Necessary and sufficient conditions for macroscopic realism from quantum mechanics

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Macroscopic realism, the classical world view that macroscopic objects exist independently of and are not influenced by measurements, is usually tested using Leggett-Garg inequalities. Recently, another necessary condition called no-signaling in time (NSIT) has been proposed as a witness for non-classical behavior. In this paper, we show that a combination of NSIT conditions is not only necessary but also sufficient for a macrorealistic description of a physical system. Any violation of macroscopic realism must therefore be witnessed by a suitable NSIT condition. Subsequently, we derive an operational formulation for NSIT in terms of positive-operator valued measurements and the system Hamiltonian. We argue that this leads to a suitable definition of “classical” measurements and Hamiltonians, and apply our formalism to some generic coarse-grained quantum measurements.

I. INTRODUCTION

Whether or not the laws of quantum mechanics are universally valid and hold on the level of macroscopic objects, is still an open question in the physics community. Some believe that the issue will be settled in favor of quantum theory by the experimental demonstration of Schrödinger cat-like states [1]. Others hold that some physical mechanism, altering the laws of quantum mechanics [2–4], guarantees a fully classical world on the macroscopic level.

In 1985, Leggett and Garg [5] have put forward macroscopic realism (or macrorealism), a world view encompassing all physical theories which enforce that macroscopic properties of macroscopic objects exist independently of and are not influenced by measurement. While setups such as superconducting devices, heavy molecules, and quantum-optical systems are promising candidates in the race towards an experimental violation of macrorealism, non-classical effects have so far only been observed for microscopic objects or microscopic properties of larger objects [6–19]. However, a genuine violation of macroscopic realism—with its reference to macroscopically distinct states—requires using solely measurements of macroscopically coarse-grained observables. Note that there are several approaches to quantifying the “macroscopicity” of quantum states and measurements [20–27]. It is also known that usually the restriction to such coarse-grained (“classical”) measurements alone already leads to the emergence of classicality [28], unless a certain type of (“non-classical”) Hamiltonian is governing the object’s time evolution [29]. Recent investigations have confirmed the intuition that these Hamiltonians are hard to engineer and require a very high control precision in the experimental setup [30–32].

A quantum violation of macrorealism (MR) is usually witnessed by the violation of a Leggett-Garg inequality (LGI), which is composed of temporal correlations between sequential measurements of an object undergoing time evolution. Recently, following earlier works [29, 33–35], another necessary condition for MR called no-signaling in time (NSIT) was proposed [36]. It can be regarded as a statistical version of the non-invasive measurability postulate.

In section [II], we start with the discussion of various instances of NSIT and show that in the correct combination they form a sufficient condition for a macrorealistic description (at a given set of possible measurement times). We also demonstrate that it is impossible to establish such a sufficient condition for a macrorealistic description by combining LGIs involving two-time measurements. Subsequently, in section [II], we derive an operational condition for NSIT, based on (projective and non-projective) measurement operators and the system Hamiltonian. In section [IV], we use these results to define the classicality of measurements based on a reference set of a-priori classical operators and to characterize the classicality of Hamiltonians. Finally, in section [IV], we apply our formalism to measurements of coherent states, quadratures, and Fock states, and quantify their invasiveness as a function of their coarse-graining.

II. NON-INVASIVE MEASUREMENTS

Let us start with the definition of macrorealism, consisting of the following postulates [37]: “(1) Macrorealism per se. A macroscopic object which has available to it two or more macroscopically distinct states is at any given time in a definite one of those states. (2) Non-invasive measurability. It is possible in principle to determine which of these states the system is in without any effect on the state itself or on the subsequent system dynamics. (3) Induction. The properties of ensembles are determined exclusively by initial conditions (and in particular not by final conditions).”

In the following, we will first show that a strong reading of non-invasive measurability implies macrorealism per se (section [IIA]). Then we will present various necessary conditions (section [IIB]) and a set of sufficient conditions...
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for a macrorealistic description.

\[ H \] values and the macrostates. Further consider measure-
the postulate of NIM in \cite{37}:

Now argue that there exist two different ways of reading
subsequent state, but MRps is fulfilled, as the
invasive, as they affect the guiding field and thus the
by the de Broglie–Bohm theory, where measurements are
counter example to the statement MRps
before \cite{36, 39, 40}, we do not adhere to this position. A
is “so natural a corollary of [MRps] that the latter is
condition for macrorealism per se. It is argued that NIM
performing the same measurement immediately again, the
all non-invasive measurements are repeatable, i.e. when
be non-invasive, the time evolution of \( Q \)
result is obtained with probability 1.
performing the same measurement immediately again, the
same outcome is obtained with probability 1.

A. Macrorealism per se following from strong non-invasive measurability

In the following, we assume that the state space of a
macroscopic object is split into macroscopically distinct
non-overlapping states (macrostates). Consider a macro-
observable \( Q(t) \) with a one-to-one mapping between its
values and the macrostates. Further consider measure-
ments of the macro-observable that enforce a definite
post-measurement macrostate and report the correspond-
ing value as the outcome.

Macrorealism per se (MRps) is fulfilled if \( Q(t) \) has a
definite value at all times \( t \), prior to and independent of
measurement:

\[
\forall t: \exists \text{ definite } Q(t).
\]

Probabilistic predictions for \( Q(t) \) are merely due to igno-
rance of the observer. Even in cases where \( Q(t) \) evolves
unpredictably (e.g. in classical chaos) or even indetermin-
istically, it is still assumed to have a definite value at all
times.

