Hamilton-Jacobi Solutions for Strongly-Coupled Gravity and Matter

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Abstract. A Green’s function method is developed for solving strongly-coupled gravity and matter in the semiclassical limit. In the strong-coupling limit, one assumes that Newton’s constant approaches infinity, $G \rightarrow \infty$. As a result, one may neglect second order spatial gradients, and each spatial point evolves like an homogeneous universe. After constructing the Green’s function solution to the Hamiltonian constraint, the momentum constraint is solved using functional methods in conjunction with the superposition principle for Hamilton-Jacobi theory. Exact and approximate solutions are given for a dust field or a scalar field interacting with gravity.

Strong-coupling expansions have proven to been useful in elucidating gauge theories of modern particle physics (see, e.g., Creutz 1983 and Polyakov 1987). A general class of Hamilton-Jacobi solutions for strongly-coupled gravity and matter will be presented here. It is hoped that these solutions will help clarify quantum features of gravity that are necessary in formulating the inflationary scenario (Hartle & Hawking 1983 and Hartle 1997).

In the context of classical gravity, the strong-coupling expansion is very similar although not exactly the same as a long-wavelength expansion. In their analysis of singularities in the early Universe, Lifshitz and Khalatnikov (1964) expanded the 3-metric in a Taylor series in what was basically a strong-coupling expansion (see also Belinski et al 1970 and Landau and Lifshitz 1975). However, their program was incomplete in that they did not give an explicit solution of the momentum constraint equation which is just the $G^0_0$ Einstein equation. Pilati (1982), Teitelboim (1982) and Henneaux et al (1982) took some initial steps in formulating a quantum treatment of strongly-coupled gravity although many difficulties still remain (see also Husain 1988). They pointed out that this system is much simpler than the case for full gravity: the Hamiltonian was similar to that of an homogeneous minisuperspace model. Solutions were given which were a product over the points on a lattice. Once again, they did not attempt to solve the momentum constraint. In the present work, I hope to improve upon
this situation by showing how to simultaneously solve both the energy and momentum
constraints in the semiclassical limit. Special cases of the semiclassical limit were given
before (Salopek 1991), but now general solutions will be given.

A long-wavelength expansion has been considered by numerous authors, including
Tomita (1975), Salopek and Stewart (1992), Deruelle and Langlois (1995) and Veneziano
(1997). However, by using Hamilton-Jacobi (HJ) methods, one can elegantly solve the
momentum constraint. In fact, Parry et al. (1994) used a spatial gradient expansion
which is manifestly invariant under spatial coordinate transformations to give a HJ
solution for gravity and matter. They described a systematic method of obtaining
terms of arbitrarily high order. Soda et al. (1995), Chiba (1995) and Darian (1997)
have extended this method to encompass Brans-Dicke gravity, n-dimensional gravity,
and gravity interacting with electromagnetism, respectively.

In a previous paper, Salopek (1991) had shown that one could construct solutions
of the energy constraint and the momentum constraint for strongly-coupled gravity by
assuming that the generating functional was of the simple form:

\[ \mathcal{S} = -2 \int d^3 x \gamma^{1/2} H[\phi(x), \chi(x)], \]

(1)

where \( \phi(x) \) and \( \chi(x) \) denote the matter fields. If the Hubble function \( H \) were a constant,
then this functional would correspond to the volume of any given 3-geometry. For
many astrophysical applications, the ansatz of eq.(1) is sufficient. Such a formalism
provides an elegant description of the long-wavelength evolution of fluctuations arising
from inflation. For example, it was used to construct inflation models that yielded
non-Gaussian fluctuations (Salopek 1992a). Such models are still of observational and
theoretical interest (Moscardini et al. 1993). However, for future extensions of the HJ
approach, it is of interest to construct a more general class of solutions. Some steps
were taken in that direction by Salopek (1991) who presented a complete solution of
the energy constraint of the long-wavelength problem. In addition, by a suitable choice
of the spatial coordinates, one could solve the momentum constraint. In the present
work, powerful geometric methods will be utilized to give a more elegant solution of the
momentum constraint.

Over the past few years, an ever increasing number of tools have been developed in
order to solve the functional Hamilton-Jacobi equation. They include:

(1) The Spatial Gradient Expansion. For centuries, Taylor series solutions have been
constructed for linear equations. Series expansion techniques can also be applied to
nonlinear differential equations including the HJ equation for general relativity (Parry et
al 1994, Salopek 1997). In this way, it is possible to decompose semiclassical superspace
into a sum of minisuperspaces.

(2) The Superposition Principle. Since the equations for quantum mechanics are linear,
the superposition principle plays an important role. By employing the stationary phase
approximation, one can also enunciate a Superposition Principle for Hamilton-Jacobi theory (see, e.g., Salopek 1997). Although it is not linear, it proves very effective in constructing complicated solutions from more elementary ones (Landau & Lifshitz 1960). It will also prove useful in solving inhomogeneous gravitational problems such as the one presented in this paper.

The Lagrangian and Hamiltonian formulations for strongly-coupled gravity and matter are given in section 1. The semiclassical solution method may be simply explained using some rudimentary ideas from quantum mechanics. A complete solution of the energy constraint for a dust field interacting with gravity is described in section 2. This solution is related to the well-known Kasner metric. The momentum constraint is solved using a superposition over the complete set of solutions. It is also shown how to construct the set of constant functionals. Explicit solutions are constructed for a single dust field in section 3 and for two dust fields in section 4. In sections 5, 6 and 7, the entire analysis is repeated for a scalar field interacting with gravity. Conclusions are given in section 8.

1. Strongly-Coupled Equations for Gravity and Matter

The action for Einstein gravity interacting with a dust field, $\chi$, and a scalar field, $\phi$, may be written as follows:

$$I = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa} (4) R - \frac{1}{2\kappa} g^{\mu\nu} \phi,_{\mu} \phi,_{\nu} - \kappa V(\phi) - \frac{n}{2} (g^{\mu\nu} \chi,_{\mu} \chi,_{\nu} + \kappa^2) \right\}, \quad (2a)$$

where

$$\kappa \equiv 8\pi G = 8\pi/m_P^2, \quad (2b)$$

is the gravitational coupling constant. $n \equiv n(t, x)$ is a Lagrange multiplier which ensures that the square of the four-velocity, $U^\mu$, is minus one:

$$U^\mu U_\mu = -1, \quad U^\mu = -g^{\mu\nu} \chi,_{\nu}/\kappa. \quad (3)$$

The above form can be obtained from the usual one by appropriate scalings of the matter fields and their coupling constants. Factors of $\kappa$ appear in key places of eq.(2a) in order to obtain the desired energy constraint:

$$H(x) = \kappa \gamma^{-1/2} (2\gamma_{ac}\gamma_{bd} - \gamma_{ab}\gamma_{cd}) \pi^{ab}\pi^{cd} + \kappa \gamma^{-1/2} \left( \pi^0 \right)^2 + \kappa \pi^x \sqrt{1 + \chi^a,_{a}/\kappa^2} - \frac{\gamma^1}{2\kappa} R + \frac{\gamma^1}{2\kappa} \phi,^a,_{a} + \kappa \gamma^{1/2} V(\phi) = 0. \quad (4a)$$

No factors of $\kappa$ appear in the momentum constraint:

$$H_i(x) = -2 \left( \gamma_{ik} \pi^{kj} \right)_{,j} + \pi^{kl} \gamma_{kl, i} + \pi^{0} \phi,_{i} + \pi^{x} \chi,_{i} = 0. \quad (4b)$$
In a classical Hamilton-Jacobi formulation of general relativity, one defines the generating functional,

\[ S \equiv S[\gamma_{ab}(x), \phi(x), \chi(x)], \quad (5a) \]

by assigning a real number to each field configuration \([\phi(x), \chi(x)]\) on a space-like hypersurface with 3-geometry given by \(\gamma_{ab}(x)\). In a semiclassical context, one allows for the possibility that \(S\) may be complex. The Hamilton-Jacobi equations are obtained from the constraints \((4a-b)\) by replacing the momenta with functional derivatives of \(S\):

\[ \pi^\chi(x) = \frac{\delta S}{\delta \chi(x)}, \quad \pi^\phi(x) = \frac{\delta S}{\delta \phi(x)}, \quad \pi^{ab}(x) = \frac{\delta S}{\delta \gamma_{ab}(x)}. \quad (5b) \]

In the limit of large gravitational coupling, \(\kappa \to \infty\), the generating functional \(S^{(s)}\) for the strongly-coupled system evolves according to the following equations:

\[ \mathcal{H}^{(s)}(x)/\kappa = \gamma^{-1/2} \left(2\gamma_{ac}\gamma_{bd} - \gamma_{ab}\gamma_{cd}\right) \frac{\delta S^{(s)}}{\delta \gamma_{bd}} \frac{\delta S^{(s)}}{\delta \gamma_{cd}} + \frac{\delta S^{(s)}}{\delta \chi^2} \]

\[ + \gamma^{-1/2} \left(\frac{\delta S^{(s)}}{\delta \phi}\right)^2 + \gamma^{1/2} V(\phi) = 0, \quad (6a) \]

\[ \mathcal{H}_i^{(s)}(x) = -2 \left(\gamma_{ik} \frac{\delta S^{(s)}}{\delta \gamma_{kj}}\right)_{,j} + \frac{\delta S^{(s)}}{\delta \gamma_{kl}} \gamma_{kl,i} + \frac{\delta S^{(s)}}{\delta \phi_i} \phi_i + \frac{\delta S^{(s)}}{\delta \chi} \chi_i = 0. \quad (6b) \]

The energy constraint is ultra-local in the sense that different spatial points are not coupled to each other. As a result, the Poisson bracket of \(\mathcal{H}^{(s)}(x)\) with \(\mathcal{H}^{(s)}(y)\) vanishes, and consistency of the Hamiltonian constraint at different spatial points is assured. Consistency or ‘integrability’ of the Hamiltonian constraint for full general relativity is more complicated. It is related to the freedom in choosing an arbitrary time foliation: see Parry \textit{et al} (1994) as well as Salopek (1995). However, the momentum constraint does couple the spatial points together because spatial derivatives appear. It is this cross-coupling of different spatial points that makes the strongly-coupled system non-trivial.

