Arbitrarily large $\mathcal{O}$-Morita Frobenius numbers $^*$

Michael Livesey$^\dagger$

Abstract

We construct blocks of finite groups with arbitrarily large $\mathcal{O}$-Morita Frobenius numbers. There are no known examples of two blocks defined over $\mathcal{O}$, with isomorphic defect groups, that are not Morita equivalent but the corresponding blocks defined over $k$ are. Therefore, the above strongly suggests that Morita Frobenius numbers are also unbounded, which would answer a question of Benson and Kessar.

1 Introduction

Let $l$ be a prime, $(K, \mathcal{O}, k)$ an $l$-modular system with $k$ algebraically closed, $H$ a finite group and $b$ a block of $\mathcal{O}H$. In this setup we always assume $K$ contains a primitive $|H|^{\text{th}}$ root of unity. We define $\longrightarrow: \mathcal{O} \to k$ to be the natural quotient map which we extend to the corresponding ring homomorphism $\longrightarrow: \mathcal{O}H \to kH$. For $n \in \mathbb{N}$, we define $\overline{b}^{(l^n)}$ to be the block of $kH$ that is the image of $\overline{b}$ under the following ring automorphism

$$kH \to kH \quad \sum_{h \in H} \alpha_h h \mapsto \sum_{h \in H} (\alpha_h)^{l^n} h. \quad (1)$$

We can also define the corresponding permutation of the blocks of $\mathcal{O}H$. In other words, $\overline{b}^{(l^n)}$ is the unique block of $\mathcal{O}H$ such that $\overline{b}^{(l^n)} = \overline{b}^{(l^n)}$. We now define the Morita Frobenius number of a block, first defined by Kessar $^\text{7}$.

**Definition 1.1.** Let $H$ be a finite group and $b$ a block of $\mathcal{O}H$. The Morita Frobenius number of $b$, denoted by $\text{mf}(b)$, is the smallest $n \in \mathbb{N}$, such that $\overline{b}$ and $\overline{b}^{(l^n)}$ are Morita equivalent as $k$-algebras. Similarly, the $\mathcal{O}$-Morita Frobenius number of $b$, denoted by $\text{mf}_{\mathcal{O}}(b)$, is the smallest $n \in \mathbb{N}$, such that $b$ and $b^{(l^n)}$ are Morita equivalent as $\mathcal{O}$-algebras.

Since a Morita equivalence between two blocks defined over $\mathcal{O}$ implies a Morita equivalence between the corresponding blocks defined over $k$, we always

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$^\dagger$School of Mathematics, University of Manchester, Manchester, M13 9PL, United Kingdom. Email: michael.livesey@manchester.ac.uk
have $\text{mf}(\mathcal{O}) \leq \text{mf}_{\mathcal{O}}(b)$. Donovan’s conjecture, which can be stated over $\mathcal{O}$ or $k$, is as follows.

**Conjecture 1.2** (Donovan). Let $L$ be a finite $l$-group. Then, amongst all finite groups $H$ and blocks $b$ (respectively $\mathcal{O}$) of $\mathcal{O}H$ (respectively $kH$) with defect groups isomorphic to $L$, there are only finitely many Morita equivalence classes.

A consequence of Donovan’s conjecture stated over $\mathcal{O}$ (respectively over $k$) is that $\mathcal{O}$-Morita Frobenius numbers (respectively Morita Frobenius numbers) are bounded in terms of a function of the isomorphism class of the defect group.

**Conjecture 1.3.** Let $L$ be a finite $l$-group. Then, amongst all finite groups $H$ and blocks $b$ (respectively $\mathcal{O}$) of $\mathcal{O}H$ (respectively $kH$) with defect groups isomorphic to $L$, $\text{mf}_{\mathcal{O}}(b)$ (respectively $\text{mf}(\mathcal{O})$) is bounded.

In [7, Theorem 1.4] Kessar proved that Donovan’s conjecture stated over $k$ is equivalent to Conjecture 1.3 stated over $k$ together with the so-called Weak Donovan conjecture.

**Conjecture 1.4** (Weak Donovan). Let $L$ be a finite $l$-group. Then there exists $c(L) \in \mathbb{N}$ such that if $H$ is a finite group and $b$ is a block of $\mathcal{O}H$ with defect groups isomorphic to $L$, then the entries of the Cartan matrix of $b$ are at most $c(L)$.

In [3, Theorem 3.11] Eaton, Eisele and the author proved that Donovan’s conjecture stated over $\mathcal{O}$ is equivalent to Conjecture 1.3 stated over $\mathcal{O}$ together with the Weak Donovan conjecture.

