A HOMOTOPY PRINCIPLE FOR MAPS WITH PRESCRIBED
THOM-BOARDMAN SINGULARITIES

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ABSTRACT. Let $N$ and $P$ be smooth manifolds of dimensions $n$ and $p$ ($n \geq p \geq 2$) respectively. Let $\Omega^I(N, P)$ denote an open subspace of $J^\infty(N, P)$ which consists of all Boardman submanifolds $\Sigma^J(N, P)$ of symbols $J$ with $J \leq I$. An $\Omega^I$-regular map $f : N \to P$ refers to a smooth map having only singularities in $\Omega^I(N, P)$ and satisfying transversality condition. We will prove what is called the homotopy principle for $\Omega^I$-regular maps in the existence level. Namely, a continuous section $s$ of $\Omega^I(N, P)$ over $N$ has an $\Omega^I$-regular map $f$ such that $s$ and $j^\infty f$ are homotopic as sections.

INTRODUCTION

Let $N$ and $P$ be smooth ($C^\infty$) manifolds of dimensions $n$ and $p$ respectively with $n \geq p \geq 2$. In [B] there have been defined what are called the Boardman manifolds $\Sigma^I(N, P)$ in $J^\infty(N, P)$ for the symbol $I = (i_1, i_2, \cdots, i_r)$, where $i_1 \geq i_2 \geq \cdots \geq i_r \geq 0$. We say that a smooth map germ $f : (N, x) \to (P, y)$ has $x$ as a Thom-Boardman singularity of the symbol $I$ if and only if $j_x^\infty f \in \Sigma^I(N, P)$. Let $\Omega^I(N, P)$ denote an open subset of $J^\infty(N, P)$ which consists of all Boardman manifolds $\Sigma^J(N, P)$ with symbols $J$ of length $r$ satisfying $J \leq I$ in the lexicographic order. It is known that $\Omega^I(N, P)$ is an open subbundle of $J^\infty(N, P)$ with the projection $\pi^\infty_N \times \pi^\infty_P$, whose fiber is denoted by $\Omega^I(n, p)$. A smooth map $f : N \to P$ is called an $\Omega^I$-regular map if and only if (i) $j^\infty f(N) \subset \Omega^I(N, P)$ and (ii) $j^\infty f$ is transverse to all $\Sigma^J(N, P)$.

We will study a homotopy theoretic condition for a given continuous map to be homotopic to an $\Omega^I$-regular map. Let $C^\infty_{\Omega^I}(N, P)$ denote the space consisting of all $\Omega^I$-regular maps equipped with the $C^\infty$-topology. Let $\Gamma_{\Omega^I}(N, P)$ denote the space consisting of all continuous sections of the fiber bundle $\pi^\infty_N | \Omega^I(N, P) : \Omega^I(N, P) \to N$ equipped with the compact-open topology. Then there exists a continuous map

$j_{\Omega^I} : C^\infty_{\Omega^I}(N, P) \to \Gamma_{\Omega^I}(N, P)$

defined by $j_{\Omega^I}(f) = j^\infty f$. It follows from the well-known theorem due to Gromov[G1] that if $N$ is a connected open manifold, then $j_{\Omega^I}$ is a weak homotopy equivalence. This property is called the homotopy principle (the terminology used in [G2]). If $N$ is a closed manifold, then it becomes a hard problem for us to prove the homotopy principle. As the primary investigation preceding [G1], we must refer to the Smale-Hirsch Immersion Theorem ([H]), $k$-mersion Theorem due to [F] and the Phillips Submersion Theorem for open manifolds ([P]). In [E1] and [E2], Eliashberg has proved the well-known homotopy principle in the 1-jet level.

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for $\Omega^{n-p+1,0}$-regular maps, say fold-maps. As for the Thom-Boardman singularities, du Plessis\cite{duP} has proved that if $i_r > n - p - d^I$, where $d^I$ is the sum of $\alpha_1, \cdots, \alpha_{r-1}$ with $\alpha_t$ being 1 or 0 depending on $i_t - i_{t+1} > 1$ or otherwise, then $j^I_{Ω}$ is a weak homotopy equivalence.

In this paper we prove the following homotopy principle in the existence level for closed manifolds.

**Theorem 0.1.** Let $n \geq p \geq 2$. Let $N$ and $P$ be connected manifolds of dimensions $n$ and $p$ respectively with $\partial N = \emptyset$. Assume that $Ω^I(N, P)$ contains $\Sigma^{n-p+1,0}(N, P)$ at least. Let $C$ be a closed subset of $N$. Let $s$ be a section of $Γ_{Ω^I}(N, P)$ which has an $Ω^I$-regular map $g$ defined on a neighborhood of $C$ into $P$, where $j^∞g = s$.

Then there exists an $Ω^I$-regular map $f : N \rightarrow P$ such that $j^∞f$ is homotopic to $s$ relative to a neighborhood of $C$ by a homotopy $s_λ$ in $Γ_{Ω^I}(N, P)$ with $s_0 = s$ and $s_1 = j^∞f$.

In \cite{A1} we have given Theorem 0.1 for the symbol $I = (n - p + 1, 1, \cdots, 1, 0)$ with a partially sketchy proof using the results in \cite{E1} and \cite{E2}. The singularities of this symbol $I$ are often called $A_r$-singularities or Morin singularities. The detailed proof are given in \cite{An4, Theorem 4.1} and \cite{An6, Theorem 0.5} for the symbol $I = (n - p + 1, 0)$. We will use these two theorems in the proof of Theorem 0.1 in this paper.

Recently it turns out that this kind of the homotopy principle has many applications. Theorem 0.1 is very important even for fold-maps in proving the relations between fold-maps, surgery theory and stable homotopy groups (\cite{An4, Theorem 1} and \cite{An5, Theorems 0.2 and 0.3}). The homotopy type of $Ω^{n-p+1,0}$ determined in \cite{An3} and \cite{An5} has played an important role. We can now readily deduce the famous theorem about the elimination of cusps in \cite{L1} and \cite{E1} (see also \cite{T}) from these theorems.

The homotopy principle in the existence level for maps and singular foliations having only what are called $A$, $D$ and $E$ singularities are proved in \cite{An2} and \cite{An7}.

In \cite{Sady} Sadykov has applied \cite[Theorem 1]{An1} to the elimination of higher $A_r$ singularities ($r \geq 3$) for Morin maps when $n - p$ is odd. This result is a strengthened version of the Chess conjecture proposed in \cite{C}.

As an application of Theorem 0.1 we prove the following theorem. We note that the simplest case is also a little stronger form of the Chess conjecture.

**Theorem 0.2.** Let $n \geq p \geq 2$, and $N$ and $P$ be connected manifolds of dimensions $n$ and $p$ respectively. Let $I = (n - p + 1, i_2, \cdots, 1, 1, 1)$ and $J = (n - p + 1, 1, i_2, \cdots, 1, r - 1, 1, 0)$ such that $n - p + 1 - i_2$ and $r$ ($r \geq 3$) are odd integers. Then if $f : N \rightarrow P$ is an $Ω^I$-regular map, then $f$ is homotopic to an $Ω^J$-regular map $g : N \rightarrow P$ such that $j^∞f$ and $j^∞g$ are homotopic in $Γ_{Ω^I}(N, P)$.

In Section 1 we explain notations which are used in this paper. In Section 2 we review the definitions and the fundamental properties of the Boardman manifolds, from which we deduce several further results about higher intrinsic derivatives in Section 3. In Section 4 we reduce the proof of Theorem 0.1 to the proof of Theorem 4.1 by the induction, and prepare a certain rotation of the tangent spaces defined around the singularities of given symbol in $N$ to deform the section $s$. In Section 5 we prepare several lemmas which are used in the deformation of the section $s$ in
We define the smooth bundle map with the exponential maps we always consider the convex neighborhoods \((K-N)\). Let \(\pi^t : F \to Y\) be another fiber bundle. A map \(b : E \to F\) is called a fiber map over a map \(\pi^t : X \to Y\) if \(\pi^t \circ b = b \circ \pi\) holds. The restriction \(\tilde{b}|(E|_C) : E|_C \to F\) (or \(F|_{b(C)}\)) is denoted by \(\tilde{b}|_C\). In particular, for a point \(x \in X\), \(E|_x\) and \(\tilde{b}|_x\) are simply denoted by \(E_x\) and \(\tilde{b}_x : E_x \to F_{b(x)}\) respectively. We denote, by \(b^*\), the induced fiber map \(b^*(F) \to F\) covering \(b\). For a map \(j : W \to X\), let \(j^*(b) : j^*E \to (b \circ j)^*F\) over \(W\) be the fiber map canonically induced from \(b\) and \(j\). A fiberwise homomorphism \(E \to F\) is simply called a homomorphism. For a vector bundle \(E\) with a metric and a positive function \(\delta\) on \(X\), let \(D_\delta(E)\) be the associated disk bundle of \(E\) with radius \(\delta\). If there is a canonical isomorphism between two vector bundles \(E\) and \(F\) over \(X = Y\), then we write \(E \cong F\).

When \(E\) and \(F\) are smooth vector bundles over \(X = Y\), \(\text{Hom}(E, F)\) denotes the smooth vector bundle over \(X\) with fiber \(\text{Hom}(E_x, F_x)\), \(x \in X\) which consists of all homomorphisms \(E_x \to F_x\).

Let \(J^k(N, P)\) denote the \(k\)-jet space of manifolds \(N\) and \(P\). Let \(\pi^k_N\) and \(\pi^k_P\) be the projections mapping a jet to its source and target respectively. The map \(\pi^k_N \times \pi^k_P : J^k(N, P) \to N \times P\) induces a structure of a fiber bundle with structure group \(L^k(m)\), where \(L^k(m)\) denotes the group of all \(k\)-jets of local diffeomorphisms of \((\mathbb{R}^m, 0)\). The fiber \((\pi^k_N \times \pi^k_P)^{-1}(x, y)\) is denoted by \(J^k_{x, y}(N, P)\).

Let \(\pi_N\) and \(\pi_P\) be the projections of \(N \times P\) onto \(N\) and \(P\) respectively. We set

\[
J^k(TN, TP) = \bigoplus_{i=1}^{k} \text{Hom}(S^i(\pi^*_N(TN)), \pi^*_P(TP))
\]

over \(N \times P\). Here, for a vector bundle \(E\) over \(X\), let \(S^i(E)\) be the vector bundle \(\bigcup_{x \in X} S^i(E_x)\) over \(X\), where \(S^i(E_x)\) denotes the \(i\)-fold symmetric product of \(E_x\). If we provide \(N\) and \(P\) with Riemannian metrics, then the Levi-Civita connections induce the exponential maps \(\exp_{N,x} : T_xN \to N\) and \(\exp_{P,y} : T_yP \to P\). In dealing with the exponential maps we always consider the convex neighborhoods \(((K-N))\). We define the smooth bundle map

\[
J^k(N, P) \to J^k(TN, TP) \quad \text{over } N \times P
\]

by sending \(z = j^k_x f \in J^k_{x,y}(N, P)\) to the \(k\)-jet of \((\exp_{P,y})^{-1} \circ f \circ \exp_{N,x}\) at \(0 \in T_xN\), which is regarded as an element of \(J^k(T_xN, T_yP)(= J^k_{x,y}(TN, TP))\) (see [K-N, Proposition 8.1] for the smoothness of exponential maps). More strictly, (1.2) gives a smooth equivalence of the fiber bundles under the structure group \(L^k(p) \times L^k(n)\). Namely, it gives a smooth reduction of the structure group \(L^k(p) \times L^k(n)\) of \(J^k(N, P)\) to \(O(p) \times O(n)\), which is the structure group of \(J^k(TN, TP)\).

