Twistor spaces for QKT manifolds

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ABSTRACT

We find that the target space of two-dimensional $(4,0)$ supersymmetric sigma models with torsion coupled to $(4,0)$ supergravity is a QKT manifold, that is, a quaternionic Kähler manifold with torsion. We give four examples of geodesically complete QKT manifolds one of which is a generalisation of the LeBrun geometry. We then construct the twistor space associated with a QKT manifold and show that under certain conditions it is a Kähler manifold with a complex contact structure. We also show that, for every $4k$-dimensional QKT manifold, there is an associated $4(k+1)$-dimensional hyper-Kähler one.
1. Introduction

The geometry of the target space of two-dimensional sigma models with extended supersymmetry is described by the properties of a metric connection with torsion [1, 2]. Rigid (4,0) supersymmetry requires that the target space of two-dimensional sigma models without Wess-Zumino term (torsion) is a hyper-Kähler (HK) manifold. In the presence of torsion the geometry of the target space becomes hyper-Kähler with torsion (HKT) [3]. Manifolds with either HK or HKT structure admit three complex structures which obey the algebra of imaginary unit quaternions and the sigma model metric is hermitian with respect to all complex structures. In addition, in HK geometry the complex structures are covariantly constant with respect to the Levi-Civita connection, while in HKT geometry the complex structures are covariantly constant with respect to a metric connection with torsion. Local (4,0) supersymmetry requires that the target space of two-dimensional sigma models with torsion be either (i) HKT or (ii) a generalisation of the standard quaternionic Kähler geometry (QK) (see [4, 5]) for which the associated metric connection has torsion [6]; we shall call this geometry quaternionic Kähler with torsion (QKT). This is unlike the case of (4,4) locally supersymmetric sigma models where it has been shown that the geometry of the target space is either of HKT type or it is standard quaternionic Kähler geometry [7]. Thus QKT geometry is not compatible with (4,4) local supersymmetry. Nevertheless, the conditions on the geometry of the target space of two-dimensional sigma models required by (4,0)-local supersymmetry can be derived from an appropriate truncation of the conditions found for the (4,4)-locally supersymmetric ones. It is well known in QK geometry that the holonomy of the Levi-Civita connection is a subgroup of $Sp(k) \cdot Sp(1)$. Similarly, QKT geometry is characterised by the fact that the holonomy of a metric connection with torsion has holonomy $Sp(k) \cdot Sp(1)$. The torsion is the exterior derivative of the Wess-Zumino term of the sigma model action and is therefore a closed three-form on the sigma model manifold, at least in the classical theory.
In this paper, we list the conditions on the target manifold of a sigma model required by (4,0)-local supersymmetry and thus derive the restrictions on a Riemannian manifold that must be satisfied in order for it to admit a QKT geometry. We shall then explore some of the properties of QKT geometry. In particular, we shall show that for every four-dimensional quaternionic Kähler manifold there is an associated class of QKT manifolds. These manifolds are parameterised by harmonic functions (possibly with delta function singularities on the QK manifold). This gives a large class of QKT manifolds since every orientable 4-manifold is QK due to the fact that $SO(4) = Sp(1) \cdot Sp(1)$. Using this method, we present four examples of complete four-dimensional QKT manifolds. Allowing $dH \neq 0$, we show that any $4k$-dimensional QKT manifold admits a twistor construction. We construct the twistor space of a QKT manifold and show that it is a Kähler manifold with a complex contact structure provided that $k > 1$ and $dH$ is $(2,2)$-form with respect to all three complex structures. In addition, we associate to every $4k$-dimensional QKT manifold a $4(k + 1)$-dimensional HK one which is a fibre bundle over the QKT manifold with fibre $\mathbb{C}^2 - 0$. In the limit that the torsion $H$ vanishes the results of Salamon [5] and Swann [8] for QK manifolds are recovered.

This paper is organised as follows: in section two we state the algebraic and differential conditions required by QKT geometry on a Riemannian manifold; in section three we present examples of QKT manifolds; in section four we give the twistor construction for QKT manifolds and show that the twistor space is Kähler with a complex contact structure; in section five we show that for any QKT manifold there is an associated HK one, and in section six we make some concluding remarks.
2. Local (4,0) supersymmetry

The multiplets required for the construction of a two-dimensional (4,0) locally supersymmetric theory coupled to sigma model matter are as follows: (i) The supergravity multiplet \((e, C, \psi)\) comprises of the graviton \(e\), a \(SO(3)\) gauge field \(\{C_r; r = 1, 2, 3\}\) and four Majorana-Weyl gravitini \(\{\psi, \psi^r; r = 1, 2, 3\}\); (ii) sigma model scalar multiplets \((\phi, \chi)\), each comprised of four real scalars \(\phi\) and four Majorana-Weyl fermions \(\chi\). The spinors of the scalar multiplet have opposite chirality to those of the supergravity one. Let \(M\) be the sigma model manifold of dimension \(4k\) with metric \(g\), Wess-Zumino 3-form \(H\), a \(\mathcal{L}SO(3)\)-valued one-form \(B\) and three almost complex structures \(\{I_r; r = 1, 2, 3\}\). The Lagrangian\(^*\) that describes the (4,0)-supergravity multiplet coupled to \(k\) scalar multiplets system is

