Extended reflection equation algebras, the braid group on a handlebody and associated link polynomials

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ABSTRACT

The correspondence of the braid group on a handlebody of arbitrary genus to the algebra of Yang-Baxter and extended reflection equation operators is shown. Representations of the infinite dimensional extended reflection equation algebra in terms of direct products of quantum algebra generators are derived, they lead to a representation of this braid group in terms of $R$-matrices. Restriction to the reflection equation operators only gives the coloured braid group. The reflection equation operators, describing the effect of handles attached to a 3-ball, satisfy characteristic equations which give rise to additional skein relations and thereby

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invariants of links on handlebodies. The origin of the skein relations is explained and they are derived from an adequately adapted handlebody version of the Jones polynomial. Relevance of these results to the construction of link polynomials on closed 3-manifolds via Heegard splitting and surgery is indicated.
1. INTRODUCTION

In this paper† we should like to explain the relation between quantum groups (QG) and the braid group on a three dimensional manifold of arbitrary genus with boundary. As a consequence we will be able to define invariants of links on such manifolds. A three-manifold having a genus $g$ Riemann surface as boundary is conventionally named a genus $g$ handlebody. The braid group on such a handlebody can be formulated in terms of the usual braid group generators $\sigma_i$ for genus zero 3-manifolds [1] plus additional generators $\tau_\alpha$ implementing windings around handles. We shall make use of the results in [2] where such a description was given, explicitly, the braid group $B^g_n$ on a handlebody comprises the following relations:

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-1 \\
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2 \\
\sigma_i \tau_\alpha = \tau_\alpha \sigma_i, \quad i \geq 2, \quad \alpha = 1, \ldots, g \tag{1.1}
\]

\[
\sigma_1 \tau_\alpha \sigma_1 \tau_\alpha = \tau_\alpha \sigma_1 \tau_\alpha \sigma_1, \\
\sigma_1 \tau_\alpha \sigma_1^{-1} \tau_\beta = \tau_\beta \sigma_1 \tau_\alpha \sigma_1^{-1}, \quad \alpha < \beta.
\]

The first two equations define the well known Artin braid group $B_n$ acting on $n$ strands in a topologically trivial 3-manifold [1]. For each handle there is a new generator $\tau_\alpha$ having nontrivial commutation relations only with $\sigma_1$ and the $\tau$ generators. The last equation is absent in the case of a solid torus, i.e. for genus one. The first equation of (1.1) corresponds to the Yang-Baxter equation written in braid form

\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}, \tag{1.2}
\]

providing a link to the theory of quantum groups as $\sigma_i$ can be represented in terms of the $\hat{R}$-matrix. The fourth equation of (1.1) is also related to quantum groups, it

† A completely rewritten and largely extended version of ref. [10]
can be considered both as a comodule invariant w.r.t. QG coaction and as a way of describing the quantum algebra, we will use it in the form

\[ R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21}. \]  

(1.3)

There exist also spectral parameter dependent versions of it which play prominent roles in quantum inverse scattering [3], describing the commutation relations of monodromy matrices. Actually, they appeared first in the study of two particle scattering on a half-line, with matrix \( K(\theta) \) describing reflection of a particle at the endpoint and \( R(\theta - \theta') \) describing two particle scattering [4]. Hence the name reflection equation (RE), suggested in [5], where also the connection of (1.3) with \( B_n^1 \) was mentioned. We should mention that there exists still another spectral parameter independent reflection equation [6, 7], which is invariant w.r.t. different QG comodule transformations compared to (1.3). The last equation of (1.1) is close to the RE, it is a compatibility condition for solutions of the RE s.t. these can be combined into new solutions of the RE. In QG language it looks like

\[ R_{12}K_1R_{12}^{-1}K'_2 = K'_2R_{12}K_1R_{12}^{-1}, \]  

(1.4)

and its properties and connection to \( B_n^g \) were discussed in [8, 9, 10] under different aspects. When discussing representations of the braid group (1.1) we will naturally be led to representations of \( \tau_\alpha \) in terms of \( R \)-matrices s.t. \( B_n^g \) can be viewed as a subgroup of \( B_{n+g} \). Equivalently, \( \tau_\alpha \) can be expressed in terms of quantum algebra generators. Indeed we will derive whole series of new solutions of both (1.3) and (1.4) in terms of quantum algebra generators, and they precisely correspond to the description of \( \tau_\alpha \) in terms of \( R \)-matrices. Furthermore, we derive quadratic characteristic equations for the matrix \( K \), and hence the additional generators \( \tau_\alpha \), similar to the Hecke algebra relation for \( \sigma_i \). They can be interpreted as an additional skein relation when considering closed braids on the handlebody and, in principle, they recursively define link invariants for closed braids on arbitrary genus 3-manifolds with boundary [10]. This in turn, via Heegaard splitting, might
be a way of constructing invariant polynomials of links on arbitrary 3-manifolds without boundary. Since we know the representation of $\tau_\alpha$ in terms of $R$-matrices we can also write down a trace formula for the link invariants, this is equivalent to using the quantum trace that is defined for the matrix $K$ [6]. Further we show that the characteristic equation for $\tau_\alpha$ is actually a consequence of the one for $\sigma_i$.

The plan of the paper is as follows. We introduce the RE in section two and discuss its properties as an associative quadratic algebra, then we extend it by (1.4) and derive new solutions of the combined system in terms of quantum algebra generators. In section three we review some results of [2] concerning the braid group on a handlebody and obtain the representation of $\tau_\alpha$ in terms of $R$-matrices and quantum algebra generators. We also discuss there the connection between the Hecke algebra relation and the quadratic equation for $\tau_\alpha$. In the fourth section we look at closed braids on handlebodies and their invariants, notably by means of new skein relations and quantum traces for the additional generators. Finally, in the fifth section we mention some implications and possible applications of our results.
2. ALGEBRAS OF REFLECTION EQUATION OPERATORS

We will study the properties of the following reflection equation

\[ RK_1 \tilde{R} K_2 = K_2 R K_1 \tilde{R}, \tag{2.1} \]

where \( \tilde{R} = P R P \) and \( P \) is the permutation operator. Its basic property and a guideline for its construction is invariance w.r.t. the QG coaction, i.e. \( K_T = T K T^{-1} \) is also a solution of this RE if all elements of \( K \) and \( T \) commute, \([K_{ij}, T_{mn}] = 0\), and \( T \) obeys the QG relations

\[ RT_1 T_2 = T_2 T_1 R. \tag{2.2} \]

Just as in the case of the defining relations (2.2) of the QG we can view (2.1) as an associative quadratic algebra. If we use the \( sl_q(2) \) \( R \)-matrix

\[ R_{ij}^{kl} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \omega = q - q^{-1} \tag{2.3} \]

being a solution of the Yang-Baxter equation

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \tag{2.4} \]

then we find that the commutation relations for the entries of the matrix \( K = \binom{a_{ij}}{c_{kl}} \)

\[ \dagger \text{We assume familiarity of the reader with basic quantum group terminology as introduced in [11, 12], for example. Throughout this paper when giving explicit examples we only use } sl_q(2) \text{ for simplicity, generalizations should be obvious.} \]
are given by
\begin{align*}
ab = q^{-2}ba, & \quad ad = da, & \quad bd - db = -q^{-1}\omega ab, \\
ac = q^{2}ca, & \quad bc - cb = q^{-1}\omega(ad - a^{2}), & \quad cd - dc = q^{-1}\omega ca.
\end{align*} \tag{2.5}

