Strong consistency of MLE for finite mixtures of location-scale distributions when the scale parameters are exponentially small

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Abstract

In a finite mixture of location-scale distributions maximum likelihood estimator does not exist because of the unboundedness of the likelihood function when the scale parameter of some mixture component approaches zero. In order to study the strong consistency of maximum likelihood estimator, we consider the case that the scale parameters of the component distributions are restricted from below by $c_n$, where $\{c_n\}$ is a sequence of positive real numbers which tend to zero as the sample size $n$ increases. We prove that under mild regularity conditions maximum likelihood estimator is strongly consistent if the scale parameters are restricted from below by $c_n = \exp(-n^d)$, $0 < d < 1$.

Key words and phrases: Mixture distribution, maximum likelihood estimator, consistency.

1 Introduction

In some finite mixture distributions maximum likelihood estimator (MLE) does not exist. Let us consider the following example. Denote a normal mixture distribution with $M$ components and parameter $\theta = (\alpha_1, \mu_1, \sigma_1^2, \ldots, \alpha_M, \mu_M, \sigma_M^2)$ by

$$f(x; \theta) = \sum_{m=1}^{M} \alpha_m \phi_m(x; \mu_m, \sigma_m^2),$$

where $\alpha_m (m = 1, \ldots, M)$ are nonnegative real numbers that sum to one and $\phi_m(x; \mu_m, \sigma_m^2)$ are normal densities. Let $x_1, \ldots, x_n$ denote a random sample of size $n \geq 2$ from the density $f(x; \theta_0)$. In view of the identifiability problem of mixture distributions discussed below,
here $\theta_0$ is a parameter value designating the true distribution. However for simplicity we just say $\theta_0$ is the true parameter from now on. The log likelihood function is

$$\sum_{i=1}^{n} \log f(x_i; \theta) = \sum_{i=1}^{n} \log \left\{ \sum_{m=1}^{M} \alpha_m \phi_m(x_i; \mu_m, \sigma_m^2) \right\}.$$ 

If we set $\mu_1 = x_1$, then the likelihood tends to infinity as $\sigma_1^2 \to 0$. Thus MLE does not exist.

But when we restrict $\sigma_m \geq c$ ($m = 1, \ldots, M$) by some positive real constant $c$, we can avoid the divergence of the likelihood. Furthermore, in this situation, it can be shown that MLE is strongly consistent if the true parameter $\theta_0$ is in the restricted parameter space.

On the other hand, the smaller $\sigma_1^2$ is, the less contribution $\phi_1(x; \mu_1 = x_1, \sigma_1^2)$ makes to the likelihood at other observations $x_2, \ldots, x_n$. Therefore an interesting question here is whether we can decrease the bound $c = c_n$ to zero with the sample size $n$ and yet guarantee the strong consistency of MLE. If this is possible, the further question is how fast $c_n$ can decrease to zero.

This question is similar to the (so far open) problem stated in [Hathaway (1985)](#), which treats mixtures of normal distributions with constraints imposed on the ratios of variances while our restriction is imposed on variances themselves. See also a discussion in section 3.8.1 of [McLachlan and Peel (2000)](#).

In the above example, the normality of the component distributions is not essential and the same difficulty exists for finite mixtures of general location-scale distributions such as mixtures of uniform distributions. Furthermore in this paper we allow that each component belongs to different location-scale families. Let $\sigma_m$ ($m = 1, \ldots, M$) denote the scale parameters of the component distributions and consider the restriction $\sigma_m \geq c_n$ ($m = 1, \ldots, M$). Then a question of interest here is whether we can decrease the bound $c_n$ to zero.

For the case of mixture of uniform distributions, in [Tanaka and Takemura (2005)](#) we proved that MLE is strongly consistent if $c_n = \exp(-n^d)$, $0 < d < 1$. Here $d$ can be arbitrarily close to 1 but fixed. In this paper, we prove that the same result holds for general finite mixtures of location-scale distributions under very mild regularity conditions (assumptions 1–4 below). We employ the same line of proof as in [Tanaka and Takemura (2005)](#), but the proof for the general finite mixture is much more difficult. As discussed in section 5 the normal density satisfies the regularity conditions and our result implies that MLE is strongly consistent for the finite normal mixture if $\sigma_m \geq c_n = \exp(-n^d)$, $0 < d < 1$, $m = 1, \ldots, M$.

Our framework is closely related to the method of sieve (Grenander (1981)). In the sieve method an objective function is maximized over a constrained subspace of parameter space and then this subspace is expanded to the whole parameter space as the sample size increases. Some applications and consistency results for the method are given in [Geman and Hwang (1982)](#). MLE based on a sieve is called a sieve MLE. The convergence rates of sieve MLE for Gaussian mixture problems are studied in [Genovese and Wasserman (2002)](#).
and Ghosal and van der Vaart (2001) and their ideas are very interesting. They obtain the convergence rates by bounding the Hellinger bracketing entropy of subsets of the function space and assume that the corresponding subsets of the parameter space are compact so that their bracketing entropy does not diverge. In the case of sieve MLE, the approximating subspaces are usually taken to be compact, whereas we treat a sequence of non-compact subsets of the parameter space expanding to the whole parameter space as the sample size increases. Therefore results on sieve MLE are not directly applicable in our framework.

The organization of the paper is as follows. In section 2 we summarize some preliminary descriptions. In section 3 we state our main results in theorems 1 and 2. Section 4 is devoted to the proof of theorems and lemmas. Finally in section 5 we give some discussions.

2 Preliminaries on strong consistency and identifiability of mixture distributions

A mixture of $M$ densities with parameter $\theta = (\alpha_1, \mu_1, \sigma_1, \ldots, \alpha_M, \mu_M, \sigma_M)$ is defined by

$$f(x; \theta) \equiv \sum_{m=1}^{M} \alpha_m f_m(x; \mu_m, \sigma_m),$$

where $\alpha_m$, $m = 1, \ldots, M$, called the mixing weights, are nonnegative real numbers that sum to one and $f_m(x; \mu_m, \sigma_m)$, called the components of the mixture, are density functions. In this paper we consider the case that the component densities are location-scale densities with the location parameter $\mu_m \in \mathbb{R}$ and the scale parameter $\sigma_m > 0$, i.e.

$$f_m(x; \mu_m, \sigma_m) = \frac{1}{\sigma_m} f_m \left( \frac{x - \mu_m}{\sigma_m}; 0, 1 \right).$$  \quad (1)

As mentioned above, we allow $f_m(x; \mu_m, \sigma_m)$ to belong to different families. For example, $f_1(x; \mu_1, \sigma_1)$ may be a normal density, $f_2(x; \mu_2, \sigma_2)$ may be a uniform density, etc. Let $\Omega_m = \mathbb{R} \times (0, \infty)$ denote the parameter space of the $m$-th component $(\mu_m, \sigma_m)$ and let $\Theta$ denote the entire parameter space:

$$\Theta \equiv \{ (\alpha_1, \ldots, \alpha_M) \in \mathbb{R}^M \mid \sum_{m=1}^{M} \alpha_m = 1, \alpha_m \geq 0 \} \times \prod_{m=1}^{M} \Omega_m.$$  \quad (2)

Let $\mathcal{K}$ be a subset of $\{1, 2, \ldots, M\}$ and let $|\mathcal{K}|$ denote the number of elements in $\mathcal{K}$. Denote by $\theta_\mathcal{K}$ a subvector of $\theta \in \Theta$ consisting of the components in $\mathcal{K}$. Then the parameter space of subprobability measures consisting of the components in $\mathcal{K}$ is

$$\tilde{\Theta}_\mathcal{K} \equiv \{ \theta_\mathcal{K} \mid \theta \in \Theta, \sum_{m \in \mathcal{K}} \alpha_m \leq 1 \}.$$  \quad (2)
Corresponding density and the set of subprobability densities are denoted by
\[
f_{\mathcal{X}}(x; \theta) \equiv \sum_{k \in \mathcal{X}} \alpha_k f_k(x; \mu_k, \sigma_k),
\]
\[
\mathcal{G}_{\mathcal{X}} \equiv \{ f_{\mathcal{X}}(x; \theta) \mid \theta \in \Theta \}.
\]
Furthermore denote the set of subprobability densities with no more than \(K\) components by
\[
\mathcal{G}_K \equiv \bigcup_{|\mathcal{X}| \leq K} \mathcal{G}_{\mathcal{X}} \quad (1 \leq K \leq M).
\]

We now briefly discuss identifiability of parameters. In mixture models, different parameters may designate the same distribution. When the component densities belong to a common location-scale family, we can permute the labels of the components and the distribution remains the same. A mixture model of \(K-1\) components can be obtained by setting one weight \(\alpha_m = 0\) (with arbitrary \(\mu_m\) and \(\sigma_m\)) in a model with \(K\) components. These are trivial cases of unidentifiability of parameters. However there are more complicated cases. Let \(U(x; a, b)\) denote the uniform density on the interval \([a,b]\). Then, for example, \(\frac{1}{3}U(x; -1, 1) + \frac{2}{3}U(x; -2, 2)\) and \(\frac{1}{2}U(x; -2, 1) + \frac{1}{2}U(x; -1, 2)\) represent the same distribution (Everitt and Hand (1981)). In this case the limiting behavior of MLE is not obvious, although the estimated density should be consistent. Therefore we first give a definition of consistency in terms of the estimated density.

**Definition 1.** An estimator \(\hat{f}_n\) is strongly consistent if
\[
\text{Prob} \left( \lim_{n \to \infty} \left\| \hat{f}_n - f_0 \right\| = 0 \right) = 1 ,
\]
where \(\| \cdot \|\) is the \(L_1\)-norm.

Although definition 1 is conceptually simple, in order to prove the strong consistency of MLE we work with the location and the scale parameters in \(\Pi\) and the mixing weights. In order to deal with the identifiability problem let us introduce a distance between two sets of parameters. Let \(\text{dist}(\theta, \theta')\) denote the ordinary Euclidean distance (or any other equivalent distance) between two parameter vectors \(\theta, \theta' \in \Theta\). For \(U, V \subset \Theta\) define
\[
\text{dist}(U, V) \equiv \inf_{\theta \in U} \inf_{\theta' \in V} \text{dist}(\theta, \theta').
\]
For a parameter \(\theta\), let
\[
\Theta(\theta) \equiv \{ \theta' \in \Theta \mid f(x; \theta') = f(x; \theta) \quad \forall x \}.
\]
Then \(\Theta_0 = \Theta(\theta_0)\) denotes the set of true parameters. Since our densities are continuous with respect to \(\theta\), by Scheffé’s theorem (Theorem 16.12 of Billingsley (1995)) \(\text{dist}(\Theta(\hat{\theta}_n), \Theta_0) \to 0\) implies \(\left\| \hat{f}_n - f_0 \right\| \to 0\).
3 Main results

We assume the following regularity conditions for strong consistency of MLE.

**Assumption 1.** There exist real constants $v_0, v_1 > 0$ and $\beta > 1$ such that

$$f_m(x; \mu_m = 0, \sigma_m = 1) \leq \min\{v_0, v_1 \cdot |x|^{-\beta}\}$$

for all $m$.

This assumption means that $f_m (m = 1, \ldots, M)$ are bounded and their tails decrease to zero faster than or equal to $|x|^{-\beta}$, which is a very mild condition.

The following three regularity conditions are standard conditions assumed in discussing strong consistency of MLE. Let $\Gamma$ denote any compact subset of $\Theta$.

**Assumption 2.** For $\theta \in \Theta$ and any positive real number $\rho$, let

$$f(x; \theta, \rho) \equiv \sup_{\operatorname{dist}(\theta', \theta) \leq \rho} f(x; \theta').$$

For each $\theta \in \Gamma$ and sufficiently small $\rho$, $f(x; \theta, \rho)$ is measurable.

