Quotient Complexities of Atoms of Regular Languages

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Abstract. An atom of a regular language $L$ with $n$ (left) quotients is a non-empty intersection of uncomplemented or complemented quotients of $L$, where each of the $n$ quotients appears in a term of the intersection. The quotient complexity of $L$, which is the same as the state complexity of $L$, is the number of quotients of $L$. We prove that, for any language $L$ with quotient complexity $n$, the quotient complexity of any atom of $L$ with $r$ complemented quotients has an upper bound of $2^n - 1$ if $r = 0$ or $r = n$, and $1 + \sum_{k=1}^{r} \binom{n-r+1}{k} \cdot \binom{h}{k}$ otherwise, where $\binom{r}{j}$ is the binomial coefficient. For each $n \geq 1$, we exhibit a language whose atoms meet these bounds.

1 Introduction

Atoms of regular languages were introduced in 2011 by Brzozowski and Tamm [3]; we briefly state their main properties here.

The \textit{(left) quotient} of a regular language $L$ over an alphabet $\Sigma$ by a word $w \in \Sigma^*$ is the language $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$. It is well known that a language $L$ is regular if and only if it has a finite number of distinct quotients, and that the number of states in the minimal deterministic finite automaton (DFA) recognizing $L$ is precisely the number of distinct quotients of $L$. Also, $L$ is its own quotient by the empty word $\varepsilon$, that is $\varepsilon^{-1}L = L$. Note too that the quotient by $u \in \Sigma^*$ of the quotient by $w \in \Sigma^*$ of $L$ is the quotient by $wu$ of $L$, that is, $u^{-1}(w^{-1}L) = (wu)^{-1}L$.

An \textit{atom}\textsuperscript{3} of a regular language $L$ with quotients $K_0, \ldots, K_{n-1}$ is any non-empty language of the form $\bar{K}_0 \cap \cdots \cap \bar{K}_{n-1}$, where $\bar{K}_i$ is either $K_i$ or $\overline{K_i}$, and

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\textsuperscript{3} The definition in [3] does not consider the intersection of all the complemented quotients to be an atom. Our new definition adds symmetry to the theory.
$K_i$ is the complement of $K_i$ with respect to $\Sigma^*$. Thus atoms of $L$ are regular languages uniquely determined by $L$ and they define a partition of $\Sigma^*$. They are pairwise disjoint, every quotient of $L$ (including $L$ itself) is a union of atoms, and every quotient of an atom is a union of atoms. Thus the atoms of a regular language are its basic building blocks. Also, $\overline{L}$ defines the same atoms as $L$.

The quotient complexity \cite{2} of $L$ is the number of quotients of $L$, and this is the same number as the number of states in the minimal DFA recognizing $L$; the latter number is known as the state complexity \cite{8} of $L$. Quotient complexity allows us to use language-theoretic methods, whereas state complexity is more amenable to automaton-theoretic techniques. We use one of these two points of view or the other, depending on convenience.

We study the quotient complexity of atoms of regular languages. Suppose that $L \subseteq \Sigma^*$ is a non-empty regular language and its set of quotients is $K = \{K_0, K_1, \ldots, K_{n-1}\}$, with $n \geq 1$. Our main result is the following:

**Theorem 1 (Main Result).**

For $n \geq 1$, the quotient complexity of the atoms with 0 or $n$ complemented quotients is less than or equal to $2^n - 1$. For $n \geq 2$ and $r$ satisfying $1 \leq r \leq n-1$, the quotient complexity of any atom of $L$ with $r$ complemented quotients is less than or equal to

$$f(n, r) = 1 + \sum_{k=1}^{r} \left( \sum_{h=k+1}^{n-r} C_n^h \cdot C_k^h \right).$$

For $n = 1$, the single atom $\Sigma^*$ of the language $\Sigma^*$ or $\emptyset$ meets the bound 1. Moreover, for $n \geq 2$, all the atoms of the language $L_n$ recognized by the DFA $D_n$ of Figure 1 meet these bounds.

![Fig. 1. DFA $D_n$ of language $L_n$ whose atoms meet the bounds.](image)

In Section 2 we derive upper bounds on the quotient complexities of atoms. In Section 3 we define our notation and terminology for automata, and present the definition of the átomaton \cite{3} of a regular language; this is a nondeterministic finite automaton (NFA) whose states are the atoms of the language. We also provide a different characterization of the átomaton. We introduce a class of DFA’s in Section 4 and study the átomata of their languages. We then prove in Section 5 that the atoms of these languages meet the quotient complexity bounds. Section 6 concludes the paper.
2 Upper Bounds on the Quotient Complexities of Atoms

We first derive upper bounds on the quotient complexity of atoms. We use quotients here, since they are convenient for this task. First we deal with the two atoms that have only uncomplemented or only complemented quotients.

**Proposition 1 (Atoms with 0 or \(n\) Complemented Quotients).**

For \(n \geq 1\), the quotient complexity of the two atoms \(A_K = K_0 \cap \cdots \cap K_{n-1}\) and \(A_\emptyset = K_0 \cap \cdots \cap K_{n-1}\) is less than or equal to \(2^n - 1\).

**Proof.** Every quotient \(w^{-1}A_K\) of atom \(A_K\) is the intersection of languages \(w^{-1}K_i\), which are quotients of \(L\):

\[
w^{-1}A_K = w^{-1}(K_0 \cap \cdots \cap K_{n-1}) = w^{-1}K_0 \cap \cdots \cap w^{-1}K_{n-1}.
\]

Since these quotients of \(L\) need not be distinct, \(w^{-1}A_K\) may be the intersection of any non-empty subset of quotients of \(L\). Hence \(A_K\) can have at most \(2^n - 1\) quotients.

The argument for the atom \(A_\emptyset = K_0 \cap \cdots \cap K_{n-1}\) with \(n\) complemented quotients is similar, since \(w^{-1}K_i = w^{-1}K_i\). \(\square\)

Next, we present an upper bound on the quotient complexity of any atom with at least one and fewer than \(n\) complemented quotients.

**Proposition 2 (Atoms with \(r\) Complemented Quotients, \(1 \leq r \leq n-1\)).**

For \(n \geq 2\) and \(1 \leq r \leq n-1\), the quotient complexity of any atom with \(r\) complemented quotients is less than or equal to

\[
f(n, r) = 1 + \sum_{k=1}^{r} \sum_{h=k+1}^{k+n-r} C^n_k \cdot C^h_k,
\]

where \(C_j^i\) is the binomial coefficient “\(i\) choose \(j\).”

