Induced quantum gravitational interaction between two objects with permanent quadrupoles in external gravitational fields

Yongshun Hu,1 Jiawei Hu,1,* and Hongwei Yu1,†

1Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha, Hunan 410081, China

Abstract

We investigate, in the framework of linearized quantum gravity, the induced quantum gravitational interaction between two ground-state objects with permanent quadrupole moments, which are subjected to an external gravitational radiation field. Compared with the nonpolar case, there exists an additional term in the leading-order field-induced interobject interaction between two polar objects. This term arises when a real graviton is scattered by the same object center with coupling between the two objects occurring via the exchange of a single virtual graviton, and the interaction is thus relevant to the number density of gravitons, the frequency and polarization of the external gravitational field, as well as the permanent quadrupoles of the objects. Due to the existence of such an additional term, the field-induced quantum gravitational interaction between two polar objects can be significantly different from that between nonpolar ones when the inter-object distance is much smaller than the wavelength of the external gravitational radiation field. Although we model the external gravitational radiation as a quantized monochromatic gravitational wave with a certain wave vector and polarization for simplicity, it is possible to generalize to more realistic situations, such as the case of a stochastic background of gravitational radiation.

* Corresponding author. jwhu@hunnu.edu.cn
† Corresponding author. hwyu@hunnu.edu.cn
I. INTRODUCTION

Gravitational waves, which are ripples of spacetime predicted naturally by general relativity [1], may be regarded as a number of gravitons propagating through the universe when gravity is quantized. In this respect, a keen interest in the possible quantum nature or quantum properties of gravitational waves arises. Unfortunately, a full theory of quantum gravity is elusive at present. Nonetheless, one can still study low energy quantum gravitational effects, since, at low energy scales, general relativity can be treated as an effective field theory. For example, the quantum correction to the Newtonian potential between two point masses has been obtained by summing one-loop Feynman diagrams with off-shell gravitons [2–7]. Also, a framework of linearized quantum gravity has been established in the study of the quantum light-cone fluctuations [8–12], the basic idea of which is to quantize the linearized perturbation propagating on a flat background spacetime in the canonical approach.

Recently, based on linearized quantum gravity, the quantum gravitational vacuum fluctuation induced interaction between two nonpointlike objects in their ground states as well as that between a nonpointlike object and a gravitational boundary have been studied in Refs. [13–18]. Later on, in analogy to the cases in electrodynamics [19–26], the behavior of the interobject vacuum-fluctuation-induced quadrupole-quadrupole interactions is found to be relative to the quantum states of the objects [27]. That is, for instance, the quantum gravitational quadrupole-quadrupole interaction behaves as $r^{-10}$ and $r^{-11}$ in the near and far regimes respectively when the two objects are in their ground states [13–16], while it behaves as $r^{-5}$ and $r^{-1}$ in the near and far regimes respectively when the two objects are in a symmetric/antisymmetric entangled state [27]. Moreover, in the presence of an external gravitational radiation field, the interobject quadrupole-quadrupole interaction between two ground-state objects is also found to be significantly different from that of the vacuum case. It has been demonstrated in Ref. [28] that the external gravitational field-induced interobject interaction behaves as $r^{-5}$ in the near regime and oscillates with a decreasing amplitude proportional to $r^{-1}$ in the far regime, and whether the interaction is attractive or repulsive depends on the propagation direction, polarization and frequency of the external gravitational field.

The quantum gravitational interaction discussed above is obtained under the assumption that the objects have no permanent quadrupole moments. Since polar objects exist
extensively in the universe, a natural question arises as to what the quantum gravitational interaction will be in the presence of external gravitational radiation if objects with permanent quadrupole moments are concerned. Similar examples in quantum electrodynamics show that, compared to the nonpolar case, in which the field-induced interatomic or intermolecular interaction arises when a real photon is scattered by the two atomic or molecular centers with coupling between the bodies occurring via the exchange of a single virtual photon, there is an additional term in the field-induced interatomic or intermolecular interaction between atoms or molecules with permanent dipole moments [29–31], which arises when a real photon is scattered by the same atomic or molecular center. Likewise, we expect that in the gravitational case, the quantum interobject gravitational interaction between objects with permanent quadrupole moments should also be discriminative from that between nonpolar objects.

