COMBINATORIAL IDENTITIES RELATED TO EIGENFUNCTION DECOMPOSITIONS OF HILL OPERATORS: OPEN QUESTIONS

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Abstract. We formulate several open questions related to enumerative combinatorics, which arise in the spectral analysis of Hill operators with trigonometric polynomial potentials.

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In analysis of asymptotics of spectral gaps and the basisness property of root function systems of Hill operators

\[ Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi, \]

with potentials that are trigonometric polynomials we faced (see [1, 2, 4, 5]) a series of Enumerative Combinatorics questions. Some of them have been settled, and this allows us to find the asymptotics of the spectral gap of Schrödinger operators with potentials of the form

\[ v(x) = a \cos 2x + b \cos 4x \]

(see [2, Theorem 7]), or to prove (see [5, Theorem 21]) that the Hill operator with potentials of the form

\[ v(x) = ae^{-2ix} + Ae^{-4ix} + be^{2ix} + Be^{4ix}, \quad a, b, A, B \neq 0, \]

and periodic (or antiperiodic) boundary conditions has a basis in \( L^2([0, \pi]) \) consisting of its root functions if and only if \(|A| = |B|\) (under the restriction that neither \(-b^2/(4B)\) nor \(-a^2/(4A)\) is an integer square).

However, many of those questions remain open. They seem to be interesting by themselves but their answers would imply series of results on spectra and spectral decompositions related to Hill operators with trigonometric potentials. Let us formulate some of them.

1. For each \( n \in \mathbb{N} \) a walk \( x \) from \(-n\) to \( n\) or from \( n\) to \(-n\) is defined by its sequence of steps

\[ x = (x(t))_{t=1}^{\nu+1}, \quad 1 \leq \nu = \nu(x) < \infty, \]
where \( x(t) \in 2\mathbb{Z} \setminus \{0\} \), and respectively,

\[
\sum_{t=1}^{\nu+1} x(t) = 2n \quad \text{or} \quad \sum_{t=1}^{\nu+1} x(t) = -2n.
\]

A walk \( x \) is called \textit{admissible} if its vertices \( j(t) = j(t, x) \) given, respectively, by

\[
j(0) = -n \quad \text{or} \quad j(0) = +n
\]

and

\[
j(t) = -n + \sum_{i=1}^{t} x(i) \quad \text{or} \quad j(t) = n + \sum_{i=1}^{t} x(i), \quad 1 \leq t \leq \nu + 1,
\]

satisfy the condition

\[
j(t) \neq \pm n \quad \text{for} \quad 1 \leq t \leq \nu.
\]

Let \( X_n \) and \( Y_n \) be, respectively, the set of all admissible walks from \(-n\) to \( n\) and from \( n\) to \(-n\), and let \( \{V(m), m \in 2\mathbb{Z}\} \) be a sequence of complex numbers such that \( V(0) = 0 \). For each walk \( x \in X_n \) or \( x \in Y_n \) we set

\[
h_1(x) = \prod_{t=1}^{\nu} [n^2 - j(t)^2]^{-1}, \quad h(x) = h_1(x) \prod_{t=1}^{\nu+1} V(x(t))
\]

2. In the sequel we consider finite sets \( F \) of permitted steps \( x(t) \). We say that a walk \( x \) is \( F\)-admissible if it is admissible and \( x(t) \in F \) for every \( t = 1, \ldots, \nu + 1 \). We define \( X^+_n(F) \) (or \( Y^-_n(F) \)) to be the set of all \( F\)-admissible walks with positive steps \( x(t) > 0 \) (or negative steps \( x(t) < 0 \)).

In these notations, the following statement holds (it is proven in [2], see formulas (109)-(113) there).

\[\textbf{Proposition 1.} \quad \text{Let} \quad F = F_2 = \{-2, -4, 2, 4\}, \quad \text{and let}
\]

\[
V(-2) = a, \quad V(2) = b, \quad V(-4) = A, \quad V(4) = B.
\]

\[\text{Then, for even} \quad n = 2m,
\]

\[
\sum_{\xi \in X^+_n} h(\xi) = \frac{1}{4^{m-1}} \prod_{j=1}^{m} [b^2/4 + (2j - 1)^2B],
\]

\[\text{and for odd} \quad n = 2m + 1
\]

\[
\sum_{\xi \in X^+_n} h(\xi) = -\frac{b}{4^{m}} \prod_{j=1}^{m} [b^2/4 + (2j)^2B].
\]
Question 2. Let \( F_m = \{ j \in 2\mathbb{Z} \setminus \{0\} : |j| \leq m \} \), and let \( z = \{V(2k)\}_1^m \in \mathbb{C}^m \). Find the polynomials

\[
P_{m,n}(z) = \sum_{x \in X_n^+ (F_m)} h(x)
\]

and/or their asymptotics for fixed \( m \) and \( n \to \infty \).

Proposition 1 answers this question for \( m = 2 \).

3. The identities (11) and (12) were discovered and proven by analyzing the combinatorial meaning of the asymptotic formulas for the spectral gaps of Hill operators with potentials of the form (2). From (11) and (12) one derives the following (Theorem 8 in [2]).

