Time-Like Constant Slope Surfaces and Space-Like Bertrand Curves in Minkowski 3-Space

ABSTRACT

Defining Lorentzian Sabban frame of the unit speed time-like curves on de Sitter 2-space $S^2_1$ and introducing space-like height function on the unit speed time-like curves on $S^2_1$, the invariants of the unit speed time-like curves on $S^2_1$ and geometric properties of de Sitter evolutes of the unit speed time-like curves on $S^2_1$ are studied. A relation between space-like Bertrand curves and helices is obtained. De Sitter Darboux images of space-like Bertrand curves are equal to de Sitter evolutes. The relations between time-like constant slope surfaces lying in the space-like cone and space-like Bertrand curves in Minkowski 3-space $\mathbb{R}^3_1$ are obtained.

Key words: Bertrand curve, de Sitter Darboux image, helix, Lorentzian Sabban frame, de Sitter evolute, time-like constant slope surface, de Sitter 2-space
1. Introduction

Huygens [1] discovered involutes (also known as evolvents) while trying to build a more accurate pendulum clock. The curve $\alpha$ is called evolute of $\tilde{\alpha}$ and the curve $\tilde{\alpha}$ is called involute of $\alpha$ if the tangent vectors are orthogonal at the corresponding points for all $s \in I \subset \mathbb{R}$. Thus the pair of $(\alpha, \tilde{\alpha})$ is called the evolute-involute pair [2].

Bertrand curves [3] are different generalization of circular helices and are particular examples of offset curves which are used in computer-aided design (CAD) and computer-aided manufacture (CAM) [4]. Izumiya and Takeuchi [5] showed that Bertrand curves can be constructed from the unit speed curves on the 2-sphere $S^2$. Also they defined spherical evolutes of the unit speed curves on $S^2$ and proved that these spherical evolutes are the locus of the centre of the curvatures of the unit speed curves on $S^2$. In addition, they showed that the spherical Darboux images of Bertrand curves are equal to the spherical evolutes of the unit speed curves on $S^2$ in Euclidean 3-space $\mathbb{R}^3$.

Babaarslan and Yayli [6] found Bertrand curves corresponding to the tangent, binormal, principal normal and Darboux images of a space curve in $\mathbb{R}^3$.

Izumiya et al. [7] defined the notion of hyperbolic evolutes and hyperbolic height function of space-like curves on the hyperbolic space $\mathbb{H}^2$. As a result, they showed that the hyperbolic evolutes are the locus of the center of geodesic curvatures in Minkowski 3-space $\mathbb{R}^3$.

As a generalization of the concept of helix, we can also think constant slope surfaces whose position vectors make a constant angle with the normals at each point on the surfaces. Munteanu [8] defined constant slope surfaces in $\mathbb{R}^3$ and showed that these surfaces can be constructed from unit speed curves on $S^2$ such as Bertrand curves.

Babaarslan and Yayli [9] investigated the relations among Bertrand curves, spherical images and constant slope surfaces in $\mathbb{R}^3$. Also, Babaarslan et al. [10, 11] found some different characterizations of constant slope surfaces and Bertrand curves with respect to the Darboux frame. In addition, Babaarslan and Yayli [12] showed that the constant slope surfaces can be reparametized by using quaternions and homothetic motions.
Fu and Yang [13] studied space-like constant slope surfaces in $\mathbb{R}^3_1$ and classified these surfaces in the same space. Thereafter Babaarslan and Yayli [14] gave the relations among split quaternions, homothetic motions and space-like constant slope surfaces in $\mathbb{R}^3_1$.

