Interference effects between two initially independent Bose-condensed gases

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When two initially independent Bose-condensed gases are allowed to overlap, we investigate the density expectation value of the whole system by using the second quantization method. In the presence of interatomic interaction, based on the exact expression of the density expectation value, it is found that there is a nonzero interference term in the density expectation value of the whole system. The evolution of the density expectation value is shown for different coupling constants. The present work shows clearly that there is an interaction-induced coherence process between two initially independent condensates.

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A. Introduction

After the experimental realization of Bose-Einstein condensates about ten years ago \cite{1,2,3}, the intensive experimental and theoretical studies have shown clearly that the Bose condensate has stable coherence property \cite{2,3,4}. After two separated condensates are allowed to overlap, the coherence property of the whole system is a very interesting problem. Interference patterns between Bose condensates can give us important information about the first-order coherence properties of the whole system. Thus, in the celebrated experiment by Andrews et al \cite{7}, the interference patterns between two separated condensates are investigated. In this experiment, the interference patterns between two coherently separated Bose condensates were observed clearly. In the same experiment, there is also a quite striking experimental result that there are high-contrast fringes even for two completely independent condensates at an initial time.

In our recent theoretical paper \cite{8}, based on the exact expression of the density expectation value calculated from the many-body wavefunction of the whole system, it is shown clearly that there is an interaction-induced coherence process when two initially independent condensates are allowed to overlap. Due to the interaction-induced coherence process between two initially independent condensates, there would be interference patterns in the density expectation value when there is an overlapping between two initially independent condensates. In the present work, we will investigate the density expectation value based on the second quantization method. It is shown that the result calculated from the second quantization method is the same as the result based on the many-body wavefunction. The calculations based on the second quantization method will give us further physical picture for the interaction-induced coherence process. In particular, to show clearly the interaction-induced coherence process, we calculate the density expectation value for different coupling constants in this work. It is shown clearly that increasing the interatomic interaction will enhance the coherence effect of the whole system.

B. The exact expression of the density expectation value

Two separated condensates can be created by a double-well trap (See for example the experiments of Refs \cite{7,9}). When the tunneling effect between two wells can be omitted, one can obtain two completely independent condensates with laser cooling and the following evaporative cooling in the presence of the double-well trap. Obviously, when the tunneling effect between two wells is obvious, the two condensates confined in the double-well trap are coherently separated. The interference patterns of the whole system can be investigated by switching off the double-well trap. As shown in Fig.1(a), there is no overlapping between two initially independent condensates. After the confined potential is switched off, two condensates will expand freely and result in the overlapping.

For two initially independent condensates comprising particle number $N_1$ and $N_2$, the corresponding state is (see also \textsuperscript{[10,11]}):

$$|N_1, N_2\rangle = \frac{C_n}{\sqrt{N_1!N_2!}}(\hat{a}_1^\dagger)^{N_1}(\hat{a}_2^\dagger)^{N_2}|0\rangle,$$

(1)
where \( C_n \) is a normalization constant to assure \( \langle N_1, N_2 | N_1, N_2 \rangle = 1 \). \( \hat{a}_1^\dagger \) (\( \hat{a}_2^\dagger \)) is a creation operator which creates a particle described by the single-particle state \( \phi_1 \) (\( \phi_2 \)) in the left (right) condensate.

One should note that for two initially coherently-separated condensates, the state is
\[
|N\rangle = \frac{1}{\sqrt{N!}} (\hat{b}_1^\dagger)^N |0\rangle,
\]
where \( \hat{b}_1^\dagger \) is a creation operator which creates a particle with the single-particle state \( (\sqrt{N_1} \phi_1 + \sqrt{N_2} \phi_2) / \sqrt{N_1 + N_2} \).

For two coherently-separated condensates, the density expectation value is
\[
n_c (r, t) = \langle N_1, N_2, t | \hat{b}_1^\dagger (r, t) \hat{b}_1 (r, t) | N_1, N_2, t \rangle
= N \left[ a_c^2 |\phi_1 (r, t)|^2 + 2b_c \times \text{Re} (\phi_1^* (r, t) \phi_2 (r, t)) + c_c^2 |\phi_2 (r, t)|^2 \right].
\]

where \( a_c = N_1 / N, b_c = \sqrt{N_1 N_2} / N \) and \( c_c = N_2 / N \) with \( N = N_1 + N_2 \). The second term in the above equation accounts for the interference effect when there is an overlapping between two condensates upon expansion. For two coherently-separated condensates, the interference patterns were investigated in several interesting theoretical works [12, 13].

