The Verlinde formula for $\text{PGL}_p$

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To the memory of
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Introduction

The Verlinde formula expresses the number of linearly independent conformal blocks in any rational conformal field theory. I am concerned here with a quite particular case, the Wess-Zumino-Witten model associated to a complex semi-simple group $G$. In this case the space of conformal blocks can be interpreted as the space of holomorphic sections of a line bundle on a particular projective variety, the moduli space $M_G$ of holomorphic $G$-bundles on the given Riemann surface. The fact that the dimension of this space of sections can be explicitly computed is of great interest for mathematicians, and a number of rigorous proofs of that formula (usually called by mathematicians, somewhat incorrectly, the “Verlinde formula”) have been recently given (see e.g. [F], [B-L], [L-S]).

These proofs deal only with simply-connected groups. In this paper we treat the case of the projective group $\text{PGL}_r$ when $r$ is prime.

Our approach is to relate to the case of $\text{SL}_r$, using standard algebro-geometric methods. The components $M^d_{\text{PGL}_r}$ ($0 \leq d < r$) of the moduli space $M_{\text{PGL}_r}$ can be identified with the quotients $M^d_r/J_r$, where $M^d_r$ is the moduli space of vector bundles on $X$ of rank $r$ and fixed determinant of degree $d$, and $J_r$ the finite group of holomorphic line bundles $\alpha$ on $X$ such that $\alpha^{\otimes r}$ is trivial. The space we are looking for is the space of $J_r$-invariant global sections of a line bundle $L$ on $M^d_r$; its dimension can be expressed in terms of the character of the representation of $J_r$ on $H^0(M^d_r, L)$. This is given by the Lefschetz trace formula, with a subtlety for $d = 0$, since $M^0_r$ is not smooth. The key point (already used in [N-R]) which makes the computation quite easy is that the fixed point set of any non-zero element of $J_r$ is an abelian variety – this is where the assumption on the group is essential. Extending the method to other cases would require a Chern classes computation on the moduli space $M_H$ for some semi-simple subgroups $H$ of $G$; this may be feasible, but goes far beyond the scope of the present paper. Note that the case of $M^1_{\text{PGL}_2}$ has been previously worked out in [P] (with an unfortunate misprint in the formula).

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2 This group is the complexification of the compact semi-simple group considered by physicists.
In the last section we check that our formulas agree with the predictions of Conformal Field Theory, as they appear for instance in [F-S-S]. Note that our results are slightly more precise (in this particular case): we get a formula for \( \dim H^0(M_{\text{PGL}}^d, \mathcal{L}) \) for every \( d \), while CFT only predicts the sum of these dimensions (see Remark 4.3).

1. The moduli space \( M_{\text{PGL}}^r \)

(1.1) Throughout the paper we denote by \( X \) a compact (connected) Riemann surface, of genus \( g \geq 2 \); we fix a point \( p \) of \( X \). Principal \( \text{PGL}_r \)-bundles on \( X \) correspond in a one-to-one way to projective bundles of rank \( r-1 \) on \( X \), i.e. bundles of the form \( \mathbf{P}(E) \), where \( E \) is a rank \( r \) vector bundle on \( X \); we say that \( \mathbf{P}(E) \) is semi-stable if the vector bundle \( E \) is semi-stable. The semi-stable projective bundles of rank \( r-1 \) on \( X \) are parameterized by a projective variety, the moduli space \( M_{\text{PGL}}^r \).

