Regularizing infinite sums of zeta-determinants

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Abstract. We present a new multiparameter resolvent trace expansion for elliptic operators, polyhomogeneous in both the resolvent and auxiliary variables. For elliptic operators on closed manifolds the expansion is a simple consequence of the parameter dependent pseudodifferential calculus. As an additional nontrivial toy example we treat here Sturm-Liouville operators with separated boundary conditions.

As an application we give a new formula, in terms of regularized sums, for the $\zeta$-determinant of an infinite direct sum of Sturm-Liouville operators. The Laplace-Beltrami operator on a surface of revolution decomposes into an infinite direct sum of Sturm-Liouville operators, parametrized by the eigenvalues of the Laplacian on the cross-section $S^1$. We apply the polyhomogeneous expansion to equate the zeta-determinant of the Laplace-Beltrami operator as a regularized sum of zeta-determinants of the Sturm-Liouville operators plus a locally computable term from the polyhomogeneous resolvent trace asymptotics. This approach provides a completely new method for summing up zeta-functions of operators and computing the meromorphic extension of that infinite sum to $s = 0$.

We expect our method to extend to a much larger class of operators.

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1. Introduction and formulation of the result

Various geometric problems involve zeta-determinants of Hodge-Laplace operators which decompose into an infinite sum of scalar Laplace-type operators. The most prominent example seems to be the discussion of analytic torsion on spaces with conical singularities, where the problem of computing the zeta-determinant of an infinite sum of scalar operators arises naturally and has motivated the work of the first author in [Les98].

The basic approach to this problem is given by summing up zeta-functions \( \zeta(s, \Delta_\lambda), \lambda \in \mathbb{N}_0 \) of the scalar Laplace-type operators \( \Delta_\lambda \) for \( \Re(s) \gg 0 \) and computing the meromorphic extension of that infinite sum to \( s = 0 \). This approach was taken by Spreafico in [Spr05, Spr06], where the intricate task of constructing a meromorphic extension is addressed for bounded cones. Compare also the discussion by Bordag, Kirsten and Dowker in [BKD96] and by the second author in [Ver09].

In this article we present a conceptually new method for computing the zeta-determinant of an infinite sum of operators, which uses a new polyhomogeneous resolvent trace expansion. Our model setup here is a surface of revolution. The spectral decomposition on \( S^1 \) decomposes the Laplace-Beltrami operator \( \Delta \) on a surface of revolution into an infinite sum of Sturm-Liouville operators \( \Delta_\lambda, \lambda \in \mathbb{N}_0 \) on a finite interval with separated boundary conditions.

We establish an expansion of the resolvent trace for \( \Delta_\lambda \) polyhomogeneous both in \( \lambda \) and the resolvent parameter, and prove that the zeta-determinant of \( \Delta \) is given by a regularized sum of zeta-determinants for \( \Delta_\lambda, \lambda \in \mathbb{N}_0 \). This avoids completely the question of constructing a meromorphic extension for the infinite sum of zeta-functions.

Moreover, the polyhomogeneous resolvent trace expansion explains the origin of the trace coefficients in the expansion of \( \text{Tr}(\Delta + z^2)^{-2} \) as \( z \to \infty \), which do not appear in the corresponding (standard) resolvent expansions of the scalar operators \( \Delta_\lambda \).

1.1. Laplace-Beltrami operator on a surface of revolution. Let \( (M = [0, 1] \times S^1, g = dx^2 \oplus r(x)^2 g_{S^1}) \) be a surface of revolution with \( r \in C^\infty([0, 1]), r > 0 \). This is a warped product metric and the associated Laplace-Beltrami operator is given by the differential expression

\[
\Delta = -\frac{\partial^2}{\partial x^2} - \frac{r'(x)}{r(x)} \frac{\partial}{\partial x} + \frac{1}{r(x)^2} \Delta_{S^1},
\]

acting on \( C_0^\infty((0, 1) \times S^1) \), the space of complex-valued smooth compactly supported functions on \( (0, 1) \times S^1 \). The natural \( L^2 \)-space with respect to the metric \( g \) is \( L^2(M, g) = L^2([0, 1] \times S^1, r(x) \, dx \, dvol(g_{S^1})) \). Under the unitary map

\[
\Phi : L^2(M, g) \to L^2([0, 1], L^2(S^1, g_{S^1})), \quad (\Phi u)(x) := u(x)/\sqrt{r(x)},
\]
the Laplacian $\Delta$ transforms into the operator

$$\Phi \Delta \Phi^{-1} = -\frac{\partial^2}{\partial x^2} + \frac{1}{r(x)} \Delta_{S^1} + \left[ \frac{r''(x)}{2r(x)} - \left( \frac{r'(x)}{2r(x)} \right)^2 \right], \quad (1.3)$$

acting in $L^2([0, 1], L^2(S^1))$. The functions $\left( \frac{1}{\sqrt{2\pi}} e^{i\lambda x} \right)_{\lambda \in \mathbb{Z}}$ form an orthonormal basis of $L^2(S^1)$ of eigenfunctions of $\Delta_{S^1}$ to the eigenvalues $\lambda^2, \lambda \in \mathbb{Z}$. The eigenvalues $\lambda^2 \neq 0$ have multiplicity two, the eigenvalue $\lambda^2 = 0$ has multiplicity one. Hence we have a decomposition

$$\Phi \Delta \Phi^{-1} = -\frac{\partial^2}{\partial x^2} + \frac{1}{r(x)} \Delta_{S^1} + \left[ \frac{r''(x)}{2r(x)} - \left( \frac{r'(x)}{2r(x)} \right)^2 \right] \quad (1.4)$$

$$= \sum_{\lambda=-\infty}^{\infty} \left( -\frac{\partial^2}{\partial x^2} + \frac{\lambda^2}{r(x)} \Delta_{S^1} + \left[ \frac{r''(x)}{2r(x)} - \left( \frac{r'(x)}{2r(x)} \right)^2 \right] \right) =: \sum_{\lambda=0}^{\infty} \Delta_\lambda,$$

into a direct sum of one-dimensional Sturm-Liouville type operators. We consider separated Dirichlet or generalized Neumann boundary conditions for $\Delta$. It is straightforward to check that under the unitary transformation $\Phi$ they correspond to separated Dirichlet or generalized Neumann boundary conditions for $\Phi \Delta \Phi^{-1}$ and that the resulting self-adjoint operator is compatible with the decomposition Eq. (1.4). By slight abuse of notation we denote the transformed self-adjoint operator again by $\Delta$; the resulting self-adjoint extensions of $\Delta_\lambda, \lambda \in \mathbb{Z}$ are again denoted by $\Delta_\lambda$. So the operators are identified with their self-adjoint extensions which does not lead to notational confusion as the boundary conditions are fixed.

1.2. Hadamard partie finie regularized sums and integrals. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^* = (0, \infty)$. Let $f \in C^\infty(\mathbb{R}_+^*, \mathbb{C})$ be a function with a (partial) asymptotic expansion

$$f(x) \sim \sum_{j=1}^{N-1} \sum_{k=0}^{M_j} a_{jk} x^{\alpha_j} \log^k(x) + \sum_{k=0}^{M_0} a_{0k} \log^k(x) + x^{\alpha_N} \log^{M_N}(x)f_N(x), \quad \text{as } x \to \infty, \quad (1.5)$$

where $N \in \mathbb{N}$ is arbitrary, the remainder $f_N(x) = o(1)$ as $x \to \infty$, $(\alpha_j) \subset \mathbb{C}$ is a sequence of complex numbers with $\text{Re}(\alpha_j) \neq 0$, ordered by descending real part and $\text{Re}(\alpha_N) \leq 0$. We define its regularized limit for $x \to \infty$ as

$$\lim_{x \to \infty} f(x) := a_{00}. \quad (1.6)$$

If $f(x)$ has a (partial) asymptotic expansion of the form Eq. (1.5) as $x \to 0$, with $\text{Re}(\alpha_N) \geq 0$, its regularized limit at zero is defined again as the constant term in the expansion. If for $N \in \mathbb{N}$ sufficiently large, the remainder $x^{\alpha_N} \log^{M_N}(x)f_N \in$
L^1[1, \infty) (i.e. if Re(\alpha_N) < -1), the integral \int_1^R f(x)\,dx also admits an asymptotic expansion of the form Eq. (1.5) and we can define its regularized integral as

$$\int_1^\infty f(x)\,dx := \lim_{R \to \infty} \int_1^R f(x)\,dx. \quad (1.7)$$

Similarly,

$$\int_0^1 f(x)\,dx := \lim_{\epsilon \to 0} \int_\epsilon^1 f(x)\,dx, \quad (1.8)$$

if this regularized limit exists. We also need a notion of a partie finie regularized sum. The Euler MacLaurin summation formula yields for f \in C^\infty(\mathbb{R}_+, \mathbb{C}) and any N \in \mathbb{N}, M \in \mathbb{N}_0

$$\sum_{\lambda=1}^N f(\lambda) = \int_1^N f(x)\,dx + \sum_{k=1}^M \frac{B_{2k}}{(2k)!} \left( f^{(2k-1)}(N) - f^{(2k-1)}(1) \right) + \frac{1}{(2M+1)!} \int_1^N B_{2M+1}(x-[x]) f^{(2M+1)}(x)\,dx + \frac{1}{2} \left( f(1) + f(N) \right), \quad (1.9)$$

where B_j denotes the j-th Bernoulli number, B_j(x) the j-th Bernoulli polynomial, and f^{(j)} denotes the j-th derivative of f \in C^\infty(\mathbb{R}_+, \mathbb{C}). Assume that f admits an asymptotic expansion of the form Eq. (1.5), which may be differentiated (2M + 1) times and 2M > Re(\alpha_1). Then Eq. (1.9) shows that also \sum_{\lambda=1}^N f(\lambda) admits an asymptotic expansion of the form Eq. (1.5) and analogously to the regularized integral we define the regularized sum as

$$\sum_{\lambda=1}^\infty f(\lambda) := \lim_{N \to \infty} \sum_{\lambda=1}^N f(\lambda) \quad (1.10)$$

$$= \int_1^\infty f(x)\,dx - \sum_{k=1}^M \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1) + \frac{1}{2} f(1) + \frac{1}{2} \lim_{N \to \infty} f(N) + \frac{1}{(2M+1)!} \int_1^\infty B_{2M+1}(x-[x]) f^{(2M+1)}(x)\,dx. \quad (1.11)$$

The second equality follows from Eq. (1.9) and Eq. (1.5).

Similarly, for a function f \in C^\infty(\mathbb{R}, \mathbb{C}) with an asymptotic expansion of the form Eq. (1.5) both as x \to +\infty and also as x \to -\infty, which may be differentiated (2M + 1) times and 2M > Re(\alpha_1), we define the regularized sum as

$$\sum_{\lambda=-\infty}^\infty f(\lambda) := \lim_{N \to \infty} \sum_{\lambda=-N}^N f(\lambda) = \lim_{N \to \infty} \left( \sum_{\lambda=1}^N f(\lambda) + \sum_{\lambda=1}^N f(-\lambda) + f(0) \right). \quad (1.12)$$
1.3. Statement of the main results. Our first main result establishes a Fubini-type theorem for regularized integrals and is one fundamental ingredient in the derivation of our main Theorem 1.4 below.

**Theorem 1.1.** Assume \( f \in C^\infty(\mathbb{R}_+^2) \) is of the form

\[
f(x, y) = \sum_{j=0}^{N-1} f_{\alpha_j}(x, y) + F_N(x, y),
\]

where each \( f_{\alpha_j} \in C^\infty(\mathbb{R}_+^2 \setminus \{(0,0)\}) \) is homogeneous of order \( \alpha_j \in \mathbb{C} \) in both variables jointly, and the remainder \( F_N \) is integrable over \( \mathbb{R}_+^2 \setminus \{(x, y) \mid |(x, y)| \leq r\} \) for any \( r > 0 \). Then, for \( a, b \geq 0 \)

\[
\int_a^\infty \int_b^\infty f(x, y) \, dy \, dx = \int_b^\infty \int_a^\infty f(x, y) \, dx \, dy + \int_0^\infty f_{-2}(1, y) \log(y) \, dy.
\]

Note that regardless of the values of \( a, b \) the integration in the correction term on the right is from 0 to \( \infty \). The integral on the right hand side of Eq. (1.14) indeed exists, see Remark 3.2.

