Every graph is eventually Turán-good

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November 15, 2022

Abstract

Let $H$ be a graph. We show that if $r$ is large enough as a function of $H$, then the $r$-partite Turán graph maximizes the number of copies of $H$ among all $K_{r+1}$-free graphs on a given number of vertices. This confirms a conjecture of Gerbner and Palmer.

1 Introduction

For a pair of graphs $G$ and $F$, say that $G$ is $F$-free if $G$ does not contain a subgraph isomorphic to $F$. Let $N(H, G)$ denote the number of copies of a graph $H$ in $G$, that is, the number of subgraphs of $G$ isomorphic to $H$, and let

$$\text{ex}(n, H, F) = \max \{N(H, G) \mid G \text{ is an } n\text{-vertex } F\text{-free graph}\}$$

be the maximum number of subgraphs isomorphic to the target graph $H$ in an $n$-vertex $F$-free graph.

For the case $H = K_2$, this function has been widely studied. Indeed, the classical theorem of Turán [22] states that the unique $n$-vertex $K_{r+1}$-free graph with $\text{ex}(n, K_2, K_{r+1})$ edges is the Turán graph $T_r(n)$: the complete $r$-partite graph with parts of size either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. The special case $r = 2$ was originally resolved by Mantel [20] in 1907. The large scale behaviour of $\text{ex}(n, K_2, F)$ was resolved by the powerful Erdős-Stone-Simonovits Theorem [6,7]. This theorem gives asymptotically tight bounds for $\text{ex}(n, K_2, F)$ in terms

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This research is a part of projects that have received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme Grant Agreement 948057.
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Research supported by the Vanier Canada Graduate Scholarships program.
of the chromatic number of $F$, for all non-bipartite graphs $F$, and shows the Turán graph is essentially best possible.

**Theorem 1.1** (Erdős-Stone-Simonovits [6,7]). Let $F$ be a graph with chromatic number $k \geq 2$. Then $$\text{ex}(n, K_2, F) = (1 + o(1))e(T_{k-1}(n)).$$

Following a number of results concerning $\text{ex}(n, H, F)$ when $H \neq K_2$ (see, for example, [2,3,5,16]), the systematic study of this function in the more general setting was initiated by Alon and Shikhelman [1]. This has sparked a number of directions of research and bounds on $\text{ex}(n, H, F)$ are now known for many families of $H$ and $F$, but exact results are rare. (See e.g. [13] for additional details.)

One difficulty in precisely determining $\text{ex}(n, H, F)$ is identifying a potential extremal graph, that is, an $n$-vertex $F$-free graph $G$ such that $\text{ex}(n, H, F) = N(H, G)$. In many cases when the problem is tangible, the extremal graph turns out to be the Turán graph. Let $F$ be a graph with chromatic number $k+1$ and say that a graph $H$ is $F$-Turán-good if for $n$ sufficiently large, $\text{ex}(n, H, F) = N(H, T_k(n))$. That is, the Turán graph $T_k(n)$ is an $n$-vertex $F$-free graph containing the maximum possible number of copies of $H$. The term Turán-good was recently introduced by Gerbner and Palmer [13], but the study of this phenomenon goes back much further to work of Györi, Pach and Simonovits [15]. See [13] for a comprehensive summary of what is known so far about $F$-Turán-good graphs.

Gerbner and Palmer [13, Conjecture 20] conjectured that for every graph $H$ there exists $r_0 = r_0(H)$ such that $H$ is $K_{r+1}$-Turán-good for every $r \geq r_0(H)$. This conjecture is known to hold for stars [4], more generally for complete multipartite graphs [13], for paths [11] and for the 5-cycle [18]. In this paper we prove the conjecture holds with $r_0 = 300v(H)^9$.

**Theorem 1.2.** Let $H$ be a graph and $r \geq 300v(H)^9$. Then $H$ is $K_{r+1}$-Turán-good.

Theorem 1.2 follows from a more technical result (Theorem 1.3 below, which also implies that for $r \geq 300v(H)^9$ Turán graphs always maximize the number of copies of $H$ among $K_{r+1}$-free graphs on any given number of vertices, i.e., the requirement that the number of vertices is large compared to $r$ is unnecessary).

In [18] it is conjectured that Theorem 1.2 should hold with $r_0(H) = \chi(H) + 1$. Counterexamples are known for all $r \leq \chi(H) + 1$, but determining the exact value of $r_0(H)$ remains open, see [10,14,17,18]. We did not attempt to optimize our bound, beyond ensuring that it is polynomial in $v(H)$, and prioritized simplicity of presentation. However, our methods are unlikely to extend to this stronger conjecture. Even a bound quadratic in $v(H)$ would likely require additional ideas.