This algebra has two central elements, the quantum trace and the quantum determinant which we set equal to one
\begin{align*}
c_{1} = q^{-1}a + qd, & \quad c_{2} = ad - q^{2}cb \equiv 1. \tag{2.6}
\end{align*}

The normalization of \( c_{1} \) is chosen such that \( K \) and \( K^{-1} \) have equal quantum trace. Using these relations the ‘antipode’ \( S(K) \equiv K^{-1} \) can be found to be
\begin{align*}
K^{-1} = \begin{pmatrix}
q^{2}d - q\omega a & -q^{2}b \\
-q^{2}c & a
\end{pmatrix}.
\tag{2.7}
\end{align*}

Then we easily establish a relation (characteristic equation) between \( K \) and \( K^{-1} \)
\begin{align*}
qK + q^{-1}K^{-1} - c_{1}I = 0, \tag{2.8}
\end{align*}
which will give rise to a skein relation later on.

A few remarks about the properties of the above algebra follow.

(i) The RE algebra (2.5) depends only on \( q^{2} \).

(ii) If \( q \) is a root of unity, \( q^{2p} = 1 \), then \( a^{p} \) is a further central element.

(iii) The \( sl_{q}(2) \) RE algebra has two constant or one-dimensional representations, one of them clearly is the identity matrix and the other one a lower-right triangular matrix with arbitrary constants \( b, c, d \). Constant solutions of (2.1) were studied in [6].

(iv) The \( K \)-matrix can be considered as a product of two suitable quantum planes [13, 14] \( x^{i}x^{j} = q^{-1}R_{kl}^{ji}x^{k}x^{l} \) and \( y_{k}y_{j} = q^{-1}y_{k}y_{l}R_{kl}^{ji} \) invariant w.r.t. the QG coaction \( x^{i'} = T_{ij}^{i}x^{j} \) and \( y_{j}' = y_{j}(T^{-1})_{ji} \). Commutation relations
between $x^i$ and $y_j$ can be determined using (2.2) as $x^i y_j = \text{(const.)} y_k x^l R^{ik}_{lj}$ and hence those of their product $K_{ij} = x^i y_j$ which coincide with (2.5). This also gives a better understanding of the comodule property $K_T = TKT^{-1}$ of the RE.

(v) If we impose suitable reality conditions on $x^i, y_j$ and hence $K_{ij}$ then a linear combination of the elements of (2.5) is just the $q$-deformed Minkowski space [15, 16], where $c_1$ is the time coordinate and $c_2$ the invariant length. Various reality conditions are discussed in [7], they parallel those of $sl_q(2)$.

(vi) Truncation of algebra (2.5) by $c_1 = 0$ can be shown to lead to the quantum 2-sphere, a quantum analogue of homogeneous spaces [17].

(vii) It is possible to introduce an index free notation for quantum planes and extend it to the $K$-matrix, such that the RE can be rewritten in exchange algebra form with four $R$-matrices on one side [7, 18].

(viii) The monodromy $M = Pexp(\frac{2\pi i}{K} \int_0^{2\pi} J(x) dx)$ of the $sl_q(2)$ Kac-Moody current satisfies the RE when regularized on a one-dimensional lattice with periodic boundary conditions [19]. As this commutation relation of the regularized monodromy holds for arbitrary numbers of sites it might be expected to survive the continuum limit.

(ix) Neither $K^{-1}$ nor $K^2$ is a solution of the RE.

The last remark leads us to a very important property of the RE, namely, given two different solutions of the RE satisfying a certain compatibility condition then one can use them to construct new solutions [8, 9, 10]. Explicitly, let $K$ and $K'$ be solutions of (2.1) then

\[(i) \quad \tilde{K} = KK' \quad \text{and} \quad (ii) \quad \tilde{\tilde{K}} = KK'K^{-1} \quad (2.9)\]

are also solutions of (2.1) provided $K$ and $K'$ commute as follows

\[RK_1 R^{-1} K'_2 = K'_2 RK_1 R^{-1}. \quad (2.10)\]

This equation is invariant under the coaction $K_T = TKT^{-1}$ and $K'_S = SK' S^{-1}$ if $S$ also obeys QG relations (2.2) and in addition $RT_1 S_2 = S_2 T_1 R$, especially we
can put $S$ equal to $T$. Note that the second composite solution in (2.9) is not a trivial consequence of the first since $R^{-1}$ does not solve the RE. This process of building up new solutions can obviously be continued using newly constructed solutions if they satisfy (2.10), but some care has to be taken to keep track of the ordering as (2.10) is not symmetric under exchange of $K$ and $K'$. This will become clearer when we discuss systems of solutions of both (2.1) and (2.10). We will sometimes refer to both equations as extended RE algebra. Equation (2.10) gives 16 commutation relations between the elements of $K$ and $K'$

\begin{align*}
a' a &= a a' - q \omega b c', \\
a' b &= b a', \\
a' c &= c a' + q \omega (a - d)c', \\
a' d &= d a' + q^{-1} \omega b c', \\
b' a &= a b' + q \omega b(a' - d'), \\
b' b &= q^2 b b', \\
b' c &= q^{-2} c b' + (1 + q^{-2}) \omega^2 b c' \\
&\quad - q^{-1} \omega (a - d)(a' - d'), \\
b' d &= d b' - q^{-1} \omega b(a' - d'),
\end{align*}

(2.11)

and they only depend on $q^2$. Note that $K$ and $K'$ are commuting for $q = 1$ even if one linearizes them, whereas the RE in this case produces the undeformed $sl(2)$ Lie algebra relations [20, 21]. The extended RE algebra was implicitly contained also in the construction of complex quantum groups recently [22], where relations (2.11) describe commutation relations among the generators of the quantum algebra and their complex conjugates, while both sets individually satisfy (2.5). Algebra (2.5) was also constructed in [18] in the framework of braided tensor categories.†

† This approach to the RE algebra takes the point of view that one has an extra braiding between elements of the two copies of the algebra in the coproduct $\Delta(K) = K \otimes K$ s.t. $(1 \otimes a) \cdot (a \otimes 1) = a \otimes a - q \omega b \otimes c$, for example, note the new term on the RHS. Denoting $1 \otimes a$ as $a'$ and $a \otimes 1$ as $a$, etc., this relation is identical with the first of (2.11). Then this
An important point is further that the central elements of $K$ and $K'$ are mutually central in both algebras, i.e.

$$[K^i_j, c'_m] = [K'^i_j, c_m] = 0, \quad m = 1, 2. \quad (2.12)$$

It is obvious that we have central elements for the combined solutions and also characteristic equations, for example

$$qKK' + q^{-1}(KK')^{-1} - C_1 I = 0, \quad (2.13)$$

where $C_1 = q^{-1}(aa' + bc') + q(cb' + dd')$.

