Bounds on the Sum of a Divergent Series

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Abstract

Given a truncated perturbation expansion of a physical quantity, one can, under certain circumstances, obtain lower or upper bounds (or both) to the sum of the full perturbation series by using the Borel transform and a variational conformal map. The method is illustrated by applying it to various mathematical toy-models for which exact results are known. One of these models is used to exemplify how non-perturbative contributions supplement the sum of a Borel-nonsummable series to give the final exact and unambiguous result. Finally, the method is applied to some physical problems. In particular, some speculations are made on the phase of quantum electrodynamics at super-high temperatures from a study of its perturbative free-energy density.

1 Introduction

After more than half a century, perturbation theory is still the best analytical tool we have for computations in quantum field theory. Unfortunately in many cases the perturbation parameter, $\lambda$, is not small. So the series, of which in practice only a few terms are known, diverges or gives a poor representation of the physical quantity. Consequently, several techniques have

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been used to estimate the full sum of a series from the small number of given terms. In this paper, one particular method which achieves this goal is described in detail. The method was introduced in Ref. [1].

Suppose one is given the perturbation expansion

$$\hat{S}_N(\lambda) = \sum_{n=0}^{N} f_n \lambda^n.$$  \hspace{1cm} (1)

Since perturbation expansions in quantum field theories have the generic behaviour $f_n \sim n!$ for $n$ large [2], it is natural to consider the Borel transform

$$B_N(z) = \sum_{n=0}^{N} \frac{f_n}{n!} z^n.$$ \hspace{1cm} (2)

The series (1) can then be recovered from the Borel integral,

$$\hat{S}_N(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-z/\lambda} B_N(z) \, dz.$$ \hspace{1cm} (3)

If the exact function $B(z) \equiv B_\infty(z)$ is known, Eq. (3) can be taken as defining the exact sum, $S = \hat{S}_\infty$, of the full perturbation series if the Borel integral is well-defined. The poor convergence of (1) is then attributed to an expansion of $B(z)$ in a power series as in (2) and the subsequent use of that series beyond its radius of convergence in (3).

Suppose $B(z)$ has only one singularity in the complex $z$-plane (the Borel plane), at $z = -1/p$, with $p$ real and positive. Then the radius of convergence of the Borel series (2) is $1/p$. Therefore in order to reconstruct an approximation to the exact sum, one must extend the domain of convergence of the partial series in Eq. (3). The method of Loeffel, Le-Guillou and Zinn-Justin [3, 4] is to use a conformal map

$$w(z) = \frac{\sqrt{1 + zp} - 1}{\sqrt{1 + zp} + 1},$$ \hspace{1cm} (4)

which maps the $z$-plane into a unit circle in the $w$-plane, with the singularity at $z = -1/p$ mapped to $w = -1$. The inverse of this map is

$$z(w) = \frac{4w}{p} \frac{1}{(1 - w)^2}.$$ \hspace{1cm} (5)

Making the change of variables (4) in (3) one obtains,
\[ \hat{S}_N(\lambda) = \frac{1}{\lambda} \int_0^1 dw \int_0^1 dw \ e^{-z(w)/\lambda} B_N(z(w)), \] (6)

where \( B_N(z(w)) \) is understood as a power series in \( w \) obtained by an expansion of (5). In terms of the variable \( w \), the series \( B_N(z(w)) \) converges for \(|w| < 1\), so that the potential problem in (8) is only at the upper limit of integration. Now, the difference between \( B_N(z(w)) \) and \( B_{N+1}(z(w)) \) begins at order \( w^{N+1} \). If one knows the coefficients \( f_n \) only up to \( n = N \), then it is consistent to keep only terms up to order \( w^N \) in \( B_N(z(w)) \). Performing this truncation in (6) and reverting back to the \( z \) variable, one obtains

\[ S_N(\lambda) \equiv \frac{1}{\lambda} \sum_{n=0}^{n=N} \frac{f_n}{n!} \left( \frac{4}{p} \right)^n \sum_{k=0}^{N-n} \frac{(2n + k - 1)!}{k!(2n - 1)!} \int_0^\infty dz \ e^{-z/\lambda} w(z)^{(k+n)}. \] (7)

\( S_N(\lambda) \) is a non-trivial resummation of the original series \( \hat{S}_N(\lambda) \). In Ref.[4, 5], Eq.(7) has been used to resum long perturbation expansions for critical exponents, with \( z = -1/p \) the location of the instanton singularity of \( \phi^4 \) field theory. Note that after a scaling, (7) may be written as

\[ S_N(\lambda) \equiv \sum_{n=0}^{n=N} \frac{f_n}{n!} \left( \frac{4}{p} \right)^n \sum_{k=0}^{N-n} \frac{(2n + k - 1)!}{k!(2n - 1)!} \int_0^\infty dz \ e^{-z/\lambda} w(\lambda z)^{(k+n)}. \] (8)

Since \( w(z) \) is a bounded and slowly varying function, this shows that the resummed expression \( S_N(\lambda) \) is a much slower varying function of the coupling than the original series \( \hat{S}_N(\lambda) \).

In recent years, the resummation (7) has also been applied to quantum chromodynamics (QCD), with \( z = -1/p \) the location of the first ultraviolet renormalon pole [6, 7]. Actually, QCD is Borel-nonsummable [8], which means that \( B(z) \) also has poles for \( z > 0 \), rendering the Borel integral ambiguous. In such a case, one can still use (3) to define the sum of the perturbation series once the integral is made definite through some prescription such as the principal value.

In all the applications of (7) in [4, 5, 6, 7], \( p \) is a known and fixed constant which determines the location of the singularity of \( B(z) \) closest to the origin. In Ref.[1], the expression (7) was used as the starting point for a new technique which can be used even in cases when the singularity structure of \( B(z) \) is unknown. In the following section, I describe in detail how the technique
of \[\Pi\] can be used to obtain lower or upper bounds for some series, and also in many cases to obtain accurate estimates of the exact sum. Then in Sec.(3) the procedure is illustrated with several toy models. In Sec.(4) the method is used to study a toy model for Borel-nonsummmable series, and the difference between the sum of the perturbation series and the exact quantity, which includes non-perturbative contributions, is emphasized. In Sec.(5) I discuss how one can simultaneously obtain both upper and lower bounds on the sum of certain series. Some physical applications are discussed in Sects.(6-8). In Sec.(6), the perturbative Euler-Heisenberg expansion for the effective action of quantum electrodynamics (QED) in a background magnetic field is resummed and the result compared with Schwinger’s exact expression. In Sec.(7), the infrared fixed point of a \[\phi^4_3\] theory is determined from the long series for its beta function and the results compared with those in the literature. In Sec.(8) the free-energy density of super-hot quantum electrodynamics is studied and some speculations made. A summary and the conclusion are in Sect.(9). Finally, the Appendix contains an approximate but very important analysis of the relevant equations which explains the empirically observed trends.

2 The Variational Conformal Map

The conventional transformation of a series

\[ \hat{S}_N(\lambda) = \sum_{n=0}^{N} f_n \lambda^n. \]  

by the Borel-conformal map method results in the reorganised expression

\[ S_N(\lambda, p) \equiv \frac{1}{\lambda} \sum_{n=0}^{N} \frac{f_n}{n!} \left(\frac{4\lambda}{p}\right)^n \sum_{k=0}^{N-n} \frac{(2n+k-1)!}{k!(2n-1)!} \int_0^\infty dz e^{-z/\lambda} w(z)^{(k+n)}, \]

with \(p\) a fixed constant. However, notice that \(p\) does not figure in (9) but enters (10) only through the conformal map (4). Thus instead of treating \(p\) as some fixed constant as in Ref.[4, 5, 6, 7], one may consider \(p > 0\) a free parameter which defines the conformal map (4) and Eq.(10) then represents a continuous family of resummations, one for each value of \(p\). Thus, from now on, \(p\) will not refer to the location of some singularity in \(B(z)\). In fact,
absolutely no knowledge about the singularity structure of \( B(z) \) is required for the method elaborated below.

Of course having liberated ourselves from the usual interpretation of \( p \) in (10), we also lose definiteness in our resummation. Therefore some new condition must be imposed to fix the value of \( p \) and hence of the expression (10). For a start, for each \( N \), choose \( p > 0 \) to be the location of an extremum of \( S_N(\lambda, p) \). Since (10) depends on \( \lambda \), the value of \( p \) in principle will also depend on \( \lambda \). However the procedure then becomes too unwieldy, and to simplify it \( p \) is determined at some reference value \( \lambda = \lambda_0 \), say at the midpoint of the range of interest:

\[
\frac{\partial S_N(\lambda_0, p)}{\partial p} = 0. \tag{11}
\]

As mentioned after (8), \( S(\lambda, p) \) is a relatively slowly varying function of \( \lambda \), hence determining \( p \) at some fixed \( \lambda = \lambda_0 \), and then using the same \( p \) in (10) for various \( \lambda \) is sufficient for practical purposes. Indeed, in many applications, one requires the sum of the series at one particular value of the coupling, or in a very narrow range, so the simplification made in (11) is both useful and sufficient.

