SYMMETRICALLY COMPLETE ORDERED SETS,
ABELIAN GROUPS AND FIELDS

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Abstract. We characterize and construct linearly ordered sets, abelian groups and fields that are symmetrically complete, meaning that the intersection over any chain of closed bounded intervals is nonempty. Such ordered abelian groups and fields are important because generalizations of Banach’s Fixed Point Theorem hold in them. We prove that symmetrically complete ordered abelian groups and fields are divisible Hahn products and real closed power series fields, respectively. We show how to extend any given ordered set, abelian group or field to one that is symmetrically complete. A main part of the paper establishes a detailed study of the cofinalities in cuts.

1. Introduction

In the paper [9], the third author introduced the notion of “symmetrically complete” ordered fields and proved that every ordered field can be extended to a symmetrically complete ordered field. He also proved that an ordered field $K$ is symmetrically complete if and only if every nonempty chain of closed bounded intervals in $K$ has a nonempty intersection; talking of chains of intervals, we refer to the partial ordering by inclusion. It is this property that is particularly interesting as it allows to prove fixed point theorems for such fields that generalize Banach’s Fixed Point Theorem, replacing the usual metric of the reals by the distance function that is derived from the ordering, see [3]. In accordance with the notion used in that paper, we will call a linearly ordered set $(I, <)$ spherically complete w.r.t. the order balls if every nonempty chain of closed bounded intervals has a nonempty intersection. Ordered fields and ordered abelian groups shall be called spherically complete w.r.t. the order balls if the underlying linearly ordered set is.

It is not a priori clear whether fields that are spherically complete w.r.t. the order balls, other than the reals themselves, do exist. At first glance, the above condition on chains of intervals seems to imply that the field is cut complete and hence isomorphic to the reals. But it is shown in [9] that
there are arbitrarily large fields with this property. Let us describe
the background in more detail; for some of the notions used, see Section 2.

A cut in a linearly ordered set $I$ is a pair $C = (D, E)$ with a lower cut set $D$ and an upper cut set $E$ if $I = D \cup E$ and $d < e$ for all $d \in D$, $e \in E$. Throughout this paper, when we talk of cuts we will mean Dedekind cuts, that is, cuts with $D$ and $E$ nonempty. By the cofinality of the cut $C$ we mean the pair $(\kappa, \lambda)$ where $\kappa$ is the cofinality of $D$, denoted by $\text{cf}(D)$, and $\lambda$ is the coinitiality of $E$, denoted by $\text{ci}(E)$. Recall that the coinitiality of a linearly ordered set is the cofinality of this set under the reversed ordering.

Recall further that cofinalities and coinitialities of ordered sets are regular cardinals.

We will call a linearly ordered set $(I, <)$ symmetrically complete if every cut $C$ in $I$ is asymmetric, that is, $\kappa \neq \lambda$. Ordered fields and ordered abelian groups shall be called symmetrically complete if the underlying linearly ordered set is. For example, the reals are symmetrically complete because every cut $C$ is principal (also called realized), that is, either $D$ has a maximal element or $E$ has a minimal element, in which case the cofinality of $C$ is either $(1, \aleph_0)$ or $(\aleph_0, 1)$. We have the following characterization of symmetrical completeness, which will be proved in Section 2:

**Proposition 1.** A linearly ordered set $I$ is spherically complete w.r.t. the order balls if and only if every nonprincipal cut in $I$ is asymmetric.

Note that $\mathbb{Z}$ has these properties, but a discretely ordered abelian group is never symmetrically complete. On the other hand, there are no cuts with cofinality $(1, 1)$ in densely ordered abelian groups and in ordered fields, so they are symmetrically complete as soon as every nonprincipal cut is asymmetric.

The main aim of this paper is to characterize the symmetrically complete ordered abelian groups and fields. Obviously, an ordered field is symmetrically complete if and only if its underlying additive ordered abelian group is. But ordered abelian groups also appear as the value groups of nonarchimedean ordered fields w.r.t. their natural valuations.

More generally, we will need the **natural valuation** of any ordered abelian group $(G, <)$, which we define as follows. Two elements $a, b \in G$ are called archimedean equivalent if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. The ordered abelian group $(G, <)$ is archimedean ordered if all nonzero elements are archimedean equivalent. If $0 \leq a < b$ and $na < b$ for all $n \in \mathbb{N}$, then “$a$ is infinitesimally smaller than $b$” and we will write $a \ll b$. We denote by $va$ the archimedean equivalence class of $a$. The set of archimedean equivalence classes can be ordered by setting $va > vb$ if and only if $|a| < |b|$ and $a$ and $b$ are not archimedean equivalent, that is, if $n|a| < |b|$ for all $n \in \mathbb{N}$. We write $\infty := v0$; this is the maximal element in the linearly ordered set of equivalence classes. The function $a \mapsto va$ is a
group valuation on \( G \), i.e., it satisfies \( va = \infty \Leftrightarrow a = 0 \) and the ultrametric triangle law
\[(UT) \quad v(a - b) \geq \min\{va, vb\}, \]
and by definition,
\[0 \leq a \leq b \implies va \geq vb.\]

The set \( vG := \{vg | 0 \neq g \in G\} \) is called the value set of the valued abelian group \((G, v)\). For every \( \gamma \in vG \), the quotient \( C_\gamma := O_\gamma / M_\gamma \), where \( O_\gamma := \{g \in G | vg \geq \gamma\} \) and \( M_\gamma := \{g \in G | vg > \gamma\} \), is an archimedean ordered abelian group (hence embeddable in the ordered additive group of the reals, by the Theorem of Hölder); it is called an archimedean component of \( G \).

The natural valuation induces an ultrametric given by \( u(a, b) := v(a - b) \).

We define the smallest ultrametric ball \( B_u(g, h) \) containing the elements \( g \) and \( h \) to be
\[B_u(a, b) := \{g | v(a - g) \geq v(a - b)\} = \{g | v(b - g) \geq v(a - b)\},\]
where the last equation holds because in an ultrametric ball, every element is a center. For the basic facts on ultrametric spaces, see [5]. Note that all ultrametric balls are cosets of convex subgroups in \( G \) (see [6]). We say that an ordered abelian group (or an ordered field) is spherically complete w.r.t. its natural valuation if every nonempty chain of ultrametric balls (ordered by inclusion) has a nonempty intersection. The ordered abelian groups that are spherically complete w.r.t. their natural valuation are precisely the Hahn products (see [6] or [7]); see Section 2.2 for the definition and basic properties of Hahn products.

If \((K, <)\) is an ordered field, then we consider the natural valuation on its ordered additive group and define \( va + vb := v(ab) \). This turns the set of archimedean classes into an ordered abelian group, with neutral element \( 0 := v1 \) and inverses \( -va = v(a^{-1}) \). In this way, \( v \) becomes a field valuation (with additively written value group). It is the finest valuation on the field \( K \) which is compatible with the ordering. The residue field, denoted by \( Kv \), is archimedean ordered, hence by the version of the Theorem of Hölder for ordered fields, it can be embedded in the ordered field \( \mathbb{R} \). Via this embedding, we will always identify it with a subfield of \( \mathbb{R} \).

**Remark 2.** In contrast to the notation for the natural valuation (in the Baer tradition) that we have used in [3], we use here the Krull notation because it is more compatible with our constructions in Section 5. In this notation, two elements in an ordered abelian group or field are close to each other when the value of their difference is large.

Every ordered field that is spherically complete w.r.t. its natural valuation is maximal, in the sense of [2]. In this paper Kaplansky shows that under certain conditions, which in particular hold when the residue field has characteristic 0, every such field is isomorphic to a power series field. In
general, a nontrivial factor system is needed on the power series field, but it is not needed for instance when the residue field is $\mathbb{R}$.

In [3], we have already proved that if an ordered abelian group $(G, <)$ is spherically complete w.r.t. the order balls, then it is spherically complete w.r.t. its natural valuation $v$. If $G$ is even an ordered field, then we proved that in addition, it has residue field $\mathbb{R}$. From this and Proposition 1, we obtain:

**Proposition 3.** If an ordered abelian group is symmetrically complete, then it is spherically complete w.r.t. its natural valuation. If an ordered field is symmetrically complete, then it is spherically complete w.r.t. its natural valuation $v$ and has residue field $Kv = \mathbb{R}$.

In the present paper, we wish to extend these results. It turns out that for an ordered abelian group to be symmetrically complete, the same must be true for the value set $vG$, and in fact, it must have an even stronger property. We will call a cut with cofinality $(\kappa, \lambda)$ in a linearly ordered set $(I, <)$ strongly asymmetric if $\kappa \neq \lambda$ and at least one of $\kappa$, $\lambda$ is uncountable. We will call $(I, <)$ strongly symmetrically complete if every cut in $I$ is strongly asymmetric, and we will call it extremely symmetrically complete if in addition, the cointiality and cofinality of $I$ are both uncountable. The reals are not strongly symmetrically complete.

In Section 4, we will prove the following results:

**Theorem 4.** An ordered abelian group $(G, <)$ is symmetrically complete if and only if it is spherically complete w.r.t. its natural valuation $v$, has a strongly symmetrically complete value set $vG$ and all archimedean components $C_\gamma$ are isomorphic to $\mathbb{R}$. It is strongly symmetrically complete if and only if in addition, $vG$ has uncountable cofinality, and it is extremely symmetrically complete if and only if in addition, $vG$ is extremely symmetrically complete.

Now we turn to ordered fields.

**Theorem 5.** An ordered field $K$ is symmetrically complete if and only if it is spherically complete w.r.t. its natural valuation $v$, has residue field $\mathbb{R}$ and a strongly symmetrically complete value group $vK$. Further, the following are equivalent:

a) $K$ is strongly symmetrically complete,

b) $K$ is extremely symmetrically complete,

c) $K$ is spherically complete w.r.t. its natural valuation $v$, has residue field $\mathbb{R}$ and an extremely symmetrically complete value group $vK$.

**Corollary 6.** Every symmetrically complete ordered abelian group is divisible and isomorphic to a Hahn product. Every symmetrically complete ordered field is real closed and isomorphic to a power series field with residue field $\mathbb{R}$ and divisible value group.
These results show a way for the construction of symmetrically complete and extremely symmetrically complete ordered fields $K$, which is an alternative to the construction given in [9]. For the former, construct a strongly symmetrically complete linearly ordered set $I$ with uncountable coinitiality. Then take $G$ to be the Hahn product with index set $I$ and all archimedean components equal to $\mathbb{R}$. Finally, take $K = \mathbb{R}((G))$, the power series field with coefficients in $\mathbb{R}$ and exponents in $G$. To obtain an extremely symmetrically complete ordered field $K$, construct $I$ such that in addition, also its cofinality is uncountable. See Section 5 for details.