### 1.1 Preliminary definitions and proof outline

Throughout the paper we use $\nu(G)$ and $\epsilon(G)$ to denote the number of vertices and edges, respectively, in a graph $G$. It will be more convenient for us to work with homomorphism counts rather than subgraphs, and we precede the statement of Theorem 1.3 with the necessary definitions.

Recall that we say a map $\varphi : V(H) \rightarrow V(G)$ is a homomorphism from $H$ to $G$ if $\varphi(u)\varphi(v) \in E(G)$ for every $uv \in E(H)$. We denote by $\text{INJ}(H, G)$ the set of all injective homomorphisms from $H$ to $G$, and let $\text{inj}(H, G) = |\text{INJ}(H, G)|$. 


The goal of this section is to prove Lemma 2.4, which bounds the difference between inj$(H, T_r(n))$ and inj$(H, G)$ for an $r$-partite graph $G$, by a function depending on the ‘density’ of $G$. To capture the notion of density that will be useful to us, we define a graph $G$ to be $\delta$-dense for some $\delta > 0$ if deg$(v) \geq (1 - \delta)v(G)$ for every $v \in V(G)$.

We begin by showing that if $\delta$ is small enough, then in every $\delta$-dense graph every partial injective homomorphism has almost the maximum number of extensions.

**Lemma 2.1.** Let $G$ and $H$ be graphs such that $G$ is $\delta$-dense. Let $H' = H[X]$ for some $X \subseteq V(H)$. Let $k = v(H) - |X|$. Then for every $\varphi' \in \text{INJ}(H', G)$ there exist at least 

$$(1 - \delta k \cdot v(H)) \cdot v(G)^k$$

Clearly, inj$(H, G) = |\text{Aut}(H)| \cdot \mathcal{N}(H, G)$, where Aut$(H)$ is the automorphism group of $H$, and so a graph $H$ is $K_{r+1}$-Turán-good if and only if inj$(H, G) \leq \text{inj}(H, T_r(n))$ for every $n$-vertex $K_{r+1}$-free graph $G$ (and $n$ is sufficiently large).

Theorem 1.2 is a direct consequence of the following theorem.

**Theorem 1.3.** For every graph $H$, every $r \geq 300v(H)^9$, and every $n$-vertex $K_{r+1}$-free graph $G$ we have

\[
\text{inj}(H, G) \leq \text{inj}(H, T_r(n)).
\]

1.2 Proof Outline

Here we present an informal outline of the proof, which will be given in Section 3. As the theorem trivially holds when $n \leq r$, we can assume that $n \geq r + 1 \geq 300v(H)^9$, and so $n$ is much larger than $v(H)$. Thus almost every $\varphi : V(H) \to V(T_r(n))$ is in $\text{INJ}(H, T_r(n))$ and most of the maps that are not injective homomorphisms fail due to a single edge of $H$ being mapped to a non-edge of $T_r(n)$, i.e., there are approximately $e(H)(n^2 - 2e(T_r(n))n^{v(H) - 2}$ failures. Similarly, if an $n$-vertex $K_{r+1}$-free graph $G$ maximizes inj$(H, G)$, approximately $e(H)(n^2 - 2e(G))n^{v(H) - 2}$ maps fail to be an injective homomorphism from $H$ to $G$. As $e(T_r(n)) \geq e(G)$, Theorem 1.3 would follow as long as we can control the error terms implicit in the term “approximately”.

Arguing along the lines of the above sketch, we additionally see that adding any edge to $G$ increases inj$(H, G)$ and removing any edge decreases inj$(H, G)$ by approximately the same amount. If $G$ could be transformed to $T_r(n)$ by changing $O(e(T_r(n)) - e(G))$ adjacencies then the naive estimates hinted at above would suffice to make the argument precise. Unfortunately, such a transformation is not always possible. However, a sharp stability version of Turán’s theorem due to Füredi [9] allows us to transform $G$ into a (not necessarily balanced) $r$-partite graph $G_0$.

**Theorem 1.4** (Füredi [9]). Every $n$-vertex $K_{r+1}$-free graph $G$ contains an $r$-partite subgraph $G_0$ such that

\[
e(G) - e(G_0) \leq e(T_r(n)) - e(G).
\]

Given this, we subsequently transform the $r$-partite graph $G_0$ into $T_r(n)$, carefully controlling the homomorphism count, which gives the required result.

The key technical lemmas for our proof will be given in Section 2, before using them to complete the proof of Theorem 1.3 in Section 3.

2 Density Lemmas

The goal of this section is to prove Lemma 2.4, which bounds the difference between inj$(H, T_r(n))$ and inj$(H, G)$ for an $r$-partite graph $G$, by a function depending on the ‘density’ of $G$. To capture the notion of density that will be useful to us, we define a graph $G$ to be $\delta$-dense for some $\delta > 0$ if deg$(v) \geq (1 - \delta)v(G)$ for every $v \in V(G)$.
injective homomorphisms $\varphi \in \text{INJ}(H, G)$ such that $\varphi|_X = \varphi'$.

