Martingale representations in progressive enlargement setting: the role of the accessible jump times

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Abstract
Let $M$ and $N$ be an $\mathbb{F}$-martingale and an $\mathbb{H}$-martingale respectively on the same probability space, both enjoying the predictable representation property. We discuss how, under the assumption of the existence of an equivalent decoupling measure for $\mathbb{F}$ and $\mathbb{H}$, the nature of the jump times of $M$ and $N$ affects the representation of the $\mathbb{F} \vee \mathbb{H}$-martingales. More precisely we show that the multiplicity in the sense of Davis and Varaiya of $\mathbb{F} \vee \mathbb{H}$ depends on the behavior of the common accessible jump times of the two martingales. Then we propose an extension of Kusuoka’s representation theorem.

Keywords: Semi-martingales, predictable representations property, enlargement of filtration, completeness of a financial market

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1 Introduction
Classical topics of stochastic analysis are martingale representation formulas and predictable representation property (in short p.r.p.) of semi-martingales. The interest in them is enforced by their crucial role in many application fields and among the others in mathematical finance (see e.g. [26], [14] and as more recent contributions, among the others, [25], [11], [12]).

In [4], under the hypotheses that $M$ and $N$ are two square-integrable martingales on a finite time interval $[0, T]$, both enjoying the p.r.p. with respect to the filtrations $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \in [0, T]}$ respectively, it is shown that, if there exists an equivalent probability measure such that $\mathbb{F}$ and $\mathbb{H}$ are independent (decoupling measure), then the triplet $(M, N, [M, N])$ enjoys the $\mathbb{F} \vee \mathbb{H}$-p.r.p.. (see Theorem 4.5 and Remark 9 of that paper). The key argument to prove this result is that there exists an equivalent decoupling measure under which $M$ and $N$ are still martingales (martingale preserving measure) and $[M, N]$ is an $\mathbb{F} \vee \mathbb{H}$-square integrable martingale strongly orthogonal to $M$ and $N$. Under

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that measure the triplet \((M, N, [M, N])\) enjoys the \(\mathbb{F} \lor \mathbb{H}\)-p.r.p. and therefore it is an \(\mathbb{F} \lor \mathbb{H}\)-basis of martingales. A multidimensional version of previous statement is presented in [5].

The result given in [4] is redundant as soon as one of the two martingales has only totally inaccessible jump times. More precisely in this case, if there exists an equivalent decoupling measure for \(\mathbb{F}\) and \(\mathbb{H}\), then, under any equivalent probability measure, \(M\) and \(N\) do not share definitely any jump time with positive probability and the process \([M, N]\) vanishes almost surely. In fact as well-known in general the covariation process of two independent martingales coincides with the sum of the common jumps, two independent martingales cannot share with positive probability any jump time which is totally inaccessible for one of them and finally the covariation process and the nature of a random time are both invariant under equivalent changes of measure.

Basically under the hypotheses of Theorem 4.5 in [4] the covariation process \([M, N]\) is identically zero whenever \(M\) or \(N\) are c\-ad\-lag quasi-left continuous martingales (see e.g. pages 121-122 in [16]). Among the others this is the case when at least one of the reference filtrations of \(M\) and \(N\) is quasi-left continuous. In fact any martingale with quasi-left continuous reference filtration is quasi-left continuous (see e.g. page 190 in [27]). Therefore if \(M\) or \(N\) belong to the class of Lévy processes without deterministic jump times, then the result in [4] is redundant.

The main contribution of our paper is to handle, under the decoupling assumption, the covariation process in the martingale representation formulas, whenever the involved processes have accessible jump times.

Processes with accessible jump times are not very often investigated, but recently they received new attention by the literature of mathematical finance. This fact is due to the interest in the behavior of real markets, which may present critical announced random times. Processes with predictable jump times naturally enter in the construction of models for such markets (see [23] and [13]), so that classical results, in particular of credit risk theory, must be revisited. In fact, if \(\mathbb{F}\) denotes the reference filtration of the risky asset price, the most common assumption on the default time \(\tau\) is the density hypothesis, namely the absolute continuity for all \(t\) of the \(\mathcal{F}_t\)-conditional law of \(\tau\) with respect to a deterministic measure without atoms (see [10]). Under last condition \(\tau\) has no atoms and therefore it is totally inaccessible with respect to the natural filtration of the default process \(I_{\tau \leq \cdot}\) (see IV-107 in [8]). Then, if the market is complete, at most two are the strongly orthogonal martingales needed for representing all the martingales with respect to the filtration which models the global information, obtained progressively enlarging \(\mathbb{F}\) by the occurrence of \(\tau\) (see [8], [19] and [4]). One of the starting points of this paper was the following question. When no density hypothesis is assumed, how can be applied the results in [4] and [5] in order to get a basis of martingales for the progressively enlarged filtration?

In the first part of this paper we deal with the multiplicity in the sense of Davis and

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1 Here and in the rest of the paper **totally inaccessible is any jump time of \(M\) which is totally inaccessible with respect to the filtration \(\mathbb{F}\)**. \(\mathbb{F}\) may differ from the natural filtration, \(\mathbb{F}^M\). Nevertheless, as will be discussed in Remark 2.4, the results of Section 2 still holds when considering totally inaccessibility of the jump times of \(M\) (\(N\)) either with respect to \(\mathbb{F}^M\) (\(\mathbb{F}^N\)) or with respect to \(\mathbb{F}\lor \mathbb{H}\).
Varaiya of the filtration $\mathbb{F} \vee \mathbb{H}$, that is the minimal number of square-integrable strongly orthogonal $\mathbb{F} \vee \mathbb{H}$-martingales needed to characterize all martingales (see [7]). We prove that under the equivalent decoupling martingale preserving measure the multiplicity is 1, 2 or 3 according to the behavior of the random measures induced by the sharp bracket processes $\langle M \rangle$ and $\langle N \rangle$. This follows combining Theorem 4.11 in [4] with some results interesting in themselves. First of all a necessary and sufficient condition for $[M, N]$ to be null. Then two general lemmas dealing with the predictable supports of the random measures induced by two predictable non-negative increasing processes.

In the second part of the paper we derive a martingale representation result for the filtration $\mathbb{F} \vee \mathbb{H}$ where $\mathbb{F}$ is the reference filtration of a semi-martingale $X$ enjoying the $\mathbb{F}$-p.r.p. and $\mathbb{H}$ is the natural filtration of the default process of a general random time $\tau$. As already known, last process after compensation enjoys the $\mathbb{H}$-p.r.p. and, for the sake of completeness, here we provide a simple proof of this fact. Our main result, in the language of credit risk theory, can be read as follows. Denote by $M$ the martingale part of the risky asset price $X$, by $H$ the compensated default process and by $H'$ the $\mathbb{F}$-conditional compensated default process. All essentially bounded payoffs can be represented under the historical probability measure as a vector stochastic integral with respect to the $\mathbb{R}^3$-valued martingale, whose components are $M$, $H'$ and the covariation process of $M$ and $H$.

When $X$ is a Brownian motion and $\tau$ satisfies the density hypothesis, Kusuoka’s representation theorem follows as a particular case.