The question of whether Morita Frobenius numbers of blocks defined over $k$ have a universal bound is one that has gained much interest in recent years. In [1] Examples 5.1,5.2 Benson and Kessar constructed blocks with Morita Frobenius number two, the first discovered to be greater than one. The relevant blocks all have a normal, abelian defect group and abelian $l'$ inertial quotient with a unique isomorphism class of simple modules. It was also proved that among such blocks the Morita Frobenius numbers cannot exceed two [1, Remark 3.3]. In work of Benson, Kessar and Linckelmann [2, Theorem 1.1] the bound of two was extended to blocks that don’t necessarily have a unique isomorphism class of simple modules. It was also shown that the bound of two applies to $\mathcal{O}$-Morita Frobenius numbers of the corresponding blocks defined over $\mathcal{O}$. Finally, Farrell [3, Theorem 1.1] and Farrell and Kessar [6, Theorem 1.1] proved that the $\mathcal{O}$-Morita Frobenius number of any block of a finite quasi-simple group is at most four.

Our main result (see Theorem 4.4) is as follows:

**Theorem.** For every prime $l$ and $n \in \mathbb{N}$, there exists an $\mathcal{O}$-block $b$ with $\text{mf}_{\mathcal{O}}(b) = n$. 

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The proof of the above theorem presented in this paper relies heavily on the fact that the blocks are defined over $\mathcal{O}$. Specifically, the result [4, Proposition 4.4] quoted in the proof of Proposition 2.2 is a result that ultimately depends on Weiss’ criterion [9]. Weiss’ criterion is a result concerning permutation modules for groups algebras of $l$-groups that does not hold over $k$.

Notwithstanding the previous paragraph, there are no known examples of two blocks defined over $\mathcal{O}$, with isomorphic defect groups, that are not Morita equivalent but the corresponding blocks defined over $k$ are. Therefore, Theorem 4.4 strongly suggests that Morita Frobenius numbers are also unbounded. This would answer two questions posed by Benson and Kessar [1, Questions 6.2, 6.3]. Note that for a fixed $l$, the blocks constructed in Theorem 4.4 do not have bounded defect. Therefore, the theorem does not contradict Conjecture 1.3.

The following notation will hold throughout this article. If $H$ is a finite group and $b$ a block of $\mathcal{O}H$, then we set $\text{Irr}(H)$ (respectively $\text{IBr}(H)$) to be the set of ordinary irreducible (respectively irreducible Brauer) characters of $H$ and $\text{Irr}(b) \subseteq \text{Irr}(H)$ (respectively $\text{IBr}(b) \subseteq \text{IBr}(H)$) the set of ordinary irreducible (respectively irreducible Brauer) characters lying in the block $b$. If $N \trianglelefteq H$ and $\chi \in \text{Irr}(N)$, then we denote by $\text{Irr}(H, \chi)$ the set of irreducible characters of $H$ appearing as constituents of $\chi \uparrow^H$. Similarly we define $\text{Irr}(b, \chi) := \text{Irr}(b) \cap \text{Irr}(H, \chi)$. $1_H \in \text{Irr}(H)$ will designate the trivial character of $H$. We use $e_b \in \mathcal{O}H$ to denote the block idempotent of $b$. Similarly if $H$ is a $p'$-group and $\varphi \in \text{Irr}(H)$, then we use $e_{\varphi} \in \mathcal{O}H$ to signify the block idempotent corresponding to $\varphi$. Finally we set $[h_1, h_2] := h_1^{-1} h_2^{-1} h_1 h_2$ for $h_1, h_2 \in H$.

The article is organised as follows. In §2 we establish some preliminaries about Morita equivalences between blocks of finite groups. §3 contains the definition of the blocks $B_{\varphi}$ that are then used in §4 to prove our main theorem.

## 2 Morita equivalences between blocks

For any free $\mathcal{O}$-module $M$, we will denote by $\text{rk}_{\mathcal{O}}(M)$ its rank as a module over $\mathcal{O}$. All $\mathcal{O}$-modules considered will have finite rank.

**Lemma 2.1.** Let $H_1$ (respectively $H_2$) be a finite group and $b_1$ (respectively $b_2$) a block of $\mathcal{O}H_1$ (respectively $\mathcal{O}H_2$) such that $\text{rk}_{\mathcal{O}}(b_1) = \text{rk}_{\mathcal{O}}(b_2)$. Then any $b_1$-$b_2$-bimodule $M$ inducing a Morita equivalence between $b_1$ and $b_2$ must satisfy $\text{rk}_{\mathcal{O}}(M) \leq \text{rk}_{\mathcal{O}}(b_1)$.

