On Reliability Estimation of Stress-Strength (S-S) Modified Exponentiated Lomax Distribution

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Abstract

This paper deals with estimation of the stress-strength reliability for modified exponentiated Lomax distribution the suggested approach biased on using different estimation methods such as, Maximum likelihood method, Moment method, Least square method and Shrinkage methods, numerical study via MATLAB software, has been done and comparison between the obtained results has been carried out according to mean square error, the results showed that the effectiveness of these estimators which evaluated using Monte-Carlo simulation study.

Keywords: Modified Exponentiated Lomax Distribution, Stress-Strength (S-S), Shrinkage Estimation, Least Square, Maximum likelihood estimation

I. Introduction

The Lomax distribution it can be called pareto distribution presented by the Lomax in (1954), It was used frequently in several statistical literature to study business failure data, It was applied in many application fields such as actuarial sciences, biological sciences and engineering [IV]. Actually the problem for estimation of the reliability has been discussed by many researchers for different distributions like, gamma, half normal, Rayleigh, power, exponential, Burr, Weibull and others [XI]. In recent years the Lomax distribution has been used in the data of business failure, so that the income has been introduced into the field of life testing [XI]. Extended and modified for Lomax distribution have been studied in different distributions such as exponentiated Lomax distribution, Poisson Lomax distribution, Weibull Lomax distribution, exponential Lomax distribution, gamma Lomax distribution, McDonald Lomax distribution and power Lomax distribution [VIII].

Assume a random variable $x$, then the (c.d.f) of exponentiated Lomax distribution will be [I]

$$F(x, \alpha, \beta, \lambda) = \left[ 1 - \left( 1 + \lambda x \right)^{-\beta} \right]^\alpha, \quad x > 0, \alpha, \beta, \lambda > 0$$

(1)
The (p.d.f) for exponentiated Lomax distribution
\[ f(x, \alpha, \theta, \lambda) = \alpha \theta \lambda [1 - (1 + \lambda x)^{-\theta}]^{-1} (1 + \lambda x)^{-\alpha (\theta - 1)} \quad x > 0 \quad \theta, \alpha, \lambda > 0 \quad (2) \]

We will study special case from exponentiated Lomax distribution when \( \theta = 2, \lambda = 1 \)

The (c.d.f) for modified exponentiated Lomax distribution will be
\[ F(x, \alpha) = \left[ 1 - \left( 1 + x \right)^{-2\alpha} \right]^\alpha \quad x > 0 \quad (3) \]

The (p.d.f) for modified exponentiated Lomax distribution
\[ f(x, \alpha) = 2 \alpha \left[ 1 - (1 + x)^{-2\alpha} \right]^{-1} (1 + x)^{-3} \quad x > 0 \quad (4) \]

Where \( \theta, \alpha \) are the scale and shape parameters respectively

II. The reliability of stress-strength (S-S) model

The reliability function can be defined as the probability for failure of process until a given time and the reliability function can be defined as the ratio for the true variance to total variance that does not contain the variance for the random measurement \( [X][V] \). The stress \( y \) and the strength \( x \) in stress-strength (S-S) model will be considered as the random variables follow the modified exponentiated Lomax distribution. Assume the two random variables \( x \) and \( y \)
\[ x \sim ME LD (2, \alpha_1) \text{ and } y \sim ME LD (2, \alpha_2) \]
then the strength and stress (S-S) reliability for \( R \) this model define as
\[ R = P \left( y < x \right) \]

\[ R = \int_0^\infty \int_0^x f(x) f(y) dy \, dx \]

\[ = \int_0^\infty \int_0^x 2 \alpha_1 \left[ 1 - (1 + x)^{-2} \right]^{-1} (1 + x)^{-3} \cdot 2 \alpha_2 \lambda \left[ 1 - (1 + y)^{-2} \right]^{-1} (1 + y)^{-3} \, dy \, dx \]

\[ = \int_0^\infty 2 \alpha_1 \left[ 1 - (1 + x)^{-2} \right]^{-1} (1 + x)^{-3} \left[ 1 - (1 + y)^{-2} \right]^{-1} \, dx \]

\[ = \int_0^\infty 2 \alpha_1 \left[ 1 - (1 + x)^{-2} \right]^{-1} (1 + x)^{-3} \, dx \]

\[ R = \frac{\alpha_1}{\alpha_1 + \alpha_2} \quad (5) \]
III. Estimation methods of \( R = P(y < x) \)

The estimation of reliability is a very common problem in the statistical literature. The widest approach applied for reliability estimation is the well-known stress–strength model. This model is used in many applications of physics and engineering such as strength failure and the system collapse. In some engineering systems, many have more than components there may fail separately or together [VII]

Maximum likelihood Estimator [VIII] [III] [II]

In this estimator method. The log likelihood function of a random sample has been considered, Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) and Let \( y_1, y_2, \ldots \)

\( y_m \) be a random sample of size \( m \) obtained as in below,

The maximum likelihood for \( \alpha_1, \alpha_2 \) will be

\[
L = f(\alpha_1, x_i, y_j) = 2^n \alpha_1^n \prod_{i=1}^{n} [1 - (1 + x_i)^{-\alpha_1}] \prod_{i=1}^{m} [1 + x_i]^{-3} 2^m \alpha_2^m
\]

\[
= m \prod_{j=1}^{m} [1 - (1 + y_j)^{-\alpha_2}] \prod_{j=1}^{m} (1 + y_j)^{-3}
\]

(6)

