The rate of convergence of new Lax–Oleinik type operators for time-periodic positive definite Lagrangian systems

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Abstract

Assume that the Aubry set of the time-periodic positive definite Lagrangian $L$ consists of one hyperbolic 1-periodic orbit. We provide an upper bound estimate of the rate of convergence of the family of new Lax–Oleinik type operators associated with $L$ introduced by the authors in Wang and Yan (2012 Commun. Math. Phys. 309 663–91). In addition, we construct an example where the Aubry set of a time-independent positive definite Lagrangian system consists of one hyperbolic periodic orbit and the rate of convergence of the Lax–Oleinik semigroup cannot be better than $O(1/t)$ as $t \to +\infty$.

Mathematics Subject Classification: 37J50

1. Introduction

In an earlier paper [15] the present authors introduced a new kind of Lax–Oleinik type operator with parameters (hereinafter referred to as new L–O operator) associated with time-periodic positive definite Lagrangian systems in the context of the weak KAM theory, and proved that the family of new L–O operators with an arbitrary continuous function as initial condition converges to a backward weak KAM solution of the corresponding Hamilton–Jacobi equation. In this paper, we study the rate of convergence of the family of new L–O operators under the assumption that the Aubry set of the time-periodic positive definite Lagrangian system consists of one hyperbolic 1-periodic orbit.

Let $M$ be a closed and connected smooth manifold of dimension $m$. Denote its tangent bundle by $TM$ and the cotangent one by $T^*M$. We choose, once and for all, a $C^\infty$ Riemannian
metric on $M$. It is classical that there is a canonical way to associate with it a Riemannian metric on $TM$. Consider a $C^\infty$ Lagrangian $L : TM \times R^1 \to R^1$, $(x, v, t) \mapsto L(x, v, t)$. We suppose that $L$ satisfies the following conditions introduced by Mather [12]:

(H1) Periodicity. $L$ is $1$-periodic in the $R^1$ factor, i.e. $L(x, v, t) = L(x, v, t + 1)$ for all $(x, v, t) \in TM \times R^1$.

(H2) Positive definiteness. For each $x \in M$ and each $t \in R^1$, the restriction of $L$ to $T_x M \times t$ is strictly convex in the sense that its Hessian second derivative is everywhere positive definite.

(H3) Superlinear growth. $\lim_{\|v\|_x \to +\infty} \frac{L(x, v, t)}{\|v\|_x} = +\infty$ uniformly on $x \in M$, $t \in R^1$, where $\| \cdot \|_x$ denotes the norm on $T_x M$ induced by the Riemannian metric on $M$.

(H4) Completeness of the Euler–Lagrange flow. The maximal solutions of the Euler–Lagrange equation, which in local coordinates is

$$\frac{d}{dt} \frac{\partial L}{\partial v}(x, \dot{x}, t) = \frac{\partial L}{\partial x}(x, \dot{x}, t),$$

are defined on all of $R^1$.

The Euler–Lagrange equation is a second order periodic differential equation on $M$ and generates a flow of diffeomorphisms $\phi_t^L : TM \times S^1 \to TM \times S^1$, $t \in R^1$, where $S^1$ denotes the circle $R^1 / Z$, defined by

$$\phi_t^L(x_0, v_0, t_0) = (x(t + t_0), \dot{x}(t + t_0), (t + t_0) \mod 1),$$

where $x : R^1 \to M$ is the maximal solution of the Euler–Lagrange equation with initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$. The completeness and periodicity conditions grant that this correctly defines a flow on $TM \times S^1$. We can associate with $L$ a Hamiltonian, as a function on $T^* M \times R^1$: $H(x, p, t) = \sup_{v \in T_x M} \{ \langle p, v \rangle_x - L(x, v, t) \}$, where $\langle \cdot, \cdot \rangle_x$ represents the canonical pairing between the tangent and cotangent space. The corresponding Hamilton–Jacobi equation is

$$u_t + H(x, u_x, t) = c(L),$$

where $c(L)$ is the Mañe critical value [11] of the Lagrangian $L$. In terms of Mather’s $\alpha$ function $c(L) = \alpha(0)$. Without loss of generality, we will from now on always assume $c(L) = 0$.

Let us first recall the definition of the L–O semigroup (hereinafter referred to as L–O semigroup) associated with $L$. The L–O semigroup is well known in several domains, such as PDE, optimization and control theory, calculus of variations and dynamical systems (especially in the weak KAM theory [9]). For each $t \geq 0$ and each $u \in C(M, R^1)$, let

$$T_t u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^1 L(\gamma(s), \gamma'(s), s) \, ds \right\}$$

for all $x \in M$, where the infimum is taken among the continuous and piecewise $C^1$ paths $\gamma : [0, t] \to M$ with $\gamma(t) = x$. For each $t \geq 0$, $T_t$ is an operator from $C(M, R^1)$ to itself. Since $L$ is time-periodic, the one-parameter semigroup associated with $L$, where $N = \{0, 1, 2, \ldots\}$.

Fathi proved [7] the convergence of the full L–O semigroup (i.e. $\{T^n_t\}_{n \geq 0}$) in the time-independent case. More precisely, he showed that for each $C^2$ superlinear and strictly convex 3 The L–O semigroup associated with a time-independent Lagrangian $L_\alpha$ is the semigroup of operators $\{T^n_t\}_{n \geq 0} : C(M, R^1) \to C(M, R^1)$ defined by

$$T^n_t u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^1 L_\alpha(\gamma(s), \gamma'(s)) \, ds \right\},$$

where the infimum is taken among the continuous and piecewise $C^1$ paths $\gamma : [0, t] \to M$ with $\gamma(t) = x$. 

\[3\] The L–O semigroup associated with a time-independent Lagrangian $L_\alpha$ is the semigroup of operators $\{T^n_t\}_{n \geq 0} : C(M, R^1) \to C(M, R^1)$ defined by

$$T^n_t u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^1 L_\alpha(\gamma(s), \gamma'(s)) \, ds \right\},$$

where the infimum is taken among the continuous and piecewise $C^1$ paths $\gamma : [0, t] \to M$ with $\gamma(t) = x$. 

\[4\] Completeness of the Euler–Lagrange flow. The maximal solutions of the Euler–Lagrange equation, which in local coordinates is

$$\frac{d}{dt} \frac{\partial L}{\partial v}(x, \dot{x}, t) = \frac{\partial L}{\partial x}(x, \dot{x}, t),$$

are defined on all of $R^1$.

\[5\] The L–O semigroup associated with a time-independent Lagrangian $L_\alpha$ is the semigroup of operators $\{T^n_t\}_{n \geq 0} : C(M, R^1) \to C(M, R^1)$ defined by

$$T^n_t u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^1 L_\alpha(\gamma(s), \gamma'(s)) \, ds \right\},$$

where the infimum is taken among the continuous and piecewise $C^1$ paths $\gamma : [0, t] \to M$ with $\gamma(t) = x$. 

\[6\] The L–O semigroup associated with a time-independent Lagrangian $L_\alpha$ is the semigroup of operators $\{T^n_t\}_{n \geq 0} : C(M, R^1) \to C(M, R^1)$ defined by

$$T^n_t u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^1 L_\alpha(\gamma(s), \gamma'(s)) \, ds \right\},$$

where the infimum is taken among the continuous and piecewise $C^1$ paths $\gamma : [0, t] \to M$ with $\gamma(t) = x$. 

\[7\] Fathi proved [7] the convergence of the full L–O semigroup (i.e. $\{T^n_t\}_{n \geq 0}$) in the time-independent case. More precisely, he showed that for each $C^2$ superlinear and strictly convex
Lagrangian $L_u : TM \to \mathbb{R}^1$ and each $u \in C(M, \mathbb{R}^1)$, the uniform limit, for $t \to +\infty$, of $T^n u + c(L_u)$ exists and the limit $\bar{u}$ is a backward weak KAM solution of the corresponding Hamilton–Jacobi equation. In the same paper Fathi raised the question as to whether the analogous result holds in the time-periodic case. This would be the convergence of $T_n u + n c(L)$, $\forall u \in C(M, \mathbb{R}^1)$, as $n \to +\infty$, $n \in \mathbb{N}$. In view of the relation between $T_n$ and the Peierls barrier $h$ (see [13] or [3, 5, 8]), if the liminf in the definition of the Peierls barrier is not a limit, then the L–O semigroup in the time-periodic case does not converge. Fathi and Mather [8] constructed examples where the liminf in the definition of the Peierls barrier is not a limit, thus answering the above question negatively.

As mentioned at the beginning of this paper, the authors [15] introduced a new kind of operator with parameters associated with $L$, called the new L–O operator, and proved the convergence of the family of new L–O operators. Let us now recall the definition of the new L–O operator and some important results in [15].

**Definition 1.1 (new L–O operator).** For each $\tau \in [0, 1]$, each $n \in \mathbb{N}$ and each $u \in C(M, \mathbb{R}^1)$, let

$$\tilde{T}_n^\tau u(x) = \inf_{\gamma \in \bar{C}(M, \mathbb{R}^1)} \inf_{y \in M} \left\{ u(y(0)) + \int_0^{r+k} L(y(s), \dot{y}(s), s) \, ds \right\}$$

for all $x \in M$, where the second infimum is taken among the continuous and piecewise $C^1$ paths $\gamma : [0, \tau + k] \to M$ with $\gamma(\tau + k) = x$.

For each $\tau \in [0, 1]$ and each $n \in \mathbb{N}$, $\tilde{T}_n^\tau$ is an operator from $C(M, \mathbb{R}^1)$ to itself. For more properties of the new L–O operator $\tilde{T}_n^\tau$, we refer the reader to [15]. For each $n \in \mathbb{N}$ and each $u \in C(M, \mathbb{R}^1)$, let $U_n^\tau(x, \tau) = \tilde{T}_n^\tau u(x)$ for all $(x, \tau) \in M \times [0, 1]$. Then $U_n^\tau$ is a continuous function on $M \times [0, 1]$.

The main result of [15] is the following theorem.

**Theorem 1.** For each $u \in C(M, \mathbb{R}^1)$, the uniform limit $\lim_{n \to +\infty} U_n^\tau$ exists and

$$\lim_{n \to +\infty} U_n^\tau(x, \tau) = \inf_{y \in M} (u(y) + h_0(\tau)(y, x))$$

for all $(x, \tau) \in M \times [0, 1]$, where $(\tau) = \tau \mod 1$ and $h$ denotes the (extended) Peierls barrier. Furthermore, let $\bar{u}(x, (\tau)) = \inf_{y \in M} (u(y) + h_0(\tau)(y, x))$. Then $\bar{u} : M \times S^1 \to \mathbb{R}^1$ is a backward weak KAM solution of the Hamilton–Jacobi equation

$$w_\tau + H(x, w_\tau, x) = 0. \quad (1.1)$$

Another important result of [15] states as follows.