In some cases, (11) will not have any solutions. For example, if all the \( f_n \) are of the same sign, \( S_N(\lambda, p) \) will be a monotonic function of \( p \). Thus for the procedure to work, at least some of the \( f_n \) must be of a different sign. This will be assumed to be the case from now on. Suppose next that \( f_1 \neq 0 \). Then for \( p \) large, and since the \( p \) dependence of \( w(z) \) is mild,

\[
S_N(p) \sim f_0 + \frac{f_1}{p}. \tag{12}
\]

Therefore for \( f_1 < 0 \), \( S_N \) will first decrease as \( 1/p \) increases and then increase when the next term \( f_n/p^n > 0 \) dominates the sum. If the last non-zero term \( f_N/p^N \) is positive then one expects \( S_N(p) \) to have a global minimum at some \( p > 0 \). However if \( f_N/p^N < 0 \) then \( S_N(p) \) can become very small as \( p \to 0^+ \) and so the minimum would likely be only a local extremum. From this heuristic argument one concludes that if the first non-zero coefficient \( f_n, (n > 0) \) is negative, Eq. (11) will have global minima solutions for some \( N \).

For conciseness, unless otherwise stated, I will discuss from now on only the case when a global minimum exists, as the arguments for the global maximum case are then obvious. For a series \( S_N(\lambda, p) \), define \( p(N) \) to be
the position of the global minima for the values of $N$ when they exist, and for other values of $N$ let $p(N)$ denote the location of the local minima. Also, let

$$S_N \equiv S_N(\lambda, p(N)).$$

(13)

Obviously the definition of $p(N)$ has been chosen because it is useful. If $p(N)$ is a location of a global minimum, then $S_N$ certainly is a lower bound on $S_N(\lambda, p)$. However this does not imply that $S_N$ is a lower bound on the exact sum $S$ of the perturbation series because for finite $N$, $S$ might lie outside of the space of resummations labelled by $p$. For each $N$, let $p^*(N)$ be the value of $p$ that is optimal, that is, it is the value which when used in (10) gives the best estimate of $S$. Define, $S_N^* = S_N(\lambda, p^*(N))$. Then for a global minima one has

$$S_N \leq S_N^*$$

(14)

Presumably $S_N^*$ converges to $S$ as $N \to \infty$. Then for those $N$ when global minima exist,

$$S_{N\to\infty} \leq S.$$

(15)

(This implicitly assumes that the sub-sequence of global minima is infinite: That is, given any positive integer $N_0$, there is some $n > N_0$ for which $f_n$ is positive.)

Though the inclusion of global minima in the definition of $p(N)$ is quite clear, the inclusion of local minima requires some explanation. As mentioned earlier, if $S_N(\lambda, p)$ has a global minimum, then $S_{N+1}(\lambda, p)$ will probably not have a global minimum if $f_{N+1}$ is negative because $S_{N+1}(\lambda, p)$ might become very small as $p \to 0^+$. However $S_{N+1}(\lambda, p)$ will still have a local minimum (and hence also a local maximum). The local minimum will occur for moderate values of $p$ close to $p(N)$, the global minimum position of $S_N(\lambda, p)$. Therefore one might expect that the local minimum for $S_{N+1}(\lambda, p)$ still gives a bound on $S_{N+1}^*$. Indeed, suppose that the inequality

$$S_N \leq S_{N+1}$$

(16)

holds for all $N$ up to $N = \infty$, then clearly

$$S_N \leq S$$

(17)

and one concludes that $S_N(\lambda, p(N))$ is a lower bound on the sum of the full perturbation series. Of course proving (16) is equivalent to knowing $f_n$
explicitly for all \( n \), which is not the situation in reality. As a practical matter, it is sufficient to draw a conclusion after observing the trend (14) (or lack of) for the available terms of the series. Arguments given in the Appendix indicate that the trend, once started, will continue.

An interesting question is whether the inequality (15) is saturated. In most of the examples studied this has been found to be the case. In fact the convergence is so rapid that one can conclude with a high degree of confidence, not only that \( S_N \) is a lower bound, but that it is close to the exact value. However a toy model in Sec.(3.6) and a physical example in Sec.(7) show that the series \( S_N \), though forming a bound on the actual value \( S \), might not converge to \( S \). In both these examples the exact result \( S(\lambda) \) had a first derivative \( dS(\lambda)/d\lambda \) that varied rapidly with \( \lambda \). An explanation of why in those cases the bounds \( S_N \) do not converge to the exact value is given in the Appendix. However, as described below, such situations can also be taken care of by the resummation procedure (10-11).

Recall that when a local minima occurs for some \( N \), one expects also a local maximum. Define \( \bar{p}_N \) to be the position of those local maxima and \( \bar{S}_N \) the corresponding value of the resummed series. It turns out that \( \bar{S}_N \) also satisfies an inequality like (16). However since for the sequence \( \bar{S}_N \) of only local extrema one does not seem to have a statement like (14-15), it is not a priori obvious that they form bounds to the exact result. Of course if one believes that the \( p^*(N) \) as defined above always take moderate values, then a statement like (14) might also be made for the sequence \( \bar{S}_N \). Furthermore, in Sec.(5) I describe how an auxiliary series \( S'_N(\lambda, p) \) defined from \( S_N(\lambda, p) \) can be used to obtain upper bounds to \( S \), when \( S_N \) gives lower bounds. Using the sequences \( S_N, \bar{S}_N, \) and \( \bar{S}'_N \), one can obtain constraints on \( S \) and, with some physical input, an estimate of \( S \) itself. In the Appendix it is explained why the alternate bounds formed from \( \bar{S}_N \) and the auxiliary series give better approximations to the exact value \( S \) itself when \( \partial S/\partial \lambda \) varies rapidly with \( \lambda \).

Although the main discussion in this paper will be for the \( p(N) \) (or \( \bar{p}(N) \) and \( p'(N) \)) as defined above, in some cases one finds (by inspection) solutions \( p_0(N) \) to (11) which have the property that as \( N \to \infty \), \( p_0(N) \to p_0 \), a constant. In such a case one is tempted to speculate that the fixed point \( p_0 \) indicates the existence of a singularity at \( z = -1/p \) in \( B(z) \). This is indeed found to be the case in the examples studied, though apparently the converse is not necessarily true: the singularities of \( B(z) \) need not show up as solutions of (11). Furthermore, the convergence of the series \( S_{(0)N} \) is not expected to
be monotonic since in general the \( p_0(N) \) refer to both maxima and minima.

Though not manifest at first sight, the analysis in the Appendix suggests that even the \( p(N) \) defined above Eq.(13) actually will converge to a fixed value as \( N \to \infty \). This trend can be observed in the examples studied.

3 Mathematical Models

In this section a number of mathematical models are studied to illustrate the resummation technique of Eqs.(10-11). The models all define Borel-summable series, that is, \( B(z) \) has no singularities at real, positive \( z \). A Borel-nonsummable example will be considered in Sec.(4). All of the examples here have global minima as solutions to the extremum condition (11) for some \( N \). (From any such series \( S_N \) one can trivially construct \( C - S_N \), with \( C \) a constant, which then gives an example of a series with global maxima solutions to (11)). The reader is reminded that the existence of global minima does not by itself imply that one has obtained a lower bound on the sum of the series. The additional condition (13) must be satisfied for a lower bound. Example (5) below shows how one can end up with an upper bound from global minima! (See also the Appendix).

It is also re-emphasized that although in these toy models the exact singularity structure of \( B(z) \) is known, that information will not be used in the resummation. The resummation of the partial series

\[
\hat{S}_N(\lambda) = \sum_{n=0}^{N} f_n \lambda^n
\]

will proceed using Eqs.(10-11). The exact \( B(z) \) will only be used to compare the resummed results with the exact sum of the full series given by the Borel integral

\[
S(\lambda) = \frac{1}{\lambda} \int_{0}^{\infty} e^{-z/\lambda} B(z) dz .
\]

3.1 \( B(z) = \frac{1}{1+z} \)

The first model is defined by the Borel function,

\[
B(z) = \frac{1}{1 + z} .
\]
Expanding $B(z)$ to $N$-th order in $z$ and using the result in (3) gives the truncated series

\[ \hat{S}_N = \sum_{n=0}^{N} (-\lambda)^n n! \]  

(21)

The divergent nature of this series is displayed in Fig.(1a). Using $f_n = (-1)^n n!$ in (10) and solving Eq.(11) at the reference value $\lambda_0 = 1$ gives the following solutions (minima): $p(2) = 2.65$, $p(3) = 5.1$ and $p(4) = 8.4$. As expected from the arguments of Sec.(2), there is no solution for $N = 1$, the solutions for $N = 2$ and $N = 4$ are global minima while that for $N = 3$ is a local minimum. The resummed series is shown in Fig.(1b). The convergence of the $S_N$ is monotonic and satisfies the condition (16). Hence the approximants $S_N$ can be argued to form lower bounds to the exact result. That this is indeed the case can be seen by inspection of the exact result in Fig.(1b) as obtained from (20) and (19). Furthermore, it is clear that the lower bounds converge to the exact value, and so may be used to estimate it. At $\lambda = 0.5$, the exact value is 0.722657, while the approximants are $S_2 = 0.704$, $S_3 = 0.709$, $S_4 = 0.711$.

