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Transport properties of superfluid phonons in neutron stars

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Abstract: We review the effective field theory associated to the superfluid phonons that we use for the study of transport properties of superfluid neutrons stars in their low temperature regime. We then discuss the shear and bulk viscosities together with the thermal conductivity coming from the collisions of superfluid phonons in neutron stars. With regards to shear, bulk and thermal transport coefficients, the phonon collisional processes are obtained in terms of the equation of state and the superfluid gap. We compare the shear coefficient due to the interaction among superfluid phonons with other dominant processes in neutron stars, such as electron collisions. We also analyze the possible consequences for the r-mode instability in neutron stars. As for the bulk viscosities, we determine that phonon collisions contribute decisively to the bulk viscosities inside neutron stars. For the thermal conductivity resulting from phonon collisions, we find that it is temperature-independent well below the transition temperature. We also obtain that the thermal conductivity due to superfluid phonons dominates over the one resulting from electron-muon interactions once phonons are in the hydrodynamic regime. As the phonons couple to the $Z$ electroweak gauge boson, we estimate the associated neutrino emissivity. We also briefly comment on how the superfluid phonon interactions are modified in the presence of a gravitational field, or in a moving background.

Keywords: effective theories, transport coefficients, superfluid phonons, neutrino emissivity

1. Introduction

Superfluidity is a property of quantum liquids related to the existence of low energy excitations that satisfy the so called Landau’s criterion \cite{1}

\[
\text{Min} \frac{e(p)}{p} \neq 0,
\]

where $e(p)$ is the dispersion law of the excitation. The lowest energy modes are usually called superfluid phonons, and they dominate the low temperature properties of the superfluid. While superfluidity was first discovered in a bosonic system, it was soon realised that it could occur in a fermionic system as well, as a consequence of Cooper’s theorem.

Migdal’s observation \cite{2} that superfluidity of neutron matter may occur in the core of compact stars has captured a lot of interest over the years. The presence of superfluidity inside neutron stars would affect different neutron star phenomena, such as its cooling, rotational properties, pulsar glitches or the hydrodynamical and oscillation modes, and thus have several observational consequences.

At low temperatures neutron matter superfluidity originates from the presence of a quantum condensate, due to neutron pairing. This condensate breaks the global $U(1)$ symmetry associated to
baryon number conservation and gives rise to the existence of the low energy modes, the superfluid phonons. At very low temperatures they dominate the thermal corrections to the thermodynamical and hydrodynamical properties of the superfluid. More precisely, the superfluid phonon contribution could be relevant for the determination of the transport properties of neutron stars, that is, the shear and bulk viscosities as well as the thermal conductivity.

In this manuscript we review the progress on the transport properties of superfluid phonons. We give an overview of the superfluid phonon contribution to the shear and bulk viscosities together with the thermal conductivity in the core of neutron stars, as discussed in Refs. [3–6]. So as to calculate the phonon contribution, it is important to determine the relevant collisions involving phonons that are responsible for the transport phenomena. By using the universal character of the effective field theory (EFT) at leading order, we can obtain a very general formulation so as to determine the leading phonon interactions, which can be fixed by the equation of state (EoS) and the superfluid gap of neutron matter [7,8]. We also discuss neutrino emission by superfluid phonons. Even if these couple to the $Z$ electroweak gauge boson, this possible channel of cooling of the star is very much suppressed, as we will show.

The paper is organized as follows. In Sec. 2 we review the EFT that describes the phonon self-interactions in a superfluid, whereas in Sec. 3 we present the specific EoS for $\beta$-stable nuclear matter as well as the superfluid gap that will be used for the computation of the transport coefficients. In Secs. 4 and 5 we show our results for the shear and bulk viscosities, and in Sec. 6 we analyze the r-mode instability window using the shear damping mechanism. Then, in Sec. 7 we discuss the thermal conductivity, and in Sec. 8 we comment on neutrino emission due to superfluid phonons. We finalize our paper with Sec. 9 where we present a discussion on how the superfluid phonon EFT changes in the presence of a gravitational field, or by considering that the superfluid medium is not at rest. The summary is given in Sec. 10. We use natural units $\hbar = c = k_B = 1$ in this work.

2. Effective field theory and the superfluid phonon

EFT techniques allow to write down the effective Lagrangian for superfluid phonons. A superfluid phonon is the Goldstone mode related to the spontaneous symmetry breaking of the particle number conservation. The effective Lagrangian is constructed as an expansion in derivatives of the superfluid phonon field, each of the terms restricted by the allowed symmetry. The coefficients of the each term in the expansion can be determined from the microscopic theory using a matching procedure. An important observation was made in [7,8], as it was pointed out that the EoS determines completely the leading-order (LO) effective Lagrangian for phonons. More particularly, the LO Lagrangian for a non-relativistic system reads

\[
\mathcal{L}_{\text{LO}} = P(X),
\]

\[
X = \mu - \partial_t \varphi - \frac{(\nabla \varphi)^2}{2m},
\]

where $P(\mu)$ and $\mu$ are the pressure and chemical potential, respectively, of the system at $T = 0$, whereas $\varphi$ is the phonon field (the phase of the fermionic condensate) and $m$ is the mass of the particles that form a condensate. Note that, after a Legendre transformation, the associated Hamiltonian can be obtained, having the same form as the one used by Landau for the phonon self-interactions of $^4\text{He}$ [8,9].

The Lagrangian for the phonon field is determined after performing a Taylor expansion of the pressure and rescaling the phonon field

\[
\varphi = \frac{\varphi}{\frac{\partial P}{\partial \mu}},
\]

(3)
in order to canonically normalized the kinetic term as

\[
\mathcal{L}_{\text{LO}} = \frac{1}{2} \left( (\partial_t \phi)^2 - v_{\text{ph}}^2 (\nabla \phi)^2 \right) - g \left( (\partial_t \phi)^3 - 3\eta_g \phi (\nabla \phi)^2 \right) + \lambda \left( (\partial_t \phi)^4 - \eta_{\lambda,1} (\partial_t \phi)^2 (\nabla \phi)^2 + \eta_{\lambda,2} (\nabla \phi)^4 \right) + \cdots
\] (4)

In Ref. [10] the phonon self-couplings of Eq. (4) were obtained as different ratios of derivatives of the pressure with respect to the chemical potential. Similarly, we can express them in terms of the speed of sound at \(T = 0\) and derivatives of the speed of sound with respect to the mass density.

