Multitime stochastic maximum principle on curvilinear integral actions

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Abstract

Based on stochastic curvilinear integrals in the Cairoli-Walsh sense and in the Itô-Udriște sense, we develop an original theory regarding the multitime stochastic differential systems. The first group of the original results refer to the complete integrable stochastic differential systems, the path independent stochastic curvilinear integral, the Itô-Udriște stochastic calculus rules, examples of path independent processes, and volumetric processes. The second group of original results include the multitime Itô-Udriște product formula, first stochastic integrals and adjoint multitime stochastic Pfaff systems. Thirdly, we formulate and we prove a multitime maximum principle for optimal control problems based on stochastic curvilinear integral actions subject to multitime Itô-Udriște process constraints. Our theory requires the Lagrangian and the Hamiltonian as stochastic 1-forms.

Key-words: stochastic curvilinear integral, multitime stochastic differential system, path independence, multitime Itô-Udriște product formula, adjoint multitime stochastic system, multitime stochastic maximum principle.

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1 Introduction

The subject of this paper risen from the intersections of basic ideas in our works [12] - [20] and those well known as stochastic literature [1], [11], [21] - [25]. The principal aim is to solve stochastic optimal control problems based on curvilinear functionals as actions and stochastic differential systems as constraints. In this paper we describe how the concepts, methods and results in [12] - [20] can be applied to give a rigorous multitime stochastic model.

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There are several reasons why one should learn more about multitime Wiener process, Itô curvilinear stochastic integrals, multitime stochastic differential systems, complete integrability conditions, path independent stochastic curvilinear integral, path independent stochastic processes, multitime Itô-Udriște product formula and adjoint stochastic multitime Pfaff systems, optimization problems with curvilinear stochastic integral functionals, and multitime stochastic maximum principle. They have a wide range of applications outside mathematics (for example, can be applied to give a rigorous mathematical models in Finance), there are many fruitful connections to other mathematical disciplines and these subjects will give a rapidly developing life of its own as a fascinating research field with many interesting unanswered questions.

In Section 2 we present a brief introduction in the theory of multitime Wiener processes (e.g., [7]). We outline in Section 3 how the introduction of stochastic curvilinear integrals in the Cairoli-Walsh sense leads to a simple, intuitive and useful understanding of multitime integration process. In Section 4 we define and study the stochastic curvilinear integrals in our sense. In Section 5 we study the complete integrable multitime stochastic differential systems, the path independent stochastic curvilinear integral, and the multitime stochastic path independent models. Section 6 contains original results regarding the multitime Itô-Udriște product formula and the adjoint stochastic multitime Pfaff systems and multitime stochastic first integrals. The stochastic control problems with curvilinear stochastic integral functionals constrained by multitime stochastic differential systems are formulated and solved in the Section 7. In this context we have obtained a stochastic multitime maximum principle.

### 2 Multitime Wiener process

Let $t = (t^\alpha)_{\alpha=1,m} \in \mathbb{R}^m_+$ be a multi-parameter of evolution or *multitime* and let $\Omega _{0T} \subset \mathbb{R}^m_+$ be the parallelepiped fixed by the diagonal opposite points $0 = (0,...,0)$ and $T = (T^1,...,T^m)$, equivalent to the closed interval $0 \leq t \leq T$, via the product order on $\mathbb{R}^m_+$, defined by

$$(t_1^\alpha, t_2^\alpha, ..., t_m^\alpha) \leq (<) (t_1^\alpha, t_2^\alpha, ..., t_m^\alpha)$$

if $t_1^\alpha \leq (<) t_2^\alpha$, for all $\alpha = 1,m$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a *complete, increasing* and *right-continuous* filtration (a complete natural history) $\{(\mathcal{F}_t): t \in \Omega _{0T}\}$. Such a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \Omega _{0T}}, \mathbb{P})$ is called *filtered probability space*. Let $I$ be any subset of $\{1,...,m\}$ and let us denote by $\mathcal{F}_I (t)$ the $\sigma$–algebra generated by the $\sigma$–algebras $\mathcal{F}_{\tau}$, where $\tau^\alpha \leq t^\alpha$, for all $\alpha \in I$. We say that the filtration satisfies the *conditional independence* property if for all bounded random variables $X$, all $t \in \mathbb{R}^m$ and $I \subset J \subset \{1,...,m\}$,

$$\mathbb{E}[X | \mathcal{F}_I (J)] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}_I (J)] | \mathcal{F}_I (J \setminus I)].$$

(2.1)

This property implies that the conditional expectations with respect to $\mathcal{F}_I (J)$ and $\mathcal{F}_I (J \setminus I)$ commute.
For a multitime process \( x = (x(t, \omega))_{t \in \Omega_\omega} \), the increment of \( x \) on an interval \((t_1, t_2] \subset \Omega_\omega\), is given by ([3, pp. 10])

\[
x ((t_1, t_2]) = \sum_{i(1)=1}^{2} \cdots \sum_{i(m)=1}^{2} (-1)^{\alpha_1} x \left( t_{i(1)}^1, \ldots, t_{i(m)}^m \right).
\] 

(2.2)

For simplicity, here and in the whole paper, we will denote by \( \xi_t \) the random variable \( \xi(t, \omega) \) at each multitime \( t \).

**Definition 2.1** (Martingale) Let \( x = (x_t)_{t \in \Omega_\omega} \) be a multitime \( \mathcal{F}_t \)-adapted process.

(i) The process \( x \) is called weak martingale if \( \mathbb{E} \left[ x((t, s]) \mid \mathcal{F}_t \right] = 0 \), for all \( t, s \in \mathbb{R}_+^m \), such that \( t \leq s \).

(ii) The process \( x \) is called martingale if \( \mathbb{E} \left[ x_s \mid \mathcal{F}_t \right] = x_t \), for all \( t, s \in \mathbb{R}_+^m \), such that \( t \leq s \).

Clearly, every martingale is a weak martingale [3].

**Definition 2.2** A multitime, real-valued, and right-continuous process \( A = (A_t : t \in \mathbb{R}_+^m) \) is said to be increasing if \( A_t = 0 \) \( \mathbb{P} \)-a.s. for \( t \in \mathbb{R}_+^m \) and if \( A((t, s]) \geq 0 \), for all subintervals \((t, s] \subset \mathbb{R}_+^m \) (see (2.2)).

**Theorem 2.3 ([3, Prop. 8, pp. 41])** If \( x \) is a squared-integrable martingale, then there exists an increasing process \( (A_t) \) such that \( (x_t^2 - A_t) \) is a weak martingale.

**Theorem 2.4 ([3, Prop. 8, pp. 41])** Let \( x \) be a strong continuous martingale such that \( \mathbb{E} \left[ |x_t|^4 \right] < \infty \). Then there exists an increasing \( \mathcal{F}_t \)-previsible process \( [x]_t \) such that \( (x_t^2 - [x]_t) \) is a martingale.

Let \( B \) be the Borel \( \sigma \)-field of \( \mathbb{R}_+^m \) and let \( \nu \) denote the Lebesgue measure. For \( t = (t^1, \ldots, t^m) \), set \( \Omega_{dt} = [0, t^1] \times [0, t^2] \times \ldots \times [0, t^m] \) and

\[
W(t) = W(t^1, \ldots, t^m) = W(\Omega_{dt}).
\]

This defines a multitime mean-zero Gaussian process [7, Ch. 5] \( W = W(t)_{t \in \Omega_\omega} \) with covariance

\[
\mathbb{E} [W_{t_1} W_{t_2}] = \nu ([0, t_1] \cap [0, t_2]) = \prod_{\alpha=1}^{m} \min \{t_1^\alpha, t_2^\alpha\}.
\]

**Definition 2.5** A stochastic process of the form \( (W_t : t \in \mathbb{R}_+^m) \) is called multitime Wiener process (starting at zero) or Brownian sheet if \( W_0 = 0 \) and if \( W_t \) is a gaussian process with \( \mathbb{E} [W_t] = 0 \) and for \( t_1 = (t_1^\alpha)_{\alpha=1}^{m} \), \( t_2 = (t_2^\alpha)_{\alpha=1}^{m} \),

\[
\mathbb{E} [W_{t_1} W_{t_2}] = \prod_{\alpha=1}^{m} \min \{t_1^\alpha, t_2^\alpha\}.
\]
Definition 2.6 The multitime stochastic process \((W_t : t \in \Omega_{0T})\) is called a \(\mathcal{F}_t\)-Wiener process if, in addition, \(E[W_s | \mathcal{F}_t] = W_t\), for all \(t, s \in \mathbb{R}_T^m\), such that \(t \leq s\).

A first example of martingale is the multitime Wiener process.

**Hypothesis right-left (RL)** Suppose a sample sheet \(x : \Omega_{0T} \rightarrow \mathbb{R}\) is continuous from the right and bounded from the left at every point. That is, for every \(t_0 \in \Omega_{0T}\), \(t \downarrow t_0\), implies \(x(t) \rightarrow x(t_0)\) and for \(t \uparrow t_0\), \(\lim_{t \uparrow t_0} x(t)\) exists, but need not be \(x(t_0)\). We use only stochastic processes \(x\) where almost all sample sheets have the RL property.

