Continuity of bilinear maps on direct sums of topological vector spaces

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Abstract

We prove a criterion for continuity of bilinear maps on countable direct sums of topological vector spaces. As a first application, we get a new proof for the fact (due to Hirai et al. 2001) that the map $f: C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$, $(\gamma, \eta) \mapsto \gamma \ast \eta$ taking a pair of test functions to their convolution is continuous. The criterion also allows an open problem by K.-H. Neeb to be solved: If $E$ is a locally convex space, regard the tensor algebra $T(E) := \bigoplus_{j \in \mathbb{N}_0} T^j(E)$ as the locally convex direct sum of projective tensor powers of $E$. We show that $T(E)$ is a topological algebra if and only if every sequence of continuous seminorms on $E$ has an upper bound. In particular, if $E$ is metrizable, then $T(E)$ is a topological algebra if and only if $E$ is normable. Also, $T(E)$ is a topological algebra if $E$ is DFS or $k_\omega$.

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Introduction and statement of results

Consider a bilinear map $\beta: \bigoplus_{i \in \mathbb{N}} E_i \times \bigoplus_{j \in \mathbb{N}} F_j \to H$, where $H$ is a topological vector space and $(E_i)_{i \in \mathbb{N}}$ and $(F_j)_{j \in \mathbb{N}}$ are sequences of topological vector spaces (which we identify with the corresponding subspaces of the direct sum). We prove and exploit the following continuity criterion:

**Theorem A.** $\beta$ is continuous if, for all double sequences $(W_{i,j})_{i,j \in \mathbb{N}}$ of 0-neighbourhoods in $H$, there exist 0-neighbourhoods $U_i$ and $R_{i,j}$ in $E_i$ and 0-neighbourhoods $V_j$ and $S_{i,j}$ in $F_j$ for $i, j \in \mathbb{N}$, such that

\[
\begin{align*}
\beta(U_i \times S_{i,j}) & \subseteq W_{i,j} \text{ for all } i, j \in \mathbb{N} \text{ such that } i < j; \text{ and } \\
\beta(R_{i,j} \times V_j) & \subseteq W_{i,j} \text{ for all } i, j \in \mathbb{N} \text{ such that } i \geq j.
\end{align*}
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As a first application, we obtain a new proof for the continuity of the bilinear map \( f: C^\infty_c(\mathbb{R}^n) \times C^\infty_c(\mathbb{R}^n) \to C^\infty_c(\mathbb{R}^n) \), \((\gamma, \eta) \mapsto \gamma \ast \eta\) taking a pair of test functions to their convolution (Corollary 3.1). This was first shown in [12, Proposition 2.3]. Our proof allows \( \mathbb{R}^n \) to be replaced with a Lie group \( G \), in which case \( f \) is continuous if and only if \( G \) is \( \sigma \)-compact [3].

For a second application of Theorem A, consider a locally convex space \( E \) over \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \). Let \( T^0_\pi(E) \) := \( \mathbb{K} \), \( T^1_\pi(E) := E \) and endow the tensor powers \( T^2_\pi(E) := E \otimes_\pi E \), \( T^{j+1}_\pi(E) := E \otimes_\pi T^j_\pi(E) \) with the projective tensor product topology (see, e.g., [20]). Topologize the tensor algebra \( T_\pi(E) := \bigoplus_{j\in\mathbb{N}_0} T^j_\pi(E) \) (see [16, XVI, §7]) as the locally convex direct sum [4]. In infinite-dimensional Lie theory, the question arose of whether \( T_\pi(E) \) always is a topological algebra, i.e., whether the algebra multiplication is continuous [18, Problem VIII.5]. We solve this question (in the negative), and actually obtain a characterization of those locally convex spaces \( E \) for which \( T_\pi(E) \) is a topological algebra.

To formulate our solution, given continuous seminorms \( p \) and \( q \) on \( E \) let us write \( p \preceq q \) if \( p \leq Cq \) pointwise for some \( C > 0 \). For \( \theta \) an infinite cardinal number, let us say that \( E \) satisfies the upper bound condition for \( \theta \) (the UBC(\( \theta \)), for short) if for every set \( P \) of continuous seminorms on \( E \) of cardinality \( |P| \leq \theta \), there exists a continuous seminorm \( q \) on \( E \) such that \( p \preceq q \) for all \( p \in P \). If \( E \) satisfies the UBC(\( \mathbb{N}_0 \)), we shall simply say that \( E \) satisfies the countable upper bound condition. Every normable space satisfies the UBC(\( \theta \)), and there also exist non-normable examples (see Section 8). We obtain the following characterization:

**Theorem B.** Let \( E \) be a locally convex space. Then \( T_\pi(E) \) is a topological algebra if and only if \( E \) satisfies the countable upper bound condition.

In particular, for \( E \) a metrizable locally convex space, \( T_\pi(E) \) is a topological algebra if and only if \( E \) is normable (Corollary 4.2).

The upper bound conditions introduced here are also useful for the theory of vector-valued test functions. If \( E \) is a locally convex space and \( M \) a paracompact, non-compact, finite-dimensional smooth manifold, let \( C^\infty_c(M, E) \)

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2For hypocontinuity of convolution \( C^\infty(\mathbb{R}^n)' \times C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \), see [21, p. 167].

3If \( g \) is a locally convex topological Lie algebra and \( T_\pi(g) \) a topological algebra, then also the enveloping algebra \( U(g) \) (which is a quotient of \( T_\pi(g) \)) is a topological algebra with the quotient topology. This topology on \( U(g) \) has been used implicitly in [19].
be the space of all compactly supported smooth $E$-valued functions on $M$. Consider the bilinear map

$$\Phi: C_c^\infty(M, \mathbb{R}) \times E \to C_c^\infty(M, E), \quad (\gamma, v) \mapsto \gamma v,$$

where $(\gamma v)(x) := \gamma(x)v$. If $M$ is $\sigma$-compact, then $\Phi$ is continuous if and only if $E$ satisfies the countable upper bound condition. If $M$ is not $\sigma$-compact, then $\Phi$ is continuous if and only if $E$ satisfies the UBC($\theta$), for $\theta$ the number of connected components of $M$ (see [10, Theorem B]).

Without recourse to the countable upper bound condition, for a certain class of non-metrizable locally convex spaces we show directly that $T_\pi(E)$ is a topological algebra. Recall that a Hausdorff topological space $X$ is a $k_\omega$-space if $X = \lim_{\to} K_n$ as a topological space for a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact spaces (a so-called $k_\omega$-sequence) with union $\bigcup_{n=1}^\infty K_n = X$ [6, 11]. For example, the dual space $E'$ of any metrizable locally convex space is a $k_\omega$-space when equipped with the compact-open topology (cf. [2, Corollary 4.7]). In particular, every Silva space (or DFS-space) is a $k_\omega$-space, that is, every locally convex direct limit of Banach spaces $E_1 \subseteq E_2 \subseteq \cdots$, such that all inclusion maps $E_n \to E_{n+1}$ are compact operators [8, Example 9.4]. For instance, every vector space of countable dimension (like $\mathbb{R}^{(\mathbb{N})}$) is a Silva space (and hence a $k_\omega$-space) when equipped with the finest locally convex topology. We show:

**Theorem C.** Let $E$ be a locally convex space. If $E$ is a $k_\omega$-space (e.g., if $E$ is a DFS-space), then $T_\pi(E)$ is a topological algebra.

To enable a proof of Theorem C, we first study tensor powers $T^j_\nu(E)$ in the category of all (not necessarily locally convex) topological vector spaces, for $E$ as in the theorem.\(^4\) We show that $T^j_\nu(E)$ and $T_\nu(E) := \bigoplus_{i \in \mathbb{N}_0} T^j_\nu(E)$ are $k_\omega$-spaces (Lemmas 5.4 and 5.7) and that $T_\nu(E) = \lim_{\to} \prod_{j=1}^\infty T^j_\nu(E)$ as a topological space (Lemma 5.7). This allows us to deduce that $T_\nu(E)$ is a topological algebra (Proposition 5.8), which entails that also the convexification $T_\pi(E) = (T_\nu(E))_{lcx}$ is a topological algebra (see Section 7).\(^5\)

The conclusion of Theorem C remains valid if $E = F_{lcx}$ for a topological vector space $F$ which is a $k_\omega$-space (Proposition 7.1). This implies, for example, that $T_\pi(E)$ is a topological algebra whenever $E$ is the free locally

\(^4\)See [22] and [7] for such tensor products, and the references therein.

\(^5\)(Quasi-)convexifications of direct limits of $k_\omega$-spaces also appear in [11], for other goals.
convex space over a $k_\omega$-space $X$ (Corollary 7.2). Combining this result with Theorem B, we deduce: If a locally convex space $E$ is a $k_\omega$-space, or of the form $E = F_{lcx}$ for some topological vector space $F$ which is a $k_\omega$-space, then $E$ satisfies the countable upper bound condition (Corollary 8.1).

Of course, also many non-metrizable locally convex spaces $E$ exist for which $T_\pi(E)$ is not a topological algebra. This happens, for example, if $E$ has a topological vector subspace $F$ which is metrizable but not normable (e.g., if $E = \mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^\mathbb{N}$). In fact, $E$ cannot satisfy the countable upper bound condition because this property would be inherited by $F$ [10, Proposition 3.1 (c)].