Recall that \(S^i(E)\) has the inclusion \(S^i(E) \to \otimes^i E\) and the canonical projection \(\otimes^i E \to S^i(E)\) ([B, Section 4] and [Mats, Ch. III, Section 2]). Let \(E_j\) be subbundles of \(E\) \((j = 1, \cdots, i)\). We define \(E_1 \circ \cdots \circ E_i = \bigcirc_{j=1}^{i} E_j\) to be the
image of $E_1 \otimes \cdots \otimes E_i = \otimes_{j=1}^i E_j \to \otimes^i E \to S^i(E)$. When $E_{j+1} = \cdots = E_{j+\ell}$, we often write $E_1 \oplus \cdots \oplus E_j \otimes \ell E_{j+1} \ominus \cdots \ominus E$ in place of $\otimes_{j=1}^i E_j$.

2. Boardman manifolds

We review well-known results about Boardman manifolds in $J^{\infty}(N,P)$ ([B], [L2] and [Math2]). Let $I = (i_1, \ldots, i_r)$ be a Boardman symbol with $i_1 \geq \cdots \geq i_r \geq 0$. For $k \leq r$, set $I_k = (i_1, i_2, \ldots, i_k)$ and $(I_k,0) = (i_1, i_2, \ldots, i_k,0)$. In the infinite jet space $J^{\infty}(N,P)$, there have been defined a sequence of the submanifolds $\Sigma^{I_i}(N,P) \supseteq \cdots \supseteq \Sigma^{I_r}(N,P)$ with the following properties. In this paper we often write $\Sigma^{I_r}$ for $\Sigma^{I_r}(N,P)$ if there is no confusion.

Let $P = (\pi_N^*)^*(TP)$ and $D$ be the total tangent bundle defined over $J^{\infty}(N,P)$.

We explain important properties of the total tangent bundle $D$, which are often used in this paper. Let $f : (N,x) \to (P,y)$ be a germ and $f$ be a smooth function in the sense of [B, Definition 1.4] defined on a neighborhood of $j^\infty f$. Given a vector field $v$ defined on a neighborhood of $x$ in $N$, there is a total vector field $D$ defined on a neighborhood of $j^\infty f$ such that $DF \circ j^\infty f = v(f \circ j^\infty f)$. It follows that $d(j^\infty f)(v)(f) = DF(j^\infty f)$ for $d(j^\infty f) : TN \to T(J^{\infty}(N,P))$ around $x$. This implies $d(j^\infty f)(v) = D$. Hence, we have $D \cong (\pi_N^*)^*(TN)$.

First we have the first derivative $d_1 : D \to P$ over $J^{\infty}(N,P)$. We define $\Sigma^{I_1}(N,P)$ to be the submanifold of $J^{\infty}(N,P)$ which consists of all jets $z$ such that the kernel rank of $d_1|_z$ is $i_1$. Since $d_1|_{\Sigma^{I_1}(N,P)}$ is of constant rank $n - i_1$, we set $K_0 = \text{Ker}(d_1)$ and $Q_1 = \text{Cok}(d_1)$, which are vector bundles over $\Sigma^{I_1}(N,P)$. Set $K_0 = D$, $P_0 = P$ and $\Sigma^{I_0}(N,P) = J^{\infty}(N,P)$. We can inductively define $\Sigma^{I_k}(N,P)$ and the bundles $K_k$ and $P_k$ over $\Sigma^{I_k}(N,P)$ ($k \geq 1$) with the properties:

1. $K_{k-1}|_{\Sigma^{I_{k-1}}(N,P)} \supseteq K_k$ over $\Sigma^{I_k}(N,P)$.
2. $K_k$ is an $i_k$-dimensional subbundle of $T(\Sigma^{I_{k-1}}(N,P))|_{\Sigma^{I_k}(N,P)}$.
3. There exists the $(k+1)$-th intrinsic derivative $d_{k+1} : T(\Sigma^{I_{k-1}}(N,P))|_{\Sigma^{I_k}(N,P)} \to P_k$ over $\Sigma^{I_k}(N,P)$, so that it induces the exact sequence over $\Sigma^{I_k}(N,P)$:

$$0 \to T(\Sigma^{I_k}(N,P)) \underset{\text{inclusion}}{\hookrightarrow} T(\Sigma^{I_{k-1}}(N,P))|_{\Sigma^{I_k}(N,P)} \xrightarrow{d_{k+1}} P_k \to 0.$$ Namely, $d_{k+1}$ induces the isomorphism of the normal bundle

$$0 \to T(\Sigma^{I_k}(N,P))|_{\Sigma^{I_{k-1}}(N,P)} \to T(\Sigma^{I_k}(N,P))$$

of $\Sigma^{I_k}(N,P)$ in $\Sigma^{I_{k-1}}(N,P)$ onto $P_k$.

4. $\Sigma^{I_{k+1}}(N,P)$ is defined to be the submanifold of $\Sigma^{I_k}(N,P)$ which consists of all jets $z$ with $\dim(\text{Ker}(d_{k+1}|_{\Sigma^{I_{k+1}}(N,P)})) = i_{k+1}$. In particular, $\Sigma^{I_k}(N,P)$ is the disjoint union $\bigcup_{j=0}^{i_k} \Sigma(I_{k,j})(N,P)$.

5. Set $K_{k+1} = \text{Ker}(d_{k+1}|_{\Sigma^{I_{k+1}}(N,P)})$ and $Q_{k+1} = \text{Cok}(d_{k+1}|_{\Sigma^{I_{k+1}}(N,P)})$ over $\Sigma^{I_{k+1}}(N,P)$. Then it follows that $(K_k|_{\Sigma^{I_{k+1}}(N,P)} \cap T(\Sigma^{I_k}(N,P))|_{\Sigma^{I_{k+1}}(N,P)}) = K_{k+1}$. We have the canonical projection $e_k : P_k|_{\Sigma^{I_{k+1}}(N,P)} \to Q_{k+1}$.

6. The intrinsic derivative

$$d(d_{k+1}|_{K_k}) : T(\Sigma^{I_k}(N,P))|_{\Sigma^{I_{k+1}}(N,P)} \to \text{Hom}(K_{k+1}, Q_{k+1}) \text{ over } \Sigma^{I_{k+1}}(N,P)$$

de of $d_{k+1}|_{K_k}$ is of constant rank $\dim(\Sigma^{I_k}(N,P)) - \dim(\Sigma^{I_{k+1}}(N,P))$. We set $P_{k+1} = \text{Im}(d(d_{k+1}|_{K_k}))$ and define $d_{k+2}$ to be

$$d_{k+2} = d(d_{k+1}|_{K_k}) : T(\Sigma^{I_k}(N,P))|_{\Sigma^{I_{k+1}}(N,P)} \to P_{k+1}$$
as the epimorphism.

(7) There exists the bundle homomorphism of constant rank
\[ u_1 : \text{Hom}(K_k \circ K_{k-1} \circ \cdots \circ K_1, P) \to \text{Hom}(K_k, Q_k) \]
over \( \Sigma^j(N, P) \) such that the image of \( u_k \) coincides with \( P_k \). We denote, by \( c_k \), the map \( u_k \) as the epimorphism onto \( P_k \). Furthermore, \( u_k \) is defined as the composition
\[ u_k = \left( \text{Hom}(K_k \circ K_{k-1} \circ \cdots \circ K_1, P) \right) \circ \text{inclusion} \circ \text{id}(K_k, e_k) \to \text{Hom}(K_k, P_k) \]
\((2.5)\) \( \text{Hom}(K_k \circ K_{k-1} \circ \cdots \circ K_1, P) \)

Remark 2.1. (1) It is known that \( \mathcal{O}^j(N, P) \) is an open subset of \( J^j(N, P) \): Let \( I = (i_1, i_2, \cdots, i_r) \). We prove that the closure of \( \Sigma^j(N, P) \) is contained in the subset which consists of all submanifolds \( \Sigma^j(N, P) \) of the symbol \( J \) of length \( r \) with \( J \geq I \). Let \( z \in J^j(N, P) \) lies in the closure of \( \Sigma^j(N, P) \). By definition, we first have \( \text{dim}(\text{Ker}(d_{i_1})) \geq i_1 \). If the symbol of \( z \) is \( J \) with \( J \neq I \), then we can inductively prove that \( z \) has a number \( k \) such that \( \text{dim}(\text{Ker}(d_{i_j} | K_j)) = i_j \) for \( 1 \leq j \leq k < r \) and \( \text{dim}(\text{Ker}(d_{i_j+1} | K_j)) \geq i_{j+1} \). This implies the assertion.

(2) If a symbol \( J \) is an infinite series \( (j_1, j_2, \cdots, j_k, \cdots) \) and \( \text{codim} \Sigma^j(N, P) \leq n \), then \( j_1, j_2, \cdots, j_k, \cdots \) are equal to 0 except for a finite number of \( j_k \)’s.

3. Polynomials

Let \( V \) and \( W \) be vector spaces with inner product of dimensions \( v \) and \( w \) respectively. Let \( e_1, e_2, \cdots, e_v \) and \( d_1, d_2, \cdots, d_w \) be orthogonal basis of \( V \) and \( W \) respectively. We introduce the inner product in \( \text{Hom}(\otimes^j V, W) \) as follows. Let \( h_i \in \text{Hom}(\otimes^j V, W) \) \((i = 1, 2)\) and let
\[ h_1(e_{i_1} \otimes \cdots \otimes e_{i_t}) = \sum_{j=1}^{w} a_{i_1 \cdots i_t}^j d_j \quad \text{and} \quad h_2(e_{i_1} \otimes \cdots \otimes e_{i_t}) = \sum_{j=1}^{w} b_{i_1 \cdots i_t}^j d_j. \]
Then we define the inner product by
\[ \langle h_1, h_2 \rangle = \sum_{j=1}^{w} \sum_{i_1 \cdots i_t} a_{i_1 \cdots i_t}^j b_{i_1 \cdots i_t}^j. \]
Let \( S \) and \( T \) be isomorphisms of \( V \) and \( W \) which preserve the inner products respectively. We define the action of \( (T, S) \) on \( \text{Hom}(\otimes^j V, W) \) by \( (T, S)h = T \circ h \circ (\otimes^j S^{-1}) \). We show by induction on \( \ell \) that this inner product is invariant with respect to this action. We represent \( S^{-1} \) by the matrix \( (s_{ij}) \) under the basis \( e_1, e_2, \cdots, e_w \).