Our second main result addresses the polyhomogeneous asymptotic expansion of the resolvent trace for \( (\Delta_\lambda + z^2)^{-1} \) jointly in \( (\lambda, z) \in \mathbb{R}_+^2 \).

**Proposition 1.2.** Consider for \( \lambda \in \mathbb{R} \) and \( V, W \in C^\infty(\mathbb{R}) \) with \( V > 0 \) the differential operator

\[
\Delta_{\lambda,0} = -\frac{\partial^2}{\partial x^2} + \lambda^2 V + W : C^\infty_0(0, 1) \to C^\infty_0(0, 1).
\]

Let \( \Delta_\lambda \) be the self-adjoint extension of \( \Delta_{\lambda,0} \), obtained by imposing separated Dirichlet or generalized Neumann boundary conditions. Then the resolvent \( (\Delta_\lambda + z^2)^{-1} \) is trace class for \( |(\lambda, z)| \geq z_0 \) large, and its trace admits the following polyhomogeneous expansion

\[
\partial^\alpha \partial^\beta \text{Tr}(\Delta_\lambda + z^2)^{-1} \sim \sum_{i=0}^{\infty} h_i(\lambda, z), \quad |(\lambda, z)| \to \infty,
\]

where each \( h_i \in C^\infty(\mathbb{R}_+^2 \setminus \{(0,0)\}) \) is homogeneous of order \( -(\gamma_i) \), \( \gamma_i := i + 1 + \alpha + \beta \).

Note that \( h_i \) depends on \( \alpha, \beta \). Moreover, the leading term \( h_0 \) comes from the interior expansion only.

In particular

\[
\text{Tr}(\Delta_\lambda + z^2)^{-2} = -(2z)^{-1} \partial_z \text{Tr}(\Delta_\lambda + z^2)^{-1} \sim \sum_{i=0}^{\infty} h_i(\lambda, z), \quad |(\lambda, z)| \to \infty,
\]

where each \( h_i \in C^\infty(\mathbb{R}_+^2 \setminus \{(0,0)\}) \) is homogeneous of order \( -(\gamma_i) \), \( \gamma_i := i + 3 \), jointly in both variables.
Proposition 1.2 implies in particular the well-known fact, that for fixed \( \lambda \) there is an asymptotic expansion as \( z \to \infty \)

\[
\text{Tr}(\Delta_\lambda + z^2)^{-1} \sim \sum_{k=0}^{\infty} b_k z^{-k-1}, \tag{1.17}
\]

which may be differentiated in \( z \), e.g.,

\[
\text{Tr}(\Delta_\lambda + z^2)^{-2} = -\frac{1}{2z} \frac{d}{dz} \text{Tr}(\Delta_\lambda + z^2)^{-1} \sim \sum_{k=0}^{\infty} \frac{k+1}{2} b_k z^{-k-3} =: \sum_{k=0}^{\infty} c_k z^{-k-3}. \tag{1.18}
\]

The leading orders in the resolvent trace asymptotics for \( \Delta := \bigoplus_{\lambda \in \mathbb{Z}} \Delta_\lambda \) and \( \Delta_\lambda \) are fundamentally different. On the one hand \( \text{Tr}(\Delta_\lambda + z^2)^{-2} = O(z^{-3}) \) whereas \( \text{Tr}(\Delta + z^2)^{-2} = O(z^{-2}) \) as \( z \to \infty \). Indeed, the resolvent trace asymptotics of \( \Delta_\lambda \) does not sum up to the asymptotics of the full resolvent trace for \( \Delta \) in an obvious way. Nevertheless we have the

**Theorem 1.3.** In the notation of Proposition 1.2 we have for the operator

\[
\Delta := \bigoplus_{\lambda = -\infty}^{\infty} \Delta_\lambda \tag{1.19}
\]

the resolvent trace expansion

\[
\text{Tr}(\Delta + z^2)^{-2} = \sum_{\lambda = -\infty}^{\infty} \text{Tr}(\Delta_\lambda + z^2)^{-2} \sim \sum_{k=2}^{\infty} a_k z^{-k}, \quad z \to \infty. \tag{1.20}
\]

Note that \( \Delta \) is just an abstract sum of operators of the form Eq. (1.15) and therefore does not necessarily have an interpretation as a realization of an elliptic boundary value problem on a surface. If, like in the case of a surface of revolution, \( \Delta \) is a realization of a local elliptic boundary value problem, then Theorem 1.3 is well-known, e.g. [Gil95, Sec. 1.11]. More important than the result itself, however, is our method of proof using the polyhomogeneous resolvent trace expansion in Proposition 1.2 and Eq. (1.9), which explains precisely the difference in the leading orders of the resolvent trace expansion of \( \text{Tr}(\Delta_\lambda + z^2)^{-2} = O(z^{-3}) \), \( z \to \infty \), and their sum \( \text{Tr}(\Delta + z^2)^{-2} = O(z^{-2}) \), \( z \to \infty \).

We now define the associated zeta-regularized determinants, following [Les98, (1.7)]. The zeta-function of \( \Delta_\lambda \) is defined for \( \text{Re}(s) \gg 0 \) by

\[
\zeta(s, \Delta_\lambda) = \sum_{\mu \in \text{Spec} \Delta_\lambda \setminus \{0\}} m(\mu) \mu^{-s}, \tag{1.21}
\]

where \( m(\mu) \) denotes the multiplicity of the eigenvalue \( \mu > 0 \). Using the identity

\[
\zeta(s, \Delta_\lambda) = \frac{2}{\pi} \frac{\sin \pi s}{\pi} \int_{0}^{\infty} z^{1-2s} \text{Tr}(\Delta_\lambda + z^2)^{-1} dz, \tag{1.22}
\]
the asymptotics Eq. (1.17) implies that \( \zeta(s, \Delta_\lambda) \) extends meromorphically to \( \mathbb{C} \) with \( s = 0 \) being a regular point. From Eq. (1.17) and Eq. (1.22) one derives the formula for \( \log \det_\zeta \Delta_\lambda = -\zeta'(0, \Delta_\lambda) \)

\[
\log \det_\zeta \Delta_\lambda = -2 \int_0^\infty z \Tr(\Delta_\lambda + z^2)^{-1} \, dz.
\] (1.23)

The resolvent \( (\Delta + z^2)^{-1} \) is not trace class and we cannot employ exactly the same formulas for the definition of \( \det_\zeta \Delta \). However, integration by parts in Eq. (1.22) yields

\[
\zeta(s, \Delta_\lambda) = 2 \frac{\sin \pi s}{\pi (1 - s)} \int_0^\infty z^{3-2s} \Tr(\Delta_\lambda + z^2)^{-2} \, dz,
\] (1.24)

and thus with Eq. (1.18)

\[
\log \det_\zeta \Delta_\lambda = -2 \int_0^\infty z^3 \Tr(\Delta_\lambda + z^2)^{-2} \, dz.
\] (1.25)

Invoking Theorem 1.3 one sees that Eq. (1.24) is still valid for \( \Delta \) instead of \( \Delta_\lambda \). Moreover, the asymptotic expansion Eq. (1.20) implies that Eq. (1.25) also holds for \( \Delta \):

\[
\log \det_\zeta \Delta = -2 \int_0^\infty z^3 \Tr(\Delta + z^2)^{-2} \, dz.
\] (1.26)

Note that unlike in the standard convention, here we do not set the zeta-determinant to zero for operators that are not invertible. Our third and final main result now reads as follows.

**Theorem 1.4.** In the notation of Proposition 1.2 and Theorem 1.3 we have for the zeta-regularized sum of \( \Delta \)

\[
\log \det_\zeta \Delta = \sum_{\lambda = -\infty}^\infty \log \det_\zeta \Delta_\lambda - 4 \int_0^\infty h_2(1, y) \log(y) \, dy,
\] (1.27)

where \( h_2 \) denotes the homogeneous term of degree \( -5 \) in the polyhomogeneous asymptotic expansion of \( \Tr(\Delta_\lambda + z^2)^{-2} \) as \( ||(\lambda, z)|| \to \infty \).

In the diploma thesis of B. Sauer [Sa13] the term \( \int_0^\infty h_2(1, y) \log(y) \, dy \) has been identified in terms of \( V, W \) and their derivatives at the boundary.

Note that by Eq. (1.16), the correction term \( h_2 \) in Theorem 1.4 is the third (local) component of the polyhomogeneous asymptotic expansion of \( \Tr(\Delta_\lambda + z^2)^{-2} \).

### 2. Polyhomogeneous expansion of the resolvent trace

In this section we establish a polyhomogeneous asymptotic expansion of the resolvent trace for \( (\Delta_\lambda + z^2)^{-1} \) jointly in \( (\lambda, z) \in \mathbb{R}^2_+ \). The discussion is separated into two parts for the interior and the boundary parametrices. We begin with the interior parametrix where the polyhomogeneous expansion is a consequence of the strongly parametric elliptic calculus.
2.1. The interior parametrix. We will use here freely the calculus of pseudo-differential operators with parameter, for a survey type exposition see [Les10, Sec. 4 and 5]. We apply this calculus to establish a polyhomogeneous asymptotic expansion for the resolvent \((\Delta_\lambda + z^2)^{-1}\) in the interior of the interval \([0, 1]\), jointly in \((\lambda, z)\).

Consider the differential operators
\[
\begin{align*}
\Delta_{\lambda,0} &= -\partial_x^2 + \lambda^2 V + W : C_0^\infty(0, 1) \to C_0^\infty(0, 1), \\
\Delta_\lambda^R &= -\partial_x^2 + \lambda^2 V + W : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}), 
\end{align*}
\]
where \(V, W \in C_0^\infty(\mathbb{R})\) with \(V > 0\). As before, \(\Delta_\lambda\) is a self-adjoint extension of \(\Delta_{\lambda,0}\) in \(L^2[0, 1]\), obtained by imposing separated Dirichlet or generalized Neumann boundary conditions. The boundary conditions will be specified in the next section. We write
\[
\Delta(\lambda, z) := \Delta_\lambda + z^2, \\
\Delta^R(\lambda, z) := \Delta_\lambda^R + z^2.
\]

Then \(\Delta^R(\lambda, z)\) is elliptic in the parametric sense with parameter \((\lambda, z)\) in the cone \(\Gamma = \mathbb{R}_+^* \times \mathbb{R}_+^*\). The space of classical parameter dependent pseudo-differential operators of order \(m\) is, as usual, denoted by \(\text{CL}^m(\mathbb{R}; \Gamma)\). By [Shu01, Sec. II.9] \(\Delta^R(\lambda, z)\) admits a parametrix \(R \equiv R(\lambda, z) \in \text{CL}^{-2}(\mathbb{R}; \Gamma)\), such that
\[
\Delta^R(\lambda, z) R - I, \ R\Delta^R(\lambda, z) - I \in \text{CL}^{-\infty}(\mathbb{R}; \Gamma).
\]

Since \(\text{ord} R + \dim \mathbb{R} = -1 < 0\) the Schwartz kernel \(k(\cdot, \cdot; \lambda, z)\) of \(\Delta^R(\lambda, z)^{-1}\) is a continuous function and on the diagonal it has an asymptotic expansion
\[
k(x, x; \lambda, z) \sim \sum_{j=0}^{\infty} e_j \left(\frac{\lambda}{|\lambda, z|}\right)^{2j} |(\lambda, z)|^{-1-j}, \ |(\lambda, z)| \to \infty, \ (\lambda, z) \in \Gamma, \tag{2.3}
\]
see [Les10, Theorem 5.1]. The functions \(e_j\) are smooth on \(\mathbb{R} \times (\Gamma \cap S^1)\) and the expansion Eq. (2.3) is uniform for \(x\) in compact subsets of \(\mathbb{R}\).