Proof. The proof proceeds by induction on $k$. The case $k = 0$ is trivial.

Suppose now $k = 1$ and let $V(H) \setminus X = \{u\}$. By the degree condition at least $(1 - \delta v(H))v(G)$ vertices are adjacent to every vertex in $\varphi'(V(H'))$ and setting $\varphi(u) = v$ for any such vertex $v$ extends $\varphi'$ to an injective homomorphism $\varphi$ from $H$ to $G$, as desired.

The induction step follows readily from the base case. Let $u \in V(H) - X$ be arbitrary and let $H'' = H[X \cup \{u\}]$ By the base case, for every $\varphi' \in \text{INJ}(H'', G)$ there exist at least $(1 - \delta v(H'')) \cdot v(G)$ maps $\varphi'' \in \text{INJ}(H'', G)$ such that $\varphi''|_X = \varphi'$. By the induction hypothesis for every such $\varphi''$ there are at least

$$ (1 - \delta(k - 1)v(H)) \cdot v(G)^{k-1} $$

injective homomorphisms $\varphi \in \text{INJ}(H, G)$ such that $\varphi|_{X \cup \{u\}} = \varphi''$. Thus we get at least

$$ (1 - \delta v(H'')) \cdot v(G) \cdot (1 - \delta(k - 1)v(H)) \cdot v(G)^{k-1} \geq (1 - \delta kv(H)) \cdot v(G)^k $$

(1) injective homomorphisms $\varphi$ from $H$ to $G$ extending $\varphi'$ as desired.

Setting $X = \emptyset$ in Lemma 2.1, i.e., letting $k = \nu(H)$, yields the following corollary.

Corollary 2.2. If $G$ and $H$ are graphs such that $G$ is $\delta$-dense, then

$$ \text{inj}(H, G) \geq (1 - \delta v(H)^2) \cdot v(G)^{v(H)}. $$

The next lemma is the key technical step in the proof of Lemma 2.4, controlling the change of homomorphism counts in a multipartite graph as we rebalance the sizes of the parts.

Lemma 2.3. Let $H$ be a graph with at least one edge. Let $0 < \delta \leq 1/4$, let $G$ be a $\delta$-dense graph, and let $A, B \subseteq V(G)$ be a pair of disjoint independent sets such that

- $|A| \geq |B| \geq 1$,
- every vertex in $A$ is adjacent to every vertex in $V(G) - A$, and
- every vertex in $B$ is adjacent to every vertex in $V(G) - B$.

Let $G_A$ and $G_B$ be obtained from $G$ by deleting one vertex in $A$ and $B$, respectively. Then

$$ \text{inj}(H, G_A) \geq \text{inj}(H, G_B) + 2e(H)(|A| - |B|)(1 - 3\delta v(H))^2 \cdot v(G)^{v(H) - 2}. $$

Proof. Let $G_0 = G \setminus (A \cup B)$ and $G_1 = G[A \cup B]$. Let $a \in A$ and $b \in B$ be such that $G_A = G \setminus a$ and $G_B = G \setminus b$.

For $S \subseteq V(H)$ and $G' \in \{G_A, G_B\}$, let $\text{INJS}(H, G')$ denote the set of injective homomorphisms $\varphi$ from $H$ to $G'$ such that $\varphi^{-1}(A \cup B) = S$. Then

$$ |\text{INJS}(H, G')| = \text{inj}(H[S], G'[A \cup B]) \cdot \text{inj}(H \setminus S, G_0). $$

Let

$$ \Delta(S) = |\text{INJS}(H, G_A)| - |\text{INJS}(H, G_B)| $$

$$ = (\text{inj}(H[S], G_1 \setminus a) - \text{inj}(H[S], G_1 \setminus b)) \cdot \text{inj}(H \setminus S, G_0). $$

(2)
Then
\[ \text{inj}(H, G_A) - \text{inj}(H, G_B) = \sum_{S \subseteq V(G)} \Delta(S), \tag{3} \]
and so it suffices to lower bound \( \sum_{S \subseteq V(G)} \Delta(S) \).

First let us lower bound the value \( \text{inj}(H \setminus S, G_0) \), which appears in (2).

**Claim 2.3.1.** \( \text{inj}(H \setminus S, G_0) \geq (1 - 2\delta v(H)^2) \cdot v(G_0)^{v(H) - |S|} \).

*Proof of Claim.* We aim to use Corollary 2.2, so we need to calculate the density of \( G_0 \). Since \( A \) is an independent set, \( v(G) - |A| \geq \deg(v) \geq (1 - \delta)v(G) \) for every \( v \in A \), and thus \( |A| \leq \delta v(G) \). Similarly, \( |B| \leq \delta v(G) \) and so
\[ v(G_0) \geq (1 - 2\delta)v(G). \tag{4} \]
In particular, \( v(G_0) \geq v(G)/2 \). As every vertex of \( G_0 \) has at most \( \delta v(G) \leq 2\delta v(G_0) \) non-neighbors in \( G_0 \), the graph \( G_0 \) is \( 2\delta \)-dense. Consequently, we obtain the claim by Corollary 2.2.