In particular, the speed of sound at \(T = 0\) is given by

\[
v_{\text{ph}} = \sqrt{\frac{\partial P}{\partial \rho}} \equiv c_s ,
\] (5)

where \(\rho\) is the mass density. And the three and four phonon self-coupling constants are obtained as

\[
\begin{align*}
g &= \frac{1 - 2u}{6c_s \sqrt{\rho}}, & \eta_g &= \frac{c_s^2}{1 - 2u}, & \lambda &= \frac{1 - 2u(4 - 5u) - 2wp}{24c_s^2 \rho}, \\
\eta_{\lambda,1} &= \frac{6c_s^2 (1 - 2u)}{1 - 2u(4 - 5u) - 2wp}, & \eta_{\lambda,2} &= \frac{3c_s^4}{1 - 2u(4 - 5u) - 2wp},
\end{align*}
\] (6)

with

\[
u = \frac{\rho c_s}{\partial \rho}, \quad w = \frac{\rho}{\partial c_s}. \] (7)

Note that the dispersion law coming from this Lagrangian at tree level is exactly \(E_p = c_s \rho\), so that phonons move at the speed of sound.

The formulation of the superfluid phonon EFT just presented is universal, and valid for different superfluid systems, either bosonic or fermionic, and no matter if those systems are weakly coupled or not. Thus, with the same formulation we can describe the superfluid phonons of either superfluid \(^4\)He, cold Fermi gases at unitarity, or the superfluid neutron matter inside neutron stars. With very minor modifications relativistic superfluids, such as the color-flavor locked quark matter phase, can be also described within the same methods [7]. With the knowledge of the EoS, many of the low temperature properties of the superfluid, mainly those associated to transport, can be immediately deduced. For astrophysical applications our treatment is very convenient, as we could provide the values associated to the transport phenomena involving superfluid phonons in terms of different EoSs.

It also is possible to construct the next-to-leading order (NLO) Lagrangian in a derivative expansion. As seen in Ref. [8] this reads

\[
\mathcal{L}_{\text{NLO}} = \partial_i X \partial_i X f_1 (X) + (\Delta^2 \theta)^2 f_2 (x) ,
\] (8)

where \(\theta = \mu t - \varphi\), and \(f_1\) and \(f_2\) are arbitrary functions. Unfortunately, there is a not a simple way of finding the value of these functions. For the cold Fermi gas at unitarity the NLO Lagrangian can be determined up to two arbitrary constants, just demanding invariance under scale transformations. However, this is not a symmetry in superfluid neutron matter, for example.

The NLO Lagrangian is relevant to study different corrections to the different scattering rates among superfluid phonons, which will be typically minor corrections. However, it is important for determining the corrections to the phonon dispersion law. The NLO phonon dispersion law is given by

\[
E_p = c_s \rho (1 + \gamma p^2) ,
\] (9)
with 

$$\gamma = -\frac{1}{2F} \left( f_1(\mu) + \frac{f_2(\mu)}{c_s^2} \right).$$  \hfill (10)$$

The sign of $\gamma$ determines whether the decay of one phonon into two (or more) phonons is kinematically allowed. More particularly, with the LO Lagrangian, energy and momentum conservation allows for the decay of one phonon into two, if the three involved phonons are collinear. If we take into account the NLO corrections, energy and momentum conservation impose that the small angle $\delta \theta$ among the two resulting phonons, with momenta $p_b$ and $p_c$, respectively is 

$$\delta \theta = \sqrt{6\gamma (p_b + p_c)}.$$  \hfill (11)$$

Thus, the process is only kinematically allowed if $\gamma > 0$.

In the computation of the transport properties of a superfluid associated to the phonons it becomes thus essential to determine the value of $\gamma$, so as to know the possible scattering processes that contribute to a particular transport coefficient. For the astrophysical applications we should have in mind that the value of $\gamma$ was computed assuming that neutron pairing is in a $^1S_0$ channel, treating the neutrons as weakly coupled system, resulting [6]

$$\gamma = -\frac{v_F^2}{45\Delta^2},$$  \hfill (12)$$

with $v_F$ being the Fermi velocity and $\Delta$ the gap value in the $^1S_0$ phase. In the following we assume that $\gamma$ takes this same value in the $^3P_2$ phase, where $\Delta$ is the angular averaged gap in that phase. Therefore, taking into account that $\gamma < 0$, the first allowed phonon scattering processes are binary collisions. It might be interesting to compute the value of $\gamma$ for more realistic neutron-neutron interactions.

3. Equation of state and the gap of neutron matter in superfluid neutron stars

The EoS in neutron stars is needed so as to determine not only the speed of sound at $T = 0$ but also the phonon self-couplings. The theoretical description of matter in neutron stars is a complicated task given that the EoS spans over a wide range of densities, temperatures and isospin asymmetries (see, for example, the recent review of Ref. [11]). More precisely, inside the core of neutron stars nuclear matter can be described using different theoretical many-body schemes. Those are usually divided between microscopic ab-initio approaches and phenomenological models. Whereas microscopic ab-initio approaches obtain the EoS solving the many-body problem from two-body and three-body interactions, that are fitted to experimental data on scattering and finite nuclei, phenomenological models rely on density-dependent interactions adjusted to nuclear observables and neutron star predictions.

Among the first ones, one finds the well-known variational method of Akmal, Pandharipande and Ravenhall (APR) [12], which is commonly used as a benchmark for other EoSs. Heiselberg and Hjorth-Jensen parametrized the APR EoS in a simple and handleable form in Ref. [13]. In this paper we will make use of this EoS. Note that the effect of neutron pairing in the EoS is not taken into account as $\Delta/\mu << 1$.

In the upper panel of Fig. 1 we show the speed of sound versus the speed of light $c_s/c$ in $\beta$-stable nuclear matter as a function of the density. We observe that relativistic effects become important for densities of the order of 1.5-2 $n_0$, as the speed of sound increases monotonically with density.

With regards to the value of the gap of superfluid matter, in the lower panel of Fig. 1 we show the two gap models we will use in this work as a function of the density. These extreme cases have been chosen so as to account for the model dependence of our results. These models are named $^1S_0(A)+^3P_2(i)$ and $^1S_0(a)+^3P_2(h)$, and take into account a wide range of gap values inside the core of neutron stars.
The $^1S_0(A) + ^3P_2(i)$ scheme results from the combination, on the one hand, of the $^1S_0$ neutron gap coming from the parametrization $A$ of Table I in Ref. [14], that takes into account the BCS approach of different nuclear interactions with a maximum gap of approximately 3 MeV at $p_F \approx 0.85\text{fm}^{-1}$, and, on the other hand, the anisotropic $^3P_2$ neutron gap from the parametrization $i$ (strong neutron superfluidity in the core) of Table I in Ref. [14], a model dependent result that goes beyond BCS theory. The $^1S_0(a) + ^3P_2(h)$ goes beyond BCS for the $^1S_0$ neutron gap as it incorporates medium polarization effects (parametrization $a$), reducing the maximum value to 1 MeV, whereas the $^3P_2$ neutron gap comes from the parametrization $h$ (strong neutron superfluidity) with a maximum value of about 0.5 MeV. For both models, the transition temperatures from the superfluid to the normal phase are $T_c \sim 1/2\Delta \gtrsim 0.25 \times 10^8\text{K}$.