Two vector bundles \( E, F \) give rise to isomorphic projective bundles if and only if \( F \) is isomorphic to \( E \otimes \alpha \) for some line bundle \( \alpha \) on \( X \). Thus a projective bundle can always be written as \( \mathbf{P}(E) \) with \( \det E = \mathcal{O}_X(dp) \), \( 0 \leq d < r \); the vector bundle \( E \) is then determined up to tensor product by a line bundle \( \alpha \) with \( \alpha^r = \mathcal{O}_X \). In particular, the moduli space \( M_{\text{PGL}}^r \) has \( r \) connected components \( M_{\text{PGL}}^d \) \((0 \leq d < r)\). Let us denote by \( M_{\text{r}}^d \) the moduli space of semi-stable vector bundles on \( X \) of rank \( r \) and determinant \( \mathcal{O}_X(dp) \), and by \( J_r \) the kernel of the multiplication by \( r \) in the Jacobian \( J_X \) of \( X \); it is a finite group, canonically isomorphic to \( H^1(X, \mathbb{Z}/(r)) \). The group \( J_r \) acts on \( M_{\text{r}}^d \), by the rule \( (\alpha, E) \mapsto E \otimes \alpha \); it follows from the above remarks that the component \( M_{\text{PGL}}^d \) is isomorphic to the quotient \( M_{\text{r}}^d/J_r \).

(1.2) We will need a precise description of the line bundles on \( M_{\text{PGL}}^r \). Let me first recall how one describes line bundles on \( M_{\text{r}}^d \) [D-N]: a simple way is to mimic the classical definition of the theta divisor on the Jacobian of \( X \) (i.e. in the rank 1 case). Put \( \delta = (r, d) \); let \( A \) be a vector bundle on \( X \) of rank \( r/\delta \) and degree \( (r(g-1)-d)/\delta \). These conditions imply \( \chi(E \otimes A) = 0 \) for all \( E \) in \( M_{\text{r}}^d \); if \( A \) is general enough, it follows that the condition \( H^0(X, E \otimes A) \neq 0 \) defines a (Cartier) divisor \( \Theta_A \) in \( M_{\text{r}}^d \). The corresponding line bundle \( \mathcal{L}_d := \mathcal{O}(\Theta_A) \) does not depend on the choice of \( A \), and generates the Picard group \( \text{Pic}(M_{\text{r}}^d) \).

(1.3) The quotient map \( q : \mathcal{M}_{\text{r}}^d \to \mathcal{M}_{\text{PGL}}^d \) induces a homomorphism \( q^* : \text{Pic}(\mathcal{M}_{\text{PGL}}^d) \to \text{Pic}(\mathcal{M}_{\text{r}}^d) \), which is easily seen to be injective. Its image is determined in [B-L-S]: it is generated by \( \mathcal{L}^\delta_d \) if \( r \) is odd, by \( \mathcal{L}^\delta_d \) if \( r \) is even.

(1.4) Let \( \mathcal{L}' \) be a line bundle on \( \mathcal{M}_{\text{PGL}}^d \). The line bundle \( \mathcal{L} := q^* \mathcal{L}' \) on
$\text{M}_d'$ admits a natural action of $J_r$, compatible with the action of $J_r$ on $\text{M}_d'$ (this is often called a $J_r$-linearization of $\mathcal{L}$). This action is characterized by the property that every element $\alpha$ of $J_r$ acts trivially on the fibre of $\mathcal{L}$ at a point of $\text{M}_d'$ fixed by $\alpha$. In the sequel we will always consider line bundles on $\text{M}_d'$ of the form $q^*\mathcal{L}'$, and endow them with the above $J_r$-linearization.

This linearization defines a representation of $J_r$ on the space of global sections; essentially by definition, the global sections of $\mathcal{L}'$ correspond to the $J_r$-invariant sections of $\mathcal{L}$. Therefore our task will be to compute the dimension of the space of invariant sections; as indicated in the introduction, we will do that by computing, for any $\alpha \in J_r$ of order $r$, the trace of $\alpha$ acting on $H^0(\text{M}_d', \mathcal{L})$.

2. The action of $J_r$ on $H^0(\text{M}_d', \mathcal{L}_d^k)$

We start with the case when $r$ and $d$ are coprime, which is easier to deal with because the moduli space is smooth.

Proposition 2.1. Assume $r$ and $d$ are coprime. Let $k$ be an integer; if $r$ is even we assume that $k$ is even. Let $\alpha$ be an element of order $r$ in $JX$. Then the trace of $\alpha$ acting on $H^0(\text{M}_d', \mathcal{L}_d^k)$ is $(k + 1)(r - 1)(g - 1)$.

