A GENERALIZED SCHWARZ LEMMA FOR TWO DOMAINS RELATED TO \( \mu \)-SYNTHESIS

SOURAV PAL AND SAMRIDDHO ROY

Abstract. We present a set of necessary and sufficient conditions that provides a Schwarz lemma for the tetrablock \( \mathcal{E} \). As an application of this result, we obtain a Schwarz lemma for the symmetrized bidisc \( \mathcal{G}_2 \). In either case, our results generalize all previous results in this direction for \( \mathcal{E} \) and \( \mathcal{G}_2 \).

1. Introduction

The aim of this article is to prove an explicit Schwarz lemma for two polynomially convex domains related to \( \mu \)-synthesis, the symmetrized bidisc \( \mathcal{G}_2 \) and the tetrablock \( \mathcal{E} \), defined as

\[
\mathcal{G}_2 = \{ (z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D} \}, \\
\mathcal{E} = \{ (x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zwx_3 \neq 0, z, w \in \mathbb{D} \}.
\]

The classical Schwarz lemma gives a necessary and sufficient condition for the solvability of a two-point interpolation problem for analytic functions from the open unit disc \( \mathbb{D} \) to itself, and describes the extremal functions. It has substantial generalizations in which the two copies of \( \mathbb{D} \) are replaced by various other domains \([9]\), typically either homogeneous or convex. In this paper, we prove a sharp Schwarz lemma for two domains which are neither homogeneous nor convex. We believe that our results will throw new lights on the spectral Nevanlinna-Pick problem, which is to interpolate from the unit disc to the set of \( k \times k \) matrices of spectral radius no greater than 1 by analytic matrix functions.

We first produce several independent necessary and sufficient conditions under which there exists an analytic function from \( \mathbb{D} \) to \( \mathcal{E} \).
that solves a two-point Nevanlinna-Pick interpolation problem for the tetrablock. This, in a way, generalizes the previous result of Abouhajar, White and Young in this direction, [1]. We mention here that the domain tetrablock was introduced by these three mathematicians in [1] and that this domain has deep connection with the most appealing and difficult problem of $\mu$-synthesis (see [5] for further details). Since then the tetrablock has attracted considerable attention [6, 11, 10, 15, 16, 5, 17, 12]. So, our first main result is the following Schwarz lemma for the tetrablock.

**Theorem 1.1.** Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$ and let $x = (a, b, p) \in \mathbb{E}$. Then the following conditions are equivalent:

1. there exists an analytic function $\varphi : \mathbb{D} \to \mathbb{E}$ such that $\varphi(0) = (0, 0, 0)$ and $\varphi(\lambda_0) = x$;
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2. 
$$\max \left\{ \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2}, \frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \right\} \leq |\lambda_0|;$$
3. either $|b| \leq |a|$ and 
$$\frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \leq |\lambda_0|$$
or $|a| \leq |b|$ and 
$$\frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \leq |\lambda_0|;$$
4. there exists a $2 \times 2$ function $F$ in the Schur class such that 
$$F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \text{ and } F(\lambda_0) = A = [a_{ij}]$$
where $x = (a_{11}, a_{22}, \det A)$.
5. either $|b| \leq |a|$ and $\|\Psi(., x)\|_{H^\infty} \leq |\lambda_0|$ 
or $|a| \leq |b|$ and $\|\Upsilon(., x)\|_{H^\infty} \leq |\lambda_0|$;
6. either $|b| \leq |a|$ and 
$$|a|^2 - |\lambda_0|^2 |b|^2 + |p|^2 + 2|\lambda_0|^2 b - \bar{a}p| \leq |\lambda_0|^2$$
or $|a| \leq |b|$ and 
$$|b|^2 - |\lambda_0|^2 |a|^2 + |p|^2 + 2|\lambda_0|^2 a - \bar{b}p| \leq |\lambda_0|^2;$$
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(7) either \( \lambda_0 - az - b\lambda_0 w + pzw \neq 0 \) for all \( z, w \in \mathbb{D} \) and \( |b| \leq |a| \)

or \( \lambda_0 - a\lambda_0 z - bw + pzw \neq 0 \) for all \( z, w \in \mathbb{D} \) and \( |a| \leq |b| \);

