INTERIOR SECOND DERIVATIVES ESTIMATES FOR NONLINEAR DIFFUSIONS

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ABSTRACT. By an extension of some estimates due to Crandall and Pierre \cite{Pierce} and Di Benedetto \cite{DiBenedetto} we derive consequences for fully nonlinear parabolic equations of the form $\partial_t v + F(t,x,D^2 v) = 0$, where $F$ can be both singular and degenerate elliptic and also non-homogeneous. Such equations appear in the theory of option pricing with market impact.

1. INTRODUCTION

The original motivation for this paper is the study of fully nonlinear parabolic partial differential equations of the form

$\partial_t v + F(t,x,\partial_{xx} v) = 0,$

where $u$ is defined in $[0,T] \times \mathbb{R}$, the terminal condition $u(T,\cdot)$ is given, and the solution is solved backwards in time. We investigate the case where $F(t,x,\gamma)$ is typically a convex function in its third argument, with its derivative $F_\gamma$ going from 0 at $-\infty$ to $+\infty$ at $\bar{\gamma}$ (potentially $\bar{\gamma} = \infty$). One example is

$\partial_t v + \frac{1}{2}\sigma^2(t,x) \left( a + \frac{b}{(1-\lambda \partial_{xx} v)^{p_1}} + \frac{c}{(1-\lambda \partial_{xx} v)^{p_2}} \right) = 0,$

for $t \in [0,T], x \in \mathbb{R}$, which comes from theory of option pricing with market impact, see \cite{1 4 5 12 3}. There, $0 < p_1 < p_2, \lambda > 0$, and $\sigma$ is a bounded Lipschitz function such that $\inf \sigma > 0$. The conditions $b, c > 0$ guarantee that the equation is parabolic as long as $\lambda \partial_{xx} v < 1$, and $a + b + c = 0$ ensures that constants are solutions.

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The equation is singular when \( \lambda \partial_{xx}v \to 1^- \) and degenerate when \( \partial_{xx}v \to -\infty \). Our aim is to obtain a priori interior estimates for the second derivatives guaranteeing that the equation is neither degenerate nor singular if we are away from the terminal time \( T \). Namely, we will prove that, if there exists a supersolution, then, for any \( \tau > 0 \), there exists some \( \varepsilon(\tau) > 0 \) such that

\[-\varepsilon^{-1} \leq \partial_{xx}v \leq \lambda^{-1} - \varepsilon \text{ for } t \leq T - \tau.\]

Consequently the equation is uniformly parabolic away from the terminal time and higher regularity follows by standard arguments.

General equations of the form (1.1) with singular behaviour are also met in some problems related to optimal transport by diffusions, see [13, 11, 10].

Some of our results are quite general and apply to solutions of

\[(1.3) \quad \partial_t v = F(t, x, -A(v)),\]

for \( A \) an accretive operator as in [6]. The most important cases will be \( A = -\partial_{xx} \), or \( A = -\Delta \) in higher dimensions. To obtain our results, we will study the equation followed by \( u = -Av \):

\[(1.4) \quad \partial_t u + A(F(t, x, u)) = 0.\]

Our paper consists of three estimates for solutions to (1.4) which have independent interest.

The first result is a generalisation of the classical estimate obtained by Aronson and Bénilan in [2] for the time derivative of non-negative solutions of (1.4) when \( A = -\Delta \) and \( F(t, x, u) = u^m, \ m > (d - 2)^+/d, \) where \( d \) is the spatial dimension. This estimate was later extended by Crandall and Pierre to the case in which \( F(t, x, u) = \phi(u) \), under some assumptions on \( \phi \), first for \( A = -\Delta \) in [7], and later for general accretive operators in [6]. Here we generalize this last result to the case in which \( F \) is not homogeneous, neither in space nor in time, giving an unconditional (i.e. independent of the initial data) information on \( \partial_t u \). It is somewhat a surprise that there is no need for any regularity of \( F \) with respect to \( x \), only with respect to \( t \) and \( u \). These results are given first in the separable case, \( F(x, t, u) = \kappa(t, x)\phi(u) \), in Theorem 2.2 and are later extended to the general non-separable case in Theorem 2.3.

