GROMOV-WITTEN INVARIANTS OF CALABI-YAU FIBRATIONS

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Abstract. We study the quasimap invariants of elliptic and K3 fibrations. Oberdieck and Pixton conjectured that the Gromov-Witten potentials of elliptic fibrations are quasi-modular forms. Analogously, we propose similar conjecture for the quasimap potentials of elliptic fibrations. We also conjecture some finite generation properties of quasimap potentials of K3 fibrations. Via wall-crossing conjecture, this will imply some quasi-modularity of the Gromov-Witten potentials of K3 fibrations. We provide some evidences for our conjectures through several examples. The method here can be further generalized to arbitrary n-dimensional Calabi-Yau fibrations.

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0. Introduction

0.1. Elliptic fibrations. Consider non-singular algebraic varieties $X$, $B$ and elliptic fibration

$$\pi : X \to B,$$

i.e. a flat morphism with fibers connected curves of arithmetic genus

1. Assume that $\pi$ has integral fibers and has a section

$$\iota : B \to X.$$

Let $N_\iota$ be the normal bundle of $\iota$. Fix a curve class $\gamma \in H_2(B, \mathbb{Z})$ and let
be the Gromov-Witten series associated to $\gamma$. The ring of quasimodular forms is the free polynomial algebra 

$$Q\text{Mod} = \mathbb{Q}[E_2, E_4, E_6],$$

where $E_k$ are the weight $k$ Eisenstein series

$$E_k(Q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1}Q^n}{1 - Q^n},$$

and $B_k$ are the Bernoulli numbers. Based on the results for Gromov-Witten invariants of elliptic curves, Oberdieck and Pixton made the following conjecture ([18]).

**Conjecture 1.** For $\gamma \in H_2(B, \mathbb{Z})$, $F_{g, \gamma}^{GW}$ is a quasimodular form:

$$F_{g, \gamma}^{GW} \in \frac{1}{\Delta(Q)^m} Q\text{Mod},$$

where $\Delta = Q \prod_{n=1}^{\infty} (1 - Q^n)^{24}$ is the modular discriminant and $m = -\frac{1}{2} c_1(N_{\gamma}) \cdot \gamma$.

Motivated by the above conjecture, we study the quasimap theory of $X$. The quasimap invariants are introduced in [5, 10] for the study of mirror symmetry. The relationship of quasimap invariants to Gromov-Witten invariants, which is called wall-crossing conjecture, were studied in [6, 8, 9, 11]. Based on the wall-crossing conjecture, a fundamental relationship between the quasimap invariants and the B-model for all local and complete intersection Calabi-Yau 3-folds $X$ is natural to propose: the quasimap invariants of $X$ exactly equal the B-model invariants of the mirror $Y$. Therefore, via quasimap theory, we study the string theoretic B-model theory directly. See introduction of [16], for more details.

For a curve class $\gamma \in H_2(B, \mathbb{Z})$, let

$$F_{g, \gamma}^{SQ}(q) = \sum_{\pi_{*} \beta = \gamma} q^{\beta} \int_{[\mathcal{M}_g(X, \beta)]^{vir}} 1 \in \mathbb{C}[[q]],$$
be the quasimap series associated to \( \gamma \). The following series in \( q \) will play basic role.

\[
L(q) = (1 - 27q)^{-\frac{1}{3}} = 1 + 9q + 162q^2 + \ldots ,
\]

\[
I_1^E(q) = 3 \sum_{n=1}^{\infty} \frac{(3d - 1)!}{(d!)^3} q^d ,
\]

\[
B'_1(q) = q \frac{\partial}{\partial q} I_1^E ,
\]

\[
X(q) = \frac{q \frac{\partial}{\partial q} B'_1}{1 + B'_1} .
\]

We define the ring of quasimap elliptic fibrations

\[
\text{QEF} := \mathbb{C}[L^{\pm 3}, B'_1, X]
\]

Define differential operator

\[
D = q \frac{d}{dq} .
\]

The following equations were obtained in [10].

\[
DL = \frac{L}{3} (L^3 - 1) ,
\]

\[
X^2 - (L^3 - 1)X + DX - \frac{2}{9} (L^3 - 1) = 0 .
\]

Via the above relations, the ring QEF is closed under the action of \( D \). Motivated by Conjecture 1, we conjecture the following.

**Conjecture 2.** For \( \gamma \in H_2(B, \mathbb{Z}) \), we have

\[
\mathcal{F}_{\gamma}^{SQ} \in \text{QEF} .
\]

The following Lemma relate the Conjecture 1 and 2.

**Lemma 3 ([2]).** Via the change of variable by the mirror map of elliptic curve

\[
Q = q \text{Exp}(I_1^E(q)) ,
\]

we have

\[
E_2 = \frac{(1 + B'_1)^2}{L^3} (12X + 4 - 3L^3) ,
\]

\[
E_4 = \frac{(1 + B'_1)^4}{L^6} (-8L^3 + 9L^6) ,
\]

\[
E_6 = \frac{(1 + B'_1)^6}{L^9} (-8L^3 + 36L^6 - 27L^9) .
\]
0.2. **K3 fibrations.** We can also consider Calabi-Yau 3-fold $X$, curve $B$ and K3 fibration
\[
\pi : X \to B,
\]
where generic fibers of $\pi$ are smooth K3 surfaces. For a fixed curve class $\gamma \in H_2(B, \mathbb{Z})$, Let
\[
\mathcal{F}_{g,\gamma}^{GW}(Q) = \sum_{\pi_* \beta = \gamma} Q^\beta \int_{\overline{M}_g(X, \beta)}^{\text{vir}} 1,
\]
\[
\mathcal{F}_{g,\gamma}^{SQ}(q) = \sum_{\pi_* \beta = \gamma} q^\beta \int_{\overline{Q}_g(X, \beta)}^{\text{vir}} 1,
\]
be the Gromov-Witten and quasimap series associated to $\gamma$, respectively. We define the following series in $q$.
\[
L(q) = (1 - 4^4 q)^{-\frac{1}{4}} = 1 + 9q + 162q^2 + \ldots,
\]
\[
I_{K^3}^1 = \frac{4 \sum_{n=1}^{\infty} \frac{(4d)!}{(d!)^4} (\sum_{r=d+1}^{d+1} \frac{1}{r}) q^d}{\sum_{n=0}^{\infty} \frac{(4d)!}{(d!)^4} q^d},
\]
\[
A'_1(q) = q \frac{\partial}{\partial q} I_{K^3}^1,
\]
\[
X(q) = \frac{q \frac{\partial}{\partial q} A'_1}{1 + A'_1}.
\]
We define the ring of quasimap K3 fibrations
\[
\text{QKF} := \mathbb{C}[L^{\pm 4}, A'_1, X].
\]
Via the following relations which are obtained in [13],
\[
DL = \frac{L}{4}(L^4 - 1),
\]
\[
X^2 - 2DX + \frac{1}{16}(12L^8 - 11L^4 - 1) = 0.
\]
the ring QEF is closed under the action of D.

**Conjecture 4.** For $\gamma \in H_2(B, \mathbb{Z})$, We have
\[
\mathcal{F}_{g,\gamma}^{SQ} \in \text{QKF}.
\]
It is an interesting question to find the modular interpretation of the generators in \( Q_{KF} \) after change of variable by mirror map
\[
Q = q E x p(I^K_1(q)) .
\]
This will imply the modularity of the Gromov-Witten series of K3-fibrations.

0.3. **Twisted theories on \( \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \).** Twisted theories associated to \( \mathbb{P}^{m} \) was introduced in [16]. We similarly define twisted theories associated to \( \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \) and study several cases.

Let \( T_{n_1+1,n_2+1} \) be the algebraic torus
\[
(\mathbb{C}^*)^{n_1+1} \times (\mathbb{C}^*)^{n_2+1}
\]
and let each component \( (\mathbb{C}^*)^{n_i+1} \) of \( T_{n_1+1,n_2+1} \) act with the standard linearization on \( \mathbb{P}^{n_i} \) with weights \( \lambda_i, \lambda_i, 0, \ldots, \lambda_i, n_i \) on the vector space \( H^0(\mathbb{P}^{n_i}, \mathcal{O}_{\mathbb{P}^{n_i}}(1)) \).

Let \( \overline{M}_g(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, (d_1, d_2)) \) be the moduli space of stable maps to \( \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \) with the canonical \( T_{n_1+1,n_2+1} \)-action, and let
\[
\mathcal{C} \rightarrow \overline{M}_g(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, (d_1, d_2)) , \ f : \mathcal{C} \rightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} , \ S_i = f^* \mathcal{O}_{\mathbb{P}^{n_i}}(-1)
\]
be the standard universal structures. Let
\[
a = (a_{11}, \ldots, a_{1r}; a_{21}, \ldots, a_{2r}) , \quad b = (b_{11}, \ldots, b_{1s}; b_{21}, \ldots, a_{2s})
\]
be vectors of positive integers satisfying the Calabi-Yau conditions
\[
\sum_{i=1}^r a_{ki} - \sum_{j=1}^s b_{kj} = n_k + 1 \quad \text{for } k = 1, 2 .
\]

The Gromov-Witten invariants of the \( (a, b) \)-twisted geometry of \( \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \) are defined by the equivariant integrals
\[
\tilde{N}^{GW}_{g,(d_1,d_2)} = \int_{\overline{M}_g(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, (d_1, d_2))} \prod_{i=1}^r e(R \pi_* S^{-a_{1i}} \otimes S^{-a_{2i}}) \prod_{j=1}^s e(-R \pi_* S^{b_{1j}} \otimes S^{b_{2j}}) .
\]

The above integral define a rational function in equivariant variables \( \lambda_{ki} \)
\[
\tilde{N}^{GW}_{g,(d_1,d_2)} \in \mathbb{C}(\lambda_{1,0}, \ldots, \lambda_{1,n_1}, \lambda_{2,0}, \ldots, \lambda_{2,n_2}) .
\]

Using the moduli space \( \overline{Q}_g(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, (d_1, d_2)) \) of stable quasimaps with the standard structures,
\[ C \to \overline{\mathcal{Q}}_g(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, (d_1, d_2)), \quad f : C \to [\mathbb{C}^{n_1+1} \times \mathbb{C}^{n_2+1}/\mathbb{C}^* \times \mathbb{C}]^*, \]

the quasimap invariants of the \((a, b)\)-twisted geometry of \(\mathbb{P}^{n_1+1} \times \mathbb{P}^{n_2+1}\) are defined by the equivariant integrals

\[
\tilde{N}^{SQ}_{g,(d_1,d_2)} = \int_{\overline{\mathcal{M}}_g(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, (d_1, d_2))} \prod_{i=1}^{r} e(R \pi_* S^{-a_{1i}} \otimes S^{-a_{2i}}) \prod_{j=1}^{s} e(-R \pi_* S^{b_{1j}} \otimes S^{b_{2j}}).
\]

Remark 5. By Quantum Lefschetz theorem [13], for genus 0 and 1, the \((a, 0)\) - twisted quasimap theories of \(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}\) recover the quasimap theories of complete intersections of degree \((a)\) in \(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}\). For higher genus, more techniques are required to obtain the theories of complete intersections from twisted theories. For quintic threefolds, this was studied using NMSP theory [4] or log GLSM theory [12]. Using these approaches we expect to be able to study the quasimap theories of Calabi-Yau fibrations for higher genus. We will return to these problems in the future.

0.4. Main results.

0.4.1. Local \(\mathbb{P}^1 \times \mathbb{P}^1\) theory. Let

\[
\mathcal{F}^{SQ}_g(q_1, q_2) = \sum_{d_1,d_2 \geq 0} N^{SQ}_{g,(d_1,d_2)} q_1^{d_1} q_2^{d_2} \in \mathbb{C}[[q_1, q_2]]
\]

be the genus g quasimap series of local \(\mathbb{P}^1 \times \mathbb{P}^1\). In order to state the theorem, we define a series in \(q_1\).

\[
X(q_1) = (1 - 8q_1)^{-1/2} = 1 + 4q_1 + 24q_1^2 + 160q_1^3 + \ldots
\]

**Theorem 6.** For the quasimap invariants of \(K_{\mathbb{P}^1 \times \mathbb{P}^1}\), we have

(i) \(q_1 \frac{\partial}{\partial q_1} \mathcal{F}^{SQ}_1 \in \mathbb{C}[[q_2]][X^2, (1 + X^2)^{-1}]\),

(ii) \(\mathcal{F}^{SQ}_g \in \mathbb{C}[[q_2]][X^2, (1 + X^2)^{-1}]\) for \(g \geq 2\).

