Localization of Hidden Attractors in Chua’s System With Absolute Nonlinearity and Its FPGA Implementation

Xianming Wu¹, Huihai Wang² and Shaobo He²*¹

¹School of Mechanical and Electrical Engineering, Guizhou Normal University, Guiyang, China, ²School of Physics and Electronics, Central South University, Changsha, China

Investigation of the classical self-excited and hidden attractors in the modified Chua’s circuit is a hot and interesting topic. In this article, a novel Chua’s circuit system with an absolute item is investigated. According to the mathematical model, dynamic characteristics are analyzed, including symmetry, equilibrium stability analysis, Hopf bifurcation analysis, Lyapunov exponents, bifurcation diagram, and the basin of attraction. The hidden attractors are located theoretically. Then, the coexistence of the hidden limit cycle and self-excited chaotic attractors are observed numerically and experimentally. The numerical simulation results are consistent with the FPGA implementation results. It shows that the hidden attractor can be localized in the digital circuit.

Keywords: Chua’s system, hidden attractor, FPGA, coexisting limit cycle, chaos

1 INTRODUCTION

According to Jenkins [1] and Kuznetsov et al. [2], there are two kinds of attractors, namely, the self-excited attractors and the hidden attractors. It can be distinguished by considering whether the basin of attraction of an attractor intersects any arbitrarily small open neighborhood of an equilibrium. If it does, the attractor is a self-excited attractor; otherwise, it is hidden. Generally, the “self-excited” attractors are generated in a system from a small vicinity of an unstable equilibrium. However, for instance, the hidden attractor can be generated by a system without equilibria [3–5] or with one stable zero equilibrium [6–8]. Meanwhile, there is a concept in the nonlinear systems, which is the “extreme multistability” [9–12]. Those attractors are obtained in a multistability system with given parameters but using different initial conditions. Although the definitions of the hidden attractors and coexisting attractors are not the same, they can be observed using different initial conditions. Currently, investigation of the multiple coexisting hidden attractors in the nonlinear systems is a research hotspot [13–16]. It means that multiple coexisting attractors and hidden attractors are both found in the system with different initial conditions [17].

Currently, the multiple coexisting attractors can be analyzed by using the basin attraction plots [18, 19] in the initial condition plane in which different initial conditions are used, and characteristics of the attractors such as size, center of gravity, and bounds are considered. Usually, different colors represent different kinds of attractors in the initial condition plane. Also, there could be some rules for choosing the initial conditions for systems, such as the symmetric coexisting chaotic systems [20–22] and the self-reproducing chaotic systems [23, 24]. However, as the basin of attraction of the hidden attractors is not connected with equilibria, it should be noted that the location of hidden
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Attractors could be a challenging task. At present, hidden attractors have been widely investigated by researchers such as Leonov et al. [8, 25, 26], Sajad et al. [27], Dudkowski et al. [28], Zhang et al. [29], and Stankevich et al. [30]. Especially, the method by Leonov et al. [8, 25, 26] can locate the exact positions of the hidden attractors for the given systems. In real applications, dynamics of the systems with hidden attractors is determined not only by the parameters but also by the initial conditions.

In 1984, a system was derived based on Chua’s circuit [31] with two capacitors, one inductor, one resistor, and one nonlinear element, and it shows that its attractor is different from that of the Lorenz system and Chen system. In 1986, the existence of chaotic attractors in Chua’s circuit was proven mathematically [32]. Because of its rich dynamic characteristic behaviors and plasticity of the nonlinear item, Chua’s circuit has aroused much interest of scholars. Zhong et al. [33] implemented Chua’s circuit with a cubic nonlinearity. Moreover, the bifurcation, chaos, hidden attractors, and synchronization of different Chua’s circuits were investigated in Refs. [34–36]. Meanwhile, designing multiscroll chaotic systems based on Chua’s system by expanding the index two saddle coke equilibrium points has aroused, increasing research interests [37, 38]. Furthermore, hidden attractors and coexisting attractors in Chua’s circuit have been widely investigated [39, 40]. By replacing the nonlinear element in Chua’s circuit with memristor, memristor-based Chua’s circuits are designed, and then dynamic analysis and applications of those circuits are carried out [41–43]. In conclusion, Chua’s circuit provides a basis for designing nonlinear circuits with complex dynamic behaviors.

