Localization for Random Operators with Non-Monotone Potentials with Exponentially Decaying Correlations

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Abstract. I consider random Schrödinger operators with exponentially decaying single site potential, which is allowed to change sign. For this model, I prove Anderson localization both in the sense of exponentially decaying eigenfunctions and dynamical localization. Furthermore, the results imply a Wegner-type estimate strong enough to use in classical forms of multi-scale analysis.

1. Introduction

In [3], Anderson proposed that randomness of the potential leads to localization phenomena in the Schrödinger equation. In [27], Fröhlich and Spencer laid the foundations of multi-scale analysis to give a mathematical justification of this phenomenon of Anderson localization. Multi-scale analysis was then improved by a sequence of people notably von Dreifus and Klein in [24] and Germinet and Klein [30]. These forms of multi-scale analysis were used to prove pure point spectrum with exponentially decaying eigenfunctions. For recent expositions of multi-scale analysis, see the book [45] by Stollmann, the lecture notes [35] by Kirsch, and the review [37] by Klein.

As pointed out by del Rio, Jitomirskaya, Last, and Simon in [23] this is not enough to conclude dynamical properties. This was first shown using the fractional moments method by Aizenman and Molchanov in [2]. Later methods using multi-scale analysis were developed by de Biévre and Germinet in [7], Damanik and Stollmann in [22], and Germinet and Klein [31].

All these proofs rely on an explicit a priori bound on the concentration of eigenvalues of the Schrödinger operator restricted to a finite box known as a Wegner estimate. For one-dimensional Schrödinger operators various techniques not relying on a Wegner estimate exist. This is mainly due to transfer matrices, see for example Carmona, Klein, and Martinelli [18], Jitomirskaya [34], Bourgain and Goldstein [13], and Damianik, Sims, and Stolz [21]. A form of multi-scale analysis for multi-dimensional Schrödinger operators without a Wegner estimate was first developed in the context of quasi-periodic Schrödinger operators by Bourgain, Goldstein, and Schlag in [14] and then improved by Bourgain in [8, 9, 11].
It was then applied by Bourgain [10] and more importantly Bourgain and Kenig in [16] to prove Anderson localization for Bernoulli-Anderson models. It was then extended by Aizenman, Germinet, Klein, and Warzel in [1] to arbitrary distributions. Then used by Germinet, Hislop, and Klein [28] for the Poisson random potential, and by Bourgain in [12] to consider certain models with vector-valued potentials. The methods for Bernoulli potentials rely on a combinatorial fact known as Sperner’s lemma and unique continuation properties of the Laplace operator. Both in [12] and this paper, analyticity properties and smooth distributions will be considered, and a certain fact about analytic functions, Cartan’s lemma, will play a key-role. This is similar to the techniques used for quasi-periodic operators.

My goal in this paper is to continue to develop the methods not relying on a Wegner estimate. I will consider certain non-monotone random models with long range correlations. There has been previous work on this by Klopp in [38] and [39], which imply localization in extreme energy regions. Models with long range correlations were considered for example by Kirsch, Stollmann, and Stolz in [36] and non-monotonous models by Elgart, Tautenhahn, and Veselic [26], Tautenhahn and Veselic [46], and Veselic [48].

Let me also point of the work by Baker, Loss, and Stolz in [5], [6] on the random displacement model. This is a model for continuum Schrödinger operators, which exhibits non-monotonic behavior.

The proof will be much in the spirit of the multi-scale analysis of von Dreifus and Klein in [24]. However, I will use the method of Bourgain from [12] based on analyticity of the potential in order to obtain Wegner estimates. Furthermore, in difference to the results of [10] and [12], the results of this paper will imply a Wegner estimate, which is strong enough to start the multi-scale analysis of Damanik and Stollmann [22] or Stollmann [45], which would imply dynamical localization and pure point spectrum with exponentially decaying eigenfunctions.

However, I have included a proof of dynamical localization at the end of the paper. This is mainly done to show that the single energy estimates of this paper are good enough to conclude it. For a discussion how the single energy estimate implies exponential decay of the eigenfunctions, I refer the reader to the work of Bourgain and Kenig [10].

2. Statement of the results

Before introducing the general assumptions on the potential $V$, I will introduce the class of alloy-type potentials which will serve as an example. Let the single-site potential $\varphi : \mathbb{Z}^d \rightarrow \mathbb{R}$ satisfy

\begin{equation}
\varphi(0) \neq 0, \quad |\varphi(n)| \leq e^{-c|n|_\infty}
\end{equation}

for some positive constant $c > 0$ and $|n|_\infty = \max_{1 \leq j \leq d} |n_j|$. Furthermore let $\left\{ \omega_x \right\}_{x \in \mathbb{Z}^d}$ be independent and identically distributed random variables in $[-\frac{1}{2}, \frac{1}{2}]$, whose density $\rho$ is bounded. For $\lambda > 0$ and $\omega$, we introduce the potential

\begin{equation}
V_{\lambda, \omega}(x) = \lambda \left( \sum_{m \in \mathbb{Z}^d} \omega_{x+m} \varphi(m) \right).
\end{equation}
Then our Schrödinger operator $H_{\lambda,\omega} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is defined by

$$H_{\lambda,\omega} = \Delta + V_{\lambda,\omega},$$

where $\Delta u(x) = \sum_{|e|=1} u(x + e)$ is the discrete Laplacian with $|n|_1 = \sum_{k=1}^d |n_k|$. Operators of this form have been studied for example in [26], [46], and [48].

Before, defining all the properties of $H_{\lambda,\omega}$, we will study, I will state the main results in the case of an alloy-type potential.

**Theorem 2.1.** Let $\lambda > 0$ be large enough, then we have

(i) There exists an interval $\Sigma_\lambda$ such that for almost every $\omega$

$$\sigma(H_{\lambda,\omega}) = \Sigma_\lambda.$$  

(ii) For almost every $\omega$, $H_{\lambda,\omega}$ exhibits Anderson localization (see Definition 2.2).

(iii) For almost every $\omega$, $H_{\lambda,\omega}$ exhibits dynamical localization (see Definition 2.3).

(iv) For $\beta > 1$, there exists a constant $C_\beta > 0$ such that the integrated density of states $N(E)$ (see Definition 2.4) obeys

$$N(E + \varepsilon) - N(E - \varepsilon) \leq C_\beta \frac{\log(\varepsilon^{-1})}{\beta}$$

for any $E \in \mathbb{R}$ and $\varepsilon \in (0, \frac{1}{2})$.

(i) holds for all $\lambda > 0$ and will be proven in Appendix B. A finite volume version of (iv) holds, see Theorem 2.9. The continuity of the integrated density of states given here is probably not optimal. In fact Veselić has shown that if the density $\rho$ is of bounded variation and $\sum_{n \in \mathbb{Z}^d} \varphi(n) \neq 0$ that (2.5) can be improved to

$$N(E + \varepsilon) - N(E - \varepsilon) \leq C \varepsilon$$

for some constant $C > 0$. That is the integrated density of states is Lipschitz continuous. This can be found in [13]. I also refer to [32], [47], [48] for discussions of earlier results on the continuity of the integrated density of states for models with alloy-type or sign changing potential. I will now define the two localization properties from Theorem 2.1.

**Definition 2.2.** The operator $H : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is said to exhibit Anderson localization, if its spectrum is pure point and there exists a constant $\gamma > 0$ such that for every eigenfunction $\psi$, we have for $n \in \mathbb{Z}^d$ that

$$|\psi(n)| \leq C_\psi e^{-\gamma |n|_\infty},$$

where $C_\psi > 0$ is a constant.

I denote by $\{e_x\}_{x \in \mathbb{Z}^d}$ the standard basis of $\ell^2(\mathbb{Z}^d)$ given by

$$e_x(n) = \begin{cases} 1, & x = n; \\ 0, & \text{otherwise}. \end{cases}$$

The function $\psi(t) = e^{-itH}e_x$ is the solution of the problem

$$\psi(0) = e_x$$

$$i\partial_t \psi(t) = H \psi(t),$$

which is known as Schrödinger’s equation. The second localization property, we are interested in is
Definition 2.3. The operator $H: \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is said to exhibit dynamical localization, if for every $x \in \mathbb{Z}^d$ and $p \geq 1$ we have

\begin{equation}
\sup_{t \in \mathbb{R}} \left( \sum_{n \in \mathbb{Z}^d} (1 + |n|_\infty)^p |\langle e_n, e^{-itH}e_x \rangle|^2 \right) < \infty.
\end{equation}

I note that this is not a very strong localization property. I believe that it is possible to show stronger localization properties than this, but have decided not to do so to keep this work at a reasonable length.

We now begin by introducing the integrated density of states. Denote by $\Lambda_r(x)$ the cube with radius $r$ and center $x$

\begin{equation}
\Lambda_r(x) = \{ n \in \mathbb{Z}^d : |n - x|_\infty \leq r \}.
\end{equation}

$H_{\lambda,\omega}^{\Lambda_r(x)}$ denotes the restriction of $H_{\lambda,\omega}$ to $\ell^2(\Lambda_r(x))$. We denote the number of eigenvalues of $H_{\lambda,\omega}^{\Lambda_r(x)}$ in the interval $[E_0, E_1]$ by

\begin{equation}
\text{tr}(P_{E_0, E_1}(H_{\lambda,\omega}^{\Lambda_r(0)})).
\end{equation}

Furthermore, $E$ denotes the expectation value, and $\#(\Lambda_r(0))$ denotes the number of elements of $\Lambda_r(0)$.

Definition 2.4. The integrated density of states $\mathcal{N}(E)$ is given by

\begin{equation}
\mathcal{N}(E) = \lim_{r \to \infty} \left( \frac{1}{\#(\Lambda_r(0))} \text{tr}(P_{-\infty, E}(H_{\lambda,\omega}^{\Lambda_r(0)})) \right).
\end{equation}

Here the limit is known to exist (see Section 5 of [35]). We will now discuss classes of potential, which include potentials of the form (2.2), but which isolate the properties needed for the proof.

Hypothesis 2.5. The potential $V_{\lambda,\omega}(x)$ is said to have exponentially decaying correlations, if the following properties hold.

(i) There exists a map

\begin{equation}
f : \left[ -\frac{1}{2}, \frac{1}{2} \right]^\mathbb{Z}^d \to \mathbb{R}.
\end{equation}

such that $V_{\lambda,\omega}(x) = \lambda f(T_x\omega)$, where $(T_x\omega)_n = \omega_{x+n}$.

(ii) There exists $c > 0$, such that if $\omega_n = \tilde{\omega}_n$ for $n \in \Lambda_r(0)$, we have

\begin{equation}
|f(\omega) - f(\tilde{\omega})| \leq e^{-cr}.
\end{equation}

(iii) There are constants $F > 0$ and $\alpha > 0$ such that for any $E \in \mathbb{R}$ and $\varepsilon > 0$, we have

\begin{equation}
\mu(\{ \omega : |f(\omega) - E| \leq \varepsilon \}) \leq F \cdot e^{\alpha}.
\end{equation}

(iv) $P$ is the product measure of a fixed probability measure $\mu$ on $[-\frac{1}{2}, \frac{1}{2}]$.

In order to see that the potential defined in (2.2) satisfies this hypothesis, define

\begin{equation}
f(\omega) = \sum_{m \in \mathbb{Z}^d} \omega_m \varphi(m).
\end{equation}

Then (i) holds, (ii) follows from $|\varphi(n)| \leq e^{-c|n|_\infty}$, and (iii) from $\varphi(0) \neq 0$. 
Assumptions (i) and (ii) imply that properties of \( H_{\Lambda,\omega}^{(x)} \) and \( H_{\Lambda,\omega}^{(y)} \) become almost independent if \( |x - y|_\infty \) is large enough. For example Lemma 5.10 is an implementation of this fact. Assumption (iii) is necessary to obtain an initial condition for multi-scale analysis, see Appendix A.

It is furthermore noteworthy that Conditions (i) and (ii) imply that the function \( f \) is Hölder continuous, if we use the metric

\[
d(\omega, \tilde{\omega}) = \sum_{n \in \mathbb{Z}^d} \frac{|\omega - \tilde{\omega}|}{2^{[n]_\infty}}
\]
on \([-\frac{1}{2}, \frac{1}{2}]^d\). This is the natural metric, since under it also the maps \( T_x \) are Hölder continuous.

The following theorem illustrates that Hypothesis 2.5 combined with a Wegner estimate is already sufficient to prove Anderson localization.

**Theorem 2.6.** Assume Hypothesis 2.5. In addition assume the Wegner estimate for \( R \geq 1 \) large enough and all \( \varepsilon > 0 \)

\[
\mathbb{E}(tr(P_{[E-\varepsilon, E+\varepsilon]}(H_{\Lambda,\omega}^{\Lambda R(0)}))) \leq \frac{C^W(\#\Lambda R(0))^b}{\log(\varepsilon^{-1})^{2(b+d+\beta)}},
\]

where \( b \geq 0 \), \( C^W \) is a \( \lambda \) independent constant, and \( \beta > 0 \) is large enough.

Then for \( \lambda > 0 \) large enough and almost every \( \omega \) Anderson localization and dynamical localization hold.

I have decided to include this theorem, since it is a good dividing line to lay out the framework of the classical parts of multi-scale analysis, which can also be found for example in the already mentioned works by Kirsch \[35\] and Stollmann \[45\]. The main difference between the proof of this theorem, and with what we will use later, is that for this theorem we only need to allow one bad cube, and later we will let this number go to infinity. The tools for this proof are developed in Section 5 to 7. The proof is then given in Section 8.

It is still possible in Theorem 2.6 that the measure \( \mu \) is a Bernouilli measure. However, it is unclear how to prove a Wegner estimate in this generality. Except in special cases like the one treated by Bourgain in \[10\]. See also Veselić \[43\] for Wegner estimates for \( \varphi \) finitely supported and \( \mu \) absolutely continuous. In fact, we will make an analyticity assumption on the potential in Hypothesis 2.7 which will serve as a replacement of (2.18). In the case of the potential defined in (2.2) the assumption reduces to the measure \( \mu \) being absolutely continuous with a bounded density.

We denote by \( D \) the disk in \( \mathbb{C} \) with center 0 and radius 6, that is

\[
D = \{ z \in \mathbb{C} : |z| < 6 \}.
\]

We are now ready for

**Hypothesis 2.7.** We say the \( V_{\lambda,\omega}(x) \) obeys the analyticity assumption, if the following hold.

(i) There exist a sequence of maps

\[
f_r : D^{\Lambda R(0)} \to \mathbb{C}.
\]
and a constant $c > 0$ satisfying the tail estimate for $R \geq 0$

$$\sum_{r=R}^{\infty} \| f_r \|_{L^\infty(D_{\Lambda_r(0)})} \leq e^{-cR}.$$  

(ii) The potential is given by

$$V_{\lambda, \omega}(x) = \lambda \left( \sum_{r=0}^{\infty} f_r(\{\omega_n\}_{n \in \Lambda_r(x)}) \right).$$

(iii) The map

$$\omega_0 \mapsto V_{\lambda, \omega}(0)$$

is non-constant for any choice of $\{\omega_x\}_{x \neq 0}$.

(iv) The measure $\mu$ has a density $\rho$, which is bounded.

One also can check that the potential defined in (2.2) satisfies this hypothesis. It is also possible to show that Hypothesis 2.7 implies Hypothesis 2.8. A proof that (iii) of Hypothesis 2.5 holds is given in Lemma A.1. The main result of this paper is

**Theorem 2.8.** Assume Hypothesis 2.7. Then for $\lambda > 0$ large enough, Anderson localization and dynamical localization hold for almost every $\omega$.

We also have

**Theorem 2.9.** Assume Hypothesis 2.7 and $\lambda > 0$ large enough. For any $\beta \geq 1$ there exists a constant $C_\beta$ and a length scale $R_\beta$ such that for all $R \geq R_\beta$ the Wegner estimate

$$E \left( \frac{1}{\#\Lambda_R(0)} \text{tr}(P_{E - \varepsilon, E + \varepsilon}(H_{\Lambda, \omega}^{\Lambda_R(0)})) \right) \leq \frac{C_\beta}{\log(\log(1/\varepsilon))^{3/2}}$$

holds for $E \in \mathbb{R}$ and $\varepsilon \in (0, 1/2)$. These two theorems imply Theorem 2.1. Actually, the results imply an estimate of the form

$$E \left( \frac{1}{\#\Lambda_R(0)} \text{tr}(P_{E - \varepsilon, E + \varepsilon}(H_{\Lambda, \omega}^{\Lambda_R(0)})) \right) \leq e^{-C \log(\log(1/\varepsilon))^{3/2}}$$

for some $C > 0$. I have decided to state it in the form of Theorem 2.9, since it is somewhat easier on the eye.

The proof of Theorem 2.8 and 2.9 proceed by a version of multi-scale analysis similar to the one of Bourgain in [12]. Let me point out some new features except that I allow for long range correlations. The control on the probabilities is super polynomial in the length scale as in the work of Germinet and Klein [29], which allows us to proof Theorem 2.9. Instead of working with elementary regions as in [12], I work only with boxes $\Lambda_r(x)$ in the proof.

At this point, I also wish to point out that the result should be easy to extend in various direction. First, one should be able to allow arbitrary background operators and not just the Laplacian $\Delta$. Second, one should be able to extend the proof to more general underlying probability spaces then $[-1/2, 1/2]$ and vector valued potentials as done in [12]. The essential point here is that one still need an analog
of Cartan’s Lemma to hold. Third, one should be able to use the methods to understand localization in Lifschitz tails. I will give some more comments on further directions in the next section.

Let me now discuss the example that motivated Hypothesis 2.7. Consider the times two map

\[ T : [0, 1] \to [0, 1] \]

\[ T x = 2x \pmod{1}. \]

(2.26)

It is relatively well known that this is an ergodic transformation. Furthermore, this transformation has attracted some attention in spectral theory \[17\], \[19\]. Using the binary expansion \[x = \sum_{j=1}^{\infty} \frac{x_j}{2^j}\], we can conjugate \(T\) to the shift on the space \([0, 1]^{\mathbb{Z}}\) with respect to the Bernoulli measure.

One can show that for an analytic and one-periodic function \(g\), one has that

\[ f(\omega) = g \left( \sum_{j=1}^{\infty} \frac{1}{2^j} \omega_j \right) \]

(2.27)

satisfies Hypothesis 2.7. So transformations like the doubling map would be in our framework, if we could relax the absolutely continuous measure to a pure point one.

In Section 4, we show how the procedure of multi-scale analysis works. As already mentioned, we lay out the basics for the multi-scale analysis with a Wegner estimate in Sections 5 to 7. Then we provide the proof of Theorem 2.6 in Section 8.