In Section 6 we will use our theorems to prove the following result, which extends the corresponding result of [9]:

**Theorem 7.** Every ordered abelian group can be extended to an extremely symmetrically complete ordered abelian group. Every ordered field can be extended to an extremely symmetrically complete ordered field.

For the proof of this theorem, we need to extend any given ordered set $I$ to an extremely symmetrically complete ordered set $J$. We do this by constructing suitable lexicographic products of ordered sets. Let us describe the most refined result that we achieve, which gives us the best control of the cofinalities of cuts in the constructed ordered set $J$.

We denote by $\text{Reg}$ the class of all infinite regular cardinals, and for any ordinal $\lambda$, by

$$\text{Reg}_{<\lambda} = \{\kappa < \lambda \mid \aleph_0 \leq \kappa = \text{cf}(\kappa)\}$$

the set of all infinite regular cardinals $< \lambda$. We define:

$$\text{Coin}(I) := \{\text{ci}(S) \mid S \subseteq I \text{ such that } \text{ci}(S) \text{ is infinite}\} \subset \text{Reg},$$

$$\text{Cofin}(I) := \{\text{cf}(S) \mid S \subseteq I \text{ such that } \text{cf}(S) \text{ is infinite}\} \subset \text{Reg}.$$

We choose any $\mu, \kappa_0, \lambda_0 \in \text{Reg}$. Then we set

$$R_{\text{left}} := \text{Cofin}(I) \cup \text{Reg}_{<\kappa_0} \cup \text{Reg}_{<\mu} \subset \text{Reg},$$

$$R_{\text{right}} := \text{Coin}(I) \cup \text{Reg}_{<\lambda_0} \cup \text{Reg}_{<\mu} \subset \text{Reg}.$$ 

All of the subsets we have defined here are initial segments of $\text{Reg}$ in the sense that if they contain $\kappa$, then they also contain every infinite regular cardinal $< \kappa$.

Further, we assume that functions

$$\varphi_{\text{left}} : \{1\} \cup \text{Reg} \to \text{Reg} \text{ and } \varphi_{\text{right}} : \{1\} \cup \text{Reg} \to \text{Reg}$$

are given. We prove in Section 5:

**Theorem 8.** Assume that $\mu$ is uncountable and that

$$(1) \quad \varphi_{\text{left}}(\{1\} \cup R_{\text{right}}) \subset R_{\text{left}} \text{ and } \varphi_{\text{right}}(\{1\} \cup R_{\text{left}}) \subset R_{\text{right}}.$$
with \( \varphi_{\text{left}}(\kappa) \neq \kappa \neq \varphi_{\text{right}}(\kappa) \) for all \( \kappa \in R_{\text{left}} \cup R_{\text{right}} \). Then \( I \) can be extended to a strongly symmetrically complete ordered set \( J \) of cofinality \( \kappa_0 \) and coinitiality \( \lambda_0 \), in which the cuts have the following cofinalities:

\[
\{ (1, \mu), (\mu, 1) \} \cup \{ (\kappa, \varphi(\kappa)) \mid \kappa \in R_{\text{left}} \} \cup \{ (\varphi(\lambda), \lambda) \mid \lambda \in R_{\text{right}} \}.
\]

If in addition \( \kappa_0 \) and \( \lambda_0 \) are uncountable, then \( J \) is extremely symmetrically complete.

Among the value groups of valued fields, not only the dense, but also the discretely ordered groups play an important role. The value groups of formally \( p \)-adic fields are discretely ordered, and the value groups of \( p \)-adically closed fields are \( \mathbb{Z} \)-groups, that is, ordered abelian groups \( G \) that admit (an isomorphic image of) \( \mathbb{Z} \) as a convex subgroup such that \( G/\mathbb{Z} \) is divisible. We wish to prove a version of the previous theorem for discretely ordered abelian groups. Note that in a discretely ordered group, every principal cut has cofinality \((1,1)\). We call an ordered abelian group \( G \) **symmetrically d-complete** if every nonprincipal cut in \( G \) is asymmetric. We will call it **extremely symmetrically d-complete** if in addition, \( G \) has uncountable cofinality (and hence also uncountable coinitiality). Note that if a nonprincipal cut is asymmetric, then it is strongly asymmetric because the only countable coinitiality/cofinality other than \( 1 \) is \( \aleph_0 \). So “symmetrically d-complete” is at the same time the discrete version of “strongly symmetrically complete”.

**Theorem 9.** For a discretely ordered abelian group \((G, <)\), the following are equivalent:

a) \((G, <)\) is symmetrically d-complete,

b) \((G, <)\) is spherically complete w.r.t. the order balls,

c) \((G, <)\) is a \( \mathbb{Z} \)-group such that \( G/\mathbb{Z} \) is strongly symmetrically complete.

Further, \((G, <)\) is extremely symmetrically d-complete if and only if \( G/\mathbb{Z} \) is extremely symmetrically complete.

Again, this shows a way of construction. To obtain a symmetrically d-complete discretely ordered abelian group \( G \), construct a strongly symmetrically complete ordered abelian group \( H \) and then take the lexicographic product \( H \times \mathbb{Z} \). If in addition the cofinality of \( H \) is uncountable, then \( G \) will even be extremely symmetrically d-complete.

Note that \( G \) is isomorphic to a Hahn product if and only if \( G/\mathbb{Z} \) is. Therefore, if \( G \) is symmetrically d-complete, then it is a Hahn product.

**Remark 10.** After the completion of this paper it was brought to our attention by Salma Kuhlmann that Hausdorff constructed already in the years 1906-8 ordered sets with prescribed cofinalities for all of its cuts (cf. \([\Pi]\)). His construction also yields extremely symmetrically complete ordered sets. However, we are convinced that the constructions we present in Section 5
are indispensable in the context of this paper (and a good service to the reader), for the following reasons:

- they represent a shortcut to the result we need, whereas the construction of Hausdorff is complicated and spread over several sections of the long paper [1]; moreover, it is written in German and in a somewhat oldfashioned notation that is not always the most elegant (according to our “modern" standards);
- Hausdorff does not construct the ordered sets so that they extend a given ordered set; convincing the reader that his construction can be adapted to accommodate this additional condition would essentially take the same effort as a direct proof.

2. Preliminaries and notations

2.1. Proof of Proposition 1. A quasicut in a linearly ordered set \( I \) is a pair \( (D, E) \) of subsets \( D \) and \( E \) of \( I \) such that \( I = D \cup E \) and \( d \leq e \) for all \( d \in D, e \in E \). In this case, \( D \cap E \) is empty or a singleton; if it is empty, then \( (D, E) \) is a cut.

Assume that every nonprincipal cut in the linearly ordered set \( I \) is asymmetric. Every nonempty chain of closed bounded intervals has a cofinal subchain \( ([d_\nu, e_\nu])_{\nu<\mu} \) indexed by a regular cardinal \( \mu \). We set \( D := \{ d \in I \mid d \leq d_\nu \text{ for some } \nu < \mu \} \) and \( E := \{ e \in I \mid e \geq e_\nu \text{ for some } \nu < \mu \} \). Then \( d \leq e \) for all \( d \in D \) and \( e \in E \). If \( D \cap E \neq \emptyset \), then \( (D, E) \) is a quasicut and the unique element of \( D \cap E \) lies in the intersection of the chain. If \( (D, E) \) is a cut, then because of \( \text{cf}(D) = \mu = \text{ci}(E) \) it must be principal, i.e., \( \mu = 1 \) and \( \{d_0, e_0\} = [d_0, e_0] \) is contained in the intersection of the chain. If \( D \cap E = \emptyset \) but \( (D, E) \) is not a cut, then the set \( \{ c \in I \mid d < c < e \text{ for all } d \in D, e \in E \} \) is nonempty and contained in the intersection of the chain. So in all cases, the intersection of the chain is nonempty.

Now assume that \( I \) is spherically complete w.r.t. the order balls. Suppose that \( (D, E) \) is a cut with \( \kappa := \text{cf}(D) = \text{ci}(E) \). Then we can choose a cofinal sequence \( (d_\nu)_{\nu<\kappa} \) in \( D \) and a coinitial sequence \( (e_\nu)_{\nu<\kappa} \) in \( E \). By assumption, the descending chain \( ([d_\nu, e_\nu])_{\nu<\kappa} \) of intervals has nonempty intersection. But this is only possible if \( \kappa = 1 \), which implies that \( (D, E) \) is principal. This proves that every nonprincipal cut is asymmetric.

2.2. Hahn products. Given a linearly ordered index set \( I \) and for every \( \gamma \in I \) an arbitrary abelian group \( C_\gamma \), we define a group called the Hahn product, denoted by \( \mathbb{H}_{\gamma \in I} C_\gamma \). Consider the product \( \prod_{\gamma \in I} C_\gamma \) and an element \( c = (c_\gamma)_{\gamma \in I} \) of this group. Then the support of \( c \) is the set \( \text{supp} \, c := \{ \gamma \in I \mid c_\gamma \neq 0 \} \). As a set, the Hahn product is the subset of \( \prod_{\gamma \in I} C_\gamma \) containing all elements whose support is a wellordered subset of \( I \), that is, every nonempty subset of the support has a minimal element). In particular, the support of every nonzero element \( c \) in the Hahn product has a minimal
element $\gamma_0$, which enables us to define a group valuation by setting $v c = \gamma_0$ and $v 0 = \infty$. The Hahn product is a subgroup of the product group. Indeed, the support of the sum of two elements is contained in the union of their supports, and the union of two wellordered sets is again wellordered.

We leave it to the reader to show that a Hahn product is divisible if and only if all of its components are.

If the components $C_\gamma$ are (not necessarily archimedean) ordered abelian groups, we obtain the ordered Hahn product, also called lexicographic product, where the ordering is defined as follows. Given a nonzero element $c = (c_\gamma)_{\gamma \in I}$, let $\gamma_0$ be the minimal element of its support. Then we take $c > 0$ if and only if $c_{\gamma_0} > 0$. If all $C_\gamma$ are archimedean ordered, then the valuation $v$ of the Hahn product coincides with the natural valuation of the ordered Hahn product. Every ordered abelian group $G$ can be embedded in the Hahn product with its set of archimedean classes as index sets and its archimedean components as components. Then $G$ is spherically complete w.r.t. the ultrametric balls if and only if the embedding is onto.

