Charge and isospin fluctuations in a non-ideal pion gas with dynamically fixed particle number

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Abstract We study the behavior of the non-ideal pion gas with the dynamically fixed number of particles, formed on an intermediate stage in ultra-relativistic heavy-ion collisions. The pion spectrum is calculated within the self-consistent Hartree approximation. General expressions are derived for the behavior of the non-ideal pion gas with small isospin imbalance \(0 < |G| \ll N\) and of a relative abundance of charged and neutral pions, \(G = (N_+ + N_-)/2 - N_0\), remain finite in the critical point. Then, fluctuations are studied in the pion gas with small isospin imbalance \(0 < |G| \ll N\) and \(0 < |Q| \ll N\) and shifts of the effective masses, chemical potentials, and values of critical temperatures are calculated for various pion species, and the highest critical temperature, \(T_{cr}\), is found, above which the pion system exists in the non-condensed phase. Various pion cross variances are calculated for \(T > \max T_{cr}^a\), which prove to be strongly dependent on the isospin composition of the system, whereas the variances of \(N\) and \(G\) are found to be independent on the isospin imbalance up to the term linear in \(G/N\) and \(Q/N\).

1 Introduction

The first hadrochemical calculations for heavy-ion collisions at energies \(\gtrsim 1\text{A GeV}\) [1,2] inspired considerations of a possibility for the Bose–Einstein condensation (BEC) of pions in baryon enriched matter. However, subsequent more detailed studies, see Ref. [3,4] for review, excluded such a possibility for heavy-ion collisions at these energies. The attention was shifted to the study of an inhomogeneous \((k \neq 0)\) pion condensate in dense warm nuclear matter, which formation could lead to a significant enhancement of the in-medium pion distributions with \(k \neq 0\) at non-zero temperature \(T \neq 0\) and baryon density \(n \gtrsim n_0\), where \(n_0 = 0.16\text{fm}^{-3}\) is the nuclear saturation density. Finally, a pion BEC in baryon enriched matter did not manifest itself in experiments at GSI energies [5].

Experimental evidence for a formation of the baryon-poor medium at midrapidity at SPS, RHIC and LHC energies [6–10] invited investigations of the properties of a dense and hot purely pion gas. Spectra of produced pions proved to be approximately exponential at intermediate transverse momenta, \(m_\pi \lesssim p_T \lesssim 7m_\pi\), but show an enhancement at low transverse momenta, here and below \(m_\pi\) is the pion mass, \(m_\pi = 139\text{MeV}\), and we imply \(\hbar = c = 1\). Already first attempts [11,12] to fit the \(p_T\) pion distributions in heavy-ion collisions at 200 \(\text{AGeV}\) by ideal-gas expressions required the pion chemical potential \(\sim 120 – 130\text{MeV}\). The magnitude of the fitted chemical potential depends on how one takes into account a flow of the medium [13] and contribution of resonances [14]. Subsequent more detailed analyzes of the SPS data [15,16], using the method proposed in [17] of extraction of the pion freeze-out density from the mid-rapidity particle densities and the femtoscopic radii, supported statement about significant enhancement of pion distributions at small transverse momenta.

Estimates [18] showed that at temperatures \(T \simeq 130 – 140\text{MeV}\) the rate of pion absorption becomes smaller than the rate of re-scattering. This implies that the total pion num-

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ber can be considered as approximately fixed during subsequent pion fireball expansion from such estimated value of the chemical freeze-out temperature \[19\], \(T = T_{\text{chem}} \simeq 130 - 140 \text{MeV}\), up to a lower temperature of kinetic (thermal) freeze-out \(T = T_{\text{kin}}\), also see \[20\].

The proximity of the pion chemical potential to the critical value of the BEC in the ideal gas initiated speculations that a Bose–Einstein condensate might be formed in the pion fireball prepared in ultrarelativistic nucleus-nucleus collisions. This idea motivated a study of the properties of a pion BEC \[21\], which could be possibly formed in collisions at highest SPS energy and then at still higher RHIC and LHC energies. Using the results of the first pion femtoscopy experiments, the value of the density of the pion system at the kinetic freeze-out was estimated as \(n \sim (1-6)n_0\). An ideal pion gas and an interacting relativistic pion gas within \(\lambda \rho^4\) interaction model were studied in \[21\] under the assumption that the number of pions of each species is dynamically fixed within the time interval between the chemical and kinetic freeze-outs. The most central collisions with high pion multiplicity were proposed as the most preferable for observation of effects of the pion BEC. The investigation of the pion BEC within the model with dynamically fixed pion particle number was continued in \[22,23\] utilizing the Weinberg Lagrangian. A role of inelastic reactions \(\pi^+ \pi^- \rightarrow 2\pi^0\), which however conserve the particle number, was also discussed.

Authors of hadronic resonance model, e.g. cf. Ref. \[24\] and references therein, assume that hadrons at RHIC and LHC conditions are produced at the hadronization temperature \(T_{\text{had}}\) after cooling of an expanding quark-gluon fireball. At the temperature of chemical freeze-out, \(T_{\text{chem}}\), all inelastic processes cease and only elastic processes are assumed to be possible. Fitting of particle yields measured at RHIC and LHC energies gives \(T_{\text{had}} \simeq T_{\text{chem}} \simeq 160 \text{MeV}\) for uniform fireball, cf. \[24\]. The chemical potentials of the conserving charges are determined by the number of the corresponding hadron species in the ideal gas state at \(T_{\text{chem}}\) plus the number that follows from decay of the resonances existed at \(T_{\text{chem}}\). At \(T = T_{\text{chem}}\) \(T_{\text{had}}\) pions are assumed to be at thermal equilibrium, i.e. having \(\mu_\alpha = 0\), \(\alpha = \pm\), 0.

Authors of a chemical non-equilibrium approach \[25,26\] performed the analysis of top-SPS and LHC data on mean particle multiplicities assuming a sudden decay of the quark-gluon medium into hadrons streaming then towards detector without a hadron re-scattering phase, i.e \(T_{\text{had}} \simeq T_{\text{chem}} \simeq T_{\text{kin}}\). For the best fits they used all particle distributions, including those for pions, with non-zero fugacities determined by the values of the quark fugacities. They extracted smaller values \(T_{\text{chem}} \simeq 140 - 145 \text{MeV}\) than \[24\] and values of pion chemical potentials \(\mu_\alpha \neq 0\) to be rather close to \(m_\pi\). Note that following \[26\] at \(T_{\text{chem}}\) about 80% of pions are hidden in hadron resonances producing “secondary pions” at the freeze-out. The later ones contribute to the total number of pions and to the extracted values \(\mu_\alpha\) together with primary pions from the ideal gas state.

As was evaluated in \[27-30\], for energies under consideration should be \(T_{\text{chem}} > T_{\text{kin}}\) and the kinetic freeze-out temperature for hadrons proves to be rather low, \(T_{\text{kin}} \simeq 100 - 120 \text{MeV}\). One assumes that between chemical and kinetic freeze-outs (within time interval \(0 < t < \tau_{\text{exp}}\) of the fireball expansion, here and below to be specific we put \(t_{\text{chem}} = 0\), \(t_{\text{kin}} = t_{\text{exp}}\)) pions can be described by a temperature \(T(t)\), a density \(n(t)\), and chemical potentials \(\mu_\alpha(t)\) for each pion species, cf. analysis \[20\] performed for SPS energies. The time scale of chemical equilibration at \(T \simeq 100 - 120 \text{MeV}\), was estimated as \(\tau_{\text{abs}} \sim 100 \text{fm}\) for SPS energies \[28,31\], being, thereby, much longer than the typical time of the thermal equilibration in the system, \(\tau_{\text{term}} \sim \text{few fm}\), and than the time of the fireball expansion, \(\tau_{\text{exp}} \sim 10 - 20 \text{fm}\) \[30,32\]. In these estimates one assumed that the absorption time \(\tau_{\text{abs}}\) is a typical time for 1 ↔ 3 processes and the thermalization time \(\tau_{\text{term}} \sim \tau_{\text{relat}}\) is characterized mainly by elastic 2 ↔ 2 processes. The relation among characteristic time scales \(\tau_{\text{abs}} \gg \tau_{\text{exp}} \gg \tau_{\text{term}}\) remains valid up to the moment of the kinetic freeze-out.

We continue to use that at \(T = T_{\text{had}}\) pions are produced from quark-gluon plasma in various non-equilibrium processes and significant part of them is hidden in hadron resonances like the \(\rho\) resonance. Then at \(T_{\text{kin}} < T(t) < T_{\text{chem}}\), for \(T_{\text{chem}} < T_{\text{had}}\), i.e. during the re-scattering phase, pions are liberated from resonances however they remain inside the fireball. The pion distributions at \(T_{\text{kin}} < T(t) < T_{\text{chem}}\) are characterized by \(T(t), \mu_\alpha(t)\) with approximately fixed total pion number (the dynamically fixed pion number, cf. \[21,23\]). Typical time for the reproduction of resonances is determined by their spectral function \(\tau_{\text{res}} \simeq A/2 \sim \text{fm.}\), cf. \[33\]. Their contribution to the number of secondary pions is maximal at \(T = T_{\text{chem}}\). At \(T = T_{\text{kin}}\) (which can be rather low \[34\]) number of resonances is already significantly smaller than at \(T = T_{\text{chem}}\) and thereby we may neglect their contribution to the final pion distribution but we include their contribution to the number of pions at \(T = T_{\text{chem}}\). Although at \(T_{\text{kin}} < T(t) < T_{\text{chem}}\) number of pions is dynamically fixed, their momentum distribution continues to change producing a bump at low momenta, the circumstance we will further exploit.

If for \(t = \tau_{\text{exp}}\) the chemical potentials of all three values \(\mu_\alpha\) remain lower than an effective pion mass, \(m_\pi^*\), there is no BEC but a Bose enhancement for low momenta in the pion distribution. If at some moment \(t_{\text{BEC}}\) \((t_{\text{BEC}} < \tau_{\text{exp}})\) the chemical potential of one of pion species reaches \(m_\pi^*\) then for \(t > t_{\text{BEC}}\) the BEC of the given species is forming till kinetic freeze-out. Although it was shown in \[35\] that in the course of the quasi-equilibrium isentropic expansion of the initially equilibrated ideal pion gas the chemical potential cannot reach the critical value, the non-equilibrium
overcooling effects may drive pions to the BEC [21,36–38]. The BEC is supported by additional injection of non-equilibrium pions from resonance decays [39], decomposition of a blurred phase of hot baryon-poor and pion-rich medium existing before the chemical freeze-out [40], sudden hadronization of supercooled quark-gluon plasma [41], and decay of the transient Bose–Einstein condensate of gluons or glueballs pre-formed at an initial stage in a heavy-ion collision, cf. [42–49]. Reference [37] demonstrated that before the formation of the Bose–Einstein condensate the initially non-equilibrium interacting pion gas still passes several stages including a wave-turbulence stage.

As we have mentioned, within the sudden hadronization model [25] the relative abundances of hadrons produced in lead-lead collisions at the SPS energies can be described if one accepts a rather large value of the pion chemical potential: for the projectile energy of $E_{\text{proj}} = 40 \text{AGeV}$ one finds $\mu_\pi \approx 128 \text{MeV}$ at $T = 129.5 \text{MeV}$, whereas for $E_{\text{proj}} = 80$ and $158 \text{AGeV}$ the best fits suggest $T \approx 136 \text{MeV}$ and the chemical potential grows up to the maximum value $\mu_\pi \simeq m_\pi$. Similarly, at the LHC in collisions with $\sqrt{s_{\text{NN}}} = 2.76 \text{TeV}$ the obtained pion spectra were fitted in [50] with the help of the ideal gas distribution and the chemical potential $\mu \simeq 134.9 \text{MeV}$ at $T = 138 \text{MeV}$. Thus, in both mentioned cases the value of the pion chemical potential suggested by the experimentally observed number of pions is very close to the critical value for BEC in an ideal pion gas. The existing estimates for the typical density of the pion fireball are contradictory [27,50] yielding values varying in a broad range, from $n \sim 0.8 n_0$ to $2.5 n_0$ for LHC energies. Recently, experimentalists in the ALICE Collaboration observed a significant suppression of three- and four-pion Bose–Einstein correlations in Pb–Pb collisions at $\sqrt{s_{\text{NN}}} = 2.76 \text{TeV}$ at the LHC [51,52]. This can be interpreted as there is a considerable degree of coherent pion emission in relativistic heavy-ion collisions [53,54]. Analysis [55] indicated that about 5% of pions could stem from the BEC. Further discussions of a BEC in heavy-ion collisions at LHC energies can be found in the review [56].

The higher is the pion multiplicity the more probable is to observe effects of the pion BEC [21,37]. The fluctuation effects ordinary are increased in the vicinity of the critical point of any phase transition. In particular, second-order phase transitions are accompanied by fluctuations of the order parameter observed in various critical opalescence phenomena in equilibrium systems [57]. If the system crosses the spinodal instability border at a first order phase transition, fluctuations begin to grow exponentially, cf. [58].

References [59,60] argued for the divergence of the normalized variance of the number of produced pions at the critical point of the BEC for the ideal pion gas. In experiments at SPS a growth of the normalized variance for the pion number with an increase of the collision energy (and the number of produced pions) was reported in Ref. [61], see also Fig. 16 in Ref. [62]. Also an enhancement of the normalized variance with an increase of the pion multiplicity was observed for $pp$ collisions in the energy range $50–70 \text{GeV}$ [63,64]. However, a care should be taken when one compares theoretical expectations for the thermal fluctuation characteristics with results of actual measurements, which incorporate background contributions, the dependence on center-of-mass energy, other dynamical effects, collision centrality, kinematic cuts, etc., cf. [60,65–67]. The most simple and still relevant description of fluctuations in a quasi-equilibrium system formed in heavy-ion collisions can be performed employing the grand-canonical ensemble formulation, since usually only a part of the system, typically around mid-rapidity, is considered. Thus energy and conserved quantum numbers may be exchanged with the rest of the system, which serves as a heat bath [67–69].

Self-consistent account for a pion-pion interaction in the Hartree approximation demonstrated that variance of the pion number in the system with an equal averaged number of pion species remains finite at the critical temperature [38] as well as the skewness and kurtosis [70]. A suppression of fluctuation effects occurs also due to the finiteness of the system [60]. Nevertheless, they remain to be enhanced near the critical point of BEC. Therefore, the appearance of a significant increase of particle number fluctuations could be considered as a signal that the pion system formed in heavy-ion collisions is approaching the BEC at some conditions.

In this work we calculate characteristics of particle number fluctuations in the non-ideal hot pion gas with the dynamically fixed number of particles considered in the grand-canonical formulation. In Sect. 2 we remind a formalism for the description of the pion non-ideal gas with the dynamically fixed particle number [38]. In Sect. 3 we apply the self-consistent Hartree approximation for an arbitrary relation between various pion fractions. The results are applicable for temperatures $T > \max T_{\text{cr}}^a$, where $T_{\text{cr}}^a$ is the critical temperature of the BEC for the pion species $a = \pm, 0$. We find general expressions for cross-variances of various pion species. Then the behavior of the cross-variances is analyzed for the temperature approaching the critical temperature of the BEC $\max T_{\text{cr}}^a$. In Sect. 4 we study fluctuations of the charge and the relative number of charged and neutral pions in a system with equal averaged number of pions for each isospin species. In Sect. 5 we consider properties of a system with a small isospin imbalance, either with a small net charge or with a small difference between the number of charged and neutral pions at zero net charge. For this, in Sect. 5.1 we calculate pion characteristics and in Sect. 5.2 we apply the results for estimations of the cross-variances in such systems. Conclusions are drawn in Sect. 6. Some details of calculations are collected in Appendices A, B, and C.
2 Non-ideal pion gas with dynamically fixed particle number. Formalism

We use the simplest model for the description of a non-ideal pion gas with the interaction $\lambda \phi \bar{\phi}$, where $\phi$ is the isospin vector in cartesian representation $\phi = (\phi_1, \phi_2, \phi_3)$. Following arguments of Ref. [38] applying this model to a pion-enriched system created on an intermediate stage of a heavy-ion collision, we keep in the Lagrangian density only the terms containing equal number of creation and annihilation operators, responsible for the absorption and production processes with a change of the number of particles, generated by the dropped terms, do not occur within the time window $0 < t < t_{\text{exp}}$. The resulting Lagrangian density reads [38]

$$\mathcal{L} = \sum_{a=\pm} \left( \partial_\mu \psi_a \partial^\mu \psi_a - m_\pi^2 \psi_a \bar{\psi}_a - \lambda (\psi_a \bar{\psi}_a)^2 \right) + \partial_\mu \bar{\psi}_0 \partial^\mu \psi_0 \bar{\psi}_0 - \frac{3}{2} \lambda (\psi_0 \bar{\psi}_0)^2 - 4\lambda (\bar{\psi}_+ \psi_+)(\bar{\psi}_- \psi_-) - 2\lambda (\bar{\psi}_0 \psi_0)(\psi_0 \bar{\psi}_0) - \lambda [\bar{\psi}_+ \bar{\psi}_- (\bar{\psi}_0)^2 + (\psi_0 \bar{\psi}_+)^2 + (\psi_0 \bar{\psi}_-)^2],$$

(1)

where $\psi_a$ and $\bar{\psi}_a$ stand for annihilation and creation operators of a pion of type $a = +, -, 0$, and $\bar{\psi}_2$ and $\bar{\psi}_3$ are defined as $\phi_2 \pm i \phi_3).$ From the comparison with the leading terms of the effective Weinberg Lagrangian [71], the coupling constant can be estimated as $\lambda = m_\pi^2/2 f_\pi^2 \simeq 1.13$, where $f_\pi = 93$ MeV is the weak pion decay constant, and $m_\pi = 139$ MeV is the free pion mass (we neglect small explicit isospin symmetry breaking).

In the Lagrangian density (1) we dropped the terms $\mathcal{L}_{3\rightarrow 1}$ containing non-equal number of creation and annihilation operators, responsible for the absorption and production processes, which are assumed to be not operative for $t < t_{\text{exp}}$. Due to this, we deal with three complex fields, whereas initial Lagrangian for pions is formulated in terms of three real fields $\phi_1, \phi_2, \phi_3$. The doubling of the degrees of freedom is related to the fact that we consider the system at time scale $t \sim t_{\text{last}}$ much less than $t_{\text{abs}}$. So, e.g., the positive and negative pions are not treated anymore as particles and antiparticles for $t \ll t_{\text{abs}}$, recall nonrelativistic Schrödinger description performed using the complex wave functions.

Interaction terms in two first lines in (1) allow for $\pi^- \pi^+$, $\pi^0 \pi^+$ and $\pi^0 \pi^0$ elastic re-scattering processes, which permit to equilibrate the energy-momentum in the corresponding $+$, $-$ and 0 pion subsystems. The corresponding terms in the third line allow an exchange of the energy-momentum in the $\pi^- \pi^+ \rightarrow \pi^- \pi^+$, and $\pi^0 \pi^0 \rightarrow \pi^+ \pi^0$ and $\pi^- \pi^0 \rightarrow \pi^- \pi^0$ reactions. The last term in (1) conserves the total number of pions but permits changes between isospin fractions in the system in reactions of the type $\pi^+ \pi^- \leftrightarrow \pi^0 \pi^0$ without a change of the total charge and isospin projection. The rates of all these binary processes are of the same order. The binary processes $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$ and associated with them multi-pion processes $\pi^+ \pi^- + k\pi \rightarrow \pi^0 \pi^0 + k\pi$ (with additional $k$ pions) tend to equilibrate the relative abundance of various pion species over the time interval between chemical and kinetic freeze-outs. If such an equilibrium is reached, one can use relation between chemical potentials of the species, $\mu_0 = \mu_+ + \mu_-$. Although, being important for kinetics of the system, the last term in (1) does not contribute to the modification of the pion spectrum within the self-consistent Hartree approximation [38], which we employ below.