Taking the logarithm of the both sides for equation (6), then

\[
\ln L(f(x, \alpha_1, \alpha_2)) = n \ln 2 + n \ln \alpha_1 + (\alpha_1 - 1) \sum_{i=1}^{n} \ln [1 - (1 + x_i)^{-\alpha_1}] - 3 \sum_{i=1}^{n} \ln (1 + x_i) +
\]

\[
m \ln 2 + m \ln \alpha_2 + (\alpha_2 - 1) \sum_{j=1}^{m} \ln [1 - (1 + y_j)^{-\alpha_2}] - 3 \sum_{j=1}^{m} \ln (1 + y_j)
\]

The partial derivative for log-function with respect to parameter \( \alpha_1 \) will become

\[
\frac{\partial \ln L}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} \ln \left[ 1 - \left(1 + \frac{x_i}{\alpha_1} \right)^{-2} \right]
\]

Equating partial derivation with zero, then

\[
\hat{\alpha}_{1, mle} = \frac{-n}{\sum_{i=1}^{n} \ln \left[ 1 - \left(1 + \frac{x_i}{\hat{\alpha}_1} \right)^{-2} \right]}
\]

(7)

And

\[
\hat{\alpha}_{2, mle} = \frac{-m}{\sum_{i=1}^{m} \ln \left[ 1 - \left(1 + \frac{y_i}{\hat{\alpha}_2} \right)^{-2} \right]}
\]

(8)
Substituting equations (7) and (8) in equation (5) then the reliability for stress-strength model by using maximum likelihood method will be as in the following

\[ R_{\text{mle}} = \frac{\hat{\alpha}_{1\text{mle}}}{\hat{\alpha}_{1\text{mle}} + \hat{\alpha}_{2\text{mle}}} \]  

(9)

**Moment method (MOM)**

The moment method has been introduced by Pearson in (1894), it can be represented as one of the oldest methods that used to estimate the parameter in this method.

Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) and let \( y_1, y_2, \ldots, y_m \) be a random sample of size \( m \).

The two parameters \( \alpha_1 \) and \( \alpha_2 \) will be estimated with using the first order population moment, hence the first moment for modified Generalized Lomax distribution will be

\[ E(x) = \alpha_1 B\left( \frac{1}{2}, \alpha_1 \right) - \frac{1}{\alpha_1} = \frac{\sum_{i=1}^{n} x_i}{n}, \quad E(y) = \alpha_2 B\left( \frac{1}{2}, \alpha_2 \right) - \frac{1}{\alpha_2} = \frac{\sum_{j=1}^{m} y_j}{m} \]

\[ \frac{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \alpha_1 \right)}{\Gamma\left( \frac{1}{2} + \alpha_1 \right)} - \frac{1}{\alpha_1} = \frac{\sum_{i=1}^{n} x_i}{\alpha_1 n}, \quad \frac{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \alpha_2 \right)}{\Gamma\left( \frac{1}{2} + \alpha_2 \right)} - \frac{1}{\alpha_2} = \frac{\sum_{j=1}^{m} y_j}{\alpha_2 m} \]

\[ \frac{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \alpha_1 \right)}{\Gamma\left( \alpha_1 + \frac{1}{2} \right)} = \frac{x}{\alpha_1} + \frac{1}{\alpha_1} \]

\[ \frac{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \alpha_1 \right)}{\Gamma\left( \alpha_1 + \frac{1}{2} \right)} = \frac{x + 1}{\alpha_1} \]

\[ \hat{\alpha}_{1\text{mle}} = \frac{\Gamma\left( \frac{1}{2} + \alpha_{01} \right) (x + 1)}{\Gamma\left( \frac{1}{2} \right) \Gamma\left( \alpha_{01} \right)} \]

(10)

And by the same way
Substituting equations (10) and (11) in equation (5) then the reliability for stress-strength model by using moment method will be as follows

\[
\hat{R}_{mom} = \frac{\hat{\alpha}_{1mom}}{\hat{\alpha}_{1mom} + \hat{\alpha}_{2mom}}
\]

**Least square method (LS)** [VI] [XIV]

To estimate the parameters \((\alpha_1)\) and \((\alpha_2)\) with using least square method the nonlinear equation must be convert to linear as in the follows

\[
F(x_i) = \left[1 - (1 + x_i)^{-2}\right]^{\frac{1}{\alpha_i}}
\]

\[
\left[ F(x_i) \right]^{\frac{1}{\alpha_i}} = 1 - (1 + x_i)^{-2}
\]

Taking the logarithm for both sides

\[
\frac{1}{\alpha_i} \text{Ln} \left[ F(x_i) \right] = \text{Ln} \left[ 1 - (1 + x_i)^{-2} \right]
\]

\[
a x + b = y_i
\]

\[
a = \frac{1}{\alpha_i}
\]

\[
b = 0
\]

\[
x_i = \text{Ln} \left[ F(x_i) \right]
\]

\[
y_i = \text{Ln} \left[ 1 - (1 + x_i)^{-2} \right]
\]

\[
a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} - \frac{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}}{n}
\]

\[
\hat{a}_{ls} = \frac{\sum_{i=1}^{n} [LnF(x_i)] - \left[\sum_{i=1}^{n} LnF(x_i)\right]^2}{\sum_{i=1}^{n} LnF(x_i) - \left[\sum_{i=1}^{n} LnF(x_i) Ln \left[1 - (1 + x_i)^{-2}\right]\right]/n}
\]