For two initially independent condensates, however, for the state given by Eq. (1), it seems that there is no interference term in the density expectation value with the following simple calculation (See also [10, 11]):
\[
n_d (r, t) = \langle N_1, N_2, t | \hat{b}_1^\dagger (r, t) \hat{b}_2^\dagger (r, t) \hat{b}_2 (r, t) \hat{b}_1 (r, t) | N_1, N_2, t \rangle
= \langle N_1, N_2, t | (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) | N_1, N_2, t \rangle
+ 2 \times \text{Re} \left( \langle N_1, N_2, t | \hat{a}_1^\dagger \hat{a}_2 | N_1, N_2, t \rangle \phi_1^* (r, t) \phi_2 (r, t) \phi_2^* (r, t) \phi_1 (r, t) \right)
= N_1 |\phi_1 (r, t)|^2 + N_2 |\phi_2 (r, t)|^2.
\]
In the above equation, the field operator \( \hat{\Psi}(r, t) \) is expanded as \( \hat{\Psi}(r, t) = \hat{a}_1 \phi_1 + \hat{a}_2 \phi_2 + \cdots \) with \( \hat{a}_1 \) and \( \hat{a}_2 \) being the annihilation operators. One should note that, to get (10) from (8), there is an implicit assumption that \( \hat{a}_1 \) and \( \hat{a}_2^\dagger \) are commutative. This holds when \( \int \phi_1(r, t) \phi_2^*(r, t) \, dV = 0 \). When \( [\hat{a}_1, \hat{a}_2^\dagger] = 0 \), it is easy to understand that the interference term (the last term in Eq. (6)) is zero in \( n_d(r, t) \).

In the present work, however, we will show that \( \phi_1 \) and \( \phi_2 \) will become non-orthogonal in the presence of interatomic interaction. Thus, we will investigate the general case for \( \int \phi_1 \phi_2^* dV = \zeta \). The operators \( \hat{a}_1 \) and \( \hat{a}_2 \) can be written as

\[
\hat{a}_1 = \int \hat{\Psi} \phi_1^* dV,
\]

and

\[
\hat{a}_2 = \int \hat{\Psi} \phi_2^* dV.
\]

Here \( \hat{\Psi}(x, t) \) is the field operator. By using the commutation relations of the field operators \([\hat{\Psi}(x, t), \hat{\Psi}(y, t)] = 0\) and \([\hat{\Psi}(x, t), \hat{\Psi}^\dagger(y, t)] = \delta(x - y)\), it is easy to get the commutation relation

\[
[\hat{a}_1, \hat{a}_2^\dagger] = \zeta^*.
\]

We see that \( \hat{a}_1 \) and \( \hat{a}_2^\dagger \) are not commutative any more for \( \int \phi_1 \phi_2^* dV \) being a nonzero value. In this case, it is obvious that one cannot get the result (10) from (8). This means that one should be very careful to get the correct density expectation value for two initially independent condensates.

It is well-known that the field operator should be expanded in terms of a complete and orthogonal basis set. Generally speaking, the field operator \( \hat{\Psi}(r, t) \) can be expanded as:

\[
\hat{\Psi}(r, t) = \hat{a}_1 \phi_1(r, t) + \hat{k} \phi_2(r, t) + \cdots,
\]

where \( \phi_1 \) and \( \phi_2 \) are orthogonal normalization wavefunctions. Assuming that \( \phi_2' = \beta (\phi_2 + \alpha \phi_1) \), based on the conditions \( \int \phi_1' \phi_2' dV = 0 \) and \( \int |\phi_2'|^2 dV = 1 \), we have \( |\beta| = (1 - |\zeta|^2)^{-1/2} \) and \( \alpha = -\zeta^* \). Based on

\[
\hat{k} = \int \hat{\Psi} (\phi_2')^* dV,
\]

we have

\[
\hat{a}_2 = \hat{k} / \beta^* + \zeta \hat{a}_1.
\]

It is easy to get the following commutation relations:

\[
[k, k^\dagger] = [k^\dagger, k^\dagger] = 0, [k, k^\dagger] = 1,
\]

\[
[\hat{a}_1, \hat{a}_1] = [\hat{a}_1^\dagger, \hat{a}_1^\dagger] = 0, [\hat{a}_1, \hat{a}_1^\dagger] = 1,
\]

\[
[k, \hat{a}_1] = [k, \hat{a}_1^\dagger] = 0.
\]

