MODULAR FORMS FROM NOETHER-LEFSCHETZ THEORY

FRANÇOIS GREER

Abstract. We enumerate smooth rational curves on very general Weierstrass fibrations over hypersurfaces in projective space. The generating functions for these numbers lie in the ring of classical modular forms. The method of proof uses topological intersection products on a period stack and the cohomological theta correspondence of Kudla and Millson for special cycles on a locally symmetric space of orthogonal type. The results here apply only in base degree 1, but heuristics for higher base degree match predictions from the topological string partition function.

1. Introduction

Locally symmetric spaces of noncompact type are special Riemannian manifolds which serve as classifying spaces for (torsion-free) arithmetic groups. As such, their geometry and topology has been studied intensely from several different perspectives. By a famous theorem of Baily and Borel, if such a manifold admits a parallel complex structure, then it is a complex quasi-projective variety. In this paper, we study more general indefinite orthogonal groups, which act on Hodge structures of even weight.

Theorem 1. Let $\Lambda$ be an integral lattice inside $\mathbb{R}^{2,l}$, and let $\Gamma$ be a congruence subgroup of $O(\Lambda)$. Then the locally symmetric space $\Gamma\backslash O(2,l)/O(2) \times O(l)$ is a quasi-projective variety.

These Hermitian symmetric examples have played a central role in classical moduli theory. For instance, moduli spaces of polarized K3 surfaces, cubic fourfolds, and holomorphic symplectic varieties are all contained within these Baily-Borel varieties as Zariski open subsets. Theorem 1 provides natural compactifications for these moduli spaces and bounds on their cohomology.

One can interpret $O(2,l)/O(2) \times O(l)$ as the set of 2-planes in $\mathbb{R}^{2,l}$ on which the pairing is positive definite. The presence of the integral lattice $\Lambda$ allows us to define a sequence of $\mathbb{R}$-codimension 2 submanifolds, indexed by $n \in \mathbb{Q}_{>0}$ and $\alpha \in \Lambda^\vee/\Lambda$.

$$C_{n,\alpha} := \Gamma \backslash \bigcup_{\substack{v \in \Lambda^\vee, \, v + \Lambda = \alpha \\text{ and } (v,v) = -n}} v^\perp$$

By another theorem of Borel, there are finitely many $\Gamma$-orbits of lattice vectors with fixed norm, so the above union is finite in the arithmetic quotient. Each $C_{n,\alpha}$ is isomorphic to a locally symmetric space for $O(2,l-1)$, so by Theorem 1, it is an algebraic subvariety of $\mathbb{C}$-codimension 1 called a Heegner divisor. The classes of
these divisors in the Picard group satisfy non-trivial relations from the Howe theta correspondence between orthogonal and symplectic groups:

**Theorem 2.** \[4\] The formal $q$-series with coefficients in

$$\text{Pic}_Q(\Gamma \backslash O(2,l)/O(2) \times O(l)) \otimes \mathbb{Q}[^V/\Lambda]$$

given by

$$e(V) e_0 + \sum_{n,\alpha} [C_{2n,\alpha}] e_\alpha q^n$$

transforms like a $\mathbb{Q}[\Lambda^V/\Lambda]$-valued modular form with respect to the Weil representation of the metaplectic group $Mp_2(\mathbb{Z})$. Here, $\{e_\alpha\}$ denotes the standard basis for $\mathbb{Q}[\Lambda^V/\Lambda]$, and $e(V)$ is the Euler class of the (dual) tautological bundle of positive definite 2-planes.

In other words, the $q$-series above lies in the finite dimensional subspace

$$\text{Mod} \left( 1 + \frac{l}{2},Mp_2(\mathbb{Z}),\mathbb{Q}[^V/\Lambda] \right) \otimes \text{Pic}_Q(\Gamma \backslash O(2,l)/O(2) \times O(l)).$$

Theorem 2 has been used to describe the Picard group of moduli spaces [12], and also has applications to enumerative geometry, initiated by [16]. In this paper, we move beyond the Hermitian symmetric space to more general symmetric spaces of orthogonal type. These no longer have a complex structure, and their arithmetic quotients are no longer algebraic, but they still have a theta correspondence, and thus an analogous modularity statement for special cycles in singular cohomology:

**Theorem 3.** [15] Assume for convenience of exposition that $\Lambda \subset \mathbb{R}^{p,l}$ is even and unimodular. Then the formal $q$-series

$$e(V) + \sum_{n \geq 1} [C_{2n}] q^n \in \mathbb{Q}[[q]] \otimes \mathbb{Q} H^p(\Gamma \backslash O(p,l)/O(p) \times O(l),\mathbb{Q})$$

lies in the finite-dimensional subspace of modular forms:

$$\text{Mod} \left( \frac{p+l}{2},SL_2(\mathbb{Z}) \right) \otimes H^p(\Gamma \backslash O(p,l)/O(p) \times O(l),\mathbb{Q}).$$

Here, $e(V)$ is the Euler class of the (dual) tautological bundle of $p$-planes.

This paper uses Theorem 3 to enumerate smooth rational curves on certain elliptically fibered varieties $X \to Y$. We give a general formula which applies Weierstrass fibrations over hypersurfaces in projective space. The answers are honest counts, not virtual integrals, and are expressed in terms of $q$-expansions of modular forms.

All period domains $D$ for smooth projective surfaces with positive geometric genus admit smooth proper fibrations

$$D \to O(p,l)/O(p) \times O(l).$$

The Noether-Lefschetz loci in $D$ are the pre-images of special sub-symmetric spaces of $\mathbb{R}$-codimension $p$. This provides a valuable link between moduli theory and the cohomology of locally symmetric spaces. We expect the ideas developed in this paper to compute algebraic curve counts on a broad class of varieties.
Let \( Y \subset \mathbb{P}^{m+1} \) be a smooth hypersurface of degree \( d \) and dimension \( m \geq 2 \). For an ample line bundle \( \mathcal{L} = \mathcal{O}_Y(k) \), a Weierstrass fibration over \( Y \) is a hypersurface

\[
X \subset \mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_Y)
\]
cut out by a global Weierstrass equation (see Section 2 for details). For general choice of coefficients, \( X \) is smooth of dimension \( m+1 \), and the morphism \( \pi : X \to Y \) is flat with generic fiber of genus one. Since \( \pi \) admits a section \( i : Y \to X \), the generic fiber is actually an elliptic curve.

The second homology group of \( X \) is given by

\[
H_2(X, \mathbb{Z}) \cong H_2(Y) \oplus \mathbb{Z}f = \mathbb{Z}\ell + \mathbb{Z}f,
\]
where \( \ell \) is the line class on \( Y \) pushed forward via \( i \), and \( f \) is the class of a fiber. We begin by posing the following naive question:

**Question 4.** How many smooth rational curves are there on \( X \) in the homology class \( \ell + nf \)?

The deformation theory of curves on \( X \) allows us to estimate when this question has a finite answer. First order deformations of \( C \subset X \) are parametrized by \( H^0(C, N_{C/X}) \), and the obstruction to lifting a deformation to all orders lies in \( H^1(C, N_{C/X}) \). Hence, the dimension of the moduli space of curves in \( X \) is bounded below by a sheaf theoretic Euler characteristic, which can be computed topologically using the Hirzebruch-Riemann-Roch formula:

\[
h^0(C, N_{C/X}) - h^1(C, N_{C/X}) = \int_C c_1(T_X) + (1-g)(\dim X - 3).
\]

The adjunction formula lets us compute \( c_1(T_X) = -K_X \) for a Weierstrass fibration:

\[
K_X = \pi^*(K_Y + c_1(\mathcal{L})).
\]

**Remark 5.** Since \( K_X \) is pulled back from \( Y \), we have \( K_X \cdot f = 0 \), so our dimension estimate is independent of \( n \). This feature holds more generally for any morphism \( \pi \) with \( K \)-trivial fibers.

We expect a finite answer to Question 4 whenever

\[
0 = -K_X \cdot (\ell + nf) + (m - 2) \iff k = 2m - d.
\]

Recall that \( \mathcal{L} = \mathcal{O}_Y(k) \) was the ample line bundle used to construct the Weierstrass fibration \( X \to Y \), so for the rest of the paper, we require that \( k = 2m - d > 0 \). Note that \( X \) is Calabi-Yau if and only if \( \dim(X) = 3 \). Our main result is the

**Theorem 6.** A very general Weierstrass model \( X \to Y \) constructed using \( \mathcal{L} = \mathcal{O}_Y(k) \) contains finitely many smooth rational curves in the class \( \ell + nf \), whose count we denote \( r_X(n) \). For \( k \leq 4 \), the generating series is given by

\[
\sum_{n \geq 1} r_X(n)q^n = \varphi(q) - \Theta(q),
\]
where \( \varphi(q) \in \text{Mod}(6k-2, SL_2(\mathbb{Z})) \), and \( \Theta(q) \in \mathbb{Q}[\theta_{A_1}, \theta_{A_2}, \theta_{A_3}]_{\leq k} \), a polynomial of weighted degree \( < k \).
Recall that for a lattice $A$, the associated theta series is given by

$$\theta_A(q) = \sum_{v \in A} q^{(v,v)/2},$$

and we assign the weight $\rho$ to the series $\theta_{A_\rho}$ for the root lattice $A_\rho$.

In short, the curve counts $r_X(n)$ are controlled by a finite amount of data, since $\text{Mod}(6k - 2, SL_2(\mathbb{Z}))$ and $\mathbb{Q}[\theta_{A_1}, \theta_{A_2}, \ldots]_{<k}$ are finite dimensional $\mathbb{Q}$-vector spaces. The series can be explicitly computed when $k \leq 3$. We record a few examples in Section 7 to illustrate the scope of applicability.

**Remark 7.** We expect that Theorem 6 can be extended to $k \leq 8$, but the statement is less tidy and involves the root systems $D_4$, $E_6$, and $E_7$.

The argument proceeds roughly as follows. The curves $C$ that we wish to count have the property that $\pi(C) \subset Y$ is a line. In other words, they can be viewed as sections of the elliptic fibration

$$\pi^{-1}(\pi(C)) \to \pi(C) \simeq \mathbb{P}^1.$$

As we vary over lines $L \subset Y$, we obtain a family of elliptic surfaces over the Fano variety of lines in $Y$:

$$\nu : \mathcal{S} \to F(Y).$$

We would like to count points $[L] \in F(Y)$ such that $\mathcal{S}[L] = \pi^{-1}(L)$ has a non-identity section, i.e. a non-trivial Mordell-Weil group. The Shioda-Tate sequence expresses the Mordell-Weil group of an elliptic surface in terms of its Néron-Severi lattice and the sublattice $V(S)$ spanned by vertical classes and the identity section:

$$0 \to V(S) \to \text{NS}(S) \to \text{MW}(S/\mathbb{P}^1) \to 0.$$  

The period domain for a given class of elliptic surfaces is related to a locally symmetric space, whence the modular form $\varphi(q)$, which counts surfaces with jumping Picard rank. To obtain the counts $r_X(n)$ we subtract contributions $\Theta(q)$ from surfaces with jumping $V(S)$, which are precisely those with $A_\rho$ singularities. Hence, the terms in the difference formula of Theorem 6 are matched to the groups in the sequence.

The paper is organized as follows. In Section 2 we review the relevant theory of elliptic fibrations and set up the tools for proving transversality of intersections in moduli. In Section 3 we review the theory of period domains and lattices in the
cohomology of elliptic surfaces. Section 4 is devoted to the deformation and resolution of $A_\nu$ singularities in families of surfaces, and we introduce the monodromy stack of such a family. Section 5 explains how Noether-Lefschetz intersection numbers on the period stack satisfy a modularity statement from Theorem 3. In Section 6 we use the fact that $k \leq 4$ to classify the singularities which occur in the family $\nu$ at various codimensions, and compute their degrees in terms of Schubert intersections. Finally, Section 7 explains how to compute the modular form $\varphi(q)$ when $k \leq 3$, and the general form of the correction term $\Theta(q)$.

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2. Elliptic Fibrations

We begin by reviewing some aspects of the Weierstrass equation for elliptic curves

\begin{equation}
    y^2z = x^3 + Axz^2 + Bz^3.
\end{equation}

The resulting cubic curve in $\mathbb{P}^2$ has a flex point at $[0 : 1 : 0]$, which is taken to be the identity of a group law in the smooth case. The curve is singular if and only if the right hand side has a multiple root, which occurs when $\Delta = 4A^3 + 27B^2 = 0$. These singular curves are all isomorphic to the nodal cubic, except for $A = B = 0$, which gives the cuspidal cubic.