Sections 9 to 11 discuss the machinery used to replace the Wegner estimate. The main result is Theorem 11.1, which essentially proofs a Wegner type estimate on a large scale from a weak assumption at the large scale and strong assumptions on a smaller scale. The main ingredient here is Cartan’s Lemma, whose different forms we review in Section 10. Sections 12 to 15 then contain the main elements of the proof of Theorem 2.8.

In Sections 16 to 18, we prove that the conclusions of our multi-scale analysis imply dynamical localization. I have decided not to include a proof of Anderson localization, since it is similar to the argument of Bourgain and Kenig \[16\]. Also it follows by running a multi-scale analysis as in Kirsch \[35\] having the Wegner estimate from Theorem 2.9 at ones disposition.

In Appendix A, I demonstrate how one can deduce the initial condition for multi-scale analysis, essentially from a largeness assumption on \(\lambda\). In Appendix B, I have included an argument that shows that the spectrum of the operators under consideration is an interval. Lastly, Appendix C demonstrates how the conclusions of multi-scale analysis imply a Wegner estimate.

### 3. Comments, Improvements, and Questions

In this section, I want to discuss some further directions the results of this paper can be improved on. I also wish to ask some questions.

It is required in Hypothesis 2.7 that all the functions \(f_r\) are defined on \(D^{\Lambda_r(0)}\), where

\[ D = \{ z : |z| < 6 \}. \] (3.1)
It would be natural to replace \( D \) in this definition by
\[
(3.2) \quad \mathcal{A}_\rho = \{ z : \text{dist}(z, [-\frac{1}{2}, \frac{1}{2}]) < \rho \}.
\]
a \( \rho \) neighborhood of \([-\frac{1}{2}, \frac{1}{2}]\). Then at least two changes are necessary first, one needs to add a covering argument to the application of Cartan’s lemma (e.g., in the proof of Theorem 11.1). Then the assumption \( (v) \) of it changes to the existence of a \( x_0 \in [-\frac{1}{2}, \frac{1}{2}]^n \) to the existence of such an \( x_0 \) in every ball of radius \( \frac{\rho}{2} \). I believe it is possible to change the probabilistic arguments to show this. Such a treatment is necessary for quasi-periodic systems and ones defined by the skew-shift, see [9]. It should also be possible to replace \([-\frac{1}{2}, \frac{1}{2}]\) by more general sets. An interesting example are Lie groups such as \( SU(n) \) as discussed by Bourgain in [12]. I believe that probably changes as above will also be necessary in the proof.

Reducing the size of the sets, where the functions \( f_r \) are analytic is a way of lowering the regularity of these functions. Similarly, one could ask if the result stays true for smooth functions or quasi-analytic functions. By the remarks following Hypothesis 2.5, we know that our assumptions imply that the function \( f \) is Hölder continuous on \([-\frac{1}{2}, \frac{1}{2}]^2 \). It is also intriguing if the assumption of exponential decay of the correlations is optimal. I would expect that one can relax condition \( (ii) \) of Hypothesis 2.5 significantly.

The main motivation for this is that Kirsch, Stollmann, and Stolz have shown in [36] that an assumption of the form \( (\varepsilon > 0 \text{ and } C > 0) \)
\[
(3.3) \quad |f(\omega) - f(\omega')| \leq \frac{C}{r^{2d+\varepsilon}}
\]
is sufficient to carry out multi-scale analysis if a Wegner estimate is available. It is known from Veselić work [48] that such an estimate holds for the alloy-type potential as long as \( \sum_{n \in \mathbb{Z}^d} \varphi(n) \neq 0 \).

This paper shows localization at large disorders. The usual proofs of localization also show it near the band-edges. It would be interesting to obtain such a result for the potentials considered here. The necessary improvement is not in the results of this paper, but to prove \textit{Lifschitz tails} to have an initial scale estimate for multi-scale analysis. Some positive results can be found in [10].

As mentioned after Theorem 2.1, I do not expect that control of the integrated density of states in this paper to be optimal. One should probably at least expect some form of Hölder continuity of it. See also the paper [48] of Veselić for further discussion and currently the best results in this direction.

4. The multi-scale scheme

The goal of this section is to discuss the main aspects of the proofs of the theorems from the introduction. As the classical multi-scale analysis, we will be concerned with showing that certain estimates on the inverses of the restrictions of \( H_{\lambda, \omega} \) to boxes \( \Lambda_r(0) \) hold with large probability. We make the required properties precise in the following definition.
Definition 4.1. An interval \([r_0, r_1]\) is called \((\gamma, \alpha)\)-acceptable for \(H_{\lambda, \omega} - E\) if for \(r_0 \leq r \leq r_1\) there exists an event \(B_r\) such that
\[
P(B_r) \leq \frac{1}{r^\alpha}
\]
and for \(\omega \notin B_r\) we have

(i) The resolvent estimate
\[
\| (H_{\lambda, \omega}^{(0)} - E)^{-1} \| \leq \frac{1}{8} e^{\sqrt{\gamma}}.
\]

(ii) For any pair \(x, y \in \Lambda_r(0)\) with \(|x - y| \geq \frac{r}{10}\)
\[
| \langle e_x, (H_{\lambda, \omega}^{(0)} - E)^{-1} e_y \rangle | \leq \frac{1}{8} \frac{1}{\#(\partial - \{\Lambda_r(0)\})} e^{-\gamma|x-y|}.
\]

Here \(\#(\Xi)\) denotes the number of elements of \(\Xi \subseteq \mathbb{Z}^d\) and \(\partial - \{\Lambda_r(0)\}\) the inner boundary defined in (5.2). We will begin by stating the initial condition of multiscale analysis. The proof uses the largeness assumption on \(\lambda > 0\) and is given in Appendix A.

Proposition 4.2. Assume Hypothesis 2.5. Let \(r \geq 1\) and \(\alpha > 0\). Then there exists \(\lambda_0 = \lambda_0(r, \alpha)\) such that for any \(E \in \mathbb{R}\) and \(\lambda \geq \lambda_0\).
\[
(4.4) \quad [1, r] \text{ is } (2, \alpha)\text{-acceptable for } H_{\lambda, \omega} - E.
\]

The proof of this is pretty standard, and only uses condition (iii) of Hypothesis 2.5. We now come to results that allow us to extend the range of the interval, that is \((\gamma, \alpha)\)-acceptable.

Theorem 4.3. Assume Hypothesis 2.5 and (2.18) for \(\beta > 1\) large enough. Let \(1 \leq \gamma \leq 2\), \(r \geq 1\) large enough, and
\[
\alpha = d \frac{2d + 1}{2d - 1}.
\]
Assume
\[
(4.6) \quad \{r\} \text{ is } (\gamma, \alpha)\text{-acceptable for } H_{\lambda, \omega} - E.
\]
Then
\[
(4.7) \quad [R_0, R_1] \text{ is } (\gamma, \alpha)\text{-acceptable for } H_{\lambda, \omega} - E,
\]
where
\[
(4.8) \quad R_0 = \left( (r)^{1+\frac{1}{2d}} \right), \quad R_1 = \left( (r)^{1+\frac{1}{2d}} \right), \quad \gamma \geq \gamma \left( 1 - \frac{200}{r^\frac{1}{2d}} \right).
\]

The proof of this statement is well known, and can for example be found in the lecture notes [35] of Kirsch or in the paper by von Dreifus and Klein [24]. We will give a proof in Sections 5 to 8. Let us now record the main consequence

Corollary 4.4. Assume Hypothesis 2.5 and (2.18) for \(\beta > 1\) large enough. Let \(\alpha = d \frac{2d + 1}{2d - 1} \). Then for \(\lambda \geq \lambda_0\) and all energies \(E\)
\[
(4.9) \quad [1, \infty) \text{ is } (1, \alpha)\text{-acceptable for } H_{\lambda, \omega} - E
\]

Proof. One needs to check that \(\tilde{\gamma}\) defined in Theorem 4.3 stays \(\geq 1\) while applying Theorem 4.3 countably many times. This can be done as in Corollary 4.6. \(\square\)
Before deriving consequences of this corollary, in particular proving Theorem 2.6, we will look at the multi-scale steps of our second approach.

**Theorem 4.5.** Assume Hypothesis 2.7 and that $r$ is large enough.

Assume for $\alpha \geq 3d + 2$ that

$$[r, r^3] \text{ is } (\gamma, \alpha)\text{-acceptable for } H_{\lambda, \omega} - E.$$ (4.10)

Then with

$$\tilde{\alpha}_r = \left\lfloor \sqrt{\frac{1}{2d} \log(r) \log(\max(4, 2 + \frac{4\gamma}{r}))} \right\rfloor$$ (4.11)

we have

$$[R_0, R_1] \text{ is } (\tilde{\gamma}, \tilde{\alpha}_r)\text{-acceptable for } H_{\lambda, \omega} - E,$$ (4.12)

where

$$R_0 = r^{3d+8}, \quad R_1 = r^{\frac{\tilde{\alpha}_r}{\tilde{\gamma}}}, \quad \tilde{\gamma} = \gamma \left(1 - \frac{2}{r}\right).$$ (4.13)

The proof of this theorem will be given in Section 15. This theorem will allow us to prove

**Corollary 4.6.** For any $\beta$, there exists $R_\beta$ such that

$$[R_\beta, \infty) \text{ is } (1, \beta)\text{-acceptable for } H_{\lambda, \omega} - E.$$ (4.14)

**Proof.** Fix $E \in \mathbb{R}$. By Proposition 4.2, we can conclude that for some large enough $r$ and $\alpha \geq 1$, we have

$$[1, r^3] \text{ is } (2, \alpha)\text{-acceptable for } H_{\lambda, \omega} - E.$$ We choose a such that $\frac{a+1}{a+1} \geq (3d+8)^3$, and $r$ so large that $\tilde{\alpha}_r \geq \alpha$. Then, we define a sequence of $R_k$

$$R_1 = r^{3d+8}, \quad R_k = (R_{k-1})^{3d+8},$$

satisfying $R_k \geq r^k$, and

$$\gamma_1 = 2, \quad \gamma_k = \gamma_{k-1} \left(1 - \frac{2}{R_{k-1}}\right).$$

We obtain by Theorem 4.5 that

$$[R_k, (R_k)^{3d+9}] \text{ is } (\gamma_k, \alpha_R)\text{-acceptable for } H_{\lambda, \omega} - E.$$ In particular $\alpha_{R_k} \to \infty$ as $k \to \infty$. It remains to check that $\gamma_k \geq 1$, but this is easy. The claim follows.

Before discussing the results on localization, let us quickly prove the result on the Wegner estimate.

**Proof of Theorem 2.9.** This follows from the previous corollary combined with the results from Section C.

We now come to

**Theorem 4.7.** Let $r_\infty \geq 1$, $\gamma > 0$. Assume Hypothesis 2.7 and

$$[r_\infty, \infty) \text{ is } (\gamma, 4d)\text{-acceptable for } H_{\lambda, \omega} - E.$$ (4.15)

Then $H_{\lambda, \omega} - E$ exhibits dynamical localization for almost every $\omega$. 

□
We can obtain Anderson localization as in [16]. We see that Theorem 2.6 and Theorem 2.8 follow.

5. Suitability and probabilistic estimates

In this section, we first introduce the notion of suitability, which quantifies the properties of restrictions to finite boxes. In order to emphasize that these notions are independent of the specific setting, we will write $H$ instead of $H_{\lambda, \omega}$. Second, we will derive the main probabilistic estimates needed for the proof of Theorem 4.5.

We denote by $\{e_x\}_{x \in \mathbb{Z}^d}$ the standard basis of $\ell^2(\mathbb{Z}^d)$, that is

$$e_x(n) = \begin{cases} 1, & n = x; \\ 0, & \text{otherwise}. \end{cases}$$

We recall that for $\Lambda_r(x) = \{n \in \mathbb{Z}^d : |x - n|_\infty \leq r\}$ a box in $\mathbb{Z}^d$, we denote by $H^{\Lambda_r(x)}$ the restriction of $H$ to $\ell^2(\Lambda_r(x))$. We will denote by $\partial_- \Lambda_r(x)$ the inner boundary, that is

$$\partial_- \Lambda_r(x) = \{n \in \Lambda_r(x) : \exists m \in \mathbb{Z}^d \setminus \Lambda_r(x) : |n - m|_1 = 1\}.$$ 

We note that $\partial_- \Lambda_r(x) = \Lambda_r(x) \setminus \Lambda_r(x - 1)$. In particular,

$$#(\partial_- \Lambda_r(x)) \leq 2d(3r)^{d-1}.$$

**Definition 5.1.** Let $\gamma > 0$, $\tau \in (0, 1)$, and $p \geq 0$ an integer. A box $\Lambda_r(x)$ is called $(\gamma, \tau, p)$-suitable for $H - E$ if the following hold.

(i) The resolvent estimate

$$\|(H^{\Lambda_r(x)} - E)^{-1}\| \leq \frac{1}{2^p} e^{\tau r}.$$ 

(ii) For $x, y \in \Lambda_r(x)$ satisfying $|x - y| \geq \frac{r}{10}$

$$|\left\langle e_x, (H^{\Lambda_r(x)} - E)^{-1} e_y \right\rangle| \leq \frac{1}{2^p \#(\partial_- \Lambda_r(x))} e^{-\gamma |x - y|}.$$ 

This definition is made in such a way, that it becomes useful for the computations in Sections 6 and 9. Furthermore, this definition is more general than we will need. In fact, it will suffice for the purposes of this paper to restrict to the case $\tau = \frac{1}{2}$ and $p \in \{0, 1, 2, 3\}$. The main motivation for the more general definition is that parameters $\tau$ close to 1 are necessary, if one wants to treat skew-shifts on tori of large dimension, see [12]. Let me make the connection to $(\gamma, \alpha)$-acceptable precise:

**Remark 5.2.** We have that $[r_0, r_1]$ is $(\gamma, \alpha)$-acceptable in the sense of Definition 4.1 if and only if for $r \in [r_0, r_1]$ we have

$$\mathbb{P}(\Lambda_r(0) \text{ is not } (\gamma, \frac{1}{2}, 3)-\text{suitable for } H_{\lambda, \omega} - E) \leq \frac{1}{r^\alpha}.$$ 

The integer $p$ will serve as a parameter, we can lower if we need to perturb $E$ or $H$ slightly. We will now proceed to make this precise.

**Lemma 5.3.** Let $\gamma > 0$, $\tau \in (0, 1)$, and $p \geq 1$ an integer. Assume $\Lambda_r(x)$ is $(\gamma, \tau, p)$-suitable for $H - E$ and

$$\|(H^{\Lambda_r(x)} - \hat{E}) - (H^{\Lambda_r(x)} - E)\| \leq \frac{1}{2^p+1} \frac{1}{\#(\partial_- \Lambda_r(x))} e^{-\gamma r - 2\tau r}.$$ 

Then $\Lambda_r(x)$ is $(\gamma, \tau, p - 1)$-suitable for $\hat{H} - \hat{E}$.
Proof. Denote \( A = H^{\Lambda_r(n)} - E \) and \( B = \tilde{H}^{\Lambda_r(n)} - \tilde{E} \). From \( B^{-1} - A^{-1} = B^{-1}(B - A)A^{-1} \), we obtain
\[
B^{-1} = A^{-1}(I + (A - B)A^{-1})^{-1}.
\]
By assumption, we have \( \|(A - B)A^{-1}\| \leq \frac{1}{2} \) and thus
\[
\|B^{-1}\| \leq 2\|A^{-1}\|,
\]
\[
|\langle e_x, B^{-1} e_y \rangle| \leq |\langle e_x, A^{-1} e_y \rangle| + 2\|A^{-1}\|^2\|B - A\|.
\]
The claim follows. 

We now specialize to the case of estimating a perturbation in the energy

**Lemma 5.4.** Assume
\[
d^{p+2}(3r)^{d-1} \leq e^{\gamma r}, \quad \gamma r^{1-\tau} \geq 1,
\]
that \( \Lambda_r(x) \) is \((\gamma, \tau, p)\)-suitable for \( H - E \), and that
\[
|E - \tilde{E}| \leq e^{-4\gamma r}.
\]
Then \( \Lambda_r(x) \) is \((\gamma, \tau, p - 1)\)-suitable for \( H - \tilde{E} \).

**Proof.** The claim follows from Lemma 5.3.

We will now begin to study properties specific to potentials obeying either Hypothesis 2.6 or 2.7. In particular (2.15) respectively (2.21) will be important. The first goal will be to understand what happens if we change \( \omega \). For \( \Xi \subseteq \mathbb{Z}^d \), we denote by \( \Xi^c = \mathbb{Z}^d \setminus \Xi \) its complement.

**Definition 5.5.** Let \( \omega, \tilde{\omega} \in \Omega \) and \( \Xi \subseteq \mathbb{Z}^d \), we will write \( \omega \equiv \tilde{\omega} \pmod{\Lambda_r(x)^c} \) if
\[
\omega_x = \tilde{\omega}_x, \quad x \in \Xi^c.
\]
Having this definition, we are ready for

**Lemma 5.6.** Let \( \omega = \tilde{\omega} \pmod{\Lambda_r(x)^c} \), then
\[
|V_{\Lambda, \omega}(x) - V_{\Lambda, \tilde{\omega}}(x)| \leq le^{-csr}.
\]

**Proof.** This follows from (2.15) respectively (2.21).

**Lemma 5.7.** Let \( r, s > 0 \), and \( \omega = \tilde{\omega} \pmod{\Lambda_{r+s}(x)^c} \). Then
\[
\|H_{\Lambda, \omega}^{\Lambda_r(x)} - H_{\Lambda, \tilde{\omega}}^{\Lambda_r(x)}\| \leq le^{-cs}.
\]

**Proof.** By assumption, we have for \( y \in \Lambda_r(x) \) that \( \omega = \tilde{\omega} \pmod{\Lambda_s(y)^c} \). The claim now follows by the previous lemma.

In order to state the next lemma, we define given \( r \) the number \( \varpi \) by
\[
\varpi = \left[ \left( 1 + \frac{4\gamma}{c} \right) r + \frac{\log(\lambda)}{c} \right],
\]
Lemma 5.8. Assume \((5.8)\) and that  
\[ \Lambda_r(x) \text{ is } (\gamma, \tau, p)\text{-suitable for } H_{\lambda, \omega} - E. \]  
Furthermore assume that  
\[ \omega = \tilde{\omega} \pmod{\Lambda_{r}(x)^c}, \]  
Then \(\Lambda_r(x)\) is \((\gamma, \tau, p - 1)\)-suitable for \(H_{\lambda, \tilde{\omega}} - E\).