And by the same way

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Substituting equations (13) and (14) in equation (5), then the reliability for stress-strength model with using least square method will become

\[
\hat{R}_{s} = \frac{\hat{\alpha}_{1s} - \hat{\alpha}_{2s}}{\hat{\alpha}_{1s} + \hat{\alpha}_{2s}}
\]  

(15)

Shrinkage methods (SH)[XII][XIII]

In the shrinkage estimation method the prior estimation will be \( \phi(\hat{\alpha}) \) and \( 0 < \phi(\hat{\alpha}) < 1 \)

\[
\hat{\alpha}_{ub} = \phi(\hat{\alpha}) \alpha_{ub} + [1 - \phi(\hat{\alpha})] \alpha_{\omega}\]

(16)

Where "we are estimating (\( \alpha \)) and we believe (\( \alpha_{0} \)) is closed to the true value of (\( \alpha \)) and something bad happens if \( \alpha = \alpha_{0} \) and we do not use \( \alpha_{0} \)"

To find \( \alpha_{ub} \)

\[
n - 1
\frac{n - 1}{n} \times \alpha_{M.L.E}
\]

\[
\alpha_{ub} = \frac{n - 1}{n} \times \alpha_{M.L.E}
\]

\[
\alpha_{ub} = \frac{(n - 1)}{\sum_{i=1}^{n} \ln \left[ 1 - \left(1 + x_{i} \right)^{-2} \right]}
\]

(17)

Shrinkage weight function (sh1)[XIII]

In this method, the Shrinkage weight factor as function

\[
n \cdot \phi(\hat{\alpha}) = \frac{\sin n}{n}, \phi(\hat{\alpha}) = \frac{\sin m}{m}
\]
We put \( \phi(\bar{\alpha}) \) in (16), will be

\[
\hat{\alpha}_{1b1} = \frac{\sin n}{n} \alpha_{1b} + \left( 1 - \frac{\sin n}{n} \right) \alpha_{01} \tag{18}
\]

And by the same way

\[
\hat{\alpha}_{2b1} = \frac{\sin m}{m} \alpha_{2b} + \left( 1 - \frac{\sin m}{m} \right) \alpha_{02} \tag{19}
\]

Substituting equations (18) and (19) in equation (5), then the reliability for stress-strength model by using shrinkage weight function will be

\[
\hat{R}_{sh1} = \frac{\hat{\alpha}_{1b1}}{\hat{\alpha}_{1b1} + \hat{\alpha}_{2b1}} \tag{20}
\]

**Constant shrinkage factor (sh\(_2\)) [XIII]**

In this method, the constant shrinkage weight factor will be assumed

\[
\phi(\hat{\alpha}_1) = K_1 = 0.001 \quad \phi(\hat{\alpha}_2) = K_2 = 0.001 \quad \text{we put } \phi(\bar{\alpha}) \text{ in (16), will be}
\]

\[
\hat{\alpha}_{1b2} = k_1 \alpha_{1b} + \left( 1 - k_1 \right) \alpha_{01} \tag{21}
\]

And by the same way

\[
\hat{\alpha}_{2b2} = k_2 \alpha_{2b} + \left( 1 - k_2 \right) \alpha_{02} \tag{22}
\]

Substituting equations (21) and (22) in equation (5), then reliability for stress-strength model by using constant shrinkage factor

\[
\hat{R}_{sh2} = \frac{\hat{\alpha}_{1b2}}{\hat{\alpha}_{1b2} + \hat{\alpha}_{2b2}} \tag{23}
\]

**Beta shrinkage factor (sh\(_3\)) [XIII]**

The beta shrinkage weight factor will be assumed

\[
\phi(\hat{\alpha}_1) = B(1, n) \quad \phi(\hat{\alpha}_2) = B(1, m) \quad \text{we put } \phi(\bar{\alpha}) \text{ in (16), will be}
\]

\[
\hat{\alpha}_{1b3} = B(1, n) \alpha_{1b} + \left( 1 - B(1, n) \right) \alpha_{01} \tag{24}
\]

And by the same way
\[
\hat{\alpha}_{2sh} = B(1,m)\alpha_{2sh} + (1 - B(1,m))\alpha_{02}
\]  \hspace{3cm} \text{(25)}

Substituting equations (24) and (25) in equation (5), then reliability for stress-strength model by using beta shrinkage factor

\[
\hat{R}_{sh3} = \frac{\hat{\alpha}_{1sh3}}{\hat{\alpha}_{1sh3} + \hat{\alpha}_{2sh3}}
\]  \hspace{3cm} \text{(26)}

IV. Simulation process

The simulation process has been done with using unlike sample size (30, 50, 100) and built on 1000 replications via mean square error (MSE). It has been used in several steps to find the performance, \( x \) according to the uniform distribution on interval \((0,1)\) as \( w_1, w_2, \ldots, w_n \) and \( y \) according to the uniform distribution on interval \((0,1)\) as \( z_1, z_2, \ldots, z_m \) as in the following:

Step 1: from equation \( F(x,\alpha) = \left[1 - \left(1 + x\right)^{-2}\right]^{\alpha} \),
\[x_{ij} = \left[1 - \left(\frac{1}{w_i}\right)^{\alpha_1}\right]^{\frac{1}{2}} - 1 \] \hspace{1cm} i=1,2,3,\ldots,n

And
\[z_{ij} = \left[1 - \left(\frac{1}{z_j}\right)^{\alpha_2}\right]^{\frac{1}{2}} - 1 \] \hspace{1cm} j=1,2,3,\ldots,m

Step 2: from equations (7) and (8), \( \hat{\alpha}_{1mle} \) and \( \hat{\alpha}_{2mle} \) have been calculated respectively.

