INVESTIGATING THE EFFECTS OF INTERVENTION STRATEGIES IN A SPATIO-TEMPORAL ANTHRAX MODEL

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Abstract. In this paper, we extend our previous work on optimal control applied in an anthrax outbreak in wild animals. We use a system of ordinary differential equation (ODE) and partial differential equations (PDEs) to track the change in susceptible, infected and vaccinated animals as well as the infected carcasses. In addition to the assumption that the infected animals and the infected carcasses are the main source of infection, we consider the animal movement by diffusion and see its effects in disease transmission. Two controls: vaccinating susceptible animals and disposing infected carcasses properly are applied in the model and these controls depend on both space and time. We formulate an optimal control problem to investigate the effect of intervention strategies in our spatio-temporal model in controlling the outbreak at minimum cost. Finally some numerical results for the optimal control problem are presented.

1. Introduction. Movements and migrations of wild animals are universal phenomenon and there may be various reasons behind these movements such as search for resources, seeking suitable climate, avoiding predation during the period of reproduction [1]. These movements and migration of the animals have both good and bad impact in disease transmission. It has been commonly accepted that spatial diffusion and environmental heterogeneity are important in the spread of any infectious diseases [1, 40]. On the one hand, the movements enhance the spread of pathogens in a specific region and also increase the chance of disease transmission within and between species [1] it may allow a host to escape from infected habitats and may reduce disease levels when infected animals do not migrate successfully [1]. Studies in migratory animal populations have shown that some amounts of host movement could prevent host extinction from malicious pathogens and also allow host resistance genes to spread [1]. This may be due to the longer stay in a habitat increasing the exposure to parasites and pathogens and which in turn increases the chances of transmission of infection. On the other hand, movements help animals to escape from an infectious environment. When animals migrate from old locations to a new place, the pathogens residing in the older habitat may disappear due to the

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lack of hosts. Some other conditions such as climates and availability of parasites in the new locations or on the migration routes may change the transmissibility of an infection \[26, 3, 34\].

Thus, understanding the animal movements and the associated risk can be very important in the application of effective control strategies for curbing animal disease and their spread. In many parts of world, animal movements have resulted in the emergence of new pathogens to previously disease free areas \[15\]. Sometimes the translocation of animals can be one of the major causes for the emergence of new diseases into new regions. On the other hand, daily movements of animals within an area in order to graze or browse may cause disease spread when susceptible animals come in contact with infected animals or a contaminated environment \[4, 7, 10, 9\]. For anthrax, it is highly likely that susceptible animals who come into contact with infected animals or who graze on carcass sites can acquire the infection through any of the inhalational, intestinal and cutaneous routes \[3, 21, 22, 30\]. As seen in the Malilangwe Wildlife Reserve in Zimbabwe, an anthrax outbreak started at a point in spread in the reserve of about 502 square kilometers within 3 months \[7, 13\]. Also, the licking behavior of animals on fresh carcasses or carcass sites, scavenging infected animal carcasses for food, and breathing in dusty air in an infected region can result in transmission of the disease \[40, 35, 34, 42, 41, 39, 14\]. As anthrax spores live in the soil for long period of time and they are spread in many parts of the world, clearing the spores from the environment is almost impossible. The commonly used intervention strategies focus on the vaccination of the susceptible animals and limiting the environmental contamination by infected carcasses. Burning and burial of the infected carcasses are the two main methods used for carcass disposal \[31, 40, 7, 42, 24\].

There have been some mathematical models for disease dynamics of anthrax infection in the animals. Hahn and Furniss \[20\] proposed a mathematical model consisting a system of ordinary differential equations for susceptible animals, environmental spores and environmental carcasses. In order to understand the impact of the movement of individuals on the persistence and extinction of anthrax, Friedman et al. \[18\] extended the model \[20\] to a PDE system with density-dependent and frequency-dependent transmission rates in a continuous spatial habitat. They calculated the basic reproduction number for the model with and without animal migration. Their work has mainly focused on evaluating the effect of parameters that reduce carcass ingestion or environmental contamination in the outbreak. Roy et al. \[33\] extended the model by Friedman et al. by adding carcass decay and animal recovery from anthrax. This paper computed the basic reproduction number and used elasticity indexes to show the parameter sensitivity to various parameters. In addition, the paper used a type reproduction number to illustrate the application of constant vaccination and/or carcass disposal policies to eradicate anthrax in the infected herbivore and carnivore animal populations. Our previous work \[31\] studied optimal control theory to most common intervention strategies: vaccination of susceptible animals and carcass disposal in a wildlife reserve under the assumption that all infected animals face inevitable death after infection. We considered a model consisting ordinary differential equations for susceptible animals, infected animals and infected carcasses that are similar to the model by Friedman et al. \[18\] (without animal movement). We considered an additional compartment for vaccinated animals. We estimated some of the model parameters using experimental data by Clegg et al. \[7\] where number of anthrax infected carcasses were collected.
and disposed in Malilangwe Wildlife Reserve, Zimbabwe for 86 days. We investigated the effect of control strategies in controlling the anthrax outbreak in wild animals.

In this paper, we extend our previous anthrax ODE model [31] to a PDE system including animal movements and state variables depending on both space and time. We aim to find the effects of intervention strategies on anthrax spread. We assume that the animal movement is mainly by diffusion, and the diffusion coefficient of infected animals is less than that of susceptible or vaccinated animals under the hypothesis that infected animals reduce their normal movement due to sickness. Two controls (depending on both time and space) representing vaccination of susceptible animals and disposal of infected carcasses are considered. A compartment of vaccinated animals is included. Our goal is to investigate control strategies that minimize the outbreak (infected animals and carcasses) and the cost of control measures. We establish the existence and uniqueness of the state system and derive the optimality system (which consists of the state system coupled with the adjoint system and optimal control characterization). The existence, uniqueness and characterisation of the optimal controls are presented. At the end, we present some numerical results for various parameter sets to illustrate various outbreak scenarios.