For this model there are other solutions, for each $N$, to the extremum condition (11) in addition to the minima. By inspection one picks out the sequence of values, $p_0(3) = 1.6$ (local maximum), $p_0(4) = 1.3$ (local minimum) and $p_0(5) = 1.15$ (local maximum) as plausibly approaching a fixed point. Indeed we already know that in this model the exact singularity of $B(z)$ is a pole at $z = -1/p = -1$, so the fixed point probably refers to the location of this singularity. It should come as no surprise that if the approximants (10) are evaluated at the values $p_0(N)$, the convergence to the exact value will be much faster, and this is indicated in Fig.(1c). At $\lambda = 0.5$ the values of the approximants are $S_{(0)3} = 0.726$, $S_{(0)4} = 0.7219$, $S_{(0)5} = 0.7228$. Notice however that in this case the convergence is not monotonic (and was not expected to be).

The alternate sequence $\bar{S}_N$ for this model will be discussed in Sect.(5).

3.2 $B(z) = \frac{1}{1+z^2}$

The second model is defined by the Borel function,

\[ B(z) = \frac{1}{1 + z^2} \]  

(22)
which has poles only on the *imaginary* $z$-axis. The truncated perturbation series corresponding to (22) is

$$ \hat{S}_N = \sum_{n=0}^{N} (-1)^n \lambda^{2n} (2n)! . \tag{23} $$

The divergent nature of this series is displayed in Fig.(2a). Using $f_n = (-1)^n (2n)!$ in (11) and solving Eq.(11) at the reference value $\lambda_0 = 1$, gives the following solutions (minima): $p(4) = 1.45$, $p(5) = 2.9$, $p(6) = 4.5$, $p(7) = 6.3$, $p(8) = 8.4$. Notice that although $f_{2n+1} = 0$, the approximants $S_{2N+1}$ do exist and are different from $S_{2N}$. This is because of the way the resummation is performed in Eq.(10). As expected on general grounds, the minima for $N = 4, 5, 8$ are global minima while those for $N = 6, 7$ are local minima.

The resummed series is shown in Fig.(2b) together with the exact result. The convergence is again monotonic, satisfies the condition (16), and the approximants $S_N$, are indeed lower bounds to the exact result. Furthermore, compared to the divergent series (23), the resummed series is close to the exact result for moderate values ($\sim 0.3)$ of the coupling. At $\lambda = 0.3$, the exact result is 0.89, while the approximants give, $S_4 = 0.86$, $S_5 = 0.868$, $S_6 = 0.871$, $S_7 = 0.873$, $S_8 = 0.874$.

### 3.3 $B(z) = e^{-z}$

The third model is defined by the Borel function,

$$ B(z) = e^{-z} . \tag{24} $$

This function is regular everywhere in the Borel plane and the exact sum of the perturbation series has the closed form $S(\lambda) = 1/(1 + \lambda)$. Thus in this case a power expansion of $S(\lambda)$ is actually convergent for $|\lambda| < 1$. However close to $\lambda = 1$, the convergence is very slow and one requires a large number of terms of the series to obtain accurate results. The utility of the resummation procedure even in this case is now demonstrated.

The truncated perturbation series corresponding to (24) is

$$ \hat{S}_N = \sum_{n=0}^{N} (-1)^n \lambda^n . \tag{25} $$

Notice that there is no factorial growth of the coefficients in (25). The slow convergence of this series for $\lambda$ close to 1 is displayed in Fig.(3a). Using
The fourth model is defined by the Borel function,

\[ B(z) = \frac{1}{1+z} + \frac{1}{1+z^2}. \]  

(26)

The truncated perturbation series corresponding to (26) is

\[ \hat{S}_N = \sum_{n=0}^{N} f_n \lambda^n, \]  

(27)

with

\[ f_{2n+1} = -(2n+1)!, \]  

(28)

\[ f_{2n} = (1+(-1)^{n})(2n)!. \]  

(29)

This is an example of a series which is not strictly alternating. The divergent series is displayed in Fig.(4a). Using the values of \( f_n \) given above and solving Eq.(11) at the reference value \( \lambda_0 = 1 \), gives the following solutions (minima): \( p(4) = 1.5 \), \( p(5) = 2.8 \), \( p(6) = 4.5 \), \( p(7) = 6.4 \). As expected from the signs of the \( f_n \), there are no solutions for \( N \leq 3 \), the solution for \( N = 4 \) is a global minimum, while those for \( N = 5, 6, 7 \) are local minima. Again note that \( S_6 \) exists and does not equal \( S_5 \) eventhough \( f_6 = 0 \).
The resummed series is shown in Fig.(4b) together with the exact result. The convergence is monotonic, satisfies the condition (16), and the approximants $S_N$ form lower bounds to the exact result as expected. The resummed series shows a much improved convergence compared to the original series (27), even for couplings as large as $\lambda = 0.5$: The exact value is 1.521, while the resummed approximants give, $S_4 = 1.397$, $S_5 = 1.426$, $S_6 = 1.437$, $S_7 = 1.443$.

In this example, no sign of the sequence $p_0(N)$ was found even though there is a pole of the Borel function $B(z)$ at $p = -1$. This is taken as support of the statement at the end of Sec.(2). Actually, a comparison of the $p(N)$ values of this model with those of the model in Sec.(3.2) shows that they are almost identical! This means that the $1/(1 + z^2)$ part of the Borel function in (26) is dominating the behaviour of $p(N)$ and thus masking the singularity at $z = -1$.

This example also displays the following curiosity: Although the series (27) is divergent, if one keeps only the first two terms $f_0$ and $f_1$, then the unresummed expression $\hat{S}_1(\lambda)$ agrees quite well with the exact result $S(\lambda)$ (See Fig.(4a)) for a large range of couplings! A similar 'accident' occurs for the free energy density of hot $SU(3)$ gauge theory, where it is found that the usual second order perturbative contribution already agrees with the full lattice result (see [1] and references therein).

### 3.5 $B(z) = \frac{-z}{\sqrt{1+z}}$

The fifth model is defined by the Borel function,

$$B(z) = \frac{-z}{\sqrt{1+z}}, \quad (30)$$

which has a branch cut beginning at $z = -1$. The truncated perturbation series corresponding to (30) is

$$\hat{S}_N = \sum_{n=1}^{N} \left( \frac{-\lambda}{2} \right)^n 2n(2n - 3)!! \quad (31)$$

with $(-1)!! \equiv 1$. The divergent series is displayed in Fig.(5a). Solving Eq.(11) at the reference value $\lambda_0 = 1$, gives (global minima): $p(2) = 1.3$, $p(4) = 1.1$, $p(6) = 1$. No solutions were found for $N = 3$ or 5. This example illustrates that although the heuristic argument of the Sec.(2) suggests a
solution to (11) in general, such a solution might fail to exist for some values of \( N \).

This example has a number of other peculiarities. First notice that the global minima sequence \( p(2) = 1.3, \ p(4) = 1.1, \ p(6) = 1 \) is identical to the sequence \( p_0(N) \) (see the end of Sec.(2)) which appears to converge to the fixed point \( p_0 = 1 \), and which happens to locate the exact singularity of \( B(z) \).

The resummed series is shown in Fig.(5b) together with the exact result. The convergence is rapid and monotonic but the approximants \( S_N \) do not form lower bounds to the exact result as might have been naively expected because \( S_2 > S_4 > S_6 \), which is opposite to what is required. Indeed, as the figure shows, the approximants approach the exact result from above and hence appear to form upper bounds eventhough the \( p(N) \) are positions of global minima! An explanation of this oddity is given in the Appendix. See also the discussion prior to Eq.(14).

For \( \lambda = 2 \), the exact result is \(-1\), while the succesive approximants give, \( S_2 = -0.907, \ S_4 = -0.986, \ S_6 = -0.996 \). Thus the approximants converge to the exact value.

### 3.6 \( S(\lambda) = -\sin(\pi \lambda) \)

The sixth model is defined by the exact expression

\[
S(\lambda) = -\sin(\pi \lambda) . \tag{32}
\]

The partial series corresponding to (32) is

\[
\hat{S}_N = \sum_{n=0}^{N} \frac{(\lambda \pi)^{2n+1}}{(2n+1)!} (-1)^{n+1} . \tag{33}
\]

This series is convergent but it has been chosen to test the resummation procedure for cases when the exact expression \( S(\lambda) \) is not a monotonic function of the coupling. The partial series is displayed in Fig.(6a).

Solving Eq.(11) at the reference value \( \lambda_0 = 1 \), gives (minima): \( p(3) = 1.5, \ p(4) = 3.1, \ p(5) = 5, \ p(6) = 7.3, \ p(7) = 10 \). Notice that solutions exist for \( N = 4 \) and \( N = 6 \) (and are different from those for \( N = 3, 5 \)) eventhough \( f_4 = f_6 = 0 \). (See the discussion for the example in Sec.(3.2).)

The resummed series is shown in Fig.(6b). Though the convergence of the \( S_N \) is rapid and monotonic, and although the approximants \( S_N \) form lower
bounds to the exact result, they do not give a good estimate of the exact result when $\lambda > 0.2$. This seems to be because the resummed expressions $S_N$ are monotonic functions of $\lambda$ while the exact expression is not (see the Appendix).

Now, recalling the general discussion of Sect.(2), one expects there to be local maxima solutions to Eq.(11) when $N = 5$ and $N = 6$. Indeed the solutions are $\bar{p}(5) = 0.15$ and $\bar{p}(6) = 0.275$. The curves for the approximants $\bar{S}_5$ and $\bar{S}_6$ are shown in Fig.(6c). These are seen to form upper bounds to the exact result and also give very good estimates! Thus it seems that one should use all the solutions to (11), that is the $p(N)$ and $\bar{p}(N)$, to get all possible bounds on the sum of the series. Then using some additional physical or theoretical information or prejudice, one can decide near which of the bounds (upper or lower), the exact result lies. A physical example is given in Sec.(7).

