THE DERIVED CATEGORY WITH RESPECT TO A GENERATOR

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ABSTRACT. Let \( \mathcal{G} \) be any Grothendieck category along with a choice of generator \( G \), or equivalently a generating set \( \{G_i\} \). We introduce the derived category \( \mathcal{D}(G) \), which kills all \( G \)-acyclic complexes, by putting a suitable model structure on the category \( \text{Ch}(\mathcal{G}) \) of chain complexes. It follows that the category \( \mathcal{D}(G) \) is always a well-generated triangulated category. It is compactly generated whenever the generating set \( \{G_i\} \) has each \( G_i \) finitely presented, and in this case we show that two recollement situations hold. The first is when passing from the homotopy category \( K(\mathcal{G}) \) to \( \mathcal{D}(G) \). The second is a \( G \)-derived analog to the recollement of Krause from [Kra05]. We illustrate with several examples ranging from pure and clean derived categories to quasi-coherent sheaves on the projective line \( \mathbb{P}^1(k) \).

1. INTRODUCTION

This paper is about doing homological algebra with respect to a given generator in a Grothendieck category. Let \( R \) be a ring and \( \text{Ch}(R) \) denote the category of chain complexes of (left) \( R \)-modules. Recall that the usual derived category \( \mathcal{D}(R) \) is defined by first constructing the homotopy category \( K(R) \) of unbounded chain complexes of \( R \)-modules, and then formally inverting the homology isomorphisms. \( R \) itself, when viewed as an \( R \)-module is a generator for \( R \)-Mod. But when \( R \) is viewed as a chain complex in degree zero, it is a weak generator for \( \mathcal{D}(R) \) which essentially means it can detect exactness. Note that for a chain complex \( X \), the standard isomorphism \( \text{Hom}_R(R, X) \cong X \) allows one to view the homology of \( X \) as \( H_n[\text{Hom}_R(R, X)] \). Similarly, homology isomorphisms can be viewed as those chain maps \( X \to Y \) in \( \text{Ch}(R) \) which become homology isomorphisms after applying \( \text{Hom}_R(R, -) \).

But sometimes the derived category \( \mathcal{D}(R) \) is not the right home for the homological algebra one is interested in. For example, there is the pure derived category of a ring \( R \) introduced in [CH02], and recently extended to any locally presented additive category in [Kra12]. Here if we take \( G = \bigoplus G_i \) where the \( G_i \) range through a set of isomorphism representatives for all finitely presented objects, then a complex \( X \) is pure acyclic if and only if \( H_n[\text{Hom}(G, X)] \) vanishes for all \( n \). Similarly, isomorphisms in the pure derived category are those chain maps \( X \to Y \) which become homology isomorphisms after applying \( \text{Hom}(G, -) = \prod \text{Hom}(G_i, -) \). So we are essentially doing homological algebra with respect to the generator \( G \).

The most important categories we encounter in homological algebra are the Grothendieck categories, which recall are the abelian categories having exact direct limits and a generator \( G \). A generator \( G \) is equivalent to a generating set \( \{G_i\} \) where \( G = \bigoplus G_i \). This paper starts by showing that given any Grothendieck category \( \mathcal{G} \)
and a fixed choice of generator $G = \oplus G_i$, we can define the derived category $\mathcal{D}(G)$. This category is obtained by inverting the $G$-homology isomorphisms, which are the chain maps $X \to Y$ in $\text{Ch}(\mathcal{G})$ such that $\text{Hom}_\mathcal{G}(G, X) \to \text{Hom}_\mathcal{G}(G, Y)$ is a homology isomorphism. Said another way, this is the category obtained from $\text{Ch}(\mathcal{G})$ by forcing the $G$-acyclic complexes, which are those complexes $X$ for which $\text{Hom}_\mathcal{G}(G, X)$ is exact, to be 0. A nice way to understand such a category is to look for a Quillen model structure on $\text{Ch}(\mathcal{G})$ whose “trivial” objects are the $G$-acyclic complexes. We construct such a model structure, generalizing the usual projective model structure on $\text{Ch}(R)$. Our method is as follows: We start by showing that the generator $G = \oplus G_i$ determines a Quillen exact structure on $\mathcal{G}$, which as we prove in Appendix B, is equivalent to a proper class of short exact sequences in the sense of [Mac63]. The short exact sequences here are precisely the usual short exact sequences which remain exact after applying $\text{Hom}_\mathcal{G}(G, -)$. We call them $G$-exact sequences and we denote this exact structure by $\mathcal{G}_G$. So then the $G$-derived category $\mathcal{D}(G)$ is really just the derived category of this Quillen exact structure. To show its existence we use the connection between model structures and cotorsion pairs from [Hov02]. Here is a more precise statement of this first result.

**Theorem A** (Existence of $G$-derived categories). Let $\mathcal{G}$ be any Grothendieck category with a generator $G = \oplus G_i$. Then there is a model structure on $\text{Ch}(\mathcal{G})$ which we call the $G$-projective model structure whose trivial objects are the $G$-acyclic complexes. We call the associated homotopy category the $G$-derived category, and denote it by $\mathcal{D}(G)$. It is always a well generated triangulated category and is compactly generated whenever each $G_i$ is finitely presented. For given objects $A, B \in \mathcal{G}$ we have $\mathcal{D}(G)(A, \Sigma^n B) = G\text{-Ext}^n_G(A, B)$ where $G\text{-Ext}^n_G(A, B)$ denotes the group of (equivalence classes of) $n$-fold $G$-exact sequences $B \to X_1 \to \cdots X_n \to A$.

**Proof.** The proof, along with a more complete description of the cofibrations, fibrations, and weak equivalences appears in Theorem 4.6 and Corollary 4.7. See also Subsection 4.5. \qed

But there are two standard Quillen model structures on $\text{Ch}(R)$ having $\mathcal{D}(R)$ as its homotopy category. Besides the projective one, there is a dual injective model. In general, if we assume each $G_i$ is finitely presented, then in addition to $\mathcal{D}(G)$ being compactly generated, we have that the dual $G$-injective model structure exists on $\text{Ch}(\mathcal{G})$. Corollary 5.12 gives the precise statement. The two model structures are “balanced” in the sense that possess the same trivial objects. This is essentially the reason behind the following theorem. It is a $G$-derived version of a well know fact about $\mathcal{D}(R)$.

**Theorem B** (Verdier localization recollement for $G$-derived categories). Suppose $\mathcal{G}$ is a Grothendieck category and that $G = \oplus G_i$ is a generator with each $G_i$ finitely presented. Let $\mathcal{D}(G)$ denote the $G$-derived category. Let $K(\mathcal{G})$ denote the homotopy category of all chain complexes and let $K_{G,\text{-ac}}(\mathcal{G})$ denote the subcategory of all $G$-acyclic complexes. Then we have a recollement of triangulated categories:

$$K_{G,\text{-ac}}(\mathcal{G}) \xrightarrow{\text{inclusion}} K(\mathcal{G}) \xrightarrow{\text{inclusion}} \mathcal{D}(G).$$

**Proof.** See Theorem 6.4 where the functors are described as well. \qed

We note that constructing the $G$-injective model structure is far more technical than constructing the $G$-projective model. To do so we use the theory of purity
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from [AR94]. In particular, the assumption that the generating set \( \{ G_i \} \) satisfies that each \( G_i \) is finitely presented is equivalent to saying that \( \mathcal{G} \) is a locally finitely presented Grothendieck category. This is precisely the setting in which a nice theory of purity holds. See [AR94],[CB94], and Appendix A. We emphasize that this still includes the most important categories we encounter in homological algebra. For instance, the category of quasi-coherent sheaves over a quasi-compact and quasi-separated scheme is a locally finitely presented Grothendieck category by [Gar10, Proposition 3.1].

The existence of the injective model structure will also lead us to the following Theorem, which is a \( G \)-version of Krause’s result from [Kra05]. Here we call an object \( G \)-injective if it is injective with respect to the \( G \)-exact sequences already mentioned above.

**Theorem C** (Krause’s recollement for \( G \)-derived categories). Let \( \mathcal{G} \) be a Grothendieck category and let \( G = \oplus G_i \) be a generator with each \( G_i \) finitely presented. Let \( D(G) \) denote the \( G \)-derived category. Let \( K_{G\text{-injective}}(\text{Inj}) \) denote the homotopy category of all complexes of \( G \)-injectives. Let \( K_{G\text{-acyclic}}(\text{Inj}) \) denote the homotopy category of all \( G \)-acyclic complexes of \( G \)-injectives. Then there is a recollement

\[
K_{G\text{-acyclic}}(\text{Inj}) \xrightarrow{\approx} K_{G\text{-injective}}(\text{Inj}) \xrightarrow{\approx} D(G) .
\]

**Proof.** See Theorem 6.3. \( \square \)

The introduction continues in Section 2 where we list several applications or examples of the above Theorems.

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2. Examples

As described in the Introduction, this paper shows that for a given set of generators \( \{ G_i \} \) in a Grothendieck category \( \mathcal{G} \), we can do homological algebra by viewing everything “through the eyes of \( G \)”. In particular, one should try to understand the proper class of \( G \)-exact sequences; those short exact sequences which remain exact after applying \( \text{Hom}_{\mathcal{G}}(G_i, -) \) for all the \( G_i \). Whenever \( G = \oplus G_i \) is projective, then this is just the usual class of short exact sequences and so \( D(G) \) is the usual derived category \( D(\mathcal{G}) \). So the interesting thing is to explore what happens for other choices of \( G \). We consider a some examples here but there is much more room to explore this theme.

2.1. Pure and \( \lambda \)-pure derived categories. In [CH02], Christensen and Hovey put a model structure on \( \text{Ch}(R) \) whose homotopy category was the \textit{pure derived category}, obtained by killing the pure acyclic complexes. More generally Krause shows in [Kra12, Theorem 4.1] that the pure derived category \( D_{\text{pur}}(\mathcal{G}) \) exists whenever \( \mathcal{G} \) is a locally finitely presentable Grothendieck category. In this case he shows there is a recollement situation when passing from \( K(\mathcal{G}) \) to \( D_{\text{pur}}(\mathcal{G}) \). This also follows from Theorem B by taking \( G = \oplus G_i \) where the \( G_i \) range through a set of
isomorphism representatives for all finitely presented objects. (However, we note that Krause does not even assume that $\mathcal{G}$ is Grothendieck, merely additive.) But now we also have the following result as an immediate consequence of our above Theorem C.

**Theorem D.** Suppose that $\mathcal{G}$ is any locally finitely presented Grothendieck category. Let $\mathcal{D}_{\text{pur}}(\mathcal{G})$ denote the pure derived category. Let $K(\text{PInj})$ denote the homotopy category of all complexes of pure-injective objects in $\mathcal{G}$. Let $K_{\text{p-ae}}(\text{PInj})$ denote the homotopy category of all pure acyclic complexes of pure-injectives. Then there is a recollement

$$K_{\text{p-ae}}(\text{PInj}) \overset{i}{\longrightarrow} K(\text{PInj}) \overset{p}{\longrightarrow} \mathcal{D}_{\text{pur}}(\mathcal{G}).$$

Theorem D is interesting even for the category of modules over a ring $R$. For instance, it indicates that there ought to be a notion of Gorenstein pure-injective modules. It also indicates that other applications of Krause’s recollement from [Kra05] might have pure analogs.

We describe in Subsection 4.6 a generalization of the pure derived category to any Grothendieck category by replacing the notion of pure with the notion of $\lambda$-pure where $\lambda$ is some large regular cardinal. We are only able to show that the projective model structure exists. But here a cofibrant replacement of an object $A \in \mathcal{G}$ is obtained by taking a $\lambda$-pure projective resolution of $A$ in the sense of [Ros09]. It is worth noting that the existence of the $\lambda$-pure derived category doesn’t appear to follow from results in [Kra12] because the $\lambda$-pure short exact sequences are not closed under filtered colimits, only $\lambda$-filtered colimits. For a similar reason, the $\lambda$-pure exact structure on $\mathcal{G}$ doesn’t appear to be, in general, of Grothendieck type in the sense of [Sto13]. We see in Subsection 4.6 that for any generator $G = \bigoplus G_i$, there is a regular cardinal $\lambda$ and a canonical functor $\mathcal{D}_{\lambda, \text{pur}}(\mathcal{G}) \to \mathcal{D}(\mathcal{G})$ where $\mathcal{D}_{\lambda, \text{pur}}(\mathcal{G})$ is the $\lambda$-pure derived category. This functor admits a left adjoint and provides a map of relative Ext groups $\lambda\text{-PExt}^n_\mathcal{G}(A, B) \to G\text{-Ext}^n_\mathcal{G}(A, B)$ which is natural in $A, B \in \mathcal{G}$.

### 2.2. Sheaves of modules on a ringed space

Let $\mathcal{O}_X$ be a ringed space, that is, a sheaf of rings on a topological space $X$. The category $\mathcal{O}_X\text{-Mod}$ of sheaves of $\mathcal{O}_X$-modules is a Grothendieck category. Let’s first recall the standard set of generators for $\mathcal{O}_X\text{-Mod}$. For each open $U \subseteq X$, extend $\mathcal{O}_U$ by 0 outside of $U$ to get a presheaf, which we denote by $\mathcal{O}_U$. Now sheafify to get an $\mathcal{O}_X$-module, which we will denote $j!(\mathcal{O}_U)$. There are standard isomorphisms $\text{Hom}(j!(\mathcal{O}_U), G) \cong \text{Hom}(\mathcal{O}_U, G) \cong G(U)$. It follows at once that the set $\{ j!(\mathcal{O}_U) \}$ forms a generating set since the modules $j!(\mathcal{O}_U)$ “pick out points”. Hence the direct sum $G = \bigoplus_{U \subseteq X} j!(\mathcal{O}_U)$ is a generator. The above isomorphisms also imply that the $G$-exact category is just $\mathcal{O}_X\text{-Mod}$ together with the proper class of short presheaf exact sequences of $\mathcal{O}_X$-modules. That is, a $G$-exact sequence is an exact sequence $0 \to F \to G \to H \to 0$ of $\mathcal{O}_X$-modules for which $0 \to F(U) \to G(U) \to H(U) \to 0$ is an exact sequence of $\mathcal{O}(U)$-modules for each open $U \subseteq X$. The $G$-derived category of Theorem A is thus the category of unbounded complexes of $\mathcal{O}_X$-modules modulo the presheaf acyclic complexes. Using, again, the above isomorphisms, it follows immediately from [Har77, Exercise II.1.11] that each $j!(\mathcal{O}_U)$ is finitely presented whenever the space $X$ is Noetherian. In particular, whenever $X = (X, \mathcal{O}_X)$ is a Noetherian scheme then $\mathcal{D}(\mathcal{G})$ is compactly generated. Also Theorems B and C...
apply in this case and the reader can interpret what they say. Just note that a G-injective \( \mathcal{O}_X \)-module here translates to one that is injective with respect to the short presheaf exact sequences. By Proposition 5.6, there are enough such G-injectives in the sense that we can find for any \( \mathcal{O}_X \)-module \( F \) a short presheaf exact sequence

\[ 0 \to F \to I \to I/F \to 0 \]

where \( I \) is G-injective.

2.3. Quasi-coherent sheaves over the projective line \( \mathbb{P}^1(k) \). Let \( k \) be a commutative ring with identity. Here we consider the category of quasi-coherent sheaves over the projective line \( \mathbb{P}^1(k) \). However, we use the quiver description of this category from [EE05], [EEGOb], [EEGOa] and [EEGR]. From this point of view, we consider the representation

\[ R \equiv k[x] \hookrightarrow k[x, x^{-1}] \hookleftarrow k[x^{-1}] \]

of the quiver \( Q \equiv \bullet \to \bullet \leftarrow \bullet \). Then \( R \) corresponds to the structure sheaf on \( \mathbb{P}^1(k) \).