The second result, Theorem 2.5, is a consequence of Theorem 2.3 for solutions to (1.3) when \( F \) can be singular for large values of \( -A(v) \), still under some structure condition on the behavior of \( F \) with respect to \( u \). We show interior \( C^2 \) regularity under the assumption of the existence of a supersolution.

The third result, Theorem 3.1, shows expansion of positivity for equations of the form

\[\partial_t v = F(t, D_x v, \Delta v),\]
with \( v \) convex, and \( F(t, p, z) \) singular for \( z \sim 0 \). This result is in the spirit of the one of Di Benedetto [8], in a case where we have gradient dependency. Under a Legendre transform, this result will imply the bound from below for \( \partial_{tt} v \) in equation (1.1).

Building on these results we deduce the interior regularity for solutions of (1.1) in Theorem 4.1.

2. Time derivative estimate and applications to the singular case

In this section we generalize the time derivative estimate obtained by Bénilan and Crandall in [6] and derive consequences for singular partial differential equations that appear in option pricing.

2.1. The operator. As in [6], we assume that:

- \( A \) is a densely defined, \( m \)-accretive in \( L^1(\mathbb{R}^d) \) linear operator.
- If \( u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) and \( \beta \) is a monotone graph in \( \mathbb{R} \times \mathbb{R} \) with \( 0 \in \beta(0), \nu \in \beta(u) \) then

\[
\int v A(u) \, dx \geq 0. \tag{2.1}
\]

Thanks to (2.1) we have a comparison principle, which will be important in the sequel.

**Lemma 2.1.** The comparison principle holds for solutions in \( L^1 \cap L^\infty \) of equation (1.4).

**Proof.** Assume that \( u(0) \geq v(0) \), take the difference of the equations (1.4) for \( u \) and \( v \), multiply by \( 1_{u \leq v} \), and use (2.1) to conclude. \( \square \)

2.2. The separable case. Let \( u \) be a non-negative solution on \( t > 0 \) to

\[
\partial_t u + A(\kappa(t,x)\phi(u)) = 0. \tag{2.2}
\]

Under an structural assumption on \( \phi \), which coincides with that in [6] for the case in which \( \kappa = 1 \), and with some regularity hypothesis on \( \kappa \), there is an unconditional estimate for the time derivative of non-negative solutions of (2.2), as we show next.

**Theorem 2.2.** Let \( u \) be a non-negative classical solution to (2.2) on \( [0,T] \) belonging to \( L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \), and assume that \( \phi \) is non-decreasing, with \( \phi(0) = 0 \) and satisfies for some \( m > 0, \theta \in \{-1, 1\} \)

\[
\inf_{u \geq 0} \left\{ \frac{\theta \phi(u) \phi''(u)}{(\phi'(u))^2} \right\} \geq m. \tag{2.3}
\]

Assume also that \( \kappa \) is positive and such that

\[
\sup_{t \in [0,T], x} \{ \kappa, \kappa^{-1}, \vert \partial_t \kappa \vert, \vert \partial_{tt} \kappa \vert \} \leq L \tag{2.4}
\]
for some constant $L > 0$. Then there exists a constant $\rho > 0$ depending only on $m, L, T$ such that

$$t \rightarrow \theta \rho^t \kappa(t) \varphi(u(t))$$

is non-decreasing in $[0, T]$.

**Proof.** We consider

$$w = t \partial_t u + \left( \theta \rho + t \frac{\partial_t \kappa}{\kappa} \right) \frac{\varphi(u)}{\varphi'(u)} \tag{2.6}$$

where $\rho > 0$ is a constant to be chosen later. Differentiating equation (2.2) with respect to time we get

$$\partial_t \varphi'(u) = \partial_t u + \left( \frac{\partial_t \kappa}{\kappa} \right) \frac{\varphi(u)}{\varphi'(u)} \tag{2.5}$$

which reads also

$$\partial_t \varphi'(u) + \frac{1}{t} A \left( \kappa \varphi'(u) \right) \left( w - \theta \rho \frac{\varphi'(u)}{\varphi(u)} \right) = 0,$$

while differentiating (2.6) we obtain

$$\partial_t w = t \partial_t u + \partial_t u + \left( \theta \rho + t \frac{\partial_t \kappa}{\kappa} \right) \partial_t u \left( 1 - \frac{\varphi(u) \varphi''(u)}{\varphi'^2(u)} \right) + \partial_t \left( t \frac{\partial_t \kappa}{\kappa} \right) \frac{\varphi(u)}{\varphi'(u)}.$$