1 Notational conventions

Throughout the article, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and topological vector spaces over $\mathbb{K}$ are considered (which need not be Hausdorff). If $q$ is a seminorm on a vector space $E$, we write $B_q^r(x) := \{y \in E : q(y - x) < r\}$ and $\overline{B}_q^r(x) := \{y \in E : q(y - x) \leq r\}$ for the open (resp., closed) ball of radius $r > 0$ around $x \in E$. We let $(E_q, \|\|_q)$ be the normed space associated with $q$, defined via

$$E_q := E/q^{-1}(0) \quad \text{and} \quad \|x + q^{-1}(0)\|_q := q(x).$$

(2)

Also, we let

$$\rho_q : E \to E_q, \quad \rho_q(x) := x + q^{-1}(0)$$

(3)

be the canonical map. If $q$ is continuous with respect to a locally convex vector topology on $E$, then $\rho_q$ is continuous. If $(E, \|\|)$ is a normed space and $q = \|\|$, we also write $B_E^r(x) := B_q^r(x)$ and $\overline{B}_E^r(x) := \overline{B}_q^r(x)$ for the balls.

A subset $U$ of a vector space $E$ is called balanced if $\overline{B}_1^E(0) \subseteq U$. We set $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

If $I$ is a countable set and $(E_i)_{i \in I}$ a family of topological vector spaces, its direct sum is the space $\bigoplus_{i \in I} E_i$ of all $(x_i)_{i \in I} \in \prod_{i \in I} E_i$ such that $x_i = 0$ for all but finitely many $i \in I$. The sets of the form

$$\bigoplus_{i \in I} U_i := \bigoplus_{i \in I} E_i \cap \prod_{i \in I} U_i,$$

for $U_i$ ranging through the set of 0-neighbourhoods in $E_i$, form a basis of 0-neighbourhoods for a vector topology on $\bigoplus_{i \in I} E_i$. We shall always equip countable direct sums with this topology (called the ‘box topology’), which
is locally convex if so is each $E_i$. Then a linear map $\bigoplus_{i \in I} E_i \to F$ to a topological vector space $F$ is continuous if and only if all of its restrictions to the $E_i$ are continuous ([14, §4.1 & §4.3]; cf. [4] for the locally convex case).

A topological algebra is a topological vector space $A$, together with a continuous bilinear map $A \times A \to A$. If $A$ is assumed associative or unital, we shall say so explicitly.

## 2 Bilinear maps on direct sums

We now prove Theorem A. Afterwards, we discuss the hypotheses of the theorem and formulate special cases which are easier to apply.

**Proof of Theorem A.** By Proposition 5 in [4, Chapter I, §1, no. 6], the bilinear map $\beta$ will be continuous if it is continuous at $(0, 0)$. To verify the latter, let $W_0$ be a 0-neighbourhood in $H$. Recursively, pick 0-neighbourhoods $W_k \subseteq H$ for $k \in \mathbb{N}$ such that $W_k + W_k \subseteq W_{k-1}$. Then

$$\forall k \in \mathbb{N} \quad W_1 + \cdots + W_k \subseteq W_0. \quad (4)$$

Let $\sigma: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection, and $W_{i,j} := W_{\sigma(i,j)}$ for $i, j \in \mathbb{N}$. By (4),

$$\bigcup_{(i,j) \in \Phi} W_{i,j} \subseteq W_0 \quad \text{for every finite subset } \Phi \subseteq \mathbb{N}^2. \quad (5)$$

For $i, j \in \mathbb{N}$, choose 0-neighbourhoods $U_i$ and $R_{i,j}$ in $E_i$ and 0-neighbourhoods $V_j$ and $S_{i,j}$ in $F_j$ such that (11) holds. For $i \in \mathbb{N}$, the set $P_i := U_i \cap \bigcap_{j=1}^i R_{i,j}$ is a 0-neighbourhood in $E_i$. For $j \in \mathbb{N}$, let $Q_j \subseteq F_j$ be the 0-neighbourhood $Q_j := V_j \cap \bigcap_{i=1}^j S_{i,j}$. We claim that

$$\forall i, j \in \mathbb{N} \quad \beta(P_i \times Q_j) \subseteq W_{i,j}. \quad (6)$$

If this is true, then $P := \bigoplus_{i \in \mathbb{N}} P_i$ is a 0-neighbourhood in $\bigoplus_{i \in \mathbb{N}} E_i$ and $Q := \bigoplus_{j \in \mathbb{N}} Q_j$ a 0-neighbourhood in $\bigoplus_{j \in \mathbb{N}} F_j$ such that $\beta(P \times Q) \subseteq W_0$, as

$$\sum_{(i,j) \in \Phi} \beta(P_i \times Q_j) \subseteq \sum_{(i,j) \in \Phi} W_{i,j} \subseteq W_0$$

for each finite subset $\Phi \subseteq \mathbb{N}^2$ (by (6) and (5)) and therefore $\beta(P \times Q) = \bigcup_{\Phi} \sum_{(i,j) \in \Phi} \beta(P_i \times Q_j) \subseteq W_0$. Thus continuity of $\beta$ at $(0, 0)$ is established,
Once (6) is verified. To prove (6), let $i,j \in \mathbb{N}$. If $i \geq j$, then $\beta(R_i \times Q_j) \subseteq \beta(R_{i,j} \times V_j) \subseteq W_{i,j}$. If $i < j$, then $\beta(P_i \times Q_j) \subseteq \beta(U_i \times S_{i,j}) \subseteq W_{i,j}$. □

The criterion from Theorem A is sufficient, but not necessary for continuity.

**Example 2.1** Let $H := \mathbb{R}^\mathbb{N}$ be the space of all real-valued sequences, equipped with the product topology, and $E_i := F_i := H$ for all $i \in \mathbb{N}$. Then $\mathbb{R}^\mathbb{N}$ is an algebra under the pointwise multiplication

$$\delta : \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}, \quad \delta((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) := (x_i)_{i \in \mathbb{N}} \odot (y_i)_{i \in \mathbb{N}} := (x_i y_i)_{i \in \mathbb{N}}.$$  

We show that the bilinear map

$$\beta : \bigoplus_{i \in \mathbb{N}} E_i \times \bigoplus_{j \in \mathbb{N}} F_j \to H, \quad \beta((f_i)_{i \in \mathbb{N}}, (g_j)_{j \in \mathbb{N}}) := \sum_{i,j \in \mathbb{N}} f_i \odot g_j$$

is continuous, but does not satisfy the hypotheses of Theorem A.

To see this, note that the seminorms

$$p_n : \mathbb{R}^\mathbb{N} \to [0, \infty[, \quad p_n((x_i)_{i \in \mathbb{N}}) := \max\{|x_i| : i = 1, \ldots, n\}$$

define the topology on $\mathbb{R}^\mathbb{N}$ for $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we have

$$(\forall f, g \in \mathbb{R}^\mathbb{N}) \quad p_n(f \circ g) \leq p_n(f)p_n(g), \quad (7)$$

entailing that $\delta$ is continuous and thus $\mathbb{R}^\mathbb{N}$ a topological algebra. Also, if $W \subseteq H$ is a 0-neighbourhood, then $B^{p_n}_\varepsilon(0) \subseteq W$ for some $n \in \mathbb{N}$ and $\varepsilon > 0$. Set $Q_i := B^{p_n}_{2^{-i}\sqrt{\varepsilon}}(0)$ for $i \in \mathbb{N}$. Then $Q_i \odot Q_j \subseteq B^{p_n}_{2^{-i-j}\varepsilon}(0)$ for all $i, j \in \mathbb{N}$ (by (7)), entailing that $Q := \bigoplus_{i \in \mathbb{N}} Q_i$ is a zero-neighbourhood in $\bigoplus_{i \in \mathbb{N}} E_i$ such that $\beta(Q \times Q) \subseteq \sum_{(i,j) \in \mathbb{N}^2} B^{p_n}_{2^{-i-j}\varepsilon}(0) \subseteq B^{p_n}_\varepsilon(0) \subseteq W$. Hence $\beta$ is continuous at $(0,0)$ and hence continuous.

On the other hand, let $r, s > 0$ and $k, m, n \in \mathbb{N}$.

If $k > n$ or $k > m$, then $(\exists f \in B^{p_n}_r(0), g \in B^{p_m}_s(0)) \ f \circ g \not\in B^{p_k}_1(0). \quad (8)$

In fact, assume that $k > m$ (the case $k > n$ is similar). Let $e_k \in \mathbb{R}^\mathbb{N}$ be the sequence whose $k$-th entry is 1, while all others vanish. Then $f := \frac{r}{2}e_k \in B^{p_n}_r(0)$, $g := \frac{s}{2}e_k \in B^{p_m}_s(0)$ (noting that $p_m(g) = 0$ since $k > m$), and $f \circ g \not\in B^{p_k}_1(0)$ as $p_k(f \circ g) = p_k(e_k) = 1$. 

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Now consider the 0-neighbourhoods $W_{i,j} := B_r^{p_{i+j}}(0)$ in $H$. Suppose there are 0-neighbourhoods $U_i, R_{i,j}, V_i$ and $S_{i,j}$ in $\mathbb{R}^N$ such that (1) holds – this will yield a contradiction. There is $n \in \mathbb{N}$ and $r > 0$ such that $B_r^{p_n}(0) \subseteq U_1$. Also, for each $j \in \mathbb{N}$ there are $m_j \in \mathbb{N}$ and $s_j > 0$ with $B_r^{p_{m_j}}(0) \subseteq S_{1,j}$. Then

$$B_r^{p_n}(0) \circ B_r^{p_{m_j}}(0) = \beta(B_r^{p_n}(0) \times B_r^{p_{m_j}}(0)) \subseteq W_{1,j} = B_r^{p_{1+j}}(0)$$

for all $j \geq 2$, by (1). Thus $n \geq 1 + j$ for all $j \geq 2$, by (8). This is impossible.

