On the Hadamard condition on Robertson-Walker spacetime

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Abstract
Using construction of adiabatic vacuum states of a free scalar field on Robertson-Walker spacetime, using results of Lüders and Roberts we prove that validity of the Hadamard condition implies smoothness of the scale factor.

1 Introduction
In the end of the sixties Parker [1], [2] investigated the particle creation in dynamic universe using adiabatic vacuum states. Later, Lüders and Roberts gave an exact mathematical definition in [3] based on the algebraic approach to QFT of Haag, Kastler [8]. In their article the Klein-Gordon equation is solved by the Fourier method. As a part of this, they found the equation for the time dependent part of the state distribution, which lead to the another equation whose solutions describes adiabatic vacuum states, and as a limit case the Hadamard states. They don’t solve this equation exactly, but by iterations. All iterative solutions were obtained under the assumption of smoothness of the scale factor.

In the first part of this paper we give a construction of adiabatic vacuum states and Hadamard states according to Lüders and Roberts [3]. In the second part we will give a statement of the Hadamard condition, in the way of characterization of the microlocal structure 2-point Wightman distribution due to Radzikowski [4]. Here we assume the validity of the Hadamard condition and its corollaries to prove that the scale factor is smooth.

2 Hadamard Condition for Linear Klein-Gordon Field
Let $M$ be a globally hyperbolic four dimensional Riemannian manifold with metric tensor $g$ and Riemannian connection $\nabla$. Consider a scalar field $\Phi : M \rightarrow \mathbb{R}$ satisfying the Klein-Gordon equation
\[(\Box_g + m^2)\Phi = 0.\] (1)
The hyperbolicity of \( M \) implies that this equation is well-posed. Denote by \( C^\infty_0(M) \) the space of smooth functions with compact support on \( M \) and by \( C^\infty(M) \) the space of smooth functions on \( M \). There are two uniquely determined continuous operators

\[
\Delta_R, \Delta_A : C^\infty_0(M) \to C^\infty(M),
\]

such that

\[
(\Box g + m^2)\Delta_R f = \Delta_R(\Box g + m^2)f = f,
\]

and similarly for \( \Delta_A \), and

\[
\text{supp} (\Delta_A f) \subset J^-(\text{supp } f),
\]

\[
\text{supp} (\Delta_R f) \subset J^+(\text{supp } f),
\]

for \( f \in C^\infty_0(M) \), where \( J^+(S) \) means the causal future and \( J^-(S) \) the causal past of a set \( S \subset M \). These operators are called the retarded \( (\Delta_R) \) and the advanced \( (\Delta_A) \) propagator. Their difference \( E = \Delta_R - \Delta_A \) is the propagator of the Klein-Gordon equation. The following is valid

\[
(\Box g + m^2)Ef = E(\Box g + m^2)f = 0,
\]

\[
\text{supp} (Ef) \subset J^+(\text{supp } f) \cup J^-(\text{supp } f),
\]

for \( f \) as above. The operators \( \Delta_R, \Delta_A \) and \( E \) have continuous extensions to operators \( \Delta'_R, \Delta'_A, E' : \mathcal{E}'(M) \to \mathcal{D}'(M) \), where \( \mathcal{D}'(M) \) resp. \( \mathcal{E}'(M) \) is the space of distributions resp. the space of distributions with compact support. Now let \( \Sigma \) be an arbitrary Cauchy hypersurface in \( M \) with unit normal field \( n^\alpha \) directed to the future cone. Then there exist operators

\[
\rho_0 : C^\infty(M) \to C^\infty(\Sigma)
\]

\[
f \mapsto f_{\mid \Sigma}
\]

\[
\rho_1 : C^\infty(M) \to C^\infty(\Sigma)
\]

\[
f \mapsto (n^\alpha \nabla_\alpha f)_{\mid \Sigma},
\]

with adjoints \( \rho_0^*, \rho_1^* : \mathcal{E}'(\Sigma) \to \mathcal{E}'(M) \), see (e. g. [3], [7]).