Combining these two identities with (2.2) and (2.6) we obtain

$$\partial_t w + A \left( \kappa \varphi'(u) w \right) = \theta \rho A \left( \kappa \varphi'(u) \right) + \partial_t u \tag{2.7}$$

Defining

$$\tilde{\rho} = \rho + \theta t \frac{\partial_t \kappa}{\kappa}, \quad Q = - \left( 1 + t \frac{\partial_t \kappa}{\kappa} \right) \rho \frac{\varphi(u) \varphi''(u)}{\varphi'^2(u)}.$$

this can be rewritten as

$$t \partial_t (\theta w) + A \left( t \kappa \varphi'(u) \theta w \right) + Q \theta w = \left( \theta t \partial_t \left( t \frac{\partial_t \kappa}{\kappa} \right) + \tilde{\rho} Q \right) \frac{\varphi(u)}{\varphi'(u)}.$$

It follows easily from hypotheses (2.3) and (2.4) that if we take $\rho$ large enough then $\tilde{\rho}$ is positive and large enough so that

$$Q > 0, \quad \theta t \partial_t \left( t \frac{\partial_t \kappa}{\kappa} \right) + \tilde{\rho} Q > 0.$$
Thus, if we multiply equation (2.7) by \( t = 1_{\{\theta w \leq 0\}} \), we get that

\[
t \partial_t \int (\theta w)^- + B + Q \int (\theta w)^- \leq 0,
\]

where \( B = \int t A (t \kappa \varphi'(u) \theta w) \). Since \( t \) is a non-decreasing function of \( \kappa \varphi'(u)(\theta w) \), property (2.1) implies \( B \geq 0 \). Hence \( \partial_t \int (\theta w)^- \leq 0 \). On the other hand, \( \theta w(0) \geq 0 \). Therefore, since \( (\theta w)^- \) is non-negative, it is identically 0 for \( t \geq 0 \), and hence \( \theta w \geq 0 \).

To conclude, we notice that

\[
\partial_t (\theta t \rho \theta^2 \kappa(t,x) \varphi(u(t,x))) = t \rho \theta^2 \kappa \varphi'(u) \theta w \geq 0,
\]

which implies (2.5).

\[ \square \]

2.3. The general (non-separable) case. The monotonicity formula (2.5) can be extended to equations in the general non-separable form (1.4)

**Theorem 2.3.** Let \( u \) be a non-negative classical solution to (1.4) on \([0,T]\), belonging to \( L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Assume that \( F(t,x,u) \) is non-decreasing in \( u \), satisfies \( F(t,x,0) = 0 \), \( F_{tt} \), \( F_{uu} \) are bounded, and for some \( m > 0 \), \( \theta \in \{-1,1\} \)

\[
\inf_{u \geq 0, x \in \mathbb{R}^d, t \in [0,T]} \left\{ \frac{\theta F_{uu} F}{F_u^2} \right\} \geq m.
\]

Then, there exists a constant \( \rho > 0 \), independent of \( u \), such that

\[
t \to \theta t^\rho F(t,x,u(t,x)) \text{ is non-decreasing in } [0,T].
\]