Our applications use the following consequence of Theorem A:

**Corollary 2.2** Let $(E_i)_{i \in \mathbb{N}}$ and $(F_j)_{j \in \mathbb{N}}$ be sequences of topological vector spaces and $H$ be a topological vector space. Then a bilinear mapping $\beta : \bigoplus_{i \in \mathbb{N}} E_i \times \bigoplus_{j \in \mathbb{N}} F_j \rightarrow H$ is continuous if there exist 0-neighbourhoods $U_i$ in $E_i$ and $V_j$ in $F_j$ for $i,j \in \mathbb{N}$, such that (a) and (b) hold:

(a) For all 0-neighbourhoods $W \subseteq H$ and $i,j \in \mathbb{N}$, there exists a 0-neighbourhood $S_{i,j}$ in $F_j$ such that $\beta(U_i \times S_{i,j}) \subseteq W$.

(b) For all 0-neighbourhoods $W \subseteq H$ and $i,j \in \mathbb{N}$, there exists a 0-neighbourhood $R_{i,j}$ in $E_i$ such that $\beta(R_{i,j} \times V_j) \subseteq W$.

**Proof.** Let $(W_{i,j})_{i,j \in \mathbb{N}}$ be a double sequence of 0-neighbourhoods in $H$. For $i,j \in \mathbb{N}$, choose 0-neighbourhoods $U_i \subseteq E_i$ and $V_j \subseteq F_j$ as described in the corollary. Then, by (a) and (b) (applied with $W = W_{i,j}$), for all $i,j \in \mathbb{N}$ there exist 0-neighbourhoods $R_{i,j} \subseteq E_i$ and $S_{i,j} \subseteq F_j$ such that

$$\beta(U_i \times S_{i,j}) \subseteq W_{i,j} \quad \text{and} \quad \beta(R_{i,j} \times V_j) \subseteq W_{i,j}.$$ 

Hence Theorem A applies. \hfill \Box

The next lemma helps to check the hypotheses of Corollary 2.2 in important cases.

**Lemma 2.3** Let $E$, $F$ and $H$ be topological vector spaces and $\beta : E \times F \rightarrow H$ be bilinear. Assume $\beta = b \circ (\text{id}_E \times \phi)$ for a continuous linear map $\phi : F \rightarrow X$ to a normed space $(X, \| \cdot \|)$ and continuous bilinear map $b : E \times X \rightarrow H$. Then $V := \phi^{-1}(B^X_1(0))$ is a 0-neighbourhood in $F$ such that, for each 0-neighbourhood $W \subseteq H$, there is a 0-neighbourhood $R \subseteq E$ with $\beta(R \times V) \subseteq W.$
Proof. Since $b^{-1}(W)$ is a 0-neighbourhood in $E \times X$, there exist a 0-neighbourhood $S \subseteq E$ and $r > 0$ such that $S \times B_r^X(0) \subseteq b^{-1}(W)$. Set $R := rS$. Using that $b$ is bilinear, we obtain $\beta(R \times V) \subseteq b(rS \times B_1^X(0)) = b(S \times rB_1^X(0)) \subseteq W$. □

If $F$ is a normed space, we can simply set $X := F$, $\phi := \text{id}_F$ and $b := \beta$ in Lemma 2.3, i.e., the conclusion is always guaranteed then (with $V = B_1^F(0)$).

Corollary 2.4 Let $(E_i)_{i \in \mathbb{N}}$ and $(F_j)_{j \in \mathbb{N}}$ be sequences of normed spaces, $H$ be a topological vector space and $\beta_{i,j} : E_i \times F_j \to H$ be continuous bilinear maps for $i, j \in \mathbb{N}$. Then the following bilinear map is continuous:

$$\beta : \bigoplus_{i \in \mathbb{N}} E_i \times \bigoplus_{j \in \mathbb{N}} F_j \to H, \quad \beta((x_i)_{i \in \mathbb{N}}, (y_j)_{j \in \mathbb{N}}) := \sum_{(i,j) \in \mathbb{N}^2} \beta_{i,j}(x_i, y_j). \quad (9)$$

Proof. Lemma 2.3 shows that the hypotheses of Corollary 2.2 are satisfied if we define $U_i$ and $V_j$ as the unit balls, $U_i := B_{1}^{E_i}(0)$ and $V_j := B_{1}^{F_j}(0)$. □

If $H$ is locally convex, then Corollary 2.4 also follows from [5, Corollary 2.1]. In the locally convex case, Theorem A can be reformulated as follows:

Corollary 2.5 Let $(E_i)_{i \in \mathbb{N}}$ and $(F_j)_{j \in \mathbb{N}}$ be sequences of locally convex spaces, $H$ be a locally convex space and $\beta_{i,j} : E_i \times F_j \to H$ be continuous bilinear maps for $i, j \in \mathbb{N}$. Assume that, for every double sequence $(P_{i,j})_{i,j \in \mathbb{N}}$ of continuous seminorms on $H$, there are continuous seminorms $p_i$ (for $i \in \mathbb{N}$) and $p_{i,j}$ on $E_i$ (for $i \geq j$) and continuous seminorms $q_j$ (for $j \in \mathbb{N}$) and $q_{i,j}$ on $F_j$ (for $i < j$), such that:

(a) $P_{i,j}(\beta_{i,j}(x,y)) \leq p_i(x)q_{i,j}(y)$ for all $i < j$ in $\mathbb{N}$, $x \in E_i$, $y \in F_j$; and

(b) $P_{i,j}(\beta_{i,j}(x,y)) \leq p_{i,j}(x)q_j(y)$ for all $i \geq j$ in $\mathbb{N}$ and all $x \in E_i$, $y \in F_j$.

Then the bilinear map $\beta$ described in (9) is continuous.

Proof. Let $W_{i,j} \subseteq H$ be 0-neighbourhoods for $i, j \in \mathbb{N}$. Then there are continuous seminorms $P_{i,j}$ on $H$ such that $B_{1}^{P_{i,j}}(0) \subseteq W_{i,j}$. Let $p_i, p_{i,j}, q_j$ and $q_{i,j}$ be as described in Corollary 2.5. Then $U_i := B_{1}^{p_i}(0)$ and $R_{i,j} := B_{1}^{p_{i,j}}(0)$ are 0-neighbourhoods in $E_i$. Also, $V_j := B_{1}^{q_j}(0)$ and $S_{i,j} := B_{1}^{q_{i,j}}(0)$ are 0-neighbourhoods in $F_j$. If $i < j$, $x \in U_i$ and $y \in S_{i,j}$, then $P_{i,j}(\beta_{i,j}(x,y)) \leq p_{i,j}(x)q_j(y)$ for all $i \geq j$ in $\mathbb{N}$ and all $x \in E_i$, $y \in F_j$. If $i < j$, $x \in U_i$ and $y \in S_{i,j}$, then $P_{i,j}(\beta_{i,j}(x,y)) \leq p_{i,j}(x)q_j(y)$ for all $i \geq j$ in $\mathbb{N}$ and all $x \in E_i$, $y \in F_j$. Therefore, $\beta_{i,j}(x,y) \in V_j$ for all $i < j$, $x \in U_i$ and $y \in S_{i,j}$. Hence, $\beta_{i,j}(x,y) \in V_j$ for all $i < j$, $x \in U_i$ and $y \in S_{i,j}$, □
Let $G$ be a Lie group, with Haar measure $\mu$. Let $b: E_1 \times E_2 \to F$ be a continuous bilinear map between locally convex spaces (where $F$ is sequentially complete), and $r, s, t \in \mathbb{N}_0 \cup \{\infty\}$ such that $t \leq r + s$. Using Corollary 2.3, it is possible to characterize those $(G, r, s, t, b)$ for which the convolution map

$$\beta: C^r_c(G, E_1) \times C^s_c(G, E_2) \to C^t_c(G, F), \quad (\gamma, \eta) \mapsto \gamma * b \eta$$

is continuous, where $(\gamma * b \eta)(x) := \int_G b(\gamma(y), \eta(y^{-1}x)) d\mu(y)$ (see [3]).

## 3 Continuity of convolution of test functions

Using the continuity criterion, we obtain a new proof for [12, Proposition 2.3]:

**Corollary 3.1** The map $C^\infty_c(\mathbb{R}^n) \times C^\infty_c(\mathbb{R}^n) \to C^\infty_c(\mathbb{R}^n)$, $(\gamma, \eta) \mapsto \gamma * \eta$ is continuous.

Before we present the proof, let us fix further notation and recall basic facts. Given an open set $\Omega \subseteq \mathbb{R}^n$, $r \in \mathbb{N}_0 \cup \{\infty\}$ and a compact set $K \subseteq \Omega$, let $C^r_c(\Omega)$ be the space of all $C^r$-functions $\gamma: \Omega \to K$ with support $\text{supp}(\gamma) \subseteq K$. Using the partial derivatives $\partial^\alpha \gamma := \frac{\partial^{\alpha}}{\partial x^\alpha}$ for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ of order $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq r$ and the supremum norm $\|\cdot\|_\infty$, we define norms $\|\cdot\|_k$ on $C^r_c(\Omega)$ for $k \in \mathbb{N}_0$ with $k \leq r$ via

$$\|\gamma\|_k := \max_{|\alpha| \leq k} \|\partial^\alpha \gamma\|_\infty,$$

and give $C^r_c(\Omega)$ the locally convex vector topology determined by these norms. We give $C^r_c(\Omega) = \bigcup_K C^r_c(\Omega)$ the locally convex direct limit topology, for $K$ ranging through the set of compact subsets of $\Omega$.