Step 3: from equations (10) and (11), \( \hat{\alpha}_{1mom} \) and \( \hat{\alpha}_{2mom} \) have been calculated respectively.

Step 4: from equations (13) and (14), \( \hat{\alpha}_{1ls} \) and \( \hat{\alpha}_{2ls} \) have been calculated respectively.

Step 5: from equations (18), (19), (21), (22), (24) and (25), \( \hat{\alpha}_{1shi} \) and \( \hat{\alpha}_{2shi} \) have been calculated for \( i = 1,2,3 \)

Step 6: from equation (9), (12), (15), (20), (23) and (26), \( \hat{R}_{mle}, \hat{R}_{mom}, \hat{R}_{ls}, \hat{R}_{shi}, \hat{R}_{shi2} \) and \( \hat{R}_{sh3} \) have been calculated respectively.

The results in tables (1), (3), (5), (7), (9), (11), (13), (15) and (17) explain the reliability of estimation, while the results in tables (2), (4), (6), (8), (10), (12), (14), (16), (18) show the mean square error (MSE).

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Bareq B. Selman et al
Table 1: Shown estimation when \( R = 0.50000 \), \( \alpha_1 = 1 \), \( \alpha_2 = 1 \)

| n | m | \( \hat{R}_{\text{mle}} \) | \( \hat{R}_{\text{mom}} \) | \( \hat{R}_{\text{sh1}} \) | \( \hat{R}_{\text{sh2}} \) | \( \hat{R}_{\text{sh3}} \) | \( \hat{R}_{\text{ls}} \) |
|---|---|---|---|---|---|---|---|
| 30 | 0.500475 | 0.492403 | 0.500026 | 0.500000 | 0.500026 | 0.496673 |
| 50 | 0.496051 | 0.491338 | 0.500002 | 0.500001 | 0.500030 | 0.497260 |
| 100 | 0.496909 | 0.494117 | 0.500004 | 0.500002 | 0.500019 | 0.499171 |
| 30 | 0.503786 | 0.515426 | 0.499938 | 0.499998 | 0.499952 | 0.502514 |
| 50 | 0.504399 | 0.494969 | 0.500000 | 0.500000 | 0.500026 | 0.502310 |
| 100 | 0.496909 | 0.494117 | 0.500004 | 0.500002 | 0.500019 | 0.499171 |

Table 2: Shown MSE values when \( R = 0.50000 \), \( \alpha_1 = 1 \), \( \alpha_2 = 1 \)

| n | m | MSE_{\text{mle}} | MSE_{\text{mom}} | MSE_{\text{sh1}} | MSE_{\text{sh2}} | MSE_{\text{sh3}} | MSE_{\text{ls}} | Best |
|---|---|---|---|---|---|---|---|---|
| 30 | 0.006002 | 0.011637 | 0.000000 | 0.000000 | 0.000000 | 0.006936 | MSE |
| 50 | 0.001903 | 0.001161 | 0.000000 | 0.000000 | 0.000000 | 0.001503 | MSE |
| 10 | 0.079360 | 0.072828 | 0.000000 | 0.000000 | 0.000000 | 0.020860 | MSE |
| 0 | 0.004959 | 0.051857 | 0.000000 | 0.000000 | 0.000000 | 0.007500 | MSE |
| 30 | 0.034849 | 0.038892 | 0.000000 | 0.000000 | 0.000000 | 0.027660 | MSE |
| 50 | 0.002636 | 0.011103 | 0.000000 | 0.000000 | 0.000000 | 0.001970 | MSE |
| 10 | 0.002636 | 0.011103 | 0.000000 | 0.000000 | 0.000000 | 0.001970 | MSE |
| 0 | 0.011340 | 0.002828 | 0.000000 | 0.000000 | 0.000000 | 0.003870 | MSE |
| 30 | 0.004530 | 0.042649 | 0.000000 | 0.000000 | 0.000000 | 0.028390 | MSE |
| 10 | 0.007907 | 0.048823 | 0.000000 | 0.000000 | 0.000000 | 0.019950 | MSE |
| 0 | 0.027419 | 0.048130 | 0.019107 | 0.007469 | 0.006365 | 0.009592 | MSE |
Table 3: Shown estimation when $R = 0.66666$, $\alpha_1= 2$, $\alpha_2= 1$

| $n$ | $m$ | $\hat{R}_{\text{mle}}$ | $\hat{R}_{\text{mom}}$ | $\hat{R}_{\text{sh1}}$ | $\hat{R}_{\text{sh2}}$ | $\hat{R}_{\text{sh3}}$ | $\hat{R}_{Ls}$ |
|-----|-----|----------------|-----------------|----------------|----------------|----------------|-------------|
| 30  | 30  | 0.670644       | 0.673761        | 0.666714       | 0.666668       | 0.666714       | 0.671054    |
| 50  | 50  | 0.663320       | 0.655894        | 0.666664       | 0.666666       | 0.666657       | 0.666083    |
| 100 | 100 | 0.659543       | 0.661369        | 0.666602       | 0.666665       | 0.666607       | 0.664609    |

