On fractional variational problems which admit local transformations

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Abstract
We extend the second Noether theorem to fractional variational problems which are invariant under infinitesimal transformations that depend upon \( r \) arbitrary functions and their fractional derivatives in the sense of Caputo. Our main result is illustrated using the fractional Lagrangian density of the electromagnetic field.

Keywords
Lagrangian systems, fractional calculus, gauge symmetries, second Noether’s theorem, electromagnetic field

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1. Introduction
In 1918 Emmy Noether published a paper that is now famous for Noether’s theorem. But, in fact, Noether proved two theorems in the 1918 paper. The first theorem explains the correspondence between conserved quantities and continuous symmetry transformations that depend on constant parameters. Such transformations are global transformations. Familiar examples from classical mechanics include the connections between: spatial translations and conservation of linear momentum; spatial rotations and conservation of angular momentum; and time translations and conservation of energy (Gelfand and Fomin 2000; Logan 1977). The second theorem, less well known, guarantees syzygies between the Euler–Lagrange equations for a variational problem which is invariant under transformations that depend on arbitrary functions and their derivatives. Such transformations are local transformations. The statement of the theorem for this case is very general and, aside from its application to general relativity, it applies in a wide variety of other cases. For example, quantum chromodynamics and other gauge field theories are theories to which it applies. From the second theorem, one has identities between Lagrange expressions and their derivatives. These identities Noether called ‘dependencies’. For example, the Bianchi identities, in the general theory of relativity, are examples of such ‘dependencies’. In electrodynamics, if the Lagrangian represents a charged particle interacting with a electromagnetic field, one finds that it is invariant under the combined action of the so-called gauge transformation of the first kind on the charged particle field, and a gauge transformation of the second kind on the electromagnetic field. As a result of this invariance it follows, from second Noether’s theorem, the conservation of charge. For a complete history of Noether’s two theorems on variational symmetries see Brading (2002) and Kosmann-Schwarzbuch (2010); see also Carinena et al. (2005), Hydon and Mansfield (2011), Logan (1974) and Torres (2003) for some other generalizations.

Fractional calculus is a discipline that studies integrals and derivatives of non-integer (real or complex) order (Kilbas et al. 2006; Klimek 2009; Podlubny 1999; Samko et al. 1993). The field was born in 1695 and became an ongoing topic with many well-known mathematicians contributing to its theory (see Tenreiro Machado et al. 2010, for a review). Fractional derivatives are nonlocal operators and are historically applied in the study of nonlocal or time-dependent processes. The first and most well-established application of

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fractional calculus in physics was in the framework of anomalous diffusion, which are related to features observed in many physical systems, e.g. in dispersive transport in amorphous semiconductor, liquid crystals, polymers, proteins, etc. Electromagnetic equations (Baleanu et al. 2009) and field theories (Almeida et al. 2010; Cresson 2007; Herrmann 2008; Tarasov 2008) have been considered recently in the context of fractional calculus. The subject is nowadays very active due to its many applications in mechanics, chemistry, biology, economics, and control theory (see Baleanu et al. (2011), Fonseca Ferreira et al. (2008), Hilfer (2000) and Tenreiro Machado et al. (2011) for a review).

One of the most remarkable applications of fractional calculus is in the context of classical mechanics. Riewe (1996, 1997) showed that a Lagrangian involving fractional time derivatives leads to an equation of motion with nonconservative forces such as friction. It is a remarkable result since frictional and nonconservative forces are beyond the usual macroscopic variational treatment and, consequently, beyond the most advanced methods of classical mechanics. Riewe generalized the usual variational calculus, by considering Lagrangians that depend on fractional derivatives, in order to deal with nonconservative forces. Recently, several approaches have been developed to generalize the least action principle and the Euler–Lagrange equations to include fractional derivatives. Investigations cover problems depending on the Caputo fractional derivatives (Agrawal 2007; Almeida et al. 2012; Baleanu and Agrawal 2006; Baleanu 2008; Malinowska and Torres 2010, 2011; Tarasov 2006), the Riemann–Liouville fractional derivatives (Almeida and Torres 2009; Atanackovic et al. 2008; Baleanu and Avk 2004; Baleanu and Muslih 2005; Herzallah and Baleanu 2011) and others (Agrawal et al. 2011; El-Nabulsi and Torres 2007; El-Nabulsi 2011; Jumarie 2007; Malinowska 2012).