Before we estimate \( \Delta(S) \), let us show one more technical claim.

**Claim 2.3.2.** Let \( x, y \) be non-negative reals and \( p \) be a non-negative integer. Then \( (1 - x)(1 - y)^p \geq 1 - x - py \).

*Proof of Claim.* Induction on \( p \). If \( p = 0 \), then the claim is trivial. Thus suppose that \( p \geq 1 \) and \( (1 - x')(1 - y')^{p-1} \geq 1 - x' - (p - 1)y' \) for all \( x', y' \geq 0 \). We have
\[ (1 - x)(1 - y)^p = (1 - x)(1 - y) \cdot (1 - y)^{p-1} = (1 - x - y + xy) \cdot (1 - y)^{p-1} \geq (1 - x - y)(1 - y)^{p-1} \geq 1 - x - py, \]
where the first inequality follows since \( xy \geq 0 \) and the second inequality follows by the inductive assumption for \( x' = x + y \) and \( y' = y \).

Now let us estimate \( \Delta(S) \) when \( |S| \leq 2 \).

**Claim 2.3.3.** \( \sum_{S \subseteq V(H), |S| \leq 2} \Delta(S) \geq 2e(H)(|A| - |B|)(1 - 2\delta v(H)^3) \cdot v(G)^{v(H) - 2}. \)

*Proof of Claim.* If \( S \) is independent in \( H \), then \( \Delta(S) = 0 \). If \( |S| = 2 \) and the vertices of \( S \) are adjacent, then a homomorphism from \( H[S] \) to \( G_1 \setminus a \) or \( G_1 \setminus b \) maps \( S \) to an edge, hence
\[ \Delta(S) = (2e(G_1 \setminus a) - 2e(G_1 \setminus b)) \cdot \text{inj}(H \setminus S, G_0) \]
\[ = 2((|A| - 1)|B| - 2(|B| - 1)|A|) \cdot \text{inj}(H \setminus S, G_0) \]
\[ = 2(|A| - |B|) \cdot \text{inj}(H \setminus S, G_0). \]

By Claim 2.3.1 and (4) we further get:
\[ \Delta(S) \geq 2(|A| - |B|)(1 - 2\delta v(H)^2) \cdot v(G_0)^{v(H) - 2} \]
\[ \geq 2(|A| - |B|)(1 - 2\delta v(H)^2)((1 - 2\delta) \cdot v(G))^{v(H) - 2} \]
\[ = 2(|A| - |B|)(1 - 2\delta v(H)^2)(1 - 2\delta)^{v(H) - 2} \cdot v(G)^{v(H) - 2}. \]
Now, applying Claim 2.3.2 for $x = 2\delta v(H)^2$, $y = 2\delta$, and $p = v(H) - 2$, we get
\[
\Delta(S) \geq 2(|A| - |B|)(1 - 2\delta v(H)^2 - 2\delta(v(H) - 2)) \cdot v(G)^{v(H)^{-2}} \\
\geq 2(|A| - |B|)(1 - 2\delta v(H)^3) \cdot v(G)^{v(H)^{-2}}.
\]
Thus
\[
\sum_{S \subseteq V(H), |S| \leq 2} \Delta(S) \geq 2e(H)(|A| - |B|)(1 - 2\delta v(H)^3) \cdot v(G)^{v(H)^{-2}},
\]
as claimed.

It remains to lower bound $\sum_{S \subseteq V(H), |S| \geq 3} \Delta(S)$.

**Claim 2.3.4.** $\sum_{S \subseteq V(H), |S| \geq 3} \Delta(S) \geq -2e(H)(|A| - |B|) \cdot \delta v(H)^3 \cdot v(G)^{v(H)^{-2}}$.

**Proof of Claim.** In order to prove the claim, we construct an injection from a large subset of $\text{INJ}(H, G_B)$ into $\text{INJ}(H, G_A)$ that maps homomorphisms in $\text{INJ}_S(H, G_B)$ to homomorphisms in $\text{INJ}_S(H, G_A)$, and lower bound the sum above by $-1$ times the number of remaining homomorphisms in $\bigcup_{|S| \geq 3} \text{INJ}_S(H, G_B)$. For example, if $\varphi$ is a homomorphism from $H$ to $G_B$, such that $a \notin \varphi(H)$, then $\varphi$ is also a homomorphism from $H$ to $G_A$, so we get a natural correspondence. If $a \in \varphi(H)$, then $\varphi$ is not a map from $H$ to $G_A$, hence we will define a correspondence in the following (for most such maps $\varphi$).