**Assumption 3.** For each $\theta \in \Gamma$, if $\lim_{j \to \infty} \theta^{(j)} = \theta$, $(\theta^{(j)} \in \Gamma)$ then $\lim_{j \to \infty} f(x; \theta^{(j)}) = f(x; \theta)$ except on a set which is a null set and does not depend on the sequence $\{\theta^{(j)}\}_{j=1}^{\infty}$.

**Assumption 4.**

$$\int |\log f(x; \theta_0)| f(x; \theta_0) dx < \infty.$$ 

Let $E_0[\cdot]$ denote the expectation under the true parameter $\theta_0$. The following theorem is essential to our argument and it is of some independent interest.

**Theorem 1.** Suppose that assumptions 1–4 are satisfied and $f_0 \in \mathcal{G}_M \setminus \mathcal{G}_{M-1}$ where $\mathcal{G}_M$ and $\mathcal{G}_{M-1}$ are defined in (5). Then there exist real constants $\kappa, \lambda > 0$ such that

$$E_0[\log \{g(x) + \kappa\}] + \lambda < E_0[\log f(x; \theta_0)]$$

for all $g \in \mathcal{G}_{M-1}$.

We now state the main theorem of this paper.

**Theorem 2.** Suppose that assumptions 1–4 are satisfied and $f_0 \in \mathcal{G}_M \setminus \mathcal{G}_{M-1}$ where $\mathcal{G}_M$ and $\mathcal{G}_{M-1}$ are defined in (5). Let $c_0 > 0$ and $0 < d < 1$. If $c_n = c_0 \cdot \exp(-n^d)$ and

$$\Theta_n \equiv \{\theta \in \Theta \mid \sigma_m \geq c_n, (m = 1, \ldots, M)\},$$

then

$$\text{Prob} \left( \lim_{n \to \infty} \operatorname{dist}(\Theta(\hat{\theta}_n), \Theta_0) = 0 \right) = 1,$$

where $\hat{\theta}_n$ is MLE restricted to $\Theta_n$. 

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As remarked at the end of the previous section theorem 2 implies the following corollary.

**Corollary 1.** Under the same assumptions of theorem 2, \( \hat{f}_n \) is strongly consistent in the sense of definition 1.

### 4 Proofs

In this section, we prove theorems stated in section 3. The organization of this section is as follows. First in subsection 4.1 we state some lemmas for theorem 1 and 2. Next in subsection 4.2 we prove theorem 1 which is also essential for theorem 2. Finally we prove theorem 2 in subsection 4.3. For convenience a list of notations used in our proofs is provided at the end of this paper.

#### 4.1 Notations and some lemmas

Fix arbitrary \( \kappa_0 > 0 \), which corresponds to \( \kappa \) in theorem 1. Define \( \tilde{\beta} \) and \( \nu(y), y > 0 \), as

\[
\tilde{\beta} \equiv \frac{\beta - 1}{\beta}, \quad \nu(y) \equiv \left( \frac{v_1}{\kappa_0} \right)^{\frac{1}{\beta}} y^{\frac{1}{\beta}},
\]

where \( v_1 \) and \( \beta \) are given in assumption 1. Noting that \( v_1 \cdot (\nu(y))^{-\beta} = \kappa_0/y \), the following lemma is easily proved and we omit its proof. See figure 1.

**Lemma 1.** Under the assumption 1, for arbitrary \( \kappa_0 > 0 \) each component density \( f_m(x; \mu, \sigma) \) is bounded by a step function

\[
f_m(x; \mu, \sigma) \leq \max \left\{ 1_{|\mu - \nu(\sigma), \mu + \nu(\sigma)|} \cdot \frac{v_0}{\sigma}, \kappa_0 \right\} \leq 1_{|\mu - \nu(\sigma), \mu + \nu(\sigma)|} \cdot \frac{v_0}{\sigma} + \kappa_0,
\]

where \( 1_U(x) \) denotes the indicator function of \( U \subset \mathbb{R} \).

![Figure 1: Each component is bounded by step function.](image-url)
From lemma 1
\[
\sum_{m=1}^{M} \alpha_m f_m(x; \mu_m, \sigma_m) \leq \sum_{m=1}^{M} 1_{[\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m))}(x) \cdot \frac{v_0}{\sigma_m} + \kappa_0.
\]  

(8)

The right-hand side of (8) is a step function. We look at this step function where the density \( f(x; \theta) \) is high, i.e. the scale parameter of some component is small.

For a given choice of \( \kappa_0 > 0 \), choose \( c_0 > 0 \) such that
\[
c_0 < \frac{v_0}{\kappa_0(M + 1)},
\]

(9)

Below we will impose additional constraints on \( \kappa_0 \) and \( c_0 \) to make \( \kappa_0 \) and \( c_0 \) sufficiently small to satisfy other conditions. For each \( \theta \), let
\[
\mathcal{K}_{\sigma \leq c_0} = \mathcal{K}_{\sigma \leq c_0}(\theta) \equiv \{ m \mid 1 \leq m \leq M, \sigma_m \leq c_0 \}
\]

denote the set of components with \( \sigma_m \leq c_0 \) and define
\[
J(\theta) \equiv \bigcup_{m \in \mathcal{K}_{\sigma \leq c_0}} [\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)).
\]

(10)

On \( J(\theta) \) the density \( f(x; \theta) \) is high. Now dividing \( J(\theta) \) according to the height of the step function on the right-hand side of (8), for \( x \in J(\theta) \) we can write the right-hand side of (8) as
\[
1_{J(\theta)}(x) \cdot \left\{ \sum_{m=1}^{M} 1_{[\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m))}(x) \cdot \frac{v_0}{\sigma_m} + \kappa_0 \right\} = \sum_{t=1}^{T(\theta)} H(J_t(\theta)) \cdot 1_{J_t(\theta)}(x),
\]

where \( J_t(\theta) \) (\( t = 1, \ldots, T(\theta) \)) are disjoint intervals, \( [\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)) \) (\( m \in \mathcal{K}_{\sigma \leq c_0} \)) are unions of some of \( J_t(\theta) \)'s and the height \( H(J_t(\theta)) \) for each \( t \) is defined by any \( x \in J_t(\theta) \) as
\[
H(J_t(\theta)) \equiv \sum_{m=1}^{M} 1_{[\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m))}(x) \cdot \frac{v_0}{\sigma_m} + \kappa_0.
\]

(11)

See figure 2. For \( x \in J_t(\theta) \), there is at least one \( m = m_t \) such that \( x \in [\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)) \) and \( H(J_t(\theta)) \geq v_0/c_0 + \kappa_0 \). Also note that the total number \( T(\theta) \) of \( J_t(\theta) \)'s satisfies \( T(\theta) \leq 2M \), because the change of the height can only occur at \( \mu_m - \nu(\sigma_m) \) or \( \mu_m + \nu(\sigma_m) \).

By (8) we have the following lemma for \( x \in J(\theta) \).

**Lemma 2.** Under the assumption 1, for each \( x \in J(\theta) \)
\[
\sum_{m=1}^{M} \alpha_m f_m(x; \mu_m, \sigma_m) \leq \sum_{t=1}^{T(\theta)} H(J_t(\theta)) \cdot 1_{J_t(\theta)}(x).
\]
A density can be high only in a small region and we want to have some explicit bound on the length $W(J_t(\theta))$ of $J_t(\theta)$ in terms of its height $H(J_t(\theta))$. Let

$$v_2 \equiv 2 \left( \frac{v_1}{\kappa_0} \right)^{\frac{1}{\beta}} \left( v_0 \cdot (M + 1) \right)^{\tilde{\beta}} , \quad \xi(y) \equiv v_2 \cdot \left( \frac{1}{y} \right)^{\tilde{\beta}} , \quad y > 0 \quad (12)$$

where $v_0, v_1$ and $\beta$ are given in assumption 1 and $\tilde{\beta}$ is defined in (7).

**Lemma 3.** Under the assumption 1 the length $W(J_t(\theta))$ of $J_t(\theta)$ for each $t$ is bounded as

$$W(J_t(\theta)) \leq v_2 \cdot \left( \frac{1}{H(J_t(\theta))} \right)^{\tilde{\beta}} = \xi(H(J_t(\theta))) \quad (13).$$

**Proof:** In (11) at any $x \in J_t(\theta)$, $H(J_t(\theta)) - \kappa_0$ consists of at most $M$ components. Thus, for each $J_t(\theta)$, there exists at least one component $m = m_t$ such that

$$\frac{v_0}{\sigma_m} \geq \frac{1}{M} \cdot (H(J_t(\theta)) - \kappa_0) .$$

Furthermore from (9) we have

$$H(J_t(\theta)) - \kappa_0 > H(J_t(\theta)) - \frac{v_0}{c_0(M + 1)}$$

$$> H(J_t(\theta)) - \frac{H(J_t(\theta))}{M + 1} = \frac{M \cdot H(J_t(\theta))}{M + 1} .$$
Therefore we have
\[
\nu(\sigma_m) = \left( \frac{v_1}{\kappa_0} \right)^{\frac{1}{\beta}} \sigma_m^\beta \leq \left( \frac{v_1}{\kappa_0} \right)^{\frac{1}{\beta}} \left( \frac{v_0(M + 1)}{H(J_t(\theta))} \right)^{\frac{1}{\beta}}.
\]
This with \(W(J_t(\theta)) \leq 2\nu(\sigma_m)\) proves the lemma.

So far we have been concerned with bounding the density at its peaks. Now we consider bounding the tail of the true density \(f(x; \theta_0)\). Write \(\bar{\mu}_0 \equiv \max(|\mu_{01}|, \ldots, |\mu_{0M}|)\) and \(\theta_0 = (\alpha_{01}, \mu_{01}, \sigma_{01}, \ldots, \alpha_{0M}, \mu_{0M}, \sigma_{0M})\). Let

\[
u_0 \equiv \sup_x f(x; \theta_0) \quad , \quad u_1 \equiv \max(u_0 \cdot (2\bar{\mu}_0)^\beta, 2^\beta v_1 \sum_{m=1}^{M} \alpha_{0m} \sigma_{0m}^{-\beta}) .
\]

(13)

**Lemma 4.** Under the assumption \(\square\), the following inequality holds.

\[
f(x; \theta_0) \leq \min \{u_0, u_1 \cdot |x|^{-\beta}\}, \quad \forall x \in \mathbb{R}.
\]

**Proof:** From assumption \(\square\)

\[
u_0 \leq \sum_{m=1}^{M} \alpha_{0m} \frac{v_0}{\sigma_{0m}}.
\]

Then for \(|x| \geq 2\bar{\mu}_0\)

\[|x - \mu_{0m}|^{-\beta} \leq (|x| - \bar{\mu}_0)^{-\beta} \leq 2^\beta |x|^{-\beta}, \quad (m = 1, \ldots, M).
\]

Therefore for \(|x| \geq 2\bar{\mu}_0\)

\[
f(x; \theta_0) \leq |x|^{-\beta} 2^\beta v_1 \sum_{m=1}^{M} \alpha_{0m} \sigma_{0m}^{-\beta-1}
\]

and

\[
f(x; \theta_0) \leq \min \{u_0, u_1 \cdot |x|^{-\beta}\}, \quad \forall x \in \mathbb{R}.
\]

\(\square\)

Based on lemma \(\square\) we can bound the behavior of the minimum and the maximum of the sample. Let \(x_1, \ldots, x_n\) denote a random sample of size \(n\) from \(f(x; \theta_0)\) and let

\[
x_{n,1} \equiv \min \{x_1, \ldots, x_n\} \quad , \quad x_{n,n} \equiv \max \{x_1, \ldots, x_n\}.
\]

The following lemma follows from the Borel-Cantelli lemma.

