A hybridizable discontinuous Galerkin method for fractional diffusion problems

Bernardo Cockburn · Kassem Mustapha

Abstract We study the use of the hybridizable discontinuous Galerkin (HDG) method for numerically solving fractional diffusion equations of order $-\alpha$ with $-1 < \alpha < 0$. For exact time-marching, we derive optimal algebraic error estimates assuming that the exact solution is sufficiently regular. Thus, if for each time $t \in [0, T]$ the approximations are taken to be piecewise polynomials of degree $k \geq 0$ on the spatial domain $\Omega$, the approximations to $u$ in the $L^{\infty}(0, T; L_{2}(\Omega))$-norm and to $\nabla u$ in the $L^{\infty}(0, T; L_{2}(\Omega))$-norm are proven to converge with the rate $h^{k+1}$, where $h$ is the maximum diameter of the elements of the mesh. Moreover, for $k \geq 1$ and quasi-uniform meshes, we obtain a superconvergence result which allows us to compute, in an elementwise manner, a new approximation for $u$ converging with a rate of $\sqrt{\log(T h^{-2/(\alpha+1)}) \ h^{k+2}}$.

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1 Introduction

In this paper, we propose and analyze a numerical method using exact integration in time and the so-called HDG method for the spatial discretization of the following anomalous, slow diffusion (sub-diffusion) model problem:

\[
\begin{align*}
  & u_t - B_\alpha \Delta u = f & \text{in } \Omega \times (0, T], \quad (1a) \\
  & u = g & \text{on } \partial\Omega \times (0, T], \quad (1b) \\
  & u|_{t=0} = u_0 & \text{on } \Omega, \quad (1c)
\end{align*}
\]

where \( \Omega \) is a convex polyhedral domain of \( \mathbb{R}^d \), where \( d = 1, 2, 3 \). Here, \( B_\alpha \) is the Riemann–Liouville fractional derivative in time defined, for \( -1 < \alpha < 0 \), by

\[
B_\alpha v(t) := \frac{\partial}{\partial t} \int_0^t \omega_{\alpha+1}(t-s)v(s) \, ds \quad \text{with} \quad \omega_{\alpha+1}(t) := \frac{t^\alpha}{\Gamma(\alpha+1)}
\]

where \( \Gamma \) denotes the usual gamma function. One may show that \( B_\alpha v \to v \) as \( \alpha \to 0 \). So, in the limiting case \( \alpha = 0 \), the problem (1) becomes nothing but an initial-boundary value problem for the classical heat equation.

Problems of the form (1) arise in a variety of physical, biological and chemical applications [17,22,28,29,37,41,44]. They describe slow or anomalous sub-diffusion and occur, for example, in models of fractured or porous media, where the particle flux depends on the entire history of the density gradient, \( \nabla u \). It is thus important to devise, efficient methods for numerically solving them.

Let us briefly review the development of numerical methods for the fractional sub-diffusion problem (1). Several authors have proposed a variety of numerical methods for this problem. For finite difference (FD) methods with convergence rates of order \( O(h^2) \) in space, where \( h \) is the maximum meshsize, see, for example, [4,5,19,20,31,46,47,50,51]. In [11], FD schemes were considered which are first-order accurate in time but \( O(h^4) \)-accurate in space provided \( u \) is sufficiently smooth including at \( t = 0 \). In [30], the second author studied a FD method in time combined with spatial piecewise linear finite elements scheme. In [26,32,34], a piecewise-constant and a piecewise-linear, discontinuous Galerkin (DG) and a postprocessed DG time-stepping methods combined with piecewise-linear finite elements for the spatial discretization were analyzed. Full convergence results were provided for variable time steps employed to compensate the lack of regularity of the exact solution near \( t = 0 \). A FD method and convolution quadrature had been studied in [10,39]. Another type of scheme involving Laplace transformation combined with a quadrature along a contour in the complex plane, provides spectral accuracy for the time discretization, but appears to offer little scope for handling nonlinear versions of (1), see [21,27].

Furthermore, various numerical methods have been applied for the following alternative representation of the fractional sub-diffusion Eq. (1a):

\[
\int_0^t \omega_{-\alpha}(t-s)u_t(s) \, ds - \Delta u(t) = f(t) \quad \text{in } \Omega \times (0, T],
\]
see [12,13,16,38,49] and the references therein. The two representations are equivalent under reasonable assumptions on the initial data, see [48], but the methods obtained for each representation are formally different.

Here, we continue the above-described effort and propose and analyze a method using exact integration in time and the HDG method for the space discretization for problem (1). The choice of the HDG methods for the problem under consideration can be easily justified. Indeed, the HDG methods are a relatively new class of DG methods introduced in [6] in the framework of steady-state diffusion which share with the classical (hybridized version of the) mixed finite element methods their remarkable convergence properties, [7–9], as well as the way in which they can be efficiently implemented, [18]. They provide approximations that are more accurate than the ones given by any other DG method for second-order elliptic problems [36].

Here we prove that, for each time \( t \in [0, T] \), the error of the HDG approximation to the solution \( u \) of (1) in the \( L_\infty(0, T; L_2(\Omega)) \)-norm and to the flux \( q := -\nabla u \) in the \( L_\infty(0, T; L_2(\Omega)) \)-norm converge with order \( h^{k+1} \) where \( k \) is the polynomial degree; see Theorem 2. We also show that a suitably defined projection of the error in \( u \) superconverges with order \( h^{k+2} \) whenever \( k \geq 1 \). This allows us to obtain, by a simple elementwise postprocessing, another approximation to \( u \) converging in the \( L_\infty(0, T; L_2(\Omega)) \)-norm with a rate of \( \sqrt{\log(T/h^2(\alpha+1))} h^{k+2} \) for quasi-uniform meshes and whenever \( k \geq 1 \); see Theorem 3. We thus obtain a much better approximation at a cost which is negligible in comparison with that of obtaining the approximate solution. These convergence results extend those obtained in [3] for the heat equation, that is for the case \( \alpha = 0 \), and hold uniformly for any \(-1 < \alpha \leq 0 \). Our error analysis extends the approach used in [3] for the heat equation. We make the full use of several important properties of the fractional derivative operator \( \mathcal{B}_\alpha \); see Lemma 1. In particular, especial care has to be used in the proof of the uniformity-in-time of the above-mentioned superconvergence property, as new, delicate regularity estimates are required by the use of a fractional duality argument.

Outline of the paper. In the next section, we define the HDG method. In Sect. 4, we prove the main convergence result, Theorem 2. Particularly relevant to this a priori error analysis is the derivation of several important properties of the fractional order operator \( \mathcal{B}_\alpha \), which we gather in Lemma 1. In Sect. 5, we prove the superconvergence result, Theorem 3. Finally, in Sect. 6, we comment on the extension of this work to other methods fitting the general formulation of the HDG methods; see [8].

2 The HDG method

We begin this section by discretizing the domain \( \Omega \) by a triangulation \( \mathcal{T}_h \) (made of simplexes \( K \)) which we take to be conforming for the sake of simplicity. We denote by \( \partial \mathcal{T}_h \) the set of all the boundaries \( \partial K \) of the elements \( K \) of \( \mathcal{T}_h \). We denote by \( E_h \) the union of faces \( F \) of the simplexes \( K \) of the triangulation \( \mathcal{T}_h \).

Next, we introduce the discontinuous finite element spaces:

\[
W_h = \{ w \in L^2(\Omega) : w|_K \in \mathcal{P}_k(K) \quad \forall \ K \in \mathcal{T}_h \}, \quad (3a)
\]

\[
V_h = \{ v \in L^2(\Omega) := [L^2(\Omega)]^d : v|_K \in \mathcal{P}_k(K) \quad \forall \ K \in \mathcal{T}_h \}, \quad (3b)
\]
\[ M_h = \{ \mu \in L^2(\Omega_h) : \mu|_{\Gamma} \in P_k(\Gamma) \quad \forall \Gamma \subset \partial \Omega_h \}, \]  

where \( P_k(K) := [P_k(K)]^d \) [the space of vector-valued functions whose entries lie on \( P_k(K) \)]. Here, \( P_k(D) \) is the space of polynomials of total degree \( \leq k \) on any spatial domain \( D \).

To describe our HDG scheme, we rewrite (1a) as a first order system as follows: \( \mathbf{q} + \nabla u = 0, u_t + \nabla \cdot \mathbf{B}_a \mathbf{q} = f \) in \( \Omega \times (0, T] \). So, the exact solution satisfies:

\[
\begin{align*}
(q, \phi) - (u, \nabla \cdot \phi) + (\mathbf{B}_a \mathbf{q}, \nabla \phi) &= 0 \quad \forall \phi \in \Pi_K \subset \mathcal{T}_h \mathbf{H} \text{(div, } K), \quad (5a) \\
(u_t, \chi) - (\mathbf{B}_a \mathbf{q}, \nabla \chi) + (\mathbf{B}_a \mathbf{q} \cdot \mathbf{n}, \chi) &= (f, \chi) \quad \forall \chi \in \Pi_K \subset \mathcal{T}_h H^1(K) \quad (5b)
\end{align*}
\]

for \( t \in (0, T] \), where \( (v, w) := \sum_{K \in \mathcal{T}_h} \langle v, w \rangle_K \) and \( \langle v, w \rangle := \sum_{K \in \partial K} \langle v, w \rangle_{\partial K} \). We write, for any domain \( D \) in \( \mathbb{R}^d \), \( (u, v)_D := \int_D uv \, dx \), and \( \langle u, v \rangle_{\partial D} := \int_{\partial D} u \, v \, d\gamma \).