In this example, even though we used a series up to $N = 7$, only for two values could we form the upper bounds. In Sec.(5) I describe how one can get additional upper bounds from the series by first constructing an auxiliary series.

4 A Borel-Nonsummable Model

For most of the physical quantities calculable from the Standard Model of particle physics, the perturbation expansion is not expected to be Borel summable. In simple terms, this means that the function $B(z)$ has poles on the positive semi-axis of the Borel plane thus rendering the Borel integral (19) ambiguous. If one chooses an $i\epsilon$ prescription then the resulting ambiguity is in the imaginary part. Sometimes these imaginary parts are of direct physical relevance [11, 12]. More generally they indicate that the perturbation series does not give the full answer, but must be supplemented with some non-perturbative contributions [3, 4]. Since the imaginary parts will be of the form $e^{-1/\lambda}$, the additional real non-perturbative terms are expected to take the same form. These expectations have been confirmed in some lower dimensional field theories (see [3] and references therein).

One can also construct mathematical models to illustrate the arguments of the last paragraph. Suppose, for simplicity, that the only singularity of $B(z)$ for $z > 0$ is a single pole at $z = q$. Then the sum of the perturbation series can be defined by the principal value prescription,
With this definition, one focuses only on the real part of the physical quantity. Suppose furthermore, again for simplicity, that $B(z)$ is integrable at infinity. Then the Eq.\((34)\) may be rewritten as

$$S_{\text{pert}} \equiv \frac{1}{\lambda} \mathcal{P} \int_{0}^{\infty} dz \ e^{-z/\lambda} B(z).$$

(34)

The first term on the right-hand-side is finite and unambiguous. Call it $S_{\text{exact}}$, and denote the second term on the right-hand-side as $S_{\text{np}}$. Thus we have

$$S_{\text{exact}}(\lambda) = S_{\text{pert}}(\lambda) - S_{\text{np}}(\lambda).$$

(36)

In this way, the exact result $S_{\text{exact}}$, has been broken into two components, $S_{\text{pert}}$ which is purely perturbative, and $S_{\text{np}}$ which is purely non-perturbative. However while $S_{\text{exact}}$ is well-defined, both $S_{\text{pert}}$ and $S_{\text{np}}$ are only defined through the principal value prescription. Though highly simplified, this model plausibly represents the situation, for example, in Quantum Chromodynamics (QCD).

In order to illustrate explicitly the resummation technique (10-11) in the Borel-nonsummable case, set

$$B(z) = \frac{1}{(1 + z)(5 - z)}.\tag{37}$$

Then,

$$S_{\text{pert}}(\lambda) = \frac{1}{\lambda} \mathcal{P} \int_{0}^{\infty} \frac{e^{-z/\lambda}}{(1 + z)(5 - z)} dz B(z)\tag{38}.$$

The truncated perturbation series corresponding to \((38)\) is

$$\hat{S}_N = \sum_{0}^{N} f_n \lambda^n,$$

(39)

with

$$f_n = \left((-1)^n + 5^{-(n+1)}\right) \frac{n!}{6}.$$

(40)

The divergent series is displayed in Fig.(7a). The solution of Eq.\((11)\) at the reference value $\lambda_0 = 1$, gives (minima): $p(2) = 2.8$, $p(3) = 5.4$, $p(4) =$
The resummed partial series is shown in Fig.(7b) together with the exact sum of the full perturbation series given by (38). The convergence is rapid, monotonic, satisfies the condition (16), and the approximants $S_N$ do indeed form lower bounds. Also, the convergence is so fast that, except for a small interval at intermediate coupling, $S_5$ is not only a lower bound, but also a very good approximation to $S_{pert}$.

Now, corresponding to the Borel function (37), one has from the definitions before Eq.(36),

$$S_{np}(\lambda) = \frac{\ln 5}{6\lambda} e^{-5/\lambda}$$

and

$$S_{exact}(\lambda) = \frac{1}{\lambda} \int_0^\infty \frac{e^{-z/\lambda} - e^{-5/\lambda}}{(1 + z)(5 - z)}.$$  (42)

In Fig.(7c), the curves for $S_{exact}$ and $S_5$ are plotted for a bigger range of $\lambda$. This shows that except for a small range of couplings, the resummed perturbative approximant $S_5$ lies above and deviates significantly from the exact result although, as seen above, $S_5$ does form a converging lower bound to the perturbative component $S_{pert}$ of the exact result. The difference is of course the non-perturbative piece $S_{np}$. In the same figure there is a plot of $(S_5 - S_{np})$ which, according to the definition (36) should approximate $S_{exact}$. Indeed the agreement is very good and, amusingly, it improves at large coupling: At $\lambda = 5$, $S_{exact} = 0.0533$ and $S_5 - S_{np} = 0.0457$, while at $\lambda = 10$, $S_{exact} = 0.0219$ and $S_5 - S_{np} = 0.0193$.

This toy model discussion illustrates concretely the arguments given in Ref.[1] for the free-energy density of thermal $SU(3)$ gauge theory. There it was found that the exact result, given by lattice data, differed significantly from the resummed perturbative result, and it was argued that the difference was caused by the Borel-nonsummability of the theory. Assuming a non-perturbative component of the form $A e^{-q/\lambda}$, the constants $A$ and $q$ were determined from the difference between $S_{exact}$ and $S_5$. A more detailed discussion of the results in [1] and their extension to full QCD and other non-Abelian gauge theories will be presented in [11].

This simple toy model exhibits the following curiosity. In Fig.(7d) the second order resummed perturbative result $S_2$ is plotted together with the exact expression $S_{exact}$. The curves agree very well over a wide range of couplings! This shows that sometimes a resummation of purely perturbative results can accidently give agreement with the exact value which contains both perturbative and non-perturbative components.
5 Lower And Upper Bounds

Most of the discussion so far has been for the approximants $S_N$ formed from the $p(N)$’s which are positions of global or local minima. However as mentioned near the end of Sect.(2), when local minima exist, so generically must local maxima. Those local maxima, located at $\bar{p}(N)$, form the series $\bar{S}(N)$ which provides additional information. In the $S(\lambda) = -\sin(\lambda \pi)$ example studied earlier, it turned out that though the $S_N$ and $\bar{S}_N$ approximants separately formed converging lower and upper bounds, the exact result was closer to the upper bounds. In the next subsection, another example, that of Sect.(3.4), is re-investigated to obtain upper bounds from the positions of its local maxima.

Then in the second subsection, another technique, that of using an auxiliary series, is introduced so that alternative bounds to a series may be obtained also from positions of global maxima. The reason why this alternative method is important is that, as explained in Sec.(2), a priori the sequence $\bar{S}_N$ does not seem to obey a condition like (14,15). Therefore in practical problems where the exact result is unknown, any conclusions drawn on the basis of $\bar{S}_N$ should be confirmed by other means, such as the auxiliary series method.

As explained in the Appendix, the bounds formed from the $S_N$ are generically monotonic functions of $\lambda$ and therefore do not approximate the exact result $S(\lambda)$ very well if the latter is not monotonic. In that case, the bounds formed from the local extrema, $\bar{S}_N$, and from the auxiliary series, are expected to give better approximations to $S$ (see Appendix).

5.1 Bounds from Local Maxima

Generally, for a short series which mainly has global and local minima solutions to (14), the number of values for which local maxima solutions exist will be even less. However for a series which is not strictly alternating, there will be more local extrema solutions (see the general discussion in Sec.(2)), and hence an opportunity to form a longer series $\bar{S}_N$. An example of this is provided by the toy-model of Sect.(3.4), with exact Borel transform

$$B(z) = \frac{1}{1 + z} + \frac{1}{1 + z^2}.$$  (43)

The perturbation series corresponding to (43) is
\[ \hat{S}_N = \sum_{0}^{N} f_n \lambda^n, \]  
(44)

with

\[ f_{2n+1} = -(2n + 1)! \]  
(45)

\[ f_{2n} = (1 + (-1)^n) (2n)! . \]  
(46)

From the arguments of Sect.(2), one expects local maxima to exist for \( N = 5, 6, 7 \). An explicit analysis confirms this and gives their location. Using the reference point \( \lambda_0 = 1 \), the solutions to (11) are: \( \bar{p}(5) = 0.3, \bar{p}(6) = 0.7, \bar{p}(7) = 1.5 \). The corresponding curves for \( S_N \) are shown in Fig.(8) and clearly form upper bounds to the exact result. At \( \lambda = 0.5 \) the values are \( \bar{S}_5 = 1.97, \bar{S}_6 = 1.80, \bar{S}_7 = 1.68 \), while the exact value is 1.521. Note the expected large curvature of the bounds.

### 5.2 Bounds from an Auxiliary Series

Given a series which has global minima solutions to (11), then by 'removing' the first nontrivial coefficient of that series one forms an auxiliary series which has global maxima solutions. In this way one can form alternative bounds to the sum of the original series complementing those obtained from \( S_N \).