### 2 Setting and basic results

Let $T$ be a fixed time horizon. On a probability space $(\Omega, \mathcal{F}, P)$ let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \in [0,T]}$ be two filtrations satisfying the usual conditions and such that $\mathcal{F}_T \subset \mathcal{F}$ and $\mathcal{H}_T \subset \mathcal{F}$. Set

$$G = \mathbb{F} \vee \mathbb{H}$$

(1)

and assume that

D) there exists $Q$ on $(\Omega, \mathcal{G}_T)$ such that $Q$ is equivalent to $P|_{\mathcal{G}_T}$ and $\mathbb{F}$ and $\mathbb{H}$ are $Q$-independent.

On $(\Omega, \mathcal{F}, P)$ let $M = (M_t)_{t \in [0,T]}$ and $N = (N_t)_{t \in [0,T]}$ be a square-integrable $(\mathbb{P}, \mathbb{F})$-martingale and a square-integrable $(\mathbb{P}, \mathbb{H})$-martingale respectively, so that

$$E^P \left[ \sup_{t \in [0,T]} M_t^2 \right] < +\infty, \quad E^P \left[ \sup_{t \in [0,T]} N_t^2 \right] < +\infty.$$

Define a new probability measure $P^*$ on $\mathcal{G}_T$ by

$$dP^* = \frac{dP|_{\mathcal{F}_T}}{dQ|_{\mathcal{F}_T}} \cdot \frac{dP|_{\mathcal{H}_T}}{dQ|_{\mathcal{H}_T}} dQ.$$  

(2)

$P|_{\mathcal{G}_T}$ is the restriction of $P$ to $\mathcal{G}_T$
Then $M$ and $N$ are independent $(P^*, \mathbb{G})$-martingales and each of them preserves under $P^*$ the same law which it has under $P$. Moreover the existence of $P^*$ assure that $\mathbb{G}$ is a right continuous filtration (see [11]).

Denote by $\mathbb{P}(M, \mathbb{F})$ the set of probability measures on $(\Omega, \mathcal{F}_T)$ under which $M$ is a martingale and which are equivalent to $P|_{\mathcal{F}_T}$. Denote by $\mathbb{P}(N, \mathbb{H})$ the analogous for $N$.

Consider the condition

**A1**) $\mathbb{P}(M, \mathbb{F}) = \{P|_{\mathcal{F}_T}\}$ and $\mathbb{P}(N, \mathbb{H}) = \{P|_{\mathcal{H}_T}\}$.

As well-known, assumption **A1** implies that $\mathcal{F}_0$ and $\mathcal{H}_0$ are trivial and that $M$ and $N$ enjoy the $(P, \mathbb{F})$-p.r.p. and $(P, \mathbb{H})$-p.r.p. respectively (see [15] and Theorem 13.4 in [16]).

Let us recall a well known result.

**Theorem 2.1.** (Theorem 1-4 p. 437 in [29])

Let $Z = (Z)_t$ be a local martingale starting from 0. Then $Z$ can be uniquely decomposed as

$$Z = Z^c + Z^{dp} + Z^{dq},$$

(3)

where $Z^c$, $Z^{dp}$ and $Z^{dq}$ are local martingale starting from 0 and where $Z^c$ has continuous trajectories, $Z^{dp}$ has only accessible jump times and is strongly orthogonal to any local martingales with only totally inaccessible jump times, $Z^{dq}$ has only totally inaccessible jump times and is strongly orthogonal to any local martingales with only accessible jump times.

We will refer to $Z^c$, $Z^{dp}$ and $Z^{dq}$ as the continuous martingale part, the accessible martingale part and the totally inaccessible martingale part of $Z$ respectively.

It is worthwhile to mention that the above decomposition depends on the reference filtration. In fact as it is well-known the nature of a random time is linked to the choice of the filtration and in particular accessibility is preserved by enlarging the filtration and viceversa total inaccessibility is preserved under restriction of the filtration.

In the light of Theorem (2.1) and referring to the filtrations $\mathbb{F}$ and $\mathbb{H}$ we get

$$M = M^c + M^{dp} + M^{dq}, \quad N = N^c + N^{dp} + N^{dq}$$

(4)

with the usual meaning of superscripts.

Next lemma and its corollary make rigorous the following roughly statement. Two independent martingales can only share accessible jump times with positive probability and their covariation process coincides with the covariation process of their accessible martingale parts.

**Lemma 2.2.** Assume condition **D**. Let $\tau$ and $\sigma$ be an $\mathbb{F}$-stopping time and an $\mathbb{H}$-stopping time respectively. If $P(\tau = \sigma) > 0$ then there exist an accessible $\mathbb{F}$-stopping time $\tau^{dp}$ and an accessible $\mathbb{H}$-stopping time $\sigma^{dp}$ such that on the set $(\tau = \sigma)$ it holds $\tau = \tau^{dp}$ and $\sigma = \sigma^{dp}$, $P$-a.s.
Proof. As well-known there exist disjoint events \(A, B\) and \(C, D\) such that \(P\) and \(Q\)-a.s. \(A \cup B = (\tau < \infty), C \cup D = (\sigma < \infty)\) and stopping times \(\tau^{dp}\) and \(\sigma^{dp}\) \(F\) and \(H\)-accessible respectively, \(\tau^{dq}\) and \(\sigma^{dq}\) \(F\) and \(H\)-totally inaccessible respectively such that

\[
\tau^{dp} = \tau \mathbb{I}_A + \infty \mathbb{I}_{A^c}, \quad \tau^{dq} = \tau \mathbb{I}_B + \infty \mathbb{I}_{B^c}, \quad \tau = \tau^{dp} \wedge \tau^{dq}
\]

and analogously

\[
\sigma^{dp} = \sigma \mathbb{I}_C + \infty \mathbb{I}_{C^c}, \quad \sigma^{dq} = \sigma \mathbb{I}_D + \infty \mathbb{I}_{D^c}, \quad \sigma = \sigma^{dp} \wedge \sigma^{dq}
\]

(see e.g. Theorem 3, page 104 [27]). Therefore

\[
\tau \mathbb{I}_{(r=\sigma)} = \tau \mathbb{I}_{(r=\sigma=+\infty)} + \tau \mathbb{I}_{(r=\sigma) \cap A} + \tau \mathbb{I}_{(r=\sigma) \cap B}
\]

and after considering that \((\tau = \sigma = +\infty) \subset A^c\) and that \((\tau = \sigma) \cap A\) coincides with \((\tau = \sigma) \cap (\tau^{dp} < +\infty)\) whereas \((\tau = \sigma) \cap B\) is equal to \((\tau = \sigma) \cap (\tau = \tau^{dq} < +\infty)\) one immediately derives that \(P\) and \(Q\)-a.s.

\[
\tau \mathbb{I}_{(r=\sigma)} = \tau^{dp} \mathbb{I}_{(r=\sigma=+\infty)} + \tau^{dp} \mathbb{I}_{(r^{dp}=\sigma) \cap A} + \tau^{dq} \mathbb{I}_{(r^{dq}=\sigma) \cap B}.
\]

Since \(Q\) decouples \(F\) and \(H\) and \(\tau^{dq}\) is totally inaccessible then \((\tau^{dq} = \sigma)\) has null measure under \(Q\) and \(P\) so that \(Q\) and \(P\)-a.s.

\[
\tau \mathbb{I}_{(r=\sigma)} = \tau^{dp} \mathbb{I}_{(r=\sigma=+\infty)} + \tau^{dp} \mathbb{I}_{(r^{dp}=\sigma) \cap A}
\]

which implies \(\tau = \tau^{dp}\) on \((\tau = \sigma)\).