Proof: The Lefschetz trace formula reads [A-S]

$$\text{Tr}(\alpha \mid H^0(\text{M}_d', \mathcal{L}_d^k)) = \int_P \text{Todd}(T_P) \lambda(N_P/M_d', \alpha)^{-1} \tilde{\text{ch}}(\mathcal{L}_d^k_{|P}, \alpha).$$

Here $P$ is the fixed subvariety of $\alpha$; whenever $F$ is a vector bundle on $P$ and $\varphi$ a diagonalizable endomorphism of $F$, so that $F$ is the direct sum of its eigen-subbundles $F_\lambda$ for $\lambda \in \mathbb{C}$, we put

$$\tilde{\text{ch}}(F, \varphi) = \sum \lambda \text{ch}(F_\lambda); \quad \lambda(F, \varphi) = \prod_{\lambda, p \geq 0} (-\lambda)^p \text{ch}(\Lambda^p F_\lambda^*) .$$

We have a number of informations on the right hand side thanks to [N-R]:

(2.1 a) Let $\pi: \tilde{X} \to X$ be the étale $r$-sheeted covering associated to $\alpha$; put $\xi = \alpha^{r(r-1)/2} \in JX$. The map $L \mapsto \pi_*(L)$ identifies any component of the fibre of the norm map $Nm: J^d\tilde{X} \to J^dX$ over $\xi(dp)$ with $P$. In particular, $P$ is isomorphic to an abelian variety, hence the term $\text{Todd}(T_P)$ is trivial.

(2.1 b) Let $\theta \in H^2(P, \mathbb{Z})$ be the restriction to $P$ of the class of the principal polarization of $J^d\tilde{X}$. The term $\lambda(N_P/M_d', \alpha)$ is equal to $r^{r(g-1)}e^{-r\theta}$.

(2.1 c) The dimension of $P$ is $N = (r - 1)(g - 1)$, and one has $\int_P \frac{q^N}{N!} = r^{g-1}$.

With our convention the action of $\alpha$ on $\mathcal{L}_d^k_{|P}$ is trivial. The class $c_1(\mathcal{L}_d_{|P})$ is equal to $r\theta$: the pull back to $P$ of the theta divisor $\Theta_\Lambda$ (1.2) is the divisor of
line bundles $L$ in $P$ with $H^0(L \otimes \pi^*A) \neq 0$; to compute its cohomology class we may replace $\pi^*A$ by any vector bundle with the same rank and degree, in particular by a direct sum of $r$ line bundles of degree $r(g-1)-d$, which gives the required formula.

Putting things together, we find

$$\text{Tr}(\alpha | H^0(M_r^d, \mathcal{L}_d^k)) = \int_P r^{-r(g-1)} e^{r\theta} e^{k\theta} = (k+1)^{(r-1)(g-1)} .$$

We now consider the degree 0 case:

**Proposition 2.2.** Let $k$ be a multiple of $r$, and of $2r$ if $r$ is even; let $\alpha$ be an element of order $r$ in $JX$. Then the trace of $\alpha$ acting on $H^0(M_r^0, \mathcal{L}_0^k)$ is $(k/r + 1)^{(r-1)(g-1)}$.

**Proof:** We cannot apply directly the Lefschetz trace formula since it is manageable only for smooth projective varieties; instead we use another well-known tool, the Hecke correspondence (this idea appears for instance in [B-S]). For simplicity we write $M_d$ instead of $M_r^d$. There exists a Poincaré bundle $E$ on $X \times M_1$, i.e. a vector bundle whose restriction to $X \times \{E\}$, for each point $E$ of $M_1$, is isomorphic to $E$. Such a bundle is determined up to tensor product by a line bundle coming from $M_1$; we will see later how to normalize it. We denote by $E_p$ the restriction of $E$ to $\{p\} \times M_1$, and by $\mathcal{P}$ the projective bundle $\mathbb{P}(\mathcal{E}_p^*)$ on $M_1$. A point of $\mathcal{P}$ is a pair $(E, \varphi)$ where $E$ is a vector bundle in $M_1$ and $\varphi : E \rightarrow \mathbb{C}_p$ a non-zero homomorphism, defined up to a scalar; the kernel of $\varphi$ is then a vector bundle $F \in M_1$, and we can view equivalently a point of $\mathcal{P}$ as a pair of vector bundles $(F, E)$ with $F \in M_0$, $E \in M_1$ and $F \subset E$. The projections $p_d$ on $M_d$ $(d = 0, 1)$ give rise to the “Hecke diagram”