To illustrate this concretely, re-consider the toy example Sec(3.1) with \( B(z) = \frac{1}{1+z} \). The original truncated series which leads to global minima and lower bounds is

\[ \hat{S}_N = \sum_{n=0}^{N} (-\lambda)^n n! . \]  
(47)

Define the auxiliary series

\[ \hat{S}_N' = \frac{\hat{S}_N - 1}{\lambda} = \sum_{n=0}^{\infty} (-1)^{n+1} (n + 1)! \lambda^n . \]  
(48)

Since the auxiliary series \( S_N' \) begins with a positive non-trivial coefficient, it will give maxima solutions to (11). The solutions at the reference point \( \lambda_0 = 1 \) are, \( p'(2) = 3.9, p'(3) = 8, p'(4) = 13.25, p'(5) = 20, p'(6) = 28 \). The curves for \( 1 + \lambda S_N'(\lambda) \) are plotted in Fig.(9). Indeed they are seen to form upper bounds to the exact result. However the convergence to the exact result is slower than that of the lower bounds \( S_N \) considered earlier.
For a second example, re-consider the example of Sec.(3.6) with $S(\lambda) = -\sin(\lambda \pi)$. The partial series is given by

$$
\hat{S}_N = \sum_{n=0}^{N} \frac{(\lambda \pi)^{2n+1}}{(2n+1)!} (-1)^{n+1}.
$$

(49)

Define the auxiliary series

$$
\hat{S}_N' = \frac{\hat{S}_N}{\lambda} = \sum_{n=0}^{N} \frac{(\lambda \pi)^{2n+1}}{\lambda (2n+1)!} (-1)^{n+1}.
$$

(50)

The auxiliary series begins with a positive non-trivial coefficient and so one will obtain maxima as solutions to the extremum condition. At the reference value $\lambda_0 = 1$, the solutions are: $p'(4) = 0.3$, $p'(5) = 0.575$, $p'(6) = 0.92$. The corresponding curves for $\lambda S'_N$ are shown in Fig.(10). Not only do they form converging upper bounds, but in this case they approximate the exact function very well compared to the approximants $S_N$ used earlier in Sect.(3.6).

A question now arises for practical problems where exact results are not known. Is it possible to determine whether the unknown exact result lies closer to the upper or lower bound? Empirically, it appears that the series $S_N$ converges to the exact result $S(\lambda)$ if $\partial^2 S(\lambda)/\partial \lambda^2$ is very small in magnitude in the entire range of interest, $0 < \lambda < \lambda_0$. Otherwise the exact result seems to be better approximated by the series $\bar{S}_N$ or by the auxiliary series method discussed above. An explanation of why the bounds $S_N$ have slowly varying first derivatives, and why the bounds $\bar{S}_N$ and those from the auxiliary series have faster varying first derivatives is given in the Appendix.

6 The Effective Action of QED in a Magnetic field

The first physics example is Schwinger’s effective Lagrangian (density) for QED in a uniform magnetic field $B$,

$$
L(\lambda) = -\frac{e_r^2 B^2}{8\pi^2} \int_{0}^{\infty} \frac{ds}{s^2} \left( \coth(s) - \frac{1}{s} - \frac{s}{3} \right) e^{-s/\sqrt{\lambda}}
$$

(51)

where $\lambda \equiv e_r^2 B^2 / m^4$, $e_r$ is the renormalized electron coupling, and $m$ the electron mass.
Define the dimensionless quantity

\[ S(\lambda) = 100 \int_0^\infty \frac{ds}{s^2} \left( \coth(s) - \frac{1}{s} - \frac{s}{3} \right) e^{-s/\sqrt{\lambda}} \]  

(52)

which is related in an obvious way to (51). An expansion of (52) is given by

\[ \hat{S}(\lambda) = 1600 \sum_{n=1}^\infty f_n \lambda^n \]  

(53)

with

\[ f_n = \frac{4^{n-1} B_{2n+2}}{2n (2n+1) (2n+2)} \]  

(54)

and where \( B_{2n} \) are the Bernoulli numbers which alternate in sign, and diverge factorially for large \( n \). Equation (53) is the Euler-Heisenberg [13] expansion which is equivalent to a sum of an infinite number of Feynman diagrams consisting of one closed fermion loop with an even number of external photons. The Euler-Heisenberg series diverges for large values of \( \lambda \) as shown in Fig.(11a). Consider now a resummation of the divergent series using (10-11). At the reference value \( \lambda = 10 \), the solutions to (11) are (global minima) \( p(2) = 0.77, p(3) = 1.4, p(6) = 2.25 \). No solutions were found for \( N = 3, 5, 7 \).

The exact result of Schwinger, given by (52), is plotted in Fig.(11b), together with the approximants \( S_2, S_4, S_6 \). As in the toy model of Sec.(3.3), the approximants form upper bounds although the \( p(N) \) are positions of global minima. See again the caution prior to Eq.(14) and the Appendix. The convergence of the approximants \( S_N \) to the exact expression is manifest. For example, at \( \lambda = 10 \), where the Euler-Heisenberg series is badly divergent, the Schwinger’s exact value is \( S(10) = -8.056 \) while the estimates are, \( S_2 = -5.9, S_4 = -6.9, S_6 = -7.3 \).

The Euler-Heisenberg series (53) is Borel summable [9], and therefore provides an example of a Borel-summable series in a theory (QED) which is generally considered Borel-nonsummable. However it should be noted that the Euler-Heisenberg series (53) represents a one-fermion-loop result whereas the ”renormalon” singularities [8] which signal Borel-nonsummability are expected to arise when higher-loop diagrams are considered. That is, Borel-nonsummability might manifest itself when multi-loop corrections to (51) are taken into account.
7 The $\phi^4_3$ Field Theory

The three-dimensional $O(N)$ symmetric $\phi^4$ field theory can be used as an effective theory (see [4, 5] and references therein) to describe the critical behaviour of many physical systems near a second-order phase transition. For this purpose, the renormalization group functions of this theory have been calculated to very high order. A detailed list of references can be found in [3, 14]. For example, the beta function for the polymer case, $N = 0$, is given by

$$\beta(\lambda) = -\lambda + \lambda^2 - 0.439815\lambda^3 + 0.389923\lambda^4 - 0.447316\lambda^5 + 0.633855\lambda^6 - 1.03493\lambda^7,$$

(55)

where $\lambda$ is a dimensionless coupling (usually denoted as $g$ or $\tilde{g}$ in the literature [5]). The objective is to find the non-trivial infrared fixed point, $\lambda^\star$, of the theory, which is given by the zero of the beta function,

$$\beta(\lambda^\star) = 0,$$  \hspace{1cm} (56)

$$\frac{\partial \beta}{\partial \lambda}|_{\lambda^\star} \equiv \omega > 0.$$  \hspace{1cm} (57)

The expression (55) is divergent for large $\lambda \sim 1$ where a nontrivial zero is expected. Using the resummation (10) with $S_N$ denoting approximants to the beta function, and choosing the reference point $\lambda_0 = 1$, one gets as solutions to Eq.(11) (minima), $p(2) = 1.3$, $p(3) = 3.2$, $p(4) = 5.6$, $p(5) = 8.6$, $p(6) = 12.25$, $p(7) = 16.5$. The curves are shown in Fig.(12a). They form rapidly converging lower bounds but intercept the $\lambda$-axis only at the origin, thus giving only a trivial zero to the beta function. The situation is similar to the $\sin(\pi \lambda)$ toy model studied earlier.

Now, since for $N = 3, 5, 7$ the minima are only local, one expects local maxima to exist also. Indeed the local maxima are located at, $\bar{p}(3) = 0.18$, $\bar{p}(5) = 0.21$, $\bar{p}(7) = 0.19$. The curves for $\bar{S}_N$ are shown in Fig(12b). Though these are local maxima, they obey the inequality $\bar{S}_{N+1} > \bar{S}_N$ and so appear to form lower bounds! Since the lower bounds due to $\bar{S}_N$ are higher than the lower bounds due to $S_N$ (Fig.(12a)), one may argue that those due to $\bar{S}_N$ provide more accurate information. The curves in Fig.(12b) do in fact have a non-trivial zero. For $N = 7$, the zero is near $\lambda = 1.425$. In this respect, the physical example here is different from the toy-model of Sec.(3.6): here both the approximants $S_N$ and $\bar{S}(N)$ provide lower bounds,
so it is simply a matter of choosing the highest lower bound to estimate the sum of the series.

Given the physical importance of this example, it is useful to perform further checks. So let us re-analyse the problem using the auxiliary series method of Sect.(5). That is, let \( S'_N(\lambda, p) = S_N(\lambda, p)/\lambda \). The auxiliary series will then obviously have maxima as solutions to \( (11) \). The solutions at the reference point \( \lambda_0 = 1 \) are \( p'(2) = 0.6, \ p'(3) = 0.875, \ p'(4) = 0.33, \ p'(5) = 0.4, \ p'(6) = 0.2525 \). (Actually, \( p'(5) \) is a point of inflexion). The curves for \( \lambda S'_N \) are shown in Fig.(12c). Again, though determined by positions of maxima, the curves appear to form rapidly converging lower bounds which have a non-trivial zero.

In order to compare with the results in the literature, more precise numbers are now quoted. Conjecturing that the curves in Fig.(12c) are indeed lower bounds to the exact result, the \( N = 6 \) curve, shown magnified near its nontrivial zero in Fig.(12d), gives an upper bound of 1.4193 to the non-trivial zero of the beta function. Re-optimizing the \( N = 6 \) equation \( (11) \) at \( \lambda_0 = 1.419 \) does not change the curve for \( \lambda S'_N \) significantly to modify that bound at the level of accuracy quoted. The slope of the beta function at the non-trivial zero can also be determined from the \( N = 6 \) curve in Fig.(12d). It is \( \omega = 0.7955 \). Since the curves appear to get steeper as \( N \) increases, this value of \( \omega \) is a lower bound. These values can now be compared with those of Ref.[5]: there it was found that \( \lambda^* = 1.413 \pm 0.006 \) and \( \omega = 0.812 \pm 0.016 \). The agreement with the bounds found here is excellent. (Comparison of results obtained by other methods and other authors may be found in [4]).