In other words, the coefficients of \(q_2^k\) in \(q_1 \frac{\partial}{\partial q_1} \mathcal{F}^{SQ}_1, \mathcal{F}^{SQ}_{g \geq 2}\) are elements in the ring

\[
\mathbb{C}[X^2, (1 + X^2)^{-1}].
\]

0.4.2. Elliptic fibration : \(K3\) surface. Let

\[
\mathcal{F}^{SQ}_g(q_1, q_2) = \sum_{d_1,d_2 \geq 0} N^{SQ}_{g,(d_1,d_2)} q_1^{d_1} q_2^{d_2} \in \mathbb{C}[[q_1, q_2]]
\]

be the \(((3; 2), (0; 0))\)-twisted genus g quasimap series of \(\mathbb{P}^2 \times \mathbb{P}^1\).
Recall the series which generate the ring $\mathbb{Q}E_F$,
\[
L(q_1) = (1 - 27q_1)^{-\frac{1}{2}} = 1 + 9q_1 + 162q_1^2 + \ldots ,
\]
\[
B_1'(q_1) = q_1 \frac{\partial}{\partial q_1} I^E_1,
\]
\[
X(q_1) = \frac{q_1 \frac{\partial}{\partial q_1} B_1'}{1 + B_1'}.
\]

**Theorem 7.** For the $((3; 2), (0; 0))$-twisted genus $g$ quasimap series of $\mathbb{P}^2 \times \mathbb{P}^1$, we have
(i) $q_1 \frac{\partial}{\partial q_1} \mathcal{F}_1^{SQ}(q_1, q_2) \in \mathbb{C}[[q_2]](L^3, B'_1, X)$,
(ii) $\mathcal{F}_g^{SQ}(q_1, q_2) \in \mathbb{C}[[q_2]](L^3, B'_1, X)$ for $g \geq 2$.

From the low degree calculations, we make the following conjecture.

**Conjecture 8.** For the $((3; 2), (0; 0))$-twisted genus $g$ quasimap series of $\mathbb{P}^2 \times \mathbb{P}^1$, we have
(i) $q_1 \frac{\partial}{\partial q_1} \mathcal{F}_1^{SQ}(q_1, 0) \in \mathbb{C}[[q_2]](L^3, B'_1, X)$,
(ii) $\mathcal{F}_g^{SQ}(q_1, 0) \in \mathbb{C}[[q_2]](L^3, B'_1, X)$ for $g \geq 2$.

For example, in genus 1 we have
\[
q_1 \frac{\partial}{\partial q_1} \mathcal{F}_1^{SQ}(q_1, 0) = -X.
\]

0.4.3. **Elliptic fibration : CY 3-fold.** Let
\[
\mathcal{F}_g^{SQ}(q_1, q_2) = \sum_{d_1, d_2 \geq 0} N_{g,(d_1, d_2)}^{SQ} q_1^{d_1} q_2^{d_2} \in \mathbb{C}[[q_1, q_2]]
\]
be the $((3; 3), (0; 0))$-twisted genus $g$ quasimap series of $\mathbb{P}^2 \times \mathbb{P}^2$. We use the same generators \([2]\) in the following theorem.

**Theorem 9.** For the $((3; 3), (0; 0))$-twisted genus $g$ quasimap series of $\mathbb{P}^2 \times \mathbb{P}^2$, we have
(i) $q_1 \frac{\partial}{\partial q_1} \mathcal{F}_1^{SQ}(q_1, q_2) \in \mathbb{C}[[q_2]](L^3, B'_1, X)$,
(ii) $\mathcal{F}_g^{SQ}(q_1, q_2) \in \mathbb{C}[[q_2]](L^3, B'_1, X)$ for $g \geq 2$.

From the genus one calculation
\[
q_1 \frac{\partial}{\partial q_1} \mathcal{F}_1^{SQ}(q_1, 0) = -\frac{1}{4}(L^3 - 1) - \frac{3}{2}X,
\]
we make the following conjecture.

**Conjecture 10.** For the $((3; 3), (0; 0))$-twisted genus $g$ quasimap series of $\mathbb{P}^2 \times \mathbb{P}^2$, we have
0.4.4. K3 fibration. Let

\[ F_{SQ}^g(q_1, q_2) = \sum_{d_1, d_2 \geq 0} N_{g, (d_1, d_2)}^S q_1^{d_1} q_2^{d_2} \in \mathbb{C}[[q_1, q_2]] \]

be the \(((4; 2), (0; 0))\)-twisted genus \(g\) quasimap series of \(\mathbb{P}^3 \times \mathbb{P}^1\). In order to state the theorem, we need the generators \([1]\) of the ring \(QKF\) and the extra generators \(E'_1, B'_2\) whose definition will appear in Section 4.2.

**Theorem 11.** For the \(((4; 2), (0; 0))\)-twisted genus \(g\) quasimap series of \(\mathbb{P}^3 \times \mathbb{P}^1\), we have

(i) \( q_1 \frac{\partial}{\partial q_1} F_{SQ}^1(q_1, q_2) \in \mathbb{C}[[q_2]][L^4, A'_1, X, E'_1, B'_2] \),

(ii) \( F_{SQ}^g(q_1, q_2) \in \mathbb{C}[[q_2]][L^4, A'_1, X, E'_1, B'_2] \) for \(g \geq 2\).

Using the argument in the proof of the theorem in Section 4, one can show that the coefficient of \(q_2^d\) in \(F_{SQ}^g\) do not have extra series \(E'_1\) and \(B'_2\) for fixed \(d\).

For example in the genus 1 case, we have

\[ q_1 \frac{\partial}{\partial q_1} F_{SQ}^1(q_1, 0) = \frac{13}{12} (1 - L^4) + 2X. \]

From this observation, we conjecture the stronger result.

**Conjecture 12.** For the \(((4; 2), (0; 0))\)-twisted genus \(g\) quasimap series of \(\mathbb{P}^3 \times \mathbb{P}^1\), we have

(i) \( q_1 \frac{\partial}{\partial q_1} F_{SQ}^1(q_1, q_2) \in \mathbb{C}[[q_2]][L^4, A'_1, X] \),

(ii) \( F_{SQ}^g(q_1, q_2) \in \mathbb{C}[[q_2]][L^4, A'_1, X] \) for \(g \geq 2\).

0.5. **Plan of the paper.** We will prove Theorem 6, 7, 9 and 11 in Section 1, 2, 3 and 4, respectively.

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1. Local $\mathbb{P}^1 \times \mathbb{P}^1$

1.1. Overview. Let $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ be the total space of the canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. The $((0; 0), (-2; -2))$-twisted theory on $\mathbb{P}^1 \times \mathbb{P}^1$ recovers the standard theory of local $\mathbb{P}^1 \times \mathbb{P}^1$. Since the quasimap invariants are independent of $\lambda_{i,k}$, we are free to use the specialization

$$\lambda_{1,1} = 1, \lambda_{1,2} = -1, \lambda_{2,1} = \sqrt{-1}, \lambda_{2,2} = -\sqrt{-1}.\quad (3)$$

The specialization (3) will be imposed for our entire study of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$. For simplicity we use following notations:

$$\alpha_0 = \lambda_{1,1}, \alpha_1 = \lambda_{1,2}, \beta_0 = \lambda_{2,1}, \beta_1 = \lambda_{2,2}.\quad (3)$$

1.2. Generators. From the small $I$-function associated to $K_{\mathbb{P}^1 \times \mathbb{P}^1}$,

$$I(q_1, q_2) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} \prod_{k=0}^{2d_1 + 2d_2 - 1} \left(-2H_1 - 2H_2 - k^2\right) \prod_{i=0}^{d_1} (H_1 - \alpha_i + k^2) \prod_{j=0}^{d_2} (H_1 - \beta_j + k^2),$$

define the series $I_{ij}$ by the equation

$$I = 1 + \left(I_{11} H_1 + I_{12} H_2\right) \frac{1}{z} + \left(I_{21} H_1^2 + I_{22} H_1 H_2 + I_{23} H_2^2 + I_{24} H_1 + I_{25} H_2\right) \frac{1}{z^2} + \left(I_{31} H_1^3 + I_{32} H_1^2 H_2 + I_{33} H_1 H_2^2 + I_{34} H_2^3 + I_{35} H_1^2 + I_{36} H_1 H_2 + I_{37} H_2^2 + I_{38} H_1 + I_{39} H_2\right) \frac{1}{z^3} + \mathcal{O}\left(\frac{1}{z^4}\right).$$

From the Birkhoff factorization which will appear in Section 1.3.2, it is natural to define the following series.

$$J_{11} = \frac{I_{11} + I_{11}^* I_{22} + I_{21}^* + I_{12}^* I_{21} - I_{12}^* I_{21}}{1 + I_{11}^* I_{12} + I_{21}^* + I_{12}^* I_{21}},$$

$$J_{12} = \frac{-I_{11} I_{12} + (1 + I_{12})(I_{12} + I_{22}) - I_{12}^* I_{22}}{1 + I_{11}^* I_{12} + I_{22}^* + I_{12}^* I_{22}},$$

$$J_{13} = \frac{-I_{12}^* I_{23} + I_{12}^* I_{23} - I_{12}^* I_{23}}{1 + I_{11}^* I_{12} + I_{22}^* + I_{12}^* I_{22}},$$

$$J_{14} = \frac{I_{24}^* + I_{12}^* I_{24} - I_{12}^* I_{24}}{1 + I_{11}^* I_{12} + I_{22}^* + I_{12}^* I_{22}},$$

$$J_{15} = \frac{I_{25}^* + I_{12}^* I_{25} - I_{12}^* I_{25}}{1 + I_{11}^* I_{12} + I_{22}^* + I_{12}^* I_{22}}.$$
Here, upper subscript ‘ and * mean the action of differential operators $q_1 \frac{\partial}{\partial q_1}$ and $q_2 \frac{\partial}{\partial q_2}$ respectively. Define the series $J_{1i}$ by the equations

$$J_{11} = J_{12}, \quad J_{12} = J_{14}, \quad J_{13} = J_{15} + s_1 J_{13}, \quad J_{14} = J_{11} - s_2 J_{13}.$$  

Here, $s_1 = \beta_0 + \beta_1$ and $s_2 = \beta_0 \beta_1$. Define the series $K_{ij}$ by the equations

$$K_{11} = \frac{-I_{11} J_{11}' + I_{12} J_{12}' + I_{21} J_{21}'}{1 + I_{11}' - I_{12}' + I_{11} J_{12}'},$$

$$K_{12} = \frac{I_{11}(1 + I_{11}') - I_{12}' (I_{11} + I_{12}') - (1 + I_{11}') I_{22}'}{1 + I_{11}' - I_{12}' + I_{12} J_{12}'},$$

$$K_{13} = \frac{I_{12} + I_{12}' I_{12}' - I_{11} J_{23}' + I_{23}'}{1 + I_{11}' - I_{12}' + I_{11} J_{12}'},$$

$$K_{14} = \frac{-I_{12}' J_{24}' + I_{24}'}{1 + I_{11}' - I_{12}' + I_{12} J_{12}'},$$

$$K_{15} = \frac{-I_{11}' J_{25}' + I_{25}'}{1 + I_{11}' - I_{12}' + I_{12} J_{12}'}.
Finally define the series $M_{1.3}$. Basic correlators.

Under the specialization (3), it is easy to check the following results.