2. The mathematical model for bounded spatial domain. Let \( s(x,t) \), \( i(x,t) \) and \( v(x,t) \) denote the density of susceptible, infected and vaccinated animals and \( c(x,t) \) be the density of infected carcasses at spatial location \( x \in \Omega \subset \mathbb{R}^2 \) and time \( t \in [0,T] \) with density being the number of animals per \( km^2 \). Also, let \( n(x,t) = s(x,t) + v(x,t) \) denote the sum of non-infected animals. We extend our previous model [31] with a system of ODEs to include animal movements. We assume that animals move by diffusion and that infected animals have a smaller diffusivity than healthy (susceptible or vaccinated) animals, \( d_1 < d \). The logistic growth rate is given as the parameter, \( r \); \( K \) is the carrying capacity of the population; \( \theta_c \) and \( \theta_i \) are the transmission rates from infected carcasses and infected animals, respectively; and \( \gamma \) is the disease induced death rate of infected animals. Additionally, \( \alpha \) is the...
rate at which the infected carcasses are eaten by scavenger animals and \( p \) is the natural decay rate of infected carcasses. The two control functions representing the vaccination rate and the carcass disposal rate are denoted by \( u_1(x,t) \) and \( u_2(x,t) \), respectively.

Our goal is to minimize the number of infected animals and carcasses while also minimizing the total cost associated with vaccination and carcasses disposal. Again, as in our previous work [31], the vaccination is applied to both susceptible and infected animals, although there is no effect of vaccination on infected animals. We do this, since vaccination programs vaccinate all animals that are found and cannot distinguish whether they are infected or not. The objective functional then becomes

\[
J(u_1, u_2) = \int_Q \left[ A_1 i(x,t) + A_2 c(x,t) + B_1 u_1 (s(x,t) + i(x,t)) + B_2 u_2 c(x,t) + C_1 u_1(x,t)^2 + C_2 u_2(x,t)^2 \right] dxdt
\]

(5)

where the weight coefficients \( A_i, B_i, C_i, i = 1, 2 \) are positive constants. The first two terms \( A_1 i + A_2 c \) represent damage related to the infected animals and the remaining terms \( B_1 u_1 (s(x,t) + i(x,t)) + B_2 u_2 c(x,t) + C_1 u_1(x,t)^2 + C_2 u_2(x,t)^2 \) represent the cost of applying the two controls. This cost is assumed to be non-linear, and quadratic terms are used here for simplicity. Note that in (5), the vaccination control is applied to both susceptible \( s(x,t) \) and infected \( i(x,t) \) animals even though there the vaccination has no effect on infected animals. In practice, it is difficult to separate these two classes of animals in the wild. Thus, all the live animals are vaccinated regardless of whether they are infected or not. To minimize the functional \( J(u_1, u_2) \), we want to characterize \( (u_1^*, u_2^*) \) such that

\[
J(u_1^*, u_2^*) = \inf_{(u_1, u_2) \in \mathcal{U}} J(u_1, u_2).
\]

where the control set \( \mathcal{U} = \{(u_1, u_2) \in (L^\infty(Q))^2 : 0 \leq u_i \leq M_i \text{ a.e. for } i = 1, 2 \} \).

3. Existence of optimal controls. Let \( \Omega \) be bounded, smooth domain in \( \mathbb{R}^2 \).

Let \( V = L^2(0,T; H^1_0(\Omega)) \) and \( V^* = L^2(0,T; H^{-1}(\Omega)) \) where \( H^{-1}(\Omega) \) is the dual of \( H^1_0(\Omega) \). For a control pair \( (u_1, u_2) \in \mathcal{U} \), we say that the functions \( s, i, v \in V \cap L^\infty(Q) \) and \( c \in L^2(Q) \) with time derivatives \( s_1, i_1, v_1 \in V^* \) and \( c_t \in L^2(Q) \) are weak solutions of our system if for any test functions \( \phi_1, \phi_2, \phi_3 \in V \cap L^\infty(Q) \) the following equations (3.1) – (3.4) are satisfied:

\[
\int_0^T < s_1, \phi_1 > dt + d \int_Q \nabla s \cdot \nabla \phi_1 dxdt = \int_Q [rn(1 - \frac{n}{K}) - \theta_isc - \theta_isi - u_1s]\phi_1 dxdt; \quad (6)
\]

\[
\int_0^T < i_1, \phi_2 > dt + d \int_Q \nabla i \cdot \nabla \phi_2 dxdt = \int_Q [\theta_isc + \theta_isi - \gamma_i]\phi_2 dxdt; \quad (7)
\]

\[
\int_0^T < v_1, \phi_3 > dt + d \int_Q \nabla v \cdot \nabla \phi_3 dxdt = \int_Q [u_1s]\phi_3 dxdt; \quad (8)
\]

\[
c(x,t) = c^0(x) + \int_0^t [\gamma_i - \alpha(s + i + v)c - (p + u_2)c]^i(x,\tau)d\tau
\]
together with initial and boundary conditions
\[ s(x, 0) = s^0(x), \quad i(x, 0) = i^0(x), \quad v(x, 0) = v^0(x), \quad c(x, 0) = c^0(x), \quad \text{for } x \in \Omega. \]
\[ s = i = v = 0; \quad \text{on } \partial \Omega \times (0, T) \]
where \(<, >\) is the duality between \(H^1_0(\Omega)\) and \(H^{-1}(\Omega)\).