First, choose any $A' \subseteq A$ such that $a \in A'$ and $|A'| = |B|$. Let $\iota : (A' \cup B) \to (A' \cup B)$ be an involution corresponding to a perfect matching between $A'$ and $B$ containing the edge $ab$, that is $\iota(a) = b$, $\iota(A') = B$ and $\iota(\iota(v)) = v$ for every $v \in A' \cup B$. We extend $\iota$ to $V(G) - (A - A')$ by setting $\iota(v) = v$ for every $v \in V(G_0)$. Define $\text{INJ}^*(H, G_B) \subseteq \text{INJ}(H, G_B)$ as the set of all injective homomorphisms $\varphi$ from $H$ to $G_B$ such that either

- $a \notin \varphi(V(H))$, or
- $\varphi(V(H)) \cap (A - A') = \emptyset$.

The map $f : \text{INJ}^*(H, G_B) \to \text{INJ}(H, G_A)$ is defined by setting $f \varphi = \varphi$ if $a \notin \varphi(V(H))$, and $f \varphi = \iota \varphi$ if $a \in \varphi(V(H))$. Note that $b \notin \varphi(V(H))$ hence $a \notin \iota \varphi(V(H))$, and in the second case, $\varphi(V(H)) \cap (A - A') = \emptyset$ and so $\iota$ is indeed well defined on $\varphi(V(H))$. It is easy to see that $f$ has the properties we stated in the previous paragraph, i.e., $f$ is an injection and $f$ maps $\text{INJ}^*(H, G_B) \cap \text{INJ}_S(H, G_B) \to \text{INJ}_S(H, G_A)$ for every $S \subseteq V(H)$. It follows that
\[
\Delta(S) = |\text{INJ}_S(H, G_A)| - |\text{INJ}_S(H, G_B)| \geq -|\text{INJ}_S(H, G_B) - \text{INJ}^*(H, G_B)|,
\]
for every $S \subseteq V(H)$, and so
\[
\sum_{S \subseteq V(H), |S| \geq 3} \Delta(S) \geq -\left|\bigcup_{|S| \geq 3} \text{INJ}_S(H, G_B) - \text{INJ}^*(H, G_B)\right|.
\]
(5)

Every homomorphism in $\bigcup_{|S| \geq 3} \text{INJ}_S(H, G_B) - \text{INJ}^*(H, G_B)$ maps some vertex of $H$ to
In this case we not only show that inj obtained from Lemma 2.4.

Let the \( \phi \) be extended to at least \( (1 - \delta v(H))^{v(H)} \) homomorphisms in \( \text{INJ}(H, G') \), yielding a total of \( 2e(H)(1 - \delta v(H)^2)\nu(G)^{v(H)-2} \) homomorphisms in \( \text{INJ}(H, G') - \text{INJ}(H, G) \). Thus

\[
\text{inj}(H, G') - \text{inj}(H, G) \geq 2e(H)(1 - \delta v(H)^2)\nu(G)^{v(H)-2},
\]

and

\[
\text{inj}(H, T_r(n)) - \text{inj}(H, G') \geq 2e(H)(\nu(T_r(n)) - \nu(G'))(1 - 3\delta v(H)^3)\nu(G)^{v(H)-2},
\]

by induction hypothesis, implying (7).

It remains to consider the case that \( G \) is complete \( r \)-partite. As \( G \) is not a Turán graph there exist parts \( A, B' \) of the \( r \)-partition of \( G \) such that \( |A| \geq |B'| + 2 \). Let the graph \( G' \) be obtained from \( G \) by deleting some vertex of \( a \in A \) and adding a new vertex \( b \) such that \( b \) is adjacent to every vertex of \( G' \) except for vertices in \( B' \), i.e., \( G' \) has an \( r \)-partition with the parts \( A \) and \( B' \) replaced by parts \( A - \{a\} \) and \( B = B' \cup \{b\} \). Note that \( e(G') = e(G) + |A| - |B| \). By Lemma 2.3 we have

\[
\text{inj}(H, G') \geq \text{inj}(H, G) + 2e(H)(|A| - |B|)(1 - 3\delta v(H)^3)\nu(G)^{v(H)-2},
\]

Combining this inequality with (8), which once again holds by the induction hypothesis, we obtain (7), as desired.

\[
\text{inj}(H, T_r(n)) - \text{inj}(H, G) \geq 2e(H)(1 - \delta v(H)^3)\nu(G)^{v(H)-2}. \tag{6}
\]

Combining (5) and (6) completes the proof of the claim.

Now the statement of the lemma follows directly from equation (3) and Claims 2.3.3 and 2.3.4.

