A solvable 2d gravity model with $\gamma > 0$

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Abstract

We consider a model of discretized 2d gravity interacting with Ising spins where phase boundaries are restricted to have minimal length and show analytically that the critical exponent $\gamma = 1/3$ at the spin transition point. The model captures the numerically observed behavior of standard multiple Ising spins coupled to 2d gravity.

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1 Introduction

Despite the progress in our understanding of 2d gravity coupled to matter, both in the continuum description \cite{1, 2, 3} and as a statistical model of random triangulations \cite{4, 5, 6}, we have still no understanding of the region which may be most interesting, i.e. matter fields with central charge \( c > 1 \) interacting with 2d gravity. One beautiful feature of 2d gravity coupled to conformal matter theories with \( c < 1 \) is that the influence of matter on the geometry of the 2d universes only depends on the central charge. This is expressed by the KPZ formula \cite{1, 2, 3}. Recent numerical simulations suggest rather surprisingly that the same is true at least for some range of \( c > 1 \) \cite{7}. It is often conjectured that \( c > 1 \) is associated with \( \gamma > 0 \), where \( \gamma \) denotes the string susceptibility. In this letter we describe a solvable model with non-trivial critical behavior for which \( \gamma = 1/3 \) and which explicitly realizes a scenario for \( c > 1 \) advocated in \cite{8} (see also \cite{9}). For other attempts to cross the \( c = 1 \) barrier we refer to \cite{10, 11}.

2 The model

A discretized model of 2d quantum gravity is defined by summing over the triangulations one can obtain by gluing together equilateral triangles along their links to form closed surfaces, and where one assigns the weight to each triangulation dictated by Regge calculus. In the following we will restrict ourselves to triangulations with spherical topology. This means that we can ignore the Einstein-Hilbert term in the action. The cosmological term is proportional to the number of triangles so the partition function is given by:

\[
Z(\mu) = \sum_{T \in \mathcal{T}} \frac{1}{C_T} e^{-\mu N_T} = \sum_{N} e^{-\mu N} \sum_{T \in \mathcal{T}_N} \frac{1}{C_T}
\]  

where \( \mathcal{T} \) denotes the class of triangulations of the sphere\footnote{We find it convenient to consider only non-degenerate triangulations, i.e. each triangle in the surface has three different vertices. In terms of the dual \( \phi^3 \) graph it means that we exclude tadpole graphs.}, \( C_T \) a combinatoric factor present for triangulations of closed surfaces, \( N_T \) the number of triangles in the triangulation \( T \) and \( \mathcal{T}_N \) the subclass of \( \mathcal{T} \) whose triangulations consist of \( N \) triangles. Since

\[
\sum_{T \in \mathcal{T}_N} \frac{1}{C_T} = e^{\mu_0 N} N^\gamma_0 -3 (1 + \mathcal{O}(1/N))
\]  

the model has a critical point \( \mu_0 \) at which the continuum limit has to be taken. It is known that \( \gamma_0 = -1/2 \).
It is easy to couple matter fields to this discretized model. As an example one can couple Ising spins to the geometry by placing one spin $\sigma_i$ on each triangle $i$. The partition function is now given by

$$Z(\mu, \beta) = \sum_{N} e^{-\mu N} \sum_{T \in T_N} \frac{1}{C_T} \sum_{\{\sigma_i\}} \exp \left( \frac{\beta}{2} \sum_{(ij)} (\sigma_i \sigma_j - 1) \right)$$  \hspace{1cm} (3)$$

where $\sum_{(ij)}$ denotes the summation over all pairs of neighboring triangles for a given triangulation $T$ and $\sum_{\{\sigma_i\}}$ the summation over the spin configurations on $T$. We can couple multiple Ising spins to the discretized model in the same way. For a fixed lattice such an extension would be trivial since the models would not interact, but here they interact via their backreaction on the geometry.

The partition function (3) depends on two coupling constants. For each $\beta$ there is a critical point $\mu_c(\beta)$ (and a $\gamma(\beta)$) where the continuum limit has to be taken. In this way the possible candidates for a continuum theory will be labelled by the spin coupling $\beta$. In the case of a single Ising model one can explicitly solve the model \cite{12, 13} and the result is as follows: There is a critical value of $\beta$, $\beta_c$, above which there is spontaneous magnetization and below which the magnetization is zero. Away from $\beta_c$ the backreaction of the Ising spins on the geometry is effectively zero ($\gamma(\beta) = \gamma_0$), while $\gamma(\beta_c) = -1/3$, in agreement with the KPZ formula for a $c = 1/2$ conformal theory. On a regular lattice the spin transition is a second order transition and the corresponding conformal field theory has $c = 1/2$. On the dynamical lattice we see a corresponding transition at $\beta_c$, but the transition is modified by the coupling to gravity, and in addition the backreaction of matter has modified the gravity theory.

