Limit theorems for the minimal position of a branching random walk in random environment

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Abstract

We consider a branching system of random walk in random environment (in location) in $\mathbb{N}$. We will give the exact limit value of $M_n$, where $M_n$ denotes the minimal position of branching random walk at time $n$. A key step in the proof is to transfer our branching random walks in random environment (in location) to branching random walks in random environment (in time), by use of Bramson’s “branching processes within a branching process”.

Keywords: Random walk in random environment, Branching random walk, Branching process, Minimal position, Jumping time;
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1 Introduction

There have been abundant works on the minimal position of branching random walks. Hammersley ([11]), Kingman ([15]), Biggins ([3]) gave the first-order limit of the minimal position of a branching random walk. When this model is extended to a random environment, both time and spatial position will affect the particle behavior. Therefore, from these two perspectives, it is possible to generate some different models of branching random walks in random environment. Greven and den Hollander ([10]) considered the model where the reproduction law of the particles depends on their locations while the transition probabilities are the same everywhere. They discussed problems on global particle density and local particle density. Comets et al. ([5]) considered the model where both the reproduction law of the particles and the transition probabilities depend on their locations in $\mathbb{N}$, and gave an appropriate classification of the transience and recurrence. Bartsch et al. ([2]) considered the model which is similar to the model in ([5]) but the movement is limited to 0 and 1, they mainly cared about problems on local survival and global survival. Devulder ([8]) consider a branching system of random walk in random environment (in location), where the particles branching with a fixed law but move as random walk in random environment, and particle’s displacements are limited to $\pm 1$. Devulder ([8]) discussed the classification criteria for the case that the upper limit of $\frac{m_n^*}{n}$ is greater than 0 and the case that the lower limit of $\frac{m_n^*}{n}$ is less than 0, where $m_n^*$ denotes the location of the rightmost particle at time $n$. But the accurate velocity of limit of $\frac{m_n^*}{n}$ has not been specified. We also note that, Huang and Liu ([13], [14]) considered the branching random walks in random environment (in time) and gave the limit theorems of the minimal position and maximal position and some related large deviation principles.

In the present paper, we discuss a model that is similar to that in Devulder ([8]) except that we restrict the displacements to be 0 or 1. We will give the exact limit value of $\frac{M_n}{n}$, where $M_n$
denotes the minimal position of branching random walk at time $n$. A key step is to transfer our branching random walks in random environment (in location) to branching random walks in random environment (in time), by use of Bramson’s “branching processes within a branching process” ([4]), and then applying the result of Huang and Liu ([13], [14]).

We consider a branching system of random walks in random environment (in location) in $\mathbb{N}$, where the particles branching with a fixed law but move as random walk in random environment. For details, let $(\omega_i)_{i \in \mathbb{N}}$ be a collection of independent and identically distributed random variables, taking values in $(0,1)$. $\eta$ is the distribution of $\omega := (\omega_i)_{i \in \mathbb{N}}$. For any realization of the environment $\omega := (\omega_i)_{i \in \mathbb{N}}$, we define a random walk in random environment $(X_n)_{n \in \mathbb{N}}$ which satisfies, $X_0 = 0$, and

$$P_\omega(X_{n+1} = i|X_n = i) = 1 - P_\omega(X_{n+1} = i + 1|X_n = i) = \omega_i; \quad (1.1)$$

We assume that there exists $\delta > 0$ such that $\omega_0 \in (\delta, 1 - \delta) \eta$-a.s. Based on this model of random walk, we construct the branching system as following,

- At time $n = 0$, there is only one particle at the origin;
- At time $n = 1$, the particle dies and reproduces $k$ offspring with probability $p_k$. Each particle moves independently to a new position as the way described in (1.1);
- Iterating this procedure, each particle reproduces and makes displacement in the same way with their ancestors. And thus we get a branching random walk in random environment (in location), write BRWiRE (in location) for short. To avoid the possibility of extinction and trivial special cases, we assume that $p_0 = 0$, $p_1 < 1$. (1.2)

This implies $m := \sum_{k=0}^{\infty} kp_k > 1$.

Let $Z_n$ represent the number of particles in generation $n$ of the BRWiRE (in location), with $X_{n,k}, k = 1, \cdots, Z_n$, being the positions of these particles, then $\{Z_n\}$ is a Galton-Watson process with $Z_0 = 1, P_\omega(Z_1 = k) = p_k$. We call $P_\omega$ the quenched law, and if $\eta$ denotes the law of the environment $(\omega_i)_{i \in \mathbb{N}}$, we call

$$\mathbb{P}(\cdot) := \int P_\omega(\cdot) \eta(dw),$$

the annealed law.