\[
2e^{-1}\mathcal{L} = g_{\mu\nu}\partial_\alpha \phi^\mu \partial^\alpha \phi^\nu - \epsilon^{\alpha\beta} b_{\mu\nu} \partial_\alpha \phi^\mu \partial_\beta \phi^\nu - i g_{\mu\nu} \bar{\chi}^\mu \gamma^\alpha \partial_\alpha \chi^\nu + i g_{\mu\nu} \bar{\chi}^\mu \gamma^\alpha \partial_\alpha \phi^\nu (\delta_\kappa^\nu \psi_\alpha - (I_r)_\kappa^\nu \psi^r_\alpha) \\
- \frac{1}{3} \bar{\chi}^\mu \gamma^\alpha \chi^\nu (3H_{\kappa[\lambda\nu}(I_r)_\mu]^{\kappa} \psi^r_\alpha + H_{\mu\lambda\nu} \psi_\alpha) \\
- \frac{1}{8} g_{\mu\kappa} \bar{\chi}^\mu \gamma^\alpha \chi^\nu ((I_r)_\nu^{\lambda} \bar{\psi}^r_\beta + \delta_\nu^{\lambda} \bar{\psi}_\beta) \bar{\gamma}^\alpha \bar{\gamma}^\beta ((I_r)_\lambda^{\kappa} \psi^r_\delta - \delta_\lambda^{\kappa} \psi^r_\delta),
\]

where

\[
\mathcal{D}_\alpha \chi^\mu = \nabla_\alpha^{(+)} \chi^\mu + B_\alpha^\rho (I_r)_{\nu} \gamma^\rho \chi^\nu - \frac{1}{2} \omega_\alpha \chi^\mu - C_\alpha^\rho (I_r)_\nu \chi^\rho,
\]

\(B_\alpha^\rho\) is the pull back of \(B_\mu^r\) with respect to \(\phi\), \(\omega_\alpha\) is the spin connection of the worldvolume and the covariant derivatives \(\nabla^{(\pm)}\) are associated with the connections

\[
\Gamma_{\nu\kappa}^{(\pm)} = \hat{\Gamma}_{\nu\kappa}^\mu \pm \frac{1}{2} H_{\nu\kappa}^\mu;
\]

\(\hat{\Gamma}\) is the Levi-Civita connection of the metric \(g\). To simplify the notation we set

\[
\Gamma = \Gamma^{(+)} (\nabla = \nabla^{(+)}).
\]

\(^*\) The letters from the beginning of the Greek alphabet \(\alpha, \beta, \gamma, \delta = 0, 1\) are worldvolume induces and the letters from the middle of the Greek alphabet are target space indices \(\lambda, \mu, \nu, \kappa = 1, \ldots, 4k\). We have also suppress spinor indices.
The conditions on the geometry of $M$ required by (4,0) local supersymmetry can be found by appropriately truncating the conditions required by (4,4) local supersymmetry [7]. The former are the following:

\[
I_r I_s = -\delta_{rs} + \epsilon_{rst} I_t \\
(I_r)_\mu^\kappa (I_r)_\nu^\lambda g_{\kappa \lambda} = g_{\mu \nu} ; \quad r = 1, 2, 3 \\
D_\mu (I_r)_\kappa^\nu = 0 \\
N_{\hat{D}}^\nu (I_r)_\mu^\kappa = 0 ,
\]

where

\[
D_\mu (I_r)_\kappa^\nu = \nabla_\mu (I_r)_\kappa^\nu + B_{\mu r}^s (I_s)_\kappa^\nu ,
\]

\[
B_{r s} = -2B^t \epsilon_{t r}^s .
\]

In addition

\[
N_{\hat{D}}^\nu (I_r)_\mu^\kappa = (I_r)_\nu^\lambda \hat{D}[\lambda (I_r)_\kappa]_\mu^\nu - (\nu \leftrightarrow \kappa)
\]

is a Nijenhuis-like tensor associated with the covariant derivative

\[
\hat{D}_\mu (I_r)_\nu^\kappa = \hat{\nabla}_\mu (I_r)_\nu^\kappa + B_{\mu r}^s (I_s)_\nu^\kappa ,
\]

where $\hat{\nabla}$ is the Levi-Civita covariant derivative. We remark that this Nijenhuis tensor is independent from the Levi Civita part of $\hat{D}$.

The first three conditions in (4) imply that (i) the almost complex structures, $I_r$, obey the algebra of imaginary unit quaternions, (ii) the metric $g$ is hermitian with respect to all almost complex structures and (iii) the holonomy of the connection $D$ is a subgroup of $Sp(k) \cdot Sp(1)$, respectively. The covariantised Nijenhuis condition, $N_{\hat{D}}^\nu (I_r)_\mu^\kappa = 0$, together with the third condition in (4) imply that the torsion is (1,2)-and (2,1)-form with respect to all almost complex structures. We remark that in the commutator of two supersymmetry transformations, apart from $N_{\hat{D}}^\nu (I_r)_\mu^\kappa$, the mixed covariantised Nijenhuis ‘tensors’, $N_{\hat{D}}^\nu (I_r, I_s)$, appear as well (see for example [9]). However they do not give independent conditions on the almost complex structures since they vanish provided that $N_{\hat{D}}^\nu (I_r) = 0$.  

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In analogy with the HKT case [3], we say that the manifold $M$ with tensors $g, I, B$ and $H$ that satisfy (4) has a weak QKT structure if no further conditions are imposed on $H$. However, if in addition we take $H$ to be a closed 3-form ($dH = 0$), we say that $M$ has a strong QKT structure, in which case we can write

$$H = 3db$$

for some locally defined two-form $b$ on $M$. Finally, if $H$ vanishes, the manifold $M$ becomes quaternionic Kähler. The target space, $M$, of a (classical) $(4,0)$ locally supersymmetric sigma model with torsion is a manifold with a strong QKT structure. The couplings of the classical action of the theory are the metric $g$, the $LSO(3)$ valued one-form $B$ and the two-form $b$. However, in the quantum theory and in particular in the context of the anomaly cancellation mechanism [10,11,12], the (classical) torsion $H$ of $(2,0)$-supersymmetric sigma models receives corrections*. The new torsion is not a closed three form. Therefore, although classically the target space of $(4,0)$-supersymmetric sigma models has a strong QKT structure, quantum mechanically this may change to a weak QKT structure, albeit of a particular type.

