The variational capacity with respect to nonopen sets in metric spaces

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Abstract. We pursue a systematic treatment of the variational capacity on metric spaces and give full proofs of its basic properties. A novelty is that we study it with respect to nonopen sets, which is important for Dirichlet and obstacle problems on nonopen sets, with applications in fine potential theory. Under standard assumptions on the underlying metric space, we show that the variational capacity is a Choquet capacity and we provide several equivalent definitions for it. On open sets in weighted $\mathbb{R}^n$ it is shown to coincide with the usual variational capacity considered in the literature.

Since some desirable properties fail on general nonopen sets, we introduce a related capacity which turns out to be a Choquet capacity in general metric spaces and for many sets coincides with the variational capacity. We provide examples demonstrating various properties of both capacities and counterexamples for when they fail. Finally, we discuss how a change of the underlying metric space influences the variational capacity and its minimizing functions.

Key words and phrases: Choquet capacity, doubling measure, metric space, Newtonian space, nonlinear, outer capacity, $p$-harmonic, Poincaré inequality, potential theory, quasi-continuous, Sobolev space, upper gradient, variational capacity.

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1. Introduction

The variational capacity $\text{cap}_p(A, \Omega)$ has been used extensively in nonlinear potential theory on $\mathbb{R}^n$, e.g. in the monographs Heinonen–Kilpeläinen–Martio [22] and Malý–Ziemer [34]. Roughly speaking it is the energy of the $p$-harmonic function in $\Omega \setminus A$ with zero boundary values on $\partial \Omega$ and boundary values 1 on $A$, but its exact definition in $\mathbb{R}^n$ is usually done in three steps: through a minimization problem for compact $A$ and then by inner and outer regularity for open and arbitrary sets $A$. Also the choice of admissible functions in the minimization problem varies in the literature.

The variational capacity is closely related to capacitory potentials and thus naturally appears in the Wiener criterion for boundary regularity of $p$-harmonic functions (Maz′ya [36], Lindqvist–Martio [32], Kilpeläinen–Malý [25]), even though in unweighted $\mathbb{R}^n$ with $p < n$ it can be equivalently replaced by the Sobolev capacity. Through the Wiener integral it also plays an important role in the definition of thinness and in nonlinear fine potential theory ([22], [34]).
In nonlinear potential theory on metric spaces, the variational capacity \( \text{cap}_p \) (Definition 3.1) has been used in different contexts in e.g. [3, 6, 7, 9, 13–17, 19, 27, 29, 30] and [35]. Very few of even the basic properties of \( \text{cap}_p \) have been given full proofs in the metric space literature, even though they must be known to experts in the field.

In this paper we pursue a systematic treatment of the variational capacity on metric spaces and give full proofs of its basic properties. (Recently some of these results were for open \( E \) included in the monograph Björn–Björn [5].) A novelty here is that we consider \( \text{cap}_p(A, E) \) when \( E \) is not open, which does not seem to have been considered earlier. This is motivated by the study of obstacle and Dirichlet problems on nonopen sets in Björn–Björn [6], where some of the results in this paper are used. Another motivation is the use of \( \text{cap}_p \) in the development of fine potential theory on metric spaces which has been touched upon in [6] and which we want to pursue in forthcoming papers.

Let \( X \) be a metric space equipped with a Borel measure \( \mu \). After presenting some background results in Section 2, we define the variational capacity \( \text{cap}_p(A, E) \) in Section 3 and establish its basic properties holding in full generality, such as countable and strong subadditivity. The definition of \( \text{cap}_p \) on metric spaces is more straightforward than in \( \mathbb{R}^n \), due to the use of Newtonian spaces which are the natural (and in some aspects better) substitutes for Sobolev spaces in the setting of metric spaces. This straightforward approach is essential when studying the variational capacity \( \text{cap}_p(A, E) \) with respect to nonopen \( E \). This on the other hand means that the outer regularity

\[
\text{cap}_p(A, E) = \inf_{G \text{ open}} \text{cap}_p(G, E) \tag{1.1}
\]

on metric spaces is not a direct consequence of the definition and can be proved only under additional assumptions. In particular, in Theorem 4.1 we show that \( \text{cap}_p, p > 1 \), is an outer capacity for sets \( A \subset \text{int} \, E \), provided that all Newtonian functions are quasicontinuous (which holds e.g. under the standard assumptions that \( X \) is complete and the measure \( \mu \) is doubling and supports a \((1, p)\)-Poincaré inequality). As a consequence of this result we obtain the equality between the capacity of a set and its fine closure. Together with other properties, which we prove here, the outer regularity implies that \( \text{cap}_p(\cdot, E) \) is a Choquet capacity and all Borel sets are capacitable, see Theorem 4.9.

This and the above outer regularity result are then used in Section 5 to show that our variational capacity coincides with the variational capacity on (weighted and unweighted) \( \mathbb{R}^n \) considered in Heinonen–Kilpeläinen–Martio [22]. Since these capacities are defined in different ways and using different admissible functions, their equality is not straightforward but relies on the above mentioned Theorems 4.1 and 4.9. Another ingredient is that the Newtonian spaces on weighted \( \mathbb{R}^n \) coincide with the usual weighted Sobolev spaces considered in [22], with equal norms. The proof of this relies on a deep result of Cheeger [18] and is given in Björn–Björn [5].

If \( E \) is open and the metric space \( X \) satisfies the above standard assumptions, then Theorems 4.1 and 4.9 imply that \( \text{cap}_p \) on \( X \) can be defined without Newtonian spaces using only elementary properties of Lipschitz functions in a similar way as in \( \mathbb{R}^n \), see Section 6. This is however not possible for nonopen \( E \) and not known in more general spaces. We also give examples showing that the outer regularity (1.1) of \( \text{cap}_p \) is not true for arbitrary \( A \subset E \). This suggests the following definition of a related capacity:

\[
\overline{\text{cap}}_p(A, E) = \inf_{G \text{ relatively open}} \text{cap}_p(G, E), \tag{1.2}
\]

which is obviously outer for all sets \( A \subset E \) and turns out to be a Choquet capacity for many \( A \subset E \) without any additional assumptions on \( X \), see Theorem 6.3. If
is locally compact then this holds for all $A \subset E$. Thus, $\tilde{\text{cap}}_p$ seems to be a “better” modification of $\text{cap}_p$, and in Proposition 6.5 we show that under certain assumptions, $\tilde{\text{cap}}_p(A, E) = \text{cap}_p(A, E)$ or $\tilde{\text{cap}}_p(A, E) = \infty$. At the same time, some natural basic properties can fail for $\tilde{\text{cap}}_p$, see Section 6. Moreover, in connection with e.g. Adams’ criterion in Björn–Björn [6], it is our capacity $\text{cap}_p$ which is needed and its role cannot be played by any other nonequivalent capacity.

We finish the paper with a short discussion on how changes of the underlying space $X$ influence $\text{cap}_p$, see Example 6.6.

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2. Notation and preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a metric space equipped with a metric $d$ and a measure $\mu$ such that

$$0 < \mu(B) < \infty$$

for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in $X$ (we make the convention that balls are nonempty and open). We also assume that $1 \leq p < \infty$ and that $E \subset X$ is a bounded set.

The $\sigma$-algebra on which $\mu$ is defined is obtained by completion of the Borel $\sigma$-algebra. The measure $\mu$ is doubling if there exists a constant $C > 0$ such that

$$0 < \mu(2B) \leq C \mu(B) < \infty,$$

where $\lambda B = B(x_0, \lambda r)$. Note that if $\mu$ is doubling then $X$ is complete if and only if $X$ is proper, i.e. all bounded closed sets are compact.

A curve is a continuous mapping from an interval. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length $ds$. We follow Heinonen and Koskela [23] in introducing upper gradients as follows (they called them very weak gradients).

**Definition 2.1.** A nonnegative Borel function $g$ on $X$ is an upper gradient of an extended real-valued function $f$ on $X$ if for all (nonconstant, compact and rectifiable) curves $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds,$$

where we make the convention that the left-hand side is $\infty$ whenever both terms therein are infinite. If $g$ is a nonnegative measurable function on $X$ and if (2.1) holds for $p$-almost every curve (see below), then $g$ is a $p$-weak upper gradient of $f$.

Here and in what follows, we say that a property holds for $p$-almost every curve if it fails only for a curve family $\Gamma$ with zero $p$-modulus, i.e. there exists $0 \leq \rho \in L^p(X)$ such that $\int_{\Gamma} \rho \, ds = \infty$ for every curve $\gamma \in \Gamma$. Note that a $p$-weak upper gradient need not be a Borel function, only measurable. It is implicitly assumed that $\int_\gamma g \, ds$ is defined (with a value in $[0, \infty]$) for $p$-almost every curve $\gamma$, although this is in fact a consequence of the measurability. For proofs of these and all other facts in this section we refer to Björn–Björn [5]. (Some of the references we mention below may not provide a proof in the generality considered here, but such proofs are given in [5].)
The $p$-weak upper gradients were introduced in Koskela–MacManus [31]. They also showed that if $g \in L^p_{\text{loc}}(X)$ is a $p$-weak upper gradient of $f$, then one can find a sequence $\{g_j\}_{j=1}^{\infty}$ of upper gradients of $f$ such that $g_j - g \to 0$ in $L^p(X)$. If $f$ has an upper gradient in $L^p_{\text{loc}}(X)$, then it has a minimal $p$-weak upper gradient $g_f \in L^p_{\text{loc}}(X)$ in the sense that for every $p$-weak upper gradient $g \in L^p_{\text{loc}}(X)$ of $f$ we have $g_f \leq g$ a.e., see Shanmugalingam [38] and Hajłasz [20]. The minimal $p$-weak upper gradient is well defined up to an equivalence class in the cone of nonnegative functions in $L^p_{\text{loc}}(X)$. Following Shanmugalingam [37], we define a version of Sobolev spaces on the metric space $X$.