**Theorem 2.** Let $\bar{u} \in C(M \times S^1, \mathbb{R}^1)$. Then the following three statements are equivalent.

- There exists $u \in C(M, \mathbb{R}^1)$ such that the uniform limit $\lim_{n \to +\infty} U_n^\tau = \bar{u}$.
- $\bar{u}$ is a backward weak KAM solution of (1.1).
- $\bar{u}$ is a viscosity solution of (1.1).

The aim of this paper is to derive the rate of convergence of $U_n^\tau$ in a special case. More precisely, we will provide an upper bound estimate of the rate of convergence of $U_n^\tau$ associated with a $C^\infty$ Lagrangian $L$, which satisfies (H1)–(H4) and the following hypothesis.

(H5) The Aubry set consists of one hyperbolic 1-periodic orbit.

Now we come to the major result of this paper.
Theorem 3. If a $C^\infty$ Lagrangian $L: TM \times R^1 \to R^1$ satisfies the hypotheses (H1)–(H5), then there exists $\rho > 0$ such that for each $u \in C(M, R^1)$ there is $K > 0$ such that

$$\|U^n_u(x, \tau) - \bar{u}(x, (\tau))\|_\infty \leq Ke^{-\rho n}, \quad \forall n \in N,$$

where $\| \cdot \|_\infty$ denotes the supremum norm in the space $C(M \times [0, 1], R^1)$.

We believe that there is a deep relation between dynamical properties of the Aubry set (Mather set) and the rates of convergence of the L–O semigroup (time-independent case) and the family of new L–O operators. We would now like to detail available related works in the literature. All these results are for time-independent Lagrangian systems.

I. Results on the rate of convergence of the L–O semigroup $\{T^a_t\}_{t \geq 0}$:

In [10], Iurriaga and Sánchez-Morgado proved that if the Aubry set consists of a finite number of hyperbolic fixed points, the L–O semigroup converges exponentially. At the end of this paper, we will construct an example (example 4.1) to show that the rate of convergence of the L–O semigroup is provided under the assumption that the Aubry set consists of a finite number of hyperbolic periodic orbits.

The authors [14] dealt with the rate of convergence problem when the Mather set consists of degenerate fixed points. More precisely, consider the Lagrangian $L_0^0(x, v) = \frac{1}{2} v^2 + V(x)$, $x \in S^1$, $v \in R^1$, where $V$ is a real analytic function on $S^1$ and has a unique global minimum point $x_0$. Without loss of generality, one may assume $x_0 = 0$, $V(0) = 0$. Then $c(L_0^0) = 0$ and $\mathcal{M}_0 = \{0, 0\}$, where $\mathcal{M}_0$ is the Mather set with cohomology class 0. An upper bound estimate of the rate of convergence of the L–O semigroup is provided under the assumption that $\{0, 0\}$ is a degenerate fixed point: for every $u \in C(S^1, R^1)$, there exists a constant $K_1 > 0$ such that

$$\|T^a_t u - \bar{u}\|_\infty \leq \frac{K_1}{\sqrt{t}}, \quad \forall t > 0,$$

where $k \in N$, $k \geq 2$ depends only on the degree of degeneracy of the minimum point of the potential function $V$.

In [15] the authors discussed the rate of convergence problem when the Aubry set is a quasi-periodic invariant torus of the Euler–Lagrange flow. Consider a class of $C^2$ superlinear and strictly convex Lagrangians on $T^n$

$$L^a_0(x, v) = \frac{1}{2} \langle A(x)(v - \omega), (v - \omega) \rangle + f(x, v - \omega), \quad x \in T^n, \quad v \in R^n,$$

(1.2)

where $A(x)$ is an $n \times n$ matrix, $\omega \in S^{n-1}$ is a given vector, and $f(x, v - \omega) = O(||v - \omega||^3)$ as $v - \omega \to 0$. It is clear that $c(L^a_0) = 0$ and $\mathcal{M}_0 = \mathcal{A}_0 = \mathcal{N}_0 = \cup_{x \in T^n} (x, \omega)$, which is a quasi-periodic invariant torus with frequency vector $\omega$ of the Euler–Lagrange flow associated with $L^a_0$, where $\mathcal{A}_0$ and $\mathcal{N}_0$ are the Aubry set and the Mañé set with cohomology class 0, respectively. For (1.2), the authors showed that for each $u \in C(T^n, R^1)$, there is a constant $K_2 > 0$ such that

$$\|T^a_t u - \bar{u}\|_\infty \leq \frac{K_2}{t}, \quad \forall t > 0.$$

(1.3)

An example was also provided in [15] to show that the above result is sharp in the sense that there exist $u \in C(T^n, R^1)$, $x^0 \in T^n$ and $t_j \to +\infty$ as $j \to +\infty$ such that

$$|T^a_{t_j}u(x^0) - \bar{u}(x^0)| = O\left(\frac{1}{t_j}\right), \quad j \to +\infty.$$
II. Results on the rate of convergence of the new L–O semigroup \( \{ \tilde{T}_t^n \}_{t \geq 0}^4 \):

The authors showed in [15] that for each \( C^2 \) superlinear and strictly convex Lagrangian \( L_u \) with \( c(L_u) = 0 \) and each \( u \in C(M, R^1) \), the uniform limit \( \lim_{t \to +\infty} \tilde{T}_t^n u \) exists and \( \lim_{t \to +\infty} \tilde{T}_t^n u = \lim_{t \to +\infty} T_t^n u = \tilde{u} \). Furthermore, \( \| \tilde{T}_t^n u - \tilde{u} \|_\infty \leq \| T_t^n u - \tilde{u} \|_\infty, \forall t \geq 0 \). It means that the new L–O semigroup converges faster than the L–O semigroup. For a specific case (1.2), the authors [15] made a more precise comparison between the rates of convergence of the L–O semigroup and the new L–O semigroup as follows.

Recall the notation for Diophantine vectors: for \( \varrho > n - 1 \) and \( \alpha > 0 \), let

\[
D(\varrho, \alpha) = \left\{ \beta \in S^{n-1} \mid \| \langle \beta, k \rangle \| \geq \frac{\alpha}{\| k \|^2}, \forall k \in Z^n \setminus \{0\} \right\},
\]

where \( |k| = \sum_{i=1}^n |k_i| \). For (1.2), the authors proved that given any frequency vector \( \omega \in D(\varrho, \alpha) \), for each \( u \in C(T^n, R^1) \), there is a constant \( K_3 > 0 \) such that

\[
\| \tilde{T}_t^n u - \tilde{u} \|_\infty \leq K_3 t^{-\frac{1+4}{2}\frac{1}{n}}, \quad \forall t \geq 0.
\]

Therefore, for the case with \( \omega \in D(\varrho, \alpha) \), from (1.4) \( O(t^{-\frac{1+4}{2}\frac{1}{n}}) \) as \( t \to +\infty \) is an upper bound estimate of the rate of convergence of the new L–O semigroup, while the rate of convergence of the L–O semigroup cannot be better than \( O(\frac{1}{t}) \) as \( t \to +\infty \) since (1.3) is sharp.

The rest of the paper is organized as follows. Section 2 includes some basic definitions and preliminary results. In section 3 we give the proof of theorem 3. Section 4 presents an example (example 4.1) where the Aubry set of a time-independent positive definite Lagrangian system consists of one hyperbolic periodic orbit and the rate of convergence of the L–O semigroup cannot be better than \( O(\frac{1}{t}) \) as \( t \to +\infty \).

2. Preliminaries

In this section we introduce the notation used in the following and review some definitions and results of Mather and weak KAM theories that we are going to use. In addition, we also prove two preliminary lemmas.

In this paper, as is usual, \( S^1 = R^1 / Z \), whose universal cover is the Euclidean space \( R^1 \).

We view \( S^1 \) as a fundamental domain in \( R^1 : I = [0, 1] \) with the two endpoints identified. The unique coordinate \( s \) of a point in \( S^1 \) will belong to \( I = [0, 1] \). The standard universal covering projection \( \pi : R^1 \to S^1 \) takes the form \( \pi(\tilde{s}) = \tilde{s} \), where \( \tilde{s} = s \mod 1 \) denotes the fractional part of \( s \). \( \tilde{s} = [s] + \tilde{\bar{s}} \), where \( [s] \) is the greatest integer not greater than \( s \). The fractional part of a real number \( s \) is sometimes denoted by \( \{s\} \), but this notation is not used in this paper due to possible confusion with the set containing the element \( \tilde{s} \). \( \| \cdot \| \) denotes the usual Euclidean norm.

As done by Mather in [13], it is convenient to introduce, for all \( t' \geq t \) and \( x, y \in M \), the following quantity:

\[
F_{t',t}(x, y) = \inf_{\gamma} \int_{t}^{t'} L(\gamma(s), \dot{\gamma}(s), s) \, ds,
\]

\[4\] In [15], the authors also introduced a new kind of Lax–Oleinik type operator \( \tilde{T}_t^n \) associated with time-independent Lagrangians. The new L–O semigroup associated with a time-independent Lagrangian \( L_u \) is the semigroup of operators \( \{ \tilde{T}_t^n \}_{t \geq 0} : C(M, R^1) \to C(M, R^1) \) defined by

\[
\tilde{T}_t^n u(x) = \inf_{v \in C^1([0,1], R^1)} \inf_{\gamma \in C^1([0,1], M)} \left\{ u(\gamma(0)) + \int_0^1 L_u(\gamma(s), \dot{\gamma}(s)) \, ds \right\},
\]

where the second infimum is taken among the continuous and piecewise \( C^1 \) paths \( \gamma : [0,\sigma] \to M \) with \( \gamma(\sigma) = x \).
Lemma 2.1. For each \( \gamma : [t, t'] \to M \) such that \( \gamma'(t) = x \) and \( \gamma'(t') = y \),

Following Mañé [11] and Mather [13], define the action potential and the extended Peierls barrier as follows.

**Action potential:** for each \( (s, s') \in S^1 \times S^1 \), let

\[
\Phi_{s,s'}(x, x') = \inf_{t' - t = \tau} F_{t,t'}(x, x')
\]

for all \( (x, x') \in M \times M \), where the infimum is taken on the set of \( (t, t') \in \mathbb{R}^2 \) such that \( s = (t) \), \( s' = (t') \) and \( t' \geq t + 1 \).

**Extended Peierls barrier:** for each \( (s, s') \in S^1 \times S^1 \), let

\[
h_{s,s'}(x, x') = \liminf_{\tau \to +\infty} F_{t,t'}(x, x')
\]

for all \( (x, x') \in M \times M \), where the liminf is restricted to the set of \( (t, t') \in \mathbb{R}^2 \) such that \( s = (t) \), \( s' = (t') \). It can be shown that the extended Peierls barrier \( h \) is Lipschitz (see [5]).

A continuous and piecewise \( C^1 \) curve \( \gamma : \mathbb{R}^1 \to M \) is called global semi-static if

\[
\int_t^{t'} L(\gamma(s), \gamma'(s)) ds = \lim_{\tau \to +\infty} \int_t^{t'} L(\gamma'(s), \gamma''(s)) ds
g for all \( [t, t'] \subseteq \mathbb{R}^1 \). An orbit (\( \gamma(s), \gamma'(s), (s) \)) is called global semi-static if \( \gamma \) is a global semi-static curve. The Mañé set \( \mathcal{N} \) is the union of \( TM \times S^1 \) of the images of global semi-static orbits.

