Analytical studies of spectrum broadcast structures in quantum Brownian motion

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Abstract
Spectrum broadcast structures are a new and fresh concept in the quantum-to-classical transition, introduced recently in the context of decoherence and the appearance of objective features in quantum mechanics. These are specific quantum state structures, responsible for the objectivization of the decohered state of a system. Recently, they have been demonstrated by means of the well-known quantum Brownian motion model of the recoilless limit (infinitely massive central system), as the principal interest lies in information transfer from the system to the environment. However, a final analysis relied on numerics. Here, after a presentation of the main concepts, we perform analytical studies of the model, showing the timescales and the efficiency of the spectrum broadcast structure formation. We consider a somewhat simplified environment, being random with a uniform distribution of frequencies.

Keywords: decoherence, quantum open systems, quantum communication

1. Introduction

We begin with a brief introduction into the problem of objectivity in quantum mechanics. The considerations below are of a general nature and serve to familiarize the reader with the subject and provide a framework the technical details in the following sections.

A measurement in quantum mechanics typically alters the state of the system, so that if several observers try to measure a certain observable they will in general interfere with each other. In this sense, system properties in quantum mechanics are subjective. This is in stark contrast with the classical world, where properties of systems such as position, momentum, etc can be in principle observed by as many observers as one wishes, and they will all agree on the results (modulo eventual reference frame transformations) without disturbing the
system. This observer-independence and non-disturbance may be taken as the basis for an intuitive definition of objectivity. Thus a problem arises: how can one explain the observed objectivity of the everyday world, using quantum mechanics? This is a problematic aspect of quantum-to-classical transition, present from the very beginning of quantum mechanics [1].

One notable attempt to address this problem is known as quantum Darwinism [2]. It is a refined and more realistic version of the decoherence theory (see e.g. [3–5]), where one realizes that often observations are made indirectly, through portions of the environment, rather than by a direct interaction with the object (e.g. an illuminated object scatters photons, which are then detected by the eyes of the observers). Hence, the environment is no longer treated merely as a source of noise and dissipation but is recognized as an important ‘information carrier’. This implies a paradigmatic shift in the main object of studies in the theory of open quantum systems—from the reduced state of the system $\rho_S$ alone [3–5] to a joint state - the system and an observed fraction $fE$ of the environment $\rho_{S|fE}$ (for an alternative approach see e.g. [6]). Objectivity is then linked to ‘information redundancy’: If the environment acquires in the course of the decoherence a large number of copies of the state of the system and this information can be read out without disturbance, then the state of the system becomes objective [2]. In pictorial terms, ‘objective’ is redefined as information about the system which not only ‘survives’ interaction with the environment, but manages to proliferate there. As a measure of this effect a family of scalar criteria based on quantum mutual information has been proposed. The criteria check for a change in quantum mutual information $I(\rho_{S|fE})$ between the system and a fraction of the environment as a function of this fraction size (e.g. a number of scattered photons taken into account in $fE$). If at some point no change is produced while increasing the fraction size (and the so-called classical plateau appears on partial information plots), it is then concluded that environment stores a largely redundant amount of information about the system state (more precisely about pointer states to which the system decoheres).

This approach has been reconsidered in [7] on the grounds that scalar criteria, and quantum mutual information in particular, might not be the right tools, and more convincing arguments for information proliferation are needed, preferably on the most fundamental level available—that of quantum states. Such an approach has indeed been proposed in [7] (see also [2, 8] for earlier attempts and [9] for a somewhat complementary approach). Starting from an intuitive definition of the objectively existing state of a system [2, 10] it has been shown that, under certain assumptions, the structure of a partially reduced system state can be singled out, and the observed fraction of the environment is compatible with objectivity as follows, using the method known as spectrum broadcast structure (SBS):

$$\rho_{S|fE} = \sum_i p_i |\tilde{i}\rangle \langle \tilde{i}| \otimes \rho^E_{i} \otimes \ldots \otimes \rho^E_{M}, \quad \rho^E_{i} \rho^E_{j} = \delta_{ij} = 0, \quad (1)$$

where $|\tilde{i}\rangle$ is the pointer basis to which the system decoheres, $p_i$ are the initial pointer probabilities and $\rho^E_i$ are some states of the fragments of the environment $E_1, \ldots, E_M \in fE$, which are supposed to be observed and hence cannot be traced out. Due to the non-overlap condition in (1), any observer measuring the supports of the states $\rho^E_i$ will be able to recover the index $i$ from the environment without disturbing either the system or other observers. Thus, the information about the state of the system—the index $i$ is encoded in a number of copies $1, \ldots, M$—in the environment can be perfectly recovered without any disturbance (on average) to the whole state $\rho_{S|fE}$. As observed above, this leads to objectivity regarding the state of the system. Surprisingly, the converse is also true as shown in [7] with the help of several assumptions, thus putting forward SBS as a possible explanation for the observed
objectivity in the classical world. One assumption is that the ‘non-disturbance’ requirement should be understood in the sense of Bohr’s reply [11] to the famous EPR paper [12], further formalized in [13]. The other important assumption is a so-called ‘strong independence condition’, demanding that the only correlation between parts of the environment is due to common information about the system. Formally, from a quantum information point of view, states (1) realize a certain weak form of state broadcasting [14], called spectrum broadcasting [15]. The spectrum \( p_i \) of the reduced state of the system \( \rho_S \) is present in many copies in environments \( E_1, \ldots, E_M \) and can be retrieved from there by projections supported by \( \rho_i^{E_i} \) (due to the non-overlap condition in (1)). This broadcasting process can be also described by a channel [15]:

\[
N^S \rightarrow^E (\rho_{0S}) = \sum_i \langle \tilde{i} | \rho_{0S} | \tilde{i} \rangle \rho_i^{E_1} \otimes \cdots \otimes \rho_i^{E_M},
\]

where \( \rho_{0S} \) is the initial state of the system. Thus, objectivity can be seen as the result of a certain broadcasting process given by the channel (2).

The strength of the above result is that it links objectivity and quantum state structures in a completely abstract, model independent way. A natural question then arises if these structures appear in the canonical models of decoherence [3]: collisional decoherence, quantum Brownian motion (boson-boson), spin-spin, and spin-boson models. So far, the first two models have been analyzed and the answer is in general affirmative [18, 23]; parameter regimes of the models, such that SBS are formed, do exist. The first studied example was the famous illuminated sphere model of collisional decoherence of Joos and Zeh [16]. Following a quantum Darwinism-inspired analysis of [17], it has been shown in [18] that spectrum broadcast structures (1) are indeed asymptotically formed in the course of evolution, even if the environment is noisy (initially in a mixed state) and appropriate timescales were given. Importantly, from the perspective of the current work, methods of confirming the presence of SBS are introduced; these will be reviewed in the next chapter. The sphere model, however illustrative, is rather simple, since the system has no self-dynamics. More realistic, and richer in this sense, is quantum Brownian motion, where a central harmonic oscillator is linearly coupled to a bath of harmonic oscillators. This is arguably one of the most popular models for describing quantum dissipative systems. Despite its long history [19], QBM is still the focus of recent research - both experimental (e.g. [20]) and theoretical. In the latter case, studies of the informational content of the environment have appeared only recently [21–24]. The first two works analyzed both numerically and analytically (in the massive central system regime) the scalar condition of quantum Darwinism, assuming an initially pure environment and a squeezed system state, and showing that indeed the characteristic classical plateau is being formed. On the other hand, in [23, 24] the model has been analyzed from a SBS perspective and, under somewhat similar conditions to those above but with a thermal environment, numerical evidence has been found that there are indeed parameter regimes where a SBS is formed. A distinctive feature of the found structure is that it is dynamic and evolves in time: The pointer basis in (1) rotates in time according to the self-Hamiltonian of the central oscillator and at any instant a SBS is being formed, encodes traces of this motion. It must be stressed that due to the mentioned paradigmatic shift in the treatment of the environment, i.e. it may contain useful information, one cannot assume it to be so inert as not to feel the presence of the system, as in the standard Born-Markov approximation and master equation approach to open quantum systems (see e.g. [3] for an introduction and standard applications). Thus, our study of quantum Brownian motion does not rely on the Born-Markov approximation and master equation methods, but rather on a direct state analysis (details are presented in
section 3). A drawback of previous studies [23] is that the analysis of SBS formation was in the end performed numerically. Here, continuing the previous research, we overcome this difficulty and show analytically that there is a parameter regime of the model such that a dynamical SBS is formed. We give analytical expressions for the decoherence and the SBS formation timescales in both low- and high-temperature regimes.

