Backward Stochastic Riccati Equation with Jumps associated with Stochastic Linear Quadratic Optimal Control with Jumps and Random Coefficients *

Fu Zhang† Yuchao Dong‡ Qingxin Meng§

Abstract

In this paper, we investigate the solvability of matrix valued Backward stochastic Riccati equations with jumps (BSREJ), which is associated with a stochastic linear quadratic (SLQ) optimal control problem with random coefficients and driven by both Brownian motion and Poisson jumps. By dynamic programming principle, Doob-Meyer decomposition and inverse flow technique, the existence and uniqueness of the solution for the BSREJ is established. The difficulties addressed to this issue not only are brought from the high nonlinearity of the generator of the BSREJ like the case driven only by Brownian motion, but also from that i) the inverse flow of the controlled linear stochastic differential equation driven by Poisson jumps may not exist without additional technical condition, and ii) how to show the inverse matrix term involving jump process in the generator is well-defined. Utilizing the structure of the optimal problem, we overcome these difficulties and establish the existence of the solution. In additional, a verification theorem for BSREJ is given which implies the uniqueness of the solution.

Keywords: dynamic programming principle, Doob-Meyer decomposition, stochastic differential equation, Poisson jump, backward stochastic Riccati equation with jumps

1 Introduction

1.1 Framework and Preliminary

We start with a stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\) with a finite time horizon \(T < \infty\) and a filtration \(\mathcal{F} := \{\mathcal{F}_t | t \in [0, T]\}\) satisfying the usual conditions of right continuity and completeness, such that we can and do take all semimartingales to have right continuous paths with left limits. For simplicity,
we assume that \( \mathcal{F}_0 \) is trivial and \( \mathcal{F} = \mathcal{F}_T \). Denote by \( \mathbb{E}[\cdot] \) the expectation under \( \mathbb{P} \). Conditional expectations with respect to a sub-\( \sigma \)-algebra \( \mathcal{G} \) of \( \mathcal{F} \) are denoted by \( \mathbb{E}^{\mathcal{G}}[\cdot] \). Let \( \mathcal{B}(\Lambda) \) denote the Borel \( \sigma \)-algebra of the topological space \( \Lambda \). Let \( W = \{ W(t) = (W^1(t), W^2(t), \ldots, W^d(t))^\top | t \in [0, T] \} \) be a \( d \)-dimensional standard Brownian motion with respect to its natural filtration under \( \mathbb{P} \). Let \((\Lambda, \mathcal{B}(\Lambda))\) be a measurable space and \( \nu \) a finite measure defined on it. Denote by \( \mu \) an integer-valued random measure

\[
\mu(de, dt) = \{ \mu(\omega, de, dt) | \omega \in \Omega \}
\]

on \([0, T] \times \Lambda, \mathcal{B}([0, T]) \otimes \mathcal{B}(\Lambda)\) induced by a stationary \( \mathcal{F}\)-Poisson point process \((\mu_i)_{i \geq 0}\) on \( \Lambda \) with the Lévy measure \( \nu \). Let \( \tilde{\mu}(de, dt) := \mu(de, dt) - \nu(de)dt \) be the compensated Poisson random measure. Suppose that the Brownian motion \( W \) and the random measure \( \tilde{\mu}(de, dt) \) are stochastically independent under \( \mathbb{P} \). Without loss of general assumptions, we assume that the filtration \( \mathcal{F} \) is the \( \mathbb{P}\)-augmentation of the natural filtration generated by the Brownian motion and the Poisson random measure.

Let \( \mathcal{P} \) be the \( \mathcal{F}\)-predictable \( \sigma \)-field on \( \Omega \times [0, T] \) and denote

\[ \tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\Lambda). \]

For a \( \tilde{\mathcal{P}}\)-measurable function \( U \) on \( \tilde{\Omega} \), define its integration with respect to \( \mu \) (analogously for \( \nu \otimes \text{Leb} \)) by

\[
\int_0^T \int_{\Lambda} U(s,e)\mu(de,ds)(\omega) = \begin{cases} \int_0^T \int_{\Lambda} U(\omega,s,e)\mu(\omega,ds,de), & \text{if finitely defined,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.1}
\]

The random measure and stochastic integrals can be referred to \([9, 23]\) for details.

### 1.2 Introduction on BSREJ

Denote by \( \mathbb{S}^n \) the space of all \( n \times n \) symmetric matrices and by \( \mathbb{S}^n_+ \) the space of all \( n \times n \) nonnegative matrices. Throughout this paper, the following standard assumptions holds. Suppose that \( A, B, C, D, E, F, Q, N \) and \( M \) are given random mappings such that \( A : [0, T] \times \Omega \to \mathbb{R}^{n \times n}, B : [0, T] \times \Omega \to \mathbb{R}^{n \times m}, C^i : [0, T] \times \Omega \to \mathbb{R}^{n \times n}, D^i : [0, T] \times \Omega \to \mathbb{R}^{m \times m}, i = 1, 2, \ldots, d; E : [0, T] \times \Omega \times \Lambda \to \mathbb{R}^{n \times n}; F : [0, T] \times \Omega \times \Lambda \to \mathbb{R}^{m \times n}, Q : [0, T] \times \Omega \to \mathbb{R}^{n \times n}, N : [0, T] \times \Omega \to \mathbb{R}^{m \times m}; M : \Omega \to \mathbb{R}^{n \times n} \) satisfies:

**Assumption 1.1.** \( A, B, C, D, N \) and \( Q \) are uniformly bounded \( \mathcal{F}\)-predictable stochastic processes. \( E \) and \( F \) are uniformly bounded \( \tilde{\mathcal{P}}\)-measurable stochastic processes. \( M \) is a uniformly bounded \( \mathcal{F}_T\)-measurable random variable. Moreover, for a.s. a.e. \( (t, \omega) \in [0, T] \times \Omega, Q \in \mathbb{S}^n_+ \) and \( N \in \mathbb{S}^n_+ \). \( M \in \mathbb{S}^n_+ \) for a.e. \( \omega \in \Omega \). And \( N \) is uniformly positive, i.e. for a.s. a.e. \( (t, \omega) \in [0, T] \times \Omega, N(t) \geq \delta I \) for some positive constant \( \delta \).

For any \((t, K, L, R(\cdot)) \in [0, T] \times \mathbb{S}^n \times (\mathbb{S}^n)^d \times \mathcal{M}^{n, 2}(\mathbb{S}^n)\) (see the meaning of the notations in subsection 2.1), define

\[
\mathcal{N}(t,K,R(\cdot)) := N(t) + \sum_{i=1}^{d} (D^i)^*(t)K D^i(t) + \int_{\Lambda} F^*(t,e)(K + R(e))F(t,e)\nu(de),
\]
\[ M(t, K, L, R(\cdot)) := KB(t) + \sum_{i=1}^{d} L_i^i D_i^i(t) + \sum_{i=1}^{d} (C^i)^*(t) KD_i^i(t) \]

\[ + \int_{\Lambda} \left[ E^*(t, e)KF(t, e) + (I + E^*(t, e))R(e)F(t, e) \right] \nu(de), \quad (1.2) \]

\[ G(t, K, L, R(\cdot)) := A^*(t)K + KA(t) + \sum_{i=1}^{d} L^i C^i(t) + \sum_{i=1}^{d} (C^i)^*(t)L^i + \sum_{i=1}^{d} (C^i)^*(t)KC^i(t) \]

\[ + \int_{\Lambda} R(e)E(t, e) \nu(de) + \int_{\Lambda} E^*(t, e)R(e) \nu(de) \]

\[ + \int_{\Lambda} E^*(t, e)(K + R(e))E(t, e) \nu(de) \]

\[ + Q(t) - M(t, K, L, R(\cdot)). \mathcal{N}^{-1}(t, K, R(\cdot)). M^*(t, K, L, R(\cdot)), \]

where \( I \) is \( n \)-th order identity matrix and \( * \) denotes the transpose of a matrix.

With the notations defined above, we introduce the following backward stochastic integral-differential equation driven by Brownian motion \( W \) and Poisson random measure \( \tilde{\mu} \):

\[
\begin{cases}
  dK(t) = -G(t, K(t^-), L(t), R(\cdot))dt + \sum_{i=1}^{d} L_i^i(t)dW_i^i(t) + \int_{\Lambda} R(t, e)\tilde{\mu}(de, dt), \\
  K(T) = M, \quad L(t) := (L^1(t), \ldots, L^d(t)).
\end{cases} \quad (1.3)
\]

with the unknown triple of stochastic processes \((K, L, R)\). Now we give the definition of the solution to BSREJ (1.3) as follows.

**Definition 1.1.** A triplet of stochastic processes \((K, L, R)\) valued in \( \mathbb{S}^n \times (\mathbb{S}^n)^d \times \mathcal{M}^{n,2}(\mathbb{S}^n) \) with \( K \) being \( \mathcal{F} \)-progressive measurable, \( L \) \( \mathcal{F} \)-predictable and \( R \) \( \mathcal{G} \)-measurable is called a solution of BSREJ (1.3) if

(i) \( \int_0^T |G(t, K(t^-), L(t), R(t))|^2 dt + \int_0^T \int_{\Lambda} |R(t, e)|^2 \nu(de) dt < \infty, \) a.s.;

(ii) \( \mathcal{N}(t, K(t^-), R(t)) \) is positive definite a.s. a.e.;

(iii) for all \( t \in [0, T] \), it a.e. holds that

\[ K(t) = M + \int_t^T G(s, K(s^-), L(s), R(s)) ds - \int_t^T \sum_{i=1}^{d} L_i^i(s) dW_i^i(s) - \int_t^T \int_{\Lambda} R(s, e)\tilde{\mu}(de, ds). \quad (1.4) \]

This is the so-called BSREJ associated with a linear quadratic optimal control problem with jumps formulated in Section 2 (See Problem 2.3). When the coefficients \( A, B, C, D, E, F, Q, N \) are all deterministic, then \( L^1 = \cdots = L^d = R = 0 \), and the BSREJ (1.3) degenerates to a deterministic Riccati integral-differential equation (see [20] for the case without jumps). If \( D = 0 \) and \( F = 0 \), i.e. the corresponding controlled differential system does not contain control in martingale integration terms, and the second and third unknown variables \((L, R)\) only have a linear structure in the generator \( G \). And in this case the solvability of BSREJ could be covered by the result of Meng [19]. Due to that the martingale integration parts of corresponding controlled system (2.3) contains control variable, and the system has non-Markovian structure, the associated BSREJ (1.3) is highly nonlinear with respect to the unknown triple of \((K, L, R)\).
1.3 Developments of BSRE and Contributions of this Paper

The study of BSREs had quite a long history. In the case of BSREs driven by only Brownian motion $W$, (1.3) will reduce to the following form:

$$
\begin{cases}
\frac{dK(t)}{dt} = -A^*(t)K(t) + K(t)A(t) + \sum_{i=1}^{d} L^i(t)C^i(t) + \sum_{i=1}^{d} C^i(t)^* \cdot (t)K(t)C^i(t) + Q \\
-\left[K(t)B(t) + \sum_{i=1}^{d} L^i(t)D^i(t) + \sum_{i=1}^{d} (C^i)^*(t)K(t)D^i(t)\right]dt + \sum_{i=1}^{d} L^i(t)dW^i(t),
\end{cases}
$$

(K(T) = M, \quad L(t) := (L^1(t), \ldots, L^d(t)).) \quad (1.5)

Historically speaking, the French mathematician Bismut [11] firstly proposed the definition of the adapted solution to (1.5), and due to the difficulty of its solvability, it is listed as an open problem by Peng [20]. Until 2013, Tang [25] generally solved this open problem applying the stochastic maximum principle and using the technique of stochastic flow for the associated stochastic Hamiltonian system. In 2015, Tang [25] gives the second but more comprehensive (seeming much simpler, by Doob-Meyer decomposition theorem and Dynamic programming principle) method to solve the general BSREs.

For earlier history on BSRE, we refer to Peng [22], Tang and Kohlmann [12] [13], Tang [25] and the plenary lecture reported by Peng [21] at the ICM in 2010. For the indefinite BSRE, the reader can be referred to [2] [30] [14] [15] [24] [4].

Equation (1.3) is very different from equation (1.5). From a direct viewpoint, Equation (1.3) is driven by both a Brownian motion $W$ and an additional compensated Poisson measure $\tilde{\mu}$. From an essential viewpoint, not only the first unknown element $K$ and but also the third unknown element $R$ are included in the nonlinear term $\mathcal{N}(t, K(t-), R(t, \cdot))^{-1}$ in BSREJ (1.3). For the BSRE driven only by a Brownian motion, the nonlinear term $\mathcal{N}(t, K(t-), R(t, \cdot))^{-1}$ degenerates into $\left[N(t) + D^*(t)K(t)D^*(t)\right]^{-1}$ which is well defined since in that case we can show that $K$ is continuous and nonnegative. But for the BSREJ (1.3), one only expects to prove the square integrability of the third unknown element $R$, but this regularity is difficult to derive the non-negativity of matrix $\mathcal{N}(t, K(t-), R(t, \cdot))$. How to show $\mathcal{N}(t, K(t-), R(t, \cdot))$ keeping to be positive is key to give the solvability of BSREJ (1.3).

As far as we know, there is very few literature related to BSREJ. In 2008, under partial information framework, Hu and Øksendal [8] studied the one-dimensional SLQ problem with random coefficients and Poisson jumps, where they presented the state feedback representation of the optimal control by an one-dimensional BSREJ, but the authors did not discuss the wellposedness of the solution to BSREJ. [19] is the first work addressed to the study of high dimensional SLQ with random coefficients, the author formally derived BSREJ (1.3) and utilized Bellman’s principle of quasi-linearization to solve a special form of BSREJ (1.3), in which the generator $G$ only linearly depends on $L$ and $R$. Li et al. [13] used so-called relax compensator to describe indefinite BSREJ and investigated the solvability BSREJ in some special cases.