Analogously one shows that \(\sigma = \sigma^{dp}\) on \((\tau = \sigma)\). \(\square\)

**Corollary 2.3.** Assume condition **D**. Then for all \(t\) in \([0, T]\) it holds \(P\)-a.s.

\[
[M, N]_t = [M^{dp}, N^{dp}]_t.
\]

**Proof.** Let \(P^*\) be the probability measure defined by [2]. \(M\) and \(N\) are \(F\)-martingale and \(H\)-martingale respectively either under \(P\) or under \(P^*\) and the decomposition ([1]) holds either under \(P\) or under \(P^*\), since the continuity of the trajectories is invariant under equivalent changes of measure and the nature of a random time do not depend on the measure. From the general formula

\[
[M, N]_t = \langle M^c, N^c \rangle_t + \sum_{s \leq t} \Delta M_s \Delta N_s,
\]

recalling that by definition \([M, N]\) is invariant under equivalent changes of measure and that \(P^*\) makes \(F\) and \(H\) independent so that \([M^c, N^c]^{P^*, G}\) is \(P^*\)-a.s. null, we derive that \(P^*\)-a.s. and therefore \(P\)-a.s.

\[
[M, N]_t = \sum_{s \leq t} \Delta M_s \Delta N_s.
\]

If \(\tau\) and \(\sigma\) are jump times of \(M\) and \(N\) respectively then they are finite stopping times with respect to the natural filtrations, \(F^M\) and \(F^N\) respectively. If \(P(\tau = \sigma) > 0\) then
by Lemma 2.2 there exists an $F^M$-accessible jump time of $M$, $\tau^{dp}$, and an $F^N$-accessible jump time of $N$, $\sigma^{dp}$, such that on the set ($\tau = \sigma$) it holds $\tau = \tau^{dp}$ and $\sigma = \sigma^{dp}$, P.a.s.. Indeed the accessible stopping times $\tau^{dp}$ and $\sigma^{dp}$ of Lemma 2.2 since the accessibility is preserved by enlargement of filtration, are also $F$-accessible and $H$-accessible respectively. Moreover they inherit from $\tau$ and $\sigma$ respectively the property of finite jump times so that on the set ($\tau = \sigma$) it holds $\Delta M^{dp}_{\tau^{dp}} > 0$ and $\Delta N^{dp}_{\sigma^{dp}} > 0$ and as a consequence $P(\Delta M^{dp}_{\tau^{dp}} \Delta N^{dp}_{\sigma^{dp}} > 0) > 0$. Viceversa if for any $F$-accessible jump time of $M$, $\tau^{dp}$, and any $H$-accessible jump time of $N$, $\sigma^{dp}$, it holds $P(\Delta M^{dp}_{\tau^{dp}} \Delta N^{dp}_{\sigma^{dp}} > 0) > 0$ then trivially $P(\Delta M^{dp}_{\tau^{dp}} \Delta N^{dp}_{\sigma^{dp}} > 0) > 0$. It follows that $P$-a.s.

$$[M, N]_t = \sum_{n,l} \mathbb{I}_{\tau^{dp}_n \leq t} \mathbb{I}_{\tau^{dp}_n = \sigma^{dp}_l} \Delta M^{dp}_{\tau^{dp}_n} \Delta N^{dp}_{\sigma^{dp}_l} = [M^{dp}, N^{dp}]_t,$$

where $(\tau^{dp}_n)_n$ is the set of $F$-accessible jump times of $M$ i.e. the jump times of $M^{dp}$ and $(\sigma^{dp}_l)_l$ is the set of $H$-accessible jump times of $N$ i.e. the jump times of $N^{dp}$.

Remark 2.4. Slightly changing the arguments in the above proof it can be shown that equality (6) is still true either if one replaces $M^{dp}$ with the $F^M$-accessible martingales parts of $M$ and $N^{dp}$ with the $F^N$-accessible martingales parts of $N$ or if one replaces $M^{dp}$ and $N^{dp}$ with the $G$-accessible martingale parts of $M$ and $N$ respectively.

3 The multiplicity of the progressively enlarged filtration

In this section we deal with the cardinality of any basis of martingales for the enlarged filtration $G$ introduced by (11). More precisely we compute the multiplicity in the sense of Davis and Varaiya of $G$ according to the following definition.

Definition 3.1. [7] Given a filtered space $(\Omega, \mathcal{A}, \mathcal{F}, R)$ the multiplicity of the filtration $\mathcal{F}$ under the measure $R$ is the smallest integer $k$ such that there exists a $k$-dimensional martingale with pairwise $(R, \mathcal{F})$-strongly orthogonal components which enjoys the $(R, \mathcal{F})$-p.r.p..

First of all we state a result which specifies how the covariation process $[M, N]$ behaves under the decoupling assumption D).

Proposition 3.2. Assume condition D) and let $M^{dp}$ and $N^{dp}$ be the martingales of decomposition (4). Then the covariation process $[M, N]$ is null P.a.s. if and only if the random measures generated by the increasing processes $\langle M^{dp} \rangle_{P,F}$ and $\langle N^{dp} \rangle_{P,H}$ are P.a.s. mutually singular.

Proof. Fixed $n \in \mathbb{N}$, let $\tau^{dp}_n$ be jump time of $M^{dp}$ and let $\{\tau_{n,m}\}_{m \in \mathbb{N}}$ be the sequence of predictable stopping times that envelop it and, fixed $l \in \mathbb{N}$, let $\sigma^{dp}_l$ be jump time of $N^{dp}$ and let $\{\sigma_{l,h}\}_{m \in \mathbb{N}}$ be the sequence of predictable stopping times that envelop it. Then by Corollary 2.3 it follows that P.a.s.

$$[M, N]_t = \sum_{n,m} \sum_{l,h} \mathbb{I}_{\tau^{dp}_n = \tau_{n,m}} \mathbb{I}_{\tau_{n,m} \leq t} \mathbb{I}_{\tau_{n,m} = \sigma_{l,h}} \Delta M^{dp}_{\tau_{n,m}} \Delta N^{dp}_{\sigma_{l,h}}. \quad (7)$$
Let us show that the process \((M^{dp})^{P,F}\) is a sum of jumps at the predictable times \(\{\tau_{n,m}\}_{n,m \in \mathbb{N}}\) and the process \((N^{dp})^{P,H}\) is a sum of jumps at the predictable times \(\{\sigma_{l,h}\}_{l,h \in \mathbb{N}}\).