Because \( k \) and \( \hat{a}_1^\dagger \) are commutative, it is convenient to calculate the density expectation value \( n_d(r, t) \) by using the operators \( \hat{k} \) and \( \hat{a}_1^\dagger \). The exact expression of the density expectation value is

\[
n_d(r, t) = \langle N_1, N_2, t | \hat{\Psi}^\dagger(r, t) \hat{\Psi}(r, t) | N_1, N_2, t \rangle = C_n^2 \left[ \alpha_d |\phi_1(r, t)|^2 + 2 \times \text{Re} (\beta_d \phi_1^*(r, t) \phi_2(r, t)) + \gamma_d |\phi_2(r, t)|^2 \right],
\]

where the coefficients are

\[
\alpha_d = \sum_{i=0}^{N_2} \frac{N_2! \cdot (N_1 + i - 1)! \cdot N_1 \left(1 - |\zeta|^2\right)^{N_2-i} |\zeta|^{2i}}{i! (N_1 - 1)! (N_2 - i)!},
\]

\[
\beta_d = \sum_{i=0}^{N_2-1} \frac{N_2! \cdot (N_1 + i)! \left(1 - |\zeta|^2\right)^{N_2-i-1} |\zeta|^{2i} \zeta}{i! (i + 1)! (N_1 - 1)! (N_2 - i - 1)!},
\]

\[
\gamma_d = \sum_{i=0}^{N_2-1} \frac{N_2! \cdot (N_1 + i)! \left(1 - |\zeta|^2\right)^{N_2-i-1} |\zeta|^{2i}}{i! N_1! (N_2 - i - 1)!}.
\]
In addition, the normalization constant is determined by

\[
C_n^2 \left( \sum_{i=0}^{N_n} \frac{N_n!}{i!} (N_1+i)! \left(1 - |\zeta(t)|^2\right)^{N_2-i} |\zeta(t)|^{2i} \right) = 1.
\]  

The above density expectation value is obtained based on the second quantization method. Although the expressions of the coefficients given by Eqs. (15), (16), and (17) are quite different from the results calculated from the many-body wavefunction (See Ref. [8]), we have proven that this density expectation value given by Eq. (6) is equal to the result calculated from the many-body wavefunction \( \Psi_{N_1,N_2}(r_1, \ldots, r_{N_1+N_2}, t) \) which satisfies the exchange symmetry of identical bosons in Ref. [8].

For two independent ideal condensates, before the overlapping of the two condensates, \( \zeta(t = 0) = 0 \). Based on the Schrödinger equation, it is easy to verify that after the double-well potential separating two condensates is removed, one has \( \zeta(t) = 0 \) at any further time. Thus \( \beta_d = 0 \), and the density expectation value given by Eq. (14) is equal to the result given by Eq. (6). In this case, the interference term is zero in the density expectation value.

In the presence of interatomic interaction, we now turn to investigate the evolution equations of \( \phi_1 \) and \( \phi_2 \). The overall energy of the whole system is

\[
E = \int dV \langle N_1, N_2, t \rangle \left( \frac{\hbar^2}{2m} \nabla\Psi^\dagger \cdot \nabla\Psi + V_{ext} \Psi^\dagger \Psi + \frac{g}{2} \Psi^\dagger \Psi \Psi^\dagger \Psi \right) |N_1, N_2, t\rangle,
\]

where \( V_{ext} \) is the external potential and \( g \) is the coupling constant. By using the ordinary action principle and the above interaction energy, one can get the following coupled evolution equations for \( \phi_1 \) and \( \phi_2 \):

\[
\begin{align*}
\frac{i\hbar}{\partial t} \frac{\partial \phi_1}{\partial t} &= \frac{1}{N_1} \frac{\delta E}{\delta \phi_1^*}, \\
\frac{i\hbar}{\partial t} \frac{\partial \phi_2}{\partial t} &= \frac{1}{N_2} \frac{\delta E}{\delta \phi_2^*},
\end{align*}
\]

where \( \delta E/\delta \phi_1^* \) and \( \delta E/\delta \phi_2^* \) are functional derivatives. With the above coupled equations, one can understand that \( \zeta(t) \) becomes nonzero after the overlapping between two condensates for \( g \) being nonzero. The reason that \( \zeta(t) \) becomes nonzero is especially due to the nonlinear interaction of the whole system.

Although generally speaking, \( |\zeta(t)| \) is much smaller than 1 because \( \phi_1 \phi_2^* \) is an oscillation function about the space coordinate, nevertheless, a nonzero value of \( |\zeta(t)| \) will give significant contribution to the density expectation value for large \( N_1 \) and \( N_2 \). It is found that for \( |\zeta| \) the order of \( N_1^{-1} \) and \( N_2^{-1} \), the interference term in Eq. (14) will play very important role in the density expectation value [8]. The coherence effect of two initially independent condensates can be shown through the value of \( |\beta_d/o_d| \). For \( \beta_d/o_d = 0 \), there is no coherence between two condensates even there is an overlapping. For \( |\beta_d/o_d| = 1 \), the two condensates can be regarded to be fully coherent.

