THE CONFORMAL FLOW OF METRICS AND THE GENERAL PENROSE INEQUALITY

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Abstract. In this note we show how to adapt Bray’s conformal flow of metrics, so that it may be applied to the Penrose inequality for general initial data sets of the Einstein equations. This involves coupling the conformal flow with the generalized Jang equation.

1. Introduction

Let \((M, g, k)\) be an asymptotically flat initial data set for the Einstein equations, with one end for simplicity. This consists of a Riemannian 3-manifold \(M\) with metric \(g\), and a symmetric 2-tensor \(k\), which satisfy the constraint equations:

\[
16\pi\mu = R + (\text{Tr}k)^2 - |k|^2,
\]

\[
8\pi J = \text{div}(k + (\text{Tr}k)g).
\]

Here \(\mu\) and \(J\) are the energy and momentum densities of the matter fields, respectively, and \(R\) is the scalar curvature of \(g\). It will be assumed that all measured energy densities are nonnegative, so that the dominant energy condition is valid, \(\mu \geq |J|\). One physical consequence of the dominant energy condition is that matter existing on the initial data cannot travel faster than the speed of light.

Recall that the strength of the gravitational field in the vicinity of a 2-surface \(S \subset M\) may be measured by the null expansions

\[
\theta_{\pm} := H_S \pm \text{Tr}Sk,
\]

where \(H_S\) is the mean curvature with respect to the unit outward normal (pointing towards spatial infinity). The null expansions measure the rate of change of area for a shell of light emitted by the surface in the outward future direction (\(\theta_+\)), and outward past direction (\(\theta_-\)). Thus the gravitational field is interpreted as being strong near \(S\) if \(\theta_+ < 0\) or \(\theta_- < 0\), in which case \(S\) is referred to as a future (past) trapped surface. Future (past) apparent horizons arise as boundaries of future (past) trapped regions and satisfy the equation \(\theta_+ = 0\) (\(\theta_- = 0\)). An outermost future (past) apparent horizon refers to a future (past) apparent horizon outside of which there is no other apparent horizon; such a horizon may have several components, each having spherical topology (\([8, 9]\)). In the setting of the initial data set formulation of the Penrose inequality, apparent horizons take the place of event horizons, in that the area of the event horizon is replaced by the least area required to enclose the outermost future (past) apparent horizon. More precisely, under the above hypotheses on the initial data, the standard version of the Penrose conjecture (\([16]\)) asserts that

\[
M_{ADM} \geq \sqrt{\frac{A_{\text{min}}}{16\pi}}
\]

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where $M_{ADM}$ is the ADM mass and $A_{min}^+ (A_{min}^-)$ is the minimum area required to enclose the outermost future (past) apparent horizon. The Riemannian version of this inequality, when $k = 0$, along with the corresponding rigidity statement in the case of equality, have been established. For a single component horizon the proof was given by Huisken and Ilmanen in [11] using the inverse mean curvature flow, and for a multiple component horizon the proof was given by Bray in [2] using the conformal flow of metrics. However for general $k$, the inequality (1.1) remains an important open problem.

In this paper we aim to study a slightly different inequality. Suppose that the initial data have boundary $\partial M$, consisting of an outermost apparent horizon. We will refer to the union of the outermost future apparent horizon and outermost past apparent horizon as the outermost apparent horizon, and it will be assumed that past and future horizon components do not intersect. Then a version of the Penrose conjecture asserts that

\begin{equation}
M_{ADM} \geq \sqrt{\frac{A_{min}}{16\pi}},
\end{equation}

where $A_{min}$ is the minimum area required to enclose $\partial M$. Moreover the rigidity statement is as follows. Equality holds in (1.2) if and only if $(M, g, k)$ arises from a spacelike slice of the Schwarzschild spacetime, with outerminimizing boundary.

In [3] and [4] Bray and the second author proposed a generalized version of the Jang equation [17], designed specifically for application to the Penrose inequality. This consists of searching for a hypersurface $\Sigma$, given by a graph $t = f(x)$, inside the warped product space $(M \times \mathbb{R}, g + \phi^2 dt^2)$, where $\phi$ is a nonnegative function defined on $M$. The choice of $\phi$ depends heavily on the application at hand, and in fact there may be several choices even for a single application, as is the case for the Penrose inequality. In this paper we will give a new choice for $\phi$, see (3.3) below. The goal or motive which leads to the generalized Jang equation is the same as in the classical case. That is, we search for a hypersurface which has the most positive scalar curvature that is possible. In order to have any chance of obtaining a positivity property for the scalar curvature, we would like the Jang surface $\Sigma$ to satisfy an equation with the same structure as in [17], namely

\begin{equation}
H_\Sigma - Tr_\Sigma K = 0,
\end{equation}

where $H_\Sigma$ is mean curvature and $Tr_\Sigma K$ denotes the trace of $K$ over $\Sigma$. Here, however, $K$ represents a nontrivial extension of the initial data $k$ to all of $M \times \mathbb{R}$, which is given by

\begin{align}
K(\partial_{x'i}, \partial_{x'i}) &= K(\partial_{x'i}, \partial_{x'i}) = k(\partial_{x'i}, \partial_{x'i}) \quad \text{for} \quad 1 \leq i, j \leq 3, \\
K(\partial_{x'i}, \partial_{t}) &= K(\partial_{t}, \partial_{x'i}) = 0 \quad \text{for} \quad 1 \leq i \leq 3, \\
K(\partial_{t}, \partial_{t}) &= \frac{\phi^2 g(\nabla f, \nabla \phi)}{\sqrt{1 + \phi^2 |\nabla f|^2}},
\end{align}

where $x^i, i = 1, 2, 3$, are local coordinates on $M$. With this particular extension, the scalar curvature of the Jang surface $\Sigma$ takes the following form (3.3)

\begin{equation}
R_{\Sigma} = 16\pi (\mu - J(w)) + |h - K|_{\Sigma}^2 + 2|q|^2 - 2\phi^{-1}d\nabla (\phi q).
\end{equation}