First of all observe that by definition

\[
[M^{dp}]_t = \sum_{s \leq t} (\Delta M_{s}^{dp})^2
\]

and therefore

\[
[M^{dp}]_t = \sum_{n \in \mathbb{N}} (\Delta M_{\tau_{n,m}}^{dp})^2 \mathbb{I}_{\tau_{n,m} \leq t} = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} (\Delta M_{\tau_{n,m}}^{dp})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mathbb{I}_{\tau_{n,m} \leq t}.
\]

As well-known, the process \((M^{dp})^{P,F}\) is the predictable compensator of \([M^{dp}]\), and, following Theorem VI-76 pg. 148 in \([9]\), for any \(\mathbb{F}\)-predictable stopping time \(\tau\)

\[
\Delta [M^{dp}]^{P,F}_{\tau} = E \left[ \Delta [M^{dp}]_{\tau} \mid \mathcal{F}_{\tau} \right]
\]

As we are going to prove the last fact joint with representation \((\mathcal{S})\) allows to write

\[
\langle M^{dp} \rangle_t = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} E \left[ (\Delta M_{\tau_{n,m}}) \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mid \mathcal{F}_{\tau_{n,m}} \right] \mathbb{I}_{\tau_{n,m} \leq t}, \quad t \in [0,T]
\]

In fact, since the process at the right hand side of the previous relation is \(\mathbb{F}\)-predictable, in force of the uniqueness of the Doob-Meyer decomposition (see Theorem 2.28 at page 43 in \([16]\)), if we verify that

\[
[M^{dp}]_t - \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} E \left[ (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mid \mathcal{F}_{\tau_{n,m}} \right] \mathbb{I}_{\tau_{n,m} \leq t}, \quad t \in [0,T]
\]

is a \((P,\mathbb{F})\)-martingale, then equality \((\mathcal{S})\) is proved.

At this aim using representation \((\mathcal{S})\), for any \(s \leq t\), we get

\[
E \left[ [M^{dp}]_t - \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} E \left[ (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mid \mathcal{F}_{\tau_{n,m}} \right] \mathbb{I}_{\tau_{n,m} \leq t} \mid \mathcal{F}_s \right] =
\]

\[
[M^{dp}]_s - \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} E \left[ (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mid \mathcal{F}_{\tau_{n,m}} \right] \mathbb{I}_{\tau_{n,m} \leq s} +
\]

\[
E \left[ \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \left( (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} - E \left[ (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mid \mathcal{F}_{\tau_{n,m}} \right] \right) \mathbb{I}_{s < \tau_{n,m} \leq t} \mid \mathcal{F}_s \right].
\]

Last term in the previous expression can be written as

\[
\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} E \left[ \left( (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} - E \left[ (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mid \mathcal{F}_{\tau_{n,m}} \right] \right) \mathbb{I}_{s < \tau_{n,m} \leq t} \mid \mathcal{F}_s \right] \mathbb{I}_{s < \tau_{n,m}}
\]

and since \(\tau_{n,m}\) is \(\mathcal{F}_{\tau_{n,m}}\)-measurable (see Theorem 3.4 point 1 in \([16]\)) the general term of the sum is null if and only if

\[
E \left[ (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mathbb{I}_{s < \tau_{n,m} \leq t} \mid \mathcal{F}_s \right] = E \left[ E \left[ (\Delta M_{\tau_{n,m}})^2 \mathbb{I}_{\tau_{n,m} = \tau_{n,m}} \mid \mathcal{F}_{\tau_{n,m}} \right] \mathbb{I}_{s < \tau_{n,m} \leq t} \mid \mathcal{F}_s \right] \mathbb{I}_{s < \tau_{n,m}}
\]

\(\mathcal{F}_{\tau} = \sigma (C \cap (s < \tau), C \in \mathcal{F}_s)\)
that is if and only if for any set $A \in \mathcal{F}_s$

$$\int_{A \cap (s < \tau_{n,m})} E \left[ (\Delta M_{\tau_{n,m}}^m)^2 \mathbb{1}_{\tau_{n,m} = \tau_{n,m}} \mathbb{1}_{\tau_{n,m} \leq t} \mid \mathcal{F}_s \right] dP =$$

$$\int_{A \cap (s < \tau_{n,m})} E \left[ E \left[ (\Delta M_{\tau_{n,m}}^m)^2 \mathbb{1}_{\tau_{n,m} = \tau_{n,m}} \mathbb{1}_{\tau_{n,m} \leq t} \mid \mathcal{F}_{\tau_{n,m}^-} \right] \mid \mathcal{F}_s \right] dP.$$ 

Last equality is true. In fact, $A \cap (s < \tau_{n,m})$ belongs either to $\mathcal{F}_s$ or to $\mathcal{F}_{\tau_{n,m}^-}$, so that both expressions coincide with $\int_{A \cap (s < \tau_{n,m})} (\Delta M_{\tau_{n,m}}^m)^2 \mathbb{1}_{\tau_{n,m} \leq t} dP$.

In a similar way

$$\langle N^{dp} \rangle_t^{P,H} = \sum_{l \in \mathbb{N}} \sum_{h \in \mathbb{N}} E \left[ (\Delta N_{\sigma_{l,h}})^2 \mathbb{1}_{\sigma_{l,h} = \tau_{l,h}} \mid \mathcal{H}_{\sigma_{l,h}^-} \right] \mathbb{1}_{\sigma_{l,h} \leq t}. \tag{10}$$

Equalities (7), (9) and (10) tell us that either $[M, N] = 0$ or the mutual singularity of the random measures generated by $\langle M^{dp} \rangle^{P,H}$ and $\langle N^{dp} \rangle^{P,H}$ are equivalent to the condition

$$P \left( \bigcup_{n,k,l,h} (\tau_{n,k} = \sigma_{l,h}) \right) = 0.$$

Following [18] Chapter 1, we recall that a random set (see page 3 in [18]) is just a subset of the product space $\Omega \times \mathbb{R}_+$, while a predictable random set (see page 16 in [18]) is a random set belonging to the predictable sigma algebra $\mathcal{P}$ on $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$. Finally, in analogy with the definition in [2] (see page 19), we call support of a measure $\mu$ on a measurable space $(E, \mathcal{E})$ any set $C \in \mathcal{E}$ such that $\mu(E \setminus C) = 0$.

**Lemma 3.3.** Let $(A_t)_{t \in [0,T]}$ be a non-negative increasing predictable process on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{P})$. Then there exists a predictable random set $C^A \in \Omega \times [0,T]$ such that $P$–a.s. the $\omega$-section of $C^A$ is a support for the measure induced on $[0,T]$ by the trajectory $t \to A_t(\omega)$.

**Proof.** For each $\omega$ denote by $\tilde{C}^{A,\omega}$ a support of the measure induced on $[0,T]$ by the trajectory $t \to A_t(\omega)$, that is $[0,T] \setminus \tilde{C}^{A,\omega}$ has $dA(\omega)$-measure zero.

Consider the measurable random set

$$\tilde{C}^A = \bigcup_{\omega} \left( (\omega, t) : t \in \tilde{C}^{A,\omega} \right)^5$$

and set

$$C^A = \left( (\omega, t) \in \Omega \times [0,T] : E[\mathbb{1}_{\tilde{C}^A}(t) \mid \mathcal{F}_{t-}] (\omega) > 0 \right). \tag{11}$$

Then $C^A$ is a predictable random set (see Definition 2.32 at page 24 in [18]) and for each $\omega$ its $\omega$-section, $C^{A,\omega}$, is a support of the measure associated to $(A_t(\omega))_{t \in [0,T]}$.

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4 In measure theory the support of a measure $\mu$ is defined as the complementary set of the union of all the $\mu$-null open sets (see, e.g. [3], III-50)

5 Measurability of $\tilde{C}^A$ follows by the fact that $(\tilde{C}^A)^c = \left\{ (t, \omega) \text{ such that } t \in \left( \tilde{C}^{A,\omega} \right)^c \right\}$ has $dP \times dA(\omega)$-measure zero, so that it is an element of the product $\sigma$-algebra $\mathcal{B}[0,T] \otimes \mathcal{F}$
In order to prove this assertion it suffices to establish that \( \int_{[0,T] \setminus C^A_\omega} dA_s(\omega) = 0 \).