On top of MRps, the assumption of non-invasive measu-
measurability (NIM) in principle allows a measurement at
every instant of time, revealing the macrostate without
disturbance. NIM guarantees that

\[
\forall t: Q(t) = Q_H(t),
\]

where \( H \) denotes the history of past non-invasive meas-
urements on the system: In order for measurements to
be non-invasive, the time evolution of \( Q \) must not de-
pend on the history of the experiment \cite{38}. Note that
all non-invasive measurements are repeatable, i.e. when
performing the same measurement immediately again, the
same outcome is obtained with probability 1.

In the literature, NIM is often treated as a necessary
condition for macrorealism per se. It is argued that NIM
is “so natural a corollary of [MRps] that the latter is
virtually meaningless in its absence” \cite{37}. As some others
before \cite{36, 39, 40}, we do not adhere to this position. A
counter example to the statement MRps \( \Rightarrow \) NIM is given
by the de Broglie–Bohm theory, where measurements are
invasive, as they affect the guiding field and thus the
subsequent (position) state, but MRps is fulfilled, as the
(position) state is well-defined at all times. In fact, we
now argue that there exist two different ways of reading
the postulate of NIM in \cite{37}:

- **Weak NIM.** Given a macroscopic object is in a
definite one of its macrostates, it is possible to
determine this state without any effect on the state
itself or on the subsequent system dynamics.

- **Strong NIM (sNIM).** It is always possible to measure
the macrostate of an object without any effect on the
state itself or on the subsequent system dynamics.

Let us now argue that sNIM actually implies MRps. As-
suming sNIM, a hypothetical non-invasive measurement
can be performed at every instant of time, determining the
value of the macro-observable \( Q \). Due to its non-invasive
nature, \( Q \) must have had a definite value already before
the measurement. This ensures that \( Q \) has a definite value
at all times, giving rise to a “trajectory” \( Q(t) \). Therefore,

\[
sNIM \Rightarrow MRps. \quad (3)
\]

Another way of establishing this implication is the fol-
lowing: Assume that MRps fails, i.e. the object is not in
a definite macrostate. A measurement leaves the object
in a definite macrostate, creating a definite state out of
an indefinite one, and therefore doesn’t satisfy sNIM. We
thus have \( \neg MRps \Rightarrow \neg sNIM \), which is equivalent to \( (3) \).

Note that \ref{37} holds even if sNIM is made less stringent,
allowing measurements to change the subsequent time
evolution, while still determining the macrostate.

In this paper, we implicitly assume induction (the ar-
row of time) \cite{37} and freedom of choice concerning the
initial states and measurement times (including whether a
measurement takes place at all). Therefore, sNIM alone is
sufficient for macrorealism, and by extension, for testable
conditions such as the Leggett-Garg inequalities or no-
signaling in time \cite{38}:

\[
sNIM \Leftrightarrow MRps \land NIM \Leftrightarrow MR \Rightarrow LGI, NSIT. \quad (4)
\]

B. Necessary conditions for macrorealism

The relationship between LGI and NSIT has previously
been discussed in the literature for a number of example
systems \cite{29, 36, 40, 41}. Here we consider the archetypal
setup depicted in fig. 1: A system starting in the initial
state \( \rho_0 \) evolves under unitary \( U_{t_0} \) from \( t_0 \) to \( t_1 \), and
under unitary \( U_{t_1} \) from \( t_1 \) to \( t_2 \). During the evolution,
dichotomic measurements may be performed at times
\( t_i \) for \( i \in \{0, 1, 2\} \). Let us call the outcomes of these
measurements \( Q_i \in \{-1,+1\} \), and define the correlations
\( C_{ij} = \langle Q_i Q_j \rangle \). Then, the simplest LGI reads

\[
\text{LGI}_{12}: C_{01} + C_{12} - C_{02} \leq 1. \quad (5)
\]

There exist many other Leggett-Garg inequalities involv-
ing more than three measurement times or more than
two outcomes (for a recent review see \cite{42}). Quantum
 mechanical experiments are able to violate ineq. \ref{5}
up to 1.5 for a qubit and, as shown in \cite{43}, up to the alge-
braic maximum 3 for higher-dimensional systems using
dichotomic measurements \( Q_i = \pm 1 \).

On the other hand, NSIT \((i,j)\) is a statistical version of
eq. \ref{2}, requiring that the outcome probabilities \( P_j(Q_j) \)
of result \( Q_j \) measured at time \( t_j \) be the same, no matter
whether or not a measurement was performed at some
earlier time \( t_i < t_j \):

\[
\text{NSIT}_{(i)j}: P_j(Q_j) = P_{ij}(Q_j) \equiv \sum_{Q_i'} P_{ij}(Q_i', Q_j). \quad (6)
\]
Note that the probability distributions on both sides of the equation, $P_i$ and $P_{ij}$, correspond to different physical experiments: While $P_f$ is established by measuring only at $t_f$, $P_{ij}$ is obtained by measuring both at $t_i$ and $t_j$. Unlike in the LGI in \[\text{(5)},\] one is not limited to only two outcomes. If it is the later measurement at $t_f$ which may or may not be performed, NSIT\(_{(i)}\) is an instance of the arrow of time and is therefore fulfilled by both macrorealism and quantum mechanics.