Proof. By Lemma 5.7 with \(s = 4\gamma c \cdot r + \frac{\log(\lambda)}{c}\) and \((5.8)\), we obtain  
\[ \|H_{\lambda, \omega} - H_{\lambda, \tilde{\omega}}\| \leq e^{-4\gamma r}. \]  
By Lemma 5.3 the claim follows.

We furthermore note

Lemma 5.9. We have  
\[ \tau \leq T_0 \max(r, \frac{\log(\lambda)}{c}), \quad T_0 = \left(2 + \frac{4\gamma}{c}\right). \]  
Proof. This follows from \((5.13)\).

We will now begin with the probabilistic arguments. For \(n \in \Lambda_R(0)\) and \(r \geq 1\), introduce \(X^r_n = X^r_n(\gamma, \tau, p, E)\) as the set of \(\omega\) such that for some \(\tilde{\omega} = \omega \pmod{\Lambda_{r}(n)^c}\) we have  
\[ \Lambda_r(n) \text{ is not } (\gamma, \tau, p - 1)\text{-suitable for } H_{\lambda, \tilde{\omega}} - E. \]  

Lemma 5.10. Assume \((5.8)\) and  
\[ \mathbb{P}(\Lambda_r(0) \text{ is not } (\gamma, \tau, p)\text{-suitable for } H_{\lambda, \omega} - E) \leq \varepsilon. \]  
Then \(\mathbb{P}(X^r_n) \leq \varepsilon\).  
Furthermore, for \(|m - n|_{\infty} \geq 2T + 1\), \(X^r_n\) and \(X^r_m\) are independent.

Proof. By assumption and Lemma 5.8 we have that \(\mathbb{P}(X^r_n) \leq \varepsilon\). The independence follows from \(\Lambda_r(n) \cap \Lambda_r(m) = \emptyset\)

This implies

Proposition 5.11. Assume \((5.8)\) and \((5.18)\). Then for \(R \geq r\) and \(K \geq 1\) there exists a set \(B^r_{1,K}\) with the following properties.  
\begin{itemize}
  \item[(i)] \(\mathbb{P}(B^r_{1,K}) \leq \frac{1}{(K + 1)^d} (3R)^d \cdot \varepsilon)^{K+1}\).
  \item[(ii)] For \(\omega \notin B^r_{1,K}\) there exist \(0 \leq L \leq K\) and \(m^1_{\omega}, \ldots, m^L_{\omega}\) such that for  
  \[ \Lambda_r(n) \subseteq \Lambda_R(0) \setminus \bigcup_{k=1}^{L} \Lambda_{2^r}(n_{m^k_{\omega}}), \]  
  we have  
  \[ \Lambda_r(n) \text{ is } (\gamma, \tau, p - 1)\text{-suitable for } H_{\lambda, \omega} - E. \]
\end{itemize}
The conclusion of this proposition has to be understood as there exist at most $K$ bad cubes with large probability. Of course in order for this statement to be true, we need that $\varepsilon < (3R)^d$.

We now begin the proof of Proposition 5.11. Let $m_k \in \Lambda_R(0)$ for $k = 1, \ldots, K + 1$, which satisfy for $k \neq \ell$

\[(5.21) \quad \Lambda_r(m_k) \cap \Lambda_r(m_\ell) = \emptyset.\]

Denote the collection $m = \{m_k\}_{k=1}^{K+1}$. Introduce $B_{r,K}^{m_r}$ as the set of $\omega$ such that for $k = 1, \ldots, K + 1$, we have

\[(5.22) \quad \Lambda_r(m_k) \text{ is not } (\gamma, \tau, p-1)-\text{suitable for } H_{\lambda,\omega - E}.\]

We have

**Lemma 5.12.** Assume (5.8), and (5.18). Let $m \in \Lambda_R(0)^{K+1}$ satisfy (5.21). Then

\[(5.23) \quad P(B_{r,K}^{m_r}) \leq \varepsilon^{K+1}.\]

**Proof.** One shows that $B_{r,K}^{m_r} \subseteq \bigcap_{k=1}^{K+1} X^{m_k}$. The claim now follows by Lemma 5.10. \[\square\]

**Proof of Proposition 5.11.** Introduce $B_{r,K}^{R}$ as the union over all possible choices of $B_{r,K}^{m_r}$. Since the number of these choices is bounded by

\[\frac{1}{(K+1)!} \left( \# \Lambda_R(0) \right)^{K+1} \leq \frac{1}{(K+1)!} (3R)^{(K+1)d},\]

we obtain that (i) holds.

Let us now check (ii). Take $\omega \notin B_{r,K}^{R}$. Denote by $\{m_j\}_{j=1}^{J}$ a maximal collection of $m \in \Lambda_R(0)$ such that (5.21) holds and

\[\Lambda_r(m_j) \text{ is not } (\gamma, \tau, p-1)-\text{suitable for } H_{\lambda,\omega - E}.\]

If $J \leq K$, we are done. Otherwise, we have a contradiction to $\omega \notin B_{r,K}^{m_j}_{j=1}^{K+1}$, which finishes the proof. \[\square\]

6. Obtaining exponential decay of the off-diagonal terms

In this section, we demonstrate how to obtain estimates on the off-diagonal elements, using an iteration of the resolvent equation. The results of this section are again independent of the specific form of $H_{\lambda,\omega}$, so we use $H$ to denote some Schrödinger operator. Introduce for $x, y \in \Lambda_r(n)$ the Green’s function

\[(6.1) \quad G_E^{\Lambda_r(n)}(x, y) = \left\langle e_x, (H^{\Lambda_r(n)} - E)^{-1} e_y \right\rangle.\]

We furthermore introduce the boundary of $\Lambda_r(n)$ in $\Lambda_R(n)$ by

\[(6.2) \quad \partial^\Lambda_R(n) \Lambda_r(n) = \{(x, y) : x \in \Lambda_r(n), y \in \Lambda_R(0) \setminus \Lambda_r(n), |x - y|_1 = 1\}.\]

For $x \in \Lambda_r(n)$ and $y \in \Lambda_R(0) \setminus \Lambda_r(n)$, we obtain from the second resolvent equation

\[(6.3) \quad G_E^{\Lambda_R(n)}(x, y) = -\sum_{(u, v) \in \partial^\Lambda_R(n) \Lambda_r(n)} G_E^{\Lambda_r(n)}(x, u) \cdot G_E^{\Lambda_R(n)}(y, v).\]
We will refer to this equation as the geometric resolvent equation. See Section 5.3 in \[35\]. We note the following consequence, which is more convenient for our applications

\[
|G_E^{\Lambda_R(0)}(x,y)| \leq \left( \max_{n \in \Lambda_{r+1} \cap \Lambda_R(0)} |G_E^{\Lambda_R(0)}(y,n)| \right) \\
\quad \cdot \left( \frac{\#(\partial_{-}^{\Lambda_R(0)}(0))}{\Lambda_{r}(0)} \max_{n \in \partial_{-}^{\Lambda_R(0)}(0)} |G_E^{\Lambda_r(n)}(x,n)| \right).
\]

Here $\partial^{\Lambda_R(0)}_{-}(0)$ denotes

\[
\partial^{\Lambda_R(0)}_{-}(n) = \{ x \in \mathbb{R} : \exists y \in \Lambda_R(0) \setminus \Lambda_r(n) : |x-y|_1 = 1 \}.
\]

We note that $\partial_{-}(n) = \partial_{-}^{\Lambda_R(0)}(n)$. In order to illustrate the use of \(6.4\), we first prove the following, which is similar to Proposition 10.4. in \[35\].

**Proposition 6.1.** Let $R \geq r \geq 1$, $\tau \in (0,1)$, $\gamma > 0$, and $p \geq 0$. Assume for $\Lambda_{r}(n) \subseteq \Lambda_R(0)$ that

\[
\Lambda_{r}(n) \text{ is } (\gamma,\tau,0)\text{-suitable for } H - E
\]

and

\[
\| (H^{\Lambda_R(0)} - E) \| \leq \frac{1}{2^p} e^{R\tau}.
\]

Then

\[
\Lambda_R(0) \text{ is } (\tilde{\gamma},\tau,p)\text{-suitable for } H - E
\]

where

\[
\tilde{\gamma} = \gamma \left( 1 - \frac{1}{r+1} - \frac{10}{R} (\gamma \tau + R \tau + d \log(3)) \right).
\]

**Proof.** We have to check that for $x, y \in \Lambda_R(0)$ with $|x-y| \geq \frac{R}{10}$, we have exponential decay of the off-diagonal elements. By \(6.4\) and \(6.6\), we obtain for any $u \in \Lambda_r(n)$ and $v \in \Lambda_R(0) \setminus \Lambda_r(n)$ that

\[
|G_E^{\Lambda_r(n)}(u,v)| \leq e^{-\gamma \tau} \max_{n \in \Lambda_{r+1} \cap \Lambda_R(0)} |G_E^{\Lambda_R(0)}(v,n)|.
\]

Suppose for $k \geq 1$ that $v \notin \Lambda_R(0) \setminus \Lambda_{k(r+1)}(u)$, then by iterating the previous equation we find

\[
|G_E^{\Lambda_r(n)}(u,v)| \leq e^{-\gamma \tau k} \max_{n \in \Lambda_{k(r+1)} \cap \Lambda_R(0)} |G_E^{\Lambda_R(0)}(v,n)|.
\]

We now apply this formula to the case when $u = x$ and $v = y$. Then, for

\[
k = \left\lfloor \frac{|x-y|}{r+1} \right\rfloor \geq \frac{|x-y|}{r+1} - 1
\]

we obtain that $y \notin \Lambda_R(0) \setminus \Lambda_{k(r+1)}(y)$. Thus using that $|G_E^{\Lambda_r(n)}(n,m)| \leq \frac{1}{2^p} e^{R\tau}$ for any $n, m \in \Lambda_R(0)$ by assumption, that

\[
|G_E^{\Lambda_r(n)}(x,y)| \leq \frac{1}{2^p} e^{-\gamma \tau |x-y| + \gamma \tau + R \tau}.
\]

The claim follows. \(\square\)
A close inspection of the argument in this proof shows that it is slightly wrong. In order to make it completely rigorous, one would need to replace $\Lambda_r(n)$ in (6.4) by the shifted cube $\Lambda^R_r(n)$ defined in Definition 7.1. That this does not create problems follows from

$$\partial^{\Lambda_n(0)} \Lambda^R_r(n) \cap \partial \Lambda_R(0) = \emptyset.$$  

Although Proposition 6.1 illustrates well, what we will do it is not good enough yet. In view of Proposition 5.11, we will need to allow for $K$ regions $\Lambda_{s_k}(m_k) \subseteq \Lambda_R(0)$ such that we only know that every

$$\Lambda_r(n) \subseteq \Lambda_R(0) \setminus \left( \bigcup_{k=1}^K \Lambda_{s_k}(m_k) \right)$$

is $(\gamma, \tau, 0)$-suitable. We will now give such a result, which is similar to Theorem 10.20 in [35].

**Theorem 6.2.** Let $a \geq 1$ and

$$s_k + ar \leq t_k.$$  

Assume the following conditions.

(i) $\Lambda_{t_k}(m_k) \subseteq \Lambda_R(0)$,

(ii) For $k \neq \ell$

$$\Lambda_{t_k + ar}(m_k) \cap \Lambda_{t_\ell + ar}(m_\ell) = \emptyset.$$  

(iii) For every subcube

$$\Lambda_r(n) \subseteq \Lambda_R(0) \setminus \left( \bigcup_{k=1}^K \Lambda_{s_k}(m_k) \right)$$

that

(iv) For $1 \leq k \leq K$

$$\| (H^{\Lambda_{t_k}(m_k)} - E)^{-1} \| \leq e^{a\gamma r}$$

(v) On the resolvent of the whole cube

$$\| (H^{\Lambda_n(0)} - E)^{-1} \| \leq \frac{1}{2^p} e^{R^*}.$$  

Then

$$\Lambda_R(0) \text{ is } (\hat{\gamma}, \tau, p)\text{-suitable for } H - E$$

where

$$\hat{\gamma} = \gamma \left( 1 - \frac{1}{r + 1} - \frac{10}{R} \left( 3 \sum_{k=1}^K t_k + \gamma r + R^* + d \log(3) \right) \right).$$

In order to prove this theorem, we will introduce some additional notation. Given $t \geq s \geq 1$, we introduce the annulus

$$\mathcal{A}^{s,t}_r(n) = (\Lambda_t(n) \setminus \Lambda_s(n)) \cap \Lambda_R(0) = \{ x \in \Lambda_R(0) : \ s + 1 \leq |x - n|_\infty \leq t \}.$$  

We also introduce the abbreviation

$$\mathcal{A}^{R}_r(n) = \mathcal{A}^{R}_{r-1,t}(n) = \{ x \in \Lambda_R(0) : \ |x - n|_\infty = t \}.$$
We note $\mathcal{A}_t^R(n) = \partial_{\Lambda^n(0)}^t \Lambda_t(n)$.

**Lemma 6.3.** Let $x \in \Lambda_{t_k}(m_k)$ and $y \in \Lambda_R(0) \setminus \Lambda_{t_k+ar}(m_k)$. Assume \textbf{6.15}, then

(6.21) $|G_{\Lambda_R}^{\Lambda_R(0)}(x,y)| \leq \max_{n \in \mathcal{A}_{-ar,0}^R} |G_{\Lambda_R}^{\Lambda_R(0)}(y,n)|$.

**Proof.** Using (6.3) and then (6.4), we compute

\[
|G_{\Lambda_R}^{\Lambda_R(0)}(x,y)| \leq \|(H^{\Lambda_{t_k}(m_k)} - E)^{-1}\| \cdot \sum_{v \in \partial_{\Lambda_R(0)} \Lambda_{t_k}(m_k)} |G_{\Lambda_R}^{\Lambda_R(0)}(y,v)|
\]

where we used that for $n$ with $|n - n_0|_{\infty} = t_k + 1$ we have $\Lambda_{t_k}(n)$ is $(\gamma, \tau, 0)$-suitable for $H - E$.

The claim follows by iterating the above procedure $a$ times. \hfill $\Box$

Given $x, y \in \Lambda_R(0)$ satisfying $|x - y| \geq \frac{R}{10}$, we introduce $L$ maximal such that $y \notin \Lambda_{Lr}(x)$ or alternatively \textbf{6.22}

\[
\Lambda_{Lr}(x) \cap \Lambda_{t_k+ar}(m_k) \neq \emptyset
\]

for some $k$. We will need $\Lambda_{t_k}(n)$ is $(\gamma, \tau, 0)$-suitable for $H - E$.

**Lemma 6.4.** We have

(6.24) $\frac{\#(\text{bad } \ell)}{|x - y|} \leq \frac{30}{R} \left( \sum_{k=1}^{K} \left( \frac{t_k}{r} + a \right) \right)$.

**Proof.** For $1 \leq k \leq K$, there are at most $3 \left( \frac{t_k}{r} + a \right)$ choices of $\ell$ such that $\Lambda_{t_k+ar}(m_k) \neq \emptyset$. Summing over $k$ and using $|x - y| \geq \frac{R}{10}$ implies the claim. \hfill $\Box$

**Proof of Theorem 6.2.** We now proceed as in the proof of Proposition \textbf{6.1} except we use Lemma \textbf{6.3} to deal with the bad $\ell$. \hfill $\Box$

7. A COMBINATORIAL RESULT

In this section, we will derive a combinatorial results about boxes in $\mathbb{Z}^d$, which will be used to ensure the geometric conditions (i) and (ii) in Theorem \textbf{6.2}. We begin by introducing some notation.

Given $R \geq 1$, $n \in \Lambda_R(0)$, and $0 \leq t \leq R$, we denote by $n_t^R \in \Lambda_R(0)$ the point

(7.1) \[
(n_t^R)_j = \begin{cases} 
-R + t, & n_j < -R + t; \\
R - t, & n_j > R - t; \\
n_j, & \text{otherwise.}
\end{cases}
\]

This choice is made such that $\Lambda_t(n_t^R) \subseteq \Lambda_R(0)$ and $\#(\Lambda_t(n_t^R) \cap \Lambda_t(n))$ is maximal under all subcubes $\Lambda_t(n) \subseteq \Lambda_R(0)$.
Definition 7.1. Given $0 \leq t \leq R$, $s \geq 0$, and $n \in \Lambda_t(n)$, we introduce the shifted cube

$$\Lambda^{R,t}_s(n) = \Lambda_s(n^t_R).$$

Furthermore, we define $\Lambda^R_t(n) = \Lambda^{R,t}_s(n)$.

We always have $\Lambda^R_t(n) \subseteq \Lambda_R(0)$, but this must not be true for $\Lambda^{R,t}_s(n)$ if $s > t$.

In order to pass from the probabilistic result to what we need in the analytic part, we will need the following proposition.

Proposition 7.2. Let $Q \geq K + 1$ and $R \geq r \geq 1$. Given $K$ points $n_1, \ldots, n_K \in \Lambda_R(0)$ and $Q$ length scales $r \leq r_1 \leq s_1 \leq \cdots \leq r_Q \leq s_Q \leq R$ satisfying

$$r_{q+1} \geq 3s_q.$$

There exists a sequence of points $m_1, \ldots, m_J$ and numbers $q_1, \ldots, q_J$ with $J \leq K$ such that

1. For $i \neq j$
   $$\Lambda^{R}_{s_{q_i}}(m_i) \cap \Lambda^{R}_{s_{q_j}}(m_j) = \emptyset$$
2. For each $1 \leq k \leq K$, there exists $j$ such that
   $$\Lambda_r(n_k) \subseteq \Lambda^{R}_{s_{q_j}}(m_j).$$

We will need the following lemma.

Lemma 7.3. Let $q \leq \tilde{q}$ and $m, \tilde{m} \in \Lambda_R(0)$. Assume $r_{q+1} \geq 3s_q$ and

$$\Lambda_{s_q}(m) \cap \Lambda_{s_q}^R(\tilde{m}) \neq 0.$$

Then

$$\Lambda_{s_q}(m) \subseteq \Lambda_{r_{q+1}}(\tilde{m}).$$

Proof. Let $x \in \Lambda_{s_q}(m) \cap \Lambda_{s_q}^R(\tilde{m})$. For any $y \in \Lambda_{s_q}(m)$, we have $|y - x|_\infty \leq 2r_q$. Thus

$$|y - \tilde{m}|_\infty \leq |y - x|_\infty + |y - \tilde{m}|_\infty \leq s_q + 2r_q \leq 3s_{q+1} \leq r_{q+1}.$$ 

This implies the claim. \qed

We now come to

Proof of Proposition 7.2. The proof of this result is constructive, and proceeds by induction.

Base case of the induction: Assume $K = 1$. Then $Q = 1$, $m_1 = n_1$, and $q_1 = 1$. It is clear that

$$\Lambda_r(n_1) \subseteq \Lambda_{r_{q_1}}(m_1).$$

This finishes the base case.