In this paper, we study the induced quantum gravitational interaction between two ground-state objects with permanent quadrupole moments in a weak external gravitational radiation field, in the framework of linearized quantum gravity. First, we give a derivation of the induced interaction energy shift between two ground-state objects with permanent quadrupole moments based on the fourth-order perturbation theory. Then, we compare the result with that in the nonpolar case given in Ref. [28]. For simplicity, we model the external gravitational radiation as a quantized monochromatic gravitational wave with a certain wave vector and polarization. However, it is possible to generalize to more realistic situations. As an example, a generalization to the case of a stochastic background of gravitational radiation is discussed. Throughout the paper, the Latin indices run from 1 to 3, the Greek indices run from 0 to 3, and the Einstein summation convention is assumed.

II. BASIC EQUATIONS

We consider two nonpointlike objects A and B coupled with the fluctuating gravitational fields in vacuum, which are endowed with permanent quadrupole moments, and are subjected to an externally applied weak gravitational radiation field. The two objects (A and B) are modeled as multilevel systems with the ground and excited energy eigenvalues being $E_0$ and $E_s$ ($s = 1, 2, 3, ...$), and the corresponding eigenstates being $|e_0\rangle$ and $|e_s\rangle$, respectively.
total Hamiltonian of the system considered is

\[ H = H_S + H_F + H_R + H_I, \]  

where \( H_S \) is the Hamiltonian of the two objects A and B, \( H_F \) the Hamiltonian of the fluctuating gravitational fields, \( H_R \) the Hamiltonian of the external gravitational radiation field, and \( H_I \) the interaction Hamiltonian between the objects and the gravitational fields, which takes the form

\[ H_I = -\frac{1}{2} Q^A_{ij} [\epsilon_{ij}(\vec{x}_A) + E_{ij}(\vec{x}_A)] - \frac{1}{2} Q^B_{ij} [\epsilon_{ij}(\vec{x}_B) + E_{ij}(\vec{x}_B)]. \]  

Here \( Q^\xi_{ij} \) is the quadrupole moment operator of object \( \xi (\xi = A, B) \), \( \epsilon_{ij}(\vec{x}) \) the gravitoelectric tensor of the external gravitational radiation field, and \( E_{ij}(\vec{x}) \) the gravitoelectric tensor characterizing the fluctuating gravitational fields in vacuum, which is defined by an analogy between the linearized Einstein field equations and the Maxwell’s equations as \( E_{ij} = -c^2 C_{0i0j} \) \cite{32–38}, where \( C_{\alpha\beta\mu\nu} \) is the Weyl tensor and \( c \) the speed of light. In the absence of external gravitational radiation, the metric tensor of the spacetime \( g_{\mu\nu} \) can be expressed as \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), with \( \eta_{\mu\nu} \) being the flat spacetime metric and \( h_{\mu\nu} \) the fluctuating gravitational fields. Then, in the transverse traceless gauge, the gravitoelectric tensor \( E_{ij} \) is found to be \( E_{ij} = \frac{1}{2} \ddot{h}_{ij} \), which can be quantized in the canonical approach as

\[ E_{ij} = -\sum_{\vec{p},\lambda} \sqrt{\frac{\hbar G \omega^3}{c^2 (2\pi)^2}} \left[ a_{\lambda}(\vec{p}) e^{(\lambda)}_{ij} e^{i(\vec{p}\cdot\vec{x} - \omega t)} + \text{H.c.} \right], \]  

where \( \hbar \) is the reduced Planck constant, and \( G \) the Newtonian gravitational constant. Here \( a_{\lambda}(\vec{p}) \) is the annihilation operator of the fluctuating gravitational fields, \( \lambda \) labels the polarization states, \( e^{(\lambda)}_{ij} \) are polarization tensors, \( \omega = c|\vec{p}| = c(p_x^2 + p_y^2 + p_z^2)^{1/2} \), and H.c. denotes the Hermitian conjugate. Assume that the weak external gravitational radiation field can be described as a quantized monochromatic gravitational wave containing \( N \) gravitons, then the corresponding gravitoelectric tensor \( \epsilon_{ij} \) can be written as \cite{28}