Throughout this chapter, we make the following assumptions:

- \(d, d_1, r, K, \theta_c, \theta_t, \gamma, \alpha, \beta\) are positive constants;
- \(s^0, i^0, v^0, c^0 \in L^\infty(\Omega)\);
- \(0 \leq \rho^0(x), \rho^0(x), \rho^0(x) \leq B < K\) for some \(B \in \mathbb{R}\) for a.e. \(x \in \Omega\).

We start our optimal control problem by stating the existence of the solutions to the state system (6)-(9). The proof is standard and can be done by using the Banach Fixed Point Theorem or as iteration method [32, 29, 6].

**Theorem 3.1.** For sufficiently small \(T\) and given \((u_1, u_2) \in \mathcal{U}\), the state system (6)-(9) together with initial and boundary conditions admits a unique solution in the corresponding solution space.

Next, we state an a priori estimate for the state solutions and their derivatives.

**Theorem 3.2.** Suppose \((s, i, v) \in V^3 \cap L^\infty(Q)\) with \((s_1, i_1, v_1) \in (V^\gamma)^3\) and \(c, c_1 \in L^2(Q)\) is a weak solution of the system (6)-(9) corresponding to control \((u_1, u_2) \in \mathcal{U}\).

Then there exists a positive constant \(K_1\), such that \(\forall (u_1, u_2) \in \mathcal{U}\),
\[
|s|_{V^\gamma}, |i|_{V^\gamma}, |v|_{V^\gamma}, |c|_{L^2(\Omega)} \leq K_1 \quad \text{and} \quad |s_1|_{V^\gamma}, |i_1|_{V^\gamma}, |v_1|_{V^\gamma}, |c_1|_{L^2(\Omega)} \leq K_1
\]
and \(\|u_1\|_{L^\infty(Q)}, \|u_2\|_{L^\infty(Q)} \leq K_1\).

3.1. Characterization and uniqueness of optimal control pair. First, we show that there exists an optimal control pair that minimizes the objective functional given in (5). Let \((s, i, v, c)(u_1, u_2)\) be the state solution of the system (6)-(9) corresponding to the controls \((u_1, u_2) \in \mathcal{U}\).

**Theorem 3.3.** There exists \((u_1^*, u_2^*) \in \mathcal{U}\) such that
\[
J(u_1^*, u_2^*) = \inf_{(u_1, u_2) \in \mathcal{U}} J(u_1, u_2).
\]

**Proof.** Since the controls and the state variables are non-negative, there exists a minimizing sequence \((u_1^n, u_2^n)\) such that
\[
\lim_{n \to \infty} J(u_1^n, u_2^n) = \inf \{J(u_1, u_2) | u_1, u_2 \in \mathcal{U}\} \geq 0.
\]
Using Theorem 3.1, let us define for each \(n\),
\[
(s_n, i_n, v_n, c_n) = (s(u_1^n, u_2^n), i(u_1^n, u_2^n), v(u_1^n, u_2^n), c(u_1^n, u_2^n))
\]
Using Theorem 3.2, we have \(|s^n|, |i^n|, |v^n|\) are uniformly bounded (independent of \(n\)) in the space \(V\) and \(|s^n_t|, |i^n_t|, |v^n_t|\) are uniformly bounded in \(V^\gamma\). Also \(|c^n|\) and \(|c^n_t|\) are uniformly bounded in \(L^2(Q)\). Then, from equation (9) and the \(L^\infty\) bounds on state variables and controls, the sequence \(\{c_n\}\) is uniformly bounded in \(L^\infty(Q)\). Hence the sequence \(\{c_n\}\) is uniformly equicontinuous in \(t\) for each \(x\). Then by the Arzela-Ascoli Theorem, the sequence has subsequence that is uniformly convergent to \(c^*\) in \(t\) at each \(x\). Now, from the boundedness of state variables in \(V, V^\gamma\) and \(L^\infty(Q)\), and the fact that \(u_1^n, u_2^n \in L^\infty(Q)\) with uniform bounds, there exist \((s^*, i^*, v^*, u_1^*, u_2^*)\) and subsequences \((s^n, i^n, v^n, u_1^n, u_2^n)\) such that
\[
s_n \to s^*, i_n \to i^* \quad \text{and} \quad v_n \to v^* \quad \text{weakly in } V
\]
\[(s_n)_t \to s^*_t, (i_n)_t \to i^*_n, (v_n)_t \to v^*_n \text{ weakly in } \mathcal{V}^* \text{ and } c_n \to c^* \]
in \(t\) for each \(x\), and
\[u^n_1 \to u^*_1 \text{ and } u^n_2 \to u^*_2 \text{ weakly in } L^2(Q).\]

Next, we show that \((s^*, i^*, v^*, c^*)\) are the states associated with the control pair \((u^*_1, u^*_2)\). To pass to the limit in the above system (6)-(9) with \(s^n, i^n, v^n\) and \(c^n\), we need stronger convergence results for the terms with products of states and controls. Since \(\{s^n\}\) is uniformly bounded in \(V\) with \(\{(s^n)_t\}\) uniformly bounded in \(\mathcal{V}^*\), then by a compactness theorem (by Simon [36]), there exists a subsequence \(\{s_n\}\) converging strongly to \(s^*\) in \(L^2(Q)\). Similarly on subsequences,
\[i_n \to i^* \text{ strongly in } L^2(Q) \text{ and } v_n \to v^* \text{ strongly in } L^2(Q).\]

We illustrate the convergence of some of the terms:
\[
\left| \int_Q s^n u^n_1 \phi_1 - s^* u^*_1 \phi_1 dxdt \right| 
\leq \int_Q u^n_1 (s^n - s^*) \phi_1 dxdt + \int_Q (u^n_1 - u^*_1) s^* \phi_1 dxdt 
\leq \int_Q u^n_1 \|s^n - s^*\| \phi_1 dxdt + \int_Q (u^n_1 - u^*_1) s^* \phi_1 dxdt \to 0 \text{ as } n \to \infty.
\]
since the first integral converges to zero as \(u^n_1\) is uniformly bounded and the strong convergence of \(s^n \to s^*\) in \(L^2(Q)\). The second integral converges to zero as the state variable, \(s^*\), is \(L^\infty\) bounded implying \(s^* \phi_1 \in L^2(Q)\) and using the weak convergence of \(u^n_1 \to u^*_1\) in \(L^2(Q)\). All the other limits follow similarly.