We choose cutoff functions \(\phi\) and \(\psi\), with \(\text{supp} \phi, \text{supp} \psi \subset (0, 1)\), such that \(\text{supp} \phi \subset \text{supp} \psi\) and \(\text{supp} \phi \cap \text{supp} \psi = \emptyset\). We define \(R^I := \psi R \phi\), which will be shown to provide an interior parametrix to \(\Delta(\lambda, z)\). Indeed
\[
\Delta(\lambda, z) R^I = [-\partial_x^2, \psi]\psi R \phi + \psi(\Delta^R(\lambda, z) R - I) \phi + \phi =: \phi + R_2(\lambda, z) \phi. \tag{2.4}
\]

Note that by the choice of cutoff functions \([-\partial_x^2, \psi]\) and \(\phi\) have disjoint support and hence \([-\partial_x^2, \psi]\psi R \phi \in \text{CL}^{-\infty}(\mathbb{R}; \Gamma)\). Moreover, \(\Delta^R(\lambda, z) R - I \in \text{CL}^{-\infty}(\mathbb{R}; \Gamma)\) and hence \(R_2(\lambda, z) \in \text{CL}^{-\infty}(\mathbb{R}, \Gamma)\), however the Schwartz kernel of \(R_2(\lambda, z)\) is compactly
supported in \((0, 1)^2\). Consequently, by Eq. (2.3) we find
\[
\text{Tr}(\Delta(\lambda, z)^{-1} \phi) = \text{Tr} R^\delta - \text{Tr}(\Delta(\lambda, z)^{-1} R_2)
\]
\[
= \text{Tr}(\psi \Delta^R(\lambda, z)^{-1} \phi) + O((|\lambda, z|)^{-\infty})
\]
\[
\sim \sum_{j=0}^{\infty} e^j \frac{(\lambda, z)}{(|\lambda, z|)^{-1-j}}, |(\lambda, z)| \to \infty.
\] (2.5)

This establishes a polyhomogeneous asymptotic expansion for the trace of the resolvent \(\Delta(\lambda, z)^{-1}\) in the interior as a consequence of the parametric pseudo-differential calculus.

### 2.2. The boundary parametrix.

We construct a parametrix to \(\Delta(\lambda, z)\) near the boundary \(x = 0\). The parametrix construction near \(x = 1\) works ad verbatim. The polyhomogeneous asymptotic expansion of the boundary parametrix together with the expansion Eq. (2.5) of the interior parametrix proves the statement in Proposition 1.2 on the polyhomogeneous asymptotic expansion of the trace of the resolvent \(\Delta(\lambda, z)^{-1}\).

Consider \(l = -\partial_x^2\) acting on \(C^\infty_\delta(\mathbb{R})\). The operator \(l\) is essentially self-adjoint in \(L^2(\mathbb{R})\) and we write \(\bar{l}\) for its self-adjoint extension. For \(0 \leq \theta < \pi\) let \(L^\theta\) be \(\bar{l}\) restricted to
\[
\mathcal{D}(L^\theta) := \{ f \in H^1(\mathbb{R}_+) \mid \cos \theta \cdot f(0) + \sin \theta \cdot f'(0) = 0 \}.
\] (2.6)

For \(\mu \in \mathbb{C}, \text{Re} \mu > 0\) the resolvent kernel of \((L^\theta + \mu^2)^{-1}\) is given by
\[
K_\theta(x, y; \mu) = \frac{1}{2\mu} \left[ e^{-\mu|x-y|} + C(\mu, \theta)e^{-\mu|x+y|} \right],
\]
\[
C(\mu, \theta) = \frac{\mu \sin \theta + \cos \theta}{\mu \sin \theta - \cos \theta}.
\] (2.7)

The kernel \(K_\mathbb{R}(\cdot, \cdot; \mu)\) of the resolvent \((\bar{l} + \mu^2)^{-1}\) is given by
\[
K_\mathbb{R}(x, y; \mu) = \frac{1}{2\mu} \exp(-\mu|x-y|).
\] (2.8)

Assume below \(\mu > 0\) for simplicity. Then
\[
|K_\theta(x, y; \mu)| \leq (1 + |C(\mu, 0)|) \frac{1}{2\mu} \exp(-\mu|x-y|)
\]
\[
= (1 + |C(\mu, 0)|)K_\mathbb{R}(x, y; \mu).
\] (2.9)

We will also need an estimate for a \(j\)-fold convolution of the resolvent kernels. Let \(K_\mathbb{R}^j(x, y; \mu)\) denote the kernel of \((\bar{l} + \mu^2)^{-1}\). From the formula
\[
\frac{\partial}{\partial \mu}(\bar{l} + \mu^2)^{-j} = 2\mu(-j)(\bar{l} + \mu^2)^{-j-1},
\] (2.10)

we infer
\[
(\bar{l} + \mu^2)^{-j} = \frac{(-1)^{j-1}}{2^{j-1}(j-1)!} \frac{1}{\mu} \left( \frac{\partial}{\partial \mu} \right)^{j-1}(\bar{l} + \mu^2)^{-1}.
\] (2.11)
From Eq. (2.11) and the explicit formula Eq. (2.8) for $K_R$ we find
\[
K^j_R(x, y; \mu) = \sum_{k=0}^{j-1} \frac{1}{k!} \left( \prod_{l=1}^{j-k-1} \frac{2l - 1}{l} \right) \frac{|x - y|^k}{2^{j-k} \mu^{j-k}} e^{-\mu|x - y|} \\
\leq \frac{1}{2} \sum_{k=0}^{j-1} \frac{|x - y|^k}{2^{j-k} \mu^{j-k}} e^{-\mu|x - y|} \\
\leq \frac{1}{2^{j-1} \mu^j} \exp \left( -\frac{\mu}{2} |x - y| \right). \tag{2.12}
\]

Consider smooth potentials $V, W \in C_0^\infty(\mathbb{R}_+)$, with $V(0) > 0$ and assume for the moment that $\text{supp}V \subset [0, \delta], \delta > 0$ sufficiently small, such that $\|V - V(0)\|_\infty \leq \frac{1}{2} V(0)$. Abbreviate $\mu^2 := \lambda^2 V(0) + z^2$ and $V := V - V(0)$. Consider
\[
(L^0 + \lambda^2 V + W + z^2)^{-1} = (1 + (L^0 + \mu^2)^{-1}(\lambda^2 \tilde{V} + W))^{-1}(L^0 + \mu^2)^{-1} = \sum_{j=0}^\infty (-1)^j \left( (L^0 + \mu^2)^{-1}(\lambda^2 \tilde{V} + W) \right)^j (L^0 + \mu^2)^{-1}.
\]

Note that the Neumann series converges in the operator norm sense, since for $\|\tilde{V}\|_\infty \leq \frac{1}{2} V(0)$ and $z \gg 0$ sufficiently large we find for the operator norm
\[
\|(L^0 + \mu^2)^{-1}(\lambda^2 \tilde{V} + W)\| \leq \frac{\lambda^2 \|\tilde{V}\|_\infty + \|W\|_\infty}{\lambda^2 V(0) + z^2} < 1. \tag{2.13}
\]

We also need to justify the corresponding Neumann series expansion for the resolvent kernel. Suppose that $V, W$ are both supported in $[0, \delta]$ and recall the estimates Eq. (2.9) and Eq. (2.12). Then for real $(\lambda, z)$ and for $0 \leq x, y < \delta$ we find
\[
\left| \left( (L^0 + \mu^2)^{-1}(\lambda^2 \tilde{V} + W) \right)^j (x, y) \right| \leq (\lambda^2 \|\tilde{V}\|_\infty + \|W\|_\infty)^j (1 + |C(\mu, \theta)|)^j \\
\times \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} K_R(x, s_1; \mu) \cdots K_R(s_{j-1}, y; \mu) \, ds_1 \cdots ds_{j-1} \\
= (\lambda^2 \|\tilde{V}\|_\infty + \|W\|_\infty)^j (1 + |C(\mu, \theta)|)^j K_R(x, y; \mu) \\
\leq \frac{\mu}{2} \left( \frac{\lambda^2 \|\tilde{V}\|_\infty + \|W\|_\infty}{\mu^2} \right)^j (1 + |C(\mu, \theta)|)^j \exp \left( -\frac{\mu}{2} |x - y| \right).
\]

Note that $|C(\mu, \theta)| \to 1$ as $\mu \to \infty$ and consequently for $\mu \geq \mu_0$ large enough, the sequence of kernels converges uniformly for $0 \leq x, y < \delta$ and
\[
\sum_{j=0}^\infty \left| \left( (L^0 + \mu^2)^{-1}(\lambda^2 \tilde{V} + W) \right)^j (x, y) \right| \\
\leq \frac{1}{2} \frac{\mu^3 \exp(-\mu|x - y|/2)}{\mu^2 - (\lambda^2 \|V\|_\infty + \|W\|_\infty)(1 + |C(\mu, \theta)|)}. \tag{2.14}
\]
Similar arguments also work for the derivatives of the kernels. Another convolution by \( K \) then yields

**Proposition 2.1.** Let \( V, W \in C^\infty_0(\mathbb{R}) \) with \( V(0) > 0 \) and \( \| V - V(0) \|_\infty \leq \frac{1}{2} V(0) \). Put \( \mu^2 = \lambda^2 V(0) + z^2 \). Then the kernel of \( (L^0 + \lambda^2 V + W + z^2)^{-1} \) satisfies uniformly for \( \mu \geq \mu_0 > 0 \) and \( \lambda \geq 0 \)

\[
|\partial_x^j (L^0 + \lambda^2 V + W + z^2)^{-1}(x, y)| \leq C(\mu_0) \mu^{j-1} \exp(-\mu|x - y|/2), \quad j = 0, 1. \tag{2.15}
\]

Consider the differential operator \( \Delta_{\lambda, 0} = -\partial_x^2 + \lambda^2 V + W \) on \( C^\infty_0(\mathbb{R}_+) \), and its self-adjoint realization \( L^0 + \lambda^2 V + W \) in \( L^2(\mathbb{R}_+) \). We can now write down a parametrix for \( \Delta^0(\lambda, z) := L^0 + \lambda^2 V + W + z^2 \). We consider two cutoff functions \( \phi \) and \( \psi \), see Figure 1, both identically one in an open neighborhood of \( x = 0 \) with compact support \( \text{supp} \phi, \text{supp} \psi \subset [0, 1) \) such that \( \text{supp} \phi \subset \text{supp} \psi \) and \( \text{supp} \phi \cap \text{supp} \psi = \emptyset \).

**Figure 1.** The cutoff functions \( \phi \) and \( \psi \).

Given \( V, W \in C^\infty[0, 1] \) with \( V(0) > 0 \) we set

\[
W_\psi = \psi W \quad \text{and} \quad V_\psi = \psi V = V(0) + \tilde{V}_\psi,
\tag{2.16}
\]

where we choose \( \text{supp} \psi \) small enough to guarantee that \( \| \tilde{V}_\psi \|_\infty \leq \frac{1}{2} V(0) \). Then we put

\[
R_\phi := \psi (L^0 + \lambda^2 V_\psi + W_\psi + z^2)^{-1} \phi.
\]

Clearly, \( R_\phi \) maps into \( \mathcal{D}(L^0) = \mathcal{D}(\Delta^0(\lambda, z)) \). Moreover we compute

\[
\Delta^0(\lambda, z) R_\phi = [-\partial_x^2, \psi] [L^0 + \lambda^2 V_\psi + W + z^2]^{-1} \phi + \phi =: \phi + R_3(\lambda, z).
\]

Note that by the choice of cutoff functions, \( [-\partial_x^2, \psi] \) and \( \phi \) have disjoint support. Let \( d > 0 \) denote the minimum of \( |x - y| \) for \( x \in \text{supp} [-\partial_x^2, \psi] \) and \( y \in \text{supp} \phi \). Then there exists a constant \( C > 0 \) such that by Proposition 2.1

\[
|R_3(\lambda, z)(x, y)| \leq C \cdot \exp(-\mu d/2) = O(\mu^{-\infty}), \quad \text{as} \ \mu \to \infty.
\]

Consequently we find

\[
\text{Tr}(\Delta^0(\lambda, z)^{-1} \phi) = \text{Tr} R_\phi + O((|\lambda, z|)^{-\infty}), \quad |(\lambda, z)| \to \infty, \tag{2.17}
\]
and hence it suffices to establish a polyhomogeneous expansion for the trace of the boundary parametrix $R_\partial$. Write

$$K_+(x, y; \mu) = -\frac{1}{2\mu} C(\mu, 0) e^{-\mu(x+y)}$$

(2.18)

so that $K_0 = K_\mathbb{R} + K_+$, cf. Eq. (2.7) and Eq. (2.8). Moreover we abbreviate

$$\lambda(V, W) := \lambda^2 \tilde{V}_\psi + W_\psi.$$  

(2.19)

Then, using $\mu^2 = \lambda^2 V(0) + z^2$, we can write

$$R_\partial = \psi(L^\theta + \lambda^2 V_\psi + W_\psi + z^2)^{-1} \phi = \psi(L^\theta + \mu^2 + \lambda(V, W))^{-1} \phi.$$  