To replicate this construction in the relative case, let $Y$ be a smooth projective variety, and $L \in \text{Pic}(Y)$ an ample line bundle. We form the $\mathbb{P}^2$ bundle

\[ \mathbb{P}(L^2 \oplus L^3 \oplus \mathcal{O}_Y) \rightarrow Y. \]

The same Weierstrass equation (1) makes sense for $x, y, z$ fiber coordinates, and

\[ A \in H^0(Y, L^4) \]
\[ B \in H^0(Y, L^6). \]

Let $X \subset \mathbb{P}(L^2 \oplus L^3 \oplus \mathcal{O})$ be the solution of the global Weierstrass equation and $\pi : X \rightarrow Y$ the morphism to the base. The fibers of $\pi$ are elliptic curves in Weierstrass form, and there is a global section $i : Y \rightarrow X$ given in coordinates by $[0 : 1 : 0]$, which induces a group law on each smooth fiber. Now, $\Delta = 4A^3 + 27B^2 \in H^0(Y, L^{12})$ cuts out a hypersurface in $Y$ whose generic fiber is a nodal cubic. The singularities of $\Delta$ occur along the smooth complete intersection $(A) \cap (B)$, and are analytically locally isomorphic to $$(\text{cusp}) \times \mathbb{C}^{m-2}.$$
The case of $m = 2$ and $d = 1$ is pictured below, with three fibers drawn over points in $\mathbb{P}^2$ in different singularity strata of $\Delta$. 

By the adjunction formula applied to $X \subset \mathbb{P}(L^2 \oplus L^3 \oplus \mathcal{O}_Y)$, 

\[ K_X = (K_{\mathbb{P}(L^2 \oplus L^3 \oplus \mathcal{O}_Y)} + [X])|_X \]

\[ = (\pi^* K_Y + 5 \pi^* c_1(L) - 3 \zeta) + (3 \zeta + 6 \pi^* c_1(L)) \]

\[ = \pi^*(K_Y + c_1(L)), \]

so the relative dualizing sheaf $\omega_{X/Y}$ is isomorphic to $\pi^* L$. By the adjunction formula applied to $Y \subset X$ through the section $i$, 

\[ K_Y = (K_X + [Y])|_Y \]

\[ = K_Y + c_1(L) + c_1(N_{Y/X}), \]

so the normal bundle $N_{Y/X}$ is isomorphic to $L^\vee$.

**Definition 8.** The parameter space for Weierstrass fibrations over $Y$ is given by the weighted projective space 

\[ W(Y, L) := (H^0(Y, L^4) \oplus H^0(Y, L^6) - \{0\}) / \mathbb{C}_{(2,3)}^\times. \]

In this paper, $X$ is always a general member of $W(Y, \mathcal{O}(k))$, where $Y \subset \mathbb{P}^{m+1}$ is a smooth hypersurface of degree $d$, and $k = 2m - d$. Since we are interested in counting curves in $X$ which lie over lines in $Y$, we will also consider elliptic surfaces $S \in W(\mathbb{P}^1, \mathcal{O}(k))$. For convenience, we gather some properties of these surfaces.

**Proposition 9.** For $S \in W(\mathbb{P}^1, \mathcal{O}(k))$ a smooth surface, 

\[ H^1(\mathcal{O}_S) = 0. \]

**Proof.** The Leray spectral sequence for the morphism $\pi : S \to \mathbb{P}^1$ yields 

\[ 0 \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(S, \mathcal{O}_S) \to H^0(\mathbb{P}^1, R^1\pi_* \mathcal{O}_S) \to 0. \]

Since $R^1\pi_* \mathcal{O}_S \simeq (\pi_* \omega_{S/\mathbb{P}^1})^\vee \simeq \mathcal{O}_{\mathbb{P}^1}(-k)$, all terms vanish. \[ \square \]

This allows us to compute the remaining Hodge numbers. Since $K_S \simeq \pi^* \mathcal{O}_{\mathbb{P}^1}(k-2)$, Noether’s formula says that 

\[ \chi(S, \mathcal{O}_S) = k = \frac{e(S)}{12}. \]

Alternatively, the topological Euler characteristic is equal to $\deg(\Delta) = 12k$, the number of singular fibers. This determines the last Hodge number 

\[ h^{1,1}(S) = 10k. \]
Theorem 10. Any elliptic surface $S \to \mathbb{P}^1$ is birational to a Weierstrass surface. Furthermore, there is a bijection between (isomorphism classes of) smooth relatively minimal surfaces and Weierstrass fibrations with rational double points.

Proof. This uses Kodaira’s classification of singular fibers. See [18]. □

It will be convenient for us to take a further quotient of $W(\mathbb{P}^1, \mathcal{O}(k))$ to account for changes of coordinates on the base.

Definition 11. The moduli space for Weierstrass surfaces over $\mathbb{P}^1$ is given by the (stack) quotient

$$W_k := W(\mathbb{P}^1, \mathcal{O}(k))/\text{PGL}(2).$$

Remark 12. Miranda showed in [17] that $\mathcal{S} \in W(\mathbb{P}^1, \mathcal{O}(k))$ is GIT stable with respect to $\text{PGL}(2)$ if and only if it has rational double points, so $W_k$ has a quasi-projective coarse space with good modular properties.

For any variety $Y \subset \mathbb{P}^{m+1}$, the locus of lines contained in $Y$ is called the Fano scheme of $Y$, and is denoted $F(Y) \subset G(1, m+1)$.

Theorem 13. For a general hypersurface $Y \subset \mathbb{P}^{m+1}$ of degree $d$, the Fano scheme is smooth of dimension $2m - d - 1 = k - 1$, for $k > 0$.

Proof. To study the general behavior, we construct an incidence correspondence

$$\Omega = \{(L, Y) : L \subset Y\} \subset G(1, m+1) \times \mathbb{P}^N.$$

The first projection $\Omega \to G(1, m+1)$ is surjective and has linear fibers of dimension $N - d - 1$, so $\Omega$ is smooth and irreducible of dimension $N + 2m - d - 1$. The second projection has fiber $F(Y)$ over $[Y] \in \mathbb{P}^N$. To get the desired dimension, it suffices to show that a general hypersurface $Y$ contains a line, so that the second projection $\Omega \to \mathbb{P}^N$ is surjective. This can be done by constructing a smooth hypersurface containing a line whose normal bundle is balanced as in [10]. □

Moreover, if we vary the hypersurface $Y$, then $F(Y)$ varies freely inside $G(1, m+1)$.

To be precise,

Definition 14. Let $Z \to B$ be a submersion of complex manifolds, and let $f : Z \to P$ be a family of immersions $\{f_b : Z_b \to P\}$. This deformation is called freely movable if for any $x_0 \in Z_0$ and any $v \in T_{f(x_0)}P$, there exists a 1-parameter subfamily $T \subset B$ and a section $x(t) \in Z_t$ such that $x(0) = x_0$ and $\frac{d}{dt}|_{t=0}(f \circ x) = v$.

The second projection $\Omega \to \mathbb{P}^N$ from Theorem 13 is generically smooth, and the family of embeddings $\Omega \to G(1, m+1)$ is freely movable because for any line $L \in F(Y_0)$ and tangent vector $v \in T_{[L]}G$, there is a curve of lines $\{L_t\}$ in the direction $v$. Since every line lies on a hypersurface, there is a deformation $Y_t$ such that $[L_t] \in F(Y_t)$. This property is useful for proving transversality statements, using the

Lemma 15. Let $(Z \to B, f : Z \to P,)$ be freely movable, and fix some subvariety $\Pi \subset P$. Then for general $b \in B$, $Z_b$ intersects $\Pi$ transversely.

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1 There is a natural scheme structure on $F(Y)$ coming from its defining equations in the Grassmannian, but for our purposes, it will always be a variety.
We argue using local holomorphic coordinates. Suppose that \( \Pi \subset P \) is simply \( \mathbb{C}^\ell \subset \mathbb{C}^n \), and that \( f_0 : D^r \to P \) is an embedding. Assume for the sake of contradiction that the locus \( \Sigma \subset D^r \times B \) where \( f_0 \) is not transverse to \( \Pi \) surjects onto \( B \). We may choose \( 0 \in B \) such that \( \Sigma \to B \) does not have a multiple fiber over \( 0 \). If \( f_0 \) is non-transverse to \( \Pi \) at \( p \in D^r \), then we have
\[
\mathbb{C}^\ell + df_0(T_p D^r) \subset \mathbb{C}^n.
\]
Let \( \vec{v} \) be a vector outside the subspace above, and use the hypothesis of free movability to find a subfamily \( f_1 : D^r \to P \) and section \( x(t) \in D^r \) such that \( x(0) = p \) and \( \frac{df}{dt}|_{t=0}(f \circ x) = \vec{v} \). Since \( \Sigma \) does not have a multiple fiber, there exists another section \( y(t) \in D^r \) such that \( y(0) = p \) and \( f_1(y(t)) \) meets \( \Pi \) non-transversely. Now
\[
f_1(y(t)) - f_0(p) = (f_1(y(t)) - f_1(x(t)) + (f_1(x(t)) - f_0(p)).
\]
At first order in \( t \), the left hand side lies in \( \mathbb{C}^\ell \), the first term on the right hand side lies in the image of \( df_0 \), and the second term on the right hand side lies in the span of \( \vec{v} \). This contradicts our choice of \( \vec{v} \). Since transversality is Zariski open, we obtain the statement for general \( b \in B \).

Any smooth curve \( C \subset X \) with class \( \ell + nf \) maps isomorphically to a line \( L \subset Y \). The pre-image \( \pi^{-1}(L) \) will contain \( C \), so to set up the enumerative problem, we consider the family of all elliptic surfaces over lines in \( Y \).

**Definition 16.** Let \( U \to F(Y) \times Y \) be the universal line, and form the fibered product
\[
\mathcal{S} := X \times_Y U.
\]
The natural morphism \( \nu : \mathcal{S} \to F(Y) \) is flat by base change and composition:
\[
\begin{array}{ccc}
\mathcal{S} & \longrightarrow & X \\
\downarrow \nu & & \downarrow \pi \\
F(Y) & \longrightarrow & Y.
\end{array}
\]
The family \( \nu \) will be our primary object of study. Its fiber over a line \([L] \in F(Y)\) is simply \( \pi^{-1}(L) \). Proposition \[39] shows that for \( k \leq 4 \), the fibers of \( \nu \) have no worse than isolated \( A_p \) singularities. We have an associated moduli map to the Weierstrass moduli space
\[
\mu_X : F(Y) \to \mathcal{W}_k
\]
by restricting the global Weierstrass equation from \( Y \) to \( L \).

**Lemma 17.** The map \( \mu_X \) is an immersion for general \( Y \) and \( X \in W(Y, \mathcal{O}(k)) \).

**Proof.** This is a statement about unordered point configurations on \( \mathbb{P}^1 \). The argument is rather technical and is relegated to Appendix B. \( \square \)

If we fix \( Y \) and vary \( X \in W(Y, \mathcal{O}(k)) \), we obtain a family of immersions \( \mu_X \).

**Proposition 18.** The family of immersions given by \( F(Y) \times W(Y, \mathcal{O}(k)) \to \mathcal{W}_k \) is freely movable in the sense of Definition \[14\].

**Proof.** This follows from surjectivity of the restriction map
\[
H^0(Y, \mathcal{O}(4k)) \oplus H^0(Y, \mathcal{O}(6k)) \to H^0(L, \mathcal{O}(4k)) \oplus H^0(L, \mathcal{O}(6k)).
\]
There is no need to vary the line \( L \), only \([A : B] \in W(Y, \mathcal{O}(k)) \). \( \square \)
By Lemma 15, we may assume after deformation that $\mu_X$ is transverse to any fixed subvariety of $W_k$. This will be applied to the Noether-Lefschetz loci inside $W_k$. 

**Definition 19.** An elliptic surface $S \in W_k$ is Noether-Lefschetz special if its relatively minimal resolution has Picard rank $> 2$.

**Theorem 20.** [7] All components of the Noether-Lefschetz locus in $W_k$ are reduced of codimension $k - 1$, except for the discriminant divisor, which is codimension 1.

The intersection of $\mu_X$ with the discriminant divisor in $W_k$ is responsible for the correction term $\Theta(q)$ in Theorem 6.