Noether’s first theorems have been extended to fractional variational problems using several approaches (Atanackovic et al. 2009; Cresson 2007; Frederico and Torres 2007, 2008, 2010). To the best of the authors’s knowledge, no second Noether-type theorem is available for the fractional setting. Such a generalization is the aim of this paper. We prove the fractional second Noether theorem for one- and multiple-dimensional Lagrangians, and we show how our results can be applied for fractional electromagnetic field. We trust that this paper will open several new directions of research and applications. Although we choose the Caputo calculus, our results are general and can be straightforward generalized to other fractional calculus, such as the Riemann–Liouville fractional calculus and others approaches.

This paper is organized in the following way. In Section 2 we recall the notion of fractional derivatives and their basic properties, that are needed in the sequel. The intended fractional second Noether-type theorem is formulated and proved in Section 3, for single (Section 3.1) and multiple (Section 3.2) integral problems. Our main result is illustrated in Section 4 using the fractional Lagrangian density of the electromagnetic field.

2. Fractional derivatives

In this section we review the necessary definitions and facts from the fractional calculus. For more on the subject we refer the reader to Kilbas et al. (2006), Klimek (2009), Podlubny (1999) and Samko et al. (1993).

Let \( \alpha \in \mathbb{R} \) and \( 0 < \alpha < 1 \), \( f \in L^1([a, b], \mathbb{R}) \). By the left Riemann–Liouville fractional integral of \( f \) on the interval \([a, b]\) we mean a function \( \mathbf{I}_a^\alpha f \) defined by

\[
\mathbf{I}_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-x)^{\alpha-1} f(t) dt, \quad x \in [a, b] \text{ a.e.}
\]

(1)

By the right Riemann–Liouville fractional integral of \( f \) on the interval \([a, b]\) we mean a function \( \mathbf{I}_b^\alpha f \) defined by

\[
\mathbf{I}_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x \in [a, b] \text{ a.e.}
\]

(2)

where \( \Gamma(\cdot) \) represents the Gamma function. For \( \alpha = 0 \), we set \( \mathbf{I}_a^0 f = \mathbf{I}_b^0 f := If \), the identity operator.

If the function \( \mathbf{I}_a^\alpha f \) is absolutely continuous on the interval \([a, b]\), then the left Riemann–Liouville fractional derivative is given by

\[
\mathbf{D}_a^\alpha f(x) = \frac{d}{dx} \mathbf{I}_a^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t) dt = \frac{d}{dx} \mathbf{I}_a^{1-\alpha} f(x).
\]

(3)

If the function \( \mathbf{I}_b^\alpha f \) is absolutely continuous on the interval \([a, b]\), then the right Riemann–Liouville fractional derivative is given by

\[
\mathbf{D}_b^\alpha f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \mathbf{I}_b^{1-\alpha} f(x) = \left(-\frac{d}{dx}\right) \mathbf{I}_b^{1-\alpha} f(x).
\]

(4)

Let \( f \in AC([a, b], \mathbb{R}) \). By the left Caputo fractional derivative of \( f \) on the interval \([a, b]\) we mean a function \( \mathbf{D}_a^\alpha f \) defined by

\[
\mathbf{D}_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{1-\alpha} \frac{d}{dt} f(t) dt = \mathbf{I}_a^{1-\alpha} \frac{d}{dx} f(x),
\]

(5)
and by the right Caputo fractional derivative of \( f \) on the interval \([a, b]\) we mean a function \( C D_b^\alpha f \) defined by

\[
C D_b^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_a^b (t-x)^{-\alpha} \frac{d}{dt} f(t) \, dt,
\]

\[
= x I_b^{-\alpha} \left( -\frac{d}{dx} \right) f(x).
\] (6)

**Remark 1.** Observe that if \( \alpha \) goes to 1, then under suitable assumptions operators \( a D_x^\alpha \) and \( C D_x^\alpha \) can be replaced with \( \frac{d}{dx} \), and operators \( x D_x^\alpha \) and \( C D_x^\alpha \) can be replaced with \( \frac{d}{dt} \) (see Podlubny 1999).

The operators (1)–(6) are obviously linear. Below we present the rules of fractional integration by parts for Riemann–Liouville fractional integral and Caputo fractional derivatives which are particularly useful for our purposes.