With above convergence results, we obtain that \((s^*, i^*, v^*, c^*)\) solves the state system (6)-(9) with controls \((u^*_1, u^*_2)\).

Thus, we can conclude
\[
J(u^*_1, u^*_2) \leq \liminf_{n \to \infty} \int_0^T \int_\Omega [A_1 i^n(x, t) + A_2 c^n(x, t) + \{B_1 u^n_1 (s^n(x, t) + i^n(x, t)) + B_2 u^n_2 c^n(x, t) + C_1 u^n_1 (s^n(x, t) + i^n(x, t))^2 + C_2 u^n_2 (c^n(x, t))^2\}] dxdt 
= \inf\{J(u_1, u_2) | (u_1, u_2) \in \mathcal{U}\},
\]
which implies that \((u^*_1, u^*_2)\) is an optimal control for the problem (5). On the quadratic terms on the controls, we used the lower semi-continuity of the \(L^2\) norm with respect to \(L^2\) weak convergence.

Next, we derive the optimality system which consists of the state system coupled with the adjoint system and optimal control characterization.

**Lemma 3.4.** Let \((u_1, u_2) \in \mathcal{U}\) with the corresponding state solution \((s, i, v, c)(u_1, u_2)\). Let \((u^*_1, u^*_2) = (u_1 + \epsilon k_1, u_2 + \epsilon k_2)\), be another control pair corresponding to the state solution \((s^*, i^*, v^*, c^*) = (s, i, v, c)(u^*_1, u^*_2)\), where \((u^*_1, u^*_2) \in \mathcal{U}\) for all sufficiently small \(\epsilon > 0\) with \(k_1, k_2 \in L^\infty(Q)\). The mapping \((u_1, u_2) \to (s, i, v, c)(u_1, u_2)\) is weakly differentiable in the directional derivative sense and there exists \(\psi_1 \in \mathcal{V}\) with \((\psi_i)_t \in \mathcal{V}^*\) for \(i = 1, 2, 3\) and \(\psi_4, (\psi_4)_t \in L^2(Q)\) such that as \(\epsilon \to 0^+\) we have that,
\[
\frac{s^* - s}{\epsilon} \to \psi_1, \frac{i^* - i}{\epsilon} \to \psi_2, \frac{v^* - v}{\epsilon} \to \psi_3 \text{ weakly in } \mathcal{V} \text{ and } \frac{c^* - c}{\epsilon} \to \psi_4 \text{ weakly in } L^2(Q).
\]
Also, for each \( x, \frac{s^t - s}{\epsilon} \rightarrow \psi_4 \) pointwise in \( t \). Furthermore, the sensitivity functions \((\psi_1, \psi_2, \psi_3, \psi_4)\) satisfy the weak form of the sensitivity system

\[
\mathcal{L}\psi = K \tag{10}
\]

where \( \mathcal{L}\psi = \begin{pmatrix} (\psi_1)_t - d\Delta \psi_1 \\ (\psi_2)_t - d_1 \Delta \psi_2 \\ (\psi_3)_t - d\Delta \psi_3 \\ (\psi_4)_t \end{pmatrix} + M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad K = \begin{pmatrix} -k_1 s \\ 0 \\ k_1 s \\ -k_2 c \end{pmatrix}, \]

where \( M = (m_{ij}) \) is a 4x4 coefficient matrix with

\[
m_{1,1} = -r + \frac{2r}{K}(s + v) + \theta_c c + \theta_i i + u_1; \quad m_{1,2} = +\theta_i s; \quad m_{1,3} = -r + \frac{2r}{K}(s + v); \\
m_{1,4} = \theta_c s; \\
m_{2,1} = -\theta_c c - \theta_i i; \quad m_{2,2} = -\theta_i s + \gamma; \quad m_{2,3} = 0; \quad m_{2,4} = -\theta_c s; \\
m_{3,1} = -u_1; \quad m_{3,2} = 0; \quad m_{3,3} = 0; \quad m_{3,4} = 0; \\
m_{4,1} = \alpha c; \quad m_{4,2} = -\gamma + \alpha c; \quad m_{4,3} = \alpha c; \quad m_{4,4} = \alpha(s + i + v) + (u_2 + p);
\]

and where the initial and boundary conditions for \( \psi_i \) are \( \psi_i(x, 0) = 0 \) for \( i = 1, ..., 4 \) and, \( \psi_1 = \psi_2 = \psi_3 = 0 \) on \( \partial \Omega \times (0, T) \).

