Z\_8 \text{ IS NOT DUALIZABLE}

CS. SZABÓ

1. INTRODUCTION

In [4] natural (strong) duality is proved for the ring \( \mathbb{Z}_4 \). They also show it for \( \mathbb{Z}_{p^2} \), where \( p \) is a prime. The next question is, whether \( \mathbb{Z}_8 \) is dualizable or not. There were several attempts to approach this problem. The closest shot was made by Louisind Sabourin, who interpreted the problem into a question of quadratic equations over vector spaces. Let \( V \) be a vector space over \( F_2 \), the two element field. A subset \( S \subseteq V \) has the property

(Q) if \( S \) is the set of solutions of a quadratic equation
(Q\_3) if \( S \cap W \) is the set of solutions of a quadratic equation for every 3-dimensional affine subspace \( W \) of \( V \).

Of course, Q implies Q\_3. Sabourin showed ([6]) that if Q\_3 implies Q (Sindi’s conjecture), then \( \mathbb{Z}_8 \) admits a natural duality.

In this paper we show that \( \mathbb{Z}_8 \) does not admit a natural duality. In fact, we show that \( 2\mathbb{Z}_8 = \{2, 4, 6, 8 | +, \cdot \} \) is not dualizable, and this will imply that the original ring is not dualizable, either. As a corollary we show that Sindi’s conjecture does not hold. Our technique will be similar to the one in [5], where non-dualizability is proved for the quaternion group.

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2. REMARKS ON DUALITY

This chapter is supposed to be skipped by the ones who have already experienced some duality before.

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For the benefit of readers not familiar with the theory of natural
dualities, we begin with a brief review of what is meant by ‘admitting
a (natural) duality’ and refer to Davey [2] or the forthcoming text Clark
and Davey [1] for a detailed account.

Let $A$ be a finite algebra and let $\tilde{A} = \langle A; F, H, R, \tau \rangle$ be a topological
structure on the same underlying set $A$, where:

(a) each $f \in F$ is a homomorphism $f : A^n \to A$ for some $n \in \mathbb{N} \cup \{0\}$,
(b) each $h \in H$ is a homomorphism $h : \text{dom}(h) \to A$ where $\text{dom}(h)$
is a subalgebra of $A^n$ for some $n \in \mathbb{N}$,
(c) each $r \in R$ is (the universe of) a subalgebra of $A^n$ for some $n \in \mathbb{N}$,
(d) $\tau$ is the discrete topology.

Whenever (a), (b) and (c) hold, we say that the operations in $F$,
the partial operations in $H$ and the relations in $R$ are
algebraic over $A$. These compatibility conditions between the structure on $A$
and the structure on $\tilde{A}$ guarantee that there is a naturally defined dual
adjunction between the quasivariety $A := \text{ISP}_A$ generated by $A$
and the topological quasivariety $X_{\tilde{A}} := \mathbb{S}_{\tilde{A}} \tilde{A}$ generated by $\tilde{A}$; if there is
no chance of confusion, we will write $X$ for $X_{\tilde{A}}$. For all
$B \in A$ the
homset $D(A) := A(B, A)$ of all homomorphisms from $B$ to $A$ is a
closed substructure of the direct power $\tilde{A}$ and for all $X \in X$
the homset $E(X) := X(\tilde{X}, \tilde{A})$ is a subalgebra of the direct power $A^X$.
It follows easily that the contravariant hom-functors $A(-, A) : A \to S$
and $X(-, A) : X \to S$, where $S$ is the category of sets, lift to
contravariant functors $D : A \to X$ and $E : X \to A$.

For each $B \in A$ there is a natural embedding $e_B$ of $B$ into $ED(B)$
given by evaluation: for each $b \in B$ and each $x \in D(B) = A(B, A)$ de-
fine $e_B(b)(x) := x(b)$. Similarly, for each $X \in X$ there is an embedding
$\varepsilon_X$ of $\tilde{X}$ into $DE(\tilde{X})$. A simple calculation shows that $e : \text{id}_A \to DE$
and $\varepsilon : \text{id}_X \to DE$ are natural transformations. If $e_B$ is an isomorphism
for all $B \in A$ we say that $\tilde{A}$ yields a (natural) duality on $A$. If there is
some choice of $F$, $H$ and $R$ such that $\tilde{A}$ yields a duality on $A$ then we
say that $\tilde{A}$ (or $A$) admits a natural duality or, briefly, is dualizable.