**Lemma 5.** For any real constant \(A_0 > 0\) and \(\zeta > 0\), define

\[
A_n \equiv A_0 \cdot n^{\frac{2 + \zeta}{\beta - 1}}.
\]

Then

\[
\text{Prob} (x_{n,1} < -A_n \quad \text{or} \quad x_{n,n} > A_n) = 0.
\]
Proof: By the Borel-Cantelli lemma and the Bonferroni inequality it suffices to show that
\[ \sum_{n=1}^{\infty} \text{Prob} \left( x_{n,1} < -A_n \right) < \infty, \quad \sum_{n=1}^{\infty} \text{Prob} \left( x_{n,n} > A_n \right) < \infty. \]

We consider the left tail. Let \( F_0(x) \) denotes the distribution function of \( f(x; \theta_0) \). Then
\[ \text{Prob} \left( x_{n,1} < -A_n \right) = 1 - (1 - F_0(-A_n))^n, \]
and
\[ F_0(-A_n) \leq \int_{-\infty}^{-A_n} u_1 \cdot |x|^{-\beta} \, dx = \frac{u_1}{\beta - 1} \cdot A_n^{-\beta + 1}. \]

By replacing \( n \) by \( n - n_0 \) with a sufficiently large \( n_0 \) if necessary, we can assume without loss of generality that
\[ \frac{u_1 A_0^{-\beta + 1}}{\beta - 1} \left( n^{\frac{2+\xi}{\beta - 1}} \right)^{-\beta + 1} < 1, \quad \forall n. \]

Then
\[
\log(1 - F_0(-A_n))^n \geq \log \left( 1 - \frac{u_1}{\beta - 1} \cdot A_n^{-\beta + 1} \right)^n = \log \left( 1 - \frac{u_1 A_0^{-\beta + 1}}{\beta - 1} \left( n^{\frac{2+\xi}{\beta - 1}} \right)^{-\beta + 1} \right)^n.
\]

Let \( u_2 \equiv u_1 A_0^{-\beta + 1} / (\beta - 1) \) and we have
\[
|\log (1 - F_0(-A_n))^n| \leq \left| \log \left( 1 - \frac{u_2}{n^{2+\xi}} \right)^n \right| = \frac{u_2}{n^{1+\xi}} \left| \log \left( 1 - \frac{u_2}{n^{2+\xi}} \right)^{\frac{n^{2+\xi}}{u_2}} \right| = O(n^{-(1+\xi)}).
\]

Hence there exists a sufficiently large \( N \) and \( u_3 > 0 \) such that
\[ |\log (1 - F_0(-A_n))^n| \leq \frac{u_3}{n^{1+\frac{\xi}{2}}} \]
for all \( n > N \). This and \( (1 - F_0(-A_n))^n \leq 1 \) imply that for \( n > N \)
\[ \log (1 - F_0(-A_n))^n \geq -\frac{u_3}{n^{1+\frac{\xi}{2}}}. \]

Hence by \( 1 - e^{-y} \leq y \), we have for \( n > N \)
\[ \text{Prob}(x_{n,1} < -A_n) = 1 - (1 - F_0(-A_n))^n \leq 1 - \exp \left( -\frac{u_3}{n^{1+\frac{\xi}{2}}} \right) \leq \frac{u_3}{n^{1+\frac{\xi}{2}}}. \]
Therefore we obtain
\[
\sum_{n>N} \text{Prob}(x_{n,1} < -A_n) = \sum_{n>N} 1 - (1 - F_0(-A_n))^n \leq \sum_{n} \frac{u_3}{n^{1+\frac{1}{2}}} < \infty.
\]

The case of the right tail \(\text{Prob}(x_{n,n} > A_n)\) is also proved by the same argument. \(\square\)

Finally we consider subprobability densities in \(G_{\mathcal{X}}\). For any positive real number \(\rho\), let
\[
f_{\mathcal{X}}(x; \theta_{\mathcal{X}}, \rho) \equiv \sup_{\text{dist}(\theta'_{\mathcal{X}}, \theta_{\mathcal{X}}) \leq \rho} f_{\mathcal{X}}(x; \theta'_{\mathcal{X}}), \quad (\theta'_{\mathcal{X}} \in \tilde{\Theta}_{\mathcal{X}}).
\]

The following lemma follows from the bounded convergence theorem.

**Lemma 6.** Let \(\Gamma_{\mathcal{X}}\) denote any compact subset of \(\tilde{\Theta}_{\mathcal{X}}\). For any real constant \(\kappa \geq 0\) and any point \(\theta_{\mathcal{X}} \in \Gamma_{\mathcal{X}}\), the following equality holds under the assumption 1 and 3.
\[
\lim_{\rho \to 0} E_0[\log \{f_{\mathcal{X}}(x; \theta_{\mathcal{X}}, \rho) + \kappa\}] = E_0[\log \{f_{\mathcal{X}}(x; \theta_{\mathcal{X}}) + \kappa\}].
\]

**Proof:** We treat the case of \(\kappa > 0\). The case of \(\kappa = 0\) is almost the same as the proof of Lemma 2 in [Wald (1949)]#ref-wald1949. From assumption 3 we have
\[
\lim_{\rho \to 0} \log \{f_{\mathcal{X}}(x; \theta_{\mathcal{X}}, \rho) + \kappa\} = \log \{f_{\mathcal{X}}(x; \theta_{\mathcal{X}}) + \kappa\} \quad \text{a.e.}
\]

Now \(\Gamma_{\mathcal{X}}\) is compact and \(\kappa > 0\). Hence by assumption 1 \(\log \{f_{\mathcal{X}}(x; \theta_{\mathcal{X}}, \rho) + \kappa\}\) is bounded. Therefore
\[
\lim_{\rho \to 0} E_0[\log \{f_{\mathcal{X}}(x; \theta_{\mathcal{X}}, \rho) + \kappa\}] = E_0[\log \{f_{\mathcal{X}}(x; \theta_{\mathcal{X}}) + \kappa\}]
\]

by the bounded convergence theorem. \(\square\)

### 4.2 Proof of theorem 1

In this section we prove theorem 1 by contradiction. Fix arbitrary proper subset \(L\) of \(\{1, \ldots, M\}\). It suffices to prove that 3 holds for all \(g \in G_{\mathcal{X}}\). Suppose that 3 does not hold for some \(G_{\mathcal{X}}\). Then for any \(\lambda, \kappa > 0\), there exists \(g \in G_{\mathcal{X}}\) such that
\[
E_0[\log \{g(x) + \kappa\}] + \lambda \geq E_0[\log f(x; \theta_0)].
\]

Here, let \(\{\lambda_j\}, \{\kappa_j\}\) be positive sequences which decrease to zero. Then for each \(\lambda_j, \kappa_j > 0\), there exists \(g_j \in G_{\mathcal{X}}\) such that
\[
E_0[\log \{g_j(x) + \kappa_j\}] + \lambda_j \geq E_0[\log f(x; \theta_0)].
\]
It follows that

$$\liminf_{j \to \infty} E_0[\log \{g_j(x) + \kappa_j\}] + \lambda_j \geq E_0[\log f(x; \theta_0)]$$

(15)

Now $g_j$ can be written as

$$g_j(x) = f_{\mathcal{L}}(x; \theta^{(j)}_{\mathcal{L}}).$$

Then the following lemma holds by compactification argument.

**Lemma 7.** There exists a subsequence of $\{\theta^{(j)}_{\mathcal{L}}\}_{j=1}^\infty \equiv \{\{\alpha_m^{(j)}, \mu_m^{(j)}, \sigma_m^{(j)} \mid m \in \mathcal{L}\}\}_{j=1}^\infty$ and disjoint subsets $\mathcal{K}_{\sigma,0}, \mathcal{K}_{\sigma,\infty}, \mathcal{K}_{\mu,\infty} \subseteq \mathcal{L}$ such that along the subsequence:

- $\sigma_m^{(j)} \to 0$ for $m \in \mathcal{K}_{\sigma,0},$
- $\sigma_m^{(j)} \to \infty$ for $m \in \mathcal{K}_{\sigma,\infty},$
- $\sigma_m^{(j)}$ converges to a finite value and $|\mu_m^{(j)}| \to \infty$ for $m \in \mathcal{K}_{\mu,\infty},$
- $(\alpha_m^{(j)}, \mu_m^{(j)}, \sigma_m^{(j)})$ converges to a finite point $(\alpha_m^{(\infty)}, \mu_m^{(\infty)}, \sigma_m^{(\infty)})$ for $m \in \mathcal{K}_{R},$

where $\mathcal{K}_R \equiv \mathcal{L} \setminus \{\mathcal{K}_{\sigma,0} \cup \mathcal{K}_{\sigma,\infty} \cup \mathcal{K}_{\mu,\infty}\}.$

**Proof:** Let

$$\mu'_m^{(j)} \equiv \arctan (\mu_m^{(j)}), \quad \sigma'_m^{(j)} \equiv \arctan (\sigma_m^{(j)}).$$

Then $\{(\alpha_m^{(j)}, \mu'_m^{(j)}, \sigma'_m^{(j)} \mid m \in \mathcal{L}\}\}_{j=1}^\infty$ is regarded as a sequence in the following compact set.

$$0 \leq \alpha_m \leq 1, \quad \sum_{m=1}^L \alpha_m \leq 1, \quad -\frac{\pi}{2} \leq \mu'_m \leq \frac{\pi}{2}, \quad 0 \leq \sigma'_m \leq \frac{\pi}{2}.$$  

(16)

Therefore there exists a subsequence of $\{(\alpha_m^{(j)}, \mu'_m^{(j)}, \sigma'_m^{(j)} \mid m \in \mathcal{L}\}_{j=1}^\infty$ that converges to a point in the set (16). Now we sort the elements in $\mathcal{L}$ according to their behaviors in this subsequence. First, we choose components such that $\sigma_m^{(j)} \to 0$, and add these $m$ to $\mathcal{K}_{\sigma,0}.$

Second, from the remainder, we choose components such that $\sigma_m^{(j)} \to \infty$, and add these $m$ to $\mathcal{K}_{\sigma,\infty}.$

Third, from the remainder, we choose components such that $|\mu_m^{(j)}| \to \infty$ and add these $m$ to $\mathcal{K}_{\mu,\infty}.$

Finally, we choose the remaining components as $\mathcal{K}_R.$ \hfill \Box

From lemma 7 we define $g_{\infty}$ as follows.

$$g_{\infty}(x) \equiv \sum_{m \in \mathcal{K}_R} \alpha_m^{(\infty)} f_m(x; \mu_m^{(\infty)}, \sigma_m^{(\infty)}) \in \mathcal{G}_{\mathcal{K}_R}.$$

For notational simplicity and without loss of generality, we replace the original sequence with this subsequence, because (15) holds for this subsequence as well. Furthermore, by considering the sequence $\{\theta^{(j)}_{\mathcal{L}}\}_{j=j_0}$ where $j_0$ is sufficiently large and replacing $j$ by $j - j_0$
if necessary, we can assume without loss of generality that there exist sufficiently small real constants \( \kappa_0 > 0 \) and \( c_0 > 0 \) such that

\[
E_0[\log f(x; \theta_0)] - E_0[\log \{ g_\infty(x) + 3\kappa_0 \}] > 0 \ , \ \kappa_0 < \frac{v_0}{c_0(M + 1)} ,
\]

\[
\sigma_m^{(j)} < c_0 \ (m \in \mathcal{K}_\sigma) , \ \sigma_m^{(j)} > \frac{v_0}{\kappa_0} \ (m \in \mathcal{K}_\sigma \cap \infty) ,
\]

\[c_0 \leq \sigma_m^{(j)} \leq \frac{v_0}{\kappa_0} \ (m \in \mathcal{K}_{|\mu| \cap \infty}) \text{ for all } j . \tag{17}
\]

From lemma 1 and 2 we have

\[
E_0 \left[ \log \left\{ f_{\mathcal{L}}(x; \theta_{\mathcal{L}}^{(j)}) + \kappa_j \right\} + \lambda_j \right]
\]

\[
\leq \int 1_J(\theta_{\mathcal{L}}^{(j)})(x) \cdot \log \left\{ \sum_{t=1}^{T(\theta_{\mathcal{L}}^{(j)})} H(J_t(\theta_{\mathcal{L}}^{(j)})) \cdot 1_J(\theta_{\mathcal{L}}^{(j)})(x) + \kappa_j \right\} f(x; \theta_0) dx
\]

\[
+ \int 1_{\mathbb{R} \setminus J}(\theta_{\mathcal{L}}^{(j)})(x) \cdot \log \left\{ f_{\mathcal{K}_0}(x; \theta_{\mathcal{K}_0}^{(j)}) + \kappa_0 + \kappa_j \right\} f(x; \theta_0) dx + \lambda_j ,
\]

(18)

where \( \mathcal{K}_\sigma \equiv \mathcal{L} \setminus \mathcal{K}_\sigma \).

