Quantum Integrable Systems in One Dimension

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1 Introduction

The field of integrable systems is a vast one that has grown almost explosively in recent years. It is not remotely possible to review all the developments in the field in one hour, so I am restricting my attention to some particular topics. I shall only consider one dimensional quantum systems, or 1+1 as it is usually said, including the time dimension, but necessarily therefore including some discussion of the statistical mechanics of two dimensional lattice systems (vertex models) which share a common mathematical structure with the 1+1 quantum systems, and have been the key to much of the understanding of the quantum systems.

This is still too big a field to review in any detail in one hour, so I shall focus on some generalizations of the spin one-half chain with nearest-neighbor coupling, for which Bethe first set down his celebrated ansatz. The simplest generalization, to fully anisotropic coupling, the so-called XYZ model, has an amazingly rich mathematical structure, parts of which are still being elucidated. A second generalization, which has been very fashionable lately, is to extend the coupling beyond nearest neighbors, specifically to a long range inverse-square type coupling between spins one-half. This also turns out to be an integrable system, although the wave functions are not of the Bethe ansatz type.

How relevant are these models to real physical systems? The simplest isotropic spin one-half XXX Heisenberg antiferromagnet is a good Hamiltonian for the quasi-one-dimensional system CPC, and one of the early experimental vindications of the Bethe ansatz was the confirmation by neutron scattering that it correctly predicted the observed elementary excitation spectrum, in contrast to the standard spin wave theory used at the time. An appropriate continuum limit of the XYZ chain gives the sine-Gordon model, and the XXZ chains are good representations of known systems. The closely related one-dimensional Hubbard model has been widely used to describe one-dimensional conductors, and may be relevant to some high temperature superconductors. One of the great successes of the Bethe ansatz has been the solution of the Kondo problem. The Luttinger liquid model, first used to analyze low
energy properties of the $XXZ$ spin chain, is proving valuable in studying one-dimensional mesoscopic conducting systems\[10\]. In fact, Bethe ansatz models are very useful for one-dimensional conducting ring systems having a threading magnetic field, because the Bohm-Aharonov phase can be incorporated as a twisted boundary condition which, it turns out, does not spoil the integrability. Another very fruitful interaction in recent years has been that between Bethe ansatz methods and conformal field theory, discussed below. Finally, the inverse-square systems\[11\] are cousins of the Bethe ansatz systems. They do not have Bethe ansatz wavefunctions, but their energy levels are given by Bethe ansatz like equations of a rather simple kind. These systems are closely related to the important edge states in the quantum Hall effect, and also, surprisingly, to level distributions in quantum chaotic systems\[12\].

2 Quick review of Bethe ansatz

The simplest Bethe ansatz system to visualize is the “billiard balls on a line” problem—classically, imagine $N$ perfectly elastic billiard balls on a ring, say, to give periodic boundary conditions. The initial set of momenta $k_1, \ldots, k_N$ is conserved in subsequent collisions. The quantum mechanical version, solved long ago by Lieb and Liniger\[13\], has $N$ bosons interacting through repulsive delta function potentials. The two particle problem is equivalent to a single particle interacting with a static delta-function potential, and the standard boundary condition at the potential leads to a two-particle wavefunction which is a sum of two plane wave terms with the momenta permuted,

$$\psi(x_1, x_2) = e^{i(k_1 x_1 + k_2 x_2)} + e^{i\theta_{12} e^{i(k_2 x_1 + k_1 x_2)}}$$

for $x_1 < x_2$, and boson symmetry determines the wavefunction for $x_1 > x_2$. The phase shift term is found to be:

$$e^{i\theta_{12}} = -\frac{c - i(k_1 - k_2)}{c + i(k_1 - k_2)}$$

where $c$ is the strength of the repulsive delta-function potential. This is a simple Bethe ansatz wavefunction. Notice that it vanishes if $k_1 = k_2$. Surprisingly, this wavefunction generalizes in a simple way to $N$ particles,

$$\psi(x_1, x_2, \ldots, x_N) = \sum a(P) e^{i\sum k_i x_i}$$

for $x_1 < x_2 < \ldots < x_N$, with other orderings given by the boson symmetry. Here $a(P)$ is a product of phase shift factors for each pair of transitions needed to get the ordering $P$ from the ordering $1, \ldots, N$. It follows immediately from the form of the phase shift term above that all the $k_i$ must be distinct, or the wavefunction vanishes identically. The total energy and total momentum
are easily seen to be given by $E = \sum k_i^2$, $K = \sum k_i$, from applying the $E$, $K$ operators in regions where no two $x_i$'s coincide.

