SAMPLING AND RECONSTRUCTION IN DIFFERENT SUBSPACES BY USING OBLIQUE PROJECTIONS

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1. Introduction

We consider an approach, where sampling and reconstruction are done in different subspaces of a Hilbert space $H$. Let $U$ be a sampling subspace, and let $G$ be a reconstruction subspace. Let $\{u_j\}_{j \in \mathbb{N}}$ be a frame for $U$. Given the scalar products $\{\langle f, u_j \rangle\}_{j \in \mathbb{N}}$ of an element $f \in H$, we want to find a stable reconstruction $\tilde{f}$ of $f$, that is close to the unknown orthogonal projection of $f$ onto the reconstruction space $G$.

The main idea of our reconstruction method is explained in the following example. Let $H = \mathbb{R}^3$, let $G$ be a one dimensional subspace of $H$, and let $U$ be the linear span of two linearly independent vectors $u_1$ and $u_2$. We intend to reconstruct $f \in H$ from $\langle f, u_1 \rangle$ and $\langle f, u_2 \rangle$. From the measurements $\langle f, u_1 \rangle$ and $\langle f, u_2 \rangle$, we can calculate $P_U f$, the projection of $f$ onto the plane $U$. Conversely, $P_U f$ determines $\langle f, u_1 \rangle$ and $\langle f, u_2 \rangle$.

Thus all the information we have about $f$ is that $f$ lies in the affine subspace $P_U f + U^\perp$, but we do not know the exact location of $f$ in this affine subspace. Let $P_{GP_U(G)}^\perp$ denote the oblique projection with range $G$ and kernel $P_U^\perp(G)$.

We assume that $f$, the element to be reconstructed, is close the reconstruction space $G$. Naturally, we now want to find $\tilde{f}$, the element of $G$ (the reconstruction space) closest to $P_U f + U^\perp$. The two spaces $P_U f + U^\perp$ and $G$ may, or may not intersect. In both cases, the element of $G$ closest to $P_U f + U^\perp$ is exactly $P_{GP_U(G)}^\perp f$.

If they intersect, then $P_{GP_U(G)}^\perp f = (P_U f + U^\perp) \cap G$, and $\langle f, u_1 \rangle = \langle \tilde{f}, u_1 \rangle$ and $\langle f, u_2 \rangle = \langle \tilde{f}, u_2 \rangle$. In this case $\tilde{f}$ is a so called consistent reconstruction of $f$.

One should remember that in this setup only the scalar products of $f$ with the frame sequence $\{u_j\}_{j \in \mathbb{N}}$ of $U$ are given. Thus we analyse the operator $Q : \{\langle f, u_j \rangle\}_{j \in \mathbb{N}} \mapsto P_{GP_U(G)}^\perp f$. An explicit formula for the mapping $Q$ is given in Theorem 2.6. We refer to the mapping $P_{GP_U(G)}^\perp$ as frame independent sampling to indicate that $P_{GP_U(G)}^\perp$ does not depend on the frame sequences $\{u_j\}_{j \in \mathbb{N}}$ and $\{g_k\}_{j \in \mathbb{N}}$ themselves, but only on their closed linear spans $U$ and $G$.

This mapping $Q$ is a generalisation of consistent sampling, which is treated in [7–9]. If the frame sequence $\{u_j\}_{j \in \mathbb{N}}$ of the sampling space $U$ is tight, this reconstruction coincides with the generalized sampling introduced in [1–3]. In the

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following, we study this operator, and compare it with the generalized sampling introduced in [1–3].

2. Stability and quasi-optimality

Let \( \{u_j\}_{j \in \mathbb{N}} \) be a frame sequence in \( H \), i.e., a frame for its closed linear span. Setting \( U := \text{span}\{u_j\}_{j \in \mathbb{N}} \), this is equivalent to the statement that there exist constants \( A, B > 0 \), such that

\[
A \|f\|_2^2 \leq \sum_{j \in \mathbb{N}} |\langle f, u_j \rangle|^2 \leq B \|f\|_2^2, \quad \text{for every } f \in U.
\]

The constant \( A \) is called lower frame bound and the constant \( B \) is called upper frame bound. We call \( U = \text{span}\{u_j\}_{j \in \mathbb{N}} \) the sampling space.

Let \( \{g_k\}_{k \in \mathbb{N}} \) be a frame sequence in \( H \). Setting \( G := \text{span}\{g_k\}_{k \in \mathbb{N}} \), this is equivalent to the statement that there exist constants \( C, D > 0 \), such that

\[
C \|f\|_2^2 \leq \sum_{k \in \mathbb{N}} |\langle f, g_k \rangle|^2 \leq D \|f\|_2^2, \quad \text{for every } f \in G.
\]

We call \( G = \text{span}\{g_k\}_{j \in \mathbb{N}} \) the reconstruction space. The operator

\[
U : l^2(\mathbb{N}) \to H, \quad U\{c_j\}_{j \in \mathbb{N}} = \sum_{j=1}^{\infty} c_j u_j
\]

is called the synthesis operator of the frame sequence \( \{u_j\}_{j \in \mathbb{N}} \). The adjoint operator

\[
U^* : H \to l^2(\mathbb{N}), \quad U^* f = \{\langle f, u_j \rangle\}_{j \in \mathbb{N}}
\]

is called the analysis operator of the frame sequence \( \{u_j\}_{j \in \mathbb{N}} \). The composition

\[
S : H \to H, \quad S f = U U^* f = \sum_{j=1}^{\infty} \langle f, u_j \rangle u_j
\]

is called the frame operator.

From now on we denote by \( U \) the synthesis operator, by \( U^* \) the analysis operator, and by \( S \) the frame operator of the frame sequence \( \{u_j\}_{j \in \mathbb{N}} \) of the sampling space \( U \).

We denote by \( G \) the synthesis operator and by \( G^* \) the analysis operator of the frame sequence \( \{g_k\}_{k \in \mathbb{N}} \) of the reconstruction space \( G \).

In the following we denote by \( \mathcal{R}(A) \) the range of the operator \( A \) and by \( \mathcal{N}(A) \) the nullspace of the operator \( A \).

As mentioned in the introduction, we want to find a mapping

\[
Q : \mathcal{R}(U^*) \to G,
\]
such that the mapping
\begin{align}
F &:= QU^*
\end{align}
\begin{align}
F : H \to G, \quad F f = QU^* f,
\end{align}
has the property that $F f$ is a good approximation to $f$ for every $f \in H$. Following [3], we use two quantities to measure the quality of the reconstruction $F = QU^*$.

**Definition 2.1.** Let $F : H \to G$ be an operator. The quasi-optimality constant $\mu = \mu(F) > 0$ is the smallest number $\mu$, such that
\begin{align}
\|f - F f\| \leq \mu \|f - P_G f\|, \quad \text{for all } f \in H,
\end{align}
where $P_G : H \to G$ is the orthogonal projection onto $G$. If there does not exist a $\mu \in \mathbb{R}$ such that $[3]$ is fulfilled, we set $\mu = \infty$.

We note that $P_G f$ is the element of $G$ closest to $f$. Thus the quasi-optimality constant $\mu(F)$ is a measure of how well $F$ performs in comparison to $P_G$.

In order to measure stability of the reconstruction, we define the quantity $\eta(F)$ as the operator norm of $Q_{|R(U^*)}$.