It is known that the RE algebra has a representation in terms of the quantum algebra generators [24]. The $sl_q(2)$ algebra dual to the QG (2.2) can similarly be written in matrix form [11]

$$\tilde{R}L_1^{\varepsilon_1}L_2^{\varepsilon_2} = L_2^{\varepsilon_2}L_1^{\varepsilon_1}\tilde{R}, \quad (\varepsilon_1, \varepsilon_2) \in \{(+, +), (+, -), (-, -)\} \quad (2.14)$$

where

$$L^+ = \begin{pmatrix} q^{H/2} & q^{-1/2}\omega X^- \\ 0 & q^{-H/2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{-H/2} & 0 \\ -q^{1/2}\omega X^+ & q^{H/2} \end{pmatrix} \quad (2.15)$$

and this gives the $sl_q(2)$ algebra

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \omega^{-1}(q^H - q^{-H}) \quad (2.16)$$

with antipode $S(H) = -H$, $S(X^\pm) = -q^\mp X^\pm$ and coproduct $\Delta(L^\pm) = L^\pm \otimes L^\pm$. It is easy to show using (2.14) that $K = S(L^-)L^+$ represents a solution of the RE.

‘braided coproduct’ for the RE algebra is compatible with the algebra relations and is of the same form as for the QG and quantum algebra. This so-called ‘braided group’ which seemingly can be associated to any QG was cast into RE form in [23].
explicitly $K$ is given by

$$K = \begin{pmatrix} q^H & q^{-1/2}\omega q H/2 X^- \\ q^{-1/2}\omega X^+ q^H/2 & q^{-H} + q^{-1/2}\omega^2 X^+ X^- \end{pmatrix}.$$  

(2.17)

This representation of the RE algebra has quantum determinant $c_2 = 1$ and the quantum trace $c_1$ is just the quadratic Casimir operator of the quantum algebra $sl_q(2)$.

We now find whole towers of new representations of the RE algebra in terms of quantum algebra generators and generalize them to the extended RE algebra. They will be useful for representing the braid group (1.1). To avoid clumsy notation we introduce the abbreviation $S^\pm \equiv S(L^\pm)$. There is a simple way to produce further representations of the RE algebra, namely by means of the coproduct $\Delta$ which gives representations on tensor products of spaces, in the simplest case of two spaces we obtain

$$\Delta(K) = \Delta(S^{-i}_k)\Delta(L^+^k_j) = K^n_m \otimes S^{-i}_m L^+_n j = \left(1 \otimes S^-ight)^i_m \left(K \otimes 1\right)_n^m \left(1 \otimes L^+\right)_j^n \equiv S^{-i}_m K^m_n L^+_n j$$

or in matrix notation simply $\Delta(K) = S^-_2 K^2_1 L^+_2$. Note that this coproduct for $K$ is a consequence of the one for $L^\pm$ and cannot be expressed as $\Delta(K) = K_1 K_2$ (cf. footnote after eqn. (2.11)). We stress that whenever we write down tensor products in the following then entries of different spaces are strictly commuting.

We can get a whole string of solutions of the RE algebra generalizing (2.18) by repeatedly applying the coproduct, and in addition embed it into the $g$-fold tensor product of the universal enveloping algebra of $sl_q(2)$ leading to the definition

$$K^0_{(m)} = S^-_g \cdots S^-_{g-(m-2)} K_{g-(m-1)}^+ L^+_{g-(m-2)} \cdots L^+_g, \quad 1 \leq m \leq g$$

(2.19)

where $K_i = S^-_i L^+_i$ and $L^+_i = 1 \otimes \cdots 1 \otimes L^+ \otimes 1 \cdots \otimes 1$ with $L^+$ inserted into the $i$-th position, etc. In the simplest case of $sl_q(2)$ all objects on the RHS of (2.19)
are $2 \times 2$ matrices, matrix multiplication being understood, and their entries take values in the $g$-fold tensor product $U^\otimes g(sl(2))$. So (2.19) defines $g$ operators $K^0_{(1)} = K_g$, $K^0_{(2)} = S_g^{-} K_{g-1} L_g^{+}$, \ldots, $K^0_{(g)} = S_g^{-} \cdots S_2^{-} K_1 L_2^{+} \cdots L_g^{+}$. We keep $g$ arbitrary but fixed, it will correspond to the genus of the handlebody later on. Formally one may envisage the limit $g \to \infty$ as one has a natural sequence of embeddings of tensor products into higher ones. Incidentally, we found that a variant of the operators $K^0_{(m)}$ has been employed in the formulation of quantum differential geometry [25].

However, there is a drawback because $K^0_{(m)}$ and $K^0_{(n)}$ do not satisfy (2.10) for $m \neq n$ but instead again the RE

$$R(K^0_{(m)})_1 \tilde{R}(K^0_{(n)})_2 = (K^0_{(n)})_2 R(K^0_{(m)})_1 \tilde{R}, \quad m \geq n \quad (2.20)$$

and hence $K^0_{(m)}$ cannot be used to represent the generators $\tau_\alpha$. Fortunately, this construction gives us a hint how to solve the problem. We define two more sets of operators $K^+_{(m)}$ and $K^-_{(m)}$ by

$$K^\pm_{(m)} = S_g^\pm \cdots S_g^{(m-2)} K_{g-(m-1)} L_g^\pm \cdots L_g^\pm, \quad (2.21)$$

they are also solutions of the RE and have the following commutation relations

$$R(K^\pm_{(m)})_1 \tilde{R}(K^\pm_{(m)})_2 = (K^\pm_{(m)})_2 R(K^\pm_{(m)})_1 \tilde{R},$$

$$R(K^+_{(m)})_1 R^{-1}(K^+_{(n)})_2 = (K^+_{(n)})_2 R(K^+_{(m)})_1 R^{-1}, \quad m < n \quad (2.22)$$

$$R(K^-_{(m)})_1 R^{-1}(K^-_{(n)})_2 = (K^-_{(n)})_2 R(K^-_{(m)})_1 R^{-1}, \quad m > n$$

$$R(K^-_{(m)})_1 R^{-1}(K^+_{(n)})_2 = (K^+_{(n)})_2 R(K^-_{(m)})_1 R^{-1}, \quad m \neq n$$

corresponding to (2.1) and (2.10) while the last equation gives the commutation relations between the two sets of operators. They also have definite commutation
relations with $K^0_{(m)}$ given by

\[
R(K^0_{(m)})_1 \tilde{R}(K^\pm_{(m)})_2 = (K^\pm_{(m)})_2 R(K^0_{(m)})_1 \tilde{R}, \quad m \geq n
\]
\[
R(K^0_{(m)})_1 R^{-1}(K^+_{(n)})_2 = (K^+_{(n)})_2 R(K^0_{(m)})_1 R^{-1}, \quad m < n \tag{2.23}
\]
\[
R(K^-_{(m)})_1 R^{-1}(K^0_{(n)})_2 = (K^0_{(n)})_2 R(K^-_{(m)})_1 R^{-1}. \quad m > n
\]