While NSIT\(_{(1)}\) is a promising condition that is usually able to detect violations of MR more reliably than LGI\(_{012}\) \[\text{[36, 41]},\] it fails for particular initial states, where the invasiveness is able to “hide” in the statistics of the experiment (see the discussion below). We can however make NSIT\(_{(1)}\) robust against such cases, by always performing a measurement at $t_0$. We call the resulting condition

\[
\text{NSIT}_{0(1)}: P_{02}(Q_0, Q_2) = P_{012}(Q_0, Q_1, Q_2) \equiv \sum_{Q_1'} P_{012}(Q_0, Q_1', Q_2). \quad \text{(7)}
\]

NSIT\(_{0(1)}\) alone is not sufficient for LGI\(_{012}\). Hence, we also introduce the condition

\[
\text{NSIT}_{0(1)2}: P_{12}(Q_1, Q_2) = P_{012}(Q_1, Q_2) \equiv \sum_{Q_2'} P_{012}(Q_0, Q_1, Q_2). \quad \text{(8)}
\]

As was recently shown in \[\text{[40]},\] a combination of NSIT\(_{0(1)2}\), NSIT\(_{0(1)}\) and the arrow of time (AoT) is sufficient for LGI\(_{012}\):

\[
\text{NSIT}_{0(1)} \land \text{NSIT}_{0(1)2} \land \text{AoT} \Rightarrow \text{LGI}_{012}. \quad \text{(9)}
\]

The inverse is not true, and the left-hand side is not sufficient for macrorealism (see discussion below).

We further remark that one can also write NSIT\(_{0(1)}\) in a more intuitive form that we call non-invaded correlations (NIC),

\[
\text{NIC}_{0(1)2}: C_{02} = C_{02(1)}, \quad \text{(10)}
\]

where $C_{02}$ denotes the correlation $\langle Q_0 Q_2 \rangle$ given that an additional measurement was performed at $t_1$. It is shown in appendix \[\text{A}\] that NIC\(_{0(1)}\) follows from NSIT\(_{0(1)2}\).

Fig. \[\text{1}\] shows a graphical summary of the conditions that have been discussed in this section.

C. NSITs as sufficient conditions for macrorealism

In the following, we will show that the combination of various NSIT conditions and the arrow of time (AoT) guarantees the existence of a unique global probability distribution $P_{012}(Q_0, Q_1, Q_2)$, which is equivalent to macrorealism evaluated at $t_0, t_1, t_2$. Let us start by writing all single-measurement probabilities in terms of $P_{012}$.

Once again, note that joint probabilities $P$ with different subscripts correspond to different experimental setups (e.g. $P_2(Q_2)$ is obtained by measuring only at $t_2$, while $P_{12}(Q_1, Q_2)$ is obtained by measuring at times $t_1$ and $t_2$):

\[
P_2(Q_2) = \sum_{Q_1'} P_{12}(Q_1', Q_2) = \sum_{Q_0'} \sum_{Q_1'} P_{012}(Q_0', Q_1', Q_2). \quad \text{(11)}
\]

where we have used NSIT\(_{(1)}\) for the first equality and NSIT\(_{(0)}\) for the second one. Furthermore,

\[
P_1(Q_1) = \sum_{Q_2'} P_{12}(Q_1, Q_2) = \sum_{Q_0'} \sum_{Q_2'} P_{012}(Q_0, Q_1, Q_2'). \quad \text{(12)}
\]

where for the first equality we assumed AoT, i.e. $Q_i$ are (statistically) independent of $Q_j$ for $j > i$, and NSIT\(_{(0)}\) for the second one. Furthermore, we see that

\[
P_0(Q_0) = \sum_{Q_1'} \sum_{Q_2'} P_{012}(Q_0, Q_1', Q_2'), \quad \text{(13)}
\]

where AoT was used twice. Next, the pairwise joint probability functions can be constructed:

\[
P_{01}(Q_0, Q_1) = \sum_{Q_2} P_{012}(Q_0, Q_1, Q_2') \quad \text{(14)}
\]

follows from AoT. Using NSIT\(_{(0)}\) one obtains

\[
P_{02}(Q_0, Q_2) = \sum_{Q_1'} P_{012}(Q_0, Q_1', Q_2). \quad \text{(15)}
\]

Finally, using NSIT\(_{(0)}\), we obtain

\[
P_{12}(Q_1, Q_2) = \sum_{Q_0} P_{012}(Q_0', Q_1, Q_2). \quad \text{(16)}
\]

We have thus shown that there exists a combination of NSIT conditions, whose fulfillment guarantees that all probability distributions in any experiment can be
As discussed above, it is possible for LGI to be written as the marginals of a unique global probability distribution \( P_{012}(Q_0, Q_1, Q_2) \). This is equivalent to the existence of a macrorealistic model for measurements at times \( t_0, t_1, t_2 \) (MR012). Note that while MR012 cannot prove the world view of MR in general, it implies that no experimental procedure (with measurements at \( t_0, t_1, t_2 \)) can detect a violation of MR. Let us now write a necessary and sufficient condition for MR012:

\[
\text{NSIT}_{(1/2)} \land \text{NSIT}_{(0/1)} \land \text{NSIT}_{(0/12)} \land \text{AoT} \leftrightarrow \text{MR012}. \tag{17}
\]

This set of conditions is not unique. We can e.g. substitute NSIT\(_{(1/2)}\) by NSIT\(_{(0/2)}\), as can easily be seen from a graphical representation of these conditions in fig. 2. We remark that even the combination of all two-time NSIT conditions, NSIT\(_{(0/1)} \land \text{NSIT}_{(1/2)} \land \text{NSIT}_{(0/12)}\), is sufficient neither for MR012 nor for LGI012. Note that LGIs only test for non-classicalities of the pairwise joint probability distributions. A smaller set of conditions is therefore sufficient for fulfilling all LGIs using two-time correlation functions or probabilities (such as ineq. (5) or the so-called Wigner LGIs [11]):

\[
\text{NSIT}_{(0/12)} \land \text{NSIT}_{(0/12)} \land \text{AoT} \leftrightarrow \text{LGI012}. \tag{18}
\]

