Phase diagram of Regge quantum gravity coupled to SU(2) gauge theory

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Abstract

We analyze Regge quantum gravity coupled to SU(2) gauge theory on $4^3 \times 2$, $6^3 \times 4$ and $8^3 \times 4$ simplicial lattices. It turns out that the window of the well-defined phase of the gravity sector where geometrical expectation values are stable extends to negative gravitational couplings as well as to gauge couplings across the deconfinement phase transition. We study the string tension from Polyakov loops, compare with the $\beta$-function of pure gauge theory and conclude that a physical limit through scaling is possible.

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I. INTRODUCTION

Regge Calculus [1] provides a nonperturbative way for investigations of Euclidean quantum gravity on a simplicial lattice and offers the possibility to construct a unified theory by coupling gauge fields to the skeleton. One remarkable finding was the discovery of an “entropy dominated”, well-defined phase, where the expectation values with respect to the pure Regge-Einstein action are stable [2–4]. The next question addressed was the physical relevance of this regime which has been tested by coupling a non-Abelian gauge field to gravity [5]. If one assumes that the world without gravity is described by a grand unified asymptotically free theory, these numerical studies investigate the relation of the hadronic scale to the Planck scale. In particular, it has already been shown that confinement exists in the coupled system [6]. In this work, we perform a nonperturbative analysis of the phase diagram of Regge quantum gravity coupled to SU(2) gauge fields on several lattice sizes in four spacetime dimensions. The stability and boundary of the well-defined phase is investigated on lattices of sizes up to $6^3 \times 4$, considerably extending the previously available data. Within this phase we find that the confinement-deconfinement transition of conventional lattice gauge theory is still present. We extract values for the string tension and gain some evidence from the $\beta$-function that the window of the well-defined phase extends to large $\beta$ values corresponding to small lattice spacings $a$ in the order of the Planck scale.

II. ENTROPY DOMINATED PHASE

In Regge Calculus the edge lengths are considered to be the dynamical degrees of freedom of the discretized spacetime manifold. In $d = 4$ dimensions the geometry is Euclidean inside of a $d$-simplex and the curvature is concentrated at the $d - 2$-subset of the lattice, the triangles. Quantization proceeds via the path integral although the choice of the gravitational measure is an unresolved issue. Investigations on regular triangulations do not favor any of them [4], but for irregular triangulations a preference for scale-invariant measures was
found \( \text{[7]} \). It may be that measures are divided into universality classes such that identical physics is obtained within one class.

In our simulations we chose a hypercubic triangulation with \( N_s^3 \times N_t \) vertices and the scale-invariant measure:

\[
D[l^2] = \prod_l \frac{dl^2}{l^2},
\]

(2.1)

where \( l \) is used to denote the link label as well as its length. The system of SU(2) gauge fields coupled to quantum gravity has the action \( \text{[5]} \)

\[
S = 2m_p^2 \sum_t A_t \alpha_t - \frac{\beta}{2} \sum_t W_t \text{Re}[\text{Tr}(1 - U_t)],
\]

(2.2)

with the first term being the Regge-Einstein part composed of the bare Planck mass \( m_p \) and \( A_t, \alpha_t \) the area and the deficit angle of the triangle \( t \). The addition of the SU(2) gauge term is straightforward and follows ordinary lattice gauge theory by assigning SU(2) matrices to the links. The elementary plaquettes become triangles on the Regge skeleton. \( \beta \) corresponds to the inverse gauge coupling and the weights \( W_t = \text{const} \times \frac{V_t}{(\alpha_t)^2} \), with a 4-volume \( V_t \) assigned to each triangle, describe the coupling of gravity to the gauge field. \( U_t \) is the ordered product of SU(2) matrices around \( t \). In contrast to the flat lattice, the unit matrix in the action is important because the weight factors are dynamical. They are constructed such that the correct continuum limit is ensured in the limit of vanishing lattice spacing \( \text{[8]} \).