2. Checking for spectrum broadcast structures

The method of detection of SBS developed so far [18, 24] is rather direct and most naturally applied to situations where the system-environment interaction is of the von Neumann measurement type:

$$\hat{H}_{SE} = \hat{X} \otimes \sum_{k=1}^{N} \hat{Y}_k,$$

where $\hat{X}$, $\hat{Y}_k$ are some observables of the system and the $k$th environment respectively, assumed for simplicity to have discrete spectra. Albeit of a special form, this class of Hamiltonians is of fundamental importance both for decoherence [3] and measurement theories and thus is worth investigating. To illustrate the method, we will neglect here the self-Hamiltonians of the system and the environment (quantum measurement limit), as one can then calculate everything explicitly. The resulting unitary evolution given by (3) is of a controlled-unitary type, where the system controls the environments through eigenvalues $\xi$ of $\hat{X}$:

$$\hat{U}(t) = \sum_{\xi} |\xi\rangle \langle \xi| \otimes \hat{U}_1(\xi; t) \otimes \cdots \otimes \hat{U}_N(\xi; t),$$

Assuming, as is usually done, a fully product initial state $\rho_{0S} \otimes \rho_{01} \otimes \cdots \otimes \rho_{0k}$, one immediately finds that after tracing a portion of the environment, denoted $(1 - f)E$ and containing the fraction $f N$, 0 < $f$ < 1 subsystems, the state reads:

$$\rho_{SE}(t) = tr_{(1 - f)E} \left[ \hat{U}(t) \rho_{0S} \otimes \bigotimes_{k=1}^{N} \rho_{0k} \hat{U}(t)^\dagger \right],$$

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where:

$$\Gamma_{\xi, \xi'}(t) = \prod_{k \in (1 - f)E} tr[\hat{U}(\xi; t) \rho_{0k} \hat{U}(\xi'; t)^\dagger] = \prod_{k \in (1 - f)E} tr[\rho_{0k} e^{-i(\xi - \xi')\hat{H}_k/\hbar}]$$

where $|\xi\rangle$, $|\xi'\rangle$ is the usual decoherence factor between states. A check for SBS (1) proceeds in two steps.

First of all the coherent part (7), containing entanglement between the system and the environment, should vanish. This is of course the usual decoherence process, controlled by $\Gamma_{\xi, \xi'}(t)$. If one is able to show $|\Gamma_{\xi, \xi'}(t)| = 0$ with time for all different pair of $\xi$, $\xi'$, one proves that decoherence has taken place and $|\xi\rangle$ becomes the pointer basis.
Second, we check if the information deposited in the environment during decoherence can be perfectly read out, i.e. if the system-dependent states of the environments:

\[ \psi_{\xi k}(t) \equiv U_k(\xi; t) \psi_{\xi} U_k^{-1}(\xi; t) \]  

have non-overlapping supports (cf (1)):

\[ \langle \psi_{\xi k}(t) | \psi_{\xi'_k}(t) \rangle = 0, \]  

and hence are perfectly one-shot distinguishable. Among different measures of distinguishability \([25]\), the most suitable turns out to be the generalized overlap (also known as the fidelity):

\[ B(\psi_1, \psi_2) \equiv \text{tr} \sqrt{\sqrt{\psi_1} \psi_2 \sqrt{\psi_1}}. \]  

This is due to the fact that the most interesting situations occur when the interaction with each individual portion of the environment in (3) is vanishingly small (see e.g. [16]). Then, one cannot expect (10) to hold at the level of single environments. On the contrary, since each of the unitaries \( U_k(\xi; t) \) weakly depends on the parameter \( \xi \), the states \( \psi_{\xi k}(t) \) are almost identical for different \( \xi \)'s. In other words, the information about \( \xi \) is diluted in the environment.

However, it can happen that by grouping subsystems of the observed part of environment \( \mathcal{E} \) into larger fractions, called macrofractions \( \text{mac} \) and introduced in [18], one can approach perfect distinguishability (10) at the level of macrofraction states \( \psi_{\xi \text{mac}}^{\text{mac}}(t) \equiv \bigotimes_{k \in \text{mac}} \psi_{\xi k}(t) \).

Generalized overlap is well suited for such tests due to its factorization with the tensor product:

\[ B_{\xi \xi'}^{\text{mac}}(t) \equiv B(\psi_{\xi \text{mac}}^{\text{mac}}(t), \psi_{\xi' \text{mac}}^{\text{mac}}(t)) = \prod_{k \in \text{mac}} B(\psi_{\xi k}(t), \psi_{\xi' k}(t)). \]  

To summarize, if one is able to prove that for some time both functions vanish

\[ \langle \psi_{\xi \xi'}(t) \rangle \approx 0, \quad B_{\xi \xi'}^{\text{mac}}(t) \approx 0, \]  

then based on ((6), (7)) this is equivalent to the formation of a spectrum broadcast structure (1):

\[ \psi_{\xi \mathcal{E} \xi'} \approx \sum_{\xi} p_{\xi}(\xi) \otimes \psi_{\xi}^{\text{mac}1} \otimes \cdots \otimes \psi_{\xi}^{\text{mac}m}, \]  

with \( \psi_{\xi}^{\text{mac}m} \) having orthogonal supports for different \( \xi \)'s and where convergence is in the trace norm.

This grouping into macrofractions (or equivalent coarse-graining of the observed environment) introduced above can be seen as a reflection of detection thresholds of real-life detectors, e.g. an eye. Since one is usually interested in a thermodynamic-type limit \( N \rightarrow \infty \), it is important that those fractions scale with \( N \) (hence the name ‘macrofractions’).

3. Spectrum broadcast structures in quantum Brownian motion

The model Hamiltonian \([4, 5, 19]\) reads:

\[ \hat{H} = \frac{\hat{p}^2}{2M} + \frac{M \Omega^2 \hat{X}^2}{2} + \sum_{k=1}^{N} \left( \frac{\hat{p}_k^2}{2m_k} + \frac{m_k \omega_k^2 \hat{x}_k^2}{2} \right) + \hat{X} \sum_{k=1}^{N} C_k \hat{x}_k, \]  

where \( \hat{X}, \hat{P} \) are the position and momentum of the central oscillator of mass \( M \) and frequency \( \Omega \), \( \hat{x}_k, \hat{p}_k \) are the positions and momenta of the bath oscillators, each with mass \( m_k \) and
frequency $\omega_k$, and $C_k$ are the coupling constants. This model can in principle be solved explicitly either directly [19] or using Wigner functions [27]. However, unlike the standard treatments, we are interested here not merely in the reduced state of the central oscillator alone, but in the joint state of the central and part of the bath oscillators, where the aforementioned exact methods do not produce manageable solutions. As already stated in the Introduction, the standard master equation methods are of no use either, since we are primarily interested in the influence of the system on the environment and not the other way around. Following this ‘inverted’ logic, one can try to eliminate, at least to a first approximation, the recoil on the system due to the environment (apart from the renormalization of the frequency). This suggests a greatly simplified assumption of a massive central system [21, 22], which we will adopt. One can then use a non-adiabatic version of the Born-Oppenheimer (Non-Born-Oppenheimer, NBO) approximation (see e.g. [28]), where the central system evolves unperturbed, according to its self-Hamiltonian $H_R = \hat{P}^2/(2M) + M\Omega^2\hat{X}^2/2$ (with the renormalized frequency $\Omega^2 \equiv \Omega_{\text{bare}}^2 - \sum_k C_k^2/(2m_k \omega_k^2)$) and the environment follows this evolution. The system propagator $K_s(X; X_0) \equiv \langle X | e^{-iHt/\hbar} | X_0 \rangle$ is re-written with the help of the classical trajectory $X(t; X_0)$, starting at $t = 0$ at $X_0$, and reaching $X$ at time $t$ (as is well-known, for the oscillator this semi-classical approximation is exact; see e.g. [29]) and this trajectory acts as a classical driving force for the environment through the coupling term, leading to a controlled evolution:

\[
i\hbar \frac{\partial}{\partial t} |\psi_E(t)\rangle = \hat{H}_E(X(t; X_0)) |\psi_E(t)\rangle,
\]

where $\hat{H}_E(X(t; X_0)) \equiv \sum_{k=1}^{N} m_k \omega_k^2 \hat{x}_k^2/2 + X(t; X_0) \sum_{k=1}^{N} C_k \hat{x}_k$. The difference from the standard, adiabatic Born-Oppenheimer approximation (see e.g. [30]) is that we solve exactly the time dependent environment evolution (16), rather than use the adiabatic approximation for the environment wavefunction $|\psi_E(t)\rangle$. The full system-environment state is then constructed using the Born-Oppenheimer type of Ansatz with an exact solution of (16):

\[
\Psi_{S,E}^{\text{NBO}}(X, x) = \int dX_0 \phi_{S0}(X_0) \langle X_0 | e^{-i\hat{H}_R t/\hbar} | X_0 \rangle \langle X | \hat{U}_E(X(t; X_0)) | \psi_{E0} \rangle.
\]

where $|\phi_{S0}\rangle$, $|\psi_{E0}\rangle$ are the initial states of the system and the environment respectively, and $\hat{U}_E(X(t; X_0))$ is a solution of (16).