**Proof.** Let \( w := t \partial_t u + (\theta \rho + t \rho^2) \frac{F}{F_t} \) with \( \rho > 0 \) to be fixed later. Differentiating (1.4) we now have

\[
\partial_t w + \frac{1}{t} A(F_u(t \frac{F_t}{F_u} + t \partial_t u)) = 0,
\]

or equivalently

\[
\partial_t w + \frac{1}{t} A \left( F_u \left( w - \theta \rho \frac{F}{F_t} \right) \right) = 0,
\]

while

\[
\partial_t w = t \partial_t w + \partial_t u + \left( (\theta \rho + t \rho^2) \left( 1 - \frac{F_{uu} F^2}{F_t^2} \right) + t \left( \frac{F_t}{F} \right) \frac{F}{t} \right) \partial_t w
\]

\[
+ \left( \theta \rho + t \rho^2 \right) \left( \frac{F}{F_t} \right) \frac{F}{t}.
\]
Combining these equations, we arrive to
\[ \partial_t w + A(F_u w) = \partial_t u \left( 1 + \frac{F_i}{F} + t \left( \frac{F_i}{F u_F} - \tilde{\rho} \frac{F_{uu} F}{F_F} \right) \right) \]
\[ + \tilde{\rho} \left( \frac{F}{F_u} \right)_t + \left( \frac{F_i}{F} \right)_t \frac{F}{F_u} \]
\[ = -\frac{1}{t} \left( w - \tilde{\rho} \frac{F}{F_u} \right) Q + \tilde{\rho} \left( \frac{F}{F_u} \right)_t + \left( \frac{F_i}{F} \right)_t \frac{F}{F_u}, \]
where
\[ \tilde{\rho} = \rho + \theta t \frac{F_i}{F}, \quad Q = - \left( 1 + \frac{F_i}{F} + \left( \frac{F_i}{F u_F} \right) \right) + \tilde{\rho} \frac{F_{uu} F}{F_F}. \]

Then
\[ t \partial_t (\theta w) + A(t F_u \theta w) + Q \theta w = \left( \tilde{\rho} Q + t \left( \frac{F}{F_u} \right) \frac{F_u}{F} + \theta t \left( \frac{F_i}{F} \right)_t \right) \frac{F}{F_u}. \]

It follows easily from the assumptions on \( F \) that if we take \( \rho \) large enough, then \( \tilde{\rho} \) is positive and large enough so that
\[ Q > 0, \quad \rho Q + t \left( \frac{F}{F_u} \right) \frac{F_u}{F} + \theta t \left( \frac{F_i}{F} \right)_t > 0, \]
and the result follows as in the proof of Theorem 2.2.

Note that
\[ \left( \frac{F}{F_u} \right)_t = \frac{F_i F_u - F_{u,t}}{F_u F} = \frac{F_i}{F} - \frac{F_{u,t}}{F_u}. \]

\[ \blacksquare \]

2.4. Consequences for fully nonlinear parabolic equations. We discuss here implications for the models studied in [1, 5].

We assume that \( v \) is a classical solution to (1.3) on \([0, T]\), and hence that \( u = -A(v) \) solves equation (1.4). We start by proving an auxiliary result.

Lemma 2.4. Let \( v \) be a locally bounded classical solution to (1.3) on \([0, T]\), with \( F \) satisfying the assumptions of Theorem 2.3 with \( \theta = 1 \), and with initial data \( v_0 \). Given \( M > 0 \), there exists \( M' > M \) such that
\[ -A(v_0) \geq M' \implies -A(v) \geq M \quad \text{for } t \in [0, T]. \]

Proof. Let \( b(t, x) = F \left( t, x, -A(v(x, t)) \right) \). Since, by assumption, \( F_i \geq -\ell F \), then
\[ \partial_t b = -F_u A(b) + F_i \geq -F_u A(b) - \ell b. \]

Therefore, \( g(t, x) = b(t, x) e^{\ell t} \) satisfies \( \partial_t g \geq -F_u A(g) \), while any constant \( k \) satisfies \( \partial_t k = -F_u A(k) \). Multiplying \( \partial_t (k - g) \) by \( 1_{g \leq k} \) and using property (2.1), we conclude that \( g \) remains larger than \( k \) if it was so at the initial time. Take \( M' \) large so that \( b(0, x) = F(t, x, -A(v_0(x, t))) \) is larger than \( k > 0 \) to be determined. Since \( F(t, x, -A(v(x, t))) \geq e^{-\ell t} k \), then \( F(t, x, -A(v(x, t))) \).
is large if \( k \) is large enough, and we conclude that \(-A(v)\) can be made as large as desired. \(\square\)

We now consider \( \tilde{v} \) solution to (1.3) such that
\[
A(\tilde{v}_0) = \min\{A(v_0), -M'\},
\]
with \( M' \) as above. Then, \( \tilde{u} = -A(\tilde{v}) \) is a solution to (1.4) and, by the comparison principle
\[
-A(v) \leq -A(\tilde{v}) \quad \text{holds for all time } t \geq 0.
\]
Now, thanks to the monotonicity formula (2.9), we will prove the interior regularity of \( v \).