2.3. Some facts about cofinalities and coinitialities. Take a nontrivial ordered abelian group $G$ and define

$$G^{>0} := \{ g \in G \mid g > 0 \} \quad \text{and} \quad G^{<0} := \{ g \in G \mid g < 0 \}.$$ 

Since $G \ni g \mapsto -g \in G$ is an order inverting bijection,

$$\text{ci}(G) = \text{cf}(G) \quad \text{and} \quad \text{cf}(G^{<0}) = \text{ci}(G^{>0}).$$

Further, we have:

**Lemma 11.** 1) The cofinality of $G$ is equal to $\max\{\aleph_0, \text{ci}(vG)\}$. Hence it is uncountable if and only if the coinitiality of $vG$ is uncountable.

2) If $G$ is discretely ordered, then $\text{ci}(G^{>0}) = \text{cf}(vG) = 1$. Otherwise, $\text{ci}(G^{>0}) = \max\{\aleph_0, \text{cf}(vG)\}$.

3) Take $\gamma \in vG$, not the largest element of $vG$, and let $\kappa$ be the coinitiality of the set $\{ \delta \in vG \mid \delta > \gamma \}$. Then $\text{cf}(M_\gamma) = \max\{\aleph_0, \kappa\}$.

**Proof:** 1): Since a nontrivial ordered abelian group has no maximal element, its cofinality is at least $\aleph_0$. If $vG$ has a smallest element, then take a positive $g \in G$ whose value is this smallest element. Then the sequence $(ng)_{n \in \mathbb{N}}$ is cofinal in $G$, so its cofinality is $\aleph_0$.

If $\kappa := \text{ci}(vG)$ is infinite, then take a sequence $(\gamma_\nu)_{\nu < \kappa}$ which is coinitial in $vG$, and take positive elements $g_\nu \in G$, $\nu < \kappa$, with $vg_\nu = \gamma_\nu$. Then the sequence $(g_\nu)_{\nu < \kappa}$ is cofinal in $G$ and therefore, $\text{cf}(G) \leq \text{ci}(vG)$. On the other hand, for every sequence $(g_\nu)_{\nu < \lambda}$ cofinal in $G$, the sequence of values $(vg_\nu)_{\nu < \lambda}$ must be coinitial in $vG$, which shows that $\text{cf}(G) \geq \text{ci}(vG)$.

2) If $G$ is discretely ordered, then it has a smallest positive element $g$ and hence, $\text{ci}(G^{>0}) = 1$. Further, $vg$ must be the largest element of $vG$, so $\text{cf}(vG) = 1$. 

If $G$ is not discretely, hence densely ordered, then the coinitiality of $G^0$ is at least $\aleph_0$. If $vG$ has a largest element $\gamma$, then we take a positive $g \in G$ with $vg = \gamma$. Then $M_\gamma = \{0\}$ and $O_\gamma$ is an archimedean ordered convex subgroup of $G$. This implies that $\text{ci}(G^0) = \text{ci}(O^0) = \aleph_0$.

If $\kappa := \text{cf}(vG)$ is infinite, then take a sequence $(\gamma_\nu)_{\nu < \kappa}$ which is cofinal in $vG$, and take positive elements $g_\nu \in G$, $\nu < \kappa$, with $vg_\nu = \gamma_\nu$. Then the sequence $(g_\nu)_{\nu < \kappa}$ is cofinal in $G^0$ and therefore, $\text{ci}(G^0) \leq \text{cf}(vG)$. On the other hand, for every sequence $(g_\nu)_{\nu < \lambda}$ cofinal in $G^0$, the sequence of values $(vg_\nu)_{\nu < \lambda}$ must be cofinal in $vG$, which shows that $\text{ci}(G^0) \geq \text{cf}(vG)$.

3: By our condition on $\gamma$, $M_\gamma$ is a nontrivial subgroup of $G$ and therefore, its cofinality is at least $\aleph_0$. If $\nu M_\gamma = \{\delta \in vG \mid \delta > \gamma\}$ has a smallest element, then take a positive $g \in G$ whose value is this smallest element. Then the sequence $(ng)_{n \in \mathbb{N}}$ is cofinal in $M_\gamma$, so $\text{cf}(M_\gamma) = \aleph_0$.

Assume that $\kappa = \text{ci}(vM_\gamma)$ is infinite. Take a sequence $(\gamma_\nu)_{\nu < \kappa}$ which is cofinal in $vM_\gamma = \{\delta \in vG \mid \delta > \gamma\}$ and take positive elements $g_\nu \in G$, $\nu < \kappa$, with $vg_\nu = \gamma_\nu$. Then the sequence $(g_\nu)_{\nu < \kappa}$ is cofinal in $M_\gamma$ and therefore, $\text{cf}(M_\gamma) \leq \kappa$. On the other hand, for every sequence $(g_\nu)_{\nu < \lambda}$ cofinal in $M_\gamma$, the sequence of values $(vg_\nu)_{\nu < \lambda}$ must be cofinal in $vM_\gamma$, which shows that $\text{cf}(M_\gamma) \geq \kappa$. □

3. Analysis of cuts in ordered abelian groups

In this section, we will use the facts outlined in Section 2.3 freely without further citation.

Take a cut $C = (D, E)$ with cofinality $(\kappa, \lambda)$ in the ordered abelian group $G$. First assume that $C$ is principal. If $D$ has largest element $g$, then the set $g + G^0$ is cofinal in $E$. Hence in this case, $C$ has cofinality $(1, \lambda)$ with $\lambda = \text{ci}(G^0)$. Symmetrically, if $E$ has smallest element $g$, then the set $g + G^0$ is cofinal in $D$. Hence in this case, $C$ has cofinality $(\kappa, 1)$ with $\kappa = \text{cf}(G^0) = \text{ci}(G^0)$.

If $G$ is discretely ordered, then $\text{ci}(G^0) = 1$ by part 2) of Lemma 11. So for every principal cut to be asymmetric, it is necessary that $G$ is not discretely, hence densely ordered. If $G$ is densely ordered, then $\text{ci}(G^0) = \max\{\aleph_0, \text{cf}(vG)\}$. So we obtain:

**Lemma 12.** Take any ordered abelian group $G$. Every principal cut in $G$ is asymmetric if and only if $G$ is densely ordered. Every principal cut in $G$ is strongly asymmetric if and only if in addition, $\text{cf}(vG)$ is uncountable.

From now on we assume that the cut $C$ in $G$ is nonprincipal. Then the only countable cardinality that can appear as coinitiality or cofinality is $\aleph_0$. This shows:

**Lemma 13.** If a nonprincipal cut is asymmetric, then it is strongly asymmetric.
We consider the ultrametric balls $B_u(d, e)$ for all $d \in D$, $e \in E$. Any two of them have nonempty intersection since this intersection will contain both a final segment of $D$ and an initial segment of $E$. Since two ultrametric balls with nonempty intersection are already comparable by inclusion, it follows that these balls form a nonempty chain. Now there are two cases: I) the chain contains a smallest ball, II) the chain does not contain a smallest ball.

First, we discuss cuts of type I). We choose $d_0 \in D$, $e_0 \in E$ such that $B_u(d_0, e_0)$ is the smallest ball. The shifted cut

$$C - d_0 := \{\{d - d_0 \mid d \in D\}, \{e - d_0 \mid e \in E\}$$

has the same cofinality as $C$. Moreover,

$$B_u(d_0, e_0) - d_0 := \{b - d_0 \mid b \in B_u(d_0, e_0)\} = B_u(0, e_0 - d_0)$$

remains the smallest ball in the new situation. Therefore, we can assume that $d_0 = 0$. Set $\gamma := v e_0$ and $I := [0, e_0]$. Then $v h \geq \gamma$ for all $h \in I$, that is, $h \in O_\gamma$. The images $D'$ of $D \cap I$ and $E'$ of $E \cap I$ in $C_\gamma = O_\gamma / M_\gamma$ are convex and satisfy $D' \leq E'$. If there were $d' \in D' \cap E'$, then it would be the image of elements $d \in D \cap I$, and $e \in E \cap I$, with $\gamma < v(e - d)$, and $B_u(d, e)$ would be a ball properly contained in $B_u(0, e_0)$, contrary to our minimality assumption. Hence, $D' < E'$. If there were an element strictly between $D'$ and $E'$, then it would be the image of an element $h - d_0$ with $h$ strictly between $D$ and $E$, which is impossible. So we see that $(D', E')$ defines a cut $C''$ in $C_\gamma$, with $D'$ a final segment of the left cut set and $E'$ an initial segment of the right cut set.

Since $C_\gamma$ is archimedean ordered, the cofinality of $C'$ can only be $(1, 1)$, $(1, R_0)$, $(R_0, 1)$, or $(R_0, R_0)$. Lifting cofinal sequences in $D'$ back into $D$, we see that if the cofinality of $D'$ is $R_0$, then so is the cofinality of $D$. Similarly, if the cofinality of $E'$ is $R_0$, then so is the cofinality of $E$. However, if $D'$ contains a last element $a'$, and if $a \in D \cap I$ is such that $a$ has image $a'$ in $C_\gamma$, then the set of all elements in $G$ that are sent to $a'$ is exactly the coset $a + M_\gamma$. This set has empty intersection with $E$ since $a' \notin E'$. This together with $a'$ being the last element of $D'$ shows that $a + M_\gamma$ is a final segment of $D$ and therefore, the cofinality of $D$ is equal to that of $M_\gamma$. Similarly, if $E'$ has a first element $b'$ coming from an element $b \in E \cap I$, then $b + M_\gamma$ is an initial segment of $E$ and therefore, the cofinality of $E$ is equal to that of $M_\gamma$, which in turn is equal to the cofinality of $M_\gamma$. If $\lambda$ denotes this cofinality, we see that the cofinality of $C$ is

a) $(\lambda, \lambda)$ if $C'$ has cofinality $(1, 1)$,

b) $(\lambda, \mathbb{N}_0)$ or $\langle \mathbb{N}_0, \lambda \rangle$ if $C'$ has cofinality $(1, \mathbb{N}_0)$ or $(\mathbb{N}_0, 1)$, and

c) $\langle \mathbb{N}_0, \lambda \rangle$ if $C'$ has cofinality $(\mathbb{N}_0, \mathbb{N}_0)$.

Cofinality $(1, 1)$ can only appear for $C'$ if $C_\gamma$ is isomorphic to $\mathbb{Z}$, and then every cut in $C_\gamma$ has this cofinality. In this case, $C$ is principal (and thus "out of scope" in our present discussion) if and only if $\lambda = 1$, which means that
\( \mathcal{M}_\gamma = \{0\} \) and thus, \( \gamma \) is the maximal element of \( vG \) and \( O_\gamma \simeq C_\gamma \simeq \mathbb{Z} \), showing that \( G \) is discretely ordered.