Here $\overline{g} = g + \phi^2 df^2$ and $h$ are the induced metric and second fundamental form of $\Sigma$, respectively. $K|_{\Sigma}$ is the restriction to $\Sigma$ of the extended tensor $K$, $d\nabla$ is the divergence operator with respect to $\overline{g}$, and $q$ and $w$ are 1-forms given by

\begin{align}
w_i &= \frac{\phi f}{\sqrt{1 + \phi^2 |\nabla f|^2}}, \\
q_i &= \frac{\phi f}{\sqrt{1 + \phi^2 |\nabla f|^2}} (h_{ij} - (K|_{\Sigma})_{ij}),
\end{align}
with \( f^i = g^{ij} f_j \). If the dominant energy condition is satisfied, then all terms appearing on the right-hand side of (1.5) are nonnegative, except possibly the last term. However the last term has a special structure, and in many applications it is clear that a specific choice of \( \phi \) will allow one to ‘integrate away’ this divergence term, so that in effect the scalar curvature is weakly nonnegative (that is, nonnegative when integrated against certain functions).

When the tensor \( k \) is extended according to (1.4), we will refer to equation (1.3) as the generalized Jang equation, and the solution \( \Sigma = \{ t = f(x) \} \) will be called the Jang surface. In local coordinates the generalized Jang equation takes the following form:

\[
(1.6) \quad \left( g^{ij} - \frac{\phi^2 f^i f^j}{1 + \phi^2 |\nabla f|^2} \right) \left( \frac{\phi \nabla_i f + \phi_i f + \phi_j f_j}{\sqrt{1 + \phi^2 |\nabla f|^2}} - k_{ij} \right) = 0.
\]

This is a quasilinear degenerate elliptic equation, which degenerates when \( f \) blows-up or \( \phi = 0 \). When \( \phi = 1 \) this reduces to the classical Jang equation studied by Schoen and Yau [17].

In [10] blow-up solutions of the generalized Jang equation have been studied in detail, and the main result may be described as follows. Set \( \tau(x) = \text{dist}(x, \partial M) \), and let \( S_\tau \) denote the level sets, that is, each point of \( S_\tau \) is of distance \( \tau \) from the boundary. Furthermore decompose \( \partial M = \partial_+ M \cup \partial_- M \), where \( \partial_+ M (\partial_- M) \) denotes the future (past) apparent horizon components. We then stipulate that near \( \partial_\pm M \),

\[
|\theta_\pm(S_\tau)| \leq cr^l
\]

for some constants \( l \geq 1 \) and \( c > 0 \). Suppose also that in a neighborhood of \( \partial M \) the following structure condition holds,

\[
\phi(x) = \tau^b(x) \tilde{\phi}(x)
\]

for some smooth (up to the boundary), strictly positive function \( \tilde{\phi} \), with \( b \geq 0 \). It is also assumed that \( \phi \) is positive on the interior of \( M \). If \( \frac{1}{2} \leq b < \frac{l+1}{2} \), then there exists a smooth solution \( f \) of the generalized Jang equation (1.6), such that \( f(x) \to \pm \infty \) as \( x \to \partial_\pm M \). More precisely, in a neighborhood of \( \partial_\pm M \)

\[
(1.7) \quad \alpha^{-1} \tau^{-1-2b} + \beta^{-1} \leq f \leq \alpha \tau^{1-2b} + \beta \quad \text{if} \quad \frac{1}{2} < b < \frac{l+1}{2},
\]

\[
-\alpha^{-1} \log \tau + \beta^{-1} \leq f \leq -\alpha \log \tau + \beta \quad \text{if} \quad b = \frac{1}{2},
\]

for some positive constants \( \alpha \) and \( \beta \). If in addition the warping factor satisfies the following asymptotics at spatial infinity

\[
\phi(x) = 1 + \frac{C}{|x|} + O\left( \frac{1}{|x|^2} \right) \quad \text{as} \quad |x| \to \infty
\]

for some constant \( C \), then

\[
(1.8) \quad |\nabla^m f|(x) = O(|x|^{-\frac{m}{2} + \frac{1}{2} - m}) \quad \text{as} \quad |x| \to \infty, \quad m = 0, 1, 2.
\]

These asymptotics ensure that the ADM energy of the Jang surface agrees with that of the initial data. Furthermore, the blow-up (1.7) implies that the Jang surface \( \Sigma \) is a manifold with boundary, when \( \frac{1}{2} \leq b < 1 \). This is in contrast to the blow-up solutions of the classical Jang equation, which approximate an infinitely long cylinder over the horizon. Moreover simple computations shows that \( \partial \Sigma \) is a minimal surface, and that the area of the boundaries of the Jang surface and initial data agree \( |\partial \Sigma| = |\partial M| \).

It is not always the case that blow-up is the correct boundary behavior for application to the Penrose inequality. Moreover instead of working with \( M \), it will become apparent that we should
focus on $M_0$, where $M_0 \subset M$ is the region outside of the outermost minimal area enclosure of $\partial M$. Assuming that the warping factor $\phi$ vanishes at $\partial M_0$ and is positive on the interior of $M_0$, the correct boundary condition for the generalized Jang equation is

$$\overline{H}_{\partial \Sigma_0} = 0$$

where $\Sigma_0$ represents the Jang surface over $M_0$, and $\overline{H}_{\partial \Sigma_0}$ is the mean curvature of $\partial \Sigma_0$. It is conjectured [4] that the following Neumann type condition at $\partial M_0$ implies that (1.9) holds:

$$\frac{\phi \partial_{\nu} f(x)}{\sqrt{1 + \phi^2 |\nabla f|^2}} = \begin{cases} +1 & \text{if } \theta^+(x) = 0 \text{ and } \theta^-(x) \neq 0, \\ -1 & \text{if } \theta^+(x) \neq 0 \text{ and } \theta^-(x) = 0, \\ 0 & \text{if } \theta^+(x) = 0 \text{ and } \theta^-(x) = 0 \end{cases}$$

where $\nu$ denotes the unit outer normal to $\partial M_0$ (pointing away from spatial infinity). Notice that since $\phi|_{\partial M_0} = 0$ we still have $|\partial \Sigma_0| = |\partial M_0|$, even if blow-up does not occur.