4. The shear viscosity of superfluid phonons

The shear viscosity $\eta$ emerges as a dissipative term in the energy-momentum tensor $T_{ij}$. If one performs small deviations from equilibrium, one finds that

$$\delta T_{ij} = -\eta \bar{V}_{ij} \equiv -\eta \left( \partial_i V_j + \partial_j V_i - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{V} \right),$$

being $\mathbf{V}$ the fluid velocity of the normal component of the system.

The superfluid phonon contribution to $T_{ij}$ is given by

$$T_{ij} = c_s^2 \int d^3 p \frac{p_ip_j}{(2\pi)^3 E_p} f(p, x),$$

where $f$ is the phonon distribution function that obeys the Boltzmann equation [9]

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial E_p}{\partial p} \cdot \nabla f = C[f],$$

assuming to be in the superfluid rest frame, and $C[f]$ being the collision term. As mentioned before, in order to obtain the shear viscosity due to superfluid phonons is enough to consider binary collisions in the collision term. Those are depicted in Fig. 2.

The shear viscosity is calculated by considering small departures from equilibrium to the phonon distribution function and, afterwards, linearizing the corresponding transport equation [3]. Then, we use variational methods so as to solve the transport equation [15–17]. The expression for the shear viscosity then reads [3]

$$\eta = \left( \frac{2\pi}{15} \right)^4 T^8 \frac{1}{c_s^5} \frac{1}{M},$$

where $M$ is a multidimensional integral that takes into account the thermally weighted scattering matrix for phonons.

On the upper panel of Fig. 3 we display the shear viscosity due to the binary phonon collisions as a function of the temperature for different densities. We find that $\eta \propto 1/T^5$. In fact, this is a universal feature that occurs in other superfluid systems, such as $^4\text{He}$ [9] or superfluid cold atoms at unitary [17,18]. The value of the $\eta$ is, however, determined by the choice of EoS.

The next question is at which densities and temperatures the hydrodynamical regime can be reached. The hydrodynamical regime is only achieved when the mean free path (mfp) is smaller than the typical macroscopic length of the system, in our case the radius of the star. Therefore, on the lower panel of Fig. 3
where we present the mfp of phonons for different densities as a function of temperature, also showing the radius of the star, which we take of 10 Km. The mfp \( l \) results from the calculation of \( \eta \) in Ref. [16]

\[
l = \frac{\eta}{n < p >},
\]

\[
< p > = 2.7 \frac{T}{c_s}, \quad n = \int \frac{d^3p}{(2\pi)^3} f_p = \xi(3) \frac{T^3}{\pi^2 c_s^3},
\]

with \( < p > \) the thermal average momentum, and \( n \) the phonon density.

We find that a hydrodynamical description starts being questionable for temperatures below \( T \sim 10^9 \) K. Note that the critical temperature for the phase transition to the normal phase is \( T_c \sim 10^{10} \) K. The value for the critical temperature as well as the values for the phonon mfp depend on the model for the EoS.

5. Bulk viscosities

Apart from the shear viscosity, another dissipative term that appears in the energy-momentum tensor is the bulk viscosity \( \zeta \). However, in superfluid matter there exist four bulk viscosity coefficients [9]. Three of them, \( \zeta_1, \zeta_3, \zeta_4 \), are associated to dissipative processes leading to entropy production related to the space–time dependent relative motion between the superfluid and normal fluid components, whereas \( \zeta_2 \) is the analogue to the one in a normal superfluid.

Following the dynamical evolution of the phonon number density, developed by Khalatnikov [9], the bulk viscosities can be determined. This method is equivalent to calculating the bulk viscosities by means of the Boltzmann equation for phonons in the relaxation time approximation, as shown in Ref. [19]. In the case of small departures from equilibrium and small values of the normal and superfluid velocities, one obtains [9]

\[
\zeta_i = \frac{T}{\Gamma_{ph}} C_i, \quad i = 1, 2, 3, 4,
\]

with \( \Gamma_{ph} \) the phonon decay rate and

\[
C_1 = C_4 = -I_1 I_2, \quad C_2 = I_2^2, \quad C_3 = I_1^2,
\]

where \( I_1 \) and \( I_2 \) are

\[
I_1 = \frac{60T^5}{7c_s^5 \pi^2} \left( \pi^2 \zeta(3) - 7\zeta(5) \right) \left( c_s \frac{\partial B}{\partial \rho} - B \frac{\partial c_s}{\partial \rho} \right),
\]

\[
I_2 = -\frac{20T^5}{7c_s^5 \pi^2} \left( \pi^2 \zeta(3) - 7\zeta(5) \right) \left( 2Bc_s + 3p \left( c_s \frac{\partial B}{\partial \rho} - B \frac{\partial c_s}{\partial \rho} \right) \right),
\]

with \( B = c_s \gamma \) and \( \zeta(n) \) the Riemann zeta function. We should indicate that one has to consider the phonon dispersion law beyond linear order so as to have non-vanishing bulk viscosities. Moreover, due the Onsager symmetry principle, \( \zeta_1 = \zeta_4 \), whereas positive entropy production leads to \( \zeta_2, \zeta_3 \geq 0 \) and \( \zeta_1^2 \leq \zeta_2 \zeta_3 \).

In the case of astrophysical phenomena, the bulk viscosities are calculated for periodic perturbations leading the system away from equilibrium. In Ref. [20] it was shown that

\[
\zeta_i(\omega) = \frac{1}{1 + \left( \omega I_1 \frac{\partial \rho}{\partial \rho} \frac{\partial \rho}{\partial \rho} \frac{T}{\Gamma_{ph}} \right)^2} \frac{T}{\Gamma_{ph}} C_i, \quad i = 1, 2, 3, 4,
\]
defining a characteristic value for the frequency $\omega_c$ for the phonon collisions as

$$\omega_c = \frac{1}{\frac{\partial^2 \rho}{\partial n \partial \mu \partial T}} \Gamma_{ph}. \tag{23}$$

Note that when $\omega \ll \omega_c$ one recovers the static bulk viscosity coefficients of Eqs. (19).