The assertion for \( \ell = 1 \) is well known. Assume that the assertion holds for \( \ell - 1 \). Under the canonical isomorphism \( \text{Hom}(\otimes^j V, W) \cong \text{Hom}(V, \text{Hom}(\otimes^{j-1} V, W)) \) we let \( h \in \text{Hom}(\otimes^j V, W) \) correspond to \( \overline{h} \), which satisfies \( \overline{h}(e_{i_1}) (e_{i_2} \otimes \cdots \otimes e_{i_t}) = \overline{h}(e_{i_1}) (e_{i_2} \otimes \cdots \otimes e_{i_t}) \).
we have that $h(e_1 \otimes e_2 \otimes \ldots \otimes e_i)$. Then we have that $\langle h_1, h_2 \rangle = \Sigma_{j=1}^{v} (\overline{h_1(e_j)}, \overline{h_2(e_j)})$. Hence, we have that

$$\langle (T, S)h_1, (T, S)h_2 \rangle = \sum_{i=1}^{v} (\langle (T, S)\overline{h_1}, (T, S)\overline{h_2} \rangle) \langle (S^{-1}(e_i)), (S^{-1}(e_i)) \rangle$$

$$= \sum_{i=1}^{v} (\overline{h_1(S^{-1}(e_i))}, \overline{h_2(S^{-1}(e_i))})$$

$$= \sum_{i=1}^{v} (\overline{h_1(\Sigma_{j=1}^{v} s_{ij} e_j)}, \overline{h_2(\Sigma_{k=1}^{v} s_{ik} e_k)})$$

$$= \sum_{i=1}^{v} (\Sigma_{j=1}^{v} s_{ij} (\overline{h_1(e_j)}, \overline{h_2(e_k)}))$$

$$= \sum_{j=1}^{v} \sum_{k=1}^{v} \delta_{jk} (\overline{h_1(e_j)}, \overline{h_2(e_k)})$$

$$= \sum_{j=1}^{v} (\overline{h_1(e_j)}, \overline{h_2(e_j)})$$

$$= \langle h_1, h_2 \rangle.$$

We recall that $\text{Hom}(\Sigma_{j=1}^{\ell} \bigcirc \bigcirc V, W)$ is identified with the set of polynomials of degree $\leq \ell$ having the constant 0 (see [Mats, Ch. III, Section 2]). Let $V$ and $W$ be smooth vector bundles with metric over a manifold $S$ with fibers $V$ and $W$ respectively. Then $\text{Hom}(\bigcirc \bigcirc V, W)$ is also a vector bundle with metric. For a point $c \in S$, take an open neighborhood $U$ around $c$ such that $V|_U$ and $W|_U$ are the trivial bundles, say $U \times V$ and $U \times W$ respectively. Then an element of $\text{Hom}(\bigcirc \bigcirc V, W)|_U$ is identified with a polynomial $\Sigma_{j=1}^{v} (\Sigma_{i=1}^{v} A_{ij}(c)e_i x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n})d_j$, $c \in U$, where $\omega = (\omega_1, \omega_2, \ldots, \omega_v)$, $\omega_i \geq 0$ ($i = 1, \ldots, v$), and $|\omega| = \omega_1 + \cdots + \omega_v$ and $A_{ij}(c)$ is a real number. If $A_{ij}(c)$ are smooth functions of $c$, then $\{A_{ij}(c)\}$ defines a smooth section of $\text{Hom}(\bigcirc \bigcirc V, W)|_U$ over $U$.

We now provide $N$ and $P$ with Riemannian metrics respectively. Then they induce the metrics on $D$ and $P$, and hence induces the metric on $\text{Hom}(K_k \otimes K_{k-1} \otimes \cdots \otimes K_1, P)$. Furthermore, we can prove inductively that $P_k$, and also $Q_{k+1}$ as the orthogonal complement of $\text{Im}(d_{k+1}|_K)$ inherit the induced metrics by (5) and (6) in Section 2 respectively. Consequently we have the induced metric on $\text{Hom}(K_{k+1}, Q_{k+1})$.

Let us recall $d_k|K_{k-1} : K_{k-1} \rightarrow P_{k-1}$ and $e_{k-1} : P_{k-1} \rightarrow Q_{k}$ over $\Sigma^{I_k}(N, P)$, which induces the commutative diagram

$$
\begin{array}{cccc}
K_k & \rightarrow & K_{k-1} & \rightarrow & P_{k-1} & \rightarrow & Q_k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & K_{k-1}/K_k & \rightarrow & \text{Hom}(K_{k-1}, Q_{k-1}).
\end{array}
$$

Since $Q_k$ is the cokernel of $d_k|K_{k-1}$, we obtain the canonical isomorphism

$$\text{j}_{Q_k} : Q_k \rightarrow \text{Im}(d_k|K_{k-1})^{\perp} \quad \text{over} \quad \Sigma^{I_k}(N, P),$$
where the symbol \( \perp \) refers to the orthogonal complement. We also use the notation 
\[ j_{Q_i} : Q_j \to \text{Hom}(K_{k-1}, Q_{k-1}). \]

Let \( k \geq 2 \). We now construct the homomorphism, for \( 1 \leq i \leq k, \)
\[
(3.1) \quad q(k)^{i+1,i+1} : T(S^{i-1}(N,P))|_{\Sigma^i_k(N,P)} \bigcirc K_i \bigcirc K_{i-1} \cdots \bigcirc K_1 \to Q_i
\]
over \( \Sigma^i_k(N,P) \) inductively by using \( d_{i+1}|_{\Sigma^i_k(N,P)} : T(S^{i-1}(N,P))|_{\Sigma^i_k(N,P)} \to P_i|\Sigma^i_k(N,P) \) as follows. By the inclusion \( P_i|\Sigma^i_k(N,P) \subset \text{Hom}(K_i, Q_i)|_{\Sigma^i_k(N,P)} \) we have the homomorphism
\[
q(k)^{i+1,i+1}_j : (T(S^{i-1}(N,P))|_{\Sigma^i_k(N,P)}) \otimes K_i \to Q_i \quad \text{over } \Sigma^i_k(N,P).
\]

Suppose that we have constructed the homomorphism, for \( j \leq i, \)
\[
q(k)^{i+1,i+1}_j : T(S^{i-1}(N,P))|_{\Sigma^i_k(N,P)} \otimes K_i \bigcirc K_{i-1} \cdots \bigcirc K_j \to Q_j
\]
over \( \Sigma^i_k(N,P) \). By using \( j_{Q_i} : Q_j \to \text{Hom}(K_{j-1}, Q_{j-1}) \) over \( \Sigma^i_k(N,P) \), we obtain the homomorphism
\[
(3.2) \quad q(k)^{i+1,i+1}_j : T(S^{i-1}(N,P))|_{\Sigma^i_k(N,P)} \otimes K_i \bigcirc K_{i-1} \cdots \bigcirc K_j \to Q_{j-1}
\]
over \( \Sigma^i_k(N,P) \). By setting \( j = 2 \), we obtain \( q(k)^{i+1,i+1}_1 \). It remains to prove that \( q(k)^{i+1,i+1}_1 \) is symmetric. This fact has been essentially stated in [B, Section 7, p.413] without proof. Following the proof of [B, Theorem 4.1] we briefly prove it.

Let \( z \in \Sigma^i_k(N,P) \). By the Riemannian metric of \( P \), we consider the convex neighborhood of \( P \) around \( \pi^P_{\Sigma}(z) = y \). Let us canonically identify \( Q_{1,z} \) with a subspace of \( T_yP \) by the isomorphism \( P_z \to T_yP \). By taking a basis of \( Q_{1,z} \) and projecting it by the exponential map, we have the local coordinates \( y_1, y_2, \ldots, y_{p-n+i} \) on the convex neighborhood of \( y \). Then we identify \( Q_{1,z} \) with \( \text{Hom}(m_y^Q/(m_y^Q)^2, R) \), where \( m_y^Q \) is the ideal generated by \( y_1, y_2, \ldots, y_{p-n+i} \). Let \( D \) and \( D_j \) be sections of \( T(S^{i-1}(N,P))|_{\Sigma^i_k(N,P)} \) and \( K_j \) defined around \( z \) and let \( \alpha \in m_y^Q \). Then (3.1) is regarded as the homomorphism induced from
\[
T(S^{j-1}(N,P))_{z} \otimes K_{i,z} \otimes K_{i-1,z} \otimes \cdots \otimes K_{1,z} \otimes m_y^Q/(m_y^Q)^2 \to R
\]
which maps \( D \otimes D_i \otimes \cdots \otimes D_j \otimes \alpha \) to \( (DD_i \cdots D_j \alpha)(z) \) (see (a) and (b) in the proof of [B, Theorem 4.1]). We have to show the following for the symmetry (consult Remark 3.1 below to avoid the infinity of the dimensions of the tangent spaces). In the expression with \( [D_j, D_{j-1}] = D_j D_{j-1} + D_{j-1} D_j \)
\[
DD_i \cdots D_j D_{j-1} \cdots D_1 \alpha - DD_i \cdots D_{j-1} D_1 \alpha = DD_i \cdots [D_j, D_{j-1}] \cdots D_1 \alpha
\]
for some \( j \) with \( 1 < j \leq i + 1 \) \((D_{i+1} = D)\), we have that \([D_j, D_{j-1}] \) is the section of \( K_{j-1} \) for \( j \leq i \) and of \( T(S^{j-1}(N,P)) \) for \( j = i + 1 \) by [B, Lemma 3.2]. Since \( K_j|_{\Sigma^i_k(N,P)} \subset K_{j-1}|_{\Sigma^i_k(N,P)} \), the length of \( DD_i \cdots [D_j, D_{j-1}] \cdots D_1 \) is \( i \), \( D \) and \([D_j, D_{j-1}] \) lie in \( T(S^{j-1}(N,P))_z \) and since \( T(S^{j-1}(N,P))_z \subset T(S^{j-2}(N,P))_z \), we have that \( (DD_i \cdots [D_j, D_{j-1}] \cdots D_1 \alpha)(z) = 0 \) by \( \text{Ker}(d_{i,z}) = T(S^{i-1}(N,P))_z \) in (2.1). This is what we want.

In particular, if \( i = 1 \) and we restrict \( T(S^{j-1}(N,P))|_{\Sigma^i_k(N,P)} \) to \( K_1 \), then we have the homomorphism \( q(k)^{2,2}_1(K_1 \bigcirc K_1) : K_1 \bigcirc K_1 \to Q_1 \) over \( \Sigma^k_k(N,P) \), which induces the nonsingular quadratic form \( (K_1/K_2) \bigcirc (K_1/K_2) \to Q_1 \) on each fiber.
Remark 3.1. We can entirely do the arguments in Sections 2 and 3 on $J^l(N,P)$ for a large $l$. We provide $N$ and $P$ with Riemannian metrics. For any points $x \in N$ and $y \in P$, we have the local coordinates $(x_1, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ on convex neighbourhoods of $x$ and $y$ associated to orthonormal basis of $T_xN$ and $T_yP$ respectively. Let us define the canonical embedding $\mu^l_\infty : J^l(TN,TP) \to J^\infty(TN,TP)$ such that $\pi_\infty \circ \mu^l_\infty = id_{J^l(TN,TP)}$ and that the $i$-th components for $i > l$ of elements of the image $\mu^l_\infty$ are the null homomorphisms of Hom($S^l(\pi_\infty^*(TN)), \pi_\infty^*(TP)$). We regard $\mu^l_\infty$ as the map to $J^\infty(N,P)$ under the identification (1.2). Any element $z \in \mu^l_\infty(J^l(TN,TP))$ is represented by a $C^\infty$ map germ $f : (N,x) \to (P,y)$ such that any $i$-th derivative of $f$ with $i > l$ vanishes under these coordinates. It is clear that we can prove that $D|_{\mu^l_\infty(J^l(TN,TP))}$ is tangent to $\mu^l_\infty(J^l(TN,TP))$. Indeed, for $\sigma = (\sigma_1, \ldots, \sigma_n)$ with non-negative integers, let us recall the functions $X_i$ and $Z_{j,\sigma}$ with $1 \leq i \leq n$ and $1 \leq j \leq p$ defined locally on a neighbourhood of $J^\infty(N,P)$ by, for $z = \sum_{j,\sigma} f_j$, $X_i(z) = x_i$ and $Z_{j,\sigma}(z) = \frac{\partial|\sigma| f (y_j \circ f)}{\partial x_i} \cdots \frac{\partial|\sigma| f (y_j \circ f)}{\partial x_n}(x)$, which constitute the local coordinates on $J^\infty(N,P)$ as described in [B, Section 1]. Let $\Phi$ be a smooth function defined locally on $\mu^l_\infty(J^l(TN,TP))$ and let $D_i \in D$ be the total tangent vector corresponding to $\partial/\partial x_i$ by the canonical identification of $D$ and $(\pi_\infty^*)^*(TN)$. Let $\sigma' = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \ldots, \sigma_n)$. Then we have $D_i(\Phi)(z) = \frac{\partial(\Phi \circ j^\infty f)}{\partial x_i}(x) = \partial \Phi(\partial x_i)(z) + \sum_{j,\sigma} \frac{\partial \Phi}{\partial Z_{j,\sigma}}(z) Z_{j,\sigma}(z)$ by [B, (1.8)]. If $z \in \mu^l_\infty(J^l(TN,TP))$, then $Z_{j,\sigma}(z)$ vanishes for $|\sigma| > l$. Hence, $D_i(\Phi)$ is a smooth function defined locally on $\mu^l_\infty(J^l(TN,TP))$. This implies that $D_i$ is tangent to $\mu^l_\infty(J^l(TN,TP))$. Since $D_z$ consists of all linear combinations of $D_1, \ldots, D_n$, we have that $D_z \subset T_z(\mu^l_\infty(J^l(TN,TP)))$. Therefore, we can do the required arguments on $\mu^l_\infty(J^l(TN,TP))$, namely also on $J^l(TN,TP)$.