A quasi-coherent sheaf of modules over \( \mathbb{P}^1(k) \) may be thought of as a representation

\[ A \equiv M \xrightarrow{f} L \xleftarrow{g} N \]

with \( M \) a \( k[x] \)-module, \( L \) a \( k[x, x^{-1}] \)-module, \( N \) a \( k[x^{-1}] \)-module, \( f \) a \( k[x] \)-linear map, and \( g \) a \( k[x^{-1}] \)-linear map; all satisfying that the localization maps \( S^{-1}f : S^{-1}M \to S^{-1}L \cong \mathbb{L} \) and \( T^{-1}g : T^{-1}N \to S^{-1}L \cong \mathbb{L} \) are \( k[x, x^{-1}] \)-isomorphisms, where \( S = \{1, x, x^2, \ldots\} \) and \( T = \{1, x^{-1}, x^{-2}, \ldots\} \). We call such an \( A \) a quasi-coherent \( R \)-module. A morphism is the obvious triple of linear maps providing commutative squares. Denote by \( \text{Qco}(R) \) the category of all quasi-coherent \( R \)-modules. Then \( \text{Qco}(R) \) is equivalent to the category of quasi-coherent sheaves on \( \mathbb{P}^1(k) \) and so it is a Grothendieck category. There is a set of generators corresponding to the line bundles of degree \( n \) over \( \mathbb{P}^1(k) \). They are the quasi-coherent \( R \)-modules

\[ R(n) \equiv k[x] \hookrightarrow k[x, x^{-1}] \overset{x^n}{\xleftarrow{k[x^{-1}]}} \]

where the map on the right is multiplication by \( x^n \). Tensor products, direct limits, and finite limits are all taken componentwise. In particular, a short exact sequence in \( \text{Qco}(R) \) is one having all three involved short sequences exact. We refer the reader to [EE05], [EEGOb], [EEGOa] and [EEGR] for more detail on all of the above.

Now given any \( A \in \text{Qco}(R) \), by regarding it as a diagram \( M \xrightarrow{f} L \xleftarrow{g} N \) of just abelian groups, we may take the pullback \( M \times_L N \). Denote this abelian group by \( PA \). Also, given an integer \( n \), denote by \( A(n) \) the twisted sheaf \( R(n) \otimes_R A \). Note that there is an obvious isomorphism \( A(n) \equiv M \xrightarrow{f} L \overset{x^n}{\xleftarrow{k[x^{-1}]}} N \). Each \( R(n) \) is flat and in particular if \( 0 \to A \to B \to C \to 0 \) is a short exact sequence in \( \text{Qco}(R) \), then so is \( 0 \to A(n) \to B(n) \to C(n) \to 0 \). Consequently we have that \( 0 \to PA(n) \to PB(n) \to PC(n) \) is exact. If each \( PB(n) \to PC(n) \) is also onto, then let’s refer to \( 0 \to A \to B \to C \to 0 \) as a twisted fibre exact sequence.

From [EEGOa] we have that \( \{ R(n) \} \) is a set of (flat) generators for \( \text{Qco}(R) \). Setting \( G = \oplus_{n \in \mathbb{Z}} R(n) \), one can show that the \( G \)-exact sequences are precisely the twisted fibre exact sequences. Indeed for each \( n \) one can check directly that the elements of \( \text{Hom}_{\text{Qco}(R)}(R(n), A) \) are in one to one correspondence with the elements of the pullback \( PA(-n) \). That is, we have natural isomorphisms of abelian groups

\[ \text{Hom}_{\text{Qco}(R)}(R(n), A) \cong PA(-n) \]

This isomorphism also can be used to show that
each $R(n)$ is finitely presented: For a direct limit $\lim\rightarrow A_i$, using that pullbacks and tensor products commute with direct limits we see

$$
\text{Hom}_{\text{Qco}(R)}(R(n), \lim\rightarrow A_i) \cong P[(\lim\rightarrow A_i)(-n)] \cong P[R(-n) \otimes_R \lim\rightarrow A_i] \cong P[\lim\rightarrow (R(-n) \otimes_R A_i)] \\
\cong \lim\rightarrow \text{Hom}_{\text{Qco}(R)}(R(n), A_i).
$$

So Theorems A, B, and C apply. Moreover, our characterization of the cofibrant and trivially cofibrant objects provided by Theorem 4.6 allows one to easily check that the model structure is monoidal so that the tensor product descends to a well-behaved tensor product on the $G$-derived category. To do this, apply Hovey’s [Hov02, Theorem 7.2] and the method of [Gil07, Theorem 5.1]; it all boils down to the fact that $R(m) \otimes_R R(n) \cong R(m + n)$ which was shown from the quiver perspective in [EEGOa, Proposition 3.3].

### 2.4. Other examples concerning modules over a ring

Let $R$ be a ring with 1, and let $G = R\text{-Mod}$ be the category of (left) $R$-modules. Note that if $S$ is any set of $R$-modules, then $S \cup \{R\}$ is a generating set for $R\text{-Mod}$. So Theorem A gives us a model structure killing the exact complexes which remain exact after applying $\text{Hom}_R(S, -)$ for all $S \in S$. Of course Theorems B and C also hold if all the $S$ are finitely presented modules. Moreover, whenever $S \subseteq T$, then in a way analogous to Corollary 4.8 we have a canonical functor $D(T) \to D(S)$ with a left adjoint. The functor provides a mapping of relative Ext groups. We give two interesting examples below.

#### 2.4.1. The clean derived category

For non-coherent rings we have the following variant of the pure derived category. An $R$-module is said to be of type $FP_\infty$ if it has a projective resolution consisting of finitely generated free modules. The category of all type $FP_\infty$ modules has a small skeleton. So we can take $S$ to be a set of isomorphism representatives. Then with $G = \oplus_{S \in S} S$ we get that the $G$-exact category $\mathcal{G}_G$ is exactly the category of $R$-modules along with the proper class of all clean exact sequences in the sense of [BGH13]. The injectives in $\mathcal{G}_G$ ought to be called clean injective modules. The projectives in $\mathcal{G}_G$ are precisely direct summands of direct sums of modules of type $FP_\infty$. Since all modules of type $FP_\infty$ are finitely presented, Theorems A, B and C apply giving recollements involving the clean derived category. We see a canonical functor from the pure derived category to the clean derived category. However, we point out that for coherent rings, a module is finitely presented if and only if it is of type $FP_\infty$. So this example only differs from the pure derived category for non-coherent rings. It seems likely that the clean derived category will generalize to some other locally finitely presented Grothendieck categories. By [Bie81, Corollary 1.6] we have that for modules over a ring, $F$ is of type $FP_\infty$ if and only if $\text{Ext}^n_R(F, -)$ preserves direct limits for all $n \geq 0$. So in the more general setting, even without enough projective objects, one could define an object $F \in \mathcal{G}$ to be of type $FP_\infty$ if $\text{Ext}^n_R(F, -)$ preserves direct limits for all $n \geq 0$. However, one needs to be sure that the objects of type $FP_\infty$ form a generating set for $\mathcal{G}$!
2.4.2. Inj-acyclic complexes. Suppose $R$ is (left) Noetherian. Recall that every injective (left) $R$-module is a direct sum of indecomposable injective modules and there is a set $S$ of (isomorphism representatives) of all indecomposable injectives. (See [Lam99, Theorem 3.48].) So taking $G$ to be the direct sum of $R$ and all the indecomposable injectives, it is easy to see that a short exact sequence is $G$-exact if and only if it remains exact after applying $\text{Hom}_G(I, -)$ where $I$ is any injective $R$-module. So these are a proper class of short exact sequences and the injective modules are projective objects with respect to these. More generally, by part (4) of Corollary 3.5, the $G$-projectives are precisely the direct summands of direct sums of modules in $S \cup \{R\}$. By Theorem A, we get a model structure for an associated derived category obtained by killing all the exact "Inj-acyclic" complexes.

3. The $G$-exact category $\mathcal{G}_G$

Throughout this section $\mathcal{G}$ will always denote a Grothendieck category with a chosen (fixed) set of generators $\{G_i\}_{i \in I}$. Furthermore, $G$ will always denote their direct sum $G = \oplus_{i \in I} G_i$. So $G$ itself is a generator for $\mathcal{G}$. The goal of this section is to give a detailed construction of an exact category, in the sense of Quillen [Qui73] and [Büh10], which we will call the $G$-exact category of $\mathcal{G}$. Being abelian, an exact structure on $\mathcal{G}$ is, as shown in Appendix B, nothing more than a proper class of short exact sequences in the sense of [Mac63]. In this case, the proper class is the class of all $G$-exact sequences. That is, the short exact sequences $0 \to A \to B \to C \to 0$ which remain exact after applying $\text{Hom}_G(G, -)$. We denote this exact category by $\mathcal{G}_G$, and see that $G$ is a projective generator for $\mathcal{G}_G$.

3.1. $G$-exact sequences and $G$-projectives. Recall that an object $G$ is a generator if $\text{Hom}_A(G, -)$ is faithful. Since $G$ is abelian this is equivalent to saying that if $f : A \to B$ is nonzero, then there exists a map $s : G \to A$ such that $fs \neq 0$. We have the following basic fact.

Lemma 3.1. Let $G$ be a generator for any abelian category $A$ and let $X$ be a chain complex in $\text{Ch}(A)$. If the complex of abelian groups $\text{Hom}_A(G, X)$ is exact, then $X$ itself must be exact.

Proof. We just need to show that $d_{n+1} : X_{n+1} \to Z_n X$ is an epimorphism, that is, right cancelable. Since $A$ is abelian we just need to show that for a map $f : Z_n X \to Y$ we have $fd_{n+1} = 0$ implies $f = 0$. By way of contradiction, say $fd_{n+1} = 0$ but $f \neq 0$. Then because $G$ is a generator we get a map $s : G \to Z_n X$ such that $fs \neq 0$. But notice $s$ determines a map in the domain of $(d_n)_* : \text{Hom}_A(G, X_n) \to \text{Hom}_A(G, X_{n-1})$ for which $(d_n)_*(s) = 0$. So by hypothesis we have $s \in \ker (d_n)_* = \text{Im} (d_{n+1})$, which ensures a map $t : G \to X_{n+1}$ such that $s = d_{n+1}t$. Now $fd_{n+1} = 0$ implies $fd_{n+1}t = 0$ implies $fs = 0$, which is the contradiction. 

Now let $\text{Hom}_G(G, G)$ be the endomorphism ring of $G$ and $\text{Mod}-R$ the category of right $R$-modules. By the Gabriel-Popescu Theorem, the functor $\text{Hom}_G(G, -) : \mathcal{G} \to \text{Mod}-R$ is fully faithful and has an exact left adjoint $T$. Therefore $\mathcal{G}$ is equivalent to the full subcategory $S = \text{Im} [\text{Hom}_G(G, -)]$ of $\text{Mod}-R$. Since the property of being a Grothendieck category is stable under equivalence of categories we know that $S$ is Grothendieck. However $S$ is not an abelian subcategory of $\text{Mod}-R$. In particular, if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence in $\mathcal{G}$, then of course

$$0 \to \text{Hom}_G(G, A) \xrightarrow{f^*} \text{Hom}_G(G, B) \xrightarrow{g^*} \text{Hom}_G(G, C)$$
Proposition 3.3. We have the following properties of a \( G \)-subobject if the inclusion map is a \( B \)-epimorphism. We wish to show that \( 0 = tg_* = (hg)_* \). Since \( \text{Hom}_G(G, C) \) is faithful we have \( hg = 0 \). But \( g \) is right cancelable, so \( h = 0 \) and this implies \( h_* = t = 0 \).

Definition 3.2. We call a pair of composeable maps \( A \xrightarrow{f} B \xrightarrow{g} C \) in \( \mathcal{G} \) a \( G \)-exact sequence if \( 0 \to \text{Hom}_G(G, A) \xrightarrow{f_*} \text{Hom}_G(G, B) \xrightarrow{g_*} \text{Hom}_G(G, C) \to 0 \) is a short exact sequence in the category of abelian groups (so also in \( \text{Mod-}R \)). We often denote a \( G \)-exact sequence by \( A \to B \to C \), and call \( A \to B \) a \( G \)-monomorphism and \( B \to C \) a \( G \)-epimorphism. We will also call a subobject \( P \subseteq A \) a \( G \)-subobject if the inclusion map is a \( G \)-monomorphism, and denote this \( P \subseteq_G A \).

We list some basic properties of \( G \)-exact sequences.

Proposition 3.3. We have the following properties of \( G \)-exact sequences.

1. Any \( G \)-exact sequence is an exact sequence in \( \mathcal{G} \).
2. The class of all \( G \)-exact sequences is closed under isomorphisms and contains all split exact sequences.
3. A pushout of a \( G \)-monomorphism is again a \( G \)-monomorphism. In fact, \( \text{Hom}_G(G, -) \) takes pushouts of \( G \)-monomorphisms to pushouts in \( \text{Mod-}R \).
   We also have that pullbacks of \( G \)-epimorphisms are again \( G \)-epimorphisms. Moreover, \( \text{Hom}_G(G, -) \) takes all pullbacks in \( \mathcal{G} \) to pullbacks in \( \text{Mod-}R \) since it is a right adjoint.
4. \( G \)-monomorphisms are closed under composition and \( G \)-epimorphisms are closed under composition.

Proof. For (1), note that in the definition of \( G \)-exact sequence we have \( 0 = g_*f_* = (gf)_* \). So \( \text{Hom}_G(G, -) \) faithful implies \( 0 = gf \). So we can view \( A \xrightarrow{f} B \xrightarrow{g} C \) as a chain complex in \( \mathcal{G} \), and so (1) follows from Lemma 3.1.

(2) is clear.

For (3), we first show that a pullback of a \( G \)-epimorphism is a \( G \)-epimorphism. Let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be a given \( G \)-exact sequence. Taking a pullback \( B \xleftarrow{g_*} C \leftarrow X \) leads to a diagram of short exact sequences.

\[
\begin{array}{c}
\text{0} \rightarrow A \xrightarrow{f} P \xrightarrow{g'} X \rightarrow 0 \\
\text{0} \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
\end{array}
\]

Applying \( \text{Hom}_G(G, -) \) to this diagram gives us a commutative diagram with the bottom row exact.

\[
\begin{array}{c}
\text{0} \rightarrow \text{Hom}_G(G, A) \xrightarrow{f_*} \text{Hom}_G(G, P) \xrightarrow{g'_*} \text{Hom}_G(G, X) \\
\text{0} \rightarrow \text{Hom}_G(G, A) \xrightarrow{f_*} \text{Hom}_G(G, B) \xrightarrow{g_*} \text{Hom}_G(G, C) \rightarrow 0
\end{array}
\]
But the functor $\text{Hom}_G(G, -) : \mathcal{G} \to \text{Mod-}R$ is a right adjoint and so it preserves limits, so in particular it preserves pullbacks. Therefore the right square is a pullback in $\text{Mod-}R$. So since $g_*$ is an epimorphism we get that $g'_*$ must also be an epimorphism. This proves $0 \to A \xrightarrow{f'} P \xrightarrow{g'} X \to 0$ is a $G$-exact sequence.

Next, we wish to show that a pushout of a $G$-monomorphism is a $G$-monomorphism. So consider a $G$-exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. Taking a pushout of $X \leftarrow A \xrightarrow{f} B$ leads to a diagram of short exact sequences.

```
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0
\downarrow \quad \downarrow \quad \| \quad \|
0 \longrightarrow X \xrightarrow{f'} P \xrightarrow{g'} C \longrightarrow 0
```

We only need to show that $g'_*$ is an epimorphism. Since $\text{Hom}_G(G, -)$ is not a left adjoint we can’t expect it to preserve all pushouts. However, note that since $\text{Hom}_G(G, -) : \mathcal{G} \to \mathcal{S}$ is an equivalence it takes pushouts in $\mathcal{G}$ to pushouts in the abelian category $\mathcal{S}$. This implies that we get the $\mathcal{S}$-diagram below with $\mathcal{S}$-exact rows and with the left square being a pushout in $\mathcal{S}$.

```
0 \longrightarrow \text{Hom}_G(G, A) \xrightarrow{f_*} \text{Hom}_G(G, B) \xrightarrow{g_*} \text{Hom}_G(G, C)
\downarrow \quad \downarrow \quad \| \quad \|
0 \longrightarrow \text{Hom}_G(G, X) \xrightarrow{f'_*} \text{Hom}_G(G, P) \xrightarrow{g'_*} \text{Hom}_G(G, C)
```

But by hypothesis, $g_*$ is an epimorphism in $\text{Mod-}R$, and so we see immediately that $g'_*$ is also an epimorphism in $\text{Mod-}R$. This shows that $X \xrightarrow{f'} P \xrightarrow{g'} C$ is a $G$-exact sequence. In fact, since the rows of the diagram above are exact in $\text{Mod-}R$, it follows that the left hand square is actually the pushout in $\text{Mod-}R$. So the functor $\text{Hom}_G(G, -) : \mathcal{G} \to \text{Mod-}R$ preserves pushouts of $G$-monomorphisms.