Cofinality \( (\aleph_0, \aleph_0) \) can only appear (and will appear) for \( C'_\gamma \) if \( C_\gamma \) has nonprincipal cuts. If \( (D'_1, E'_1) \) is a cut in \( C_\gamma \) with cofinality \( (\aleph_0, \aleph_0) \), then we set

\[
D_1 := \{ d \in G \mid d \leq d_1 \text{ for some } d_1 \in O_\gamma \text{ with } d_1 + M_\gamma \in D'_1 \},
E_1 := \{ e \in G \mid e \geq e_1 \text{ for some } e_1 \in O_\gamma \text{ with } e_1 + M_\gamma \in E'_1 \}.
\]

This defines a cut in \( G \) with cofinality \( (\aleph_0, \aleph_0) \).

We conclude that the only choice for the components \( C_\gamma \) that prevents cofinality \( (\lambda, \lambda), \lambda \geq \aleph_0 \), for \( C \) is: \( C_\gamma \simeq \mathbb{R} \) for all \( \gamma \in vG \), or \( C_\gamma \simeq \mathbb{Z} \) if \( \gamma \) is the smallest element of \( vG \) and \( C_\gamma \simeq \mathbb{R} \) otherwise.

If this condition is satisfied, then all nonprincipal cuts of type I have cofinalities \( (\lambda, \aleph_0) \) or \( (\aleph_0, \lambda) \). Hence all of them are asymmetric if and only if for all \( \gamma \in vG \) not the last element of \( vG \), the cofinality of \( M_\gamma \) is uncountable. By part 3) of Lemma 11 this happens if and only if the cut

\[ \gamma^+ := (\{ \delta \in vG \mid \delta \leq \gamma \}, \{ \delta \in vG \mid \delta > \gamma \}) \]

in \( vG \) has cofinality \( (1, \lambda) \) with \( \lambda \) uncountable.

Note that every cut in \( vG \) of cofinality \( (1, \lambda) \) is of the form \( \gamma^+ \), in which case the upper cut set will be the value set \( vM_\gamma \) of \( M_\gamma \). Then the cut

\[ (\{ d \in G \mid d < M_\gamma \}, \{ e \in G \mid e \geq c \text{ for some } c \in M_\gamma \}) \]

in \( G \) will have cofinality \( (\aleph_0, \lambda') \) with \( \lambda' = \aleph_0 \) if \( \lambda = 1 \) and \( \lambda' = \lambda \) otherwise.

So for every nonprincipal cut of type I to be asymmetric it is also necessary that for every cut of cofinality \( (1, \lambda) \) in \( vG \), \( \lambda \) is uncountable.

We summarize our discussion so far:

**Lemma 14.** Take any ordered abelian group \( G \). Then every nonprincipal cut of type I is (strongly) asymmetric if and only if the following conditions are satisfied:

a) \( C_\gamma \simeq \mathbb{R} \) for all \( \gamma \in vG \), or \( C_\gamma \simeq \mathbb{Z} \) if \( \gamma \) is the largest element of \( vG \) and \( C_\gamma \simeq \mathbb{R} \) otherwise.

b) for every cut in \( vG \) of cofinality \( (1, \lambda) \), \( \lambda \) is uncountable.

Now we discuss nonprincipal cuts \( C \) of type II). We assume in addition that the ordered abelian group \( G \) is spherically complete w.r.t. its natural valuation \( v \). Then there is some \( g \in G \) such that

\[ g \in \bigcap_{d \in D, e \in E} B_v(d, e) \]

Replacing the cut \( C \) by the shifted cut \( C - g \) as we have done before already, we can assume that \( g = 0 \). Since \( C \) is nonprincipal by assumption, there
must be \( d_0 \in D, e_0 \in E \) such that \( d_0 \leq 0 \leq e_0 \) does not hold, and we have two cases:

A) \( e_0 < 0 \), 

B) \( 0 < d_0 \).

Again, we set \( I := [d_0, e_0] \). We set \( \tilde{D} = \{vd \mid d \in D \cap I\} \subseteq vG \) and \( \tilde{E} = \{ve \mid e \in E \cap I\} \subseteq vG \).

Let us first discuss case A). We claim that \( \tilde{D} < \tilde{E} \). We observe that “\( \leq \)” holds since \( d < e < 0 \) for \( d \in D \cap I \) and \( e \in E \cap I \). Suppose that \( \tilde{D} \cap \tilde{E} \neq \emptyset \), that is, \( vd = ve \) for some \( d \in D \cap I \) and \( e \in E \cap I \). Then \( v(e - d) \geq vd \) by the ultrametric triangle law, and since there is no smallest ball by assumption, we can even choose \( d, e \) such that \( v(e - d) > vd \). But then, \( 0 \) would not lie in \( B_{va}(d, e) \), a contradiction. We have proved our claim. Now if there were an element \( \alpha \) strictly between the two sets, then there were some \( \alpha \in I \) with \( va = \alpha \) and \( a > 0 \). This would yield that \( d < a < e \) for all \( d \in D \cap I \) and \( e \in E \cap I \) and thus, \( D \subset a < E \), a contradiction.

We conclude that \( (\tilde{D}, \tilde{E}) \) defines a cut \( \tilde{C} \) in \( vG \), with \( \tilde{D} \) a final segment of the left cut set, and \( \tilde{E} \) an initial segment of the right cut set. Denote by \( (\tilde{\kappa}, \tilde{\lambda}) \) its cofinality. We have that \( vd < ve \) and consequently \( vd = v(e - d) \) for all \( d \in D \cap I \) and \( e \in E \cap I \). Since by assumption there is no smallest ball, there is no largest value \( v(e - d) \). This shows that \( \tilde{D} \) has no largest element and therefore, \( \tilde{\kappa} \) is infinite. Lifting cofinal sequences in \( \tilde{D} \) to coinitial sequences in \( D \), we see that \( \kappa = \tilde{\kappa} \). By the same argument, if \( \tilde{\lambda} \) is infinite, then \( \lambda = \tilde{\lambda} \). If on the other hand \( \tilde{\lambda} = 1 \), then we take \( \gamma \in G \) to be the smallest element of \( \tilde{E} \). The preimage of \( \gamma \) under the valuation is \( \mathcal{O}_\gamma \setminus \mathcal{M}_\gamma \), and this set is coinitial in \( \tilde{E} \). The cofinality of \( \mathcal{O}_\gamma \setminus \mathcal{M}_\gamma \) is equal to the cofinality of \( \mathcal{O}_\gamma \), which in turn is equal to the cofinality \( \aleph_0 \) of the archimedean ordered group \( \mathcal{C}_\gamma \). Hence in this case, \( \lambda = \aleph_0 \).

From this discussion it follows that \( C \) is asymmetric if and only if \( \tilde{C} \) is strongly asymmetric.

Now we consider case B). Since \( 0 < d < e \) for \( d \in D \cap I \) and \( e \in E \cap I \), we now obtain that \( \tilde{E} \leq \tilde{D} \). It is proven as in case A) that \( (\tilde{E}, \tilde{D}) \) defines a cut \( \tilde{C} \) in \( vG \), and that \( \tilde{E} \) has no largest element. In this case the argument is the same as before, but with \( \tilde{D} \) and \( \tilde{E} \) interchanged, and the conclusion is the same as in case A). We note that in both cases, the cofinality of the left cut set of \( C \) must be infinite.

If we have a cut \( \tilde{C} = (\tilde{D}, \tilde{E}) \) in \( vG \) of cofinality \( (\tilde{\kappa}, \tilde{\lambda}) \) with \( \tilde{\kappa} \) infinite, then we can associate to it a nonprincipal cut of type II as follows. We set \( D = \{d \in G \mid d < 0 \text{ and } vd \in \tilde{D}\} \) and \( E = \{e \in G \mid e > 0 \text{ or } ve \in \tilde{E}\} \). This is a cut in \( G \), and it is of type II since for all \( d \in D \) and \( e \in E \), \( e < 0 \), we have that \( vd < ve \) and hence \( v(e - d) = vd \in \tilde{D} \), which has no largest element. Then \( C \) induces the cut \( \tilde{C} \) in the way described under case A).
Thus for every cut of type II to be asymmetric, it is necessary that every cut in $vG$ of cofinality $(\tilde{\kappa}, \tilde{\lambda})$ with $\tilde{\kappa}$ infinite is strongly asymmetric.

We summarize:

**Lemma 15.** Take any ordered abelian group $G$ which is spherically complete w.r.t. its natural valuation $v$. Then every nonprincipal cut of type II is asymmetric (and hence strongly asymmetric) if and only if every cut in $vG$ of cofinality $(\kappa, \lambda)$ with $\kappa$ infinite is strongly asymmetric.

4. Proof of the main theorems

**Proof of Theorem 4**

Take any ordered abelian group $G$. Assume first that it is symmetrically complete. Then by Proposition 3, $G$ is spherically complete w.r.t. its natural valuation. By Lemma 12, $G$ is densely ordered. Thus it cannot have an archimedean component $C_\gamma \simeq \mathbb{Z}$ with $\gamma$ the largest element of $vG$ because otherwise, it would have a convex subgroup isomorphic to $\mathbb{Z}$ and would then be discretely ordered. Hence by Lemma 14, every archimedean component of $G$ is isomorphic to $\mathbb{R}$ and for every cut in $vG$ of cofinality $(1, \kappa)$, $\kappa$ is uncountable. Finally by Lemma 15, every cut in $vG$ of cofinality $(\lambda, \kappa)$ with $\lambda$ infinite is strongly asymmetric. Altogether, every cut in $vG$ is strongly asymmetric. This proves that $vG$ is strongly symmetrically complete.

If $G$ is spherically complete w.r.t. its natural valuation, every archimedean component of $G$ is isomorphic to $\mathbb{R}$ and $vG$ is strongly symmetrically complete, then in particular, $G$ is densely ordered, and it follows from Lemmas 12, 14 and 15 that $G$ is symmetrically complete.

From Lemma 13 we see that for a symmetrically complete $G$ to be strongly symmetrically complete it suffices that every principal cut is strongly asymmetric, which by Lemma 12 holds if and only if in addition to the other conditions, the cofinality of $vG$ is uncountable.