We write

$$M_n := \min_{1 \leq k \leq Z_n} X_{n,k}. \quad (1.3)$$

This model is just the one considered by Bramson ([4]) (and then by Dekking and Host ([6])) if the displacement transition probability in (1.1) is constant. We can also classify three different cases in terms of $\omega_{\max} := \sup\{x : x \in \text{Supp} \ \omega_0\},$

$$\omega_{\max} \begin{cases} > \frac{1}{m}, & \text{supercritical;} \\ = \frac{1}{m}, & \text{critical;} \\ < \frac{1}{m}, & \text{subcritical.} \end{cases}$$
which is consistent with the classification in Dekking and Host ([6]). The limit behavior of $M_n$ can be obtained accordingly as follows.

**Theorem 1.1** (Supercritical case) If $\omega_{\text{max}} > \frac{1}{m}$ then there exists an almost surely finite random variable $M$ such that

$$M_n \to M \quad \mathbb{P}\text{-a.s.}$$

**Remark 1** The result of Theorem 1.1 is consistent with Theorem 3.1* in [2], indeed the concept of local survival in [2] is equivalent to finiteness of the limit random variable $M$. Our proof is based on the 0-1 law and with a full classification in Lemma 2.2.

**Theorem 1.2** (Subcritical case) If $\omega_{\text{max}} < \frac{1}{m}$ and $m^{(2)} := \sum_{k=0}^{\infty} k^2 p_k < \infty$ then there exists a constant $\gamma > 0$ satisfies

$$\frac{M_n}{n} \to \gamma \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

where $\gamma = \frac{1}{\mathbb{E}\left[\frac{1}{1-m\omega_0 e^t}\right]}$, and $t_+ = \sup\{t < \log \frac{1}{m\omega_{\text{max}}}: \mathbb{E}\left[\frac{t}{1-m\omega_0 e^t}\right] - \mathbb{E}\left[\log \frac{m(1-\omega_0)e^t}{1-m\omega_0 e^t}\right] \leq 0\}$.

**Theorem 1.3** (Critical case) If $\omega_{\text{max}} = \frac{1}{m}$ and $\eta(\omega_0 = \frac{1}{m}) > 0$ then

$$M_n \to \infty \text{ and } \frac{M_n}{n} \to 0 \quad \mathbb{P}\text{-a.s.}$$

**Remark 2** The proof of Theorem 1.1 is based on the 0-1 law and the nondecreasing of $M_n$, which will figure out in section 2. In section 3, we will prove Theorem 1.2 and Theorem 1.3, where a key step is to transfer our branching random walks in random environment (in location) to branching random walks in random environment (in time), by use of Bramson’s “branching processes within a branching process” ([4]), and then applying the result of Huang and Liu ([13], [14]).

**Remark 3** Hammersley ([11]), Kingman ([15]), Biggins ([3]) gave the first-order limit of the minimal position of a branching random walk in non-random environment. That is

$$\frac{M_n}{n} \to \gamma_1 \tag{1.5}$$

almost surely, where $\gamma_1 = \inf\{a : \mu(a) \geq 1\}$ with $\mu(a) = \inf\{e^{\theta a} \phi(\theta) : \theta \geq 0\}$, and $\phi(\theta) = \mathbb{E} \sum_{|x|=1} e^{-\theta V(x)}$ satisfies $\phi(\theta) < \infty$ for some $\theta > 0$ ($V(x)$ denotes the position of $x$).

We will show that our result in (1.4) of Theorem 1.2 is consistent with the classical Hammersley-Kingman-Biggins Theorem for the branching random walk in non-random environment, i.e.,
when the displacement transition probability in (1.1) is a constant \( \omega_0 \equiv p < \frac{1}{m} \). To this end, on the one hand, \( \phi(\theta) = mp + m(1 - p)e^{-\theta} \), and

\[
\mu(a) = \begin{cases} m, & \text{if } a \geq 1 - p; \\ \frac{mp}{1-a} \left[ \frac{(1-p)(1-a)}{pa} \right]^a, & \text{if } 0 < a < 1 - p, \end{cases}
\]

by Hammersley-Kingman-Biggins Theorem,

\[
\gamma_1 = \inf \left\{ a > 0 : \frac{mp}{1-a} \left[ \frac{(1-p)(1-a)}{pa} \right]^a \geq 1 \right\}. \tag{1.6}
\]

On the other hand, if \( \omega_0 \equiv p < \frac{1}{m} \), by use of the result in Theorem 1.2,

\[
t_+ = \sup \left\{ t < \log \frac{1}{mp} : \frac{t}{1 - mpe^t} - \log \frac{m(1-p)e^t}{1 - mpe^t} \leq 0 \right\},
\]

\[
1 - mpe^{t_+} = \inf \left\{ 1 - mpe^t > 0 : \frac{t}{1 - mpe^t} - \log \frac{m(1-p)e^t}{1 - mpe^t} \leq 0 \right\}
\]

\[
= \inf \left\{ a > 0 : \frac{\log \frac{1-a}{mp}}{a} - \log \frac{(1-p)(1-a)}{ap} \leq 0 \right\},
\]

thus

\[
\gamma = \inf \left\{ a > 0 : \frac{\log \frac{1-a}{mp}}{a} - \log \frac{(1-p)(1-a)}{ap} \leq 0 \right\}. \tag{1.7}
\]

Since \( \frac{\log \frac{1-a}{mp}}{a} - \log \frac{(1-p)(1-a)}{ap} \leq 0 \) is equivalent to \( \frac{mp}{1-a} \left[ \frac{(1-p)(1-a)}{pa} \right]^a \geq 1 \), we get that \( \gamma_1 = \gamma \) by (1.6) and (1.7).