Next we prove that a strengthening of Theorem 1.3 holds if \( G \) is \( r \)-partite, i.e., in this case we not only show that \( \text{inj}(H, G) \leq \text{inj}(H, T_r(n)) \), but give an essentially optimal bound on \( \text{inj}(H, T_r(n)) - \text{inj}(H, G) \) in terms of \( e(T_r(n)) - e(G) \).

**Lemma 2.4.** For every graph \( H \) with at least one edge, every \( 0 < \delta \leq 1/4 \), and every \( r \)-partite \( \delta \)-dense graph \( G \) we have

\[
\text{inj}(H, T_r(n)) - \text{inj}(H, G) \geq 2e(H)(1 - 3\delta v(H)^3)(e(T_r(n)) - e(G))\nu(G)^{v(H)-2}. \tag{7}
\]

**Proof.** We prove the lemma by induction on the difference \( e(T_r(n)) - e(G) \). By Turán’s theorem \( e(G) \leq e(T_r(n)) \) for every \( K_{r+1} \)-free \( n \)-vertex graph \( G \) and if \( e(G) = e(T_r(n)) \) then \( G \) is isomorphic to \( T_r(n) \), implying that the lemma holds when \( e(T_r(n)) - e(G) \leq 0 \).

For the induction step we assume \( e(T_r(n)) - e(G) \geq 1 \). Suppose first that \( G \) is not a complete \( r \)-partite graph, i.e., there exists an \( r \)-partite graph \( G' \) such that \( G \) is obtained from \( G' \) by deleting an edge between some pair of vertices \( v_1, v_2 \in V(G') \). By Lemma 2.1 each map \( \varphi' \) that maps the ends of some edge of \( H \) to \( v_1 \) and \( v_2 \) can be extended to at least \( (1 - \delta v(H)^2)\nu(G)^{v(H)-2} \) homomorphisms in \( \text{INJ}(H, G') \), yielding a total of \( 2e(H)(1 - \delta v(H)^2)\nu(G)^{v(H)-2} \) homomorphisms in \( \text{INJ}(H, G') - \text{INJ}(H, G) \). Thus

\[
\text{inj}(H, G') - \text{inj}(H, G) \geq 2e(H)(1 - \delta v(H)^2)\nu(G)^{v(H)-2},
\]

and

\[
\text{inj}(H, T_r(n)) - \text{inj}(H, G') \geq 2e(H)(e(T_r(n)) - e(G'))(1 - 3\delta v(H)^3)\nu(G)^{v(H)-2},
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by induction hypothesis, implying (7).

It remains to consider the case that \( G \) is complete \( r \)-partite. As \( G \) is not a Turán graph there exist parts \( A, B' \) of the \( r \)-partition of \( G \) such that \( |A| \geq |B'| + 2 \). Let the graph \( G' \) be obtained from \( G \) by deleting some vertex of \( a \in A \) and adding a new vertex \( b \) such that \( b \) is adjacent to every vertex of \( G' \) except for vertices in \( B' \), i.e., \( G' \) has an \( r \)-partition with the parts \( A \) and \( B' \) replaced by parts \( A - \{a\} \) and \( B = B' \cup \{b\} \). Note that \( e(G') = e(G) + |A| - |B| \). By Lemma 2.3 we have

\[
\text{inj}(H, G') \geq \text{inj}(H, G) + 2e(H)(|A| - |B|)(1 - 3\delta v(H)^3)\nu(G)^{v(H)-2},
\]

Combining this inequality with (8), which once again holds by the induction hypothesis, we obtain (7), as desired.

\[
\text{inj}(H, T_r(n)) - \text{inj}(H, G) \geq 2e(H)(1 - \delta v(H)^3)\nu(G)^{v(H)-2}. \tag{6}
\]
3 Proof of Theorem 1.3

In this section we complete the proof of Theorem 1.3. This result follows from
Theorem 1.4 and Lemma 2.4. The remaining technical difficulty is to show that the extremal graph $G$ must be $\delta$-dense for sufficiently small $\delta$.

Proof of Theorem 1.3. Theorem 1.3 trivially holds if $e(H) = 0$, so we assume $e(H) \geq 1$. Similarly, the theorem holds if $n \leq r$ as $T_r(n) = K_n$ for such $n$ and so we assume $n \geq r + 1$.

