On-State Commutativity of Measurements and Joint Distributions of Their Outcomes

Jan Czajkowski* and Alex B. Grilo†

*QuSoft, University of Amsterdam
†Sorbonne Université, CNRS, LIP6

Abstract

In this note, we analyze joint probability distributions that come from the outcomes of quantum measurements performed on sets of quantum states. First, we identify the properties of these distributions that need to be fulfilled to recover a classical behavior. Secondly, we connect the existence of a joint distribution with the “on-state” permutability (commutativity of more than two operators) of measurement operators. By “on-state” we mean properties of operators that can hold only on a subset of states in the Hilbert space. Then, we disprove a conjecture proposed by Carstens, Ebrahimi, Tabia, and Unruh (eprint 2018), which states that partial on-state permutation imply full on-state permutation. We disprove such a conjecture with a counterexample where pairwise “on-state” commutativity does not imply on-state permutability, unlike in the case where the definition is valid for all states in the Hilbert space.

Finally, we explore the new concept of on-state commutativity by showing a simple proof that if two projections almost on-state commute, then there is a commuting pair of operators that are on-state close to the originals. This result was originally proven by Hasting (Communications in Mathematical Physics, 2019) for general operators.
1 Introduction

In this work we propose a basic formalism for studying classical distributions that come from joint measurements on quantum states.

Our initial motivation comes from studying a conjecture proposed in a recent paper by Carstens, Ebrahimi, Tabia, and Unruh \cite{CETU18}. Their result on quantum indifferentiability relies on a conjecture proposed by them, which informally states that commutation of projectors with respect to a fixed quantum state implies a classical joint distribution of their measurement outcomes. More concretely, they conjecture the following.

**Conjecture 1 (Informal).** If we have a set of \( N \) measurements \( P_1, \ldots, P_N \) that commute on a quantum state \( |\psi\rangle \) then there exist random variables \( X_1, \ldots, X_N \) drawn from a distribution \( D \) such that the marginals of this distribution on \( X_{i_1}, \ldots, X_{i_t} \) correspond to measuring \( |\psi\rangle \) with measurements \( P_{i_1}, \ldots, P_{i_t} \).

Motivated by this conjecture, our goal is to study the behavior of \( N \) random variables \( X_1, X_2, \ldots, X_N \) corresponding to the outcomes of a sequence of quantum measurements that commute on a set of quantum states \( F \subseteq D(H) \). Surprisingly, such type of results has only been carried on for \( F = D(H) \), i.e. measurements commuting on all quantum states.

The focal point of this note is to study which are the necessary and sufficient properties of the quantum setup, so that such a probability distribution is well-defined. With this in hand, we then have two applications. First, we disprove Conjecture \cite{CETU18} Secondly, we show a simpler proof for a variant of the result by Hastings \cite{Has09} on operators that almost-commute on specific states.

To be able to explain our contributions in more details, we will first start with a detour to very basic properties of probability distributions that arise from classical processes. Then, we discuss how these properties could be defined in the quantum setting (but, unfortunately, they do not hold for general quantum setups), and finally we state our results and discuss related works.

1.1 Classical Distributions

We discuss here properties of classical distributions that may be obvious at first but are crucial and not trivial in the quantum world.

In the following, we let \( A, B, C \) be events that come from a classical experiment. We denote the event corresponding to \( A \) not happening as \( \overline{A} \), the probability that \( A \) and \( B \) happen together as \( \mathbb{P}[A, B] \), and the probability that \( A \) happens conditioned on the fact that event \( B \) happens as \( \mathbb{P}[A | B] = \frac{\mathbb{P}[A, B]}{\mathbb{P}[B]} \) (assuming \( \mathbb{P}[B] \neq 0 \)).

The first property that we want to recall on classical distributions is that we can compute the marginals of the distribution when given the a joint distribution:

**Property 1 (Classical Marginals).** \( \mathbb{P}[A \mid C] = \mathbb{P}[A, B \mid C] + \mathbb{P}[A, B \mid C] \).

A second property that we want to recall is that the probability that \( A \) and \( \overline{A} \) occur is 0, even when considering other events:

**Property 2 (Classical Disjointness).** \( \mathbb{P}[A, B \mid \overline{A}] = \mathbb{P}[A, B, \overline{A}] = 0 \).

---

1Indifferentiability is a strong security notion capturing security of cryptographic constructions such as hash functions, where we require that any polynomial-time adversary cannot distinguish if it has access to a cryptographic hash function or an ideal random function, even if she has access to some internal auxiliary functions used to construct the hash function. The quantum version assumes the adversary makes quantum queries.

2See Conjecture \cite{CETU18} for the formal statement.

3Informally, two operators \( A \) and \( B \) commute on \( |\psi\rangle \) if \( [A, B]|\psi\rangle = 0 \).
Another property that we have classically is reducibility, which says that probability of events \( A \) and \( A \) happening is the same as the probability of \( A \).

**Property 3** (Classical Reducibility). \( \mathbb{P}[A, B \mid A] = \mathbb{P}[A, B, A] = \mathbb{P}[A, B] \).

Finally, the last property we study is sequential independence of random variables. Roughly, this property just says that the probability that event \( A \) happens and that event \( B \) happens is the same as the probability that event \( B \) happens and that event \( A \) happens.

**Property 4** (Classical Sequential Independence). \( \mathbb{P}[A \mid B] \mathbb{P}[B] = \mathbb{P}[A, B] = \mathbb{P}[B \mid A] \mathbb{P}[A] \).

We stress that these properties hold trivially for all classical distributions.

### 1.2 Quantum Distributions and their Properties

Our goal is to find necessary conditions for existence of a classical description of the experiment where we perform a sequence of \( N \) general measurements, despite the order. More concretely, we aim to find the properties of measurement operators \( Q_1, \ldots, Q_N \) on specific subsets of quantum states \( \mathcal{F} \) so that there is a joint distribution of \( X_1, X_2, \ldots, X_N \) such that marginals of this distribution on \( X_{i_1}, \ldots, X_{i_t} \) correspond to measuring a state \( |\psi\rangle \in \mathcal{F} \) with measurements \( Q_{i_1}, \ldots, Q_{i_t} \). In this case, we call it a quantum distribution.

The main obstacle in this task is the fact that quantum measurements do not commute, unlike in the classical world: the chosen order for performing the measurements influences the final probability distribution of the joint measurement outcomes. Because of that, we will consider the quantum analog of Properties 1 to 4 and study when such properties hold in the quantum case, and their implication for having such a joint distribution. Our connections closely follow \cite{ME84}, where they show that the existence of a joint distribution for two arbitrary quantum observables (Hermitian operators) on every quantum state is equivalent to their commutation. In this work, we show how to extend their analysis in two ways: we are interested in multiple observables and we consider specific sets of quantum states. In order to carry on this analysis, we extend the properties described in Section 1.1 to quantum measurements and study their connections. We leave the formal definitions of the quantum analogs of these classical properties to Section 3.1.