The contributions of our paper is to establish the solvability of the general BSREJ (1.3). Adapting the method proposed by Tang [24], with the help of control problem and dynamic programming principle, we use the value function and Doob-Meyer decomposition to construct the triple process $(K(t), L(t), R(t, \cdot))$ and later show it is nothing but the solution of BSREJ (1.3). Conversely, we also could utilize the solution of BSREJ (1.3) to depict the optimal control in a feedback form.

One advantage of above method is to avoid the proof of the positive definiteness of the matrix process $\mathcal{N}$ at the beginning. In our approach, we show not only the positive definiteness of $\mathcal{N}$,
but also that of \( \int A^* F(t, e) (K(t -) + R(t, e)) F(t, e) \nu(de) \). The proof is based on an observation that:
\[
\int A R(t, e) \mu(de, \{t\})
\]
is nothing but the jump measure of \( K(t) \). Hence the value \( \int A^* F(t, e) (K(t -) + \int R(t, e)) F \mu(de, \{t\}) \) vanishes except at the jump time, then it coincides with
\[
\int A^* F(t, e) K(t) F(t, e) \mu(de, \{t\})
\]  
(1.6)
since the jump \( \Delta K_i = K(t) - K(t-) = R(t, \Delta p_i) \), where \( \Delta p_i \) is the jump of underlying Poisson process. Obviously \( \text{1.6} \) is positive once the positive definiteness of \( N \) obtained.

The inverse flow of the controlled stochastic differential equation on interval \([0, T]\) is a key technique in Tang’s method in [26] to give the representation of the BSREJ. In some literature about stochastic differential with jumps [7, 16, 27, 3], the authors give a technical condition to guarantee its inverse flow exists on \([0, T]\) (using the notation of SDE (2.3))
\[
I + E(t, e) \geq \delta I, \quad \text{a.e.a.s., for some } \delta > 0.
\]  
(1.7)
But this condition is not necessary for the LQ control problem. In our approach, to overcome the difficulty brought from the absence of condition \( \text{1.7} \), we deal with SDE (2.3) in every stochastic sub-interval between every two adjacent jumping time (\( (\tau_i, \tau_{i+1}) \)), on which SDE (2.3) has continuous trajectory solution and subsequently inverse flow without the help of condition \( \text{1.7} \). Then we use the semi-martingale property of \( K \) to integrate all the sub-intervals to obtain the representation of BSREJ on the whole interval \([0, T]\).

The rest of this article is organized as follows. In Section 2 we introduce some useful notations, preliminary results and the SLQ problem with jumps. In Section 3 we list the preliminary results and the controlled SLQ problem. Section 4 gives some basic properties of the value function \( V \), and also the semimartingale property of \( V \) by dynamic programming principle. In Section 5 with the help of results in Section 4 we show the existence of BSREJ \( \text{1.3} \). In Section 6 we show the verification theorem which gives the uniqueness of the solution for BSREJ, and use the solution of BSREJ to describe the optimal control and valuation of the SLQ problem.

## 2 Preliminary Results and SLQ Problem

### 2.1 Notations

Let \( H \) be a Hilbert space. The inner product in \( H \) is denoted by \( \langle \cdot, \cdot \rangle \), and the norm in \( H \) is denoted by \( |\cdot|_H \) or \( |\cdot| \) if there is no danger of confusion. Let \( p \geq 1 \). Let \( \mathcal{F} \) denote the totality of all \( \mathcal{F} \)-stopping times taking values in \([0, T]\). Define \( \mathcal{F}_\tau := \{ \gamma \in \mathcal{F} : \gamma \geq \tau, \text{ P-a.s.}\} \) for \( \tau \in \mathcal{F} \). Given \( \tau \in \mathcal{F} \) and \( \gamma \in \mathcal{F}_\tau \), the following spaces will be frequently used in this paper:

- \( S^p_{\mathcal{F}}(\tau, \gamma; H) \): the set of all \( H \)-valued \( \mathcal{F} \)-adapted right continuous left limit (RCLL) processes \( f \in \mathcal{F} \) such that \( \|f\|_{S^p_{\mathcal{F}}(\tau, \gamma; H)} := \left\{ \mathbb{E}\left[ \sup_{\tau \leq t \leq \gamma} |f(t)|_H^p \right]^\frac{1}{p} \right\} < \infty \);

- \( M^p_{\mathcal{F}}(\tau, \gamma; H) \): the set of all \( H \)-valued \( \mathcal{F} \)-progressively measurable processes \( f \in \mathcal{F} \) such that \( \|f\|_{M^p_{\mathcal{F}}(\tau, \gamma; H)} := \left\{ \mathbb{E}\left[ \int_\tau^\gamma |f(t)|_H^p dt \right]^\frac{1}{p} \right\} < \infty \);
Let \(\text{Lemma 2.1.}\) random variables in a probability space (see, e.g. Karatzas and Shreve [11, Appendix A]).

In the following we recall a classical theorem for the essential infimum of a family of nonnegative random variables in a probability space (see, e.g. Karatzas and Shreve [11, Appendix A]).

**Definition 2.1.** For any \(\tau \in \mathcal{T}\), \(\mathcal{F}_\tau\)-measurable random variable \(X\) such that \(\|X\|_{\mathcal{F}_\tau}\) is a random variable satisfying \(\|X\|_{\mathcal{F}_\tau} \leq X\) a.s. for all \(\tau \in \mathcal{T}\), \(\mathcal{F}_\tau\)-measurable random variable \(\xi\) defined on \((\Omega, \mathcal{F}, P)\) such that \(\|\xi\|_{L^p(\Omega, \mathcal{G}, P; H)} := \left\{ \mathbb{E}[|\xi|^p] \right\}^{\frac{1}{p}}\) where \(\mathcal{G}\) is a subalgebra of \(\mathcal{F}\).

In the following we recall a classical theorem for the **essential infimum** of a family of nonnegative random variables in a probability space (see, e.g. Karatzas and Shreve [11, Appendix A]).

**Lemma 2.1.** Let \(\mathcal{X}\) be a family of nonnegative integrable random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). Then there exists an \(\mathcal{F}\)-measurable random variable \(X^*\) such that

1. for all \(X \in \mathcal{X}\), \(X \geq X^*\) a.s.;
2. if \(Y\) is a random variable satisfying \(X \geq Y\) a.s. for all \(X \in \mathcal{X}\), then \(X^* \geq Y\) a.s.

This random variable, which is unique a.s., is called the essential infimum of \(\mathcal{X}\), and is denoted by \(\text{ess inf } \mathcal{X}\) or \(\text{ess inf }_{X \in \mathcal{X}} X\). Furthermore, if \(\mathcal{X}\) is closed under pairwise minimum (i.e. \(X, Y \in \mathcal{X}\) implies \(X \wedge Y \in \mathcal{X}\)), then there exists a nondecreasing sequence \(\{Z_n\}_{n \in \mathbb{N}}\) of random variables in \(\mathcal{X}\) such that \(X^* = \lim_{n \to \infty} Z_n\) a.s. Moreover, for any sub-algebra \(\mathcal{G}\) of \(\mathcal{F}\), the \(\mathcal{G}\)-conditional expectation is interchangeable with the essential infimum:

\[ \mathbb{E}[\text{ess inf } X | \mathcal{G}] = \text{ess inf } \mathbb{E}[X | \mathcal{G}]. \]

### 2.2 Some Basic Definition and Results on \(\mathcal{T}\)-System

For any \(\tau_1, \tau_2 \in \mathcal{T}\), with \(\tau_1 \leq \tau_2\) almost surely and \(P(\tau_1 < \tau_2) > 0\), let

\[ \mathcal{T}[\tau_1, \tau_2] := \{ \tau \in \mathcal{T} | \tau_1 \leq \tau \leq \tau_2 \ P\text{-a.s.} \}. \]

The following classical result of aggregation of supmartingale system could be found in [6].

**Definition 2.1.** A family of random variables \(\mathcal{K} := \{ \mathcal{K}(\tau), \tau \in \mathcal{T} \}\) indexed by \(\mathcal{T}\) is said to be \(\mathcal{T}\)-system if it satisfies

1. for all \(\tau \in \mathcal{T}\), \(\mathcal{K}(\tau)\) is \(\mathcal{T}\)-measurable random variable;
2. for all \(\tau_1, \tau_2 \in \mathcal{T}\), \(\mathcal{K}(\tau_1) = \mathcal{K}(\tau_2)\) a.s. on \(\{\tau_1 = \tau_2\}\) for \(\tau_1, \tau_2 \in \mathcal{T}\).
Definition 2.2. We call a $\mathcal{T}$-system $\{\mathcal{K}(\tau), \tau \in \mathcal{T}\}$ a submartingale system if the following two properties hold:
(i) $\mathcal{K}(\tau)$ is integrable for any $\tau \in \mathcal{T}$;
(ii) $\mathbb{E}^{\mathcal{T}_t} [\mathcal{K}(\tau_2)] \geq \mathcal{K}(\tau_1)$, $\mathbb{P}$-a.s., for all $\tau_1 \in \mathcal{T}, \tau_2 \in \mathcal{T}_t$.

We call $\mathcal{T}$-system $\mathcal{K} := \{\mathcal{K}(\tau), \tau \in \mathcal{T}\}$ is said to be a supermartingale system if $-\mathcal{K}$ is a submartingale system, and call it a martingale system if it is both a $\mathcal{T}$-supermartingale and a $\mathcal{T}$-submartingale system.

Definition 2.3. A $\mathcal{T}$-system $\{\mathcal{K}(\tau), \tau \in \mathcal{T}\}$ is called right-(resp., left-) continuous along times in expectation (RCE (resp., LCE)) if for any sequences of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\tau_n \uparrow \tau$ a.s. (resp., $\tau_n \nearrow \tau$), one has $\mathbb{E}[\mathcal{K}(\tau)] = \lim_{n \to \infty} \mathbb{E}[\mathcal{K}(\tau_n)]$.

Definition 2.4. We call an process $X = \{X(t), t \in [0, T]\}$ aggregates the $\mathcal{T}$-system $\{\mathcal{K}(\tau), \tau \in \mathcal{T}\}$, if for any $\tau \in \mathcal{T}$, it holds $X(\tau) = \mathcal{K}(\tau)$, $\mathbb{P}$-a.s.

The following result could be found in [5, subsection 2.14 on p.112], or adapted from [10, Theorem 3.13 in Chapter 1].

Proposition 2.2. Let a $\mathcal{T}$-system $\{\mathcal{K}(\tau), \tau \in \mathcal{T}\}$ be a supermartingale system which is RCE and such that $\mathcal{K}(0) < \infty$. There then exists a RCLL adapted process denoted by $\{K(t)\}_{t \in [0, T]}$ which aggregates $\mathcal{T}$-system $\{\mathcal{K}(\tau), \tau \in \mathcal{T}\}$.

Proof. Consider a supermartingale process $(\mathcal{K}(t))_{0 \leq t \leq T}$, by Theorem 3.13 in [10, Chapter 1], it has a RCLL modification $K(t) := \lim_{s \downarrow t} \mathcal{K}(s)$. For any stopping time $\tau$, define $\tau_n(\omega) := \frac{i}{\tau}$, if $\tau(\omega) \in (\frac{i-1}{\tau}, \frac{i}{\tau}]$ for some integer $i > 0$. It is easy to see that $\mathcal{K}(\tau_n) = K(\tau_n)$. Then by REC of $\mathcal{K}$ and uniform convergence of $\{K(\tau_n)\}$ (see Remark 3.12 in [10, Chapter 1]), passing $n$ to infinity, we have $\mathcal{K}(\tau) = K(\tau)$ a.e. Thus $\{K(t)\}_{t \in [0, T]}$ aggregates $\mathcal{T}$-system $\{\mathcal{K}(\tau), \tau \in \mathcal{T}\}$. \hfill $\square$

For future purposes, we shall consider the "conditional" extension of $\mathcal{T}$-system. More precisely, for a family of random variables $\mathcal{K} := \{\mathcal{K}(\sigma), \sigma \in \mathcal{T}\}$ indexed by $\mathcal{T}$, it is called a $\mathcal{T}$-system if it satisfies
1. for all $\sigma \in \mathcal{T}$, $\mathcal{K}(\sigma)$ is $\mathcal{T}_\sigma$-measurable random variable.
2. for all $\sigma_1, \sigma_2 \in \mathcal{T}$, $\mathcal{K}(\sigma_1) = \mathcal{K}(\sigma_2)$ a.s. on $\{\sigma_1 = \sigma_2\}$ for $\sigma_1, \sigma_2 \in \mathcal{T}$.

Naturally, Definitions 2.3 and 2.4 can be adapted for the $\mathcal{T}$-system. Given a $\mathcal{T}$-system $\mathcal{K}$, one can extend it to be a $\mathcal{T}$-system, still denoted by $\mathcal{K}$, in the following way:

$$\mathcal{K}(\sigma) := \mathcal{K}(\sigma) \chi_{\{\sigma \geq \tau\}} + \mathbb{E}[\mathcal{K}(\tau) \chi_{\{\sigma < \tau\}} | \mathcal{T}_\tau] \chi_{\{\sigma < \tau\}}.$$ 

If the original $\mathcal{T}$-system $\mathcal{K}$ is a submartingale (resp. supermartingale) system, then the extension is also a submartingale (resp. supermartingale) system. Moreover, the RCE (or LCE) property holds for the extension. Hence, according to Proposition 2.2, if $\mathcal{K}$ is a supermartingale $\mathcal{T}$-system which is RCE and $\mathbb{E}[\mathcal{K}(\tau)] < +\infty$, then there exists a RCLL adapted process $K$ defined on the random interval $[\tau, T]$ which aggregates $\mathcal{K}$, i.e., for any $\sigma \in \mathcal{T}$,

$$K(\sigma) = \mathcal{K}(\sigma), \mathbb{P} - a.s..$$
2.3 Preliminary Results for Linear SDE with Jumps

Let $p \geq 2$. For any $(\tau, \xi) \in \mathcal{T} \times L^p(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$, consider the following linear SDE with jumps

$$
\begin{aligned}
dX(t) &= [A(t)X(t^-) + f(t)]dt + \sum_{i=1}^{d}[C^n(t)X(t^-) + g^i(t)]dW^i(t) \\
&\quad + \int_{\Lambda}[E(t,e)X(t^-) + h(t,e)]\bar{\mu}(de, dt), \quad \tau \leq t \leq T;
X(0) &= x,
\end{aligned}
$$

(2.1)

where the coefficients satisfy the following basic assumption:

**Assumption 2.1.** The matrix-valued processes $A : [0, T] \times \Omega \to \mathbb{R}^{n \times n}$, $B : [0, T] \times \Omega \to \mathbb{R}^{n \times m}$; $C_i : [0, T] \times \Omega \to \mathbb{R}^{n \times n}$, $i = 1, 2, \ldots, d$ are uniformly bounded and $\mathcal{F}$-predictable. The matrix process $E : [0, T] \times \Omega \times \Lambda \to \mathbb{R}^{n \times n}$ is uniformly bounded and $\mathcal{F}$-measurable. The stochastic processes $f(\cdot), g^i(\cdot)$ belong to $\mathcal{M}_{\mathcal{F}}^p(0, T; \mathbb{R}^n)$ and $h(\cdot, \cdot)$ belongs to $\mathcal{M}_{\mathcal{F}}^p(0, T; \mathbb{R}^n)$.