**Lemma 3.2** (a) The pointwise multiplication $C^r_c(\Omega) \times C^r_c(\Omega) \to C^r_c(\Omega)$, $(\gamma, \eta) \mapsto \gamma \eta$ is continuous.

(b) Let $(h_i)_{i \in \mathbb{N}}$ be a locally finite, smooth partition of unity on $\Omega$, such that each $h_i$ has compact support $K_i := \text{supp}(h_i) \subseteq \Omega$. Then the linear map $\Phi: C^r_c(\Omega) \to \bigoplus_{i \in \mathbb{N}} C^r_c(K_i), \gamma \mapsto (h_i \gamma)_{i \in \mathbb{N}}$ is continuous.

---

6See, e.g., [15] Chapter II, §3, Corollary 3.3.
Proof. (a) Let $E := (1, \ldots, 1) \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$ such that $k \leq r$. By the Leibniz Rule, $\|\partial^\alpha(\gamma \eta)\|_\infty \leq \sum_{\beta \leq \alpha} (\alpha) \|\partial^\beta \gamma\|_\infty \|\partial^{\alpha-\beta} \eta\|_\infty$, using multi-index notation. Since $\sum_{\beta \leq \alpha} (\alpha) = (E + E)^\alpha = 2^{\|\alpha\|} \leq 2^k$ if $|\alpha| \leq k$ and $\|\partial^\beta \gamma\|_\infty \|\partial^{\alpha-\beta} \eta\|_\infty \leq \|\gamma\|_k \|\eta\|_k$, we deduce that $\|\gamma \eta\|_k \leq 2^k \|\gamma\|_k \|\eta\|_k$. Hence multiplication is continuous at $(0, 0)$ and hence continuous, being bilinear.

(b) To see that the linear map $\Phi$ is continuous, it suffices to show that its restriction $\Phi_K$ to $C^r_K(\Omega)$ is continuous, for each compact set $K \subseteq \Omega$. As $K$ is compact and $(K_i)_{i \in \mathbb{N}}$ locally finite, the set $F := \{ i \in \mathbb{N} : K \cap K_i \neq \emptyset \}$ is finite. Because the image of $\Phi_K$ is contained in the subspace $\bigoplus_{i \in F} C^r_{K_i}(\Omega) \cong \prod_{i \in F} C^r_{K_i}(\Omega)$ of $\bigoplus_{i \in F} C^r_{K_i}(\Omega) \cong \bigoplus_{i \in F} C^r_{K_i}(\Omega) \oplus \bigoplus_{i \in \mathbb{N} \setminus F} C^r_{K_i}(\Omega)$, the map $\Phi_K$ will be continuous if its components with values in $C^r_{K_i}(\Omega)$ are continuous for all $i \in F$. But these are the maps $C^r_{K_i}(\Omega) \to C^r_K(\Omega)$, $\gamma \mapsto h_i \gamma$, which are continuous as restrictions of the maps $C^r_{K \cup K_i}(\Omega) \to C^r_{K \cup K_i}(\Omega)$, $\gamma \mapsto h_i \gamma$, whose continuity follows from (a).

If $\gamma \in C^0_c(\mathbb{R}^n)$ and $\eta \in C^0_c(\mathbb{R}^n)$, it is well-known that $\gamma * \eta \in C^0_c(\mathbb{R}^n)$, with $\text{supp}(\gamma * \eta) \subseteq \text{supp}(\gamma) + \text{supp}(\eta)$ (see [13], 1.3.11). Moreover,

$$\|\gamma \eta\|_\infty \leq \|\gamma\|_\infty \|\eta\|_{L^1} \quad \text{and} \quad \|\gamma * \eta\|_\infty \leq \|\gamma\|_{L^1} \|\eta\|_\infty,$$

(10)

since $|\gamma * \eta|(x) = \int_{\mathbb{R}^n} |\gamma(y)| |\eta(x - y)| \, d\lambda(y) \leq \|\gamma\|_\infty \int_{\mathbb{R}^n} |\gamma(y)| \, d\lambda(y)$. If $\gamma \in C^\infty_0(\mathbb{R}^n)$ and $\eta \in C^0_c(\mathbb{R}^n)$, then $\gamma * \eta \in C^\infty_0(\mathbb{R}^n)$ and

$$\partial^\alpha(\gamma * \eta) = (\partial^\alpha \gamma) * \eta$$

(11)

for all $\alpha \in \mathbb{N}_0^n$ (see 1.3.5 and 1.3.6 in [13]). Likewise, $\gamma * \eta \in C^\infty_0(\mathbb{R}^n)$ for all $\gamma \in C^0_c(\mathbb{R}^n)$ and $\eta \in C^\infty_0(\mathbb{R}^n)$, with $\partial^\alpha(\gamma * \eta) = \gamma * \partial^\alpha \eta$. By (11) and (10),

$$\|\gamma * \eta\|_k \leq \|\gamma\|_k \|\eta\|_{L^1}$$

(12)

for all $\gamma \in C^\infty_0(\mathbb{R}^n)$, $\eta \in C^0_c(\mathbb{R}^n)$ and $k \in \mathbb{N}_0$. Likewise, $\|\eta * \gamma\|_k \leq \|\eta\|_{L^1} \|\gamma\|_k$. We shall also use the obvious estimate

$$\|\eta\|_{L^1} \leq \lambda(\text{supp}(\eta)) \|\eta\|_\infty \quad \text{for} \ \eta \in C^0_c(\mathbb{R}^n).$$

(13)

Hence, given compact sets $K, L \subseteq \mathbb{R}^n$, we have $\|\gamma * \eta\|_k \leq \lambda(L) \|\gamma\|_k \|\eta\|_\infty$ for all $\gamma \in C^\infty_K(\mathbb{R}^n)$, $\eta \in C^0_L(\mathbb{R}^n)$ and $k \in \mathbb{N}_0$. This entails the first assertion of the next lemma, and the second can be proved analogously:
Lemma 3.3 The following bilinear maps are continuous:

\[ C^\infty_K(\mathbb{R}^n) \times C^0_L(\mathbb{R}^n) \rightarrow C^\infty_{K+L}(\mathbb{R}^n), \quad (\gamma, \eta) \mapsto \gamma \ast \eta \quad \text{and} \]
\[ C^0_K(\mathbb{R}^n) \times C^\infty_L(\mathbb{R}^n) \rightarrow C^\infty_{K+L}(\mathbb{R}^n), \quad (\gamma, \eta) \mapsto \gamma \ast \eta. \]

Proof of Corollary 3.1. Choose a locally finite, smooth partition of unity \((h_i)_{i \in \mathbb{N}}\) on \(\mathbb{R}^n\) such that each \(h_i\) has compact support \(K_i := \text{supp}(h_i)\). Set \(E_i := F_i := C^\infty_K(\mathbb{R}^n)\) for \(i \in \mathbb{N}\). Then \(X_i := (C^0_{K_i}(\mathbb{R}^n), \|\cdot\|_\infty)\) is a normed space and inclusion \(\phi_i: E_i \rightarrow X_i, \gamma \mapsto \gamma\) is continuous linear. Let \(\beta_{i,j}: E_i \times E_j \rightarrow C^\infty_c(\mathbb{R}^n)\), \(\mu_{i,j}: E_i \times X_j \rightarrow C^\infty_c(\mathbb{R}^n)\), and \(\nu_{i,j}: X_i \times E_j \rightarrow C^\infty_c(\mathbb{R}^n)\) be convolution \((\gamma, \eta) \mapsto \gamma \ast \eta\) for \(i,j \in \mathbb{N}\). Lemma 3.3 implies that \(\beta_{i,j}, \mu_{i,j}, \text{and } \nu_{i,j}\) are continuous bilinear. Since \(\beta_{i,j} = \mu_{i,j} \circ (\text{id}_{E_i} \times \phi_j) = \nu_{i,j} \circ (\phi_i \times \text{id}_{E_j})\),

Lemma 2.3 shows that the bilinear map \(\beta: \bigoplus_{i \in \mathbb{N}} E_i \times \bigoplus_{j \in \mathbb{N}} E_j \rightarrow C^\infty_c(\mathbb{R}^n)\) from (6) obtained from the above \(\beta_{i,j}\) satisfies the hypotheses of Corollary 2.2 with \(U_i := V_i := \phi_i^{-1}(B^X_1(0))\). Hence \(\beta\) is continuous. But the convolution map \(f: C^\infty_c(\mathbb{R}^n) \times C^\infty_c(\mathbb{R}^n) \rightarrow C^\infty_c(\mathbb{R}^n)\) can be expressed as
\[ f = \beta \circ (\Phi \times \Phi) \tag{14} \]
with the continuous linear map \(\Phi\) introduced in Lemma 3.2(b) (for \(r = \infty\)), as we shall presently verify. Hence, being a composition of continuous maps, \(f\) is continuous. To verify (14), let \(\gamma, \eta \in C^\infty_c(\mathbb{R}^n)\). Since \(\gamma\) has compact support, only finitely many terms in the sum \(\gamma = \sum_{i \in \mathbb{N}} h_i \gamma\) are non-zero, and likewise in \(\eta = \sum_{j \in \mathbb{N}} h_j \eta\). Hence
\[ f(\gamma, \eta) = \sum_{(i,j) \in \mathbb{N}^2} f(h_i \gamma, h_j \eta) = \sum_{(i,j) \in \mathbb{N}^2} \beta_{i,j}(h_i \gamma, h_j \eta) = \beta((h_i \gamma)_{i \in \mathbb{N}}, (h_j \eta)_{j \in \mathbb{N}}), \]
which coincides with \((\beta \circ (\Phi \times \Phi))(\gamma, \eta)\). The proof is complete. \(\square\)

4 Proof of Theorem B

We now prove Theorem B, and then discuss the case of metrizable spaces.