These results show that the Jang surface $(\Sigma, g)$ may be interpreted as a deformation of the initial data $(M, g, k)$, which preserves the geometric quantities involved in the Penrose inequality, and yields weakly nonnegative scalar curvature. All of this suggests that in order to establish (1.2), one should try to apply the techniques used to prove the Riemannian Penrose inequality, inside the Jang surface. In fact such a method was outlined in [3] and [4], for the inverse mean curvature flow. This leads to a coupling of the inverse mean curvature flow with the generalized Jang equation, through a specific choice of warping factor $\phi$. Unfortunately, this choice of warping factor lacks regularity and vanishes identically when the weak inverse mean curvature flow jumps, which in turns leads to a very degenerate generalized Jang equation. Thus, it is of significant interest to find a better choice for $\phi$, that leads to a better system of equations. In [3] and [4], it was conjectured that such a $\phi$ exists, and may be found by coupling the generalized Jang equation to the conformal flow of metrics. It is the purpose of this paper to confirm this, and to give an explicit and simple expression for $\phi$ (see (3.3)). This choice of warping factor is positive on the interior of $M$, and has better regularity properties than that arising from the inverse mean curvature flow. It will also be shown that (3.3) reduces to the correct expression, that is, the warping factor for the Schwarzschild spacetime, in the case of equality in (1.2).

Beyond the Penrose inequality, the generalized Jang equation has been applied to other related geometric inequalities in [7], [12], [13], [14]. Moreover in [5], [6], a further modification of the Jang equation has been adapted to treat inequalities for which the model spacetime is stationary. Lastly, a charged version of the conformal flow has been developed in [15], and was used to establish the Penrose inequality with charge for multiple black holes, in the time-symmetric case. It is likely that the methods of the current note may be extended to this setting, yielding a coupling of the charged conformal flow with the generalized Jang equation, which may be applied to the Penrose inequality with charge in the non-time-symmetric case.

2. THE CONFORMAL FLOW OF METRICS AND A REFINED VERSION OF THE RIEMANNIAN PENROSE INEQUALITY

The purpose of this section is to review the basic properties of the conformal flow of metrics [2], and to obtain a strengthened version of the Riemannian inequality. We will also slightly modify a portion of Bray’s proof of the Riemannian Penrose inequality, so as to make the conformal flow applicable to the setting described in the introduction.

Let $(M, g)$ be a 3-dimensional Riemannian manifold, which is asymptotically flat (having one end), and has an outerminimizing minimal surface boundary consisting of a finite number of components.
Unlike \cite{2}, however, we do not assume that the scalar curvature $R_g$ is nonnegative. The conformal flow of metrics is given by $g_t = u_t^4 g$ where

\begin{equation}
\frac{d}{dt} u_t = v_t u_t,
\end{equation}

and

\begin{equation}
\Delta g_t v_t = 0, \quad \text{on } M_t,
\end{equation}

\begin{equation}
v_t|_{\partial M_t} = 0, \quad v_t(x) \to -1 \quad \text{as } |x| \to \infty.
\end{equation}

Here $M_t$ denotes the region outside of the outermost minimal surface (denoted $\partial M_t$) in $(M, g_t)$. Note that $u_t(x) = \exp \left( \int_0^t v_s(x) ds \right)$. Moreover if $L_g$ denotes the conformal Laplacian, then by a standard formula

\begin{equation}
u_t^5 R_{g_t} = -8 L_g u_t := -8 \left( \Delta u_t - \frac{1}{8} R_g u_t \right),
\end{equation}

so that

\begin{equation}
\frac{d}{dt} (u_t^5 R_{g_t}) = -8 L_g \frac{d}{dt} u_t
= -8 L_g (v_t u_t)
= -8 u_t^5 L_{g_t} v_t
= -8 u_t^5 \left( \Delta_{g_t} v_t - \frac{1}{8} R_{g_t} v_t \right)
= (u_t^5 R_{g_t}) v_t.
\end{equation}

We then have

\begin{equation}
u_t^5 R_{g_t} = \exp \left( \int_0^t v_s ds \right) R_g = u_t R_g.
\end{equation}

Let $\tilde{M}_t = M_t^- \cup M_t^+$ denote the doubled manifold, reflected across the minimal surface $\partial M_t$. Here $M_t^-$ and $M_t^+$ each represent a copy of $M_t$, and we endow each of these copies with the metric $g_t^\pm = (w_t^\pm)^4 g_t$ where $w_t^\pm = (1 \pm v_t)/2$. The metric on $\tilde{M}_t$ will be denoted by $\tilde{g}_t = g_t^- \cup g_t^+$. Let $c : Cl(TM_t) \to \text{End}(\mathcal{G})$ be the usual representation of the Clifford algebra on the bundle of spinors $\mathcal{G}$, so that

\begin{equation}
c(X)c(Y) + c(Y)c(X) = -2g(X,Y).
\end{equation}

Choose orthonormal frame fields $e_i$, $i = 1, 2, 3$ for $g$, and observe that $e_i^\pm = (w_t^\pm u_t)^{-2} e_i$, $i = 1, 2, 3$ are the corresponding orthonormal frames for $g_t^\pm$. There exists a positive definite inner product on $\mathcal{G}$, denoted by $\langle \cdot, \cdot \rangle$, with respect to which $c(e_i)$ is anti-hermitian

\begin{equation}
\langle c(e_i) \psi, \eta \rangle = -\langle \psi, c(e_i) \eta \rangle.
\end{equation}

