Algebraically determined topologies on permutation groups

Taras Banakh, Igor Guran, Igor Protasov

Kielce-Lviv-Kyiv

SPM 2012, Caserta
For a set $X$ by $S(X)$ we denote the *symmetric group*, i.e., the group all permutations (＝bijections) of $X$.

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**Theorem (Cayley, 1854)**

*Each group $G$ is isomorphic to a subgroup of the symmetric group $S(G)$.*

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The symmetric group $S(X)$ contains the normal subgroup $S_\omega(X)$ consisting of all permutations $f : X \to X$ that have finite support

$$\text{supp}(f) = \{x \in X : f(x) \neq x\}.$$

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If $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, then $f \circ g = g \circ f$. 
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On each permutation group $G \subset S(X) \subset X^X$ we can consider the \textit{topology of pointwise convergence} $\mathcal{T}_p$ inherited from the Tychonoff power $X^X$ of $X$ endowed with the discrete topology.

\textbf{Fact}

The topology $\mathcal{T}_p$ turns $G$ into a Hausdorff topological group. In other words, $\mathcal{T}_p$ is a \textit{Hausdorff group topology} on $G$.

A neighborhood base of the topology $\mathcal{T}_p$ at the neutral element $1_G$ consists of the subgroups

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# Extremal properties of the topology $\mathcal{T}_p$

**Theorem (Dierolf-Schwanengel, 1977)**

For any group $G$ with $S_\omega(X) \subset G \subset S(X)$, the topology $\mathcal{T}_p$ is a minimal Hausdorff group topology on $G$.

**Theorem (Gaughan, 1967)**

For the group $G = S(X)$, the topology $\mathcal{T}_p$ is the smallest Hausdorff group topology on $G$.

**Problem (Dikranjan, 2010)**

Let $G$ be a group such that $S_\omega(X) \subset G \subset S(X)$. Is $\mathcal{T}_p$ the smallest Hausdorff group topology on $G$?

**Answer (B-G-P, 2011)**

Yes!
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Various sorts of topologized groups

A group $G$ endowed with a topology $\mathcal{T}$ is called

- a **topological group** if the binary operation $(x, y) \mapsto xy^{-1}$ is continuous;
- a **quasi-topological group** if the binary operation $(x, y) \mapsto xy^{-1}$ is separately continuous;
- a **semi-topological group** if the binary operation $(x, y) \mapsto xy$ is separately continuous;
- a **[quasi]-topological group** if the binary operations $(x, y) \mapsto xy^{-1}$ and $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$ are separately continuous;
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Interplay between various sorts of topologized groups

Fact

A group $G$ with topology $T$ is [semi]-topological if and only if for any $a, b \in G$

- the shift $s_{a,b} : x \mapsto axb$ and
- the conjugator $\gamma_a : x \mapsto xax^{-1}$

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Main result answering the Dikranjan’s Problem

Theorem (B-G-P, 2011)

For any group $G$ with $S_\omega(X) \subset G \subset S(X)$, the topology $T_p$ is the smallest $T_1$-topology turning $G$ into a [semi]-topological group.
Proof of Theorem

Let $S_\omega(X) \subset G \subset S(X)$ and $\mathcal{T}$ be a $T_1$-topology on $G$ such that $(G, \mathcal{T})$ is a [semi]-topological group.

Our aim: To prove that $\mathcal{T}_p \subset \mathcal{T}$.
This is trivial if $X$ is finite. So, we assume that $X$ is infinite.

Observe that the subgroups

$$G_A = \{ g \in G : g|A = \text{id} \}, \quad |A| < \infty$$

form a neighborhood base of the topology $\mathcal{T}_p$ at $1_G$, while the family

$$\{ G_A : A \subset X, \ |A| = 3 \}$$

is a neighborhood subbase of $\mathcal{T}_p$ at $1_G$.

So, to prove the theorem, it suffices to check that for each 3-element subset $A \subset X$ the subgroup $G_A$ is $\mathcal{T}$-open.
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Lemma

For each 3-element subset $A \subset X$ the subgroup $G_A$ is $T$-closed.

Proof.

Given any permutation $g \notin G_A$, find a point $a \in A$ with $g(a) \neq a$. Choose any $b \in A \setminus \{a, g(a)\}$ and consider the transposition $t : X \to X$ such that $\text{supp}(t) = \{a, b\}$. Then $t \circ g \neq g \circ t$ as $g \circ t(a) = g(b)$ while $t \circ g(a) = g(a)$.