\[ \epsilon_{ij} = -\sqrt{\frac{\hbar G \omega^3 \rho^N}{c^2 (2\pi)^2 N}} \left[ b(\vec{k}) e^{(\varepsilon)}_{ij} e^{i(\vec{k}\cdot\vec{x} - \omega_R t)} + \text{H.c.} \right], \]  

where \( \rho_N \) is the number density of gravitons, \( b(\vec{k}) \) the annihilation operator of the quantized gravitational wave, \( e^{(\varepsilon)}_{ij} \) the polarization tensors, and \( \omega_R = c|\vec{k}|. \)
The initial state $|\phi\rangle$ of the whole system is taken as

$$|\phi\rangle = |e_{\alpha}^{A}\rangle|e_{\alpha}^{B}\rangle|0\rangle|N\rangle,$$

where $|0\rangle$ denotes the vacuum state of the fluctuating gravitational field, and $|N\rangle$ is the number state of the external gravitational radiation field. Denote the initial energy of the whole system as $E_{\phi}$. Then, the leading-order field-induced interaction energy considered here can be obtained through a direct fourth-order perturbation calculation, i.e.,

$$\Delta E_{AB} = \sum_{I_{1},I_{2},I_{3}} \frac{\langle \phi|H_{I_{1}}|I_{1}\rangle\langle I_{3}|H_{I_{2}}|I_{2}\rangle\langle I_{2}|H_{I_{1}}|I_{1}\rangle\langle I_{1}|H_{I_{1}}|\phi\rangle}{(E_{\phi} - E_{I_{1}})(E_{\phi} - E_{I_{2}})(E_{\phi} - E_{I_{1}})},$$

(6)

which contains 96 time-ordered diagrams corresponding to two different kinds of physical processes. The first kind of processes is the scattering of a real graviton by the two object centers and an exchange of a single virtual graviton between them. For a typical time-ordered diagram, see Fig. 1 in Ref [28]. The field-induced interaction associated with this kind of processes is exactly what we have investigated in Ref. [28], so we do not repeat it here. On the other hand, when the objects have permanent quadrupole moments, there is an additional field-induced interobject interaction, which occurs through the coupling between the permanent quadrupole in one object and the external field-induced quadrupole in the other. That is, a real graviton is now scattered by the same object center with coupling between the two objects occurring via the exchange of a single virtual graviton. There are 48 time-ordered diagrams of this kind, two typical ones of which are shown in Fig. 1. Summing up all the 48 possible intermediate processes (see Table I in Appendix A), the additional field-induced interobject interaction energy between two polar objects can then be expressed as

$$\Delta E_{AB} = \frac{\hbar G \alpha_{R}^{2} \rho N}{32\pi^{2} c^{2}} e_{\alpha l}^{(e)} e_{\alpha b}^{(e)} Q_{mn}^{B 00} s_{t} \left\{ \frac{\hat{Q}_{ij}^{f 0 t} \hat{Q}_{hl}^{f 0 s} \hat{Q}_{ij}^{f 0 s}}{(E_{i 0} - h\omega_{R})(E_{s 0} - h\omega_{R})} + \frac{\hat{Q}_{hl}^{f 0 l} \hat{Q}_{ij}^{f 0 s} \hat{Q}_{ij}^{f 0 s}}{(E_{i 0} - h\omega_{R})(E_{s 0} - h\omega_{R})} + \frac{\hat{Q}_{ij}^{A 0 t} \hat{Q}_{ij}^{A 0 s} \hat{Q}_{ij}^{A 0 s}}{(E_{i 0} + h\omega_{R})(E_{s 0} + h\omega_{R})} + \frac{\hat{Q}_{ij}^{A 0 t} \hat{Q}_{ij}^{A 0 s} \hat{Q}_{ij}^{A 0 s}}{(E_{i 0} + h\omega_{R})(E_{s 0} + h\omega_{R})} \right\} V_{ijmn}(|\vec{r}|) + A \equiv B \text{ terms},$$