To illustrate these conditions for a qubit, in table I we show the individual conditions evaluated for a Mach-Zehnder setup (reflectivities \( R_1, R_2 \), phase plate \( \varphi \) in one arm) with arbitrary initial state and time evolution. The three possible measurements are which-path measurements before the first beam splitter (\( t_0 \)), between the two beamsplitters (\( t_1 \)), and after the second beamsplitter (\( t_2 \)), respectively. We can easily find cases where LGI012 is always fulfilled, but various NSIT conditions still witness a violation of MR, e.g. for \( R_1 = R_2 = 1/2, \varphi \neq (n+1/2)\pi \). As discussed above, it is possible for LGI012 to be violated with NSIT\(_{(1/2)}\) fulfilled, e.g. for \( R_1 = 1/4, R_2 = 3/4, q = 1/2, \varphi = \pi \). For mixed initial states, NSIT\(_{(0/1)}\) reduces to the condition \( \varphi = (n+1/2)\pi \) with \( n \in \mathbb{N}_0 \) and is sufficient for MR012, as no interference is possible in this case. For general superposition states, NSIT\(_{(0/12)}\) can be violated with NSIT\(_{(0/1)}\) fulfilled. Moreover, NSIT conditions still allow detecting violations of MR if \( R_1 = 0, 1 \) or \( R_2 = 0, 1 \).

FIG. 2. Different combinations of NSIT and AoT conditions are sufficient for guaranteeing that all probability distributions \( P_{ij} \) are the marginals of a unique global probability distribution \( P_{012} \). There are multiple ways of obtaining a sufficient set. The black arrows correspond to one particular choice, additional conditions are printed for completeness in blue. Note that the existence of a classical explanation for the pairwise joint probabilities \( P_{ij} \) is sufficient for fulfilling LGI012, but not for MR012.

III. NSIT FOR QUANTUM MEASUREMENTS

In the following, we will look at NSIT\(_{(0/12)}\) in an archetypical quantum experiment. A system \( \hat{\rho}(t) \) has been prepared at \( t = 0 \) in an initial state \( \hat{\rho}(0) = \hat{\rho}_0 \). Then, at \( t = 0 \), a POVM \( \{ \hat{A}_a, \hat{A}_b \}_a \) with outcomes \( a \) is carried out. After the measurement, the system evolves according to a unitary \( \hat{U} = e^{-iHt} \). At time \( t = T \) a second, possibly different POVM \( \{ \hat{B}_b, \hat{B}_b \}_b \) with outcomes \( b \) is performed.

To determine the effect of the first measurement \( \hat{A}_a, \hat{A}_b \) on the system’s state and its subsequent dynamics, we will compare the results of the final measurement with a different experiment, where no measurement was performed at \( t = 0 \) (or, equivalently, a measurement \( \hat{A}_a = \hat{1} \) was performed). The two setups are shown in fig. 3.

The probabilities for obtaining outcome \( b \) in the two setups are called \( P_B(b) \) and \( P_B|A(b) \). They can be calculated as

\[
P_B(b) = \text{tr}(\hat{B}_b \hat{U}_T \hat{\rho}_0 \hat{U}_T^\dagger \hat{B}_b), \tag{19}
\]

\[
P_B|A(b) = \int \text{d}a \text{tr}(\hat{B}_b \hat{U}_T \hat{A}_a \hat{\rho}_0 \hat{A}_a^\dagger \hat{U}_T^\dagger \hat{B}_b), \tag{20}
\]

with the integral replaced by a sum if the number of outcomes is countable. NSIT\(_{(0/12)}\) is fulfilled if the test measurement has no detectable effect on the system, i.e. if \( P_B = P_B|A \):

\[
\forall b: \text{tr}(\hat{B}_b \hat{U}_T \hat{\rho}_0 \hat{U}_T^\dagger \hat{B}_b) = \int \text{d}a \text{tr}(\hat{B}_b \hat{U}_T \hat{A}_a \hat{\rho}_0 \hat{A}_a^\dagger \hat{U}_T^\dagger \hat{B}_b). \tag{21}
\]

Note that the equality sign in eq. (21) will often be fulfilled only approximately, even by non-invasive measurements. In practice, one can choose from a variety of error measures and corresponding reasonable error thresholds.

FIG. 3. A system evolves from \( t = 0 \) to \( t = T \) under Hamiltonian \( H \). In the first setup only a final measurement \( \hat{B}_b, \hat{B}_b \) is performed, in the second setup measurements \( \hat{A}_a, \hat{A}_a \) and \( \hat{B}_b, \hat{B}_b \) are performed at \( t = 0 \) and \( t = T \), respectively.
TABLE I. Different necessary conditions for macrorealism evaluated for a Mach-Zehnder (qubit) experiment [44]. The reflectivity of the first beamsplitter is $R_1$, of the second $R_2$. In one path of the interferometer, a phase $\phi$ is added. Which-path measurements may be performed before, between and after the beamsplitters. The initial states are $\rho_{\text{mix}} = (0\; 1 - q)$ and $\rho_{\text{sup}} = (q\; 1 - q)$. The symbol “$\checkmark$” means that the condition holds for all values of the free parameters. For brevity, $\alpha \equiv \sqrt{R_1 R_2 (1 - R_1)(1 - R_2)}$. Equation [*] reads $\langle 2i\sqrt{3}e + 6q - 3 \rangle \cos \phi - 2i\sqrt{3} \Re(e) \cos \phi - 2i \sin \phi = 0$, equation [**] reads $\cos \phi[2(4 - 1)\alpha + ic(1 - 2R_1) \sqrt{-(R_1 - R_2)} + i \sqrt{-(R_1 - R_2)}(2R_1 - 1) \cos \phi + i \sin \phi] = 0$. See main text for discussion.

However, to simplify notation, we will continue to use the equality sign in the following calculations.