**Proof.** Since \((s^t, i^t, v^t, c^t)\) is the state corresponding to the control pair \((u_1^t, u_2^t)\), the system (3.1) – (3.4) with control pair \((u_1^t, u_2^t)\) is used to get the weak formulation

\[
\int_0^T \left< \frac{s^t - s}{\epsilon}, \phi_1 \right> dt + d_1 \int_Q \nabla \left( \frac{s^t - s}{\epsilon} \right) \cdot \nabla \phi_1 dx \, dt = \int_Q \frac{1}{\epsilon} \left[ n^r(1 - \frac{n^r}{K}) - \theta_c s^t c^t - \theta_i s^t i^t - u_1 s^t \right] \phi_1 dx \, dt; \tag{11}
\]

\[
\int_0^T \left< \frac{i^t - i}{\epsilon}, \phi_2 \right> dt + d_1 \int_Q \nabla \left( \frac{i^t - i}{\epsilon} \right) \cdot \nabla \phi_2 dx \, dt = \int_Q \frac{1}{\epsilon} \left[ \theta_c s^t c^t + \theta_i s^t i^t - \gamma i^t \right] \phi_2 dx \, dt; \tag{12}
\]

\[
\int_0^T \left< \frac{v^t - v}{\epsilon}, \phi_3 \right> dt + d_1 \int_Q \nabla \left( \frac{v^t - v}{\epsilon} \right) \cdot \nabla \phi_3 dx \, dt = \int_Q \frac{1}{\epsilon} \left[ u_1 s^t - u_1 s \right] \phi_3 dx \, dt; \tag{13}
\]

\[
\int_0^T \left< \frac{c^t - c}{\epsilon}, \phi_4 \right> dt = \int_Q \left[ \frac{c^t - c}{\epsilon} \right] \phi_4 dx \, dt; \tag{14}
\]

for any test functions \( \phi_1, \phi_2, \phi_3 \in \mathcal{V} \) and \( \phi_4 \in L^2(Q) \) and together with initial and boundary conditions

\[
\frac{s^t - s}{\epsilon}(x, 0) = 0, \quad \frac{i^t - i}{\epsilon}(x, 0) = 0, \quad \frac{v^t - v}{\epsilon}(x, 0) = 0, \quad \frac{c^t - c}{\epsilon}(x, 0) = 0; \quad \text{for } x \in \Omega,
\]

\[
\frac{s^t - s}{\epsilon}(x, t) = \frac{i^t - i}{\epsilon}(x, t) = \frac{v^t - v}{\epsilon}(x, t) = \frac{c^t - c}{\epsilon}(x, t) = 0 \quad \text{for } (x, t) \in \partial \Omega \times (0, T).
\]

By the standard PDE estimates,

\[
\sup_{\tau \in [0, T]} \left( \int_{\Omega} \left[ \left( \frac{s^t - s}{\epsilon} \right)^2 + \left( \frac{i^t - i}{\epsilon} \right)^2 + \left( \frac{v^t - v}{\epsilon} \right)^2 + \left( \frac{c^t - c}{\epsilon} \right)^2 \right] dx \right) + d_0 \int_Q \left[ \left| \nabla \frac{s^t - s}{\epsilon} \right|^2 + \left| \nabla \frac{i^t - i}{\epsilon} \right|^2 + \left| \nabla \frac{v^t - v}{\epsilon} \right|^2 \right] dx \, dt \leq \tilde{C} \int_Q (k_1^2 + k_2^2) dx \, dt \tag{15}
\]

where \( d_0, \tilde{C} \) depends on the bounds of the states, controls, and coefficients. Using similar techniques as in the Theorem 3.3, we get the quotients \( \left\{ \frac{s^t - s}{\epsilon}, \frac{i^t - i}{\epsilon}, \frac{v^t - v}{\epsilon} \right\} \)
converge strongly to \( \{ \psi_1, \psi_2, \psi_3 \} \) in \( L^2(Q) \) and \( \{ \epsilon \} \) converges strongly to \( \psi_4 \) in \( L^\infty(Q) \) and \( \psi_1, \psi_2, \psi_3, \psi_4 \) satisfy system (10). \( \square \)

Next, we derive the system for adjoint functions which will be used to characterize our optimal control. The adjoint operator \( \mathcal{L}^* \) is related to the operator \( \mathcal{L} \), which is associated with the sensitivity system (10), formally as using weak formulation on both sides:

\[
\int_0^T \int_\Omega L \tilde{\psi} dx dt = \int_0^T \int_\Omega \tilde{\psi} \mathcal{L}^* \tilde{\lambda} dx dt,
\]

with \( M \) defined as in Lemma 4.1 and \( \tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathcal{V}^3 \times L^2(Q) \) with \((\lambda_1)_t, (\lambda_2)_t, (\lambda_3)_t, (\lambda_4)_t \in (\mathcal{V}^*)^3 \times L^2(Q)\).

We use integration by parts and use the initial and boundary conditions for the sensitivity system (10) and obtain

\[
\mathcal{L}^* \left( \begin{array}{l}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{array} \right) = \left( \begin{array}{cccc}
\mathcal{L}_1^* & \mathcal{L}_2^* & \mathcal{L}_3^* & \mathcal{L}_4^* \\
0 & 0 & 0 & 0
\end{array} \right) + M^* \left( \begin{array}{l}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{array} \right) = \left( \begin{array}{l}
-\langle \lambda_1 \rangle_t - d(\lambda_1) \\
-\langle \lambda_2 \rangle_t - d(\lambda_2) \\
-\langle \lambda_3 \rangle_t - d(\lambda_3) \\
-(\lambda_4)_t
\end{array} \right) + M^* \left( \begin{array}{l}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{array} \right),
\]

with the final time condition \( \lambda_i(x, T) = 0 \) for \( i = 1, 2, 3, 4 \) (transversality condition) and zero boundary conditions. Now for the adjoint system, use source term from the derivative of the integrand in equation (2.5) with respect to the state variable to obtain

\[
\mathcal{L}_{\lambda}^* \tilde{\lambda} = \left( \begin{array}{c}
B_1 u_1 \\
A_1 + B_2 u_1 \\
0 \\
A_2 + B_2 u_2
\end{array} \right)
\]