Let us note that using these operators we can construct, if there are given Cauchy initial conditions \( u_0, u_1 \in C^\infty_0(\Sigma) \), solutions of Klein-Gordon equation (for the details see [4]).

This makes it to possible describe elements of the phase space using initial data on the Cauchy surface \( \Sigma \). Suppose that \( d^3\sigma \) is the volume form on this surface and define the real symplectic space \( (\Gamma, \varsigma) \), where \( \Gamma = C^\infty(\Sigma) \otimes C^\infty(\Sigma) \) is formed by the initial data of [1] and the real valued symplectic form \( \varsigma \) is given by

\[
\varsigma \left( \left( \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right) \right) = -\int_\Sigma [u_1 p_2 - u_2 p_1] \ d^3\sigma.
\]

To this symplectic space \( (\Gamma, \varsigma) \) there exists an associated Weyl algebra \( \mathcal{A}[\Gamma, \varsigma] \), generated by elements \( W(F), \ F \in \Gamma \), subject to the relations

\[
W(F)^* = W(F)^{-1} = W(-F),
\]

2
\[ W(F_1)W(F_2) = e^{-i\sigma(F_1,F_2)}W(F_1 + F_2), \quad \text{for all } F_1, F_2 \in \Gamma. \]

This Weyl algebra is a local algebra of observables in the sense of Haag and Kastler [8]. States on the algebra \( \mathcal{A} \) are defined as complex-valued functionals on \( \mathcal{A} \).

Now we can define the \textit{quasifree state} \( \omega_\mu \) associated with \( \mu \) by

\[ \omega_\mu(W(F)) = e^{-\frac{1}{2}\mu(F,F)}, \]

where \( \mu \) is a real-valued scalar product on \( \Gamma \) satisfying

\[ \frac{1}{4} |\varsigma(F_1,F_2)|^2 \leq \mu(F_1,F_1)\mu(F_2,F_2). \]

So we are now in a position to introduce two point function

\[ \lambda^{(2)}(F_1,F_2) = \mu(F_1,F_2) + \frac{i}{2}\varsigma(F_1,F_2), \]

and the Wightman the two-point distribution

\[ \Lambda^{(2)}(f_1,f_2) = \lambda^{(2)}\left(\rho_{0,Ef_1}\rho_{1,Ef_1},\rho_{0,Ef_2}\rho_{1,Ef_2}\right), \]

which enters into the definition the Hadamard state.

To introduce the latter, we use the microlocal formulation discovered by Radzikowski in [4]. (An overview of the theory of pseudodifferential operators, microlocal analysis and wavefront sets can be found in the book [9].) Denote by \( T^*\mathcal{M} \) the cotangent bundle of \( \mathcal{M} \). For \((x_i,\xi_i) \in T^*\mathcal{M}, i = 1,2\), writing \((x_1,\xi_1) \sim (x_2,\xi_2)\) means that there is a null geodesic \( \gamma \) such that \( x_1, x_2 \in \gamma \), with \( \xi^0_1 \) tangent to \( \gamma \) at \( x_1 \) and \( \xi^0_2 \) is obtained from \( \xi_1 \) by a parallel transport to \( x_2 \) along \( \gamma \). We use Theorem 3.9 of [6] as the microlocal formulation of a the Hadamard condition.

**Definition 1** A \textit{quasifree state} of a Klein-Gordon field on a globally hyperbolic spacetime is a Hadamard state if and only if the wavefront set of the two-point distribution \( \Lambda^{(2)} \) has the form

\[ WF(\Lambda^{(2)}) = \{(x_1,\xi_1;x_2,\xi_2) \in T^* (\mathcal{M} \times \mathcal{M}) \setminus \{0\}; (x_1,\xi_1) \sim (x_2,\xi_2), \xi_1^0 \geq 0 \}. \]

The same microlocal structure has 2-point distribution of the Hadamard state.

Note that the Hadamard state is defined locally, but according to the "local-to-global singularity theorem" the local Hadamard condition implies the global Hadamard condition [10].