Define the respective spin connections by

\begin{equation}
\nabla_{e_i} = e_i + \frac{1}{4} \sum_{j,l=1}^3 \Gamma_{ij}^{\ell} c(e_j) c(e_{\ell}), \quad \nabla_{e_i}^\pm = e_i^\pm + \frac{1}{4} \sum_{j,l=1}^3 \Gamma_{ij}^{\ell} c(e_j^\pm) c(e_{\ell}^\pm),
\end{equation}
where $\Gamma^l_{ij}$ and $\Gamma^{\pm l}_{ij}$ are Levi-Civita connection coefficients for $g$ and $g^\pm_l$. The corresponding Dirac operators are given by

$$\mathcal{D} = \sum_{i=1}^{3} c(e_i) \nabla_{e_i}, \quad \mathcal{D}^\pm = \sum_{i=1}^{3} c(e^\pm_i) \nabla^\pm_{e^\pm_i}.$$ 

Let $\tilde{\psi}_t$ be a harmonic spinor on $\tilde{M}_t$ which converges to a constant spinor of unit norm at spatial infinity; for the present discussion it is assumed that such a spinor exists. We will denote the restriction of $\tilde{\psi}_t$ to $M^\pm_t$ by $\psi^\pm_t$, then $\psi^\pm_t$ is a harmonic spinor on $(M^\pm_t, g^\pm_t)$. We may now apply the Lichnerowicz formula \cite{1} on each part of the doubled manifold to find that

\begin{equation}
4\pi \tilde{E}_{ADM}(t) - \int_{\partial M^+_t} \langle \psi^+_t, c(e^+_3) \sum_{i=1}^{2} c(e^+_i) \nabla^+_e \psi^+_t \rangle = \int_{M^+_t} \left( |\nabla^+_\psi^+_t|^2 + \frac{1}{4} R^+_t |\psi^+_t|^2 \right) d\omega^+_t,
\end{equation}

and

\begin{equation}
- \int_{\partial M^-_t} \langle \psi^-_t, c(e^-_3) \sum_{i=1}^{2} c(e^-_i) \nabla^-_e \psi^-_t \rangle = \int_{M^-_t} \left( |\nabla^-_\psi^-_t|^2 + \frac{1}{4} R^-_t |\psi^-_t|^2 \right) d\omega^-_t,
\end{equation}

where $e_i, i = 1, 2$ are tangent and $e_3$ is normal (pointing towards spatial infinity) to the boundary $\partial M_t$, and $\tilde{E}_{ADM}(t)$ is the ADM energy of $(M_t, \tilde{g}_t)$. A standard calculation shows that

$$c(e^\pm_3) \sum_{i=1}^{2} c(e^\pm_i) \nabla^\pm_{e^\pm_i} \psi^\pm_t = D_{\partial M^\pm_t} \psi^\pm_t - \frac{1}{2} H^\pm_t \psi^\pm_t,$$

where $H^\pm_t$ are the mean curvatures of $\partial M^\pm_t$ with respect to the metrics $g^\pm_t$, and the boundary Dirac operator is given by

$$D_{\partial M^\pm_t} = c(e^\pm_3) \sum_{i=1}^{2} c(e^\pm_i) \nabla^\pm_{e^\pm_i}$$

with

$$\nabla^\pm_{e^\pm_i} = e^\pm_i + \frac{1}{4} \sum_{j,l=1}^{2} \Gamma^{\pm l}_{ij} c(e^\pm_j) c(e^\pm_l), \quad i = 1, 2.$$

We claim that the sum of the boundary terms in (2.4) and (2.5) all cancel. To see this note that $\psi^-|_{\partial M^-_t} = \psi^+_|_{\partial M^+_t}$, and since $v_t|_{\partial M_t} = 0$ we have $g^-_t|_{\partial M^-_t} = g^+_t|_{\partial M^+_t}$. Moreover, as $\partial M^-_t$ and $\partial M^+_t$ represent the same surface in $\tilde{M}_t$ we have that $e^+_3 = -e^-_3$, and thus

\begin{equation}
D_{\partial M^-_t} = -D_{\partial M^+_t}.
\end{equation}

Now consider the mean curvature terms. According to a standard formula for the change of mean curvature under conformal deformation

$$H^\pm_t = (w^\pm_t)^{-2} [H_t + 4u^-_t e^-_3 (\log w^\pm_t)] = \pm 4u^-_t e^-_3 (v_t),$$

where $H_t = 0$ is the mean curvature of $\partial M_t$ with respect to $g_t$. It follows that

\begin{equation}
H^-_t = -H^+_t.
\end{equation}
We may now add \( (2.4) \) and \( (2.5) \), and apply \( (2.6) \) and \( (2.7) \), to obtain
\[
4\pi \tilde{E}_{ADM}(t) = \int_{M_t} \left( |\nabla^- \psi_t^-|^2 + \frac{1}{4} R_{g_t}^- |\psi_t^-|^2 \right) d\omega_{g_t^-} \\
+ \int_{M_t^+} \left( |\nabla^+ \psi_t^+|^2 + \frac{1}{4} R_{g_t^+} |\psi_t^+|^2 \right) d\omega_{g_t^+}.
\]
Moreover, since \( v_t \) is harmonic we find
\[
R_{g_t^\pm} = -8(w_t^\pm)^{-5} \left( \Delta_{g_t} \left( \frac{1 \pm v_t}{2} \right) - \frac{1}{8} R_{g_t} \left( \frac{1 \pm v_t}{2} \right) \right) = (w_t^\pm u_t)^{-4} R_g.
\]
It follows that
\[
4\pi \tilde{E}_{ADM}(t) = \int_{M_t} \left[ (w_t^- u_t)^6 |\nabla^- \psi_t^-|^2 + (w_t^+ u_t)^6 |\nabla^+ \psi_t^+|^2 \right] d\omega_g \\
+ \int_{M_t} \frac{1}{4} \left( (w_t^-)^2 |\psi_t^-|^2 + (w_t^+)^2 |\psi_t^+|^2 \right) u_t^2 R_g d\omega_g,
\]
\( (2.8) \)