In the self-consistent Hartree approximation the modification of the spectrum of pions of type $a$ is reduced to the replacement of the vacuum pion mass by an effective mass $m^2 (\mu, T)$, which may depend on the species under consideration, $\omega_a(p) = \sqrt{m^2 (\mu, T) + p^2}$.$m^2 (\mu, T)$ is reduced to the $m^2 (\mu, T)$ value by$ \Pi$.

$$\Pi_a = 2\lambda \sum_{b=+, -, 0} [\mathcal{K}]_{ab} \Pi_b,$$$\mathcal{K} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

(2)

where $a, b = (+, -, 0)$, and $d_b = d(m^2_b, \mu_b, T)$ are dimensionless functions of the effective mass $m^2_b$, chemical potential $\mu_b$ and temperature $T$.

$$d(m^2, \mu, T) = \int \frac{\frac{\partial^3}{(2\pi)^3} 1}{2m^2 \sqrt{m^2 + p^2} - 1} \times \frac{1}{e^{(\sqrt{m^2 + p^2} - \mu)/T} - 1}.$$ 

(3)

The ensemble-averaged pion densities are expressed through the chemical potentials as

$$n_a(\mu_a, T) = \frac{1}{V} \int \bar{N}_a = h_a (m^2 (\mu_+, \mu_-, \mu_0, T), \mu_a, T),$$

(4)

where $V$ is the volume of the system and $m^2 (\mu_+, \mu_-, \mu_0, T)$ is the solution of Eqs. (2) and the function $h$ is defined as

$$h(m^2, \mu, T) = \int \frac{1}{(2\pi)^3 e^{(\sqrt{m^2 + p^2} - \mu)/T} - 1}.$$ 

(5)

Notice that relation (4) makes sense only if $m_a \geq \mu_a$, otherwise there appears a pole in the particle momentum distribution. The critical temperature of such an instability (we call it the Bose–Einstein instability), $T_{\text{c}}$, is determined by equation $m_a (T_{\text{c}}) = \mu_a (T_{\text{c}})$. Thus, our consideration here and below is valid only for temperatures $T > \max_a T_{\text{c}}$. For $T < \max_a T_{\text{c}}$, Eq. (2) becomes invalid, since it does not
take into account a contribution of the Bose-Einstein condensate for the pions of that sort $a$, for which $T_{\text{cr}}^{a} = \max_{a} T_{\text{cr}}^{a}$.

3 Variances and cross-variances of pion numbers for system of arbitrary composition

In [38] it was argued that the minimum of free energy is realized in an interacting pion gas for a system with equal values of chemical potentials, i.e. with the equal average numbers of pions of different species. A variance of the pion number was studied for the symmetrical system when averaged densities for all three pion fractions are equal. In heavy-ion collisions of symmetric nuclei in some events pion subsystems can be produced with compositions deviating from that corresponding to the most energetically favorable case. Also, the processes changing the relative abundances of pionic species can have not enough time to equilibrate the system completely. Besides, for collision of initially not symmetric nuclei the pion gas at chemical freeze out is not symmetric, i.e. chemical potentials for various pion species are different. Therefore, it would be useful to extend analysis of [38,70] for such systems. For that, we introduce three independent chemical potentials, $\mu_{a}$. Then the partition function in the grand-canonical ensemble is equal to

$$Z(\mu_{+}, \mu_{-}, \mu_{0}, T) = \text{Tr} e^{-\hat{H} - \sum_{a} \mu_{a} \hat{N}_{a}}/T,$$

where $\hat{H}$ is the Hamiltonian describing the system with conserved number of pions of each sort and which yields the pion spectrum with the effective mass (2) in the Hartree approximation. $\hat{N}_{a}$ is the operator of the number of pions of type $a$, $\text{Tr}$ means ensemble averaging and integration over the phase volume.

Besides the ensemble-averaged number of particles for the given species

$$\langle \hat{N}_{a} \rangle = T \frac{\partial}{\partial \mu_{a}} \log Z \bigg|_{T, V},$$

one can calculate the higher order cumulants of the particle number operators as derivatives of $Z$ with respect to the chemical potentials. The cross-variances of the number of pions of various sorts are determined as [72,73]

$$\langle \hat{N}_{a} \hat{N}_{b} \rangle - \langle \hat{N}_{a} \rangle \langle \hat{N}_{b} \rangle = T^{2} \frac{\partial}{\partial \mu_{a}} \frac{\partial}{\partial \mu_{b}} \log Z \bigg|_{T, V},$$

and normalized cross-variances are given by

$$\sigma_{ab} = \frac{\langle \hat{N}_{a} \hat{N}_{b} \rangle - \langle \hat{N}_{a} \rangle \langle \hat{N}_{b} \rangle}{\sqrt{\langle \hat{N}_{a} \rangle \langle \hat{N}_{b} \rangle}} = \frac{T}{\sqrt{\hat{n}_{a} \hat{n}_{b}} \frac{\partial \mu_{a}}{\partial \mu_{b}}} \frac{\partial \hat{n}_{a}}{\partial \mu_{b}} \bigg|_{T, V},$$

with the obvious symmetrical relation $\sigma_{ab} = \sigma_{ba}$. These general expressions allow to express the normalized variance of the total number of particles, $N = N_{+} + N_{-} + N_{0}$ with $N_{a}$ standing for the number of pions of species $a$ in the given event, as

$$\sigma_{N} = \frac{\langle \hat{N}^{2} \rangle - \langle \hat{N} \rangle^{2}}{\langle \hat{N} \rangle} = \sum_{a,b=\pm,0} \frac{\sqrt{c_{a} c_{b} \sigma_{ab}}}{\sigma_{ab}},$$

where we introduced a relative fraction of pions of sort $a$:

$$c_{a} = \frac{\langle \hat{N}_{a} \rangle}{\langle \hat{N} \rangle} = \frac{n_{a}}{n_{+} + n_{-} + n_{0}}.$$

Fluctuations of the charge in the system, $Q = N_{+} - N_{-}$, are characterized by the quantity

$$\sigma_{Q} = 2 \frac{\langle \hat{Q}^{2} \rangle - \langle \hat{Q} \rangle^{2}}{\langle \hat{N}_{\text{ch}} \rangle + \langle \hat{N}_{0} \rangle} = \sum_{a,b=\pm,0} \frac{\sqrt{c_{a} c_{b} \sigma_{ab}(2\delta_{ab} - 1)}}{1 + c_{0}} \sigma_{ab}(1 - 3\delta_{ab}) (1 - 3\delta_{0b}).$$

Imbalance of charged versus neutral pions, $G = N_{\text{ch}} - N_{0}$, where $N_{\text{ch}} = \frac{1}{2}(N_{+} + N_{-})$, is characterized by

$$\sigma_{G} = 2 \frac{\langle \hat{G}^{2} \rangle - \langle \hat{G} \rangle^{2}}{\langle \hat{N}_{\text{ch}} \rangle + \langle \hat{N}_{0} \rangle} = \sum_{a,b=\pm,0} \frac{\sqrt{c_{a} c_{b} \sigma_{ab}(1 - 3\delta_{ab})(1 - 3\delta_{0b})}}{1 + c_{0}} \sigma_{ab}(1 - 3\delta_{ab}) (1 - 3\delta_{0b}).$$

As follows from Eqs. (8) and (9), the normalized cross-variance $\sigma_{ab}$ can be associated with fluctuations of particle densities

$$\sigma_{ab} = V \frac{\langle \hat{n}_{a} \hat{n}_{b} \rangle - \langle \hat{n}_{a} \rangle \langle \hat{n}_{b} \rangle}{\sqrt{\langle \hat{n}_{a} \rangle \langle \hat{n}_{b} \rangle}} = \frac{T}{\sqrt{\hat{n}_{a} \hat{n}_{b}} \frac{\partial \mu_{a}}{\partial \mu_{b}}} \frac{\partial \hat{n}_{a}}{\partial \mu_{b}} \bigg|_{T, V},$$

given by Eq. (4) and through the effective pion masses, being functions of all three chemical potentials. Using relations $\delta \hat{n}_{a} = -N_{0} \delta V / V^{2}$ one may introduce variance of the volume for fixed $N_{0}$ and $T$.

Speaking about fluctuations of intensive (not depending on $V$) and extensive (depending on $V$) variables one has to take into account that measuring characteristics of fluctuations at different experimental conditions may reflect different moments of the fireball evolution, like the chemical freeze-out, ($n_{\text{chem}}, T_{\text{chem}}$), and the kinetic one, ($n_{\text{kin}}, T_{\text{kin}}$). Right after the chemical freeze-out (for $t > 0$) the pion annihilation and creation processes cease and the total number of pions $N$ does not change, therefore. Thus, if we consider an ideal detector with full $4\pi$ geometry, fluctuations of the
total pion number reflect the state of the system at the chemical freeze-out. So, expression (10) describes fluctuations of the total pion number at $T(0) = T_{\text{chem}}$, $V(0) = V(T_{\text{chem}})$. However, the same quantity, $\sigma_N$, taken for $T = T_{\text{kin}}$ and $V(T_{\text{kin}})$ also characterizes fluctuations of the volume of the pion fireball at the kinetic freeze-out at measurements done in the $4\pi$ geometry.

The chemical potentials of pions for all three species evolve from moment of the chemical freeze-out ($t = 0$) until the kinetic freeze-out occurring at $T$ in the 4145 total pion number at $T_{\text{chem}}$. Depending on the specifics of the measurement.

If one measures correlations between pions emitted at different angles and in various momentum bins. Therefore, there exists a kind of thermodynamic reservoir for the subsystem of pions, which later reach the pion distribution. Since estimates show that thermalization time for mentioned processes is short we assume that the total number of pions remains approximately unchanged and neglect their contribution to the pion number. During the cooling process the pion system may approach the critical point of the BEC, at which various fluctuation characteristics may significantly grow. Although in the time interval between chemical and kinetic freeze-outs the total number of pions remains approximately unchanged an exchange of particles between pion species continues. Thus, if pions are measured in experiments with incomplete geometry and/or in a restricted momentum range, then the elastic pion-pion reactions and processes of the type $\pi^0\pi^0 \leftrightarrow \pi^+\pi^-$ change populations of pions of different isotopic species and in different momentum bins. Therefore, there exists a kind of thermodynamic reservoir for the subsystem of pions, which later reach the pion fireball at the kinetic freeze-out. Moreover, the quantities $\sigma_G$, $\sigma_G$ characterize fluctuations in the system at the kinetic freeze-out.

However, any case one should bear in mind that comparison of the results of idealized calculations and real measurements is very uncertain without a detailed study of experimental conditions. Thus we may say that, only if indeed a significant growth of fluctuation characteristics were observed, it could be associated with a closeness to the pion BEC either at the chemical freeze-out or at the thermal freeze-out, depending on the specifics of the measurement.

To calculate cross-variances (9) we need the derivatives of the densities

$$\frac{\partial n_a}{\partial \mu_b} = \frac{\partial h_a}{\partial \mu_b} \delta_{ab} + \frac{\partial h_a}{\partial m_a^2} \frac{\partial m_a^2}{\partial \mu_b},$$

where enter derivatives of the effective pion masses with respect to chemical potentials. From Eq. (2) taking into account the dependence of $d_a$ on $m_a^2$ and $\mu_a$ we get

$$\frac{\partial m_a^2}{\partial \mu_b} = 2\lambda \sum_{c=\pm,0} \left[ \frac{\partial (d_c, m_c^2)}{\partial \mu_b} + \frac{\partial (d_c, m_c^2)}{\partial m_c^2} \frac{\partial m_c^2}{\partial \mu_b} \right] K_{ca},$$

$$\frac{\partial h_a}{\partial \mu_a} = m_a^2 I_1^a, \quad \frac{\partial h_a}{\partial m_a^2} = -m_a^2 I_3^a,$$

$$m_a^2 \frac{\partial d_a}{\partial \mu_a} = \frac{1}{2} I_3^a, \quad m_a^2 \frac{\partial d_a}{\partial m_a^2} = -\frac{1}{4} I_2^a,$$

through auxiliary dimensionless quantities, $I_n^{(a)}$, being functions of $m_a^2$, $\mu_a$ and $T$,

$$\{I_1^1, I_2^1, I_3^a\} = \int \frac{d^3 p}{(2\pi)^3} \frac{\omega_a^2(p) + p^2}{m_a^2 \omega_a(p)^2} \frac{1}{\omega_a(p)^{1/2} m_a^2},$$

with $\omega_a(p) = \sqrt{m_a^2 + p^2}$. All integrals (18) diverge at the critical point of the induced BEC of pions of sort $a$, $T_{\text{cr}}^a$, determined by the equation

$$m_a^2(T_{\text{cr}}^a) = \mu_a(T_{\text{cr}}^a).$$

Indeed, for $\mu_a \to m_a^* - 0$, we get

$$I_n^{(a)} \mid_{\mu_a \to m_a^* - 0} \to \frac{T}{2^{3/2} \pi \sqrt{m_a^*} \sqrt{m_a^* - \mu_a}}.$$

Finally solving (16) we find the cross-variances:

$$\sigma_{\pm \pm} = \frac{m_a^2 T}{n_\pm} \left[ I_1^\pm - \frac{\lambda[I_3^\pm]^2}{D} (4 + 5 \lambda I_2^\pm) \right],$$

$$\sigma_{\pm \mp} = \frac{m_a^2 m_\pm^* T}{\sqrt{n_\pm n_\mp}} \frac{\lambda I_3^\pm I_3^\mp}{D} (4 + 5 \lambda I_2^\pm),$$

$$\sigma_{\pm 0} = \frac{m_a^2 T}{n_0} \left[ I_1^0 - \frac{\lambda[I_3^0]^2}{D} (6 + 5 \lambda(I_2^+ + I_2^-)) \right],$$

$$\sigma_{\pm 0} = \sigma_{0 \pm} = \frac{m_a^2 m_\pm^* T}{\sqrt{n_\pm n_0}} \left[ 2 \lambda I_3^+ I_3^\pm \right] (4 + 5 \lambda I_2^0),$$

$$D = \lambda I_2^0 + (4 + 5 \lambda I_2^0)[1 + \lambda(I_2^+ + I_2^-)].$$

For an ideal pion gas ($\lambda = 0$) fluctuations of pions of different species are independent, $\sigma_{ab} = \sigma_{aa} \delta_{ab}$, and the normalized variances of the particle number are given by one simple expression $\sigma_{aa} = T I_a^0 / n_a$. Thus, the particle number fluctuations in an ideal Bose gas diverge at the critical point of the BEC. The presence of this divergence in the variance of the particle number puts in doubt [74–76] the applicability of the grand canonical description of an ideal Bose gas at
temperatures close to the critical one. However, if the interaction is self-consistently taken into account, as it is done here within the Hartree approximation, the divergence disappears [38, 70]. For example, in a system with only one sort of particles we obtain the following expression for the normalized variances

$$\sigma_{aa} = \frac{n_a e^2 T}{m_a^2 T} \left[ I_1^a - \frac{\xi_a \lambda [I_1^a]^2}{1 + \xi_a \lambda I_2^a} \right], \quad \xi_\pm = 1, \quad \xi_0 = \frac{3}{2}. \quad (21)$$

This expression can be rewritten as

$$\frac{n_a e^2 T}{m_a^2 T} \sigma_{aa} = I_1^a - \frac{[I_1^a]^2}{I_2^a} + \frac{[I_3^a / I_2^a]^2}{\xi_a \lambda + 1/I_2^a}. \quad (22)$$

Taking into account that integrals $I_1^a$ diverge in a correlated way, see Eq. (19), we show that the divergent parts in the first two terms in Eq. (22) cancel each other exactly. The ratio of two divergent integrals in the numerator of the third term also proves to be finite. To identify the finite remainder one can use the following identities among quantities $I_n^a, n = 1, 2, 3,$

\[
\begin{align*}
I_1^a - I_2^a &= 2 (\tilde{d}_a + d_a), \\
I_2^a - I_3^a &= 2 (\tilde{d}_a - d_a),
\end{align*}
\]

where $d$ is defined in Eq. (3) and $\tilde{d}$ is given by

$$\tilde{d} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2m^2(\omega(p) + m^2)} e^{i \omega(p) - \mu} - 1. \quad (24)$$

Both $d_a$ and $\tilde{d}_a$ remain finite at the critical temperature $T^a_{ct}$. Now, expressing, e.g., $I_1^a$ and $I_2^a$ through $I_3^a$ and substituting in the first two terms in (22) we obtain the finite limiting value at $T \to T^a_{ct}$ equal to

\[
\left( I_1^a - \frac{[I_1^a]^2}{I_2^a} \right)_{T=T^a_{ct}} = 4 \tilde{d}_{ct,a}, \quad (25)
\]

where $\tilde{d}_{ct,a} \equiv \tilde{d}_a|_{T=T^a_{ct}}$. The final expression for the normalized variance $\sigma_{aa}$ at $T^a_{ct}$ reads

$$\sigma_{aa}(T^a_{ct}) = \frac{\mu^2_{ct,a}}{\xi_{\lambda a}^2 \lambda n_a} \left( 1 + 4 \xi_{\lambda a} \tilde{d}_{ct,a} \right). \quad (26)$$

Note that although $d_a, \tilde{d}_a$ and $h_a$ are functions of three variables $m^a_{\pm}, T_{\lambda} \mu_a$ and $T$, the quantities $\tilde{d}_{ct,a}, d_{ct,a}$ defined at $T = T^a_{ct}$ and $h_a|_{T = T^a_{ct}}$ are already functions of only one variable $t_{ct,a} = T^a_{ct}/\mu_{ct,a}$, where $\mu_{ct,a} = \mu_a(T^a_{ct}) = m^a_{\pm}(T^a_{ct})$.

Although the integrals in $d_a$ and $\tilde{d}_a$ cannot be evaluated analytically at arbitrary $T$, at $T = T^a_{ct}$ we can write expansions in terms of

\[
\begin{align*}
d_{ct,a} &= \frac{i_{ct,a}^{3/2}}{4\sqrt{2\pi}^{3/2}} \\
&\times \left( \xi(\lambda) + \frac{27}{8} t_{ct,a} \xi(\lambda) - \frac{15}{128} t_{ct,a}^{2} \xi(\frac{\lambda}{2}) + \cdots \right), \\
\tilde{d}_{ct,a} &= \frac{i_{ct,a}^{3/2}}{8\sqrt{2\pi}^{3/2}} \\
&\times \left( \xi(\lambda) + \frac{9}{8} t_{ct,a} \xi(\lambda) - \frac{75}{128} t_{ct,a}^{2} \xi(\frac{\lambda}{2}) + \cdots \right),
\end{align*}
\]

which rapidly converge for $t_{ct,a} \ll 1$. Our numerical evaluations show that in a gas consisting of pions of one species at $\lambda \approx 1$ we have $t_{ct,a} < 1$ for $0 < n \lesssim 1.5n_0$. Range of densities, for which $t_{ct,a} < 1$, is increased, if more pion species are present, e.g., for the gas with two species of equal fractions $t_{ct,a} < 1$ for $0 < n \lesssim 3n_0$ and for the isospin-symmetrical gas with three species, $t_{ct,a} < 1$ for densities $0 < n \lesssim 5n_0$ (see discussion in the next Sect. 4). An increase of $\lambda$ also extends this interval up to higher densities. Thus, expansion (27) in $t_{ct,a}$ is indeed useful.