(7)

where $\hat{Q}_{ij}^{f 00} = \langle e_{\alpha l}^{(e)}|Q_{ij}^{f 0 l}|e_{\alpha b}^{(e)}\rangle$ is the permanent quadrupole moment of object $\xi$, $\hat{Q}_{ij}^{f 0 s} = \langle e_{\alpha l}^{(e)}|Q_{ij}^{f 0 s}|e_{\alpha b}^{(e)}\rangle$ the quadrupole transition moments, and $E_{s 0} = E_{s} - E_{0}$ the energy level spacing. Here
FIG. 1. Two typical time-ordered diagrams (each with 23 further permutations) for the calculation of the interobject interaction in the existence of an external gravitational radiation field. The blue solid line represents a real graviton, while the dotted one represents a virtual graviton.

$V_{ijmn}(\vec{r})$ is a tensor describes the quadrupole-quadrupole interaction, which is

$$V_{ijmn}(\vec{r}) = \int d^3 p \sum_{\lambda} e^{(\lambda)}_{ij} e^{(\lambda)}_{mn} \frac{G \omega^2}{4 \pi^2 c^2} e^{i \vec{p} \cdot \vec{r}},$$

where $\vec{r} = \vec{x}_A - \vec{x}_B$. In the transverse traceless gauge, the summation of polarization tensors gives [10]

$$\sum_{\lambda} e^{(\lambda)}_{ij} e^{(\lambda)}_{mn} = \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} - \delta_{ij} \delta_{mn} - \frac{1}{p^2} H_{ijmn} + \frac{1}{p^4} P_{ijmn},$$

where

$$H_{ijmn} = \partial_i \partial_j \delta_{mn} + \partial_m \partial_n \delta_{ij} - \partial_i \partial_m \delta_{jn} - \partial_i \partial_n \delta_{jm} - \partial_j \partial_m \delta_{in} - \partial_j \partial_n \delta_{im} - \delta_{ij} \delta_{mn}, \quad P_{ijmn} = \partial_i \partial_j \partial_m \partial_n.$$ (10)

Substituting Eq. (9) into Eq. (8) and performing the integral, it is easy to obtain

$$V_{ijmn}(\vec{r}) = \frac{G}{2r^5} \left[ 3(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} + \delta_{ij} \delta_{mn}) - 15(\hat{r}_i \hat{r}_j \delta_{mn} + \hat{r}_i \hat{r}_m \delta_{jn} + \hat{r}_i \hat{r}_n \delta_{jm}) + \hat{r}_j \hat{r}_m \delta_{in} + \hat{r}_j \hat{r}_n \delta_{im} + \hat{r}_m \hat{r}_n \delta_{ij} + 105 \hat{r}_i \hat{r}_j \hat{r}_m \hat{r}_n \right],$$ (11)

where $\hat{r}_i$ is the $i$th component of the unit vector $\vec{r}/r$. This shows that, the additional field-induced interobject interaction between two polar objects in the presence of a weak external gravitational radiation field behaves as $r^{-5}$ in all distance regimes. Notice that, the above result is nonretarded, which is in contrast to the case considered in Ref. [28]. This is understandable since, in our case, a real graviton is scattered by the same object center, the
process of which is unrelated to the interobject distance, while in the case of Ref. [28], a real graviton is scattered by both the two object centers, the process of which is related to the interobject distance and may lead to a change of the behavior of the interobject interaction due to the finite speed of graviton propagation. Moreover, from Eq. (7), we find that such an interaction is related to the number density of gravitons, the frequency and polarization of the external gravitational radiation field, as well as the permanent quadrupoles of the two objects. Note here that, the above calculations are applicable only when the frequency of the external gravitational radiation field is in the off-resonant range.