where the terms in the right hand side of the above matrix are the derivative of the integrand in objective functional (5) with respect to \( s, i, v, c \) respectively. Thus, the adjoint system is

\[
\left( \begin{array}{cccc}
-\langle \lambda_1 \rangle_t - d(\lambda_1) & -\langle \lambda_2 \rangle_t - d(\lambda_2) & -\langle \lambda_3 \rangle_t - d(\lambda_3) & -(\lambda_4)_t \\
\mathcal{L}_1^* & \mathcal{L}_2^* & \mathcal{L}_3^* & \mathcal{L}_4^*
\end{array} \right) + M^* \left( \begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{array} \right) = \left( \begin{array}{c}
B_1 u_1 \\
A_1 + B_2 u_1 \\
0 \\
A_2 + B_2 u_2
\end{array} \right) \tag{17}
\]

with the final time conditions \( \lambda_i(x, T) = 0 \) and \( i = 1, 2, 3, 4 \) for \( x \in \Omega \) and the boundary conditions \( \lambda_i(x, t) = 0 \) for \( i = 1, 2, 3 \) and \( (x, t) \in \partial \Omega \times (0, T) \).

**Theorem 3.5.** Given an optimal control \( u = (u_1, u_2) \in \mathcal{U} \), there exists a solution \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in (\mathcal{V}^3 \cap (L^\infty(Q))^{4 \times L^2(Q)})\) with \((\lambda_1)_t, (\lambda_2)_t, (\lambda_3)_t, (\lambda_4)_t \in (\mathcal{V}^*)^{3 \times L^2(Q)}\) to the adjoint system (17). Furthermore,

\[
u^*_1(x, t) = \min \left( M_1, \max \left( -\frac{(\lambda_3 - \lambda_1 + B_1)s^* + B_1 i^*}{2C_1}, 0 \right) \right)
\]

\[
u^*_2(x, t) = \min \left( M_2, \max \left( -\frac{(B_2 - \lambda_4)c^*}{2C_2}, 0 \right) \right)
\]

**Proof.** Since the adjoint system (17) is linear in its variables, its solution exists by standard results. For the optimal control characterization, we compute the directional derivative of \( J(u_1^*, u_2^*) \) with respect to \((u_1^*, u_2^*)\) in the direction of \((k_1, k_2)\) at \((s^*, i^*, v^*, c^*)\).
Since \( J(u_1^*, u_2^*) \) is the minimum, then for \((u_1^*, u_2^*) = (u_1^1, u_2^1) + \epsilon (k_1, k_2) \) with corresponding state solution \((s^*, i^*, v^*, c^*) = (s, i, v, c)(u_1^1, u_2^1) \), we have

\[
0 \leq \lim_{\epsilon \to 0^+} \frac{J(u_1^1, u_2^1 + \epsilon k_2) - J(u_1^1, u_2^1)}{\epsilon}
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_Q A_1 i^* + A_2 c^* + B_1 (u_1^1 + \epsilon k_1)(s^* + i^*) + B_2 (u_2^1 + \epsilon k_2)c^* + C_1 (u_1^1 + \epsilon k_1)^2 + C_2 (u_2^1 + \epsilon k_2)^2\ dx dt
\]

\[
= \int_Q (\psi_1, \psi_2, \psi_3, \psi_4) \left( \begin{array}{c} B_1 u_1^1 \\ A_1 + B_1 u_1^1 \\ 0 \\ A_2 + B_2 u_2^1 \end{array} \right) \ dx dt + \int_Q (B_1 k_1 (s^* + i^*) + B_2 k_2 c^* + 2C_1 k_1 u_1^1 + 2C_2 k_2 u_2^1) \ dx dt
\]

\[
= \int_Q (\lambda_1, \lambda_2, \alpha, \lambda_4) \left( \begin{array}{c} -k_1 s^* \\ 0 \\ k_1 s^* \\ -k_2 c^* \end{array} \right) \ dx dt + \int_Q (B_1 k_1 (s^* + i^*) + B_2 k_2 c^* + 2C_1 k_1 u_1^1 + 2C_2 k_2 u_2^1) \ dx dt
\]

where we used sensitivities and adjoint systems in the weak sense. Now, using standard variation techniques, we get the characterizations (3.13)-(3.14) [25].

\( \square \)

Our optimality system consists of the system of state equations (1)-(4), the adjoint system (17), and the characterization of controls (3.13) – (3.14). Using the techniques used in [29, 6], we obtain the following uniqueness result:

**Theorem 3.6.** For sufficiently small \( T \), the solution to the optimality system is unique, which implies the uniqueness of the optimal control.