For completeness we give also expansions for integrals $I_n^a$ for $T \to T^a_{ct}$:

\[
I_n^a(T \to T^a_{ct}) = \frac{T^a_{ct}}{2\pi \sqrt{\alpha_a (T - T^a_{ct})}} + \frac{1}{2\pi \sqrt{\alpha_a}} + \delta I_n^a_{ct,a} + O((T - T^a_{ct})/T^a_{ct}). \quad (28)
\]

To derive first two terms in (28) we used expansion at $T$ near $T^a_{ct}$:

\[
m^a_{\pm}(T) - \mu_a(T) \approx \frac{1}{2} (\alpha_a/\mu_{ct,a}) (T - T^a_{ct})^2, \quad (29)
\]

where $\alpha_a$ is a coefficient taken at $T = T^a_{ct}$. In Appendix A we demonstrate calculation of the coefficient $\alpha_a$ on example of the isospin-symmetrical medium. Finite parts, $\delta I_n^a_{ct,a}$, can be expressed with the help of Eq. (23) through the quantities $d_{ct,a}, \tilde{d}_{ct,a}$ given by (27) and $\delta I_{ct,a}^a$ given by

\[
\delta I_{ct,a}^3 = \frac{i_{ct,a}^{3/2}}{(2\pi)^{3/2}} \xi(\frac{\lambda}{2}) + \frac{3i_{ct,a}^{3/2}}{8(2\pi)^{3/2}} \left[ \xi(\frac{\lambda}{2}) - \frac{5}{16} t_{ct,a} \xi(\lambda) - \frac{35}{128} t_{ct,a}^{2} \xi(\frac{\lambda}{2}) + \cdots \right]. \quad (30)
\]

We note that for $T \to T^a_{ct}$ all infinite terms as well as a part of finite terms present in the integrals $I_n^a(T \to T^a_{ct})$ cancel away in the functions $d_a$ and $\tilde{d}_a$, up to the order $t_{ct,a}^{\lambda/2}$, more precisely, they cancel away the first two terms in (28) and the first term in (30). Therefore, it is preferable to express the quantities remaining finite at $T = T^a_{ct}$, through functions $d_{ct,a}$ and $\tilde{d}_{ct,a}$.

At the end, we should remind that although formally Eq. (28) is valid for $T \to T^a_{ct}$ for all $a = \pm, 0$, in reality, our
Fig. 1 The effective pion mass, \( m^* \) (solid lines), and the chemical potential \( \mu \) (dashed lines) as functions of temperature calculated for three values of the coupling constant \( \lambda = 1, 2, \) and \( 3, \) and several values of the pion gas density. Dash-dotted lines show the chemical potentials for the ideal pion gas (\( \lambda = 0 \)).

consideration should be modified for temperatures below
\[
\max T_{\text{cr}}^a = T_{\text{cr}}^a, \quad \text{since for such temperatures we already need to include the BEC for the species } a.
\]

4 Fluctuations in the isospin-symmetrical gas

In this section, we consider the isospin-symmetrical pion gas, where \( \mu_a = \mu \) and \( m_a^* = m^* \) for \( a = \pm, 0. \) The dependences of the effective pion mass \( m^* \) and the chemical potential \( \mu \) on the temperature \( T \) and the particle density \( n \) are determined by the set of equations, cf. Eq. (2),

\[
\begin{align*}
    m^* &= m_\pi^2 + 10\lambda m^n^2 d(m^*, \mu, T), \\
    n &= 3 h(m^*, \mu, T).
\end{align*}
\]  

(31)

Solutions of the system of Eqs. (31) for the effective pion mass and the chemical potential for various temperatures and densities are shown in Fig. 1. In the system with the interaction the effective pion mass is larger than the free pion mass. In Fig. 1 we see that the effective pion mass and chemical potential decrease with increase of the temperature, and grow with increase of the density and the coupling constant \( \lambda. \)

As found in [38], the critical temperature, \( T_{\text{cr}} \), is a monotonously increasing function of the density and it decreases with an increase of the interaction constant \( \lambda. \) The ratios \( T_{\text{cr}}/m_\pi \) and \( t_{\text{cr}} = T_{\text{cr}}/\mu_{\text{cr}} \) are shown in Fig. 2 by solid and dashed lines, respectively, as functions of a particle density for various values of \( \lambda. \) Since \( m^*(T) > m_\pi \) for all temperatures and densities, we have also \( t_{\text{cr}} < T_{\text{cr}}/m_\pi. \) As it is seen in Fig. 2, \( t_{\text{cr}} \) depends more weakly on the density than \( T_{\text{cr}}/m_\pi. \) For \( n \approx (1–2)n_0, \) \( t_{\text{cr}} \) begins to flatten out. Note also that for a given value of \( \lambda, t_{\text{cr}} \) is limited from above. Indeed, using Eqs. (31) we can combine an equation relating \( t_{\text{cr}} \) and \( n: \)

\[
\frac{n}{3m_\pi^3} = \frac{\tilde{h}(t_{\text{cr}})}{(1 - 10\lambda d_{\text{cr}})^{3/2}},
\]  

(32)

where \( \tilde{h}(t_{\text{cr}}) \equiv h(\mu_{\text{cr}}^3, \mu_{\text{cr}}, T_{\text{cr}})/\mu_{\text{cr}}^3. \) The right hand side of this equation depends only on \( t_{\text{cr}} \) and must be positive. Therefore we have the constraint \( d_{\text{cr}} < 1/(10\lambda). \) Hence, since \( d_{\text{cr}} \) is an increasing function of \( t_{\text{cr}}, \) we have the constraint \( t_{\text{cr}} < t_{\text{cr}}^{(\text{max})}, \) where \( t_{\text{cr}}^{(\text{max})} \) is the solution of equation

\[
d_{\text{cr}}(t_{\text{cr}}) = 1/(10\lambda).
\]

For example, for \( \lambda = 1, 2, \) and \( 3, \) we have \( t_{\text{cr}}^{(\text{max})} = 1.023, 0.6646, \) and \( 0.5145, \) respectively. Thus, Fig. 2 allows us to determine applicability range and precision of expansions (27). As we see, for densities \( n < 5n_0 \) the value \( t_{\text{cr}} \) does not exceed 0.8 for \( \lambda = 1 \) and 0.4 for \( \lambda = 3. \) For these values the expansion (27) converges very rapidly and first three terms are enough to reproduce the full value of
with a deviation on the level of 0.3% for $d_{\text{cr}}$ and of 1.4% for $\tilde{d}_{\text{cr}}$.

Now we apply the results derived in the previous section to study fluctuations of various quantities in the isospin-symmetrical pion gas, which properties are described by Eqs. (31). In this case $I^{+}_{a} = I^{-}_{a} = I^{0}_{a} = I_{a}$ and relations (20) are essentially simplified.

Consider now fluctuations of the number of pions of a particular species (14). For charged pions we get

$$\sigma_{\pm} = \frac{3m^2T}{n}\left(I_{1} - \frac{\lambda I_{2}^{0}(4 + 5\lambda I_{2})}{2(1 + \lambda I_{2})(2 + 5\lambda I_{2})}\right), \quad (33)$$

and thereby

$$\sigma_{\pm}(T \to T_{\text{cr}}) \rightarrow \frac{3T_{\text{cr}}\mu_{\omega}^{2}}{2n}I_{3}(T \to T_{\text{cr}}) \rightarrow \infty.$$ \hspace{1cm}

For neutral pions we find

$$\sigma_{00} = \frac{3Tm^2}{n}\left(I_{1} - \frac{\lambda I_{2}^{0}(3 + 5\lambda I_{2})}{(1 + \lambda I_{2})(2 + 5\lambda I_{2})}\right). \quad (34)$$

At $T = T_{\text{cr}}$ we obtain that

$$\sigma_{00}(T_{\text{cr}}) = \frac{12T_{\text{cr}}\mu_{\omega}^{2}}{5\lambda n}\left(1 + 5\lambda \tilde{d}_{\text{cr}}\right),$$

so the variance of the number of neutral pions remains finite in the critical point. For the cross-variances of charged pions we obtain a negative quantity

$$\sigma_{\pm \mp} = -3\frac{Tm^2}{n}\frac{\lambda I_{2}^{0}(4 + 5\lambda I_{2})}{2(1 + \lambda I_{2})(2 + 5\lambda I_{2})}. \quad (35)$$

For $T \to T_{\text{cr}}$ we get

$$\sigma_{\pm \mp}(T \to T_{\text{cr}}) \rightarrow -\frac{3T_{\text{cr}}\mu_{\omega}^{2}}{2n}I_{3}(T \to T_{\text{cr}}).$$

At the end, for the cross-variances of the charged and neutral pions we find

$$\sigma_{\pm 0} = -3\frac{Tm^2}{n}\frac{\lambda I_{2}^{0}}{(1 + \lambda I_{2})(2 + 5\lambda I_{2})}. \quad (36)$$

At $T = T_{\text{cr}}$ this quantity remains finite,

$$\sigma_{\pm 0}(T \to T_{\text{cr}}) \rightarrow -\frac{3T_{\text{cr}}\mu_{\omega}^{2}}{5\lambda n}.$$\hspace{1cm}

Thus, we see that all results for variances involving neutral pions, $\sigma_{00}$ and $\sigma_{\pm 0}$, remain finite at $T_{\text{cr}}$, whereas the variances associated with only charged pions, $\sigma_{\pm \pm}$ and $\sigma_{\pm \mp}$, diverge, whereas the combinations $\sigma_{\pm \pm} + \sigma_{\pm \mp}$ remain finite. Also from Eqs. (33), (34), (35) and (36) we find useful relation

$$\sigma_{00} = \sigma_{\pm \pm} + \sigma_{\pm \mp} - \sigma_{\pm 0}. \quad (37)$$

Now we apply the above formulas for fluctuations of the observables, which can be more directly accessed in experiments. First, we consider the normalized variance of the total number of pions, $N$, which is defined in (10). For isospin symmetrical case under consideration in this section $c_{a} = 1/3$ and therefore

$$\sigma_{N} = \frac{1}{3} \sum_{a,b} \sigma_{ab}. \quad (38)$$

Substituting here the results (33), (34), (35), and (36) we obtain

$$\sigma_{N} = \frac{3Tm^2}{n}\left(I_{1} - \frac{5\lambda I_{2}^{0}}{2 + 5\lambda I_{2}}\right), \quad (39)$$

recovering thereby the expression derived in [38]. Note that Eq. (39) has the same form as Eq. (21) but with $\xi_{a} \rightarrow \xi_{N} = \frac{I_{3}}{2}$. Making use the similarity between Eqs. (39) and (21) we can write expression for $\sigma_{N}(T_{\text{cr}})$ substituting in Eq. (26) $\xi_{N}$ instead of $\xi_{a}$. Then we have

$$\sigma_{N}(T_{\text{cr}}) = \frac{6T_{\text{cr}}\mu_{\omega}^{2}}{5\lambda n}\left[1 + 10\lambda \tilde{d}_{\text{cr}}\right]. \quad (40)$$

Now we turn to another quantity, $G$, which characterizes imbalance between charged and neutral pions. Its variance is defined in Eq. (13). If all pion species are equally populated, the average of $G$ vanishes, $\langle G \rangle = 0$. Then normalized variance of this quantity can be written through the partial fluctuations (14) as follows

$$\sigma_{G} = \frac{1}{4}(\sigma_{++} + 2\sigma_{+-} + \sigma_{--}) + \sigma_{00} - \sigma_{+0} - \sigma_{-0}.$$

(41)

With the help of relations (20) and (37) one can show that

$$\sigma_{G} = \frac{3}{2}(\sigma_{N} - 3\sigma_{\pm 0}). \quad (42)$$

Since $\sigma_{\pm 0} < 0$, we immediately conclude that the quantity $\sigma_{G}$ is larger than $\sigma_{N}$. Replacing Eqs. (33), (34), (35), and (36) in Eq. (41) we obtain

$$\sigma_{G} = \frac{9Tm^2}{2n}\left[I_{1} - \frac{\lambda I_{2}^{0}}{(1 + \lambda I_{2})}\right]. \quad (43)$$

This result is similar to that given by Eq. (21), but now with $\xi_{a} \to 1$. For $T \to T_{\text{cr}}$ we immediately obtain

$$\sigma_{G}(T_{\text{cr}}) = \frac{9T_{\text{cr}}\mu_{\omega}^{2}}{2\lambda n}\left[1 + 4\lambda \tilde{d}_{\text{cr}}\right]. \quad (44)$$

Thus the variance for $G$ remains finite at $T \to T_{\text{cr}}$. \hspace{1cm}

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Note that the value of the normalized variance of the imbalance quantity $G$ significantly differs from the normalized variance of the total particle number $N$. We emphasize that the result for $\sigma_G$, Eq. (43), ought to be used in analysis of experimental data of [64], where fluctuations of neutral pions were studied in selected events with a fixed total pion number.

Now consider fluctuations of the charge, $Q$, in the system. In the neutral isospin-symmetrical system, the average of this quantity vanishes, \( \langle Q \rangle = 0 \). The normalized variance of the charge, which we define in (12), can be expressed as

$$\sigma_Q = \sigma_{++} + \sigma_{--} - 2\sigma_{+-}.$$  \hfill (45)

Substituting (33) and (35) in (45) we find

$$\sigma_Q = 6 \frac{T}{n} \frac{\partial h}{\partial \mu} = 6 \frac{T m^2}{n} - I_1.$$  \hfill (46)

This result is similar to that for the ideal pion gas, as all the terms explicitly dependent on \( \lambda \) canceled out. Thus, \( \sigma_G \) is divergent at \( T \to T_{cr} \). Applying Eqs. (28) and (23) we have for \( T \to T_{cr} \): \( n \ll \lambda \)

$$\sigma_Q = \frac{3 \mu_0^2 T_{cr}}{\pi \sqrt{\alpha n}} \left[ \frac{T}{T - T_{cr}} + 2\pi \sqrt{\alpha \left( \delta I_{cr,3} + 2(\delta d + d_{cr}) \right)} \right],$$  \hfill (47)

where \( \alpha \) is given in Eq. (A28) in Appendix A, cf. also Eq. (29).

One may ask why self-consistent inclusion of the interaction renders the variances \( \sigma_N \) and \( \sigma_G \) finite but \( \sigma_Q \) divergent? In Appendix B we showed that the variance (45) can be written in general case as

$$\sigma_Q = \frac{6L I_1}{1 + \frac{\alpha}{\lambda} I_1} \frac{\partial}{\partial t} \left[ \sinh(\omega(t)/T) - 2 \frac{\delta(\omega(t)-\omega(p))}{\partial n_Q} \right]_{n_Q=0},$$  \hfill (48)

where the integral in the denominator includes a derivative of the difference of spectra of positive and negative pions with respect to the charge density. For our isospin symmetric Lagrangian (1) the spectra of \( \pi^\pm \) are the same, see also expansion (55) below, and the derivative vanishes and we recover Eq. (46). The inclusion of the Coulomb interaction will break the isospin symmetry, the difference between \( \pi^+ \) and \( \pi^- \) spectra is proportional to the charge density, and \( \partial(\omega(t)-\omega(p))/\partial n \big|_{n_Q=0} = \alpha \neq 0 \). Then, the variance \( \sigma_Q \) in (48) becomes finite for \( T \to T_{cr} \),

$$\sigma_Q = \frac{6L I_1}{1 + \frac{\alpha}{\lambda} I_1} \to \frac{T}{\alpha_Q n}.$$  \hfill (49)

The detailed account for electromagnetic interaction and calculation of \( \lambda Q \) go beyond the scope of this paper and have to be done carefully taking into account the modification of electromagnetic interaction in medium and preserving the gauge invariance. Nevertheless we can qualitatively estimate how the system will react on an increase of the charge fluctuations. In Appendix B we also showed that the variance (45) can be written through the derivatives of the free-energy density \( F \) with respect to the charge density \( n_Q = n_+ - n_- \), for \( n_Q \to 0 \), cf. Eq. (B13). The Coulomb contribution to the free-energy density, which appears because of the formation of a fluctuation characterized by a constant value of charge density \( n_Q \) in a sphere of radius \( R_{\|} \), is equal to

$$\delta F_{\text{Coul}} = \frac{3}{5} n_Q^2 V_0 \frac{e^2}{R_{\|}^2}, \quad V_0 = \frac{4\pi}{3} R_{\|}^3,$$  \hfill (50)

where \( e^2 \approx 1/137 \). Replacing this result in Eq. (B13) of Appendix B we find for \( T \to T_{cr} \) that

$$\sigma_Q(T \to T_{cr}) = 3 \frac{T_{cr}}{n} \left( \frac{\partial^2 F_{\text{Coul}}}{\partial n_Q^2} \bigg|_{n_Q=0} \right)^{-1} = \frac{15 T_{cr}}{8\pi R_{\|}^2 e^2 n}.$$  \hfill (51)

Here we took into account that for \( T \to T_{cr} \) there remains only the Coulomb term in the free energy depending on \( n_Q \). Variance \( \sigma_N \) becomes finite after the account for the pion self-interaction, \( \sigma_N \propto 1/\lambda \). Magnitude of \( \sigma_Q \) is, in turn, determined by the electromagnetic interaction and a typical size of fluctuations, \( R_{\|} \). Thus, we get that \( \sigma_Q/\sigma_N \sim \lambda/(e^2 \mu^2 R_{\|}^2) \). Since \( e^2 \ll \lambda \), for relevant values of \( R_{\|} \sim \text{several} 1/m_\pi \) the resulting quantity \( \sigma_Q \) could be considerably larger than \( \sigma_N \). Because in our study in this paper we disregard effects associated with the electromagnetic interaction compared with the effects of the strong interaction, we may employ \( \sigma_Q \) as follows from Eqs. (46), (47), being divergent for \( T \to T_{cr} \).

In terms of observables, the result, \( \sigma_Q \gg \sigma_N \), means that the stronger the multiplicity of an event deviates from the expected mean value the higher will be the probability that the numbers of positive and negative pions are different in this event. In view of the estimate (51) one may expect that the magnitude of the charge fluctuations seen in an experiment to be sensitive to a size of the system created in a nuclear collision, and therefore to the impact parameter.

Figure 3 demonstrates that variances for all three quantities, \( N, G \) and \( Q \), increase with a decrease of the temperature, whereby the hierarchy of fluctuation variances is \( \sigma_N < \sigma_G < \sigma_Q \), and an increase of \( \lambda \) leads to a reduction of the variances.

To compare the variances shown in Fig. 3 with experimental data from HIC we assume that the temperature on the horizontal axis corresponds to the temperature of kinetic
freeze-out, \( T_{\text{kin}} \), whereas the pion density is determined by the number of pions in the fireball at the chemical freeze-out, \( T = T_{\text{chem}} \), and the final volume of the fireball at \( T_{\text{kin}} \). These quantities can be calculated, e.g., within a statistical model fit of the experimental particle abundances. In Fig. 3 we see that for densities \( n \sim 1-2 n_0 \) the maximal value of the particle number variance proves to be not very high, even at the critical point of BEC, \( \sigma_N \lesssim 2-2.5 \) for \( \lambda = 1 \). For larger values of \( \lambda \) the variance would be even smaller.

The experimental information about the particle number fluctuations is rather controversial. Results on the centrality dependence of charged hadron multiplicity fluctuations in Pb+Pb collisions obtained by the NA49 [77] and WA98 [78] collaborations at top SPS energy show an increase of multiplicity fluctuations (up to \( \sigma_{N_{ch}} \sim 2 \)) with decreasing centrality of the collision in the forward hemisphere. An overview of charged particle fluctuations measured in a broad range of collision energies can be found in Ref. [79]. Reference [80] shows that the results are sensitive to the rapidity window and fluctuations of negatively charged hadrons are smaller than those for all charged hadrons, \( \sigma_{N_{ch}} \sim 1.05 \). Applying new methods for the particle identification and fluctuation analysis, Ref. [61] reported the fluctuations of the pion number of the order \( \sigma_{N_{\pi}} \approx 2 \) for the projectile energy 158 AGeV and \( \approx 1.6 \) for 80 AGeV. Preliminary results of recent experiment NA61/SHINE [81] show that variances of the charged particle number are essentially different for various nucleus pairs colliding at similar energies, e.g., at 150-158 AGeV in Be+Be collisions \( \sigma_{N_{ch}} \approx 1.2 \), whereas in Ar+Sc collisions \( \sigma_{N_{ch}} \approx 0.83 \). Analysis in [82] reveals strong dependence of the results for variances on the experimental acceptance and the centrality selection.