Fu and Wang [15] gave a complete classification of time-like constant slope surfaces in $\mathbb{R}^3_1$. They showed that $S \subset \mathbb{R}^3_1$ is a time-like constant slope surface lying in the space-like cone if and only if it can be parametrized by

$$x(u,v) = u \sin \theta \left( \cos \xi(u)f(v) + \sin \xi(u)f(v) \wedge f'(v) \right),$$

where $\theta$ is a constant satisfying $\theta \in (0, \pi/2]$, $\xi(u) = \cot \theta \ln u$ and $f$ is a unit speed time-like curve on de Sitter 2-space $S^2_1$. Also, there are some different characterizations of time-like constant slope surfaces in [15] since the results of them are similar to that we found [16], we only investigate the above characterization.

Constant slope surfaces have nice shapes and they are interesting in terms of differential geometry in both Euclidean and Minkowski 3-space [8].

In the present paper, we define the notions of Lorentzian Sabban frame and de Sitter evolutes of the unit time-like curves on $S^2_1$ and study the invariants of the unit speed time-like curves on $S^2_1$ and geometric properties of de Sitter evolutes of the unit speed time-like curves on $S^2_1$. The relations among space-like Bertrand curves, helices, de Sitter Darboux images and time-like constant slope surfaces lying in the space-like cone in $\mathbb{R}^3_1$ are also given.

2. Basic notations, definitions and formulas

Let $\mathbb{R}^3_1$ denote the Minkowski 3-space, that is, the real vector space $\mathbb{R}^3$ endowed with the standard Lorentzian metric [17]

$$<x, y> = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x, y \in \mathbb{R}^3_1$. An arbitrary vector $x \in \mathbb{R}^3_1$ is called space-like if $<x, x> > 0$ or $x = 0$, time-like if $<x, x> < 0$ and light-like (null) if $<x, x> = 0$ and $x \neq 0$. The norm (length) of a vector $x$ is given by $||x|| = \sqrt{<x, x>}$. 

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We say that a regular curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ is space-like, time-like or light-like if all of its velocity vector $\alpha'(t)$ is space-like, time-like or light-like, respectively. $\alpha$ is parametrized by the arc-length parameter if $\|\alpha'(s)\|=1$ for all $s \in I \subset \mathbb{R}$. In this case, we say that $\alpha$ is a unit speed curve.

We can define the notion of the Lorentzian cross-product as follows:

$$\wedge : \mathbb{R}_1^3 \times \mathbb{R}_1^3 \to \mathbb{R}_1^3$$

$$(x_1, x_2, x_3, y_1, y_2, y_3) \to \begin{vmatrix} i & j & -k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

As the cross-product in Euclidean 3-space, the Lorentzian cross-product has similar algebraic and geometric properties:

(i) $< x \wedge y, z > = \det(x, y, z)$;

(ii) $x \wedge y = -y \wedge x$;

(iii) $(x \wedge y) \wedge z = -< x, z > y + < y, z > x$;

(iv) $< x \wedge y, x > = 0$ and $< x \wedge y, y > = 0$;

(v) $< x \wedge y, x \wedge y > = - < x, x > < y, y > + (< x, y >)^2$ for all $x, y, z$ in $\mathbb{R}_1^3$.

Given a unit speed curve $\alpha$ in $\mathbb{R}_1^3$, it is possible to define a Frenet frame $\{T(s), N(s), B(s)\}$ associated for each point $s$. Here $T$, $N$ and $B$ are the tangent, principal normal and binormal vector fields, respectively.

Let $\alpha$ be a unit speed space-like curve in $\mathbb{R}_1^3$. We assume that $T'(s)$ is time-like. Then we have $N(s) = \alpha''(s) / \kappa(s)$ and $B(s) = T(s) \wedge N(s)$. Thus the Frenet formulae is

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where $\kappa(s) = -< T'(s), T'(s) >$ and $\tau(s) = -< N'(s), B(s) >$ are the curvature and torsion of the unit speed space-like curve, respectively.