Based on the type of coupling in (15) and the analysis of the previous section, it follows that candidates for the pointer states will be related to the position eigenstates. Hence, initial system states with large coherences in position are of the greatest interest, and for the purpose of this study we have chosen, as the initial system state, a momentum-squeezed ground state, which has position uncertainty larger than the ground state zero-point width (studies from [23] of the initially position-squeezed state, i.e. with position uncertainty lower than the ground state zero-point width, suggest that there is no SBS formation in the considered regime; see also [22]). This reduces the classical trajectories to $X(t; X_0) = X_0 \cos \Omega t$ [21, 22] and in this case the evolution (17) can be formally re-written using [23]:

\[
\hat{U}_{S,E}(t) = \int dX_0 e^{-i\hat{H}_R t/\hbar} |X_0\rangle \langle X_0| \otimes \hat{U}_E(X_0 \cos \Omega t).
\]

The driven evolution of the environment is easily solved and gives [23]:

\[
\hat{U}_E(X_0 \cos \Omega t) = \bigotimes_{k=1}^{N} \hat{U}_k(X_0 \cos \Omega t) = \bigotimes_{k=1}^{N} \hat{U}_k(X_0; t),
\]

where $\hat{U}_k(X_0; t)$ is the environment propagator with the bare coupling $C_k$. This is a solution of the time-dependent Hamiltonian $\hat{H}_E(X_0 \cos \Omega t)$, and the analysis of the previous section, it follows from the standard, adiabatic Born-Oppenheimer approximation that there is no SBS formation in the considered regime; see also [22]). This reduces the classical trajectories to $X(t; X_0) = X_0 \cos \Omega t$ [21, 22] and in this case the evolution (17) can be formally re-written using [23]:

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\]
where \( \zeta_k(t) \) is a phase factor (unimportant for our considerations), and \( \hat{H}_k \equiv \hat{p}_k^2/(2m_k) + m_k\omega_k^2\hat{x}_k^2/2 \), \( \hat{D}(\alpha) \equiv e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \) is the displacement operator, and:

\[
\alpha_k(t) \equiv \frac{C_k}{2\sqrt{2/m_k\omega_k}} \left[ \frac{e^{i(\omega_k + \Omega)t} - 1}{\omega_k + \Omega} + \frac{e^{i(\omega_k - \Omega)t} - 1}{\omega_k - \Omega} \right].
\]

The evolution (19) is formally a controlled-unitary type (4), in which the environment evolves accordingly to the initial position \( X_0 \) of the central oscillator. To study our central object – the partially reduced state (5), one should be careful with the integral in (18) so as not to lose the diagonal part (6). We rewrite the integral using a finite division of the real line of \( X_0 \) into intervals \( \{ \Delta_i \} \) with \([X_0] \langle X_0 \rangle \) replaced by orthogonal projectors \( \hat{1}_\Delta \) on the intervals \( \Delta \) (continuous distribution of \( X_0 \) is recovered in the limit of these divisions see e.g. [31]). The partially traced state then reads:

\[
\rho_{S|E}(t) = \sum_{\Delta} e^{-i\hat{H}_\Delta t/\hbar} \hat{1}_\Delta \langle \phi_0 \rangle \langle \phi_0 \rangle e^{i\hat{H}_\Delta t/\hbar} \bigotimes_{k=1}^{\frac{\partial}{\partial t}} \hat{\rho}_k(X_\Delta; t) + \sum_{\Delta < \Delta'} \Gamma_{X_\Delta X_{\Delta'}}(t) e^{-i\hat{H}_{\Delta'} t/\hbar} \hat{1}_{\Delta'} \langle \phi_0 \rangle \langle \phi_0 \rangle e^{i\hat{H}_{\Delta'} t/\hbar} \bigotimes_{k=1}^{\frac{\partial}{\partial t}} \hat{\rho}_k(X_{\Delta'}; t) | \phi_{\Delta'} \rangle \langle \phi_{\Delta'} | \rho_{\Delta'}(X_{\Delta'}; t),
\]

where \( X_{\Delta} \) is some position within an interval \( \Delta \),

\[
\hat{\rho}_k(X_0; t) \equiv \hat{U}_k(X_0; t) \hat{\rho}_{0k} \hat{U}_k(X_0; t)^\dagger
\]

are the system-dependent states of the environments (9), and:

\[
\Gamma_{X_\Delta X_{\Delta'}}(t) \equiv \prod_{k \in \{1, \ldots, \frac{\partial}{\partial t} \}} tr \left[ \hat{U}_k(X_0; t) \hat{\rho}_{0k} \hat{U}_k(X_0'; t)^\dagger \right] \equiv \prod_{k \in \{1, \ldots, \frac{\partial}{\partial t} \}} \Gamma_{X_\Delta X_{\Delta'}}^{(k)}(t),
\]

is the decoherence factor. We note that since the driven evolution of the environment is in this case given by the displacement operators (19), the decoherence factor (23) is in fact a characteristic function of the unobserved environment’s initial state (or more precisely, due to the presence of the phase factor \( e^{i\zeta_k(t)X_0^2} \) is a non-commutative characteristic function on the Heisenberg group [32]). Following the general procedure of section 2, one has to calculate it together with the generalized overlap (11) for the states (22): \( B^{\Delta}(\hat{\rho}_k(X_\Delta; t), \hat{\rho}_k(X_\Delta'; t)) \) as they serve as the indicator functions for the formation of spectrum broadcast structures; cf (13). Assuming the environment oscillators are initially in thermal states with the same temperature, the form of the decoherence factor is known [5]:

\[
-\log[\Gamma_{X_\Delta X_{\Delta'}}^{(k)}(t)] = \frac{(X_0 - X_0')^2}{2} |\alpha_k(t)|^2 \coth (\gamma_T \omega_k) = (X_0 - X_0')^2 \frac{C_2 \omega_k \coth (\gamma_T \omega_k)}{4/m_k (\omega_k^2 - \Omega^2)^2} \left[ (\cos \omega_k t - \cos \Omega t)^2 + (\sin \omega_k t - \Omega \sin \Omega t)^2 \right],
\]

\[
\equiv \frac{|X_0 - X_0'|^2}{2} f_T^\Delta (t; \omega_k)
\]

where \( \gamma_T \equiv \hbar/(2k_B T) \) is the thermal time. We have introduced a function \( f_T^\Delta (t; \omega_k) \equiv |\alpha_k(t)|^2 \coth (\gamma_T \omega_k) \) for later convenience. The generalized overlap, in turn, for
thermal (and also more general Gaussian) environments has been obtained in [23]:

$$-\log B_{X_0, X_0'}^{(k)}(t) = \frac{|X_0 - X_0'|^2}{2} \left[ |0_k(t)|^2 \tanh (\gamma \omega_k) \right]$$

$$= \frac{|X_0 - X_0'|^2}{2} \int d^B (t; \omega_k).$$

(26)

We note that the factor \( \coth (\gamma \omega_k) \) appearing in the decoherence factor is related to the mean initial energy of the environmental oscillators at temperature \( T \), \( \coth (\gamma \omega_k) = \langle E(\omega, T) \rangle / E_0(\omega) \), where \( E_0(\omega) \equiv h \omega / 2 \) is the zero-point energy, while \( \tanh (\gamma \omega_k) \), appearing in the generalized overlap, is nothing else but the purity \( tr(\rho_{0k}^2) \) of the initial thermal state \( \rho_{0k} \), which in turn is related to the linear entropy \( S_{lin}(\rho_{0k}) = 1 - tr(\rho_{0k}^2) \). Thus, the effectiveness of the decoherence depends on the initial energy of the environment, while information accumulation depends on environmental purity.

To proceed further with the analysis, one has to specify the environment. The standard procedure [3–5, 19, 21, 22] is to pass to a continuum limit of frequencies \( \omega_k \) and encode the properties of the environment in a specific continuous approximation to the spectral density function \( J(\omega) = \sum_k C_k^2 / (2m_k \omega_k) \delta (\omega - \omega_k) \) (e.g. in [21, 22] the Ohmic spectral density has been chosen). By contrast, [23] proposes a somewhat different idea - to keep the environment discrete but random, with frequencies \( \omega_k \) chosen from a given ensemble. Randomness is needed to effectively induce decoherence in the spirit of [33], as the environment remains finite-dimensional. For the sake of definiteness, the simplest case has been studied, where the frequencies \( \omega_k \) are independently, identically distributed (i.i.d.) with a uniform distribution over a finite interval \([\omega_L, \omega_U]\). The interval is chosen so that the environment is off-resonant (cf (24), (26)), to avoid decohering of the system by a single environment, and ‘fast’:

$$\omega_U, \omega_L \gg \Omega.$$  

(27)

This choice of environment may be considered as direct, or ‘mechanistic’, as opposed to the more common ‘field’ treatment of the environment: the bath is a collection of identical mechanical oscillators with masses \( m_k \) and random frequencies \( \omega_k \). It leads to a complication in the study of the conditions (13) as ((23), (24)) and ((12), (26)) the macroscopic indicators, associated with the traced-over part of the environment \((1 - f)E\) and an observed macroscopic fraction \( \text{mac} \) respectively:

$$|\Gamma_{X_0, X_0'}(t)| = \exp \left[ -\frac{|X_0 - X_0'|^2}{2} \sum_{k \in (1 - f)E} f_{0k}^E (t; \omega_k) \right].$$

$$B_{X_0, X_0'}(t) = \exp \left[ -\frac{|X_0 - X_0'|^2}{2} \sum_{k \in \text{mac}} f_{0k}^B (t; \omega_k) \right].$$

(28)

(29)

become almost periodic functions of time. Previously [23] those functions were studied only numerically, indicating that there is indeed a parameter regime where the time averages, over very long times of the above (non-negative) functions, simultaneously vanish, indicating small typical fluctuations above zero. This in turn implies that the partially traced state (21) approaches dynamical spectrum broadcast form with respect to the initial position \( X_0 \):

$$\rho_{\text{fast}}(t) \approx \int dX_0 \ |X_0| \rho_{0k}(\varphi_{0k})^j e^{-i\hat{H}_f t} |X_0\rangle \langle X_0| e^{i\hat{H}_f t} \otimes \rho_{\text{mac}, k}(X_0; t) \otimes \cdots \otimes \rho_{\text{mac}, j}(X_0; t),$$

(30)
with \( L_{\text{mac}}(X_0; t) \) having non-overlapping supports. The dynamical character of the above structure is manifest in the time dependence of all the appearing states: the pointer basis is rotating according to the system self-Hamiltonian (thanks to the recoil-free assumption \( \{X(t)\} \equiv e^{-iHt}X_0) \), and this motion modulates the evolution of the environment in such a way that an instantaneous SBS is formed at any moment. In a sense, the environment encodes the motion of the central oscillator. The main parameters this process is dependent on through ((28), (29)) are time, temperature, separation \( \Delta X_0 \equiv |X_0 - X_0'| \) [24], and macrofraction size \( N_{\text{mac}} \). Trade-offs between them dictate if and when the structure (30) will be formed. In what follows we study this behavior analytically, assuming large macrofraction size \( N_{\text{mac}} \).