Proof of Theorem B. Let \(\beta_{0,i}: \mathbb{K} \times T^i_n(E) \rightarrow T^i_n(E), (z, v) \mapsto zv\) and \(\beta_{i,0}: T^i_n(E) \times \mathbb{K} \rightarrow T^i_n(E), \beta_{i,0}(v, z) := zv\) be multiplication with scalars, for
for all \( x_1, \ldots, x_i, y_1, \ldots, y_j \in E \). As we are using projective tensor topologies, all of the bilinear maps \( \beta_{i,j}, i,j \in \mathbb{N}_0 \) are continuous, which is well known.

We first consider the special case of a normable space \( E \). Then the multiplication \( \beta: T_\pi^j(E) \times T_\pi(E) \to T_\pi^j(E) \) of the tensor algebra is the map \( \beta \) from (9), hence continuous by Corollary 2.4.

Next, let \( E \) be an arbitrary locally convex space satisfying the countable upper bound condition. Let \( U \) be a 0-neighbourhood in \( T_\pi^j(E) \). After shrinking \( U \) to a box neighbourhood, we may assume that \( U = \bigoplus_{j \in \mathbb{N}_0} \overline{B}_1^{q_j}(0) \) for continuous seminorms \( q_j \) on \( T_\pi^j(E) \). For \( j \in \mathbb{N}_0 \), let \( H_j := ((T_\pi^j(E))_{q_j}, \| \cdot \|_{q_j}) \) be the normed space associated to \( q_j \), and \( \rho_{q_j}: T_\pi^j(E) \to H_j \) the canonical map (see (2) and (3)). Let \( V := \bigoplus_{j \in \mathbb{N}_0} \overline{B}_1^\|q_j\|(0) \subseteq \bigoplus_{j \in \mathbb{N}_0} H_j \). Then

\[
\rho: T_\pi(E) \to \bigoplus_{j \in \mathbb{N}_0} H_j, \quad \rho((x_j)_{j \in \mathbb{N}_0}) := (\rho_{q_j}(x_j))_{j \in \mathbb{N}_0}
\]

is a continuous linear map, and \( \rho^{-1}(V) = U \). If we can show that \( \rho \circ \beta \) is continuous, then \( (\rho \circ \beta)^{-1}(V) = \beta^{-1}(\rho^{-1}(V)) = \beta^{-1}(U) \) is a 0-neighbourhood in \( T_\pi(E) \times T_\pi(E) \), entailing that the bilinear map \( \beta \) is continuous at 0 and hence continuous.

To this end, recall that the \( j \)-linear map \( \tau_j: E^j \to T_\pi^j(E) \) taking \((v_1, \ldots, v_j)\) to \( v_1 \otimes \cdots \otimes v_j \) is continuous, for each \( j \in \mathbb{N} \). Hence, for each \( j \in \mathbb{N} \), there exists a continuous seminorm \( p_j \) on \( E \) such that

\[
(\forall v_1, \ldots, v_j \in E) \quad q_j(\tau_j(v_1, \ldots, v_j)) \leq p_j(v_1) \cdots p_j(v_j). \tag{15}
\]

By the countable upper bound condition, there exists a continuous seminorm \( q \) on \( E \) such that \( p_j \leq q \) for all \( j \in \mathbb{N} \), say

\[
p_j \leq C_j q \tag{16}
\]

with \( C_j > 0 \). We let \((E_q, \| \cdot \|_q)\) be the normed space associated with \( q \), and \( \rho_q: E \to E_q \) be the canonical map. For each \( j \in \mathbb{N} \), consider the map

\[
\tau_j': (E_q)^j \to T_\pi^j(E_q), \quad (v_1, \ldots, v_j) \mapsto v_1 \otimes \cdots \otimes v_j,
\]

for \( j \in \mathbb{N} \).
and the direct product map \((\rho_q)^j = \rho_q \times \cdots \times \rho_q : E^j \to (E_q)^j\). Then \(\tau'_j \circ (\rho_q)^j : E^j \to T^j_\pi(E_q)\) is continuous \(j\)-linear, and hence gives rise to a continuous linear map \(\phi_j := T^j_\pi(\rho_q) : T^j_\pi(E) \to T^j_\pi(E_q)\), determined by
\[
\phi_j \circ \tau_j = \tau'_j \circ (\rho_q)^j. \tag{17}
\]

Also, define \(\phi_0 := \text{id}_K\). Then the linear map
\[
\phi := T_\pi(\rho_q) : T_\pi(E) \to T_\pi(E_q), \quad (x_j)_{j \in \mathbb{N}_0} \mapsto (\phi_j(x_j))_{j \in \mathbb{N}_0} \tag{18}
\]
is continuous (being continuous on each summand). For each \(j \in \mathbb{N}\), there exists a continuous \(j\)-linear map
\[
\theta_j : (E_q)^j \to T^j_\pi(E), \quad \text{such that} \quad \theta_j \circ (\rho_q)^j = \rho_{q_j} \circ \tau_j,
\]
as follows from (15) and (16). Now the universal property of \(T^j_\pi(E_q)\) provides a continuous linear map \(\psi_j : T^j_\pi(E) \to H_j\), determined by
\[
\psi_j \circ \tau'_j = \theta_j. \tag{19}
\]

Define \(\psi_0 := \rho_{q_0} : \Omega \to H_0\). Then the linear map
\[
\psi : T_\pi(E_q) \to \bigoplus_{j \in \mathbb{N}_0} H_j, \quad (x_j)_{j \in \mathbb{N}_0} \mapsto (\psi_j(x_j))_{j \in \mathbb{N}_0} \tag{20}
\]
is continuous. By the special case of normed spaces already discussed, the algebra multiplication \(\beta' : T_\pi(E_q) \times T_\pi(E_q) \to T_\pi(E_q)\) is continuous. We now verify that the diagram
\[
\begin{array}{ccc}
T_\pi(E) \times T_\pi(E) & \xrightarrow{\rho \circ \beta} & \bigoplus_{j \in \mathbb{N}_0} H_j \\
\downarrow \phi \times \phi & & \uparrow \psi \\
T_\pi(E_q) \times T_\pi(E_q) & \xrightarrow{\beta'} & T_\pi(E_q)
\end{array}
\tag{21}
\]
is commutative. If this is true, then \(\rho \circ \beta = \psi \circ \beta' \circ (\phi \times \phi)\) is continuous, which implies the continuity of \(\beta\) (as observed above). Since both of the maps \(\rho \circ \beta\) and \(\psi \circ \beta' \circ (\phi \times \phi)\) are bilinear, it suffices that they coincide on \(S \times S\) for a subset \(S \subseteq T_\pi(E)\) which spans \(T_\pi(E)\). We choose \(S\) as the union
of $\mathbb{K}$ and $\bigcup_{j \in \mathbb{N}} \tau_j(E^j)$. For $i, j \in \mathbb{N}$ and $v_1, \ldots, v_i, w_1, \ldots, w_j \in E$, we have

$$\psi(\beta'((\phi(v_1 \otimes \cdots \otimes v_j), \phi(w_1 \otimes \cdots \otimes w_j))) = \psi(\beta'((\rho_q(v_1) \otimes \cdots \otimes \rho_q(v_i), \rho_q(w_1) \otimes \cdots \otimes \rho_q(w_j))) = \psi_{i+j}(\rho_q(v_1) \otimes \cdots \otimes \rho_q(v_i)) \otimes \rho_q_w(w_1) \otimes \cdots \otimes \rho_q(w_j)) = \theta_{i+j}(\rho_q(v_1), \ldots, \rho_q(v_i), \rho_q(w_1), \ldots, \rho_q(w_j)) = \rho_{q+i}(v_1 \otimes \cdots \otimes v_i \otimes w_1 \otimes \cdots \otimes w_j) = (\rho \circ \beta)(v_1 \otimes \cdots \otimes v_i, w_1 \otimes \cdots \otimes w_j),$$

as required. For $x, y \in \mathbb{K}$, we have $\psi(\beta'(\phi(x), \phi(y))) = \psi_q(x, y) = \rho(x, y)$. For $x \in \mathbb{K}$ and $w_1, \ldots, w_j \in E$, we have

$$\psi(\beta'(\phi(x), \phi(w_1 \otimes \cdots \otimes w_j))) = \psi(\beta'(x, \rho_q(w_1) \otimes \cdots \otimes \rho_q(w_j))) = x \psi(\rho_q(w_1) \otimes \cdots \otimes \rho_q(w_j)) = x \theta_j(\rho_q(w_1), \ldots, \rho_q(w_j)) = x \rho_{q_j}(w_1 \otimes \cdots \otimes w_j) = \rho(\beta(x, w_1 \otimes \cdots \otimes w_j)).$$

Likewise, $\psi(\beta'(\phi(v_1 \otimes \cdots \otimes v_i), \phi(y))) = \rho(\beta(v_1 \otimes \cdots \otimes v_i, y))$ for $v_1, \ldots, v_i \in E$ and $y \in \mathbb{K}$. Hence $\beta$ commutes, and hence $\beta$ is continuous.