Suppose that $G$ is a $K_{r+1}$-free $n$-vertex graph that maximizes $\text{inj}(H, G)$ among all such graphs. As every vertex of $T_r(n)$ has degree at least $n - n/r - 1 \geq (1 - 2/r)n$, the graph $T_r(n)$ is $(2/r)$-dense. Thus by Corollary 2.2 we have

$$\text{inj}(H, G) \geq \text{inj}(H, T_r(n)) \geq \left(1 - \frac{2}{r}v(H)^2\right)n^{v(H)}.$$  

For $v \in V(G)$ let $\text{inj}(v)$ denote, for brevity, the number of homomorphisms in $\text{INJ}(H, G)$ that contain $v$ in its image. By averaging there exists $v_0 \in V(G)$ such that

$$\text{inj}(v_0) \geq \frac{v(H)}{n} \text{inj}(H, G) \geq v(H) \left(1 - \frac{2}{r}v(H)^2\right)n^{v(H)-1}. \quad (9)$$

We now loosely upper bound $\text{inj}(v)$ for arbitrary $v \in V(G)$ in terms of $\deg(v)$, as follows. There are $v(H)n^{v(H)-1}$ (not necessarily injective) maps $\varphi : V(H) \to V(G)$ that contain $v$ in its image. On the other hand, if the map $\varphi : V(H) \to V(G)$ is such that for some fixed edge $uv' \in E(H)$ we have $\varphi(u) = v$ and $\varphi(u')$ is not a neighbor of $v$, then $\varphi$ is not a homomorphism from $H$ to $G$. There are $(n - \deg(v))n^{v(H)-2}$ such (not necessarily injective) maps, implying

$$\text{inj}(v) \leq v(H)n^{v(H)-1} - (n - \deg(v))n^{v(H)-2}. \quad (10)$$

Construct a graph $G'$ from $G \setminus v$ by adding a copy of $v_0$, i.e., adding a vertex $v_1$ such that $v_1w \in E(G')$ if and only if $v_0w \in E(G)$ for $w \in V(G) \setminus \{v\}$. Note that $G'$ is $K_{r+1}$-free as no clique contains $v_0$ and $v_1$ and replacing $v_1$ with $v_0$ in any clique of $G'$ gives a clique in $G$ of the same size. As at most $v(H)^2n^{v(H)-2}$ homomorphisms in $\text{INJ}(H, G)$ contain both $v_0$ and $v$ in their image, we have

$$\text{inj}(H, G) \geq \text{inj}(H, G') \geq \text{inj}(H, G) - \text{inj}(v) + (\text{inj}(v_0) - v(H)^2n^{v(H)-2}),$$

and so $\text{inj}(v) \geq \text{inj}(v_0) - v(H)^2n^{v(H)-2}$. Thus by (9) and (10) we have

$$v(H)n^{v(H)-1} - (n - \deg(v))n^{v(H)-2} \geq v(H) \left(1 - \frac{2}{r}v(H)^2\right)n^{v(H)-1} - v(H)^2n^{v(H)-2},$$

and so

$$\deg(v) \geq n - \frac{2}{r}v(H)^3n - v(H)^2 \geq \left(1 - \frac{3}{r}v(H)^3\right)n \geq \left(1 - \frac{1}{100v(H)^6}\right)n$$

for every $v \in V(G)$, where we use $r \geq 300v(H)^6$. It follows that $G$ is $\frac{1}{100v(H)^6}$-dense and $e(G) \geq (1 - \frac{1}{100v(H)^6})n^2/2$. 

8
Theorem 1.4 gives an $r$-partite subgraph $G_0$ of $G$ such that $e(G_0) \geq 2e(G) - e(T_r(n))$. Note that every injective homomorphism $\varphi \in \text{INJ}(H, G) - \text{INJ}(H, G_0)$ must map the ends of some edge $e \in E(H)$ to the ends of some edge $f \in G \setminus G_0$. There are $e(H)$ choices of $e$, $(e(G) - e(G_0))$ choices of $f$, two choices of bijection between their ends, and at most $n^{v(H)^{-2}}$ choices of values of $\varphi$ on the remaining vertices of $H$. Putting this together yields

$$\text{inj}(H, G_0) \geq \text{inj}(H, G) - 2e(H)(e(G) - e(G_0))n^{v(H)^{-2}}. \quad (11)$$

We assume without loss of generality that $G_0$ is a maximal $r$-partite subgraph of $G$. We have

$$e(G_0) \geq e(G) - (e(T_r(n)) - e(G)) \geq e(G) - (n^2/2 - e(G)) \geq \left(1 - \frac{1}{50v(H)^6}\right) \frac{n^2}{2}.$$ 

Let $A_1, A_2, \ldots, A_r$ be some partition of $V(G_0)$ into $r$ independent sets. As $2e(G_0) \leq n^2 - \sum_{i=1}^r |A_i|^2$ we have $|A_i|^2 \leq \frac{1}{50v(H)^6} \cdot n^2$ for every $i$, and so $\max_{1 \leq i \leq r} |A_i| \leq \frac{n}{50v(H)^6}$. By maximality of $G_0$, every edge in $E(G) - E(G_0)$ has both ends in some $A_i$, implying

$$\deg_{G_0}(v) \geq \deg_G(v) - \max_{1 \leq i \leq r} |A_i| \geq \left(1 - \frac{1}{100v(H)^6} - \frac{1}{7v(H)^3}\right) n \geq \left(1 - \frac{1}{6v(H)^3}\right) n$$