The following classical estimate could be found in lots of literature (see [23, 17]), the proof based on the Itô formula, Gronwall’s inequality and BDG inequality is standard.

**Lemma 2.3.** Let Assumptions 2.1 be satisfied. Then the SDE (2.1) has a unique strong solution $X(\cdot) \in \mathcal{S}^p_\mathcal{F}(\tau, T; \mathbb{R}^n)$ and there is a constant $C_p > 0$ such that for any stopping time $\tau < T$,

$$
\mathbb{E}^{\mathbb{F}}\left[\sup_{\tau \leq t \leq T}|X(t)|^p\right] \leq C_p \mathbb{E}^{\mathbb{F}}\left[|\xi|^p + \left(\int_{\tau}^{T}|f(t)|^2 dt\right)^{\frac{p}{2}} + \left(\int_{\tau}^{T}\sum_{i=1}^{d}|g^i(t)|^2 dt\right)^{\frac{p}{2}} + \int_{\tau}^{T}\int_{\Lambda}|h(t,e)|^p \nu(de)dt\right].
$$

(2.2)

2.4 Formulation on SLQ Problem

In this section, we formulate the SLQ problem with jumps. We first give the following definition of admissible control.

**Definition 2.5.** Let $\tau \in \mathcal{T}$. An $\mathcal{F}$-predictable process $u(\cdot)$ is said to be an admissible control on the random interval $[\tau, T]$, if $u(\cdot) \in \mathcal{M}_{\mathcal{F}}^{p}(\tau, T; \mathbb{R}^n)$. The set of all admissible control is denoted by $\mathcal{U}_{\tau}$.

For any given admissible control $u(\cdot) \in \mathcal{U}_{\tau}$, consider the following controlled linear SDE with jumps:

$$
\begin{aligned}
dX(t) &= [A(t)X(t^-) + B(t)u(t)]dt + \sum_{i=1}^{d}[C^n(t)X(t^-) + D^i(t)u(t)]dW^i(t) \\
&\quad + \int_{\Lambda}[E(t,e)X(t^-) + F(t,e)u(t)]\bar{\mu}(de, dt), \\
X(0) &= x,
\end{aligned}
$$

(2.3)

with the cost functional

$$
J(u(\cdot); 0, x) := \mathbb{E}\left[\langle MX(T), X(T)\rangle + \int_{0}^{T}\langle (Q(t)X(t), X(t)) + \langle N(t)u(t), u(t)\rangle\rangle dt\right].
$$

(2.4)
Here $A, B, C, D, E, F, Q, N$ and $M$ are given random mappings such that $A : [0, T] \times \Omega \to \mathbb{R}^{n \times n}; B : [0, T] \times \Omega \to \mathbb{R}^{n \times m}; C^i : [0, T] \times \Omega \to \mathbb{R}^{n \times n}, i = 1, 2, \ldots, d; E : [0, T] \times \Omega \times \Lambda \to \mathbb{R}^{n \times n}; F : [0, T] \times \Omega \times \Lambda \to \mathbb{R}^{n \times n}; Q : [0, T] \times \Omega \to \mathbb{R}^{n \times n}, N : [0, T] \times \Omega \to \mathbb{R}^{m \times m}; M : \Omega \to \mathbb{R}^{n \times n}$ satisfying Assumption 1.1.

By Lemma 2.3 for any $u(\cdot) \in \mathcal{U}_0$, it follows that the SDE (2.3) admits a unique strong solution in the space $\mathcal{S}_F(0, T; \mathbb{R}^n)$, denoted by $X^{0,x,u(\cdot)}(\cdot)$. We call $X(\cdot) \triangleq X^{0,x,u(\cdot)}(\cdot)$ the state process corresponding to the control process $u(\cdot)$ and call $(u(\cdot); X(\cdot))$ the admissible pair. Furthermore, Assumption 1.1 and the a priori estimate (2.2) imply that

$$|J(u(\cdot); 0, x)| < \infty.$$ 

Then our SLQ problem can be stated as follows.

**Problem 2.4.** Find an admissible control process $\bar{u}(\cdot) \in \mathcal{U}_0$ such that

$$J(\bar{u}(\cdot); 0, x) = \inf_{u(\cdot) \in \mathcal{U}_0} J(u(\cdot); 0, x).$$

(5.2)

The admissible control $\bar{u}(\cdot)$ satisfying (2.5) is called an optimal control process of Problem 2.4. Correspondingly, the state process $\bar{X}(\cdot)$ associated with $\bar{u}(\cdot)$ is called an optimal state process and $(\bar{u}(\cdot); \bar{X}(\cdot))$ is called an optimal pair of Problem 2.4.

### 3 Dynamical Programming Principle and the Semimartingale Property of the Value Process

#### 3.1 Initial-Data-Parameterized SLQ Problem

This subsection is devoted to introducing the initial-data-parameterized SLQ Problem. For simplicity, we define the random function

$$f(t, x, u) := \langle Q(t)x, x \rangle + \langle N(t)u, u \rangle, \quad \forall (t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m.$$ 

Fixed initial data $(\tau, \xi) \in \mathcal{T} \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$, for any given admissible control $u(\cdot) \in \mathcal{U}_\tau$, denote by $X^{\tau, \xi; u}$ the solution of following state equation

$$\begin{aligned}
\frac{dX(t)}{dt} &= [A(t)X(t) + B(t)u(t)]dt + \sum_{i=1}^d [C^i(t)X(t) + D^i(t)u(t)]dW^i(t)
+ \int_\Lambda [E(t, \omega)X(t) + F(t, \omega)u(t)]\tilde{\mu}(d\omega, dt),

X(\tau) &= \xi.
\end{aligned}$$

(3.1)

The cost functional is defined as the following conditional expectation:

$$J(u(\cdot); \tau, \xi) := \mathbb{E}_{\mathbb{P}}[\int_\tau^T f(s, X^{\tau, \xi; u(\cdot)}(s), u(s))ds + \langle MX^{\tau, \xi; u(\cdot)}(T), X^{\tau, \xi; u(\cdot)}(T) \rangle].$$

(3.2)

Then the corresponding initial-data-parameterized SLQ Problem is stated as follows:
Problem 3.1. Find an admissible control process \( \hat{u}(\cdot) \in \mathcal{U}_\tau \) such that

\[
J(\hat{u}(\cdot); \tau, \xi) = \inf_{u(\cdot) \in \mathcal{U}_\tau} J(u(\cdot); \tau, \xi).
\] (3.3)

We also denote the above optimal control problem by Problem \( \mathcal{P}_{\tau, \xi} \) to stress the dependence on the parameter \( (\tau, \xi) \). Clearly, for any initial data \( (\tau, \xi) \in \mathcal{T} \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \) and admissible control \( u(\cdot) \in \mathcal{U}_\tau \), the state equation (3.1) has a unique strong solution \( X(\cdot) = X^{\tau, \xi; u(\cdot)} \) and (3.3) is well-defined. Furthermore, we can define the following conditional minimal value system

\[
\mathcal{V}(\tau, \xi) := \inf_{u(\cdot) \in \mathcal{U}_\tau} J(u(\cdot); \tau, \xi).
\] (3.4)

It is obvious that \( \mathcal{V}(\tau, \xi) \) is \( \mathcal{F}_\tau \)-measurable random variable for any \( (\tau, \xi) \in \mathcal{T} \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \).

The following two results Proposition 3.2 and Theorem 3.3 are needed in our approach. The description and their proofs are more or less standard in the context of SLQ problem. We just give a sketch of the proof in the case of jumps since it is similar to that in the case of Brownian motion. We suggest the reader to visit Sections 2 and 3 in [26] for full details.

Proposition 3.2. Let Assumption 1.1 hold.

(i) There is a positive constant \( \lambda \) such that for any \( (\tau, \xi) \in \mathcal{T} \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \), it has

\[
0 \leq \mathcal{V}(\tau, \xi) \leq J(0; \tau, \xi) \leq \lambda |\xi|^2.
\] (3.5)

(ii) For any given initial data \( (\tau, \xi) \in \mathcal{T} \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \), Problem \( \mathcal{P}_{\tau, \xi} \) has a unique optimal control \( \hat{u}(\cdot) \in \mathcal{U}_\tau \), i.e.,

\[
\mathcal{V}(\tau, \xi) = J(\hat{u}(\cdot); \tau, \xi), \text{ } \mathbb{P}\text{-a.s.}
\]

(iii) The value functional \( \mathcal{V}(\tau, \xi) \) is quadratic with respect to \( \xi \). Moreover, there is an \( \mathbb{R}^n \)-valued family \( \mathcal{V} := \{ \mathcal{V}(\tau), \tau \in \mathcal{T} \} \) such that \( \mathcal{V}(\tau) \) is essentially bounded for any \( \tau \in \mathcal{T} \) and \( \xi \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \)

\[
\mathcal{V}(\tau, \xi) = \langle \mathcal{V}(\tau) \xi, \xi \rangle.
\] (3.6)

(iv) For each \( x \in \mathbb{R}^n \), define the family

\[
\mathcal{V}_x := \{ \mathcal{V}(\tau, x), \tau \in \mathcal{T} \}.
\]

Then it is a \( \mathcal{T} \)-system. Moreover, the family \( \mathcal{V} := \{ \mathcal{V}(\tau), \tau \in \mathcal{T} \} \) is also a \( \mathcal{T} \)-system.

Proof. (i) Noting Assumption 1.1 and 3.1, it is sufficient to show \( J(0; \tau, \xi) \leq \lambda |\xi|^2 \). In fact, from the a priori estimate (2.2), we get that

\[
J(0; \tau, \xi) \leq C \mathbb{E}^\mathcal{F}_\tau \left[ \int_\tau^T |X^{\tau, \xi; 0}(t)|^2 dt + |X^{\tau, \xi; 0}(T)|^2 \right]
\]

\[
\leq C \mathbb{E}^\mathcal{F}_\tau \left[ \sup_{\tau \leq t \leq T} |X^{\tau, \xi; 0}(t)|^2 \right]
\]

\[
\leq C |\xi|^2.
\]
(ii) Let \((\tau, \xi) \in \mathcal{T} \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\). For any \(u_1(\cdot), u_2(\cdot) \in \mathcal{U}_\tau\), define
\[
\hat{u}(\cdot) := u_1(\cdot) \chi_{\{J(u_1(\cdot); \tau, \xi) \leq J(u_2(\cdot); \tau, \xi)\}} + \mathbb{I}_{\{u_2(\cdot) \in \mathcal{U}_\tau\}} \chi_{\{J(u_1(\cdot); \tau, \xi) > J(u_2(\cdot); \tau, \xi)\}}.
\]
Then \(X^{\tau, \xi; \hat{u}(\cdot)} = X^{\tau, \xi; u_1(\cdot)} \chi_{\{J(u_1(\cdot); \tau, \xi) \leq J(u_2(\cdot); \tau, \xi)\}} + X^{\tau, \xi; u_2(\cdot)} \chi_{\{J(u_1(\cdot); \tau, \xi) > J(u_2(\cdot); \tau, \xi)\}}\). Hence
\[
J(\hat{u}(\cdot); \tau, \xi) = J(u_1(\cdot); \tau, \xi) \chi_{\{J(u_1(\cdot); \tau, \xi) \leq J(u_2(\cdot); \tau, \xi)\}} + J(u_2(\cdot); \tau, \xi) \chi_{\{J(u_1(\cdot); \tau, \xi) > J(u_2(\cdot); \tau, \xi)\}}
\]
\[
= \min\{J(u_1(\cdot); \tau, \xi), J(u_2(\cdot); \tau, \xi)\}.
\]
That is \(\{J(u(\cdot); \tau, \xi) : u(\cdot) \in \mathcal{U}_\tau\}\) is closed under pairwise minimum. By Lemma 2.1, there is a sequence \(\{u_k(\cdot)\}_{k=1}^\infty \subset \mathcal{U}_\tau\), such that
\[
J(u_k(\cdot); \tau, \xi) \searrow \mathcal{V}(\tau, \xi), \quad \text{as } k \to \infty. \tag{3.7}
\]

By the parallelogram equality,
\[
2J\left(\frac{1}{2}(u_k(\cdot) - u_l(\cdot)); \tau, \xi\right) + 2\mathcal{V}(\tau, \xi) \leq 2J\left(\frac{1}{2}(u_k(\cdot) - u_l(\cdot)); \tau, \xi\right) + 2J\left(\frac{1}{2}(u_k(\cdot) + u_l(\cdot)); \tau, \xi\right)
\]
\[
= J(u_k(\cdot); \tau, \xi) + J(u_l(\cdot); \tau, \xi).
\]
Let \(k, l \to \infty\) in the following inequality,
\[
0 \leq 2J\left(\frac{1}{2}(u_k(\cdot) - u_l(\cdot)); \tau, \xi\right) \leq J(u_k(\cdot); \tau, \xi) + J(u_l(\cdot); \tau, \xi) - 2\mathcal{V}(\tau, \xi) \to 0,
\]
which means \(\{u_k(\cdot)\}_{k=1}^\infty\) is Cauchy sequence in \(\mathcal{M}_\tau^2(\tau, T; \mathbb{R}^m)\). And it is easy to check that \(\hat{u}(\cdot) := \lim_{k \to \infty} u_k(\cdot)\) is the unique optimal control for problem \(\mathcal{P}_{\tau, \xi}\).