Similarly define the series $K_{1i}$ by the equations

$$K_{11} = K_{12}, \quad K_{12} = K_{14}, \quad K_{13} = K_{15} + s_1 K_{13}, \quad K_{14} = K_{11} - s_2 K_{13}.$$

Finally define the series $M_{1i}$ by the equations

$$M_{11} = \frac{J_{12}^* J_{12} + J_{12}^* J_{14} + (J_{23}^* + J_{11}) s_1}{1 + J_{11}^*},$$

$$M_{12} = \frac{J_{21}^* J_{12} + J_{28}^* - J_{12}^* J_{14} - J_{13}^* K_{12} - (J_{23}^* + J_{11}) s_2}{1 + J_{11}^*},$$

$$M_{13} = \frac{J_{22}^* J_{29}^* - J_{12}^* J_{14} - J_{12}^* J_{14} - J_{13}^* K_{12} + (J_{27} + J_{13}) s_1 + J_{24}^* (s_1^2 - s_2)}{1 + J_{11}^*},$$

$$M_{14} = \frac{J_{25}^* - J_{12}^* J_{14} - J_{13}^* K_{12} + (J_{27} + J_{13}) s_1 - J_{24}^* s_1 s_2}{1 + J_{11}^*}.$$

Under the specialization (3), it is easy to check the following results.

$$J_{12} = J_{13} = K_{12} = K_{13} = M_{11} = M_{12} = 0.$$  

In the next section, we will find the relations between the series defined above.

1.3. Basic correlators.
1.3.1. **Light markings.** Moduli of quasimaps can be considered with $n$ ordinary (weight 1) markings and $k$ light (weight 0+) markings,

$$ \mathcal{Q}_{g,n;k}^{0+,0+}(\mathbb{P}^1 \times \mathbb{P}^1,(d_1,d_2)). $$

See [7] for more explanations. For $\gamma_i \in H^*_T(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\delta_j \in H^*_T(\mathbb{C}^2/(\mathbb{C}^* \times \mathbb{C}^*))$, we define series for the $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ geometry,

$$ \left(\gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; \delta_1, \ldots, \delta_k\right)_{g,n,k,(d_1,d_2)}^{0+,0+} = \int_{[\mathcal{Q}_{g,n;k}^{0+,0+}(\mathbb{P}^1 \times \mathbb{P}^1,(d_1,d_2))]|_{\text{vir}}} e(\text{Obs}) \cdot \prod_{i=1}^n ev^*_i(\gamma_i) \psi^{a_i} \cdot \prod_{j=1}^k \hat{ev}^*_j(\delta_j), $$

$$ \left(\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle \right)_{0,n}^{0+,0+} = \sum_{d \geq 0} \sum_{k \geq 0} \frac{q_1^{d_1} q_2^{d_2}}{k!} \left(\gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n}; t, \ldots, t\right)_{0,n,k,(d_1,d_2)}^{0+,0+}, $$

where, in the second series, $t \in H^*_T(\mathbb{C}^2/(\mathbb{C}^* \times \mathbb{C}^*))$.

For each $T$-fixed point $p_{ij} \in \mathbb{P}^1 \times \mathbb{P}^1$, let

$$ e_{ij} = e(T_{p_{ij}}(\mathbb{P}^1 \times \mathbb{P}^1)) \cdot (-2\alpha_i - 2\beta_j) $$

be the equivariant Euler class of the tangent space of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ at $p_{ij}$. Let

$$ \phi_{ij} = \frac{(-2\alpha_i - 2\beta_j)(H_1 - \alpha_{i+1})(H_2 - \beta_{j+1})}{e_{ij}}, \quad \phi^{ij} = e_{ij} \phi_{ij} \in H^*_T(\mathbb{P}^1 \times \mathbb{P}^1) $$

be the cohomology classes. The following series will play an important role.

$$ S_{ij}(\gamma) := e_{ij} \langle \left(\frac{\phi_{ij}}{z - \psi}\right)^{0+,0+} \rangle_{0,2} $$

We also write

$$ S(\gamma) := \sum_{i=0}^1 \sum_{j=0}^1 \phi_{ij} S_{ij}(\gamma). $$

1.3.2. **I-function and Birkhoff factorization.** Via the geometry of weighted quasimap graph space

$$ \mathcal{QG}_{g,n;k,(d_1,d_2)}^{0+,0+}(\mathbb{C}^2/(\mathbb{C}^* \times \mathbb{C}^*)) $$

the big $I$-function is defined in [7]. See also [16, Section 3.4] for brief introduction.
The $I$-function can be evaluated explicitly using the arguments in [7, Section 5].

**Proposition 13.** For $t = t_1H_1 + t_2H_2 \in H^*_T([[\mathbb{C}^2 \times \mathbb{C}^2]/(\mathbb{C}^* \times \mathbb{C}^*)], \mathbb{Q})$, 

\[
I(t) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} e^{t_1(H_1 + d_1z)/z + t_2(H_2 + d_2z)/z} 
\cdot \frac{\prod_{k=0}^{2d_1+2d_2-1}(-2H_1 - 2H_2 - kz)}{\prod_{i=0}^{d_1} (H_1 - \alpha_i + kz) \prod_{j=0}^{d_2} (H_1 - \beta_j + kz)}.
\]

Using Birkhoff factorization, an evaluation of the series $S(H_1H_2^d)$ can be obtained from $I$-function, see [14]:

\[
S(1) = I,
\]

\[
S(H_1) = E_{11} \cdot z \frac{\partial}{\partial t_1} S(1) + E_{12} \cdot z \frac{\partial}{\partial t_2} S(1),
\]

\[
S(H_2) = E_{21} \cdot z \frac{\partial}{\partial t_1} S(1) + E_{22} \cdot z \frac{\partial}{\partial t_2} S(1),
\]

\[
S(H_1H_2) = E_{31} \cdot z \frac{\partial}{\partial t_1} S(H_2) + E_{32} \cdot S(1).
\]

Here, $E_{ij}$ are the series defined by

\[
E_{11} = \frac{1 + I_{12}^*}{1 + I_{11}' - I_{11}'I_{12}^* + I_{12}^* + I_{11}'I_{12}^*},
\]

\[
E_{12} = \frac{-I_{12}^*}{1 + I_{11}' - I_{11}'I_{12}^* + I_{12}^* + I_{11}'I_{12}^*},
\]

\[
E_{21} = \frac{-I_{11}^*}{1 + I_{11}' - I_{11}'I_{12}^* + I_{12}^* + I_{11}'I_{12}^*},
\]

\[
E_{22} = \frac{1 + I_{11}'}{1 + I_{11}' - I_{11}'I_{12}^* + I_{12}^* + I_{11}'I_{12}^*},
\]

\[
E_{31} = \frac{1}{1 + K_{11}^*},
\]

\[
E_{32} = \frac{-K_{14}'}{1 + K_{11}'},
\]

1.3.3. **Picard-Fuchs equations and asymptotic expansion.** The function $I$ satisfies the following Picard-Fuchs equations.
\[
\left( \left( \frac{d}{dt_1} \right)^2 - 1 - q_1 \left( 2 \left( \frac{d}{dt_1} \right) + 2 \left( \frac{d}{dt_2} \right) \right) \left( 2 \left( \frac{d}{dt_1} \right) + 2 \left( \frac{d}{dt_2} \right) + z \right) \right) I = 0,
\]
\[
\left( \left( \frac{d}{dt_2} \right)^2 + 1 - q_2 \left( 2 \left( \frac{d}{dt_1} \right) + 2 \left( \frac{d}{dt_2} \right) \right) \left( 2 \left( \frac{d}{dt_1} \right) + 2 \left( \frac{d}{dt_2} \right) + z \right) \right) I = 0.
\]

Define small \(I\)-function
\[
I(q_1, q_2) \in H^*_T(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})[[q_1, q_2]]
\]
by the restriction
\[
I(q_1, q_2) = I(q_1, q_2, t_1, t_2)|_{t_1=t_2=0}
\]
Define differential operators
\[
D_1 = q_1 \frac{d}{dq_1}, \quad D_2 = q_2 \frac{d}{dq_1}, \quad M_1 = H_1 + zD_1, \quad M_2 = H_2 + zD_2.
\]

The small \(I\)-function satisfies following Picard-Fuchs equations.
\[
(6) \quad \left( M_1^2 - 1 - q_1 (2M_1 + 2M_2) (2M_1 + 2M_2 + z) \right) I = 0,
\]
\[
\left( M_2^2 + 1 - q_1 (2M_1 + 2M_2) (2M_1 + 2M_2 + z) \right) I = 0.
\]

The restriction \(I|_{H_1=\alpha_i, H_2=\beta_j}\) admits following asymptotic form
\[
(7) \quad I|_{H_1=\alpha_i, H_2=\beta_j} = e^{U_{ij}/z} \left( R_{0,ij} + R_{1,ij} z + R_{2,ij} z^2 + \ldots \right)
\]
with series \(U_{ij}, R_{k,ij} \in \mathbb{C}[[q_1, q_2]]\). Define series \(L_{ij}\) and \(UD_{ij}\) for \(0 \leq i \leq 1, 0 \leq j \leq 1\) by
\[
L_{ij} = \alpha_i + q_1 \frac{d}{dq_1} U_{ij}, \quad UD_{ij} = \beta_j + q_2 \frac{d}{dq_2} U_{ij}.
\]
Let \(L_{ij}\) be the series in \(q_1\) defined by constant term with respect to \(q_2\). The series \(L_{ij}\) is found by solving differential equations obtained from the coefficient of \(z^k\).
\[
(8) \quad L_{ij}(q_1) = \frac{4\beta_j q_1 + \alpha_i \sqrt{1 + 4(-1 + \beta_j^2)q_1}}{1 - 4q_1}.
\]

Define the series \(u_{k,ij}\) and \(R_{n,k,ij}\) by the following equations.
\[
(9) \quad U_{ij} := u_{0,ij} + u_{1,ij} q_2 + u_{2,ij} q_2^2 + \ldots,
\]
\[
(10) \quad R_{n,ij} := R_{n,0,ij} + R_{n,1,ij} q_2 + R_{n,2,ij} q_2^2 + \ldots.
\]
Denote by \(G_{ij}\) the subring of \(\mathbb{C}[[q_1]]\) generated by \(L_{ij}\) and \(\frac{1}{\sqrt{1+\beta_j L_{ij}}}\).
\[ G_{ij} = \mathbb{C}[L_{ij}, \frac{1}{\sqrt{1 + \beta_j L_{ij}}}] \subset \mathbb{C}[[q_1]]. \]

By analyzing the differential equations from the coefficient of \( z^k \), we obtain the following results.

**Lemma 14.** We have
\[ L_{ij}, UD_{ij}, R_{n,ij} \in G_{ij}[[q_2]]. \]

**Proof.** To simplify the notations, we will only prove the lemma for the case \((i,j) = (0,0)\) and omit the index \((i,j)\) for the series \(L_{ij}, UD_{ij}, R_{k,ij}\). The proof for other \((i,j)\) follows from the same argument. If we apply the equations (6) to the asymptotic form (7), the coefficients of \( z \) in each equation yield the following equation,
\[ 1 + UD^2 - 4q_2(L + UD)^2 = 0. \]

By applying (9) to above equation, we obtain the equation Eq\(k\) from the coefficient of \( q^{k+1} \). Each term of equation Eq\(k\) is a monomial of \((D_1)^l u_m\) for \( l \geq 0 \). Furthermore one can easily check that the equations Eq\(k\) are linear in \( u_k \) with coefficient \( 2k\sqrt{-1} \) and has no \((D_1)^l u_m\) term for \( m \geq k + 1 \). The statements of the lemma for \(L\) and \(UD\) follow from following equation which is easy to check from (8),
\[ D_1 L = \frac{(\sqrt{-1} + L)(-1 + L^2)}{2(1 + \sqrt{-1}L)}. \]

The same argument for the coefficient of \( z^{n+1} \) in (6) gives the proof of the statement of the lemma for \(R_n\).

\[ \square \]

1.3.4. **Relations on generators.** We prove some relations on the generators defined in Section 1.2.