3. ACCESORIES

We wish to prove that for no choice of $F$, $H$ and $R$ does $\tilde{Z}_8$ yield a
duality on $G$, the quasivariety generated by $\mathbb{Z}_8$. For this, it is enough
to show that there is no duality when $F = H = \emptyset$ and $R$ consists of all
subgroups of all finite powers of $\mathbb{Z}_8$, the so-called brute force duality,
see [1] or [2]. In order to prove that there is no brute force duality, we
need to find a (necessarily infinite) group $D \in G$ such that $e_D$ is not
onto $ED(D)$. We will use the ghost element method. We will choose $D$
to be a subring of $\mathbb{Z}_8^Z$ and will define a continuous structure preserving map $\Phi : D(D) \to \mathbb{Z}_8$ such that $\{\Phi(\pi_i) \mid i \in \mathbb{Z}\}$ is not an element of $D$, implying that $\Phi$ is not the evaluation map for any element of $D$ and therefore that $e_D$ is not onto $ED(D)$. Here $\pi_i$ is the (restriction to $D$ of the) $i$-th projection of $\mathbb{Z}_8^Z$ onto $\mathbb{Z}_8$.

For a subring $D$ of $R^Z$ let $D_{\text{fin}}$ denote the elements of $D$ with finitely many nonzero coordinates. We say that an element $\bar{v}$ of $D$ has eventual value $v$ (is eventually $v$) if all its coordinates but finitely many ones are equal to $v$. So $D_{\text{fin}}$ is the set (subring) of eventually 0 elements of $D$.

3.1. The ring. The ring $\mathbb{Z}_8$ is an algebra with two binary operations and a constant.

$$\mathbb{Z}_8 = \{1, 2, 3, 4, 5, 6, 7, 0 \mid +, \cdot, 0\}.$$

The Jacobson(nil)-radical of $\mathbb{Z}_8$,

$$\text{J}(\mathbb{Z}_8) = 2\mathbb{Z}_8 = \{0, 2, 4, 6 \mid +, \cdot, 0\},$$

is a two-class nilpotent ring.

First we construct a subring $D$ of $2\mathbb{Z}_8^Z$ with the ghost-vector missing, and after that we examine the possible homomorphisms from $D$ to $\mathbb{Z}_8$.

The vectors $\bar{b} = (2, 2, 0)$, $\bar{a} = (0, 2, 2)$ generate a subring $R$ of $\mathbb{Z}_8^3$, isomorphic to the 2 generated free ring in the variety generated by $\text{J}(\mathbb{Z}_8)$. Define $\bar{b}$ and $\bar{a}_i$ for $i \in \mathbb{Z}$, elements of $R^Z$ as follows:

$$\bar{b}_i = \bar{b} \quad \text{and} \quad \bar{a}_{ij} = \begin{cases} \bar{b} & \text{if } i = j \\ \bar{0} & \text{if } |i - j| = 1 \\ \bar{0} & \text{if } |i - j| > 1 \end{cases}$$

Let $D_1 = \langle \bar{b}, \bar{a}_i | i \in \mathbb{Z}\rangle$. Moreover, let $\bar{e}_i = \bar{a}_i - \bar{a}_{i-1}$, then

$$\bar{e}_{ij} = \begin{cases} \bar{a} & \text{if } j = i - 2 \\ -\bar{b} & \text{if } j = i - 1 \\ \bar{b} & \text{if } i = j \\ -\bar{a} & \text{if } j = i + 1 \\ \bar{0} & \text{otherwise} \end{cases},$$

and so, $D_1 = \langle \bar{b}, \bar{a}_0, \bar{e}_i | i \in \mathbb{Z}\rangle$, i.e., $D_1$ is generated by the vectors of the form:

$\bar{b} = (\ldots, \bar{b}, \bar{b}, \bar{b}, \ldots)$,

$\bar{a}_0 = (\ldots, \bar{a}, \bar{a}, \bar{a}, 0, \bar{b}, 0, \bar{a}, \bar{a}, \ldots)$,
\[ \bar{e}_i = (0, 0, \bar{a}, -\bar{b}, \bar{b}, \bar{a}, 0, 0, \ldots). \]

Let \( D_2 = \langle x^2 | x \in R^2 \rangle_{\text{fin}}, \) the ring generated by the squares of the elements in \( R^2 \) containing finitely many nonzero coordinates. Observe that \( D_2 = \langle \bar{a}^2, \bar{b}^2 \rangle_{\text{fin}} \), and \( D_2 \cdot R^2 = 0 \) holds. Finally, let \( D = \langle D_1, D_2 \rangle = D_1 + D_2. \)

3.2. The ghost-vector. Our ghost vector will be \( \overline{ab} \), where \( \overline{ab}_i = \bar{a} \cdot \bar{b} = \bar{a} \bar{b} \). First we have to show that \( \overline{ab} \) is not in \( D \), but it can be ‘approximated’ by vectors from \( D \).