In order to go back to equilibrium after an expansion of the superfluid, the system needs to change the number of phonons. Therefore, $\Gamma_{ph}$ includes the first number-changing mechanisms allowed by kinematics. In the case of phonons with a negative $\gamma$ in the dispersion law, those are $2 \leftrightarrow 3$ collisions. The different $2 \leftrightarrow 3$ processes are given in Figs. 4 and 5, and can also be found in Ref. [5]. Those are indicated as Type I, given in Fig. 4, and Type II, given in Fig. 5, using the phonon propagators with a NLO dispersion law, as discussed in Ref. [5].

The frequency–dependent $\zeta_2$ coefficient at $4\hbar_0$ for frequencies of $\omega = 10^3 - 10^5 \text{s}^{-1}$, typical for stellar pulsations, is shown as a function of temperature in Fig. 6. On the l.h.s. of Fig. 6 we show the $\zeta_2$ coefficient when using the $^1S_0(A) + ^3P_2(i)$ neutron gap model. The value of $\zeta_2$ for $\omega = 10^3 \text{s}^{-1}$ differs by more than 10% from its static value for only $T \gtrsim 10^{10}\text{K}$, whereas the difference is larger for $T \gtrsim 10^9\text{K}$ in the case of $\omega = 10^5\text{s}^{-1}$. As for the case when using the $^1S_0(a) + ^3P_2(h)$ model on the r.h.s. of Fig. 6, we see that $\zeta_2$ strongly depends on the frequency. We did not plot $\zeta_1$ and $\zeta_3$ coefficients, as we expect a similar behaviour with frequency.

The frequency–dependent $\zeta_2$ bulk viscosity coefficient resulting from the collisions of superfluid phonons has to be compared to contributions coming from direct Urca [21] and modified Urca [22]. We see that phonon collisions are the leading contribution to the bulk viscosities in the core for $T \sim 10^9\text{K}$ and for typical radial pulsations. The different conclusion reached in Ref. [5] is due to the fact that the comparison was performed only for Urca processes in normal matter, as the phonon collisions are the most important contribution until the opening of the Urca processes.

6. The r-mode instability window

R-mode oscillations (non-radial fluid oscillations whose dynamics is dominated by rotation) of neutron stars have been extensively studied because they appear to be subject to the Chandrasekhar-Friedman-Schutz gravitational radiation instability in realistic astrophysical conditions (see, for example, reviews [23] and [24]). A rapidly rotating neutron star could emit a significant fraction of its rotational energy and angular momentum as gravitational waves, which can be detected by interferometers. If the timescales due to viscous processes in neutron-star matter are shorter than the gravitational-radiation driving timescale, r-modes are damped. There is a typical instability region at relatively high frequencies, whereas the star is stable for low frequencies, or at very low or high temperatures, due to viscous damping mechanisms [25,26]. Given that the spin rate of various neutron stars falls in the instability window, the r-mode studies are trying to find new solutions to this puzzle.

The r-mode instability window is determined by the computation of the different time scales associated to gravitational wave emission $\tau_{GR}$, and to the different dissipative processes that could damp the r-mode. In this paper we review the effect of the shear viscosity, $\tau_\eta$, in the r-mode instability window. The analysis of time scales involving the bulk viscosities due to superfluid phonons are more involved, as not only one but three independent bulk viscosities appear, and are matter of future works. Thus, the different time scales for gravitational wave emission and shear viscosity are given by

$$-\frac{1}{|\tau_{GR}(\Omega)|} + \frac{1}{\tau_\eta(T)} = 0, \tag{24}$$
with
\[
\frac{1}{\tau_{\text{GR}}(\Omega)} = \frac{32 \pi G \Omega^{2l+2}}{c^{2l+3}} \frac{(l-1)^{2l}}{(2l+1)!!} \left( \frac{l+2}{l+1} \right)^{2l+2} \int_0^R \rho r^{2l+2} dr ,
\]
(25)
and
\[
\frac{1}{\tau_\eta(T)} = (l-1)(2l+1) \int_{R_c}^R \eta r^{2l} dr \left( \int_0^R \rho r^{2l+2} dr \right)^{-1} ,
\]
(26)
with the typical time scales summarized in [25] (in C.G.S. units).

In this paper we concentrate our discussion on the study of the dominant r-modes with \( l = 2 \) [23,25] and the dominant damping mechanisms in the core of neutron stars. In fact, the shear viscosity coming from electron collisions is one of the most efficient damping mechanisms in neutron stars [27]. Therefore, in this paper we review the shear viscosity as arising from electrons as well as phonons. With regards to the characteristic time associated to the phonon shear viscosity, we only consider the phonon contribution in the hydrodynamical regime. We then introduce a temperature-dependent lower cut \((R_c)\) in the integral for the shear viscosity in Eq. (26).

The r-mode instability window coming from the different shear viscous damping mechanisms is shown in Fig. 7. The results are displayed for 1.4 \( M_\odot \) and 1.93 \( M_\odot \) mass configurations. The quantity \( \Omega_0 = \sqrt{G \pi \bar{\rho}} \) is the critical frequency, given in terms of the average mass density of the star, \( \bar{\rho} \). For the case of electrons, we show the estimate coming from the longitudinal and transverse plasmon exchange in superconducting matter with a transition temperature of \( T_{cp} \sim 10^9 \text{K} \) [27] (dashed lines). As for the hydrodynamical phonons, we indicate our results in dashed-double-dotted lines. And, finally, we add both contributions, indicated in solid lines. By analyzing those results, we find that the electron contribution dominates for low temperatures, whereas the contribution of hydrodynamic phonon dissipation is relevant for \( T \gtrsim 7 \times 10^8 \text{K} \) for a star of 1.4 \( M_\odot \) and \( T \gtrsim 10^9 \text{K} \) for 1.93 \( M_\odot \).

7. Thermal conductivity

Another transport coefficient that we would like to discuss is the thermal conductivity due to superfluid phonon collisions. The thermal conductivity \( \kappa \) relates the heat flux with the temperature gradient in the hydrodynamical regime
\[
q = -\kappa \nabla T .
\]
(27)

In order to obtain this coefficient, we use variational methods to solve the transport equation, as done for the shear viscosity. In this case, the thermal conductivity is given by [6]
\[
\kappa \geq \left( \frac{4a_1^2}{3T^2} \right) A_1^2 M_{11}^{-1} ,
\]
(28)
with \( M_{11}^{-1} \) the \((1,1)\) element of the inverse of a matrix of \( N \times N \) dimensions, being \( N \) a variational parameter. This matrix contains different elements that are multidimensional integrals with thermally weighted phonon scattering matrix. Note that, as in the case of bulk viscosities, the thermal conductivity requires the dispersion law at NLO, because the thermal conductivity vanishes with a linear dispersion law [28].