### 1.3 Our Results

Using the formalism described in the previous section, we achieve the following connections between joint quantum distributions and the measurement operators.

First, we show that quantumly, the marginal property also implies the sequential independence one.

**Result 1** (Informal statement of Theorem 1). If a joint distribution has the quantum marginal property, then it also has the quantum sequential independence property.

Then, we show that in the on-state case, we have that there is a quantum joint distribution iff all operators permute.\(^4\) This is a generalization of the classic results from \cite{Nel67, Fin73, Fin82, ME84} to the on-state case.

**Result 2** (Informal statement of Theorem 3). Fix a set of quantum states \( \mathcal{F} \). A set of measurements yield a quantum joint distribution on each state in \( \mathcal{F} \) iff these operators permute on every state in \( \mathcal{F} \).

\(^4\)Informally, a set of operators permutes on \( |\psi\rangle \) if applied in any order they yield the same state: \( A_1 \cdots A_N |\psi\rangle = A_{\sigma(1)} \cdots A_{\sigma(N)} |\psi\rangle \), where \( \sigma \) is a permutation.
Then, we show that pairwise on-state commutation does not imply full on-state permutation, unlike in the case of permutation on all states. This fact—that we prove via a numerical example—together with Result 3 implies that Conjecture 1 is false.

**Result 3.** Conjecture 1 is false.

Finally, our last result is a simpler proof for a restricted version of Theorem 1 in [Has09], which states that if two operators \( A \) and \( B \) almost-commute, we can find commuting operators \( A' \) and \( B' \) that are close to \( A \) and \( B \), respectively. In our case, we consider on-state commutation instead of the regular one, and unlike in [Has09], our proof works only for projectors.

**Result 4** (Making almost commuting projectors commute). Given any two projectors \( P_1 \) and \( P_2 \) and a state \( |\psi\rangle \) we have that if \( \|P_1 P_2 - P_2 P_1|\psi\rangle\| = \epsilon \) then there is a projector \( P' \) that is close to the original projector on the state \( \|P' P_2 - P_2 P'\| \leq \sqrt{2}\epsilon \) and \([P_1, P'] = 0\).

1.4 Related Work

The result that is prominent in the literature is that a joint distribution for a set of measurements exists iff all the operators pairwise commute. Different versions of this result were previously proven: In [Nel67] the author considers the case of continuous variables and \( N \) observables. A similar result but without specifying the Hilbert space is achieved with different mathematical tools in [Fin82]. In the specific case where we have only two observables, we mention three works; In [Fin73] and [MES4] the authors prove the classic problem in a way similar to each other, but using different mathematical tools. All but the first work mentioned here focus on the joint distribution as a functional from the space of states. An approach using \(*\)-algebras was presented by Hans Maassen in [Maa06, Maa10].

The authors of [GN02] analyze the case of general measurements but prove that the measurement operators pairwise commute iff the square root operators permute (Corollaries 3 and 6 in [GN02]), in the sense of our Definition 3 (for all states in \( \mathcal{H} \)). In general the problem of conditional probabilities in Quantum Mechanics was discussed by Cassinelli and Zanghi in [CZ83].

In [Lin97, FR96] the authors prove that for any two Hermitian matrices if their commutator has a small norm, then there are operators close to the origins that fully commute. In [Has09] Hastings proves how close the new operators are in terms of the norm of the commutator.

Organization

In Section 2 we provide some preliminaries. Then in Section 3 we discuss quantum distributions and their properties. Finally, in Section 4 we discuss the almost-commuting case.

Acknowledgements

JC thanks Dominique Unruh and Christian Schaffner for helpful discussions. JC was supported by a NWO VIDI grant (Project No. 639.022.519). Most of this work was done while AG was affiliated to CWI and QuSoft.

2 Preliminaries

2.1 Notation

In this work, we are going to use calligraphic letters (\( \mathcal{S}, \mathcal{R}, \ldots \)) to denote sets. We denote \([N] := \{1, 2, \ldots, N\} \). For \( S \subseteq [N] \), we denote by \( S(i) \) the \( i \)-th element of the set \( S \) in the ascending order. For some fixed sets \( \mathcal{X}_1, \ldots, \mathcal{X}_N \), we denote \( \vec{x} \) an element of \( \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \) and for \( S \subseteq [N] \) we have \( \vec{x}_S := (x_{S(1)}, \ldots, x_{S(|S|)}) \). We denote the set of all \( t \)-element permutations by \( \Sigma_t \). For some complex number \( c = a + bi \), we define \( \Re(c) = a \) as its real part.
2.2 Quantum Measurements

We briefly review some concepts in quantum computation/information and we refer to [NC11] for a more detailed introduction to these topics.

Quantum states are represented by positive semi-definite operators with unit trace, i.e., $\rho \succeq 0$, $\text{Tr}(\rho) = 1$. We denote the set of all density operators on some Hilbert space $\mathcal{H}$ by $\mathcal{D}(\mathcal{H})$.

To describe general measurements, we use the notion of Positive Operator Valued Measure (POVM). The only requirement of POVMs is that they consist of positive operators and sum up to the identity operator. More formally, a POVM with set of outcomes $\mathcal{X}$ is described by a set of operators $\mathcal{M} = \{Q^x\}_{x \in \mathcal{X}}$, where $\forall x \in \mathcal{X} : Q^x \succeq 0$, and $\sum_{x \in \mathcal{X}} Q^x = \mathbb{1}$.

We denote the probability of getting the outcome $x$ when measuring $\rho$ with the measurement $\mathcal{M}$ by $P[x \leftarrow \mathcal{M}(\rho)] := \text{Tr}(Q^x \rho)$. To describe the post-measurement state, we can write down operators of $\mathcal{M}$ as products of linear operators on $\mathcal{H}$ (denoted by $\mathcal{L}(\mathcal{H})$), $Q^x = A^x A^x \dagger$, where $A^x \in \mathcal{L}(\mathcal{H})$ (such a decomposition is always possible since $Q^x \succeq 0$). The post-measurement state when the outcome of $\mathcal{M}$ on $\rho$ is $x$ is given by

$$\rho_x := \frac{A^x \rho A^x \dagger}{\text{Tr}(Q^x \rho)}.$$ (1)

The operator $A^x$ is called the square root of $Q^x$.

3 Quantum Distributions

In this section, we study the description of the statistics of outcomes derived from a sequence of measurements. Our approach is to consider the quantum version of the classical properties described in Section 1.1. Since quantum measurements do not commute in general, these quantum properties do not always hold. We then study the connection of properties of the measurements and the properties of their outcome distribution.