Claim The vector \( \overline{ab} = (\ldots, \bar{a} \bar{b}, \bar{a} \bar{b}, \ldots) \) is not contained in \( D \), but \( \overline{i} \overline{ab} = (\ldots, \bar{a} \bar{b}, 0, 0, \bar{a} \bar{b}, \ldots) \), where 0 is at the \( i \)-th place is in \( D \).

Proof of the claim. The variety generated by \( 2Z_8 \) satisfies the identities \( 2x = x^2 \) and \( 4x = (2x^2) = 0 \). Thus a generator set of \( \langle \bar{a} \bar{b} \rangle^2 \cap D \) can be obtained on the following way: Take the product of all pairs of generators and substitute each coordinate equal to \( \bar{a}^2 \) or \( \bar{b}^2 \) with 0. All these elements satisfy the following ‘parity check condition’: Every vector is eventually 0 or eventually \( \bar{a} \bar{b} \). In the first case the sum of the coordinates is 0 (there are even many \( \bar{a} \bar{b} \)-s), in the second case there are odd many 0-s. Obviously, this property is preserved by addition of these elements and so, the property holds for \( \langle \bar{a} \bar{b} \rangle^2 \cap D \), proving the claim. \( \square \)

As \( R \) has a natural embedding into \( Z_8 \), \( R^2 \) has a natural embedding to \( (Z_8)^2 \), that gives a natural embedding of \( D \) into \( R^2 \). For this subring and for its elements we shall use the notations above, we shall denote the elements of \( D \) and their images at the embedding on the same way. (e.g. \( \bar{b} = (\ldots, 0, 2, 2, 0, 2, 2, \ldots) \) and \( \overline{ab} = (\ldots, 0, 4, 0, 0, 4, 0, \ldots) \). Thus \( D \leq Z_8^2 \).

3.3. The maps. First we examine the possible maps from \( D \) to \( Z_8 \). Let \( f \in \text{hom}(D, Z_8) \). Since \( D \) satisfies the identity \( 4x = 0 \), the same holds for its image, hence \( D \) is mapped to \( 2Z_8 \). We are interested in the action of \( f \) on the set \( \{ i \overline{ab} \mid i \in Z \} \), in fact, we will show that \( f(i \overline{ab}) = f(j \overline{ab}) \) for almost all \( i, j \in Z \). Note that \( \overline{ab} - j \overline{ab} \in D_{\text{fin}} \).

Moreover, \( \bar{e}_i \cdot \bar{e}_{i+2} = (\ldots, 0, 0, 0, \bar{a} \bar{b}, \bar{a} \bar{b}, 0, 0, \ldots) \) and so, \( \langle \bar{a} \bar{b} \rangle^2 \cap D_{\text{fin}} = \langle \bar{e}_i, \bar{e}_{i+2} \mid i \in Z \rangle \).

First case: the image of \( D_{\text{fin}} \) at \( f \) is a zeroring. Then \( f(x)f(y) = f(xy) = 0 \) for any \( x, y \in D_{\text{fin}}, \) so \( f(\bar{e}_i \cdot \bar{e}_{i+2}) = 0. \) Thus \( f(\langle \bar{a} \bar{b} \rangle^2 \cap D_{\text{fin}}) = \{0\} \) holds, hence \( f(i \overline{ab}) = f(j \overline{ab}) \) for every \( i, j \in Z \).
**Last case:** the image of $D_{fin}$ is not a zeroring. Then there is an $i \in Z$ such that $f(\bar{e}_i) = 2$ or 6. As $\bar{e}_i \cdot \bar{e}_j = 0$ if $|i - j| > 3$, $f(\bar{e}_i)$ is contained in $\{0, 4\}$, the annihilator of $2Z_8$ for $|i - j| > 3$. Thus there is a smallest $i$ such that $f(\bar{e}_i) = 2$ or 6, and for $i < j$, $f(\bar{e}_j)Z_8 = 0$. Without loss of generality we may assume that $f(\bar{e}_i) = 2$ but $f(\bar{e}_i) \in \{0, 4\}$ for $i < 1$. As $\bar{e}_i \bar{e}_j = 0$ for $j \geq 3$, $f(\bar{e}_j)$ annihilates $2Z_8$ in this case, too. From this easily can be derived that for $\bar{h} \in \langle \bar{a}\bar{b}\rangle Z \cap D_{fin}$, if $\bar{h}_i = 0$ for $i < 2$ or $\bar{h}_i = 0$ for $i > 2$, then $f(\bar{h}) = 0$. Thus $f(i \bar{a}\bar{b}) = f(j \bar{a}\bar{b})$ if 2 is not between $i$ and $j$.