If $T_\pi(E)$ is a topological algebra, let $(p_j)_{j \in \mathbb{N}}$ be any sequence of continuous seminorms on $E$. Omitting only a trivial case, we may assume that $E \neq \{0\}$. For each $j \in \mathbb{N}$, we then find a continuous seminorm $q_j \neq 0$ on $T_\pi^j(E)$. Let $Q_0(x) := |x|$ for $x \in \mathbb{K}$. For $j \in \mathbb{N}$, let $Q_j$ be a continuous seminorm on $T_\pi^j(E) = E \otimes \pi T_\pi^{j-1}(E)$ such that

$$Q_j(x \otimes y) = p_j(x)q_{j-1}(y) \quad \text{for all } x \in E \text{ and } y \in T_\pi^{j-1}(E)$$

(see, e.g., [20, III.6.3]). Then $W := \bigoplus_{j \in \mathbb{N}} B_1^{Q_j}(0)$ is a 0-neighbourhood in $T_\pi(E) = \bigoplus_{j \in \mathbb{N}} T_\pi^j(E)$. Since $\beta$ is assumed continuous, there exists a box neighbourhood $V \subseteq T_\pi(E)$, of the form $V = \bigoplus_{j \in \mathbb{N}} V_j$ with 0-neighbourhoods $V_j \subseteq T_\pi^j(E)$, such that $\beta(V \times V) \subseteq W$ and hence

$$(\forall j \in \mathbb{N}) \beta_{1, j-1}(V_1 \times V_{j-1}) \subseteq B_1^{Q_j}(0). \quad (22)$$

For $j \in \mathbb{N}$, pick $x_j \in V_{j-1} \subseteq T_\pi^{j-1}(E)$ such that $q_{j-1}(x_j) \neq 0$. Then $1 \geq Q_j(\beta_{1, j-1}(v, x_j)) = Q_j(v \otimes x_j) = p_j(v)q_{j-1}(x_j)$ for all $v \in V_1$. Hence

$$p_j(V_1) \subseteq [0, 1/q_{j-1}(x_j)] \quad (23)$$
for all \( j \in \mathbb{N} \). Let \( q \) be a continuous seminorm on \( E \) such that \( \overline{B}_q^1(0) \subseteq V_1 \). Then (23) implies that \( p_j \leq \frac{1}{q_{j-1}(x_j)} q \) for each \( j \in \mathbb{N} \), and thus \( p_j \leq q \). Hence \( E \) satisfies the countable upper bound condition.

**Lemma 4.1** Let \( E \) be a metrizable locally convex space. Then \( E \) satisfies the countable upper bound condition if and only if \( E \) is normable.

**Proof.** If the topology on \( E \) comes from a norm \( \| \cdot \| \), then \( p \leq \| \cdot \| \) for each continuous seminorm \( p \) on \( E \), entailing that \( E \) satisfies the countable upper bound condition (and UBC(\( \theta \)) for each infinite cardinal \( \theta \)). Conversely, let \( E \) satisfy the countable upper bound condition. Let \( p_1 \leq p_2 \leq \cdots \) be a sequence of seminorms defining the topology of \( E \). Then there exists a continuous seminorm \( q \) on \( E \) such that \( p_j \leq q \) for all \( j \in \mathbb{N} \), say \( p_j \leq C_j q \) with \( C_j > 0 \). It is clear from this that the balls \( B_q^r(0) \) form a basis of 0-neighbourhoods in \( E \) for \( r > 0 \). Hence \( q \) is a norm and defines the topology of \( E \). \( \square \)

In view of Lemma 4.1, Theorem B has the following immediate consequence:

**Corollary 4.2** Let \( E \) be a metrizable locally convex space. Then \( T_\pi(E) \) is a topological algebra if and only if \( E \) is normable. \( \square \)

### 5 Tensor products beyond local convexity

We shall deduce Theorem C from new results on tensor products in the category of general (not necessarily locally convex) topological vector spaces.

**Definition 5.1** Given topological vector spaces \( E_1, \ldots, E_j \) with \( j \geq 2 \), we write \( E_1 \otimes_\nu \cdots \otimes_\nu E_j \) for the tensor product \( E_1 \otimes \cdots \otimes E_j \), equipped with the finest vector topology \( \mathcal{O}_\nu \) making the ‘universal’ \( j \)-linear map

\[
\tau: E_1 \times \cdots \times E_j \to E_1 \otimes \cdots \otimes E_j, \quad (x_1, \ldots, x_j) \mapsto x_1 \otimes \cdots \otimes x_j \tag{24}
\]

continuous.

**Remark 5.2** By definition of \( \mathcal{O}_\nu \), a linear map \( \phi: E_1 \otimes_\nu \cdots \otimes_\nu E_j \to F \) to a topological vector space \( F \) is continuous if and only if \( \phi \circ \tau \) is continuous. If
\(E_1, \ldots, E_j\) are Hausdorff, then also \(E_1 \otimes \nu \cdots \otimes \nu E_j\) is Hausdorff. If \(E_1, \ldots, E_j\) are locally convex Hausdorff or their dual spaces separate points, this follows from the continuity of the identity map \(E_1 \otimes \nu \cdots \otimes \nu E_j \rightarrow (E_1)_w \otimes \pi \cdots \otimes \pi (E_j)_w\), using weak topologies. In general, the Hausdorff property follows by an induction from the case \(j = 2\) in \([22]\) (see \([7]\) Proposition 1 (d)).

**Lemma 5.3** Let \((E, \mathcal{O})\) be a Hausdorff topological vector space and \(K_n \neq \emptyset\) be compact, balanced subsets of \(E\) such that \(E = \bigcup_{n \in \mathbb{N}} K_n\) and \(K_n \cup K_n \subseteq K_{n+1}\) for all \(n \in \mathbb{N}\). Let \(T\) be the topology on \(E\) making it the direct limit \(\lim_{\rightarrow} K_n\) as a topological space. Then \((E, T)\) is a topological vector space.

**Proof.** Consider the continuous addition map \(\alpha : (E, \mathcal{O}) \times (E, \mathcal{O}) \rightarrow (E, \mathcal{O})\) and the addition map \(\alpha' : (E, T) \times (E, T) \rightarrow (E, T)\). Because \(K_n \cup K_n \subseteq K_{n+1}\) and \(T\) induces the given topology on \(K_{n+1}\), the restriction \(\alpha|_{K_n \times K_n} = \alpha|_{K_n \times K_n}: K_n \times K_n \rightarrow K_{n+1} \subseteq (E, \mathcal{T})\) is continuous. Since \((E, T) \times (E, T) = \lim_{\rightarrow} (K_n \times K_n)\) as a topological space \([22]\) Theorem 4.1, we deduce that \(\alpha'\) is continuous as a map \((E, T) \times (E, T) \rightarrow (E, T)\). Next, consider the continuous scalar multiplication \(\mu : \mathbb{K} \times (E, \mathcal{O}) \rightarrow (E, \mathcal{O})\) and the scalar multiplication \(\mu' : \mathbb{K} \times (E, T) \rightarrow (E, T)\). To see that \(\mu'\) is continuous, it suffices to show that its restriction to a map \(\overline{B}_{2j}^{K}(0) \times (E, T) \rightarrow (E, T)\) is continuous for each \(j \in \mathbb{N}\). Since \(\overline{B}_{2j}^{K}(0) \times (E, T) = \lim_{\rightarrow} \overline{B}_{2j}^{K}(0) \times K_n\) as a topological space, we need only show that the restriction of \(\mu'\) to \(\overline{B}_{2j}^{K}(0) \times K_n\) is continuous. But \(\mu'(\overline{B}_{2j}^{K}(0) \times K_n) = 2^j K_n \subseteq K_{n+j}\), and \(T\) induces the given topology on \(K_{n+j}\). Hence \(\mu'|_{\overline{B}_{2j}^{K}(0) \times K_n} = \mu|_{\overline{B}_{2j}^{K}(0) \times K_n}\) is continuous. \(\square\)

**Lemma 5.4** If the topological vector spaces \(E_1, \ldots, E_j\) are \(k_\omega\)-spaces, then also \(E_1 \otimes \nu \cdots \otimes \nu E_j\) is a \(k_\omega\)-space.