### 3 Stochastic curvilinear integrals in the Cairoli-Walsh sense

Let \(\Gamma\) be an oriented piecewise \(C^1\) curve in \(\mathbb{R}_T^m\) given by parametric representation \(t = \gamma(\tau), \tau \in [0, 1]\). The curve \(\hat{\Gamma}\) of the parametric representation \(t = \hat{\gamma}(\tau) = \gamma(1 - \tau), \tau \in [0, 1]\) has opposite orientation.

**Definition 3.1** A piecewise \(C^1\) curve is called of pure type if each component of the tangent vector field \(\frac{d\gamma}{d\tau}\) preserves its sign.

Given a process \(W = W(t)_{t \in \Omega_{0T}}\) and a pure type curve \(\Gamma\), we can define \(m\) processes \(W^\Gamma_\alpha\), \(\alpha = 1, m\), on \(\Gamma\), which may be thought of as coming from increments in the direction of each \(Ox_\alpha\) axis. A suggestive notation for this would be \(dW^\Gamma_\alpha\).

If \(\frac{d\gamma^\alpha}{d\tau}(\tau) \geq 0\), i.e., the component \(\gamma^\alpha\) is nondecreasing function and if \(\gamma(0) = t_0\) and \(t = \gamma(\sigma) \in \Gamma\), then we introduce the subset

\[
D^\alpha_t = \{s \in \mathbb{R}_T^m | t^\alpha_0 < s^\alpha \leq t^\alpha, 0 \leq s^\beta \leq \gamma^\beta(\sigma), \beta \neq \alpha, 0 < \tau \leq \sigma\}.
\]

For a squared-integrable martingale \(W\) and a curve \(\Gamma\) of pure type joining the initial point \(t_0\) and the final point \(t_f\), with \(\frac{d\gamma^\alpha}{d\tau}(\tau) \geq 0\), we define \(W^\Gamma_\alpha(t_f) = W(D^\alpha_{t_f}) = W^\Gamma_\alpha(t) = \mathbb{E}\{W(D^\alpha_{\tau}) | \mathcal{F}^\alpha_t\}, t \in \Gamma\). Then each \(W^\Gamma_\alpha = \{W^\Gamma_\alpha(t), \mathcal{F}^\alpha_t, t \in \Gamma\}\) is a one-parameter square-integrable martingale. Consequently, one can define the Itô curvilinear integral of a process \(\phi = \{\phi(t) : t \in \Omega_{0T}\}\) with respect to \(W^\Gamma_\alpha\), in the usual way, and denote it by

\[
\int_{\Gamma} \phi(t) \partial_\alpha W.
\]

If the component \(\alpha\) of the tangent vector field satisfy the condition \(\frac{d\gamma^\alpha}{d\tau}(\tau) \leq 0\) (nonincreasing function), then we define

\[
\int_{\Gamma} \phi(t) \partial_\alpha W = -\int_{\Gamma} \phi(t) \partial_\alpha W.
\]
Finally, if $\Gamma$ is of pure type curve, we let
$$\int_{\Gamma} \phi(t) \, dW = \sum_{\alpha=1}^{m} \int_{\Gamma} \phi(t) \, d_{\alpha}W.$$ 

4 Stochastic curvilinear integrals in the Itô-Udrişte sense

In our theory we need a curve
$$\gamma : [0, 1] \rightarrow \Omega_{0T} \subset \mathbb{R}^m_t, \quad t = t(\tau), \quad \tau \in [0, 1],$$
where $\tau$ is the curvilinear abscissa. The curve $\gamma$ is called increasing if
$$\gamma(\tau) \leq \gamma(\tau') \quad \text{in } \mathbb{R}^m_t, \quad \text{if } \tau \leq \tau' \text{ in } [0, 1].$$

The curve $\gamma$ is called piecewise $C^1$ if there exists a curve $\gamma' : [0, 1] \rightarrow \Omega_{0T}$ with finitely many discontinuities, satisfying the (RL) hypotheses, so that
$$\gamma(\tau) = \int_{0}^{\tau} \gamma'(\lambda) d\lambda, \quad \forall \tau \in [0, 1].$$

Let $t = (t^1, \ldots, t^m) \in \Omega_{0T}$ be a multitime Wiener process. Then ([3 Th. 3, pp. 45]) $W^2$ decomposes into the sum of a weak martingale and an increasing process (see Definition 2.1 and Definition 2.2). According to Dozzi ([3 Lemma 6, pp. 151]), the increasing process associated with $(W_t)_{t \in \Omega_{0T}}$ is denoted by $(\langle W \rangle_t) = t^1 t^2 \ldots t^m$ and it means that for $h = (h^1, \ldots, h^m)$, we have
$$\mathbb{E} \left[ W^2_{t+h} - \prod_{\alpha=1}^{m} (t^\alpha + h^\alpha) \mid \mathcal{F}_t \right] = W^2_t - \prod_{\alpha=1}^{m} t^\alpha.$$

In other words, for the multitime $t = (t^1, \ldots, t^m) \in \Omega_{0T}$, the stochastic process
$$\left( W^2_t - \prod_{\alpha=1}^{m} t^\alpha \right)_{t \in \Omega_{0T}}$$
is a (continuous) martingale. This leads to the construction of stochastic curvilinear integral with respect to $W$. The case $m = 2$ was explicitly given in [2 §7, pp. 157].

**Definition 4.1** Let $\phi = \{ \phi(t) : t \in \Omega_{0T} \}$ be a real $\mathcal{F}_t$-predictible process, such that
$$\mathbb{E} \left[ \int_{\gamma_{0T}} \phi^2_t \, d\langle W \rangle_t \right] < \infty.$$ 

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Let \( \gamma_{0T} \) be a \( C^1 \) increasing curve. The real number

\[
I(\phi) = \int_{\gamma_{0T}} \phi(t) \, dW_t = \int_0^1 \phi(\gamma(\tau)) \, dB_\tau
\]

(4.3)

is called the stochastic curvilinear integral of the process \( \phi = \{\phi(t) : t \in \Omega_{0T}\} \) along the curve \( \gamma \) with respect to \( (W_t)_{t \in \Omega_{0T}} \), where \( B_\tau \overset{def}{=} W_{\gamma(\tau)} \) is a stochastic processes normal distributed of mean 0 and variance \( \text{vol}(\Omega_{0\gamma(\tau)}) \).

For \( m = 2 \), see \([2]\).

**Definition 4.2** Let \( \phi = \{(\phi_a(t))_{a=1}^{d} : t \in \Omega_{0T}\} \) be an \( \mathcal{F}_t \)-predictible process with values in \( \mathbb{R}^d \). Let \( \gamma_{0T} \) be a \( C^1 \) increasing curve. The real number

\[
I(\phi) = \int_{\gamma_{0T}} \phi_a(t) \, dW_{t}^a = \int_0^1 \phi_a(\gamma(\tau)) \, dB_a^\tau
\]

(4.4)

(i.e., a sum of Itô classical integrals) is called the stochastic curvilinear integral of process \( \phi = \{(\phi_a(t))_{a=1}^{d}\} \), along the curve \( \gamma \) with respect to the Wiener process \( W = (W_t^a)_{a=1}^{d} \), where \( B_a^\tau \overset{def}{=} W_{\gamma(\tau)}^a \), for each \( a = 1, \ldots, d \), is a stochastic processes as in the above definition.

Here and in the whole article, for any given Euclidean space \( H \), we denote by \( \langle \cdot, \cdot \rangle \) (resp. \( \| \cdot \| \)) the inner product (resp. norm) of \( H \). Also, we use Einstein summation convention.

**Remark 4.3** With the previous definition \([4.4]\), we have, by Itô’s isometry property

\[
\mathbb{E}[|I(\phi)|^2] = \mathbb{E} \left[ \int_{\gamma_{0T}} \|\phi(t)\|^2 \, d(t^1 \ldots t^m) \right]
\]

\[
= \mathbb{E} \left[ \int_0^1 \|\phi(t(\tau))\|^2 \frac{d}{d\tau} (t^1(\tau) \ldots t^m(\tau)) \, d\tau \right].
\]

A particular case \((n = 1, m = 2)\) of the previous formula is proved in \([2]\). It is obvious that \( \mathbb{E}[|I(\phi)|] = 0 \).

### 5 Multitime stochastic differential systems

Let us change the single-time approach of stochastic theory (see, for example \([9]\)) to a new approach issuing from the papers of the first author. We use a multitime parameter of evolution \( t = (t^1, \ldots, t^m) \in \Omega_{0T} \) and we introduce
the multitime stochastic differential systems \((mSDS)\). For that, let \(f(t,x_t) = (f_i^\alpha(t,x_t))_{i = 1}^n\) be an \(n \times m\) matrix of previsible processes with \(\mathbb{E}\left[\int_{\gamma_0t}^t \|f(s,x_{s(\tau)})\| \, d\tau\right] < \infty\) for all \(t \in \Omega_{0T}\),

where \(\tau\) is the curvilinear abscissa on \(\gamma_0t\), let \(g(t,x_t) = (g_i^a(t,x_t))_{i = 1}^n\) be an \(n \times d\) matrix of previsible processes, such that

\[
\mathbb{E}\left[\int_{\gamma_0t}^t \|g(s,x_s)\|^2d(s^1...s^m)\right] < \infty \quad \text{for all} \quad t \in \Omega_{0T},
\]

and let \(W = (W_t^{\alpha})_t, a = 1,d\), be a multitime Wiener process with values in \(\mathbb{R}^d\).