Table 4: Shown MSE values when $R = 0.66666$, $\alpha_1= 2$, $\alpha_2= 1$

| $n$ | $m$ | $\text{mse}_{\text{mle}}$ | $\text{mse}_{\text{mom}}$ | $\text{mse}_{\text{sh1}}$ | $\text{mse}_{\text{sh2}}$ | $\text{mse}_{\text{sh3}}$ | $\text{mse}_{Ls}$ | Best | $\text{mse}$ |
|-----|-----|----------------|----------------|----------------|----------------|----------------|----------------|-----|-------------|
| 30  | 30  | 0.028234       | 0.233989       | 0.000014       | 0.000000       | 0.000015       | 0.000012       | 0.036490 | 0.12771     |
| 30  | 50  | 0.015703       | 0.130395       | 0.000002       | 0.000000       | 0.000003       | 0.000001       | 0.093      |
| 100 | 100 | 0.013217       | 0.061803       | 0.000019       | 0.000000       | 0.000017       | 0.000004       | 0.234732 |
| 50  | 50  | 0.132166       | 0.054519       | 0.000002       | 0.000000       | 0.000016       | 0.000002       | 0.083      |
| 10  | 10  | 0.013282       | 0.054519       | 0.000002       | 0.000000       | 0.000004       | 0.000001       | 0.234732 |
| 30  | 30  | 0.022039       | 0.054519       | 0.000002       | 0.000000       | 0.000016       | 0.000002       | 0.083      |
| 10  | 10  | 0.257713       | 0.083168       | 0.000001       | 0.000000       | 0.000001       | 0.000001       | 0.386141 |
| 50  | 50  | 0.013217       | 0.061803       | 0.000002       | 0.000000       | 0.000004       | 0.000001       | 0.16311   |
Table 5: Shown estimation when R = 0.33333, alpha1= 1, alpha2= 2

| n  | m  | $\hat{R}_{\text{mle}}$ | $\hat{R}_{\text{mom}}$ | $\hat{R}_{\text{sh1}}$ | $\hat{R}_{\text{sh2}}$ | $\hat{R}_{\text{sh3}}$ | $\hat{R}_{\text{Ls}}$ |
|-----|----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 30  | 30 | 0.331588        | 0.323635        | 0.333354        | 0.333334        | 0.333354        | 0.326357        |
|     | 50 | 0.327932        | 0.322604        | 0.333334        | 0.333334        | 0.333354        | 0.327064        |
| 100 | 100| 0.334910        | 0.324680        | 0.333369        | 0.333334        | 0.333354        | 0.330783        |

Table 6: Shown MSE values when R = 0.33333, alpha1= 1, alpha2= 2

| n  | m  | $\text{mse}_{\text{mle}}$ | $\text{mse}_{\text{mom}}$ | $\text{mse}_{\text{sh1}}$ | $\text{mse}_{\text{sh2}}$ | $\text{mse}_{\text{sh3}}$ | $\text{mse}_{\text{Ls}}$ | Best         |
|-----|----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------|
| 30  | 30 | 0.005357        | 0.01744         | 0.000000        | 0.000000        | 0.000000        | 0.000000        | 0.01021       |
|     | 50 | 0.003170        | 0.01366         | 0.000000        | 0.000000        | 0.000000        | 0.000000        | 0.00457       |
| 100 | 100| 0.000728        | 0.005008        | 0.000000        | 0.000000        | 0.000000        | 0.000000        | 0.03354       |

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Bareq B.Selman et al
Table 7: Shown estimation when R = 0.500000, alpha1= 2, alpha2= 2

| n  | m  | mle  | mom  | sh1  | sh2  | sh3  | Ls   |
|----|----|------|------|------|------|------|------|
| 30 | 30 | 0.500475 | 0.493166 | 0.500026 | 0.500000 | 0.500026 | 0.496673 |
| 50 | 50 | 0.496051 | 0.492136 | 0.500002 | 0.500001 | 0.500026 | 0.497260 |
| 100 | 100 | 0.489019 | 0.485763 | 0.499928 | 0.499998 | 0.499928 | 0.499872 |
| 30 | 30 | 0.504675 | 0.506444 | 0.500044 | 0.499999 | 0.500010 | 0.500231 |
| 50 | 50 | 0.505597 | 0.502586 | 0.500027 | 0.500005 | 0.500105 | 0.504625 |
| 100 | 100 | 0.493941 | 0.493004 | 0.499989 | 0.499997 | 0.499984 | 0.494804 |
| 30 | 30 | 0.506682 | 0.506464 | 0.499916 | 0.499997 | 0.499964 | 0.499963 |
| 50 | 50 | 0.500619 | 0.501888 | 0.499984 | 0.499996 | 0.499963 | 0.496296 |
| 100 | 100 | 0.497709 | 0.497540 | 0.499987 | 0.499997 | 0.499976 | 0.496810 |