**Remark 4** Compared with the model discussed by Devulder ([8]), our model is simpler but we give the exact limit value of \( \frac{M_n}{n} \). It should be an interesting task to investigate the accurate velocity of limit of \( \frac{m_n}{n} \) for that of Devulder ([8]), but a more complicated Bramson’s “branching processes within a branching process” ([4]) should be constructed at first, which we are now going on.

**Remark 5** Compared with the (one particle) RWRE driven by (1.1), the minimal position of the “branching system” goes slowly. Recall ([16]) the velocity of the RWRE being \( \frac{1}{1-\omega_0} \), which is strictly bigger than \( \gamma = \frac{1}{\xi_{1-\omega_0}} \) in (1.4) of Theorem 1.2, the first-order limit of the minimal position of the BRWiRE (in location) because of \( t_+ > 0 \) and \( m > 1 \), as it should be.

## 2 0-1 law and the proof of Theorem 1.1

First we give a classification criterion for the supercritical case.

**Lemma 2.1** Let \( \pi_\omega = P_\omega(M_n \rightarrow \infty) \), then for \( \eta \)-a.e. \( \omega \), \( \pi_\omega = 0 \) or \( \pi_\omega = 1 \).
Proof We denote by $P^x_\omega$ the law of the particle system conditionally on the environment $\omega$ and start from the position $x$ instead of 0. $\theta$ is the shift operator, given by $(\theta \omega)_i := \omega_{i+1}$. Then we have

$$P^i_\omega(M_n \to \infty) = P^{\theta^i \omega}(M_n \to \infty).$$

Since $\omega_i$ is i.i.d., then sequence $\{P^{\theta^i \omega}(M_n \to \infty)\}_{i \in \mathbb{Z}}$ is a stationary sequence. Moreover, by a simple coupling argument, it is also a nondecreasing sequence. Thus it is constant, i.e., for $\eta$-a.e. $\omega$,

$$P^{\theta^i \omega}(M_n \to \infty) = P_\omega(M_n \to \infty), \forall i \in \mathbb{Z}.$$ (2.1)

Let $M_n^{(j)}$ be the minimal displacement starting from the $j^{th}$ particle in the first generation, $j = 1, \cdots, Z_1$. Then

$$\{M_n \to \infty\} = \bigcap_{j=1}^{Z_1} \{M_n^{(j)} \to \infty\}.$$

Let $N_i(j)$ be the number of particles at position $j$ at time $i$. Since $M_n^{(j)}$ are independent for different $j$, we have

$$P_\omega(M_n \to \infty) = E_\omega \left[ P^1_\omega(M_n \to \infty)^{N_i(1)} P^0_\omega(M_n \to \infty)^{N_i(0)} \right]$$

$$= E_\omega P_\omega(M_n \to \infty)^{N_i(1)+N_i(0)}$$

$$= E_\omega P_\omega(M_n \to \infty)^{Z_1},$$

the second equality is by (2.1). Recall the assumption (1.2) we know that $Z_1 \geq 1$ and $Z_1 > 1$ with positive probability. Then $P_\omega(M_n \to \infty) = 0$ or 1, i.e., $\pi_\omega = 0$ or 1, $\eta$-a.e.. \hfill $\square$

Lemma 2.2 (i) If $\omega_{\max} > \frac{1}{m}$ then $\mathbb{P}(M_n \to \infty) = 0$.
(ii) If $\omega_{\max} \leq \frac{1}{m}$ then $\mathbb{P}(M_n \to \infty) = 1$.

Proof (i) Let $i_\omega := \min\{j \geq 0 : m\omega_j > 1\}$, $\mathcal{D} = \{\omega : P_\omega(i_\omega < \infty) = 1\}$, $N_i(j)$ is the number of particles at position $j$ at time $i$. If $\omega_{\max} > \frac{1}{m}$, then $\mathbb{P}(\mathcal{D}) = 1$. For every $\omega \in \mathcal{D}$, there exists $N_\omega$ satisfies

$$1 - \pi_\omega \geq P_\omega(M_n = i_\omega, \forall n \geq N_\omega)$$

$$= P_\omega \left( \lim_{n \to \infty} N_n(i_\omega) > 0 \right)$$

$$> 0.$$ (2.2)

The last inequality is due to the fact that when a particle reaches $i_\omega$, this particle and its descendants which stay at $i_\omega$ form a Galton-Watson process with mean offspring $m\omega_{i_\omega} > 1$, so it is a supercritical branching process and has positive probability to exist forever.