Now we evaluate the first term on the right-hand side of (18). From lemma 3

\[
\int 1_J(\theta_{\mathcal{L}}^{(j)})(x) \cdot \log \left\{ \sum_{t=1}^{T(\theta_{\mathcal{L}}^{(j)})} H(J_t(\theta_{\mathcal{L}}^{(j)})) \cdot 1_J(\theta_{\mathcal{L}}^{(j)})(x) + \kappa_j \right\} f(x; \theta_0) dx
\]

\[
\leq \sum_{t=1}^{T(\theta_{\mathcal{L}}^{(j)})} W(J_t(\theta_{\mathcal{L}}^{(j)})) \cdot \log \left\{ H(J_t(\theta_{\mathcal{L}}^{(j)})) + \kappa_j \right\} \cdot u_0 \rightarrow 0 \ , \ (n \rightarrow \infty) , \tag{19}
\]

where \( u_0 = \sup_x f(x; \theta_0) \) defined in (13). Next we evaluate the second term on the right-hand side of (18). Let

\[
A^{(j)} \equiv \min_{m \in \mathcal{K}_\mu \cap \infty} \left\{ \min \left\{ |\mu_m^{(j)} + \nu(\sigma_m^{(j)})|, |\mu_m^{(j)} - \nu(\sigma_m^{(j)})| \right\} \right\} . \tag{20}
\]

Then

\[
f_{\mathcal{K}_\sigma \cap \infty}(x; \theta_{\mathcal{K}_\sigma \cap \infty}^{(j)}) = f_{\mathcal{K}_\sigma \cap \infty}(x; \theta_{\mathcal{K}_\sigma \cap \infty}^{(j)}) + f_{\mathcal{K}_\mu \cap \infty}(x; \theta^{(j)}_{\mathcal{K}_\mu \cap \infty}) + f_{\mathcal{R}}(x; \theta_{\mathcal{R}}^{(j)}),
\]

\[
f_{\mathcal{K}_\sigma \cap \infty}(x; \theta_{\mathcal{K}_\sigma \cap \infty}^{(j)}) \leq \kappa_0 \quad \text{for all } x ,
\]

\[
f_{\mathcal{K}_\sigma \cap \infty}(x; \theta_{\mathcal{K}_\sigma \cap \infty}^{(j)}) + f_{\mathcal{K}_\mu \cap \infty}(x; \theta_{\mathcal{K}_\mu \cap \infty}^{(j)}) \leq \kappa_0 \quad \text{for } x \in [-A^{(j)}, A^{(j)}) \setminus J(\theta_{\mathcal{L}}^{(j)}) .
\]

13
Therefore the following inequality holds.

\[
\int_{\mathbb{R} \setminus J(\theta^*_j)} (x) \cdot \log \left\{ f_{\mathcal{X}_{\theta_0}}(x; \theta^*_j) + \kappa_0 + \kappa_j \right\} f(x; \theta_0) \, dx \leq \int_{[-A(j), A(j)] \setminus J(\theta^*_j)} (x) \cdot \log \left\{ f_{\mathcal{X}_R}(x; \theta^*_j) + 2\kappa_0 + \kappa_j \right\} f(x; \theta_0) \, dx \\
+ \int_{(\mathbb{R} \setminus [-A(j), A(j)]) \setminus J(\theta^*_j)} (x) \cdot \log \left\{ f_{\mathcal{X}_{|\mu|\uparrow \infty}}(x; \theta^*_j) \cdot \log \left\{ f_{\mathcal{X}_{\mathcal{X}_{|\mu|\uparrow \infty}}}(x; \theta^*_j \mathcal{X}_{|\mu|\uparrow \infty}) + 2\kappa_0 + \kappa_j \right\} f(x; \theta_0) \, dx
\]

\[
= I^{(j)}_1 + I^{(j)}_2 \quad (\text{say}). \quad (21)
\]

By the bounded convergence theorem, we obtain

\[
I^{(j)}_1 \to \int \log \left\{ g_\infty(x) + 2\kappa_0 \right\} f(x; \theta_0) \, dx , \quad I^{(j)}_2 \to 0. \quad (22)
\]

From (18), (19), (21), (22), we have

\[
E_0[\log f(x; \theta_0)] \leq \limsup_{j \to \infty} E_0[\log \{g_j(x) + \kappa_j\}] + \lambda_j \leq E_0 \left[ \log \left\{ g_\infty(x) + 2\kappa_0 \right\} \right].
\]

This is a contradiction to (17). This completes the proof of theorem \( \Box \).

4.3 Proof of the main theorem

We choose real constants \( \kappa \) and \( \lambda \) to satisfy (6) by using theorem \( \Box \). Having chosen these constants, from now on we follow the line of the proof in [Tanaka and Takemura (2005)], although the details of the proof here is much more complicated. For the sake of readability we divide our proof into further sections.

4.3.1 Setting up constants

For \( \kappa, \lambda \) satisfying (6), let \( \kappa_0, \lambda_0 \) be real constants such that

\[
0 < 4\kappa_0 \leq \kappa , \quad 0 < 4\lambda_0 \leq \lambda .
\]

Note that \( 4\kappa_0, 4\lambda_0 \) also satisfy (6). Define

\[
B \equiv \frac{\lambda_0}{\kappa_0} > \max \{ \sigma_{01}, \ldots, \sigma_{0M} \}. \quad (23)
\]

If \( \sigma_m \geq B \), then the density of the \( m \)-th component is almost flat and makes little contribution to the likelihood. In section 4.3.2 we partition the parameter space according to this property.
Because \( \{c_n\} \) is decreasing to zero, by replacing \( c_0 \) by some \( c_n \) if necessary, we can assume without loss of generality that \( c_0 \) is sufficiently small to satisfy the following conditions,

\[
\begin{align*}
(v_0/c_0)^\beta &> e, \\
c_0 &< \min\{\sigma_{01}, \ldots, \sigma_{0M}\}, \\
3M \cdot u_0 \cdot 2 \nu(c_0) \cdot |\log \kappa_0| &< \lambda_0, \\
3 \cdot 2M \cdot u_0 \cdot \xi(v_0/c_0) \cdot \log(v_0/c_0) &< \lambda_0, \\
\kappa_0 &< \frac{v_0}{c_0(M + 1)}
\end{align*}
\]

(24)

where \( \tilde{\beta}, \nu(\cdot) \) and \( \xi(\cdot) \) are defined in (7) and (12).

For any subset \( V \subset \mathbb{R} \), let \( P_0(V) \) denote the probability of \( V \) under the true density

\[
P_0(V) \equiv \int_V f(x; \theta_0)dx.
\]

(25)

Let \( A_0 > 0 \) be a positive constant which satisfies

\[
P_0(\mathcal{A}_0) \cdot \log \left( \frac{v_0/c_0 + 2\kappa_0}{3\kappa_0} \right) < \lambda_0,
\]

(26)

where

\[
\mathcal{A}_0 \equiv (-\infty, -A_0] \cup [A_0, \infty).
\]

(27)

Let \( A_n \equiv A_0 \cdot n^{\frac{2+\xi}{\beta-1}} \) as in lemma 5. Define a subset \( \Theta'_n \) of \( \Theta_n \) in theorem 2 by

\[
\Theta'_n \equiv \{ \theta \in \Theta_n \mid \exists m \text{ s.t. } c_n \leq \sigma_m \leq c_0 \text{ or } |\mu_m| > A_0 \} \subset \Theta_n,
\]

and let

\[
\Gamma_0 \equiv \{ \theta \in \Theta \mid c_0 \leq \sigma_m \leq B, |\mu_m| \leq A_0, (m = 1, \ldots, M) \}.
\]

Note that \( \Theta_0 \subset \Gamma_0 \), where \( \Theta_0 \) is the set of true parameters.

### 4.3.2 Partitioning the parameter space

In view of theorems in Wald (1949), Redner (1981), for the strong consistency of MLE on \( \Theta_n \) under assumption 1, 2, 3 and 4, it suffices to prove that

\[
\lim_{n \to \infty} \sup_{\theta \in \Gamma \cup \Theta'_n} \prod_{i=1}^n f(x_i; \theta) / \prod_{i=1}^n f(x_i; \theta_0) = 0, \quad \text{a.e.}
\]

for all closed \( \Gamma \subset \Gamma_0 \) not intersecting \( \Theta_0 \). Note that for all \( \Gamma \) and \( \{x_i\}_{i=1}^n \),

\[
\sup_{\theta \in \Gamma \cup \Theta'_n} \prod_{i=1}^n f(x_i; \theta) = \max \left\{ \sup_{\theta \in \Gamma} \prod_{i=1}^n f(x_i; \theta), \sup_{\theta \in \Theta'_n} \prod_{i=1}^n f(x_i; \theta) \right\}.
\]
Furthermore
\[ \lim_{n \to \infty} \sup_{\theta \in \Gamma} \prod_{i=1}^{n} f(x_i; \theta) \prod_{i=1}^{n} f(x_i; \theta_0) = 0, \quad a.e. \]
holds by theorems in [Wald 1949], [Redner 1981]. Therefore it suffices to prove
\[ \lim_{n \to \infty} \sup_{\theta \in \Theta'_n} \prod_{i=1}^{n} f(x_i; \theta) \prod_{i=1}^{n} f(x_i; \theta_0) = 0, \quad a.e. \]

Note that in the argument above the supremum of the likelihood function over \( \Gamma \cup \Theta'_n \) is considered separately for \( \Gamma \) and \( \Theta'_n \). \( \Gamma \) and \( \Theta'_n \) form a covering of \( \Gamma \cup \Theta'_n \). In our proof, we consider finer and finer finite coverings of \( \Theta'_n \). As above, it suffices to prove that the ratio of the supremum of the likelihood over each member of the covering to the likelihood at \( \theta_0 \) converges to zero almost everywhere.

Let \( \theta \in \Theta'_n \). Let \( \mathcal{K}_{\sigma \leq c_0}, \mathcal{K}_{\sigma \geq B}, \mathcal{K}_{|\mu| \geq A_0} \) represent disjoint subsets of \( \{1, \ldots, M\} \) and define
\[ \mathcal{K}_R \equiv \{1, \ldots, M\} \setminus \{\mathcal{K}_{\sigma \leq c_0} \cup \mathcal{K}_{\sigma \geq B} \cup \mathcal{K}_{|\mu| \geq A_0}\}. \]
For any given \( \mathcal{K}_{\sigma \leq c_0}, \mathcal{K}_{\sigma \geq B}, \mathcal{K}_{|\mu| \geq A_0} \), we define a subset of \( \Theta'_n \) by
\[ \Theta'_{n,\mathcal{K}} \equiv \{\theta \in \Theta'_n \mid \sigma_m \leq c_0, (m \in \mathcal{K}_{\sigma \leq c_0}); \sigma_m \geq B, (m \in \mathcal{K}_{\sigma \geq B}); \]
\[ c_0 < \sigma_m < B, |\mu_m| \geq A_0, (m \in \mathcal{K}_{|\mu| \geq A_0}); \]
\[ c_0 < \sigma_m < B, |\mu_m| < A_0, (m \in \mathcal{K}_R). \]
(28)

As above, it suffices to prove that for each choice of disjoint subsets \( \mathcal{K}_{\sigma \leq c_0}, \mathcal{K}_{\sigma \geq B}, \mathcal{K}_{|\mu| \geq A_0} \),
\[ \lim_{n \to \infty} \sup_{\theta \in \Theta'_{n,\mathcal{K}}} \prod_{i=1}^{n} f(x_i; \theta) \prod_{i=1}^{n} f(x_i; \theta_0) = 0, \quad a.e. \]
We fix \( \mathcal{K}_{\sigma \leq c_0}, \mathcal{K}_{\sigma \geq B}, \mathcal{K}_{|\mu| \geq A_0} \), from now on.

