Students’ Factual Understanding on the concept of Limit

N Arsyad¹, Y. Ramadhana¹, S. F. Assagaf¹
Mathematics Education Department, Universitas Negeri Makassar, Indonesia

*Email: said.fachry.assagaf@unm.ac.id

Abstract. This research aims to describe the factual understanding of the concept of the formal definition of limit. This is a descriptive qualitative research with 8 students. The data gathered by using test and interview. The instruments is a conceptual understanding test. We divide the factual understanding into three main concepts: (1) students’ understanding on the symbol of $\varepsilon$ and $\delta$ in the definition, (2) students’ understanding on the meaning of the absolute value in the definition, and (3) students’ understanding on the implication statement in the definition. Each elaborate to some categories to get the ideas on how students understand about the concept. The results reveals that most students have incomplete understanding about the definition of limit function.

Keywords: factual understanding, limit, absolute value, calculus.

1. Introduction

Learning mathematics is not only related to the skills in calculating and memorizing as many mathematical formulas as possible, but also understanding the concept. It is one of the goals in learning mathematics. It also help students not only memorize formulas, but can correctly understand the meaning of mathematics ideas [1]. Learning mathematics with conceptual understanding requires a higher reasoning skill due to the abstraction of the mathematical objects. Therefore, learning mathematics should be directed at understanding concepts that will lead individuals to think mathematically based on logical and systematical rules rather than working on formulas and calculate the result.

One of the important concepts in mathematics, especially in the university is a calculus. Various mathematical problems can be solved by calculus such as optimization, modeling, and problems related to physics. Juter [2] suggests that the concept of limit functions is the most important part of calculus. He also stated that how could a student understand the concept of derivatives and integrals if the student did not understand the concept of limits. The concept of limit functions is a crucial concept and a prerequisite concept in learning various other mathematical concepts [3][4]. [5] also stated that without limits, calculus could not exist. In addition, [6] states that without the concept of limit functions, the most important branch of mathematics called analysis would also be not exist. This is in line with the opinion of [7] that the concept of limit is a basic concept for calculus and mathematical analysis. This illustrates the importance of the limit function concept. Therefore, a good understanding of the concept of limit function is needed.
However, there are still some students who do not really understand the concept of the limit function [8], especially the formal definition of function limit [9]. [10] argues that there are still students who have difficulty understanding the concept of limit function, especially in the formal definition of function limit so that their understanding of the concept is still lacking. In addition, there are also students who have not been able to apply the formal definition of function limit to prove the truth of the limit function value [11].

This research will examine students' understanding of the concept of limit function. This study aims to describe students' understanding of the concept of limit function at one point. The focus is the factual understanding of the concept. Factual understanding in this article refers to the basic elements that students must know to understand a concept [12]. In this study, factual understanding is an understanding of the prerequisite topic for the concept of limit functions, the formal definition of limit. The prerequisite material in the formal definition of the limit is absolute value and mathematical logic. The understanding of the prerequisite materials is divided into several parts, namely: (1) understanding of the meanings of ε and δ; (2) understanding the meaning of the absolute value $0 < |xc| < \delta$ and $|f(x) - L| < \varepsilon$; (3) understanding the meaning of the implication statement $0 < |xc| < \delta \implies |f(x) - L| < \varepsilon$.

2. Method
This research is a qualitative research with a descriptive approach. There were 8 subjects in this study who were students majoring in Mathematics in semester 5. Subjects were selected based on the subject's willingness to participate in the collection and the subject's ability to communicate his thoughts. The research instrument was conceptual understanding test to identify the subject's understanding related to the concept of limit function. The instrument validated by 2 experts. The subjects then interviewed about their answer in the test. Interview were being transcribed and analyzed to get the students' factual understanding on limit function. The factual understanding consisted of three concepts: (1) understanding the meaning of ε and δ, (2) understanding the meaning of $0 < |x - c| < \delta$ and $|f(x) - L| < \varepsilon$, and (3) the meaning of the statement $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$. We break out every concepts into several categories to draw the conclusion on student’s factual understanding of the limit function.

3. Result and Discussion
We divide the factual understanding into three main concepts: (1) students’ understanding on the symbol of ε and δ in the definition, (2) students’ understanding on the meaning of the absolute value in the definition, and (3) students’ understanding on the implication statement in the definition. Each elaborate to some categories to get the ideas on how students understand about the concept.

There are 5 categories of understanding related to the meanings of ε and δ as shown in Table 1. In the first category (F.1.1), the subject did not express his understanding either regarding the meaning of ε or δ. In the second category (F.1.2), the subject described the meaning of ε and δ by relating to the meaning of the quantifier sentence in the definition of the limit function. The subject understood that ε is any positive real number. This is because in the definition of the limit function, it is preceded by the word "for every $\varepsilon > 0". Meanwhile, δ is a positive real number which corresponds to the given ε.