In the real applications, to find hidden chaotic attractors, we need to use the particular initial conditions. Thus, analog circuit implementation for hidden attractors becomes a key issue. Although analog circuit implementation of chaotic systems can generate real chaos, the initial conditions cannot be set, artificially and accurately. Compared with analog circuit implementation, digital circuit implementation of chaotic systems has the advantages such as reproducibility, good stability, and good controllability. Meanwhile, the initial conditions can be set accordingly. As a result, digital circuit implementation including DSP implementation [44], FPGA implementation [45], and microcontroller implementation [46] of chaotic systems has aroused much research interest. However, there are few reports regarding digital circuit implementation of hidden chaotic systems with observation of hidden attractors in the oscilloscope.

Motivated by the above discussions, in this article, a Chua’s system with an absolute item is investigated. Then, the dissipativity, equilibria, stability, and Hopf bifurcation diagram are studied, and the coexistence hidden attractors are investigated theoretically and demonstrated by numerical simulation and FPGA circuit realization. Specifically, we will use the method proposed by Leonov et al. [8, 25, 26] to find the initial conditions for the hidden attractors. Phase diagram and basin attraction plots are used to verify the effectiveness of the analyses.

The rest of this article is organized as follows. In Section 2, the modified Chua’s system is presented, and its equilibrium analysis, Hopf bifurcation analysis, bifurcation diagram, Lyapunov exponents, coexisting attractors, and basin attractions are analyzed. In Section 3, location of the hidden attractors in Chua’s circuit system is investigated theoretically and numerically. In Section 4, FPGA implementation of Chua’s system is carried out, and hidden attractors are observed. Finally, the results are summarized in Section 5.

2 THE MODIFIED CHUA’S SYSTEM AND ITS DYNAMICS

The absolute circuit $x|\mathbf{x}|$ can increase the nonlinearity of the circuit. For instance, Tang et al. [47, 48] introduced the absolute function $x|\mathbf{x}|$ to Chua’s system and Chen system and show the role of the function as a chaos generator in nonautonomous systems. Here, a Chua’s system with absolute item is investigated, and the system is defined by [47, 48]

$$
\begin{cases}
\dot{x} = \alpha (y + mx - x|\mathbf{x}|) \\
\dot{y} = x - cy + z \\
\dot{z} = -\beta y
\end{cases}
$$

where $\alpha, \beta, c$, and $m$ are the system parameters, and $x, y, z$ are the state variables.

2.1 Basic Properties and Equilibria

2.1.1 Dissipativity

According to system (1), we have

$$
\mathbf{VV} = \nabla V \cdot \mathbf{V} = \nabla V \cdot \mathbf{V} = \alpha \left( m - |x| - x\text{sgn}(x) - \frac{1}{\alpha} \right).
$$

Obviously, when $m < \frac{1}{\alpha}$, we get $\mathbf{V} \mathbf{V} < 0$. It means that the system is dissipative. At this point, the solution of system (1) is bounded and could be chaotic.

2.1.2 Symmetry

As system (1) satisfies $(x, y, z) \rightarrow (-x, -y, -z)$, it is symmetric to the origin in the state variable space.

2.1.3 Equilibria and Stability

The equilibrium points of the system are $(0, 0, 0)$ and $(\pm m, 0, \mp m)$.

**Theorem 1.** If $\alpha > 0, \beta > 0, m > 0$ hold, the equilibria point $O(0, 0, 0)$ is unstable.

**Proof.** The Jacobian matrix of system (1) at the equilibria point $O(0, 0, 0)$ is given by

$$
J_{O} = \begin{bmatrix}
\alpha m & \alpha & 0 \\
1 & -c & 1 \\
0 & -\beta & 0
\end{bmatrix}.
$$

Then, its characteristic equation is denoted as

$$
P_{O}(\lambda) = \lambda^{3} + \lambda^{2} (c - \alpha m) + \lambda (\beta - \alpha - \alpha m) - \alpha \beta m = 0.
$$