**Definition 5.1** Let \(i = 1,n,\alpha = 1,m\) and \(a = 1,d\). The multitime stochastic differential system

\[
 dx_i^\alpha = f_i^\alpha(t,x_t) \, dt^\alpha + g_i^a(t,x_t) \, dW_t^a, \quad t \in \Omega_{0T}
\]

is called Itô - Pfaff stochastic system.

The coefficients \(f_i^\alpha(t,x_t)\) are called drift coefficients and \(g_i^a(t,x_t)\) are the diffusion coefficients.

**Definition 5.2** A multitime stochastic differential system is called completely integrable if there exists an \(m\)-sheet \(x(t)\) satisfying the stochastic integral relation

\[
x_i^\alpha(t) = x_i^\alpha(0) + \int_{\gamma_0t}^t f_i^\alpha(s,x_s) \, ds^\alpha + \int_{\gamma_0t}^t g_i^a(s,x_s) \, dW_s^a, \quad t \in \Omega_{0T},
\]

independent of the selection of the \(C^1\) increasing curve \(\gamma_0t\).

Thus, the right hand side of an Itô - Pfaff stochastic system is well-defined as a stochastic curvilinear integral, under suitable assumptions on the functions \(f_i^\alpha\) and \(g_i^a\).

The first integral can be interpreted as an ordinary curvilinear integral. The second integral, i.e., the stochastic curvilinear integral, cannot be treated as such, since the sheet-wise \(W_t\) is nowhere differentiable. Of course, the complete integrability conditions for the first curvilinear integral are contained in the classical books while, for the stochastic curvilinear integral, only partial results can be found in \([2],[3]\).

If a multitime Itô - Pfaff stochastic differential system is not completely integrable, given the increasing \(C^1\) curve \(\gamma_0t : s^\alpha = s^\alpha(\tau), \tau \in [0,\tau_0], s(0) = 0\), a curve \(x(\tau)\) which satisfies

\[
x_i^\alpha(\tau) = x_i^\alpha(0) + \int_0^\tau f_i^\alpha(s(\lambda),x(\lambda)) \frac{ds^\alpha}{d\lambda} d\lambda + \int_0^\tau g_i^a(s(\lambda),x(\lambda)) dB_\lambda^a
\]

is called solution.

Having in mind some ideas in \([2],[3]\), completing with our ideas, let us praise a fundamental path independent stochastic curvilinear integral.
5.1 Path independent stochastic curvilinear integral

Here is for the first time when is presented a curvilinear stochastic integral independent of the path.

**Theorem 5.3** Let $\gamma_{0t}$ be an increasing curve in $\Omega_{0T}$. The stochastic curvilinear integral (primitive) $\int_{\gamma_{0t}} W_s dW_s$ has the value $\frac{W^2_t - W^2_0}{2} - \frac{1}{2} t^1 \cdots t^m$. It can be written as

$$\frac{1}{2} W^2_t = \frac{1}{2} W^2_0 + \frac{1}{2} \int_{\gamma_{0t}} d(s^1 \cdots s^m) + \int_{\gamma_{0t}} W_s dW_s.$$ 

Obviously, $W_0 = 0$ and the stochastic curvilinear integral $\int_{\gamma_{0t}} W_s dW_s$ is path independent.

**Remark 5.4** Our point of view requires the volume written as a special curvilinear integral,

$$\int_{\gamma_{0t}} d(s^1 \cdots s^m) = \text{vol}(\Omega_{0t}), \quad \int_{\gamma_{st}} d(\tau^1 \cdots \tau^m) = \text{vol}(\Omega_{0t}) - \text{vol}(\Omega_{0s}), \ s \leq t.$$

**Proof (Ioneț Țevy)** For simplicity, we refer to $m = 2$. Let $\mathcal{M}$ be the random measure in $\mathbb{R}^2_+$ which assigns to each Borel set $A$ a Gaussian random variable of mean zero and variance $\mu(A)$, where $\mu$ is Lebesgue measure and which assigns independent random variables to disjoint sets.

Define a process $(W(z) : z \in \mathbb{R}^2_+)$ by $W(z) = W(\Omega_{0z})$, where $\Omega_{0z}$ is the rectangle whose lower left hand corner is the origin $O(0,0)$ and whose upper right hand corner is $z = (s,t)$. The process $(W(z) : z \in \mathbb{R}^2_+)$ is called a two-parameter Wiener process.

For $z_1(s_1,t_1)$ and $z_2(s_2,t_2)$ with $z_1 < z_2$, note $L_{z_1z_2} = \Omega_{0z_2} \setminus \Omega_{0z_1}$. Let $[a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2_+$, and let

$$P^n = \{a = s^n_0 < s^n_1 < \ldots < s^n_{m_n} = b\}, \quad Q^n = \{c = t^n_0 < t^n_1 < \ldots < t^n_{m_n} = d\}$$

be partitions of the segment $[a,b]$ and, respectively, $[c,d]$ with $|P^n| \to 0$, $|Q^n| \to 0$ as $n \to \infty$. Let us denote $\alpha = (a,c)$, $\beta = (b,d)$, $z^n_k = (z^n_k, t^n_k)$, $k = 1, m_n$. Then

$$|L^n| = \max_k \left(\text{area} (L_{z^n_{k+1}z^n_{k+1}})\right) \to 0 \quad \text{as} \quad n \to \infty.$$ 

**Lemma 5.5 (Quadratic variation)** The limit

$$\sum_{k=0}^{m_n-1} (W(z^n_{k+1}) - W(z^n_k))^2 \to \text{area} (L_{\alpha\beta}),$$

as $n \to \infty$, holds in $L^2$. 

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Proof. Set
\[
R_n = \sum_{k=0}^{m_n-1} \left( W \left( z_{k+1}^n \right) - W \left( z_k^n \right) \right)^2.
\]
Then
\[
R_n - \text{area} \left( L_{\alpha\beta} \right) = \sum_{k=0}^{m_n-1} \left[ \left( W \left( z_{k+1}^n \right) - W \left( z_k^n \right) \right)^2 - \text{area} \left( L_{z_k^n z_{k+1}^n} \right) \right].
\]
Denoting,
\[
\rho_k = \left( W \left( z_{k+1}^n \right) - W \left( z_k^n \right) \right)^2 - \text{area} \left( L_{z_k^n z_{k+1}^n} \right),
\]
we find
\[
\mathbb{E} \left[ (R_n - \text{area} \left( L_{\alpha\beta} \right))^2 \right] = \sum_{k=0}^{m_n-1} \sum_{j=0}^{m_n-1} \mathbb{E}(\rho_k \rho_j).
\]
For \( k \neq j \), the term in the double sum equals 0, according to the independent increments, as \( W(v) - W(u) \) and \( N(0, \text{area} \left( L_{uv} \right)) \). Hence
\[
\mathbb{E} \left[ (R_n - \text{area} \left( L_{\alpha\beta} \right))^2 \right] = \sum_{k=0}^{m_n-1} \mathbb{E} \left[ (Y_k^2 - 1)^2 \text{area} \left( L_{z_k^n z_{k+1}^n} \right) \right],
\]
where
\[
Y_k = Y_k^n = \frac{W \left( z_{k+1}^n \right) - W \left( z_k^n \right)}{\sqrt{\text{Area} \left( L_{z_k^n z_{k+1}^n} \right)}},
\]
is a Gaussian process \( N(0,1) \). Therefore, for some constant \( C \), we have
\[
\mathbb{E} \left[ (R_n - \text{area} \left( L_{\alpha\beta} \right))^2 \right] \leq C \sum_{k=0}^{m_n-1} \left[ \text{area} \left( L_{z_k^n z_{k+1}^n} \right) \right]^2 \leq C |L^n| \text{area} \left( L_{\alpha\beta} \right) \to 0, \text{ as } n \to \infty.
\]

Let \( \Gamma^n = \{ 0 < z_1 < ... < z_{m_n} = z \} \) be a partition of the curve \( \gamma_0z \), i.e., \( z_k^n (s_k^n, t_k^n) \in \gamma_0z \), and \( |\Gamma^n| = \max_k \text{area} \left( \Omega_{z_k^n z_{k+1}^n} \setminus \Omega_{z_k^n} \right) \). Note, according Itô definition,
\[
R_n = \sum_{k=0}^{m_n-1} W \left( z_k^n \right) \left( W \left( z_{k+1}^n \right) - W \left( z_k^n \right) \right).
\]
Then, we have
\[
R_n = \frac{W^2(z)}{2} - \frac{1}{2} \sum_{k=0}^{m_n-1} \left( W \left( z_{k+1}^n \right) - W \left( z_k^n \right) \right)^2.
\]
When $|\Gamma_n| \to 0$, as $n \to \infty$, based on the foregoing Lemma, we find

$$
\sum_{k=0}^{m_n-1} (W(z^n_{k+1}) - W(z^n_k))^2 \to st
$$
in $L^2$, as $n \to \infty$, and the result is established.

The following examples are based on the complete integrability notion for the stochastic curvilinear integrals and on Itô-Udriște stochastic calculus rules

$$
dW^a_t dW^b_t = \delta^{ab} c_\alpha(t) dt^\alpha, dW^a_t dt^\alpha = dt^\alpha dW^a_t = 0, dt^\alpha dt^\beta = 0,
$$
for any $a, b = 1, \ldots, m$, where $\delta^{ab}$ is the Kronecker symbol, $c_\alpha(t) = \partial^\alpha(t_1 \cdots t_m)$ and the tensorial product $\delta^{ab} c_\alpha(t)$ represents the correlation coefficients.