(In order to simplify the notation, \(S\) will be used to denote \(S^{(s)}\) for the remainder of this paper.)

\subsection*{1.1. Analogy from Elementary Quantum Mechanics}

The general solution for the strongly-coupled system may appear quite complicated so it is instructive to consider a well-known example from elementary quantum mechanics which illustrates the essential features.
1.2. Rotationally Symmetric Solutions to the Schrodinger Equation for a Free Particle

The two-dimensional Schrodinger equation for a free particle with zero angular momentum is given by:

\[
\frac{i}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right), \quad (7a)
\]

\[
L_z \psi = i \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \psi = 0. \quad (7b)
\]

In solving these two equations, one would ordinarily invoke polar coordinates, \((r, \theta)\), and then write the solution as \(\psi \equiv \psi(r)\). Instead, I will utilize a circuitous method which is of interest because the same technique may be generalized to the case of strongly-coupled gravity.

One notes immediately that the first equation \((7a)\) can be solved generally without any reference to the second by using a superposition of plane waves:

\[
\Psi = \int d^2k \ f(\vec{k}) \ e^{i(\vec{k} \cdot \vec{x} - \omega t)} \quad w = (k_1^2 + k_2^2)/(2m), \quad (8)
\]

where \(f(\vec{k})\) is an arbitrary function of the wave-vector \(\vec{k}\). Please note that a plane wave \(e^{i(\vec{k} \cdot \vec{x} - \omega t)}\) is not a solution of the symmetry condition, eq.(7b). However, by suitably restricting the function \(f\) so that it is function only of the magnitude of \(k = \sqrt{k_1^2 + k_2^2}\),

\[
f(\vec{k}) \equiv f(k) \quad (9)
\]

one can indeed satisfy the symmetry condition.

In a semiclassical gravitational context, the energy constraint \((6a)\) will be analogous to the Schrodinger equation \((7a)\) and the momentum constraint \((6b)\) will be analogous to the symmetry condition \((7b)\). In general relativity, it is not yet known how to explicitly solve the momentum constraint such that one obtains a set of reduced variables (physical variables) analogous to solving the \(L_z\) constraint and then deducing that \(\psi\) is solely a function of \(r\). Instead, one solves the strongly-coupled system by first obtaining a general class of solutions for the energy constraint. By considering a suitable superposition over these solutions, one then constructs a solution to the momentum constraint.

Shortly after Dirac (1958) gave general relativity its Hamiltonian form, Higgs (1958) pointed out that the momentum constraint of general relativity implied that the wavefunctional was invariant under spatial coordinate transformations. Although, the Hamilton-Jacobi equation was written explicitly by Peres (1962), only quite recently (Salopek and Stewart 1992, Parry et al 1994) has the spatial coordinate invariance been exploited to construct explicit solutions of the full HJ equation. They utilized a spatial gradient expansion.
2. Strongly-Coupled Solutions: Dust Interacting with Gravity

In the strongly-coupled limit, the energy constraint for gravity and a dust field is:
\[ \mathcal{H}^{(s)}(x)/\kappa = \gamma^{-1/2} \left( 2\gamma_{ac}\gamma_{bd} - \gamma_{ab}\gamma_{cd} \right) \frac{\delta S}{\delta \gamma_{ab}} \frac{\delta S}{\delta \gamma_{cd}} + \frac{\delta S}{\delta \chi} = 0. \]  

(10)

2.1. Green’s Function Solution for the Strongly-Coupled HJ Equation

A complete solution, \( G \), to the above equation is:
\[ G[\gamma^{(0)}_{ab}(x), \chi^{(0)}(x)|\gamma^{(0)}_{ab}(x), \chi^{(0)}(x)] = \frac{4}{3} \int d^3x \frac{1}{(\chi(x) - \chi^{(0)}(x))} \left[ 2\gamma^{1/4} \gamma^{1/4}_{(0)} \cosh(\sqrt{\frac{3}{8}z}) - \gamma^{1/2} - \gamma^{1/2}_{(0)} \right], \]  

(11a)

where
\[ z = \frac{1}{2} \sqrt{\text{Tr} \left\{ \ln \left[ [h][h^{-1}(0)] \right] \ln \left[ [h][h^{-1}(0)] \right] \right\}}, \]  

(11b)

and \([h]\) and \([h^{(0)}]\) are matrices with components given by
\[ [h]_{ab} = \gamma^{-1/3}\gamma_{ab}, \quad [h^{(0)}]_{ab} = \gamma^{-1/3}_{(0)}\gamma^{(0)}_{ab}. \]  

(11c)

This solution will be referred to as the Green’s function. It depends on 7 parameter fields, \( \chi^{(0)}(x) \) and \( \gamma^{(0)}_{ab}(x) \), which may be interpreted as the initial fields for the dust and 3-metric. Although the variable \( z \) had been defined in an earlier paper (Salopek 1991), the present form for the generating functional \( G \) is new. It is superior to the previous formulation in that it is symmetric upon interchange of the metric \( \gamma_{ab}(x) \) with the initial metric \( \gamma^{(0)}_{ab}(x) \):
\[ G[\gamma_{ab}(x), \chi(x)|\gamma^{(0)}_{ab}(x), \chi^{(0)}(x)] = G[\gamma^{(0)}_{ab}(x), \chi(x)|\gamma_{ab}(x), \chi^{(0)}(x)]. \]  

(12)

One may verify by differentiation that eq.(11a) is indeed a solution to eq.(10) but in its present form it does not satisfy the momentum constraint,
\[ \mathcal{H}^{(s)}(x) = -2 \left( \gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}} \right)_{ij} + \frac{\delta S}{\delta \gamma_{kl}} \gamma_{kl,i} + \frac{\delta S}{\delta \chi} \chi,i = 0, \]  

(13)

because the parameter fields are spatially dependent and hence they carry momentum.

2.1.1. Too Many Parameter Fields  There is an additional problem with the Green’s function solution eq.(11a) as it stands: there is one too many parameter fields. This subtle fact is not immediately apparent but it becomes painfully clear after one explicitly tries to implement the superposition principle in Section 2.2. One can resolve this problem by arbitrarily setting one of the parameter fields to zero, say \( \chi^{(0)}(x) = 0 \).
brief explanation of this point now follows although it is best to skip over the next two paragraphs upon first reading of this paper.

For each spatial point, there were originally seven degrees of freedom associated with $\chi(x)$ and the symmetric 3-metric $\gamma_{ab}(x)$. Because of the symmetry between the initial fields, $[\gamma_{ab}(0)(x), \chi(0)(x)]$ and the original fields, there is an equal number of degrees of freedom associated with the former. I will now demonstrate that the initial fields are not really independent. First note that there is a relationship between

$$
\pi^\chi_{(0)}(x) \equiv -\frac{\delta G}{\delta \chi(0)(x)} \quad \text{and} \quad \pi^{ab}_{(0)}(x) \equiv -\frac{\delta G}{\delta \gamma^{(0)}_{ab}(x)},
$$

(14)

because the Hamiltonian constraint is also valid when expressed in terms of the initial fields:

$$
0 = \mathcal{H}^{(s)}(x)/\kappa = \gamma^{-1/2}(0) \left(2\gamma^{(0)}_{ac}\gamma^{(0)}_{bd} - \gamma^{(0)}_{ab}\gamma^{(0)}_{cd}\right) \pi^{cd}_{(0)} \pi^{ab}_{(0)} + \pi^\chi_{(0)}.
$$

(15)

If one were to apply the superposition principle in Section 2.2 by minimizing $G + Q$ with respect to $\gamma^{(0)}_{ab}(x)$ and $\chi(0)(x)$, one would find that

$$
0 = \delta G / \delta \gamma^{(0)}_{ab} + \delta Q / \delta \gamma^{(0)}_{ab},
$$

(16a)

$$
0 = \delta G / \delta \chi(0) + \delta Q / \delta \chi(0).
$$

(16b)

As a result, the momenta

$$
\pi^\chi_{(0)}(x) = \frac{\delta Q}{\delta \chi(0)(x)},
$$

(17a)

$$
\pi^{ab}_{(0)}(x) = \frac{\delta Q}{\delta \gamma^{(0)}_{ab}(x)},
$$

(17b)

are functions of $\gamma^{(0)}_{ab}(x)$ and $\chi(0)(x)$. Hence eq. (15) implicitly defines a relationship amongst the initial fields $[\gamma^{(0)}_{ab}(x), \chi(0)(x)]$: they are not independent variables.