### 4. Analytical estimates of the SBS formation

As mentioned in the previous section, we are working with random environments with i.i.d. frequencies \( \omega_k \), and with some distribution \( P(\omega) \). As a consequence, the functions \( f_{\Gamma,b}^0(t; \omega_k) \) and \( f_{\Gamma}^{1,2}(t; \omega_k) \), appearing in the SBS indicator functions ((28) (29)), also become i.i.d. random variables for a fixed time \( t \) and temperature \( T \). Analytical study of their sums over a macrofraction \( \sum_{k=1}^{N_{\text{mac}}} f_{\Gamma,b}^0(t; \omega_k) \) (we assume for simplicity that both the unobserved macrofraction \( 1 - f \)E as well as each of the observed ones have the same size \( N_{\text{mac}} \)) is possible in the limit of a large macrofraction size \( N_{\text{mac}} \to \infty \) using the Law of Large Numbers (LLN) [34]. This will be our main tool. It states (in its strong form) that the macrofraction averages \( 1/N_{\text{mac}} \sum_{k=1}^{N_{\text{mac}}} g_{\Gamma,b}(t; \omega_k) \) converge a.s., i.e. with probability one, to their expectation values:

\[
\frac{1}{N_{\text{mac}}} \sum_{k=1}^{N_{\text{mac}}} g_{\Gamma,b}(t; \omega_k) \stackrel{a.s.}{\longrightarrow} \int d\omega P(\omega) f_{\Gamma}(t; \omega) \equiv \langle \langle f_{\Gamma}(t; \omega) \rangle \rangle
\]

(31)

(we will be neglecting the superscripts \( \Gamma, B \) unless it leads to a confusion) and, according to the large deviation theory, the probability of error is exponentially small in \( N_{\text{mac}} \) with the rate governed by the so-called rate function (which we will not be interested in here, only assuming that it exists and is non-zero). This allows us to approximate the sums \( \sum_{k=1}^{N_{\text{mac}}} f_{\Gamma}(t; \omega_k) \) with \( N_{\text{mac}} \langle \langle f_{\Gamma}(t; \omega) \rangle \rangle \).

We note that the invocation of LLN is, in this context, effectively equivalent to the continuous limit for the macrofractions of the environment with \( P(\omega) \) determining the spectral density. In other words, we divide the environment into fractions of such a large size that the LLN may be applied. Following our approach, explained in the previous section, instead of standard spectral densities, such as e.g. Ohmic, we will use here a much simpler, uniform probability distribution over an interval \( [\omega_L, \omega_U] \) for ease of analysis:

\[
\langle \langle f_{\Gamma}(t; \omega) \rangle \rangle = \frac{1}{\Delta \omega} \int_{\omega_L}^{\omega_U} d\omega f_{\Gamma}(t; \omega),
\]

(32)

where \( \Delta \omega = \omega_U - \omega_L \). In what follows we analyze the short- and long-time behavior of this expression within certain limits of high and low temperature. This will enable us to estimate the macrofraction size \( N_{\text{mac}} \) needed in order for the functions ((28), (29)) to attain asymptotically values close to zero within a given error margin, as well as give the timescales of their initial decays, observed numerically in [23].

#### 4.1. Low temperature

Let us first assume that the temperature is so low that the associated thermal energy is much lower than the lowest oscillator energy: \( k_B T \ll \hbar \omega_L \). Then in the leading order the temperature dependence can be neglected \( \coth(\hbar \omega_L / 2k_B T) \approx \tanh(\hbar \omega_L / 2k_B T) \approx 1 \) and the
behavior of decoherence and orthogonalization becomes identical:

\[ f^T(t; \omega_k) \approx f^B(t; \omega_k) \approx |\alpha_k(t)|^2 \equiv f_0(t; \omega_k) \]  

(33)

with \( \alpha_k(t) \) given by (20). The calculation of the ensemble mean of \( f_0(t; \omega_k) \) is rather lengthy and is presented in appendix C, with an assumption that the interaction strengths \( C_k \) obey (cf [21, 22]) \( C_k = 2 \sqrt{(Mm_k \gamma_0) / \pi} \), where \( \gamma_0 \) is a constant.

Firstly, we are interested in the short-time behavior, valid for times much shorter than the shortest timescale of the full Hamiltonian, which in this case is \( t \ll \omega_L^{-1} \) (we recall that we assume \( \Omega \) to be much lower than the environmental frequencies in order to be in the off-resonant regime, so that collections rather than individual environments matter). By expanding the expression for \( \langle f_0(t; \omega) \rangle \) in power series with respect to time, we find, after extensive calculations (for details see appendix C equation (C.10)) that:

\[
\langle f_0(t; \omega) \rangle = \frac{2M\gamma_0}{\hbar \pi \Delta \omega} \log \left( \frac{\omega_L}{\omega_k} \right) t^2 + O(t^4),
\]

(34)

which immediately implies that the initial behavior of both decoherence and orthogonalization factors is a Gaussian decay (cf ((24), (26))):

\[
|\Gamma_{x_0,x_0}(t)| \approx B_{X_0,x_0}(t) \approx \exp \left[ -N_{\text{max}} \left( \frac{t}{\tau_0} \right)^2 \right],
\]

(35)

with a common timescale:

\[
\frac{\tau_0}{\sqrt{N_{\text{max}}}}, \quad \tau_0 = \frac{\hbar \pi \Delta \omega}{\Delta X_0 M \gamma_0^2} \log^{-1} \left( 1 + \frac{\Delta \omega}{\omega_L} \right),
\]

(36)

(this behavior can be related the Quantum Central Limit theorem [35]). We note that this depends on the macrofraction size and the separation through the product \( \Delta X_0 \sqrt{N_{\text{max}}} \). Thus, in order to keep the same timescale for small separations, the macrofraction size should increase quadratically with decreasing separation.

The initial Gaussian decay (35) by no means guarantees that the functions will stay close to zero with negligible fluctuations-revivals are possible, as shown in [23]. Thus, a long-time analysis is needed, governed in our case by the condition \( t \gg 1/\omega_L - \Omega \approx 1/\omega_L \) as \( \Omega \ll \omega_L \). The detailed calculation is tedious and is given in the appendix C, equation (C.11). The result reads:

\[
\langle f_0(t; \omega) \rangle = \frac{2M\gamma_0}{\hbar \pi \Delta \omega} \left( A_0 \cos^2(\Omega t) + B_0 \right),
\]

(37)

where:

\[
A_0 = -\frac{1}{2\Omega^2} \left( 2 \log \frac{\omega_U}{\omega_L} - \log \frac{\omega_U^2 - \Omega^2}{\omega_L^2 - \Omega^2} \right),
\]

(38)

\[
B_0 = \frac{1}{\omega_L^2 - \Omega^2} - \frac{1}{\omega_U^2 - \Omega^2} - A_0.
\]

(39)

Interestingly, for large times the mean has an oscillatory part with the system frequency \( \Omega \), but for fast environments (27) this part is vanishingly small as \( A_0 \approx 0 \). The above formulae allow us to solve a very important problem in the context of SBS—namely, how large selected macrofractions should be in order to get decoherence and orthogonalization with a prescribed error \( \epsilon \) (common in the low \( T \) limit for both functions):

\[
|\Gamma_{x_0,x_0}(t)|, \quad B_{X_0,x_0}(t) < \epsilon.
\]

(40)
This in turn will determine the trace norm distance of the actual state $\rho_{S:B}(t)$ to the spectrum broadcast form. From (37) and ((28), (29)) we immediately obtain that if:

$$\Delta X_0^2 N_{\text{mac}} > \frac{\hbar \Delta \omega}{M^2 \omega_0 B_0} \log \frac{1}{\epsilon} \approx \frac{\hbar \pi \omega_L^2 \omega_L^2}{M^2 \omega_0 (\omega_L + \omega_L)} \log \frac{1}{\epsilon},$$

then the functions will be bounded by (40) for all times $t \gg 1/(\omega_L - \Omega)$. This result can be treated as an analytical proof of SBS formation in the studied regime. Similarly to short-time decay (35), the asymptotic behavior of $[\Gamma_{X_0:Y_0}(t)]$, $B_{X_0:Y_0}(t)$ is governed by the product $\Delta X_0^2 N_{\text{mac}}$, so that the increase of the macrofraction size is quadratic with decreasing the spatial resolution of the SBS. This finite spatial resolution of the SBS for a given error level and macrofraction size is a manifestation of the ‘macroscopic objectivity’ idea, introduced in [24] for simplified models of QBM. Namely, for a given tolerance $\epsilon$ and a macrofraction size, the objective state of the system appear only on the length scales greater than those given by (41).