$$\begin{array}{ccc}
M_1 & \xleftarrow{p_1} & \mathcal{P} & \xrightarrow{p_0} & M_0
\end{array}$$

**Lemma 2.3.** The Poincaré bundle $\mathcal{E}$ can be normalized (in a unique way) so that $\det \mathcal{E}_p = \mathcal{L}_1$; then $\mathcal{O}_\mathcal{P}(1) \cong p_0^* \mathcal{L}_0$.

**Proof:** Let $E \in M_1$. The fibre $p_1^{-1}(E)$ is the projective space of non-zero linear forms $\ell : E_p \rightarrow \mathbb{C}$, up to a scalar. The restriction of $p_0^* \mathcal{L}_0$ to this projective space is $\mathcal{O}(1)$ (choose a line bundle $L$ of degree $g-1$ on $X$; if $E$ is general enough, $H^0(X, E \otimes L)$ is spanned by a section $s$ with $s(p) \neq 0$, and the condition that the bundle $F$ corresponding to $\ell$ belongs to $\Theta_L$ is the vanishing of $\ell(s(p))$). Therefore
$p_0^*\mathcal{L}_0$ is of the form $\mathcal{O}_P(1) \otimes p_1^*\mathcal{N}$ for some line bundle $\mathcal{N}$ on $M_1$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{N}$ we ensure $\mathcal{O}_P(1) \cong p_0^*\mathcal{L}_0$.

An easy computation gives $K_P = p_1^*\mathcal{L}_1^{-1} \otimes p_0^*\mathcal{L}_0^{-r}$ ([B-L-S], Lemma 10.3). On the other hand, since $\mathcal{P} = \mathcal{P}(\mathcal{E}_p)$, one has $K_P = p_1^*(K_{M_1} \otimes \det \mathcal{E}_p) \otimes \mathcal{O}_P(-r)$; using $K_{M_1} = \mathcal{L}_1^{-2}$ [D-N], we get $\det \mathcal{E}_p = \mathcal{L}_1$.

We normalize $\mathcal{E}$ as in the lemma; this gives for each $k \geq 0$ a canonical isomorphism $p_1*p_0^*\mathcal{L}_0^k \cong S^k\mathcal{E}_p$. Let $\alpha$ be an element of order $r$ of $\text{JX}$. It acts on the various moduli spaces in sight; with a slight abuse of language, I will still denote by $\alpha$ the corresponding automorphism. There exists an isomorphism $\alpha^*\mathcal{E} \cong \mathcal{E} \otimes \alpha$, unique up to a scalar ([N-R], lemma 4.7); the induced isomorphism $u : \alpha^*\mathcal{E} \cong \mathcal{E}$ induces the action of $\alpha$ on $\mathcal{P}$. Imposing $\alpha^r = \text{Id}$ determines $u$ up to a $r$-th root of unity, hence determines completely $S^k u$ when $k$ is a multiple of $r$. Since the Hecke diagram is equivariant with respect to $\alpha$, it gives rise to a diagram of isomorphisms

$$
\begin{array}{ccc}
\text{H}^0(\mathcal{P}, p_0^*\mathcal{L}_0^k) & \xrightarrow{p_1^*} & \text{H}^0(\mathcal{M}_1, S^k\mathcal{E}_p) \\
\xrightarrow{p_0^*} & & \xleftarrow{\text{H}^0(\mathcal{M}_0, \mathcal{L}_0^k)}
\end{array}
$$

which is compatible with the action of $\alpha$; in particular, the trace we are looking for is equal to the trace of $\alpha$ on $\text{H}^0(\mathcal{M}_1, S^k\mathcal{E}_p)$.