A. NSIT without time evolution

Let us start by considering the case $T = 0$ (NSIT(0)), i.e. the final measurement is performed immediately after the test measurement. In this setup, NSIT can be regarded as a case of joint measurability, a condition previously discussed in the context of compatibility of quantum measurements [45–52]. To rewrite eq. (21) we use that $\int d\alpha \tr(A_0^\dagger A_0) = 1$. This yields

$$P_{\hat{B}}(b) - P_{\hat{B} | \hat{A}}(b) = \int d\alpha \tr[(\hat{A}_0^\dagger \hat{B}_b \hat{B}_\alpha - \hat{B}_b^\dagger \hat{A}_0^\dagger \hat{A}_0 \hat{B}_b)\hat{\rho}_0].$$

The trace in the above equation can be interpreted as the expectation value of the Hermitian operator $(\hat{A}_0^\dagger \hat{B}_b \hat{B}_\alpha - \hat{B}_b^\dagger \hat{A}_0^\dagger \hat{A}_0 \hat{B}_b)$. For NSIT(0) to be universally valid, we require that it is zero for all initial states $\hat{\rho}_0$. Thus, the operator itself has to be zero,

$$\forall \hat{\rho}_0 : \text{NSIT}^{(0)0} \iff \forall b : \int d\alpha (\hat{A}_0^\dagger \hat{B}_b \hat{B}_\alpha - \hat{B}_b^\dagger \hat{A}_0^\dagger \hat{A}_0 \hat{B}_b) = 0. \tag{23}$$

This equation can be further simplified to

$$\int d\alpha \hat{A}_0^\dagger \hat{B}_b \hat{B}_\alpha \hat{A}_0 - \hat{B}_b^\dagger \hat{A}_0^\dagger \hat{A}_0 \hat{B}_b = 0.$$

Comparing eq. (27) and eq. (28), we see that $|\langle a|b\rangle|^2 = |\langle a|b\rangle|^4$ can only be fulfilled if it is non-zero for exactly one $a$. Thus, $|b\rangle$ is an eigenstate of $\hat{A}_a$. Note that for Hermitian operators $\hat{A}_a = \hat{A}_a^\dagger$, $\hat{B}_b = \hat{B}_b^\dagger$ we can rewrite (23) using the commutator

$$\forall \hat{\rho}_0 : \text{NSIT}^{(0)0} \iff \forall b : \int d\alpha [\hat{A}_0 \hat{B}_b, \hat{B}_b \hat{A}_0] = 0. \tag{24}$$

Furthermore, we have as sufficient conditions the vanishing commutators

$$\forall a, b : [\hat{A}_a \hat{B}_b, \hat{B}_b \hat{A}_a] = 0 \Rightarrow \forall \hat{\rho}_0 : \text{NSIT}^{(0)0}. \tag{25}$$

and, consequently,

$$\forall a, b : [\hat{A}_a, \hat{B}_b] = 0 \Rightarrow \forall \hat{\rho}_0 : \text{NSIT}^{(0)0}. \tag{26}$$

It is interesting to note that both of these commutator conditions are, generally, only sufficient but not necessary for NSIT(0). In fact, a formulation of NSIT(0) must inherently have an asymmetry [51] between the test and final measurements, but both (25) and (26) are symmetric under exchange of $\hat{A}$ and $\hat{B}$ [53].

We can, however, show that vanishing commutators in (25) and (26), are sufficient and necessary when $\hat{A}_a, \hat{B}_b$ are von Neumann projective measurements ($\hat{A}_a^2 = \hat{A}_a$, $\hat{B}_b^2 = \hat{B}_b$). Let us start by rewriting the equality in (23) using $\hat{A}_a = |a\rangle\langle a|$ and $\hat{B}_b = |b\rangle\langle b|$:

$$\int d\alpha |\langle a|b\rangle|^2 |\langle a|\rangle = |b\rangle|.$$  

(27)

Since $|b\rangle\langle b|$ is a projector, squaring the integral on the left-hand side must leave it unchanged. Using the fact that in order to sum up to identity, the $\hat{A}_a$ must be orthogonal projectors, and therefore $\langle a|a'\rangle = \delta(a - a')$, we obtain

$$\int d\alpha |\langle a|b\rangle|^2 |\langle a|\rangle = |b\rangle|.$$  

(28)

Comparing eq. (27) and eq. (28), we see that $|\langle a|b\rangle|^2 = |\langle a|b\rangle|^4$ can only be fulfilled if it is non-zero for exactly one $a$. Thus, $|b\rangle$ is an eigenstate of $\hat{A}_a$, and the commutator is $[\hat{A}_a, \hat{B}_b] = 0$. We have therefore shown that for von Neumann measurements (but not for general POVMs), vanishing commutators in (25) and (26) are both sufficient and necessary for NSIT(0).

B. NSIT with time evolution

Let us now consider NSIT(0) with unitary time evolution $\hat{U} = e^{-i\hat{H}t}$. Analogous to the derivation of (23) and...
defining $\tilde{B}_b^T = \tilde{U}_T \tilde{B}_b \tilde{U}_T$, we obtain

$$\forall \rho_0: \text{NSIT}_{(0)T} \Leftrightarrow \forall \rho_0: \text{NSIT}_{(0)T},$$

and, if $\hat{A}_a, \hat{B}_b$ are Hermitian operators,

$$\forall \rho_0: \text{NSIT}_{(0)T} \Leftrightarrow \forall \rho_0: \text{NSIT}_{(0)T}.$$  

Comparing (23) and (29), we can apply the results for NSIT derived above, namely

$$\forall a, b: [\hat{A}_a, \hat{B}_b^T] = 0 \Rightarrow \forall \rho_0: \text{NSIT}_{(0)T},$$

and

$$\forall a, b: [\hat{A}_a, \hat{B}_b^T] = 0 \Rightarrow \forall \rho_0: \text{NSIT}_{(0)T}.$$  

Furthermore, one obtains

$$\forall a, b: [\hat{A}_a, \hat{B}_b] = [\hat{A}_a, \hat{U}_T] = 0 \Rightarrow \forall \rho_0: \text{NSIT}_{(0)T}.$$  

If $\hat{A}_a, \hat{B}_b$ are von Neumann operators, we have $(\hat{B}_b^T)^2 = \hat{U}_T \hat{B}_b \hat{U}_T \hat{U}_T \hat{B}_b \hat{U}_T = \hat{U}_T \hat{B}_b \hat{U}_T$. Thus, the results from section IIIA apply here too: For projectors (but not for general POVMs), vanishing commutators in (32) are sufficient and necessary for NSIT$_{(0)T}$.