**Definition 2.2.** The *Newtonian space* on $X$ is

$$N^{1,p}(X) = \{ u : \|u\|_{N^{1,p}(X)} < \infty \},$$

where

$$\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p d\mu + \int_X g^p u d\mu \right)^{1/p}$$

for an everywhere defined measurable function $u : X \to \mathbb{R}$ having an upper gradient in $L^p_{\text{loc}}(X)$.

The space $N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [37]. For a measurable set $E \subset X$, the space $N^{1,p}(E)$ is defined by considering $E$ as a metric space on its own. Let us here point out that we assume that functions in Newtonian spaces are defined everywhere, and not just up to equivalence classes in $L^p$.

If $u, v \in N^{1,p}_{\text{loc}}(X)$, then their minimal $p$-weak upper gradients coincide a.e. in the set $\{ x \in X : u(x) = v(x) \}$, in particular $g_{\min(u,c)} = g_u \chi_{\{u < c\}}$ a.e. for $c \in \mathbb{R}$. Moreover, $g_{uv} \leq |g_u| + |v|g_u$.

**Definition 2.3.** The *Sobolev capacity* of a set $A \subset X$ is the number

$$C_p(A) = \inf \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on $A$.

We say that a property holds *quasi-everywhere* (q.e.) if the set of points for which it fails has capacity zero.

The Sobolev capacity was introduced and used for Newtonian spaces in Shanmugalingam [37]. It is countably subadditive and the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$ and $v : X \to \mathbb{R}$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e. The proofs of properties for $C_p$ are similar or easier than the proofs of the corresponding properties for the variational capacity $\text{cap}_p$, presented in this paper. Note also that if $C_p(E) = 0$, then $p$-almost every curve in $X$ avoids $E$, by e.g., Lemma 3.6 in Shanmugalingam [37] or Proposition 1.48 in Björn–Björn [5].

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. We let

$$N^{1,p}_0(E) = \{ f \in E : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus E \}.$$

One can replace the assumption “$f = 0$ on $X \setminus E$” with “$f = 0$ q.e. on $X \setminus E$” without changing the obtained space $N^{1,p}_0(E)$. Functions from $N^{1,p}_0(E)$ can be extended by zero q.e. in $X \setminus E$ and we will regard them in that sense if needed. Note that if $C_p(X \setminus E) = 0$, then $N^{1,p}_0(E) = N^{1,p}(E) = N^{1,p}(X)$, since $p$-almost every curve in $X$ avoids $X \setminus E$.

The following Poincaré inequality is often assumed in the literature. Because of the dilation $\lambda$ in the right-hand side, it is sometimes called weak Poincaré inequality.
Definition 2.4. We say that $X$ supports a $(q, p)$-Poincaré inequality, $q \geq 1$, if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$ and all integrable $u \in N^{1,p}(X)$,

$$\left( \int_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C (\text{diam } B) \left( \int_{\lambda B} g_u^p \, d\mu \right)^{1/p},$$  

(2.2)

where $u_B := \int_B u \, d\mu := \int_B u \, d\mu/\mu(B)$.

Using the above-mentioned results on $p$-weak upper gradients from Koskela–MacManus [31], it is easy to see that (2.2) can equivalently be required for all upper gradients $g$ of $u$. If $X$ supports a $(1, p)$-Poincaré inequality and $\mu$ is doubling, then by Theorem 5.1 in Hajłasz–Koskela [21], it supports a $(q, p)$-Poincaré inequality for some $q > p$, and in particular a $(p, p)$-Poincaré inequality. If $X$ is moreover complete then Lipschitz functions are dense in $N^{1,p}(X)$, see Shanmugalingam [37], and functions in $N^{1,p}(X)$ as well as in $N^{1,p}(\Omega)$ are quasicontinuous (see Theorem 2.5 below). It also follows that $N^{1,p}_0(\Omega)$ for open $\Omega$ can equivalently be defined as the closure of Lipschitz functions with compact support in $\Omega$, see Shanmugalingam [38] or Theorem 5.45 in Björn–Björn [5]. For a general set $E$ this is not always possible and our definition of $N^{1,p}_0(E)$ seems to be the natural one.

Moreover, if $X$ is unweighted $\mathbb{R}^n$ and $u \in N^{1,p}(X)$, then $g_u = |\nabla u|$ a.e., where $\nabla u$ is the distributional gradient of $u$. This means that in the Euclidean setting, $N^{1,p}(\Omega)$ for open $\Omega \subset \mathbb{R}^n$ is the refined Sobolev space as defined on p. 96 of Heinonen–Kilpeläinen–Martio [22]. See Hajłasz [20] or Appendix A.1 in [5] for a full proof of this fact for unweighted $\mathbb{R}^n$, and Appendix A.2 in [5] for a proof for weighted $\mathbb{R}^n$ (requiring $p > 1$). See also Theorem 5.1 and its proof for further details.

A function $u : X \to \mathbb{R}$ is quasicontinuous if for every $\varepsilon > 0$ there is an open set $G$ with $C_p(G) < \varepsilon$ such that $u|_{X \setminus G}$ is real-valued continuous.

Theorem 2.5. (Björn–Björn–Shanmugalingam [10]) Let $X$ be proper, $\Omega \subset X$ be open, and assume that continuous functions are dense in $N^{1,p}(X)$ (which in particular holds if $X$ is complete and supports a $(1, p)$-Poincaré inequality, and $\mu$ is doubling). Then every $u \in N^{1,p}(\Omega)$ is quasicontinuous in $\Omega$.

In several of our results the main assumption needed in the proof is that all functions in $N^{1,p}(X)$ are quasicontinuous. The theorem above is the main result guaranteeing this. See Björn–Björn [5], Section 5.1, for several examples, not supporting Poincaré inequalities, when this holds. Moreover, in the other extreme situation when there are no curves in $X$, then $N^{1,p}(X) = L^p(X)$ and thus the quasicontinuity follows directly from Luzin’s theorem. (Incidentally, Luzin’s theorem (on $\mathbb{R}$) was first obtained by Vitali [39] in 1905, while Luzin [33] obtained it in 1912.) In fact, there is no example of a nonquasicontinuous Newtonian function, see Open problems 5.34 and 5.35 in [5].

3. Definition of $\text{cap}_p$ and basic properties

Recall that we assume that $E \subset X$ is a bounded set.

Definition 3.1. For $A \subset E$ we define the variational capacity

$$\text{cap}_p(A, E) = \inf \int_X g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}_0(E)$ such that $u \geq 1$ on $A$.

(Here and later we use the usual convention that $\inf \emptyset = \infty$.)
The infimum can equivalently be taken over all nonnegative \( u \in N_{0}^{1,p}(E) \) such that \( u = 1 \) on \( A \). If \( E \) is measurable we may also equivalently integrate over \( E \) instead of \( X \).

Note that as \( N_{0}^{1,p}(E) \subset N^{1,p}(X) \), it is natural to consider the minimal \( p \)-weak upper gradient \( g_u \) with respect to \( X \). On the other hand, by Proposition 3.10 in Björn–Björn [6], \( g_u \) is also minimal as a \( p \)-weak upper gradient on \( E \) (if \( E \) is measurable).

The variational capacity \( \text{cap}_p(A, E) \) has been used and studied earlier for bounded open \( E \) in metric spaces by e.g. Björn–MacManus–Shanmugalingam [17] and J. Björn [13], [15]. It can also be regarded as the condenser capacity \( \text{cap}_p(X \setminus E, A, X) \), in which the test functions satisfy \( u = 0 \) in \( X \setminus E \) and \( u = 1 \) on \( A \). Such a capacity has been studied on metric spaces by Heinonen–Koskela [23], Kallunki–Shanmugalingam [24] and Adamowicz–Björn–Björn–Shanmugalingam [1].

A novelty here is that we consider nonopen \( E \). However most of the results below have not been given full proofs in the Newtonian literature even for open \( E \). The following result shows that, under quite general assumptions, the zero sets of \( C_p \) and \( \text{cap}_p \) are the same. It generalizes Lemma 3.3 in [13].

**Lemma 3.2.** Assume that \( X \) supports a \((p, p)\)-Poincaré inequality for \( N_{0}^{1,p} \), see below, and that \( C_p(X \setminus E) > 0 \). Let \( A \subset E \). Then \( C_p(A) = 0 \) if and only if \( \text{cap}_p(A, E) = 0 \).

Observe that if \( C_p(X \setminus E) = 0 \) (and thus \( X \) is bounded), then \( 1 \in N_{0}^{1,p}(E) \), making \( \text{cap}_p(A, E) = 0 \) for all sets \( A \subset E \).

The proof shows that the necessity holds without any assumptions on \( X \). For the sufficiency we need a Poincaré inequality, but it is enough with a considerably weaker Poincaré inequality than the one in Definition 2.4. Note also that doubling is not needed.