**Theorem 2.5.** Let \( v \in L^\infty_{\text{loc}}([0, T) \times \mathbb{R}) \) be a classical solution to (1.3) with \( F \) satisfying (2.8) with \( \theta = 1 \) on \([M, +\infty)\) for some \( M > 0 \), and the rest of the conditions of Theorem 2.3. If \( \tilde{v} \in L^\infty_{\text{loc}}([0, T) \times \mathbb{R}) \), then
\[
F^+(t, x, -A(v)) \in L^\infty_{\text{loc}}([0, T) \times \mathbb{R}),
\]
with bounds that depend only on \( \tilde{v}_0, \tilde{v}, m, \) and \( t \).

Assuming moreover that either \( A(v) \) is bounded from above or that \( F_u(t, x, u) \) is bounded away from \( 0 \) and \( +\infty \) for \( u < 0 \), then \( v(t, \cdot) \in C^{2,\alpha} \) uniformly on \([\tau, T]\) for \( \tau > 0 \).

**Proof.** If \( \tilde{v} \) is locally bounded, it follows from the auxiliary lemma that \( F(t, x, \cdot) \) satisfies the assumptions of Theorem 2.3 at \(-A(\tilde{v})\) for all \( t \in [0, T] \).

Therefore, Theorem 2.3 applies. Using the monotonicity formula (2.9) with \( \theta = 1 \) for \( 0 < t_1 \leq t_2 \leq T \),
\[
\tilde{v}(t_2, x) = \tilde{v}(t_1, x) + \int_{t_1}^{t_2} F(s, x, -A(\tilde{v})) \, ds \\
\geq \tilde{v}(t_1, x) + F(t_1, x, -A(\tilde{v}(t_1))) \int_{t_1}^{t_2} \frac{t^0}{s^0} \, ds \\
\geq \tilde{v}(0, x) + F(t_1, x, -A(\tilde{v}(t_1))) \int_{t_1}^{t_2} \frac{t^0}{s^0} \, ds,
\]
which yields the stated boundedness of \( F(t_1, x, -A(\tilde{v})) \).

The second point follows from the first, as, now, \( F \) is uniformly elliptic, and standard theory applies. \(\square\)

3. Expansion of positivity and application to the degenerate case

We consider the case
\[
\partial_t v = F(t, x, D^2 v) = \tilde{F}(t, x, A_{ij}(M - D^2 v)^{ij}),
\]
where $A$ and $M$ are symmetric positive matrices and $(M - D^2v)^{ij}$ is the inverse of $M - D^2v$. By elementary affine transformations one can assume $A = M = I$ the identity matrix. We also assume that $\tilde{F} = \tilde{F}(t,x,z)$ satisfies

$$\tilde{F}_z(t,x,z) \sim |z|^{m-1} \text{ as } z \sim 0$$

for some $m \in [0, 1]$, that $\tilde{F}$ is smooth with respect to the other variables, and

$$F(t,x,D^2v) \leq C \text{ on } B_r(0).$$

We further assume that

$$F_x \in L^\infty_{loc}((0,T) \times \mathbb{R}^d; L^\infty(\mathbb{R}_+));$$

that is, for compact sets $K \subset (0,T) \times \mathbb{R}^d$, $F_x \in L^\infty(K \times \mathbb{R}_+)$. The problem is defined for $(I - D^2)v$ non-negative. Hence, $v = |x|^2/2 - v$ is convex, and we can consider its lower semi-continuous Legendre transform

$$v^*(y) = \sup_x (x \cdot y - v(x)).$$

When $v$ is lower semi-continuous and its supremum is attained at a point $(x,y)$ where $v$ is twice differentiable, then

$$x = Dv^*(y), \quad D^2v^*(y) = [D^2v(x)]^{-1}.$$

Moreover, if $v$ depends smoothly on $t$,

$$\partial_t v(t,x) + \partial_y v^*(t,y) = 0.$$

The equation satisfied by $v^*$ is now

$$\partial_t v^* = \tilde{F}(t,Dv^*, Du^*).$$

Note that (3.2) implies that $0 \leq D^2v^* \leq C$ on $B_r(0)$. Here we establish an independent result for this parabolic equation, on the condition that the solution is convex.

