Modified Gauge Invariance Einstein Maxwell Gravity and Stability of Spherical Stars with Magnetic Monopoles

Hossein Ghaffarnejad\textsuperscript{1} and Leyla Naderi\textsuperscript{2}

\textit{Faculty of Physics, Semnan University, P.C. 35131-19111, Semnan, Iran}

Abstract

In this paper we consider an extended Einstein-Maxwell gravity model containing a gauge invariance property and studied stability of a spherically symmetric static star made from a Magnetic monopole. Extension of the theory is assumed to be directional coupling between spatial electric and magnetic fields with the Ricci tensor as gravity side of the model. In particle physics, a magnetic monopole is a hypothetical elementary particle which is proposed by Paul Dirac at a first time from quantum theory of matter. Dynamical field equations in this model become nonlinear second order differential equations and so we had to solve the equations through dynamical systems approach. In the latter approach, the first step the eigenvalues of the Jacobi matrix are determined and then solve alternative linear differential equations near the critical points of the system. At last step to study stability nature of the obtained solutions near the critical point we should determine sign of the eigenvalues.

1 Introduction

High energy astronomical compact objects in cosmic scales are considered as excellent laboratories for investigating astrophysical phenomena, and their relationship with nuclear and elementary particles physics has opened a new approach to modern astrophysics. High energy astronomical compact objects include neutron stars), quark stars, boson stars, white dwarfs, and black holes can be formed when a massive star runs out of its fuel and therefore cannot remain stable against its own gravity and collapses \cite{1,2}. Depending on the mass of the star, the collapse changes the star’s configuration and initiates a new structure. In general a star is stable when the pressure force from the gas atoms is equal to the its gravitational force and otherwise will be

\textsuperscript{1}E-mail address: hghafarnejad@semnan.ac.ir
\textsuperscript{2}E-mail address: l.naderi@semnan.ac.ir
unstable. The stability of the star can be investigated in the presence of both electric and magnetic fields. Solving the Einstein-Maxwell field equations for compact stars with the charged anisotropic fluid model gives more stable solutions than for neutral stars. The presence of charges create repulsive forces against the gravitational force, and this factor causes to be denser stable stars and to have higher maximum mass and larger redshift. In the \( f(R,T) \) gravitational model, changes in the background gravity increase the radius of the star, which decreases the surface electric field, so it increases the stability of charged stars. In the core of neutron stars, there is a possibility of hadron-quark phase transition. Charged quarks can create more stable quark stars than neutron nuclei. The charge-mass relation for a charged star is similar to the extreme Reissner-Nordstrom black hole. Compact stars such as neutron stars in addition to creating quark core stars allow the emergence of exotic physics phenomena. In theories beyond the standard model, the effect of dark matter on the internal structure of the neutron stars suggests that the neutron stars is mixed with dark matter in the core and it is surrounded by a shell. This feature affects the stellar mass-radius relation such that dark matter effects are responsible for reducing the stellar mass, while the main effect of the shell is to increase the stellar radius. In a compact objects mixed with normal matter and dark matter, as the central pressure of dark matter increases, the neutron stars becomes unstable and the white dwarfs will have unusual masses and radii. Therefore, the resulting object will have unusually small mass and radii. When enough non-destructive dark matter accumulates on a neutron stars, it creates a central degenerate star. If the mass of the dark matter in the star reaches the Chandrasekhar mass limitation of the star, the dark matter leads to collapse the mixed neutron stars. The stability can also be investigate for compact stars that are affected by strong magnetic fields which can affect the process of stellar evolution. Surface Magnetic fields observed in stars can be divided into two categories: the fossil and dynamo hypothesis. The fossil hypothesis is used to explain magnetism in massive stars, and the dynamo hypothesis, which is used for the inner space of stars, shows the effects of a strong Magnetic field on the propagation of gravitational waves. Evolution of the different astrological objects, consisting of stability and instabilities of them, has attracted many attentions and explorations. The question of stability in stars has provided via many models in which the main question is whether a small perturbation can rapidly decay in comparison to the model’s parameters or not. Instability in stars includes dynamical, vibrational and secular.
instabilities [9]. Stability and equilibrium of stars has been investigated in extended theories of gravity also [10]. Hydrodynamical simulations in full general relativity have been applied to investigate the dynamical stability of differentially rotating neutron stars [11]. Dynamical instability of a star undergoing a dissipative collapse, considering the role of pressure anisotropy, has been explored [12]. However, in some stars, magnetic field has a main role in evolution and stability of them. For instance sunspots are the largest concentration of complex magnetic flux [13]. Energy source of emission from magnetars is magnetic field [14],[15]. The origin and dynamics of magnetic fields in the surface of massive stars have been studied [16]. Extensive study of the evolution of magnetic field has been performed in differentially rotating radiative zones of intermediate-mass stars [17]. In Ref. [18], stability of the gaseous stars has been investigated, applying a fundamental hypothesis that all gaseous stars generate an internal magnetic field which prevents them from collapsing gravitationally and exploding due to radiation. The theory of magnetohydrodynamics (MHD) is applied to study the stability of the stars in the presence of a magnetic field. In this theory the motion of an electrically conducting fluid is explored in the presence of a magnetic field [19]. In Ref [20], stability properties of neutron stars and magnetic field influence in their dynamical behavior have been studied applying MHD equations in presence of general theory of relativity model of gravity. Neutron stars are considered in presence of magnetic fields providing effective pressure for greater mass [21]. Accretion on to magnetized stars with tilted rotational and magnetic axes about rotational axis of the disc has been investigated, applying 3D MHD simulations [22] where the disc accreting stars have strong magnetic fields. Due to the importance of the hypothesis of magnetic monopoles, which we considered in this article as an effective source in the stability of a spherical star, we conclude the introduction section with a brief statement about the history of magnetic monopoles.