(iii) One can show that (see [6] or [26, Lemma 3.2]), for any real number \(\eta > 0\), \(x, y \in \mathbb{R}^n\),
\[
\mathcal{V}(\tau, \eta x) = \eta^2 \mathcal{V}(\tau, x),
\]
\[
\mathcal{V}(\tau, x + y) + \mathcal{V}(\tau, x - y) = 2\mathcal{V}(\tau, x) + 2\mathcal{V}(\tau, y).
\]
So \(\mathcal{V}(\tau, x)\) is a quadratic form. Let
\[
\mathcal{K}(\tau) = \frac{1}{4}(\mathcal{V}(\tau, e_i + e_j) - \mathcal{V}(\tau, e_i - e_j))_{i,j=1}^n, \tag{3.8}
\]
then we have \([3, 10]\).

(iv) Verifying Definition \(2.1\) directly, we shall prove that \(\mathcal{V}\) is \(\mathcal{T}\)-system and and consequently so does \(\mathcal{K}\).

\[\square\]

3.2 Dynamical Programming Principle and the Semimartingale Property

The following result is the dynamical programming principle for Problem \(\mathcal{P}_{\tau, \xi}\).

**Theorem 3.3.** Let Assumption \(1.1\) hold. (i) For \(\tau \in \mathcal{T}, \sigma \in \mathcal{T}_\tau\), and \(\xi \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\),
\[
\mathcal{V}(\tau, \xi) = \text{ess inf}_{u(\cdot) \in \mathcal{U}_\tau} \mathbb{E}_\mathcal{P} \left[ \int_\tau^\sigma f(s, X^{\tau, \xi; u(\cdot)}(s), u(s)) ds + \mathcal{V}(\sigma, X^{\tau, \xi; u(\cdot)}(\sigma)) \right]. \tag{3.9}
\]
And it holds that
\[
\mathcal{V}(\tau, \xi) = \mathbb{E}^\mathcal{F}_\tau \left[ \int_{\tau}^{\sigma} f(s, X^{\tau, \xi; u(\cdot)}(s), \bar{u}(s))ds + \mathcal{V}(\sigma, X^{\tau, \xi; \bar{u}(\cdot)}(\sigma)) \right]
\] (3.10)
for the optimal control \(\bar{u}(\cdot)\) in \(\mathcal{U}_\tau\) of Problem \(\mathcal{P}_{\tau, \xi}\).

(ii) For any \(\tau \in \mathcal{I}\) and \((x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}_\tau\), the family \(\mathcal{J}^{\tau, x; u(\cdot)} = \{ \mathcal{J}^{\tau, x; u(\cdot)}(\sigma), \sigma \in \mathcal{I}_\tau \}\) is a \(\mathcal{F}\)-submartingale, where
\[
\mathcal{J}^{\tau, x; u(\cdot)}(\sigma) := \mathcal{V}(\sigma, X^{\tau, x; u(\cdot)}(\sigma)) + \int_{\tau}^{\sigma} f(r, X^{\tau, x; u(\cdot)}(r), u(r))dr, \quad \sigma \in \mathcal{I}_\tau; \quad (3.11)
\]
And the family \(\mathcal{J}^{\tau, x; \bar{u}(\cdot)}\) is a \(\mathcal{F}\)-martingale for the optimal control \(\bar{u}(\cdot)\) in \(\mathcal{U}_\tau\) of problem \(\mathcal{P}_{\tau, x}\).

Besides,
\[
\mathcal{J}^{\tau, x; u(\cdot)}(\sigma) = \underset{v(\cdot) \in \mathcal{U}_\sigma^{u(\cdot)}}{\text{ess inf}} \mathbb{E}^\mathcal{F}_\tau \left[ \int_{\tau}^{\sigma} f(r, X^{\tau, x; v(\cdot)}(r), v(r)) + \langle M X^{\tau, x; v(\cdot)}(T), X^{\tau, x; v(\cdot)}(T) \rangle, u \in \mathcal{U}_\tau, \right]
\]
where
\[
\mathcal{U}_\sigma^{u(\cdot)} := \{ v(\cdot) \in \mathcal{U}_\sigma | v(\cdot) = u(\cdot) \text{ on } [\tau, \sigma] \}.
\]
(iii) If \(u(\cdot) \in \mathcal{U}_\tau\) such that \(\mathcal{J}^{\tau, x; u(\cdot)}\) is a \(\mathcal{F}\)-martingale, then \(\bar{u}(\cdot)\) is optimal for Problem \(\mathcal{P}_{\tau, x}\).

Proof. (i) Similar as (3.7), there is a minimizing sequence \(\{v_m(\cdot)\} \subset \mathcal{U}_\sigma\) of Problem \(\mathcal{P}_{\sigma, X^{\tau, \xi; u(\cdot)}(\sigma)}\) such that, then we have for any \(v(\cdot) \in \mathcal{U}_\sigma\),
\[
\mathbb{E}^\mathcal{F}_\tau \left[ J(v(\cdot); \sigma, X^{\tau, \xi; u(\cdot)}(\sigma)) \right] \geq \mathbb{E}^\mathcal{F}_\tau \left[ \mathcal{V}(\sigma, X^{\tau, \xi; \bar{u}(\cdot)}(\sigma)) \right]
\]
\[
= \mathbb{E}^\mathcal{F}_\tau \left[ \inf_m J(v_m(\cdot); \sigma, X^{\tau, \xi; u(\cdot)}(\sigma)) \right]
\]
\[
= \underset{m}{\text{ess inf}} \mathbb{E}^\mathcal{F}_\tau \left[ J(v_m(\cdot); \sigma, X^{\tau, \xi; u(\cdot)}(\sigma)) \right]
\]
\[
\geq \underset{v(\cdot) \in \mathcal{U}_\sigma}{\text{ess inf}} \mathbb{E}^\mathcal{F}_\tau \left[ J(v(\cdot); \sigma, X^{\tau, \xi; u(\cdot)}(\sigma)) \right].
\]
Taking \(\text{ess inf}_{v(\cdot) \in \mathcal{U}_\sigma}\) on the left hand side of above inequality, then the inequalities turn to equalities. We have
\[
\underset{v(\cdot) \in \mathcal{U}_\sigma}{\text{ess inf}} \mathbb{E}^\mathcal{F}_\tau \left[ J(v(\cdot); \sigma, X^{\tau, \xi; u(\cdot)}(\sigma)) \right] = \mathbb{E}^\mathcal{F}_\tau \left[ \mathcal{V}(\sigma, X^{\tau, \xi; \bar{u}(\cdot)}(\sigma)) \right].
\]
Furthermore for any \(u(\cdot) \in \mathcal{U}_\tau\),
\[
\mathbb{E}^\mathcal{F}_\tau \left[ \int_{\tau}^{\sigma} f(s, X^{\tau, \xi; u(\cdot)}(s), u(s))ds + \mathcal{V}(\sigma, X^{\tau, \xi; \bar{u}(\cdot)}(\sigma)) \right]
\]
\[
= \underset{v(\cdot) \in \mathcal{U}_\sigma}{\text{ess inf}} \mathbb{E}^\mathcal{F}_\tau \left[ \int_{\tau}^{\sigma} f(s, X^{\tau, \xi; u(\cdot)}(s), u(s))ds + J(v(\cdot); \sigma, X^{\tau, \xi; u(\cdot)}(\sigma)) \right]
\]
\[
= \underset{v(\cdot) \in \mathcal{U}_\sigma}{\text{ess inf}} \mathbb{E}^\mathcal{F}_\tau \left[ J(u(\cdot) \otimes v(\cdot); \tau, \xi) \right].
\]
where \( u(\cdot) \otimes v(\cdot) = u(\cdot) \) on \([\tau, \sigma]\), and \( u(\cdot) \otimes v(\cdot) = v(\cdot) \) on \([\sigma, T]\). (3.12) is the result of taking \( \text{ess inf}_{u(\cdot) \in \mathcal{U}_r} \) on both sides of above equality.

If \( \bar{u}(\cdot) \in \mathcal{U}_r \) is the optimal control for \( \mathcal{P}_{\tau, \xi} \), then its restriction \( \bar{u}|_{[\sigma, T]}(\cdot) \) is the optimal control for \( \mathcal{P}_{\tau, \xi}(u(\cdot)|_{[\sigma, T]}(\cdot)) \). Then (3.10) follows. Then assertion (i) holds.

In view of (i), it is easy to check that (ii) and (iii) hold.

\[ \square \]

**Lemma 3.4.** Let Assumptions 1.1 be satisfied. Then for each \( n \in \mathbb{N} \), the \( \mathcal{F} \)-systems \( \mathcal{V}_x \) and \( \mathcal{K} = \{ \mathcal{K}(\tau), \tau \in \mathcal{F} \} \) are RCE.

**Proof.** For any \( \tau \in \mathcal{F}_0, \tau_m \in \mathcal{F} \) satisfying that \( \tau_m \searrow \tau \) a.s. as \( m \to \infty \). By (i) of Theorem 3.3 for the optimal control \( \bar{u}(\cdot) \) of Problem \( \mathcal{P}_{\tau, x} \),

\[ \mathcal{V}(\tau, x) = \mathbb{E}^{\mathcal{F}_x} \left[ \int_{\tau}^{\tau_m} f(s, X^{\tau, x; \bar{u}(\cdot)}(s), \bar{u}(s)) ds + \mathcal{V}(\tau_m, X^{\tau, x; \bar{u}(\cdot)}(\tau_m)) \right]. \]

Since \( \mathcal{K} \) is uniformly bounded,

\[ \mathbb{E}^{\mathcal{F}_x} \left[ \mathcal{V}(\tau_m, X^{\tau, x; \bar{u}(\cdot)}(\tau_m)) - \mathcal{V}(\tau_m, x) \right] \]

\[ = \mathbb{E}^{\mathcal{F}_x} \left[ \langle \mathcal{K}(\tau_m)X^{\tau, x; \bar{u}(\cdot)}(\tau_m), X^{\tau, x; \bar{u}(\cdot)}(\tau_m) \rangle - \langle \mathcal{K}(\tau_m)x, x \rangle \right] \]

\[ \leq \lambda \left( \mathbb{E}^{\mathcal{F}_x} \left[ |x| + X^{\tau, x; \bar{u}(\cdot)}(\tau_m) \right]^2 \right)^{\frac{1}{2}} \left( \mathbb{E}^{\mathcal{F}_x} \left[ X^{\tau, x; \bar{u}(\cdot)}(\tau_m) - x \right]^2 \right)^{\frac{1}{2}}, \]

and

\[ \mathbb{E}^{\mathcal{F}_x} \left[ \int_{\tau}^{\tau_m} f(s, X^{\tau, x; \bar{u}(\cdot)}(s), \bar{u}(s)) ds \right] \leq C \mathbb{E}^{\mathcal{F}_x} \left[ \int_{\tau}^{\tau_m} \left( |X^{\tau, x; \bar{u}(\cdot)}(s)\right|^2 + |\bar{u}(s)|^2 \right] ds, \]

Then by (3.12), the estimate (2.2) and the dominate control theorem, we get

\[ \mathbb{E}^{\mathcal{F}_x} \left[ |\mathcal{V}(\tau, x) - \mathcal{V}(\tau_m, x)| \right] \]

\[ \leq \mathbb{E}^{\mathcal{F}_x} \left[ \int_{\tau}^{\tau_m} |f(s, X^{\tau, x; \bar{u}(\cdot)}(s), \bar{u}(s)) ds| + \left| \mathcal{V}(\tau_m, X^{\tau, x; \bar{u}(\cdot)}(\tau_m)) - \mathcal{V}(\tau_m, x) \right| \right] \]

\[ \to 0, \quad \text{as} \quad m \to \infty. \]

This is the RCE of \( \mathcal{V}_x \). The RCE of \( \mathcal{K} \) is a direct inference of that of \( \mathcal{V}_x \) and (3.6). \( \square \)

**Theorem 3.5.** Let Assumptions 1.1 be satisfied.

(i) For any \( \tau \in \mathcal{F} \) and \((x, u(\cdot)) \in \mathbb{R}^n \times \mathcal{U}_r \), the \( \mathcal{F}_\tau \)-system \( \mathcal{J}^{\tau, x, u(\cdot)} \) is RCE and aggregated by a RCLL \( \mathcal{F} \)-submartingale denoted by \( \{ \mathcal{J}^{\tau, x, u(\cdot)}(t), t \in [\tau, T]\} \). For the optimal control \( \bar{u}(\cdot) \in \mathcal{U}_r \) of Problem \( \mathcal{P}_{\tau, x} \), the corresponding \( \mathcal{F}_\tau \)-system \( \mathcal{J}^{\tau, x, \bar{u}(\cdot)} \) is aggregated by a RCLL \( \mathcal{F} \)-martingale denoted by \( \{ \mathcal{J}^{\tau, x, \bar{u}(\cdot)}(t), t \in [\tau, T]\} \).

(ii) The \( \mathcal{F}_\tau \)-system \( \{ \mathcal{K}(\tau), \tau \in \mathcal{F} \} \) is RCE and aggregated by a RCLL process denoted by \( \{ K(t), t \in [0, T]\} \). \( K \) is essentially bounded and \( \mathcal{S}_2 \)-valued. We have for any \( t \in [0, T] \)

\[ K(t) = K(0) - \int_0^t dk(s) + \sum_{i=1}^d \int_0^t L_i(s) dW_i^s + \int_0^t \int_{\Lambda} R(s, e) d\bar{\mu}(de, ds), \quad K(T) = M, \quad (3.13) \]
where \( k \) is an \( \mathbb{S}^n \)-valued predictable process of bounded variation, \( L \) an \( \mathbb{S}^n \)-valued predictable process and \( R \) a \( \mathcal{F} \)-measurable process.