It is well known that all $4k$-dimensional QK manifolds are Einstein, i.e.

$$R_{\mu\nu} = \Lambda g_{\mu\nu},$$

and that

$$G^\tau_{\mu\nu} = (dB + B \wedge B)^\tau_{\mu\nu} = -\frac{\Lambda}{k+2}(I^\tau)_{\mu\nu},$$

where $\Lambda$ is a constant and $G$ is the curvature of the $B$ connection. There is no direct analogue of these statements in the context of QKT geometry. However, one can show that if the curvature $G$ of the $B$ connection satisfies (10) then the

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* Apart from the sigma model anomalies, these models have also two-dimensional gravitational anomalies.
torsion $H$ vanishes. To show this, we first differentiate (10) with respect to the $\nabla_\kappa$ connection and then antisymmetrise in all three $i,j,k$ indices. Then, using the fact that $DI_r = 0$ we find that the right hand side of the equation can be expressed in terms of $B$ and $I_r$. Using (10) once more we find that the left hand side of the equation is expressed in terms of the torsion and a term similar to that of the right hand side. Finally, one gets

$$ (I_r)_{[\kappa}^{\lambda} H_{\mu\nu]\lambda} = 0. $$ (11)

Using this together with the fact that $H$ is a (2,1) and (1,2) tensor on $M$, we conclude that $H$ vanishes. Thus equation (10) excludes torsion. A consequence of this is that the (4,0) locally supersymmetric models constructed in [13] have zero torsion.

3. Examples

To construct examples of QKT geometry, we generalize the ansatz used in [14] to find HKT geometries from HK ones. As we have already mentioned in the introduction, any oriented four-dimensional manifold is QK. Let $M$ be such a manifold with metric $h$, connection $B$ and compatible almost complex structures $I_r$. The volume form, $\Omega$, of $M$ can be written in terms of the almost complex structures as

$$ \Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r, $$ (12)

where

$$ \omega_r(X,Y) = h(X,I_rY) $$ (13)

are the Kähler-like forms of the almost complex structures $I_r$. We remark that $\Omega$ is covariantly constant with respect to the Levi-Civita connection. We also mention
for later use that†

\[ \sum_{r=1}^{3} (\omega_r)_{\mu\nu} (\omega_r)_{\kappa\lambda} = \Omega_{\mu\nu\kappa\lambda} + h_{\mu\kappa}h_{\nu\lambda} - h_{\nu\kappa}h_{\mu\lambda}. \] (14)

To construct four-dimensional QKT manifolds, we use the ansatz

\[ g = F h, \quad H = \frac{1}{2} \ast dF, \] (15)

where \( \ast \) is the Hodge dual with respect to \( \Omega \). The manifold \( M \) with metric \( g \), torsion \( H \), almost complex structures \( I_r \) and connection \( B \) is a weak QKT manifold. To show the covariant constancy condition of the almost complex structures in (4), we use the equation (14) and the ansatz (15). The remaining conditions in (4) are straightforwardly satisfied. For \( M \) to have a strong QKT structure, \( H \) must be closed which in turn implies that \( F \) must be a harmonic function on \( M \) with respect to the \( h \) metric, i.e.

\[ d \ast dF = 0. \] (16)

We shall allow \( F \) to have delta function singularities on \( M \). There always exist non-trivial solutions of (16) on any four-dimensional manifold. So we conclude that there is a family of QKT manifolds associated to every four-dimensional QK manifold labeled by the harmonic functions of the latter‡.

Due to the singularities of \( F \), the associated QKT metric may be geodesically incomplete. This in fact is the case for some choices of harmonic function for the compact four-dimensional Wolf spaces \( S^4 \) and \( CP^2 \). However there are examples of complete QKT geometries. Here we shall present four non-singular QKT manifolds starting from the QK manifolds, \( \mathbb{R} \times dS_3 \), \( dS_4 \), the Tolman wormhole and a LeBrun like metric, respectively, where \( dS_n \) is \( n \)-dimensional de Sitter space.

† Although the 4-form \( \Omega \) can be defined for QK manifolds of any dimension, this identity holds only for four-dimensional manifolds.

‡ Note that the QKT manifold with metric \( g \) is also QK with respect to the same metric, as four-dimensional manifold, but with a different set of almost complex structures.
The metric $h$ on $\mathbb{R} \times dS(3)$ is

$$ds^2 = du^2 + dv^2 + \cosh^2 v d\Omega^2_{(2)},$$

(17)

where $-\infty < u, v < \infty$ and $d\Omega^2_{(2)}$ is the $SO(3)$ invariant metric on $S^2$. Supposing that the harmonic function, $F$, depends only on $v$, we get

$$F = \lambda_1 \tanh(v) + \lambda_2,$$

(18)

where $\lambda_1$ and $\lambda_2$ are real numbers. It is straightforward to compute the metric and the torsion of the QKT manifold to find that

$$ds^2_F = (\lambda_1 \tanh(v) + \lambda_2) \left[ du^2 + dv^2 + \cosh^2 v d\Omega^2_{(2)} \right]$$

$$H = \lambda_1 \sin \theta du \wedge d\theta \wedge d\phi,$$

(19)

where $\theta, \phi$ are the angular coordinates on $S^2$. This QKT metric is geodesically complete if we choose $\lambda_2 > |\lambda_1|$.