### 5.2 Examples of path independent processes

1) **Stock prices.** The idea to reconsider applications in Finance via the multitime stochastic calculus was inspired by the work given in [9]. Let $(P_{t})_{t \in \mathbb{R}^2}$ denote the price of a stock at two-time $t = (t_1, t_2) \in \mathbb{R}^2$, where $t_1$ means time and $t_2$ represents a "space" variable (as example, showing the price evolution as function of the distance between supplier and seller). In this way, the stochastic perturbations involved in price dynamics are modelled by a two-time (time-space) Brownian sheet. We model the evolution of the price $P_t$ supposing that the relative change $dP_t/P_t$ in price $P_t$ is involved in the SDE

$$
dP_t = P_t \mu_\alpha dt^\alpha + P_t \sigma_\alpha dW^\alpha,
$$
for constant drift vector $\mu_\alpha > 0$ and constant diffusion vector $\sigma_\alpha$. Hence

$$
dP_t = P_t \mu_\alpha dt^\alpha + P_t \sigma_\alpha dW^\alpha.
$$

Using Itô-Udriște formula

$$
d[\ln P_t] = \frac{dP_t}{P_t} - \frac{1}{2} \left( \frac{P_t^2}{P_t^2} \sigma_\alpha \sigma_\beta \delta^{\alpha\beta} c_\lambda(t) \right) dt^\lambda + \sigma_\alpha dW^\alpha,
$$
we find

$$
P_t = p_0 e^{\mu_\lambda t^\lambda - \frac{1}{2} \sigma_\alpha \sigma_\beta \delta^{\alpha\beta} t^\lambda + \sigma_\alpha W^\alpha}.
$$
The price $P_t$ is always positive, if $p_0 > 0$. Since

$$
P_t = p_0 + \int_{\gamma_0} P_s \mu_\alpha ds^\alpha + \int_{\gamma_0} P_s \sigma_\alpha dW^\alpha
$$
and $\mathbb{E} \left[ \int_{\gamma_0} P_s \sigma_\alpha dW^\alpha \right] = 0$, we find

$$
\mathbb{E}[P_t] = p_0 + \int_{\gamma_0} \mathbb{E}[P_s] \mu_\alpha ds^\alpha.
$$
Applying the Itô-Udrisite Lemma for satisfies and hence is called a volumetric stochastic process.

Hence if the (usual and stochastic) curvilinear integrals are path independent, i.e.,

$$\int x \, dy$$

A completely integrable stochastic process

$$dP_t = P_t \mu_t dt.$$  

2) Let \( t = (t_1, t_2) \in \mathbb{R}^2_+, x \in \mathbb{R} \) and \( g_\alpha : \mathbb{R}^2_+ \to \mathbb{R}, \alpha = 1, 2 \) be two continuous functions. Let \( c_1(t) = t_2, c_2(t) = t_1 \) and \( \gamma_0 \) be an increasing curve in \( \Omega_0 \). The unique solution of the stochastic differential equation \( dx_t = x_t g_\alpha(t) dW_t^\alpha, x(0) = 1 \) is

$$x_t = e^{-\frac{1}{2} \int_{\gamma_0} g_\alpha(s) g_\beta(s) \delta^\alpha \beta \lambda(s) ds} + \int_{\gamma_0} g_\alpha(s) dW_s^\alpha,$$

if the (usual and stochastic) curvilinear integrals are path independent, i.e., \( g_\alpha(t) = h_\alpha(t_1 t_2) \). To verify the solution, we note that the stochastic process

$$y_t = -\frac{1}{2} \int_{\gamma_0} g_\alpha(s) g_\beta(s) \delta^\alpha \beta \lambda(s) ds + \int_{\gamma_0} g_\alpha(s) dW_s^\alpha$$

satisfies

$$dy_t = -\frac{1}{2} g_\alpha(t) g_\beta(t) \delta^\alpha \beta \lambda(t) dt + g_\alpha(t) dW_t^\alpha.$$

Applying the Itô-Udrisite Lemma for \( u(x) = e^x \), we obtain

$$dx_t = \frac{\partial u}{\partial x} dy_t + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} g_\alpha(t) g_\beta(t) \delta^\alpha \beta \lambda(t) dt$$

$$= e^{y_t} \left( -\frac{1}{2} g_\alpha(t) g_\beta(t) \delta^\alpha \beta \lambda(t) dt + g_\alpha(t) dW_t^\alpha + \frac{1}{2} g_\alpha(t) g_\beta(t) \delta^\alpha \beta \lambda(t) dt \right)$$

and hence

$$dx_t = x_t g_\alpha(t) dW_t^\alpha.$$

3) The formula

$$\frac{1}{2} W_t^2 = \frac{1}{2} W_0^2 + \frac{1}{2} \int_{\gamma_0} d(s^1 \cdots s^m) + \int_{\gamma_0} W_s dW_s,$$

and the notations \( x_t = \frac{1}{2} W_t^2, u(t) = \frac{1}{2}, v(t) = W_t \) motivate the following

**Definition 5.6** A completely integrable stochastic process \( x_t \) of the form

$$x_t = x_0 + \int_{\gamma_0} u(s, \omega) d(s^1 \cdots s^m) + \int_{\gamma_0} v(s, \omega) dW_s,$$

where \( u(t), v(t) \) satisfy the integrability conditions

$$\mathbb{E} \int_{\gamma_0} |u(s)||c(s)|| d\sigma < \infty, \mathbb{E} \int_{\gamma_0} v^2(s) d(s^1 \cdots s^m) < \infty,$$

is called a volumetric stochastic process.
The volumetric stochastic process can be written also as a stochastic Pfaff equation
\[ dx_t = u(t) \, d(t^1 \cdots t^m) + v(t) \, dW_t. \]
To motivate the adjective "volumetric" we need the following

**Lemma 5.7** The curvilinear integrals
\[ \int_{t_0}^t u(s, \omega) \, d(s^1 \cdots s^m) \quad \text{and} \quad \int_{t_0}^t v(s, \omega) \, dW_s \]
are path independent if and only if \( u(s, \omega) = \varphi(s^1 \cdots s^m, \omega) \) respectively \( v(s, \omega) = \psi(s^1 \cdots s^m, \omega) \), i.e., the functions \( u \) and \( v \) depend on the point \( (s^1, ..., s^m) \) only through the product of components \( s^1 \cdots s^m \).

**Proof.** The first curvilinear integral is path independent if and only if
\[ \frac{\partial}{\partial s^\beta}(u(s)c^\alpha(s)) = \frac{\partial}{\partial s^\alpha}(u(s)c^\beta(s)). \]
Since
\[ \frac{\partial c^\alpha}{\partial s^\beta}(s) = \frac{\partial c^\beta}{\partial s^\alpha}(s), \]
it follows
\[ \frac{\partial u}{\partial s^\beta}(s)c^\alpha(s) = \frac{\partial u}{\partial s^\alpha}(s)c^\beta(s), \]
i.e., \( u(s) = \varphi(s^1 \cdots s^m) \).

As was shown by Cairoli and Walsh [2], the (second) stochastic curvilinear integral is path independent if and only if \( v(s) = \psi(s^1 \cdots s^m) \).

In the following theorem we state a formula which is very useful for computing curvilinear It\'\=o integrals. It can be thought as the fundamental theorem of multitime stochastic calculus.

**Theorem 5.8** Let \( x_t, t = (t^1, ..., t^m) \in \mathbb{R}^m_+ \) be a volumetric stochastic process, i.e.,
\[ dx_t = u(t) \, d(t^1 \cdots t^m) + v(t) \, dW_t. \]
If \( g(t, x) \) is a \( C^2 \) function, then the stochastic process \( y_t = g(t, x_t) \) is completely integrable, with
\[ dy_t = \frac{\partial g}{\partial t^\alpha}(t, x_t) dt^\alpha + \frac{\partial g}{\partial x}(t, x_t) dx_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, x_t)(dx_t)^2, \]
where for computing \((dx_t)^2\) we use the following formal rules [12]
\[ dW_t \, dW_t = d(t^1 \cdots t^m) = c_\lambda(t) \, dt^\lambda, \quad dW_t \, dt^\alpha = dt^\alpha \, dW_t = 0, \quad dt^\alpha \, dt^\beta = 0. \]
Explicitly,

\[
dy_t = \left( \frac{\partial g}{\partial t}(t, x_t) + \frac{\partial g}{\partial x}(t, x_t)u(t)c_\alpha(t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, x_t)v^2(t)c_\alpha(t) \right) dt^\alpha + \frac{\partial g}{\partial x}(t, x_t)v(t)dW_t.
\]

In terms of curvilinear integrals it reads as follows

\[
g(t, x_t) = g(0, x_0) + \int_{\gamma_0}^t \left( \frac{\partial g}{\partial s}(s, x_s) + \frac{\partial g}{\partial x}(s, x_s)u(s)c_\alpha(s) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(s, x_s)v^2(s)c_\alpha(s) \right) ds^\alpha + \int_{\gamma_0}^t \frac{\partial g}{\partial x}(s, x_s)v(s)dW_s.
\]