(iii) The condition minimal value system \( \mathcal{V}_x \) for \( x \in \mathbb{R}^n \) is aggregated by the following RCML semimartingale

\[
V(t, x) := \langle K(t)x, x \rangle, \quad t \in [0, T].
\]

**Proof.** In view of (3.11), the REC of family \( \mathcal{J}^x,u(\cdot) \) comes from that of the \( \mathcal{F} \)-system \( \mathcal{V}_x \) and the a.s. right continuity of maps \( t \mapsto X_t^x,u(\cdot) \) and \( t \mapsto \int_s^t f(s, X(s), u(s))ds \). Using Proposition 2.2 we prove the first part of assertion (i). From the second part of Theorem 3.3 we see that \( \mathcal{J}^x,u(\cdot) \) is a \( \mathcal{F} \)-martingale.

Now we begin to show the assertion (ii). Denote by \( \tau_k \) the \( n \)-th jump time of the Poisson point process. Recall that \( e_i \) is the unit column vector whose \( i \)-th component is the number 1 for \( i = 1, \ldots, n \). We see that for \( x = e_i + e_j, e_i - e_j \) with \( i, j = 1, \ldots, n \), the process \( J^{k,x}(t) := J_{\tau_k \wedge T,x}(t), t \in [\tau_k \wedge T, T] \) is a right-continuous submartingale and \( \mathcal{J}^{k,x}(\cdot) - J^{k,x}(\tau_k \wedge T) \) is of class \( D \). Hence by Doob-Meyer decomposition (see [23, Theorem 11 in Section III.3]), it could be decomposed to an increasing, predictable process and a uniformly integrable martingale. Consider an \( \mathbb{S}^n \)-valued \( \mathcal{F}_T \wedge T \)-system \( \Gamma_k := \{ \Gamma_k(\tau), \tau \in \mathcal{F}_{\tau_k \wedge T} \} \) defined as follows:

\[
\Gamma_k(\tau) := \frac{1}{4} \left( \mathcal{J}^{\tau_k \wedge T, e_i + e_j, 0}(\tau) - \mathcal{J}^{\tau_k \wedge T, e_i - e_j, 0}(\tau) \right)_{1 \leq i, j \leq n}, \quad \tau \in \mathcal{F}_{\tau_k \wedge T}. \tag{3.14}
\]

In view of \( X^{\tau_k \wedge T, e_i + e_j, 0}(\tau) = X^{\tau_k \wedge T, e_i, 0}(\tau) + X^{\tau_k \wedge T, e_j, 0}(\tau), (3.6) \) and the proof of (3.8), we have

\[
\begin{align*}
&\left( \mathcal{V}(\tau, X^{\tau_k \wedge T, e_i + e_j, 0}(\tau)) - \mathcal{V}(\tau, X^{\tau_k \wedge T, e_i - e_j, 0}(\tau)) \right)_{1 \leq i, j \leq n} \\
= &\left( X^{\tau_k \wedge T, e_i, 0}(\tau), \ldots, X^{\tau_k \wedge T, e_n, 0}(\tau) \right) \mathcal{K}(\tau) \left( X^{\tau_k \wedge T, e_1, 0}(\tau), \ldots, X^{\tau_k \wedge T, e_n, 0}(\tau) \right).
\end{align*}
\]

This together with (3.11) and (3.14) yields

\[
\Gamma_k(\tau) = \Phi_k^*(\tau) \mathcal{K}(\tau) \Phi_k(\tau) + \int_{\tau_k \wedge T} \Phi_k^*(r) Q(r) \Phi_k(r) dr,
\]

where \( \Phi_k(t) \) is the solution of the following linear SDE:

\[
\begin{align*}
&d\Phi(t) = A(t)\Phi(t) - dt + \sum_{i=1}^d C^i(t)\Phi(t) - dW^i(t) + \int_{\mathcal{A}} E(t, e)\Phi(t) - \mu(de, dt), \\
&\Phi(\tau_k \wedge T) = I, \quad t \in (\tau_k \wedge T, \tau_{k+1} \wedge T). \tag{3.15}
\end{align*}
\]

The \( \mathcal{F}_{\tau_k \wedge T} \)-system \( \Gamma_k \) is aggregated by the following process still denoted by \( \{ \Gamma_k(t), t \in [\tau_k \wedge T, T] \} \):

\[
\Gamma_k(t) := \frac{1}{4} \left( \mathcal{J}^{\tau_k, e_i + e_j}(t) - \mathcal{J}^{\tau_k, e_i - e_j}(t) \right)_{1 \leq i, j \leq n}, t \in [\tau_k \wedge T, T],
\]

which is a right-continuous semimartingale with predictable of bounded variational part. We see that \( \Phi_k(t) \) is reversible for \( t \in (\tau_k \wedge T, \tau_{k+1} \wedge T) \) and its inverse \( \Psi_k(t) := \Phi_k^{-1}(t) \) satisfying

\[
\begin{align*}
&d\Psi_k(t) = \Psi_k(t) \left[ - A(t) + C^2(t) + \int_{\mathcal{A}} E(t, e) \nu(de) \right] dt - \sum_{i=1}^d \Psi_k(t) \cdot C^i(t) dW^i(t), \\
&\Psi_k(\tau_k \wedge T) = I, \quad t \in (\tau_k \wedge T, \tau_{k+1} \wedge T). \tag{3.16}
\end{align*}
\]

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It is obvious that $\Psi_k(t)$ is continuous at $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ and has left-limit at $\tau_{k+1} \wedge T$. Define

$$K_k(t) := \Psi_k^*(t) \Gamma_k(t) \Psi_k(t) - \Psi_k^*(t) \int_{\tau_k \wedge T}^{t} \Phi_k^*(s) Q(s) \Phi_k(s) ds \Psi_k(t), \quad t \in [\tau_k \wedge T, \tau_{k+1} \wedge T].$$

It is continuous on $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ and has left-limit at $\tau_{k+1} \wedge T$. By Itô formula, $K_k$ is a semimartingale, i.e.

$$K_k(t) = K_k(\tau_k \wedge T) + \tilde{M}_k(t) + \hat{A}_k(t), \quad t \in [\tau_k \wedge T, \tau_{k+1} \wedge T]$$

where $\tilde{M}_k$ with $\tilde{M}_k(\tau_k \wedge T) = 0$ is a local martingale and $\hat{A}$ with $\hat{A}(\tau_k \wedge T) = 0$ a predictable process with finite variation. We see that $\mathcal{K}(\tau) = K_k(\tau)$ for $\tau_k \wedge T \leq \tau < \tau_{k+1} \wedge T$. Thus $\mathcal{K}$ is aggregated by the process

$$K(t) := \sum_{k=0}^{\infty} K_k(t) \chi_{\{\tau_k \wedge T \leq t < \tau_{k+1} \wedge T\}}$$

$$= \left( \sum_{\tau_k \wedge T \leq t} \tilde{M}_k((\tau_{k+1} \wedge T) -) + \tilde{M}_i(t) \right) + \left( \sum_{\tau_k \wedge T \leq t} \hat{A}_k((\tau_{k+1} \wedge T) -) + \hat{A}_i(t) \right)$$

$$+ \sum_{\tau_k \leq t, k>1} \left( K_k(\tau_k \wedge T) - K_{k-1}((\tau_k \wedge T) -) \right),$$

where $i$ is the maximal integer with $\tau_i \leq t$. It is easy to observe that the first term of the right hand of above equality is a continuous martingale, the second term is continuous bounded variational process, and the third term is a pure jump process. By localizing method, it is easy to know the first part of last term is a local martingale, second part a finite variational predictable process.

According to $K_k$ is uniformly bounded, Theorem 35 in [28, Section III.7] yields the pure jump process $\sum_{\tau_k \leq t, k>1} (K_k(\tau_k \wedge T) - K_{k-1}((\tau_k \wedge T) -))$ is a special semimartingale. Thus $K$ could be canonically decomposed into the sum of an $\mathcal{F}$-predictable process $K_t$ with finite variation and an $\mathcal{F}$-martingale process on the whole time interval $[0, T]$. By martingale representation theorem (see [23, Section 5, Chapter IV] or [28, Lemma 2.3] for a easier version), we know that $K$ can be written as (3.13).

At last, the assertion (iii) is just a result of (3.11). Thus we finish the proof.

**Remark 3.1.** In [25, 26], the inverse flow of the controlled SDE is the key technique to show $K_t$ to be the fist part of the triple processes solution of BSRE. And in [25], the author pays lots of calculus to prove that the inverse flow of the solution $X$ for SDE associated with the corresponding optimal control exists on the whole time interval. For the SDE with jump, its inverse flow may not exists on whole time $[0, T]$ without additional condition, e.g.,

$$I + E \geq \delta I, \quad a.e., a.s.$$  \hspace{1cm} (3.17)

However, condition (3.17) is not necessary for the original control problem. So we insist on not introducing the condition (3.17) in the formulation of our BSREJ.

We observe that in the form of optimal feedback (see (5.3)), $K$ is independent of the state of the controlled equation, which hint us to represent $K$ by different state process in different time interval. Hence to overcome the difficulty of absence of (3.17), we can piece-wisely represent $K$ by the inverse flow on sub-interval between two adjacent jump time, on which the SDE (3.10) has continuous trajectories hence an inverse flow. After that we integrated the representation of $K$ from piece-wise to whole process on $[0, T]$ by the semimartingale property.
4 Existence of Solutions to BSREJ

This section is devoted to showing that $(K, L, R)$ given by Theorem 3.3 is nothing other than the solution of BSREJ (1.3), and to giving their estimates. Thus we establish the existence of solution for BSREJ (1.3).

**Theorem 4.1.** Let Assumptions (1.3) be satisfied. Then $(K, L, R)$ given by Theorem 3.3 satisfies BSREJ (1.3). And there is a deterministic constant $C$ such that the following estimate holds:

$$
\mathbb{E}\left( \int_0^T \sum_{i=1}^d |L_i(t)|^2 ds \right) + \mathbb{E}\left( \int_0^T \int_\Lambda |R(t,e)|^2 \nu(de) dt \right) \leq C. \quad (4.1)
$$

Hence $\int_0^T L_i(s)dW_s^i + \int_0^T \int_\Lambda R(t,e)\mu(de, ds)$ is a BMO martingale. Moreover $\int_\Lambda F^*(t,e)(K(t) + R(t,e))F(t,e)\nu(de)$ is nonnegative for almost all $t$, P-a.s.

**Proof.** Firstly, we show that $(K, L, R)$ satisfies satisfies (1.3) a.e.a.s. Define the functional

$$
\mathbb{F}(t,x,u,K(t),L(t),R(t,\cdot))
$$

$$
:=2\langle K(t)x,A(t)x+B(t)u \rangle + 2 \sum_{i=1}^d \langle L_i(t)x,C_i(t)x+D_i(t)u \rangle + \sum_{i=1}^d \langle K(t)(C_i(t)x+D_i(t)u), C_i(t)x+D_i(t)u \rangle
$$

$$
+ 2 \int_\Lambda \left( R(t,e)x,E(t,e)x+F(t,e)u \right) \nu(de) + \int_\Lambda \left( (K(t)+R(t,e))(E(t,e)x+F(t,e)u), E(t,e)x+F(t,e)u \right) \nu(de).
$$

For $\tau \in \mathcal{F}$, $\sigma \in \mathcal{F}$ and $u(\cdot) \in \mathcal{B}_\tau$, applying Itô formula to $V(t,X^{\tau,x,u(\cdot)}(t)) = \langle K(t)X^{\tau,x,u(\cdot)}(t), X^{\tau,x,u(\cdot)}(t) \rangle$, we get

$$
V(\sigma,X^{\tau,x,u(\cdot)}(\sigma)) = V(\tau,x) - \int_\tau^\sigma \langle dk(t)X(t),X(t) \rangle + \int_\tau^\sigma \mathbb{F}(t,x,u(t),K(t),L(t),R(t,\cdot)) dt
$$

$$
+ \sum_{i=1}^d \int_\tau^\sigma \left[ \langle L_i(t)X(t),X(t) \rangle + 2\langle K(t)X(t),C_i(t)X+D_i(t)u(t) \rangle \right] dW^i(t)
$$

$$
+ \int_\tau^\sigma \int_\Lambda \langle R(t,e)(X+E(t,e)x+F(t,e)u(t)),X+E(t,e)x+F(t,e)u(t) \rangle \mu(dt,de)
$$

$$
+ \int_\tau^\sigma \int_\Lambda \langle (K(t)+R(t,e))(E(t,e)x+F(t,e)u(t)), E(t,e)x+F(t,e)u(t) \rangle \mu(dt,de)
$$

$$
+ 2 \int_\tau^\sigma \int_\Lambda \langle K(t)X(t),E(t,e)x+F(t,e)u(t) \rangle \mu(dt,de),
$$