So,

$$U = \{f \in G : f \circ t \neq t \circ f\} = \{f \in G : f \circ t \circ f^{-1} \neq t\} = \gamma_t^{-1}(G\{t\})$$

is a $T$-open neighborhood of $g$, which is disjoint with $G_A$. \qed
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Lemma

For some 3-element subset \( A \subset X \) the subgroup \( G_A \) is \( T \)-open.

Proof. Assume not. Then for each 3-element subset \( A \subset X \) the subgroup \( G_A \) is not open and being closed is nowhere dense in \((G, T)\).

Claim

For any 3-element subset \( A \subset X \) and any finite set \( B \subset X \) the set

\[
G(A, B) = \{ g \in G : g(A) \subset B \}
\]

is closed and nowhere dense in \((G, T)\).

Proof. Since the set of maps \( A \to B \) is finite, we can choose a finite subset \( F \subset G(A, B) \) such that for each \( g \in G(A, B) \) there is \( f \in F \) with \( f|A = g|A \). Then \( f^{-1} \circ g \in G_A \) and hence \( g \in f \circ G_A \). So, \( G(A, B) = \bigcup_{f \in F} f \circ G_A \) is closed and nowhere dense as a finite union of closed nowhere dense subspaces.
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Choose two disjoint 3-element subsets $A, B \subset X$ and consider the nowhere dense subset $G(A, A \cup B) \cup G(B, A \cup B)$ in $(G, \mathcal{T})$.

For any distinct points $a, b \in A \cup B$ let $t_{a,b} \in S_\omega(X) \subset G$ be the transposition with $\text{supp}(t_{a,b}) = \{a, b\}$.

Put $T = \{t_{a,b} : a, b \in A \cup B\}$.

For every $t \in T$ the set

$$V_t = \{u \in G : u \circ t \neq t \circ u\} = \gamma_t^{-1}(G \setminus \{t\})$$

is $\mathcal{T}$-open and contains each transposition $s \in T$ with $s \circ t \neq t \circ s$.

Then the set

$$U_{s,t} = \gamma_s^{-1}(V_t) = \{u \in G : (usu^{-1})t \neq t(usu^{-1})\}$$

is a $\mathcal{T}$-open neighborhood of $1_G$ and so is the intersection

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$$U = \bigcap\{U_{s,t} : s, t \in T, t \circ s \neq s \circ t\}.$$
Choose two disjoint 3-element subsets $A, B \subset X$ and consider the nowhere dense subset $G(A, A \cup B) \cup G(B, A \cup B)$ in $(G, \mathcal{T})$. For any distinct points $a, b \in A \cup B$ let $t_{a,b} \in S_\omega(X) \subset G$ be the transposition with $\text{supp}(t_{a,b}) = \{a, b\}$. Put $T = \{t_{a,b} : a, b \in A \cup B\}$. For every $t \in T$ the set

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Choose any point $c \in B \setminus \{b\}$ and consider two non-commuting permutations $t = t_{a,c}$ and $s = t_{a,b}$.

It follows from $u \in U \subset U_{s,t} = \gamma_s^{-1}(V_t)$ that the permutation $v = usu^{-1} = \gamma_s(u) \in V_t$ and hence $v \circ t \neq t \circ v$.

On the other hand, $\text{supp}(v) = u(\text{supp}(s)) = u(\{a, b\})$ does not intersect $\{a, b\} = \text{supp}(t_{a,b})$ and hence $v$ commutes with $t$.

This contradiction shows that, the subgroup $G_A$ is $\mathcal{T}$-open for some 3-element subset $A \subset X$. 
Choose a permutation $u \in U \setminus \left( G(A, A \cup B) \cup G(B, A \cup B) \right)$ and observe that $u(a), u(b) \notin A \cup B$ for some points $a \in A$ and $b \in B$.

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Claim

For each 3-element subset $B \subset X$ the subgroup $G_B$ is $T$-open.

Proof. Choose any permutation $f \in S_\omega(X) \subset G$ with $f(A) = B$ and observe that $G_B = f \circ G_A \circ f^{-1}$ is $T$-open, being a two-sided shift of the $T$-open subgroup $G_A$. 
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What we have just proved

**Theorem (B-G-P, 2011)**

For any group $G$ with $S_\omega(X) \subset G \subset S(X)$, the topology $\mathcal{T}_p$ is the smallest $T_1$-topology turning $G$ into a [semi]-topological group.

