All-genus open-closed mirror symmetry
for affine toric Calabi–Yau 3-orbifolds

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Abstract

The remodeling conjecture proposed by Bouchard–Klemm–Mariño–Pasquetti relates all-genus open and closed Gromov–Witten invariants of a semi-projective toric Calabi–Yau 3-manifold/3-orbifold $X$ to the Eynard–Orantin invariants of the mirror curve of $X$. In this paper, we present a proof of the remodeling conjecture for open-closed orbifold Gromov–Witten invariants of an arbitrary affine toric Calabi–Yau 3-orbifold relative to a framed Aganagic–Vafa Lagrangian brane. This can be viewed as an all-genus open-closed mirror symmetry for affine toric Calabi–Yau 3-orbifolds.

1. Introduction

1.1 Background and motivation

Mirror symmetry relates the A-model topological string theory on a Calabi–Yau 3-fold $X$ to the B-model topological string theory on another Calabi–Yau 3-fold $\tilde{X}$, the mirror of $X$. The genus $g$ free energy of the topological A-model on $X$ is mathematically defined as a generating function $F^X_{g}$ of genus $g$ Gromov–Witten invariants of $X$ which is a function on a (formal) neighborhood of the large radius limit in the (complexified) Kähler moduli of $X$. The genus $g$ free energy of the topological B-model on $\tilde{X}$ is a section of $L^{-2g}$, where $L$ is a line bundle over the complex moduli of $\tilde{X}$, so locally it is a function $\tilde{F}^{\tilde{X}}_{g}$ on the complex moduli of $\tilde{X}$. A mathematical consequence of mirror symmetry is $\tilde{F}^{\tilde{X}}_{0} = F^X_{0} + \sum a_{g} a_{1}$ under the mirror map, where $a_{0}$ (respectively, $a_{1}$) is a cubic (respectively, linear) function in Kähler parameters. The mirror map and $\tilde{F}^{\tilde{X}}_{0}$ are determined by period integrals of a holomorphic 3-form on $\tilde{X}$. Period integrals of Calabi–Yau complete intersections in toric manifolds can be expressed in terms of explicit hypergeometric functions.

Let $Q$ be the quintic 3-fold, which is a Calabi–Yau hypersurface in $\mathbb{P}^4$, and let $\tilde{Q}$ be the mirror of $Q$. Candelas, de la Ossa, Green, and Parkes computed $F^Q_{0}$ and the mirror map explicitly and obtained a conjectural formula for the number of rational curves of arbitrary degree in $Q$; this formula was first proved independently by Givental [Giv96] and Lian–Liu–Yau [LLY97], who later extended their results to Calabi–Yau complete intersections in projective toric manifolds [Giv98, LLY99a, LLY99b]. Their proofs rely on a good understanding of genus zero Gromov–Witten theory of $Q$. The mirror formula for $F^Q_{1}$ was conjectured by Bershadsky–Cecotti–Ooguri–
Vafa (BCOV for short) [BCOV93] and first proved by Zinger [Zin09]. Combining the techniques of BCOV, results of Yamaguchi–Yau [YY04], and boundary conditions, Huang–Klemm–Quacken [HKQ09] proposed a mirror conjecture on $F^g_Q$ up to $g = 51$. Maulik–Pandharipande provided a mathematical determination of Gromov–Witten invariants of $Q$ in all genera and degrees [MP06]. In general, it is very difficult to evaluate higher-genus Gromov–Witten invariants of a compact Calabi–Yau 3-fold.

In contrast, higher-genus Gromov–Witten invariants of toric Calabi–Yau 3-folds (which must be non-compact) are much better understood. In general, Gromov–Witten invariants are defined for projective manifolds (or, more generally, compact almost-Kähler manifolds), but in the toric case one may use localization to define Gromov–Witten invariants of certain non-compact toric manifolds. By virtual localization [GP99], all-genus Gromov–Witten invariants of toric manifolds can be reduced to Hodge integrals, which can be evaluated by effective algorithms. When the toric manifold $X$ is a Calabi–Yau 3-fold, the topological vertex [AKMV05, LLLZ09, MOOP11] provides a much more efficient algorithm for computing Gromov–Witten invariants of certain non-compact toric manifolds. By virtual localization [GP99], all-genus Gromov–Witten invariants of $X$ can be reduced to integrals of 1-forms on the mirror curve along loops, whereas the generating function $F_{g,0}^X$ of genus zero open Gromov–Witten invariants (which count holomorphic disks in $X$ bounded by $L$) corresponds to integrals of 1-forms on the mirror curve along paths [AV00, AKV02]. On the basis of the work of Eynard–Orantin [EO07] and Mariño [Mar08], Bouchard–Klemm–Mariño–Pasquetti [BKMP09] proposed a new formalism of the topological B-model on $X$ in terms of the Eynard–Orantin invariants $\omega_{g,n}$ of the mirror curve and conjectured a precise correspondence, known as the remodeling conjecture, between $\omega_{g,n}$ (where $n > 0$) and the generating function $F_{g,n}^{X,L}$ of open Gromov–Witten invariants counting holomorphic maps from bordered Riemann surfaces with $g$ handles and $n$ holes to $X$ with boundaries in $L$. This can be viewed as a version of all-genus open mirror symmetry; the closed sector of the remodeling conjecture relates $\omega_{g,0}$ to $F_{g}^{X}$. The open string part of the remodeling conjecture for $\mathbb{C}^3$ was proved independently by Chen [Che18] and Zhou [Zho09a, Zho09b]. The free energy part of the remodeling conjecture for $\mathbb{C}^3$ [BS12] was proved independently by Bouchard–Catuneanu–Marchal–Sulkowski [BCMS13] and Zhu [Zhu15]. Eynard and Orantin provided a proof of the remodeling conjecture for general smooth semi-projective toric Calabi–Yau 3-folds in [EO15].

Bouchard–Klemm–Mariño–Pasquetti have extended the remodeling conjecture to toric Calabi–Yau 3-orbifolds [BKMP10]. Topological string theory on orbifolds was constructed decades ago by physicists [DHVW85, DHVW86], and many works followed in both mathematics and physics (for example, [HV87, DFMS87, BR87, Kni87, Roa90, CV92]). Zaslow discussed orbifold quantum cohomology [Zas93] along with many examples, in both abelian and non-abelian quotients. Later, the mathematical definition of orbifold Gromov–Witten theory and quantum cohomology was laid by Chen–Ruan [CR02] in the symplectic setting and Abramovich–Graber–Vistoli [AGV02, AGV08] in the algebraic setting. Orbifold Gromov–Witten invariants of a toric Calabi–Yau 3-orbifold $X$ and open orbifold Gromov–Witten invariants of $X$ relative to an Aganagic–Vafa Lagrangian brane in $X$ can be expressed in terms of the orbifold Gromov–Witten vertex, which is a generating function of abelian Hurwitz–Hodge integrals [Ros14]. The algorithm for the topological vertex is equivalent to the Gromov–Witten (GW)/Donaldson–Thomas (DT) correspondence.
for smooth toric Calabi–Yau 3-folds [MNOP06]. The GW/DT correspondence has been conjectured for Calabi–Yau 3-orbifolds satisfying the hard Lefshetz condition [BCY12] and proved for toric Calabi–Yau 3-orbifolds with transverse $A_n$-singularities [Zon15, RZ13, RZ15, Ros15]. This provides an efficient algorithm for computing closed and open Gromov–Witten invariants of toric Calabi–Yau 3-orbifolds with transverse $A_n$-singularities, in all genera and degrees. The precise statement of the GW/DT correspondence for toric Calabi–Yau 3-orbifolds which do not satisfy the hard Lefshetz condition (for example, $[C^3/\mu_3]$, where $\mu_3$ acts diagonally) is not known even conjecturally. The remodeling conjecture provides a recursive algorithm to compute closed and open Gromov–Witten invariants of all semi-projective toric Calabi–Yau 3-orbifolds in all genera. Explicit predictions of open orbifold Gromov–Witten invariants of $[C^3/\mu_3]$ are given in [BKMP10].

1.2 Statement of the main result and outline of the proof

In this paper, we study the remodeling conjecture for all affine toric Calabi–Yau 3-orbifolds $[C^3/G]$, where $G$ can be any finite subgroup of the maximal torus of $SL(3, \mathbb{C})$. We consider a general framed Aganagic–Vafa brane $(\mathcal{L}, f)$, where $\mathcal{L} \cong [(S^1 \times \mathbb{C})/G]$ and $f \in \mathbb{Z}$. We define open-closed orbifold Gromov–Witten invariants of $X = [C^3/G]$ relative to $(\mathcal{L}, f)$ as certain equivariant relative orbifold invariants of a toric Calabi–Yau 3-orbifold $\mathcal{Y}_f$ relative to a divisor $D_f = [C^2/\mu_m]$, where $\mu_m \cong \mathbb{Z}_m$ is the stabilizer of the $G$-action on $S^1 \times \{0\} \subset S^1 \times \mathbb{C}$, and the coarse moduli space $C^2/\mu_m$ of $D_f$ is the $A_{m-1}$ surface singularity. We define generating functions of open-closed Gromov–Witten invariants

$$F_{g,n}^{X,(\mathcal{L},f)}(\tau; X_1, \ldots, X_n),$$

which are $H^*_{CR}(B\mu_m; \mathbb{C})\otimes \tau^n$-valued formal power series in A-model closed string coordinates $\tau = (\tau_1, \ldots, \tau_p)$ and A-model open string coordinates $X_1, \ldots, X_n$; here $H^*_{CR}(B\mu_m; \mathbb{C}) \cong \mathbb{C}^m$ is the Chen–Ruan orbifold cohomology of the classifying space $B\mu_m$ of $\mu_m$. We use the Eynard–Orantin invariants $\omega_{g,n}$ of the framed mirror curve to define B-model potentials

$$\hat{F}_{g,n}(\tau; X_1, \ldots, X_n),$$

which are $H^*_{CR}(B\mu_m; \mathbb{C})\otimes \tau^n$-valued functions in B-model closed string flat coordinates $\tau = (\tau_1, \ldots, \tau_p)$ and B-model open string coordinates $X_1, \ldots, X_n$, analytic in an open neighborhood of the origin in $\mathbb{C}^p \times \mathbb{C}^n$. The mirror map relates the B-model flat coordinates $(\tau_1, \ldots, \tau_p)$ to the complex parameters $(q_1, \ldots, q_p)$ of the framed mirror curve. Our main result is the following.

**Theorem 7.19.** For any $g \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$,

$$\hat{F}_{g,n}(\tau; X_1, \ldots, X_n) = (-1)^{g-1+n} F_{g,n}^{X,(\mathcal{L},f)}(\tau; X_1, \ldots, X_n). \quad (1.1)$$

This is indeed more general than the original conjecture in [BKMP10], which covers the $m = 1$ case, that is, the case when $\mathcal{L}$ is on an effective leg.

We now give an outline of our proof of Theorem 7.19. For simplicity, we consider the stable case $2g - 2 + n > 0$ in this outline. (The unstable cases $(g,n) = (0,1), (0,2)$ will be treated separately.) The proof consists of three steps:

1. **(A-model graph sum)** In [FLZ16], we used Tseng’s orbifold quantum Riemann–Roch theorem [Tse10] to write down a graph sum formula for the total descendant equivariant Gromov–Witten potential of $X$. We use this formula and localization to derive a graph
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sum formula for the A-model potential $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}$:

$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)} = \sum_{\bar{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{w_A(\bar{\Gamma})}{|\text{Aut}(\bar{\Gamma})|},$$

where $\Gamma_{g,n}(\mathcal{X})$ is a certain set of decorated graphs, $\text{Aut}(\bar{\Gamma})$ is the automorphism group of the decorated graph $\bar{\Gamma}$, and $w_A(\bar{\Gamma})$ is the A-model weight of the decorated graph $\bar{\Gamma}$ defined by (5.9).

2. \textbf{(B-model graph sum)} The Eynard–Orantin invariants $\omega_{g,n}$ can be expressed as sums over labeled graphs [KO10, Eyn11, Eyn14, DOSS14]. We use special geometry to obtain the Taylor series expansion of the graph sum formula in [DOSS14, Theorem 3.7] in B-model closed string flat coordinates $\tau = (\tau_1, \ldots, \tau_p)$ at $\tau = 0$ and derive a graph sum formula for the B-model potential $\tilde{F}_{g,n}$:

$$\tilde{F}_{g,n} = \sum_{\bar{\Gamma} \in \Gamma_{g,n}(\mathcal{X})} \frac{w_B(\bar{\Gamma})}{|\text{Aut}(\bar{\Gamma})|},$$

where $w_B(\bar{\Gamma})$ is the B-model weight of the decorated graph $\bar{\Gamma}$ defined by (7.14).

3. \textbf{(Comparison of weights)} For each decorated graph $\bar{\Gamma} \in \Gamma_{g,n}(\mathcal{X})$, we prove the following identity relating A-model and B-model weights:

$$w_B(\bar{\Gamma}) = (-1)^{g-1+n} w_A(\bar{\Gamma}).$$

1.3 Remarks on the BKMP remodeling conjecture in the general case

In [FLZ19], the authors provide a proof of the BKMP remodeling conjecture for all semi-projective toric Calabi–Yau 3-orbifolds. The proof in [FLZ19] relies on (i) the quantization formula for the total descendant potential of equivariant GW theory of GKM orbifolds [Zon16], (ii) the B-model graph sum formula in [DOSS14, Theorem 3.7], (iii) the genus zero mirror theorem for toric DM stacks [CCIT15, CCK15], and (iv) the genus zero open mirror theorem for semi-projective toric Calabi–Yau 3-orbifolds [FLT12]. Item (i) is used to derive the A-model graph sum formula, whereas items (ii) and (iv) are used to match the A-model and B-model graph sums. In the affine case, the proof in this paper is more direct than the specialization of the proof in [FLZ19] to the affine case: the proof in this paper relies on Tseng’s orbifold quantum Riemann–Roch theorem [Tse10] and item (ii), but not on items (i), (iii), (iv). The proof in [FLZ19] also relies on the computation of oscillating integrals on a mirror curve in this paper; we obtain the desired result by integrating in the Landau–Ginzburg model and dimensional reduction (Theorem 7.8).

1.4 Overview of the paper

In Section 2, we describe affine toric Calabi–Yau 3-orbifolds and their mirror curves. In Section 3, for each affine toric Calabi–Yau 3-orbifold $\mathcal{X}$ and a framed Aganagic–Vafa A-brane $(\mathcal{L},f)$, we construct a relative Calabi–Yau 3-orbifold $(\mathcal{Y}_f, \mathcal{D}_f)$, where $\mathcal{X} = \mathcal{Y}_f \setminus \mathcal{D}_f$. In Section 4, we describe the Chen–Ruan orbifold cohomology of the classifying space $BG$ of the finite abelian group $G$ and the $\bar{T}$-equivariant Chen–Ruan orbifold cohomology of $\mathcal{X}$ and $\mathcal{Y}_f$. In Section 5, we give the precise definition of the A-model partition functions $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}$ as generating functions of equivariant relative Gromov–Witten invariants of $(\mathcal{Y}_f, \mathcal{D}_f)$ and derive the A-model graph sum formula (Theorem 5.5). In Section 6, we study the geometry and topology of the mirror curve.
In particular, we clarify the choice of A-cycles and B-cycles and the definition of the B-model flat coordinates. In Section 7, we give the precise definition of the B-model partition functions $\tilde{F}_{g,n}$, derive the B-model graph sum formula (Theorem 7.15), and complete the proof of the main result (Theorem 7.19).

2. Affine toric Calabi–Yau 3-orbifolds and their mirrors

2.1 The A-model geometry

Let $T = (\mathbb{C}^*)^3$ and $N = \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^3$, and let $M = \text{Hom}(T, \mathbb{C}^*) = \text{Hom}(N, \mathbb{Z})$. Let $\sigma \subset N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^3$ be a simplicial cone spanned by $b_1, b_2, b_3 \in N$ such that the simplicial affine toric variety $X_\sigma := \text{Spec } \mathbb{C}[\sigma^\vee \cap M]$ has trivial canonical divisor. Then there exists a $u \in M$ such that $\langle u, b_1 \rangle = \langle u, b_2 \rangle = \langle u, b_3 \rangle = 1$. We may choose a $\mathbb{Z}$-basis $u_1, u_2, u_3$ of $M$ such that $u_3 = u$. Let $e_1, e_2, e_3$ be the dual $\mathbb{Z}$-basis of $N$. Then

$$b_1 = re_1 - se_2 + e_3, \quad b_2 = me_2 + e_3, \quad b_3 = e_3,$$

where $r$ and $m$ are positive integers and $s \in \{0, 1, \ldots, r - 1\}$.

We have a short exact sequence of abelian groups

$$1 \to G \to \tilde{T} = (\mathbb{C}^*)^3 \xrightarrow{\phi} T = (\mathbb{C}^*)^3 \to 1,$$

where $\phi(t_1, t_2, t_3) = \left(\frac{t_1}{r}, \frac{t_1}{s}, \frac{t_1}{s^m}\right)$. For $i = 1, 2, 3$, let $\chi_i: G \to \mathbb{C}^*$ be the projection to the $i$th factor. Then $\chi_1, \chi_2, \chi_3 \in G^*$ are $\chi_1, \chi_2, \chi_3 \in \text{Hom}(G, \mathbb{C}^*)$, and $\chi_1 \chi_2 \chi_3 = 1$. Given a positive integer $r$, let $\mu_r := \{z \in \mathbb{C}^*: z^r = 1\} \cong \mathbb{Z}_r$ be the group of $r$th roots of unity. The image of $\chi_1$ is $\mu_r$, and the kernel of $\chi_1$ is isomorphic to $\mu_m$.

We have a short exact sequence of finite abelian groups

$$1 \to \mu_m \to G \xrightarrow{\chi_1} \mu_r \to 1.$$

Then $X_\sigma = \mathbb{C}^3/G$, which is the coarse moduli space of $X := [\mathbb{C}^3/G]$. Let $N_\sigma = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \mathbb{Z}b_3$. Define

$$\text{Box}(\sigma) = \{v \in N: v = c_1b_1 + c_2b_2 + c_3b_3, 0 \leq c_i < 1\}.$$

There is a bijection $\text{Box}(\sigma) \to N/N_\sigma$ given by $v \mapsto v + N_\sigma$; there is a bijection $\text{Box}(\sigma) \to G$ given by

$$c_1b_1 + c_2b_2 + c_3b_3 \mapsto \left(e^{2\pi \sqrt{-1}c_1}, e^{2\pi \sqrt{-1}c_2}, e^{2\pi \sqrt{-1}c_3}\right). \quad (2.1)$$

For example, for $j \in \{1, \ldots, m - 1\}$, we have

$$(0, j, 1) = \frac{j}{m} b_2 + \left(1 - \frac{j}{m}\right) b_3 \in \text{Box}(\sigma), \quad \left(1, e^{2\pi \sqrt{-1}j/m}, e^{-2\pi \sqrt{-1}j/m}\right) \in G.$$

Define age: $\text{Box}(\sigma) \to \{0, 1, 2\}$ by

$$c_1b_1 + c_2b_2 + c_3b_3 \mapsto c_1 + c_2 + c_3.$$

Then we have a disjoint union

$$\text{Box}(\sigma) = \{0\} \cup \{v \in \text{Box}(\sigma): \text{age}(v) = 1\} \cup \{v \in \text{Box}(\sigma): \text{age}(v) = 2\}.$$

Given $h \in G$ and $i \in \{1, 2, 3\}$, define $c_i(h) \in [0, 1)$ and age$(h) \in \{0, 1, 2\}$ by

$$e^{2\pi \sqrt{-1}c_i(h)} = \chi_i(h), \quad \text{age}(h) = c_1(h) + c_2(h) + c_3(h). \quad (2.2)$$

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The inverse map \( G \to \Box(\sigma) \) of (2.1) is given by \( h \mapsto c_1(h)b_1 + c_2(h)b_2 + c_3(h)b_3 \).

2.2 The B-model geometry
There exist \((m_1, n_1), \ldots, (m_p, n_p) \in \mathbb{Z}^2 \) such that
\[
\{ v \in \Box(\sigma) : \text{age}(v) = 1 \} = \{ (m_a, n_a, 1) : a = 1, \ldots, p \}.
\]
We define \( b_{3+a} = (m_a, n_a, 1) \) for \( a = 1, \ldots, p \) and define
\[
H_f(X,Y,q) = X^rY^{s-rf} + Y^m + 1 + \sum_{a=1}^{p} q_a X^{ma}Y^{na-ma}f.
\]
The mirror of \( X \) is a hypersurface in \( \mathbb{C}^2 \times (\mathbb{C}^*)^2 \):
\[
\tilde{X}_q = \{(u,v,X,Y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : H_f(X,Y,q) - uv = 0 \}.
\]
This is a family of non-compact Calabi–Yau 3-folds parametrized by \( q = (q_1, \ldots, q_p) \). Different framings \( f \in \mathbb{Z} \) give isomorphic \( \tilde{X}_q \) but different superpotentials \( X : \tilde{X}_q \to \mathbb{C}^* \). The framed mirror curve is
\[
\Sigma_q = \{(X,Y) \in (\mathbb{C}^*)^2 : H_f(X,Y,q) = 0 \}.
\]
Then
(i) the hypersurface \( \tilde{X}_q \) is smooth if and only if \( \Sigma_q \) is smooth;
(ii) all the branch points of \( X : \Sigma_q \to \mathbb{C}^* \) are simple if and only if the critical points of \( X : \tilde{X}_q \to \mathbb{C}^* \) are isolated and non-degenerate.

Note that \( \Sigma_0 = \{(X,Y) \in (\mathbb{C}^*)^2 : X^rY^{s-rf} + Y^m + 1 = 0 \} \) is smooth and all the branch points of \( X : \Sigma_0 \to \mathbb{C}^* \) are simple. So properties (i) and (ii) hold for small enough \( q \).

Remark 2.1. When \( m = 1, G = \mu_r \), and \( q_a = 0 \), the framed mirror curve is given by \( X^rY^{s-rf} + Y + 1 = 0 \) or, equivalently,
\[
Y^{s+rf}(1 + Y) + X^r = 0. \tag{2.4}
\]
Up to the sign, which is a matter of convention, (2.4) agrees with the following framed mirror curve given by equation (3.28) of [BHL14]:
\[
Y^{s+rf}(1 - Y) - X^r = 0.
\]
In this paper, we assume that the framing \( f \) is a positive integer.

We define
\[
w_1 = \frac{1}{r}, \quad w_2 = \frac{s + rf}{rm}, \quad w_3 = -w_1 - w_2 = \frac{-s - rf - m}{rm}. \tag{2.5}
\]

For later convenience, we fix two explicit bijections (which are not necessarily group homomorphisms) \( \iota : \mathbb{Z}_r \times \mathbb{Z}_m \to G \) and \( \iota^* : G^* \to \mathbb{Z}_r \times \mathbb{Z}_m \):
- \( \iota : \mathbb{Z}_r \times \mathbb{Z}_m \to G \), \( (j, \ell) \mapsto \eta_1 \eta_2^\ell \), where
  \[
  \eta_1 = (e^{2\pi \sqrt{-1}w_1}, e^{2\pi \sqrt{-1}w_2}, e^{2\pi \sqrt{-1}w_3}), \quad \eta_2 = (1, e^{2\pi \sqrt{-1}/m}, e^{-2\pi \sqrt{-1}/m}). \tag{2.6}
  \]
- \( \iota^* : \mathbb{Z}_r \times \mathbb{Z}_m \to G^* \), \( (j, \ell) \mapsto \chi_1^j \chi_2^\ell \), where \( \chi_1, \chi_2 \in G^* \) are defined as in Section 2.1.
3. The 1-leg framed orbifold topological vertex

Let \( L = \{(z_1, z_2, z_3) \in \mathbb{C}^3: |z_1|^2 - |z_2|^2 = c, |z_1|^2 - |z_3|^2 = c, \Im(z_1z_2z_3) = 0\} \), where \( c > 0 \) and \( \Im(z) \) means the imaginary part of \( z \). Then \( L \) is a special Lagrangian of \( \mathbb{C}^3 \) (Harvey–Lawson [HL82]), and \( \mathcal{L} := [L/G] \) is a special Lagrangian sub-orbifold of the affine toric Calabi–Yau 3-orbifold \( \mathcal{X} = [\mathbb{C}^3/G] \). A framed Aganagic–Vafa A-brane of \( \mathcal{X} \) is a pair \((\mathcal{L}, f)\), where \( \mathcal{L} \subset \mathcal{X} \) is as above and \( f \in \mathbb{Z} \). The 1-leg framed orbifold topological vertex is a relative toric Calabi–Yau 3-orbifold \((\mathcal{Y}_f, \mathcal{D}_f)\) associated with \( (\mathcal{X}, \mathcal{L}, f) \).