where $X$ is short for $X^{\tau,x,u(\cdot)}(t-)$. Taking conditional expectation with $\mathcal{F}_\tau$ on both sides of the above relation and noting the fact that the conditional expectation of the stochastic integrals w.r.t. the Brownian motion $W$ and the Poisson random measure $\mu$ vanishes by the localization with the
stopping time, we obtain
\[E^F_r [V(\sigma, X^{\tau,x;u}(\cdot)(\sigma))] = V(\tau, x) + E^F_r \left[ \int_\tau^\sigma \Pi(t, X^{\tau,x;u}(t-)) dt \right] - E^F_r \left[ \int_\tau^\sigma f(t, X^{\tau,x;u}(t-), u(t)) dt \right], \tag{4.3} \]
where
\[\Pi(dt; \tau, x, u(\cdot)) := - \langle dk(t) X^{\tau,x;u}(t-), X^{\tau,x;u}(t-) \rangle + \mathbb{E} \left[ f(t, X^{\tau,x;u}(t-), u(t), K(t-), L(t), R(t, \cdot)) dt \right] + f(t, X^{\tau,x;u}(t-), u(t)) dt. \tag{4.4} \]
This implies that
\[E^F_r \left[ \int_\tau^\sigma \Pi(dt; \tau, x, u(\cdot)) dt \right] = E^F_r [V(\sigma, X^{\tau,x;u}(\cdot))] + E^F_r \left[ \int_\tau^\sigma f(t, X^{\tau,x;u}(t-), u(t)) dt \right] - V(\tau, x). \tag{4.5} \]
From the dynamic programming principle, we have
\[\text{ess. inf}_{u(\cdot) \in \mathcal{U}_x} E^F_r \left[ \int_\tau^\sigma \Pi(dt; \tau, x, u(\cdot)) dt \right] = \text{ess. inf}_{u(\cdot) \in \mathcal{U}_x} \left\{ E^F_r [V(\sigma, X^{\tau,x;u}(\cdot))] + E^F_r \left[ \int_\tau^\sigma f(t, X^{\tau,x;u}(t-), u(t)) dt \right] \right\} - V(\tau, x) \tag{4.6} \]
Choose \( \tau_k \) as the \( k \)-th jump time of the Poisson point process. This implies that the measure \( \Pi(ds; \tau_k \land T, x, u(\cdot)) dx dP \) is nonnegative on \( \{(t, x, \omega) : t \in (\tau_k(\omega) \land T), x \in \mathbb{R}^n, \omega \in \Omega\} \) for any \( u(\cdot) \in \mathcal{U}_x \). Therefore, for any essentially bounded nonnegative predictable field \( \eta \) defined on \([0, T] \times \mathbb{R}^n \times \Omega\), we have
\[E \int_{\tau_k \land T}^{\tau_{k+1} \land T} \int_{\mathbb{R}^n} \eta(s, X^{\tau_k \land T,x;u}(s)) \det(\Phi_k(s)) \Pi(ds; \tau_k \land T, x, u(\cdot)) \geq 0, \quad \forall u(\cdot) \in \mathcal{U}_0 \]
with \( \Phi_k(s) \) being the Jacobian matrix of flow transformation \( x \rightarrow X^{\tau_k \land T,x;u}(s) \) for any \( u(\cdot) \in U_{\tau_k \land T} \). Note that before the next jump time \( \tau_{k+1}, \Phi(s) \) is inversible, i.e., \( \det(\Phi(s)) > 0 \) \( \mathbb{P} \)-a.s. Via a transformation of state variable \( x \), we have
\[E \int_{\tau_k \land T}^{\tau_{k+1} \land T} \int_{\mathbb{R}^n} \eta(s, x) \Pi(ds; \tau_k \land T, Y^{\tau_k \land T,x;u}(s), u(\cdot)) \geq 0, \quad \forall u(\cdot) \in \mathcal{U}_0, \]
where \( Y^{\tau_k \land T,x;u}(s) \) is the inverse of the flow \( x \rightarrow X^{\tau_k \land T,x;u}(s) \) for \( \tau_k \land T \leq s < \tau_{k+1} \land T \). Incorporating \( \Pi(ds; 0, \cdot, u(\cdot)) \geq 0 \) with the inverse flow \( Y^{\tau_k \land T,x;u}(s), x \in \mathbb{R}^n \) we have
\[0 \leq \Pi(dt; \tau_k \land T, Y^{\tau_k \land T,x;u}(t); u(\cdot)) = - \langle dk(t)x, x \rangle + F(t, x, u(t), K(t-), L(t), R(t, \cdot)) dt + f(t, x, u(t)) dt \]
on \( \{ (t, \omega) : t \in (\tau_k(\omega) \wedge T, \tau_{k+1}(\omega) \wedge T), \omega \in \Omega \} \). In a similar way, we have for a.e. a.s. \( (t, \omega) \in [0, T] \times \Omega \),

\[
0 = \Pi(dt; \tau_k \wedge T, Y^{\tau_k \wedge T; x; \bar{u}(-)}(t), \bar{u}(\tau_k \wedge T, Y^{\tau_k \wedge T; x; \bar{u}(-)}(t))) \\
= -\langle dk(t)x, x \rangle + \mathbb{F}(t, x, \bar{u}(\tau_k \wedge T, Y^{\tau_k \wedge T; x; \bar{u}(-)}(t)), K(t-), L(t), R(t, \cdot))dt \\
+ f(t, x, \bar{u}(\tau_k \wedge T, Y^{\tau_k \wedge T; x; \bar{u}(-)}(t)))dt.
\]

Therefore, we have

\[
\langle dk(t)x, x \rangle = \min_{v \in \mathbb{R}^n} \left[ \mathbb{F}(t, x, v, K(t-), L(t), R(t, \cdot)) + f(t, x, v) \right]dt, \quad t \in \{ \tau_k \wedge T, \tau_{k+1} \wedge T \}.
\]

Since \( k \) is a predictable process, it does not have a jump at the inaccessible time \( \tau_k \). Thus \( dk \) does not contain singular measure, in other word, any \( t \in [0, T] \),

\[
\langle dk(t)x, x \rangle = \min_{v \in \mathbb{R}^n} \left[ \mathbb{F}(t, x, v, K(t-), L(t), R(t, \cdot)) + f(t, x, v) \right]dt.
\] (4.7)

In view of assertion (ii) in Proposition 3.2, the right hand side of (4.7) has a unique minimal point \( \bar{u}(t) \), hence the minimum value is nothing but \( G(t, K(t-), L(t), R(t, \cdot)) \) and \( \mathcal{N}(t, K(t), R(t, \cdot)) \) is invertible, which together with \( 3.13 \) implies that \( (K, L, R) \) satisfies \( 1.3 \) a.s.

Next we prove the BMO martingale property and \( 1.1 \). Using \( 1.2 \) for \( u(\cdot) = 0 \) and \( X = X_{\tau, x:0} \), we have

\[
\begin{align*}
\{ &dV(t, X(t)) = -\langle dk(t)X(t), X(t) \rangle + \mathbb{F}(t, X(t-), 0, K(t-), L(t), R(t, \cdot))dt \\
&+ \sum_{i=1}^{d} \left[ \langle L^i(t)X(t-), X(t-) \rangle + 2\langle K(t-)X(t-), C^i(t)X(t-) \rangle \right]dW^i(t) \\
&+ \int_{\Lambda} \langle R(t, e)(X(t-)) + E(t, e)X(t-), X + E(t, e)X(t-) \rangle \hat{\mu}(de, dt) \\
&+ \int_{\Lambda} \langle K(t-)E(t, e)X(t-), E(t, e)X(t-) \rangle \hat{\mu}(de, dt) \\
&+ 2\int_{\Lambda} \langle K(t-)X(t-), E(t, e)X(t-) \rangle \hat{\mu}(de, dt) \\
&V(T, X(T)) = \langle M\hat{X}(T), X(T) \rangle.
\end{align*}
\] (4.8)

Applying Itô formula to \( |V(t, X(t))|^2 \), we have

\[
\begin{align*}
\int_{\tau}^{T} \int_{\Lambda} \left[ \langle R(I + E)X, (I + E)X \rangle + \langle K(2I + E)X, EX \rangle \right]^2 \hat{\mu}(de, dt) \\
+ \int_{\tau}^{T} \sum_{i=1}^{d} \left[ \langle L^iX, X \rangle + 2\langle KX, C^iX \rangle \right]^2 dt \\
= |\langle M\hat{X}(T), X(T) \rangle|^2 - \langle K(\tau)x, x \rangle + 2\int_{\tau}^{T} \langle KX, X \rangle \left[ \langle dkX, X \rangle - \mathbb{F}(t, X, 0, K, L, R)dt \right] \\
- 2\int_{\Lambda} \langle KX, X \rangle \left[ \langle R(I + E)X, (I + E)X \rangle + \langle K(2I + E)X, EX \rangle \right] \hat{\mu}(de, dt)
\end{align*}
\] (4.9)
where \(X\) means \(X^{\tau,\tau\equiv 0}\) and \(K\) means \(K^{\tau}\).

In the following estimates the constant \(C\) may change line by line. Since \(V(t, X(t)) > 0\) and the measure \(\Pi(dt; \tau \wedge T, x, u)dxP\) (see (4.3)) is nonnegative, we have a.e.

\[
- 2 \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \left[ \langle L^i X, X \rangle + 2 \langle KX, C^i X \rangle \right] dW^i(t),
\]

\[
\text{where } X \text{ means } X^{\tau,\tau\equiv 0}(t-), \text{ and } K \text{ means } K(t-).
\]

Thanks to inequality \(\frac{1}{2}a^2 - b^2 \leq (a + b)^2\), and the boundness of \(K\), we have

\[
\int_\tau^T \int_\Lambda \left| \langle R(I + E)X, (I + E)X \rangle + \langle K(2I + E)X, EX \rangle \right|^2 \mu(de, dt) + \int_\tau^T \sum_{i=1}^d \left| \langle L^i X, X \rangle + 2 \langle KX, C^i X \rangle \right|^2 dt
\]

\[
\leq |M|^2 |X(T)|^4 + 2 \int_\tau^T \langle KX, X \rangle f(t, X, 0) dt
\]

\[
- 2 \int_\tau^T \int_\Lambda \left| \langle K(2I + E)X, EX \rangle \right|^2 \mu(de, dt) + 2 \int_\tau^T \sum_{i=1}^d \left| \langle L^i X, X \rangle + 2 \langle KX, C^i X \rangle \right|^2 dt
\]

\[
+ 2 |M|^2 |X|^4 + 4 \int_\tau^T \langle KX, X \rangle f(t, X, 0) dt
\]

\[
- 4 \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \left[ \langle L^i X, X \rangle + 2 \langle KX, C^i X \rangle \right] dW^i(t)
\]

\[
- 4 \int_\tau^T \int_\Lambda \left[ \langle R(I + E)X, (I + E)X \rangle + \langle K(2I + EX, EX) \rangle \right] \bar{\mu}(de, dt)
\]

\[
\leq C \sup_{\tau \leq t \leq T} |X|^4 + 4 \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \langle L^i X, X \rangle dW^i(t)
\]

\[
+ 8 \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \langle KX, C^i X \rangle dW^i(t)
\]

\[
+ 4 \int_\tau^T \int_\Lambda \langle R(I + E)X, (I + E)X \rangle \bar{\mu}(de, dt)
\]

\[
+ 4 \int_\tau^T \int_\Lambda \langle K(2I + EX, EX) \rangle \bar{\mu}(de, dt).
\]
This means that
\[
\mathbb{E}^\mathcal{F} \left[ \int_\tau^T \int_{\Lambda} |\langle R(I + E)X, (I + E)X \rangle_\mu| dt \right] + \mathbb{E}^\mathcal{F} \left[ \int_\tau^T \sum_{i=1}^d |\langle L^i X, X \rangle| dt \right]
\]
\[
\leq C \mathbb{E}^\mathcal{F} \left[ \sup_{\tau \leq t \leq T} |X(t)|^4 \right] + C_p \mathbb{E}^\mathcal{F} \left[ \left| \int_\tau^T \left( \langle KX, X \rangle \sum_{i=1}^d \langle L^i X, X \rangle dW^i(t) \right) \right| \right] \tag{4.11}
\]
\[
+ C \mathbb{E}^\mathcal{F} \left[ \left| \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \langle KX, C^i X \rangle dW^i(t) \right| \right] + C \mathbb{E}^\mathcal{F} \left[ \left| \int_\tau^T \langle K(I + E)X, EX \rangle \mu dt \right| \right].
\]

Using BDG inequality, Hölder inequality, boundness of \(K\) and the estimation Lemma \[20\], we have the following estimation about the every terms in right hand side of \[4.10\],
\[
\mathbb{E}^\mathcal{F} \left[ \left| \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \langle L^i X, X \rangle dW(t) \right| \right] \leq \mathbb{E}^\mathcal{F} \left[ \left| \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \langle L^i X, X \rangle \right|^2 \frac{1}{2} dt \right]
\]
\[
\leq C \mathbb{E}^\mathcal{F} \left[ \sup_{\tau \leq t \leq T} |X|^4 \left( \int_\tau^T \sum_{i=1}^d \langle L^i X, X \rangle^2 dt \right)^{\frac{1}{2}} \right] \tag{4.12}
\]
\[
\leq \frac{C}{\varepsilon} \mathbb{E}^\mathcal{F} \left[ \left| \sup_{\tau \leq t \leq T} |X|^4 \right| + \varepsilon \mathbb{E}^\mathcal{F} \left[ \left( \int_\tau^T \sum_{i=1}^d \langle L^i X, X \rangle^2 dt \right)^{\frac{1}{2}} \right] \right]
\]
\[
\leq \frac{C}{\varepsilon} |X|^4 + \varepsilon \mathbb{E}^\mathcal{F} \left[ \int_\tau^T \sum_{i=1}^d \langle L^i X, X \rangle^2 dt \right],
\]
\[
\mathbb{E}^\mathcal{F} \left[ \left| \int_\tau^T \langle KX, X \rangle \sum_{i=1}^d \langle KX, C^i X \rangle dW^i(t) \right| \right] \leq \mathbb{E}^\mathcal{F} \left[ \left( \int_\tau^T \left| \langle KX, X \rangle \sum_{i=1}^d \langle KX, C^i X \rangle \right|^2 dt \right)^{\frac{1}{2}} \right] \tag{4.13}
\]
\[
\leq C_p \mathbb{E}^\mathcal{F} \left[ \sup_{\tau \leq t \leq T} |X|^4 \right] \leq C_p |X|^4,
\]
\[ E_{\tau}^\mathcal{F} \left[ \left| \int_\tau^T \int_\Lambda \langle KX, X \rangle \langle K(2I + E)X, EX \rangle \bar{\mu}(de, dt) \right| \right] \]
\[ \leq E_{\tau}^\mathcal{F} \left[ \left( \int_\tau^T \int_\Lambda \langle KX, X \rangle \langle K(2I + E)X, EX \rangle \right)^2 \mu(de, dt) \right]^{1/2} \]
\[ \leq C E_{\tau}^\mathcal{F} \left[ \sup_{\tau \leq t \leq T} |X|^4 \right] \]
\[ \leq C|x|^4, \]
\[ E_{\tau}^\mathcal{F} \left[ \int_\tau^T \int_\Lambda \langle R(I + E)X, (I + E)X \rangle \bar{\mu}(de, dt) \right] \]
\[ \leq C E_{\tau}^\mathcal{F} \left[ \left\{ \int_\tau^T \int_\Lambda \langle R(I + E)X, (I + E)X \rangle \right)^2 \mu(de, dt) \right]^{1/2} \]
\[ \leq C E_{\tau}^\mathcal{F} \left[ \left\{ \int_\tau^T \int_\Lambda \langle R(I + E)X, (I + E)X \rangle \right)^2 \mu(de, dt) \right] \]
\[ \leq \frac{C}{\varepsilon} E_{\tau}^\mathcal{F}\left[ \sup_{\tau \leq t \leq T} |X|^4 \right] + \varepsilon E_{\tau}^\mathcal{F} \left[ \int_\tau^T \int_\Lambda \langle R(I + E)X, (I + E)X \rangle \mu(de, dt) \right] \]
\[ \leq \frac{C}{\varepsilon} |x|^4 + \varepsilon E_{\tau}^\mathcal{F} \left[ \int_\tau^T \int_\Lambda \langle R(I + E)X, (I + E)X \rangle \mu(de, dt) \right]. \]

