On the upper open detour monophonic number of a graph

K. Krishna Kumari¹*, S. Kavitha² and D. Nidha³

Abstract
An open detour monophonic set M in a connected graph G is called a minimal open detour monophonic set if no proper subset of M is an open detour monophonic set of G. The upper open detour monophonic number odm+(G) of G is the maximum cardinality of a minimal open detour monophonic set of G. Some general properties satisfied by this concept are studied. The upper open detour monophonic number of some standard graphs are determined. Connected graphs of order n with upper open detour monophonic number 2 or 3 or n are characterized. It is shown that for every pair a and b of integers a and b with 2 ≤ a ≤ b, there exists a connected graph G such that odm(G) = a and odm+(G) = b, where odm(G) is the open detour monophonic number of a graph.

Keywords
detour number, open detour number, monophonic number, open monophonic number, upper open detour monophonic number.

AMS Subject Classification
05C12, 05C38.

1 Introduction

By a graph G = (V, E), we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m, respectively. For basic graph theoretical terminology, we refer to [1]. The neighborhood of a vertex v is the set N(v) consisting of all vertices u which are adjacent with v. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A chord of a path P in G is an edge connecting two non adjacent vertices of P. For two vertices u and v in a connected graph G, a u-v path P is called a monophonic path P is a chordless path [2-11]. A longest u-v monophonic path is called an u-v detour monophonic path. A u-v monophonic path with its length equal to dm(u, v) is known as a uv monophonic. For any vertex v in a connected graph G, the monophonic eccentricity of v is eM(v) = max{dm(u, v): u ∈ V}. A vertex u of G such that dm(u, v) = ei (v) called a monophonic eccentric vertex of v. The monophonic radius and monophonic diameter of G are defined by radM = min{eG (v) : v ∈ V} and diamM = max{eG (v) : v ∈ V}, respectively. We denote radM by rM and diamM by dm. Two vertices u and v are said to be detour antipodal if dm(u, v) = dm.The monophonic distance of a graph was studied in [16]. For two vertices u, v ∈ V, let Jdm[u, v] denotes the set of all vertices that lies in u-v detour monophonic path including u and v, and Jdm(u, v) denotes the set of all internal vertices that lies in u-v detour monophonic path. For M ⊆ V, let Jdm[M] = ∪(u,v∈M) Jdm(u, v). A set M ⊆ V is a detour monophonic set if Jdm[M] = V. The minimum cardinality of a detour monophonic set of G is the detour monophonic number of G and is denoted by dm(G). The detour monophonic set of cardinality dm(G) is called dmset. The detour monophonic number of a graph was studied...

1. Introduction

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A vertex \( x \) of a connected graph \( G \) is said to be a **detour simplicial vertex** of \( G \) if \( x \) is not an internal vertex of any \( u \)-\( v \) detour monophonic path for every \( u, v \in V \). Each extreme vertex of \( G \) is a detour monophonic simplicial vertex of \( G \). In fact there are detour monophonic simplicial vertices which are not extreme vertices of \( G \) [12,14]. For the graph \( G \) given in the Figure 1.1, \( v_9 \) and \( v_{10} \) are the only two detour monophonic simplicial vertices of \( G \). Every extreme vertex of \( G \) is a detour monophonic simplicial vertex of \( G \). In fact there are detour monophonic simplicial vertices which are not extreme vertices of \( G \). For the graph \( G \) given Figure 1.1, \( v_9 \) and \( v_{10} \) is a detour monophonic simplicial vertex of \( G \), which is not an extreme vertex of \( G \). A set \( M \subseteq V \) is called an **open detour monophonic set** of \( G \) if for each vertex \( x \) in \( G \), (1) either \( x \) is a detour monophonic simplicial vertex of \( G \) and \( x \in M \) or (2) \( x \in J_{dm}(u,v) \) for some \( u,v \in M \). An open detour monophonic set of minimum cardinality is called a minimum open detour monophonic set and this cardinality is the open detour monophonic number of \( G \), denoted by \( odm(G) \). An open detour set of cardinality \( odm(G) \) is called an \( odm \)-set of \( G \). The open detour monophonic number of a graph was studied in [13]. Throughout the following \( G \) denotes a connected graph with at least two vertices. The following Theorems are used in the sequel.