**Proposition 15.** The series

\[ J'_{11}, J^{\bullet}_{11}, J'_{14}, J^{\bullet}_{14}, K'_{11}, K^{\bullet}_{11}, K'_{14}, K^{\bullet}_{14}, M'_{12}, M^{\bullet}_{12}, M'_{13}, M^{\bullet}_{13} \]

are representable as rational functions in \( I'_{11}, I^{\bullet}_{11}, L, UD \).

**Proof.** First consider the quantum product by
\[ (H_1 + q_1 \frac{\partial}{\partial q_1}), (H_2 + q_2 \frac{\partial}{\partial q_2}). \]

In the basis \( \{1, H_1, H_2, H_1 H_2\} \), they can be considered as matrix multiplications and the eigenvalues of these matrices are \( L_{ij} \) and \( UD_{ij} \), respectively.
From the commutativity of quantum product, we also get some relations. Finally, using the fact that \(q_1 \frac{\partial}{\partial q_1}\) and \(q_2 \frac{\partial}{\partial q_2}\) commute, we get some relations by repeatedly applying \(q_1 \frac{\partial}{\partial q_1}\) and \(q_2 \frac{\partial}{\partial q_2}\) to S-operators.

By the relations we get from above, we can find following explicit equations. Since the results are independent of the index \((i, j)\), we will omit the index \((i, j)\) from the series \(L_{ij}\) and \(UD_{ij}\).

\[
J'_{11} = -\frac{(L + UD)^2(-1 + L^2 + I_{11}^*(-1 + L^2) - I_{11}' L \cdot UD)}{2(1 + I_{11} + I_{11}^*)^2(-1 + L^2 - UD^2)},
\]

\[
J_{11}^* = \frac{1}{2(1 + I_{11} + I_{11}^*)^2(-1 + L^2 - UD^2)} \left( 2 - 2L^2 - L^3UD + 2UD^2 - 2L^2UD^2 - L \cdot UD^3 + (I_{11}')^2(2 - 2L^2 + 2UD^2) + (I_{11}^*)^2(2 - 2L^2 + 2UD^2) - 4(1 + UD^2) + 2L^2(2 + UD^2) + I_{11}'(4 + 5UD^2 + UD^4 + L^2(-3 + UD^2) + I_{11}^*(4 - 4L^2 + 4UD^2) + 2L(UD + UD^3)) \right),
\]

\[
J_{14} = \frac{1}{2(1 + I_{11} + I_{11}^*)^2(-1 + L^2 - UD^2)} \left( (1 + I_{11}^*)(-1 + L^2)(-2 + L^2 - UD^2 + 2I_{11}^*(-1 + L^2 + L \cdot UD)) + I_{11}'(-3L^3UD + 4(1 + UD^2) - 2L^2(2 + UD^2) + L \cdot UD(4 + UD^2) - 4I_{11}^*(-1 + L^2)(1 + L \cdot UD + UD^2)) + 2(I_{11}')^2(1 + UD^2 + L^2(-1 + UD^2) + L(UD + UD^3)) \right),
\]

\[
J_{14}^* = \frac{1}{2(1 + I_{11} + I_{11}^*)^2(-1 + L^2 - UD^2)} \left( (1 + I_{11}^*)LUD(-2 + L^2 - UD^2 + 2I_{11}^*(-1 + L^2 + L \cdot UD)) + 2(I_{11}')^2(L^2 + UD^2 + UD^4 + L(UD + UD^3)) + I_{11}^*(UD^2 + UD^4 + L^2(1 - (3 + 4I_{11}^*)UD^2) - 2(1 + 2I_{11}^*)L(UD + UD^3)) \right),
\]
\[ K'_{11} = \frac{1}{2(1 + I'_{11} + I^*_1)^2(-1 + L^2 - UD^2)} \left( 2 - 2L^2 - L^3UD + 2UD^2 ight. \right. \\
\left. \left. - 2L^2UD - L \cdot UD^3 + (I'_{11})^2(2 - 2L^2 + 2UD^2) + (I^*_1)^2(2 - 2L^2 \\
+ 2UD^2) + I'_{11}(4 + L^4 - 2L \cdot UD + 2L^3UD + 3UD^2 + L^2(-5 + UD^2)) \\
- I'_{11}(L^3UD + L \cdot UD^3 + 4I^*_1(-1 + L^2 - UD^2) - 4(1 + UD^2) \\
+ 2L^2(2 + UD^2)) \right) , \]

\[ K^*_{11} = -\frac{1}{2(1 + I'_{11} + I^*_1)^2(-1 + L^2 - UD^2)} \left( (L + UD)^2(1 + I'_{11} \\
- I^*_1) \cdot UD + UD^2 + I'_{11} UD^2 \right) \]

\[ K'_{14} = \frac{1}{2(1 + I'_{11} + I^*_1)^2(-1 + L^2 - UD^2)} \left( -2(I^*_1)^2(-L^2 + L^4 \\
- L \cdot UD + L^3UD - UD^2) - (1 + I'_{11})L \cdot UD(2 - L^2 + UD^2 \\
+ 2I'_{11}(1 + L \cdot UD + UD^2)) + I^*_1(-L^4 + UD^2 - 2L(UD \\
+ 2I'_{11}UD) + 2L^3(UD + 2I'_{11}UD) + L^2(1 + (3 + 4I'_{11})UD^2)) \right) , \]

\[ K^*_{14} = \frac{1}{2(1 + I'_{11} + I^*_1)^2(-1 + L^2 - UD^2)} \left( (-2 + L^2 - UD^2)(1 + UD^2) \\
- 2(I^*_1)^2(1 + UD^2)(1 + L \cdot UD + UD^2) + I'_{11}(1 + UD^2)((-4 + L^2 \\
- 2L \cdot UD - 3UD^2 + 4I^*_1(-1 + L^2 + L \cdot UD)) - 2(I^*_1)^2(1 - L \cdot UD \\
+ L^3UD + UD^2 + L^2(-1 + UD^2)) + I^*_1(-L^3UD - 4(1 + UD^2) \\
+ 2L^2(2 + UD^2) + L \cdot UD(4 + 3UD^2)) \right) , \]

\[ M'_{12} = -\frac{1}{(L + UD)^2} \left( (1 + 3I'_{11} + 2(I^*_1)^2)L^2 + 2(I^*_1 + (I^*_1)^2 \\
- 2I'_{11}I^*_1 - I^*_1(2 + I^*_1))L \cdot UD + (1 + 3I'_{11} + 2(I^*_1)^2)UD^2 \right) , \]

\[ M^*_{12} = \frac{1}{(L + UD)^2} \left( (1 + I^*_1 + 2(I^*_1)^2)L^2 - 2((I^*_1)^2 - (-1 + I^*_1)I^*_1 \\
+ 2I'_{11}(1 + I^*_1))L \cdot UD + (1 + I^*_1 + 2(I^*_1)^2)UD^2 \right) . \]
The series $r_{ij}$ satisfies following differential equation.

\[
D_1 r_{ij} = \frac{(\sqrt{-1} + r_{ij}^2)(-1 + r_{ij}^4)}{4(r + \sqrt{-1}r_{ij}^3)}.
\]
From (8), we obtain the following equation.

\[
1 + D_1 a_0 = \frac{(1 + \sqrt{-1}) + (1 - \sqrt{-1}) r_{ij}^2}{2 r_{ij}} \tag{15}
\]

Lemma 16. For \( k \geq 1 \), We have

\[
a_k \in r_{ij} \cdot \mathbb{C}[r_{ij}^2, r_{ij}^{-2}, (1 + \beta_j r_{ij})^{-1}] .
\]

Proof. If we apply the forms (7) and (13) to the equation (12), we get the equation Eq\(_k\) from the coefficient of \( q_k^2 \). Each term of Eq\(_k\) is monomial of \((D_1)^l L, (D_1)^l a_m, (D_1)^l u_m\). Furthermore Eq\(_k\) is linear in \( a_k \) with coefficient

\[
(2k)^2 (1 + D_1 a_0) L_{ij}^2 (-\sqrt{-1} + L_{ij}) (\sqrt{-1} + L_{ij})^2
\]

and has no \((D_1)^l a_m\) with \( m \geq k + 1 \). The statement of the proposition follows from (14) and (15). \( \square \)

1.4. Higher genus series.

1.4.1. Graphs. Let \( g \geq 2 \). A decorated graph \( \Gamma \in G_g \) consist of the data \((V, E, g, p)\) such that

(i) \( V \) is the vertex set,
(ii) \( E \) is the edge set (possibly including self-edges),
(iii) \( g : V \to \mathbb{Z}_{\geq 0} \) is a genus assignment satisfying

\[
g = \sum_{v \in V} g(v) + h^1(\Gamma)
\]

and for which \((V, E, g)\) is stable graph,

(iv) \( p : V \to (\mathbb{P}^1 \times \mathbb{P}^1)^T \) is an assignment of a \( T \)-fixed point \( p(v) \) to each vertex \( v \in V \).

1.4.2. Localization formula. We summarize the localization formula for the \( K_{\mathbb{P}^1 \times \mathbb{P}^1} \) quasimap theories.

We write the localization formula as

\[
\sum_{d_1, d_2 \geq 0} [\overline{Q}_g(K_{\mathbb{P}^1 \times \mathbb{P}^1}, (d_1, d_2))]_{\text{vir}} q_1^{d_1} q_2^{d_2} = \sum_{\Gamma \in G_g} \text{Cont}_\Gamma .
\]

Proposition 17. We have

\[
\text{Cont}_\Gamma = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{A \in \mathbb{Z}^F_{>0}} \prod_{v \in V} \text{Cont}^A(v) \prod_{e \in E} \text{Cont}^A(e),
\]
where the vertex and edge contributions with incident flag A-values $(a_1, \ldots, a_n)$ and $(b_1, b_2)$ respectively are

$$\text{Cont}_A^\Gamma(v) = \mathcal{P}\left[\psi_1^{a_1-1}, \ldots, \psi_n^{a_n-1}|H_p^{(v)}\right]_{g(v), n}^{p(v), 0+},$$

$$\text{Cont}_A^\Gamma(e) = \left[(-1)^{b_1+b_2}e^{-\frac{v_{p_1}}{y}-\frac{v_{p_2}}{y}} \sum_{0 \leq i, 0 \leq j \leq 1} \bar{S}_{p_1}(\phi_{ij})|_{z=x}\bar{S}_{p_2}(\phi_{ij})|_{z=y}\right] x^{b_1}y^{b_2-1-x}+y^{b_2-2-z}+\cdots+(-1)^{b_1-1}x^{b_1}+b_2-1.$$

For the precise definition of notations in the vertex contribution, see [16]. In this paper we do not need the exact definition of vertex contribution. We only need following results for vertex contribution

$$\text{Cont}_A^\Gamma(v) \in \mathbb{C}[R_{0,ij}, R_{1,ij}, R_{2,ij}, \ldots],$$

which follows from the definition.

The subscript in the edge contribution signifies a (signed) sum of the respective coefficients.

1.4.3. Proof of Theorem 6 In the proof of Lemma 14 we can actually check that $\frac{1}{\sqrt{1+\alpha_iL_{ij}}}$ only appears in $R_{k,ij}$. And we can easily check in the decomposition formula in Proposition 17 that the order of factor $\frac{1}{\sqrt{1+\alpha_iL_{ij}}}$ in $\mathcal{F}_{SQ}^g$ is even. (Precisely, $2g(v) - 2$ at the vertex $v$.) By Lemma 14, (5) and Lemma 16, Proposition 17 immediately yields

$$\mathcal{F}_{SQ}^g \in \mathbb{C}[q_2][r_{00}, r_{00}^{-1}, (1 + \beta_0 r_{01})^{-1}, r_{01}, r_{01}^{-1}, (1 + \beta_1 r_{01})^{-1}, r_{10}, r_{10}^{-1}, (1 + \beta_0 r_{10})^{-1}, r_{11}, r_{11}^{-1}, (1 + \beta_1 r_{11})^{-1}].$$

Now we need to show that the order of $r_{ij}$ in $\mathcal{F}_{SQ}^g$ is always even. By [15] we can represent each edge contribution in the Proposition 17 formally as rational function in variables $D_1^mD_2^mI_{11}$, $D_1^mD_2^mU_{k,ij}$ and $D_1^mD_2^mR_{k,ij}$. By direct calculations we can check that the sum of order of the factors $D_1^mD_2^mR_{k,ij}$ is always even. Therefore we conclude that order of $r_{ij}$ in $\mathcal{F}_{SQ}^g$ in each edge contribution is always even from [14], [15] and Lemma 16.