**Proof.** Consider an intersection of complemented and uncomplemented quotients that constitutes an atom. Without loss of generality, we arrange the terms in the intersection in such a way that all complemented quotients appear on the right. Thus let \(A_i = K_0 \cap \cdots \cap K_{n-r-1} \cap \overline{K_{n-r}} \cap \cdots \cap \overline{K_{n-1}}\) be an atom of \(L\) with \(r\) complemented quotients of \(L\), where \(1 \leq r \leq n-1\). The quotient of \(A_i\) by any word \(w \in \Sigma^*\) is

\[
w^{-1}A_i = w^{-1}(K_0 \cap \cdots \cap K_{n-r-1} \cap \overline{K_{n-r}} \cap \cdots \cap \overline{K_{n-1}})
= w^{-1}K_0 \cap \cdots \cap w^{-1}K_{n-r-1} \cap w^{-1}K_{n-r} \cap \cdots \cap w^{-1}K_{n-1}.
\]

Since each quotient \(w^{-1}K_j\) is a quotient, say \(K_{i_j}\), of \(L\), we have

\[
w^{-1}A_i = K_{i_0} \cap \cdots \cap K_{i_{n-r-1}} \cap \overline{K_{i_{n-r}}} \cap \cdots \cap \overline{K_{i_{n-1}}}.
\]

The cardinality of a set \(S\) is denoted by \(|S|\). Let the set of distinct quotients of \(L\) appearing in \(w^{-1}A_i\) uncomplemented (respectively, complemented) be \(X\)
(respectively, \(Y\)), where \(1 \leq |X| \leq n - r\) and \(1 \leq |Y| \leq r\). If \(X \cap Y = \emptyset\), then \(w^{-1}A_i = \emptyset\). Therefore assume that \(X \cap Y = \emptyset\), and that \(|X \cup Y| = h\), where \(2 \leq h \leq n\); there are \(C^n_h\) such sets \(X \cup Y\). Suppose further that \(|Y| = k\), where \(1 \leq k \leq r\). There are \(C^n_k\) ways of choosing \(Y\). Hence there are at most \(\sum_{h=k+1}^{k+n-r} C^n_h \cdot C^n_k\) distinct intersections with \(k\) complemented quotients. Thus, the total number of intersections of uncomplemented and complemented quotients can be at most \(\sum_{k=1}^{k+n-r} C^n_h \cdot C^n_k\).

Adding 1 for the empty quotient of \(w^{-1}A_i\), we get the required bound. \(\square\)

We now consider the properties of the function \(f(n, r)\).

**Proposition 3 (Properties of Bounds).** For any \(n \geq 2\) and \(1 \leq r \leq n - 1\),

1. \(f(n, r) = f(n, n - r)\).
2. For a fixed \(n\), the maximal value of \(f(n, r)\) occurs when \(r = \lfloor n/2 \rfloor\).

**Proof.** Since \(f(n, r) = 1 + \sum_{k=1}^{r} \sum_{h=k+1}^{k+n-r} C^n_h \cdot C^n_k\), and the following equations hold:

\[
\sum_{k=1}^{r} \sum_{h=k+1}^{k+n-r} C^n_h \cdot C^n_k = \sum_{h=k+1}^{r} \sum_{l=1}^{n-r} C^n_h \cdot C^{k+l}_k
\]

there are at most \(\sum_{k=1}^{k+n-r} C^n_h \cdot C^n_k\) distinct intersections with \(k\) complemented quotients. Thus, the total number of intersections of uncomplemented and complemented quotients can be at most \(\sum_{k=1}^{k+n-r} C^n_h \cdot C^n_k\).

Adding 1 for the empty quotient of \(w^{-1}A_i\), we get the required bound. \(\square\)

For the second part, we will assume that \(1 \leq r < \lfloor n/2 \rfloor\), and show that \(f(n, r + 1) > f(n, r)\) for this case. We find \(f(n, r + 1) - f(n, r)\) as follows:

\[
f(n, r + 1) - f(n, r) = 1 + \sum_{k=1}^{r+1} \sum_{h=k+1}^{k+n-r-1} C^n_h \cdot C^n_k
\]

\[
= \sum_{h=r+1}^{n} C^n_h \cdot C^n_{r+1} - \sum_{k=1}^{r} \sum_{h=k+1}^{k+n-r} C^n_h \cdot C^n_k
\]

Since the first summation can be written as

\[
\sum_{h=r+2}^{n} C^n_h \cdot C^n_{r+1} = \sum_{h=r+2}^{n-r} C^n_h \cdot C^n_{r+1} + \sum_{h=n-r+1}^{n} C^n_h \cdot C^n_{r+1}
\]

\[
= \sum_{h=r+2}^{n} C^n_h \cdot C^n_{r+1} + \sum_{k=1}^{r} \sum_{h=k+n-r}^{k+n-r} C^n_h \cdot C^n_{k+n-r+1}
\]
we get

\[ f(n, r + 1) - f(n, r) = \sum_{h=r+2}^{n-r} C_n^h \cdot C_{r+1}^h + \sum_{k=1}^{r} C_n^{n-r} \cdot C_{r+1}^{k+n-r} - \sum_{k=1}^{r} C_{k+n-r}^n \cdot C_k^{k+n-r}. \]

Assuming \(1 \leq k \leq r\), we will show that \(C_{r+1}^{k+n-r} > C_k^{k+n-r}\). We can express the ratio \(C_{r+1}^{k+n-r} / C_k^{k+n-r}\) as follows:

\[
\frac{C_{r+1}^{k+n-r}}{C_k^{k+n-r}} = \frac{(k + n - r)!}{(r + 1)!(k + n - 2r - 1)!} \cdot \frac{(k + n - r)!}{k!(n - r)!} \\
= \frac{(k + n - r)!}{(r + 1)!(k + n - 2r - 1)!} \\
= \frac{k!(n - r) \cdots (n - 2r + k)(n - 2r + k - 1)!}{(r + 1) \cdots (k + 1)k!(n - 2r + k - 1)!} \\
= \frac{(n - r) \cdots (n - 2r + k)}{(r + 1) \cdots (k + 1)}.
\]

Note that there are \(r - k + 1\) factors both in the numerator and the denominator of the obtained fraction. Therefore, we can write

\[
\frac{C_{r+1}^{k+n-r}}{C_k^{k+n-r}} = \frac{n - r}{r + 1} \cdot \frac{n - r - 1}{r} \cdots \frac{n - 2r + k}{k + 1}.
\]

The condition \(1 \leq r < \lfloor n/2 \rfloor\) implies that \(n > 2r + 1\); consequently we have

\[ n - r > r + 1, \ n - r - 1 > r, \ldots, \ n - 2r + k > k + 1. \]

Therefore \(C_{r+1}^{k+n-r} / C_k^{k+n-r} > 1\), which implies that \(C_{r+1}^{k+n-r} > C_k^{k+n-r}\).

It follows that

\[
\sum_{k=1}^{r} C_{k+n-r}^n \cdot C_{r+1}^{k+n-r} > \sum_{k=1}^{r} C_k^{n-r} \cdot C_k^{k+n-r},
\]

and \(f(n, r + 1) - f(n, r) > 0\). So, if \(1 \leq r < \lfloor n/2 \rfloor\), then \(f(n, r + 1) > f(n, r)\).