It is important to note the significant conceptual difference between the methodology used in this paper and that employed in Ref.[5]. In [4] the authors used a Borel-Leroy transformation with a variable parameter \( b \) but they used the conventional conformal map [1] with the value of \( p \) fixed at the precise location of the instanton singularity, \( p^* = 0.166246 \) (the value for \( N = 0 \)). The parameter \( b \), together with some other parameters introduced in [5] were used to check the convergence of the series and to estimate their errors. Here instead the usual Borel transform [2] is used but the conformal map has a single variational parameter \( p \) determined according to the condition \( (11) \). From the numbers quoted above, it is clear that the values \( p(N) \) used here are not the same as the value \( p^* \). Furthermore, in the approach of this paper, no assumption about the analyticity structure of the Borel transform is made except that in order for the resummed perturbation series to faithfully represent the physical quantity, the series should be Borel summable (which
I simply take to mean that there are no poles on the positive Borel axis, and the Borel integral converges at its upper limit. Borel summability of the $\phi^4_3$ theory has been established in [15].

From this example it is clear that the novel resummation presented here, with the parameter $p$ determined from (11), can be used to complement the analysis done in [5].

8 Hot Quantum Electrodynamics

The fine structure constant $\alpha$ of QED is so small ($\sim 1/137$) that in practice the perturbation series converges even without any resummation. However, renormalization group arguments indicate that $\alpha$ will increase at high energies. Since the growth is only logarithmic, the energy scale must be exponentially high before the perturbation series fails to converge.

The free-energy density of QED at very high temperature ($T$) has been computed up to order $\alpha^{5/2}$ [16, 17]. The temperature was assumed to be high enough so that the electron mass could be neglected. Let $\lambda = (\alpha/\pi)^{1/2}$. Then the normalised free-energy density of QED at the $\overline{MS}$ renormalisation scale $\bar{\mu} = 2\pi T$, is given by [16, 17]

$$F/F_0 = 1 - 1.13636\lambda^2 + 2.09946\lambda^3 + 0.488875\lambda^4 - 6.34112\lambda^5.$$ (58)

where $F_0 = 11\pi^2T^4/180$ is the free-energy density of a non-interacting plasma. Figure (13a) shows the plot of (58) at different orders. At large coupling (super-high temperatures) the series diverges, exhibiting a behaviour similar to that of Yang-Mills theory at low-temperatures. The convergence at large coupling can be improved by using the resummation technique (10-11).

Using the coefficients from (58), the solutions of (14) at the reference value $\lambda_0 = 0.5$ are (minima): $p(3) = 0.7$, $p(4) = 1.75$, $p(5) = 3$.

The resummed approximants are shown in Fig.(8b). The convergence is clearly much better than in Fig.(8a), suggesting that the curves not only form lower bounds but also good estimates to the perturbative free-energy density. The emphasis on 'perturbative' is because QED, like QCD, is probably Borel non-summable, and so the large coupling perturbative results, even when resummed, might differ significantly from the exact result by non-perturbative contributions of the form $e^{-q/\lambda}/\lambda$.

If one assumes that the potential non-perturbative contributions lower the perturbative result, as happens in QCD (see [1] and references therein)
and in the toy model of the last section, or are negligible, then the conclusion would be that the free-energy density of QED decreases significantly at super high temperatures, suggesting a phase transition [16]. This high-temperature phase of QED might then be analogous to the low-temperature phase of QCD: one might have bound states of the electrons, positrons and photons. Or, the high-temperature phase might be due to the formation of other structures, such as magnetic strings [18]. Of course at the moment all this is speculative as one knows neither the sign nor magnitude of the non-perturbative corrections.

9 Conclusion

A method has been developed which enables one to obtain bounds on the full perturbation expression, $S(\lambda)$, of a physical quantity eventhough the only information available is its partial perturbation series

$$\hat{S}_N(\lambda) = \sum_{n=0}^{N} f_n \lambda^n. \quad (59)$$

The first requirement is that not all the $f_n$ be of the same sign. Then solutions exist to the extremum condition (11). If the first nontrivial coefficient $f_n(n > 0)$ is negative, then the solutions to (11) will be global or local minima, depending on the sign of $f_N$. The approximants $S_N$ as defined through (13) then form lower bounds to $S(\lambda)$ if the inequality (16) is satisfied for all $N$ (and if the plausible assumptions leading to (15) are admitted). The inequality (16) must first be explicitly tested for the available terms of the series, then the arguments given in the Appendix indicate that the trend will continue. Therefore even with partial information as in (59), one can deduce lower bounds to the exact value $S(\lambda)$. Additional bounds may be obtained by using the auxiliary series method described in Sec.(5).

Sometimes one finds minima solutions to (11) but for which the $S_N$ obey an inequality opposite to (16). In such cases, even though the convergence is monotonic, it is not a priori obvious that the $S_N$ are actually upper bounds to the exact value. For problems where the exact result is unknown, additional input from theory or physics is required before a definitive statement can be made. However in all the examples encountered of this type, the $S_N$ did actually bound the exact result. A similar caveat concerns the approximants $\bar{S}$ defined near the end of Sec.(2).
The observed rapid convergence of the bounds has been explained in the Appendix. The obvious question is whether the bounds converge to the exact value itself. Empirically, it is found that the bounds formed by $S_N$ actually converge to the exact value if $S(\lambda)$ has a slowly varying first derivative, $\partial S(\lambda)/\partial \lambda$. Otherwise the complementary bounds formed by $\bar{S}(\lambda)$ or the auxiliary series method give better approximations to the exact result. The reason for this phenomena has been given in the Appendix.

Currently the most popular resummation methods for divergent series are the Pade’ or Borel-Pade’, see for example [10, 19]. Those methods usually do not converge monotonically, and so do not provide bounds, but nevertheless can sometimes be used to estimate the sum of a series. Those estimates can probably be constrained by using bounds obtained through the method of this paper.

Although most of the observed features of the resummation method developed here have been explained in the Appendix, at least in a semi-quantitative way, more patterns were detected than could be explained. In order to highlight some of these apparently universal trends, I summarize them as three questions: (i) Is it true that in all cases, as $N \to \infty$, $c(N) \equiv p(N+1)/p(N) \to 1$ (Similarly for the $\bar{p}(N)$) ? (ii) Is it true that in all cases the approximants $S_N$ and $\bar{S}_N$ form bounds to the full perturbative result ? (iii) Is it true that the bounds $S_N$ always approximate the exact result $S$ well if $|\frac{\partial^2 S}{\partial \lambda^2}/\partial S|$ is small throughout the range, $0 < \lambda < \lambda_0$, of interest, and otherwise the alternative bounds $\bar{S}_N$ or those from the auxiliary series give better approximations ?

The technique itself can be improved in several ways. For example, if the large order behaviour of $f_n$ is $\sim (2n)!$, instead of $n!$, then a generalised Borel transform can be used to take advantage of that fact. Alas, in order to keep this paper itself from growing out of bound, some of these refinements and further physical applications have to be discussed elsewhere [11].
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Appendix

A1

The resummed perturbation series with a variable parameter $p$ is given by

$$S_N(\lambda, p) \equiv \sum_{n=0}^{N} \frac{f_n}{n!} \left(\frac{4}{p}\right)^n \sum_{k=0}^{N-n} \frac{(2n+k-1)!}{k!(2n-1)!} \int_0^\infty dz e^{-z} w(z\lambda)^{(k+n)}. \quad (60)$$

Write the equation above in the compact form

$$S_N(\lambda, p) = \sum_{n=0}^{N} \frac{A_n(N)}{p^n}, \quad (61)$$

where $A_n(N)$ is a $p$ and $\lambda$ dependent constant that can be read off from (60). In particular note that the sign of $A_n(N)$ is the same as the sign of $f_n$. The extremum condition (11) applied to (61) results in

$$\lambda \frac{\partial S_N}{\partial \lambda}|_{\lambda_0} = \sum_{n=1}^{N} \frac{nA_n(N)}{p(N)^n}, \quad (62)$$

where use has been made of the form $w(z\lambda)$ in transforming the $p$ derivative of $A_n(N)$ into a $\lambda$ derivative. At the solution $p = p(N)$ of (11), one has

$$S_N \equiv \sum_{n=0}^{N} \frac{A_n(N)}{p(N)^n} \quad (63)$$

Since for a given $N$ there is in general more than one solution to the extremum equation (11), here $p(N+1)$ and $p(N)$ will refer to solutions at
consecutive orders which are both positions of minima or both positions of maxima. Now define

\[ p(N+1) = c(N) p(N) \] (64)

where \( c(N) \) is some function of \( N \). Then from equation (62) and the corresponding one at order \( N+1 \), one easily deduces

\[
λ \frac{∂(S_{N+1} - S_N)}{∂λ}|_{λ₀} = \sum_{n=1}^{N} \frac{n}{p(N)^n} \left( \frac{A_n(N + 1)}{c(N)^n} - A_n(N) \right) + \frac{(N + 1)A_{N+1}(N + 1)}{c(N)^{N+1}p(N)^{N+1}}.
\] (65)