4. Primary obstruction

Let $\Gamma^{tr}_0(N,P)$ denote the subspace of $\Gamma_0(N,P)$ consisting of all continuous sections of $\pi_0^*\Omega^l(N,P) : \Omega^l(N,P) \to N$ which are transverse to each $\Sigma^l(N,P)$. For $s \in \Gamma^{tr}_0(N,P)$, we set $s^I(s) = s^{-1}(\Sigma^l(N,P)), s^I,0(s) = s^{-1}(\Sigma^l,0(N,P)), (s|s^I(s))^* (K_j) = K_j(s^I(s))$ and $(s|s^I(s))^{-1} Q_1 = Q(s^I(s))$. We often write $s^I(s)$ as $s^I$ if there is no confusion.

Let $L = (\ell_1, \ell_2, \ldots, \ell_r)$ and $I = (i_1, i_2, \ldots, i_k, 0, \ldots, 0, \ldots)$ such that $I_r \leq L$, codim $\Sigma^l(n,p) \leq n$ and codim $\Sigma^l(n,p) \leq n$, where $k$ may be larger than $r$. Let $C(I^-)$ (resp. $C(I)$) refer to the union $C \cup (\cup_{j \geq 1} S^I(s))$ (resp. $C \cup (\cup_{j \geq 1} S^I(s))$), where $J$ are symbols of infinite length and $C$ is a closed subset of $N$. We show in this section that it is enough for the proof of Theorem 0.1 to prove Theorem 4.1.

Theorem 4.1. Let $n \geq p \geq 2$. Let $N$ and $P$ be connected manifolds of dimensions $n$ and $p$ respectively with $\partial N = \emptyset$. Let $L$ and $I = (i_1, i_2, \ldots, i_k, 0)$ be as above. Assume that $\Omega^l(N,P)$ contains $\Sigma^{n-p+1,0}(N,P)$ at least. Let $s$ be a section of $\Gamma^{tr}_0(N,P)$ which has an $\Omega^L$-regular map $g(I^+)$ defined on a neighborhood of $C(I^+)$
into $P$, where $j^\infty g(I^+) = s$. Then there exists a homotopy $s_\lambda \in \Gamma^r_{tr}(N, P)$ relative to a neighborhood of $C(I^+)$ with the following properties.

1. $s_0 = s$ and $s_1 \in \Gamma^r_{tr}(N, P)$.
2. There exists an $\Omega^r$-regular map $g_I$ defined on a neighborhood of $C(I)$, where $j^\infty g_I = s_1$ holds.
3. $s^{-1}(\Sigma^I(N, P)) = (j^\infty g_I)^{-1}(\Sigma^I(N, P))$.

The case $I = (n - p + 1, 0)$ of Theorem 4.1 follows from Theorem 1 of [An1], where a partially sketchy proof was given and the detailed proof was given in [An4, Theorem 4.1] and [An6, Theorem 0.5]. Let us explain how it follows. In fact, we have $\Omega^{n-p+1,0}(N, P) = \Sigma^{n-p}(N, P) \cup \Sigma^{n-p+1,0}(N, P)$, and if we set $N_0 = S^{n-p}(s) \cup S^{n-p+1,0}(s)$, then $C((n - p + 1, 0)^+) = C \cup (N \setminus \Omega N_0)$. Let $U$ and $U'$ be closed neighborhoods of $U \cap (N \setminus N_0)$ with $U \subset IntU'$, where $g((n - p + 1, 0)^+)$ is defined. Since $s \in \Gamma^r_{tr}(N, P)$, $s|N_0$ is a section of $\Gamma^r_{\Omega^r}(N, P)$ and $g((n - p + 1, 0)^+)(U' \setminus \Omega N_0)$ is an $\Omega^{n-p+1,0}$-regular map. Hence, we obtain a homotopy $u_\lambda \in \Gamma^r_{\Omega^r}(N, P)$ relative to a neighborhood of $U \cap N_0$ and an $\Omega^{n-p+1,0}$-regular map $f_0: N_0 \to P$ such that $s_0|N_0 = u_0$ and $u_1 = j^\infty f_0$. Then we obtain a required homotopy $s_\lambda$ by defining $s_\lambda N_0 = u_\lambda$ and $s_\lambda U = j^\infty g((n - p + 1, 0)^+)$.

We will prove Theorem 4.1 for $I > (n - p + 1, 0)$ in Section 6.

We now prove Theorem 0.1 for $\Omega^L$ for this symbol $L = (\ell_1, \ell_2, \ldots, \ell_r)$ in place of $\Omega^I$ by using Theorem 4.1. In Sections 4, 5 and 6 we use the notation $\Omega$ for $\Omega^L$.

**Proof of Theorem 0.1.** Suppose that the section $s$ given in Theorem 0.1 lies in $\Gamma^r_{tr}(N, P)$. Let $I = (i_1, i_2, \ldots, i_k, 0, \ldots, 0, \ldots)$ be the largest symbol such that $I \leq L$ and codim $\Sigma^I(n, p) \leq n$. We can choose such a symbol $I$ by using Section 2 (4) and Remark 2.1 (2). Then we first set $C(I^+) = C$ and $g(I^+) = g$. By Theorem 4.1 there exists an $\Omega$-regular map $g_t$ defined on a neighborhood of $C(I)$, where $j^\infty g_t = s$ holds. If we note Remark 2.1 (2), then we can prove Theorem 0.1 by the downward induction on the symbols $I$ in the lexicographic order.

We begin by preparing several notions and results, which are necessary for the proof of Theorem 4.1. For the map $g(I^+)$ and the closed subset $C(I^+)$, we take an open neighborhood $U(C(I^+))'$ of $C(I^+)$, where $j^\infty g(I^+) = s$. Without loss of generality we may assume that $N \setminus U(C(I^+))'$ is nonempty. Take a smooth function $h_{C(I^+)}: N \to [0, 1]$ such that

$$
\begin{align*}
&h_{C(I^+)}(x) = 1 && \text{for } x \in C(I^+), \\
&h_{C(I^+)}(x) = 0 && \text{for } x \in N \setminus U(C(I^+))', \\
&0 < h_{C(I^+)}(x) < 1 && \text{for } x \in U(C(I^+))' \setminus C(I^+).
\end{align*}
$$

By the Sard Theorem ([H2]) there is a regular value $r$ of $h_{C(I^+)}$ with $0 < r < 1$. Then $h_{C(I^+)}^{-1}(r)$ is a submanifold and we set $U(C(I^+)) = h_{C(I^+)}^{-1}([r, 1])$. We decompose $N \setminus \text{Int}U(C(I^+))$ into the connected components, say $L_1, \ldots, L_j, \ldots$.

It suffices to prove Theorem 4.1 for each $L_j \cup \text{Int}U(C(I^+))$. Since $\partial N = \emptyset$, we have that $N \setminus U(C(I^+))$ has empty boundary. If $L_j$ is not compact, then Theorem 4.1 holds for $L_j \cup \text{Int}U(C(I^+))$ by Gromov’s theorem ([G1, Theorem 4.1.1]). Therefore, it suffices to consider the special case where

- (C1) $N \setminus \text{Int}U(C(I^+))$ is compact, connected and nonempty;
- (C2) $\partial U(C(I^+))$ is a submanifold of dimension $n - 1$;
- (C3) for the smooth function $h_{C(I^+)}: N \to [0, 1]$ satisfying (4.1) there is a sufficiently small positive real number $\varepsilon$ with $r - 2\varepsilon > 0$ such that $r - t\varepsilon$ ($0 \leq t \leq 2$)
are all regular values of $h_{C(I^+)}$. We have that $h_{C(I^+)}^{-1}([r - 2\varepsilon, 1])$ is contained in $U(C(I^+))^t$.

We set $U(C(I^+))_t = h_{C(I^+)}^{-1}([r - (2-t)\varepsilon, 1])$. In particular, we have $U(C(I^+))^2 = U(C(I^+))$. Furthermore, we may assume that

(C4) $s \in \Gamma^+_s(N, P)$ and $S^l(s)$ is transverse to $\partial U(C(I^+))_0$ and $\partial U(C(I^+))^2$.

In what follows we choose and fix a Riemannian metric of $N$, which satisfies

**Orthogonality Condition:** for the symbol $I$, $K_{j-1}(S^l(s))/K_j(S^l(s))$ is orthogonal to $S^l_{j-1}(s)$ for $k \leq j \leq 1$ on $S^l(s)$ ($S^0(s) = N$).

Let $\nu(\Sigma^l)$ be the normal bundle $(T(J^{\infty}(N,P))|_{\Sigma^l})/T(\Sigma^l(N,P))$ and let $c(I) = \dim \nu(\Sigma^l)$. Let us fix a direct sum decomposition

\begin{equation}
\nu(\Sigma^l) = \oplus_{k=1}^k (I_j \subset I_{j-1})|_{\Sigma^l(N,P)},
\end{equation}

and the direct sum decomposition $K_1 = \oplus_{j=1}^{k-1} (K_j/K_{j+1}) \oplus K_k$ over $\Sigma^l(N,P)$. Let $j_{K_1} : K_1 \to \nu(\Sigma^l)$ over $\Sigma^l(N,P)$ be the composition of the inclusion $K_1 \to T(J^{\infty}(N,P))$ and the projection $T(J^{\infty}(N,P))|_{\Sigma^l(N,P)} \to \nu(\Sigma^l)$. We have the monomorphism

$j_{K_1} \circ (s|S^l)^{K_1} : K_1(S^l(s)) \to K_1|_{\Sigma^l(N,P)} \to \nu(\Sigma^l)$.

For $s \in \Gamma^+_s(N, P)$, let $n(s, I)$ or simply $n(I)$ be the orthogonal normal bundle of $S^l(s)$ in $N$. Let $n(s, I_j \subset I_{j-1})$ be the orthogonal normal bundle of $S^l_{j-1}(s)$ in $S^l_{j-1}(s)$ over $S^l(s)$. Then we have the canonical direct sum decomposition such as

\begin{equation}
n(s, I) = \oplus_{j=1}^k n(s, I_j \subset I_{j-1}),
\end{equation}

Furthermore, we obtain the bundle map

\[ ds(n(s, I)) : n(s, I) \to \nu(\Sigma^l) \]

covering $s|S^l : S^l(s) \to \Sigma^l(N,P)$. Let $i_{n(s,I)} : n(s, I) \subset TN|_{S^l}$ denote the inclusion. We define $\Psi(n(s, I)) : K_1(S^l(s)) \to n(s, I) \subset TN|_{S^l}$ to be the composition

\[ i_{n(s,I)} \circ ((s|S^l)^*(ds(n(s, I))))^{-1} \circ ((s|S^l)^*(j_{K_1} \circ (s|S^l)^{K_1}))) : K_1(S^l(s)) \to (s|S^l)^*\nu(\Sigma^l) \to n(s, I) \to TN|_{S^l}. \]

We note that this homomorphism does not use the decomposition in (4.2) and we can take the direct sum decompositions in (4.2) to be compatible with those in (4.3). Let $i_{K_1(S^l(s))} : K_1(S^l(s)) \to TN|_{S^l}$ be the inclusion.