**Proof.** Let $M$ be such a bimodule and $\sigma : \text{Irr}(b_1) \rightarrow \text{Irr}(b_2)$ the corresponding bijection of characters. Then

$$K \otimes_{\mathcal{O}} M \cong \bigoplus_{\chi \in \text{Irr}(b_1)} V_{\chi} \otimes_K V_{\sigma(\chi)}^*,$$
where \( V_\chi \) denotes a \( KH_1 \)-module affording \( \chi \) and \( V_{\sigma(\chi)}^* \) denotes the dual of a \( KH_2 \)-module affording \( \sigma(\chi) \). Therefore,

\[
\text{rk}_\mathcal{O}(M) = \sum_{\chi \in \text{Irr}(b_1)} \chi(1)\sigma(\chi)(1) \leq \sqrt{\left( \sum_{\chi \in \text{Irr}(b_1)} \chi(1)^2 \right) \left( \sum_{\psi \in \text{Irr}(b_2)} \psi(1)^2 \right)}
\]

\[
= \sqrt{\text{rk}_\mathcal{O}(b_1) \text{rk}_\mathcal{O}(b_2)} = \text{rk}_\mathcal{O}(b_1),
\]

where the inequality follows from the Cauchy-Schwarz inequality.

Let \( b \) be a block of \( \mathcal{O}H \), for some finite group \( H \) and \( Q \) a normal \( p \)-subgroup of \( H \). We denote by \( b^Q \) the direct sum of blocks of \( \mathcal{O}(H/Q) \) dominated by \( b \), that is those blocks not annihilated by the image of \( e_b \) under the natural \( \mathcal{O} \)-algebra homomorphism \( \mathcal{O}H \to \mathcal{O}(H/Q) \). Also, for any pair of finite groups \( H_1, H_2 \) and \( \mathcal{O}H_1, \mathcal{O}H_2 \)-bimodule \( M \) we routinely view \( M \) as an \( \mathcal{O}(H_1 \times H_2) \) via \( (h_1, h_2).m = h_1 m h_2^{-1} \), for \( h_1 \in H_1, h_2 \in H_2 \) and \( m \in M \).

**Proposition 2.2.** Let \( H_1 \) (respectively \( H_2 \)) be a finite group, with \( Q_1 \trianglelefteq H_1 \) (respectively \( Q_2 \trianglelefteq H_2 \)) a normal \( l \)-subgroup and \( b_1 \) (respectively \( b_2 \)) a block of \( \mathcal{O}H_1 \) (respectively \( \mathcal{O}H_2 \)).

1. Suppose \( M \) is a \( b_1,b_2 \)-bimodule inducing a Morita equivalence between \( b_1 \) and \( b_2 \) such that the corresponding bijection \( \text{Irr}(b_1) \to \text{Irr}(b_2) \) restricts to a bijection \( \text{Irr}(b_1,1_{Q_1}) \to \text{Irr}(b_2,1_{Q_2}) \). Then \( Q_1 M = M^{Q_2} \), the set of fixed points of \( M \) under the left action of \( Q_1 \) (respectively the right action of \( Q_2 \)), induces a Morita equivalence between \( b_1^{Q_1} \) and \( b_2^{Q_2} \).

2. If, in addition, \( Q_1 M = M^{Q_2} \) has trivial source, when considered as an \( \mathcal{O}((H_1/Q_1) \times (H_2/Q_2)) \)-module, then \( M \) has trivial source, when considered as an \( \mathcal{O}(H_1 \times H_2) \)-module.

**Proof.** This is proved in [4, Propositions 4.3,4.4], with the added assumption that \( H_1 = H_2, Q_1 = Q_2 \) and \( b_1 = b_2 \). However, the proof in this more general setting is identical. \( \square \)

### 3 Definition of \( B_\varphi \)

In this section we provide a number of necessary preliminary results before defining the blocks \( B_\varphi \) that will ultimately form the family of blocks with arbitrarily large \( \mathcal{O} \)-Morita Frobenius numbers in the proof of Theorem 4.4.

Until further notice we fix a prime \( p \neq l \) such that \( p - 1 \) is not a power of \( l \). We set \( a := v_l(p - 1) \), the largest power of \( l \) dividing \( p - 1 \). For \( t \in \mathbb{N} \), we define \( \Omega_t \) to be the direct product of \( p \) copies of \( C_l \) indexed by the elements of \( \mathbb{F}_p \),

\[
\Omega_t := \prod_{x \in \mathbb{F}_p} C_{lt}.
\]
We also define following the subgroups of $\Omega_t$,

$$D_t := \left\{ (g_x)_{x \in \mathbb{F}_p} \in \Omega_t \middle| \prod_{x \in \mathbb{F}_p} g_x = 1 \right\}, \quad \Lambda_t := \{ (g, \ldots, g) \mid g \in C_t \}.$$