(8) either \(|b| \leq |a|\) and

\[
|\lambda_0| |a - \bar{b}p| + ||\lambda_0|^2 b - \bar{a}p| + |p|^2 \leq |\lambda_0|^2
\]

or \(|a| \leq |b|\) and

\[
|\lambda_0| |b - \bar{a}p| + ||\lambda_0|^2 a - \bar{b}p| + |p|^2 \leq |\lambda_0|^2;
\]

(9) either \(|b| \leq |a|\), \(|p| \leq |\lambda_0|\) and

\[
|a|^2 + |\lambda_0 b|^2 - |p|^2 + 2|\lambda_0| |ab - p| \leq |\lambda_0|^2
\]

or \(|a| \leq |b|\), \(|p| \leq |\lambda_0|\) and

\[
|b|^2 + |\lambda_0 a|^2 - |p|^2 + 2|\lambda_0| |ab - p| \leq |\lambda_0|^2
\]

(10) either \(|b| \leq |a|\), \(|p| \leq |\lambda_0|\) and there exist \( \beta_1, \beta_2 \in \mathbb{C} \) with \( |\beta_1| + |\beta_2| \leq 1 \) such that

\[
a = \beta_1 \lambda_0 + \bar{\beta}_2 p \quad \text{and} \quad b\lambda_0 = \beta_2 \lambda_0 + \bar{\beta}_1 p
\]

or \(|a| \leq |b|\), \(|p| \leq |\lambda_0|\) and there exist \( \beta_1, \beta_2 \in \mathbb{C} \) with \( |\beta_1| + |\beta_2| \leq 1 \) such that

\[
a\lambda_0 = \beta_1 \lambda_0 + \bar{\beta}_2 p \quad \text{and} \quad b = \beta_2 \lambda_0 + \bar{\beta}_1 p.
\]

The proof of the theorem is given in Section 3. The functions \( \Psi \) and \( \Upsilon \) play major role here. These two functions were introduced in [1] and we shall briefly describe them in Section 2. Being armed with the Schwarz lemma for the tetrablock, we apply the result to the symmetrized bidisc to obtain a generalized Schwarz lemma for \( G_2 \), which is another main result of this paper and is presented as Theorem 4.4 in Section 4. This is possible because of the underlying relationship between the two domains \( \mathbb{E} \) and \( G_2 \) which has been described in Lemma 4.1 and Theorem 4.3 in Section 4. This result generalizes the Schwarz lemma obtained by Agler, Young in [3] and also independently by Nokrane and Ransford in [13].

2. Background material

We recall from [1] the following two rational functions \( \Psi, \Upsilon \) which play central role in the study of complex geometry of \( \mathbb{E} \).
Definition 2.1. For \( z \in \mathbb{C} \) and \( x = (x_1, x_2, x_3) \in \mathbb{C}^3 \) the functions \( \Psi \) and \( \Upsilon \) are defined as

\[
\Psi(z, x) = \frac{x_3z - x_1}{x_2z - 1},
\]

\( (1) \)

\[
\Upsilon(z, x) = \Psi(z, (x_2, x_1, x_3)) = \frac{x_3z - x_2}{x_1z - 1}.
\]

\( (2) \)

Also we define

\[
D(x) = \sup_{z \in \mathbb{D}} |\Psi(z, x)| = ||\Psi(., x)||_{H^\infty}.
\]

\( (3) \)

We interpret \( \Psi(., x) \) to be the constant function equal to \( x_1 \) in the event that \( x_1x_2 = x_3 \); thus \( \Psi(z, x) \) is defined when \( zx_2 \neq 1 \) or \( x_1x_2 = x_3 \). The quantity \( D(x) \) is finite (and \( \Psi(., x) \in H^\infty \)) if and only if either \( x_2 \in \mathbb{D} \) or \( x_1x_2 = x_3 \). Indeed, for \( x_2 \in \mathbb{D} \), the linear fractional function \( \Psi(., x) \) maps \( \mathbb{D} \) to the open disc with centre and radius

\[
\frac{x_1 - x_2x_3}{1 - |x_2|^2}, \quad \frac{|x_1x_2 - x_3|}{1 - |x_2|^2}
\]