Now for large \( N \), \( A_n(N) \sim A_n(N+1) \), and write this simply as \( A_n \). Also define, \( \Delta S_N ≡ S_{N+1} - S_N \). Assume now that \( p(N) \) is large and \( c(N) > 1 \). Also assume that all the coefficients \( A_n \) are generically of the same order, or at least do not increase rapidly with \( n \) (the factorial growth of \( f_n \) has already been taken care of by the Borel transform). Then keeping terms needed to solve for \( 1/c(N) \) at leading order, (65) simplifies to

\[
λ \frac{∂ΔS_N}{∂λ}|_{λ₀} = \frac{A_1}{p(N)} \left( \frac{1}{c(N)} - 1 \right) - \frac{2A_2}{p(N)^2}.
\] (66)

Similarly, from eq.(63) and the corresponding one at order \( N+1 \) one deduces at large \( N \) and to leading order in \( 1/p(N) \),

\[
ΔS_N = \frac{A_1}{p(N)} \left( \frac{1}{c(N)} - 1 \right),
\] (67)

where it is implicit in this equation that \( λ = λ₀ \). Now, if \( λ₀ \) is varied, then the leading change in (67) comes from \( A_1 \). Thus using (67), Eq.(66) can be simplified to

\[
\frac{α}{p(N)} \left( \frac{1}{c(N)} - 1 \right) \approx \frac{2A_2}{p(N)^2},
\] (68)

where \( α = (A_1 - λ \frac{∂A_1}{∂λ})|_{λ₀} \). Thus,

\[
\frac{1}{c(N)} \approx 1 + \frac{2A_2}{αp(N)}.
\] (69)

This solution will be self-consistent with the initial assumption \( c(N) > 1 \) only if \( A_2/α \) is negative, which then requires that that \( f_2 \) and \( f_1 \) should be of opposite signs (since \( A_1 \) is larger than \( \frac{∂A_1}{∂λ} \)). Note that for \( N = 2 \) the solution
\( p(2) \) exists in the first place only if \( f_2 \) and \( f_1 \) are of opposite signs. Hence one concludes that \( c(2) > 1 \) is generically expected. Comparing (69) with the corresponding equation at the next order \( N + 1 \) one obtains at large \( N \) the recursive relation
\[
\frac{1}{c(N + 1)} = 1 - \frac{1}{c(N)} + \frac{1}{c(N)^2},
\]
which can also be written as
\[
\frac{1}{c(N + 1)} - \frac{1}{c(N)} = \left(1 - \frac{1}{c(N)}\right)^2
\]
showing that
\[
c(N + 1) < c(N).
\]
Indeed the large \( N \) solution of (70) is
\[
\frac{1}{c(N)} \approx 1 + \frac{1}{N + K},
\]
with \( K \) a constant. Therefore \( c(N \to \infty) \to 1^+ \), that is, though \( p(N) \) will increase with \( N \), it will approach a constant as \( N \to \infty \). Now, since \( p(N) \) increases with \( N \), this means that the various approximations that led from (60) to (70,72) will become increasingly accurate. What is remarkable is that in the examples studied with \( c(N) > 1 \), (72) is already satisfied at low \( N \) and furthermore the relation (70) too is a reasonable approximation.

The above relations (70,72) were obtained under the assumption \( c(N) > 1 \). If \( c(N) < 1 \) then there are apparently no simple relations. For the example in Sec.(3.5), one had \( c(N) < 1 \) for all \( N \). For the auxiliary series of the beta function in Sec.(7), \( c(N) \) alternated between being slightly larger than one and much smaller than one.

Consider now the Eq.(67) which is valid at large \( N \) and large \( p(N) \),
\[
\Delta S_N = \frac{A_1}{p(N)} \left( \frac{1}{c(N)} - 1 \right),
\]
Remarkably, this simple equation summarizes most of the observed trends. When \( c(N) > 1 \), (74) implies that for \( A_1 < 0 \), which corresponds to \( f_1 < 0 \) and minima solutions to (11), \( \Delta S_N > 0 \), which is indeed observed in the
examples studied and leads to lower bounds. Similarly if $A_1 > 0$, which corresponds to $f_1 > 0$ and maxima solutions, $\Delta S_N < 0$, implying upper bounds. The only exception observed so far is the Euler-Heisenberg series in Sec.(6) which had $c(N) > 1$, $A_1 < 0$, and yet gave upper bounds. Presumably for that example the approximation (74) is not appropriate. Now suppose that $c(N) < 1$, as happens in the model of Sec.(3.5) and (roughly) for the auxiliary series in Sec.(7). Then Eq.(74) implies that minima, corresponding to $f_1 < 0$ and hence $A_1 < 0$, give $\Delta S < 0$, which explains some of the oddities observed.

The equation (74) also explains the rapid convergence of the bounds. Indeed one sees that convergence can be achieved simply by one of two ways. Firstly, if $c(N) \to 1$ as $N \to \infty$, that is $p(N) \to p_0$. This type of behaviour, is manifested, for example, by the toy-model in Sec.(3.5). The second way is for $c(N) > 1$ for all $N$, which leads to $p(N)$ increasing with $N$. This is what has been generically observed. Actually, as shown above, the second behaviour also gives $c(N) \to 1^+$, and hence the convergence is doubly fast. Explicitly, from Eqns.(69,73,74) one deduces for the $c(N) > 1$ case,

$$\Delta S \approx \frac{\alpha A_1}{2A_2} \left( \frac{1}{c(N)} - 1 \right)^2$$

(75)

$$= \frac{\alpha A_1}{2A_2} \frac{1}{(N + K)^2} .$$

(76)

In summary, for practical problems, the strongest statements can be made for the case when $c(N) > 1$ is observed for the given terms of a partial perturbative series, that is when $p(N)$ increases with $N$. Then the various approximations leading to the above equations become increasingly accurate at large $N$, allowing one to make assertions about all $N$. Firstly, one deduces $c(N \to \infty) \to 1^+$. Secondly, for $S_N$ which are global (and local) minima, if $S_N < S_{N+1}$ for the given terms of the series, the trend will continue for larger $N$, the convergence of the $S_N$ will be rapid, and they will form lower bounds to the exact result. However if $S_N > S_{N+1}$ is observed for the global (and local) minima, (as in Sec.(6)), then though the trend will continue and though the convergence of the $S_N$ will be rapid, it is not a priori obvious that they will be upper bounds to the exact result, because then key pieces (14-15) of the argument are missing. (The same loophole occurs for the approximants $\bar{S}_N$ formed from the $\bar{p}(N)$’s.)

If it is observed that $c(N) < 1$ for the given terms of a series, then the various equations above are not necessarily accurate at higher $N$. In that
case one can only conjecture that the observed monotonic and rapid conver-
gence will continue at higher orders.

A2

Consider now the slope of the approximants $S_N(\lambda)$. From (60), as $\lambda \to 0$, $S_N(\lambda) \to f_0 + f_1 \lambda$, so that

$$\left. \frac{\partial S_N}{\partial \lambda} \right|_{\lambda=0} = f_1 . \tag{77}$$

Thus all the bounds approach the origin with the same value $(f_0)$ and slope $(f_1)$ independent of $p$ and $N$. This fact can be seen in all the figures. Consider next Eq. (62) for large $p(N)$,

$$\left. \frac{\partial S_N}{\partial \lambda} \right|_{\lambda=0} \sim \frac{1}{\lambda_0} \frac{A_1}{p(N)} . \tag{78}$$

Since the sign of $A_1$ is the same as the sign of $f_1$, this shows that the curves $S_N(\lambda)$ are not expected to change direction as $\lambda$ varies. This is indeed observed, and explains why the bounds $S_N(\lambda)$ to $S(\lambda)$ are also good estimates of $S(\lambda)$ itself only when the latter has a slowly varying first derivative.

As was observed in the main text, the approximants $\bar{S}(\lambda)$ formed from the local extrema $\bar{p}(N)$ had a more varying slope. In order to understand this, set for simplicity $f_0 = 0$ so that $\bar{S}_N(0) = 0$ and let us demand that

$$\bar{S}_N(\lambda_0) = 0 , \tag{79}$$

so that $\bar{S}_N(\lambda)$ curves back to its value at the origin, and so can better approximate functions like $\sin(\lambda \pi)$. The condition (79) then leads from (61) to

$$0 = \sum_{n=1}^{N} \frac{A_n}{\bar{p}(N)^n} . \tag{80}$$

Note that since the $\bar{p}(N)$ also have to satisfy the extremum condition (11), eqns. (11,80) are actually two coupled equations for $\lambda_0$ and $\bar{p}(N)$ which we would like to analyse for consistency. Firstly, (80) can be used to eliminate the leading term in the 'barred' version of (62), so that now

$$\lambda \left. \frac{\partial S_N}{\partial \lambda} \right|_{\lambda=0} = \sum_{n=2}^{N} \frac{(n-1)A_n}{p(N)^n} \tag{81}$$
which at large $\bar{p}(N)$ gives the slope
\begin{equation}
\frac{\partial \bar{S}_N}{\partial \lambda} \bigg|_{\lambda_0} \sim \frac{1}{\lambda_0 p(N)^2} A_2.
\end{equation}

If $f_2$ is opposite in sign to $f_1$ then by comparing (82) with (77) one sees that indeed the approximants $\bar{S}$ can change direction. Of course this just shows that the approximate large $\bar{p}(N)$ analysis above is self-consistent. Now, in Sec.(2), for $f_1 < 0$, $f_2 > 0$, the $\bar{p}(N)$ have been defined as positions of local maxima when the $p(N)$ are positions of local minima. We can compare the relative magnitudes of the two values as follows. For large $\bar{p}(N)$, (83) has the approximate solution
\begin{equation}
\bar{p}(N) \sim \frac{-A_2}{A_1}.
\end{equation}

By contrast the $p(N)$ are approximate solutions of (62) with the left-hand-side deleted, which is equivalent to ignoring the mild $p$ dependence of the $A_n$’s,
\begin{equation}
p(N) \sim -\frac{2A_2}{A_1}.
\end{equation}

Thus the values of $\bar{p}(N)$ are expected to be smaller than those of $p(N)$, and the interested reader may verify from the given examples that this is indeed the case. Reversing the logic of the argument above, one concludes as follows. For that $N$ when $p(N)$ is the position of a local minimum, one expects a local maximum at $\bar{p}(N)$. While the $S_N$ curve is expected to be monotonic in $\lambda$, that of $\bar{S}(\lambda)$ will not be monotonic if the value of $\bar{p}(N)$ is smaller than that of $p(N)$.