In order to avoid the above problem I will now arbitrarily fix one of the parameter fields. In the present work, I will choose $\chi(0)(x) = 0$, while $\gamma^{(0)}_{ab}(x)$ will remain a free variable. The subsequent analysis is indeed consistent, although other choices are presumably possible.

2.2. Superposition Principle for HJ Theory and Solution of the Momentum Constraint

Assuming $\chi(0)(x) = 0$ in the general solution to the Hamiltonian constraint, eq.(11a-c), one can construct a solution $S$ to the momentum constraint through a superposition principle:

$$
S[\gamma_{ab}(x), \chi(x)] = G[\gamma_{ab}(x), \chi(x)| \gamma^{(0)}_{ab}(x), \chi(0)(x) = 0] + Q[\gamma^{(0)}_{ab}],
$$

(18a)
where $\gamma_{ab}^{(0)}(x)$ has been chosen to minimize $\mathcal{G} + \mathcal{Q}$,

$$0 = \frac{\delta \mathcal{G}}{\delta \gamma_{ab}^{(0)}} + \frac{\delta \mathcal{Q}}{\delta \gamma_{ab}^{(0)}}. \quad (18b)$$

Here $\mathcal{Q}$ is an arbitrary “gauge-invariant” functional of the initial 3-metric, $\gamma_{ab}^{(0)}$:

$$0 = -2 \left( \frac{\gamma_{ik}^{(0)}}{\delta \gamma_{kj}^{(0)}} \right)_{,j} + \frac{\delta \mathcal{Q}}{\delta \gamma_{kl}^{(0)}} \gamma_{kl,i}. \quad (18c)$$

Construction of solutions to the strongly-coupled problem through the superposition principle eq.(18a-c) will be known as the Semiclassical Green’s Function Method.

An elementary discussion of the Superposition Principle for HJ theory was given by Salopek (1997). As will be demonstrated in section 3.3, the functional $\mathcal{Q}[\gamma_{ab}^{(0)}]$ may be interpreted as the initial state for $\mathcal{S}$ when $\chi(x) = 0$:

$$\mathcal{S}[\gamma_{ab}(x), \chi(x) = 0] = \mathcal{Q}[\gamma_{ab}]. \quad (18d)$$

Since the Poisson bracket of $\mathcal{H}_i^{(s)}(x)$ with $\mathcal{H}^{(s)}(x)$ vanishes (weakly), the momentum constraint is preserved upon evolution: if $\mathcal{S}$ satisfies $\mathcal{H}_i^{(s)}(x) = 0$ for $\chi(x) = 0$, it is guaranteed to solve it for arbitrary $\chi(x)$.

2.3. Classical Evolution

Classical evolution is obtained from the minimization prescription eq.(18b) by solving for $\gamma_{ab}(x)$ in terms of $\gamma_{ab}^{(0)}(x)$ and $\chi(x)$:

$$\left( \frac{\gamma}{\gamma^{(0)}} \right)^{1/2} = \left( 1 - \frac{1}{2} \chi \gamma^{-1/2} \pi^{(0)} \right)^2 - \frac{3}{2} \chi \gamma^{-1/2} \pi^{(0)} \pi^{(0)}_{ab}, \quad (19a)$$

\[ z = \sqrt{\frac{8}{3}} \tanh^{-1} \left\{ \sqrt{\frac{3}{2}} \chi \left( \gamma^{-1/2} \pi^{(0)} \right) \left( \gamma^{-1/2} \pi^{(0)}_{ab} \right)^{1/2} \left( 1 - \frac{1}{2} \chi \gamma^{-1/2} \pi^{(0)} \right) \right\}, \quad (19b) \]

$$[h] = [h^{(0)}] \exp \left[ \frac{2z \left[ \pi^{(0)} \right] \left[ \gamma^{(0)} \right]}{\left[ \pi^{(0)} \right]^{1/2}} \right] \quad (19c)$$

Matrix notation denoted by $[\ ]$ has been used to simplify the above expression. For e.g., $[\gamma^{(0)}]$ denotes the 3-metric with components $\gamma_{ab}^{(0)}$ and $[h]$ denotes $h_{ab}$, etc. In addition $[\pi^{(0)}(x)]$ is a matrix whose components are just the functional derivative of $\mathcal{Q}$ with respect to $\gamma_{ab}^{(0)}(x)$,

$$[\pi^{(0)}(x)]^{ab} \equiv \pi^{(0)}_{ab}(x) = -\frac{\delta \mathcal{G}}{\delta \gamma_{ab}^{(0)}(x)} = \frac{\delta \mathcal{Q}}{\delta \gamma_{ab}^{(0)}}. \quad (19d)$$
where the last equality follows from eq.(18b). $\pi(0)$ is its traceless counterpart where $\pi(0)$ denotes the trace:

$$\pi(0) = \gamma_{ab} \tilde{\pi}^{ab}(0), \quad \tilde{\pi}^{ab}(0) = \pi_{ab}(0) - \frac{1}{3} \gamma_{ab} \pi(0), \quad \tilde{\pi}_{ab} = \gamma_{ac} \gamma_{bd} \tilde{\pi}^{cd}(0).$$

In eq.(19c), the exponential of a matrix $[A]$ is defined in terms of a Taylor series expansion,

$$\exp[A] = [I] + [A] + \frac{1}{2!} [A] [A] + \ldots$$

At this point it is important to remember that $[\chi(x), \gamma_{ab}(x)]$ and $[\gamma_{ab}^{(0)}(x), \pi_{ab}^{(0)}(x)]$ are spatially dependent. Eq.(19a-d) represents a general solution to the strongly-coupled system consisting of gravity and dust.

2.4. Relation to Kasner Metric

Using semiclassical methods, Francisco and Pilati (1985) have shown how to derive the Kasner metric describing pure gravity (see also Salopek and Stewart 1993). Here, it will be shown that the case of dust interacting with gravity admits a solution where the metric behaves like a Kasner metric at two separate epochs.

$(\gamma/\gamma(0))^{1/2}$ is a quadratic polynomial in $\chi$,

$$\left(\frac{\gamma}{\gamma(0)}\right)^{1/2} = \left(1 - \frac{\chi}{\chi(1)}\right) \left(1 - \frac{\chi}{\chi(2)}\right),$$

with roots $(\chi(1), \chi(2))$,

$$\chi(1) = \left\{ \frac{1}{2} \gamma(0)^{-1/2} \pi(0) + \sqrt{\frac{3}{2} \left[ \gamma(0)^{-1} \tilde{\pi}^{(0)}_{ab} \tilde{\pi}^{ab}(0) \right]^{1/2}} \right\}^{-1},$$

$$\chi(2) = \left\{ \frac{1}{2} \gamma(0)^{-1/2} \pi(0) - \sqrt{\frac{3}{2} \left[ \gamma(0)^{-1} \tilde{\pi}^{(0)}_{ab} \tilde{\pi}^{ab}(0) \right]^{1/2}} \right\}^{-1},$$

which are spatially dependent: $\chi(1) \equiv \chi(1)(x)$ and $\chi(2) \equiv \chi(2)(x)$. The average of the two roots,

$$\frac{\chi(1) + \chi(2)}{2} = \frac{2 \gamma^{1/2} \pi(0)}{\left(\pi(0)^2 - 6 \pi_{ab} \tilde{\pi}^{(0)}_{ab}\right)}$$

gives the time when the extrema of $(\gamma/\gamma(0))^{1/2}$ is reached. $z$ becomes

$$z = \sqrt{\frac{2}{3}} \ln \left( \frac{1 - \frac{\chi(2)}{\chi(1)}}{1 - \frac{\chi(2)}{\chi(3)}} \right).$$
Note that the equation for \( h \) may be put in the following form

\[
[h] = [h_{(0)}^{1/2}] \exp \left\{ \frac{2\pi [\gamma_{(0)}^{1/2}] [\pi_{(0)}] [\gamma_{(0)}^{1/2}]}{(\pi_{ab}^{(0)} \pi_{ab}^{(0)})^{1/2}} \right\} [h_{(0)}^{1/2}],
\]  