Now from the definitions of $r_{ij}$, we obtain

$$\mathcal{F}_{SQ}^g \in \mathbb{C}[X, (\sqrt{-1} + X)^{-1}, (\sqrt{-1} - X)^{-1}].$$

Since all coefficients of $q_1^{k_1}q_2^{k_2}$ in $\mathcal{F}_{SQ}^g$ are real numbers by definition, the statement of the Theorem 6

$$\mathcal{F}_{SQ}^g \in \mathbb{C}[X, (1 + X^2)^{-1}].$$
follows immediately from (16).

To finish the proof, we need to show that
\[ F_{SQ} \in \mathbb{C}[X^2, (1 + X^2)^{-1}] . \]

For this, we need the mirror symmetry argument which was explained by Iritani.

Using the Givental’s equivariant mirror \( K_{\mathbb{P}^1 \times \mathbb{P}^1} \), the asymptotic form of \( \mathbb{I}_{H_1=\alpha_i, H_2=\beta_j} \) can be calculated using the oscillatory integral associated to the mirror of \( K_{\mathbb{P}^1 \times \mathbb{P}^1} \). See [3, Proposition 6.9] for the precise statement. Using this oscillatory integral, it is easy to see that the presentation of \( R_{n,ij} \) in terms of \( L_{ij} \) do not depend on the choice of \( (i, j) \). In other words, \( R_{n,ij} \) in (7) have same polynomial expressions in terms of \( L_{ij} \) for all \( (i, j) \). For example, this was explained explicitly for the case of \( K_{\mathbb{P}^2} \) in [17, Appendix A]. The argument in [17, Appendix A] applies to the case of \( K_{\mathbb{P}^1 \times \mathbb{P}^1} \) to yield the similar result. From this observation and Proposition 17, we conclude that \( F_{SQ} \) is symmetric rational function in \( L_{ij} \). Then the statement of the Theorem follows from the fact that \( L_{0i} \) and \( L_{1i} \) are the two roots of the equations for \( i = 0, 1 \),

\[ L_i^2 - \frac{(-1)^i\sqrt{-18}q_1}{1 - 4q_1} L_i - 1 = 0 . \]

2. **Elliptic fibration: Surface**

2.1. **Overview.** We study the \((3, 2), (0, 0))\)-twisted theory on \( \mathbb{P}^2 \times \mathbb{P}^1 \). This theory recover the standard theory of K3 surface \( X \), defined by the general section of the anti-canonical bundle over \( \mathbb{P}^2 \times \mathbb{P}^1 \) for genus zero and one. For the rest of the section, the specialization

\[ \lambda_{1,k} = e^{2\pi i k}, \lambda_{2,1} = 1, \lambda_{2,2} = -1 \]

will be fixed. Since the argument of the proof is parallel to that of Section 1, we mostly omit the proofs whose arguments appeared in Section 1.

2.2. **Generators.** From the small \( I \)-function associated to \((3, 2), (0, 0))\)-twisted theory on \( \mathbb{P}^2 \times \mathbb{P}^1 \),

\[ I(q_1, q_2) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} \frac{\prod_{k=1}^{3d_1 + 2d_2} (3H_1 + 2H_2 + kz)}{\prod_{i=0}^{d_1} \prod_{k=1}^{d_1} (H_1 - \lambda_{1,i} + kz) \prod_{j=0}^{1} \prod_{k=1}^{d_2} (H_1 - \lambda_{2,j} + kz)} , \]

we get the big \( I \)-function using the argument in [7, Section 5].
Proposition 18. For $t = t_1 H_1 + t_2 H_2 \in H^+_1((\mathbb{C}^3 \times \mathbb{C}^2)/((\mathbb{C}^* \times \mathbb{C}^*)), \mathbb{Q})$, 
\begin{equation}
I(t) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} e^{t_1 (H_1 + d_1 z)/z + t_2 (H_2 + d_2 z)/z} \cdot \frac{\prod_{k=1}^{3d_1+2d_2} (3H_1 + 2H_2 + k z)}{\prod_{i=0}^{d_1} \prod_{k=1}^{d_1} (H_1 - \lambda_{1,i} + kz) \prod_{j=0}^{1} \prod_{k=1}^{d_2} (H_1 - \lambda_{2,j} + kz)}.
\end{equation}

Using Birkhoff factorization ([14]), an evaluation of the series $S(H_1 H_2)$ can be obtained from I-function similar to (17).

We define the series $A_i, B_i, \ldots, G_i$ in $q_1, q_2$ by the following equations.

\begin{align*}
S(1) &= 1 + \frac{A_1 H_1 + A_2 H_2}{z} + O\left(\frac{1}{z}\right), \\
S(H_1) &= H_1 + \frac{B_1 H_1^2 + B_2 H_1 H_2 + B_3 1}{z} + O\left(\frac{1}{z}\right), \\
S(H_2) &= H_2 + \frac{C_1 H_1^2 + C_2 H_1 H_2 + C_3 1}{z} + O\left(\frac{1}{z}\right), \\
S(H_1^2) &= H_1^2 + \frac{E_1 1 + E_2 H_1^2 H_2 + E_3 H_1 + E_4 H_2}{z} + O\left(\frac{1}{z}\right), \\
S(H_1 H_2) &= H_1 H_2 + \frac{F_1 1 + F_2 H_1^2 H_2 + F_3 H_1 + F_4 H_2}{z} + O\left(\frac{1}{z}\right), \\
S(H_2^2) &= H_2^2 + \frac{G_1 H_1 + G_2 H_2 + G_3 H_1^3 + G_4 H_1 H_2 + G_5 1}{z} + O\left(\frac{1}{z}\right).
\end{align*}

2.2.1. Picard-Fuchs equations and asymptotic expansion. The function $I$ satisfies the Picard-Fuchs equations.

\begin{align*}
\left(\left(\frac{d}{dt_1}\right)^3 - 1 - q_1 \prod_{k=1}^{3} \left(3\left(\frac{d}{dt_1}\right) + 2\left(\frac{d}{dt_2}\right) + k z\right)\right) I &= 0, \\
\left(\left(\frac{d}{dt_2}\right)^2 - 1 - q_2 \prod_{k=1}^{2} \left(3\left(\frac{d}{dt_1}\right) + 2\left(\frac{d}{dt_2}\right) + k z\right)\right) I &= 0.
\end{align*}

Denote the small I-function by 
\[ \overline{I} := I|_{t_1=0, t_2=0}. \]

The restriction \( \overline{I}|_{H_1=\lambda_{1,i}, \ H_2=\lambda_{2,j}} \) admits the asymptotic form,
\[ \overline{I}|_{H_1=\lambda_{1,i}, \ H_2} = e^{\frac{\psi_{ij}}{z}} \left(R_{0,ij} + R_{1,ij}z + R_{2,ij}z^2 + \ldots\right). \]
with series $U_{ij}, R_{k,ij} \in \mathbb{C}[[q_1, q_2]]$. Define series $L_{ij}$ and $UD_{ij}$ by

$$L_{ij} = \lambda_{1,i} + q_1 \frac{d}{dq_1} U_{ij}, \quad UD_{ij} = \lambda_{2,j} + q_2 \frac{d}{dq_2} U_{ij}.$$  

Let $L_{ij}$ be the series in $q_1$ defined by the constant term with respect to $q_2$. The argument of Lemma 14 yields the following lemma.

**Lemma 19.** We have

$$L_{ij}, UD_{ij}, R_{n,ij} \in \mathbb{C}[[q_2]][L_{ij}^\pm].$$

2.3. **Relations.** Using the argument in Section 1.3.4, we can find the relations among the series $A_i, B_i, \ldots, G_i$. Since this yields complicated expressions, we instead find the relations among the series which are coefficient of $q_2^k$ in $A_i, B_i, \ldots$. Define the series in $q_1$ by the following equations.

$$A_i(q_1, q_2) = A_i(q_1) + \sum_{k=1}^{\infty} A_{i,k}(q_1) q_2^k,$$

$$B_i(q_1, q_2) = B_i(q_1) + \sum_{k=1}^{\infty} B_{i,k}(q_1) q_2^k,$$

$$\ldots$$

$$G_i(q_1, q_2) = G_i(q_1) + \sum_{k=1}^{\infty} G_{i,k}(q_1) q_2^k.$$

We get the following results from the argument of Proposition 15.

**Proposition 20.** The series $A'_n, B'_n, \ldots, G'_n$ and $A_{n,k}, B_{n,k}, \ldots, G_{n,k}$ can be represented as rational functions in $B'_1, L_{ij}$ for fixed $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$,

$$A'_n, B'_n, \ldots G'_n, A_{n,k}, B_{n,k}, \ldots, G_{n,k} \in \mathbb{C}(B'_1, L_{ij}).$$
We give the explicit results for the convenience of the reader.

\[
\begin{align*}
A'_1 &= -\frac{1}{(35 + 36L + 54L^2)(1 + B'_1)^2}(27 - 27L^3 + 70B'_1 + 35B'^2_1 + 36LB'_1(2 + B'_1) + 54L^2B'_1(2 + B'_1)), \\
A'_2 &= \frac{1}{3(35 + 36L + 54L^2)(1 + B'_1)^2}(54(-1 + L^3) + 2(-46 + 36L + 54L^2 + 81L^3)B'_1 + (-46 + 36L + 54L^2 + 81L^3)B_1^2), \\
B'_2 &= -\frac{1}{3(35 + 36L + 54L^2)(1 + B'_1)^2}(108 + 302B'_1 + 256B'^2_1 + 70B'^3_1 + 36LB'_1(4 + 5B'_1 + 2B'^2_1) + 54L^2B'_1(4 + 5B'_1 + 2B'^2_1) - 27L^3(4 + 6B'_1 + 3B'^2_1)) \\
B'_3 &= -\frac{1}{9(35 + 36L + 54L^2)(1 + B'_1)^2}((36LB'_1^2(6 + 8B'_1 + 3B'^2_1) + 54L^2B'_1(6 + 8B'_1 + 3B'^2_1) - 27L^3(8 + 36B'_1 + 54B'^2_1 + 36B'^3_1 + 9B'^4_1) + 4(54 + 243B'_1 + 417B'^2_1 + 313B'^3_1 + 87B'^4_1)) \\
C'_1 &= 0 \\
C'_2 &= -\frac{1}{(35 + 36L + 54L^2)(1 + B'_1)^2}(27 - 27L^3 + 70B'_1 + 35B'^2_1 + 36LB'_1(2 + B'_1) + 54L^2B'_1(2 + B'_1)) \\
C'_3 &= \frac{1}{3(35 + 36L + 54L^2)(1 + B'_1)^2}(36LB'_1(2 + B'_1) + 54L^2B'_1(2 + B'_1) + 27L^3(2 + 6B'_1 + 3B'^2_1)) \\
E'_1 &= B'_1 \\
E'_2 &= B'_1 \\
E'_3 &= \frac{1}{9(35 + 36L + 54L^2)(1 + B'_1)^2}(108(-1 + L^3) + 324(-1 + L^3)B'_1 - 6(62 + 36L + 54L^2 - 27L^3)B'_1^2 - 4(35 + 36L + 54L^2)B'_1^3) \\
E'_4 &= -\frac{1}{27(35 + 36L + 54L^2)(1 + B'_1)^2}(72LB'_1^2(4 + 3B'_1) + 108L^2B'_1^3(4 + 3B'_1) - 54L^3(2 + 6B'_1 + 3B'^2_1)^2 + 8(27 + 162B'_1 + 324B'^2_1 + 278B'^3_1 + 87B'^4_1)) \\
F'_1 &= 0 \\
F'_2 &= B'_1 \\
F'_3 &= -\frac{1}{3(35 + 36L + 54L^2)(1 + B'_1)^2}(108 + 302B'_1 + 256B'^2_1 + 70B'^3_1 + 36LB'_1(4 + 5B'_1 + 2B'^2_1) + 54L^2B'_1(4 + 5B'_1 + 2B'^2_1) - 27L^3(4 + 6B'_1 + 3B'^2_1)) \\
F'_4 &= -\frac{1}{9(35 + 36L + 54L^2)(1 + B'_1)^2}((36LB'_1^2(6 + 8B'_1 + 3B'^2_1) + 54L^2B'_1(6 + 8B'_1 + 3B'^2_1) - 27L^3(8 + 36B'_1 + 54B'^2_1 + 36B'^3_1 + 9B'^4_1) + 4(54 + 243B'_1 + 417B'^2_1 + 313B'^3_1 + 87B'^4_1)) \\
G'_1 &= 0 \\
G'_2 &= B'_1 \\
G'_3 &= \frac{2B'_1}{3} \\
G'_4 &= \frac{1}{9(35 + 36L + 54L^2)(1 + B'_1)^2}(108(-1 + L^3) + 324(-1 + L^3)B'_1 - 6(62 + 36L + 54L^2 - 27L^3)B'_1^2 - 4(35 + 36L + 54L^2)B'_1^3) \\
G'_5 &= -\frac{1}{27(35 + 36L + 54L^2)(1 + B'_1)^2}(72LB'_1^2(4 + 3B'_1) + 108L^2B'_1^3(4 + 3B'_1) - 54L^3(2 + 6B'_1 + 3B'^2_1)^2 + 8(27 + 162B'_1 + 324B'^2_1 + 278B'^3_1 + 87B'^4_1))
\end{align*}
\]
2.4. **Proof of Theorem [7]** The theorem follows from the argument in Section 1.4.3 together with Lemma 19 and Proposition 20.