In the given section we have studied behavior of fluctuations in the isospin-symmetrical pion gas but disregarded the Coulomb effects. In a realistic situation owing to the fact that colliding nuclei are positively charged, the pion gas has most probably a positive charge imbalance. In some less probable events the charge imbalance might be also negative. In Sect. 3 we have derived general expressions valid for the system of arbitrary isospin composition. Now we are at the position to study in a more detail a situation, when at the moment of the chemical freeze-out the pion system is formed with an isospin imbalance and the equilibrium among the pion species is not completely established till the kinetic freeze-out.

5 Fluctuations in the pion gas with small isospin imbalance

Let the numbers of pions of each species are \( n_+ \), \( n_- \) and \( n_0 \) and the total density is \( n = n_+ + n_- + n_0 \). Let \( |\delta n_a| \ll n/3 \), where

\[
\delta n_a = n_a - n/3. \tag{52}
\]

We will calculate shifts of critical temperatures for various pion species, \( T_{cr}^a \), from the value for the isospin-symmetrical gas to determine the lowest temperature, \( \max T_{cr}^a \), up to which our consideration is valid.

5.1 Effective mass, chemical potential and critical temperature

The condition \( |\delta n_a|/n \ll 1 \) implies that the effective masses and the chemical potentials \( \mu_a = \mu + \delta \mu_a \) and \( m_a^2 = m^2 + \delta m_a^2 \), where \( \mu \) and \( m^2 \) satisfy Eqs. (31), differ only a little from their values in the symmetrical case. Expanding Eq. (4) up to linear order in \( \delta n_a \), \( \delta m_a^2 \) and \( \delta \mu_a \), we find the relation

\[
\delta \mu_a = \left( \frac{\partial h}{\partial \mu} \right)^{-1} \left[ \delta n_a - \frac{\partial h}{\partial m^2} \delta m_a^2 \right] = \frac{\delta n_a}{m^2 I_1} + \frac{I_3}{2 I_1} \frac{\delta m_a^2}{m^*}. \tag{53}
\]

A variation of the effective mass is given by the derivatives of the polarization operator, see Eq. (2), \( \delta m_a^2 = \sum_b \frac{\partial \Pi_{ab}}{\partial m_b^2} \delta m_b^2 + \sum_b \frac{\partial \Pi_{ab}}{\partial \mu} \delta \mu_b \), or explicitly

\[
\delta m_a^2 = 2 \lambda \frac{\partial (dm_a^2)}{\partial m^2} \sum_b [K]_{ab} \delta m_b^2 + 2 \lambda \frac{\partial (dm_a^2)}{\partial \mu} \sum_b [K]_{ab} \delta \mu_b. \tag{54}
\]

In Eqs. (53) and (54) all derivatives are taken for \( \delta n_a = 0 \), \( \delta m_a^2 = 0 \) and \( \delta \mu_a = 0 \), corresponding to the isospin-symmetrical state. Substituting (53) in (54) and solving the
system of equations for the effective mass-shifts we find
\[
\delta m^2_+ = \delta m^2_- = \frac{I_3}{I_1} \frac{2(5C)(\delta n_+ + \delta n_-) + \delta n_0}{m^*(1+2C)(1+5C)}, \\
\delta m^2_0 = \frac{\lambda}{I_1} \frac{\delta n_+ + \delta n_- + (3+10C)\delta n_0}{m^*(1+2C)(1+5C)},
\]
with \( C = \frac{1}{2} (I_2-I_3^2/I_1) \). Substituting (55) in (53), we recover changes of the chemical potentials. Quantities \( I_1, I_3 \) and \( C \) in Eqs. (53) and (55) are calculated for the isospin-symmetrical matter.

The shift of the critical temperatures, \( \delta T^a_{cr} = T^a_{cr} - T_{cr} \), because of the variation of the particle densities is determined from the relation \( \mu_a(T_{cr} + \delta T^a_{cr}) = m^*(T_{cr} + \delta T^a_{cr}), \) which we rewrite as \( \mu_a(T_{cr} + \delta T^a_{cr}) + \mu_a(T_{cr}) = m^*(T_{cr} + \delta T^a_{cr}) + \delta m^*_a(T_{cr}). \) For \( |\delta T^a_{cr}| \ll T_{cr} \) in linear approximation we find
\[
\delta T^a_{cr} = \frac{\delta m^2_0(T_{cr}) - 2m^*(T_{cr})\delta \mu_a(T_{cr})}{2m^*(T_{cr})\left(\frac{\delta \mu_a}{\delta T} - \frac{\delta m^*_a}{\delta T}\right)}_{T_{cr}}.
\]
Partial derivatives appearing in the denominator can be written as
\[
\left(\frac{\partial \mu_a}{\partial T} - \frac{\partial m^*_a}{\partial T}\right) = -\frac{n \chi}{Tm^2a I_1(1+5C)},
\]
where the function \( \chi(T) \) is defined in Eq. (A17) of Appendix A and its limiting value at \( T \to T_{cr} \) is given in (A18). Knowing the critical temperatures for various pion species we can find the maximal one, \( T_{cr}^a = \max_a T_{cr}^a \). Our consideration is valid only for temperatures \( T \geq T_{cr}^a \), since already slightly below the temperature \( T_{cr}^a \) one has to take into account presence of the Bose–Einstein condensate of the given pion species \( \tilde{a} \). Thus, for \( T < T_{cr}^a \) the pion excitation spectra for all species must be recalculated. Therefore, the values of other two critical temperatures \( T_{cr}^b \), for \( b \neq \tilde{a} \), being calculated without inclusion of the BEC of the species \( \tilde{a} \) prove to be physically irrelevant.

Consider first the system at a fixed density \( n \). Then
\[
\sum_a \delta n_a = 0 \quad (58)
\]
and we can rewrite
\[
\delta n_\pm = \frac{1}{3} \delta n_G \pm \frac{1}{2} \delta n_Q, \\
\delta n_0 = -\frac{2}{3} \delta n_G,
\]
where \( \delta n_Q \) is a charged density of the system and \( \delta n_G \) characterizes the excess of the number of charged pions above the neutral ones. Now two variables \( \delta n_Q \) and \( \delta n_G \) can be considered as independent ones, instead of the two variables chosen from \( \delta n_\pm \) and \( \delta n_0 \), with the relation (58) between them. Fluctuations of the quantities \( Q \) and \( G \) introduced in Eqs. (12) and (13), respectively, are characterized by the variances \( \sigma_Q \) and \( \sigma_G \).

Consider specific variations:
(i) Variations of \( \delta n_Q \) at \( \delta n_G = 0 \). Then \( \delta n_\pm = \pm \delta n_Q/2 \) and \( \delta n_0 = 0 \), and from Eqs. (53) and (55) we obtain
\[
\delta m^2_\pm = \delta \mu_0 = 0, \\
\delta \mu_\pm = \pm \frac{\delta n_Q}{2n} \eta^{(Q)}_\mu, \\
\eta^{(Q)}_\mu = \frac{n}{m^3a I_1},
\]
where we introduced the susceptibility \( \eta^{(Q)}_\mu \). Thus, the variation of the charge of the system, while keeping the total number of particles fixed, does not lead to a change of the effective pion mass, and only chemical potentials of charged pions change to accommodate the difference in \( \pi^+ \) and \( \pi^- \) concentrations.

The quantity \( \eta^{(Q)}_\mu \) is shown in Fig. 4 as a function of the temperature for two values of the density and three values of the coupling constant. As we see, \( \eta^{(Q)}_\mu \) decreases monotonously with the temperature decrease and vanishes at the critical temperature of the BEC. Note that the lines do not cross at one point, but there are three crossings at values of the temperature \( T \) separated by \( \simeq \pm 2 \) MeV.

The variation of the critical temperature is obtained after substitution of Eq. (60) in (56) and using Eq. (57),
\[
\frac{\delta T^a_{cr}}{\delta n_a} = \pm \frac{\delta n_a}{2n} \eta^{(Q)}_{\mu a}, \\
\delta T^0_{cr} = 0, \\
\eta_a = 1 - \frac{2\mu^a_A}{n} (d_{cr} + d_{ct}), \\
\eta^{(Q)}_{\mu a}(T_{cr}) = 1 + \frac{\lambda}{\lambda_{cr}} (d_{cr} + d_{ct}) - 20 \frac{\mu^a_A}{n} (d_{cr} + d_{ct})^2.
\]

The factor 2 \( \eta_a \) is separated so that for \( \lambda = 0 \) and \( \eta^{(Q)}_{\mu a} = \eta^{(Q)}_{\mu a}(T = T_{cr}) = 1 \) Eq. (61) reduces (after the replacement
\( \mu = m^* \rightarrow m_\pi \) to the one following directly from the variation of the second Eq. (31) at \( m^* = m_\pi, \mu = m_\pi \).

The functions \( \eta_{1}\) and \( \eta_n \) are shown in Fig. 5. They are growing functions of \( n \) and both are limited from above in view of the constraint \( d_{cr} < 1/(10a) \) following from the relation (32). The lower limits of these functions are realized for \( n \rightarrow 0 \) when \( d_{cr} \rightarrow 0 \) and \( \delta_{1}/d_{cr} \rightarrow \frac{1}{2} \eta \) and we have

\[
(\mu_{cr}d_{cr}/n)|_{n \rightarrow 0} \rightarrow 1/6, \tag{62}
\]

that follows from the expression for the critical temperature

\[
(T_{cr}/\mu_{cr})^{3/2} = \left(\frac{2\pi}{3}\right)^{3/2} n^{1/3} m^3
\]

valid in the non-relativistic limit.

(ii) Variations of \( \delta n_G \) at \( \delta n_Q = 0 \). Then \( \delta n_{\pm} = \delta n_G/3 \) and \( \delta n_0 = -2\delta n_G/3 \). Expressions (53) and (55) yield now

\[
\frac{\delta m_{\pm}}{m^*} = \frac{1}{2} \frac{\delta m_0^n}{m^*} = \frac{\delta n_G}{3n} \eta_m^{(G)}, \quad \eta_{m}^{(G)} = \frac{2\lambda n m}{m^* 3^{1/3} (1 + 2C)},
\]

\[
\frac{\delta m_{\pm}}{m^*} = \frac{1}{2} \frac{\delta m_0}{m^*} = \frac{\delta n_G}{3n} \eta_m^{(G)}, \quad \eta_{m}^{(G)} = \frac{1}{m^{1/3} I_1} \left[ 1 + \frac{4\lambda I_2}{1 + 2C} \right]. \tag{63}
\]

The susceptibilities \( \eta_m^{(G)} \) and \( \eta_{m}^{(G)} \) are plotted in Fig. 6 as functions of the temperature for \( n = n_0 \) and \( 2n_0 \) and for \( \lambda = 1, 2 \) and 3. We see that the susceptibility parameter \( \eta_m^{(G)} \) decreases very weakly with a temperature increase and increases with an increase of \( \lambda \) and \( n \). On the other hand, the quantity \( \eta_{m}^{(G)} \) is rapidly and monotonically increasing function of \( T \). At \( T = T_{cr} \) we have \( \eta_m^{(G)}(T_{cr}) = 0 \). Notice also that, as in Fig. 4, the lines for \( \eta_{m}^{(G)} \) do not cross in one point.

The imbalanced system with \( \delta n_G \neq 0 \) changes dynamically its composition in reactions \( \pi^+ + \pi^- \leftrightarrow 2\pi^0 \). These reactions are controlled by the difference of chemical potentials \( \Delta \mu = \mu_+ + \mu_- - 2\mu_0 = \delta \mu_+ + \delta \mu_- - 2\delta \mu_0 \). As follows from (63), \( \Delta \mu = \gamma \delta n_G \), where \( \gamma = \frac{4}{3} m^* \eta^{(G)}_m > 0 \). Hence, if \( \delta n_G > 0 \) (this means an excess of charged pions) then \( \Delta \mu > 0 \) and the reaction balance is shifted to the conversion of charged pions into the neutral ones. Oppositely, if \( \delta n_G < 0 \) and there are more neutral pions than charged ones, the neutral pions are converted into the charged ones, since \( \Delta \mu < 0 \). Thus, the interacting isospin-symmetrical pion gas is stable with respect to deviations from the equilibrium between charged and neutral pions, i.e. fluctuations in the quantity \( G \) do not grow spontaneously.

Shifts of the critical temperatures induced by the variations of \( \delta n_G \) are obtained by substituting Eq. (63) in (56),

\[
\frac{\delta T_{cr}^+}{T_{cr}} = - \frac{1}{2} \frac{\delta T_{cr}^0}{T_{cr}} = \frac{\delta n_G}{3n} \eta_{T_{cr}}^{(G)},
\]

\[
\eta_{T_{cr}}^{(G)} = \eta_m^{(G)} / X, \tag{64}
\]

where

\[
X = \frac{1 + 4\lambda \delta_{1}/d_{cr}}{1 + 2\lambda (d_{cr} - d_{cr})}. \tag{65}
\]

This quantity is plotted in Fig. 7 as a function of the density for three values of the coupling constant \( \lambda \). We see that the ratio is a monotonically increasing function of \( n \) and it increases with an increase of \( \lambda \). Taking into account the constraint \( d_{cr} < 1/(10\lambda) \) following from (32) and that \( \frac{1}{2} < d_{cr}/d_{cr} < 1 \) as follows from definitions (3) and (24) we can limit \( X \) from above as

\[
X < 1 + 4\lambda \delta_{1}/d_{cr}, X_{max} = \frac{14}{9}. \tag{66}
\]

Now consider a general case when both \( \delta n_Q \) and \( \delta n_G \) can be nonzero. Variations can be constructed as a linear
Let us study regions characterized by different relations between densities of the species. The regions are determined by three inequalities comparing the density variations \( \delta n_{a,-0} \). Using the relation \( \delta n_0 = -(\delta n_+ + \delta n_-) \) we can reduce inequalities among three \( \delta n_a \) to inequalities between \( \delta n_+ \) and \( \delta n_- \) and obtain

\[
\begin{align*}
\delta n_+ - \delta n_- & \geq 0, \\
\delta n_+ - \delta n_0 & \geq 0 \rightarrow \delta n_+ \geq -\frac{1}{2} \delta n_-, \\
\delta n_- - \delta n_0 & \geq 0 \rightarrow \delta n_- \geq -2 \delta n_-.
\end{align*}
\]

From the analysis of these inequalities we find three regions on the \((\delta n_-, \delta n_+)\) plane, where \( \pi^+ \), either \( \pi^- \) or \( \pi^0 \) are the most abundant species. These regions are shown in Fig. 8a by different hatching. For \( \delta n_- > 0 \) the solid line in Fig. 8a, \( \delta n_+ = \delta n_- \), divides the regions with maximal concentrations of \( \pi^+ \) (above the line) and \( \pi^- \) (below). For \( \delta n_- < 0 \) the dashed line, \( \delta n_+ = -\frac{1}{2} \delta n_- \), divides regions of \( \pi^+ \) and \( \pi^0 \) dominance (above and below the line, respectively). Also, the dash-dotted line, \( \delta n_+ = -2 \delta n_- \), separates regions of \( \pi^- \) and \( \pi^0 \) dominance.

To find regions of the maximal \( \delta T_{cr}^a \), we consider three differences of the critical temperature shifts,

\[
\begin{align*}
\delta T_{cr}^+ - \delta T_{cr}^- &= \frac{m^* \eta_{Ta}^G}{2n \eta_{\pi^+}^a} (\delta n_+ - \delta n_-), \\
\delta T_{cr}^+ - \delta T_{cr}^0 &= \frac{m^* \eta_{Ta}^G}{2n \eta_{\pi}^a} [(X + 3) \delta n_+ - (X - 3) \delta n_-], \\
\delta T_{cr}^- - \delta T_{cr}^0 &= \frac{m^* \eta_{Ta}^G}{2n \eta_{\pi}^a} [(X + 3) \delta n_- - (X - 3) \delta n_+],
\end{align*}
\]

where we used (69) and (65). These relations hold at least in the range of the applicability of the linear approximation that we use in this section.

We can reduce three inequalities among \( \delta T_{cr}^a \) to inequalities between \( \delta n_+ \) and \( \delta n_- \),

\[
\begin{align*}
\delta T_{cr}^+ - \delta T_{cr}^- & \geq 0 \rightarrow \delta n_+ \geq \delta n_-, \\
\delta T_{cr}^+ - \delta T_{cr}^0 & \geq 0 \rightarrow \delta n_+ \geq -\frac{1}{2} \frac{3(3X - 1)}{(3 + X)} \delta n_-, \\
\delta T_{cr}^- - \delta T_{cr}^0 & \geq 0 \rightarrow \delta n_- \geq -2 \frac{3(3X - 1)}{(3 - X)} \delta n_-, 
\end{align*}
\]

where we employed that \( X < X_{\text{max}} < 3 \). We brought here inequalities (74) and (75) in the form closely resembling inequalities (71) and (72), respectively. Therefore, we can directly see how the border lines between regions with maximal critical temperature for a certain pion species may differ from the regions, where the given species is most abundant.

For \( \lambda = 0 \) we have \( X = 1 \) and inequalities in (73), (74), and (75) become identical to inequalities in (70), (71), and

![Image](https://example.com/image.png)
Fig. 8 The $(\delta n_-, \delta n_+)$ plane showing a state with an arbitrary small isospin imbalance at a condition $\sum_\lambda \delta n_\lambda = 0$. Since the results for the ratios $\delta n_+/n$ are valid only for small density variations we introduce a scaling factor $S \ll 1$. Panel (a): Regions with maximal density of pion species $a$. Solid, dashed and dash-dotted lines are the zero lines for inequalities (70), (71) and (72), respectively. Panel (b): Regions where the maximal critical temperatures are realized for pions of a specific species. Short-dashed and dash-dot-dotted lines are the zero lines for inequalities (74) and (75), calculated for $\lambda = 1$ and $n = 3n_0$, which corresponds to $X = 1.17$. Solid, dashed and dash-dotted lines are the same as on panel (a). Doubly-hatched regions correspond to the case when the neutral pions are most abundant in the system but the maximal critical temperature is $T^+_{cr}$ in the upper region and $T^-_{cr}$ in the lower region. On panel (c) it is shown the same as in panel (b), but for $X = X_{\text{max}}$ given in Eq. (66).

The critical temperature $T^+_{cr}$ is maximal for the most abundant pion species $\tilde{\alpha}$. For $\lambda \neq 0$ we have $X > 1$, since $0 < \tilde{d} < d$ following Eqs. (3), (24), see also Fig. 7. We see that the slope of the border line [Eq. (74)] between the regions with maximal $T^+_{cr}$ and $T^-_{cr}$ decreases, and for $\delta n_- < 0$ this border lies below the border [Eq. (71)], which separates regions of dominance of $\pi^+$ and $\pi^0$ meson, respectively, cf. dashed and short-dashed lines in Fig. 8b. On the other hand, the slope of border line [Eq. (75)] separating regions with maximal $T^-_{cr}$ and $T^+_{cr}$ becomes steeper and for $\delta n_- > 0$ this line falls below the line [Eq. (72)], which separates the regions of the $\pi^-$ and $\pi^0$ dominance, cf. dash-dotted and dash-dot-dotted lines in Fig. 8b. Thus, there appear two regions, where although the most abundant species is $\pi^0$, the maximal critical temperature is realized for $\pi^0$ mesons in one region and for $\pi^- \pi^0$ mesons in the other one. Both regions are marked by double hatching. Figure 8b is calculated for $X = 1.17$ corresponding to the density $n = 3n_0$ and $\lambda = 1$. With an increase of $X$ the anomalous regions grow and the case of $X = X_{\text{max}} = 14/9$ is shown in Fig. 8c.