A strongly symmetrically complete $G$ is extremely symmetrically complete if and only if in addition, its cofinality (which is equal to its coinitiality) is uncountable. By part 1) of Lemma 11 this holds if and only if the coinitiality of $vG$ is uncountable. Hence by what we have just proved before, a symmetrically complete $G$ is extremely symmetrically complete if and only if in addition, $vG$ is extremely symmetrically complete.

**Proof of Theorem 5**

Considering the additive ordered abelian group of the ordered field $K$, the first assertion of Theorem 5 follows readily from that of Theorem 4 if one takes into account that through multiplication, all archimedean components are isomorphic to the ordered additive group of the residue field.

Similarly, the equivalence of b) and c) follows from the third case of Theorem 4. Since $vK$ is an ordered abelian group, its cofinality is equal to its coinitiality, so the condition that it is strongly symmetrically complete with
uncountable cofinality already implies that it is extremely symmetrically complete. Hence, by the second case of Theorem 4] a) is equivalent with c). □

Proof of Corollary 6
The assertion for ordered abelian groups follows from the facts that have been mentioned before. For ordered fields, it remains to show that a power series field with residue field \( \mathbb{R} \) and divisible value group is real closed. Since every power series field is henselian under its canonical valuation, this follows from \([8, \text{ Theorem (8.6)}]\). □

Proof of Theorem 9
The equivalence of a) and b) follows directly from Proposition 1. It remains to prove the equivalence of a) and c).

If \( G \) is discretely ordered, then \( vG \) must have a largest element \( vg \) (where \( g \) can be chosen to be the smallest positive element of \( G \)) with archimedean component \( O_{vg} \cong C_{vg} \cong \mathbb{Z} \). We identify the convex subgroup \( O_{vg} \) with \( \mathbb{Z} \).

We will now prove that \( G \) is symmetrically d-complete if and only if \( G/\mathbb{Z} \) is strongly symmetrically complete.

Take any cut \( (D, E) \) in \( G \). Since the canonical epimorphism \( G \to G/\mathbb{Z} \) preserves \( \le \), the image \( (\overline{D}, \overline{E}) \) of \( (D, E) \) in \( G/\mathbb{Z} \) is a quasicut. If \( \overline{D} \) and \( \overline{E} \) have a common element \( \overline{d} \), then there is \( d \in D \) and \( z \in \mathbb{Z} \) such that \( d + z \in E \). In this case, the cofinality of \( (D, E) \) is \( (1, 1) \). Now suppose that \( \overline{D} \) and \( \overline{E} \) have no common element. Then for all \( d \in D \) and \( e \in E \), we have that \( d + z = \{ d + z \mid d \in D, z \in \mathbb{Z} \} \subset D \) and \( e + z \subset E \). Hence if \( D' \subset D \) is a set of representatives for \( \overline{D} \) and \( E' \subset E \) is a set of representatives for \( \overline{E} \), then \( D = D' + \mathbb{Z} = \{ d + z \mid d \in D', z \in \mathbb{Z} \} \) and \( E = E' + \mathbb{Z} \). This yields that

\[
\begin{align*}
(2) & \quad \left\{ \begin{array}{l}
\text{cf}(D) = \max\{\text{cf}(D'), \aleph_0\} = \max\{\text{cf}(\overline{D}), \aleph_0\}, \\
\text{ci}(E) = \max\{\text{ci}(E'), \aleph_0\} = \max\{\text{ci}(\overline{E}), \aleph_0\}.
\end{array} \right.
\end{align*}
\]

Suppose that \( G/\mathbb{Z} \) is strongly symmetrically complete and that \( (D, E) \) is a cut in \( G \) of cofinality \( \neq (1, 1) \). Then by what we have just shown, \( (\text{cf}(D), \text{ci}(E)) = (\max\{\text{cf}(\overline{D}), \aleph_0\}, \max\{\text{ci}(\overline{E}), \aleph_0\}) \). By our assumption on \( G/\mathbb{Z} \), \( (\overline{D}, \overline{E}) \) is strongly asymmetric, which yields that \( \text{cf}(D) \) and \( \text{ci}(E) \) are not equal and at least one of them is uncountable.

is equal to \( \aleph_0 \) and the other is uncountable, or that In each case, \( (D, E) \) is strongly asymmetric. This proves that \( G \) is symmetrically d-complete.

For the converse, assume that \( G \) is symmetrically d-complete and that \( (\overline{D}, \overline{E}) \) is any cut in \( G/\mathbb{Z} \). Then we pick a set \( D' \subset G \) of representatives for \( \overline{D} \) and a set \( E' \subset E \) of representatives for \( \overline{E} \). With \( D = D' + \mathbb{Z} \) and \( E = E' + \mathbb{Z} \) we obtain a nonprincipal cut \( (D, E) \) in \( G \) with image \( (\overline{D}, \overline{E}) \) in \( G/\mathbb{Z} \). As before, \( (2) \) holds. By our assumption on \( G \), the cut \( (D, E) \) is strongly asymmetric. This implies that at least one of \( \text{cf}(\overline{D}) \) and
are ordered sets, then \( I \) disjoint union symmetrically d-complete, then it cannot be isomorphic to \( \mathbb{Z} \) symmetrically complete.

For any linearly ordered set \( I = (I,<) \), we denote by \( I^c \) its completion. Note that \( \text{Coin}(I^c) = \text{Coin}(I) \) and \( \text{Cofin}(I^c) = \text{Cofin}(I) \). Further, we denote by \( I' \) the set \( I \) endowed with the inverted ordering \( <^* \), where \( i <^* j \Leftrightarrow j < i \).

If \( I' \) is another ordered set, then \( I + I' \) is the sum in the sense of order theory, that is, the orderings of \( I \) and \( I' \) are extended to \( I \cup I' \) in the unique way such that \( i < i' \) for all \( i \in I \) and \( i' \in I' \).

We choose some ordered set \( I \) (where \( I = \emptyset \) is allowed) and infinite regular cardinals \( \mu \) and \( \kappa \), \( \lambda \), for all \( \nu < \mu \). We define

\[
I_0 := \lambda^*_0 + I^c + \kappa_0 \quad \text{and} \quad I_\nu := \lambda^*_\nu + \kappa_\nu \quad \text{for} \quad 0 < \nu < \mu.
\]

Note that all \( I_\nu \), \( \nu < \mu \), are cut complete. Note further that if \( C \) is a cut in \( I_0 \) with cofinality \( (\kappa, \lambda) \), then \( \kappa \in \text{Cofin}(I) \cup \text{Reg}_{<\kappa_0} \) and \( \lambda \in \text{Coin}(I) \cup \text{Reg}_{<\lambda_0} \).

We define \( J \) to be the lexicographic product over the \( I_\nu \) with index set \( \mu \); that is, \( J \) is the set of all sequences \( (\alpha_\nu)_{\nu<\mu} \) with \( \alpha_\nu \in I_\nu \) for all \( \nu < \mu \), endowed with the following ordering: if \( (\alpha_\nu)_{\nu<\mu} \) and \( (\beta_\nu)_{\nu<\mu} \) are two different sequences, then there is a smallest \( \nu_0 < \mu \) such that \( \alpha_{\nu_0} \neq \beta_{\nu_0} \) and we set \( (\alpha_\nu)_{\nu<\mu} < (\beta_\nu)_{\nu<\mu} \) if \( \alpha_{\nu_0} < \beta_{\nu_0} \).

**Theorem 16.** The cofinalities of the cuts of \( J \) are:

\[
(1, \mu), (\mu, 1), (\kappa_1, \lambda), (\kappa, \lambda_1) \quad \text{for} \quad \lambda \in \text{Coin}(I) \cup \text{Reg}_{<\kappa_0}, \kappa \in \text{Cofin}(I) \cup \text{Reg}_{<\kappa_0},
\]

\[
(\kappa_\nu+1, \lambda), (\kappa, \lambda_\nu+1) \quad \text{for} \quad 0 < \nu < \mu \quad \text{and} \quad \kappa < \kappa_\nu, \lambda < \lambda_\nu \quad \text{regular cardinals,}
\]

\[
(\kappa_\nu, \lambda_\nu) \quad \text{for} \quad \nu < \mu \quad \text{a successor ordinal, and}
\]

\[
(\kappa_\nu, \mu'), (\mu', \lambda_\nu) \quad \text{for} \quad \nu < \mu \quad \text{a limit ordinal and} \quad \mu' < \mu \quad \text{its cofinality.}
\]

Further, the cofinality of \( J \) is \( \kappa_0 \) and its coinitiality is \( \lambda_0 \).

**Proof:** Take any cut \( (D, E) \) in \( I \). Assume first that \( D \) has a maximal element \( (\alpha_\nu)_{\nu<\mu} \). By our choice of the linearly ordered sets \( I_\nu \) we can choose, for every \( \nu < \mu \), some \( \beta_\nu \in I_\nu \) such that \( \beta_\nu > \alpha_\nu \). For \( \rho < \mu \) we define \( \beta^*_\rho := \beta_\rho \) and \( \beta^*_\nu := \alpha_\nu \) for \( \nu \neq \rho \). Then the elements \( (\beta^*_\nu)_{\nu<\mu}, \rho < \mu \), form a strictly decreasing coinitial sequence of elements in \( E \). Since \( \mu \) was chosen
to be regular, this shows that the cofinality of \((D, E)\) is \((1, \mu)\). Similarly, it is shown that if \(E\) has a minimal element, then the cofinality of \((D, E)\) is \((\mu, 1)\).

Now assume that \((D, E)\) is nonprincipal. Take \(S\) to be the set of all \(\nu' < \mu\) for which there exist \(a_{\nu'} = (\alpha_{\nu', \nu})_{\nu < \mu} \in D\) and \(b_{\nu'} = (\beta_{\nu', \nu})_{\nu < \mu} \in E\) such that \(\alpha_{\nu', \nu} = \beta_{\nu', \nu}\) for all \(\nu \leq \nu'\). Note that \(S\) is a proper initial segment of the set \(\mu\). We claim that \(\nu_1 < \nu_2 \in S\) implies that

\[
\alpha_{\nu_1, \nu'} = \alpha_{\nu_2, \nu'} \quad \text{for all } \nu \leq \nu_1,
\]
or in other words, \((\alpha_{\nu_1, \nu})_{\nu \leq \nu_1}\) is a truncation of \(a_{\nu_2}\). Indeed, suppose that this were not the case. Then there would be some \(\nu' < \nu_1\) such that

\[
\beta_{\nu_1, \nu'} = \alpha_{\nu_1, \nu'} \neq \alpha_{\nu_2, \nu'} = \beta_{\nu_2, \nu'}.
\]

Suppose that \(\nu'\) is minimal with this property and that the left hand side is smaller. But then, \((\beta_{\nu_1, \nu'})_{\nu < \mu} < (\alpha_{\nu_2, \nu'})_{\nu < \mu}\), so \(b_{\nu_1} \in D\), a contradiction. A similar contradiction is obtained if the right hand side is smaller.