In the third category (F.1.3), the subject did not only describe these symbols based on the quantifier sentence, but also by relating to the intuitive meaning of limit function. In dealing with the intuitive meaning of limit function, the subject described that the value of ε chosen is usually a small value. In the fourth category (F.1.4), the subject also described the meaning of each symbol by linking them to the meaning of the quantifier. In addition, the subject can also describe each of these symbols by relating them to the concept of distance. In describing the meaning of each symbol as distance, the subject expressed them as the distance on the coordinate axes as stated in [13]. The subject argued that $\varepsilon$ refers to the distance between $f(x)$ and $L$ on the y-axis and $\delta$ refers to the distance between $x$ and $c$ on the x-axis. The subject understanding in the fifth category (F.1.5) was the most complete category compared to the previous ones. In this category, the subject described the meaning of each symbol by linking
them to the meaning of quantifier sentence. In addition, the subject also explained them by relating to the intuitive meaning of the limit function and can express them as a distance on each coordinate axis.

Table 1. Understanding the meaning of $\varepsilon$ and $\delta$

| No   | Categories                                                                 | Subjects |
|------|---------------------------------------------------------------------------|----------|
|      | Do not state the meaning of $\varepsilon$ and $\delta$, and their relation in quantifier sentence. | ✓        |
| F.1.1| Describe the meaning of the symbols by relating with the quantifier sentence in the definition. | ✓ ✓ ✓    |
| F.1.2| Describe the meaning of each symbols by relating them with quantifier sentence and intuitive meaning of limit function. | ✓        |
| F.1.3| Describe the meaning of the symbols in quantifier sentence and relate them with the distance concept. | ✓ ✓      |
| F.1.4| Describe the meaning of the symbols in quantifier sentence, relate them with intuitive meaning of limit and use the distance concept in explain the meaning of the symbols. | ✓        |

The subject's understanding of absolute value consists of 5 categorises as shown in Table 2. In the first category (F.2.1), the subject did not express his understanding regarding the meaning of absolute value, neither $0 < |x - c| < \delta$ nor $|f(x) - L| < \varepsilon$. In the second category (F.2.2), the subject described the meaning of each absolute value based only on how the absolute value is written. The subject described the absolute value of the difference between $x$ and $c$ is always less than $\delta$ and the absolute value of the difference between $f(x)$ and $L$ is always less than $\varepsilon$. The subject described them without associating with other mathematical objects. In the third category (F.2.3), the subject explain the meaning of the absolute value based on the intuitive meaning of limit function. The subject understood that $x$ is never the same as $c$. This is because in the intuitive meaning of the limit function, it talks about the points that are close to $c$ and not at the point $c$ so that $x$ is never the same as $c$. The subject also understood that $f(x)$ minus $L$ can be equal to zero or in other word $f(x) = L$.

In the fourth category (F.2.4), the subject described the meaning of absolute value based on the meaning of distance. The subject described that the distance between $x$ and $c$ is more than zero and less than $\delta$, so $x$ cannot be equal to $c$ and the distance between $f(x)$ and $L$ is less than $\varepsilon$. In the fifth category (F.2.5), the understanding of the subject includes the two previous categories. In addition, subjects explained the meaning of absolute value based on the meaning of the difference by stating that the differences of $x$ and $c$ are more than zero and less than $\delta$ and the differences of $f(x)$ and $L$ are less than $\varepsilon$.

The subject was basically describing the meaning of the absolute value $0 < |x - c| < \delta$ and $|f(x) - L| < \varepsilon$ by associating them with the distance, the difference and relating them to the intuitive meaning of limit function. In addition, there were also subjects who described them based on how it is
written. The description of the absolute value based on the meaning of distance is basically the same as that stated in [14]. It explains that \( |x - c| \) means the distance between \( x \) and \( a \).

| Table 2. Understand the meaning of \( 0 < |x - c| < \delta \) and \( |f(x) - L| < \varepsilon \) |
| --- |
| No | Categories | Subject |
| --- | --- | --- |
| F.2.1 | Do not understand the meaning of \( 0 < |x - c| < \delta \) and \( |f(x) - L| < \varepsilon \) | S11, S16, S17, S18 |
| F.2.2 | Describe the meaning of every absolute values based on the written symbol. | ✓ |
| F.2.3 | Describe the absolute value based on the intuitive meaning of limit function. | ✓ |
| F.2.4 | Describe the absolute value based on the distance concept. | ✓ |
| F.2.5 | Describe the absolute value based on the distance, the difference, and the intuitive meaning of limit function. | ✓, ✓ |