The above results show that a non-classical “resource” is required for an experimental violation of NSIT, namely either highly non-classical states (equivalent to non-classical measurements used in their preparation) or non-classical Hamiltonians (usually requiring an extremely large experimental “control precision” as discussed in [30,32]).

IV. CLASSICALITY

As we have indicated in the introduction, the coarse-graining of “sharp” quantum measurement operators into “fuzzy” classical measurements, plays a crucial role in the transition from quantum mechanics to classical physics [28]. However, not every coarse-grained operator can be called classical. As an example, the parity operator (e.g. for large spins or photonic states) only differentiates two macrostates, but is in fact highly non-classical. Generally speaking, a suitable coarse-graining should “lump” together neighbouring eigenvalues, independent of a (quantum) experiment’s Hamiltonian. However, Hilbert spaces in quantum mechanics possess no inherent measure for the distance between orthogonal states. Such a measure must thus arise solely out of interaction Hamiltonians. Effectively, any definition of classicality must therefore depend on Hamiltonians spontaneously realized by nature, which define a natural order and closeness of states. In the following, this closeness is established with an a priori choice of suitable reference operators. With this reference set, we can write a definition for classical operators and classical Hamiltonians:

(I) A measurement operator is called classical with respect to a reference set iff it fulfills the equality in (23) pairwise with every member of the set.

(II) A Hamiltonian is called classical with respect to a reference set iff the equality in (29) is fulfilled for each combination of measurement operators from the set.

A natural choice for the reference set are coarse-grained versions of quantum operators in phase space. Phase space inherently involves the necessary definition of closeness in a suitable and intuitive way. Several exemplary candidates for different experiments are discussed in the next section.

V. CLASSICALITY OF QUANTUM MEASUREMENTS

In the following, we will apply our results to a number of physical systems. We will focus on the classicality of operators—condition (I) from the previous section—and always assume either an immediate test measurement, or free time evolution in-between. To measure the overlap of the undisturbed [19] and the disturbed [20] probability distributions, we make use of the Bhattacharyya coefficient [31], as defined by

$$V = \int \, db \, \sqrt{P_B(b)P_{\tilde{B}_{\beta}}(b)} \in [0,1].$$

The extreme cases of $V = 0$ and $V = 1$ correspond to orthogonal and identical probability distributions, respectively.

A. Quadrature measurements

Let us start with quadrature measurements on pure coherent initial states $\hat{\rho} = |\gamma\rangle\langle\gamma|$. We investigate coarse-grained measurements with unsharpness $\delta$ in the X-quadrature, and unsharpness $\kappa$ in the P-quadrature, as described by the (dimensionless) operators

$$\hat{X}^\delta_x = \frac{1}{(\delta^2\pi)^{1/4}} \exp\left(-\frac{1}{2\delta^2}(x - \hat{X})^2\right),$$

$$\hat{P}^\kappa_p = \frac{1}{(\kappa^2\pi)^{1/4}} \exp\left(-\frac{1}{2\kappa^2}(p - \hat{P})^2\right).$$

Note that for $\hat{B}_\beta = \pi^{-1}|\beta\rangle\langle\beta|$, we recover the well-known Husimi Q-distribution [55], since $P_B(\beta) = \pi^{-2} \text{tr}(|\beta\rangle\langle\beta|\hat{\rho}_0|\beta\rangle\langle\beta|) = \pi^{-1} |\beta\rangle\langle\beta|\hat{\rho}_0|\beta\rangle = Q(\beta)$. As an example, choosing $\hat{A} = \hat{X}^\delta$ and $\hat{B}_\beta = \pi^{-1}|\beta\rangle\langle\beta|$, the Husimi distribution $P_{\hat{B}_{\beta}\hat{A}}$ is shown in fig. 4 for several values of $\delta$.

The behavior for different combinations of $\hat{A}, \hat{B} \in \{\hat{X}^\delta, \hat{P}^\kappa\}$ are printed in table I detailed analytic values for the overlaps are listed in appendix I.
The importance of selecting a complete set of classical reference operators becomes clear when looking at different combinations of coarse-grained $X^\delta, P^\kappa$. In particular, even a sharp $X$ measurement is revealed by a second (coarse-grained) $X$ measurement only after time evolution. Therefore, $P^\kappa$ has to be a member of the reference set. On the other hand, a sharp measurement in $P$ can never be detected by another measurement in $P$ under free time evolution $H = P^2/(2m)$. Therefore, $X^\delta$ needs to be a member of the set. For $X^\delta$ and $P^\kappa$ to fulfill the consistency condition, we further require sufficiently large $\delta \gg 1$ and $\kappa \gg 1$, such that $[X^\delta, P^\kappa] \approx 0$.