Then, by Eq. (2.14) we may expand the boundary parametrix $R_\partial$ as a Neumann series as follows

$$R_\partial = \sum_{j=0}^{\infty} (-1)^j \psi \left[ K_\Theta \lambda(V, W) \right] j K_\Theta \phi$$

$$= \sum_{j=0}^{\infty} (-1)^j \psi \left[ [K_\Theta \lambda(V, W)]^j K_\Theta - [K_\mathbb{R} \lambda(V, W)]^j K_\mathbb{R} \right] \phi$$

$$+ \sum_{j=0}^{\infty} (-1)^j \psi \left[ K_\mathbb{R} \lambda(V, W) \right] j K_\mathbb{R} \phi =: R^0_\partial + R^1_\partial.$$  

By similar arguments as in the previous subsection, $\text{Tr}(R^1_\partial) = \text{Tr}(R^1) + O(\mu^{-\infty})$ and hence a polyhomogeneous expansion of the boundary parametrix follows from such an expansion of $R^0_\partial$. We write

$$R^0_\partial = \sum_{j=0}^{\infty} (-1)^j \psi \left[ [K_\Theta \lambda(V, W)]^j K_\Theta - [K_\mathbb{R} \lambda(V, W)]^j K_\mathbb{R} \right] \phi =: \sum_{j=0}^{\infty} R^0_j.$$  

Before we proceed we note that for elements $a, b$ in a not necessarily commutative ring we have the identity

$$(a + b)^n - b^n = \sum_{j=0}^{n-1} b^j a (a + b)^{n-1-j},$$  

(2.20)

as one checks by induction. Consequently

$$R^0_j = (-1)^j \psi \left[ [K_\Theta \lambda(V, W)]^j K_\Theta - [K_\mathbb{R} \lambda(V, W)]^j K_\mathbb{R} \right] \phi$$

$$= (-1)^j \psi \sum_{k=0}^{j-1} (K_\mathbb{R} \lambda(V, W))^k (K_+) \lambda(V, W) (K_\Theta \lambda(V, W))^{j-k-1} K_\mathbb{R} \phi$$

$$+ (-1)^j \psi (K_\Theta \lambda(V, W))^j K_+ \phi.$$  

(2.21)
To expand $R_0^0$ we write for a fixed $M \in \mathbb{N}$
\[
R_0^0 = \sum_{j=0}^{M-1} R_0^0 j + \sum_{j=M}^{\infty} R_0^j.
\]

The first task is to show that the trace of the second sum in Eq. (2.22) decays sufficiently fast, more concretely $O(\mu^{-M-3/2})$, $\mu \to \infty$. This is the content of the next proposition, cf. also [Ver13, Cor. 4.2] where a parallel result is obtained for elliptic boundary value problems by a different method. The second task, which will occupy the whole Subsection 2.3, then is to show that the first sum in Eq. (2.22) has a polyhomogeneous expansion. Since we may choose $M$ as large as we please we will then obtain Proposition 1.2.

**Proposition 2.2.** Let $M \in \mathbb{N}$, $\gamma, \nu \in \mathbb{N}_0$ be fixed. For $\mu_0$ sufficiently large there exist constants $C > 0, 0 < q < 1$ such that for $N \geq M$ and $\mu \geq \mu_0$
\[
\|\partial_\gamma^\nu \partial_\nu^\gamma R_0^N\|_{tr} \leq C \cdot N \cdot q^{N-M} \cdot \mu^{-M-\gamma-\nu-3/2},
\]
and consequently
\[
\|\partial_\gamma^\nu \partial_\nu^\gamma \sum_{j=M}^{\infty} R_0^j\|_{tr} = O(\mu^{-M-\gamma-\nu-3/2}), \text{ as } \mu \to \infty.
\]

Here $\| \cdot \|_{tr}$ denotes the trace norm.

**Proof.** We treat the case $\gamma = \nu = 0$. The case of general $\gamma, \nu$ follows easily since, e.g.,
\[
\partial_\nu (L^0 + \mu^2)^{-1} = -2\nu (L^0 + \mu^2)^{-2},
\]
resp.
\[
\partial_\lambda (L^0 + \mu^2)^{-1} = -2\nu V(0) (L^0 + \mu^2)^{-2},
\]
and similarly for the other involved kernels.

Each of the $N$ summands of $R_0^N$ is of the form
\[
P_N = \psi K_0 \prod_{j=1}^{N} \lambda(V,W) K_j \phi,
\]
where $K_j, j = 0, \ldots, N$, is either $K_{\omega}, K_\gamma$, or $K_\phi$. Note that due to the factor $\lambda(V,W)$ all kernels (may be assumed to have) support $\subset [0, \delta]$.

In view of Eq. (2.13) we may choose $\mu_0 > 0$ sufficiently large such that there exists a $0 < q < 1$ such that
\[
\|\lambda(V,W) K_j\| \leq q
\]
for \( \mu \geq \mu_0 \) and all \( j \). Thus we may estimate the trace norm of \( P_N \) by

\[
\|P_N\|_{tr} \leq \|\psi K_0\|_{HS} \cdot q^{N-M} \cdot \prod_{j=1}^{M} \lambda(V,W) K_j \|_{HS},
\]

(2.27)

where \( \| \cdot \|_{tr} \), \( \| \cdot \|_{HS} \) denote the trace norm resp. the Hilbert-Schmidt norm.

Of the \( K_j \) in Eq. (2.27) at least one equals \( K_+ \) (cf. Eq. (2.21)) and by choosing those factors whose norm we estimate by \( q \) appropriately we can arrange that in Eq. (2.27) at least one of the \( K_j, j = 1, \ldots, M \) equals \( K_+ \). So we have

1. All \( K_j, j = 0, \ldots, N \) satisfy the estimate

\[
|K_j(x, y; \mu)| \leq C_1 \frac{1}{\mu} e^{-\mu |x-y|},
\]

(2.28)

2. At least one kernel \( K_j \) satisfies

\[
|K_j(x, y; \mu)| \leq C_1 \frac{1}{\mu} e^{-\mu (x+y)}.
\]

(2.29)

For the Hilbert-Schmidt norm of \( \psi K_0 \) we have

\[
\|\psi K_0\|_{HS}^2 \leq \frac{C_2^2}{\mu^2} \int_0^1 \int_0^1 e^{-2\mu |x-y|} dx dy = O(\mu^{-3}), \text{ as } \mu \to \infty.
\]

(2.30)

For \( Q_M := \prod_{j=1}^{M} \lambda(V,W) K_j \) we claim that

\[
|Q_M(x, y; \lambda, \mu)| \leq \sum_{\alpha, \beta} c_{\alpha \beta} \cdot \mu^{-\alpha} \cdot \max(x, y)^\beta \cdot e^{-\mu (x+y)},
\]

(2.31)

where the sum is over finitely many \( \alpha, \beta \) with the restriction \( \alpha + \beta \geq M - 1 \).

The Hilbert-Schmidt norm square of each summand on the right of Eq. (2.31) can be estimated by

\[
2\mu^{-2\alpha} \int_0^1 \int_0^y y^{2\beta} e^{-2\mu (x+y)} dx dy \\
\leq 2\mu^{-2\alpha} \int_0^\infty y^{2\beta} e^{-2\mu y} dy \cdot \int_0^\infty e^{-2\mu x} dx \\
= O(\mu^{-2\alpha - 2\beta - 2}) = O(\mu^{-2M}), \text{ as } \mu \to \infty,
\]

(2.32)

thus Eq. (2.31) implies

\[
\|Q_M\|_{HS} = O(\mu^{-M}),
\]

(2.33)

and Eq. (2.27), Eq. (2.30) and Eq. (2.33) give the claim. It therefore remains to prove Eq. (2.31), which we single out separately below. \( \Box \)
2.2.1. **Proof of Eq. (2.31)**. We proceed by induction on \( M \in \mathbb{N} \). Recall from Eq. (2.19) \( \lambda(V,W) = \lambda^2 \hat{V}_\psi + W_\psi \), \( \hat{V}_\psi = \psi V - V(0) \). Recall furthermore from Eq. (2.16) that \( \psi \) was chosen such that \( \|\hat{V}_\psi\|_\infty \leq \frac{1}{2} V(0) \). Moreover, since \( \hat{V}_\psi(0) = 0 \) and \( \hat{V} \) is smooth, we have \( |\hat{V}_\psi(x)| \leq c \cdot x \) for some \( c > 0 \). Thus
\[
|\lambda(V,W)(x)| \leq c(\lambda^2 \cdot x + 1),
\]
and hence
\[
|\lambda(V,W)K_1(x, y; \lambda, \mu)| \leq c(\mu \cdot x + \mu^{-1})e^{-\mu|x-y|},
\]
resp., for at least one \( j \), \( e^{-\mu(x+y)} \) instead of \( e^{-\mu|x-y|} \). This establishes Eq. (2.31) for \( M = 1 \).

For the inductive step we treat the case \( x \leq y \). Though the kernels are not symmetric, the estimates for \( x \geq y \) are similar. We pick one of the summands on the right of Eq. (2.31)
\[
k_1(x, y; \mu) = \mu^{-\alpha} \max(x, y)^\beta e^{-\mu(x+y)}, \quad \alpha + \beta \geq M - 1,
\]
and
\[
k_2(x, y; \mu) = (\mu \cdot x + \mu^{-1})e^{-\mu|x-y|}.
\]
We split the integral \( \int_0^y k_1(x, z; \mu)k_2(z, y; \mu)dz \) into the two parts \( \int_0^y \) and \( \int_y^1 \). In the first case \( z \in [0, y] \) we find
\[
\int_0^y |k_1(x, z; \mu)k_2(z, y; \mu)|dz \\
\leq \mu^{-\alpha}(\mu y + \mu^{-1})e^{-\mu(x+y)}y^\beta \int_0^y 1dz \\
\leq C_2(\mu^{-\alpha-1}y^{\beta+2} + \mu^{-\alpha+1}y^{\beta+1})e^{-\mu(x+y)},
\]
certainly \( \alpha - 1 + \beta + 2 \geq (M + 1) - 1, \alpha + 1 + \beta + 1 \geq (M + 1) - 1 \).

Secondly,
\[
\int_y^1 |k_1(x, z; \mu)k_2(z, y; \mu)|dz \\
\leq \mu^{-\alpha}e^{-\mu(x-y)} \int_y^1 z^\beta(\mu z + \mu^{-1})e^{-2\mu z}dz \\
\leq \mu^{-\alpha+1+\beta}e^{-\mu(x-y)} \int_{\mu y}^\infty z^\beta(\mu z + \mu^{-1})e^{-2\mu z}dz \\
\leq C_3(\mu^{-\alpha+1+\beta+1} + \mu^{-\alpha+2+1} + \mu^{-\alpha+1} + \mu^{-\alpha+2})e^{-\mu(x+y)}.
\]
In the last step we have used that for \( \gamma > 0 \)
\[
\int_\mathbb{R}^\infty z^\delta e^{-2\mu z}dz \leq C(\delta)(1 + R^\delta)e^{-2R}, \quad 0 \leq R < \infty.
\]
The last line of Eq. (2.34) is indeed of the form as the right hand side of Eq. (2.31) with \( \alpha + \beta + 2 \geq \alpha + \beta + 1 \geq (M + 1) - 1 \). This establishes the inductive step and Eq. (2.31) is proved. \( \square \)

### 2.3. The polyhomogeneous expansion of the boundary parametrix

It follows from Proposition 2.2 that a polyhomogeneous expansion of the trace of the boundary parametrix \( R^0_\partial \) up to a given order \( O(\mu^{-M-3/2}) \) follows from a polyhomogeneous expansion of the trace of the finitely many summands

\[
\sum_{j=0}^{M-1} R_{\partial}^{0j} = \sum_{j=0}^{M-1} (-1)^j \psi \left( [K_\partial \lambda(V,W)]^j K_\partial - [K_\mathbb{R} \lambda(V,W)]^j K_\mathbb{R} \right) \phi.
\]

Since \( M \) can be chosen arbitrarily this in fact establishes a full asymptotic expansion of the trace of \( R^0_\partial \). We establish a polyhomogeneous expansion of the finitely many summands above using the microlocal formalism of blowups.