We set $F := \mathbb{F}_p \times \mathbb{F}_p^\times$, with multiplication given by

$$(x, \alpha)(y, \beta) = (x + \alpha y, \alpha \beta),$$

for $x, y \in \mathbb{F}_p$ and $\alpha, \beta \in \mathbb{F}_p^\times$. Note that $F$ acts on $\mathbb{F}_p$ via

$$(x, \alpha).y = x + \alpha y,$$

for all $x, y \in \mathbb{F}_p$ and $\alpha \in \mathbb{F}_p^\times$. Therefore, $F$ acts on $\Omega_t$ by permuting indices and $\Omega_t = D_t \times \Lambda_t$ is an $F$-stable direct decomposition of $\Omega_t$.

**Lemma 3.1.** Let $t \in \mathbb{N}$ and $\theta \in \text{Irr}(\Omega_t)$. Viewing $\mathbb{F}_p \leq F$, $\theta$ is $\mathbb{F}_p$-stable if and only if $\theta = 1_{D_t} \otimes \theta_{\Lambda_t}$, for some $\theta_{\Lambda_t} \in \text{Irr}(\Lambda_t)$. In particular, the only irreducible, $\mathbb{F}_p$-stable character of $D_t$ is $1_{D_t}$.

**Proof.** Certainly $\theta = 1_{D_t} \otimes \theta_{\Lambda_t}$ is $\mathbb{F}_p$-stable. For the converse, suppose $\theta \in \text{Irr}(\Omega_t)$ is $\mathbb{F}_p$-stable and $(g_x)_{x \in \mathbb{F}_p} \in D_t$. Then

$$\theta(g_0, g_1, \ldots) = \theta(g_0, 1, 1, \ldots) \theta(1, g_1, 1, \ldots) \ldots \theta(1, \ldots, 1, g_{p-1})$$

$$= \theta \left( \prod_{x \in \mathbb{F}_p} g_x, 1, 1, \ldots, 1 \right) = 1.$$

In other words, $D_t$ is contained in the kernel of $\theta$. The claim follows. \hfill $\square$

From now on we fix a generator $\lambda$ of $\mathbb{F}_p^\times$. We define the group $E$ to be

$$\mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p^\times \times \mathbb{F}_p^\times \times \mathbb{F}_p^\times$$

as a set, with multiplication given by

$$(x_1, y_1, \lambda^{m_1}, \lambda^{n_1}, \mu_1), (x_2, y_2, \lambda^{m_2}, \lambda^{n_2}, \mu_2)$$

$$= (x_1 + \lambda^{m_1}x_2, y_1 + \lambda^{n_1}y_2, \lambda^{m_1+m_2}, \lambda^{n_1+n_2}, \mu_1 \mu_2 \lambda^{n_1+m_2}),$$

for $x_1, y_1, x_2, y_2 \in \mathbb{F}_p$, $\mu_1, \mu_2 \in \mathbb{F}_p^\times$ and $m_1, n_1, m_2, n_2 \in \mathbb{N}_0$. Setting $Z := \mathbb{F}_p^\times$, we have the short exact sequence

$$1 \longrightarrow Z \xrightarrow{\eta} E \xrightarrow{\phi} F_1 \times F_2 \longrightarrow 1,$$  \hfill (2)

where $F_1 \cong F_2 \cong F$,

$$\eta(\mu) = (0, 0, 1, 1, \mu) \quad \text{and} \quad \phi(x, y, \lambda^m, \lambda^n, \mu) = ((x, \lambda^m), (y, \lambda^n)),$$
for all $x, y \in \mathbb{F}_p$, $\mu \in \mathbb{F}_p^\times$ and $m, n \in \mathbb{N}_0$. We identify $Z$ with its image under $\eta$. Note that

$$[(x, \lambda^m), (y, \lambda^n)] = \lambda^{-mn} \in Z,$$

(3)

where $(x, \lambda^m) \in F_1$, $(y, \lambda^n) \in F_2$ and the tildes denote lifts to $E$. Let $\xi$ be a generator of $\mathbb{F}_p^\times$. We set

$$P_1 := \{(x, 0, 1, 1) \in E| x \in \mathbb{F}_p\}, \ P_2 := \{(0, y, 1, 1) \in E| y \in \mathbb{F}_p\}, \ P := P_1 \times P_2 \in \text{Syl}_p(E),$$

$$E_{l'} := O_{l'}(E) = \{(x, y, \xi^m, \xi^n, \xi^{m'}) \in E|m, n, r \in \mathbb{Z}\},$$

$$Z_{l'} := O_{l'}(Z) = E_{l'} \cap Z.$$ We set $F_{i,l'} := O_{l'}(F_i)$, for $i = 1, 2$, so $F_{1,l'} \times F_{2,l'}$ is the image of $E_{l'}$ under $\phi$. Note that, since $p - 1$ is not a power of $l$, $\phi(P_i) \leq F_{i,l'}$, for $i = 1, 2$.