**Lemma 2.1 (Love and Young 1938).** Let \( 0 < \alpha < 1, \rho \geq 1, \) \( q \geq 1, \) and \( 1/p + 1/q \leq 1 + \alpha. \) If \( g \in L_p((a, b]) \) and \( f \in L_q([a, b]), \) then

\[
\int_a^b g(x) a D_x^\alpha f(x) \, dx = \int_a^b f(x) a I_x^\alpha g(x) \, dx.
\] (7)

**Lemma 2.2.** [cf. Klimek (2009)] Let \( 0 < \alpha < 1. \) If \( f, \)
\( g \in AC([a, b]), \) then

\[
\int_a^b g(x) C D_x^\alpha f(x) \, dx = \left. f(x) C D_x^\alpha g(x) \right|_{x=a}^{x=b} + \int_a^b f(x) C D_x^\alpha g(x) \, dx,
\]

\[
\int_a^b g(x) C D_b^\alpha f(x) \, dx = - \left. f(x) C D_b^\alpha g(x) \right|_{x=a}^{x=b} + \int_a^b f(x) C D_b^\alpha g(x) \, dx.
\] (8)

**Proof.** Formulas could be derived using Equations (3)–(6), the identity (7) and performing standard integration by parts.

Partial fractional integrals and derivatives are a natural generalization of the corresponding one-dimensional fractional integrals and derivatives, being taken with respect to one or several variables. For \( (x_1, \ldots, x_n), \) \( (\alpha_1, \ldots, \alpha_n), \) where \( 0 < \alpha_i < 1, \) \( i = 1, \ldots, n \) and \( [a_1, b_1] \times \cdots \times [a_n, b_n], \) the partial Riemann–Liouville fractional integrals of order \( \alpha_k \) with respect to \( x_k \) are defined by

\[
a_k I_{x_k}^\alpha f(x_1, \ldots, x_n) = \frac{1}{\Gamma(1-\alpha_k)} \int_{a_k}^{b_k} (x_k - t_k)^{-\alpha_k} \frac{\partial}{\partial t_k} f(x_1, \ldots, x_n) \, dt_k,
\]

\[
f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n) \, dt_k, \quad x_k > a_k,
\]

\[
x_k I_{x_k}^\alpha f(x_1, \ldots, x_n) = \frac{1}{\Gamma(1-\alpha_k)} \int_{a_k}^{b_k} (t_k - x_k)^{-\alpha_k} \frac{\partial}{\partial t_k} f(x_1, \ldots, x_n) \, dt_k,
\]

\[
f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n) \, dt_k, \quad x_k < b_k.
\]

Partial Riemann–Liouville and Caputo derivatives are defined by

\[
a_k D_{x_k}^\alpha f(x_1, \ldots, x_n) = \frac{\partial}{\partial x_k} I_{x_k}^{\alpha_k} f(x_1, \ldots, x_n),
\]

\[
f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n) \, dt_k,
\]

\[
x_k D_{x_k}^\alpha f(x_1, \ldots, x_n) = \frac{\partial}{\partial x_k} I_{x_k}^{\alpha_k} f(x_1, \ldots, x_n),
\]

\[
f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n) \, dt_k,
\]

\[
a_k C D_{x_k}^\alpha f(x_1, \ldots, x_n) = \frac{\partial}{\partial x_k} I_{x_k}^{\alpha_k} f(x_1, \ldots, x_n),
\]

\[
f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n) \, dt_k,
\]

\[
x_k C D_{x_k}^\alpha f(x_1, \ldots, x_n) = \frac{\partial}{\partial x_k} I_{x_k}^{\alpha_k} f(x_1, \ldots, x_n),
\]

\[
f(x_1, \ldots, x_{k-1}, t_k, x_{k+1}, \ldots, x_n) \, dt_k.
\]

3. **Main results**

In this section we formulate and prove the fractional second Noether-type theorem, for single (Section 3.1) and multiple (Section 3.2) integral problems.

3.1 **Single integral case**

Consider a system characterized by a set of functions

\[
x'(t), \quad i = 1, \ldots, n,
\] (9)

depending on time \( t. \) We can simplify the notation by interpreting (9) as a vector function \( x = (x^1, \ldots, x^n). \) Define the action functional in the form

\[
J(x) = \int_a^b L(t, x(t), \frac{\partial}{\partial t} x(t)) \, dt,
\] (10)
where:

(i) \( C_i^D p_i x(t) := \left( C_i^D p_i x^1(t), \ldots, C_i^D p_i x^n(t) \right) \), \( 0 < \alpha_i \leq 1, i = 1, \ldots, n \);
(ii) \( x \in C^1([a, b], \mathbb{R}^n) \);
(iii) \( L \in C^1([a, b] \times \mathbb{R}^n, \mathbb{R}) \);
(iv) \( t \rightarrow \frac{dx}{p_i} \in AC([a, b]) \) for every \( x \in C^1([a, b], \mathbb{R}^n) \), \( k = 1, \ldots, n \).