The kernels \( K_\partial \) and \( K_\mathbb{R} \) are functions on \( \mathbb{R}^+_{1/\mu} \times (\mathbb{R}^+)^2_{|x,y|} \) with non-uniform behaviour at the diagonal \( D := \{ \mu = \infty, x = y \} \) and the highest codimension corner \( A := \{ \mu = \infty, x = y = 0 \} \). This non-uniform behaviour is resolved by considering an appropriate blowup \( M^j_\partial \) of \( \mathbb{R}^+_1 \times \mathbb{R}^+_1 \) at \( A \) and \( D \), a procedure introduced by Melrose, see [Mel93], such that both kernels lift to polyhomogeneous distributions on the manifold with corners \( M^j_\partial \) in the sense of the following definition.

**Definition 2.3.** Let \( X \) be a manifold with corners, with embedded boundary defining functions \( \{(H_i,\rho_i)\}_{i=1}^N \). For any multi-index \( b = (b_1, \ldots, b_N) \in \mathbb{C}^N \) we denote \( \rho^b = \rho_1^{b_1} \cdots \rho_N^{b_N} \). Let \( \mathcal{V}_b(X) \) denote the space of smooth vector fields on \( X \) which lie tangent to all boundary faces, the \( b \)-vector fields. We consider distributions on \( X \) that are locally restrictions of distributions defined across the boundaries of \( X \). A distribution \( \omega \) on \( X \) is said to be conormal if \( \omega \in \rho^b \mathcal{C}^\infty(X) \) for some \( b \in \mathbb{C}^N \), and \( \mathcal{V}_i \omega \in \rho^b_i \mathcal{C}^\infty(X) \) for all \( \mathcal{V}_i \in \mathcal{V}_b(X) \) and for every \( \ell \geq 0 \). An index set \( E_1 = \{(\gamma,p)\} \subset \mathbb{C} \times \mathbb{N} \) satisfies the following hypotheses:

1. \( \text{Re}(\gamma) \) accumulates only at plus infinity,
2. For each \( \gamma \) there is \( \rho_\gamma \in \mathbb{N}_0 \), such that \( (\gamma,p) \in E_1 \) iff \( p \leq \rho_\gamma \),
3. If \( (\gamma,p) \in E_1 \), then \( (\gamma+j,p') \in E_1 \) for all \( j \in \mathbb{N} \) and \( 0 \leq p' \leq p \).

An index family \( E = (E_1, \ldots, E_N) \) is an \( \mathbb{N} \)-tuple of index sets. Finally, we say that a conormal distribution \( \omega \) is polyhomogeneous on \( X \) with index family \( E \), we write \( \omega \in \mathcal{A}^{E}_{\text{phg}}(X) \), if \( \omega \) is conormal and if in addition, near each \( H_i \),

\[
\omega \sim \sum_{(\gamma,p) \in E_i} a_{\gamma,p} \rho_\gamma \log(\rho_i)^p, \quad \text{as } \rho_i \to 0,
\]

with coefficients \( a_{\gamma,p} \) conormal on \( H_i \), polyhomogeneous with index \( E_j \) at any \( H_i \cap H_j \).

In view of the non-uniform behaviour of \( K_\mathbb{R} \) across the diagonal, we also need to consider polyhomogeneous distributions on a manifold with corners \( X \) that are conormal to an embedded submanifold \( Y \subset X \). The definition is drawn from
[Maz91]. The basic space $I^m(\mathbb{R}^n, \{0\})$ consists of compactly supported distributions with the Fourier transform given by a symbol of order $(m-n/4)$. $I^m(\mathbb{R}^n, \{0\})$ is invariant under local diffeomorphisms and thus makes sense on any manifold around an isolated point.

For an embedded $k$-submanifold $Y \subset X$, any point in $Y$ admits an open neighborhood $U$ in $X$ which can be locally decomposed as a product $U = X' \times X''$ so that $U \cap Y = X' \times \{p\}, p \in X''$. The space $I^m(X, Y)$ is defined (locally) as the space of smooth functions on $X'$ with values $I^{m+\dim X'/4}(X'', \{p\})$. The normalization is chosen to give pseudo-differential operators their expected orders. All distributions in $I^m(X, Y)$ are locally restrictions of distributions on an ambient space, which are conormal to any smooth extension of $Y$ across $\partial X$.

Choosing now index sets $E$ for each boundary face of $X$ as in Definition 2.3, we define a space $A_{phg}^E(X, Y)$ as the space of distributions conormal to $Y$, with polyhomogeneous expansions as in Eq. (2.36) at all boundary faces and with coefficients conormal to the intersection of $Y$ with each boundary face.

We now continue with the definition of a blowup $M_2^b$, so that the kernels $K_+, K_\mathbb{R}$ lift to polyhomogeneous distributions conormal to an embedded submanifold. Blowing up $\mathbb{R}_+ \times \mathbb{R}^2_+$ at $A$ and $\mathcal{D}$ amounts in principle to introducing polar coordinates in $\mathbb{R}_+ \times \mathbb{R}^2_+$ at $A$ and $\mathcal{D}$ together with a unique minimal differential structure with respect to which these coordinates are smooth. Similar construction has been employed in [Moo99] and [MV12] with the difference that the blowups there are parabolic in time direction.

We first perform a blowup of $A$. The resulting space $[\mathbb{R}_+ \times \mathbb{R}_+^2, A]$ is defined as the union of $\mathbb{R}_+ \times \mathbb{R}_+^2 \setminus A$ with the interior spherical normal bundle of $A$ in $\mathbb{R}_+ \times \mathbb{R}_+^2$. The blowup $[\mathbb{R}_+ \times \mathbb{R}_+^2, A]$ is endowed with the unique minimal differential structure with respect to which smooth functions in the interior of $\mathbb{R}_+ \times \mathbb{R}_+^2$ and polar coordinates on $\mathbb{R}_+ \times \mathbb{R}_+^2$ around $A$ are smooth. This blowup introduces four new boundary hypersurfaces; we denote these by ff (the front face), rf (the right face), lf (the left face) and tf (the temporal face).

![Figure 2](image.png)

**Figure 2.** The reduced blowup space $M_{rb}^2 = [\mathbb{R}_+ \times \mathbb{R}_+^2, A]$. 
The actual blowup space $M_2^b$ is obtained by a blowup of $M_2^c$ along the lift of the diagonal $D$. The resulting blowup space $M_2^c$ is defined as before by cutting out the submanifold and replacing it with its spherical normal bundle. The second blowup introduces an additional boundary hypersurface $td$, the temporal diagonal. $M_2^b$ is a manifold with boundaries and corners, visualized in the Figure 3.

Figure 3. The blowup space $M_2^b$.

Projective coordinates on $M_2^b$ are given as follows. Near the top corner of $ff$ away from $tf$ the projective coordinates are given by

$$\rho = \frac{1}{\mu}, \quad \xi = \frac{x}{\rho}, \quad \tilde{\xi} = \frac{y}{\rho},$$ (2.37)

where in these coordinates $\rho, \xi, \tilde{\xi}$ are the defining functions of the faces $ff, rf$ and $If$ respectively. For the bottom corner of $ff$ away from $If$ the projective coordinates are given by

$$\tau = (\mu y)^{-1}, \quad s = \frac{x}{y}, \quad y,$$ (2.38)

where in these coordinates $\tau, s, y$ are the defining functions of $tf, rf$ and $ff$ respectively. For the bottom corner of $ff$ away from $If$ the projective coordinates are obtained by interchanging the roles of $x$ and $y$. The projective coordinates on $M_2^b$ near the top of $td$ away from $tf$ are given by

$$\eta = \tau, \quad S = \frac{s - 1}{\eta}, \quad y.$$ (2.39)

In these coordinates $tf$ is the face in the limit $|S| \to \infty$, $ff$ and $td$ are defined by $y, \eta$, respectively. The blowup space $M_2^b$ is related to the original space $\mathbb{R}_+ \times \mathbb{R}^2_+$ via the obvious ‘blow-down map’

$$\beta : M_2^b \to \mathbb{R}_+ \times \mathbb{R}^2_+,$$

which is in local coordinates simply the coordinate change back to $(1/\mu, x, y)$. The only difference between $M_2^b$ and the heat space for incomplete conical or
edge singularities in [Moo99] and [MV12] is that here the blowup is not parabolic in \( \mu^{-1} \)-direction.

One can easily check in local projective coordinates above that the kernels \( K_+ \) and \( K_R \) both lift to polyhomogeneous distributions on \( \mathbb{M}^2_b \), the latter being conormal to \( \beta^* (x = y) \). Put for any \( k \in \mathbb{N}_0 \)

\[
E_k := \{(j, \theta) \in \mathbb{N} \times \mathbb{N} \mid j \geq k \}.
\]

Then the index set of \( \beta^* K_R \) is given by \( E_1 \) at \( \text{ff} \) and \( E_0 \) at \( \text{rf} \) and \( \text{lf} \). The index sets of \( K_+ \) are the same at \( \text{ff} \), \( \text{rf} \) and \( \text{lf} \), and given by \( E_\infty \) at \( \text{tf} \), i.e. \( \beta^* K_+ \) is vanishing to infinite order at the temporal face \( \text{tf} \).

We denote by \( A_{\text{phg}}^{lp,E_0,E_\infty} (\mathbb{M}^2_b, \beta^* (x = y)) \) the space of polyhomogeneous distributions on \( \mathbb{M}^2_b \) conormal up to \( \beta^* (x = y) \), with index set \( E_1, l \in \mathbb{N} \) at \( \text{ff} \), the index set \( E_p, p \in \mathbb{N} \) at \( \text{td} \), index sets \( (E_{H}, E_{\text{rf}}) \) at \( \text{lf} \) and \( \text{rf} \), respectively, and vanishing to infinite order at \( \text{td} \). The space \( A_{\text{phg}}^{lp,E_0,E_\infty} (\mathbb{M}^2_b) \) denotes the subspace of polyhomogeneous distributions that are smooth across \( \beta^* (x = y) \).

Clearly, \( K_R \in A_{\text{phg}}^{1,1,E_0,E_\infty} (\mathbb{M}^2_b, \beta^* (x = y)) \), as the Schwartz kernel of \( (I + \mu^2)^{-1} \) inside the strongly parametric calculus. Moreover \( K_+ \in A_{\text{phg}}^{1,\infty,E_0,E_\infty} (\mathbb{M}^2_b) \). We need to establish polyhomogeneity for the various compositions of \( K_+ \) and \( K_R \), or more generally for any \( K_A \in A_{\text{phg}}^{1,p,E_0,E_\infty} (\mathbb{M}^2_b) \) and \( K_B \in A_{\text{phg}}^{1,\infty,E_0,E_\infty} (\mathbb{M}^2_b) \). We employ the pushforward theorem, similar to the presentation in [MV12, Appendix], in the following way. Consider the composition

\[
K_C (x, y; \mu) = \int_{\mathbb{R}_+} K_A (x, z; \mu)K_B (z, y; \mu) dz. \tag{2.41}
\]

In order to present this composition as a pushforward of some polyhomogeneous distributions, we consider the triple-space \( \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^3)_{(x,y)} \), and the three projections

\[
\begin{align*}
\pi_C : \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^3)_{(x,y)} &\rightarrow \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^2)_{(x,y)}, \\
\pi_L : \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^3)_{(x,y)} &\rightarrow \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^2)_{(x,z)}, \\
\pi_R : \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^3)_{(x,y)} &\rightarrow \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^2)_{(z,y)}. \tag{2.42}
\end{align*}
\]

Multiplying the kernels with suitable volume forms as specified below, we can rewrite Eq. (2.41) as

\[
K_C = (\pi_C^*)_+ (\pi_L^*)_+ K_A (\pi_R^*)_+ K_B .
\]

The basic idea of the present discussion is the definition of a blowup \( \mathbb{M}^3_b \) of \( \mathbb{R}_+^{1/\mu} \times (\mathbb{R}_+^3)_{(x,y)} \) with a blowdown map \( \beta^{(3)} \), constructed by a sequence of blowups, such that there are maps

\[
\Pi_L, \Pi_C, \Pi_R : \mathbb{M}^3_b \rightarrow [\mathbb{R}_+^{1/\mu} \times \mathbb{R}_+^2, \mathcal{A}] = \mathbb{M}^2_{\text{rb}}
\]
which ‘cover’ the three projections above, in the sense that \( \pi_{C,L,R} \circ \beta^{(3)} = \beta^{(2)} \circ \Pi_{C,L,R} \), where \( \beta^{(2)} : M^2_b \to \mathbb{R}_+ \times \mathbb{R}_+^2 \) is the blowdown map for the reduced blowup space. Writing \( \kappa_{A,B,C} := \beta^{(2)} \kappa_{A,B,C} \), we obtain the central formula

\[
\kappa_C = (\Pi_C)_* (\Pi_{A}^i \kappa_{A} \Pi_B^i \kappa_B).
\]

(2.43)

By this formula it suffices to show that if \( \kappa_A \) and \( \kappa_B \) are polyhomogeneous, then so are their lifts to \( \mathcal{M}^2_b \), so that the product of these lifts is polyhomogeneous and its pushforward by \( \Pi_C \) is again polyhomogeneous on \( M^2_b \) by the Pushforward theorem below.