Now take \(\mu_0\) to be the minimum of \(\mu \setminus S\); in fact, \(S\) is is equal to the set \(\mu_0\). We define

\[
D_{\mu_0} := \{ \alpha \in I_{\mu_0} \mid \exists (\alpha_{\nu})_{\nu < \mu} \in D : \alpha_{\mu_0} = \alpha \text{ and } \alpha_{\nu} = \alpha_{\nu, \nu} \text{ for } \nu < \mu_0 \},
\]

\[
E_{\mu_0} := \{ \beta \in I_{\mu_0} \mid \exists (\beta_{\nu})_{\nu < \mu} \in E : \beta_{\mu_0} = \beta \text{ and } \beta_{\nu} = \alpha_{\nu, \nu} \text{ for } \nu < \mu_0 \}.
\]

By our definition of \(\mu_0\), these two sets are disjoint, and it is clear that their union is \(I_{\mu_0}\) and every element in \(D_{\mu_0}\) is smaller than every element in \(E_{\mu_0}\). However, one of the sets may be empty, and we will first consider this case. Suppose that \(E_{\mu_0} = \emptyset\). Then \(D_{\mu_0} = I_{\mu_0}\) and since this has no last element, the cofinality of \(D\) is the same as that of \(I_{\mu_0}\), which is \(\kappa_{\mu_0}\). In order to determine the cofinality of \(E\), we proceed as in the beginning of this proof. Observe that since \(E_{\mu_0} = \emptyset\), for an element \((\beta_{\nu}')_{\nu < \mu}\) to lie in \(E\) it is necessary that \(\beta_{\nu}' > \alpha_{\nu, \nu}\) for some \(\nu < \mu_0\). For all \(\nu < \mu_0\), choose some \(\beta_\nu \in I_\nu\) such that \(\beta_\nu > \alpha_{\nu, \nu}\); then for all \(\rho < \mu_0\) we define \(\beta_\rho := \beta_\rho', \beta_\nu := \alpha_{\nu, \nu} \) for \(\nu < \rho\), and choose \(\beta_\rho\) arbitrarily for \(\rho < \nu < \mu\). Then the elements \((\beta_{\nu}')_{\nu < \mu}, \rho < \mu_0\), form a strictly decreasing coinitial sequence in \(E\).

If \(\mu'\) denotes the cofinality of \(\mu_0\), this shows that the coinitiality of \(E\) is \(\mu'\), and the cofinality of \(\mu\) is \((\kappa_{\mu_0}, \mu')\). Since \(\mu\) was chosen to be regular, we have that \(\mu' < \mu\).

Similarly, it is shown that if \(D_{\mu_0} = \emptyset\), then the cofinality of \((D, E)\) is \((\mu', \lambda_{\mu_0})\). Note that \(D_{\mu_0}\) or \(E_{\mu_0}\) can only be empty if \(\mu_0\) is a limit ordinal. Indeed, if \(\mu_0 = 0\) and \((\alpha_{\nu})_{\nu < \mu} \in D\), \((\beta_{\nu})_{\nu < \mu} \in E\), then \(\alpha_0 \in D_0\) and \(\beta_0 \in E_0\); if \(\mu_0 = \mu' + 1\), then with \((\alpha_{\nu'})_{\nu < \mu} \in D\) and \((\beta_{\mu', \nu})_{\nu < \mu} \in E\) chosen as before, it follows that \(\alpha_{\mu', \nu_0} \in D_{\mu_0}\) and \(\beta_{\mu', \nu_0} \in E_{\mu_0}\).

From now on we assume that both \(D_{\mu_0}\) and \(E_{\mu_0}\) are nonempty. Since \(I_{\mu_0}\) is complete, \(D_{\mu_0}\) has a maximal element or \(E_{\mu_0}\) has a minimal element.

Suppose that \(D_{\mu_0}\) has a maximal element \(\tilde{\alpha}\). Then for all \(\rho \in \kappa_{\mu_0 + 1} \subseteq I_{\mu_0 + 1}\), we define \(\alpha_\rho = \alpha_{\nu, \nu} \) for \(\nu < \mu_0\), \(\alpha_{\mu_0} = \tilde{\alpha}\), \(\alpha_{\mu_0 + 1} = \rho\), and choose an
arbitrary element of $I_0$ for $\alpha'_\nu$ when $\mu_0 + 1 < \nu < \mu$. Then the elements $(\alpha'_\nu)_{\nu < \mu}$, $\rho \in \kappa_{\mu_0 + 1}$, form a strictly increasing cofinal sequence in $D$. Since $\kappa_{\mu_0 + 1}$ was chosen to be a regular cardinal, this shows that the cofinality of $D$ is $\kappa_{\mu_0 + 1}$.

Suppose that $E_{\mu_0}$ has a minimal element $\tilde{\beta}$. Then for every $\sigma \in \lambda^*_{\mu_0 + 1} \subseteq I_{\mu_0 + 1}$, we define $\beta^\sigma_\nu = \alpha_{\nu, \nu}$ for $\nu \leq \mu_0$, $\beta^\sigma_{\mu_0} = \tilde{\beta}$, $\beta^\sigma_{\mu_0 + 1} = \sigma$, and choose an arbitrary element of $I_\nu$ for $\beta^\sigma_\nu$ when $\mu_0 + 1 < \nu < \mu$. Then the elements $(\beta^\sigma_\nu)_{\nu < \mu}$, $\sigma \in \lambda^*_{\mu_0 + 1}$, form a strictly decreasing coinitial sequence in $E$. Since $\lambda_{\mu_0 + 1}$ was chosen to be a regular cardinal, this shows that the coinitiality of $E$ is $\lambda_{\mu_0 + 1}$.

If $D_{\mu_0}$ has a maximal element and $E_{\mu_0}$ has a minimal element, then we obtain that the cofinality of $(D, E)$ is $(\kappa_{\mu_0 + 1}, \lambda_{\mu_0 + 1})$.

Now we deal with the case where $D_{\mu_0}$ does not have a maximal element. Since $I_{\mu_0}$ is complete, $E_{\mu_0}$ must then have a smallest element, and by what we have already shown, we find that $E$ has coinitiality $\lambda_{\mu_0 + 1}$. Denote the cofinality of $D_{\mu_0}$ by $\kappa$. We choose a sequence of elements $\alpha^\rho_\nu$, $\rho < \kappa$, cofinal in $D_{\mu_0}$. For all $\rho < \kappa$, we define $\alpha^\rho_\nu = \alpha_{\nu, \nu}$ for $\nu < \mu_0$ and choose an arbitrary element of $I_\nu$ for $\alpha^\rho_\nu$ when $\mu_0 + 1 < \nu < \mu$. Then the elements $(\alpha^\rho_\nu)_{\nu < \mu}$, $\rho < \kappa$, form a strictly increasing cofinal sequence in $D$. Hence, $(D, E)$ has cofinality $(\kappa, \lambda_{\mu_0 + 1})$ with $\kappa$ the coinitiality of a lower cut set in $I_{\mu_0}$, i.e., $\kappa \in \text{Cofin}(I) \cup \text{Reg}_{< \kappa_0}$ if $\mu_0 = 0$, and $\kappa \in \text{Reg}_{< \kappa_{\mu_0}}$ otherwise.

If $E_{\mu_0}$ does not have a minimal element, then a symmetrical argument shows that the cofinality of $(D, E)$ is $(\kappa_{\mu_0 + 1}, \lambda)$ for some $\lambda$ the coinitiality of an upper cut set in $I_{\mu_0}$, i.e., $\lambda \in \text{Coin}(I) \cup \text{Reg}_{< \lambda_0}$ if $\mu_0 = 0$, and $\lambda \in \text{Reg}_{< \lambda_{\mu_0}}$ otherwise.

We have now proved that the cofinalities of the cuts in $J$ are all among those listed in the statement of the theorem. By our arguments it is also clear that all listed cofinalities do indeed appear.

Finally, the easy proof of the last statement of the theorem is left to the reader. \qed

The following result is an immediate consequence of the theorem:

**Corollary 17.** Assume that

- (a) $\kappa_1 \notin \text{Coin}(I) \cup \text{Reg}_{< \lambda_0}$ and $\lambda_1 \notin \text{Cofin}(I) \cup \text{Reg}_{< \kappa_0}$,
- (b) $\kappa_{\nu + 1} \geq \lambda_\nu$ and $\lambda_{\nu + 1} \geq \kappa_\nu$ for all $\nu < \mu$,
- (c) $\kappa_\nu \neq \lambda_\nu$ for $\nu < \mu$ a successor ordinal, and
- (d) $\kappa_\nu \geq \mu$ and $\lambda_\nu \geq \mu$ for $\nu < \mu$ a limit ordinal.

Then $J$ is symmetrically complete. If in addition $\mu$ is uncountable, then $J$ is strongly symmetrically complete, and if also $\kappa_0$ and $\lambda_0$ are uncountable, then $J$ is extremely symmetrically complete.
It is easy to choose our cardinals by transfinite induction in such a way that all conditions of this corollary are satisfied. We choose

- \( \kappa_0 \) and \( \lambda_0 \) to be arbitrary uncountable regular cardinals,
- \( \mu > \max\{\kappa_0, \lambda_0, \text{card}(I)\} \),
- \( \kappa_{\nu} = \mu \) and \( \lambda_{\nu} = \mu^+ \) for \( \nu = 1 \) or \( \nu < \mu \) a limit ordinal,
- \( \kappa_{\nu+1} = \kappa_{\nu}^+ \) and \( \lambda_{\nu+1} = \lambda_{\nu}^+ \) for \( 0 < \nu < \mu \).

Sending an element \( \alpha \in I \) to an arbitrary element \( (\alpha_{\nu})_{\nu<\mu} \in J \) with \( \alpha_0 = \alpha \) induces an order preserving embedding of \( I \) in \( J \). So we obtain the following result:

**Corollary 18.** Every linearly ordered set \( I \) can be embedded in an extremely symmetrically complete ordered set \( J \).

Our above construction can be seen as a “brute force” approach. We will now present a construction that offers more choice for the prescribed cofinalities.