3. **Elliptic fibration : Threefold**

3.1. **Overview.** We study the \(((3, 3), (0, 0))\)-twisted theory on \(\mathbb{P}^2 \times \mathbb{P}^2\). This theory recover the standard theory of elliptic fibered Calabi-Yau 3-fold \(X\), defined by the general section of the anti-canonical bundle over \(\mathbb{P}^2 \times \mathbb{P}^2\) for genus zero and one.

For the rest of the section, the specialization

\[
\lambda_{1,k} = e^{\frac{2\pi i k}{3}}, \quad \lambda_{2,k} = e^{\frac{2\pi i k}{3}}
\]

will be fixed. Since the argument of the proof is parallel to that of Section 1, we mostly omit the proofs whose arguments appeared in Section 1.

3.2. **Generators.** From the small \(I\)-function associated to \(((3, 3), (0, 0))\)-twisted theory on \(\mathbb{P}^2 \times \mathbb{P}^2\),

\[
I(q_1, q_2) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} \frac{\prod_{k=1}^{3d_1+2d_2} (3H_1 + 3H_2 + kz)}{\prod_{i=0}^{d_1} \prod_{k=1}^{d_1} (H_1 - \lambda_{1,i} + kz) \prod_{j=0}^{d_2} \prod_{k=1}^{d_2} (H_1 - \lambda_{2,j} + kz)}
\]

we get the big \(I\)-function using the argument in [7] Section 5.

**Proposition 21.** For \(t = t_1 H_1 + t_2 H_2 \in H_\ast^+((\mathbb{C}^3 \times \mathbb{C}^3)/\langle \mathbb{C}^* \times \mathbb{C}^* \rangle), \mathbb{Q}\),

\[
(I(t) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} e^{t_1(H_1+d_1z)/z+t_2(H_2+d_2z)/z} \prod_{i=0}^{d_1} \prod_{k=1}^{d_1} (H_1 - \lambda_{1,i} + kz) \prod_{j=0}^{d_2} \prod_{k=1}^{d_2} (H_1 - \lambda_{2,j} + kz))
\]

Using Birkhoff factorization ([14]), an evaluation of the series \(S(H_1^i H_2^j)\) can be obtained from \(I\)-function similar to (5).

We define the series \(A_i, B_i, \ldots, J_i\) in \(q_1, q_2\) by the following equations.
\[
S(1) = 1 + \frac{A_1 H_1 + A_2 H_2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_1) = H_1 + \frac{B_1 H_1^2 + B_2 H_1 H_2 + B_3 H_2^2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_2) = H_2 + \frac{C_1 H_1^2 + C_2 H_1 H_2 + C_3 H_2^2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_1^2) = H_1^2 + \frac{E_1 I + E_2 H_1^2 H_2 + E_3 H_1 H_2^2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_1 H_2) = H_1 H_2 + \frac{F_1 I + F_2 H_1 H_2 + F_3 H_1^2 H_2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_2^2) = H_2^2 + \frac{G_1 I + G_2 H_1^2 H_2 + G_3 H_1 H_2^2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_1^2 H_2) = H_1^2 H_2 + \frac{H_1 H_1 + H_2 H_2 + H_3 H_1^2 H_2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_1 H_2^2) = H_1 H_2^2 + \frac{I_1 H_1 + I_2 H_2 + I_3 H_1^2 H_2}{z} + O\left(\frac{1}{z^2}\right),
\]
\[
S(H_1^2 H_2^2) = H_1^2 H_2^2 + \frac{J_1 H_1^2 + J_2 H_1 H_2 + J_3 H_2^2}{z} + O\left(\frac{1}{z^2}\right).
\]

### 3.2.1. Picard-Fuchs equations and asymptotic expansion.

The function \(I\) satisfies the Picard-Fuchs equations.

\[
\left((z \frac{d}{dt_1})^3 - 1 - q_1 \prod_{k=1}^3 \left(3 \left(z \frac{d}{dt_1}\right) + 3 \left(z \frac{d}{dt_2}\right) + k z\right)\right)I = 0,
\]
\[
\left((z \frac{d}{dt_2})^3 - 1 - q_2 \prod_{k=1}^3 \left(3 \left(z \frac{d}{dt_1}\right) + 3 \left(z \frac{d}{dt_2}\right) + k z\right)\right)I = 0.
\]

Denote the small \(I\)-function by

\[
\overline{I} := I_{|t_1 = 0, t_2 = 0}.
\]

The restriction \(\overline{I}|_{H_1 = \lambda_1, i, H_2 = \lambda_2, j}\) admits the asymptotic form,

\[
\overline{I}|_{H_1 = \lambda_1, i, H_2} = e^{\frac{U_{ij}}{z}} \left(R_{0,ij} + R_{1,ij} z + R_{2,ij} z^2 + \ldots\right)
\]

with series \(U_{ij}, R_{k,ij} \in \mathbb{C}[[q_1, q_2]]\). Define series \(L_{ij}\) and \(UD_{ij}\) by

\[
L_{ij} = \lambda_{1, i} + q_1 \frac{d}{dq_1} U_{ij}, \quad UD_{ij} = \lambda_{2, j} + q_2 \frac{d}{dq_2} U_{ij}.
\]
Let $L_{ij}$ be the series in $q_1$ defined by the constant term with respect to $q_2$. The argument of Lemma 14 yields the following lemma.

**Lemma 22.** We have

\[ L_{ij}, UD_{ij}, R_{n,ij} \in \mathbb{C}[[q_2]][L_{ij}^\pm]. \]

3.3. Relations. Using the argument in Section 1.3.4, we can find the relations among the series $A_i, B_i, \ldots, J_i$. Since this yields complicated expressions, we instead find the relations among the series which are coefficient of $q_2^k$ in $A_i, B_i, \ldots, J_i$. Define the series in $q_1$

\[
A_i(q_1, q_2) = A_i(q_1) + \sum_{k=1}^{\infty} A_{i,k}(q_1) q_2^k,
\]

\[
B_i(q_1, q_2) = B_i(q_1) + \sum_{k=1}^{\infty} B_{i,k}(q_1) q_2^k,
\]

\[
\ldots
\]

\[
J_i(q_1, q_2) = J_i(q_1) + \sum_{k=1}^{\infty} J_{i,k}(q_1) q_2^k.
\]

We get the following results from the argument of Proposition 15.

**Proposition 23.** The series $A_i', B_i', \ldots, J_i'$ and $A_{n,k}, B_{n,k}, \ldots, J_{n,k}$ can be represented as rational functions in $B_1', L_{ij}$ for fixed $i, j \in \{1, 2, 3\}$,

\[ A_i', B_i', \ldots, J_i', A_{n,k}, B_{n,k}, \ldots, J_{n,k} \in \mathbb{C}(B_1', L_{ij}). \]

We give the explicit results in Appendix A.1 for the convenience of the reader.

3.4. **Proof of Theorem 9** The theorem follows from the argument in Section 1.4.3 together with Lemma 22 and Proposition 23.

4. K3 fibration

4.1. Overview. We study the $((4,2), (0,0))$-twisted theory on $\mathbb{P}^3 \times \mathbb{P}^1$. This theory recover the standard theory of K3-fibered Calabi-Yau 3-fold $X$, defined by the general section of the anti-canonical bundle over $\mathbb{P}^3 \times \mathbb{P}^1$ for genus zero and one.

For the rest of the section, the specialization

\[ \lambda_{1,k} = \sqrt{-1}^k, \lambda_{2,1} = 1, \lambda_{2,2} = -1 \]

will be fixed. Since the argument of the proof is parallel to that of Section 1 we mostly omit the proofs whose arguments appeared in Section 1.
4.2. Generators. From the small $I$-function associated to $((4;2), (0;0))$-twisted theory on $\mathbb{P}^3 \times \mathbb{P}^1$,

$$I(q_1, q_2) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} \frac{\prod_{k=1}^{4d_1+2d_2}(4H_1 + 2H_2 + k\tau)}{\prod_{i=0}^{3} H_{i} \prod_{j=0}^{1} (H_1 - \lambda_{1,i} + k\tau) \prod_{k=1}^{d_2} (H_1 - \lambda_{2,j} + k\tau)},$$

we get the big $I$-function using the argument in [7, Section 5].

**Proposition 24.** For $t = t_1 H_1 + t_2 H_2 \in H^*_+(\mathbb{C}^3 \times \mathbb{C}^2/\mathbb{C}^* \times \mathbb{C}^*)$, $Q$,

$$I(t) = \sum_{d_1, d_2 \geq 0} q_1^{d_1} q_2^{d_2} e^{t_1(H_1 + d_1\tau)/\tau + t_2(H_2 + d_2\tau)/\tau} \frac{\prod_{k=1}^{4d_1+2d_2}(4H_1 + 2H_2 + k\tau)}{\prod_{i=0}^{3} H_{i} \prod_{j=0}^{1} (H_1 - \lambda_{1,i} + k\tau) \prod_{k=1}^{d_2} (H_1 - \lambda_{2,j} + k\tau)}.$$  

Using Birkhoff factorization ([14]), an evaluation of the series $S(H_1 H_2^2)$ can be obtained from $I$-function similar to [5].