In particle physics, a magnetic monopole is a hypothetical elementary particle that is an isolated magnet with only one magnetic pole [23],[24]. The being of the magnetic monopoles comes in fact from notably the grand unified and superstring theories, which predict their existence [25],[26]. Magnetism in bar magnets and electromagnets is not caused by magnetic monopoles, and indeed, there is no known experimental or observational evidence that magnetic monopoles exist. Some condensed matter systems contain effective (non-isolated) magnetic monopole quasi-particles [27] or contain phenomena that are mathematically analogous to magnetic monopoles [28]. Gauss’s law
for magnetism, one of Maxwell’s equations, is the mathematical statement that magnetic monopoles do not exist. Nevertheless, Pierre Curie pointed out in 1894 [29] that magnetic monopoles could conceivably exist, despite not having been seen so far. From quantum theory of matter, Paul Dirac [30] showed that if any magnetic monopoles exist in the universe, then all electric charge in the universe must be quantized (Dirac quantization condition) [31]. Since Dirac’s paper, several systematic monopole searches have been performed. Experiments in 1975 [32] and 1982 [33] produced candidate events that were initially interpreted [32] as monopoles, but are now regarded as inconclusive. Therefore, it remains an open question whether monopoles exist. Further advances in theoretical particle physics, particularly developments in grand unified theories and quantum gravity, have led to more compelling arguments that monopoles do exist. Joseph Polchinski, provided an argument from string theory, confirming the existence of magnetic unipolarity, which has not yet been observed by experimental physics [34]. These theories are not necessarily inconsistent with the experimental evidence. In some theoretical models, magnetic monopoles are unlikely to be observed, because they are too massive to create in particle accelerators, and also too rare in the Universe to enter a particle detector with much probability [34]. Layout of this paper is as follows.

In section two we present our proposed modified Einstein-Maxwell gravity model where the modified term is a directional coupling between electric and magnetic spatial vector fields and the Ricci tensor (the gravity side). As an magnetic source to produce a spherically symmetric static metric of a stellar object we consider a magnetic monopole charge field and generate Euler-Lagrange equations of the metric fields. We see that the magnetic monopole generates a radial magnetostatic field which same as the field of an electric monopole the intensity of the fields change by inverse square of radial distance from the monopoles. We see that the Euler-Lagrange equations of the metric fields are nonlinear second order differential equations and so we must use dynamical system approach to obtain metric field solutions near the critical points in phase space. This is done in the section of third. Also we plotted arrow diagrams to interpret that which of the obtained critical points describe stable nature for the obtained metric field and which of them are not. The last part is devoted to conclusions and prospects for the development of the work.
2 Magnetostatics stellar objects

As an gauge invariance electromagnetic field interacting with gravity let us we propose the following form of generalized Einstein Maxwell gravity.