For vector-valued functions \( \mathbf{v} \) and \( \mathbf{w} \), the notation is similarly defined with the integrand being the dot product \( \mathbf{v} \cdot \mathbf{w} \).

The HDG method provides a scalar approximation \( u_h(t) \in W_h \) to \( u(t) \), a vector-valued approximation \( \mathbf{q}_h(t) \in V_h \) to the flux \( \mathbf{q}(t) \), and a scalar approximation \( \hat{u}_h(t) \in M_h \) to the trace of \( u(t) \) on element boundaries for each time \( t \in [0, T] \), which are determined by requiring that the equations

\[
\begin{align*}
(q_h, r) - (u_h, \nabla \cdot r) + (\hat{u}_h, r \cdot \mathbf{n}) &= 0, \quad (5a) \\
(\partial_t u_h, w) - (\mathbf{B}_a q_h, \nabla w) + (\mathbf{B}_a \hat{q}_h \cdot \mathbf{n}, w) &= \langle f, w \rangle, \quad (5b) \\
\langle \hat{u}_h, \mathbf{n} \rangle_{\partial \Omega} &= \langle g, \mathbf{n} \rangle_{\partial \Omega}, \quad (5c) \\
\langle \mathbf{B}_a \hat{q}_h \cdot \mathbf{n}, \mu \rangle_{\partial \Omega} - (\mathbf{B}_a \hat{q}_h, \mathbf{n}) \langle \mu \rangle_{\partial \Omega} &= 0, \quad (5d) \\
\hat{u}_h|_{t=0} &= \Pi_W u_0, \quad (5e) \\
\end{align*}
\]

hold for all \( r \in V_h \), \( w \in W_h \), and \( \mu \in M_h \). Here, \( \partial_t u_h \) is nothing but the partial derivative of \( u_h \) with respect to time. We take the numerical trace for the flux as

\[
\hat{\mathbf{q}}_h = \mathbf{q}_h + \tau (u_h - \hat{u}_h) \mathbf{n} \quad \text{on } \partial \mathcal{T}_h, \quad (5f)
\]

for some nonnegative stabilization function \( \tau \) defined on \( \partial \mathcal{T}_h \); we assume that, for each element \( K \in \mathcal{T}_h \), \( \tau |_{\partial K} \) is constant on each of its faces. How to choose this stabilization function in order to achieve optimal convergence properties is discussed later. Note that the first two equations are inspired in the weak form satisfied by the exact solution, (4). The operator \( \Pi_W \) is the one introduced in [7] and will be defined later.

Let us briefly describe the feature of the HDG method which renders it efficiently implementable. Note that the form of the numerical trace given by (5d) allows us to express \( (u_h, \mathbf{q}_h, \hat{\mathbf{q}}_h) \) elementwise in terms of \( \hat{u}_h \), \( f \) and \( u_0 \) by using Eqs. (5a), (5b), (5e) and (5f). Then, \( \hat{u}_h \) is determined by as the solution of the transmission condition (5d), which enforces the single-valuedness of the normal component of the numerical trace \( \mathbf{B}_a \hat{\mathbf{q}}_h \), and the boundary condition (5c). Thus, the only globally-coupled degrees of freedom are those of the numerical trace \( \hat{u}_h \).

Let us end this subsection by noting that the existence and uniqueness of the approximation provided by the HDG method justintroduced follows from the corresponding

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results for linear systems of fractional differential equations. In particular, see [17] in page 139 the result for the Cauchy-problem for the linear system (3.1.29).

3 Properties of the operator $B_\alpha$

We begin the analysis by collecting several crucial properties of the operator $B_\alpha$. They involve the adjoint operators $B_\alpha^*$ and $I_{-\alpha}^*$ of $B_\alpha$ and $I_{-\alpha}$, respectively, where $I_{-\alpha}$ is the Riemann–Liouville fractional integral;

$$I_{-\alpha}v(t) = \int_0^t \omega_{-\alpha}(t-s)v(s)\,ds \quad \text{for } -1 < \alpha < 0$$

As we pointed out in the Introduction, these properties are essential for the analysis because they allow us to extend the approach used for the error analysis of the HDG method applied to the heat equation considered in [3].

For convenience, we introduce the following notation. Starting from the definition of the adjoint operators $B_\alpha^*$ and $I_{-\alpha}^*$,

$$\int_0^T v(t) B_\alpha w(t)\,dt = \int_0^T B_\alpha^* v(t) w(t)\,dt, \quad (6a)$$

$$\int_0^T v(t) I_{-\alpha} w(t)\,dt = \int_0^T I_{-\alpha}^* v(t) w(t)\,dt, \quad (6b)$$

One can show that for $\alpha \in (-1, 0)$ and $t \in (0, T]$, see [34, Lemma 3.1], that

$$B_\alpha^* v(t) = -\frac{\partial}{\partial t} \int_t^T \omega_{1+\alpha}(s-t)v(s)\,ds \quad \text{for any } v \in C^1(0, T), \quad (7a)$$

$$I_{-\alpha}^* v(t) = \int_t^T \omega_{-\alpha}(s-t)v(s)\,ds \quad \text{for any } v \in C^0(0, T). \quad (7b)$$

Moreover, since

$$B_\alpha^* I_{-\alpha}^* v(t) = -\frac{\partial}{\partial t} \int_t^T \omega_{1+\alpha}(s-t) \int_s^T \omega_{-\alpha}(q-s)v(q)\,dq\,ds$$

$$= -\frac{\partial}{\partial t} \int_t^T v(q) \int_q^T \omega_{1+\alpha}(s-t) \omega_{-\alpha}(q-s)\,ds\,dq,$$

and since $\int_t^q \omega_{1+\alpha}(s-t) \omega_{-\alpha}(q-s)\,ds = 1$, it is easy to see that $I_{-\alpha}^*$ is the right-inverse of $B_\alpha^*$, that is,

$$B_\alpha^* I_{-\alpha}^* v = v. \quad (8)$$

We gather in the following result several key properties we use in our analysis. They are expressed by using a notation we introduce next. First, we set
respectively. Note that, we drop out Lemma 1

\[ T \]

we replace \( \| \cdot \| \) and \( (\cdot, \cdot) \) with \( \| \cdot \| \) and \( (\cdot, \cdot) \), respectively. Note that, we drop out \( \tilde{t} \) from the above definitions when \( \tilde{t} = T \).

Lemma 1 Let \( c_\alpha = \frac{\cos(\alpha \pi/2)}{\pi^\alpha} \frac{|a|^{-\alpha}}{(1-\alpha)^{1-\alpha}} \) and \( d_\alpha = 1/\cos(\alpha \pi/2) \). Then, for any \( v, w \in \mathcal{C}^1(0, T) \) and any \( \alpha \in (-1, 0) \), we have

(i) \( |v|_\alpha^2 \geq c_\alpha T^\alpha \int_0^T v^2(t) \, dt \),
(ii) \( \int_0^T v(t) w(t) \, dt \leq d_\alpha |v|_\alpha |w|_{-\alpha} \),
(iii) \( \int_0^T \mathcal{J}_\alpha v(t) \, dt \leq d_\alpha |v|_{-\alpha} |w|_{-\alpha} \),
(iv) \( \lim_{t \to 0} \int_0^t \mathcal{J}_\alpha v(s) \, ds = v^2(0) \).

Proof The coercivity property (i) was proven in [24, Theorem A.1] by using the Laplace transform and Plancherel Theorem. Using a similar technique and the fact that \( \mathcal{J}_\alpha \) is the right-inverse of \( \mathcal{B}_\alpha \), see (8), property (ii) can also be obtained, see [35, Lemma 3.1]. Properties (iii) and (iv) easily follow from property (ii) and again from the fact that \( \mathcal{J}_\alpha \) is the right-inverse of \( \mathcal{B}_\alpha \).