**Definition 3.3.** We say that \( X \) supports a \((p, p)\)-Poincaré inequality for \( N_{0}^{1,p} \) if for every bounded \( E \subset X \) with \( C_p(X \setminus E) > 0 \) there exists \( C_E > 0 \) such that for all \( u \in N_{0}^{1,p}(E) \) (extended by \( 0 \) outside \( E \)),

\[
\int_X |u|^p \, d\mu \leq C_E \int_X g_u^p \, d\mu. \tag{3.1}
\]

A direct consequence is that \( \|u\|_{N_{1,p}(X)}^p \leq \tilde{C}_E \|g_u\|_{L^p(X)}^p \) for \( u \in N_{0}^{1,p}(E) \).

See Björn–Björn [6] for further discussion of this Poincaré inequality, in particular a proof that it follows from the \((p, p)\)-Poincaré inequality.

**Proof of Lemma 3.2.** Assume first that \( C_p(A) = 0 \). Then \( \chi_A \in N_{0}^{1,p}(X) \) and consequently \( \chi_A \in N_{0}^{1,p}(E) \). Since \( g_{\chi_A} = 0 \) a.e., it follows that \( \text{cap}_p(A, E) = 0 \).

Conversely, assume that \( \text{cap}_p(A, E) = 0 \) and let \( \varepsilon > 0 \). Then there is \( u \in N_{0}^{1,p}(E) \) such that \( u = 1 \) on \( A \) and \( \int_X g_u^p \, d\mu < \varepsilon \). The \((p, p)\)-Poincaré inequality for \( N_{0}^{1,p} \) then yields that

\[
C_p(A) \leq \|u\|_{N_{1,p}(X)}^p \leq \tilde{C}_E \int_X g_u^p \, d\mu < \tilde{C}_E \varepsilon.
\]

Letting \( \varepsilon \to 0 \) concludes the proof.

Let us collect the main general properties of the capacity \( \text{cap}_p \). Observe that these properties all hold in full generality (apart from the requirement \( p > 1 \) in (vi)).

**Theorem 3.4.** Assume that \( A_1, A_2, \ldots \subset E \). Then the following properties hold:
We may assume that the right-hand side is finite. Let
\[ \text{cap}_p(A \cup A_2, E) + \text{cap}_p(A_1 \cap A_2, E) \leq \text{cap}_p(A_1, E) + \text{cap}_p(A_2, E); \]

(v) \( \text{cap}_p \) is countably subadditive (and is also an outer measure), i.e.
\[ \text{cap}_p \left( \bigcup_{i=1}^{\infty} A_i, E \right) \leq \sum_{i=1}^{\infty} \text{cap}_p(A_i, E); \]

(vi) if \( 1 < p < \infty \) and \( A_1 \subset A_2 \subset \ldots \subset E \), then
\[ \text{cap}_p \left( \bigcup_{i=1}^{\infty} A_i, E \right) = \lim_{i \to \infty} \text{cap}_p(A_i, E); \]

(vii) if \( F \subset E \) is closed (as a subset of \( X \)), then \( \text{cap}_p(F, E) = \text{cap}_p(\partial F, E) \).

Even if \( E \) is open, (vi) is not true (in general) for \( p = 1 \). We refer the reader to Björn–Björn [5] for a counterexample due to Korte [28] (it also applies to \( \text{cap}_p \) from (1.2)).

To prove (v) we need the following simple lemma, which probably belongs to folklore. It is a special case of Lemma 1.52 in [5], but can be proved more easily along the lines of the proof of Lemma 1.28 in [5].

**Lemma 3.5.** Let \( u_i \leq M \in \mathbb{R} \), \( i = 1, 2, \ldots \), be functions with \( p \)-weak upper gradients \( g_i \). Let further \( u = \sup_i u_i \) and \( g = \sup_i g_i \). Then \( g \) is a \( p \)-weak upper gradient of \( u \).

**Proof of Theorem 3.4.** (i)–(iii) These statements are trivial.

(iv) We may assume that the right-hand side is finite. Let \( \varepsilon > 0 \). We can thus find \( u_j \in N_p^1, \rho'(E), \chi_{A_j} \leq u_j \leq 1 \), such that \( \|g_{u_j}\|_{L^p(X)} < \text{cap}_p(A_j, E) + \varepsilon, j = 1, 2 \).

Let \( v = \max\{u_1, u_2\} \) and \( w = \min\{u_1, u_2\} \). Then \( g_v = g_{u_1} \chi_{\{u_1 > u_2\}} + g_{u_2} \chi_{\{u_2 > u_1\}} \) and \( g_w = g_{u_2} \chi_{\{u_2 > u_1\}} + g_{u_1} \chi_{\{u_1 > u_2\}} \). Since \( v \) and \( w \) are admissible in the definition of the variational capacity of \( A_1 \cup A_2 \) and \( A_1 \cap A_2 \), respectively, we obtain
\[ \text{cap}_p(A_1 \cup A_2, E) + \text{cap}_p(A_1 \cap A_2, E) \leq \int_X (g_{u_1}^p + g_{u_2}^p) \, d\mu = \int_X (g_{u_1}^p + g_{u_2}^p) \, d\mu \leq \text{cap}_p(A_1, E) + \text{cap}_p(A_2, E) + 2\varepsilon. \]

Letting \( \varepsilon \to 0 \) completes the proof of (iv).

(v) We may assume that the right-hand side is finite. Let \( \varepsilon > 0 \). Choose \( u_i \in N_p^1, \rho'(E) \) with \( \chi_{A_i} \leq u_i \leq 1 \) such that
\[ \|g_{u_i}\|_{L^p(X)} \leq \text{cap}_p(A_i, E) + \frac{\varepsilon}{2}. \]

Let \( u = \sup_i u_i \) and \( g = \sup_i g_{u_i} \). By Lemma 3.5, \( g \) is a \( p \)-weak upper gradient of \( u \). Clearly \( u \geq 1 \) on \( \bigcup_{i=1}^{\infty} A_i \). Hence
\[ \text{cap}_p \left( \bigcup_{i=1}^{\infty} A_i, E \right) \leq \int_X \left( \sup_i g_{u_i} \right)^p \, d\mu \leq \int_X \sum_{i=1}^{\infty} g_{u_i}^p \, d\mu = \sum_{i=1}^{\infty} \int_X g_{u_i}^p \, d\mu \leq \sum_{i=1}^{\infty} \left( \text{cap}_p(A_i, E) + \frac{\varepsilon}{2} \right) = \varepsilon + \sum_{i=1}^{\infty} \text{cap}_p(A_i, E). \]
Letting $\varepsilon \to 0$ completes the proof of (v).

(vi) Let $A = \bigcup_{i=1}^{\infty} A_i$. That $\lim_{i \to \infty} \operatorname{cap}_p(A_i, E) \leq \operatorname{cap}_p(A, E)$ follows from monotonicity, and monotonicity also shows that the limit always exists. Conversely, assume that $\lim_{i \to \infty} \operatorname{cap}_p(A_i, E) < \infty$.

We can find $u_i \in N_0^{1,p}(E)$ with $\chi_{A_i} \leq u_i \leq 1$ and such that

$$
\|g_{u_i}\|_{L^p(X)}^p < \operatorname{cap}_p(A_i, E) + 1/i.
$$

By Lemma 3.2 in Björn–Björn–Parviainen [8] (or Lemma 6.2 in Björn–Björn [5]), there are $u, g \in L^p(X)$, finite convex combinations $v_j = \sum_{i=1}^{\infty} a_{ij}u_i$ and a strictly increasing sequence of indices $\{i_k\}_{k=1}^{\infty}$ such that both $u_{i_k} \to u$ and $g_{u_{i_k}} \to g$ weakly in $L^p(X)$, as $k \to \infty$, $v_j \to u$ q.e., as $j \to \infty$, and $g$ is a $p$-weak upper gradient of $u$. Without loss of generality $u = 0$ outside of $E$.

It is clear that $v_j \geq \chi_{A_i}$ and thus $u \geq \chi_A$ q.e. Let $v := \max\{u, \chi_A\} = u$ q.e. Then $g$ is a $p$-weak upper gradient also of $v$. As $v \geq \chi_A$, we obtain

$$
\operatorname{cap}_p(A, E) \leq \|g\|_{L^p(X)}^p \leq \liminf_{k \to \infty} \|g_{u_{i_k}}\|_{L^p(X)}^p
\leq \lim_{k \to \infty} (\operatorname{cap}_p(A_{i_k}, E) + 1/i_k) = \lim_{i \to \infty} \operatorname{cap}_p(A_i, E).
$$

(vii) Let $u$ be admissible in the definition of $\operatorname{cap}_p(\partial F, E)$. Without loss of generality we can assume that $0 \leq u \leq 1$ and that $u = 0$ in $X \setminus E$. Let

$$
v = \begin{cases} 
1 & \text{in } F, \\
u & \text{in } X \setminus F.
\end{cases}
$$

Then $\|v\|_{L^p(X)} \leq \|u\|_{L^p(X)} + \mu(F) < \infty$. Let $\gamma : [0,l_\gamma] \to X$ be a curve such that (2.1) holds for $u$ and $g_u$ on $\gamma$ and all its subcurves. If $\gamma \subset F$ or $\gamma \subset X \setminus F$, then it is straightforward that (2.1) holds for $v$ and $g_u$ on $\gamma$. If $\gamma$ intersects both $F$ and $X \setminus F$, we can, by splitting $\gamma$ into parts if necessary, and possibly reversing the direction, assume that $x = \gamma(0) \in F$ and $y = \gamma(l_\gamma) \in X \setminus F$. Letting $t = \sup\{\tau : \gamma(\tau) \in F\}$ we have that $\gamma(t) \in \partial F$ and hence

$$
|v(x) - v(y)| = |u(\gamma(t)) - u(y)| \leq \int_{[\gamma(t),y]} g_u \, ds \leq \int_{\gamma} g_u \, ds,
$$

i.e. (2.1) holds for $v$ and $g_u$ on $\gamma$ as well. Thus $g_u$ is a $p$-weak upper gradient of $v$ and hence $v \in N^{1,p}(X)$.