4) **Integration by parts** Let \( \gamma_{0t} \) be an increasing curve in \( \Omega_{0T} \). If \( g(t, x) = (t_1 \cdots t_m) x \), then the stochastic process \( y_t = g(t, x_t) \) satisfies

\[
dy_t = x_t d(t_1 \cdots t_m) + (t_1 \cdots t_m) dx_t.
\]

Replacing \( x_t = W_t \), we find the "integration by parts" formula

\[
(t_1 \cdots t_m) W_t = \int_{\gamma_0}^t W_s d(s_1 \cdots s_m) + \int_{\gamma_0}^t (s_1 \cdots s_m) dW_s.
\]

Generally, if \( g(t, x) = \varphi(t) x \), where \( \varphi(t) \) is continuous and of bounded variation in \([0, t]\), then the stochastic process \( y_t = g(t, x_t) \) satisfies

\[
dy_t = x_t d\varphi(t) + \varphi(t) dx_t.
\]

Replacing \( x_t = W_t \), we find the integration by parts formula

\[
\varphi(t) W_t = \int_{\gamma_0}^t W_s d\varphi(s) + \int_{\gamma_0}^t \varphi(s) dW_s.
\]

5) **Geometric Brownian Sheet** It is a stochastic process of the form

\[
S_t = e^{\mu x + \sigma B_t}, \quad t = (t_1, \ldots, t_m) \in \Omega_{0T} \subset \mathbb{R}_+^m,
\]

where \( \mu = (\mu_\alpha) \in \mathbb{R}^m \), \( \sigma > 0 \) and \( (B_t)_{t \in \Omega_{0T}} \) is a standard multitime Wiener process. Since \( S_t = f(t, B_t) \), for \( f(t, x) = e^{\mu x + \sigma x} \) and

\[
\frac{\partial f}{\partial t^\alpha} = f\mu_\alpha, \quad \frac{\partial f}{\partial x} = f\sigma, \quad \frac{\partial^2 f}{\partial x^2} = f\sigma^2,
\]

the Itô-Udriște formula shows that the geometric Brownian sheet is a solution of the multitime stochastic Pfaff equation

\[
dS_t = \sigma S_t dB_t + S_t \mu_\alpha dt^\alpha + \frac{1}{2} \sigma^2 S_t d(t_1 \cdots t_m).
\]
Definition 5.9  (i) Let \( x(t, \omega) \) be an \( m \)-sheet solution of (5.1), \( t \in \Omega_{0\Gamma}, \omega \in \Omega \). Sheetwise uniqueness of \( x(\cdot, \omega) \) means that if \( \pi(\cdot, \omega) : \Omega_{0\Gamma} \to \mathbb{R}^n \) is also an \( m \)-sheet solution of (5.1), on the filtered probability space endowed with the same Wiener process and initial random variable, then

\[
P \left[ x(t, \omega) = \pi(t, \omega), \forall t \in \Omega_{0\Gamma} \right] = 1.
\]

(ii) Let \( x(\cdot, \omega) \) be a curve solution of (5.1) with \( \omega \in \Omega \). Pathwise uniqueness of \( x(\cdot, \omega) \) means that if \( \pi(\cdot, \omega) \) is also a curve solution of (5.1), on the filtered probability space endowed with the same Wiener process and initial random variable, then

\[
P \left[ x(\tau, \omega) = \pi(\tau, \omega), \forall \tau \in [0, \tau_0] \right] = 1.
\]

6 Multitime Itô - Udrişte product formula and adjoint stochastic multitime Pfaff systems

Let \( \Omega_{0\Gamma} \) be the parallelepiped fixed by the diagonal opposite points \( 0 = (0,...,0) \) and \( T = (T^1,...,T^m) \) and \( t \in \Omega_{0\Gamma} \) be the multitime. Given a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \Omega_{0\Gamma}}, \mathbb{P}) \) satisfying the usual conditions, on which a Wiener process \( W(\cdot, \omega) \) with values in \( \mathbb{R}^d \) is defined, consider a controlled multitime stochastic differential system

\[
\begin{align*}
\left\{
\begin{array}{l}
dx_t^\alpha = \mu^\alpha_t (t, x_t, u_t) dt + \sigma^\alpha_t (t, x_t, u_t) dW_t^a, \\
x(0) = a \in \mathbb{R}^n,
\end{array}
\right.
\end{align*}
\]

(6.1)

where

\[
\mu(\cdot, x(\cdot, \omega), u(\cdot, \omega)) = (\mu^\alpha_t) : \Omega_{0\Gamma} \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m},
\]

\[
\sigma(\cdot, x(\cdot, \omega), u(\cdot, \omega)) = (\sigma^\alpha_t) : \Omega_{0\Gamma} \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}
\]

and, for simplicity, we denote \( x(t, \omega), \) respectively \( u(t, \omega), \) by \( x_t \) and \( u_t. \) Here, \( u_t \in U \subset \mathbb{R}^k \) is a parameter whose value we can choose in the given Borel set \( U \) at any instant multitime \( t \) in order to control the process \( x_t. \) Thus, \( u_t = u(t, \omega) \) is a stochastic process, called control (vector-valued) variable or, simplified, control. Since our decision at multitime \( t \) must be based upon what has happened up to multitime \( t, \) the function \( \omega \to u(t, \omega) \) must (at least) be measurable w.r.t. \( \mathcal{F}_t, \) i.e. the process \( u_t \) must be \( \mathcal{F}_t \)-adapted. We also assume that \( u(t, \omega) \) is satisfying RL hypothesis. In addition we require that \( u(t, \omega) \) gives rise to a unique solution \( x(t) = x^{(u)}(t) \) of (6.1) for \( t \in \Omega_{0\Gamma}, \) i.e., the system (6.1) is completely integrable. Let us denote by \( \mathcal{A} \) the set of all controls with the above properties. Any \( u(\cdot, \omega) \in \mathcal{A} \) is called also a feasible control.

Definition 6.1 Let \( \left( \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P} \right) \) be given satisfying the usual conditions and let \( W(t) \) be a given standard \( (\mathcal{F}_t)_{t \in \mathbb{R}^+} \)-Wiener process with values in \( \mathbb{R}^d. \) A control \( u(\cdot, \omega) \) is called admissible, and the pair \( (x(\cdot, \omega), u(\cdot, \omega)) \) is called admissible, if
1. $u(\cdot, \omega) \in \mathcal{A}$;
2. $x(\cdot, \omega)$ is the unique solution of system (6.1);
3. some additional convex constraint on the terminal state variable are satisfied, e.g.
   \[ x(T, \omega) \in K, \]
   where $K$ is a given nonempty convex subset in $\mathbb{R}^n$.

The set of all admissible controls is denoted by $\mathcal{A}_{ad}$. We assume:

(H1) $\mu_\alpha$, $\sigma_\alpha$, and $f_\alpha$ are continuous in their arguments and continuously differentiable in $(x, u)$, for every $i = 1, n$, $\alpha = 1, m$, $a = 1, d$;

(H2) the derivatives of $\mu_\alpha$ and $\sigma_\alpha$ in $(x, u)$ are bounded for every $i = 1, n$, $\alpha = 1, m$, $a = 1, d$;

(H3) the derivatives of $f_\alpha$ in $(x, u)$ are bounded by $C (1 + |x| + |u|)$, for every $\alpha = 1, m$ and the derivative of $\Psi$ in $x$ is bounded by $C (1 + |x|)$.

Then, for a given $u(\cdot, \omega) \in \mathcal{A}_{ad}$, there exists a unique solution $x(\cdot, \omega)$ which solves the system (6.1).

6.1 Multitime Itô - Udrişte product formula

In order to prove the multitime stochastic maximum principle using the ideas rising from the papers [12], [14], [16], [17], we need the following auxiliary result, which is a special case of the multitime Itô - Udrişte formulas [12]. The theory covers both the problems formulated with complete integrable stochastic systems and problems based on nonintegrable systems.

**Lemma 6.2 (Itô - Udrişte product formula)** Suppose the Itô process $(x_i^t)_{t \in \Omega_{0T}}$, $i = 1, n$ is solution of the multitime stochastic Pfaff system
\[
\begin{aligned}
\text{d}x_i^t &= \mu_i^t (t, x(t, \omega), u(t, \omega)) \, dt + \sigma_i^t (t, x(t, \omega), u(t, \omega)) \, dW_i^a, \\
x(0, \omega) &= x \in \mathbb{R}^n,
\end{aligned}
\]
and the Itô process $(p_i (t))_{t \in \Omega_{0T}}$, $i = 1, n$ is solution of the multitime stochastic Pfaff system
\[
\begin{aligned}
\text{d}p_i (t) &= a_{i\alpha} (t, x(t, \omega), u(t, \omega)) \, dt + q_{i\alpha} (t, x(t, \omega), u(t, \omega)) \, dW_i^a, \\
p(0, \omega) &= p \in \mathbb{R}^n,
\end{aligned}
\]
where the coefficients in both evolutions are predictable processes and $u(\cdot, \omega)$ is an admissible control. Then the interior product $p_i(t)x_i^t(t)$ is an multitime Itô process and
\[ d \left( p_i (t) x_i^t (t) \right) = p_i \, dx_i^t + x_i^t \, dp_i + q_{i\alpha} \sigma_i^t \delta^{ab} c_{\alpha} (t) \, dt. \]

This equality can be called the stochastic differentiation formula for the interior product.
6.2 Adjoint multitime stochastic Pfaff systems

For simplicity, we will omit $\omega$ as argument of processes.

