A BRIEF REVIEW OF THE SINGULARITIES IN 4D AND 5D VISCOUS COSMOLOGIES NEAR THE FUTURE SINGULARITY

I. Brevik

Department of Energy and Process Engineering, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

O. Gorbunova

Tomsk State Pedagogical University, Tomsk, Russia

Abstract

Analytic properties of physical quantities in the cosmic fluid such as energy density $\rho(t)$ and Hubble parameter $H(t)$ are investigated near the future singularity (Big Rip). Both 4D and 5D cosmologies are considered (the Randall-Sundrum II model in the 5D case), and the fluid is assumed to possess a bulk viscosity $\zeta$. We consider both Einstein gravity and modified gravity, where in the latter case the Lagrangian contains a term $R^\alpha$ with $\alpha$ a constant. If $\zeta$ is proportional to the power $(2\alpha - 1)$ of the scalar expansion, the fluid can pass from the quintessence region into the phantom region as a consequence of the viscosity. A property worth noticing is that the 4D singularity on the brane becomes carried over to the bulk region.

1 Introduction

The possibility of crossing the $w = -1$ barrier in dark energy cosmology has recently become a topic of considerable interest. One usually assumes that the equation of state for the cosmic fluid can be written in the form

$$p = w\rho,$$

(1)

---

1E-mail: iver.h.brevik@ntnu.no
2E-mail: gorbunovaog@tspu.edu.ru
where \( w \) is a constant. If \( w = -1 \) the fluid is called a "vacuum fluid", with peculiar thermodynamic properties such as negative entropy \([1]\). More general forms for the equation of state can be envisaged, such as

\[
p = w(\rho)\rho = -\rho - f(\rho),
\]

which is a form that we shall consider below. As is known, cosmological observations indicate that the present universe is accelerating. Recent discussions on the actual value of \( w \) can be found, for instance, in refs. \([2, 3, 4]\). Perhaps, \( w \) is even an oscillating function in time. For discussions on time-dependent values of \( w \), one may consult Refs. \([5, 6, 7]\). The possibility of crossing from the quintessence region \((-1 < w < -1/3)\) into the phantom region \( w < -1 \), is obviously of physical interest. It may be noted that both quintessence and phantom fluids lead to the inequality \( \rho + 3p \leq 0 \), thus breaking the strong energy condition.

Once being in the phantom region, the cosmic fluid will inevitably be led into a future singularity, called the Big Rip \([8, 9, 10, 11]\). And this brings us to the main theme of the present paper, namely to give an overview of the behavior of central physical quantities near the future singularity. This is the case of main interest. We think that such an exposition should be useful, not least so because the situation is rather complex. Namely, there is a variety of different factors at play here: (i) the thermodynamic parameter \( w(\rho) \), (ii) the possible time dependence of the bulk viscosity, \( \zeta = \zeta(t) \), and (iii) the adoption of Einstein’s gravity, or a version of the so-called modified gravity. (For an introduction to modified gravity theories, one may consult Refs. \([12, 13]\).)

To begin with, it is convenient to quote from Ref. \([11]\) the classification of possible future singularities:

(i) Type I ("Big Rip"): For \( t \to t_s, \ a \to \infty, \ \rho \to \infty, \ \text{and} \ \ |p| \to \infty \), or \( p \) and \( \rho \) are finite at \( t = t_s \).

(ii) Type II ("sudden"): For \( t \to t_s, \ a \to a_s, \ \rho \to \rho_s, \ \text{and} \ \ |p| \to \infty \),

(iii) Type III: For \( t \to t_s, \ a \to a_s, \ \rho \to \infty, \ \text{and} \ \ |p| \to \infty \),

(iv) Type IV: For \( t \to t_s, \ a \to a_s, \ \rho \to 0, \ \text{or} \ p \ \text{and} \ \rho \ \text{are finite.} \)

Higher order derivatives of \( H \) diverge.