We define the series $A_i, B_i, \ldots, I_i$ in $q_1, q_2$ by the following equations.

\[
S(1) = 1 + \frac{A_1 H_1 + A_2 H_2}{z} + O\left(\frac{1}{z^2}\right),
S(H_1) = H_1 + \frac{B_1 H_1^2 + B_2 H_1 H_2 + B_3 1}{z} + O\left(\frac{1}{z^2}\right),
S(H_2) = H_2 + \frac{C_1 H_1^2 + C_2 H_1 H_2 + C_3 1}{z} + O\left(\frac{1}{z^2}\right),
S(H_1^2) = H_1^2 + \frac{E_1 H_1^3 + E_2 H_1^2 H_2 + E_3 H_1 + E_4 H_2}{z} + O\left(\frac{1}{z^2}\right),
S(H_1 H_2) = H_1 H_2 + \frac{F_1 H_1^3 + F_2 H_1^2 H_2 + F_3 H_1 + F_4 H_2}{z} + O\left(\frac{1}{z^2}\right),
S(H_2^3) = H_2^3 + \frac{G_1 1 + G_2 H_1^2 H_2 + G_3 H_1^2 + G_4 H_1 H_2}{z} + O\left(\frac{1}{z^2}\right),
S(H_1^2 H_2) = H_1^2 H_2 + \frac{H_1 1 + H_2 H_1^2 H_2 + H_3 H_1^2 + H_4 H_1 H_2}{z} + O\left(\frac{1}{z^2}\right),
S(H_1^3 H_2) = H_1^3 H_2 + \frac{I_1 H_1 + I_2 H_2 + I_3 H_1^3 + I_4 H_1^2 H_2}{z} + O\left(\frac{1}{z^2}\right).
\]

4.2.1. Picard-Fuchs equations and asymptotic expansion. The function $I$ satisfies the Picard-Fuchs equations.
\[
\left(\left(\frac{z}{dt_1}\right)^4 - 1 - q_1 \prod_{k=1}^{4} \left(4 \left(\frac{z}{dt_1}\right) + 2 \left(\frac{z}{dt_2}\right) + k z\right)\right) I = 0,
\]
\[
\left(\left(\frac{z}{dt_2}\right)^2 - 1 - q_2 \prod_{k=1}^{2} \left(4 \left(\frac{z}{dt_1}\right) + 2 \left(\frac{z}{dt_2}\right) + k z\right)\right) I = 0.
\]

Denote the small I-function by
\[
\overline{I} := I|_{t_1=0, t_2=0}.
\]

The restriction \(\overline{I}|_{H_1=\lambda_1,i, H_2=\lambda_2,j}\) admits the asymptotic form,
\[
\overline{I}|_{H_1=\lambda_1,i, H_2=\lambda_2,j} = e^{U_{ij}} \left(\text{R}_{0,ij} + \text{R}_{1,ij} z + \text{R}_{2,ij} z^2 + \ldots\right)
\]
with series \(U_{ij}, \text{R}_{k,ij} \in \mathbb{C}[[q_1, q_2]].\) Define series \(L_{ij}\) and \(UD_{ij}\) by
\[
L_{ij} = \lambda_{1,i} + q_1 \frac{d}{dq_1} U_{ij}, \quad UD_{ij} = \lambda_{2,j} + q_2 \frac{d}{dq_2} U_{ij}.
\]

Let \(L_{ij}\) be the series in \(q_1\) defined by the constant term with respect to \(q_2.\) The argument of Lemma 14 yields the following lemma.

**Lemma 25.** We have
\[
L_{ij}, UD_{ij}, R_{n,ij} \in \mathbb{C}[[q_2]][L_{ij}^{\pm 1}].
\]

### 4.3. Relations

Using the argument in Section 1.3.4 we can find the relations among the series \(A_i, B_i, \ldots, I_i.\) Since this yields complicated expressions, we instead find the relations among the series which are coefficient of \(q_2^k\) in \(A_i, B_i, \ldots.\) Define the series in \(q_1\)
\[
A_i(q_1, q_2) = A_i(q_1) + \sum_{k=1}^{\infty} A_{i,k}(q_1) q_2^k,
\]
\[
B_i(q_1, q_2) = B_i(q_1) + \sum_{k=1}^{\infty} B_{i,k}(q_1) q_2^k,
\]
\[
\ldots
\]
\[
I_i(q_1, q_2) = I_i(q_1) + \sum_{k=1}^{\infty} I_{i,k}(q_1) q_2^k.
\]

We get the following results from the argument of Proposition 15.
Proposition 26. The series $A'_k, B'_k, \ldots, I'_k$ and $A_{n,k}, B_{n,k}, \ldots, I_{n,k}$ can be represented as rational functions in $A'_1, B'_2, E'_1, L_{ij}$ for fixed $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2\}$,

$$A_n', B_n', \ldots, I_n', A_{n,k}, B_{n,k}, \ldots, I_{n,k} \in \mathbb{C}(A'_1, B'_2, E'_1, L_{ij}).$$

In the proof of Proposition 26, we can show that the series $A'_1, E'_1, L_{ij}$ satisfy the following relations.

$$2E'_1 + E'_2 + 2A'_1(1 + E'_1)^2 + A'_2(1 + E'_1)^2 - \frac{16(-1 + L_{ij}^4)}{16 + 1 + (-1)^j8L_{ij} + 24L_{ij}^2 + (-1)^j32L_{ij}^3} = 0.$$

4.4. Proof of Theorem 11. The theorem follows from the argument in Section 1.4.3 together with Lemma 25 and Proposition 26.

Appendix A. Relations on the generators

A.1. Elliptic fibration. Recall the series $A'_k, B'_k \ldots J'_k$ defined in Section 3.3. They satisfy the following equations.

$$A'_1 = \frac{1}{(1 + B'_1)^2(2 + 3L + 3L^2)}(-1 + L^3 - 2B'_1(2 + 3L + 3L^2) - B'_1^2(2 + 3L + 3L^2)), $$

$$A'_2 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-2 + 2L^3 + B'_1^2(-1 + 3L + 3L^2 + 3L^3) + B'_1(-2 + 6L + 6L^2 + 6L^3)), $$

$$B'_2 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-4 + 4L^3 - 2B'_1^2(2 + 3L + 3L^2) + B'_1(-13 - 15L - 15L^2 + 3L^3) + 2B'_1(-7 - 6L - 6L^2 + 3L^3)), $$

$$B'_3 = \frac{1}{4(1 + B'_1)^2(2 + 3L + 3L^2)}(-8 + 8L^3 + 36B'_1(-1 + L^3) + 3B'_1^4(-5 - 3L - 3L^2 + 3L^3) + 4B'_1^3(-13 - 6L - 6L^2 + 9L^3) + 6B'_1^2(-11 - 3L - 3L^2 + 9L^3)), $$

$$C'_1 = 0, $$

$$C'_2 = \frac{1}{(1 + B'_1)^2(2 + 3L + 3L^2)}(-1 + L^3 - B'_1^2(2 + 3L + 3L^2) - B'_1^2(4 + 6L + 6L^2)), $$

$$C'_3 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-2 + 2L^3 + B'_1^2(-1 + 3L + 3L^2 + 3L^3) + B'_1(-2 + 6L + 6L^2 + 6L^3)), $$

$$C'_4 = \frac{1}{4(1 + B'_1)^2(2 + 3L + 3L^2)}(-4 + 4L^3 - 2B'_1^2(2 + 3L + 3L^2) + B'_1(-13 - 15L - 15L^2 + 3L^3) + 2B'_1(-7 - 6L - 6L^2 + 3L^3)).$$
\[ E'_2 = B'_1, \]
\[ E'_3 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-2 + 2L^3 - 2B'_1^3(2 + 3L + 3L^2) + 6B'_1(-1 + L^3) + 3B'_1^2(-3 - 3L - 3L^2 + L^3)), \]
\[ F'_1 = \frac{1}{4(1 + B'_1)^2(2 + 3L + 3L^2)}(-8 + 8L^3 + 36B'_1(-1 + L^3) + 3B'_1^2(-5 - 3L - 3L^2 + 3L^3) + 6B'_1^2(-13 - 6L - 6L^2 + 9L^3) + 6B'_1^2(-11 - 3L - 3L^2 + 9L^3)), \]
\[ F'_2 = B'_1, \]
\[ F'_3 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-4 + 4L^3 - 2B'_1^3(2 + 3L + 3L^2) + B'_1^2(-13 - 15L - 15L^2 + 3L^3) + 2B'_1(-7 - 6L - 6L^2 + 3L^3)), \]
\[ G'_1 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-2 + 2L^3 + B'_1^2(-1 + 3L + 3L^2 + 3L^3) + B'_1(-2 + 6L + 6L^2 + 6L^3)), \]
\[ G'_2 = 0, \]
\[ G'_3 = \frac{1}{(1 + B'_1)^2(2 + 3L + 3L^2)}(-1 + L^3 - B'_1^2(2 + 3L + 3L^2) - B'_1(4 + 6L + 6L^2)), \]
\[ H'_1 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-2 + 2L^3 - 2B'_1^3(2 + 3L + 3L^2) + 6B'_1(-1 + L^3) + 3B'_1^2(-3 - 3L - 3L^2 + L^3)), \]
\[ H'_2 = \frac{1}{4(1 + B'_1)^2(2 + 3L + 3L^2)}(-4 + 4L^3 + 36B'_1(-1 + L^3) + 3B'_1^2(-5 - 3L - 3L^2 + 3L^3) + 6B'_1^2(-13 - 6L - 6L^2 + 9L^3) + 6B'_1^2(-11 - 3L - 3L^2 + 9L^3)), \]
\[ H'_3 = B'_1, \]
\[ I'_1 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-4 + 4L^3 - 2B'_1^3(2 + 3L + 3L^2) + B'_1^2(-13 - 15L - 15L^2 + 3L^3) + 2B'_1(-7 - 6L - 6L^2 + 3L^3)), \]
\[ I'_2 = \frac{1}{4(1 + B'_1)^2(2 + 3L + 3L^2)}(-8 + 8L^3 + 36B'_1(-1 + L^3) + 3B'_1^2(-5 - 3L - 3L^2 + 3L^3) + 6B'_1^2(-13 - 6L - 6L^2 + 9L^3) + 6B'_1^2(-11 - 3L - 3L^2 + 9L^3)), \]
\[ I'_3 = B'_1, \]
\[ J'_1 = B'_1, \]
\[ J'_2 = \frac{1}{2(1 + B'_1)^2(2 + 3L + 3L^2)}(-2 + 2L^3 - 2B'_1^3(2 + 3L + 3L^2) + 6B'_1(-1 + L^3) + 3B'_1^2(-3 - 3L - 3L^2 + L^3)), \]
\[ J'_3 = \frac{1}{4(1 + B'_1)^2(2 + 3L + 3L^2)}(-4 + 4L^3 + 36B'_1(-1 + L^3) + 3B'_1^2(-5 - 3L - 3L^2 + 3L^3) + 6B'_1^2(-13 - 6L - 6L^2 + 9L^3) + 6B'_1^2(-11 - 3L - 3L^2 + 9L^3)). \]

Here \( L = L_{0,0} \). Similar equations hold for other \((i,j) \neq (0,0)\).
A.2. **K3 fibration.** The series $A'_k, B'_k \ldots I'_k$ defined in Section 3.3 satisfy the following equations.