**Remark 4.2.** If $f$ is an $\Omega$-regular map, then it follows from the definition of $D$ that $i_{K_1(S^l(j^\infty f)))} = \Psi(j^\infty f, I)$.

In what follows let $M = S^l(s) \cap \text{Int}(U(C(I^+)))$. Let $\text{Mono}(K_1(S^l(s))|_M, TN|_M)$ denote the subset of $\text{Hom}(K_1(S^l(s))|_M, TN|_M)$ which consists of all monomorphisms $K_1(S^l(s))_c \to T_c N, c \in M$. We denote the bundle of the local coefficients $\mathcal{B}(\pi_j(\text{Mono}(K_1(S^l(s))_c, T_c N)), c \in M, \mathcal{B}(\pi_j))$, which is a covering space over $M$ with fiber $\pi_j(\text{Mono}(K_1(S^l(s))_c, T_c N))$ defined in [Ste, 30.1]. From the obstruction theory due to [Ste, 36.3], it follows that the obstructions for $i_{K_1(S^l(s))}|_M$ and $\Psi(s, I)|_M$ to be homotopic are the primary differences $d(i_{K_1(S^l(s))}|_M, \Psi(s, I)|_M)$, which are defined in $H^0(M, \partial M; \mathcal{B}(\pi_j))$ with the local coefficients. We show that all of them vanish. In fact, note $I > (n - p + 1, 0)$. If $i_1 = n - p + 1$, then we have

\[ \dim M < \dim S^{n_1} = n - i_1(p - n + i_1) = n - i_1. \]
If $i_1 > n - p + 1$, then

$$\dim M \leq \dim S^{i_1} = n - i_1(p - n + i_1) < n - i_1.$$  

Since $\text{Mono}(\mathbb{R}^n, \mathbb{R}^n)$ is identified with $\text{GL}(n)/\text{GL}(n - i_1)$, it follows from [Ste, 25.6 38.2] that $\pi_j(\text{Mono}(\mathbb{R}^n, \mathbb{R}^n)) \cong \{0\}$ for $j < n - i_1(\leq p - 1)$. Hence, there exists a homotopy $\psi^M(s, I)_\lambda : K_1(S^I(s))|_M \to TN|_M$ relative to $M \cap U(C(I^+)_{\lambda})$ in $\text{Mono}(K_1(S^I(s))|_M, TN|_M)$ such that $\psi^M(s, I)_{\lambda} = i_{K_1(S^I(s))}|_M$ and $\psi^M(s, I)_0 = \Psi(s, I)|_M$. Let $\text{Iso}(TN|_M, TN|_M)$ denote the subspace of $\text{Hom}(TN|_M, TN|_M)$ which consists of all isomorphisms of $T_c N$, $c \in M$. The restriction map

$$r_M : \text{Iso}(TN|_M, TN|_M) \to \text{Mono}(K_1(S^I(s))|_M, TN|_M)$$

defined by $r_M(h) = h_{|K_1(S^I(s))|_M}$, for $h \in \text{Iso}(T_c N, T_c N)$, induces a structure of a fiber bundle with fiber $\text{Iso}([n, \infty) \times \text{Hom}(\mathbb{R}^{n-i_1}, \mathbb{R}^{n-i_1}) \times \text{Hom}(\mathbb{R}^{n-i_1}, \mathbb{R}^{n-i_1}))$. By applying the covering homotopy property of the fiber bundle $r_M$ to the sections $id_{TN|_M}$ and the homotopy $\psi^M(s, I)_\lambda$, we obtain a homotopy $\Psi^M(s, I)_\lambda : TN|_M \to TN|_M$ such that $\Psi^M(s, I)_{\lambda} = id_{TN|_M}$ and $\psi^M(s, I)_\lambda \circ \psi^M(s, I)_0 = i_{K_1(S^I(s))}|_M$ for all $c \in M \cap U(C(I^+)_{\lambda})$ and $r_M \circ \psi^M(s, I)_\lambda = \psi^M(s, I)_\lambda$. We define $\Phi(s, I)_\lambda : TN|_M \to TN|_M$ by $\Phi(s, I)_\lambda = (\psi^M(s, I)_\lambda)^{-1}$.

5. LEMMAS

Let $I$ be the symbol in Theorem 4.1. In the proof of the following lemma, $\Phi(s, I)_\lambda|_c$ $(c \in M)$ is regarded as a linear isomorphism of $T_c N$. Let $r_0$ be a small positive real number with $r_0 < 1/10$.

**Lemma 5.1.** Let $s \in \Gamma^+_{M}(N, P)$ be a section satisfying the hypotheses of Theorem 4.1. Then there exists a homotopy $s_\lambda$ relative to $U(C(I^+)_{\lambda})_0$ in $\Gamma^+_{M}(N, P)$ with $s_0 = s$ satisfying

1. for any $\lambda$, $S^I(s) = S^I(s)$ and $\pi_\lambda^\infty \circ s_\lambda|S^I(s) = \pi_\lambda^\infty \circ s|S^I(s)$,
2. for any point $c \in S^I(s)$, we have $i_{K_1(S^I(s))} = \Psi(s, I)$. In particular, $K_1(S^I(s))|_c \subseteq n(I)|_c$.

**Proof.** Recall the exponential map $\exp_{N,c} : T_{N,c} N \to N$ defined near $0 \in T_{N,c} N$. We write an element of $n(I)|_c$ as $v_c$. There exists a small positive number $\delta$ such that the map

$$e : D_\delta(n(I))|_M \to N$$

defined by $e(v_c) = \exp_{N,c}(v_c)$ is an embedding, where $c \in M$ and $v_c \in D_\delta(n(I)|_c)$ (note that $e|M$ is the inclusion). Let $\rho : [0, \infty) \to \mathbb{R}$ be a decreasing smooth function such that $0 \leq \rho(t) \leq 1$, $\rho(t) = 1$ if $t \leq \delta/10$ and $\rho(t) = 0$ if $t \geq \delta$.

If we represent $s(x) \in \Omega(N, P)$ by a jet $j_{\infty}^c \sigma_x$ for a germ $\sigma_x : (N, x) \to (P, \pi_\lambda^\infty \circ s(x))$, then we define the homotopy $s_\lambda$ of $\Gamma^+_{M}(N, P)$ using $\Phi(s, I)_\lambda$ by

$$s_\lambda\left(e(v_c)\right) = j_{\infty}^c(\sigma_x \circ e(v_c)) \circ \Phi(s, I)_\lambda|_c \circ \exp_{N,c}^{-1} \circ \Phi(s, I)_0|_c \circ e(v_c) \quad \text{if } c \in M \text{ and } \|v_c\| \leq \delta,$$

$$s_\lambda(x) = s(x) \quad \text{if } x \notin \text{Im}(e).$$

Here, $\Phi(s, I)_\rho|_c$ refers to $\ell(\nu)_c \circ \Phi(s, I)_\rho|_c \circ (\nu)_c$, where $\ell(\nu)$ denotes the parallel translation defined by $\ell(\nu)(a) = a + v$. If $\|v_c\| \geq \delta$, then $\Phi(s, I)_\rho|_c = \Phi(s, I)_0|_c$, and if $c \in S^I \cap U(C(I^+)_{\lambda})_0$, then $\Phi(s, I)_\lambda|_c = \Phi(s, I)_\lambda|_0|_c$. Hence, $s_\lambda$ is well defined. It follows from (5.1) that

$$\pi_\lambda^\infty \circ s_\lambda(x) = \pi_\lambda^\infty \circ s(x),$$
We regard \( \tilde{\Theta} \), place of \( \tilde{\Theta} \), since \( T \rightarrow \Omega \), manifold \( \Sigma^I(N,P) \) is a monomorphism. By the Hirsch Immersion Theorem (\cite[Theorem 5.7]{H1}), there exists a homotopy \( s \), relative to \( U(C(I^+))_2 \), \( \Gamma \) in \( \pi \). Then 

(1) \( S^I(s) = S^I(s) \) for any \( \lambda \),

(2) \( \pi_{\lambda} \) is an immersion into \( \pi \) such that \( d(\pi_{\lambda} \circ s \circ d_1) \circ (s(I^+) \times [0,1]) = \pi_{\lambda} \circ s \circ d_1 \).

In what follows we set \( d(s, I) = (s[I^+] \circ d_1) \). We also choose and fix a Riemannian metric of \( \pi \) and identify \( Q(S^I(s)) \) with the orthogonal complement of \( \operatorname{Im}(d(s, I)) \) in \( (\pi_{\lambda} \circ s \circ d_1)(TP) \).

**Lemma 5.2.** Let \( s \) be a section of \( \Gamma^I_\Omega(N,P) \) satisfying the property (2) for \( s \) in place of \( s_0 \) of Lemma 5.1. Then there exists a homotopy \( s_0 \), relative to \( U(C(I^+))_2 \), \( \Gamma \) in \( \pi \) such that \( \rho_\lambda \) of homomorphisms of constant rank \( n \) from \( \rho \) is transverse to \( \Sigma \).

**Proof.** Since \( K_1 \cap T(\Sigma^I(N,P)) = \{0\} \), it follows that \( (\pi_{\lambda} \circ s \circ d_1)(TP) \) is a homomorphism. By the Hirsch Immersion Theorem (\cite[Theorem 5.7]{H1}) there exists a homotopy of monomorphisms \( \rho_\lambda : T(S^I) \rightarrow TP \) covering a homotopy \( \rho_\lambda : S^I \rightarrow TP \) such that \( \rho_\lambda = (\pi_{\lambda} \circ s \circ d_1)(TP) \) and that \( \rho_\lambda \) is an immersion with \( d(\rho_\lambda) = \rho \). Then we can extend \( \rho_\lambda \) to a homotopy \( \rho_\lambda : TN |_{S^I} \rightarrow TP \) of homomorphisms of constant rank \( n - i_1 \) relative to \( U(C(I^+))_2 \). In fact, let \( m : S^I \times [0,1] \rightarrow P \times [0,1] \) and \( \rho' : T(S^I) \times [0,1] \rightarrow TP \times [0,1] \) be the maps defined by \( m(c, \lambda) = (m_\lambda(c), \lambda) \) and \( \rho'(v, \lambda) = (m_\lambda'(v), \lambda) \) respectively. Let \( m'(m') : T(S^I) \times [0,1] \rightarrow TP \times [0,1] \) be the canonical monomorphism induced from \( \rho \) by \( m \). Let \( F_1 = \operatorname{Im}(m'(m')) \) and \( F_2 \) be the orthogonal complement of \( F_1 \) in \( m'(TP) \times [0,1] \). Since \( F_2 \) is isomorphic to \( (F_2[I^+] \times [0,1]) \), we obtain a homomorphism of \( m_\rho \) over \( S^I \times [0,1] \).

\[
 j_F : \operatorname{Im}(d(s, I)|n(I)) \times [0,1] \rightarrow F_2 \quad \text{over} \ S^I \times [0,1].
\]

Since \( d(s, I)|n(I) \times [0,1] \) is of constant rank \( n - i_1 \), it induces the homomorphism of kernel rank \( i_1 \)

\[
d : n(I) \times [0,1] \rightarrow \operatorname{Im}(d(s, I)|n(I)) \times [0,1] \xrightarrow{j_F} F_2.
\]

We define \( m' \) to be the composition

\[
 T(N |_{S^I} \times [0,1] \cong (T(S^I) \oplus n(I)) \times [0,1]) \xrightarrow{m'(m') \oplus d} F_1 \oplus F_2 \xrightarrow{\operatorname{Im}(m'(m')) \oplus \operatorname{Cok}(m'(m'))} \operatorname{Im}(m'(TP) \times [0,1]) \xrightarrow{m'(TP \times [0,1])} TP \times [0,1].
\]

We define \( \tilde{\rho} \) to be \( (\tilde{\rho} \circ \rho_\lambda)(v, \lambda) = m'(v, \lambda) \).