In fact \( \int_{[0,T] \setminus C^A_\omega} dA_s(\omega) \geq 0 \) and moreover the following equalities hold

\[
E\left[ \int_{[0,T] \setminus C^A_\omega} dA_s(\cdot) \right] = E\left[ \int_{[0,T] \setminus C^A_\omega} \mathbb{I}_\tilde{C}_\omega(s) dA_s(\cdot) \right] = E\left[ \int_{[0,T] \setminus C^A_\omega} \mathbb{I}_\tilde{C}_\omega(s) \mathbb{I}_{\tilde{C}}(\cdot, s) dA_s(\cdot) \right],
\]

and, due to the fact that the process \( (A_t)_{t \in [0,T]} \) is predictable, the term at the right hand side turns to be equal to (see Theorem VI.57 in [9])

\[
E\left[ \int_{[0,T]} \mathbb{I}_{[0,T] \setminus C^A_\omega}(s) \mathbb{I}_{\tilde{C}}(\cdot, s) dA_s(\cdot) \right] =
E\left[ \int_{[0,T]} \mathbb{I}_{[0,T] \setminus C^A_\omega}(s) E[\mathbb{I}_{\tilde{C}}(\cdot, s) | \mathcal{F}_s^-] dA_s(\cdot) \right] =
E\left[ \int_{[0,T] \setminus C^A_\omega} E[\mathbb{I}_{\tilde{C}}(\cdot, s) | \mathcal{F}_s^-] dA_s(\cdot) \right].
\]

Finally the last term is equal to zero since by definition on the set \( [0,T] \setminus C^A_\omega \) the function \( s \rightarrow E[\mathbb{I}_{\tilde{C}}(\cdot, s) | \mathcal{F}_s^-] (\omega) \) is null. \( \square \)

**Lemma 3.4.** Let \( (A_t)_{t \in [0,T]} \) and \( (B_t)_{t \in [0,T]} \) be two non-negative increasing predictable processes \((\Omega, \mathcal{F}, \mathbb{F}, P)\). Assume that the associated random measures on \([0,T] \) are \( P \)-a.s. mutually singular. Then there exist two predictable sets \( C^A \) and \( C^B \) such that, for all \( \omega \), their \( \omega \)-sections \( C^A_\omega \) and \( C^B_\omega \) are disjoint supports respectively of the measure induced on \([0,T] \) by \( t \rightarrow A_t(\omega) \) and \( t \rightarrow B_t(\omega) \).

**Proof.** By the assumption of singularity, for each \( \omega \) there exist two measurable sets \( \tilde{C}^A_\omega \) and \( \tilde{C}^B_\omega \) such that

\[
\int_{\tilde{C}^B_\omega} dA_t(\omega) = 0 \quad \text{and} \quad \int_{\tilde{C}^A_\omega} dB_t(\omega) = 0
\]

with \( \tilde{C}^A_\omega \cap \tilde{C}^B_\omega = \emptyset \), \( \tilde{C}^A_\omega \cup \tilde{C}^B_\omega = [0,T] \). Obviously \( \tilde{C}^B_\omega = [0,T] \setminus \tilde{C}^A_\omega \).

Consider the predictable random set \( C^A \) constructed in (11). Then the thesis follows as soon as one prove that \( P \)-a.s.

\[
\int_{C^A_\omega} dB_t(\omega) = 0.
\]

In fact \( \int_{\tilde{C}^A_\omega} dB_t(\omega) = 0 \) so that

\[
0 = E\left[ \int_{\tilde{C}^A} dB_t \right] = E\left[ \int_{[0,T]} \mathbb{I}_{\tilde{C}^A}(t) dB_t(\cdot) \right] = E\left[ \int_{[0,T]} E[\mathbb{I}_{\tilde{C}^A}(t) | \mathcal{F}_t^-] dB_t \right]
\]

that is \( P(d\omega) \otimes dB_t(\omega) \)-a.s. \( E[\mathbb{I}_{\tilde{C}^A}(t) | \mathcal{F}_t^-] (\omega) = 0 \). Then, by definition of \( C^A \) it follows that \( dB_t(\omega) \)-a.s. it holds \( \mathbb{I}_{C^A_\omega}(\cdot) = 0 \), which ends the proof. \( \square \)

Before announcing the main theorem of this section let us state a lemma.

**Lemma 3.5.** Assume **A1** and **D**. Let \( P^* \) be the probability measure defined by \([\Omega, \mathcal{F}, \mathbb{F}, P] \). The random measures on \([0,T] \) induced by the predictable processes \( \langle M \rangle_{P^*}^\mathcal{F} \) and \( \langle N \rangle_{P^*}^\mathcal{F} \) are \( P^* \)-a.s. mutually singular if and only if \( M + N \) is a \( (P^*, \mathbb{G}) \)-basis of martingales.
Proof. ⇒ Under $P^*$ the triplet $(M, N, [M, N])$ is a $(P^*, \mathcal{G})$-basis (see Theorem 4.5 in [4]). If the random measures on $[0, T]$ induced by the predictable processes $(M)^{P,F}$ and $(N)^{P,H}$ are $P^*$-a.s. mutually singular then the same holds for the random measures induced by $(M^{dp})^{P,F}$ and $(N^{dp})^{P,H}$ and, since $P^* \sim P$, by Proposition 3.2 the covariation process $[M, N]$ is $P$-a.s. null so that the $(P^*, \mathcal{G})$-basis of martingales reduces to the pair $(M, N)$. As a consequence if $V \in L^2(\Omega, \mathcal{G}, P^*)$ then there exist two $\mathcal{G}$-predictable processes $(\gamma)_t \in [0, T]$ which satisfies $E^{P^*} \left[ \int_0^T \gamma_t^2 d[M]_t \right] < +\infty$ and $(\eta)_t \in [0, T]$ which satisfies $E^{P^*} \left[ \int_0^T \eta_t^2 d[N]_t \right] < +\infty$ such that

$$V = v_0 + \int_0^T \gamma_s dM_s + \int_0^T \eta_s dN_s. \quad (12)$$

Moreover by the construction of $P^*$ it follows that $(M)^{P,F} = (M)^{P^*,\mathcal{G}}$ and $(N)^{P,H} = (N)^{P^*,\mathcal{G}}$ so Proposition 3.3 applies to prove the existence of two $\mathcal{G}$-predictable sets $C^M$ and $C^N$ such that $P^*$-a.s.

$$\int_{C^M} d\langle M \rangle_t^{P^*,\mathcal{G}} = 0, \quad \int_{C^N} d\langle N \rangle_t^{P^*,\mathcal{G}} = 0$$

with $C^M \cap C^N = \emptyset$, $C^M \cup C^N = \Omega \times [0, T]$. Let $(\lambda)_t \in [0, T]$ be defined at time $t$ by

$$\lambda_t = \gamma_t \mathbb{1}_{C^M}(t) + \eta_t \mathbb{1}_{C^N}(t).$$

$\lambda$ is a $\mathcal{G}$-predictable process such that $E^{P^*} \left[ \int_0^T (\lambda_t - \gamma_t)^2 d[M + N]_t \right] < +\infty$. Moreover

$$E \left[ \int_0^T (\lambda_t - \gamma_t)^2 d\langle M \rangle_t^{P^*,\mathcal{G}} \right] = E \left[ \int_0^T (\lambda_t - \eta_t)^2 d\langle N \rangle_t^{P^*,\mathcal{G}} \right] = 0$$

so that

$$\int_0^T \lambda_s dM_s = \int_0^T \gamma_s dM_s, \quad \int_0^T \lambda_s dN_s = \int_0^T \eta_s dN_s.$$

Then (12) can be rewritten as

$$V = v_0 + \int_0^T \lambda_s d(M_s + N_s) \quad (13)$$

and by the arbitrariness of $V$ we derive that $M + N$ enjoys the $(P^*, \mathcal{G})$-p.r.p.