**Definition 6.3** (Variational multitime stochastic Pfaff system) Let $u(\cdot, \omega)$ be an admissible control. Let

$$dx^i_t = \mu^i_{\alpha}(t, x_t, u_t) \, dt^\alpha + \sigma^i_{\alpha}(t, x_t, u_t) \, dW^\alpha_t$$

be a multitime stochastic Pfaff evolution. The multitime stochastic system

$$d\xi^i_t = \left( \mu^i_{\alpha j}(t, x_t, u_t) \, dt^\alpha + \sigma^i_{\alpha j}(t, x_t, u_t) \, dW^\alpha_t \right) \xi^j(t, \omega)$$

is called stochastic variational multitime Pfaff system with control $u(\cdot, \omega)$.

**Definition 6.4** (Adjoint multitime stochastic Pfaff system) Consider a multitime stochastic Pfaff evolution as in (6.2). A linear multitime stochastic system of the form

$$dp^i_j(t) = \left( a^i_{\alpha j}(t, x_t, u_t) \, dt^\alpha + q^i_{\beta j}(t, x_t, u_t) \, dW^\beta_t \right) p^i(t), \quad b = 1, d, \quad i, j = 1, n$$

is called adjoint multitime stochastic Pfaff system if the interior product $p^i_k(t) \xi^k(t)$ is a global multitime stochastic first integral.

**Theorem 6.5** The multitime stochastic Pfaff system

$$dp^i_j(t, \omega) = \left( \mu_{\alpha j}^i(t, x_t, u_t) \, dt^\alpha + q_{\beta j}^i(t, x_t, u_t) \, dW^\beta_t \right) p^i(t), \quad i, j = 1, n$$

is the adjoint multitime stochastic Pfaff system with respect to the variational multitime stochastic Pfaff system.

**Proof.** Let

$$d\xi^i_t = \left( \mu^i_{\alpha j}(t, x_t, u_t) \, dt^\alpha + \sigma^i_{\alpha j}(t, x_t, u_t) \, dW^\alpha_t \right) \xi^j(t, \omega)$$

be the variational stochastic multitime Pfaff system. Denote the adjoint multitime stochastic Pfaff system by

$$dp^i_j(t) = \left( a^i_{\alpha j}(t, x_t, u_t) \, dt^\alpha + q^i_{\beta j}(t, x_t, u_t) \, dW^\beta_t \right) p^i(t), \quad i, j = 1, n.$$ 

We determine the coefficients $a^i_{\alpha j}$ and $q^i_{\beta j}$ such that $p^i_k(t) \xi^k(t)$ to be a multitime stochastic first integral, i.e.,

$$d \left( p^i_k(t) \xi^k(t) \right) = 0,$$

where $d$ is the stochastic differential. Imposing the identity

$$p^i_k(t) \xi^k(t) = p^i_k(0) \xi^k(0), \quad \text{for any } t \in \Omega_{0T},$$

for $i, k = 1, n$. 

16
or

\[ 0 = p_i (t) \xi^j (t) \approx_k (t) (\mu_{i, t} (t, x_t, u_t) + a_{i, j}^j (t, x_t, u_t) + \int q_{a, k}^j (t, x_t, u_t) \sigma_{a, k}^j (t, x_t, u_t) \sigma_{a, c}^j (t) dt + p_i (t) \xi^j (t) (\sigma_{i, x}^j (t, x_t, u_t) + q_{a, i}^j (t, x_t, u_t)) dW_0^a, \]

we obtain

\[ a_{i, j}^j (t, x_t, u_t) = -\mu_{i, x}^j (t, x_t, u_t) - q_{a, k}^j (t, x_t, u_t) \sigma_{a, x} (t), \]

\[ q_{a, j}^j (t, x_t, u_t) = -\sigma_{a, x}^j (t, x_t, u_t). \]

### 7 Optimization problems with \(\) stochastic curvilinear integral functionals

**Multitime stochastic optimal control problems with terminal conditions** have some features: there is a **multitime diffusion system**, which is described by a multitime Itô - Pfaff stochastic differential system; there are some **constraints** that the decisions and/or the state are subject to; there is a **criterion** that measures the performance of the decisions. The goal is to **optimize** the criterion by selecting a **nonanticipative** decision among the ones satisfying all the constraints.

#### 7.1 Multitime stochastic maximum principle

The idea to use curvilinear integrals in stochastic control theory is very recent \[12, 6\]. Our paper \[12\] refers to multitime stochastic maximum principle, Itô - Udrisë formulas and Hamilton-Jacobi-Bellman approach, while the paper \[6\] obtains a stochastic curvilinear integral via Hamilton-Jacobi-Bellman approach. Also, some geometrical methods used in the single-time stochastic theory (see \[5\]), can be extended to multitime stochastic techniques.

The cost functionals of stochastic economical and/or mechanical work type are very important for applications. This motivates to solve the multitime stochastic optimal control problem

\[
\max_{u(t)} \mathbb{E} \left[ \int_{\gamma T} f_n (t, x_t, u_t) dt + \Psi (x (T, \omega)) \right]
\]

subject to a multitime Itô process constraint

\[
dx_i = \mu_{i, a} (t, x_t, u_t) dt + \sigma_{i, a} (t, x_t, u_t) dW_i^a, \quad x(0) = x_0, \quad x(T) = x_T.
\]

where \( x_i = (x_t)_{t=0}^T \) is the **multitime state variable**, \( u(t) \in U, \forall t \in \Omega_0 \) is the **multitime closed-loop control variable**, \( W_t = (W_t^1, ..., W_t^d) \) is a **standard multitime Wiener process**.
The problem (7.1)-(7.2) can often be handled by applying a kind of “Lagrangian multiplier” method, as in holonomic and nonholonomic approach of deterministic case [10], respectively, [17]. The stochastic running cost \( \eta = f_\alpha (t, x_t, u_t) \, dt^\alpha \) is a stochastic Lagrangian 1-form. We introduce the stochastic Lagrange multiplier \( p_i (t, \omega) \in L^2_T (\Omega_{0T}, \mathbb{R}^n), i = 1, \ldots, n \), where \( L^2_T (\Omega_{0T}, \mathbb{R}^n) \) is the space of all \( \mathbb{R}^n \)-valued adapted processes \((\phi(t))_{t \in \Omega_{0T}}\) such that

\[
\mathbb{E} \left[ \int_{\gamma_{0T}} \| \phi (s(\tau), x_{s(\tau)}) \| \, d\tau \right] < \infty \text{ for all } t \in \Omega_{0T},
\]

where \( \tau \) is the curvilinear abscissa on the increasing curve \( \gamma_{0T}, t \in \Omega_{0T} \). To use geometrical methods in control theory, let us suppose \((p_i (t, \omega))_{t \in \Omega_{0T}} \) as a multitime Itô process, i.e.,

\[
dp_i (t) = a_{ia} (t, x_t, u_t) \, dt^\alpha + q_{ia} (t, x_t, u_t) \, dW_t^a,
\]

where \([a_{ia} (t, x_t, u_t)]_{t \in \Omega_{0T}} \) and \([q_{ia} (t, x_t, u_t)]_{t \in \Omega_{0T}} \) are matrices of previsible processes. The adjoint process \((p_i (t, \omega))_{t \in \Omega_{0T}} \) is required to be \((\mathcal{F}_t)_{t \in \Omega_{0T}} \)-adapted, for any \( t \in \Omega_{0T} \).

To solve the foregoing problem, we use the stochastic Lagrangian 1-form

\[
\mathcal{L} (t, x_t, u_t, p_t) = f_\alpha (t, x_t, u_t) \, dt^\alpha + p_t \left[ \mu_i^\alpha (t, x_t, u_t) \, dt^\alpha + \sigma_i^\alpha (t, x_t, u_t) \, dW_t^a - dx_t^i \right].
\]

The foregoing multitime stochastic optimization problem, constrained by a contact distribution, with stochastic perturbations, (7.1)-(7.2) can be change into another free multitime stochastic optimization problem

\[
\max_{u(\cdot, \omega) \in \mathcal{A}_{ad}} \mathbb{E} \left[ \int_{\gamma_{0T}} \mathcal{L} (t, x_t, u_t, p_t) + \Psi (x (T, \omega)) \right],
\]

subject to

\[
p (t, \omega) \in \mathcal{P}, \forall t \in \Omega_{0T}, \ x (0, \omega) = x_0 \in \mathbb{R}^n,
\]

where the set \( \mathcal{P} \) will be defined as the set of adjoint multitime stochastic processes in an appropriate context. The problem (7.3) can be rewritten as

\[
\max_{u(\cdot, \omega) \in \mathcal{A}_{ad}} \mathbb{E} \left\{ \int_{\gamma_{0T}} [f_\alpha (t, x_t, u_t) + p_t (t) \mu_i^\alpha (t, x_t, u_t)] \, dt^\alpha \right. \]

\[
+ \left. \int_{\gamma_{0T}} p_t (t) \sigma^i_\alpha (t, x_t, u_t) \, dW_t^a - \int_{\gamma_{0T}} p_t (t) \, dx_t^i + \Psi (x (T, \omega)) \right\},
\]

subject to

\[
p (t, \omega) \in \mathcal{P}, \forall t \in \Omega_{0T}, \ x (0, \omega) = x_0 \in \mathbb{R}^n, \ i = 1, \ldots, n.
\]