For $c > 1$, i.e. for more than two Ising models coupled to gravity, no analytical solution of the model is known. However, numerical simulations indicate the following:

1) There is still a critical point $\beta_c$ below which there is no magnetization and above which the system is magnetized. \cite{13, 16, 17}

2) Above $\beta_c$ the geometry seems to be that of pure gravity. Below $\beta_c$ there is a region where the situation is unclear, especially for a large number of Ising spins, but there seems to be a range of values of $\beta$ for which the fractal structure of the surface is very pronounced. For sufficiently small $\beta$ the geometry again is that of pure 2d gravity \cite{15, 16}.

The mere existence of a critical point in the discretized model does of course not ensure that there is an underlying continuum field theory.
3) For a sufficiently large number of Ising spins $\gamma(\beta_c)$ will be positive \[14, 18, 7\].

4) The geometry at $\beta_c$ seems to be a function of central charge $c$ of the matter system alone, at least for some range of $c > 1$. By this we mean that the effective distribution of geometries is the same no matter whether we consider $2N$ Ising spins at their critical point or $N$ gaussian fields or any other conformal field theory with central charge $c = N$, coupled to gravity \[7\].

In view of point 2) above it seems reasonable to attempt a description of the model (for a large number of Ising copies) in a region around $\beta_c$ and for large $\beta$ in terms of an effective model where baby universes, i.e. parts of the surface connected to the rest by a small loop, are completely magnetized and where a transition should reflect the alignment of spins in different baby universes. Below we shall define such a model (see also \[8\]) which can be solved explicitly by very simple means and show that it, indeed, captures the features listed above except for the pure gravity region for $\beta$ small.

The model is obtained by restricting the summation in (3) over Ising spin configurations as follows: Given a configuration $\{\sigma_i\}$ on a triangulation $T$ the corresponding spin clusters consist of the maximal connected subsurfaces of $T$ on whose triangles the spins are aligned and we shall require that the boundary components of all spin clusters are minimal, i.e. of length 2. In other words, we require all phase boundaries separating spins of opposite sign to be loops of length 2. We shall henceforth indicate summation over spin configurations with this property by $\sum'\{\sigma_i\}$. Spin clusters with a boundary of length 2 are clearly the excitations of the spins with lowest energy. Our summation $\sum'\{\sigma_i\}$ represent an self-consistent iteration of such spin excitations.

Thus the one-point (or one-loop) function $G(\mu, \beta)$ is defined as

$$G(\mu, \beta) = \sum_{T \in T_1} e^{-\mu N_T} \sum'_{\{\sigma_i\}} \exp \left( \frac{\beta}{2} \sum_{(ij)} (\sigma_i \sigma_j - 1) \right)$$

(4)

where $T_1$ denotes the class of triangulations whose boundary is a loop consisting of two (marked) links and for later convenience the spins on the two boundary triangles are fixed to one. Similarly, one defines $n$-point functions that are essentially derivatives w.r.t. $\mu$ of the one-point function.

We note that $G(\mu, \beta)$ is well defined and finite in a region of the $(\mu, \beta)$-plane that contains the domain of definition for the full Ising model coupled to 2d gravity and is contained in the half plane $\mu > \mu_0$, where $\mu_0$ is the critical point of pure 2d
gravity, whose one-point function is given by
\[ G_0(\mu) = \sum_{T \in T_1} e^{-\mu N_T}, \]  
(5)

More specifically, it follows by standard arguments that there exists a critical curve
\[ (\mu'_c(\beta), \beta) \] with
\[ \mu_0 \leq \mu'_c(\beta) \leq \mu_c(\beta) \]  
(6)
such that \( G(\mu, \beta) \) is analytic in \( \mu, \beta \) on the right of this curve. It is also easy to show that \( \mu'_c(\beta) \to \mu_0 \) as \( \beta \to \infty \).

The susceptibility \( \chi(\mu, \beta) \) is defined as
\[ \chi(\mu, \beta) = -\frac{\partial G}{\partial \mu} \]  
(7)
and the string susceptibility exponent \( \gamma(\beta) \) is given by
\[ \chi(\mu, \beta) = f(\mu - \mu'_c(\beta), \beta) + \frac{c(\beta)}{(\mu - \mu'_c(\beta))^\gamma(\beta)} + \text{less singular terms}, \]  
(8)
where \( f \) is an analytic function.

The corresponding quantities in pure 2d gravity will be denoted by \( \chi_0(\mu) \) and \( \gamma_0 \). In particular \( \gamma_0 \) is given by eq. (2) and, as mentioned above, equals \(-1/2\).