The Darboux vector of this space-like curve $\alpha$ is given by $D(s) = \tau(s)T(s) - \kappa(s)B(s)$. Then de Sitter Darboux image of $\alpha$ is
\[
\mathbf{C} : I \rightarrow \mathbb{S}^2_1 \\
s \rightarrow C(s) = \frac{\mathbf{D}(s)}{\|\mathbf{D}(s)\|}
\]

For a general parameter \( t \) of a space-like space curve \( \alpha \), we can calculate the curvature and torsion as follows:

\[
\kappa(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{\|\alpha'(t)\|^2}, \quad \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \wedge \alpha''(t)\|^2}.
\] (2.1)

If a time-like or space-like curve \( \alpha \) in \( \mathbb{R}^3 \) is a helix, then \( \tau/\kappa \) is a constant function. Conversely, let \( \alpha \) be a time-like or a space-like curve with non-null principal normal vector. If \( \tau/\kappa \) is constant, then \( \alpha \) is a helix. Furthermore, a time-like or space-like curve \( \alpha \) is a Bertrand curve if and only if there are non-zero real constants \( A, B \) such that \( A\kappa(s) + B\tau(s) = 1 \) for any \( s \in I \subset \mathbb{R} \).

We can define de Sitter 2-space and hyperbolic space in \( \mathbb{R}^3_1 \), respectively as follows:

\[
\mathbb{S}^2_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 : x_1^2 + x_2^2 - x_3^2 = 1\},
\]

\[
\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 : x_1^2 + x_2^2 - x_3^2 = -1\}.
\]

Now we define a pseudo-orthonormal frame along a time-like curve on \( \mathbb{S}^2_1 \). Let \( f : I \rightarrow \mathbb{S}^2_1 \) be a unit speed time-like curve. We denote \( v \) as the arc-length parameter of \( f \). Let us denote \( t(v) = f'(v) \) and we call \( t(v) \) as the unit tangent vector of \( f \) at \( v \). We set a vector \( s(v) = f(v) \wedge t(v) \) and as a consequence \( s(v) \wedge t(v) = -f(v) \), where \( f \) denotes the position vector of the curve. By definition of the time-like curve \( f \), we have a Lorentzian Sabban frame \{\( f(v), t(v), s(v) \)\} along \( f \). Then we have the following pseudo-spherical Frenet-Serret formulae of \( f \):

\[
\begin{align*}
    f'(v) &= t(v), \\
    t'(v) &= f(v) + \mathbf{K}_g(v)s(v), \\
    s'(v) &= \mathbf{K}_g(v)t(v),
\end{align*}
\] (2.2)

where \( \mathbf{K}_g(v) \) is the geodesic curvature of the unit speed time-like curve \( f \) on \( \mathbb{S}^2_1 \) which is given by \( \mathbf{K}_g(v) = \det(f(v),t(v),t'(v)) \).

Also we define a curve on \( \mathbb{S}^2_1 \) as follows:
3. Space-like height function of unit speed time-like curves on $\mathbb{S}^2_1$

We introduce a function on a time-like curve $f : I \rightarrow \mathbb{S}^2_1$ [7]. Now we define a function $H^S : I \times \mathbb{S}^2_1 \rightarrow \mathbb{R}$ by $H^S(v,u) = \langle f(v), u \rangle$. We call $H^S$ as the space-like height function of the time-like curve $f$ and denoted by $(h^S_u)(v) = H^S(v,u)$.

Thus, we have the following proposition:

**Proposition 3.1.** Let $f : I \rightarrow \mathbb{S}^2_1$ be a unit speed time-like curve. For any $(v,u) \in I \times \mathbb{S}^2_1$:

(a) $(h^S_u)'(v) = 0$ if and only if $u \in \text{span}\{f(v), s(v)\}$,

(b) $(h^S_u)'(v) = (h^S_u)''(v) = 0$ if and only if $u = \pm \left(\kappa_g(v)f(v) - s(v)\right)/\sqrt{\kappa_g^2(v) + 1}$.