Since \(f(n, r) = f(n, n - r)\), the maximum of \(f(n, r)\) occurs when \(r = \lfloor n/2 \rfloor\).

To better illustrate the properties of \(f(n, r)\), we derive explicit formulas for the first three values of \(r\). Using the well-known identity

\[
\sum_{h=k}^{n} C_n^h \cdot C_k^h = 2^{n-k}C_n^k, \quad (2)
\]
we find
\[
\begin{align*}
 f(n, 1) & = n2^{n-1} - n + 1, \\
 f(n, 2) & = n2^{n-1} - 2n + \frac{n(n-1)}{2} (2^{n-2} - 1) + 1, \\
 f(n, 3) & = n2^{n-1} - (n^2 + n) + \frac{n(n-1)(n+1)}{6} (2^{n-3} - 1) + 1.
\end{align*}
\]

Some numerical values of \( f(n, r) \) are shown in Table 1. The figures in boldface type are the maxima for a fixed \( n \). The row marked \( \text{max} \) shows the maximal quotient complexity of the atoms of \( L \). The row marked \( \text{ratio} \) shows the value of \( f(n, \lfloor n/2 \rfloor) / f(n - 1, \lfloor (n - 1)/2 \rfloor) \), for \( n \geq 2 \). It appears that this ratio converges to 3. For example, for \( n = 100 \) it is approximately 3.0002.

| \( n \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | \ldots |
|-------|----|----|----|----|----|----|----|----|----|-----|        |
| \( r=0 \) | 1  | 3  | 7  | 15 | 31 | 63 | 127 | 255 | 511 | 1,023 |        |
| \( r=1 \) | 1  | 3  | 10 | 29 | 76 | 187 | 442 | 1,017 | 2,296 | 5,111 |        |
| \( r=2 \) | *  | 3  | 10 | 43 | 141 | 406 | 1,086 | 2,773 | 6,859 | 16,576 |        |
| \( r=3 \) | *  | *  | 7  | 29 | 141 | 501 | 1,548 | 4,425 | 12,043 | 31,681 |        |
| \( r=4 \) | *  | *  | *  | 15 | 76 | 406 | 1,548 | 5,083 | 15,361 | 44,071 |        |
| \( r=5 \) | *  | *  | *  | *  | 31 | 187 | 1,086 | 4,425 | 15,361 | 48,733 |        |
| \( \text{max} \) | 1  | 3  | 10 | 43 | 141 | 501 | 1,548 | 5,083 | 15,361 | 48,733 |        |
| \( \text{ratio} \) | −  | 3  | 3.33 | 4.30 | 3.28 | 3.55 | 3.09 | 3.28 | 3.02 | 3.17 |        |

### 3 Automata and Átomata of Regular Languages

If \( \Sigma \) is a non-empty finite alphabet, then \( \Sigma^* \) is the free monoid generated by \( \Sigma \). A word is any element of \( \Sigma^* \), and the empty word is \( \varepsilon \). A language over \( \Sigma \) is any subset of \( \Sigma^* \). The reverse of a language \( L \) is denoted by \( L^R \) and defined as \( L^R = \{ w^R \mid w \in L \} \), where \( w^R \) is \( w \) spelled backwards.

A *nondeterministic finite automaton* (NFA) is a quintuple \( \mathcal{N} = (Q, \Sigma, \eta, I, F) \), where \( Q \) is a finite, non-empty set of states, \( \Sigma \) is a finite non-empty alphabet, \( \eta : Q \times \Sigma \to 2^Q \) is the transition function, \( I \subseteq Q \) is the set of initial states, and \( F \subseteq Q \) is the set of final states. As usual, we extend the transition function to functions \( \eta' : Q \times \Sigma^* \to 2^Q \), and \( \eta'' : 2^Q \times \Sigma^* \to 2^Q \). We do not distinguish these functions notationally, but use \( \eta \) for all three.

The *language accepted* by an NFA \( \mathcal{N} \) is \( L(\mathcal{N}) = \{ w \in \Sigma^* \mid \eta(I, w) \cap F \neq \emptyset \} \). Two NFA’s are equivalent if they accept the same language. The *right language* of a state \( q \) of \( \mathcal{N} \) is \( L_{q,F}(\mathcal{N}) = \{ w \in \Sigma^* \mid \eta(q, w) \cap F \neq \emptyset \} \). The *right language* of a set \( S \) of states of \( \mathcal{N} \) is \( L_{S,F}(\mathcal{N}) = \bigcup_{q \in S} L_{q,F}(\mathcal{N}) \); hence \( L(\mathcal{N}) = L_{I,F}(\mathcal{N}) \).
A state is *empty* if its right language is empty. Two states of an NFA are *equivalent* if their right languages are equal. The *left language* of a state $q$ of $\mathcal{N}$ is $L_{t,q} = \{w \in \Sigma^* \mid q \in \eta(I,w)\}$. A state is *unreachable* if its left language is empty. An NFA is *trim* if it has no empty or unreachable states.

A *deterministic finite automaton* (DFA) is a quintuple $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$, where $Q$, $\Sigma$, and $F$ are as in an NFA, $\delta : Q \times \Sigma \to Q$ is the transition function, and $q_0$ is the initial state. A DFA is an NFA in which the set of initial states is $\{q_0\}$ and the range of $\delta$ is restricted to singletons $\{q\}, q \in Q$. Note that an empty state of $\mathcal{N}$ is an unreachable state of $\mathcal{N}^\exists$ and vice versa.

We use the following operations on automata:

1. The *determinization* operation $\square$ applied to an NFA $\mathcal{N}$ yields a DFA $\mathcal{N}^\exists$ obtained by the well-known subset construction, where only subsets reachable from the initial subset of $\mathcal{N}^\exists$ are used and the empty subset, if present, is included.

2. The *reversal* operation $\mathbb{R}$ applied to an NFA $\mathcal{N}$ yields an NFA $\mathcal{N}^\mathbb{R}$, where sets of initial and final states of $\mathcal{N}$ are interchanged and each transition between any two states is reversed.

From now on we consider only non-empty regular languages. Let $L$ be any such language, and let its set of quotients be $K = \{K_0, \ldots, K_{n-1}\}$. One of the quotients of $L$ is $L$ itself; this is called the *initial quotient* and is denoted by $K_0$. A quotient is *final* if it contains the empty word $\varepsilon$. The set of final quotients is $F = \{K_i \mid \varepsilon \in K_i\}$.

In the following definition we use a one-to-one correspondence $K_i \leftrightarrow K_j$ between quotients $K_i$ of a language $L$ and the states $K_i$ of the quotient DFA $\mathcal{D}$ defined below. We refer to the $K_i$ as *quotient symbols*.