Now, let us discuss the case in the limit of a large graviton number, i.e., \( N \gg 1 \), which may correspond to the case of classical gravitational waves. In this regard, the interaction energy shift Eq. (7) can be expressed as

\[
\Delta E_{AB} \simeq -\frac{I_R}{16}\epsilon^{(c)}_{ijkl}e_{ab}^A\hat{Q}^{i00}_{ijl}\hat{Q}^{j00}_{kmn}V_{ijmn}(\vec{r}),
\]

where \( I_R \) is the intensity of the external gravitational radiation field, which is defined as [28]

\[
I_R = \langle N|\epsilon_i^2|N \rangle = \frac{\hbar G\omega R^3\rho_N}{4N\pi^2c^2}(2N + 1) \simeq \frac{\hbar G\omega R^3\rho_N}{2\pi^2c^2},
\]

and \( \beta_{ijl}^\xi \) is the hyperpolarizability of object \( \xi \), which is defined in analogy to the electromagnetic case [29–31] as

\[
\beta_{ijl}^\xi(\omega_R) = \sum_{s,t} \left[ \frac{\hat{Q}^\xi_{ij}^{\xi0t}\hat{Q}^{\xi0s}_{jl}\hat{Q}^{\xi0}_{ab}}{E_{00}(E_{s0} - \hbar\omega_R)} + \frac{\hat{Q}^{\xi0t}_{ij}\hat{Q}^{\xi0s}_{jl}\hat{Q}^{\xi0}_{ab}}{(E_{00} - \hbar\omega_R)(E_{s0} - \hbar\omega_R)} \right] + \left[ \frac{\hat{Q}^{\xi0t}_{ij}\hat{Q}^{\xi0s}_{jl}\hat{Q}^{\xi0}_{ab}}{E_{00}(E_{s0} - \hbar\omega_R)} + \frac{\hat{Q}^{\xi0t}_{ij}\hat{Q}^{\xi0s}_{jl}\hat{Q}^{\xi0}_{ab}}{(E_{00} + \hbar\omega_R)(E_{s0} + \hbar\omega_R)} \right] + \left[ \frac{\hat{Q}^{\xi0t}_{ij}\hat{Q}^{\xi0s}_{jl}\hat{Q}^{\xi0}_{ab}}{(E_{00} - \hbar\omega_R)E_{s0}} + \frac{\hat{Q}^{\xi0t}_{ij}\hat{Q}^{\xi0s}_{jl}\hat{Q}^{\xi0}_{ab}}{(E_{00} + \hbar\omega_R)E_{s0}} \right],
\]

Thus, in the large graviton number limit, the interobject interaction is now determined by the permanent quadrupole moments and the hyperpolarizabilities of the two objects, as well as the intensity and polarization of the external gravitational wave. In this aspect, such an interaction can be understood intuitively from the point of induced quadrupole moments. That is, the externally applied gravitational wave induces a quadrupole moment which is quadratic in the field in one object, which then interacts with the permanent quadrupole moment in the other object via the static quadrupole-quadrupole interaction tensor (i.e., \( V_{ijmn} \)), and an interaction energy is thus obtained.
A few comments are now in order. First, we compare the field-induced interobject interaction between two polar objects $\Delta E_{\text{polar}}$ with that between nonpolar ones $\Delta E_{\text{non}}$. As discussed before, for the polar case, there exists an additional term which is related to the hyperpolarizability $\beta_{ijklab}$ and here labeled as $\Delta E_{\beta}$, due to the scattering of a real graviton by the same object center and an exchange of a single virtual graviton, i.e., $\Delta E_{\text{polar}} = \Delta E_{\text{non}} + \Delta E_{\beta}$.