![Graph Image](image)

**Theorem 1.1.** [8] Every open detour monophonic set of \( G \) contains its detour monophonic simplicial vertices. Also, if the set \( M \) of all detour monophonic simplicial vertices is an open detour monophonic set of \( G \), then \( M \) is the unique minimum open detour monophonic set of \( G \).

**Theorem 1.2.** [8] Let \( G \) be a connected graph of order \( n \geq 4 \). If \( G \) contains no detour monophonic simplicial vertices, then \( odm(G) \leq 4 \).

### 2. On The Upper Open Detour Monophonic Number of a Graph

**Definition 2.1.** An open detour monophonic set \( M \) in a connected graph \( G \) is called a minimal open detour monophonic set if no proper subset of \( M \) is an open detour monophonic set of \( G \). The upper open detour monophonic number \( odm^+(G) \) of \( G \) is the maximum cardinality of a minimal open detour monophonic set of \( G \).

**Example 2.2.** For the graph \( G \) given in Figure 1.1, \( M_1 = \{v_1,v_9,v_{10}\} \), \( M_2 = \{v_4,v_9,v_{10}\} \), \( M_3 = \{v_1,v_4,v_9,v_{10}\} \), \( M_4 = \{v_3,v_6,v_9,v_{10}\} \) and \( M_5 = \{v_7,v_8,v_9,v_{10}\} \). \( M_6 = \{v_2,v_3,v_9,v_{10}\} \) are the only minimal open detour monophonic sets of \( G \) so that \( odm(G) = 3 \) and \( odm^+(G) = 4 \).

**Remark 2.3.** Every minimum open detour monophonic set of \( G \) is a minimal open detour monophonic set of \( G \) and the converse is not true. For the graph \( G \) given in Figure 2.1, \( \{v_1,v_4,v_9,v_{10}\} \) is a minimal open detour monophonic set of \( G \) but not a minimum open detour monophonic set of \( G \). Here, \( M_1 = \{v_1,v_9,v_{10}\} \) is a minimal open detour monophonic set of \( G \) so that \( odm(G) = 3 \).

**Theorem 2.4.** Every minimal open detour monophonic set of \( G \) contains its detour monophonic simplicial vertices. Also, if the set \( M \) of all detour monophonic simplicial vertices is an open detour monophonic set of \( G \), then \( M \) is the unique minimal open detour monophonic set of \( G \).

**Proof.** Let \( M \) be a monophonic set of \( G \) and \( v \) be an open detour simplicial vertex of \( G \). Let \( \{v_1,v_2,...,v_k\} \) be the neighbors of \( v \) in \( G \). Suppose that \( v \notin M \). Then \( v \in P \) such for a detour monophonic path \( P : x = x_1,x_2,...,x_i,v_1,v_j,...,x_m = y \), where \( x,y \in M \). Since \( v_1v_j \) is a chord of \( P \) and so \( P \) is not a detour monophonic path, which is a contradiction. Hence it follows that \( v \notin M \).

**Theorem 2.5.** Let \( G \) be a connected graph with cut-vertices and \( M \) be a monophonic set of \( G \). If \( v \) is a cut-vertex of \( G \), then every component of \( G-v \) contains at least two element of \( M \).