It should be noted that a formula similar to \( (2.8) \) may be obtained without the use of spinors. To see this, solve the zero scalar curvature equation
\[
\Delta_{\tilde{g}_t} z_t - \frac{1}{8} \tilde{R}_t z_t = 0 \quad \text{on} \quad M_t, \quad z_t \to 1 \quad \text{as} \quad |x| \to \infty,
\]
where \( \tilde{R}_t \) is the scalar curvature of \( \tilde{g}_t \); for the present discussion it is assumed that such a solution exists. Then the conformally related metric \( z_t^4 \tilde{g}_t \) has zero scalar curvature. It follows that the ADM energy is nonnegative \( E_{ADM}(z_t^4 \tilde{g}_t) \geq 0 \). Moreover, the solution has an asymptotic expansion of the form
\[
z_t = 1 + \frac{C_t}{|x|} + O \left( \frac{1}{|x|^2} \right) \quad \text{as} \quad |x| \to \infty.
\]
A small calculation then shows that
\[
0 \leq E_{ADM}(z_t^4 \tilde{g}_t) = \tilde{E}_{ADM}(t) + 2C_t.
\]
Hence, integrating by parts produces
\[
2\pi \tilde{E}_{ADM}(t) \geq -4\pi C_t
\]
\( (2.9) \)
\[
= \int_{S_{\infty}} z_t \partial_v z_t \\
= \int_{\tilde{M}_t} \left( |\nabla_{\tilde{g}_t} z_t|^2 + \frac{1}{8} \tilde{R}_t z_t^2 \right) d\omega_{\tilde{g}_t} \\
= \int_{M_t} \left[ (w_t^- u_t)^6 |\nabla_{\tilde{g}_t} z_t^-|^2 + (w_t^+ u_t)^6 |\nabla_{\tilde{g}_t} z_t^+|^2 \right] d\omega_g \\
+ \int_{M_t} \frac{1}{8} \left( (w_t^-)^2 |z_t^-|^2 + (w_t^+)^2 |z_t^+|^2 \right) u_t^2 R_g d\omega_g,
\]
where \( z_t^\pm \) denotes \( z_t \) restricted to \( M_t^\pm \). This formula is analogous to \( (2.8) \), and was obtained without the use of spinors. Note that part of the integrand on the right-hand side of \( (2.9) \) satisfies an elliptic equation with respect to the metric \( g \), even though the solution depends on \( t \):
\[
0 = (w_t^\pm u_t)^5 L_{\tilde{g}_t^\pm} z_t^\pm = L_g (w_t^\pm z_t^\pm u_t)
\]
where \( \tilde{g}_t^\pm = (z_t^\pm)^4 g_t^\pm \).
We mention that an expression similar to (2.8) and (2.9) may also be obtained using the inverse mean curvature flow starting from a point in $M_t$.

Equations (2.8) and (2.9) imply that $\tilde{E}_{\text{ADM}}(t) \geq 0$ under the hypothesis of nonnegative scalar curvature $R_g \geq 0$. Actually, this fact may be considered the crux of the argument for Bray’s proof of the Riemannian Penrose inequality. Although the explicit expression above for the energy $\tilde{E}_{\text{ADM}}(t)$ is not necessary for Bray’s proof, it will however be needed in the next section where we outline an argument for the general Penrose inequality. The purpose of (2.8) and (2.9) is to show that the ADM energy of $(M_t, g_t)$, which will only be shown to be decreasing on the interval $[0, \infty)$, is decreasing instantaneously, that is the derivative $\frac{dm_t}{dt}$ will be negative for all $t$. However for the general Penrose inequality, where one does not necessarily have nonnegative scalar curvature, $m_t$ will only be shown to be decreasing on the interval $[0, \infty)$, that is $m_0 \geq \lim_{t \to \infty} m_t$. In order to obtain such a result, let us first conclude this review of the proof of the Riemannian Penrose inequality.

As in [2], we may assume without loss of generality, that the initial data metric $g$ is scalar flat and conformal to the Euclidean metric, with a conformal factor that approaches a positive constant at infinity. In this outer region we may then write

\[(2.10)\]

\[g = \left(1 + \frac{m}{r} + O \left(\frac{1}{r^2}\right)\right)^4 \delta.\]

Recall that according to a standard formula, if $w = 1 + \frac{c}{r} + \cdots$ at spatial infinity, then $E_{\text{ADM}}(w^4 g) = E_{\text{ADM}}(g) + 2c$. Therefore the parameter $m$ in (2.10) represents half the mass of $g$, or rather $m_0 = 2m$.