We are now in the situation of Prop. 2.1, and the Lefschetz trace formula gives:

$$
\text{Tr}(\alpha | \text{H}^0(\mathcal{M}_1, S^k\mathcal{E}_p)) = \int_P \text{Todd}(T_P) \lambda(N_{P/M_1}, \alpha)^{-1} \tilde{\chi}(S^k\mathcal{E}_p|_P, \alpha).
$$

The only term we need to compute is $\tilde{\chi}(S^k\mathcal{E}_p|_P, \alpha)$. Let $\mathcal{N}$ be the restriction to $\hat{X} \times P$ of a Poincaré line bundle on $\hat{X} \times \text{JX}$; let us still denote by $\pi : \hat{X} \times P \to X \times P$ the map $\pi \times \text{Id}_P$. The vector bundles $\pi_*(\mathcal{N})$ and $\mathcal{E}|_{X \times P}$ have the same restriction to $X \times \{\gamma\}$ for all $\gamma \in P$, hence after tensoring $\mathcal{N}$ by a line bundle on $P$ we may assume they are isomorphic ([R], lemma 2.5). Restricting to $\{p\} \times P$ we get $\mathcal{E}_{p|_P} = \bigoplus_{\pi(q)=p} \mathcal{N}_q$, with $\mathcal{N}_q = \mathcal{N}|_{\{q\} \times P}$.

We claim that the $\mathcal{N}_q$’s are the eigen-sub-bundles of $\mathcal{E}_{p|_P}$ relative to $\alpha$. By (2.1 a), a pair $(E, F) \in \mathcal{P}$ is fixed by $\alpha$ if and only if $E = \pi_*L$, $F = \pi_*L'$, with $\text{Nm}(L) = \xi(p)$, $\text{Nm}(L') = \xi$; because of the inclusion $F \subset E$ we may take $L'$ of the form $L(-q)$, for some point $q \in \pi^{-1}(p)$. In other words, the fixed locus of $\alpha$ acting on $\mathcal{P}$ is the disjoint union of the sections $(\sigma_q |_{\pi^{-1}(p)})$ of the fibration $p_1^{-1}(P) \to P$ characterized by $\sigma_q(\pi_*L) = (\pi_*L, \pi_*(L(-q)))$. Viewing $\mathcal{P}$ as $\mathcal{P}(\mathcal{E}_p|_P)$, the section $\sigma_q$ corresponds to the exact sequence

$$
0 \to \pi_*(\mathcal{N}(-q))|_{\{p\} \times P} \to \pi_*(\mathcal{N})|_{\{p\} \times P} \cong \mathcal{E}|_{\{p\} \times P} \to \mathcal{N}_q \to 0.
$$
Therefore on each fibre $P(E_p)$, for $E \in P$, the automorphism $\alpha$ has exactly $r$ fixed points, corresponding to the $r$ sub-spaces $N_{(q,E)}$ for $q \in \pi^{-1}(p)$; this proves our claim.

The line bundles $N_q$ for $q \in \tilde{X}$ are algebraically equivalent, and therefore have the same Chern class. We thus have $c_1(E_p|_P) = r c_1(N_q)$. On the other hand we know that $\det E_p = L_1$ (lemma 2.3), and that $c_1(L_1|_P) = r \theta$ (proof of Prop. 2.1). By comparison we get $c_1(N_q) = \theta$. Putting things together we obtain

$$\tilde{\text{ch}}(S^k E_p|_P, \alpha) = \int_P \text{Tr} S^k D_r \ e^{k \theta} r^{-r(g-1)} e^{r \theta}$$

where $D_r$ is the diagonal $r$-by-$r$ matrix with entries the $r$ distinct $r$-th roots of unity.

