IMAGES OF IDEALS UNDER DERIVATIONS AND \(\varepsilon\)-DERIVATIONS OF UNIVARIATE POLYNOMIAL ALGEBRAS OVER A FIELD OF CHARACTERISTIC ZERO

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Abstract. Let \(K\) be a field of characteristic zero and \(x\) a free variable. A \(K\)-\(\varepsilon\)-derivation of \(K[x]\) is a \(K\)-linear map of the form \(I - \phi\) for some \(K\)-algebra endomorphism \(\phi\) of \(K[x]\), where \(I\) denotes the identity map of \(K[x]\). In this paper we study the image of an ideal of \(K[x]\) under some \(K\)-derivations and \(K\)-\(\varepsilon\)-derivations of \(K[x]\). We show that the LFED conjecture proposed in [Z4] holds for all \(K\)-\(\varepsilon\)-derivations and all locally finite \(K\)-derivations of \(K[x]\). We also show that the LNED conjecture proposed in [Z4] holds for all locally nilpotent \(K\)-derivations of \(K[x]\), and also for all locally nilpotent \(K\)-\(\varepsilon\)-derivations of \(K[x]\) and the ideals \(uK[x]\) such that either \(u = 0\), or \(\deg u \leq 1\), or \(u\) has at least one repeated root in the algebraic closure of \(K\). As a bi-product, the homogeneous Mathieu subspaces (Mathieu-Zhao spaces) of the univariate polynomial algebra over an arbitrary field have also been classified.

1. Introduction

Let \(K\) be a field and \(A\) a commutative \(K\)-algebra. We denote by \(1_A\) or simply 1 the identity element of \(A\), if \(A\) is unital, and \(I_A\) or simply \(I\) the identity map of \(A\), if \(A\) is clear in the context.

A \(K\)-linear endomorphism \(\eta\) of \(A\) is said to be locally nilpotent (LN) if for each \(a \in A\) there exists \(m \geq 1\) such that \(\eta^m(a) = 0\), and locally finite (LF) if for each \(a \in A\) the \(K\)-subspace spanned by \(\eta^i(a)\) \((i \geq 0)\) is finite dimensional over \(K\).

A \(K\)-derivation \(D\) of \(A\) is a \(K\)-linear map \(D : A \rightarrow A\) that satisfies \(D(ab) = D(a)b + aD(b)\) for all \(a, b \in A\). A \(K\)-\(\varepsilon\)-derivation \(\delta\) of \(A\) is

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a $K$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$ the following equation holds:

$$\delta(ab) = \delta(a)b + a\delta(b) - \delta(a)\delta(b).$$

(1.1)

It is easy to verify that $\delta$ is an $R$-$\mathcal{E}$-derivation of $\mathcal{A}$, if and only if $\delta = I - \phi$ for some $R$-algebra endomorphism $\phi$ of $\mathcal{A}$. Therefore an $R$-$\mathcal{E}$-derivation is a special so-called $(s_1, s_2)$-derivation introduced by N. Jacobson [J] and also a special semi-derivation introduced by J. Bergen in [B]. $R$-$\mathcal{E}$-derivations have also been studied by many others under some different names such as $f$-derivations in [E1, E2] and $\phi$-derivations in [BFF, BV], etc..

Next, we recall the following two notions of associative algebras that were introduced in [Z2, Z3]. Since all algebras in this paper are commutative, here we recall only the cases for commutative algebras over a field.

**Definition 1.1.** Let $K$ be a field and $\mathcal{A}$ a commutative $K$-algebra. A $K$-subspace $V$ of $\mathcal{A}$ is said to be a Mathieu subspace (MS) of $\mathcal{A}$ if for all $a, b \in \mathcal{A}$ with $a^m \in V$ for all $m \geq 1$, we have $a^mb \in V$ for all $m \gg 0$.

Note that a MS is also called a Mathieu-Zhao space in the literature (e.g., see [DEZ, EN, EH], etc.), as suggested by A. van den Essen [E3]. The introduction of this notion is mainly motivated by the study in [M, Z1] of the well-known Jacobian conjecture (see [K, BCW, E2]). See also [DEZ]. But, a more interesting aspect of the notion is that it provides a natural but highly non-trivial generalization of the notion of ideals.

**Definition 1.2.** [Z3, p. 247] Let $V$ be a $K$-subspace (or a subset) of a $K$-algebra $\mathcal{A}$. We define the radical $r(V)$ of $V$ to be

$$r(V) := \{ a \in \mathcal{A} | a^m \in V \text{ for all } m \gg 0 \}.$$

(1.2)

Next we recall the cases of the so-called LFED and LNED conjectures proposed in [Z4] for commutative algebras. For the study of some other cases of these two conjectures, see [EWZ, Z4–Z7].

**Conjecture 1.3.** Let $K$ be a field of characteristic zero, $\mathcal{A}$ a commutative $K$-algebra and $\delta$ a LF (locally finite) $K$-derivation or a LF $K$-$\mathcal{E}$-derivation of $\mathcal{A}$. Then the image $\text{Im} \delta := \delta(\mathcal{A})$ of $\delta$ is a MS of $\mathcal{A}$.

**Conjecture 1.4.** Let $K$ be a field of characteristic zero, $\mathcal{A}$ a commutative $K$-algebra and $\delta$ a LN (locally nilpotent) $K$-derivation or a LN $K$-$\mathcal{E}$-derivation of $\mathcal{A}$. Then $\delta$ maps every ideal of $\mathcal{A}$ to a MS of $\mathcal{A}$.
Throughout the paper we refer Conjecture 1.3 as the (commutative) LFED conjecture, and Conjecture 1.4 the (commutative) LNED conjecture.

In this paper, among some other results, we show the following two theorems regarding the commutative LFED and LNED conjectures, respectively.

**Theorem 1.5.** Let $K$ be a field of characteristic zero and $x$ a free variable. Let $\delta$ be an arbitrary $K$-derivation or $K$-$E$-derivation of $K[x]$. Then $\text{Im} \, \delta$ is a MS of $K[x]$. In particular, the LFED conjecture 1.3 holds for $K[x]$.

**Theorem 1.6.** Let $K$ be a field of characteristic zero, $I$ an ideal of $K[x]$ and $\delta$ a $K$-derivation or $K$-$E$-derivation of $K[x]$. Then $\delta I$ is a MS of $K[x]$ if one of the following conditions holds:

1) $\delta$ is a locally nilpotent $K$-derivation of $K[x]$;
2) $\delta = 1 - \phi$ for some $K$-algebra endomorphism $\phi$ of $K[x]$ such that $\deg \phi(x) \geq 2$;
3) $\delta = 1 - \phi$ for some $K$-algebra endomorphism $\phi$ of $K[x]$ which maps $x$ to $x + c$ ($c \in K$), and $I = uK[x]$ such that either $u = 0$, or $\deg u \leq 1$, or $u$ has at least one repeated root in the algebraic closure of $K$.