Let us now do the induction step. So assume we have a solution for $K - 1$, and we are given the point $n_K \in \Lambda_R(0)$. Then there are several cases, depending on the location of $\Lambda_r(n_K)$ with respect to the $\Lambda_{r_{q_j}}(m_j)$.

Case 1: Assume there exists $j$ such that

$$\Lambda_r(n_K) \subseteq \Lambda_{r_{q_j}}(m_j).$$

Then we do nothing.
**Case 2:** Assume that for $1 \leq j \leq J$, we have

$$\Lambda_{s_1}(n_K) \cap \Lambda_{s_{q_j}}(m_j) = \emptyset. \quad (7.10)$$

Then, we add $n_k$ and 1 to the $m_j$ and $q_j$.

**Case 3:** There exists $1 \leq i \leq J$ such that

$$\Lambda_{s_1}(n_K) \cap \Lambda_{s_{q_i}}(m_i) \neq \emptyset. \quad (7.11)$$

Then, we choose an $i$ satisfying the previous condition such that $q_i$ is maximal. We then increase $q_i$ by 1. It is clear that now our second condition holds since we will have $\Lambda_{r}(n_K) \subseteq \Lambda_{r_{q_i+1}}(m_i)$. But it is not clear that the $\Lambda_{s_{q_j}}(m_j)$ are still disjoint from $\Lambda_{s_{q_i+1}}(m_i)$.

If for some $j \neq i$

$$\Lambda_{s_{q_j}}(m_j) \cap \Lambda_{s_{q_i+1}}(m_i) \neq \emptyset,$$

then we increase the larger $q_u$ by 1, and remove the other cube. This is possible by the previous lemma. It is clear that this process terminates, and generates the goal of the inductive scheme.

We have already seen the existence. Let us now discuss the bound on $Q$. Each time we increase one of the $q_j$, we remove one of the $n_k$ from the list of $m_j$. Since, we only have $K$ many $n_k$, we see that we can increase $q_j$ at most $K$ times. This finishes the proof. $\square$

### 8. Under the assumption of a Wegner estimate

In this section, we will give the multi-scale argument under the assumption of a Wegner estimate. We will choose $\tau = \frac{1}{2}$, $p = 3$,

$$K = 1, \quad R \in \left[ r^{1 + \frac{d}{2}}, r^{1 + \frac{d}{2}} \right], \quad \alpha = \frac{2d + 2}{2d - 1}. \quad (8.1)$$

The first step is the following probabilistic estimate

**Lemma 8.1.** Assume that for $r \geq r_0$ and some fixed $C > 0$, $b \geq 0$, and $\sigma > 0$.

$$\mathbb{E} \left( \text{tr} \left( P_{[E - \varepsilon, E + \varepsilon]} \left( H_{\Lambda,\omega}^{\Lambda_r(n)} \right) \right) \right) \leq \frac{C \cdot (\# \Lambda_r(n))^b}{(\log(\varepsilon^{-1}))^{2\alpha + 2d(\alpha + 1) + \sigma}}. \quad (8.2)$$

Assume $R \geq R_0(d, C, \sigma)$. Then there exists a set $\mathcal{B}_W^R$ with the properties

(i) $\mathbb{P}(\mathcal{B}_W^R) \leq \frac{1}{2} \cdot \frac{1}{R^\sigma}$.

(ii) For $\omega \notin \mathcal{B}_W^R$ and $\Lambda_{s}(n) \subseteq \Lambda_{R}(0)$, we have

$$\|(H_{\Lambda,\omega}^{\Lambda_r(n)} - E)^{-1}\| \leq \frac{1}{8} e^{\sqrt{\sigma}}. \quad (8.3)$$

**Proof.** The number of subcubes $\Lambda_{s}(n) \subseteq \Lambda_{R}(0)$ is bounded by

$$R \cdot \#(\Lambda_{R}(0)) \leq 3d \cdot R^{d+1}.$$

By Markov’s inequality, we have

$$\mathbb{P} \left( \|(H_{\Lambda,\omega}^{\Lambda_r(n)} - E)^{-1}\| \leq \frac{1}{8} e^{\sqrt{\sigma}} \right) \leq \mathbb{E} \left( \text{tr} \left( P_{[E - \varepsilon, E + \varepsilon]} \left( H_{\Lambda,\omega}^{\Lambda_r(0)} \right) \right) \right),$$

where $\varepsilon = 8e^{-\sqrt{\sigma}}$. The claim follows after some computations. $\square$
Introduction

\[ B^R = B_{1}^R \cup B_{W}^R, \]

where \( B_{1}^R \) denotes the set from Proposition 5.11. If \( r \geq 1 \) is large enough, we have

\[ P(B_{r}) \leq \frac{1}{R^{\alpha}}. \]

We introduce in view of Theorem 6.2

\[ a = \left\lceil \frac{\sqrt{R}}{\gamma r} \right\rceil. \]

We now introduce a sequence of scales

\[ r_1 = 2r, \quad s_q = r_q, \quad r_{q+1} = \max(s_q + ar, 3s_q). \]

We now obtain

**Lemma 8.2.** Let \( \omega \not\in B^R \). There exist \( \tilde{m}_1, \ldots, \tilde{m}_J \) and \( q_1, \ldots, q_J \) such that

(i) For \( \Lambda_r(n) \subseteq \Lambda_R(0) \setminus \bigcup_{j=1}^{J} \Lambda_{r_{q_j}}(\tilde{m}_j) \) we have

\[ \Lambda_r(n) \text{ is } (\gamma, \frac{1}{2}, 1)\text{-suitable for } H_{\lambda, \omega} - E. \]

(ii) For \( i \neq j \), we have

\[ \Lambda_{r_{q_i} + ar}(\tilde{m}_i) \cap \Lambda_{r_{q_j} + ar}(\tilde{m}_j) = \emptyset. \]

**Proof.** Denote by \( m^1_\omega, \ldots, m^K_\omega \) the sites from conclusion (ii) of Proposition 5.11. The claim then follows by applying Proposition 7.2 to these and the length scales \( r_q \) and \( s_q \) defined above. \( \square \)

**Proof of Theorem 4.3.** Let \( \omega \not\in B^R \). Then by Theorem 6.2 and some computations, we have

\[ \Lambda_R(0) \text{ is } (\hat{\gamma}, \frac{1}{2}, 2)\text{-suitable for } H_{\lambda, \omega} - E, \]

where

\[ \hat{\gamma} = \gamma \left( 1 - \frac{200}{r^2} \right). \]

This finishes the proof. \( \square \)

9. Estimating the norm of the resolvent

In this section, we will discuss how to obtain estimates on the norm of the resolvent. The key point is that, we will develop methods that will allow us to exploit the knowledge obtained from Proposition 5.11 to find a replacement of the Wegner estimate. In particular, we will be able to show that the norm of the resolvent of \( H_{\lambda, \omega} - E \) restricted to the union of \( (\gamma, \tau, p) \)-suitable cubes of size \( r \) is bounded by \( e^{4r^2} \). Since the results of this section are independent of the specific form of the potential, we will again write \( H \) instead of \( H_{\lambda, \omega} \).
We will need to introduce some notation. Given two sets \( \Lambda \subseteq \Xi \subseteq \mathbb{Z}^d \), the boundary \( \partial \Xi | \Lambda \) of \( \Lambda \) in \( \Xi \) is defined by
\[
(9.1) \quad \partial \Xi | \Lambda = \{(x,y): x \in \Lambda, y \in \Xi \setminus \Lambda, |x-y|_1 = 1\}.
\]
The relevance of the boundary comes from that if \( u \) solves \( H \Xi u = Eu \), then we have for \( n \in \Lambda \) that
\[
(9.2) \quad u(n) = -\sum_{(x,y) \in \partial \Xi | \Lambda} \langle e_n, (H^\Lambda - E)^{-1} e_x \rangle u(y).
\]
We furthermore introduce the distance of \( n \in \mathbb{Z}^d \) to \( \partial \Xi | \Lambda \) by
\[
(9.3) \quad \text{dist}(n, \partial \Xi | \Lambda) = \min_{(x,y) \in \partial \Xi | \Lambda} |n-x|_\infty.
\]
We are now ready to introduce the following geometric condition.

**Definition 9.1.** A set \( \Xi \subseteq \mathbb{Z}^d \) is called \( r \)-acceptable, if the following conditions hold.

(i) The set \( \Xi \) is finite.
(ii) For any \( x \in \Xi \) there exists a cube \( \Lambda_r(n) \subseteq \Xi \) such that
\[
(9.4) \quad \text{dist}(x, \partial \Xi | \Lambda_r(n)) \geq \frac{r}{10}.
\]

If a set \( \Xi \) is \( r \)-acceptable, we can apply (9.2) to every \( n \in \Xi \) such that the off-diagonal decay condition in the definition of suitability is meaningful. Furthermore an \( r \)-acceptable set is always the union of cubes of size \( r \). We are now ready for

**Theorem 9.2.** Let \( \Xi \) be \( r \)-acceptable. Assume for any cube \( \Lambda_r(n) \subseteq \Xi \) that
\[
(9.5) \quad \Lambda_r(n) \text{ is } (\gamma, \tau, 0)\text{-suitable for } H - E,
\]
and the inequalities
\[
(9.6) \quad 4\#(\partial \Xi | \Lambda_r(n)) \leq e^{(r)^7}, \quad 40 \leq \gamma r, \quad r^7 \geq \log(2).
\]
Then
\[
(9.7) \quad \|(H \Xi - E)^{-1}\| \leq e^{3(r)^7}.
\]
It is worth pointing out that this estimate is independent of the size of \( \Xi \). This is for example not the case in the similar estimate of Lemma 2.2. in [14]. By the results of Section 6 we obtain

**Corollary 9.3.** Let \( R \geq r \geq 1, \tau \in (0,1), \) and \( p \geq 0 \). Assume (9.6), \( 3r^7 + p \log(2) \leq R^7 \), and for every subcube \( \Lambda_r(n) \subseteq \Lambda_R(0) \) that
\[
(9.8) \quad \Lambda_r(n) \text{ is } (\gamma, \tau, 0)\text{-suitable for } H - E
\]
Then
\[
(9.9) \quad \Lambda_R(0) \text{ is } (\tilde{\gamma}, \tau, p)\text{-suitable for } H - E
\]
where \( \tilde{\gamma} = \gamma \cdot \left(1 - \frac{1}{r+1} - \frac{10}{P} (\gamma r + R^7 + d \log(3))\right) \).

**Proof.** Use Proposition 6.1. \( \square \)

We furthermore have the following variant, which loosens the requirement on all subcubes of size \( r \) to be suitable. However, we will need to at least require control on the norm of the resolvent on the exceptional regions, which can be larger then size \( r \).
Theorem 9.4. Given $R \geq 1$, $\gamma > 0$, $\tau \in (0, 1)$, $m_1, \ldots, m_Q \in \Lambda_R(0)$, and $t_q \geq s_q + r$ for $1 \leq q \leq Q$. Assume the following

(i) For $1 \leq q \leq Q$, we have $\Lambda_q(m_q) \subseteq \Lambda_R(0)$.

(ii) The set

$$\Lambda_R(0) \setminus \left( \bigcup_{q=1}^{Q} \Lambda_{s_q+r}(m_q) \right)$$

is $r$-acceptable.

(iii) For $p \neq q$, we have

$$\Lambda_{t_p+2r+1}(m_p) \cap \Lambda_{t_q+2r+1}(m_q) = \emptyset.$$

(iv) For any cube

$$\Lambda_r(n) \subseteq \Lambda_R(0) \setminus \left( \bigcup_{q=1}^{Q} \Lambda_{s_q}(m_q) \right),$$

we have

$$\Lambda_r(n) \text{ is } (\gamma, \tau, 0)\text{-suitable for } H - E.$$

(v) For $1 \leq q \leq Q$, we have

$$\|(H^{\Lambda_q(m_q)} - E)^{-1}\| \leq e^{(t_q)^\tau}.$$

(vi) The inequalities

$$4\#(\Lambda_R(0)) \leq e^{t_\infty \tau}, \quad \gamma r \geq 10(t_\infty)^\tau + 10 \log(2\#(\partial_-(\Lambda_r(0)))),$$

where $t_\infty = \max_{1 \leq q \leq Q} t_q$.

Then

$$\|(H^{\Lambda_R(0)} - E)^{-1}\| \leq e^{8(t_\infty)^\tau}.$$

It should be noted that (i), (ii), and (iii) restrict the geometry of the sets $\Lambda_R(0)$, $\Lambda_q(m_q)$ and $\Lambda_q(m_q)$. Whereas (iv) and (v) consist of the assumptions on the operator $H - E$.

Remark 9.5. Theorem 9.2 and 9.4 remain valid if the potential of the Schrödinger operator is complex valued, since the proof only uses that $H$ is normal.

We now start with the proof of Theorem 9.2 and Theorem 9.4 for which one needs to specialize $\Xi = \Lambda_R(0)$. We first recall that

$$\|(H^\Xi - E)^{-1}\| = \frac{1}{\text{dist}(E, \sigma(H^\Xi))}.$$

Hence, there exists $\hat{E} \in \sigma(H^\Xi)$ such that

$$\|(H^\Xi - E)^{-1}\| = \frac{1}{|E - \hat{E}|}$$

and a solution $u$ of $H^\Xi u = \hat{E}u$ with $\|u\|_{\ell^2(\Xi)} = 1$. The strategy of the proof will to show that for $|E - \hat{E}|$ small no such solution can exist.
Lemma 9.6. Assume \( \Lambda_r(n) \) is \((\gamma, \tau, 0)\)-suitable for \( H - E \) and (9.6). Let
\[
|\hat{E} - E| \leq e^{-3(r)^\tau}.
\]
Then for \( x, y \in \Lambda_r(n) \) satisfying \(|x - y| \geq \frac{r}{10}\)
\[
|\langle e_x, (H^{\Lambda_r(n)} - \hat{E})^{-1} e_y \rangle| \leq \frac{1}{2\#(\partial \ldots \Lambda_r(0))}.
\](9.20)

Proof. By a computation as done in the proof of Lemma 5.3, we find
\[
\|e_x\| \leq e^{-3(r)^\tau}.
\]
\[
|\langle e_x, (H^{\Lambda_r(n)} - \hat{E})^{-1} e_y \rangle| = |\langle e_x, (H^{\Lambda_r(n)} - E)^{-1} e_y \rangle| + 2|E - \hat{E}||H^{\Lambda_r(n)} - E|^{-1}||^2.
\]

By (9.6) and the condition on \( \hat{E} \), we find
\[
|E - \hat{E}||H^{\Lambda_r(n)} - E|^{-1}||^2 \leq e^{-(r)^\tau} \leq \frac{1}{4} \frac{1}{\#(\partial \ldots \Lambda_r(0))}.
\]

The claim follows. \( \square \)

Proof of Theorem 9.2. Let \( \hat{E} \) be such that
\[
|\hat{E} - E| \leq e^{-3(r)^\tau}.
\]
Let \( u : \Xi \to \mathbb{C} \) solve \( H \Xi u = \hat{E}u \) with possibly \( u = 0 \). Since \( \Xi \) is finite, we can choose \( x \) such that
\[
|u(x)| = \max_{y \in \Xi} |u(y)|.
\]

Let \( \Lambda_r(n) \) be the set from the \( r \)-acceptable property of \( \Xi \) for \( x \). By (9.2) and the previous lemma, we can conclude
\[
|u(x)| \leq \frac{1}{2} \sup_{(\tilde{z}, \tilde{y}) \in \partial \ldots \Lambda_r(n)} |u(\tilde{y})| \leq \frac{1}{2} |u(x)|.
\]

This is only possible if \( u = 0 \). Hence, we see that we must have \( \text{dist}(E, \sigma(H \Xi)) \geq e^{-3(r)^\tau} \). By (9.15), the claim follows. \( \square \)

Similarly to Lemma 9.6 one can show

Lemma 9.7. Assume \( \Lambda_r(n) \) is \((\gamma, \tau, 0)\)-suitable for \( H - E \) and (9.15). Let
\[
|\hat{E} - E| \leq e^{-5(t)^\tau}.
\](9.21)

Then for \( x, y \in \Lambda_r(n) \) satisfying \(|x - y| \geq \frac{t}{10}\)
\[
|\langle e_x, (H^{\Lambda_r(n)} - \hat{E})^{-1} e_y \rangle| \leq \frac{1}{\#(\partial \ldots \Lambda_r(n))} e^{-2(t)^\tau}.
\](9.22)

Let \( \hat{E} \) satisfy
\[
|E - \hat{E}| \leq e^{-5(t)^\tau}.
\]

Assume \( u \) solves \( H^{\Lambda_r(0)} u = \hat{E}u \) and \( \|u\|_{L^2(\Lambda_r(0))} = 1 \). We can thus find \( x_0 \in \Lambda_R(0) \)
such that
\[
|u(x_0)| = \max_{x \in \Lambda_R(0)} |u(x)|.
\]

Since \( \|u\|_{L^2(\Lambda_R(0))} = 1 \), we have
\[
|u(x_0)|^2 \geq \frac{1}{\#(\Lambda_R(0))}.
\]

(9.25)
Our first goal will to localize $x_0$.

**Lemma 9.8.** Assume (ii), (iv), and (vi). Let

$$x \in \Lambda_R(0) \setminus \left( \bigcup_{q=1}^{Q} \Lambda_{s+r}(m_q) \right).$$

Then, we have

$$|u(x)| \leq e^{-2(t_\infty)}.$$  

**Proof.** By (ii) and (iv), we can apply Lemma 9.7 to $\Lambda_{s+r}(x)$. The claim follows from (9.2). \qed

**Proof of Theorem 9.4.** By the previous lemma, there is $1 \leq q \leq Q$ such that $x_0 \in \Lambda_{s+r}(m_q) \subseteq \Lambda_{t_q}(m_q)$. Define $v$ by

$$v(x) = \begin{cases} u(x), & x \in \Lambda_{s+r}(m_q); \\ 0, & \text{otherwise}. \end{cases}$$

We can now compute

$$\|(H^{\Lambda_{t_q}(m_q)} - E)v\| \leq \frac{1}{2} \sqrt{\#(\partial_{\Lambda_{t_q}(m_q)})} e^{-2(t_\infty)r} \leq \frac{1}{2} \|v\| e^{-t_\infty r}.$$

By (9.18), this implies $\|(H^{\Lambda_{t_q}(m_q)} - E)^{-1}\| \geq 2e^{t_\infty r}$. This contradicts (9.14) showing no such $u$ can exist. The claim follows. \qed

## 10. Cartan’s Lemma

In this section, we derive a simple matrix valued version of Cartan’s lemma. Cartan’s lemma was first used in the spectral theory context by Goldstein and Schlag \[33\] and in the form we use it by Bourgain, Goldstein, and Schlag in \[14\]. Then further improved on by Bourgain in \[8\], \[9\], \[11\], and \[12\]. In particular \[12\] is important for us, since it contains a version of Cartan’s lemma where the constant depends nicely on the dimension. We introduce for $r > 0$

$$D_r = \{ z \in \mathbb{C} : |z| < r \}$$

and we recall $D = D_6$.