If $\alpha > 0, \beta > 0, m > 0$, we have $-\alpha \beta m < 0$. According to the Routh–Hurwitz criterion, Eq.(4) has at least one positive real solution. It means that the system is unstable at the point $O(0, 0, 0)$ if $\alpha, \beta,$ and $m$ are positive real numbers. End proof.
For the equilibria point \( P_\pm = (\pm m, 0, \mp m) \), the Jacobian matrix is

\[
J_\pm = \begin{bmatrix}
-\alpha m & \alpha & 0 \\
1 & -c & 1 \\
0 & 0 & -\beta 
\end{bmatrix},
\]

and its characteristic equation is

\[
\lambda^3 + \lambda^2 (c + \alpha m) + \lambda (\beta - \alpha + acm) + \alpha \beta m = 0.
\]

Let \( p = c + \alpha m, q = \beta - \alpha + acm, \) and \( r = \alpha \beta m \), we have

\[
\lambda^3 + p\lambda^2 + q\lambda + r = 0.
\]

As \( \alpha > 0, \beta > 0, c > 0, \) and \( m > 0, \) thus \( r > 0 \), thus Eq. 7 has at least one negative root. Let \( f(\lambda) = \lambda^3 + p\lambda^2 + q\lambda + r \), thus \( \frac{df(\lambda)}{d\lambda} = 3\lambda^2 + 2p\lambda + q \). When \( \Delta = \lambda^2 - 3q \) is defined, the following theorem is obtained.

**Theorem 2.** If \( \Delta < 0 \), then the system has no positive roots and it is asymptotically stable. If \( \Delta \geq 0, \lambda_0 = \frac{1}{2} \left( -p + \sqrt{\Delta} \right) > 0 \) and \( f(\lambda_0) \leq 0 \), the system has two positive roots, then \( P_\pm \) are unstable saddle points.

### 2.1.4 Hopf Bifurcation Analysis

According to the above section, the equilibria point \( O(0, 0, 0) \) is unstable. Thus, we focus on the bifurcation near the equilibria point \( P_\pm (\pm m, 0, \mp m) \). Suppose that there is a pair of pure imaginary roots \( i\omega(\omega > 0) \); according to Eq. 7, we have

\[
\begin{bmatrix}
-\omega^3 + \omega (\beta - \alpha + acm) = 0 \\
-\omega^2 (c + \alpha m) + \alpha \beta m = 0
\end{bmatrix}.
\]

Set the system parameter \( c \) as the bifurcation parameter, based on Eq. 8, we obtain

\[
a^2 (cm^2 - m) + a(-c + c^2 m) + c\beta = 0.
\]

When \( \alpha > 0 \), we have the following conclusions: (1) if \( cm \geq 1 \), then Eq. 7 has no positive roots; (2) if \( cm < 1 \), then Eq. 6 has one positive root.

As \( \alpha > 0 \) is the bifurcation parameter, the critical point \( \alpha_{HI} \) satisfies the following equation:

\[
\begin{cases}
\alpha_{HI} = -c (cm - 1) - \sqrt{\Delta} \\
\omega = \sqrt{\frac{\alpha \beta m}{c + \alpha m}}
\end{cases},
\]

where \( \Delta = (-c + c^2 m)^2 - 4(-m + cm^2)c\beta \).

Differentiating Eq. 6 versus \( c \), we have

\[
\frac{d\lambda}{dc} = -\frac{\lambda^2m + \lambda (cm - 1) + \beta m}{3\lambda^2 + 2\lambda (c + \alpha m) + \lambda (\beta - \alpha + acm)}.
\]

Substituting \( \alpha_{HI} \) to Eq. 11 and \( \lambda(\alpha_{HI}) = i\omega \), the calculation result is

\[
\frac{dRe(\lambda)}{da} \bigg|_{a=\alpha_{HI}} = \frac{-\omega^2 (cm - 1) (c + \alpha \beta m)}{2w^4 + 2w^3 (c + \alpha \beta m)^2}.
\]

According to Eq. 12, when \( cm < 1 \), \( \frac{dRe(\lambda)}{da} \bigg|_{a=\alpha_{HI}} > 0 \). Thus, we have the result as presented in Theorem 3.

**Theorem 3.** When the parameter \( \alpha \) passes through the critical value \( \alpha_{HI} \), system (1) undergoes a Hopf bifurcation at equilibria \( P_\pm (\pm m, 0, \mp m) \).