It remains to prove property (v). We have, for small enough \( \tau > 0 \), that

\[
\omega_{\alpha+2}^{-1}(t) \int_0^t v(s) \mathcal{B}_\alpha v(s) \, ds = \omega_{\alpha+2}^{-1}(t) \int_0^t \omega_{\alpha+1}(s) v(s) \omega_{\alpha+1}^{-1}(s) \mathcal{B}_\alpha v(s) \, ds
\]

\[
= \left[ \omega_{\alpha+2}^{-1}(t) \int_0^t \omega_{\alpha+1}(s) \, ds \right] v(t^*) \omega_{\alpha+1}^{-1}(t^*) \mathcal{B}_\alpha v(t^*)
\]

\[
= v(t^*) \omega_{\alpha+1}^{-1}(t^*) \mathcal{B}_\alpha v(t^*)
\]

for some \( t^* \in (0, t) \). From the definition of \( \mathcal{B}_\alpha \), (2), we have that

\[
\mathcal{B}_\alpha v(t^*) = \omega_{\alpha+1}(t^*) v(0) + \int_0^{t^*} \omega_{\alpha+1}(s) v'(t^* - s) \, ds.
\]

Since \( \int_0^{t^*} \omega_{\alpha+1}(s)|v'(t^* - s)| \, ds < \infty \), the desired result follows. \qed

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4 Error estimates

In this section, we carry out the first part of our a priori error analysis of the HDG method. To be able to do this, we carefully use several crucial properties of the operators \( \mathcal{B}_\alpha \) and \( \mathcal{I}_{-\alpha} \) introduced in the previous section.

4.1 Projections

Given \( q \in H^1(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H^1(K) \) and \( u \in H^1(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H^1(K) \), the projections \( \Pi_q q \in V_h \) and \( \Pi_w u \in W_h \) are on each simplex \( K \in \mathcal{T}_h \) as the solutions of the following equations:

\[
\begin{align*}
(\Pi_q q, v)_K &= (q, v)_K \quad \text{for all } v \in \mathcal{P}_{k-1}(K), \quad (9a) \\
(\Pi_w u, w)_K &= (u, w)_K \quad \text{for all } w \in \mathcal{P}_{k-1}(K), \quad (9b) \\
\langle \Pi_q q \cdot n + \tau \Pi_w u, \mu \rangle_F &= \langle q \cdot n + \tau u, \mu \rangle_F \quad \text{for all } \mu \in \mathcal{P}_k(F), \quad (9c)
\end{align*}
\]

for all faces \( F \) of the simplex \( K \). This is the projection introduced in [7] to study HDG methods for the steady-state diffusion problem. Its approximation properties are described in the following result. For convenience, we introduce the following notation:

\[
e_q := \Pi_q q - q \quad \text{and} \quad e_u := \Pi_w u - u.
\]

We use \( \Vert \cdot \Vert_D \) to denote the \( L^2(D) \)-norm. The norm on any other Sobolev space \( X \) is denoted by \( \Vert \cdot \Vert_X \). We also denote \( \Vert \cdot \Vert_{X(0,T;Y(D))} \) by \( \Vert \cdot \Vert_{X(Y(D))} \) and omit \( D \) whenever \( D = \Omega \).

**Theorem 1** ([7]) Suppose \( \tau |_{\partial K} \) is nonnegative and \( \tau^\text{max}_K := \max \tau |_{\partial K} > 0 \). Then the system (9) is uniquely solvable for \( \Pi_q q \) and \( \Pi_w u \). Furthermore, there is a constant \( C \) independent of \( K \) and \( \tau \) such that

\[
\begin{align*}
\Vert e_q \Vert_K &\leq C h^{k+1}_K \left( |q|_{H^{k+1}(K)} + \tau^*_K |u|_{H^{k+1}(K)} \right), \\
\Vert e_u \Vert_K &\leq C h^{k+1}_K \left( |u|_{H^{k+1}(K)} + |\nabla \cdot q|_{H^k(K)} / \tau^\text{max}_K \right).
\end{align*}
\]

Here \( \tau^*_K := \max \tau |_{\partial K \setminus F^*} \), where \( F^* \) is a face of \( K \) at which \( \tau |_{\partial K} \) is maximum.

Note that the approximation error of the projection is of order \( k+1 \) provided that the stabilization function is such that both \( \tau^*_K \) and \( 1/\tau^\text{max}_K \) are uniformly bounded and the exact solution is sufficiently regular. For example, we can take \( \tau \) to be a positive constant. Another possible choice is to take it zero on all but one face of the simplex \( K \), so that \( \tau^*_K = 0 \), and then take it equal to \( 1/h_K \) on the remaining face, so that \( 1/\tau^\text{max}_K = h_K \).

4.2 The equations of the projection of the errors

Setting

\[
(\varepsilon_q^h, \varepsilon_u^h, \varepsilon_{\tilde{u}}^h, \varepsilon_{\tilde{q}}^h) := (\Pi_q q - q_h, \Pi_w u - u_h, P_M u - \tilde{u}_h, P_M q - \tilde{q}_h),
\]

\( \varepsilon_q^h, \varepsilon_u^h, \varepsilon_{\tilde{u}}^h, \varepsilon_{\tilde{q}}^h \) are the projection of the errors.
where \( P_M \) denotes the \( L^2 \)-orthogonal projection onto \( M_h \), and \( P_M \) denotes the vector-valued projection each of whose components are equal to \( P_M \). The projection of the errors satisfy the following equations:

**Lemma 2** We have

\[
\begin{align*}
(\varepsilon_h^q, r) - (\varepsilon_u^q, \nabla \cdot r) \beta_h + (\varepsilon_h^\hat{q}, r \cdot n) &= (e_q, r), \\
(\partial_t \varepsilon_h^u, w) - (\mathcal{B}_\alpha \varepsilon_h^q, \nabla w) \beta_h + (\mathcal{B}_\alpha \varepsilon_h^\hat{q} \cdot n, w) &= (e_u, w), \\
(\mathcal{B}_\alpha \varepsilon_h^\hat{q} \cdot n, \mu) - (\mathcal{B}_\alpha \varepsilon_h^\hat{q} \cdot n, \mu)_{\partial \Omega} &= 0,
\end{align*}
\]

for all \( r \in V_h \), \( w \in W_h \), and \( \mu \in M_h \), where

\[
\varepsilon_h^\hat{q} \cdot n := \varepsilon_h^q \cdot n + \tau (\varepsilon_h^u - \varepsilon_h^\hat{u}) \quad \text{on } \partial \Omega.
\]

**Proof** From (4), we know that the exact solution \( \{q, u\} \) satisfies the equations

\[
\begin{align*}
(q, r) - (u, \nabla \cdot r) + (u, r \cdot n) &= 0 \quad \text{for all } r \in V_h, \\
(u_t, w) - (\mathcal{B}_\alpha q, \nabla w) + (\mathcal{B}_\alpha q \cdot n, w) &= (f, w) \quad \text{for all } w \in W_h.
\end{align*}
\]

By using the orthogonality properties of the projections \( \Pi_V, \Pi_W, \) and \( P_M \), we can rewrite these equations as follows:

\[
\begin{align*}
(\Pi_V q, r) - (\Pi_u, \nabla \cdot r) + (P_M u, r \cdot n) &= (e_q, r), \\
(\Pi_u u_t, w) - (\mathcal{B}_\alpha \Pi_V q, \nabla w) + (\mathcal{B}_\alpha (\Pi_V q \cdot n + \tau (\Pi_u - P_M u)), w) &= (f + e_{u_t}, w),
\end{align*}
\]

for all \( r \in V_h \) and \( w \in W_h \). Indeed, the fact that \( P_M \) is the \( L^2 \)-projection into \( M_h \) was used in the third term of the left-hand side of the first equation, and the orthogonality property (9c) was used in the third term of the left-hand side of the second equation. To deal with that term, we also used the fact that

\[
\langle \tau (P_M u - u), \mu \rangle = 0 \quad \text{for all } \mu \in M_h,
\]

given that, for each element \( K \in \partial \Omega \), \( \tau \) is constant on each face \( e \) of \( K \). Subtracting the Eqs. (5a) and (5b) from the above ones, respectively, we obtain Eqs. (11a) and (11b), respectively.

The Eq. (11c) follows directly from the Eq. (5c) and (1c). To prove (11d), we note that, by definition of \( \varepsilon_h^\hat{q} \), (10), we have

\[
\langle \mathcal{B}_\alpha \varepsilon_h^\hat{q} \cdot n, \mu \rangle - \langle \mathcal{B}_\alpha \varepsilon_h^\hat{q} \cdot n, \mu \rangle_{\partial \Omega} = \left[ \langle \mathcal{B}_\alpha q \cdot n, \mu \rangle - \langle \mathcal{B}_\alpha q \cdot n, \mu \rangle_{\partial \Omega} \right] - \left[ \langle \mathcal{B}_\alpha q_h \cdot n, \mu \rangle - \langle \mathcal{B}_\alpha q_h \cdot n, \mu \rangle_{\partial \Omega} \right].
\]
since $P_M$ is the $L^2$-projection into $M_h$. The first term of the right-hand side is equal to zero because $B_\alpha q$ is in $H(\text{div}, \Omega)$ and the second because the normal component of $B_\alpha \hat{q}_h$ is single valued by the Eq. (5d). Hence, the identity (11d) holds.