A similar discussion can be carried out for the curves formed from an auxiliary series $S'_N$. Again set for simplicity $f_0 = 0$, and define
\begin{equation}
\hat{S}'_N \equiv \frac{\hat{S}'_N}{\lambda}.
\end{equation}

The extremization (11) in $p$ is done with respect to the resummed auxiliary series $S'_N$, and one deduces for large $p'(N)$, from an equation analogous to (62), that
\begin{equation}
\text{sign} \left( \lambda \frac{\partial S'_N}{\partial \lambda} \right)_{\lambda_0} = \text{sign}(f_2).
\end{equation}

Now demanding
\begin{equation}
S_N(\lambda_0) = 0,
\end{equation}

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for the reconstructed resummed series $S_N \equiv \lambda S'_N$ gives,

$$\frac{\partial S_N}{\partial \lambda}|_{\lambda_0} = \lambda_0 \left( \frac{\partial S'_N}{\partial \lambda} \right)_{\lambda_0},$$

which when combined with (86) shows that for $f_1$ and $f_2$ of opposite signs, the slope of the reconstructed $S_N$ at $\lambda = \lambda_0$ is opposite in sign to its slope at the origin which is given by (77).

Thus if $f_1$ and $f_2$ are of opposite signs, then the auxiliary series may be expected to give a reconstructed $S_N$ with a slope that has variation in sign, compared with the slope of the approximant $S_N$ which is obtained by direct means, if the two conditions (11) and (87) for $S'$ have a consistent solution $p' (N)$ for some $\lambda_0$.

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Figure Captions

Figure (1a): Plots of the divergent series $\hat{S}_N(\lambda)$ for the model in Sec.(3.1), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $\hat{S}_5$, $\hat{S}_3$, $S$, $\hat{S}_2$, $\hat{S}_4$.

Figure (1b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.1), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_2$, $S_3$, $S_4$, $S$.

Figure (1c): Plots of the resummed series $S_{(0)}(\lambda)$ for the model in Sec.(3.1), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_4$, $S$, $S_5$, $S_3$. Compared to Fig.(1b), the convergence is faster but not monotonic.

Figure (2a): Plots of the divergent series $\hat{S}_N(\lambda)$ for the model in Sec.(3.2), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $\hat{S}_6$, $\hat{S}_2$, $S$, $\hat{S}_4$.

Figure (2b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.2), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_4$, $S_5$, $S_6$, $S_7$, $S_8$, $S$.

Figure (3a): Plots of the series $\hat{S}_N(\lambda)$ for the model in Sec.(3.3), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $\hat{S}_3$, $S$, $\hat{S}_4$, $\hat{S}_2$.

Figure (3b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.3), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_2$, $S_3$, $S_4$, $S$.

Figure (4a): Plots of the divergent series $\hat{S}_N(\lambda)$ for the model in Sec.(3.4), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $\hat{S}_3$, $\hat{S}_1$, $S$, $\hat{S}_4$.

Figure (4b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.4), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_4$, $S_5$, $S_6$, $S_7$, $S$. 
Figure (5a): Plots of the divergent series $\hat{S}_N(\lambda)$ for the model in Sec.(3.5), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $\hat{S}_3$, $S$, $\hat{S}_2$, $\hat{S}_4$.

Figure (5b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.5), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S$, $S_6$, $S_4$, $S_2$. The approximants approach the exact result from above. (The curves for $S_6$ and $S$ are indistinguishable).

Figure (6a): Plots of the series $\hat{S}_N(\lambda)$ for the model in Sec.(3.6). Starting from the lowest curve and moving upwards, one has $\hat{S}_1$, $\hat{S}_5$, $\hat{S}_9$, $\hat{S}_7$, $\hat{S}_3$.

Figure (6b): Plots of the resummed series $S_N(\lambda)$ for the model in Sec.(3.6), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S$, $\bar{S}_6$, $\bar{S}_5$, $S_6$, $S_4$, $S_2$.

Figure (6c): Plots of the series $S_N(\lambda)$ for the model in Sec.(3.6). Starting from the lowest curve and moving upwards, one has $S$, $\bar{S}_6$, $\bar{S}_5$. The approximants approach the exact result from above.

Figure (7a): Plots of the divergent perturbation series $\hat{S}_N(\lambda)$ for the model in Sec.(4). Starting from the lowest curve and moving upwards, one has $\hat{S}_5$, $\hat{S}_3$, $\hat{S}_2$, $\hat{S}_4$.

Figure (7b): Plots of the resummed perturbation series $S_N(\lambda)$ for the model in Sec.(4), together with the full perturbative result $S_{\text{pert}}(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_2$, $S_3$, $S_4$, $S_5$, $S_{\text{pert}}$.

Figure (7c): Plots of $S_{\text{exact}}(\lambda)$, $S_5(\lambda)$ and $S_5(\lambda) - S_{\text{np}}(\lambda)$ as defined in Sec.(4). At $\lambda = 10$, the lowest curve is of $S_5 - S_{\text{np}}$, and the highest one is $S_5$.

Figure (7d): Plots of the resummed perturbative result $S_2(\lambda)$ and the exact (sum of perturbative and non-perturbative) value $S_{\text{exact}}(\lambda)$ as defined in Sec.(4). At $\lambda = 2$, the lower curve is of $S_2$.

Figure (8): Plots of the resummed series $\bar{S}_N(\lambda)$ for the model in Sec.(5.1), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S$, $\bar{S}_7$, $\bar{S}_6$, $\bar{S}_5$. The approximants approach the exact
result from above.

Figure (9): Plots of the resummed series $S_N(\lambda)$ corresponding to the auxiliary series of (43) in Sec.(5.2), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S$, $S_6$, $S_5$, $S_4$, $S_3$, $S_2$. The approximants approach the exact result from above.

Figure (10): Plots of the resummed series $S_N(\lambda)$ corresponding to the auxiliary series of (44) in Sec.(5.2), together with the exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S$, $S_6$, $S_5$, $S_4$. The approximants approach the exact result from above.

Figure (11a): Plots of the divergent Euler-Heisenberg series $\hat{S}_N(\lambda)$ given in Sec.(6). Starting from the lowest curve and moving upwards, one has $\hat{S}_3$, $\hat{S}_2$, $\hat{S}_4$.

Figure (11b): Plots of the resummed Euler-Heisenberg series $S_N(\lambda)$, together with Schwinger’s exact result $S(\lambda)$. Starting from the lowest curve and moving upwards, one has $S$, $S_6$, $S_4$, $S_2$. The approximants approach the exact result from above.

Figure (12a): Plots of the resummed beta function $S_N(\lambda)$ of Sec.(7). Starting from the lowest curve and moving upwards, one has $S_7$, $S_6$, $S_5$, $S_4$, $S_3$, $S_2$.

Figure (12b): Plots of the resummed series $\bar{S}_N(\lambda)$ for beta function in Sec.(7). Starting from the lowest curve and moving upwards, one has $\bar{S}_3$, $\bar{S}_5$, $\bar{S}_7$. The approximants appear to form upper bounds.

Figure (12c): Plots of the resummed beta function $S_N(\lambda)$ of Sec.(7), obtained through the auxiliary series. Starting from the lowest curve and moving upwards, one has $S_2$, $S_3$, $S_4$, $S_5$, $S_6$. The approximants appear to form upper bounds. The curves for $N = 2$ and $N = 3$ are indistinguishable, and similarly, those for $N = 4$ and $N = 5$ are very close.

Figure (12d): Magnification of the $N = 6$ curve of Fig.(12c) near its non-trivial zero.

Figure (13a): Plots of the divergent perturbative free-energy density of
QED, $\hat{S}_N(\lambda)$ given in Sec.(8). Starting from the lowest curve and moving upwards, one has $\hat{S}_2, \hat{S}_5, \hat{S}_3, \hat{S}_4$.

Figure (13b): Plots of the resummed perturbative free-energy density of QED, $S_N(\lambda)$. Starting from the lowest curve and moving upwards, one has $S_3, S_4, S_5$. 
Fig. 13a

Fig. 13b