Two alternatives of spontaneous chiral symmetry breaking in QCD

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Considering QCD in an Euclidean box, the mechanism of spontaneous breaking of chiral symmetry (SBχS) is analyzed in terms of average properties of lowest eigenstates of the Dirac operator. A formal analogy between the pion decay constant and conductivity in disordered systems is established. It follows that SBχS results from a subtle balance between the density of Euclidean quark states and their mobility. SBχS can be realized either with \( <\bar{q}q> = 0 \), provided the low density of states is compensated by a high mobility, or with a non-vanishing condensate, provided the mobility is suppressed. It is conjectured that the first case corresponds to extended whereas the latter case to (weakly) localized quark states.

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The subject of this note is the mechanism of flavour symmetry breaking in confining vector-like gauge theories. We refer to QCD, but we specify neither the gauge nor the representation to which belong Dirac fermions \( q(x) \). This representation is assumed to be complex and there are \( N \) identical fermion species (flavours) \( q_i(x) \) with a common mass \( m \). Under these conditions the vector symmetry \( U_V(N) \) remains unbroken \( (\square) \) and for \( m \to 0 \) one recovers the chiral symmetry \( SU_L(N) \times SU_R(N) \times U_V(1) \). We assume the latter to be spontaneously broken. In QCD with at least three flavours, SBχS follows from the absence of coloured physical states \( (\square) \). One proves that anomalous Ward identities imply \( (\square) \) the existence of massless Goldstone bosons (pions) coupled to the conserved axial currents, \( <0|A_i^\mu|\pi^j, p> = i\delta^{ij} F_0 p_\mu \). The standard wisdom however goes well beyond this established fact, assuming, in addition, that SBχS is triggered by a formation of a large condensate \( <\bar{q}q> \) \( (\square) \). No theoretical proof of this assumption is available and the first clear experimental test of the existence of a non-negligible \( \bar{q}q \) condensate in the vacuum is still awaited \( (\square) \). Some theoretical evidence does exist of \( \bar{q}q \) condensation on a lattice, but its interpretation is complicated by a triple extrapolation procedure, by the quenching and, perhaps, by the problem of fermion doublers, which obscures the description of SBχS within the lattice regularization.

In the following, the possible interplay between SBχS and \( \bar{q}q \) condensation is reconsidered from the onset, exploiting the analogy between vector-like gauge theories in continuum Euclidean space-time and disordered systems. It will be shown that the formation of a \( \bar{q}q \) condensate is not the only possible mechanism of SBχS: The asymmetric vacuum and massless Goldstone bosons coupled to the axial currents can occur even if \( <\bar{q}q> = 0 \). Nature provides examples of a similar situation: There is no symmetry reason forcing the spontaneous magnetization of an anti-ferromagnet to vanish, and, yet, it does vanish as a consequence of a particular magnetic order in the ground state. Different order parameters are then necessary to describe the system. Similarly, a complete description of SBχS in QCD may require order parameters other than \( <\bar{q}q> \). We shall concentrate on the two-point left-right correlation function

\[
\Delta_{\mu\nu}(q)\delta^{ij} = i \int dx e^{iqx} <\Omega|T L^{\mu}_i(x) R^{\nu}_j(0)|\Omega>,
\]

where \( L = \frac{1}{2}(V - A) \) and \( R = \frac{1}{2}(V + A) \) are the Noether currents generating left and right chiral transformations respectively. \( (\Omega > \) denotes the vacuum of the massive theory and \( |0> \) stands for its limit \( m \to 0 \).) In the chiral limit, the correlator \( (\square) \) must vanish, unless the vacuum \( |0> \) is asymmetric: \( \Delta_{\mu\nu} \) can be expressed as a commutator of the axial charge and represents a (non-local) order parameter. On the other hand, if there is a Goldstone boson coupled to the axial current, it necessarily shows up as a pole in the correlator \( (\square) \). In particular, taking in Eq. \( (\square) \) \( q_\mu = 0 \) and going to the chiral limit, one obtains

\[
\lim_{m \to 0} \Delta_{\mu\nu}(0) = \frac{1}{4} \eta_{\mu\nu} F_0^2.
\]

Consequently, the expression \( (\square) \) represents an order parameter of a particular type: Its nonvanishing is not only a sufficient but also a necessary condition of SBχS.