**Theorem 3.1.** Let $\tilde{F} \in C^1_{loc}([0,T] \times \mathbb{R}^d \times \mathbb{R})$ having the behaviour (3.1) for some $m \in [0, 1]$. Assume that $v^*$ is a convex solution to (3.4) such that $Du^*$ is bounded from above, and not identically 0 until time $T$. Then for $t > 0$, $v^*$ is $C^2$ smooth in $y$ and $Du^*$ is bounded away from 0 locally uniformly on $(0,T) \times \mathbb{R}^d$.

**Proof.** If $m = 1$ the problem is uniformly elliptic, and the result is well known, so we assume $m \neq 1$.

Let $u = Du^*$. Then,

$$\partial_t u = \text{div}(\tilde{F}_z D^2v^*) + \text{div}(\tilde{F}_z \nabla u).$$

The proof is done by Moser iterations. We follow the technique of [12] that we adapt from the elliptic to the parabolic case. We first observe that from
the convexity of $v^*$ and the fact that $\Delta v^*$ is bounded, $D^2 v^*$ is bounded, and $D_{ij} v^* \leq u$. Multiplying (3.5) by $\eta^2(y)u^\beta$ for $\beta < 0$, $\beta \neq -m, -1$ we obtain

$$|\beta| \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left( \eta \partial_y (u^{\beta+m}) \right)^2 \left( \frac{2}{\beta + m} \right)^2 \, dy \leq C \left( \frac{1}{\beta + 1} \int_{\mathbb{R}^n} \eta^2 u^{\beta+1} (t,y) \, dy \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (\partial_y \eta)^2 (u^{\beta+m} + u^{\beta+1}) + \eta^2 u^{\beta+2-m} \right),$$

(3.6)

where $C$ depends on our assumptions on $\tilde{F}$ and the bound on $u$. If $\beta = -m$ we obtain:

$$m \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (\eta \partial_y (\ln u))^2 \leq C \left( \frac{1}{1-m} \int_{\mathbb{R}^n} \eta^2 u^{1-m} (t,y) \, dy \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (\partial_y \eta)^2 (1 + u^{1-m}) + \eta^2 u^{2-2m} \right).$$

Following [9] Section 8.6] the second bound yields that

$$\int_{[t_1, t_2]} \int_{B_r(0)} |\partial_y (\ln u)| \, dy \leq C r^{n-1},$$

and hence by [9] Theorem 7.21] that for some $p_0 > 0$ and $l = \frac{1}{|B_r|} \int_{B_r} \ln u$

there holds

$$\int_{[t_1, t_2]} \int_{B_r} e^{p_0 |\ln u - l|} \, dy \leq D.$$ 

Note that $C, D$ here might depend on $\|u\|_{L^\infty([t_1, t_2] \times B_r)}$, which we control anyway. This in turn implies

$$\left( \int_{[t_1, t_2]} \int_{B_r} u^{p_0} \right) \left( \int_{[t_1, t_2]} \int_{B_r} u^{-p_0} \right) \leq D,$$

which gives a bound on $\int_{[t_1, t_2]} \int_{B_r} u^{-p_0}$ depending also on $(\int_{[t_1, t_2]} \int_{B_r} u^{p_0})^{-1}$.