Table 8: Shown MSE values when R = 0.500000, alpha1= 2, alpha2= 2

| n  | m  | mle  | mom  | sh1  | sh2  | sh3  | Ls   | Best |
|----|----|------|------|------|------|------|------|------|
| 30 | 30 | 0.00600 | 0.01646 | 0.00000 | 0.00000 | 0.00000 | 0.00693 | sh2  |
| 50 | 50 | 0.00190 | 0.01097 | 0.00000 | 0.00000 | 0.00000 | 0.00150 | sh2  |
| 100 | 100 | 0.01299 | 0.02227 | 0.00000 | 0.00000 | 0.00000 | 0.00371 | sh2  |
| 30 | 30 | 0.00312 | 0.01336 | 0.00000 | 0.00000 | 0.00000 | 0.00282 | sh2  |
| 50 | 50 | 0.00544 | 0.00248 | 0.00000 | 0.00000 | 0.00000 | 0.00430 | sh2  |
| 100 | 100 | 0.00442 | 0.00564 | 0.00000 | 0.00000 | 0.00000 | 0.00523 | sh2  |
| 30 | 30 | 0.00570 | 0.01297 | 0.00000 | 0.00000 | 0.00000 | 0.00282 | sh2  |
| 50 | 50 | 0.00105 | 0.00349 | 0.00000 | 0.00000 | 0.00000 | 0.00314 | sh2  |
| 100 | 100 | 0.00122 | 0.00178 | 0.00000 | 0.00000 | 0.00000 | 0.00172 | sh2  |
| 0  | 0  | 0.6984 | 3573 | 0032 | 001 | 0126 | 1086 | sh2  |
Table 9: Shown estimation when R = 0.750000, alpha1 = 3, alpha2 = 1

| n  | m  | mle     | mom     | sh1     | sh2     | sh3     | Ls     |
|----|----|---------|---------|---------|---------|---------|--------|
| 30 | 30 | 0.755734| 0.748044| 0.750082| 0.750002| 0.750083| 0.754089|
| 30 | 50 | 0.747484| 0.743238| 0.749927| 0.749998| 0.749938| 0.755993|
| 100| 75 | 0.746065| 0.745587| 0.750015| 0.750001| 0.750019| 0.749192|
| 30 | 50 | 0.755921| 0.756221| 0.749998| 0.750000| 0.750001| 0.752759|
| 100| 75 | 0.750889| 0.749384| 0.749999| 0.749999| 0.749984| 0.753031|
| 30 | 100| 0.749510| 0.747731| 0.750004| 0.750004| 0.750001| 0.752929|
| 50 | 50 | 0.757137| 0.757004| 0.750000| 0.750000| 0.750001| 0.754089|
| 100| 75 | 0.750857| 0.756221| 0.749998| 0.750000| 0.750001| 0.752759|

Table 10: Shown MSE values when R = 0.750000, alpha1 = 3, alpha2 = 1

| n  | m  | mle     | mom     | sh1     | sh2     | sh3     | sh2     |
|----|----|---------|---------|---------|---------|---------|---------|
| 30 | 30 | 0.03613 | 0.02020 | 0.000001| 0.000001| 0.000001| 0.02901 |
| 30 | 50 | 0.01684 | 0.07156 | 0.000002| 0.000001| 0.000001| 0.06948 |
| 100| 75 | 0.02225 | 0.02365 | 0.000000| 0.000000| 0.000000| 0.00488 |
| 30 | 50 | 0.03978 | 0.08559 | 0.000000| 0.000000| 0.000000| 0.02936 |
| 50 | 10 | 0.01059 | 0.00710 | 0.000000| 0.000000| 0.000000| 0.01672 |
| 50 | 50 | 0.01059 | 0.00710 | 0.000000| 0.000000| 0.000000| 0.01672 |
| 50 | 10 | 0.00469 | 0.01384 | 0.000000| 0.000000| 0.000000| 0.01987 |
| 10 | 30 | 0.07653 | 0.06751 | 0.000001| 0.000001| 0.000001| 0.09079 |
| 0  | 30 | 0.02142 | 0.00599 | 0.000000| 0.000000| 0.000000| 0.05540 |
| 0  | 50 | 0.00368 | 0.00761 | 0.000000| 0.000000| 0.000000| 0.01119 |
| 0  | 60 | 0.04367 | 0.151    | 0.000005| 0.0588   | 0.4837   | sh2     |
Table 11: Shown estimation when R = 0.250000, alpha1= 1, alpha2= 3

| n  | m   | $\hat{m}_{\text{mle}}$ | $\hat{m}_{\text{mom}}$ | $\hat{m}_{\text{sh1}}$ | $\hat{m}_{\text{sh2}}$ | $\hat{m}_{\text{sh3}}$ | $\hat{m}_{Ls}$ |
|-----|-----|------------------------|------------------------|------------------------|------------------------|------------------------|----------------|
| 30  | 30  | 0.247618               | 0.240610               | 0.250016               | 0.250000               | 0.250016               | 0.242459       |
| 50  | 30  | 0.244659               | 0.239707               | 0.250000               | 0.250001               | 0.250017               | 0.243124       |
| 100 | 30  | 0.236585               | 0.236032               | 0.249934               | 0.249995               | 0.249920               | 0.239052       |
| 30  | 50  | 0.252024               | 0.243497               | 0.250011               | 0.250000               | 0.250022               | 0.246713       |
| 50  | 50  | 0.250985               | 0.245903               | 0.250014               | 0.250002               | 0.250055               | 0.249077       |
| 100 | 50  | 0.248124               | 0.247139               | 0.250011               | 0.250002               | 0.250030               | 0.247603       |
| 30  | 100 | 0.256303               | 0.248820               | 0.250063               | 0.250001               | 0.250063               | 0.249796       |
| 50  | 100 | 0.250757               | 0.253537               | 0.249995               | 0.249999               | 0.249983               | 0.248375       |
| 100 | 100 | 0.250037               | 0.249814               | 0.250004               | 0.250000               | 0.250009               | 0.247633       |