### 3. Noether-Lefschetz Theory

Let $S$ be a smooth elliptic surface over $\mathbb{P}^1$ with $p_g(S) = k - 1$. Its Picard rank is automatically $\geq 2$ because its Néron-Severi group $\text{NS}(S)$ contains the fiber class $f$ and the class of the identity section $z$. Any section class has self-intersection $-k$ by adjunction, so $\text{NS}(S)$ will always contain the rank 2 lattice

$$\langle f, z \rangle = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}.$$ 

We refer to this sublattice as the polarization, and it comes naturally from the fibration structure.

**Remark 21.** Except for the K3 case $(k = 2)$, the elliptic fibration structure on $S$ is canonical: $K_S$ is a nonzero multiple of the fiber class.

Noether-Lefschetz theory is the study of Picard rank jumping in families of surfaces. The short exact sequence of Shioda-Tate for an elliptic surface clarifies the two potential sources of jumping:

$$0 \to V(S) \to \text{NS}(S) \to \text{MW}(S/\mathbb{P}^1) \to 0.$$  

Here, $V(S)$ is the sublattice spanned by the zero section class and all vertical classes, and $\text{MW}(S/\mathbb{P}^1)$ is the Mordell-Weil group of the generic fiber, which is an elliptic curve over $\mathbb{C}(\mathbb{P}^1)$. If $S$ is a smooth Weierstrass fibration, then all fibers are integral, so $V(S)$ is simply the polarization sublattice. The group $\text{MW}(S/\mathbb{P}^1)$ will be torsion-free in the cases that concern us (see Appendix A), so the intersection form on $\text{NS}(S)$ splits the short exact sequence (2). In particular, the orthogonal projection $\Pi : \text{NS}(S) \to V(S)^\perp$ induces an isomorphism of groups

$$\text{MW}(S/\mathbb{P}^1) \to V(S)^\perp.$$ 

**Lemma 22.** If $\sigma \in \text{NS}(S)$ is the class of a section curve, then its orthogonal projection to $V(S)^\perp$ has self-intersection

$$-2(z \cdot \sigma + k).$$ 

**Proof.** Since $\text{MW}(S/\mathbb{P}^1)$ is torsion-free, $\sigma$ misses all exceptional curves. The projection can be computed by applying Gram-Schmidt to the polarization sublattice $\langle f, z \rangle$. \hfill $\square$

**Lemma 23.** If $\sigma$ is a section curve, and $\sigma^m$ its $m$-th power with respect to Mordell-Weil group law, then the class of $\sigma^m$ in $\text{NS}(S)$ is given by

$$[\sigma^m] = m\sigma - (m - 1)z + (z \cdot \sigma + k)m(m - 1)f.$$ 

In particular $(\sigma^m) \cdot z$ grows quadratically with $m$. 

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Proof. The class on the generic fiber is computed using the Abel-Jacobi map for elliptic curves. To determine the coefficient of \( f \), use the fact that any section curve has self-intersection \(-k\) in \( \text{NS}(S) \).

Lemma 24. If \( \sigma \) is the class of a section curve, and \( \iota : S \rightarrow X \) is the inclusion morphism, then

\[
\iota_*(\sigma) = \ell + (z \cdot \sigma + k) f \in H_2(X, \mathbb{Z})
\]

Proof. The class can be computed by intersecting with complementary divisors. The global section \( \iota(Y) \subset X \) has normal bundle \( \mathcal{O}_Y(-k) \), whence the shift. \( \square \)

Setting \( \text{NS}_0(S) := \langle f, z \rangle \perp \subset \text{NS}(S) \) and \( V_0(S) := \text{NS}_0(S) \cap V(S) \), the sequence

\[
0 \rightarrow V_0(S) \rightarrow \text{NS}_0(S) \rightarrow \text{MW}(S/P^1) \rightarrow 0
\]

is also split exact. This polarized sequence is more germane to future considerations.

The lattice \( V_0(S) \) will be a root lattice spanned by the classes of exceptional curves.

To set up the Noether-Lefschetz jumping phenomenon, we consider the full polarized cohomology lattice

\[
\Lambda(S) := \langle f, z \rangle \perp \subset H^2(S, \mathbb{Z}).
\]

Theorem 25. As abstract lattices, \( \Lambda(S) \simeq H^{2k-2} \oplus E_8(-1)^\oplus k \), where \( H \) denotes the rank 2 hyperbolic lattice, and \( E_8(-1) \) denotes the \( E_8 \) lattice with signs reversed.

Proof. By Poincaré duality, the pairing on \( H^2(S, \mathbb{Z}) \) is unimodular, and the Hodge Index Theorem gives its signature to be \((2k-1, 10k-1)\). The polarization sublattice \( \langle f, z \rangle \) is unimodular, so its orthogonal complement is as well. The Wu formula for Stiefel-Whitney classes reads

\[
\alpha \cdot \alpha \equiv \alpha \cdot K_S \pmod{2},
\]

so \( \alpha \cdot \alpha \in 2\mathbb{Z} \) for \( \alpha \in \langle z, f \rangle \perp \). By the classification of indefinite unimodular lattices, there is a unique even lattice of signature \((2k-2, 10k-2)\), namely the one above. \( \square \)

By the Lefschetz \((1, 1)\) Theorem, we have

\[
\text{NS}_0(S) \simeq (H^{2,0}(S) \oplus H^{0,2}(S))_R \perp \cap \Lambda(S),
\]

so the Picard rank jumping can be detected from the Hodge structure of \( S \). There is a period domain which parametrizes polarized Hodge structures on the abstract lattice \( \Lambda \). A weight 2 Hodge structure can be interpreted as a real representation of the Deligne torus \( S^1 = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^\times \):

\[
\psi : S^1 \rightarrow O(\Lambda_R),
\]

such that \( \psi(t) = t^2 \) for \( t \in \mathbb{R}^\times \). The Hodge decomposition comes from extending linearly to \( \Lambda_{\mathbb{C}} \), and setting

\[
H^{p,q}(\psi) = \{ v \in \Lambda_{\mathbb{C}} : \psi(z) \cdot v = z^p z^q v \}.
\]

For fixed Hodge numbers, the group \( O(\Lambda_R) \) acts transitively via conjugation on the set of such representations. This realizes the relevant period domain as a homogeneous space:

\[
D \simeq O(2k-2, 10k-2)/ \pm U(k-1) \times O(10k-2).
\]

\(^2\)For the rest of the paper, all Hodge structures are assumed polarized.
This can alternatively be viewed as an open orbit inside a complex flag variety, so in particular it has a complex structure. It contains a sequence of Noether-Lefschetz loci, which are given by

\[ \tilde{\text{NL}}_{2n} := \bigcup_{\beta \in \Lambda, (\beta,\beta) = -2n} \{ \psi \in D : H^{2,0}(\psi) \subset \beta^\perp \}, \]

each of which is simultaneously a homogeneous space

\[ \text{NL}_{2n} \simeq O(2k - 2, 10k - 3)/\pm U(k - 1) \times O(10k - 3) \]

and a complex submanifold of \( C \)-codimension \( k - 1 \). These loci parametrize Hodge structures on \( \Lambda \) which potentially come from a surface \( S \) with \( \text{NS}_0(S) \neq 0 \), since \( \beta \in \text{NS}_0(S) \).

The family \( \mathcal{S} \rightarrow F(Y) \) is generically smooth, so we have a holomorphic period map

\[ j : F(Y) \rightarrow \Gamma \backslash D, \]

defined away from the singular locus, where \( \Gamma \subset O(\Lambda) \) is the monodromy group of the smooth locus.

**Proposition 26.** The period map \( j \) is an immersion.

**Proof.** Combine Lemma 17 with the infinitesimal Torelli theorem of M. Saito [21] for deformations of elliptic surfaces.

By Proposition 39, singularities in the fibers of \( \nu \) are ADE type when \( k \leq 4 \), so the local monodromy of the smooth family is finite order. This allows us to extend \( j \) over all of \( F(Y) \) on general grounds [22]. The extension can be understood explicitly in terms of a simultaneous resolution (see Theorem 28). The period image of a singular surface is the Hodge structure of its minimal resolution.

The Noether-Lefschetz numbers of the family \( \mathcal{S} \rightarrow F(Y) \) are morally the intersections of \( j_*[F(Y)] \) with

\[ \text{NL}_{2n} := \Gamma \backslash \tilde{\text{NL}}_{2n} \subset \Gamma \backslash D. \]

However, since \( \Gamma \) contains torsion elements, the period space \( \Gamma \backslash D \) has singularities. To compute the topological intersection product we consider instead the smooth analytic stack quotient \( [\Gamma \backslash D] \). The period map \( j \) does not lift to this stack, so in Section 4 we construct a stack \( \mathfrak{g}(Y) \) with coarse space \( F(Y) \), admitting a map

\[ \mathfrak{g}(Y) \rightarrow [O(\Lambda) \backslash D], \]

lifting the classical period map.

Lastly, we note that the Noether-Lefschetz loci \( \text{NL}_{2n} \subset [O(\Lambda) \backslash D] \) are irreducible after fixing the divisibility of the lattice vector. This is a formal consequence of a statement about indefinite lattices:

**Theorem 27.** [8] A primitive embedding of an even lattice of signature \((m_+, m_-)\) into an even unimodular lattice of signature \((n_+, n_-)\) exists if

\[ m_+ + m_- \leq \min(n_+, n_-). \]

When this inequality is strict, the embedding is unique up to isomorphism.
The locus \( \text{NL}_{2n} \subset (O(\Lambda) \setminus D) \) decomposes into components indexed by \( m \in \mathbb{N} \) such that \( m^2 | n \). Let \( v_m \in \Lambda \) be a lattice vector of self-intersection \( 2n/m^2 \) so that \( mv_m \) has self-intersection \( 2n \). Then we can write

\[
\text{NL}_{2n} = \bigcup_{m^2 | n} O(\Lambda) \setminus \{ \psi \in D : H^2,0(\psi) \subset v_m^\perp \}.
\]

4. Simultaneous Resolution

In this section, we study flat families of surfaces with rational double points, focusing on the \( A_\rho \) case.

**Theorem 28.** \([6]\) Let \( \pi : X \to B \) be a flat family of surfaces over a smooth variety \( B \), such that each fiber \( X_b \) has at worst ADE singularities. Then after a finite base change \( B' \to B \) in the category of analytic spaces, the new family \( \pi' : X' \to B' \) admits a simultaneous resolution, a proper birational morphism \( \tilde{X}' \to X' \) which restricts to a minimal resolution on each fiber of \( \pi \).

Étale locally it suffices to consider a versal family. In the case of the \( A_\rho \) singularity \((x^2 + y^2 = z^\rho + 1)\), this family is given by

\[
x^2 + y^2 = z^\rho + 1 + s_1 z^{\rho - 1} + s_2 z^{\rho - 2} + \cdots + s_\rho
\]

in the deformation coordinate \( \bar{s} \in S = \mathbb{C}^\rho \). The base change required is given by the elementary symmetric polynomials

\[
s_i = \sigma_{i+1}(\bar{t}),
\]

where \( \bar{t} \in T = \text{Spec} \mathbb{C}[t_0, t_1, \ldots, t_\rho]/\sum t_i \simeq \mathbb{C}^\rho \). The cover is Galois with deck group \( \mathfrak{S}_{\rho + 1} \), branched over the discriminant hypersurface. After base change, the equation can be factored

\[
x^2 + y^2 = \prod_{i=0}^{\rho} (z + t_i).
\]

Singularities in the fibers occur over the big diagonal in \( T \), which is the hyperplane arrangement dual to the root system \( A_\rho \simeq \mathbb{Z}^\rho \subset \mathbb{C}^\rho \). The \( \mathfrak{S}_3 \) covering \( T \to S \) in the case of \( A_2 \) is pictured below.