From now on, the theory will be considered in an Euclidean box \( L \times L \times L \times L \) with (anti)periodic boundary conditions (up to a gauge), and the integration over fermions will be performed first. This leads to a quantum mechanical problem of a single quark in a random gluonic background \( G_\mu(x) \), which is defined by the hermitean Hamiltonian

\[
H = \gamma_\mu (\partial_\mu + i G_\mu^a t^a).
\]

The result of the integration over quarks can be expressed in terms of eigenvalues \( \lambda_n \) and orthonormal eigenvectors \( \phi_n(x) \) of the Dirac Hamiltonian \( H \). The spectrum is symmetric around the origin : \( \gamma_5 \phi_n = \phi_{-n}, \lambda_n = -\lambda_{-n} \).

The resulting expression should then be averaged over all gluon configurations:

\[
<< X[G] >> = \int d[G] \exp(-S_Y[M[G]]) \prod_{\lambda_n > 0} (m^2 + \lambda_n^2)^N X[G].
\]
The fact that this integral involves a positive probability measure suggests a possible analogy with disordered systems. Following this way, the $\bar{q}q$ condensate can be written as

$$< \bar{q}q >= -\lim_{m \to 0, L \to \infty} \frac{1}{L^2} <\sum_{n} \frac{m}{m^2 + \lambda_n^2}>,$$  \hspace{1cm} (5)$$

whereas the order parameter (2) becomes

$$F_0^2 = \lim_{m \to 0, L \to \infty} \frac{1}{L^4} <\sum_{k_n} \frac{m}{m^2 + \lambda_k^2 + \lambda_n^2} J_{kn}>,\hspace{1cm} (6)$$

where

$$J_{kn} = \frac{1}{4} \sum_{\mu} \int dx \phi_k^\dagger(x) \gamma_{\mu} \phi_n(x)^2.\hspace{1cm} (7)$$

One observes that both order parameters (5) and (6) are merely sensitive to the infrared end of the Dirac spectrum, $|\lambda_n| < \epsilon$. In particular, the possible ultraviolet divergences in the sum over eigenstates of $H$ become irrelevant in the chiral limit.

A possible interpretation of Eq. (5) emerges, if one considers the Dirac Hamiltonian (5) as a generator of evolution in a fictitious time $t$ added to the four Euclidean space coordinates $x_{\mu}$. In the corresponding 4+1 dimensional space-time one can switch on a homogeneous colour singlet electric field $E_{\mu} \cos(\omega t)$, adding to $H$ a time-dependent perturbation $\delta H = i\gamma_{\mu} E_{\mu} \sin(\omega t)$, where $E_{\mu}$ is a constant. $J_{kn}$ is then related to the probability of the transition $|k| \to |n|$ induced by the perturbation $\delta H$, $|\omega| \to 0$ at the end). This suggests that in the fictitious 4+1 dimensional space-time, the dynamics of $SB\chi S$ might be related to transport properties of massless quarks in a random medium characterized by a coloured magnetic type disorder $G^\mu_\nu(x)$ with the probability distribution (5). A deeper insight into this formal analogy may be obtained reexpressing the order parameters (5) and (6) in terms of global characteristics of the infrared part of the Dirac spectrum. Denoting by $\sum_{\epsilon}^\epsilon$ the sum over all states with “energy” $|\lambda| < \epsilon$, Eq. (5) can be written as

$$F_0^2 = \pi^2 \lim_{\epsilon, \epsilon' \to 0, L \to \infty} \frac{1}{4 \epsilon \epsilon' L^4} <\sum_{k} \sum_{n} \epsilon' J_{kn}>>. \hspace{1cm} (8)$$

Setting $\epsilon = \epsilon'$ in this equation, then dividing and multiplying by the square of the average number of states with $|\lambda| < \epsilon$, $N(\epsilon, L) = \epsilon' \sum_{n} >>$, one finally obtains

$$F_0^2 = \pi^2 \lim_{\epsilon \to 0, L \to \infty} L^4 J(\epsilon, L) \rho^2(\epsilon, L). \hspace{1cm} (9)$$

Here, $J(\epsilon, L)$ stands for the transition probability (5) averaged over all pairs of initial and final states with energy $|\lambda| < \epsilon$ and over the disorder,

$$J(\epsilon, L) = \frac{1}{N^2(\epsilon, L)} <\sum_{nk} J_{nk}>>. \hspace{1cm} (10)$$

$J$ measures the “mobility” of quarks at the infrared edge of the Dirac spectrum, subject to the action of the electric field. $\rho(\epsilon, L)$ in Eq. (6) denotes the density of states, i.e. the number of states per unit “energy” $\epsilon$ and unit volume: $\rho = N/2\pi L^4$. It is well known (10) that the density of states defines the condensate (5):

$$< \bar{q}q >= -\pi \lim_{\epsilon \to 0, L \to \infty} \rho(\epsilon, L). \hspace{1cm} (11)$$