### 4.2. High temperature

Here we consider the opposite situation of a hot environment: $k_B T \gg \hbar \omega_0$. Intuitively, a formation of SBS should be quite compromised now, as high temperature, while increasing the decoherence power of the environment through the increase of its energy appearing in (24), decreases its information capacity, by decreasing the purity, on which the orthogonalization factor (26) depends. Indeed, this is what we show below. In the leading order $\tanh(\tau_T \omega) = [\coth(\tau_T \omega)]^{-1} \approx \tau_T \omega$, (24) and (26) read:

$$f_T^+ (t; \omega_k) \approx \frac{1}{\tau_T \omega_k} \langle \hat{a}_k(t) \rangle^2,$$

$$f_T^B (t; \omega_k) \approx \tau_T \omega_k \langle \hat{a}_k(t) \rangle^2.$$

The relevant means (32) can be calculated analytically again; see appendix D.1 and D.2. For short timescales $t \ll \omega_0^{-1}$ we obtain the following behavior (for details see appendix D, equations (D.12), (D.23)):

$$\langle \langle f_T^+(t; \omega) \rangle \rangle = \frac{2M^2 \omega_0}{\hbar \pi \omega_L \omega_L \tau_T} t^2 + O(t^4),$$

$$\langle \langle f_T^B(t; \omega) \rangle \rangle = \frac{2M^2 \omega_0 \tau_T}{\hbar \pi} t^2 + O(t^4),$$

resulting again in the initial Gaussian decay:

$$|\Gamma_{X,Y}(t)| \approx \exp \left[ -N_{\text{mac}} \left( \frac{t}{\tau_{\text{dec}}} \right)^2 \right]$$

$$B_{X,Y}(t) \approx \exp \left[ -N_{\text{mac}} \left( \frac{t}{\tau_{\text{ort}}} \right)^2 \right]$$

However, this time the timescales are different. For the decoherence one obtains (cf (D.12)):

$$\frac{\tau_{\text{dec}}}{\sqrt{N_{\text{mac}}}}, \quad \tau_{\text{dec}} = \frac{\hbar \pi \omega_L \omega_L}{\Delta X_0 M^2 \omega_0}.$$
whereas for generalized overlap (cf (D.23)) the characteristic time is:

$$\frac{\tau_{\text{ort}}}{\sqrt{N_{\text{mac}}}}, \quad \tau_{\text{ort}} = T^{-1} \frac{\hbar}{\Delta X_0 M_{\gamma_0}}.$$  \hspace{1cm} (49)

As one would expect, the key difference lies in the temperature dependence through the thermal time $\tau_T = \hbar/(2k_B T)$. While $\tau_{\text{dec}}$ decreases as $T^{-1}$ indicating faster decoherence with higher temperature, $\tau_{\text{ort}} \sim T$ so that it may even happen that the orthogonalization timescale $\tau_{\text{ort}}/\sqrt{N_{\text{mac}}}$ is greater than the validity of the short-time approximation $t \ll \omega^{-1}_U$. Keeping $\tau_{\text{dec}}/\sqrt{N_{\text{mac}}} \ll \omega^{-1}_U$ so that the short-time approximation, and hence the Gaussian decay, is valid, puts a constraint on the temperature, the macrofraction size, and the separation to be discriminated:

$$\frac{T}{\Delta X_0 \sqrt{N_{\text{mac}}}} < \frac{M_{\gamma_0}}{2\pi k_B \omega_U}.$$  \hspace{1cm} (50)

To get some insight into possible revivals of the decoherence and orthogonalization factors, we perform long-time analysis. In appendix D it is shown that for $t \gg 1/(\omega_L - \Omega) \approx 1/\omega_L$ the asymptotic expression for $\langle \langle f^g_T(t; \omega) \rangle \rangle$ reads:

$$\langle \langle f^g_T(t; \omega) \rangle \rangle = \frac{2M_{\gamma_0}}{\hbar \pi \Delta \omega \tau_T} (A_T \cos^2(\Omega t) + B_T) + O(t^{-1})$$  \hspace{1cm} (51)

with:

$$A_T \equiv -\frac{1}{4\Omega^2} \left[ \frac{\Delta \omega}{\omega_L \omega_U} + \frac{1}{2\Omega} \log \left( \frac{\omega_U + \Omega}{\omega_L - \Omega} \right) \right],$$

$$B_T \equiv \frac{1}{4\Omega^2} \left( \frac{\omega_L}{\omega_L^2 - \Omega^2} - \frac{\omega_U}{\omega_U^2 - \Omega^2} \right) = A_T.$$  \hspace{1cm} (52)

while for generalized overlap it is:

$$\langle \langle f^g_T(t; \omega) \rangle \rangle = \frac{2M_{\gamma_0} T}{\hbar \pi \Delta \omega} (A_B \cos^2(\Omega t) + B_B) + O(t^{-1}),$$  \hspace{1cm} (53)

where:

$$A_B \equiv \frac{1}{2\Omega} \log \left( \frac{\omega_U - \Omega}{\omega_L + \Omega} \right),$$

$$B_B \equiv \frac{\omega_L}{\omega_L^2 - \Omega^2} - \frac{\omega_U}{\omega_U^2 - \Omega^2}.$$  \hspace{1cm} (54)

We observe that, unlike in a low $T$ regime, the decoherence asymptotic keeps oscillating with the system frequency $\Omega$ even for fast environments (27), as $A_T \approx \Delta \omega/(4\Omega^2 \omega_U \omega_L)$, while $A_B \approx 0$. We are now ready to solve the problem of SBS formation in a high-temperature regime—for a given temperature $T$, how large macrofraction sizes should be, in order to achieve decoherence and distinguishability, and hence SBS, on a length-scale $\Delta X_0$ within given errors:

$$|\Gamma_{\lambda_{\omega},\omega}^\gamma(t)| < \epsilon_{\text{dec}}, \quad B_{\lambda_{\omega},\omega}(t) < \epsilon_{\text{ort}}.$$  \hspace{1cm} (56)

Equations (51) and (53) give us the answer:

$$T \Delta X_0^2 \sqrt{N_{\text{mac}}} > \frac{\hbar^2 \pi \Delta \omega}{2 M_{\gamma_0} T} \log \frac{1}{\epsilon_{\text{dec}}} \approx \frac{\hbar^2 \pi \Omega^2 \omega_U \omega_L}{M_{\gamma_0} T} \log \frac{1}{\epsilon_{\text{dec}}},$$  \hspace{1cm} (57)
\[ \Delta x^2 N_{mac}^B T > \frac{2\pi k_B \Delta \omega}{M \gamma_0 B_0} \log \frac{1}{\epsilon_{dec}} \approx \frac{2\pi k_B \omega_U \omega_L}{M \gamma_0} \log \frac{1}{\epsilon_{ort}}, \]  

where \( N_{mac}^B \) is the size of the traced-over part of the environment \((1 - f)E\), and \( N_{mac}^{pl} \) is the size of (each of) the observed macrofraction. As predicted, keeping all other parameters fixed, the observed macrofraction size in high temperature must be much larger than the unobserved one in order to come close to SBS. Indeed, from the above results these sizes scale like thermal-to-central-system energies:

\[ \frac{N_{mac}^B}{N_{mac}^{pl}} > 2 \left( \frac{k_B T}{\hbar \Omega} \right)^2 \frac{\log \epsilon_{ort}}{\log \epsilon_{dec}}, \]  

and the latter factor is huge for the considered fast environments, since \( k_B T \gg \hbar \Omega \gg \hbar \Omega \).

5. Conclusions

We have studied the process of formation of spectrum broadcast structures using the quantum Brownian motion model, continuing the research initiated in [23]. Being interested in the information gained about the system from environment, we have considered a rather non-standard recoilless limit with a massive central system (initially in the momentum-squeezed state, i.e. with a large position uncertainty), and somewhat simplified random environments with i.i.d. uniformly distributed frequencies. Assuming the environment to be sufficiently coarse-grained, the use of the Law of Large Numbers allowed us to obtain analytical proofs of the dynamical spectrum broadcast structure formation with respect to the initial system position (30) in this model in terms of bounds on the error between the actual state and the ideal SBS in both low-temperature (41) and high-temperature ((57), (58)) regimes. The results of [7], provide an objectivization (within a given error) of the initial system position, with other details of the motion also present in the environment, and reflected by the dynamical character of the SBS (30).

More specifically, we have investigated the short-time behavior of decoherence and generalized overlap factors in low and high temperatures. In a low-temperature regime, we have shown that both factors admit Gaussian decay with the same timescale, which depends on frequencies of unobserved environmental oscillators only. In a high-temperature regime they also decay in a Gaussian way. However, the resulting timescales are different functions of temperature; the decoherence rate is proportional to the temperature, whereas the rate of decay of generalized overlap is inversely proportional to temperature. This explains in a quantitative way previous numerical simulations showing rapid decoherence and vanishing orthogonalization of remaining environmental states in studies using a model with growing temperature for a fixed number of environmental systems.