The structure of our proofs follows [MES4], where they show that for two Hermitian observables there is a joint distribution for the outcomes of their joint measurement if they commute. We stress that the result in [MES4] only works for measurements that commute on every quantum state and our result extends it to the case of joint distributions defined on a limited set of states.

In the following, we denote the observables with $Q$ and their square-roots with $R$. In Section 3.1 we define the quantum analogues of the classical properties of distributions defined in Section 1.1. Then, in Section 3.2, we state and prove the main result of this section, where we show a connection between existence of a distribution and permutability—a generalization of commutativity—of the corresponding measurement operators.

3.1 Quantum Distributions

We analyse a functional from the set of density operators and measurable sets of $N$ random variables $X_1, X_2, \ldots, X_N$ to reals $W_{[N]} : \mathcal{D}(\mathcal{H}) \times (X_1 \times \cdots \times X_N) \rightarrow [0,1]$. We define this functional as

$$W^\rho_{[N]}(\vec{x}) := \text{Tr} \left( Q^x_{[N]} \rho \right),$$ (2)

where $Q^x_{[N]}$ is a positive semidefinite operator corresponding to the outcome $\vec{x} \in X_1 \times \cdots \times X_N$. The subscript of $W$ denotes the set of random variables that the distribution is defined on. Such definition is similar to the one proposed by [MES4].

Given the definition of $W$, we can state conditions so that it can be seen as a joint quantum distribution:

---

5Note that the superscript of $W^\rho_{[N]}(\vec{x})$ denotes the first input to the functional, so we have $W_{[N]}(\rho, \vec{x})$. 
We start with a sequence of general measurements (i.e. POVMs): measurements corresponding to $Q_i$ are a partition of $\mathcal{X}_i$, where $\mathcal{X}_i$ define the conditional distribution for any sequence $\vec{x}$. We then consider the post-measurement states generated by sequences of measurements, for a set $\mathcal{F} \subseteq \mathcal{D}(\mathcal{H})$, and for $\mathcal{U} \subseteq [N]$, we define $Q_{\mathcal{U}}^\mathcal{F}$ to be measurement operators where $\vec{y} \in \mathcal{X}_{\mathcal{S}(1)} \times \cdots \times \mathcal{X}_{\mathcal{S}(|\mathcal{S}|)}$

Given $Q_{\mathcal{S}}^\mathcal{F}$ and their corresponding square-roots $R_{\mathcal{S}}^\mathcal{F}$, we have that $\text{Tr} \left( R_{\mathcal{S}}^\mathcal{F} \rho R_{\mathcal{S}}^{\mathcal{F}} \right) \neq 0$, then we define the conditional distribution for any sequence $\vec{x}$ as

$$W_{[N]}^\rho(\vec{x} | \vec{y}) := W_{[N]}^{\rho R_{\mathcal{S}}^\mathcal{F}}(\vec{x})/\text{Tr}R_{\mathcal{S}}^{\mathcal{F}} \rho R_{\mathcal{S}}^{\mathcal{F}}.$$  

For all those measurements, for a set $\mathcal{F} \subseteq \mathcal{D}(\mathcal{H})$, and for $\mathcal{U} \subseteq [N]$, we define the “orbit” of the post-measurement states. For any $\mathcal{T} \subseteq [N]$ of size $t$, we take $s \leq t$ sets $\mathcal{S}_1, \ldots, \mathcal{S}_s$ that are a partition of $\mathcal{T}$. We then consider the post-measurement states generated by sequences of measurements corresponding to $\mathcal{S}$:

$$G_{\mathcal{U}}(\mathcal{F}) := \left\{ R_{\mathcal{S}_1}^{\mathcal{F}} \cdots R_{\mathcal{S}_s}^{\mathcal{F}} \psi R_{\mathcal{S}_1}^{\mathcal{F} \dagger} \cdots R_{\mathcal{S}_s}^{\mathcal{F} \dagger} / \text{Tr} \left( R_{\mathcal{S}_1}^{\mathcal{F}} \cdots R_{\mathcal{S}_s}^{\mathcal{F}} \psi R_{\mathcal{S}_1}^{\mathcal{F} \dagger} \cdots R_{\mathcal{S}_s}^{\mathcal{F} \dagger} \right) : \psi \in \mathcal{F}, \mathcal{T} \subseteq \mathcal{U}, \right\}$$

where $R_{\mathcal{S}}^{\mathcal{F}}$ are the square root operators of $Q_{\mathcal{S}}^{\mathcal{F}} = R_{\mathcal{S}}^{\mathcal{F} \dagger} R_{\mathcal{S}}^{\mathcal{F}}$. The subscript of $G$ denotes the set we take the subsets of, usually it is $[N]$ but later we also consider $[N] \setminus \mathcal{S}$ for some $\mathcal{S}$.

With our quantum marginals property, we require that the operator we get after we sum over a subset of variables is still a valid measurement operator.

**Property 5 (Quantum Marginals).** We say that the joint distribution $W$ has the quantum marginals property if for every $\mathcal{S} \subseteq [N]$, there is a measurement $\mathcal{M}_\mathcal{S} = \{Q_{\vec{y}}^{\mathcal{S}}\}_{\vec{y} \in \mathcal{X}_\mathcal{S}}$ such that for every value $\vec{x} \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N$, for which we have $\vec{x} := (x_1, x_2, \ldots, x_N)$ and for every density operators $\rho \in G_{[N]}(\mathcal{F})$ defined as in Equation (4) we have that

$$W_{\mathcal{S}}^\rho(\vec{x} | \vec{y}) := \text{Tr} \left( Q_{\vec{x} \mathcal{S}}^{\mathcal{S} \mathcal{F}} \rho \right) = \sum_{x_i \in \mathcal{X}_i, i \in [N] \setminus \mathcal{S}} W_{[N]}^\rho(\vec{x}).$$

Additionally for $|\mathcal{S}| = 1$ the operators $Q_{\vec{x}}^{\mathcal{S}}$ are the operators from $\mathcal{M}_{\mathcal{I}}$.

**Disjointness.** It follow from the definition of POVMs that the quantum measurement operators need not be orthogonal, and this implies that the disjointness property does not hold in generality quantumly.

Disjointness is a property that concerns a post measurement state of a set $\mathcal{S}$ of variables. To ensure the existence of a measurement operator corresponding to $\mathcal{S}$, we need assume Property 5.