\[ I = - \int dx^4 \sqrt{g} \left[ R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha}{2} F_{\mu\lambda} R^{\lambda}_{\eta\mu} F_{\eta\mu} \right], \tag{2.1} \]

where \( g \) is absolute value of determinant of the metric field and dimensions in the coupling constant \( \alpha \) is square of length and we write the action in the geometric units \( c = G = 1 \). \( R_{\mu\nu}(R) \) is Ricci tensor (scalar) and anti-symmetric electromagnetic tensor field \( F_{\mu\nu} \) is defined versus the partial derivatives of the four vector electromagnetic potential \( A_\mu \) (in the torsion free Riemannian geometry) such that

\[ F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.2} \]

In the differential geometric formulation of the electromagnetic field, the above antisymmetric Faraday tensor can be considered as the Faraday 2-form \( F \). In this view, one of Maxwell’s two equations is \( dF = 0 \), where \( d \) is the exterior derivative operator. This equation is completely coordinate and metric independent and says that the electro-magnetic flux through a closed two dimensional surface in space time is topological, more precisely, depends only on its homology class (a generalization of the integral form of Gauss law and Maxwell-Faraday equation as the homology class in Minkowski space is automatically 0). By the Poincare lemma, this equation implies, (at least locally) that there exists a 1-form \( A \) satisfying \( F = dA \). The other Maxwell equation is \( d^* F = J \). In this context, \( J \) is the current 3-form (or even more precise, twisted three form), the asterisk * denotes the Hodge star operator, and \( d \) is the exterior derivative operator. The dependence of Maxwell’s equation on the metric of spacetime lies in the Hodge star operator * on two forms, which is conformally invariant. Written this way, Maxwell’s equation is the same in any space time, manifestly coordinate invariant, and convenient to use (even in Minkowski space or Euclidean space and time especially with curvilinear coordinates) [35]. In this view we have

\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = B + E \wedge dx^0 = dA + E \wedge dx^0 \tag{2.3} \]

in which

\[ E = E_i dx^i \tag{2.4} \]
is 1-form electric field and
\[ B = \frac{1}{2} \epsilon_{ijk} B^i dx^j \wedge dx^k \] (2.5)
is 2-form magnetic field. In fact they are spatial vector fields and \( i, j = 1, 2, 3 \) correspond to spatial coordinates while \( x^0 \) denote to time coordinates in the curved background spacetime. In the above equation \( \epsilon_{ijk} \) is third rank totally antisymmetric Levi Civita tensor density. Its numeric value is +1(−1) for \( \{i, j, k\} = \{1, 2, 3\} \) and for any even (odd) permutations while it takes zero value for any two indices are equal. We can rewrite also the Maxwell tensor field \( F_{\mu\nu} \) with the following form [12].
\[ F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu + \epsilon_{\mu\nu\eta\lambda} B^\eta n^\lambda \] (2.6)
where \( n_\mu \) is a unit time-like vector field and it is normal to the spatial 3D hypersurface with constant time coordinate \( x^0 = \text{const} \). It can be written as \( n^\mu = -\nabla_\mu x^0 / ||\nabla_\mu x^0|| \). Consequently the electric and magnetic fields components \( \{E_i, B_i\} \) measured by a normal observer aligned to \( n_\mu \) and so they are absolutely spatial vector fields \( E_\mu n^\mu = 0 = B_\mu n^\mu \). From ADM formalism in the 1 + 3 decomposition of the curved background 4D space time metric the whole of space time can be foliated into hypersurfaces with constant time coordinates with spatial 3-metric \( h_{ij} = g_{ij} \) on the space-like hypersurfaces. In other words the general form of line element \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) reads
\[ ds^2 = -\alpha^2 dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \] (2.7)
in which \( \alpha \) is shift function and \( \beta^i \) is shift vector. In the definition (2.6) \( \epsilon_{\mu\nu\eta\lambda} \) is fourth rank totally antisymmetric Levi Civita tensor density. Its numeric value is +1(−1) for \( \{\mu, \nu, \eta, \lambda\} = \{0, 1, 2, 3\} \) and for any even (odd) permutations while it takes zero value for any two indices are equal. For time independent static curved spacetimes the line element (2.7) takes a simpler form because \( \beta^i = 0 \) and \( \alpha \) and \( h_{ij} \) take on spatial coordinates \( x^i \) only. In the latter case we can apply a suitable coordinates transformation to remove all non-diagonal components of \( h_{ij} \) such that
\[ ds^2 = -(\alpha dt)^2 + (\gamma_i dq^i)^2 \] (2.8)
in which \( d\ell_i = \gamma_i dq_i \) has length dimension in the used curvilinear coordinates \( dq^i \) in the 3D subspace [36]. For the line element (2.8) the identity (2.3) reads
\[ F_{it} = \sqrt{\alpha} \gamma_i E_i, \quad F_{ijk} = \epsilon_{ijk} B_i \gamma_j \gamma_k \] (2.9)
in which repeated indexes for $\epsilon_{ijk}$ do not follow the Einstein summation rule and just follows permutation cycles. In this paper we consider a spherically symmetric static spacetime whose line element in general form is given by