We show the variational calculation of the thermal conductivity up to order \( N = 6 \) for nuclear saturation density \( n_0 \) as a function of temperature on the upper panel of Fig. 8. The last value \( N \) is determined once the final results do not deviate more than 10% from those in the previous iteration. The value \( T_c \) is the end temperature, which turns out to be \( T_c = 0.57 \Delta(n_0) = 3.4 \times 10^9 \text{K} \), with \( \Delta(n) \) from Fig. 1.
for the $^1S_0(A) + ^3P_2(i)$ gap model. We find that for $T \lesssim 10^9$ K, well below $T_c$, the thermal conductivity scales as $\kappa \propto 1/\Delta^6$. The proportionality factor depends on the EoS. This temperature-independent behaviour was also observed for the color-flavor-locked superfluid [28]. Note that close to $T_c$, higher-order corrections in energy and momentum might be expected in the phonon dispersion law and self-interactions.

Associated to the thermal conductivity, one can also determine the mfp for phonons, which is different from the one coming from the shear viscosity (see Eqs. (17,18) to compare). The mfp resulting from the thermal conductivity is given by

$$ l = \frac{\kappa}{\frac{1}{3} c_v c_s}, $$

$$ c_v = \frac{2\pi^2}{15 c_s^2} \left( T^3 + \frac{25\gamma (2\pi)^2}{7} c_s^2 T^5 \right), $$

with $c_v$ the heat capacity for phonons [9].

On the lower panel of Fig. 8 we display the mfp of phonons in $\beta$-stable neutron star matter for different densities ($0.5n_0$ in the left panel, $n_0$ in the middle panel and $2n_0$ in the right panel) as a function of the temperature in the case of the $^1S_0(A) + ^3P_2(i)$ gap model. For comparison, we also show the radius of the star of 10 Km with an horizontal line. We find that for $n = 0.5n_0$ the superfluid phonon mfp stays below the radius of the star. This is also the case for $n = n_0$ and $T \gtrsim 6 \times 10^8$ K and for $2n_0$ and $T \gtrsim 3 \times 10^9$ K. Moreover, $l \propto 1/T^3$, coming from the temperature-independent behaviour of the thermal conductivity. For smaller values of the gap, as the $^1S_0(a) + ^3P_2(h)$ case, the thermal conductivity would be orders of magnitude higher than the previous case, being away from the hydrodynamical regime.

Again, our results must be compared to those coming from electrons and muons in neutron stars. The electron-muon contribution to the thermal conductivity was analyzed in Ref. [29]. Compared to these results, we find that phonons in the hydrodynamical regime dominate the thermal conductivity in neutron stars [6]. We also conclude that if the contribution of electrons-muons and phonons to the thermal conductivity get comparable, electron-phonon collisions could play an important role. Simple estimates have been performed in Ref. [30]. This topic deserves further studies.

8. Neutrino emissivity and the superfluid phonon

The cooling of a neutron star is very much affected if superfluidity is achieved in its core. Close to the superfluid phase transition, the neutrino emissivity is dominated by the formation/breaking of Cooper pairs [31–33]. However, at much lower temperatures, these processes are exponentially suppressed, and the neutrino emissivity is dominated by scatterings involving the Goldstone modes of the system, as these couple with the $Z^0$ electroweak gauge boson. In Refs.[34,35] the neutrino emissivity involving the angulons, the Goldstone modes associated to the spontaneous breaking of the rotational symmetry that occur in a $^3P_2$ neutron superfluid phase were considered, while the contribution associated to the superfluid phonons was not taken into account.

The superfluid phonon interacts with the electroweak $Z^0$ gauge boson, which, in turn, can decay into a neutrino-antineutrino pair. Standard EFT techniques can be used to determine the Lagrangian associated to these electroweak processes [34]

$$ \mathcal{L}_{EW} = -f_0 C_V Z_0^0 \partial_0 \phi + \cdots + g_Z \bar{\nu} \gamma^\mu (1 - \gamma_5) \nu, $$

(31)
where

\[ C_{V}^2 = \frac{G_{F}M_{Z}^2}{2\sqrt{2}}, \quad g_{Z\bar{\nu}\nu}^2 = \frac{G_{F}M_{Z}^2}{2\sqrt{2}}, \quad (32) \]

where \( G_{F} \) is the Fermi constant, and \( M_{Z} \) is the mass of the \( Z_{\mu} \) electroweak gauge boson, and

\[ f_{0} = 2\frac{\sqrt{\rho}}{c_{s}m}, \quad (33) \]

where \( \rho \) is the mass density, and \( m \) is the neutron mass. Note that in Ref. [34] \( f_{0} \) is the superfluid decay constant, different from the constant used here. The difference comes in how the superfluid field is normalized with respect to the phase of neutron-neutron condensate, which in our case is not the same as that used in Ref. [34], see Eq. (3).

Energy and momentum conservation prevent the possibility that a superfluid phonon may decay into a neutrino-antineutrino pair. However, the process

\[ \phi + \phi \rightarrow \phi + \nu + \bar{\nu} \quad (34) \]

is kinematically allowed. This process has been considered in the color-flavor lock superfluid quark matter [36]. A rough estimate of the neutrino emissivity associated to this process reveals that it is very much suppressed, involving the \( \phi\phi \) binary scattering, and with dependence of the four coupling constant. Parametrically, a naive dimensional analysis gives

\[ \mathcal{E} \sim G_{F}^2 T^9 \frac{T^8}{m^4 \rho}. \quad (35) \]

In the temperature regime where the Goldstone modes might be relevant (say \( T < 10^9K \)), the ratio in Eq. (35) suppresses very much the strength of this channel, in front of that involving only the angulons, which parametrically behave as \( \mathcal{E} \sim G_{F}^2 T^9 \) [34], for example. Superfluid phonons do not play thus any relevant role in the cooling of the star by neutrino emission.

9. Superfluid phonons in the presence of gravity and in a moving background

In the computation of the transport coefficients in superfluid neutron stars associated to superfluid phonons we have assumed that these move in a static medium, and we have ignored the effect of the gravitational field. These two effects might be easily incorporated in our EFT approach, let us explain how.