**Property 6 (Quantum Disjointness).** Let $W$ be a joint distribution for which Property 5 holds. We say that $W$ has the quantum disjointness property if for every subset $\mathcal{S} \subseteq [N]$, for every density operator $\rho \in G_{[N] \setminus \mathcal{S}}(\mathcal{F})$, and for every value $\vec{x} \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N$ and $\vec{y} \in \prod_{i \in \mathcal{S}} \mathcal{X}_i$, we have that $\vec{y} \neq \vec{x} \mathcal{S}$, then

$$W_{[N]}^\rho(\vec{x} | \vec{y}) W_{[N]}^\rho(\vec{y}) = \text{Tr} \left( Q_{\vec{x} \mathcal{S}}^{\mathcal{S} \mathcal{F}} R_{\mathcal{S}}^{\mathcal{F}} \rho R_{\mathcal{S}}^{\mathcal{F}} \right) = 0.$$
Reducibility. Recallicity is a similar property to the disjointness but with the key difference that we condition on the same event:

**Property 7** (Quantum Reducibility). Let \( W \) be a joint distribution for which Property 6 holds. We say that \( W \) has the quantum reducibility property if for every subset \( S \subseteq [N] \), every density operator \( \rho \in \mathcal{G}_{[N]}(\mathcal{S}(F)) \), and value \( \vec{x} \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \), we have that

\[
W_{[N]}(\vec{x} | \vec{x}S)\mathcal{W}_{[N]}(\vec{x}S) = \text{Tr} \left( Q_{[N]}^\vec{x} R_{\vec{x}S}^N \rho R_{\vec{x}S}^{\vec{x}+1} \right) = \text{Tr} \left( Q_{[N]}^\vec{x} \rho \right). \tag{7}
\]

Note that the last two properties together allow us to interpret that the operators are (morally) on-state projections: Property 6 plays the role of different projectors being orthogonal and Property 7 that projecting twice to the same space does not change the resulting state.

More concretely, we say that \( R_S \) are no-state projectors on \( \psi \in F \) if for all \( S \subseteq [N] \), all \( \vec{x} \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \), all \( \vec{y} \in \prod_{i \in S} \mathcal{X}_i \), and for \( R_{\vec{y}}^S := R_{S(1)}^y R_{S(2)}^y \cdots R_{S(t)}^y \) and \( Q_{[N]}^\vec{y} := R_{\vec{y}}^S R_{\vec{y}}^{\vec{x}+1} \) (similarly for \([N] \)), we have

\[
\text{Tr} \left( R_{\vec{y}}^S Q_{[N]}^\vec{x} R_{\vec{x}S}^N \psi \right) = \delta_{\vec{y},\vec{x}} \text{Tr} \left( Q_{[N]}^\vec{x} \psi \right). \tag{8}
\]

**Sequential Independence** As previously discussed, the notion of time order in the quantum setting is much more delicate as the probabilistic events no longer commute. Let us go back to the example of the simple sequence \((A, B)\) from Section 1.1 but now consider \(A\) and \(B\) as quantum observables measured with the state \( \rho \). Let us assume for simplicity that \(A\) and \(B\) are projections. The probability of measuring \(a\) with \(A\) is \(\text{Tr}(A\rho)\) and the state after this measurement if \(\rho_a := \frac{A\rho A}{\text{Tr}(A\rho)}\) so the probability of measuring the sequence \((a, b)\) equals

\[
\mathbb{P}[b \leftarrow B(a)] = \mathbb{P}[a \leftarrow A(\rho)] = \text{Tr} \left( B \frac{A\rho A}{\text{Tr}(A\rho)} \right) \text{Tr}(A\rho) = \text{Tr}(A B A\rho). \tag{9}
\]

On the other hand the probability of measuring the sequence \((b, a)\) equals \(\text{Tr}(B A B \rho)\) which is in general different than Equation (9). This simple example shows that sequential independence is not attained by all quantum joint probabilities. More formally, the notion of sequential independence from [GN02] for quantum joint probabilities can be stated as follows.

**Property 8** (Quantum Sequential Independence). Let \( W \) be a joint distribution for which Property 6 holds. We say that \( W \) has the quantum sequential independence property if for every density operator \( \psi \in F \), for any \( T \subseteq [N] \) of size \( t \), for all \( s \leq t \), partition \( S_1, \ldots, S_s \) of \( T \), permutation \( \sigma \in \Sigma_s \), and \( \vec{x} \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N \) such that \( \vec{x} := (x_1, x_2, \ldots, x_N) \) and \( \vec{y}_i := (x_{S_i(1)}, x_{S_i(2)}, \ldots, x_{S_i(|S_i|)}) \), we have that

\[
\text{Tr} \left( Q_{S_{\sigma(1)}}^\vec{y}_1 R_{S_{\sigma(1)-1}}^{\vec{y}_{\sigma(1)-1}} \cdots R_{S_{\sigma(t)}}^\vec{y}_t \psi R_{S_{\sigma(t)}^{\vec{y}_{\sigma(t)}}} \cdots R_{S_{\sigma(t)-1}}^{\vec{y}_{\sigma(t)-1}} \right)
\]

\[
= \text{Tr} \left( Q_{S_{\sigma(1)}}^\vec{y}_1 R_{S_{\sigma(1)-1}}^{\vec{y}_{\sigma(1)-1}} \cdots R_{S_{\sigma(t)}}^\vec{y}_t \psi R_{S_{\sigma(t)}^{\vec{y}_{\sigma(t)}}} \cdots R_{S_{\sigma(t)-1}}^{\vec{y}_{\sigma(t)-1}} \right). \tag{10}
\]

**Theorem 1** (Marginals imply Sequential Independence). If a joint distribution \( W \) has Property 5, 6, and 7, then it also has Property 8.

**Proof.** First note that Properties 5, 6 and 7 directly imply Equation (8).
We now prove the statement for two sets of indices and then argue how to extend it for \(s\) sets. In our restricted case, we have that
\[
\text{Tr} \left( Q_{\vec{y}_1} R_{\vec{y}_2}^\dagger \psi R_{\vec{y}_2} \right) = \text{Tr} \left( \sum_{\vec{y}} Q_{\vec{y}} R_{\vec{S}_1 \cup \vec{S}_2} R_{\vec{y}_2}^\dagger \right)
\]
(11)
\[
= \text{Tr} \left( Q_{\vec{S}_1 \cup \vec{S}_2} R_{\vec{y}_2}^\dagger \psi R_{\vec{y}_2} \right)
\]
(12)
\[
= \text{Tr} \left( Q_{\vec{S}_1 \cup \vec{S}_2} \psi \right),
\]
(13)
where the first equality comes from the marginals property, the second equality comes from the disjointness property if we take \(\vec{y}\) that agrees with \(\vec{y}_2\) on \(\vec{S}_2\), and the last equality comes from the reducibility property.