Since $\nu_t$ is a harmonic function, both $u_t$ and $v_t$ have similar asymptotic expansions at spatial infinity

\[(2.11)\]

\[u_t = \alpha_t + \frac{\beta_t}{r} + O \left(\frac{1}{r^2}\right), \quad v_t = -1 + \frac{\gamma_t}{r} + O \left(\frac{1}{r^2}\right),\]

where $\alpha_0 = 1$ and $\beta_0 = 0$. It follows that

\[g_t = u_t^4 g = \left(\alpha_t + \frac{\beta_t + m \alpha_t}{r} + O \left(\frac{1}{r^2}\right)\right)^4 \delta,\]

and hence

\[m_t = 2\alpha_t (\beta_t + m \alpha_t).\]

In order to calculate the derivative $m'_t = \frac{dm_t}{dt}$, observe that (2.11) and (2.11) imply

\[\alpha'_t = -\alpha_t, \quad \beta'_t = \alpha_t \gamma_t - \beta_t \quad \Rightarrow \quad \alpha_t = e^{-t}, \quad \beta_t = e^{-t} \int_0^t \gamma_s ds.\]

From this we obtain

\[m'_t = 2\alpha'_t (\beta_t + m \alpha_t) + 2\alpha_t (\beta'_t + m \alpha'_t) = 2e^{-2t} \gamma_t - 4e^{-t} (\beta_t + e^{-t} m) = -2(\beta_t - e^{-2t} \gamma_t).\]

However if we set $\bar{g}_t = \left(1 - \frac{m_t}{2}\right)^4 g_t$ then

\[\bar{g}_t = \left(\alpha_t + \frac{\beta_t + m \alpha_t - \alpha_t \gamma_t/2}{r} + O \left(\frac{1}{r^2}\right)\right)^4 \delta,\]

and thus the energy of $\bar{g}_t$ is given by

\[\bar{m}_t = 2\alpha_t \left(\beta_t + m \alpha_t - \frac{\alpha_t \gamma_t}{2}\right) = m_t - e^{-2t} \gamma_t,\]
which yields

\[ m_t' = -2\tilde{m}_t. \]

Therefore upon integrating and rewriting in terms of the previous notation, we find that

\[ E_{ADM}(0) - E_{ADM}(\infty) = 2 \int_0^\infty \tilde{E}_{ADM}(t) dt \tag{2.12} \]

where \( E_{ADM}(t) \) denotes the total energy of \((M_t, g_t)\).

Together, (2.8) (or (2.9)) and (2.12) show that the function \( E_{ADM}(t) \) is decreasing over all time. There are two other crucial properties of the conformal flow which yield the Penrose inequality. First, due to the boundary condition (2.2) and the fact that \( \partial M_t \) is a minimal surface, it follows that the area function \( A(t) \) remains constant in time, where \( A(t) \) denotes the area of \( \partial M_t \). Second, it is shown in [2] that after rescaling \((M_t, g_t)\) converges to the exterior region of a constant time slice of the Schwarzschild spacetime, so that \( E_{ADM}(\infty) = \sqrt{\frac{A(\infty)}{16\pi}} \). By combining all these results together we obtain the following refined version of the Penrose inequality

\[ E_{ADM} - \sqrt{\frac{|\partial M|}{16\pi}} = 2 \int_0^\infty \tilde{E}_{ADM}(t) dt. \tag{2.13} \]

In the case of equality, when \( E_{ADM} = \sqrt{\frac{|\partial M|}{16\pi}} \), standard arguments combined with (2.8) (or (2.9)) and (2.12) show that \((M, g)\) is conformally equivalent to \((\mathbb{R}^3 - B_1(0), \delta)\) with zero scalar curvature.

Since the boundary \( \partial M \) is a minimal surface, it follows that \((M, g)\) is isometric to a constant time slice of the Schwarzschild spacetime.

### 3. Coupling the Conformal Flow to the Generalized Jang Equation

Let \( \Sigma_0 \) be a solution of the generalized Jang equation (1.6) satisfying boundary condition (1.9), as described in Section 1. Then the boundary \( \partial \Sigma_0 \) is a minimal surface, and is in fact the outermost minimal surface. To see this, suppose that a surface \( S \) encloses but is not equal to \( \partial \Sigma_0 \) and satisfies \( |S| \leq |\partial \Sigma_0| \). Since \( g \geq \tilde{g} \), we have \( |S| \leq |\tilde{S}| \), where \( \tilde{S} \subset M \) is the projection of \( S \). Thus \( |S| \leq |\partial \Sigma_0| = |\partial M_0| \), which is impossible since \( \partial M_0 \) is the outermost minimal area enclosure of \( \partial M \).

We will now perform the conformal flow of metrics on the Jang surface \( \Sigma_0 \), and show how this leads naturally to a choice for the warping factor \( \phi \). Following the notation from the previous section let \((g_t = u_t^4\tilde{g}, \Sigma_t)\) be a conformal flow, where \( \Sigma_t \) denotes the region outside of the outermost minimal surface \( \partial \Sigma_t \). Then from (2.3) the scalar curvature evolves by

\[ R_{g_t} = u_t^{-4}\tilde{R}. \]

Moreover, since the ADM energies of \((M, g, k)\) and \((\Sigma, \tilde{g})\) agree, we may apply (2.8) (or (2.9)) and (2.12) to obtain

\[ E_{ADM} - \sqrt{\frac{|\partial \Sigma_0|}{16\pi}} \geq 2 \int_0^\infty \tilde{E}_{ADM}(t) dt, \tag{3.1} \]

where \( \tilde{E}_{ADM}(t) \) denotes the total energy of \((\tilde{\Sigma}_t, g_t^- \cup g_t^+)\) and is given by

\[ 4\pi \tilde{E}_{ADM}(t) = \int_{\Sigma_t} [(w_t^- u_t)^6 |\nabla^- \psi_t^-|^2 + (w_t^+ u_t)^6 |\nabla^+ \psi_t^+|^2] d\omega_{\tilde{g}} \]

\[ + \int_{\Sigma_t} \frac{1}{4} ((w_t^-)^2 |\psi_t^-|^2 + (w_t^+)^2 |\psi_t^+|^2) u_t^2 \tilde{R} d\omega_{\tilde{g}}, \tag{3.2} \]
or
\[
4\pi \overline{E}_{ADM}(t) = \int_{\Sigma_t} \left[ 2(\omega^{-}_{\ell} u_{\ell}^0)^2 |\nabla \tilde{\eta}^{-}_{\ell} \psi_{\ell}^{-}|^2 + (\omega^{+}_{\ell} u_{\ell}^0)^2 |\nabla \tilde{\eta}^{+}_{\ell} \psi_{\ell}^{+}|^2 \right] d\omega_{\overline{\gamma}}
+ \int_{\Sigma_t} \frac{1}{4} \left[ (\omega^{-}_{\ell} \psi_{\ell}^{-})^2 + (\omega^{+}_{\ell} \psi_{\ell}^{+})^2 \right] u_{\ell}^2 R d\omega_{\overline{\gamma}}.
\]