For (4), we first show that $G$-epimorphisms are closed under composition. Say $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are each $G$-epimorphisms. Since each is an epimorphism, so is the composition $hg$. Then $0 \to \ker hg \to B \xrightarrow{hg} D \to 0$ must be a $G$-exact sequence since $(hg)_* = h_*g_*$ is an epimorphism.

Finally, we wish to show that $G$-monomorphisms are closed under composition. So let $A \xrightarrow{i} B$ and $B \xrightarrow{j} C$ each be $G$-monomorphisms. Taking the pushout of
$B/A \leftarrow B \overset{j}{\rightarrow} C$ leads to a diagram of short exact sequences.

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow \\
A & A \\
\downarrow i & \downarrow j_i \\
0 \rightarrow B & 0 & \rightarrow C & \overset{\pi}{\rightarrow} \overset{j}{\rightarrow} C/B & \rightarrow 0 \\
\downarrow g & \downarrow & \downarrow & \downarrow & \\
0 \rightarrow B/A & 0 & \rightarrow P & \overset{\pi'}{\rightarrow} \overset{j'}{\rightarrow} C/B & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

Since the row $0 \rightarrow B \overset{j}{\rightarrow} C \overset{\pi}{\rightarrow} C/B \rightarrow 0$ is $G$-exact, we have by what was proved already that the pushout row $0 \rightarrow B/A \overset{j'}{\rightarrow} P \overset{\pi'}{\rightarrow} C/B \rightarrow 0$ must also be $G$-exact. So applying $\text{Hom}_G(G, -)$ yields a commutative diagram with exact rows and columns.

\[
\begin{array}{cccccccccc}
0 & 0 \\
\downarrow & \downarrow \\
\text{Hom}_G(G, A) & \text{Hom}_G(G, A) \\
\downarrow i_* & \downarrow (ji)_* \\
0 \rightarrow \text{Hom}_G(G, B) & \overset{j_*}{\rightarrow} \text{Hom}_G(G, C) & \overset{\pi_*}{\rightarrow} \text{Hom}_G(G, C/B) & \rightarrow 0 \\
\downarrow g_* & \downarrow & \downarrow & \downarrow & \\
0 \rightarrow \text{Hom}_G(G, B/A) & \overset{j'_*}{\rightarrow} \text{Hom}_G(G, P) & \overset{\pi'_*}{\rightarrow} \text{Hom}_G(G, C/B) & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

We are trying to show that $g_*$ is an epimorphism in $\text{Mod-}R$, and now the snake lemma shows that it is. \qed

We show in Appendix B that when working in abelian categories, Quillen’s notion of an exact category from [Qui73] coincides with the notion of a proper class of short exact sequences from [Mac63, Chapter XII.4].

**Corollary 3.4.** Let $\mathcal{G}$ be a Grothendieck category with generator $G$. Let $\mathcal{E}$ denote the class of all $G$-exact sequences. Then $(\mathcal{G}, \mathcal{E})$ is an exact category. Equivalently, $\mathcal{E}$ is a proper class of short exact sequences. We will let $\mathcal{G}_G = (\mathcal{G}, \mathcal{E})$ denote this exact category and we will call it the **$G$-exact category** of $\mathcal{G}$. The functor $\text{Hom}_G(G, -) : \mathcal{G}_G \rightarrow \text{Mod-}R$ is exact.

**Proof.** The four properties of Proposition 3.3 are the axioms of an exact category in [Büh10]. It is clear from definitions that the functor $\text{Hom}_G(G, -) : \mathcal{G}_G \rightarrow \text{Mod-}R$
is exact. We refer the reader to Appendix B for the equivalence with proper classes.

The generator \( G = \oplus_{i \in I} G_i \) is not just a generator for \( \mathcal{G} \). It is easy to see that it is also a generator for \( \mathcal{G}_G \), but we first explain what we mean by this.

In [Hov02], Hovey worked with abelian categories along with a proper class of short exact sequences in the sense of [Mac63, Chapter XII.4]. There he defined an object \( U \) to be a generator for a proper class \( \mathcal{P} \) if for all maps \( f, \text{Hom}_\mathcal{G}(U, f) \) surjective implies \( f \) is a \( \mathcal{P} \)-epimorphism. Also here, a set \( \{U_i\} \) generates \( \mathcal{P} \) if \( U = \oplus U_i \) is a generator for \( \mathcal{P} \). On the other hand, in [SS11] and [Sto13], the authors work with exact categories and define a set \( \{U_i\} \) to be generating if for any object \( A \), there is an admissible epimorphism \( \pi : U \to A \) where \( U \) is some set-indexed direct sum of objects from \( \{U_i\} \). The following corollary shows that \( G \) is a generator for \( \mathcal{G}_G \) in both senses. We therefore can feel free to reference the above authors’ results.

**Corollary 3.5.** \( G = \oplus_{i \in I} G_i \) is a projective generator for the \( G \)-exact category \( \mathcal{G}_G \). In particular, the following hold:

1. By definition, an object \( P \) is projective in \( \mathcal{G}_G \) if the functor \( \text{Hom}_\mathcal{G}(P, -) \) takes \( G \)-exact sequences to short exact sequences. We will call such an object \( G \)-projective. Notice that the construction of the \( G \)-exact category immediately forces \( G \) and each \( G_i \) to be \( G \)-projective.
2. \( G \) is a generator for \( \mathcal{G}_G \). That is, if \( \text{Hom}_\mathcal{G}(G, A) \xrightarrow{f} \text{Hom}_\mathcal{G}(G, B) \) is surjective, then \( f \) is a \( G \)-epimorphism.
   Equivalently, \( \{G_i\} \) is a set of generators for \( \mathcal{G}_G \). That is, if \( \text{Hom}_\mathcal{G}(G_i, A) \xrightarrow{f} \text{Hom}_\mathcal{G}(G_i, B) \) is surjective for all \( G_i \), then \( f \) is a \( G \)-epimorphism.
3. \( \mathcal{G}_G \) has enough projectives. In particular, for each \( A \in \mathcal{G} \), we can find a \( G \)-epimorphism \( \oplus_{i \in I} G \to A \). Equivalently, we can find a \( G \)-epimorphism \( X \to A \) where \( X \) is a direct sum of copies of some of the \( G_i \).
4. An object \( P \) is \( G \)-projective if and only if it is a direct summand of a direct sum of copies of some of the \( G_i \).

**Proof.** For (2), let \( f : A \to B \) be such that \( \text{Hom}_\mathcal{G}(G, A) \xrightarrow{f} \text{Hom}_\mathcal{G}(G, B) \) is surjective. Since \( G \) is a generator for \( \mathcal{G} \) this implies \( f \) is an epimorphism and so there is a short exact sequence \( 0 \to K \to A \xrightarrow{f} B \to 0 \). By definition this sequence is \( G \)-exact, so we are done. In terms of the generating set \( \{G_i\} \), just note that \( \text{Hom}_\mathcal{G}(G, A) \xrightarrow{f} \text{Hom}_\mathcal{G}(G, B) \) is surjective if \( \text{Hom}_\mathcal{G}(G_i, A) \xrightarrow{f} \text{Hom}_\mathcal{G}(G_i, B) \) is surjective for all \( G_i \).

For (3), in the usual way, take \( I = \text{Hom}_\mathcal{G}(G, A) \), and define \( \oplus_{i \in I} G \to A \) in component \( t : G \to A \in I \). It is immediate that this is a \( G \)-epimorphism. \( \oplus_{i \in I} G \) is indeed a \( G \)-projective object, since in any exact category, direct sums of projectives are again projectives by [Buh10, Corollary 11.7]. For (4), we see that the \( G \)-epimorphism \( \oplus_{i \in I} G \to P \) splits if and only if \( P \) is \( G \)-projective by [Buh10, Corollary 11.6].

3.2. \( G \)-subobjects. Here we go on to list more properties of \( G \)-monomorphisms, but we state them in terms of \( G \)-subobjects. This is the form in which we will use them later. Note that they are analogous to properties of pure submodules.
Recall that we write $P \subseteq_G A$ to mean that $P$ is a $G$-subobject of $A$, that is, $\text{Hom}_G(G, A) \rightarrow \text{Hom}_G(G, A/P)$ is surjective.

**Proposition 3.6.** Consider subobject $A \subseteq B \subseteq C$ in $\mathcal{G}$.

1. If $A \subseteq_G B$ and $B \subseteq_G C$ then $A \subseteq_G C$.
2. If $A \subseteq_G C$ then $A \subseteq_G B$.
3. If $A \subseteq_G C$ and $B/A \subseteq_G C/A$ then $B \subseteq_G C$.

**Proof.** (1) has already appeared as part (4) of Proposition 3.3. (2) follows from general facts about admissible monomorphisms in (weakly idempotent complete) exact categories. See [Büh10, Prop. 7.6 or Prop. 2.16].

For (3), all we need to check is that the map $\text{Hom}_G(G, C) \rightarrow \text{Hom}_G(G, C/B)$ is an epimorphism. But this is just the composite

$$\text{Hom}_G(G, C) \rightarrow \text{Hom}_G(G, C/A) \rightarrow \text{Hom}_G(G, (C/A)/(B/A)) \cong \text{Hom}_G(G, C/B),$$

and these are epimorphisms by hypothesis. \hfill \Box

4. THE $G$-DERIVED CATEGORY

Again let $\mathcal{G}$ be a Grothendieck category and let $G = \oplus G_i$ where $\{G_i\}$ is a set of generators. In this section we construct the derived category $\mathcal{D}(G)$. It is the derived category of the $G$-exact category $\mathcal{G}_G$ and we obtain it by putting a suitable model structure on $\text{Ch}(\mathcal{G})$. Following the general definition of an exact chain complex from [Büh10, Definition 10.1], the exact complexes in $\mathcal{G}_G$ are the $G$-acyclic complexes. That is, those chain complexes $X$ for which $\text{Hom}_G(G, C) \rightarrow \text{Hom}_G(G, (C/A)/(B/A)) \cong \text{Hom}_G(G, C/B)$.

4.1. The category $\text{Ch}(\mathcal{G})_G$. Our convention when working with chain complexes is that the differential lowers degree, so $\cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots$ is a chain complex. Given $X \in \text{Ch}(\mathcal{G})$, the $n$th suspension of $X$, denoted $\Sigma^n X$, is the complex given by $(\Sigma^n X)_k = X_{k-n}$ and $(\delta_k \Sigma^n X)_k = (-1)^n d_{k-n}$. Given two chain complexes $X$ and $Y$ we define $\text{Hom}(X, Y)$ to be the complex of abelian groups

$$\cdots \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n-1}) \rightarrow \cdots,$$

where $(\delta_n f)_k = d_{k+n} f_k - (-1)^n f_{k-1} d_k$. This gives a functor $\text{Hom}(X, -) : \text{Ch}(A) \rightarrow \text{Ch}(\mathbb{Z})$. Note that this functor takes exact sequences to left exact sequences, and it is exact if each $X_n$ is projective. Similarly the contravariant functor $\text{Hom}(-, Y)$ sends exact sequences to left exact sequences and is exact if each $Y_n$ is injective. It is an exercise to check that the homology satisfies $H_n[\text{Hom}(X, Y)] = \text{Ch}(\mathcal{G})(X, \Sigma^{-n} Y)/ \sim$ where $\sim$ is the usual relation of chain homotopic maps.

For a given $A \in \mathcal{G}$, we denote the $n$-disk on $A$ by $D^n(A)$. This is the complex consisting only of $A \xrightarrow{1} A$ concentrated in degrees $n$ and $n - 1$. We denote the $n$-sphere on $A$ by $S^n(A)$, and this is the complex consisting of $A$ in degree $n$ and 0 elsewhere.

Recall that $\mathcal{G}_G$ is the same category as $\mathcal{G}$, with the same morphisms, but with an exact structure coming from the proper class of $G$-exact sequences. In the same way, we let $\text{Ch}(\mathcal{G})_G$ denote the category of all chain complexes, with the usual chain maps, but considered as an exact category where the short exact sequences are $G$-exact in each degree. We will call these degreewise $G$-exact sequences. It is indeed a general fact that for any exact category $\mathcal{A} = (A, \mathcal{E})$, the category $\text{Ch}(\mathcal{A})$
becomes an exact category when considered along with the short exact sequences which degreewise lie in $\mathcal{E}$. So one might argue that the proper notation in our case is $\text{Ch}(\mathcal{G}_G)$, rather than $\text{Ch}(\mathcal{G})_G$. However, we have the following lemma.

**Lemma 4.1.** Consider the standard generating set $\{D^n(G_i)\}$ in $\text{Ch}(\mathcal{G})$ and let $G = \oplus D^n(G_i)$ be the direct sum, taken over all $n \in \mathbb{Z}$ and $i \in I$. Then the $G$-exact category $\text{Ch}(\mathcal{G})_G$ of Corollary 3.4 coincides with $\text{Ch}(\mathcal{G}_G)$. That is, the proper class of $G$-exact sequences in $\text{Ch}(\mathcal{G})$ (here $G = \oplus D^n(G_i)$) coincides with the class of all short exact sequences which degreewise are $G$-exact sequences (here $G = \oplus G_i$) in $\mathcal{G}$.

**Proof.** Consider a short sequence $X \rightarrow Y \rightarrow Z$ of complexes. Then it is $G$-exact iff

$$\text{Hom}_{\text{Ch}(\mathcal{G})}(G, X) \rightarrow \text{Hom}_{\text{Ch}(\mathcal{G})}(G, Y) \rightarrow \text{Hom}_{\text{Ch}(\mathcal{G})}(G, Z)$$

is a short exact sequence of abelian groups, iff

$$\prod_{n,i} \text{Hom}(D^n(G_i), X) \rightarrow \prod_{n,i} \text{Hom}(D^n(G_i), Y) \rightarrow \prod_{n,i} \text{Hom}(D^n(G_i), Z)$$

is short exact, iff $\prod_{n,i} \text{Hom}_G(G_i, X_n) \rightarrow \prod_{n,i} \text{Hom}_G(G_i, Y_n) \rightarrow \prod_{n,i} \text{Hom}_G(G_i, Z_n)$ is short exact, iff $X \rightarrow Y \rightarrow Z$ is degreewise $G$-exact (where here $G = \oplus G_i$) in $\mathcal{G}$. \hfill $\square$

Being an exact category, $\text{Ch}(\mathcal{G})_G$ comes with a Yoneda Ext group, which in this case is the group of (equivalence classes of) degreewise $G$-exact sequences $Y \rightarrow Z \rightarrow X$, with addition defined by the Baer sum. We will denote this bifunctor by $\text{G-Ext}^1_{\text{Ch}(\mathcal{G})}$, and note that for given chain complexes $X$ and $Y$, $\text{G-Ext}^1_{\text{Ch}(\mathcal{G})}(X, Y)$ is a subgroup of the usual Yoneda Ext $\text{Ext}^1_{\text{Ch}(\mathcal{G})}(X, Y)$. We sometimes will also call an element of $\text{G-Ext}^1_{\text{Ch}(\mathcal{G})}(X, Y)$ a **degreewise $G$-extension**. We also denote by $\text{G-Ext}^1_{\mathcal{G}}$, the group of $G$-extensions in the ground category $\mathcal{G}_G$. We have the following $G$-versions of standard isomorphisms.

**Lemma 4.2.** Let $A \in \mathcal{G}$ and $X \in \text{Ch}(\mathcal{G})$. Then we have the following natural isomorphisms.