**Definition 1.** The quotient DFA of $L$ is $\mathcal{D} = (K, \Sigma, \delta, K_\text{in}, F)$, where $K = \{K_0, \ldots, K_{n-1}\}$, $K_\text{in}$ corresponds to $K_0$, $F = \{K_i \mid K_i \in F\}$, and $\delta(K_i, a) = K_j$ if and only if $a^{-1}K_i = K_j$, for all $K_i, K_j \in K$ and $a \in \Sigma$.

In a quotient DFA the right language of $K_i$ is $K_i$, and its left language is $\{w \in \Sigma^* \mid w^{-1}L = K_i\}$. The latter is the equivalence class of the Nerode equivalence [5]. The language $L(\mathcal{D})$ is the right language of $K_{n-1}$, and hence $L(\mathcal{D}) = L$. Also, DFA $\mathcal{D}$ is minimal, since all quotients in $K$ are distinct.

It follows from the definition of an atom, that a regular language $L$ has at most $2^n$ atoms. An atom is *initial* if it has $L$ (rather than $\overline{T}$) as a term; it is *final* if it contains $\varepsilon$. Since $L$ is non-empty, it has at least one quotient containing $\varepsilon$. Hence it has exactly one final atom, the atom $\overline{K_0} \cap \cdots \cap \overline{K_{n-1}}$, where $\overline{K}_i = K_i$ if $\varepsilon \in K_i$, and $\overline{K}_i = \overline{K_i}$ otherwise. Let $A = \{A_0, \ldots, A_{m-1}\}$ be the set of atoms of $L$. By convention, $I$ is the set of initial atoms and $A_{m-1}$ is the final atom.

As above, we use a one-to-one correspondence $A_i \leftrightarrow A_i$ between atoms $A_i$ of a language $L$ and the states $A_i$ of the NFA $\mathcal{A}$ defined below. We refer to the $A_i$ as *atom symbols*.
Definition 2. The átomaton\(^4\) of \(L\) is the NFA \(A = (A, \Sigma, \eta, I, \{A_{m-1}\})\), where \(A = \{A_i \mid A_i \in A\}\), \(I = \{A_i \mid A_i \in I\}\), \(A_{m-1}\) corresponds to \(A_{m-1}\), and \(A_j \in \eta(A_i, a)\) if and only if \(aA_j \subseteq A_i\), for all \(A_i, A_j \in A\) and \(a \in \Sigma\).

Example 1. Let \(L_2 \subseteq \{a, c\}^*\) be defined by the quotient equations below (left) and recognized by the DFA \(D_2\) of Fig. 2 (a). The equations for the atoms of \(L_2\) are below (right), and the átomaton \(A_2\) is in Fig. 2 (b); here each atom is denoted by \(A_P\), where \(P\) is the set of uncomplemented quotients. Thus \(K_0 \cap K_1\) becomes \(A_{\{0\}}\), etc., and we represent the sets in the subscripts without brackets and commas. The reverse \(D_2^R\) of \(D_2\) is in Fig. 2 (c). The determinized reverse \(D_2^{RD}\) is in Fig. 2 (d); this is the minimal DFA for \(L_2^R\), the reverse of \(L_2\). The reverse \(A_2^R\) of the átomaton is in Fig. 2 (e). Note that \(D_2^{RD}\) and \(A_2^R\) are isomorphic.

\[K_0 = aK_1 \cup cK_0, \quad K_0 \cap K_1 = a(K_0 \cap K_1) \cup c((K_0 \cap K_1) \cup (K_0 \cap K_1)),\]
\[K_1 = aK_0 \cup cK_0 \cup \varepsilon, \quad K_0 \cap \overline{K_1} = a(K_0 \cap K_1),\]
\[K_0 \cap K_1 = a(K_0 \cap K_1) \cup \varepsilon, \quad \overline{K_0 \cap K_1} = a(K_0 \cap K_1) \cup c((K_0 \cap K_1) \cup (K_0 \cap K_1)).\]

The next theorem from [1], also discussed in [3], will be used several times.

**Theorem 2 (Determinization).** If an NFA \(N\) has no empty states and \(N^R\) is deterministic, then \(N^\circ\) is minimal.

\(^4\) In [3], the intersection \(A_{\emptyset} = K_0 \cap \cdots \cap K_{m-1}\) was not considered an atom. It was shown that the right language of state \(A_i\) is the atom \(A_i\), the left language of \(A_i\) is non-empty, the language of the átomaton \(A\) is \(L\), and \(A\) is trim. If the intersection \(A_{\emptyset}\) of all the complemented quotients is non-empty, then \(A_{\emptyset}\) is an atom and \(A\) is no longer trim because state \(A_{\emptyset}\) is not reachable from any initial state.
It was shown in [3] that the átomaton $A$ of $L$ with reachable atoms only is isomorphic to the trimmed version of $D^{R^{\lor}}$, where $D$ is the quotient DFA of $L$. With our new definition, $A$ is isomorphic to $D^{R^{\lor}}$. We now study this isomorphism in detail, along with the isomorphism between $A^R$ and $D^{RD}$. We deal with the following automata:

1. Quotient DFA $D = (K, \Sigma, \delta, K_{in}, F)$ of $L$ whose states are quotient symbols.
2. The reverse $D^R = (K, \Sigma, \delta^R, F, \{K_{in}\})$ of $D$. The states in $K$ are still quotient symbols, but their right languages are no longer quotients of $L$.
3. The determinized reverse $D^{R^D} = (S, \Sigma, \alpha, F, G)$, where $S \subseteq 2^K$ and $G = \{S_i \in S \mid K_{in} \in S_i\}$. The states in $S$ are sets of quotient symbols, i.e., subsets of $K$. Since $(D^R)^R = D$ is deterministic and all of its states are reachable, $D^R$ has no empty states. By Theorem 2, DFA $D^{R^D}$ is minimal and accepts $L^R$; hence it is isomorphic to the quotient DFA of $L^R$.
4. The reverse $D^{R^{D^R}} = (S, \Sigma, \alpha^R, G, \{F\})$ of $D^{D^R}$; here the states are still sets of quotient symbols.
5. The átomaton $A = (A, \Sigma, \eta, I, \{A_{m-1}\})$, whose states are atom symbols.
6. The reverse $A^R = (A, \Sigma, \eta^R, A_{m-1}, I)$ of $A$, whose states are still atom symbols, though their right languages are no longer atoms.

The results from [3] and our new definition of atoms imply that $A^R$ is a minimal DFA that accepts $L^R$. It follows that $A^R$ is isomorphic to $D^{RD}$. Our next result makes this isomorphism precise.