From Lemma 2.1 we know that $\pi_\omega = 0$ or 1, combined with (2.2) we deduce that for any $\omega \in \mathcal{D}$, $\pi_\omega = 0$, then

$$\mathbb{P}(M_n \to \infty) = E\pi_\omega = 0.$$
(ii) When $\omega_{\text{max}} \leq \frac{1}{m}$, we need to prove that for $\mathbb{P}$-a.e. $\omega$, $P_\omega(M_n \to \infty) = 1$. Let $B := \{\omega : P_\omega(M_n \to \infty) = 0\}$. Suppose that $\mathbb{P}(B) > 0$. Since

$$\{M_n \not\to \infty\} = \bigcup_p \{M_n \to p\} = \bigcup_p \bigcup_M \{n \geq M : M_n = p\},$$

there exists $p_\omega, M_\omega$ such that

$$P_\omega(M_n = p_\omega : n \geq M_\omega) > 0.$$

As a result we obtain that the sub-branching process which stays at $p_\omega$ is supercritical, that is to say $m_\omega p_\omega > 1$, but $m_\omega p_\omega > 1$ contradicts to the condition that $\omega_{\text{max}} \leq \frac{1}{m}$. So the assumption can not be true. Combined with Lemma 2.1 gives

$$P_\omega(M_n \to \infty) = 1$$

is true for $\mathbb{P}$-a.e. $\omega$. Hence, $\mathbb{P}(M_n \to \infty) = 1$. □

Lemma 2.2 tells us that only when $\omega_{\text{max}} > \frac{1}{m}$ can the limit of the minimal position of branching random walk be finite. In this case, we call the branching random walk supercritical.

**Proof of Theorem 1.1** From Lemma 2.2 (i) we know that if $\omega_{\text{max}} > \frac{1}{m}$, $P_\omega(M_n \to \infty) = 0$ a.s. Since $M_n$ is nondecreasing, for any $\omega \in D$ (defined in Lemma 2.2), there exists an almost surely finite random variable $M(\omega)$ such that $M_n(\omega) \to M(\omega)$ $P_\omega$-a.e., i.e.,

$$P_\omega\left(M_n(\omega) \to M(\omega)\right) = 1, \ \forall \ \omega \in D.$$

For any $\omega \in D^c$, let $M(\omega) = 0$. Then we have $\mathbb{P}(M_n \to M) = \mathbb{E}P_\omega(M_n \to M) = 1$ and $\mathbb{P}(M < \infty) = 1$. □

### 3 Proof of Theorem 1.2 and Theorem 1.3

Firstly, when $\omega_{\text{max}} < \frac{1}{m}$, we can easily see that $\liminf_{n \to \infty} \frac{M_n}{n} > 0$, $\mathbb{P}$-a.s.. Indeed, if we set $\rho := (\omega_{\text{max}}, \cdots, \omega_{\text{max}}, \cdots)$, i.e., for any $i$, $(\rho)_i = \omega_{\text{max}}$, it is known (3) that

$$P_\rho\left(\lim_{n \to \infty} \frac{M_n}{n} = \gamma_\rho\right) = 1,$$

where $\gamma_\rho > 0$. By coupling method we conclude that for $\eta$-a.e. $\omega$,

$$P_\omega\left(\liminf_{n \to \infty} \frac{M_n}{n} > 0\right) \geq P_\rho\left(\liminf_{n \to \infty} \frac{M_n}{n} > 0\right) = 1.$$

#### 3.1 From a BRWiRE (in location) to a BRWiRE (in time)

Inspired by the method of proving the law of large numbers for random walk in random environment ([16]), for exploring the limit behavior of $\frac{M_n}{n}$, we will transfer our branching random walks in random environment (in location) to branching random walks in random environment (in time), by use of Bramson’s “branching processes within a branching process” ([4]).
In the model of branching random walk (in non-random environment) considered by Bramson \(^4\) (and then by Dekking and Host \(^6\)), i.e., the displacement transition probability in \([1,1]\) is constant. Bramson \(^4\) intelligently proposed that \(\{Y_j\}\) is also a branching process, where \(\{Y_j\}\) refers to the number of particles that jump from location \(j - 1\) to \(j\) at some time, the generating function of \(\{Y_j\}\) is \(\phi_Y(s) = \phi_Z((1 - p)s + p\phi_Y(s))\), where \(1 - p\) is the probability of the particle jumps one step up and \(p\) is the probability that the particle stays in place (here \(\phi_W\) denotes the generating function of the first generation distribution \(W_1\) of the branching process \(\{W_j\}\)). For details, we need to introduce some notations,

- \(X(a_1, \ldots, a_k)\) represents the relative displacement of the \(a_k\)th individual of the \(k\)th generation with forbears \((a_1), (a_1, a_2), \ldots, (a_1, \ldots, a_{k-1})\).
- \(S(a_1, \ldots, a_k)\) represents the position of individual \((a_1, \ldots, a_k)\). Accordingly, \(S(a_1, \ldots, a_k) = \sum_{i=1}^{k} X(a_1, \ldots, a_i), M_n = \min_{a_1 \ldots a_n} S(a_1, \ldots, a_n)\).
- \(I_j = \{(a_1, \ldots, a_n) : S(a_1, \ldots, a_{n-1}) = j - 1, S(a_1, \ldots, a_n) = j\}\).
- \(Y_j = |I_j|\), the cardinality of \(I_j\).