These relations can be verified because we obviously have

\[
\tilde{R}(L^\pm_1)_1 (L^\pm_2)_2 = (L^\pm_2)_2 (L^\pm_1)_1 \tilde{R},
\]
\[
(L^\pm_1)_1 (L^\pm_2)_2 = (L^\pm_2)_2 (L^\pm_1)_1, \quad m \neq n \tag{2.24}
\]
as a consequence of (2.14). Thus there are two sets of operators expressed in terms of quantum algebra generators that can be used to represent the braid group generators $\tau_n$. Their meaning will be clarified in the next section. In principle (2.22) constitutes an infinite dimensional algebra and it might have representations other than by quantum algebra generators. Some of the relations in (2.22) and (2.23) have the form of (2.10) and so the new solutions of the RE in (2.9) can be built. For example, if one considers only the $K^+_{(m)}$ series then it can be seen that the product $K^+_{(m)} K^+_{(n)}$, $n > m$, has also commutation relation (2.10) with $K^+_{(p)}$ if $p > n$, and this behaviour persists for operators with appropriate ordering. This is most easily seen in terms of diagrams to be introduced in the next section.

For these new solutions (but not for $K^0_{(m)}$) the characteristic equation (2.8) holds as well

\[
q K^\pm_{(m)} + q^{-1} (K^\pm_{(m)})^{-1} - c_1 I = 0, \tag{2.25}
\]

with \((K^\pm_{(m)})^{-1} = S^g \cdots S_{g-(m-2)}^g K_{g-(m-1)}^1 L_{g-(m-2)}^g \cdots L_i^g\) and \(K^{-1}_i = S^+_i L^-_i\). The central element $c_1$ as the quadratic casimir operator of $sl_q(2)$ remains unchanged. Finally, as representations of the extended RE algebra their commutation relations are automatically invariant w.r.t. the comodule transformation $(K^\pm_{(m)})_T = T K^\pm_{(m)} T^{-1}$. 

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We remark that instead of (2.18) we could have used the permuted coproduct
\[ \Delta'(K) = S_1^- K_2 L_1^+ \]
having a generalization as
\[ K_{(m)}^0 = S_1^- \cdots S_{m-1}^- K_m L_{m-1}^+ \cdots L_1^+ , \]
and hence
\[ K_{(m)}^\pm = S_1^\pm \cdots S_{m-1}^\pm K_m L_{m-1}^\pm \cdots L_1^\pm . \]
They correspond to a reverse ordering of spaces and differ from (2.21), but do also satisfy (2.20), (2.22) and (2.23). However, \( K_{(m)}^\pm \) in this simpler form is not consistent with our conventions in the next section, so we do not consider this further.

The RE algebra therefore plays different roles, it is a comodule w.r.t. the QG and on the other hand it acts via (2.17) on representations of the quantum algebra dual to the QG. Further applications of the RE algebra were mentioned in [6].

3. REPRESENTATIONS OF THE BRAID GROUP ON HANDLEBODIES

The braid group \( B_n^g \) on a solid handlebody \( H_g \) of genus \( g \) was described in [2]. In addition to the generators \( \sigma_i, i = 1, \ldots, n - 1 \) of the braid group \( B_n \) defined on a 3-dimensional manifold of genus zero there are generators \( \tau_\alpha, \alpha = 1, \ldots, g \) implementing windings around the \( g \) handles. The algebra is given by

\[
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i &= 1, \ldots, n - 1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| &\geq 2, \\
\sigma_i \tau_\alpha &= \tau_\alpha \sigma_i, & i &\geq 2, \quad \alpha = 1, \ldots, g, \\
\sigma_1 \tau_\alpha \sigma_1 \tau_\alpha &= \tau_\alpha \sigma_1 \tau_\alpha \sigma_1, \\
\sigma_1 \tau_\alpha \sigma_1^{-1} \tau_\beta &= \tau_\beta \sigma_1 \tau_\alpha \sigma_1^{-1}, & \alpha &< \beta
\end{align*}
\]

and the first two relations define the well known Artin braid group [1]. We refer to [2] for details and references, here we only explain conventions briefly which should make (3.1) fairly transparent.

On the handlebody (Fig.1) it is possible, without loss of generality, to prescribe a fixed ordering of the points where the strands begin (resp. end) having coordinates \( P^{(1)}_i = (\frac{i}{n+1}, \frac{1}{2}, 1) \) (resp. \( P^{(0)}_j = (\frac{j}{n+1}, \frac{1}{2}, 0) \), \( i, j = 1, \ldots, n \) in a lefthanded
$(x, y, z)$-coordinate system. So the unit cube in the positive octant is contained in $H_g$ and the usual braids are obtained by connecting points $P_i^{(1)}$ and $P_j^{(0)}$ by strands confined to the unit cube. The braid diagram is obtained by projecting on the $x$-$z$-plane. The handles are positioned, say, to the left of the unit cube around coordinates $h_\alpha = \left( -\frac{\alpha}{g+1}, y, 1 - \frac{\alpha}{g+1} \right)$, $\alpha = 1, \ldots, g$. For the braid group on $H_g$ the strands are allowed to leave the unit cube at height $z$ and go around the handle $h_\alpha$ counterclockwise for $\tau_\alpha$ (clockwise for $\tau_\alpha^{-1}$) and then come back to the unit cube at height $z - \delta$, $\delta$ small. The convention† is that strands leaving or entering the unit cube at height $z_1$ should be over those doing so at $z_2$ in the projection onto the $x$-$z$-plane if $z_1 > z_2$. Within the unit cube strands can only go downward in the negative $z$-direction. This definition can be further formalized, but everything is rather intuitive.

Fig.1: The 2-braid $\tau_2^{-1}\sigma_1^2\tau_1$ and its closure (dotted lines)

† We have chosen conventions slightly different from [2], especially in the last equation of (3.1) the condition in [2] is $\alpha > \beta$, and also 'strands should be over if $z_2 > z_1$'.
For our arguments it is more appropriate to think of piercing long bars through the handles and after that forget about them. Then, if we rotate the bars by $\pi/4$ around the $x$-axis counterclockwise to $h'_\alpha = \left(\frac{-\alpha}{g+1}, \frac{\alpha}{g+1} - 1, z\right)$ we can depict the braiding in a more systematic way by projecting on the $x$-$z$-plane. For example, the fourth equation of (3.1) can be represented graphically as in Fig.2 where, as usual, $\sigma_1$ has been represented by a crossing of two strands.

![Fig.2: Graphical representation of the reflection equation (for later convenience numbering of spaces is indicated)](image)

Similarly, the last equation of (3.1) can be represented as in Fig.3 and is proven easily this way by pulling lines appropriately.