Taking conditional expectation on both sides of (4.11), putting (4.13)–(4.15) into it, and then letting \( \varepsilon = 1/4 \), we get
\[ E_{\tau}^\mathcal{F} \left[ \int_\tau^T \int_\Lambda \langle R(I + E)X, (I + E)X \rangle \mu(de, dt) \right] + E_{\tau}^\mathcal{F} \left[ \int_\tau^T \sum_{i=1}^d \langle L^i X, X \rangle \mu(de, dt) \right] \leq C|x|^4, \]

the constant \( C \) is independent of \( \tau \) and \( x \). Then we have
\[ E_{\tau}^\mathcal{F} \left[ \int_\tau^T \int_\Lambda \Phi^*(I + E)^*R(I + E)\Phi \mu(de, dt) \right] + E_{\tau}^\mathcal{F} \left[ \int_\tau^T \sum_{i=1}^d \Phi L^i \Phi \mu(de, dt) \right] \leq C, \]
where \( \Phi \) is the solution of matrix equation (3.15) on \( [\tau \wedge T, T] \) with initial data \( \Phi(\tau \wedge T) = I \).

Recall that \( \tau_k \) as the \( k \)-th jump time of the Poisson point process. For any stopping time \( \gamma \leq T \), denote by \( \tilde{\tau}_k := \gamma \vee \tau_k \) the \( n \)-th jump time after the stopping time \( \gamma \). Applying (4.10) for \( \tau = \tilde{\tau}_k \wedge T \) and noting that \( \Phi \) is invertible on time \( [\tilde{\tau}_k \wedge T, \tilde{\tau}_k+1 \wedge T] \) and \( E_{\tilde{\tau}_k \wedge T} \left[ \sup_{t \in [\tilde{\tau}_k \wedge T, \tilde{\tau}_k+1 \wedge T]} \Phi^{-1}(t) \right] \) is bounded by a constant only depending on the bound of the coefficients and \( T \) (see (3.16) for details), we see that for any \( k \geq 1 \),
\[ E_{\tilde{\tau}_k \wedge T} \left[ \int_{\tilde{\tau}_k \wedge T}^{\tilde{\tau}_{k+1} \wedge T} |L^i|^2 dt \right] \]
\[ \leq E_{\tilde{\tau}_k \wedge T} \left[ \int_{\tilde{\tau}_k \wedge T}^{\tilde{\tau}_{k+1} \wedge T} |(\Phi^*)^{-1} \Phi^i L^i \Phi^i \Phi^{-1}|^2 dt \right] \]
Then, using estimates (4.17) and (4.18), we have

\[
\begin{align*}
\leq & \mathbb{E}_{\mathcal{F}_{\tau_k \wedge T}} \left[ \sup_{t \in [\tau_k \wedge T, \tau_{k+1} \wedge T]} |\Phi^{-1}(t)|^2 \int_{\tau_k \wedge T}^{\tau_{k+1} \wedge T} \sum_i |\Phi L^i| dt \right] \\
\leq & \left\{ \mathbb{E}_{\mathcal{F}_{\tau_k \wedge T}} \left[ \sup_{t \in [\tau_k \wedge T, \tau_{k+1} \wedge T]} |\Phi^{-1}(t)|^2 \int_{\tau_k \wedge T}^{\tau_{k+1} \wedge T} \sum_i |\Phi L^i| dt \right] \right\}^{\frac{1}{2}} \\
\leq & C.
\end{align*}
\] (4.17)

Similarly, we have

\[
\mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \int_{\gamma \wedge T}^{\tau_{n+1} \wedge T} |L|^2 dt \right] \leq C. \quad (4.18)
\]

Then, using estimates (4.17) and (4.18), we have

\[
\begin{align*}
\mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \int_{\gamma \wedge T}^{\tau_{n+1} \wedge T} |L|^2 dt \right] &= \mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \int_{\gamma \wedge T}^{\tau_{n+1} \wedge T} \sum_i |L|^2 dt \right] \\
&= \mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \int_{\gamma \wedge T}^{\tau_{n+1} \wedge T} \sum_i |L|^2 dt \right] \\
&= \mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \chi_{\{\gamma < T\}} \mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \int_{\gamma \wedge T}^{\tau_{n+1} \wedge T} \sum_i |L|^2 dt \right] + \sum_{n=1}^{\infty} \chi_{\{\tau_n < T\}} \mathbb{E}_{\mathcal{F}_{\tau_n \wedge T}} \left[ \int_{\tau_n \wedge T}^{\tau_{n+1} \wedge T} \sum_i |L|^2 dt \right] \right] \\
&\leq \mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \chi_{\{\gamma < T\}} + \sum_{n=1}^{\infty} \chi_{\{\tau_n < T\}} \right] C \\
&= C \mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \mu(\{\gamma, T\} \times \Lambda) \right].
\end{align*}
\]

In view of the independent increment property of the Poisson point process \( \{p_t\}_{t \geq 0} \), \( \mu(\{\gamma, T\} \times \Lambda) \) is independent of \( \mathcal{F}_{\gamma} \). So we have \( \mathbb{E}_{\mathcal{F}_{\gamma}} \left[ \mu(\{\gamma, T\} \times \Lambda) \right] = \mathbb{E} \left[ \mu(\{\gamma, T\} \times \Lambda) \right] \leq \mathbb{E} \left[ \mu([0, T] \times \Lambda) \right] = T \nu(\Lambda) \). Hence we obtain that for any stopping time \( \gamma \) valued in \([0, T]\),

\[
\mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \left\{ \sum_i \int_{\gamma}^{T} L_i W^i(t) \right\}^2 \right] \leq \mathbb{E}_{\mathcal{F}_{\gamma \wedge T}} \left[ \int_{\gamma}^{T} \sum_i |L|^2 dt \right] \leq C, \quad (4.19)
\]

which means \( \int_0^T L_i(s)dW^i_s \) is a BMO martingale, \( i = 1, \ldots, d \).

For \( \eta_k := \int_0^T R(e, t) \mu(de, dt) \), we see that it is a purely continuous martingale whose jumps coincide with those of \( K \). Since \( K \) is uniformly bounded by some constant \( \lambda \), jumps of \( \eta \) is also uniformly bounded by \( 2\lambda \). Hence we have

\[
|\eta|_{T} - |\eta|_{\gamma} \leq \sum_{\gamma \leq s \leq T} |\Delta \eta_s| = \sum_{\gamma \leq s \leq T} |\Delta \eta_s|^2 \leq 4\lambda^2 \mu(\{\gamma, T\} \times \Lambda).
\]
Thus
\[ \mathbb{E}_F \left[ \eta_T - \eta_{\gamma-} \right] \leq 4 \lambda^2 \mathbb{E}_F \left[ \mu([\gamma, T] \times \Lambda) \right] \leq C T \nu(\Lambda) < \infty, \quad (4.20) \]
which means that \( J \) is also a BMO martingale.

Let \( \gamma = 0 \) in (4.19) and (4.20), we have
\[ \mathbb{E} \left[ \int_0^T \sum_i |L_i|^2 dt \right] \leq C, \quad (4.21) \]
and
\[ \mathbb{E} \left[ \int_0^T \int_\Lambda R^2 \nu(de) dt \right] = \mathbb{E}[\eta_T] \leq C, \quad (4.22) \]
we have estimate (4.1).

Last we show the nonnegativity of \( \int_\Lambda F^*(t, e)(K(t-) + R(t, e))F(t, e)\nu(de) \). First note that the pure jump process \( \zeta_t := \int_0^t \int_\Lambda F^*(s, e)(K(t-) + R(s, e))F(s, e)\mu(de, ds) \) only changes its value at the jumping time of Poisson process and \( \Delta \zeta_t = \int_0^t F^*(s, e)(K(s-) + R(s, e))F(s, e)\mu(de, \{t\}) \).

Since at the jumping moment \( R(s, p_s) \) is equivalent to \( K(s) - K(s-) \), it is easy to know that \( K(s-) + R(s, e) = K(s) \) is nonnegative definite (here \( e \) is the jumping amplitude at the moment), therefore for any \( y \in \mathcal{M}_F(0, T; \mathbb{R}^m) \), we have
\[ \int_0^T \int_\Lambda y^*(s)F^*(s, e)(K(s-) + R(s, e))F(s, e)y(s)\mu(de, ds) \geq 0, \quad P-a.e. \]
In view of the martingale property,
\[ \mathbb{E} \left[ \int_0^T \int_\Lambda y^*(s)F^*(s, e)(K(s-) + R(s, e))F(s, e)y(s)\mu(ds, de) \right] = 0. \]
Hence
\[ \mathbb{E} \left[ \int_0^T \int_\Lambda y^*(s)F^*(s, e)(K(s-) + R(s, e))F(s, e)y(s)\nu(de)ds \right] = \mathbb{E} \left[ \int_0^T \int_\Lambda y^*(s)F^*(s, e)(K(s-) + R(s, e))F(s, e)y(s)[\mu(de, ds) - \tilde{\mu}(de, ds)] \right] \geq 0. \]
By the arbitrariness of \( y \), we have \( \int_\Lambda F^*(t, e)(K(t-) + R(t, e))F(t, e)\nu(de) \) is nonnegative for almost all \( t, \mathbb{P}\)-a.s. \( \omega \). Thus, the proof is complete.

\[ \square \]

**Remark 4.1.** If we have the condition (3.17) in hand, (4.22) could be obtained from (4.16) directly like the way of (4.17)-(4.19). In our case, observing the structure of BSREJ and utilizing the relationship between the jump of \( K \) and \( R \), we can prove (4.22) by the estimate of \( K \), and this way seemed to be easier.
5 Verification theorem

In section 4 we exploit Problem 2.4 and the dynamic programming principle to show the existence of solution for BSREJ (1.3). In this section we will deal with the problem from an inverse aspect – if the BSREJ (1.3) has a solution, how to describe the corresponding optimal control problem? The following Theorem 5.1 tells us that the existence of solution for BSREJ (1.3) means the existence of the optimal control for problem (5.3). Besides, the optimal control could be depicted as a linear feedback by the solution of BSREJ (1.3).

**Theorem 5.1.** Let Assumptions 1.1 be satisfied. And assume BSREJ (1.3) has a solution $(K, L, R)$ in the meaning of Definition 1.1. Then the linear SDE

\[
\begin{cases}
    d\bar{X}^{t,x}(s) = [A(s) - B(s)\mathcal{N}^{-1}(s, K(s-), R(s, \cdot))\mathcal{M}^s(s, K(s-), L(s), R(s, \cdot))]\bar{X}^{t,x}(s)ds \\
    + \sum_{i=1}^{d_i} C_i(s) - D_i(s)\mathcal{N}^{-1}(s, K(s-), R(s, \cdot))\mathcal{M}^s(s, K(s-), L(s), R(s, \cdot))]\bar{X}^{t,x}(s)ds dW^i(s) \\
    + \int_{\Lambda} E(s, e) - F(s, e)\mathcal{N}^{-1}(s, K(s-), R(s, \cdot))\mathcal{M}^s(s, K(s-), L(s), R(s, \cdot))]\bar{X}^{t,x}(s)\mu(ds, de), \\
    \bar{X}(t) = x, \quad s \in [t, T]
\end{cases}
\]

has a unique solution $\bar{X}^{t,x}(\cdot)$ such that

\[E^\mathcal{F}_t \left[ \sup_{s \in [t,T]} |\bar{X}^{t,x}(s)|^2 \right] < C_x, \quad (5.2)\]

where the constant $C_x$ is independent of initial time $t$.

(ii) The given process

\[\bar{u}^{t,x}(s) := -\mathcal{N}^{-1}(s, K(s-), R(s, \cdot))\mathcal{M}^s(s, K(s-), L(s), R(s, \cdot))\bar{X}(s), \quad s \in [t, T] \quad (5.3)\]

belongs to $\mathcal{M}_2^\mathcal{F}(t, T; \mathbb{R}^m)$, and is the optimal control for the problem (5.2) for the initial data $(\tau, \xi) = (t, x)$.