**Proposition 10.1.** Let $f : \mathbb{D}_{2^n}^{2n} \rightarrow \mathbb{C}$ be an analytic function satisfying

$$\|f\|_{L^\infty(D_{2^n})} \leq 1, \quad |f(0)| \geq \varepsilon$$

Then for $s > 0$

$$\{|x \in [-1,1]^n : |f(x)| \leq e^{-s}\| \leq 60e^{3/2}2^n \exp \left( -\frac{s}{\log(\varepsilon^{-1})} \right).$$

This proposition is a variant of Lemma 1 in \[12\]. We note that the $2^n$ on the right hand side corresponds to the measure of $[-1,1]^n$. So that the constant has a slight dependence on the dimension. It would be interesting to determine the optimal value. We now turn to the proof of Proposition 10.1. For this, we first recall the one dimensional version of Cartan’s lemma.
**Theorem 10.2.** Let \( f : \mathbb{D}_{2e} \to \mathbb{C} \) be analytic satisfying
\[
\sup_{|z| \leq 2e} |f(z)| \leq 1, \quad |f(0)| \geq \epsilon.
\]
Then for \( s > 0 \)
\[
|\{ x \in [-1, 1] : \ |f(x)| \leq e^{-s} \}| \leq 30e^3 \exp \left( -\frac{s}{\log(\epsilon^{-1})} \right).
\]

**Proof.** We apply Theorem 11.3.4. in Levin’s book \[43\] to \( g(z) = \frac{1}{\epsilon} f(z) \) with \( R = 1, \ M_g(2e) \leq \frac{1}{\epsilon} \) and
\[
\log(\eta) = -\frac{s}{\log(\epsilon^{-1})} + \log(15e^3).
\]
The claim follows using \( |f(x)| \leq |g(x)| \). \( \square \)

We now come to

**Proof of Proposition 10.1.** We can write \( x \in \mathbb{R}^n \) as \( x = r\vartheta \), where \( \vartheta \in S^{n-1} \) and \( r > 0 \). Define \( r^{\max}_\vartheta \) as the maximal choice of \( r > 0 \) such that \( r\vartheta \in [-1, 1]^n \). Then one finds that
\[
2^n = |[-1, 1]^n| = c_n \int_{S^{n-1}} \int_0^{r^{\max}_\vartheta} r^{n-1} dr d\vartheta = c_n \int_{S^{n-1}} \frac{1}{n} (r^{\max}_\vartheta)^n d\vartheta.
\]
In particular that \( c_n \int_{S^{n-1}} r(\vartheta)^n d\vartheta = n2^n \). Introduce
\[
E_s = \{ x \in [-1, 1]^n : \ |f(x)| \leq e^{-s} \}.
\]
We have
\[
|E_s| = c_n \int_{S^{n-1}} \int_0^{r^{\max}_\vartheta} \chi_{E_s}(r\vartheta)r^{n-1} dr d\vartheta.
\]
We will now analyze the inner integral for fixed \( \vartheta \in S^{n-1} \). Define \( g_\vartheta(z) = f(\vartheta r^{\max}_\vartheta z) \), then we have that
\[
\int_0^{r^{\max}_\vartheta} \chi_{E_s}(r\vartheta)r^{n-1} dr \leq (r^{\max}_\vartheta)^n \int_0^1 \chi_{F_s(\vartheta)}(x) (xr^{\max}_\vartheta)^{n-1} dx
\]
\[
\leq (r^{\max}_\vartheta)^n \int_0^1 \chi_{F_s(\vartheta)}(x) dx = (r^{\max}_\vartheta)^n |F_s(\vartheta)|
\]
where
\[
F_s(\vartheta) = \{ x \in [-1, 1] : \ |g_\vartheta(x)| \leq e^{-s} \}.
\]
Observe now that \( r^{\max}_\vartheta \vartheta \mathbb{D}_{2e} \subseteq \mathbb{D}_{2e}, \ |g_\vartheta(0)| \geq \epsilon, \) and \( \sup_{|z| \leq 2e} |g_\vartheta(z)| \leq 1. \) Hence, we obtain by Cartan’s lemma that
\[
|F_s| \leq 30e^3 \exp \left( -\frac{s}{\log(\epsilon^{-1})} \right).
\]
The claim now follows using that \( r^{\max}_\vartheta \leq \sqrt{2n}. \) \( \square \)

We will now derive a matrix-valued version of this result. The main idea is to apply Proposition 10.1 to the determinant of the matrix.
Theorem 10.3. Let $B \geq N$, $D \geq 1$, $s > 0$, and $A : \mathbb{D}^n \to \mathbb{C}^{N \times N}$ be an analytic function. Assume there exists $x_0 \in \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ such that
\begin{equation}
\sup_{z \in \mathbb{D}^n} \|A(z)\| \leq B, \quad \text{and} \quad \|A(x_0)^{-1}\| \leq D.
\end{equation}
Assume
\begin{equation}
\frac{s}{N} \geq 20n \log(B \cdot D).
\end{equation}
Then
\begin{equation}
\left|\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^n : \|A(x)\| \geq e^s\}\right| \leq \exp\left(-\frac{1}{2} \frac{s}{N \log(BD)}\right).
\end{equation}

In order to pass from the determinant to information on the matrix, we use the following lemma.

Lemma 10.4. Let $A$ be an $N \times N$ matrix.
\begin{itemize}
  \item[(i)] $|\det(A)| \leq \|A\|^N$.
  \item[(ii)] $|\det(A)| \geq \frac{1}{\|A^{-1}\|^N}$.
  \item[(iii)] $\|A^{-1}\| \leq \frac{N\|A\|^N}{|\det(A)|}$.
\end{itemize}

Proof. Denote by $\lambda_j$ the eigenvalues of $A$ and order them by modulus $0 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_N|$. By $\det(A) = \prod_{j=1}^N \lambda_j$, $|\lambda_N| \leq \|A\|$, and $|\lambda_1| \geq \frac{1}{\|A^{-1}\|}$, the first two claim follows. For the third claim, use the first one to show that for any minor $\tilde{A}$ of $A$, we have $|\det(\tilde{A})| \leq \|\tilde{A}\|^{n-1}$. □

Proof of Theorem 10.3 Define $\varphi(z) = \det(A(z))$. By Lemma 10.4 we have that
\begin{equation}
\sup_{z \in \mathbb{D}^n} |\varphi(z)| \leq B^N \quad \text{and} \quad |\varphi(x_0)| \geq \frac{1}{D^N}.
\end{equation}
Hence, we can apply Proposition 10.1 to the function
\begin{equation*}
f(z) = \frac{1}{B^N} \varphi(z + x_0).
\end{equation*}
Since $2e + \frac{1}{2} < 6$, we obtain $|f(z)| \leq 1$ for $z \in (\mathbb{D}2e)^n$, and $|f(0)| \geq \varepsilon = \left(\frac{1}{BD}\right)^N$. By Proposition 10.1 we obtain
\begin{equation*}
\left|\{x \in [-\frac{1}{2}, \frac{1}{2}]^n : |f(x)| \leq e^{-s+2N \log(B)}\}\right| \leq 60e^3n^{3/2}2^s \exp\left(-s - 2N \log(BD)\right).
\end{equation*}
By the previous lemma and $B \geq N$, we have
\begin{equation*}
\|A(x)^{-1}\| \leq \frac{B^N}{|\det(A(x))|} = \frac{B^{2N}}{|f(x)|}.
\end{equation*}
The claim follows by some computations. □

Let us now discuss, what happens if we replace the Lebesgue measure $|.|$ by an absolutely continuous measure with density $\rho$. First note, that then the measure in the case $n = 1$ of a set is just
\begin{equation}
\mu(A) = \int_A \rho(x)dx.
\end{equation}
In particular, when the density $\rho$ is bounded, and supported in $[-\frac{1}{2}, \frac{1}{2}]$, we obtain the simple estimate
\begin{equation}
(10.10) \quad \mu(A) \leq \|\rho\|_{\infty} \cdot |A|.
\end{equation}
Of course, if we do this now for $A \subseteq [-\frac{1}{2}, \frac{1}{2}]^n$, we obtain
\begin{equation}
(10.11) \quad \mu_{\otimes n}(A) \leq (\|\rho\|_{\infty})^n \cdot |A|.
\end{equation}
Hence, we see that our results remain valid.

**Theorem 10.5.** Let $B \geq N$, $D > 0$, $s > 0$, $A : \mathbb{D}^n \to \mathbb{C}^{N \times N}$ be an analytic function, and $\mu$ an absolutely continuous measure with bounded density $\rho$ and $\text{supp}(\mu) \subseteq [-\frac{1}{2}, \frac{1}{2}]$. Assume there exists $x_0 \in [-\frac{1}{2}, \frac{1}{2}]^n$ such that
\begin{equation}
(10.12) \quad \sup_{z \in \mathbb{D}^n} \|A(z)\| \leq B, \quad \text{and} \quad \|A(x_0)^{-1}\| \leq D.
\end{equation}
Assume
\begin{equation}
(10.13) \quad \frac{s}{N} \geq 25n \log(\|\rho\|_{\infty}) \cdot \log(B \cdot D).
\end{equation}
Then
\begin{equation}
(10.14) \quad \mu_{\otimes n}(\{x \in [-\frac{1}{2}, \frac{1}{2}]^n : \|A(x)\| \geq e^s\}) \leq \exp\left(\frac{-1}{4N} \log(BD)\right).
\end{equation}

11. **Cartan’s lemma for Schrödinger operators**

In this section, we will state a version of the results of the last section suited for our application. In particular, we will combine them with the results of Section 9. We will denote by $B(\ell^2(\Lambda_R(0)))$ the Banach space of Schrödinger operators $\ell^2(\Lambda_R(0)) \to \ell^2(\Lambda_R(0))$ with not necessarily real potential. Given $A \in B(\ell^2(\Lambda_R(0)))$, we denote by $A^\Xi$ the restriction of $A$ to $\ell^2(\Xi)$, where $\Xi \subseteq \Lambda_R(0)$. Similar results can be found in the work of Bourgain, see for example Lemma 2 in [12]. We recall $\mathbb{D} = \{z : \ |z| < 6\}$.

**Theorem 11.1.** Let
\begin{equation}
(11.1) \quad H : \mathbb{D}^n \to B(\ell^2(\Lambda_R(0)))
\end{equation}
be an analytic function taking values in the normal operators. Given $\Lambda_{r,j}(m_j) \subseteq \Lambda_R(0)$ for $1 \leq j \leq J$. Assume
\begin{enumerate}[(i)]
\item The set
\begin{equation}
(11.2) \quad \Xi = \Lambda_R(0) \setminus \left( \bigcup_{j=1}^J \Lambda_{r,j}(m_j) \right)
\end{equation}
is r-acceptable.
\item For $i \neq j$
\begin{equation}
(11.3) \quad \Lambda_{r,i+2r}(m_i) \cap \Lambda_{r,j+2r}(m_j) = \emptyset.
\end{equation}
\item The bound
\begin{equation}
(11.4) \quad \sup_{z \in \mathbb{D}^n} \|H(z)\| \leq e^{e^r}.
\end{equation}
\item For any subcube $\Lambda_r(n) \subseteq \Xi$ and $z \in \mathbb{D}^n$
\begin{equation}
(11.5) \quad \Lambda_r(n) \text{ is } (\gamma, r, 1)-\text{suitable for } H(z).
\end{equation}
\end{enumerate}
There exists \( x_0 \in \left[ -\frac{1}{2}, \frac{1}{2} \right]^n \) such that for \( 1 \leq j \leq J \)
\[
\| H^{\Lambda, r_j(m_j)}(x_0)^{-1} \| \leq e^{\sigma(r_j)\tau}.
\]

The measure \( \mu \) is absolutely continuous with bounded density \( \rho \).

Let \( r_{\infty} = \max_{1 \leq j \leq J} r_j \). Let \( S \geq T > 0 \). The inequalities
\[
\frac{S}{T} \geq 1200 \cdot 3^d J(r_{\infty})^{d+2\tau},
\]
\[
S \geq \max \left( 10000 J 3^d (r_{\infty})^{d+2\tau} n \log(\| \rho \|_\infty), 2(p+4) \log(2) + 10r^\tau \right).
\]
Then
\[
\rho^{\otimes n} \left( \left\{ x \in \left[ -\frac{1}{2}, \frac{1}{2} \right]^n : \| H(x) \| \geq \frac{1}{2^p e^{S}} \right\} \right) \leq e^{-T}.
\]

In our applications, we will have \( n \approx \#(\Xi) \). This is the big difference to Schrödinger operators with quasi-periodic or skew-shift potential, where \( n \) is a small fixed number.

We will now begin with the setup of the proof. As usually, it takes some notation. Introduce
\[
\Theta = \bigcup_{j=1}^J \Lambda_{r_j(m_j)}.
\]
We note that \( \Lambda_R(0) = \Xi \cup \Theta \). By the results of Section \ref{sec:setup}, we can conclude that

**Lemma 11.2.** We have that
\[
(i) \text{ For } z \in \mathbb{D}^n, \text{ we have }
\]
\[
\| (H^\Xi(z))^{-1} \| \leq e^{3r\tau}.
\]
\[
(ii) \text{ We have }
\]
\[
\| (H(x_0))^{-1} \| \leq e^{5(r_{\infty})\tau}.
\]

**Proof.** These are consequences of Theorem \ref{thm:approximation} and Theorem \ref{thm:continuity} \( \square \)

We now write our operator as a block matrix
\[
H(z) = \begin{pmatrix} H^\Xi(z) & \Gamma_1(z) \\ \Gamma_2(z) & H^\Theta(z) \end{pmatrix}.
\]

From (iii), we have that
\[
\| H^\Xi(z) \|, \| \Gamma_1(z) \|, \| \Gamma_2(z) \|, \| H^\Theta(z) \| \leq e^{r\tau},
\]
for \( z \in \mathbb{D}^n \). We now recall the Schur complement formula.

**Lemma 11.3** (Schur complement). Assume \( A \) is invertible. Then
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1},
\]
is invertible, if and only if
\[
S = D - CA^{-1}B
\]
is invertible. Furthermore then
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}.
\]
In particular, if we have
\[(11.18) \quad \|S^{-1}\| \leq \left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \right\|\]
and
\[(11.19) \quad \left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \right\| \leq (1 + \|S^{-1}\|)(1 + \|A^{-1}\|)^2(1 + \|B\|)(1 + \|C\|).
\]
We now switch to our explicit setting with
\[(11.20) \quad A = H^\Xi(z), \quad B = \Gamma_1(z), \quad C = \Gamma_2(z), \quad D = H^\Theta(z).
\]
We then have
\[(11.21) \quad S(z) = H^\Theta(z) - \Gamma_2(z)(H^\Xi(z))^{-1}\Gamma_1(z).
\]
By (iii), we have that
\[(11.22) \quad \sup_{z \in \Delta^n} \|S(z)\| \leq 2e^{5r^7},
\]
and from (11.18) that \(\|S(x_0)^{-1}\| \leq e^{5(r_\infty)^7}.
\]

**Proof of Theorem 11.1.** We have that \(S(z)\) satisfies the assumptions of Theorem 10.5. We apply it with
\[s = S - 5r^7 - (p + 4) \log(2), \quad N \leq 3dJ(r_\infty)^d, \quad B \leq 2e^{5(r_\infty)^7}, \quad D \leq e^{5(r_\infty)^7}.
\]
We obtain a set
\[B \subseteq \left[ -\frac{1}{2}, \frac{1}{2} \right]^n
\]
satisfying
\[\mu^{\otimes n}(B) \leq \exp \left( -\frac{1}{4} \frac{r^7}{N \log(B \cdot D)} \right).
\]
such that for \(x \notin B\), we have
\[\|S(x)^{-1}\| \leq \frac{1}{16} e^{-5r^7} \frac{1}{2^p} e^S
\]
and By (11.19), we have
\[\|H(x)\| \leq 16e^{5r^7} \cdot \|S(x)\|.
\]
The claim follows. \(\square\)

12. Further probabilistic estimates and combinatorial results

In this section, we return to result specific for the operator \(H_{\lambda, \omega}\). We will first prove a variant of the probabilistic estimate, Proposition 6.11 which will allow to obtain (9.14). Second, we will improve on Proposition 7.2 in order to obtain the geometric conditions from Theorem 9.4. After this section, we will have to combine all our results to prove the multi-scale step.

We now prove the second probabilistic estimate, which follows ideas from Bourgain’s work [12].
Proposition 12.1. Given $s_1 \geq -1$, $t_1 \geq \max(r + \tau, s_1 + \tau)$, and for $2 \leq k \leq K$ (12.1) 
\[ s_k \geq t_{k-1}, \quad t_k \geq s_k + \tau. \]
Assume (2.21), (5.8), and for $r \leq u \leq t_K$ that (12.2) 
\[ \mathbb{P}(\Lambda_n(0) \text{ is not } (\gamma, \tau, p)\text{-suitable for } H_{\lambda,\omega} - E) \leq \varepsilon. \]
Then there exists a set $B^{R,K}_2$ with the following properties.

(i) $\mathbb{P}(B^{R,K}_2) \leq (3R)^d e^K$.

(ii) Let $\omega \notin B^{R}_2$ and $n \in \Lambda_R(0)$. There exists $1 \leq k \leq K$ such that for some (12.3) 
\[ \hat{\omega} = \omega \pmod{\Lambda^{R,t_k-1}_k(n)} \]
we have (12.4) 
\[ \Lambda_k^n(n) \text{ is } (\gamma, \tau, p-1)\text{-suitable for } H_{\lambda,\hat{\omega}} - E. \]

The proof is similar to the one of Proposition 5.11. For $n \in \Lambda_R(0)$, introduce the set $B^n$ such that (ii) fails.

Lemma 12.2. Assume (12.2) and let $n \in \Lambda_R(0)$. Then we have $\mathbb{P}(B^n) \leq \varepsilon^K$.