A simultaneous resolution can be obtained by taking a small resolution of the total space. This is far from unique; one can write the total space as an affine toric variety, and then choose any simplicial subdivision of the single cone:

\[
\mathbb{Z}_+(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{\rho+2}, \vec{e}_1 - \vec{e}_2 + \vec{e}_3, \vec{e}_1 - \vec{e}_2 + \vec{e}_4, \ldots, \vec{e}_1 - \vec{e}_2 + e_{\rho+2}) \subset \mathbb{Z}^{\rho+2}.
\]

The smooth fiber in both families is diffeomorphic to a resolved \( A_\rho \) singularity, and thus has cohomology lattice

\[
H^2(X_s, \mathbb{Z}) \simeq A_\rho(-1).
\]
The Gauss-Manin local system on $S - \Delta$ corresponds to the representation
\[ \pi_1(S - \Delta) \simeq Br_{\rho+1} \to \mathfrak{S}_{\rho+1} \to O(A_{\rho}), \]
where $Br_{\rho+1}$ denotes the braid group, and $\mathfrak{S}_{\rho+1}$ is the Weyl group of $A_{\rho}$, which acts by reflections. The base change morphism $T \to S$ can be interpreted as the quotient $\mathbb{C}^o \to \mathbb{C}^o/\mathfrak{S}_{\rho+1} \simeq \mathbb{C}^o$ by extending the Weyl group action $\mathbb{C}$-linearly to $A_{\rho} \otimes \mathbb{C}$. The quotient map is ramified along the dual hyperplane arrangement and branched over $\Delta$.

Artin framed Theorem 28 in the language of representable functors.

**Theorem 29.** \[ [1] \] For $X \to B$ as above, let $\text{Res}_{X/B}$ be the functor from $\text{Sch}/B \to \text{Set}$ which sends $[B' \to B] \mapsto \{\text{simultaneous resolutions } \tilde{X}' \to X' = X \times_B B'\}$.

Then $\text{Res}_{X/B}$ is represented by a locally quasi-separated algebraic space.

The space is often not separated, even in the case of an ordinary double point $A_1$:

**Example 30.** Let $X \to \mathbb{C}$ be the versal deformation of $A_1$: $x^2 + y^2 + z^2 = t$. To build a simultaneous resolution, we base change by $t \mapsto t^2$, and then take a small resolution of the threefold singularity $x^2 + y^2 + z^2 = t^2$. There are two choices of small resolution, differing by the Atiyah flop. Hence, the algebraic space $\text{Res}_{X/\mathbb{C}}$ is isomorphic to $A_1$, with the étale equivalence relation $\mathcal{R} = \Delta \cup \{(x, -x) : x \neq 0\} \subset \mathbb{A}^1 \times \mathbb{A}^1$.

To do intersection theory, we want a nicer base for the simultaneous resolution, ideally a smooth Deligne-Mumford stack. From the perspective of periods we only need a resolution at the level of cohomology lattices. Let $\Delta \subset B$ be the discriminant locus of the family, and $j : U \hookrightarrow B$ its complement. Assuming that $U$ is nonempty, we have a Gauss-Manin local system $R^2\pi_{U,*}(\mathbb{Z})$ on $U(\mathbb{C})$ whose stalk at $b \in U$ is $H^2(X_b, \mathbb{Z})$ equipped with the cup product pairing.

**Remark 31.** In our situation, we will consider instead the primitive cohomology $R^2\pi_{U,*}(\mathbb{Z})_{\text{prim}} \subset R^2\pi_{U,*}(\mathbb{Z})$ whose stalk at $b \in U$ is isomorphic to the orthogonal complement of the polarization sublattice, as defined in Section [3].

The pushforward $\mathcal{H} := j_*R^2\pi_{U,*}(\mathbb{Z})_{\text{prim}}$ is a constructible sheaf on $B(\mathbb{C})$ whose stalk at $b \in \Delta(\mathbb{C})$ consists of classes invariant under the local monodromy action.

**Definition 32.** Let $\Lambda$ be the stalk of $\mathcal{H}$ over a smooth point $b \in U(\mathbb{C})$. A cohomological simultaneous resolution is an embedding $\mathcal{H} \hookrightarrow \mathcal{L}$ into a local system $\mathcal{L}$ on $B(\mathbb{C})$ with stalk $\Lambda$.

If $X \to B$ admits a simultaneous resolution, then the local monodromy action is trivial so $\mathcal{H}$ is a local system already. If it does not, then the cohomological resolutions are representable by a stack:

**Definition 33.** The monodromy stack $\mathcal{M}$ over $B$ is the following category fibered in groupoids. An object of $\mathcal{M}$ is given by a pair $$(f : B' \to B, i' : f^*\mathcal{H} \hookrightarrow \mathcal{L}'),$$
where $L'$ is a local system on $B'(\mathbb{C})$ with stalk $\Lambda$. A morphism from $(B', f, L', i')$ to $(B'', g, L'', i'')$ is a map $h : B' \to B''$ such that $f = g \circ h$, and an isomorphism $\phi : h^* L \to L$ such that $i' = \phi \circ h^* i''$.

To check that $\mathfrak{B}$ is a stack for the étale topology on $B$, we must verify that isomorphisms form a sheaf, and that objects satisfy descent. Both of these follow from the corresponding facts for local systems and the fact that étale morphisms of $\mathbb{C}$-schemes induce local isomorphisms on their underlying analytic spaces. Automorphisms of an object $(B', f, L', i')$ are the automorphisms of $L'$ which fix $i'(f^* H)$.

When $B'$ is a point $p$, then $H_p$ is the space of local invariant cycles, so the automorphism group is generated by reflections in the vanishing cycle classes.

**Theorem 34.** The stack $\mathfrak{B}$ is Deligne-Mumford.

**Proof.** We have natural morphisms
\[
\text{Res}_{X/B} \to \mathfrak{B} \to B,
\]
since a simultaneous resolution induces a cohomological one. Hence, $\mathfrak{B}$ is an algebraic stack by Theorem 29. The Deligne-Mumford property follows from the fact that simple surface singularities have finite monodromy. \[\square\]

**Proposition 35.** If $X \to S$ is the versal family of the $A_\rho$ singularity as described in (3), then its monodromy stack is isomorphic to $[T/\mathfrak{S}_{\rho+1}]$, which has coarse space is $S$.

**Proof.** We define an equivalence of categories fibered over $\text{Sch}/S$. An object of $[T/\mathfrak{S}_{\rho+1}]$ consists of a principal $\mathfrak{S}_{\rho+1}$-bundle $E \to Y$ with an equivariant map $\tilde{f} : E \to T$. Composing with the coarse space map $[T/\mathfrak{S}_{\rho+1}] \to S$ produces a map $f : Y \to S$. Let $L$ be the sheaf of sections of the associated bundle
\[
E \times_{\mathfrak{S}_{\rho+1}} A_\rho \to Y,
\]
which has fiber $A_\rho$. We can describe $\mathcal{H}$ as the sheaf of sections of
\[
T \times_{\mathfrak{S}_{\rho+1}} A_\rho \to S.
\]
To be single-valued in a neighborhood of $s \in S$, a section of $\mathcal{H}$ must send $s$ to the class of $(t, a)$, where $a$ is fixed by all $g \in \text{Stab}_{\mathfrak{S}_{\rho+1}}(t)$. Equivalently,
\[
a \in (t^\perp \cap R(A_\rho))^\perp.
\]
With this in mind, we define
\[
\tilde{F} := \left\{ (e, a) : a \in \left(\tilde{f}(e)^\perp \cap R(A_\rho)\right)^\perp \right\} \subset E \times A_\rho.
\]
This is $\mathfrak{S}_{\rho+1}$-invariant, so it descends to
\[
F \subset E \times_{\mathfrak{S}_{\rho+1}} A_\rho
\]
over $Y$. The equivariant map $\tilde{f} : E \to T$ induces a map
\[
E \times_{\mathfrak{S}_{\rho+1}} A_\rho \to (T \times_{\mathfrak{S}_{\rho+1}} A_\rho) \times_S Y
\]
If $\mathcal{F}$ is the sheaf of sections of $F \to Y$, then (1) gives an isomorphism
\[
\mathcal{F} \to f^* \mathcal{H}.
\]
Morphisms in $[T/\mathfrak{S}_{p+1}]$ are Cartesian diagrams of principal bundles with commuting equivariant maps, which induce morphisms of the above data. This defines a functor from $[T/\mathfrak{S}_{p+1}]$ to the monodromy stack. Recall that the action of $\mathfrak{S}_{p+1}$ on $A_\rho$ gives an equivalence from the category of principal $\mathfrak{S}_{p+1}$-bundles to the category of $A_\rho$-local systems. The data of commuting maps to $T$ corresponds to the coincidence of cohomology subsheaves, so our functor is fully faithful.

To show essential surjectivity, let $(f : Y \to S, i : f^*\mathcal{H} \to \mathbb{L})$ be an object of the monodromy stack. The bundle $E$ of Weyl chambers in the stalks of $\mathbb{L}$ is a principal $\mathfrak{S}_{p+1}$-bundle over $Y$. Let $\Delta' \subset \Delta$ be the smallest singular stratum containing the image of $f$, which corresponds to a partition of $\rho + 1$, or equivalently a conjugacy class of subgroup $G \subset \mathfrak{S}_{p+1}$. The general stalk of $f^*\mathcal{H}$ is isomorphic to the invariant sublattice $(A_\rho)^G$. We can form the associated bundle

$$E := E \times_{\mathfrak{S}_{p+1}} \mathfrak{S}_{p+1}/G$$

The fact that $f^*\mathcal{H}$ extends to a local system over $Y$ implies that it is trivial, so $E$ is a trivial bundle. Any choice of lift $Y \to T$ gives rise to an equivariant map from $E \to T$. Lifting this to an equivariant map $E \to T$ is automatic because each $G$-coset maps to the same point of $T$. □

More generally, if $X \to B$ has only $A_\rho$ singularities in the fibers, then étale locally on $B$, we have a morphism

$$B \to \prod_j S_j$$

to the versal bases of the isolated singularities in the central fiber. There is an embedding of the associated root lattice $A \subset \Lambda$, which induces an embedding of the Weyl group $W(A) \subset O(\Lambda)$. The monodromy of the family lies in $W(A)$, which implies that the diagram

$$\begin{array}{ccc}
\mathfrak{B} & \longrightarrow & \prod_j [T_j/\mathfrak{S}_{p_j+1}] \\
\downarrow & & \downarrow \\
B & \longrightarrow & \prod_j S_j
\end{array}$$

of stacks is 2-Cartesian. In particular, $B$ is the coarse space of $\mathfrak{B}$. If the family has maximal variation at each singularity then $\mathfrak{B}$ is smooth.

We apply Theorem 34 to the family $\nu : \mathcal{X} \to F(Y)$ to obtain a Deligne-Mumford stack $\mathcal{F}(Y)$ such that the (primitive) Gauss-Manin system on the smooth locus extends to a local system $\mathcal{L}$ on all of $\mathcal{F}(Y)$ with stalk $\Lambda$. Let $\mathcal{E}$ be the principal $O(\Lambda)$-bundle on $\mathcal{F}(Y)$ of isomorphisms from $\mathcal{L}$ to the constant sheaf $\Lambda$. There is an equivariant map from $\mathcal{E} \to D$ sending a point in $\mathcal{E}$ to the Hodge structure on $\Lambda$ obtained by identifying the stalk of $\mathcal{L}$ with $\Lambda$ via the isomorphism. This data gives the desired stacky period map

$$\mathcal{F}(Y) \to [O(\Lambda)\backslash D].$$
5. Modularity Statement

The work of Kudla-Millson produces a modularity statement for intersection numbers in a general class of locally symmetric spaces \( M \) of orthogonal type. In this section, we summarize\(^3\) the material in [15], and adapt it to our situation.

Let \( M \) be the double quotient \( \Gamma \backslash O(p, l)/K \) of an orthogonal group on the left by a torsion-free arithmetic subgroup preserving an even unimodular lattice \( \Lambda \subset \mathbb{R}^{p+l} \), and on the right by a maximal compact subgroup. This is automatically a manifold, since any torsion-free discrete subgroup acts freely on the compact cosets. We can interpret \( O(p, l)/K \) as an open subset of the real Grassmannian \( \text{Gr}(p, p+l) \) consisting of those \( p \)-planes \( Z \subset \mathbb{R}^{p+l} \) on which the form is positive definite. For any negative definite line \( \langle v \rangle \subset \mathbb{R}^{p+l} \), set

\[
\tilde{C}_{\langle v \rangle} := \{ Z \in O(p, l)/K : Z \subset \langle v \rangle^{\perp} \}
\]

which is \( \mathbb{R} \)-codimension \( p \). Indeed, the normal bundle to \( \tilde{C}_{\langle v \rangle} \) has fiber at \( Z \) equal to \( \text{Hom}(Z, \langle v \rangle) \). While the image of \( \tilde{C}_{\langle v \rangle} \) in \( M \) may be singular, it can always be resolved if we instead quotient by a finite index normal subgroup of \( \Gamma \). Furthermore, [15] gives a coherent way of orienting the \( \tilde{C}_{\langle v \rangle} \), so that it makes sense to take their classes in the Borel-Moore homology group \( H_{BM}^{pl-p}(M) \cong H^p(M) \).