We define the admissible set of functions \( A([a, b]) \) by

\[
A([a, b]) := \{ x \in C^1([a, b], \mathbb{R}^n) : x(a) = x_a, x(b) = x_b, x_a, x_b \in \mathbb{R}^n \}.
\]

Theorem 3.1 (Agrawal 2007). A necessary condition for the function \( x \in A([a, b]) \) to provide an extremum for the functional (10) is that its components satisfy the \( n \) fractional equations

\[
\frac{\partial L}{\partial x^k} + \Delta_i^D p_i \frac{\partial L}{\partial \Delta_i^D x^k} = 0, \quad k = 1, \ldots, n
\]

for \( t \in [a, b] \).

Define

\[
E_i^k(L) := \frac{\partial L}{\partial x^k} + \Delta_i^D p_i \frac{\partial L}{\partial \Delta_i^D x^k}.
\]

We shall call \( E_i^k(L) \) the fractional Lagrange expressions.

The invariance transformations that we shall consider are infinitesimal transformations that depend upon arbitrary functions and their fractional derivatives in the sense of Caputo. Let

\[
\begin{cases}
\tau = t, \\
\tilde{x}^k(t) = \chi^k(t) + T_t^1(p_1(t)) + \cdots + T_t^k(p_k(t)), \quad k = 1, \ldots, n
\end{cases}
\]

where \( T_t^k \) are linear fractional differential operators and \( p_i, s = 1, \ldots, r \) are arbitrary, independent \( C^1 \) functions defined on \( [a, b] \). Then, we consider four types of fractional differential operators:

(I) Operator of the first kind

\[
T_t^{ks} = T_t^{ks} := a_0^{ks}(t) + a_1^{ks}(t)C_i^D p_i^{ks} + \cdots + a_r^{ks}(t)C_i^D p_i^{ks}, \quad 0 < \beta_{ks} \leq 1,
\]

and \( C_i^D p_s \in C^1([a, b]), a_i^{ks} \in C^1([a, b], \mathbb{R}), s = 1, \ldots, r, i = 1, \ldots, l \).

(II) Operator of the second kind

\[
T_t^{ks} = T_2^{ks} := a_0^{ks}(t) + a_1^{ks}(t)C_i^D p_i^{ks} + \cdots + a_r^{ks}(t)C_i^D p_i^{ks}, \quad 0 < \beta_{ks} \leq 1,
\]

and \( C_i^D p_s \in C^1([a, b]), a_i^{ks} \in C^1([a, b], \mathbb{R}), s = 1, \ldots, r, i = 1, \ldots, l \).

(III) Operator of the third kind

\[
T_t^{ks} = T_3^{ks} := a_0^{ks}(t) + a_1^{ks}(t)C_i^D p_i^{ks} + \cdots + a_r^{ks}(t)C_i^D p_i^{ks}, \quad 0 < \beta_{ks} \leq 1,
\]

and \( C_i^D p_s \in C^1([a, b]), a_i^{ks} \in C^1([a, b], \mathbb{R}), s = 1, \ldots, r, i = 1, \ldots, l \).

(IV) Operator of the fourth kind

\[
T_t^{ks} = T_4^{ks} := a_0^{ks}(t) + a_1^{ks}(t)C_i^D p_i^{ks} + \cdots + a_r^{ks}(t)C_i^D p_i^{ks}, \quad 0 < \beta_{ks} \leq 1,
\]

and \( C_i^D p_s \in C^1([a, b], \mathbb{R}) \) and \( p \in C^1([a, b], \mathbb{R}) \) and \( C_i^D p_s \in C^1([a, b], \mathbb{R}) \).
\[ \int_a^b q T_k^s(p_s) dt = \int_a^b p_s \tilde{T}_k^s(q) \]
\[ = \int_a^b p_s \left( a_k^s q + \sum_{i=0}^{l-1} a_i^s p_{s+i} a_{i+i+1}^s q \right) dt + \left[ \right]_{t=a}^{t=b}. \]
(15)

\[ \int_a^b q T_k^s(p_s) dt = \int_a^b p_s \tilde{T}_k^s(q) \]
\[ = \int_a^b p_s \left( a_k^s q + \sum_{i=0}^{l-1} a_i^s p_{s+i} a_{i+i+1}^s q \right) dt + \left[ \right]_{t=a}^{t=b}. \]
(16)