For any \(\nu \in I_j\), \(|\nu|\) denotes the generation of \(\nu\).

Let \(\tau_{j_1} \leq \tau_{j_2} \leq \tau_{j_3} \leq \cdots\) denote the generation of all individuals in \(I_j\) and rank them in ascending order. Denote by

\[
L_j := \max\{|\nu| : \nu \in I_j\}
\]

the latest generation time that particles jump from \(j - 1\) to \(j\).

Bramson has already proved that \(\{Y_j\}\) is a branching process, where \(j\) represents location information in our original process. But now we need to take a different perspective to view \(j\) as time, and treat \(\nu\) (\(\nu \in I_j\)) information originally representing time, as the location information of the new branching random walk, that is to say, when we care about these quantities of \(\tau_{j_i}\), we get a new branching random walk. This new branching random walk can be considered as being constructed as follows,

- At time 0, there is one particle \(\emptyset\) at the origin;
- At time 1, this particle splits into a random number \(Y(\emptyset)\) particles, and these particles move to the position \(\tau_1, \tau_2, \cdots, \tau_{Y(\emptyset)}\), where \(\tau_{il}\) are integer-valued random variables (may not be independent of each other) and the distribution of the random vector \(X(\emptyset) := (Y(\emptyset), \tau_1, \tau_2, \cdots, \tau_{Y(\emptyset)})\) is \(\xi(\omega)\) (when given the environment \(\omega\)), which is determined by

\[
\begin{align*}
m_0(t) &:= E_{\omega} \left[ \sum_{i=1}^{\infty} e^{\tau_{il}} \right] = E_{\omega} \left[ \sum_{i=1}^{\infty} Y(\emptyset, i)e^{t_{il}} \right] \\
&= \sum_{i=1}^{\infty} m^i \omega_0^{i-1}(1 - \omega_0)e^{t_{il}}, \quad (3.1)
\end{align*}
\]

where \(Y(\emptyset, i)\) represents the number of particle \(\emptyset\)'s children which locate in position \(i\).

\[\cdots \cdots\]

- At time \(n\), the particle \(\nu\) (\(|\nu| = n - 1\)) located at position \(k\) splits into a random number \(Y(\nu)\) particles, and these particles move to \(\tau_{n1}, \tau_{n2}, \cdots, \tau_{nY(\nu)}\), where the distribution of \((Y(\nu), \tau_{n1} - \cdots - \tau_{n1})\) ...
\( k, \tau_{n2} - k, \cdots, \tau_n Y(\nu) - k \) is \( \xi_{n-1} = \xi(\omega_{n-1}) \), which is determined by

\[
m_{n-1}(t) := E_{\omega} \left[ \sum_{l=1}^{Y(\nu)} e^{t(\tau_{nl}-k)} \right] = E_{\omega} \left[ \sum_{i=1}^{\infty} Y(\nu, i)e^{ti} \right] = \sum_{i=1}^{\infty} m'_i \omega_{n-1}(1 - \omega_{n-1})e^{ti},
\]

(3.2)

where \( Y(\nu, i) \) represents the number of particle \( \nu \)'s children which locate in position \( k + i \).

- Iterating this procedure and we get a new branching random walk with a random environment in time.

If we use \( Y(\nu, i) \) to denote the number of particle \( \nu \)'s children which locate in position \( i + V(\nu) \) \((V(x) \text{ denotes the position of } x)\), then \( Y(\nu) = \sum_{i=1}^{\infty} Y(\nu, i) \). By the structure of our model, we know that the descendants of \( \emptyset \) that stay at 0 form a branching process with mean offspring \( m_0 \), we use \( \{N_n(0)\} \) and \( \phi_{N(0)}(s) \) to denote this process and its generating function respectively. From its branching structure we have \( \phi_{N(0)}(s) = \phi_Z(\omega_0 s + 1 - \omega_0) \).

The corresponding relationship between first two generations of the new branching random walk and the previous one is shown in the figure,

Then \( Y_j \) represent the number of particles in generation \( j \) of the BRWiRE (in time), with \( S_{j;k}, k = 1, \cdots, Y_j, \) being the positions of these particles, and

\[
L_j := \max_{1 \leq k \leq Y_j} S_{j;k}. \tag{3.3}
\]

3.2 Relationship between \( M_n \) in (1.3) and \( L_n \) in (3.3)

Lemma 3.1 If \( \lim_{n \to \infty} \frac{L_n}{n} = \alpha > 0, \) \( \mathbb{P} \)-a.s. then \( \lim_{n \to \infty} \frac{M_n}{n} = \frac{1}{\alpha}, \) \( \mathbb{P} \)-a.s..