From Ref. [28], we know that the interaction between two nonpolar objects can be either attractive or repulsive depending on the propagation direction and polarization of the external gravitational field. However, as our main concern here is the relative magnitude of the extrema of the interaction potentials in the polar and nonpolar cases, we rewrite the extremum of the interaction potential $\Delta E_{\text{non}}$ as

$$\Delta E_{\text{non}} \sim \begin{cases} -\frac{\hbar c G^2 \rho_N}{\lambda_R^5} \alpha^2(\omega_R), & r \ll \lambda_R, \\ -\frac{\hbar c G^2 \rho_N}{\lambda_R^7} \alpha^2(\omega_R), & r \gg \lambda_R, \end{cases} \quad (15)$$

where $\lambda_R$ is the wavelength of the external gravitational field, and $\alpha(\omega_R)$ is the isotropic gravitational quadrupole polarizability of the object, which takes the form

$$\alpha(\omega_R) = \sum_s \frac{E_{s0} \hat{Q}^{s0} \hat{Q}^{s0}}{E_{s0}^2 - (\hbar \omega_R)^2} = \sum_s \frac{\alpha_s(0)}{1 - (\hbar \omega_R/E_{s0})^2}, \quad (16)$$

with $\alpha(0) = \sum_s \alpha_s(0)$ being the static polarizability (i.e., the case when $k = 0$). From Eq. (12), $\Delta E_{\beta}$ can be approximately written as

$$\Delta E_{\beta} \sim -\frac{\hbar c G^2 \rho_N}{\lambda_R^5} \beta(\omega_R) \hat{Q}^{00}, \quad (17)$$

where $\hat{Q}^{00}$ is the permanent quadrupole moment, and $\beta(\omega_R)$ denotes the leading (second) term of the gravitational quadrupole hyperpolarizability given in Eq. (14), which can be approximately expressed as

$$\beta(\omega_R) \sim \sum_{s,t} \frac{\beta_{s,t}(0)}{(1 - \hbar \omega_R/E_{s0})(1 - \hbar \omega_R/E_{t0})}, \quad (18)$$

with $\beta(0) = \sum_{s,t} \beta_{s,t}(0)$ being the static hyperpolarizability. In the near regime where the interobject distance is much smaller than the wavelength of the external gravitational radiation field, the ratio of $\Delta E_{\text{polar}}$ to $\Delta E_{\text{non}}$ can be approximately obtained as

$$\frac{\Delta E_{\text{polar}}}{\Delta E_{\text{non}}} \sim 1 + \frac{\beta(0) \hat{Q}^{00}}{\alpha^2(0)}. \quad (19)$$
Obviously, when the permanent quadrupole \( \hat{Q}_{00} \) is much larger than the characteristic quadrupole described by the ratio of two polarizabilities, i.e., \( \frac{\alpha^2(0)}{\beta(0)} \), the external gravitational field-induced interaction between two polar objects is significantly different from that between the nonpolar ones. In the far regime, since \( \Delta E_{\text{non}} \) behaves as \( r^{-1} \) while \( \Delta E_\beta \) behaves as \( r^{-5} \), the difference between the field-induced interactions in the two cases is usually negligible. However, at some special positions where \( \Delta E_{\text{non}} \) is zero, the interaction between nonpolar objects vanishes but that between polar ones are not.

Second, let us note here that although we have modeled the external gravitational radiation as a quantized monochromatic gravitational wave with a certain wave vector and polarization for simplicity, it is possible to generalize to more realistic situations. As an example, in the following, we consider the case of a stochastic background of gravitational radiation. Summing all the propagation directions, polarizations and frequencies of the gravitational waves, the interaction potential between two objects with permanent quadrupole moments induced by a stochastic background of gravitational waves can be formally expressed as

\[
\Delta E_{\text{stoch}} = \int f(\omega R) d\omega R \int d\Omega_\vec{k} \sum_\epsilon \Delta E_{\text{polar}}
\]

\[
= -\frac{1}{8} \int I_R(\omega R) f(\omega R) d\omega R \int d\Omega_\vec{k} \sum_\epsilon \left[ \alpha_{ijab}^A \alpha_{hlmn}^B e_{ab}^{(e)} e_{mn}^{(e)} \cos (\vec{k} \cdot \vec{r}) V_{ijkl}(\omega R, \vec{r}) \right.
\]

