Remaining Useful Life Prediction of \( r \)-out-of-\( n \): G System

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Abstract. Remaining useful life (RUL) prediction has been gradually concerned in reliability engineering. In addition, \( r \)-out-of-\( n \): G system is one of the most typical systems. Therefore, it is meaningful and significant to predict the RUL of \( r \)-out-of-\( n \): G system with high precision. Motivated by this problem, relationship between the reliability and the RUL is discussed in this paper. Further, on the assumption that the components are independently exponentially or Weibull distributed, the closed-forms for RUL of \( r \)-out-of-\( n \): G system are derived, respectively. For components following more complex distributions, a simulation method has also been provided. The numerical examples prove that the proposed method is rather accurate and efficient, and it is worth learning in engineering practice.

1. Introduction

Prognostics and health management (PHM) has received considerable attention during the last several decades [1]. Remaining useful life (RUL) prediction results with high precision can provide effective basis for formulating maintenance schedules, and then decrease their risk [1]. As a consequence, RUL prediction has been considered as a significant procedure in PHM [2].

On the other hand, redundancy is an effective way to improve system reliability. As the most common form of redundancy, \( r \)-out-of-\( n \): G system has attracted substantial research interest because of its wide application in both military and industrial field [3]. A \( r \)-out-of-\( n \): G system consists of \( n \) components, and operates iff at least \( r \) of these components could function [4]. It is obvious that in case where \( r=1 \), it is a parallel system and in case where \( r=n \), it is known as a series system [5].

Actually, extensive literature exists with regard to analysis and discussion of \( r \)-out-of-\( n \): G system. After exhaustive reviews on all these developments, conclusions can be drawn that significant improvements closely relevant to this system have been made in reliability evaluation. The most recent research work in this direction can be seen in [3,6,7]. Little literature has focused on the residual life estimation of this system.

Recently, RUL prediction has received more and more consideration in reliability field [8] and is of paramount importance. Let \( F(t) \) designate the cumulative distribution function (CDF) of lifetime, then the CDF of RUL at current time point \( \tau \) takes the following form [8].

\[
F_{RUL}(t) = \frac{F(t + \tau) - F(\tau)}{1 - F(\tau)}
\]  

(1)
And the probability density function (PDF) of RUL is the derivation of the CDF, which can be given by

\[ f_{RUL}(t) = \frac{dF_{RUL}(t)}{dt} = \frac{f(t + \tau)}{1 - F(\tau)} \]  

(2)

Then the RUL prediction at current time \( \tau \) can be calculated by

\[ f_{RUL}(t) = \frac{dF_{RUL}(t)}{dt} = \frac{f(t + \tau)}{1 - F(\tau)} \]  

(3)

According to existing literature, RUL can also be obtained by [9]

\[ RUL_0 = \frac{\int_{\tau}^{\infty} R(t) dt}{R(\tau)} \]  

(4)

Where \( R(t) \) represents the reliability function, and we have \( R(t) = 1 - F(t) \).

System RUL prediction when components were exponentially distributed with different failure rates has been discussed in [9]. However, r-out-of-n: G system hasn’t been mentioned due to the complexity of its system structure. Further, r-out-of-n: G system with Weibull distributed components has been considered in [10], and it has been assumed that all the components were identical, which limited its application. Further, the distribution and expected number of working components under the condition that the weighted r-out-of-n: G system are working at current time have been studied. By fusing test data and available prior information, Bayesian tolerance limits have been analyzed [11]. And great contributions have been made in [5] and the mean residual life (MRL) function of r-out-of-n: G system has been studied when all components were identical and performed at current time. Based on this research, Bayramoglu et al. [12,13] considered the case where components were nonidentical. Further, some properties of mean conditional MRL of voting system are working have been obtained [14]. And for more generalized models, readers can refer to [15-17].

After exhaustive reviews on all these developments, conclusions can be drawn that the discussion of RUL prediction for voting system is insufficient. Although quite a bit of work has been done, we have not come across any research on the derivation of RUL prediction of r-out-of-n: G system when components follow exponential distribution or Weibull distribution. Motivated by this gap, we predict the RUL of r-out-of-n: G system with exponentially or Weibull distributed components. For components following more complex distribution, an effective and precise simulation method will be proposed as well.