(25)

where the argument of the exponential is explicitly a symmetric matrix. Following Salopek (1992), we diagonalize the argument using an orthogonal matrix \([O]\),

\[
\frac{[\gamma_{(0)}^{1/2}] [\pi_{(0)}] [\gamma_{(0)}^{1/2}]}{(\pi_{ab}^{(0)} \pi_{ab}^{(0)})^{1/2}} = [O]^T [D] [O],
\]  

(26a)

where

\[
[O]^T [O] = [O] [O]^T = [I] \quad \text{(identity matrix)},
\]  

(26b)

\[
[D] = \text{Diag}[d_1, d_2, d_3], \quad d_1 + d_2 + d_3 = 0, \quad d_1^2 + d_2^2 + d_3^2 = 1.
\]  

(26c)

\([h]\) then admits the following simple form:

\[
[h] = [h_{(0)}^{1/2}] [O]^T \text{Diag} \left[ \left( \frac{1 - \chi_{(2)}}{1 - \chi_{(1)}} \right)^{2p_1^{(1)}} \left( \frac{1 - \chi_{(2)}}{1 - \chi_{(1)}} \right)^{2p_2^{(1)}} \right] \left( \frac{1 - \chi_{(2)}}{1 - \chi_{(1)}} \right)^{2p_3^{(1)}} 
\]  

\[ [O] [h_{(0)}^{1/2}], \]

(27)

and \([\gamma]\)

\[
[\gamma] = [\gamma_{(0)}^{1/2}] [O]^T \text{Diag} \left[ \left( \frac{1 - \chi_{(2)}}{\chi_{(1)}} \right)^{2p_1^{(2)}} \left( \frac{1 - \chi_{(2)}}{\chi_{(1)}} \right)^{2p_2^{(2)}} \right] \left( \frac{1 - \chi_{(2)}}{\chi_{(1)}} \right)^{2p_3^{(2)}} 
\]  

\[ [O] [\gamma_{(0)}^{1/2}], \]

(28)

with

\[
p_1^{(i)} = \frac{1}{3} - \sqrt{\frac{2}{3}} d_i, \quad p_2^{(i)} = \frac{1}{3} + \sqrt{\frac{2}{3}} d_i.
\]  

(29a)
Given a spatial point \( x \), for times \( \chi \) close to \( \chi(2)(x) \), the diagonal part of the 3-metric in eq.(28) evolves like a Kasner universe, with Kasner exponents, \( p_2^{(1)}, p_2^{(2)}, p_2^{(3)} \):

\[
p_2^{(1)} + p_2^{(2)} + p_2^{(3)} = 1, \quad \left( p_2^{(1)} \right)^2 + \left( p_2^{(2)} \right)^2 + \left( p_2^{(3)} \right)^2 = 1.
\]

(29b)

For times close to \( \chi(1)(x) \), the 3-metric also evolves like a Kasner universe, but with Kasner exponents, \( p_1^{(1)}, p_1^{(2)}, p_1^{(3)} \). (See Belinski et al 1970 and Salopek and Stewart 1993.)

2.5. Constant Functionals

Given a generating functional \( S \) describing a HJ flow, \( C[\gamma_{ab}(x), \chi(x)] \) is a Constant Functional if for all choices of the lapse \( N \) and shift \( N^i \), \( \dot{C} \) vanishes:

\[
0 = \dot{C} \equiv \int d^3 x \left\{ \frac{\delta C}{\delta \chi(x)} \dot{\chi}(x) + \frac{\delta C}{\delta \gamma_{ab}(x)} \dot{\gamma}_{ab}(x) \right\}
\]

(30)

\[
= \int d^3 x \left\{ \frac{\delta C}{\delta \chi(x)} \left( N + N^i \chi_i \right) + 2 N \gamma^{-1/2} \left[ 2 \gamma_{ac}(x) \gamma_{bd}(x) - \gamma_{ab}(x) \gamma_{cd}(x) \right] \frac{\delta S}{\delta \gamma_{cd}(x)} \right.
\]

\[
\left. + N_{a|b} + N_{b|a} \right\} \frac{\delta C}{\delta \gamma_{ab}(x)} \}
\]

where the last equality follows from the HJ flow equations,

\[
\left( \dot{\chi} - N^i \chi_i \right) / N = 1,
\]

(31a)

\[
\left( \dot{\gamma}_{ab} - N_{a|b} - N_{b|a} \right) / N = 2 \gamma^{-1/2} \left( 2 \gamma_{ac}(x) \gamma_{bd}(x) - \gamma_{ab}(x) \gamma_{cd}(x) \right) \frac{\delta S}{\delta \gamma_{cd}(x)}.
\]

(31b)

Since \( N, N_i \) are arbitrary, the constant functional \( C \) obeys the following:

\[
\frac{\delta C}{\delta \chi(x)} + 2 \left[ 2 \gamma_{ac}(x) \gamma_{bd}(x) - \gamma_{ab}(x) \gamma_{cd}(x) \right] \frac{\delta S}{\delta \gamma_{cd}(x)} \frac{\delta C}{\delta \gamma_{ab}(x)} = 0,
\]

(32a)

\[
-2 \left( \gamma_{ik} \frac{\delta C}{\delta \gamma_{kj}} \right)_{;j} + \frac{\delta C}{\delta \gamma_{kl}} \gamma_{kl,i} + \frac{\delta C}{\delta \chi} \chi_i = 0.
\]

(32b)

One may show that if

\[
C \equiv C[\gamma_{ab}^{(0)}(x)]
\]

(33)

is an arbitrary gauge-invariant functional of the initial metric \( \gamma_{ab}^{(0)}(x) \) defined by the minimization prescription eq.(18b), then \( C \) is indeed a constant functional. Two simple examples are,

\[
C_1 = \int d^3 x \gamma_{(0)}^{1/2}, \quad \text{etc.}
\]

(34a)

\[
C_2 = \int d^3 x \gamma_{(0)}^{1/2} R_{(0)}, \quad \text{etc.}
\]

(34b)
Eq. (33) is an elegant result which has wide ranging implications. For example, a complete list of constant functionals could be used to define a particular universe within the ensemble of universes whose evolution is described by the generating functional $S$. In fact, a space-time transformation of that particular universe would not change the numerical values of the complete list of constant functionals.

3. Semiclassical Evolution: Dust and Gravity

Solving the HJ equations, the energy constraint and the momentum constraint, brings us one step closer to understanding the quantum theory of the gravitational field because the generating functional $S$, eq. (18a), may be interpreted as the phase of the wavefunctional in the semiclassical approximation,

$$\Psi[\gamma_{ab}(x), \chi(x)] \sim e^{iS[\gamma_{ab}(x), \chi(x)]/\hbar},$$

where Planck’s constant $\hbar$ is assumed to be tiny. Explicit solutions for $S \equiv S[\gamma_{ab}(x), \chi(x)]$, are now discussed.

Semiclassical evolution is found by inverting the classical evolution eqs. (19a-d) to express the initial metric, $\gamma_{cd}(0) \equiv \gamma_{cd}^{(0)}(\gamma_{cd}, \chi)$, in terms of the original fields, and then substituting to obtain $S$ in eq. (18a). Typically, the inversion is very difficult because one must solve nonlinear, partial differential equations. Some special cases are given below and in Section 4 where the inversions may be performed explicitly without resorting to an approximation method. In the section denoted General Case below, one obtains semiclassical solutions by inverting the classical evolution equations through a Taylor series in $\chi(x)$.

3.1. Elementary Example

In the special case where the initial functional $Q$ is proportional to the volume of the initial 3-geometry,

$$Q = C \int d^3x \gamma^{1/2}_0,$$

where $C$ is a homogeneous constant

$$Q = C \int d^3x \gamma^{1/2}_0, \quad \text{where C is a homogeneous constant} \quad (36)$$

it is easy to perform the necessary inversion to compute $S$. Since $\pi_{(0)}^{ab}$ vanishes, one finds that $z = 0$ and that

$$h_{ab}^{(0)} = h_{ab}, \quad (37a)$$

$$\gamma^{1/2}_0 = \frac{\gamma^{1/2}}{(1 - \frac{2}{3}C\chi)^2}. \quad (37b)$$

The generating functional $S$ is then given by eq. (18a)

$$S[\gamma_{ab}(x), \chi(x)] = -\frac{4}{3} \int d^3x \gamma^{1/2} \frac{1}{(\chi(x) - \frac{4}{3C})}. \quad (38)$$
This solution had been given previously by Salopek and Stewart (1992), and it is of the form given in eq. (39). It is mentioned here to show that the new formalism of this paper encompasses the earlier work.

If \( C \) is real, one may define the classical Hubble parameter, \( H \), through

\[
H(x) \equiv -\frac{\gamma^{-1/2}}{3} \frac{\delta S}{\delta \gamma_{ab}},
\]

(39)

giving

\[
H(x) = \frac{2}{3} \frac{1}{(\chi(x) - a)}.
\]

(40)

Hence, if \( C < 0 \) and \( \chi(x) \geq 0 \), the Universe is expanding (locally) at each spatial point, whereas for \( C > 0 \) and \( \chi(x) \geq 0 \), it is contracting locally until it reaches a singularity at \( \chi(x) = 4/(3C) \). The appearance of the singularity is problematic for the classical theory. \( C = 0 \) yields the trivial solution \( S = 0 \).