![Fig.3: Graphical representation of compatibility condition (2.10)](image)
The strands leaving the unit cube to wind around the bars always belong to the first space $V_1$ of the tensor product $V(n) = V_1 \otimes \cdots \otimes V_n$ on which the $\sigma_i$ act, and this explains why only $\sigma_1$ is non-commuting with $\tau_\alpha$. It also means that $\tau_\alpha$ is acting non-trivially only in $V_1$ by some operator $K_{(\alpha)}$, so we put

$$\sigma_i = q 1 \otimes \cdots 1 \otimes \hat{R}_{i,i+1} \otimes 1 \cdots 1, \quad i = 1, \ldots, n - 1$$

$$\tau_\alpha = q^3 K_{(\alpha)} \otimes 1 \cdots 1, \quad \alpha = 1, \ldots, g$$

(3.2)

where as usual $\sigma_i$ is acting non-trivially only in $V_i \otimes V_{i+1}$ as $\sigma_i = PR_{i,i+1} \equiv \hat{R}_{i,i+1}$. The factor $q^3$ is inserted for later convenience to match the factor $q$ in the definition of $\sigma_i$. Using this we can show explicitly that (3.1) is equivalent to the Yang-Baxter equation and to the extended RE algebra, by identifying $\sigma_1 = \hat{R}_{12}$ and $\tau_\alpha = (K_{(\alpha)})_1$, plus two other rather obvious consistency conditions as given in (3.1). This equivalence is also explained in [8, 9].

Thus $\sigma_i$ has an explicit matrix representation, but what about $\tau_\alpha$? Because $\sigma_i$ is represented by a $R$-matrix one should expect that the same holds true for $\tau_\alpha$, and this is supported by Fig.2 which suggests to represent the effect of a handle on a strand going around it by the square of a $R$-matrix. Let us consider first the genus one case, i.e. only the RE has to be taken into account. As shown in the previous section $K = S^- L^+$ is a solution of the RE and this tells us how to represent $\tau_\alpha$ because $S^-$ and $L^+$ are related to the $R$-matrix. In fact, looking at the universal $R$-matrix of $sl_q(2)$ acting on $V_1 \otimes V_2$

$$R_U = q^{\frac{1}{2} H \otimes H} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n; q^{-2}]!} (q^{-\frac{1}{2} H} X^-)^n \otimes (q^{\frac{1}{2} H} X^+)^n, \quad [n; q] = \frac{(1 - q^n)}{(1 - q)}$$

(3.3)

we can represent the $sl_q(2)$ generators either on $V_1$ or $V_2$ (the fundamental repre-
sentation of $sl_q(2)$ is $\rho_f(H) = (1 \ 0 \ 0 \ 0)$, $\rho_f(X^+) = (0 \ 1 \ 0 \ 0)$, $\rho_f(X^-) = (0 \ 0 \ 1 \ 0)$ giving

$$\rho_f(RU)|_{V_1} = \begin{pmatrix} q^{H/2} & 0 \\ q^{-1/2} \omega X^+ & q^{-H/2} \end{pmatrix} = S^-, \quad (3.4)$$

$$\rho_f(RU)|_{V_2} = \begin{pmatrix} q^{H/2} & q^{-1/2} \omega X^- \\ 0 & q^{-H/2} \end{pmatrix} = L^+.$$  

And further representing in a second step the ‘semiuniversal’ operators $S^-$ and $L^+$ we get

$$\rho_f(S^-) = q^{-1/2} \hat{R}, \quad \rho_f(L^+) = q^{-1/2} \hat{R}, \quad \rho_f(S^- L^+) = q^{-1} \hat{R}^2 \quad (3.5)$$

where $\hat{R} = P \hat{R} P = RP$. This means that we have to represent a strand ‘interacting’ with the handle like in Fig.2 by $K^{(1)}_{(1) j} = q^{-1} (\hat{R}^2)_{ij} m_n \equiv q^{-1} (\hat{R}^2 m_n)_{ij}$, where the $(ij)$ indices are in the first space of $V(n)$ and therefore $K^{(1)}_{(1) j} = q^{-1} (\hat{R}^2 m_n)_{ij}$ suits the graphical representation in Fig.2. In effect we have translated a topological property of the solid torus into a quantum algebra operator acting on $V_1$ and an additional ‘internal’ space $V_1'$, embedded into the space $V(1; n) = V_1 \otimes V_1 \otimes \cdots V_n$. As a consequence we have a two-dimensional representation of the RE algebra given by matrices

$$a^n_m = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad b^n_m = \begin{pmatrix} 0 & 0 \\ q^{-1} \omega & 0 \end{pmatrix}, \quad c^n_m = \begin{pmatrix} 0 & q^{-1} \omega \\ 0 & 0 \end{pmatrix}, \quad d^n_m = \begin{pmatrix} q^{-1} (1 + \omega^2) & 0 \\ 0 & q \end{pmatrix} \quad (3.6)$$

which indeed satisfy (2.5). The operator $K^{(1)}_{(1)} = S^- L^+|_{\rho}$ appeared also in the context of conformal field theory [24] and was used there, for example, in connection with topology changing amplitudes in Chern-Simons field theory.
It is easy to check from (2.3) that both \( \hat{R} \) and \( \hat{\tilde{R}} \) obey a quadratic equation of the form

\[
\hat{R}^2 - \omega \hat{R} - I = 0.
\] (3.7)

This, in turn, leads to a quadratic equation for \( \hat{R}^2 \)

\[
\hat{R}^2 + \hat{R}^{-2} - (q^2 + q^{-2})I = 0,
\] (3.8)

and this is nothing but the characteristic equation (2.8) for \( K_{(1)} = q^{-1} \hat{R}^2_{1,1}. \) Comparing with (2.8) we can identify \( c_1(K) = q^2 + q^{-2} \) (discarding the \( 2 \times 2 \) identity matrix), and this can be verified by calculating the quantum trace of the explicit representation (3.6). Hence, for this representation of \( K_{(1)} \) its characteristic equation follows from the one for the \( R \)-matrix. Therefore the somewhat ad hoc assumption of considering the characteristic equation of the \( K \)-matrix as a skein relation for lines going around handles in [10] is justified because (3.7) has an interpretation as a skein relation. Even more so as we have seen that handles themselves can be considered as kind of lines in topologically trivial regions, we will focus on this in the next section.