**Definition 3.2.** The functional (10) is invariant under transformations (11) if and only if for all \( x \in C^1([a, b], \mathbb{R}^n) \) we have

\[ \int_a^b L(t, \tilde{x}(t), \frac{\partial}{\partial \tilde{x}} \tilde{x}(t)) dt = \int_a^b \tilde{L}(t, x(t), \frac{\partial}{\partial x} x(t)) dt. \]

**Theorem 3.3.** If functional (10) is invariant under transformations (11), then there exist \( r \) identities of the form

\[ \sum_{k=1}^{n} \tilde{T}_k^s \left( E_k^s(L) \right) = 0, \quad s = 1, \ldots, r, \]
(17)

where \( \tilde{T}_k^s \) is the adjoint of \( T_k^s \).

**Proof.** We give the proof only for the case \( T_k^s = T_1^s \); other cases can be proved similarly. By Definition 3.2 we have

\[ 0 = \int_a^b L(t, \tilde{x}(t), \frac{\partial}{\partial \tilde{x}} \tilde{x}(t)) dt - \int_a^b \tilde{L}(t, x(t), \frac{\partial}{\partial x} x(t)) dt \]
\[ = \int_a^b \left( L(t, \tilde{x}(t), \frac{\partial}{\partial \tilde{x}} \tilde{x}(t)) - L(t, x(t), \frac{\partial}{\partial x} x(t)) \right) dt. \]

Then, by the Taylor formula

\[ 0 = \sum_{k=1}^{n} \int_a^b \left[ \frac{\partial}{\partial \tilde{x}} T_k^s(p_s) \frac{\partial}{\partial \tilde{x}} \tilde{T}_1^s(p_s) \right] dt, \]
(18)

where \( T_k^s(p_s) = \sum_{i=1}^{r} T_k^s(p_s) \). The second term in the integrand may be integrated by parts (see the first formula of (8)):

\[ \int_a^b \frac{\partial}{\partial \tilde{x}} T_k^s(p_s) \frac{\partial}{\partial \tilde{x}} \tilde{T}_1^s(p_s) \]
\[ = \int_a^b T_1^s(p_s) \frac{\partial}{\partial \tilde{x}} \tilde{T}_1^s(p_s) dt. \]
(19)

Since \( p_s \) are arbitrary, we may choose \( p_s \) such that

\[ p_s(0) = p_s(b) = 0 \]
\[ \text{and} \quad \frac{\partial}{\partial \tilde{x}} \tilde{T}_1^s(p_s) \text{ are arbitrary}. \]

Therefore, the boundary term in (19) vanishes and substituting (19) into (18) we get

\[ 0 = \sum_{k=1}^{n} \int_a^b \left[ \frac{\partial}{\partial \tilde{x}} T_k^s(p_s) \frac{\partial}{\partial \tilde{x}} \tilde{T}_1^s(p_s) \right] dt. \]

Using the definition of the adjoint operator \( \tilde{T}_1^s \) of a fractional differential operator \( T_1^s \), that is, Equation (13), we get

\[ 0 = \sum_{k=1}^{n} \int_a^b \left[ \frac{\partial}{\partial \tilde{x}} T_k^s(p_s) \frac{\partial}{\partial \tilde{x}} \tilde{T}_1^s(p_s) \right] dt. \]

Again appealing to the arbitrariness of \( p_s \), we can force the boundary term to vanish, and finally by the fundamental lemma of calculus of variations we conclude that

\[ \sum_{k=1}^{n} \tilde{T}_k^s \left( E_k^s(L) \right) = 0, \quad s = 1, \ldots, r. \]

**Remark 2.** Note that if we put \( \beta_k = 1 \) in transformations of the third or the fourth kind, then we obtain infinitesimal transformations:

\[ \begin{pmatrix} \tilde{t} = t \\ \tilde{x}^k(t) = x^k + B^k(s) \end{pmatrix}, \]

where

\[ B^k = b_0^k(t) + b_1^k(t) \frac{d}{dt} + b_2^k(t) \frac{d^2}{dt^2} + \cdots + b_r^k(t) \frac{d^r}{dt^r}, \quad k = 1, \ldots, n. \]
In this case the adjoint operator \( \tilde{B}^{ks} \) of the differential operator \( B^{ks} \) is given by

\[
\tilde{B}^{ks}(q) = b_0^{ks} q + \sum_{i=1}^{l} (-1)^i \frac{d^{i}}{dt^i} (b_i^{ks} q), \quad k = 1, \ldots, n
\]

and the identities (17) take the form

\[
\sum_{k=1}^{n} b_0^{ks}(E_k(L)) + \sum_{k=1}^{n} \sum_{i=1}^{l} (-1)^i \frac{d}{dt} (b_i^{ks} E_k(L)) = 0,
\]

where

\[
s = 1, \ldots, r,
\]

which are exactly the Noether identities (see Brading 2002; Logan 1974).