The metric $h$ on $dS(4)$ is

$$ds^2 = dv^2 + \cosh^2 v d\Omega^2_{(3)},$$

(20)

where $-\infty < v < \infty$ and $d\Omega^2_{(3)}$ is the $SO(4)$ invariant metric on $S^3$. Supposing that the harmonic function, $F$, depends only on $v$, we get

$$F = \lambda_1 \left[ \frac{\sinh(v)}{\cosh^2(v)} + \arctan(\sinh(v)) \right] + \lambda_2,$$

(21)

where $\lambda_1$ and $\lambda_2$ are real numbers. It is straightforward to compute the metric and the torsion of the QKT manifold to find that

$$ds^2_F = \left( \lambda_1 \left[ \frac{\sinh(v)}{\cosh^2(v)} + \arctan(\sinh(v)) \right] + \lambda_2 \right) \left[ dv^2 + \cosh^2 v d\Omega^2_{(3)} \right]$$

$$H = \frac{\lambda_1}{2} \sin^2 \theta \sin \phi d\theta \wedge d\phi \wedge d\psi,$$

(22)

where $\theta, \phi, \psi$ are the angular coordinates on $S^3$. This QKT metric is geodesically complete, if we choose $\lambda_2 > \frac{\pi}{2} \lambda_1 > 0$. 
The metric of the Tolman wormhole is

\[ ds^2 = dv^2 + (a^2 + v^2) d\Omega^2_{(3)} , \]  

(23)

where \(-\infty < v < \infty\), \(a\) is a real non-zero constant, and \(d\Omega^2_{(3)}\) is the \(SO(4)\) invariant metric on \(S^3\). This metric is the analytic continuation of the FRW model of a universe filled with a perfect fluid with pressure equal to \(1/3\) of its density. Using the Einstein equations, we find the Tolman wormhole metric has zero scalar curvature. In addition, it is conformally flat

\[ ds^2 = (1 + \frac{a^2}{4r^2})^2 (dr^2 + r^2 d\Omega^2_{(3)}) , \]  

(24)

as can be easily seen using the coordinate transformation

\[ v = r - \frac{a^2}{4r} . \]  

(25)

Supposing that the harmonic function, \(F\), depends only on \(v\), we get

\[ F = \lambda_1 \frac{v}{a^2 \sqrt{a^2 + v^2}} + \lambda_2 , \]  

(26)

where \(\lambda_1\) and \(\lambda_2\) are real numbers. It is straightforward to compute the metric and the torsion of the QKT manifold to find that

\[ ds_F^2 = (\lambda_1 \frac{v}{a^2 \sqrt{a^2 + v^2}} + \lambda_2) [dv^2 + (a^2 + v^2)d\Omega^2_{(3)}] \]  

\[ H = \frac{\lambda_1}{2} \sin^2 \theta \sin \phi \ d\theta \wedge d\phi \wedge d\psi , \]  

(27)

where \(\theta, \phi, \psi\) are the angular coordinates on \(S^3\). This QKT metric is geodesically complete, if we choose \(\lambda_2 > (1/a^2)|\lambda_1|\).
All the examples of QKT geometries presented so far are conformally flat and therefore their Weyl tensor vanishes. For reasons that will become apparent in the twistor construction of four-dimensional QKT manifolds, we give an example of a QKT geometry with non-vanishing but self-dual Weyl tensor. To do this we begin with the four-dimensional metric

\[ ds^2 = V^{-1}(d\tau + \omega)^2 + Vds^2_{(3)} , \]  

(28)

where

\[ ds^2_{(3)} = \frac{1}{q^2}(dx^2 + dy^2 + dq^2) , \]  

(29)

is the hyperbolic 3-metric and

\[ d\omega = *dV \]  

(30)

with the Hodge duality operation taken with respect to the metric \( ds^2_{(3)} \). The equation (30) is just the magnetic monopole equation in a hyperbolic background. The function \( V \) is harmonic with respect to the hyperbolic 3-metric. Solving (30) for one monopole we get

\[ V = 1 + \frac{1}{2}(\coth \rho - 1) \]
\[ \omega_x = -\frac{1}{2} \frac{y}{x^2 + y^2} \coth \rho \]
\[ \omega_y = \frac{1}{2} \frac{x}{x^2 + y^2} \coth \rho \]
\[ \omega_z = 0 , \]  

(31)

where

\[ \coth \rho = \frac{x^2 + y^2 + q^2 + q_0^2}{\sqrt{(x^2 + y^2 + q^2 + q_0^2)^2 - 4q^2q_0^2}} . \]  

(32)

To construct the associated QKT geometry, let us suppose that the harmonic
function $F$ depends only on the coordinate $q$. Then we find that

$$F = q^2 .$$

(33)

Therefore the associated QKT geometry is

$$ds^2_F = q^2[V^{-1}(d\tau + \omega)^2 + Vds_{(3)}^2]$$

$$H = d\tau \wedge dx \wedge dy .$$

(34)

The metric $ds^2_F$ is the LeBrun metric which has been shown to be complete in [15]. It is also known to have a non-vanishing but self-dual Weyl tensor.