Notice that $f(\bar{e}_0)f(\bar{e}_1) + f(\bar{e}_0)f(\bar{e}_2) = 0$ as each summand contains the image of a vector with index smaller than 1. On the other hand
\[f(\bar{e}_0)f(\bar{e}_1) + f(\bar{e}_0)f(\bar{e}_2) = f(\bar{e}_1)f(\bar{e}_2 + \bar{b}).\]
Since $f(\bar{e}_1)$ generates $2Z_8$, the element $f(\bar{e}_2) + f(\bar{b}) \in Ann(Z_8)$, and so $f(\bar{e}_2)(f(\bar{e}_2) + f(\bar{b})) = 0$. Also, by examining the indices, $f(\bar{e}_2)f(\bar{e}_{-1}) + f(\bar{e}_2)f(\bar{e}_0) = 0$. Adding the last two equalities we get $f(\bar{e}_2)(f(\bar{e}_2) + f(\bar{e}_{-1}) + f(\bar{e}_3) + f(\bar{b})) = 0$. This is a vector with two $\bar{a}\bar{b}$ entries on different sides of the critical 2nd coordinate, and so if non of $i$ and $j$ equals 2, $f(i \bar{a}\bar{b}) = f(j \bar{a}\bar{b})$. We showed that for each $f$, which $f(D_{fin}) = 2Z_8$ holds for, there is a coordinate $i(f)$, such that $f(i \bar{a}\bar{b}) = f(m \bar{a}\bar{b})$ holds if $m$ and $l$ are different from $i(f)$. We call $i(f)$ the critical coordinate of the map $f$.

4. The results

Now, we know everything to prove our main theorem:

**Theorem 4.1.** The ring $Z_8$ does not admit a natural duality.

**Proof.** Let $\phi$ be the following map form $\text{hom}(D, Z_8)$:

\[\phi(f) = \begin{cases} f(0 \bar{a}\bar{b}) & \text{if } f \text{ belongs to the first case,} \\ f(i+1 \bar{a}\bar{b}) & \text{if } i \text{ is the critical coordinate of } f. \end{cases}\]

In order to show that $\phi$ is structure preserving, for any finite set of homomorphisms $f_1, \ldots, f_n$ from $D$ to $Z_8$ we have to find an element $\bar{v}$ of $D$ such that $\phi(f_i) = f_i(\bar{v})$ for $i = 1, \ldots, n$. Let $m$ be different (if any exists) from the critical coordinates of the $f_i$-s.

Then by Section 3.3 $\phi(f_i) = f(m \bar{a}\bar{b})$ for all these maps.

For showing that $\phi$ is continuous, it is enough (and easy) to see that if two map agrees on $0 \bar{a}\bar{b}, i \bar{a}\bar{b}$ and $j \bar{a}\bar{b}$, then they have the same value at $\phi$.

On the other hand, $\phi(\pi_i) = \bar{a}\bar{b}$ and $\bar{a}\bar{b}$ is not in $D$, hence $\phi$ is not an evaluation map. \qed
Corollary 4.2. Sindi’s conjecture fails. There is a vector space $V$ over $F_2$ and a subset $S$ of $V$, such that for every 3-dimensional subspace $W$ of $V$, the intersection $W \cap S$ is the solution set of some quadratic equation, but there is no quadratic equation with $S$ as its the solution set.

Several times rings are considered as algebras with a unit element 1. In that case

$$Z_8 = \{1, 2, 3, 4, 5, 6, 7, 0 \mid +, \cdot, 0, 1 \}$$

an algebra with two constants, 0 and 1. For us, the main difference is that in this case $D$ has to contain the $1 = (\ldots, 1, 1, \ldots)$ constant 1 vector. If we try to add it to our construction, the ring turns out to contain $\overline{ab}$, the ghost vector. But, if at the beginning we take $a = (2, 2, 0, 0, \ldots)$ and $b = (0, 2, 2, 0)$, the whole idea of the construction can be saved, and the proof goes exactly the same way.

Corollary 4.3. $Z_8$ with 1 is not dualizable.

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Fields Institute/ McMaster University
E-mail address: csaba@cs.elte.hu