The above derivation can be repeated for any other two sets \(\vec{S}_1'\) and \(\vec{S}_2'\) such that \(\vec{S}_1' \cup \vec{S}_2' = \vec{S}_1 \cup \vec{S}_2\). Any pair like this will yield \(\text{Tr} \left( Q_{\vec{S}_1' \cup \vec{S}_2'} \psi \right)\) that equals \(\text{Tr} \left( Q_{\vec{S}_1' \cup \vec{S}_2'} \psi \right)\) for all other sets \(\vec{S}_1'\) and \(\vec{S}_2'\), hence we have proven Property \(8\) for any two subsets.

In general, we prove a slightly stronger statement than Property \(8\). We prove that not only different sequences give the same probabilities but also that these probabilities equal \(\text{Tr} \left( Q_{\vec{y}} \psi \right)\) for some operator \(Q_{\vec{y}}\).

The general case holds by taking \(\rho \in G_{[N]} \setminus S_1(F)\) instead of \(\psi\) and use it in the above calculation. To prove Property \(8\) for any \(s\) sets we consider \(\rho = R_{\vec{S}_2} \cdots R_{\vec{y}_{s-1}} \psi R_{\vec{y}_1}^\dagger \cdots R_{\vec{y}_{s-1}}^\dagger\) and \(\text{Tr} \left( Q_{\vec{y}_1} \rho \right)\). We shave off operators from \(\rho\) one by one using Equation (13). After \(s\) steps we have that \(\text{Tr} \left( Q_{\vec{y}_1} \rho \right) = \text{Tr} \left( Q_{\vec{y}_1} \psi \right)\). Again, repeating this procedure for a different \(\rho'\) with sets that sum to the same \(T\) yields the claimed result. \(\square\)

### 3.2 Joint Distributions and Permutability

In this section, we state the definition of a joint distribution that describes a sequence of quantum measurements done on states from a small set. We also define a generalization of commutativity and prove that a joint distribution exists if and only if the measurement operators are permutable.

**Definition 2 (Quantum Joint Distribution On State).** A joint distribution of \(N\) random variables \(X_1, \ldots, X_N\) that describe outcomes of general quantum measurements of states \(\psi \in F\) is defined as a positive, normalized, and linear functional
\[
W_{[N]} : \mathcal{D}(\mathcal{H}) \times (\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_N) \rightarrow [0, 1],
\]
for which

1. Quantum Marginals on states in \(F\), Property \(5\) holds,
2. Quantum Disjointness on states in \(F\), Property \(6\) holds,
3. Quantum Reducibility on states in \(F\), Property \(7\) holds,
4. Quantum Sequential Independence on states in \(F\), Property \(8\) holds.

Note that we need a strong assumption on the set of states the marginals exist for, the set \(G\) is defined in Equation (14). Next we show the connection of existence of joint distributions with requirements on the measurement operators. But first let us define the notion of on-state permutability.
**Definition 3** (On-state permutator, (fully) permutable operators). For any $s$ operators $R_i$ and a permutation of the $s$-element set $\sigma \in \Sigma_s$ the permutator on $\psi$ is defined as

$$[R_1, R_2, \ldots, R_s]_\psi(\sigma) := \text{Tr} \left( R_s R_{s-1} \cdots R_1 \psi R_1^\dagger R_2^\dagger \cdots R_s^\dagger \right) - \text{Tr} \left( R_{\sigma(s)} R_{\sigma(s-1)} \cdots R_{\sigma(1)} \psi R_{\sigma(1)}^\dagger R_{\sigma(2)}^\dagger \cdots R_{\sigma(s)}^\dagger \right).$$

(15)

We say that the operators $R_1, \ldots, R_s$ are permutable if $[R_1, \ldots, R_s]_\psi(\sigma) = 0$ for all $\sigma \in \Sigma_s$. Moreover, we call a set of measurements fully permutable if all operators $R_i^{ij}$ of all measurements $M_i$ permute on $\psi \in \mathcal{F}$ for all $\sigma \in \Sigma_s$.

Now we state the theorem connecting existence of joint distributions with permutability of the measurement operators. This statement extends Theorem 3.2 of [MES1], where they prove that if the joint distribution satisfies the marginals property (they use the name “nondisturbance”), then the operators pairwise commute.

**Theorem 4** (Quantum Joint Distribution and Permutability). There is a quantum joint distribution on states $\psi \in \mathcal{F}$ describing measurement outcomes of $N$ observables $X_1, \ldots, X_N$ if and only if all square roots $R_i^x$ of $Q_i^x$ operators of measurements $M_i$ permute on $\psi \in \mathcal{F}$ and are on-state projectors according to Equation (9).

**Proof.** ($\Rightarrow$) It follows directly from Property [S] for $N$ single-element sets $S_i = \{i\}$.

The other direction of the proof follows by setting the measurement operators to $Q_i^x := R_i^{x1} S_i^{x1} \cdots R_i^{xT} S_i^{xT}$, for every set $S \subseteq [N]$ with $|S| = t$. Similarly we define $R_S^{ij} := R_i^{1j} S_i^{1j} \cdots R_i^{Tj} S_i^{Tj}$.

The marginals property follows from the fact that $\sum_{x_i \in X_i} Q_i^{x_i} = 1$:

$$\sum_{x_j \in X_j, i \in [N] \setminus S} \text{Tr} \left( Q_i^{x_i} R_i^{xj} \rho R_i^{xj} \right) = \sum_{x_j \in X_j, i \in [N] \setminus (S \cup \{i\})} \text{Tr} \left( Q_i^{xj} R_i^{xj} \rho R_i^{xj} \right) = \cdots = \text{Tr} \left( Q_S^{ij} \rho \right).$$

(16)

Properties [S] and [T] are a natural consequence of Equation (9). The only difference between the properties and on-state projections is the set $G_{[N] \setminus S}(\mathcal{F})$ versus just $\mathcal{F}$. Nonetheless, with our definition of $R_S^{ij}$, Equation (9) implies disjointness and reducibility.

In Theorem 3.3 we have already proved that marginals together with disjointness and reducibility imply sequential independence, which concludes our proof.

3.3 Pairwise on-state commutation does not imply full commutation

We now investigate whether full permutability is the weakest assumption we can have for joint distributions to exist.

When we consider this question for the full Hilbert space $\mathcal{F} = D(\mathcal{H})$, this problem has been considered by a number of works in the literature [Nel67, Fin73, Fin82, MES1] and it is well-known that it suffices for the measurement operators to pairwise commute, i.e., pairwise commutation on all possible quantum states implies permutability of the operators.

Our goal in this section is to consider the case where $\mathcal{F} \subseteq D(\mathcal{H})$. In particular, in CETU18, in order connect perfect quantum indifferentiability to classical indifferentiability with stateless simulators, they relied on the following conjecture.