1. $\text{G-Ext}^1_{\text{Ch}(\mathcal{G})}(D^n(A), X) \cong \text{G-Ext}^1_{\mathcal{G}}(A, X_n)$
2. $\text{G-Ext}^1_{\text{Ch}(\mathcal{G})}(X, D^{n+1}(A)) \cong \text{G-Ext}^1_{\mathcal{G}}(X_n, A)$

**Proof.** The point is that the standard isomorphisms take degreewise $G$-extensions to $G$-extensions. For example, for (1), the standard mapping $\text{Ext}^1_{\text{Ch}(\mathcal{G})}(D^n(A), X) \rightarrow \text{Ext}^1_{\mathcal{G}}(A, X_n)$ takes a short exact sequence $0 \rightarrow X \rightarrow Z \rightarrow D^n(A) \rightarrow 0$ to $0 \rightarrow X_n \rightarrow Z_n \rightarrow A \rightarrow 0$. Its inverse is formed by taking an extension $0 \rightarrow X_n \rightarrow Z \rightarrow A \rightarrow 0$ and forming the pushout of $X_{n-1} \xrightarrow{d_{n-1}} X_n \rightarrow Z$. Since pushouts of $G$-monomorphisms are again $G$-monomorphisms, we see that the isomorphisms restrict nicely between $G$-extensions. This shows (1). The isomorphism (2) is dual, using that pullbacks of $G$-epimorphisms are again $G$-epimorphisms. \hfill $\square$

There is one more exact category that will be of use. We denote by $\text{Ch}(\mathcal{G})_{dw}$ the category of all chain complexes along with the proper class of all degreewise split short exact sequences. We denote its Yoneda Ext bifunctor by $\text{Ext}^1_{\text{Ch}(\mathcal{G})_{dw}}$. We note that we have subgroup containments

$$\text{Ext}^1_{\text{Ch}(\mathcal{G})_{dw}}(X, Y) \subseteq \text{G-Ext}^1_{\text{Ch}(\mathcal{G})}(X, Y) \subseteq \text{Ext}^1_{\text{Ch}(\mathcal{G})}(X, Y),$$
and we have the following well-known connection between $\text{Ext}^1_{dw}$ and the functor $\text{Hom}$.

**Lemma 4.3.** For chain complexes $X$ and $Y$, we have isomorphisms:

\[
\text{Ext}^1_{dw}(X, \Sigma^{-n-1}Y) \cong H_n \text{Hom}(X, Y) = \text{Ch}(\mathcal{G})(X, \Sigma^{-n}Y)/\sim.
\]

In particular, for chain complexes $X$ and $Y$, $\text{Hom}(X, Y)$ is exact iff for any $n \in \mathbb{Z}$, any chain map $f : \Sigma^n X \to Y$ is homotopic to 0 (or iff any chain map $f : X \to \Sigma^n Y$ is homotopic to 0).

We note also that the functor $\text{Hom}(X, -) : \text{Ch}(\mathcal{G}) \to \text{Ch}(\mathbb{Z})$ takes degreewise $G$-exact sequences to short exact sequences if each $X_n$ is $G$-projective. Similarly the contravariant functor $\text{Hom}(-, Y)$ sends degreewise $G$-exact sequences to short exact sequences if each $Y_n$ is $G$-injective.

4.2. **G-acyclic complexes.** Following definition [Bühl10, Definition 10.1]), an acyclic chain complex with respect to the exact structure $\mathcal{G}_G$ ought to be a chain complex $X$ for which its differentials each factor as $G$-chain complex with respect to the exact structure $\text{Ch}(-)$ in the exact category $G$-projective in the exact category $\text{Ch}(\mathcal{G})$. That is, an exact sequence $X$ in $\text{Ch}(\mathcal{G})$ is exact iff for any $n \in \mathbb{Z}$, any chain map $f : \Sigma^n X \to Y$ is homotopic to 0.

**Lemma 4.4.** We have the following properties of $G$-acyclic complexes.

1. Let $X$ be a chain complex. Then the following are equivalent:
   (a) $X$ is $G$-acyclic.
   (b) $X$ is exact and $Z_nX \subseteq G X_n$ is a $G$-subobject for each $n$.
   (c) $\text{Hom}_G(G, X)$ is exact.
   (d) Each $\text{Hom}_G(G_i, X)$ is exact.

   Note in particular that any $G$-acyclic complex is exact in the usual sense.

2. If $X$ is contractible, meaning $1_X \sim 0$, then $X$ is $G$-acyclic.

3. The class of $G$-acyclic complexes is thick in $\text{Ch}(\mathcal{G})_G$. That is, it is closed under retracts and for any exact $X \hookrightarrow Y \twoheadrightarrow Z$ in $\text{Ch}(\mathcal{G})_G$, if two out of three terms are $G$-acyclic then so is the third.

**Proof.** For (1), we clearly have (a) $\implies$ (b) $\implies$ (c) $\implies$ (d). (d) and (c) are equivalent because $\text{Hom}_G(G, X) \cong \prod_{i \in \mathbb{Z}} \text{Hom}_G(G_i, X)$ (and a product of exact complexes is exact in $\text{Ab}$). Using Lemma 3.1 we see (c) implies (b).

For (2), recall that having $1_X \sim 0$ means there exists maps \{s_n : X_n \to X_{n+1}\} such that $sd + ds = 1$. Applying the additive functor $\text{Hom}_G(G, -)$ to this equation shows that $\text{Hom}_G(G, X)$ is also contractible. In particular it is exact.

For (3), note that if $X \hookrightarrow Y \twoheadrightarrow Z$ is a short exact sequence in $\text{Ch}(\mathcal{G})_G$, then since it is degreewise $G$-acyclic we get a short exact sequence of complexes of abelian groups $0 \to \text{Hom}_G(G, X) \to \text{Hom}_G(G, Y) \to \text{Hom}_G(G, Z) \to 0$. If any two out of three of these are exact then so is the third. For retracts, note that any additive functor preserves retracts. So this is true since a retract of an exact complex of abelian groups is again an exact complex.

4.3. **Projectives in $\text{Ch}(\mathcal{G})_G$.** Here we classify the projective objects of $\text{Ch}(\mathcal{G})_G$.

**Lemma 4.5.** Call a chain complex $X$ in $\text{Ch}(\mathcal{G})$ a $G$-projective complex if it is projective in the exact category $\text{Ch}(\mathcal{G})_G$. The following are equivalent:

1. $X$ is $G$-projective.
2. $X$ is $G$-acyclic with each $Z_nX$ a $G$-projective.
(3) \( X \) is isomorphic to a split exact complex with \( G \)-projective components. That is, \( X \cong \oplus_{n \in \mathbb{Z}} D^n(P_n) \) where each \( P_n \) is a \( G \)-projective.

(4) \( X \) is a contractible complex with each \( X_n \) \( G \)-projective.

Proof. Using part (3) of Corollary 3.5 and [Gil13, Corollary 2.7] we can find, for any chain complex \( X \), a \( G \)-epimorphism \( \oplus_{n \in \mathbb{Z}} D^n(P_n) \to X \) in which each \( P_n \) is \( G \)-projective. If \( X \) is \( G \)-projective, then this is a split epi. Then (2),(3), and (4) all follow and are equivalent by standard arguments. On the other hand, the isomorphism \( G\text{-Ext}^1_{\text{Ch}(G)}(D^n(A), X) \cong G\text{-Ext}^1_{\mathcal{P}}(A, X_n) \) of Lemma 4.2 tells us that a disk \( D^n(A) \) is \( G \)-projective if and only if \( A \) is \( G \)-projective in \( \mathcal{G}_G \). Moreover, in any exact category a direct sum is projective if and only if each summand is projective by [Büh10, Corollary 11.7]. \( \square \)

4.4. The \( G \)-derived category. We now construct the \( G \)-derived category by putting a cofibrantly generated “projective” model structure on \( \text{Ch}(G)_G \). The model structure follows as a Corollary to the next theorem. The proof relies on Quillen’s small object argument. We refer to the version in [Hov99, Theorem 2.1.14] and in particular we refer the reader there for the definition of the notation \( I \)-cell and \( I \)-inj. We also refer the reader to [Hov02] for the language of cotorsion pairs.

We define two sets of maps which will respectively be the generating cofibrations and generating trivial cofibrations:

\[
I = \{0 \to D^n(G_i)\} \cup \{S^{n-1}(G_i) \to \text{D}^n(G_i)\}, \quad \text{and} \quad J = \{0 \to D^n(G_i)\}.
\]

We also define the following set of objects which will cogenerate the cotorsion pair:

\[
S = \{D^n(G_i)\} \cup \{S^n(G_i)\}.
\]

Note that \( S = \text{cok} I = \{\text{cok } i \mid i \in I\} \). We leave it to the reader to check the easy fact that a chain complex \( X \) satisfies \( X \in S^\perp \) if and only if \( (X \to 0) \in I \)-inj. (The “perp” here is taken with respect to the degreewise \( G \)-exact sequences. So use \( G\text{-Ext}^1_{\text{Ch}(G)} \) and the fact that the \( D^n(G_i) \) are \( G \)-projective.)

**Theorem 4.6.** Let \( \mathcal{G} \) be any Grothendieck category with a generator \( G = \oplus G_i \). Let \( \mathcal{W} \) denote the class of all \( G \)-acyclic complexes. Then the set \( S = \{D^n(G_i)\} \cup \{S^n(G_i)\} \) cogenerates a cotorsion pair \( (\mathcal{P}, \mathcal{W}) \) in the exact category \( \text{Ch}(G)_G \) with the following properties.

1. \( (\mathcal{P}, \mathcal{W}) \) is complete. In fact, for any chain complex \( X \) there is a \( G \)-exact sequence \( W \to P \to X \) where \( W \in \mathcal{W} \) and \( P \in \mathcal{P} \) is a transfinite (degreewise-split) extension of \( S \). In particular, each \( P_n \) is a direct sum of copies of the \( G_i \).

2. \( P \in \mathcal{P} \) if and only if \( P \) is a retract of a transfinite (degreewise-split) extension of \( S \). We will call a complex in \( \mathcal{P} \) a \( G \)-semiprojective complex.

3. \( \mathcal{W} \) is thick and \( \mathcal{P} \cap \mathcal{W} \) coincides with the class of projective complexes in \( \text{Ch}(G)_G \). (See Lemma 4.5.)

For \( \mathcal{G} = \text{R-Mod} \) and \( G = \text{R} \), this recovers the usual projective model structure on \( \text{Ch}(\text{R}) \) where the cofibrant complexes are the DG-projective complexes. Some authors call these complexes semiprojective, and since DG-\( G \)-projective looks odd we use semiprojective.

Our proof of Theorem 4.6 is based on the proof of [Hov02, Theorem 6.5]. Indeed for the case when \( \mathcal{G} \) is locally finitely presentable (that is, \( G = \oplus G_i \) where the
\(G_i\) are finitely presented, we only need the first paragraph of the proof below, combined with Corollary 5.3 and [Hov02, Theorem 6.5].

**Proof.** Since each \(G_i\) is \(G\)-projective we have an equality

\[
G\text{-Ext}^1_{\text{Ch}(G)}(S^n(G_i), X) = \text{Ext}^1_{\text{dw}}(S^n(G_i), X).
\]

So \(X \in \{S^n(G_i)\}^\perp\) if and only if for each \(n\) we have vanishing of

\[
\text{Ext}^1_{\text{dw}}(S^n(G_i), X) = H_{n-1}\text{Hom}(S^0(G_i), X) = H_{n-1}\text{Hom}_G(G_i, X).
\]

So \(X \in \{S^n(G_i)\}^\perp\) if and only if \(X\) is \(G\)-acyclic. So indeed \(S\) cogenerates a cotorsion pair \((\mathcal{P}, \mathcal{W})\) in the exact category \(\text{Ch}(\mathcal{G})G\).

To show this cotorsion pair is complete we apply the small object argument from [Hov99, Theorem 2.1.14]. We can do this since every object in a Grothendieck category is small. The small object argument provides, for a given map \(X \to Y\), a functorial factorization \(X \to Z \to Y\) where \((X \to Z) \in I\text{-cell} and (Z \to Y) \in I\text{-inj}\).

To prove this, say we have such a \(p: Z \to Y\) in \(I\text{-inj}\). Then for each \(n\) and \(i\) we have a lift in the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & Z \\
\downarrow & & \downarrow p \\
D^n(G_i) & \longrightarrow & Y
\end{array}
\]

This implies that each \(p_n\) is a \(G\)-epimorphism as claimed. So \(K \to Z \to Y\) is \(G\)-exact where \(K = \ker p\). It is left to show \(K\) is \(G\)-acyclic. For any set of maps \(I\), it is an easy exercise to check that \(I\text{-inj}\) is closed under pullbacks. Since \(K \to 0\) lies in the pullback square

\[
\begin{array}{ccc}
K & \longrightarrow & X \\
\downarrow & & \downarrow p \\
0 & \longrightarrow & Y
\end{array}
\]

we see \((K \to 0) \in I\text{-inj}\). But as pointed out above the statement of the theorem, this is equivalent to saying \(X \in S^\perp\). So \(X\) is \(G\)-acyclic.

**Claim:** If \((p: Z \to Y) \in I\text{-inj}\), then \(p\) is a degreewise \(G\)-epimorphism with \(G\)-acyclic kernel.

To prove this, say we have such a \(p: Z \to Y\) in \(I\text{-inj}\). Then for each \(n\) and \(i\) we have a lift in the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & Z \\
\downarrow & & \downarrow p \\
D^n(G_i) & \longrightarrow & Y
\end{array}
\]

This implies that each \(p_n\) is a \(G\)-epimorphism as claimed. So \(K \to Z \to Y\) is \(G\)-exact where \(K = \ker p\). It is left to show \(K\) is \(G\)-acyclic. For any set of maps \(I\), it is an easy exercise to check that \(I\text{-inj}\) is closed under pullbacks. Since \(K \to 0\) lies in the pullback square

\[
\begin{array}{ccc}
K & \longrightarrow & X \\
\downarrow & & \downarrow p \\
0 & \longrightarrow & Y
\end{array}
\]

we see \((K \to 0) \in I\text{-inj}\). But as pointed out above the statement of the theorem, this is equivalent to saying \(X \in S^\perp\). So \(X\) is \(G\)-acyclic.

**Claim:** If \((f: X \to Z) \in I\text{-cell}\), then \(f\) is a degreewise split monomorphism with cokernel a transfinite extension of \(S\).

To prove this, say \(f: X \to Z\) is in \(I\text{-cell}\). By definition, \(f\) is a transfinite composition of pushouts of maps of the form \(0 \to D^n(G_i)\) or \(S^{n-1}(G_i) \to D^n(G_i)\). Note that such pushouts are necessarily degreewise split monomorphisms whose cokernels are in \(S\). This means \((f: X \to Z) = (X_0 \xrightarrow{f} \lim_{\gamma < \lambda} X_\alpha)\) is a transfinite (degreewise split) extension of \(X = X_0\) by \(S\). Since transfinite extensions of split monomorphisms are again split monomorphisms, we conclude that \(f\) too is a degreewise split monomorphism. We now look at \(\text{cok} f\). Since direct limits are exact we have a short exact sequence \(X_0 \xrightarrow{f} \lim_{\gamma \leq \lambda} X_\alpha \to \lim_{\gamma \leq \lambda} (X_\alpha/X_0)\). In particular, \(\text{cok} f \cong \lim_{\gamma \geq \lambda} (X_\alpha/X_0)\) is a transfinite extension of

\[
0 \to X_1/X_0 \to X_2/X_0 \to X_3/X_0 \to \cdots \to X_\alpha/X_0 \to \cdots
\]
But \((X_{\alpha+1}/X_0)/(X_{\alpha}/X_0) \cong X_{\alpha+1}/X_{\alpha} \in \mathcal{S}\). This proves \(f\) is a degreewise split monomorphism with cokernel a transfinite extension of \(\mathcal{S}\).