$$ds^2 = -X(r)dt^2 + Y(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$  \hspace{1cm} (2.10)

for which the equations (2.9) reads

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\sqrt{XY}E_r & -\sqrt{X}r E_\theta & -\sqrt{X}r \sin \theta E_\varphi \\ \sqrt{XY}E_r & 0 & \sqrt{Y}r B_\varphi & -\sqrt{Y}r \sin \theta B_\theta \\ \sqrt{X}r E_\theta & -\sqrt{Y}r B_\varphi & 0 & r \sin \theta B_r \\ \sqrt{X}r \sin \theta E_\varphi & \sqrt{Y}r \sin \theta B_\theta & -r^2 \sin \theta B_r & 0 \end{pmatrix}.$$  \hspace{1cm} (2.11)

It is easy to check that the only magnetic field having property of spherical symmetry is corresponded just to the magnetic monopole with assumed charge $q_m$, which is defined by the magnetic potential

$$A_\varphi(\theta) = -q_m \cos \theta.$$  \hspace{1cm} (2.12)

By regarding (2.2) one can show that the corresponding non-vanishing component of the Maxwell tensor field for (2.12) is

$$F_{\theta\varphi} = \partial_\theta A_\varphi = q_m \sin \theta$$  \hspace{1cm} (2.13)

which by substituting into (2.11) we obtain that non-vanishing component of the magnetic field is only radial such that

$$B_r(r) = \frac{q_m}{r^2}.$$  \hspace{1cm} (2.14)

This is similar to electric field of an electric monopole $E_r(r) = \frac{q_m}{r^2}$. By substituting (2.10) and (2.13) we integrate on the 2-sphere $\sin \theta d\theta d\varphi$ and also integrate by parts for $\frac{d^2X}{d\tau^2}$, which is come from Ricci scalar and then we remove the corresponding divergence-less term in the action functional (2.1) and at last we obtain an alternative or effective Lagrangian for this magneto-statics stellar object such that

$$L_m = 4\pi \left\{ \frac{2\dot{X}}{\sqrt{XY}} - \frac{\ddot{X}}{2\sqrt{XY}} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \right) + \frac{1}{2} \sqrt{\frac{X}{Y}} \left[ \frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + 4(Y - 1) - \frac{q_m^2 e^{-2\tau}}{D^2} - \frac{\alpha q_m^2 e^{-4\tau}}{D^4} \left( \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} + 2(1 - Y) \right) \right] \right\}$$  \hspace{1cm} (2.15)
where \( \dot{\cdot} \) means the derivative of the fields with respect to a logarithmic radial coordinate such that
\[
\dot{\cdot} = \frac{d}{d\tau} = \frac{d}{d\ln (r/D)} = r \frac{d}{dr}.
\] (2.16)

In the above definition we bring \( D \) as a particular length scale to make dimensionless the argument of the logarithm function. We should point that other cases for non-vanishing spatial component of the four-vector potential which may be generate spherically symmetric static spacetime are \( A_\theta (r) \neq 0 \) and \( A_\phi (r) \neq 0 \). We will consider these situations as our next work to investigate that whether or not they can give us some spherically symmetric non-singular (singular) metric field for a star(black hole)? However we continue our work to obtain the Euler-Lagrange equations for the metric fields \( X(r) \) and \( Y(r) \) such that
\[
2 \ddot{X} + \frac{(Y - 1) \dot{Y}}{2} Y = \frac{3}{4} \left( \frac{\dot{X}}{X} \right)^2 + \dot{Y} + \frac{1}{4} \left( \frac{\dot{Y}}{Y} \right)^2 + \frac{\dot{X} \dot{Y}}{XY} - 1 + Y \quad \text{(2.17)}
\]
\[
- \frac{q_m^2 e^{-2\tau}}{8D^2} - \frac{2\alpha q_m^2 e^{-4\tau}}{D^4} = 0
\]
is for \( X(r) \) and
\[
- \frac{\dot{X}}{X} + 1 + Y + \frac{q_m^2 e^{-2\tau}}{8D^2} + \frac{3\alpha q_m^2 e^{-4\tau}}{D^4} = 0 \quad \text{(2.18)}
\]
is for \( Y(r) \) respectively. We are now in position to solve these equations to obtain metric solutions which we do in the subsequent section.