**Remark**

The [semi]-topological cannot be replaced by semi-topological as the group $G = S_\omega(\mathbb{Z})$ admits a shift-invariant Hausdorff topology $\mathcal{T}$ which is incomparable with $\mathcal{T}_p$.

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The group $G = S_\omega(X)$ is $\sigma$-discrete in any $T_2$-topology turning $G$ into a semi-topological group.
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**Definition**

A group $G$ is *topologizable* if $G$ admits a non-discrete Hausdorff group topology.

**Remark**

Each infinite abelian group $G$ is topologizable as $G$ embeds in $\mathbb{T}^{|G|}$.

**Problem (Markov, 1946)**

Is each infinite group topologizable?

**Answer**

There exist:
- an uncountable non-topologizable group (Hesse, 1979);
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For a group $G$

- the *Markov topology* $\mathcal{M}_G$ is the intersection of all Hausdorff groups topologies on $G$;
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- $\mathcal{Z}_G \subset \mathcal{M}_G \subset T$ for each group $T_2$-topology $T$ on $G$.
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**Theorem**

\[ \mathcal{Z}_G = \mathcal{M}_G \text{ if the group } G \text{ is:} \]

- countable (Markov, 1946);
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**Theorem (Hesse, 1979)**

There is an uncountable non-topologizable group \( G \) with \( \mathcal{M}_G \neq \mathcal{Z}_G \) (so, \( \mathcal{M}_G \) is discrete while \( \mathcal{Z}_G \) is not).

**Problem (Dikranjan-Shakhmatov, 2007 (OPIT2))**

Is \( \mathcal{Z}_G = \mathcal{M}_G \) for each symmetric group \( G = S(X) \)?

**Answer (B-G-P, 2011)**

Yes: \( \mathcal{Z}_G = \mathcal{M}_G = T_p \) for each group \( G \) with \( S_\omega(X) \subset G \subset S(X) \).
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Let $G$ be a group with $\mathcal{Z}_G = \mathcal{M}_G$. Is $\mathcal{Z}_H = \mathcal{M}_H$ for each subgroup $H$ of $G$?

Answer (B-G-P, 2011)

No!

Proof.

Take Hesse's non-topologizable group $H$ with $\mathcal{Z}_H \neq \mathcal{M}_H$ and using Cayley theorem, embed $H$ into the permutation group $G = S(H)$. Then $G$ is a group with $\mathcal{Z}_G = \mathcal{M}_G$ containing the subgroup $H \subset G$ with $\mathcal{Z}_H \neq \mathcal{M}_H$. \qed
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Two natural topologies on $S(X)$

Each discrete space $X$ has two natural compactifications:
- $\alpha X$, the *Aleksandrov* one-point compactifications;
- $\beta X$, the *Čech-Stone* compactification.

**Fact**

*Each bijection $f : X \rightarrow X$ can be uniquely extended to homeomorphisms $\alpha f : \alpha X \rightarrow \alpha X$ and $\beta f : \beta X \rightarrow \beta X$.***

Consequently, the group $S(X)$ can be identified with the homeomorphisms groups $\mathcal{H}(\alpha X)$ and $\mathcal{H}(\beta X)$ of the compactifications $\alpha X$ and $\beta X$.

This identification allows us to introduce the compact-open topologies $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ on $S(X)$. 
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$\mathcal{T}_\alpha = \mathcal{T}_p$. Consequently, $S_\omega(X)$ is a dense subgroup of the topological group $(S_\omega(X), \mathcal{T}_\alpha) = \mathcal{H}(\alpha X)$.

Theorem (B-G-P, 2011)

The subgroup $S_\omega(X)$ is closed and nowhere dense in the topological group $(S(X), \mathcal{T}_\beta) = \mathcal{H}(\beta X)$. Consequently, the quotient topological group $S(X)/S_\omega(X)$ is not discrete and thus is topologizable.
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An Open Problem

**Definition**
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topologizable → [quasi]-topologizable → [semi]-topologizable
                ↓     ↓
quasi-topologizable → semi-topologizable
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**Theorem (Zelenyuk, 2000)**
Each infinite group is quasi-topologizable.

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Is each infinite group [quasi]-topologizable? [semi]-topologizable?
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T. Banakh, I. Guran, I. Protasov, *Algebraically determined topologies on permutation groups*, Topology Appl. **159** (2012) 2258-2268.

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Thanks!
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