Remark 3. The fractional differential operators \( T_1^{ks}, T_2^{ks}, T_3^{ks} \) and \( T_4^{ks} \) can of course be combined, that is, we can consider infinitesimal transformations that depend upon arbitrary functions and their fractional derivatives in the sense of Caputo: left and right with various orders.

### 3.2 Multiple integral case

Consider a system characterized by a set of functions

\[
u^j (t, x_1, \ldots, x_m), \quad j = 1, \ldots, n,
\]

depending on time \( t \) and the space coordinates \( x_1, \ldots, x_m \). We can simplify the notation by interpreting (20) as a vector function \( \bar{u} = (u^1, \ldots, u^n) \) and writing \( t = x_0, \bar{x} = (x_0, x_1, \ldots, x_m), dx = dx_0 \ dx_1 \cdots \ dx_m \). Then (20) becomes simply \( u(x) \) and is called a vector field.

Define the action functional in the form

\[
\mathcal{J}(u) = \int_{\Omega} \mathcal{L}(x, u, \partial^\nu \bar{u}) \ dx,
\]

where \( \Omega = R \times [a_0, b_0], \mathcal{R} = [a_1, b_1] \times \ldots \times [a_m, b_m] \), \( \partial^\nu \bar{u} \) is the operator

\[
\left( \begin{array}{c}
\partial^\alpha_{a_0} \partial^\beta_{a_1} \partial^\gamma_{a_2} \cdots \partial^\nu_{a_m} \bar{u} \\
\end{array} \right),
\]

where \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m), \quad 0 < \alpha_i \leq 1, \quad i = 0, \ldots, m. \)

The action functional \( \mathcal{L}(x, u, \partial^\nu \bar{u}) \) is called the fractional Lagrangian density of the field. We assume that:

(i) \( u^j \in C^1(\Omega, \mathcal{R}), \ j = 1, \ldots, n; \)

(ii) \( \mathcal{L} \in C^1(\mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^{(n+1)}; \mathcal{R}); \)

(iii) \( x \to \frac{\partial \mathcal{L}}{\partial u^j_{\partial^\alpha \partial^\beta \partial^\gamma \cdots \partial^\nu \bar{u}}} \) for every \( u^j \in C^1(\Omega, \mathcal{R}) \) are \( C^1 \) functions, \( j = 1, \ldots, n, i = 0, \ldots, m. \)

Define the admissible set of functions \( A(\Omega) \) by

\[
A(\Omega) := \{ u : \Omega \to \mathcal{R}^n : u(x) = \varphi(x) \quad \text{for} \quad x \in \partial \Omega \},
\]

where \( \varphi : \partial \Omega \to \mathcal{R}^n \) is a given function.

Applying the principle of stationary action to (21) we obtain the multidimensional fractional Euler–Lagrange equations for the field.

**Theorem 3.4. [cf. Cresson 2007]** A necessary condition for the function \( u \in A(\Omega) \) to provide an extremum for the action functional (21) is that its components satisfy the n multidimensional fractional Euler–Lagrange equations:

\[
\frac{\partial \mathcal{L}}{\partial u^j} + \sum_{i=0}^{m} x_i D^\mu_{b_i} \left( \frac{\partial \mathcal{L}}{\partial \partial^\alpha \partial^\beta \partial^\gamma \cdots \partial^\nu \bar{u}^j} \right) = 0, \quad j = 1, \ldots, n.
\]

As before we define

\[
E^j_j(\mathcal{L}) := \frac{\partial \mathcal{L}}{\partial u^j} + \sum_{i=0}^{m} x_i D^\mu_{b_i} \left( \frac{\partial \mathcal{L}}{\partial \partial^\alpha \partial^\beta \partial^\gamma \cdots \partial^\nu \bar{u}^j} \right),
\]

which are called the fractional Lagrange expressions.