In what follows we deal with the case of a relativistic superfluid, where all the EFT techniques used for the non-relativistic case also apply, with minor changes. More particularly, we use \( X = (g^{\mu\nu}D_{\mu}\phi D_{\nu}\phi)^{1/2} \), with \( D_{\mu}\phi = \partial_{\mu}\phi - \mu_{\mu}, A_{\mu}, \) and \( A_{\mu} = (1, 0, 0, 0) \), and \( g^{\mu\nu} \) is the gravitational metric. The non-relativistic case can be obtained from the relativistic case [37], by simply approximating the \( X \) function, taking into account also that a non-relativistic potential is defined as \( \mu = \mu_{\mu} - m \). In this case, it is only necessary to keep the \((0,0)\) component of the metric, which is related to the Newtonian gravitational potential by \( \delta g_{00} = -2\Phi \). Thus, in the presence of a gravitational potential Eq. (2) is replaced by

\[ X_{g} = \mu - \partial_{\mu} \phi - \frac{\left(\nabla \phi\right)^2}{2m} - m\Phi. \quad (36) \]

Thus, even if the superfluid phonons are massless, they are affected by the presence of a gravitational field, even in the non-relativistic case.

Even if we ignore the effects of a gravitational metric, as we will do in what follows, taking into account the effects of a moving superfluid medium in the superfluid phonons is much more conveniently
done with the use of a gravitational analogue model [38–40]. It was first suggested in Ref. [41], but we will follow a different approach here, using the superradial phonon EFT. From the expression of the phonon Lagrangian in terms of $X$ it is possible to derive the EFT of the phonons moving in the background of the superfluid. The superfluid phonon is the Goldstone boson associated to the breaking of the $U(1)$ symmetry and it can be introduced as the phase of quantum condensate. However, according to Landau’s two fluid model of superfluidity, the gradient of the phase of the condensate defines the velocity of the pure superfluid component. Then it should be possible to decompose the phase of the condensate into two fields, the first describing the hydrodynamical variable, the second describing the quantum fluctuations associated to the phonons. Thus, we write

$$\varphi(x) = \bar{\varphi}(x) + \phi(x).$$ (37)

This splitting implies a separation of scales - the background field $\bar{\varphi}(x)$ is associated to the long-distance and long-time scales, while the fluctuation $\phi(x)$ is associated to rapid and small scale variations, and is identified with the superfluid phonon. The gradient of $\bar{\varphi}(x)$ is proportional to the hydrodynamical velocity

$$v_\rho = -\frac{D_\rho \bar{\varphi}}{\bar{\mu}}, \quad \bar{\mu} \equiv (D_\rho \bar{\varphi} D^\rho \bar{\varphi})^{1/2}. \quad (38)$$

The classical equations of motion associated to $\bar{\varphi}(x)$ can be conveniently expressed as the hydrodynamical equations of a perfect relativistic fluid

$$\partial_\nu (\bar{n}_0 v^\nu) = 0,$$ (39)

where

$$\bar{n}_0 = \frac{dP}{d\mu} |_{\mu = \bar{\mu}}$$ (40)

is interpreted as the particle density. The energy-momentum tensor can be written in terms of the velocity defined in Eq. (38) and Noether’s energy-density $\rho_0$,

$$T^{\rho\sigma}_{0 \rho} = (\bar{n}_0 \bar{\mu}) v^\rho v^\sigma - \eta^{\rho\sigma} P_0 = (\rho_0 + P_0) v^\rho v^\sigma - \eta^{\rho\sigma} P_0,$$ (41)

where we have written $\rho_0 + P_0 = \bar{n}_0 \bar{\mu}$, with $P_0$ the pressure evaluated at $\bar{\mu}$. The energy-momentum tensor is conserved

$$\partial_\rho T^{\rho\sigma}_{0 \rho} = 0,$$ (42)

and traceless $T^{\rho\sigma}_{0 \rho} = 0$.

From the low energy effective action of the system

$$S[\varphi] = \int d^4 x \, \mathcal{L}_{\text{eff}}[\partial \varphi],$$ (43)

we deduce the effective action for the phonon field expanding around the stationary point corresponding to the classical solution $\bar{\varphi}$

$$S[\varphi] = S[\bar{\varphi}] + \frac{1}{2} \int d^4 x \, \frac{\partial^2 \mathcal{L}_{\text{eff}}}{\partial (\partial_\mu \varphi) \partial (\partial_\nu \varphi)} \bigg|_{\varphi = \bar{\varphi}} \partial_\mu \varphi \partial_\nu \varphi + \cdots, \quad (44)$$

$$= = \frac{1}{2} \int d^4 x \sqrt{-G} \, G^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi, \quad (45)$$
where we have defined the acoustic metric tensor
\[ \mathcal{G}^{\mu\nu} = \eta^{\mu\nu} + \frac{1}{c_s^2} \mathbf{v}^\mu \mathbf{v}^\nu , \tag{46} \]
and the determinant \( \mathcal{G} = 1/\det|\mathcal{G}^{\mu\nu}|. \) Thus, in the background of a moving superfluid, the superfluid phonon propagate as in the background of the so-called acoustic metric. Note that in the presence of a real gravitational field one simply has to write the corresponding metric \( g^{\mu\nu} \) above instead of the Minkowskian metric \( \eta^{\mu\nu} \).

The transport equation associated to the superfluid phonons should then incorporate the effect of the acoustic metric. It should be written as
\[ L[f] = p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^\nu_{\mu\alpha} p^\mu p^\gamma \frac{\partial f}{\partial p^\gamma} = C[f] \tag{47} \]
that is the general relativistic version of the Boltzmann equation. The Christoffel symbols of the phonon transport equation are those related to the acoustic metric. Quite interestingly, from this approach it is possible to deduce that even in the collisionless limit, the number of phonons might not be conserved, as they are covariantly conserved
\[ \partial_\nu n^\nu_{\text{ph}} + \Gamma^\mu_{\mu\nu} n^\nu_{\text{ph}} = 0 . \tag{48} \]
This means that even in collisionless processes the number of phonons might not be conserved. This is reflected in the second term in the l.h.s of the above equation where a Christoffel symbol appears signaling that propagation of phonons is taking place in “curved” space-time. Since we can write
\[ \Gamma^\mu_{\mu\alpha} = \frac{1}{\sqrt{-\mathcal{G}}} \partial_\alpha \sqrt{-\mathcal{G}} = \frac{1}{c_s} \frac{\partial c_s}{\partial x^\alpha} . \tag{49} \]
Thus, the variation of the speed of sound inside the star might act as a source, or sink, of superfluid phonons.

In all our developments presented in this review article we have ignored all the above effects to simplify the computations. We might expect that if the mfp of the phonons is shorter than the variation of both the gravitational potential, or of the speed of sound, we can simply ignore the effects discussed in this Section. Unfortunately, the phonon mfp tends to increase when the temperature drops. Thus, the evaluation of transport should be reformulated along the lines here discussed.