From Theorem 1.6 above and also its proof it is easy to see that the LNED conjecture 1.4 is established for $K[x]$ except for the case that $\delta = 1 - \phi$ for some $K$-algebra endomorphism $\phi$ of $K[x]$ that maps $x$ to $x + c$ ($0 \neq c \in K$), and $I = uK[x]$ such that $\deg u \geq 2$ and $u$ has no repeated root in the algebraic closure of $K$.

Theorems 1.5 and 1.6 are shown case by case. For some cases, e.g., the $K$-$E$-derivation case of Theorem 1.5 and Theorem 1.6, 2), etc., certain stronger results are actually proved. Furthermore, as a bi-product of the proof of Theorem 1.6, 2) all homogeneous MSs of the univariate polynomial algebra over a field of arbitrary characteristic also are classified (see Proposition 3.4).

**Arrangement.** In Section 2 we first show that the image of every $K$-derivation of $K[x]$ is a MS (see Lemma 2.1), and then show that every LN $K$-derivation $D$, i.e., $D = a \frac{d}{dx}$ ($a \in K$), maps each ideal of $K[x]$ to a MS of $K[x]$ (see Proposition 2.3). Consequently, the $K$-derivation cases of Theorems 1.5 and 1.6 are established. We also give an example, Example 2.4, to show that the LN condition in the LNED conjecture 1.4 cannot be replaced by the LF condition.
In Section 3 we let $K$ be a field of arbitrary characteristic and show the LFED conjecture 1.3 for all the $K$-$\mathcal{E}$-derivations $\delta$ that are not LN (the LN case of the conjecture also holds and follows from Lemma 4.1 in Section 4). In subsection 3.1 we consider the case $\delta = I - \phi$ for some $K$-algebra endomorphism $\phi$ of $K[x]$ that maps $x$ to $qx$ with $q \in K$. As a bi-product of the proof for this case we also obtain a classification of all homogeneous MSs of $K[x]$ (see Proposition 3.4). In subsection 3.2 we consider the case $\delta = I - \phi$ such that $\phi$ maps $x$ to $w(x)$ with $\deg w(x) \geq 2$. In particular, we show in Proposition 3.7 that $\delta$ in this case actually maps every $K$-subspace to a MS of $K[x]$, even though $\delta$ itself in this case is not LF (nor LN).

In Section 4 we consider all LN $K$-$\mathcal{E}$-derivations $\delta$ of $K[x]$, i.e., $\delta = I - \phi$ for some $K$-algebra endomorphism $\phi$ of $K[x]$ that maps $x$ to $x + c$ ($c \in K$), and prove Theorem 1.6 3). Among all the cases studied in the paper, the proof of this case is the most involved, in which the Bernoulli polynomials; the Bernoulli numbers; and the Clausen-von Staudt Theorem 4.3 that was found independently by Thomas Clausen [Cl] and Karl von Staudt [St] in 1840, all unexpectedly play some crucial roles.

2. The Case of $K$-Derivations of $K[x]$

Throughout this section $K$ stands for a field of characteristic zero and $x$ a free variable. We denote by $\partial$ the $K$-derivation $d/dx$ of the univariate polynomial algebra $K[x]$.

Now, let $D$ be a nonzero $K$-derivation of $K[x]$. Then $D = a(x)\partial$ for some $a(x) \in K[x]$. It is easy to see that $D$ is LF (locally finite), if and only if $\deg a \leq 1$, and $D$ is LN (locally nilpotent), if and only if $\deg a = 0$.

We first show Theorem 1.5 for all $K$-derivations of $K[x]$.

**Lemma 2.1.** Let $D = a(x)\partial$ with $a(x) \in K[x]$. Then $\text{Im } D = a(x)K[x]$. In particular, the LFED conjecture 1.3 holds for all $K$-derivations of $K[x]$.

**Proof:** Since $\text{Im } \partial = K[x]$, we have $\text{Im } D = a(x)K[x]$, which is an ideal of $K[x]$, and hence also a MS of $K[x]$. \(\square\)

Next, we consider LN (locally nilpotent) $K$-derivations. First, let us recall the following:

**Theorem 2.2.** Let $a \neq b \in K$ and set

$$V_{a,b} := \left\{ f \in K[x] \mid \int_a^b f(x)dx = 0 \right\}. \quad (2.3)$$
Then \( r(V_{a,b}) = 0 \).

For an algebraic proof of the theorem above, see [FPYZ, Theorem 4.1], and for a complex analytic proof with some slightly stronger condition, see [P, Corollary 4.3]. Although the theorem is proven in [FPYZ] and [P] over the complex field \( \mathbb{C} \), by the Lefschetz rule or from the proof of [FPYZ, Theorem 4.1] it is easy to see that the theorem also holds over all the fields of characteristic zero.

Now we show Theorem 1.6 for LN-derivations \( D \) of \( K[x] \). Since the only LN-derivations of \( K[x] \) are \( a \partial \) with \( a \in K \), it suffices to consider the case that \( D = \partial \), for the case \( D = 0 \) is trivial.

**Proposition 2.3.** Let \( I \) be a nonzero ideal of \( K[x] \). Write \( I = (u(x)) \) for some \( u(x) \in K[x] \). Then the following statements hold:

1) if \( u(x) = (x - c)^n \) for some \( c \in K \) and \( n \geq 0 \), then the image \( \partial I \) of \( I \) under \( \partial \) is an ideal of \( K[x] \). More precisely,

\[
\partial I = \begin{cases} 
K[x] & \text{if } n \leq 1; \\
(x - c)^{n-1}K[x] & \text{if } n \geq 2.
\end{cases}
\]

(2.4)

2) if \( u(x) \neq (x - c)^n \) for any \( c \in K \) and \( n \geq 0 \), then the radical \( r(\partial I) = \{0\} \), whence \( \partial I \) is a MS of \( K[x] \).

Consequently, the LNED conjecture 1.4 holds for all LN-derivations of \( K[x] \).

**Proof:** Since \( (x - c)^n \ (n \geq 0) \) form a \( K \)-linear basis of \( K[x] \), it is easy to see that statement 1) holds.