Next we construct a homotopy \( s_\lambda : N \rightarrow \Omega(N,P) \) from \( \tilde{\rho} \). Recall the submanifold \( \bar{\Sigma}^I(N,P) \) of \( J^I(TN,TP) \) which corresponds to \( \Sigma^I(N,P) \) in Section 2 (9). Then \( \pi_{\lambda} \) of \( \bar{\Sigma}^I(N,P) : \Sigma^I(N,P) \rightarrow \bar{\Sigma}^I(N,P) \) becomes a fiber bundle. We regard \( \tilde{\rho} \) as a homotopy \( S^I \rightarrow \bar{\Sigma}^I(N,P) \). By the covering homotopy property to \( s \) and \( \rho_\lambda \), we obtain a homotopy \( \rho'_\lambda : S^I \rightarrow \bar{\Sigma}^I(N,P) \) covering \( \rho_\lambda \) relative to \( U(C(I^+))_2 \) such that \( \rho'_\lambda = s[I^+] \).

By using the transversality of \( s \) and the homotopy extension property to \( s \) and \( \rho_\lambda \), we first extend \( \rho'_\lambda \) to a homotopy defined on a tubular neighborhood of \( S^I \) and
then to a required homotopy $s_\lambda \in \Gamma^j_\Omega(N,P)$, which satisfies $s_0 = s$, $s_\lambda|S^I = s'_\lambda$ and $s_\lambda|U(C(I^+))_{2-3r_0} = s|U(C(I^+))_{2-3r_0}$.

Here we give two lemmas necessary for the proof of Theorem 4.1. Let $\pi : E \to S$ be a smooth $c(I)$-dimensional vector bundle with a metric over an $(n - c(I))$-dimensional manifold, where $S$ is identified with the zero-section. Then we can identify $\exp_E|D_\varepsilon(E) = id_{D_\varepsilon(E)}$.

**Lemma 5.3.** Let $\pi : E \to S$ be given as above. Let $f_i : E \to P$ $(i = 1,2)$ be $\Omega$-regular maps which have singularities of the symbol $I$ exactly on $S$ such that

(i) $f_1|S = f_2|S$, which are immersions,

(ii) $S = S^I(j^\infty f_1) = S^I(j^\infty f_2)$,

(iii) $K_1(S^I(j^\infty f_1)) = K_1(S^I(j^\infty f_2))$, are tangent to $\pi^{-1}(c)$,

(iv) $T_c(S^{I-1}(j^\infty f_1)) = T_c(S^{I-1}(j^\infty f_2))$, $(j^\infty f_1)^*P_j = (j^\infty f_2)^*P_j$ and $(j^\infty f_1)^*(d_{j+1} \circ d(j^\infty f_2)) = (j^\infty f_2)^*(d_{j+1} \circ d(j^\infty f_2))$ for each number $j$ and any $c \in S$.

Let $\eta : S \to [0,1]$ be any smooth function. Let $\varepsilon : S \to R$ be a sufficiently small positive smooth function. Let $f^\varepsilon(v_\lambda)$ denote $\exp_{P,f_i(c)}((1-\eta(c))\exp_{P,f_i(c)}(f_i(v_\lambda)) + \eta(c)\exp_{P,f_i(c)}(f_2(v_\lambda)))$ for any $c \in S$ and any $v_\lambda \in \pi^{-1}(c)$ with $\|v_\lambda\| \leq \varepsilon(c)$.

Then the map $f^\varepsilon : D_\varepsilon(E) \to P$ is a well-defined $\Omega$-regular map such that

(1) $f^\varepsilon|S = f_1|S = f_2|S$,

(2) $S = S^I(j^\infty f^\varepsilon)$,

(3) $T_c(S^{I-1}(j^\infty f^\varepsilon)) = T_c(S^{I-1}(j^\infty f_1))$, $(j^\infty f^\varepsilon)^*P_j = (j^\infty f_1)^*P_j$ and $(j^\infty f^\varepsilon)^*(d_{j+1} \circ d(j^\infty f^\varepsilon)) = (j^\infty f_1)^*(d_{j+1} \circ d(j^\infty f_1))$ for each number $j$ and any $c \in S$.

**Proof.** Let us take a Riemannian metric on $E$ which is compatible with the metric of the vector bundle $E$ over $S$. In particular, $S$ is a Riemannian submanifold of $E$. Furthermore, take a Riemannian metric on $P$ such that $f_i(S) \cap P$ is a Riemannian submanifold of $P$ around $f(c)$. Then the local coordinates of $\exp_{\Sigma}(K_1,c)$ and $\exp_{P,f_i(c)}(Q_c)$ are independent of the coordinates of $S$, where $Q_c$ is regarded as a subspace of $T_{f_i(c)}P$.

We may consider $\eta(c)$ as a constant when dealing with higher intrinsic derivatives in the lemma by the identification (1.2) and the property of the total tangent bundle $D$ given in the beginning of Section 2. Then the assertion follows from the assumptions and the properties of $\Sigma^j(N,P)$.

The proof of the following lemma is elementary, and so is left to the reader.

**Lemma 5.4.** Let $\pi : E \to S$ be given as above. Let $\Omega, \Sigma$ be a pair of a smooth manifold and its submanifold of codimension $c(I)$. Let $\varepsilon : S \to R$ be a sufficiently small positive smooth function. Let $h : D_\varepsilon(E) \to (\Omega, \Sigma)$ be a smooth map such that $S = h^{-1}(\Sigma)$ and that $h$ is transverse to $\Sigma$. Then there exists a smooth homotopy $h_\lambda : (D_\varepsilon(E), S) \to (\Omega, \Sigma)$ between $h$ and $\exp_{\Omega, h(c)}|D_\varepsilon(E)$ such that

(1) $h_\lambda|S = h_0|S$, $S = h_\lambda^{-1}(\Sigma) = h_0^{-1}(\Sigma)$ for any $\lambda$,

(2) $h_\lambda$ is smooth and is transverse to $\Sigma$ for any $\lambda$,

(3) $h_0 = h$ and $h_1(v_\lambda) = \exp_{\Omega, h(c)}|D_\varepsilon(E)$ for $c \in S$ and $v_\lambda \in D_\varepsilon(E)$.}

6. **Proof of Theorem 4.1**

Consider the bundles $n(I) (= n(s, I))$ and $Q (= Q(S^I(s)))$. For a point $c \in S^I(s)$, take an open neighborhood $U$ around $c$ such that $n(I)|_U$ and $Q|_U$ are the trivial
Let us consider the direct sum decompositions covering the immersion \( i \) and the inclusion \( U \) bundles, say \( U \times \mathbb{R}^{c(I)} \) and \( U \times \mathbb{R}^{p-n+i_1} \) respectively, where \( \mathbb{R}^{c(I)} \) has coordinates \((x_1, \ldots, x_{c(I)})\) and \( \mathbb{R}^{p-n+i_1} \) has \((y_1, \ldots, y_{p-n+i_1})\). Then we identify an element of \( \text{Hom}(\mathbb{O}^j n(I), Q)|_U \) with polynomials \( y_i(c) = \sum_{|\omega|=j} a_\omega^i(c) x_1^{\omega_1} x_2^{\omega_2} \cdots x_{c(I)}^{\omega_{c(I)}}, \ c \in U, \) where \( \omega = (\omega_1, \omega_2, \ldots, \omega_{c(I)}), \ \omega_i \geq 0 \ (i = 1, \ldots, p-n+i_1), \) and \( |\omega| = \omega_1 + \cdots + \omega_{c(I)} \) and \( a_\omega^i(c) \) are real numbers. If \( a_\omega^i(c) \) are smooth functions of \( c, \) then \( a_\omega^i(c) \) defines a smooth section of \( \text{Hom}(\mathbb{O}^j n(I), Q)|_S^I \) over \( U. \)

We first introduce several homomorphisms between vector bundles over \( S^I(s) \), which are used for the construction of the required \( \Omega \)-regular map in Theorem 4.1.

Let \( s \in \Gamma^p_{I_1}(N, P). \) By deforming \( s \) if necessary, we may assume without loss of generality that \( s \) satisfies (2) of Lemma 5.1 and (2) of Lemma 5.2, where \( s_1 \) is replaced by \( s. \)

In the following, let \( K_j \) \((j \geq 1)\) refer to \( K_j(S^I(s)) \). Let \( K_j/K_{j+1} \) refer to the orthogonal complement of \( K_{j+1} \) in \( K_j, T^I_j \) refer to the orthogonal complement of \( K_j/K_{j+1} \) in \( n(s, I_j \subset I_{j-1}) \), and \( P^I_j = (s|S^I)^* P_j. \) Then we have the following isomorphism by (2.2)
\[
(6.1) \quad (s|S^I)^* (d_{s} \circ ds|n(s, I_j \subset I_{j-1})): \quad n(s, I_j \subset I_{j-1}) = K_j/K_{j+1} \oplus T^I_j \rightarrow P^I_j \quad (1 \leq j \leq k).
\]

We first define the section \( q'(s, I)^1 \) of \( \text{Hom}(n(I), \text{Im}(d_1(s, I))) \) over \( S^I(s) \) defined by \( q'(s, I)^1 = d_1(s, I)|n(I) \) which vanishes on \( K_1|S^I \) and gives an isomorphism of \( \oplus_{j=1}^{k} T^I_j \) onto \( \text{Im}(d_1(s, I)). \)

For \( 1 \leq j \leq k, q(k)^j|s|^{j+1,j+1} \) in (3.1) induces the homomorphism
\[
(6.2) \quad q(s, I)^{j+1}: n(s, I_j \subset I_{j-1}) \cap K_j \cap K_{j-1} \cap \cdots \cap K_1 \rightarrow Q \quad \text{over } S^I(s) \text{ defined as the composition}
\]
\[
((s|S^I)^* q(k)^{j+1,j+1}) \circ ((n(s, I_j \subset I_{j-1})) \cap n(s, I_j \subset I_{j-1})) \circ id_{K_j \cap K_{j-1} \cap \cdots \cap K_1}.
\]

Furthermore, we define the following section of \( \text{Hom}(\Sigma_{j=1}^{k+1} n(I), Q) \)
\[
(6.3) \quad q'(s, I) = \Sigma_{j=1}^{k} q(s, I)^{j+1} \quad \text{over } S^I(s).
\]

Let us consider the direct sum decompositions
\[
n(s, I) = \oplus_{j=1}^{k} n(s, I_j \subset I_{j-1}), \quad n(s, I_j \subset I_{j-1}) = K_j/K_{j+1} \oplus T^I_j,
\]
\[
K_1 = \oplus_{j=1}^{k-1} K_j/K_{j+1} \oplus K_k, \quad (\pi_{\infty}^P \circ s|S^I)^* (TP) = Q \oplus Q^\perp
\]
and the inclusion \( i_Q: Q \rightarrow (\pi_{\infty}^P \circ s|S^I)^* (TP). \) Then we obtain the smooth fiber map
\[
(6.4) \quad q(s, I) = (\pi_{\infty}^P \circ s|S^I)^* TP \circ (i_Q \circ q'(s, I) + q'(s, I)^1): n(s, I) \rightarrow TP
\]
covering the immersion \( \pi_{\infty}^P \circ s|S^I(s): S^I(s) \rightarrow P \) such that for any \( c \in S^I(s), \ q(s, I)_c \) is regarded as \( p - n + c(I) \) polynomials of \( c(I) \) variables with constant 0.