### 2.2 Dynamic Analysis

Set \( \beta = 100, m = 0.2, \) and \( c = 1.6 \), bifurcation diagram and Lyapunov exponents of the system with \( \alpha \) are illustrated in Figure 1. For the bifurcation diagrams, the initial conditions for the red are \([x_0, y_0, z_0] = [0.01, 0, 0.01] \); the initial conditions for the blue is \([x_0, y_0, z_0] = [-0.01, 0, -0.01] \). It shows that when \( \alpha < 70 \), the system has coexisting attractors and enters to chaos through period doubling bifurcation. After this, the system has periodic windows with the increase of \( \alpha \). It indicates that the system has rich dynamics.

### 2.3 Multistability With Symmetry

First, fix \( \alpha = 50, \beta = 100, m = 0.2, \) and \( c = 1.6 \). \( x_0 \) varies from \(-2 to 2 \) with step size of 0.0268, \( y_0 \) varies from \(-0.2 to 0.2 \) with step size of 0.0027, and \( z_0 \) varies from \(-4 to 4 \) with step size of 0.0537. Set \( z_0 = -1 \) and \( \alpha = 50 \). As \( x_0 \) varies from \(-1 to 1 \), set \( x_0 \), \( y_0 \), and \( z_0 \) are illustrated in Figures 2B,C, respectively. Moreover, when \( \alpha = 69 \), let \( z_0 = -1, y_0 = 0.1 \) and \( x_0 = 0.3 \), and \( z_0 \) is coexisting basins of attraction in the \( x_0 - y_0 \) plane, \( x_0 - z_0 \) plane, and \( y_0 - z_0 \) are shown in Figures 2D,E,F, respectively. It shows that the system has three coexisting states with different initial conditions.

Moreover, phase diagrams of Chua’s system with different parameters and initial conditions are presented. The initial conditions for the red attractors are \([x_0, y_0, z_0] = [0.01, 0, 0.01], \) and the initial conditions for the blue attractors are \([x_0, y_0, z_0] = [-0.01, 0, -0.01] \). Phase diagrams with \( \alpha = 60, \alpha = 63, \alpha = 63.8, \alpha = 69, \) and \( \alpha = 75 \) are shown in Figure 3. Coexisting symmetric periodic-1, periodic-2, and periodic-4 attractors are found in Figures 3A–C; coexisting chaotic attractors are observed in Figure 3D. Figures 3E,F do not show coexisting attractors, but different chaotic attractor and periodic circle are found. However, it shows only two coexisting symmetric attractors. Meanwhile, it shows that there are third regions in the basin attractions; thus, there should be one more attractor that is hidden. In the next section, hidden attractors in Chua’s system are investigated.

### 3 HIDDEN ATTRACTORS

#### 3.1 Localization of Hidden Attractors

Leonov et al. [8, 25, 26] proposed a scheme that is used to find the location of the hidden attractors in the chaotic systems. We here use this method to find the initial conditions for the hidden attractors in the system (1). According to the scheme, the system is rewritten as a lure’s system, which is given by
\[
\frac{dx}{dt} = Px + q \psi(r^* x), x \in R^3, \quad (13)
\]
where \( P = \begin{pmatrix} \alpha m & \alpha & 0 \\ 1 & -c & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \quad q = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi(\sigma) = \sigma|\sigma|.
\]
Let \( k \) be the coefficient of harmonic linearization, and \( \varepsilon \) be an infinitesimal number, the system (13) can be rewritten as
\[
\frac{dx}{dt} = P_0 x + q \varepsilon \delta(r^* x), \quad (14)
\]
where \( P_0 = \begin{pmatrix} \alpha (m+k) & \alpha & 0 \\ 1 & -c & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \quad \lambda_{1,2} = \pm i\omega_0, \quad \lambda_3 = -d < 0 \) and
\[ \delta(\sigma) = \psi(\sigma) - k \sigma. \]

Using nonsingular linear transformation \( x = Sy \), the system (14) can be transformed as
\[
\frac{dy}{dt} = H x + b \varphi(u^* y), \quad (15)
\]
where \( H = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix} \) and \( u = \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix}. \)

The transfer function of the system (15) can be expressed as
\[
W_H(p) = \frac{-b_1 p + b_2 \omega_0}{p^2 + \omega_0^2} + \frac{h}{p + d} \quad (16)
\]