The actual values of the $k_i$ appearing in the wavefunction are determined by the boundary conditions, just as they are for free particles. For periodic boundary conditions, the total change in the phase of the wave function on taking a particle around the system must be $2\pi I$, where $I$ is an integer. This phase is part kinetic, from the $e^{i k x}$ term, and part phase shifting from passing the other particles so the total phase shift

$$k_j L + \sum_{i \neq j} \theta_{ji} = 2\pi I_j$$

(4)

Now writing the phase shift in terms of the momenta, we get a set of coupled nonlinear equations for the $k_i$'s for a given set of quantum numbers $I_i$. Since the $k_i$ must all be distinct, the ground state is fermi-sea like, with a set of $I_j$ equal to 0, $\pm 1, \pm 2, ...$. Elementary excitations are given by gaps (holes) in the sequence of integers, or single integers above the sea (particles). It is found that for an attractive delta function potential, the ground state for $N$ particles is a string of pure imaginary momenta having uniform spacing $i\epsilon$. Actually, this attractive boson gas is a pathological system in the thermodynamic limit where all the particles bind together infinitely tightly. However, these strings of bound momenta prove to be a common (usually nonpathological) feature of Bethe ansatz systems, as discussed below.

Let us now turn to the spin one-half $XXZ$ chain having Hamiltonian:

$$H_{XXZ} = \sum_{j=1}^{N} \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)$$

(5)

where we shall restrict our attention to the gapless regime, $|\Delta| < 1$, and write $\Delta = \cos \gamma$. The eigenstates of the Hamiltonian are described by Bethe ansatz wavefunctions having the same general structure as in the boson model, the bosons being replaced by magnons. A magnon is created by applying the operator $a_k^+ = \sum e^{i k n} S_n^+$ to the all-spins-down “ferromagnetic” state. It is convenient to parameterize the magnon momenta $k_j$ in terms of so-called rapidity variables $v_j$:

$$e^{ik_j} = \frac{\sinh(v_j + i\gamma/2)}{\sinh(v_j - i\gamma/2)}$$

(6)

The reason for the change of variables is that the magnon-magnon phase shift is a function of rapidity difference. It was not a function of momentum difference—the lattice spoils Galilean invariance. The equations are much easier to handle in the new variables. The boundary condition for allowed rapidities analogous to (4) (or, rather, its exponential) is:

$$\left( \frac{\sinh(v_j + i\gamma/2)}{\sinh(v_j - i\gamma/2)} \right)^N = -\prod_{i=1}^{n} \frac{\sinh(v_j - v_i + i\gamma)}{\sinh(v_j - v_i - i\gamma)}$$

(7)
One significant difference from the (repulsive) boson system is that the magnon rapidities given by solving these equations include in general some grouped in strings of certain lengths, the allowed lengths depending on $\gamma$, and also showing a very complicated system size dependence for finite systems.

One of the most important applications of the Bethe ansatz method is finding thermodynamic properties. Roughly speaking, this is done by summing over an ensemble of systems having the same macroscopic quantum number density with appropriate entropic weighting factors. For the spin chains, the presence of so many string-like excitations makes this a formidable task even in the infinite limit [14], and for finite systems, where string lengths vary with system size, it is even more tricky. Fortunately, new techniques have recently been discovered which circumvent these difficulties [15] [16]. We shall discuss them below.

3 Luttinger liquids and conformal fields

One real problem with the above analysis is that, although all the energy levels of the Bethe ansatz system can be found and the wavefunctions written down, it is very difficult in practice to find matrix elements of operators between these states, and hence connect the known excitation spectrum with what is observed in, say, neutron scattering from a spin chain. The first significant step in this direction was made by Luther and Peschel [17], who argued that the low energy excitations of the gapless $XXZ$ spin chain could be described using the Luttinger model, solved by Mattis and Lieb [18]. These ideas were extended and put on a firm footing by Haldane [9], who coined the term Luttinger liquid. Essentially, the method is to transform the magnons to spinless fermions using the Jordan-Wigner transformation, whereupon the $XX$ term is seen to be a kinetic fermion hopping contribution to the Hamiltonian, and the $\Delta$ term is a four-fermion interaction. The fermion energy-momentum curve is then linearized near the “fermi surface” so that density fluctuations can be represented by bosons. With this approximation, the whole low-energy Hamiltonian is quadratic in the boson operators, and can be diagonalized and solved. The fermion creation operator is found to be closely related to an exponential function of boson creation operators. The critical exponents can be found in terms of bulk susceptibilities which can be calculated by conventional Bethe ansatz methods. This Luttinger liquid theory was a forerunner of, and is in fact a special case of, modern conformal techniques.