Table 12: Shown MSE values when R = 0.250000, alpha1= 1, alpha2= 3

| n  | m   | $\text{MSE}_{\text{mle}}$ | $\text{MSE}_{\text{mom}}$ | $\text{MSE}_{\text{sh1}}$ | $\text{MSE}_{\text{sh2}}$ | $\text{MSE}_{\text{sh3}}$ | $\text{MSE}_{Ls}$ | Best |
|-----|-----|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|------------------|------|
| 30  | 30  | 0.00678                 | 0.01620                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.00260 | sh2  |
| 50  | 30  | 0.00303                 | 0.01180                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.00520 | sh2  |
| 100 | 30  | 0.19019                 | 0.20284                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.13883 | sh1  |
| 30  | 50  | 0.1452                  | 0.05898                 | 0.00001                 | 0.00000                 | 0.00000                 | 0.00000         | 0.03043 | sh2  |
| 50  | 50  | 0.00077                 | 0.00027                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.00263 | sh1  |
| 100 | 50  | 0.00077                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.00131 | sh2  |
| 30  | 100 | 0.01100                 | 0.00263                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.00133 | sh2  |
| 50  | 100 | 0.00110                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.00058 | sh1  |
| 100 | 100 | 0.00050                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000                 | 0.00000         | 0.00099 | sh2  |
Table 13: Shown estimation when \( R = 0.600000 \), \( \alpha_1=3 \), \( \alpha_2=2 \)

| n  | m  | \( \hat{\mu}_{\text{mle}} \) | \( \hat{\mu}_{\text{mom}} \) | \( \hat{\mu}_{\text{sh1}} \) | \( \hat{\mu}_{\text{sh2}} \) | \( \hat{\mu}_{\text{sh3}} \) | \( \hat{\mu}_{Ls} \) |
|----|----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 30 | 30 | 0.599965        | 0.594228        | 0.5999956       | 0.5999956       | 0.599956        | 0.604850        |
| 50 | 50 | 0.597252        | 0.593312        | 0.599938       | 0.600000       | 0.599984        | 0.598718        |
| 100| 100| 0.591781        | 0.589941        | 0.600003       | 0.599999       | 0.600002        | 0.601835        |
| 30 | 30 | 0.608563        | 0.603401        | 0.600004       | 0.600001       | 0.600036        | 0.607182        |
| 50 | 50 | 0.600595        | 0.597543        | 0.5999997      | 0.5999999      | 0.599999       | 0.600587        |
| 100| 100| 0.601205        | 0.609301        | 0.600005       | 0.600001       | 0.600027        | 0.607343        |

Table 14: Shown MSE values when \( R = 0.600000 \), \( \alpha_1=3 \), \( \alpha_2=2 \)

| n  | m  | \( \text{MSE}_{\text{mle}} \) | \( \text{MSE}_{\text{mom}} \) | \( \text{MSE}_{\text{sh1}} \) | \( \text{MSE}_{\text{sh2}} \) | \( \text{MSE}_{\text{sh3}} \) | \( \text{MSE}_{Ls} \) | Best |
|----|----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----|
| 30 | 30 | 0.029099        | 0.09293         | 0.000002       | 0.000002       | 0.04937        | 0.000000        | sh2 |
| 30 | 50 | 0.025258        | 0.06023         | 0.00001        | 0.00000        | 0.00001       | 0.02344        | sh2 |
| 30 | 100| 0.008476        | 0.01200         | 0.00000       | 0.00000        | 0.00000       | 0.00150        | sh2 |
| 30 | 0  | 0.011628        | 0.00593         | 0.00000       | 0.00000        | 0.00000       | 0.00961        | sh2 |
| 50 | 30 | 0.016424        | 0.04118         | 0.00000       | 0.00000        | 0.00000       | 0.02832        | sh2 |
| 50 | 50 | 0.0621          | 0.4519          | 0.0425        | 0.015         | 0.6185        | 2354           | sh2 |
| 50 | 100| 0.001053        | 0.01375         | 0.00000       | 0.00000        | 0.00000       | 0.00880        | sh2 |
| 50 | 0  | 0.02501         | 0.9339          | 0.0066        | 0.002         | 0.0564        | 2075           | sh2 |
| 30 | 30 | 0.017939        | 0.03557         | 0.00000       | 0.00000        | 0.00000       | 0.00959        | sh2 |
| 30 | 100| 0.006720        | 0.01504         | 0.00000       | 0.00000        | 0.00000       | 0.01266        | sh2 |
| 10 | 50 | 0.006451        | 0.01795         | 0.00000       | 0.00000        | 0.00000       | 0.01342        | sh2 |
| 0  | 8820| 0.0117          | 0.004           | 0.0459        | 2827           |                |                |     |
Table 15: Shown estimation when $R = 0.40000$, $\alpha_1 = 2$, $\alpha_2 = 3$

| n  | m  | $\hat{\alpha}_{\text{mle}}$ | $\hat{\alpha}_{\text{mom}}$ | $\hat{\alpha}_{\text{sh1}}$ | $\hat{\alpha}_{\text{sh2}}$ | $\hat{\alpha}_{\text{sh3}}$ | $\hat{\alpha}_{\text{Ls}}$ |
|----|----|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 30 | 30 | 0.399050                    | 0.391347                    | 0.400023                    | 0.400000                    | 0.400023                    | 0.394188                    |
| 50 | 50 | 0.394979                    | 0.390309                    | 0.400011                    | 0.400001                    | 0.400023                    | 0.394875                    |
| 100| 100| 0.391577                    | 0.391983                    | 0.400039                    | 0.400011                    | 0.400041                    | 0.398444                    |
| 30 | 30 | 0.402963                    | 0.398266                    | 0.399979                    | 0.399999                    | 0.399984                    | 0.399179                    |
| 50 | 50 | 0.403020                    | 0.406418                    | 0.400019                    | 0.400003                    | 0.400073                    | 0.400784                    |
| 100| 100| 0.398825                    | 0.397170                    | 0.400015                    | 0.400002                    | 0.400067                    | 0.399961                    |