**Proof.** Suppose that there is a component \( G_1 \) of \( G-v \) such that \( G_1 \) contains no vertex of \( S \). By Theorem 2.4, \( G_1 \) does not contain any end-vertex of \( G \). Thus \( G_1 \) contains at least one vertex, say \( u \). Since \( M \) is a monophonic set, there exists vertices \( x,y \in M \) such that \( z \) lies on the \( x-y \) detour monophonic path \( P : x = u_0,u_1,u_2,....,u_i,y \) in \( G \). Let \( P_1 \) be a \( x-u \) sub path of \( P \) and \( P_2 \) be a \( u-y \) subpath of \( P \). Since \( v \) is a cut-vertex of \( G \), both \( P_1 \) and \( P_2 \) contain \( v \) so that \( P \) is not a path, which is a contradiction. Thus every component of \( G-v \) contains an element of \( M \).

**Theorem 2.6.** For any connected graph \( G \), no cut-vertex of \( G \) belongs to any minimal open detour monophonic set of \( G \).

**Proof.** Let \( M \) be a minimal open detour monophonic set of \( G \) and \( v \in M \) be any vertex. We claim that \( v \) is not a cut vertex of \( G \). Suppose that \( v \) is a cut vertex of \( G \). Let \( G_1,G_2,...,G_r \) \((r \geq 2)\) be the components of \( G-v \). By Theorem 2.5, each component \( G_i \) \((1 \leq i \leq r)\) contains an element of \( M \). We
claim that $M_1 = M - \{v\}$ is also a open detour monophonic set of $G$. Let $x$ be a vertex of $G$. Since $M$ is a monophonic set, $x$ lies on a monophonic path $P$ joining a pair of vertices $u$ and $v$ of $M$. Assume without loss of generality that $u \in G_1$. Since $v$ is adjacent to at least one vertex of each $G_i$ ($1 \leq i \leq r$), assume that $v$ is adjacent to $z$ in $G_{i_k}$. Since $M$ is a open detour monophonic set, $z$ lies on a detour monophonic path joining a vertex $v$ and a vertex $w$ of $M$ such that $w$ must necessarily belongs to $G_k$. Thus $w \neq v$. Now, since $v$ is a cut vertex of $G$, $P \cup Q$ is a path joining $u$ and $v$ in $M$ and thus the vertex $x$ lies on this detour monophonic path joining two vertices $u$ and $w$ of $M_1$. Thus we have proved that every vertex that lies on a detour monophonic path joining a pair of vertices $u$ and $v$ of $M$ also lies on a detour monophonic path joining two vertices of $M_1$. Hence it follows that every vertex of $G$ lies on a detour monophonic path joining two vertices of $M_1$, which shows that $M_1$ is a open detour monophonic set of $G$.

Since $M_1 \subset M$, this contradicts the fact that $M$ is a minimal open detour monophonic set of $G$. Hence $v \notin M$ so that no cut vertex of $G$ belongs to any minimal open detour monophonic set of $G$.

Corollary 2.7. For any non-trivial tree $T$, the monophonic number $odm^+(T) = odm(T) = k$, where $k$ is number of end vertices of $T$.

Proof. This follows from Theorems 2.4 and 2.6.

Corollary 2.8. For the complete graph $K_n$ ($n \geq 2$), $odm(K_n) = odm^+(K_n) = n$.

Proof. Since every vertex of the complete graph $K_n$ ($n \geq 2$) is an open detour simplicial vertex, the vertex set of $K_n$ is the unique detour monophonic set of $K_n$. Thus $odm^+(K_n) = odm(K_n) = n$.

Theorem 2.9. For the cycle $G = C_n$ ($n \geq 4$), $odm^+(G) = 4$.

Proof. If $n$ is even, then let $x$ and $y$ be two monophonic antipodal vertices of $G = C_n$, and $z$ be a detour monophonic antipodal vertex of $y$. Then $M = \{x, y, w, z\}$ is a minimal open detour monophonic set of $G$ and so $odm^+(G) \geq 4$. We show that $odm^+(G) = 4$. Suppose that $odm^+(G) \geq 5$. Then there exists a minimal open detour monophonic set $M_1$ such that $|M_1| \geq 5$. Suppose that $M_1$ is a set of independent vertices of $G$. Then $M_2 = M_1 - \{u\}$, where $u \in M_1$ is an open detour monophonic set of $G$ with $M_2 \subset M_1$, which is a contradiction. Therefore $G[M_1]$ contains at least one edge. Let $uv$ be an edge of $G[M_1]$. Then either $M_1 - \{u\}$ or $M_1 - \{v\}$ is an open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction. Therefore $M_1$ is not a minimal open detour monophonic set of $G$, which is a contradiction.