**Definition 2.2.** Let $F : H \to G$ be an operator such that, for each $f \in H$, $F f$ depends only on the measurements $U^* f$, i.e., $F = QU^*$ for some operator $Q : l^2(\mathbb{N}) \to G$. We define $\eta = \eta(F)$ by
\begin{align}
\eta(F) := \sup_{U^* f \neq 0} \frac{\|QU^* f\|}{\|U^* f\|} = \|Q_{|R(U^*)}\|.
\end{align}

If $\eta(F)$ is small, we call $F$ a well-conditioned mapping, and otherwise ill-conditioned.

In section [7] we show that for both, the oblique projection $P_{GP_G(G)^\perp}$ and for the oblique projection $P_{G_S(G)^\perp}$, the mapping introduced in [1][3], $\eta(F) = \|Q\|$ holds. In this case the following lemma applies.

**Lemma 2.3.** Let $F : H \to G$ be an operator that can be decomposed into the form $F = QU^*$ for some operator $Q : l^2(\mathbb{N}) \to G$. Let $f \in H$ and $c \in l^2(\mathbb{N})$. If $\eta(F) = \|Q\|$, then
\begin{align}
\|P_G f - Q(U^* f + c)\| \leq \|f - P_G f\| \sqrt{\mu^2 - 1} + \|c\| \eta. \tag{11}
\end{align}

**Proof.** Using the Pythagorean theorem, Definition 2.1 and Definition 2.2 we obtain
\begin{align*}
\|P_G f - Q(U^* f + c)\| &\leq \|P_G f - F f\| + \|Qc\| \\
&= \sqrt{\|f - F f\|^2 - \|f - P_G f\|^2} + \|c\| \eta \\
&\leq \sqrt{\mu^2 \|f - P_G f\|^2 - \|f - P_G f\|^2} + \|c\| \eta \\
&= \|f - P_G f\| \sqrt{\mu^2 - 1} + \|c\| \eta.
\end{align*}

Equation (11) bounds the distance between $P_G f$ and $F(f)$, where $F(f)$ is calculated from the perturbed measurements $U^* f + c$. We can obtain a good error estimate if $f$ is not close to the reconstruction space, provided that $\|c\|$ is small and $\mu$ is close to one.
Definition 2.4. Let $S$ and $W$ be closed subspaces of a Hilbert space $H$ and let $P_W : H \to W$ be the orthogonal projection onto $W$. We define the subspace angle $\varphi = \varphi_{SW} \in [0, \frac{\pi}{2}]$ between $S$ and $W$ by

$$\cos(\varphi_{SW}) = \inf_{s \in S \atop \|s\|=1} \|P_W s\|.$$

Let $S = UU^*$ denote the frame operator of the frame $\{u_j\}_{j \in \mathbb{N}}$, the frame for the sampling space. In [3] it is shown for finite dimensional $G$, that if $\cos(\varphi_{GU}) > 0$, then the oblique projection with range $G$ and nullspace $S(G)^\perp$, denoted by $\mathcal{P}_G S(G)^\perp$, exists, and can be written in the form (see [3, section 4.4])

$$\mathcal{P}_G S(G)^\perp = \mathcal{G}(U^* G)^\dagger U^*.$$

Therefore if $Q_1 := G(U^* G)^\dagger$, then the oblique projection factors as

$$\mathcal{P}_G S(G)^\perp = Q_1 U^*.$$

This shows that the oblique projection $\mathcal{P}_G S(G)^\perp f$ can be calculated from the measurements $\{(f, u_j)\}_{j \in \mathbb{N}}$.

Formula (13) is equivalent to

$$\mathcal{P}_G S(G)^\perp f = \mathcal{G} \hat{c},$$

where $\hat{c}$ is the minimal norm element of the set

$$\mathcal{G}(U^* U)^\dagger (U^* U)^\dagger U^* f = \mathcal{G} \hat{c},$$

where $\hat{c}$ is the minimal norm element of the set

$$\mathcal{G}(U^* U)^\dagger (U^* U)^\dagger U^* f = \mathcal{G} \hat{c},$$

The following theorem can be found in [3, Theorem 6.2].

Theorem 2.5. Let $\{u_j\}_{j=1, \ldots, m}$ and $\{g_k\}_{k=1, \ldots, n}$ be finite sequences in $H$, and $\cos(\varphi_{GU}) > 0$. Let $F : H \to G$ be a mapping that can be decomposed into $F = Q_2 U^*$ for some mapping $Q_2 : \mathcal{R}(U^*) \to G$. If $F(f) = f$ for all $f \in G$, then

$$\eta(F) \geq \eta(\mathcal{P}_G S(G)^\perp).$$

If the quasi-optimality constant $\mu(F) < \infty$, then $F(f) = f$ for all $f \in G$. In this case Theorem 2.5 states that $\mathcal{P}_G S(G)^\perp$ has the smallest possible $\eta(F)$ among all $F = QU^*$.

The main theorems of this paper are Theorems 2.6 and 2.8.

Theorem 2.6. If $\cos(\varphi_{GU}) > 0$, then $\mathcal{P}_{GP(U(G)^\perp}$, the oblique projection with range $G$ and kernel $P_U(G)$ exists and

$$\mathcal{G} \left((U^* U)^\dagger U^* G\right)^\dagger (U^* U)^\dagger U^* = \mathcal{P}_{GP(U(G)^\perp}.$$

Equivalently,

$$\mathcal{P}_{GP(U(G)^\perp} f = \mathcal{G} \hat{c},$$

where $\hat{c}$ is the minimal norm element of the set

$$\mathcal{G} \left((U^* U)^\dagger U^* G\right)^\dagger (U^* U)^\dagger U^* = \mathcal{G} \hat{c},$$

where $\hat{c}$ is the minimal norm element of the set

$$\mathcal{G}(U^* U)^\dagger (U^* U)^\dagger U^* f = \mathcal{G} \hat{c},$$

where $\hat{c}$ is the minimal norm element of the set
Theorem 2.7. If \( \cos(\varphi_{GU}) > 0 \), then
\[
\|P_{GP\perp(G)}\| = \frac{1}{\cos(\varphi_{GU})}
\]
and
\[
\|f - P_Gf\| \leq \|f - P_{GP\perp(G)}f\| \leq \frac{1}{\cos(\varphi_{GU})}\|f - P_Gf\|.
\]

The bound in (17) is sharp.

Theorem 2.8. Let \( \cos(\varphi_{GU}) > 0 \). If the mapping \( F = QU^* : H \to H \), satisfies
\[
\|f - Ff\| \leq \alpha\|f - P_Gf\|, \quad \text{for all } f \in H,
\]
for some \( \alpha > 0 \), then \( \alpha \geq \mu(P_{GP\perp(G)}) \).

Theorem 2.7 shows how the oblique projection \( P_{GP\perp(G)}f \) can be calculated from the measurements \( \{\langle f, u_j \rangle\}_{j \in \mathbb{N}} \). Specifically, setting
\[
Q := \mathcal{G}\left((U^\ast U)^\frac{1}{2}U^\ast G\right)^\dagger(U^\ast U)^\frac{1}{2},
\]
this projection is given by
\[
P_{GP\perp(G)} = QU^*.
\]

Theorem 2.7 shows that the quasi optimality constant of this projection is \( \frac{1}{\cos(\varphi_{GU})} \), and Theorem 2.8 states that this is smallest possible quasi-optimality constant.

A key property of the mapping \( P_{GP\perp(G)} \) is that \( \mu(P_{GP\perp(G)}) \) and \( \eta(P_{GP\perp(G)}) \) can be calculated. In section 7 we state explicit formulas for them.