---

*Roughly, we say that $A$ is classical (quantum) indifferentiable from $B$ iff we can map classical (resp. quantum) attacks on $A$ to classical (resp. quantum) attacks on $B$ using simulators. Moreover, we say that the simulator is stateless if it does not store any internal state.*
Conjecture 2 (Conjecture 2 from [CETU18]). Consider N binary measurements described by projectors $P_1, \ldots, P_N$, and a quantum state $|\Psi\rangle$.

Assume that any $t$ out of $N$ measurements permute on state $|\Psi\rangle$. That is, for any $I$ with $|I| = t$, if $P'_1, \ldots, P'_t$ and $P''_1, \ldots, P''_t$ are the projectors $\{P_i\}_{i \in I}$ (possibly in different order), then $P'_t, \ldots, P'_1|\Psi\rangle = P''_t, \ldots, P''_1|\Psi\rangle$.

Then there exist random variables $X_1, \ldots, X_N$ with a distribution $D$ such that for any $I = \{i_1, \ldots, i_t\}$ the joint distribution of $X_{i_1}, \ldots, X_{i_t}$ is the distribution of the outcomes when we measure $|\Psi\rangle$ with measurements $P_{i_1}, \ldots, P_{i_t}$.

Conjecture 2 states that if any t measurement operators permute then there is a joint distribution. From Theorem 4, we know that if there is a joint distribution then the operators fully permute. Hence, the key point of the conjecture is that if any $t$ operators permute on a state, then they fully permute on it. However, we show here that Conjecture 2 is not true, in general.

Lemma 5. There is a set of four projectors $\{P_1, P_2, P_3, P_4\}$ and a state $|\phi\rangle$ such that the projectors are 2-permutable (they pairwise commute) on state $|\phi\rangle$ and they are not 4-permutable on $|\phi\rangle$.

Proof. To prove the statement we found an example of such operators and a state numerically by random constrained search. We consider 4 projectors $P_i$ and a state $|\phi\rangle$ of dimension 8. The constraints we put are

$$\forall i, j \neq i \ [P_i, P_j]|\phi\rangle = 0,$$

moreover operators $P_i$ are projectors: $\forall i P_i^2 = P_i$ and $|\phi\rangle$ is a unit norm complex vector.

We look for an example to the statement that 2-permutability (commutativity) does not imply 4-permutability, so that

$$(P_1 P_2 P_3 P_4 - P_3 P_4 P_1 P_2)|\phi\rangle \neq 0.$$

To find such example we used software for symbolic computing to define the problem and—using random search—maximize $\|(P_1 P_2 P_3 P_4 - P_3 P_4 P_1 P_2)|\phi\rangle\|$, where we maximize over operators $P_i$ and the state $|\phi\rangle$.

The result of our optimization can be found in Appendix A. Note that however we consider vector equalities—instead of just traces like in Theorem 4—our example provides a stronger argument for the necessity of full permutability.

As one can notice looking at the optimization problem that we have it is not a semidefinite problem, nor has any other structure that is easy to exploit. For that reason finding larger instances is computationally very expensive.

We notice that Lemma 5 actually disproves a slightly stronger version of Conjecture 2. In the use of Conjecture 2 in [CETU18], they implicitly assume that we can replace $P_i$ by $1 - P_i$ and the permutation still holds. While this modification gives a slightly stronger assumption, our counterexample in Lemma 5 works just as well. Another important observation is that Conjecture 2 regards vector equalities and Theorem 4 regards measurement outcomes. The latter is “easier” than the former and our counterexample works with vector equalities, hence we indeed disprove Conjecture 2.

4 Almost On-State Commutation

In the last part of the paper we discuss almost commutativity in the on-state case. In particular, we show here that if we have two projectors that almost commute on a state then we can define
a projector that fully commutes with one of the original operators and is on-state close to the second one.

The main tool that we need to prove this result is the Jordan’s lemma.

**Lemma 6** (Jordan’s lemma [Jor75]). Let $P_1$ and $P_2$ be two projectors with rank $r_i := \text{rank}(P_i)$ for $i \in \{1, 2\}$. Then both projectors can be decomposed simultaneously in the form $P_i = \bigoplus_{k=1}^{r_i} P_i^k$, where $P_i^k$ denote rank-1 projectors acting on one- or two-dimensional subspaces. We denote the one- and two-dimensional subspaces by $S_1, \ldots, S_l$ and subspaces by $T_1, \ldots, T_r$, respectively. The eigenvectors $|v_{i,1}\rangle$ and $|v_{i,2}\rangle$ of $P_i^1$ and $P_i^2$ respectively are related by:

$$
|v_{k,2}\rangle = \cos \theta_k |v_{k,1}\rangle + \sin \theta_k |v_{k,1}^\perp\rangle, |v_{k,1}\rangle = \cos \theta_k |v_{k,2}\rangle - \sin \theta_k |v_{k,2}^\perp\rangle.
$$

(19)

We can now prove our result.

**Theorem 7** (Making almost commuting projectors commute). Given any two projectors $P_1$ and $P_2$ and a state $|\psi\rangle$ we have that if $\|(P_1 P_2 - P_2 P_1)|\psi\rangle\| = \varepsilon$ then there is a projector $P'_2$ that is close to the original projector on the state $\|(P'_2 - P_2)|\psi\rangle\| \leq \sqrt{2\varepsilon}$ and $[P_1, P'_2] = 0$.

**Proof.** By Jordan’s lemma (Lemma 6), there exist scalars $\lambda_{i,1}, \lambda_{i,2} \in \{0, 1\}$ and vectors $|u_1\rangle, \ldots, |u_m\rangle$ and $|v_{1,1}\rangle, |v_{1,2}\rangle, \ldots, |v_{l,1}\rangle, |v_{l,2}\rangle$, such that

1. $P_1 = \sum_{i \in [m]} \lambda_{i,1} |u_i\rangle \langle u_i| + \sum_{i \in [l]} |v_{i,1}\rangle \langle v_{i,1}|$ and $P_2 = \sum_{i \in [m]} \lambda_{i,2} |u_i\rangle \langle u_i| + \sum_{i \in [l]} |v_{i,2}\rangle \langle v_{i,2}|$;
2. $\langle u_i | u_k \rangle = 0$ and $\langle u_i | v_{j,b} \rangle = 0$ for all $b, i, j$ and $k \neq i$;
3. $\langle v_{j,b} | v_{j,b} \rangle = 0$ for $i \neq j$ and any $b, b'$;
4. $0 < \langle v_{1,1} | v_{1,1} \rangle < 1$.

