Elastic beam line with noticeable deflection

Vsevolod Krepkogorskiy

1Kazan State University of Architecture and Engineering
E-mail: vkrepko@mail.ru

Abstract. Two differential equations are considered in the article. They describe the elasticity line of a curved beam. The second is obtained from the first if the derivative of the deviation function of the beam axis from the straight line is negligible. The question of the proximity of both solutions is studied. The literature considers many options for deviations from ordinary conditions, such as composite beams, complex deformations, too much bending. In our case, the hypothesis of Kirchhoff is supposed to be fulfilled.

The following cases are considered: 1) a beam supported by two supports, and 2) a cantilever beam. The load is distributed evenly. Graphs of solutions are constructed for both equations at different load densities and beam stiffness. A parameter is found, knowing which, we can indicate from the table below how many percent these two solutions differ in. Our task is to find out the limits of application of conventional calculation methods for strong beam bending.

Keywords: elastic line; differential equations of beam bending.

1 Introduction

Although Euler studied the beam bending, this topic continues to interest researchers. In the latest issues of such journals as the International Journal of Concrete Structures and Materials, Scientific Reports, the International Journal of Solids and Structures and others, dozens of articles on this subject can be easily found. The equation of the elastic line of the beam has the form:

\[ EJ \cdot \frac{y''}{[1 + (y')^2]^{3/2}} = M(x). \]  

(1)

Here E is the Young's modulus of elasticity, J is the moment of inertia of the cross section of the beam in position x relative to the horizontal line passing through the center of mass of the cross section. In most cases, the value of y' is small, and can be considered equal to zero. Then we obtain the approximate equation instead of Eq. (1):

\[ EJ \cdot y'' = M(x). \]  

(2)

Traditional methods for calculating beam bending are based on a theory following from Eq. (2). A large number of methods and calculators are known, the justification of which assumes the validity of Eq. (2). Our goal is to determine the boundaries of the application of these methods. The main criterion is as follows: the solutions of Eq. (1) and the approximate Eq. (2) are compared. If close results are obtained, then the use of Eq. (2) is acceptable. Parameter values are estimated at which the difference is negligible. The relative error of the approximate solution is calculated.

The classical theory of bending of thin plates is based on the Kirchhoff hypotheses. In this case, transverse shear deformations are not taken into account. Ignoring lateral shear deformations is possible only for materials with a high shear modulus in the transverse direction. Isotropic materials,
for example, metal alloys, to some extent meet this condition, but composite materials, as a rule, have low shear stiffness, so transverse shear deformations must be taken into account.

Most of the modern articles on this issue are devoted to the study of bending in cases where the main hypotheses are not satisfied. For example, this is the case with composite materials. Concrete composite materials with various methods of reinforcement are considered. Materials surface hardened with a stretched wire [1], reinforced with fiberglass [2], bamboo [3], and other materials [4-6]. Comparison of various methods for calculating beam bending in articles [7, 8].

The study of the deformation of complex structures composed of elastic elements [9, 10-12], the bending of multilayer beams [13] and the study of the bending of a plate of variable thickness [14]. Special cases are considered: composite beams subjected to cyclical freezing and thawing [15]; study of bending equations with parameter [16]. Another problem may be too much beam bending. The first works in this direction appeared in the middle of the 20th century [17], then the issue was studied by various authors [18-20]. In [21] the large bending of a beam subjected to concentrated load on the tip is studied.

2 Materials and methods
A theoretical solution is proposed. The solutions of two equations are compared. Moreover, Eq. (1) cannot be solved analytically. Numerical methods are used. Calculations are performed using the program for analytical calculations "Maxima". The results are presented in the form of graphs and tables. Conclusions are made about the parameter values at which both methods give similar results.

1. Flexible beam resting by two supports. The beam, which has a cylindrical shape, rests on two supports. One of them is articulated-motionless (in the right end A), the second is a articulated-motion support (point B). The beam carrying a uniformly distributed vertical load of intensity w.

\[ M(x) = \frac{px^2}{2} - \frac{px\ell}{2}, \quad 0 \leq x \leq \ell. \]  

Let be

\[ \frac{M(x)}{Ej} = \frac{p}{2Ej} \cdot (x^2 - x\ell) = k \cdot (x^2 - x\ell), \]  

where \( k \) is the coefficient:

\[ k = \frac{p}{2Ej}. \]  

We get the equation:

\[ y'' = k(x^2 - \ell x). \]  