**Proposition 4 (Isomorphism).** Let $\varphi : A \to S$ be the mapping assigning to state $A_j$, given by $A_j = K_{i_0} \cap \cdots \cap K_{i_{n-r-1}} \cap K_{i_{n-r}} \cap \cdots \cap K_{i_{n-1}}$ of $A^R$, the set $\{K_{i_0}, \ldots, K_{i_{n-r-1}}\}$. Then $\varphi$ is a DFA isomorphism between $A^R$ and $D^{RD}$.

**Proof.** The initial state $A_{m-1}$ of $A^R$ is mapped to the set of all quotients containing $\varepsilon$, which is precisely the initial state $F$ of $D^{RD}$. Since the quotient $L$ appears uncomplemented in every initial atom $A_i \in I$, the image $\varphi(A_i)$ contains $L$. Thus the set of final states of $A^R$ is mapped to the set of final states of $D^{RD}$.

It remains to be shown, for all $A_i, A_j \in A$ and $a \in \Sigma$, that $\eta^R(A_j, a) = A_i$ if and only if $\alpha(\varphi(A_j), a) = \varphi(A_i)$.

Consider atom $A_i$ with $P_i$ as the set of quotients that appear uncomplemented in $A_i$. Also define the corresponding set $P_j$ for $A_j$. If there is a missing quotient $K_h$ in the intersection $a^{-1}A_i$, we use $a^{-1}A_i \cap \{K_h \cup \bar{K}_h\}$. We do this for all missing quotients until we obtain a union of atoms. Hence $A_j \in \eta(A_i, a)$ can hold in $A$ if and only if $P_j \supset \delta(P_i, a)$ and $P_j \cap \delta(Q \setminus P_i, a) = \emptyset$. It follows that in $A^R$ we have $\eta^R(A_j, a) = A_i$ if and only if $P_j \supset \delta(P_i, a)$ and $P_j \cap \delta(Q \setminus P_i, a) = \emptyset$.

Now consider $D^{RD}$. Let $P_i$ be any subset of $Q$: then the successor set of $P_i$ in $D$ is $\delta(P_i, a)$. Let $\delta(P_i, a) = P_k$. So in $D^R$, we have $P_i \in \delta^R(P_k, a)$. But suppose that state $q$ is not in $\delta(Q, a)$; then $\delta^R(q, a) = \emptyset$. Consequently, we also have $P_i \in \delta^R(P_k \cup \{q\}, a)$. It follows that for any $P_j$ containing $\delta(P_i, a)$ and satisfying $P_j \cap \delta(Q \setminus P_i, a) = \emptyset$, we also have $\alpha(P_j, a) = P_i$.

We have now shown that $\eta^R(A_j, a) = A_i$ if and only if $\alpha(P_j, a) = P_i$, for all subsets $P_i, P_j \in S$, that is, if and only if $\alpha(\varphi(A_j), a) = \varphi(A_i)$. □
Corollary 1. The mapping $\varphi$ is an NFA isomorphism between $A$ and $D^{\text{RDR}}$.

In the remainder of the paper it is more convenient to use the $D^{\text{RDR}}$ representation of átomata, rather than that of Definition 2.

4 The Witness Languages and Automata

We now introduce a class $\{L_n \mid n \geq 2\}$ of regular languages defined by the quotient DFA’s $D_n$ given below; we shall prove that the atoms of each language $L_n = L(D_n)$ in this class meet the worst-case quotient complexity bounds.

**Definition 3 (Witness).** For $n \geq 2$, let $D_n = (Q, \Sigma, \delta, q_0, F)$, where $Q = \{0, \ldots, n - 1\}$, $\Sigma = \{a, b, c\}$, $q_0 = 0$, $F = \{n - 1\}$, $\delta(i, a) = i + 1 \mod n$, $\delta(0, b) = 1$, $\delta(1, b) = 0$, $\delta(i, b) = i$ for $i > 1$, $\delta(i, c) = i$ for $0 \leq i \leq n - 2$, and $\delta(n - 1, c) = 0$. Let $L_n$ be the language accepted by $D_n$.

For $n \geq 3$, the DFA of Definition 3 is illustrated in Fig. 1, and $D_2$ is the DFA of Example 1 ($a$ and $b$ coincide). The DFA $D_n$ is minimal, since for $0 \leq i \leq n - 1$, state $i$ accepts $a^{n-1-i}$, and no other state accepts this word.

A transformation of a set $Q$ is a mapping of $Q$ into itself. If $t$ is a transformation of $Q$ and $i \in Q$, then it is the image of $i$ under $t$. The set of all transformations of a finite set $Q$ is a semigroup under composition, in fact, a monoid $T_Q$ of $n^n$ elements. A permutation of $Q$ is a mapping of $Q$ onto itself. A transposition $(i, j)$ interchanges $i$ and $j$ and does not affect any other elements. A singular transformation, denoted by $(i^j)$, has $it = j$ and $ht = h$ for all $h \neq i$.

In 1935 Piccard [6] proved that three transformations of $Q$ are sufficient to generate $T_Q$. Dénes [4] studied more general generators; we use his formulation:

**Theorem 3 (Transformations).** The transformation monoid $T_Q$ can be generated by any cyclic permutation of $n$ elements together with any transposition and any singular transformation.

In any DFA $D = (Q, \Sigma, \delta, q_0, F)$, each word $w$ in $\Sigma^+$ performs a transformation on $Q$ defined by $\delta(\cdot, w)$. The set of all these transformations is the transformation semigroup of $D$. By Theorem 3, the transformation semigroup of our witness $D_n$ has $n^n$ elements, since $a$ is a cyclic permutation, $b$ is a transposition and $c$ is a singular transformation.

The following result of Salomaa, Wood and Yu [7] concerning reversal is restated in our terminology.

**Theorem 4 (Transformations and Reversal).** Let $D$ be a minimal DFA with $n$ states accepting a language $L$. If the transformation semigroup of $D$ has $n^n$ elements, then the quotient complexity of $L^R$ is $2^n$.

**Corollary 2 (Reversal).** For $n \geq 2$, the quotient complexity of $L^R_n$ is $2^n$.

**Corollary 3 (Number of Atoms of $L_n$).** The language $L_n$ has $2^n$ atoms.
To find the quotient complexity of \(A\) properly represented by the \(\alpha\) atomaton. Since, however, the \(\alpha\) atomaton is an NFA, their sets of uncomplemented quotients. To simplify the notation, we represent \(\subseteq\) state, states, transitions, and final state as the \(\alpha\) atomaton, but has only one initial as in Definition 3.