3. Existence of the Oblique Projection
The following lemma follows from [11, Thm. 2.1] and [11, (2.2)].

Lemma 3.1. If \( S \) and \( W \) are closed subspaces of a Hilbert space \( H \), then \( \cos(\varphi_{SW\perp}) > 0 \) if and only if \( S \cap W = \{0\} \) and \( S \oplus W \) is closed in \( H \).

The proof of the following lemma can be found in [4, Theorem 1].

Lemma 3.2. If \( S \cap W = \{0\} \) and \( H_1 := S \oplus W \) is a closed subspace of \( H \), then the oblique projection \( P_{SW} : H_1 \to S \) with range \( S \) and kernel \( W \) is well defined and bounded.

The following theorem can be found in [31, Corollary 3.5].

Theorem 3.3. Let \( S \) and \( W \) be closed subspaces of \( H \) with \( \cos(\varphi_{SW\perp}) > 0 \) and let \( H_1 := S \oplus W \). If \( P_{SW} : H_1 \to S \) is the oblique projection with range \( S \) and kernel \( W \), then
\[
\|P_{SW}\| = \frac{1}{\cos(\varphi_{SW\perp})}
\]
and
\[
\|f - P_Sf\| \leq \|f - P_{SW}f\| \leq \frac{1}{\cos(\varphi_{SW\perp})}\|f - P_Sf\|,
\]
for all \( f \in H_1 \). The upper bound in (21) is sharp.

We make use of the following well known lemma.

**Lemma 3.4.** Let \( L \) and \( H \) be Hilbert spaces, and let \( \mathcal{U} : H \to L \) be a bounded operator. If there exists an \( A > 0 \), such that

\[
A\|c\| \leq \|\mathcal{U}c\| \quad \text{for all } c \in \mathcal{N}(\mathcal{U})^\perp,
\]

then the operator \( \mathcal{U} \) has a closed range.

In the following, we use the notation

\[
P_U(G) := \{P_Ug : g \in G\}.
\]

**Lemma 3.5.** Let \( G \) and \( U \) be closed subspaces of a Hilbert space \( H \). If \( \cos(\varphi_{GU}) > 0 \), then the subspace \( P_U(G) \) is closed. Furthermore

\[
(23) \quad \cos(\varphi_{GP_U(G)}) = \cos(\varphi_{GU}).
\]

**Proof.** From (12), it follows that

\[
\|g\| \cos(\varphi_{GU}) \leq \|P_Ug\| \quad \text{for all } g \in G.
\]

The closedness of the subspace \( P_U(G) \) follows from the fact that \( \mathcal{N}(P_U)^\perp = (U^\perp)^\perp = \overline{U} = U \), using Lemma 3.4. The second statement follows from

\[
\begin{align*}
\cos(\varphi_{GP_U(G)}) &= \inf_{u \in G} \sup_{\|v\|=1, \|v\|=1} \langle u, v \rangle = \inf_{u \in G} \sup_{\|v\|=1} \langle P_Uu, v \rangle = \\
&= \inf_{u \in G} \|P_Uu\| = \cos(\varphi_{GU}),
\end{align*}
\]

using (12) for the last equality. \( \square \)

**Theorem 3.6.** If \( \cos(\varphi_{GU}) > 0 \), then \( H = G \oplus P_U(G)^\perp \), and the oblique projection \( P_{GP_U(G)^\perp} : H \to G \) is well defined and bounded.

**Proof.** By assumption \( \cos(\varphi_{GU}) > 0 \) and thus by Lemma 3.5

\[
\cos(\varphi_{GP_U(G)}) = \cos(\varphi_{GU}) > 0.
\]

By Lemma 3.1 and Lemma 3.2 the oblique projection \( \mathcal{P}_{GP_U(G)^\perp} \) is well defined and bounded as a mapping from \( G \oplus P_U(G)^\perp \) onto \( G \). We prove that

\[
G \oplus P_U(G)^\perp = H.
\]

Lemma 3.1 implies that \( G \oplus P_U(G)^\perp \) is closed. Consequently, it is sufficient to show that \( (G \oplus P_U(G)^\perp)^\perp = \{0\} \). By assumption \( \cos(\varphi_{GU}) > 0 \) and thus by Lemma 3.5 \( P_U(G) \) is closed, and

\[
(G \oplus P_U(G)^\perp)^\perp = G^\perp \cap \overline{P_U(G)} = G^\perp \cap P_U(G).
\]

Let \( h \in G^\perp \cap P_U(G) \). Using that \( h = P_Ug \) for some \( g \in G \) and that \( h \in G^\perp \), we conclude that for all \( s \in G \)

\[
(24) \quad 0 = \langle h, s \rangle = \langle g, P_U s \rangle.
\]
If \( g \neq 0 \), from (24), it follows that
\[
\inf_{g \in G} \sup_{s \in G} \frac{\langle g, P_U s \rangle}{\|g\| \|P_U s\|} = 0.
\]
This is a contradiction to \( \cos(\varphi_{GP_U(G)}) > 0 \). Consequently \( g = 0 \) and \( h = P_U g = 0 \). □

**Lemma 3.7.** Assume that \( \{u_j\}_{j \in \mathbb{N}} \) is a frame sequence in \( H \) and \( \{g_k\}_{k \in \mathbb{N}} \) is a Riesz sequence in \( H \). Then \( G \cap U^\perp = \{0\} \) if and only if the operator \( U^* G \) is injective.

**Proof.** Let \( G \cap U^\perp = \{0\} \). This implies that for every \( g \in G \) with \( U^* g = 0 \), it follows that \( g = 0 \). For every \( g \in G \) there exists a \( c \in l^2(\mathbb{N}) \) such that \( g = Gc \), and consequently for every \( c \in l^2(\mathbb{N}) \) with \( U^* Gc = 0 \), it follows that \( Gc = 0 \). Since \( \{g_k\}_{k \in \mathbb{N}} \) is a Riesz sequence, \( Gc = 0 \) if and only if \( c = 0 \), which shows that the operator \( U^* G \) is injective.

Let the operator \( U^* G \) be injective. Using that \( \{g_k\}_{k \in \mathbb{N}} \) is a Riesz sequence, we conclude that for every \( c \in l^2(\mathbb{N}) \) with \( U^* Gc = 0 \), it follows that \( Gc = 0 \). Since \( \{g_k\}_{k \in \mathbb{N}} \) is a frame for \( G \), this implies that for every \( g \in G \) with \( U^* g = \{0\} \), it follows that \( g = 0 \). Consequently \( G \cap U^\perp = \{0\} \). □

Lemma 3.7 implies, that for finite sequences \( \{u_j\}_{j \in J} \) and \( \{g_k\}_{k \in K} \) with \( g_k, k \in K \), linearly independent, \( \cos(\varphi_{GU}) > 0 \) if and only if \( U^* G \) is injective.

4. **Frames and the PseudoInverse**

We make use of the following version of the spectral theorem.

**Theorem 4.1.** Let \( A \) be a bounded operator on \( H \). If \( A \) is normal, there exists a measure space \((X, \Sigma, \mu)\) and a real-valued essentially bounded measurable function \( f \) on \( X \) and a unitary operator \( U : H \to L^2_\mu(X) \) such that
\[
U^* TU = A,
\]
where
\[
[Tg](x) = f(x)g(x)
\]
and \( \|T\| = \|f\|_\infty \).