3. Some Results On The Upper Open Detour Monophonic Number of a Graph

Theorem 3.1. For a connected graph $G$, $2 \leq odm(G) \leq odm^+(G) \leq n$.

Proof. Any open detour monophonic set needs at least two vertices and so $odm(G) \geq 2$. Since every minimal open detour monophonic set is an open detour monophonic set, $odm(G) \leq odm^+(G)$. Also, since $V(G)$ is an open detour monophonic set of $G$, it is clear that $odm^+(G) \leq n$. Thus $2 \leq odm(G) \leq odm^+(G) \leq n$.

Theorem 3.2. For a connected graph $G$ of order $n$, $odm^+(G) = 2$ if and only if $odm(G) = 2$.

Proof. Let $odm^+(G) = 2$. Then by Theorem 3.1, $odm(G) = 2$. Conversely, let $odm(G) = 2$. Let $M$ be an $odm$-set of $G$ with $|M| = 2$. Then by Theorem 1.1, $M$ consists of two detour simplicial vertices. By Theorem 1.1, $M$ is a subset of every open detour monophonic set of $G$. Hence it follows that $M$ is the unique minimal open detour monophonic set of $G$. Therefore $odm^+(G) = 2$.

Theorem 3.3. Let $G$ be a connected graph. If $G$ has a minimal open detour monophonic set $M$ of cardinality three, then all the vertices in $M$ are detour monophonic simplicial vertices.

Proof. Let $M = \{x, y, z\}$ be a minimal open detour monophonic set of $G$. On the contrary suppose that $z$ is not a detour monophonic simplicial vertex of $G$. We consider the following three cases.

Case(1) $x$ and $y$ are non-detour simplicial vertices of $G$. Then $M$ is the set of $M$ contains no detour monophonic simplicial vertices. By Theorem 1.2, $odm(G) \geq 4$, which is a contradiction.

Case(2) $x$ is a detour monophonic simplicial vertex of $G$ and $y$ is not detour monophonic simplicial vertex of $G$. Since $M$ is an open detour monophonic set of $G$, we have $y \in J_{d_m}(x, z)$ and $z \in J_{d_m}(x, y)$. Then we have $d_m(x, z) > d_m(x, y)$ and $d_m(x, y) > d_m(x, z)$. Hence $d_m(x, y) > d_m(x, y)$, which is a contradiction.

Case(3) $x$ and $y$ are detour monophonic simplicial vertices of $G$. Since $M$ is an open detour monophonic simplicial vertex of $G$. We have $z \in J_{d_m}(x, y)$. Let $d_m(u, v) = l$ and $P$ be an $u - v$ detour monophonic path of length $k$. Let us assume that $d(x, z) = k_1$ and $d(y, z) = k_2$. Then $k_1 + k_2 \leq k$. Let $P'$ be a $x - z$ subpath of $P$ and $P''$ be a $z - y$ subpath of $P$. We prove that $M' = \{x, y\}$ is an open detour monophonic set of $G$. Let $v \in V - M'$. Then $v$ is non-detour simplicial vertex of $G$. It follows that $x$ and $y$ are the only two detour monophonic vertices of $G$. We have $v \in J_{d_m}(x, y)$ or $v \in J_{d_m}(x, z)$. If $v \in J_{d_m}(x, z)$, then nothing to prove. Let us assume that $v \in J_{d_m}(x, y)$. Let $v$ be an internal vertex of detour monophonic $x - z$, say $R$. Let $a$ be the $x - y$ walk obtained from $R$ followed by $P'$. Then $|R| = k$ and so $R$ is a $x - y$ detour monophonic containing $v$. Thus $v \in J_{d_m}(x, y)$. Similarly, if $v \in J_{d_m}(x, z)$, we can prove that
For positive integers $m$ and $n$, let $G$ be a connected graph of order $n$. For a connected graph $G$ of order $n$, $\text{odm}^+(G) = 3$ if and only if $\text{odm}(G) = 3$.