**Proof.** Let \(\mathcal{O}_\nu\) be the topology on \(E := E_1 \otimes \nu \cdots \otimes \nu E_j\). For \(i \in \{1, \ldots, j\}\), pick a \(k_\omega\)-sequence \((K_{i,n})_{n \in \mathbb{N}}\) for \(E_i\). After replacing \(K_{i,n}\) with \(\overline{B}_1^{K}(0)K_{i,n}\), we may assume that each \(K_{i,n}\) is balanced. Let

\[
K_n := \sum_{i=1}^{2^n} (K_{1,n} \otimes \cdots \otimes K_{j,n}).
\]

\(\text{Thus } U \subseteq E\) is open if and only if \(U \cap K_n\) is relatively open in \(K_n\) for each \(n \in \mathbb{N}\).
Then each $K_n$ is a compact, balanced subset of $E$, and $E = \bigcup_{n \in \mathbb{N}} K_n$. Since $K_n + K_n \subseteq K_{n+1}$ by definition, Lemma 5.3 shows that the topology $\mathcal{T}$ making $E$ the direct limit topological space $\lim_{\to} K_n$ is a vector topology. As the inclusion maps $K_n \to (E, \mathcal{O}_\nu)$ are continuous, it follows that $\mathcal{O}_\nu \subseteq \mathcal{T}$. Note that $\tau$ from (24) maps $L_n := K_{1,n} \times \cdots \times K_{j,n}$ into $K_n$. Since $\mathcal{T}$ and $\mathcal{O}_\nu$ induce the same topology on $K_n$ and $\tau$ is continuous as a map to $(E, \mathcal{O}_\nu)$, it follows that each restriction $\tau_{|L_n}: L_n \to K_n \subseteq (E, \mathcal{T})$ is continuous. Thus $\tau$ is continuous to $(E, \mathcal{T})$ (as $E_1 \times \cdots \times E_j = \lim_{\to} L_n$ by [12, Theorem 4.1]) and hence $\mathcal{T} \subseteq \mathcal{O}_\nu$. Thus $\mathcal{O}_\nu = \mathcal{T}$, whence $E$ is the $k_\omega$-space $\lim_{\to} K_n$. \hfill $\square$

**Lemma 5.5** Consider topological vector spaces $E_1, \ldots, E_i$ and $F_1, \ldots, F_j$, $E := E_1 \otimes_\nu \cdots \otimes_\nu E_i$ and $F := F_1 \otimes_\nu \cdots \otimes_\nu F_j$, and the bilinear map

$$\kappa: E \times F \to E_1 \otimes_\nu \cdots \otimes_\nu E_i \otimes_\nu \cdots \otimes_\nu F_j =: H$$

determined by $\kappa(x_1 \otimes \cdots \otimes x_i, y_1 \otimes \cdots \otimes y_j) = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j$. If $E_1, \ldots, E_i, F_1, \ldots, F_j$ are $k_\omega$-spaces, then $\kappa$ is continuous and the linear map

$$\tilde{\kappa}: E \otimes_\nu F \to H \text{ determined by } \tilde{\kappa}(v \otimes w) = \kappa(v, w) \quad (25)$$

is an isomorphism of topological vector spaces.

**Proof.** Let $\tau: E_1 \times \cdots \times E_i \to E$, $\tau': F_1 \times \cdots \times F_j \to F$, $\tilde{\tau}: E \times F \to E \otimes_\nu F$ and $\tau''$: $E_1 \times \cdots \times E_i \times F_1 \times \cdots \times F_j \to H$ be the universal maps. It is known from abstract algebra that $\tilde{\kappa}$ is an isomorphism of vector spaces. Moreover, $\tilde{\kappa}^{-1} \circ \tau'' = \tilde{\tau} \circ (\tau \times \tau')$ is continuous, whence $\tilde{\kappa}^{-1}$ is continuous (see Remark 5.2). Thus $\tilde{\kappa}$ will be a topological isomorphism if $\tilde{\kappa}$ is continuous, which will be the case if we can show that $\kappa$ is continuous, as $\tilde{\kappa} \circ \tilde{\tau} = \kappa$ (see Remark 5.2). To this end, pick $k_\omega$-sequences $(K_{a,n})_{n \in \mathbb{N}}$ and $(K'_{b,n})_{n \in \mathbb{N}}$ of balanced sets for the spaces $E_a$ and $F_b$, respectively. Then $K_n := \sum_{k=1}^{2^n} (K_{1,n} \otimes \cdots \otimes K_{i,n})$ and $K'_n := \sum_{k=1}^{2^n} K'_{1,n} \otimes \cdots \otimes K'_{j,n}$ define $k_\omega$-sequences $(K_n)_{n \in \mathbb{N}}$ and $(K'_n)_{n \in \mathbb{N}}$ for $E$ and $F$, respectively (see proof of Lemma 5.4). Moreover, $(K_n \times K'_n)_{n \in \mathbb{N}}$ is a $k_\omega$-sequence for $E \times F$ (cf. [12, Theorem 4.1]), entailing that $\kappa$ will be continuous if we can show that $\kappa|_{K_n \times K'_n}$ is continuous for each $n \in \mathbb{N}$. Consider the map

$$q_n: (K_{1,n} \times \cdots \times K_{i,n} \times K'_{1,n} \times \cdots \times K'_{j,n})^{2^n} \to K_n \times K'_n,$$

$$(x_1, k, \ldots, x_i, k, y_1, k, \ldots, y_j, k)^{2^n} \mapsto \left(\sum_{k=1}^{2^n} x_1, k \otimes \cdots \otimes x_i, k, \sum_{\ell=1}^{2^n} y_{1, \ell} \otimes \cdots \otimes y_{j, \ell}\right).$$

Then $q_n$ is a continuous map from a compact space onto a Hausdorff space and
hence a topological quotient map. Hence $\kappa|_{K_\alpha \times K'_\alpha}$ is continuous if and only if $\kappa \circ q_\alpha$ is continuous. But $\kappa \circ q_\alpha$ is the map taking $(x_{1, k}, \ldots, x_{i, k}, y_{1, k}, \ldots, y_{j, k})_{k=1}^{2^n}$ to $\sum_{k, l=1}^{2^n} x_{1, k} \otimes \cdots \otimes x_{i, k} \otimes y_{1, l} \otimes \cdots \otimes y_{j, l}$, and hence continuous (because $\tau''$ is continuous). \hfill \Box

**Remark 5.6** Although $\nu$-tensor products fail to be associative in general \cite{7}, this pathology is absent in the case of $k_\omega$-spaces $E_1$, $E_2$, $E_3$. In fact, the natural vector space isomorphism $(E_1 \otimes_\nu E_2) \otimes_\nu E_3 \rightarrow E_1 \otimes_\nu (E_2 \otimes_\nu E_3)$ is an isomorphism of topological vector spaces in this case as it can be written as a composition $(E_1 \otimes_\nu E_2) \otimes_\nu E_3 \rightarrow E_1 \otimes_\nu E_2 \otimes_\nu E_3 \rightarrow E_1 \otimes_\nu (E_2 \otimes_\nu E_3)$ of isomorphisms of the form discussed in Lemma 5.5.

Our next lemma is a special case of \cite{11} Corollary 5.7.

**Lemma 5.7** Let $E$ be a topological vector space. If $E$ is a $k_\omega$-space, then the box topology makes $T_\nu(E) := \bigoplus_{j \in \mathbb{N}_0} T^j_\nu(E)$ a $k_\omega$-space, and $T_\nu(E) = \lim_{\longrightarrow} \prod_{j=0}^k T^j_\nu(E)$ as a topological space. \hfill \Box

**Proposition 5.8** Let $E$ be a topological vector space. If $E$ is a $k_\omega$-space, then $T_\nu(E)$ is a topological algebra, which satisfies a universal property:

For every continuous linear map $\phi: E \rightarrow A$ to an associative, unital topological algebra $A$, there exists a unique continuous homomorphism $\tilde{\phi}: T_\nu(E) \rightarrow A$ of unital associative algebras such that $\tilde{\phi}|_E = \phi$.

**Proof.** Define bilinear maps $\beta_{i,j}: T^j_\nu(E) \times T^j_\nu(E) \rightarrow T^{i+j}_\nu(E)$ for $i, j \in \mathbb{N}_0$ and the algebra multiplication $\beta: T^j_\nu(E) \times T^j_\nu(E) \rightarrow T^j_\nu(E)$ as in Section 4. Since countable direct limits and twofold direct products of $k_\omega$-spaces can be interchanged by \cite{11} Proposition 4.7, we have $T^j_\nu(E) \times T^j_\nu(E) = \lim_{\longrightarrow} P_k$ as a topological space, with $P_k := \prod_{i,j=1}^k T^j_\nu(E) \times T^j_\nu(E)$ for $k \in \mathbb{N}$. Hence $\beta$ will be continuous if $\beta|_{P_k}$ is continuous for each $k \in \mathbb{N}$. But $\beta(x_1, \ldots, x_k, y_1, \ldots, y_k) = \sum_{i,j=1}^k \beta_{i,j}(x_i, y_j)$ is a continuous function of $(x_1, \ldots, x_k, y_1, \ldots, y_k) \in P_k$, because $\beta_{i,j}: T^j_\nu(E) \times T^j_\nu(E) \rightarrow T^{i+j}_\nu(E) \subseteq T^j_\nu(E)$ is continuous by Lemma 5.5. Thus, $T^j_\nu(E)$ is a topological algebra. For $\phi$ as described in the proposition, there is a unique homomorphism $\tilde{\phi}: T^j_\nu(E) \rightarrow A$ of unital associative algebras such that $\tilde{\phi}|_E = \phi$ (as is well known from abstract algebra). For $j \in \mathbb{N}$, let $\tau^j: E^j \rightarrow T^j_\nu(E)$ be the universal $j$-linear map. By the universal property of the direct sum, $\tilde{\phi}$ will be continuous if $\tilde{\phi}|_{T^j_\nu(E)}$ is continuous for each $j \in \mathbb{N}_0$, which holds if and only if $\tilde{\phi} \circ \tau^j$ is continuous for each $j \in \mathbb{N}$ (continuity is trivial if $j = 0$). But $\tilde{\phi} \circ \tau^j$ is the map $E^j \rightarrow A, (x_1, \ldots, x_j) \mapsto \phi(x_1) \cdots \phi(x_j)$, which indeed is continuous. \hfill \Box
6 Observations on convexifications

Recall that each topological vector space $Y$ admits a finest locally convex topology $\mathcal{O}_{\text{lcx}}$ which is coarser than the given topology. We call $Y_{\text{lcx}} := (Y, \mathcal{O}_{\text{lcx}})$ the convexification of $Y$. Convex hulls of 0-neighbourhoods in $Y$ form a basis of 0-neighbourhoods for a locally convex vector topology on $Y$, and it is clear that this topology coincides with $\mathcal{O}_{\text{lcx}}$.