Next we consider coverings of \( \Theta_{\mathcal{K}_R} \). Recall that \( \Theta_{\mathcal{K}}, f_{\mathcal{K}}(x; \theta_{\mathcal{K}}) \) and \( f_{\mathcal{K}}(x; \theta_{\mathcal{K}}, \rho) \) are defined in (2), (3) and (12). The following lemma follows from lemma 6 and compactness of \( \Theta_{\mathcal{K}_R} \).

**Lemma 8.** Let \( \mathcal{B}(\theta, \rho(\theta)) \) denote the open ball with center \( \theta \) and radius \( \rho(\theta) \). Then \( \Theta_{\mathcal{K}_R} \) can be covered by a finite number of balls \( \mathcal{B}(\theta^{(1)}_{\mathcal{K}_R}, \rho(\theta^{(1)}_{\mathcal{K}_R})), \ldots, \mathcal{B}(\theta^{(S)}_{\mathcal{K}_R}, \rho(\theta^{(S)}_{\mathcal{K}_R})) \) such that
\[ E_0 [\log \{ f_{\mathcal{K}_R}(x; \theta^{(s)}_{\mathcal{K}_R}, \rho(\theta^{(s)}_{\mathcal{K}_R})) + \kappa_0 \}] + \lambda_0 < E_0 [\log f(x; \theta_0)] \quad (s = 1, \ldots, S). \]

**Proof:** From lemma 6 we have
\[ \lim_{\rho \to 0} E_0 [\log \{ f_{\mathcal{K}_R}(x; \theta_{\mathcal{K}_R}, \rho) + \kappa_0 \}] = E_0 [\log \{ f_{\mathcal{K}_R}(x; \theta_{\mathcal{K}_R}) + \kappa_0 \}]. \]
For each \( \theta_{\mathcal{K}_R} \in \Theta_{\mathcal{K}_R} \)
\[ E_0 [\log \{ f_{\mathcal{K}_R}(x; \theta_{\mathcal{K}_R}) + \kappa_0 \}] + \lambda_0 < E_0 [\log f(x; \theta_0)] \]
holds. Therefore for each \( \theta_{K^R} \in \bar{\Theta}_{K^R} \), there exists a radius \( \rho(\theta_{K^R}) > 0 \) such that

\[
E_0[\log \{ f_{K^R}(x; \theta_{K^R}, \rho(\theta_{K^R})) + \kappa_0 \}] + \lambda_0 < E_0[\log f(x; \theta_0)].
\]

Since

\[
\bar{\Theta}_{K^R} \subset \bigcup_{\theta_{K^R} \in \bar{\Theta}_{K^R}} B(\theta_{K^R}, \rho(\theta_{K^R}))
\]

and the compactness of \( \bar{\Theta}_{K^R} \), there exists a finite number of balls \( B(\theta_{K^R}^{(1)}, \rho(\theta_{K^R}^{(1)})) \), \ldots, \( B(\theta_{K^R}^{(S)}, \rho(\theta_{K^R}^{(S)})) \) which cover \( \bar{\Theta}_{K^R} \).

Based on lemma 8, we partition \( \Theta'_{n, \mathcal{X}} \). Recall that we denote by \( \theta_{\mathcal{X}} \) the subvector of \( \theta \in \Theta \) consisting of the components in \( \mathcal{X} \). Define a subset of \( \Theta'_{n, \mathcal{X}} \) by

\[
\Theta'_{n, \mathcal{X}, s} \equiv \{ \theta \in \Theta'_{n, \mathcal{X}} \mid \theta_{K^R} \in B(\theta_{K^R}^{(s)}, \rho(\theta_{K^R}^{(s)})) \}.
\] (29)

Then \( \Theta'_{n, \mathcal{X}} \) is covered by \( \Theta'_{n, \mathcal{X}, 1}, \ldots, \Theta'_{n, \mathcal{X}, S} \):

\[
\Theta'_{n, \mathcal{X}} = \bigcup_{s=1}^{S} \Theta'_{n, \mathcal{X}, s}.
\]

Again it suffices to prove that for each choice of \( \mathcal{X}_{\sigma \leq c_0}, \mathcal{X}_{\sigma \geq B}, \mathcal{X}_{|\mu| \geq A_0}, s \)

\[
\lim_{n \to \infty} \sup_{\theta \in \Theta'_{n, \mathcal{X}, s}} \frac{\prod_{i=1}^{n} f(x_i; \theta)}{\prod_{i=1}^{n} f(x_i; \theta_0)} = 0, \ a.e.
\] (30)

We fix \( \mathcal{X}_{\sigma \leq c_0}, \mathcal{X}_{\sigma \geq B}, \mathcal{X}_{|\mu| \geq A_0} \) and \( s \) from now on. Because

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log f(x_i; \theta_0) = E_0[\log f(x; \theta_0)], \ a.e.
\]

(30) is implied by

\[
\lim_{n \to \infty} \sup_{\theta \in \Theta'_{n, \mathcal{X}, s}} \sum_{i=1}^{n} \log f(x_i; \theta) < E_0[\log f(x; \theta_0)], \ a.e.
\] (31)

Therefore it suffices to prove (31), which is a new intermediate goal of our proof hereafter.

### 4.3.3 Bounding the likelihood by four terms

In this section we bound the likelihood function by four terms depending on the positions of the observations \( x_1, \ldots, x_n \). Let \( R_n(V) \) denote the number of observations which belong to a set \( V \subset \mathbb{R} \).
Lemma 9. For \( \theta \in \Theta'_{n, \mathcal{X}_s} \)

\[
\frac{1}{n} \sum_{i=1}^{n} \log f(x_i; \theta) \leq \frac{1}{n} \sum_{i=1}^{n} \log \{f_{X,R}(x_i; \theta, \rho(\theta, X_R)) + 3\kappa_0 \}
+ \frac{1}{n} R_n(\mathcal{X}_0) \cdot \log \left( \frac{M v_0/c_0 + 2\kappa_0}{3\kappa_0} \right)
+ \frac{1}{n} R_n(J(\theta)) \cdot (-\log \kappa_0) + \frac{1}{n} \sum_{x_i \in J(\theta)} \log f(x_i; \theta) .
\] (32)

Proof: Let \( \mathcal{X}_{\sigma > c_0} = \{1, \ldots, M\} \setminus \mathcal{X}_{\sigma \leq c_0} \) and \( \mathcal{X}_{c_0 < \sigma < B} = \{1, \ldots, M\} \setminus (\mathcal{X}_{\sigma \leq c_0} \cup \mathcal{X}_{\sigma \geq B}) \).

For \( x \notin J(\theta) \), \( f(x; \theta) \leq f_{X,Y > c_0}(x; \theta, X_{\sigma > c_0}) + \kappa_0 \) holds. Therefore

\[
\frac{1}{n} \sum_{i=1}^{n} \log f(x_i; \theta) \leq \frac{1}{n} \sum_{x_i \notin J(\theta)} \log f(x_i; \theta) + \frac{1}{n} \sum_{x_i \in J(\theta)} \log \{f_{X,Y > c_0}(x_i; \theta, X_{\sigma > c_0}) + \kappa_0 \}
= \frac{1}{n} \sum_{i=1}^{n} \log \{f_{X,Y > c_0}(x_i; \theta, X_{\sigma > c_0}) + \kappa_0 \}
+ \frac{1}{n} \sum_{x_i \in J(\theta)} \left[ \log f(x_i; \theta) - \log \{f_{X,Y > c_0}(x_i; \theta, X_{\sigma > c_0}) + \kappa_0 \} \right].
\] (33)

Consider the second term on the right-hand side. We have

\[
\frac{1}{n} \sum_{x_i \in J(\theta)} \left[ \log f(x_i; \theta) - \log \{f_{X,Y > c_0}(x_i; \theta, X_{\sigma > c_0}) + \kappa_0 \} \right]
\leq \frac{1}{n} \sum_{x_i \in J(\theta)} \log f(x_i; \theta) - \frac{1}{n} R_n(J(\theta)) \cdot \log \kappa_0 .
\]

This takes care of the third and the fourth term of (32).

Now consider the first term on the right-hand side of (33). Note that

\[
\frac{1}{n} \sum_{i=1}^{n} \log \{f_{X,Y > c_0}(x_i; \theta, X_{\sigma > c_0}) + \kappa_0 \} \leq \frac{1}{n} \sum_{i=1}^{n} \log \{f_{X,Y > c_0 < B}(x_i; \theta, X_{\sigma < c_0 < B}) + 2\kappa_0 \}
\]

For \( x \notin \mathcal{X}_0 \)

\[
f_{X,Y > \mu > A_0}(x; \theta, X_{\mu > A_0}) \leq \kappa_0 .
\]
Therefore we obtain

\[
\frac{1}{n} \sum_{i=1}^{n} \log \left\{ f_{\mathcal{X}_{c_0} < \sigma < B}(x_i; \theta_{\mathcal{X}_{c_0} < \sigma < B}) + 2\kappa_0 \right\} \\
= \frac{1}{n} \sum_{x_i \notin A_0} \log \left\{ f_{\mathcal{X}_{c_0} < \sigma < B}(x_i; \theta_{\mathcal{X}_{c_0} < \sigma < B}) + 2\kappa_0 \right\} + \frac{1}{n} \sum_{x_i \in A_0} \log \left\{ f_{\mathcal{X}_{c_0} < \sigma < B}(x_i; \theta_{\mathcal{X}_{c_0} < \sigma < B}) + 2\kappa_0 \right\} \\
\leq \frac{1}{n} \sum_{x_i \notin A_0} \log \left\{ f_{\mathcal{X}_{R}}(x_i; \theta_{\mathcal{X}_{R}}) + 3\kappa_0 \right\} + \frac{1}{n} \sum_{x_i \in A_0} \log \left\{ f_{\mathcal{X}_{c_0} < \sigma < B}(x_i; \theta_{\mathcal{X}_{c_0} < \sigma < B}) + 2\kappa_0 \right\} \\
= \frac{1}{n} \sum_{i=1}^{n} \log \left\{ f_{\mathcal{X}_{R}}(x_i; \theta_{\mathcal{X}_{R}}) + 3\kappa_0 \right\} \\
\quad \quad + \frac{1}{n} \sum_{x_i \in A_0} \left[ \log \left\{ f_{\mathcal{X}_{c_0} < \sigma < B}(x_i; \theta_{\mathcal{X}_{c_0} < \sigma < B}) + 2\kappa_0 \right\} - \log \left\{ f_{\mathcal{X}_{R}}(x_i; \theta_{\mathcal{X}_{R}}) + 3\kappa_0 \right\} \right] \\
\leq \frac{1}{n} \sum_{i=1}^{n} \log \left\{ f_{\mathcal{X}_{R}}(x_i; \theta_{\mathcal{X}_{R}}) + 3\kappa_0 \right\} + \frac{1}{n} R_n(A_0) \cdot \log \left( \frac{v_0/c_0 + 2\kappa_0}{3\kappa_0} \right).
\]

(34)

Note that \( f_{\mathcal{X}_{c_0} < \sigma < B}(x; \theta_{\mathcal{X}_{c_0} < \sigma < B}) \leq v_0/c_0 \) from lemma 1. Therefore

The r.h.s of (34)

\[
\leq \frac{1}{n} \sum_{i=1}^{n} \log \left\{ f_{\mathcal{X}_{R}}(x_i; \theta_{\mathcal{X}_{R}}) + 3\kappa_0 \right\} + \frac{1}{n} \sum_{x_i \in A_0} \left[ \log \left\{ v_0/c_0 + 2\kappa_0 \right\} - \log 3\kappa_0 \right]
\]

\[
\leq \frac{1}{n} \sum_{i=1}^{n} \log \left\{ f_{\mathcal{X}_{R}}(x_i; \theta_{\mathcal{X}_{R}}) + 3\kappa_0 \right\} + \frac{1}{n} R_n(A_0) \cdot \log \left( \frac{v_0/c_0 + 2\kappa_0}{3\kappa_0} \right).
\]

This takes care of the first and the second term of (32).