3.1.1. Evolution Beyond a Singularity

Complex values of the parameter \( C \) are of interest because they help to explain how a universe may evolve beyond a singularity. Let us write

\[
\frac{4}{3C} = a - ib, \text{ where } a, b \text{ are real},
\]

(41)

and then split \( S \) into real and imaginary parts:

\[
S = S_R + iS_I,
\]

(42a)

\[
S_R = -\frac{4}{3} \int d^3x \gamma^{1/2} \left[ \frac{\chi(x) - a}{(\chi(x) - a)^2 + b^2} \right],
\]

(42b)

\[
S_I = \frac{4b}{3} \int d^3x \gamma^{1/2} \left[ \frac{1}{(\chi(x) - a)^2 + b^2} \right].
\]

(42c)

The wavefunctional then becomes

\[
\Psi = \exp\left(-S_I/\hbar\right) \exp\left(iS_R/\hbar\right).
\]

(43)

As a result, one can define a classical Hubble parameter using the real part of the generating functional, \( S_R \), in eq. (39):

\[
H(x) = \frac{2}{3} \left[ \frac{\chi(x) - a}{(\chi(x) - a)^2 + b^2} \right].
\]

(44)

(Here I am assuming that \( H(x) \) or some closely related function can be represented by some Hermitean operator. In ordinary quantum mechanics, the expectation value
<\hat{p}> of some Hermitean operator \(\hat{p}\) is guaranteed to be real. Hence for a wavefunction which is tightly peaked about a classical trajectory, the expectation value of \(\hat{p}\) may be computed \textit{approximately} using the \textit{real} part of the phase.)

For \([\chi(x) - a]\) large and negative in eq.\((44)\), a universe is contracting (locally), whereas for \([\chi(x) - a]\) large and positive, the same universe is expanding (locally). Apparently, that universe was able to pass through the singularity at \(\chi(x) = a\), and bounce from a contracting phase to an expanding phase. If \(b > 0\), universes with large volumes are exponentially suppressed:

\[
|\Psi|^2 = \exp\left\{-\frac{8b}{3\hbar} \int d^3x \frac{1}{(\chi(x) - a)^2 + b^2}\right\}. \tag{45}
\]

The above analysis was valid for arbitrary \(\chi(x)\). These qualitative results concerning the bounce from a singularity are in agreement with a quantum formulation given by Salopek (1992b) where he assumed that \(\chi(x)\) was homogeneous.

### 3.2. Intermediate Example

The case of where the initial functional \(Q\) is a function of the initial volume is partially tractable:

\[
Q = f\left[V_0\right], \quad \text{with} \quad V_0 = \int d^3x \gamma_{(0)}^{1/2}. \tag{46}
\]

After minimizing with respect to \(\gamma_{(0)}^{(0)}(x)\), the classical evolution equations give

\[
z = 0, \quad h_{ab}(x) = h_{ab}^{(0)}(x), \tag{47a}
\]

\[
\gamma^{1/2} = \left(1 - \frac{3f'[V_0]}{4}\chi\right)^2 \gamma_{(0)}^{1/2}, \quad \tag{47b}
\]

where \(f'[V_0]\) denotes the derivative of the function \(f[V_0]\) with respect to the variable \(V_0\). Hence, only the conformal factor of the 3-metric evolves. Bringing the conformal factor to the other side, and integrating over all \(x\), one finds that, given a 3-metric \(\gamma_{ab}(x)\) and field configuration \(\chi(x)\), \(V_0\) is defined implicitly through

\[
V_0 = \int d^3x \gamma^{1/2} \frac{1}{\left[1 - \frac{3f'[V_0]}{4}\chi(x)\right]^2}. \tag{48a}
\]

The resulting generating functional for arbitrary \(\chi(x)\) is,

\[
\mathcal{S} = -\frac{4}{3} \int d^3x \gamma^{1/2} \frac{1}{\left(\chi - \frac{4}{3f'[V_0]}\right)} + f[V_0] - V_0 f'[V_0]. \tag{48b}
\]
By invoking a Legendre transformation,
\[ g = f - \frac{df}{dV(0)} V(0), \] (49a)
\[ b = \frac{df}{dV(0)}, \] (49b)
one can simplify eqs. (48a-b) even further,
\[ S = -\frac{4}{3} \int d^3 x \gamma^{1/2} \frac{1}{(\chi - \frac{4}{3}b)^2} + g(b), \] (50a)
\[ \frac{dg}{db} = -\int d^3 x \gamma^{1/2} \frac{1}{\left[1 - \frac{3b}{4\chi(x)}\right]^2}. \] (50b)

One verifies that the second eq. (50b) is a consequence of minimizing the first eq. (50a) with respect to the single independent variable \( b \). Finally, using one more set of substitutions,
\[ a = \frac{4}{3b}, \quad j(a) = g(b) \] (51)
one can express the generating functional in its simplest form,
\[ S[\gamma_{ab}(x), \chi(x)] = -\frac{4}{3} \int d^3 x \gamma^{1/2} \frac{1}{[\chi(x) - a]} + j(a), \] (52a)
\[ 0 = -\frac{4}{3} \int d^3 x \gamma^{1/2} \frac{1}{[\chi(x) - a]^2} + \frac{dj}{da}. \] (52b)

Once again, the second eq. (52b) is obtained from the first by minimizing \( S \) with respect to \( a \) where \( j(a) \) is an arbitrary function of \( a \).

One could have derived this result more simply by starting with the solution,
\[ \mathcal{T}[\gamma_{ab}, \chi(x)|a] = -\frac{4}{3} \int d^3 x \gamma^{1/2} \frac{1}{[\chi(x) - a]} \] (53)
of the energy constraint and the momentum constraint, and by then constructing another solution \( S \) by taking a superposition over the homogeneous parameter \( a \),
\[ S[\gamma_{ab}(x), \chi(x)] = \mathcal{T}[\gamma_{ab}, \chi(x)|a] + j(a), \] (54a)
which means that one chooses \( a \) according to the minimization prescription:
\[ 0 = \frac{\partial \mathcal{T}}{\partial a} + \frac{dj}{da}. \] (54b)

This solution was actually suggested by Salopek and Bond (1990) who gave a much different derivation. Here the aim was to show that the Green’s function method encompasses previous results. In addition, the Green’s function method has the advantage that it yields the classical evolution equations eqs. (47a-b).
3.2.1. Explicit Computation for Intermediate Example  If the arbitrary function \( j(a) \) is assumed to be linear,
\[
j(a) = a, \tag{55a}
\]
one may readily compute the generating functional eq.(52a), if \( \chi(x) = \bar{\chi} \) is homogeneous,
\[
\mathcal{S}[\gamma_{ab}(x), \chi(x) = \bar{\chi}] = \bar{\chi} + \left[ \frac{16}{3} \int d^3x \gamma^{1/2} \right]^{1/2} \text{(exact)}, \tag{55b}
\]
since one may invert eq.(52b) to give,
\[
a = \bar{\chi} + \left[ \frac{4}{3} \int d^3x \gamma^{1/2} \right]^{1/2} \text{(exact)} . \tag{55c}
\]
The sign before the square root in eq.(55c) is arbitrary, and I have chosen it to be positive. (Choosing a negative sign yields another solution of the HJ equation.)

One may obtain an approximate form which is valid for mildly inhomogeneous fields \( \chi(x) \), by assuming that \( a \) is large, and then expanding the implicit equation (52b) in powers of \( \Delta \chi(x)/a \) where
\[
\Delta \chi(x) = \chi(x) - \bar{\chi}, \quad \bar{\chi} = \frac{1}{V} \int d^3x \gamma^{1/2} \chi(x), \quad V = \int d^3x \gamma^{1/2} . \tag{56}
\]
The first few terms of the implicit equation (52b) yield,
\[
a = \bar{\chi} + \sqrt{\frac{4V}{3}} \left[ 1 + \frac{9}{8V} < (\chi - \bar{\chi})^2 > \right] + \ldots . \tag{57}
\]
The generating functional then becomes
\[
\mathcal{S} = \sqrt{\frac{16V}{3}} + \bar{\chi} + \sqrt{\frac{3}{4V}} < (\chi - \bar{\chi})^2 > + \ldots , \tag{58a}
\]
with
\[
< (\chi - \bar{\chi})^2 > = \frac{1}{V} \int d^3x \gamma^{1/2} (\chi(x) - \bar{\chi})^2 . \tag{58b}
\]
If \( \chi(x) = \bar{\chi} \) is homogeneous, then one recovers the earlier result, eq.(55b).