**Lemma 2.4.** The trace of $S^k D_r$ is $1$ if $r$ divides $k$ and $0$ otherwise.

Consider the formal series $s(T) := \sum_{i \geq 0} T^i \text{Tr} S^i u$ and $\lambda(T) := \sum_{i \geq 0} T^i \text{Tr} \Lambda^i u$. The formula $s(T)\lambda(-T) = 1$ is well-known (see e.g. [Bo], § 9, formula (11)). But

$$\lambda(-T) = \sum_{i=0}^r (-T)^i \text{Tr} \Lambda^i u = \prod_{\zeta^r = 1} (1 - \zeta T) = 1 - T^r,$$

hence the lemma. Using (2.1 c) the Proposition follows. ■

### 3. Formulas

In this section I will apply the above results to compute the dimension of the space of sections of the line bundle $L^k_d$ on the moduli space $M^d_{\text{PGL}_r}$. Let me first recall the corresponding Verlinde formula for the moduli spaces $M^d_r$. Let $\delta = (r,d)$; we write $L_d = D^{r/\delta}$, with the convention that we only consider powers of $D$ which are multiple of $r/\delta$ (the line bundle $D$ actually makes sense on the moduli stack $M^d_r$, and generates its Picard group). We denote by $\mu_r$ the center of $\text{SL}_r$, i.e. the group of scalar matrices $\zeta I_r$ with $\zeta^r = 1$.

**Proposition 3.1.** Let $T_k$ be the set of diagonal matrices $t = \text{diag}(t_1, \ldots, t_r)$ in $\text{SL}_r(\mathbb{C})$ with $t_i \neq t_j$ for $i \neq j$, and $k^{r+1} \in \mu_r$; for $t \in T_k$, let $\delta(t) = \prod_{i<j} (t_i - t_j)$.

Then

$$\dim H^0(M^d_r, D^k) = r^{g-1}(k + r)^{(r-1)(g-1)} \sum_{t \in T_k/\mu_r} \frac{((-1)^{r-1/r} k^{r+1})^d}{|\delta(t)|^{2g-2}}.$$
Proof: According to [B-L], Thm. 9.1, the space $H^0(M_r^d, D^k)$ for $0 < d < r$ is canonically isomorphic to the space of conformal blocks in genus $g$ with the representation $V_{k \varpi_{r-d}}$ of $\text{SL}_r$ with highest weight $k \varpi_{r-d}$ inserted at one point. The Verlinde formula gives therefore (see [B], Cor. 9.8\(^1\)):

$$\dim H^0(M_r^d, D^k) = r^{g-1}(k+r)^{(r-1)(g-1)} \sum_{t \in T_k / \mathfrak{S}_r} \frac{\text{Tr}_{V_{k \varpi_{r-d}}}(t)}{|\delta(t)|^{2g-2}};$$

this is still valid for $d = 0$ with the convention $\varpi_r = 0$.

The character of the representation $V_{k \varpi_{r-d}}$ is given by the Schur formula (see e.g. [F-H], Thm. 6.3):

$$\text{Tr}_{V_{k \varpi_{r-d}}}(t) = \frac{1}{\delta(t)} \begin{vmatrix} t_1^{k+r-1} & t_1^{k+r-2} & \ldots & t_1^2 & t_1^1 \\ t_2^{k+r-1} & t_2^{k+r-2} & \ldots & t_2^2 & t_2^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_r^{k+r-1} & t_r^{k+r-2} & \ldots & t_r^2 & t_r^1 \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix}.$$ 

Writing $t^{k+r} = \zeta I_r \in \mu_r$, the big determinant reduces to $\zeta^{r-d}(-1)^{(d-r)} \det(t_{j-i}^{d-i})$, and finally, since $\prod t_i = 1$, to $((-1)^{r-1}\zeta)^{-d}\delta(t)$, which gives the required formula. ■

Corollary 3.2.-- Let $T'_k$ be the set of matrices $t = \text{diag}(t_1, \ldots, t_r)$ in $\text{SL}_r(\mathbb{C})$ with $t_i \neq t_j$ if $i \neq j$, and $t^{k+r} = (-1)^{r-1}I_r$. Then

$$\sum_{d=0}^{r-1} \dim H^0(M_r^d, D^k) = r^g(k+r)^{(r-1)(g-1)} \sum_{t \in T'_k / \mathfrak{S}_r} \frac{1}{|\delta(t)|^{2g-2}} .$$ ■