(iii) Isospin imbalance with a variation of the density. Let us show how the relations derived above at the fulfilled condition (58) are applied in the case when this condition is not satisfied. Assume we have an isospin-symmetrical system with a density $n$, i.e. at $n_+ = n_0 = n_0 = n/3$. Let us change the densities of $\pi^+$, $\pi^0$ and $\pi^0$ by small quantities $\Delta n_+$, $\Delta n_-$ and $\Delta n_0$, respectively, with $\sum_\lambda \Delta n_\lambda \neq 0$. The total density is now $n' = n + \sum_\lambda \Delta n_\lambda$. In order the quantity $n'$ would be the density of the isospin-symmetrical system characterized by the densities of each pion species $n'/3$, the deviations from this equilibrium density should satisfy equations

$$\delta n_+ = \frac{n}{3} + \Delta n_+ - \frac{n'}{3} = \frac{2}{3} \Delta n_+ - \frac{1}{3} (\Delta n_+ + \Delta n_0),$$

$$\delta n_- = \frac{n}{3} + \Delta n_- - \frac{n'}{3} = \frac{2}{3} \Delta n_- < \frac{1}{3} (\Delta n_+ + \Delta n_0),$$

$$\delta n_0 = \frac{n}{3} + \Delta n_0 - \frac{n'}{3} = \frac{2}{3} \Delta n_0 - \frac{1}{3} (\Delta n_+ + \Delta n_-).$$

In terms of $\delta n_Q$ and $\delta n_G$ we have, respectively

$$\delta n_Q = \Delta n_+ - \Delta n_-, \quad \delta n_G = \frac{\Delta n_+ + \Delta n_-}{2} - \Delta n_0. \quad (77)$$

We see that the densities (76) satisfy now the condition $\sum_\lambda \delta n_\lambda = 0$ and the expressions derived above, (67), (68), and (69), are valid after the replacements $n \to n'$ with $m^*$, $\mu$ and all $\eta$’s now evaluated at the density $n'$.

For completeness let us now recalculate effective masses, chemical potentials and the critical temperature for new densities in the isospin-symmetrical case. We can still apply expressions (53), (55) and (56) for a small variation of the total particle density without any change of the isospin composition $\delta n_\lambda = \delta n/3 = \frac{1}{3} \sum_\lambda \Delta n_\lambda$. Then we find

$$\frac{\delta m^2}{m^2} = \frac{\delta n}{3n} \eta_m, \quad \frac{\delta n'}{3n'} \eta_m(N) = \frac{5\lambda I_3 n}{m^2 I_1(1 + 5C)},$$

$$\frac{\delta \mu_a}{m^*} = \frac{\delta n}{3n} \eta_\mu, \quad \frac{\delta \mu_a(N)}{m^*} = \frac{n(1 + \frac{5}{2}\lambda I_2)}{m^2 I_1(1 + 5C)}. \quad (78)$$
Certainly, the same relations, which we derived here from expansions of Eq. (4), could be obtained directly from the variations performed in Eqs. (31).

Quantities $\eta_m^{(N)}$ and $\eta_{\mu}^{(N)}$ are shown in Fig. 9 as functions of a temperature. In general $\eta_m^{(N)}$ decreases rather weakly with increase of $T$. The quantity $\eta_{\mu}^{(N)}$ demonstrates much stronger dependence on $T$, than $\eta_m^{(N)}$, increasing by a factor $3 - 4$ with an increase of $T$ from $T \approx T_{ct}$ to $\sim 1.5T_{ct}$.

It is instructive to compare the $\eta_{\mu}^{(N)}$ parameter with the corresponding parameter for the ideal pion gas

$$\eta_{\mu}^{(N),id} = \frac{n}{m^2 \pi I_1},$$

(79)

which is shown in Fig. 9 by the dash-dotted line. We see the qualitative difference in the behaviour of this parameter, when the temperature approaches $T_{ct}$ from above and the interaction is switched on. Since $I_1$ increases strongly for $T \rightarrow T_{ct}$, the quantity $\eta_{\mu}^{(N),id}$ also decreases strongly and tends to zero in contrast to $\eta_{\mu}^{(N)}$ in Eq. (78), where the divergency of $I_1$ in the denominator is compensated by the divergency of $I_2$ in the numerator. We also have $\eta_m^{(N)}(T_{ct}) = 2n\eta_{\mu}^{(N)}(T_{ct})$, and therefore

$$\left(\frac{\delta m^*}{\delta n} - \frac{\delta \mu}{\delta n}\right)|_{T \rightarrow T_{ct}} \rightarrow 0.$$  

(80)

The shift of the critical temperature, which in the given case is the same for all pion species, can be presented with the help of Eq. (56) as

$$\frac{\delta T_{ct}}{T_{ct}} = \frac{\delta n}{3n} \eta_{\mu}^{(N)}, \quad \eta_{\mu} = 1 - \frac{2\mu_{ct}}{n}(d_{ct} + d_{cr}),$$

$$\eta_m^{(N)} = 1 - 10 \lambda \frac{(d_{ct} + d_{cr})^2}{\mu_{ct} n \chi_{ct}}.$$  

(81)

Now we present the results for shifts of effective masses, chemical potentials and critical temperature for the density variations (76):

for mass shifts

$$\delta m_{\pm}^2 = \frac{m^*}{3n} \left[ \left( \eta_{\mu}^{(N)} + 2\eta_m^{(N)} \right) \eta_{\mu}^{(Q)} \right],$$

$$\delta m_{\pm}^2 = \frac{m^*}{3n} \left[ \left( \eta_{\mu}^{(N)} - \eta_m^{(G)} \right) \Delta n_+ + (2\eta_m^{(G)} + \eta_m^{(N)}) \Delta n_0 \right],$$

(82)

for chemical potentials

$$\delta \mu_{\pm} = \frac{m^*}{2n} \left[ \left( \eta_{\mu}^{(G)} + 2\eta_m^{(N)} \right) \Delta n_+ + (\eta_{\mu}^{(N)} - \eta_m^{(G)}) \Delta n_0 \right],$$

(83)

and for critical temperatures

$$\delta T_{ct}^\pm = \frac{m^*}{2n\eta_{\mu}} \left[ \left( \eta_{\mu}^{(G)} + 2\eta_m^{(N)} \right) \eta_{T_{ct}}^{(Q)} \right],$$

$$\delta T_{ct}^0 = \frac{m^*}{3n\eta_{\mu}} \left[ \left( \eta_{T_{ct}}^{(N)} - \eta_{T_{ct}}^{(G)} \right) \Delta n_+ + (2\eta_m^{(G)} + \eta_m^{(N)}) \Delta n_0 \right].$$

(84)

5.2 Variances and cross-variances of the particle number

Here we apply the relations derived above to analyze the particle number fluctuations. We assume that the relation $\sum_1 \delta n_1 = 0$ is fulfilled. We calculate the variances in special cases considered above.

(i) Variations of $\delta n_Q \delta n_G = 0$. Consider first a slightly charged gas at $\delta n_G = 0$. Then pion densities are $n_{\pm} = n/3 \pm \delta n_Q/2$, $n_0 = n/3$, and $\delta n_0 = 0$. In this case $\delta n_+ = -\delta n_-$. and, as one can see, in Fig. 8 (b,c) the line $\delta n_+ = -\delta n_-$ passes in the 2nd and 4th quadrants through the regions corresponding to the maximal critical temperature $T_{ct}^+$, if $\delta n_Q = -\delta n_+ > 0$ and $T_{ct}^-$, if $\delta n_Q < 0$.

To be specific let first $\delta n_Q > 0$. We are interested to find characteritics of fluctuations for $T > T_{ct}^+$, especially when
$T$ approaches $T_{cr}^+$ from above. In this case, only quantities $I_n^+(T \rightarrow T_{cr}^+)$ diverge as in (19) and others, $I_n^-(T \rightarrow T_{cr}^-)$ and $I_0^+(T \rightarrow T_{cr}^-)$, remain finite. Then, setting in relations (20) $I_n^+ \rightarrow \infty$, taking into account Eq. (25) and keeping terms $I_n^0(T_{cr}^+)$, we obtain

\[
\sigma_+ (T_{cr}^+) = \frac{\mu_{cr}^2 + m_0^2 (T_{cr}^+)}{\lambda n_+} \left[ 1 + 4 \lambda \tilde{a}_+ (T_{cr}^+) \right] + \frac{\lambda \mu_{cr}^2 I_{cr}^0 (T_{cr}^+)}{4 + 5 \lambda} + \frac{\lambda^2 I_{cr}^0 (T_{cr}^+)}{5 + 5 \lambda},
\]

\[
\sigma_- (T_{cr}^+) = \frac{T_{cr}^+ m_0^2 (T_{cr}^+)}{n_0} I_1 (T_{cr}^+),
\]

\[
\sigma_+ (T_{cr}^+) = \sigma_- (T_{cr}^+) = -\frac{T_{cr}^+ \mu_{cr}^2 + m_0^2 (T_{cr}^+)}{\sqrt{n_+}} - \frac{5 \lambda I_{cr}^0 (T_{cr}^+)}{4 + 5 \lambda},
\]

\[
\sigma_{00} (T_{cr}^+) = \frac{T_{cr}^+ m_0^2 (T_{cr}^+)}{n_0} \left[ I_1 (T_{cr}^+) - \frac{5 \lambda [I_{cr}^0 (T_{cr}^+)]^2}{4 + 5 \lambda} \right],
\]

\[
\sigma_0+ (T_{cr}^+) = \sigma_{0+} (T_{cr}^+) = -\frac{27 \lambda^2 \mu_{cr}^2 + m_0^2 (T_{cr}^+)}{4 + 5 \lambda},
\]

\[
\sigma_0- (T_{cr}^+) = \sigma_{0-} (T_{cr}^+) = 0.
\]

where $\mu_{cr,+} = m_0^2 (T_{cr}^+)$. We have to emphasize that these expressions can be used for any isospin composition of the system (without invoking smallness of $n_0$ and the expansions derived in Sect. 5.1) with the only constraint $T > T_{cr}^+$, where $T_{cr}^+$ is the maximal critical temperature. We note that for a non-vanishing value of $\delta n_0$ all partial variances in (85) take finite values at $T_{cr}^+$ (if $\delta n_0 > 0$). In contrast, in the isospin-symmetrical case the variances $\sigma_{\pm \pm}$ and $\sigma_{\pm \mp}$ in Eqs. (33) and (35) diverge when the temperature tends to $T_{cr}$.

Now let us analyze relations (85) for $\delta n_0 \ll n$. The result proves to be dependent on the order, whether first $T \rightarrow T_{cr}^+$ for fixed $\delta n_0$ and then $\delta n_0 \rightarrow 0$ or $\delta n_0 \rightarrow 0$ at fixed $T$ and then $T \rightarrow T_{cr}^+$. In the former case expressions in (85) are not valid but we may use the original relations (33), (34), (35), (36). After letting $\delta n_0 \rightarrow 0$ we reduce temperature down to $T_{cr}$ and recover the results for the isospin-symmetrical gas.

If we put first $T \rightarrow T_{cr}^+$ at fixed $\delta n_0 > 0$ to evaluate changes of variances at decreasing values of $\delta n_0$, we may use results of Appendix C for expansions of $I_1 (T_{cr}^+)$ and $I_0 (T_{cr}^+)$ given in Eqs. (C9) and (C10), respectively, expressions for the effective masses $m_{-0}^2 (T_{cr}^+)$ given by Eqs. (C3), expressions for the chemical potential $\mu_{cr,+}$ given by Eqs. (C4) and (C8), and the expression for the critical temperature $T_{cr}$ from Eq. (C5). Then, keeping terms $\propto 1/\delta n_0$ and those not depending on $\delta n_0$ we obtain

\[
\sigma_+ (T_{cr}^+) = \frac{3 \mu_{cr}^2 T_{cr}^+}{n} \left( \beta (Q) \frac{n}{\delta n_0} + b_{++} (Q) \right) + O(\delta n_0),
\]

\[
\sigma_- (T_{cr}^+) = \frac{3 \mu_{cr}^2 T_{cr}^+}{n} \left( \beta (Q) \frac{n}{\delta n_0} + b_{-+} (Q) \right) + O(\delta n_0),
\]

\[
\sigma_0+ (T_{cr}^+) = \frac{3 \mu_{cr}^2 T_{cr}^+}{n} \left( \beta (Q) \frac{n}{\delta n_0} + b_{0+} (Q) \right) + O(\delta n_0),
\]

\[
\sigma_0- (T_{cr}^+) = \frac{3 \mu_{cr}^2 T_{cr}^+}{n} \left( \beta (Q) \frac{n}{\delta n_0} + b_{0-} (Q) \right) + O(\delta n_0),
\]

where

\[
\rho (Q) = \frac{\mu_{cr}^2 T_{cr}^+}{2 \pi^2 a_{cr}^2} \frac{\eta_n}{\delta n_0},
\]

and the background terms are equal to

\[
b_{0+} (Q) = b_{+} = \delta I_{cr,1} + \delta I_{cr,2} + 2 - \delta I_{cr,3} = 4 \delta a_{cr} + b_{00}^0 - \delta I_{cr,3},
\]

\[
b_{+} (Q) = b_{+}^0 - \delta I_{cr,3},
\]

\[
b_{+} (Q) = \frac{\beta (Q) \eta_{T_{cr}}}{2 \eta_n} \left( \frac{5}{2} + \frac{5}{2} \frac{b_{+} (Q)}{\mu_{cr}^2} + \pi \alpha_{cr}^{1/2} \delta I_{cr,1} \right),
\]

\[
b_{+} (Q) = b_{+}^0 - \frac{3}{2} \beta (Q) + b_{cr} + \delta I_{cr,2},
\]

\[
b_{+} (Q) = \frac{3}{2} \beta (Q) + b_{cr} + \delta I_{cr,1}.
\]

There are three sources for background terms: variation of the density for $\frac{3}{2} \beta (Q)$; variation of the critical temperature for $b_{cr}$; and the finite part of the integrals (18) for $\delta I_n$ and $b_{1}$. All quantities on the right-hand sides in Eqs. (86) are calculated for the isospin-symmetrical pion gas at the critical temperature $T_{cr}$ (although we continue to consider $T > T_{cr}^+$).

In (86) we separated explicitly potential long terms controlling the magnitude of variances $\propto 1/\delta n_0$. Also explicitly are separated terms $\propto 1/\lambda$ to recover the limit of the ideal gas. Note that the pole terms, $\propto 1/\delta n_0$ and $\propto 1/\lambda$, cancel out, e.g., in the combinations $\sigma_{++} = \sigma_{00} - \sigma_{0+} + \sigma_{\pm \pm}$ and $\sigma_{--} + \sigma_{\pm \mp}$.

The obtained values for the cross-variances (86) differ essentially from the corresponding expressions (33), (34), (35), (36) for the isospin-symmetrical case. The expressions for the isospin-symmetrical case are not recovered in the limit $\delta n_0 \rightarrow 0$.

Substituting expressions (86) and (88) in the definitions of the variances $\sigma_N$ and $\sigma_G$, Eqs. (10) and (13), we obtain

\[
\sigma_N (T_{cr}^+) = \frac{\mu_{cr}^2 T_{cr}^+}{n} \left( \frac{6}{5 \lambda} + 12 \delta a_{cr} + O(\delta n_0) \right),
\]

\[
\sigma_G (T_{cr}^+) = \frac{\mu_{cr}^2 T_{cr}^+}{2 \lambda n} \left( 1 + 4 \lambda \delta a_{cr} + O(\delta n_0) \right).
\]
In the limit $\delta n_Q \to 0$ these expressions reduce to those found for the isospin-symmetrical case at $T = T_{cr}$, (40) and (44). The differences may appear only at the first order in $\delta n_Q$. Oppositely, the variance of the total charge in the system, which behaves as $1/(T - T_{cr})$ for $T \to T_{cr}$ in the isospin-symmetrical system, see Eq. (46), proves to be finite for $T \to T_{cr}^+$ in a slightly asymmetrical system,

$$\sigma_Q(T_{cr}^+) = \frac{3\mu_Q^2 T_{cr}}{n} \left( 4\beta(nQ) - \frac{6}{5\lambda} + 2b_1 \right) + 3\delta I_{cr,3} + 4b_3 T_{cr}^+ + O(\delta n_Q), \quad (90)$$

for small but finite values of $\delta n_Q$. Interestingly, the critical value of the variance (90) depends now explicitly on the value of the self-interaction constant $\lambda$, that was not the case in the purely isospin-symmetrical case, see Eqs. (46) and (47).

So far we assumed $\delta n_Q > 0$. If the system is slightly negatively charged, i.e. $\delta n_Q < 0$, the maximal critical temperature is $T_{cr}^-$ and Eqs. (86), (89), and (90) can be used after the replacements $T_{cr}^+ \to T_{cr}^-$ and $\delta n_Q \to -\delta n_Q$.

(ii) Variations of $\delta n_G$ at $\delta n_Q = 0$. Consider now another case of isospin asymmetry, when the number of charged and neutral pions, $(n_+ + n_-)/2 - n_0$, differs by $\delta n_Q$,

$$n_+ = n_- = \frac{n + \delta n_Q}{3}, \quad n_0 = \frac{n - 2\delta n_Q}{3}. \quad (91)$$

As we show below, the values of variances are essentially different in dependence on the sign of $\delta n_Q$. Therefore, we consider separately the cases $\delta n_Q < 0$ and $\delta n_Q > 0$.

First, let us consider the case $\delta n_Q < 0$. As we can see in Fig. 8, along the line $\delta n_Q = 0$ the maximal critical temperature is realized for neutral pions, $T_{cr}^0$. Then the quantities $I_{1,2,3}(T \to T_{cr}^0)$ diverge, whereas $I_{1,2,3}(T_c^0)$ are finite for finite values of $|\delta n_G|$. Then in the limit $T \to T_{cr}^0$, expressions (20) take the following forms:

$$\sigma_{\pm}(T_{cr}^0) = \frac{3\mu_{cr,0}^2}{n} \left( \frac{1}{\sqrt{n+} - \sqrt{n-}} \left( I_{1+}^2(T_{cr}^0) \right) \right),$$

$$\sigma_{\mp}(T_{cr}^0) = \frac{3\mu_{cr,0}^2}{n} \left( \frac{1}{\sqrt{n+} + \sqrt{n-}} \left( I_{1-}^2(T_{cr}^0) \right) \right),$$

$$\sigma_{00}(T_{cr}^0) = \frac{2\mu_{cr,0}^2}{3n} \left( 1 + 6\lambda \delta_0(T_{cr}^0) \right),$$

$$\sigma_{\pm,0}(T_{cr}^0) = \frac{3\mu_{cr,0}^2}{n} \left( \frac{1}{\sqrt{n+} + \sqrt{n-}} \left( I_{1\pm}^2(T_{cr}^0) \right) \right). \quad (92)$$

Here $\mu_{cr,0} = m_0^*(T_{cr})$. Take into account that for $\delta n_Q = 0$ we have $\mu_+ = \mu_- = \mu_\infty$, and, therefore, put $I_{n}^{(+)} = I_{n}^{(-)}$. These expressions do not rely on the smallness of $|\delta n_G|$ and can be used for any neutral system with an access of $\pi^0$ mesons, when the maximal critical temperature is $T_{cr}^0$. For a non-vanishing value of $\delta n_G$ all partial variances in (92) take finite values at $T_{cr}^0$.