We shall study infinitesimal transformations that depend upon arbitrary functions of independent variables and their partial fractional derivatives in the sense of Caputo. Let

\[
\begin{align*}
\bar{x} &= x, \\
\bar{u}^j(x) &= u^j(x) + T^{ij}(p_j(x)) + \cdots + T^{ir}(p_r(x)), \quad j = 1, \ldots, n,
\end{align*}
\]

where \( T^{ij} \) are linear fractional differential operators and \( p_s, s = 1, \ldots, r, \) are \( r \) arbitrary, independent \( C^1 \) functions defined on \( \Omega \). Then we consider two types of fractional differential operators:

(I) Operator of the first kind

\[
T^{ij} = T^{ij}_1 := c^{ij}(x) + \sum_{i=0}^{m} c^{ij}_i(x) D^\mu_{b_i}, \quad 0 < \beta_{bij} \leq 1,
\]

and \( c^{ij}_i D^\mu_{b_i} p_s, \ c^{ij}, \ c^{ij}_i \) are \( C^1 \) functions defined on \( \Omega, s = 1, \ldots, r, \ i = 1, \ldots, m. \)

(II) Operator of the second kind

\[
T^{ij} = T^{ij}_2 := c^{ij}(x) + \sum_{i=0}^{m} c^{ij}_i(x) D^\mu_{b_i}, \quad 0 < \beta_{bij} \leq 1,
\]

and \( c^{ij}_i D^\mu_{b_i} p_s, \ c^{ij}, \ c^{ij}_i \) are \( C^1 \) functions defined on \( \Omega, s = 1, \ldots, r, \ i = 1, \ldots, m. \)

We define invariance similarly to the one-dimensional case.
Definition 3.5. The functional (21) is invariant under transformations (22) if and only if for all \( u \in C^2(\Omega, \mathbb{R}^n) \) we have

\[
\int_{\Omega} \mathcal{L}(x, \tilde{u}, c^a \nabla^a \tilde{u}) \, dx = \int_{\Omega} \mathcal{L}(x, u, c^a \nabla^a u) \, dx.
\]

Theorem 3.6. If functional (21) is invariant under transformations (22), then there exist \( r \) identities of the form

\[
\sum_{j=1}^{\infty} \tilde{T}^p_j \left( E_j (\mathcal{L}) \right) = 0, \quad s = 1, \ldots, r,
\]

where \( \tilde{T}^p_j \) is the adjoint of \( T^p_j \).

Proof. We give the proof only for the case \( T^p_1 = T^p_1 \); the other case can be proved similarly. By Definition 3.5 we have

\[
0 = \int_{\Omega} \mathcal{L}(x, \tilde{u}, c^a \nabla^a \tilde{u}) \, dx - \int_{\Omega} \mathcal{L}(x, u, c^a \nabla^a u) \, dx
\]

Then, by the Taylor formula

\[
0 = \sum_{j=1}^{\infty} \int_{\Omega} \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} \, dx,
\]

(23)

where \( T^p_1 (p_s) = \sum_{j=1}^{\infty} T^p_j (p_s) \). The Fubini theorem allows us to rewrite integrals as the iterated integrals so that we can use the integration by parts formula (8):

\[
\int_{\Omega} \sum_{j=1}^{\infty} \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} \, dx = \int_{\Omega} \sum_{j=1}^{\infty} \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} \, dx + [\cdot]_{\partial \Omega}, \quad j = 1, \ldots, m,
\]

(24)

where \([\cdot]_{\partial \Omega}\) represent the boundary terms: \( m \)-volumes integrals. Since \( p_s \) are arbitrary, we may choose \( p_s \) such that \( p_s(x)_{\partial \Omega} = 0 \) and \( \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} p_s(0)_{\partial \Omega} = 0, \quad s = 1, \ldots, r, \quad i = 0, \ldots, m \). Therefore, the boundary term in (24) vanishes and substituting (24) into (23) we get

\[
0 = \sum_{j=1}^{\infty} \int_{\Omega} \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} \, dx
\]

Now we proceed as in the one-dimensional case and define the adjoint operator \( \tilde{T}^p_1 \) of a fractional differential operator \( T^p_1 \) by

\[
\int_{\Omega} q(x) T^p_1 (p_s(x)) \, dx = \int_{\Omega} p_s(x) \tilde{T}^p_1 (q(x)) \, dx + \{\cdot\}_{\partial \Omega}, \quad j = 1, \ldots, n, \quad s = 1, \ldots, r,
\]

where we use the Fubini theorem. Again appealing to the arbitrariness of \( p_s \) we can force the boundary term to vanish (by putting \( p_s(x)_{\partial \Omega} = 0 \)). Therefore,

\[
0 = \sum_{j=1}^{n} \int_{\Omega} \sum_{i=1}^{r} \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} + \left[ \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} \right]_{\partial \Omega}, \quad j = 1, \ldots, n,
\]