3.1 The toric graph

The lattice \( N_\sigma = \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \oplus \mathbb{Z}b_3 \) is a sublattice of \( N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \) of index \( rm \), and \( \{b_1, b_2, b_3\} \) is a \( \mathbb{Q} \)-basis of \( N_\sigma := N \otimes \mathbb{Z} \mathbb{Q} \). Let \( \{w_1, w_2, w_3\} \) be the dual \( \mathbb{Q} \)-basis of \( M_\mathbb{Q} := M \otimes \mathbb{Z} \mathbb{Q} \). Then

\[
\begin{align*}
w_1 &= \frac{1}{r} u_1, \quad w_2 = \frac{s}{rm} u_1 + \frac{1}{m} u_2, \quad w_3 = -\frac{s + m}{rm} u_1 - \frac{1}{m} u_2 + u_3.
\end{align*}
\]

Note that

\[
w_1 + w_2 + w_3 = u_3
\]

is the weight of the \( T \)-action on \( T_p Y_f \), where \( p = BG \) is the unique \( T \)-fixed point in the toric 3-orbifold \( Y_f \). Let \( T' \) be the kernel of \( u_3 \in M = \text{Hom}(T, \mathbb{C}^*) \).

The relative formal toric Calabi–Yau (FTCY) graph of the 1-leg framed orbifold topological vertex is shown in Figure 1. This generalizes the 1-leg framed topological vertex [LLLZ09].

![Figure 1. The toric graph of the 1-leg framed orbifold topological vertex](image_url)

3.2 The fan

The toric graph in Figure 1 defines a relative toric Calabi–Yau 3-orbifold \( Y_f \) which is a partial compactification of \( \mathcal{X} \). We now describe the fan \( \Sigma_f \) defining the coarse moduli space \( Y_f \) of \( Y_f \). Let

\[
\begin{align*}
v_1 &= b_1 = re_1 - se_2 + e_3, \quad v_2 = b_2 = me_2 + e_3, \quad v_3 = b_3 = e_3, \quad v_4 = -e_1 - fe_2.
\end{align*}
\]

Let \( \sigma \) be the 3-cone spanned by \( v_1, v_2, v_3 \), as before; let \( \sigma' \) be the 3-cone spanned by \( v_2, v_3, v_4 \). Let \( \tau \) be the 2-cone \( \sigma \cap \sigma' \), and let \( \rho_i \) be the 1-cone spanned by \( v_i \). Then

\[
\Sigma_f(1) = \{\rho_1, \rho_2, \rho_3, \rho_4\}, \quad \Sigma_f(3) = \{\sigma, \sigma'\}.
\]

The coarse moduli space \( D_f \) is the \( T \)-invariant divisor in \( Y_f \) associated with \( \rho_4 \). The pair \((\mathcal{Y}_f, \mathcal{D}_f)\) is a relative toric Calabi–Yau 3-orbifold, where the relative Calabi–Yau condition is \( K_{\mathcal{Y}_f} + \mathcal{D}_f = 0 \).
Let $D_i$ be the $T$-invariant divisor associated with $\rho_i$. Then

$$K_{\mathcal{Y}_f} = -D_1 - D_2 - D_3 - D_4, \quad K_{\mathcal{Y}_f} + D_f = -D_1 - D_2 - D_3.$$ 

Note that $K_{\mathcal{X}_f} + D_f$ is the principal $T$-divisor associated with $\chi^{u_3} \in \mathbb{C}[M]$.

### 3.3 The root construction

In this subsection, we give another description of the relative toric 3-orbifold $(\mathcal{Y}_f, D_f)$. Let $p_0 = [0, 1]$ and $p_\infty = [1, 0]$ be the two torus-fixed points in $\mathbb{P}(1, r)$, where $\mathbb{P}(1, r)$ is the weighted projective line with an orbifold point of order $r$. Then $p_0$ is the unique stacky point in $\mathbb{P}(1, r)$, and any $\mathbb{C}^*$-equivariant line bundle on $\mathbb{P}(1, r)$ is of the form

$$O_{\mathbb{P}(1, r)}(s p_0 + f p_\infty),$$

where $a, b \in \mathbb{Z}$. We have

$$\deg O_{\mathbb{P}(1, r)}(s p_0 + f p_\infty) = \frac{s}{r} + f.$$

Let $p : I \to \mathbb{P}(1, r)$ be the $\mathbb{Z}_m$-gerbe obtained by applying the $m$th root construction to the equivariant line bundle $O_{\mathbb{P}(1, r)}(s p_0 + f p_\infty)$. Let $L_2$ be the tautological line bundle over $I$, so that

$$L_2^\otimes_m = p^* O_{\mathbb{P}(1, r)}(s p_0 + f p_\infty).$$

Then $\deg L_2 = s/rm + f/m = w_2$. Let

$$L_3 := L_2^{-1} \otimes p^* O_{\mathbb{P}(1, r)}(-p_0).$$

Then $\deg L_3 = -s/rm - f/m - 1/r = w_3$. The toric 3-orbifold $\mathcal{Y}_f$ is the total space of the rank 2 vector bundle

$$V_f := L_2 \oplus L_3 \to I.$$

The divisor $D_f$ is the fiber of $V_f$ over the stacky point $p^{-1}(p_\infty)$ in $I$.

When $G$ is trivial, we have $r = m = 1, s = 0, I = \mathbb{P}(1, r) = \mathbb{P}^1$, and $V_f = O_{\mathbb{P}^1}(f) \oplus O_{\mathbb{P}^1}(-f-1) \to \mathbb{P}^1$.

### 4. Chen–Ruan orbifold cohomology

In this section, we describe the Chen–Ruan orbifold cohomology [CR04] of the classifying space $BG$ of the finite abelian group $G$ and the $\tilde{T}$-equivariant Chen–Ruan orbifold cohomology of $\mathcal{X} = [\mathbb{C}^3/G]$ and $\mathcal{Y}_f$. The Chen–Ruan orbifold cohomology of the classifying space of any finite group is described in [JK02].

#### 4.1 Chen–Ruan orbifold cohomology of $BG$

The inertia stack of $BG$ is

$$\mathcal{I}BG = \bigcup_{h \in G} (BG)_h, \quad \text{where} \quad (BG)_h = [\{h\}/G] \cong BG.$$

As a graded vector space over $\mathbb{C}$,

$$H^*_\text{CR}(BG; \mathbb{C}) = H^*(\mathcal{I}BG; \mathbb{C}) = \bigoplus_{h \in G} H^0((BG)_h; \mathbb{C}),$$

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where $H^0((BG)_h; \mathbb{C}) = \mathbb{C}1_h$. The orbifold Poincaré pairing of $H^*_{CR}(BG; \mathbb{C})$ is given by

$$\langle 1_h, 1_{h'} \rangle = \frac{\delta_{h^{-1}, h'}}{|G|}.$$  

The orbifold cup product of $H^*_{CR}(BG; \mathbb{C})$ is given by $1_h \star 1_{h'} = 1_{hh'}$.

We now define a canonical basis for the semi-simple algebra $H^*_{CR}(BG; \mathbb{C})$. Given a character $\gamma \in G^* = \text{Hom}(G, \mathbb{C}^*)$, define

$$\phi_\gamma := \frac{1}{|G|} \sum_{h \in G} \chi_\gamma(h^{-1})1_h.$$  

Then

$$H^*_{CR}(BG; \mathbb{C}) = \bigoplus_{h \in G} \mathbb{C}1_h = \bigoplus_{\gamma \in G^*} \mathbb{C}\phi_\gamma.$$  

Recall that we have the orthogonality of characters:

1. For any $\gamma, \gamma' \in G^*$, we have $\frac{1}{|G|} \sum_{h \in G} \chi_\gamma(h^{-1})\chi_{\gamma'}(h) = \delta_{\gamma, \gamma'}$.

2. For any $h, h' \in G$, we have $\frac{1}{|G|} \sum_{\gamma \in G^*} \chi_\gamma(h^{-1})\chi_{\gamma}(h') = \delta_{h, h'}$.

Therefore, $\langle \phi_\gamma, \phi_{\gamma'} \rangle = \delta_{\gamma, \gamma'}/|G|^2$ and $\phi_\gamma \star \phi_{\gamma'} = \delta_{\gamma, \gamma'}\phi_\gamma$. Then $\{\phi_\gamma : \gamma \in G^*\}$ is a canonical basis of $H^*_{CR}(BG; \mathbb{C})$.

### 4.2 Equivariant Chen–Ruan orbifold cohomology of $\mathcal{X} = [\mathbb{C}^3/G]$

Given any $h \in G$, define $c_i(h) \in [0, 1) \cap \mathbb{Q}$ and $\text{age}(h) \in \{0, 1, 2\}$ by equations (2.2) and (2.3) in Section 2.2, respectively. Let $(\mathbb{C}^3)^h$ denote the $h$-invariant subspace of $\mathbb{C}^3$. Then

$$\text{dim}_{\mathbb{C}} (\mathbb{C}^3)^h \leq \sum_{i=1}^{3} \delta_{c_i(h), 0}.$$  

The inertial stack of $\mathcal{X}$ is

$$\mathcal{I}\mathcal{X} = \bigcup_{h \in G} \mathcal{X}_h,$$  

where $\mathcal{X}_h = [(\mathbb{C}^3)^h/G]$.

In particular, $\mathcal{X}_1 = [\mathbb{C}^3/G] = \mathcal{X}$. As a graded vector space over $\mathbb{C}$,

$$H^*_{CR}(\mathcal{X}; \mathbb{C}) = \bigoplus_{h \in G} H^*(\mathcal{X}_h; \mathbb{C})[2\text{age}(h)] = \bigoplus_{h \in G} \mathbb{C}1_h,$$  

where $\text{deg}(1_h) = 2\text{age}(h) \in \{0, 2, 4\}$. The orbifold Poincaré pairing of the (non-equivariant) Chen–Ruan orbifold cohomology $H^*_{CR}(\mathcal{X}; \mathbb{C})$ is given by

$$\langle 1_h, 1_{h'} \rangle_{\mathcal{X}} = \frac{1}{|G|} \delta_{h^{-1}, h'} \cdot \delta_{0, \text{dim}_{\mathbb{C}} \mathcal{X}_h}.$$  

Let $\mathcal{R} = H^*(B\tilde{T}; \mathbb{C}) = \mathbb{C}[w_1, w_2, w_3]$, where $w_1, w_2, w_3$ are the first Chern classes of the universal line bundles over $B\tilde{T}$. The $\tilde{T}$-equivariant Chen–Ruan orbifold cohomology $H^*_{CR, \tilde{T}}([\mathbb{C}^3/G]; \mathbb{C})$ is an $\mathcal{R}$-module. Given $h \in G$, define

$$e_h := \prod_{i=1}^{3} w_i^{\delta_{c_i(h), 0}} \in \mathcal{R}.$$  

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In particular, \( e_1 = w_1w_2w_3 \). Then the \( \widetilde{T} \)-equivariant Euler class of \( 0_h := [0/G] \) in \( X_h = \left( (\mathbb{C}^3)^h / G \right) \) is

\[
e^{\tilde{T}}(0_h X_h) = e_h f h \in H^4_T(X_h; \mathbb{C}) = R 1_h.
\]

Let \( \chi_1, \chi_2, \chi_3 \in G^* \) be defined as in Section 2.1. The image of \( \chi_i : G \to \mathbb{C}^* \) is \( \mu_{r_i} \) for some positive integer \( r_i \). In particular, \( r_1 = r \). Define

\[
R' = \mathbb{C}[w_1^{1/r_1}, w_2^{1/r_2}, w_3^{1/r_3}],
\]

which is a finite extension of \( R \). Let

\[
Q = \mathbb{C}(w_1, w_2, w_3) \quad \text{and} \quad Q' = \mathbb{C}(w_1^{1/r_1}, w_2^{1/r_2}, w_3^{1/r_3})
\]

be the fractional fields of \( R \) and \( R' \), respectively. The \( \widetilde{T} \)-equivariant Poincaré pairing of \( H^*_{\tilde{T}, CR}(\mathcal{X}; \mathbb{C}) \otimes_R Q \) (which is isomorphic to \( H^*_{CR}(\mathcal{X}; \mathbb{Q}) \) as a vector space over \( \mathbb{Q} \)) is given by

\[
\langle (1_h, 1_{h'}), \mathcal{X} \rangle = \frac{1}{|G|} \cdot \frac{\delta_{h^{-1}, h'}}{e_h} \in Q.
\]

The \( \widetilde{T} \)-equivariant orbifold cup product of \( H^*_{\tilde{T}, CR}(\mathcal{X}; \mathbb{C}) \otimes_R Q \) is given by

\[
1_h \star_{\mathcal{X}} 1_{h'} = \left( \prod_{i=1}^{3} w_i^{c_{i}(h)+c_{i}(h')-c_{i}(hh')} \right) 1_{hh'}.
\]

Define

\[
\bar{1}_h := \frac{1_h}{\prod_{i=1}^{3} w_i^{c_{i}(h)}} \in H^*_{\tilde{T}, CR}(\mathcal{X}; \mathbb{C}) \otimes_R Q'.
\]

Then

\[
\langle \bar{1}_h, \bar{1}_{h'} \rangle, \mathcal{X} = \frac{\delta_{h^{-1}, h'}}{|G|w_1w_2w_3}, \quad \bar{1}_h \star_{\mathcal{X}} \bar{1}_{h'} = \bar{1}_{hh'}.
\]

We now define a canonical basis for the semi-simple algebra \( H^*_{\tilde{T}, CR}(\mathcal{X}; \mathbb{C}) \otimes_R Q' \). Given \( \gamma \in G^* \), define

\[
\tilde{\phi}_{\gamma} := \frac{1}{|G|} \sum_{h \in G} \chi_1(h^{-1}) 1_h.
\]

Then

\[
\langle \tilde{\phi}_{\gamma}, \tilde{\phi}_{\gamma'} \rangle, \mathcal{X} = \frac{\delta_{\gamma\gamma'}}{|G|w_1w_2w_3}, \quad \tilde{\phi}_{\gamma} \star_{\mathcal{X}} \tilde{\phi}_{\gamma'} = \delta_{\gamma\gamma'} \tilde{\phi}_{\gamma'},
\]

So \( \{ \tilde{\phi}_{\gamma} : \gamma \in G^* \} \) is a canonical basis of \( H^*_{\tilde{T}, CR}(\mathcal{X}; \mathbb{C}) \otimes_R Q' \).

4.3 Equivariant Chen–Ruan orbifold cohomology of \( \mathcal{Y}_f \)

We use the notation in Sections 3.2 and 3.3: the coarse moduli space \( Y_f \) of the toric 3-orbifold \( \mathcal{Y}_f \) is defined by a simplicial fan \( \Theta_f \) with two 3-dimensional cones \( \sigma \) and \( \sigma' \). Let \( \mathcal{X} = [\mathbb{C}^3 / G] \) and \( \mathcal{X}' = [\mathbb{C}^3 / \mu_m] \) be the affine toric sub-orbifolds of \( \mathcal{Y}_f \) associated with \( \sigma \) and \( \sigma' \), respectively. As \( Q \)-vector spaces,

\[
H^*_{\tilde{T}, CR}(\mathcal{Y}_f; \mathbb{C}) \otimes_R Q = H^*_{CR}(\mathcal{Y}_f; \mathbb{Q}) = H^*_{CR}(\mathcal{X}; \mathbb{Q}) \oplus H^*_{CR}(\mathcal{X}'; \mathbb{Q}),
\]

where

\[
H^*_{CR}(\mathcal{X}; \mathbb{Q}) = \bigoplus_{h \in G} Q 1_h, \quad H^*_{CR}(\mathcal{X}'; \mathbb{Q}) = Q 1' \oplus \bigoplus_{k=1}^{m-1} Q 1'_{k/m}.
\]

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We have the following isomorphisms of $Q$-vector spaces (which do not preserve the grading and the orbifold cup product):

$$H^*_\text{CR}(Y_f; Q) \cong H^*_\text{CR}(\mathcal{X}; Q), \quad H^*_\text{CR}(\mathcal{X}; Q) \cong H^*_\text{CR}(BG; Q),$$

$$H^*_\text{CR}(\mathcal{X}; Q) \cong H^*_\text{CR}(D_f; Q) \cong H^*_\text{CR}(p_\infty; Q) = H^*_\text{CR}(B\mu_m; Q).$$

5. A-model topological string

5.1 Relative orbifold Gromov–Witten invariants of $(Y_f, D_f)$

In this section, we define open-closed orbifold Gromov–Witten invariants of $\mathcal{X}$ as certain equivariant relative orbifold Gromov–Witten invariants of $(Y_f, D_f)$. Recall from Section 3.3 that the orbifold $Y_f$ can be viewed as the total space of the rank 2 vector bundle $V = L_2 \oplus L_3 \to \mathcal{I}$, where $\mathcal{I}$ is a $\mu_m$-gerbe over $\mathbb{P}(1, r)$, so equivariant relative orbifold Gromov–Witten invariants of $(Y_f, D_f)$ are twisted equivariant relative orbifold Gromov–Witten invariants of $(\mathcal{I}, p_\infty)$.

Relative orbifold Gromov–Witten theory has been constructed in algebraic geometry by Abramovich–Fantechi [AF16] and in symplectic geometry by Chen–Li–Sun–Zhao [CLSZ11]. We refer to [AF16, CLSZ11] for the precise definitions of moduli spaces of relative stable maps to smooth DM stacks.

We first introduce some notation. We fix non-negative integers $g$ and $\ell$ and an integer $\bar{\mu} = ((\mu_1, k_1), \ldots, (\mu_n, k_n))$, where $\mu_j \in \mathbb{Z}_{>0}$ and $k_j \in \{0, 1, \ldots, m-1\}$.

1. Let $\overline{M}_{g, \ell}(Y_f/D_f, \bar{\mu})$ and $\overline{M}_{g, \ell}(\mathcal{I}/p_\infty, \bar{\mu})$ be the moduli spaces of genus $g$ relative stable maps to $(Y_f, D_f)$ and $(\mathcal{I}, p_\infty)$, respectively, with $\ell$ marked points and $n$ relative points, where the relative points are ordered, and the ramification index of the $j$th relative point is $\mu_j$ and its monodromy is $k_j$.

2. Let $\text{ev}_i: \overline{M}_{g, \ell}(Y_f/D_f, \bar{\mu}) \to \mathcal{I}Y_f$ and $\text{ev}_i: \overline{M}_{g, \ell}(\mathcal{I}/p_\infty, \bar{\mu}) \to \mathcal{I}$ be the evaluation maps at the $i$th marked point, where $i = 1, \ldots, \ell$.

3. Let $\pi: \mathcal{U} \to \overline{M}_{g, \ell}(\mathcal{I}/p_\infty, \bar{\mu})$ be the universal curve, $\mathcal{T} \to \overline{M}_{g, \ell}(\mathcal{I}/p_\infty, \bar{\mu})$ be the universal target, and $F: \mathcal{U} \to \mathcal{T}$ be the universal map.

4. There is a contraction map $\mathcal{T} \to \mathcal{I}$. Let $\tilde{F}: \mathcal{U} \to \mathcal{I}$ be the composition of this contraction map and the universal map $F: \mathcal{U} \to \mathcal{T}$.

5. Let $D = \bigcup_{1 \leq j \leq n} D_j \subset \mathcal{U}$ be the universal relative points corresponding to $\bar{\mu}$, and let $D_0 = \bigcup_{k_j=0} D_j \subset D$.

6. Let $\mathbb{T}_m = \{ (t, \frac{1}{r}, 1) : t \in \mathbb{C}^* \} \subset \mathbb{T} \cong (\mathbb{C}^*)^3$. This induces

$$\phi_f: H^*(BT; Q) = Q[u_1, u_2, u_3] = Q[w_1, w_2, w_3] \to H^*(BT_f; Q) = Q[v],$$

$$u_1 \mapsto v, \quad u_2 \mapsto fv, \quad u_3 \mapsto 0$$
or, equivalently,

$$w_1 \mapsto \frac{1}{r}v = w_1v, \quad w_2 \mapsto \frac{s + rf}{rm}v = w_2v, \quad w_3 \mapsto \frac{-s - rf - m}{rm}v = w_3v.$$We extend $\phi_f$ to $\phi_f: Q = \mathbb{C}(u_1, u_2, u_3) = \mathbb{C}(w_1, w_2, w_3) \to \mathbb{C}(v)$.

For any $\gamma_1, \ldots, \gamma_\ell \in H^*_\text{CR}(\mathcal{X}; Q) = \bigoplus_{h \in G} Q_{1h} \subset H^*_\text{CR}(Y_f; Q) \cong H^*_\text{CR}(\mathcal{I}; Q)$,
we define the $\tilde{T}$-equivariant relative orbifold Gromov–Witten invariant

$$\langle \gamma_1, \ldots, \gamma_{\ell} \rangle_{g, \mu, \tilde{\mu}}: = \frac{1}{|\text{Aut}(\tilde{\mu})|} \cdot \phi_f \left( \int_{[M_{g,\ell}(X_{\ell}/D,\tilde{\mu})]^{\text{vir}}} (u_2 - f u_1)^{\sum_{j=1}^{n} \delta_{k_{j,0}}} \prod_{i=1}^{\ell} \text{ev}_i^* \gamma_i \right)$$

$$= \frac{1}{|\text{Aut}(\tilde{\mu})|} \cdot \phi_f \left( \int_{[M_{g,\ell}(l/p,\tilde{\mu})]^{\text{vir}}} e_\tilde{T}(-R\pi_* (\hat{F}^* L_2 (D_0) \oplus \hat{F}^* L_3)) \prod_{i=1}^{\ell} \text{ev}_i^* \gamma_i \right) \in \mathbb{Q}(v).$$

Note that $\phi_f(u_2 - f u_1) = 0$. The factor $(u_2 - f u_1)^{\sum_{j=1}^{n} \delta_{k_{j,0}}}$ is included to cancel such a factor in $e_\tilde{T}(N^{\text{vir}})$.

If $\gamma_1, \ldots, \gamma_{\ell}$ are homogeneous, then

$$\langle \gamma_1, \ldots, \gamma_{\ell} \rangle_{g, \mu, \tilde{\mu}} = \nu^{\sum_{i=1}^{\ell} (\deg \gamma_i/2 - 1)} \cdot \langle \gamma_1, \ldots, \gamma_{\ell} \rangle_{g, \mu, \tilde{\mu}} \big|_{v=1}. \quad (5.1)$$

### 5.2 Descendant orbifold Gromov–Witten invariants of $X$

In this subsection, we use localization to compute $\langle \gamma_1, \ldots, \gamma_{\ell} \rangle_{g, \mu, \tilde{\mu}}$ and obtain an expression in terms of cubic abelian Hurwitz–Hodge integrals.

We first give some definitions. For $i = 1, 2, 3$, let $\chi_i \in G^* = \text{Hom}(G, \mathbb{C}^*)$ be the characters of $G$ defined in Section 2.1, and let $L_{\chi_i}$ be the line bundle over $BG$ associated with the irreducible character $\chi_i \in G^*$. Then $X = [\mathbb{C}^3/G]$ is the total space of

$$\bigoplus_{i=1}^{3} L_{\chi_i} \to BG.$$

Let $\overline{M}_{g,n}(BG)$ denote the moduli space of genus $g$, $n$-pointed twisted stable maps to $BG$. Let $\pi: C_{g,n} \to \overline{M}_{g,n}(BG)$ be the universal curve, and let $F: C_{g,n} \to BG$ be the universal map. Define

$$e_{g,n} := \phi_f \left( e_\tilde{T} \left( R\pi_* F^* \left( \bigoplus_{i=1}^{3} L_{\chi_i} \right) \right) \right) = \prod_{i=1}^{3} (w_i)^{(\text{rank} R\pi_* F^* L_{\chi_i})} c_{1/w_i}(R\pi_* F^* L_{\chi_i}), \quad (5.2)$$

where $c_{\ell}(E) = 1 + t c_1(E) + t^2 c_2(E) + \cdots$ denotes the Chern polynomial of a complex vector bundle $E$ (or, more generally, an element in the $K$-theory). The rank of $R\pi_* F^* L_{\chi_i}$ is constant on a connected component of $\overline{M}_{g,n}(BG)$ but might be different on different connected components of $\overline{M}_{g,n}(BG)$.