Proposition 3. (a) If \( k \) and \( m, 1 \leq k \leq m, \) are fixed, then

\[
\sum_{i=1}^{k} \prod_{s=1}^{k} (m^2 - i_s^2) = \sum_{1 \leq j_1 < \cdots < j_k \leq m} \prod_{s=1}^{k} (2j_s - 1)^2,
\]

where the left sum is over all indices \( i_1, \ldots, i_k \) such that

\[-m < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \geq 2 \text{ if } s \neq r.\]

(b) If \( k \) and \( m, 1 \leq k \leq m - 1, \) are fixed, then

\[
\sum_{i=1}^{k} \prod_{s=1}^{k} [(2m - 1)^2 - (2i_s - 1)^2] = \sum_{1 \leq j_1 < \cdots < j_k \leq m-1} \prod_{s=1}^{k} (4j_s)^2,
\]

where the left sum is over all indices \( i_1, \ldots, i_k \) such that

\[-m + 1 < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \geq 2 \text{ if } s \neq r.\]

4. A few years later D. Zagier gave a direct combinatorial proof of (14) and (15). These identities are used in an essential way by J. M. Borwein, A. Straub, J. Wan and W. Zudilin in [7], [8]. Moreover, an appendix to [8] presents the proof of D. Zagier.

Just recently an elegant "elementary" proof was given by S. Rosenberg [11]; it is based on information about the spectrum \( \sigma(M_n) = \{n - 2\ell : 0 \leq \ell \leq n\} \) of Sylvester-Kac matrices (see [12] or [9]).

5. Good information about the polynomials (13) would help to find asymptotics of the sequences

\[
\beta_n^+ = \sum_{x \in X_n^+ (F_m)} h(x), \quad \beta_n^- = \sum_{y \in X_n^- (F_m)} h(y)
\]

\[1\] We have asked many colleagues working in combinatorics for a direct proof of (14) and (15). Finally, such a proof was given by D. Zagier in August 2010 when we met him by chance at Mathematisches Forschungsinstitut Oberwolfach.
as well because the following holds.

**Proposition 4.**

\[(17) \beta_n^+ = \left( \sum_{x \in X_n^+} h(x) \right) \left[ 1 + O(\log n/n) \right] \]

\[(18) \beta_n^- = \left( \sum_{x \in Y_n^+} h(x) \right) \left[ 1 + O(\log n/n) \right] \]

if, respectively,

\[(19) \sum_{x \in X_n^+} |h(x)| \leq C(V) \sum_{x \in X_n^+} h(x) \]

or

\[(20) \sum_{x \in Y_n^-} |h(x)| \leq C(V) \sum_{x \in Y_n^-} h(x) \]

**Question 5.** Is it true that \[(19)\] holds for almost all \(\{V\} = z \in \mathbb{C}^m\)? The structure of the exceptional set \(W = \{ z : C(z) = \infty \}\) is of special interest also.

6. Suppose \(F\) has just two elements, say \(F = \{-2R, +2S\}\) and \(V(-2R) = a, V(2S) = b\). We are concerned on the polynomials

\[(21) P_\kappa(a, b) = \sum_{x \in X_n(\kappa)} h(x), \]

where \(X_n(\kappa)\) is the set of all admissible walks from \(-n\) to \(n\) with \(\kappa\) negative steps, i.e., \(x \in X_n(\kappa)\) if and only if

\[(22) \#\{t : x(t) = -2R\} = \kappa. \]

If

\[(23) q = \#\{t : x(t) = 2S\}, \]

then \(-2Rp + 2Sq = 2n\), or

\[(24) Sq = n + R\kappa. \]

Of course, if \(q = \frac{1}{S}(n + R\kappa) \notin \mathbb{N}\), then \(X_n(\kappa) = \emptyset\) and \(P_\kappa(a, b) \equiv 0\). Otherwise,

\[P_\kappa(a, b) = a^\kappa b^q B_\kappa(n), \quad B_\kappa(n) = \sum_{x \in X_n(\kappa)} h_1(x)\]
and we need to analyze just the sequence $B_\kappa(n)$.

Different pairs $\{R, S\}$ bring completely different problems related to solvability of (24) in $\mathbb{N}$. We will mention just one question from this series.

Let $F = \{-2, +4\}$, i.e., $R = 1, S = 2$, and let $n = 2m + 1$. Then (24) becomes

$$2q = 2m + 1 + \kappa,$$

so $X_n^+(\kappa) = \emptyset$ if $\kappa$ is even. Unfortunately,

$$H_0 = B_1(2m + 1) = \sum_{x \in X_n^+(1)} h_1(x) = 0$$

as we realized in [5, Proposition 25], see formulas (145) - (154). (As a matter of fact, (26) is equivalent to the fundamental identity for Catalan numbers – see in [10] p.117, formulas (14.10) - (14.12.).

But the next level $X_n^+(3)$ gives some hopes.

**Question 6.** Find an asymptotic formula for the sequence $B_3(2m + 1)$, or at least explain that for some absolute constant

$$|B_3(2m + 1)| \geq C^m / m!.$$

(It is not difficult to see that the upper estimate

$$|B_3(2m + 1)| \leq \sum_{x \in X_n^+(3)} |h_1(x)| \leq C_1^m / m!$$

holds.)

The inequality (27) would imply that the root function system of the Hill operator (1) with potentials of the form

$$v(x) = ae^{-2ix} + be^{4ix}, \quad a, b \neq 0,$$

subject to antiperiodic boundary conditions does not contain a basis in $L^2([0, \pi])$.

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