Until further notice we fix $t_1, t_2 \in \mathbb{N}$. Since $F$ acts on $D_t$, for any $t \in \mathbb{N}$, we have a natural action of $F_1 \times F_2$ on $D_{t_1} \times D_{t_2}$.

**Lemma 3.2.** $F_1 \times F_2$ acts faithfully on $D_{t_1} \times D_{t_2}$ and

$$N_{\text{Aut}(P_{t_1} \times P_{t_2})}(F_{1,l'} \times F_{2,l'}) = \begin{cases} ((C_1 \times F_1) \times (C_2 \times F_2)) \times \langle s \rangle & \text{if } t_1 = t_2, \\ (C_1 \times F_1) \times (C_2 \times F_2) & \text{otherwise,} \end{cases}$$

where $s \in \text{Aut}(D_{t_1} \times D_{t_2})$ is defined via $s(g, h) = (h, g)$, for all $(g, h) \in D_{t_1} \times D_{t_2}$ and $C_i := C_{\text{Aut}(P_i)}(F_{i,l'})$, for $i = 1, 2$.

**Proof.** To show $F_1 \times F_2$ acts faithfully on $D_{t_1} \times D_{t_2}$, we need only show that $F$ acts faithfully on $D_t$, for any $t \in \mathbb{N}$. However, $\Lambda_t = C_{\Omega_t}(F)$ and $\Omega_t = D_t \times \Lambda_t$, so it suffices to show that $F$ acts faithfully on $\Omega_t$. This follows since the action of $F$ on $\mathbb{F}_p$ is faithful.

Again, consider the action of $F$ on $\Omega_t$ for some $t \in \mathbb{N}$. Since $\Lambda_t = C_{\Omega_t}(\mathbb{F}_p)$, $C_{D_t}(\mathbb{F}_p)$ is trivial and so we can determine $D_{t_1}$ and $D_{t_2}$ as the unique non-trivial subgroups of $D_{t_1} \times D_{t_2}$ occurring as the centraliser of some non-trivial $p$-element of $F_{1,l'} \times F_{2,l'}$. Therefore, any element of $N_{\text{Aut}(D_{t_1} \times D_{t_2})}(F_{1,l'} \times F_{2,l'})$ must respect the decomposition $D_{t_1} \times D_{t_2}$. If $t_1 = t_2$, then $s$ swaps $D_{t_1}$ and $D_{t_2}$ and if $t_1 \neq t_2$, then every element of $N_{\text{Aut}(D_{t_1} \times D_{t_2})}(F_{1,l'} \times F_{2,l'})$ must leave each $D_{t_i}$ invariant, for $i = 1, 2$.

To complete the claim we need to prove that $N_{\text{Aut}(D_t)}(F_{i,l'}) = C_i \times F_{i,t}$, for $i = 1, 2$. However, this follows from the claim that $\text{Aut}(F_{i,t}) \cong F_i$, where $F_{i,t} := O_{l'}(F) \cong F_{i,l'}$. We first show that $F$ acts faithfully on $F_{i,t}$ via conjugation. Suppose that $g \in C_F(F_{i,t})$, then $g \in C_F(\mathbb{F}_p) = \mathbb{F}_p$. As noted before the lemma, $\mathbb{F}_p$ is a proper subgroup of $F_{i,t}$ and so $g \in C_{\mathbb{F}_p}(h) = \{1\}$, where $h \in F_{i,t} \setminus \mathbb{F}_p$. Therefore, we have proved that $F \leq \text{Aut}(F_{i,t})$. Now let $\zeta \in \text{Aut}(F_{i,t})$. Certainly
$\text{Aut}(\mathbb{F}_p) \cong \mathbb{F}_p^\times$ and so to prove that $\zeta$ is induced by some element of $F$ we may assume that $\zeta$ fixes $\mathbb{F}_p$ pointwise. By the Schur-Zassenhaus theorem, we may, in addition, assume that $\zeta$ leaves $((\mathbb{F}_p^\times)_p$ invariant. Finally, since it fixes $\mathbb{F}_p$ pointwise, it must also fix $((\mathbb{F}_p^\times)_p$ pointwise.