Stating the pushforward theorem requires some terminology. Let \( X \) and \( X' \) be two compact manifolds with corners and their codimension one boundary faces \( \{H_i\} \) and \( \{H'_i\} \), respectively. Let \( \rho_i, \rho'_i \) be global defining functions for \( H_i \), resp. \( H'_i \). We say that a smooth map \( f : X \to X' \) is a b-map if for each \( i \)

\[
f^* \rho'_i = A_{ij} \prod_j \rho_j^{e(i,j)}, \quad A_{ij} > 0, \quad e(i,j) \in \mathbb{N} \cup \{0\}.
\]

The map \( f \) is called a b-submersion if \( f_* \) induces a surjective map between the b-tangent bundles* of \( X \) and \( X' \). If, moreover for each \( j \) there is at most one \( i \) such that \( e(i,j) \neq 0 \), then \( f \) is called a b-fibration. For a geometric interpretation of these conditions we refer the reader to [Mel93].

Consider a volume form \( \nu_b \) on \( X \) that is smooth up to all boundary faces. A smooth b-density \( \nu_b \) is, by definition, any density of the form \( \nu_b = \nu_0(\Pi \rho_i)^{-1} \). Let us fix smooth nonvanishing b-densities \( \nu_b \) on \( X \) and \( \nu'_b \) on \( X' \). Then we may state the pushforward theorem as follows.

**Proposition 2.4.** [Mel93]. Let \( f : X \to X' \) be a b–fibration. Consider a polyhomogeneous function \( \omega \) on \( X \) with index sets \( E_j \) at the faces \( H_j \) of \( X \). Assume \( \text{Re} z > 0 \) for each \( (z, p) \in E_j \) if the index \( j \) satisfies \( e(i,j) = 0 \) for all \( j \). Then the pushforward \( f_*(\omega \nu_b) \) is well-defined and equals \( h \nu'_b \) where \( h \) is polyhomogeneous on \( X' \) and has a specified index family \( f_*(\mathcal{E}) \).

We make the index set \( f_*(\mathcal{E}) \) explicit in a special case that is enough for the present situation. If \( H_{j_1} \) and \( H_{j_2} \) are both mapped to a face \( H'_{j'} \), and if \( H_{j_1} \cap H_{j_2} = \emptyset \), then their contribution to the index set of \( h \) at \( H'_{j'} \) is given by \( E_{j_1} \cup E_{j_2} \). If however \( H_{j_1} \cap H_{j_2} \neq \emptyset \), then the contribution is given by the extended union

\[
E_{j_1} \cup E_{j_2} := E_{j_1} \cup E_{j_2} \cup \{(z, p + q + 1) : \exists (z, p) \in E_{j_1}, \text{ and } (z, q) \in E_{j_2}\}.
\]

For any two index sets \( E_{j_1}, E_{j_2} \) we also write

\[
E_{j_1} + E_{j_2} := \{(z_1 + z_2, p_1 + p_2) : (z_1, p_1) \in E_{j_1}, \text{ and } (z_2, p_2) \in E_{j_2}\}.
\]

Moreover, \( E > a, a \in \mathbb{R} \) if for any \( (z, p) \in E, z > a \). We then have the following fundamental composition result.

*The b-tangent space at a point \( p \) of \( \partial X \) on a codimension \( k \) corner is spanned locally by the sections \( x_1 \partial x_1, \ldots, x_k \partial x_k, \partial y_i \), where \( x_1, \ldots, x_k \) are the defining functions for the faces meeting at \( p \) and the \( y_i \) are local coordinates on the corner through \( p \).
Proposition 2.5. For index sets $E_L$ and $E'_R$ such that $E_L + E'_R > -1$, we have

$$\mathcal{A}_{phg}^{1+p,E_L,E_R}(M_b^2 \circ \mathcal{A}_{phg}^{1',\infty,E'_L,E'_R}(M_b^2, \beta^*\{x = y\})) \subset \mathcal{A}_{phg}^{1+1',+1,\infty,E'_L,E'_R}(M_b^2).$$

Proof. Consider $K_A \in \mathcal{A}_{phg}^{1+p,E_L,E_R}(M_b^2)$, $K_B \in \mathcal{A}_{phg}^{1',\infty,E'_L,E'_R}(M_b^2, \beta^*\{x = y\})$ and

$$K_C(x, y; \mu) = \int_{\mathbb{R}^+} K_A(x, z; \mu) K_B(z, y; \mu) \, dz.$$

The construction of the aforementioned triple space $M_b^3$ is strictly dictated by the requirement that the maps $\pi_L, \pi_C, \pi_R$ all lift to $b$-fibrations $\Pi_L, \Pi_C, \Pi_R$. This is reminiscent of the triple space construction for the heat space calculus for conical singularities, see [Moo99], but differs from the latter since there is no convolution in the parameter $\mu^{-1}$ variable and the blowups are not parabolic in the $\mu^{-1}$ direction. First we blow up the submanifold

$$F = \{\mu = \infty, x = z = y = 0\}.$$

Then we blow up the resulting space $[\mathbb{R}^{1+}_{1/\mu} \times (\mathbb{R}^+)^3_{(x,z,y)}, F]$ at the lifts of each of the three submanifolds

$$F_C = \{\mu = \infty, x = y = 0\},$$
$$F_L = \{\mu = \infty, z = y = 0\},$$
$$F_R = \{\mu = \infty, x = z = 0\}.$$

Thus altogether,

$$M_b^3 := [\mathbb{R}^{1+}_{1/\mu} \times (\mathbb{R}^+)^3_{(x,z,y)}, F, F_C, F_L, F_R].$$

If we ignore the $\mu^{-1}$-direction, the spatial part of $M_b^3$ is exactly the same as the triple space appearing in the elliptic theory of edge operators, see [Maz91], which can be visualized as in Figure 4 below.

![Figure 4](image-url)  

Figure 4. The spatial component of the triple space $M_b^3$.
faces created by blowing up \( F_C, F_L \) and \( F_R \) are denoted by \((101),(011)\) and \((110)\), respectively. We write \( \rho_{ij} \) for the defining function of the face \((ijk)\).

The projections \( \pi_C, \pi_L \) and \( \pi_R \), as defined in Eq. (2.42), lift to maps \( \Pi_C, \Pi_L \) and \( \Pi_R \) from \( M_b^3 \) to the reduced blowup space \( M_{rb}^3 \). One can check that the choice of submanifolds \( F, F_L,R,C \) that have been blown up ensures that these maps are in fact \( b \)-fibrations.

In order to describe the action of \( \Pi_{C,L,R} \), denote the defining functions for \( rf, ff \) and \( lf \) in \( M_{rb}^2 \) by \( \{ \rho_{10}, \rho_{11}, \rho_{01} \} \), respectively. These lift via \( \Pi_{C,L,R} \) as follows

\[
\begin{align*}
\Pi_C^\tau(\rho_{ij}) &= \rho_{i0j}\rho_{0ij}, \\
\Pi_L^\tau(\rho_{ij}) &= \rho_{ij0}\rho_{0ij}, \\
\Pi_R^\tau(\rho_{ij}) &= \rho_{0ij}\rho_{ij0}.
\end{align*}
\]  

We also need their action on the defining function \( T \) of the temporal face \( tf \) in \( M_{rb}^2 \). Let \( \tau \) be the defining function for the boundary face in \( M_b^3 \) corresponding to \( \{ \mu = \infty \} \). Then we find

\[
(\beta^{(3)})^*(\mu^{-1}) = \tau\rho_{111}\rho_{110}\rho_{011}\rho_{011}.
\]  

If \( \beta^{(2)} \) is the blowdown map for the reduced blowup space, the lifts \((\beta^{(2)})^*(\mu^{-1})\) to \( \Pi^*\left(M_b^3\right) = M_{rb}^3 \) are equal to \( T\rho_{111} \). Note

\[
\Pi^*_{C,L,R} \circ \left(\beta^{(2)}\right)^* = \left(\beta^{(3)}\right)^* \circ \pi^*_{C,L,R}.
\]

Consequently, in view of Eq. (2.45) and Eq. (2.46), we conclude

\[
\begin{align*}
\Pi_C^*(T) &= \tau\rho_{101}\rho_{011}, \\
\Pi_L^*(T) &= \tau\rho_{101}\rho_{011}, \\
\Pi_R^*(T) &= \tau\rho_{101}\rho_{011}.
\end{align*}
\]

We write \( t := \mu^{-1} \) and instead of working in terms of half-densities we choose to reinterpret both kernels as ‘right densities’, \( K_A(x,z;t)dz \) and \( K_B(z,y;t)dy \). Then their product on \( \mathbb{R}_t^+ \times (\mathbb{R}_+)^3 \) is

\[
K_A(x,z;t)K_B(z,y;t)dz dy.
\]

The integral over \( dz \) gives \( K_{A\cdot B}(x,y;t)dy \). To put this into the same form required in the pushforward theorem, multiply this expression by \( dt dx \).

Blowing up a submanifold of codimension \( n \) corresponds in local coordinates to introducing polar coordinates, with the polar coordinate being a defining function of the resulting front face. Applying such a coordinate transformation to a volume form yields an \( (n-1) \)st power of the radial function. Hence we compute the lift

\[
(\beta^{(3)})^*(dt dx dz dy) = \rho_{111}\rho_{011}^2\rho_{011}^2\nu^{(3)} = \rho_{111}\rho_{011}^2\rho_{011}^2\tau(\Pi_{\rho_{ij}})^{\nu_b^{(3)}}.
\]
where $\nu^{(3)}$ is a density on $\mathcal{M}^3$, smooth up to all boundary faces and everywhere nonvanishing; $\nu_b^{(3)}$ is a b-density, obtained from $\nu^{(3)}$ by dividing by a product of all defining functions on $\mathcal{M}^3_b$. Let $\kappa_A = (\beta^{(2)})^* \kappa_A$ and $\kappa_B = (\beta^{(2)})^* \kappa_B$ vanish to infinite order at $T = 0$. Hence $(\Pi^*_t \kappa_A)(\Pi^*_r \kappa_B)$ vanishes to infinite order in $\tau_{001010001}$. Altogether, we obtain that

$$(\Pi^*_t \kappa_A)(\Pi^*_r \kappa_B) (\beta^{(3)})^* (dt \, dx \, dz \, dy) = \rho_{111}^{\ell + \ell' + 3} (\Pi \rho_{ijk}) G \nu_b^{(3)},$$

where $G$ is a bounded polyhomogeneous function on $\mathcal{M}^3_b$, vanishing to infinite order in $\tau_{001010001}$, with index sets $E'_{lf}, E_{rf}$ and $E_{lf} + E_{rf}$ at the faces $(001), (100)$ and $(010)$, respectively.

Note that since $\kappa_A$ does not vanish to infinite order on $\tau_{001}$ in $\mathcal{M}^3_b$, the lift $\Pi^*_t \kappa_A$ is not polyhomogeneous on $\mathcal{M}^3_b$. However, the other factor $\kappa_B$ does vanish to infinite order there, and hence the product $\Pi^*_t \kappa_A \cdot \Pi^*_r \kappa_B$ is indeed polyhomogeneous on $\mathcal{M}^3_b$. Applying Theorem 2.4 now gives

$$(\beta^{(2)})^* (K_{A_0 B}x, y; t) dt \, dx \, dy) = (\Pi_C)^* (\Pi^*_t \kappa_A)(\Pi^*_r \kappa_B) (\beta^{(3)})^* (dt \, dx \, dz \, dy),$$

where $\nu_b^{(2)}$ is a b-density on $\mathcal{M}^2_{\tau b}$ and $G'$ is a bounded polyhomogeneous function on $\mathcal{M}^2_{\tau b}$, which vanishes to infinite order in $T$, is of leading order $(\ell + \ell' + 3)$ at the front face and has the index sets $(E'_{lf}, E_{rf})$ at the left and right boundary faces. By [EMM91, Proposition B7.20] the pushforward is again smooth across $\beta^*(x = y)$.