As is discussed further in the next section, the $1+1$ dimensional Bethe ansatz systems are intimately related to two dimensional statistical systems. Essentially, they are analytic continuations of each other from time to imaginary time. Thus techniques developed for one can be readily applied to the other.

Macroscopic thermodynamic systems at the critical point have been long known to be scale invariant. Polyakov [19] first pointed out that they are also conformally invariant, that is, invariant under any transformation that is locally
a scale change plus a rotation. In two dimensions, this constraint dramatically
reduces possible varieties of critical behavior. The conformal transformations of
the plane, under which the system must be invariant, correspond to the set of all
analytic functions, and the generators of these transformations are the elements
of the so-called Virasoro algebra. Representations of this algebra, corresponding
to possible critical systems, are labelled by a central charge $c$ and other
parameters usually denoted $\Delta_i$. For the statistical systems, the central charge
measures the finite-size dependence of the free energy (or the low temperature
specific heat for the spin chain), and the other parameters measure correlation
exponents for both systems. It was shown by Friedan, Qiu and Shenker\cite{20} that
unitarity considerations only allow certain values of $c$ and the $\Delta_i$, and these
values determines possible excitation energies of the system, in particular the
low-lying states. But these are precisely the energy levels that can be evaluated
fairly straightforwardly using the Bethe ansatz, so by this backdoor method we
can use Bethe ansatz results to learn about correlation functions without trying
to evaluate matrix elements! A great deal of numerical work has been done to
classify the conformal theories corresponding to continuum limits of integrable
spin chain systems.\cite{21}

The Luttinger liquid technique (with some conformal ideas added) has been
applied successfully to the Hubbard model by Frahm and Korepin\cite{22}. This is
a more complicated situation, in that there are two sets of low energy excita-
tions, charge and spin density waves, having different speeds. This means that
conformal methods need to be extended. The basic tool used is the dressed
charge matrix, a generalized susceptibility, the elements of which can be cal-
culated using the Bethe ansatz. This immediately gives asymptotic correlation
functions in terms of coupling strength and magnetic field. Recently, Kane and
Fisher\cite{10} have applied Luttinger liquid techniques to predicting conductivity
in a one-dimensional system with barriers.

4 Transfer matrices and analyticity

Our growing understanding of the $XXZ$ (and fully anisotropic $XYZ$) spin one
half chains has come about in large part through their connections with the
six (and eight) vertex models. These are classical two dimensional statistical
models defined on a square lattice, such that on each bond there is an arrow
which can point either way. The energy, and therefore the statistical weight,
of each vertex depends on its arrow configuration. In the six vertex model,
originally conceived to describe hydrogen-bonded crystals (the arrow direction
indicates the end of the bond where the hydrogen atom sits) electrostatic energy
considerations dictate that each vertex has two and only two hydrogen atoms
nearby, so two arrows point in and two out at each vertex. This gives six possible
vertices, three arrow-reversed pairs. (The eight vertex model allows all in or all
out arrows.) Taking arrow reversed pairs to have the same energy, there are
three distinct statistical weights, usually denoted $a$, $b$, and $c$. The free energy of these lattice systems can be found by transfer matrix techniques—the matrix is defined between adjacent rows of $N$ vertical bonds, each matrix element giving the combined statistical weight of all allowed vertex configurations on the row of horizontal bonds sandwiched between the two adjacent rows of specified vertical bonds. This is a $2^N \times 2^N$ matrix with elements having row and column labels corresponding to the $2^N$ possible spin configurations on each of two successive rows of vertical bonds. Denoting this matrix by $T$, the free energy of the $N \times N$ system is given by $TrT^N$, so the matrix multiplication sums over all possible vertical bond configurations, with the appropriate Boltzmann weighting factor supplied by the matrix. For large $N$, the trace is dominated by the largest eigenvalue of $T$, so this gives the free energy. Finding correlation functions of the system requires knowledge of the next leading eigenvalues.