**Proof of Theorem 4.1.** By Lemmas 5.1 and 5.2 we may assume that \( s \) satisfies (2) of Lemma 5.1 and (2) of Lemma 5.2, where \( s_1 \) is replaced by \( s. \) We define \( E(S^I) \) to be the union of all \( \exp_N D_{\delta \alpha}(n(I)) \), where \( \delta : \Sigma^I(N, P) \rightarrow \mathbb{R} \) is a sufficiently small positive function such that \( \delta \circ s|(S^I \setminus \text{Int} U(C(I^+)))_2) \) is constant. This is a tubular neighborhood of \( S^I. \)

It is enough for the proof of Theorem 4.1 to prove the following assertion:

(A) There exists a homotopy \( H_A \) relative to \( U(C(I^+))_{2-r_0} \) in \( \Gamma_{\Omega}^p(N, P) \) with \( H_0 = s \) satisfying the following.
(1) \( S^l(H_\lambda) = S^l \) for any \( \lambda \).

(2) We have an \( \Omega \)-regular map \( G \) defined on a neighborhood of \( U(C(I^+))_{2-r_0} \to P \) such that \( j^\infty G = H_1 \) on \( U(C(I^+))_{2-r_0} \cup E(S^l) \).

By the Riemannian metric on \( P \), we identify \( Q \) with the orthogonal \( p - n + \imath_1 \) dimensional bundle of \( \text{Im}(d_1(s, I)) \) in \( (\pi^\infty_P \circ s|S^l)^\ast(T P) \). Then the map \( \text{exp}_P \circ (\pi^\infty_P \circ s|S^l)^\ast(T P) | D_\gamma(Q) \) is an immersion for some small positive function \( \gamma \). In the proof we express a point of \( E(S^l) \) as \( v_c \), where \( c \in S^l \), \( v_c \in N(I) \), and \( \| v_c \| \leq \delta(s(c)) \). In the proof we say that a smooth homotopy

\[
k_\lambda : (E(S^l), \partial E(S^l)) \to (\Omega(N, P), \Omega(N, P) \setminus \Sigma^l(N, P))
\]

has the property (C) if it satisfies that for any \( \lambda \)

(C-1) \( k_\lambda^\ast|\Sigma^l(N, P) = S^l \), and \( \pi^\infty_P \circ k_\lambda|S^l = \pi^\infty_P \circ k_0|S^l \) and, \( k_\lambda \) is smooth and transverse to \( \Sigma^l(N, P) \).

If we choose \( \delta \) sufficiently small compared with \( \gamma \), then we can define the \( \Omega \)-regular map \( g_0 : E(S^l) \to P \) by

\[
g_0(v_c) = \text{exp}_{P, \pi^\infty_P \circ s|c} \circ q(s, I) \circ \text{exp}^{-1}_{N, c}(v_c).
\]

It follows from Section 2 that \( g_0 \) has each point \( c \in S^l \) as a singularity of the symbol \( I \) and vice versa. Now we need to modify \( g_0 \) by using Lemma 5.3 so that \( g_0 \) is compatible with \( g(I^+) \). Let \( \eta : S^l \to R \) be a smooth function such that

(i) \( 0 \leq \eta(c) \leq 1 \) for \( c \in S^l \),

(ii) \( \eta(c) = 0 \) for \( c \in S^l \cap U(C(I^+))_{2-r_0} \),

(iii) \( \eta(c) = 1 \) for \( c \in S^l \setminus U(C(I^+))_{2-r_0} \).

Then consider the map \( G : U(C(I^+))_{2-r_0} \cup E(S^l) \to P \) defined by

\[
\begin{cases}
G(x) = g(I^+)(x) & \text{if } x \in U(C(I^+))_{2-r_0}, \\
G(v_c) = (1 - \eta(c))g(I^+)(v_c) + \eta(c)g_0(v_c) & \text{if } v_c \in E(S^l).
\end{cases}
\]

It follows from Lemma 5.3 that \( G \) is an \( \Omega \)-regular map defined \( U(C(I^+))_{2-r_0} \cup E(S^l) \), that \( G|E(S^l) \) has the singularities of the symbol \( I \) exactly on \( S^l \), and that for any \( c \in S^l \), the assumptions (i)-(iv) of Lemma 5.3 holds for \( f_1 = g(I^+) \) and \( f_2 = g_0 \).

Set \( \text{exp}_0 = \text{exp}_{\Omega(N, P)} \) for short. Let \( h_1^0 \) and \( h_0^3 \) be the maps \( (E(S^l), S^l) \to (\Omega(N, P), \Sigma^l(N, P)) \) defined by

\[
h_1^0(v_c) = \text{exp}_{\Omega, s|c} \circ d_c \circ (\text{exp}_{N, c})^{-1}(v_c),
\]

\[
h_0^3(v_c) = \text{exp}_{\Omega, j^\infty G} \circ d_c \circ (j^\infty G) \circ (\text{exp}_{N, c})^{-1}(v_c).
\]

By applying Lemma 5.4 to the section \( s \) and \( h_1^o \), we first obtain a homotopy \( h_\lambda \in \Gamma^N_0(E(S^l), P) \) between \( h_0^3 = s \) and \( h_1^0 \) on \( E(S^l) \) satisfying the properties (1), (2) and (3) of Lemma 5.4. Similarly we obtain a homotopy \( h_\lambda^3 \in \Gamma^N_0(E(S^l), P) \) between \( h_0^3 \) and \( h_\lambda^3 = j^\infty G \) on \( E(S^l) \) satisfying the properties (1), (2) and (3) of Lemma 5.4.

Next we construct a homotopy of bundle maps \( n(I) \to \nu(\Sigma^l) \) covering a homotopy \( S^l \to \Sigma^l(N, P) \) between \( ds|n(I) \) and \( d(j^\infty G)|n(I) \). Let us recall the additive structure of \( J^\infty(N, P) \) in (1.2). Then we have the homotopy \( \kappa_\lambda : S^l \to J^\infty(N, P) \) defined by

\[
\kappa_\lambda(c) = (1 - \lambda)s(c) + \lambda j^\infty G(c) \quad \text{covering } \pi^\infty_P \circ s|S^l : S^l \to P,
\]
where \( \pi^C_\infty \circ s | S^I \) is the immersion as in (2) of Lemma 5.2. We show that \( \kappa_\lambda \) is actually a homotopy of \( S^I \) into \( \Sigma^I(N, P) \). Under the identification \( (s)^* P \cong (\pi^C_\infty \circ s)^*(T P) \) and \( s^* D \cong TN \), it follows from the decomposition of \( n(I) \) in (4.3) that

\[
(6.6) \quad (s|S^I)^*(d_{j+1} \circ d|n(I_j \subset I_{j-1})) = (j^\infty G|S^I)^*(d_{j+1} \circ d(j^\infty G)|n(I_j \subset I_{j-1}))
\]

to \( S^I \). These formulas are the direct consequence of the construction of \( q(s, I) \) used in the definition of \( G \) and the definition of the intrinsic derivatives in Sections 2 and 3. By (6.6) we have that \( n(\kappa_\lambda, I)_c = n(I)_c \) and \( Q(\kappa_\lambda)_c = Q_c \) for any \( c \in S^I \). Hence, it follows that the equalities of the homomorphisms in (6.6) also hold when \( s \) is replaced by \( \kappa_\lambda \). This implies that \( \kappa_\lambda \) is a homotopy into \( \Sigma^I(N, P) \).

Hence, the homotopy \( (\kappa_\lambda)^*(\nu(\Sigma^I)) : \kappa_\lambda^*(\nu(\Sigma^I)) \to \nu(\Sigma^I) \), \( ds \) and \( d(j^\infty G) \) induce the homotopy of bundle maps \( \tilde{\kappa}_\lambda : n(I) \to \nu(\Sigma^I) \) covering \( \kappa_\lambda \) such that \( \tilde{\kappa}_0 = ds \) and \( \tilde{\kappa}_1 = d(j^\infty G) \). We define the homotopy \( h^\lambda_2 : (E(S^I), S^I) \to (\Omega(N, P), \Sigma^I(N, P)) \) by

\[
h^\lambda_2(v_c) = \exp_{\Omega, s(c)}(\tilde{\kappa}_\lambda) \circ (\exp_{N, c})^{-1}(v_c).
\]

Then we have that \( h_0^\lambda(v_c) = h_1^\lambda(v_c) = \exp_{\Omega, s(c)}(\tilde{\kappa}_\lambda) \circ (\exp_{N, c})^{-1}(v_c) \) and \( h_0^\lambda(v_c) = h^\lambda_2(v_c) = \exp_{\Omega, s(c)}(\tilde{\kappa}_\lambda) \circ (\exp_{N, c})^{-1}(v_c) \) for any \( v_c \in E(S^I) \).

By (6.6) we have that \( h^\lambda_2(v_c) = \exp_{\Omega, s(c)}(\tilde{\kappa}_\lambda) \circ (\exp_{N, c})^{-1}(v_c) \) and \( h_0^\lambda(v_c) = h^\lambda_2(v_c) = \exp_{\Omega, s(c)}(\tilde{\kappa}_\lambda) \circ (\exp_{N, c})^{-1}(v_c) \) on \( E(S^I) \).

Let \( \tilde{h}_\Lambda \in \Gamma^r_{\Omega}(E(S^I) \cup U(C(I^+)_{2-3r_0}), P) \) be the homotopy which is obtained by pasting \( h^\Lambda_1 \), \( h^\Lambda_2 \) and \( h^\Lambda_3 \). The homotopies \( h^\Lambda_1 \) and \( h^\Lambda_3 \) are not homotopies relative to \( E(S^I) \cap U(C(I^+)_{2-3r_0}) \) in general. By using the above properties of \( h^\Lambda_1 \), \( h^\Lambda_2 \) and \( h^\Lambda_3 \), we can modify \( \tilde{h}_\Lambda \) to a homotopy \( h_\Lambda \in \Gamma^r_{\Omega}(E(S^I), P) \) satisfying the property (C) such that \( h_\Lambda(v_c) = h_0(v_c) = s(v_c) \) for any \( \lambda \) and any \( v_c \in E(S^I) \cap U(C(I^+)_{2-2r_0}) \).

By (1), we can extend \( h_\Lambda \) to the homotopy \( H_\Lambda \in \Gamma^r_{\Omega}(E(S^I) \cup U(C(I^+)_{2-3r_0}), P) \) defined by \( H_\Lambda|E(S^I) = h_\Lambda \) and \( H_\Lambda|U(C(I^+)_{2-2r_0}) = s(U(C(I^+)_{2-2r_0}) \).

By the transversality of \( H_\Lambda \) and the homotopy extension property to \( s \) and \( H_\Lambda \), we obtain an extended homotopy

\[
H_\Lambda : (N, S^I) \to (\Omega(N, P), \Sigma^I(N, P))
\]

relative to \( U(C(I^+)_{2-3r_0}) \) with \( H_0 = s \). Furthermore, we replace \( \delta \) and \( E(S^I) \) by smaller ones. Then \( H_\Lambda \) is a required homotopy in \( \Gamma^r_{\Omega}(N, P) \) in the assertion (A). \( \square \)

7. Proof of Theorem 0.2

In this section we prove Theorem 0.2 by applying Theorem 0.1.

**Proposition 7.1.** Under the same assumption of Theorem 0.2, any section \( s \in \Gamma^r_{\Omega}(N, P) \) has a homotopy \( s_\lambda \in \Gamma^r_{\Omega}(N, P) \) such that

1. \( s_0 = s \),
2. \( s_1 \) is a section of \( \Omega^I(N, P) \) over \( N \),
3. \( S^I(s_\lambda) = S^I(s) = S^I(s_1) \) for any \( \lambda \).

We need the following lemma for the proof of Proposition 7.1.
Lemma 7.2. Assume the same assumption of Theorem 0.2. Then, we have

1. \( Q_1|_{\Sigma^r(N,P)} \) and \( \bigcirc^2 K_r|_{\Sigma^r(N,P)} \) are trivial line bundles equipped with the canonical orientations respectively.