3 Metric field solutions

By mixing the equations (2.17) and (2.18) one can obtain
\[
Y(Y - 1)\ddot{Y} + 2Y(2 + 3Y)\dot{Y} + \frac{Y^2}{2} + \frac{Y^2}{4} + 7Y^3 + \frac{5}{2}Y^4
\]
\[
+ \frac{\epsilon e^{-2\tau} Y^2}{4} + \epsilon^2 e^{-4\tau} \left[ \left( 15\lambda - \frac{6661}{128} \right) Y^2 + 6Y \dot{Y} + 15\lambda Y^3 \right]
\]
\[
+ \frac{15\epsilon^3 \lambda e^{-6\tau} Y^2}{8} + \frac{45\epsilon^4 \lambda^2 \alpha^2 e^{-8\tau} Y^2}{2} = 0
\]
in which we defined

\[ \epsilon = \frac{q_m^2}{D^2}, \quad \lambda = \frac{\alpha}{q_m^2}. \]  

(3.2)

To solve the nonlinear second order differential equation (3.1) via dynamical system approach we make it as two set of first order nonlinear differential equations such that

\[ \dot{Y} = W, \]  

(3.3)

and

\[ \dot{W} = -2 \left( \frac{2 + 3Y}{Y - 1} \right) W - \frac{W^2}{2Y(Y - 1)} - \frac{Y}{4(Y - 1)} - \frac{7Y^2}{(Y - 1)} \]  

(3.4)

\[ -\frac{5Y^3}{2(Y - 1)} - \frac{\epsilon We^{-2\tau}}{4(Y - 1)} - \frac{\epsilon^2 e^{-4\tau}}{8(Y - 1)} \left\{ 6W + \left( 15\lambda - \frac{6661}{128} \right) Y + 15\lambda Y^2 \right. \]

\[ + \left. \frac{15\lambda eYe^{-2\tau}}{8} + 45(\epsilon)^2 Ye^{-4\tau} \right\}. \]

In the first step, we should obtain critical points in the phase space \((Y, W)\) by solving the equations \(\dot{Y} = 0, \dot{W} = 0\) for which we obtain

\[ W_c = 0, \quad Y_c = \{0, 1, h \pm \sqrt{k/5}\} \]  

(3.5)

in which we set \(\tau_c = 0\) which reads \(r_c = D\) and also we defined

\[ h = -7/5 - 3\lambda^2 \]  

(3.6)

and

\[ k = 93/2 + \epsilon^2(60\lambda + 33305/64) - 150\lambda \epsilon^3/8 - 450\lambda^2 \epsilon^4 \]  

(3.7)

It is easy to infer that the critical values \(Y_c = 0\) and \(Y_c = 1\) are not physical because they make singular right side of the equation \(\dot{W}\) given by (3.4). Thus the third kind critical point \(Y_c = h \pm \sqrt{k/5}\) is used to calculate the metric solutions about the critical point and also we investigate its stability nature as follows. We should obtain Jacobi matrix of the dynamical equations (3.3) and (3.4) by calculating \(J_{ij} = \frac{\partial P_i}{\partial Q_j}\) at the critical point in which \(P_i \equiv \{Y, \dot{W}\}\) and \(Q_j = \{Y, W\}\) such that

\[ J_{ij} = \begin{pmatrix} 0 & 1 \\ J_{21} & J_{22} \end{pmatrix} \]  