Now the transfer matrix $T$ operates on a space of $N$ up or down arrows, which is the same as the configuration space of the spin-one-half chain Hamiltonian. This was first appreciated by Lieb, who solved the ice problem by using a Bethe ansatz wavefunction for the leading eigenstate of the transfer matrix. However, the relationship between $T$’s and $H_{XXZ}$’s is not one-to-one. Taking vertex weights with all arrows reversed to be equal, and factoring out an overall normalization, we still have a two-parameter family of vertex models, with transfer matrices conveniently labelled $T(v, \gamma)$, where $T(v, \gamma)$ has the same set of eigenstates (but not of course eigenvalues) as the member $H_{XXZ}(\gamma)$ of the one-parameter $XXZ$ spin-chain family. That is to say, if we take $\gamma$ as fixed, there is a one-parameter family of transfer matrices $T(v)$ which all commute with $H_{XXZ}(\gamma)$, and in fact they all commute with each other, since they have the same set of eigenstates. In the notation of Klümper et al., the ratio of the three vertex statistical weights $a : b : c$ becomes $\sin(\gamma/2 + iv) : \sin(\gamma/2 - iv) : \sin \gamma$, and it can easily be shown that for the particular parameter values $v = \pm i\gamma/2$, the transfer matrices reduce to simple shift operators (because $a$ or $b$ become zero), and their logarithmic derivatives at these points are just the $XXZ$ Hamiltonian,

$$T(\mp(i\gamma/2 - v)) = e^{\pm ivP - vH_{XXZ} + O(v^2)}$$

It might appear at this point that introducing this family of commuting transfer matrices does not obviously look like progress in understanding the properties of the $XXZ$ spin chain. In fact, though, the existence of the family $T(v)$ makes it evident that the $XXZ$ chain is integrable, in the sense of having a sequence of higher conserved quantities—i.e., beyond total momentum and energy—one could, for example, find the higher logarithmic derivatives of $T(v)$ at $v = \pm i\gamma/2$. However, the real utility of the $T(v)$ arises from their analytic properties. As Baxter makes clear throughout his classic book, the analyticity of $T(v)$ as a function of $v$ constrains the properties of these systems, and renders
them solvable, in a truly amazing way. As an example of the connection between
the Bethe Ansatz equations and analyticity, we mention Baxter’s result that
each eigenvalue $\Lambda(v)$ of the transfer matrix satisfies the equation:

$$\Lambda(v)q(v) = \Phi(v + i\gamma/2)q(v - i\gamma) + \Phi(v - i\gamma/2)q(v + i\gamma)$$  \hspace{1cm} (9)

with $\Phi(v) = (\sinh v)^N$, and $q(v) = \prod_{j=1}^{N} \sinh(v - v_j)$. The rapidities $v_j$ satisfy
the Bethe ansatz equations $p(v_j) = -1$ where

$$p(v) = \frac{\Phi(v - i\gamma/2)q(v + i\gamma)}{\Phi(v + i\gamma/2)q(v - i\gamma)} \hspace{1cm} (10)$$

These are of course the same equations as in (7) above, but in the context
of the properties of the transfer matrix eigenvalue as a function of $v$, we see
that the zeros of $q(v)$ coincide with zeros of the right hand side of equation (9),
ensuring that $\Lambda(v)$ has no poles. Furthermore, for the largest eigenvalue, it is
known that the $v_i$ are all real, from which it follows that $\Lambda(v)$ is analytic
and nonzero in a strip of width $\gamma$ centered on the real axis. This is a very strong
constraint, and makes it possible to use Cauchy’s theorem to derive a nonlinear
integral equation for the function $1/p(x - i\gamma/2)$, which is exact even for finite
systems. Although the equation cannot be completely solved, it can be used to
find the $1/N$ corrections to the eigenvalues for large systems. This means we
can evaluate critical exponents analytically [24] [1] without directly solving
the Bethe ansatz equations.

As mentioned earlier, the standard method for evaluating thermodynamic
properties of Bethe Ansatz systems leads to formidably difficult calculations
involving large numbers of string-like excitation densities, even the completeness
of the set of states is sometimes hard to establish. Fortunately, it has been
realized quite recently that the free energy can also be evaluated more directly
using the analyticity properties. One needs to evaluate $Tr(e^{-\beta H_{XXZ}})$. Writing
this in discretized fashion as

$$Tr\left(e^{-\frac{\beta}{N}H_{XXZ}}\right)^N \hspace{1cm} (11)$$

at first glance it appears to be dominated by the largest eigenvalue of the operator in the bracket. This is of course incorrect, because as $N$ goes to infinity,
all the eigenvalues degenerate to unity. The trick to getting around this is
to use crossing symmetry [15] [14]. A neat presentation was given recently by Kliümper [25]. From equation (8) above,

$$T(i\gamma/2 - \beta/N)T(-i\gamma/2 + \beta/N) = e^{-\frac{\beta}{N}H_{XXZ}} \hspace{1cm} (12)$$