### 3 Adiabatic Vacuum and Hadamard States on Robertson-Walker Spacetimes

Let \( \mathcal{M} = \mathbb{R} \times \Sigma \) be a Lorentz manifold equipped with the Robertson-Walker metric

\[ ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \]

(3)
where $\phi \in [0, 2\pi]$, $\theta \in [0, \pi]$, $r \in [0, \infty)$ if $\kappa = -1, 0$ and $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$, $r \in [0, 1)$ if $\kappa = 1$ with $a(t) \in C^2(\mathbb{R})$, $a(t) > 0$ for all $t \in \mathbb{R}$. The function $a(t)$ is called a scale factor. The Riemannian manifolds $\Sigma^\kappa$ are defined as

\[\Sigma^+ = \{ x \in \mathbb{R}^4; (x^0)^2 + \sum_{i=0}^{3} (x^i)^2 = 1 \},\]

\[\Sigma^0 = \{ x \in \mathbb{R}^4; (x^0) = 0 \},\]

\[\Sigma^- = \{ x \in \mathbb{R}^4; (x^0)^2 - \sum_{i=0}^{3} (x^i)^2 = 1, x^0 > 0 \} .\]

On $\Sigma^\kappa$ we consider the metric tensor

\[s_{ij} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & r^2 & r^2 \sin^2 \theta \\ 0 & r^2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\]

Cauchy surfaces are of the form $\Sigma_t = \{ t \} \times \Sigma$ with $n_\alpha = (1, 0, 0, 0)$. The hypersurfaces $\Sigma_t = \{ t \} \times \Sigma$ are therefore equipped with the metrics $h_{ij} = a^2(t)s_{ij}$ in which we will study the Klein-Gordon equation on the $\Sigma_t$.

\[(\Box_g + m^2)\Phi = \frac{\partial^2 \Phi}{\partial t^2} + 3\frac{\dot{a}(t)}{a(t)} \frac{\partial \Phi}{\partial t} + (-3\Delta_h + m^2)\Phi = 0,
\]

where $^{(3)}\Delta_h$ is the Laplace-Beltrami operator on the Cauchy surface $\Sigma$,

\[^{(3)}\Delta_h = \frac{1}{a^2(t)} \left\{ \left(1 - r^2\right) \frac{\partial^2}{\partial r^2} + \frac{2 - 3r^2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta(\theta, \varphi) \right\}\]

\[\Delta(\theta, \varphi) = \frac{1}{\sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \varphi^2} \right].\]

This is a linear partial differential equation solvable by the Fourier method. By separation of variables we get

\[\Phi(t, r, \theta, \varphi) = \int T_k(t)\phi_k(r, \theta, \varphi)d\mu(k),\]

where $T_k(t)$ is obtained as the solution of the ordinary differential equations

\[\ddot{T}_k(t) + 3\frac{\dot{a}(t)}{a(t)} \dot{T}_k(t) + \omega_k^2(t)T_k(t) = 0,\]

\[\omega_k^2(t) = \frac{E(k)}{a^2(t)} + m^2. \ k = 0, 1, 2, \ldots,\]

and the measure is defined by

\[\int d\mu(\vec{k}) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{m=-l}^{l} \delta(\vec{k} - (k, l, m), E(k) = k(k + 2) \text{ for } \kappa = 1, \]

\[\int d\mu(\vec{k}) = \int_{\mathbb{R}^3} d^3k, \ k \in \mathbb{R}^3, \ E(k) = k^2 \text{ for } \kappa = 0, \]

\[\int d\mu(\vec{k}) = \int_{\mathbb{R}^3} d^3k, \ k \in \mathbb{R}^3, \ E(k) = k^2 + 1 \text{ for } \kappa = -1.\]
Each of these Riemannian manifolds have their own generalized eigenfunctions (for details see [6]). This system of functions is complete and orthonormal, hence we can define a generalized Fourier transform

$$\hat{\cdot}: L^2(\Sigma) \to L^2(\Sigma),$$

$h \mapsto \hat{h}(\vec{k}) = \int_\Sigma \phi_{\vec{k}}(\vec{y}) h(\vec{y}) d^3\sigma,$

where $d^3\sigma = \sqrt{|s|} dy = (1 - \kappa r^2)^{-\frac{3}{2}} r^2 \sin \theta dr d\theta d\varphi.$

The phase space $(\Gamma, \varsigma)$ of the initial data $\Gamma = C^\infty_0(\Sigma) \times a^3(t) C^\infty_0(\Sigma)$ has the symplectic form

$$\varsigma(F_1, F_2) = -a^3(t) \int_\Sigma [q_1 p_2 - q_2 p_1] d^3\sigma,$$

for $F_i = (a^3(t), p_i) \in \Gamma, i = 1, 2.$

Following a theorem in [6] (for the proof see [3]) we introduce two parameters which will serve to describe certain states.