**Proof.** Let $\text{odm}(G) = 3$. Let $M$ be an open detour monophonic set of $G$. Since every minimum open detour monophonic set of $G$ is a minimal open detour monophonic set of $G$, by Theorem 3.3, all the vertices of $M$ are detour simplicial vertices. Then by Theorem 1.1, $\text{odm}^+(G) = 3$. Conversely, let $\text{odm}^+(G) = 3$. Let $S$ be a minimal open detour monophonic set of $G$. Then by Theorem 1.1, all the vertices of $M$ are detour simplicial vertices. Hence it follows from Theorem 1.1 that $\text{odm}(G) = 3$.

**Theorem 3.5.** For a connected graph $G$ of order $n$, $\text{odm}^+(G) = n$ if and only if $\text{odm}(G) = n$.

**Proof.** Let $\text{odm}^+(G) = n$. Then $M = V(G)$ is the unique minimal open detour monophonic set of $G$. Since no proper subset of $M$ is an open detour monophonic set, it is clear that $M$ is the unique minimum open detour monophonic set of $G$ and so $\text{odm}(G) = n$. The converse follows from Theorem 3.1.

**Theorem 3.6.** For positive integers $r_m, d_m$ and $l \geq 4$ with $r_m < d_m$, there exists a connected graph $G$ with $\text{rad}_m G = r_m$, $\text{diam}_m G = d_m$ and $\text{odm}^+(G) = l$.

**Proof.** For convenience, we assume $r_m = r$ and $d_m = d$. When $r = 1$, we let $G = K_1$. Then the result follows from Corollary 2.9. Let $r \geq 2$. Let $C_{r+2}v_1, v_2, \ldots, v_{r+2}$ be a cycle of length $r + 2$ and let $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path. Let $H$ be a graph obtained from $C_{r+2}$ and $P_{d-r+1}$ by identifying $v_1$ in $C_{r+2}$ and $u_0$ in $P_{d-r+1}$. Now add $l - 3$ new vertices $w_1, w_2, \ldots, w_{l-3}$ to $H$ and join each $w_i$ ($1 \leq i \leq l-3$) to the vertex $u_{d-r+1}$ to obtain the graph $G$ of Figure 3.1. Then $\text{rad}_m G = r$ and $\text{diam}_m G = d$. Let $W = \{w_1, w_2, \ldots, w_{l-3}, u_{d-r}\}$ be the set of all end-vertices of $G$. Then by Theorem 1.1, $W$ is contained in every open detour monophonic set of $G$. $M_1 = W \cup \{v_1, v_{r+1}\}$. Then $M_1$ is an open detour monophonic set of $G$. If $M_1$ is not a minimal open detour monophonic set of $G$, then there is a proper subset $T$ of $M_1$ such that $T$ is an open detour monophonic set of $G$. Then there exists $v \in M_1$ such that $v \notin T$. By Theorem 1.1, $v \notin w_i$ ($1 \leq i \leq l-3$) and $v \notin u_{d-r}$. Therefore $v$ is either $v_3$ or $v_{r+2}$. If $v = v_3$, then $v_1$ does not lie on a detour monophonic path joining some vertices of $T$. If $v = v_{r+2}$, then $v_{r+1}$ does not lie on a detour monophonic path joining some vertices of $T$ and so $T$ is not an open detour monophonic set of $G$, which is a contradiction. Thus $M_1$ is a minimal open detour monophonic set of $G$ and so $\text{odm}^+(G) \geq l$. We show that $\text{odm}^+(G) = l$. Suppose that $\text{odm}^+(G) \geq l + 1$. Let $T'$ be a minimal open detour monophonic set of $G$ with $|T'| \geq l + 1$. By Theorem 2.4, $W \subseteq T'$. Since $W \cup \{v_j\}$ ($3 \leq i \leq r + 1$) is an open detour monophonic set of $G$, $v_i \notin T'$ ($3 \leq i \leq r + 1$).