To solve it, we make the substitution \( y' = z \). Then

\[ \frac{z'}{[1 + z^2]^{\frac{3}{2}}} = k(x^2 - \ell x) \Rightarrow \frac{dz}{[1 + z^2]^{\frac{3}{2}}} = k(x^2 - \ell x) \, dx. \]  

We denote \( z = \tan t \). Then \( dz = \frac{dt}{\cos^2 t} \) and

\[ \frac{dt}{[1 + z^2]^{\frac{3}{2}}} = \frac{dt}{\cos^2 t \cdot \left( \frac{1}{\cos^2 t} \right)^{\frac{3}{2}}} = \cos t \, dt. \]  

Substituting into Eq. (7) we obtain the equation:

\[ \cos t \, dt = k(x^2 - \ell x) \, dx. \]

Then integrating both parts we get:

\[ \sin t = k \left( \frac{x^3}{3} - \ell \frac{x^2}{2} \right) + C. \]
Back to the old variable:
\[ \sin t = \pm \sqrt{1 - \cos^2 t} = \pm \sqrt{1 - \frac{1}{1 + \sin^2 t}} = \pm \sqrt{1 - \frac{1}{1 + \tan^2 t}} = \] (10)
\[ \pm \frac{\tan t}{\sqrt{1 + \tan^2 t}} = \frac{\pm \tan t}{\sqrt{1 + \tan^2 t}} = \frac{\pm z}{\sqrt{1 + z^2}} \] (11)

From Eq. (9):
\[ \frac{z}{\sqrt{1 + z^2}} = \pm k \left( \frac{x^3}{3} - \ell \frac{x^2}{2} \right) + C. \] (12)

For reasons of symmetry of the source data at \( x = \ell / 2 \), the derivative \( y' = z \) is equal to zero and should change sign from the minus sign to the plus. Substituting \( \ell / 2 \) in the right-hand side, we obtain the condition:
\[ k \left( \frac{\ell/2)^3}{3} - \ell \frac{(\ell/2)^2}{2} \right) + C = 0 \Rightarrow C = k \frac{\ell^3}{12}. \] (13)

Let be
\[ H(x) = k \left( \frac{x^3}{3} - \ell \frac{x^2}{2} + \ell^3 \right). \] (14)

Substitute in Eq. (9). Then
\[ \frac{y'}{\sqrt{1 + (y')^2}} = H(x). \] (15)

Let's square both sides:
\[ \frac{(y')^2}{1 + (y')^2} = H^2(x) \Rightarrow (y')^2 = (1 + (y')^2) \cdot H^2(x) \Rightarrow \]
\[ H^2(x) = (y')^2 \left( 1 - H^2(x) \right) \]
\[ y' = \frac{H(x)}{\sqrt{1 - H^2(x)}}. \] (18)

We use the condition \( y(0) = 0 \). Then
\[ y(x) = \int_0^x \frac{H(x)}{\sqrt{1 - H^2(x)}} \, dx. \] (19)

Usually, when solving Eq. (3), it is assumed that the values of the derivative \( y' \) are negligible. Then we get the equation:
\[ y'' = k(x^2 - \ell z). \] (20)

Since the beam does not bend at the ends, \( y(0) = 0 \) and \( y'(0) = 0 \). Solving this problem, we obtain an approximate solution of the elastic line (center line):
\[ y_1(x) = \frac{p}{24EI} \left( x^4 - 2\ell x^3 + \ell^3 x \right) = \frac{k}{12} \left( x^4 - 2\ell x^3 + \ell^3 x \right). \] (21)

We solve Eq. (1) and Eq. (2) using the program for analytical calculations “Maxima”. The function \( y(x) \) is the solution of the exact Eq. (1), and \( y_1(x) \) is the solution of the approximate equation. When searching for \( y(x) \), integral (19) can only be calculated numerically. Nevertheless, the result can be declared a function of \( x \) and plot.
Figure 1 shows graphs of the functions \( y(x) \) and \( y_1(x) \) in the case when the beam length \( l = 20 \) m for various values of \( k = p / 2EJ \). The x-axis represents the distance from the left end of the beam in meters, and the y-axis represents the deflection in meters. It can be seen from Figure 2 that the difference between the exact and approximate solutions is small if \( k \leq 0.001 \). With increasing \( k \), the magnitude of the deflection and the difference between the exact and approximate solutions increases rapidly.