To employ expressions (92) at $0 < -\delta n_G < n$ we use expansions of $I_{n,\pm}^+(T_{cr}^0)$ given by Eq. (C19) in Appendix C and expansions for the critical temperature (C13), the effective mass (C17) and the chemical potential (C18). Substituting them in Eqs. (92), we obtain

$$\sigma_{\pm}(T_{cr}^0) = \frac{3\mu_{cr,0}^2}{n} \left( \frac{\beta(G)}{2} \frac{n}{\delta n_G} - \frac{3}{10\lambda} - b_{\pm}^G + O(\delta n_G) \right),$$

$$\sigma_{\mp}(T_{cr}^0) = \frac{3\mu_{cr,0}^2}{n} \left( \frac{\beta(G)}{2} \frac{n}{\delta n_G} + \frac{3}{10\lambda} + b_{\mp}^G + O(\delta n_G) \right),$$

$$\sigma_{00}(T_{cr}^0) = \frac{3\mu_{cr,0}^2}{n} \left( \frac{4}{5\lambda} + b_{00}^G \right) + O(\delta n_G),$$

$$\sigma_{\pm,0}(T_{cr}^0) = \frac{3\mu_{cr,0}^2}{n} \left( \frac{1}{5\lambda} + O(\delta n_G) \right), \quad (93)$$

where

$$\beta(G) = \beta(Q) \frac{\sqrt{3} \eta_{cr}^E}{2 \eta_{Tcr}^E}, \quad (94)$$

and the background terms are

$$b_{\pm}^G = \frac{1}{2} b_I + b_{Tcr}^G + \frac{1}{2} b_{cr,1} + \frac{3}{4} \beta(G) - b_m,$$

$$b_{\mp}^G = \frac{1}{2} b_I - b_{Tcr}^G - \frac{1}{2} b_{cr,1} - \frac{3}{4} \beta(G) + b_m,$$

$$b_{00}^G = b_I, \quad b_m = \frac{b_{\mp}^G \lambda n}{4\mu_{cr,0}^2 (1 + 2C)}. \quad (95)$$

The quantities $b_I$ and $b_{Tcr}$ are the same as in Eq. (88). We see the qualitative difference in the structure of limiting variances for the temperature approaching the temperature of the Bose–Einstein instability for variations $n_Q$ and $n_G$, cf. Eqs. (86) and (93). The variances $\sigma_{-}$ and $\sigma_{\mp}$ as functions of $\delta n_Q$ do not contain terms $\propto 1/\lambda$ in contrast to the same variances as functions of $\delta n_G$. Also $\sigma_{0-}$ and $\sigma_{-0}$ vanish as functions of $\delta n_Q$ but remain finite as functions of $\delta n_G$. Nevertheless, $\sigma_{0+}$ and $\sigma_{+0}$ are equal for both cases, since $b_{00}^G = b_{00}^G$. In spite of these differences, if we combine variances (93) in the variances $\sigma_N$ and $\sigma_G$ defined by Eqs. (38) and (41), we obtain

$$\sigma_N(T_{cr}^0) = \frac{\mu_{cr,0}^2}{n} \left( \frac{6}{5\lambda} + 12\lambda \delta_0 + O(\delta n_G) \right),$$

$$\sigma_G(T_{cr}^0) = \frac{9\mu_{cr,0}^2}{2\lambda n} \left( 1 + 4\lambda \delta_0 + O(\delta n_G) \right). \quad (96)$$
We see that the leading terms are the same as for the isospin symmetric system (40) and (44) and also for the case of $\delta n_G$ variations (89). On the other hand the expression for the variance $\sigma_G$ is quite different,

$$\sigma_G(T_{cr}^0) = \frac{6 \mu^2_{cr} T_{cr}}{n} \left( - \beta (G) \frac{n}{\delta n_G} + 2 b^G_{+1} - b_I \right). \tag{97}$$

This expression does not contain terms $\propto 1/\lambda$ in difference with the result (90).

We turn now to the case $\delta n_G > 0$. Then $T_{cr}^+ = T_{cr}^-$ is the largest critical temperature, and $I_{1,2,3}^\pm$ diverge at $T \to T_{cr}^+ + 0$. Expressions (20) yield in this case:

$$\sigma_{\pm \pm}(T_{cr}^+) = \frac{T_{cr}^+ \mu^2_{cr} + I_{1}^+(T \to T_{cr}^+)}{2n_+} + \frac{\mu^2_{cr} + 3}{4n_+} \left( 1 + 8 \tilde{\lambda} (T_{cr}^+) \frac{\delta n_{cr}}{\lambda} + \frac{\lambda I_1^0(T_{cr}^+)^2}{4 + 5 \lambda I_0^0(T_{cr}^+)} \right),$$

$$\sigma_{\pm \mp}(T_{cr}^+) = \sigma_{\pm \pm}(T_{cr}^+) - \frac{T_{cr}^+ \mu^2_{cr} + I_{1}^+(T \to T_{cr}^+)}{2n_+} + \frac{\mu^2_{cr} + 3}{4n_+} \left( 1 + 8 \tilde{\lambda} (T_{cr}^+) \frac{\delta n_{cr}}{\lambda} + \frac{\lambda I_1^0(T_{cr}^+)^2}{4 + 5 \lambda I_0^0(T_{cr}^+)} \right),$$

$$\sigma_{00}(T_{cr}^+) = \sigma_{00}(T_{cr}^-) = \frac{T_{cr}^+ \mu^2_{cr} + I_{1}^+(T \to T_{cr}^+)}{n_0} \left( I_1^0(T_{cr}^+) \frac{\delta n_{cr}}{\lambda} + \frac{5 \lambda I_1^0(T_{cr}^+)^2}{4 + 5 \lambda I_0^0(T_{cr}^+)} \right),$$

$$\sigma_{\pm 0}(T_{cr}^+) = \sigma_{\pm 0}(T_{cr}^-) = -\frac{T_{cr}^+ \mu^2_{cr} + I_{1}^+(T \to T_{cr}^+)}{n_0} \left( I_1^0(T_{cr}^+) \frac{\delta n_{cr}}{\lambda} + \frac{5 \lambda I_1^0(T_{cr}^+)^2}{4 + 5 \lambda I_0^0(T_{cr}^+)} \right). \tag{98}$$

Here, $I_1^0(T \to T_{cr}^+)$ is a divergent term given by expansion (28). We see that the variances (98) obtained for $\delta n_G > 0$ differ from the variances (85) and (92) for $\delta n_Q$ and $\delta n_G < 0$ variations. Particularly, in the present case the variances for charged pions, $\sigma_{\pm \pm}$ and $\sigma_{\mp \mp}$, are divergent. However, the sums $\sigma_{\pm \mp} + \sigma_{\mp \pm}$ remain finite. As Eqs. (85) and (92), these expressions follow directly from (20) and can be used for arbitrary isospin composition, provided the charge of the system in zero and the maximal critical temperature is realized for $\pi^\pm$ mesons.

To exploit relations (98) in the limit of a small isospin imbalance $0 < \delta n_G \ll n$ we need the expansion in $\delta n_G$ for the critical temperature (C21), the effective mass (C26) and the chemical potential (C27). Substituting these relations in Eq. (98) together with the expansion for $I_1^0(T_{cr}^+)$ given by Eq. (C28) we obtain

$$\sigma_{\pm \pm}(T_{cr}^+) = \frac{1}{2} \sigma_{\infty} + \frac{3 \mu^2_{cr} T_{cr}}{n} \left( \frac{3}{10 \lambda} + \frac{1}{2} b_{00}^G \right) + O(\delta n_G),$$

$$\sigma_{\pm \mp}(T_{cr}^+) = -\frac{1}{2} \sigma_{\infty} + \frac{3 \mu^2_{cr} T_{cr}}{n} \left( \frac{3}{10 \lambda} + \frac{1}{2} b_{00}^G \right) + O(\delta n_G),$$

$$\sigma_{00}(T_{cr}^+) = \frac{3 \mu^2_{cr} T_{cr}}{n} \left( \frac{4}{5 \lambda} + b_{00}^G \right) + O(\delta n_G),$$

$$\sigma_{\pm 0}(T_{cr}^+) = \sigma_{0 \pm}(T_{cr}^+) = -\frac{3 \mu^2_{cr} T_{cr}}{n} \left( \frac{1}{5 \lambda} + \frac{1}{2} b_{00}^G \right), \tag{99}$$

where we denoted the divergent term as

$$\sigma_{\infty} = \frac{3 \mu^2_{cr} T_{cr}}{n} \left( \frac{T_{cr}}{2\pi \sqrt{\alpha(T - T_{cr})}} + \frac{1}{2\pi \sqrt{\alpha}} + \delta I_{cr} \right).$$

We see that these results are quite different from the corresponding results (93) obtained for $\delta n_G < 0$. Particularly, expressions (99) do not contain terms $\propto 1/\delta n_G$. Nevertheless, if we substitute expressions (99) in the definitions (38) and (41), we obtain

$$\sigma_N(T_{cr}^+) = \frac{\mu^2_{cr} T_{cr}}{n} \left( \frac{6}{5 \lambda} + 12 \tilde{\lambda} d_{cr} + O(\delta n_G) \right),$$

$$\sigma_G(T_{cr}^+) = \frac{9 \mu^2_{cr} T_{cr}}{2 \lambda n} \left( 1 + 4 \lambda d_{cr} + O(\delta n_G) \right), \tag{100}$$

i.e., the leading terms are exactly the same as in Eq. (89) up to linear terms in $\delta n_G$ and $\delta n_Q$ and agree with the results for the isospin symmetrical system. For the variance of the charge of the system defined by Eq. (45) we find from (99)

$$\sigma_Q(T_{cr}^+) = 2 \sigma_{\infty},$$

that agrees with the result (47) for the isospin symmetrical case.

The main result of this section is that the partial variances of the numbers of various pion species, $\sigma_{ab}$, are quite sensitive to a small isospin imbalance of the system. So, the variances of charged pions, $\sigma_{\pm \pm}$ and $\sigma_{\mp \mp}$, and the variance of the total charge of the system, $\sigma_Q$, turn out to be finite at the critical point of the BEC for finite values of $\delta n_G \neq 0$ or $\delta n_G < 0$, whereas they are divergent in the isospin symmetrical case and if $\delta n_G$ is finite but positive. On the other hand, the variances $\sigma_N$ and $\sigma_G$ are weakly sensitive to the isospin imbalance and up to terms $O(\delta n_{Q,G})$ coincide with those for the isospin symmetrical case.

6 Conclusion

Experimental evidence for formation of the baryon-poor medium at midrapidity at SPS, RHIC and LHC energies [6–10] calls for investigations of properties of a dense and hot pion gas. In this paper, we studied fluctuations in the self-interacting pion gas, which could be formed at an intermediate or latest stage at the heavy-ion collisions of ultrarelativistic energies. Characteristics of fluctuations measured at different experimental conditions may give an information on different moments of the fireball evolution, like the chemical freeze-out, at the total density and the temperature ($T_{chem}$, $T_{chem}$), and the kinetic one, at ($T_{kin}$, $T_{kin}$). Right after the chemical freeze-out the pion annihilation and creation processes cease and the total number of pions $N$ almost
does not change. Thus, if we consider an ideal detector with full 4π geometry, a variance of the total pion number reflects the state of the system at the chemical freeze-out. The same quantity, taken for \( T = T_{\text{kin}} \) and the volume \( V(T_{\text{kin}}) \), characterizes fluctuations of the volume of the pion fireball at the kinetic freeze-out at measurements done in the 4π geometry.

Although in the time interval between chemical and kinetic freeze-outs the total number of pions remains fixed, an exchange of particles between pion species continues owing to the 2 ↔ 2 reactions. Thus, if pions are measured in experiments with incomplete geometry and/or in a restricted momentum range, then the elastic pion-pion reactions and processes of the type \( \pi^0 \pi^0 \leftrightarrow \pi^+ \pi^- \) change populations of pions of different isospin species and in different momentum bins. Therefore, there exists a kind of thermodynamic reservoir for the subsystem of pions, which reach the detector later, and the grand-canonical formulation can be relevant in such a situation. If one measures correlations between pions emitted at different angles and in various momentum bins, one may get an information about the state of the pion fireball at the kinetic freeze-out.

The temperature decreases when the pion system evolves from the chemical to the kinetic freeze-out. Chemical potentials \( \mu_a \) of pions for all three species \( a \) grow towards their effective masses \( m^*_a \) and the system may reach the critical point of the Bose–Einstein condensation, first for one of the species and with a further decrease of the temperature for other pion species, cf. [21]. If a significant growth of fluctuation characteristics were observed, it could be associated with a closeness to the pion Bose–Einstein condensation either at the chemical freeze-out or the thermal freeze-out, depending on the specifics of the measurement.

In the given paper, continuing our recent studies [38,70] we consider behavior of the strongly interacting pion gas with a dynamically fixed number of particles within the self-consistent Hartree approximation in the \( \lambda \phi^4 \) model. Within the grand-canonical approach for the pion system of arbitrary particle composition we calculated normalized cross-variances of the numbers of pions of various species, \( \sigma_{ab} \), \( a, b = \pm, 0 \) (defined in Eq. (9)) and variances of the total number, \( N = N_{\pi^+} + N_{\pi^-} + N_{\pi^0} \), the charge number, \( Q = N_{\pi^+} - N_{\pi^-} \), and the imbalance between charged and neutral pions, \( G = (N_{\pi^+} + N_{\pi^-})/2 - N_{\pi^0} \) (defined in Eqs. (10), (12), (13)).

Then we focused on the description of the isospin-symmetrical system with equal densities of \( \pi^+ \), \( \pi^- \) and \( \pi^0 \), being considered above the critical temperature of the Bose–Einstein condensation, \( T_{\text{cr}} \), which value in this case is one and the same for \( \pi^+ \), \( \pi^- \) and \( \pi^0 \). We found that various variances show different behaviors when the temperature approaches \( T_{\text{cr}} \): the quantities describing only the charged particles, \( \sigma_{\pm \pm} \) and \( \sigma_{\pm \mp} \), diverge at \( T_{\text{cr}} \), whereas variances involving neutral pions, \( \sigma_{\pm 0} \) and \( \sigma_{00} \), remain finite. Also the particular combination of variances, \( \sigma_{\pm \pm} + \sigma_{\pm \mp} \), does not diverge at \( T \to T_{\text{cr}} \). Then we analyzed variances of the total particle number, the charge and the difference between charged and neutral numbers \( \sigma_{N,G} \) (see Eqs. (39), (43), and (46), respectively). All these quantities increase with a decrease of the temperature keeping a hierarchy \( \sigma_G < \sigma_G < \sigma_Q \), see Fig. 3. When the system approaches the critical temperature, variances \( \sigma_{N,G} \) and \( \sigma_G \) remain finite because of the self-consistent account of a pion interaction, whereas \( \sigma_Q \) diverges. The results for \( \sigma_G \) ought to be used in the analysis of the experimental data, where fluctuations of neutral pions are studied in selected events with a fixed total pion number. To understand why the self-consistent inclusion of the interaction renders the variances \( \sigma_G \) and \( \sigma_G \) finite but \( \sigma_Q \) divergent we showed that the variance of the charge can be written through the derivatives of the free-energy density \( F \) with respect to the charge density \( n_Q = n_+ - n_- \), for \( n_0 \to 0 \), when only the Coulomb term remains in the free energy depending on \( n_Q \). It proved to be that the variance of the total number \( \sigma_{N,G} \) does not depend on the size of the fluctuation region whereas the Coulomb contribution in \( \sigma_Q \) depends on the size of the fluctuation, \( R_0 \), because of the long-range Coulomb force. Thus we get that \( \sigma_Q / \sigma_N \sim \lambda / e^2 R_0^2 \). Since \( e^2 = 1/137 \ll \lambda \), for relevant values of \( R_0 \) ~ several \( 1/m_\pi \) the resulting quantity \( \sigma_Q \) could be considerably larger than \( \sigma_N \). Thus, disregarding effects associated with the electromagnetic interaction compared with the effects of the strong interaction we may employ \( \mu_G \), being divergent for \( T \to T_{\text{cr}} \).

In the actual heavy-ion collisions initial nuclei have a charge and may have unequal numbers of protons and neutrons, i.e. an isospin imbalance. Therefore, in a realistic situation, the created pion-enriched system could have some charge imbalance and an imbalance between the numbers of charged and neutral pions. To address this situation we studied a pion system formed with a small isospin imbalance in more details. First we considered variations for the system at a fixed total density \( n \) allowing for a small variation of the charge density, \( \delta n_Q \), and the isospin density, \( \delta n_G = (n_+ + n_-)/2 - \delta n_0 \). We considered specific cases: (i) a variation of \( \delta n_Q \) at \( \delta n_G = 0 \) and (ii) a variation of \( \delta n_G \) at \( \delta n_Q = 0 \). The variation of \( \delta n_Q \) only, case (i), does not change the effective pion mass, whereas the variation of \( \delta n_G \), case (ii), results in a change of the pion mass: an increase of \( \delta n_G \) leads to an increase of masses of charged pions and a reduction of the neutral pion mass, (63). Variations of chemical potentials of pions were parameterized in both cases as \( \delta \mu_a / m^* = (\delta n_a / n) \eta_a^{(N,G)} \), where \( \delta n_a \) is a variation of the density of pions of type \( a = \pm, 0 \). The susceptibility parameters \( \eta_a^{(N,G)} \) decrease with a temperature decrease, and \( \eta_a^{(G)} \) vanishes at \( T_{\text{cr}} \), whereas \( \eta_a^{(G)} \) remains finite, see Figs. 4 and 6, respectively. Then we cal-
culated shifts of critical temperatures of Bose–Einstein insta-
rities (determined by the equation \( m^a_c(T^a_c) = \mu_a(T^a_c) \)) with respect to the critical temperature of the Bose–Einstein insta-
ility, \( T^a_c \) for an isospin symmetrical pion gas, with the den-
sity \( n \). In both cases defined above, the shift \( \delta T^a_c = T^a_c - T^a \)
can be written as \( \delta T^a_c = (\delta n_a/n) \eta^{(G)}_{T^a_c} \), where sus-
sceptibilities \( \eta^{(N)}_{T^a_c} \) are illustrated in Figs. 5 and 7. Using this result and exploiting established relations between \( \eta^{(G)}_{T^a_c} \) and \( \eta^{(Q)}_{T^a_c} \), Eq. (65), we were able to determine what pion species
would have a largest \( T^a_c \) in the case of an arbitrary isospin imbalance. For an ideal gas the answer is obvious: the most
abundant species will have the largest \( T^a_c \). For an interacting
system we have found two regions, where neutral pions are
most abundant but the maximal critical temperature is realized either for positive pions or for negative pions, see Fig. 8.

Having determined the maximal among three values of the critical temperatures, \( T^a = \max_a T^a_c \), we can specify the temperature above which our consideration is valid, since already slightly below the temperature \( T^a_c \) one has to take into account presence of the Bose–Einstein condensate of the given pion species \( a \) that we did not do.

As next, we addressed the question of how the pion prop-
erties will change, if in an isospin-symmetrical system with density \( n \) the densities of various pion species are changed by some small amounts, \( \delta n_a \). The total density of the isospin-
symmetrical system changes in this case, and we calculated susceptibilities of the effective mass, the chemical potential and the critical temperature to small variations of the density, \( \eta^{(N)}_{m, \mu, T^a_c} \), see Fig. 9 and Eq. (81). Then, Eqs. (82), (83), and (84) provide responses to the changes of \( m^*_a, \mu_a \) and \( T^a_c \) in general case.