From (3.6) using the boundedness of $u$ and fixing some $\theta \in (0, 1)$ we deduce

$$\int_{t_1}^{t_2} \int_{B_{\theta r}} (\partial_y (u^{\beta+m}))^2 \, dy \leq \frac{C}{|\beta| |\beta + 1|} \int_{B_r} u^{\beta+1} (t,y) \, dy \left|_{t_1}^{t_2} + r^{-2} \int_{t_1}^{t_2} \int_{B_r} u^{\beta+m} \, dy \right.$$

Sobolev’s inequality will then yield a control on $\|u\|_{q(m)}$, for

$$q(m) = \frac{\beta + m}{2} \frac{2d}{d - 2} = \frac{(\beta + m)d}{d - 2}$$

if $d \geq 2$ and $+\infty$ otherwise. By starting with $\beta + m = -p_0$ above, and classically iterating Sobolev’s injection this gives a bound of the form

$$\sup_{y \in B_r} u(t_3, y) \leq C(r, m, \|u\|_{L^\infty([t_1, t_2] \times B_r)}, \inf_{[t_1, t_2]} \{ \|u\|_{L^p(B_r)} \}) \frac{1}{\inf_{[t_1, t_2]} \{ \|u\|_{L^p(B_r)} \}},$$

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for $t_3 \in [t_1 + v, t_2 - v]$. Equation (3.5) becomes now uniformly elliptic, and we obtain that $u \in C^\alpha$. As $u = \Delta v^*$, classical elliptic regularity then yields $v^* \in C^{2,\alpha}_\gamma$.

Remarks. (i) When $d > 1$, this theorem does not imply that $D^2 v$ is uniformly positive.

(ii) Equation (3.5) and our result is somehow similar to the porous medium like equation addressed in [5]; see equation 5.1 of Chapter 3, and the proof in Proposition 7.2 of Chapter 4 about expansion of positivity for singular porous medium equations. However in our present case the a priori knowledge that $D^2 v^*$ is positive and bounded considerably simplifies the estimates.

(iii) The presence of the term $\frac{1}{\inf_{[t_1,t_2]} \|u\|_{L^p(B_r)}}$ in the estimate implies that it is valid up to extinction. Indeed, before extinction, there exists always $R$ large enough so that $\|u\|_{L^p(B_r)}$ is bounded away from 0. Extinction in our case means that $\Delta v^* \equiv 0$, hence that $\Delta v \equiv -\infty$ which does not occur if there is a bounded subsolution to (1.3).

(iv) If we remain in a class of solutions to (2.2) in which the comparison principle holds, then the expansion of positivity result of Theorem 3.1 should remain valid without assuming that $\Delta v^*$ is bounded from above. Equivalently, one can write that $\min\{v, C/\varphi^{-1}(\varphi)\}$ is a supersolution to (2.2) and proceed with the estimates.

As a corollary, we have an interior lower bound for Laplacian of solutions to (1.3).

**Theorem 3.2.** Let $v$ be a solution to (1.3). Assume that $F$, $\bar{F}$ and $v$ satisfy (3.1)–(3.3). Then $\Delta v$ admits an interior lower bound in $B_{\theta r}(0)$ for $\theta < 1$.

**Proof.** Theorem 3.1 implies that $v$ is bounded away from $+\infty$, and hence that the $D^2 v$ as a matrix is bounded from below (i.e. its eigenvalues are bounded away from $-\infty$).

4. Consequence for fully non-linear Hamilton-Jacobi-Bellman equations

This section is motivated by the papers [1, 12, 3] of the first author, where fully non-linear versions of the Black-Scholes equation are considered in the context of financial derivatives pricing with market impact. We are in dimension $d = 1$, $A = -\partial_{xx}$, and $F(t, x, \gamma) : ([0, T] \times \mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ satisfies the assumptions of Theorem 2.3 for $\gamma > 0$ and such that $F_\gamma \sim \gamma^{m-1}$ for $\gamma < 0$ with $m \in [0, 1]$.

Considering again equation (1.3), but backwards in time (as is usually the case for stochastic control problems)

(4.1) \[ \partial_t v + F(t, x, \partial_{xx} v) = 0, \]
for which, we assume that the classical solution \( u \) is locally bounded. By combining Theorems 2.3 and 3.1 we obtain the following interior regularity result.