The rest of this paper is structured as follows. RUL prediction of r-out-of-n: G system with nonidentical components following exponentially distribution is introduced in next section. In Section 3, Weibull distributed components are considered. A simulation method will be provided in Section 4. In the following section, numerical examples are illustrated and Section 6 summarizes the conclusion.

2. RUL prediction of r-out-of-n: G system with exponentially distributed components

According to the definition of r-out-of-n: G system, the reliability function can be denoted by.

\[ R_k(t) = \sum_{k=0}^{n} \prod_{i=0}^{k} (1 - F_{ij}) \prod_{j=k+1}^{n} F_{ij} \]  

(5)

Where \( i_1, i_2, \ldots, i_k \) is the permutation of any \( k \) numbers of 1,2,3,\ldots,n as defined in [9] and \( (j) \) represents the remaining \( (n-k) \) numbers. In existing literature, it is often assumed that all the components in the system are identical, therefore, the reliability function can be simplified as
In reliability engineering, exponential distribution has been widely used with the reliability function

\[ R(t) = \exp(-\lambda t) \]  

According to (6), when all the components are nonidentical and exponentially distributed with failure rates \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \), the reliability function of this \( r \)-out-of-\( n \): G system can be calculated by

\[ R_s(t) = \sum_{m=r}^{n} \binom{n}{m} (-1)^{n-m} \left[ 1 - R(t) \right]^{m-n} \]  

Assuming that this system performs at current time \( \tau \), then according to (4), the useful lifetime (UL) prediction of this system is

\[ E_s(T) = \int_{0}^{\infty} R_s(t) \, dt = \sum_{m=r}^{n} \binom{n}{m} (-1)^{n-m} \left[ \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_m} \right] \]  

An example of a high priority freight train, which is structured as 3-out-of-4: G system with four locomotives, can be given here again [18]. This system operates iff at least three locomotives. Obviously, in this case, \( n = 4 \) and \( r = 3 \), then the UL of this system obtained by (9) can be shown by

\[ E_s(T) = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} + \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_{r+1}} + \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_{r+1} + \lambda_{r+2}} - \frac{3}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} \]  

This expression is consistent with the conclusion in [12,13].

Meanwhile the system RUL is

\[ E_s(T_r) = \int_{0}^{\infty} \sum_{m=r}^{n} \binom{n}{m} (-1)^{n-m} \left[ \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_m} \right] \, dt = \sum_{m=r}^{n} \binom{n}{m} (-1)^{n-m} \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_m} \]  

Specially, when failure rates are all equal ( \( \lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_n = \lambda \) ), the reliability function can be obtained by

\[ R_s(t) = \sum_{m=r}^{n} \binom{n}{m} (-1)^{n-m} \binom{n-m}{k} \left( \frac{1}{\lambda} \right)^{n-m} \]  

Then the UL is [19]

\[ E_s(T) = \frac{1}{\lambda} \]  

And the RUL prediction can be obtained by simplifying (11), which is

\[ E_s(T_r) = \frac{1}{\lambda^2} \]
As aforementioned in the introduction, series and parallel system are special cases of r-out-of-n: G system with r = n and r = 1. By substituting the values of r in (11), the RUL of series and parallel system with nonidentical components are

\[
E_{r,s}(T_s) = \frac{1}{\sum_{i=1}^{n} \lambda_i}
\]

(15)

And

\[
E_{r,p}(T_p) = \frac{\sum_{k=1}^{n} (-1)^{k+1} \sum_{i_{1},\ldots,i_{k}} \frac{1}{\lambda_{i_{1}} + \cdots + \lambda_{i_{k}}} e^{\left(-\left(\lambda_{i_{1}} + \cdots + \lambda_{i_{k}}\right) r\right)}}{\sum_{i=1}^{n} (-1)^{i+1} \sum_{i_{1},\ldots,i_{n}} e^{\left(-\left(\lambda_{i_{1}} + \cdots + \lambda_{i_{n}}\right) r\right)}}
\]