We present Monte Carlo (MC) results concerning the boundary and stability of the well-defined phase. For this purpose large statistics simulations were performed on \( 4^3 \times 2 \) and \( 6^3 \times 4 \) lattices, covering a variety of \( (m_p^2, \beta) \) values. The accumulated statistics is summarized in Tables \( \text{[I]} \) and \( \text{[II]} \).

The stability of the coupled system was analyzed from MC-time histories of the Regge action. Examples are presented in Fig. \( \text{[I]} \). Long runs on the larger lattice show that it is difficult to decide whether the well-defined phase is stable or just metastable. In the latter case an additional curvature term of higher order \( \text{[9]} \) may stabilize the system.

The deconfinement transition was studied from the behavior of the Polyakov loops \( P = \frac{1}{2} \text{Tr}(U_1 U_2 \ldots U_{N_t}) \). For \( 4^3 \times 2 \) and \( 6^3 \times 4 \) lattices Fig. \( \text{[2]} \) shows MC-time histories of
\(P\) and \(l\) in the confined and deconfined phase for gravitational couplings in the well-defined phase. The link lengths are largely independent of fluctuations of the order parameter. Notable is the long equilibration time in the deconfinement phase.

Extracted phase diagrams of the gauge-gravity system are displayed in Fig. 3. The dotted lines are to guide the eyes and rely on the depicted stable versus unstable data points. From the \(4^3 \times 2\) lattices we have numerical evidence that the stable phase extends to \(\beta = 3.0\) (Table I). The \(6^3 \times 4\) lattices indicate a glitch in the well-defined–ill-defined boundary when passing from confinement to deconfinement. However, the present runs do not decide the question conclusively. It may be accidental that at \(\beta = 1.6\) the \(m^2_p = 0.025\) data did not run away. For \(\beta = 1.55\) tunneling into the ill-defined phase happened for these \(m^2_p\) values only after more than 100k sweeps. Our systems exhibit a small shift of the deconfinement phase transition from \(\beta_c(N_t = 2) = 1.525\) to \(\beta_c(N_t = 4) = 1.575\) with error bars less than 0.025.

III. STRING TENSION

The Polyakov loop \(P(R)\) in the short extent \(L_t\) of the lattice describes the propagation of a static quark. We introduce a quark source and a sink separated by a distance \(R\) and calculate the correlations of the Polyakov loops at these points. As in conventional SU(2) lattice gauge theory at finite temperature \(T\) [10,11], we extract from the correlation function

\[
\langle P(0)P^\dagger(R) \rangle = \exp\left[-\frac{1}{T}V(R)\right],
\]

the quantity \(V(R)\) corresponding to the potential between the static quark-antiquark pair. In the confinement phase \(V(R)\) should grow linearly for large \(R\) due to an infinite free energy of isolated quarks:

\[
V_c(R) = -\frac{\alpha}{R} + \sigma R + C,
\]

where \(\alpha\) is the Coulomb parameter, \(\sigma\) the string tension, and \(C\) a constant.
The correct distance between two points should be measured using geodesic distances. We take the distance $R$ between the source and sink to be equal to the index distance along the main axes of the skeleton. This seems a reasonable approximation in the well-defined phase with small curvature fluctuations. Using scalar field propagation $[12]$ one may calculate corrections which are expected to be small for our purposes.

To extract a reliable value for the string tension, we simulated the system on an $8^3 \times 4$ lattice. Our data rely on 30000 measurements after equilibration for the coupled system as well as for the pure gauge system on a flat simplicial lattice.

Figure 4(a) presents the data points for the confinement potentials for several gauge couplings in the presence of gravity with $m_p^2 = 0.005$ in the “entropy dominated” phase, while Fig. 4(b) depicts the situation with gravity switched off. The dotted lines correspond to fits to the correlations in Eq. (3.1) according to the potential of Eq. (3.2) with the Coulomb parameter fixed to $\alpha = \pi/12$ and a mirror term included.