**Definition 3.3.** We define $\tilde{G} := (D_{t_1} \times D_{t_2}) \rtimes E$, where the action of $E$ on $D_{t_1} \times D_{t_2}$ is given via $\phi$. In addition we set $G := (D_{t_1} \times D_{t_2}) \rtimes E \subseteq \tilde{G}$ and for each $\phi \in \text{Irr}(Z_l)$ we set $B_\phi := O \text{Ge}_\phi$.

Note that, since $D := D_{t_1} \times D_{t_2} \triangleleft G$, any block idempotent of $O \text{Ge}$ is supported on $C_G(D) = D \times Z_l$. Therefore, $B_\phi$ is a block of $O \text{Ge}$ with defect group $D$.

4 Arbitrarily large $O$-Morita Frobenius numbers

Adopting the notation of Lemma 3.2 we have the following.

**Lemma 4.1.** Let $\phi \in \text{Irr}(Z_l)$.

1. For each $\zeta \in (C_1 \times F_1) \times (C_2 \times F_2)$, there exists $\delta \in \text{Aut}(G)$ such that $\delta \downarrow_D = \zeta$ and $\delta \downarrow_{Z_l} = \text{Id}_{Z_l}$. In particular, $\delta$ induces an $O$-algebra automorphism of $B_{\phi}$.

2. If $t_1 = t_2$, there exists $\delta \in \text{Aut}(G)$ such that $\delta \downarrow_D = s$ and $\delta \downarrow_{Z_l}$ is given by inversion. In particular, $\delta$ induces an $O$-algebra homomorphism $B_{\phi} \to B_{\phi^{-1}}$.

**Proof.**

1. For each $\zeta \in C_1 \times C_2$, we define $\delta \in \text{Aut}(G)$ via $\delta \downarrow_D = \zeta$ and $\delta \downarrow_{E_l} = \text{Id}_{E_l}$. For each $\zeta \in F_1 \times F_2$, we define $\delta \in \text{Aut}(G)$ to be given by conjugating by some $g \in E \subseteq \tilde{G}$, a lift of $\zeta$.

2. We define $\delta \in \text{Aut}(G)$ via $\delta \downarrow_D = s$ and

$$\delta \downarrow_{E_l} (x, y, \lambda^m, \lambda^n, \lambda^r) = (y, x, \lambda^n, \lambda^m, \lambda^{mn-r}),$$

for all $x, y \in \mathbb{F}_p$ and $m, n, r \in l^a \mathbb{N}_0$, on $E_l$.

**Lemma 4.2.** Any $\chi \in \text{Irr}(G)$ reduces to an irreducible Brauer character if and only if $\chi \in \text{Irr}(G, 1_D)$.

**Proof.** Since $D \triangleleft G$ and $G/D \cong E_l$ is an $l'$-group, $D$ is contained in the kernel of every simple $kG$-module and every irreducible Brauer character is determined by its restriction to $E_l$. Therefore, we have a bijection between $\text{IBr}(G)$ and $\text{Irr}(E_l)$ given by restriction to $E_l$ and through this bijection we can identify the decomposition map

$$\mathbb{Z} \text{Irr}(G) \to \mathbb{Z} \text{IBr}(G)$$
with the restriction map
\[ Z \text{Irr}(G) \to Z \text{Irr}(E_\nu). \]

It therefore remains to show that for any \( \chi \in \text{Irr}(G) \), \( \chi \downarrow_{E_\nu} \) is irreducible if and only if \( \chi \in \text{Irr}(G, 1_D) \).

If \( \chi \in \text{Irr}(G, 1_D) \), then certainly \( \chi \downarrow_{E_\nu} \) is irreducible.

For the converse let \( 1_D \neq \theta \in \text{Irr}(D) \). By Lemma 3.1 \( \text{Stab}_P(\theta) = \{ 1 \} \), \( P_1 \) or \( P_2 \). Therefore, any \( \chi \in \text{Irr}(G, \theta) \) must satisfy the following condition. Either \( \chi \downarrow_{P_1} \) has trivial and non-trivial, irreducible constituents or \( \chi \downarrow_{P_2} \) has trivial and non-trivial, irreducible constituents. However, by considering orbits of \( \text{Irr}(P) \) under the action of \( E_\nu \), we get that for any \( \xi \in \text{Irr}(E_\nu) \), \( \xi \downarrow_{P_1} \) does not have both trivial and non-trivial, irreducible constituents and \( \xi \downarrow_{P_2} \) does not have both trivial and non-trivial, irreducible constituents. Therefore, \( \chi \downarrow_{E_\nu} \) cannot be irreducible.

**Proposition 4.3.** Let \( \varphi, \theta \in \text{Irr}(Z_\nu) \). Then \( B_\varphi \) is Morita equivalent to \( B_\theta \) if and only if \( \varphi = \theta \) or \( t_1 = t_2 \) and \( \varphi = \theta^{-1} \).