An interesting observation has been formulated in Ref. [42]. While it was generally believed that sound waves do not transport mass, in that reference it is claimed that they do carry gravitational mass if non-linear order effects are considered. That is to say, they are affected by gravity, as we have just seen, and also generate a tiny gravitational field. In particular, this applies to the superfluid phonon field. The superfluid phonon can be considered as the quanta of sound waves, and even in the non-relativistic case, and associated wave packet with energy \( E \) carries a gravitational mass
\[ M \approx -\rho \frac{dc_s}{c_s} E \frac{E}{c_s} \tag{50} \]
It is however not obvious how this mass might have an effect in the transport phenomena of neutron stars, as claimed in [42].
10. Summary

We have presented an overview of the computation of the shear and bulk viscosities together thermal conductivity due to superfluid phonons inside neutron stars, based on an effective field theory for the interaction among superfluid phonons. The effective field theory approach is universal, valid for different superfluid systems. As the superfluid phonons couple to the $Z$ electroweak gauge boson, they open a channel to neutrino emissivity, but we have checked that it is very much suppressed, so it can hardly affect the cooling of the star.

With regards to the shear viscosity, we have found that the shear viscosity coming from binary collisions of superfluid phonons scales with $1/T^5$, as seen in $^4$He and cold Fermi gases at unitary. Whether the temperature dependence is a universal feature, the evolution of the shear viscosity with density is determined by equation of state under beta-stable equilibrium.

As for the bulk viscosities in superfluid neutron matter, we have seen that the bulk viscosity coefficients are highly dependent on the superfluid neutron matter gap. Nevertheless, phonon-phonon collisions rule the bulk viscosity over the Urca and modified Urca.

We have further studied the r-mode instability in neutron stars and the consequences of the shear viscosity coming from superfluid phonons. We have determined that the r-mode instability window would be modified for $T \gtrsim 10^8 - 10^9$ K, depending on the exact neutron star mass configuration.

Also, we have determined a temperature-independent behaviour of the thermal conductivity due to phonons well below the transition temperature, while scaling as $1/\Delta^6$, similarly to color-flavor-locked phase. Furthermore, the thermal conductivity is dominated by phonon-phonon interactions in comparison with electron-muon collisions for densities in the core of neutron stars.

We have finally discussed how the superfluid phonon effective field theory, and ultimately their interactions, is modified in the presence of a gravitational field, or by taking into account that the superfluid is not at rest. In particular, given that the mean free path of phonons tends to increase when temperature drops, it would be interesting to evaluate the effect of a gravity on the transport coefficients we have reviewed.

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References
1. Landau, L.D. The theory of superfluidity of helium II. J. Phys. (USSR) 1941, 5, 71–100.
2. Migdal, A. Soviet Physics JETP 1960, 10, 176.
3. Manuel, C.; Tolos, L. Shear viscosity due to phonons in superfluid neutron stars. Phys. Rev. D 2011, 84, 123007, [arXiv:astro-ph.SR/1110.0669]. doi:10.1103/PhysRevD.84.123007.
4. Manuel, C.; Tolos, L. Shear viscosity and the r-mode instability window in superfluid neutron stars. Phys. Rev. D 2013, 88, 043001, [arXiv:astro-ph.SR/1212.2075]. doi:10.1103/PhysRevD.88.043001.
5. Manuel, C.; Tarrus, J.; Tolos, L. Bulk viscosity coefficients due to phonons in superfluid neutron stars. JCAP 2013, 07, 003, [arXiv:astro-ph.HE/1302.5447]. doi:10.1088/1475-7516/2013/07/003.
6. Manuel, C.; Sarkar, S.; Tolos, L. Thermal conductivity due to phonons in the core of superfluid neutron stars. Phys. Rev. C 2014, 90, 055803, [arXiv:astro-ph.SR/1407.7431]. doi:10.1103/PhysRevC.90.055803.
7. Son, D. Low-energy quantum effective action for relativistic superfluids 2002. [hep-ph/0204199].
8. Son, D.; Wingate, M. General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas. *Annals Phys.* 2006, 321, 197–224, [cond-mat/0509786]. doi:10.1016/j.aop.2005.11.001.
9. Khalatnikov, I.M.; Khalatnikov, I.M. *An introduction to the theory of superfluidity*; Frontiers in Physics, Benjamin: New York, NY, 1965. Trans. from the Russian.
10. Escobedo, M.A.; Manuel, C. Effective field theory and dispersion law of the phonons of a non-relativistic superfluid. *Phys. Rev. A* 2010, 82, 023614, [arXiv:cond-mat.quant-gas/1004.2567]. doi:10.1103/PhysRevA.82.023614.
11. Tolos, L.; Fabbietti, L. Strangeness in Nuclei and Neutron Stars. *Prog. Part. Nucl. Phys.* 2020, 112, 103770, [arXiv:nucl-ex/2002.09223]. doi:10.1016/j.ppnp.2020.103770.
12. Akmal, A.; Pandharipande, V.; Ravenhall, D. The Equation of state of nucleon matter and neutron star structure. *Phys. Rev. C* 1998, 58, 1804–1828, [nucl-th/9804027]. doi:10.1103/PhysRevC.58.1804.
13. Heiselberg, H.; Hjorth-Jensen, M. Phases of dense matter in neutron stars. *Phys. Rept.* 2000, 328, 237–327, [nucl-th/9902033]. doi:10.1016/S0370-1573(99)00110-6.
14. Andersson, N.; Comer, G.; Glampedakis, K. How viscous is a superfluid neutron star core? *Nucl. Phys. A* 2005, 763, 212–229, [astro-ph/0411748]. doi:10.1016/j.nuclphysa.2005.08.012.
15. Manuel, C.; Dobado, A.; Llanes-Estrada, F.J. Shear viscosity in a CFL quark star. *JHEP* 2005, 09, 076, [hep-ph/0406058]. doi:10.1088/1126-6708/2005/09/076.
16. Alford, M.G.; Braby, M.; Mahmoodifar, S. Shear viscosity due to kaon condensation in color-flavor locked quark matter. *Phys. Rev. C* 2010, 81, 025202, [arXiv:nucl-th/0910.2180]. doi:10.1103/PhysRevC.81.025202.
17. Haensel, P.; Levenfish, K.; Yakovlev, D. Bulk viscosity in superfluid neutron star cores. 2. Modified Urca processes in npe mu matter. *Astron. Astrophys.* 2000, 357, 1157–1169, [astro-ph/0004183].
18. Andersson, N.; Kokkotas, K.D. The R mode instability in rotating neutron stars. *Int. J. Mod. Phys. D* 2001, 10, 381–442, [gr-qc/0010102]. doi:10.1142/S0218271801001062.
19. Lindblom, L. Neutron star pulsations and instabilities. *ICTP Lect. Notes Ser.* 2001, 3, 257–276, [astro-ph/0101136].
20. Lindblom, L.; Owen, B.J.; Morsink, S.M. Gravitational radiation instability in hot young neutron stars. *Phys. Rev. Lett.* 1998, 80, 4843–4846, [gr-qc/9803053]. doi:10.1103/PhysRevLett.80.4843.
21. Andersson, N.; Kokkotas, K.D.; Schutz, B.F. Gravitational radiation limit on the spin of young neutron stars. *Astrophys. J.* 1999, 510, 846, [astro-ph/9805225]. doi:10.1086/306625.
22. Shpelnin, P.; Yakovlev, D. Shear viscosity in neutron star cores. *Phys. Rev. D* 2008, 78, 063006, [arXiv:astro-ph/0808.2018]. doi:10.1103/PhysRevD.78.063006.
23. Shpelnin, P.; Yakovlev, D. Electron-muon heat conduction in neutron star cores via the exchange of transverse plasmons. *Phys. Rev. D* 2007, 75, 103004, [arXiv:astro-ph/0705.1963]. doi:10.1103/PhysRevD.75.103004.
30. Bedaque, P.F.; Reddy, S. Goldstone modes in the neutron star core. *Phys. Lett. B* 2014, 735, 340–343, [arXiv:nucl-th/1307.8183]. doi:10.1016/j.physletb.2014.06.033.