As $v = u = 0$ in $X \setminus E$, we have that $v \in N_0^{1,p}(E)$ and

$$
\operatorname{cap}_p(F, E) \leq \int_X g_u^p \, d\mu \leq \int_X g_u^p \, d\mu.
$$

Taking infimum over all $u$, we see that $\operatorname{cap}_p(F, E) \leq \operatorname{cap}_p(\partial F, E)$. The converse inequality is trivial.

4. Outer and Choquet capacity

In $\mathbb{R}^n$, the capacity is usually defined in a way which automatically makes it an outer capacity. Our definition using Newtonian functions is more direct, but $\operatorname{cap}_p$ is an outer capacity only under some additional assumptions on the underlying space. The following theorem shows this.

**Theorem 4.1.** Assume that all functions in $N^{1,p}(X)$ are quasicontinuous (which in particular holds if $X$ is complete and supports a $(1,p)$-Poincaré inequality, and
\( \mu \) is doubling. If \( p > 1 \), then \( \text{cap}_p \) is an outer capacity for sets in \( \text{int} \ E \), i.e. for every \( A \subset \text{int} \ E \),
\[
\text{cap}_p(A, E) = \inf_{G \text{ open} \subset G \subset \text{int} \ E} \text{cap}_p(G, E). \tag{4.1}
\]
If \( p = 1 \), then (4.1) holds for all \( A \subset E \) with \( \text{dist}(A, X \setminus E) > 0 \).

When \( p = 1 \) we do not know if (4.1) holds for arbitrary \( A \subset \text{int} \ E \).

In Björn–Björn–Shanmugalingam [10], p. 1199, (4.1) was observed, under the more restrictive assumptions that \( X \) is proper, \( E \) is open and \( A \subset E \) (which is however not enough to obtain Theorem 5.1). In Theorem 4.1, the requirement \( A \subset E \) has been weakened, and for open \( E \) the result now holds without any additional assumptions on \( A \). The proof is a fair bit more involved in this case and uses e.g. the strong subadditivity of the capacity and Theorem 3.4(vi) (it is here that \( p > 1 \) is needed).

Note that it is possible to have \( \text{cap}_p(A, E) < \infty \) even if \( A \) “reaches” to the boundary \( \partial E \) and both \( A \) and \( E \) are open, see Example 4.2. Note also that there are no known examples of Newtonian functions which are not quasicontinuous.

For general \( A \subset E \) (i.e. such that \( A \not\subset \text{int} \ E \)), (4.1) is impossible (unless \( \text{cap}_p(A, E) = \infty \) as there are no open sets \( G \supset A \) such that \( \text{cap}_p(G, E) \) is defined. In this case the natural question would be if
\[
\text{cap}_p(A, E) = \inf_{G \text{ relatively open} \subset A \subset G \subset \text{int} \ E} \text{cap}(G, E). \tag{4.2}
\]

This is not true in general, see Example 4.3, but can be true also when \( E \) is nonopen, see Example 4.4 and Proposition 6.5 below. Of course for open \( E \) it follows from Theorem 4.1 (provided that \( p > 1 \) and all functions in \( N^{1,p}(X) \) are quasicontinuous).

**Example 4.2.** Let \( E = (-1, 1) \times (0, 1) \subset X = \mathbb{R}^2 \) (unweighted) with \( 1 < p < 2 \), and let \( A = \{(x, y) \in E : |x| < y < \frac{1}{2} \} \). Then the function \( u(x, y) = \min\{y/|x|, 1\} \) multiplied by the cut-off function \( \eta(x, y) = \min\{2 - 4 \max\{|x|, y\}, 1\} \) belongs to \( N^{1,p}_0(E) \) and is admissible in the definition of \( \text{cap}_p(A, E) \).

**Example 4.3.** Let \( E = [-1, 1] \times [0, 1] \subset X = \mathbb{R}^2 \) (unweighted) with \( 1 < p \leq 2 \), and let \( A = \{(0, 0)\} \). Then any relatively open set \( G \supset A \) contains an open subinterval of the real axis. Since all functions in \( N^{1,p}(X) \) are absolutely continuous on \( p \)-almost every curve (by Shanmugalingam [37]), it follows that there is no function \( u \in N^{1,p}_0(E) \) such that \( u = 1 \) on \( G \). Thus the right-hand side in (4.2) is infinite. The left-hand side is however 0, by Lemma 3.2.

**Example 4.4.** Let \( E = B(0, 1) \setminus D \subset X = \mathbb{R}^2 \) (unweighted) with \( 1 < p \leq 2 \), where \( D \) is a countable dense subset of \( B(0, 1) \). As \( \text{int} \ E = \emptyset \), Theorem 4.1 is directly applicable only for \( A = \emptyset \). At the same time \( \text{cap}_p(D) = 0 \), which shows that \( N^{1,p}_0(E) = N^{1,p}_0(B(0, 1)) \), and thus that \( \text{cap}_p(A, E) = \text{cap}_p(A, B(0, 1)) \) for \( A \subset B(0, 1) \). Hence, Theorem 4.1 applied to \( \text{cap}_p(\cdot, B(0, 1)) \) shows that (4.2) is in fact true in this case.

It may be worth pointing out that \( N^{1,p}_0(E) \) is closely related to the fine interior of \( E \), see Björn–Björn [6]. In fact, \( N^{1,p}_0(E) = N^{1,p}_0(B(0, 1)) \) in Example 4.4 holds because the set \( E \) therein is finely open.

To prove Theorem 4.1, we shall need the following simple lemma which is based on the strong subadditivity of the capacity.

**Lemma 4.5.** Let \( A_1 \subset A_2 \subset \ldots \subset E \) be arbitrary and such that \( \text{cap}_p(A_j, E) < \infty \) for all \( j = 1, 2, \ldots \). For each \( j \) let moreover \( G_j \subset E \) be such that \( G_j \supset A_j \) and
\[
\text{cap}_p(G_j, E) \leq \text{cap}_p(A_j, E) + \varepsilon_j,
\]
where $\varepsilon_j > 0$ are arbitrary. Let $\tilde{G}_k = \bigcup_{j=1}^k G_j$. Then for all $k = 1, 2, \ldots$,

$$\text{cap}_p(\tilde{G}_k, E) \leq \text{cap}_p(A_k, E) + \sum_{j=1}^k \varepsilon_j.$$  

Proof. The lemma is clearly true for $k = 1$. Assume that it holds for some $k \geq 1$. We then have by the strong subadditivity of $\text{cap}_p$, see Theorem 3.4 (iv), that

$$\text{cap}_p(\tilde{G}_{k+1}, E) = \text{cap}_p(\tilde{G}_k \cup G_{k+1}, E) \leq \text{cap}_p(\tilde{G}_k, E) + \text{cap}_p(G_{k+1}, E) - \text{cap}_p(\tilde{G}_k \cap G_{k+1}, E).$$

Since $\tilde{G}_k \cap G_{k+1} \supset A_k$, this together with the induction assumption yields

$$\text{cap}_p(\tilde{G}_{k+1}, E) \leq \text{cap}_p(A_k, E) + \sum_{j=1}^k \varepsilon_j + \text{cap}_p(A_{k+1}, E) + \varepsilon_{k+1} - \text{cap}_p(A_k, E)$$

$$= \text{cap}_p(A_{k+1}, E) + \sum_{j=1}^{k+1} \varepsilon_j. \quad \square$$

Proof of Theorem 4.1. That

$$\text{cap}_p(A, E) \leq \inf_{G \text{ open}} \text{cap}_p(G, E)$$

follows from the monotonicity of the capacity. The converse inequality is trivial if $\text{cap}_p(A, E) = \infty$. Assume therefore that $\text{cap}_p(A, E) < \infty$.