Proof. We first discuss the statement of the analog of Lemma 5.10 in this context. Let $1 \leq k \leq K$. Introduce $Y^{k,q}$ as the set of $\omega$ such that for every $\hat{\omega} = \omega \pmod{\Lambda^{R,t_k-1}_k(n)}$, we have 
\[ \Lambda^{R}_k(n) \text{ is not } (\gamma, \tau, q)\text{-suitable for } H_{\lambda,\hat{\omega}} - E. \]
By (12.2), we have $\mathbb{P}(Y^{k,q}) \leq \varepsilon$. Denote by $X^k$ the set of $\omega$ such that there exists $\hat{\omega} = \omega \pmod{\Lambda^{R,t_k-1}_k(n)}$ such that 
\[ \hat{\omega} \in Y^{k,p-1}. \]
One should note that $\omega \in X^k$ and 
\[ \hat{\omega} = \omega \pmod{\Lambda^{R,t_k-1}_k(n) \cup \Lambda^{R}_k(n)} \]
then also $\hat{\omega} \in X^k$.

By Lemma 5.8 we have that $X^k \subseteq Y^{k,p}$, which implies $\mathbb{P}(X^k) \leq \varepsilon$. As in Lemma 5.10 one checks that that $X^k$ and $X^\ell$ are independent for $k \neq \ell$.

We also have 
\[ B^n \subseteq \bigcap_{k=1}^K X^k, \]
which implies the claim. \qed

Proof of Proposition 12.1 Define 
\[ B^{R}_2 = \bigcup_{n \in \Lambda_R(0)} B^n. \]
From the definition of $B^n$, we have that (ii) holds. By the previous lemma (i) follows. \qed

We are now done with the probabilistic part of this section, and now move on to the combinatorics. We will show a variant of Proposition 7.2 which also allows us to prove the geometric conditions of Theorem 9.4.
The main difference is that we will not only ensure that the sets \( \Lambda_{s_{q_j}}(m_j) \) are disjoint, but we will also wish to ensure that the set

\[
\Lambda_R(0) \setminus \left( \bigcup_{j=1}^{J} \Lambda_{\tilde{r}_j}(m_j) \right)
\]

is \( r \)-acceptable for any choice \( r_{q_j} \leq \tilde{r}_j \leq s_{q_j} - 2r - 1 \).

**Theorem 12.3.** Let \( Q \geq (d+1)K+1 \) and \( R \geq r \geq 1 \). Given \( K \) points \( n_1, \ldots, n_K \in \Lambda_R(0) \) and \( Q \) length scales \( r \leq r_1 \leq s_1 \leq \cdots \leq s_Q \leq R \) satisfying (13.3) and (12.6)

\[
s_q \geq r_q + 3r.
\]

Then there exists a sequence of points \( m_1, \ldots, m_J \) and numbers \( q_1, \ldots, q_J \) with \( J \leq K \) such that (i) and (ii) from Proposition 7.2 and

(iii) For any choice \( r_{q_j} \leq \tilde{r}_j \leq s_{q_j} - 2r - 2 \), we have that the set

\[
\Lambda_R(0) \setminus \left( \bigcup_{j=1}^{J} \Lambda_{\tilde{r}_j}(m_j) \right)
\]

is \( r \)-acceptable.

In order to prove this theorem, we prove two preliminary lemmas, which study properties of being \( r \)-acceptable from Definition 9.1.

**Lemma 12.4.** Let \( 1 \leq k \leq K \). There exists a set \( Q^k \) such that

(i) \( \#(Q^k) \leq d \).

(ii) For \( q \not\in Q^k \) and \( r_q \leq \tilde{r} \leq s_q - r - 1 \), we have that

\[
\Lambda_R(0) \setminus \Lambda^R_q(n)
\]

is \( r \)-acceptable.

**Proof.** Choose \( Q^k \) as the set of \( q \) such that for some \( 1 \leq j \leq d \)

\[|n_j| + r_q \leq R \leq |n_j| + s_q.\]

Clearly we have at most \( d \) such choices, and if neither of the two conditions is the case, then the above set is \( r \)-acceptable. \( \square \)

**Lemma 12.5.** Suppose that

\[
\Lambda_R(0) \setminus \left( \bigcup_{i=1}^{I-1} \Lambda^R_{r_i}(n_i) \right)
\]

and

\[
\Lambda_R(0) \setminus \Lambda^R_{r_I}(n_I).
\]

are \( r \)-acceptable. Furthermore suppose that for \( i \neq I \)

\[
\Lambda^R_{r_{I+2r+1}}(n_I) \cap \Lambda^R_{r_I+2r+1}(n_I) = \emptyset.
\]

Then

\[
\Lambda_R(0) \setminus \left( \bigcup_{i=1}^{I} \Lambda^R_{r_i}(n_i) \right)
\]

is \( r \)-acceptable.
Proof. Let
\[ \Xi = \Lambda_R(0) \setminus \left( \bigcup_{i=1}^{I} \Lambda_R(n_i) \right). \]
Let \( x \in \Lambda_{r_i+2r+1}(n_I) \cap \Xi \). By (12.10), we can find \( \Lambda_r(n) \subset \Xi \) such that
\[ \text{dist}(x, \partial \Xi \Lambda_r(n)) \geq \frac{r}{10}. \]

For \( x \in \Xi \setminus \Lambda_{r_i+2r+1}(n_I) \), the claim follows from (12.9). \( \square \)

**Proof of Theorem 12.3.** By the first lemma, we can eliminate \( Q \cdot K \) choices of the \( r_q, s_q \), so that the individual cubes would all satisfy the \( r \)-acceptability assumption.

By Proposition 7.2, we can now find a choice \( m_1, \ldots, m_J, q_1, \ldots, q_J \) such that (i) and (ii) hold. Then, by the previous lemma, we have that (iii) holds. \( \square \)

### 13. Setup for the proof of Theorem 4.5

In this section, we will begin the proof of Theorem 4.5 and for this reason no longer work in full generality. In particular, we will begin specializing to

\[ (13.1) \quad p = 3, \quad \tau = \frac{1}{2} \]

The main reason is that this way the numerical inequalities become somewhat more transparent. We will furthermore define

\[ (13.2) \quad \tilde{t} = \max(t, 3t, t + 3r), \]

which is motivated by the conditions of Theorem 12.3.

**Theorem 13.1.** Let \( K \geq 1, r \geq 1 \) and assume

\[ (13.3) \quad r \geq \left( \max(4, 2 + \frac{4\gamma}{c}) \right)^{2dK^2}. \]

Assume for \( r \leq u \leq r^3 \) that

\[ (13.4) \quad \mathbb{P}(\Lambda_u(0) \text{ is not } (\gamma, 1, 2, 3)\text{-suitable for } H_{\lambda, \omega} - E) \leq \epsilon. \]

Then for \( R \geq r^4 \) there exists a set \( B^R \) satisfying

(i) We have

\[ (13.5) \quad \mathbb{P}(B^R) \leq 3^d \left( \frac{3dK}{(K+1)!} + 2dK \right) R^{dK} \epsilon K. \]

(ii) For each \( \omega \notin B^R \) there exist \( 0 \leq L \leq K, \ m \in \Lambda_R(0)^L, s_\ell \geq 0, \) and \( \frac{1}{c} \leq t_\ell \leq r^3 \)

such that the following hold.

(a) The set

\[ (13.7) \quad \Lambda_R(0) \setminus \left( \bigcup_{\ell=1}^{L} \Lambda_{r_\ell+r}(m_\ell) \right) \]

is \( r \)-acceptable.

(b) For \( k \neq \ell \), we have

\[ (13.8) \quad \Lambda_{r_k}(m_k) \cap \Lambda_{r_\ell}(m_\ell) = \emptyset. \]
Correlated Random Operators

For

\[ \Lambda_r(n) \subseteq \Lambda_R(0) \setminus \left( \bigcup_{\ell=1}^{L} \Lambda_R^{\ell}(m_\ell) \right) \]

we have

\[ \Lambda_r(n) \text{ is } (\gamma, \frac{1}{2}, 2)-\text{suitable for } H_{\lambda, \omega} - E. \]

For \( 1 \leq \ell \leq L \), there exists \( 0 \leq \tilde{\ell} \leq s_\ell \)

\[ \tilde{\omega}_\ell = \omega \pmod{\Lambda_{s_\ell}(m_\ell)} \]

such that

\[ \| (H_{\lambda, \tilde{\omega}_\ell}^{s_\ell}(m_\ell) - E)^{-1} \| \leq \frac{1}{2} e^{\sqrt{t_\ell}}. \]

Furthermore the possible number of choices \( s_q, t_q \) in (ii) is bounded by \( r \).

The conditions in (ii) are chosen in such a way that the ones of Theorem 9.4 hold.

The proof of this theorem will proceed by combining the results of the last section with the ones from Section 5. Denote by \( B_{1,R,K}^{1} \) the set from Proposition 5.11. We see that (ii.c) holds for \( \omega \not\in B_{1,R,K}^{1} \) as long as for \( m_\omega \), we have

\[ \bigcup_{k=1}^{K} \Lambda_r(m_k^\omega) \subseteq \bigcup_{q=1}^{Q} \Lambda_R^{s_q}(m_q). \]

Next, we wish to apply Proposition 12.1. Introduce

\[ Q = (d + 1)K + 1, \quad s_1^1 = \tau \]

and

\[ t_1^Q = s\_k^Q, \quad s_1^Q = s\_k^{Q-1}, \quad t_1^Q = s\_k^{Q-1}. \]

We will write \( t_1^Q = \{ t_1^Q \}_k^{K} \) and \( s_1^Q = \{ s_1^Q \}_k^{K} \). These are chosen such that Theorem 12.1 will be applicable.

**Lemma 13.2.** The number of \( t_1^Q \) is bounded by \( 2dK^2 \). Furthermore, we have that

\[ t_1^Q \leq r^3. \]

**Proof.** The first claim follows by the number being bounded by \( QK \). For the second claim, we have by Lemma 5.10

\[ s_1^Q \leq T_0t_1^{Q-1}, \quad s_1^Q \leq T_0t_1^{Q-1}, \quad t_1^Q \leq T_0s_1^Q, \]

as long as the terms on the right hand side are greater than \( \frac{\log(\lambda)}{\epsilon} \). The claim follows since \( \tau \leq r^2 \). \( \square \)

Denote by \( B_{2,q}^{R,K} \) the set resulting by applying Proposition 12.1 to \( s_1^Q \) and \( t_1^Q \). We then introduce

\[ B_R = B_{1,R,K}^{1} \cup \left( \bigcup_{q=1}^{Q} B_{2,q}^{R,K} \right). \]

One can easily check that (i) holds.
Let now $\omega \notin B_{R,K}$. We have already seen that (ii.c) holds. Let’s now check (ii.a) and (ii.b). For this, we introduce
\begin{equation}
\hat{r}_q = s_q, \quad \hat{s}_q = t_q.
\end{equation}
Now we can apply Theorem 12.3 apply to the sequence $m^k$ and the $\hat{r}_q$ and $\hat{s}_q$. We see that (ii.a) and (ii.b) hold as long as we choose
\begin{equation}
s_\ell \in s_q, \quad t_\ell \in t_q.
\end{equation}
Since $\omega \notin B_{R,K}$, there is a choice of $k$ such that for
\begin{equation}
s_k \leq t_k \leq r_3,
\end{equation}
we have that (ii.d) holds. This finishes the proof of Theorem 13.1.

14. Application of Cartan’s lemma

The idea of this section is to study the measure of the set of $\omega$ such that conclusions (ii.a) to (ii.d) of Theorem 13.1 hold but
\begin{equation}
\Lambda_R(0) \text{ is not } (\gamma, 1/2, 3)-suitable for } H_{\lambda,\omega} - E
\end{equation}
for some fixed $\gamma$. In order to do this, we will show that the Schrödinger valued Cartan Theorem, Theorem 11.1, implies that the measure of $\omega$, such that the necessary resolvent estimates to apply Theorem 6.2 do not hold, is small.

Definition 14.1. We see that $m, s, t$ obey the geometric conditions for $\Lambda_R(0)$, if the following hold:
\begin{enumerate}
\item For $1 \leq \ell \leq L$, we have $m_\ell \in \Lambda_R(0)$ and $0 \leq \tilde{t}_\ell \leq s_\ell \leq t_\ell$.
\item We have
\begin{equation}
s_k \leq t_k \leq r_3.
\end{equation}
\item The set
\begin{equation}
\Xi = \Lambda_R(0) \setminus \left( \bigcup_{\ell=1}^{L} \Lambda_{r_3}^R(m_\ell) \right)
\end{equation}
is $r$-acceptable.
\item For $k \neq \ell$
\begin{equation}
\Lambda_{r_3}^R(m_k) \cap \Lambda_{r_3}^R(m_\ell) = \emptyset.
\end{equation}
\end{enumerate}
These purely geometric conditions correspond to (ii.a) and (ii.b) of Theorem 13.1.

We also note that (i) and (iii) ensure the conditions (ii) and (iv) of Theorem 6.2 and that (i) - (iv) also imply the conditions (i), (ii), and (iii) of Theorem 13.1.

We will now turn to introduce the conditions on the operator $H_{\lambda,\omega}$, which will depend on the random parameter $\omega$. Let $m, s, t$ obey geometric conditions for $\Lambda_R(0)$. Introduce the set $\Omega_{\gamma, \frac{1}{2}, 3}$ as the set of $\omega$ satisfying
\begin{enumerate}
\item For
\begin{equation}
\Lambda_r(n) \subseteq \Lambda_R(0) \setminus \left( \bigcup_{\ell=1}^{L} \Lambda_{r_3}^R(m_\ell) \right),
\end{equation}
\end{enumerate}
we have
\begin{equation}
\Lambda_r(n) \text{ is } (\gamma, 1/2, 3)-\text{suitable for } H_{\lambda,\omega} - E.
\end{equation}

(ii) For $1 \leq \ell \leq L$ there exists
\begin{equation}
\hat{\omega}_\ell = \omega \pmod{\Lambda R, \tilde{t}_\ell s_\ell (m_\ell)}
\end{equation}
such that
\begin{equation}
\| (H_{\lambda,\hat{\omega}_\ell (m_\ell)} - E)^{-1} \| \leq \frac{1}{2} e^{\sqrt{t}}.
\end{equation}

Remark 14.2. Theorem 13.1 implies that \begin{equation}
P \left( \Omega \setminus \bigcup_{m,s,\tilde{t},t \in \mathbb{C}} \Omega_{m,s,\tilde{t},t} \right) \lesssim (\varepsilon R)^K.
\end{equation}

Define \begin{equation}
t_\infty = \max_{1 \leq \ell \leq L} t_\ell.
\end{equation}

We are now ready for the main result of this section, which will use the analyticity of the map $\omega \mapsto H_{\lambda,\omega}$ for the first time.

**Theorem 14.3.** Assume Hypothesis 2.7, $\gamma \geq 1$
\begin{equation}
r \geq \max(20000 \cdot K \cdot 3^d, \log(\|\rho\|_\infty), 2^{dK}, (\log(2d+\lambda))^2)
\end{equation}
\begin{equation}
R \geq r^{3d+8}
\end{equation}
\begin{equation}
t_\infty \leq r^3.
\end{equation}
There exists a set $B_{m,s,\tilde{t},t}^{C}$ of measure \begin{equation}
P(B_{m,s,\tilde{t},t}^{C}) \leq e^{-Kr}
\end{equation}
such that for
\begin{equation}
\omega \in \Omega_{m,s,\tilde{t},t}^{C} \setminus B_{m,s,\tilde{t},t}^{C}
\end{equation}
we have
\begin{equation}
\Lambda_\gamma(0) \text{ is } (\hat{\gamma}, 1/2, 3)-\text{suitable for } H_{\lambda,\omega} - E,
\end{equation}
where $\hat{\gamma} = \gamma(1 - \frac{2}{r}).$

We now proceed to give the prove of this theorem. For this, we will need to switch from the probabilistic notions in statement to the more analytic notions in Theorem 11.1.

We fix some $\omega_0 \in \Omega_{m,s,\tilde{t},t}^{C}$. Introduce
\begin{equation}
\mathcal{M} = \bigcup_{\ell=1}^{L} \Lambda_{s_\ell}^{R,\tilde{t}_\ell (m_\ell)}, \quad \overline{\mathcal{M}} = \bigcup_{\ell=1}^{L} \Lambda_{t_\ell}^{R}(m_\ell).
\end{equation}
We introduce the restricted probability space $\widehat{\Omega}$ by
\begin{equation}
\widehat{\Omega} = \{ \omega : \omega = \omega_0 \pmod{\mathcal{M}} \}.
\end{equation}
In particular, we see that $\widehat{\Omega}$ is now $\#(\mathcal{M})$ dimensional. Using Fubini, we see that it is sufficient to prove the following variant of the main theorem.
Proposition 14.4. Assume $\gamma \geq 1$ (14.11), (14.12), and (14.13). There exists a set $\hat{B}$ of measure
\begin{equation}
\mu \otimes \#(M)(\hat{B}) \leq e^{-Kr}
\end{equation}
such that for $\omega \in \hat{\Omega} \setminus \hat{B}$, we have
\begin{equation}
\Lambda_R(0) \text{ is } (\hat{\gamma}, \frac{1}{2}, 3)\text{-suitable for } H_{\lambda, \omega} - E,
\end{equation}
and it will furthermore be convenient to identify
\begin{equation}
\hat{\Omega} \cong \left[ \frac{1}{2}, \frac{1}{2} \right]^{\nu},
\end{equation}
where
\begin{equation}
\nu = \#(M) \leq 3^d L \left( \max_{1 \leq \ell \leq L} s_\ell \right)^d.
\end{equation}
We now check the conditions of Theorem 11.1.

Lemma 14.5. The map $\omega \mapsto H_{\lambda, \omega} - E$ extends to analytic map
\begin{equation}
D^{\nu} \ni z \mapsto H(z).
\end{equation}
The following properties hold
(i) The bound
\begin{equation}
\sup_{z \in D^{\nu}} \|H(z)\| \leq e^{\sqrt{r}}.
\end{equation}
(ii) For
\begin{equation}
\Lambda_r(n) \subseteq \Lambda_R(0) \setminus \mathcal{M}
\end{equation}
we have
\begin{equation}
\Lambda_r(n) \text{ is } (\gamma, \tau, 0)\text{-suitable for } H(z).
\end{equation}

Proof. The existence of the analytic extension is a consequence of Hypothesis 2.7. From (2.21), we can conclude that
\begin{equation}
\sup_{z \in D^{\nu}} \|H(z)\| \leq 2d + \lambda.
\end{equation}
(i) now follows from (14.11).