We now can prove that \((\mathcal{P}, \mathcal{W})\) is complete. So suppose \(Y\) is an arbitrary chain complex and use the small object argument to factor \(0 \to Y\) as \(0 \to Z \xrightarrow{p} Y\) where \(0 \to Z \in I\)-cell and \(Z \xrightarrow{p} Y\) is in \(I\)-inj. Then \(K \hookrightarrow Z \to Y\) is a degreewise \(G\)-exact sequence with \(K\) a \(G\)-acyclic complex. Also, \(Z\) must be a transfinite extension of \(\mathcal{S}\). But by [Hov02, Lemma 6.2] (taking the \(G\)-exact sequences as the proper class of short exact sequences) we have that \(\mathcal{P}\) is closed under retracts and transfinite extensions. Therefore \(Z \in \mathcal{P}\) and \((\mathcal{P}, \mathcal{W})\) has enough projectives in the way we claim in (1). To see that \((\mathcal{P}, \mathcal{W})\) has enough injectives we instead factor \(X \to 0\) as \(X \xrightarrow{f} Z \to 0\) where \(X \xrightarrow{f} Z \in I\)-cell and \(Z \to 0\) is in \(I\)-inj. Then \(f\) is a degreewise split monomorphism split monomorphism (so a \(G\)-mono) with \(\text{cok} f \in \mathcal{P}\), and \(Z \in \mathcal{W}\).

Next, statement (2). As mentioned above, \(\mathcal{P}\) is closed under retracts. Statement (2) is then a result of the following observation: Given \(Q \in \mathcal{P}\), write a \(G\)-exact sequence \(W \to P \to Q\) where \(W \in \mathcal{W}\) and \(P \in \mathcal{P}\) is a transfinite (degreewise-split) extension of \(\mathcal{S}\). This \(G\)-exact sequence is an element of \(\text{G-Ext}_{\text{Ch}(\mathcal{G})}^1(Q, W) = 0\). So it splits and \(Q\) is a retract of \(P\) as desired.

For (3), we see from Lemma 4.4 that \(\mathcal{W}\) is thick and contains all contractible complexes. So in particular \(\mathcal{W}\) contains the projective objects of \(\text{Ch}(\mathcal{G})_G\); by Lemma 4.5. Since \((\mathcal{P}, \mathcal{W})\) is complete with \(\mathcal{W}\) thick and containing the projectives, the result follows by the argument in [BGH13, Proposition 3.4].

In the language of [Gil11] and [Gil12], parts (1) and (3) of the above Theorem say that \((\mathcal{P}, \mathcal{W})\) is a projective cotorsion pair in \(\text{Ch}(\mathcal{G})_G\). Such a cotorsion pair is equivalent to a model structure for which every object is fibrant. The following corollary records some basic facts about this model structure.

**Corollary 4.7.** Let \(\mathcal{G}\) be any Grothendieck category with a generator \(G = \oplus G_i\). Then there is a model structure on \(\text{Ch}(\mathcal{G})\) which we call the **\(G\)-projective model structure** whose trivial objects are the \(G\)-acyclic complexes. This gives us a model for the **\(G\)-derived category**, which we denote by \(\mathcal{D}(G)\). The model structure satisfies the following:

1. The fibrations are precisely the \(G\)-epimorphisms. That is, the chain maps which are \(G\)-epimorphisms in each degree.
2. The trivial fibrations are the \(G\)-epimorphisms with \(G\)-acyclic kernel.
3. The cofibrations are the degreewise split monomorphisms whose cokernel is a \(G\)-semiprojective complex.
4. The trivial cofibrations are the degreewise split monomorphisms whose cokernel is a \(G\)-projective complex.
5. The weak equivalences are the **\(G\)-homology isomorphisms**. That is, the chain maps \(f : X \to Y\) for which \(\text{Hom}_{\mathcal{G}}(G, f) : \text{Hom}_{\mathcal{G}}(G, X) \to \text{Hom}_{\mathcal{G}}(G, Y)\) is a homology isomorphism.
6. The model structure is cofibrantly generated. The sets \(I\) and \(J\) from above are respectively the generating cofibrations and generating trivial cofibrations. Thus \(\mathcal{D}(G)\) is well generated in the sense of [Nee01].
7. If each \(G_i\) is finitely presented, then the model structure is finitely generated and so in this case \(\mathcal{D}(G)\) is compactly generated.
Proof. In the exact category \( \text{Ch}(G)_G \), we have the complete cotorsion pair \((\mathcal{P}, \mathcal{W})\).

We also have the categorical \((\mathcal{Q}, \mathcal{A})\) where \(\mathcal{Q}\) is the class of \(G\)-projective complexes of Lemma 4.5 and \(\mathcal{A}\) is the class of all complexes. Theorem 4.6 along with the main theorem of [Hov02] imply that we automatically have the model structure with (trivial) fibrations and (trivial) cofibrations as described.

In the correspondence between cotorsion pairs and model structures, the weak equivalences are precisely the maps which factor as a trivial cofibration followed by a trivial fibration. We wish to see that such maps are exactly the \(G\)-homology isomorphisms. First, given any \(f : X \to Y\), let \(\phi\) denote the composite functor \(H_n[\text{Hom}_G(G, f)]\) simply by \(H_nf_\ast\). Using the model structure we can apply the factorization axiom and write \(f = pi\) where \(p\) is a fibration and \(i\) is a trivial cofibration.

We have \(H_nf_\ast = H_n(p_i \circ H_ni_\ast)\). Since \(i\) is a degreewise split monomorphism with \(G\)-projective (so \(G\)-acyclic) cokernel, we see \(H_ni_\ast\) is an isomorphism. So \(H_nf_\ast\) is an isomorphism if and only if \(H_np_i\) is an isomorphism. Since \(p\) is a \(G\)-epimorphism, we see \(H_np_i\) is an isomorphism (for all \(n\)) if and only if \(\text{ker} p = G\)-acyclic. That is, \(\text{ker} p\) is a trivial fibration. We have now shown that \(f\) factors as a trivial cofibration followed by a trivial fibration if \(H_nf_\ast\) is an isomorphism for all \(n\).

It is easy to see that \(J\text{-inj}\) is the class of \(G\)-epimorphisms. This means that \(J\) is the set of generating trivial cofibrations. We also showed that everything in \(I\text{-inj}\) is a \(G\)-epimorphism with \(G\)-acyclic kernel. So it is left to show that every \(G\)-epimorphism with \(G\)-acyclic kernel is in \(I\text{-inj}\). So let \(X \xrightarrow{f} Y\) be a \(G\)-epimorphism with kernel \(K \in \mathcal{W}\). Being a \(G\)-epimorphism we know that there is a lift in any diagram of the form

\[
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow & & \downarrow p \\
D^n(G_i) & \longrightarrow & Y
\end{array}
\]

So all we need to show is that there is a lift for any diagram

\[
\begin{array}{ccc}
S^{n-1}(G_i) & \longrightarrow & X \\
\downarrow i & & \downarrow \psi \\
D^n(G_i) & \longrightarrow & Y
\end{array}
\]

But again, we may start by finding an \(D^n(G_i) \xrightarrow{\psi} X\) such that \(ph = g\). We check that \((f - hi)\) lands in the kernel \(K\). Now since \(\text{G-Ext}_G^1(S^{n-1}(G_i), K) = 0\), we see that the map \((f - hi)\) extends to some \(D^n(G_i) \xrightarrow{\psi} K\). That is, \(\psi i = (f - hi)\). So now we check that \((h + j\psi)\), where \(j : K \hookrightarrow X\) is the desired lift. (i) \(p(h + j\psi) = ph + 0 = ph = g\). (ii) \((h + j\psi)i = hi + j\psi i = hi + (f - hi) = f\).

Since we have a cofibrantly generated model structure on a locally presentable (pointed) category, a main result from [Ros05] assures us that \(D(G) = \text{Ho}(\text{Ch}(G))\) is well generated in the sense of [Nee01] and [Kra01]. In the case that \(G = \oplus G_i\) has each \(G_i\) finitely presented, then the \(G_i\) are finite in the sense of [Hov02, Section 7.4]. We then see that our model structure is finitely generated and so [Hov02, Corollary 7.4.4] tells us that \(D(G) = \text{Ho}(\text{Ch}(G))\) has a set of small weak generators. In other words, it is compactly generated. \(\square\)
Remark 1. Recall that by definition, a set $S$ of objects in a triangulated category such as $D(G)$ is called a set of weak generators if $X = 0$ in $D(G)$ if and only if $D(G)(\Sigma^n S, X) = 0$ for all $n$ and $S \in S$. It is easy to see directly that $\{G_i = S^0(G_i)\}$ is a set of weak generators for $D(G)$. Indeed we wish to see that $X$ is $G$-acyclic if and only if $D(G)(S^n(G_i), X) = 0$ for all $n$ and $i$. But in the $G$-projective model structure we have that each $S^n(G_i)$ is cofibrant and every $X$ is fibrant, so we get that $D(G)(S^n(G_i), X) \cong Ch(G)(S^n(G_i), X)/\sim$ and the homotopy relation $\sim$ is the usual relation of chain homotopic maps. So it all boils down to checking that $X$ is $G$-acyclic if and only if $Ch(G)(S^n(G_i), X)/\sim = 0$ for all $n$ and $i$. But it is easy to see that this condition is equivalent to the statement that each $\text{Hom}_G(G_i, X_{n+1}) \to \text{Hom}_G(G_i, Z_n X)$ is an epimorphism. By Lemma 4.4 this in turn is equivalent to saying that $\text{Hom}_G(G, X)$ is exact.

4.5. Computation of $G$-$\text{Ext}^n_G$. We have already seen an obvious analogy: $G$ is to $G_G$ as $R$ is to $R$-$\text{Mod}$. This analogy extends to the calculation of $G$-$\text{Ext}^n_G(A, B)$, as the existence of the $G$-projective model structure formalizes the fact that one can do homological with respect to $G$. In more detail, according to Corollary 3.5, given any $A \in \mathcal{G}$, we may take a $G$-projective resolution

$$P \to A \equiv \cdots \to P_2 \to P_1 \to P_0 \to A.$$ 

By this we mean it is $G$-acyclic and each $P_n$ is $G$-projective. Then all the usual definitions and theorems hold for $G$-projective resolutions. For example, they are unique up to chain homotopy and one can define $G$-$\text{Ext}^n_G(A, B)$ via such resolutions. We obtain long exact sequences, starting with $G$-exact sequences, etc. Moreover $G$-$\text{Ext}^n_G(A, B)$ can alternately be defined using Yoneda’s method: as equivalence classes of $G$-exact sequences $B \to L_1 \to \cdots \to L_n \to A$. (See also [CH02, Sections 1.2 and 2.1]; it is easy to see that the $G$-projectives and $G$-epimorphisms form a projective class.)

Our point here is that for a $G$-projective resolution $P \to A$, we have a $G$-exact sequence of chain complexes $K \to P \to S^0(A)$, where $K = \ker(P \to S^0(A))$. Moreover $K$ is $G$-acyclic and $P$ is $G$-semiprojective (since it is built up as a transfinite extension by consecutively attaching the $G$-semiprojective spheres $S^0(P_0), S^1(P_1), S^2(P_2), \ldots$) So $P$ is a cofibrant replacement of $S^0(A)$ in the $G$-projective model structure. Hence using the fundamental theorem of model categories we have

$$D(G)(A, \Sigma^n B) = Ch(G)(P, S^n(B))/\sim = H^n[\text{Hom}_G(P, B)] = G$-$\text{Ext}^n_G(A, B).$$

4.6. The $\lambda$-pure derived category. In [CH02, Section 5.3] we see the construction of a model structure for the pure derived category of a ring $R$ and a canonical adjunction between the pure derived category of $R$ and the usual derived category $D(R)$. We describe now a natural extension of this fact to the $G$-derived category.

In any Grothendieck category $\mathcal{G}$, all objects are $\lambda$-presentable for some regular cardinal $\lambda$. In particular, for any choice of generator $G = \oplus G_i$ there is a $\lambda$ such that all the $G_i$ are $\lambda$-presentable. It follows that $\mathcal{G}$ is locally $\lambda$-presentable. (See Appendix A and [AR94, page 22] for language in this Subsection.) In fact, we see from [AR94, page 22] that $\mathcal{G}$ is locally $\lambda$-presentable if and only if it has a generating set consisting of $\lambda$-presentable objects. Moreover, the category of all $\lambda$-presentable objects in $\mathcal{G}$ has a small skeleton. So we can find a set $\{\Lambda_i\}$ of representatives from each isomorphism class, and we set $\Lambda = \oplus \Lambda_i$. Since $\{\Lambda_i\}$ contains $\{G_i\}$, it is also a generating set for $\mathcal{G}$. From Lemma 4.4 and Proposition A.1 the $\lambda$-acyclic
complexes are characterized as the exact complexes $X$ for which each $Z_nX \subseteq X_n$ is $\lambda$-pure. Such complexes are called $\lambda$-pure acyclic. We call $D(\Lambda)$ the $\lambda$-pure derived category of $\mathcal{G}$, and its model structure from Corollary 4.7 we call the $\lambda$-pure projective model structure. The extension groups of Subsection 4.5 we denote by $\lambda\text{-PExt}_\mathcal{G}$. We easily get the following.

**Corollary 4.8.** Let $\mathcal{G}$ be any Grothendieck category with $G$ and $\Lambda$ as above. There is a canonical functor $D(\Lambda) \to D(G)$ that is the identity on objects from the $\lambda$-pure derived category to the $G$-derived category. It induces a map $\lambda\text{-PExt}_\mathcal{G}(A,B) \to G\text{-Ext}_\mathcal{G}(A,B)$ which is natural in $A, B \in \mathcal{G}$. Moreover, $D(\Lambda) \to D(G)$ admits a left adjoint.

**Proof.** First note that the identity functor $\text{Ch}(\mathcal{G}) \xrightarrow{id} \text{Ch}(\mathcal{G})$ is left adjoint to itself. Since $\{G_i\} \subseteq \{\Lambda_i\}$, the identity functor takes $G$-semiprojective complexes (complexes built from all the $S^n(G_i)$) to $\Lambda$-semiprojective complexes (complexes built from all the $S^n(\Lambda_i)$). Similarly it takes $G$-projective complexes (those built from the $D^n(G_i)$) to $\Lambda$-projective complexes (built from the $D^n(\Lambda_i)$). This directly leads us to conclude the identity functor is a left Quillen functor from the $G$-projective model structure to the $\lambda$-pure projective model structure. This automatically provides an adjunction $D(G) \xrightarrow{L(id)} D(\Lambda)$, taking a complex $X$ to its $G$-semiprojective cofibrant replacement. Its right adjoint $D(\Lambda) \xrightarrow{R(id)} D(G)$ is the identity on objects since every object is fibrant. Since the functor $R(id)$ is identity on objects, the functor provides, for all $A, B \in \mathcal{G}$, a natural map $D(G)(\Lambda)(A, \Sigma^nB) \to D(G)(A, \Sigma^nB)$. But from Subsection 4.5 we see this translates to a natural map $\lambda\text{-PExt}_\mathcal{G}(A, B) \to G\text{-Ext}_\mathcal{G}(A, B)$. □

5. THE INJECTIVE MODEL FOR LOCALLY FINITELY PRESENTABLE CATEGORIES

In the previous section, we constructed the $G$-derived category of any pair $(\mathcal{G}, G)$ where $\mathcal{G}$ is a Grothendieck category and $G = \oplus G_i$ is a generator. We constructed a model structure for $D(G)$ in which the cofibrant complexes were built from $G$-projective objects. Our goal in this section is to construct a dual model structure for $D(G)$, whose fibrant complexes are based on the $G$-injective objects. In order to do this we need to assume each $G_i$ is finitely presented, or equivalently, that $\mathcal{G}$ is locally finitely presentable ($= \text{locally } \omega\text{-presentable as defined in Appendix A}$). Indeed from [AR94, Theorem 1.11] we have that $\mathcal{G}$ is locally finitely presentable if and only if $\mathcal{G}$ has a set of generators $\{G_i\}_{i \in I}$ for which each $G_i$ is finitely presented ($= \omega\text{-presented}$). Having different models for the same category is often useful. For example, the existence of the injective model structure implies the two recollement situations presented in Section 6.