Finally by the fundamental lemma of calculus of variations we conclude that

\[
\sum_{j=1}^{n} \tilde{T}^p_1 \left( E_j (\mathcal{L}) \right) = 0, \quad s = 1, \ldots, r.
\]

Remark 4. The adjoints of \( T^p_1, i = 1, 2 \), are given by expressions:

\[
\tilde{T}^p_1 = c^p q + \sum_{j=0}^{m} \left[ \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} \right]_{\partial \Omega}, \quad j = 1, \ldots, n
\]

\[
\tilde{T}^p_2 = c^p q + \sum_{j=0}^{m} \left[ \frac{\partial \mathcal{L}}{(\partial u) c^a \nabla^a u} \right]_{\partial \Omega}, \quad j = 1, \ldots, n
\]

Remark 5. The fractional differential operators \( T^p_1 \) and \( T^p_2 \) can of course be combined, that is, we can consider infinitesimal transformations that depend upon arbitrary functions and their partial fractional derivatives in the sense of Caputo: left and right with various orders.

4. Example

In order to illustrate our result we will use the Lagrangian density for the electromagnetic field (see Gelfand and Fomin 2000):

\[
\mathcal{L} = \frac{1}{8\pi} (E^2 - H^2),
\]

(25)

where \( E \) and \( H \) are the electric field vector and the magnetic field vector, respectively. Following Baleanu et al. (2009), we shall generalize (25) to the fractional Lagrangian density by changing classical partial
derivatives by fractional. Let \( x = (x_0, x_1, x_2, x_3) \in \Omega \) and \( A(x) = (A_1(x), A_2(x), A_3(x)) \), \( A_0(x) \) be a vector potential and a scalar potential, respectively. They are defined by setting

\[
E = \nabla^{(\alpha_1, \alpha_2, \alpha_3)} A_0 - C \frac{\partial^{\alpha_3}}{\partial x_3} A, \quad H = \text{curl} A,
\]

where

\[
\nabla^{(\alpha_1, \alpha_2, \alpha_3)} A_0 = \frac{C}{\alpha_1} \frac{\partial}{\partial x_1} A_0 + \int \frac{C}{\alpha_2} \frac{\partial}{\partial x_2} A_0 + k \frac{C}{\alpha_0} \frac{\partial}{\partial x_0} A_0,
\]

\[
\text{curl} A = i \left( \frac{C}{\alpha_1} \frac{\partial}{\partial x_1} A_3 - C \frac{\partial}{\partial x_3} A_1 + C \frac{\partial}{\partial x_3} A_2 \right) + k \left( \frac{C}{\alpha_1} \frac{\partial}{\partial x_1} A_2 - C \frac{\partial}{\partial x_2} A_1 \right).
\]

Replacing \( E \) and \( H \) in (25) by their expressions (26) we obtain the fractional Lagrangian density

\[
L = \frac{1}{8\pi} \left[ (\nabla^{(\alpha_1, \alpha_2, \alpha_3)} A_0 - C \frac{\partial^{\alpha_3}}{\partial x_3} A) - (\text{curl} A)^2 \right].
\]

Note that, similarly to the integer case, the potential \( (A_0, \mathbf{A}) \) is not uniquely determined by the vectors \( \mathbf{E} \) and \( \mathbf{H} \). Namely, \( \mathbf{E} \) and \( \mathbf{H} \) do not change if we make a gauge transformation:

\[
\tilde{A}_j(x) = A_j(x) + \sum_{\gamma} D_{\gamma j} f(x), \quad j = 0, \ldots, 3,
\]

where \( f : \Omega \to \mathbb{R} \) is an arbitrary function of class \( C^2 \) in all of its argument. Therefore, the Lagrangian density (27), and hence the action functional, is invariant under transformation (28). By Theorem (3.6), we conclude that

\[
\sum_{j=0}^{3} \frac{\partial}{\partial x_j} \left( E_j^0(\mathcal{L}) \right) = 0,
\]

where \( E_j^0(\mathcal{L}) \) are Lagrange expressions corresponding to (27). Equations \( E_j^0(\mathcal{L}) = 0 \) do not uniquely determine the potential \( (A_0, \mathbf{A}) \) and to avoid this lack of uniqueness, the fractional Lorentz condition can be imposed on \( (A_0, \mathbf{A}) \).

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