**Theorem 1** The homogeneous, isotropic quasifree states for the free Klein-Gordon field in a Robertson-Walker spacetime are given by the following 2-point function

$$\lambda^{(2)}(F_1, F_2) = \int_\Sigma \langle \hat{F}_1(\vec{k}), S(k) \hat{F}_2(\vec{k}) \rangle,$$

where $p(k)$ and $q(k)$ are (essentially bounded measurable) complex valued functions satisfying

$$q(k)p(k) - q(k)p(k) = -i.$$

Using these parameters we can according to [3] write down the formulas defining the adiabatic vacuum states, which in limit $n \to \infty$ gives the Hadamard states.

**Definition 2** An adiabatic vacuum state of order $n$ is a homogeneous, isotropic Fock state whose 2-point function is given by the functions $q(k) = T_k(t), p(k) = a^3(t) \dot{T}_k(t)$ where $T_k(t)$ is a solution of the differential equation (4) with initial conditions at time $t$

$$T_k(t) = W_k^{(n)}(t) \quad (5)$$

$$\dot{T}_k(t) = W_k^{(n)}(t). \quad (6)$$

Here,

$$W_k^{(n)}(t) = \frac{e^{-i \int_0^t \Omega_k^{(n)}(t') dt'}}{a^{3/2}(t) \sqrt{2\Omega_k^{(n)}(t)}} \quad (7)$$
is iteratively defined by

$$
(\Omega_k^{[0]}(t))^2 = \omega_k^2(t) = \frac{E(k)}{a^2(t)} + m^2
$$

(8)

$$
(\Omega_k^{[n+1]}(t))^2 = \omega_k^2(t) = \frac{3}{4} \left( \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{3}{2} \frac{\ddot{a}(t)}{a(t)} + \frac{3}{4} \left( \frac{\dot{\Omega}_k(t)}{\Omega_k(t)} \right)^2 - \frac{1}{2} \frac{\ddot{\Omega}_k(t)}{\Omega_k(t)}.
$$

(9)

The functions $\Omega_k^{[n]}(t)$ are iterative solutions of the equation

$$
\Omega_k^2(t) = \omega_k^2(t) = \frac{3}{4} \left( \frac{\dot{a}(t)}{a(t)} \right)^2 - \frac{3}{2} \frac{\ddot{a}(t)}{a(t)} + \frac{3}{4} \left( \frac{\dot{\Omega}_k(t)}{\Omega_k(t)} \right)^2 - \frac{1}{2} \frac{\ddot{\Omega}_k(t)}{\Omega_k(t)},
$$

which is closely linked to the equation (4), see [3]. From this definition we see that the adiabatic vacuum states are essentially dependent on the order of the iteration and on the initial time $t$.

4 Main Result

Now we will say that the Hadamard condition is valid if

$$
\Omega_k^{[n]}(t) \text{ exists for each natural number } n. \quad (H1)
$$

This means, in particular, that

$$
\Omega_k^{[n]}(t) \text{ is twice continuously differentiable} \quad (H2)
$$

and

$$
\Omega_k^{[n]}(t) > 0, \forall t \in \mathbb{R} \text{ for any } n. \quad (H3).
$$

**Theorem 2** Let $(M, g)$ be a Robertson-Walker spacetime with $a(t) \in C^2(\mathbb{R})$. Suppose that the Hadamard condition is valid. Then $a(t)$ is a smooth function.