Eq. (10) can be viewed as an ultrarelativistic ($m = 0$) version of the Kubo-Greenwood formula for electric conductivity (see e.g. (2), (3), (4)). The latter usually deals with the non-relativistic current $\phi \nabla_\mu \phi/m$ (instead of $\phi^\dagger \gamma_\mu \phi$ in Eq. (3)) and it involves the neighbourhood of the Fermi energy instead of the infrared end of the Dirac spectrum. The relativistic transition probability $J_{kn}$ satisfies for each gauge field configuration a completeness sum rule $\sum_{n} J_{kn} = 1$ implying that the “mobility” $J(\epsilon, L)$ should be bounded by the inverse number of states:

$$J(\epsilon, L) \leq \frac{1}{N(\epsilon, L)}. \hspace{1cm} (12)$$

There is no analogous restriction in the non-relativistic theory of conductivity.

We now turn to the double limit that appears in the expressions (8) and (11) of the order parameters $F_0^2$ and $< \bar{q}q >$. $SB\chi S$ can only occur if the infinite volume limit is performed first. The result then depends on the degree of accumulation of small eigenvalues $\lambda_n$ around 0 as $L \to \infty$. The latter may be characterized by an index $\kappa$ indicating how rapidly the n-th (positive) eigenvalue averaged over the disorder vanishes, if $L \to \infty$ and $n$ is kept fixed: We shall say that $\lambda_n$ belongs to the “$\kappa$-band” if $<\lambda_n>>$ behaves for large $L$ as $L^{-\kappa}$. We assume that the spectrum of the Dirac operator consists of several “$\kappa$-bands”. The case $\kappa = 1$ is characteristic of perturbation theory and it has been shown (5) that $\kappa \geq 1$. We are interested in small eigenvalues that belong to the maximal $\kappa$-band. The corresponding index $\kappa$ determines the leading behavior of the number of states $N(\epsilon, L)$ for large $L$ and small $\epsilon$: It follows from the definition that the number of states in a $\kappa$-band becomes a function of a single scaling variable $\epsilon L^\kappa$. Since on the other hand, one expects the number of states $N(\epsilon, L)$ to be proportional to the volume $L^4$, one should have

$$N(\epsilon, L) = (2\epsilon/\mu)^{1/\kappa}(\mu L)^4 + \ldots, \hspace{1cm} (13)$$

where $\mu$ is a mass scale and the dots stand for higher powers of $\epsilon$. This formula has three important consequences: i) The first one concerns the condition of existence of $SB\chi S$, i.e. the requirement that the limit in Eq. (10) be non-vanishing and finite. Using inside Eq. (10) the unitarity bound (12), one obtains
\[
F_0^2 \leq \pi^2 \mu^2 \lim_{\epsilon \to 0} \left( \frac{2\epsilon}{\mu} \right)^{4/\kappa - 2}.
\]

Hence, SBS\'S can take place only provided \( \kappa \geq 2 \). ii) The second consequence concerns the existence of a non-zero quark condensate. Using Eq. (13) in the formula (11), one gets

\[
< \bar{q}q > = -\pi \mu^3 \lim_{\epsilon \to 0} \left( \frac{2\epsilon}{\mu} \right)^{4/\kappa - 1}.
\]

Hence, the chiral condensate is formed if and only if \( \kappa = 4 \), a known result obtained in [10]. The case \( \kappa > 4 \) would lead to an infinite condensate and it can likely be excluded [11]. Indeed, the well known Ward identity leads to the expression

\[
< \bar{q}q > \delta^{ij} = \lim_{m \to 0} k_i \int dxe^{ikx} < A^i_{\mu}(x)P^j(0) >
\]