Raising this to the $N^{th}$ power, one has a product of transfer matrices for a sixvertex model having two different vertex strengths, appearing on alternate rows,
a horizontally striped lattice. To evaluate this in the large $N$ limit, we switch
to the equivalent product of \textit{column} transfer matrices. The column transfer matrices are all identical, but down each column the vertex strengths alternate, as it crosses all the stripes. The eigenvalues of this \textit{inhomogeneous} transfer matrix do \textit{not} all collapse to unity as $N$ becomes large—the limit is well defined, and leads to an integral equation from which one can find the leading eigenvalue. Therefore, by this method we can find the thermodynamic behavior of the spin chain in a very direct way, without the headache of summing over many different string excitations, and concerns about completeness. The new approach proves to be very effective computationally.

\section{Inverse-square type systems}

At this point, we back up to our $N$ billiard balls on a line model (meaning point bosons), but this time add an inverse-square repulsive potential between pairs of billiard balls. This is a famous classical integrable system, solved by Calogero\cite{26}, and others, using a Lax pair of matrices. One result that emerged was that if initially the balls are far apart with momenta $p_1, p_2, ..., p_N$ then the final state has them far apart with the same set of momenta. This suggested to Sutherland\cite{11} the idea of an \textit{asymptotic Bethe ansatz} for the quantum system. The wavefunction is certainly not of the standard Bethe ansatz form of a sum of products of plane waves, because the smoothly varying potential energy between particles will give functions of smoothly varying wavelength, destroying the plane wave property. Nevertheless, the dilute system will be close to plane wave form in most of configuration space, and if the scattering phases are factorizable so that diffraction does not occur, the Bethe ansatz might make sense in this dilute limit. Actually, the phase shifting is particular simple for the $1/r^2$ potential, since there is no scale, it is independent of relative momentum, except for the sign. To be precise, for a potential of strength $2\lambda(\lambda - 1)/r^2$, the phase shift is $\pi \lambda k / |k|$. This makes the Bethe ansatz equations like (4) above extremely easy to solve! Of course, by going to equations like (4), we have switched to periodic boundary conditions, but it turns out this can be accomplished by replacing the inverse-square potential by the periodic version, $2\lambda(\lambda - 1)/\sin^2 r$, without spoiling the integrability. In fact, the ground state wavefunction for this potential is known exactly. It is $\prod_{i<j} |\sin(x_i - x_j)|^\lambda$, quite unlike a Bethe ansatz wavefunction. Yet, remarkably, the asymptotic Bethe ansatz equations give the exact ground state energy of the system even in this manifestly nondilute limit. They also give all the excitation energies correctly—in other words, the full spectrum and the thermodynamics. Note that for $2\lambda = 1, 2$ or 4 the ground state correlation function is exactly the distribution of eigenstates of random matrices found by Dyson\cite{27} for orthogonal, unitary and symplectic ensembles.

The high degeneracy of the spectra indicates the presence of new types of symmetries\cite{28}. There has been much recent work on the algebraic structures
of these systems, see for example the review of Bernard[29]. An illuminating interpretation has been suggested by Haldane[30], who argues that the magnon excitations of the inverse-square spin one-half chain are actually a gas of free anyons, specifically spin one-half semions, so the dynamics of the system follows entirely from the statistical “interaction”.

As a simple example of the power of the algebraic approach, we present here a new exchange operator formalism invented by Polychronakos[31], which gives an easy way to prove the integrability of some of these systems. For $N$ particles on a line with positions and momenta $x_i, p_i$ he introduces an operator $M_{ij}$ that permutes the positions and momenta of particles $i, j$, then defines a generalized momentum operator

$$\pi_i = p_i + i \sum_{i \neq j} V(x_i - x_j)M_{ij}$$

with $V$ as yet unspecified. He then requires the Hamiltonian to have free-particle form in the $\pi_i$’s,

$$H = \frac{1}{2} \sum \pi_i^2$$

which gives

$$H = \frac{1}{2} \sum p_i^2 + \frac{1}{2} \sum_{i \neq j} \left( iV_{ij}(p_i - p_j)M_{ij} + V_{ij}'M_{ij} + V_{ij}^2 \right) - \frac{1}{6} \sum V_{ijk}M_{ijk}$$

with

$$V_{ijk} = V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij}$$

and $M_{ijk}$ generates cyclic permutations, $M_{ijk} = M_{ij}M_{jk}$.