To show statement 2), note first that by the assumption we have that \( \deg u \geq 2 \) and has at least two distinct roots \( a \) and \( b \) in the algebraic closure \( \bar{K} \) of \( K \). Now for every \( f \in \partial I \), write \( f = \partial(ug) \) for some \( g \in K[x] \). Then we have \( \int_a^b f dx = \int_a^b \partial(ug)dx = (ug)|_a^b = 0 \). Therefore \( \partial I \subseteq V_{a,b} := \{ f \in \bar{K}[x] \mid \int_a^b f dx = 0 \} \). Applying Theorem 2.2 to \( \bar{K}[x] \) we get \( r(V_{a,b}) = \{0\} \), whence \( r(\partial I) = \{0\} \) and statement 2) follows.  

It is worthy to point out that the LNED conjecture 1.4 can not be generalized to all LF-derivations of \( K[x] \), which can be seen from the following:

**Example 2.4.** Let \( D = x \partial \) and \( I = (x^2 - 1)K[x] \). Then the image \( DI \) of \( I \) under \( D \) is not a MS of \( K[x] \).
Proof: Let $V = DI$. Then for all $k \geq 0$, since $D(x^{k+2} - x^k) = (k + 2)x^{k+2} - kx^k \in V$, we have
\[(2.5) \quad (k + 2)x^{k+2} \equiv kx^k \mod V.\]
In particular, $x^2 \in V$ (by letting $k = 0$) and, inductively by Eq. (2.5), so are $x^{2n}$ for all $n \geq 1$.

On the other hand, $x \notin V$ since each nonzero element of $V$ has degree at least 2, and by Eq. (2.5) neither are $x^{2n+1}$ for all $n \geq 1$. Therefore $(x^2)^m x \notin V$ for all $m \geq 1$, whence $V$ is not a MS of $K[x]$. \[\square\]

We end this section with the following remark on an application of the results proved in this section.

Remark 2.5. Let $u, v \in K[x]$, $D = u\partial$, $I = v(x)K[x]$ and $\Lambda$ the differential operator of $K[x]$ that maps $f \in K[x]$ to $u\partial(vf)$, i.e.,
\[
\Lambda := u(v\partial + v')
\]
Then it is easy to see that $\text{Im} \Lambda = DI$. Therefore by Lemma 4.2 and Proposition 2.3 we see that $\text{Im} \Lambda$ is a MS of $A$, if $\deg u \leq 1$. For example, by letting $u = 1$ we see that for all $v(x) \in K[x]$ the image of the differential operators $v(x)\partial + v'(x)$ is a MS of $K[x]$.

For some other differential operators with the image being a MS, see \[Z1, Z2, EZ, EWZ\].

3. The Case of $E$-Derivations of $K[x]$  

Through this section $K$ denotes a field of arbitrary characteristic, and $\phi$ a $K$-algebra endomorphism of $K[x]$, and $\delta = I - \phi$. Since the case $\phi = 0$ is trivial, we assume $\phi \neq 0$. Denote by $w(x)$ the image of $x \in K[x]$ under $\phi$, i.e., $w(x) = \phi(x)$. Then $\phi$ is completely determined by $w(x)$. More precisely, for each $f(x) \in K[x]$, we have $\phi(f) = f(w(x)).$

We start with the following

Lemma 3.1. Assume $w(x) = ax + b$ with $a, b \in K$ and $a \neq 0, 1$. Let $\psi$ be the $K$-algebra automorphism of $K[x]$ which maps $x$ to $x + (1-a)^{-1}b$. Then $\psi \circ \phi \circ \psi^{-1}$ is the $K$-algebra automorphism of $K[x]$ which maps $x$ to $ax$. 
Proof: Note that the inverse map \( \psi^{-1} \) is the \( K \)-algebra automorphism of \( K[x] \) which maps \( x \) to \( x - (1 - a)^{-1}b \). Then we have
\[
\psi \circ \phi \circ \psi^{-1}(x) = (\psi \circ \phi)(x - (1 - a)^{-1}b) = \psi(ax + b - (1 - a)^{-1}b) \\
= a(x + (1 - a)^{-1}b) + b - (1 - a)^{-1}b \\
= ax + b + (a(1 - a)^{-1} - (1 - a)^{-1})b \\
= ax + b - b = ax.
\]
Hence the lemma follows. \( \square \)

Note that \( K \)-algebra automorphisms preserve ideals and MSs, and conjugations by \( K \)-algebra automorphisms preserve (LF or LN) derivations and \( E \)-derivations. By the lemma above the proofs of Theorem 1.5 and 1.6 for \( K \)-\( E \)-derivations of \( K[x] \) can be divided into the following four (exhausting) cases:

I) \( \deg w = 0 \), i.e., \( w(x) = c \text{ for some } c \in K \);
II) \( w(x) = x + c \text{ for some } c \in K \);
III) \( w(x) = qx \text{ for some nonzero } q \in K \);
IV) \( \deg w(x) \geq 2 \).

For Case I it is easy to verify, or by the more general [Z4, Proposition 5.2], that we have the following:

Lemma 3.2. Let \( c \in K \) and \( \phi \) the \( K \)-algebra endomorphism that maps \( f \in K[x] \) to \( f(c) \). Then the image \( \text{Im}(I - \phi) = \text{Ker} \phi = (x - c)K[x] \), and hence is a MS of \( K[x] \).

In the rest of this section we consider Case III in subsection 3.1 and Case IV in subsection 3.2.

3.1. Case III: \( w(x) = qx \). Note that this case has been shown in [Z7, Corollary 3.15] for multivariate polynomial algebras over a field of characteristic zero. Here we give a more straightforward proof over the field \( K \) (of arbitrary characteristic). As a by-product the homogeneous MSs of \( K[x] \) are also classified.

Lemma 3.3. Let \( 0 \neq q \in K \) and \( \phi \) the \( K \)-algebra endomorphism of \( K[x] \) that maps \( x \) to \( qx \). Set \( \delta = I - \phi \). Then the following statements hold:

1) If \( q = 1 \), then \( \text{Im} \delta = 0 \).
2) If \( q \) is not a root of unity in \( K \), then \( \text{Im} \delta = xK[x] \).
3) If \( q \) is a root of unity in \( K \), then \( \nu(\text{Im} \delta) = \{0\} \).

In all the cases above \( \delta I \) is a MS of \( K[x] \).
Proof. 1) In this case $\phi = 1$ and $\delta = 0$. Hence $\text{Im} \delta = 0$.

2) For all $n \geq 1$, we have

\[(3.6) \quad \delta x^n = x^n - (qx)^n = (1 - q^n)x^n.\]

Since $q$ is not a root of unity, we have $1 - q^n \neq 0$, and hence $x^n \in \text{Im} \delta$, for all $n \geq 1$. Since $\delta 1 = 0$, we have $\text{Im} \delta = xK[x]$, i.e., statement 2) follows.