For each \( n \in \mathbb{N} \) and each \( (t, t', x, x') \in [0, 1] \times [0, 1] \times M \times M \), let

\[ F_n(t, t', x, x') = \inf_{i \in \mathbb{N}} F_{i, t,t'}(x, x') \]

Then from proposition 3.5 in [15],

\[
\lim_{n \to +\infty} F_n(t, t', x, x') = h_{[t], [t']}(x, x')
\]

uniformly on \( (t, t', x, x') \in [0, 1] \times [0, 1] \times M \times M \).

Now we prove a preliminary result:

**Lemma 2.1.** For each \( n \in \mathbb{N} \) and each \( \tau \in [0, 1] \),

1. \( h_{0,0}(x, z) = F_0,0(x, y) + h_{0,0}(y, z) \), \( \forall x, y, z \in M \);
2. \( h_{0,\tau}(x, z) \leq h_{0,0}(x, y) + F_{0,\tau}(y, z) \), \( \forall x, y, z \in M \).

**Proof.**

1. Since \( h_{0,0}(y, z) = \lim \inf_{i \to +\infty} F_{0,\tau}(y, z) \), then there exist \( \{k_i\}_{i=1}^{\infty} \subseteq \mathbb{N} \) such that \( k_i \to +\infty \) and \( F_{0,k_i}(y, z) \to h_{0,0}(y, z) \) as \( i \to +\infty \). For each \( n \in \mathbb{N} \) and each \( k_i \), in view of the definition of \( F_{i,t'} \), we have

\[
F_{0,n,k_i}(x, z) \leq F_{0,n}(x, y) + F_{n,n,k_i}(y, z).
\]

Hence,

\[
h_{0,0}(x, z) \leq \lim \inf_{i \to +\infty} F_{0,n+k_i}(x, z) \leq F_{0,n}(x, y) + h_{0,0}(y, z).
\]

2. Since \( h_{0,0}(x, y) = \lim \inf_{i \to +\infty} F_{0,k_i}(x, y) \), then there exist \( \{k_i\}_{i=1}^{\infty} \subseteq \mathbb{N} \) such that \( k_i \to +\infty \) and \( F_{0,k_i}(x, y) \to h_{0,0}(x, y) \) as \( i \to +\infty \). For each \( n \in \mathbb{N} \), each \( \tau \in [0, 1] \) and each \( k_i \), we have

\[
F_{0,\tau,n+k_i}(x, z) \leq F_{0,k_i}(x, y) + F_{k_i,\tau,n+k_i}(y, z).
\]

It follows that

\[ h_{0,\tau}(x, z) \leq \lim \inf_{i \to +\infty} F_{0,\tau,n+k_i}(x, z) \leq h_{0,0}(x, y) + F_{0,\tau}(y, z). \]
Following Fathi [6], as done by Contreras et al in [5], we give the definition of the weak KAM solution as follows.

**Definition 2.1.** A backward weak KAM solution of the Hamilton–Jacobi equation (1.1) is a function \( w : M \times S^1 \to \mathbb{R}^1 \) such that

1. \( w \) is dominated by \( L \), i.e.
   \[
   w(x_1, s_1) - w(x_2, s_2) \leq \Phi_{x_2,s_2}(x_2, x_1), \quad \forall (x_1, s_1), (x_2, s_2) \in M \times S^1.
   \]
2. For every \((x, s) \in M \times S^1\) there exists a curve \( \gamma : (-\infty, \tilde{s}] \to M \) with \( \gamma(\tilde{s}) = x \) and \( \tilde{s} = s \) such that
   \[
   w(x, s) - w(\gamma(t), \gamma'(t)) = \int_t^\tilde{s} L(\gamma(s), \gamma'(s)) \, ds, \quad \forall t \in (-\infty, \tilde{s}].
   \]

Similarly, we say that \( w : M \times S^1 \to \mathbb{R}^1 \) is a forward weak KAM solution of (1.1) if \( w \) is dominated by \( L \) and for every \((x, s) \in M \times S^1\) there exists a curve \( \gamma : [\tilde{s}, +\infty) \to M \) with \( \gamma(\tilde{s}) = x \) and \( \tilde{s} = s \) such that
\[
\int_{\tilde{s}}^s L(\gamma(s), \gamma'(s)) \, ds, \quad \forall \tilde{s} \in [\tilde{s}, +\infty).
\]

We denote by \( \mathcal{S}_- \) (\( \mathcal{S}_+ \)) the set of backward (forward) weak KAM solutions. The following well-known result [5] will be used later.

**Lemma 2.2.** Given \((x_0, s_0) \in M \times S^1\), define
\[
w^*(x, s) := h_{s_0, x}(x_0, x), \quad w_*(x, s) := -h_{s_0, x}(x_0, x)
\]
for \((x, s) \in M \times S^1\). Then \( w^* \in \mathcal{S}_- \), \( w_* \in \mathcal{S}_+ \).

Define the projected Aubry set \( \mathcal{A}_0 \) as follows:
\[
\mathcal{A}_0 := \{(x, s) \in M \times S^1 \mid h_{s, x}(x, x) = 0\}.
\]

Note that \( \mathcal{A}_0 = \Pi \tilde{\mathcal{A}}_0 \), where \( \Pi : TM \times S^1 \to M \times S^1 \) denotes the projection and \( \tilde{\mathcal{A}}_0 \) denotes the Aubry set in \( TM \times S^1 \). Define an equivalence relation on \( \mathcal{A}_0 \) by saying that \((x_1, s_1)\) and \((x_2, s_2)\) are equivalent if and only if
\[
\Phi_{x_1, s_1}(x_1, x_2) + \Phi_{x_2, s_1}(x_2, x_1) = 0.
\]

The equivalent classes of this relation are called static classes. Let \( \mathcal{A} \) be the set of static classes. For each static class \( \Gamma \in \mathcal{A} \) choose a point \((x, 0) \in \Gamma\) and let \( \mathcal{A}_0 \) be the set of such points. Since the Lagrangian \( L \) in theorem 3 satisfies (H5), then \( \mathcal{A}_0 \) consists of only one point, denoted by \((p, 0) \in \mathcal{A}_0 \).

From a result of Contreras et al [5], for each backward weak KAM solution \( w \) of (1.1), we have
\[
w(x, s) = \min_{(q, 0) \in \mathcal{A}_0} (w(q, 0) + h_{s, q}(q, x)) = w(p, 0) + h_{0, x}(p, x)
\]
for all \((x, s) \in M \times S^1\).

Given \( u \in C(M, \mathbb{R}^1) \), for each \( n \in \mathbb{N} \), each \( \tau \in [0, 1] \) and each \( x \in M \),
\[
\tilde{T}^*_\tau u(x) = \inf_{\gamma \in \mathcal{A}_{[\tau, \tau+k],[0, n]}} \inf_{s \in [\tau, \tau+k]} \left\{ u(\gamma(0)) + \int_0^{\tau+k} L(\gamma(s), \gamma'(s), s) \, ds \right\}
\]
where the second infimum is taken among the continuous and piecewise \( C^1 \) paths \( \gamma : [0, \tau + k] \to M \) with \( \gamma(\tau + k) = x \). In view of (2.3), it is easy to see that there exist
We conclude that \( y_{x,t,n} : [0, t + k_{x,t,n}] \to M \) such that \( y_{x,t,n}(x + k_{x,t,n}) = x \) and
\[
\tilde{T}_n^s u(x) = u(y_{x,t,n}(0)) + \int_0^{t+k_{x,t,n}} L(y_{x,t,n}(s), \dot{y}_{x,t,n}(s), s) \, ds.
\]
Let
\[
A(y_{x,t,n}) := \int_0^{t+k_{x,t,n}} L(y_{x,t,n}(s), \dot{y}_{x,t,n}(s), s) \, ds.
\]
Obviously, we have
\[
A(y_{x,t,n}) = F_{0, t+k_{x,t,n}}(y_{x,t,n}(0), x) = F_n(0, x, y_{x,t,n}(0), x).
\]
Then there exists \( T > 0 \) such that if \( n \geq T, n \in \mathbb{N} \), then
\[
(y_{x,t,n}(s), \dot{y}_{x,t,n}(s), (s)) \|_{\frac{T}{4}, \frac{T}{3}} \subset W, \quad \forall (x, \tau) \in M \times [0, 1].
\]
Under the assumptions of theorem 3, let \( A_0 \) in \( TM \times S^1 \). Then there exists \( T > 0 \) such that if \( n \geq T, n \in \mathbb{N} \), then
\[
\text{The following result for minimizers } y_{x,t,n} \text{ will be used in the proof of theorem 3.}
\]
**Lemma 2.3.** Under the assumptions of theorem 3, let \( W \) be a neighbourhood of the Aubry set \( A_0 \) in \( TM \times S^1 \). Then there exists \( T > 0 \) such that if \( n \geq T, n \in \mathbb{N} \), then
\[
(y_{x,t,n}(s), \dot{y}_{x,t,n}(s), (s)) \|_{\frac{T}{4}, \frac{T}{3}} \subset W, \quad \forall (x, \tau) \in M \times [0, 1].
\]
**Proof.** To prove the lemma, we argue by contradiction. For, otherwise, there would be \( \{n_i\}_{i=1}^\infty \subset \mathbb{N} \) with \( n_i \to +\infty \) as \( i \to +\infty \), \( \{(x_{n_i}, \tau_{n_i})\}_{i=1}^\infty \subset M \times [0, 1] \), \( \{k_{n_i}\}_{i=1}^\infty \subset \mathbb{N} \) with \( n_i \leq k_{n_i} \leq 2n_i, \) a sequence \( y_{n_i} : [0, \tau_{n_i} + k_{n_i}] \to M, i = 1, 2, \ldots \) of minimizers satisfying
\[
y_{n_i}(\tau_{n_i} + k_{n_i}) = x_{n_i} \quad \text{and} \quad \tilde{T}_{n_i}^s u(x_{n_i}) = u(y_{n_i}(0)) + \int_0^{k_{n_i}} L(y_{n_i}(s), \dot{y}_{n_i}(s), s) \, ds \quad \text{and} \quad \{k_{n_i}\}_{i=1}^\infty \text{ with } \frac{n_i}{3} \leq t_{n_i} \leq \frac{2n_i}{3}
\]
with
\[
(y_{n_i}(t_{n_i}), \dot{y}_{n_i}(t_{n_i}), (t_{n_i})) \notin W, \quad i = 1, 2, \ldots ,
\]
where we used \( k_{n_i} \) and \( y_{n_i} \) to denote \( k_{x_n, \tau_n, n} \) and \( y_{x_n, \tau_n, n} \), respectively. For each positive integer \( i \), we set \( y_{n_i} = y_{n_i}(0) \). Passing as necessary to a subsequence, we may suppose that \( x_{n_i} \to x_0, y_{n_i} \to y_0 \) and \( \tau_{n_i} \to \tau_0 \) as \( i \to +\infty \), where \( x_0, y_0 \in M \) and \( \tau_0 \in [0, 1] \).