If an index set \( I \) is not well ordered, then the lexicographic product of ordered abelian groups \( G_i, i \in I \), is defined to be the subset of the product consisting of all elements \( (g_i)_{i \in I} \) with well ordered support \( \{i \in I \mid g_i \neq 0\} \). Likewise, the lexicographic sum is defined to be the subset consisting of all elements \( (g_i)_{i \in I} \) with finite support \( \{i \in I \mid g_i \neq 0\} \). The problem with ordered sets is that they usually do not have distinguished elements (like neutral elements for an operation). The remedy used in [4] is to fix distinguished elements in all linear orderings we wish to use for our lexicographic sum. Hausdorff ([1]) does this in quite an elegant way: he observes that the full product is still partially ordered. Singling out one element in the product then determines the distinguished elements in the ordered sets (being the corresponding components of the element), and in this manner one obtains an associated maximal linearly ordered subset of the full product.

While the index sets we use here are ordinals and hence well ordered, which makes a condition on the support unnecessary for the work with lexicographic products, we will use the idea (as apparent in the definition of the lexicographic sum) that certain elements can be singled out by means of their support.

We choose infinite regular cardinals \( \mu, \kappa_0 \) and \( \lambda_0 \). Further, we denote by \( \text{On} \) the class of all ordinals and set

\[
I_0 := \lambda_0^* + I^c + \kappa_0 \quad \text{and} \quad I_\nu := \text{On}^* + \mu + \{\emptyset\} + \mu^+ + \text{On} \quad \text{for} \ 0 < \nu < \mu,
\]

assuming that \( \emptyset \) does not appear in \( I^c \) or any ordinal or reversed ordinal. Note that \( \text{On} \) can be replaced by a large enough cardinal; its minimal size depends on the choice of \( I, \mu, \kappa_0 \) and \( \lambda_0 \). But the details are not essential for our construction, so we skip them.
We define $J^\circ$ to consist of all elements of the lexicographic product over the $I_\nu$ with index set $\mu$ whose support

$$\text{supp}(\alpha_\nu)_{\nu<\mu} = \{ \nu \mid \nu < \mu \text{ and } \alpha_\nu \neq 0 \}$$

is an initial segment of $\mu$ (i.e., an ordinal $\leq \mu$).

A further refinement of our construction uses the idea to define suitable subsets of $J^\circ$ by restricting the choice of the coefficient $\alpha_\nu$ in dependence on the truncated sequence $(\alpha_\rho)_{\rho<\nu}$.

For every $\nu < \mu$ we consider the following set of truncations:

$$J_\nu^\circ := \{ (\alpha_\nu)_{\rho \leq \nu} \mid (\alpha_\rho)_{\rho < \mu} \in J^\circ \}.$$

By induction on $\nu < \mu$ we define subsets

$$J_\nu \subset J_\nu^\circ$$

as follows:

(J1) $J_0 := I_0$.

(J2) If $J_{\nu'}$ for all $\nu' < \nu$ are already constructed, then we first define the auxiliary set

$$J_{<\nu} := \{ (\alpha_\rho)_{\rho < \nu} \mid (\alpha_\rho)_{\rho < \nu'} \in J_{\nu'} \text{ for all } \nu' < \nu \}.$$

For $a = (\alpha_\rho)_{\rho < \nu} \in J_{<\nu}$ we set

$$\kappa_a := \text{cf}(\{ b \in J_{<\nu} \mid b < a \}) \text{ and } \lambda_a := \text{ci}(\{ b \in J_{<\nu} \mid b > a \}),$$

and

(J3) if $\alpha_\nu = 0$ for $\nu < \nu' < \mu$, then $\alpha_{\nu'} = 0$.

After having defined $J_\nu$ for all $\nu < \mu$, we set

$$J := \{ (\alpha_\rho)_{\rho < \mu} \in J^\circ \mid (\alpha_\rho)_{\rho \leq \nu} \in J_\nu \text{ for all } \nu < \mu \}.$$

The following is our first step towards the proof of Theorem 19.

**Theorem 19.** With the sets $R_{\text{left}}$ and $R_{\text{right}}$ defined as in the introduction, assume that (J) holds. Then the cofinalities of the cuts of $J$ are:

$$\{(1, \mu), (\mu, 1)\} \cup \{ (\kappa, \varphi_{\text{right}}(\kappa)) \mid \kappa \in R_{\text{left}} \} \cup \{ (\varphi_{\text{left}}(\lambda), \lambda) \mid \lambda \in R_{\text{right}} \}.$$

Further, the cofinality of $J$ is $\kappa_0$ and its coinitiality is $\lambda_0$.

**Proof:** First, we observe that for each $\nu < \mu$ we obtain an embedding

$$\iota_\nu : J_\nu \hookrightarrow J$$

by sending $(\alpha_\rho)_{\rho \leq \nu}$ to $(\beta_\rho)_{\rho \leq \mu}$, where $\beta_\rho = \alpha_\rho$ for $\rho \leq \nu$ and $\beta_\rho = 0$ for $\nu < \rho < \mu$. 


We start by proving that the principal cuts in \( J \) have cofinalities \((1, \mu)\) and \((\mu, 1)\). Take \((\alpha_\rho)_{\rho \leq \mu} \in J \) and assume first that its support is smaller than \( \mu \). Set \( \nu := \min\{\rho < \mu \mid \alpha_\rho = \emptyset\} \geq 1 \). Then the second case of definition \( \text{(3)} \) applies and therefore, the cofinalities of the principal cuts generated by \( \emptyset \) in \( I_\nu \) are \((1, \mu)\) and \((\mu, 1)\) and therefore, the cofinalities of the principal cuts generated by \((\alpha_\rho)_{\rho \leq \nu} \) in \( J_\nu \) are also \((1, \mu)\) and \((\mu, 1)\). By means of the embeddings \( \iota_\nu \) it follows that the cofinalities of the principal cuts generated by \((\alpha_\rho)_{\rho < \mu} \) in \( J \) are again \((1, \mu)\) and \((\mu, 1)\).

Now assume that the support of \( a := (\alpha_\rho)_{\rho < \mu} \) is \( \mu \). For each \( \nu < \mu \) there are elements \( \beta_\nu, \gamma_\nu \in I_\nu((\alpha_\rho)_{\rho < \nu}) \) with \( \beta_\nu < \alpha_\nu < \gamma_\nu \). We set \( \beta_\rho := \gamma_\rho := \alpha_\rho \) for \( \rho < \nu \), and define

\[
b_\nu := \iota_\nu((\beta_\rho)_{\rho \leq \nu}) \quad \text{and} \quad c_\nu := \iota_\nu((\gamma_\rho)_{\rho \leq \nu}).
\]

Then we find that whenever \( \nu < \nu' < \mu \), then

\[
b_\nu < b_{\nu'} < a < a_{\nu'} < c_\nu.
\]

This proves that again, the cofinalities of the principal cuts generated by \((\alpha_\rho)_{\rho < \mu} \) in \( J \) are \((1, \mu)\) and \((\mu, 1)\).

Now take any nonprincipal cut \((D, E)\) in \( J \). By restricting the elements to index set \( \nu + 1 = \{\rho \mid \rho \leq \nu\} \), this cut induces a quasicut \((D_\nu, E_\nu)\) in \( J_\nu \), that is, \( J_\nu = D_\nu \cup E_\nu, D_\nu < E_\nu \), and therefore, \( D_\nu \cap E_\nu \) contains at most one element.

Assume that \( \nu < \mu \) is such that \( \iota_\nu(D_\nu) \) is not a cofinal subset of \( D \) and \( \iota_\nu(E_\nu) \) is not a cofinal subset of \( E \). Then we have one of the following cases:

- \( \iota_\nu(D_\nu) \cap E \neq \emptyset \) or \( \iota_\nu(E_\nu) \cap D \neq \emptyset \);
- there are \( d_\nu \in D \) and \( e_\nu \in E \) such that \( \iota_\nu(D_\nu) < d_\nu < e_\nu < \iota_\nu(E_\nu) \), which yields that the restrictions of \( d_\nu \) and \( e_\nu \) to index set \( \nu + 1 \) are equal and lie in \( D_\nu \cap E_\nu \).

In both cases, \( D_\nu \cap E_\nu \neq \emptyset \). This implies that also \( D_{\nu'} \cap E_{\nu'} \neq \emptyset \) for all \( \nu' < \nu \), with the element in \( D_{\nu'} \cap E_{\nu'} \) being the restriction of the element in \( D_\nu \cap E_\nu \).

Now we show that there is some \( \nu < \mu \) such that \( \iota_\nu(D_\nu) \) is cofinal in \( D \) or \( \iota_\nu(E_\nu) \) is cofinal in \( E \). Suppose that the contrary is true. Then \( D_\nu \cap E_\nu \neq \emptyset \) for all \( \nu < \mu \) and there is a unique element \( a \in J \) whose restriction to index set \( \nu + 1 \) lies in \( D_\nu \cap E_\nu \), for all \( \nu < \mu \). It follows that \( a \) is either the largest element of \( D \) or the smallest element in \( E \). But this contradicts our assumption that \((D, E)\) is nonprincipal.

We take \( \nu \) to be minimal with the property that \( \iota_\nu(D_\nu) \) is cofinal in \( D \) or \( \iota_\nu(E_\nu) \) is cofinal in \( E \). From what we have shown above, it follows that \( D_{\nu'} \cap E_{\nu'} \neq \emptyset \) for all \( \nu' < \nu \) and there is \((\alpha_\rho)_{\rho < \nu} \in J_{<\nu} \) whose restriction to \( \nu + 1 \) lies in \( D_{\nu'} \cap E_{\nu'} \), for all \( \nu' < \nu \). Therefore, there must be elements in both \( D_\nu \) and \( E_\nu \) whose restrictions to \( \nu \) are equal to \((\alpha_\rho)_{\rho < \nu} \). Consequently, with
\[\overline{D}_\nu := \{\alpha_\nu \in I_\nu((\alpha_\rho)_{\rho<\nu}) \mid (\alpha_\rho)_{\rho \leq \nu} \in D_\nu\}\] and
\[\overline{E}_\nu := \{\alpha_\nu \in I_\nu((\alpha_\rho)_{\rho<\nu}) \mid (\alpha_\rho)_{\rho \leq \nu} \in E_\nu\},\]

\((\overline{D}_\nu, \overline{E}_\nu)\) is a cut in \(I_\nu((\alpha_\rho)_{\rho<\nu})\). But \(I_\nu((\alpha_\rho)_{\rho<\nu})\) is cut complete, and so there is some \(\alpha_\nu \in I_\nu((\alpha_\rho)_{\rho<\nu})\) such that \(a = (\alpha_\rho)_{\rho \leq \nu}\) is either the largest element of \(D_\nu\) or the smallest element of \(E_\nu\). We note that \(\alpha_\nu \neq 0\); otherwise, the element \((\alpha_\rho)_{\rho<\mu}\) with \(\alpha_\rho = 0\) for \(\nu \leq \rho < \mu\), which is the unique element in \(J\) whose restriction to \(\nu + 1\) is \(a\), would be the largest element of \(D\) or the smallest element of \(E\) in contradiction to our assumption on \((D, E)\). Hence by construction, for every

\[
\alpha \in I_{\nu+1}((\alpha_\rho)_{\rho \leq \nu}) = \varphi_{\text{right}}(\kappa_\rho)^* + \mu + \{0\} + \mu^* + \varphi_{\text{left}}(\rho) \]

there is an element \((\alpha_\rho)_{\rho<\mu}\) with \(\alpha_{\nu+1} = \alpha\) whose restriction to \(\nu + 1\) is \(a\).