For an essentially bounded function \( f \in L^2_\mu(X) \) we use the notation \( M_f \) for the multiplication operator
\[
M_fg(x) := f(x)g(x) \text{ for } g \in L^2_\mu(X).
\]

The following theorem can be found in [10, Theorem 2.1]

**Theorem 4.2.** Let \( f \in L^2_\mu(X) \) be essentially bounded. Then \( M_f \) has a closed range if and only if \( f \) is bounded away from zero on \( X \setminus \{x \in X : f(x) = 0\} \).

We need the definition of the pseudoinverse in a Hilbert space.
Lemma 4.3. Let $H$ and $L$ be Hilbert spaces. If $U : L \to H$ is a bounded operator with a closed range $\mathcal{R}(U)$, then there exists a unique bounded operator $U^\dagger : H \to L$ such that

\begin{align*}
(26) \quad \mathcal{N}(U^\dagger) &= \mathcal{R}(U)^\perp, \\
(27) \quad \mathcal{R}(U^\dagger) &= \mathcal{N}(U)^\perp, \quad \text{and} \\
(28) \quad UU^\dagger x &= x, \quad x \in \mathcal{R}(U).
\end{align*}

We call the operator $A^\dagger$ the pseudoinverse of $A$. The proof of the following lemma is straightforward and thus skipped.

Lemma 4.4. Let $f \in L^2_\mu(X)$ be an essentially bounded function. If $f$ is bounded away from zero on $X \setminus \{x \in X : f(x) = 0\}$, then the pseudoinverse of the multiplication operator $M_f$ is a bounded function and given by

\[ M_f^\dagger g(x) = \begin{cases} \frac{g(x)}{f(x)} & \text{for } x \in X \setminus \{x \in X : f(x) = 0\}, \\ 0 & \text{otherwise}. \end{cases} \]

Lemma 4.5. If $U : L \to H$ is an operator with a closed range, then the following identities hold,

\[ \mathcal{R}(U^\ast) = \mathcal{R}(U), \quad \mathcal{N}(U^\ast) = \mathcal{N}(U), \]
\[ \mathcal{R}(U^\ast U) = \mathcal{R}(U^\ast), \quad \mathcal{N}(U^\ast U) = \mathcal{N}(U). \]

Proof. Obviously $\mathcal{R}(U^\ast) \subset \mathcal{R}(U)$. Let $f \in \mathcal{R}(U)$, $f = Ud$ for some $d \in L$. Since $\mathcal{R}(U^\ast) \subset \mathcal{R}(U)$ is closed (if $\mathcal{R}(U)$ is closed and onlay if $\mathcal{R}(U)$ is closed), $d$ can be decomposed into $d = d_{\mathcal{R}(U^\ast)} + d_{\mathcal{R}(U^\ast)^\perp}$, where $d_{\mathcal{R}(U^\ast)} = P_{\mathcal{R}(U^\ast)}c \in \mathcal{R}(U^\ast) = \mathcal{R}(U^\ast)^\perp$ and $d_{\mathcal{R}(U^\ast)^\perp} = d - P_{\mathcal{R}(U^\ast)}d \in \mathcal{R}(U^\ast)^\perp = \mathcal{N}(U)$. Consequently $f = Ud_{\mathcal{R}(U^\ast)}$ and $f \in \mathcal{R}(U^\ast)$. The proofs of the other statements are similar. \hfill \Box

We observe that (11) can be written in the form

\[ A \leq \frac{\langle Sf, f \rangle}{\langle f, f \rangle} \leq B, \quad \text{for every } f \in \mathcal{R}(U). \]

Since by Lemma 4.5 $\mathcal{R}(U) = \mathcal{N}(U^\ast)^\perp = \mathcal{N}(U^\ast)^\perp = \mathcal{N}(U)^\perp$, this ensures that, except of zero, the spectrum of the operator $S$ is bounded away from zero. Using Theorem 4.1, Theorem 4.2 and Lemma 4.3 we obtain that the pseudoinverse of the operator $S^\dagger$ exists and is a bounded operator on $H$. Since $S \geq 0$ also $S^\dagger \geq 0$. For every positive operator there exists a unique positive square root. Therefore we can define the operator

\[ S^{\frac{1}{2}} := (S^\dagger)^{\frac{1}{2}}. \]

The following lemma can be found in [5, Lemma 5.4.5]

Lemma 4.6. Let $\{u_j\}_{j \in \mathbb{N}}$ be a sequence in $H$. The sequence $\{u_j\}_{j \in \mathbb{N}}$ is a frame sequence with frame bounds $A$ and $B$ if and only if the synthesis operator $U$ is well defined on $l^2(\mathbb{N})$ and

\[ A\|c\|^2 \leq \|Uc\|^2 \leq B\|c\|^2 \quad \text{for all } c \in \mathcal{N}(U)^\perp \]

(29)
We observe, that (29) can be written in the form
\[ A \leq \frac{\langle U^*Uc, c \rangle}{\langle c, c \rangle} \leq B \] for every \( c \in \mathcal{N}(U) \perp \).

Since by Lemma 4.5 \( \mathcal{N}(U) \perp = \mathcal{N}(U^*U) \perp \), this ensures that except of zero, the spectrum of the operator \( U^*U \) is bounded away from zero. Using Theorem 4.1, Theorem 4.2 and Lemma 4.4 we obtain that the pseudoinverse of the operator \( (U^*U)^\dagger \) exists and is a bounded operator on \( l^2(\mathbb{N}) \). Since \( U^*U \succeq 0 \) also \( (U^*U)^\dagger \succeq 0 \).

For every positive operator there exists a unique positive square root. Therefore we can define the operator
\[ (U^*U)^\frac{1}{2} := \left((U^*U)^\dagger\right)^{\frac{1}{2}}. \]

The following theorem is a slightly modified version of [5, Theorem 5.3.4] and thus omitted.

**Theorem 4.7.** Let \( \{u_j\}_{j \in \mathbb{N}} \) be a frame sequence in \( H \). If \( S \) is the corresponding frame operator, then
\[ \{S^\frac{1}{2}u_j\}_{j \in \mathbb{N}} \]
forms a tight frame for \( U \) with frame bound equal to 1. Let \( M = S^\frac{1}{2}U \) denote the synthesis operator of the tight frame sequence \( \{S^\frac{1}{2}u_j\}_{j \in \mathbb{N}} \). Then
\[ P_U = MM^* = S^\frac{1}{2}SS^\frac{1}{2}. \]

**Lemma 4.8.** Let \( \{u_j\}_{j \in \mathbb{N}} \) be a frame sequence in \( H \). The operator \( (U^*U)^\frac{1}{2}U^* \) is the analysis operator of the frame sequence \( \{S^\frac{1}{2}u_j\}_{j \in \mathbb{N}} \). Equivalently,
\[ (U^*U)^\frac{1}{2}U^* = U^*(UU^*)^\frac{1}{2}. \]

**Proof.** Obviously for \( k \in \mathbb{N} \)
\[ (U^*U)^kU^* = U^*(UU^*)^k. \]
Therefore
\[ \gamma(U^*U)U^* = U^*(U^*)^\gamma \]
for every polynomial \( \gamma \). Taking limits, it follows that
\[ f(U^*U)U^* = U^* f(U^*) \]
for every continous function \( f \), in particular for
\[ f(A) = A^\frac{1}{2}. \]