The transfer functions of system (14) can be expressed as
\[
W_{P_0}(p) = r^* (P_0 - pI)^{-1} q \quad (17)
\]
where \( p \) is complex variables; \( \omega_0 \) is the initial frequency, which can be calculated by \( \text{Im}W_H(\omega_0) = 0 \); \( k \) is the harmonic linearization coefficient, which can be calculated by \( k = -(R_c W_H(\omega_0))^{-1}. \)
From the equivalence of the transfer functions of system (14) and system (15), then it can be concluded:

\[
\begin{align*}
    k &= (\beta - \alpha(1 + cm) - \omega_0^2)/ac \\
    d &= (\omega_0^2 - \beta + \alpha + c^2)/c \\
    h &= (\alpha \beta - acd + ad^2)/(\omega_0^2 + d^2) \\
    b_1 &= (\alpha \beta - acd - \alpha \omega_0^2)/(\omega_0^2 + d^2) \\
    b_2 &= (\alpha \omega_0^2 + \alpha \beta d - ad \omega_0^2)/(\omega_0^2(\omega_0^2 + d^2))
\end{align*}
\]

(18)

The system (14) is transformed by nonsingular linear transformation; it can be concluded that:

\[
\begin{align*}
    H &= S^{-1}P_0S \\
    b &= S^{-1}q \\
    u^* &= r^* S
\end{align*}
\]

(19)

Let \( S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \), we can obtain

\[
\begin{align*}
    s_{11} &= 1, s_{12} = 0, s_{13} = -h \\
    s_{21} &= -(m + k) \\
    s_{22} &= \omega_0/\alpha \\
    s_{23} &= -h(\alpha(m + k) + d)/\alpha \\
    s_{31} &= \beta/\alpha, s_{32} = -\beta(m + k)/(\alpha \omega_0) \\
    s_{33} &= h(\alpha(m + k) + d)/ad
\end{align*}
\]

(20)

For the infinitesimal number \( \epsilon \), the initial value of system (15) is

\[
\begin{align*}
    y(0) &= \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} a_0 \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
\]

(21)

From Eq. 9, the relationship between the initial values of system (14) and (15) can be obtained

\[
\begin{align*}
    x(0) = S y(0) = S \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} a_0 s_{11} \\ a_0 s_{21} \\ a_0 s_{31} \end{pmatrix}.
\end{align*}
\]

(22)

In this way, the initial value of the system (1) is as follows:

\[
\begin{align*}
    x(0) &= a_0 \\
    y(0) &= -a_0 (m + k) \\
    z(0) &= -a_0 \beta/\alpha
\end{align*}
\]

(23)

here, the description function of \( a_0 \) can be calculated as follows:

\[
\Phi(a) = \int_0^{2\pi/\omega_0} (\varphi_1(t) + \varphi_2(t)) dt
\]

(24)
where \( \varphi_1(t) = \delta_1((\cos \omega_0 t)a, (\sin \omega_0 t)a, 0)\cos \omega_0 t \) and \( \varphi_2(t) = \delta_2((\cos \omega_0 t)a, (\sin \omega_0 t)a, 0)\sin \omega_0 t \). The description function satisfies \( \Phi(a_0) = 0 \) and \( <b/b> <b/b> \cdot \frac{d\Phi(a_0)}{da}|_{a=a_0} \neq 0 <b/b> <b/b> \).

### 3.2 Numerical Verification

Set \( \beta = 100, m = 0.2, \) and \( c = 1.6 \), the initial conditions of hidden attractors for different \( a \) are presented in Table 1. Meanwhile, values of \( \omega_0 \) and \( k \) are listed correspondingly. Based on the initial condition obtained, the coexisting hidden attractors, limited circles, and chaotic attractors are presented in Figure 4. It shows that the hidden attractors are not chaotic, but they are periodic circles.

### 4 FPGA IMPLEMENTATION

In this section, FPGA implementation of Chua’s system is carried out. We implement Chua’s system in Altera DE2-115 with EP4CE115F29C7. Figure 5 shows the physical implementation platform of Chua’s system with absolution item, where the system is solved by the fourth-order Runge-Kutta algorithm. Then, the output of the Altera DE2-115 contains two 16-bit current and voltage signals, which are converted by the DAC8552 chip. The obtained attractors are displayed in the oscilloscope and captured for further analysis.