(3.8)
where we defined

\[ J_{21} = \left. \frac{\partial \dot{W}}{\partial Y} \right|_{W_c=0,Y_c=h\pm\sqrt{k/5}} = -\frac{(20Y_c^3 - 2Y_c^2 - 56Y_c - 1)}{4(Y_c - 1)^2} + \frac{\epsilon^2}{128} \times (3.9) \]

\[ \left( -\frac{1920\epsilon\lambda Y_c^2 + 5760\epsilon^2\lambda + 3840\epsilon\lambda Y_c - 6661\epsilon + 240\epsilon^2\lambda + 1920\epsilon\lambda}{\epsilon(Y_c - 1)^2} \right) \]

and

\[ J_{22} = \left. \frac{\partial \dot{W}}{\partial W} \right|_{W_c=0,Y_c=h\pm\sqrt{k/5}} = -2 \left( \frac{2 + 3Y_c}{Y_c - 1} \right) - \frac{\epsilon}{4(Y_c - 1)} - \frac{6\epsilon^2}{Y_c - 1}. \quad (3.10) \]

To investigate stability of the metric solutions of the magneto static stellar object we should solve secular equation of the above Jacobi matrix, i.e. \( \det(J_{ij} - \delta_{ij}E) = 0 \) to determine sign of the eigenvalues \( E \) such that

\[ E_\pm = \frac{J_{22} \pm \sqrt{\Delta}}{2}, \quad \Delta = J_{22}^2 + 4J_{21}. \quad (3.11) \]

These eigenvalues have parametric forms and so to determine their sign we must be fix numeric values of the parameters \( \epsilon, \lambda \). Near the above acceptable critical points one can obtain metric solutions by solving

\[ \frac{d}{d\tau} \begin{pmatrix} Y \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} Y \\ W \end{pmatrix} \quad (3.12) \]

which by mixing two components of the above systems of equations we obtain a second order linear differential equation such that

\[ \ddot{Y} - J_{22} \dot{Y} - J_{21} Y = 0 \quad (3.13) \]

with the following general solution

\[ Y(\tau) = T_+ e^{E_+\tau} + T_- e^{E_-\tau} \quad (3.14) \]

where \( E_\pm \) are eigenvalues given by (3.11). The constant parameters \( T_\pm \) in the above solution should be determined by initial conditions. It is appropriate to choose initial conditions as \( W_c = W(0) = \dot{Y}(0) = 0 \) and \( Y(0) = Y_c = h \pm \sqrt{k/5} \) for which the above general solution reads

\[ Y(\tau) = \left( \frac{Y_c}{E_- - E_+} \right) (E_- e^{E_+\tau} - E_+ e^{E_-\tau}). \quad (3.15) \]
By substituting this particular solution into the equation $2.18$ and after by integrating we obtain

\[ \ln \left( \frac{X}{X_c} \right) = \tau + \left( \frac{Y_c}{E_- - E_+} \right) \left[ \frac{E_-}{E_+} (e^{E_+ \tau} - 1) - \frac{E_+}{E_-} (e^{E_- \tau} - 1) \right] \]

(3.16)

\[-\frac{\epsilon}{16} (e^{-2\tau} - 1) - \frac{3\lambda \epsilon^2}{4} (e^{-4\tau} - 1)\]

in which

\[ \tau = \ln \left( \frac{r}{D} \right) \]

There is not any real root for apparent horizon equation $g^{rr} = 1/Y(r) = 0$ and event horizon equation $g_{tt}(r) = X(r) = 0$ which means that this metric solutions is regular at all positions of the space time and so it should be describe geometry of a magnetic spherically symmetric static star with magnetic monopole source. Now we interpret which values of the parameter $\alpha$ can control stability nature of the obtained metric solutions for this magnetic monopole star from point of view of dynamical system approach as follows.

In this approach for eigenvalues $E$ of the Jacobi matrix of the dynamical equations in phase space if they take on some real values then for $E < 0(> 0)$ the system is stable(unstable) and for particular choice $E = 0$ the system is degenerate (undetermined). If the eigenvalues take on complex numbers (in this work $\Delta < 0$ in the equation (3.11)) then for stable nature of the solutions real part of the complex eigenvalues should take one some negative values ($J_{22} < 0$ in the equation (3.11)) and if $J_{22} > 0$ then the system will be unstable (see [37] for more discussion).

4 Conclusion

In this paper we considered a modified Einstein-Maxwell gravity where modification is the directional dependence of coupling between the electromagnetic field with Ricci tensor (or geometry). We solved dynamical equations near some critical points in which numeric values of the coupling constant plays important role in stability of the obtained metric solutions. In this model we assumed that just azimuthal component of the vector potential to be nonzero and it depends just to polar angle coordinate of the space time. We assumed that source of this vector potential to be magnetic monopole which generates
a radial magnetic field whose intensity decreases with inverse square of radial distance from the charge position. Arrow diagrams in phase space at constant magnetic charge for different values of the coupling constant, show that the magnetostatic star reaches from spiral stable phase to a saddle (quasi-stable) phase by raising the numeric values for the coupling constant. Also numeric values of the magnetic monopole charge restrict permissible values for the coupling constant. As an extension of this work we like to investigate effects of radial dependent for spatial components of the vector potential of the Maxwell field in our future work and seek stabilization of an magnetic star.