A moduli point in $\overline{M}_{g,\ell}(l/p, \tilde{\mu})$ is represented by a morphism

$$\rho: (\mathcal{C}, x_1, \ldots, x_{\ell}, y_1, \ldots, y_n) \to \{k\} = I \cup \Delta_1 \cup \cdots \cup \Delta_k,$$

where each $\Delta_i$ is a trivial $\mu_m$-gerbe over $\mathbb{P}^1$. Let

$$\overline{M}_{0,\ell}(l/p, \tilde{\mu}) \subset \overline{M}_{g,\ell}(l/p, \tilde{\mu})$$

be the substack where the target is $\{0\} = I$. Computations in [RZ13] show that the contribution from a $\tilde{T}$-fixed locus outside $\overline{M}_{0,\ell}(l/p, \tilde{\mu})$ vanishes. Therefore,

$$\langle \gamma_1, \ldots, \gamma_{\ell} \rangle_{g, \mu, \tilde{\mu}} = \frac{1}{|\text{Aut}(\tilde{\mu})|} \cdot \phi_f \left( \int_{[\overline{M}_{0,\ell}(l/p, \tilde{\mu})]^{\text{vir}}} e_\tilde{T}(-R\pi_* (\hat{F}^* L_2 (D_0) \oplus \hat{F}^* L_3)) \prod_{i=1}^{\ell} \text{ev}_i^* \gamma_i \right).$$

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Suppose that \( \rho: (\mathcal{C}, x_1, \ldots, x_\ell, y_1, \ldots, y_n) \to I \) represents a point in \( \overline{\mathcal{M}}_{g,n+\ell}(\mathcal{F}, \underline{p}, \underline{\mu}) \). Then \( \mathcal{C} = C_0 \cup \bigcup_{j=1}^n C_j \), where \( C_0 \) is an orbicurve of genus \( g \), the components \( C_1, \ldots, C_n \) are 1-dimensional toric orbifolds, \( x_1, \ldots, x_\ell \in C_0 \) and \( y_j \in C_j \), and \( C_0 \) and \( C_j \) intersect at a node \( z_j \) for \( j = 1, \ldots, n \). We view \( z_j \) as a point on \( C_j \) in order to determine its monodromy. Let \( \rho_j := \rho|_{C_j}: C_j \to I \). Then \( \rho_0 \) is a constant map to \( p_0 = [0, 1] \), and we have the following diagram for \( 1 \leq j \leq n \):

\[
\begin{array}{ccc}
C_j & \xrightarrow{\rho_j} & I \\
\downarrow & & \downarrow \\
p^1 & \xrightarrow{\rho_j} & p^1,
\end{array}
\]

where \( \tilde{\rho}_j([x, y]) = [x^{\mu_j}, y^{\mu_j}] \) is the map between coarse moduli spaces, the monodromy around \( y_j \) is \( e^{2\pi \sqrt{-1}k_j/m} \in \mathbb{P}_m \), and the monodromy around \( z_j \) is

\[
\left( e^{2\pi \sqrt{-1}k_1w_1}, e^{2\pi \sqrt{-1}(\mu_jw_2-k_j/m)}, e^{2\pi \sqrt{-1}(\mu_jw_3+k_j/m)} \right) = \eta_1^{\mu_j} \eta_2^{-k_j} \in G.
\]

For \( (d_0, k) \in \mathbb{Z} \times \mathbb{Z}_m \), define \( h(d_0, k) := \eta_1^{d_0} \eta_2^{-k} \in G \), and define \( D'(d_0, k) \) to be

\[
-\frac{r}{v}(-1)^{d_0w_3+k/m} \left( \frac{v}{d_0} \right)^{\text{age}(h(d_0,k))} \frac{\Gamma(d_0(w_1+w_2)+c_3(h(d_0,k)))}{\Gamma(d_0w_1-c_1(h(d_0,k))+1)\Gamma(d_0w_2-c_2(h(d_0,k))+1)}.
\]

Then \( D'(d_0, k) \in \mathbb{C}^{\text{age}(h(d_0,k))} \). Up to the sign, \( D'(d_0, k) \) is essentially \( |G| \) times the disk function in \( \text{[Ros14, Section 3.3]} \). Note that

\[
(-1)^{d_0w_3+k/m} = (-1)^{d_0w_3-c_3(h(d_0,k))+k/m}.
\]

By virtual localization, we have the following result.

**Proposition 5.1.** We have

\[
\langle \chi_{(\mathcal{C},f)} \rangle_{0,(d_0,k)} = \frac{\delta_{1,h(d_0,k)}}{|G|} D'(d_0, k) \cdot \left( \frac{v}{d_0} \right)^2,
\]

\[
\langle \gamma \rangle_{0,(d_0,k)} = \langle \gamma, e_{h(d_0,k)}1_{h(d_0,k)-1} \rangle_{\mathcal{X}} \cdot D'(d_0, k) \cdot \frac{v}{d_0},
\]

\[
\langle \chi_{(\mathcal{C},f)} \rangle_{0,(\mu_1,k_1),(\mu_2,k_2)} = \langle e_{h(\mu_1,k_1)}1_{h(\mu_1,k_1)-1}, e_{h(\mu_2,k_2)}1_{h(\mu_2,k_2)-1} \rangle_{\mathcal{X}} \cdot \frac{D'(\mu_1,k_1)D'(\mu_2,k_2)}{|\text{Aut}((\mu_1,k_1),(\mu_2,k_2))|} \cdot \frac{v}{\mu_1 + \mu_2}.
\]

If \( \tilde{\mu} = ((\mu_1, k_1), \ldots, (\mu_n, k_n)) \) and \( 2g - 2 + \ell + n > 0 \), then

\[
\langle \gamma_1, \ldots, \gamma_\ell \rangle_{g,\tilde{\mu}} = \frac{1}{|\text{Aut}(\tilde{\mu})|} \prod_{j=1}^n D'(\mu_j, k_j) \int_{\mathcal{M}_{g,\ell+n}(BG)} \prod_{j=1}^n \text{ev}_{\ell+j}^*(\gamma_1) \prod_{j=1}^n \text{ev}_{\ell+j}^*(e_{h(\mu_j,k_j)}1_{h(\mu_j,k_j)-1}) 1 - (\mu_j/v)^{\psi_{\ell+j}}.
\]

Introduce variables \( X_1, \ldots, X_n \), and let

\[
\tau = \sum_{a=1}^p \tau_a 1_{h_a},
\]

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where \( h_a \in G \) corresponds to \((m_a,n_a,1) \in \{ v \in \text{Box}(\sigma) : \text{age}(v) = 1 \}\). For \( k \in \{0,1,\ldots,m-1\} \), let \( 1'_{-k/m} := 1'_{1-h_0-k/m} \in H^*_\text{CR}(\mathcal{B} \mu_{m}; \mathbb{C}) = \bigoplus_{k=0}^{m-1} \mathbb{C} \mathbf{1}'_{k/m} \). Define

\[
F^X_{g,n}(\tau; X_1, \ldots, X_n) = \sum_{\mu_1, \ldots, \mu_n > 0} \sum_{k_1, \ldots, k_n = 0} \frac{\langle \tau^\ell \rangle X_{g, \langle (\mu_1, k_1), \ldots, (\mu_n, k_n) \rangle}}{\ell!} \prod_{j=1}^n \langle X_j \rangle_{\mu_j} \left( -1 \right)^{-k_j/m} 1'_{-k_j/m} \otimes \cdots \otimes \left( -1 \right)^{-k_n/m} 1'_{-k_n/m},
\]

which is an \( H^*_\text{CR}(\mathcal{B} \mu_{m}; \mathbb{C}) \otimes \mathbb{C}^n \)-valued function, where \( H^*_\text{CR}(\mathcal{B} \mu_{m}; \mathbb{C}) = \bigoplus_{k=0}^{m-1} \mathbb{C} \mathbf{1}'_{k/m} \).

**Remark 5.2.** For each \( k \in \{0,1,\ldots,m-1\} \), the set \( \{ \mu_j : k_j = k \} \) determines a partition \( \mu^k = (\mu^k_1 \geq \cdots \geq \mu^k_\ell(k^\ell) > 0) \). We have

\[
\text{Aut}(\langle (\mu_1, k_1), \ldots, (\mu_n, k_n) \rangle) = \prod_{k=0}^{m-1} \text{Aut}(\mu^k),
\]

\[
\ell(\mu^0) + \cdots + \ell(\mu^{m-1}) = n.
\]

The correlator \( \langle \tau^\ell \rangle X_{g, \langle (\mu_1, k_1), \ldots, (\mu_n, k_n) \rangle} \) is invariant under permutation of \( (\mu_1, k_1), \ldots, (\mu_n, k_n) \), so it depends only on the \( m \)-tuple of partitions \( \mu^0, \ldots, \mu^{m-1} \). So \( F^X_{g,n}(\tau; X_1, \ldots, X_n) \) is a symmetric function in \( X_1, \ldots, X_n \) and can be rewritten as

\[
F^X_{g,n}(\tau; X_1, \ldots, X_n) = \sum_{\mu^0, \ldots, \mu^{m-1} \in \mathcal{P}} \sum_{\ell(\mu^0) + \cdots + \ell(\mu^{m-1}) = n} \frac{\langle \tau^\ell \rangle X_{g, \mu^0, \ldots, \mu^{m-1}}}{\ell!} X_{\mu^0, \ldots, \mu^{m-1}},
\]

where \( \mathcal{P} \) is the set of partitions and

\[
X_{\mu^0, \ldots, \mu^{m-1}} = \sum_{\sigma \in \mathcal{S}_n} \prod_{k=0}^{m-1} \prod_{j=\ell(\mu^0) + \cdots + \ell(\mu^{k-1})+1}^{\ell(\mu^k)} X_{\sigma(j)} \left( -1 \right)^{-k_j/m} 1'_{-k_j/m} \otimes \ell(\mu^k).
\]

We introduce some notation.

(1) Given \( h \in G \), define

\[
\Phi_h^0(X) := \frac{1}{|G|} \sum_{\substack{(d_0,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m \\ h(d_0,k) = h}} D'(d_0,k) X_{d_0} \left( -1 \right)^{-k/m} 1'_{-k/m}
\]

\[
= - \sum_{\substack{(d_0,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m \\ h(d_0,k) = h}} \frac{1}{\mathcal{N}} e^{\sqrt{-1} \pi (d_0 w_3 - c_3(h))} \left( \frac{v}{d_0} \right)^{\text{age}(h)-1} \frac{\Gamma(d_0 (w_1 + w_2) + c_3(h))}{\Gamma(d_0 w_1 - c_1(h) + 1) \Gamma(d_0 w_2 - c_2(h) + 1)} X_{d_0} 1'_{-k/m}.
\]

Then \( \Phi_h^0(X) \) takes values in \( \bigoplus_{k=0}^{m-1} \mathbb{C} \mathbf{1}_k \). For \( a \in \mathbb{Z} \) and \( h \in G \), we define

\[
\Phi_a^0(X) := \frac{1}{|G|} \sum_{\substack{d_0 > 0 \\ h(d_0,k) = h}} D'(d_0,k) \left( \frac{d_0}{v} \right)^a X_{d_0} \left( -1 \right)^{-k/m} 1'_{-k/m}.
\]
Then $\Phi^h_a(X)$ takes values in $\bigoplus_{k=0}^{m-1} \mathbb{C} v^{a(e(v)) - 2 - a} 1^k_{1/m}$, and
\[
\Phi^h_{a+1}(X) = \left( \frac{1}{v} \frac{d}{dX} \right) \Phi^h_a(X).
\]

(2) For $a \in \mathbb{Z}$ and $\gamma \in G^*$, we define
\[
\tilde{\xi}^a_\ell(X) := |G| \sum_{h \in G} \chi_\gamma(h^{-1}) \left( \prod_{i=1}^{3} (w_i v)^{1-c_i(h)} \right) \Phi^h_a(X).
\]

Then $\tilde{\xi}^a_\ell(X)$ takes values in $\bigoplus_{k=0}^{m-1} \mathbb{C} v^{-a} 1^k_{1/m}$.

(3) Given $\gamma_1, \gamma_2 \in H^*_\text{CR}(X; \mathbb{C}) \otimes \mathbb{C}(v)$ and $a_1, a_2 \in \mathbb{Z}_{\geq 0}$, define
\[
\langle \tau_{a_1}(\gamma_1), \tau_{a_2}(\gamma_2) \rangle_{0,1} = \delta_{a_1,0}(1, \gamma_1) \chi \cdot v^2,
\]
\[
\langle \tau_{a_1}(\gamma_1), \tau_{a_2}(\gamma_2) \rangle_{0,2} = \delta_{a_1,0} \delta_{a_2,0} \cdot v(\gamma_1, \gamma_2) \chi.
\]

(4) Given $\gamma_1, \ldots, \gamma_n \in H^*_\text{CR}(X; \mathbb{C}) \otimes \mathbb{C}(v)$ and $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$, define
\[
\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{0,n} = \int_{\mathcal{M}_{g,n}(BG)} \frac{1}{\mathcal{e}_{g,n}} \prod_{i=1}^{n} \left( \epsilon v_i^\gamma(\gamma_i) \tilde{\xi}_i^{a_i} \right).
\]

(5) Given $h \in G$, define $1^*_h = |G| \phi_f(e_h 1_{h^{-1}})$.

(6) In the rest of this paper, we consider $T_f$-equivariant cohomology and write $\phi_\gamma$ and $\tilde{\phi}_\gamma$ instead of $\phi_f(\phi_\gamma)$ and $\phi_f(\tilde{\phi}_\gamma)$, respectively. (Recall that $\phi_f(w_i) = w_i v$.)

**Proposition 5.3.** We have the following equations:

(i) (Disk invariants)
\[
F^X_{0,1}(\tau; X) = F^1_{-2}(X) + \sum_{a=1}^{p} \tau_a \Phi^{h_{a-1}}(X) + \sum_{a \in \mathbb{Z}_{\geq 0}} \sum_{h \in G} \sum_{\ell \geq 2} \frac{\langle \tau^\ell, \tau_a(1^*_h) \rangle_{0,\ell+1} \chi^{X,f}_{\ell}}{\ell!} \Phi^{h}_a(X)
\]
\[
= \left. \frac{1}{|G|^2 w_1 w_2 w_3} \left( \sum_{\gamma \in G^*} \tilde{\xi}^{\gamma}_2(X) + \sum_{a=1}^{p} \tau_a \prod_{i=1}^{3} w_i^{c_i(h_a)} \sum_{\gamma \in G^*} \chi(\gamma(h_a) \tilde{\xi}^{\gamma}_1(X)) \right) \right|_{v=1}
\]
\[
+ \sum_{a \in \mathbb{Z}_{\geq 0}} \sum_{\gamma \in G^*} \sum_{\ell \geq 2} \frac{\langle \tau^\ell, \tau_a(\tilde{\phi}_\gamma) \rangle_{0,\ell+1} \tilde{\xi}^{\gamma}_a(X)}{\ell!}.
\]

(ii) (Annulus invariants)
\[
F^X_{0,2}(\tau; X_1, X_2) - F^X_{0,2}(0; X_1, X_2)
\]
\[
= \sum_{a_1, a_2 \in \mathbb{Z}_{\geq 0}} \sum_{h_1, h_2 \in G} \sum_{\ell \geq 1} \frac{\langle \tau^\ell, \tau_a(1^*_h) \tau_b(1^*_h) \rangle_{0,\ell+2} \chi^{X,f}_{\ell}}{\ell!} \Phi^{h_{a_1}}_{a_1}(X_1) \Phi^{h_{a_2}}_{a_2}(X_2)
\]
\[
= \sum_{a_1, a_2 \in \mathbb{Z}_{\geq 0}} \sum_{\gamma_1, \gamma_2 \in G^*} \sum_{\ell \geq 1} \frac{\langle \tau^\ell, \tau_a(\tilde{\phi}_\gamma) \tau_b(\tilde{\phi}_\gamma) \rangle_{0,\ell+2} \chi^{X,f}_{\ell}}{\ell!} \tilde{\xi}^{\gamma_1}_{a_1}(X_1) \tilde{\xi}^{\gamma_2}_{a_2}(X_2),
\]
where
\[
\left( X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} \right) F_{0,2}^{X,\mathcal{L},f}(0; X_1, X_2) = |G| \left( \sum_{h\in G} \mathbf{e}_h \Phi_0^h (X_1) \Phi_0^{h^{-1}} (X_2) \right) \bigg|_{v=1} = \frac{1}{|G|^2 w_1 w_2 w_3} \left( \sum_{\gamma \in G^*} \widetilde{\xi}_0^\gamma (X_1) \widetilde{\xi}_0^\gamma (X_2) \right) \bigg|_{v=1} \ .
\]
(5.5)

(iii) For \(2g - 2 + n > 0\),
\[
F_{g,n}^{X,\mathcal{L},f}(\tau; X_1, \ldots, X_n) = \sum_{a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}} \sum_{h_1, \ldots, h_n \in G} \sum_{\ell \geq 0} \frac{\langle \tau, \tau a_1 (1_{h_1}) \cdots \tau a_n (1_{h_n}) \rangle_{X,T_f}}{\ell!} g_{a,\ell+n} \prod_{j=1}^n \Phi_{a_j}^h (X_j)
\]
\[
= \sum_{a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}} \sum_{\gamma_1, \ldots, \gamma_n \in G^*} \sum_{\ell \geq 0} \frac{\langle \tau, \tau a_1 (\bar{\phi}_{\gamma_1}) \cdots \tau a_n (\bar{\phi}_{\gamma_n}) \rangle_{X,T_f}}{\ell!} g_{a,\ell+n} \prod_{j=1}^n \xi_{a_j}^\gamma (X_j) \ .
\]

Remark 5.4. The function \(F_{0,2}^{X,\mathcal{L},f}(0; X_1, X_2)\) is an \(H^*(\mathcal{B}_{\mu_m}; \mathbb{C})\)\(\otimes \mathbb{Q}\)-valued power series in \(X_1\) and \(X_2\) which vanishes at \((X_1, X_2) = (0, 0)\), so it is determined by (5.5).

5.3 A-model graph sum

Introduce formal variables
\[
\hat{u} = \sum_{a \geq 0} \hat{u}_a z^a = \sum_{a \geq 0} \sum_{\beta \in G^*} u_a^\beta \phi_\beta z^a = \sum_{\beta \in G^*} u^\beta (z) \phi_\beta ,
\]
where
\[
u^\beta \in \mathbb{Q} , \quad \hat{u}_a = \sum_{\beta \in G^*} u_a^\beta \phi_\beta \in H^*_{\text{CR}} (\mathcal{X}; \mathbb{Q}) , \quad u^\beta (z) = \sum_{a \geq 0} u_a^\beta z^a .
\]
For \(j = 1, \ldots, n\), define \(\hat{u}_j\) and \(u^\beta_j (z)\) similarly. Define
\[
\langle \hat{u}^\ell, \hat{u}_1, \ldots, \hat{u}_n \rangle_{g,\ell+n}^{X,T_f} = \sum_{a_1, \ldots, a_\ell, b_1, \ldots, b_n \in \mathbb{Z}_{\geq 0}} \langle \tau a_1 (\hat{u}_1) \cdots \tau a_\ell (\hat{u}_\ell) \tau b_1 (\bar{\hat{u}}_1) \cdots \tau b_n (\bar{\hat{u}}_n) \rangle_{g,\ell+n}^{X,T_f} .
\]

By [FLZ16, Theorem 4.2], the term \((1/\ell!) \langle \hat{u}^\ell, \hat{u}_1, \ldots, \hat{u}_n \rangle_{g,\ell+n}^{X,T_f}\) can be written as a graph sum. By Proposition 5.3,
\[
F_{g,n}^{X,\mathcal{L},f}(\tau; X_1, \ldots, X_n) = \sum_{\ell \geq 0} \frac{1}{\ell!} \langle \tau^\ell, \bar{\xi}(X_1), \ldots, \bar{\xi}(X_n) \rangle_{g,\ell+n}^{X,T_f} ,
\]
(5.6)
where \(\bar{\xi}(X) = \sum_{a \in \mathbb{Z}_{\geq 0}} \sum_{\beta \in G^*} \bar{\xi}_a^\beta (X) \phi_\beta z^a\). Therefore, a graph sum of \(F_{g,n}^{X,\mathcal{L},f}(\tau; X_1, \ldots, X_n)\) can be obtained by the following specialization of [FLZ16, Theorem 2]:
\[
\hat{u} = \tau = \sum_{a=1}^p \tau a 1_{h_a} , \quad \hat{u}_j = \bar{\xi}(X_j) , \quad w_i = w_i v .
\]
By (5.1), we may let \(v = 1\); that is, \(w_i = w_i\). In the rest of this subsection, we give the precise statement of this graph sum.

Given a connected graph \(\Gamma\), we introduce the following notation:

(1) \(V(\Gamma)\) is the set of vertices in \(\Gamma\).

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(2) $E(\Gamma)$ is the set of edges in $\Gamma$.
(3) $H(\Gamma)$ is the set of half-edges in $\Gamma$.
(4) $L^O(\Gamma)$ is the set of open leaves in $\Gamma$. The open leaves are ordered: $L^O(\Gamma) = \{l_1, \ldots, l_n\}$, where $n$ is the number of open leaves. (Open leaves correspond to *ordered* ordinary leaves in [FLZ16].)
(5) $L^o(\Gamma)$ is the set of primary leaves in $\Gamma$. The primary leaves are unordered. (Primary leaves correspond to *unordered* ordinary leaves in [FLZ16].)
(6) $L^1(\Gamma)$ is the set of dilaton leaves in $\Gamma$. The dilaton leaves are not ordered.

With the above notation, we introduce the following labels:

1. (genus) $g: V(\Gamma) \to \mathbb{Z}_{\geq 0}$,
2. (marking) $\alpha: V(\Gamma) \to G^*$ (This induces $\alpha: L(\Gamma) = L^O(\Gamma) \cup L^o(\Gamma) \cup L^1(\Gamma) \to G^*$ as follows: if $l \in L(\Gamma)$ is a leaf attached to a vertex $v \in V(\Gamma)$, define $\alpha(l) = \alpha(v)$.),
3. (height) $k: H(\Gamma) \to \mathbb{Z}_{\geq 0}$.

Given a vertex $v \in V(\Gamma)$, let $H(v)$ denote the set of half-edges emanating from $v$. The valency $\text{val}(v)$ of the vertex $v$ is equal to the size $|H(v)|$ of the set $H(v)$. A labeled graph $\vec{\Gamma} = (\Gamma, g, \alpha, k)$ is stable if $2g(v) - 2 + \text{val}(v) > 0$ for all $v \in V(\Gamma)$.