for every $v \in V(G)$. Thus $G_0$ is $\frac{1}{6v(H)^{3}}$-dense and $\frac{1}{6v(H)^{3}} < 1/4$. By Lemma 2.4 we have

$$\text{inj}(H, T_r(n)) \geq \text{inj}(H, G_0) + 2e(H) \left(1 - 3 \left(\frac{1}{6v(H)^3}\right) v(H)^3\right) (e(T_r(n)) - e(G_0))n^{v(H)^{-2}} \geq \text{inj}(H, G) - 2e(H)(e(G) - e(G_0))n^{v(H)^{-2}} + e(H) (e(T_r(n)) - e(G_0))n^{v(H)^{-2}} = \text{inj}(H, G) + e(H)(e(G_0) + e(T_r(n)) - 2e(G))n^{v(H)^{-2}} \geq \text{inj}(H, G),$$

as desired. \hfill $\Box$

## 4 Further Questions

We conclude with a few questions that our current approach does not resolve. A classic strengthening of Turán’s Theorem due to Erős and Simonovits [8, 21] is the Stability Theorem: if $G$ is a $K_{r+1}$-free graph on $e(n, K_2, K_{r+1}) - o(n^2)$ edges, then $G$ has edit distance $o(n^2)$ from the Turán graph, which is to say $G$ can be transformed into the Turán graph by adding and subtracting $o(n^2)$ edges. It is natural to ask if the generalized Turán problem also exhibits stability.

We say that a graph $H$ is $F$-Turán-stable if any $K_{r+1}$-free graph $G$ on $n$ vertices with $N(H, T_r(n)) - o(n^{v(H)})$ copies of $H$ has edit distance $o(n^2)$ from the Turán graph.

**Question 4.1.** Fix any graph $H$ and let $r$ be large enough that $H$ is $K_{r+1}$-Turán-good. Does it follow that $H$ is $K_{r+1}$-Turán-stable?

Stability for generalized Turán problems has been considered. Ma and Qiu [19] proved that $K_r$ is $F$-Turán-stable for $\chi(F) > r$. Gerbner [12] showed that if $\chi(F) = \chi(H) + 1 = r + 1$ and $H$ is both $K_{r+1}$-Turán-good and $K_{r+1}$-Turán-stable, then $H$ is $F$-Turán-good.
Our method shows such a graph $G$ has edit distance at most $O(n^2/r)$ from $T_r(n)$ but our techniques are not sufficient to show a sub-quadratic bound.

Another natural question is whether the generalized Turán problem is monotonic in the following sense:

**Question 4.2.** Fix a graph $H$ and suppose $H$ is $K_r$-Turán-good. Does it follow that $H$ is also $K_{r+1}$-Turán-good? In other words, if $T_{r-1}(n)$ maximizes $\mathcal{N}(H, G)$ among $K_r$-free graphs, does it follow that $T_r(n)$ maximizes $\mathcal{N}(H, G)$ among $K_{r+1}$-free graphs?

One approach to proving monotonicity might be to start with a graph $G$ that is $K_{r+1}$-free and then remove edges to get $G'$ that is $K_r$-free. If this can be done in such a way that $e(G) - e(G') < e(T_r(n)) - e(T_{r-1}(n))$, then the methods described in this paper suggest

$$\mathcal{N}(H, G) - \mathcal{N}(H, G') \leq \mathcal{N}(H, T_r(n)) - \mathcal{N}(H, T_{r-1}(n))$$

for sufficiently large $r$, and thus $\mathcal{N}(H, G') \leq \mathcal{N}(H, T_{r-1}(n))$ is sufficient to demonstrate monotonicity.

Any $K_{r+1}$-free graph can be made $K_r$-free by removing at most $n^2/r^2$ edges as demonstrated by the following argument: If $S \subseteq E(G)$ is a minimal set of edges such that $G - S$ is $K_r$-free, then for each $e \in S$ there is an $r$-clique intersecting $S$ only at $e$; otherwise, $S - e$ is a smaller set intersecting every $K_r$. Thus, using Turán’s Theorem,

$$\binom{r}{2} |S| \leq e(G) \leq \binom{n}{2} \left(\frac{n}{r} \right)^2 \implies |S| \leq \frac{n^2}{r^2}.$$

Unfortunately,

$$e(T_r(n)) - e(T_{r-1}(n)) \approx \frac{n^2}{2r(r-1)}$$

is slightly smaller. This suggests a more delicate approach may be necessary where edges are added to $G$ in addition to being removed. Tracking the change in $\mathcal{N}(H, G)$ in such a process is beyond the scope of this paper.

**Acknowledgements.**

This research was partially completed at the Cross-community collaborations in combinatorics workshop (22w5107) at the Banff International Research Station, 29 May-3 June 2022. We thank the organizers and other participants. The authors also thank Dániel Gerbner for his comments on an earlier draft.

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