Long-time analysis gave us the efficiency of the spectrum broadcast structure formation, in terms of the required observed/unobserved macrofraction sizes necessary to obtain decoherence, and environmental state distinguishability, within given errors. In low temperatures these sizes are equal, but, as one would expect, in high-temperature regimes they have the opposite temperature dependence, as hot environments decohere the central system efficiently but encode a vanishingly small amount of information due to high noise.
An obvious generalization of the present work would be an analysis of more standard environment models, e.g. the Ohmic model with a cut-off. The resulting functions will be more complicated but we believe still analyzable, at least in certain approximate regimes, similar to those studied above. Another, much more demanding, generalization would be to allow for some recoil on the central system. We then suspect that in such a case a regime would exist where an interplay between partial coherence restoration and decoherence will lead to the formation of dynamical SBS but with respect to the full system trajectory. This is a subject for future studies.

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Appendix A. Appendix content

Appendix B is devoted to Sine and Cosine Integrals. In appendix B.1 we introduce notation for particular combinations of Sine and Cosine integrals that appear in formulas. In appendix B.2 and B.4 formulas for short-time and long-time behavior of these functions are presented. Appendix C contains details of computations for low-temperature quantum Brownian motion. The high-temperature case is treated in appendix D, with appendix D.1, D.2 devoted to decoherence factor and generalized overlap respectively.

Appendix B. Sine and Cosine integrals

B.1. Notation

It will prove beneficial for the sake of clarity to introduce the following notation for combinations of Sine and Cosine integrals:

\[
F_{S_i}(±, ±, ±, ±) = ±1, ±1, ±1, ±1 \cdot \left[ \text{Si}((\omega_L - \Omega)t), \text{Si}((\omega_U - \Omega)t), \text{Si}((\omega_L + \Omega)t), \text{Si}((\omega_U + \Omega)t) \right]^T
\]

(B.1)

\[
F_{C_i}(±, ±, ±, ±) = ±1, ±1, ±1, ±1 \cdot \left[ \text{Ci}((\omega_L - \Omega)t), \text{Ci}((\omega_U - \Omega)t), \text{Ci}((\omega_L + \Omega)t), \text{Ci}((\omega_U + \Omega)t) \right]^T
\]

(B.2)

where \([±1, ... ±1]\) is a vector, \(\cdot\) denotes vector product and \(T\) stands for transposition. The argument of \(F_{S_i}(±, ±, ±, ±)\) and \(F_{C_i}(±, ±, ±, ±)\) specifies signs’ pattern of functions e.g.:

\[
F_{S_i}(+, -, +, -) = \text{Si}((\omega_L - \Omega)t) - \text{Si}((\omega_U - \Omega)t) + \text{Si}((\omega_L + \Omega)t) - \text{Si}((\omega_U + \Omega)t)
\]
B.2. Short-time behavior

In the short-time regime, i.e. for $t \ll \omega^{-1}$, we can approximate relevant functions as follows:

$$F_{S}(+, -, +, +) = 2(\omega_{L} - \omega_{U})t + \frac{t^{3}}{9}(\omega^{3}_{U} - \omega^{3}_{L} + 3\Omega^{2}\omega_{U} - 3\Omega^{2}\omega_{L}) + O(t^{5})$$

$$F_{S}(+, -, +, +) = \frac{t^{3}}{3}\Omega(\omega^{2}_{L} - \omega^{2}_{U}) + O(t^{5})$$

$$F_{C}(+, -, +, +) = \log\left(\frac{\omega^{2}_{L} - \omega^{2}_{U}}{2\omega^{2}_{U}}\right) + \frac{1}{2}(\omega^{2}_{L} - \omega^{2}_{U})t^{2} + o(t^{4})$$

$$F_{C}(+, -, +, +) = \log\left(\frac{(\omega_{L} - \Omega)(\omega_{L} + \Omega)}{(\omega_{L} + \Omega)(\omega_{L} - \Omega)}\right) + \Omega(\omega_{L} - \omega_{U})t^{2} + O(t^{4}), \quad (B.3)$$

B.3. Long-time behavior

On the other hand, the asymptotic of relevant functions is given by:

$$tF_{S}(+, -, +, +) = 2\left(\frac{\cos(\omega_{U}t)\cos(\Omega t)}{\omega^{2}_{U} - \Omega^{2}} + \frac{\sin(\omega_{U}t)\sin(\Omega t)}{\omega^{2}_{L} - \Omega^{2}}\right) + O(t^{-1})$$

$$tF_{S}(+, -, +, +) = 2\left(\frac{\sin(\omega_{U}t)\cos(\Omega t)}{\omega^{2}_{L} - \Omega^{2}} + \frac{\cos(\omega_{U}t)\sin(\Omega t)}{\omega^{2}_{U} - \Omega^{2}}\right) + O(t^{-1})$$

$$tF_{C}(+, l, +, +) = 2\left(\frac{\sin(\omega_{U}t)\cos(\Omega t)}{\omega^{2}_{L} - \Omega^{2}} + \frac{\cos(\omega_{U}t)\sin(\Omega t)}{\omega^{2}_{U} - \Omega^{2}}\right) + O(t^{-1})$$

$$tF_{C}(+, -, +, +) = 2\left(\frac{\Omega(\omega_{U}t)\cos(\Omega t)}{\omega^{2}_{L} - \Omega^{2}} + \frac{\omega_{L}\cos(\omega_{U}t)\sin(\Omega t)}{\omega^{2}_{U} - \Omega^{2}}\right) + O(t^{-1}). \quad (B.4)$$

Appendix C. Low temperature

Here, we present details of computing $\langle \langle f_{g}^{x}(t; \omega) \rangle \rangle$ and $\langle \langle f_{g}^{y}(t; \omega) \rangle \rangle$ in the low-temperature regime. As mentioned in the main text, in this case $\langle \langle f_{g}^{x}(t; \omega) \rangle \rangle \approx \langle \langle f_{g}^{y}(t; \omega) \rangle \rangle \approx \langle \langle f_{g}(t; \omega) \rangle \rangle$.

$$\langle \langle f_{g}(t; \omega) \rangle \rangle = \frac{2M_{\text{B}}}{\hbar \pi \Delta \omega} \int_{\omega_{l}}^{\omega_{U}} \frac{\omega_{k}}{(\omega^{2}_{k} - \Omega^{2})^{2}} \left(1 - \cos^{2} \Omega t\right) + \frac{\Omega^{2}}{\omega^{2}}(1 + \cos^{2} \Omega t)$$

$$- 2\cos \Omega t \cos \omega t - \frac{2\Omega}{\omega_{k}} \sin \Omega t \sin \omega t$$

$$= \frac{2M_{\text{B}}}{\hbar \pi \Delta \omega} (I_{1} + I_{2} - 2I_{1} - 2I_{2}). \quad (C.1)$$
where the mean consists of four integrals. Results of each integration are given by:

\[
I_1 = \int_{\omega_l}^{\omega_u} \frac{\omega}{(\omega^2 - \Omega^2)} (1 + \cos^2(\Omega t)) = \frac{1}{2} \left( \frac{1}{\omega_L^2 - \Omega^2} - \frac{1}{\omega_U^2 - \Omega^2} \right) (1 + \cos^2(\Omega t))
\]

(C.2)

\[
I_2 = \int_{\omega_l}^{\omega_u} \frac{\Omega^2}{\omega(\omega^2 - \Omega^2)} (1 - \cos^2(\Omega t))
\]

\[= \left[ \frac{1}{4\Omega^2} \left( 4 \log \frac{\omega_U}{\omega_L} - 2 \log \frac{\omega_U^2 - \Omega^2}{\omega_L^2 - \Omega^2} \right) + \frac{1}{2(\omega_L^2 - \Omega^2)} - \frac{1}{2(\omega_U^2 - \Omega^2)} \right] (1 - \cos^2(\Omega t))
\]

(C.3)

\[
I_3 = \int_{\omega_l}^{\omega_u} d\omega \frac{\omega}{(\omega^2 - \Omega^2)^2} \cos \omega t \cos \Omega t
\]

\[= \frac{1}{4\Omega} \cos \Omega t \left[ \frac{\Omega \cos \omega_L t}{\omega_L^2 - \Omega^2} - \frac{\Omega \cos \omega_U t}{\omega_U^2 - \Omega^2} + t \cos \Omega t F_{Si}(+, - , -) + t \sin \Omega t F_C (+ , - , +) \right]
\]

(C.4)

\[
I_4 = \int_{\omega_l}^{\omega_u} d\omega \frac{\Omega}{(\omega^2 - \Omega^2)^2} \sin \omega t \sin \Omega t
\]

\[= \frac{1}{4\Omega} \sin \Omega t \left\{ \frac{2\omega_L \sin \omega_L t}{\omega_L^2 - \Omega^2} - \frac{2\omega_U \sin \omega_U t}{\omega_U^2 - \Omega^2} + t \left[ \cos(\Omega t) F_{Ci}(-, + , +) + \sin(\Omega t) F_{Si}(+ , - , +) \right] - \Omega^{-1} [F_{Si}(- , + , +) - F_{Ci}(- , + , +)] \right\}
\]