**Theorem 4.1.** Under the above assumptions, the solution to (4.1) belongs to \( C^{2,\alpha}(\mathbb{R}) \) for \( 0 \leq t < T - \tau \) for any \( \tau > 0 \). In particular, the result applies to the solution of (1.2) if \( 0 < p_1 \leq 1, p_1 \leq p_2, \kappa \) satisfies the assumptions of Theorem 2.2, and \( \partial_x \kappa \) is bounded.

This bound also has probabilistic interpretation: We consider the associated stochastic differential equation

\[
    dX_t = \sigma(t, X_t) dW_t, \quad \sigma^2(t, X_t) = 2\kappa(t, X_t) \varphi'(\partial_{xx} v(t, X_t)),
\]

which corresponds to the linearized equation. As done in [1, 12, 3], we have

\[
    \partial_t (\kappa(t, x) \varphi(\partial_{xx} v)) + \kappa \varphi'(\partial_{xx} v) \partial_{xx} (\kappa \varphi(\partial_{xx} v)) = \frac{\partial_t \kappa}{\kappa}(\kappa(t, x) \varphi(\partial_{xx} v)).
\]

We thus have (under assumptions that guarantee that the representation formula holds) that for \( V_t = \kappa \varphi(\partial_{xx} v)(t, X_t) \),

\[
    V(t, x) = \mathbb{E}_{t, x} \left( V(T, X_T^{t,x}) e^{-\int_t^T \partial_x \kappa / \kappa} \right).
\]

The interior bound on \( \varphi(\partial_{xx} v) \) implies that the stochastic differential equation is well defined on \([0, T)\), and that

\[
    \mathbb{P}(\varphi(\partial_{xx} v(T, X_T)) = +\infty) = 0.
\]

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**References**

[1] Frédéric Abergel and Grégoire Loeper. Pricing and hedging contingent claims with liquidity costs and market impact. To appear in the proceedings of the International Workshop on Econophysics and Sociophysics, Springer, New Economic Window, 2016.

[2] Donald G. Aronson and Philippe Bénilan. Régularité des solutions de l’équation des milieux poreux dans \( \mathbb{R}^N \). *C. R. Acad. Sci. Paris Sér. A-B*, 288(2):A103–A105, 1979.

[3] Bruno Bouchard, Grégoire Loeper, Halil Mete Soner, and Chao Zhou. Second order stochastic target problems with generalized market impact. *arXiv preprint arXiv:1806.08533*, 2018.

[4] Bruno Bouchard, Grégoire Loeper, and Yiyi Zou. Almost-sure hedging with permanent price impact. *Finance Stoch.*, 20(3):741–771, 2016.

[5] Bruno Bouchard, Grégoire Loeper, and Yiyi Zou. Hedging of covered options with linear market impact and gamma constraint. *SIAM J. Control Optim.*, 55(5):3319–3348, 2017.
[6] Michael Crandall and Michel Pierre. Regularizing effects for $u_t + A\phi(u) = 0$ in $L^1$. *J. Funct. Anal.*, 45(2):194–212, 1982.

[7] Michael G. Crandall and Michel Pierre. Regularizing effects for $u_t = \Delta \phi(u)$. *Trans. Amer. Math. Soc.*, 274(1):159–168, 1982.

[8] Emmanuele DiBenedetto, Ugo Gianazza, and Vincenzo Vespri. *Harnack’s inequality for degenerate and singular parabolic equations*. Springer Monographs in Mathematics. Springer, New York, 2012.

[9] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[10] Ivan Guo and Grégoire Loeper. Path dependent optimal transport and model calibration on exotic derivatives. *arXiv preprint arXiv:1812.03526*, 2018.

[11] Ivan Guo, Grégoire Loeper, and Shiyi Wang. Local volatility calibration by optimal transport. *arXiv preprint arXiv:1709.08075*, 2017.

[12] Grégoire Loeper. Option pricing with linear market impact and nonlinear Black-Scholes equations. *Ann. Appl. Probab.*, 28(5):2664–2726, 2018.

[13] Xiaolu Tan and Nizar Touzi. Optimal transportation under controlled stochastic dynamics. *Ann. Probab.*, 41(5):3201–3240, 2013.

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