Proof Take \( k_n \) as a unique integer to satisfy,

\[
L_{k_n} \leq n < L_{k_n+1}. \tag{3.4}
\]
Recalling the definition of $L_n$ we have $k_n \leq M_n < k_n + 1$, i.e. $\frac{k_n}{n} \leq \frac{M_n}{n} < \frac{k_n + 1}{n}$. As a consequence

$$\lim_{n \to \infty} \frac{M_n}{n} = \lim_{n \to \infty} \frac{k_n}{n}.$$ 

Given the condition $\lim_{n \to \infty} \frac{L_n}{n} = \alpha$, then

$$\lim_{n \to \infty} \frac{L_{k_n}}{k_n} = \alpha \quad \text{and} \quad \lim_{n \to \infty} \frac{L_{k_n + 1}}{k_n + 1} = \alpha,$$

(3.5) since $k_n \to \infty$ as $n \to \infty$. Combining (3.4) and (3.5) we get

$$\lim_{n \to \infty} \frac{n}{k_n} \geq \alpha \quad \text{and} \quad \lim_{n \to \infty} \frac{n}{k_n + 1} \leq \alpha.$$

As a result, we obtain

$$\lim_{n \to \infty} \frac{n}{k_n} = \alpha \quad \text{and} \quad \lim_{n \to \infty} \frac{M_n}{n} = \frac{1}{\alpha}.$$ 

\[\square\]

3.3 Proof of Theorem 1.2

Let

$$\Lambda(t) := \mathbb{E}[\log m_0(t)],$$

$$B := \{t : \Lambda(t) < \infty\}.$$

In order to use the result in branching random walk in a random environment (in time) ([13], [14]), we have to ensure the following conditions (1)-(4):

1. $\mathbb{E}\log m_0(0) \in (0, \infty)$, (2) $0 \in \check{B}$, (3) $\mathbb{E}\tau_{11} < \infty$, (4) $\mathbb{E}\frac{Y}{m_0(0)} \log^+ Y < \infty$.

**Condition (1)** Since $\omega_{\max} < \frac{1}{m}$, $m > 1$, use (3.1) we can calculate that $\mathbb{E}\log m_0(0) = \mathbb{E}[\log m_0(1-\omega_0)] \in (0, \infty)$.

**Condition (2)** If $t > \log \frac{1}{\omega_{\max}}$, i.e. $m\omega_{\max} e^t > 1$, then there exists a set $A$ satisfies $\eta(A) > 0$ and for any $\omega \in A, m\omega e^t \geq 1$, therefore from (3.1) we see that $m_0(t) = \infty$ on $A$. $\Lambda(t) = \int_A \log m_0(t) d\eta + \int_{A^c} \log m_0(t) d\eta = \infty$; if $t < \log \frac{1}{\omega_{\max}}$, then there exists a constant $c$ such that $m\omega_{\max} e^t \leq c < 1$, in that case $\Lambda(t) = \mathbb{E}[\log m_0(1-\omega_0) e^t] \leq \mathbb{E}[\log \frac{m e^t}{1-c}] = \log \frac{m e^t}{1-c} \leq \infty$.

Therefore $\check{B} = \{t : t < \log \frac{1}{\omega_{\max}}\}$. Since $\log \frac{1}{\omega_{\max}} > 0$, $0 \in \check{B}$ obviously.

**Condition (3)** Note that $\omega_{\max} < \frac{1}{m}$ ensures that $\{N_n(0)\}$ is a subcritical branching process and $\mathbb{E}T_0 < \infty$, where $T_0$ denotes the extinction time of the branching process that always stays at 0, then $\mathbb{E}\tau_{11} \leq \mathbb{E}T_0 < \infty$.

**Condition (4)** Let $N_\infty(0)$ denotes the total population of $\{N_n(0)\}$. We number all the particles in $\{N_n(0)\}$ from 1 to $N_\infty(0)$, let $N(0, i)$ represents the number of offspring of the $i^{th}$ particle of
\{N_\omega(0)\} that jumps from 0 to 1. When given the environment \(\omega\), \(N(0, i)\) are i.i.d., the generating function of \(N(0, i)\) is \(\phi_{N(0,i)}(s) = \phi_Z(\omega_0 + (1 - \omega_0)s)\).

\[
E_\omega Y^2 = E_\omega \left( \sum_{i=1}^{N_\omega(0)} N(0, i) \right)^2 = E_\omega \left[ E_\omega \left( \sum_{i=1}^{N_\omega(0)} N(0, i)^2 \right) | N_\omega(0) \right] 
= E_\omega \left[ \sum_{i=1}^{N_\omega(0)} N(0, i)^2 \right] = E_\omega \left[ \sum_{1 \leq i \neq j \leq N_\omega(0)} N(0, i)N(0, j) \right] 
\leq E_\omega \left[ \sum_{1 \leq i \neq j \leq N_\omega(0)} E_\omega[N(0, i)]E_\omega[N(0, j)] \right] 
\leq E_\omega\left[ (N_\omega(0))^2 \right] \left[ E_\omega(N(0, 1))^2 \right] (3.6)
\]