Note also that the pushforward by $\Pi_C$ does not introduce logarithmic terms in the front face expansion of $K_{A_0 B}$, since $G$ is vanishing to infinite order at $(101), (110), (011)$. Hence, for $\kappa_A$ and $\kappa_B$ with integer exponents in their front face expansions, same holds for their composition.

By an argument similar to Eq. (2.48), we compute

$$(\beta^{(2)})^* (dt \, dx \, dy) = \rho_{11}^2 (\rho_{10} \rho_{10} T) (\beta^*)^{(0)} \nu_b^{(2)}.$$  

Consequently, combining Eq. (2.49) and Eq. (2.50), we deduce that

$$\kappa_{A_0 B} \equiv (\beta^{(2)})^* \kappa_{A_0 B} = \rho_{11}^{-2} G',$$

vanishes to infinite order in $T$, is of leading order $(\ell + \ell' + 1)$ at the front face and has the index sets $(E'_{lf}, E_{rf})$ at the left and right boundary faces. This proves the statement.

**Proposition 2.6.** For any $K \in \mathcal{A}_{phg}^{1, E_{lf}, E_{rf}}(\mathcal{M}^3_b)$ and any cutoff function $\phi \in C^\infty_b(\mathbb{R}_+)$ with $\phi \equiv 1$ in an open neighborhood of zero, we find for $(E_{lf} + E_{rf} + 1) > -1$
that

\[ \text{Tr}(K\phi) = \int_0^\infty K(x, x; \mu) \phi(x) \, dx \sim \sum_{j=0}^{\infty} a_j \mu^{-(1+1)-j}, \quad \mu \to \infty. \quad (2.51) \]

**Proof.** The lift \( \beta^*K \) restricts to a polyhomogeneous distribution on \( \mathcal{B} := \beta^*(x = y) \subset M_b^2 \), which itself is a blowup of \( \mathbb{R}^{1/\mu}_+ \times \mathbb{R}^+ \) at \((0, 0)\) with the blowdown map denoted by \( \beta_\mathcal{B} \). We refer to the restrictions of ff, lf and td in \( M_b^2 \) to \( \mathcal{B} \) as the front face, left face and temporal diagonal again.

The restriction of \( \beta^*K \) to \( \mathcal{B} \) is polyhomogeneous with the index set \( E_f \) at the front face, index set \( (E_{lf} + E_{tf}) \) at the left face and vanishes to infinite order at the temporal diagonal. Consider the projection \( \pi : \mathbb{R}^{1/\mu}_+ \times \mathbb{R}^+ \). Then \((t = 1/\mu)\)

\[ (\pi \circ \beta_\mathcal{B})_\ast (\beta^*K|_\mathcal{B}) |_{\mathcal{B}} \cdot \beta^+_\mathcal{B}(dt \, dx) = \int_0^\infty K(x, x; \mu) \phi(x) \, dx \, dt. \]

Note that

\[ \beta^*(K\phi)|_\mathcal{B} \cdot \beta^+_\mathcal{B}(dt \, dx) = \rho_{H}^{1+2} G_{\nu}, \]

where \( \nu_b \) is a b-density on \( \mathcal{B} \), \( \rho_{H} \) is the defining function of the front face and \( G \) is a bounded polyhomogeneous distribution with the index set \( E_0 \) at ff, index set \((E_{lf} + E_{tf} + 1)\) at the lef face and vanishes to infinite order at the temporal diagonal in \( \mathcal{B} \). By the pushforward theorem we find

\[ (\pi \circ \beta_\mathcal{B})_\ast (\beta^*K|_\mathcal{B}) |_{\mathcal{B}} \cdot \beta^+_\mathcal{B}(dt \, dx) = (\pi \circ \beta_\mathcal{B})_\ast (\rho_{H}^{1+2} G_{\nu_b}) \sim \sum_{j=0}^{\infty} t^{(1+2)j+1} \frac{dt}{t}. \]

This proves the statement. \( \square \)

We can now employ Proposition 2.5 together with Proposition 2.6 in order to derive a polyhomogeneous expansion of the finite sum (recall \( \lambda(V, W) = \lambda^2 V_\psi + W_\psi \))

\[ \sum_{j=0}^{M-1} (-1)^j \psi \left( |K_0 \lambda(V, W)|^j K_0 - |K_R \lambda(V, W)|^j K_R \right) \phi = \sum_{j=0}^{M-1} R_j^0. \]

Each \( R_j^0 \) is finite number of summands of the form \( K(j,p), j \leq (M - 1) \), which are given by a convolution of \((j + 1)\) kernels \( K_R \) and \( K_+ \), with at least one \( K_+ \) and \( p(\leq j) \) times \( \lambda^2 V_\psi \). Note that \( V_\psi(x) = O(x) \) as \( x \to 0 \), and is smooth so that

\[ K_R V_\psi \in A^{2,1,1,E_1,E_0} \left( M_b^2, \beta^*(x = y) \right), \quad K_+ V_\psi \in A^{2,\infty,E_1,E_0} \left( M_b^2 \right). \]

Consequently we find by Proposition 2.5

\[ K(j, p) \in \lambda^{2p} A^{2j+1+p,\infty,E_0} \left( M_b^2 \right). \]
Proposition 2.6 now implies
\[ \text{Tr } K(j, p) \sim \sum_{i=0}^{\infty} a_i \frac{\lambda^{2p}}{\mu^{2(j+1)+p+i}} =: \sum_{i=0}^{\infty} a_i^{jp}(\lambda, z), \]

where each \( a_i^{jp} \) is homogeneous in \((\lambda, z)\) of homogeneity degree \((p - 2(j + 1) - i)\). Consequently, overall we obtain
\[ \text{Tr } \sum_{j=0}^{M-1} R_0^{(j)} \sim \sum_{i=0}^{\infty} e_i(\lambda, z), \quad ||(\lambda, z)|| \to \infty, \quad (2.52) \]

where each \( e_i \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\}) \) is homogeneous of order \((-2 - i)\) jointly in both variables. We have now all ingredients to prove Proposition 1.2.

### 2.4. Proof of Proposition 1.2

This is now a consequence of the interior expansion Eq. (2.5), Proposition 2.2 and Eq. (2.52). Comparing Eq. (2.2) and Eq. (2.52) we see that the leading term in the polyhomogeneous expansion of \( \text{Tr}(\Delta + z^2)^{-1} \) indeed comes from the interior.

### 2.5. Proof of Theorem 1.3

As an application of the polyhomogeneity of the resolvent trace we now prove Theorem 1.3 and clarify in which sense the trace expansions of \( \text{Tr}(\Delta + z^2)^{-2} \) sum up to the trace expansion of \( \text{Tr}(\Delta + z^2)^{-2} \). Note that \( \text{Tr}(\Delta + z^2)^{-2} = O(z^{-3}) \), whereas \( \text{Tr}(\Delta + z^2)^{-2} = O(z^{-2}) \), as \( z \to \infty \). So the trace expansions clearly do not sum up in an obvious way.

We apply the Euler MacLaurin formula Eq. (1.11) to \( f(\lambda) = \text{Tr}(\Delta + z^2)^{-2} \). Note that the sum is convergent and that \( \lim_{N \to \infty} f(N) = 0 \). We find for \( M \in \mathbb{N} \)

sufficiently large

\[
\sum_{\lambda=1}^{\infty} \text{Tr}(\Delta + z^2)^{-2} = \int_{1}^{\infty} \text{Tr}(\Delta + z^2)^{-2} d\lambda + \frac{1}{2} \text{Tr}(\Delta + z^2)^{-2} \lambda_1 = 1
\]

\[
- \sum_{k=1}^{M} \frac{B_{2k}}{[2k]!} \delta^{(2k-1)}_{\lambda} \text{Tr}(\Delta + z^2)^{-2} \lambda_1 = 1
\]

\[
+ \frac{1}{(2M + 1)!} \int_{1}^{\infty} B_{2M+1}(\lambda - [\lambda]) \delta^{(2M+1)}_{\lambda} \text{Tr}(\Delta + z^2)^{-2} d\lambda.
\]

We need to establish asymptotic behaviour of each of the terms above as \( z \to \infty \).

The standard resolvent trace expansion, cf. Eq. (1.16) yields
\[
\frac{1}{2} \text{Tr}(\Delta + z^2)^{-2} - \sum_{k=1}^{M} \frac{B_{2k}}{[2k]!} \delta^{(2k-1)}_{\lambda} \text{Tr}(\Delta + z^2)^{-2} \lambda_1 = \sum_{i=0}^{\infty} a_i z^{-3-i}, \quad z \to \infty. \quad (2.54)
\]
Moreover, Proposition 1.2 implies
\[
\frac{1}{(2M+1)!} \int_1^\infty |B_{2M+1}(\lambda - [\lambda]) \partial^{2M+1}_\lambda \text{Tr}(\Delta + z^2)^{-2}| \, d\lambda \\
\leq C \cdot \int_1^\infty (\lambda + z)^{-4-2M} \, d\lambda = O(z^{-3-2M}), \ z \to \infty.
\] (2.55)

It remains to derive an asymptotic expansion for the first integral term in Eq. (2.53). Proposition 1.2 implies
\[
\text{Tr}(\Delta + z^2)^{-2} = (2z)^{-1} \partial_z \text{Tr}(\Delta + z^2)^{-1} \sim \sum_{i=0}^\infty h_i(\lambda, z), \ |(\lambda, z)| \to \infty,
\]
where \( \gamma_i := (i + 3) \) and each \( h_i \in C^\infty(\mathbb{R}^2_+ \setminus ((0,0))) \) is homogeneous of order \( -\gamma_i \) jointly in both variables. The asymptotic expansion of the first integral term in Eq. (2.53) now follows from
\[
\int_1^\infty h_i(\lambda, z) \, d\lambda = z^{-\gamma_i} \int_1^\infty h_i(\lambda/z, 1) \, d\lambda = z^{-\gamma_i + 1} \int_1^\infty h_i(v, 1) \, dv.
\]
The \( \nu \)-integral is finite, since as a consequence of smoothness of \( h_i(1, \cdot) \) at \( z = 0 \) and homogeneity, \( h(\nu, 1) = O(\nu^{-\gamma_i}), \gamma_i \geq 3 \), cf. Remark 3.2 below. This proves our second main result Theorem 1.3. \( \square \)

3. Fubini theorem for regularized integrals

The following Fubini-type result for homogeneous functions is the main technical tool for proving the general Fubini Theorem 1.1 for polyhomogeneous functions.