We now show that the upper bounds derived in Section 2 are tight by proving

**Proposition 5 (Transitions of the \(\alpha\) Atomaton).** Let \(D_n = (Q, \Sigma, \delta, q_0, F)\) be the DFA of Definition 3. The \(\alpha\) atomaton of \(L = L(D_n)\) is the NFA \(A_n = (2^Q, \Sigma, \eta, I, \{n - 1\})\), where

1. If \(S = \{\emptyset\}\), then \(\eta(S, a) = \{\emptyset\}\). Otherwise,
   \(\eta(s_1, \ldots, s_k, a) = s_1 + 1, \ldots, s_k + 1\), where the addition is modulo \(n\).
2. If \(\{0, 1\} \cap S = \emptyset\), then
   (a) \(\eta(S, b) = S\),
   (b) \(\eta(\{0\} \cup S, b) = \{1\} \cup S\),
   (c) \(\eta(\{1\} \cup S, b) = \{0\} \cup S\),
   (d) \(\eta(\{0, 1\} \cup S, b) = \{0, 1\} \cup S\).
3. If \(\{0, n - 1\} \cap S = \emptyset\), then
   (a) \(\eta(S, c) = \{S, \{n - 1\} \cup S\}\),
   (b) \(\eta(\{0, n - 1\} \cup S, c) = \{\{0, n - 1\} \cup S, \{0\} \cup S\}\),
   (c) \(\eta(\{0\} \cup S, c) = \emptyset\),
   (d) \(\eta(\{n - 1\} \cup S, c) = \emptyset\).

Proof. The reverse of DFA \(D_n\) is the NFA \(D_n^R = (Q, \Sigma, \delta^R, \{n - 1\}, \{0\})\), where \(\delta^R\) is defined by \(\delta^R(i, a) = i - 1 \mod n\), \(\delta^R(i, b) = \delta(i, b)\), \(\delta^R(0, c) = \{0, n - 1\}\), \(\delta^R(n - 1, c) = \emptyset\), and \(\delta^R(i, c) = i\), for \(0 < i < n - 1\). After applying determinization and reversal to \(D_n^R\), the claims follow by Corollary 1.

### 5 Tightness of the Upper Bounds

We now show that the upper bounds derived in Section 2 are tight by proving that the atoms of the languages \(L_n\) of Definition 3 meet those bounds.

Since the states of the \(\alpha\) atomaton \(A_n = (A, \Sigma, \eta, I, \{A_{m-1}\})\) are atom symbols \(A_i\), and the right language of each \(A_i\) is the atom \(A_i\), the languages \(A_i\) are properly represented by the \(\alpha\) atomaton. Since, however, the \(\alpha\) atomaton is an NFA, to find the quotient complexity of \(A_i\), we need the equivalent minimal DFA.

Let \(D_n\) be the \(n\)-state quotient DFA of Definition 3 for \(n \geq 2\), and recall that \(L(D_n) = L_n\). In the sequel, using Corollary 1, we represent the \(\alpha\) atomaton \(A_n\) of \(L_n\) by the isomorphic NFA \(D_n^{\alpha R} = (S, \Sigma, \alpha^R, G, \{F\})\), and identify the atoms by their sets of uncomplemented quotients. To simplify the notation, we represent atoms by the subscripts of the quotients, that is, by subsets of \(Q = \{0, \ldots, n - 1\}\), as in Definition 3.

In this framework, to find the quotient complexity of an atom \(A_P\), with \(P \subseteq Q\), we start with the NFA \(A_P = (S, \Sigma, \alpha^R, \{P\}, \{F\})\), which has the same states, transitions, and final state as the \(\alpha\) atomaton, but has only one initial state, \(P\), corresponding to the atom symbol \(A_P\). Because \(A_P^D\) is deterministic.
and $A_P$ has no empty states, $A_P^D$ is minimal by Theorem 2. Therefore, $A_P^D$ is the quotient DFA of the atom $A_P$. The states of $A_P^D$ are certain sets of sets of quotient symbols; to reduce confusion we refer to them as collections of sets. The particular collections appearing in $A_P^D$ will be called “super-algebras”.

Let $U$ be a subset of $Q$ with $|U| = u$, and let $V$ be a subset of $U$ with $|V| = v$. Define $⟨V⟩_U$ to be the collection of all $2^{n−v}$ subsets of $U$ containing $V$. There are $C_u^v C_v^n$ collections of the form $⟨V⟩_U$, because there are $C_u^v$ ways of choosing $U$, and for each such choice there are $C_v^n$ ways of choosing $V$. The collection $⟨V⟩_U$ is called the super-algebra of $U$ generated by $V$. The type of a super-algebra $⟨V⟩_U$ is the ordered pair $⟨|V|, |U|⟩ = (v, u)$.

The following theorem is a well-known result of Picard [6] about the group—known as the symmetric group—of all permutations of a finite set:

**Theorem 5 (Permutations).** The symmetric group of size $n!$ of all permutations of a set $Q = \{0, \ldots, n−1\}$ is generated by any cyclic permutation of $Q$ together with any transposition.

**Lemma 1 (Strong-Connectedness of Super-Algebras).** Super-algebras of the same type are strongly connected by words in $\{a, b\}^*$.

**Proof.** Let $⟨V_1⟩_{U_1}$ and $⟨V_2⟩_{U_2}$ be any two super-algebras of the same type. Arrange the elements of $V_1$ in increasing order, and do the same for the elements of the sets $V_2, U_1 \setminus V_1, U_2 \setminus V_2, Q \setminus U_1$, and $Q \setminus U_2$. Let $\pi : Q \to Q$ be the mapping that assigns the $i$th element of $V_2$ to the $i$th element of $V_1$, the $i$th element of $U_2 \setminus V_2$ to the $i$th element of $U_1 \setminus V_1$, and the $i$th element of $Q \setminus U_2$ to the $i$th element of $Q \setminus U_1$. For any $R_1$ such that $V_1 \subseteq R_1 \subseteq U_1$, there is a corresponding subset $R_2 = \pi(R_1)$, where $V_2 \subseteq R_2 \subseteq U_2$. Thus $\pi$ establishes a one-to-one correspondence between the elements of the super-algebras $⟨V_1⟩_{U_1}$ and $⟨V_2⟩_{U_2}$. Also, $\pi$ is a permutation of $Q$, and so can be performed by a word $w \in \{a, b\}^*$ in $D_n$, in view of Theorem 5. Thus every set $R_2$ defined as above is reachable from $R_1$ by $w$. So $⟨V_2⟩_{U_2}$ is reachable from $⟨V_1⟩_{U_1}$. \qed

**Lemma 2 (Reachability).** Let $⟨V⟩_U$ be any super-algebra of type $(v, u)$. If $v \geq 2$, then from $⟨V⟩_U$ we can reach a super-algebra of type $(v−1, u)$. If $u \leq n−2$, then from $⟨V⟩_U$ we can reach a super-algebra of type $(v, u+1)$.