We see that interatomic interaction plays an essential role in the emergence of the interference effect for two initially independent condensates. In the presence of interatomic interaction, the two initially independent condensates will become coherent after the overlapping between two condensates. This interference effect is a natural result of the commutation relation \( \left[ a_1, a_2^\dagger \right] = \zeta^* \) which is nonzero for \( \zeta^* \) being nonzero in the presence of interatomic interaction. Generally speaking, increasing the particle number will enhance the interference effect in the density expectation value. Based on Eq. (20), increasing the coupling constant \( g \) will have the effect of increasing \( \zeta(t) \). Thus, increasing the interaction between particles will enhance the coherence effect. To show more clearly the interaction-induced coherence process between two initially independent condensates, here we will investigate the density expectation value for different coupling constants. In the last few years, the rapid experimental advances of Feshbach resonance [14] where the scattering length can be tuned from positive to negative make this sort of experiment be possible.

C. Evolution of the density expectation value for different coupling constants

To give a clear presentation, we consider the evolution of the density expectation value for one-dimensional case. At \( t = 0 \), to give a general comparison for different coupling constants, the initial wavefunctions for two independent condensates are assumed as

\[
\phi_1(x_1, t = 0) = \frac{1}{\pi^{1/4} \sqrt{\Delta_1}} \exp \left[ -\frac{(x_1 - x_1^*)^2}{2\Delta_1^2} \right],
\]

where \( \Delta_1 \) and \( \Delta_2 \) are determined by the above interaction energy, one can get the following coupled evolution equations for \( \phi_1 \) and \( \phi_2 \):

\[
\begin{align*}
\frac{i\hbar}{\partial t} \frac{\partial \phi_1}{\partial t} &= \frac{1}{N_1} \frac{\delta E}{\delta \phi_1^*}, \\
\frac{i\hbar}{\partial t} \frac{\partial \phi_2}{\partial t} &= \frac{1}{N_2} \frac{\delta E}{\delta \phi_2^*},
\end{align*}
\]
FIG. 2: After two initially independent condensates are allowed to expand freely, shown is the evolution of the density expectation value \( n_d(x, \tau) \) (in unit of \( N_1 + N_2 \)) for different coupling constants \( g_l \). Shown in the inset of each figure is the relation between \( \beta_d/\alpha_d \) and dimensionless time \( \tau \). For two ideal condensates shown in figure 2(a), we see that there is no interference pattern even there is an overlapping between two condensates. For the case of \( g_l = 1 \) shown in figure 2(b), we see that low-contrast interference patterns begin to emerge due to the interaction-induced coherence process. In figures 2(c) and 2(d), we see that there are high-contrast interference patterns. In particular, in figure 2(d) for \( g_l = 20 \), two initially independent condensates can be regarded to be fully coherent near \( \tau = 1 \).

\[
\phi_2(x_1, t = 0) = \frac{1}{\pi^{1/4}/\sqrt{\Delta_2}} \exp \left[ -\frac{(x_1 - x_2)^2}{2\Delta_2^2} \right]. \tag{22}
\]

In the above wavefunction, we have introduced a dimensionless variable \( x_l = x/l \) with \( l \) being a length. In the present work, we assume that \( \Delta_1 = \Delta_2 = 0.5 \), and \( x_2 - x_1 = 4.5 \). For these parameters, at \( t = 0 \), the two condensates are well separated. In the numerical calculations of the coupled equations given by Eq. (20), it is useful to introduce the dimensionless variable \( \tau = E_l t/\hbar \) with \( E_l = \hbar^2/2ml^2 \) and dimensionless coupling constant \( g_l = N_1 g/E_l l \). In addition, the particle number is assumed as \( N_1 = N_2 = 1.0 \times 10^5 \). In real experiments, interatomic interaction will play very important role in the ground-state wavefunction of the condensates confined in the double-well trap. However, in principle, one can prepare the state given by Eqs. (21) and (22) by adjusting the double-well trap for different coupling constants. In the present theoretical work, the identical initial wavefunction for different coupling constants will be helpful in the comparison of the density expectation value for different coupling constants.

With these parameters, one can get the evolution of \( \phi_1 \) and \( \phi_2 \) based on the numerical calculations of Eq. (20) after the double-well potential is switched off. From \( \phi_1 \) and \( \phi_2 \), we can get \( \zeta \), and thus the density expectation value based on Eq. (14). Shown in figure 2 is the evolution of the density expectation value for different coupling constants. It is shown clearly that increasing the coupling constant will enhance the coherence effect.

D. Summary and discussion

In summary, by using the second quantization method, we give the exact expression of the density expectation value when two initially independent condensates are allowed to overlap. The exact expression of the density expectation
value is the same as the result calculated from the many-body wavefunction [8]. In the presence of interatomic interaction, the wavefunctions of two condensates will become non-orthogonal after the overlapping. For large particle number, the non-orthogonal property of the wavefunctions plays very important role in the interference effect between two condensates. Through the numerical calculations of the density expectation value of the whole system for different coupling constants, we show the interaction-induced coherence process between two initially independent condensates.

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