Assume first that $\text{dist}(A, X \setminus E) > 0$ and let $0 < d < \frac{1}{2}\text{dist}(A, X \setminus E)$. Let $0 < \varepsilon < 1$ and find $u \in N^1_0(E)$ such that $u \geq \chi_A$ and

$$\|g_u\|_{L^p(X)} < \text{cap}_p(A, E) + \varepsilon.$$

As $u$ (extended by zero outside $E$) is quasicontinuous in $X$, there is an open set $V$ with $C_p(V)^{1/p} < \varepsilon$ such that $u|_{X \setminus V}$ is continuous. Thus, there is an open set $U$ such that

$$U \setminus V = \{x : u(x) > 1 - \varepsilon\} \setminus V \supset A \setminus V.$$

We can also find $v \geq \chi_V$ with $\|v\|_{N^1_1(X)} < \varepsilon$. Let $\eta(x) = \min\{1, 2 - \text{dist}(x, A)/d\}.$ Note that $\eta \in N^1_0(E)$, $0 \leq \eta \leq 1$, $g_\eta \leq 1/d$ and $\eta = 1$ in the open neighbourhood $W := \{x \in E : \text{dist}(x, A) < d\}$ of $A$. Then

$$\|g_{\eta v}\|_{L^p(X)} \leq \|g_v\|_{L^p(X)} + \frac{1}{d}\|v\|_{L^p(X)} \leq \left(1 + \frac{1}{d}\right)\|v\|_{N^1_1(X)} < \varepsilon + \frac{\varepsilon}{d}.$$  

Let $w = u/(1 - \varepsilon) + \eta v$, so that $w \in N^1_0(E)$ and $w \geq 1$ on

$$((U \setminus V) \cup V) \cap W = (U \cup V) \cap W,$$

which is an open set containing $A$. It follows that

$$\inf_{G \text{ open}} \text{cap}_p(G, E)^{1/p} \leq \text{cap}_p((U \cup V) \cap W, E)^{1/p} \leq \|g_w\|_{L^p(X)}$$

$$\leq \frac{\|g_u\|_{L^p(X)}}{1 - \varepsilon} + \|g_{\eta v}\|_{L^p(X)} < \frac{(\text{cap}_p(A, E) + \varepsilon)^{1/p}}{1 - \varepsilon} + \varepsilon + \frac{\varepsilon}{d}.$$  

Letting $\varepsilon \to 0$ completes this part of the proof.
Let now $A \subset \text{int} E$ be arbitrary and $p > 1$. For $j = 1, 2, \ldots$, let

$$A_j = \{ x \in A : \text{dist}(x, X \setminus E) \geq 1/j \}.$$ 

Then the first part of the proof applies to $A_j$. Let $\varepsilon > 0$ and for each $j = 1, 2, \ldots$, find an open set $G_j \subset E$ such that $A_j \subset G_j$ and

$$\text{cap}_p(G_j, E) \leq \text{cap}_p(A_j, E) + \varepsilon_j,$$

where $\varepsilon_j = 2^{-j}\varepsilon$. Let $\tilde{G}_k = \bigcup_{j=1}^k G_j$ and $G = \bigcup_{k=1}^\infty \tilde{G}_k = \bigcup_{j=1}^\infty G_j$. Then $G$ is open and $A \subset G \subset E$. Lemma 4.5 shows that for all $k = 1, 2, \ldots$,

$$\text{cap}_p(\tilde{G}_k, E) \leq \text{cap}_p(A_k, E) + \sum_{j=1}^k \varepsilon_j.$$ 

Finally, by Theorem 3.4 (vi) (it is here that we need that $p > 1$) we get that

$$\text{cap}_p(G, E) = \lim_{k \to \infty} \text{cap}_p(\tilde{G}_k, E) \leq \lim_{k \to \infty} \text{cap}_p(A_k, E) + \sum_{j=1}^\infty \varepsilon_j \leq \text{cap}_p(A, E) + \varepsilon.$$

Since $\varepsilon$ was arbitrary, this finishes the proof. 

Let us now draw some consequences of the fact that $\text{cap}_p$ is an outer capacity. We start by the following characterization of our variational capacity, whose proof we leave to the reader.

**Corollary 4.6.** Assume that all functions in $N^{1,p}(X)$ are quasicontinuous (which in particular holds if $X$ is complete and supports a $(1,p)$-Poincaré inequality, and $\mu$ is doubling). Assume further that $A \subset \text{int} E$, if $p > 1$, or $\text{dist}(A, X \setminus E) > 0$, if $p = 1$. Then

$$\text{cap}_p(A, E) = \inf_u \int_X g_0^p d\mu,$$

where the infimum is taken over all functions $u \in N^{1,p}_0(E)$ such that $u \geq 1$ in an open set containing $A$.

If $E$ is measurable, then one may equivalently integrate over $E$ instead.

Again, if we consider $A \subset E$ and replace “open” by “relatively open” the result is false in general but true sometimes, see Examples 4.3 and 4.4 and Proposition 6.5.

Another consequence is the following equality between the capacity of a set and its $p$-fine closure. For open $E$ and $A \subset E$ this was proved in J. Björn [15], Corollary 4.5 (and is also included as Corollary 11.39 in [5]). (See e.g. [5] for the definition of the fine topology and other concepts used in the proof below.)

**Corollary 4.7.** Assume that $p > 1$, that $X$ is complete and supports a $(1,p)$-Poincaré inequality, and that $\mu$ is doubling. Let $A \subset \text{int} E$ and $\overline{A}^p$ be the $p$-fine closure of $A$, i.e. the smallest $p$-finely closed set containing $A$. Then $\overline{A}^p \subset \text{int} E$ and

$$\text{cap}_p(\overline{A}^p, E) = \text{cap}_p(A, E).$$

**Proof.** If $\text{cap}_p(A, E) = \infty$, then $\text{cap}_p(\overline{A}^p, E) = \infty$. We can therefore assume that $\text{cap}_p(A, E) < \infty$. One inequality is trivial. To prove the other one, assume first that $A$ is open and let $u$ be a solution of the obstacle problem on $E$ with zero boundary data and obstacle $\chi_A$, i.e. $u \in N^{1,p}_0(E)$ satisfies $u \geq 1$ q.e. on $A$ and minimizes the energy integral in the definition of $\text{cap}_p(A, E)$. Such a minimizer exists, and is unique up to sets of $C_p$-capacity zero, by Theorem 4.2 in Björn–Björn [6]. Note
that $\text{cap}_p(A, E) = \int_E g_p^0 \, d\mu$. It is easily verified that $u$ is a superminimizer in $\text{int} \, E$, and hence by Theorem 5.1 in Kinnunen–Martio [26] (or Theorem 8.22 in [5]), it can be redefined on a set of zero $C_p$-capacity so that it becomes lower semicontinuously regularized, i.e. 

$$u(x) := \lim_{t \to 0^+} \text{ess inf} \, u_t.$$ 

Proposition 7.6 in [26] (or Proposition 9.4 in [5]) then implies that $u$ is superharmonic in $\text{int} \, E$, and Theorem 4.4 in J. Björn [15] (or Theorem 11.38 in [5]) shows that it is $p$-finely continuous in $\text{int} \, E$. Thus, the set $\{x \in X : u(x) \geq 1\}$ is $p$-finely closed and contains $A$, and thus also $\overline{A}$. It follows that $u$ is admissible in the definition of $\text{cap}_p(\overline{A}^p, E)$ and hence 

$$\text{cap}_p(\overline{A}^p, E) \leq \int_E g_p^0 \, d\mu = \text{cap}_p(A, E),$$

proving the corollary for open $A$. For general $A \subset \text{int} \, E$, Theorem 4.1 yields 

$$\text{cap}_p(A, E) = \inf_{G \text{ open}} \text{cap}_p(G, E) = \inf_{G \subset G \subset E} \text{cap}_p(G^p, E) \geq \text{cap}_p(\overline{A}^p, E).$$

**Theorem 4.8.** Assume that all functions in $N^{1,p}(X)$ are quasicontinuous (which in particular holds if $X$ is complete and supports a $(1,p)$-Poincaré inequality, and $\mu$ is doubling). Let $K_1 \supset K_2 \supset \ldots \supset K := \bigcap_{j=1}^\infty K_j$ be compact subsets of $\text{int} \, E$. If $p = 1$, we further require that $\text{dist}(K, X \setminus E) > 0$. Then 

$$\text{cap}_p(K, E) = \lim_{j \to \infty} \text{cap}_p(K_j, E). \quad (4.4)$$

It is natural to ask what happens if we merely require that $K_1 \supset K_2 \supset \ldots$ are compact subsets of $E$. In the situation described in Example 4.4 it follows from those arguments that (4.4) is true even if $K \not\subset \text{int} \, E$. On the other hand, if we let $K_j = [0, 1/j]^2$ and $K = \{0, 0\}$ in the situation described in Example 4.3, we see that $\text{cap}_p(K_j, E) = \infty$ for $j = 1, 2, \ldots$, while $\text{cap}_p(K, E) = 0$ for $1 < p \leq 2$.

**Proof.** That $\text{cap}_p(K, E) \leq \lim_{j \to \infty} \text{cap}_p(K_j, E)$ follows directly from monotonicity.

Conversely, let $G \supset K$ be open. Then $G \cup \bigcup_{j=1}^\infty (X \setminus K_j)$ is an open cover of the compact set $K_1$. Thus, there is a finite subcover, i.e. an $N$ such that 

$$K_1 \subset G \cup \bigcup_{j=1}^N (X \setminus K_j) = G \cup (X \setminus K_N).$$

As $K_N \subset K_1$, it follows that $K_N \subset G$. So $\lim_{j \to \infty} \text{cap}_p(K_j, E) \leq \text{cap}_p(G \cap \text{int} \, E, E)$. By Theorem 4.1 we obtain the equality sought for.

A set function satisfying the conditions in Theorem 3.4(ii), (vi) and Theorem 4.8 is a Choquet capacity. More precisely, $\text{cap}_p(\cdot, E)$ is a Choquet capacity for subsets of $\text{int} \, E$. An important consequence is the following result.