Due to properties of stochastic curvilinear integrals (Remark 4.3), the terms containing \( dW_t^a \) vanish. Evaluating \( \int_{\gamma_{0T}} p_t (t, \omega) \, dx_t^i \), via multitime stochastic
integration by parts, it appears the control Hamiltonian multitime stochastic \( 1 \)-form
\[
\mathcal{H}(t, x_t, u_t, p_t) = (f^i(t, x_t, u_t) + p_i(t) \mu^i_\alpha(t, x_t, u_t))
- p_i(t) \sigma^i_{x} \delta^{ab} c_{\alpha}(t) dt^\alpha.
\]
It verifies a modified multitime stochastic Legendrian duality, i.e.,
\[
\mathcal{H} = \mathcal{L} + p_i(t) dx^i_t - p_i(t) \sigma^i_{\alpha x} \delta^{ab} c_{\alpha}(t) dt^\alpha
- p_i(t) \sigma^i_{x} dW^a_t.
\]

The key point to the derivation of the necessary conditions of optimality is that the multitime Legendre stochastic transformation of the multitime Lagrangian stochastic 1-form to be minimized into a multitime Hamiltonian stochastic 1-form converts a functional minimization problem into a static optimization problem on the 1-form \( \mathcal{H}(t, x_t, u_t, p_t) \).

**Theorem 7.1 (Stochastic maximum principle)** We assume the conditions H1 – H3. Suppose that the problem of maximizing the functional (7.1) constrained by (7.2) over \( \mathcal{A}_{ad} \) has an interior optimal solution \( u^*(t) \), which determines the multitime stochastic optimal evolution \( x(t) \). Let \( \mathcal{H} \) be the Hamiltonian multitime stochastic \( 1 \)-form. Then there exists an adapted process \( (p(t, \omega))_{t \in \Omega_0T} \) (adjoint process) satisfying:

(i) the initial multitime stochastic differential system,
\[
dx^i(t) = \frac{\partial \mathcal{H}}{\partial p_i}(t, x_t, u^*_t, p_t) + \sigma^i_{\alpha x} \delta^{ab} c_{\alpha}(t) dt^\alpha
+ \sigma^i_{x} dW^a_t;
\]

(ii) the linear multitime stochastic adjoint system,
\[
dp_i(t) = - \mathcal{H}_{x^i}(t, x_t, u^*_t, p_t) - p_j(t) \sigma^j_{\alpha x} \delta^{ab} c_{\alpha}(t) dW^a_t,
p_i(T) = \Psi_{x^i}(x_T), \ i = 1, n;
\]

(iii) the critical point condition,
\[
\mathcal{H}_{u^c}(t, x_t, u^*_t, p_t) = 0, \ c = 1, k.
\]

**Proof.** In the whole this proof, we will omit \( \omega \) as argument of processes. Suppose that there exists a continuous control \( u^*(t, \omega) \) over the admissible controls in \( \mathcal{A}_{ad} \), which is an optimum point in the previous problem. Consider a variation
\[
u(t, \varepsilon) = u^*(t) + \varepsilon h(t),
\]
where, by hypothesis, \( h(t) \) is an arbitrary multitime stochastic process. Since \( u^*(t, \omega) \in \mathcal{A}_{ad} \) and a continuous function over a compact set \( \Omega_0T \) is bounded, there exists a number \( \varepsilon_h > 0 \) such that
\[
u(t, \varepsilon) = u^*(t) + \varepsilon h(t) \in \mathcal{A}_{ad}, \ \forall |\varepsilon| < \varepsilon_h.
\]
This $\varepsilon$ is used in our variational arguments. 

Now, let us define the contact distribution with stochastic multitime perturbations, corresponding to the control variable $u(t, \varepsilon)$, i.e.,

$$dx^i(t, \varepsilon) = \mu^i_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dt^\alpha + \sigma^i_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dW^\alpha_t,$$

for $i = 1, n$, or,

$$x^i(t, \varepsilon) = x^i(0, \varepsilon) + \int_{\gamma_0}^{t} \mu^i_\alpha(s, x(s, \varepsilon), u(s, \varepsilon)) ds^\alpha$$

$$+ \int_{\gamma_0}^{t} \sigma^i_\alpha(s, x(s, \varepsilon), u(s, \varepsilon)) dW^\alpha_s, \quad \forall t \in \Omega_0T, \quad \forall i = 1, n$$

and $x(0, \varepsilon) = x_0 \in \mathbb{R}^n$. For $|\varepsilon| < \varepsilon_h$, we define the function

$$J(\varepsilon) = \mathbb{E}\left[ \int_{\gamma_0T} f_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dt^\alpha + \Psi(x(T, \varepsilon)) \right].$$

For any adapted process $(p_t)_{t \in \Omega_0T}$, we have

$$\int_{\gamma_0T} p_i(t) \left[ \mu^i_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dt^\alpha - dx^i(t, \varepsilon) \right]$$

$$+ \int_{\gamma_0T} p_i(t) \sigma^i_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dW^\alpha_t = 0, \quad i = 1, n.$$

To solve the foregoing constrained multitime stochastic optimization problem, we transform it into a free multitime stochastic optimization problem (16). For that, we use the Lagrange multitime stochastic 1-form which includes the variations

$$\mathcal{L}(t, x(t, \varepsilon), u(t, \varepsilon), p_t) = f_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dt^\alpha$$

$$+ p_i(t) \left[ \mu^i_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dt^\alpha + \sigma^i_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) dW^\alpha_t - dx^i(t, \varepsilon) \right],$$

where $i = 1, n$. We have to optimize now the function

$$\tilde{J}(\varepsilon) = \mathbb{E}\left[ \int_{\gamma_0T} \mathcal{L}(t, x(t, \varepsilon), u(t, \varepsilon), p_t) + \Psi(x(T, \varepsilon)) \right],$$

with doubt any constraints. If the control $u^*(t)$ is optimal, then

$$\tilde{J}(\varepsilon) \leq \tilde{J}(0), \quad \forall |\varepsilon| < \varepsilon_h.$$ 

Explicitly,

$$\tilde{J}(\varepsilon) = \mathbb{E}\left[ \int_{\gamma_0T} \left[ f_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) + p_i(t) \mu^i_\alpha(t, x(t, \varepsilon), u(t, \varepsilon)) \right] dt^\alpha \right]$$

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\[
-\mathbb{E}\left[\int_{\gamma_{0T}} p_i(t,\omega) \, dx^i(t,\omega,\varepsilon) + \Psi(x(T,\varepsilon))\right], \quad i = 1, n.
\]

To evaluate the integral

\[
\int_{\gamma_{0T}} p_i(t) \, dx^i(t,\varepsilon),
\]

we integrate by parts, via Itô-Udrițe product formula. Taking into account that \((p_t)_{t\in\Omega_{0T}}\) is a multitime Itô process, we obtain

\[
\tilde{J}(\varepsilon) = \mathbb{E}\left[\int_{\gamma_{0T}} f_\alpha(t,x(t,\varepsilon),u(t,\varepsilon)) + p_i(t) \mu^i_\alpha(t,x(t,\varepsilon),u(t,\varepsilon)) \right] dt^\alpha
\]

\[
-\mathbb{E}\left[ p_i(t) x^i(t,\varepsilon)\big|_0^T - \int_{\gamma_{0T}} x^i(t,\varepsilon) \, dp_i(t) \right] + \mathbb{E}\Psi(x(T,\varepsilon))
\]

\[
-\mathbb{E}\int_{\gamma_{0T}} p_j(t) \sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon)) \sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon)) \delta^{ab} c_\alpha dt^\alpha,
\]

with \(\sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon))\big|_{\varepsilon=0} = \sigma^j_{ax^i}(t,x_t,u_t)\), for all \(i,j = 1, n\) and \(a = 1, d\) and where \(\gamma_{0T}\) is an inverse image of the projection \(\pi \circ \gamma_{0T} = \gamma_{0T}\). Then,

\[
\tilde{J}(\varepsilon) = \mathbb{E}\left[\int_{\gamma_{0T}} f_\alpha(t,x(t,\varepsilon),u(t,\varepsilon)) \right] dt^\alpha
\]

\[
+\mathbb{E}\int_{\gamma_{0T}} [p_i(t) \mu^i_\alpha(t,x(t,\varepsilon),u(t,\varepsilon))
\]

\[
-p_j(t) \sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon)) \sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon)) \delta^{ab} c_\alpha \right] dt^\gamma
\]

\[
+\mathbb{E}\left[\int_{\gamma_{0T}} x^i(t,\varepsilon) \, dp_i(t) - p_i(t) x^i(t,\varepsilon)\big|_0^T \right] + \mathbb{E}\Psi(x(T,\varepsilon)).
\]