4. Twistor Spaces

Let $M$ be a 4k-dimensional weak QKT manifold. Since the connection $\Gamma$ of $M$ has holonomy $Sp(k) \cdot Sp(1)$, the tangent bundle is associated to a principal $Sp(k) \cdot Sp(1)$ bundle. In particular this implies that the complexified tangent bundle $T_cM = T_{2k} \otimes T_2$ with the first subbundle associated with $Sp(k)$ and the second associated with $Sp(1)$. Next we introduce a frame $e^{ai}$ and write the metric as

$$ds^2 = e^{bj} \otimes e^{ai} \eta_{ab} \epsilon_{ij} ,$$

(35)

where $\eta$ is the invariant $Sp(k)$ symplectic form $(a, b = 1, \ldots, 2k)$ and $\epsilon$ is the invariant $Sp(1)$ symplectic form $(i, j = 1, 2)$. The reality condition for a vector $X$ in this frame is

$$\bar{X}_{ai} = X^{bj} \eta_{ba} \epsilon_{ji} ,$$

(36)

which can be extended to tensors in a straightforward way. A basis for the almost

* In principle $T_2$ and $T_{2k}$ are only locally defined, but we shall assume that they exist globally for simplicity; this is similar to demanding the existence of a spin structure and means that one can define a principal $Sp(k) \times Sp(1)$ bundle. See [5] for a discussion.
complex structures in this frame is

\[(I_r)_{ai}{}^{bj} = -i \delta_a{}^b (\tau_r)_{ij}, \quad (37)\]

where the \(\tau_r\) are the Pauli matrices; the almost complex structures are real tensors. The connection-form \(\Gamma\) can be written in this basis as

\[\Gamma_{ai}{}^{bj} = \delta_i{}^j A_a{}^b + \delta_a{}^b B_i{}^j, \quad (38)\]

where \(A_a{}^b\) is the \(Sp(k)\) connection and \(B_i{}^j\) is the \(Sp(1)\) connection introduced in equation (5). Similarly the curvature can be decomposed as

\[R_{ai}{}^{bj} = \delta_i{}^j F_a{}^b + \delta_a{}^b G_i{}^j. \quad (39)\]

The twistor space, \(Z\), can be defined either as the projective bundle of \(T_2\) or as the quotient \(U(1)\backslash P\) of the principal \(Sp(1)\) subbundle, \(P\), of the principal \(Sp(k) \times Sp(1)\) bundle. (We take the group of a principal bundle to act from the left.) We shall work mainly with \(P\). Functions on twistor space are \(U(1)\) invariant functions on \(P\) while \(U(1)\) equivariant functions on \(P\) correspond to sections of \(U(1)\) line bundles over \(Z\) associated to \(P\) considered as a \(U(1)\) principal bundle over \(Z\). This allows us to work with \(P\) and then reduce our results to \(Z\). For this we introduce “coordinates” \((x, u)\) on \(P\), where \(x\) are coordinates on the base space \(M\) and \(u \in SU(2)\). We write \(u\) as \(u_I{}^i\) (with inverse \(u_i{}^I\)), \(i = 1, 2\) and \(I = 1, 2\) with the local \(Sp(1)\) gauge transformations acting from the right, i.e. on the index \(i\), and the rigid \(Sp(1)\) transformations act from the left, i.e. on the index \(I\), as we have already mentioned.\(\dagger\) Since the structure group of \(P\) as a principal bundle over \(Z\) is \(U(1)\), it will be appropriate to split up the capital \(I\) indices into two

\(\dagger\) The equivariant formalism used here has been called “harmonic space” formalism elsewhere; it was applied to QK geometry in [16].
indicating the $U(1)$ charges. The right-invariant one-forms on the fibre (of $P \to M$) in these coordinates are

$$ e_I^J = du_I^i u_i^J \quad (40) $$

with

$$ e_I^I = 0 \quad (41) $$
as a consequence of the fact that $\det u = 1$. The dual right-invariant vector fields $D_I^J$ satisfy

$$ D_I^J u_K^i = \delta_K^J u_I^i - \frac{1}{2} \delta_I^J u_K^i \quad (42) $$

and the algebra of vector fields is

$$ [D_I^J, D_K^L] = \delta_K^J D_I^L - \delta_I^L D_K^J \quad (43) $$

which is isomorphic to the $\mathcal{L}Sp(1)$ Lie algebra. To see this, we note that $D_I^I = 0$ and set

$$ D_0 = D_1^1 - D_2^2 \quad (44) $$

It is then easy to verify that $\{D_0, D_1^2, D_2^1\}$ satisfy the familiar Lie algebra commutator relations of $SU(2)$. We shall take the vector field $D_0$ to be tangent to the orbits of $U(1)$ subgroup of $SU(2)$ acting on $P$ from the left which we have used to define the twistor space $Z = U(1) \setminus P$. We also note that

$$ D_0 u_1^i = u_1^i $$
$$ D_0 u_2^i = -u_2^i \quad (45) $$

In the following we shall use the properties of the torsion and the curvature of the $Sp(1)$ connection

$$ T_{aib1c1} \equiv u_i^j u_1^k H_{aibjek} = 0 $$
$$ G_{aibjk} = (\epsilon_{ik} \epsilon_{j\ell} + \epsilon_{i\ell} \epsilon_{jk}) G_{ab} \quad (46) $$

respectively. The latter condition holds provided that $k \geq 2$ and that $dH$ is $(2,2)$
with respect to all almost complex structures. An outline of the proof of the above properties is given in the appendix.

Now we can state the properties of the twistor space $Z$ associated with a QKT manifold $M$:

(i) $Z$ is a complex manifold provided that $k \geq 2$.

(ii) $Z$ has a real structure.

(iii) $Z$ admits a complex contact structure provided that $k \geq 2$, $dH$ is $(2,2)$ with respect to all almost complex structures and $\det(G_{ab}) \neq 0$.

(iv) $Z$ is a Kähler manifold provided that $(-\epsilon_{ij}G_{ab})$ is positive definite, $k \geq 2$ and $dH$ is $(2,2)$ with respect to all almost complex structures as in the previous property.

The real structure is induced on $Z$ from the antipodal map on each two-sphere fibre of $Z$ over $M$ in exactly the same way as in hyper-Kähler and quaternionic Kähler geometry, so we refer the reader to the literature for discussions of this point [5,17].