\( (4) \)

respectively. Hence

\[
D(x) = \begin{cases} 
\frac{|x_1 - x_2x_3| + |x_1x_2 - x_3|}{1 - |x_2|^2} & \text{if } |x_2| < 1 \\
|x_1| & \text{if } x_1x_2 = x_3 \\
\infty & \text{otherwise}.
\end{cases}
\]

\( (5) \)

Similarly, if \( x_1 \in \mathbb{D} \), \( \Upsilon(., x) \) maps \( \mathbb{D} \) to the open disc with centre and radius

\[
\frac{x_2 - x_1x_3}{1 - |x_1|^2}, \quad \frac{|x_1x_2 - x_3|}{1 - |x_1|^2}
\]

\( (6) \)

respectively. It was shown in [1] that the closure \( \overline{E} \) of the tetrablock is the following set

\[
E = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zwx_3 \neq 0 \text{ whenever } |z| < 1, |w| < 1\}.
\]

We interpret \( \Psi(., x) \) to be the constant function equal to \( x_1 \) in the event that \( x_1x_2 = x_3 \); thus \( \Psi(z, x) \) is defined when \( zx_2 \neq 1 \) or \( x_1x_2 = x_3 \). The quantity \( D(x) \) is finite (and \( \Psi(., x) \in H^\infty \)) if and only if either \( x_2 \in \mathbb{D} \) or \( x_1x_2 = x_3 \). Indeed, for \( x_2 \in \mathbb{D} \), the linear fractional function \( \Psi(., x) \) maps \( \mathbb{D} \) to the open disc with centre and radius

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\frac{x_1 - x_2x_3}{1 - |x_2|^2}, \quad \frac{|x_1x_2 - x_3|}{1 - |x_2|^2}
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respectively. Hence

\[
D(x) = \begin{cases} 
\frac{|x_1 - x_2x_3| + |x_1x_2 - x_3|}{1 - |x_2|^2} & \text{if } |x_2| < 1 \\
|x_1| & \text{if } x_1x_2 = x_3 \\
\infty & \text{otherwise}.
\end{cases}
\]

\( (5) \)

Similarly, if \( x_1 \in \mathbb{D} \), \( \Upsilon(., x) \) maps \( \mathbb{D} \) to the open disc with centre and radius

\[
\frac{x_2 - x_1x_3}{1 - |x_1|^2}, \quad \frac{|x_1x_2 - x_3|}{1 - |x_1|^2}
\]

\( (6) \)

respectively. It was shown in [1] that the closure \( \overline{E} \) of the tetrablock is the following set

\[
E = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zwx_3 \neq 0 \text{ whenever } |z| < 1, |w| < 1\}.
\]

In [1], the points in the sets \( E \) and its closure \( \overline{E} \) are characterized in the following way.

**Theorem 2.2** (Abouhajar, White, Young). For \( x = (x_1, x_2, x_3) \) in \( \mathbb{C}^3 \), the following are equivalent.

1. \( x \in E \) (respectively \( x \in \overline{E} \)),
2. \( ||\Psi(., x)||_{H^\infty} < 1 \) (respectively \( \leq 1 \)),
3. \( ||\Upsilon(., x)||_{H^\infty} < 1 \) (respectively \( \leq 1 \)),
shall prove here the rest parts.