**Lemma 4.9.** If \( \cos(\varphi_{GU}) > 0 \), then
\[ \mathcal{N}(G) = \mathcal{N}((U^*U)^\frac{1}{2}U^*G), \]
and
\[ \sqrt{C} \cos(\varphi_{GU}) \|c\| \leq \|(U^*U)^\frac{1}{2}U^*Gc\| \leq \sqrt{D} \|c\| \]
for all \( c \in \mathcal{N}(G) \perp \).
Proof. We define
\begin{equation}
L := \mathcal{U}(\mathcal{U}^* \mathcal{U})^\frac{1}{2}.
\end{equation}
By Theorem 4.8, the operator \(L^*\) is the analysis operator of the tight frame sequence \(\{\mathcal{S}^* u_j\}_{j \in \mathbb{N}}\) with frame bound equal to one. Therefore
\begin{equation}
\|L^* f\| = \|f\| \quad \text{for all } f \in \mathcal{U}.
\end{equation}
Clearly \(P_U \mathcal{U} = \mathcal{U}\) and consequently
\begin{equation}
\mathcal{U}^* = \mathcal{U}^* P_U.
\end{equation}
Using (36), (35) and (29), we obtain
\begin{equation}
\|L^* \mathcal{G} c\| = \|L^* P_U (\mathcal{G} c)\| = \|P_U (\mathcal{G} c)\| \leq \|\mathcal{G} c\| \leq \sqrt{D} \|c\|.
\end{equation}
Using the definition of \(\cos(\varphi_{GU})\) and (29), it follows that for \(c \in \mathcal{N}(\mathcal{G})^\perp\)
\begin{equation}
\|P_U (\mathcal{G} c)\| \geq \cos(\varphi_{GU}) \|\mathcal{G} c\| \geq \cos(\varphi_{GU}) \sqrt{C} \|c\|,
\end{equation}
which proves (33).
Trivially \(\mathcal{N}(\mathcal{G}) \subset \mathcal{N}(L^* \mathcal{G})\). Let \(c \in \mathcal{N}(L^* \mathcal{G})\). We decompose \(c\) into \(c = c_1 + c_2\), with \(c_1 := P_{\mathcal{N}(\mathcal{G})} c \in \mathcal{N}(\mathcal{G})\) and \(c_2 := c - P_{\mathcal{N}(\mathcal{G})} c \in \mathcal{N}(\mathcal{G})^\perp\). Using (33), we obtain that
\(0 = \|L^* \mathcal{G} c\| = \|L^* \mathcal{G} c_2\| \geq \sqrt{C} \cos(\varphi_{GU}) \|c_2\|\),
which implies that \(c_2 = 0\), and consequently \(c \in \mathcal{N}(\mathcal{G})\). \(\square\)

The following lemma is a part of [5, Lemma 2.5.2].

**Lemma 4.10.** Let \(\mathcal{U} : L \to H\) be a bounded operator. If \(\mathcal{U}\) has a closed range \(\mathcal{R}(\mathcal{U})\), then the following holds:

1. The orthogonal projection of \(H\) onto \(\mathcal{R}(\mathcal{U})\) is given by \(\mathcal{U} \mathcal{U}^\dagger\).
2. The orthogonal projection of \(L\) onto \(\mathcal{R}(\mathcal{U}^\dagger)\) is given by \(\mathcal{U}^\dagger \mathcal{U}\).
3. The operator \(\mathcal{U}^\dagger\) has closed range, and \((\mathcal{U}^\dagger)^\dagger = (\mathcal{U}^\dagger)^*\).

The following Lemma can be found in [6, Corollary 1.]

**Lemma 4.11.** If \(\mathcal{U} : L \to H\) is a bounded operator with a closed range, then
\begin{equation}
\mathcal{N}(\mathcal{U}^\dagger) = \mathcal{N}(\mathcal{U}^*)
\end{equation}
and
\begin{equation}
\mathcal{R}(\mathcal{U}^\dagger) = \mathcal{R}(\mathcal{U}^*).
\end{equation}

**Proof.** By (26),
\(\mathcal{N}(\mathcal{U}^\dagger) = \mathcal{R}(\mathcal{U})^\perp = \mathcal{N}(\mathcal{U}^*)\).
Since \(\mathcal{R}(\mathcal{U})\) is closed, if and only if \(\mathcal{R}(\mathcal{U}^*)\) is closed, by (27),
\(\mathcal{R}(\mathcal{U}^\dagger) = \mathcal{N}(\mathcal{U})^\perp = \mathcal{R}(\mathcal{U}^*) = \mathcal{R}(\mathcal{U}^*)\). \(\square\)
5. Proofs

Proof of Theorem 2.6. We set \( R := \mathcal{G} \left( (U^* U)^\dagger U^* \mathcal{G} \right)^\dagger (U^* U)^\dagger U^* \). We show

(1) \( R \) is well defined,
(2) \( \mathcal{R}(R) = G \),
(3) \( \mathcal{N}(R) = P_U(G) \perp \) and
(4) \( R^2 = R \).

(1.) The lower bound of (33) ensures that the operator \((U^* U)^\dagger U^* \mathcal{G}\) has a closed range. The upper bound of (33) shows that the operator \((U^* U)^\dagger U^* \mathcal{G}\) is bounded and by Lemma 4.3 this proves the existence of the operator \( (U^* U)^\dagger U^* \mathcal{G} \). Therefore \( R \) is a well defined operator from \( H \) to \( G \).

(2.) Clearly \( \mathcal{R}(R) \subset G \). The lower bound in Lemma 4.9 implies that \( \mathcal{R}(U^* U)^\dagger U^* \mathcal{G} \) is closed. From Lemma 4.10 (2.) and Lemma 4.11, it follows that \( \mathcal{R}(G) = G \). From Theorem 4.7 and Lemma 4.8, we deduce that \( P_U(G) = \mathcal{G} \mathcal{P}_{\mathcal{N}(\mathcal{G})} \). Since \( \{ g_k \}_{k \in \mathbb{N}} \) is a frame for \( G \) and \( \mathcal{G} \) is the corresponding synthesis operator, it holds \( \mathcal{R}(\mathcal{G}) = G \) and consequently

\[
(41) \quad P_U(G)^\perp = \mathcal{R}(U^* U)^\dagger U^* \mathcal{G} = \mathcal{N}(G^* U(U^* U)^\dagger U^*).
\]

(3.) Let \( L \) be defined by (34). From Theorem 4.7 and Lemma 4.8, we deduce that \( P_U = LL^* = U(U^* U)^\dagger U^* \). Since \( \{ g_k \}_{k \in \mathbb{N}} \) is a frame for \( G \) and \( G \) is the corresponding synthesis operator, it holds \( \mathcal{R}(\mathcal{G}) = G \) and consequently

\[
(42) \quad \mathcal{N}(G^* U(U^* U)^\dagger U^*).
\]

From (39), it follows that

\[
(43) \quad \mathcal{N}(G^* U(U^* U)^\dagger U^*) = \mathcal{N}(G^* U(U^* U)^\dagger U^*).
\]

Combining (41) and (43), we obtain

\[
P_U(G)^\perp \subset \mathcal{N}(G \left( (U^* U)^\dagger U^* \mathcal{G} \right)^\dagger (U^* U)^\dagger U^*) = \mathcal{R}(G).
\]

In Theorem 3.6 it is shown that \( G \oplus P_U(G)^\perp = H \). Let \( f \in \mathcal{N}(R) \). We decompose \( f \) into \( f = f_1 + f_2 \), where \( f_1 \in G \) and \( f_2 \in P_U(G)^\perp \). Since \( f \in \mathcal{N}(R) \) and \( P_U(G)^\perp \subset \mathcal{N}(R) \),

\[
0 = Rf = R(f_1 + f_2) = Rf_1 = f_1,
\]

and consequently \( f = f_2 \in P_U(G)^\perp \).
that satisfies the equations
\[ F f (44) \]
which implies that
\[ \mu \]
value
\[ U \]
3.3.
From (18), it follows that
\[ F h (45) \]
otherwise \( \mu = \infty \), and consequently \( Q U^* g = g \). From (14), we deduce that
\[ F(g + u^⊥) = g \quad \text{for all } g \in G \text{ and } u \in U^⊥. \]
This means that \( F_{(g \in U^⊥)} = P_{GU^⊥} \). From (21), it follows that
\[ \mu(F_{(g \in U^⊥)}) = \frac{1}{\cos(\varphi_{GU})}, \]
which implies that \( \mu(F) \geq \frac{1}{\cos(\varphi_{GU})} \).