**Proof.** If $v \geq 2$, then by Lemma 1, from $⟨V⟩_U$ we can reach a super-algebra $⟨V'⟩_{U'}$ of type $(v, u)$ such that $\{0, n−1\} \subseteq V'$. By input $c$ we reach $⟨V' \setminus \{n−1\}⟩_{U'}$ of type $(v−1, u)$. For the second claim, if $u \leq n−2$, then by Lemma 1, from $⟨V⟩_U$ we can reach a super-algebra $⟨V'⟩_{U'}$ of type $(v, u)$ such that $\{0, n−1\} \cap V' = \emptyset$. By input $c$ we reach $⟨V'⟩_{U'} \cup \{n−1\}$ of type $(v, u+1)$. \qed

The next proposition holds for $n \geq 1$ if we let $L_1 = \Sigma^*$.

**Proposition 6 (Atoms with 0 or $n$ Complemented Quotients).**
For $n \geq 1$, the quotient complexity of the atoms $A_Q$ and $A_\emptyset$ of $L_n$ is $2^n − 1$. 

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Proof. Let \( A_Q (A_\emptyset) \) be the modified atomaton with only one initial state, \( Q (\emptyset) \).
By the considerations above, \( A_Q^0 (A_\emptyset^0) \) is the quotient DFA of \( A_Q (A_\emptyset) \); hence it suffices to prove the reachability of \( 2^n - 1 \) collections.

For \( A_Q \), the initial state of \( A_Q^0 \) is the collection \( \{Q\} \), which is the super-algebra \( \langle Q \rangle_Q \) of \( Q \) generated by \( Q \). Now suppose that we have reached a super-algebra of type \( (v, n) \). By Lemma 1, we can reach every other super-algebra of type \( (v, n) \). If \( v \geq 2 \), then by Lemma 2 we can reach a super-algebra of type \( (v - 1, n) \). Thus we can reach all super-algebras \( \langle V \rangle_Q \) of \( Q \), one for each non-empty subset \( V \) of \( Q \). Since there are at most \( 2^n - 1 \) collections and that many can be reached, no other collection can be reached.

For \( A_\emptyset \), the initial state of \( A_\emptyset^0 \) is the empty collection, which is the super-algebra \( \langle \emptyset \rangle \) of \( \emptyset \) generated by \( \emptyset \). Now suppose we have reached a super-algebra of type \( (0, u) \). By Lemma 1, we can reach every other super-algebra of type \( (0, u) \).

If \( u \leq n - 2 \), then by Lemma 2 we can reach a super-algebra of type \( (0, u + 1) \). Thus we can reach all super-algebras \( \langle \emptyset \rangle_U \), one for each non-empty subset \( U \) of \( Q \). Since there are at most \( 2^n - 1 \) collections and that many can be reached, no other collection can be reached.

Hence the proposition holds. \( \square \)

**Proposition 7 (Tightness).** For \( n \geq 2 \) and \( 1 \leq r \leq n - 1 \), the quotient complexity of any atom of \( L_n \) with \( r \) complemented quotients is \( f(n, r) \).

Proof. Let \( A_P \) be an atom of \( L_n \) with \( n - r \) uncomplemented quotients, where \( 1 \leq r \leq n - 1 \), that is, let \( P \) be the set of subscripts of the uncomplemented quotients.

Let \( A_P \) be the modified atomaton with the initial state \( P \). As discussed above, \( A_P^0 \) is minimal; hence it suffices to prove the reachability of \( f(n, r) \) collections.

We start with the super-algebra \( \langle P \rangle_P \) with type \( (n - r, n - r) \). By Lemmas 1 and 2, we can now reach all super-algebras of types

\[
(n - r, n - r), (n - r - 1, n - r), \ldots, (1, n - r),
\]
\[
(n - r, n - r + 1), (n - r - 1, n - r + 1), \ldots, (1, n - r + 1),
\]
\[
\ldots
\]
\[
(n - r, n - 1), (n - r - 1, n - 1), \ldots, (1, n - 1).
\]

Since the number of super-algebras of type \( (v, u) \) is \( C_u^n C_v^n \), we can reach

\[
g(n, r) = \sum_{u=n-r}^{n-1} \sum_{v=1}^{n-r} C_u^n C_v^n
\]

algebras. Changing the first summation index to \( k = n - u \), we get

\[
g(n, r) = \sum_{k=1}^{r-1} \sum_{v=1}^{n-r} C_{n-k}^n C_v^n - k.
\]

Note that \( C_{n-k}^n C_v^n - k = C_{n-k}^n C_{v+k}^{n+k} \), because \( C_{n-k}^n C_v^n - k = \frac{n!}{(n-k)!k!} \cdot \frac{(n-k)!}{v!(n-k-v)!} = \frac{n!}{k!(n-k-v)!} \), and \( C_{n-k}^n C_{v+k}^{n+k} = \frac{n!}{(k+v)!(n-k-v)!} \cdot \frac{(k+v)!}{k!v!} = \frac{n!}{(n-k-v)!k!v!} \). Now, we can
write $g(n, r) = \sum_{k=1}^{r} \sum_{v=1}^{n-r} C_{k+v}^r \cdot C_k^r$, and changing the second summation index to $h = k + v$, we have

$$g(n, r) = \sum_{k=1}^{r} \sum_{h=k+1}^{k+n-r} C_h^r \cdot C_k^r.$$ 

We notice that $g(n, r) = f(n, r) - 1$. From the super-algebra $\langle V \rangle_V$, where $V = \{0, 1, \ldots, n - r - 1\}$, we reach the empty quotient by input $c$, since $V$ contains 0, but not $n - 1$.

Since we can reach $f(n, r)$ super-algebras, no other collection can be reached, and the proposition holds.

The entire process of finding the complexity of atoms is illustrated in the example below for $n = 3$.

**Example 2.** Let $L_3$ be the language accepted by the quotient DFA $D_3$ of Definition 3 and Table 2, where the initial state is identified by an incoming arrow and the final state, by an outgoing arrow. The first column consists of states $q$, and the remaining columns give the values of $\delta(q, x)$ for each $x \in \Sigma$. Let the quotients of $L_3$ be $K_0 = L_3 = \varepsilon^{-1} L_3$, $K_1 = a^{-1} L_3$, and $K_2 = (aa)^{-1} L_3$. The states of $D_3$ are subscripts of quotient symbols.

Reversing $D_3$, we obtain the NFA $D^R_3$ of Table 3. The states of $D^R_3$ are the same as those of $D_3$, but the transitions are to sets of states, and 02 stands for $\{0, 2\}$, 0 stands for $\{0\}$, etc.