(3) $|x_1 - x_2 x_3| + |x_1 x_2 - x_3| < 1 - |x_2|^2$ (respectively $\leq 1 - |x_2|^2$),
(3') $|x_2 - \bar{x}_1 x_3| + |x_1 x_2 - x_3| < 1 - |x_1|^2$ (respectively $\leq 1 - |x_1|^2$),
(4) $|x_1|^2 - |x_2|^2 + |x_3|^2 + 2 x_2 - \bar{x}_1 x_3| < 1$ (respectively $\leq 1$) and $|x_2| < 1$ (respectively $\leq 1$),
(4') $-|x_1|^2 + |x_2|^2 + |x_3|^2 + 2 |x_1 - x_2 x_3| < 1$ (respectively $\leq 1$) and $|x_1| < 1$ (respectively $\leq 1$),
(5) $|x_1|^2 + |x_2|^2 - |x_3|^2 + 2 x_1 x_2 - x_3| < 1$ (respectively $\leq 1$) and $|x_3| < 1$ (respectively $\leq 1$),
(6) $|x_1 - x_2 x_3| + |x_2 - \bar{x}_1 x_3| < 1 - |x_3|^2$ (respectively $\leq 1 - |x_3|^2$),
(7) there exists a matrix $A = (a_{i,j}) \in \mathbb{M}_2$ with $|A| < 1$ (respectively $\leq 1$) such that $x = (a_{11}, a_{22}, \text{det } A)$,
(8) there exists a symmetric matrix $A = (a_{i,j}) \in \mathbb{M}_2$ with $|A| < 1$ (respectively $\leq 1$) such that $x = (a_{11}, a_{22}, \text{det } A)$,
(9) $|x_3| < 1$ (respectively $\leq 1$) and there are complex numbers $\beta_1$ and $\beta_2$ with $|\beta_1| + |\beta_2| < 1$ (respectively $\leq 1$) such that
$$x_1 = \beta_1 + \beta_2 x_3 \text{ and } x_2 = \beta_2 + \beta_1 x_3.$$  

3. Proof of Theorem 1

The equivalence of (1) – (4) was established in Theorem 1.2 of [1]. We shall prove here the rest parts.

(3) $\iff$ (5). Condition (3) can be written as

either $\frac{|a - \bar{b} p| + |a b - p|}{1 - |b|^2} \leq |\lambda_0|$

or $\frac{|b - \bar{a} p| + |a b - p|}{1 - |a|^2} \leq |\lambda_0|.$

Since

$$\|\Psi(., x)\|_{H^\infty} = \frac{|a - \bar{b} p| + |a b - p|}{1 - |b|^2} \quad \text{and} \quad \|\Upsilon(., x)\|_{H^\infty} = \frac{|b - \bar{a} p| + |a b - p|}{1 - |a|^2}$$

we have that condition (3) is equivalent to condition (5).

(5) $\iff$ (6). From the definition of $\Psi(., x)$ (see [1]), it is evident by an application of the Maximum Modulus Principle that $\|\Psi(., x)\|_{H^\infty} \leq |\lambda_0|$ holds if and only if

$$\frac{|p z - a|}{|b z - 1|} \leq |\lambda_0| \quad \text{for all } z \in \mathbb{T}.$$
Now for all \( z \in T \),
\[
\frac{|pz - a|}{|bz - 1|} \leq |\lambda_0|
\]
\( \iff |pz - a|^2 \leq |\lambda_0|^2|bz - 1|^2 \)
\( \iff |pz|^2 + |a|^2 - 2 \text{Re}(\bar{a}pz) \leq |\lambda_0|^2(|bz|^2 + 1 - 2 \text{Re}(bz)) \)
\( \iff |a|^2 - |\lambda_0|^2|b|^2 + |p|^2 - |\lambda_0|^2 + 2 \text{Re} \left( z(|\lambda_0|^2b - \bar{a}p) \right) \leq 0 \)
\( \iff |a|^2 - |\lambda_0|^2|b|^2 + |p|^2 - |\lambda_0|^2 + 2 \left| |\lambda_0|^2b - \bar{a}p \right| \leq 0. \)

We obtained the last step by using the fact that \( |x| < k \iff \text{Re}(zx) < k \), for all \( z \in T \).

Similarly we obtain
\[
\|\Upsilon(.,x)\|_{H^\infty} \leq |\lambda_0|
\]
\( \iff \frac{|pz - b|}{|az - 1|} \leq |\lambda_0| \quad \text{for all } z \in T \)
\( \iff |b|^2 - |\lambda_0|^2|a|^2 + |p|^2 - |\lambda_0|^2 + 2 \left| |\lambda_0|^2a - \bar{b}p \right| \leq 0. \)

So, \((5) \iff (6)\) is evident now.