For any positive integer \( n \), the action of \( \Gamma \) on the lattice vectors in \( \Lambda \) of norm \( -2n \) has finitely many orbits [5]. Choose orbit representatives \( \{v_1, \ldots, v_k\} \), and set

\[
\tilde{C}_{2n} := \bigcup_{i=1}^k \tilde{C}_{\langle v_i \rangle}.
\]

The image of \( \tilde{C}_{2n} \) in the arithmetic quotient \( M \) is denoted by \( C_{2n} \). Locally, \( \tilde{C}_n \) is a union of smooth (real) codimension \( p \) cycles meeting pairwise transversely, one for each lattice vector of norm \( -2n \) orthogonal to \( Z \). We quote the following result from [15]:

**Theorem 36.** For any homology class \( \alpha \in H_p(M) \), the series

\[
\alpha \cap e(V^\vee) + \sum_{n=1}^{\infty} (\alpha \cap [C_{2n}]) e^{2\pi in\tau}
\]

is a classical modular form for \( \tau \in \mathbb{H} \) of weight \( (p+l)/2 \). The constant term is the integral of the Euler class of the (dual) tautological bundle of \( p \)-planes.

The proof of Theorem 3 utilizes the cohomological theta correspondence, which relates automorphic forms for orthogonal groups and symplectic groups. In our case, \( Sp(1) \simeq SL(2) \) gives a classical modular form, but their result is framed in the context of Siegel modular forms.

To apply this statement to our situation, we note that the further quotient map

\[
g : D \to O(2k-2, 10k-2)/O(2k-2) \times O(10k-2)
\]

We match the notation of [15] for the most part, but all instances of positive (resp. negative) definiteness are switched.
is a smooth proper fiber bundle with fiber $SO(2k - 2)/U(k - 1)$. Given a positive definite real $(2k - 2)$-plane $Z \subset \Lambda_{\mathbb{R}}$, a polarized Hodge structure is given by a choice of splitting

$$Z_\mathbb{C} \simeq H^{0,2} \oplus H^{2,0} \subset \Lambda_\mathbb{C}$$

into a pair of conjugate complex subspaces, isotropic with respect to the form. A fiber of $g$ over $[Z]$ corresponds to this choice, which does not affect the orthogonal complement of $Z$. Thus, the Noether-Lefschetz loci in $D$ are pulled back from the symmetric space:

$$\tilde{\text{NL}}_{2n} = g^{-1} \left( \tilde{C}_{2n} \right).$$

The constant term of the series can be interpreted in terms of the Hodge bundle on $\Gamma \backslash D$. Indeed, if $V$ is the tautological bundle of $p$-planes,

$$g^*V \otimes \mathbb{C} = V^{0,2} \oplus V^{2,0}$$

where each summand is isomorphic to $g^*V$ as a real vector bundle. There is a natural complex structure on $V^{0,2}$ coming from the Hodge filtration, so we can take its Chern class:

$$c_{\text{top}}(V^{0,2}) = g^*e(V).$$

Theorem 36 is only proved for $\Gamma$ torsion-free, so that $M$ is a manifold. For our application, we need to allow general arithmetic groups, since ADE singularities have finite order monodromy. For convenience, we will take $\Gamma = O(\Lambda)$.

**Lemma 37.** $O(\Lambda)$ contains a finite index normal subgroup which is torsion-free.

**Proof.** Consider the congruence subgroup $\Gamma(3)$, consisting of automorphisms congruent to the identity modulo $3\Lambda$. It is normal and finite index because the quotient $O(\Lambda)/\Gamma(3)$ lies in the automorphism group of $\Lambda/3\Lambda$. Suppose that $T \in \Gamma(3)$ is torsion, say of prime order $p$. Write $1 - T = 3A$ for some $A \in \text{End}(\Lambda)$. Since $T$ has an eigenvalue $\lambda$ of order $p$, $A$ has an eigenvalue $\omega$ with $1 - \lambda = 3\omega$. Now take the norm $N_{\mathbb{Q}[\mathbb{Q}]}/\mathbb{Q}$ of this equation to obtain

$$p = 3^{p-1} \cdot N(\omega)$$

which is never true in $\mathbb{Z}$. \hfill \Box

Hence, the analytic stack $[O(\Lambda) \backslash D]$ can be realized as the quotient of a complex manifold by a finite group $G$. The spaces described above are related by

$$\overline{D} := \Gamma(3) \backslash D$$

$$\begin{array}{ccc}
\overline{D} & \xrightarrow{h} & [O(\Lambda) \backslash D] \\
\downarrow g & & \downarrow m \\
M, & & \\
\end{array}$$

where $h$ is a $G$-cover, and $g$ is a proper fiber bundle. The modularity statement carries over to intersections on the stack as follows. If $\alpha \in H_\text{p}([O(\Lambda) \backslash D], \mathbb{Q})$ is a rational homology class, and $\text{NL}_{2n} \subset \overline{D}$ is the Noether-Lefschetz locus in the manifold $\overline{D}$,

$$\alpha \cap h_*[\text{NL}_{2n}] = h^*\alpha \cap [\text{NL}_{2n}]$$

$$= h^*\alpha \cap g^*[C_{2n}]$$

$$= g_*h^*\alpha \cap [C_{2n}],$$
by repeated applications of the push-pull formula, which is valid for smooth stacks of Deligne-Mumford type \([3]\). Since the locus \(\mathbb{N}L_{2n}\) inside \(\overline{D}\) is \(O(\Lambda)\)-invariant, its pushforward under \(h\) acquires a multiplicity of \(|G|\). This overall factor can be divided out from the generating series. With these adjustments, we modify Theorem 36 to fit our Hodge theoretic situation:

**Theorem 38.** For any homology class \(\alpha \in H_p([O(\Lambda) \setminus D], \mathbb{Q})\), the series

\[
\varphi(q) = c_0 + \sum_{r=1}^{\infty} (\alpha \cap [\mathbb{N}L_{2n}]) q^r
\]

is a modular form of weight \(6k-2\) and level \(\text{SL}_2(\mathbb{Z})\).

We apply this statement to the \(\alpha = j_*[\mathfrak{H}(Y)]\), as defined in Section 4. To compute the intersection product, we spread out the period map to a section of a smooth fiber bundle over \(\mathfrak{H}(Y)\). Using the principal \(O(\Lambda)\)-bundle \(E \to \mathfrak{H}(Y)\) in the definition of \(j\), we set

\[
D(L) := D \times_{O(\Lambda)} E = \left( D \times E \right)/O(\Lambda),
\]

which admits a section \(s : \mathfrak{H}(Y) \to D(L)\) coming from the graph of the period map. The Noether-Lefschetz loci can be spread out similarly

\[
\mathbb{N}L_{2n}(L) := \mathbb{N}L_{2n} \times_{O(\Lambda)} E.
\]

By Lemma 2.1 of [15], after further shrinking \(\Gamma(3)\), we may assume that \(\mathbb{N}L_{2n} \subset \overline{D}\) has only normal crossing singularities. The section \(s : \mathfrak{H}(Y) \to D(L)\) is a local regular embedding of stacks, so it admits a Gysin map:

\[
j^*[\mathfrak{H}(Y)] \cap \mathbb{N}L_{2n} = s_*[\mathfrak{H}(Y)] \cap \mathbb{N}L_{2n}(L) = \deg s^! [\mathbb{N}L_{2n}(L)].
\]

Using Vistoli’s formalism [24], the Gysin map is given in terms of its normal cone. Consider the 2-Cartesian square

\[
\begin{array}{ccc}
W & \xrightarrow{s} & \mathbb{N}L_{2n}(L) \\
\downarrow{g} & & \downarrow{s} \\
\mathfrak{H}(Y) & \xrightarrow{s} & D(L)
\end{array}
\]

Let \(N = g^* N_s\), containing the normal cone \(C_{W/\mathbb{N}L_{2n}(L)}\). The intersection number is given by the Gysin pullback of this normal cone via the zero section of \(N\).

\[
s^! [\mathbb{N}L_{2n}(L)] := 0_N^! [C_{W/\mathbb{N}L_{2n}(L)}].
\]

This can be computed étale locally on \(\mathfrak{H}(Y)\). If \(f : Z \to \mathfrak{H}(Y)\) is an étale morphism from a scheme, we can form the fibered product

\[
\overline{D}(L)_Z := Z \times_{\mathfrak{H}(Y)} \overline{D}(L) \to Z
\]

which also admits a section \(s_Z : Z \to \overline{D}(L)_Z\). Lastly, we form \(W_Z = Z \times_{\mathfrak{H}(Y)} W\) and \(\mathbb{N}L_{2n}(L)_Z = \overline{D}(L)_Z \times_{\overline{D}(L)} \mathbb{N}L_{2n}(L)\). This spaces fit into the cubic diagram
below, where each face is 2-Cartesian.

\[
\begin{array}{c}
W_Z \\
\downarrow \downarrow \\
\rightarrow \\
\end{array}
\xrightarrow{f'}

\begin{array}{c}
NL_{2n}(L)_Z \\
\downarrow \downarrow \\
\rightarrow \\
\end{array}
\xrightarrow{f'}

\begin{array}{c}
W \\
\downarrow \downarrow \\
\rightarrow \\
\end{array}
\xrightarrow{s_Z}

\begin{array}{c}
NL_{2n}(L) \\
\downarrow \downarrow \\
\rightarrow \\
\end{array}
\xrightarrow{s_Z}

\begin{array}{c}
Z \\
\downarrow \downarrow \\
\rightarrow \\
\end{array}
\xrightarrow{f}

\begin{array}{c}
\delta(Y) \\
\downarrow \downarrow \\
\rightarrow \\
\end{array}
\xrightarrow{f}

\begin{array}{c}
NL_{2n}(L) \\
\downarrow \downarrow \\
\rightarrow \\
\end{array}
\xrightarrow{d(L)}

The front and lateral sides are 2-Cartesian by construction, the bottom uses the fact that \(s\) is a monomorphism, and the top and back are proven by repeated application of the universal property. All the diagonal morphisms are \(\acute{e}tale\), by invariance under base change. As a result, the normal cones are related by

\[N_{W_Z/\NL_{2n}(L)_Z} = f'^*N_{W/\NL_{2n}(L)}\]

Assuming that \(Z\) covers the support of \(W\), we have

\[\deg s^*_Z[\NL_{2n}(L)_Z] = \deg(f) \cdot \deg s[\NL_{2n}(L)]\]

We use this formula in Section 7 to compute the contributions of 0-dimensional intersections to the \(\Theta(q)\) correction term.

6. Discriminant Hypersurfaces

In this section, we study the cuspidal hypersurface cut out by \(\Delta = 4A^3 + 27B^2\) of degree \(12k\) inside \(\mathbb{P}^{m+1}\). The multiplicity of a line \(L \subset \mathbb{P}^{m+1}\) intersects \(\Delta\) dictates the singularities in the surface \(\pi^{-1}(L)\).

**Proposition 39.** The fibers of \(\nu : \mathcal{S} \to F(Y)\) have isolated rational double points of type \(A_\rho\).

**Proof.** The surface \(\pi^{-1}(L)\) can only be singular at the singular points of the cubic fibers, since elsewhere it is locally a smooth fiber bundle.

- If \(L\) intersects \(\Delta_{sm}\) at a point \(p\) with multiplicity \(\mu\), then the local equation of \(\pi^{-1}(L)\) near the node in the fiber \(\pi^{-1}(p)\) is
  \[x^2 + y^2 = t^\mu.\]
  This is an \(A_{\rho-1}\) singularity.\(^4\) We refer to such lines as Type I.

- If \(L\) intersects \(\Delta_{sing} = (A) \cap (B)\) at a point \(p\), then the fiber \(\pi^{-1}(p)\) has a cusp. The local equation of \(\pi^{-1}(L)\) there depends on the intersection multiplicity \(\alpha\), resp. \(\beta\), of \(L\) the hypersurface \((A)\), resp. \((B)\):
  \[x^3 + y^2 = t^\alpha x + t^\beta.\]
  This singularity can be wild for large values of \(\alpha\) and \(\beta\), but the codimension of this phenomenon is \(\alpha + \beta - 1\). For \(k \leq 4\), only \(A_1\) and \(A_2\) singularities can occur for \(L \in F(Y)\). We refer to such lines as Type II.