## 3 Critical behavior

In order to determine the critical behavior of the model we establish a self-consistency equation as follows. Let \( T \in T_1 \) be a triangulation with a spin configuration contributing to \( G(\mu, \beta) \). Each phase boundary (of length 2) separates a baby universe from the rest of the triangulation and clearly any two such baby universes are either disjoint or one contains the other. Thus we may in a unique way cut off maximal baby universes bounded by phase boundaries and close up the corresponding boundary loops of length 2 in the remaining part of \( T \) to obtain a triangulation \( \bar{T} \in T_1 \) (with the same boundary as \( T \)) all of whose spins are aligned. Conversely, we obtain a \( T \in T_1 \) and a spin configuration \( \{\sigma_i\} \) contributing to \( G(\mu, \beta) \) by starting with a \( \bar{T} \in T_1 \) with all spins aligned, cutting it open along a set of links and finally gluing on baby universes with appropriate spin configurations along the corresponding loops of length 2 such that the spin on the two boundary triangles are oppositely oriented to those of \( \bar{T} \).

Using this decomposition procedure and summing first over baby universes with spin configurations we obtain
\[ G(\mu, \beta) = \sum_{T \in T_1} e^{-\mu N_T} \left( 1 + e^{-2\beta G(\mu, \beta)} \right)^{L_T-2} \]  
(9)
where $L_T$ denotes the number of links in $T$, i.e. $L_T = 1 + \frac{3}{2}N_T$. In eq. (9) the factor $e^{-2\beta}$ represents the coupling of a baby universe across the phase boundary to the rest of the surface and the factor 1 in the parentheses originates from the empty baby universe. We can now reexpress eq. (9) as

$$G(\mu, \beta) = \sum_{T \in \mathcal{T}} e^{-\bar{\mu}N_T} = G_0(\bar{\mu}), \quad \bar{\mu} = \mu - \frac{3}{2} \log \left(1 + e^{-2\beta}G(\mu, \beta)\right). \quad (10)$$

Note that the last equation can be written as

$$\mu = \bar{\mu} + \frac{3}{2} \log \left(1 + e^{-2\beta}G_0(\bar{\mu})\right) \quad (11)$$

which expresses $\mu$ in terms of known functions of $\bar{\mu}$ and $\beta$ since pure gravity can be solved.

From eq. (10) and (11) we get

$$\chi(\mu, \beta) = \chi_0(\bar{\mu}) \frac{\partial \bar{\mu}}{\partial \mu}, \quad (12)$$

$$\frac{\partial \bar{\mu}}{\partial \mu} = \frac{e^{2\beta} + G_0(\bar{\mu})}{e^{2\beta} - (\frac{3}{2}\chi_0(\bar{\mu}) - G_0(\bar{\mu}))} \quad (13)$$

Since the string susceptibility exponent $\gamma_0 = -1/2 < 0$ in the case of pure gravity both $G_0(\mu_0)$ and $\chi_0(\mu_0)$ are finite. This implies that there exists a $\beta_c$, given by

$$e^{2\beta_c} = \frac{3}{2}\chi_0(\mu_0) - G_0(\mu_0) \quad (14)$$

such that the denominator in (13) is different from zero for all $\mu \geq \mu_c(\beta)$ provided $\beta > \beta_c$. By differentiating $\chi(\mu, \beta)$ $n$ times with respect to $\mu$ we know from (8) that $\chi^{(n)}(\mu, \beta)$ will be singular for $\mu \rightarrow \mu_c(\beta)$ provided $n$ is sufficiently large. Performing the same differentiation on the rhs of (12), using again (13), we see that the only chance for a singular behavior is that $\bar{\mu}(\mu_0) = \mu_0$ and in this case the leading singularity has to come from $\chi_0^{(n)}(\bar{\mu})$. We conclude:

$$\bar{\mu}(\mu_c(\beta)) = \mu_0 \quad \text{and} \quad \gamma(\beta) = \gamma_0 \quad \text{for} \quad \beta > \beta_0. \quad (15)$$

This is the phase where the model is magnetized, where the spin fluctuations are small and where the geometry of the surfaces is not affected by the spins.

The number $\beta_c$ (given by (14)) is characterized by being the largest $\beta$ for which $\partial \mu/\partial \bar{\mu}$ equals zero for some $\bar{\mu} \geq \mu_0$, i.e. (by (13)) the largest $\beta$ for which the equation

$$e^{2\beta} = \frac{3}{2}\chi_0(\bar{\mu}) - G_0(\bar{\mu}) \quad (= \sum_{T \in \mathcal{T}} (3N_T/2 - 1)e^{-\bar{\mu}N_T}) \quad (16)$$

}\footnote{Strictly speaking we have added a factor $(1 + e^{-2\beta}G(\mu, \beta))$ on the rhs of (4) since $L_T - 2 = \frac{3}{2}N_T - 1$. However, all arguments presented in the follows are valid even if we did not include this factor. We have chosen to add it in order to simplify the formulas.}
has a solution for \( \bar{\mu} \geq \mu_0 \). If we define \( \bar{\mu}_c(\beta) \) by

\[
\frac{\partial \mu}{\partial \bar{\mu}} \bigg|_{\bar{\mu}_c(\beta)} = 0
\]