**Theorem 4.9.** (Choquet’s capacitability theorem) Let $p > 1$. Assume that $X$ is locally compact and that all functions in $N^{1,p}(X)$ are quasicontinuous (which in particular holds if $X$ is complete and supports a $(1,p)$-Poincaré inequality, and $\mu$ is doubling).

Then, all Borel sets (and even all Suslin sets) $A \subset \text{int} \, E$ are capacitable, i.e. 

$$\text{cap}_p(A, E) = \sup_{K \text{ compact}} \text{cap}_p(K, E) = \inf_{G \text{ open}} \text{cap}_p(G, E). \quad (4.5)$$
Suslin sets are sometimes called analytic sets (although analytic sets in complex analysis is an entirely different concept). The interested reader should look elsewhere for more on Suslin sets, e.g. in Aikawa–Essén [2], Part 2, Section 10.

**Proof.** To obtain the first equality in (4.5), we apply Choquet’s capacitability theorem in its usual abstract formulation (for which we need that intE is locally compact), see e.g. Theorem 10.1.1 in [2], Part 2. The second equality follows from Theorem 4.1.

5. Equivalence with the definition in \( \mathbb{R}^n \)

Our aim in this section is to show that our definition of the variational capacity based on Newtonian spaces is equivalent to the definitions based on usual (and weighted) Sobolev spaces used in \( \mathbb{R}^n \). This probably belongs to folklore but does not seem to be written down anywhere. The proof in fact depends on a deep result due to Cheeger [18] and on Choquet’s capacitability theorem (Theorem 4.9) together with Theorem 4.1. In unweighted \( \mathbb{R}^n \), the use of Cheeger’s theorem can be avoided by more elementary methods, see e.g. Appendix A.1 in Björn–Björn [5].

**Theorem 5.1.** Let \( \mathbb{R}^n \) be equipped with a \( p \)-admissible weight \( w \), \( p > 1 \), and let \( \Omega \subset \mathbb{R}^n \) be a nonempty bounded open set. Then our variational capacity \( \operatorname{cap}_{p,\mu}(\cdot, \Omega) \) coincides with the variational capacity \( \operatorname{cap}_{p,\mu}(\cdot, \Omega) \) in Heinonen–Kilpeläinen–Martio [22], where \( d\mu = w \, dx \).

An arbitrary nonnegative function \( w \) on \( \mathbb{R}^n \) is a \( p \)-admissible weight, \( p > 1 \), if \( d\mu := w \, dx \) is doubling and \( \mathbb{R}^n \) equipped with \( \mu \) supports a \( (1, p) \)-Poincaré inequality, see Corollary 20.9 in [22] (which is only in the second edition). The \( p \)-Poincaré inequality used there differs somewhat from our Definition 2.4, but by Proposition A.17 in Björn–Björn [5] it is equivalent to it.

Let us recall how the capacity \( \operatorname{cap}_{p,\mu}(\cdot, \Omega) \) is defined in [22], p. 27. For compact \( K \subset \Omega \) one lets

\[
\operatorname{cap}_{p,\mu}(K, \Omega) = \inf_u \int_\Omega |\nabla u|^p \, d\mu, \tag{5.1}
\]

where the infimum is taken over all \( u \in C_0^\infty(\Omega) \) such that \( u \geq 1 \) on \( K \). The capacity is first extended to open \( G \subset \Omega \) by letting

\[
\operatorname{cap}_{p,\mu}(G, \Omega) = \sup_{K \text{ compact} \subset G} \operatorname{cap}_{p,\mu}(K, \Omega) \tag{5.2}
\]

and then to arbitrary \( A \subset \Omega \) by

\[
\operatorname{cap}_{p,\mu}(A, \Omega) = \inf_{G \text{ open} \subset \Omega} \operatorname{cap}_{p,\mu}(G, \Omega). \tag{5.3}
\]

**Proof of Theorem 5.1.** Let \( X \) be \( \mathbb{R}^n \) equipped with the measure \( d\mu = w \, dx \). By Propositions A.12 and A.13 in [5] (whose proofs depend on a deep result of Cheeger [18]), we have \( g_u = |\nabla u| \) a.e. for all \( u \in N^{1,p}(X) \), where \( \nabla u \) is the weak Sobolev gradient of \( u \) as defined in [22]. (If \( \mathbb{R}^n \) is unweighted, then \( \nabla u \) is the distributional gradient.)

Let \( \Omega \subset X \) be a bounded open set and \( K \subset \Omega \) compact. Theorem 6.19 (x) in [5] (or Theorem 1.1 in Kallunki–Shanmugalingam [24]) shows that

\[
\operatorname{cap}_p(K, \Omega) = \inf_u ||g_u||_{L^p(X)}^p, \tag{5.4}
\]

where the infimum is taken over all Lipschitz functions \( u \) on \( X \) such that \( u \geq 1 \) on \( K \) and \( u = 0 \) in \( X \setminus \Omega \). Replacing each such \( u \) by \((1 - \varepsilon)^{-1}(u - \varepsilon)_+ \) and letting \( \varepsilon \to 0 \)
implies that the infimum can equivalently be taken over all Lipschitz functions \( u \) with compact support in \( \Omega \) and \( u \geq 1 \) on \( K \). Since \( C^\infty_0(\Omega) \)-functions are Lipschitz, we can directly conclude that \( \operatorname{cap}_p(K, \Omega) \leq \operatorname{cap}_{p, \mu}(K, \Omega) \).

Conversely, it follows from the comments on pp. 27–28 in [22] that

\[
\operatorname{cap}_{p, \mu}(K, \Omega) = \inf_u \int_{\Omega} |\nabla u|^p \, d\mu,
\]

where the infimum is taken over all continuous \( u \in H^{1,p}_0(\Omega, \mu) \) such that \( u \geq 1 \) on \( K \). Here \( H^{1,p}_0(\Omega, \mu) \) is the closure of \( C^\infty_0(\Omega) \) in the Sobolev norm \( \|u\|_{L^p(\Omega, \mu)} + \|\nabla u\|_{L^p(\Omega, \mu)} \). As Lipschitz functions with compact support in \( \Omega \) belong to \( H^{1,p}_0(\Omega, \mu) \), by Lemma 1.25 in [22], we immediately get from (5.4) and (5.5) that \( \operatorname{cap}_p(K, \Omega) \geq \operatorname{cap}_{p, \mu}(K, \Omega) \). Thus, the capacities coincide for compact sets.

For open and arbitrary subsets of \( \Omega \), the result now follows from the definitions (5.2) and (5.3) together with Choquet’s capacitability theorem (Theorem 4.9) and Theorem 4.1.

In Malý–Ziemer [34], p. 63, the variational capacity on unweighted \( \mathbb{R}^n \) is defined directly for arbitrary \( A \subset \Omega \) by taking the infimum in the \( p \)-energy integral over all \( u \) in the Sobolev space \( H^{1,p}_0(\Omega) \) defined above, such that \( u \geq 1 \) in a neighbourhood of \( A \). A similar definition can be made for weighted \( \mathbb{R}^n \) as well. Using this definition, the equivalence with our capacity \( \operatorname{cap}_p(A, \Omega) \) can be proved without the use of Choquet’s capacitability theorem. All that is needed is the equality \( g_u = |\nabla u| \) (provided essentially by Cheeger’s theorem) and the fact that

\[
H^{1,p}_0(\Omega, \mu) = \{ u : u = v \text{ a.e. for some } v \in N^{1,p}_0(\Omega) \},
\]

i.e. that \( N^{1,p}_0(\Omega) \) consists exactly of the quasicontinuous representatives of functions from \( H^{1,p}_0(\Omega, \mu) \), see Proposition A.13 in [5] and Theorem 4.5 in [22].

6. Other definitions and applications of capacity

In \( \mathbb{R}^n \), capacity is often defined without using Sobolev spaces, as e.g. in (5.1)–(5.3). This is sometimes possible also on metric spaces, when \( E = \Omega \) is open.

If \( X \) is complete and supports a \( (1, p) \)-Poincaré inequality, \( \mu \) is doubling and \( p > 1 \), then Theorem 1.1 in Kallunki–Shanmugalingam [24] (or Theorem 6.19 (x) in [5]) shows that for compact sets \( K \subset \Omega \), the capacity \( \operatorname{cap}_p(K, \Omega) \) can be defined using only Lipschitz functions with compact support in \( \Omega \), i.e. \( u \in \operatorname{Lip}_r(\Omega) \). Theorem 6.1 in Cheeger [18] shows that under the same assumptions, \( g_u = \operatorname{Lip}_u = \operatorname{lips}_u \) a.e., where

\[
\operatorname{Lip}_u(x) := \limsup_{r \to 0} \sup_{y \in B(x, r)} \frac{|u(y) - u(x)|}{r}
\]

and

\[
\operatorname{lips}_u(x) := \liminf_{r \to 0} \sup_{y \in B(x, r)} \frac{|u(y) - u(x)|}{r}
\]

are the upper and lower pointwise dilations of \( u \), respectively. Thus, \( g_u \) in the definition of \( \operatorname{cap}_p(K, \Omega) \) can be replaced by \( \operatorname{Lip}_u \) or \( \operatorname{lips}_u \) and \( \operatorname{cap}_p(K, \Omega) \) can be defined using only elementary properties of Lipschitz functions, i.e.

\[
\operatorname{cap}_p(K, \Omega) = \inf_{u \geq 1 \text{ on } K} \int_{\Omega} (\operatorname{Lip}_u)^p \, d\mu = \inf_{u \geq 1 \text{ on } K} \int_{\Omega} (\operatorname{lips}_u)^p \, d\mu.
\]
Equivalently, \( \text{Lip}_p(\Omega) \) can be replaced by \( \text{Lip}_0(E) := \{ f \in \text{Lip}(X) : f = 0 \text{ on } X \setminus \Omega \} \). The capacity can then be extended to open and arbitrary sets as in (5.2) and (5.3). By Theorems 4.1 and 4.9, this is equivalent to Definition 3.1, provided that \( p > 1 \), \( X \) is complete and supports a \((1,p)\)-Poincaré inequality, and \( \mu \) is doubling.