(16)

3. RUL prediction of r-out-of-n: G system with Weibull distributed components

Weibull distribution is also widely used in reliability engineering with reliability function [20]

\[
R(t) = \exp \left(-\frac{t}{\eta}^\beta\right)
\]

(17)

Similarly, when all components are identical and follow Weibull distribution, the reliability function of this r-out-of-n: G system can be calculated by (6)

\[
R_s(t) = \sum_{m=0}^{n} \sum_{i=0}^{n-m} (-1)^{-m} C_{n-m}^m [R(t)]^m \]

(18)

Then the UL of this system is

\[
E_s(T_s) = \int_0^\infty \sum_{m=0}^{n} \sum_{i=0}^{n-m} (-1)^{-m} C_{n-m}^m [R(t)]^m dt = \sum_{m=0}^{n} \sum_{i=0}^{n-m} (-1)^{-m} C_{n-m}^m i \beta^{-1} E(T)
\]

(19)

where \(E(T) = \eta \Gamma(1+\frac{1}{\beta})\) is the lifetime expectation of Weibull distribution. Further, the RUL of this system is

\[
E_{s,T}\{T_s\} = \frac{\sum_{m=0}^{n} \sum_{i=0}^{n-m} (-1)^{-m} C_{n-m}^m \int_0^\infty [R(t)]^m dt}{\sum_{m=0}^{n} \sum_{i=0}^{n-m} (-1)^{-m} C_{n-m}^m [R(t)]^m E_s(T)}
\]

(20)

where \(E_{s,T}\{T_s\}\) denotes the RUL of series system consisting of Weibull distributed components. Obviously, the core problem is the calculation of \(E_{s,T}\{T_s\}\). The derivation of \(E_{s,T}\{T_s\}\) has been given in [10], which is

\[
E_{s,T}\{T_s\} = \int_0^\frac{1}{\beta} E(T) - \int_0^\frac{1}{\beta} R(\mu) d\mu
\]

(21)

\[
\int_0^\frac{1}{\beta} R(\mu) d\mu\] can be solved numerically, e.g. by using trapezoidal rule. Specially, when \(\frac{1}{\beta}\) is small enough, it is reasonable to assume that \(R(\mu)\) changes linearly in interval \((0, \frac{1}{\beta})\), then we have

\[
\int_0^\frac{1}{\beta} R(\mu) d\mu = \frac{1}{2} \int_0^\frac{1}{\beta} R(\mu) d\mu
\]
\[
\int_0^\infty R(\mu)d\mu = \frac{1}{\tau^\beta} \left( \frac{1}{2} (1 + R(\tau^\beta)) \right)
\]

Therefore, the approximate solution of \( E_{s}(T; i) \) can be written as

\[
E_{s}(T; i) = \frac{1}{\tau^\beta} \left( \frac{1}{2} (1 + R(\tau^\beta)) \right)
\]

It is important to mention that (22) can also be calculated by using numerical methods and a more precise result can be obtained.

4. Simulation method for RUL prediction of r-out-of-n: G system

Methods provided in Section 2 and 3 for RUL prediction of r-out-of-n: G system is especially intractable when the distributions of components are more complicated. The simulation method apparently can be a useful tool to tackle the problem. In this section, the method for drawing UL sample of r-out-of-n: G system is firstly proposed. Then RUL sample can be generated from UL sample. Further, the RUL will be predicted by the RUL sample.

The steps of this simulation method for calculating the RUL prediction of r-out-of-n: G system has been listed in Algorithm 1.

Algorithm 1

Given lifetime distribution of \( n \) components \( F_i(t) \), simulation number of times \( S \), current time \( \tau \) and \( count = 0 \).

Step 1: Draw UL sample values of all these components \( L_i \) \((i = \ldots, n)\) according to the corresponding \( F_i(t) \).

Step 2: Sort the lifetime sample \( L_i \) \((i = \ldots, n)\) in an ascending order.

Step 3: The UL of this system in this simulation is \( UL_{\text{system}} = L_{\text{r-out-of-}n+1} \).

Step 4: If \( UL_{\text{system}} > \tau \), then \( count = count + 1 \) and \( RUL(count) = UL_{\text{system}} - \tau \).