Here the notation is standard, \( a \) meaning the scale factor and \( t_s \) referring to the instant of the singularity. The above classification was introduced in the context of ideal, i.e., nonviscous, cosmology. We can however make use of the same classification also in the viscous case.
In the following, we will present salient features of 4D, respective 5D, viscous cosmology theory, and thereafter focus on the classification of the various alternatives.

2 Viscous 4D theory: Basics

We include this basic material mainly for reference purposes. We consider the standard FRW metric,

\[ ds^2 = -dt^2 + a^2(t)dx^2, \]

and set the spatial curvature \( k \), as well as the 4D cosmological constant \( \Lambda_4 \), equal to zero. The Hubble parameter is \( H = \dot{a}/a \), the scalar expansion is \( \theta = U^\mu \gamma_\mu = 3H \) with \( U^\mu \) the four-velocity of the fluid, and \( \kappa_4^2 = 8\pi G_4 \) is the gravitational coupling. Of main interest are the \((tt)\) and \((rr)\) components of the Friedmann equations. They are

\[ \theta^2 = 3\kappa_4^2 \rho, \]

\[ \frac{2\ddot{a}}{a} + H^2 = -\kappa_4^2 \tilde{p}, \]

where \( \tilde{p} = p - \zeta \theta \) is the effective pressure. From the differential equation for energy, \( T^0_{\nu \gamma} = 0 \), we get

\[ \dot{\rho} + (\rho + p)\theta = \zeta \theta^2. \]

As a consequence of positive entropy change in an irreversible process, we must require the value of \( \zeta \) to be non-negative. From the equations above we obtain the following differential equation for the scalar expansion,

\[ \dot{\theta} - \frac{f(\rho)}{2\rho} \theta^2 - \frac{3}{2} \kappa_4^2 \zeta \theta = 0. \]

In view of the relationship \( \dot{\theta} = (\sqrt{3} \kappa_4/2)\sqrt{\rho} / \sqrt{\rho} \) we can alternatively reformulate this equation as an equation for the density,

\[ \dot{\rho} - \kappa_4 \sqrt{3\rho f(\rho)} - 3\kappa_4^2 \zeta \rho = 0. \]

The solution is (cf. Eq. (9) in [14])

\[ t = \frac{1}{\sqrt{3} \kappa_4} \int_{\rho_*}^{\rho} \frac{d\rho}{\sqrt{\rho f(\rho)[1 + \kappa_4 \zeta \sqrt{3\rho} f(\rho)]}}. \]
We here let \( t = 0 \) be the initial (present) time, and let the corresponding initial density be \( \rho_* \). The functional form of the bulk viscosity \( \zeta \) is so far unspecified. The shear viscosity is omitted, due to the assumed spatial isotropy in the cosmic fluid. Note the dimensions: \( [\kappa_4^2] = \text{cm}^2, \ [f(\rho)] = [\rho] = \text{cm}^{-4}, \ [\zeta] = \text{cm}^{-3} \). Viscous cosmology are treated at various places, for instance, in Refs. [3, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

3 Specific cases in 4D

We are now in a position to discuss various cases in 4D explicitly. We have to distinguish between several alternatives: (i) use of Einstein or modified gravity; (ii) possible density dependence of the thermodynamic parameter \( w = w(\rho) \); and (iii) possible time dependence of the bulk viscosity \( \zeta(t) \).

3.1 Einstein gravity, \( w \) and \( \zeta \) being constants

Let \( f(\rho) = \alpha \rho \), with \( \alpha \) a constant. The equation of state is then
\[
p = w\rho = -(1 + \alpha)\rho.
\]
(10)

We can now solve explicitly for the Hubble parameter [14, 26],
\[
H(t) = \frac{H_* e^{t/t_c}}{1 - \frac{3\alpha}{2} H_* t_c (e^{t/t_c} - 1)},
\]
(11)

where \( H_* \) is the present-time value of \( H \) and \( t_c \) is the 'viscosity time',
\[
t_c = \left( \frac{3}{2} \kappa_4^2 \right)^{-1} \zeta.
\]
(12)