We now consider the moduli space $M_{\text{PGL}_r}$. We know that the line bundle $\mathcal{D}^k$ on $M_r^d$ descends to $M_{\text{PGL}_r}^d = M_r^d / J_r$ exactly when $k$ is a multiple of $r$ if $r$ is odd, or of $2r$ if $r$ is even (1.3). When this is the case we obtain a line bundle on $M_{\text{PGL}_r}^d$, that we will still denote by $\mathcal{D}^k$; its global sections correspond to the $J_r$-invariant sections of $H^0(M_r^d, D^k)$.

We will assume that $r$ is prime, so that every non-zero element $\alpha$ of $J_r$ has order $r$. Then Prop. 2.1 and 2.2 lead immediately to a formula for the dimension of the $J_r$-invariant subspace of $H^0(M_r^d, D^k)$ as the average of the numbers $\text{Tr}(\alpha)$ for $\alpha$ in $J_r$. Using Prop. 3.1 we conclude:

\(^1\) There is a misprint in the first equality of that corollary, where one should read $T_\ell^{\text{reg}} / W$ instead of $T_\ell^{\text{reg}}$; the second equality (and the proof!) are correct.
Proposition 3.3. Assume that $r$ is prime. Let $k$ be a multiple of $r$; if $r = 2$ assume $4 \mid k$. Then

$$\dim H^0(M_{PGL_r}, D^k) = r^{-2g} \dim H^0(M_d, D^k) + (1 - r^{-2g}) \left( \frac{k}{r} + 1 \right)^{(r-1)(g-1)}$$

$$= r^{-2g} \left( \frac{k}{r} + 1 \right)^{(r-1)(g-1)} \sum_{t \in T_k/\mathbb{S}_r} \frac{1}{\delta(t)^{2g-2}} + r^{2g} - 1.$$ 

Summing over $d$ and plugging in Cor. 3.2 gives the following rather complicated formula:

Corollary 3.4. The dimension is given by

$$\dim H^0(M_{PGL_r}, D^k) = r^{1-2g} \left( \frac{k}{r} + 1 \right)^{(r-1)(g-1)} \sum_{t \in T_k/\mathbb{S}_r} \frac{1}{\delta(t)^{2g-2}} + r^{2g} - 1.$$ 

As an example, if $k$ is an integer divisible by 4, we get

$$(3.5) \quad \dim H^0(M_{PGL_2}, D^k) = 2^{1-2g} \left( \frac{k}{2} + 1 \right)^{g-1} \left( \sum_{0 < i < k+2} \frac{1}{\left( \sin \frac{i\pi}{k+2} \right)^{2g-2}} + 2^{2g} - 1 \right).$$

4. Relations with Conformal Field Theory

(4.1) According to Conformal Field Theory, the space $H^0(M_{PGL_r}, D^k)$ should be canonically isomorphic to the space of conformal blocks for a certain Conformal Field Theory, the WZW model associated to the projective group. This implies in particular that its dimension should be equal to $\sum_{j} |S_{0j}|^{2-2g}$, where $(S_{ij})$ is a unitary symmetric matrix. For instance in the case of the WZW model associated to $SL_2$, one has

$$S_{0j} = \frac{\sin \left( \frac{(j+1)\pi}{k+2} \right)}{\sqrt{\frac{k+2}{2} + 1}}, \quad \text{with} \quad 0 \leq j \leq k,$$

where the index $j$ can be thought as running through the set of irreducible representations $S^1, \ldots, S^k$ of $SL_2$ (or equivalently $SU_2$), with $S^j := S^j(C^2)$.