(iii) The value field $V$ is given by

\[V(t, x) = \langle K(t)x, x \rangle, \quad (t, x) \in [0, T] \times \mathbb{R}^n. \quad (5.4)\]

**Proof.** Since the coefficients of the optimal SDE (5.1) are square integrable w.r.t. $t$ a.s., it admits a unique strong solution $\bar{X}(\cdot)$. For a sufficiently large integer $j$, define the stopping time $\gamma_j$ as follows:

\[\gamma_j := T \land \inf\{s \geq t \mid |\bar{X}^{t,x}(s)| \geq j\}\]

with the convention that $\inf \emptyset = \infty$. It is obvious that $\gamma_j \uparrow T$ almost surely as $j \uparrow \infty$. Then by Itô formula we have

\[\langle K(t)x, x \rangle = E^\mathcal{F}_t \left[ \langle K(\gamma_j^{t,x})\bar{X}^{t,x}(\gamma_j^{t,x}), \bar{X}^{t,x}(\gamma_j^{t,x}) \rangle + \int_t^{\gamma_j^{t,x}} f(s, \bar{X}^{t,x}(s), \bar{u}^{t,x}(s))ds \right] \quad (5.5)\]

Noting that $K$ is positive and bounded by $\lambda$, and $N > \delta I$ for some constant $\delta$ (see Assumption 1.1), (5.5) implies

\[E^\mathcal{F}_t \left[ \int_t^{\gamma_j^{t,x}} (\bar{u}^{t,x})^2(s)ds \right] \leq \frac{1}{\delta} E^\mathcal{F}_t \left[ \int_t^{\gamma_j^{t,x}} f(s, \bar{X}^{t,x}(s), \bar{u}^{t,x}(s))ds \right] \leq \frac{1}{\delta} \langle K(t)x, x \rangle \leq \frac{\lambda}{\delta} |x|^2. \]
Using Fatou’s lemma, we have \(\bar{u}^{t,x}(\cdot) \in \mathcal{M}^2_\mathbb{F}(0,T;\mathbb{R}^m)\). Then we have the estimation (5.2) from Lemma 2.3. Thus, Assertion (i) and the first part of the assertion (ii) have been proved.

Now we prove the optimality of \(\bar{u}^{t,x}(\cdot)\) and the assertion (iii). By (5.2), we know for any stopping time \(\tau\) valued in \([t,T]\),

\[
\mathbb{E}^{\mathbb{F}_t}\left[|\tilde{X}_{t,x}(\tau)|^2\right] \leq \mathbb{E}^{\mathbb{F}_t}\left[\sup_{s\in[t,T]}|\tilde{X}_{t,x}(s)|^2\right] < C_x,
\]

hence \(|\tilde{X}_{t,x}|^2\) is uniformly integrable. Besides (5.2) together with Chebyshev inequality shows that for any positive integer \(j\),

\[
\mathbb{P}\left(\sup_{s\in[t,T]}|\tilde{X}_{t,x}(s)| \geq j\right) \leq \frac{\mathbb{E}^{\mathbb{F}_t}\left[\sup_{s\in[t,T]}|\tilde{X}_{t,x}(s)|^2\right]}{j^2} \to 0, \quad \text{as} \quad j \to \infty.
\]

It follows that \(\mathbb{P}(\gamma_{j}^{t,x} = T) \nearrow 1\). Combining the dominate convergence theorem and the boundness of \(K\), we have the first term in right hand of (5.3)

\[
\mathbb{E}^{\mathbb{F}_t}\left[f_t^{\gamma_{j}^{t,x}} f(s,\tilde{X}_{t,x}(s),\bar{u}^{t,x}(s))ds\right] \to \mathbb{E}^{\mathbb{F}_t}\left[\int_t^T f(s,\tilde{X}_{t,x}(s),\bar{u}^{t,x}(s))ds\right] \quad \text{as} \quad j \to \infty.
\]

Hence (5.5) yields

\[
\langle K(t)x,x \rangle = \lim_{j \to \infty} \mathbb{E}^{\mathbb{F}_t}\left[(K(\gamma_{j}^{t,x})\tilde{X}_{t,x}(\gamma_{j,t}^{t,x}),\bar{X}_{t,x}(\gamma_{j,t}^{t,x})) + \int_{\gamma_{j}^{t,x}}^{t} f(s,\tilde{X}_{t,x}(s),\bar{u}^{t,x}(s))ds\right]
\]

\[
= \mathbb{E}^{\mathbb{F}_t}\left[(K(T)\tilde{X}_{t,x}(T),\bar{X}_{t,x}(T)) + \int_{t}^{T} f(s,\tilde{X}_{t,x}(s),\bar{u}^{t,x}(s))ds\right] = J(\bar{u}^{t,x}(\cdot);0,x).
\]

To obtain the optimality of \(\bar{u}^{t,x}(\cdot)\), it remain to show

\[
J(u(\cdot);t,x) \geq \langle K(t)x,x \rangle, \quad \forall u(\cdot) \in \mathcal{M}^2_\mathbb{F}(t,T;\mathbb{R}^m).
\]

To do this, for any \(u(\cdot) \in \mathcal{M}^2_\mathbb{F}(t,T;\mathbb{R}^m)\), define the stopping times

\[
\gamma_{j}^{t,x;u(\cdot)} = T \wedge \inf\{s \geq t|X_{t,x;u(\cdot)}(s)| \geq j\}, \quad j \in \mathbb{Z}_+.
\]

Same as \(\gamma_{j}^{t,x}, \gamma_{j}^{t,x;u(\cdot)} \nearrow T\) and \(\mathbb{P}\{\gamma_{j}^{t,x;u(\cdot)} = T\} \nearrow 1\) as \(j \to \infty\). Define

\[
\tilde{u}(s) := -\mathcal{N}^{-1}(s,K(s,\cdot),R(s,\cdot))\mathcal{M}(s,K(s,\cdot),L(s),R(s,\cdot))X^{0,x,u(\cdot)}(s,-), \quad s \in [t,T].
\]

Obviously, \(\mathbb{E}\left[\int_t^{\gamma_{j}^{t,x;u(\cdot)}} |\tilde{u}(t)|^2dt\right] < \infty\). Then applying Itô formula to \(\langle K(t)X_{t,x;u(\cdot)}(t),X_{t,x;u(\cdot)}(t)\rangle\) and by straightforward computing, we get that

\[
\mathbb{E}^{\mathbb{F}_t}\left[(K(\gamma_{j}^{t,x;u(\cdot)})X_{t,x;u(\cdot)}(\gamma_{j}^{t,x;u(\cdot)}),X_{t,x;u(\cdot)}(\gamma_{j}^{t,x;u(\cdot)})) + \int_{\gamma_{j}^{t,x;u(\cdot)}}^{t} f(s,X_{t,x;u(\cdot)}(s),u(s))ds\right] \quad (5.7)
\]
\[ \langle K(t)x, x \rangle + \mathbb{E}^\mathbb{P}_t \left[ \int_t^T \langle \mathcal{N}^{-1}(s, K(s), R(s, \cdot))(u(s) - \tilde{u}(s)), u(s) - \tilde{u}(s) \rangle \right] \geq \langle K(t)x, x \rangle. \]

Since \( u(\cdot) \in \mathcal{M}_2^x(t, T; \mathbb{R}^m) \), according to the estimate \( 5.2 \), similar to the limitation in \( 5.6 \), we take limit in \( 5.7 \)

\[
J(u(\cdot); t, x) = \mathbb{E}^\mathbb{P}_t \left[ \langle K(T)X^{t,x,u}(T), X^{t,x,u}(T) \rangle + \int_t^T f(s, X^{t,x,u}(s), u(s))ds \right] 
\geq \langle K(t)x, x \rangle.
\]

According to the above verification theorem, we immediately have the following uniqueness of the solution for BSREJ \((1.3)\).

**Theorem 5.2.** Let Assumptions 1.1 be satisfied. Let \((\tilde{K}, \tilde{L}, \tilde{R})\) be another solution of BSREJ \((1.3)\) in the meaning of Definition 1.1. Then \((\tilde{K}, \tilde{L}, \tilde{R}) = (K, L, R)\).

**Proof.** In view of \((5.3)\), the uniqueness of value function \(V\) leads to that of first unknown variable \(K\) of solution for BSREJ \((1.3)\), hence \(\tilde{K} = K\). By the expression of BSREJ \((1.3)\), the integration w.r.t. \(\tilde{\mu}\) is just pure jump martingale, hence

\[
\sum_{s \leq t} \Delta K_s = \int_0^t \int_{\Lambda} \tilde{R}(t, e) \mu(de, ds),
\]

\[
\sum_{s \leq t} \Delta \tilde{K}_s = \int_0^t \int_{\Lambda} \tilde{R}(t, e) \mu(de, ds).
\]

Comparing the above two equality, taking the quadratic variation (the bracket), and then taking expectation on both sides, we have

\[
0 = \mathbb{E} \left[ \sum_{s \leq t} \Delta (K_s - \tilde{K}_s), \sum_{s \leq t} \Delta (K_s - \tilde{K}_s) \right] = \mathbb{E} \left[ \int_0^t \int_{\Lambda} (R - \tilde{R})^2 \nu(de)ds \right].
\]

This means \(\tilde{R} = R\).

With the uniqueness of the first and third unknown variables \((K, R)\) in hand, the uniqueness of the optimal control and its feedback form \((5.3)\) yields the uniqueness of the second unknown variable \(L\).

**References**

[1] J. M. Bismut. Linear quadratic optimal stochastic control with random coefficients. *SIAM J. Control Optim.*, 14(3):419–444, 1976.
[2] S. Chen, X. Li, and X. Y. Zhou. Stochastic linear quadratic regulators with indefinite control weight costs. *SIAM J. Control Optim.*, 36(5):1685–1702, 1998.

[3] S. Chen and S. Tang. Semi-linear backward stochastic integral partial differential equations driven by a Brownian motion and a Poisson point process. *Math. Control Relat. Fields*, 5(3):401–434, 2015.

[4] K. Du. Solvability conditions for indefinite linear quadratic optimal stochastic control problems and associated stochastic Riccati equations. *SIAM J. Control Optim.*, 53(6):3673–3689, 2015.

[5] N. El Karoui. Les aspects probabilistes du contrôlé stochastique. In *Ninth Saint Flour Probability Summer School—1979 (Saint Flour, 1979)*, volume 876 of *Lecture Notes in Math.*, pages 73–238. Springer, Berlin-New York, 1981.

[6] P. Faurre. Sur les points conjugués en commande optimale. *C. R. Acad. Sci. Paris Sér. A-B*, 266:A1294–A1296, 1968.

[7] T. Fujiwara and H. Kunita. Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group. *J. Math. Kyoto Univ.*, 25(1):71–106, 1985.

[8] Y. Hu and B. Øksendal. Partial information linear quadratic control for jump diffusions. *SIAM J. Control Optim.*, 47(4):1744–1761, 2008.

[9] N. Ikeda and S. Watanable. *Stochastic Differential Equations and Diffusion Processes*. North-Holland/Kodansha, Amsterdam, Oxford, New York, 1989.

[10] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[11] I. Karatzas and S. E. Shreve. *Methods of mathematical finance*, volume 39 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1998.

[12] M. Kohlmann and S. Tang. New developments in backward stochastic Riccati equations and their applications. In *Mathematical finance (Konstanz, 2000)*, Trends Math., pages 194–214. Birkhäuser, Basel, 2001.

[13] M. Kohlmann and S. Tang. Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging. *Stochastic Process. Appl.*, 97(2):255–288, 2002.

[14] M. Kohlmann and S. Tang. Minimization of risk and linear quadratic optimal control theory. *SIAM J. Control Optim.*, 42(3):1118–1142, 2003.

[15] M. Kohlmann and S. Tang. Multidimensional backward stochastic Riccati equations and applications. *SIAM J. Control Optim.*, 41(6):1696–1721, 2003.

[16] H. Kunita. Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and stochastic analysis*, Trends Math., pages 305–373. Birkhäuser Boston, Boston, MA, 2004.
[17] J. Li and S. Peng. Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations. Nonlinear Anal., 70(4):1776–1796, 2009.

[18] N. Li, Z. Wu, and Z. Yu. Indefinite stochastic linear-quadratic optimal control problems with random jumps and related stochastic Riccati equations. Sci. China Math., 61(3):563–576, 2018.

[19] Q. Meng. General linear quadratic optimal stochastic control problem driven by a Brownian motion and a Poisson random martingale measure with random coefficients. Stoch. Anal. Appl., 32(1):88–109, 2014.

[20] S. Peng. Open problems on backward stochastic differential equations. In Control of distributed parameter and stochastic systems (Hangzhou, 1998), pages 265–273. Kluwer Acad. Publ., Boston, MA, 1999.

[21] S. Peng. Backward stochastic differential equation, nonlinear expectation and their applications. In Proceedings of the International Congress of Mathematicians. Volume I, pages 393–432. Hindustan Book Agency, New Delhi, 2010.

[22] S. G. Peng. Stochastic Hamilton-Jacobi-Bellman equations. SIAM J. Control Optim., 30(2):284–304, 1992.

[23] P. E. Protter. Stochastic integration and differential equations, volume 21 of Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin, 2005. Second edition. Version 2.1, Corrected third printing.

[24] Z. Qian and X. Y. Zhou. Existence of solutions to a class of indefinite stochastic Riccati equations. SIAM J. Control Optim., 51(1):221–229, 2013.

[25] S. Tang. General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations. SIAM J. Control Optim., 42(1):53–75, 2003.

[26] S. Tang. Dynamic programming for general linear quadratic optimal stochastic control with random coefficients. SIAM J. Control Optim., 53(2):1082–1106, 2015.

[27] S. Tang and S-H. Hou. Optimal control of point processes with noisy observations: the maximum principle. Appl. Math. Optim., 45(2):185–212, 2002.

[28] S. Tang and X. Li. Necessary conditions for optimal control of stochastic systems with random jumps. SIAM J. Control Optim., 32:1447–1475, 1994.

[29] W. M. Wonham. On a matrix Riccati equation of stochastic control. SIAM J. Control, 6:681–697, 1968.

[30] X. Y. Zhou and D. Li. Continuous-time mean-variance portfolio selection: a stochastic LQ framework. Appl. Math. Optim., 42(1):19–33, 2000.