\((5) \iff (7)\). It is evident from part-(2) of Theorem 2.2 that
\[
\|\Psi(.,x)\|_{H^\infty} \leq |\lambda_0|
\]
\( \iff \frac{|pz - a|}{|bz - 1|} \leq |\lambda_0| \quad \text{for all } z \in T \)
\( \iff \frac{|zp/\lambda_0 - a/\lambda_0|}{|bz - 1|} \leq 1 \quad \text{for all } z \in T \)
\( \iff (a/\lambda_0, b, p/\lambda_0) \in \overline{E} \)
\( \iff \lambda_0 - az - b\lambda_0 w + pzw \neq 0 \quad \text{for all } z, w \in \mathbb{D}. \)

Similarly from part-(2’) of Theorem 2.2 we can conclude that
\( \|\Upsilon(.,x)\|_{H^\infty} \leq |\lambda_0| \iff \lambda_0 - a\lambda_0 z - bw + pzw \neq 0 \quad \text{for all } z, w \in \mathbb{D}. \)

\((7) \iff (8)\). It is evident from the equation \((6)\) in Section 2 that \((7)\) holds if and only if \((a/\lambda_0, b, p/\lambda_0) \in \overline{E} \) or \((a, b/\lambda_0, p/\lambda_0) \in \overline{E} \). Part-(6) of Theorem 2.2 tells us that
\[
(a/\lambda_0, b, p/\lambda_0) \in \overline{E} \iff |\lambda_0||a - \bar{b}p| + ||\lambda_0|^2b - \bar{a}p| + |p|^2 \leq |\lambda_0|^2. \]
Also by part-(6) of Theorem 2.2
\[(a, b/\lambda_0, p/\lambda_0) \in \overline{E} \iff |\lambda_0||b - \bar{a}p| + ||\lambda_0|^2 a - \bar{b}p| + |p|^2 \leq |\lambda_0|^2.\]
Therefore, (7) ⇔ (8) holds.

(7) ⇔ (9). We apply part-(5) of Theorem 2.2 to get
\[
\lambda_0 - az - b\lambda_0 w + pzw \neq 0, \quad \text{for all } z, w \in \mathbb{D}
\]
⇔ \[(a/\lambda_0, b, p/\lambda_0) \in \overline{E}
\]
⇔ \[|a|^2 + |\lambda_0 b|^2 - |p|^2 + 2|\lambda_0||ab - p| \leq |\lambda_0|^2.\]
Similarly one obtains
\[
\lambda_0 - a\lambda_0 z - bw + pzw \neq 0, \quad \text{for all } z, w \in \mathbb{D}
\]
⇔ \[(a, b/\lambda_0, p/\lambda_0) \in \overline{E}
\]
⇔ \[|b|^2 + |\lambda_0 a|^2 - |p|^2 + 2|\lambda_0||ab - p| \leq |\lambda_0|^2.\]
Therefore, (7) is equivalent to (9).

(7) ⇔ (10). We again mention that (7) holds if and only if \((a/\lambda_0, b, p/\lambda_0) \in \overline{E}\) or \((a, b/\lambda_0, p/\lambda_0) \in \overline{E}\). So, by part-(9) of Theorem 2.2, 7 holds if and only if either \(|p/\lambda_0| \leq 1\) and there exist \(\beta_1, \beta_2 \in \mathbb{C}\) with \(|\beta_1| + |\beta_2| \leq 1\) such that
\[a/\lambda_0 = \beta_1 + \bar{\beta}_2(p/\lambda_0) \text{ and } b = \beta_2 + \bar{\beta}_1(p/\lambda_0)\]
which is same as saying that \(|p| \leq |\lambda_0|\) and
\[a = \beta_1 \lambda_0 + \bar{\beta}_2 p \text{ and } b\lambda_0 = \beta_2 \lambda_0 + \bar{\beta}_1 p;\]
or, \(|p/\lambda_0| \leq 1\) and there exist \(\beta_1, \beta_2 \in \mathbb{C}\) with \(|\beta_1| + |\beta_2| \leq 1\) such that
\[a = \beta_1 + \bar{\beta}_2(p/\lambda_0) \text{ and } b/\lambda_0 = \beta_2 + \bar{\beta}_1(p/\lambda_0)\]
which is equivalent to saying that \(|p| \leq |\lambda_0|\) and
\[a\lambda_0 = \beta_1 \lambda_0 + \bar{\beta}_2 p \text{ and } b = \beta_2 \lambda_0 + \bar{\beta}_1 p.\]