To prove (i) we introduce the horizontal lift basis on $P$:

$$\tilde{E}_I^J = D_I^J$$
$$\tilde{E}_a I = \hat{e}_a I - B_{aI,J}^K D_K^J$$

with dual basis given by

$$E_I^J = e_I^J + e^{aK}B_{aK,I}^J$$
$$E^a I = e^{ai}u_i^I,$$

where we convert $i,j,k$ indices to $I,J,K$ indices using $u_i^I$ or $u_I^j$ as appropriate and where $\hat{e}_ai$ are the basis vector fields on $M$ dual to $e^{ai}$. We then find that

$$dE_I^J = -E_I^K \wedge E_K^J + G_I^J$$
$$dE^a I = -E^aJ \wedge E_J^I + T^aI - E^{bI} \wedge A_b^a,$$

where $G_I^J$ is the $Sp(1)$ curvature and $T^aI$ is the torsion in the $\{E_I^J, E^a I\}$ frame.
We claim that the set of vector fields \( \{ \tilde{E}_{a1}, \tilde{D}_1^2 \} \) spans an integrable distribution up to a \( U(1) \) translation and therefore defines a complex structure in \( Z \). To show this, we write the second equation in (49) in the dual form

\[
[\tilde{E}_{aI}, \tilde{E}_{bJ}] = -T_{aIbJ}^{\ cK} \tilde{E}_{cK} + A_{aIbJ}^{\ c} \tilde{E}_{cJ} - A_{bJ,a}^{\ c} \tilde{E}_{cI} - G_{aIbJ,K}^{\ L} \tilde{D}_L^{\ K} .
\] (50)

Setting \( I = J = 1 \) we find that the commutator \([\tilde{E}_{a1}, \tilde{E}_{b1}]\) closes on terms linear in \( \{ \tilde{E}_{a1}, D_1^2 \} \) and \( D_0 \) provided that

\[
T_{a1b1}^{\ c2} = 0
\]
\[
G_{a1b1,1}^{\ 2} = 0 .
\] (51)

Similarly, the commutator \([\tilde{E}_{a1}, D_1^2]\) closes on terms linear in \( D_1^2 \) and \( D_0 \). The first condition in (51) is equivalent to the first condition in (46). For \( k \geq 2 \), the second condition in (51) is a special case of the second condition in (46) which holds for any weak QKT space, even if \( dH \) is not \((2,2)\) with respect to all almost complex structures (see the appendix). Therefore for \( k \geq 2 \), the twistor space \( Z \) is always a complex manifold. For \( k = 1 \), the second condition in (51) must be imposed in addition to the conditions required by \((4,0)\) local supersymmetry on the geometry of \( M \). In particular, for the examples that we have presented in section 3, this condition always holds because the Weyl tensor is self-dual.

To show (iii), we first note that a complex contact structure is defined locally by a \((1,0)\) form \( \beta \) such that

\[
\beta \wedge (\partial \beta)^k \neq 0 .
\] (52)

In our case we choose

\[
\beta = E_{1}^{2} .
\] (53)

Using the definition of \( E_{1}^{2} \) and the second condition in (46), we find that

\[
dE_{1}^{2} = -E_{b2}^{k} \wedge E_{a2}^{2} G_{ab} - (E_{1}^{1} - E_{2}^{2}) \wedge E_{1}^{2} .
\] (54)
So
\[ \partial \beta = -E^{b2} \wedge E^{a2} G_{ab}, \tag{55} \]
and the condition (52) is satisfied provided that
\[ det(G_{ab}) \neq 0. \tag{56} \]
As we have already mentioned, for \( k \geq 2 \) the second condition in (46) always holds provided that \( dH \) is \((2,2)\) with respect to all almost complex structures. For \( k = 1 \), the second condition in (46) must be imposed in addition to those required by \((4,0)\) local supersymmetry on \( M \). For the examples that we have presented in section 3, this always holds since the Weyl tensor is self-dual. Note that for \( k = 1 \), \( dH \) is always \((2,2)\) with respect to all almost complex structures.

It remains to show (iv). Since we have already shown that \( Z \) is complex, it is enough to find the appropriate Kähler form \( \Omega \). The metric can then be constructed from the Kähler form and the complex structure. We choose as Kähler form
\[ \Omega = 2i (E_1^{2} \wedge E_2^{1} + E^{b2} \wedge E^{a1} G_{ab}) . \tag{57} \]
Clearly, \( \Omega \) is \((1,1)\) with respect to the chosen complex structure so it remains to show that it is closed. For this, using (46) we find that
\[ d\Omega = 2i (E^{c2} \wedge E^{b1} \wedge E^{a1} \nabla_{a1} G_{bc} + E^{c2} \wedge E^{b1} \wedge E^{a2} \nabla_{a2} G_{bc} \]
\[ + E^{c2} \wedge T^{b1} G_{bc} - T^{c2} \wedge E^{b1} G_{bc}) . \tag{58} \]
Expanding \( T \) using the \( E \) basis, we find that the term involving \( E^{c2} \wedge E^{b1} \wedge E^{a1} \) in the above equation is proportional to
\[ \nabla_{a1} G_{bc} + \frac{1}{2} T_{a1b1} d^{1} G_{dc} - T_{a1c2} d^{2} G_{db} , \tag{59} \]
antisymmetrised on \( a \) and \( b \), which vanishes because of the second Bianchi identity
\[ \nabla_{a1} G_{bJcK,LM} + T_{a1b1} d^{N} G_{dNcK,LM} + \text{cyclic in } (aI, bJ, cK) = 0 . \tag{60} \]
Similarly, the term proportional to \( E^{c2} \wedge E^{b1} \wedge E^{a2} \) in (58) vanishes and therefore
Ω is closed. The metric is non-degenerate and positive definite provided that
\((-\epsilon_{ij}G_{ab})\) is non-degenerate and positive definite.