Next, let us prove (11e). By the Eq. (5e) defining the HDG method, $u_h |_{t=0} = \Pi_w u_0$, and so $\varepsilon^u_h |_{t=0} = \Pi_w u_0 - u_h |_{t=0} = \Pi_w u_0 - \Pi_w u_0 = 0$. It remains to prove the identity (11f). We have

$$
\varepsilon^q_h \cdot n = P_M(q \cdot n) - (q_h \cdot n + \tau (u_h - \hat{u}_h)) = (\Pi_v q \cdot n + \tau (\Pi_w u - P_M u)) - (q_h \cdot n + \tau (u_h - \hat{u}_h)) \quad \text{by (10) and (5f),}
$$

$$
= \varepsilon^q_h \cdot n + \tau (\varepsilon^u_h - \varepsilon^\alpha u_h)
$$

This completes the proof.

4.3 A first error bound

**Lemma 3** For any $T \geq 0$, we have

$$
\left( \|\varepsilon^u_h(T)\|_2^2 + \|\varepsilon^q_h\|_\alpha^2 + 2\|\sqrt{\tau}(\varepsilon^u_h - \varepsilon^\alpha u_h)\|_\alpha^2 \right)^{1/2} \leq \|e_u\|_{L^1(L^2)} + d_\alpha \max_{t \in (0, T)} \|e_q\|_{\alpha, t}.
$$

**Proof** Taking $\mathbf{r} = B_\alpha \varepsilon^q_h$ in (11a), $\mathbf{w} = \varepsilon^u_h$ in (11b), $\mu = -B_\alpha \varepsilon^q_h \cdot \mathbf{n}$ in (11c) and $\mu = -\varepsilon^\alpha u_h$ in (11d), and adding the resulting four equations, we get

$$
\frac{1}{2} \frac{d}{dt} \|\varepsilon^u_h\|_2^2 + (\langle B_\alpha \varepsilon^q_h, \varepsilon^q_h \rangle + \Psi_h = (e_q, B_\alpha \varepsilon^q_h) + (e_u, \varepsilon^u_h),
$$

where by the definition of $\varepsilon^q_h$, (11f),

$$
\Psi_h := -\langle \varepsilon^u_h, \nabla \cdot B_\alpha \varepsilon^q_h \rangle + \langle \varepsilon^\alpha u_h, B_\alpha \varepsilon^q_h \cdot \mathbf{n} \rangle - (B_\alpha \varepsilon^q_h, \nabla \varepsilon^u_h)_{\mathcal{H}} + (\langle B_\alpha \varepsilon^\alpha u_h \cdot \mathbf{n}, \varepsilon^u_h - \varepsilon^\alpha u_h \rangle
$$

$$
= -\langle \varepsilon^u_h, B_\alpha \varepsilon^q_h \cdot \mathbf{n} \rangle + \langle \varepsilon^\alpha u_h, B_\alpha \varepsilon^q_h \cdot \mathbf{n} \rangle + (\langle B_\alpha \varepsilon^\alpha u_h \cdot \mathbf{n}, \varepsilon^u_h - \varepsilon^\alpha u_h \rangle
$$

$$
= (\langle B_\alpha \varepsilon^\alpha u_h \cdot \mathbf{n}, \varepsilon^u_h - \varepsilon^\alpha u_h \rangle = \langle B_\alpha (\sqrt{\tau}(\varepsilon^u_h - \varepsilon^\alpha u_h)), \sqrt{\tau}(\varepsilon^u_h - \varepsilon^\alpha u_h) \rangle.
$$

Integrating over the time interval $(0, T)$, and using the fact that $\varepsilon^u_h(0) = 0$ by (11e),

$$
\|\varepsilon^u_h(T)\|_2^2 + \|\varepsilon^q_h\|_\alpha^2 + 2\|\sqrt{\tau}(\varepsilon^u_h - \varepsilon^\alpha u_h)\|_\alpha^2 = 2 \int_0^T (e_q, B_\alpha \varepsilon^q_h) + 2 \int_0^T (e_u, \varepsilon^u_h).
$$

Since $2 \int_0^T (e_q, B_\alpha \varepsilon^q_h) \leq 2d_\alpha \|e_q\|_\alpha \|\varepsilon^q_h\|_\alpha \leq d_\alpha^2 \|e_q\|_\alpha^2 + \|\varepsilon^q_h\|_\alpha^2$, by the property (iii) of Lemma 1, and since $\int_0^T (e_u, \varepsilon^u_h) \leq \int_0^T \|e_u\|_\alpha \|\varepsilon^u_h\|_\alpha$,

$$
\|\varepsilon^u_h(T)\|_2^2 + \|\varepsilon^q_h\|_\alpha^2 + 2\|\sqrt{\tau}(\varepsilon^u_h - \varepsilon^\alpha u_h)\|_\alpha^2 \leq d_\alpha^2 \|e_q\|_\alpha^2 + 2 \int_0^T \|e_u\|_\alpha \|\varepsilon^u_h\|_\alpha \quad \text{for} \quad T > 0.
$$
The result now easily follows from Lemma 4 below with \( A(t) := d^2_\alpha \| \epsilon_q \|_{\alpha,t}^2 \), \( B(t) := \| \epsilon_u(t) \| \) and with \( E^2(t) := \| \epsilon_h^q(t) \|^2 + \| \epsilon_h^q \|_{\alpha,t}^2 + 2 \| \epsilon_h^q \|_{\alpha,t}^2 \).

**Lemma 4 (An integral inequality)** Suppose that, for any \( t \geq 0 \), we have that \( E^2(t) \leq A(t) + 2 \int_0^t B(s) E(s) \, ds \), for some nonnegative functions \( A \) and \( B \). Then, for any \( T > 0 \), \( E(T) \leq \max_{t \in (0,T)} A^{1/2}(t) + \int_0^T B(s) \, ds \).

**Proof** Setting \( X(t) = \max_{t \in [0,T]} A(t) + 2 \int_0^t B(s) E(s) \, ds \), we see that, for \( t \in (0,T) \), \( \frac{d}{dt} X(t) = 2 B(t) E(t) \leq 2 B(t) \sqrt{X(t)} \), and so \( \frac{d}{dt} \sqrt{X(t)} \leq B(t) \). This implies that \( \sqrt{X(t)} \leq \sqrt{X(0)} + \int_0^t B(s) \, ds \), and the result follows.

### 4.4 A second error bound

We derive next an estimate of \( \epsilon_h^q \) in the \( L^\infty(0,T;L^2(\Omega)) \) norm.

**Lemma 5** Let \( S_h^2 := (\tau(\epsilon_h^u - \epsilon_h^u), (\epsilon_h^u - \epsilon_h^u)) \). For any \( T > 0 \), we have

\[
\left( \| \epsilon_h^q(T) \|^2 + S_h^2(T) + 2 \| \partial_t \epsilon_h^q \|^2_{\alpha} \right)^{1/2} \leq \left( \| \epsilon_h^q(0) \|^2 + S_h^2(0) \right)^{1/2} + d_\alpha \max_{t \in (0,T)} \| \epsilon_u \|_{\alpha,t} + \| \epsilon_q \|_{L^1(\Omega^2)}.
\]

**Proof** By the adjoint property (6) and the identity property (8),

\[
2 \int_0^T (B_\alpha \epsilon_h^q, I_{-\alpha} \partial_t \epsilon_h^q) = 2 \int_0^T (\epsilon_h^q, B_\alpha^* I_{-\alpha} \partial_t \epsilon_h^q)
= 2 \int_0^T (\epsilon_h^q, \partial_t \epsilon_h^q) = \| \epsilon_h^q(T) \|^2 - \| \epsilon_h^q(0) \|^2.
\]

Now, applying the operator \( I_{-\alpha} \partial_t \) to the first equation of the errors, (11a), and taking \( r := B_\alpha \epsilon_h^q \), we obtain

\[
(I_{-\alpha} \partial_t \epsilon_h^q, B_\alpha \epsilon_h^q) - (I_{-\alpha} \partial_t \epsilon_h^q, \nabla \cdot B_\alpha \epsilon_h^q)
+ (I_{-\alpha} \partial_t \epsilon_h^q, \eta_h \cdot n) = (I_{-\alpha} \epsilon_u, B_\alpha \epsilon_h^q).
\]

Integrating in time from 0 to \( T \) and using the identity of the previous step, we get

\[
\frac{1}{2} \| \epsilon_h^q(T) \|^2 - \int_0^T (I_{-\alpha} \partial_t \epsilon_h^q, \nabla \cdot B_\alpha \epsilon_h^q) + \int_0^T (I_{-\alpha} \partial_t \epsilon_h^q, \eta_h \cdot n)
\]
\[
= \frac{1}{2} \| \epsilon_h^q(0) \|^2 + \int_0^T (I_{-\alpha} \epsilon_u, B_\alpha \epsilon_h^q).
\]
Now, taking \( w := \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h} \) in Eq. (11b), and integrating from 0 to \( T \),
\[
\int_0^T \left[ (\partial_t \varepsilon^u_{h}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) - (\mathcal{B}_\alpha \varepsilon^q_{h}, \nabla \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) + (\mathcal{B}_\alpha \varepsilon_{h}^\hat{q} \cdot \mathbf{n}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) \right] dt
= \int_0^T (e_{u_t}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}).
\]