Table 16: Shown MSE values when $R = 0.40000$, $\alpha_1 = 2$, $\alpha_2 = 3$

| n  | m  | $\text{MSE}_{\text{mle}}$ | $\text{MSE}_{\text{mom}}$ | $\text{MSE}_{\text{sh1}}$ | $\text{MSE}_{\text{sh2}}$ | $\text{MSE}_{\text{sh3}}$ | $\text{MSE}_{\text{Ls}}$ | Best |
|----|----|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|------|
| 30 | 30 | 0.00583                   | 0.01812                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00927                   | sh2  |
| 50 | 50 | 0.00283                   | 0.01283                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00335                   | sh2  |
| 100| 100| 0.00900                   | 0.00827                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00229                   | sh2  |
| 30 | 30 | 0.00256                   | 0.00363                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00184                   | sh2  |
| 50 | 50 | 0.00114                   | 0.00705                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00075                   | sh2  |
| 100| 100| 0.00276                   | 0.00267                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00481                   | sh2  |
| 30 | 30 | 0.01073                   | 0.00253                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00552                   | sh2  |
| 100| 100| 0.00185                   | 0.03012                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00210                   | sh2  |
| 50 | 50 | 0.00136                   | 0.00308                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00418                   | sh2  |
| 100| 100| 0.00136                   | 0.00308                   | 0.00000                   | 0.00000                   | 0.00000                   | 0.00418                   | sh2  |

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Bareq B. Selman et al
Table 17: Shown estimation when R = 0.50000, alpha1= 3, alpha2= 3

| n  | m  | mle  | mom  | sh1  | sh2  | sh3  | Ls  | Best   |
|----|----|------|------|------|------|------|-----|--------|
| 30 | 30 | 0.500475 | 0.493457 | 0.500026 | 0.500000 | 0.500026 | 0.496673 | sh2    |
| 50 | 50 | 0.496051 | 0.492439 | 0.500002 | 0.500001 | 0.500026 | 0.497260 | sh2    |
| 100| 100| 0.492299 | 0.493380 | 0.500042 | 0.500001 | 0.500044 | 0.500258 | sh2    |
| 30 | 30 | 0.504675 | 0.507075 | 0.500044 | 0.499999 | 0.500010 | 0.500231 | sh2    |
| 50 | 50 | 0.494800 | 0.546276 | 0.499972 | 0.499994 | 0.499896 | 0.496510 | sh2    |
| 100| 100| 0.495996 | 0.495441 | 0.500001 | 0.500015 | 0.498652 | 0.505113 | sh2    |

Table 18: Shown MSE values when R = 0.50000, alpha1= 3, alpha2= 3

| n  | m  | mle_m  | mom_m | sh1_m | sh2_m | sh3_m | Ls_m | Best_m |
|----|----|-------|-------|-------|-------|-------|------|--------|
| 30 | 30 | 0.00600 | 0.01651 | 0.00000 | 0.0000000 | 0.00000 | 0.00693 | sh2   |
| 50 | 50 | 0.00190 | 0.01077 | 0.00000 | 0.0000000 | 0.00000 | 0.00150 | sh2   |
| 10 | 10 | 0.00793 | 0.00630 | 0.00000 | 0.0000000 | 0.00000 | 0.00208 | sh2   |
| 0  | 0  | 0.00073 | 0.00000 | 0.00000 | 0.0000000 | 0.00000 | 0.00020 | sh2   |
| 30 | 30 | 0.00312 | 0.01471 | 0.00000 | 0.0000000 | 0.00000 | 0.00282 | sh2   |
| 50 | 50 | 0.00007 | 0.00823 | 0.00000 | 0.0000000 | 0.00000 | 0.00004 | sh2   |
| 10 | 10 | 0.00201 | 0.00269 | 0.00000 | 0.0000000 | 0.00000 | 0.00086 | sh2   |
| 0  | 0  | 0.00131 | 0.00623 | 0.00000 | 0.0000000 | 0.00000 | 0.00406 | sh2   |
| 30 | 30 | 0.00153 | 0.00318 | 0.00000 | 0.0000000 | 0.00000 | 0.00197 | sh2   |
| 10 | 10 | 0.00187 | 0.00040 | 0.00000 | 0.0000000 | 0.00000 | 0.00439 | sh2   |
| 0  | 0  | 0.00277 | 0.499872 | 0.500013 | 0.500002 | 0.50026 | 0.504741 | sh2   |
V. Numerical result

The result of simulation refer to all methods have been used to estimate reliability for modified exponentiated Lomax distribution have had good results, but we have to find best method, from the tables for all \( n=(30,50,100), m=(30,50,100) \), for each \( \alpha \), It is clear that, the best method is shrinkage estimation method \([\text{constant shrinkage weight factor } (\overline{h}_2)]\), and always follow by\([\text{shrinkage weight function } (\overline{h}_1)]\).

VI. Conclusion

All the results are listed in tables. Montecarlo simulation study shows that if we compare methods estimator. Find that for all samples size their shrinkage estimators produce better results anchor methods, and the shrinkage estimator using constant shrinkage weight factor \((\overline{sh}_2)\) has minimum statistic indicator \((\text{MSE})\) in at most cases.

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