(C.5)

To investigate the short-time behavior of \( \langle \langle \omega(t; \omega) \rangle \rangle \) we expand the above expressions up to the second order in time. This is a good approximation for \( t \ll \omega_U^{-1} \). As a result we obtain

\[
I_1 = \left[ \frac{1}{4\Omega^2} \left( 4 \log \frac{\omega_U}{\omega_L} - 2 \log \frac{\omega_U^2 - \Omega^2}{\omega_L^2 - \Omega^2} \right) + \frac{1}{2(\omega_L^2 - \Omega^2)} - \frac{1}{2(\omega_U^2 - \Omega^2)} \right] \Omega^2 t^2 + O(t^4)
\]

(C.6)

\[
I_2 = \left( \frac{1}{\omega_L^2 - \Omega^2} - \frac{1}{\omega_U^2 - \Omega^2} \right) (2 - \Omega^2 t^2) + O(t^4)
\]

(C.7)

\[
I_3 = \frac{1}{4\Omega} \left[ \frac{2\Omega}{\omega_L^2 - \Omega^2} \left( 1 - \frac{\omega_L^2 t^2}{2} - \frac{\Omega^2 t^2}{2} \right) - \frac{2\Omega}{\omega_U^2 - \Omega^2} \left( 1 - \frac{\omega_U^2 t^2}{2} - \frac{\Omega^2 t^2}{2} \right) + \Omega^2 t^2 \log \frac{\omega_L^2 - \Omega^2}{\omega_U^2 - \Omega^2} \right] + O(t^4)
\]

(C.8)

\[
I_4 = \frac{1}{4\Omega} \left( \frac{2\omega_L \omega_U t}{\omega_L^2 - \Omega^2} \Omega_L t^2 + \frac{2\omega_L \omega_U t}{\omega_U^2 - \Omega^2} \Omega_U t^2 \right) + O(t^4)
\]

(C.9)
As a result, the expression for the mean valid in the short-time regime is
\[
\langle \langle f_0(t; \omega) \rangle \rangle = \frac{2M\gamma_0}{\hbar \tau \Delta \omega} \log \left( \frac{\omega_U}{\omega_L} \right) t^2 + O(t^4),
\] (C.10)
which leads to (35) in the main text.

On the other hand, using asymptotic formulas from appendix B.3 one can show that for 
\[t \gg (\omega_L - \Omega)^{-1} I_3 \approx 0\] and 
\[I_4 \approx 0\] so the only relevant terms are \(I_1\) and \(I_2\), which results in the following expression for long-time behavior of the mean:
\[
\langle \langle f_0(t; \omega) \rangle \rangle = \frac{2M\gamma_0}{\hbar \tau \Delta \omega} (A_0 \cos^2(\Omega t) + B_0) + O(t^{-1}),
\] (C.11)
where
\[
A_0 \equiv - \frac{1}{2\Omega^2} \left( 2 \log \frac{\omega_U}{\omega_L} - \log \frac{\omega_U^2 - \Omega^2}{\omega_L^2 - \Omega^2} \right),
\]
\[
B_0 \equiv \frac{1}{\omega_L^2 - \Omega^2} - \frac{1}{\omega_L^2 - \Omega^2} + A_0.
\]
Minimization of (C.11) is straightforward, yields minimal value \(\frac{2M\gamma_0 \gamma_1}{\hbar \tau \omega U}\), which was used to derive formula (37) in the main text.

**Appendix D. High temperature**

In the case of high temperatures one can approximate functions appearing in decoherence factor and generalized overlap as
\[
\langle \langle f_0^A (t; \omega_k) \rangle \rangle \approx \frac{1}{\tau \omega_k} f_0^A (t; \omega_k), \quad \langle \langle f_0^B (t; \omega_k) \rangle \rangle \approx \tau \omega_k f_0^B (t; \omega_k).
\]

**D.1. Decoherence**

We begin our considerations with the decoherence factor. The mean is given by the expression:
\[
\langle \langle f_0^A (t; \omega_k) \rangle \rangle = \frac{2M\gamma_0}{\hbar \tau \omega_U \omega L} \int_{\omega_U}^{\omega_L} \frac{1}{(\omega_k^2 - \Omega^2)^2} \left( 1 - \cos^2(\Omega t) + \frac{\omega_k^2}{\omega_k} (1 + \cos^2(\Omega t)) \right)
\]
\[
- 2 \cos \Omega t \cos \omega_k t - \frac{2 \Omega}{\omega_k} \sin \Omega t \sin \omega_k t \right) =
\]
\[
\frac{2M\gamma_0}{\hbar \tau \omega_U \omega L} (I_1 + I_2 - 2I_3 - 2I_4) \right) \right.
\] (D.3)
Computing integrals are obtained by:
\[
I_1 = \int_{\omega_U}^{\omega_L} \frac{\Omega^2}{\omega^2} \frac{1}{(\omega^2 - \Omega^2)^2} (1 - \cos^2(\Omega t))
\]
\[
= \frac{(1 - \cos^2(\Omega t))}{\Omega^2} \left( \frac{\omega_U - \omega_k}{\omega_U \omega L} - \frac{\omega_U - \omega_k}{2(\omega_U^2 - \Omega^2)} + \frac{\omega_k}{2(\omega_L^2 - \Omega^2)} \right)
\]
\[
+ \frac{3}{4\Omega} \log \frac{(\omega_U + \Omega)(\omega_L - \Omega)\left(\omega_U - \Omega)\left(\omega_L + \Omega)\right)}{(\omega_U - \Omega)(\omega_L + \Omega)} \right)
\] (D.4)
Now we turn our attention to short-time behavior of the mean. We expand the above expressions up to the second order in time. This is a good approximation for $t \ll \omega_U^{-1}$. As a result we obtain:

\begin{align}
I_1 &= \frac{1}{2\Omega t} \sin \Omega t \left[ 2\text{Si}(\omega_U t) - \text{Si}(\omega_U^* t) - \cos(\Omega t) F_{\text{Si}}(-,+,+,+) + t \sin(\Omega t) F_{\text{Si}}(+,-,+,+) \right] \quad (D.6)
\end{align}

\begin{align}
I_2 &= \frac{1}{4\Omega^2} \left( \frac{2\omega_U^*}{\omega_U^* - \Omega} - \frac{2\omega_U}{\omega_U^* - \Omega^2} + \frac{\omega_U}{\omega_U^* - \Omega} \right) t^2 + O(t^4) \\
I_3 &= \frac{\Omega^2}{4} \left( \frac{2\omega_U^*}{\omega_U^* - \Omega} - \frac{2\omega_U}{\omega_U^* - \Omega^2} + \frac{\omega_U}{\omega_U^* - \Omega} \right) \left( \frac{\omega_U + \Omega}{\omega_U - \Omega} \right) t^2 + O(t^4) \\
I_4 &= \frac{1}{4\Omega^2} \left( \frac{1}{\omega_U^* - \Omega^2} - \frac{1}{\omega_U^* - \Omega} \right) + \frac{1}{4\Omega} \log \left( \frac{\omega_U + \Omega}{\omega_U - \Omega} \right) t^2 + O(t^4) \\
\end{align}

Now we turn our attention to short-time behavior of the mean. We expand the above expressions up to the second order in time. This is a good approximation for $t \ll \omega_U^{-1}$. As a result we obtain:

\begin{align}
I_1 &= \frac{1}{2\Omega t} \sin \Omega t \left[ 2\text{Si}(\omega_U t) - \text{Si}(\omega_U^* t) - \cos(\Omega t) F_{\text{Si}}(-,+,+,+) + t \sin(\Omega t) F_{\text{Si}}(+,-,+,+) \right] \quad (D.6)
\end{align}

\begin{align}
I_2 &= \frac{1}{4\Omega^2} \left( \frac{2\omega_U^*}{\omega_U^* - \Omega} - \frac{2\omega_U}{\omega_U^* - \Omega^2} + \frac{\omega_U}{\omega_U^* - \Omega} \right) t^2 + O(t^4) \\
I_3 &= \frac{\Omega^2}{4} \left( \frac{2\omega_U^*}{\omega_U^* - \Omega} - \frac{2\omega_U}{\omega_U^* - \Omega^2} + \frac{\omega_U}{\omega_U^* - \Omega} \right) \left( \frac{\omega_U + \Omega}{\omega_U - \Omega} \right) t^2 + O(t^4) \\
I_4 &= \frac{1}{4\Omega^2} \left( \frac{1}{\omega_U^* - \Omega^2} - \frac{1}{\omega_U^* - \Omega} \right) + \frac{1}{4\Omega} \log \left( \frac{\omega_U + \Omega}{\omega_U - \Omega} \right) t^2 + O(t^4) \\
\end{align}
Finally, we can add the terms to obtain:

\[ \langle \langle f_T^L(t; \omega) \rangle \rangle = \frac{2M_{\gamma 0}^T}{\hbar \pi \gamma T \omega_L \omega_U} t^2 + O(t^4), \]  

which leads to the equation (44) in the main text.