Finally, we studied how the pion number variances, \( \sigma_{ab} \)
behave when the temperature approaches \( T^a_c \) from above, i.e.,
when the system is still in a non-condensed phase, for the sys-
tem with an isospin imbalance. Equations (85), (92) and (98)
provide the results for the cases (i) and (ii), specified above,
without invoking smallness of the isospin imbalance. It was
found that the variances are quite sensitive to isospin imbalance.
For example the variances of charged pions, \( \sigma_{\pm \pm} (T^a_c) \),
\( \sigma_{\pm \pm} (T^a_c) \), and the variance of the total charge of the system,
\( \sigma_Q (T^a_c) \), turn out to be finite, if \( \delta n_Q \neq 0 \) or \( \delta n_G < 0 \),
whereas they are divergent (provided the Coulomb interaction
is not taken into account in calculation of \( \sigma_Q \)) in the
isospin-symmetrical medium and, if \( \delta n_Q \) is finite but posi-
tive. Interestingly, the variances \( \sigma_N \) and \( \sigma_G \) are only weakly
sensitive to the isospin imbalance and are equal to those for the
isospin-symmetrical case up to terms \( O(\delta n_Q, G) \).

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\[
\frac{\partial \Pi}{\partial \mu} = 5\lambda m^* I_3, \quad \frac{\partial h}{\partial m^*} = \frac{m^*}{2} I_3, \\
\frac{\partial h}{\partial \mu} = m^2 I_1 = m^2 (I_3 + 2(\tilde{d} + d)).
\]

(A5)

Now, using these expressions we can simplify the denominators in Eqs. (A2) and (A3) as

\[
\left(1 - \frac{\partial \Pi}{\partial m^*} \right) \frac{\partial h}{\partial \mu} + \frac{\partial \Pi}{\partial \mu} \frac{\partial h}{\partial m^*} = \frac{\partial h}{\partial \mu} + 5Cm^2 I_1 = m^2 (I_3 + 2(\tilde{d} + d))(1 + 5C).
\]

(A6)

where the quantity

\[
C = \frac{\lambda}{2} \left( I_2 - \frac{I_1^2}{I_1} \right) = 2\lambda \frac{d}{\tilde{d}} I_3 + (\tilde{d}^2 - d^2),
\]

(A7)

wherefrom for \( T \to T_{cr} \) using (25) we get

\[
C(T_{cr}) = 2\lambda \tilde{d}_{cr}.
\]

(A8)

From Eq. (A5) we find another useful relations for the derivatives of \( \Pi \) and \( h \):

\[
\frac{\partial \Pi}{\partial m^*} + \frac{\partial \Pi}{\partial \mu} = -10\lambda m^*(\tilde{d} - d), \\
\frac{\partial h}{\partial m^*} + \frac{\partial h}{\partial \mu} = 2m^2(\tilde{d} + d).
\]

(A9)

These expressions are finite at \( T = T_{cr} \). Oppositely, differences of these derivatives are divergent at \( T \to T_{cr} \) and the leading terms are

\[
\left( \frac{\partial \Pi}{\partial \mu} - \frac{\partial \Pi}{\partial m^*} \right) \bigg|_{T \to T_{cr}} \to 10\lambda \mu_{cr} I_3(T \to T_{cr}), \\
\left( \frac{\partial h}{\partial \mu} - \frac{\partial h}{\partial m^*} \right) \bigg|_{T \to T_{cr}} \to 2\mu_{cr}^2 I_3(T \to T_{cr}).
\]

(A10)

Now we can express temperature derivatives of \( m^* \) and \( \mu \) as

\[
\frac{\partial \mu}{\partial T} \bigg|_n = -\frac{n \chi_{\mu}(T)}{Tm^2 I_1(1 + 5C)}, \quad \frac{\partial m^*}{\partial T} \bigg|_n = \frac{-n \chi_{m}(T)}{Tm^2 I_1(1 + 5C)},
\]

(A11)

where

\[
\chi_{\mu}(T) = \frac{5}{2} \lambda I_3 \left( 1 - 2(\mu + m^*) \frac{m^2(\tilde{d} + d)}{n} \right) + \left( 1 + 5\lambda (\tilde{d} - d) \right) \\
\times \left( \frac{m^2 (m^* - \mu)}{I_3} + 1 - \frac{2\mu m^2}{n} (\tilde{d} + d) \right).
\]

(A12)

At \( T \to T_{cr} \) we have \( \chi_{m} = \chi_{\mu} \propto I_1 \) since \( (m - \mu) I_3 \to 0 \) in view of (19). Therefore both derivatives (A11) equal to each other and are finite,

\[
\beta = \frac{\partial \mu}{\partial T} \bigg|_{n, T_{cr}} = \frac{\partial m^*}{\partial T} \bigg|_{n, T_{cr}} = \frac{5\lambda n}{2T_{cr} \mu_{cr}^2} \left( 1 - \frac{4m^3}{n} (\tilde{d}_{cr} + d_{cr}) \right).
\]

(A14)

We see that at \( T \to T_{cr} \) the divergence in denominator (\( \propto I_1 \)) is canceled by the divergence in numerator (\( \propto I_3 \)).

Note that for the ideal gas, when \( \lambda = 0 \), both derivatives (A14) vanish. Thus, the finiteness of the derivatives (A11) is another manifestation of the effect of the self-consistent account of the interaction. Using Eq. (27) we can write the expression of the coefficient \( \beta \) for \( \mu_{cr} \ll T_{cr} \) as

\[
\beta = \frac{-5\lambda n}{2T_{cr} \mu_{cr}^2} \left[ 1 - \frac{\xi(3)}{2(2\pi)^{3/2}} \frac{m^3}{\tilde{d}_{cr} + d_{cr}} \right] \bigg( \frac{\mu_{cr}^3}{n} + \frac{3}{\tilde{d}_{cr} + d_{cr}} \bigg) \\
+ \frac{5\lambda n}{\mu_{cr}^3} \left[ 1 + \frac{9}{8} \frac{\xi(3)}{2(2\pi)^{3/2}} \left( \frac{\mu_{cr}^4}{n} + \frac{5}{6} \right) \right] \\
+ O(t_{cr}^{-2/3}).
\]

(A15)

We turn now to the combinations of partial derivatives appearing in Eq. (56). From Eq. (A11) we can construct

\[
\frac{\partial (\mu - m)}{\partial T} \bigg|_n = -\frac{n \chi(T)}{Tm^2 I_1(1 + 5C)},
\]

where the function \( \chi(T) \) is

\[
\chi(T) = \chi_{\mu}(T) - \chi_{m}(T) \\
= \left( 1 + 5\lambda (\tilde{d} - d) \right) \frac{m^2 (m^* - \mu)}{n} I_3 \\
+ \left( 1 + 10\lambda \tilde{d} \right) \left( 1 - \frac{2\mu m^2}{n} (\tilde{d} + d) \right) \\
+ 10\lambda \frac{m^2(\tilde{d} + d)^2}{n}.
\]

(A17)

We observe that the terms in \( \chi_{\mu} \) and \( \chi_{m} \) divergent at \( T \to T_{cr} \) cancel each other exactly and, therefore, in the limit \( T \to T_{cr} \) the function \( \chi \) reduces to the finite quantity

\[
\chi_{cr} = 1 + 5\lambda (\tilde{d}_{cr} - d_{cr}) - \frac{2\mu_{cr}^3}{n} (\tilde{d}_{cr} + d_{cr})(1 - 10\lambda d_{cr}).
\]

(A18)

The lower limit of \( \chi_{cr} \) is realized for \( n \to 0 \) when \( d_{cr} \to 0 \) and \( \tilde{d}_{cr}/d_{cr} \to \frac{1}{2} \) and using Eq. (62) we have \( \chi_{cr}(n \to 0) \to \).
The quantity \( \chi_{cr} \) is illustrated in Fig. 10. As we see, it exhibits a rather weak dependence on the coupling constant \( \lambda \) and on the pion density \( n \).

Since quantity \( C \) remains finite at \( T = T_{cr} \) and \( I_1 \) diverges at \( T \to T_{cr} \) the derivative \( \partial (\mu - m^*)/\partial T \) at \( T = T_{cr} \) vanishes. The same can be seen also directly from (A14). Thus, if we want to estimate the dependence of \( \mu - m^* \) on the temperature for \( T \) close to \( T_{cr} \) we have to calculate the second derivatives.

After differentiating Eq. (A1) second time with respect to \( T \), we obtain

\[
\left( 2m^* - \frac{\partial \Pi}{\partial m^*} \right) \frac{\partial^2 m^*}{\partial T^2} - \frac{\partial \Pi}{\partial \mu} \frac{\partial^2 \mu}{\partial T^2} = \frac{\partial^2 \Pi}{\partial T^2} + A,
\]

\[
\frac{\partial h}{\partial m^*} \frac{\partial^2 m^*}{\partial T^2} \bigg|_n + \frac{\partial h}{\partial \mu} \frac{\partial^2 \mu}{\partial T^2} \bigg|_n = -\frac{\partial^2 h}{\partial T^2} - m^* B,
\]

(A19)

where

\[
A = \hat{D}_T \frac{\partial \Pi}{\partial T} + \hat{D}_T^2 (\Pi - m^*),
\]

\[
m^* B = \hat{D}_T \frac{\partial h}{\partial T} + \hat{D}_T^2 h
\]

(A20)

with \( \hat{D}_T \) standing for the differential operator

\[
\hat{D}_T = \frac{\partial (\mu - m^*)}{\partial T} \hat{D}_- + \frac{\partial (\mu + m^*)}{\partial T} \hat{D}_+,
\]

\[
\hat{D}_\pm = \frac{1}{2} \left( \frac{\partial}{\partial \mu} \pm \frac{\partial}{\partial m^*} \right).
\]

(A21)

From Eq. (A4) second derivatives with respect to \( T \) can be written as

\[
\frac{\partial^2 \Pi}{\partial T^2} = -\frac{1}{T} \frac{\partial h}{\partial T} + \frac{5\lambda}{T} \left( m^* \frac{\partial I_3}{\partial T} - \mu m^* \frac{\partial I_1}{\partial T} \right),
\]

(A22)

\[
\frac{\partial^2 h}{\partial T^2} = -\frac{1}{T} \frac{\partial h}{\partial T} + \frac{m^*}{T} \left( m^* \frac{\partial I_3}{\partial T} - \mu m^* \frac{\partial I_1}{\partial T} \right).
\]

The full evaluation of this expression is very cumbersome because of the necessity to calculate additional derivatives in Eqs. (A20) and (A22). However, the task is simplified, if we take into account the value of \( \rho - n \) at \( T_{cr} \). Thus, only terms with the \( \hat{D}_\pm \) operator can be potentially divergent and, therefore, survive in (A23) at \( T_{cr} \). An additional analysis shows that the terms linear in \( \hat{D}_- \) produce at the end terms proportional to \( \frac{I_3}{T} \frac{\partial (\mu - m^*)}{\partial T} \approx I_3/I_1^2 \), which vanish at \( T \to T_{cr} \). Concluding, we see that in the limit \( T \to T_{cr} \) we may keep in Eq. (A23) only the terms quadratic in \( \hat{D}_- \). As the result we find

\[
\frac{\partial^2 (m^* - \mu)}{\partial T^2} \bigg|_{n,T \to T_{cr}} \approx \frac{1}{m^*} \left[ \frac{\partial (\mu - m^*)}{\partial T} \right]^2 
\]

\[
\times \left( 1 + 5\lambda (\tilde{d} - d) \right) \frac{m^*}{T} \frac{I_3}{I_1(1 + 5C)} \hat{D}_- h + \frac{(\tilde{d} + d)}{I_1(1 + 5C)} \hat{D}_+ \Pi \bigg|_{T_{cr}}.
\]

(A24)
or after account for Eq. (A10) we have

\[ \frac{\partial^2 (m^* - \mu)}{\partial T^2} \bigg|_{n,T_{cr}} \approx \frac{n^2 \xi^2}{T_{cr}^2 \mu_0^4 (1 + 10 \lambda^2 d)^2} \left[ \frac{D - I_3}{I_3^2} \right] \bigg|_{T \rightarrow T_{cr}}, \]

(A25)

where

\[ \left[ \frac{D - I_3}{I_3^2} \right] \bigg|_{T \rightarrow T_{cr}} = \frac{4\pi \mu_{cr}}{T_{cr}^2}. \]

(A26)

Combining all terms we write the expansion of \( m^* - \mu \) for \( T \) close to \( T_{cr} \) as

\[ m^* - \mu \approx \frac{\alpha}{2\mu_{cr}} (T - T_{cr})^2, \]

(A27)

with

\[ \alpha = \frac{4\pi n^2 \xi^2}{\mu_{cr}^2 T_{cr}^4 (1 + 10 \lambda^2 d)^2}. \]

(A28)

The dependence of the quantity \( T_{cr} \alpha/\mu_{cr} \) on a density is shown in Fig. 10. We see that this product is finite for \( n \rightarrow 0 \).

### Appendix B: \( \sigma_Q \) in isospin-symmetrical system

In Sect. 4 we have obtained that the variance of the total charge, \( \sigma_Q \), diverges at \( T \rightarrow T_{cr} \) in the isospin-symmetrical system even with taking into account of a strong pion-pion interaction within \( \lambda \phi^4 \) model. In this Appendix we study, under which conditions this divergency can be eliminated. For this we rewrite the variances derived in Sect. 3, as the derivatives of the densities with respect to chemical potentials through the derivatives of the chemical potentials with respect to densities. Below we do not indicate explicitly the dependence on the number of \( \pi^0 \) mesons, assuming that it is fixed.

The chemical potentials \( \mu_a \), and the particle densities are connected by Eq. (4). We consider this relation as an equation for \( n_a(\mu_+, \mu_-) \). Then we can calculate derivatives in (15). On the other hand, the system of equations (4) implicitly defines functions \( \mu_a = \mu_a(n_+, n_-) \). The relation among the partial derivatives of direct and inverse function implies in this case

\[ \hat{\delta}_{ab} = \sum_{c=\pm} \frac{\partial n_a}{\partial \mu_c} \frac{\partial \mu_c}{\partial n_b}, \quad a, b = \pm. \]

(B1)

Terms with \( \pi^0 \) vanish here, as we assume that \( n_{\pi^0} \) is fixed. Then the letter relation can be written as inversion of a \( 2 \times 2 \) matrix,

\[ \frac{\partial n_a}{\partial \mu_b} = [M^{-1}]_{ab}, \quad M = \begin{bmatrix} \frac{\partial \mu_+}{\partial n_+} & \frac{\partial \mu_+}{\partial n_-} \\ \frac{\partial \mu_-}{\partial n_+} & \frac{\partial \mu_-}{\partial n_-} \end{bmatrix}. \]

(B2)

or explicitly as

\[ \frac{\partial n_\pm}{\partial \mu_\mp} = \frac{1}{E} \frac{\partial n_\pm}{\partial \mu_\mp} = -\frac{1}{E} \frac{\partial \mu_\mp}{\partial n_\pm}, \]

(E)

Using these relations the charge variance (45) can be written as

\[ \sigma_Q = 3 \frac{T}{E} \left( \frac{\partial \mu_+}{\partial n_+} + \frac{\partial \mu_+}{\partial n_-} + \frac{\partial \mu_-}{\partial n_-} + \frac{\partial \mu_-}{\partial n_+} \right) \]

\[ = 3 \frac{T}{E} \left( \frac{\partial}{\partial n_+} + \frac{\partial}{\partial n_-} \right) (\mu_+ + \mu_-). \]

(B4)

Now we can introduce densities \( n_ch = n_+ + n_- \) and \( n_Q = n_+ - n_- \) and the corresponding derivatives

\[ \frac{\partial}{\partial n_Q} = \frac{\partial}{\partial n_ch} \pm \frac{\partial}{\partial n_Q}. \]

(B5)

So, the quantity \( E \) in (B3) can be rewritten as

\[ E = \frac{\partial (\mu_+ - \mu_-)}{\partial n_Q} \frac{\partial (\mu_+ + \mu_-)}{\partial n_ch} - \frac{\partial (\mu_+ - \mu_-)}{\partial n_ch} \frac{\partial (\mu_+ + \mu_-)}{\partial n_Q}. \]

(B6)

Since we consider fluctuations in the isospin-symmetrical system we need these derivatives for \( n_Q = 0 \) and \( n_ch = \frac{2}{3} n \). Since \( \mu_+ - \mu_- \) and \( \mu_+ + \mu_- \) are odd and even functions of \( n_Q \), respectively, the last term in (B6) vanishes. Then from Eqs. (B4) and (B6) we find a simple relation

\[ \sigma_Q = 6 \frac{T}{n} \left( \frac{\partial (\mu_+ - \mu_-)}{\partial n_Q} \right)_{n_Q=0}^{-1}. \]

(B7)

In order to proceed further we calculate the partial derivatives in Eq. (B7). We start with the equation relating the change density and the chemical potentials,

\[ n_Q = \left[ f_+ \right]_p - \left[ f_- \right]_p. \]

(B8)

1 Symmetry properties of the functions \( g_1(n_ch, n_Q) = \mu_+ + \mu_- \) and \( g_2(n_ch, n_Q) = \mu_+ - \mu_- \) can be verified, if we formally rename positive and negative pions, i.e. interchange all minuses and pluses (“+” ↔ “-”), that leads to the relations

\[ g_1(n_ch, \pm n_Q) = (\mu_+ + \mu_-) = g_1(n_ch, n_Q). \]

\[ g_2(n_ch, \pm n_Q) = (\mu_+ - \mu_-) = -g_2(n_ch, n_Q). \]
We introduce here a short notation for the momentum integration \([\ldots]_p = \int \frac{d^3p}{(2\pi)^3}\ldots\), and the Bose–Einstein occupation factor

\[ f_\pm = (e^{(\omega_\pm(p,n,n_{ch},n_Q) - \mu_\pm(n,n_{ch},n_Q))/T} - 1)^{-1}, \tag{B9} \]

where \(\omega_\pm(p,n,n_{ch},n_Q)\) is the spectrum of a \(\pi^\pm\) meson, which in general case can depend on the total density of pions, \(n\), the density of charged pions, \(n_{ch}\), and the charge density \(n_Q\). Differentiating Eq. (B8) with respect to \(n_Q\) keeping other densities fixed we get

\[ 1 = \left[ \frac{f_-(1 + f_-) \partial (\omega_- - \mu_-)}{T} \right]_p - \left[ \frac{f_+(1 + f_+) \partial (\omega_+ - \mu_+)}{T} \right]_p \tag{B10} \]

Hence for \(n_Q = 0\) we obtain

\[ \frac{\partial (\mu_+ - \mu_-)}{\partial n_Q} \bigg|_{n_Q=0} = \frac{T}{[f(1 + f)]_p} + \frac{[f(1 + f) \frac{\partial (\omega_+ - \omega_-)}{\partial n_Q}]_{n_Q=0}}{[f(1 + f)]_p} \tag{B11} \]

Substituting this relation in Eq. (B7) and taking into account that \([f(1 + f)]_p = I_1 T\), we obtain the relation

\[ \sigma_Q = \frac{6T}{n} \frac{I_1}{1 + \frac{T}{f(1 + f) \frac{\partial (\omega_+ - \omega_-)}{\partial n_Q}]_{n_Q=0}} \tag{B12} \]

which can be rewritten as Eq. (48).