Since
\[
\lim_{i \to +\infty} F_{n_i}(0, \tau_{n_i}, y_{n_i}, x_{n_i}) = h_{0, (\tau_0)}(y_0, x_0)
\]
and
\[
\lim_{i \to +\infty} A(y_{n_i}) = h_{0, (\tau_0)}(y_0, x_0),
\]
In view of (2.4) and (2.6), we have
\[
\lim_{i \to +\infty} A(y_{n_i}) = h_{0, (\tau_0)}(y_0, x_0).
\]
For each \( i \), we set \( (\tilde{x}_{n_i}, \dot{\tilde{x}}_{n_i}, \sigma_{n_i}) = (y_{n_i}(t_{n_i}), \dot{y}_{n_i}(t_{n_i}), (t_{n_i})) \). By (2.5), \( (\tilde{x}_{n_i}, \dot{\tilde{x}}_{n_i}, \sigma_{n_i}) \notin W, \forall i \). Since \( y_{n_i} \) are minimizing extremal curves, using the *a priori* compactness lemma 3.4 in [15], we conclude that \( \{(\tilde{x}_{n_i}, \dot{\tilde{x}}_{n_i}, \sigma_{n_i})\}_{i=1}^\infty \) are contained in a compact subset of \( TM \times S^1 \). So we may assume upon passing if necessary to a subsequence that \( (\tilde{x}_{n_i}, \dot{\tilde{x}}_{n_i}, \sigma_{n_i}) \to (\tilde{x}, \dot{\tilde{x}}, \sigma) \in TM \times S^1 \) as \( i \to +\infty \). Since \( (\tilde{x}_{n_i}, \dot{\tilde{x}}_{n_i}, \sigma_{n_i}) \notin W, \forall i \), then \( (\tilde{x}, \dot{\tilde{x}}, \sigma) \notin A_0 \).

Let \( (\gamma(s), \dot{\gamma}(s), (s)) = \phi_{\tilde{x}, \tilde{\gamma}, \sigma}^{\tilde{x}, \tilde{\gamma}, \sigma}(\tilde{x}, \tilde{\gamma}, \sigma), s \in R^1 \). We assert that the orbit \( (\gamma(s), \dot{\gamma}(s), (s)) \) is global semi-static, i.e. \( \gamma \) is a global semi-static curve. If this assertion is true, then \( (\tilde{x}, \dot{\tilde{x}}, \sigma) \in \mathcal{A}_0 \). By our assumption that \( \mathcal{A}_0 \) consists of one hyperbolic 1-periodic orbit, it is easy to see that \( \mathcal{M}_0 = \mathcal{A}_0 = \mathcal{N}_0 \). Thus, we deduce that \((\tilde{x}, \dot{\tilde{x}}, \sigma) \notin \mathcal{A}_0 \), which is impossible since \( (\tilde{x}, \dot{\tilde{x}}, \sigma) \notin \mathcal{A}_0 \). This contradiction proves the lemma.
Based on the above arguments, it is sufficient to show that $\gamma$ is a global semi-static curve. We prove it by contradiction. Otherwise, there would be $j_1, j_2 \in N$ such that

$$A(\gamma'|_{\sigma-j_1, \sigma+j_2}) > \Phi_{\sigma, \sigma}(\gamma'(\sigma - j_1), \gamma'(\sigma + j_2)).$$

It implies that there exist $j_1', j_2' \in N$ with $\sigma - j_1' + 1 \leq \sigma + j_2'$ and a minimizing curve $\tilde{\gamma} : [\sigma - j_1', \sigma + j_2'] \to M$ satisfying $\tilde{\gamma}(\sigma - j_1') = \gamma(\sigma - j_1)$ and $\tilde{\gamma}(\sigma + j_2') = \gamma(\sigma + j_2)$ such that

$$A(\gamma'|_{\sigma-j_1, \sigma+j_2}) > A(\tilde{\gamma}|_{\sigma-j_1', \sigma+j_2'}).$$

Thus, there exists $\Delta > 0$ such that

$$A(\tilde{\gamma}|_{\sigma-j_1', \sigma+j_2'}) \leq A(\gamma'|_{\sigma-j_1, \sigma+j_2}) - \Delta. \tag{2.8}$$

Since $(\tilde{x}_{n_i}, \tilde{x}_{n_i}, \sigma_{n_i}) \to (\tilde{x}, \tilde{x}, \sigma) \in TM \times S^1$ as $i \to +\infty$, then, for every $\varepsilon > 0$, by the differentiability of the solutions of the Euler–Lagrange equation with respect to initial values, we have

$$d((\gamma(s), \tilde{\gamma}(s), \langle \rangle), (\gamma_{n_i}(s + t_{n_i} - \sigma), \gamma_{n_i}(s + t_{n_i} - \sigma), (s + t_{n_i} - \sigma)) < \varepsilon, \tag{2.9}$$

for all $s \in [\sigma - j_1, \sigma + j_2]$ and $i$ large enough. Using the periodicity of $L$, we have

$$A(\gamma_{n_i}|_{t_{n_i}-j_1, t_{n_i}+j_2}) = \int_{\sigma-j_1}^{\sigma+j_2} L(\gamma_{n_i}(s + t_{n_i} - \sigma), \gamma_{n_i}(s + t_{n_i} - \sigma), (s + t_{n_i} - \sigma))ds, \tag{2.10}$$

and

$$A(\gamma'|_{\sigma-j_1, \sigma+j_2}) = \int_{\sigma-j_1}^{\sigma+j_2} L(\gamma(s), \gamma(s), \langle \rangle)ds. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we have

$$|A(\gamma_{n_i}|_{t_{n_i}-j_1, t_{n_i}+j_2}) - A(\gamma'|_{\sigma-j_1, \sigma+j_2})| \leq C\varepsilon \tag{2.12}$$

for some constant $C > 0$ independent of $\varepsilon$ and sufficiently large $i$. Since $\varepsilon$ may be taken arbitrary small, from (2.8) and (2.12) we obtain

$$A(\gamma_{n_i}|_{t_{n_i}-j_1, t_{n_i}+j_2}) \geq A(\gamma'|_{\sigma-j_1, \sigma+j_2}) - C\varepsilon \geq A(\tilde{\gamma}|_{\sigma-j_1', \sigma+j_2'}) + \frac{3\Delta}{4}, \tag{2.13}$$

provided $i$ is large enough.

We set $\tilde{x} = \tilde{\gamma}(\sigma - j_1') = \gamma(\sigma - j_1)$ and $\tilde{x} = \tilde{\gamma}(\sigma + j_2') = \gamma(\sigma + j_2)$. For $i$ large enough, consider the following curves on $M$. Let $\alpha_1^i : [0, t_{n_i} - j_1] \to M$ with $\alpha_1^i(0) = y_{n_i}$ and $\alpha_1^i(t_{n_i} - j_1) = \tilde{x}$ be a Tonelli minimizer such that

$$A(\alpha_1^i) = F_{0, t_{n_i}-j_1}(y_{n_i}, \tilde{x}).$$

Let $\alpha_1^2 : [t_{n_i} - j_1 + j_1' + j_2', \tau_{n_i} + k_{n_i} - j_1 - j_2 + j_1' + j_2'] \to M$ with $\alpha_1^2(t_{n_i} - j_1 + j_1' + j_2') = \tilde{x}$ and $\alpha_1^2(\tau_{n_i} + k_{n_i} - j_1 - j_2 + j_1' + j_2') = x_{n_i}$ be a Tonelli minimizer such that

$$A(\alpha_1^2) = F_{t_{n_i}-j_1+j_1'+j_2', \tau_{n_i}+k_{n_i}-j_1-j_2+j_1'+j_2'}(\tilde{x}, x_{n_i}).$$

Let

$$\tilde{\gamma}_{n_i}(s) = \begin{cases} \alpha_1^1(s), & s \in [0, t_{n_i} - j_1], \\ \tilde{\gamma}(s - t_{n_i} + j_1 + \sigma - j_1'), & s \in [t_{n_i} - j_1, t_{n_i} - j_1 + j_1' + j_2'], \\ \alpha_1^2(s), & s \in [t_{n_i} - j_1 + j_1' + j_2', \tau_{n_i} + k_{n_i} - j_1 - j_2 + j_1' + j_2']. \end{cases}$$

It is clearly that $\tilde{\gamma}_{n_i} : [0, \tau_{n_i} + k_{n_i} - j_1 - j_2 + j_1' + j_2'] \to M$ is a continuous and piecewise $C^1$ curve connecting $y_{n_i}$ and $x_{n_i}$. 

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We set \( \tilde{x}_n = \gamma_n(t_n - j_1) \) and \( \underline{x}_n = \gamma_n(t_n + j_2) \). For \( i \) large enough, compare \( A(\tilde{y}_n) \) with \( A(\gamma_n) \) as follows. In view of (2.9), we have

\[
|A(\tilde{y}_n \mid [t_0 - j_1, t_0 - j_1 + j'_1 + j'_2]) - A(\gamma_n \mid [t_0 - j_1, t_0 + j_2])|
= \int_{\sigma - j'_1}^{\sigma + j'_2} L(\tilde{F}(s), \tilde{\gamma}(s), s + \sigma_n - \sigma) \, ds - A(\gamma_n \mid [t_0 - j_1, t_0 + j_2]).
\]

Since \( \sigma_n \to \sigma \) as \( i \to +\infty \), then

\[
|A(\tilde{F} \mid [\sigma - j'_1, \sigma + j'_2]) - \int_{\sigma - j'_1}^{\sigma + j'_2} L(\tilde{F}(s), \tilde{\gamma}(s), s + \sigma_n - \sigma) \, ds| \leq \frac{\Delta}{4}
\]

for \( i \) large enough. Hence, from (2.13), (2.15) and (2.16) we have

\[
A(\tilde{y}_n \mid [t_0 - j_1, t_0 - j_1 + j'_1 + j'_2]) - A(\gamma_n \mid [t_0 - j_1, t_0 + j_2]) \leq - \frac{\Delta}{2}.
\]

From the Lipschitz property of \( F_{t, t'} \) and (2.9), we find

\[
|A(\tilde{y}_n \mid [t_0 - j_1, t_0 - j_1 + j'_1 + j'_2]) - A(\gamma_n \mid [t_0 + j_2, \tau_n])| \leq \frac{D_{\text{Lip}}}{2}.
\]

Since \( \epsilon \) may be taken arbitrary small, from (2.14), (2.17) and (2.18), we have

\[
A(\tilde{y}_n) \leq A(\gamma_n) - \frac{\Delta}{4}
\]

for \( i \) large enough.