\(^4\)By convention, \(A_0\) means a smooth point.
Lastly, we must rule out the possibility of lines \( L \subset Y \) lying completely inside \( \Delta \), because otherwise \( \pi^{-1}(L) \) would not be normal. For this, consider the incidence correspondence

\[
\Omega = \{(L, [A : B]) : (4A^3 + 27B^2)|_L = 0 \} \subset F(Y) \times W(P^{m+1}, \mathcal{O}(k)).
\]

The first projection \( \Omega \to F(Y) \) is surjective and has irreducible fibers. To see this, note that if \( (4A^3 + 27B^2)|_L = 0 \), then

\[
A|_L = -3f^2,
B|_L = 2f^3
\]

for some \( f \in H^0(P^1, \mathcal{O}(2k)) \), by unique factorization. The codimension of this locus in \( W(P^1, \mathcal{O}(k)) \) is greater than

\[
8k > k - 1 = \dim F(Y).
\]

Hence, the second projection is not dominant, so for general \( A \) and \( B \), there are no lines on \( Y \) contained inside the discriminant hypersurface. \( \Box \)

We introduce tangency schemes to record how these \( A_\rho \) singularities appear,

**Definition 40.** Given a partition \( \mu \) of \( 12k \), let

\[
T_\mu(\Delta) := \{ (L, \Delta_{sm} \cap L = \sum j \mu_j p_j) \} \subset \mathbb{G}(1, m+1).
\]

**Proposition 41.** For general \( A \) and \( B \), \( T_\mu(\Delta) \) has codimension

\[
\sum_{j=1}^t (\mu_j - 1)
\]

when the latter is \( \leq k \leq 4 \).

**Proof.** For the partition \( (\mu_1, 1, \ldots, 1) \), consider the incidence correspondence

\[
\Omega_{\mu_1} = \{(L, p, [A : B]) : p \in L, \text{mult}_p(A^3 + B^2)|_L \geq \mu_1 \} \subset U \times W(P^{m+1}, \mathcal{O}(k)).
\]

The first projection \( \Omega_{\mu_1} \to U \) has fiber cut out by \( \mu_1 \) equations on \( W \), which we wish to be independent. Setting \( A(t) = a_0 + a_1 t + \ldots \) and \( B(t) = b_0 + b_1 t + \ldots \) for \( t \) the uniformizer at \( p \), the differential of the multiplicity condition is given by

\[
\begin{pmatrix}
3a_0^2 & 0 & 0 & \ldots & 2b_0 & 0 & 0 & \ldots & 0 \\
6a_0a_1 & 3a_0^2 & 0 & \ldots & 2b_1 & 2b_0 & 0 & \ldots & 0 \\
6a_0a_2 + 3a_1^2 & 6a_0a_1 & 3a_0^2 & 0 & \ldots & 2b_2 & 2b_1 & 2b_0 & \ldots & 0 \\
\vdots & & & & & & & & & \vdots
\end{pmatrix}
\]

The rows are independent unless \( a_0 = b_0 = 0 \). In this case, it is easy to check that \( A(t) \), resp. \( B(t) \), is actually divisible by \( t^{\lceil \mu_1/3 \rceil} \), resp. \( t^{\lceil \mu_1/2 \rceil} \), when \( \mu_1 \leq 6 \). There is an entire irreducible component

\[
\Omega_{\mu_1, I} = \{(L, p, A, B) : p \in L, \text{mult}_p(A) \geq \lceil \mu_1/3 \rceil, \text{mult}_p(B) \geq \lceil \mu_1/2 \rceil \} \subset \Omega
\]

whose fiber over a pair \( (L, p) \) is a linear subspace of \( W \). For \( \mu_1 < 6 \), \( \Omega_{\mu_1} \) is equidimensional with two irreducible components:

\[
\Omega = \Omega_{\mu_1, I} \cup \Omega_{\mu_1, II}.
\]
Now consider the second projection \( \Omega_{\mu, I} \rightarrow W \). It is dominant by a dimension count, so the general fiber must have codimension \( \mu_1 - 1 \). The case of a general partition is similar; gather only the multiplicities \( \mu_j \) greater than 1, and consider

\[
\Omega_\mu = \{(L, p_1, \ldots, p_l, A, B) : p_j \in L, (A^2 + B^3)|_L = \sum \mu_j p_j\}
\]

\[
\subset (U \times_G \cdots \times_G U) \times W(\mathbb{P}^{n+1}, \mathcal{O}(k)) =: \mathcal{W} \times W.
\]

Consider the fiber at \( (L, p_1, \ldots, p_l) \in \mathcal{W} \) for distinct points \( p_j \in L \). If \( A \) and \( B \) do not simultaneously vanish at any of the points, then the matrix of differentials can be row reduced to a matrix with blocks of the form

\[
\frac{1}{n!} \partial^n_x(1, x, x^2, x^3, \ldots)|_{x=p_j}; \quad (0 \leq n \leq \mu_j).
\]

This is a generalized Vandermonde matrix which has independent rows since the points are distinct. If \( A \) and \( B \) vanish simultaneously at some \( p_j \), then in fact they vanish maximally, which is a linear condition on \( W \) of the expected codimension. Hence, the general fiber of \( \Omega_\mu \rightarrow \mathcal{W} \) has \( 2^l \) irreducible components, which collapse over the big diagonal. Since the monodromy is trivial, we conclude that \( \Omega_\mu \) itself has \( 2^l \) components. We are only interested in one of them, \( \Omega_{\mu, I} \), which gives the desired codimension for \( T_\mu(\Delta) \).

Next, we observe that Type II lines \( L \in F(Y) \) are always limits of Type I lines (for \( k \leq 4 \)), so we can effectively ignore them. Trailing 1’s in the partitions are suppressed for convenience.

**Lemma 42.** Let \( T_{\alpha, \beta}(A, B) \subset \mathbb{G}(1, m + 1) \) be the locus lines \( L \) meeting \( (A) \), resp. \( (B) \), with multiplicity \( \alpha \), resp. \( \beta \), at a common point \( p \in (A) \cap (B) \). Then we have

\[
T_{1, 2}(A, B) \subset T_2(\Delta);
\]

\[
T_{2, 2}(A, B) \subset T_3(\Delta).
\]

These are the only Type II lines which show up after intersecting with \( F(Y) \).

**Proof.** Since \( T_{1, 2}(A, B) \) is codimension 2, we intersect \( \Delta \) with a general \( \mathbb{P}^2 \subset \mathbb{P}^{n+1} \) containing \( p \). The Type II lines are those which pass through the cusps of \( \mathbb{P}^2 \cap \Delta \) in the preferred direction. They lie in the dual plane curve. Since \( T_{2, 2}(A, B) \) is codimension 3, we intersect \( \Delta \) with a general \( \mathbb{P}^3 \subset \mathbb{P}^{n+1} \) containing \( p \). The Type II lines are those tangent to the curve \( \mathbb{P}^3 \cap (A) \cap (B) \). They are always limits of flex lines at smooth points of \( \mathbb{P}^3 \cap \Delta \), by a local calculation.

With these rough dimension results in hand, we turn to degree computations.

**Proposition 43.** The class of \( T_2(\Delta) \) in \( H_{4m-2}(\mathbb{G}(1, m + 1)) \) is Poincaré dual to \( 12k(6k - 1) \cdot \sigma_1 \in H^2(\mathbb{G}(1, m + 1)) \).

**Proof.** We intersect \([T_2(\Delta)]\) with the complementary Schubert class \( \sigma^1 \), which is represented by a pencil of lines in a general \( \mathbb{P}^2 \subset \mathbb{P}^{n+1} \). Since \( \mathbb{P}^2 \cap \Delta \) is a curve of degree \( d = 12k \) with \( c = 4k \cdot 6k \) cusps, the intersection number is given by the Plücker formula for the dual degree:

\[
d^* = d(d - 1) - 3c.
\]
Proposition 44. The class of $T_3(\Delta)$ in $H_{4m-4}(\mathbb{G}(1, m+1))$ is Poincaré dual to
$24k(10k - 3) \cdot \sigma_{11} + 24k(6k - 1)(4k - 1) \cdot \sigma_2 \in H^4(\mathbb{G}(1, m + 1))$.

Proof. First, we intersect $[T_3(\Delta)]$ with the class $\sigma^{11}$, which is represented by all
the lines in a general $\mathbb{P}^2 \subset \mathbb{P}^{n+1}$. The intersection number is again given by the
Plücker formula for flex lines applied to $\mathbb{P}^2 \cap \Delta$:
$$c^* = 3d(d - 2) - 8c.$$

Next, we intersect with the class $\sigma^2$, which is represented by all lines through a
general point in a general $\mathbb{P}^3 \subset \mathbb{P}^{n+1}$. To compute this number we find the top Chern class
of a bundle of principal parts. Consider the universal line $U \to \mathbb{G}(1, 3) \times \mathbb{P}^3$, and let $\mathcal{P}$
denote the bundle whose fiber at a point $(L, p)$ is the space of 2nd order germs at
$p$ of sections of $O_L(12k)$. The surface $\mathbb{P}^3 \cap \Delta$ induces a section of $\mathcal{P}$, by restriction,
vanishing at each flex. By standard arguments in [10], $\mathcal{P}$ admits a filtration with
successive quotients given by
$$\pi^*_2O_{\mathbb{P}^3}(12k), \pi^*_2O_{\mathbb{P}^3}(12k) \otimes \Omega^1_{U/G}, \pi^*_2O_{\mathbb{P}^3}(12k) \otimes \text{Sym}^2\Omega^1_{U/G}.$$ By the Whitney sum formula, its top Chern class is given by
$$c_3(\mathcal{P}) = (12k\zeta)((12k - 2)\zeta + \sigma_1)((12k - 4)\zeta + 2\sigma_1)$$
$$= 96k(6k - 1)(3k - 1)\zeta^3 + 48k(9k - 2)\zeta^2\sigma_1 + 24k\zeta\sigma_1^2,$$
where $\zeta$ denotes the relative hyperplane class on $U \to \mathbb{G}(1, 3)$ as a projective bundle.
We intersect this class with $\sigma_2 \in H^4(\mathbb{G}(1, 3))$ to get the number of flex lines for $\Delta$
passing through a general point in $\mathbb{P}^3$.
$$c_3(\mathcal{P}) \cdot \sigma_2 = 96k(6k - 1)(3k - 1) + 48k(9k - 2) + 24k.$$ This number is too high, since lines meeting $(A) \cap (B)$ tangent to $(B)$ have vanishing
principal part, and each contribute multiplicity 8 by a local calculation. If $\theta : (B) \to 
\mathbb{P}^{3*}$ is the Gauss map associated the hypersurface $(B)$, the number of such lines is
$$\deg(\theta|_{(A)\cap(B)}) = 4k \cdot 6k(6k - 1).$$ Subtracting this cuspidal correction from the Chern class gives the desired number.

Proposition 45. The class of $T_{2,2}(\Delta)$ in $H_{4m-4}(\mathbb{G}(1, m+1))$ is Poincaré dual to
$108k(3k - 1)(8k^2 - 1) \cdot \sigma_{11} + 36k(6k - 1)(4k - 1)(3k - 1) \cdot \sigma_2 \in H^4(\mathbb{G}(1, m + 1))$.