Requiring the Hamiltonian to have only two-body potentials, $V(x)$ must satisfy

$$V(x)V(y) + V(y)V(z) + V(z)V(x) = W(x) + W(y) + W(z), x + y + z = 0$$

The commutator of the $\pi_i$’s is

$$[\pi_i, \pi_j] = \sum V_{ijk}(M_{ijk} - M_{jik})$$

At this point, it is very easy to establish the integrability of the Calogero system, in other words, to find the $N$ independent constants of motion of the system. Assume $V(x) = l/x$. Then the Hamiltonian has the form

$$H = \frac{1}{2} \sum p_i^2 + \sum_{i > j} \frac{l(l - M_{ij})}{(x_i - x_j)^2}$$

and since on the boson system $M_{ij} = 0$, this is just the Calogero system. Now for $V(x) = l/x$, from (16) $V_{ijk} = 0$, so from (18) the $\pi_i$ all commute, and, trivially, so do the symmetric quantities

$$I_n = \sum \pi_i^n$$
Assuming now $V(x) = l \cot ax$ gives the periodic system. The $\pi_i$ no longer all commute, but by adding $l \sum_{j \neq i} M_{ij}$ to each we generate a new set of $I_n$ that do commute, so again we establish integrability with a minimum of effort!

In fact, this route leads directly to the integrability of the Haldane-Shastry spin chain system. All we have to do is to give the bosons a spin of one-half, and take the classical (infinitely heavy boson) limit. (This is a mathematical model, so we do not worry about spin and statistics.) On these bosons, the operator $M_{ij}$ above is essentially equivalent to a spin exchange operator. In the classical limit, the bosons form a lattice ground state, and the spin Hamiltonian is identical to the Haldane-Shastry model—the momentum term disappears in the expression for $H$ above, establishing the integrability of the spin chain.\[32\]

The analogous argument applied to the Calogero system of bosons confined by an external harmonic potential gives a Haldane-Shastry type system in which the spins are located at points corresponding to zeros of an Hermite polynomial. Again, the eigenstates are highly degenerate.\[33\]

Another inverse-square integrable system that has been extensively investigated recently\[34\] is the Hubbard model with long-range hopping:

$$H = \sum t_{m,n} c^+_m c_n + U \sum n_{m\uparrow} n_{m\downarrow}$$  \hspace{1cm} (21)

where

$$t_{m,n} = \frac{i t (-1)^{m-n}}{\pi \sin \frac{\pi}{2}}$$  \hspace{1cm} (22)

This system also has highly degenerate energy levels, given by Bethe ansatz like equations which are much more tractable than those for the standard Hubbard model, so this is an interesting easy model for exploring properties of one-dimensional conductors.

Finally, we should at least mention the intriguing connection between integrability and random matrices, although this will be discussed in more detail by Altshuler at this Conference. As stated above, the ground state wavefunction for the periodic inverse-square boson system with interaction parameter $\lambda$, $\prod_{i<j} |\sin(x_i - x_j)|^\lambda$, is for $2\lambda = 1, 2$ or $4$, the distribution function of eigenvalues found by Dyson for orthogonal, unitary and symplectic random matrix ensembles. Can any meaning be given to the time-dependent correlation functions of the boson system in terms of random systems? Simons, Lee and Altshuler\[12\] explored how the eigenvalues of a random system (electronic energy levels in a disordered conductor) respond to a steadily growing external potential (or a twisting boundary condition). For the disordered conductor, they were able to use supersymmetry methods developed by Efetov\[35\] to evaluate eigenvalue correlations. They argue that for a potential

$$H = H_0 + \lambda V$$  \hspace{1cm} (23)

where $H_0$ is a random potential, containing many noninteracting electrons, writing $\lambda^2 = 2it$, the movement of the eigenvalues as $\lambda$ is increased exactly mirrors
the dynamics of the bosons with the inverse-square type interaction. Perhaps it is a suitable note to end on that one of the most ordered systems imaginable, an integrable system with many higher symmetries reflected in its simple spectrum, is intimately connected mathematically to a totally disordered system.

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