Step 5: Repeat Steps 1-4 \( S \) times and RUL sample of this r-out-of-n: G system with sample size \( count \) can be obtained.

Then the RUL prediction can be calculated by

\[
RUL_0 = \frac{1}{count} \sum_{i=1}^{count} RUL(count)
\]

5. Case study

To verify the proposed methods in this paper, 3-out-of-4: G systems with components following exponential distribution and Weibull distribution are taken as examples in this section, respectively. It should be noted that our experiment is made on a PC with Intel(R) Core(TM) i5-8250U CPU@1.60GHz and 8G RAM.

As described in Section 2, the UL of this system with \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.01h^{-1} \) can be calculated by (13), which is 58.33h. The RUL of this system when \( \tau = 20h \) can be obtained by (14), which is 46.59h. To verify the UL and RUL prediction results, the simulation method provided in Section 3 can be used and the UL and RUL sample drawn by simulation (simulation time \( S = 5000 \)) can be depicted, respectively.
The UL and RUL prediction calculated by simulation are 58.481h and 46.73h, respectively. It is evident that the analytical method is more efficient and precise.

Further, when components of this system are nonidentical and exponentially distributed with $\lambda_1 = 0.01 h^{-1}$, $\lambda_2 = 0.02 h^{-1}$, $\lambda_3 = 0.03 h^{-1}$ and $\lambda_4 = 0.04 h^{-1}$, the UL prediction and RUL prediction with $\tau = 20h$ can be calculated by (10) and (11), which are 24.56h and 17.37h. Similarly, the UL and RUL sample can be depicted in Figs. 3 and 4.

The UL and RUL prediction calculated by simulation are 24.66h and 18.19h, respectively. It can also be seen that the UL and RUL decrease with larger failure rates. This conclusion is consistent with the results in [13].

Further, the proposed method can also deal with components following Weibull distribution. Assume that all the components of are identically Weibull distributed, the RUL prediction results
can be obtained by using analytical method (using (20) and (21)), approximate method (using (20) and (23)) and simulation method (given in Algorithm 1). Comparisons of RUL predictions under different parameter settings are listed in Table 1.

| n  | r  | η  | β  | Method                  |
|----|----|----|----|-------------------------|
|    |    |    |    | Analytical method       |
| 4  | 3  | 100| 2  | 61.76                   |
| 4  | 3  | 100| 2  | 43.71                   |
| 4  | 3  | 200| 2  | 113.81                  |
| 5  | 2  | 200| 2  | 188.94                  |
|    |    |    |    | Approximate method      |
| 4  | 3  | 100| 2  | 61.81                   |
| 4  | 3  | 100| 2  | 44.70                   |
| 4  | 3  | 200| 2  | 114.10                  |
| 5  | 2  | 200| 2  | 189.12                  |
|    |    |    |    | Simulation method       |
| 4  | 3  | 100| 2  | 60.79                   |
| 4  | 3  | 100| 2  | 43.97                   |
| 4  | 3  | 200| 2  | 114.64                  |
| 5  | 2  | 200| 2  | 189.08                  |

From Table 1, conclusions can be drawn as follows.

- RUL of system can be influenced by many factors, including system structure (n and r), components characteristic (e.g. distribution and parameters values), and current time.
- The analytical method apparently outperforms the other two methods with regard to efficiency and precision. And simulation method is satisfactory at the cost of time consuming. The approximate method performs well when current time r is small, and with the increasing of r , the results get worse.
- The simulation method has advantages of strong adaptability and convenient using. It can deal with components following more complex distributions, e.g. normal distribution, gamma distribution.

6. Conclusions
In this paper, the remaining useful life prediction of r-out-of-n: G system has been discussed. For exponentially and Weibull distributed components, the closed-forms of RUL prediction have been derived. Further, when components follow more complicated distributions, a simulation method has been also provided. Then the accuracy of these proposed methods has been validated in case study and the results prove that this method is feasible and easy to be implemented in engineering practice. Furthermore, similar analytical methods would be explored to solve RUL of cold standby system in our future work.

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