**Proof.** By using Eq. (2.2), we have

(i) $(h^S_u)'(v) = \langle f(v), u \rangle$,

(ii) $(h^S_u)''(v) = \langle f(v) + \kappa_g(v)s(v), u \rangle$.

The assertion (a) can be found from the formula (i). By using assertion (a), there exist $\lambda, \mu \in \mathbb{R}$ such that $u = \lambda f(v) + \mu s(v)$. From formula (ii), we obtain

$$0 = \langle f(v) + \kappa_g(v)s(v), \lambda f(v) + \mu s(v) \rangle$$

$$= \lambda \langle f(v), f(v) \rangle + \mu \kappa_g(v) < s(v), s(v) >$$

$$= \lambda + \mu \kappa_g(v).$$

Thus we get $u = -\frac{\mu}{\mu \kappa_g(v)}(\kappa_g(v)f(v) - s(v))$. Since $\langle u, u \rangle = 1$, we have

$$\mu = \mp \frac{1}{\sqrt{\kappa_g^2(v) + 1}}.$$
\[ u = \pm \frac{1}{\sqrt{\kappa_g^2(v)} + 1} \left( \kappa_g(v)f(v) - s(v) \right). \]

Conversely, substituting \( u \) into (i) and (ii), respectively, we find \((h_u^v)'(v) = (h_u^v)'(v) = 0\). This completes the proof.

4. Spherical invariants of unit speed time-like curves on \( S_1^2 \)

We study the geometric properties of de Sitter evolutes of the unit speed time-like curves on \( S_1^2 \) [7]. For any \( r \in \mathbb{R} \) and \( u_0 \in S_1^2 \), we denote \( PS^i(u_0, r) = \{ u \in S_1^2 : < u, u_0 > = r \} \). We call \( PS^i(u_0, r) \) as a pseudo-circle whose center is \( u_0 \) on \( S_1^2 \).

Then we have the following proposition:

**Proposition 4.1.** Let \( f : I \to S_1^2 \) be a unit speed time-like curve. Then \( \kappa_g'(v) = 0 \) if and only if \( u_0 = \pm \left( \kappa_g(v)f(v) - s(v) \right)/\sqrt{\kappa_g^2(v) + 1} \) are constant vectors. Under this condition, \( f \) is a part of a pseudo-circle whose center is \( u_0 \) on \( S_1^2 \).

**Proof.** Let we denote
\[ P_\pm(v) = \pm u_0 = \pm \frac{1}{\sqrt{\kappa_g^2(v)} + 1} \left( \kappa_g(v)f(v) - s(v) \right). \]

Taking the derivative of this equation with respect to \( v \), we have
\[ P'_\pm(v) = \pm \kappa'_g(v) \frac{f(v) + \kappa_g(v)s(v)}{\left( \kappa_g^2(v) + 1 \right)^{1/2}}. \]

Thus \( P'_\pm(v) = 0 \) if and only if \( \kappa'_g(v) = 0 \).

Under this condition, we put \( r = \pm \kappa_g(v)/\sqrt{\kappa_g^2(v) + 1} \) and \( u_0 = \pm \left( \kappa_g(v)f(v) - s(v) \right)/\sqrt{\kappa_g^2(v) + 1} \).

Then \( f(v) \) is a part of the pseudo-circle \( PS^i(u_0, r) \). This completes the proof.

Let \( f : I \to S_1^2 \) be a unit time-like curve. For any \( v_0 \in I \), we consider the pseudo-circle \( PS^i(u_0, r_0^v) \), where \( u_0 = d_f(v_0) \) and \( r_0 = \kappa_g(v_0)/\sqrt{\kappa_g^2(v_0) + 1} \).
Thus we have the following proposition:

**Proposition 4.2.** Under the above notations, $f$ and $PS^1(u_0, r_0)$ have at least a 3-point contact at $f(v_0)$.

**Proof.** Proposition 3.1 (b) says that $f$ and $PS^1(u_0, r_0)$ have at least a 3-point contact at $f(v_0)$. Thus, the proof is completed.