Adding this equation to the one obtained in the last step and rearranging terms,
\[
\|e^q_{h}(T)\|^2 + 2 \int_0^T (\partial_t \varepsilon^u_{h} - e_{u_t}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) + 2 \Phi_h
= \|e^q_{h}(0)\|^2 + 2 \int_0^T (\mathcal{I}_\alpha^* e_{q}, \mathcal{B}_\alpha \varepsilon^q_{h})
= \|e^q_{h}(0)\|^2 + 2 \int_0^T (e_{q}, \varepsilon^q_{h}), \quad \text{by the properties (6) and (8),}
\]

where
\[
\Phi_h := - \int_0^T (\mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}, \nabla \cdot \mathcal{B}_\alpha \varepsilon^q_{h}) \mathcal{B} + \int_0^T (\mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}, \mathcal{B}_\alpha \varepsilon^q_{h} \cdot \mathbf{n})
- \int_0^T (\mathcal{B}_\alpha \varepsilon^q_{h}, \nabla \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) \mathcal{B} + \int_0^T (\mathcal{B}_\alpha \varepsilon_{h}^\hat{q} \cdot \mathbf{n}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h})
= \int_0^T \left[ -(\mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}, \mathcal{B}_\alpha \varepsilon^q_{h} \cdot \mathbf{n}) + (\mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}, \mathcal{B}_\alpha \varepsilon^q_{h} \cdot \mathbf{n}) + (\mathcal{B}_\alpha \varepsilon_{h}^\hat{q} \cdot \mathbf{n}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) \right]
= \int_0^T \langle \mathcal{B}_\alpha (\varepsilon_{h}^{\hat{q}} - \varepsilon^q_{h}) \cdot \mathbf{n}, \mathcal{I}_\alpha^* \partial_t (\varepsilon^u_{h} - \varepsilon^\hat{u}_{h}) \rangle \quad \text{by equations (11c) and (11d),}
= \int_0^T \langle (\varepsilon_{h}^\hat{q} - \varepsilon^q_{h}) \cdot \mathbf{n}, \mathcal{B}_\alpha \mathcal{I}_\alpha^* \partial_t (\varepsilon^u_{h} - \varepsilon^\hat{u}_{h}) \rangle \quad \text{by the adjoint property (6),}
= \int_0^T \langle \tau (\varepsilon^u_{h} - \varepsilon^\hat{u}_{h}), \partial_t (\varepsilon^u_{h} - \varepsilon^\hat{u}_{h}) \rangle = \frac{1}{2} \mathcal{S}_{h}^2(T) - \frac{1}{2} \mathcal{S}_{h}^2(0)
\]

by the identity property (8) and the error Eq. (11f). Therefore, for any \( T > 0 \),
\[
\|e^q_{h}(T)\|^2 + \mathcal{S}_{h}^2(T) + 2 \|\partial_t \varepsilon^u_{h}\|^2_\alpha = \|e^q_{h}(0)\|^2 + \mathcal{S}_{h}^2(0)
+ 2 \int_0^T \left[ (e_{q}, \varepsilon^q_{h}) + (e_{u_t}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) \right].
\]

But, by property (iv) of Lemma 1,
\[
2 \int_0^T (e_{u_t}, \mathcal{I}_\alpha^* \partial_t \varepsilon^u_{h}) \leq d^2_\alpha \|e_{u_t}\|^2_\alpha + \|\partial_t \varepsilon^u_{h}\|^2_\alpha,
\]

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and since $\int_0^T (e_{qh}, e_h^q) \leq \int_0^T \|e_{qh}\| \|e_h^q\|$, we have, that, for any $T > 0$,

$$
\|e_h^q(T)^2 + S_h^2(T) + 2 \|\partial_t e_h^u\|_{-\alpha}^2 \leq \|e_h^q(0)^2 + S_h^2(0) + d_{\alpha}^2 \|e_{ut}\|_{-\alpha}^2
$$

$$
+ 2 \int_0^T \|e_{qh}\| \|e_h^q\|.
$$

Finally, the desired inequality follows from Lemma 4 with $B(t) := \|e_{qh}(t)\|$ and

$$
A(t) := \|e_h^q(0)^2 + S_h^2(0) + d_{\alpha}^2 \max_{t \in (0, T)} \|e_{ut}\|_{-\alpha, t}^2,
$$

$$
E^2(t) := \|e_h^q(t)^2 + S_h^2(t) + 2 \|\partial_t e_h^u\|_{-\alpha, t}^2.
$$

We still need to estimate the term $\|e_h^q(0)^2 + S_h^2(0)$ in Lemma 5.

**Lemma 6** We have that $\|e_h^q(0)^2 + S_h^2(0) \leq \frac{d_{\alpha}^2}{c_{\alpha} T (\alpha + 2)} \|e_q(0)\|^2$, provided $e_{ut} \in \mathscr{C}^0(0, \varepsilon; L^2(\Omega))$ and $e_q \in \mathscr{C}^1(0, \varepsilon; L^2(\Omega))$ for some positive $\varepsilon$.

**Proof** Setting $\Theta_h(t) := \|e_h^q(t)^2 + S_h^2(t)$, we get, by the coercivity property (i) of Lemma 1, that

$$
\left( c_{\alpha} t^\alpha \int_0^t \Theta_h \right)^{1/2} \leq \left( \|e_h^q\|_{\alpha, t}^2 + \|\sqrt{r}(e_h^u - \tilde{e}_h^u)\|_{\alpha, t}^2 \right)^{1/2}
$$

$$
\leq \int_0^t \|e_{ut}\| + d_{\alpha} \max_{t^* \in (0, t)} \|e_q\|_{\alpha, t^*}
$$

by Lemma 3. Then

$$
\Theta_h^{1/2}(0) = \lim_{t \downarrow 0} t^{-(1+\alpha)/2} \left( t^\alpha \int_0^t \Theta_h(s) \, ds \right)^{1/2} \leq c_{\alpha}^{-1/2} (T_1 + d_{\alpha} T_2),
$$

where $T_1 := \lim_{t \downarrow 0} t^{-(1+\alpha)/2} \int_0^t \|e_{ut}\| = 0$, by the assumption on $e_{ut}$,

$$
T_2 := \lim_{t \downarrow 0} t^{-(1+\alpha)/2} \max_{t^* \in (0, t)} \|e_q\|_{\alpha, t^*} = \frac{1}{T^{1/2}(\alpha + 2)} \|e_q(0)\|,
$$

by property (v) of Lemma 1. This completes the proof.

4.5 The error estimates

We are now ready to obtain our HDG error estimates. By Lemmas 3, 5, and 6, we get

$$
\|(u - u_h)(T)\| \leq \|e_u(T)\| + \|\mathbb{I} [e_q, e_{ut}]\|_{1, \alpha}
$$

and

$$
\|(q - q_h)(T)\| \leq \|e_q(T)\| + \|\mathbb{I} [e_q, e_{ut}]\|_{2, \alpha},
$$
where

\[
\| [q, u] \|_{1, \alpha} := \| u_t \|_{L^1(L^2)} + d_\alpha \max_{t \in (0, T)} \| q \|_{\alpha, t},
\]

\[
\| [q, u] \|_{2, \alpha} := \frac{d_\alpha}{c_\alpha^{1/2} \Gamma^{1/2}(\alpha + 2)} \| q(0) \| + \| q_t \|_{L^1(L^2)} + d_\alpha \max_{t \in (0, T)} \| u_t \|_{-\alpha, t}.
\]

Note that when \( \alpha = 0 \), we recover the error estimates for the HDG methods for the heat equation of [3, Theorem 2.1] since in this case \( d_0 = 1, c_0 = 1 \) and \( \Gamma(2) = 1 \). If we now use the approximation properties of the projections \( \Pi_V \) and \( \Pi_W \) of Theorem 1, we obtain our optimal HDG error estimates.

**Theorem 2** Assume that \( u \in C^1(0, T; H^{k+1}(\Omega)) \) and \( q \in C^1(0, T; H^{k+1}(\Omega)) \). Assume also that \( \tau_\star^K \) and \( 1/\tau_{\text{max}}^K \) are bounded by \( C \). Then we have that

\[
\| (u - u_h)(T) \| \leq C_1 h^{k+1} \quad \text{and} \quad \| (q - q_h)(T) \| \leq C_2 h^{k+1}.
\]

The constant \( C_i, i = 1, 2 \), only depends on \( C, \alpha, \| u \|_{C^1(H^{k+1})} \), and on \( \| q \|_{C^1(H^{k+1})} \).

Note that, provided that the exact solution is smooth, the above error estimates are uniform for \( \alpha \in [\alpha^*, 0] \) provided \( \alpha^* > -1 \). This is not true for \( \alpha^* = -1 \) since the coefficients \( d_\alpha \) and \( 1/c_\alpha \) behave like \( 1/(\alpha + 1) \) as \( \alpha \) goes to \(-1\). Note also that these results hold even when the domain \( \Omega \) is not convex.