Proof. The intersection of $[T_{2,2}(\Delta)]$ with the class $\sigma^{11}$ is given by the Plücker formula for bitangent lines applied to $\mathbb{P}^2 \cap \Delta$:
$$d^* = \frac{d^*(d^* - 1) - d - 3c^*}{2}.$$ Next, we intersect with the class $\sigma^2$, which counts bitangent lines to $S = \mathbb{P}^3 \cap \Delta$
passing through a general point $p \in \mathbb{P}^3 \subset \mathbb{P}^{n+1}$. Consider the projection from $p
\Pi_p : S \to \mathbb{P}^2$.
The normalization of $S$ is a smooth surface $\tilde{S}$, and we write
$$\tilde{\Pi}_p : \tilde{S} \to \mathbb{P}^2$$
for the composition. The ramification curve $R = R' \cup R''$ has two components: $R'$
is the pre-image of the cusp curve, and $R''$ is the closure of the ramification locus.
for $\Pi_p : S_{sm} \to \mathbb{P}^2$. These lie over the branch locus $B = B' \cup B'' \subset \mathbb{P}^2$. The degree of $B''$ was already computed in Proposition 43
\[
\deg(B'') = 12k(6k - 1),
\]
so we know its arithmetic genus. Nodes of $B''$ correspond to a bitangent lines to $S$ through $p$, and cusps of $B''$ correspond to a flex lines to $S$ through $p$, which we already counted in Proposition 6. There are no worse singularities in $B''$ for a general choice of $p$, so it suffices to compute
\[
\delta(B'') = p_a(B'') - g(R'').
\]
The Riemann-Hurwitz formula says that
\[
K_{\tilde{S}} = \Pi_p^*K_{\mathbb{P}^2} + R = -3H + R.
\]
Realizing $\tilde{S}$ inside the blow up of $\mathbb{P}^3$ along $\Sigma = \mathbb{P}^3 \cap (A) \cap (B)$, the adjunction formula reads
\[
K_{\tilde{S}} = (12k - 4)H - E = (12k - 4)H - 2R'.
\]
Combining these equations, we deduce that
\[
R'' = (12k - 1)H - 3R'.
\]
This is enough to determine the genus of $R''$, using
\[
H \cdot H = 12k; \quad H \cdot R' = 24k^2; \quad R' \cdot R' = 48k^3.
\]
The latter can be computed inside the projective bundle $\mathbb{P}N_{\Sigma/\mathbb{P}^3}$, where $R'$ is the class of $\mathbb{P}N_{\Sigma/(B)}$. As an additional check, observe that
\[
R' \cdot R'' = 24k^2(6k - 1) = \deg(\theta|_{(A) \cap (B)}),
\]
which agrees with the cuspidal correction from Proposition 6. Finally, we use the genus formula on $R'':$
\[
2g - 2 = (K_{\tilde{S}} + R'') \cdot R''; \quad \delta(B'') = 12k(9k - 1)(6k - 1)(4k - 1).
\]
Subtracting the flex line count from Proposition 6 leaves the desired number. □

In the sequel, we will use the notation
\[
t_\mu := [T_\mu(\Delta)] \cdot F(Y)
\]
when this intersection is 0 dimensional, that is
\[
\sum_{j=1}^t (\mu_j - 1) = k - 1
\]
By the results of Section 4, the stack $\mathcal{S}(Y)$ will have isotropy group
\[
\prod_{j=1}^t \mathcal{S}_{\mu_j}
\]
at these isolated points.
7. Counting Curves

We begin by discussing the case $k = 1$, which is trivial because the period domain is a point. Since $F(Y)$ is 0-dimensional, let $N_m = \# F(Y)$, and we may assume that for each $[L] \in F(Y)$, the rational elliptic surface $S = \pi^{-1}(L)$ is smooth. Every class in the polarized Néron-Severi lattice $\text{NS}_0(S) \simeq \mathbb{E}_8$ is the orthogonal projection of a section curve. The degree shifts from Lemmas \ref{lemma:deg_shift} and \ref{lemma:deg_shift2} match, so we have

$$\sum_{n \geq 1} r_X(n)q^n = N_m \theta_{\mathbb{E}_8}(q) - N_m$$

and $\theta_{\mathbb{E}_8}(q)$ is the modular form of weight 4, as desired.

For $2 \leq k \leq 4$, recall the family of elliptic surfaces $\nu : \mathcal{S} \to F(Y)$ defined by the diagram

$$\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\nu} & X \\
\pi' \downarrow & & \pi \\
F(Y) & \leftarrow U & \to Y.
\end{array}$$

The period map $j : F(Y) \to O(\Lambda) \setminus D$ lifts to an immersion of stacks $j : \mathfrak{F}(Y) \to [O(\Lambda) \setminus D]$ of Deligne-Mumford type. Theorem \ref{thm:period_map} applied to $\alpha = j_*[\mathfrak{F}(Y)]$ yields a classical modular form $\varphi(q)$ of weight $6k - 2$ and full level. Our first task is to determine this modular form. We start by computing the constant term as the top Chern class of the dual Hodge bundle:

**Definition 46.** The Hodge bundle of the family $\nu : \mathcal{S} \to F(Y)$ is defined as the pushforward of the relative dualizing sheaf:

$$\lambda := \nu_* (\omega_{\mathcal{S}/F(Y)}).$$

**Proposition 47.** Let $S$ be the tautological rank 2 bundle on $F(Y)$. Then

$$\lambda \simeq \text{Sym}^{k-2}(S^\vee) \otimes \mathcal{O}_{F(Y)}(\sigma_1).$$

**Proof.** Writing $\nu$ as the composition $u \circ \pi'$, we compute the pushforward in stages.

$$\pi'_* (\omega_{\mathcal{S}/F(Y)}) = \pi'_* (\omega_{\mathcal{S}/U} \otimes \pi'^* \omega_{U/F(Y)})$$

$$= \pi'_* (\omega_{\mathcal{S}/U}) \otimes \omega_{U/F(Y)}$$

$$= \mathcal{O}_U(k\zeta) \otimes \omega_{U/F(Y)}$$

$$= \mathcal{O}_U(k\zeta) \otimes \mathcal{O}_U(-2\zeta) \otimes u^* \mathcal{O}_{F(Y)}(\sigma_1)$$

using the fact that $\pi'$ is a Weierstrass fibration, and $u$ is the restriction to $F(Y)$ of the projective bundle $\mathbb{P}(S) \to \mathcal{G}(1,m + 1)$. Next, we compute

$$u_* (\mathcal{O}_U((k - 2)\zeta) \otimes u^* \mathcal{O}_{F(Y)}(\sigma_1)) = \text{Sym}^{k-2}(S^\vee) \otimes \mathcal{O}_{F(Y)}(\sigma_1).$$

The top Chern class of $\lambda$ is enough to determine the constant term of $\varphi(q)$ when $k = 2$. Indeed,

**Proposition 48.** For $k = 2, 3, 4$, the space of modular forms $\text{Mod}(6k - 2, SL_2(\mathbb{Z}))$ has dimension $1, 2, 2$, respectively.

**Proof.** This follows from the presentation of the ring $\text{Mod}(\bullet, SL_2(\mathbb{Z}))$ as a free polynomial ring on the Eisenstein series $E_4$ and $E_6$. \qed
The positive degree terms of $\varphi(q)$ are Noether-Lefschetz intersections. By the split sequence of Shioda-Tate, there are two sources of jumping Picard rank in the family of Hodge structures.

- Resolved singular surfaces have non-trivial $V_0(S)$. If $e$ is the class of an exceptional curve, then $\langle f, z, e \rangle$ has intersection matrix
  
  $\begin{pmatrix}
  0 & 1 & 0 \\
  1 & -k & 0 \\
  0 & 0 & -2
  \end{pmatrix}$.

- Surfaces with extra sections have non-trivial Mordell-Weil group $\text{MW}(S/\mathbb{P}^1)$. If $\sigma$ is the class of a section, then $\langle f, z, \sigma \rangle$ has intersection matrix
  
  $\begin{pmatrix}
  0 & 1 & 1 \\
  1 & -k & z \cdot \sigma \\
  z \cdot \sigma & -k
  \end{pmatrix}$.

By the shift in Lemma 22, the Mordell-Weil jumping starts to contribute at order $q^k$, which matches the shift in Lemma 24 for the rational curve class on $X$. Thus, terms of order $< k$ are determined by enumerating the singular surfaces.

**Remark 49.** Every vector in the sublattice $\text{MW}(S/\mathbb{P}^1)$ corresponds to the class of some section curve. On the other hand, classes in $V_0(S)$ are less geometric: they are arbitrary $\mathbb{Z}$-linear combinations of exceptional curve classes, which are accounted for by the theta series $\Theta(q)$.

For $k = 3$, we can compute the full generating series. The $q^1$ term is given by

$[\varphi_1] = j_*[\mathfrak{F}(Y)] \cdot [\text{NL}_2],$

which is an excess intersection along $T_2(\Delta) \cap F(Y)$. If $v \in \Lambda$ lies in $Z^\perp$ with $v^2 = -2$, then any integer multiple $mv \in \Lambda$ lies in $Z^\perp$ with $(mv)^2 = -2m^2$, and the corresponding component of $\text{NL}_{2m^2}$ meets $\mathfrak{F}(Y)$ with isomorphic normal cone. As a result this excess intersection contributes $\theta_1(q)$ to the generating series $\varphi(q)$. Next, the $q^2$ term is given by

$[\varphi_2] = j_*[\mathfrak{F}(Y)] \cdot \text{NL}_4,$

which is a 0-dimensional intersection along $T_{2,2}(\Delta) \cap F(Y)$. Since $\mathfrak{F}(Y)$ has isotropy group $\mathfrak{S}_2 \times \mathfrak{S}_2$ there, we can compute

$[\varphi_2] = \frac{1}{4} t_{2,2}.$

This completely determines the modular form $\varphi(q)$, so we can solve for $[\varphi_1]$. The only remaining singular surfaces which contribute are the 0-dimensional intersection along $T_3(\Delta) \cap F(Y)$. Since $\mathfrak{F}(Y)$ has isotropy group $\mathfrak{S}_3$ there, we have

$\varphi(q) = \frac{1}{2} [\varphi_1] \theta_1(q) + \frac{1}{4} t_{2,2} (\theta_1(q)^2 - 2\theta_1(q)) + \frac{1}{6} t_3 (\theta_2(q) - 3\theta_1(q)) + \sum_{n \geq 3} r_X(n) q^n$

For each successive singular stratum, we subtract the root lattice vectors which are limits of previous strata. There are 3 copies of $A_1$ in $A_2$, and there are 2 copies of $A_1$ in $A_1 \times A_1$, so we subtract the double counted exceptional classes.
For $k = 4$, there are too many undetermined excess intersections (denoted $a_i$) to determine $\varphi(q)$, but we still have the general form

$$\varphi(q) = \Theta(q) + \sum_{n \geq 4} r_X(n)q^n,$$

where the theta correction term is given by:

$$\Theta(q) = a_1 \theta_1(q) + a_2 \left( \theta_1(q)^2 - 2\theta_1(q) \right) + a_3 \left( \theta_2(q) - 3\theta_1(q) \right) + t_4 \left( \theta_3(q) - 4\theta_2(q) - 3\theta_1(q)^2 + 18\theta_1(q) \right) + t_{2,2,2} \left( \theta_1(q)^3 - 3\theta_1(q) \right) + t_{2,3} \left( \theta_1(q) \theta_2(q) - 4\theta_1(q) \right).$$

We conclude with two examples of the full computation:

**Example 50.** $m = d = 2$.

The quadric surface $Y \subset \mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and the Weierstrass model $X$ is a Calabi-Yau threefold sometimes called the STU model. Its curve counts were computed previously, and can be found in [14].

$$\varphi(q) = -2E_4E_6; \quad \Theta(q) = -266 + 264 \theta_1.$$  

**Example 51.** $m = 2$, $d = 1$.

The hyperplane $Y \subset \mathbb{P}^3$ is isomorphic to $\mathbb{P}^2$, and the Weierstrass model $X$ is a Calabi-Yau threefold, which can also be realized as the resolution of the weighted hypersurface

$$X_{18} \subset \mathbb{P}(1, 1, 1, 6, 9).$$

The topological string partition function of $X$ is computed in [13], but our formula is the first in the mathematical literature.

$$\varphi(q) = \frac{31}{48} E_4^4 + \frac{113}{48} E_4 E_6^2; \quad \Theta(q) = 47253 - 93582 \theta_1 + 46008 \theta_1^2 + 324 \theta_2.$$  

**Example 52.** $m = 3$, $d = 3$.

The hypersurface $Y \subset \mathbb{P}^4$ is a cubic threefold, and the Weierstrass model $Y$ is a (non-CY) fourfold.

$$\varphi(q) = \frac{433}{16} E_4^4 + \frac{1439}{16} E_4 E_6^2; \quad \Theta(q) = 2089215 - 4107510 \theta_1 + 1969272 \theta_1^2 + 49140 \theta_2.$$  

8. Future Directions

Theorem [4] gives a generating series for true counts of smooth rational curves on $X$ lying over lines in $Y$. One can ask how this relates to the generating series for genus 0 Gromov-Witten invariants, which virtually count nodal rational curves. When $X$
is a threefold, the work of Oberdieck-Shen [19] on stable pairs invariants of elliptic fibrations implies via the GW/pairs correspondence [20] that
\[
\sum_{n=0}^{\infty} GW^X_0 (\ell + nf) q^n = \varphi(q) \cdot \eta(q)^{-12k},
\]
where \( \eta(q) \) is the Dedekind eta function. We cannot yet recover this formula directly from the Gromov-Witten side, but the \( \Theta(q) \) correction term from Theorem 6 yields formulas for the local contributions of \( A_\rho \) singular surfaces \( S \subset X \) to the stable pairs moduli space; compare with [23].