3) If $q = 1$, the statement follows from statement 1). Assume $q \neq 1$ and let $r$ be the order of $q$, i.e., the least positive integer such that $q^r = 1$. Then $r \geq 2$. Let $\{n_i \mid i \geq 1\}$ be the sequence of all positive integers $n$ such that $r \nmid n$. Then by Eq. (3.6) it is easy to see that $\text{Im} \delta$ is the homogeneous $K$-subspace spanned by the monomials $x^{n_i} \ (i \geq 1)$ over $K$. Note that for each integer $d \geq 1$, we have $dr \notin \{n_i \mid i \geq 1\}$. Then the statement immediately follows from the lemma below.  

Lemma 3.4. Let $\{n_i \mid i \geq 1\}$ be a strictly increasing (infinite) sequence of positive integers such that $n_{i+1} - n_i \neq 1$ for infinitely many $i \geq 1$. Let $V$ be the (homogeneous) $K$-subspace of $K[x]$ spanned by $x^{n_i} \ (i \geq 1)$ over $K$. Then the following three statements are equivalent:

1) $r(V) = \{0\}$;
2) $V$ is a MS of $K[x]$;
3) there exists no integer $d \geq 1$ such that $md \in \{n_i \mid i \geq 1\}$ for all $m \geq 1$.

Proof. 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 3): Assume otherwise. Let $d \geq 1$ such that $md \in \{n_i \mid i \geq 1\}$ for all $m \geq 1$. Hence $x^{md} \in V$ for all $m \geq 1$. If $d = 1$, then the sequence $\{n_i \mid i \geq 1\}$ contains all positive integers, which contradicts to the assumption on the sequence $\{n_i \mid i \geq 1\}$. So we have $d \geq 2$.

Since $(x^d)^m = x^{md} \in V$ for all $m \geq 1$ and $V$ is a MS of $K[x]$, for each $0 \leq r \leq d - 1$ there exists $N_r \geq 1$ such that for all $m \geq N_r$, we have $x^{md+r} = (x^d)^m x^r \in V$, and hence $md + r \in \{n_i \mid i \geq 1\}$. Let $N = \max\{N_r \mid 0 \leq r \leq d - 1\}$. Then for all $k \geq Nd$ we have $x^k \in V$, whence $k \in \{n_i \mid i \geq 1\}$, which contradicts again to the assumption on the sequence $\{n_i \mid i \geq 1\}$. Hence statement 3) follows.

3) $\Rightarrow$ 1): Assume otherwise. Let $0 \neq f(x) \in r(V)$, i.e., $f^m(x) \in V$ when $m \gg 0$. Since $1 \notin V$, we have $d := \deg f(x) \geq 1$, and since $V$ is homogeneous, we further have $x^{dm} \in V$. Hence $md \in \{n_i \mid i \geq 1\}$, for all $m \gg 0$. Replacing $d$ by a multiple of $d$ we have $md \in \{n_i \mid i \geq 1\}$ for all $m \geq 1$, which contradicts to statement 3).  

\[\square\]
One bi-product of the lemma above is the following classification of homogeneous MSs of univariate polynomial algebra over an arbitrary field.

**Proposition 3.5.** Let \( V \) be a homogeneous subspace of \( K[x] \). Then \( V \) is a MS of \( K[x] \), if and only if one of the following conditions holds:

1) \( V = K[x] \);
2) \( \dim_K V < \infty \) and \( 1 \not\in V \);
3) \( 1 \not\in V \) and there exists \( N \geq 1 \) such that \( (x^N) \subseteq V \);
4) \( V \) is spanned by \( x^{n_i} \) over \( K \) for a strictly increasing (infinite) sequence of positive integers \( \{n_i \mid i \geq 1\} \) such that there exists no integer \( d \geq 1 \) with \( md \in \{n_i \mid k \geq 1\} \) for all \( m \geq 1 \).

**Proof:** (\( \Leftarrow \)) First, if \( V \) satisfies statements 1), 2) or 3), then it is easy to check directly by Definition 1.1 that \( V \) is indeed a MS of \( \mathcal{A} \). If \( V \) satisfies statement 4), then it is easy to see that \( x^{n_i+1} - x^{n_i} \neq 1 \) for infinitely many \( i \geq 1 \), and by Lemma 3.4 \( V \) is a MS of \( K[x] \).

(\( \Rightarrow \)) Assume that statement 1) fails, i.e., \( V \neq K[x] \). Consider the case \( \dim_K V < \infty \). If \( 1 \in V \), then by Definition 1.1 we have \( V = K[x] \), contradiction. Hence in this case statement 2) holds.

Consider the case \( \dim_K V = \infty \). If \( 1 \in V \), then by Definition 1.1 we have \( V = K[x] \), contradiction again. So \( 1 \not\in V \) and there exists an infinite increasing sequence \( \{n_i \mid i \geq 1\} \) of positive integers such that \( V \) is spanned over \( K \) by \( x^{n_i} (i \geq 1) \).

If statement 3) does not hold, then there are infinitely many \( i \geq 1 \) such that \( n_{i+1} - n_i \neq 1 \), and by Lemma 3.4 statement 4) holds. \( \blacksquare \)

Next, we give the following example to show that the LNED conjecture 1.4 can not be generalized to all LF \( \mathcal{E} \)-derivations.

**Example 3.6.** Let \( 0 \neq q \in K \), \( \phi \) the \( K \)-algebra endomorphism of \( K[x] \) that maps \( x \) to \( qx \), and \( I \) the ideal of \( K[x] \) generated by \( x^2 - 1 \). Assume that \( q \) is not a root of unity. Set \( \delta := 1 - \phi \). Then \( \delta \) is LF but the image \( \delta I \) of \( I \) under \( \delta \) is not a MS of \( K[x] \).