Since we have extracted string tensions for several $\beta$ values, we are in a position to study its scaling behavior. For pure gauge theory $\beta$-functions are derived in the literature also for a simplicial lattice $[13]$. We fit our string-tension data to the function

$$\sigma_{\text{flat}} = \frac{\sigma_{\text{phys}}}{\Lambda_{\text{flat}}^2} \left( \frac{6\pi^2\sqrt{5}\beta}{11} \right)^{102/121} \exp\left( -\frac{6\pi^2\sqrt{5}\beta}{11} \right)$$

and obtain a value for $\Lambda_{\text{flat}} = 0.0102(1)\sqrt{\sigma_{\text{phys}}}$. This is in good agreement with an analysis of Wilson-loop ratios in the $T = 0$ case, yielding $\Lambda_{\text{flat}} = 0.008(1)\sqrt{\sigma_{\text{phys}}}$. For the fluctuating lattice to our knowledge a $\beta$-function is not worked out, we are aware only of a recent study for the pure gravity case within dynamical triangulation $[14]$. Thus, we used as a starting point the above SU(2) function Eq. (3.3). In Fig. 5 we compare the string-tension scaling for the flat and the fluctuating lattices.

To have both systems on the same scale, we rescaled the inverse gauge coupling $\beta \rightarrow \langle W_t \rangle \beta$ of the fluctuating system. We find $\Lambda_{\text{grav}} = 0.0090(1)\sqrt{\sigma_{\text{phys}}}$ which is very similar to the flat case. Assuming that the pure-gauge $\beta$-function is a reasonable approximation for the full renormalization relation, our data show that the scaling window opens already at the
\( \beta \) values considered, similar to the pure SU(2) case \([13]\). As a consequence, the continuum limit could be performed along \( \beta \to \infty \), eventually stopping at a finite value corresponding to a lattice spacing equal to the Planck length. The existence of a corridor to large \( \beta \)'s is indicated in the \((m_p, \beta)\) diagrams of Fig. 3.

**IV. CONCLUSION**

The boundary between the well-defined and the ill-defined phase of Regge quantum gravity coupled to SU(2) gauge theory was studied with the largest so far available statistics on \(6^3 \times 4\) lattices. These lattices are already very CPU time intensive, because the gravitational dynamics is very slow. Altogether, evidence was gained that the well-defined phase is stable with the increase from \(4^3 \times 2\) to \(6^3 \times 4\).

Within the well-defined phase we find that the confinement mechanism from the non-Abelian gauge fields is not spoiled by quantum gravitational effects. This is not trivial, because it is was not clear how quantum gravity affects a gauge theory. In the confined phase we observed a potential linearly rising with \(R\). Extracting string tension values for both the coupled system and for the pure gauge theory on a simplicial lattice without gravity we found a very similar scaling behavior. It indicates that gravity effects do not destroy the physics of conventional asymptotically free field theories, even if the gravity-gauge coupling is large as in our situation. This gives hope that the physical limit can be approached through scaling from the investigated region towards the Planck length. Additionally, to reproduce Newton’s law one should demonstrate a diverging gravitational correlation length of graviton propagators when approaching the boundary of the well-defined phase.
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FIG. 1. MC-time histories of the Regge action $\langle A_t \delta_t \rangle$ for $4^3 \times 2$ (upper) and $6^3 \times 4$ (lower) lattices, with one time unit representing 1k sweeps. The Planck masses $m_p^2$ used in the simulations are indicated in brackets beside the curves. The plots correspond to simulations at the $\beta$ values given in the titles.
FIG. 2. Time histories of Polyakov loops $P$ and link lengths $l$ on $4^3 \times 2$ (upper) and $6^3 \times 4$ (lower) lattices, with one time unit representing 1k sweeps. For reasons of comparison link lengths are normalized with respect to the maximum of all four runs. In the lower figures $P$ is multiplied by a factor of two. The $(m_{p}^{2}, \beta)$ values are given in the titles.
FIG. 3. Phase diagrams extracted from $4^3 \times 2$ (a) and $6^3 \times 4$ (b) lattices. The bold vertical lines indicate the confinement-deconfinement transition.
FIG. 4. Static quark potentials on an $8^3 \times 4$ fluctuating lattice with $m_p^2 = 0.005$ (a) and for a flat simplicial lattice with gravity switched off (b). The curves are fits to a confined potential with periodicity taken into account. Error bars arise from a jackknife procedure.
FIG. 5. Scaling of the string tension $\sigma$ for the flat and the fluctuating lattice ($m_P^2 = 0.005$).