Let us compare the solutions of Eq. (1) and Eq. (2) in magnitude. Assuming that Eq. (1) gives the exact value of the deflection, we estimate the relative error of the function \( y_1 \). We calculate it using the formula:

\[
\frac{\left( y(x) - y_1(x) \right)}{y\left( \frac{L}{2} \right)} \cdot 100\% \tag{22}
\]

for different values of the coefficients \( k \). The graphs of the relative error in solving the approximate equation of the elastic line for various \( k \) are shown in Figure 2.
2.1 Cantilever bending

Consider a beam which is fixed rigidly at one end $A$ and is quite free at its remote end $B$. The origin is placed at point $A$, the axis $Ox$ is directed to the right, and the axis $Oy$ is vertically upward. The beam bearing uniformly distributes load of density $q$. The calculations are carried out similarly to the case of a beam supported at two ends. The equation of the center line has the form:

$$EJ \cdot \frac{y'}{[1 + (y')^2]^2} = M(x).$$

(23)

In this case

$$M(x) = -q \frac{(\ell - x)^2}{2},$$

(24)

Consequently

$$\frac{y''}{[1 + (y')^2]^2} = -\frac{q}{2EJ} \cdot (\ell - x)^2.$$

(25)

Let $k = \frac{q}{2EJ}$. Then

$$\frac{y''}{[1 + (y')^2]^2} = -k \cdot (\ell - x)^2$$

(26)

Solving as in the previous case, we arrive at the equation:

$$\frac{y'}{\sqrt{1 + (y')^2}} = -\int k(\ell - x)^2 dx = k \frac{(\ell - x)^3}{3} + C.$$

(27)

Define a function:

$$W(x) = k \left( \frac{(\ell - x)^3}{3} - \frac{x^3}{3} \right).$$

(28)

Then, acting as in the previous case, we obtain the formula for the desired function:

$$y(x) = \int_0^x \frac{W(x)}{\sqrt{1 - W^2(x)}} dx.$$

(29)

Usually, instead of Eq. (22), an approximate equation is considered:

$$y'' = -k \cdot (\ell - x)^2.$$

(30)

The result is well known:

$$y_1(x) = -\frac{q\ell^2 x^2}{24EJ} \left( 6 - 4 + \frac{x^2}{\ell^2} \right) = -\frac{k\ell^2 x^2}{12} \left( 6 - 4 + \frac{x^2}{\ell^2} \right).$$

(31)

By numerical calculating the integral (29), we construct graphs of the functions $y(x)$ and $y_1(x)$ for various values of the parameters $k, k_1, k_2, k_3$ (figure 3).

To estimate the relative error in the calculation using an approximate formula, we construct graphs of functions (figure 4).

$$\frac{(y(x) - y_1(x))}{y(x)} \cdot 100.$$  

(32)

Note that the quantities in the expression $(y(x) - y_1(x)) / y(x) \cdot 100$ depend on the length of the beam. We will try to find a formula for calculating $y_1(x)$ so that the relative error does not depend on $l$. To do this, we transform the formulas for calculating $y(x)$ and $y_1(x)$.

$$y_1(x) = -q\frac{\ell^2 x^2}{24EJ} \left( 6 - 4 + \frac{x^2}{\ell^2} \right) = -\frac{q\ell^2}{6EJ} \cdot \frac{1}{4} \left( 6 - 4 \frac{x^2}{\ell^2} + \frac{x^4}{\ell^4} \right).$$

(33)

Put

$$r = \frac{q\ell^3}{6EJ} = \frac{k\ell^3}{3} \quad u = \frac{x}{\ell}.$$  

(34)
Then \( y_1(x) = -r \cdot \ell \cdot 0.25 \cdot (6u^2 - 4u^3 + u^4) \).

![Graphs of the functions \( y(x) \) and \( y_1(x) \).]

![Relative error of the approximate function \( y_1(x) \).]