**Proof:** Note first that for all \( k \geq 0 \), we have

\[
\delta(x^{k+2} - x^k) = (1 - q^{k+2})x^{k+2} - (1 - q^k)x^k \in \delta I.
\]

Hence

\[
(3.7) \quad (1 - q^{k+2})x^{k+2} \equiv (1 - q^k)x^k \mod \delta I.
\]

In particular, by letting \( k = 0 \) we have \( x^2 \in \delta I \), for \( q \) is not root of unity. Then by Eq. (3.7) inductively \( x^{2n} \in \delta I \) for all \( n \geq 1 \).
On the other hand, \( x \not\in \delta I \), for \( q \) is not root of unity and each nonzero element of \( \delta I \) has the degree at least 2. Then by Eq. \((3.7)\) neither are \((x^2)^m x = x^{2m+1}\) for all \( m \geq 1 \). Hence \( \delta I \) is not a MS of \( K[x] \). \( \square \)

3.2. Case IV: \( \deg w(x) \geq 2 \). In this subsection we fix a \( K \)-algebra endomorphism \( \phi \) of \( K[x] \) that maps \( x \) to \( w(x) \) with \( d = \deg w(x) \geq 2 \). Set \( \delta := I - \phi \). Write \( w(x) = \sum_{i=0}^{d} a_i x^i \) with \( d \geq 2 \), \( a_d \neq 0 \) and all \( a_i \)'s in \( K \). Then Theorem \((1.6)\) 2) for \( \delta \) immediately follows from the following:

**Proposition 3.7.** Let \( f \in K[x] \) such that \( f^i \in \text{Im} \delta \) for all \( 1 \leq i \leq 3 \). Then \( f = 0 \). Consequently, \( \nu(\text{Im} \delta) = 0 \) and \( \delta \) maps every \( K \)-subspace of \( K[x] \) to a MS of \( K[x] \).

To prove the proposition above we first fixed the following notations.

Let \( W = K[w(x)] \), \( \Lambda = \mathbb{N} \setminus d \mathbb{N} \) and \( U \) be \( K \)-subspace of \( K[x] \) spanned by \( x^m \) \( (m \in \Lambda) \) over \( K \). Since \( d = \deg w \geq 2 \), it is easy to see that for each \( f \in K[x] \) there exist unique \( f_1 \in U \) and \( f_2 \in W \) such that \( f = f_1 + f_2 \). In this case we set \( \ell(f) = \deg f_1 \) if \( f_1 \neq 0 \), and \( \ell(f) = 0 \), otherwise.

With the setting above the following lemma (with assumption \( d \geq 2 \)) can be easily verified.

**Lemma 3.8.** 1) \( \deg f \geq \ell(f) \) for all nonzero \( f \in K[x] \).

2) For all \( f, g \in K[x] \) with \( f \equiv g \mod W \) we have \( \ell(f) = \ell(g) \).

We also need the following two lemmas.

**Lemma 3.9.** Let \( 0 \neq f \in K[x] \) with \( d \nmid \deg f \). Then \( \ell(f) = \deg f \).

**Proof:** Assume otherwise. Then there exists \( g \in U \) such that \( f - g \in W \) and \( \deg f \neq \ell(f) = \deg g \). Then \( f - g \neq 0 \) and \( \deg(f - g) \) is a multiple of \( d \). On the other hand, \( \deg(f - g) \) is equal to either \( \deg f \) or \( \deg g \). Therefore, either \( \deg f \) or \( \deg g \) is a multiple of \( d \), which is a contradiction. \( \square \)

**Lemma 3.10.** For all nonzero \( f \in \text{Im} \delta \), the following statements hold:

1) if \( f \in W \), say \( f(x) = \tilde{f}(w(x)) \), then \( \tilde{f} \in \text{Im} \delta \);

2) if \( f \not\in W \), then \( \deg f \geq d \ell(f) \geq d \).

**Proof:** 1) Write \( f(x) = \delta u = u(x) - u(w(x)) \) for some \( u \in K[x] \). Then \( \tilde{f}(w) = u(x) - u(w) \) and \( u(x) = f(w) + u(w) \). Hence \( u(x) = \tilde{u}(w) \) with \( \tilde{u} = \tilde{f} + u \), and

\[
\tilde{f}(x) = \tilde{u}(x) - u(x) = \tilde{u}(x) - \tilde{u}(w) = \delta \tilde{u}.
\]
Therefore \( \tilde{f} \in \text{Im} \delta \).

2) Since \( f \notin W \), we have \( f \neq 0 \) and \( \ell(f) \geq 1 \). Write \( f(x) = u(x) - u(w(x)) \) for some nonzero \( u \in K[x] \). Since \( d = \deg w \geq 2 \), we have \( \deg f = d \deg u \). By Lemma \ref{lem:deg} we also have \( \deg u \geq \ell(u) \) and \( \ell(f) = \ell(u) \).

Therefore we have

\[
\deg f = d \deg u \geq d \ell(u) = d \ell(f) \geq d.
\]

\( \square \)

**Proof of Proposition \ref{prop:dege}** Assume otherwise, i.e., \( f \neq 0 \). If \( f \in W \), say \( f = \tilde{f}(w) \), then \( f^2, f^3 \in W \) and by applying lemma \ref{lem:deg} 1) to \( f^i \) \((1 \leq i \leq 3)\) we have \( \tilde{f}^i \in \text{Im} \delta \) for all \( 1 \leq i \leq 3 \). Since \( \deg f = d \deg \tilde{f} > \deg \tilde{f} \), by repeating the procedure that replaces \( f \) by \( \tilde{f} \), whenever it is possible, we may assume that \( f \notin W \) and \( f^i \in \text{Im} \delta \) \((1 \leq i \leq 3)\). In particular, \( f \neq 0 \). Furthermore, since \( d = \deg w \geq 2 \), \( \text{Im} \delta \) obviously does not contain any nonzero constant polynomials. Therefore we also have \( \deg f \geq 1 \).

Write \( f(x) = u(x) - u(w) \) for some nonzero \( u(x) \notin W \) with \( \deg u(x) \geq 1 \). Then \( \ell(u) \geq 1 \) and

\begin{equation}
(3.8) \quad \deg f = d \deg u.
\end{equation}

Assume first that char. \( K = 0 \) or \( p > 2 \). Then applying Lemma \ref{lem:deg} 1) to \( f^2 \) we have

\begin{equation}
(3.9) \quad 2 \deg f \geq d \ell(f^2).
\end{equation}

On the other hand, we also have

\[
f^2 = (u(x) - u(w))^2 = u^2(x) - 2u(x)u(w) + u^2(w).
\]

Then by Lemma \ref{lem:deg} 2) and Lemma \ref{lem:deg2} as well as the assumption \( d = \deg w \geq 2 \) we get

\begin{equation}
(3.10) \quad \ell(f^2) = \deg u(x)u(w) = (d + 1) \deg u.
\end{equation}

Combining Eqs. \ref{eq:degf}–\ref{eq:ellf2} we have

\[
2 \deg f \geq d(d + 1) \deg u = (d + 1) \deg f,
\]

which is a contradiction, for \( d \geq 2 \) and \( \deg f \geq 1 \).