Acknowledgement
This work was supported in part by the Semnan University Grant No. 1678-2021 for Scientific Research

References

[1] J. Kumar and P. Bharti, ‘The classification of interior solutions of anisotropic fluid configurations’, arXiv:2112.12518v2 [gr-qc]

[2] P. Bhar, ‘Charged strange star with Krori Barua potential in f(R,T) gravity admitting Chaplygin equation of state’, Eur. Phys. J. Plus 135, 757 (2020)

[3] B. Dayanandan, S.K. Maurya and Smitha T. T,‘Modeling of charged anisotropic compact stars in general relativity’, Eur. Phys. J. A 53, 141 (2017); arXiv:1611.00320 [gr-qc]

[4] F. Rocha, G. A. Carvalho, D. Deb and M. Malheiro, ‘Study of the charged super-Chandrasekhar limiting mass white dwarfs in the f(R,T) gravity’,Phys. Rev. D 101, 104008 (2020); arXiv:1911.08894 [physics.gen-ph]

[5] J. D. V. Arbail and M. Malheiro, ‘Equilibrium and stability of charged strange quark stars’, Phys. Rev. D 92, 084009 (2015); arXiv:1509.07692 [astro-ph.SR]

[6] J. C. Jimnez and E. S. Fraga, ‘Radial oscillations in neutron stars from QCD’, Phys. Rev. D 104, 014002 (2021); arXiv:2104.13480 [hep-ph]
[7] B. Kain, ‘Fermion-charged-boson stars’, Phys. Rev. D 104, 043001 (2021); arXiv:2108.01404 [gr-qc]

[8] X. Lai, Ch. Xia and R. Xu, ‘Bulk Strong Matter; the Trinity’; ADVANCES IN PHYSICS, X, 8, 1, 2137433 (2023); arXiv: 2210.01501[hep-ph]

[9] R. Kippenhahn, A. Weigret and A. Weiss, ‘Stellar Structure and Evolution’ (Springer-Verlag, Berlin Heidelberg, 2012).

[10] A. Wojnar and H. Velta, ‘Equilibrium and stability of relativistic stars in extended theories of gravity’, Eur. Phys. J. C. 76, 697 (2016).

[11] P. L. Espino, V. Paschalidis, T. W. Baumgarte, and S. L. Shapiro, ‘Dynamical stability of quasitoroidal differentially rotating neutron stars’; Phys. Rev. D 100, 043014 (2019).

[12] R. S. Bogadi, M. Govender and S. Moyo,’Dynamical (in)stability analysis of a radiating star model, cast from an initial static configuration’, Eur. Phys. J. Plus. 135, 170 (2020).

[13] J.H. Thomas and N. O. Weiss; Sunspots: Theory and Observations 139 (Springer Science+Business Media Dordrecht, 1992).

[14] C. Thompson, and R. C. Duncan, ‘The soft gamma repeaters as very strongly magnetized neutron stars I. Radiative mechanism for outbursts’, MNRAS 275, 255 (1995).

[15] C. Thompson C. and R. C. Duncan,’The Soft Gamma Repeaters as Very Strongly Magnetized Neutron Stars. II. Quiescent Neutrino, X Ray, and Alfven Wave Emission’APJ 473 (1996) 322.

[16] J. MacDonald and D. J. Mullan,’Magnetic fields in massive stars: dynamics and origin ‘ MNRAS 348, 702, (2004).

[17] M. Gaurat, L. Jouve, F. Lignieres and T. Gastine, ‘Evolution of a magnetic field in a differentially rotating radiative zone ‘,Astron. Astrophys. 580, A103 (2015).

[18] Fierros Palacios A., ‘The Magnetic Field in the Stability of the Stars. Journal of High Energy Physics, Gravitation and Cosmology ‘,JHEP, Gravitation and Cosmology, 1,88 (2015).
[19] J. H. Thomas and N. O. Weiss, ‘Sunspots and Starspots’ (Cambridge University Press, Cambridge, 2008)

[20] S. L. Liebling, L. Leheur, D. Neilsen and C. Palenzuela, ‘Evolutions of magnetized and rotating neutron stars’, Phys. Rev. D 81, 124023 (2010).