Proof: We use the mathematical induction. Calculating the first iteration we get

$$
(\Omega_k^{[1]}(t))^2 - \frac{1}{4a^6(t)} \omega_k^6(t) - 3a^4(t)\dot{a}^2(t)\omega_k^4(t) + 6a^5(t)\ddot{a}(t)\omega_k^4(t) + 3[E(k)\dot{a}(t)]^2 + 2E(k)(\dot{a}(t)a(t) - \dot{a}^2(t))a^2(t)\omega_k^2(t) - 2E(k)a^6(t)(\ddot{a}^2(t) - 2E(k)a^6(t)\ddot{a}(t)) = 0,
$$

(10)

The expression (10) is linearly dependent on the highest derivative of $a(t)$ so we can express

$$
\ddot{a}(t) = \frac{1}{2a^2(t)(3a^5(t) + E(k)a^3(t))} \left[ 4a^6(t)\omega_k^6(t) - 3\dot{a}^2(t)a^4(t)\omega_k^4(t) - 3E^2(k)\dot{a}^2(t) \right]
$$
\[-2E(k)\dot{a}^2(t)a^2(t)\omega_2^2(t) - 2E(k)a^2(t)\dot{a}^6(t) - 4a^{6}(t)(\Omega^{[1]}_{k}(t))^2(t)\omega_2^2(t)]. \tag{11}\]

Since \(a(t) \in C^2(\mathbb{R})\) by hypothesis, the right-hand side has a continuous derivative, hence so has the left-hand side, i.e. \(a(t) \in C^3(\mathbb{R})\). This in turn means by differentiating (11) again that right-hand side is in \(C^2(\mathbb{R})\) hence \(a(t) \in C^4(\mathbb{R})\).

From (10) we have
\[
\left(\Omega^{[1]}_{k}(t)\right)^2 = (2E(k)a^3(t)\ddot{a}(t)\omega_2^4(t) - 6a^5(t)\omega_2^2(t)) \ddot{a}(t) + 
\text{(terms involving only } a(t), \dot{a}(t), \text{)},
\]
in particular, \(\left(\Omega^{[1]}_{k}(t)\right)^2\) depends linearly on \(\ddot{a}(t)\).

Now assume \(a(t) \in C^{2n}(\mathbb{R})\) and consider the \(n\)-th iteration \((n \geq 2)\)
\[
(\Omega^{[n]}_{k})^2 = \omega_2^2(t) - \frac{3}{4} \left(\frac{\ddot{a}(t)}{a(t)}\right)^2 - \frac{3}{2} \frac{\ddot{a}(t)}{a(t)} + \frac{3}{4} \left(\frac{\dot{\Omega}^{[n-1]}_{k}(t)}{\Omega^{[n-1]}_{k}(t)}\right)^2 - \frac{1}{2} \frac{\ddot{\Omega}^{[n-1]}_{k}(t)}{\Omega^{[n-1]}_{k}(t)}, \tag{12}\]
where
\[
\left(\dot{\Omega}^{[n-1]}_{k}(t)\right)^2 = F_n(a(t), \dot{a}(t), \ldots, a^{(2n-2)}(t)),
\]
and assume that \(F_n\) depends on \(a^{(2n-2)}(t)\) linearly,
\[
\left(\Omega^{[n-1]}_{k}(t)\right)^2 = f_n(a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t))a^{(2n-2)}(t) + 
\text{(terms involving only } a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t)\text{)}, \tag{13}\]
This implies
\[
\left[\left(\Omega^{[n-1]}_{k}(t)\right)^2\right]'' = f_n(a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t))a^{(2n)}(t) + 
\text{(terms involving only } a(t), \dot{a}(t), \ldots, a^{(2n-1)}(t)\text{)}, \tag{14}\]
with \(f_n \in C^{\infty}\).