**Proposition 3.1.** Let \( f_\alpha \in C^\infty(\mathbb{R}^2_+ \setminus ((0,0))) \) be homogeneous of degree \( \alpha \in \mathbb{C} \) in both variables jointly. Then the following Fubini-identity holds
\[
\int_1^\infty \int_1^\infty f_\alpha(x, y) \, dy \, dx - \int_1^\infty \int_1^\infty f_\alpha(x, y) \, dx \, dy = \\
\begin{cases} 
\int_0^\infty f_\alpha(1, y) \log(y) \, dy, & \alpha = -2, \\
0, & \alpha \neq -2. 
\end{cases} \tag{3.1}
\]

**Remark 3.2.** The smoothness of \( f_\alpha \) on \( \mathbb{R}^2_+ \setminus ((0,0)) \) implies the smoothness of \( f_\alpha(1, \cdot) \) and \( f_\alpha(\cdot, 1) \) up to \( 0 \). Furthermore, the homogeneity and the Taylor expansion of \( f_\alpha(\cdot, 1) \) at \( 0 \) give for \( y \to \infty \)
\[
f_\alpha(1, y) = y^\alpha f_\alpha(1/y, 1) \sim \sum_{j=0}^\infty \frac{\delta^{(j)}_f}{j!} f_\alpha(0, 1) y^{\alpha-j},
\]
ensuring the existence of the integral on the right hand side of Eq. (3.1).
Proof. For any $R \geq 1$ we write

\[
\int_{1}^{\infty} \int_{1}^{\infty} f_{\alpha}(x, y) \, dy \, dx = \int_{1}^{\infty} \int_{R}^{\infty} f_{\alpha}(x, y) \, dy \, dx + \int_{1}^{R} \int_{1}^{\infty} f_{\alpha}(x, y) \, dy \, dx
\]

where in the last equation we used the Fubini theorem for finite integrals and the easy to verify fact that for homogeneous functions the regularized limit commutes with finite integrals. Consequently

\[
\int_{1}^{\infty} \int_{1}^{\infty} f_{\alpha}(x, y) \, dy \, dx - \int_{1}^{\infty} \int_{1}^{\infty} f_{\alpha}(x, y) \, dx \, dy
\]

(3.4)

We separate the integral on the right hand side of Eq. (3.4)

\[
\int_{1}^{\infty} \int_{R}^{\infty} f_{\alpha}(1, y/x) \, dy \, dx = \int_{1}^{\infty} \int_{xR}^{\infty} f_{\alpha}(1, y/x) \, dy \, dx + \int_{1}^{R} \int_{1}^{\infty} f_{\alpha}(1, y/x) \, dy \, dx
\]

\[
= I(R) + II(R).
\]

For the first summand we use the asymptotic expansion Eq. (3.2). Let $A \in \mathbb{C}$ denote the coefficient of $y^{-1}$ in this expansion as $y \to \infty$. Then we find

\[
I(R) = \lim_{T \to \infty} \int_{1}^{T} x^{\alpha} \int_{xR}^{\infty} \left( f_{\alpha}(1, y/x) - \sum_{j=0}^{[\text{Re}(\alpha)]+1} \frac{\partial_{\alpha}^{(j)} f_{\alpha}(0, 1)}{j!} (y/x)^{\alpha-j} \right) \, dy \, dx
\]

\[
+ \sum_{j=0}^{[\text{Re}(\alpha)]+1} \lim_{T \to \infty} \int_{1}^{T} x^{\alpha+1} \int_{xR}^{\infty} \frac{\partial_{\alpha}^{(j)} f_{\alpha}(0, 1)}{j!} y^{\alpha-j} \, dy \, dx
\]

(3.5)

\[
= \int_{1}^{\infty} x^{\alpha+1} \int_{R}^{\infty} \left( f_{\alpha}(1, y) - \sum_{j=0}^{[\text{Re}(\alpha)]+1} \frac{\partial_{\alpha}^{(j)} f_{\alpha}(0, 1)}{j!} y^{\alpha-j} \right) \, dy
\]

\[
+ \sum_{j=0}^{[\text{Re}(\alpha)]+1} \lim_{T \to \infty} \int_{1}^{T} x^{\alpha+1} \int_{xR}^{\infty} \frac{\partial_{\alpha}^{(j)} f_{\alpha}(0, 1)}{j!} y^{\alpha-j} \, dy \, dx.
\]
Taking the regularized limit of $I(R)$ as $R \to \infty$ we obtain

$$LIM_{R \to \infty} I(R) = \lim_{R \to \infty} \sum_{j=0}^{\Re(\alpha)+1} \lim_{T \to \infty} \int_1^T x^j \int_{xR}^\infty \frac{\partial_1^{(j)} f_\alpha(0,1)}{j!} y^{\alpha-j} dy \, dx$$

$$= \lim_{R \to \infty} \sum_{j=0}^{\Re(\alpha)+1} \delta_1^{(j)} f_\alpha(0,1) \int_1^\infty (-1)^j \frac{x^{\alpha+1}}{(\alpha-j+1)} \, dx$$

$$- \delta_1, \alpha+1 \lim_{R \to \infty} \int_1^\infty x^{\alpha+1} \lambda (\log x + \log R) \, dx,$$

where in the last summand we have used the change of variables rule for regularized integrals [Les97, Lemma 2.1.4]. In total we obtain

$$LIM_{R \to \infty} I(R) = -A \int_1^\infty x^{\alpha+1} \log x \, dx = \begin{cases} 0, & \text{if } \alpha = -2, \\ -A(\alpha + 2)^{-2}, & \text{if } \alpha \neq -2. \end{cases} \quad (3.7)$$

For the second component $II(R)$ we find

$$II(R) = \lim_{T \to \infty} \int_1^T x^{\alpha+1} \int_{R/x}^R f_\alpha(1,y) \, dy \, dx = \lim_{T \to \infty} \int_{R/T}^R f_\alpha(1,y) \int_{R/y}^T x^{\alpha+1} \, dx \, dy$$

$$= \lim_{T \to \infty} \int_{R/T}^R f_\alpha(1,y) \left\{ \begin{array}{ll} \log T - \log R + \log y, & \text{if } \alpha = -2, \\ (\alpha+2)^{-1} \left( T^{\alpha+2} - (R/y)^{\alpha+2} \right), & \text{if } \alpha \neq -2. \end{array} \right\} dy$$

$$= \left\{ \begin{array}{ll} \int_0^R f_\alpha(1,y) (\log y - \log R) \, dy, & \text{if } \alpha = -2, \\ \lim_{T \to \infty} \int_{R/T}^R f_\alpha(1,y) \left( \frac{T^{\alpha+2}}{\alpha+2} \right) \, dy - \left[ \frac{(R/y)^{\alpha+2}}{\alpha+2} \right] \, dy, & \text{if } \alpha \neq -2. \end{array} \right\}$$

In the case $\alpha = -2$ above we could omit $\log T$ as the corresponding expression has no constant term as $T \to \infty$. In order to evaluate the regularized limit of $II(R)$ as $R \to \infty$ we consider the integrals in its expression separately. First, we clearly have

$$\lim_{R \to \infty} \int_0^R f_\alpha(1,y) (\log y - \log R) \, dy = \int_0^\infty f_\alpha(1,y) \log y \, dy. \quad (3.8)$$

Moreover, we find using integration by parts for $\alpha \neq -2$

$$\lim_{T \to \infty} \int_{R/T}^R f_\alpha(1,y) \, dy \cdot T^{\alpha+2} = -\lim_{T \to \infty} \int_0^R f_\alpha(1,y) \, dy \cdot T^{\alpha+2}$$

$$= \lim_{T \to \infty} \sum_{j=0}^k (-1)^{j+1} \frac{\partial_2^{(j)} f_\alpha(1,R/T)}{(j+1)!} \left( \frac{R}{T} \right)^{j+1} T^{\alpha+2}$$

$$+ (-1)^k \lim_{T \to \infty} \int_{R/T}^R y^{k+2} \partial_2^{k+1} f_\alpha(1,y) \, dy \cdot T^{\alpha+2}.$$
Since $\alpha \neq -2$, taking $k \geq \Re(\alpha) + 1$, the second integral does not exhibit constant terms in $T$ and hence vanishes in the regularized limit. Consequently we find

$$\lim_{T \to \infty} \int_{R/T}^{R} f_{\alpha}(1, y) \, dy \cdot T^{\alpha+2} = \lim_{T \to \infty} \sum_{j=0}^{k} (-1)^{j+1} \frac{\partial_{2}^{(j)} f_{\alpha}(1, R/T)}{(j+1)!} \left( \frac{R}{T} \right)^{j+1} T^{\alpha+2}$$

$$= \begin{cases} (-1)^{\alpha} R^{\alpha+2} \frac{\partial_{2}^{(\alpha+1)} f_{\alpha}(1, 0)}{\alpha + 2}, & \text{if } (\alpha + 1) \in \mathbb{Z}_+, \\ 0, & \text{if } (\alpha + 1) \not\in \mathbb{Z}_+. \end{cases}$$  \hspace{1cm} (3.10)

Finally, using Eq. (3.2) we find for $\alpha \neq -2$

$$\lim_{k \to \infty} \int_{1}^{R} f_{\alpha}(1, y) \frac{(R/y)^{\alpha+2}}{(\alpha + 2)} \, dy = \lim_{k \to \infty} \int_{1}^{R} f_{\alpha}(1, y) \frac{(R/y)^{\alpha+2}}{(\alpha + 2)} \, dy + \lim_{k \to \infty} \int_{0}^{1} f_{\alpha}(1, y) \frac{(R/y)^{\alpha+2}}{(\alpha + 2)} \, dy$$

$$= \lim_{k \to \infty} A \frac{R^{\alpha+2}}{(\alpha + 2)} \int_{1}^{R} y^{-3-\alpha} \, dy = -\frac{A}{(\alpha + 2)^2}. \hspace{1cm} (3.11)$$

We finally obtain

$$\lim_{R \to \infty} II(R) = \begin{cases} \int_{0}^{\infty} f_{\alpha}(1, y) \log y \, dy, & \text{if } \alpha = -2, \\ A(\alpha + 2)^{-2}, & \text{if } \alpha \neq -2. \end{cases}$$  \hspace{1cm} (3.12)

The statement follows from Eq. (3.7) and Eq. (3.12). \square

Theorem 1.1 is now an immediate consequence of Proposition 3.1. Note that the lower bounds $(1, 1)$ of the integrals on the left hand side of (3.1) can be replaced by any $(a, b) \in \mathbb{R}_{+}^2$ without changing the right hand side of the relation, since for iterated integrals of continuous functions over finite intervals the order of integration can be interchanged by Fubini.
4. Proof of Theorem 1.4

\[
\log \det_\zeta \Delta = -2 \int_0^\infty z^3 \text{Tr}(\Delta + z^2)^{-2} dz
\]

\[
= -4 \int_0^\infty z^3 \sum_{\lambda=1}^\infty \text{Tr}(\Delta_\lambda + z^2)^{-2} dz - 2 \int_0^\infty z^3 \text{Tr}(\Delta_0 + z^2)^{-2} dz
\]

\[
= \log \det_\zeta \Delta_0 - 4 \int_1^\infty z^3 \sum_{\lambda=1}^\infty \text{Tr}(\Delta_\lambda + z^2)^{-2} dz
\]

\[
- 4 \int_0^1 z^3 \sum_{\lambda=1}^\infty \text{Tr}(\Delta_\lambda + z^2)^{-2} dz =: A + B + C.
\]

The sum and the integral in C commute and the only intricate summand is B. We employ the Euler MacLaurin summation formula Eq. (1.9) and find

\[
B = -4 \int_1^\infty \int_1^\infty z^3 \int_1^\infty \text{Tr}(\Delta_\lambda + z^2)^{-2} d\lambda \, dz - 4 \int_0^\infty h_2(1, y) \log(y) dy
\]

\[
+ 2B_2 \int_1^\infty z^3 \partial_\lambda \text{Tr}(\Delta_\lambda + z^2)^{-2} |_{\lambda=1} dz
\]

\[
- \frac{2}{3} \int_1^\infty z^3 \int_1^\infty B_3(\lambda - |\lambda|) \partial_\lambda^{(3)} \text{Tr}(\Delta_\lambda + z^2)^{-2} d\lambda \, dz
\]

\[
- 2 \int_1^\infty z^3 \text{Tr}(\Delta_1 + z^2)^{-2} dz.
\]

For the first summand above we employ Theorem 1.1. For the second and third summand we subtract the leading \(a_\lambda z^{-3}\) coefficient of \(\text{Tr}(\Delta_\lambda + z^2)^{-2}\) as \(z \to \infty\), since it has zero contribution in the regularized limit. This allows us to replace the regularized integral by the usual and to interchange differentiation with integration. Overall we find

\[
B = -4 \int_1^\infty \int_1^\infty z^3 \text{Tr}(\Delta_\lambda + z^2)^{-2} dz \, d\lambda - 4 \int_0^\infty h_2(1, y) \log(y) dy
\]

\[
+ 2B_2 \partial_\lambda \int_1^\infty z^3 \left( \text{Tr}(\Delta_\lambda + z^2)^{-2} - a_\lambda z^{-3} \right) |_{\lambda=1} dz
\]

\[
- \frac{2}{3} \int_1^\infty B_3(\lambda - |\lambda|) \partial_\lambda^{(3)} \int_1^\infty z^3 \left( \text{Tr}(\Delta_\lambda + z^2)^{-2} - a_\lambda z^{-3} \right) dz \, d\lambda
\]

\[
- 2 \int_1^\infty z^3 \text{Tr}(\Delta_1 + z^2)^{-2} dz.
\]
The desired statement follows now from Eq. (4.1) and a second application of the Euler MacLaurin summation formula so that

$$B = -4 \sum_{\lambda=1}^{\infty} \int_{1}^{\infty} z^3 \text{Tr}(\Delta_{\lambda} + z^2)^{-2} \, dz - 4 \int_{0}^{\infty} h_2(1, y) \log(y) \, dy.$$  \hfill (4.4)

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