Let $\Gamma(\mathcal{B}G)$ denote the set of all stable labeled graphs $\vec{\Gamma} = (\Gamma, g, \alpha, k)$. The genus of a stable labeled graph $\vec{\Gamma}$ is defined to be

$$g(\vec{\Gamma}) := \sum_{v \in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + 1 = \sum_{v \in V(\Gamma)} (g(v) - 1) + \sum_{e \in E(\Gamma)} 1 + 1.$$ 

Define

$$\Gamma_{g,\ell,n}(\mathcal{B}G) = \{\vec{\Gamma} = (\Gamma, g, \alpha, k) \in \Gamma(\mathcal{B}G): g(\vec{\Gamma}) = g, |L^o(\Gamma)| = \ell, |L^O(\Gamma)| = n\}.$$ 

Let

$$R(z) = \prod_{i=1}^{3} \exp \left( \sum_{m \geq 1} \frac{(-1)^m}{m(m+1)} A^i_m \left( \frac{z}{w_i} \right)^m \right),$$

where $A^i_m$ is the operator defined by

$$A^i_m: H^*(\mathcal{B}G; \mathbb{C}) \to H^*(\mathcal{B}G; \mathbb{C}), \quad 1_h \mapsto B_m(c_i(h))1_h$$

and $B_m(x)$ is the Bernoulli polynomial. Then, relative to the basis $\{\phi_\gamma: \gamma \in G^*\}$,

$$R(z)_\beta = \frac{1}{|G|} \sum_{h \in G} \chi_\alpha(h) \chi_\beta(h^{-1}) \prod_{i=1}^{3} \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^m}{m(m+1)} B_m(c_i(h)) \left( \frac{z}{w_i} \right)^m \right). \quad (5.7)$$

Given $\beta \in G^*$, define

$$\tilde{\xi}_\beta(z, X) = \sum_{a \geq 0} z^a \xi^\beta_a(X)|_{v=1}.$$ 

We assign weights to leaves, edges, and vertices of a labeled graph $\vec{\Gamma} \in \Gamma(\mathcal{Y})$ as follows:
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(1) (Open leaves) To each open leaf \( l_j \in L^O(\Gamma) \) with \( \alpha(l_j) = \alpha \in G^* \) and \( k(l_j) = k \in \mathbb{Z}_{\geq 0} \), we assign
\[
(L^k)_{\alpha}(l_j) = [z^k] \left( \frac{1}{|G| \sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R(-z)^\beta \chi^{\beta}(z, X_j) \right).
\]

(2) (Primary leaves) To each primary leaf \( l \in L^o(\Gamma) \) with \( \alpha(l) = \alpha \in G^* \) and \( k(l) = k \in \mathbb{Z}_{\geq 0} \), we assign
\[
(L^k)^\alpha(l) = [z^k] \left( \frac{1}{|G| \sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} \prod_{a=1}^p \sum_{\alpha} \prod_{i=1}^3 w_i c_i(h_a) R(-z)^\beta \chi^{\beta}(h_a) \tau_a \right).
\] (5.8)

By the orthogonality of the characters, we have
\[
\sum_{\beta \in G^*} \sum_{a=1}^p \prod_{i=1}^3 w_i c_i(h_a) R(-z)^\beta \chi^{\beta}(h_a) \tau_a
= \sum_{a=1}^p \prod_{i=1}^3 w_i c_i(h_a) \chi^{\alpha}(h_a) \tau_a \exp \left( \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m(m+1)} B_{m+1}(c_i(h_a)) \left( \frac{z}{w_i} \right)^m \right).
\] Here we used the facts that \( B_m(1-x) = (-1)^m B_m(x) \), that \( B_m = 0 \) when \( m \) is odd, and that \( c_i(h) + c_i(h^{-1}) = 1 - \delta_{c_i(h),0} \).

(3) (Dilaton leaves) To each dilaton leaf \( l \in L^1(\Gamma) \) with \( \alpha(l) = \alpha \in G^* \) and \( 2 \leq k(l) = k \in \mathbb{Z}_{\geq 0} \), we assign
\[
(L^1)^\alpha(l) = [z^{k-1}] \left( -\frac{1}{|G| \sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R(-z)^\beta \right).
\]

(4) (Edges) To an edge connecting a vertex marked by \( \alpha \in G^* \) and a vertex marked by \( \beta \in G^* \), and with heights \( k \) and \( l \) at the corresponding half-edges, we assign
\[
E_{k,l}^{\alpha,\beta}(e) = [z^k w^l] \left( \frac{1}{z+w} \left( \delta_{\alpha,\beta} - \sum_{\gamma \in G^*} R(-z)^\gamma \right) \right).
\]

(5) (Vertices) To a vertex \( v \) with genus \( g(v) = g \in \mathbb{Z}_{\geq 0} \) and with marking \( \alpha(v) = \gamma \in G^* \), with \( n \) primary or open leaves and half-edges attached to it with heights \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0} \) and \( m \) dilaton leaves with heights \( k_{n+1}, \ldots, k_{n+m} \in \mathbb{Z}_{\geq 0} \), we assign
\[
\left( |G| \sqrt{w_1 w_2 w_3} \right)^{2g-2+\text{val}(v)} \prod_{e \in E(v)} \psi_e^{k_e} \cdot \prod_{e \in E(\Gamma)} E_{k(e)}^{\alpha(e),\alpha(e)}(e).
\]

Given a labeled graph \( \bar{\Gamma} \in \Gamma_{g,l,n}(BG) \) with \( L^O(\Gamma) = \{l_1, \ldots, l_n\} \), we define its A-model weight to be
\[
w_A(\bar{\Gamma}) = \prod_{e \in V(\Gamma)} \left( |G| \sqrt{w_1 w_2 w_3} \right)^{2g-2+\text{val}(v)} \prod_{h \in H(v)} \tau_k(h) \prod_{g \in G} \prod_{e \in E(\Gamma)} E_{k(e)}^{\alpha(e),\alpha(e)}(e)
\cdot \prod_{l \in L^o(\Gamma)} (L^o)_{k(l)}^{\alpha(l)}(l) \prod_{j=1}^n (L^k)_{k(l_j)}^{\alpha(l_j)}(l_j) \prod_{l \in L^1(\Gamma)} (L^1)_{k(l)}^{\alpha(l)}(l)
\] (5.9)
Setting \( \mathbf{u} = \tau, \mathbf{u}_j = \tilde{\xi}(X_j) \), and \( w_i = w_i \) in [FLZ16, Theorem 4.2], we obtain

\[
\frac{1}{\mathcal{L}^!} \left( \tau^j, \tilde{\xi}(X_1), \ldots, \tilde{\xi}(X_n) \right)_{g,\ell+n}^X = \sum_{\mathbf{v} \in \Gamma_{g,\ell,n}(\mathcal{B}G)} \frac{w_A(\mathbf{v})}{|\text{Aut}(\mathbf{v})|}. \tag{5.10}
\]

Let \( \mathbf{\Gamma}_{g,n}(X) := \bigcup_{\ell \geq 0} \mathbf{\Gamma}_{g,\ell,n}(\mathcal{B}G) \). Equations (5.6) and (5.10) and Proposition 5.3 imply the following A-model graph sum formulae.

**Theorem 5.5 (A-model graph sum).** We have the following equations:

(i) (Disk invariants)

\[
F_{0,1}^X(\tau; X) = \Phi_{-2}^1(X) + \sum_{a=1}^{p} \tau_a \Phi_{-1}^a(X) + \sum_{\mathbf{v} \in \Gamma_{0,1}(X)} \frac{w_A(\mathbf{v})}{|\text{Aut}(\mathbf{v})|},
\]

where

\[
\Phi_{-2}^1(X) = \sum_{a=1}^{p} \tau_a \Phi_{-1}^a(X) = \frac{1}{|G|^2 w_1 w_2 w_3} \left( \sum_{\gamma \in G^+} \tilde{\xi}_{-2}(X) \right) + \sum_{a=1}^{p} \tau_a \prod_{i=1}^{3} w_{i_a} \sum_{\gamma \in G^+} \chi_{\gamma}(h_a) \tilde{\xi}_{-1}(X) \right|_{v=1}.
\tag{5.11}
\]

(ii) (Annulus invariants)

\[
F_{0,2}^X(\tau; X_1, X_2) = F_{0,2}^X(0; X_1, X_2) + \sum_{\mathbf{v} \in \Gamma_{0,2}(X)} \frac{w_A(\mathbf{v})}{|\text{Aut}(\mathbf{v})|},
\]

where \( F_{0,2}^X(0; X_1, X_2) \) is determined by equation (5.5).

(iii) For \( 2g - 2 + n > 0 \),

\[
F_{g,n}^X(\tau; X_1, \ldots, X_n) = \sum_{\mathbf{v} \in \Gamma_{g,n}(X)} \frac{w_A(\mathbf{v})}{|\text{Aut}(\mathbf{v})|}.
\]

**6. The mirror curve and its compactification**

The equation of the framed mirror curve is given by

\[
X^r Y^{-s-v} + Y^m + 1 + \sum_{a=1}^{p} q_a X^{m_a} Y^{n_a-m_a} = 0,
\]

where

\[
m_a = r c_1(h_a), \quad n_a = -s c_1(h_a) + mc_2(h_a).
\]

Let \( \triangle \) be the triangle on \( \mathbb{R}^2 \) with vertices \((r, -s), (0, m), \text{and} (0, 0)\). The compactified mirror curve \( \Sigma_q \) is embedded in the toric surface \( S_\triangle \) defined by the polytope \( \triangle \). In this section, we assume that \( q_1, \ldots, q_p \in \mathbb{C} \) are sufficiently small so that the compactified mirror curve \( \bar{\Sigma}_q \) is a smooth compact Riemann surface. Let \( g \) be the genus of \( \bar{\Sigma}_q \), and let \( n \) be the number of points in \( \bar{\Sigma}_q \setminus \Sigma_q \).
6.1 The Liouville 1-form

Let \( \tilde{\Sigma} := \pi^{-1}(\Sigma_q) \subset \mathbb{C}^2 \) be the preimage of \( \Sigma_q \subset (\mathbb{C}^*)^2 \) under the (universal) covering map \( \pi: \mathbb{C}^2 \to (\mathbb{C}^*)^2 \) defined by \((x,y) \mapsto (X = e^{-x}, Y = e^{-y})\). Then \( \pi_q := \pi|_{\tilde{\Sigma}_q}: \tilde{\Sigma}_q \to \Sigma_q \) is a regular \( \mathbb{Z}^2 \)-cover. The 1-form \( \log Y \, dX/X \) is multi-valued on \( (\mathbb{C}^*)^2 \), but \( \pi^*(\log Y \, dX/X) = ydx \) is a well-defined holomorphic 1-form on \( \mathbb{C}^2 \); indeed, it is the Liouville 1-form of the cotangent bundle \( T^*\mathbb{C} = (\mathbb{C})^2 \):

\[
d(ydx) = dy \wedge dx.
\]

We define

\[
\Phi_q := \left( \log \frac{dX}{X} \right)|_{\Sigma_q},
\]

which is a multi-valued 1-form on \( \Sigma_q \). Then \( \pi_q^*\Phi_q = ydx|_{\tilde{\Sigma}_q} \) is a well-defined holomorphic 1-form on \( \tilde{\Sigma}_q \).

Given a \( q \in \mathbb{C}^p \) such that \( \Sigma_q \) is smooth, we choose a base point \( p_q \in \Sigma_q \) and a point \( \tilde{p}_q \in \pi_q^{-1}(p_q) \). We have a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(\tilde{\Sigma}_q, \tilde{p}_q) & \xrightarrow{\pi_q^*} & \pi_1(\Sigma_q, p_q) & \xrightarrow{I_q^*} & \pi_1((\mathbb{C}^*)^2, p_q) & \longrightarrow & 1 \\
\alpha & | & \downarrow & & \downarrow & & \downarrow & & \alpha \\
H_1(\tilde{\Sigma}_q; \mathbb{Z}) & \xrightarrow{\pi_q^*} & H_1(\Sigma_q; \mathbb{Z}) & \xrightarrow{I_q^*} & H_1((\mathbb{C}^*)^2; \mathbb{Z}) & \longrightarrow & 0.
\end{array}
\]

(6.1)

In the above diagram, the following hold:

- The map \( I_q^* \) is induced by the inclusion map \( I_q: \Sigma_q \hookrightarrow (\mathbb{C}^*)^2 \).
- The first row is a short exact sequence of multiplicative groups.
- The second row is exact at \( H_1(\Sigma_q; \mathbb{Z}) = \mathbb{Z}^{2g+n-1} \) and \( H_1((\mathbb{C}^*)^2; \mathbb{Z}) = \mathbb{Z}^2 \).
- The maps \( \tilde{\alpha} \) and \( \alpha \) are surjective group homomorphisms given by abelianization, and \( \bar{\alpha} \) is a group isomorphism.
- The group \( \pi_1(\Sigma_q, p_q) \) is a free group generated by \( 2g + n - 1 \) elements.

We define

\[
K_1(\Sigma_q; \mathbb{Z}) := \text{Ker}(I_q^*): H_1(\Sigma_q; \mathbb{Z}) \to H_1((\mathbb{C}^*)^2; \mathbb{Z})) = \text{Im}(\pi_q^*: H_1(\tilde{\Sigma}_q; \mathbb{Z}) \to H_1(\Sigma_q; \mathbb{Z})).
\]

Then \( K_1(\Sigma_q; \mathbb{Z}) \cong \mathbb{Z}^{2g+n-3} \). There is a well-defined map

\[
\pi_1(\tilde{\Sigma}_q, \tilde{p}_q) \to \mathbb{C}, \quad \tilde{A} \mapsto \int_{\tilde{A}} \pi_q^*\Phi_q.
\]

Given any \( A \in K_1(\Sigma_q; \mathbb{Z}) \), there exists an \( \tilde{A} \in \pi_1(\tilde{\Sigma}_q, \tilde{p}_q) \) such that \( A = \pi_q^*\tilde{\alpha}(\tilde{A}) = \alpha \circ \pi_q^* (\tilde{A}) \). If \( A_1, A_2 \in \pi_1(\tilde{\Sigma}_q, \tilde{p}_q) \) and \( \alpha \circ \pi_q^* (A_1) = \alpha \circ \pi_q^* (A_2) = A \), then \( \pi_q^* (A_2) = \pi_q^* (A_1)B_1B_2B_1^{-1}B_2^{-1} \) for some \( B_1, B_2 \in \pi_1(\Sigma_q, p_q) \). If \( I_*(B_1) = (m_1, n_1) \) and \( I_*(B_2) = (m_2, n_2) \), where \( m_1, n_1, m_2, n_2 \in \mathbb{Z} \), then

\[
\int_{\tilde{A}_2} \pi_q^*\Phi = \int_{\tilde{A}_1} \pi_q^*\Phi + (2\pi\sqrt{-1})^2 (m_1n_2 - m_2n_1).
\]

So we have a map

\[
K_1(\Sigma_q; \mathbb{Z}) \to \mathbb{C}/(2\pi\sqrt{-1})^2 \mathbb{Z}, \quad A \mapsto \int_{\tilde{A}} \pi_q^*\Phi_q + (2\pi\sqrt{-1})^2 \mathbb{Z},
\]

(6.2)
where $\tilde{A}$ is any element in $(\pi_q \circ \tilde{a})^{-1}(A) \subset \pi_1(\tilde{\Sigma}_q, \tilde{p}_q)$. Let $\int_{\tilde{A}} \Phi$ denote the image of $A \in K_1(\Sigma_q; \mathbb{Z})$ under the above map. Then $\int_{\tilde{A}} \Phi_q \in \mathbb{C}/(2\pi \sqrt{-1})^2\mathbb{Z}$ is independent of choice of $p_q$ and $\tilde{p}_q$.

Let $B_\epsilon$ denote the open ball in $\mathbb{C}^g$ with radius $\epsilon > 0$ and center 0. There exists an $\epsilon > 0$ such that $\Sigma_q$ is smooth for all $q \in B_\epsilon$. For $q \in B_\epsilon$, there are canonical isomorphisms

$$H_1(\Sigma_q; \mathbb{Z}) \cong H_1(\Sigma_0; \mathbb{Z}), \quad K_1(\Sigma_q; \mathbb{Z}) \cong K_1(\Sigma_0; \mathbb{Z}).$$

Given an $A \in K_1(\Sigma_0; \mathbb{Z}) \cong K_1(\Sigma_q; \mathbb{Z})$, the integral $\int_{\tilde{A}} \Phi_q$ can be viewed as an element in $\Gamma(B_\epsilon, \mathbb{C})/(2\pi \sqrt{-1})^2\mathbb{Z}$, where $\Gamma(B_\epsilon, \mathbb{C})$ is the space of holomorphic functions on $B_\epsilon$ and $(2\pi \sqrt{-1})^2\mathbb{Z}$ is identified with the set of constant functions from $B_\epsilon$ to $(2\pi \sqrt{-1})^2\mathbb{Z}$.

The inclusion $J_q: \Sigma_q \hookrightarrow \tilde{\Sigma}_q$ induces a surjective map

$$J_{q*}: H_1(\Sigma_q; \mathbb{C}) \cong \mathbb{C}^{2g+n-1} \rightarrow H_1(\tilde{\Sigma}_q; \mathbb{C}) \cong \mathbb{C}^{2g},$$

which restricts to a surjective $\mathbb{C}$-linear map

$$K_1(\Sigma_q; \mathbb{C}) := K_1(\Sigma_q; \mathbb{Z}) \otimes \mathbb{C} \cong \mathbb{C}^{2g+n-3} \rightarrow H_1(\tilde{\Sigma}_q; \mathbb{C}) \cong \mathbb{C}^{2g}.$$ \hspace{1cm} (6.3)

The complex vector spaces $K_1(\Sigma_q; \mathbb{C}), H_1(\Sigma_q; \mathbb{C})$, and $H_1(\tilde{\Sigma}_q; \mathbb{C})$ form flat complex vector bundles $K$, $H_1$, and $\tilde{H}$ over $B_\epsilon$ of rank $2g+n-3$, $2g+n-1$, and $2g$, respectively. Given $A \in K_1(\Sigma_q; \mathbb{C})$ which is a flat section of $K$ with respect to the Gauss–Manin connection, $\int_{\tilde{A}} \Phi_q$ can be viewed as an element in $\Gamma(B_\epsilon, \mathbb{C})/\mathbb{C}$, where $\mathbb{C}$ is identified with the space of constant functions from $B_\epsilon$ to $\mathbb{C}$.

6.2 Holomorphic 1-forms on $\Sigma_q$

For any integers $m,n$,

$$\omega_{m,n} := \text{Res}_{h_f=0} \frac{X^m Y^{n-f_m}}{H_f} \cdot \frac{dX}{X} \wedge \frac{dY}{Y} = \frac{-X^m Y^{n-f_m}}{Y \partial H_f(X,Y,q)/\partial Y} \frac{dX}{X}$$

is a holomorphic 1-form on $\Sigma_q$. We view $X$ as a flat coordinate independent of $q$ and use implicit differentiation to obtain

$$\frac{\partial Y}{\partial q_a} = \frac{-X^m a Y^{n-a-f_m_a}}{\partial H/\partial Y}.$$ 

We define

$$\nabla_{\partial/\partial q_a} \Phi := \left. \frac{\partial \Phi}{\partial q_a} \right|_{X=\text{constant}} = \frac{\partial \Phi}{\partial q_a} \frac{dX}{Y X}.$$ 

Then $\nabla_{\partial/\partial q_a} \Phi = \omega_{m_a, n_a}$ for $a = 1, \ldots, p$.

6.3 Differentials of the first kind on $\tilde{\Sigma}_q$

Recall that a differential of the first kind is a holomorphic 1-form. By results in [BC94], the holomorphic 1-form $\omega_{m,n}$ on $\Sigma_q$ extends to a holomorphic 1-form on $\tilde{\Sigma}_q$ if and only if $(m,n)$ is in the interior of the triangle $\Delta$ with vertices $(r,-s)$, $(0,m)$, $(0,0)$. Without loss of generality, we may assume that $(m_a,n_a)$ is in the interior of $\Delta$ for $1 \leq a \leq g$ and is on the boundary of $\Delta$ for $g+1 \leq a \leq p$; note that $g$ can be zero. Let $n = p-g+3$ be the number of lattice points on the boundary of $\Delta$. Then $\Sigma_q$ is a Riemann surface of genus $g$ with $n$ punctures, and $\tilde{\Sigma}_q$ is a compact Riemann surface of genus $g$. We have

$$p = \dim_{\mathbb{C}} H^2_{CR}(X; \mathbb{C}), \quad g = \dim_{\mathbb{C}} H^4_{CR}(X; \mathbb{C}),$$

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and
\[-\chi(\Sigma_q) = 2g - 2 + n = 1 + p + g = |G| = \dim_{\mathbb{C}} H^*_C(X; \mathbb{C}).\]

A basis of $H^0(\Sigma_q, \omega_{\Sigma_q}) = H^{1,0}(\Sigma_q, \mathbb{C})$ is given by \{\(\nabla_{\partial/\partial q_a} \Phi = \omega_{m_a n_a} \): \(a = 1, \ldots, g\)\}.

Let $B_\epsilon = \{(q_1, \ldots, q_p) \in \mathbb{C}^p : |q| < \epsilon\}$ be an open ball in $\mathbb{C}^p$ with radius $\epsilon > 0$ centered at the origin. Let $\Sigma_\epsilon = \{(X, Y, q) \in \mathbb{C}^x \times \mathbb{C}^* \times \mathbb{C}^p : H_f(X, Y, q) = 0\}$ be the family of mirror curves over $B_\epsilon$, and let $\pi_\epsilon : \Sigma_\epsilon \to B_\epsilon$ be given by $(X, Y, q) \mapsto q$, so that $\pi_\epsilon^{-1}(q) = \Sigma_q$. Then $G^*$ acts on the total space $\Sigma_\epsilon$ by
\[
\alpha \cdot (X, Y, q_a) = \left(\chi_\alpha(\eta_1)X, \chi_\alpha(\eta_2)Y, \chi_\alpha(h_\alpha^{-1})q_a\right),
\]
where $\alpha \in G^*$, $\chi_\alpha : G \to \mathbb{C}^*$ is the associated character, $\eta_1, \eta_2 \in G$ are defined as in equation (2.6), and $h_\alpha \in G$ corresponds to $(m_\alpha, n_\alpha, 1) \in \text{Box}(\sigma)$ for $a = 1, \ldots, p$. The $G^*$-action on $\Sigma_\epsilon$ extends to the family of compactified mirror curve $\bar{\pi}_\epsilon : \bar{\Sigma}_\epsilon \to B_\epsilon$, where $\bar{\pi}_\epsilon^{-1}(q) = \bar{\Sigma}_q$. In particular, $G^*$ acts on $\bar{\Sigma}_0$, and the induced $G^*$-action on $H_1(\bar{\Sigma}_0; \mathbb{C})$ preserves the intersection pairing $\cdot \mid H_1(\bar{\Sigma}_0; \mathbb{C})$. We choose a symplectic basis $\bar{A}_1, \bar{B}_1, \ldots, \bar{A}_g, \bar{B}_g$ of $H_1(\bar{\Sigma}_0; \mathbb{C})$ such that
\[
\bar{A}_a \cdot \bar{A}_b = \bar{B}_a \cdot \bar{B}_b = 0, \quad \bar{A}_a \cdot \bar{B}_b = \delta_{ab}, \quad a, b = 1, \ldots, g.
\]
and
\[
\frac{1}{2\pi \sqrt{-1}} \int_{\bar{A}_a} \left(\nabla_{\partial/\partial q_b} \Phi\right)|_{q = 0} = \delta_{ab}, \quad a, b = 1, \ldots, g.
\]

The $G^*$-action on $\bar{A}_a$ is given by $\alpha \cdot \bar{A}_a = \chi_\alpha(h_\alpha^{-1})\bar{A}_a$, where $\alpha \in G^*$ and $1 \leq a \leq g$. We may choose $B_\epsilon$ such that $\alpha \cdot \bar{A}_a = \chi_\alpha(h_\alpha^{-1})\bar{B}_a$ for $\alpha \in G^*$ and $1 \leq a \leq g$.

The inclusion $I_0 : \Sigma_0 \to (\mathbb{C}^*)^2$ is $G^*$-equivariant, where $\alpha \in G^*$ acts on $(\mathbb{C}^*)^2$ by $\alpha \cdot (X, Y) = (\chi_\alpha(\eta_1)X, \chi_\alpha(\eta_2)Y)$. So the $G^*$-actions on $H_1((\mathbb{C}^*)^2; \mathbb{Z})$ and $H_1((\mathbb{C}^*)^2; \mathbb{C})$ are trivial, and the $G^*$-action on $H_1(\Sigma_0; \mathbb{C})$ preserves $K_1(\Sigma_0; \mathbb{C}) = \text{Ker}(I_* : H_1(\Sigma_0; \mathbb{C}) \to H_1((\mathbb{C}^*)^2; \mathbb{Z}))$.