**Proof.** Certainly if \( \varphi = \theta \), then \( B_\varphi \) is Morita equivalent to \( B_\theta \). Also, if \( t_1 = t_2 \) and \( \varphi = \theta^{-1} \), then \( B_\varphi \) and \( B_\theta \) are isomorphic via part (2) of Lemma 4.1.

Conversely, suppose \( M \) is an \( B_\varphi-B_\theta \)-bimodule inducing a Morita equivalence between \( B_\varphi \) and \( B_\theta \) and \( \sigma : \text{Irr}(B_\varphi) \to \text{Irr}(B_\theta) \) the corresponding bijection of characters. Then, by Lemma 4.2 \( \sigma \) restricts to a bijection \( \text{Irr}(B_\varphi, 1_D) \to \text{Irr}(B_\theta, 1_D) \) and by part (1) of Proposition 2.2 \( D M = M^D \) induces a Morita equivalence between \( B^D_\varphi \) and \( B^D_\theta \). However, \( G/D \) is an \( l' \)-group and so \( D M = M^D \) certainly has trivial source. Therefore, by part (2) of Proposition 2.2 \( M \) must also have trivial source.

It now follows from [8, 7.6] that \( M \) is a direct summand of \( O_\Delta \gamma \uparrow^{G \times G} \), for some \( \gamma \in \text{Aut}(D) \), where \( \Delta \gamma := \{ (d, \gamma(d)) | d \in D \} \) and \( O_\Delta \gamma \) denotes the trivial \( O(\Delta \gamma) \)-module. Therefore, \( M \) is a direct summand of
\[
eps_{\varphi}(O_{\Delta \gamma} \uparrow^{(D \times Z_\nu) \times (D \times Z_\nu)} \uparrow^{G \times G})_{\eps_{\varphi}} \cong (\eps_{\varphi}(O_{\Delta \gamma} \uparrow^{(D \times Z_\nu) \times (D \times Z_\nu)})_{\eps_{\varphi}} \uparrow^{G \times G} \\
\cong (\gamma(O D) \otimes_{O} \varphi O_\theta) \uparrow^{G \times G},
\]
where \( \gamma(O D) \) denotes the \( O D-O D \)-bimodule \( O D \), with the canonical right action of \( O D \) and the left action of \( O D \) given via \( \gamma \) and \( \varphi O_\theta \) denotes the \( O Z_{\nu} e_{\varphi} O Z_{\nu} e_{\varphi} \)-bimodule \( O \), with the canonical left \( O Z_{\nu} e_{\varphi} \) and right \( O Z_{\nu} e_{\varphi} \) actions.

We now analyse \( S := \text{Stab}_{G \times G}(\gamma(O D) \otimes_{O} \varphi O_\theta) \). First note that \( M \cong N \uparrow^{G \times G} \), for some indecomposable \( O S \)-module \( N \) such that \( \gamma(O D) \otimes_{O} \varphi O_\theta \) is a direct
summand of \( N \downarrow_{(D \times Z_{\nu}) \times (D \times Z_{\nu})} \). Therefore,
\[
|D| = \text{rk}_{\mathcal{O}}(\gamma(\mathcal{O}D) \otimes_{\mathcal{O}} \varphi_{\mathcal{O}}) \leq \text{rk}_{\mathcal{O}}(N),
\]
\[
\text{rk}_{\mathcal{O}}(N)[G \times G : S] = \text{rk}_{\mathcal{O}}(M) \leq \text{rk}_{\mathcal{O}}(B_{\varphi}) = [G : Z_{\nu}],
\]
where the inequality on the second line follows from Lemma 3.4. In particular,
\[
[G \times G : S] \leq [G : D \times Z_{\nu}],
\]
which rearranges to
\[
[S : (D \times Z_{\nu}) \times (D \times Z_{\nu})] \geq [G : D \times Z_{\nu}]
\]
with equality if and only if we have equality throughout (4). Next let \((g, h) \in S\). Since \(\gamma(\mathcal{O}D)\) is \(S\)-stable and \(\Delta_{\gamma}\) is the unique vertex of \(\gamma(\mathcal{O}D)\), we have
\[(g, h)(\Delta_{\gamma})(g, h)^{-1} = \Delta_{\gamma}.
\]
In other words,
\[
gdg^{-1} = \gamma^{-1}(h \gamma(d) h^{-1}),
\]
for all \(d \in D\). Now, by Lemma 3.3, \(F_{1, \nu} \times F_{2, \nu} \cong G/(D \times Z_{\nu})\) acts faithfully on \(D\) and so \(g(D \times Z_{\nu})\) determines \(h(D \times Z_{\nu})\). In particular,
\[
[S : (D \times Z_{\nu}) \times (D \times Z_{\nu})] \leq [G : D \times Z_{\nu}].
\]
Together with (4) and the subsequent sentence, this tells us that we have equality throughout (1) and (3). Equality in (5) and the sentence following (6) give that there exists \(\zeta \in \text{Aut}(G/(D \times Z_{\nu}))\) such that
\[
S = \{(g, h) \in G \times G | \zeta(g(D \times Z_{\nu})) = h(D \times Z_{\nu})\}.
\]
Furthermore, equality throughout (1) tells us that \(\gamma(\mathcal{O}D) \otimes_{\mathcal{O}} \varphi_{\mathcal{O}}\) extends to an \(\mathcal{O}\mathcal{S}\)-module. Now (10) implies that \(\gamma \in N_{\text{Aut}(D)}(F_{1, \nu} \times F_{2, \nu})\) and \(\zeta\) is the corresponding automorphism of \(F_{1, \nu} \times F_{2, \nu}\), once we have identified \(G/(D \times Z_{\nu})\) with \(F_{1, \nu} \times F_{2, \nu} \leq \text{Aut}(D)\).