We assume first that \(\iota_\nu(D_\nu)\) is cofinal in \(D\). Since \((D, E)\) is nonprincipal, \(D\) and hence also \(D_\nu\) has no largest element. So \(a\) is the smallest element of \(E_\nu\). Consequently,

\[
\iota_{\nu+1}((\alpha_\rho)_{\rho \leq \nu+1}) = \varphi_{\text{right}}(\kappa_\rho) + \mu + \{0\} + \mu^* + \varphi_{\text{left}}(\rho) = \varphi_{\text{left}}(\lambda_\rho),\]

is coinitial in \(E\). We observe that \(\kappa_\rho = \text{cf}(D_\nu) = \text{cf}(D) \neq 1\). By construction, the coinitiality of \(I_{\nu+1}((\alpha_\rho)_{\rho \leq \nu})\) is \(\varphi_{\text{right}}(\kappa_\rho)\). This proves that the cofinality of \((D, E)\) is \((\kappa_\rho, \varphi_{\text{right}}(\kappa_\rho))\).

If on the other hand, \(\iota_\nu(E_\nu)\) is coinitial in \(E\), then \(a\) is the largest element of \(D_\nu\) and one shows along the same lines as above that the cofinality of \((D, E)\) is \((\varphi_{\text{left}}(\lambda_\rho), \lambda_\rho)\) with \(\lambda_\rho = \text{ci}(E_\nu) = \text{ci}(E) \neq 1\).

We have to prove that the cardinals \(\kappa_\rho\) and \(\lambda_\rho\) that appear in the construction, i.e., in definition (3), are elements of \(\{1\} \cup R_{\text{left}}\) and \(\{1\} \cup R_{\text{right}}\), respectively. We observe that \(\kappa_\rho\) and \(\lambda_\rho\) appear in definition (3) only if \(a = (\alpha_\rho)_{\rho<\nu} \in J_{\rho<\nu}\) is such that \(\alpha_\rho \neq 0\) for all \(\rho < \nu\). We show our assertion by induction on \(1 \leq \nu \leq \mu\). We do this for \(\kappa_\rho\); for \(\lambda_\rho\) the proof is similar.

First, we consider the successor case \(\nu = \sigma + 1\). We set \(\overline{a} = (\alpha_\rho)_{\rho<\sigma}\). If \(\sigma \geq 1\), then our induction hypothesis states that our assertion is true for \(\kappa_{\overline{a}}\) and \(\lambda_{\overline{a}}\). We observe that because the second case of definition (3) applies to \(\overline{a}\),

\[
\kappa_\rho = \text{cf}((\beta_\rho)_{\rho \leq \sigma} \in J_\sigma \mid \beta_\rho = \alpha_\rho \text{ for } \rho < \sigma \text{ and } \beta_\sigma < \alpha_\sigma) \\
= \text{cf}((\beta \in I_\sigma(\overline{a}) \mid \beta < \alpha_\sigma)).
\]

This is the cofinality of a lower cut set of a cut in \(I_\sigma\). Therefore, if \(\kappa_\rho\) is infinite, it is an element of \(\text{Cofin}(I) \cup \text{Reg}_{<\kappa_\rho} \cup \text{Reg}_{<\mu} = R_{\text{left}}\) if \(\sigma = 0\), and of \(\text{Reg}_{<\varphi_{\text{left}}(\lambda_{\overline{a}})} \cup \text{Reg}_{<\mu}\) otherwise. In the latter case, \(\lambda_{\overline{a}} \in R_{\text{right}}\) by induction hypothesis, hence \(\varphi_{\text{left}}(\lambda_{\overline{a}}) \in R_{\text{left}}\) by (1), which yields that \(\text{Reg}_{<\varphi_{\text{left}}(\lambda_{\overline{a}})} \cup \text{Reg}_{<\mu} \subseteq R_{\text{left}}\). Altogether, we have proved that \(\kappa_\rho \in \{1\} \cup R_{\text{left}}\).

Now we consider the case of \(\nu\) a limit ordinal. Let \(\mu'\) be its cofinality. Then \(\mu' \in \text{Reg}_{<\mu}\). With a similar construction as in the beginning of the
proof one shows that the principal cuts generated by elements in $J_{<\nu}$ have cofinalities $(\mu', 1)$ and $(1, \mu')$. This yields that $\kappa_a \in R_{\text{left}}$ and $\lambda_a \in R_{\text{right}}$.

It remains to prove that all cofinalities listed in the assertion of our theorem actually appear as cofinalities of cuts in $J$. Since for all cardinals $\kappa \in \text{Cofin}(I) \cup \text{Reg}_{<\kappa_0}$, there is a cut in $I_0$ with cofinality $(\kappa, 1)$, our construction at level $\nu = 1$ shows that $(\kappa, \varphi_{\text{right}}(\kappa))$ appears as the cofinality of a cut in $J$. Similarly, one shows that $(\varphi_{\text{left}}(\lambda), \lambda)$ appears as the cofinality of a cut in $J$ for every $\lambda \in \text{Coin}(I) \cup \text{Reg}_{<\lambda_0}$.

Now take any regular cardinal $\mu' < \mu$. For an arbitrary $a = (\alpha_0) \in J_0$ we see that the second case of definition (3) applies to $I_1(a)$, so that there is a cut in $I_1$ with cofinality $(\mu', 1)$. Our construction at level $\nu = 2$ then shows that $(\mu', \varphi_{\text{right}}(\mu'))$ appears as the cofinality of a cut in $J$. Similarly, one shows that $(\varphi_{\text{left}}(\mu'), \mu')$ appears as the cofinality of a cut in $J$.

The proof of the last statement of the theorem is again left to the reader. \hfill \Box

The following result is an immediate consequence of Theorem 19, and it proves Theorem 8:

**Corollary 20.** Assume in addition to the previous assumptions that $\varphi_{\text{left}}(\kappa) \neq \kappa \neq \varphi_{\text{right}}(\kappa)$ for all $\kappa \in R_{\text{left}} \cup R_{\text{right}}$. Then $J$ is a symmetrically complete extension of $I$. If in addition $\mu$ is uncountable, then $J$ is strongly symmetrically complete.

**Remark 21.** In both constructions that we have given in this section, every element in the constructed ordered set has, in the terminology of Hausdorff, character $(\mu, \mu)$.

6. CONSTRUCTION OF SYMMETRICALLY COMPLETE ORDERED EXTENSIONS

Take any ordered abelian group $G$. We wish to extend it to an extremely symmetrically complete ordered abelian group. We use the well known fact that $G$ can be embedded in a suitable Hahn product $H_0 = \mathbb{H}_I \mathbb{R}$, for some ordered index set $I$. By Corollary 18 there is an embedding $\iota$ of $I$ in an extremely symmetrically complete linearly ordered set $J$. We set $H = \mathbb{H}_J \mathbb{R}$ and note that there is a canonical order preserving embedding $\varphi$ of $H_0 = \mathbb{H}_I \mathbb{R}$ in $H = \mathbb{H}_J \mathbb{R}$ which lifts $\iota$ by sending an element $(r_\gamma)_{\gamma \in I}$ to the element $(r'_\delta)_{\delta \in J}$ where $r'_\delta = r_\gamma$ if $\delta = \iota(\gamma)$ and $r'_\delta = 0$ if $\delta$ is not in the image of $\iota$. By Theorem 4, $H$ is an extremely symmetrically complete ordered abelian group. We have now proved the first part of Theorem 7.

Take any ordered field $K$. We wish to extend it to an extremely symmetrically complete ordered field. First, we extend $K$ to its real closure $K^{\text{rc}}$. From [Ka] we know that $K^{\text{rc}}$ can be embedded in the power series field...
\[ \mathbb{R}((G)) \] where \( G \) is the value group of \( K^{rc} \) under the natural valuation. By what we have already shown, \( G \) admits an embedding \( \psi \) in an extremely symmetrically complete ordered abelian group \( H \). By a definition analogous to the one of \( \varphi \) above, one lifts \( \psi \) to an order preserving embedding of the power series field \( \mathbb{R}((G)) \) in the power series field \( \mathbb{R}((H)) \). By Theorem 5, \( \mathbb{R}((H)) \) is an extremely symmetrically complete ordered field. We have thus proved the second part of Theorem 7.

**References**

[1] Hausdorff, F.: Grundzüge einer Theorie der geordneten Mengen, Mathematische Annalen 65 (1908), 435–505  
[2] Kaplansky, I.: Maximal fields with valuations I, Duke Math. J. 9 (1942), 303–321  
[3] Kuhlmann, F.-V. — Kuhlmann, K.: A common generalization of metric, ultrametric and topological fixed point theorems, to appear in Forum Math.  
[4] Kuhlmann, F.-V. — Kuhlmann, S. — Shelah, S.: Functorial Equations for Lexicographic Products, Proc. Amer. Math. Soc. 131 (2003), 2969–2976  
[5] F.-V. Kuhlmann, Maps on ultrametric spaces, Hensel’s Lemma, and differential equations over valued fields, Comm. in Alg. 39 (2011), 1730–1776.  
[6] Kuhlmann, F.-V.: Valuation theory, book in preparation. Preliminary versions of several chapters available at \url{http://math.usask.ca/~fvk/Fvkbook.htm}  
[7] Kuhlmann, S.: On the structure of nonarchimedean exponential fields I, Archive for Math. Logic 34 (1995), 145–182  
[8] Prestel, A.: Lectures on Formally Real Fields, Springer Lecture Notes in Math. 1093, Berlin–Heidelberg–New York–Tokyo (1984)  
[9] Shelah, S.: Quite Complete Real Closed Fields, Israel J. Math. 142 (2004), 261–272

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