By (9)
\[
(\Omega^{[n]}_{k})^2 - \left[\omega_2^2(t) - \frac{3}{4} \left(\frac{\ddot{a}(t)}{a(t)}\right)^2 - \frac{3}{2} \frac{\ddot{a}(t)}{a(t)} + \frac{3}{4} \left(\frac{\dot{\Omega}^{[n-1]}_{k}(t)}{\Omega^{[n-1]}_{k}(t)}\right)^2 - \frac{1}{2} \frac{\ddot{\Omega}^{[n-1]}_{k}(t)}{\Omega^{[n-1]}_{k}(t)}\right] = 0. \tag{15}\]
Since
\[
\frac{\dot{\Omega}^{[n-1]}_{k}(t)}{\Omega^{[n-1]}_{k}(t)} - \frac{1}{2} \left[\left(\Omega^{[n-1]}_{k}(t)\right)^2\right]'' = 0,
\]

7
and

\[
\frac{\dddot{\Omega}_k^{[n-1]}(t)}{\Omega_k^{[n-1]}(t)} = \frac{1}{2} \left[ \left( \frac{\dddot{\Omega}_k^{[n-1]}(t)}{\Omega_k^{[n-1]}(t)} \right)^2 - \left( \frac{\dddot{\Omega}_k^{[n-1]}(t)}{\Omega_k^{[n-1]}(t)} \right)^2 \right],
\]

it follows from (13) and (14) that the left hand side of (15) depends linearly on \( a^{(2n)}(t) \),

\[
\left( \frac{\dddot{\Omega}_k^{[n]}(t)}{\Omega_k^{[n]}(t)} \right)^2 = \text{(terms involving only } a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t) - \frac{1}{4} \frac{f_n(a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t))}{\Omega_k^{[n-1]}(t)} a^{(2n)}(t). \tag{16}
\]

Thus

\[
a^{(2n)}(t) = \text{(terms involving only } a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t), \Omega_k^{[n-1]}(t)) - \frac{4}{f_n(a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t))}. \tag{17}
\]

The induction hypothesis \( a(t) \in C^2(\mathbb{R}) \) together with (H2) implies that the right hand side of (17) is in \( C^2(\mathbb{R}) \). Thus \( a^{(2n)}(t) \in C^2(\mathbb{R}) \), i.e. \( a(t) \in C^{2n+2}(\mathbb{R}) \).

Besides (17) shows that \( \left( \frac{\dddot{\Omega}_k^{[n]}(t)}{\Omega_k^{[n]}(t)} \right)^2 \) depends on \( a^{(2n)}(t) \) linearly, i.e. (13) holds for \( n + 1 \) in the place of \( n \). Consequently, by induction on \( n \), we conclude that \( a(t) \in C^\infty(\mathbb{R}) \).

To make the passage from (16) to (17) completely rigorous, it remains to check that the denominator in (17) does not vanish, i.e.

\[
f_n(a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t)) \neq 0.
\]

Observe that, by (11),

\[
f_2(a(t), \dot{a}(t)) = \omega_k^2(t)(3a^5(t) - E(k)a^3(t)),
\]

while by (16)

\[
f_{n+1}(a(t), \dot{a}(t), \ldots, a^{(2n-1)}(t)) = -\frac{4}{f_n(a(t), \dot{a}(t), \ldots, a^{(2n-3)}(t))}.
\]

Iteratively the last relation gives \( n \geq 1 \)

\[
f_{n+1}(a(t), \dot{a}(t), \ldots, a^{(2n-1)}(t)) = \left( -\frac{4}{f_2(a(t), \dot{a}(t))} \prod_{i=1}^{n-1} \frac{1}{\Omega_k^{[i]}(t)^2} \right)^{n-1}.
\]

The last denominator is nonzero by (H3).

Remark: Observe that it follows from (13) and Theorem 2 that \( \Omega_k^{[n]}(t) \), are in fact, not only \( C^2 \) but \( C^\infty \).
5 Conclusion

We have proved that on the Robertson-Walker spacetime the validity of the Hadamard condition for the free scalar quantum fields implies smoothness of the scale factor, i.e. together with the paper [3] we can state that in our case the validity of the Hadamard condition is a sufficient condition for the smoothness of the scale factor of the Robertson-Walker spacetime. There are still some open questions, for instance, whether we can derive the same result for different kinds of quantum fields, e.g. Hermite scalar fields, spinor fields, etc., or whether it is valid also for others spacetimes than the Robertson-Walker spacetime.

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