(17)

for \( \beta \leq \beta_c \), then \( \bar{\mu}_c(\beta) \) obviously solves (16) and we have

\[
\bar{\mu}_c(\beta_c) = \mu_0, \quad \bar{\mu}_c(\beta) > \mu_0 \quad \text{for} \quad \beta < \beta_c.
\]

(18)

Let us now assume that \( \beta < \beta_c \). If we use eq. (12) and (13) this implies that \( \chi(\mu, \beta) \) will be singular for \( \bar{\mu} \to \bar{\mu}_c(\beta) \) due to the vanishing denominator on the lhs of eq. (13). In fact, since \( \bar{\mu}_c(\beta) > \mu_0 \) both \( \chi_0(\bar{\mu}) \) and \( G_0(\bar{\mu}) \) will be regular around \( \bar{\mu}_c(\beta) \) and we can Taylor expand the lhs of (12):

\[
\chi(\mu, \beta) \sim \frac{c}{\bar{\mu} - \bar{\mu}_c(\beta)} \sim \frac{\tilde{c}}{\sqrt{\mu - \mu_c'(\beta)}}
\]

(19)

To derive the last equation we have used (13) and (17) which tell us that

\[
\mu = \mu_c'(\beta) + \text{const}((\bar{\mu} - \bar{\mu}_c(\beta))^2 + \cdots.
\]

(20)

We conclude that \( \gamma(\beta) = 1/2 \) for \( \beta < \beta_c \). In this phase baby universes are dominant. Effectively we have branched polymers and the total magnetization of the system is zero [15].

Let us finally consider the system at the critical point \( \beta_c \). This point is characterized by the fact that \( \bar{\mu}_c(\beta_c) \) coincides with \( \mu_0 \). Although the singularity of \( \chi(\mu, \beta_c) \) for \( \mu \to \mu_c'(\beta_c) \) is still dominated by the zero of \( \partial \mu/\partial \bar{\mu} \) we can no longer Taylor expand \( \mu(\bar{\mu}, \beta_c) \) around \( \bar{\mu}_c(\beta_c) \) (\( \mu_0 \)) since the functions in (13) are singular in \( \mu_0 \). On the other hand we can use the known singular behavior of the pure gravity functions \( G_0 \) and \( \chi_0 \) at \( \mu_0 \) to deduce from (13), remembering that \( \gamma_0 < 0 \),

\[
\frac{\partial \mu}{\partial \bar{\mu}} \sim (\bar{\mu} - \bar{\mu}_c(\beta_c))^{-\gamma_0}, \quad \text{i.e.} \quad \mu - \mu_c'(\beta_c) \sim (\bar{\mu} - \bar{\mu}_c(\beta_c))^{-\gamma_0 + 1}.
\]

(21)

From (12) we finally get, using (21)

\[
\chi(\mu, \beta_c) \sim \frac{c}{(\bar{\mu} - \bar{\mu}_c(\beta_c))^{-\gamma_0}} \sim \frac{\tilde{c}}{(\mu - \mu_c'(\beta_c))^{-\gamma_0 / (-\gamma_0 + 1)}},
\]

(22)

and we have derived the remarkable relation:

\[
\gamma(\beta_c) = \frac{-\gamma_0}{-\gamma_0 + 1} = \frac{1}{3}.
\]

(23)
4 Conclusion

We have seen above that our toy model has two generic phases, a magnetized phase for large $\beta$ where the distribution of geometries coincides with that of pure gravity, as one naively would expect for large $\beta$, and a phase where $\gamma(\beta) = 1/2$. The exponent $\gamma = 1/2$ is that of branched polymers and the total magnetization of such a system is zero due to the linear structure of branched polymers [15]. At the critical point $\beta_c$ for magnetization we have $\gamma(\beta_c) = 1/3$.

Our model shows how the interaction between matter and geometry can lead to a string susceptibility $\gamma > 0$. It agrees remarkably well qualitatively with numerical simulation of multiple Ising models except for a region of small $\beta$ which shrinks with increasing multiplicity of the Ising models. The model is closely related to the matrix models studied in [19] and to the $c \to \infty$ limit of multiple Ising models studied in [21]. However, our approach has the virtue of being simple and avoids any use of matrix models by working directly with the spin excitations on the surfaces. Moreover, it highlights the general nature of eqs. (12) and (13).

It is a natural assumption that the full multiple spin model will have logarithmic corrections to $\gamma = 1/3$ (if that is indeed the correct exponent) and that these will increase as the central charge $c$ decreases from infinity. There is even some numerical support for this conjecture [7].

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