Now assume char. \( K = 2 \). Then applying Lemma \ref{lem:deg} 1) to \( f^3 \) we have

\begin{equation}
(3.11) \quad 3 \deg f \geq d \ell(f^3).
\end{equation}

On the other hand, we also have

\[
f^3 = (u(x) - u(w))^3 = u^3(x) + u^2(x)u(w) + u(x)u^2(w) + u^3(w).
\]
Then by Lemma 3.8 2) and Lemma 3.9 as well as the assumption $d = \deg w \geq 2$ we get

\[ \ell(f^3) = \deg u(x)u^2(w) = (2d + 1) \deg u. \tag{3.12} \]

Combining Eqs. (3.8), (3.11) and (3.12) we have

\[ 3 \deg f \geq d(2d + 1) \deg u = (2d + 1) \deg f, \]

which is a contradiction, for $d \geq 2$ and $\deg f \geq 1$. Hence the proposition follows. □

Remark 3.11. By Lemmas 3.2 and 3.3, Proposition 3.7 and also Lemma 4.1 in the next section we see that Theorem 1.5 holds for all $K$-\(\mathcal{E}\)-derivations (not necessarily LF) of $K[x]$. Furthermore, Proposition 3.7 also implies statement 2) of Theorem 1.6.

4. The Locally Nilpotent \(\mathcal{E}\)-Derivation Case

In this section we let $K$ be a field of characteristic zero. We consider the LN (locally nilpotent) $K$-\(\mathcal{E}\)-derivations $K[x]$ and give a proof for statement 3) of Theorem 1.6. From the exhausting list on page 7 it is easy to see that the only nonzero LN $K$-\(\mathcal{E}\)$-derivations $\delta$ of $K[x]$ are those in Case II of the list, i.e., $\delta = I - \phi$, where $\phi$ is the affine translation that maps $x$ to $x + c$ for some $c \in K$.

Note that, if $c = 0$, we have $\phi = I$ and $\delta$ is the zero map, which is a trivial case. So throughout this section we assume $\delta = I - \phi$, and $\phi$ maps $x$ to $x + c$ for a fixed $0 \neq c \in K$.

We first consider the images of the following ideals of $K[x]$ under $\delta$.

Lemma 4.1. Let $\phi, \delta$ be as above and $I = K[x]$ or $(x-a)K[x]$ for some $a \in K$. Then $\delta I = K[x]$.

Proof: It suffices to show the case that $I = (x-a)$, for $\delta I \subseteq \delta(K[x])$.

First, since $\delta(x-a) = -c$ and $c \neq 0$, we see that $1 = x^0 \subset \delta I$. Assume that for some $n \geq 1$ all polynomial $f(x) \in K[x]$ with $\deg f \leq n - 1$ lie in $\delta I$. Consider $\delta((x-a)^{n+1}) \in \delta I$:

\[ \delta((x-a)^{n+1}) = (x-a)^{n+1} - (x+c-a)^{n+1} = -(n+1)cx^n + h(x) \]

for some $h(x) \in K[x]$ with $\deg h \leq n - 1$. By the induction assumption above we have $h \in \delta I$. Therefore $-(n+1)c x^n \in \delta I$ and hence so does $x^n$. Therefore all polynomial $f(x) \in K[x]$ with $\deg f \leq n$ lie in $\delta I$, whence by induction the lemma follows. □
In order to consider the images under $\delta$ of the ideals of $K[x]$ generated by polynomials of the degree $\geq 2$, we first need to recall some well-known facts on the Bernoulli polynomials and the Bernoulli numbers (e.g., see [Wiki], [Ber] and the references therein).

First, the Bernoulli polynomials $\{B_n(t) \mid n \geq 0\}$ are defined by the following generating function:

$$\frac{ue^{tu}}{e^u - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{u^n}{n!}.$$ (4.13)

For example, the first four Bernoulli polynomials are given as follows:

$$B_0(t) = 1,$$

(4.14)

$$B_1(t) = t - \frac{1}{2},$$

$$B_2(t) = t^2 - t + \frac{1}{6},$$

$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$$

For every $n \geq 0$, the following identities of the Bernoulli polynomials hold:

(4.15) $B_n(t + 1) - B_n(t) = nt^{n-1};$

(4.16) $\frac{d}{dt} B_{n+1}(t) = (n + 1)B_n(t).$

(4.17) $\sum_{k=0}^{n} \binom{n + 1}{k} B_k(t) = (n + 1)t^n,$

(4.18) $(-1)^n B_n(-t) = B_n(t) + nt^{n-1}.$

One remark on the Bernoulli polynomials is the following:

**Proposition 4.2.** 1) the $K$-subspace spanned by the Bernoulli polynomials $B_n(x)$ $(n \geq 1)$ coincides with the $K$-subspace $V_{0,1}$ defined in Eq. (2.3) with $a = 0$ and $b = 1$.

2) Let $\Lambda$ be a non-empty set of positive integers and $W$ the $K$-subspace of $K[x]$ spanned by the Bernoulli polynomials $B_i(x)$ with $i \in \Lambda$. Then $W$ is a MS of $K[x]$ with $\nu(W) = \{0\}$. 

Proof: 1) By Eq. (4.15) with $t = 0$ we have $B_n(1) = B_n(0)$ for all $n \geq 2$. Then by Eq. (4.16) we have for all $n \geq 1$

\[
\int_0^1 B_n(x)dx = \frac{1}{n+1}B_{n+1}(x)|_0^1 = 0.
\]

Hence $B_n(x) \in V_{0,1}$ for all $n \geq 1$.

Conversely, by Eqs. (4.16) and the fact $B_0(x) = 1$ we have $\deg B_n(x) = n$ for all $n \geq 0$. Hence the Bernoulli polynomials $B_n(x)$ ($n \geq 0$) form a $K$-linear basis of $K[x]$. In particular, every $f(x) \in K[x]$ can be written uniquely as $f(x) = \sum_{i=0}^d c_i B_i(x)$ with $c_i$'s in $K$. Then by Eqs. (4.19) and the fact $B_0(x) = 1$ we see that $f(x) \in V_{0,1}$, if and only if $c_0 = 0$. Hence statement 1) follows.

2) By statement 1) we have $W \subseteq V_{0,1}$. Then by Theorem 2.2 we have $r(W) = \{0\}$, whence $W$ is a MS of $K[x]$. $\square$

The constant term $B_n$ of the Bernoulli polynomial $B_n(t)$ is called the $n^{th}$ Bernoulli number, i.e., $B_n := B_n(0)$. The Bernoulli polynomials $B_n(t)$ can be expressed in terms of the Bernoulli numbers $B_n$ ($n \geq 0$) as follows:

\[
B_n(t) = \sum_{k=0}^n \binom{n}{k} B_k t^{n-k}.
\]

For the Bernoulli numbers we have the following remarkable theorem, which was found independently by Thomas Clausen ([Cl], 1840) and Karl von Staudt ([St], 1840). See also [Wiki2].