For each sufficiently large \( i \), we choose \( m_i \in N \) such that

\[
m_i \leq k_n - j_1 - j_2 + j'_1 + j'_2 \leq 2m_i.
\]

Since \( n_i \leq k_n \leq 2m_i \), \( n_i \to +\infty \) as \( i \to +\infty \), then \( m_i \to +\infty \) as \( i \to +\infty \). By (2.20), for each \( i \) large enough, we have

\[
A(\tilde{y}_n) \geq F_{0, \tau_n} \gamma_n(\tau_n, x_n) \geq \mathcal{F}_{m_i}(0, \tau_n, y_n, x_n).
\]

Since

\[
|\mathcal{F}_{m_i}(0, \tau_n, y_n, x_n) - h_{0, (\tau_n)}(y_0, x_0)| \leq |\mathcal{F}_{m_i}(0, \tau_n, y_n, x_n) - h_{0, (\tau_n)}(y_n, x_n)| + |h_{0, (\tau_n)}(y_n, x_n) - h_{0, (\tau_n)}(y_0, x_0)|,
\]

then from (2.1) and the Lipschitz property of \( h \), we have

\[
\lim_{i \to +\infty} \mathcal{F}_{m_i}(0, \tau_n, y_n, x_n) = h_{0, (\tau_n)}(y_0, x_0).
\]

Combining (2.7), (2.19), (2.21) and (2.22), we have

\[
h_{0, (\tau_n)}(y_0, x_0) - \frac{\Delta}{4} = \lim_{i \to +\infty} A(\gamma_n) - \frac{\Delta}{4}
\]

then

\[
\lim_{i \to +\infty} A(\tilde{y}_n) \geq \lim_{i \to +\infty} \mathcal{F}_{m_i}(0, \tau_n, y_n, x_n)
= h_{0, (\tau_n)}(y_0, x_0),
\]
a contradiction. This contradiction shows that \( y \) is global semi-static. \( \square \)

Remark 2.1. The above result is independent of \( a \in C(M, R^3) \). Moreover, from the proof, it is easy to see that the conclusion of lemma 2.3 holds with \([\frac{1}{4}, \frac{3}{2}]\) replaced by \([an, bn]\) for arbitrary \( 0 < a < b < 1 \).
3. Proof of the main result

In this section we prove theorem 3. Let \((p, v_p, 0)\) be the unique point in \(\mathcal{A}_0\) with \(\Pi(p, v_p, 0) = (p, 0) \in \mathcal{A}_0\), where \(\Pi : TM \times S^1 \to M \times S^1\) denotes the projection. By (H5) the Aubry set \(\mathcal{A}_0\) consists of one hyperbolic 1-periodic orbit, denoted by \(\Gamma : \phi_t^0(p, v_p, 0) = (\gamma_p(s), \dot{\gamma}_p(s), (s))\), \(s \in R^1\).

**Proof of theorem 3.** Our purpose is to show that there exists \(\rho > 0\) such that for each \(u \in C(M, R^1)\) there is \(K > 0\) such that the following two inequalities hold.

\[
\tilde{u}(x, (\tau)) - U_n^u(x, \tau) \leq K e^{-\rho n}, \quad \forall n \in \mathbb{N}, \forall (x, \tau) \in M \times [0, 1]; \tag{11}
\]

\[
U_n^{\tilde{u}}(x, \tau) - \tilde{u}(x, (\tau)) \leq K e^{-\rho n}, \quad \forall n \in \mathbb{N}, \forall (x, \tau) \in M \times [0, 1]. \tag{12}
\]

Since the proof is rather long, it is convenient to divide it into two steps.

**Step 1.** We first prove the inequality (11). In view of lemma 2.2 and (2.2), for any given \(y \in M\), \(h_{0, y}(\cdot, \cdot)\) is a backward weak KAM solution of (1.1) and

\[
h_{0, y}(y, x) = h_{0, 0}(y, p) + h_{0, y}(p, x) \tag{3.1}
\]

for all \((x, \tau) \in M \times [0, 1]\). Given \(u \in C(M, R^1)\) and \((x, \tau) \in M \times [0, 1]\), from theorem 1 and (3.1) we have

\[
\tilde{u}(x, (\tau)) = \inf_{y \in M} (u(y) + h_{0, y}(y, x)) = \inf_{y \in M} (u(y) + h_{0, 0}(y, p) + h_{0, y}(p, x)). \tag{3.2}
\]

By the arguments in section 2, for each \(n \in \mathbb{N}\) there exist \(n \leq k_{x, \tau, n} \leq 2n, k_{x, \tau, n} \in \mathbb{N}\) and a minimizing extremal curve \(\gamma_{x, \tau, n} : [0, \tau + k_{x, \tau, n}] \to M\) such that \(\gamma_{x, \tau, n}(\tau + k_{x, \tau, n}) = x\) and

\[
\check{L}_n u(x) = u(\gamma_{x, \tau, n}(0)) + \int_{0}^{\tau + k_{x, \tau, n}} L(\gamma_{x, \tau, n}(s), \dot{\gamma}_{x, \tau, n}(s), s) \, ds. \tag{3.3}
\]

In what follows we use \(k_n\) and \(\gamma_n\) to denote \(k_{x, \tau, n}\) and \(\gamma_{x, \tau, n}\), respectively.

From (3.2) and lemma 2.1, we have

\[
\tilde{u}(x, (\tau)) \leq u(\gamma_n(0)) + h_{0, 0}(\gamma_n(0), p) + h_{0, y}(p, x)
\]

\[
\leq u(\gamma_n(0)) + F_{0, n}(\gamma_n(0), \gamma_n(s)) + h_{0, 0}(\gamma_n(s), p)
\]

\[
+ h_{0, 0}(p, \gamma_n(s)) + F_{0, n, \tau}(\gamma_n(s), x) \tag{3.4}
\]

for all \(s \in [0, \tau + k_n]\) and all \(n_1, n_2 \in \mathbb{N}\). For \(n \in \mathbb{N}\) large enough, let \(j_n = \lfloor \frac{\tau}{2} \rfloor - \lfloor \frac{\tau}{3} \rfloor - 1\). Taking \(n_1 = \lfloor \frac{\tau}{2} \rfloor + 1\) and \(n_2 = k_n - n_1\), by (3.4) we obtain

\[
\tilde{u}(x, (\tau)) \leq u(\gamma_n(0)) + \int_{0}^{\tau + k_n} L(\gamma_n, \dot{\gamma}_n, s) \, ds + 2C_{Lip} d \left( \gamma_n \left( \left\lfloor \frac{j_n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + 1, p \right) \right). \tag{3.5}
\]

where \(C_{Lip} > 0\) is a Lipschitz constant of \(\dot{h}\). From (3.3) and (3.5) we have

\[
\tilde{u}(x, (\tau)) - U_n^{\tilde{u}}(x, \tau) \leq 2C_{Lip} d \left( \gamma_n \left( \left\lfloor \frac{j_n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + 1, p \right) \right). \tag{3.6}
\]

We now estimate the term in the right-hand side of (3.6). Consider the Poincaré map for the time-periodic Lagrangian system \(L:\)

\[
\psi_{1,0} : TM \to TM, \quad (x_0, v_0) \mapsto \psi_{1,0}(x_0, v_0),
\]

where \(\psi_{1,0}(x_0, v_0) = (x(t), \dot{x}(t))\) and \(x(t)\) denotes the solution to the Euler–Lagrange equation with initial conditions \(x(0) = x_0, \dot{x}(0) = v_0\). Obviously, \(\phi^1_{c}(x_0, v_0, 0) = (\psi_1, 0(x_0, v_0), (t))\).
It is easy to see that $(p, v_p)$ is a hyperbolic fixed point of $\varphi_{1,0}$. According to the Hartman–Grobman theorem the Poincaré map $\varphi_{1,0}$ is locally conjugate to its linear part at the hyperbolic fixed point $(p, v_p)$. More precisely, there exist a neighbourhood $V(p, v_p)$ of $(p, v_p)$ in $TM$ and a homeomorphism $f : V(p, v_p) \rightarrow U(0)$, such that
\[
D\varphi_{1,0}(p, v_p) \circ f = f \circ \varphi_{1,0}.
\] (3.7)
Furthermore, there exists $0 < \alpha < 1$ such that $f$ and $f^{-1}$ are $\alpha$-Hölder continuous [1, 2].

Denote for brevity $P = (p, v_p)$. As the problem here is a local one we can, using a local chart, suppose that $\varphi_{1,0}$ is a map from $R^{2m}$ to itself with $P$ as a hyperbolic fixed point.

Let $B(P)$ be a sufficiently small neighbourhood of $P$ in $R^{2m}$ such that $B(P) \subset V(p, v_p)$. We choose a tubular neighbourhood $W_\Gamma$ of $\Gamma$ such that for each $(q, v, (s)) \in \Gamma$, $d((q, v, (s)), \partial W_\Gamma) = \delta$, where $\partial W_\Gamma$ denotes the boundary of $W_\Gamma$ and $\delta$ is a positive constant small enough such that for each $(q, v, 0) \in W_\Gamma$, we have $(q, v) \in B(P)$. For the tubular neighbourhood $W_\Gamma$, by lemma 2.3, there exists $T > 0$ such that for $n \in N$ with $n \geq T$, we have
\[
\left(\gamma_n(s), \gamma_n(s), (s)\right)_{|\mathbb{R}^2} \subset W_\Gamma.
\]
It follows that
\[
\left(\gamma_n\left(\frac{n}{3} + 1\right), \gamma_n\left(\frac{n}{3} + 1\right), 0, \ldots, \gamma_n\left(\frac{2n}{3}\right), \gamma_n\left(\frac{2n}{3}\right), 0\right) \in W_\Gamma.
\]
Thus, we have
\[
\left(\gamma_n\left(\frac{n}{3} + 1\right), \gamma_n\left(\frac{n}{3} + 1\right), \ldots, \gamma_n\left(\frac{2n}{3}\right), \gamma_n\left(\frac{2n}{3}\right)\right) \in B(P),
\]
i.e.
\[
\varphi_{1,0}^{[t]}(P_i^0), \ldots, \varphi_{1,0}^{[t]}(P_n^0) \in B(P).
\] (3.8)
where $P_i^0 = (\gamma_i(0), \gamma_i(0))$. By (3.7) and (3.8) we have
\[
Af(P_2^n) = f \circ \varphi_{1,0}^{[t]}(P_i^0), \ldots, A^h f(P_2^n) = f \circ \varphi_{1,0}^{[t]}(P_n^0),
\]
where $A = D\varphi_{1,0}(P)$ and $P_i^0 = \varphi_{1,0}^{[t]}(P_i^n)$. In view of (3.8), we obtain
\[
A_i f(P_i^n) \in U(0), \quad i = 0, 1, \ldots, j_n.
\]
Hence, there exists $\bar{\lambda} > 0$ such that
\[
\|A_i f(P_i^n)\| \leq \bar{\lambda}, \quad i = 0, 1, \ldots, j_n.
\] (3.9)