\[
+ \frac{1}{2} e_{hl}^{(e)} e_{ab}^{(e)} (\beta_{ijkl}^A \hat{Q}_{mn}^B \hat{Q}_{00} + \beta_{ijkl}^B \hat{Q}_{00} \hat{Q}_{mn}^B) V_{ijkl}(\vec{r}) \left. \right], \tag{20}
\]

where \( f(\omega R) \) is the frequency distribution function of the stochastic gravitational waves, \( \int d\Omega_\vec{k} \) denotes the integral over solid angle, \( V_{ijkl}(\omega R, \vec{r}) \) is the dynamic quadrupole-quadrupole interaction tensor given in Ref. \[28\] (See Eq. (18) in Ref. \[28\]), and \( V_{ijkl}(\vec{r}) \) is the static quadrupole-quadrupole interaction tensor given in Eq. (11). That is, one may treat the spectrum of the stochastic gravitational waves as a frequency-dependent distribution function and the induced interobject interaction potential can then be obtained via an integral over the frequencies as well as the summations of the propagation directions and polarizations.

**III. SUMMARY**

Within the framework of linearized quantum gravity, the induced quantum gravitational interaction between two ground-state objects with permanent quadrupole moments, which
are coupled with the fluctuating gravitational fields in vacuum and subjected to a weak external gravitational radiation field, is investigated based on the leading-order perturbation theory. Compared with the field-induced interaction between nonpolar objects, there exists an additional term in the leading-order field-induced interobject interaction between two polar objects, which behaves as \( r^{-5} \) in all distance regimes and arises from the scattering of a real graviton by the same object center with coupling between the two objects occurring via the exchange of a single virtual graviton. Such an additional interaction is relevant to the number density of gravitons, the polarization and frequency of the external gravitational field, as well as the permanent quadrupoles of the objects. In the limit of a large number of gravitons, the interaction can be viewed as the interaction between the induced gravitational quadrupole in one object with the permanent quadrupole moment in the other object, which can be characterized by the permanent quadrupole moments and the hyperpolarizabilities of the two objects, as well as the intensity and polarization of the external gravitational wave. Due to the existence of such an additional term, the field-induced quantum gravitational interaction between two polar objects can be significantly different from that between nonpolar ones when the interobject distance is much smaller than the wavelength of the external gravitational radiation field. Although we model the external gravitational radiation as a quantized monochromatic gravitational wave with a certain wave vector and polarization for simplicity, it is possible to generalize to more realistic situations, such as the case of a stochastic background of gravitational radiation.

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**Appendix A: Intermediate processes**

The intermediate states and their associated energy denominators of Eq. (7) are followed.
| Case  | [I]                  | [II]                  | [III]                  | Denominator                                                                 |
|-------|----------------------|----------------------|-----------------------|----------------------------------------------------------------------------|
| (1)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_1 = (E_{m0}^{a} + \hbar \omega)E_{m0}^{a}\hbar \omega$ |
| (2)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_2 = (E_{m0}^{a} - \hbar \omega)E_{m0}^{a}\hbar \omega$ |
| (3)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_3 =\ h_\omega E_{m0}^{a} + \hbar \omega$ |
| (4)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_4 = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (5)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_5 = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (6)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_6 = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (7)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_7 = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (8)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_8 = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (9)   | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_9 = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (10)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{10} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (11)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{11} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (12)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{12} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (13)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{13} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (14)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{14} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (15)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{15} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (16)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{16} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (17)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{17} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (18)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{18} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (19)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{19} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (20)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{20} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (21)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{21} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (22)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{22} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (23)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{23} = \h_\omega E_{m0}^{a} + \hbar \omega$ |
| (24)  | $|e_{m0}^{a}|(0)^{0}\langle 0|N - 1$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $|e_{m0}^{b}|(0)^{0}\langle 0|N$ | $D_{24} = \h_\omega E_{m0}^{a} + \hbar \omega$ |

TABLE I. Intermediate states and the corresponding energy denominators for the case when a real graviton is scattered by object A. Similar processes for object B can be obtained by exchanging the label A and B.

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