**Remark 4.3.** We call $PS^1(u_0, r_0)$ in proposition 4.2 as the pseudo-circle of geodesic curvature and its center $u_0$ is called as the center of geodesic curvature. As a result, the de Sitter evolute is the locus of the center of geodesic curvature.

5. **Time-like constant slope surfaces lying in the space-like cone and space-like Bertrand curves**

In this section, we give the relations among space-like Bertrand curves, helices, de Sitter Darboux images and time-like constant slope surfaces lying in the space-like cone in $\mathbb{R}^3_1$.

Now we can express the following lemma:

**Lemma 5.1.** Let $f : I \to \mathbb{S}^2_1$ be a unit speed time-like curve. Then

$$\tilde{y}(v) = a \int_a^v f(t) dt + a \tan \xi \int_a^v f(t) \wedge f'(t) dt$$

is a space-like Bertrand curve, where $a$ and $\xi = \xi(u) = \cot \theta \ln u$ are constant numbers, and $\theta$ is a constant satisfying $\theta \in (0, \pi/2]$. Moreover, all space-like Bertrand curves can be constructed by using this method.
Proof. We now calculate the curvature and the torsion of $\tilde{\gamma}(v)$. Taking the derivatives of Eq. (5.1) three times with respect to $v$, we have
\begin{align*}
\tilde{\gamma}'(v) &= a\left(f(v) + \tan \xi s(v)\right), \\
\tilde{\gamma}''(v) &= a\left(1 + \tan \xi \kappa_\parallel(v)\right)t(v), \\
\tilde{\gamma}'''(v) &= a\left(\left(1 + \tan \xi \kappa_\parallel(v)\right)f(v) + \tan \xi \kappa'_\parallel(v)t(v) + \left(\kappa_\parallel(v) + \tan \xi \kappa^2_\parallel(v)\right)s(v)\right).
\end{align*}
Therefore, by using Eq. (2.1), we obtain $\kappa(v)$ and $\tau(v)$ as follows:
\begin{align*}
\kappa(v) &= \varepsilon \frac{\cos^2 \xi \left(1 + \tan \xi \kappa_\parallel(v)\right)}{a} \\
\tau(v) &= \frac{\cos^2 \xi \left(\kappa_\parallel(v) - \tan \xi \xi\right)}{a},
\end{align*}
where $\varepsilon = \pm 1$. It follows from these formulae that $a(\varepsilon \kappa(v) - \tan \xi \tau(v)) = 1$, thus $\tilde{\gamma}(v)$ is a Bertrand curve. Also, since $<\tilde{\gamma}'(v), \tilde{\gamma}''(v) >= a^2 / \cos^2 \xi > 0$, $\tilde{\gamma}(v)$ is a space-like Bertrand curve.

Conversely, let $\tilde{\gamma}(s)$ be a space-like Bertrand curve. Thus, there exist real constants $A$, $B$ different from zero such that $A\kappa(s) + B\tau(s) = 1$. Here we put $A = a$ and $B = -a \tan \xi$. Assume that $a > 0$ and choose $\varepsilon = \pm 1$ with $\varepsilon \cos \xi / a > 0$.

Let us consider the Frenet frame $\{T(s), N(s), B(s)\}$ for the space-like Bertrand curve $\tilde{\gamma}(s)$. In this trihedron $T(s)$ and $B(s)$ are space-like vectors, $N(s)$ is a time-like vector. For these vectors, we have
\begin{align*}
T(s) \wedge N(s) = B(s) \quad \text{and} \quad B(s) \wedge N(s) = -T(s).
\end{align*}

Now we define a time-like curve on $S^1_1$ as
\begin{align*}
f(s) = \varepsilon \left(\cos \xi T(s) - \sin \xi B(s)\right).
\end{align*}
Thus we have
\begin{align*}
f'(s) &= \varepsilon \cos \xi \left(\kappa(s) - \tan \xi \tau(s)\right)N(s) = \frac{\varepsilon}{a} \cos \xi N(s). \\
\end{align*}