and one usually argues that the zero mass limit of the two point function on the r.h.s. of this equation should exist at least for some (non-exceptional) momentum transfer \( k_\mu \). Hence, \( \kappa > 4 \) would represent a too strong singular singularity incompatible with general properties of a field theory. iii) The third consequence of Eq (13) concerns the properties of the effective theory which describes the low energy dynamics in terms of Goldstone boson fields, whenever SBS\'S occurs, i.e. for \( \kappa \geq 2 \). One usually assumes that the corresponding effective Lagrangian [12], [13], is analytic in Goldstone boson fields and in scalar sources, i.e. in the quark mass. In this case, the leading small \( m \) behaviour of the partition function (given by the tree approximation) is necessarily analytic and the same should be true for the leading small \( \epsilon \) behavior of the number of states \( N(\epsilon, L) \). This requires \( 4/\kappa = \text{integer} \), selecting the values \( \kappa = 4 \) and \( \kappa = 2 \) as the two candidates leading to SBS\'S and to an analytic effective Lagrangian. For \( \kappa = 2 \), the density of states is suppressed, \( \rho(\epsilon, L) = 2\epsilon \mu^2 \) and, consequently, the condensate \( < \bar{q}q > \) vanishes. On the other hand, \( \kappa = 4 \) is characterized by a constant \( \rho \) leading to a non-vanishing condensate \( < \bar{q}q > = -\pi \mu^3 \). A given \( \kappa \)-band \( (\kappa = 2 \text{ or } 4) \) contributes to \( F_0^2 \) if the corresponding density of states is modulated by an appropriate quark mobility \( J(\epsilon, L) \) following Eq. (1): For \( \kappa = 2 \), SBS\'S can take place despite \( < \bar{q}q > = 0 \), provided the small density of states is compensated by a high mobility, behaving typically like the inverse number of states. In the \( \kappa = 4 \)-band, the mobility should on the other hand be suppressed by a factor of volume and should be independent of \( \epsilon \). One can, at least, conceive a class of models which realize the different infrared behavior of \( J(\epsilon, L) \) as required by the two alternatives of SBS\'S. The guideline will be a possible connection between mobility and localization properties [14] of Euclidean low energy Dirac eigenstates \( \phi_n(x) \) suggested by the non-relativistic theory of transport in disordered systems (see [15] for a recent review).

1) The \( \kappa = 2 \)-band. Since the density of states is suppressed, it is more convenient to use a particular form of Eq. (11):

\[
F_0^2 = \pi^2 \mu^2 \lim_{\epsilon \to 0} \lim_{L \to \infty} N(\epsilon, L)J(\epsilon, L)
\]

which holds for \( \kappa = 2 \). Notice that \( \sigma(\epsilon, L) = NJ \) represents the total probability of transition from a given initial state \( i > \) with \( |\lambda| < \epsilon \) to any state with \( |\lambda| < \epsilon \). The upper bound (12) implies \( \sigma \leq 1 \). Now, let us assume that the transition between any two states from the \( \kappa = 2 \)-band can occur with a non-negligible intensity \( J_{kn} \), provided the two states are close enough in energy. This is conceivable for extended (delocalized) states. Furthermore, transitions between the infrared edge of the \( \kappa = 2 \)-band and the perturbative \( \kappa = 1 \)-band (which is always present) should be suppressed because of a large difference in energy. Under these circumstances, one may expect that for \( |\lambda| < \epsilon \) the sum \( \sum_f J_{IJ} \) will be dominated by states with energy \( |\lambda| < \epsilon \), whereas the contribution of remaining states will be relatively suppressed as \( N(\epsilon, L) \to \infty \). In this case, the bound \( \sigma(\epsilon, L) \leq 1 \) should be nearly saturated : \( \sigma(\epsilon, L) = 1 - 0(1/N) \). Only a much weaker form of this assumption is actually needed: It is sufficient that for large \( L \) and small \( \epsilon \), \( \sigma(\epsilon, L) \) admits an expansion

\[
\sigma(\epsilon, L) = NJ = w(1/N) + 2\hat{w}(1/N)\epsilon + \cdots
\]

with \( 0 < w(0) \leq 1 \). (The particular case described above corresponds to \( w(0) = 1 \).) Notice that this expansion, as well as previous reasoning, only applies if the number of states \( N(\epsilon, L) \) is sufficiently large. In particular, Eq. (15) contains no information about the limit \( \epsilon \to 0 \), \( L \to \infty \), in which \( \sigma \) should vanish (chiral symmetry should be restored) since there are no states with \( |\lambda| < \epsilon \). The final result then reads \( F_0^2 = \pi^2 \mu^2 w(0) \), where \( w(0) \) is the probability defined in Eq. (13) and \( L \) is the mass scale that controls the leading dependence of the number of states \( N \) on \( \epsilon \) and on \( L \). Assuming the latter to be of the order of the intrinsic QCD scale \( \Lambda_{QCD} \sim 300 \text{ MeV} \), one obtains \( w(0) \sim 0.01 \), whereas \( w(0) \leq 1 \) corresponds to \( \mu \geq 30 \text{ MeV} \).