### 4. Numerical results.

We present some numerical results of our spatio-temporal anthrax model with control strategies for various parameter sets. We consider one dimensional spatial domain, a line segment \( \Omega = [0, 35] \), and the temporal domain \([0, 86]\). We try to mimic the anthrax outbreak scenario presented in the work of Clegg et al. [7], where the number of anthrax deaths were recorded over 86 days in the Malilangwe Wildlife Reserve, Zimbabwe in 2004. The size of the Reserve is 124000 hectares [13] which is 1240 km². Assuming the square shape of the domain, each side of the reserve is about 35 km. Since we are considering one dimensional spatial domain, the number of animals in the reserve at time \( t \) is represented by the number of animals in the one dimensional spatial domain \([0, 35]\). The time interval \([0, 86]\) is the same as the time interval for the outbreak data presented in Clegg et al. [7]. In the outbreak, more than 19 animal species were infected but 5 of them: Kudu, Nyala, Waterbuck, Bushbuck and Roan antelope were badly affected constituting more than 80% of the total reported deaths. To prevent endangered species in the park, vaccination program targeting those species was conducted. The total number of animals from these 5 species was estimated to be 901 with 709 of these dying due to the anthrax. Note that while finding the estimates for total number of susceptible animals for each animal species, we excluded those that were vaccinated against anthrax under the assumption that the vaccinated animals get immunity to the disease for at least one year. Since different animal species
have different growth and death rates, we calculated the weighted average of the rates taking their numbers as the weights. The growth rate of the animals were taken from various sources [38, 2, 5, 8, 13]. The weighted average for the intrinsic growth rate, $r$, is $5.0518 \times 10^{-4}$ per day. In our model, the animal movement is only by diffusion and for simplicity, we consider constant diffusion rates. Also, separate diffusion coefficients are considered for healthy and infected animals: $d$ is the diffusion coefficient for healthy animals (susceptible and vaccinated), and $d_1$ is for infected animals. Since anthrax infected animals are too sick to move normally, $d_1 < d$. The parameters value for disease induced death rate, $\gamma$, is taken from our previous work [31].

We assume that the initial population of susceptible animals are approximately uniformly distributed around the domain except on or close to the boundaries where they are assumed to be zero. In the Malilangwe Wildlife Reserve, the infection initiated near one of the corners of the reserve with a single carcass and some infected animals. To mimic this scenario, we assume that one initial carcass present close to an end of our one dimensional domain $[0, 35]$. Also, we assume that there were about 36 infected animals initially (as approximated from data in [31]) that were uniformly distributed around the location of the initial carcass site. Thus, the initial values $s(x, 0)$, $i(x, 0)$ and $v(x, 0)$ satisfy $\int_0^{35} s(x, 0)dx = 860$, $\int_0^{35} i(x, 0)dx = 36$, $\int_0^{35} v(x, 0)dx = 0$ and $\int_0^{35} c(x, 0)dx = 1$. These initial values are chosen in such a way that the areas under these curves are approximately equal to the initial numbers of susceptible and infected animals respectively. Thus, we consider,

$$s(x, 0) = \begin{cases} 24.93 & 1 \leq x \leq 34 \\ 0 & \text{otherwise}, \end{cases}$$

$$i(x, 0) = \begin{cases} 9 & 27 \leq x \leq 31 \\ 0 & \text{otherwise}, \end{cases}$$

$$v(x, 0) = 0 \text{ for all } x$$

and

$$c(x, 0) = \begin{cases} 1 & 29 \leq x \leq 30 \\ 0 & \text{otherwise}. \end{cases}$$

We assume that the environmental carrying capacity $K$ is 2000. The carcass feeding rate is assumed to be $\alpha = 0$ because all of the five animal species considered are herbivore mammals. Anthrax incidence is formulated as the mass action terms $\theta_c s i$ and $\theta_c sc$ with rate of infections $\theta_i$ from infected animals and $\theta_c c$ from infected carcasses. Since we do not have spatial data to estimate the transmission rates, we assume values for the transmission rates to be $\theta_c = 1.65 \times 10^{-3}$, $\theta_i = 2.05 \times 10^{-2}$. These values are chosen in such a way that the number of animals surviving at the end of the outbreak under our model is 200. Note that the value of $\theta_c$ is assumed to be smaller than the value of $\theta_i$ due to the fact that more transmissions from infected animals are likely because of their movement. Finally, the carcass decay rate is assumed to be 0.02816. All the parameter values are summarized in the Table 1.

In our problem, the vaccination rate at space $x$ and time $t$ is denoted by $u_1(x, t)$ and the carcass disposal rate is denoted by $u_2(x, t)$. As discussed in section 2, the optimal control problem under consideration is finding the optimal control pair $(u_1^*, u_2^*) \in U$ that minimizes the objective functional (5), where the cost of applying
Table 1. The model parameters, their description, values and units.

| Parm. | Description                              | Values                  | Units       |
|-------|------------------------------------------|-------------------------|-------------|
| $r$   | Intrinsic growth rate of healthy animals | $5.052 \times 10^{-4}$  | day$^{-1}$  |
| $\gamma$ | Disease induced death rate of infecteds | $1.75 \times 10^{-1}$  | day$^{-1}$  |
| $\alpha$ | Carcass feeding rate by scavengers | 0                       | animal$^{-1}$ day$^{-1}$ |
| $K$   | Carrying capacity of animals             | 2000                    | animal      |
| $p$   | Carcass decay rate                       | 0.02816                 | day$^{-1}$  |
| $d$   | Diffusion rate of healthy animals        | 0.12                    | km$^2$ day$^{-1}$ |
| $d_i$ | Diffusion rate of infected animals       | 0.024                   | km$^2$ day$^{-1}$ |
| $\theta_c$ | Disease transmission rate from carcasses | $1.65 \times 10^{-3}$  | carcass$^{-1}$ day$^{-1}$ |
| $\theta_i$ | Disease transmission rate from infected animals | $2.05 \times 10^{-2}$  | animal$^{-1}$ day$^{-1}$ |

the controls is considered to be a nonlinear function of $u_1$ and $u_2$. For all numerical simulations, we used a finite difference method for solving optimality system with backward-forward sweep scheme [19].