5.1. $G$-HOMOLOGY IN LOCALLY FINITELY PRESENTABLE CATEGORIES. For a chain complex $X$, we define its $G$-homology as $H_n[\text{Hom}_\mathcal{G}(G, X)]$. So the $G$-homology vanishes if and only if $X$ is $G$-acyclic. Recall that in a general Grothendieck category, a product of acyclic complexes need not again be acyclic. This is the point of Grothendieck’s (AB4*) axiom. However, Theorem 4.6 tells us that the $G$-acyclic complexes are closed under products, since they are the right half of a cotorsion pair. The point of this subsection is to collect other useful properties that hold under the added assumption that each $G_i$ is finitely presented. These properties will be used to construct the injective model structure on $\text{Ch}(\mathcal{G})$. 


Lemma 5.1. Assume each $G_i$ is finitely presented. Up to a product, $G$-homology commutes with direct limits. That is, if $\{ X_j \}_{j \in J}$ is a directed system of complexes, then

$$H_n[\text{Hom}_G(G, \lim_{j \in J} X_j)] \cong \prod_{i \in I} \lim_{j \in J} H_n[\text{Hom}_G(G_i, X_j)]$$

If the set of generators $\{ G_i \} = \{ G_1, G_2, \ldots, G_n \}$ is finite, then since direct limits commute with finite products we have

$$H_n[\text{Hom}_G(G, \lim_{j \in J} X_j)] \cong \lim_{j \in J} H_n[\text{Hom}_G(G, X_j)]$$

Proof. For complexes of abelian groups, homology commutes with products and direct limits. Also, the $G_i$ are assumed finitely presented, so we have isomorphisms:

$$H_n[\text{Hom}_G(G, \lim_{j \in J} X_j)] \cong \prod_{i \in I} H_n[\text{Hom}_G(G, \lim_{j \in J} X_j)] \cong \prod_{i \in I} \lim_{j \in J} H_n[\text{Hom}_G(G_i, X_j)].$$

Proposition 5.2. Assume each $G_i$ is finitely presented. Then the following hold.

1. The $G$-acyclic complexes are closed under direct limits.
2. Direct limits of $G$-monomorphisms are again $G$-monomorphisms.

Proof. For the second statement, suppose $f$ is a monomorphism sitting in an exact sequence $\mathcal{E} : 0 \to A \xrightarrow{j} B \xrightarrow{g} C \to 0$ which happens to be a directed limit of $G$-exact sequences $\mathcal{E}_j : 0 \to A_j \to B_j \to C_j \to 0$. Interpreting each $\mathcal{E}_j$ as a $G$-acyclic chain complex the result follows from the first statement.

By a transfinite composition we mean a map of the form $X_0 \xrightarrow{f_0} \lim_{\alpha} X_\alpha$ where $X : \lambda \to \mathcal{G}$ is a colimit-preserving functor and $\lambda$ is an ordinal. In this case $f$ is the transfinite composition of the $X_\alpha \to X_{\alpha+1}$. If each of these $X_\alpha \to X_{\alpha+1}$ is a $G$-monomorphism then $f$ is a transfinite composition of $G$-monomorphisms.

Furthermore, in this case we say that $\lim_{\alpha} X_\alpha$ is a transfinite $G$-extension of all the objects $X_0, X_{\alpha+1}/X_\alpha$.

Corollary 5.3. Assume each $G_i$ is finitely presented. Then the following hold.

1. The $G$-acyclic complexes are closed under transfinite $G$-extensions and direct sums.
2. An arbitrary transfinite composition of $G$-monomorphisms is again a $G$-monomorphism.

Proof. The $G$-acyclic complexes are always closed under $G$-extensions by Lemma 4.4. So they are closed under transfinite $G$-extensions by Proposition 5.2. Direct sums are special cases of transfinite $G$-extensions.

For the second statement, we first note that a finite composition $(\lambda = n \in \mathbb{N})$ of $G$-monomorphisms is again a $G$-monomorphism by part (4) of Proposition 3.3. For $\lambda = \omega$, we want the map $X_0 \xrightarrow{f_0} \lim_{n<\omega} X_n$ to also be a $G$-monomorphism. So we want the short exact sequence

$$\mathcal{E} : 0 \to X_0 \xrightarrow{f_0} \lim_{n<\omega} X_n \to (\lim_{n<\omega} X_n)/X_0 \to 0$$
to be \(G\)-exact. But this is the direct limit of the short exact sequences
\[
\mathcal{E}_n : \quad 0 \to X_0 \xrightarrow{f_n} X_n \to X_n/X_0 \to 0
\]
and these are \(G\)-exact because this is the finite case \(\lambda = n\). So the \(\lambda = \omega\) case holds by Proposition 5.2. We see the result follows by transfinite induction. \(\square\)

5.2. **Complete cotorsion pairs.** The result here is taken, with only a few small adjustments for our situation, from the original source [Hov02]. We again use the notion of a small cotorsion pair from [Hov02] as well as the notation \(I\)-cell and \(I\)-inj from [Hov99].

**Proposition 5.4.** Consider the \(G\)-exact category \(\mathcal{G}_G\) in the case that each \(G_i\) is finitely presented. Then a cotorsion pair \((\mathcal{F}, \mathcal{C})\) in \(\mathcal{G}_G\) is cogenerated by a set \(S\) if and only if it is small with generating monomorphisms the set
\[
I = \{0 \to G_i\}_{i \in I} \cup \{K_S \to P_S \to S\}_{S \in S}.
\]
Here we have chosen for each \(S \in S\), a \(G\)-exact sequence \(K_S \to P_S \to S\) with \(P_S\) a \(G\)-projective object. Such a cotorsion pair \((\mathcal{F}, \mathcal{C})\) satisfies each of the following:

1. \((\mathcal{F}, \mathcal{C})\) is functorially complete.
2. \(\mathcal{F}\) consists precisely of retracts of transfinite \(G\)-extensions of \(S\).
3. \(I\)-inj is precisely the class of all \(G\)-epimorphisms with kernel in \(\mathcal{C}\).

**Proof.** Note that we can find the \(G\)-exact sequences \(K_S \to P_S \to S\) with each \(P_S\) a \(G\)-projective by using Corollary 3.5. We see that the functors \(\text{G-Ext}^i_{\text{Ch}(\mathcal{G})}(P_S, -)\) and \(\text{G-Ext}^i_{\text{Ch}(\mathcal{G})}(G_i, -)\) vanish. So it is easy to see that \(S\) cogenerated the cotorsion pair iff the given set \(I\) forms a set of generating monomorphisms in the sense of [Hov02, Definition 6.4].

By Corollary 5.3 we have that transfinite compositions of \(G\)-monomorphisms are again \(G\)-monomorphisms. So by [Hov02, Theorem 6.5] we get that \((\mathcal{F}, \mathcal{C})\) is a functorially complete cotorsion pair. The proof there shows that \(\mathcal{F}\) consists precisely of retracts of transfinite \(G\)-extensions of objects in \(S\).

It is left to see that \(I\)-inj is precisely the class of all \(G\)-epimorphisms with kernel in \(\mathcal{C}\). Showing that everything in \(I\)-inj is a \(G\)-epimorphism with kernel in \(\mathcal{C}\) is formally similar to the first claim in the proof of Theorem 4.6. The converse is similar to the argument given in the last paragraph of the proof of Corollary 4.7. We leave the details to the reader. \(\square\)

**Remark 2.** We note that Proposition 5.4 applies not just to \(\mathcal{G}_G\) but also to \(\text{Ch}(\mathcal{G})_G\) by Lemma 4.1. This is because each \(D^n(G_i)\) is a finitely presented complex whenever each \(G_i\) is finitely presented. In particular, any cotorsion pair in \(\text{Ch}(\mathcal{G})_G\) that is cogenerated by a set is complete.

5.3. **Injectives in \(\mathcal{G}_G\) and \(\text{Ch}(\mathcal{G})_G\).** We need to show that the exact categories \(\mathcal{G}_G\) and \(\text{Ch}(\mathcal{G})_G\) have enough injective objects. Following our language for the projective case, we will call these objects \(G\)-injective. We will use the theory of purity summarized in Appendix A. The appendix shows that when \(\mathcal{G}\) is locally finitely presentable (= locally \(\omega\)-presentable) we have a well-behaved notion of pure (= \(\omega\)-pure) subobjects \(P \subseteq X\) in \(\mathcal{G}\). In particular, we get that pure monomorphisms are closed under directed colimits (= \(\omega\)-directed colimits) in \(\mathcal{G}\) by Proposition A.1.

Note that any pure exact sequence \(0 \to A \to B \to C \to 0\) in \(\mathcal{G}\) is automatically a \(G\)-exact sequence. This follows from Proposition A.1, our assumption that each \(G_i\)
is finitely presented, and the fact that direct products of short exact sequences (of abelian groups) are still short exact sequences. In particular, any pure subobject is automatically a $G$-subobject.

**Remark 3.** For any Grothendieck category $\mathcal{G}$ there exist arbitrarily large regular cardinals $\lambda$ such that the $\lambda$-presented objects coincide with the $\lambda$-generated objects. The author thanks Jiří Rosický for providing the following reason for this statement: Let $\mathcal{G}_{\text{mono}}$ denote the category consisting of the same objects as $\mathcal{G}$ but with morphisms only the monomorphisms of $\mathcal{G}$. Then for any $\lambda$, the $\lambda$-presented objects of $\mathcal{G}_{\text{mono}}$ coincide exactly with the $\lambda$-generated objects of $\mathcal{G}$. Moreover we note $\mathcal{G}_{\text{mono}}$ is an accessible category by [AR94, Local Generation Theorem 1.70]. The embedding functor $\mathcal{G}_{\text{mono}} \to \mathcal{G}$ is an accessible functor in the sense of [AR94, Definition 2.16]. Therefore, the Uniformization Theorem [AR94, Theorem 2.19 and Remark] applies which means there are arbitrarily large regular cardinals $\lambda$ for which this embedding is $\lambda$-accessible and preserves $\lambda$-presented objects. This means exactly that there exist arbitrarily large regular cardinals $\lambda$ such that the $\lambda$-presented objects coincide with the $\lambda$-generated objects. In fact, it follows from [AR94, Remark 2.20] that if $\gamma$ is a regular cardinal for which $\lambda \prec \gamma$, that is $\lambda$ is sharply smaller than $\gamma$ in the sense of [AR94, Definition 2.12], then the $\gamma$-presented objects coincide with the $\gamma$-generated objects too.

Note that for any $\gamma$ as in Remark 3 the notion of $\gamma$-presented (= $\gamma$-generated) becomes a substitute for “cardinality $< \gamma$”. In particular, the class of $\gamma$-presented objects is closed under quotients and subobjects. We also have that, up to isomorphism, there is just a set of $\gamma$-presented objects.

**Setup 5.5.** We now specify for our locally finitely presentable category $\mathcal{G}$ a regular cardinal $\gamma$ which will be of use. We fix a regular cardinal $\gamma$ with each of the following properties:

1. The $\gamma$-presented objects coincide with the $\gamma$-generated objects.
2. Whenever we have a subobject $S \subseteq X$ where $S$ is $\gamma$-generated, there exists a pure subobject $P \subseteq X$ which is also $\gamma$-generated and which contains $S$.

Let $S$ be a set of isomorphic representatives for the class of all $\gamma$-presented objects. Then $S$ cogenerates the injective
cotorsion pair \((A, \mathcal{I})\) in \(G_G\). That is, \(A\) consists of all objects of \(G\), while \(\mathcal{I} = S^\perp\) is precisely the class of injective objects of \(G_G\). We call these objects \(G\)-injective. \((A, \mathcal{I})\) is complete, meaning \(G_G\) has enough \(G\)-injectives.

**Proof.** By Proposition 5.4 we know that \(S\) cogenerated a complete cotorsion pair \((S^\perp, S^\perp)\) where \(S^\perp\) consist precisely of retracts of transfinite \(G\)-extensions of \(S\). Letting \(A\) denote the class of all objects of \(G\) we will be done if we can show \(A \subseteq S^\perp\). By [Hov02, Lemma 6.2] it suffices to show that every object in \(A\) is a transfinite \(G\)-extension of objects in \(S\). But since each \(G_i\) is finitely presented, we note that pure exact sequences are automatically \(G\)-exact. So it is enough to show that any object is a transfinite pure-extension of \(\gamma\)-presented objects.

So let \(M\) be any given object. First note that assuming \(M \neq 0\), we can always find a nonzero pure subobject \(P_0 \subseteq M\) with \(P_0\) \(\gamma\)-presented. Assuming \(P_0 \neq M\), we can do the same to \(M/P_0\) to get a pure \(P_1/P_0 \subseteq M/P_0\) with \(P_1/P_0\) \(\gamma\)-presented. Assuming we are not done, we continue to construct a strictly increasing \(0 \neq P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots\). Note each \(P_n \subseteq M\) is pure by part (3) of Proposition A.2. Then set \(P_n = \cup_{\alpha \leq n} P_\alpha\) and note it is also pure by part (4) of Proposition A.2. In this way we continue by transfinite induction to get \(M = \cup_{\alpha < \lambda} P_\alpha\) for some \(\lambda\) where each \(P_\alpha \subseteq P_{\alpha+1}\) is pure. \(\Box\)

**Remark 4.** No matter what our choice is for the generator \(G = \oplus_{i \in I} G_i\), it is the same set \(S\) cogenenerating the injective cotorsion pair \((A, \mathcal{I})\) (as long as each \(G_i\) is finitely presented). But a different choice of generating set \(\{G_i\}\) will of course change the proper class of short exact sequences in \(G_G\). Consequently, this changes the class \(S^\perp\) of \(G\)-injectives. (It of course also changes the \(G\)-projectives).

Note that because of Lemma 4.1, the above Proposition 5.6 also applies to the chain category complex \(\text{Ch}(G)_G\). That is, there are enough \(G\)-injective complexes. As in Lemma 4.5 we have the following classification of \(G\)-injective complexes.

**Lemma 5.7.** Call a chain complex \(X\) in \(\text{Ch}(G)_G\) a \(G\)-injective complex if it is injective in the exact category \(\text{Ch}(G)_G\). The following are equivalent:

1. \(X\) is \(G\)-injective.
2. \(X\) is \(G\)-acyclic with each \(Z_nX\) a \(G\)-injective.
3. \(X\) is isomorphic to a split exact complex with \(G\)-injective components. That is, \(X \cong \oplus_{n \in \mathbb{Z}} D^n(I_n)\) where each \(I_n\) is a \(G\)-injective.
4. \(X\) is a contractible complex with each \(X_n\) \(G\)-injective.

We note that there are enough \(G\)-injective complexes. This follows from Proposition 5.6 and Lemma 4.1.

5.4. **The injective model structure.** We now wish to construct an injective model structure for the \(G\)-derived category, assuming each \(G_i\) is finitely presented. The following lemma, which holds for arbitrary Grothendieck categories, will be used in the main proof.

**Lemma 5.8.** Let \(G\) be a locally \(\lambda\)-presentable Grothendieck category. Given an epimorphism \(g: X \to Y\) where \(Y\) is \(\lambda\)-generated, there exists a \(\lambda\)-generated subobject \(X' \subseteq X\) for which \(g_{|X'}: X' \to Y\) is an epimorphism.

**Proof.** Any locally \(\lambda\)-presentable Grothendieck category is also locally \(\lambda\)-generated. This means that, up to isomorphism, there is a set of \(\lambda\)-generated objects and that every object is a \(\lambda\)-directed union of its \(\lambda\)-generated subobjects. (The proof
of this goes by writing the given object \( X = \varproj limit \ X_i \) as a \( \lambda \)-directed colimit of \( \lambda \)-presented \( X_i \). Then factor each \( X_i \rightarrow X \) as an epi followed by a mono. Each \( \text{Im} \ X_i \) is \( \lambda \)-generated and \( X_i \) is the \( \lambda \)-directed union of the \( \text{Im} \ X_i \). So we may write \( X = \sum_{i \in I} X_i \) as a \( \lambda \)-directed union of \( \lambda \)-generated subobjects of \( X \). Since \( g \) is an epimorphism, \( Y = \sum_{i \in I} g(X_i) \), and this too is a \( \lambda \)-directed union. Now we must have \( Y = g(X_i) \) for some \( i \in I \) since \( Y \) is \( \lambda \)-generated. So \( g|_{X_i} : X_i \rightarrow Y \) is an epimorphism.