⇐ Let now $Z$ a $(P^*, \mathcal{G})$-basis. Then there exist two $\mathcal{G}$-predictable processes $(\alpha)_t \in [0, T]$ which satisfies $E^{P^*} \left[ \int_0^T \alpha_t^2 d[Z]_t \right] < +\infty$ and $(\beta)_t \in [0, T]$ which satisfies $E^{P^*} \left[ \int_0^T \beta_t^2 d[Z]_t \right] < +\infty$ such that

$$M_t = \int_0^t \alpha_s dZ_s, \quad N_t = \int_0^t \beta_s dZ_s. \quad (14)$$

Moreover assumptions imply that $M$ and $N$ are orthogonal $(P^*, \mathcal{G})$-martingales, so that 6

$$0 = \langle M, N \rangle^{P^*,\mathcal{G}} = \int_{0}^{\infty} \alpha_s \beta_s d\langle Z \rangle^{P^*,\mathcal{G}}$$

6 In fact, as shown by the equality $C^M = \{(t, \omega) : \mathbb{1}_{C^M}(t) = 1\}$, the indicator functions $\mathbb{1}_{C^M}(t)$ and $\mathbb{1}_{C^N}(t)$ of the predictable sets $C^M$ and $C^N$ are predictable processes.
that is \( \alpha, \beta = 0 \) \( d(Z)_t^{P, G} \) a.s.. Then the mutual singularity of the measures associated to \( (M)^{P, G} \) and \( (N)^{P, G} \) follows since

\[
\langle M \rangle^{P, F} = \langle M \rangle^{P, G} = \int_0^* \alpha_s^2 d(Z)^{P, G}_s,
\]
\[
\langle N \rangle^{P, H} = \langle N \rangle^{P, G} = \int_0^* \beta_s^2 d(Z)^{P, G}_s,
\]

then the direct part ends the proof. \( \square \)

**Theorem 3.6.** Assume A1) and D). Then the multiplicity of \( G \) under \( P^* \) is

i) equal to 1 if and only if the random measures on \([0, T]\) induced by the processes \( (M)^{P, F} \), \( (N)^{P, H} \) are \( P \)-a.s. mutually singular;

ii) equal to 2 if and only if the random measures on \([0, T]\) induced by the processes \( (M^{dp})^{P, F} \) and \( (N^{dp})^{P, H} \) are \( P \)-a.s. mutually singular but the random measures on \([0, T]\) induced by the processes \( (M)^{P, F} \), \( (N)^{P, H} \) are not \( P \)-a.s. mutually singular;

iii) equal to 3 if and only if the random measures generated by the processes \( (M^{dp})^{P, F} \) and \( (N^{dp})^{P, H} \) are not \( P \)-a.s. mutually singular.

**Proof.** Under the condition on the sharp brackets of point i), as stated by previous lemma, \( M + N \) has the \( (P^*, G) \)-p.r.p. Under the condition of point ii), as stated by Proposition 3.2 \([M, N] = 0\) so that, following Theorem 4.5 in [I], \( (M, N) \) has the \( (P^*, G) \)-p.r.p. and \( M \) and \( N \) are \( (P^*, G) \)-strongly orthogonal. Finally under the condition of point iii), again by Theorem 4.5 in [I], the triplet \( (M, N, [M, N]) \) has the \( (P^*, G) \)-p.r.p. and the three martingales \( M, N \) and \([M, N]\) are \( (P^*, G) \)-strongly orthogonal. \( \square \)

**Remark 3.7.** The notion of basis of martingales strictly depends on the probability measure and therefore the statement above could fail changing \( P^* \) with \( P \). Nevertheless this should not happen substituting the multiplicity of Davis and Varaiya of \( G \) with the analogous notion formulated in terms of basis of local martingales and assuming some regularity for

\[
\tilde{L}_t^* := \frac{dP}{dP^*} \bigg|_{\mathcal{G}_t}, \quad t \in [0, T].
\]

Let us now give an idea of how to check point iii) under \( P \). In a similar way we could obtain point i) and point ii) under \( P \).

Assume that \( \tilde{L}^* \) is in \( L^2_{loc}(P^*) \). Let use denote the triplet \( (M, N, [M, N]) \) with \( (M^1, M^2, M^3) \). By the invariance of the p.r.p. under equivalent martingale measure (see Lemma 2.4 in [21]) we get that the triplet of \( (P, G) \)-martingales \( (\tilde{M}^1, \tilde{M}^2, \tilde{M}^3) \) defined by

\[
\tilde{M}_t^i := M_t^i - \int_0^t \frac{1}{L^*_{s-}} d(\tilde{L}^*, M^i)^{P^*, G}_s, \quad i = 1, 2, 3
\]

is a set of local martingales with the \( (P, G) \)-p.r.p. but not necessarily with pairwise \( (P, G) \)-orthogonal components. By using the Kunita-Watanabe decomposition for local martingales, we can easily get a set of three orthogonal local martingales with the \( (P, G) \)-p.r.p.. A
(P, G)-basis of local martingales with less than three elements cannot exist, otherwise by the same but reverse procedure we could construct a (P*, G)-basis of local martingales with less than three elements.

Remark 3.8. Lemma 3.5 and Theorem 3.6 have been inspired by Theorem 9.5.2.4, Theorem 9.5.2.5 and subsequent arguments in [22] and in particular by the following example. A Brownian motion B and a compensated Poisson martingale Π are considered. Then the process dXt = f(t) dBt + g(t) dΠt, with f and g deterministic functions such that fg = 0 enjoys the p.r.p. with respect to its natural filtration.

4 P.r.p. of the default process of a general random time

On a filtered probability space (Ω, F, P) let τ be a non negative random variable and let H be the natural filtration of the default process \{I_{\{\tau \leq t\}}, t \in [0,T]\}. Obviously τ is an H-stopping time and the default process is a positive (P, H)-submartingale of class (D). Then Doob-Meyer decomposition Theorem applies to prove that there exists a unique increasing H-martingale of class (D). Then Doob-Meyer decomposition Theorem applies to prove that for any \(\mu\) there exists a unique increasing \(H\)-martingale of class (D).

Remark 3.8. According to the law \(\mu\) of τ, τ itself is either H-totally inaccessible if and only if \(\mu\) admits density, or H-accessible but not H-predictable if and only if \(\mu\) is atomic (for a detailed proof of this assertion we refer to [8], Theorem IV-107). It is to stress that τ can never be H-predictable.

In the light of previous remark, the martingale \(H\) cannot be the trivial one. Furthermore it always enjoys the \((P, H)\)-p.r.p., as proved in [8], Proposition 2. For the sake of clarity, we propose an alternative proof of this property in Proposition 4.3, based on the following simplified version of Theorem 3.4 in [17].