We extend $\bar{A}_1, \bar{B}_1, \ldots, \bar{A}_g, \bar{B}_g$ to a symplectic basis of $H_1(\bar{\Sigma}_q; \mathbb{C})$ such that $\bar{A}_i$ and $\bar{B}_i$ are flat sections of $\mathcal{H}$ with respect to the Gauss–Manin connection. Note that the composition

$$K_1(\Sigma_q; \mathbb{C}) \to H_1(\Sigma_q; \mathbb{C}) \xrightarrow{\iota^*} H_1(\bar{\Sigma}_q; \mathbb{C})$$

is surjective. For $a = 1, \ldots, g$, we lift $\bar{A}_a, \bar{B}_a \in H_1(\bar{\Sigma}_q; \mathbb{C})$ to $A_a, B_a \in K_1(\Sigma_q; \mathbb{C})$. We choose $A_a, B_a \in K_1(\Sigma_q; \mathbb{C})$ such that they are flat sections of $\mathcal{K}$ with respect to the Gauss–Manin connection and are eigenvectors of the $G^*$-action on $K_1(\Sigma_0; \mathbb{C})$ at $q = 0$.

### 6.4 Differentials of the third kind on $\bar{\Sigma}_q$

Recall that a differential of the third kind is a meromorphic 1-form with only simple poles.

Let $E_1, E_2, E_3$ be the edges of the triangle $\Delta$ opposite to the vertices $(r, s), (0, m), (0, 0)$, respectively. Let $n_i + 1$ be the number of lattice points on $E_i$ (including the end points). Then $G_i := \{h \in G : \chi_i(h) = 1\} \cong \mathbf{\mu}_{n_i}$; in particular, $n_1 = m$. For $i = 1, 2, 3$, we have short exact sequences of abelian groups

$$1 \to G_i = \mathbf{\mu}_{n_i} \to G \xrightarrow{\chi_i} \mathbf{\mu}_{n_i} \to 1, \quad 1 \to \mathbf{\mu}_{n_i}^* \to G^* \to G_i^* \to 1.$$  

The $G^*$-action on $\bar{\Sigma}_0$ preserves the finite set $\bar{\Sigma}_0 \cap D_i$, where $D_i \subset S_\Delta$ is the torus-invariant divisor associated with the edge $E_i$. The $G^*$-action on $\bar{\Sigma}_0 \cap D_i$ induces a free and transitive $G_i^*$-action on $\bar{\Sigma}_0 \cap D_i$. So we may label the point in $\bar{\Sigma}_0 \cap D_i$ (non-canonically) by elements in $G_i^* = \mathbf{\mu}_{n_i}^*$; that is, $\bar{\Sigma}_0 \cap D_i = \{\bar{p}_i^\ell : \ell \in \mathbf{\mu}_{n_i}^*\}$.

If $g + 1 \leq a \leq p$, then $h_a \in G_i \setminus \{1\} \cong \mathbf{\mu}_{n_i} \setminus \{1\}$ for some $i \in \{1, 2, 3\}$. The meromorphic 1-form $\nabla_{\partial/\partial q_a} \Phi$ has a simple pole at each point in $\{\bar{p}_i^\ell : \ell \in G_i^*\}$ and is holomorphic elsewhere.
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on the compactified mirror curve \( \bar{\Sigma}_q \), so it is a differential of the third kind on \( \bar{\Sigma}_q \). We have

\[
\text{Res}_{p \to \bar{p}_i} \left( \nabla_{\partial/\partial q_a} \Phi \right) |_{q=0} = \frac{e^{i\pi \sqrt{-1}k/n_i}}{n_i} \chi(k),
\]

where \( h_a = e^{2\pi i k/n_i} \in \mathbb{C}_{\mathbb{A}_1} \) for \( 1 \leq k \leq n_i - 1 \) and \( \chi : G_i \to \mathbb{C}^* \) is the character associated with \( \ell \in G_i^* \). Let \( C^\ell \in H_1(\Sigma_0; \mathbb{Z}) \) be the class of a small circle around \( \bar{p}_i \). Then there exists an \( A_a \in K_1(\Sigma_0; \mathbb{C}) (a = g + 1, \ldots, p) \) such that

- \( A_a \) is an eigenvector of the \( G^* \)-action on \( K_1(\Sigma_0; \mathbb{C}) \);
- \( A_a \) is a \( \mathbb{C} \)-linear combination of \( C^\ell \), with \( \ell \in G_i^* \);
- for \( a, b = 1, \ldots, p \),

\[
\frac{1}{2\pi i} \int_{A_a} \left( \nabla_{\partial/\partial q_a} \Phi \right) |_{q=0} = \delta_{ab}.
\]

We extend \( A_{g+1}, \ldots, A_p \) to flat sections of \( K_1(\Sigma_q; \mathbb{C}) \). Then for \( a, b \in \{1, \ldots, p\} \),

\[
\frac{1}{2\pi i} \int_{A_a} \Phi = \tau_a(q) + \mathbb{C} \in \Gamma(B_\ell, \mathbb{C})/\mathbb{C}.
\]

Then \( \tau_a(q) = q_a + O(|q|^2) \) and

\[
\frac{1}{2\pi i} \int_{A_a} \nabla_{\partial/\partial q_a} \Phi = \delta_{ab}, \quad a, b = 1, \ldots, p.
\]

The B-model closed string flat coordinates are related to \( q_a \) for \( a = 1, \ldots, p \) via the following hypergeometric formulae:

\[
\tau_a(q) = q_a \left( \prod_{d_i \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^3 \Gamma\left( -\left\{ c_i(h_a) \right\} + 1 \right) \cdot \frac{1}{\prod_{b=1}^p d_b! \prod_{b=1}^p d_b} \right).
\]

(6.5)

Here \( \{x\} \) denotes the fractional part of \( x \). This mirror map is obtained by solving the Gelfand–Kapranov–Zelevinsky-type (GKZ-type for short) Picard–Fuchs equations. Iritani explains these GKZ-operators and explicitly writes down the mirror map for general toric orbifolds [Iri09]. The integrals over A-cycles on mirror curves as flat coordinates and their hypergeometric expressions for toric Calabi–Yau 3-folds are discussed in [CKYZ99] together with genus zero closed Gromov–Witten mirror symmetry.

Let \( D_q^\infty = \bar{\Sigma}_q \setminus \Sigma_q = \bigcup_{i=1}^3 \{ \bar{p}_i, \ldots, \bar{p}_{n_i} \} \). By Lefschetz duality, there is a perfect pairing

\[
H_1(\Sigma_q; \mathbb{C}) \times H_1(\Sigma_q, D_q^\infty; \mathbb{C}) \to \mathbb{C}.
\]

(6.6)

The inclusion \( \Sigma_q \subset \bar{\Sigma}_q \) induces a surjective \( \mathbb{C} \)-linear map

\[
H_1(\Sigma_q; \mathbb{C}) \cong \mathbb{C}^{2g+n-1} \to H_1(\Sigma_q, D_q^\infty; \mathbb{Z}) \cong \mathbb{C}^{2g}.
\]

Our choice of \( A_1, B_1, \ldots, A_g, B_g \) gives a splitting

\[
j_1 : H_1(\bar{\Sigma}_q; \mathbb{C}) \to H_1(\Sigma_q; \mathbb{C}).
\]

(6.7)
The long exact sequence of relative homology groups of the pair \((\bar{\Sigma}_q, D_q^\infty)\) gives an injective \(\mathbb{C}\)-linear map
\[
j_2 : H_1(\bar{\Sigma}_q; \mathbb{C}) \to H_1(\bar{\Sigma}_q, D_q^\infty; \mathbb{C}).
\] (6.8)
Under the inclusion maps \(j_1\) and \(j_2\), the perfect pairing (6.6) restricts to the intersection pairing
\[
H_1(\bar{\Sigma}_q; \mathbb{C}) \times H_1(\bar{\Sigma}_q; \mathbb{C}) \to \mathbb{C},
\]
which is perfect by Poincaré duality. For \(g + 1 \leq a \leq p\), choose \(B_a \in H_1(\bar{\Sigma}_q, D_q^\infty; \mathbb{C})\) such that
- \((A_1, \ldots, A_p, B_1, \ldots, B_g)\) and \((B_1, \ldots, B_p, -A_1, \ldots, -A_g)\) are dual under the pairing (6.6);
- \(\alpha \cdot B_a = \chi_{\alpha}(h_a^{-1})B_a\) for \(\alpha \in G^*\), and \(g + 1 \leq a \leq p\).

Then for \(a = 1, \ldots, p\),
\[
(\nabla_{\partial/\partial r_a} \Phi)(z) = \int_{z' \in B_a} B(z', z),
\]
where \(B(z', z)\) is the fundamental normalized differential of the second kind on \(\bar{\Sigma}_q\) (see Section 6.6).

### 6.5 Critical points and Lefschetz thimbles

We choose a framing \(f\) such that \(X : \Sigma_0 \to \mathbb{C}^*\) has only simple branch points. Then \(X : \Sigma_q \to \mathbb{C}^*\) has only simple branch points for \(q\) sufficiently small.

The critical points of \(X : \Sigma_0 = \{(X, Y) \in (\mathbb{C}^*)^2 : X^r Y^{-s-rf} + Y^m + 1 = 0\} \to \mathbb{C}^*\) are
\[
\{(X_{j\ell}, Y_{j\ell}) : j \in \{0, 1, \ldots, r - 1\}, \ell \in \{0, 1, \ldots, m - 1\}\},
\]
where
\[
X_{j\ell} = \exp \left( 2\pi \sqrt{-1} \left( \frac{j}{r} + \frac{\ell s + rf}{rm} \right) \right) m^{1/r} (s + rf)^{(s+rf)/rm} (-m - s - rf)^{(-m-s-rf)/rm}
\]
\[
= (\chi_1^j \chi_2^\ell)(\eta_1) \prod_{i=1}^3 (G|w_i)^{w_i},
\]
\[
Y_{j\ell} = \exp \left( 2\pi \sqrt{-1} \left( \frac{\ell}{m} \right) \right) \left( \frac{s + rf}{-m - s - rf} \right)^{1/m} = (\chi_1^j \chi_2^\ell)(\eta_2) \left( \frac{w_2}{w_3} \right)^{1/m}.
\]
If \(\alpha = \chi_1^j \chi_2^\ell \in G^*\), then we define \((X_\alpha, Y_\alpha) = (X_{j\ell}, Y_{j\ell})\). The critical points of \(X : \Sigma_0 \to \mathbb{C}^*\) are \(\{(X_\alpha, Y_\alpha) : \alpha \in G^*\}\). We define
\[
a_\alpha := -\log(X_\alpha) = -\sum_{i=1}^3 w_i \log w_i - \log(\chi_{\alpha}(\eta_1)),
\] (6.9)
\[
b_\alpha := -\log(Y_\alpha) = \frac{1}{m} \log(\frac{w_3}{w_2}) - \log(\chi_{\alpha}(\eta_2)).
\] (6.10)

In equation (6.9), we use the identity \(\sum_{i=1}^3 w_i \log(|G|w_i) = \sum_{i=1}^3 w_i \log w_i\), which holds since \(\sum_{i=1}^3 w_i = 0\). Note that
\[
\log(\chi_{\alpha}(\eta_1)) = \sqrt{-1} \vartheta\alpha, \quad \log(\chi_{\alpha}(\eta_2)) = \sqrt{-1} \varphi\alpha
\]
for some \(\vartheta\alpha, \varphi\alpha \in \mathbb{R}\). We may assume \(\vartheta\alpha, \varphi\alpha \in [0, 2\pi]\).

Let \(X = e^{-x}\) and \(Y = e^{-y}\). Around each critical point \(p_\alpha\), we set up the following local
coordinates

\[ x = a_\alpha(q) + \zeta_\alpha(q)^2, \quad y = b_\alpha(q) + \sum_{d=1}^{\infty} h_d^\alpha(q) \zeta_\alpha(q)^d, \]

where

\[ h_1^\alpha(q) = \sqrt{\frac{2}{(d^2x/dy^2)(b_\alpha(q))}}. \]

Let \( \gamma_\alpha \) be the Lefschetz thimble of the superpotential \( x = -\log X : \Sigma_q \to \mathbb{C} \), so that \( x(\gamma_\alpha) = a_\alpha + \mathbb{R}^+ \). Then \( \{ \gamma_\alpha : \alpha \in G^* \} \) is a basis of the relative homology group

\[ H_1(\Sigma_q, \{(X,Y) \in \Sigma_q : \log X \ll 0\}; \mathbb{Z}) \cong \mathbb{Z}^{|G^*|}. \]

### 6.6 Differentials of the second kind on \( \bar{\Sigma}_q \)

Recall that a differential of the second kind is a meromorphic 1-form with no residues.

We choose a symplectic basis \( \{A_1, B_1, \ldots, A_g, B_g\} \) of \( H_1(\bar{\Sigma}_q, \mathbb{C}) \) as in Section 6.3. Let \( B(p_1, p_2) \) be the fundamental normalized differential of the second kind on \( \bar{\Sigma}_q \) (see, for example, [Fay73]), which is characterized by the following properties:

1. The differential \( B(p_1, p_2) \) is bilinear symmetric meromorphic on \( \bar{\Sigma}_q \times \bar{\Sigma}_q \).
2. The differential \( B(p_1, p_2) \) is holomorphic everywhere except for a double pole along the diagonal \( p_1 = p_2 \). If \( z_1, z_2 \) are local coordinates on \( \bar{\Sigma}_q \times \bar{\Sigma}_q \), then

\[ B(z_1, z_2) = \left( \frac{1}{(z_1 - z_2)^2} + f(z_1, z_2) \right) dz_1 dz_2, \]

where \( f(z_1, z_2) \) is holomorphic.
3. \( \int_{p_i \in A_i} B(p_1, p_2) = 0 \) for \( i = 1, \ldots, g \).

Let \( p_\alpha(q) = (X_\alpha(q), Y_\alpha(q)) \) be the branch point of \( X : \bar{\Sigma}_q \to \mathbb{C}^* \) such that

\[ \lim_{q \to 0}(X_\alpha(q), Y_\alpha(q)) = (X_\alpha, Y_\alpha). \]

Following [Eyn14, EO15], given \( \alpha \in G^* \) and \( d \in \mathbb{Z}_{\geq 0} \), define

\[ \theta_d^\alpha(p) := -(2d - 1)!2^{-d} \text{ Res}_{p' \sim p_\alpha} B(p, p') \zeta_\alpha^{-2d-1}. \quad (6.11) \]

(In this paper, we use the symbol \( \theta_d^\alpha \) instead of the symbol \( d\xi_{\alpha,d} \) in [Eyn11, EO15] because the 1-form defined by the right-hand side of (6.11) is not necessarily exact.) Then \( \theta_d^\alpha \) satisfies the following properties:

1. The form \( \theta_d^\alpha \) is a meromorphic 1-form on \( \bar{\Sigma}_q \) with a single pole of order \( 2d + 2 \) at \( p_\alpha \).
2. In local coordinate \( \zeta_\alpha \) near \( p_\alpha \),

\[ \theta_d^\alpha = \left( -\frac{(2d + 1)!!}{2^d \zeta_\alpha^{2d+2}} + f(\zeta_\alpha) \right) d\zeta_\alpha, \]

where \( f(\zeta_\alpha) \) is analytic around \( p_\alpha \). The residue of \( \theta_d^\alpha \) at \( p_\alpha \) is zero, so \( \theta_d^\alpha \) is a differential of the second kind.
3. \( \int_{A_i} \theta_d^\alpha = 0 \) for \( i = 1, \ldots, g \).
The meromorphic 1-form $\theta^\alpha_d$ is characterized by the above properties; $\theta^\alpha_d$ can be viewed as a section in $H^0(\Sigma_q, \omega_{\Sigma_q}((2d + 2)p_\alpha))$.

**Lemma 6.2.** Suppose that $f$ is a meromorphic function on $\tilde{\Sigma}_q$ with simple poles at the ramification points $\{p_\beta: \beta \in G^*\}$ and is holomorphic on $\Sigma_q \setminus \{p_\beta: \beta \in G^*\}$. Then

$$df = \sum_{\beta \in G^*} c_\beta \theta^\beta_0,$$

where $c_\beta = \lim_{p \to p_\beta} f \zeta_\beta$.

**Proof.** Let $\Delta \omega = df - \sum_{\beta \in G^*} c_\beta \theta^\beta_0$. Then $\Delta \omega$ is a holomorphic 1-form on $\tilde{\Sigma}_\alpha$ and $\int_{A_i} \Delta \omega = 0$ for $i = 1, \ldots, g$. So $\Delta \omega = 0$.

**Remark 6.3.** In Lemma 6.2, different choice of $A$-cycles $A_1, \ldots, A_g$ will give different meromorphic 1-forms $\theta^\beta_0$, but the equality

$$df = \sum_{\beta \in G^*} c_\beta \theta^\beta_0$$

still holds, where the coefficients $c_\beta = \lim_{p \to p_\beta} f \zeta_\beta$ do not depend on the choice of $A$-cycles.

We make the following observations:

1. The form $dx = -dX/X$ is a meromorphic 1-form on $\tilde{\Sigma}_q$ which is holomorphic on $\Sigma_q$. It has a simple zero at each of the $|G|$ ramification points $\{p_\alpha: \alpha \in G^*\}$ and a simple pole at each of the $n$ punctures $\tilde{p}_1, \ldots, \tilde{p}_n$.

2. The form $dy = -dY/Y$ is a meromorphic 1-form on $\tilde{\Sigma}_q$ and a holomorphic 1-form on $\Sigma_q$. It is non-zero at each of the $|G|$ ramification points and has at most a simple pole at each of the $n$ punctures.

3. For $a = 1, \ldots, g$, the form $\nabla_{\partial/\partial q_a} \Phi$ is a holomorphic 1-form on $\tilde{\Sigma}_q$ which is non-zero at each of the $|G|$ ramification points.

4. For $a = g + 1, \ldots, p$, the form $\nabla_{\partial/\partial q_a} \Phi$ is a meromorphic 1-form on $\tilde{\Sigma}_q$ which is holomorphic on $\Sigma_q$. It is non-zero at each of the $|G|$ ramification points and has at most a simple pole at each of the $n$ punctures.

On the basis of the above observations, the following are meromorphic function on $\tilde{\Sigma}_q$ satisfying the assumption of Lemma 6.2:

$$\frac{dy}{dx}, \quad \frac{\nabla_{\partial/\partial q_a} \Phi}{dx} = \frac{\partial y}{\partial q_a},$$

where $a = 1, \ldots, p$ and $\alpha \in G^*$.

**Proposition 6.4.** We have

$$d \left( \frac{dy}{dx} \right) = \frac{1}{2} \sum_{\beta \in G^*} h_1^\beta(q) \theta^\beta_0. \tag{6.12}$$

For $a = 1, \ldots, p$,

$$d \left( \frac{\nabla_{\partial/\partial q_a} \Phi}{dx} \right) = \frac{1}{2} \sum_{\beta \in G^*} \frac{X_\beta(q)^m X_\beta(q)^{m_a - f_{m_a}}}{(\partial H/\partial x)(X_\beta(q), Y_\beta(q), q)} h_1^\beta(q) \theta^\beta_0. \tag{6.13}$$

Equation (6.12) was proved in [EO15, Appendix D]; we include it for completeness.
Proof of Proposition 6.4. Near $p_\beta$, 

$$\frac{dy}{dx} \zeta_\beta = \sum_{k=1}^\infty h_k^\beta(q)k(\zeta_\beta)k^sd\zeta_\beta = \frac{1}{2} \sum_{k=1}^\infty h_k^\beta(q)k(\zeta_\beta)^{k-1},$$

$$\nabla_{\partial/\partial q_\alpha} \Phi = \frac{-X^{ma}Y^{na-f_{ma}}}{(\partial H/\partial q)(X,Y,q)} \zeta_\beta = \frac{X^{ma}Y^{na-f_{ma}}}{(\partial H/\partial x)(X,Y,q)} \frac{dy}{dx} \zeta_\beta.$$ 

So 

$$\lim_{p\to p_\beta} \frac{dy}{dx} \zeta_\beta = \frac{h_1^\beta(q)}{2}, \quad \lim_{p\to p_\beta} \frac{\nabla_{\partial/\partial q_\alpha} \Phi}{dx} \zeta_\beta = \frac{X_\beta(q)^{ma}Y_\beta(q)^{na-f_{ma}}}{2}. $$

The proposition follows from Lemma 6.2. \qed

Proposition 6.5. We have 

$$d \left( \frac{\nabla_{\partial/\partial r_\alpha} \Phi}{dx} \right) = \frac{1}{2} \sum_{\beta \in G^*} \left( \frac{\partial H}{\partial \tau_a} / \partial H \right) \left|_{X=X_\beta(q),Y=Y_\beta(q)} \cdot \frac{h_1^\beta(q)}{2} \theta_0^\beta, \right. \tag{6.14}$$

$$\nabla_{\partial/\partial r_\alpha} \nabla_{\partial/\partial r_\beta} \Phi = \frac{1}{2} \sum_{\beta \in G^*} \left( \frac{\partial H}{\partial \tau_a} / \partial H \right) \cdot \left( \frac{\partial H}{\partial \tau_b} / \partial H \right) \left|_{X=X_\beta(q),Y=Y_\beta(q)} \cdot \frac{h_1^\beta(q)}{2} \theta_0^\beta. \right. \tag{6.15}$$

Proof. Equation (6.14) follows from equation (6.13). It remains to prove (6.15).

By a special geometry property of the topological recursion ([EO15, Theorem 4.4], proved in [EO07]), 

$$\left( \nabla_{\partial/\partial r_\alpha} \nabla_{\partial/\partial r_\beta} \Phi \right)(p) = \int_{p_1 \in B_a} \int_{p_2 \in B_b} \omega_{0,3}(p_1,p_2,p),$$

where (see Example 7.13) 

$$\omega_{0,3}(p_1,p_2,p) = \sum_{\beta \in G^*} \frac{-1}{2h_1^\beta} \theta_0^\beta(p_1) \theta_0^\beta(p_2) \theta_0^\beta(p).$$

So $\nabla_{\partial/\partial r_\alpha} \nabla_{\partial/\partial r_\beta} \Phi$ is a linear combination of $\theta_0^\beta$: 

$$\nabla_{\partial/\partial r_\alpha} \nabla_{\partial/\partial r_\beta} \Phi = \sum_{\beta \in G^*} c_\beta \theta_0^\beta,$$

where the coefficient $c_\beta$ is given by $c_\beta = \text{Res}_{\zeta_\beta \to 0} \left( \zeta_\beta \cdot \nabla_{\partial/\partial r_\alpha} \nabla_{\partial/\partial r_\beta} \Phi \right)$. We have 

$$\nabla_{\partial/\partial r_\alpha} \nabla_{\partial/\partial r_\beta} \Phi = \sum_{c,d} \frac{\partial q_c}{\partial \tau_a} \frac{\partial q_d}{\partial \tau_b} \nabla_{\partial/\partial q_c} \nabla_{\partial/\partial q_d} \Phi + \sum_c \frac{\partial^2 q_c}{\partial \tau_a \partial \tau_b} \nabla_{\partial/\partial q_c} \Phi,$$

where $\nabla_{\partial/\partial q_c} \Phi$ is holomorphic on $\Sigma_q$, and 

$$\nabla_{\partial/\partial q_c} \nabla_{\partial/\partial q_d} \Phi = \left( \frac{\partial H}{\partial q_a} \frac{\partial H}{\partial q_b} \frac{\partial^2 H}{\partial q_c \partial q_d} \partial y/ \partial y^2 \right) \left( \frac{\partial H}{\partial y} \right)^3 + \left( \frac{\partial^2 H}{\partial q_c \partial q_d} \frac{\partial \partial H}{\partial \partial y} \frac{\partial H}{\partial \partial y} \frac{\partial H}{\partial \partial y} / \partial H / \partial y^2 \right) \left( \frac{\partial H}{\partial y} \right)^2,$$

So 

$$c_\beta = \frac{\partial H}{\partial \tau_a} \frac{\partial H}{\partial \tau_b} \bigg|_{X=X_\beta(q),Y=Y_\beta(q)} \cdot \text{Res}_{\zeta_\beta \to 0} \left( \zeta_\beta \left(- \frac{\partial^2 H}{\partial y^2} \right) / \partial \partial y \right)^3 \left( \frac{\partial H}{\partial y} \right)^2 \bigg|_{X=X_\beta(q),Y=Y_\beta(q)} \cdot \frac{h_1^\beta(q)}{2}. \qed
Given $\alpha, \beta \in G^*$ and $k, l \in \mathbb{Z}_{\geq 0}$, define $B_{k,l}^{\alpha,\beta}$ to be the coefficients of the expansion of $B(p_1, p_2)$ near $(p_\alpha, p_\beta) \in \Sigma_q \times \Sigma_q$ in coordinates $\zeta_1 = \zeta_\alpha(p_1)$ and $\zeta_2 = \zeta_\beta(p_2)$. Then
\[ B(p_1, p_2) = \left( \frac{\delta_{\alpha,\beta}}{\zeta_1 - \zeta_2} + \sum_{k,l \in \mathbb{Z}_{\geq 0}} B_{k,l}^{\alpha,\beta} \zeta_1^k \zeta_2^l \right) d\zeta_1 d\zeta_2. \]
We define
\[ \tilde{B}_{k,l}^{\alpha,\beta} = \frac{(2k - 1)!!(2l - 1)!!}{2^{k+l+1}} B_{2k,2l}^{\alpha,\beta}. \] (6.16)

The following lemma is the differential of [Eyn11, Equation (D.4)] and holds globally on the compactified mirror curve.