Using the notation $\hat{X}_{c.g.}(P_{c.g.})$ for a sufficiently coarse-grained $X$ ($P$) measurement, and $\hat{X}_{sh.}(P_{sh.})$ for a sharp, invasive measurement, we can write some candidate reference sets:

- \{\hat{X}_{c.g.}\} or \{\hat{X}_{sh.}\} do not constitute reference sets, since they cannot detect the invasiveness of a $\hat{X}_{sh.}$ measurement.
- \{\hat{X}_{sh.}, \hat{P}_{c.g.}\} is not a reference set, since the operators do not fulfill (23).
- \{\hat{X}_{c.g.}, \hat{P}_{c.g.}\} is a possible reference set.

For further discussion about the joint measurability and coexistence of coarse-grained phase space operators we refer the reader to references [50, 58].

### B. Coherent state measurements

As another example, let us now consider coarse-grained operators in coherent state space,

$$\hat{A}_a = \frac{1}{\pi} \int d\alpha f_a(\alpha) |\alpha\rangle\langle\alpha|,$$  \hspace{1cm} (37)

where $f_a(\alpha)$ are some real and positive envelope functions that define the coarse-grained regions. Again, we consider coherent initial states $\hat{\rho} = |\gamma\rangle\langle\gamma|$ and final measurements $\hat{B}_\beta = \pi^{-1}|\beta\rangle\langle\beta|$. An analytical result can be obtained for a measurement $f_a(\alpha) = \delta(\alpha - a)$ for $a \in \mathbb{C}$, yielding $\hat{A}_a = \pi^{-1}|\alpha\rangle\langle\alpha|$. We can now calculate the overlap for $T = 0$:

$$V = \frac{1}{\pi} \int d\beta \left[ |\langle\beta|\gamma\rangle|^2 \int d\alpha |\langle\beta|\alpha\rangle\langle\alpha|\gamma\rangle|^2 \right]^\frac{1}{2}$$  \hspace{1cm} (38)

This overlap provides us with a lower bound, that applies to all coarse-grained measurements based on coherent states. As an example, numerically evaluated overlaps for a ring-like coarse-graining ($f_a(r)$ is non-zero for $ad \leq r < (a+1)d$, with $a \in \mathbb{N}_0$ and $d$ the ring width) are plotted in fig. 5.

A choice of reference set, alternative to the previously discussed \{\hat{X}_{c.g.}, \hat{P}_{c.g.}\}, can be made using the coarse-grained coherent state measurements from eq. (37), i.e. \{\hat{A}_a\} with suitable envelope functions $f_a$ such that $[\hat{A}_a, \hat{A}_{a'}] \approx 0$. 

\[
\begin{array}{|c|c|c|}
\hline
\hat{A} = \hat{X}^\delta & \hat{A} = \hat{P}^\kappa \\
\hline
\hat{B} = \hat{X}^\delta & V(0) = 1 & V(0) < 1 \\
& V(T \to \infty) < 1 & V(T \to \infty) = 1 \\
\hat{B} = \hat{P}^\kappa & V(t) = \text{const} < 1 & V(t) = 1 \\
\hline
\end{array}
\]

FIG. 4. Husimi distribution in the complex plane (mesh with interval 1), immediately after a quadrature measurement with decreasing unsharpness $\delta$. Sharp measurements (small $\delta$) completely destroy the initial state, while unsharp measurements (large $\delta$) keep it intact.

FIG. 5. Overlap $V$ vs coarse-graining ring width $d$. For coherent initial states in the center of the second region $|\gamma = 3d/2\rangle$ the overlap approaches unity as more of the state’s probability distribution lies in the region. For initial states located on a border $|\gamma = d\rangle$ the overlap approaches a value close to 0.997. This is due to the artificial sharp boundary between the coarse-grained regions.
The resulting overlap for different choices of right combination of various NSIT conditions serves not correlation functions. In fact, we demonstrated that the Leggett-Garg inequalities, which are based on two-time for the violation of macrorealism than the well-known the statistical level, is in general a more reliable witness over, no-signaling in time (NSIT), i.e. non-invasiveness on strong interpretation of non-invasive measurability. More-

 increasing sharp measurements in coherent state space. regions in Fock space,
c
sufficiently large
n. Quadratic border functions lead to the classicality of measurements, already requires the notion of “closeness” or “neighborhood” of eigenvalues, and thereby an understanding of classical phase space. This notion itself stems from Hamiltonians that are spontaneously realized in nature and govern our physical world. The present definition of classicality mitigates this circularity with the choice of an a-priori set of classical measurements. However, it is an open question whether the presupposition of classical phase space can be avoided, or whether it is a fundamental requirement for understanding the quantum-to-classical transition.

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Appendix A: Proof that NSIT$_{0(1)2}$ is sufficient for NIC$_{0(1)2}$

Let us use the short notation $P_i(\pm) \equiv P_i(Q_i = \pm)$. Then, the correlations in NIC$_{0(1)2}$ can be written as

$$C_{02} = + P_{02}(+0, +2) + P_{02}(-0, -2)$$

$$- P_{02}(+0, -2) - P_{02}(-0, +2), \quad (A1)$$

and, for the variant with a measurement at $t_1$,

$$C_{02|1} = + P_{012}(+0, +2) + P_{012}(-0, -2)$$

$$- P_{012}(+0, -2) - P_{012}(-0, +2). \quad (A2)$$

Using NSIT$_{0(1)2}$, i.e. $P_{02}(Q_0, Q_2) = P_{012}(Q_0, Q_2)$, we immediately see that NSIT$_{0(1)2}$ is sufficient for $C_{02} = C_{02|1}$, and therefore for NIC$_{0(1)2}$.