It is natural to ask if (6.1) may be extended to nonopen sets, i.e. if \( \Omega \) can be replaced by an arbitrary \( E \) in (6.1). (If \( E \) is not measurable we take the integrals over \( X \).) If \( K \not\subset \text{int} \, E \), then the equality can hold only when \( \text{cap}_p(K, E) = \infty \), as there are no Lipschitz functions satisfying the requirements in the infima. The following example shows that the equality is not true (in general) even for \( K \subset \text{int} \, E \). Thus for general \( E \) we are better off using Newtonian functions in the definition of \( \text{cap}_p(A, E) \).

**Example 6.1.** Let \( E = \Omega \setminus D \subset X = \mathbb{R}^n \) (unweighted), \( 1 < p \leq n \), where \( D \subset \Omega \) is a countable set whose closure has positive Lebesgue measure. Assume also that \( \text{int} \, E = \Omega \setminus D \neq \varnothing \) and let \( K \subset \text{int} \, E \) be compact. As in Example 4.4 we see that \( \text{cap}_p(K, E) = \text{cap}_p(K, \Omega) \), while every Lipschitz function with compact support in \( E \) must vanish on \( D \) and hence

\[
\inf_{u \geq 1 \text{ on } K} \int_{\Omega} (\text{Lip} \, u)^p \, d\mu = \inf_{u \geq 1 \text{ on } \text{int} \, E} \int_{\Omega} (\text{Lip} \, u)^p \, d\mu = \text{cap}_p(K, \text{int} \, E),
\]

and similarly for \( u \in \text{Lip}_1(E) \). Since for most compact sets \( K \subset \text{int} \, E \) we have \( \text{cap}_p(K, \Omega) < \text{cap}_p(K, \text{int} \, E) \), this shows that (6.1) cannot extend to the nonopen case.

Let us now return to the capacity \( \text{cap}_p \) from (1.2) in the introduction. By definition, it is an outer capacity, in the sense that

\[
\text{cap}_p(A, E) = \inf_{A \subset G \subset E} \text{cap}_p(G, E).
\]

holds for every \( A \subset E \).

It is fairly easy to establish (i), (ii) and (iv)–(vii) of Theorem 3.4 for \( \text{cap}_p \). The parts (i) and (ii) are trivial, (iv) and (v) follow from the corresponding properties for \( \text{cap}_p \), while (vii) is proved in the same way as in Theorem 3.4 using relatively open \( G_i \supset A_i \) with \( \text{cap}_p(G_i, E) < \text{cap}_p(A_i, E) + 2^{-i} \epsilon \) and functions \( u_i \in N_1^{1,p}(E) \) such that \( \chi_{G_i} \leq u_i \leq 1 \) and \( \| u_i \|_{L_\infty(X)} < \text{cap}_p(A_i, E) + 2^{-i} \epsilon \), and the proof of (vii) is similar using open \( G \supset \partial F \) and the technique in the proof of Theorem 3.4. We omit the details here.

The strong subadditivity (Theorem 3.4 (iv)) for \( \text{cap}_p \) also implies that Lemma 4.5 holds for \( \text{cap}_p \). Lemma 3.2 is however not true for \( \text{cap}_p \), see Example 4.3. The following example shows that Theorem 3.4 (iii) for \( \text{cap}_p \) is not true either in general. However, if \( E_1 \) is relatively open in \( E_2 \) then every \( G \supset A \) which is relatively open in \( E_1 \) is also relatively open in \( E_2 \) and hence Theorem 3.4 (iii) holds for \( \text{cap}_p \), by the same property for \( \text{cap}_p \).

**Example 6.2.** Let \( 1 < p \leq 2 \),

\[
X = \{(x, y) \in [-2, 2]^2 : xy \geq 0 \}, \quad A = \{(0, 0)\},
\]

\[
E_1 = [0, 1)^2, \quad E_2 = E_1 \cup \{(x, y) \in X : x \leq y \leq 0\}.
\]

Then \( \text{cap}_p(A, E_1) = 0 \) since \( C_p(A) = 0 \). At the same time, every open \( G \supset A \) must contain a segment from the boundary \( \partial E_2 \) and since functions in \( N_1^{1,p}([-2, 0]^2) \) are absolutely continuous on \( p \)-almost every curve, we see that \( \text{cap}_p(G, E_2) = \infty \) and hence \( \text{cap}_p(A, E) = \infty \).
Note that in this example, all $u \in N^{1,p}(X)$ are quasicontinuous (by e.g. Example 5.6 and Theorem 5.29 in [5]), but the zero $p$-weak upper gradient property fails at the origin, cf. Proposition 6.5 below.

If $K_1 \supset K_2 \supset \ldots \supset K := \bigcap_{j=1}^{\infty} K_j$ are compact subsets of $E$, then the inner regularity

$$\text{cap}_p(K, E) = \lim_{j \to \infty} \text{cap}_p(K_j, E)$$

can be shown in the same way as Theorem 4.8, where we use (6.2) instead of Theorem 4.1 (and that is also why we can allow for $K_j \subset E$ here rather than only $K_j \subset \text{int} \, E$ as in Theorem 4.8). Thus $\text{cap}_p$ is a Choquet capacity if $p > 1$, and we can establish Choquet’s capacitability theorem in the following form. Note that to obtain these properties for $\text{cap}_p$ there is no need to assume that all functions in $N^{1,p}(X)$ are quasicontinuous, since outer regularity of $\text{cap}_p$ comes for free rather than from Theorem 4.1.

**Theorem 6.3.** (Choquet’s capacitability theorem for $\text{cap}_p$) Let $p > 1$. Then, all Borel sets (and even all Suslin sets) $A \subset E$, for which there exists a locally compact set $F$ such that $A \subset F \subset E$, are capacitable, i.e.

$$\text{cap}_p(A, E) = \sup_{K \text{ compact } K \subset A} \text{cap}_p(K, E) = \inf_{G \text{ relatively open } \subset G \subset E} \text{cap}_p(G, E).$$  \hspace{1cm} (6.3)

Note that if $E$ is locally compact, in particular if $E$ is open or compact, then (6.3) holds for all $A \subset E$ (provided that $p > 1$).

**Proof.** Restrict $\text{cap}_p(\cdot, E)$ to subsets of $F$. It is then clear that this restricted capacity is a Choquet capacity on $F$. We can now apply Choquet’s capacitability theorem in its usual abstract formulation (for which we need that $F$ is locally compact), see e.g. Theorem 10.1.1 in Aikawa–Essén [2], Part 2. This gives the first equality in (6.3), while the second equality is just (6.2).

**Remark 6.4.** It follows directly from the definition of $\text{cap}_p$ that $\text{cap}_p(A, E) = \text{cap}_p(A, E)$ for relatively open subsets $A$ of $E$, but not for general subsets of $E$, see Examples 4.3 and 6.2.

If all functions in $N^{1,p}(X)$ are quasicontinuous, then $\text{cap}_p(A, E) = \text{cap}_p(A, E)$ if $A \subset \text{int} \, E$ and $p > 1$, or $\text{dist}(A, X \setminus E) > 0$ and $p = 1$, by Theorem 4.1 and the fact that $G \cap \text{int} \, E$ is open for every relatively open $G \subset E$.

In fact, we have the following result which sheds some more light on the equality $\text{cap}_p = \text{cap}_p$ and the question posed in (4.2). It depends on the zero $p$-weak upper gradient property, which was introduced in A. Björn [4], where it was also shown that it follows from the $(1, p)$-Poincaré inequality.

By definition, $X$ has the zero $p$-weak upper gradient property if every measurable function $f$, which has zero as a $p$-weak upper gradient in some ball $B(x, r)$, is essentially constant in some (possibly smaller) ball $B(x, \delta)$, which can depend both on $f$ and $B(x, r)$. (It is equivalent to require this for bounded measurable functions, see Remark 5.8 in Björn–Björn [6].)

**Proposition 6.5.** Let $p > 1$. Assume that all functions in $N^{1,p}(X)$ are quasicontinuous and that $X$ has the zero $p$-weak upper gradient property (both of which hold in particular if $X$ is complete and supports a $(1, p)$-Poincaré inequality, and $\mu$ is doubling). If $A \subset E$ then $\text{cap}_p(A, E) = \text{cap}_p(A, E)$ or $\text{cap}_p(A, E) = \infty$.

