The Quenched Continuum Limit

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We show that all current formalisms for quarks in lattice QCD are consistent in the quenched continuum limit, as they should be. We improve on previous extrapolations to this limit, and the understanding of lattice systematic errors there, by using a constrained fit including both leading and sub-leading dependence on \( a \).

1. INTRODUCTION

Our increasing ability to carry out unquenched simulations means that the quenched approximation is no longer required. There is an outstanding issue associated with the quenched continuum limit, however, which we address here. At the same time we point out the ingredients that are necessary to obtain an accurate result in the continuum limit, since these ingredients will also be important for unquenched simulations.

The outstanding issue is the question of whether all quark formulations give the same result in the quenched continuum limit. It is relatively trivial to demonstrate that they should, because this is the limit in which it is easy to analyse the formulations. At Lattice 2000, however, it seemed not to be the case \cite{1}. A plot was made of the ratio of nucleon to vector meson mass at a fixed physical quark mass, as a function of the lattice spacing. The physical quark mass was chosen as the point at which the pseudoscalar meson mass divided by the vector meson mass was 0.5 (in those days quite a low mass to reach). The lattice spacing was given in units of the vector meson mass. Because of the instability of vector mesons this is not a plot that allows for a precision test against experimental results, even if it used a quark mass value from the real world and was not in the quenched approximation. However, it can be used as a comparison of different formalisms, provided the results have good statistical precision and are on large enough volumes.

The original plot showed a 6\( \sigma \) disagreement between the results extrapolated to \( a = 0 \) from the Wilson formulation \cite{2} and the unimproved staggered formulation \cite{3}. The Wilson results were extrapolated linearly in \( a \) and the staggered results quadratically. Clover results were also included but their variation suggests a statistical error that means they are not particularly useful so we drop those results in this discussion.

The disagreement, if true, would represent a major problem for lattice calculations and call the whole approach into question. With many large dynamical quark projects well underway and others planned, it is therefore time to readdress this issue, following on from \cite{4}. In addition, there are more quark formulations in widespread use than there were in 2000, so the comparison of results can be significantly extended.

2. NEW ANALYSIS AND RESULTS

Figure \cite{1} shows an updated plot of the Wilson and unimproved staggered results from Lattice 2000, along with new results from improved staggered (asqtad) quarks on improved glue \cite{5} and the perfect action \cite{6}. The new results were chosen for small statistical errors, being available at the quark mass required here (with interpola-
Figure 1. $m_N/m_V$ for $m_{PS}/m_V = 0.5$ for a variety of quark formalisms in the quenched approximation. The curves are for a fit, described in the text, with a single continuum limit, marked with a filled circle. The filled grey triangle at $a=0$ shows the previous continuum limit for Wilson quarks obtained with a purely linear fit.

Figure 2. $m_V$ in units of $r_1$ at $m_{PS} r_1 = 0.807$. Results are from different quark formalisms and the fit has a joint continuum limit, as described in the text.

...and can therefore place constraints on the higher order terms. Taking the continuum limit is just such a case. We expect the scale of discretisation errors to be set by the size of internal momenta inside the hadron. This should be roughly a few hundred MeV. Here the priors on the coefficients in the polynomial in $m_V a$ were taken to be $\pm 0.5$, and 5 terms were included in the expansion for each formulation (a constant plus 4 appropriate powers).

The result for the joint continuum limit is 1.373(10), $2\sigma$ below the previous unimproved staggered continuum limit but well above the Wilson one. With hindsight this is not surprising because a purely linear fit to Wilson results neglects terms which are not small for these results. We would expect the quadratic terms to appear as, say, $(m_V a/2)^2$ and this is 5% for the coarsest lattice used. A systematic error has to be added to a purely linear extrapolation to take this into account. For the improved formalisms this is not...
so much of an issue because the terms neglected by a leading order extrapolation are $O(m_V a/2)^4$, 1% for the improved staggered and perfect actions at their coarsest spacing. In all cases, however, a Bayesian analysis allowing for higher order terms is useful in assessing the systematic errors.

We now turn to other scaling plots, Figures 2 and 3, for which a lot more data is available. These are plots of $m_N$ and $m_V$ in units of $r_1$, vs $(a/r_1)^2$ [8]. We include results from Wilson [2], clover [9,10], perfect action [6], and domain wall quarks [11] as well as a variety of staggered results with improved and unimproved glue. Again a good fit with a single continuum limit is readily obtained and the curves for this are shown. The results are for a fixed quark mass given by $m_{PS} r_0 = 1.127$ using $r_0/r_1 = 1.397$ in the quenched continuum limit [5].

The authors would welcome additional results for any of these graphs.

3. CONCLUSION

We show, reassuringly, that all current quark formalisms give the same answer for nucleon and rho masses, at fixed arbitrary quark mass, in the quenched continuum limit. We used an analysis of discretisation effects which allows the inclusion of leading and higher order terms in the lattice spacing. This is particularly necessary for unimproved actions, emphasising once again the importance of the improvement programme in reducing systematic errors in lattice calculations.

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