From Eq. (11) it is seen that \( H(t) \) becomes singular when the denominator vanishes. Let us first for reference purposes set \( \zeta = 0 \):

The nonviscous case. If \( t_{s0} \) designates the singularity time, we have
\[
t_{s0} = \frac{2}{3\alpha H_*}.
\]
(13)

Then [26],
\[
H(t) = \frac{H_* t_{s0}}{t_{s0} - t},
\]
(14)

4
\[ a(t) = \frac{a_s t^{2/3\alpha}}{(t_{s0} - t)^{2/3\alpha}}, \quad (15) \]
\[ \rho(t) = \frac{\rho_s t^2 t_{s0}}{(t_{s0} - t)^2}. \quad (16) \]

The viscous case. If now \( t_{s\zeta} \) denotes the singularity time, we get from Eq. (11)
\[ t_{s\zeta} = t_c \ln \left[ 1 + \frac{2}{3\alpha H_s t_c} \right]. \quad (17) \]
and
\[ H(t) \to \frac{H_s t_{s0}}{t_{s0} - t}, \quad t \to t_{s\zeta}. \quad (18) \]
Close to the singularity we thus obtain the same singular behavior as in the nonviscous case. Moreover, we get the following forms,
\[ a(t) \sim (t_{s\zeta} - t)^{-2/3\alpha}, \quad t \to t_{s\zeta}, \quad (19) \]
\[ \rho(t) \sim (t_{s\zeta} - t)^{-2}, \quad t \to t_{s\zeta}. \quad (20) \]
The viscosity tends to shorten the singularity time,
\[ t_{s\zeta} < t_{s0}, \quad (21) \]
but it does not modify the exponents in the singularity. The singularity is of Type I if \( \alpha > 0 \), and of Type II if \( \alpha < 0 \).

3.2 Einstein gravity, \( f(\rho) = A\rho^\beta \), and \( \zeta \) being constant

We shall assume that \( \beta \geq 1 \). From Eq. (9) it is apparent that the last term in the denominator dominates for large \( \rho \). Near the singularity we obtain the form
\[ \rho(t) \sim (t_{s\zeta} - t)^{-2/\beta - 1}, \quad t \to t_{s\zeta}, \quad (22) \]
which generalizes Eq. (20) and reduces to it when \( \beta = 1 \). Thus \( \rho \to \infty \) implying, according to Eq. (2), that also \( |p| \to \infty \). The Hubble parameter becomes
\[ H(t) \sim (t_{s\zeta} - t)^{\frac{1}{2\beta - 1}}. \quad (23) \]
If \( \beta > 1 \), \( a \to a_s \) (a finite value) when \( t \to t_{s\zeta} \). The singularity is of Type III. If \( \beta = 1 \), the singularity is of Type I.

The material of this subsection was discussed also in Ref. [26], whereas the equation of state corresponding to (22) and (23) was discussed by Nojiri and Odintsov [27].
3.3 Modified gravity, \( w \) being constant, and \( \zeta = \tau \theta^{2\alpha - 1} \)

Consider now the following gravity model,

\[
S = \frac{1}{2\kappa^2_4} \int d^4x \sqrt{-g}(f_0 R^\alpha + L_m),
\]

(24)

where \( f_0 \) and \( \alpha \) are constants, \( L_m \) being the matter Lagrangian. This model has been considered before; cf., for instance, Refs. [28, 29, 30, 23]. The case \( f_0 = 1 \) and \( \alpha = 1 \) yields Einstein’s gravity. The equations of motion following from the action above are

\[
-\frac{1}{2} f_0 g_{\mu\nu} R^\alpha + \alpha f_0 R_{\mu\nu} R^{\alpha - 1} - \alpha f_0 \nabla_\mu \nabla_\nu R^{\alpha - 1} + \alpha f_0 g_{\mu\nu} \nabla^2 R^{\alpha - 1} = \kappa^2_4 T_{\mu\nu},
\]

(25)

where \( T_{\mu\nu} \) corresponds to the term \( L_m \) in the Lagrangian. For the cosmic fluid we have

\[
T_{\mu\nu} = \rho U_\mu U_\nu + \tilde{p} h_{\mu\nu},
\]

(26)

where \( h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \) is the projection tensor and \( \tilde{p} = p - \zeta \theta \) the effective pressure. In comoving coordinates, \( U^0 = 1, U^i = 0 \). We assume now the simple equation of state given in Eq. (1).