Theorem 4.2. (17)
Let \(P\) a probability measure on (Ω, F) such that \(P|_{\mathcal{H}_0} = P'|_{\mathcal{H}_0}\) and the \((P, H)\)-compensator and the \((P', \mathbb{H})\)-compensator of the default process coincide. Then \(P|_{\mathcal{H}_T} = P'|_{\mathcal{H}_T}\).

Proposition 4.3. \((H_t)_{t \in [0,T]}\) enjoys the \((P, H)\)-p.r.p..

Proof. We show that \(\mathbb{P}(H, \mathbb{H}) = \{P\}\), so that the Second Fundamental Theorem of Asset Pricing applies. By Theorem 4.2 it suffices to prove that for any \(Q \in \mathbb{P}(H, \mathbb{H})\), the \((Q, H)\)-compensator of the default process, \(A^{Q, H}\), coincides with \(A^{P, H}\) in (16). In fact the process \(H\) is a \((Q, H)\)-martingale as well as the process \(\hat{H} = (\hat{H}_t)_{t \in [0,T]}\) defined by

\[
\hat{H}_t = I_{\{\tau \leq t\}} - A^{Q, H}_t.
\]

Then also the process \(H - \hat{H}\) is a \((Q, H)\)-martingale. \(H - \hat{H} \equiv 0\) since it is a predictable martingale of finite variation (see Lemma 22.11 in [24]). As a consequence \(A^{P, H} = A^{Q, H}\).
Remark 4.4. Previous result is well-known when the stopping time $\tau$ admits a density (see Theorem 7.2.5.1 p.416 in [22]).

Remark 4.5. When $\tau$ is an $\mathbb{H}$-accessible stopping time, then there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{H}$-predictable stopping times such that $\bigcup_{n\in\mathbb{N}}\{\tau = \tau_n\} = \Omega$ and the default process admits the representation

$$ I_{\{\tau \leq t\}} = \sum_{n\in\mathbb{N}} I_{\{\tau = \tau_n\}} I_{\{\tau_n \leq t\}}. \quad (17) $$

By following a similar procedure as in the proof of Lemma 3.2, we derive the explicit expression of its $(P, \mathbb{H})$ compensator

$$ A^p_{H} = \sum_{n\in\mathbb{N}} E[I_{\{\tau = \tau_n\}} | \mathcal{H}_{\tau_n^-}] I_{\{\tau_n \leq t\}}. \quad (18) $$

Then (17) can be rewritten as

$$ H_t = \sum_{n\in\mathbb{N}} (I_{\{\tau = \tau_n\}} - E[I_{\{\tau = \tau_n\}} | \mathcal{H}_{\tau_n^-}]) I_{\{\tau_n \leq t\}}. \quad (19) $$

5 Kusuoka-like representation theorem for a general random time

This section is devoted to prove an extension of Kusuoka’s representation theorem (see Theorem 7.5.5.1 in [22]). In the language of mathematical finance our result can be roughly announced as follows: starting from a complete and arbitrage free market with risky asset price which is a semi-martingale (not necessarily continuous), enlarging progressively the filtration by the occurrence of a non trivial random time, under a suitable decoupling assumption, we derive a martingale representation under the historical measure.

On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ consider the semimartingale $X = (X_t)_{t\in[0,T]}$ (not necessarily of dimension one), in the space $S^2(P, \mathbb{F})$ with martingale part $M$ (see [9], VII-98). Let $\tau$ be a general random time, $H$ the associated default process and $\mathbb{H}$ its natural filtration. Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ by $\mathbb{H}$ (see (1)).

Consider the analogous of assumption A1) for $X$.

A1') $P(X, \mathbb{F}) = \{P^X\}$.

Proposition 5.1. Assume A1') and D). Let $P^*$ be the probability measure defined by [4]. Then $P^* \in P(M, \mathbb{G})$ and the triplet $(M, H, [M, H])$ is a $(P^*, \mathbb{G})$-basis of martingales.

Proof. $M$ enjoys the $(P|_{\mathcal{F}_T}, \mathbb{F})$-p.r.p. (see Proposition 2.1 in [5]). Moreover the $(P^*, \mathbb{G})$-martingales $M$ and $H$ are strongly orthogonal so that the result follows by Theorem 3.5 in [5], since $H$ enjoys the $(P|_{\mathcal{H}_T}, \mathbb{H})$-p.r.p. (see Proposition 4.3). \QED

Remark 5.2. Let $\tau$ be an $\mathbb{H}$-accessible stopping time, and denote by $(\tau_n)_{n\in\mathbb{N}}$ the sequence of $\mathbb{H}$-predictable stopping times that envelops $\tau$. Then the covariation process $[M, H]$ admits the representation

$$ [M, H]_t = \sum_{n\in\mathbb{N}} \Delta M_{\tau_n} (I_{\{\tau = \tau_n\}} - E[I_{\{\tau = \tau_n\}} | \mathcal{H}_{\tau_n^-}]) I_{\{\tau_n \leq t\}}, $$

(see Remark 4.5).
We denote by $A^{P,G}$ the $(P,G)$-compensator of the default process, so that the $\mathbb{F}$-\-conditional compensated default process $H' = (H'_t)_{t \in [0,T]}$ defined by

$$H'_t = \mathbb{1}_{\{\tau \leq t\}} - A^{P,G}_t$$

is a $(P,G)$-martingale. Note that $H'$ is a $(P^*,G)$-semi-martingale with martingale part equal to $H$. Now we state the announced extension of Kusuoka’s Theorem.

**Theorem 5.3.** Assume A1') and D. Let $P^*$ be the probability measure defined by (2) and let $P$ be the minimal martingale measure for $H'$ on $(\Omega, \mathcal{F}, G, P^*)$, i.e. each $(P^*,G)$-martingale orthogonal to $H'$ is a $(P,G)$-martingale. Then the triplet $(M, H', [M, H])$ enjoys the $(P,G)$-p.r.p.

**Proof.** Denote by $L$ the $(P^*,G)$-martingale $L$ defined by $L_t = \frac{dP}{dP^*} \bigg|_{G_t}$. Following Proposition 5.1 the triplet $(M, H, [M, H])$ is a $(P^*,G)$-basis of martingales, so that $L$ belongs to $S(M) \oplus S(H) \oplus S([M, H])$, the stable subspace generated by the strongly orthogonal $(P^*,G)$-martingales $M$, $H$ and $[M, H]$ into the space of square integrable $(P^*,G)$-martingales.

By discussion at page 15 in [28], since under $P^*$ $H'$ is the martingale part of $H'$, we get

$$L_t = 1 - \int_0^t \gamma_s L_s - d\langle L, M \rangle_{P^*,G}$$

where $\gamma$ is a suitable $G$-predictable process. Then $L$ belongs to $S(H)$ which in turn coincides with $(S(M) \oplus S([M, H]))^1$, so that $LM$ and $L[M, H]$ are $(P^*,G)$-martingales, and, as a consequence, $M$ and $[M, H]$ are also $(P,G)$-martingales.