System chart based on the FPGA implementation is shown in Figure 6. There are three modules including system module, float number to DAC input number module, and DAC output control module. As a result, the generated signals are sent to DAC8552. The total resources occupied by the system are presented in Table 2. The total logic elements, total memory bits, and embedded multiplier 9-bit elements costs are 4%, 0.14%, and 7%, respectively. In addition, the designed system requires very little logic, register, and memory resources.

The obtained phase diagrams are presented in Figure 7. And the parameter and initial conditions for Figure 7A are \( a = 60 \) and \([x_0, y_0, z_0] = [0.3200, -0.1517, 1.0667]\); those for Figure 7B are

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**FIGURE 4** | Coexisting hidden attractors of Chua’s system, where the red attractors and blue attractors are those from Figure 3, and the green attractors are the hidden attractors. (A) \( a = 50, [x_0, y_0, z_0] = [0.3800, -0.2065, 1.5600] \); (B) \( a = 60, [x_0, y_0, z_0] = [0.3200, -0.1517, 1.0667] \); (C) \( a = 63, [x_0, y_0, z_0] = [0.3100, -0.1440, 0.9841] \); (D) \( a = 63.8, [x_0, y_0, z_0] = [0.3100, -0.1433, 0.9718] \); (E) \( a = 69, [x_0, y_0, z_0] = [0.2900, -0.1305, 0.8406] \); (F) \( a = 72, [x_0, y_0, z_0] = [0.2900, -0.1288, 0.8056] \).
\[ \alpha = 63, [x_0, y_0, z_0] = [0.3100, -0.1440, 0.9841]; \text{ those for Figure 7C are } \alpha = 69, [x_0, y_0, z_0] = [0.2900, -0.1305, 0.8406]; \text{ and those for Figure 7D are } \alpha = 72, [x_0, y_0, z_0] = [0.2900, -0.1288, 0.8056]. \]

Meanwhile, for the given parameters, the coexisting periodic circles or chaotic attractors are presented with initial conditions \([x_0, y_0, z_0] = [0.01, 0, 0.01]\) and \([x_0, y_0, z_0] = [-0.01, 0, -0.01]\). For a given \(\alpha\), the captured phase diagrams are put together in the same figure, and coexisting hidden attractors are observed in the FPGA digital circuit.

### TABLE 2 | Compilation resource report

|                   | Values                          |
|-------------------|--------------------------------|
| Total logic elements | 5,052/114,480 (4%)            |
| Total registers   | 3,369                          |
| Total pins        | 6/529 (1%)                     |
| Total memory bits | 5,418/3,981,312 (< 1%)        |
| Embedded multiplier 9-bit elements | 39/532 (7%)       |

\[ \alpha = 63, [x_0, y_0, z_0] = [0.3100, -0.1440, 0.9841]; \text{ those for Figure 7C are } \alpha = 69, [x_0, y_0, z_0] = [0.2900, -0.1305, 0.8406]; \text{ and those for Figure 7D are } \alpha = 72, [x_0, y_0, z_0] = [0.2900, -0.1288, 0.8056]. \]
The main difficulty for analog circuit implementation of hidden attractor chaotic system is to locate the initial conditions. Thus, it is almost impossible to find the target hidden attractors in the analog circuits. However, when the hidden chaotic system is realized in the FPGA circuit, the initial condition can be set; thus, hidden attractors can be observed.

5 CONCLUSION

In this article, dynamics and hidden attractors and modified Chua’s circuit with an absolute item are investigated. The dynamics of the system is analyzed by means of stability analysis, Hopf bifurcation, Lyapunov exponents, bifurcation diagrams, phase diagrams, and basin of attraction plots. It shows that the system has rich dynamics with the variation of the system parameter \( \alpha \), and the system enters to chaos through period-doubling bifurcation. The locations of the hidden periodic circles are found by using the describing function method. The coexisting hidden periodic circles, chaotic attractors, and periodic circles are observed. Finally, the FPGA digital circuit of the system is realized. It shows that the FPGA results are consistent with the numerical simulation results.

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DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

XW, system analysis, draft written; HW FPGA implementation, manuscript revisiement; SH, numerical analysis, checking the whole analysis.

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