31. Flowers, E.; Ruderman, M.; Sutherland, P. Neutrino pair emission from finite-temperature neutron superfluid and the cooling of young neutron stars. *Astrophys. J.* 1976, 205, 541. doi:10.1086/154308.

32. Voskresensky, D.; Senatorov, A. Description of Nuclear Interaction in Keldysh’s Diagram Technique and Neutrino Luminosity of Neutron Stars. (In Russian). *Sov. J. Nucl. Phys.* 1987, 45, 411.

33. Yakovlev, D.; Kaminker, A.; Levenfish, K. Neutrino emission due to Cooper pairing of nucleons in cooling neutron stars. *Astron. Astrophys.* 1999, 343, 650, [astro-ph/9812366].

34. Bedaque, P.F.; Rupak, G.; Savage, M.J. Goldstone bosons in the 3P(Z) superfluid phase of neutron matter and neutrino emission. *Phys. Rev. C* 2003, 68, 065802, [nucl-th/0305032]. doi:10.1103/PhysRevC.68.065802.

35. Bedaque, P.; Sen, S. Neutrino emissivity from Goldstone boson decay in magnetized neutron matter. *Phys. Rev. C* 2014, 89, 035808, [arXiv:nucl-th/1312.6632]. doi:10.1103/PhysRevC.89.035808.

36. Jaikumar, P.; Prakash, M.; Schäfer, T. Neutrino emission from Goldstone modes in dense quark matter. *Phys. Rev. D* 2002, 66, 063003, [astro-ph/0203088]. doi:10.1103/PhysRevD.66.063003.

37. Nicolis, A.; Penco, R. Mutual Interactions of Phonons, Rotons, and Gravity. *Phys. Rev. B* 2018, 97, 134516, [arXiv:hep-th/1705.08914]. doi:10.1103/PhysRevB.97.134516.

38. Manuel, C.; Llanes-Estrada, FJ. Bulk viscosity in a cold CFL superfluid. *JCAP* 2007, 08, 001, [arXiv:hep-ph/0705.3909]. doi:10.1088/1475-7516/2007/08/001.

39. Mannarelli, M.; Manuel, C. Transport theory for cold relativistic superfluids from an analogue model of gravity. *Phys. Rev. D* 2008, 77, 103014, [arXiv:hep-ph/0802.0321]. doi:10.1103/PhysRevD.77.103014.

40. Mannarelli, M.; Grassi, D.; Trabucco, S.; Chiofalo, M.L. Hawking temperature and phonon emission in acoustic holes 2020. [arXiv:gr-qc/2011.00019].

41. Volovik, G. Superfluid analogies of cosmological phenomena. *Phys. Rept.* 2001, 351, 195–348, [gr-qc/0005091]. doi:10.1016/S0370-1573(00)00139-3.

42. Esposito, A.; Krichevsky, R.; Nicolis, A. Gravitational Mass Carried by Sound Waves. *Phys. Rev. Lett.* 2019, 122, 084501, [arXiv:gr-qc/1807.08771]. doi:10.1103/PhysRevLett.122.084501.
Figure 1. Upper panel: \( c_s/c \) for \( \beta \)-stable nuclear matter as a function of particle density. Lower panel: The combination of \( ^1S_0 \) and angle-averaged \( ^3P_2 \) neutron gaps as a function of density for two different gap cases. Figures adapted from [3,5].

Figure 2. \( 2 \leftrightarrow 2 \) phonon scattering processes contributing to the shear viscosity. Figure adapted from [3].
Figure 3. Upper panel: Shear viscosity in β-stable neutron matter for three densities as a function of temperature. Lower panel: Phonon mean free path in β-stable matter as function of temperature for three densities. The horizontal line indicates a radius of the neutron star of $R = 10$ Km. Figures adapted from [3].
Figure 4. Type I diagrams are formed by one 3-phonon vertex and one 4-phonon vertex of $L_{LO}$. Figure adapted from [5].
Figure 5. Type II diagrams are formed by three 3-phonon vertices of $\mathcal{L}_{LO}$. Figure adapted from [5].
Figure 6. $\zeta_2$ frequency–dependent bulk viscosity coefficient as a function of the temperature for $4n_0$ and frequencies between $10^3 - 10^5 \text{s}^{-1}$ for the two neutron gap models. Figure adapted from [5].

Figure 7. R-mode instability window for superfluid neutron stars. The critical (normalized) frequency is displayed for two neutron star mass (1.4 $M_\odot$ and 1.93 $M_\odot$) as a function of temperature for the electron, phonon and electron+phonon dissipative processes. Here $\omega_0 = \sqrt{G\bar{\rho}}$, where $\bar{\rho}$ is the average mass density of the star. Figure adapted from [4].
Figure 8. Upper panel: Thermal conductivity due to superfluid phonons by means of a variational calculation up to order $N = 6$ for $n_0$ as a function of temperature. We use the $^1S_0(A) + ^3P_2(i)$ gap model. Lower panel: Mean free path due to the phonon thermal conductivity using the $^1S_0(A) + ^3P_2(i)$ model for the gap as a function of temperature for three different densities. The critical temperature shown is $T_c = 0.57 \Delta$, whereas we compare the mean free path to the radius of the star ($R = 10$ Km). Figures adapted from [6].