We now transform Eq. (9). Notice, that

\[
W(x) = k \left( \frac{(\ell - x)^3}{3} - \frac{\ell^3}{3} \right) = \frac{k}{3} - 3\ell x + 3\ell^2 - x^3
\]

(35)

\[
= -\frac{k\ell^3}{3} \left( 3 \left( \frac{x}{\ell} \right) - 3 \left( \frac{x}{\ell} \right)^2 + \left( \frac{x}{\ell} \right)^3 \right).
\]

(36)

If \( H(u) = 3 \left( \frac{x}{\ell} \right) - 3 \left( \frac{x}{\ell} \right)^2 + \left( \frac{x}{\ell} \right)^3 = 3u - 3u^2 + u^3 \) then

\[
W(x) = -\frac{k\ell^3}{3} H(u) = -r H(u).
\]

(37)

To calculate the integral (22), we replace the variable \( x = u \ell, dx = \ell du, x = v \Rightarrow u = x/\ell \)

\[
\int_{0}^{\nu} \frac{W(x)}{\sqrt{1 - W^2(x)}} \, dx = -\frac{k\ell^3}{3} \int_{0}^{\nu} \frac{H(u)}{\sqrt{1 - (rH(u))^2}} \ell du =
\]

(38)

\[
= -r\ell \int_{0}^{\nu} \frac{H(u)}{\sqrt{1 - (rH(u))^2}} \, du.
\]

(39)
Function $U(v)$ is defined by

$$U(v) = \int_{0}^{v} \frac{H(u)}{\sqrt{1 - (rH(u))^2}} \, du \quad (40)$$

Obviously $y = -rU(v)$ and relative error $y_1$ equal:

$$\left(\frac{y(x) - y_1(x)}{y(x)}\right) \cdot 100 = \frac{-rU(v) - r \cdot \ell \cdot 0.25 \cdot (6u^2 - 4u^3 + u^4)}{-rU(v)}$$

$$= 1 + \frac{0.25 \cdot (6u^2 - 4u^3 + u^4)}{U(v)} \quad (41)$$

The maximum deflection is obtained at the free end of the beam at $x = \ell \Rightarrow u = 1$. In this case:

$$0.25 \cdot (6u^2 - 4u^3 + u^4) = 0.25 \cdot 3 = 0.75 \quad (43)$$

$$U(1) = \int_{0}^{1} \frac{H(u)}{\sqrt{1 - (rH(u))^2}} \, du \quad (44)$$

The relative error of the approximate formula is:

$$\left(\frac{y(x) - y_1(x)}{y(x)}\right) \cdot 100 = \left(1 + \frac{0.75}{U(1)}\right) \cdot 100 = \delta \quad (45)$$

Note that the quantity $\delta$ does not depend on the beam length $l$, but is determined only by the value $r$. The dependence of $\delta$ on the coefficient $r$ is described in Table 1.

### Table 1. The relative error of the approximate formula.

| $r$  | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 |
|------|-----|-----|-----|-----|-----|-----|-----|
| $\delta$ in percents | 0.387 | 1.559 | 3.558 | 6.456 | 10.377 | 15.527 | 22.272 |

### 3 Results

The article compares estimates of the deflection of the beam obtained using the differential equation of the midline of the beam.

$$EJ \cdot \frac{y^\prime \prime}{[1 + (y')^2]^2} = M(x) \quad (46)$$

and the approximate equation

$$EJ \cdot y^\prime \prime = M(x) \quad (47)$$

For specific tasks, graphs of solutions are built. The comparison results are shown in Fig. 1-4. Assuming that the solutions of the Eq(46) are exact, estimates of the relative error of the approximate formula are obtained. For the cantilever, universal estimates of the relative error are obtained, which are independent of the beam length. The relative error depends only on the parameter:

$$r = \frac{q \ell^3}{6EJ} \quad (48)$$

Its values are given in Table 1.

### 4 Conclusion

Using table 1 and graphs, it is possible to find the limits of application of the approximate bending equation of the elastic beam line for a given value of relative error.

### References

[1] Askandar N H, Mahmood A D 2020 Torsional Strengthening of RC Beams with Continuous Spiral Near-Surface Mounted Steel Wire Rope. *International Journal of Concrete Structures and Materials* 14(1), 7. doi: 10.1186/s40069-019-0386-4I

[2] Ahmed H Q, Jaf D K, Yaseen S A 2020 Flexural Capacity and Behaviour of Geopolymer
Concrete Beams Reinforced with Glass Fibre-Reinforced Polymer Bars. *International Journal of Concrete Structures and Materials* **14**(1), 14. doi: 10.1186/s40069-019-0389-1

[3] Wang Z, Wei Y, Li N, Zhao K, Ding M 2020 Flexural behavior of bamboo-concrete composite beams with perforated steel plate connections. *Journal of Wood Science* **66**(1), 4. doi: 10.1186/s10086-020-1854-9

[4] Qiu M, Shao X, Wille K, Yan B, Wu J 2020 Experimental Investigation on Flexural Behavior of Reinforced Ultra High Performance Concrete Low-Profile T-Beams. *International Journal of Concrete Structures and Materials* **14**(1), 5. doi: 10.1186/s40069-019-0380-x