Of main interest is the (00)-component of Eq. (25). Using that \( R = 6(\dot{H} + 2H^2), T_{00} = \rho \), as well as the energy conservation equation which in turn follows from \( \nabla^\nu T_{\mu\nu} = 0 \), we obtain

\[
\frac{3}{2} \gamma f_0 R^\alpha + 3\alpha f_0 [2\dot{H} - 3\gamma(\dot{H} + H^2)] R^{\alpha - 1} + 3\alpha (\alpha - 1) f_0 [(3\gamma - 1)H \ddot{R} + \dddot{R}] R^{\alpha - 2}
\]

\[
+ 3\alpha (\alpha - 1)(\alpha - 2) f_0 \dot{R}^2 R^{\alpha - 3} = 9\kappa^2_4 \zeta H,
\]

(27)

with \( \gamma = w + 1 \). The important point now is that this complicated equation for \( H(t) \) is satisfied with the following form

\[
H = H_*/X, \quad \text{where} \quad X = 1 - BH_*/t,
\]

(28)

\( B \) being a nondimensional parameter. For Big Rip to occur, \( B \) has to be positive.

Taking the bulk viscosity to have the form

\[
\zeta = \tau \theta^{2\alpha - 1} = \tau (3H)^{2\alpha - 1}
\]

(29)

with \( \tau \) a positive constant, the time-dependent factors in Eq. (27) drop out. There remains an algebraic equation, determining \( B \).
Of main interest is the time-dependent forms

$$\zeta = \tau(3H_*/X)^{2\alpha-1}, \quad \rho = \rho_*/X^{2\alpha}. \quad(30)$$

As an example, the case $\alpha = 2$ turns out to yield a cubic equation for $B$. There is one positive root (assuming $f_0$ positive), leading to a viscosity-generated Big Rip. If $\alpha < 0$, typically $\alpha = -1$, there may still be positive solutions for $B$ implying that $H = H_*/X$ is diverging. By contrast, $\zeta \propto X^{-(2\alpha-1)}$ and $\rho \propto X^{-2\alpha}$ go to zero.

4 Relationship to 5D viscous theory

Let us investigate the possible link between the 4D theory above and the analogous viscous theory in 5D space. To this end we consider a spatially flat ($k = 0$) brane located at the fifth dimension $y = 0$, surrounded by an anti-de-Sitter (AdS) space. If the 5D cosmological constant, called $\Lambda$, is negative, the configuration is that of the Randall-Sundrum II model (RSII) [31]. The 5D coordinates are denoted $x^A = (t, x, y)$, and the 5D gravitational coupling is $\kappa_5^2 = 8\pi G_5$. The Einstein equations are

$$R_{AB} - \frac{1}{2}g_{AB}R + g_{AB}\Lambda = \kappa_5^2 T_{AB}, \quad(31)$$

and the metric is

$$ds^2 = -n^2 dt^2 + a^2 \delta_{ij} dx^i dx^j + dy^2, \quad(32)$$

where $n(t, y)$ and $a(t, y)$ are to be determined from the Einstein equations.