As $A : R^{2m} \rightarrow R^{2m}$ is hyperbolic, there exists an invariant splitting $R^{2m} = E^s \oplus E^u$. For each $z \in R^{2m}$, we have $z = z_s + z_u$, $z_s \in E^s$, $z_u \in E^u$ and $Az = A_s z_s + A_u z_u$, where $A_s = A|E^s$ and $A_u = A|E^u$. Let $f(P_i^n) = \gamma_i^n + z_i^n$, $\gamma_i^n \in E^s$, $z_i^n \in E^u$. Then $|\lambda_i| < 1$ for $i = 1, \ldots, m$. Since $A$ is similar to a symplectic matrix, then $\frac{1}{A_s}, \ldots, \frac{1}{A_u}$ are the eigenvalues of $A_u$. Set $\lambda_{\max} = \max_{1 \leq i \leq m} |\lambda_i|$. It is a standard result that for arbitrary $\varepsilon > 0$, we have
\[
\|A_i z_i^n\| \leq (\lambda_{\max} + \varepsilon)^i \|z_i^n\|, \quad \forall z_i^n \in E^s, \quad i = 0, 1, \ldots, j_n.
\] (3.10)
for $i \in N$ large enough. We choose $\varepsilon_0 > 0$ small enough such that $\lambda_{\max} + \varepsilon_0 < 1$. Then from (3.10) we have
\[
\|A_i^{[t]} z_i^n\| \leq (\lambda_{\max} + \varepsilon_0)^{|t|} \|z_i^n\| \leq (\lambda_{\max} + \varepsilon_0)^{|t|} \bar{\lambda},
\] (3.11)
for $n$ large enough. Similarly, we have
\[ \|A_{\alpha}^{\frac{1}{2}}y\|_n = \|A_{\alpha}^{\frac{1}{2}}(\frac{\pm}{\pm})\|_n \leq (\lambda_{\text{max}} + \epsilon_0)\|A_{\alpha}^{\frac{1}{2}}\|_n \leq (\lambda_{\text{max}} + \epsilon_0)\frac{1}{2}\Delta \]
(3.12)
for $n$ large enough. By (3.11) and (3.12), we obtain
\[ \|A_{\alpha}^{\frac{1}{2}}f(P_2^n)\| \leq \|A_{\alpha}^{\frac{1}{2}}y\|_n + \|A_{\alpha}^{\frac{1}{2}}z\|_n \leq 2\Delta(\lambda_{\text{max}} + \epsilon_0)\frac{1}{2}\]
(3.13)
for $n$ large enough. Since $j_n = \frac{\pm}{\pm} - \frac{1}{2}$, then from (3.13) we have
\[ \|A_{\alpha}^{\frac{1}{2}}f(P_2^n)\| \leq 2\Delta(\lambda_{\text{max}} + \epsilon_0)\frac{1}{2} \]
(3.14)
for $n$ large enough. Note that $A_{\alpha}^{\frac{1}{2}}f(P_2^n) = f \circ \varphi_{\frac{\pm}{\pm}}(P_2^n)$ and $f(P) = 0$. Since $f^{-1}$ is $\alpha$-Hölder continuous, from (3.14) we have
\[ \|\varphi_{\frac{\pm}{\pm}}(P_2^n) - P\| = \|f^{-1} \circ A_{\alpha}^{\frac{1}{2}}f(P_2^n) - f^{-1}(0)\|
\leq C_1\|A_{\alpha}^{\frac{1}{2}}f(P_2^n) - 0\|^\alpha
\leq C_1\|\Delta(\lambda_{\text{max}} + \epsilon_0)\|^\alpha \]
(3.15)
for $n$ large enough, where $C_1 > 0$ is a constant. Therefore, there exists a constant $C_2 > 0$ independent of $u \in C(M, R^d)$ and $(x, \tau) \in M \times [0, 1]$ such that
\[ d(y_n(\frac{j_n}{2}) + \frac{n}{3} + 1, P) \leq C_2(\lambda_{\text{max}} + \epsilon_0)\frac{1}{2} \]
(3.16)
for $n$ large enough. Note that the above estimate is independent of $(x, \tau)$. By (3.6) and (3.16), for sufficiently large $n$, we have
\[ \bar{u}(x, (\tau)) - U_n^\alpha(x, \tau) \leq 2C_{\text{Lip}}C_2(\lambda_{\text{max}} + \epsilon_0)\frac{1}{2}, \quad \forall (x, \tau) \in M \times [0, 1]. \]

Hence, there exists a constant $C_3 > 0$ such that
\[ \bar{u}(x, (\tau)) - U_n^\alpha(x, \tau) \leq C_3(\lambda_{\text{max}} + \epsilon_0)\frac{1}{2}, \quad \forall n \in N, \forall (x, \tau) \in M \times [0, 1], \]
where the constant $C_3$ depends on $u$. Since $0 < \lambda_{\text{max}} + \epsilon_0 < 1$, there exists $\rho_1 > 0$ such that $(\lambda_{\text{max}} + \epsilon_0)\frac{1}{2} = e^{-\rho_1}$. Thus, we have
\[ \bar{u}(x, (\tau)) - U_n^\alpha(x, \tau) \leq C_3e^{-\rho_1}, \quad \forall n \in N, \forall (x, \tau) \in M \times [0, 1]. \]
(3.17)

**Step 2.** We now prove the inequality (12). Given $u \in C(M, R^d)$ and $(x, \tau) \in M \times [0, 1]$, by (3.2) we have
\[ \bar{u}(x, (\tau)) = \inf_{z \in M} (u(z) + h_{0,0}(z, p) + h_{0,(\tau)}(p, x)). \]
(3.18)

Thus, there exists $y \in M$ such that
\[ \bar{u}(x, (\tau)) = u(y) + h_{0,0}(y, p) + h_{0,(\tau)}(p, x). \]

To prove (12), it suffices to show that for $n \in N$ large enough, we can find a curve $\eta : [0, \tau + \tilde{k}_n] \rightarrow M$ with $\eta(0) = y$ and $\eta(\tau + \tilde{k}_n) = x$, where $n \leq \tilde{k}_n \leq 2n, \tilde{k}_n \in N$, such that
\[ u(\eta(0)) + \int_0^{\tau + \tilde{k}_n} L(\eta, \dot{\eta}, s)ds - \bar{u}(x, (\tau)) \leq \tilde{C}e^{-\theta n} \]
(3.19)
for some constants $\tilde{C}, \theta > 0$ independent of $u \in C(M, R^d), (x, \tau) \in M \times [0, 1]$ and $n \in N$. In fact, for $n \in N$ large enough, if such a curve exists, then by the definition of $U_n^\alpha(x, \tau)$, we have
\[ U_n^\alpha(x, \tau) - \bar{u}(x, (\tau)) \leq u(\eta(0)) + \int_0^{\tau + \tilde{k}_n} L(\eta, \dot{\eta}, s)ds - \bar{u}(x, (\tau)) \leq \tilde{C}e^{-\theta n}, \]
which immediately implies the desired inequality (12).
Our task now is to construct the curve mentioned above. Since $h_0, (\cdot, \cdot)$ is a backward weak KAM solution of (1.1), then there is a curve $\beta_{s, (\tau)} : (-\infty, \tau] \to M$ with $\beta_{s, (\tau)}(\tau) = x$ and $(\tilde{x}) = (\tau)$ such that
\[
h_{0, (\tau)}(p, x) - h_{0, (\tau)}(p, \beta_{s, (\tau)}(\tau)) = \int_{\tau}^{\tilde{x}} L(\beta_{s, (\tau)}, \dot{\beta}_{s, (\tau)}, s) ds, \quad \forall \tau \in (-\infty, \tilde{x}).
\]
(3.20)
It is clear that $\beta_{s, (\tau)}$ is a minimizing curve. From [3, lemma 3.9], the $\alpha$-limit set for any minimizing orbit is contained in the Aubry set $A_0$. Since $A_0$ consists of one hyperbolic 1-periodic orbit $\Gamma$, then the $\alpha$-limit set for $(\beta_{s, (\tau)}(s), \dot{\beta}_{s, (\tau)}(s), (s))$ is exactly $\Gamma$. Similarly, since $-h_0, (\cdot, \cdot)$ is a forward weak KAM solution of (1.1), then there exists a curve $\omega_y : [\tilde{\alpha}, +\infty) \to M$ with $\omega_y(\tilde{\alpha}) = y$ and $(\tilde{\alpha}) = 0$ such that
\[
h_{0, 0}(y, p) - h_{0, 0}(\omega_y, 0(t), p) = \int_{\tilde{\alpha}}^{t} L(\omega_y, \dot{\omega}_y, s) ds, \quad \forall t \in [\tilde{\alpha}, +\infty).
\]
(3.21)
Moreover, $\omega_y, 0$ is a minimizing curve and the $\omega$-limit set for $(\omega_y, 0(s), \dot{\omega}_y, 0(s), (s))$ is also the hyperbolic 1-periodic orbit $\Gamma$ [3, 3.9 lemma].

Since $\Gamma$ is a hyperbolic 1-periodic orbit, then for the tubular neighbourhood $W_\Gamma$ there exist constants $T_1 > 0$ and $C_4 > 0$, such that
\[
d((\omega_y, 0(s + \tilde{\alpha}), \dot{\omega}_y, 0(s + \tilde{\alpha}), (s + \tilde{\alpha})), (\gamma_p(s), \dot{\gamma}_p(s), (s))) \leq C_4 e^{-\mu s}
\]
(3.22)
for all $s > T_1$, and
\[
d((\beta_{s, (\tau)}(s + \tau), \dot{\beta}_{s, (\tau)}(s + \tau), (s + \tau)), (\gamma_p(s + (\tau)), \dot{\gamma}_p(s + (\tau)), (s + (\tau)))) \leq C_4 e^{\mu s}
\]
(3.23)
for all $s < -T_1$, where $T_1$ and $C_4$ depend only on $W_\Gamma$, and $\mu$ denotes the smallest positive Lyapunov exponent of $\Gamma$ (see, for example, [4]).