6. An abstract definition of our reconstruction

The oblique projection \( P_{GU(G)^⊥} \) is characterized as follows. Most part of this proof is similar to the proof of [3 Theorem 4.2].

**Theorem 6.1.** Let \( \cos(\varphi_{GU}) > 0 \). The mapping \( P_{GU(G)^⊥} \) is the unique operator \( F \) that satisfies the equations
\[ \langle P_U F f, g_k \rangle = \langle P_U f, g_k \rangle, \quad j \in \mathbb{N}, \ f \in H. \]

**Proof.** In Theorem [3.6 it is shown that \( H = G \oplus P_U(G)^⊥ \) and that the oblique projection \( F = P_{GU(G)^⊥} : H \to G \) is well defined and bounded. We show next that \( P_{GU(G)^⊥} \) satisfies equation (46). From the self adjointness of \( P_U \), and the fact that \( \{g_j\}_{j \in \mathbb{N}} \) is a frame sequence for \( G \), it follows that (46) is equivalent to
\[ \langle F f, \Phi \rangle = \langle f, \Phi \rangle \quad \text{for all } \Phi \in P_U(G), \ f \in H. \]
We have to show that
\[ \langle P_{GU(G)^⊥} f, \Phi \rangle = \langle f, \Phi \rangle \quad \text{for all } \Phi \in P_U(G), \ f \in H. \]
Using that \( H = G \oplus P_U(G)^⊥ \), every \( f \in H \) can be decomposed into \( f = f_G + f_{P_U(G)^⊥} \), where \( f_G \in G \) and \( f_{P_U(G)^⊥} \in P_U(G)^⊥ \). Thus
\[ \langle P_{GU(G)^⊥} f_G + f_{P_U(G)^⊥}, \Phi \rangle = \langle f_G, \Phi \rangle = \langle f_G + f_{P_U(G)^⊥}, \Phi \rangle = \langle f, \Phi \rangle. \]
Next we show the uniqueness. We assume that there are two mappings $F_1, F_2 : H \to G$ that satisfy (17). This means for all $f \in H$ and $\Phi \in \mathcal{P}_U(G)$

$$
(48) \quad \langle F_1 f, \Phi \rangle = \langle f, \Phi \rangle = \langle F_2 f, \Phi \rangle.
$$

From (18), it follows that $\mathcal{R}(F_1 - F_2) \subset \mathcal{P}_U(G)^\perp$. We know that $\mathcal{R}(F_1) \subset G$ and that $\mathcal{R}(F_2) \subset G$ and thus $\mathcal{R}(F_1 - F_2) \subset G \cap \mathcal{P}_U(G)^\perp$. From Lemma 3.5 in combination with Lemma 3.1 it follows that $G \cap \mathcal{P}_U(G)^\perp = \{0\}$, and consequently $F_1 = F_2$. \hfill $\square$

### 7. Stability and quasi optimality of $\mathcal{P}_{\mathcal{P}_U(G)^\perp}$

In this section we give formulas for the calculation of $\eta(\mathcal{P}_{\mathcal{P}_U(G)^\perp})$ and $\mu(\mathcal{P}_{\mathcal{P}_U(G)^\perp})$. We also estimate the condition number of the operator $(\mathcal{U}^* \mathcal{U})^{\frac{1}{2}} \mathcal{U}^* \mathcal{G}$ (see (15)) in terms of the condition number of $\mathcal{G}$ and $\cos(\varphi_{GU})$.

The following theorem is similar to [2, Lemma 2.13], but for the convenience we include a proof.

**Theorem 7.1.** Let $\{u_j\}_{j=1}^n$ and $\{g_k\}_{k=1}^m$ be finite sequences in $H$. If the vectors $g_k, k = 1, \ldots, m$, are linearly independent, then

$$
\cos^2(\varphi_{GU}) = \lambda_{\min}(G^*G)^{-1}G^*\mathcal{U}U^\dagger G.
$$

**Proof.** By definition

$$
\cos(\varphi_{GU}) = \inf_{c \neq 0} \frac{\|\mathcal{P}_U(Gc)\|}{\|Gc\|}.
$$

From $P_U^2 = P_U$ and the self adjointness of $P_U$, it follows that

$$
(49) \quad \left(\frac{\|\mathcal{P}_U(Gc)\|}{\|Gc\|}\right)^2 = \frac{\langle G^*\mathcal{U}U^\dagger G, c \rangle}{\langle G^*G, c \rangle}.
$$

Since the vectors $g_k, k = 1, \ldots, m$, are linearly independent $G^*G$ is invertible, and so is $(G^*G)^{\frac{1}{2}}$. Substituting $a := (G^*G)^{\frac{1}{2}} c$ in (49), we obtain

$$
\left(\frac{\|\mathcal{P}_U(Gc)\|}{\|Gc\|}\right)^2 = \frac{\langle (G^*G)^{-\frac{1}{2}}G^*\mathcal{U}U^\dagger G(G^*G)^{-\frac{1}{2}}, a \rangle}{\langle a, a \rangle}.
$$

From the self adjointness of $(G^*G)^{-\frac{1}{2}}G^*\mathcal{U}U^\dagger G(G^*G)^{-\frac{1}{2}}$ it follows that

$$
\cos^2(\varphi_{GU}) = \lambda_{\min}((G^*G)^{-\frac{1}{2}}G^*\mathcal{U}U^\dagger G(G^*G)^{-\frac{1}{2}}).
$$

The two operators $(G^*G)^{-\frac{1}{2}}G^*\mathcal{U}U^\dagger G(G^*G)^{-\frac{1}{2}}$ and $(G^*G)^{-1}G^*\mathcal{U}U^\dagger G$ have the same spectrum, because they are similar. This finishes the proof. \hfill $\square$

**Theorem 7.2.** Set $Q = G \left(\mathcal{U}^* \mathcal{U} \right)^{\frac{1}{2}} \mathcal{U}^* \mathcal{G}^\dagger$ and $Q_1 := G(\mathcal{U}^* \mathcal{G})^\dagger$. Then

$$
(50) \quad \eta(\mathcal{P}_{\mathcal{P}_U(G)^\perp}) = \|Q_{\mathcal{R}(\mathcal{U}^*)}\| = \|Q\|, \text{ and}
$$

$$
(51) \quad \eta(\mathcal{P}_{\mathcal{G}^\perp}) = \|Q_{1\mathcal{R}(\mathcal{U}^*)}\| = \|Q_1\|,
$$
Proof. Since by Theorem 2.6 $P_{GP_U(G)\perp} = QU^*$, we have

$$\eta(P_{GP_U(G)\perp}) = \sup_{U^*f \neq 0} \frac{\|QU^*f\|}{\|U^*f\|} = \|Q\|_R.$$ 

Since $R(U^*)$ is closed, every element in $c \in l^2(N)$ can be decomposed into $c_{R(U^*)} + c_{R(U^*)\perp}$ with $c_{R(U^*)} := P_{R(U^*)}c \in R(U^*)$ and $c_{R(U^*)\perp} := c - P_{R(U^*)}c \in R(U^*)\perp$. If we can prove $R(U^*)\perp \subset N(Q)$, then (50) follows. We show that $R(U^*)\perp = N((U^*U)^\frac{1}{2})$. Using Lemma 4.5 and (26) we obtain

$$N((U^*U)^\frac{1}{2}) = N((U^*U)^\dagger) = R(U^*)\perp = R(U^*)\perp.$$ 

The proof of (51) is similar. □

For the calculation of the coefficients $\hat{c}$ of the least squares problem (15), it is important to know the condition number of the operator $(U^*U)^\dagger$. The following statement gives some hints.