In order to explain the case of arbitrary genus it is sufficient to look at \( g = 2. \) Now we also need to take into account the last equation of (3.1). From Fig.3 we can guess that \( K_{(1)} \) is as before, but \( K_{(2)} \) should be represented by a product of four \( R \)-matrices acting on \( V(2; n) = V_2 \otimes V_1' \otimes V_1 \otimes \cdots \otimes V_n \) as

\[
K_{(1)} = q^{-1}I_2 \otimes \hat{R}^2_{1,1}, \quad K_{(2)} = q^{-1}\hat{R}^{-1}_{1,1} \hat{R}^2_{2,1} \hat{R}_{2,1}. \] (3.9)

The indices characterizing operators \( a, b, c, d \) belong to \( V_1, \) all primed indices refer to ‘internal’ spaces related to the handles of the manifold. Thus we can read off
from (3.9) the explicit four-dimensional representation using (2.3)

\[ a(1) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & q^{-1} \end{pmatrix}, \]
\[ b(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \end{pmatrix}, \]
\[ c(1) = \begin{pmatrix} 0 & q^{-1} \omega & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \omega \\ 0 & 0 & q^{-1} & 0 \end{pmatrix}, \]
\[ d(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-1}(1 + \omega^2) & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1}(1 + \omega^2) & q \end{pmatrix}, \]

\[ a(2) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & -\omega^2 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \]
\[ b(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-2} & 0 & 0 & 0 \\ q^{-2} & 0 & 0 & 0 \\ 0 & \omega & -\omega^2 & 0 \end{pmatrix}, \]
\[ c(2) = \begin{pmatrix} 0 & 0 & q^{-2} \omega \\ 0 & 0 & 0 & q^{-2} \omega \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ d(2) = \begin{pmatrix} q^{-1}(1 + \omega^2) & 0 & 0 & 0 \\ 0 & q^{-1}(1 + \omega^2) & q^{-2} \omega^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{3.10} \]

These matrices do not only satisfy (2.5) but really provide a non-trivial explicit representation of (2.11) (with \( K(1) = K, \ K(2) = K' \)), proving that (3.9) is indeed a representation of (3.1). Furthermore, using the representation \( \rho_f \) of \( H \) and \( X^\pm \) it can be verified that (3.10) may equally well be obtained from quantum algebra solution (2.21) of the extended RE algebra

\[ K(1) = \rho_f(K_2) \equiv \rho_f(K^+_1), \quad K(2) = \rho_f(S_2^+K_1L_2^+) \equiv \rho_f(K_2^+). \tag{3.11} \]

As shown in section 2 all quantum algebra solutions satisfy the same characteristic equation, so \( K(1) \) and \( K(2) \) do satisfy (3.8) which is obvious from (3.9) anyway. Then one might wonder what the meaning is of \( K_{(m)}^- \); it can be seen that it plays
the same role as $K_{(m)}^+$ but corresponds to a different incompatible set of conventions compared to those given in the beginning (i.e. $\alpha > \beta$ in (3.1), $z_2 > z_1$ for strands leaving or entering the unit cube, clockwise rotation of bars corresponding to handles, interchange of figures for $K$ and $K^{-1}$). This choice gives nothing new and needs not to be considered, for example, $K_{(2)} = \rho_f(K_{(2)}^-)$ is the same as in (3.10) but with all four matrices transposed and $b_{(2)}$ interchanged with $c_{(2)}$. The last relation of (3.1) in this case can be depicted as in Fig.4, the major difference being some lines now going under the bars.

![Fig.4: Compatibility condition (2.10) with $\tau_\alpha$ represented by the $K_{(m)}^-$ series](image)

We still have to explain the meaning of (2.9) in terms of the braid group generators. Property (i) relates to successive application of $K_{(1)}$ and $K_{(2)}$ leading to a new move encircling both bars as displayed in Fig.5.

![Fig.5: The product $\tau_1 \tau_2$ (equivalent to representing $\tau_2$ by $K_{(2)}^0$)](image)

It is immediately clear that the move $K_{(1)}K_{(2)}$ again satisfies the RE as can be seen just by inserting an additional bar appropriately into Fig.2. Property (ii) can be understood similarly by looking at $K_{(1)}K_{(2)}^-K_{(1)}^{-1}$ shown in Fig.6. If the results
of multiplying (products of) braid group generators obey the braid group relations again it then graphically comes down to being able to ‘pull lines’ in a simple way.

Fig.6: The product $\tau_1 \tau_2 \tau_1^{-1}$ (equivalent to representing $\tau_2$ by $K_{(2)}^-$)

The results of figs.5,6 show that although we have chosen the $K_{(m)}^+$ series to represent the braid group generators $\tau_\alpha$, we now see the $K_{(m)}^-$ and $K_{(m)}^0$ series appearing because the combined solutions are precisely given by $K_{(2)}^0$ and $K_{(2)}^-$ as can be checked by formulas. Even though we start from $K_{(m)}^+$ solely its properties as an extended RE algebra force us to consider the whole system (2.20), (2.22) and (2.23). As a side remark we mention here that if we may use graphs for both $K_{(m)}^+$ and $K_{(n)}^-$ it is easy to see why they have no commutation relation for $m = n$, it is just not possible to disentangle the lines.

Now it should be clear how this generalizes to the case of arbitrary genus. We will have $g$ bars corresponding to the handles and $\tau_\alpha$ is represented by a strand going from first space over the first $(\alpha - 1)$ bars to wind around the one corresponding to the handle $h_\alpha$ and going back again to $V_1$ over the first $(\alpha - 1)$ bars. An example is shown in Fig.7. It is obvious that the $\tau_\alpha$ satisfy the defining relations (3.1) and this can be proven analogously to figs.3,4.
It is clear that $\tau_\alpha$ acting on $V(g; n) = V_{g'} \otimes \cdots V_{1'} \otimes V_1 \otimes \cdots V_n$ is represented as in (3.2) with the non-trivial part of $K^{(\alpha)}$ given by (identify indices $0' \equiv 1$ where necessary)

$$K^{(\alpha)} = q^{-1}\hat{R}_{1'}^{-1}\hat{R}_{2'}^{-1}\cdots\hat{R}_{(\alpha-1)'}^{-1}\hat{R}_{(\alpha-2)'}^{2}\hat{R}_{(\alpha-1)'}^{\alpha'}\hat{R}_{(\alpha-2)'}^{(\alpha-1)'}\cdots\hat{R}_{2'}^{1}\hat{R}_{1'}$$

$$= \rho_f(K^{(\alpha)}_+), \quad (3.12)$$

and $K^{(\alpha)}_+$ is expressed in terms of quantum algebra generators as in (2.21). Note that numbering the primed spaces is by convention, and we have chosen a more natural one with opposite ordering compared to (2.21) where it is fixed (cf. remark at end of section 2). All that was said about the $g = 2$ case above can be generalized to arbitrary genus, there are no new features emerging.

We finish this section by noting that the generators $\tau_\alpha$ in the representation (3.12) can be considered as a subgroup of the coloured braid group $C_{g+1}$. By definition, the coloured braid group $C_{g+1}$ is the kernel of the mapping from the Artin braid group $B_{g+1}$ to the permutation group $P_{g+1}$ having $\frac{1}{2}g(g+1)$ elements $\kappa_{\alpha\beta}$ that can be taken in our notation as

$$\kappa_{\alpha\beta} = \sigma_{(\beta+1)'}^{-1}\cdots\sigma_{(\alpha-1)'}^{-1}\sigma_{(\alpha-1)'}^{2}\sigma_{(\alpha-1)'}^\alpha\cdots\sigma_{(\beta+1)'}^\beta, \quad 0 \leq \beta < \alpha \leq g \quad (3.13)$$