Alternatively to Eq. (48), which explicitly depends on the model for particle spectra, we can rewrite \(\sigma_Q\) through the free-energy density \(F(n_+, n_-, T)\). The chemical potentials as functions of densities can be obtained as partial derivatives of the free energy, \(\mu_\pm = \frac{\partial F}{\partial n_\pm}\), at fixed \(T\) and \(V\). Then we have \(\mu_+ - \mu_- = 2 \frac{\partial F}{\partial n_Q}\), and therefore Eq. (B7) takes the form

\[ \sigma_Q = 3 \frac{T}{n} \left( \frac{\partial^2 F}{\partial n_Q^2} \right)_{n_Q=0}^{-1} \tag{B13} \]

Appendix C: \(I_n(T \rightarrow T_{cr}^\omega)\) at non-zero isospin imbalance

In this Appendix we consider pion gas with a small isospin imbalance, \(\delta n_Q \neq 0\) or \(\delta n_G \neq 0\), so the critical temperature of the BEC is largest for the species \(\tilde{a}\), i.e., \(T_{cr}^\omega = \max_{\tilde{a}} \{T_{cr}^\omega\}\). We are interested in quantities \(I_n^b(T \rightarrow T_{cr}^\omega)\) for \(b \neq \tilde{a}\) in the case of very small imbalance \(\delta n_{Q,G} \ll n\), i.e. when \(|T_{cr}^\omega - T_{cr}| \ll T_{cr}\), where \(T_{cr}\) is the critical temperature for the isospin-symmetrical system with the density \(n\). The derived results are needed for expansion of Eqs. (85), (92), and (98). To find leading and next-to-leading terms in the \(\delta n_{Q,G}\) expansion of \(I_n^b(T_{cr}^\omega)\) for small \(\delta n_{Q,G}\), we first separate the divergent part (19), determined by the small momenta in the integrals (18). Then we have

\[ I_n^b(T_{cr}^\omega) \approx \frac{T_{cr}^\omega}{2^{3/2} \pi \sqrt{m_p^2(T_{cr}^\omega) [m_n^2(T_{cr}^\omega) - \mu_b(T_{cr}^\omega)]}} + \delta I_{cr,n}, \tag{C1} \]

where in the regular term \(\delta I_{cr,n}\) we can take all quantities in the isospin symmetrical limit.

1. Variation \(\delta n_Q\) at \(\delta n_G = 0\)

Here we consider variations \(\delta n_Q \neq 0\) at \(\delta n_G = 0\), studied in Sect. 5.2, point (i). The condition \(\delta n_G = 0\) implies \(\delta n_+ = -\delta n_-\) and along this line, as we see in Fig. 8 (b,c), the maximal critical temperature is realized for the most abundant charged species, i.e. \(T_{cr}^{+} \leq T_{cr}^{-}\). provided \(\delta n_+ \leq \delta n_-\), respectively. To be specific consider \(\delta n_Q > 0\), then the critical temperature of positively charged pions, \(T_{cr}^{+}\), is the highest one.

Our aim here is to find quantities \(I_n^0(T_{cr}^{+})\) for \(\delta n_Q \ll n\) and, therefore, for \(\delta T_{cr}^{+} = T_{cr}^{+} - T_{cr}^{-} \ll T_{cr}^{-}\), where \(T_{cr}^{-}\) is the critical temperature of the isospin-symmetrical pion gas with density \(n\). To use Eq. (C1) we have to evaluate the differences \(m_n^{+}(T_{cr}^{+}) - \mu_0(T_{cr}^{+})\) for \(a = 0, -\). Taking into account (60), we find

\[ m_n^{+}(T_{cr}^{+}) - \mu_0(T_{cr}^{+}) \approx 2 \delta n_Q \mu_{T_{cr}^{+}} I_1(T_{cr}^{+}), \tag{C2} \]

where we used that the effective masses do not depend on \(\delta n_Q\), i.e. \(m_{T_{cr}^{+}}^{*}(T_{cr}^{+}) = m_{T_{cr}^{+}}^{*}(T_{cr}^{+})\) and corrections to the chemical potentials are given by Eq. (60). Also we can write the expansion for the effective mass and the chemical potential

\[ m_n^{+}(T_{cr}^{+}) \approx m_0^{*}(T_{cr}^{+}) \approx m_0^{*}(T_{cr}^{+}) = \mu_+(T_{cr}^{+}) \tag{C3} \]

\[ \mu_+(T_{cr}^{+}) \approx \mu_0(T_{cr}^{+}) + \frac{\delta n_Q}{2 \mu_{T_{cr}^{+}}^2 I_1(T_{cr}^{+})} \]

\[ \approx \mu_{cr} + \beta T_{cr} \frac{\delta n_Q}{2n} + \frac{\delta n_Q}{2 \mu_{T_{cr}^{+}}^2 I_1(T_{cr}^{+})}, \tag{C4} \]
where we used Eqs. (A14) and that the critical temperature $T_{c}^{+}$ is shifted with respect to $T_{c}$ according to Eq. (61),

$$T_{c}^{+} = T_{c} \left(1 + \frac{\delta n_{Q} \eta_{c}^{(Q)}}{2n \eta_{n}} \right). \quad (C5)$$

In Eqs. (C2) and (C4) the quantity $I_{1}(T_{c}^{+})$ is calculated with effective masses and chemical potentials computed for the isospin-symmetrical pion gas but at the temperature $T_{c}^{+}$, i.e. $m^{*}(T_{c}^{+})$ and $\mu(T_{c}^{+})$, respectively.

To evaluate $I_{1}(T_{c}^{+})$ we use (C1), where we replace $m^{*}_{a}$ and $\mu_{a}$ to $m^{*}$ and $\mu$, respectively and take into account that $m^{*}(T_{c}^{+}) \approx \mu_{c} + \beta T_{c} \frac{\delta n_{Q} \eta_{c}^{(Q)}}{2n \eta_{n}} \quad (C6)$

according to Eqs. (A14) and (61). The mass difference in the numerator of the singular term in $I_{1}(T_{c}^{+})$ can be rewritten as

$$m^{*}(T_{c}^{+}) - \mu(T_{c}^{+}) = m^{*}(T_{c} + \delta T_{c}^{+}) - \mu(T_{c} + \delta T_{c}^{+}) \approx \frac{1}{2} \alpha \frac{\mu_{c}}{\delta T_{c}^{+}} \approx \frac{1}{2} \frac{\alpha}{\mu_{c}} \left(\frac{\delta n_{Q} \eta_{c}^{(Q)}}{4n^{2}} \eta_{c}^{(Q)} \right) \quad (C7)$$

where we used Eq. (A27) with the coefficient $\alpha$ given in Eq. (A28). Thus, using Eqs. (C5), (C6), and (C7) we obtain

$$I_{1}(T_{c}^{+}) \approx \frac{1}{\pi \alpha_{c}^{1/2}} \left(\frac{n}{\delta n_{Q} \eta_{c}^{(Q)}} \frac{\eta_{n}}{\eta_{c}} + \frac{\beta T_{c}}{4 \mu_{c}} \right) + \delta I_{c.r,1}. \quad (C8)$$

Now taking $I_{1}^{a}(T_{c}^{+})$ from (C1) we can write for $a = \ldots$:

$$I_{1}^{a}(T_{c}^{+}) \approx \beta^{(Q)} n \frac{\delta n_{Q} \eta_{c}^{(Q)}}{\delta n_{Q} \eta_{c}} \left(1 + \frac{\beta T_{c}}{4 \mu_{c}} + \frac{3}{2} \frac{\mu_{c}}{\alpha_{c}^{1/2}} \delta I_{c.r,1} \right) + \delta I_{c.r,n}. \quad (C9)$$

where

$$\beta^{(Q)} = \frac{\mu_{c} \delta T_{c}^{+} \eta_{n}}{(2\pi)^{3} \alpha_{c}^{1/2} n \eta_{c}}. \quad$$

Analogously, for neutral pions we get

$$I_{1}^{0}(T_{c}^{+}) \approx \sqrt{2} I_{1}^{(-)}(T_{c}^{+}) - \delta I_{c.r,n} + \delta I_{c.r,n}. \quad (C10)$$

Some comments about parameters of our expansions are in order. Expressions for shifts of pion effective masses and chemical potentials obtained in Sect. 5.1 are derived as expansions in $\delta n_{a}$ up to linear terms $O(\delta n_{a})$, see Eqs. (53) and (55). The results are valid for any temperature $T \geq \max_{a}[T_{c}^{a}]$. The difference between the effective mass and the chemical potential, e.g. in Eq. (C2), is $O(\delta n_{Q} / I_{1}^{(T_{c})})$. Formally for arbitrary temperatures $T > T_{c}^{+}$ this result is of the order $O(\delta n_{Q})$. However, for $T \rightarrow T_{c}^{+} = T_{c} + O(\delta n_{Q})$ the quantity $I_{1}$ behaves at the leading order like $I_{1}(T_{c}^{+}) \propto 1/\delta n_{Q}$, see Eq. (C8). Therefore, the expansion (C2) is effectively of the order $O(\delta n_{Q}^{2})$. The same expansion order is explicitly seen in the difference $m^{*} - \mu$ at $T_{c}^{+}$ for the isospin symmetrical medium, Eq. (C7), which is based on the expansion (A27) independently on the $\delta n_{Q}$ and that $\delta T_{c}^{+} \propto \delta n_{Q}$. The final expressions of this section (C8), (C9), and (C10) hold up to terms linear in $\delta n_{Q}$.

2. Variation $\delta n_{Q}$ at $\delta n_{G} = 0$

(a) Let now $\delta n_{Q} = 0$ and $\delta n_{G} < 0$. In this case, the maximal is the critical temperature of the BEC for neutral pions, $T_{c}^{0}$. In order to expand relations (92) in small quantity $-\delta n_{G} \ll n$ we need the corresponding expansions of $I_{1}^{+}(T_{c}^{0})$. We can use a relation analogous to (C1) and expand the mass-chemical potential difference in the denominator with the help of Eq. (63) as follows

$$m^{*}_{+}(T_{c}^{0}) - \mu_{+}(T_{c}^{0}) \approx m^{*}_{n}(T_{c}^{0}) + 3 \delta m^{*}_{+}(T_{c}^{0}) - 3 \delta \mu_{+}(T_{c}^{0}) = 3 \delta \mu_{+}(T_{c}^{0}) \quad \quad (C11)$$

We observe that although each of the quantities $\eta_{n}^{(G)}(T_{c}^{0})$ and $\eta_{\mu}^{(G)}(T_{c}^{0})$ remains finite, if $T_{c}^{0} \rightarrow T_{c}$, see Eq. (63), their difference in (C11) vanishes and hence integrals $I^{+}_{1}(T_{c}^{0})$ are enhanced, when $\delta n_{G} \rightarrow 0$ and $T_{c}^{0} \rightarrow T_{c}$. Being interested only in terms $\sim O(\delta n_{G})$ in the small-$\delta n_{G}$ expansion, we can write

$$m^{*}_{+}(T_{c}^{0}) - \mu_{+}(T_{c}^{0}) \approx \frac{\delta n_{G} \eta_{c}^{(G)}}{m^{2} I_{1}(T_{c}^{0})} + \frac{1 + \lambda(I_{2} - I_{3})}{1 + 2C} \quad \quad (C12)$$

where in the last equation we used Eqs. (23), (64), (65), and (A8). Next-to-leading terms in these expansions are of the order $O((\delta n_{G} / I_{1}(T_{c}^{0}))^{2})$. For the integral $I_{1}(T_{c}^{0})$ we can use the expansion (C1), where we replace $m^{*}_{a}$ and $\mu_{a}$ by $m^{*}$ and $\mu$, respectively. According to Eq. (64) we have

$$T_{c}^{0} = T_{c} \left(1 - \frac{2}{3} \frac{\delta n_{G} \eta_{c}^{(G)}}{\eta_{c}} + O((\delta n_{G})^{2}). \quad (C13)$$
The expression for the $m - \mu$ difference can be written in analogy to Eq. (C7) using Eqs. (A27) and (C13):

$$m^*(T^0_{ct}) - \mu(T^0_{ct}) \approx \frac{\alpha_{ct}}{2\mu_{ct}} \frac{4}{9} \frac{\delta n_G}{n^2} \frac{\eta^{G(G)}_{gG}}{\eta_{\mu}} T^2_{ct}. \tag{14}$$

Since, as we show below, $I_1(T^0_{ct}) \propto 1/\delta n_G$, the expansion (C12) is of the same quadratic order in $\delta n_G$ as expansion (C14). Additionally, to get the expansion $I_1(T^0_{ct})$ we need the expansion for the effective mass,

$$m^*(T^0_{ct}) = m^*(T_{ct}) + \beta \delta T^0_{ct} + O((\delta T^0_{ct})^2)$$

$$= \mu_{ct} - \beta T_{ct} \frac{2}{3} \frac{\delta n_G}{\eta_{\mu}} \frac{\eta^{G(G)}_{gG}}{n} + O((\delta n_G)^2) \tag{15}$$

where we used (A14) and (C13).

Thus, we obtain

$$I_1(T^0_{ct}) = -\frac{1}{\pi a_{ct}^{1/2}} \left( \frac{3n}{\delta n_G} \frac{\eta_{\mu}}{\eta^{G(G)}_{gG}} - \frac{1}{2} + \frac{\beta}{4} \frac{T_{ct}}{\mu_{ct}} \right)$$

$$+ \delta I_{ct,1} + O((\delta n_G)^2). \tag{16}$$

To derive the expansion of $I^+_n(T^0_{ct})$ we also need the expansion for the critical temperature $m^+_n(T^0_{ct})$, which we obtain using relations (63),

$$m^+_n(T^0_{ct}) \approx m^+_n(T^0_{ct}) + 3\delta m^+_n(T^0_{ct})$$

$$= \mu(T^0_{ct}) - 2\delta T^0_{ct} + O((\delta T^0_{ct})^2)$$

$$\approx \mu_{ct} + \beta \delta T^0_{ct} - 2\mu_{ct} + 3\delta m^+_n(T^0_{ct})$$

$$\approx \mu_{ct} \left( 1 - \frac{2}{3} \frac{\delta n_G}{\eta_{\mu}} \frac{\eta^{G(G)}_{\mu}}{n} T_{ct} \right)$$

$$+ \delta \eta^{G(G)}_{\mu} (T_{ct} - 2\eta^{G(G)}_{\mu}(T_{ct})) \right)$$

$$\approx \mu_{ct} \left( 1 - \frac{2}{3} \frac{\delta n_G}{\eta_{\mu}} \frac{\eta^{G(G)}_{\mu}}{n} T_{ct} \right)$$

$$+ O((\delta n_G)^2). \tag{17}$$

Now substituting Eqs. (C12), (C13), and (C17) in Eq. (C1) we can expand

$$I^+_n(T^0_{ct}) \approx -\frac{\lambda}{\delta n_G} \frac{n}{\mu_{ct}} \left[ 1 - \frac{\lambda}{2} \frac{n}{\delta n_G} \right]$$

$$- \frac{2}{3} \frac{\delta n_G}{\eta_{\mu}} \frac{\eta^{G(G)}_{gG}}{n} \frac{2}{3} - \frac{3}{4} \frac{\beta T_{ct}}{\mu_{ct}} + \pi a_{ct}^{1/2} O(I_{ct1}(T_{ct})) \right]$$

$$+ \delta I_{ct,n} + O((\delta n_G)^2), \tag{19}$$

where

$$\beta^G = \beta(Q) \frac{\sqrt{3}}{2} \frac{\eta^{G(G)}_{gG}}{\eta^{G(G)}_{\mu}}. \tag{20}$$

(b) Now we consider the case $\delta n_G > 0$. The maximal critical temperature is now $T^+_{ct}$. To expand relations (98) we have to find expansions of $I^+_n(T^+_{ct})$ for $\delta n_G \ll n$. For this we need the expansion for the critical temperature,

$$T^+_{ct} = T_{ct} \left( 1 + \frac{\delta n_G}{3} \frac{\eta^{G(G)}_{gG}}{\eta_{\mu}} \right) + O((\delta n_G)^2), \tag{21}$$

and for the mass difference

$$m^+_n(T^+_{ct}) - \mu_0(T^+_{ct}) \approx 3(\delta \mu_+(T^+_{ct}) - \delta m^+_n(T^+_{ct}))$$

$$= \frac{\delta n_G}{\mu_{ct}^2} \frac{\eta^{G(G)}_{gG}}{\eta^{G(G)}_{\mu}}. \tag{22}$$

which we derived in a similar way as in Eqs. (C11) and (C12). To expand $I_1(T^+_{ct})$ we need the difference

$$m^+(T^+_{ct}) - \mu(T^+_{ct}) \approx \frac{a}{2\mu_{ct}} (\delta n_G)^2 \frac{\eta^{G(G)}_{gG}}{\eta_{\mu}} T_{ct}^2, \tag{23}$$

obtained using Eqs. (A27) and (21), and the relation for the effective mass,

$$m^*(T^+_{ct}) \approx \mu_{ct} + \beta T_{ct} \frac{1}{3} \frac{\delta n_G}{\eta_{\mu}} \frac{\eta^{G(G)}_{gG}}{n} + O((\delta n_G)^2), \tag{24}$$

where we used Eq. (A14) and (21). As we have argued above for the cases described by Eqs. (C2) and (C7) and Eqs. (C12) and (C14), the differences between an effective mass and chemical potentials in Eqs. (22) and (23) prove to be of the order $(\delta n_G)^2$, that is seen after taking into account that $I_1(T^+_{ct}) \propto 1/\delta n_G$.

Now substituting Eqs. (23) and (21) in Eq. (C1) with the pion mass and the chemical potential taken as in the isospin symmetrical matter, we obtain

$$I_1(T^+_{ct}) \approx \frac{1}{\pi a_{ct}^{1/2}} \left( \frac{3n}{2\delta n_G} \frac{\eta^{G(G)}_{gG}}{\eta_{\mu}} + \frac{1}{2} - \frac{\beta T_{ct}}{4 \mu_{ct}} \right). \tag{25}$$
\[ + \delta I_{cr,1} + O(\delta n_G). \]  

(C25)

Now using this result we can evaluate Eq. (C22) and substitute it in Eq. (C1) together with the effective mass

\[ m_0^c(T^+_c) = \mu_{cr} \left( 1 + \frac{1}{3} \frac{\delta n_G}{n} \frac{\eta_{cr}}{\eta_{n}} \frac{T_{cr}}{T_{cr}} - \frac{\lambda \delta n_G}{3n} \right) \]

\[ = \mu_{cr} \left( 1 + \frac{1}{3} \frac{\delta n_G}{n} \frac{\eta_{cr}}{\eta_{n}} \frac{T_{cr}}{T_{cr}} - \frac{\lambda \delta n_G}{3 \mu_{cr}^2 (1 + 2C)} \right) \]

\[ + O((\delta n_G)^2), \]  

(C26)

and the chemical potential

\[ \mu_{cr,+} = \mu_{cr} \left( 1 + \frac{1}{3} \frac{\delta n_G}{n} \frac{\eta_{cr}}{\eta_{n}} \frac{T_{cr}}{T_{cr}} + \frac{\lambda \delta n_G}{3 \mu_{cr}^2 (1 + 2C)} \right) \]

\[ + O((\delta n_G)^2), \]  

(C27)

obtained in the same way as Eqs. (C18) and (C18), and the critical temperature (C21). Finally we obtain

\[ \rho_n(T^+_c) \approx \sqrt{2} \beta G \frac{n}{\delta n_G} \left[ 1 + \frac{\lambda \delta n_G}{6 \mu_{cr}^2 (1 + 2C)} \right] \]

\[ + \frac{\delta n_G}{3n} \frac{\eta_{cr}}{\eta_{n}} \left( \frac{3}{2} - \frac{3}{4} \frac{\beta}{\mu_{cr}} + \pi \alpha \frac{1}{2} I_{cr,1} \right) \]

\[ + \delta I_{cr,n} + O(\delta n_G). \]  

(C28)

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