Recall (see Lemma 4.4), that a chain complex \( X \) is \( G \)-acyclic if and only if it is exact and each \( Z_n X \) is a \( G \)-subobject of \( X_n \). This means the inclusion \( Z_n X \rightarrow X_n \) is a \( G \)-monomorphism, and we write \( Z_n X \subseteq_G X_n \).

**Lemma 5.9.** Let \( \gamma \) be as in Setup 5.5. Given any nonzero \( G \)-acyclic complex \( E \) there exists a degreewise \( G \)-exact sequence \( P \rightarrow E \rightarrow E/P \) where \( P \) is a nonzero \( G \)-acyclic subcomplex with each \( P_n \) \( \gamma \)-presented.

**Proof.** (Step 1) We first prove the following: For any given \( n \) and exact \( S \subseteq E \) with each \( S_i \) \( \gamma \)-presented, there exists an exact \( T \subseteq E \) satisfying the following:

1. \( S \subseteq T \) and each \( T_i \) is \( \gamma \)-presented.
2. \( Z_n T \subseteq_G T_n \) is a \( G \)-subobject.
3. \( S_n \subseteq P \subseteq T_n \subseteq E_n \) for some \( G \)-subobject \( P \subseteq E_n \).

Indeed as in Setup 5.5 we can find a \( \gamma \)-presented pure \( P \subseteq E_n \) containing \( S_n \). Then set \( T_{n-1} = S_{n-1} + d(P) \) and note that it is \( \gamma \)-presented and that \( \text{ker} \ d|_{T_{n-1}} = d(P) \).

We set \( T_{n-2} = S_{n-2}, T_{n-3} = S_{n-3}, \) etc. going downward. This gives us a portion of a subcomplex we are building

\[ \cdots P \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow T_{n-3} \rightarrow \cdots \]

which we note is exact in degrees \( n-1 \) and below. We wish to extend upwards to an exact complex.

Note that \( \text{ker} \ d|_P \) is also \( \gamma \)-presented. So there exists a \( \gamma \)-presented pure subobject \( P' \subseteq Z_n E \) containing \( \text{ker} \ d|_P \). Now let \( T_n = P + P' \), and note that we still have exactness in degrees \( \leq n-1 \) in the (still unfinished) subcomplex shown

\[ \cdots T_n \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow T_{n-3} \rightarrow \cdots \]

Moreover, since \( \text{ker} \ d|_{T_n} = P' \) is pure in \( Z_n E \), it is a \( G \)-subobject \( \text{ker} \ d|_{T_n} = P' \subseteq_G Z_n E \). We also have \( Z_n E \subseteq_G E_n \) by assumption, and so from part (1) of Proposition 3.6 we have \( \text{ker} d|_{T_n} = P' \subseteq_G E_n \). But then from part (2) of Proposition 3.6 we have \( \text{ker} d|_{T_n} = P' \subseteq_G T_n \). (Here we have arranged conditions (2) and (3) in the subcomplex \( T \) that we are constructing.)

Now since \( P' \) is \( \gamma \)-presented, we can use Lemma 5.8 to find a \( \gamma \)-presented subobject \( S'_{n+1} \subseteq E_{n+1} \) for which \( d|_{S'_{n+1}} : S'_{n+1} \rightarrow P' \) is an epimorphism. We set \( T_{n+1} = S_{n+1} + S'_{n+1} \) and note that

\[ \cdots T_{n+1} \rightarrow T_n \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow T_{n-3} \rightarrow \cdots \]

is now exact in degrees \( n \) and below. Repeatedly using Lemma 5.8 in this way we can continue upward to obtain an exact subcomplex \( T \subseteq E \) which contains \( S \), which has each \( T_i \) \( \gamma \)-presented, has \( Z_n T = P' \subseteq_G T_n \), and has \( S_n \subseteq P \subseteq T_n \subseteq E_n \) where \( P \subseteq_G E_n \).

(Step 2) We now complete the proof. For the construction just described in (Step 1), let's say that the complex \( T \) was obtained by applying a “degree \( n \) operation
Proposition 5.10. Let $\gamma$ be as in Setup 5.5. Each $G$-acyclic complex is a transfinite $G$-extension of $\gamma$-presented $G$-acyclic complexes.

Proof. Suppose $E \neq 0$ is $G$-acyclic and use Lemma 5.9 to find a nonzero $\gamma$-presented $G$-acyclic subcomplex $0 \neq P_0 \subseteq E$ which is a $G$-subobject in each degree. Then applying $\text{Hom}_G(G, -)$ to $P_0 \to E \to E/P_0$ leaves an exact sequence of complexes and it follows that $E/P_0$ is $G$-acyclic also. Assuming this complex is not zero find another nonzero $\gamma$-presented $G$-acyclic complex $P_1/P_0 \subseteq E/P_0$ which is a $G$-subobject in each degree. Since $P_0 \subseteq E$ is a $G$-subobject in each degree, we get that $P_0 \subseteq P_1$ is also a $G$-subobject in each degree by Proposition 3.6, part (2). Then part (3) of that same Proposition tells us that $P_1 \subseteq E$ is a $G$-subobject in each degree. Assuming $P_1 \neq E$, we continue to find an increasing sequence $0 \neq P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$ of $G$-acyclic subcomplexes of $E$ with each $P_n \subseteq E$ a $G$-subobject in each degree. Then set $X_\omega = \bigcup_{n < \omega} P_n$ and we see that $X_\omega$ is too a $G$-acyclic complex by Proposition 5.2 and also $X_\omega \subseteq E$ is a $G$-subobject in each degree, again by Proposition 5.2. Therefore $E/P_\omega$ is also $G$-acyclic and we can continue with transfinite induction to end up with $E$ displayed as a transfinite $G$-extension of $\gamma$-presented $G$-acyclic complexes.

Theorem 5.11. Let $\mathcal{G}$ be a Grothendieck category with a generator $G = \bigoplus_{i \in I} G_i$ where each $G_i$ is finitely presented. Let $W$ be the class of all $G$-acyclic complexes. Then there is an injective cotorsion pair $(W, \mathcal{I})$ in $\text{Ch}(\mathcal{G})_G$. That is, it is a complete cotorsion pair in $\text{Ch}(\mathcal{G})_G$ for which $W$ is thick in $\text{Ch}(\mathcal{G})_G$ and $W \cap \mathcal{I}$ coincides with the class of injective complexes in $\text{Ch}(\mathcal{G})_G$. We call the complexes in $\mathcal{I}$ the $G$-semi-injective complexes.
Proof. Let \( \gamma \) be as in Setup 5.5 and take \( S \) to be a set of isomorphism representatives for the class of all \( \gamma \)-presented complexes in \( \mathcal{W} \). So everything in \( S \) is a \( G \)-acyclic complex \( S_n \) with each \( S_n \) being \( \gamma \)-presented. We will show that \( S \) cogenerates \((\mathcal{W}, \mathcal{I})\) in \( \operatorname{Ch}(G) \). Recall that cotorsion pairs in \( \operatorname{Ch}(G) \) are with respect to \( G \)-Ext\(^1\). By Remark 2 which follows Proposition 5.4, we know that \( S \) cogenerates a complete cotorsion pair \((\perp(S^\perp), S^\perp)\) in \( \operatorname{Ch}(G) \) where \( \perp(S^\perp) \) consists precisely of retracts of transfinite \( G \)-extensions of \( S \). We wish to show that \( \mathcal{W} = \perp(S^\perp) \). But we already know that \( \mathcal{W} \) is thick in \( \operatorname{Ch}(G) \) by Lemma 4.4 and closed under transfinite \( G \)-extensions by Corollary 5.3. So \( \mathcal{W} \supseteq \perp(S^\perp) \). On the other hand, \( \mathcal{W} \subseteq \perp(S^\perp) \) was proved in Proposition 5.10. So \( (\mathcal{W}, \mathcal{I}) \) is a complete cotorsion pair where \( \mathcal{I} = S^\perp \).

Since we already know \( \mathcal{W} \) is thick, all that is left is to show that \( \mathcal{W} \cap \mathcal{I} \) coincides with the class of injective complexes in \( \operatorname{Ch}(G) \). But by the argument in [BGH13, Proposition 3.3] it is enough to show that the injectives in \( \operatorname{Ch}(G) \) are contained in \( \mathcal{W} \). Since the injective complexes are precisely the contractible complexes with \( G \)-injective components by Lemma 5.7, these are in \( \mathcal{W} \) by lemma 4.4. \( \square \)

The following corollary now follows from the main result in [Hov02].

Corollary 5.12. Let \( G \) be a Grothendieck category with a generator \( G = \oplus_{i \in I} G_i \) where each \( G_i \) is finitely presented. Then there is a model structure on \( \operatorname{Ch}(G) \) which we call the \( G \)-injective model structure whose trivial objects are the \( G \)-acyclic complexes. The model structure satisfies the following:

1. The cofibrations are precisely the \( G \)-monomorphisms. That is, the chain maps which are \( G \)-monomorphisms in each degree.
2. The trivial cofibrations are the \( G \)-monomorphisms with \( G \)-acyclic cokernel.
3. The fibrations are the degreewise split epimorphisms whose kernel is a \( G \)-semi-injective complex.
4. The trivial fibrations are the degreewise split epimorphisms whose kernel is a \( G \)-injective complex.
5. The weak equivalences are the \( G \)-homology isomorphisms.
6. The model structure is cofibrantly generated. Sets of generating cofibrations and generating trivial cofibrations can be found using Proposition 5.4.
7. The homotopy category is equivalent to \( \mathcal{D}(G) \), and this is a compactly generated triangulated category by Corollary 4.7.

6. The recollement situations

Again, \( G \) is a locally finitely presentable Grothendieck category with generator \( G = \oplus_{i \in I} G_i \) where each \( G_i \) is finitely presented. Here we wish to prove the two recollement situations from Theorems B and C of the Introduction.

We will use the correspondence between injective (resp. projective) cotorsion pairs and recollements situations from [Gil12] and [Gil13]. By definition, a cotorsion pair \((\mathcal{P}, \mathcal{W})\) in \( G \) (or \( \operatorname{Ch}(G) \)) is a \textbf{projective cotorsion pair} if it is complete, \( \mathcal{W} \) is \( G \)-thick, and if \( \mathcal{P} \cap \mathcal{W} \) coincides with the class of \( G \)-projective objects. Since the category \( G \) has enough projectives this makes the triple \((\mathcal{P}, \mathcal{W}, \mathcal{A})\), where \( \mathcal{A} \) represents the class of all objects, correspond to a model structure on \( G \) via Hovey’s correspondence [Hov02, Theorem 2.2]. For example, the cotorsion pair of Theorem 4.6 is a projective cotorsion pair in \( \operatorname{Ch}(G) \) and corresponds to the model structure of Corollary 4.7. On the other hand, we showed in Proposition 5.6 that \( G \) (and so \( \operatorname{Ch}(G) \)) also has enough injectives and so it also makes sense to say...
of injective cotorsion pairs are the dual. For example, the cotorsion pair of Theorem 5.11 is an injective cotorsion pair in \( \text{Ch}(G)_G \) and gave us the model structure of Corollary 5.12.

**Proposition 6.1.** Assume each \( G_i \) is finitely presented. There is an injective model structure \((\mathcal{W}_1, \mathcal{F}_1)\) in \( \text{Ch}(G)_G \) where \( \mathcal{F}_1 \) is the class of all complexes of \( G \)-injective complexes.

*Proof.* From Proposition 5.4 and Remark 2 which follows it, we know that any set of complexes cogenerates a complete cotorsion pair in \( \text{Ch}(G)_G \). Here we let \( \mathcal{S}_1 = \{ D^n(S) \mid S \in \mathcal{S} \} \) where \( \mathcal{S} \) is the set in Proposition 5.6 which cogenerates the injective cotorsion pair \((\mathcal{A}, \mathcal{I})\) in \( \mathcal{G}_G \). So \( \mathcal{I} \) is the class of \( G \)-injectives. By Lemma 4.2 we have \( G\text{-Ext}^1_{\text{Ch}(G)}(D^n(S), X) \cong G\text{-Ext}^1_G(S, X_n) \). It follows that \( \mathcal{S}_1^\perp = \mathcal{F}_1 \) in \( \text{Ch}(G)_G \). So we get a complete cotorsion pair \((\mathcal{W}_1, \mathcal{F}_1)\) in \( \text{Ch}(G)_G \) where \( \mathcal{F}_1 \) is the class of all complexes of \( G \)-injective complexes.

To show it is an injective cotorsion pair in \( \text{Ch}(G)_G \), we only need to show that \( \mathcal{W}_1 \) is \( G \)-thick and contains the injectives. Note that for any complex \( W \) and \( F \in \mathcal{F}_1 \) we have \( G\text{-Ext}^1_{\text{Ch}(G)}(W, F) = \text{Ext}^1_{\text{dg}}(W, F) \). So by Lemma 4.3, \( W \in \mathcal{W}_1 \) if and only if \( \text{Hom}(W, F) \) is exact. So to see that \( \mathcal{W}_1 \) is \( G \)-thick we consider a degreewise exact sequence of complexes \( 0 \to X \to Y \to Z \to 0 \). Then as noted earlier, for any complex \( F \) of \( G \)-injectives, applying \( \text{Hom}(-, F) \) will give us a short exact sequence \( 0 \to \text{Hom}(Z, F) \to \text{Hom}(Y, F) \to \text{Hom}(X, F) \to 0 \). So if two out of the three complexes are exact, then so is the third. This proves thickness of \( \mathcal{W}_1 \) in \( \text{Ch}(G)_G \). If \( I \) is an injective complex in \( \text{Ch}(G)_G \), then by Lemma 5.7 it is a split exact complex with \( G \)-injective components. In particular, it is contractible. So for such an \( I \) we have \( \text{Hom}(I, F) \) is exact for any \( F \in \mathcal{F}_1 \).

**Proposition 6.2.** Assume each \( G_i \) is finitely presented. There is an injective model structure \((\mathcal{W}_2, \mathcal{F}_2)\) in \( \text{Ch}(G)_G \) where \( \mathcal{F}_2 \) is the class of all \( G \)-acyclic complexes of \( G \)-injectives.

*Proof.* Take \( \mathcal{S}_1 \) from the proof of Proposition 6.1 and let \( \mathcal{S}_2 = \mathcal{S}_1 \cup \{ S^n(G) \} \). We claim that \( \mathcal{S}_2^\perp = \mathcal{F}_2 \) in \( \text{Ch}(G)_G \). Indeed if \( X \in \mathcal{S}_2^\perp \) then \( X \) is a complex of \( G \)-injectives for which \( 0 = G\text{-Ext}^1_{\text{Ch}(G)}(S^n(G), X) = \text{Ext}^1_{\text{dg}}(S^n(G), X) = H_{n-1}\text{Hom}(S^0(G), X) = H_{n-1}\text{Hom}(G, X) \). So \( X \) is \( G \)-acyclic. Conversely, if \( X \) is \( G \)-acyclic with \( G \)-injective components then \( X \in \mathcal{S}_2^\perp \). So we get a complete cotorsion pair by again applying Proposition 5.4 and Remark 2 which follows it. The fact that \( \mathcal{W}_2 \) is thick and contains the \( G \)-injective complexes follows just like in Proposition 6.1.

**Theorem 6.3** (Krause’s recollement for \( G \)-derived categories). Assume each \( G_i \) is finitely presented. Let \( D(G) \) denote the \( G \)-derived category. Let \( K_G(\text{Inj}) \) denote the homotopy category of all complexes of \( G \)-injectives. Let \( K_G(\text{ac}(\text{Inj})) \) denote the homotopy category of all \( G \)-acyclic complexes of \( G \)-injectives. Then there is a recollement

\[
K_G(\text{ac}(\text{Inj})) \rightleftharpoons K_G(\text{Inj}) \rightleftharpoons D(G)
\]

*Proof.* Take \((\mathcal{W}_1, \mathcal{F}_1)\) to be the injective cotorsion pair from Proposition 6.1. Take \((\mathcal{W}_2, \mathcal{F}_2)\) to be the injective cotorsion pair from Proposition 6.2. Take \((\mathcal{W}_3, \mathcal{F}_3) = (\mathcal{W}, \mathcal{I})\) to be the \( G \)-semi-injective cotorsion pair from Theorem 5.11. Since \( \mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1 \) and \( \mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2 \) the result is automatic from [Gil13, Theorem 3.4].
In particular we immediately obtain Theorem D from Section 2, as well as the sheaf examples of Section 2. We also have the following general result which applies to those examples.