In Figures 1-2, we present two different simulation results: without control strategy (i.e., $u_1 = 0, u_2 = 0$), and with optimal vaccination and carcass disposal rates. The value of the objective functional (5) and the total number of deaths due to anthrax ($\int_\Omega \gamma I dx dt$) are evaluated for each of the simulations considered. In the objective functional (5), the higher weights are chosen for infected animals than for infected carcasses because of the fact that the infected animals may cause more infection due to their ability to move from place to place. Also, a higher weight is chosen for the vaccination rate than the carcass disposal rate because of the higher cost associated with the vaccination process for wild animals. We consider the following weight parameters:

$$A_1 = 5, \quad A_2 = 1, \quad B_1 = 0.1, \quad B_2 = 0.01, \quad C_1 = 0.1, \quad C_2 = 0.1.$$  

From Figure 1, we see that without any control strategies, the infection spreads from the initial location of infected animals and gradually expands with time and spreads to almost throughout the domain. As the susceptible and infected animals move towards the center of the region, the disease spreads quickly towards that direction. By the end of 75 days, the infection spreads throughout the region. The effect of the disease is severe in the initial infected region and its neighboring regions at the beginning and eventually spreads along more than two thirds of the region over time. The spreading wave of the infection can be seen moving towards the other end infecting many other susceptible animals and the death of these infected animals contaminate the surrounding environment with infected carcasses. The decayed carcasses keep the environment contaminated for longer time increasing the risk of new infection. The total cost associated with the infected animals, infected carcasses as well as the cost of applying the controls in this case is $J = 3.3806 \times 10^4$. Also, the total number of animals dying due to anthrax is about 551 and the total number of animals surviving is 200. This number is the same as the number of animals that were surviving at the end of the anthrax outbreak in Malilangwe Wildlife Reserve, Zimbabwe in 2004.

Next, we apply the optimal vaccination and optimal carcass disposal strategies to our model (1)-(4). The maximum possible value of vaccination rate is assumed to be 0.028 which implies that if we apply vaccination with maximum effort for a month, about 90% (about 785 out of 860) of the animals will get vaccinated. Also,
Figure 1. Simulation results for model (1)-(4) without control $u_1 = u_2 = 0$. The initial population of susceptible and infected animals are considered to be uniformly distributed in $1 \leq x \leq 34$ and $27 \leq x \leq 31$ respectively while only one initial carcass is considered near an end of the domain, $29 \leq x \leq 30$. The figures in the first row show the plots for susceptible (left) and infected (right) animals; and the figure in the second row represents the carcasses.

For the carcass disposal rate, we choose the maximum value to be 0.5. Thus, 

$$0 \leq u_1(x, t) \leq 0.028, \quad 0 \leq u_2(x, t) \leq 0.5.$$

Figure 2 shows the simulation results for the optimal intervention strategies of the vaccination and carcass disposal. From the figure, we see that few animals in the domain around the initial location of the infected carcass get infected and that number decays after about a week. With the optimal intervention strategies, about one third of the region stays uninfected throughout the outbreak and after a month, the infection is seen to be cleared from the area that had initial outbreak. Most of the susceptible animals in the infection region are vaccinated with continued vaccination up to almost 70 days. Most of the vaccinated animals are in the middle of the domain. Significant reduction of infected animals and infected carcasses can be seen at as early as two weeks of the outbreak. The vaccination should be started from the site of the infection covering about two thirds of the domain at the beginning and then the area of the vaccination can be reduced as the size of the outbreak starts to shrink. Vaccination should be continued till 70 days. On the other hand, the carcass disposal should be started with full effort near the location.
Figure 2. Simulation results for model (1)-(4) with optimal rates of vaccination and optimal carcass disposal rates \(0 \leq u_1(x,t) \leq 0.027\), and \(0 \leq u_2(x,t) \leq 0.5\). The initial population of susceptible and infected animals are considered to be uniformly distributed in \(1 \leq x \leq 34\) and \(27 \leq x \leq 31\) respectively while only one initial carcass is considered near an end of the domain, \(29 \leq x \leq 30\). The two plots in the first row represent the concentrations of susceptible(left) and infected (right) animals. The plots in the second row represents the concentrations of the infected carcasses (left) and the vaccinated animals(right). The last row represents the vaccination (left) and carcass disposal(right) rates.

of the initial infection site. The carcass search and disposal process should widen to almost half the domain over time and should be continued up to 60 days and then can be tapered down. Since the disease does not spread in the one third of the region at the other end, the controls are not needed in that part of the domain. The value of the objective functional in this case is \(J = 8.5677 \times 10^3\) which is about 74\% lower than the without control case and the total number of animals surviving to the end is 566 (273 susceptible and 292 vaccinated) which is about 280\% more animals surviving than in the without control case. Also, the total number of animals dying due to anthrax is about 216 which is about 60\% less than in no control case.
5. Conclusions and discussion. This work extended our previous work which consisted of an ODE model discussed for anthrax outbreak in wild animals. The major difference is the spatial dependence of the model variables and controls and animal movements governed by diffusion. The optimal control analysis for controlling the spread among anthrax in wild animals modeled by a system of ODE/PDEs was completed to obtain existence, uniqueness and characterizations of the optimal controls. We presented the derivation of adjoint system and the characterization of optimal control that depend on coefficients in the objective functional as well as the state and adjoint variables.

Our numerical results show that without optimal control, the infection spreads almost throughout the region within 86 days causing more deaths. Also more infected carcasses will remain in the region for a longer period of time posing risk of new infection. The application of optimal rates of vaccination and carcass disposal is a reasonable strategy to lower the density of infected animals and infected carcasses load in significantly lower cost. The optimal intervention strategies considered here is able to slow down the spread of the infection within a couple of weeks of the start of the infection and also able to vaccinate and save more animals. The vaccination program should start from an area covering the half of the region where the initial infection was observed and slowly cover smaller areas and should be continued until 70 days. The carcass search and disposal should be started from the initial carcass site and should be continued in bigger surrounding area towards the middle of the region. This process should be continued with highest effort until about 60 days and can be slowed down.

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