Since $M_1$ is an open detour monophonic set of $G, v_2, v_{r+2} \notin T'$. By Theorem 2.6, $u_i \notin T'$ ($0 \leq i \leq d - r - 1$). Hence no such $T'$ exists. Therefore $\text{odm}^+(G) = l$.

In view of Theorem 3.1, we have the following realization result.

**Theorem 3.7.** For every pair $a$ and $b$ of positive integers with $2 \leq a \leq b$, there exists a connected graph $G$ such that $\text{odm}(G) = a$ and $\text{odm}^+(G) = b$.

**Proof.** Let $P_7 : v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be a path on seven vertices. Let $K$ be a graph obtained from $K_{a-2}$ and $K_{b-a}$ with $V(K_{a-2}) = \{z_1, z_2, \ldots, z_{a-2}\}$ and $V(K_{b-a}) = \{h_1, h_2, \ldots, h_{b-a}\}$ by joining each $z_i$ ($1 \leq i \leq a-2$) with $v_i$ and each $h_i$ ($1 \leq i \leq b-a$). Let $G$ be the graph obtained from $H$ and $K$ by joining each $h_i$ ($1 \leq i \leq b-a$) with $v_i$. The graph $G$ is obtained in Figure 3.2.

First we show that $\text{odm}(G) = a$. Let $Z = \{z_1, z_2, \ldots, z_{a-2}\}$ be the set of all end-vertices of $G$. Then by Theorem 1.1, $Z$ is a subset of every open detour monophonic set of $G$ and so $\text{odm}(G) \geq a - 2$. It is easily verified that $Z$ or $Z \cup \{x\}$, where $u \notin Z$ is not an open detour monophonic set of $G$ so that $\text{odm}(G) \geq a$. Let $M = Z \cup \{v_i, v_j\}$. Then $M$ is an open detour monophonic set of $G$ so that $\text{odm}(G) = a$.

Next we prove that $\text{odm}^+(G) = b$. Let $M = Z \cup \{h_1, h_2, \ldots, h_{b-a}\} \cup \{v_2, v_7\}$. Then $M$ is an open detour monophonic set of $G$. We prove that $M$ is a minimal open detour monophonic set of $G$. On the contrary suppose $M$ is not a minimal open detour monophonic set of $G$. Then there exists an open detour monophonic set $M'$ of $G$ such that $M' \subset M$.
Then there exists $x \in M$ such that $x \notin M'$. By Theorem 1.1, $x \neq z_i$ for all $i$ ($1 \leq i \leq a - 2$). If $x = h_i$ for some ($1 \leq i \leq b - a$), then $x \in J_{dm}(M)$, if $x = v_2$ then $v_2, h_i \notin I(M)$ for all $i$ ($1 \leq i \leq b - a$). If $x = v_\gamma$, then $v_\gamma, h_i \notin I(M)$ for all $i$ ($1 \leq i \leq b - a$). Therefore $M'$ is not an open detour monophonic set of $G$, which is a contradiction. Therefore $M$ is a minimal open detour monophonic set of $G$ and so $odm^+ (G) = b$. Suppose that $odm^+(G) \geq b + 1$. Then there exist a minimal open detour monophonic set $M'$ of $G$ such that $|M'| \geq b + 1$. Then by similar argument as above, we prove that $M'$ is not a minimal open detour monophonic set of $G$. Therefore $odm^+ (G) = b$. 

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