Of main interest are the $(tt)$ and $(yy)$ components of the field equations. They are

$$3 \left\{ \left( \frac{\dot{a}}{a} \right)^2 - n^2 \left[ \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 \right] \right\} - \Lambda n^2 = \kappa_5^2 T_{tt}, \quad(33)$$

$$3 \left\{ \frac{a'}{a} - \frac{\dot{n}}{n} \right\} - \frac{1}{n^2} \left[ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} \right] \right\} + \Lambda = \kappa_5^2 T_{yy} \quad(34)$$

(cf. for instance, Refs. [32, 33, 34, 19, 26]). Overdots and primes mean derivatives with respect to $t$ and $y$ respectively. On the brane $y = 0$ we assume
there is a constant tension $\sigma$, and an isotropic fluid with time-dependent energy density $\rho = \rho(t)$. The energy-momentum tensor is now

$$T_{AB} = \delta(y) (-\sigma \delta_{\mu\nu} + \rho U_\mu U_\nu + \bar{\rho} h_{\mu\nu}) \delta^\mu_A \delta^\nu_B.$$  \hspace{1cm} (35)

Applying the junction conditions across the brane we obtain, for arbitrary $y$, after integration with respect to $y$ \[35\],

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{1}{6} \Lambda + \left(\frac{a'}{a}\right)^2 + \frac{C}{a^2}. \hspace{1cm} (36)$$

$$H_0^2 = \frac{1}{6} \Lambda + \frac{\kappa_5^4}{36} (\sigma + \rho)^2. \hspace{1cm} (37)$$

We here let subscript zero refer to the brane. On the brane, $n_0(t) = 1$. Recall that $\Lambda$ and $\sigma$ are constants, and that Eq. \[37\] is a 5D, not a 4D, equation. Its essential new feature is that it contains a $\rho^2$ term. The equation functions as a bridge between 4D and 5D cosmologies.

We observe the solution for $a_0(t)$ if $\rho = 0$:

$$a_0(t) = e^{\sqrt{\lambda t}}, \hspace{1cm} \lambda = \frac{1}{6} \Lambda + \frac{1}{36} \kappa_5^4 \sigma^2; \hspace{1cm} (38)$$

normalized such that $a_0(0) = 1$.

Inserting $\rho = \rho_*/X^{2\alpha}$ into Eq. \[37\] we get

$$H_0^2 = \frac{1}{6} \Lambda + \frac{\kappa_5^4}{36} \left[\sigma + \frac{\rho_*}{(1 - BH_* t)^{2\alpha}}\right]^2. \hspace{1cm} (39)$$

Near the Big Rip, $t_s = 1/(BH_*)$, the quantities $\Lambda$ and $\sigma$ become unimportant, and we get

$$a_0(t) \sim \exp \left[\frac{(\kappa_5^2/6) \rho_*}{(2\alpha - 1)(BH_*)^{2\alpha}(t_s - t)^{2\alpha - 1}}\right], \hspace{1cm} (40)$$

showing that if $\alpha > 1/2$, $a_0(t)$ has an essential singularity. Einstein’s gravity corresponds to $\alpha = 1$. The singularity becomes stronger, the higher is the value of $\alpha$. If $\alpha < 1/2$, $a_0(t)$ does not diverge at $t_s$. From Eq. \[36\],

$$a^2(t, y) = \frac{1}{2} a_0^2(t) \left[\left(1 + \frac{\kappa_5^4 \sigma^2}{6\Lambda}\right) + \left(1 - \frac{\kappa_5^4 \sigma^2}{6\Lambda}\right) \cosh(2\mu y) - \frac{\kappa_5^2 \sigma}{3\mu} \sinh(2\mu |y|)\right], \hspace{1cm} (41)$$
with $\mu = \sqrt{-\Lambda/6}$. The important point here is that the Big Rip divergence 
onto the brane becomes transferred to the bulk. The bulk scale factor $a(t, y)$
diverges for arbitrary $y$ at $t = t_s$ if $a_0(t)$ diverges at $t_s$. There is no funda-
mental difference between an Einstein fluid and a modified gravity fluid in
this respect; their behavior is essentially the same.