From lemma 8 and the strong law of large numbers the first term on the right hand side of (32) converges to the expectation of a density which has less than \( M \) components and the expectation is less than that of the true density by theorem 1. The second term converges to a small value because the relative frequency on \( \mathcal{A}_0 \) is very small. The third term also converges to small value because the relative frequency on \( J(\theta) \) is very small. The fourth term is somewhat complicated. The component in \( \mathcal{X}_{\sigma \leq c_0} \) may have high peaks. However the widths of the peaks are very narrow and the relative frequency on the interval is very small. Hence the fourth term makes little contribution to the likelihood. Therefore the mean log likelihood (the left hand side of (32)) converges to a value which is less than that of the true density. In the following we consider the details.

The first term and the second term are easy.

The first term: By lemma 8 and the strong law of large numbers we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log \left\{ f_{\mathcal{X}_{R}}(x_i; \theta_{\mathcal{X}_{R}}, \rho(\theta_{\mathcal{X}_{R}})) + 4\kappa_0 \right\} < E_0[\log f(x; \theta_0)] - 4\lambda_0, \quad a.e.
\]

(35)
The second term: By (26) and the strong law of large numbers we have

\[
\lim_{n \to \infty} \frac{1}{n} R_n(\mathcal{A}_0) \cdot \log \left( \frac{v_0/c_0 + 2\kappa_0}{3\kappa_0} \right) < \lambda_0, \quad \text{a.e.}
\]  

(36)

Note that we have \(-4\lambda_0\) from the first term and \(\lambda_0\) from the second term. In the rest of our proof we show that both the third term and the fourth term can be bounded by \(\lambda_0\).

4.3.4 Bounding the third term

The third term can be bounded by dividing the interval \([-A_n, A_n]\) into short intervals of length \(2\nu(c_0)\).

Lemma 10.

\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta_n} \frac{1}{n} R_n(J(\theta)) \leq 3M \cdot u_0 \cdot 2\nu(c_0), \quad \text{a.e.}
\]

Proof: Let \(\epsilon > 0\) be arbitrarily fixed and let \(J_0^{(n)} \equiv [-A_n, A_n]\). We divide \(J_0^{(n)}\) from \(-A_n\) to \(A_n\) by short intervals of length \(2\nu(c_0)\). In right end of the intervals of \(J_0^{(n)}\), overlap of two short intervals of length \(2\nu(c_0)\) is allowed and the right end of a short interval coincides with the right end of \(J_0^{(n)}\). See Figure 3. Let \(k_n(c_0)\) be the number of short intervals and let \(I_1^{(n)}(c_0), \ldots, I_{k_n(c_0)}^{(n)}(c_0)\) be the divided short intervals. Then we have

\[
k_n(c_0) \leq \frac{2A_n}{2\nu(c_0)} + 1 = \frac{A_n}{\nu(c_0)} + 1 = \frac{A_0 \cdot n^{\beta - 1}}{\nu(c_0)} + 1.
\]

(37)

Note that any interval in \(J_0^{(n)}\) of length \(2\nu(c_0)\) is covered by at most 3 small intervals from \(\{I_1^{(n)}(c_0), \ldots, I_{k_n(c_0)}^{(n)}(c_0)\}\). Now consider \(J(\theta) = \bigcup_{m=1}^{K} [\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)]\). Since
\( [\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)] \), \((m = 1, \ldots, K)\), are intervals of length less than or equal to \(2\nu(c_0)\). \(J(\theta)\) is covered by at most 3 short intervals. Then
\[
\sup_{\theta \in \Theta_n} \frac{1}{n} R_n(J(\theta)) - 3M \cdot u_0 \cdot 2\nu(c_0) > \epsilon
\]
\[
= \left\{ x_{n,1} < -A_n \text{ or } x_{n,n} > A_n \right\}
\]
\[
\text{or}
\left\{ 1 \leq \exists k \leq k_n(c_0), \frac{1}{n} R_n(I_k(c_0)) - u_0 \cdot 2\nu(c_0) > \frac{\epsilon}{3M} \right\}.
\]
By lemma 5, \(\sum_n \text{Prob}(x_{n,1} < -A_n \text{ or } x_{n,n} > A_n) < \infty\) and the first event on the right-hand side of (38) can be ignored. We only need to consider the second event. We will use the same logic in the proofs of lemmas 11 and 12 below. Then
\[
\text{Prob}
\left( \sup_{\theta \in \Theta_n} \frac{1}{n} R_n(J(\theta)) - 3M \cdot u_0 \cdot 2\nu(c_0) > \epsilon \right)
\]
\[
\leq \sum_{k=1}^{k_n(c_0)} \text{Prob}
\left( \frac{1}{n} R_n(I_k(c_0)) - u_0 \cdot 2\nu(c_0) > \frac{\epsilon}{3M} \right).
\]
Recall that, for any set \(V \subset \mathbb{R}\), we denote by \(P_0(V)\) the probability of \(V\) under the true density in (25) and denote by \(R_n(V)\) the number of observations which belong to \(V\) as in lemma 9. Since
\[
P_0(I_k(c_0)) \leq u_0 \cdot 2\nu(c_0), \quad (k = 1, \ldots, k_n(\theta)),
\]
\(R_n(V) \sim \text{Bin}(n, P_0(V))\) and from Okamoto’s inequality (Okamoto (1958)), we obtain
\[
\text{Prob}
\left( \frac{1}{n} R_n(I_k(c_0)) - u_0 \cdot 2\nu(c_0) > \frac{\epsilon}{3M} \right)
\]
\[
\leq \text{Prob}
\left( \frac{1}{n} R_n(I_k(c_0)) - P_0(I_k(c_0)) > \frac{\epsilon}{3M} \right)
\]
\[
\leq \exp\left( -\frac{2n\epsilon^2}{9M^2} \right).
\]
Therefore from (37)
\[
\text{Prob}
\left( \sup_{\theta \in \Theta_n} \frac{1}{n} R_n(J(\theta)) - 3M \cdot u_0 \cdot 2\nu(c_0) > \epsilon \right)
\]
\[
\leq \left( \frac{A_0 \cdot n^{2+\epsilon}}{\nu(c_0)} + 1 \right) \cdot \exp\left( -\frac{2n\epsilon^2}{9M^2} \right).
\]
When we sum this over \(n\), the resulting series on the right converges. Hence by the Borel-Cantelli lemma, we have
\[
\text{Prob}
\left( \sup_{\theta \in \Theta_n} \frac{1}{n} R_n(J(\theta)) - 3M \cdot u_0 \cdot 2\nu(c_0) > \epsilon \right. \quad \text{i.o.}
\]
\[
= 0.
\]
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Because $\epsilon > 0$ was arbitrary, we obtain
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta_{n, x, s}} \frac{1}{n} R_n(J(\theta)) \leq 3M \cdot u_0 \cdot 2\nu(c_0), \ a.e.
\]

By this lemma and (24) we have
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta_{n, x, s}} \frac{1}{n} R_n(J(\theta)) \cdot \left( -\log \kappa_0 \right) \leq 3M \cdot u_0 \cdot 2\nu(c_0) \cdot |\log \kappa_0| < \lambda_0 \ a.e. \quad (39)
\]
This bounds the third term on the right-hand side of (32) from above.

### 4.3.5 Bonding the fourth term

Finally we bound the fourth term on the right-hand side of (32) from above. From lemma 2 we have
\[
1_{J(\theta)}(x) \cdot \sum_{m=1}^{M} \alpha_m f_m(x; \mu_m, \sigma_m) \leq \sum_{t=1}^{T(\theta)} H(J_t(\theta)) \cdot 1_{J_t(\theta)}(x) \quad (x \in J(\theta)) \quad (40)
\]
We now classify the intervals $J_t(\theta)$, $t = 1, \ldots, T(\theta)$, by the height $H(J_t(\theta))$. Let
\[
c_n' \equiv c_0 \cdot \exp\left(-n^{1/4}\right) \quad (41)
\]
and define $\tau_n(\theta)$ and $\tau_n'(\theta)$ by
\[
\tau_n(\theta) \equiv \left\{ t \in \{1, \ldots, T(\theta)\} \mid H(J_t(\theta)) \leq Mv_0/c_n' \right\}, \ \tau_n'(\theta) \equiv \left\{1, \ldots, T(\theta)\right\} \setminus \tau_n(\theta). \quad (42)
\]

See Figure 4

Now suppose that the following inequality holds.
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta_{n, x, s}} \left[ \sum_{t=1}^{T(\theta)} \frac{1}{n} R_n(J_t(\theta)) \log H(J_t(\theta)) \right.
\]
\[
- 3 \left\{ \sum_{t \in \tau_n(\theta)} u_0 \cdot \xi(H(J_t(\theta))) \cdot \log H(J_t(\theta)) + \sum_{t \in \tau_n'(\theta)} \frac{2}{n} \log H(J_t(\theta)) \right\} \leq 0, \ a.e. \quad (43)
\]

From (24), and noting that $\log y/y^{3}$ is decreasing in $y^{3} \geq e$, we have
\[
3 \cdot \sum_{t \in \tau_n(\theta)} u_0 \cdot \xi(H(J_t(\theta))) \cdot \log H(J_t(\theta)) \leq 3 \cdot 2M \cdot u_0 \cdot \xi(v_0/c_0) \cdot \log(v_0/c_0) < \lambda_0,
\]
\[
3 \cdot \sum_{t \in \tau_n'(\theta)} \frac{2}{n} \log H(J_t(\theta)) \leq 3 \cdot 2M \cdot \frac{2}{n} \cdot \log \frac{Mv_0}{c_n} \to 0. \quad (44)
\]
Then from (40), (43) and (44), the fourth term on the right-hand side of (32) is bounded from above as
\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{\theta} \sum_{t \in \tau_n} \log f(x_i; \theta) \leq \lambda_0 \quad \text{a.e.} \quad (45)
\]
Combining (32), (35), (36), (39) and (45) we obtain
\[
\limsup_{n \to \infty} \sup_{\theta} \frac{1}{n} \sum_{i=1}^{n} \log f(x_i; \theta) \leq E_0[\log f(x; \theta_0)] - \lambda_0, \quad \text{a.e.}
\]
and (31) is satisfied. Therefore it suffices to prove (43), which is a new goal of our proof.

We now consider further finite covering of \(\Theta'_{n, \mathcal{X}, s}\). For any \(T (1 \leq T \leq 2M)\) and \(\tau \subset \{1, \ldots, T(\theta)\}\), define a subset of \(\Theta_{n, \mathcal{X}, s}\) by
\[
\Theta'_{n, \mathcal{X}, s, T, \tau} \equiv \{\theta \in \Theta'_{n, \mathcal{X}, s} \mid T(\theta) = T , \tau_n(\theta) = \tau\}. \quad (46)
\]
Then (43) is derived from the following two lemmas.