Since $t_\ell \geq s_\ell + r - r$, we have
\begin{equation}
\Lambda_r(n) \subseteq \Lambda_R(0) \setminus \mathcal{M}.
\end{equation}
Thus we have $z = \omega_0 \mod \Lambda_r(n)$. So (ii) follows by Lemma 5.8. \qed

Lemma 14.6. There exists $\hat{\omega} \in \hat{\Omega}$ such that for $1 \leq \ell \leq L$, we have
\begin{equation}
\|(H_{\lambda, \hat{\omega}}^{\Lambda_r(n)} - E)^{-1}\| \leq e^{\sqrt{r}}
\end{equation}
or $\|H_{\lambda, \hat{\omega}}^{\Lambda_r(n)}(\hat{\omega})\| \leq e^{\sqrt{r}}.$
Proof. Observe that for \( k \neq \ell \), we have
\[
\Lambda_{R,t}^{k} (m_k) \cap \Lambda_{R,t}^{\ell} (m_\ell) = \emptyset.
\]
Define \( \hat{\omega} \) by
\[
\hat{\omega}_x = \begin{cases}
(\hat{\omega}_\ell)_x, & x \in \Lambda_{R,t}^{\ell} (m_\ell); \\
\omega_0, & \text{otherwise}.
\end{cases}
\]
It is easy to see that \( \hat{\omega} \in \hat{\Omega} \). The claim now follows by Lemma 5.8.

We also obtain

Lemma 14.7. Assume
\[
(14.28) \quad R^{\frac{1}{2}} \geq \max \left( 25 \cdot 3^d K \left( \max_{1 \leq \ell \leq L} t_\ell \right)^{d+\frac{1}{2}}, 100 \log(\|\rho\|_\infty) \sqrt{\max_{1 \leq \ell \leq L} t_\ell} \right).
\]

We have
\[
(14.29) \quad \mu^\otimes \nu (\{ \omega : \| (H_{L,0}^{\Lambda_t} - E)^{-1} \| > \frac{1}{8} e^{\sqrt{R}} \}) \leq e^{-R^{\frac{1}{2}}}.
\]

Proof. We will apply Theorem 11.1 here with
\[
S = R^{\frac{1}{2}}, \quad T = R^{\frac{1}{2}}.
\]
Then the claim follows after some computations.

We now proceed to prove the resolvent estimates on the cubes \( \Lambda_t (m_\ell) \). The main difficulty is that in order to apply Theorem 6.2, we will need to ensure (6.15) with some \( a \) such that \( t_\ell - s_\ell \geq ar \). Define
\[
a = r^{3d+3}.
\]

Lemma 14.8. Assume \( K \leq r, \gamma \geq 1 \)
\[
(14.30) \quad r \geq \max(20000 \cdot K \cdot 2^d, \log(\|\rho\|_\infty), 2^{2dK^2})
\]
\[
(14.31) \quad t_\infty \leq r^3.
\]

With \( S = r^3ar \) and \( T = Kr \), (11.1) holds.

Proof. A computation shows that (11.7) is equivalent to
\[
S \geq r^{3d+4}, \quad \frac{S}{T} \geq r^{3d+2}.
\]
The claim follows since \( T \leq r^2 \).

Let \( Q = (d+1)K + 1 \) and apply Theorem 12.3 with the \( r \) from it being
\[
(14.32) \quad t_\infty = \max_{1 \leq \ell \leq L} t_\ell
\]
and to the length scales
\[
(14.33) \quad r_q = t_\infty + (2q - 1)ar, \quad s_q = t_\infty + 2qar.
\]
Denote the resulting choice by \( \hat{t}_\ell \) and \( \hat{m}_\ell \). We now come to

Lemma 14.9. We have that
\[
(14.34) \quad \mu^\otimes \nu \left( \{ \omega : \| (H_{L,0}^{\Lambda_t} (m_\ell) - E)^{-1} \| > \frac{1}{8} e^{\sqrt{r}} \} \right) \leq e^{-K \cdot r}.
\]
Proof. Apply Theorem 11.1 with
\[ S = a\gamma r, \quad T = K \cdot r \]
and the claim follows by Lemma 14.8. □

We are now ready for

Proof of Proposition 14.4 First we see that using Lemma 14.9 and 14.7, we need to eliminate a set of measure
\[ L \cdot e^{-r} + e^{-R^{1/4}} \leq e^{-1/2r}. \]
Here, we used \( r \geq \frac{1}{2} \log(2K) \).

We are now in a situation, we can apply Theorem 6.2. We obtain that
\[ \frac{\gamma}{\gamma} \geq 1 - \frac{1}{r} - \frac{10}{R} \left( 2K(t_{\infty} + Kar) + \sqrt{R} + d\log(3) \right). \]
For \( R \geq \max(160, 40d\log(3))r \), this reduces to
\[ \frac{\gamma}{\gamma} \geq 1 - \frac{3}{2} \frac{1}{r} \frac{10}{R} (2K(t_{\infty} + Kar)) \]
By Lemma 13.2 this can be further reduced to
\[ \frac{\gamma}{\gamma} \geq 1 - \frac{3}{2} \frac{1}{r} \frac{1600K2^{dK^2}ar}{R} \]
By assumptions \( 10000K2^{dK^2}ar^2 \leq R \), it follows that \( \frac{\gamma}{\gamma} \geq \gamma(1 - \frac{3}{2}) \), which is what we wanted. □

15. Proof of Theorem 4.5
Our goal in this section is to complete the proof of Theorem 4.5. Before jumping into the proof, to take the time to check various inequalities. We define
\[ (15.1) \quad K = \left[ \frac{1}{2d \log(\max(4, 2 + \frac{\log(r)}{e}))} \right] \]
such that the last condition of (14.11) holds. We even obtain

Lemma 15.1. Define \( K \) by (15.1). Assume
\[ (15.2) \quad r \geq \log(2^{200d}, \|\rho\|_{\infty}). \]
Then (14.11) holds.
Furthermore, for \( r \geq 2^{4ad} \) we have \( K \geq a \)

Proof. The last two conditions hold by definition or assumption. From \( r \geq 2^{200d} \), we obtain by (15.1) that \( K \geq 10 \). Thus, we have
\[ r \geq 2^{2dK^2} \geq 2^{20K} \cdot 2^{2K} \cdot 2^{2d}, \]
since \( K \geq 10 \) and \( d \geq 1 \). This implies the first condition.
The last claim is a computation. □

We collect the result in the following lemma
Lemma 15.2. There exists a set $B^R_1$ such that
\begin{equation}
\mathbb{P}(B^R_1) \leq \frac{1}{2} \frac{1}{R^K}
\end{equation}
Furthermore the conclusions from (ii) of Theorem 13.1 hold for $\omega \notin B^R_1$.

Proof. We can apply Theorem 13.1 with $\varepsilon = \frac{1}{(r_0)^\alpha}$. To obtain a set $B^R_1$ satisfying
\begin{equation}
\mathbb{P}(B^R_1) \leq 3^d \left( \frac{3dK}{(K+1)!} + 2dK \right) \cdot \left( \frac{R^d}{(r_0)^\alpha} \right)^K
\end{equation}
By assumption and the choice of $R$, we have
\begin{equation}
3^d \left( \frac{3dK}{(K+1)!} + 2dK \right) \frac{1}{(r_0)^K} \leq \frac{1}{2}, \quad \frac{R^d}{(r_0)^\alpha-1} \leq \frac{1}{R}.
\end{equation}
The claim follows. \qed

Second, we will want to apply Theorem 14.3. For this, we first note

Lemma 15.3. Denote the number of choices of $m, s, \tilde{t}, t$ in Theorem 13.1 by $\#(\text{choices})$. We have that
\begin{equation}
\#(\text{choices}) \leq 2dK^2 (3R)^dK.
\end{equation}
Proof. The number of choices for $m$ is bounded by $(3R)^dK$. The claim now follows from the last statement in Theorem 13.1. \qed

Next, we apply Theorem 14.3 to all possible choice of $m, s, \tilde{t}, t$ to obtain a set $B^R_2$.

Lemma 15.4. For $r \geq 1$ large enough, we have that
\begin{equation}
\mathbb{P}(B^R_2) \leq \frac{1}{2} \frac{1}{R^K}
\end{equation}
Proof. This follows since $r$ is large enough to estimate
\begin{equation}
2dK^2 (3R)^dK e^{-rK} \leq \frac{1}{2} \frac{1}{R^K}.
\end{equation}
This finishes the proof. \qed

We now finally come to

Proof of Theorem 4.5. Comparing the conclusions of Theorem 13.1 and Theorem 14.3 we then obtain for \( \omega \notin B^R = B^R_1 \cup B^R_2 \) that
\begin{equation}
\Lambda_R(0) \text{ is } (\hat{\gamma}, \frac{1}{2} \cdot 3)-\text{suitable for } H_{\lambda, \omega} - E,
\end{equation}
where $\hat{\gamma} = \gamma(1 - \frac{2}{r^2})$.

Now again for $r$ large enough, we now have
\begin{equation}
\mathbb{P}(B^R) \leq \frac{1}{R^K}.
\end{equation}
This finishes the proof. \qed
16. A FIRST STEP TOWARDS LOCALIZATION

In this section, we begin to draw conclusions from the results of multi-scale analysis. The main topic will be to draw conclusions from the knowledge that the estimate
\begin{equation}
\Pr(\Lambda_{\eta}(0) \text{ is not } (\gamma, \tau, 2)\text{-suitable for } H_{\lambda, \omega} - E) \leq \frac{1}{r^{d/2}}
\end{equation}
holds for all energies $E$. In this section and the following sections, we will address questions about the spectral type of $H_{\lambda, \omega}$ and the dynamics of the time evolution $e^{-itH_{\lambda, \omega}}$. In Appendix C, we discuss what continuity property of the integrated density of states this implies.

I will begin by introducing the concept of generalized eigenfunction. We call a solution $\psi \neq 0$ of
\begin{equation}
H_{\lambda, \omega} \psi = E\psi
\end{equation}
interpreted as a formal difference equation a generalized eigenfunction, if it obeys the growth condition
\begin{equation}
|\psi(n)| \leq (1 + |n|)^{2d}.
\end{equation}
We call $E$ the generalized eigenvalue. Given $\varepsilon > 0$ and $R \geq 1$, we introduce $E_{\varepsilon, R}^{\varepsilon, R}$ as the set of all generalized eigenvalues $E$ of $H_{\lambda, \omega}$, where the generalized eigenfunction $\psi$ obeys
\begin{equation}
\sum_{x \in \Lambda_{R}(0)} |\psi(x)|^2 \geq \varepsilon.
\end{equation}
We introduce the set $E_{\lambda, \omega}$ of all generalized eigenvalues as
\begin{equation}
E_{\lambda, \omega} = \bigcup_{\varepsilon > 0} \bigcup_{R \geq 1} E_{\varepsilon, R}^{\varepsilon, R}.
\end{equation}
We furthermore, recall that given $\varphi \in \ell^2(\mathbb{Z})$, the associated spectral measure $\mu_{\varphi}^{\varepsilon, R}_{\lambda, \omega}$ is characterized by
\begin{equation}
\langle \varphi, (H_{\lambda, \omega} - z)^{-1} \varphi \rangle = \int \frac{1}{t - z} \, d\mu_{\varphi}^{\varepsilon, R}_{\lambda, \omega}(t).
\end{equation}
The importance of the generalized eigenfunctions comes from
\begin{proposition}
For any $\varphi \in \ell^2(\mathbb{Z})$, we have that
\begin{equation}
\mu_{\varphi}^{\varepsilon, R}_{\lambda, \omega}(\mathbb{R} \setminus E_{\lambda, \omega}) = 0.
\end{equation}
This means that all the spectral measures are supported on $E_{\lambda, \omega}$.
\end{proposition}
\begin{proof}
This is Proposition 7.4 in [35].
\end{proof}

With a slight abuse of notation, we will often write $\psi \in E_{\lambda, \omega}^{\varepsilon, R}$, if there exists $E \in E_{\lambda, \omega}^{\varepsilon, R}$ such that (16.1) holds and $\psi$ obeys (16.3) and (16.4). The main reason is that most of the following statements are concerned with the eigenfunctions and not the eigenvalues.

In particular, we obtain the following corollary
\begin{corollary}
Suppose we can show for any $\psi \in E_{\lambda, \omega}$ that $\psi \in \ell^2(\mathbb{Z}^d)$. Then the spectrum of $H_{\lambda, \omega}$ is pure point.
\end{corollary}
We will now turn towards investigating the set $\mathcal{E}_{\lambda,\omega}^{\varepsilon,R}$ for fixed $\varepsilon > 0$ and $R \geq 1$. This understanding will be important in order to be able to prove dynamical localization. The first result is

**Proposition 16.3.** Assume for $r \geq 2k_1$

\begin{equation}
\mathbb{P}(\Lambda_r(0) \text{ is not } (\gamma, \tau, 2)\text{-suitable for } H_{\lambda,\omega} - E) \leq \frac{1}{r^{3d}}.
\end{equation}

Let $k \geq k_1$ and

\begin{equation}
k \geq \frac{1}{\log(2)} \max \left(3\gamma, \log \left(\frac{8d}{\gamma}\right), \log \left(\frac{R}{2}\right)\right) + 1.
\end{equation}

Then there exists a set $B^L_k$ satisfying

(i) The measure estimate

\begin{equation}
\mathbb{P}(B^L_k) \leq 3\lambda e^{-2k}.
\end{equation}

(ii) Let $r = 2^k$. For $\omega_0 \notin B^L_k$, we have for

\begin{equation}
\omega = \omega_0 \pmod{\Lambda_{4r}(0)^c}
\end{equation}

that for $\varepsilon \geq e^{-\frac{2}{2}r}$

\begin{equation}
\text{dist}(\mathcal{E}_{\lambda,\omega}^{\varepsilon,R}, \sigma(H_{\lambda,\omega}^{\Lambda_{r}(0)}) \leq e^{-\frac{2}{2}r}.
\end{equation}

The proof of this proposition proceeds in several steps. Consider for $t$ and $E$ the set $B^L_t(E)$ of all $\omega$ such that for every

\begin{equation}
\tilde{\omega} = \omega \pmod{\Lambda_{t-r}(0) \cup \Lambda_{t+r}(0)^c}
\end{equation}

we have that for every $n$ with $|n|_\infty = t$

\begin{equation}
\Lambda_r(n) \text{ is } (\gamma, \tau, 1)\text{-suitable for } H_{\lambda,\omega} - E.
\end{equation}

We have the following lemma

**Lemma 16.4.** Assume \textbf{[16.5].} For $2d \leq t \leq r^3$, we have that

\begin{equation}
\mathbb{P}(B^L_t(E)^c) \leq \frac{1}{r^d}.
\end{equation}

Furthermore, we have for

\begin{equation}
|t_1 - t_2| \geq 2r + 1
\end{equation}

that $B^L_t(E)$ and $B^L_{t'}(E)$ are independent.

**Proof.** Denote by $X^r_n$ the set from Lemma \textbf{[5.10].} We have

\begin{equation}
B^L_t \subseteq \bigcap_{|n|_\infty = t} X^r_n.
\end{equation}

Since the number of such $t$ is bounded by $2dt^{d-1} \leq r^{3d}$, the claim follows. \qed

Define $t_j$ by

\begin{equation}
t_j = (1 + 2j)r.
\end{equation}

Introduce $B_k(E)$ as the set of $\omega$ such that for every $1 \leq j \leq r$, we have

\begin{equation}
\omega \in B^L_{t_j}(E).
\end{equation}
Furthermore, we have $t_r + r + 1 \leq 3r\tau$. Because of independence of the $B^j_r(E)$, we have that

$$B_k(E) = \bigcap_{j=1}^r B^j_r(E)$$

and thus by the previous lemma

$$\mathbb{P}(B_k(E)) \leq e^{-d \log(r)r}.$$  

We obtain

**Lemma 16.5.** There exists a set $B_k(E)$ with the following properties

(i) $\mathbb{P}(B_k(E)) \leq e^{-d \log(r)r}$.

(ii) For $\omega \notin B_k(E)$ there exists $1 \leq j \leq r$ such that for $|E - \tilde{E}| \leq e^{-3\gamma r}$ and

$$\tilde{\omega} = \omega \pmod{\Lambda_4(0)^c}$$

we have for $|n|_\infty = t_j$

$$\Lambda_r(n) \text{ is } (\gamma, \tau, 0)\text{-suitable for } H_{\lambda, \omega} - \tilde{E}.$$

**Proof.** This follows from the discussion preceding the statement and Lemma 5.4 to perturb $E$. Here, we used that (5.8) holds for $k$ large enough. $\square$

Since $\sigma(H_{\lambda, \omega}) \subseteq [-3\lambda, 3\lambda]$, we can introduce

$$B_k = \bigcup_{\ell = -\frac{3\lambda e^{3\gamma r}}{4\lambda e^{3\gamma r}}}^{2\lambda e^{3\gamma r}} B_k(2\ell e^{-3\gamma r}).$$

Then for $\omega \notin B_k$ the conclusion (ii) of the previous lemma holds for all $\tilde{E}$. Furthermore, we have that

$$\mathbb{P}(B_k) \leq 3\lambda e^{(3\gamma - d \log(r))r},$$

which is small as long as $d \log(r) > 3\gamma$. Since $\log(r) = k \log(2)$, we obtain that we must have

$$k \geq \frac{3\gamma}{\log(2)} + 1.$$

For the proof, we will furthermore need the elementary inequality

$$(1 + x)^p \leq e^{px}$$

for $x \geq \frac{2p}{e}$. $\square$

**Proof of Proposition 16.3.** Let $E \in \mathcal{E}_0^{B,r}$ and $\omega \notin B_k$ constructed above. Then we can find $t$ such that for every $n$ with $|n|_\infty = t$, we have

$$\Lambda_r(n) \text{ is } (\gamma, \tau, 0)\text{-suitable for } H_{\lambda, \omega} - E.$$ 

Hence, we obtain for these $n$ by (9.2) and (16.3) for $r \geq \frac{2p}{e}$ that

$$|\psi(n)| \leq e^{-\frac{3\gamma r}{2}}.$$

Consider the test function

$$u(x) = \begin{cases} \psi(x), & x \in \Lambda_{t-1}(0); \\ 0, & \text{otherwise}, \end{cases}$$
which satisfies \( \| (H_{\lambda, \omega}^{r(0)} - E)u \| \leq e^{-\frac{2}{\gamma}r} \), by the choice of \( r \). The claim follows since
\[
\| u \|_{L^2(\Lambda_R(0))} \geq e^{-\frac{2}{\gamma}r},
\]
by assumption.

\[\square\]

17. Super polynomial decay of the eigenfunctions

We will show

**Theorem 17.1.** Assume \([16.3]\). There exists a set \( \Omega_1 \) and a constant \( \hat{\gamma} > 0 \) satisfying

(i) \( \mathbb{P}(\Omega_1) = 1 \).

(ii) For \( \omega \in \Omega_1 \) the spectrum of \( H_{\lambda, \omega} \) is pure point.