3.3. General Case

It is straightforward to write down the generating functional for \( \mathcal{S} \) in terms of \( \chi(x) \) and the initial 3-metric:
\[
\mathcal{S} = \int d^3x \gamma_{ab}^{-1/2} \chi(x) \left\{ 2\pi_{ab}^{(0)} \pi_{ab}^{(0)} - \frac{1}{3} \pi^{2}_{(0)} \right\} + Q_{\gamma_{ab}^{(0)}} , \tag{59a}
\]
\[
\pi_{ab}^{(0)} = \frac{\delta Q}{\delta \gamma_{ab}^{(0)}(x)} . \tag{59b}
\]
Using the roots \((\chi(1), \chi(2))\) of \((\gamma/\gamma(0))^{1/2}\) defined in eqs.(22b-c), one may rewrite the generating functional in the elegant form:

\[
S = -\frac{4}{3} \int d^3x \frac{\chi^{1/2}}{\chi(1) \chi(2)} + Q[\gamma^{(0)}_{ab}].
\]  

(60)

For some applications, one would like to express this as a function of the original 3-metric \(\gamma_{ab}(x)\). In general, the necessary inversion of eqs.(19a-19d) is very difficult. However, in the limit as \(\chi(x) \rightarrow 0\), this inversion is straightforward since

\[
\gamma_{ab}(x) \rightarrow \gamma^{(0)}_{ab}(x) \quad \text{as} \quad \chi(x) \rightarrow 0.
\]  

(61)

Hence from eq.(59a), one concludes that

\[
S \rightarrow Q \quad \text{as} \quad \chi(x) \rightarrow 0.
\]  

(62)

which justifies the claim made earlier in eq.(18d).

3.4. Advanced Example

In order to be concrete, I will illustrate the main features of the General Case by assuming that

\[
Q = \int d^3x \gamma^{1/2} (0) \left[ C + ER(0) \right],
\]

(63a)

in which case

\[
\gamma^{-1/2}(0) \pi(0) = \frac{1}{2} \left( 3C + ER(0) \right) \quad \text{and} \quad \gamma^{-1/2}(0) \pi^{ab}(0) = -E\overline{R}^{ab}.
\]  

(63b)

3.4.1. Early Time Behavior  

Expanding the classical solutions (19a-c) in a Taylor series in \(\chi(x)\), one finds

\[
\left( \frac{\gamma}{\gamma(0)} \right)^{1/2} = 1 - \frac{1}{2} \left( 3C + R(0) \right) \chi + \ldots,
\]  

(64a)

\[
z = 2E \sqrt{\overline{R}^{ab}(0) \overline{\gamma}^{(0)}_{ab}} \chi + \ldots,
\]  

(64b)

\[
[h] = [h^{(0)}] \left( 1 - 4E[\overline{R}(0)] [\gamma^{(0)}] \chi + \ldots \right),
\]  

(64c)

where \([\overline{R}(0)]\) denotes the traceless Ricci tensor with contravariant indices:

\[
[\overline{R}(0)]^{ab} \equiv R^{ab}(0) - \frac{1}{3} \overline{\gamma}^{ab}(0) R(0).
\]  

(65)

Only first order is shown here although one could expand to arbitrary order. Using an iterative process, one can invert these equations to obtain \(\gamma^{(0)}_{ab}\) as a function of \(\chi\) and \(\gamma\),

\[
\gamma^{(0)}_{ab}(x) = \gamma_{ab}(x) + \chi(x) \left\{ C\gamma_{ab}(x) + E \left[ 4R_{ab}(x) - R(x) \gamma_{ab}(x) \right] \right\} + \ldots
\]  

(66)
Substituting into eq.(59a), $S$ becomes

$$S = \int d^3x \gamma^{1/2} (C + ER) + \int d^3x \gamma^{1/2} \chi(x) \left[ E^2 \left( \frac{3}{4} R^2 - 2 R^{ab} R_{ab} \right) + \frac{1}{2} CER + \frac{3}{4} C^2 \right] + \ldots. \quad (67)$$

This agrees with a similar calculation by Salopek (1997).

3.4.2. Late Time Behavior

In order to determine the late time behavior of $S$, one introduces a factor of $\gamma^{1/2}$ in eq.(59a):

$$S = \int d^3x \gamma^{1/2} \chi \left[ \frac{2 \pi^{ab} \pi_{ab}^{(0)} - \frac{1}{3} \pi^{(0)}_{ab}}{(1 - \frac{1}{2} \chi \gamma^{(0)} \pi^{(0)}_{ab} - \frac{3}{2} \chi \gamma^{1/2} \pi^{(0)}_{ab} \pi_{ab}^{(0)}} \right] + Q[\gamma_{ab}^{(0)}], \quad (68)$$

In the limit that $\chi$ is large and positive, $S$ assumes the very simple form,

$$S[\gamma_{ab}(x), \chi(x)] = -\frac{4}{3} \int d^3x \gamma^{1/2} \frac{1}{\chi(x)} . \quad (69)$$

However, it may not always be possible to take the limit to large $\chi$ because one may first reach a singularity at a finite value of $\chi$. For this case, the form of the generating functional in terms of the original variables has not as yet been determined.

4. Strongly-Coupled Solutions: Two Dust Fields Interacting with Gravity

A non-trivial example which illustrates the main features of the Green’s function solution method is provided by two dust fields, $\chi_1$ and $\chi_2$, interacting with gravity. The constraint equations for the long-wavelength system are:

$$H^{(s)}(x)/\kappa = \gamma^{-1/2} (2 \gamma_{ac} \gamma_{bd} - \gamma_{ab} \gamma_{cd}) \frac{\delta S}{\delta \gamma_{cd}} \frac{\delta S}{\delta \gamma_{ac}} + \frac{\delta S}{\delta \chi_1} + \frac{\delta S}{\delta \chi_2} = 0 , \quad (70a)$$

$$H_i^{(s)}(x) = -2 \left( \frac{\gamma_{ik}}{\delta \gamma_{kj}} \right) \frac{\delta S}{\delta \gamma_{kl}} \gamma_{kl,i} + \frac{\delta S}{\delta \chi_1} \gamma_{1,i} + \frac{\delta S}{\delta \chi_2} \gamma_{2,i} = 0 . \quad (70b)$$

After defining the average field, $\chi(x)$, and the static field, $y(x)$, through

$$\chi = \frac{\chi_2 + \chi_1}{2}, \quad y = \frac{\chi_2 - \chi_1}{2}, \quad (71)$$

the constraints become

$$H^{(s)}(x)/\kappa = \gamma^{-1/2} (2 \gamma_{ac} \gamma_{bd} - \gamma_{ab} \gamma_{cd}) \frac{\delta S}{\delta \gamma_{cd}} \frac{\delta S}{\delta \gamma_{ac}} + \frac{\delta S}{\delta \chi} = 0 , \quad (72a)$$

$$H_i^{(s)}(x) = -2 \left( \frac{\gamma_{ik}}{\delta \gamma_{kj}} \right) \frac{\delta S}{\delta \gamma_{kl}} \gamma_{kl,i} + \frac{\delta S}{\delta \chi} \gamma_{i,i} + \frac{\delta S}{\delta y} y_{i,i} = 0 . \quad (72b)$$
The static field $y(x)$ appears in the momentum constraint eq.(72b) but it is absent in
the energy constraint eq.(72a) which is identical to that of a single dust field, eq.(11). For all intents, one can describe the two-dust system with gravity, which consists of the fields $(\chi_1, \chi_2, \gamma_{ab})$, as a system containing a single dust field interacting with gravity, $\chi$ and $\gamma_{ab}$, and containing a static field $y(x)$ which appears in the generating functional $S$ but which does not evolve. Hence given the initial functional,

$$Q \equiv Q[\gamma_{ab}^{(0)}(x), y(x)]$$  \hspace{1cm} (73)

which is invariant upon reparametrization of the spatial coordinates,

$$-2 \left( \gamma_{ik}^{(0)} \frac{\delta Q}{\delta \gamma_{kj}^{(0)}} \right)_j + \frac{\delta Q}{\delta \gamma_{kl}^{(0)}} \gamma_{ki}^{(0)} + \frac{\delta Q}{\delta y} y_i = 0,$$  \hspace{1cm} (74)

the generating functional at later times is given by

$$S[\gamma_{ab}, \chi(x), y(x)] = G[\gamma_{ab}(x), \chi(x)|\gamma_{ab}^{(0)}(x), \chi^{(0)}(x) = 0] + Q[\gamma_{ab}^{(0)}(x), y(x)]$$ \hspace{1cm} (75a)

where $G$ is just the Green’s function for dust with gravity defined in eq.(11a) with $\chi^{(0)}(x) = 0$, and $\gamma_{ab}^{(0)}(x)$ chosen to minimize $G + Q$:

$$0 = \frac{\delta G}{\delta \gamma_{ab}^{(0)}} + \frac{\delta Q}{\delta \gamma_{ab}^{(0)}}.$$  \hspace{1cm} (75b)

The minimization prescription leads to the classical evolution equations (19a-d) which carry over directly to the present situation.