FIG. 6. Overlap $V$ (cf. eq. 34) vs initial state $|\gamma\rangle$ for coarse-grained Fock measurements with different border functions $g(m)$, from top: $100m^2$, $10m^2$, $2m^2$, $m^2$, $2m, m$. Quadratic border functions are coarse in the coherent state space and therefore not as invasive. Linear border functions lead to increasingly sharp measurements. The oscillations are caused by the fact that the presented type of coarse-graining works better when the initial state is located in the center of a bin. Dips in the overlap occur when the initial state sits at the border between two bins.

VI. CONCLUSION AND OUTLOOK

In contrast to a still widespread belief, we showed that the assumption of macrorealism per se is implied by a strong interpretation of non-invasive measurability. Moreover, no-signaling in time (NSIT), i.e. non-invasiveness on the statistical level, is in general a more reliable witness for the violation of macrorealism than the well-known Leggett-Garg inequalities, which are based on two-time correlation functions. In fact, we demonstrated that the right combination of various NSIT conditions serves not only as necessary but also sufficient condition for a macro-realistic model for measurements at the predefined time instants accessible in the experiment. We then derived operational criteria for the measurement operators and the system Hamiltonian, whose fulfillment guarantees that no violation of macrorealism can in principle be observed. We argued that these conditions can be used to define the “classicality” of measurements, and by extension, of the system’s time evolution. Finally, we showed that the classicality of measurements is arbitrarily well fulfilled by suitably coarse-grained versions of quantum measurements.

While our results suggest that an experimental demonstration of non-classicalities requires either very precise measurements or a complex time evolution, a general proof of this trade-off (in terms of experimental control parameters) is still missing. Moreover, coarse-graining, which leads to the classicality of measurements, already requires the notion of “closeness” or “neighborhood” of eigenvalues, and thereby an understanding of classical phase space. This notion itself stems from Hamiltonians that are spontaneously realized in nature and govern our physical world. The present definition of classicality mitigates this circularity with the choice of an a-priori set of classical measurements. However, it is an open question whether the presupposition of classical phase space can be avoided, or whether it is a fundamental requirement for understanding the quantum-to-classical transition.
Appendix B: Overlaps for quadrature measurements

In the following we will give analytical values for the overlap for different combinations of coarse-grained \( \hat{X}^\delta \) and \( \hat{P}^\kappa \) measures, as defined by eq. (35) and eq. (36), acting on a particle with initial state \( \langle x|\psi \rangle = \pi^{1/2} \exp(-x^2/(2\sigma^2)) \). In between the measurements we apply a unitary generated by a free Hamiltonian \( U_T = \exp(-itp^2/2m) \). There are four combinations:

- \( \hat{A} = \hat{X}^\delta, \hat{B} = \hat{X}^\delta \). Here the overlap starts at \( V(0) = 1 \), but approaches the value

\[
\lim_{t \to \infty} V(t) = \frac{4\delta^2(\delta^2 + \sigma^2)}{(2\delta^2 + \sigma^2)^2}.
\]

(B1)

The effect of the measurement only becomes apparent with time evolution.

- \( \hat{A} = \hat{P}^\kappa, \hat{B} = \hat{X}^\delta \). The overlap starts at

\[
V(0) = \frac{4\kappa^2(\delta^2 + \sigma^2)[\kappa^2(\delta^2 + \sigma^2) + 1]}{[2\kappa^2(\delta^2 + \sigma^2) + 1]^2}.
\]

(B2)

and approaches 1 for \( t \to \infty \). The momentum measurement changes the spatial distribution once, but with wave packet expansion the impact becomes less apparent.

- \( \hat{A} = \hat{X}^\delta, \hat{B} = \hat{P}^\kappa \). The overlap is constant in time at the value

\[
V = \frac{4\delta^2(\kappa^2\sigma^2 + 1)[\delta^2(\kappa^2\sigma^2 + 1) + \sigma^2]}{[2\delta^2(\kappa^2\sigma^2 + 1) + \sigma^2]^2}.
\]

(B3)

since \( [\hat{P}^\kappa, \hat{H}] = 0 \).

- \( \hat{A} = \hat{P}^\kappa, \hat{B} = \hat{P}^\kappa \). The overlap is constant at 1, a measurement in \( \hat{P} \) cannot be detected by a second \( \hat{P} \) measurement, as again, \( [\hat{P}^\kappa, \hat{H}] = 0 \).

These examples reaffirm the importance of the selection of multiple final measurements.

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Let us now assume the existence of hidden parameters \( \lambda(t) \) that define all physical properties. MRps is fulfilled if the macro-observable is a deterministic function \( Q = Q(\lambda(t)) \). There are two conceivable scenarios: (i) Deterministic time evolution of \( \lambda \), causing deterministic time evolution of the macro-observable \( Q(\lambda(t)) \). (ii) Stochastic time evolution of \( \lambda \), where some intrinsic randomness generates random jumps in \( \lambda \). We still have a deterministic dependency \( Q(\lambda) \), but \( Q(\lambda(t)) \) appears stochastic. In both cases MRps is fulfilled, since the system is in a single macrostate, as described by \( Q = Q(\lambda(t)) \), at all times. The condition for NIM then reads \( Q(\lambda(t)) = Q(\lambda_H(t)) \), where \( \lambda_H(t) \) are the hidden parameters after a history \( H \) of non-invasive measurements.

A simple example for this are the Pauli matrices with \( \hat{A} = \hat{\sigma}_x, \hat{B} = \hat{\sigma}_y \). Then, \( [\hat{A}, \hat{B}] = 2i\hat{\sigma}_z \) and \( [\hat{A}\hat{B}, \hat{B}\hat{A}] = 0 \). Although the first commutator is non-zero, NSIT \( (0)0 \) is trivially fulfilled. The physical interpretation of a \( \hat{\sigma}_x \) measurement (or rather, its corresponding POVM element 1) is a single-qubit operation without a measurement outcome.

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