We are now in a position to construct the curve $\eta$. For $n \in \mathbb{N}$ large enough such that $\frac{2n}{3} > \max\{T_1, 2\}$, choose $0 \leq d_1 < 1$ so that $(\gamma_p(\frac{2n}{3} + d_1), \dot{\gamma}_p(\frac{2n}{3} + d_1), (\frac{2n}{3} + d_1)) = (p, v_p, 0)$. Then from (3.22) we obtain
\[
d\left((\omega_y, 0\left(\frac{2n}{3} + \tilde{\alpha} + d_1\right), \dot{\omega}_y, 0\left(\frac{2n}{3} + \tilde{\alpha} + d_1\right), \left(\frac{2n}{3} + \tilde{\alpha} + d_1\right)), (p, v_p, 0)\right) \leq C_4 e^{-\mu \frac{2n}{3}}.
\]
(3.24)
From $(\tilde{\alpha}) = 0$ and the property of $F_{t, t'}$, we have
\[
F_{0, \frac{2n}{3} + d_1}(\gamma, \omega_y, 0\left(\frac{2n}{3} + \tilde{\alpha} + d_1\right)) = F_{\gamma, \omega_y, 0\left(\frac{2n}{3} + \tilde{\alpha} + d_1\right)}(\gamma, \omega_y, 0\left(\frac{2n}{3} + \tilde{\alpha} + d_1\right)) = \int_{\gamma}^{\omega_y, 0\left(\frac{2n}{3} + \tilde{\alpha} + d_1\right)} L(\omega_y, \dot{\omega}_y, s) ds,
\]
(3.25)
where the last equality holds since $\omega_y, 0$ is a minimizing curve. Let $\eta_1 : [0, \frac{2n}{3} + d_1] \to M$ with $\eta_1(0) = y$ and $\eta_1(\frac{2n}{3} + d_1) = p$ be a Tonelli minimizer such that
\[
F_{0, \frac{2n}{3} + d_1}(\gamma, p) = \int_{0}^{\frac{2n}{3} + d_1} L(\eta_1, \dot{\eta}_1, s) ds.
\]
(3.26)
Then, in view of (3.24), (3.25) and (3.26), we have
\[
\left| \int_{0}^{\frac{2n}{3} + d_1} L(\eta_1, \dot{\eta}_1, s) ds - \int_{\gamma}^{\omega_y, 0\left(\frac{2n}{3} + \tilde{\alpha} + d_1\right)} L(\omega_y, \dot{\omega}_y, s) ds \right|
\leq D_{Lip} C_4 e^{-\mu \frac{2n}{3}},
\]
(3.27)
where $D_{Lip} > 0$ is a Lipschitz constant of $F_{t, t'}$ which is independent of $t, t'$ with $t + 1 \leq t'$. 

\[\text{Equation (3.20)}\]

\[\text{Equation (3.21)}\]

\[\text{Equation (3.22)}\]

\[\text{Equation (3.23)}\]

\[\text{Equation (3.24)}\]

\[\text{Equation (3.25)}\]

\[\text{Equation (3.26)}\]

\[\text{Equation (3.27)}\]
For the above sufficiently large \( n \in \mathbb{N} \) with \( \frac{2n}{3} > \max\{T_1, 2\} \), choose \( 0 \leq d_2 < 1 \) so that \( (\gamma_p(-\frac{2n}{3} + \bar{\tau} - d_2), \gamma_p(-\frac{2n}{3} + \bar{\tau} - d_2), (-\frac{2n}{3} + \bar{\tau} - d_2)) = (p, v_p, 0) \). From (3.23) we have
\[
\begin{aligned}
&d \left( \left( \beta_{x, (\bar{\tau})} \left( -\frac{2n}{3} + \bar{\tau} - d_2 \right), \dot{\beta}_{x, (\bar{\tau})} \left( -\frac{2n}{3} + \bar{\tau} - d_2 \right) \right), \left( -\frac{2n}{3} + \bar{\tau} - d_2 \right), (p, v_p, 0) \right) \\
&\leq C_d e^{-\mu \Phi}.
\end{aligned}
\]  
(3.28)

Since \( \beta_{x, (\bar{\tau})} \) is a minimizing curve, then
\[
F_{-\frac{2n}{3} + \bar{\tau} - d_2, \bar{\tau}} \left( \beta_{x, (\bar{\tau})} \left( -\frac{2n}{3} + \bar{\tau} - d_2 \right), x \right) = \int_{-\frac{2n}{3} + \bar{\tau} - d_2}^{\bar{\tau}} L(\beta_{x, (\bar{\tau})}, \dot{\beta}_{x, (\bar{\tau})}, s) \, ds.
\]  
(3.29)

Let \( \eta_2 : [-\frac{2n}{3} + \bar{\tau} - d_2, \bar{\tau}] \to M \) with \( \eta_2(-\frac{2n}{3} + \bar{\tau} - d_2) = p \) and \( \eta_2(\bar{\tau}) = x \) be a Tonelli minimizer such that
\[
F_{-\frac{2n}{3} + \bar{\tau} - d_2, \bar{\tau}}(p, x) = \int_{-\frac{2n}{3} + \bar{\tau} - d_2}^{\bar{\tau}} L(\eta_2, \dot{\eta}_2, s) \, ds.
\]  
(3.30)

Then, by (3.28), (3.29) and (3.30), we obtain
\[
\left| \int_{-\frac{2n}{3} + \bar{\tau} - d_2}^{\bar{\tau}} L(\eta_2, \dot{\eta}_2, s) \, ds - \int_{-\frac{2n}{3} + \bar{\tau} - d_2}^{\bar{\tau}} L(\beta_{x, (\bar{\tau})}, \dot{\beta}_{x, (\bar{\tau})}, s) \, ds \right| \\
\leq D_{Lip} C_d e^{-\mu \Phi}.
\]  
(3.31)

Define a curve \( \tilde{\eta}_2 : [-\frac{2n}{3} + d_1 + \frac{2n}{3} + d_1] \to M \) by \( \tilde{\eta}_2(\varsigma) = \eta_2(\varsigma - \frac{2n}{3} - d_1 - d_2 + \bar{\tau}) \) for \( \varsigma \in [-\frac{2n}{3} + d_1 + \frac{2n}{3} + d_1 + d_2] \). Then
\[
\int_{-\frac{2n}{3} + \bar{\tau} - d_2}^{\bar{\tau}} L(\tilde{\eta}_2, \dot{\tilde{\eta}}_2, s) \, ds = \int_{\frac{2n}{3} + d_1}^{\frac{2n}{3} + d_1} L(\tilde{\eta}_2, \dot{\tilde{\eta}}_2, s) \, ds.
\]

Set \( \bar{k}_n = \frac{2n}{3} + d_1 + d_2 - \bar{\tau} \). By the choices of \( d_1 \) and \( d_2 \), we have \( \bar{k}_n \in \mathbb{N} \) and \( n \leq \bar{k}_n \leq 2n \).

Consider the curve \( \eta : [0, \bar{k}_n] \to M \) connecting \( y \) and \( x \) defined by
\[
\eta(s) = \begin{cases} 
\eta_1(s), & s \in [0, \frac{2n}{3} + d_1], \\
\tilde{\eta}_2(s), & s \in [\frac{2n}{3} + d_1, \bar{k}_n].
\end{cases}
\]  
(3.32)

Now it remains to show that the curve defined by (3.32) is just the one we need. For \( n \in \mathbb{N} \) large enough, from (3.18) we obtain
\[
u(\eta(0)) + \int_0^{\bar{k}_n} L(\eta, \dot{\eta}, s) \, ds - \tilde{u}(x, Y)
\]
\[
= u(\eta(0)) + \int_0^{\bar{k}_n} L(\eta, \dot{\eta}, s) \, ds - u(y) - h_{0,0}(y, p) - h_{0, (\bar{\tau})}(p, x)
\]  
(3.33)

In view of (3.33), (3.27) and (3.31), we have
\[
u(\eta(0)) + \int_0^{\bar{k}_n} L(\eta, \dot{\eta}, s) \, ds - \tilde{u}(x, Y)
\]
\[
\leq \int_0^{\frac{2n}{3} + d_1} L(\omega_{y, 0}, \dot{\omega}_{y, 0}, s) \, ds + \int_{-\frac{2n}{3} + \bar{\tau} - d_2}^{\bar{\tau}} L(\beta_{x, (\bar{\tau})}, \dot{\beta}_{x, (\bar{\tau})}, s) \, ds
\]
\[
+ 2D_{Lip} C_d e^{-\mu \Phi} - h_{0,0}(y, p) - h_{0, (\bar{\tau})}(p, x).
\]
From (3.34), (3.20) and (3.21), we have
\[
\begin{align*}
\frac{d}{ds} u(\eta(s)) + \int_0^s L(\eta(0), \eta, \dot{\eta}, s) \, ds - \bar{u}(x, \langle \tau \rangle) & \leq -h_{0,0}(\omega_{x,0}\left(\frac{2n}{3} + \bar{\delta} + d_1\right), p) \\
& \leq (C_{lip} + D_{lip})C_4 e^{-\mu \frac{s}{2}},
\end{align*}
\]
where the last inequality follows from \(h_{0,0}(p, p) = 0\), (3.24) and (3.28), \(C_{lip} > 0\) is a Lipschitz constant of \(h\) and \(D_{lip} > 0\) is a Lipschitz constant of \(F_{s,t}\) which is independent of \(t, t'\) with \(t + 1 \leq t'\). Let \(C_5 = 2(C_{lip} + D_{lip})C_4\). Note that \(C_5\) and \(\mu\) are independent of \((x, \tau) \in M \times [0, 1], u \in C(M, R^1)\) and \(n \in N\), which means that (3.19) holds.

Thus, for \(n \in N\) large enough, we have
\[
U_n^u(x, \tau) - \bar{u}(x, \langle \tau \rangle) \leq C_5 e^{-\mu \frac{\tau}{2}}, \quad \forall (x, \tau) \in M \times [0, 1].
\]
Hence, there exists a constant \(C_6 > 0\) such that
\[
\|U_n^u(x, \tau) - \bar{u}(x, \langle \tau \rangle)\|_\infty \leq C_6 e^{-\mu \frac{\tau}{2}}, \quad \forall n \in N, \forall (x, \tau) \in M \times [0, 1], \tag{3.34}
\]
where the constant \(C_6\) depends on \(u\).

Let \(\rho_1 = \frac{2}{3} \mu, K = \max(C_5, C_6)\) and \(\rho = \min(\rho_1, \rho_2)\). Then from (3.17) and (3.34), we have
\[
\|U_n^u(x, \tau) - \bar{u}(x, \langle \tau \rangle)\|_\infty \leq K e^{-\rho \tau}, \quad \forall n \in N. \tag*{□}
\]

4. An example

In this section we provide an example showing that, even though the Aubry set of the time-independent Lagrangian system consists of one hyperbolic periodic orbit, the rate of convergence of the L–O semigroup cannot be better than \(O(\frac{1}{t})\) as \(t \to +\infty\).

**Example 4.1.** Consider the following Lagrangian

\[ L_0 : \mathbb{T}^3 \to R^1, \quad (x, y, \dot{x}, \dot{y}) \mapsto \frac{1}{2} \dot{x}^2 + 1 - \cos(2\pi x) + \frac{1}{2} (\dot{y} - c)^2, \quad c \in R^1. \]

The associated Hamiltonian \(H_0 : T^*\mathbb{T}^3 \to R^1\) is given by \(H_0(x, y, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + c p_2 - 1 + \cos(2\pi x)\). Observe that \(\Gamma_c = \{0\} \times S^1 \times \{0\} \times \{c\}\) is a hyperbolic periodic orbit of the Lagrangian system \(L_0\). It is easy to check that the Mañé critical value \(c(L_0) = 0\) and that the probability measure evenly distributed along \(\Gamma_c\)—which we shall denote \(\nu\)—is an invariant probability measure of the Lagrangian system \(L_0\). Moreover, \(\nu\) is the unique minimal measure (0-action minimizing measure) for \(L_0\). Thus, we can conclude that \(\tilde{\mathcal{M}}_0 = \Gamma_c = \{0\} \times S^1 \times \{0\} \times \{c\}\). Note that in this example the Aubry set \(\tilde{\mathcal{A}}_0\) coincides with the Mather set \(\mathcal{M}_0\), which implies that \(\tilde{\mathcal{A}}_0\) consists of one hyperbolic periodic orbit \(\Gamma_c\) and \(\tilde{\mathcal{A}}_0 = \{0\} \times S^1\).