4.1. Exact Solution for Two Dust Fields with Gravity

Explicit solutions arise when the initial generating functional is of the form

$$Q[\gamma_{ab}^{(0)}(x), y(x)] = \int d^3x \gamma_1^{1/2} \left[ A + \left( \gamma_{ab}^{(0)} y, a, b \right)^{1/2} \right],$$  \hspace{1cm} (76)

where $A$ is a constant. Using the Green’s function method described above, I will show in section 4.2 that the resulting generating functional $S$ at arbitrary $\chi(x)$ is

$$S[\gamma_{ab}(x), \chi(x), y(x)] = -\frac{4}{3} \int d^3x \gamma^{1/2} \left( 1 - \frac{3A}{4} \chi \right)^{-4/3} \left( \gamma_{ab}^{(0)} y, a, b \right)^{1/2}.$$  \hspace{1cm} (77)

This solution is a special case considered by Salopek, Stewart and Parry (1993). One may also demonstrate the validity of this solution by direct substitution into the HJ eq.(72a).
4.2. Derivation of Exact Solution

The explicit solution (77) for two dust fields interacting with gravity will be justified using the Green’s function method of Section 2.

The momentum conjugate to $\gamma(0)_{ab}$ is given by eq. (19d):

$$\gamma_{(0)}^{-1/2} \pi_{(0)ab} = \frac{1}{2} \left[ \gamma_{(0)}^{ab} (A + s^{1/2} - s^{-1/2} y^a y^b) \right] , \quad (78a)$$

where

$$s = \gamma_{(0)}^{ab} y_a y_b \quad \text{and} \quad y^a = \gamma_{(0)}^{ab} y_b . \quad (78b)$$

This example is qualitatively different from the exact solution given in Section 3 in that the traceless part of $\pi_{(0)ab}$ is non-vanishing,

$$\gamma_{(0)}^{-1/2} \pi_{(0)ab} = -\frac{1}{2s^{1/2}} \left[ y^a y^b - s^{1/2} \gamma_{(0)}^{ab} \right] , \quad (79a)$$

while the trace is given by

$$\gamma_{(0)}^{-1/2} \pi_{(0)} = \frac{3A}{2} + s^{1/2} . \quad (79b)$$

The square root of the determinant $\gamma$ evolves as a quadratic function of $\chi$,

$$\left( \frac{\gamma}{\gamma_{(0)}} \right)^{1/2} = \left( 1 - \frac{\chi}{\chi(1)} \right) \left( 1 - \frac{\chi}{\chi(2)} \right) , \quad (80a)$$

with roots at

$$\chi(1) = \frac{1}{\frac{3A}{4} + s^{1/2}} , \quad (80b)$$

$$\chi(2) = \frac{4}{3A} . \quad (80c)$$

Please note that $\chi(1) \neq \chi_1$; $\chi(1)$ denotes the first root of $(\gamma/\gamma_{(0)})^{1/2}$ whereas $\chi_1$ denotes the first dust field. The 3-metric evolves according to

$$\gamma_{ab} = \left( 1 - \frac{\chi}{\chi(2)} \right)^{4/3} B_{ac} \gamma_{(0)cd} B_{db} , \quad (81a)$$

where the symmetric matrix $B_{ac}$ has components

$$B_{ac} = \gamma_{(0)ac} - \frac{1}{s} \left( \frac{\chi}{\chi(1)} - \frac{\chi}{\chi(2)} \right) y_a y_c . \quad (81b)$$

The inverse 3-metric has components

$$\gamma^{ab} = \left( 1 - \frac{\chi}{\chi(2)} \right)^{-4/3} C^{ac} \gamma_{(0)cd} C^{db} . \quad (82a)$$
where $C^{ac}$ is a symmetric matrix:

$$C^{ac} = \gamma_{(0)}^{ac} + \frac{1}{s} \left( \frac{\chi_{(1)} - \chi_{(2)}}{1 - \chi_{(1)}} \right) y^a y^c. \quad (82b)$$

One concludes that

$$y|^{a} y_a \equiv \gamma^{ab} y_a y_b = \left( \frac{1 - \chi_{(2)}}{1 - \chi_{(1)}} \right)^{2/3} \frac{\gamma_{(0)}^{ab} y_a y_b}{\gamma_{(0)}^{ab} y_a y_b}. \quad (83)$$

One may readily invert this relation to find $s$, eq.(78b), as function of the original fields:

$$s^{1/2} = \frac{\left( 1 - \frac{3A}{4} \chi \right)}{\chi + \left( 1 - \frac{3A}{4} \chi \right)^{1/3} \left( y^{a} y_a \right)^{-1/2}}. \quad (84)$$

Using eq.(81), one can write $S$ in terms of $\chi(x)$ and the initial 3-metric,

$$S = \int d^3x \gamma_{(0)}^{1/2} \left[ A \left( 1 - \frac{3A}{4} \chi \right) + s^{1/2} (1 - A \chi) \right] \quad (85)$$

Eliminating the initial variables in favor of the original ones, one recovers the claimed result eq.(77).

5. Strongly-Coupled Solutions: Scalar Field Interacting with Gravity

5.1. Massless Scalar Field, Gravity and Cosmological Constant

The strongly-coupled system of gravity interacting with a scalar field is also tractable. Most of the techniques that applied to dust and gravity can also be applied here although the details are quite different. The main results will be stated without much elaboration.

In the strongly-coupled limit, the Hamiltonian density for gravity and a scalar field with cosmological constant is:

$$\mathcal{H}^{(s)}(x)/\kappa = \gamma^{-1/2} \left( 2 \gamma_{ac} \gamma^{bd} - \gamma_{ab} \gamma_{cd} \right) \frac{\delta S}{\delta \gamma_{ab}} \frac{\delta S}{\delta \gamma_{cd}} + \frac{1}{2} \gamma^{-1/2} \left( \frac{\delta S}{\delta \phi} \right)^2 + \gamma^{1/2} V_0 = 0. \quad (86)$$

A complete solution ("Green’s function solution") for the generating functional describing a massless scalar field with cosmological constant term interacting with gravity is:

$$\mathcal{G}[\gamma_{ab}(x), \phi(x)| \gamma_{ab}^{(0)}(x), \phi^{(0)}(x)] = -\sqrt{\frac{4V_0}{3}} \int d^3x \gamma + \gamma^{(0)} - 2\gamma^{1/2} \gamma^{1/2} \cosh \left( \frac{3}{2} \sqrt{z^2 + (\phi - \phi^{(0)})^2} \right)^{1/2}, \quad (87)$$
where \( z \) was defined in eq.(11b). The sign of the Green’s function is arbitrary and a minus sign was chosen in eq.(87). Hereafter, I will assume that \( \phi(0) = 0 \) for similar reasons that were given for a dust field in Sections 2.1 and 2.2.

In order to satisfy the momentum constraint,

\[
\mathcal{H}^{(s)}_{i}(x) = -2 \left( \gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}} \right)_{,j} + \frac{\delta S}{\delta \gamma_{kt}} \gamma_{kl,i} + \frac{\delta S}{\delta \phi} \phi_{,i} = 0 ,
\]

one constructs a solution \( S \) using the Superposition Principle:

\[
S[\gamma_{ab}(x), \phi(x)] = \mathcal{G}[\gamma_{ab}(x), \phi(x)| \gamma_{ab}^{(0)}(x), \phi(0)(x) = 0] + \mathcal{Q}[\gamma_{ab}^{(0)}] ,
\]

where \( \gamma_{ab}^{(0)}(x) \) has been chosen to minimize \( \mathcal{G} + \mathcal{Q} \),

\[
0 = \frac{\delta \mathcal{G}}{\delta \gamma_{ab}^{(0)}} + \frac{\delta \mathcal{Q}}{\delta \gamma_{ab}^{(0)}} .
\]

\( \mathcal{Q} \) is an arbitrary gauge-invariant functional of \( \gamma_{ab}^{(0)} \). Please note that \( S \) coincides with \( \mathcal{Q} \) when \( \phi(x) = 0 \):

\[
S[\gamma_{ab}(x), \phi(x) = 0] = \mathcal{Q}[\gamma_{ab}(x)] .
\]

5.2. Classical Evolution

Classical evolution is given by:

\[
(\gamma/\gamma_0)^{1/2} = \frac{1}{\cosh \theta + \frac{4}{\sqrt{A^2 - 1}} \sinh \theta} ,
\]

\[
z = \phi \sqrt{\frac{\gamma_0^{-1} \frac{1}{2} \pi_0^2 - V_0}{2 \gamma_0^{-1} \pi_{ab}^{(0)} \pi_{ab}^{(0)}}} - 1 ,
\]

\[
[h] = [h^{(0)}] \exp \left[ \frac{2z}{\pi^{(0)}} \left( \frac{\pi_{ab}^{(0)}}{\pi^{(0)} \gamma_{ab}^{(0)}} \right)^{1/2} \right] .
\]

\( A \) and \( \theta \) are simply abbreviations for the following expressions:

\[
A = \frac{\gamma_0^{-1/2} \pi_0^{(0)}}{\sqrt{3} V_0} ,
\]

\[
\theta = \sqrt{\frac{3}{2}} \phi \sqrt{\frac{\gamma_0^{-1} \frac{1}{2} \pi_0^2 - V_0}{2 \gamma_0^{-1} \pi_{ab}^{(0)} \pi_{ab}^{(0)} - V_0}} .
\]
6. Semiclassical Evolution: Scalar Field, Gravity and Cosmological Constant

It is straightforward to write $S$ in terms of $\phi(x)$ and the initial 3-metric,

$$S = -\sqrt{\frac{4V_0}{3}} \int d^3 x \gamma^{1/2} \left[ \frac{\sinh \theta}{(\sqrt{A^2} - 1 \cosh \theta + A \sinh \theta)} + Q[\gamma^{(0)}_{ab}] \right], \quad (93)$$

with $A$ and $\theta$ defined in eqs.(92a,b), but it is very difficult to find an explicit expression in terms of the original variables. I will be content to illustrate the ‘early’ and ‘late’ time behavior.