Now, using the invariance of the martingale representation property under equivalent change of measure (see [21] Lemma 2.4), we derive the $(P,G)$-p.r.p. for the triplet $(\tilde{M}, \tilde{H}, \tilde{K})$ defined by

$$\tilde{M}_t = M_t - \int_0^t \frac{1}{L_{s^-}} d\langle L, M \rangle_{P^*,G}^s,$n$$

$$\tilde{H}_t = H_t - \int_0^t \frac{1}{L_{s^-}} d\langle L, H \rangle_{P^*,G}^s,$n$$

$$\tilde{K}_t = [M, H]_t - \int_0^t \frac{1}{L_{s^-}} d\langle L, [M, H] \rangle_{P^*,G}^s.$$  \hspace{1cm} (21)

$\tilde{M}$, $\tilde{H}$, $\tilde{K}$ are the martingale parts of the $(P,G)$-decomposition of the $(P,G)$-semi-martingales $M$, $H$, $[M, H]$ (see also [16] Theorem 12.13) so that, by previous considerations, we immediately get $\tilde{M} = M$ and $\tilde{K} = [M, H]$. As far as $\tilde{H}$ is concerned the same escamotage used in the proof of Proposition 4.3 allows to show that

$$\tilde{H} = \mathbb{1}_{\{\tau \leq t\}} - A^{P,G} = H'.$$  \hspace{1cm} (22)

□

**Remark 5.4.** Under the hypotheses of Theorem 5.3, it is to note that, unless $[M, H]$ is identically zero, the triplet $(M, H', [M, H])$ is not a $(P,G)$-basis of martingales. In fact
$M[M, H]$ is a $(P, \mathcal{G})$-martingale if and only if $LM[M, H]$ is a $(P^*, \mathcal{G})$-martingale, with $L$ as in (20). Moreover

$$L_t M_t = \int_0^t L_s^{-} dM_s + \int_0^t M_s^{-} \gamma_s^+ dH_s + \int_0^t \gamma_s^+ [H, M]_s$$

so that

$$L_t M_t[H, M]_t = \int_0^t L_s^{-} d[M, [M, H]]_s + \int_0^t M_s^{-} \gamma_s^+ d[H, [M, H]]_s + \int_0^t \gamma_s^+ [H, [M, H]]_s$$

which is a $(P^*, \mathcal{G})$-martingale if and only if $[M, H]$ is identically zero. Similarly it follows that $H'M[M, H]$ is not a $(P, \mathcal{G})$-martingale.

Nevertheless, by means of an orthogonalization procedure, it is easy to construct a $(P, \mathcal{G})$-basis of martingales.

In any case next result holds.

**Proposition 5.5.** Under the hypotheses of previous theorem, $M$ and $H'$ are strongly orthogonal $(P, \mathcal{G})$-martingales.

**Proof.** Let $H'$ be the process defined by (22). Next we verify that the $(P^*, \mathcal{G})$-martingale $LH'$ belongs to $(\mathcal{S}(M) \oplus \mathcal{S}([M, H]))^\perp$ and as a consequence $MLH'$ turns out to be a $(P^*, \mathcal{G})$-martingale, and this will end the proof.

By the product formula we get

$$L_t H'_t = \int_0^t L_s^{-} dH'_s + \int_0^t H'_s^{-} dL_s + [L, H']_t$$

and, using representation (21) for $H'$

$$L_t H'_t = \int_0^t L_s^{-} dH'_s - \langle L, H \rangle_t^{P^*, \mathcal{G}} + \int_0^t H'_s^{-} dL_s + [L, H]_t - \left[ L, \int_0^t \frac{1}{L_u^{-}} d\langle L, H \rangle_u^{P^*, \mathcal{G}} \right]_t.$$

(23)

Now, observe that, by Proposition 5.1 and representation (20), $\int_0^t L_s^{-} dH'_s$ and $\int_0^t H'_s^{-} dL_s$ belong to $(\mathcal{S}(M) \oplus \mathcal{S}([M, H]))^\perp$. Moreover, denoting by $B$ the predictable process $\int_0^t \frac{1}{L_u^{-}} d\langle L, H \rangle_u^{P^*, \mathcal{G}}$ and by using Example 9.4 n.1 at page 229 in [16], we get

$$[L, B]_t = \int_0^t \Delta B_u dL_u$$

which still belongs to $(\mathcal{S}(M) \oplus \mathcal{S}([M, H]))^\perp$. Finally, by (21) it follows

$$[L, H]_t - \langle L, H \rangle_t^{P^*, \mathcal{G}} = \int_0^t \gamma_s^+ L_s^{-} d([H]_s - \langle H \rangle_s^{P^*, \mathcal{G}}).$$

Now $P^*$ decouples $\mathbb{F}$ and $\mathbb{H}$, so that $[H] - \langle H \rangle_{P^*, \mathcal{G}}$ is a $(P, \mathbb{H})$-martingale since $\langle H \rangle_{P^*, \mathcal{G}} = \langle H \rangle_{P^*, \mathbb{H}} = \langle H \rangle_{P, \mathbb{H}}.$
Moreover by Proposition 4.3, \( H \) enjoys the \((P, \mathbb{H})\)-p.r.p. so that there exists an \( \mathbb{H} \)-predictable process \( \eta \) which satisfies
\[
E^P \left[ \int_0^T \eta_t^2 \, d[H]_t \right] < +\infty \text{ such that } [H] - \langle H \rangle^P = \int_0^\cdot \eta_s \, dH_s.
\]
Then
\[
[L, H] - \langle L, H \rangle^P = \int_0^\cdot \gamma_s L_s - \eta_s \, dH_s
\]
which is in turn a \((P^*, G)\)-martingale strongly orthogonal to \( M \) and \([M, H]\), so that the thesis is achieved.

Remark 5.6. When \( \tau \) is \( \mathbb{H} \)-totally inaccessible, the proof of the previous proposition can be done directly by noting that \( \tau \) is also \( \mathbb{G} \)-totally inaccessible. 
In fact by assumption \( \mathcal{D} \), it follows that \( H \) is both a \((Q, \mathbb{H})\)-martingale and a \((Q, \mathbb{G})\)-martingale. In particular this implies that the \((Q, \mathbb{H})\)-compensator of the default process is also its \((Q, \mathbb{G})\)-compensator. Moreover it is a continuous process since \( \tau \) is \( \mathbb{H} \)-totally inaccessible, so that \( \tau \) is also \( \mathbb{G} \)-totally inaccessible under any probability measure equivalent to \( Q \) and in particular under \( P \).
In such a situation also the process \( \{A_{t}^{P, G}, \, t \in [0, T]\} \) turns out to be continuous so that
\[
[M, H'] = [M, I_{\{\tau \leq \cdot\}}] \equiv 0,
\]
that is \( M \) and \( H' \) are strongly orthogonal \((P, \mathbb{G})\)-martingales. Then, as proved in [5], Proposition 4.5, the pair \((M, H')\) is a \((P, \mathbb{G})\)-basis of martingales.

Remark 5.7. Following Proposition 6.3 in [20], if the process
\[
\int_0^t \frac{dP[\tau \leq u | \mathcal{F}_u]}{1 - P[\tau < u | \mathcal{F}_u]}
\]
is predictable, then the process \( A_{t}^{P, G} \) can be characterized a.s. as
\[
A_{t}^{P, G} = \int_0^t \frac{dP[\tau \leq u | \mathcal{F}_u]}{1 - P[\tau < u | \mathcal{F}_u]}.
\]

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