Use the expression of \(\phi_{N(0,1)}(s) = \phi_Z(\omega_0 + (1 - \omega_0)s)\) and \(\phi_{N(0)}(s) = \phi_Z(\omega_0s + 1 - \omega_0)\), we can calculate that \(E_\omega N(0, 1) = m(1 - \omega_0)\), \(E_\omega[N(0, 1)^2] = (m^2) + m(1 - \omega_0)^2 + m(1 - \omega_0)\), \(E_\omega N(0) = m\omega_0 < m\omega_{\max} < 1\), \(E_\omega[N(0)^2] = (m^2) + m\omega_0^2 + m\omega_0\). Then from (3.6),

\[
E_\omega Y^2 \leq E_\omega N_\omega(0)(m^2) + E_\omega[N_\omega(0)^2]m^2 
\leq E[N_\omega(0)]\omega_0 = \omega_{\max}[(m^2) + 2m] + E[N_\omega(0)^2]\omega_0 = \omega_{\max}m^2 (3.7)
\]

Since from our assumption \(m\omega_{\max} < 1\), \(m^2 < \infty\), \(E[N_\omega(0)^2]\omega_0 = \omega_{\max} < \infty\). From Lemma 3.1 in [17] we have \(E[N_\omega(0)^2]\omega_0 = \omega_{\max} < \infty\). Combined with (3.7),

\[
EY^2 = \int E_\omega Y^2 d\eta \leq E[N_\omega(0)]\omega_0 = \omega_{\max}[(m^2) + 2m] + E[N_\omega(0)^2]\omega_0 = \omega_{\max}m^2 < \infty,
\]

thus condition (4) satisfies obviously.

For \(t \in \bar{B}\), we have

\[
\Lambda(t) := E[\log m_0(t)] = E \left[ \log \frac{m(1 - \omega_0)e^t}{1 - m\omega_0e^t} \right] < \infty,
\]

\[
\Lambda'(t) = E \left[ \frac{m_0'(t)}{m_0(t)} \right] = E \left[ \frac{1}{1 - m\omega_0e^t} \right] < \infty.
\]

Let

\[
\rho(t) := t\Lambda'(t) - \Lambda(t), \quad t \in \bar{B}.
\]

\[
t_+ := \sup \{t \in \bar{B} : t\Lambda'(t) - \Lambda(t) \leq 0\}.
\]

Notice that \(\rho'(t) = t\Lambda''(t)\) and \(\Lambda''(t) \geq 0\) for \(t \geq 0\). Therefore, \(\rho(t)\) increases on \([0, \infty)\). Since \(\rho(0) = -\Lambda(0) < 0\), \(\rho(t)\) is continuous on \(B\), we obtain that \(t_+ > 0\).

From Theorem 3.4 in [13] we get the result that \(\lim_{m \to \infty} \frac{L_m}{m} = \Lambda'(t_+) = E[\frac{1}{1 - m\omega_0e^{t_+}}]\), where \(t_+ = \sup \{t < \log \frac{1}{m\omega_0} : E[\frac{t}{1 - m\omega_0e^t}] - E[\log \frac{m(1 - \omega_0)e^t}{1 - m\omega_0e^t}] \leq 0\}\). As a consequence, the theorem follows from Lemma 3.1.

\[\square\]

**Remark 6** We require \(m^{(2)} < \infty\) to simplify our proof to ensure \(EY^2 < \infty\), so that condition (4) is satisfied, this assumption may be relaxed since condition (4) is much weaker than \(EY^2 < \infty\).
3.4 Proof of Theorem 1.3

Proof of Theorem 1.3 $M_n \to \infty \mathbb{P}$-a.s. has already been proved in Lemma 2.2, we only need to prove that $\frac{M_n}{n} \to 0 \mathbb{P}$-a.s. We still use the idea to transform the original branching random walk into a new branching random walk with random environment in time. In this situation, we can not use (14) any more since $\mathbb{E}|\log m_0(0)| = \infty$, but Lemma 3.1 is still valid i.e.

\[
\lim_{n \to \infty} \frac{L_n}{n} = \lim_{n \to \infty} \frac{n}{M_n}
\]

is still valid if the limit of $\frac{L_n}{n}$ exists. Thus we just need to show that

\[
\lim_{n \to \infty} \frac{L_n}{n} = \infty \quad \mathbb{P}$-a.s.
\]

Since in the new branching random walk, the position of the first generation individual is the time when the original branching random walk individual jumps from 0 to 1, then the maximum value of the first generation’s position is the maximum moment when the original branching random walk jumps from 0 to 1. On account of $p_0 = 0$ and the descendants of $\varnothing$ that stay at 0 form a Galton-Watson process with mean offspring $m_0\omega_0$, the maximal jumping moment is the extinction time of this Galton-Watson process, that is,

Let $W_k^{(u)}$ denote the Galton-Watson process formed by particle $u$ and its descendants that stay at $V(u)$ ($V(u)$ denotes the location of $u$). And let

\[
T_0 := \{ n : W_n^{(\varnothing)} \geq 1, \ W_n^{(\varnothing)} = 0 \}.
\]

Thus the maximum relative displacement of particles from $\varnothing$ is

\[
\max_{1 \leq i \leq Y^{(\varnothing)}} \tau_{1i} = T_0.
\]

If $u_1$ denotes the particle that jumps from 0 to 1 at time $T_0$, similar to previous analysis we know that the descendants of $u_1$, that stay at 1 form a Galton-Watson process with mean offspring $m_1\omega_1$.