5. HK structures from QKT manifolds

As in the previous section, let \(M\) be a \(4k\)-dimensional weak QKT manifold. As we have already mentioned the tangent bundle can be written as \(TM = T_{2k} \otimes T_2\). The main task of this section is to show that \(\hat{T}_2\), which is defined to be \(T_2\) with the zero section is removed, is a HK manifold provided that \(k \geq 2\), \(dH\) is (2,2) with respect to all three almost complex structures and \((-\epsilon_{ij}G_{ab})\) is non-degenerate and positive definite. Introducing complex coordinates \(\{y^i; i = 1, 2\}\) along the fibres of \(\hat{T}_2\), we define a set of \(2k + 2\) complex one-forms as follows:

\[
E^i = dy^i + y^j B^i_{\ j},
\]

\[
E^a = e^{ai} y_i,
\]

(61)

where \(y_i = y^j \epsilon_{ji}\). We claim that this set of forms defines a complex structure on \(\hat{T}_2\), i.e., that it defines a basis set of (1,0) forms. To show this we use the differential form version of Frobenius’ theorem which states, in the current context, that the exterior derivative of any (1,0) form should be a sum of two-forms each one of which has a (1,0) factor. Differentiating (61) we find

\[
dE^a = -E^b \wedge A^a_{\ b} + e^{ai} \wedge E_i + T^{ai} y_i
\]

\[
dE^i = -E^j \wedge B^i_{\ j} + y^j G^i_{\ j}.
\]

(62)

Since \(H\) is (2,1) and (1,2) with respect to all almost complex structures, we can write

\[
T_{\ aibjck} = H_{\ aibjck} = \epsilon_{ij} H_{\ ab,ck} + \epsilon_{ki} H_{\ ca,bj} + \epsilon_{jk} H_{\ bc,ai},
\]

(63)

where \(H_{\ ab,ck} = H_{\ ba,ck}\) and \(H_{(ab,c)k} = 0\), so that

\[
T^{ai} y_i = 2 e^{cj} \wedge E^b (H_{bc,\ a j} - H_{bc,\ cj}^a).
\]

(64)
Then, using the expression in (46) for the $Sp(1)$ curvature $G$, we find

$$y^j G_j = -e^{bi} \wedge E^a G_{ab}.$$  

(65)

Hence the right-hand sides of both of equations (62) have the required structure for Frobenius’ theorem to hold.

We choose the first complex structure to be diagonal with respect to this integrable distribution, i.e. $(IE)^i = iE^i$ and $(IE)^a = iE^a$. To find the metric and the rest of the hyper-Kähler structure, it is enough to determine two of the three Kähler forms, $\{\Omega_r, r = 1, 2, 3\}$. As we are working in a basis in which one of the complex structures is diagonal, one of the Kähler forms, say $\Omega_1$, is a $(1,1)$-form with respect to the chosen complex structure while the other two are $(2,0)$ plus $(0,2)$ with respect to the same complex structure.

The first Kähler form is

$$\Omega_1 = 2i\left(\bar{E}_i \wedge E^i - \bar{E}_b \wedge E^a G_{ab}\right),$$  

(66)

where the bars denote complex conjugation.* In particular, the frame $\{\bar{E}_a, \bar{E}_i\}$ is

$$\bar{E}_i = d\bar{y}_i - B_{ij} \bar{y}_j$$

$$\bar{E}_a = -e_{ai} \bar{y}^i.$$  

(67)

The connection forms $\{B_{ij}, A_{ab}\}$ are skew-hermitian (e.g. $\bar{B}_{ij} = -B_{ji}$) and the basis forms $e^{ai}$ real with respect to the reality condition (36). It is clear that $\Omega$ is $(1,1)$ with respect to the chosen complex structure, so it remains to show that it is closed. That this is so follows on using the second Bianchi identity for $G$, $DG_{ij} = 0$, where $D$ is the $Sp(1)$ covariant exterior derivative, and contracting it with $y^i \bar{y}_j$.

* Note that $Sp(1)$ and $Sp(k)$ indices are raised or lowered by complex conjugation as well as the corresponding symplectic invariant tensors.
Next we choose the second almost complex structure $J$ to be
\[
J(E^a) = \bar{E}_b \eta^{ba}, \quad J(\bar{E}_a) = E^b \eta_{ba},
\]
\[
J(E^i) = \bar{E}_j \epsilon^{ji}, \quad J(\bar{E}_i) = E^j \epsilon_{ji}.
\]

(68)

The almost complex structure $J$ is integrable as may easily be seen by observing that a basis of $(1,0)$ forms for $J$ is \{\(E^a + i\bar{E}^a, E^i + i\bar{E}^i\)\} and then by using the Frobenius’ theorem. The $J$ complex structure anticommutes with the $I$ complex structure as required. The $(2,0)$ part of the Kähler form of the $J$ complex structure is
\[
\Omega' = E^j \wedge E^i \epsilon_{ij} - \bar{E}^b \wedge E^a G_{ab}.
\]

(69)

The proof that this form is closed is similar to that for $\Omega_1$ with the difference that one must use the second Bianchi identity for $G$ contracted with $y^i y^j$. The Kähler forms $\Omega_2, \Omega_3$ are the real and imaginary parts of $\Omega'$,
\[
\Omega' = \frac{1}{2} (\Omega_2 + i\Omega_3).
\]

This shows that $\hat{T}_2$ is an HK manifold since the third complex structure can be constructed from the first two and its integrability is also implied by the integrability of the first two. The metric is
\[
d s^2 = 2\bar{E}^i \otimes E_i - 2\bar{E}^b \otimes E^a G_{ab}.
\]

(70)

It is hermitian with respect to all three complex structures and is non-degenerate and positive definite provided that $-\epsilon_{ij} G_{ab}$ is non-degenerate and positive definite.
6. Concluding Remarks

We have shown that the QKT geometry of manifolds that arise in the context of two-dimensional (4,0) locally supersymmetric sigma models is determined by the properties of a metric connection with torsion. This connection has holonomy $Sp(k)$-$Sp(1)$ so that the corresponding geometry is a generalization of QK geometry. QKT manifolds admit a twistor construction. The twistor space is a $(4k + 2)$-dimensional Kähler manifold with a complex contact structure. In addition, for every QKT manifold there is a $(4k + 4)$-dimensional hyper-Kähler manifold which is obtained from a vector bundle over the QKT manifold with fibre $\mathbb{C}^2$ associated to the $Sp(1)$ principal bundle by omitting the zero section.