Notice that we cannot conclude as in Section 2 that \( \overline{E}_{ADM}(t) \geq 0 \), since the scalar curvature \( \overline{R} \) of the Jang surface is not necessarily nonnegative. According to (1.5), it is a divergence term which is the possible obstruction to nonnegativity. However by choosing \( \phi \) appropriately, this divergence term may be integrated away so that the right-hand side of (2.13) is nonnegative. To accomplish this let
\[
\chi_t(x) = \begin{cases} 
1 & \text{if } x \in \Sigma_t, \\
0 & \text{if } x \in \Sigma_0 - \Sigma_t,
\end{cases}
\]
and set
\[
(3.3) \quad \phi = 2 \int_0^\infty \chi_t((\omega^{-}_{\ell} \psi_{\ell}^{-})^2 + (\omega^{+}_{\ell} \psi_{\ell}^{+})^2) u_{\ell}^2 dt,
\]
or
\[
(3.4) \quad \phi = 2 \int_0^\infty \chi_t [(\omega^{-}_{\ell} \psi_{\ell}^{-})^2 + (\omega^{+}_{\ell} \psi_{\ell}^{+})^2] u_{\ell}^2 dt.
\]
Notice that \( \phi \) should be strictly positive away from \( \partial \Sigma_0 \),
\[
(3.5) \quad \phi(x) \to 1 \quad \text{as} \quad |x| \to \infty, \quad \text{and} \quad \phi|_{\partial \Sigma_0} = 0,
\]
where we have used (2.2) and the fact that \( u_{\ell} \to e^{-t} \) as \( |x| \to \infty \). By applying Fubini’s Theorem, using the boundary condition (3.5), as well as the scalar curvature formula for the Jang surface (1.5), we may calculate
\[
\int_0^\infty \int_{\Sigma_t} ((\omega^{-}_{\ell} \psi_{\ell}^{-})^2 + (\omega^{+}_{\ell} \psi_{\ell}^{+})^2) u_{\ell}^2 R d\omega_{\overline{\gamma}} dt = \frac{1}{2} \int_{\Sigma_0} \phi R d\omega_{\overline{\gamma}}
\geq \frac{1}{2} \int_{\Sigma_0} \phi \left( 2|q|^2 - \frac{2}{\phi} \text{div}(\phi q) \right) d\omega_{\overline{\gamma}}
= \int_{\Sigma_0} \phi |q|^2 d\omega_{\overline{\gamma}} + \int_{\partial \Sigma_0} \phi q (\overline{N}) d\sigma_{\overline{\gamma}}
= \int_{\Sigma_0} \phi |q|^2 d\omega_{\overline{\gamma}},
\]
where \( \overline{N} \) is the unit normal to \( \partial \Sigma_0 \) pointing towards spatial infinity. We could also have used (3.4).

The above calculation also uses the reasonable assumption that \( |q| \) remains uniformly bounded and decays very fast at spatial infinity; this behavior is consistent with that of solutions to the classical Jang equation [17]. This may be combined with (1.5), (3.1), (3.2), and (3.6) to yield
\[
E_{ADM} \geq \frac{1}{2\pi} \int_{\Sigma_0} \frac{|\partial \Sigma_0|}{16\pi}
+ \frac{1}{2\pi} \int_{\Sigma_t} [(\omega^{-}_{\ell} u_{\ell}^0)^2 |\nabla \tilde{\eta}^{-}_{\ell} \psi_{\ell}^{-}|^2 + (\omega^{+}_{\ell} u_{\ell}^0)^2 |\nabla \tilde{\eta}^{+}_{\ell} \psi_{\ell}^{+}|^2] d\omega_{\overline{\gamma}} dt
+ \frac{1}{8\pi} \int_{\Sigma_0} ((\omega^{-}_{\ell} \psi_{\ell}^{-})^2 + (\omega^{+}_{\ell} \psi_{\ell}^{+})^2) u_{\ell}^2 (16\pi (\mu - J(w)) + |h - K|_{\Sigma}^2 + 2|q|^2) d\omega_{\overline{\gamma}}.
\]
The Penrose inequality (1.2) (with ADM energy replacing ADM mass) now follows. This is due to the fact that the Jang metric $\mathcal{J}$ measures areas to be at least as large as those measure by $g$, and hence $|\partial \Sigma_0| \geq A_{\text{min}}$. We remark that because of the asymptotics (1.8), only the ADM energy is encoded in the solution of the generalized Jang equation. At this time it is unknown how to encode the ADM linear momentum into a solution of Jang’s equation.