### 5 Superconvergence and post-processing

In this section, we carry out the second part of our a priori error analysis. We prove superconvergence results which will allow us to compute a new, better approximation to \( u \) by means of an element-by-element postprocessing. We begin by describing such approximation. Then, we show how to get our superconvergence result by a duality argument.

Following [3,14,42,43], for each fixed \( t \in [0, T] \), we define the postprocessed HDG approximation \( u_h^* \in \mathcal{P}_{k+1}(K) \) to \( u \) for each simplex \( K \in \mathcal{T}_h \), as follows:

\[
(u_h^*(t), 1)_K = (u_h(t), 1)_K
\]

\[
(\nabla u_h^*(t), \nabla w)_K = -(q_h(t), \nabla w)_K \quad \text{for all } w \in \mathcal{P}_{k+1}(K).
\]

It is not difficult to obtain the following result:

\[
\| u(t) - u_h^*(t) \|_K \leq C h^{k+2} |u(t)|_{H^{k+2}(K)} + \| P_0 e_h^u(t) \|_K + C h \| e_h^q(t) \|_K.
\]

Here \( P_0 \) is the \( L^2(\Omega) \)-projection into the space of functions which are constant on each element \( K \in \mathcal{T}_h \).
5.1 A first estimate of $\|P_0 \varepsilon_h^u(T)\|$ by duality argument

We see that if the term $\|P_0 \varepsilon_h^u\|$ is of order $O(h^{k+2})$, we would have that the post-processed approximation $u_h^*$ converges faster than the original approximation $u_h$. To obtain such an estimate, the traditional duality approach consists in, since we can write $\|P_0 \varepsilon_h^u(T)\| = \sup_{\varphi \in C_0^\infty(\Omega)} \frac{(P_0 \varepsilon_h^u(T), \varphi)}{\|\varphi\|}$, estimating the expression $[P_0 \varepsilon_h^u(T), \Theta]$ by using the solution of the dual problem

$$
\begin{align*}
\Phi + \nabla \Psi &= 0 & \text{on } \Omega \times (0,T), \\
\Psi_t - \nabla \cdot \mathcal{B}_\alpha^* \Phi &= 0 & \text{on } \Omega \times (0,T), \\
\Psi &= 0 & \text{on } \partial \Omega \times (0,T), \\
\Psi(T) &= \Theta & \text{on } \Omega.
\end{align*}
$$

(15a)

(15b)

(15c)

(15d)

In the next result, we give an expression for the quantity $(P_0 \varepsilon_h^u(T), \Theta)$ in terms of the errors $\mathcal{B}_\alpha e_h^q, \varepsilon_h^u$ and the solution of the dual problem. In it, $I_h$ is any interpolation operator from $L^2(\Omega)$ into $W_h \cap H_0^1(\Omega)$, $P_w$ is the $L^2$-projection into $W_h$ and $\Pi_{BDM}$ is the well-known projection associated to the lowest-order Brezzi–Douglas–Marini (BDM) space, see [2].

**Lemma 7** Assume that $k \geq 1$. Then, for any $T > 0$, $(P_0 \varepsilon_h^u(T), \Theta)$ equals

$$
\int_0^T \left[ (e_h^q, \mathcal{B}_\alpha^* (-\Pi_{BDM} \nabla \Psi + \nabla I_h \Psi)) + (e_q, \mathcal{B}_\alpha^* (\Pi_{BDM} \nabla \Psi - \nabla P_w \Psi)) + (\partial_t \varepsilon_h^u - e_{u_t}, P_0 \Psi - I_h \Psi) \right].
$$

(15e)

Proof Since $\Psi(T) = \Theta$ by (15d) and $\varepsilon_h^u(0) = 0$ by (11e), we have

$$
(P_0 \varepsilon_h^u(T), \Theta) = \int_0^T \left[ (\partial_t P_0 \varepsilon_h^u, \Psi) + (P_0 \varepsilon_h^u, \Psi_t) \right]
$$

$$
= \int_0^T \left[ (\partial_t \varepsilon_h^u, P_0 \Psi) + (\varepsilon_h^u, P_0 \nabla \mathcal{B}_\alpha^* \Phi) \right]
$$

by the definition of the $L^2$-projection $P_0$ and by (15b).

Let us work on the last term of the right-hand side. By the commutativity property $P_0 \nabla \cdot = \nabla \cdot \Pi_{BDM}$, we have $(e_h^q, P_0 \nabla \mathcal{B}_\alpha^* \Phi) = (e_h^q, \nabla \cdot \Pi_{BDM} \mathcal{B}_\alpha^* \Phi)$. Since $k \geq 1$, we can take $r := \mathcal{B}_\alpha^* \Pi_{BDM} \Phi$ in the first error Eq. (11a), to get

$$
(e_h^q, P_0 \nabla \cdot \mathcal{B}_\alpha^* \Phi) = (e_h^q, \nabla \cdot \Pi_{BDM} \mathcal{B}_\alpha^* \Phi),
$$

$$
= (e_h^q, \mathcal{B}_\alpha^* \Pi_{BDM} \Phi) + (\hat{\varepsilon}_h^q, \mathcal{B}_\alpha^* \Pi_{BDM} \Phi \cdot n) - (e_q, \Pi_{BDM} \mathcal{B}_\alpha^* \Phi)
$$

$$
= (e_h^q, \mathcal{B}_\alpha^* \Pi_{BDM} \Phi) - (e_q, \Pi_{BDM} \mathcal{B}_\alpha^* \Phi),
$$

since $(\varepsilon_h^u, \mathcal{B}_\alpha^* \Pi_{BDM} \Phi \cdot n)_{\partial \Omega} = 0$ because $\mathcal{B}_\alpha^* \Pi_{BDM} \Phi \in H(div, \Omega)$ and $\varepsilon_h^u = 0$ on $\partial \Omega$ by (11c).
Integrating in time from 0 to $T$ and using the adjoint property (6), we get

$$
\int_0^T (e_h^q, \mathcal{P}_\alpha^* (\Pi_{\text{BDM}} \Phi)) = \int_0^T (e_h^q, \mathcal{P}_\alpha^* (-\Pi_{\text{BDM}} \nabla \Psi + \nabla I_h \Psi)) - \int_0^T (\mathcal{P}_\alpha e_h^q, \nabla I_h \Psi).
$$

But, by the error Eq. (11b) with $w := I_h \Psi$,

$$(\mathcal{P}_\alpha e_h^q, \nabla I_h \Psi) = (\partial_t \varepsilon_h^\mu - e_{u_t}, I_h \Psi) - (\mathcal{P}_\alpha e_h^q \cdot n, I_h \Psi) = (\partial_t \varepsilon_h^\mu - e_{u_t}, I_h \Psi)$$

since $(\mathcal{P}_\alpha e_h^q \cdot n, I_h \Psi) = (\mathcal{P}_\alpha e_h^q \cdot n, I_h \Psi)_{\partial \Omega} = 0$ because the normal component of $\mathcal{P}_\alpha e_h^q$ is single valued by (11d) and $I_h \Psi = 0$ on $\partial \Omega$ by the boundary condition (15c).

Then, putting together all the above intermediate steps, $(P_0 \varepsilon_h^\mu(T), \Theta)$ equals

$$
\int_0^T [(e_h^q, \mathcal{P}_\alpha^* (\nabla I_h \Psi - \Pi_{\text{BDM}} \nabla \Psi)) - (e_q, \mathcal{P}_\alpha^* \Pi_{\text{BDM}} \Phi) + (\partial_t \varepsilon_h^\mu, P_0 \Psi - I_h \Psi) + (e_{u_t}, I_h \Psi)].
$$

Therefore, the desired result now follows after noting that

$$
\int_0^T (e_q, \mathcal{P}_\alpha^* \Pi_{\text{BDM}} \Phi) = \int_0^T (e_q, \mathcal{P}_\alpha^* (\Pi_{\text{BDM}} \nabla \Psi - \nabla P_w \Psi)),
$$

and that $(e_{u_t}, I_h \Psi) = (e_{u_t}, I_h \Psi - P_0 \Psi)$, by (15a), the definition of $P_w \Psi$ and the orthogonality property of the projection $P_\nu$, (9a); and by the definition of $P_0 \Psi$ and the orthogonality property of the projection $P_w$, (9b).