**Theorem 4.3 (The Clausen-von Staudt Theorem).** For each $n \geq 1$, we have

\[
B_{2n} + \sum_{q \text{ prime}} \frac{1}{q} \in \mathbb{Z}.
\]

From the Clausen-von Staudt Theorem we immediately have the following

**Corollary 4.4.** Let $p$ be an odd prime and $\nu_p(\cdot)$ the $p$-valuation on $\mathbb{Q}$. Then the following statements hold:

1) $\nu_p(B_n) \geq 0$ for all $1 \leq n \leq p-2$;

2) $\nu_p(B_{p-1}) < 0$, i.e., $p$ divides the denominator but not the numerator of the reduced fraction form of $B_{p-1}$.

Now we get back to the image of an ideal $I$ under the fixed $K$-$\mathcal{E}$-derivation $\delta$ of $K[x]$. 
Lemma 4.5. Let $I$ be the ideal of $K[x]$ generated by the polynomial $x^2 - ax$ for some $a \in K$ and $\beta := a/c$. Set for all $n \geq 0$

\begin{equation}
D_n(t) := \frac{B_{n+1}(t) - B_{n+1}}{(n+1)t}.
\end{equation}

Then for all $n \geq 0$ we have $D_n(t) \in \mathbb{Q}[t]$ and

\begin{equation}
x^n \equiv D_n(\beta)c^n \mod \delta I.
\end{equation}

Note that by Eqs. (4.15) and (4.20), $D_n(t)$ ($n \geq 0$) are actually given by

\begin{equation}
D_n(t) = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i t^{n-i}.
\end{equation}

\textbf{Proof of Lemma 4.5} First, for each $n \geq 1$, we have $x^{n+1} - ax^n \in I$, and hence $\delta x^{n+1} \equiv ax^n \mod \delta I$. Consequently, $\delta x^{n+1} \equiv a^n \delta x \mod \delta I$. Since $\delta x = -c$, we have

\begin{equation}
(x + c)^{n+1} - x^{n+1} \equiv a^n c \mod \delta I.
\end{equation}

\begin{equation}
\sum_{k=0}^{n} \binom{n+1}{k} c^{n+1-k} x^k \equiv a^n c \mod \delta I.
\end{equation}

\begin{equation}
\sum_{k=0}^{n} \binom{n+1}{k} c^{-k} x^k \equiv a^n / c^n = \beta^n \mod \delta I.
\end{equation}

From the equation above it is easy to see recursively that for each $k \geq 0$, we have $x^k \equiv E_k c^k$ for some $E_k \in K$, which is a polynomial in $c$, $c^{-1}$ and $a$ with coefficients in $\mathbb{Q}$. Furthermore, since $\delta I$ obviously does not contain any nonzero constant, $E_k$ ($k \geq 0$) are actually unique. In particular, $E_0 = 1$.

Now we plug the relations $x^k \equiv E_k c^k \mod \delta I$ ($k \geq 1$) into Eq. (4.24) and get

\begin{equation}
\sum_{k=0}^{n} \binom{n+1}{k} E_k \equiv \beta^n \mod \delta I.
\end{equation}

Since $\delta I$ does not contain any nonzero constant, the equation above is the same as

\begin{equation}
\sum_{k=0}^{n} \binom{n+1}{k} E_k = \beta^n.
\end{equation}
On the other hand, taking $\int_0^t$ to Eq. (4.17) and applying Eq. (4.16) we get
\begin{equation}
\sum_{k=0}^{n} \binom{n+1}{k} \frac{B_{k+1}(t) - B_{k+1}}{k+1} = t^{n+1}.
\end{equation}

Since $B_{k+1} = B_{k+1}(0)$, $B_{k+1}(t) - B_{k+1}$ is divisible by $t$. Then by Eq. (4.21) we have $D_k(t) \in \mathbb{Q}[t]$, for $B_k(t) \in \mathbb{Q}[t]$, for all $k \geq 0$. Furthermore, the equation above can be re-written as
\begin{equation}
\sum_{k=0}^{n} \binom{n+1}{k} D_k(t) = t^n.
\end{equation}
Replacing $t$ by $\beta$ we see that $D_k(\beta) (k \geq 0)$ also satisfy the recurrent relation that satisfied by $E_k (k \geq 0)$ in Eq. (4.25). Furthermore, by Eqs. (4.14) and (4.21) we also have $D_0 = 1 = E_0$. Since every solution of the recurrent relation in Eq. (4.25) is completely determined by the initial value for $E_0$, we see that $E_k = D_k(\beta)$ for all $k \geq 1$, whence the lemma follows. \(\square\)

Note that for the ideal $I$ in Lemma 4.5 it is easy to see that $\delta I$ does not contain any nonzero constant. From this fact we immediately have the following

**Corollary 4.6.** Let $I$, $\beta$ be as in Lemma 4.5 and $f(x) = \sum_{i=0}^{d} a_i x^i \in K[x]$. Then $f(x) \in \delta I$, if and only if the following equation holds:
\begin{equation}
\sum_{i=0}^{d} a_i D_i(\beta) c^i = 0.
\end{equation}

More generally, for the image under $\delta$ of an ideal $I = uK[x]$ with $\deg u \geq 2$, we have the following:

**Remark 4.7.** Assume that $K$ is algebraically closed. Let $u \in K[x]$ with $d := \deg u \geq 2$, $I = uK[x]$, and $r_i (1 \leq i \leq d)$ be all the roots of $u$ in $K$. Set $u_{ij} := (x-r_i)(x-r_j)$ and $I_{ij} := u_{ij}K[x]$ for all $1 \leq i < j \leq d$. Then we have
\begin{equation}
I = \bigcap_{1 \leq i < j \leq d} I_{ij},
\end{equation}
\begin{equation}
\delta I = \bigcap_{1 \leq i < j \leq d} \delta I_{ij}.
\end{equation}

On the other hand, let $T_i (1 \leq i \leq d-1)$ be the affine translation of $K[x]$ that maps $x$ to $x + r_i$. Then $T_i$ commutes with $\delta$ and maps the ideal $I_{ij}$ $(1 \leq j \leq d)$ to the ideal generated by $x(x - a_{ij})$, where
Then by Corollary 4.6 and Eq. (4.30) above we see that the polynomials \( f \in \delta I \) up to the translations \( T_i \) (1 \( \leq i \leq d - 1 \)) are characterized by a system of equations as the one in Eq. (4.29).