acting on $V(g; 1) = V_{g'} \otimes \cdots V_{1'} \otimes V_1$. This corresponds to $g$ bars plus one strand. Of course, in our case the bars cannot wind around each other (but see the interpretation of bars as lines in the next section) so we have to fix $\beta = 0$ and let
1 ≤ α ≤ g, s.t. \( \kappa_{\alpha_0} \) gives precisely the \( g \) generators in (3.12) as a subset of the generators of \( C_{g+1} \). Therefore we can also think of \( B^g_n \) as a subgroup of the braid group \( B_{g+n} \) having generators \( \sigma_1', \ldots, \sigma_1, \sigma_1, \ldots, \sigma_{n-1} \). From our experiences in section 2 it is obvious how to represent the full coloured braid group in terms of quantum algebra generators, namely the equivalent of (3.13) is given by

\[
K^+_{(j;m)} = S^+_{g-j} \cdots S^+_{g-j-(m-2)} K^+_{g-j-(m-1)} L^+_{g-j-(m-2)} \cdots L^+_{g-j}, \quad j = 0, \ldots, g-1
\]

\[
m = 1, \ldots, g-j
\]

and similarly for \( K^-_{(j;m)} \) which gives (3.13) but with all braid group generators except \( \sigma_1^2 \) replaced by their inverses. For each fixed value of \( j \) they have commutation relations (2.22). Of course, we can write down a similar formula for \( K^0_{(j;m)} \) but they do not represent the coloured braid group. It was already noted in [25] that the coloured braid group has a representation within tensor products of the universal enveloping algebra of \( sl_q(N) \).

4. INVARIANTS OF LINKS ON HANDLEBODIES

We will now define links on the handlebody \( H_g \) and then try to find invariant polynomials. A \( g \)-link \( L_g \) on \( H_g \) is obtained as the closure of a \( g \)-braid by connecting \( P_{i}^{(0)} \) with \( P_{i}^{(1)} \) outside the unit cube in the \( x > 0 \) region (Fig.1). Citing a theorem [2], every \( g \)-link can be obtained as the closure of a \( g \)-braid. Markov moves for \( B^g_n \) are defined in the same manner as the usual ones for \( B_n \), i.e. \( B \rightarrow B'B'(B')^{-1} \) for arbitrary \( B' \in B^g_n \) (Markov I) and \( B \rightarrow B\sigma_{n+1} \) with \( \sigma_{n} \in B^g_{n+1} \) (Markov II). Then the Markov theorem would state that two \( g \)-braids have equivalent closures iff there is a finite sequence of Markov moves of type I and II taking one \( g \)-braid to the other. However, the Markov theorem for \( g > 1 \) was only stated as a conjecture in [2], it holds for \( g = 1 \) (we shall not need it in what follows).

There are several approaches to the construction of link polynomials, one may roughly distinguish them in the following way (a convenient access to original literature is [26], basic accounts of knot theory and the relation to quantum groups...
are e.g.\([27, 28]\)). It is well known that the expression of \(\sigma_i\) in terms of \(\hat{R}\) gives rise to a Hecke algebra representation of the braid group \(B_n\) and the characteristic equation of the \(\hat{R}\)-matrix together with the first two equations of (3.1) comprise just the relations of the Hecke algebra \(H(q^2, n)\) with generators \(\sigma_i\) (we would have to rescale \(q \to q^{1/2}\) to make contact with the usual convention). One defines a linear functional on \(H(q^2, n)\), the Ocneanu trace, which is the main ingredient in the definition of the invariant link polynomial [29]. This is just the quantum trace of the braid group generators represented by \(\hat{R}\) and the last step is then proving invariance of it w.r.t Markov moves [30, 31, 32]. Further it is possible to define link polynomials recursively using skein relations [33, 34, 35]. Finally, there is the Chern-Simons field theory approach [36, 37]. In view of this we might expect that the explicit representation (3.2) can be used to define an invariant link polynomial on \(H_q\) by means of quantum traces of generators \(\sigma_i\) and \(\tau_\alpha\). Also their characteristic equations should give rise to skein relations.

We recall the definition of the Jones polynomial [29]

\[
V(B) = q^{-3w(\hat{B})}Tr|_{V(n)}(B\mu^{\otimes n}),
\]

which is a class function on the braid group \(B_n\) that is invariant w.r.t Markov moves of type I and II. Because of inclusion of type II moves it is an ambient isotopic invariant, i.e. locks in a line are irrelevant. In (4.1) we denoted the closure of the braid \(B \in B_n\) by \(\hat{B}\), and \(w(\hat{B})\), the writhe or Tait number, is the number of overcrossings minus the number of undercrossings in \(\hat{B}\). Finally, the matrix \(\mu\) is defined via an element of the quantum algebra

\[
\mu = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \equiv \rho_f(q^{-H}),
\]

so the trace in (4.1) could equally well be called the quantum trace of \(B\). For instance, the quantum trace of the RE algebra can be expressed as \(c_1 = Tr(K\mu)\). Invariance of (4.1) under Markov I rests on the property \([\sigma_i, \mu^{\otimes n}] = 0\), and for
Markov II relies on the key property of the quantum trace $Tr_{V_{n+1}}(\sigma_n^{\pm 1}(I^\otimes n \otimes \mu)) = q^{\pm 3}I^\otimes n$. The Jones polynomial satisfies a skein relation which is obtained from the characteristic equation of $\sigma_i$

$$q^{-1} \sigma_i - q\sigma_i^{-1} - \omega I = 0 \quad (4.3)$$

by using linearity of the trace, the result is for arbitrary $B, B' \in B_n$

$$q^2 V(B\sigma_i B') - q^{-2} V(B\sigma_i^{-1} B') - (q - q^{-1}) V(B B') = 0. \quad (4.4)$$

This can be depicted as

$$q^2 \begin{array}{c}
\includegraphics[width=0.2\textwidth]{skein_relation}
\end{array} - q^{-2} \begin{array}{c}
\includegraphics[width=0.2\textwidth]{skein_relation}
\end{array} - (q-q^{-1}) \begin{array}{c}
\includegraphics[width=0.2\textwidth]{skein_relation}
\end{array} = 0,$$

Fig.8: Skein relation of the Jones polynomial

where each pictogram means the polynomial of the closed braid differing only at the crossing between $B$ and $B'$ as indicated. By simply closing the lines the normalization of the unknot is obtained as $N = q + q^{-1}$, this is the same as in [36] up to rescaling of $q$ and corresponds to standard framing (see the discussion in [38] on the effects of framing).

There is a simple possibility to make use of this formalism in our context as we have a representation of $\tau_\alpha$ in terms of $\hat{R}$-matrices. We just extend the definition of the Jones polynomial from $V(n)$ to $V(g; n)$ and get an ambient isotopy invariant polynomial for $g$-links on $H_g$ in terms of the Jones polynomial of links in $\mathbb{R}^3$ (or a 3-ball, for that matter). Proof of invariance w.r.t. Markov I,II works as before. We are forced to interpret bars corresponding to the handles as lines and, because of the trace in $V(B)$, close them to obtain the link in $\mathbb{R}^3$ which is associated to the $g$-link on $H_g$. Because we have an description of the crossings by $\hat{R}$-matrices
the direction of the bars is fixed downwards. This procedure is obviously well defined as the equivalence class of a \( g \)-link is mapped uniquely onto the class of the associated ordinary link [2]. The polynomial for \( g \)-links is then given by

\[
V_g(B) = q^{-3w(B)} Tr \left|_{V(g;n)} (B \mu^{\otimes (g+n)}) \right.,
\]

(4.5)

where \( B \in B_n^g \) is a word in the generators \( \sigma_i \) and \( \tau_\alpha \), its closure is obtained via the representation (3.2) and (3.12) of \( \tau_\alpha \). It is easy to find examples of \( g \)-links which belong to different classes but share the same value of the polynomial.