Now, as a direct consequence of the previous lemma and by property (ii) of Lemma 1, we have that

$$
|P_0 \varepsilon_h^\mu(T), \Theta| \leq \|e_h^q\|_{\dot{L}^\infty(L^2)} \|\mathcal{P}_\alpha^* (\Pi_{\text{BDM}} \nabla \Psi - \nabla I_h \Psi)\|_{\dot{L}^1(L^2)} + \|e_q\|_{\dot{L}^\infty(L^2)} \|\mathcal{P}_\alpha^*(\Pi_{\text{BDM}} \nabla \Psi - \nabla P_w \Psi)\|_{\dot{L}^1(L^2(\Omega))}
$$

$$
+ (\|\partial_t \varepsilon_h^\mu\|_{-\alpha} + \|e_{u_t}\|_{-\alpha}) \|I_h \Psi - P_0 \Psi\|_\alpha.
$$

This implies the following estimate of $\|P_0 \varepsilon_h^\mu(T)\|$;

$$
\|P_0 \varepsilon_h^\mu(T)\| \leq H_1(\Theta) (\|e_h^q\|_{\dot{L}^\infty(L^2)} + \|e_q\|_{\dot{L}^\infty(L^2)}) + H_2(\Theta) (\|\partial_t \varepsilon_h^\mu\|_{-\alpha} + \|e_{u_t}\|_{-\alpha}),
$$

(16)

where

$$
H_1(\Theta) := \sup_{\Theta \in \mathcal{C}^\infty_0(\Omega)} \left\{ \frac{\|\mathcal{P}_\alpha^*(\Pi_{\text{BDM}} \nabla \Psi - \nabla I_h \Psi)\|_{\dot{L}^1(L^2)}}{\|\Theta\|}, \frac{\|\mathcal{P}_\alpha^*(\Pi_{\text{BDM}} \nabla \Psi - \nabla P_w \Psi)\|_{\dot{L}^1(L^2(\Omega))}}{\|\Theta\|} \right\},
$$

$$
H_2(\Theta) := \sup_{\Theta \in \mathcal{C}^\infty_0(\Omega)} \frac{\|P_0 \Psi - I_h \Psi\|_\alpha}{\|\Theta\|}.
$$

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The quantity $H_1(\Theta)$ can be bounded by

$$C h \sup_{\Theta \in \mathcal{C}_0^\infty(\Omega)} \frac{\|B_\alpha^* \Psi\|_{L^1(H^2)}}{\|\Theta\|} \leq C h \sup_{\Theta \in \mathcal{C}_0^\infty(\Omega)} \frac{\|B_\alpha^* \Delta \Psi\|_{L^1(L^2)}}{\|\Theta\|} = C h \sup_{\Theta \in \mathcal{C}_0^\infty(\Omega)} \frac{\|\Psi_t\|_{L^1(L^2)}}{\|\Theta\|},$$

where, to get the inequality, we used the well-known elliptic regularity property

$$\|v\|_{H^2(\Omega)} \leq C \|\Delta v\| \quad \text{for any } v \in H^1_0(\Omega) \cap H^2(\Omega),$$

which holds for convex polyhedral domains.

The quantity $H_2(\Theta)$ can be bounded by

$$C h \sup_{\Theta \in \mathcal{C}_0^\infty(\Omega)} \frac{1}{\|\Theta\|} \left( \int_0^T \|\nabla \Psi\| \|B_\alpha^* \nabla \Psi\| \right)^{1/2}.$$

Our next task is to obtain estimates of $\int_0^T \|\Psi_t\|$ and $\int_0^T \|\nabla \Psi\| \|B_\alpha^* \nabla \Psi\|$.

### 5.2 A priori estimates for the dual solution

The estimates we need are gathered in the following result.

**Lemma 8** For any $\Theta \in H^1_0(\Omega)$ and any $\delta \in (0, T)$, we have that

$$\int_0^T \|\nabla \Psi\| \|B_\alpha^* \nabla \Psi\| \leq C \|\Theta\| \left( \ell(\delta) \|\Theta\| + \delta^{(\alpha+1)/2} \|\nabla \Theta\| \right),$$

where $\ell(\delta) = \log(T/\delta)$. The constant $C$ is independent of $\Psi, T$ and $\alpha$.

**Proof** First, we define the auxiliary function $v$: for each time $t \in [0, T]$,

$$\Delta v(t) := \mathcal{R}\Psi(t) \quad \text{in } \Omega \quad \text{and} \quad v(t)|_{\partial \Omega} = 0,$$

where $\mathcal{R}$ is the time-reversal operator for the interval $[0, T]$, that is, $\mathcal{R}\psi(t) = \psi(T-t)$.

For the moment, we assume the following properties of the function $v$:

$$t^{(1-\alpha)/2} \|\Delta v_t(t)\| + \|\nabla(\Delta v(t))\| \leq C \min\{t^{-(\alpha+1)/2} \|\Theta\|, \|\nabla \Theta\|\},$$

$$t^{-\alpha} \|v_t(t)\| + \|\Delta v(t)\| \leq C \|\Theta\|,$$

$$\int_0^T t \|\Delta v_t\|^2 \, dt \leq \frac{C}{(1+\alpha)^2} \|\Theta\|^2.$$
Using the relation \( \Delta v(t) = \mathcal{R}\Psi(t) \) and the above inequalities, we obtain

\[
(T - t)^{(1-\alpha)/2} \|\Psi_t(t)\| + \|\nabla \Psi(t)\| \leq C \min\{(T - t)^{-\alpha/2} \|\Theta\|, \|\nabla \Theta\|\},
\]

\[
\|\mathcal{R}^* \Psi(t)\| \leq C (T - t)^\alpha \|\Theta\|
\]

\[
\int_0^T (T - t) \|\Psi_t\|^2 \, dt \leq \frac{C}{(1 + \alpha)^2} \|\Theta\|^2.
\]

This implies

\[
\|\mathcal{R}^* \nabla \Psi\|^2 = -(\mathcal{R}^* \Delta \Psi, \mathcal{R}^* \Psi) = (\Psi_t, \mathcal{R}^* \Psi) \leq \|\Psi_t\| \|\mathcal{R}^* \Psi\| \leq C (T - t)^{\alpha - 1} \|\Theta\|^2.
\]

and so \( \nabla \Psi \| \|\mathcal{R}^* \nabla \Psi\| \leq C \min\{(T - t)^{-1} \|\Theta\|^2, (T - t)^{\alpha - 1/2} \|\Theta\| \|\nabla \Theta\|\}

Hence

\[
\int_0^T \|\Psi_t\| \leq \int_0^{T-\delta} \|\Psi_t\| + \int_{T-\delta}^T \|\Psi_t\|
\]

\[
\leq \sqrt{\log(T/\delta)} \left( \int_0^{T-\delta} (T - t) \|\Psi_t\|^2 \right)^{1/2} + C \int_{T-\delta}^T (T - t)^{(\alpha - 1)/2} \|\nabla \Theta\|
\]

\[
\leq C \sqrt{\log(T/\delta)} \frac{\|\Theta\|}{\alpha + 1} + C \frac{\delta^{(\alpha + 1)/2}}{\alpha + 1} \|\nabla \Theta\|,
\]

and

\[
\int_0^T \|\nabla \Psi\| \|\mathcal{R}^* \nabla \Psi\| \leq \int_0^{T-\delta} \|\nabla \Psi\| \|\mathcal{R}^* \nabla \Psi\| + \int_{T-\delta}^T \|\nabla \Psi\| \|\mathcal{R}^* \nabla \Psi\|
\]

\[
\leq C \int_0^{T-\delta} (T - t)^{-1} \|\Theta\|^2 + C \int_{T-\delta}^T (T - t)^{(\alpha - 1)/2} \|\Theta\| \|\nabla \Theta\|
\]

\[
\leq C \log(T/\delta) \|\Theta\|^2 + C \frac{\delta^{(\alpha + 1)/2}}{\alpha + 1} \|\Theta\| \|\nabla \Theta\|.
\]

Therefore, the remaining task is to show the inequalities (18), (19), and (20). Using the fact that \( \mathcal{R} \partial_t = -\partial_t \mathcal{R} \) and that \( \mathcal{R} \mathcal{R}^* = \mathcal{R}^* \mathcal{R} \), we see that

\( v_t - \mathcal{R} \Delta v(t) = 0 \) in \( \Omega \times (0, T) \), \( v = 0 \) on \( \partial \Omega \times (0, T) \), and \( v(0) = \Delta^{-1} \Theta \).

