negligible in both limits. Based on the analysis for the non-relativistic limit of free massive particles [16, 17], we present how to take the infinite mass limit for those particles.

A.1 Scalar field

Let us begin by a Lagrangian for a free massive scalar

\[
L^{\text{scalar}} = D_0 \bar{Z} D_0 Z - D_i \bar{Z} D_i Z - m^2 \bar{Z} Z .
\]

(A.1)

Considering a particle mode in the scalar field \( Z \)

\[
Z = \frac{1}{\sqrt{2m}} \phi(t, x)e^{-imt} ,
\]

(A.2)

the above Lagrangian in the limit \( m \to \infty \) becomes

\[
L^{\text{scalar}}_{\text{massive}} = i \bar{\phi} D_0 \phi + O(1/m) ,
\]

(A.3)

where we suppress the irrelevant terms.

A.2 Fermion field

The Lagrangian for a free massive fermion takes the following form

\[
L^{\text{fermion}} = -i \bar{\Psi} \gamma^\mu D_\mu \Psi \mp im \bar{\Psi} \Psi .
\]

(A.4)

Keeping only the particles again, one can expand the fermion field \( \Psi \) as

\[
\Psi(t, x) = \left( u_+ \psi_-(t, x) + u_- \psi_+(t, x) \right) e^{-imt} ,
\]

(A.5)

where \( \psi_\pm \) are single-component Grassmann fields and \( u_\pm \) are orthonormal two-component constant spinors (2.13). Defining \( D_\pm = D_1 \pm iD_2 \) and \( A_\pm = A_1 \pm iA_2 \), the fermionic Lagrangian can be rewritten as

\[
L^{\text{fermion}} = \bar{\psi}_+ \left( iD_0 \psi_- + m(1 \mp 1) \psi_- - iD_- \psi_+ \right) + \bar{\psi}_- \left( iD_0 \psi_+ + m(1 \pm 1) \psi_+ - iD_+ \psi_- \right) .
\]

(A.6)
Using the equation of motion for $\bar{\psi}$ up to the leading order, one can show that one of the components $\psi_{\pm}$ is completely determined by the other

$$\begin{cases} 
\psi_+ = \frac{i}{2m} D_0 \psi_+ - \frac{i}{2m} D_t \psi_+ & \simeq O(1/m) \quad \text{for upper sign}, \\
\psi_- = \frac{i}{2m} D_0 \psi_- - \frac{i}{2m} D_t \psi_- & \simeq O(1/m) \quad \text{for lower sign}.
\end{cases} \quad (A.7)$$

It implies that the spin of dynamical modes in the infinite mass limit is correlated with the sign of the mass. Inserting the above relations, the Lagrangian becomes

$$L_{\text{fermion massive}} = \begin{cases} 
i \bar{\psi}_+ D_0 \psi_+ + O(\frac{1}{m}) & \text{for upper sign}, \\
i \bar{\psi}_- D_0 \psi_+ + O(\frac{1}{m}) & \text{for lower sign}.
\end{cases} \quad (A.8)$$

### A.3 W-boson in Chern-Simons theory

Let us then discuss the massive W-bosons. It is well-known that, in the broken phase of Chern-Simons-matter theories, there is a single massive W-boson that propagates. We will first discuss how to obtain the polarization vector for such a propagating mode in the infinite mass limit.

In the broken phase of Chern-Simons-matter theories, the free Lagrangian for W-boson $W_\mu$ can take the following form

$$L^{W\text{-boson}} = \pm \frac{k}{2\pi} \epsilon^{\mu \nu \rho} W_\mu \partial_\nu W_\rho - v^2 W_\mu \partial_\mu W^\mu , \quad (\epsilon^{012} = 1) \quad (A.9)$$

from which one can derive the equation of motions for W-bosons

$$\frac{k}{2\pi} \epsilon^{\mu \nu \rho} D_\nu W_\rho = \pm v^2 W^\mu . \quad (A.10)$$

Here $v$ stands for the vacuum expectation value for the Higgs scalar. The equation of motion for $\mu = 0$ tells us that the temporal part of vector field behaves like $W_0 \sim O(1/v^2)$. The rest of the equation of motion then reduces to

$$\epsilon^{ij} D_0 W_j = \pm \frac{2\pi}{k} v^2 W^i + O(1/v) . \quad (A.11)$$

It implies that the helicity of propagating modes in the infinite mass limit $v \rightarrow \infty$ is again correlated with the sign of mass as follows:

$$\widehat{W} \sim \begin{cases} 
\sqrt{\frac{\pi}{k}} E_\mp w_\pm(t, x)e^{-imt} + O(1/v) \\
\sqrt{\frac{\pi}{k}} E_\mp w_\mp(t, x)e^{imt} + O(1/v)
\end{cases} , \quad (A.12)$$

where $E_\pm = (1, \pm i)$ denote the polarization vector with definite helicity $\pm 1$. The Lagrangian of the particle/anti-particle mode with $k > 0$ then becomes

$$L_{\text{massive}}^{W\text{-boson}} = i \bar{w}_\mp D_0 w_\pm + O(1/m) . \quad (A.13)$$
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