Let $\theta_i$ be the angle between $|v_{i,1}\rangle$ and $|v_{i,2}\rangle$ (i.e. $\cos \theta_i = \langle v_{i,1} | v_{i,2} \rangle$), and $|v_{i,1}^\perp\rangle$ be the state orthogonal to $|v_{i,1}\rangle$ in subspace spanned by these two vectors. Since the non-commuting part of $P_1$ and $P_2$ must come from the pairs $|v_{i,1}\rangle, |v_{i,2}\rangle$, we will define $P'_2$ by removing the non-commuting part of $P_2$, shi
ting the vector $|v_{i,2}\rangle$, to either $|v_{i,1}\rangle$ or $|v_{i,1}^\perp\rangle$:

$$
P'_2 = \sum_{i \in [m]} \lambda_{i,2} |u_i\rangle \langle u_i| + \sum_{i \in [l] : |\theta_i| \leq \frac{\pi}{4}} |v_{i,1}\rangle \langle v_{i,1}| + \sum_{i \in [l] : |\theta_i| > \frac{\pi}{4}} |v_{i,1}^\perp\rangle \langle v_{i,1}^\perp|.
$$

We have clearly that $[P_1, P'_2] = 0$ since the two projectors are simultaneously diagonalizable and we now want to prove that

$$
\|(P'_2 - P_2)|\psi\rangle\| \leq \sqrt{2\varepsilon}.
$$

Notice that

$$
\|(P'_2 - P_2)|\psi\rangle\|^2
= \left\| \sum_{i \in [l] : |\theta_i| \leq \frac{\pi}{4}} (|v_{i,1}\rangle \langle v_{i,1}| - |v_{i,2}\rangle \langle v_{i,2}|) + \sum_{i \in [l] : |\theta_i| > \frac{\pi}{4}} (|v_{i,1}^\perp\rangle \langle v_{i,1}^\perp| - |v_{i,2}\rangle \langle v_{i,2}|) \right\|^2
= \sum_{i \in [l] : |\theta_i| \leq \frac{\pi}{4}} \| |v_{i,1}\rangle \langle v_{i,1}| - |v_{i,2}\rangle \langle v_{i,2}| \|^2 + \sum_{i \in [l] : |\theta_i| > \frac{\pi}{4}} \| |v_{i,1}^\perp\rangle \langle v_{i,1}^\perp| - |v_{i,2}\rangle \langle v_{i,2}| \|^2,
$$

(20)

(21)

where in the last step we used that $\langle v_{i,b} | v_{j,b} \rangle = 0$ for $i \neq j$. 

11
Using that $|v_{i,2}⟩ = \cos θ_i|v_{i,1}⟩ + \sin θ_i|v_{i,1}^⊥⟩$, we have that if $θ_i \leq \frac{π}{4}$, then
\[
\|⟨v_{i,1}|ψ⟩|v_{i,1}⟩ − ⟨v_{i,2}|ψ⟩|v_{i,2}⟩\|^2
= \sin^4 θ_i|⟨v_{i,1}|ψ⟩|^2 − 2 \sin^3 θ_i \cos θ_i \Re(⟨v_{i,1}^⊥|ψ⟩⟨v_{i,1}|ψ⟩) + \sin^2 θ_i \cos^2 θ_i|⟨v_{i,1}^⊥|ψ⟩|^2
+ \sin^4 θ_i|⟨v_{i,1}^⊥|ψ⟩|^2 + 2 \sin^3 θ_i \cos θ_i \Re(⟨v_{i,1}|ψ⟩⟨v_{i,1}^⊥|ψ⟩) + \sin^2 θ_i \cos^2 θ_i|⟨v_{i,1}|ψ⟩|^2
\leq 2 \sin^2 \cos^2 θ_i(|⟨v_{i,1}^⊥|ψ⟩|^2 + |⟨v_{i,1}|ψ⟩|^2),
\]
where in the inequality we used our assumption that $θ_i \leq \frac{π}{4}$ which implies that $\sin θ_i \leq \cos θ_i$. Using similar calculations, we have that if $θ_i \geq \frac{π}{4}$
\[
\|⟨v_{i,1}^⊥|ψ⟩|v_{i,1}⟩ − ⟨v_{i,2}|ψ⟩|v_{i,2}⟩\|^2
\leq 2 \sin^2 \cos^2 θ_i(|⟨v_{i,1}∥ψ⟩|^2 + |⟨v_{i,1}|ψ⟩|^2).
\]
We will show now that
\[
\sum_i \sin^2 \cos^2 θ_i(|⟨v_{i,1}^⊥|ψ⟩|^2 + |⟨v_{i,1}|ψ⟩|^2) = \varepsilon^2,
\]
which finishes the proof:
\[
\varepsilon^2 = \|(P_2P_1 − P_1P_2)|ψ⟩\|^2
\leq \sum_i \sin θ_i \left(\cos θ_i⟨v_{i,1}|ψ⟩ + \sin θ_i⟨v_{i,1}^⊥|ψ⟩\right)|v_{i,1}⟩ − ⟨v_{i,2}|ψ⟩\left(\cos θ_i|v_{i,1}⟩ + \sin θ_i|v_{i,1}^⊥⟩\right)\|^2
\leq \sum_i \sin θ_i \cos θ_i \left(|⟨v_{i,1}^⊥|ψ⟩|v_{i,1}⟩ − ⟨v_{i,1}|ψ⟩|v_{i,1}^⊥⟩\right)\|^2
\leq \sum_i \sin^2 θ_i \cos^2 θ_i \left(|⟨v_{i,1}^⊥|ψ⟩|^2 + |⟨v_{i,1}|ψ⟩|^2\right).
\]
where in the second equality we again use that $|v_{i,2}⟩ = \cos θ_i|v_{i,1}⟩ + \sin θ_i|v_{i,1}^⊥⟩$ and in the fourth equality we use the fact that $⟨v_{i,1}|ψ|v_{j,1}⟩ = 0$ for $i ≠ j$.

Our proof relies only of the Jordan’s Lemma, note that this is enough only if we analyze commutation of projectors. Results that show how to make any Hermitian matrices commute [PR96] [Has09] are much more complicated to prove and it is not clear how to translate them to the “on-state” case.

We stress that our proof only works for two projectors, since the Jordan’s lemma does not generalize for three or more projectors. Therefore, we leave as an open problem (dis)proving a generalized version of Theorem 7 for more projectors.

References

[CETU18] Tore Vincent Carstens, Ehsan Ebrahimi, Gelo Noel Tabia, and Dominique Unruh. On quantum indifferentiability. Technical report, Cryptology ePrint Archive, Report 2018/257, 2018. https://eprint.iacr.org/2018/257, 2018.