Thus, by [23, Theorems 4.1 and 4.2], (18) and (19) immediately follow. To prove inequality (20), we use the identity

\[
\int_0^T t \|\Delta v_t\|^2 \, dt = t (\Delta v_t(t), \Delta v(t)) \bigg|_0^T - \frac{1}{2} \|\Delta v(t)\|^2 \bigg|_0^T - \int_0^T t (\Delta v_{tt}, \Delta v) \, dt,
\]

and the inequalities (18) and (19), to get

\[
\int_0^T t \|\Delta v_t\|^2 \, dt \leq C \|\Theta\|^2 + \int_0^T |t (\Delta v_{tt}, \Delta v)| \, dt.
\]
It remains to estimate the second term of the right-hand side. To do that, we first note that, since the operator \(-\Delta_1\) (with homogeneous Dirichlet boundary conditions) has a complete orthonormal eigensystem \(\{\lambda_m, \phi_m\}_{m=1}^\infty (\phi_m \in H_0^1(\Omega) \text{ and } 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots )\), one may show that the solution \(v\) is given by the Duhamel formula

\[
v(t) = \sum_{m=1}^{\infty} E_\mu(-\lambda_m t^\mu)(v(0), \phi_m)\phi_m \quad \text{with } \mu = \alpha + 1,
\]

where \(E_\mu(t) := \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(\mu p + 1)}\), is the Mittag-Leffler function; see [25]. Thus,

\[
t(\Delta v_{tt}, \Delta v) = \sum_{m=1}^{\infty} G_{m,\mu}(t) (\Delta v(0), \phi_m)^2 = \sum_{m=1}^{\infty} G_{m,\mu}(t) (\Theta, \phi_m)^2,
\]

where \(G_{m,\mu}(t) := t E_\mu(-\lambda_m t^\mu) \frac{d^2}{dt^2}(E_\mu(-\lambda_m t^\mu))\). Since, by the proof of Theorems 4.1 and 4.2 in [23], we have that \(|G_{m,\mu}(t)| \leq C \min\{\lambda_m t^{\mu-1}, \lambda_m^{2} t^{-2\mu-1}\}\), we get

\[
\int_0^T |G_{m,\mu}(t)| \, dt \leq C \lambda_m \int_0^{\lambda_m^{-1/\mu}} t^{\mu-1} \, dt + C \lambda_m^{-2} \int_{\lambda_m^{-1/\mu}}^{\lambda_m^{-2\mu-1}} t^{-2\mu-1} \, dt \leq \frac{C}{(\alpha + 1)^2},
\]

and therefore,

\[
\int_0^T |t(\Delta v_{tt}, \Delta v)| \, dt \leq \sum_{m=1}^{\infty} \int_0^T |G_{m,\mu}(t)| \, dt (\Theta, \phi_m)^2 \leq \frac{C}{(\alpha + 1)^2} \|\Theta\|^2.
\]

This completes the proof. \(\square\)

5.3 Compensating for the lack of regularity of \(\Theta\)

Note that the a priori estimates of Lemma 8 do use the \(H_0^1(\Omega)\) - seminorm of \(\Theta\) whereas the bounds of the quantities \(H_t(\Theta)\) can only use its \(L^2(\Omega)\) - norm. To remedy this lack of regularity, we take advantage of the fact that \(P_0 e_h^u(T)\) lies in a finite dimensional space.

Let \(T_h'\) be a triangulation of \(\Omega\) obtained by refining each of the simplices of the triangulation \(T_h\), and let \(W_{h'}^c\) be the space of continuous functions which are polynomials of degree \(k\) on each element of \(T_h'\). Finally let \(P_{h'}\) be the \(L^2\)-projection from \(W_h\) to \(W_{h'}^c\). Then, we have the following result.

**Lemma 9** ([3, Appendix A.3]) \textit{For any triangulation \(T_h\) of \(\Omega\), we can always find a refinement \(T_{h'}\) for which we have}
∥∇P_h′θ∥ ≤ \frac{C_{k,d}}{ρ} ∥θ∥ ∀ θ ∈ W_h, and

∥ε∥ ≤ 2 \sup_{θ ∈ W_h} \frac{(ε, P_h′θ)}{∥θ∥} ∀ ε ∈ W_h.

Here the constant C_{k,d} depends solely on the polynomial degree k and the dimension d of the spatial domain Ω, and ρ := min_{K ∈ Th} ρ_K where ρ_K denotes the radius of the largest ball included in the simplex K.

Roughly speaking, the second inequality gives us an alternative manner to estimate the $L^2(Ω)$-norm of ε := P_0ε u_h(T). Indeed, it allows us to take Θ1 of the form P_h′θ only. The first inequality takes care of the lack of smoothness of Θ1 but at the price of the appearance of the factor ρ in the denominator. We can now modify the a priori inequalities of Lemma 8 as follows.

Lemma 10 Let (Φ, Ψ) be the solution of the dual problem with Θ := P_h′θ where θ ∈ W_h and P_h′ satisfies Lemma 9. Then

\[ \int_0^T ∥Ψ_t∥ ≤ \frac{C}{α + 1} \sqrt{\log κ} ∥θ∥ \] and

\[ \int_0^T ∥∇Ψ∥∥B^α∇Ψ∥ ≤ \frac{C}{α + 1} \log κ ∥θ∥^2, \]

where, κ > 1 is the solution of $κ^{α+1} \log κ = C_{k,d}^2 T^{α+1}/ρ^2$. Here ρ := min_{K ∈ Th} ρ_K and ρ_K denotes the radius of the largest ball included in the simplex K.

Proof We prove the first estimate; the proof of the second is almost identical. From the first inequality of Lemma 8 with Θ := P_h′θ, the fact that P_h′ is an $L^2$-projection, and the first inequality of Lemma 9, we obtain

\[ \int_0^T ∥Ψ_t∥ ≤ \frac{C}{α + 1} \left( \sqrt{\log(T/δ)} + δ^{(α+1)/2} \frac{C_{k,d}}{ρ} \right) ∥θ∥ = \frac{2C}{α + 1} \sqrt{\log(κ)} ∥θ∥, \]

if we take δ := T/κ and use the definition of κ. This completes the proof. □

5.4 The estimate of the postprocessed approximation

We can now insert the estimates of the previous corollary in the first estimate of ∥P_0ε^u_h(T)∥, (16), to obtain the superconvergence estimate we sought. Note that, since Ω is convex, we can use the elliptic regularity inequality (17).

Theorem 3 Assume that $u ∈ C^1(0, T; H^{k+2}(Ω))$ and $q ∈ C^1(0, T; H^{k+1}(Ω))$. Assume also that τ_K^* and 1/τ_K^max are bounded by C. Then, for k ≥ 1, we have that

\[ ∥(u - u^h_h)(T)∥ ≤ C_3 \sqrt{\log κ} h^{k+2}. \]

where the constant C_3, only depends on C, α, $∥u∥_{C^1(H^{k+2})}$, and on $∥q∥_{C^1(H^{k+1})}$.  

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Let us relate $\kappa$ to $T$ and the maximum diameter of the simplexes of the mesh, $h$.
For $\log \kappa > 1$,
\[
\kappa^{\alpha+1} < \frac{\log \kappa}{T^{\alpha+1}/\rho^2} \leq C \frac{C_{k,d}^2 T^{\alpha+1}/h^2}{\kappa^{\alpha+1}},
\]
when the mesh is quasi-uniform. We then easily see that $\log \kappa < C \log (Th^{-2/(\alpha+1)})$ for $\log \kappa > 1$. Therefore, $\sqrt{\log \kappa} \leq \max \{ 1, C \sqrt{\log (Th^{-2/(\alpha+1)})} \}$.

**Proof** From the first estimate of $\| P_0 \varepsilon_h^u(T) \|$, (16), we have that
\[
\| P_0 \varepsilon_h^u(T) \| \leq H_1(\Theta) \left( \| \varepsilon_h^q \|_{L^\infty(L^2)} + \| e_q \|_{L^\infty(L^2)} \right) + H_2(\Theta) \left( \| \partial_t \varepsilon_h^u \|_{-\alpha} + \| e_{ut} \|_{-\alpha} \right)
\]
\[
\leq C h \frac{\sqrt{\log \kappa}}{\alpha+1} \left( \| \varepsilon_h^q \|_{L^\infty(L^2)} + \| e_q \|_{L^\infty(L^2)} + \| \partial_t \varepsilon_h^u \|_{-\alpha} + \| e_{ut} \|_{-\alpha} \right) \| \theta \|,
\]
by the estimates of the dual solution of the previous lemma. Using these estimates in (14), we obtain
\[
\| u - u_h^\star \| \leq C h^{k+2} \| u \|_{H^{k+2}(\Omega)} + C h \frac{\sqrt{\log \kappa}}{\alpha+1} \left( \| \varepsilon_h^q \|_{L^\infty(L^2)} + \| e_q \|_{L^\infty(L^2)} \right)
\]
\[
+ \| \partial_t \varepsilon_h^u \|_{-\alpha} + \| e_{ut} \|_{-\alpha} + C h \| \varepsilon_h^q \|_{L^\infty(L^2)}.
\]
The result now follows by using the error estimates of Theorem 2. \qed

**6 Summary and concluding remarks**

We have carried out the a priori error analysis of a semi-discrete HDG method for the spatial discretization to problem (1). Assuming that the exact solution is sufficiently regular, we proved optimal error estimates of the approximations to $u$ in the $L^\infty(0, T; L^2(\Omega))$-norm and to $-\nabla u$ in the $L^\infty(0, T; L^2(\Omega))$-norm over a regular triangular meshes. Moreover, for quasi-uniform meshes, by a simple elementwise post-processing, we obtained a faster approximation for $u$ with a superconvergence rate. All the results obtained in this paper can be extended almost word-by-word to other superconvergent HDG methods as well as to the mixed methods that fit the general formulation of the HDG methods; see [8].

The devising of time-space fully discrete DG methods able to deal in an efficient manner with the memory term constitutes the subject of ongoing research.

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