By Lemmas 3.2 and 4.1 we may assume that \(\gamma = \text{Id}_{D}\). In particular, \(\zeta = \text{Id}_{F_{1, \nu} \times F_{2, \nu}}\) and so \(\mathcal{O}D \otimes_{\mathcal{O}} \varphi_{\mathcal{O}}\) extends to a module for
\[
\mathcal{O}_{((D \times Z_{\nu}) \times (D \times Z_{\nu})).(\Delta_{E_{\nu}})}.
\]
So \(kD \otimes_{k} \varphi_{k_{\theta}}\) extends to a module for
\[
k_{((D \times Z_{\nu}) \times (D \times Z_{\nu})).(\Delta_{E_{\nu}})},
\]
where
\[
kD \otimes_{k} \varphi_{k_{\theta}} := k \otimes_{\mathcal{O}} (\mathcal{O}D \otimes_{\mathcal{O}} \varphi_{\mathcal{O}}),
\]
which we identify with \(kD\) in the obvious way. As a \(k((D \times Z_{\nu}) \times (D \times Z_{\nu}))-\)
module, the radical of \(kD\) is \(J(kD)\), the Jacobson radical of \(kD\) as a ring. We
now study the 1-dimensional $k(\Delta E')$-module $kD/J(kD)$.

Viewing $\lambda \in \mathbb{Z}$, (3) gives that $\lambda^{-2a} \in [E', E']$ but $\lambda^{-2a}$ generates $Z'$ so $Z' \leq [E', E']$ and $(kD/J(kD)) \downarrow_{\Delta Z'}$ must be the trivial $k(\Delta Z')$-module. However, $(kD/J(kD)) \downarrow_{\Delta Z'}$ is also the 1-dimensional $k(\Delta Z')$-module corresponding to $\Delta Z' \rightarrow k^\times$, $(z, z) \mapsto \varphi \cdot \vartheta^{-1}(z)$. Since $Z'$ is an $l'$-group, this implies $\varphi = \vartheta$, as required.

**Theorem 4.4.** For every prime $l$ and $n \in \mathbb{N}$, there exists an $\mathcal{O}$-block $b$ with $mf_{\mathcal{O}}(b) = n$.

**Proof.** Let $p$ be a prime different from $l$ such that $p \equiv 1 \mod (l^n - 1)$, the existence of which is guaranteed by the Dirichlet prime number theorem. In particular, $p - 1$ is not a power of $l$ and so we can adopt all the notation from this and the previous section.

Let $t_1 \neq t_2 \in \mathbb{N}$ and $\varphi$ a faithful character of $Z'$, in particular, $\varphi$ has order $(p - 1)/l^n$ which is divisible by $l^n - 1$. Now set $\vartheta := \varphi^{(p-1)/(l^n(l^n-1))}$ so $\vartheta$ has order $l^n - 1$. One can quickly verify that $e_{\vartheta^{(l^n)}} = e_{\vartheta^{l^n}}$ and hence that $B_{\vartheta^{(l^n)}} = B_{\vartheta^{l^n}}$, for all $m \in \mathbb{N}$. Proposition 4.3 now implies that $mf_{\mathcal{O}}(B_{\vartheta})$ is the smallest $m \in \mathbb{N}$ such that $\vartheta^{l^m} = \vartheta$. Therefore, $mf_{\mathcal{O}}(B_{\vartheta}) = n$.

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