**Table 2.** Quotient DFA $D_3$ of $L_3$. **Table 3.** NFA $D^R_3$ for $L^R_3$.

|   | $a$ | $b$ | $c$ |
|---|----|----|----|
| $\rightarrow$ | 0  | 1  | 1  |
|   | 1  | 2  | 0  |
|   | 2  | 0  | 2  |

|   | $a$ | $b$ | $c$ |
|---|----|----|----|
| $\rightarrow$ | 0  | 2  | 1  |
| 02 | 1  | 0  | 0  |
| $\rightarrow$ | 2  | 1  | 2  |

Next, we perform the subset construction on $D^R_3$ to determinize it and get the DFA $D^R_{3R}$, the quotient DFA for $L^R_3$. Since $D^R_3$ is trim, and $(D^R_3)^R = D^R_3$ is deterministic, the DFA $D^R_{3R}$ shown in Table 4 is minimal by Theorem 2.

The states of $D^R_{3R}$ are sets of (subscripts of) quotient symbols. Now we reverse $D^R_{3R}$ to get $D^{R_{3R}}_3$ of Table 5, which is isomorphic to the atomaton $A_3$. The states of $D^{R_{3R}}_3$ are still sets of (subscripts of) quotient symbols. Note that the empty set $\emptyset$ of quotient symbols is a state of $D^{R_{3R}}_3$, and hence also of $A_3$. It is not to be confused with the empty set of transitions associated with states 0, 2, 01, and 12 under input $c$ indicated by $\rightarrow$. 

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Table 4. DFA $D^\text{RD}_3$ for $L^R_3$.

|   | $a$ | $b$ | $c$ |
|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 0 | 0 | 2 | 1 |
| $\rightarrow$ | 0 | 0 | 1 |
| 1 | 1 | 2 | $\emptyset$ |
| $\rightarrow$ | 0 | 2 | 0 |
| 2 | 1 | 2 | $\emptyset$ |
| $\rightarrow$ | 01 | 02 | 01 |
| 01 | 02 | 01 | 01 |
| $\rightarrow$ | 02 | 12 | 12 |
| 02 | 12 | 12 | 02 |
| $\rightarrow$ | 12 | 01 | 02 |
| 12 | 01 | 02 | 1 |
| $\rightarrow$ | 012 | 012 | 012 |
| 012 | 012 | 012 | 012 |

Table 5. Atómato $A_3 = D^\text{RD}_3$.

|   | $a$ | $b$ | $c$ |
|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 0 | 0 | 2 | 1 |
| $\rightarrow$ | 0 | 1 | 1 |
| 1 | 2 | 0 | 1,12 |
| $\rightarrow$ | 1 | 2 | 0 |
| 2 | 1 | 2 | $\emptyset$ |
| $\rightarrow$ | 01 | 12 | 01 |
| 02 | 01 | 12 | 02 |
| $\rightarrow$ | 02 | 01 | 12 |
| 012 | 012 | 012 | 012 |

Table 6. DFA $D_{012}$ of $A_{012}$.

|   | $a$ | $b$ | $c$ |
|---|---|---|---|
| 012 | 012 | 012 | 012 |
| $\rightarrow$ | 012 | 012 | 012 |
| 01,012 | 12,012 | 01,012 | 01,012 |
| 02,012 | 02,012 | 01,012 | 01,012 |
| 01,012 | 12,012 | 01,012 | 01,012 |
| 0,01,02,012 | 1,01,12,012 | 1,01,12,012 | 0,01,02,012 |
| 0,01,12,012 | 0,01,02,012 | 0,01,02,012 |
| 1,01,12,012 | 1,01,12,012 | 1,01,12,012 |
| 2,02,12,012 | 2,02,12,012 | 2,02,12,012 | 0,01,02,012 | $\rightarrow$ |

Atom $A_{012} = K_0 \cap K_1 \cap K_2$ is the language accepted by $A_3$ started in state 012. The states of $D_{012}$, the minimal DFA of $A_{012}$, are collections of sets of quotients. As seen from Table 6, the quotient complexity of $A_{012}$ is seven.

Atom $A_{01} = K_0 \cap K_1 \cap K_2$ is accepted by $A_3$ started in state 01. The minimal DFA $D_{01}$ of $A_{01}$ is shown in Table 7, and the quotient complexity of $A_{01}$ is ten. Since 01, 12 and 02 are strongly connected by $a$, the same collections are reached from these states, and so the quotient complexity of $A_{12}$ and $A_{02}$ is also ten.

Atom $A_2 = K_0 \cap K_1 \cap K_2$ is accepted by $A_3$ started in state 2. The minimal DFA $D_2$ of $A_2$ is shown in Table 8, and the quotient complexity of $A_2$ is ten. Since 0, 1 and 2 are strongly connected by $a$, the same collections are reached from these states, and so the quotient complexity of $A_0$ and $A_1$ is also ten.

Finally, atom $A_\emptyset = K_0 \cap K_1 \cap K_2$ is accepted by $A_3$ started in state $\emptyset$. The minimal DFA $D_\emptyset$ is shown in Table 9, and the quotient complexity of $A_\emptyset$ is seven. Note that $D_{012}$ and $D_\emptyset$ have isomorphic transition tables, if we ignore final states. The isomorphism is $\psi : 2^{2^Q} \rightarrow 2^{2^Q}$ defined as follows: If $C \subseteq 2^Q$ is a collection of subsets of $Q$, then $\psi(C) = \{Q \setminus S \mid S \in C\}$.
Table 7. DFA $D_{01}$ of $A_{01}$.

|   | a   | b   | c   |
|---|-----|-----|-----|
|   | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\rightarrow$ | $01$ | $12$ | $01$ |
| $02$ | $01$ | $12$ | $02$ |
| $12$ | $02$ | $02$ | $\emptyset$ |
| $0.01$ | $1.12$ | $1.01$ | $\emptyset$ |
| $0.02$ | $1.01$ | $1.12$ | $0.02$ |
| $1.01$ | $2.12$ | $0.01$ | $1.12$ |
| $1.12$ | $2.02$ | $0.02$ | $1.12$ |
| $2.02$ | $0.01$ | $2.12$ | $0.02$ |
| $2.12$ | $0.02$ | $2.02$ | $\emptyset$ |

Table 8. DFA $D_2$ of $A_2$.

|   | a   | b   | c   |
|---|-----|-----|-----|
|   | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\rightarrow$ | $0$ | $1$ | $1$ |
| $1$ | $2$ | $0$ | $1.12$ |
| $2$ | $0$ | $2$ | $\emptyset$ |

Table 9. DFA $D_3$ of $A_3$.

|   | a   | b   | c   |
|---|-----|-----|-----|
|   | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\rightarrow$ | $0$ | $0$ | $0$ |
| $0$ | $1$ | $1$ | $\emptyset$ |
| $1$ | $2$ | $0$ | $1.12$ |
| $2$ | $0$ | $2$ | $\emptyset$ |

6 Conclusions

The atoms of a regular language $L$ are its basic building blocks. We have studied the quotient complexity of the atoms of $L$ as a function of the quotient complexity of $L$. We have computed an upper bound for the quotient complexity of any atom with $r$ complemented quotients, and exhibited a class $\{L_n\}$ of languages whose atoms meet this bound.

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