Let $v$ be the arc-length parameter of $f$, then we have $dv / ds = \varepsilon \cos \xi / a$. Moreover, we get
\begin{align*}
av(s) \frac{dv}{ds} &= \cos \xi \left(\cos \xi T(s) - \sin \xi B(s)\right) \tag{5.3}
\end{align*}
and
\begin{align*}
a \tan \xi f(s) \wedge \frac{df}{dv} \frac{dv}{ds} &= \sin \xi \left(\cos \xi B(s) + \sin \xi T(s)\right). \tag{5.4}
\end{align*}
By using Eq. (5.3) and (5.4), we obtain
\[
a \int_0^y f(t)dt + a \tan \xi \int_0^y f(t) \wedge f'(t)dt = \int_{\xi_0}^s \cos \xi \left( \cos \xi T(t) - \sin \xi B(t) \right) dt \\
+ \int_{\xi_0}^s \sin \xi \left( \cos \xi B(t) + \sin \xi T(t) \right) dt \\
= \int_{\xi_0}^s T(t)dt = \gamma(s).
\]
This completes the proof.

As a consequence of this lemma, we can give a relation between space-like Bertrand curves and helices.

**Corollary 5.2.** The unit speed time-like curve \( f \) on \( S^2_1 \) is a part of a pseudo-circle if and only if the corresponding space-like Bertrand curve is a helix.

**Proof.** Taking the derivative of Eq. (5.2) with respect to \( v \), we have
\[
\kappa'(v) = \frac{\sin 2 \xi \kappa'_g(v)}{2a} \quad \text{and} \quad \tau'(v) = \frac{\cos^2 \xi \kappa'_g(v)}{a}.
\]
From proposition 4.1, the unit speed time-like curve \( f \) on \( S^2_1 \) is a part of a pseudo-circle if and only if \( \kappa'_g(v) \equiv 0 \). This condition is equivalent to the condition that both \( \kappa(v) \) and \( \tau(v) \) are constants. The proof is completed.

Then we have the following proposition:

**Proposition 5.3.** Let \( f : I \to S^2_1 \) be a unit speed time-like curve and \( \gamma : I \to \mathbb{R}^3_1 \) be a space-like Bertrand curve corresponding to \( f \). Then the de Sitter Darboux image of \( \gamma \) is equal to the de Sitter evolute of \( f \).

**Proof.** From Eq. (5.2), we have
\[
\kappa(v) = \varepsilon \frac{\cos^2 \xi \left( 1 + \tan \xi \kappa'_g(v) \right)}{a} \quad \text{and} \quad \tau(v) = \varepsilon \frac{\cos \xi \left( \kappa'_g(v) - \tan \xi \right)}{a}.
\]
For the space-like curve \( \gamma \), we obtain
\[
T(v) = a \left( f(v) + \tan \xi s(v) \right) \frac{dv}{ds} \quad \text{and} \quad N(v) = \varepsilon t(v).
\]
Then we get
\[ B(v) = T(v) \wedge N(v) = e a \frac{dv}{ds} \left( s(v) - \tan \xi f(v) \right). \]

We can easily show that
\[ D(v) = \tau(v)T(v) - \kappa(v)B(v) = \frac{dv}{ds} \left( \kappa_s(v) f(v) - s(v) \right). \]

Therefore we have \( C(v) = D(v)\|D(v)\| = d_f(v). \) This completes the proof.

We have the following theorem:

**Theorem 5.4.** Let \( f : I \rightarrow \mathbb{S}^2_1 \) be a unit speed time-like curve and \( \tilde{\gamma} : I \rightarrow \mathbb{R}^3_1 \) be a space-like Bertrand curve corresponding to \( f \). Then \( \tilde{\gamma}'(v) \) lies on the time-like constant slope surface \( x(u,v) \) lying in the space-like cone.