**Lemma 4.1.** For each \(u \in C(T^3, R^1)\) which satisfies \(\min_{\tilde{\mathcal{A}}_0} u = \min_{T^3} u\), we have

\[ \bar{u} \big|_{\tilde{\mathcal{A}}_0} \equiv \min u = \min_{T^3} u, \]

where \(\bar{u} = \lim_{t \to +\infty} T^u_t u\) is a backward weak KAM solution of \(H_0(x, y, u_x, u_y) = 0\).
**Proof.** For each \((0, y) \in A_0\), from the definition of \(T^u_i\) we have

\[
\bar{u}(0, y) = \lim_{i \to +\infty} T^u_i u(0, y) = \lim_{i \to +\infty} \inf_{(x', y') \in T^2} \left\{ u(x', y') + \int_0^{t_i} L_u(y, \gamma_i) ds \right\},
\]

where \(\gamma : [0, 1] \to T^2\) with \(\gamma(0) = (x', y')\) and \(\gamma(t) = (0, y)\) is a Tonelli minimizer. Since \(L_u \geq 0\), then \(\bar{u}(0, y) \geq \min_{T^2} u = \min_{A_0} u\).

It suffices to show that \(\bar{u}(0, y) \leq \min_{T^2} u = \min_{A_0} u\). Take \((0, y_*) \in A_0\) with \(u(0, y) = \min_{A_0} u\). Let \(\tilde{y}_* \in R^1\) be an arbitrary point in the fibre over \(y_*\). For \(t > 0\), consider the following two curves

\[
\gamma_{2,c} : [0, t] \to R^1, \quad s \mapsto ct \tilde{y}_*,
\]

and

\[
\gamma_{2,c'} : [0, t] \to R^1, \quad s \mapsto c't \tilde{y}_*,
\]

with \(\gamma_{2,c}(t) = \tilde{y}_*\), where \(\tilde{y}_*\) is a point in the fibre over \(y\) such that \(\tilde{y}_*\) and \(\gamma_{2,c}(t)\) are in the same fundamental domain in \(R^1\). Let \(r = \gamma_{2,c}(t) - \gamma_{2,c'}(t) = (c' - c)t\). Then \(|r| \leq 1\) and \(|c' - c| \leq \frac{1}{t}\). Let \(\gamma_c = (0, \gamma_{2,c}) : [0, t] \to R^2\) and \(\gamma_{c'} = \pi \gamma_{2,c'}\), where \(\pi : R^2 \to T^2\) is the standard universal covering projection. Then \(\gamma_c : [0, t] \to T^2\) is a curve connecting \((0, y_*\)) and \((0, y)\). Hence, we have

\[
T^u_i u(0, y) \leq u(\gamma_c(0)) + \int_0^{t_i} L(\gamma_c, \gamma_{c'}) ds = u(0, y_*) + \frac{1}{2} \int_0^{t_i} (c' - c)^2 ds \leq u(0, y_*) + \frac{1}{2t}.
\]

Let \(t \to +\infty\), to deduce

\[
\bar{u}(0, y) = \lim_{i \to +\infty} T^u_i u(0, y) \leq u(0, y_*) = \min_{A_0} u = \min u.
\]

In the following we show that there exist \(u \in C(T^2, R^1), (x_0, y_0) \in T^2\) and \(\{t_n\}_{n=1}^{+\infty}\) with \(t_n \to +\infty\) as \(n \to +\infty\) such that

\[
|T^u_{t_n} u(x_0, y_0) - \bar{u}(x_0, y_0)| \geq O\left(\frac{1}{t_n}\right), \quad n \to +\infty.
\]

Set \((x_0, y_0) = (0, \frac{1}{2}) \in T^2\). Let \((\tilde{x}_0, \tilde{y}_0) \in R^2\) denote a generic point in the fibre over \((x_0, y_0)\), i.e. \(\pi(\tilde{x}_0, \tilde{y}_0) = (x_0, y_0)\). Define a continuous function on \(R^2\) as follows: for each \((x, \tilde{y}) \in R^2\)

\[
\bar{u}(x, \tilde{y}) = \begin{cases} 
\delta - |\tilde{y} - \tilde{y}_0|, & |\tilde{y} - \tilde{y}_0| \leq \delta, \\
0, & \text{otherwise},
\end{cases}
\]

where \(0 < \delta < \frac{1}{2}\). Then we can define a continuous function on \(T^2\) as \(u(x, y) = \bar{u}(x, \tilde{y})\) for all \((x, y) \in T^2\), where \((x, \tilde{y})\) is an arbitrary point in the fibre over \((x, y)\). From lemma 4.1, we have \(\bar{u}(0, y) = \min_{T^2} u = \min_{A_0} u = 0\).

Now fix a point \((0, \tilde{y}_0) \in R^2\) in the fibre over \((0, y_0)\). Then there exist \((0, \tilde{y}_0)\) \(t_n \to +\infty\) as \(n \to +\infty\) such that \(|\tilde{y}_0 - ct_n - \tilde{y}_0| \leq \frac{1}{2}\). Let \(\tilde{z} = \tilde{y}_0 - ct_n, \forall n\). Then \(|\tilde{z} - \tilde{y}_0| \leq \frac{1}{2}, \forall n\). For each \(t_n\) there is \((x_n, \xi_n) \in T^2\) such that

\[
T^u_{t_n} u(0, y_0) = u(x_n, \xi_n) + \int_0^{t_n} L_u(y_0, \gamma_{t_n}) ds,
\]

where \(\gamma_{t_n} = (y_{1,n}, y_{2,n}) : [0, t_n] \to T^2\) with \(y_{0,n}(0) = (x_n, \xi_n)\) and \(y_{0,n}(t_n) = (0, y_0)\) is a Tonelli minimizer. We assert that \(x_n = 0, \forall n\), i.e. \((x_n, \xi_n) \in A_0, \forall n\). For, otherwise, there would be \(x_n \neq 0\) for some \(n\). Then we have

\[
u(x_n, \xi_n) + \frac{1}{2} \int_0^{t_n} (\gamma_{2,n} - c)^2 ds < u(x_n, \xi_n) + \int_0^{t_n} \left(\frac{1}{2} \nu_{1,n}^2 + 1 - \cos(2\pi y_{1,n})\right) + \frac{1}{2} (\gamma_{2,n} - c)^2 ds.
\]
Let \( y'_n = (0, \gamma_{2,n}) : [0, t_n] \rightarrow T^2 \). Then \( y'_n \) is a curve in \( T^2 \) connecting \((0, \xi_n)\) and \((0, y_0)\). In view of (4.2), we have

\[
T^u_n u(0, y_0) \leq u(0, \xi_n) + \frac{1}{2} \int_0^{t_n} (\gamma_{2,n} - c)^2 \, ds
\]

which implies that

\[
\lim_{n \to \infty} T^u_n u(0, y_0) = u(0, \xi_n) + \frac{1}{2} \int_0^{t_n} (\gamma_{2,n} - c)^2 \, ds,
\]

which contradicts (4.1). Hence \( x_0 = 0, \forall n \). It is easy to see that

\[
T^u_n u(0, y_0) = u(0, \xi_n) + \frac{1}{2} \int_0^{t_n} (\gamma_{2,n} - c)^2 \, ds, \quad n = 1, 2, \ldots \quad (4.3)
\]

In view of the lifting property of the covering projection, for each \( n \) there is a unique curve \( \tilde{y}_{2,n} : [0, t_n] \rightarrow \mathbf{R}^1 \) with \( \pi \tilde{y}_{2,n} = y_{2,n} \) and \( \tilde{y}_{2,n}(t_n) = \tilde{y}_0^n \). Set \( \tilde{\xi}^n = \tilde{y}_{2,n}(0) \). Then \( \pi \tilde{\xi}^n = \xi_n \). Moreover, \( \tilde{y}_{2,n} \) has the form \( \tilde{y}_{2,n}(s) = c_n s + \tilde{\xi}^n, s \in [0, t_n], c_n \in \mathbf{R}^1 \). It is clear that

\[
\tilde{\xi}^n = \tilde{y}_0^n - c_n t_n.
\]

If \( |\tilde{\xi}^n - \tilde{\xi}_0^n| \leq \frac{\delta}{4} \), then from \( |\tilde{z}^n - \tilde{y}_0^n| \leq \frac{\delta}{2} \), we have \( |\tilde{\xi}^n - \tilde{y}_0^n| \leq \frac{\delta}{2} \). Therefore, in view of (4.3),

\[
T^u_n u(0, y_0) = u(0, \xi_n) + \frac{1}{2} \int_0^{t_n} (\gamma_{2,n} - c)^2 \, ds \geq u(0, \xi_n) = \tilde{u}(0, \tilde{\xi}^n) \geq \frac{\delta}{4}. \quad (4.4)
\]

By (4.4) we may deduce that there can be only a finite number of \( \tilde{\xi}^n \)'s such that \( |\tilde{\xi}^n - \tilde{z}^n| \leq \frac{\delta}{4} \).

Suppose not. There are \( \{t_n\}_{n=1}^{\infty} \) and \( \{\tilde{\xi}^n\}_{n=1}^{\infty} \) such that \( T^u_n u(0, y_0) \geq \frac{\delta}{4}, i = 1, 2, \ldots \), which contradicts \( \lim_{n \to +\infty} T^u_n u(0, y_0) = \tilde{u}(0, y_0) = 0 \).

For \( \tilde{\xi}^n \) with \( |\tilde{\xi}^n - \tilde{\xi}_0^n| \geq \frac{\delta}{4} \), we have

\[
\frac{\delta}{4} < |\tilde{\xi}^n - \tilde{\xi}_0^n| = |(c - c_n)t_n|,
\]

which implies that \( |c - c_n| > \frac{\delta}{4t_n} \). Then

\[
T^u_n u(0, y_0) = u(0, \xi_n) + \frac{1}{2} \int_0^{t_n} (\gamma_{2,n} - c)^2 \, ds > \frac{\delta^2}{32t_n}.
\]

Therefore,

\[
|T^u_n u(0, y_0) - \tilde{u}(0, y_0)| = |T^u_n u(0, y_0)| > \frac{\delta^2}{32t_n},
\]

i.e.

\[
|T^u_n u(0, y_0) - \tilde{u}(0, y_0)| \geq O \left( \frac{1}{t_n} \right), \quad n \to +\infty.
\]

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