**Theorem 7.3.** If $\cos(\varphi_{GU}) > 0$, then

$$\cos(\varphi_{GU})\|Gc\| \leq \|(U^*U)^\frac{1}{2}U^*Gc\| \leq \|Gc\| \quad \text{for all } c \in l^2(N),$$

and

$$\cos(\varphi_{GU})\kappa(G) \leq \kappa((U^*U)^\dagger U^*G) \leq \frac{1}{\cos(\varphi_{GU})}\kappa(G).$$

**Proof.** Equation (52) follows from (37) and (38). Equation (53) is a direct consequence of (52). □

The following Theorem and its proof is similar to [3, Corollary 4.7]

**Theorem 7.4.** Let $\cos(\varphi_{GU}) > 0$. If $Q$ is defined by (19), then

$$\frac{1}{\sqrt{B}} \leq \eta(P_{GP_U(G)^\perp}) = \|Q\| \leq \frac{1}{\sqrt{A}\cos(\varphi_{GU})},$$

and

$$\mu(P_{GP_U(G)^\perp}) = \frac{1}{\cos(\varphi_{GU})}.$$ 

**Proof.** Equation (55) follows from (17). From the definition of $\cos(\varphi_{GU})$ we know that

$$\|g\| \cos(\varphi_{GU}) \leq \|P_Ug\| \quad \text{for all } g \in G,$$

Furthermore, from the Cauchy-Schwarz inequality, and (46), it follows that for $\tilde{f} = P_{GP_U(G)^\perp}f$

$$\langle P_U\tilde{f}, \tilde{f} \rangle = \langle P_Uf, f \rangle \frac{1}{2}\langle P_U\tilde{f}, \tilde{f} \rangle \frac{1}{2}.$$ 

This yields

$$\|P_U\tilde{f}\| \leq \|P_Uf\|.$$
From the definition of a frame sequence,
\begin{equation}
(59) \quad A\|u\|^2 \leq \|U^*u\|^2 \leq B\|u\|^2, \quad \text{for } u \in U.
\end{equation}
From (59), it follows that
\begin{equation}
(60) \quad \sqrt{A} \|P_Uf\| \leq \|U^*f\|_2.
\end{equation}
We combine (56), (58) and (60) and obtain
\begin{equation}
(61) \quad \|\tilde{f}\| \cos(\varphi_{GU}) \leq \|P_U\tilde{f}\| \leq \frac{1}{\sqrt{A}} \|U^*f\|.
\end{equation}
The lower bound of (54) follows from
\begin{equation}
(62) \quad \|\tilde{f}\| \geq \|P_U\tilde{f}\| \geq \frac{1}{\sqrt{B}} \|U^*P_Up\tilde{f}\| = \frac{1}{\sqrt{B}} \|U^*\tilde{f}\|,
\end{equation}
where we use (59) for the second inequality.

8. Comparison with generalized sampling

We review some important properties of the oblique projection \(P_{G S(G)\perp}\) that was introduced in [1–3], and we compare it with the oblique projection \(P_{GP_U(G)\perp}\).

**Definition 8.1.** Let \(\cos(\varphi_{GU}) > 0\). We call the oblique projection \(P_{G S(G)\perp}\) generalized sampling.

We recall that calculating the coefficients for the oblique projection \(P_{GP_U(G)\perp}\) amounts to computing the minimal norm element of (15). By contrast, for calculating the coefficients of generalized sampling, we have to calculate the minimal norm element of (14). Thus generalized sampling does not require the additional calculation of \((U^*U)^\dagger\).

While in general the oblique projections \(P_{GP_U(G)\perp}\) and \(P_{G S(G)\perp}\) are rather different, they coincide in several situations.

The following Lemma can be found in [3, Lemma 3.7]

**Lemma 8.2.** Let \(G\) and \(U\) be finite dimensional subspaces of \(H\) with \(\dim(G) = \dim(U)\). If \(\cos(\varphi_{GU}) > 0\), then \(G \oplus U^\perp = H\).

The following Lemma can be found in [3, Lemma 4.1].

**Lemma 8.3.** Let \(G\) and \(U\) be finite dimensional with \(\dim(G) = \dim(U)\). If \(\cos(\varphi_{GU}) > 0\), then
\[ P_{G S(G)\perp} = P_{GU\perp}, \]
i.e. the generalized sampling is exactly the consistent reconstruction.

Similarly to Lemma 8.3, we prove the following lemma.

**Lemma 8.4.** Let \(G\) and \(U\) be finite dimensional with \(\dim(G) = \dim(U)\). If \(\cos(\varphi_{GU}) > 0\), then
\[ P_{G P_U(G)\perp} = P_{GU\perp}, \]
i.e. the frame independent generalized sampling is exactly the consistent reconstruction.
Proof. From Lemma 8.2, we infer that \( P_{GU} \) is a well defined and bounded mapping. Clearly \( P_U(G) \subset U \). If \( P_Ug = 0 \) for some \( g \in G \), then \( g = 0 \), because otherwise \( \cos(\varphi_{GU}) = 0 \). From the injectivity, it follows that \( P_U(G) \) is a \( n \)-dimensional subspace of \( U \). Since the dimension of \( U \) is also \( n \), we deduce \( P_U(G) = U \).

Lemma 8.5. Let \( \{u_j\}_{j=1,...,n} \) and \( \{g_k\}_{k=1,...,m} \) be finite sequences and \( H \) and let \( \cos(\varphi_{GU}) > 0 \). If \( \{u_j\}_{j=1,...,m} \) is a tight frame sequence, then \( P_G S(G) = P_g P_U(G) \).

Proof. Let \( S \) denote the frame operator of \( \{u_j\}_{j=1,...,m} \) and \( A \) the frame bound. Since \( \{u_j\}_{j=1,...,m} \) is a tight frame sequence for \( U \), we have

\[
Su = Au \quad \text{for } u \in U.
\]

Every element \( g \in G \) can be decomposed into \( g = gu + gu^\perp \) with \( gu := P_Ug \in G \) and \( gu^\perp := g - P_Ug \in U^\perp \). From (63) we deduce that \( Su = Agu \). Consequently, \( AP_Ug = Su \) and \( S(G) = P_U(G) \).

Lemma 8.3 in combination with Lemma 8.4 show that if \( \dim(G) = \dim(U) \), then generalized sampling and frame independent generalized sampling coincide. Lemma 8.5 shows that they coincide, whenever \( \{u_j\}_{j=1,...,m} \) is a tight frame sequence. This is important, because in this case the calculation of \( (U^*U)^{\perp} \) is not necessary.