4. APPLICATION TO THE SYMMETRIZED BIDISC

The symmetrized bidisc \(\mathbb{G}_2\) was introduced by a group of control theorists and later have been extensively studied by the complex analysts, geometers and operator theorists during past three decades. We list here a few of the numerous references about the symmetrized bidisc for the readers, \([3, 4, 7, 8, 14, 15, 18]\). This is another domain which has root in the \(\mu\)-synthesis problem. The following result provides a beautiful connection between the two domains \(E\) and \(\mathbb{G}_2\).
Lemma 4.1 (Lemma 3.2, [6]). Let \((x_1, x_2, x_3) \in \mathbb{C}^3\). Then \((x_1, x_2, x_3) \in \mathbb{E}\) if and only if \((x_1 + zx_2, zx_3) \in \mathbb{G}_2\) for all \(z \in \mathbb{T}\).

Among the various characterizations of the points of \(\mathbb{G}_2\) (see Theorem 1.1, [4]), the following is very elegant.

Theorem 4.2 (Theorem 1.1, [4]). Let \((s, p) \in \mathbb{C}^2\). Then \((s, p) \in \mathbb{G}_2\) if and only if \(|p| < 1\) and there exists \(\beta \in \mathbb{C}\) with \(|\beta| < 1\) such that
\[
s = \beta + \bar{\beta}p.
\]

The following result will play central role in determining the interpolating function in the Schwarz lemma for the symmetrized bidisc.

Theorem 4.3. The symmetrized bidisc \(\mathbb{G}_2\) can be analytically embedded inside the tetrablock via the following map
\[
f : \mathbb{G}_2 \to \mathbb{E} \\
(s, p) \mapsto \left(\frac{s}{2}, \frac{s}{2}, p\right).
\]

Conversely, the map
\[
g : \mathbb{E} \to \mathbb{G}_2 \\
(a, b, p) \mapsto (a + b, p)
\]
maps the tetrablock analytically onto the symmetrized bidisc and when restricted to \(f(\mathbb{G}_2)\), becomes the inverse of the map \(f\). Moreover, both \(f\) and \(g\) map the origin to the origin.

Proof. The map
\[
f : \mathbb{G}_2 \to \mathbb{E} \\
(s, p) \mapsto \left(\frac{s}{2}, \frac{s}{2}, p\right),
\]
is clearly a one-one and holomorphic map. All we need to show is that it maps \(\mathbb{G}_2\) into the tetrablock. Let \((s, p) \in \mathbb{G}_2\). Then by Theorem 1.2
\[
s = \beta + \bar{\beta}p,
\]
where
\[
\beta = \frac{s - \bar{sp}}{1 - |p|^2}\quad\text{and}\quad|\beta| < 1.
\]

Now
\[
\frac{s}{2} = \frac{\beta}{2} + \frac{\bar{\beta}}{2}p
\]

Together with \(|\frac{\beta}{2}| + |\frac{\bar{\beta}}{2}| < 1\) imply by part-(9) of Theorem 2.2 that \(\left(\frac{s}{2}, \frac{s}{2}, p\right) \in \mathbb{E}\). Hence we are done.
Conversely, the function \( g \) maps \( \mathbb{E} \) to \( \mathbb{G}_2 \) by Lemma 4.1 and is evidently holomorphic. Now
\[
f(G) = \{(a, b, p) \in \mathbb{E} : a = b\}.
\]
Because, if \( (a, a, p) \in \mathbb{E} \), then by Lemma 4.1, \((2a, p) \in \mathbb{G}_2\) and by the previous part \((a, a, p) = f(2a, p)\). This also proves that \( g \mid_{f(\mathbb{G}_2)} \) is the inverse of \( f \). Needless to prove that both \( f \) and \( g \) map the origin to the origin.

\[\textbf{4.1. A generalized Schwarz lemma for the symmetrized bidisc.}\]

In this subsection, we apply Theorem 1.1 to obtain a Schwarz lemma for the symmetrized bidisc. This result generalizes the previous results of Agler, Young and Nokrane and Ransford [3, 13]. We transfer the results of the tetrablock to the symmetrized bidisc via the maps \( f \) and \( g \) and obtain the following (generalized) Schwarz lemma for the symmetrized bidisc.