Now consider the case of equality in (1.2). We will use (3.3), although a similar argument holds for (3.4). First note that in this case $|\partial \Sigma_0| = A_{\text{min}}$. Next, according to (3.7), both spinors $\psi^\pm_i$ are parallel. This implies that $|\psi^\pm_i| \equiv 1$, in light of the boundary condition at spatial infinity. It follows that

$$\mu = |J| \equiv 0, \quad h = K|_{\Sigma}, \quad q \equiv 0,$$

where we have used the dominant energy condition and the fact that the Jang metric $\mathcal{J}$ yields an isometric embedding of the initial data ($\Sigma_0, g, k$). Moreover, since $h$ is parallel. This implies that $|w| < 1$ away from $\partial M$. Hence $R \equiv 0$. Moreover the existence of a basis of parallel spinors shows that $(\Sigma_t, g^- \cup g^+)$ is isometric to $(\mathbb{R}^3, \delta)$. The Jang surface $\Sigma_0$ is then conformally flat with zero scalar curvature, and since $\partial \Sigma_0$ is minimal, we find that $(\Sigma_0, \mathcal{J})$ is isometric to the exterior region of the $t = 0$ slice of the Schwarzschild spacetime

$$\mathbb{S}^4 \times (\mathbb{R}^3 - B_{m_0}(0)), -\phi_{SC}^2 dt^2 + g_{SC},$$

where $r = |x|$ and

$$\phi_{SC}(x) = \frac{1 - m_0}{1 + \frac{m_0}{2r}}, \quad g_{SC} = \left(1 + \frac{m_0}{2r}\right)^4 \delta.$$

We will show that $\phi = \phi_{SC}$. Since $\mathcal{J}$ is isometric to the spatial Schwarzschild metric, the conformal flow of metrics may be given explicitly, namely

$$v_t(x) = \begin{cases} \frac{-e^{-t} + \frac{m_0}{2r}e^t}{e^{-t} + \frac{m_0}{2r}e^t} & \text{if } r \geq \frac{m_0}{2}e^{2t} \\ 0 & \text{if } r < \frac{m_0}{2}e^{2t} \end{cases} \quad u_t(x) = \begin{cases} \frac{e^{-t} + \frac{m_0}{2r}e^t}{1 + \frac{m_0}{2r}} & \text{if } r \geq \frac{m_0}{2}e^{2t} \\ \frac{\sqrt{2m_0}e^{-t}}{1 + \frac{m_0}{2r}} & \text{if } r < \frac{m_0}{2}e^{2t} \end{cases}.$$

It follows that on $\Sigma_t$,

$$w^-_t = \frac{1 - v_t}{2} = \frac{e^{-t} - \frac{m_0}{2r}e^t}{e^{-t} + \frac{m_0}{2r}e^t}, \quad w^+_t = \frac{1 + v_t}{2} = \frac{\frac{m_0}{2r}e^t}{e^{-t} + \frac{m_0}{2r}e^t},$$

and therefore

$$\phi(x) = 2 \int_0^{\log \frac{\sqrt{2m_0}}{m_0}} \left(\frac{e^{-t} - \frac{m_0}{2r}e^t}{e^{-t} + \frac{m_0}{2r}e^t}\right)^2 + \left(\frac{\frac{m_0}{2r}e^t}{e^{-t} + \frac{m_0}{2r}e^t}\right)^2 \left(\frac{e^{-t} + \frac{m_0}{2r}e^t}{1 + \frac{m_0}{2r}}\right)^2 \, dt$$

$$= 2 \int_0^{\log \frac{\sqrt{2m_0}}{m_0}} \frac{e^{-2t} + \left(\frac{m_0}{2r}\right)^2 e^{2t}}{(1 + \frac{m_0}{2r})^2} \, dt$$

$$= \left(\frac{1 - \frac{m_0}{2r}}{1 + \frac{m_0}{2r}}\right) = \phi_{SC}.$$

Consider now the graph map $G : M \to \mathbb{S}^4$ given by $x \mapsto (x, f(x))$. Since $\phi \equiv \phi_{SC}$, the map $G$ yields an isometric embedding of the initial data $(M_0, g, k)$ into the Schwarzschild spacetime, as

$$g = \mathcal{J} - \phi^2 g_{SC}^2 = g_{SC} - \phi_{SC}^2 df^2.$$

Moreover since $h = K|_{\Sigma}$, a calculation [3], [4] guarantees that the second fundamental form of this embedding agrees with $k$. Lastly observe that since $\phi$ vanishes at $\partial M_0$, we must have that $\partial M_0$ is
an apparent horizon. However $\partial M$ is an outermost apparent horizon, so in fact $\partial M_0 = \partial M$, and hence $M_0 = M$.

**Remark 3.1.** Without further modification, the arguments of this paper can only yield the inequality

$$E_{\text{ADM}} \geq \sqrt{\frac{A_{\text{min}}}{16\pi}},$$

where the ADM mass is replaced by the ADM energy in $\text{(1.2)}$. The rigidity statement for $\text{(3.8)}$ is as follows, if equality holds then $(M,g,k)$ arises from a spacelike slice of the Schwarzschild spacetime, with outerminimizing boundary. Note that the only if part is not included. That is, we cannot say that if $(M,g,k)$ arises from a spacelike slice of the Schwarzschild spacetime with outerminimizing boundary, then equality holds. This is due to the difference between the ADM mass and energy.

A proof of the only if part in the rigidity statement for the inequality $\text{(1.2)}$ is as follows. Suppose that $(M,g,k)$ is a spacelike slice of the Schwarzschild spacetime with outerminimizing boundary. Since $\phi_{\text{SC}}|_{\partial M} = 0$, we have that $|\partial M| = |\partial \hat{M}|$, where $(\hat{M},g_{\text{SC}},0)$ is the $t = 0$ slice of the Schwarzschild spacetime. Furthermore

$$M_{\text{ADM}} = M_{\text{ADM}}(\hat{M}) = E_{\text{ADM}}(\hat{M}),$$

so that

$$M_{\text{ADM}} = E_{\text{ADM}}(\hat{M}) = \sqrt{\frac{|\partial \hat{M}|}{16\pi}} = \sqrt{\frac{|\partial M|}{16\pi}} \geq \sqrt{\frac{A_{\text{min}}}{16\pi}}.$$

It follows that equality holds in $\text{(1.2)}$ since $\partial M$ is outerminimizing, that is $|\partial M| = A_{\text{min}}$. These same arguments also show why the boundary must be outerminimizing in the case of equality. Namely, in light of $\text{(3.9)}$, the only way that equality can hold in $\text{(1.2)}$ is if $|\partial M| = A_{\text{min}}$.

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