Differentiating with respect to \(\varepsilon\) (and this is possible, because the derivative with respect to \(\varepsilon\), for \(\varepsilon = 0\), exists in mean square sense; see, for example [21]), it follows

\[
\tilde{J}'(\varepsilon) = \mathbb{E}\int_{\gamma_{0T}} \{(f_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon)) + p_i(t) \mu^i_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon))
\]

\[
-p_j(t) \sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon)) \sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon)) \delta^{ab} c_\alpha 
\]

\[
-p_j(t) \sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon)) \sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon)) \delta^{ab} c_\alpha \right] dt^\alpha + dp_k(t) \} x^k_c
\]

\[
+\mathbb{E}\int_{\gamma_{0T}} [f_{au^c}(t,x(t,\varepsilon),u(t,\varepsilon)) + p_i(t) \mu^i_{au^c}(t,x(t,\varepsilon),u(t,\varepsilon))
\]

\[
-p_j(t) \sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon)) \sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon)) \delta^{ab} c_\alpha \right] dt^\gamma
\]

\[
-\mathbb{E}\left[\int_{\gamma_{0T}} x^i(t,\varepsilon) \, dp_i(t) - p_i(t) x^i(t,\varepsilon)\big|_0^T \right] + \mathbb{E}\Psi(x(T,\varepsilon)).
\]
\[-p_j(t) \sigma^j_{\alpha x^i}(t, x(t, \varepsilon), u(t, \varepsilon)) \delta^{ab} \sigma^b_{\alpha x^c}(t, x(t, \varepsilon), u(t, \varepsilon)) \delta^{ab} c_a] h^c(t) \ dt^a + \mathbb{E} \Psi_{x^k}(x(T), \varepsilon) x^k(T, \varepsilon), c = 1, 2, k.\]

Evaluating at \( \varepsilon = 0 \), we find
\[
\mathcal{J}'(0) = \mathbb{E} \int_{\gamma_0T} \left\{ \left[ \int_{\gamma_0T} \left[ f_{\alpha x^k}(t, x^k(t, u^k_t)) + p_i(t) \mu^i_{\alpha x^k}(t, x^k(t, u^k_t)) + \mathbb{E} \Psi_{x^k}(x(T), \varepsilon) x^k(T, \varepsilon), c = 1, 2, k. \right] \right] \left[ f_{\alpha x^k}(t, x^k(t, u^k_t)) + p_i(t) \mu^i_{\alpha x^k}(t, x^k(t, u^k_t)) + \mathbb{E} \Psi_{x^k}(x(T), \varepsilon) x^k(T, \varepsilon), c = 1, 2, k. \right] \right\} x^k(t, 0) \]
\[
- p_j(t) \sigma^j_{\alpha x^i}(t, x(t, \varepsilon), u(t, \varepsilon)) \delta^{ab} \sigma^b_{\alpha x^c}(t, x(t, \varepsilon), u(t, \varepsilon)) \delta^{ab} c_a] h^c(t) \ dt^a + \mathbb{E} \Psi_{x^k}(x(T), \varepsilon) x^k(T, \varepsilon), c = 1, 2, k.\]

where \( x(t) \) is the multitime state variable corresponding to the optimal control \( u^*(t) \).

We need \( J'(0) = 0 \) for all \( h(t) = (h^c(t))_{c=1,2} \). On the other hand, the functions \( x^j(t, 0) \) are involved in the Cauchy problem
\[
dx^i_{x^j}(t, 0) = \left( \mu^i_{\alpha x^j}(t, x(0, 0), u(t, 0)) \right) dt^a + \sigma^j_{\alpha x^i}(t, x^i(t, 0)) dW^a_t x^j(t, 0) \]
\[
+ \left( \mu^i_{\alpha x^j}(t, x(0, 0), u(t, 0)) \right) dt^a + \sigma^j_{\alpha x^i}(t, x^i(t, 0)) dW^a_t h^c(t), \]
\[
t \in \Omega_0 T, \quad x^j(0, 0) = 0 \in \mathbb{R}^n\]

and hence they depend on \( h(t) \). The functions \( x^j_{x^k}(t, 0) \) are eliminated by selecting \( \mathcal{P} \) as the adjoint contact distribution
\[
dp_k(t) = -[f_{\alpha x^k}(t, x^k(t, u^k_t)) + p_i(t) \mu^i_{\alpha x^k}(t, x^k(t, u^k_t)) - p_j(t) \sigma^j_{\alpha x^i}(t, x^i(t, u^i_t)) \sigma^b_{\alpha x^c}(t, x^c(t, u^c_t)) \delta^{ab} c_a] h^c(t) \ dt^a \]
\[
- [p_j(t) \sigma^j_{\alpha x^i}(t, x^i(t, u^i_t)) \sigma^b_{\alpha x^c}(t, x^c(t, u^c_t)) \delta^{ab} c_a] dt^a + dp_k(t) dW^a_t, \]
for any multitime \( t \in \Omega_0 T \), with stochastic perturbations terminal value problem (see, e.g. [13])
\[
p_k(T) = \Psi_{x^k}(x_T), \quad k = 1, n.\]

The relation \( (7.3) \) shows that
\[
a_{ia}(t, x^i(t, u^i_t)) = -[f_{\alpha x^k}(t, x^k(t, u^k_t)) - p_i(t) \mu^i_{\alpha x^k}(t, x^k(t, u^k_t)) + [p_j(t) \sigma^j_{\alpha x^i}(t, x^i(t, u^i_t)) \sigma^b_{\alpha x^c}(t, x^c(t, u^c_t)) c_a] + p_j(t) \sigma^j_{\alpha x^i}(t, x^i(t, u^i_t)) \sigma^b_{\alpha x^c}(t, x^c(t, u^c_t)) c_a] \delta^{ab}.\]
It follows
\[
dp_i(t, \omega) = -H_{x^i}(t, x^t, u^*_t, p_t) - p_j(t) \sigma_{ax^j}(t, x^t, u^*_t) dW^a_t,
\]
\[\forall t \in \Omega_{0T}, i = 1, n,\]
and
\[H_{u^c}(t, x^t, u^*_t, p_t) = 0, \forall t \in \Omega_{0T}, \text{ for all } c = 1, k.\]

Moreover,
\[
dx^i(t) = \frac{\partial H}{\partial p_i}(t, x^t, u^*_t, p_t) + \sigma_{ax^j}(t, x^t, u^*_t) \sigma_{b}(t, x^t, u^*_t) \delta^{ab}_{c} dt^\alpha
+ \sigma_{b}(t, x^t, u^*_t) dW^a_t, \forall t \in \Omega_{0T}, x(0) = x_0, \forall i = 1, n.
\]

Example (Deterministic problem, continuous two-time version, see [19]) A mine owner must decide at what rate to extract a complex ore from his mine. He owns rights to the ore from two-time date \(0 = (0^0, 0^1)\) to two-time date \(T = (T^0, T^1)\). A two-time date can be the pair (date, useful component frequency). At two-time date \(0\) there is \(x_0 = (x_0^i)\) ore in the ground, and the instantaneous stock of ore \(x^i(t) = (x^i(t))\) declines at the rate \(u^i(t) = (u^i(t))\) the mine owner extracts it. The mine owner extracts ore at cost \(q_i u^i(t)\) and sells ore at a constant price \(p = (p_i)\). He does not value the ore remaining in the ground at time \(T\) (there is no "scrap value"). He chooses the rate \(u^i(t)\) of extraction in two-time to maximize profits over the period of ownership with no two-time discounting.

**Stochastic model** Since there must be some risk in the investment, the deterministic evolution \(\frac{\partial x}{\partial t^\gamma}(t) = -u^\gamma(t)\) can be changed into a stochastic model. Hence, the manager want to maximizes the profit (curvilinear integral)
\[
P(u(\cdot)) = \int_{\gamma_{0T}} \left( p_i u^i_\alpha(t) - q_i u^i_\alpha(t)^2 \frac{1}{x^i(t)} \right) dt^\alpha
\]
subject to the stochastic law of evolution
\[
\frac{\partial x}{\partial t^\gamma}(t) = -u^\gamma(t) + W^\gamma_t,
\]
where \(W_t = (W^\gamma_t)\) is a matrix Wiener process. Form the Hamiltonian 1-form
\[
\mathcal{H} = \left( p_i u^i_\alpha(t) - q_i u^i_\alpha(t)^2 \frac{1}{x^i(t)} - \lambda_i(t) u^i_\alpha(t) \right) dt^\alpha,
\]
differentiate and write the equations
\[
\frac{\partial \mathcal{H}}{\partial u^i_\beta} = \left( p_i - 2q_i \frac{u^i_\alpha(t)}{x^i(t)} - \lambda_i \right) \delta^i_\alpha dt^\alpha = 0, \text{ no sum after the index } i;
\]
23
\[
d\lambda_i(t) = -\frac{\partial H}{\partial x^i} = -q_i \left( \frac{u^i_\alpha(t)}{x^i(t)} \right)^2 dt^\alpha, \text{ no sum after the index } i.
\]

As the mine owner does not value the ore remaining at time \( T \), we have \( \lambda_i(T) = 0 \). Using the above equations, it is easy to solve for the differential equations governing the control vector \( u(t) \) and the dual vector \( \lambda_i(t) \):

\[
2q_i \frac{u^i_\alpha(t)}{x^i(t)} = p_i - \lambda_i(t), \quad \frac{\partial \lambda_i}{\partial t^\alpha} = -q_i \left( \frac{u^i_\alpha(t)}{x^i(t)} \right)^2 \text{ no sum after } i
\]

and using the initial and turn-\( T \) conditions, the equations can be solved numerically.

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