[CZ83] Gianni Cassinelli and N Zanghi. Conditional probabilities in quantum mechanics. 1. – conditioning with respect to a single event. Il Nuovo Cimento B (1971-1996), 73(2):237–245, 1983.
Jan Czajkowski. Github repository “joints-counterexample”, 2021.

Arthur Fine. Probability and the interpretation of quantum mechanics. *The British Journal for the Philosophy of Science*, 24(1):1–37, 1973.

Arthur Fine. Joint distributions, quantum correlations, and commuting observables. *Journal of Mathematical Physics*, 23(7):1306–1310, 1982.

Peter Friis and Mikael Rørdam. Almost commuting self-adjoint matrices-a short proof of huaxin lin’s theorem. *Journal fur die Reine und Angewandte Mathematik*, 479:121–132, 1996.

Stan Gudder and Gabriel Nagy. Sequentially independent effects. *Proceedings of the American Mathematical Society*, 130(4):1125–1130, 2002.

Matthew B Hastings. Making almost commuting matrices commute. *Communications in Mathematical Physics*, 291(2):321–345, 2009.

Camille Jordan. Essai sur la géométrie à $n$ dimensions. *Bulletin de la Société mathématique de France*, 3:103–174, 1875.

Huaxin Lin. Almost commuting selfadjoint matrices and applications. *Fields Inst. Commun*, 13:193–233, 1997.

Hans Maassen. Quantum Probability and Quantum Information Theory. [https://www.math.ru.nl/~maassen/lectures/Trieste.pdf](https://www.math.ru.nl/~maassen/lectures/Trieste.pdf), 2006.

Hans Maassen. Quantum probability and quantum information theory. In *Quantum information, computation and cryptography*, pages 65–108. Springer, 2010.

WM de Muynck and JPHW van den Eijnde. A derivation of local commutativity from macrocausality using a quantum mechanical theory of measurement. *Foundations of physics*, 14(2):111–146, 1984.

Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 10th edition, 2011.

Edward Nelson. *Dynamical theories of Brownian motion*, volume 3. Princeton university press, 1967.
A Numerical values

Below we present the state and the projectors that are claimed in the proof of Lemma 5. The script used to generate these values can be found in [Cza21]. The state is

\[ |\phi\rangle := \begin{pmatrix} -0.135381 - 0.0503468i \\ 0.325588 - 0.222403i \\ -0.209447 - 0.0404665i \\ -0.418336 + 0.130098i \\ -0.503693 - 0.299414i \\ 0.379842 + 0.205081i \\ -0.179291 - 0.0381456i \\ 0.0840381 - 0.125995i \end{pmatrix} . \] (24)

We define the projectors by their eigenvectors:

\[ P_1 = |\pi^1\rangle\langle\pi^1|, \quad P_2 = |\pi_2^2\rangle\langle\pi_2^2| + |\pi_3^2\rangle\langle\pi_3^2|, \]

\[ P_3 = |\pi_2^3\rangle\langle\pi_2^3| + |\pi_3^3\rangle\langle\pi_3^3|, \quad P_4 = |\pi_1^4\rangle\langle\pi_1^4| + |\pi_2^4\rangle\langle\pi_2^4| . \] (25)

The eigenvector of \( P_1 \) is:

\[ |\pi^1\rangle := \begin{pmatrix} 0.440777 + 0.168408i \\ 0.208781 - 0.37351i \\ 0.247514 + 0.0276065i \\ -0.297971 + 0.0252308i \\ 0.118798 + 0.112225i \\ -0.293428 + 0.270889i \\ -0.193073 + 0.218869i \\ -0.41405 \end{pmatrix} . \] (27)

The eigenvectors of \( P_2 \) are:

\[ |\pi_1^2\rangle := \begin{pmatrix} -0.497016 - 0.094035i \\ 0.417527 - 0.0737062i \\ -0.00125303 + 0.35123i \\ 0.166569 - 0.187245i \\ -0.373202 + 0.205633i \\ 0.318452 - 0.251475i \\ -0.107473 - 0.123987i \\ -0.0711523 \end{pmatrix} , \quad |\pi_2^2\rangle := \begin{pmatrix} 0.365906 + 0.0620997i \\ 0.418728 - 0.2059i \\ 0.229457 + 0.0557421i \\ -0.140393 + 0.0945029i \\ -0.199205 - 0.188139i \\ 0.103617 + 0.279644i \\ -0.546498 + 0.147197i \\ 0.275295 \end{pmatrix} . \] (28)
The eigenvectors of $P_3$ are:

\[ |\pi_3^1\rangle := \begin{pmatrix} -0.453059 + 0.181543i \\ -0.452841 + 0.0154095i \\ -0.17948 - 0.222827i \\ -0.230355 - 0.0526756i \\ 0.242416 - 0.126917i \\ 0.300832 - 0.287566i \\ 0.315259 \end{pmatrix}, \quad |\pi_3^2\rangle := \begin{pmatrix} -0.0586669 - 0.269559i \\ -0.280155 + 0.373271i \\ -0.150758 - 0.158539i \\ 0.158793 - 0.0454731i \\ 0.165888 + 0.362832i \\ -0.110453 - 0.310755i \\ -0.0918752 - 0.250754i \end{pmatrix}, \quad (29) \]

\[ |\pi_3^3\rangle := \begin{pmatrix} -0.182739 - 0.114718i \\ 0.246775 - 0.134678i \\ -0.513357 - 0.193655i \\ -0.10451 + 0.421294i \\ 0.111183 + 0.122625i \\ -0.200917 - 0.25897i \\ -0.0290851 + 0.398494i \end{pmatrix}, \quad (30) \]

The eigenvectors of $P_4$ are:

\[ |\pi_4^1\rangle := \begin{pmatrix} -0.464187 + 0.213035i \\ -0.364421 + 0.119836i \\ -0.324984 - 0.23097i \\ -0.256841 + 0.0478513i \\ 0.146148 - 0.225755i \\ 0.243944 - 0.284786i \\ 0.331272 \end{pmatrix}, \quad |\pi_4^2\rangle := \begin{pmatrix} 0.111757 + 0.151275i \\ 0.236223 - 0.323279i \\ 0.157312 - 0.115385i \\ -0.30864 + 0.0990552i \\ -0.260931 - 0.236239i \\ 0.240497 + 0.13559i \\ -0.453404 + 0.12357i \end{pmatrix}, \quad (31) \]

\[ |\pi_4^3\rangle := \begin{pmatrix} 0.111757 + 0.151275i \\ 0.236223 - 0.323279i \\ 0.157312 - 0.115385i \\ -0.30864 + 0.0990552i \\ -0.260931 - 0.236239i \\ 0.240497 + 0.13559i \\ -0.453404 + 0.12357i \end{pmatrix}, \quad (31) \]