Lemma 11.
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta'_{n, \mathcal{X}, s, T, \tau}} \left[ \sum_{t \in \tau} \frac{1}{n} R_n(J_t(\theta)) \cdot \log H(J_t(\theta)) - 3 \sum_{t \in \tau'} \frac{2}{n} \log H(J_t(\theta)) \right] \leq 0 \quad \text{a.e.},
\]
where \(\tau' = \{1, \ldots, T\} \setminus \tau\).
Proof: Let $\delta > 0$ be any fixed positive real constant and let $\mu'_t(\theta)$ denote the middle point of $J_t(\theta)$. Here, we consider the probability of the event that

$$
\sup_{\theta \in \Theta_{n,\mathcal{X},s,T,\tau}'} \left[ \sum_{t \in \tau'} \frac{1}{n} R_n(J_t(\theta)) \cdot \log H(J_t(\theta)) - 3 \sum_{t \in \tau'} \frac{2}{n} \log H(J_t(\theta)) \right] > 2M\delta. \quad (47)
$$

Since $H(J_t(\theta)) > Mv_0/c'_n$ holds for $t \in \tau'$, we obtain by lemma 3

$$
W(J_t(\theta)) \leq v_2 \cdot \left( \frac{c'_n}{Mv_0} \right)^{\tilde{\beta}} = v_2 \cdot \left( \frac{c_0}{Mv_0} \right)^{\tilde{\beta}} \cdot \exp \left( -\tilde{\beta} \cdot n^{1/4} \right).
$$

Let

$$
v_3 \equiv v_2 \cdot \left( \frac{c_0}{Mv_0} \right)^{\tilde{\beta}} ,
$$

$$
w_n \equiv \frac{v_3}{2} \cdot \exp \left( -\tilde{\beta} \cdot n^{1/4} \right) ,
$$

$$
R_n[\mu, w] \equiv R_n([\mu - w, \mu + w]) .
$$

Noting that for $t \in \tau'$, the length of $J_t(\theta)$ is less than or equal to $2w_n$, the following relation holds.

The event $(47)$ occurs.

$$
\Rightarrow \sup_{\theta \in \Theta_{n,\mathcal{X},s,T,\tau}'} \left[ \sum_{t \in \tau'} \left( \frac{1}{n} R_n[\mu'_t(\theta), w_n] - 3 \cdot \frac{2}{n} \log \frac{Mv_0}{c_n} \right) \cdot \log \frac{Mv_0}{c_n} \right] > 2M\delta
$$

$$
\Rightarrow \exists \theta \in \Theta_{n,\mathcal{X},s,T,\tau}' \text{ s.t. } (\frac{1}{n} R_n[\mu'_t(\theta), w_n] - 3 \cdot \frac{2}{n} \log \frac{Mv_0}{c_n}) > \delta
$$

$$
\Rightarrow \exists \theta \in \Theta_{n,\mathcal{X},s,T,\tau}' \text{ s.t. } R_n[\mu'_t(\theta), w_n] \geq 6
$$

$$
\Rightarrow \sup_{-\infty < \mu' < \infty} R_n[\mu', w_n] \geq 6 . \quad (50)
$$

Below, we consider the probability of the event that $(50)$ occurs. We divide $J^{(n)}_0 = [-A_n, A_n]$ from $-A_n$ to $A_n$ by short intervals of length $2w_n$ as in the proof of lemma 10. Let $k(w_n)$ be the number of short intervals and let $I_1(w_n), \ldots, I_{k(w_n)}(w_n)$ be the divided short intervals. Then we have

$$
k(c'_n) \leq \frac{2A_n}{2w_n} + 1 = \frac{A_0 \cdot n^{2/3'}}{\nu(c_0)} + 1 . \quad (51)
$$

Since any interval in $J_0$ of length $2w_n$ is covered by at most 3 small intervals from $I_1(w_n), \ldots, I_{k(w_n)}(w_n)$ and from lemma 5

$$
\sup_{-\infty < \mu' < \infty} R_n[\mu', w_n] \geq 6 \Rightarrow 1 \leq \exists k \leq k(w_n) , R_n(I_k(w_n)) \geq 2 .
$$
Note that $R_n(I_k(w_n)) \sim \text{Bin}(n, P_0(I_k(w_n)))$ and $P_0(I_k(w_n)) \leq 2w_nu_0$. Therefore from (51) we have

$$
\sum_{k=1}^{k(w_n)} \text{Prob} (R_n(I_k(w_n)) \geq 2) \leq \left( \frac{A_n}{w_n} + 1 \right) \cdot \left\{ \max_{1 \leq k \leq k(w_n)} \text{Prob}(R_n(I_k(w_n)) \geq 2) \right\}
$$

$$
\leq \left( \frac{A_n}{w_n} + 1 \right) \sum_{k=2}^{n} \left( \frac{n}{k} \right) (2w_nu_0)^k (1 - 2w_nu_0)^{n-k}
$$

$$
\leq \left( \frac{A_n}{w_n} + 1 \right) \sum_{k=2}^{n} \frac{n^k}{k!} (2w_nu_0)^k \leq \left( \frac{A_n}{w_n} + 1 \right) (2nw_nu_0)^2 \sum_{k=0}^{n} \frac{1}{k!} (2nw_nu_0)^k
$$

$$
\leq \left( \frac{A_n}{w_n} + 1 \right) (2nw_nu_0)^2 \exp(2nw_nu_0).
$$

When we sum this over $n$, resulting series on the right converges. Hence by the Borel-Cantelli lemma and the fact that $\delta > 0$ was arbitrary, we obtain

$$
\limsup_{n \to \infty} \sup_{\theta \in \Theta_n, \chi, s, T, \tau} \left[ \sum_{t \in \tau} \frac{1}{n} R_n(I_t(\theta)) \cdot \log H(I_t(\theta)) - 3 \sum_{t \in \tau} \frac{2}{n} \log H(I_t(\theta)) \right] \leq 0 \text{ a.e.}
$$

\hfill \Box

**Lemma 12.**

\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta_n, \chi, s, T, \tau} \left[ \sum_{t \in \tau} \frac{1}{n} R_n(I_t(\theta)) \cdot \log H(I_t(\theta)) - 3 \sum_{t \in \tau} u_0 \cdot \xi(H(I_t(\theta))) \cdot \log H(I_t(\theta)) \right] \leq 0 \text{ a.e.}
\]

**Proof:** Let $\delta > 0$ be any fixed positive real constant and let

$$
h_n \equiv \frac{\delta}{12} \left\{ u_0 \cdot \log \frac{Mv_0}{c_n'} \right\}^{-1}.
$$

(52)

Since $v_0/c_0 \leq H(I_t(\theta)) \leq Mv_0/c_n$, we have $\xi(Mv_0/c_n) \leq H(I_t(\theta)) \leq \xi(v_0/c_0)$. We divide the interval $[\xi(Mv_0/c_n')$, $\xi(v_0/c_0)]$ from $\xi(c_0/v_0)$ to $\xi(Mv_0/c_n')$ by short intervals of length $h_n$. In the left end $\xi(Mv_0/c_n')$ of the interval $[\xi(Mv_0/c_n')$, $\xi(v_0/c_0)]$, overlap of two short intervals of length $h_n$ is allowed and the left end of a short interval is equal to $\xi(Mv_0/c_n')$. Let $l_n$ be the number of short intervals of length $h_n$ and define $w^{(n)}_l$ by

$$
2w^{(n)}_l = \begin{cases} 
\xi(v_0/c_0) - (l - 1)h_n, & 1 \leq l \leq l_n, \\
\xi(Mv_0/c_n'), & l = l_n + 1.
\end{cases}
$$

(53)
Then we have
\[ l_n \leq \frac{\xi(v_0/c_0)}{h_n} + 1. \]  

(54)

Let
\[ \psi(y) \equiv \xi^{-1}(y) = \left( \frac{v_2}{y} \right)^{1/\beta}, \quad (y > 0), \]

where \( \xi^{-1}(\cdot) \) is the inverse function of \( \xi(\cdot) \). Next we consider the probability of the event that
\[ \sup_{\theta \in \Theta_n' \times \{t\}} \left[ \sum_{t \in \tau} \frac{1}{n} R_n(J_t(\theta)) \cdot \log H(J_t(\theta)) \right. \]
\[ \left. -3 \sum_{t \in \tau} u_0 \cdot \xi(H(J_t(\theta))) \cdot \log H(J_t(\theta)) \right] > 2M\delta. \]  

(55)

For this event the following relation holds.

The event \((55)\) occurs.

\[ \Rightarrow \exists \theta \in \Theta_n' \times \{t\}, \ \forall t \in \tau, \ 1 \leq \exists l(t) \leq l_n \quad \text{s.t.} \]
\[ \psi(2w^{(n)}_{l(t)}) \leq H(J_t(\theta)) \leq \psi(2w^{(n)}_{l(t)+1}) \quad \text{and} \]
\[ \sum_{t \in \tau} \left( \frac{1}{n} R_n[\mu'_{l(t)}(\theta), w^{(n)}_{l(t)}] - 3u_0 \cdot 2w^{(n)}_{l(t)+1} \right) \cdot \log \psi(2w^{(n)}_{l(t)+1}) > 2M\delta \]

\[ \Rightarrow \exists \theta \in \Theta_n' \times \{t\}, \ \exists t \in \tau, \ 1 \leq \exists l(t) \leq l_n \quad \text{s.t.} \]
\[ \psi(2w^{(n)}_{l(t)}) \leq H(J_t(\theta)) \leq \psi(2w^{(n)}_{l(t)+1}) \quad \text{and} \]
\[ \left( \frac{1}{n} R_n[\mu'_{l(t)}(\theta), w^{(n)}_{l(t)}] - 3u_0 \cdot 2w^{(n)}_{l(t)+1} \right) \cdot \log \psi(2w^{(n)}_{l(t)+1}) > \delta \]

\[ \Rightarrow \ 1 \leq \exists l \leq l_n \quad \text{s.t.} \]
\[ \sup_{-\infty < \mu' < \infty} \left\{ \left( \frac{1}{n} R_n[\mu', w^{(n)}_{l}] - 3u_0 \cdot 2w^{(n)}_{l+1} \right) \cdot \log \psi(2w^{(n)}_{l+1}) \right\} > \delta \]

\[ \Rightarrow \ 1 \leq \exists l \leq l_n \quad \text{s.t.} \]
\[ \sup_{-\infty < \mu' < \infty} \left\{ \left( \frac{1}{n} R_n[\mu', w^{(n)}_{l}] - 3u_0 \cdot 2w^{(n)}_{l} \right) \cdot \log \psi(2w^{(n)}_{l}) \right\} + 3u_0 \cdot (2w^{(n)}_{l} - 2w^{(n)}_{l+1}) \cdot \log \psi(2w^{(n)}_{l+1}) \right\} > \delta \]  

(56)
Then from (52) and lemma 5, the following relation holds.