$$A'_2 = \frac{1}{4}(7A'_1 + 4A'_1^2 - 2B'_1 + 8E'_1 + 16A'_1E'_1 + 8A'_1^2E'_1 + 4E'_1^2 + 8A'_1E'_1^2 + 4A'_1^2E'_1^2),$$

$$B'_1 = A'_1,$$

$$B'_2 = \frac{1}{32(1 + A'_1)}(16A'_1^4(1 + E'_1)^4 + 8A'_1^3(1 + E'_1)^3(9 + 16E'_1 + 8E'_1^2)) + A'_1^2(105 + 400E'_1 + 584E'_1^2 + 384E'_1^3 + 96E'_1^4 + 16B'_2(1 + E'_1)^2) + 4A'_1(13 + 60E'_1 + 94E'_1^2 + 64E'_1^3 + 16E'_1^4 + B'_2(5 + 16E'_1 + 8E'_1^2)) + 4(-3B'_2^2 + B'_2(-2 + 8E'_1 + 4E'_1^2) + 2E'_1(6 + 11E'_1 + 8E'_1^2 + 2E'_1^3))),$$

$$C'_1 = 0,$$

$$C'_2 = A'_1,$$

$$C'_3 = \frac{1}{4}(7A'_1 + 4A'_1^2 - 2B'_1 + 8E'_1 + 16A'_1E'_1 + 8A'_1^2E'_1 + 4E'_1^2 + 8A'_1E'_1^2 + 4A'_1^2E'_1^2),$$

$$E'_2 = \frac{1}{4}(9A'_1 + 4A'_1^2 - 2B'_1 + 6E'_1 + 16A'_1E'_1 + 8A'_1^2E'_1 + 4E'_1^2 + 8A'_1E'_1^2 + 4A'_1^2E'_1^2),$$

$$E'_3 = \frac{1}{32(1 + A'_1)}(16A'_1^4(1 + E'_1)^4 + 8A'_1^3(1 + E'_1)^3(7 + 16E'_1 + 8E'_1^2)) + A'_1^2(53 + 304E'_1 + 536E'_1^2 + 384E'_1^3 + 96E'_1^4 + 16B'_2(1 + E'_1)^2) + 4A'_1(4 + 38E'_1 + 82E'_1^2 + 64E'_1^3 + 16E'_1^4 + B'_2(11 + 16E'_1 + 8E'_1^2)) + 4(-3B'_2^2 + 4B'_2(1 + E'_1)^2 + 2E'_1(3 + 9E'_1 + 8E'_1^2 + 2E'_1^3))),$$

$$E'_4 = \frac{1}{64(1 + A'_1)^2}(8B'_2^2 + 384A'_1^5(1 + E'_1)^6 + 64A'_1^6(1 + E'_1)^6 - 32B'_2E'_1(1 + E'_1)^2(2 + E'_1)) - 4B'_2^2(1 + 8E'_1 + 4E'_1^2) + 8E'_1(5 + 43E'_1 + 104E'_1^2 + 106E'_1^3 + 48E'_1^4 + 8E'_1^5) + 4A'_1^2(1 + E'_1)^2(213 + 904E'_1 + 1412E'_1^2 + 960E'_1^3 + 240E'_1^4 - 8B'_2(1 + E'_1)^2) - 2A'_1^2(-425 - 2960E'_1 - 8264E'_1^2 - 11904E'_1^3 - 9376E'_1^4 - 3840E'_1^5 - 640E'_1^6 + 64B'_2(1 + E'_1)^2) + A'_1^2(359 + 3216E'_1 + 10444E'_1^2 + 16512E'_1^3 + 13728E'_1^4 + 5760E'_1^5 + 960E'_1^6 + 16B'_2(1 + E'_1)^2) - 2B'_2(83 + 352E'_1 + 566E'_1^2 + 384E'_1^3 + 96E'_1^4)) - 4A'_1(8B'_2^2(1 + E'_1)^2 + B'_2(19 + 96E'_1 + 176E'_1^2 + 128E'_1^3 + 32E'_1^4) - 2(5 + 92E'_1 + 399E'_1^2 + 736E'_1^3 + 664E'_1^4 + 288E'_1^5 + 48E'_1^6))),$$

$$E'_1 = 0,$$

$$E'_2 = A'_1,$$

$$E'_3 = B'_2,$$

$$E'_4 = \frac{1}{32(1 + A'_1)}(16A'_1^4(1 + E'_1)^4 + 8A'_1^3(1 + E'_1)^3(9 + 16E'_1 + 8E'_1^2)) + A'_1^2(105 + 400E'_1 + 584E'_1^2 + 384E'_1^3 + 96E'_1^4 + 16B'_2(1 + E'_1)^2) + 4A'_1(13 + 60E'_1 + 94E'_1^2 + 64E'_1^3 + 16E'_1^4 + B'_2(5 + 16E'_1 + 8E'_1^2)) + 4(-3B'_2^2 + B'_2(-2 + 8E'_1 + 4E'_1^2) + 2E'_1(6 + 11E'_1 + 8E'_1^2 + 2E'_1^3))),$$
G_1' = \frac{1}{128(1 + A_1'^2)}(64A_1'^6(1 + E_1')^6 + 16A_1'^8(1 + E_1')^4(23 + 48E_1' + 24E_1'^2) + A_1'^3(733 + 5376E_1' + 15616E_1'^2 + 23168E_1'^3 + 18592E_1'^4 + 7680E_1'^5 + 1280E_1'^6)
- 16B_2'(1 + E_1')^2(9 + 16E_1' + 8E_1'^2) - 4A_1'^4(8B_2'(1 + E_1')^4 - 3(1 + E_1')^2(65 + 288E_1' + 464E_1'^2 + 320E_1'^3 + 80E_1'^4))
- 2A_1'^2(-137 - 1444E_1' - 4790E_1'^2 - 7936E_1'^3 - 6784E_1'^4 - 2880E_1'^5 - 480E_1'^6
+ 8B_2'^2(1 + E_1')^2 + B_2'(105 + 400E_1' + 584E_2'^2 + 384E_1'^3 + 96E_1'^4))
+ 8(B_2'^4 + B_2'^2(1 - 4E_1' - 2E_1'^2) - 2B_2'E_1'(6 + 11E_1' + 8E_1'^2 + 2E_1'^3 + 2E_1'(9 + 17E_1' + 48E_1'^2 + 52E_1'^3 + 24E_1'^4 + 4E_1'^5))
- 4A_1'(B_2'^2(5 + 16E_1' + 8E_1'^2) + 2B_2'(13 + 60E_1' + 94E_2'^2 + 64E_1'^3 + 16E_1'^4)
- 2(2 + 102E_1' + 349E_1'^2 + 696E_1'^3 + 654E_1'^4 + 288E_1'^5 + 48E_1'^6)),
G_2' = \frac{E_1'}{2},
G_3' = \frac{1}{8}(-2B_2' + 4A_1'^3(1 + E_1')^2 + 2E_1'(3 + 2E_1') + A_1'(7 + 16E_1' + 8E_1'^2)),
G_4' = \frac{1}{64(1 + A_1'^2)}(16A_1'^4(1 + E_1')^4 + 8A_1'^3(1 + E_1')^2(7 + 16E_1' + 8E_1'^2)
+ A_1'^4(61 + 304E_1' + 536E_1'^2 + 384E_1'^3 + 96E_1'^4 + 16B_2'(1 + E_1')^2)
+ 4A_1'(6 + 38E_1' + 82E_1'^2 + 64E_1'^3 + 16E_1'^4 + B_2'(7 + 16E_1' + 8E_1'^2))
+ 4(-3B_2' + 4B_2'E_1'(2 + E_1') + 2E_1'(3 + 9E_1' + 8E_1'^2 + 2E_1'^3))),
H_1' = \frac{1}{64(1 + A_1'^2)}(8B_2'^3 + 384A_1'^4(1 + E_1')^6 + 64A_1'^6(1 + E_1')^6 - 32B_2'E_1'(1 + E_1')^2(2 + E_1')
- 4B_2'^2(1 + 8E_1' + 4E_1'^2) + 8E_1'(5 + 43E_1' + 104E_1'^2 + 106E_1'^3 + 48E_1'^4 + 8E_1'^5)
+ 4A_1'^4(1 + E_1')^2(213 + 904E_1' + 1412E_1'^2 + 960E_1'^3 + 240E_1'^4 - 8B_2'(1 + E_1')^2)
- 2A_1'^3(-425 - 2960E_1' - 8264E_1'^2 - 11904E_1'^3 - 9376E_1'^4 - 3840E_1'^5 - 640E_1'^6 + 64B_2'(1 + E_1')^4)
+ A_1'^2(359 + 3216E_1' + 10444E_1'^2 + 16512E_1'^3 + 13728E_1'^4 + 5760E_1'^5 + 960E_1'^6 - 16B_2'^2(1 + E_1')^2)
- 2B_2'(83 + 352E_1' + 560E_1'^2 + 384E_1'^3 + 96E_1'^4)) - 4A_1'(8B_2'^2(1 + E_1')^2 + B_2'(19 + 96E_1' + 176E_1'^2 + 128E_1'^3 + 32E_1'^4 - 2(5 + 92E_1' + 399E_1'^2 + 736E_1'^3 + 664E_1'^4 + 288E_1'^5 + 48E_1'^6))),
H_2' = E_1',
H_3' = \frac{1}{4}(9A_1' + 4A_1'^2 - 2B_2' + 6E_1' + 16A_1'E_1' + 8A_1'^2E_1' + 4E_1'^2 + 8A_1'E_1'^2 + 4A_1'^2E_1'^2).
H_4' = \frac{1}{32(1 + A_1'^2)}(16A_1'^4(1 + E_1')^4 + 8A_1'^3(1 + E_1')^2(7 + 16E_1' + 8E_1'^2) + A_1'^2(53 + 304E_1' + 536E_1'^2 + 384E_1'^3 + 96E_1'^4)
+ 16B_2'(1 + E_1')^2 + 4A_1'(4 + 38E_1' + 82E_1'^2 + 16E_1'^3 + 16E_1'^4 + B_2'(11 + 16E_1' + 8E_1'^2))
+ 4(-3B_2'^2 + 4B_2'(1 + E_1')^2 + 2E_1'(3 + 9E_1' + 8E_1'^2 + 2E_1'^3))),
I_1' = \frac{1}{64(1 + A_1'^2)}(16A_1'^4(1 + E_1')^4 + 8A_1'^3(1 + E_1')^2(7 + 16E_1' + 8E_1'^2) + A_1'^2(61 + 304E_1' + 536E_1'^2 + 384E_1'^3 + 96E_1'^4)
+ 16B_2'(1 + E_1')^2 + 4A_1'(6 + 38E_1' + 82E_1'^2 + 16E_1'^3 + 16E_1'^4 + B_2'(7 + 16E_1' + 8E_1'^2))
+ 4(-3B_2'^2 + 4B_2'E_1'(2 + E_1') + 2E_1'(3 + 9E_1' + 8E_1'^2 + 2E_1'^3))).
\[ I'_2 = \frac{1}{128(1 + A'_1)^2}(64A'_1^6(1 + E'_1)^6 + 16A'_1^7(1 + E'_1)^4(23 + 48E'_1^1 + 24E'_1^2)) \\
+ A'_1^3(733 + 5376E'_1^1 + 15616E'_1^2 + 23168E'_1^3 + 18592E'_1^4 + 7680E'_1^5 + 1280E'_1^6 - 16B'_2(1 + E'_1)^2(9 + 16E'_1^1 + 8E'_1^2)) \\
- 4A'_1(8B'_2(1 + E'_1)^4 - 3(1 + E'_1)^2(65 + 288E'_1^1 + 464E'_1^2 + 320E'_1^3 + 80E'_1^4)) \\
- 2A'_1^2(-137 - 1444E'_1^1 - 4790E'_1^2 - 7936E'_1^3 - 6784E'_1^4 - 2880E'_1^5 - 480E'_1^6 + 8B'_2^2(1 + E'_1)^2) \\
+ B'_2(105 + 400E'_1^1 + 584E'_1^2 + 384E'_1^3 + 96E'_1^4)) + 8(B'_2^2 + B'_2(1 - 4E'_1^1 - 2E'_1^2) - 2B'_2E'_1(6 + 11E'_1^1 + 8E'_1^2 + 2E'_1^3) \\
+ 2E'_1^1(9 + 17E'_1^1 + 48E'_1^2 + 52E'_1^3 + 24E'_1^4 + 4E'_1^5)) - 4A'_1(8B'_2^2(5 + 16E'_1^1 + 8E'_1^2)) \\
+ 2B'_2(13 + 60E'_1^1 + 94E'_1^2 + 64E'_1^3 + 16E'_1^4) - 2(2 + 102E'_1^1 + 349E'_1^2 + 696E'_1^3 + 654E'_1^4 + 288E'_1^5 + 48E'_1^6)) \]

\[ I'_3 = E'_1^2 \hat{A} \]
\[ I'_4 = \frac{1}{8}(-2B'_2 + 4A'_1^2(1 + E'_1)^2 + 2E'_1(3 + 2E'_1) + A'_1(7 + 16E'_1^1 + 8E'_1^2)) \]

Here \( L = L_{0,0} \). Similar equations hold for other \( (i, j) \neq (0, 0) \).

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