There are various limits in the twistor construction for QKT manifolds in which one or more tensors associated with this structure vanish. In the limit that the torsion vanishes, as we have already mentioned, the QKT structure degenerates to a QK one and one recovers the results of Salamon [5] and Swann [8] for QK manifolds. In another limit where the torsion does not vanish but the holonomy becomes $Sp(k)$ the manifold becomes HKT for which the twistor construction was given in [3]. Finally, if both the torsion vanishes and the holonomy is $Sp(k)$, then the manifold is HK for which the twistor construction was given in [17].

In this paper we have not investigated the applications of the twistor construction in (4,0) supergravity coupled to sigma model matter system. However, it is likely that the sigma model maps can be thought as holomorphic maps from a harmonic extension of the (4,0) superspace to the twistor space of $M$, thus generalizing a similar property of (4,0) superfields for the models with rigid supersymmetry [18, 3].

It would also be of interest to find more examples of QKT manifolds in $4k$-dimensions for $k > 1$. For example, there might be locally symmetric spaces with a QKT structure in direct analogy to the Wolf spaces for QK manifolds [19] or to the group manifold examples for HKT manifolds [20]. New QKT manifolds may also be constructed starting from QK manifolds with an isometry that respects the
QK structure and then performing a T-duality transformation along the Killing direction. By this means one might expect to develop relationships between QK and QKT manifolds similar to those that hold between HK and HKT manifolds [21, 22].

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APPENDIX

Here we shall show that

\[ T_{a_1b_1c_2} = 0 \]
\[ G_{aibj,k\ell} = (\epsilon_{ik}\epsilon_{j\ell} + \epsilon_{i\ell}\epsilon_{jk})G_{ab} . \]  

The first condition is equivalent to

\[ T_{a_1b_1c_1} \equiv u_1^iu_1^ju_1^kH_{aibjck} = 0 . \]  

Then the first condition in (71) follows by contracting the expression for \( H \) in (63) with \( u_1^i \) as in (72).

To show the second condition in (71), one uses the Bianchi identity

\[ R_{\mu[\nu\rho\sigma]} = \frac{1}{3} \nabla_\mu H_{\nu\rho\sigma} - 2P_{\mu\nu\rho\sigma} , \]

where \( P_{\mu\nu\rho\sigma} = 3\partial_\mu H_{\nu\rho\sigma} \). We first write this Bianchi using the \( ai \) coordinate description and then contract all four \( Sp(1) \) indices \( i, j, k, \ell \) with \( u_1^i, u_1^j, u_1^k, u_1^\ell \).
This gives

\[ R_{a1[b1c1d1]} = \frac{1}{3} \nabla a1 T_{b1c1d1} - 2 P_{a1b1c1d1} . \]  

(74)

From the first condition in (71), we find that the left-hand-side of (74) vanishes. But

\[ R_{abj,ckd\ell} = \epsilon_{k\ell} F_{abj,cd} + \eta_{cd} G_{abj,k\ell} \]  

(75)

and so we find that

\[ R_{a1b1c1d1} = \eta_{cd} G_{a1b1,11} \equiv \eta_{cd} G^{'}_{ab} , \]  

(76)

where \( F_{ab} \) is the curvature of the \( Sp(k) \) connection \( A_{a}^{b} \) in (38). Substituting this in (74) we find that

\[ G_{ab}^{'} = 0 , \]  

(77)

provided that \( k \geq 2 \). Since \( G_{ab}^{'} = 0 \) for any \( u \) and \( G_{ab}^{'} = u_{1}^{i} u_{1}^{j} u_{1}^{k} u_{1}^{\ell} G_{aibj,k\ell} \), this implies that

\[ G_{ab,(ijk\ell)} = 0 . \]  

(78)

We remark that this condition is enough to show that the twistor space is a complex manifold.

Next we contract the Bianchi identity (73) with \( u_{2}^{i}, u_{1}^{j}, u_{1}^{k}, u_{1}^{\ell} \) and we get

\[ R_{a2[b1c1d1]} = \frac{1}{3} \nabla a2 T_{b1c1d1} - 2 P_{a2b1c1d1} . \]  

(79)

The right-hand-side of the above equation vanishes provided that \( P_{a2b1c1d1} = 0 \) which is precisely the condition for \( dH \) to be \((2,2)\) with respect to all three almost complex structures. Using (75) for the curvature, we find that

\[ R_{a2b1,c1d1} = \eta_{cd} G_{a2b1,11} . \]  

(80)

Substituting this back into (79), we find that both \( G_{(ab)21,11} \) and \( G_{[ab]21,11} \) vanish provided that \( k \geq 2 \). Since this is again the case for any \( u \), the first condition
implies that $G_{(ab)ij}$ vanishes and the second together with (78) imply the second condition in (71). We remark that for QK manifolds $G_{ab} = \lambda \eta_{ab}$ where $\lambda$ is a real constant.

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