**Lemma 6.6.** We have
\[ \theta_{k+1}^\alpha = -d \left( \frac{\theta_k^\alpha}{dx} \right) - \sum_{\beta \in G^*} \tilde{B}_{k,0}^{\alpha,\beta} \theta_0^\beta. \]

**Proof.** We have the following Laurent series expansion of the meromorphic function $\theta_k^\alpha/dx$ near $p_\beta$ in the local coordinate $\zeta_\beta$:
\[ \frac{\theta_k^\alpha}{dx} = \delta_{\alpha,\beta} - \frac{(2d + 1)!!}{2^{d+1} \zeta_\beta^{2d+3}} \tilde{B}_{k,0}^{\alpha,\beta} + h(\zeta_\beta), \]
where $h(\zeta_\beta)$ is a power series in $\zeta_\beta$. Set $\Delta \omega = \theta_{k+1}^\alpha + d \left( \theta_k^\alpha/dx \right) + \sum_{\beta \in G^*} \tilde{B}_{k,0}^{\alpha,\beta} \theta_0^\beta$. Then $\Delta \omega$ is a holomorphic 1-form on $\bar{\Sigma}_q$, and $\int_{A_i} \Delta \omega = 0$ for $i = 1, \ldots, g$. So $\Delta \omega = 0$. \hfill \Box

Given $\alpha \in G^*$ and $k \in \mathbb{Z}_{>0}$, we define
\[ \hat{\xi}_{\alpha,k} := (-1)^k \left( \frac{d}{dx} \right)^{k-1} \left( \frac{\theta_0^\alpha}{dx} \right) = \left( X \frac{d}{dX} \right)^{k-1} \left( X \frac{\theta_0^\alpha}{dX} \right), \] (6.17)
which is a meromorphic function on $\bar{\Sigma}_q$. We define $d\hat{\xi}_{\alpha,0} := \theta_0^\alpha$.

**Lemma 6.7.** We have
\[ \theta_k^\alpha = d\hat{\xi}_{\alpha,k} - \sum_{i=0}^{k-1} \sum_{\beta \in G^*} \tilde{B}_{k-1-i,0}^{\alpha,\beta} d\hat{\xi}_{\beta,i}. \]

**Proof.** We prove the result by induction on $k$. When $k = 0$, we have $\theta_0^\alpha = d\hat{\xi}_{\alpha,0}$ by definition. Suppose that the lemma holds for $k = d$. By Lemma 6.6,
\[ \theta_{d+1}^\alpha = -d \left( \frac{\theta_d^\alpha}{dx} \right) - \sum_{\beta \in G^*} \tilde{B}_{d,0}^{\alpha,\beta} \theta_0^\beta \]
\[ = -d \left( \frac{d\hat{\xi}_{\alpha,d}}{dx} - \sum_{i=0}^{d-1} \sum_{\beta \in G^*} \tilde{B}_{d-1-i,0}^{\alpha,\beta} \frac{d\hat{\xi}_{\beta,i}}{dx} \right) - \sum_{\beta \in G^*} \tilde{B}_{d,0}^{\alpha,\beta} d\hat{\xi}_{\beta,0} \]
\[ = d\hat{\xi}_{\alpha,d+1} - \sum_{i=0}^{d-1} \sum_{\beta \in G^*} \tilde{B}_{d-1-i,0}^{\alpha,\beta} d\hat{\xi}_{\beta,i+1} - \sum_{\beta \in G^*} \tilde{B}_{d,0}^{\alpha,\beta} d\hat{\xi}_{\beta,0} \]
\[ = d\hat{\xi}_{\alpha,d+1} - \sum_{i=0}^{d} \sum_{\beta \in G^*} \tilde{B}_{d-i,0}^{\alpha,\beta} d\hat{\xi}_{\beta,i}. \]
The second equality follows from the induction hypothesis. So the lemma holds for $k = d + 1$. \hfill \Box

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7. B-model topological string

7.1 Laplace transform

The Laplace transform of a meromorphic 1-form $\lambda$ along a Lefschetz thimble $\gamma_\alpha$ is given by $\int_{z \in \gamma_\alpha} e^{-ux} \lambda$.

**Definition 7.1.** We have $f_\beta^\alpha(u,q) := \frac{e^{u\alpha}}{2\pi i u} \int_{z \in \gamma_\alpha} e^{-ux} \theta_0^\beta$.

It is straightforward to check that $f_\beta^\alpha(u,q) = \delta_{\alpha\beta} + O(u^{-1})$.

The Eynard–Orantin invariants $\{\omega_{g,n}\}$ of the mirror curve can be expressed as a graph sum involving $f_\beta^\alpha(u,q)$, $\theta_0^\beta$, and descendant integrals over moduli spaces of stable curves [KO10, Eyn11, Eyn14, DOSS14]. The goal of this subsection and the next two subsections is to relate $f_\beta^\alpha(u,q)$ and $\theta_0^\beta$ to terms in the A-model graph sum. The following is our strategy:

1. In this section (Section 7.1), we relate $f_\beta^\alpha(u,q)$ to the following Laplace transforms:

$$\int_{\gamma_\alpha} e^{-ux} \Phi, \quad \int_{\gamma_\alpha} e^{-ux} \nabla_{\partial/\partial \tau_a} \Phi, \quad \int_{\gamma_\alpha} e^{-ux} \nabla_{\partial/\partial \tau_b} \Phi, \quad \int_{\gamma_\alpha} e^{-ux} \nabla_{\partial/\partial \tau_a} \nabla_{\partial/\partial \tau_b} \Phi. \quad (7.1)$$

2. In Section 7.2, we evaluate the oscillatory integrals in the Landau–Ginzburg mirror of $\mathcal{X} = \mathbb{C}^3/G$ for any small $q$. These oscillatory integrals can be identified with the Laplace transforms in (7.1) by dimensional reduction.

3. In Section 7.3, we expand an antiderivative of $\theta_0^\beta$ near $X = 0$ and relate it to $\bar{\theta}^\beta_0$ from Section 5.2.

By equation (6.12),

$$\int_{z \in \gamma_\alpha} e^{-ux} \Phi = u^{-2} \int_{z \in \gamma_\alpha} e^{-ux} \left( \frac{dy}{dx} \right) = \frac{1}{2} u^{-2} \sum_{\beta \in G^*} h_1^\beta \int_{z \in \gamma_\alpha} e^{-ux} d\xi_{\beta,0} = \sqrt{\pi} u^{-3/2} e^{-u\alpha} \sum_{\beta \in G^*} h_1^\beta f_\beta^\alpha(u,q).$$

By equation (6.14),

$$\int_{z \in \gamma_\alpha} e^{-ux} \nabla_{\partial/\partial \tau_a} \Phi = -u^{-1} \int_{z \in \gamma_\alpha} e^{-ux} \left( \nabla_{\partial/\partial \tau_a} \Phi \right) = -\frac{1}{2} u^{-1} \sum_{\beta \in G^*} \left( \frac{\partial H}{\partial \tau_a} \frac{\partial H}{\partial x} \right) \bigg|_{X=X_\beta,Y=Y_\beta} \cdot h_1^\beta \int_{z \in \gamma_\alpha} e^{-ux} \theta_0^\beta$$

$$= -\sqrt{\pi} u^{-1/2} e^{-u\alpha} \sum_{\beta \in G^*} \left( \frac{\partial H}{\partial \tau_a} \frac{\partial H}{\partial x} \right) \bigg|_{X=X_\beta,Y=Y_\beta} \cdot h_1^\beta (q) f_\beta^\alpha(u,q).$$

By equation (6.15),

$$\int_{z \in \gamma_\alpha} e^{-ux} \nabla_{\partial/\partial \tau_b} \nabla_{\partial/\partial \tau_a} \Phi = \frac{1}{2} \sum_{\beta \in G^*} \left( \frac{\partial H}{\partial \tau_a} \frac{\partial H}{\partial x} \right) \cdot \left( \frac{\partial H}{\partial \tau_b} \frac{\partial H}{\partial x} \right) \bigg|_{X=X_\beta,Y=Y_\beta} \cdot h_1^\beta \int_{z \in \gamma_\alpha} e^{-ux} \theta_0^\beta$$

$$= \sqrt{\pi} u^{1/2} e^{-u\alpha} \sum_{\beta \in G^*} \left( \frac{\partial H}{\partial \tau_a} \frac{\partial H}{\partial x} \right) \cdot \left( \frac{\partial H}{\partial \tau_b} \frac{\partial H}{\partial x} \right) \bigg|_{X=X_\beta,Y=Y_\beta} \cdot h_1^\beta (q) f_\beta^\alpha(u,q).$$
In the remainder of this subsection, we consider the limit \( q \to 0 \). We have

\[
\lim_{q \to 0} h^\beta_1(q) = \frac{1}{|G|} \sqrt{-\frac{2}{w_1 w_2 w_3}},
\]

where \( h_a \in G \) corresponds to \((m_a, n_a, 1) \in \text{Box}(\sigma)\), so that \( \text{age}(h_a) = 1 \).

We introduce some notation. For \( h \in G \), let

\[
\nabla_h \Phi = \begin{cases} 
\Phi, & h = 1, \\
\nabla_{\partial/\partial r_a} \Phi, & h = h_a, \\
\nabla_{\partial/\partial r_b} \nabla_{\partial/\partial r_a} \Phi, & h = h_a h_b \text{ and age}(h) = 2.
\end{cases}
\]

Then

\[
\lim_{q \to 0} \int_{z \in \gamma_\alpha} e^{-ux} \nabla_h \Phi = \frac{\sqrt{-2\pi}}{|G|} (-1)^{\text{age}(h)} u^{\text{age}(h)-3/2} \left( \prod_{i=1}^3 w_i^{c_i(h)-1/2} \right) e^{u \left( \sqrt{-1} \vartheta_a + \sum_{i=1}^3 w_i \log w_i \right)} \sum_{\beta \in G^*} f_\beta^a(u, 0) \chi_\beta(h).
\]  

We conclude the following.

**Proposition 7.2.**

\[
f_\beta^a(u, 0) = \frac{1}{\sqrt{-2\pi}} \exp \left( - \left( -1 \vartheta_a + \sum_{i=1}^3 w_i \log w_i \right) u \right) \cdot \sum_{h \in G} \chi_\beta(h^{-1}) (-1)^{\text{age}(h)} u^{3/2-\text{age}(h)} \prod_{i=1}^3 w_i^{-c_i(h)+1/2} \lim_{q \to 0} \int_{\gamma_\alpha} e^{-ux} \nabla_h \Phi.
\]

We will evaluate \( \lim_{q \to 0} \int_{\gamma_\alpha} e^{-ux} \nabla_h \Phi \) in the next subsection.

### 7.2 Oscillatory integrals

The equivariant Landau–Ginzburg mirror of a general toric orbifolds has been studied by Iritani [Iri09].

The mirror B-model to \( X = [\mathbb{C}^3/G] \) is a Landau–Ginzburg model \( W_q : (\mathbb{C}^*)^3 \to \mathbb{C} \), where

\[
W_q = X_1^r X_2^r X_3^s - r f X_3 + X_2^m X_3 + X_3 + \sum_{a=1}^p q_a X_1^{m_a} X_2^{n_a} - m_a f X_3.
\]

Define \( H = W/X_3 \). Following Iritani [Iri09], the equivariantly perturbed B-model superpotential \( \widetilde{W}_q \) is

\[
\widetilde{W}_q = W_q - u \log X_1.
\]

We assume \( w_1, w_2, u > 0 \) and \( w_3 < 0 \) as usual. Define

\[
t_1 = X_1^r X_2^r X_3, \quad t_2 = X_2^m X_3, \quad \hat{t}_1 = X_1^r X_2^r X_3, \quad \hat{t}_2 = X_3^m.
\]
We use the notation in Section 6.5. For each critical point $p_\alpha(q) = (a_\alpha(q), b_\alpha(q))$ on the mirror curve, we have
\[
\Im(a_\alpha(0)) = -\pi w_3 - \vartheta_\alpha, \quad \Im(b_\alpha(0)) = \frac{\pi}{m} - \varphi_\alpha, \quad e^{\sqrt{-1}\vartheta_\alpha} = \chi_\alpha(\eta_1), \quad e^{\sqrt{-1}\varphi_\alpha} = \chi_\alpha(\eta_2),
\]
where $\Im(z)$ is the imaginary part of $z$. Notice that we have a preferred choice of the labeling of branch points by the elements in $G^*$ with $\vartheta_1 = \varphi_1 = 0$. Let $C = -1 + \sqrt{-1}\mathbb{R} \subset \mathbb{C}^*$, and define (Lagrangian) cycles $\Gamma^{\text{red}}_{\alpha,q} \subset (\mathbb{C}^*)^2$ and $\Gamma_{\alpha,q} \subset (\mathbb{C}^*)^3$ by
\[
\Gamma^{\text{red}}_{\alpha,q} := (\mathbb{R} + e^{-a_\alpha(q)}) \times (\mathbb{R} + e^{-b_\alpha(q)}), \quad \Gamma_{\alpha,q} := \Gamma^{\text{red}}_{\alpha,q} \times C \subset (\mathbb{C}^*)^3.
\]
In the perturbed superpotential $\tilde{W}_q$, the logarithm is defined in the following way: when $q = 0$, we have $X_3 < 0$, $\Im(\log X_1) = \pi w_3 + \vartheta_\alpha$, $\Im(\log X_2) = -\pi/m + \varphi_\alpha$, $\Im(\log X_3) = \pi$. Since the cycle $\Gamma_{\alpha,q}$ is contractible and deforms continuously with respect to $q$, the choice is fixed by these conditions on $\Gamma_{\alpha,q}$.

Define the oscillatory integral of $\tilde{W}_q$ to be
\[
\tilde{I}^Y_\alpha(u) = \int_{\Gamma_{\alpha,q}} e^{-\tilde{W}_q} \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dX_3}{X_3}.
\]
Define $L_\alpha := \{(q_1, \ldots, q_p) \in \mathbb{C}^p : q_\alpha X_\alpha A B^{n_\alpha - m_\alpha} \in \mathbb{R}\}$, which is a totally real linear subspace of $\mathbb{C}^p$. We will view $\tilde{I}^Y_\alpha(u)$ as a function of $q \in L_\alpha \cong \mathbb{R}^p$ and consider the power series expansion at $q = 0$.

**Lemma 7.3.** If $q \in L_\alpha$ is sufficiently small, then $\Gamma^{\text{red}}_{\alpha,q} = \Gamma^{\text{red}}_{\alpha,0}$.

**Proof.** Consider the change of variables $X = X_\alpha A$, $Y = Y_\alpha B$, and $q_\alpha = X_\alpha A B^{n_\alpha - m_\alpha} \epsilon_\alpha$ for $a = 1, \ldots, p$. Then $q = (q_1, \ldots, q_p) \in L_\alpha$ if and only if $\epsilon = (\epsilon_1, \ldots, \epsilon_p) \in \mathbb{R}^p$. We have
\[
\hat{t}_1 = \frac{w_1}{w_3} A^r B^{-s-rf}, \quad \hat{t}_2 = \frac{w_2}{w_3} B^m, \quad H(X,Y,q) = F(A,B,\epsilon), \quad Y \frac{\partial H}{\partial Y} = G(A,B,\epsilon),
\]
where
\[
F(A,B,\epsilon) = \frac{w_1}{w_3} A^r B^{-s-rf} + \frac{w_2}{w_3} B^m + 1 + \sum_{a=1}^p \epsilon_\alpha A^{m_\alpha} B^{n_\alpha - f_\alpha},
\]
\[
G(A,B,\epsilon) = \frac{G}{w_3} \left(-A^r B^{-s-rf} + B^m\right) + \sum_{a=1}^p \left(n_\alpha - f_\alpha\right) \epsilon_\alpha A^{m_\alpha} B^{n_\alpha - f_\alpha}
\]
are $\mathbb{R}$-valued real analytic functions on $\mathbb{R}^+ \times \mathbb{R}^p$. We have $F(1,1,0) = G(1,1,0) = 0$, and the Jacobian $|\partial(F,G)/\partial(A,B)(1,1,0)| = -|G|^2 w_1 w_2$ is non-zero. By the implicit function theorem, there exist real analytic functions $c,d : U \to \mathbb{R}^+$, where $U$ is an open neighborhood of 0 in $\mathbb{R}^p$, such that $c(0) = d(0) = 1$ and $F(c(\epsilon),d(\epsilon),\epsilon) = G(c(\epsilon),d(\epsilon),\epsilon) = 0$. If $\epsilon \in U$, then $e^{-a_\alpha(q)} = e^{-a_\alpha(0)} c(\epsilon)$ and $e^{-b_\alpha(q)} = e^{-b_\alpha(0)} d(\epsilon)$, so $\Gamma^{\text{red}}_{\alpha,q} = \Gamma^{\text{red}}_{\alpha,0}$. 

By Lemma 7.3 and its proof, if $q \in L_\alpha$ and $(X_1, X_2) \in \Gamma^{\text{red}}_{\alpha,q}$ then $\hat{t}_1, \hat{t}_2 \in \mathbb{R}^-$ and $H \in \mathbb{R}$. (Recall that $w_1, w_2 > 0$ and $w_3 = -w_1 - w_2 < 0$.) We first integrate out $\hat{t}_1, \hat{t}_2$, and then integrate out $X_3 \in C$. Recall that $h_1, \ldots, h_p$ are age 1 elements in $G$. Given $h = (r_1, \ldots, r_p) \in \mathbb{Z}^{\mathbb{R}^p}$, where $r_\alpha > 0$, define $c_\alpha(h) = \sum_{a=1}^p r_\alpha c_\alpha(h_a) \in \mathbb{Q}$, and define $\chi_\alpha(h) = \chi_\alpha \left(\prod_{a=1}^p h_a^{r_\alpha}\right) \in U(1)$.
Proof. Hankel’s representation of the reciprocal Gamma function says

\[ \int_{C_{\delta}} \left( \exp \left( - \sum_{a=1}^{p} q_a X_1^{m_a} X_2^{n_a - m_a} X_3 - t_1 - t_2 - X_3 \right) (-\hat{t}_1)^{u w_1} (-\hat{t}_2)^{u w_2} e^{u (w_1 + w_2 + w_3)} \log X_3 - \sqrt{1 - \pi} \right) e^{\sqrt{1 - \pi}(\theta_a + \pi w_3) u} d(-\hat{t}_1) d(-\hat{t}_2) dX_3 \]

\[ \frac{-1}{|G|} e^{\sqrt{1 - \pi}(\theta_a + \pi w_3) u} \sum_{\tilde{h}=(r_1,...,r_p)}^{r_\alpha \in \mathbb{Z}_{\geq 0}} e^{\sqrt{1 - \pi} c_3(h)} \chi_a(\tilde{h}) \prod_{a=1}^{p} \left( \frac{-q_a}{r_a!} \right) \]

\[ \cdot \int_{1 > 0} e^{-X_3 \hat{t}_1} (-\hat{t}_1)^{c_1(\hat{h}) + uw_1 - 1} d(-\hat{t}_1) \left( \int_{1 > 0} e^{-t_2} (-\hat{t}_2)^{c_2(\hat{h}) + uw_2 - 1} d(-\hat{t}_2) \right) \]

\[ \cdot e^{X_3 e^{(\log X_3 - \sqrt{1 - \pi})(c_1(\hat{h}) + c_2(\hat{h}) + c_3(\hat{h}) + u(w_1 + w_2 + w_3) - 1)} e^{-\hat{t}_1 X_3 - \hat{t}_2 X_3 - X_3} \right) d(-\hat{t}_1) d(-\hat{t}_2) dX_3 \]

\[ \frac{-1}{|G|} e^{\sqrt{1 - \pi}(\theta_a + \pi w_3) u} \sum_{\tilde{h}=(r_1,...,r_p)}^{r_\alpha \in \mathbb{Z}_{\geq 0}} e^{\sqrt{1 - \pi} c_3(h)} \chi_a(\tilde{h}) \prod_{a=1}^{p} \left( \frac{-q_a}{r_a!} \right) 

\Gamma(u w_1 + c_1(\tilde{h})) \Gamma(u w_2 + c_2(\tilde{h})) \]

\[ \cdot \int_{X_3 \in C} e^{-X_3 e^{(\log X_3 - \sqrt{1 - \pi})(c_1(\hat{h}) + u w_3 - 1)} dX_3 \}

\[ \frac{2\pi - 1}{|G|} e^{\sqrt{1 - \pi}(\theta_a + \pi w_3) u} \sum_{\tilde{h}=(r_1,...,r_p)}^{r_\alpha \in \mathbb{Z}_{\geq 0}} e^{\sqrt{1 - \pi} c_3(h)} \chi_a(\tilde{h}) \prod_{a=1}^{p} \left( \frac{-q_a}{r_a!} \right) 

\Gamma(u w_1 + c_1(\tilde{h})) \Gamma(u w_2 + c_2(\tilde{h})) \]

\[ \frac{2\pi - 1}{|G|} e^{\sqrt{1 - \pi}(\theta_a + \pi w_3) u} \sum_{\tilde{h}=(r_1,...,r_p)}^{r_\alpha \in \mathbb{Z}_{\geq 0}} e^{\sqrt{1 - \pi} c_3(h)} \chi_a(\tilde{h}) \prod_{a=1}^{p} \left( \frac{-q_a}{r_a!} \right) 

\Gamma(u w_1 + c_1(\tilde{h})) \Gamma(u w_2 + c_2(\tilde{h})) \]

Here we use the following identity.

**Lemma 7.4.** If \( \Re z > 0 \), then

\[ \sqrt{-1} \frac{1}{2\pi} \left( \int_{C} e^{-z(\log(s) - \sqrt{1 - \pi}) e^{-s} ds} \right) = \frac{1}{\Gamma(z)} . \]

**Proof.** Hankel’s representation of the reciprocal Gamma function says

\[ \sqrt{-1} \frac{1}{2\pi} \left( \int_{C_{\delta}} e^{-z(\log(t) - \sqrt{1 - \pi}) e^{-t} dt} \right) = \frac{1}{\Gamma(z)} , \]

where \( C_{\delta} \) is a Hankel contour (see Figure 2). The integrand is holomorphic on \( \mathbb{C} \setminus [0, \infty) \), and

\[ |e^{-z(\log(t) - \sqrt{1 - \pi}) e^{-t}}| = e^{-\Re(z) \log |s| + \Im(z) (\arg(t) - \pi) - \Re(t) \leq e^{\pi \Im(z) |t| - \Re(z)} e^{-\Re(z) t} , \]

so we may deform \( C_{\delta} \) to the contour \( C_{a,b} \) (see Figure 2) without changing the value of the contour.
integral. Therefore,
\[
\frac{\sqrt{-1}}{2\pi} \left( \int_{C_{a,b}} e^{-z(\log(t) - \sqrt{-1}\pi)} e^{-t} dt \right) = \frac{1}{\Gamma(z)}
\] (7.6)
for any \(a, b \in (0, \infty)\). The estimate (7.5) implies that contributions to the above contour integral from the two horizontal rays in \(C_{a,b}\) tend to zero as \(a, b \to +\infty\). The lemma follows from taking the limit of (7.6) as \(a, b \to +\infty\).

\[
-1 + a\sqrt{-1}
\]
\[
-1 - b\sqrt{-1}
\]
\[
\Gamma_{\delta}
\]
\[
C_{a,b}
\]

**Figure 2.** A Hankel contour \(C_{\delta}\) and the contour \(C_{a,b}\) \((a, b > 0)\)

**Remark 7.5.** Let \(f(t)\) be the inverse Laplace transform of \(F(s) = \Gamma(z)/s^z\), where \(z \in \mathbb{C}\) is a constant with \(\Re z > 0\). Lemma 7.4 implies that \(f(t) = t^{z-1}\) for \(t > 0\).