Denote

\[
T_1 := \{ n : W_n^{(u_1)} \geq 1, \ W_n^{(u_1)} = 0 \}.
\]

Then the maximum relative displacement of particles from $u_1$ is

\[
\max_{1 \leq i \leq Y(u_1)} (\tau_{2i} - T_0) = T_1,
\]

where $\tau_{2i}(1 \leq i \leq Y(u_1))$ denotes the position of the descendants of $u_1$.

Iterating this procedure we obtain

\[
L_n \geq \sum_{i=0}^{n-1} T_i.
\]

Hence,

\[
\frac{L_n}{n} \geq \frac{1}{n} \sum_{i=0}^{n-1} T_i.
\]
where $T_i$ is the extinction time of a Galton-Watson process with mean offspring $m\omega_i$. Recalling that $\omega_i$ is i.i.d, then $T_i$ is also i.i.d.

Besides,

$$E(T_0) = \int_{\{\omega_0 = \frac{1}{m}\}} E_\omega(T_0) d\eta + \int_{\{\omega_0 < \frac{1}{m}\}} E_\omega(T_0) d\eta.$$  

Note that on the set $\{\omega_0 = \frac{1}{m}\}$, $E_\omega(T_0) = \infty$, and on the set $\{\omega_0 < \frac{1}{m}\}$, $0 < E_\omega(T_0) < \infty$. Combined with the assumption $\eta(\{\omega_0 = \frac{1}{m}\}) > 0$, we get that $E(T_0) = \infty$. By (9) Theorem (7.2)

$$\frac{1}{n} \sum_{i=0}^{n-1} T_i \to \infty \text{ \text{P-a.s., as } n \to \infty.}$$

(3.8) suggests that $\lim_{n \to \infty} \frac{L_n}{n} = +\infty \text{ \text{P-a.s. and the conclusion follows from Lemma 3.1.}} \Box$

References

[1] Athreya, K.B. and Ney, P.E. (1972) Branching Processes. Springer, Berlin.

[2] Bartsch, C., Gantert, N. and Kochler, M. (2009) Survival and growth of a branching random walk in random environment. Markov Process. Related Fields. 15, 525-548.

[3] Biggins, J.D. (1976) The first- and last-birth problems for a multitype age-dependent branching process. Adv. Appl. Probab. 8, 446-459.

[4] Bramson, M.D. (1978) Minimal displacement of branching random walk. Z. Wahrscheinlichkeitstheor. Verw. Geb. 45, 89-108.

[5] Comets, F., Menshikov, M.V. and Popov, S.Y. (1998) One-dimensional branching random walk in a random environment: A classification. Markov Process. Related Fields. 4, 465-477.

[6] Dekking, F.M. and Host, B. (1991) Limit distributions for minimal displacement of branching random walks. Probab. Theory Relat. Fields. 90, 403-426.

[7] Dembo, A., Peres, Y. and Zeitouni, O. (1996) Tail estimates for one-dimensional random walk in random environment. Comm. Math Phys. 181, 667-683.

[8] Devulder, A. (2007) The speed of a branching system of random walks in random environment. Prob. Stat. Letters. 77, 1712-1721.

[9] Durrett, R. (2004) Probability: Theory and Examples. 3rd ed. Belmont: Duxbury Press.

[10] Greven, A. and den Hollander, F. (1992) Branching random walk in random environment: phase transition for local and global growth rates. Probab. Theory Relat. Fields. 91, 195-249.

[11] Hammersley, J.M. (1974) Postulates for subadditive processes. Ann. Probab. 2, 652-680.
[12] Hu, Y. and Yoshida, N. (2009) Localization for branching random walks in random environment. *Stoc. Proc. Appl.* **119**, 1632-1651.

[13] Huang, C., Liang, X. and Liu, Q. (2014) Branching random walks with random environments in time. *Front. Math. China* **9**, 835-842.

[14] Huang, C. and Liu, Q. (2014) Branching random walk with a random environment in time. Arxiv: 1407.7623.

[15] Kingman, J.F.C. (1975) The first birth problem for an age-dependent branching process. *Ann. Probab.* **3**, 790-801.

[16] Zeitouni, O. (2004) Random walks in random environments. Lecture Notes in Mathematics **1837**. Springer-Verlag, Berlin, pp. 189-312.