The event (56) occurs.
\[ \text{sup}_{-\infty < \mu' < \infty} \frac{1}{n} (R_n[\mu', w_l^{(n)}] - 3u_0 \cdot 2w_l^{(n)}) \cdot \log \psi(2w_l^{(n)} + 1) > \frac{\delta}{2} \]

\[ \Rightarrow 1 \leq \exists \leq l_n \quad \text{s.t.} \quad \text{sup}_{-\infty < \mu' < \infty} \frac{1}{n} (R_n[\mu', w_l^{(n)}] - 3u_0 \cdot 2w_l^{(n)}) \cdot \log \psi(2w_l^{(n)} + 1) > \frac{\delta}{2} \]

Below, we consider the probability of the event that (57) occurs. We divide \( J_0^{(n)} \) from \( -A_n \) to \( A_n \) by short intervals of length \( 2w_l^{(n)} \) as in the proof of lemma 10. Let \( k(w_l^{(n)}) \) be the number of short intervals and let \( I_1(w_l^{(n)}), \ldots, I_{k(w_l^{(n))}}(w_l^{(n)}) \) be the divided short intervals. Then we have
\[ k(w_l^{(n)}) \leq \frac{2A_n}{2w_l^{(n)}} + 1. \quad (58) \]

Since any interval in \( J_0 \) of length \( 2\sigma_l^{(n)} \) is covered by at most 3 small intervals from \( \{ I_1(w_l^{(n)}), \ldots, I_{k(w_l^{(n))}}(w_l^{(n)}) \} \), we have
\[ \text{sup}_{-A_n \leq \mu' \leq A_n} \left( \frac{1}{n} R_n[I_k(w_l^{(n))}) - u_0 \cdot 2w_l^{(n)} \right) > \frac{\delta}{3} \left( \log \psi(2w_l^{(n)} + 1) \right)^{-1} \]
\[ \Rightarrow \max_{k=1, \ldots, k(w_l^{(n))}} \left( \frac{1}{n} R_n[I_k(w_l^{(n))}) - u_0 \cdot 2w_l^{(n)} \right) > \frac{1}{3} \cdot \frac{\delta}{2} \left( \log \psi(2w_l^{(n)} + 1) \right)^{-1}. \quad (59) \]

Note that \( R_n[I_k(w_l^{(n))}) \sim \text{Bin}(n, P_0(I_k(w_l^{(n)}))) \) and \( P_0(I_k(w_l^{(n)})) \leq u_0 \cdot 2w_l^{(n)} \). Therefore from (58) and Okamoto’s inequality \((\text{Okamoto (1958)})\) we have
\[
\text{Prob} \left( \max_{k=1, \ldots, k(w_l^{(n))}} \frac{1}{n} R_n[I_k(w_l^{(n))}) - u_0 \cdot 2w_l^{(n)} \right) > \frac{1}{3} \cdot \frac{\delta}{2} \left( \log \psi(2w_l^{(n)} + 1) \right)^{-1} \right) \\
\leq \left( \frac{2A_n}{2w_l^{(n)}} + M \right) \cdot \exp \left[ -2n \cdot \frac{\delta^2}{36} \left( \log \psi(2w_l^{(n)} + 1) \right)^{-2} \right] \\
\leq \left( \frac{A_n}{\xi(Mv_0/c_n')} + M \right) \cdot \exp \left[ -2n \cdot \frac{\delta^2}{36} \left( \log (Mv_0/c_n') \right)^{-2} \right]. \quad (60) \]

From (51), (53), (57), (59), and (60) we obtain
\[
\sum_{t=1}^{l_n} \text{Prob} \left( \sup_{-A_n < \mu' < A_n} \frac{1}{n} R_n[\mu', w_l^{(n)}] - 3u_0 \cdot 2w_l^{(n)} \right) \cdot \log \psi(2w_l^{(n)} + 1) > \frac{\delta}{2} \\
\leq \left( \frac{\xi(v_0/c_0')}{h_n} + 1 \right) \cdot \left( \frac{A_n}{\xi(Mv_0/c_n')} + M \right) \cdot \exp \left[ -2n \cdot \frac{\delta^2}{36} \left( \log (Mv_0/c_n') \right)^{-2} \right] \\
27
When we sum this over \( n \), the resulting series on the right converges. Hence by the Borel-Cantelli lemma and the fact that \( \delta > 0 \) is arbitrary, we have

\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta'} \left[ \sum_{t \in \tau} \frac{1}{n} R_n(J_t(\theta)) \cdot \log H(J_t(\theta)) \right. \\
\left. - 3 \sum_{t \in \tau} u_0 \cdot \xi(H(J_t(\theta))) \cdot \log H(J_t(\theta)) \right] \leq 0 \quad \text{a.e.}
\]

This completes the proof of theorem 2.

\[\square\]

## 5 Discussions

In this paper we consider the strong consistency of MLE for mixtures of location-scale distributions. We treat the case that the scale parameters of the component distributions are restricted from below by \( c_n = \exp(-nd), 0 < d < 1 \), and give the regularity conditions for the strong consistency of MLE.

As in the case of the uniform mixture in Tanaka and Takemura (2005), it is readily verified that if \( c_n \) decreases to zero faster than \( \exp(-n) \), then the consistency of MLE fails. Therefore the rate of \( c_n = \exp(-nd), 0 < d < 1 \), obtained in this paper is almost the lower bound of the order of \( c_n \) which maintains the strong consistency.

Although we treat the univariate case in this paper, it is clear that the result obtained in this paper can be extended to the multivariate case under the condition that components are bounded and their tails decrease to zero fast enough if the minimum singular values of the scale matrices of the components are restricted from below by \( c_n \).

Finally let us consider some sufficient conditions for the regularity conditions. For \( \theta_m \in \Omega_m \) and any positive real number \( \rho \), let

\[
f_m(x; \theta_m, \rho) \equiv \sup_{\text{dist}(\theta_m', \theta_m) \leq \rho} f_m(x; \theta_m').
\]

Let \( \Gamma \) be any compact subset of \( \Omega_m \). Consider the following two conditions.

**Assumption 5.** For each \( \theta_m \in \Gamma \) and sufficiently small \( \rho \), \( f_m(x; \theta_m, \rho) \) is measurable.

**Assumption 6.** For each \( \theta_m \in \Gamma \), if \( \lim_{j \to \infty} \theta_m^{(j)} = \theta_m \), then \( \lim_{j \to \infty} f_m(x; \theta_m^{(j)}) = f_m(x; \theta_m) \) for all \( x \).

If assumptions 5 and 6 hold, then it is easily verified that assumptions 2 and 3 hold. Thus assumptions 1, 4, 5 and 6 are sufficient conditions for regularity conditions and assumptions 4 and 6 are checked more easily. For example, finite mixture density which consists of normal density, \( t \)-density and uniform density on an open interval satisfies assumptions 1, 4, 5 and 6.
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Table 1: List of notations  \((\theta \in \Theta; \mathcal{X} \subset \{1, \ldots, M\}; V \subset \mathbb{R}; y, \rho, \mu, w \in \mathbb{R})\)

| Notation | Definition or description |
|----------|-----------------------------|
| \(M\) | Number of components |
| \(\theta_{\mathcal{X}}\) | Subvector of \(\theta \in \Theta\) consisting of the components in \(\mathcal{X}\) |
| \(\Theta_{\mathcal{X}}\) | \(\Theta_{\mathcal{X}} \equiv \{\theta_{\mathcal{X}} \mid \theta \in \Theta\}\); Parameter space of \(\theta_{\mathcal{X}}\); See [2] |
| \(f_{\mathcal{X}}(x; \theta_{\mathcal{X}})\) | \(f_{\mathcal{X}}(x; \theta_{\mathcal{X}}) \equiv \sum_{k \in \mathcal{X}} \alpha_k f_k(x; \mu_k, \sigma_k)\); See [3] |
| \(G_{\mathcal{X}}(x; \theta_{\mathcal{X}}, \rho)\) | \(G_{\mathcal{X}}(x; \theta_{\mathcal{X}}, \rho) \equiv \sup_{\mathcal{X}} \{f_{\mathcal{X}}(x; \theta) \mid \theta_{\mathcal{X}} \in \Theta_{\mathcal{X}}\}\); See [4] |
| \(G_{\mathcal{K}}\) | \(G_{\mathcal{K}} \equiv \bigcup_{|\mathcal{X}| \leq K} G_{\mathcal{X}}\); See [5] |
| \(v_0, v_1, \beta\) | \(f_m(x; \mu = 0, \sigma_m = 1) \leq \min\{v_0, v_1 \cdot |x|^{-\beta}\}\); See Assumption [6] |
| \(c_0, c_n, d, \Theta_n\) | \(c_n = c_0 \cdot \exp(-n\theta)\); See theorem [2] |
| \(B\) | \(B \equiv v_0/\kappa_0\); See [23] |
| \(\tilde{\beta}, \nu(y)\) | \(\tilde{\beta} \equiv \frac{\beta - 1}{\beta}, \nu(y) \equiv \left(\frac{v_1}{v_0}\right)^{\frac{\beta}{\beta - 1}} y^\beta\); See [7] |
| \(J(\theta)\) | \(J(\theta) \equiv \bigcup_{m \in \mathcal{X}, \beta < 0} \left[\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)\right]\); See [10] |
| \(J_t(\theta)\) | Interval of step function; See lemma [2] |
| \(H(J_t(\theta))\) | Height of step function in \(J_t(\theta)\); See lemma [2] |
| \(W(J_t(\theta))\) | Width of \(J_t(\theta)\); See lemma [2] |
| \(T(\theta)\) | Number of steps; See lemma [2] |
| \(v_2, \xi(y)\) | \(v_2 \equiv 2 \left(\frac{u_0}{v_0}\right)^{\frac{\beta}{\beta - 1}} \left(v_0(M + 1)\right)^{\frac{1}{\beta}}, \xi(y) \equiv v_2 \cdot \left(\frac{y}{v_2}\right)^{\frac{1}{\beta}}\); See [12] |
| \(u_0, u_1\) | \(f(x; \theta_0) \leq \min\{u_0, u_1 \cdot |x|^{-\beta}\}\); See lemma [1] |
| \(x_{n,1}, x_{n,n}\) | \(x_{n,1} \equiv \min\{x_1, \ldots, x_n\}, x_{n,n} \equiv \max\{x_1, \ldots, x_n\}\); See lemma [5] |
| \(A_0, \zeta, A_n\) | \(A_n \equiv A_0 \cdot n^{\frac{2\alpha - 1}{\alpha + 1}}\); See lemma [5] |
| \(A^{(\iota)}\) | \(A^{(\iota)} \equiv \min_{m \in \mathcal{X} \uparrow \infty} \left\{\min\{|\mu_m^{(\iota)} + \nu(\sigma_m^{(\iota)})|, |\mu_m^{(\iota)} - \nu(\sigma_m^{(\iota)})|\}\right\}\); See [20] |
| \(\mathcal{A}\) | \(\mathcal{A} \equiv (-\infty, -A_0] \cup [A_0, \infty)\); See [24] |
| \(\mathcal{K}_{\sigma \leq 0}, \mathcal{K}_{\sigma \geq 0}, \mathcal{K}_{|\mu| \geq A_0}\) | Disjoint subset of \(\mathcal{L}\); See lemma [4] |
| \(\mathcal{K}_{\sigma \leq 0}, \mathcal{K}_{\sigma \geq 0}, \mathcal{K}_{|\mu| \geq A_0}\) | Disjoint subset of \(\{1, \ldots, M\}\); See [28] |
| \(\mathcal{K}_R\) | \(\mathcal{K}_R \equiv \mathcal{L} \setminus \{\mathcal{K}_{\sigma \leq 0} \cup \mathcal{K}_{\sigma \geq 0} \cup \mathcal{K}_{|\mu| \geq A_0}\}\) in subsection [12] |
| \(P_0(V)\) | \(P_0(V) \equiv \int_V f(x; \theta_0)dx\); See [25] |
| \(R_n(V)\) | Number of observations which belong to a set \(V\) |
| \(B(\theta, \rho(\theta))\) | Open ball with center \(\theta\) and radius \(\rho(\theta)\) |
| \(c_n\) | \(c_n \equiv c_0 \cdot \exp(-n^{1/4})\); See [11] |
| \(\tau_n(\theta), \tau_n(\theta)^*\) | See [42] |
| \(\Theta_n\) | \(\Theta_n \equiv \{\theta \in \Theta \mid \exists m \text{ s.t. } c_m \leq \sigma_m \leq c_0 \text{ or } |\mu_m| > A_0\}\) |
| \(\Theta'_{n,\mathcal{X}}\) | See [28] |
| \(\Theta'_{n,\mathcal{X},s}\) | \(\Theta'_{n,\mathcal{X},s} \equiv \{\theta \in \Theta'_{n,\mathcal{X}} \mid \theta_{\mathcal{X},s} \in B(\theta_{\mathcal{X},s}, \rho(\theta_{\mathcal{X},s}))\}\); See [29] |
| \(\Theta'_{n,\mathcal{X},s,T,t}\) | \(\Theta'_{n,\mathcal{X},s,T,t} \equiv \{\theta \in \Theta'_{n,\mathcal{X},s} \mid T(\theta) = T, \tau_n(\theta) = \tau\}\); See [46] |
| \(\mu'_1(\theta)\) | \(\mu'_1(\theta)\) denote the middle point of \(J_t(\theta)\) |
| \(w_0\) | See [48] |
| \(w_0^{(t)}\) | See [53] |
| \(R_n[\mu, w]\) | \(R_n[\mu, w] \equiv R_n[\mu - w, \mu + w]\); See [49] |