In terms of \( \cos(\varphi_{GU}) \), a bound for the quasi-optimality constant \( \mu(P_G S(G)) \), is stated in the following lemma, see [3] Corollary 4.3.

Lemma 8.6. If \( \cos(\varphi_{GU}) > 0 \), then

\[
1 \leq \mu(P_G S(G)) \leq \frac{\sqrt{B}}{\sqrt{A} \cos(\varphi_{GU})}.
\]

In contrast to \( P_G S(G) \), the nullspace of \( P_{GP}(G) \) does not depend on the frame \( \{u_j\}_{j \in \mathbb{N}} \), and consequently \( \mu(P_{GP}(G)) \) is independent of the frame \( \{u_j\}_{j \in \mathbb{N}} \).

The following examples illustrate the difference between \( P_{GP}(G) \) and \( P_G S(G) \). Let \( H = \mathbb{R}^2 \), \( u_1 = (0,1) \), \( u_2 = \left( \frac{4}{5}, 1 \right) \), \( g = (1,0) \) and \( p = (3,5) \). With that choice, \( G \) is the \( x \)-axis, \( U \) is the whole space \( \mathbb{R}^2 \) and, consequently, \( P_{GP}(G) = P_G \), the orthogonal projection onto \( G \). The ellipse in Figure 1 is the set \( E = \{ x \in \mathbb{R}^2 : ||U^*(p-x)|| \leq 1 \} \). We recall, that

\[
\eta(P_{GP}(G)) = ||Q_{\pi(U^*)}||,
\]

where \( Q \) is defined by (19). We observe that

\[
||Q_{\pi(U^*)}|| = \sup_{||U^*f|| = 1} ||QU^*f|| = \sup_{||U^*f|| = 1} ||QU^*p - QU^*(p-f)||
\]

\[
= \sup_{||U^*(p-x)|| = 1} ||QU^*p - QU^*(x)||
\]

\[
= \sup_{||U^*(p-x)|| = 1} ||P_{GP}(G) - P_{GP}(G) - x||,
\]

\[
\eta(P_{GP}(G)) = ||Q_{\pi(U^*)}||,
\]

where \( Q \) is defined by (19). We observe that
which shows that half of the length of \(P_{GP_U(G)} \perp (E)\) (red bold segment on the x-axis) is \(\eta(P_{GP_U(G)})\). Similarly it is shown that half of the length of \(P_G S(G) \perp (E)\) (blue bold segment on the x-axis) is \(\eta(P_G S(G))\).

The length of \(P_{GP_U(G)} \perp (E)\) is greater than the length of \(P_G S(G) \perp (E)\), which shows that

\[
(69) \quad \eta(P_{GP_U(G)}) = \|Q_1 R(U^*)\| > \|Q_1 R(U^*)\| = \eta(P_G S(G)).
\]

The mapping \(P_{GP_U(G)}\) is closer to the orthogonal projection \(P_U\) than \(P_G S(G)\) (in fact in this example \(P_{GP_U(G)} = P_U\)), which shows that \(\mu(P_{GP_U(G)}) < \mu(P_G S(G))\).

In this example \(G \subset U\), and consequently \(\cos(\varphi_{GU}) = 1\).

\[\begin{array}{|c|c|c|}
\hline
 & \eta & \mu \\
\hline
P_{GP_U(G)} & 1.77 & 1 \\
P_G S(G) & 1.25 & 1.6 \\
\hline
\end{array}\]

Table 1. The quantities \(\eta\) and \(\mu\) of \(P_{GP_U(G)}\) and \(P_G S(G)\) for \(u_1 = (0,1)\) and \(u_2 = (\frac{4}{5},1)\).

Since \(\cos(\varphi_{GU}) = 1\), Lemma 8.6 implies that the large \(\mu(P_G S(G))\) is only possible due to the large value of \(\frac{\sqrt{B}}{\sqrt{A}}\). A large value of \(\frac{\sqrt{B}}{\sqrt{A}}\) need not cause a big \(\mu(P_G S(G))\). This is illustrated in the following example. Let \(H = \mathbb{R}^2\), \(u_1 = (1,0)\), \(u_2 = (1,\frac{4}{5})\), \(g = (1,0)\) and \(p = (3,5)\). Like in the previous example \(\cos(\varphi_{GU}) = 1\).
Figure 2. $\mathcal{P}_{GP_u(G)^\perp}$ versus $\mathcal{P}_G S(G)^\perp$ for $u_1 = (0,1)$ and $u_2 = (\frac{4}{5},1)$

|          | $\eta$ | $\mu$ |
|----------|--------|-------|
| $\mathcal{P}_{GP_u(G)^\perp}$ | 1      | 1     |
| $\mathcal{P}_G S(G)^\perp$    | 0.71   | 1.08  |

Table 2. The quantities $\eta$ and $\mu$ of $\mathcal{P}_{GP_u(G)^\perp}$ and $\mathcal{P}_G S(G)^\perp$ for $u_1 = (0,1)$ and $u_2 = (\frac{4}{5},1)$.

We observe that in this example the values $\mu(\mathcal{P}_G S(G)^\perp)$, $\eta(\mathcal{P}_G S(G)^\perp)$ and $\eta(\mathcal{P}_{GP_u(G)^\perp})$ are all smaller than in the previous example. This can be explained by the fact that $u_1$ and $u_2$ are closer to $G$, and consequently the major axis of the ellipse rotates into the direction of the $y$ axis.
References

[1] B. Adcock and A. C. Hansen. A Generalized Sampling Theorem for Stable Reconstructions in Arbitrary Bases. *Journal of Fourier Analysis and Applications*, 18(4):685–716.

[2] B. Adcock and A. C. Hansen. Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon. *Applied and Computational Harmonic Analysis*, 32(3):357–388, 2012.

[3] B. Adcock, A. C. Hansen, and C. Poon. Beyond consistent reconstructions: Optimality and sharp bounds for generalized sampling, and application to the uniform resampling problem. Technical report, 2012.

[4] D. Buckholtz. Hilbert space idempotents and involutions. *Proceedings of the American Mathematical Society*, 128(5):pp. 1415–1418, 2000.

[5] O. Christensen. *Frames and Bases: An Introductory Course (Applied and Numerical Harmonic Analysis)*. Birkhäuser, 2008.

[6] C. Desoer and B.H.Whalen. A note on pseudoinverses. *J. Soc. INDUST. APPL. MATH.*, 11(2):442–447, 1963.

[7] Y. C. Eldar. Sampling with Arbitrary Sampling and Reconstruction Spaces and Oblique Dual Frame Vectors. *Journal of Fourier Analysis and Applications*, 9(1):77–96, 2003.

[8] Y. C. Eldar. Sampling Without Input Constraints: Consistent Reconstruction in Arbitrary Spaces. In *Sampling, Wavelets, and Tomography*, Applied and Numerical Harmonic Analysis, pages 33–60. Birkhäuser Boston, 2004.

[9] Y. C. Eldar and T. Werther. General framework for consistent sampling in Hilbert spaces. *International Journal of Wavelets, Multiresolution and Information Processing*, 03(03):347–359, 2005.

[10] R. Singh and A. Kumar. Multiplication operators and composition operators with closed ranges. *Bulletin of the Australian Mathematical Society*, 16:247–252, 1977.

[11] W.-S. Tang. Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces. *Proceedings of the American Mathematical Society*, 128(2):pp. 463–473, 2000.

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