**Proof.** Taking the derivative of Eq. (5.1) with respect to \( v \), we obtain
\[ \tilde{\gamma}'(v) = af(v) + a \tan \xi f(v) \wedge f'(v). \]

In this equation, we can take \( a \) as \( a = u \sin \theta \cos \xi \) and so \( a \tan \xi = u \sin \theta \sin \xi \), where \( u, \ \theta \) are constants. Thus by Eq. (1.1), \( \tilde{\gamma}'(v) \) is \( v \)-parameter curve of time-like constant slope surface \( x(u,v) \) lying in the space-like cone and \( \tilde{\gamma}'(v) \) lies on it. This completes the proof.

We now state the relation between time-like constant slope space-like surfaces lying in the space-like cone and space-like Bertrand curves.

**Theorem 5.5.** Let \( x : S \rightarrow \mathbb{R}^3_1 \) be a time-like constant slope surface immersed in \( \mathbb{R}^3_1 \) and \( x \) lies in the space-like cone. If \( x(v) \) is \( v \)-parameter curve of time-like constant slope surface \( x(u,v) \) lying in the space-like cone, then \( \int_0^1 x(v)dv \) is a space-like Bertrand curve.

**Proof.** From Eq. (1.1), we have
\[ x(v) = u \sin \theta \cos \xi f(v) + u \sin \theta \sin \xi f(v) \wedge f'(v) \]
for \( u \) = constant, where \( \xi = \xi(u) = \cot \theta \ln u \). By integrating \( x(v) \), we have the equation as
\[ \int_a^v x(v)dv = u \sin \theta \cos \xi \int_a^v f(v)dv + u \sin \theta \sin \xi \int_a^v f(v) \wedge f'(v)dv. \]

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Since the coefficients of \( f(v) \) and \( f(v) \wedge f'(v) \) are constants, here we can take \( u \sin \theta \cos \xi \) as \( u \sin \theta \cos \xi = a \) and so \( u \sin \theta \sin \xi = a \tan \xi \). Thus we obtain
\[
\int_0^v x(v)dv = a \int_0^v f(v)dv + a \tan \xi \int_0^v f(v) \wedge f'(v)dv.
\]
From lemma 5.1, \( \int_0^v x(v)dv \) is a space-like Bertrand curve. This completes the proof.

Now we give an example of time-like constant slope surfaces and space-like Bertrand curves and draw the corresponding pictures via Mathematica.

**Example 5.6.** By using Eq. (1.1), we may choose the unit speed time-like curve as \( f(v) = (\cosh v, 0, \sinh v) \) on \( S^2_1 \). Then we have \( f(v) \wedge f'(v) = (0, -1, 0) \). Thus, the time-like constant slope surface lying in the space-like cone is
\[
x(u, v) = u \sin \theta \left( \cos(\cot \theta \ln u) \cosh v, -\sin(\cot \theta \ln u), \cos(\cot \theta \ln u) \sinh v \right).
\]
For \( \theta = \pi / 4 \), the picture of this surface is given in Fig. 1. Thus, for \( u = e \), the space-like Bertrand curve is
\[
\int_0^v x(v)dv = \frac{\sqrt{2}}{2} e \left( \cos(1) \sinh v, -\sin(1)v, \cos(1)(\cosh v - 1) \right).
\]
Since the time-like curve \( f(v) \) is a part of a pseudo-circle, from corollary 5.2, this space-like Bertrand curve is a helix. The picture of this curve is given in Fig. 2.
Fig. 1. Time-like constant slope surface lying in the space-like cone, \( f(v) = (\cosh v, 0, \sinh v) \),
\[ \theta = \frac{\pi}{4} \]

Fig. 2. Space-like Bertrand curve, \( \theta = \frac{\pi}{4}, u = e \)
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