By Hori–Iqbal–Vafa [HIV00], this oscillatory integral could be reduced to a Laplace transform on the mirror curve. The Landau–Ginzburg model on \((\mathbb{C}^*)^3\) is equivalent to a 5-dimensional Landau–Ginzburg model, and the 5D model is again reduced to a Calabi–Yau threefold without potential. Further dimensional reduction reduces it to the mirror curve.

Introduce two variables \(v^+, v^- \in \mathbb{C}\) and the cycles
\[
\Gamma_{\alpha,q} = \Gamma_{\alpha,q} \times \{v^+ = \overline{v^-}\}, \quad \Gamma_{\text{red},\alpha,q} = \Gamma_{\text{red},\alpha,q} \times \{v^+ = \overline{v^-}\}.
\]
The equation \(H(X_1, X_2, q_1, \ldots, q_p) = 0\) prescribes the mirror curve. Define the holomorphic volume form
\[
\Omega = \frac{dX_1}{X_1} \frac{dX_2}{X_2} \frac{dv^-}{v^-} = dx dy \frac{dv^-}{v^-}.
\]

We reduce the oscillatory integral to the mirror curve as follows:
\[
\tilde{I}_\alpha^X(u) = \frac{-1}{2\sqrt{-1}\pi} \int_{\Gamma_{\alpha,q}} e^{-X_3(H-v^+v^-)} e^{-ux} \frac{dX_1}{X_1} \frac{dX_2}{X_2} dX_3 dv^+ dv^-
\]
\[
= \int_{\Gamma_{\text{red},\alpha,q}} \delta(H-v^+v^-) e^{-ux} \frac{dX_1}{X_1} \frac{dX_2}{X_2} dv^+ dv^-
\]
\[
= \int_{\Gamma_{\text{red},\alpha,q} \cap \{H-v^+v^- = 0\}} e^{-ux} \frac{dX_1}{X_1} \frac{dX_2}{X_2} dv^-.
\]
The first identity comes from integrating out \(v^+\) and \(v^-\) (notice that on \(C\), we have \(\Re(X_3) < 0\)), using
\[
\int_{v^+ = v^-} e^{X_3v^+v^-} dv^+ dv^- = -\frac{2\pi \sqrt{-1}}{X_3}.
\]
The second identity is the Fourier transform
\[
\int_{C} e^{-X_3(H-v^+v^-)} dX_3 = \int_{-\infty}^{+\infty} e^{(1-s\sqrt{-1})(H-v^+v^-)} (-\sqrt{-1}) ds = -2\pi \sqrt{-1} \delta (H - v^+v^-).
\]

For the third identity, fixing \(X_1\) and \(X_2\) such that \(H \geq 0\), and letting \(v^+ = re^{\sqrt{-1}\theta}\), we have
\[
\int_{v^+ = v^-} \delta (H - v^+v^-) d\nu^+ d\nu^- = 2\pi \sqrt{-1} \int_{0}^{\infty} \delta (H - r^2) 2r dr = 2\pi \sqrt{-1} = -\int_{|v^-| = H} \frac{dv^-}{v^-}.
\]

This integration is further reduced to the mirror curve \(H(e^{-x}, e^{-y}) = 0\) as follows:
\[
\tilde{T}_\alpha^\nu (u) = -\int_{\tilde{\gamma}_a \cap \{H=v^-\}=0} e^{-ux} dxdy \frac{dv^-}{v^-} = 2\pi \sqrt{-1} \int_{\gamma_\alpha = \tilde{\gamma}_a \cap \{H=v^-\}=0} e^{-ux} ydx.
\]

Notice that we use the fact that \(d(e^{-ux} ydx dv^-) = -e^{-ux} \Omega\) near \(\tilde{\gamma}_a \cap \{H=v^-\}=0\).

Therefore, we obtain the following formula.

**Theorem 7.6.** The integral \(\int_{\gamma_\alpha} e^{-ux} \Phi\) is equal to
\[
\frac{1}{|G|} e^{\sqrt{-1}(\theta_\alpha + \pi w_3)u} \sum_{\tilde{h}=(r_1, \ldots, r_p) \atop \nu \in \mathbb{Z}_{\geq 0}} e^{\pi \sqrt{-1} c_3(\tilde{h})} \chi_\alpha(\tilde{h}) \prod_{a=1}^{p} \left( \frac{\Gamma(uw_1 + c_1(\tilde{h})) \Gamma(uw_2 + c_2(\tilde{h}))}{\Gamma(-uw_3 - c_3(\tilde{h}) + 1)} \right).\]

**Corollary 7.7.** We have
\[
\lim_{q \to 0} \int_{\gamma_\alpha} e^{-ux} \nabla_h \Phi = \frac{e^{\sqrt{-1}(\theta_\alpha + \pi w_3)u}}{|G|} \frac{(-1)^{age(h)} e^{\sqrt{-1} \pi c_3(h)} \chi_\alpha(h) \Gamma(uw_1 + c_1(h)) \Gamma(uw_2 + c_2(h))}{\Gamma(-uw_3 - c_3(h) + 1)}.\]

**Theorem 7.8.** We have
\[
f_\beta^\alpha(u, 0) = \sum_{h \in G} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^{3} B_{m+1}(c_i(h))(w_i u)^{-m} \right).\]

**Proof.** By Proposition 7.2 and Corollary 7.7,
\[
f_\beta^\alpha(u, 0) = \sum_{h \in G} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \cdot C_h \cdot \frac{\Gamma(uw_1 + c_1(h)) \Gamma(uw_2 + c_2(h))}{\Gamma(-uw_3 - c_3(h) + 1)},\]

where
\[
C_h = \frac{1}{\sqrt{-2\pi}} \exp \left( -u \sum_{i=1}^{3} w_i \log w_i \right) \cdot u^{\frac{3}{2} - \text{age}(h)} \prod_{i=1}^{3} w_i^{-c_i(h) + 1/2} \cdot e^{\sqrt{-1} \pi (w_3 u + c_3(h))}.
\]
By the Stirling formula [KP11],
\[
\frac{\Gamma(w_1u + c_1(h))\Gamma(w_2u + c_2(h))}{\Gamma(-w_3u + 1 - c_3(h))} = \left( w_1u + c_1(h) - \frac{1}{2} \right) \log(w_1u) + \left( w_2u + c_2(h) - \frac{1}{2} \right) \log(w_2u)
- \left( -w_3u - c_3(h) + \frac{1}{2} \right) \log(-w_3u) + \frac{1}{2} \log(2\pi)
+ \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^{3} B_{m+1}(c_i(h))(w_iu)^{-m}
= \log \sqrt{-2\pi} - \sqrt{-1}\pi(w_3u + c_3(h)) + \left( \text{age}(h) - \frac{3}{2} \right) \log u + \sum_{i=1}^{3} \left( c_i(h) - \frac{1}{2} \right) \log w_i
\]
\[
\frac{\Gamma(w_1u + c_1(h))\Gamma(w_2u + c_2(h))}{\Gamma(-w_3u + 1 - c_3(h))} = \sqrt{-2\pi}e^{-\sqrt{-1}\pi(w_3u+c_3(h))u} \text{age}(h)-3/2 \prod_{i=1}^{3} w_i^{c_i(h)-1/2} \exp \left( u \sum_{i=1}^{3} w_i \log w_i \right)
\cdot \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^{3} B_{m+1}(c_i(h))(w_iu)^{-m} \right)
= C_h^{-1} \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^{3} B_{m+1}(c_i(h))(w_iu)^{-m} \right).
\]
Therefore,
\[
f_\beta^\alpha(u, 0) = \sum_{h \in G} \frac{\chi_\alpha(h)\chi_\beta((h^{-1})}{|G|} \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^{3} B_{m+1}(c_i(h))(w_iu)^{-m} \right).
\]

Define
\[
h_\alpha(u, q) := \frac{u^{3/2}}{-2\pi} e^{uw_\alpha} \int_{z \in \gamma_\alpha} e^{-uw_\Phi} = \sum_{\beta \in G^*} f_\beta^\alpha(u, q) h_\beta^\alpha. \quad (7.7)
\]

**Corollary 7.9.**
\[
h_\alpha(u, 0) = \frac{1}{|G|} \sqrt{-2 \over w_1w_2w_3} \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^{3} B_{m+1}(w_iu)^{-m} \right).
\]

**Proof.** Recall that
\[
\lim_{q \to 0} h_\beta^\alpha = \frac{1}{|G|} \sqrt{-2 \over w_1w_2w_3}
\]
for any \( \beta \in G^* \), so
\[
h_\alpha(u, 0) = \frac{1}{|G|} \sqrt{-2 \over w_1w_2w_3} \sum_{\beta \in G^*} f_\beta^\alpha(u, 0).
\]

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By Theorem 7.8,

\[
\sum_{\beta \in G^*} f^\beta_\theta(u, 0) = \sum_{h \in G} \frac{\chi_\alpha(h)}{|G|} \left( \sum_{\beta \in G^*} \chi_\beta(h^{-1}) \right) \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(h))(w_i u)^{-m} \right),
\]

where

\[
\sum_{\beta \in G^*} \chi_\beta(h^{-1}) = \sum_{\beta \in G^*} \chi_\beta(h^{-1}) \chi_\beta(1) = |G| \delta_{h^{-1}, 1}.
\]

So

\[
\tilde{h}^\alpha(u, 0) = \frac{1}{|G|} \sqrt{-\frac{2}{w_1 w_2 w_3}} \chi_\alpha(1) \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(c_i(1))(w_i u)^{-m} \right)
\]

\[
= \frac{1}{|G|} \sqrt{-\frac{2}{w_1 w_2 w_3}} \exp \left( \sum_{m \geq 1} \frac{(-1)^{m+1}}{m(m+1)} \sum_{i=1}^3 B_{m+1}(w_i u)^{-m} \right). \quad \square
\]

### 7.3 Inverse Laplace transform and expansion at \( X = 0 \)

Recall that on the Lefschetz thimble \( \gamma_\alpha \), the local coordinate is \( \zeta_\alpha(q) \), and \( x = a_\alpha(q) + \zeta_\alpha(q)^2 \). We choose \( \zeta_\alpha(q) \) such that when \( \zeta_\alpha(0) \to -\infty \), the corresponding point on the mirror curve \( \Sigma_0 \) approaches a point with \( X = 0 \) and \( Y^m = -1 \) (this is consistent with our choice \( h^\alpha(0) > 0 \)). Furthermore, we require that for any element \( \alpha' \in G^* \), its action on the compactified mirror curve \( \Sigma_0 \) moves the end point \( \zeta_\alpha = -\infty \) on \( \gamma_\alpha \) to the end point \( \zeta_{\alpha'} = -\infty \) on \( \gamma_{\alpha'} \). (Here \( \alpha' \in G^* \) is the product of \( \alpha, \alpha' \in G^* \); that is, \( \chi_{\alpha'}(h) = \chi_\alpha(h) \chi_{\alpha'}(h) \) for all \( h \in G \).)

We label the \( m \) points with \( X = 0 \) and \( Y^m = -1 \) on \( \overline{\Sigma}_0 \) by elements in \( \mu_m^* \cong \mathbb{Z}_m \) as follows:

\[
\tilde{p}_\ell := (0, e^{\pi \sqrt{-1} (2\ell + 1)/m}) \quad \text{for} \quad \ell \in \mathbb{Z}_m.
\]

With our convention, if \( \chi_\alpha = \chi_1^\ell \chi_2^\ell' \), then the end point \( \zeta_\alpha = -\infty \) is \( \tilde{p}_{\ell'-1} \).

Every 1-form on \( \gamma_\alpha \) is exact, so there exists a function \( \xi_\alpha^\beta, 0 \) on \( \gamma_\alpha \) such that \( d\xi_\alpha^\beta, 0 = \theta_0^\beta|_{\gamma_\alpha} \).

The Laplace transform of \( \theta_0^\beta \) along \( \gamma_\alpha \) is

\[
f_\beta^\alpha(u) = \frac{e^{u a_\alpha}}{2\sqrt{\pi} u} \int_{z \in \gamma_\alpha} e^{-u(x)} \theta_0^\beta = \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{z \in \gamma_\alpha} e^{-u(x) - a_\alpha} \xi_\beta^\alpha 0 d(x) - a_\alpha
\]

\[
= \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{x - a_\alpha \in \mathbb{R}^+} e^{-u(x - a_\alpha)} (\xi_{\beta, 0}^\alpha(x) - \xi_{\beta, 0}^\alpha(x)) d(x - a_\alpha),
\]

where \( \xi_{\beta, 0}^\alpha(x) = \xi_{\beta, 0}^\alpha(x, y(\pm \sqrt{x - a_\alpha})) \in L^2(0, \infty) \). From the proof of Theorem 7.8,

\[
f_\beta^\alpha(u, 0) = \frac{e^{-u \sum_{i=1}^3 w_i \log w_i}}{\sqrt{-2\pi}} \sum_{h \in G} \left( e^{\sqrt{-1} (w_3 u + c_3(h))} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \right) u^{3/2 - \text{age}(h)} \prod_{i=1}^3 \frac{1/2 - c_i(h) \Gamma(w_1 u + c_1(h)) \Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))}.
\]

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So the “classical Laplace transform” is
\[
\int_{x-a_\alpha \in \mathbb{R}^+} e^{-u(x-a_\alpha)} (\xi_{\beta,0}^\alpha |_{q=0}) d(x-a_\alpha) \\
= -\sqrt{-2} e^{-u} \sum_{i=1}^3 w_i \log w_i \sum_{h \in G} \left( e^{\sqrt{-1} \pi (w_3 u + c_3(h))} \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \cdot u^{1-\text{age}(h)} \prod_{i=1}^3 w_i^{1/2 - c_i(h)} \Gamma(w_1 u + c_1(h)) \Gamma(w_2 u + c_2(h)) \right) \frac{\Gamma(w_1 u + c_1(h)) \Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))}.
\]

By the inverse Laplace transform formula,
\[
-\lim_{q \to 0} \xi_{\beta,0}^{-\alpha} = -\sqrt{-2} \sum_{(d_0, h) \in \mathbb{Z} \times G} \text{Res}_{u=-d_0} (\Gamma(w_1 u + c_1(h))) \left( e^{u(x-a_\alpha - \sum_{i=1}^3 w_i \log w_i)} e^{\sqrt{-1} \pi (w_3 u + c_3(h))} \cdot \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \prod_{i=1}^3 w_i^{1/2 - c_i(h)} \Gamma(w_2 u + c_2(h)) \right) \frac{\Gamma(w_2 u + c_2(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \bigg|_{u=-d_0},
\]
\[
\lim_{q \to 0} \xi_{\beta,0}^{\alpha+} = -\sqrt{-2} \sum_{(d_0, h) \in \mathbb{Z} \times G} \text{Res}_{u=-d_0} (\Gamma(w_2 u + c_2(h))) \left( e^{u(x-a_\alpha - \sum_{i=1}^3 w_i \log w_i)} e^{\sqrt{-1} \pi (w_3 u + c_3(h))} \cdot \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} \prod_{i=1}^3 w_i^{1/2 - c_i(h)} \Gamma(w_1 u + c_1(h)) \right) \frac{\Gamma(w_1 u + c_1(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \bigg|_{u=-d_0}.
\]

Remark 7.10. We obtain \(\xi_{\beta,0}^{\alpha+}\) and \(\xi_{\beta,0}^{-\alpha}\) by taking residues around the poles of \(\Gamma(w_1 u + c_1(h))\) and \(\Gamma(w_2 u + c_2(h))\). When both \(w_1 u + c_1(h)\) and \(w_2 u + c_2(h)\) are non-positive integers, the sum of the respective terms in \(\xi_{\beta,0}^{\alpha+} - \xi_{\beta,0}^{-\alpha}\) is the residue.

We are interested in the expansion of \(\xi_{\beta,0}^{\alpha-}\) at \(\zeta = -\infty\), that is, the expansion of \(\xi_{\beta,0}^{\alpha-}\) at \(X = 0\),
\[
\lim_{q \to 0} \xi_{\beta,0}^{\alpha-} = \sqrt{-2} \sum_{(d_0, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m} e^{-d_0 x} \chi_\alpha(\eta_1^{-d_0}) e^{-\sqrt{-1} \pi (d_0 w_3 - c_3(h))} \cdot \frac{\chi_\alpha(h) \chi_\beta(h^{-1})}{|G|} (-d_0)^{1-\text{age}(h)} \prod_{i=1}^3 w_i^{1/2 - c_i(h)} \frac{(-1)^{\lfloor d_0/r \rfloor} \Gamma(-d_0 w_2 + c_2(h))}{\Gamma(d_0 w_1 - c_1(h) + 1) \Gamma(d_0 w_3 - 1 - c_3(h))} \frac{\Gamma(w_1 u + c_1(h))}{\Gamma(-w_3 u + 1 - c_3(h))} \frac{1}{d_0^{\text{age}(h)-1}}.
\]
\[
\lim_{q \to 0} \xi_{\beta,0}^{\alpha-} = \sqrt{-2} \sum_{(d_0, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m} X^{d_0} e^{-\sqrt{-1} \pi (d_0 w_3 - c_3(h))} \frac{\chi_\beta(h^{-1})}{m} \frac{1}{d_0^{\text{age}(h)-1}} \prod_{i=1}^3 w_i^{1/2 - c_i(h)} \frac{\Gamma(d_0 (w_1 + w_2) + c_3(h))}{\Gamma(d_0 w_1 - c_1(h) + 1) \Gamma(d_0 w_2 - c_2(h) + 1)} \chi_\alpha(\eta_2^{-k}).
\]
This is the expansion of \(\lim_{q \to 0} \xi_{\beta,0}^{\alpha-}\) at \(X = 0\) and \(Y = e^{-\sqrt{-1} \pi / m} \chi_\alpha(\eta_2)\). If \(\alpha = \chi_1 \chi_2^f\), then
\[
\chi_\alpha(\eta_2^{-k}) = e^{-2\sqrt{-1} \pi k \ell / m}
\]
depends only on \(\ell \in \{0, 1, \ldots, m-1\}\). Let \(\xi_{\beta,0}^{\alpha-}\) be the expansion of \(\xi_{\beta,0}^{\alpha-}\) at \(\bar{p}_\ell = (0, e^{\pi \sqrt{-1} (2\ell+1)/m})\)
for $\alpha = \chi^j_1 \chi^k_2$. Define
\[
\psi_{\ell} := \frac{1}{m} \sum_{k=0}^{m-1} \omega_m^{-k\ell} \mathbf{1}_{k/m}, \quad \ell = 0, \ldots, m - 1,
\]
where $\omega_m = e^{2\pi \sqrt{-1}/m}$. Then $\{\psi_0, \ldots, \psi_{m-1}\}$ is a canonical basis of $H^*_\text{CR}(B\mu_m; \mathbb{C})$. Define
\[
\xi^\beta_0 := \sum_{\ell=0}^{m-1} \xi^\beta_\ell \psi_{\ell},
\]
(7.8)
which takes values in $H^*_\text{CR}(B\mu_m; \mathbb{C})$. Compare with the definition of $\xi^\beta_0$ in Section 5.2; we obtain the following identity.

**Proposition 7.11.** We have
\[
\lim_{q \to 0} \xi^\beta_0 = -\frac{1}{|G|} \sqrt{-2} w_1 w_2 w_3 \xi^\beta_0 |_{\nu=1}.
\]

**7.4 Eynard–Orantin topological recursion**

Let $\omega_{g,n}$ be defined recursively by the Eynard–Orantin topological recursion [EO07]:
\[
\omega_{0,1} = 0, \quad \omega_{0,2} = B(z_1, z_2).
\]

When $2g - 2 + n > 0$,
\[
\omega_{g,n}(p_1, \ldots, p_n) = \sum_{\alpha \in G^*} \text{Res}_{p=p_0} \frac{f^p_{x=p} B(p, x)}{2(\Phi(p) - \Phi(p))} \left( \omega_{g-1,n+1}(p, \bar{p}, p_1, \ldots, p_{n-1}) + \sum_{g_1+g_2=\nu \cup \nu = \{1, \ldots, n-1\} \cap \nu \cap \nu = \emptyset} \omega_{g_1, |\nu|+1}(p, p_\nu) \omega_{g_2, |\nu|+1}(\bar{p}, p_\nu) \right),
\]
where $p, \bar{p} \to p_\alpha$, $X(p) = X(\bar{p})$, and $p \neq \bar{p}$.

The B-model invariants $\omega_{g,n}$ can be expressed as sums over labeled graphs involving intersection numbers on moduli spaces of stable curves [KO10, Eyn11, Eyn14, DOSS14]. We will use the formula stated in [DOSS14, Theorem 3.7], which is equivalent to the formula in [Eyn11, Theorem 5.1]. To state this graph sum, we introduce some definitions. Following Eynard [Eyn11], define the Laplace transform of the Bergman kernel $B(z_1, z_2)$
\[
\tilde{B}^{\alpha,\beta}(u, v, q) := \frac{uv}{u + v} \delta_{\alpha,\beta} + \frac{uv}{2\pi} \epsilon_{\alpha,\beta} \int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} B(z_1, z_2) e^{-ux(z_1) - vx(z_2)},
\]
(7.9)
where $\alpha, \beta \in G^*$. By [Eyn11, equation (B.9)],
\[
\tilde{B}^{\alpha,\beta}(u, v, q) = \frac{uv}{u + v} \left( \delta_{\alpha,\beta} - \sum_{\gamma \in G^*} f^{\alpha}_\gamma (u, q) f^{\beta}_\gamma (v, q) \right).
\]
(7.10)
Let $\tilde{h}^{\alpha,\beta}_k(q)$ be defined by (6.16). Then $\tilde{B}^{\alpha,\beta}(u, v, q) = \sum_{k,l} \tilde{B}^{\alpha,\beta}_{k,l}(q) u^{-k} v^{-l}$. Let $\tilde{h}^{\alpha}_k(q)$ be defined by
\[
\tilde{h}^{\alpha}(u, q) = \sum_{k} \tilde{h}^{\alpha}_k(q) u^{-k},
\]
where $\tilde{h}^{\alpha}$ is defined by (7.7).
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Given a labeled graph $\tilde{\Gamma} = (\Gamma, g, \alpha, k) \in \Gamma_{g,0,n}(BG)$ with $L^O(\Gamma) = \{l_1, \ldots, l_n\}$, we define its weight to be

$$w(\tilde{\Gamma}) = (-1)^{g(\tilde{\Gamma})-1} \prod_{v \in V(\Gamma)} \left( \frac{h_1^v}{\sqrt{-2}} \right)^{2-2g-\text{val}(v)} \left( \prod_{h \in H(v)} \tau_{k(h)} \right) \prod_{e \in E(\Gamma)} \hat{B}_e^{q(e),e}(e) \prod_{j=1}^n \frac{1}{\sqrt{-2}} \theta_{k,l_j}(z_j) \prod_{l \in L^1(\Gamma)} \left( -\frac{1}{\sqrt{-2}} \right) \hat{h}_{k,l}(l).$$

In our notation, [DOSS14, Theorem 3.7] is equivalent to the following.