**Theorem 6.4** (Verdier localization recollement for $G$-derived categories). Assume each $G_i$ is finitely presented. Let $D(G)$ denote the $G$-derived category. Let $K(G)$ denote the homotopy category of all chain complexes and let $K_{G,\text{ac}}(G)$ denote the subcategory of all $G$-acyclic complexes. Then there is a recollement

$$
\begin{array}{ccc}
K_{G,\text{ac}}(G) & \xrightarrow{I} & K(G) \\
\cap & & \cap \\
C(W,KT) & \xleftarrow{\lambda = C(KP,W)} & D(G) \\
\cap & & \cap \\
E(W,KI) & \xleftarrow{\rho = E(W,KT)} & 
\end{array}
$$

Here, $W$ is the class of $G$-acyclic complexes, and the complexes in $KP$ are the $G$-analog of Spaltenstein’s $K$-projective complexes. The functor $C(KP,W)$ is the functor taking $X$ to its $KP$-precover since $(KP,W)$ turns out to be a complete cotorsion pair in $\text{Ch}(\mathcal{G})$. Similarly $KI$ is analogous to the class of $K$-injective complexes and $E(W,KI)$ is the functor taking $X$ to its $KI$-preenvelope.

**Proof.** The basic idea is that the existence of the $G$-projective model $(P,W)$ of Section 4 provides a left adjoint to the inclusion $K_{G,\text{ac}}(G) \to K(G)$, and in fact a colocalization sequence $K_{G,\text{ac}}(G) \to K(G) \to D(G)$. On the other hand, the existence of the $G$-injective model $(W,Z)$ of Section 5 provides a right adjoint to the inclusion $K_{G,\text{ac}}(G) \to K(G)$, and in fact a localization sequence $K_{G,\text{ac}}(G) \to K(G) \to D(G)$. Together this is a recollement. The formalization in terms of model structures follows immediately from work in [Gil13, Section 6]. The theory there is all written in terms of weakly idempotent complete exact categories, and so applies to our current setting. In full detail, we apply [Gil13, Theorem 6.3] to the $G$-injective model structure $(W,Z)$ to obtain a Quillen equivalent model structure $(W,KT)$ in the exact category $\text{Ch}(\mathcal{G})$ of chain complexes with degreewise split short exact sequences. The complexes in $KT$ are the $G$-analog of Spaltenstein’s $K$-injective complexes and in fact are, by [Gil13, Proposition 6.4], precisely the complexes that are chain homotopy equivalent to a $G$-semi-injective complex. The dual of [Gil13, Theorem 6.3] applied to the $G$-projective model structure $(P,W)$ gives us a similar model $(KP,W)$. All together $(KP,W,KT)$ is localizing cotorsion triple in the sense of [Gil13, Section 4.1] and so by [Gil13, Corollary 4.5] we obtain the recollement. □

**Appendix A. $\lambda$-purity in Grothendieck categories**

Every Grothendieck category $\mathcal{G}$ is locally presentable. This means there exists a regular cardinal $\lambda$ and a set $\mathcal{S}$ of $\lambda$-presented objects such that every object of $\mathcal{G}$ is a $\lambda$-directed colimit of objects of $\mathcal{S}$. In this case we say $\mathcal{G}$ is locally $\lambda$-presentable and it is true that for any regular cardinal $\lambda' > \lambda$, we have $\mathcal{G}$ is locally $\lambda'$-presentable as well. See [AR94, Theorem 1.20 and the Remark].
Now following [AR94], a morphism $f$ is called $\lambda$-pure if for each commutative diagram
\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
\downarrow u & & \downarrow v \\
A & \xrightarrow{f} & B
\end{array}
\]
with $A', B'$ $\lambda$-presented there is a map $t$ such that $u = tf'$. Assuming the category is locally $\lambda$-presentable we have from [AR94, Proposition 2.29] that a $\lambda$-pure morphism must be a monomorphism. In fact, they are characterized in [AR94, Proposition 2.30 and its Corollary] as being precisely the $\lambda$-directed colimits (in the category of morphisms) of split monomorphisms. Since Grothendieck categories are abelian we are lead naturally to speak instead of $\lambda$-pure short exact sequences, which we now characterize.

**Proposition A.1** ($\lambda$-purity in Grothendieck categories). Let $\mathcal{G}$ be a locally $\lambda$-presentable Grothendieck category and let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. Then the following are equivalent and characterize what we mean by saying $\mathcal{E}$ is a $\lambda$-pure short exact sequence.

1. $f$ is a $\lambda$-pure morphism.
2. $\text{Hom}_\mathcal{G}(L, \mathcal{E})$ is a short exact sequence of abelian groups for any $\lambda$-presented object $L$.
3. $\mathcal{E}$ is a $\lambda$-directed limit of split short exact sequences $\mathcal{E}_i : 0 \to A_i \to B_i \to C_i \to 0$ ($i \in I$).

**Proof.** As already pointed out above, we have from [AR94, Proposition 2.30 and Corollary] that the $\lambda$-pure morphisms are precisely the $\lambda$-directed colimits of split monomorphisms. In particular, if $f$ is a $\lambda$-pure morphism, we get that the short exact sequence
\[
\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0
\]
must be a $\lambda$-directed colimit of split short exact sequences
\[
\mathcal{E}_i : 0 \to A_i \to B_i \to C_i \to 0.
\]
So (1) if and only if (3). But if (3) holds, then we clearly have that each $\text{Hom}_\mathcal{G}(L, \mathcal{E}_i)$ is exact for any $L$. If $L$ is $\lambda$-presented then $\text{Hom}_\mathcal{G}(L, \mathcal{E}) \cong \lim \text{Hom}_\mathcal{G}(L, \mathcal{E}_i)$ is also exact. So (3) implies (2).

Now we show (2) implies (3). Using that $\mathcal{G}$ is locally $\lambda$-presentable, write $C = \lim \text{C}_i$ as a $\lambda$-directed colimit of $\lambda$-presented $C_i$. For each $\gamma_i : C_i \to C$, form the pullback
\[
\begin{array}{ccc}
\mathcal{E}_i : & 0 & \to A & \xrightarrow{f} & B_i & \xrightarrow{g} & C_i & \to 0 \\
\downarrow & & \downarrow & & \downarrow \gamma_i \\
\mathcal{E} : & 0 & \to A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0
\end{array}
\]
If (2) holds, then $\gamma_i$ lifts over $g$. This implies that $\mathcal{E}_i$ splits. One can check that $\mathcal{E} \cong \lim \mathcal{E}_i$. \[\square\]

**Proposition A.2.** Let $\mathcal{G}$ be a locally $\lambda$-presentable Grothendieck category and $A \subseteq B \subseteq C$.

1. If $A \subseteq B$ is $\lambda$-pure and $B \subseteq C$ is $\lambda$-pure then $A \subseteq C$ is $\lambda$-pure.
(2) If $A \subseteq C$ is $\lambda$-pure then $A \subseteq B$ is $\lambda$-pure.
(3) If $A \subseteq C$ is $\lambda$-pure and $B/A \subseteq C/A$ is $\lambda$-pure, then $B \subseteq C$ is $\lambda$-pure.
(4) $\lambda$-pure monomorphisms are closed under $\lambda$-directed colimits.

Proof. (1) and (2) follow easy from the definition of $\lambda$-pure via the commutative diagram. For (3), let $L$ be $\lambda$-presented. All we need to check is that the map $\Hom_G(L, C) \to \Hom_G(L, C/B)$ is an epimorphism. But this is just the composite

$$\Hom_G(L, C) \to \Hom_G(L, C/A) \to \Hom_G(L, (C/A)/(B/A)) \cong \Hom_G(L, C/B),$$

and these are epimorphisms by hypothesis. Finally, a proof of (4) appears in [AR94, Proposition 2.30 (1)].

### Appendix B. Exact categories vs. proper classes

We show here that if $\mathcal{A}$ is an abelian category, an exact category $(\mathcal{A}, \mathcal{E})$ in the sense of [Qui73] and [Bühl10] is the same thing as a proper class of short exact sequences in the sense of [Mac63, Chapter XII.4] and [Hov02]. See also the Historical Notes and Appendix B of [Bühl10] for the equivalence to Heller’s axioms for an “abelian class of short exact sequences”.

**Proposition B.1.** Let $\mathcal{A}$ be an abelian category. Then $(\mathcal{A}, \mathcal{E})$ is an exact category in the sense of [Qui73] if and only if $\mathcal{E}$ is a proper class of short exact sequences in the sense of [Mac63, Chapter XII.4].

Proof. Say $(\mathcal{A}, \mathcal{E})$ is an exact category. We wish to see that $\mathcal{E}$ is a proper class.

The only thing that is not immediate from first definitions or properties of exact categories is Mac Lane’s axiom (P-4), and the dual (P-4’). But abelian categories are weakly idempotent complete and so these follow from [Bühl10, Proposition 7.6] which states: whenever $gf$ is an admissible monomorphism (resp. epimorphism) then $f$ (resp. $g$) is an admissible monomorphism (resp. epimorphism).

On the other hand, say $\mathcal{E}$ is a proper class in $\mathcal{A}$. To see $(\mathcal{A}, \mathcal{E})$ is an exact category we just need to check the pullback/pushout axioms. But any, say pullback, exists, and pulling back along an $\mathcal{E}$-epimorphism $p$ yields a diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & A & \overset{i'}{\longrightarrow} & P & \overset{p'}{\longrightarrow} & C' & \longrightarrow & 0 \\
\bigg| & & \bigg| & & \bigg\downarrow f' & & \bigg\downarrow f & & \bigg| \\
0 & \longrightarrow & A & \overset{i}{\longrightarrow} & B & \overset{p}{\longrightarrow} & C & \longrightarrow & 0
\end{array}
$$

Since $i$ is an $\mathcal{E}$-monomorphism, so is $i = f'i'$. We wish to “cancel” $f'$ to conclude $i'$ is an $\mathcal{E}$-monomorphism. However, axiom (P-4) of [Mac63, Chapter XII.4] only allows this when $f'$ is monic. But we now remedy this by imitating the argument that can be found within the proof of [Mac63, XII.4 Theorem 4.3]. First, recall that the pullback $(P, f', p')$ can be constructed (see [Mac63, XII.4 Theorem 1.1]) so that $P$ is the kernel in the left exact sequence $0 \to P \overset{v}{\to} B \oplus C' \overset{p_1 - f'p_2}{\longrightarrow} C$ and the maps $f'$ and $p'$ satisfy $f' = \pi_1 v$ and $p' = \pi_2 v$. We see that

$$v'i' = 1v'i' = (i_1\pi_1 + i_2\pi_2)v'i' = i_1(\pi_1v)i' + i_2(\pi_2v)i' = i_1f'i' + i_2p'i' = i_1i.$$

Since $i_1$ is an $\mathcal{E}$-monomorphism by (P-2), we see that $i_1i$ is an $\mathcal{E}$-monomorphism by (P-3). So $v'i' = i_1i$ is an $\mathcal{E}$-monomorphism, and by (P-4) we may now conclude $i'$ is an $\mathcal{E}$-monomorphism.

\qed
References

[AR94] J. Adánek and J. Rosický, *Locally presentable and accessible categories*, Number 189 in London Mathematical Society Lecture Note Series, Cambridge University Press, 1994.

[Bie81] Robert Bieri, *Homological dimension of discrete groups*, second ed., Queen Mary College Mathematical Notes, Queen Mary College Department of Pure Mathematics, London, 1981.

[BGH13] Daniel Bravo, James Gillespie and Mark Hovey, *The stable module category of a general ring*, submitted.

[Bühl10] T. Bühler, *Exact Categories*, Expo. Math. vol. 28, no. 1, 2010, pp. 1–69.

[CH02] J. Daniel Christensen and Mark Hovey, *Quillen model structures for relative homological algebra*, Math. Proc. Camb. Phil. Soc. vol. 133, 2002, pp. 261–293.

[CB94] W. Crawley-Boevey, *Locally finitely presented additive categories*, Communications in Algebra vol. 22, no. 5, 1994, pp. 1641–1674.

[EE05] E. Enochs, S. Estrada, *Relative homological algebra in the category of quasi-coherent sheaves*, Advances in Mathematics vol. 194, no. 2, 2005, pp.284–295.

[EEGR] E. Enochs, S. Estrada, J.R. García-Rozas, *Locally projective monoidal model structure for complexes of quasi-coherent sheaves on P1(k)*, J. Lond. Math. Soc. (2) vol. 77, no. 1, 2008 pp. 253–269.

[EEGOa] E. Enochs, S. Estrada, J.R. García Rozas and L. Oyonarte, *Flat covers in the category of quasi-coherent sheaves over the projective line*, Communications in Algebra vol. 32, no. 4, 2004, pp. 1497–1508.

[EEGOd] E. Enochs, S. Estrada, J.R. García Rozas and L. Oyonarte, *Flat cotorsion quasi-coherent sheaves. Applications*, Algebr. Represent. Theory vol. 7, no. 4, 2004, pp. 441-456.

[Gar01] Grigory Garkusha, *Classifying finite localizations of quasi-coherent sheaves*, St. Petersburg Math. J. vol. 21, no. 3, 2010, pp. 433-458.

[Gil07] James Gillespie, *Kaplansky classes and derived categories*, Math. Zeit. vol. 257, no. 4, 2007, pp.811-843.

[Gil11] James Gillespie, *Model structures on exact categories*, Journal of Pure and Applied Algebra, vol. 215, 2011, pp. 2892–2902.

[Gil12] James Gillespie, *Gorenstein complexes and recollements from cotorsion pairs*, arXiv:1210.0196v2.

[Gil13] James Gillespie, *Exact model structures and recollements*, arXiv:1310.7530.

[Har77] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics vol. 52, Springer-Verlag, New York, 1977.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs vol. 63, American Mathematical Society, 1999.

[Hov02] Mark Hovey, *Cotorsion pairs, model category structures, and representation theory*, Mathematische Zeitschrift, vol. 241, 2002, pp. 553–592.

[Kra01] Henning Krause, *On Neeman’s well generated triangulated categories*, Doc. Math. 6, 2001, pp. 121-126.

[Kra05] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math., vol. 141, no. 5, 2005, pp. 1128–1162.

[Kra12] Henning Krause, *Approximations and adjoints in homotopy categories*, Math. Ann. vol. 353, no. 3, 2012, pp. 765-781.

[Lam99] T.Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics vol. 189, Springer-Verlag, New York, 1999.

[Mac63] Saunders MacLane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, vol.114, Springer-Verlag, 1963.

[Nee01] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies vol. 148, Princeton University Press, Princeton, NJ, 2001.

[Qui73] D. Quillen, *Higher Algebraic K-theory I*, SLNM vol. 341, Springer-Verlag, 1973, pp. 85–147.

[Ros09] J. Rosický, *Generalized purity, definability and Brown representability [Lecture Slides]*, Some Trends in Algebra, Prague 2009. Retrieved from http://www.math.muni.cz/~rosicky/2009/2011/2009.09.05/Rosicky.pdf

[Ros05] J. Rosický, *Generalized Brown representability in homotopy categories*, Theory Appl. Categ. vol. 14, no. 19, 2005, pp. 451-479.

[ŠŠ11] Manuel Saorín and Jan Štovíček, *On exact categories and applications to triangulated adjoints and model structures*, Adv. Math. vol. 228 , no. 2, 2011, pp. 968–1007.
[Sto13] Jan Šťovíček, *Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves*, Advances in Representation Theory of Algebras (ICRA Bielefeld, Germany, 8-17 August, 2012), EMS Series of Congress Reports, European Mathematical Society Publishing House, 2014, pp. 297–367.

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