6.1. Behavior for small $\phi(x)$

To this aim, note that for small $\phi(x)$ the original 3-metric evolves according to

$$\gamma^{(0)}_{ab} = \gamma_{ab} + \frac{\sqrt{2} \phi \gamma^{(0)}}{\sqrt{1 - \frac{1}{3} \gamma_{(0)}^{-1} \pi_{(0)}^{2} - 2 \gamma_{(0)}^{-1} \pi_{(0)}^{ab} \frac{\pi_{(0)}^{ab}}{V_0}}} \left[ 2\pi_{(0)}^{ab} - \pi_{(0)} \gamma^{(0)}_{ab} \right] + \ldots \quad (94)$$

This may be inverted by iteration to give $\gamma_{ab}^{(0)}(x)$ as a function of $\phi(x)$ and $\gamma_{ab}(x)$:

$$\gamma_{ab}^{(0)} = \gamma_{ab} - \frac{\sqrt{2} \phi \gamma^{-1/2}}{\sqrt{1 - \frac{1}{3} \gamma_{-1}^{-1} \pi_{2}^{2} - 2 \gamma_{-1}^{-1} \pi_{ab} \pi_{ab} - V_0}} \left[ 2\pi_{ab} - \pi \gamma_{ab} \right] + \ldots \quad (95)$$

Substituting into eq.(93), one finds to first order in $\phi(x)$ that

$$S[\gamma_{ab}(x), \phi(x)] = Q[\gamma_{ab}(x)] + \sqrt{2} \int d^3 x \gamma^{1/2} \phi \left[ \frac{1}{3} \gamma_{-1}^{-1} \pi_{2}^{2} - 2 \gamma_{-1}^{-1} \pi_{ab} \pi_{ab} - V_0 + \ldots \right] \quad (96a)$$

(small $\phi(x)$ behavior)

with

$$\pi^{ab} = \frac{\delta Q}{\delta \gamma_{ab}}, \quad \pi = \gamma_{ab} \pi^{ab}, \quad \text{and} \quad \pi^{ab} = \pi^{ab} - \frac{1}{3} \pi \gamma_{ab}, \quad (96b)$$

which is in agreement with Salopek (1997).

6.2. Behavior for large $\gamma^{1/2}(x)$

In order to determine the behavior at large $\gamma^{1/2}$, one rewrites the generating functional $S$ as

$$S = -\sqrt{\frac{4V_0}{3}} \int d^3 x \gamma^{1/2} \left[ 1 + \frac{\gamma^{(0)}}{\gamma} - 2 \left( \frac{\gamma^{(0)}}{\gamma} \right)^{1/2} \cosh \theta \right]^{1/2} + Q[\gamma^{(0)}_{ab}]. \quad (97)$$
Provided \(A^2 - 1 > 0\), \(\gamma\) becomes very large as \(\theta \to \theta_{\text{crit}}\) where \(\theta_{\text{crit}}\) is given by

\[
\theta_{\text{crit}} = -\tanh\left(\frac{\sqrt{A^2 - 1}}{A}\right).
\]

(98)

In this limit, \(\gamma(0)/\gamma \to 0\), and \(S\) is proportional to the volume of any given 3-geometry,

\[
S[\gamma_{ab}(x), \phi(x)] = -\sqrt{\frac{4V_0}{3}} \int d^3x \, \gamma^{1/2}, \quad \text{(large \(\gamma^{1/2}(x)\) behavior)}.
\]

(99)

For the sake of brevity, other cases will not be considered here.

7. Massless Scalar Field and Gravity

A complete solution for the generating functional describing gravity and a massless scalar field without cosmological constant is:

\[
G[\gamma_{ab}(x), \phi(x)\mid \gamma_{ab}^{(0)}(x), \phi^{(0)}(x)] = -\int d^3x \, \gamma^{1/4} \gamma^{(0)}_{ab} \exp\left[\frac{\sqrt{3}}{8} \sqrt{\frac{\pi^{(0)}(0)}{\pi^{(0)}_{ab}}} \phi - \frac{3}{8} \sqrt{\pi^{(0)}(0)} - 6\pi^{(0)}_{ab} \pi^{(0)}_{ab} \right],
\]

(100)

where \(z\) is still defined by eq.(11b). I will assume again that \(\phi^{(0)} = 0\).

7.1. Classical Evolution

Classical evolution is given by

\[
\left(\frac{\gamma}{\gamma^{(0)}}\right)^{1/4} = \frac{4}{3} \gamma^{(0)}_{ab} \pi^{(0)} \exp\left[\frac{\sqrt{3}}{8} \frac{\pi^{(0)}(0) \phi}{\left(\pi^{(0)}_{ab} \pi^{(0)}_{ab}\right)^{1/2}}\right],
\]

(101a)

\[
z = \frac{\phi}{\sqrt{\frac{\pi^{(0)}}{(\pi^{(0)}_{ab} \pi^{(0)}_{ab})} - 1}},
\]

(101b)

\[
[h] = [h^{(0)}] \exp\left[\frac{2z \left[\pi^{(0)}\right]}{\left(\pi^{(0)}_{ab} \pi^{(0)}_{ab}\right)^{1/2}}\right].
\]

(101c)

In particular for

\[
Q = \int d^3x \, \gamma^{1/2} \left[C + ER^{(0)}\right],
\]

(102a)

the classical evolution is given by,

\[
\left(\frac{\gamma}{\gamma^{(0)}}\right)^{1/4} = 2 \left(C + \frac{E}{3} R^{(0)}\right) \exp\left(\frac{\sqrt{3}}{8} \phi \left[\left(1 - \frac{8E^2 R^{(0)}_{ab} R^{(0)}_{ab}}{(C + \frac{E}{3} R^{(0)})^2}\right)^{1/2}\right]\right),
\]

(102b)
\[ z = \frac{\phi}{\left[ 3 \left( C + \frac{E}{3} R_{(0)} \right)^2 \right]^{1/2} / \frac{8E^2 R_{(0)}^{ab} R_{(0)}^{ab}}{3}} \, , \] (102c)

\[ [h] = [h^{(0)}] \exp \left[ -\sqrt{\frac{32}{3} E} \frac{R_{(0)}^{ab} \gamma^{(0)} ab}{\left( C + \frac{E}{3} R_{(0)} \right)^2 - \frac{8E^2 R_{(0)}^{ab} R_{(0)}^{ab}}{3}} \right] \, . \] (102d)

Semiclassical evolution is very similar to the case with cosmological constant, and it will not be discussed further.

8. Conclusions

In the semiclassical approximation, it was shown how to solve the system of strongly-coupled gravity and matter using a Green’s function method. There are two steps in implementing this program:

1. **Computing the Green’s function.** Explicit Green’s function solutions of the energy constraint in a Hamilton-Jacobi context were given for systems describing Einstein gravity interacting with either a dust field or a scalar field.

2. **Constructing the General Solution through the Superposition Principle.** One can construct a general semiclassical solution to the momentum constraint as well as the energy constraint by a superposition over the parameter fields of the Green’s function.

The number of parameter fields appearing in the Green’s function is a tricky issue. Apparently, this number should be one less than the number of fields originally in the problem. For example, in the case of a dust field \( \chi(x) \) interacting with the 3-metric \( \gamma_{ab}(x) \), there are 7 degrees of freedom per spatial point \( x \): one for the dust field and 6 for the symmetric \( 3 \times 3 \) matrix \( \gamma_{ab} \). Hence, there should be at most 6 parameter fields in the Green’s function. If there are more, one should set the additional parameter fields to zero. Exact solutions demonstrate the validity of this approach.

The problem of strongly-coupled gravity interacting with matter is not exceedingly complicated, although the actual details may be intricate. Many of the exact solutions presented in this paper had been derived earlier using other techniques. However, the Green’s function method is very general and in some sense all solutions to strongly-coupled gravity and matter may be derived using it. The general methodology may perhaps be useful in analyzing a wide range of gravitational phenomena.

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