In the case of long-time behavior, one reaches a similar qualitative conclusion as for low-temperature. Namely, for \( t \gg (\omega_L - \Omega)^{-1} \) \( I_3 \approx 0 \) and \( I_4 \approx 0 \), so the only relevant terms are \( I_1 \) and \( I_2 \), which results in the following expression for long-time behavior of the mean:

\[ \langle \langle f_T^L(t; \omega) \rangle \rangle = \frac{2M_{\gamma 0}^T}{\hbar \pi \gamma T \Delta \omega} (A_T \cos^2 (\Omega t) + B_T) + O(t^{-1}) \]  

where this time

\[ A_T = -\frac{1}{4 \Omega^2} \left( \frac{\omega_U - \omega_L}{\omega_U \omega_L} + \frac{1}{2\Omega} \log \left( \frac{\omega_U + \Omega}{\omega_U - \Omega} \right) \right) \]
\[ B_T = \frac{1}{4 \Omega^2} \left( \frac{\omega_L}{\omega_L^2 - \Omega^2} - \frac{\omega_U}{\omega_U^2 - \Omega^2} \right) - A_T, \]

To obtain equation (56) in the main text, one performs a straightforward minimization of (D.13).

### D.2. Generalized overlap

In the case of generalized overlap the mean is given by:

\[ \langle \langle f_T^L(t; \omega) \rangle \rangle = \frac{2M_{\gamma 0}^T}{\hbar \pi \gamma T \Delta \omega} \int_{\omega_L}^{\omega_U} \frac{\omega^2}{(\omega^2 - \Omega^2)^2} \left( 1 + \cos^2 \Omega t + \frac{\Omega^2}{\omega_k^2} (1 - \cos^2 \Omega t) \right. \]
\[ \left. - 2 \cos \Omega t \cos \omega_k t t - \frac{2\Omega}{\omega_k} \sin \Omega t \sin \omega_k t \right) \]  

\[ = \frac{2M_{\gamma 0}^T}{\hbar \pi \gamma T \Delta \omega} (I_1 + I_2 - 2I_3 - 2I_4) \]  

The results of integration are:

\[ I_1 = \int_{\omega_L}^{\omega_U} d\omega \frac{\omega^2}{(\omega^2 - \Omega^2)^2} (1 + \cos^2 \Omega t) \]
\[ = \frac{\omega_L}{2(\omega_L^2 - \Omega^2)} - \frac{\omega_U}{2(\omega_U^2 - \Omega^2)} + \frac{1}{4\Omega} \log \left( \frac{\omega_L - \Omega}{\omega_L + \Omega} \right) \left( \frac{\omega_U - \Omega}{\omega_U + \Omega} \right) \]  

\[ I_2 = \int_{\omega_L}^{\omega_U} d\omega \frac{\Omega^2}{(\omega^2 - \Omega^2)^2} (1 - \cos^2 \Omega t) \]
\[ = \frac{\omega_L}{2(\omega_L^2 - \Omega^2)} - \frac{\omega_U}{2(\omega_U^2 - \Omega^2)} + \frac{1}{4\Omega} \log \left( \frac{\omega_L + \Omega}{\omega_L - \Omega} \right) \left( \frac{\omega_U + \Omega}{\omega_U - \Omega} \right) \]
\[ I_3 = \int_0^{\omega_U} \frac{\omega^2}{(\omega_L^2 - \Omega^2)^2} \cos \omega t \cos \Omega t = \frac{1}{4\Omega} \left( \frac{2\omega_L \cos \omega_L t}{\omega_L^2 - \Omega^2} - \frac{2\omega_U \cos \omega_U t}{\omega_U^2 - \Omega^2} \right) + t \cos(\Omega t) \sin(\Omega t) \left( +, -, +, - \right) \]
\[ + \frac{1}{\Omega} \cos(\Omega t) \sin(\Omega t) \left( +, -, +, - \right) \]
\[ + \frac{1}{\Omega} \cos(\Omega t) \sin(\Omega t) \left( +, +, +, + \right) \]
\[ + \frac{1}{\Omega} \cos(\Omega t) \sin(\Omega t) \left( +, +, +, + \right) \]
\[ + \frac{1}{\Omega} \cos(\Omega t) \sin(\Omega t) \left( +, +, +, + \right) \]
(D.17)

\[ I_4 = \int_0^{\omega_U} \frac{\omega \Omega}{(\omega_L^2 - \Omega^2)^2} \sin \omega t \sin \Omega t = \frac{1}{4} \sin(\Omega t) \left( \frac{2\Omega \sin \omega_L t}{\omega_L^2 - \Omega^2} - \frac{2\Omega \sin \omega_U t}{\omega_U^2 - \Omega^2} \right) \]
\[ + t \cos(\Omega t) \sin(\Omega t) \left( +, +, +, - \right) \]
\[ + \frac{1}{\Omega} \cos(\Omega t) \sin(\Omega t) \left( +, +, +, + \right) \]
\[ + \frac{1}{\Omega} \cos(\Omega t) \sin(\Omega t) \left( +, +, +, + \right) \]
\[ + \frac{1}{\Omega} \cos(\Omega t) \sin(\Omega t) \left( +, +, +, + \right) \]
(D.18)

With regard to short-time behavior of the mean, we expand the above expressions up to the second order in time. This is a good approximation for \( t \ll \omega_U^{-1} \). As a result we obtain:

\[ I_1 = - \frac{\omega_L}{\omega_L^2 - \Omega^2} - \frac{\omega_U}{\omega_U^2 - \Omega^2} - \frac{1}{2\Omega} \log \frac{(\omega_L - \Omega)(\omega_U + \Omega)}{(\omega_L + \Omega)(\omega_U - \Omega)} \]
\[ + \frac{1}{2\Omega} \left( \frac{\omega_L}{\omega_L^2 - \Omega^2} + \frac{\omega_U}{\omega_U^2 - \Omega^2} \right) \]
\[ + \frac{3}{4} \left( \frac{(\omega_U - \Omega)}{(\omega_U + \Omega)} + \frac{(\omega_L - \Omega)}{(\omega_L + \Omega)} \right) \]
\[ + \frac{1}{2\Omega} \left( \frac{(\omega_U - \Omega)}{(\omega_U + \Omega)} + \frac{(\omega_L - \Omega)}{(\omega_L + \Omega)} \right) \]
\[ + O(t^4) \]
(D.19)

\[ I_2 = \frac{\Omega^2}{2} \left( \frac{\omega_L}{\omega_L^2 - \Omega^2} - \frac{\omega_U}{\omega_U^2 - \Omega^2} + \frac{1}{2\Omega} \log \frac{(\omega_L - \Omega)(\omega_U + \Omega)}{(\omega_L + \Omega)(\omega_U - \Omega)} \right) \]
\[ + \frac{1}{2} \left( \frac{\omega_L}{\omega_L^2 - \Omega^2} + \frac{\omega_U}{\omega_U^2 - \Omega^2} \right) \]
\[ + \frac{3}{4} \left( \frac{(\omega_U - \Omega)}{(\omega_U + \Omega)} + \frac{(\omega_L - \Omega)}{(\omega_L + \Omega)} \right) \]
\[ + \frac{1}{2\Omega} \left( \frac{(\omega_U - \Omega)}{(\omega_U + \Omega)} + \frac{(\omega_L - \Omega)}{(\omega_L + \Omega)} \right) \]
\[ + O(t^4) \]
(D.20)

\[ I_3 = \frac{\omega_L}{\omega_L^2 - \Omega^2} - \frac{\omega_U}{\omega_U^2 - \Omega^2} - \frac{1}{2\Omega} \log \frac{(\omega_L - \Omega)(\omega_U + \Omega)}{(\omega_L + \Omega)(\omega_U - \Omega)} \]
\[ + \frac{1}{4} \left( \frac{(\omega_U - \Omega)}{(\omega_U + \Omega)} + \frac{(\omega_L - \Omega)}{(\omega_L + \Omega)} \right) \]
\[ + \frac{3}{2} \left( \frac{(\omega_U - \Omega)}{(\omega_U + \Omega)} + \frac{(\omega_L - \Omega)}{(\omega_L + \Omega)} \right) \]
\[ + O(t^4) \]
(D.21)

Finally, we can add the terms to obtain:

\[ \langle \langle \langle f \rangle \rangle \rangle \rangle = \frac{2M_{\rho_0}^2}{\hbar \pi \Delta \omega} \cos \Omega t + O(t^4), \]
(D.23)

which leads to the equation (45) in the main text.

Taking into account that for \( t \gg (\omega_L - \Omega)^{-1} \) \( I_1 \approx 0 \) and \( I_4 \approx 0 \), the only relevant terms are \( I_1 \) and \( I_2 \), one obtains the asymptotic formula for the mean:

\[ \langle \langle \langle f \rangle \rangle \rangle \rangle = \frac{2M_{\rho_0}^2}{\hbar \pi \Delta \omega} (A_B \cos^2 (\Omega t) + B_B) + O(t^{-1}), \]
(D.24)

with

\[ A_B \equiv \frac{1}{2\Omega} \log \frac{(\omega_U - \Omega)(\omega_U + \Omega)}{(\omega_L - \Omega)(\omega_U + \Omega)} \]
(D.25)
\[ B_B \equiv \frac{\omega_L}{\omega_L^2 - \Omega^2} - \frac{\omega_U}{\omega_U^2 - \Omega^2} \]
(D.26)

After straightforward minimization of (D.14) one arrives at equation (57) in the main text.
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