$\sigma$ was extracted from fits to Eq. (3.2) using $\alpha = \pi/12$. 
TABLE I. Statistics for the $4^3 \times 2$ lattices in units of $1k$ sweeps.

| $m_p^2 \backslash \beta$ | 0.0  | $m_p^2 \backslash \beta$ | 1.0  | 1.5  | 1.55 | 1.6  | 1.7  | 1.8  | 3.0  |
|--------------------------|------|--------------------------|------|------|------|------|------|------|------|
| +0.0225                  | 200  | +0.05                    | 100  |      |      |      |      |      |      |
| +0.02                    | 200  | +0.04                    | 40   |      |      |      |      |      |      |
| +0.0                     | 200  | +0.0350                  | 167  |      |      |      |      |      |      |
| −0.005                   | 40   | +0.03                    | 200  | 100  | 100  |      |      |      |      |
| −0.01                    | 40   | +0.025                   | 100  | 200  | 140  | 200  |      |      |      |
| −0.025                   | 40   | +0.0225                  | 30   | 200  | 200  | 200  | 200  |      |      |
| −0.03                    | 40   | +0.005                   |      |      |      |      |      |      | 100  |
| −0.04                    | 40   | +0.0                     |      |      |      |      |      | 35   |      |
| −0.05                    | 40   | −0.0025                  |      |      |      |      |      |      | 200  |
| −0.06                    | 40   | −0.005                   |      |      |      |      |      |      |      | 100  |
| −0.06375                 | 100  | −0.0075                  |      |      |      |      |      |      |      | 100  |
| −0.06425                 | 200  | −0.008                   |      |      |      |      |      |      |      | 100  |
| −0.065                   | 18   | −0.009                   |      |      |      |      |      |      |      | 200  |
|                          |      | −0.0105                  |      |      |      |      |      |      |      | 100  |
TABLE II. Statistics for the $6^3 \times 4$ lattices in units of 1k sweeps.

| $m_p^2 \backslash \beta$ | 0.0  | 0.073 | 0.5  | 0.8  | 1.0  | 1.2  | 1.4  | 1.55 | 1.6  | 1.65 |
|--------------------------|------|-------|------|------|------|------|------|------|------|------|
| +0.03                    | 24   |       |      |      |      |      |      |      |      |      |
| +0.0275                  | 72   |       |      |      |      |      |      |      |      |      |
| +0.026                   |      | 48    |      |      |      |      |      |      |      |      |
| +0.025                   |      |       |      |      |      |      |      | 136  | 200  |      |
| +0.023                   |      |       |      |      |      |      | 208  |      | 144  | 208  |
| +0.02                    |      | 56    |      |      |      |      |      |      |      | 208  |
| +0.0175                  |      |       |      |      |      |      |      |      | 23   |      |
| +0.015                   | 208  |      |      |      |      |      |      | 56   | 48   |      |
| +0.01                    |      |       |      |      |      |      |      |      |      | 56   |
| -0.005                   |      |       |      |      |      |      |      |      | 208  |      |
| -0.01                    | 24   | 40    | 160  | 40   | 24   | 40   |      |      |      |      |
| -0.015                   |      |      |      |      |      |      |      |      |      |      |
| -0.02                    | 24   | 40    |      |      |      |      |      |      |      |      |
| -0.04                    |      |      |      |      |      |      |      |      |      |      |
| -0.05                    |      |      |      |      |      |      |      | 144  |      |      |
| -0.06                    |      |      |      |      |      |      |      | 40   |      |      |