In summary, we have discussed viscous dark energy as a particular rep-
resentative of inhomogeneous equation-of-state fluids and the appearance of
finite-time future singularities for such energies. It is of interest to note
that due to the relationship between modified gravity and inhomogeneous
equation-of-state ideal fluids [36], our findings may be useful in the study of
future singularities in modified gravity [37].

References

[1] I. Brevik, S. Nojiri, S. D. Odintsov and L. Vanzo, Phys. Rev. D 70,
043520 (2004).

[2] A. Vikman, Phys. Rev. D 71, 023515 (2005).

[3] S. Capozziello, V. F. Cardone, E. Elizalde, S. Nojiri and S. D. Odintsov,
Phys. Rev. D 73, 043512 (2006).

[4] H. Wei, Preprint arXiv: 0809.0057 [astro-ph].

[5] S. Nojiri and S. D. Odintsov, Phys. Lett. B637, 139 (2006).

[6] I. Brevik, O. G. Gorbunova and A. V. Timoshkin, Eur. Phys. J. C 51,
179 (2007).

[7] I. Brevik, E. Elizalde, O. G. Gorbunova and A. V. Timoshkin, Eur.
Phys. J. C 52, 223 (2007).

[8] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett.
91, 071301 (2003).

[9] B. McInnes, J. High Energy Phys. 0208, 029 (2002).

[10] J. D. Barrow, Class. Quant. Grav. 21, L79 (2004).

[11] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D 71, 063004
(2005).
[12] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007) [hep-th/0601213]; Preprint arXiv: 0807.0685 [hep-th].

[13] G. Cognola, E. Elizalde, S. D. Odintsov, P. Tretyakov and S. Zerbini, Preprint arXiv: 0810.4989 [gr-qc].

[14] I. Brevik and O. Gorbunova, Gen. Relativ. Gravit. 37, 2039 (2005).

[15] S. Weinberg, Astrophys. J. 168, 175 (1971).

[16] T. Padmanabhan and S. M. Chitre, Phys. Lett. A 120, 433 (1987).

[17] Ø. Grøn, Astrophys. Space Sci. 173, 191 (1990).

[18] I. Brevik and L. T. Høen, Astrophys. Space Sci. 219, 99 (1994).

[19] I. Brevik and A. Hallanger, Phys. Rev. D 69, 024009 (2004).

[20] I. Brevik, K. Børkje and J. P. Morten, Gen. Relativ. Gravit. 36, 2021 (2004).

[21] I. Brevik, J.-M. Børven and S. Ng, Gen. Relativ. Gravit. 38, 907 (2006).

[22] J. Ren and X. H. Meng, Phys. Lett. B 633, 1 (2006).

[23] I. Brevik, Gen. Relativ. Gravit. 38, 1317 (2006).

[24] R. Sussman, Preprint arXiv: 0801.3324 [gr-qc].

[25] I. Brevik, Gravitation and Cosmology (Moscow) 14, 332 (2008).

[26] I. Brevik and O. Gorbunova, Eur. Phys. J. C 56, 425 (2008).

[27] S. Nojiri and S. D. Odintsov, Phys. Rev. D 70, 103522 (2004).

[28] M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quant. Grav. 22, L35 (2005).

[29] I. Brevik, O. Gorbunova and Y. A. Shaido, Int. J. Mod. Phys. D 14, 1899 (2005).

[30] I. Brevik, Int. J. Mod. Phys. D 15, 767 (2006).

[31] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).
[32] P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477, 285 (2000).

[33] P. Binétruy, C. Deffayet and D. Langlois, Nucl. Phys. B 565, 269 (2000).

[34] I. Brevik, K. Ghoroku, S. D. Odintsov and M. Yahiro, Phys. Rev. D 66, 064016 (2002).

[35] I. Brevik, Eur. Phys. J. C 56, 579 (2008).

[36] S. Nojiri and S. D. Odintsov, Phys. Rev. D 72, 023003 (2005).

[37] K. Bamba, S. Nojiri and S. D. Odintsov, Preprint arXiv: 0807.2575 [hep-th].