(iii) For every \( \omega \in \Omega_1 \), there exists \( \ell \geq 1 \) such that for \( k \geq \ell \), \( \| \psi \| = 1 \) solving
\[
H_{\lambda, \omega} \psi = E\psi
\]
with
\[
\sum_{x \in \Lambda_{2^{k-2}(c)}} |\psi(x)|^2 \geq e^{-\frac{2}{\gamma}2^k},
\]
we have for \( \|n\|_\infty \geq 2^k \) that
\[
|\psi(n)| \leq e^{-c\sqrt{|n|}_\infty}.
\]

Let \( B_L^k \) be the set from Proposition \([16.3]\). Introduce
\[
B_L = \bigcap_{\ell \geq 1} \bigcup_{k \geq \ell} B_L^k.
\]
It follows that \( \mathbb{P}(B_L) = 0 \). Let now \( \omega_0 \notin B_L \). Then by Proposition \([16.3]\) we have that there exists some \( k_0(\omega) \) such that for \( k \geq k_0(\omega) \), we have with \( r = 2^k \) for
\[
\omega = \omega_0 \pmod{4r} \quad \text{and} \quad \varepsilon \geq e^{-\frac{2}{\gamma}r} \quad \text{that}
\]
\[
\text{dist}(\mathcal{E}_{\lambda, \omega}^\varepsilon, \sigma(H_{\lambda, \omega_0}^{A_{4r}(0)})) \leq e^{-\frac{2}{\gamma}r}.
\]

Denote by \( B_c^L(E) \) the set constructed before Lemma \([16.4]\). We introduce
\[
B_c^L(\omega_0) = \bigcup_{E \in \sigma(H_{\lambda, \omega_0}^{A_{4r}(0)})} B_c^L(E).
\]
We introduce
\[
B_c^S(\omega_0) = \bigcup_{4r \leq k \leq 16r} \begin{cases} B_c^L(\omega_0), & k \geq k_0(\omega_0); \\ \emptyset, & \text{otherwise}. \end{cases}
\]
We now define a set \( B_c^S \) as follows. Define a map
\[
[-\frac{1}{2}, \frac{1}{2} \Lambda_{4r}(0) \rightarrow [-\frac{1}{2}, \frac{1}{2} \Lambda_{4r}(0)
\]
by mapping \( x \) to some \( \omega \in B_L \) satisfying
\[
\omega = x \pmod{4r} \quad \text{if such an} \ \omega \ \text{exists, otherwise to any} \ \omega \ \text{satisfying this condition. Then, we define}
\]
\( B_c^S(\omega) \) as the union over the set \( B_c^S(\omega) \) with \( \omega \) constructed above.
Lemma 17.2. We have that
\[ P(B^S_k) \leq e^{-2k-2}. \]

**Proof.** We have that \( B^S_k(\omega_0) \) is the intersection of less than \( 2^{k+1} \cdot (32^{k+1})^d \) many sets of measure \( \leq e^{-2k} \). Hence
\[ P(B^S_k(\omega_0)) \leq e^{-2k-2}. \]
The claim now follows by Fubini. \( \square \)

We can now introduce
\[ B^S = \bigcap_{\ell \geq 1} \left( \bigcup_{k \geq \ell} B^S_k \right), \]
which satisfies \( P(B^S) = 0 \) by a Borel–Cantelli argument. Define
\[ \Omega_1 = \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap (B^S \cup B^L). \]
Assume now that \( \psi \in \mathcal{E}_{\lambda, \omega} \) for some \( \omega \in \Omega_1 \). Then there exists \( \ell \geq 1 \) such that for \( k \geq \ell \)
\[ \omega \notin (B^S_k \cup B^L_k). \]
In particular, we obtain that
\[ |\psi(n)| \leq e^{-c\sqrt{|n|}} \]
for some \( c \) once \( |n|_\infty \geq 2^l \).

**Proof of Theorem 17.1.** It is easy to see that, we can conclude pure point spectrum, that is (i). Now (ii) follows after some computations. \( \square \)

18. Dynamical Localization

In this section, we will adapt the machinery of the last section to prove dynamical localization. The proof follows the strategy of Bourgain and Jitomirskaya from [15], where it was used to prove dynamical localization for a certain quasi-periodic band model.

Recall that \( \{e_x\}_{x \in \mathbb{Z}^d} \) denotes the standard basis of \( \ell^2(\mathbb{Z}) \), that is
\[ e_x(n) = \begin{cases} 1, & x = n; \\ 0, & \text{otherwise.} \end{cases} \]
For simplicity, we will consider the Schrödinger equation with initial condition \( e_0 \), that is
\[ i\partial_t \psi(t) = H_\omega \psi(t) \]
\[ \psi(0) = e_0. \]
It would be somewhat more tedious to consider more general initial states. For \( p \geq 1 \), consider the moment operator
\[ X(p, \omega) = \sup_{t \geq 0} \left( \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^p |\psi(t, n)|^2 \right), \]
where $|n|^2 = \sum_{k=1}^{d}(n_k)^2$. We will show

**Theorem 18.1.** Let $p \geq 1$ then for almost every $\omega$, we have

\begin{equation}
X(p, \omega) < \infty.
\end{equation}

We now begin the proof of this theorem. We will in fact show that the almost sure set, is the same as in Theorem 17.1. So let $\omega \in \Omega_1$. The first step will be to rewrite the time evolution (18.2) in terms of the eigenfunctions of $H_{\lambda, \omega}$.

Denote by $E_{\alpha}$ and $\varphi_{\alpha}^\omega$ the orthonormal basis of $L^2(\mathbb{Z}^d)$ consisting of eigenfunctions of $H_{\lambda, \omega}$. We have that

\begin{equation}
\psi(t) = \sum_{\alpha} \varphi_{\alpha}^\omega(0) \cdot \varphi_{\alpha}^\omega(t).
\end{equation}

In particular

\begin{equation}
|\psi(t, n)|^2 \leq \sum_{\alpha} |\varphi_{\alpha}^\omega(0)|^2 \cdot |\varphi_{\alpha}^\omega(n)|^2.
\end{equation}

Hence, it suffices to show that that

\begin{equation}
\sum_{\alpha} \left( |\varphi_{\alpha}^\omega(0)|^2 \cdot \left( \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^p |\varphi_{\alpha}^\omega(n)|^2 \right) \right)
\end{equation}

is finite.

Introduce for $s \geq 0$ the set

\begin{equation}
A_{\omega,s} = \{ \alpha : \frac{1}{2^{s+1}} < |\varphi_{\alpha}^\omega(0)|^2 \leq \frac{1}{2^s} \}.
\end{equation}

Clearly, this set is finite. Furthermore, our task reduces to showing that for almost every $\omega$ the sequence

\begin{equation}
\frac{1}{2^s} \sum_{\alpha \in A_{\omega,s}} \left( \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^p |\varphi_{\alpha}^\omega(n)|^2 \right)
\end{equation}

is summable.

**Proposition 18.2.** There exists $c > 0$. For $\omega \in \Omega_1$, there exists $R_\omega$. Let $\alpha \in A_{\omega,s}$, then for $|n|_\infty \geq \max(4s^2, R_\omega)$, we have

\begin{equation}
|\varphi_{\alpha}^\omega(n)| \leq e^{-c\sqrt{|n|_\infty}}.
\end{equation}

Furthermore, we have for $R \geq \max(4s^2, R_\omega)$ that

\begin{equation}
\sum_{n \in \Lambda_R(0)} |\varphi_{\alpha}^\omega(n)|^2 \geq \frac{1}{2}.
\end{equation}

**Proof.** The first part is a consequence of (iii) of Theorem 17.1. The second part follows from

\begin{equation}
\sum_{n \in \Lambda_R} e^{-c\sqrt{|n|_\infty}} \to 0
\end{equation}

as $R \to \infty$ and possibly enlarging $R_\omega$. \hfill $\square$

**Lemma 18.3.** Let $\omega \in \Omega_1$, we have

\begin{equation}
\#(A_{\omega,s}) \leq 2 \cdot 9^d (\max(4s^2, R_\omega))^{2d}.
\end{equation}
Proof. Denote by $R_{\Lambda_{rs}(0)}$ the restriction operator to $\Lambda_{rs}(0)$. We have

$$\sum_{\alpha} \|R_{\Lambda_{rs}(0)} \varphi_{\omega}^0\|^2 = \|R_{\Lambda_{rs}(0)}\|_{HS}^2 = (\#(\Lambda_{rs}(0)))^2 \leq (3r_s)^{2d}.$$  

Next, we have for any $\alpha \in A_{\omega,s}$ that

$$\|R_{\Lambda_{rs}(0)} \varphi_{\omega}^0\|^2 \geq \frac{1}{2}\|\varphi_{\omega}^0\|^2.$$  

In particular, we also obtain

$$\#(A_{\omega,s}) = \sum_{\alpha \in A_{\omega,s}} \|\varphi_{\omega}^0\|^2 \leq 2 \sum_{\alpha \in A_{\omega,s}} \|R_{\Lambda_{rs}(0)} \varphi_{\omega}^0\|^2 \leq 2 \sum_{\alpha} \|R_{\Lambda_{rs}(0)} \varphi_{\omega}^0\|^2.$$  

This implies the claim. \hfill \Box

We are now ready for

Proof of Theorem 18.1. We first observe that the previous lemmas imply that

$$\sum_{\alpha \in A_{\omega,s}} \left( \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^p |\varphi_{\omega}^0(n)|^2 \right) \leq C(4s)^{4(d+p)}$$  

for some $C \geq 1$. We also have that

$$\sum_{s \geq 1} (4s)^{4(d+p)} < \infty.$$  

The claim follows. \hfill \Box

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Parts of the result are slight generalizations from work from my PhD thesis [42]. I thank my advisor David Damanik for his support during the process of writing it.

APPENDIX A. THE INITIAL CONDITION

In this section, I wish to discuss how to obtain the initial scale estimate at large coupling. A main motivation is that main standard proofs rely on the Wegner estimate, see [48] or [35], which is not available in our context. I will begin by showing that Hypothesis 2.7 implies (ii) of Hypothesis 2.5. We will use the notation

$$f(\omega) = \sum_{r=0}^{\infty} f(\omega_n)_{n \in \Lambda_{rs}(0)}.$$  

Lemma A.1. Assume Hypothesis 2.7. Then there exist constants $F$ and $\alpha > 0$ such that

$$\mathbb{P}(\omega \in [-\frac{1}{2}, \frac{1}{2}]^{2d} : |f(\omega) - E| \leq \varepsilon) \leq F \cdot \varepsilon^\alpha.$$  

(A.1)

$$f(\omega) = \sum_{r=0}^{\infty} f(\omega_n)_{n \in \Lambda_{rs}(0)}.$$  

(A.2)
Proof. For $\tilde{\omega} = \{\omega_x\}_{x \in \mathbb{Z} \setminus \{0\}}$, we define
\[
g(\tilde{\omega}) = \max_{x \in [-\frac{1}{2}, \frac{1}{2}]^d} (f(\{\tilde{\omega}, x\}) - \min_{x \in [-\frac{1}{2}, \frac{1}{2}]^d} (f(\{\tilde{\omega}, x\})).
\]
Then by condition (iii) of Hypothesis 2.7 we have $g(\tilde{\omega}) > 0$. Since $g$ is defined on a compact space, there exists $\eta > 0$ such that $g(\tilde{\omega}) \geq \eta$ for all choices of $\tilde{\omega}$.

For fixed $\tilde{\omega}$, introduce
\[
f_{\tilde{\omega}, E}(x) = f(\{x, \tilde{\omega}\}) - E,
\]
so $x$ plays no role of $\omega_0$. We have that $f_{\tilde{\omega}, E}$ obeys the assumption of Cartan’s lemma, Theorem III.12 with $\varepsilon = \frac{1}{2^5}$. So we may conclude that
\[
|\{x \in [-\frac{1}{2}, \frac{1}{2}]: |f(x)| \leq e^{-s}\}| \leq 30e^3 \exp\left(\frac{1}{\log(2\eta - 1)^s}\right).
\]
By Fubini, we thus see that the claim holds with $F = 30e^3$ and $\alpha = \frac{1}{\log(2\eta - 1)}$. □

We now begin with the proof of the initial condition. The strategy will be to exhibit a large gap in the spectrum of $H_{\Lambda_R}(0)$. To conclude the decay of the Green’s function, we will use the Combes–Thomas estimate, whose consequence, we now recall.

**Proposition A.2.** There exists an universal constant $c_0 > 0$. Let $H : \ell^2(\Lambda_r(0)) \to \ell^2(\Lambda_r(0))$ be a Schrödinger operator and $\tau \in (0, 1)$. Assume that
\[
(A.3) \quad \text{dist}(E, \sigma(H)) \geq \delta
\]
and $r \geq \frac{1}{\tau} + 10$. Then $\Lambda_r(0)$ is $(c_0 \log(1 + \delta), \tau, 3)$-suitable for $H - E$.

**Proof.** This is an application of the Combes–Thomas estimate [20], [35]. □

We now exhibit a gap in the spectrum. Here, we use the notation $(T_x \omega)_n = \omega_{x+n}$. Although this is of little importance for the proof, $T_x$ could be any measure preserving map.

**Lemma A.3.** Assume there are constants $F > 0$ and $\alpha > 0$ such that for every $E \in \mathbb{R}$
\[
(A.4) \quad \mathbb{P}(\{\omega : |f(\omega) - E| \leq \varepsilon\}) \leq Fe^{\varepsilon^\alpha}.
\]
Let $V_{\lambda, \omega}(x) = \lambda f(T_x \omega)$. Then for $p > 0$, $E \in \mathbb{R}$, $\delta > 0$, and $R \geq 1$, there exists $\lambda_0 = \lambda_0(F, \alpha, p, R, \delta)$ such that for $\lambda > \lambda_0$, we have
\[
(A.5) \quad \mathbb{P}(\{\omega : \text{dist}(E, \sigma(H_{\Lambda_R(0)}^{\lambda, \omega})) \leq \delta\}) \leq \frac{1}{Re^p}.
\]

**Proof.** Introduce the set $\Omega_E$ as the set of $\omega \in \Omega_E$ satisfying for $x \in \Lambda_R(0)$ that
\[
\text{dist}(E, V_{\lambda, \omega}(x)) > 2d + \delta.
\]
By assumption, we have that
\[
\mathbb{P}(\Omega_E) \leq (3R)^d F \cdot \left(\frac{2d + \delta}{\lambda}\right)^\alpha,
\]
which is $\leq \frac{1}{Re^p}$ for $\lambda > 0$ large enough. Furthermore, one sees that for $\omega \in \Omega_E$, we have
\[
\text{dist}(E, \sigma(H_{\Lambda, \omega})) > \delta,
\]
since $\|\Delta\| \leq 2d$. The claim follows. □
Combining this lemma with Proposition A.2 and an appropriate choice of $\delta > 0$, we obtain

**Theorem A.4.** Assume Hypothesis 2.5. For any $\alpha, r_0 < r_1$ there exists $\lambda_0 = \lambda_0(\alpha, r_0, r_1, f) > 0$ such that

\[ [r_0, r_1] \text{ is } (1, \alpha)-\text{acceptable for } H_{\lambda, \omega} - E \]

in the sense of Definition 4.7.

This implies Proposition 4.2.

**Appendix B. The spectrum**

The following argument originates from a discussion with Ivan Veselić.

**Theorem B.1.** The spectrum of $H_{\lambda, \omega}$ is almost surely an interval.

**Proof.** Denote by $\Sigma$ the almost sure spectrum of $H_{\lambda, \omega}$. Define $\omega_c = 0$ for $x \in \mathbb{Z}^d$.

We then have that

\[ \sigma(H_{\lambda, \omega_c}) = [-2d + f(\omega^c), 2d + f(\omega^c)]. \]

In particular, it is an interval. Let now $\omega \in \Omega_r$ and define a continuous path $\gamma : [0,1] \to \Omega$ by

\[ \gamma(t)x = t \cdot \omega_x. \]

We then have that $\omega^c = \gamma(0)$ and $\omega = \gamma(1)$. We clearly have that

\[ \bigcup_{t \in [0,1]} \sigma(H_{\lambda, \gamma(t)}) \subseteq \Sigma. \]

Since $\sigma(H_{\lambda, \gamma(t)})$ depends continuously on $t$, we obtain that this set is an interval, and so also $\Sigma$. \hfill $\blacksquare$

**Appendix C. On Wegner’s estimate**

We now discuss that the conclusions of multi-scale analysis imply Wegner estimates. This is not new and can for example be found in [44] by Schlag.

**Proposition C.1.** Let $\tau \in (0,1)$ and $\psi(r)$ be a decreasing function satisfying $\lim_{r \to \infty} \psi(r) = 0$. Assume for all $E$ and $r \geq r_0$ that

\[ \mathbb{P}(\Lambda_r(0) \text{ is not } (\gamma, \tau, 0)-\text{suitable for } H_{\omega} - E) < \psi(r). \]

Then, we have for $R \geq R_0$ that

\[ \mathbb{E}\left( \frac{1}{\# \Lambda_R(0)} \text{tr} \left( P_{[E-\varepsilon, E+\varepsilon]}(H_{\omega}^{\Lambda_R(0)}) \right) \right) \leq 7\psi\left( \frac{1}{3} \left( \log(\varepsilon^{-1}) \right)^{1/2} \right)^{1+\frac{1}{d+1}}. \]

**Proof.** Define

\[ s = \left\lfloor \frac{1}{3 \cdot (\psi(r))^{1/d+1}} \right\rfloor. \]

By Theorem B.2 combined with a trivial probabilistic estimate, we obtain

\[ \mathbb{P}(\|H_{\omega}^{\Lambda_r(0)} - E\|^{-1} > e^{3\tau r}) < \psi(r)^{1+\frac{1}{d+1}}. \]

In particular also

\[ \mathbb{E}\left( \frac{1}{\# \Lambda_s(0)} \text{tr} \left( P_{[E-\varepsilon, E+\varepsilon]}(H_{\omega}^{\Lambda_s(0)}) \right) \right) \leq 2\psi(r)^{1+\frac{1}{d+1}} \left( 1 + e^{3\tau r} \right). \]
We choose
\[ r = \left\lfloor \frac{1}{3} \left( \log(\varepsilon^{-1}) \right)^{1/2} \right\rfloor. \]

Hence, we can conclude that for any \( R \geq R_0 \) that
\[ E \left( \frac{1}{\# \Lambda_R(0)} \operatorname{tr} \left( \mathcal{P}_{E-\varepsilon,E+\varepsilon} \left( H_{\omega}^{\Lambda_R(0)} \right) \right) \right) \leq 7 \psi \left( \frac{1}{3} \left( \log(\varepsilon^{-1}) \right)^{1/2} \right)^{x+1}. \]

This is the claim. \( \square \)

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