**Theorem 4.4.** Let \( \lambda_0 \in \mathbb{D}\backslash \{0\} \) and let \((s, p) \in \mathbb{G}_2\). Then the following are equivalent:

1. there exists an analytic function \( \psi : \mathbb{D} \to \mathbb{G}_2 \) such that \( \varphi(0) = (0, 0) \) and \( \psi(\lambda_0) = (s, p) \);
2. there exists an analytic function \( \psi : \mathbb{D} \to \mathbb{G}_2 \) such that \( \varphi(0) = (0, 0) \) and \( \psi(\lambda_0) = (s, p) \);
3. \[
\frac{2|s - \bar{s}p| + |s^2 - 4p|}{4 - |s|^2} \leq |\lambda_0|;
\]
4. \[
\| \Psi(., (\frac{s}{2}, \frac{s}{2}, p)) \|_{H^\infty} \leq |\lambda_0|;
\]
5. \[
(1 - |\lambda_0|^2)|s|^2 + 4|p|^2 + 4|\lambda_0|^2 s - \bar{s}p| \leq 4|\lambda_0|^2;
\]
6. \[
2\lambda_0 - (z + \lambda_0 w)s + 2pz \neq 0 \text{ for all } z, w \in \mathbb{D};
\]
7. \[
|\lambda_0||s - \bar{s}p| + |\lambda_0|^2|s - \bar{s}p| + 2|p|^2 \leq 2|\lambda_0|^2;
\]
8. there exists a \( 2 \times 2 \) function \( F \) in the Schur class such that \( F(0) = \begin{bmatrix} 0 & \ast \\ 0 & 0 \end{bmatrix} \) and \( F(\lambda_0) = A = [a_{ij}] \), where \( a_{11} = a_{22} = \frac{s}{2} \) and \( p = \det A \).
$|p| \leq |\lambda_0|$ and
\[(1 + |\lambda_0|^2)|s|^2 - 4|p|^2 + 2|\lambda_0||s|^2 - 4p| \leq 4|\lambda_0|^2;\]

(9) $|p| \leq |\lambda_0|$ and there exist $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ such that

either $s = 2\beta\lambda_0 + 2\bar{\beta}p$ or $\lambda_0 s = 2\beta\lambda_0 + 2\bar{\beta}p$.

Proof. The proof follows easily from Theorem 4.3 by applying the functions $f$ and $g$. For $(s, p) \in G_2$, we have $f((s, p)) = (\frac{s}{2}, \frac{s}{2}, p) \in \mathbb{E}$ and Theorem 1.1 guarantees the existence of an analytic function $\phi$ from $\mathbb{D}$ to $\mathbb{E}$ or to $\overline{\mathbb{E}}$ that maps 0 to 0 and $\lambda_0$ to $(\frac{s}{2}, \frac{s}{2}, p)$, when the conditions (2) – (10) are satisfied with $a = b = \frac{s}{2}$. The conditions (2) – (10) of Theorem 1.1 when $a = b = \frac{s}{2}$, have been represented as conditions (2) – (9) in this theorem. Hence under the conditions (2) – (9), we obtain the analytic function $\psi = g \circ \phi$ that maps 0 to 0 and $\lambda_0$ to $(s, p)$.

The converse is also true. That is, if there is a function $\phi$ from $\mathbb{D}$ to $G_2$ such that $\phi(0) = 0$ and $\phi(\lambda_0) = (s, p)$, then $f \circ \phi : \mathbb{D} \to \mathbb{E}$ maps 0 to 0 and $\lambda_0$ to $(\frac{s}{2}, \frac{s}{2}, p)$. So, the conditions (2) – (10) of Theorem 1.1 are satisfied with $a = b = \frac{s}{2}$, which is same as saying that the conditions (2) – (9) hold. Hence the proof is complete.

\[\blacksquare\]

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(Sourav Pal) **MATHEMATICS DEPARTMENT, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI - 400076, INDIA.**

*E-mail address: sourav@math.iitb.ac.in*

(Samriddho Roy) **MATHEMATICS DEPARTMENT, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI - 400076, INDIA.**

*E-mail address: sroy@math.iitb.ac.in*