The understanding of the implication statement consists of 4 categories as shown in Table 3. In the first category (F.3.1), the subject did not express his understanding regarding the meaning of the implication statement. In the second category (F.3.2), the subject understood the statement as an implication (cause and effect). The subject described that if \( 0 < |x - c| < \delta \) then \( |f(x) - L| < \varepsilon \). The subject did not say further about the implication statement. In the third category (F.3.3), the subject described the statement not only based on what is written but also emphasized the neighbourhood points around the \( c \) with certain conditions. The condition are that the difference with \( c \) is less than \( \delta \) and more than zero. This condition results in the value of the function from that point having a difference with \( L \) which less than \( \varepsilon \) as a consequence in the implication statement. The thought process is included in the process domain suggested by [15].

| Table 3. Understanding the sentence \( 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \) |
| --- |
| No | Categories | Subject |
| --- | --- | --- |
| F.3.1 | Do not understand the meaning of \( 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \) | S11, S16, S17, S18 |
| F.3.2 | Describe the statement \( 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \) based on how it is written as an implication statement. | ✓, ✓, ✓ |
| F.3.3 | Describe the statement by focusing in the points around a certain point \( c \) with some conditions. | ✓ |
| F.3.4 | Describe the statement by relating it with \( \varepsilon \) and \( \delta \). | ✓ |
4. Conclusion

This article describes students' factual understanding on the definition of limit function. (1) understanding the meaning of $\varepsilon$ and $\delta$, (2) understanding the meaning of $0 < |x - c| < \delta$ and $|f(x) - L| < \varepsilon$, and (3) the meaning of the statement $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$. We categorized every concept into several categories to draw the conclusion on students' factual understanding of the limit function. In factual understanding, it is obtained that the understanding varies in each section. Most students have incomplete understanding about the definition of limit function. Therefore, some further research may focus on how to develop students’ understanding on limit function. Some may also work on the other understanding such as how students’ understanding on proofing the limit value by using the limit definition. These all studies may go to develop students understanding on the formal definition on limit function.

References

[1] Pitaloka, Y., Susilo, B., & Mulyono, M. (2013). Keefektifan Model Pembelajaran Matematika Realistik Indonesia terhadap Kemampuan Pemahaman Konsep Matematika. Unnes Journal of Mathematics Education, 1(2)

[2] Juter, K. (2005). Limits of Functions – how do students handle them. Phytogoras, 11-20.

[3] Karatas, I., Guven, B., & Cekmez, E. (2011). A Cross-Age Study of Students’ Understanding of Limit and Continuity Concepts. Bolema, Rion Claro (SP), 24(38). 245-264.

[4] Cetin, I. (2009). Students’ Understanding of Limit Concept: An APOS Perspective. Unpublished Master’s Thesis. Middle East Technical University, Ankara, Turkey.

[5] Salas, G.L. & Hille, E. (1990). Calculus: One and several variables, 6th Ed. New York: John Wileyand Sons.

[6] Denbel, D. G. (2014). Students’ Misconceptions of the Limit Concept in a First Calculus Course. Journal of Education and Practice, 5(34). 24-40.

[7] Kim, D. J., Kang, H., & Lee, H. J. (2015). Two Different Epistemologies about Limit Concepts. International Education Studies, 8(3). 138-145.

[8] Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. Journal of Mathematical Behavior, 15(2). 167-192.

[9] Nurdin, Assagaf, S. F., & Arwadi, F. (2021, February). Students’ Understanding on Formal Definition of Limit. In Journal of Physics: Conference Series (Vol. 1752, No. 1, p. 012082). IOP Publishing.

[10] Roh, K. H. (2005). College Students’ Intuitive Understanding of The Concept of Limit and Their Level of Reverse Thinking. Unpublished doctoral dissertation, The Ohio State University.

[11] Bahar E. E., Rahman, A. & Minggi, I. (2012). Analisis Pemahaman Mahasiswa terhadap Konsep Limit Fungsi di satu Titik (Studi Kasus pada Mahasiswa Jurusan Matematika FMIPA UNM. Jurnal Sainsmat, 1(2). 181-190.

[12] Krathwohl, D. R. (2002). A revision of Bloom's taxonomy: An overview. Theory into practice, 41(4). 212-218.

[13] Hughes-Hallett, D., Gleason, A.M., McCallum, W.G. dkk. (2012). Calculus Sigle and Multivariable 6th Edition. New York: JohnWiley & Sons. Inc.

[14] Purcell, E.J., Varberg, D., & Ringdon, S.E. (2004). Kalkulus Edisi 8 Varberg, Purcell, Ringdon. Diterjemahkan oleh: I Nyoman Susila. Jakarta: Erlangga.

[15] Sierpineska, A., Bobos, G., & Pruncut, A. (2011). Teaching absolute value inequalities to mature students. Educational Studies in Mathematics. 78(3). 275-305.