2. The homomorphisms \( e_j|_{\text{Hom}(\bigcirc^j K_r, Q_1)} : \text{Hom}(\bigcirc^j K_r, Q_1) \rightarrow P_j \) (\( 1 \leq j \leq r \)) and \( e_{j-1} \circ c_{j-1} : \text{Hom}(\bigcirc^{j-1} K_r, Q_1) \rightarrow Q_j \) (\( 1 < j \leq r \)) are injective over \( \Sigma^l(N,P) \).

Proof. (1) By Section 2 (5), \( d_2|K_1 : K_1 \rightarrow \text{Hom}(K_1, Q_1) \) induces the isomorphism

\[
K_1/K_2 \rightarrow \text{Hom}(K_1/K_2, Q_1) \quad \text{over } \Sigma^l(N,P).
\]

This yields \( q : K_1/K_2 \circ K_1/K_2 \rightarrow Q_1 \) over \( \Sigma^l(N,P) \), which is a nonsingular quadratic form on each fiber. Since \( \dim K_1/K_2 = n - p + 1 - i_2 \) is odd, we choose the unique orientation of \( Q_1 \), expressed by the unit vector \( e_p \), so that the index (the number of the negative eigenvalues) of \( q_z \), \( z \in \Sigma^l(N,P) \) is less than \( (n - p + 1 - i_2)/2 \).

Since \( K_r|_{\Sigma^r(N,P)} \) is a line bundle, \( \bigcirc^2 K_r|_{\Sigma^r(N,P)} \) has the canonical orientation.

(2) We prove the assertion by induction on \( j \) (\( r \geq 3 \)). Let \( j = 1 \). Since the kernel of \( d_2|K_1 \) is \( K_2 \), we have that \( c_1 = u_1 \) induces the inclusion \( \text{Hom}(K_r, Q_1)|_{\Sigma^r} \subset \text{Hom}(K_1, Q_1)|_{\Sigma^r} = P_1|_{\Sigma^r} \) and \( e_1|_{\Sigma^r} : P_1|_{\Sigma^r} \rightarrow Q_2|_{\Sigma^r} \) is identified with the restriction \( \text{Hom}(K_1, Q_1)|_{\Sigma^r} \rightarrow \text{Hom}(K_2, Q_1)|_{\Sigma^r} \). Hence, \( e_1 \circ c_1 (\text{Hom}(K_r, Q_1)|_{\Sigma^r}) \) is injective. Suppose that \( e_{j-2} \circ c_{j-2}|_{\text{Hom}(\bigcirc^{j-2} K_r, Q_1)} \) is injective into \( Q_{j-1} \) over \( \Sigma^l(N,P) \) for \( j - 2 < r \). Then it follows from the definition of \( u_{j-1} \) that \( c_{j-1}|_{\text{Hom}(\bigcirc^{j-1} K_r, Q_1)} \) is injective into \( \text{Hom}(K_{j-1}, Q_{j-1}) \) over \( \Sigma^l(N,P) \). Since the image of \( c_{j-1} \) is \( P_{j-1} \), the map \( c_{j-1}|_{\text{Hom}(\bigcirc^{j-1} K_r, Q_1)} \) is injective into \( P_{j-1} \). Since \( d_j|K_r \) vanishes for \( j \leq r \) over \( \Sigma^l(N,P) \) and \( d_j|K_r \) is symmetric, the composition \( e_{j-1} \circ c_{j-1}|_{\text{Hom}(\bigcirc^{j-1} K_r, Q_1)} \) is injective into \( Q_j \) over \( \Sigma^l(N,P) \) for \( j \leq r \). Thus the map \( c_j|_{\text{Hom}(\bigcirc^K K_r, Q_1)} \) is injective into \( P_j \) for \( j < r \). This proves the lemma.

Proof of Proposition 7.1. In the proof we identify \( J^k(N,P) \) with \( J^k(T^l(N,P)) \) by (1.2). By (9) in Section 2, there exists the open subbundles \( \Omega^l(N,P) \) of \( J^k(N,P) \) such that \( (\pi^o)^{-1}(\Omega^l(N,P)) = \Omega^l(N,P) \) for \( L \) with length \( k \). It follows that \( (\pi^o \circ s)(N \setminus (S^l(s))) \subset \Omega^l(N \setminus (S^l(s)), P) \).

We now construct a new section \( \tilde{u} : N \rightarrow \Omega^l(N,P) \) as follows.

Let \( e_j(Q) \) and \( e(\bigcirc^{r+1}(K_r)_{\Sigma^r}) \) be the oriented vectors induced from \( e_j(Q)_{\Sigma^r} \), \( e(\bigcirc^{r+1}(K_r)_{\Sigma^r}) \) by \( s \) respectively. Then we define the section \( \phi^j : S^l(s) \rightarrow \text{Hom}(\bigcirc^{r+1} K_r, Q) \) by \( \phi^j(e)(e(\bigcirc^{r+1} K_r)) = e_j(Q)_{\Sigma^r} \). Then we can extend \( \phi^j \) to a section \( u_{\phi^j} : S^l(s) \rightarrow \text{Hom}(\bigcirc^{r+1}((\pi^o \circ s)(T(N)), (\pi^o \circ s)^*(TP))) \) such that \( u_{\phi^j}(c) \bigcirc^{r+1} K_r \) is \( \phi^j \) for \( c \in S^l(s) \). Since \( S^l(s) \) is a closed submanifold and since \( \text{Hom}(\bigcirc^{r+1}((\pi^o \circ s)^*(T(N)), (\pi^o \circ s)^*(TP))) \) is a vector bundle, we extend \( u_{\phi^j} \) arbitrarily to the section \( u_{\phi^j} : N \rightarrow \text{Hom}(\bigcirc^{r+1}((\pi^o \circ s)^*(T(N)), (\pi^o \circ s)^*(TP))) \). Then we define \( \tilde{u} \) by \( \tilde{u} = \pi^o \circ s + \tilde{u}_{\phi^j} \) as the section of \( J^{r+1}(N,P) = J^{r+1}(T^l(N,P)) \). We lift \( \tilde{u} \) to the section \( s^j \) of \( J^k(N,P) \) over \( N \). Then we have that \( \pi^o \circ s^j = \tilde{u} \) and \( \pi^o \circ s^j = \pi^o \circ s \). Furthermore, we define the homotopy \( s_\lambda \in \Gamma_{\Omega^l}(N,P) \) by

\[
s_\lambda = (1 - \lambda)s + \lambda s^j.
\]

It follows from \( \pi^o \circ s_\lambda = \pi^o \circ s = \pi^o \circ s \) that \( s_\lambda \) is transverse to \( \Sigma^l(N,P) \) and \( S^l(s_\lambda) = S^l(s^j) = S^l(s) \).
We prove that \( s^J \in \Omega^J(N,P) \). For any point \( c \in S^J(s) \), let \( U_c \) be a convex neighborhood of \( c \) and let \( k \) and \( y_p \) be the coordinates of \( \exp_{N,c}(K_r,c) \) and \( \exp_{P,s^J(c)}((\pi_P^\infty \circ s^J)^TP(Q)c) \) respectively. Let \( D_k \) denote the vector of the total tangent bundle \( D \) which corresponds \( k \) defined in \([B, \text{definition 1.6}]\). It follows from the definition of \( D \) that
\[
(\bigcirc^{r+1}D_k)y_p|_{s^J(c)} = \partial^{r+1}y_p/\partial k^{r+1}(c) \neq 0 \quad \text{for} \quad c \in S^J(s).
\]

Then it follows from Lemma 7.2 (2) that
\[
d_{r+1,s^J(c)} \big| K_{r,s^J(c)} : K_{r,s^J(c)} \to P_{r,s^J(c)} \supset \text{Hom}(\bigcirc^r K_{r,s^J(c)}, Q_{s^J(c)})
\]
is injective. Hence, we have that \( s^J(S^J(s)) \subset \Sigma^J(N,P) \). Since \( S^J(N \setminus (S^J(s)) \subset \Omega^{J-1,0}(N,P) \), the assertion is proved. This proves the proposition.

**Proof of Theorem 0.2.** By the assumption, \( j^\infty f \) is the section \( N \to \Omega^J(N,P) \). By Proposition 7.1, we have the section \( s^J : N \to \Omega^J(N,P) \) such that \( \pi_P^\infty \circ s \) and \( \pi_P^\infty \circ s^J \) are homotopic. By Theorem 0.1 we obtain an \( \Omega^J \)-regular map \( g \) such that \( j^\infty g \) and \( s^J \) are homotopic. This proves the assertion.

**Corollary 7.3.** Let \( n \geq p \geq 2 \), and \( N \) and \( P \) be as above. Let \( I = (n-p+1,1,1) \) and \( J = (n-p+1,1,0) \) such that \( n-p \) and \( r \) \((r \geq 3)\) are odd integers. Then if \( f : N \to P \) is an \( \Omega^J \)-regular map with \( j^\infty f \in \Gamma^J(N,P) \), then \( f \) is homotopic to an \( \Omega^J \)-regular map \( g : N \to P \) such that \( j^\infty f \) and \( j^\infty g \) are homotopic in \( \Gamma^J(N,P) \) and that \( S^J(j^\infty f) = S^J(j^\infty g) \).

This corollary proves the Chess conjecture ([C]). Sadykov [Sady] has actually proved this corollary for \( J = (n-p+1,1,0) \) in the case of \( N \) and \( P \) being orientable.

Let \( \pi_0(X) \) be the arcwise connected components of \( X \). Theorem 0.1 asserts that
\[
(j_{\Omega^J})_* : \pi_0(C^\infty_{\Omega^J}(N,P)) \to \pi_0(\Gamma_{\Omega^J}(N,P))
\]
is surjective. However, \((j_{\Omega^J})_*\) is not necessarily injective. Let \( N = S^2 \), \( P = \mathbb{R}^2 \) and \( I = (1,0) \). Then we have by [An3] that \( \Omega^{1,0}(2,2) \) is homotopy equivalent to \( SO(3) \). It follows from [Ste, 36.4] that every two sections of \( \Omega^{1,0}(S^2, \mathbb{R}^2) \) over \( S^2 \) are mutually homotopic. Namely, \( \pi_0(\Gamma_{\Omega^J}(N,P)) \) consists of a single element. On the other hand, let \( f_\lambda : S^2 \to \mathbb{R}^2 \) be a homotopy of fold-maps. Define \( F : S^2 \times [0,1] \to \mathbb{R}^2 \) by \( F(x, \lambda) = f_\lambda(x) \) so that if \( \lambda \) is sufficiently small, then \( F(x, \lambda) = f_0(x) \) and \( F(x, 1 - \lambda) = f_1(x) \). By a very small perturbation of \( F \) fixing \( f_0 \) and \( f_1 \), we may assume that \( F \) is smooth and \( f_\lambda \) is still an \( \Omega^{1,0} \)-regular map for any \( \lambda \). Furthermore, the map \( F : S^2 \times [0,1] \to \mathbb{R}^2 \times [0,1] \) becomes an \( \Omega^{1,0} \)-regular map, and \( S^1,0(F) \) is a submanifold of \( S^2 \times [0,1] \). By the Jacobian matrix of \( F \) we know that the kernel line bundle \( K_1(j^\infty F) \) over \( S^1,0(F) \) is independent with \( \partial/\partial \lambda \), and \( T(S^1,0(F)) \cap K_1(j^\infty F) = \{0\} \). This implies that \( S^1,0(F) \) is regularly projected onto \([0,1]\). Hence, \( S^1,0(f_0) \) must be diffeomorphic to \( S^1,0(f_1) \). Thus we conclude that \( \pi_0(C^\infty_{\Omega^J}(S^2, \mathbb{R}^2)) \) is an infinite set.

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