**Theorem 7.12.** For $2g - 2 + n > 0$, we have

$$\omega_{g,n} = \sum_{\Gamma \in \Gamma_{g,0,n}(BG)} \frac{w(\tilde{\Gamma})}{|\text{Aut}(\tilde{\Gamma})|}.$$  

**Example 7.13** (Pair of pants).

$$\omega_{0,3}(p_1, p_2, p_3) = \sum_{\alpha \in G^*} \frac{1}{2h_1^\alpha} \theta_0^\alpha(p_1) \theta_0^\alpha(p_2) \theta_0^\alpha(p_3).$$

We now consider the unstable case $(g, n) = (0, 2)$. Recall that $dx = -dX/X$ is a meromorphic 1-form on $\tilde{\Sigma}$, and $d/dx = -Xd/dX$ is a meromorphic vector field on $\tilde{\Sigma}$. Define

$$C(z_1, z_2) := \left( -\frac{\partial}{\partial x}(z_1) - \frac{\partial}{\partial y}(z_2) \right) \left( \frac{\omega_{0,2}}{dx_1dx_2} \right) (z_1, z_2) d(x(z_1))(dx(z_2)). \quad (7.11)$$

Then $C(z_1, z_2)$ is meromorphic on $(\tilde{\Sigma}_q)^2$ and holomorphic on $(\tilde{\Sigma}_q \setminus \{p_\alpha : \alpha \in G^*\})^2$. It only has double poles on $\{p_\alpha : \alpha \in G^*\} \times \tilde{\Sigma}_q$ and $\tilde{\Sigma}_q \times \{p_\beta : \beta \in G^*\}$.

**Lemma 7.14.** We have

$$C(z_1, z_2) = \frac{1}{2} \sum_{\gamma \in G^*} \theta_0^\gamma(z_1) \theta_0^\gamma(z_2).$$

**Proof.** First, we show that the Laplace transforms of both sides are asymptotically equal. For any $\alpha, \beta \in G^*$,

$$\int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} e^{-u(x(z_1) - a_\alpha) - v(x(z_2) - a_\beta)} C(z_1, z_2)$$

$$= (-u - v) \int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} e^{-u(x(z_1) - a_\alpha) - v(x(z_2) - a_\beta)} \omega_{0,2}$$

$$\sim 2\pi \sqrt{uv} \sum_{\gamma \in G^*} f_\gamma^\alpha(u, q) f_\gamma^\beta(v, q)$$

$$\sim \frac{1}{2} \int_{z_1 \in \gamma_\alpha} \int_{z_2 \in \gamma_\beta} e^{-u(x(z_1) - a_\alpha) - v(x(z_2) - a_\beta)} \theta_0^\gamma(z_1) \theta_0^\gamma(z_2).$$

Define their difference

$$\omega = C(z_1, z_2) - \frac{1}{2} \sum_{\gamma \in G^*} \theta_0^\gamma(z_1) \theta_0^\gamma(z_2);$$

its Laplace transform at $\gamma_\alpha \times \gamma_\beta$ for any $\alpha, \beta \in G^*$ vanishes. For $i = 1, \ldots, g$,

$$\int_{p_2 \in A_i} \omega_{0,2}(p_1, p_2) = 0, \quad \int_{A_i} \theta_0^\alpha = 0.$$
Then
\[ \int_{p_2 \in A_i} \omega(p_1, p_2) = 0, \quad \int_{p_2 \in A_i, p_1 \mapsto p_0} \text{Res} \, \zeta_{1, \alpha}(p_1) \omega(p_1, p_2) = 0 \]
for all \( i = 1, \ldots, g \). Notice that the 1-form
\[ \text{Res}_{p_1 \mapsto p_0} \zeta_{1, \alpha}(p_1) \omega(p_1, p_2) = \left[ u^0 \right] \int_{p_1 \in \gamma_\alpha} e^{-u(x(p_1) - a_\alpha)} \omega(p_1, p_2) \]
is a well-defined holomorphic form on \( \bar{\Sigma}_q \). It has no poles; otherwise, any possible double pole at \( p_\beta \) implies a non-zero Laplace transform of \( \omega \) at \( \gamma_\alpha \times \gamma_\beta \). It follows from the vanishing A-periods of \( \omega \), we know that \( \omega = 0 \). \( \square \)

7.5 B-model potentials

Choose \( \delta > 0 \) and \( \epsilon > 0 \) sufficiently small that for \( |q| < \epsilon \), the meromorphic function \( X : \Sigma_q \to \mathbb{C} \cup \{ \infty \} \) restricts to an isomorphism
\[ X^\delta : D^\delta_q \to D_\delta = \{ X \in \mathbb{C} : |X| < \delta \}, \]
where \( D^\delta_q \) is an open neighborhood of \( \bar{p}_\ell \in X^{-1}(0) \) and \( \ell = 0, \ldots, m - 1 \). Define
\[ \rho^1_{q, \ell_1, \ell_2} := (X^{\ell_1}_q)^{-1} \times \cdots \times (X^{\ell_2}_q)^{-1} : (D_\delta)^n \to D^\delta_q \times \cdots \times D^\delta_q \subset (\bar{\Sigma}_q)^n. \]

1. (Disk invariants) At \( q = 0 \), we have \( Y(\bar{p}_\ell)^m = -1 \) for \( \ell = 0, \ldots, m - 1 \). When \( \epsilon \) and \( \delta \) are sufficiently small, \( Y(\rho^\ell_q(X)) \in \mathbb{C} \setminus [0, \infty) \). Choose a branch of logarithm \( \log : \mathbb{C} \setminus [0, \infty) \to (0, 2\pi) \), and define \( y^\ell_q(X) = -\log Y(\rho^\ell_q(X)) \). The function \( y^\ell_q(X) \) depends on the choice of logarithm, but \( y^\ell_q(X) - y^\ell_q(0) \) does not. The form \( dx = -dX/X \) is a meromorphic 1-form on \( \mathbb{C} \) with a simple pole at \( X = 0 \), and \( (y^\ell_q(X) - y^\ell_q(0))dx \) is a holomorphic 1-form on \( D_\delta \). Define the B-model disk potential by
\[ F_{0,1}(q; X) := \sum_{\ell \in \mathbb{Z}_m} \int_0^X (y^\ell_q(X') - y^\ell_q(0)) \left( -\frac{dX'}{X'} \right) \psi_\ell; \]
this takes values in \( H^*(\mathcal{B}\mu_m; \mathbb{C}) \).

2. (Annulus invariants) The form
\[ \left( \rho^1_{q, \ell_1, \ell_2} \right)^* \omega_{0,2} - \frac{dX_1dX_2}{(X_1 - X_2)^2} \]
is holomorphic on \( D_\delta \times D_\delta \). Define the B-model annulus potential by
\[ F_{0,2}(q; X_1, X_2) := \sum_{\ell_1, \ell_2 \in \mathbb{Z}_m} \int_0^{X_1} \int_0^{X_2} \left( \left( \rho^1_{q, \ell_1, \ell_2} \right)^* \omega_{0,2} - \frac{dX_1dX_2}{(X_1 - X_2)^2} \right) \psi_{\ell_1} \otimes \psi_{\ell_2}; \]
this takes values in \( H^*(\mathcal{B}\mu_m; \mathbb{C}) \otimes^2 \).

3. For \( 2g - 2 + n > 0 \), the form \( \left( \rho^1_{q, \ell_1, \ell_n} \right)^* \omega_{g,n} \) is holomorphic on \( (D_\delta)^n \). Define
\[ F_{g,n}(q; X_1, \ldots, X_n) := \sum_{\ell_1, \ldots, \ell_n \in \mathbb{Z}_m} \int_0^{X_1} \cdots \int_0^{X_n} \left( \rho^1_{q, \ell_1, \ell_n} \right)^* \omega_{g,n} \psi_{\ell_1} \otimes \cdots \otimes \psi_{\ell_n}; \]

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this takes values in $H^*(B\mu_m; C)^{\otimes n}$.

For $g \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$, the function $\hat{F}_{g,n}(q; X_1, \ldots, X_n)$ is holomorphic on $B_q \times (D_\delta)^n$ when $\epsilon, \delta > 0$ are sufficiently small. By construction, the power series expansion of $\hat{F}_{g,n}(q; X_1, \ldots, X_n)$ involves only positive powers of $X_i$.

For $\beta \in G^*$, let $\xi^\beta_{\alpha,0}$ be defined as in Section 7.3. Then $\xi^\beta_{\alpha,0}(X) = \int_0^X (\rho_0)^* \theta^\beta_0$.

### 7.6 Special geometry and B-model graph sum

By the special geometry property of the topological recursion ([EO15, Theorem 4.4], proved in [EO07]),

$$\nabla_{\partial/\partial r_{\alpha}} \omega_{g,n}(z_1, \ldots, z_n) = \int_{z_{n+1} \in B_{\alpha}} \omega_{g,n+1}(z_1, \ldots, z_{n+1}),$$

where $B_{\alpha}$ is defined as in Section 6.4. At $q = 0$,

$$\int_{z_{n+1} \in B_{\alpha}} \theta^\beta_k(z_{n+1}) = [u^{-k}] \int_{z_{n+1} \in B_{\alpha}} \left( -\frac{\sqrt{u} e^{u \alpha}}{\sqrt{\pi}} \right) \int_{z' \in \gamma} B(z_{n+1}, z') e^{-u x(z')} = [u^{-k}] \frac{1}{[G]} \sqrt{-2} \sum_{\beta \in G^*} \sum_{a=1}^p \left( \prod_{i=1}^3 w_i^{c_i(a)} \sum_{\beta \in G^*} f^a_{\beta}(u) \chi_{\beta}(h(a)) \right).$$

Define the B-model primary leaf to be

$$(\hat{\tau}^\alpha_k)_{l} = [u^{-k}] \left( \frac{1}{[G]} \sqrt{-2} \sum_{\beta \in G^*} \sum_{a=1}^p \left( \prod_{i=1}^3 w_i^{c_i(a)} \sum_{\beta \in G^*} f^a_{\beta}(u) \chi_{\beta}(h(a)) \right) \right). \quad (7.12)$$

For any $\alpha \in G^*$, $k \in \mathbb{Z}_{\geq 0}$, and $\ell \in \mathbb{Z}_m$, the form $(\rho_0)^* \theta^\alpha_k$ is a holomorphic $1$-form on $D_\delta$ for small enough $\delta > 0$. To each open leaf $l_j$ of a labeled graph $\Gamma \in \Gamma_n(\mathcal{X})$, we assign

$$(\mathcal{L}^\alpha_k(l_j)) = \frac{1}{\sqrt{-2}} \sum_{\ell \in \mathbb{Z}_m} \left( \int_0^X (\rho_0)^* \theta^\alpha_k \right) \psi_{\ell}. \quad (7.13)$$

The Taylor series expansion of $(\rho_0)^{l_1, \ldots, l_n} \omega_{g,n}$ at $\tau = 0$ is

$$(\rho_0)^{l_1, \ldots, l_n} \omega_{g,n} = \sum_{l_1, \ldots, l_p \in \mathbb{Z}_{\geq 0}}^\infty \left( \prod_{i=1}^p \frac{\tau_i}{l_i!} \right) \left( \prod_{i=1}^p \frac{\tau_i}{l_i!} \right).$$

Recall that $\Gamma_{g,n}(\mathcal{X})$ is the set of stable genus $g$ graphs with $n$ open leaves and any number of primary leaves. Given $\Gamma = (\Gamma, g, \alpha, k) \in \Gamma_{g,n}(\mathcal{X})$ with open leaves $l_1, \ldots, l_n$, we define its B-model weight to be

$$w_B(\Gamma) = (-1)^{g(\Gamma)-1} \prod_{v \in V(\Gamma)} \left( \frac{h_1^{\alpha}(\tau = 0)}{\sqrt{-2}} \right) \left( \prod_{h \in H(\mathcal{V})} \tau_k(h) \left( \prod_{g(v) \in E(\mathcal{V})} B^{\alpha(v_1(e)), \alpha(v_2(e))}(e) \right) \right)_{\tau = 0}.$$
The lemma follows from (7.15), (7.16), and (5.11).

**Proposition 7.16** All-genus open-closed mirror symmetry

**Proof.** From the definition,

\[
\frac{\partial}{\partial \tau_a} \frac{\partial F_{0,1}}{\partial \tau_a}(0; X) = \Phi_{1,0}(X) + \sum_{n=1}^{\infty} \tau_a \frac{\partial F_{0,1}^{(n)}}{\partial \tau_a}(0; X) + \sum_{\Gamma \in \Gamma_0,1(X)} \frac{w_{B}(\Gamma)}{|\Aut(\Gamma)|}.
\]

(i) (Disk invariants)

\[
F_{0,1}(\tau; X) = \sum_{n=1}^{\infty} \tau_a \frac{\partial F_{0,1}^{(n)}}{\partial \tau_a}(0; X) = \phi_{1,0}(X) + \sum_{\Gamma \in \Gamma_0,1(X)} \frac{w_{B}(\Gamma)}{|\Aut(\Gamma)|}.
\]

(ii) (Annulus invariants)

\[
F_{0,2}(\tau; X_1, X_2) = \sum_{n=1}^{\infty} \frac{\partial F_{0,1}^{(n)}}{\partial \tau_a}(0; X) + \sum_{\Gamma \in \Gamma_0,2(X)} \frac{w_{B}(\Gamma)}{|\Aut(\Gamma)|}.
\]

(iii) For \(2g - 2 + n > 0\),

\[
F_{g,n}(\tau; X_1, \ldots, X_n) = \sum_{\Gamma \in \Gamma_{g,n}(X)} \frac{w_{B}(\Gamma)}{|\Aut(\Gamma)|}.
\]

**7.7 All-genus open-closed mirror symmetry**

**Proposition 7.17.** We have

\[
F_{0,0}(0; X) + \sum_{\Gamma \in \Gamma_0,1(X)} \frac{w_{B}(\Gamma)}{|\Aut(\Gamma)|} = \Phi_{1,0}(X) + \sum_{\Gamma \in \Gamma_0,1(X)} \frac{w_{B}(\Gamma)}{|\Aut(\Gamma)|}.
\]

**Proof.** From the definition,

\[
\frac{\partial}{\partial \tau_a} \frac{\partial F_{0,1}}{\partial \tau_a}(0; X) = \Phi_{1,0}(X) + \sum_{\Gamma \in \Gamma_0,1(X)} \frac{w_{B}(\Gamma)}{|\Aut(\Gamma)|}.
\]

By Proposition 6.4, equations (6.13) and (7.2), and Proposition 7.11,

\[
\left( X \frac{d}{dX} \right)^2 \frac{\partial F_{0,1}}{\partial \tau_a}(0; X) = \sum_{\beta \in G^*} \frac{1}{|G|} \frac{\partial}{\partial \tau_a} \frac{\partial \xi_0^\beta}{\partial \tau_a}(X) = \sum_{\beta \in G^*} \frac{1}{|G|^2 w_1 w_2 w_3} \lim_{q \to 0} \xi_0^\beta(X)
\]

where

\[
\left( X \frac{d}{dX} \right) \frac{\partial F_{0,1}}{\partial \tau_a}(0; X) = \sum_{\beta \in G^*} \frac{1}{|G|^2 w_1 w_2 w_3} \lim_{q \to 0} \xi_0^\beta(X)
\]

The functions \(F_{0,1}(0; X)\) and \((\partial F_{0,1}/\partial \tau_a)(0; X)\) are power series in \(X\) with no constant term, so

\[
\frac{\partial F_{0,1}}{\partial \tau_a}(0; X) = \sum_{\beta \in G^*} \frac{1}{|G|^2 w_1 w_2 w_3} \lim_{q \to 0} \xi_0^\beta(X).
\]

The lemma follows from (7.15), (7.16), and (5.11).

**Proposition 7.17.** We have \(F_{0,2}(0; X_1, X_2) = -F_{0,2}(0; X_1, X_2)\).

**Proof.** Let \(C = C(z_1, z_2)\) be defined by (7.11). Then

\[
\left( X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} \right) F_{0,2}(0; X_1, X_2) = \sum_{\ell_1, \ell_2 \in \mathbb{Z}_m} \int_0^{X_1} \int_0^{X_2} (\rho_{\ell_1, \ell_2})^* C \psi_{\ell_1} \otimes \psi_{\ell_2}
\]

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By definition,

\[ c \in B \]

Here we have used the fact that \( \psi \) is odd and \( \epsilon \) is even. It follows from (7.17) that

\[ (7.17) \]

where the second equality follows from Lemma 7.11, and the last equality follows from (5.5). Both \( W_{0,2}(0; X_1, X_2) \) and \( F_{0,2}^{x,(L,f)}(0; X_1, X_2) \) are \( H^{*}_{\text{CR}}(B_{\mu}; \mathbb{C}) \)-valued power series in \( X_1 \) and \( X_2 \) which vanish at \( (X_1, X_2) = (0, 0) \), so

\[ F_{0,2}(0; X_1, X_2) = -F_{0,2}^{x,(L,f)}(0; X_1, X_2). \]

**Proposition 7.18.** For any \( \bar{\Gamma} \in \Gamma_{g,n}(\mathcal{X}) \), we have

\[ w_B(\bar{\Gamma}) = (-1)^{g(\bar{\Gamma})-1+n}w_A(\bar{\Gamma}), \]

where \( w_A(\bar{\Gamma}) \) is defined in Section 5.3 and \( w_B(\bar{\Gamma}) \) is defined in Section 7.6.

**Proof.** We fix \( \bar{\Gamma} \in \Gamma_{g,n}(\mathcal{X}) \). From (5.7) and Theorem 7.8, for any \( \alpha, \beta \in G^* \),

\[ R(-z)^\alpha = f_\beta^\alpha \left( \frac{1}{z}, 0 \right). \]

(7.17)

Here we have used the fact that \( B_m(1 - x) = (-1)^m B_m(x) \), \( B_m = 0 \) when \( m \) is odd and \( c_i(h) + c_i(h^{-1}) = 1 - \delta_{c_i(h),0} \).

1. (Vertex) By equation (7.2), we have

\[ \lim_{\tau \to 0} h_1^\alpha = \frac{1}{|G| \sqrt{w_1 w_2 w_3}}. \]

(7.18)

2. (Edge) By equation (7.10) (that is, [Eyn11, equation (B.9)]), we have

\[ B_{k,l}^{\alpha,\beta} \big|_{\tau=0} = \left[ u^{-k} v^{-l} \right] \left( \frac{w}{u+v} \delta_{\alpha,\beta} - \sum_{\gamma \in G^*} f_{\gamma}^\alpha(u,0) f_{\gamma}^\beta(v,0) \right) \]

\[ = \left[ z^{-k} w^l \right] \left( \frac{1}{z+w} \left( \delta_{\alpha,\beta} - \sum_{\gamma \in G^*} f_{\gamma}^\alpha \left( \frac{1}{z},0 \right) f_{\gamma}^\beta \left( \frac{1}{w},0 \right) \right) \right). \]

By definition,

\[ E_{k,l}^{\alpha,\beta} = \left[ z^{-k} w^l \right] \left( \frac{1}{z+w} \left( \delta_{\alpha,\beta} - \sum_{\gamma \in G^*} R(-z)_{\alpha}^\gamma R(-w)_{\beta}^\gamma \right) \right). \]

It follows from (7.17) that

\[ B_{k,l}^{\alpha,\beta} \big|_{\tau=0} = E_{k,l}^{\alpha,\beta}. \]

(7.19)

3. (Primary leaf) By equation (7.12),

\[ \frac{1}{\sqrt{-2}} (h^\tau)^k = \left[ u^{-k} \right] \left( \frac{1}{|G| \sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} \sum_{a=1}^p \prod_{i=1}^3 w_i^{c_i(h_a)} f_{\beta}^\alpha(u) \chi_{\beta}(h_a) \tau_a \right). \]
By equation (5.8),
\[
(\mathcal{L}^\alpha)_k = [z^k] \left( \frac{1}{|G'\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} \sum_{\alpha = 1}^p \prod_{i = 1}^3 w_i^{c_i(h_\alpha)} R(-z)^\beta R((-z)^\beta a) \right).
\]

It follows from (7.17) that
\[
\frac{1}{\sqrt{-2}} (\hat{h}^\alpha)_k = (\mathcal{L}^\alpha)_k.
\] (7.20)

(4) (Open leaf) Given \( \beta \in G^* \) and \( k \in \mathbb{Z}_{\geq 0} \), define
\[
\hat{\xi}_k^\beta(X) := \sum_{\ell = 0}^{m-1} \int_X \rho_0^* (d\hat{\xi}_\beta, k) \psi_\ell,
\] (7.21)

which is an \( H^1_{\text{CR}}(B\mu; \mathbb{C}) \)-valued holomorphic function on \( D_\delta \). Then \( \hat{\xi}_k^\beta(X) = (X d/dX)^k \hat{\xi}_0^\beta(X) \) and \( \hat{\xi}_0^\beta(X) = \lim_{\varphi \to \theta} \hat{\xi}_0^\beta \), where \( \hat{\xi}_0^\beta \) is defined by equation (7.8). By Proposition 7.11 and the definitions of \( \hat{\xi}_k^\beta(X) \) and \( \hat{\xi}_k^\beta(X) \),
\[
\hat{\xi}_k^\beta(X) = \frac{1}{|G|} \sqrt{-2} \frac{w_1 w_2 w_3}{w_1 w_2 w_3} \hat{\xi}_k^\beta(X).
\] (7.22)

By (7.13), (7.21), (7.22), and Lemma 6.7,
\[
(\mathcal{L}^\alpha)_k(l_j) = \frac{-1}{|G'\sqrt{w_1 w_2 w_3}} \left( \hat{\xi}_k^\alpha - \sum_{\ell = 0}^{m-1} \int_X \rho_0^* (d\hat{\xi}_\alpha, k) \psi_\ell \right).
\]

By item (2) above, for \( k \in \mathbb{Z}_{\geq 0} \),
\[
B_{k,0}^\alpha, \beta \bigg|_{\tau = 0} = [z^k w^0] \left( \frac{1}{z + w} \left( \delta_{\alpha, \beta} - \sum_{\gamma \in G^*} R(-z)^\gamma R(-w)^\gamma \right) \right) = [z^{k+1}] \left( (R(-z)^\beta) \right).
\]

We also have \([z^0] (R(-z)^\beta) = \delta_{\alpha, \beta} \). Therefore,
\[
(\mathcal{L}^\alpha)_k(l_j) = [z^k] \left( \frac{-1}{|G'\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R(-z)^\beta \hat{\xi}_k^\beta(z, X_j) \right).
\]

Comparing the above expression with the definition of the A-model open leaf \( \mathcal{L}^\alpha_k(l_j) \), we conclude that
\[
(\mathcal{L}^\alpha)_k(l_j) = -(\mathcal{L}^\alpha_k(l_j)).
\] (7.23)

(5) (Dilaton leaf)
\[
\left(\frac{-1}{\sqrt{-2}}\right) \hat{h}_k^0 \bigg|_{\tau = 0} = [u^{1-k}] \left( \frac{-1}{\sqrt{-2}} \right) \hat{h}_0^0 \bigg|_{\tau = 0} = [u^{1-k}] \frac{-1}{|G'\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} f_\beta^0(u, 0),
\]

where the second identity follows from Corollary 7.9. By definition,
\[
(\mathcal{L}^1)_k = [z^{k-1}] \frac{-1}{|G'\sqrt{w_1 w_2 w_3}} \sum_{\beta \in G^*} R_\beta^0(-z).
\]

It follows from (7.17) that
\[
\left(\frac{-1}{\sqrt{-2}}\right) \hat{h}_k^0 \bigg|_{\tau = 0} = (\mathcal{L}^1)_k.
\] (7.24)
By (5.9), (7.14), (7.18), (7.19), (7.20), (7.23), and (7.24),

\[ w_B(\vec{\Gamma}) = (-1)^{g(\vec{\Gamma}) - 1 + n} w_A(\vec{\Gamma}). \]

Combining Theorem 5.5 (A-model graph sum), Theorem 7.15 (B-model graph sum), and Propositions 7.16, 7.17, and 7.18, we obtain the final result.

**Theorem 7.19 (All-genus open-closed mirror symmetry).** We have

\[ \tilde{F}_{g,n}(\tau; X_1, \ldots, X_n) = (-1)^{g - 1 + n} F_{g,n}(\tau; X_1, \ldots, X_n). \]

As a consequence, the \( H^*(\mathcal{B}_{\mu_m}^{\otimes n}) \)-valued formal power series \( F_{g,n}(\tau; X_1, \ldots, X_n) \) converges in an open neighborhood of the origin in \( \mathbb{C}^p \times \mathbb{C}^n \).

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