It might appear as if one were back to the usual situation where one deals with the generators \( \sigma_i \) only. However, keeping generators \( \tau_\alpha \) is a great advantage, the reason is that they are very special expressions in \( \sigma_i \). They all obey the characteristic equation

\[
q^{-2} \tau_\alpha + q^2 \tau_\alpha^{-1} - c_1 I = 0,
\]

(4.6)

which, as before, leads to a skein relation that follows from (4.5)

\[
q^4 V_g(B \tau_\alpha B') + q^{-4} V_g(B \tau_\alpha^{-1} B') - (q^2 + q^{-2}) V_g(B B') = 0
\]

(4.7)

in addition to the \( \sigma_i \) skein relation for \( V_g(B) \). Here we inserted the value \( c_1 = q^2 + q^{-2} \) for the fundamental representation of \( sl_q(2) \). Above relation can be depicted as

\[
q^4 \begin{array}{c} \downarrow \end{array} + q^4 \begin{array}{c} \downarrow \end{array} - (q^2 + q^{-2}) \begin{array}{c} \downarrow \end{array} = 0,
\]

Fig.9: The additional skein relation for generators \( \tau_\alpha \)

where we displayed only the \( g = 1 \) case. It is known [33, 34] that by recursively using skein relations the invariant polynomial can be calculated uniquely. So it is
reasonable to suggest that both skein relations (4.4) and (4.7) suffice to calculate a well defined polynomial for any $g$-link [10], knowing the origin of the additional skein relation it is obvious that this statement is correct.

If we use skein relations to calculate the polynomial we can fix the procedure as follows. First untie the knot in the topological trivial region using (4.4), this is clearly possible and it eventually gives unknots going around the bars. If an unknot winds around a bar $n$ times, $(\tau_\alpha)^n$, then it always can be reduced to $n = 1$ with the help of (4.7), regardless whether $n$ is positive or negative. Similarly if it winds around several bars in the right order by using the analogue (2.13) of (4.7) for a product of several generators, otherwise the correct order must be established first by using (4.7). This way the reduction process can be much simplified, but nevertheless in the end (4.4) has to be used again to untie the simple loops around the $S^1$-factors (closed bars). If there is only one loop simple rules can be established as indicated in Fig.10a.

\[
\begin{array}{c}
\includegraphics{fig10a.png} \\
(a) \\
= N \left(q^{-3} c_1 \right)^n
\end{array}
\quad
\begin{array}{c}
\includegraphics{fig10b.png} \\
= q^6
\end{array}
\]

Fig.10: (a) Evaluation of a simple loop encircling $n$ (closed) bars ($N = q + q^{-1}$ is the normalization of the unknot), (b) Relation between $\tau_\alpha^{-1}$ and $\tau_\alpha$

The virtue of Fig.10a is that it connects loops going around handles to loops in topologically trivial regions, we can just reinsert instead of the factor $N$ an unknot in $\mathbb{R}^3$ (if there are bars not being encircled by the loop they contribute factors of $N$ on the RHS). In a way, it looks as if this were related to the surgery method described in [36], see also [39, 40]. One can also express a loop originating from $\tau_\alpha^{-1}$ directly in terms of one originating from $\tau_\alpha$ (only the $g = 1$ case is shown in Fig.10b), if the loop encircles $n$ closed bars corresponding to a ordered product of
$n$ generators $\tau_{\alpha}$ the factor will be $q^{6n}$. This can be taken a little further but as we do not have concrete calculations in mind we do not elaborate on it.

Of course, even though it is possible to define the above invariant polynomial, we would have preferred to define it intrinsically on the handlebody keeping the information about the topology strictly, i.e. without transforming holes into lines carrying some representations. But this is not so easy, because if we use (4.1) defined on $V(n)$ only, but with $g$-braids $B$ containing generators $\tau_{\alpha}$, then the trace is no longer invariant with respect to Markov I. This follows from the fact that $[K, \mu] \neq 0$, the only difference occurs in first space and Markov II is still valid. We are presently looking for a modification of the polynomial that would employ only the algebraic properties of the RE algebra and make no use of the representation discussed above, whether this is possible and whether it would lead to an inequivalent invariant is an open question.

5. DISCUSSION

There are a few topics that can be mentioned in connection with the present work. The motivation in [2] was to define invariant polynomials of links intrinsically on any closed 3-manifold $M$. The prerequisite for this is a polynomial defined on handlebodies, it would then be neccessary to investigate how it transforms w.r.t. the Heegard homeomorphism $\psi : \partial H_g \to \partial H_g$ since any closed compact 3-manifold $M$ can be obtained by the Heegard decomposition $M = H_g \cup_{\psi} H'_g$, $H_g \cap H'_g = \partial H_g = \partial H'_g$. Every link in $M$ is isotopic to a closed braid in $H_g$, but the braid depends on the Heegard splitting. One would then need to use (a subgroup of) the mapping class group of a genus $g$ Riemann surface in order to study the behaviour of the polynomial w.r.t. the Heegard homeomorphism, maybe the approach in [41] could be useful where the homeomorphisms of a handlebody were expressed essentially in terms of $R$-matrices. Related to this subject is the surgery method because the Heegard splitting could be used also to transform invariants that are defined on a certain closed 3-manifold to a different one. Whether such an approach
would be more tractable compared to [36] is not clear a priori. The presence of the generators $\tau_{\alpha}$ suggests the idea of an ‘algebraization’ of the surgery method.

A further problem is whether there exist representations of $B_n^q$ other than in terms of $R$-matrices. The RE algebra as written in (2.5), especially existence of central elements and characteristic equations, are a consequence of the Hecke algebra representation of the generators $\sigma_i$. It is not clear whether it is possible to represent $\tau_{\alpha}$ differently from $\sigma_i$. The problem to find representations of $B_n^q$ is of course related to the difficulty in defining invariant link polynomials, because the Markov trace is dependent on classes of (irreducible) representations. In this context it is worthwhile noting that at least $B_n^1$ is a Coxeter group [2].

Also, we would like to draw attention again to the (infinite dimensional) extended RE algebra and the new representations of it that we constructed in terms of quantum algebra generators which satisfy the system of commutation relations (2.22). This might have some applications other than discussed here, e.g. in the description of differential geometry on quantum groups or quantum spin chains. During typesetting this manuscript we came across two preprints which bear some similarity to our work. In [42] the monodromies of flat connections around the cycles of a Riemann surface with marked points are considered, they obey some variant of the extended RE algebra. In [43] a quantum group invariant $n$-state vertex model on a torus is constructed which has a topological interaction of the vertices with the interior of the torus, the graphical notation is also reminiscent of ours.
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