[5] Liu X, Xin W, Xie K, Wu Z, Li F 2020 Bond Behavior of Basalt Fiber-Reinforced Polymer Bars Embedded in Concrete Under Mono-tensile and Cyclic Loads. *International Journal of Concrete Structures and Materials* v. **14**, 19. doi: 10.1186/s40069-020-0394-4

[6] Park J, Park S-K, Hong S 2020 Experimental study of flexural behavior of reinforced concrete beam strengthened with prestressed textile-reinforced mortar. *Materials* **13**(5), 1137.

[7] Karathanasopoulos N, Dos Reis F, Diamantopoulou M, Ganghoffer J-F 2020 Mechanics of beams made from chiral metamaterials: Tuning deflections through normal-shear strain couplings. *Materials and Design* **189**, 108520. doi: 10.1016/j.matdes.2020.108520

[8] Liu H, Wang P, Liu Y, Dai J, Yang J 2020 A new theoretical method for calculating front abutment stress during coal mining. *Energy Science and Engineering* **8**(3), pp 836-848. doi: 10.1002/ese3.554

[9] Bejan A, Ferber L, Lorente S 2020 Convergent Evolution of Boats with Sails. *Scientific Reports* **10**(1), 2703. doi: 10.1038/s41598-020-58940-5

[10] Sakai Y, Ohsaki M, Adriaenssens S 2020 A 3-dimensional elastic beam model for form-finding of bending-active gridshells. *International Journal of Solids and Structures* **193-194**, pp 328-337. doi: 10.1016/j.ijsolstr.2020.02.034

[11] Abdelwahed B. 2020 Nonlinear numerical simulation for reinforced concrete elements with explicit time integration procedure. *Case Studies in Construction Materials*. 12, e00344. doi: 10.1016/j.cscem.2020.e00344

[12] Chen Y, Dong J, Tong Z, Jiang R, Yue Y 2020 Flexural behavior of composite box girders with corrugated steel webs and trusses *Engineering Structures* **209**, 110275. doi: 10.1016/j.engstruct.2020.110275

[13] Szeptyński P 2020 Comparison and experimental verification of simplified one-dimensional linear elastic models of multilayer sandwich beams. *Composite Structures* **241**, 112088. doi: 10.1016/j.compstruct.2020.112088

[14] Firsanov V V 2019 The basic stress-strain state of a circular plate of variable thickness based on a nonclassical theory. *Journal of Machinery Manufacture and Reliability* **48**(1) pp 54-60. doi: 10.3103/S1052618819010072

[15] Ge W, Ashour A F, Lu W, Cao D 2020 Flexural Performance of Steel Reinforced ECC Concrete Composite Beams Subjected to Freeze-Thaw Cycles. *International Journal of Concrete Structures and Materials* **14**(1), 11. doi: 10.1186/s40069-019-0385-5

[16] Wang H, Zhang L 2020 Local existence and uniqueness of increasing positive solutions for non singular and singular beam equation with a parameter. *Boundary Value Problems* (1), 10. doi: 10.1186/s13661-019-01320-4

[17] Bisshopp K E, Drucker D C 1945 Large deflection of cantilever beams. *Q. Appl. Math* **3** pp 272-275.

[18] Banerjee A., Bhattacharya B, Mallik A K 2008 Large deflection of cantilever beams with geometric non-linearity: Analytical and numerical approaches. *International Journal of Non-Linear Mechanics* **43**(5) pp 366-376. doi: 10.1016/j.ijnonlinmec.2007.12.020

[19] Ibrahim Abu-Alshaikh, Hashem S Alkhaldi, Nabil Beithou 2018 Large Deflection of Prismatic Cantilever Beam Exposed to Combination of End Inclined Force and Tip Moment. *Modern Applied Science* **12**(1) pp 98-111. doi: 10.5539/mas.v12n1p98

[20] Nageswara Rao B, Venkateswara Rao G 1989 Large deflections of a cantilever beam subjected
to a rotational distributed loading. *Forschung im Ingenieurwesen* A 55, pp 116-120. doi: 10.1007/BF02574981

[21] Singhal D, Narayanamurthy V 2019 Large and Small Deflection Analysis of a Cantilever Beam. *Journal of The Institution of Engineers* (India): Series A 100 pp 83-96. doi: 10.1007/s40030-018-0342-3