For situations when $\text{cap}_p(A, E) < \infty = \text{cap}_p(A, E)$ see Examples 4.3 and 6.2. In both examples we have $\text{cap}_p(A, E) = 0$ but by adding an open set $V \subset E$ with $\text{cap}_p(V, E) < \infty$ to $A$ we get

$$0 < \text{cap}_p(A \cup V, E) < \infty = \text{cap}_p(A \cup V, E).$$
Note that by the proof below we see that \( \text{cap}_p(A, E) = \overline{\text{cap}}_p(A, E) \) in case 2, while \( \overline{\text{cap}}_p(A, E) = \infty \) in case 1.

**Proof.** It is clear that \( \overline{\text{cap}}_p(A, E) \geq \text{cap}_p(A, E) \) for all \( A \subset E \). To prove the converse inequality, assume that \( \text{cap}_p(A, E) < \infty \). We shall distinguish two cases:

**Case 1.** There exists \( x \in A \) such that for all \( r > 0 \) both \( C_p(B(x, r) \setminus E) > 0 \) and \( C_p(B(x, r) \cap E) > 0 \). We shall show that in this case, \( \text{cap}_p(A, E) = \infty \). Let \( G \subset X \) be an arbitrary open set containing \( A \) and find a ball \( B = B(x, r) \subset G \). Assume that \( u \in N_0^{1,p}(E) \) such that \( u = 1 \) in \( G \cap E \). Then \( u \in N_0^{1,p}(B) \), \( u = 0 \) in \( B \setminus E \) and \( u = 1 \) in \( B \cap E \). In particular, \( g_u = 0 \) a.e. in \( B \). The zero \( p \)-weak upper gradient property implies that \( u \) is essentially (and thus q.e.) constant in some smaller ball \( B(x, \delta) \). This contradicts the choice of \( x \) and \( u \) and hence there are no \( u \in N_0^{1,p}(E) \) admissible in the definition of \( \text{cap}_p(G \cap E, E) \), i.e. \( \text{cap}_p(G \cap E, E) = \infty \). Since \( G \) was arbitrary, we conclude that \( \overline{\text{cap}}_p(A, E) = \infty \).

**Case 2.** For every \( x \in A \) there exists a ball \( B_x \ni x \) such that \( C_p(B_x \setminus E) = 0 \) or \( C_p(B_x \cap E) = 0 \). As \( X \) is separable, the Lindelöf property, see Proposition 1.6 in Björn–Björn [5], implies that \( A \) can be covered by countably many of these balls, i.e. \( A \subset \bigcup_{n=1}^{\infty} B_{x_n} \). Let \( G' \) be the union of the balls \( B_{x_i} \) for which \( C_p(B_{x_i} \setminus E) = 0 \), and \( G'' \) be the union of the remaining balls \( B_{x_i} \) in the countable subcover. Then \( C_p(G' \setminus E) = 0 \) and \( C_p(G'' \cap E) = 0 \).

As \( A \cap G' \subset \text{int}(E \cup G') \), we have by Theorem 3.4(ii) and (iii), Remark 6.4 (it is here we use that \( p > 1 \)) and the definition of \( \overline{\text{cap}}_p \) that

\[
\text{cap}_p(A, E) \geq \text{cap}_p(A \cap G', E \cup G') = \overline{\text{cap}}_p(A \cap G', E \cup G') = \inf_{\text{G open } A \cap G' \subset G} \text{cap}_p(G \cap (E \cup G'), E \cup G').
\]

Let \( G \subset G' \cap G'' \) be an open set in \( X \) containing \( A \cap G' \). We shall show that

\[
\text{cap}_p(G \cap E, E) \leq \text{cap}_p(G \cap (E \cup G'), E \cup G').
\]

Together with (6.4) this then yields

\[
\text{cap}_p(A, E) \geq \inf_{\text{G open } A \cap G' \subset G} \text{cap}_p(G \cap E, E) = \overline{\text{cap}}_p(A \cap G', E).
\]

Thus, for every \( \varepsilon > 0 \) there exists an open set \( G \supset A \cap G' \) and \( u \in N_0^{1,p}(E) \) such that \( u \geq 1 \) in \( G \cap E \) and

\[
\int_X g_u^p \, d\mu \leq \text{cap}_p(A, E) + \varepsilon.
\]

Since \( C_p(G'' \cap E) = 0 \), we can modify \( u \) on \( G'' \cap E \) to get \( u = 1 \) on the relatively open set \( (G \cap E) \cup (G'' \cap E) \supset A \). Thus, \( \overline{\text{cap}}_p(A, E) \leq \text{cap}_p(A, E) + \varepsilon \) and letting \( \varepsilon \to 0 \) will prove the proposition.

It remains to show (6.5). Let \( u \in N_0^{1,p}(E \cup G') \) be such that \( u = 1 \) in \( G \cap (E \cup G') \). Since \( C_p(G \setminus E) = 0 \), we see that \( u \in N_0^{1,p}(E) \) and is thus admissible in the definition of \( \text{cap}_p(G \cap E, E) \). Taking infimum over all such \( u \) proves (6.5) and finishes the proof. \( \square \)

The variational capacity \( \text{cap}_p(A, E) \) depends very much on the underlying metric space \( X \), even though we have refrained from making this dependence explicit in the notation. Let us however define \( \text{cap}_p(A, E; X) := \text{cap}_p(A, E) \) and see how changing \( X \) can be of use.

If \( A \subset E \subset X_1 \subset X_2 \) then \( N_0^{1,p}(X_2) \subset N_0^{1,p}(X_1) \) and we immediately obtain that \( \text{cap}_p(A, E; X_1) \leq \text{cap}_p(A, E; X_2) \). The inequality can be strict and in particular
it can happen that \( \text{cap}_p(A, E; X_1) = 0 < \text{cap}_p(A, E; X_2) \), even for open \( E \). Since
the definition of \( N^{1,p}(X) \) depends on curves in \( X \), the capacity \( \text{cap}_p \) is influenced
by the path-connectedness properties of the underlying space. In Björn–Björn–Shanmugalingam [11], similar phenomena for Sobolev capacities are used to obtain
new resolutivity results for the Dirichlet problem for \( p \)-harmonic functions. We refer
the reader to the examples therein.

In the following example we briefly comment on some other properties of \( \text{cap}_p \)
with respect to different underlying spaces, as well as on the influence of the underlying
space on the minimizers in the definition of \( \text{cap}_p \).

**Example 6.6.** Let for instance \( X \) be an open set \( G \subset \mathbb{R}^n \), equipped with the
induced metric and measure, where \( \mathbb{R}^n \) may be unweighted or weighted using a
\( p \)-admissible weight. Let further, for simplicity, \( K \subset \Omega \subset G \), where \( K \) is compact
and \( \Omega \) is open and bounded. If \( \partial_{\mathbb{R}^n} \Omega \subset G \), then it is fairly easy to see that
\( \text{cap}_p(K, \Omega; G) = \text{cap}_p(K, \Omega; \mathbb{R}^n) \). On the other hand, when \( \partial_{\mathbb{R}^n} \Omega \setminus G \) is substantial,
the situation becomes different, as we shall now see.

Usually, when calculating the variational capacity one more or less solves a
Dirichlet problem with zero boundary values on \( \partial \Omega \) and boundary values \( 1 \) on \( K \).
When regarding \( \text{cap}_p(K, \Omega; G) \) as a problem in \( \mathbb{R}^n \), it can be seen that it corresponds
to a mixed boundary value problem of the following type: zero boundary values on
\( \partial_{\mathbb{R}^n} \Omega \), boundary values \( 1 \) on \( K \), and zero Neumann boundary condition on \( \partial_{\mathbb{R}^n} \Omega \setminus \partial_{\mathbb{R}^n} \Omega \), provided that \( \Omega \) is smooth enough as a subset of \( \mathbb{R}^n \). If it is less smooth,
then the same is true in a generalized sense, making it possible to study problems
with zero Neumann boundary condition in very general situations. See e.g. the
discussion in Section 1.7 and Example 8.18 in Björn–Björn [5]. Since \( X = G \) is not
complete (unless \( G = \mathbb{R}^n \)) this gives a further motivation for studying nonlinear
total potential theory on noncomplete spaces.

In this situation one may also consider \( X = \overline{G} \) as the underlying metric space,
where the closure is taken with respect to \( \mathbb{R}^n \). The above discussion is more or less
the same for \( G \) and \( \overline{G} \) (but more care has to be taken in the formulations near the
boundary \( \partial_{\mathbb{R}^n} G \)).

An advantage of \( \overline{G} \) is that it is complete. At the same time, both \( G \) and \( \overline{G} \) may
fail to support a Poincaré inequality, and the measure may fail to be doubling on
\( G \) or \( \overline{G} \). We can still of course use the properties in Theorem 3.4, since they hold
in full generality. Also Theorem 6.3 holds on \( G \) and \( \overline{G} \).

But for \( X = \overline{G} \), Theorems 4.1, 4.8, 4.9 and Corollary 4.6 are not available in
general. On the other hand, by Theorem 2.5 (applied with \( \mathbb{R}^n \) and \( G \) in place of
\( X \) and \( \Omega \)), we have all of Theorems 4.1, 4.8, 4.9 and Corollary 4.6 available for
\( X = G \). If \( G \) moreover has the zero \( p \)-weak upper gradient property, then also
Proposition 6.5 is available for \( X = G \).

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