Lemma 6.1 If $\theta : E_1 \times \cdots \times E_j \to Z$ is a continuous $j$-linear map between topological vector spaces, then $\theta$ is also continuous as a mapping from $(E_1)_{\text{lcx}} \times \cdots \times (E_j)_{\text{lcx}}$ to $Z_{\text{lcx}}$.

Proof. If $W \subseteq Z$ is a 0-neighbourhood, there are 0-neighbourhoods $U_i \subseteq E_i$ for $i \in \{1, \ldots, j\}$ with $\theta(U_1 \times \cdots \times U_j) \subseteq W$. If $x = (x_1, \ldots, x_{j-1})$ is an element of $U_1 \times \cdots \times U_{j-1}$, then $\theta(x, U_j) \subseteq W$ implies $\theta(x, \text{conv}(U_j)) \subseteq \text{conv}(W)$. Inductively, $\theta(x_1, \ldots, x_{i-1}, \text{conv}(U_i) \times \cdots \times \text{conv}(U_j)) \subseteq \text{conv}(W)$ for all $i = j, j-1, \ldots, 1$. Thus $\theta(\text{conv}(U_1) \times \cdots \times \text{conv}(U_j)) \subseteq \text{conv}(W)$. $\square$

Lemma 6.2 If $A$ is a topological algebra, with multiplication $\theta : A \times A \to A$, then also $(A_{\text{lcx}}, \theta)$ is a topological algebra.

Proof. Apply Lemma 6.1 to the bilinear map $\theta$. $\square$

Lemma 6.3 $(E_1 \otimes_\nu \cdots \otimes_\nu E_j)_{\text{lcx}} = (E_1)_{\text{lcx}} \otimes_\pi \cdots \otimes_\pi (E_j)_{\text{lcx}}$, for all topological vector spaces $E_1, \ldots, E_j$. In particular, $(T_j^j(E))_{\text{lcx}} = T_j^j(E_{\text{lcx}})$ for each topological vector space $E$.

Proof. Let $\mathcal{O}_\nu$ be the topology on $E_1 \otimes_\nu \cdots \otimes_\nu E_j$ and $\mathcal{O}_\pi$ be the topology on $(E_1)_{\text{lcx}} \otimes_\pi \cdots \otimes_\pi (E_j)_{\text{lcx}}$. Since $\mathcal{O}_\pi$ is locally convex and coarser than $\mathcal{O}_\nu$, it follows that $\mathcal{O}_\pi \subseteq (\mathcal{O}_\nu)_{\text{lcx}}$. The universal $j$-linear map $\tau$ from (24) is continuous as a map $E_1 \times \cdots \times E_j \to E_1 \otimes_\nu \cdots \otimes_\nu E_j$ and hence also continuous as a map $(E_1)_{\text{lcx}} \times \cdots \times (E_j)_{\text{lcx}} \to (E_1 \otimes_\nu \cdots \otimes_\nu E_j)_{\text{lcx}}$, by Lemma 6.1 Hence $(\mathcal{O}_\nu)_{\text{lcx}} \subseteq \mathcal{O}_\pi$, and hence both topologies coincide. $\square$

Lemma 6.4 $\left( \bigoplus_{j \in \mathbb{N}} E_j \right)_{\text{lcx}} = \bigoplus_{j \in \mathbb{N}} (E_j)_{\text{lcx}}$, for all topological vector spaces $E_j$.

Proof. Both spaces coincide as abstract vector spaces, and the topology on the right hand side is coarser. But it is also finer, because for all balanced 0-neighbourhoods $U_j \subseteq E_j$ and $U := \bigoplus_{j \in \mathbb{N}} U_j$, we have $\text{conv}(U_j) \subseteq \text{conv}(U)$ for each $j$ and thus $\bigoplus_{j \in \mathbb{N}} 2^{-j} \text{conv}(U_j) \subseteq \text{conv}(U)$. $\square$
7 Proof of Theorem C

Taking \( F := E \), Theorem C follows from the next result:

**Proposition 7.1** Let \( E \) be a locally convex space. If \( E = F_{\text{lcx}} \) for a topological vector space \( F \) which is a \( k_\omega \)-space, then \( T_\pi(E) \) is topological algebra.

**Proof.** By Proposition 5.8, \( T_\nu(F) \) is a topological algebra. Hence also \( (T_\nu(F))_{\text{lcx}} \) is a topological algebra, by Lemma 6.2. But

\[
(T_\nu(F))_{\text{lcx}} = \bigoplus_{j \in \mathbb{N}_0} (T_\nu^j(F))_{\text{lcx}} = \bigoplus_{j \in \mathbb{N}_0} T_\pi^j(F)_{\text{lcx}} = \bigoplus_{j \in \mathbb{N}_0} T_\pi^j(E)
\]

coincides with \( T_\pi(E) \) (using Lemma 6.4 for the second equality and Lemma 6.3 for the third).

The notion of a free locally convex space goes back to [17]. Given a topological space \( X \), let \( \mathbb{K}(X) \) be the free vector space over \( X \). Write \( V(X) \) for \( \mathbb{K}(X) \), equipped with the finest vector topology making the canonical map \( \eta_X : X \to \mathbb{K}(X), \ x \mapsto \delta_x \), continuous. Write \( L(X) \) for \( \mathbb{K}(X) \), equipped with the finest locally vector topology making \( \eta_X \) continuous. Call \( V(X) \) and \( L(X) \) the free topological vector space over \( X \), respectively, the free locally convex space over \( X \).

**Corollary 7.2** Let \( E = L(X) \) be the free locally convex space over a \( k_\omega \)-space \( X \). Then \( T_\pi(E) \) is a topological algebra.

**Proof.** As is clear, \( L(X) = (V(X))_{\text{lcx}} \). It is well known that \( V(X) \) is \( k_\omega \) if so is \( X \) (see, e.g., [9, Lemma 5.5]). Hence Proposition 7.1 applies.

8 Some spaces with upper bound conditions

Recall from the proof of Lemma 4.1 that every normable space satisfies the \( \text{UBC}(\theta) \) for each infinite cardinal \( \theta \). Combining Theorem B and Proposition 7.1, we obtain further examples of spaces with upper bound conditions:

**Corollary 8.1** Let \( E \) be a locally convex space. If \( E \) is a \( k_\omega \)-space or \( E = F_{\text{lcx}} \) for some topological vector space which is a \( k_\omega \)-space, then \( E \) satisfies the countable upper bound condition.
Let $\theta$ be an arbitrary infinite cardinal now. Then there exists a non-normable space satisfying the UBC($\theta$), but not the UBC($\theta'$) for any $\theta' > \theta$:

**Example 8.2** Let $X$ be a set of cardinality $|X| > \theta$, and $\mathcal{Y}$ be the set of all subsets $Y \subseteq X$ of cardinality $|Y| \leq \theta$. Let $E := \ell^\infty (X)$ be the vector space of bounded $\mathbb{K}$-valued functions on $X$, equipped with the (unusual) vector topology $O_\theta$ defined by the seminorms

$$\| \cdot \|_Y : E \to [0, \infty[, \quad \| \gamma \|_Y := \sup\{|\gamma(x)| : x \in Y\}$$

for subsets $Y \in \mathcal{Y}$. Note that a function $\gamma : X \to \mathbb{K}$ is bounded if and only if all of its restrictions to countable subsets of $X$ are bounded. Hence $E$ can be expressed as the projective limit

$$\lim_{\leftarrow Y \in \mathcal{Y}} (\ell^\infty (Y), \| \cdot \|_\infty)$$

of Banach spaces (with the apparent restriction maps as the bonding maps and limit maps), and thus $E$ is complete. For each $Y \in \mathcal{Y}$, we have $Y \neq X$ by reasons of cardinality, whence an $y \in X \setminus Y$ exists. Define $\delta_y : X \to \mathbb{K}$, $\delta_y (y) := \delta_{x,y}$ using Kronecker’s $\delta$. Then $\delta_y \neq 0$ and $\| \delta_y \|_Y = 0$, whence $\| \cdot \|_Y$ is not a norm. As a consequence, $E$ is not normable. To see that $E$ satisfies the UBC($\theta$), let $(p_j)_{j \in J}$ be a family of continuous seminorms on $E$ such that $|J| \leq \theta$. For each $j \in J$, there exists a subset $Y_j \subseteq X$ with $|Y_j| \leq \theta$ and $C_j > 0$ such that $p_j \leq C_j \| \cdot \|_{Y_j}$. Set $Y := \bigcup_{j \in J} Y_j$. Then $|Y| \leq |J| \theta \leq \theta \theta = \theta$. Hence $q := \| \cdot \|_Y$ is a continuous seminorm on $E$, and $p_j \leq C_j q$ for all $j$. Finally, let $Z \subseteq X$ be a subset of cardinality $\theta < |Z| \leq \theta'$. Suppose we could find a continuous seminorm $p$ on $E$ such that $\| \cdot \|_{\{z\}} \leq p$ for all $z \in Z$. We may assume that $p = \| \cdot \|_Y$ for some $Y \in \mathcal{Y}$. But then $z \in Y$ for all $z \in Z$ and hence $|Y| \geq |Z| > \theta$, contradiction.

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