We deduce from (3.5) an analogous expression for $PGL_2$: we restrict ourselves to even indices and write

$$S'_{0j} = 2S_{0j} \quad \text{for} \quad j \text{ even } < k/2; \quad S'_{0,j/2}^{(1)} = S'_{0,j/2}^{(2)} = S_{0k/2}.$$ 

In other words, we consider only those representations of $SL_2$ which factor through $PGL_2$ and we identify the representation $S^{2j}$ with $S^{k-2j}$, doubling the coefficient
when these two representations are distinct, and counting twice the representation which is fixed by the involution (this process is well-known, see e.g. [M-S]).

(4.2) The case of $\text{SL}_r$ is completely analogous; we only need a few more terminology from representation theory (we follow the notation of [B]). The primary fields are indexed by the set $P_k$ of dominant weights $\lambda$ with $\lambda(H_{\theta}) \leq k$, where $H_{\theta}$ is the matrix $\text{diag}(1,0,\ldots,0,-1)$. For $\lambda \in P_k$, we put $t_\lambda = \exp 2\pi i \frac{\lambda + \rho}{k + r}$ (we identify the Cartan algebra of diagonal matrices with its dual using the standard bilinear form); the map $\lambda \mapsto t_\lambda$ induces a bijection of $P_k$ onto $T_k/\mathcal{S}_r$ ([B], lemma 9.3 c)). In view of Prop. 3.1, the coefficient $S_{0\lambda}$ for $\lambda \in P_k$ is given by

$$S_{0\lambda} = \frac{\delta(t_\lambda)}{\sqrt{r(k + r)^{(r-1)/2}}}.$$ 

Passing to $\text{PGL}_r$, we first restrict the indices to the subset $P'_k$ of elements $\lambda \in P_k$ such that $t_\lambda$ belongs to $T'_k$; this means that $\lambda$ belongs to the root lattice, i.e. that the representation $V_\lambda$ factors through $\text{PGL}_r$. The center $\mu_r$ acts on $T_k$ by multiplication; this action preserves $T'_k$, and commutes with the action of $\mathcal{S}_r$. The corresponding action on $P_k$ is deduced, via the bijection $\lambda \mapsto \frac{\lambda + \rho}{k + r}$, from the standard action of $\mu_r$ on the fundamental alcove $A$ with vertices $\{0, \varpi_1, \ldots, \varpi_{r-1}\}$.  

We identify two elements of $P'_k$ if they are in the same orbit with respect to this action. The action has a unique fixed point, the weight $\frac{k}{r}\rho$, which corresponds to the diagonal matrix $D_r$ (2.4); we associate to this weight $r$ indices $\nu^{(1)}, \ldots, \nu^{(r)}$, and put

$$S'_{0\lambda} = rS_{0\lambda} \quad \text{for } \lambda \in P'_k/\mu_r \; , \; \lambda \neq \frac{k}{r}\rho \; ; \quad S'_{0, \nu^{(i)}} = S_{0, \frac{k}{r}\rho} \quad \text{for } i = 1, \ldots, r \; .$$

One deduces easily from Cor. 3.4 the formula

$$\text{dim} \, H^0(M_{\text{PGL}_r}, D^k) = \sum |S'_{0\lambda}|^2 - 2g \; ,$$

where $\lambda$ runs over $P'_k/\mu_r \cup \{\nu^{(1)}, \ldots, \nu^{(r)}\}$.

Remark 4.3. It is not clear to me what is the physical meaning of the space $H^0(M^d_{\text{PGL}_r}, D^k)$, in particular if its dimension can be predicted in terms of the S-matrix. It is interesting to observe that the number $N(g)$ given by Prop. 3.3, which is equal to $\text{dim} \, H^0(M^d_{\text{PGL}_r}, D^k)$ for $g \geq 2$, is not necessarily an integer for $g = 1$: for $d = 0$ one finds $N(1) = 1 + \frac{(k + 1)r^{r-1} - 1}{r^2}$, which is not an integer unless $r^2 | k$.  

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1 The element $\exp \varpi_1$ of the center gives the rotation of $A$ which maps $0$ to $\varpi_1$, $\varpi_1$ to $\varpi_2$, ..., and $\varpi_{r-1}$ to $0$. 
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