Counting curves in base degree \( e > 1 \) presents challenges involving degenerations of Hodge structures. The analog of the family \( \nu \) in this situation is
\[
\begin{array}{ccc}
\mathcal{S} & \longrightarrow & X \\
\nu \downarrow & & \downarrow \pi \\
\mathcal{M}_0(Y, e) & \leftarrow & C & \rightarrow & Y.
\end{array}
\]
where the Fano variety is replaced by the Kontsevich space of stable maps. At nodal curves
\[
C_1 \cup C_2 \rightarrow Y,
\]
the fiber of \( \nu \) is a surface with normal crossings:
\[
S = \pi^{-1}(C_1) \cup \pi^{-1}(C_2),
\]
which does not have a pure Hodge structure. Thus, the associated period map
\[
\mathcal{M}_0(Y, e) \longrightarrow \Gamma\backslash D
\]
does not extend over all of \( \mathcal{M}_0(Y, e) \). Since the Noether-Lefschetz loci in \( \Gamma\backslash D \) are non-compact, we must compactify the period map to a larger target in order to have a topological intersection product. Such a target \( (\Gamma\backslash D)^* \) is provided in [11], which satisfies the Borel Extension property for all period maps coming from algebraic families. The boundary of the partial completion \( (\Gamma\backslash D)^* \) consists of products of period spaces of lower dimension, whose Noether-Lefschetz classes satisfy a modularity statement. Motivated by this observation and computations in the Hermitian symmetric case, we make a conjecture for higher base degrees.

Let \( X \) be a Weierstrass fibration in \( W(Y, \mathcal{O}(k)) \), where \( Y \subset \mathbb{P}^{m+1} \) is a hypersurface of degree \( d \), and
\[
k = \left( \frac{e+1}{e} \right) m - d + \left( 2 - \frac{2}{e} \right).
\]
Note: \( m \equiv 2 \pmod{e} \) is equivalent to integrality of this expression.

**Conjecture 53.** Let \( r_X(e, n) \) be the number of smooth rational curves on \( X \) in the homology class \( e\ell + nf \). Then for \( ke \leq 4 \) or \( m = 2 \),
\[
\sum_{n \geq 1} r_X(e, n) q^n = \varphi(q) - \Theta(q),
\]
where \( \varphi(q) \in \text{QMod} (\bullet, SL_2(\mathbb{Z})) \), and \( \Theta(q) \in \mathbb{Q}[\theta_1, \theta_3, \theta_3]_{\leq ke} \).
APPENDIX A: TORSION IN THE MORDELL-WEIL GROUP

Let $S \to \mathbb{P}^1$ be a regular minimal elliptic surface. The vertical sublattice

$$V_0(S) \subset \text{NS}_0(S)$$

is a direct sum of ADE root lattices, one for each singular fiber. The discriminant group $d_i$ of each root lattice is the component group of the Néron model for the degeneration. A torsion element of $\text{MW}(S/\mathbb{P}^1)$ restricts to a non-identity component on some fiber, so we have an embedding of finite abelian groups

$$\text{TMW}(S/\mathbb{P}^1) \hookrightarrow \bigoplus_i d_i$$

Furthermore, $\text{TMW}(S/\mathbb{P}^1)$ is totally isotropic with respect to the quadratic form on the discriminant group. For the surfaces appearing in this paper (with $k \leq 4$), the discriminant group is sufficiently small that there are no non-trivial isotropic subgroups, so $\text{MW}(S/\mathbb{P}^1)$ is torsion-free.

APPENDIX B: CONFIGURATIONS OF POINTS ON A LINE

We study configurations of $6k$ unordered points on $\mathbb{P}^1$. The moduli space of point configurations is given by

$$M_{6k} := \mathbb{P}^{6k}/\text{PGL}(2).$$

We will often ignore phenomena in codimension $\geq k$, since $\dim F(Y) = k - 1$ and $F(Y) \subset \mathbb{G}(1,m+1)$ is freely movable. Away from codimension $k$, there are at least

$$6k - 2(k - 1) = 4k + 2$$

singleton points. In particular, all such configurations are GIT-stable. For a fixed general hypersurface $B \subset \mathbb{P}^{m+1}$ of degree $6k$, we have a morphism

$$\phi_B : \mathbb{G}(1,m+1) \to M_{6k},$$

given by intersecting with $B$. First we show that when $m$ is large, $\phi_B$ has large rank.

**Lemma 54.** If $2m \geq 4k$, then $d\phi$ has rank $\geq 2k$ at lines $L$ meeting $B$ in $\geq 4k + 2$ reduced points.

**Proof.** Consider the incidence correspondence

$$\Omega := \{(B,L) : \text{rank}(d\phi_B) < 2k\} \subset \mathbb{P}^N \times \mathbb{G}(1,m+1).$$

The second projection $\Omega \to \mathbb{G}(1,m+1)$ is dominant, and we study the fiber of this morphism. Assume that $B$ intersects $L$ transversely at $q,o,p_1,p_2,\ldots,p_{4k}$. Pick coordinates on $\mathbb{P}^{m+1}$ such that $T_qB \cong \mathbb{P}^{m}$ is the hyperplane at infinity, $o$ is the origin, and $T_oB$ is orthogonal to $L$. The remaining points are nonzero scalars, so we assume that $p_1 = 1$. Pick coordinates on $\mathbb{G}(1,m+1)$ near $L$ by taking pencils based at $q \in L$ (resp. $o \in L$) in a set of $m$ general $\mathbb{P}^2$’s containing $L$. At each marked point $p_i$, the transverse tangent space $T_{p_i}B$ is given by some slope vector

$$\vec{\lambda}_i \in \mathbb{C}^m.$$
The \((4k - 1) \times (2m)\) matrix for \(d \phi\) restricted to these points is given by

\[
d \phi = \begin{pmatrix}
p_2 \lambda_1^1 - \lambda_2^1 & \lambda_1^1 - \lambda_2^1 & p_2 \lambda_1^2 - \lambda_2^2 & \lambda_1^2 - \lambda_2^2 & \ldots & p_2 \lambda_1^m - \lambda_2^m & \lambda_1^m - \lambda_2^m \\
p_3 \lambda_1^1 - \lambda_2^1 & \lambda_1^1 - \lambda_2^1 & p_3 \lambda_1^2 - \lambda_2^2 & \lambda_1^2 - \lambda_2^2 & \ldots & p_3 \lambda_1^m - \lambda_2^m & \lambda_1^m - \lambda_2^m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{4k} \lambda_1^1 - \lambda_2^1 & \lambda_1^1 - \lambda_2^1 & p_{4k} \lambda_1^2 - \lambda_2^2 & \lambda_1^2 - \lambda_2^2 & \ldots & p_{4k} \lambda_1^m - \lambda_2^m & \lambda_1^m - \lambda_2^m 
\end{pmatrix}.
\]

The coordinates of \(\tilde{X}_i\) are free, and the small rank variety is cut out by the \((2k) \times (2k)\) minors of this matrix. This is a linear section of a determinantal variety, defined by a 1-generic matrix of linear forms in the sense of \cite{[9]}. By the principal result of \cite{[9]}, it has the expected codimension:

\[
c = (2k)(2m - 2k + 1) > 2m.
\]

Hence, the projection \(\Omega \to \mathbb{P}^N\) is not dominant for dimension reasons. \(\square\)

To understand the tangent space to \(F(Y)\) inside \(G(1, m + 1)\), we use the short exact sequence of normal bundles

\[
0 \to N_{L/Y} \to N_{L/\mathbb{P}^{m+1}} \to N_{Y/\mathbb{P}^{m+1}|L} \to 0
\]

and together they give the map. In coordinates, the matrix looks like

\[
\begin{pmatrix}
g_{11} & 0 & g_{21} & 0 & \ldots & 0 \\
g_{12} & g_{11} & g_{22} & g_{21} & \ldots & g_{m1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g_{1d} & g_{1(d-1)} & g_{2d} & g_{2(d-1)} & \ldots & g_{m(d-1)} \\
0 & g_{1d} & 0 & g_{2d} & \ldots & g_{md}
\end{pmatrix}.
\]

Only the residuals matter when determining \(H^0(N_{L/Y})\), and any set of residuals forms \(g_i \in H^0(L, \mathcal{O}(d - 1))\) comes from a hypersurface \(Y\) containing \(L\).

**Proposition 55.** When \(2m \geq 4k\), the morphism

\[
\mu_X : F(Y) \to M_{6k}
\]

is an immersion for general \(Y\) and \(X \in W(Y, \mathcal{O}(k))\).

**Proof.** Consider the incidence correspondence

\[
\Omega := \{(Y, L) : L \subset Y, d \mu_X \text{ is not an immersion}\} \subset \mathbb{P}^N \times G(1, m).
\]

The second projection \(\Omega \to G(1, m + 1)\) is surjective, and we study the fiber of this morphism. The condition \(Y \subset L\) is codimension \(d + 1\), so we need \(k\) more independent conditions for the codimension to exceed \(2m\). For this, we use another incidence correspondence: let \(K = \ker(d \phi_{[L]}) \subset \mathbb{C}^{2m}\) which has dimension \(\leq 2m - 2k\) by Lemma \ref{lem51} and consider

\[
\Omega' := \{(M, w) : w \subset \ker(M)\} \subset \mathbb{P}^{md-1} \times \mathbb{P}K,
\]

where \(\mathbb{C}^{md} \subset \text{Hom}(\mathbb{C}^{2m}, \mathbb{C}^{d+1})\) is the subspace of matrices which come from residual forms. It suffices to prove that \(\pi_1(\Omega') \subset \mathbb{P}^{md-1}\) has codimension \(\geq k\). The fibers
of the second projection $\Omega' \to \mathbb{P}K$ are cut out by $d + 1$ linear conditions in $md$ variables. In terms of the coordinates of $w$, the conditions are

$$
\begin{pmatrix}
w_1 & 0 & 0 & 0 & \ldots & w_3 & 0 & 0 & 0 & \ldots \\
w_2 & w_1 & 0 & 0 & \ldots & w_4 & w_3 & 0 & 0 & \ldots \\
0 & w_2 & w_1 & 0 & \ldots & 0 & w_4 & w_3 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}.
$$

The degeneracy loci of this matrix are high codimension in $\mathbb{P}K$ by explicit calculation with minors, so we have the dimension count:

$$
\dim \Omega' = \dim \mathbb{P}K + (md - 1) - (d + 1) \\
\leq (2m - 2k - 1) + (md - 1) - (2m - k + 1) \\
\text{codim} \pi_1(\Omega') \geq (2m - k + 1) - (2m - 2k - 1) = k + 2.
$$

□

Proposition 56. When $2m < 4k$, the morphism

$$
\mu_X : F(Y) \to W_k
$$

is an immersion for general $Y$ and $X \in W(Y, \mathcal{O}(k))$.

Proof. Since $2m - 6k - 1 < 2m - 4k - 1 < 0$, general forms $A \in H^0(\mathbb{P}^{m+1}, 4k)$ and $B \in H^0(\mathbb{P}^{m+1})$ do not vanish on any line. Hence, we have a morphism

$$
\phi_{A,B} : \mathbb{G}(1, m + 1) \to M_{4k} \times M_{6k}
$$

which we claim is an immersion on $F(Y)$. Since $F(Y) \subset \mathbb{G}(1, m)$ is freely movable, we may assume that each line intersects $A$ and $B$ at $\geq 8k + 2$ reduced points. Consider the incidence correspondence

$$
\Omega = \{([A : B], L) : d\mu[L] \text{ is not injective}\} \subset W(\mathbb{P}^{m+1}, \mathcal{O}(k)) \times \mathbb{G}(1, m + 1).
$$

The fiber of the second projection $\Omega \to \mathbb{G}(1, m + 1)$ is a linear section of a determinantal variety, as in Lemma 54. By [9], it has the expected codimension:

$$
c = (8k + 2) - 2m + 1 > 2m.
$$

Hence, the projection $\Omega \to W(\mathbb{P}^{m+1}, \mathcal{O}(k))$ is not dominant for dimension reasons.

□

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