[21] M. Bocquet, S. Bonazzola, E. Gourgoulhon and J. Novak, ‘Rotating neutron star models with a magnetic field’, Astron. Astrophys. 301, 757 (1995); arXiv:gr-qc/9503044

[22] M. M. Romanova, A. V. Koldoba, G. V. Ustyugova, A. A. Blinova, and D. Lai, ‘3D MHD simulations of accretion on to stars with tilted magnetic and rotational axes’, MNRAS 506 (2021) 372.

[23] D. Hooper, ‘Dark Cosmos: In Search of Our Universe’s Missing Mass and Energy’, Harper Collins. ISBN:9780061976865, (2009).

[24] S. Eidelman, K.G. Hayes, K.A. Olive, ... R.Y. Zhu ‘Review of Particle Physics’, (Particle Data Group), Phys. Lett. B 592, 1 (2004) (URL: http://pdg.lbl.gov)

[25] G. X. Wen, E. Witten, ‘Electric and magnetic charges in superstring models’, Nucl. Phys. B, Vol261, 651 (2004).

[26] S. Coleman, The Magnetic Monopole 50 years Later, reprinted in Aspects of Symmetry, DOI: 10.1007/978-1-4613-3655-62

[27] C. Castelnovo, R. Moessner and S. L. Sondhi, ‘Magnetic monopoles in spin ice’, Nature 451, 7174, 42 (2008); arXiv:0710.5515 [cond-mat.str-el]

[28] M. W. Ray, E. Ruokokoski, S. Kandel M. Mottonen and D. S. Hall, ‘Observation of Dirac monopoles in a synthetic magnetic field’, Nature. 505, 7485, 657 (2014); arXiv:1408.3133 [cond-mat.quant-gas]

[29] P. Curie, ‘Sur la possibilite d’existence de la conductibilite magnetique et du magnetisme libre’ [On the possible existence of magnetic conductivity and free magnetism], Seances de la Societe Francaise de Physique (in French). Paris: 76, (1894).

[30] P. Dirac, ‘Quantised Singularities in the Electromagnetic Field’, Proc. Roy. Soc. (London) A 133, 60 (1931).
[31] Lecture notes by Robert Littlejohn, University of California, Berkeley, 2007, 8

[32] P. B. Price, E.K. Shirk, W. Z. Osborne, and L. S. Pinsky, ‘Evidence for Detection of a Moving Magnetic Monopole’, Phys. Rev. Lett. 35 (8), 487 (1975).

[33] B. Cabrera, ‘First Results from a Superconductive Detector for Moving Magnetic Monopoles‘, Phys. Rev. Lett. 48 (20), 1378 (1982).

[34] J. Polchinski, ‘Monopoles, Duality, and String Theory‘, Int. J. Mod. Phys. A. 19 (supp01), 145 (2004); arXiv:hep-th/0304042

[35] J. Baez and J. P. Munian, (‘Gauge Fields, Knots and Gravity‘, World Scientific publishing Co. Pte. Lid, 1994)

[36] G. Arfken, ‘Mathematical methos for Physicists‘, Academic press Inc, (1985).

[37] H. Ghaffarnejad, E. Yaraie, ‘Dynamical system approach to scalar-vector-tensor cosmology‘, Gen Relativ Gravit 49, 49 (2017); arXiv:1604.06269 [physics.gen-ph]
Figure 1: Permissible values for coupling constant $\alpha$ (a), Critical points $Y_c^\pm$ vs $\alpha$ (b) Stability regions ($J_{22} < 0$) of the solutions vs $\alpha$ (c) and spiral stable nature $\Delta < 0$ for the solutions vs $\alpha$ (d)
Figure 2: Arrow diagrams for $\alpha = 0$, (a) quasi (saddle) stable and (b) spiral stable

Figure 3: Arrow diagrams for $\alpha = 0.5$, (a) quasi (saddle) stable and (b) spiral stable
Figure 4: Arrow diagrams for \( \alpha = 0.9 \), (a) quasi (saddle) stable and (b) spiral stable.

Figure 5: Arrow diagrams for \( \alpha = 1 \), (a) and (b) quasi (saddle) stable.