2) The \( \kappa = 4 \)-band. Since the density of states stays constant, Eq (11) requires \( L^4J(\epsilon, L) \) to be independent of \( \epsilon \) and of \( L \). This suppression of mobility by the factor of volume hardly fits into the previous picture, in which all states with \( |\lambda| < \epsilon \) participated in the transition. Instead, the intensity \( J_{kn} \) should now be suppressed for most pairs of initial and final states, even if they are close in energy. The following scenario attempts to explain naturally such a suppression as a consequence of a kind of localization of states in the \( \kappa = 4 \)-band.

i) Let us first assume that to each state \( |n \rangle \), one can associate a “center” \( C_n \) localized at a point \( x_n \), all centers forming a (random) lattice \( \mathbb{Z}^2 \) covering the hypercube \( L \times L \times L \). The centers \( C_n \) may be viewed
as localized defects or instantons their precise nature and origin are at this stage irrelevant.

ii) Next, we assume that there is a one-to-one correspondence between states from the infrared edge of the $\kappa = 4$ band and the points of the five-dimensional lattice $Z_5 \times Z_2^2 \times Z_2$ collecting the levels $\lambda_n$. Notice that i) and ii) yield the correct number of states: $N(\epsilon, L) = 2\epsilon l^4 L^4$.

iii) Finally, $J_{kn}$ will be assumed to be independent of energy and sufficiently suppressed if the distance $r_{kn} = |\vec{x}_k - \vec{x}_n|$ between the corresponding two centers $C_k$ and $C_n$ grows. Only a rather weak form of this suppression is needed: Writing, for example, $J_{kn} = F(r_{kn})$, the function $F$ should be integrable in the infinite volume limit, $\lim_{L \to \infty} \int d^4xF(r) = l^4$. The new length scale $l$ characterizes the localization of the low $\lambda$ states from the $\kappa = 4$ band. It is now straightforward to use Eq. (10) and to estimate the mobility

$$J(\epsilon, L) = 2\epsilon l^4 \frac{l^4}{N} = \frac{l^4}{L^4},$$

making apparent the required suppression by the factor of volume. The same result can be obtained from the expansion (18) setting $\mu(1/N) = 0$ and writing $\mu(0) = \mu l^4$. Hence, the above argument of localization provides a justification of the absence of the first term in the expansion (18) in the case $\kappa = 4$. The Kubo-type formula (9) now reduces to the simple relation

$$F_0 = \pm <\bar{q}q> l^2,$$

where $l$ is the characteristic length of localization defined above. If the latter happens to diverge $(l \sim L)$, states become delocalized, the condensate must vanish and one recovers the $\kappa = 2$ case discussed before.

It is worth noting that the alternative of vanishing condensate $\bar{q}q$ (i.e. $\kappa = 2$) does not imply the vanishing of all local condensates of the type $\bar{q}q$. This is best illustrated by the case of the dimension 5 “mixed condensate” which can be expressed by a formula similar to Eq. (11)

$$<q\sigma_{\mu\nu}G_{\mu\nu}q> = -\pi \lim_{\epsilon \to 0} \lim_{L \to \infty} G_{\parallel}(\epsilon, L) \rho(\epsilon, L),$$

where $G_{\parallel}$ denotes the mean value of $\sigma_{\mu\nu}G_{\mu\nu}(x)$ in a state $\phi_n(x)$ with $|\lambda_n| < \epsilon$, averaged over all such states and over the disorder. For $\kappa = 2$, the mixed condensate will remain non-zero, provided $G_{\parallel}(\epsilon, L)$ blows up as $1/\epsilon$, indicating a possible importance of a large gauge field configurations parallel to the quark spin at the infrared edge of the $\kappa = 2$ band. (In the case $\kappa = 4$, a constant $G_{\parallel}$ is sufficient to guarantee a finite mixed condensate.) Let us recall that a non-vanishing of some local condensates of the type (21) is required by the consistency of the pattern of $SB_4\chi S$ with the short distance properties of the theory in the limit $N_c \to \infty$.

In QCD, the theoretical question of the importance of $<\bar{q}q>$ is of a direct phenomenological relevance. The alternative of vanishing (or tiny) condensate is at present not ruled out by available experimental data (see references therein). The decisive test of the role of $<\bar{q}q>$ in $SB_4\chi S$ will hopefully come from new high precision low energy $\pi\pi$ scattering experiments currently under preparation. We believe that the existence of theoretical alternatives described in this Letter demonstrates the fundamental importance of these and similar experimental tests.

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