Now we are ready to show the following crucial lemma.

**Lemma 4.8.** Let \( u(x) = x(x - a) \) for some \( a \in K \) and \( I = uK[x] \). Set \( \beta := a/c \). Then the following statements hold:

1) if \( \beta = 1 \), then \( \delta I = (x) = xK[x] \);
2) if \( \beta = -1 \), then \( \delta I = (x + a)K[x] \);
3) if \( \beta = 0 \), then \( r(\delta I) = \{0\} \).

In all the cases above \( \delta I \) is a MS of \( K[x] \).

**Proof:**

1) By Eq. (4.15) we have \( B_n(1) = B_n(0) \) for all \( n \geq 2 \), and by Eq. (4.21) \( D_n(1) = 0 \) for all \( n \geq 1 \). Furthermore by Eq. (4.23), \( D_0(1) = 1 \). Then by Lemma 4.5 the statement follows.

2) Let \( I_1 \) and \( I_2 \) be the ideals of \( K[x] \) generated respectively by \( x^2 - ax \) and \( x^2 + ax \). Denote by \( T \) the affine translation of \( K[x] \) that maps \( x \) to \( x + a \). Then \( T \) maps the principal ideal \( I_1 \) to \( I_2 \). Since \( T \) is a \( K \)-algebra automorphism of \( K[x] \) and commutes with \( \delta \), we have \( \delta I_2 = \delta(TI_1) = T(\delta I_1) \). Since \( \delta I_1 = (x) \) by statement 1), we have \( \delta I_2 = (x + a) \), as desired.

Another proof of this statement is to use Eqs. (4.18) and (4.21) first to show \( D_k(-1) = (-1)^k \) for all \( k \geq 0 \), and then apply Lemma 4.5.

3) Assume otherwise and let \( 0 \neq f(x) \in r(\delta I) \), i.e., \( f^m \in \delta I \) for all \( m \gg 0 \). Since \( \delta I \) does not contain any nonzero constant, \( d := \deg f(x) \geq 1 \). Furthermore, we may assume that \( f \) is monic, and by replacing \( f \) by a power of \( f \), that \( f^m \in \delta I \) for all \( m \geq 1 \). Write

\[
(4.31) \quad f(x) = x^d + \sum_{i=0}^{d-1} a_i x^i,
\]

and for all \( m \geq 1 \),

\[
(4.32) \quad f^m(x) = x^{md} + \sum_{i=0}^{md-1} \Gamma_{m,j} x^i,
\]

where \( \Gamma_{m,j} \)'s are some polynomials in \( a_i \)'s over \( \mathbb{Z} \).

Applying Corollary 4.6 to \( f^m \in \delta I \) (\( m \geq 1 \)) we get

\[
(4.33) \quad D_{md}(0)c^{md} + \sum_{j=0}^{md-1} D_j(0)\Gamma_{m,j} c^j = 0.
\]
Since by Eq. (4.23) we have $D_n(0) = B_n$ for all $n \geq 1$, the equation above becomes

\begin{equation}
B_{md}c^{md} + \sum_{j=0}^{md-1} B_j \Gamma_{m,j}c^j = 0.
\end{equation}

Next, we make the following reduction. Note that for all $m \geq 1$ the equations above are polynomial equations over $Q$ in $c$ and $a_i$'s. We may apply a similar reduction as in the proof of [FPYZ, Theorem 4.1] to assume that $K$ is a subfield of the algebraic closure $\bar{Q}$ of the rational field $Q$. Therefore, for each prime $p$ the $p$-valuation $\nu_p$ of $Q$ can be extended to $K$, which we will still denote by $\nu_p$.

Now, by Dirichlet’s prime number theorem there exist infinitely many $m \geq 1$ such that $md + 1$ is a prime number. Furthermore, it is well-known in Algebraic Number Theory (e.g., see [W, Theorem 4.1.7]) that for all but finitely primes $p$, the values of $\nu_p$ at $c$ and $a_i$ $(0 \leq i \leq d - 1)$ are equal to 0. Therefore, we may choose an $m \geq 1$ such that the following properties hold:

\begin{enumerate}
  \item $p := md + 1$ is an odd prime (In particular, $md$ is even);
  \item $\nu_p(c) = \nu_p(a_i) = 0$ for all $0 \leq i \leq d - 1$ such that $a_i \neq 0$.
\end{enumerate}

Consequently, for all $0 \leq j \leq md - 1 = p - 2$ we have $\nu_p(\Gamma_{m,j}) \geq 0$ and by Corollary 4.4, \(1\), $\nu_p(B_j) \geq 0$. Then by Eq. (4.34) $\nu_p(B_{p-1}c^{md}) \geq 0$. Since $\nu_p(c) = 0$, we have $\nu_p(B_{p-1}) \geq 0$. But this contradicts to Corollary 4.4, \(2\). Therefore statement 3) follows. \(\square\)

\textbf{Lemma 4.9.} Let $I = uK[x]$ such that $u$ has at least one repeated root in $\bar{K}$. Then $\mathfrak{v}(\delta I) = \{0\}$. In particular, $\delta I$ is a MS of $K[x]$.

\textbf{Proof:} Let $\bar{K}$ be the algebraic closure of $K$. We view $K[x]$ as a $K$-subalgebra of $\bar{K}[x]$ in the canonical way and denote by $\bar{\delta}$ the $K$-linear extension of $\delta$ from $\bar{K}[x]$ to $\bar{K}[x]$.

Let $r$ be a repeated root of $u$ in $\bar{K}$, and $T$ the affine translation of $\bar{K}[x]$ that maps $x$ to $x + r$. Denote by $I_1$ and $I_2$ the ideals of $\bar{K}[x]$ generated respectively by $x^2$ and $(x - r)^2$. Then $TI_1 = I_2$.

Applying Lemma 4.8, \(3\) to $\bar{\delta}$ and $I_1$, we see that the radical $\mathfrak{v}(\bar{\delta}I_1)$ in $\bar{K}[x]$ is equal to $\{0\}$. Since $T$ commutes with $\bar{\delta}$ and is a $\bar{K}$-algebra automorphism, we have $\mathfrak{v}(T\bar{\delta}I_1) = T(\mathfrak{v}(\bar{\delta}I_1)) = \{0\}$. Since $\delta I \subseteq \bar{\delta}I_2$, the radical $\mathfrak{v}(\delta I)$ in $K[x]$ is also equal to $\{0\}$, whence the lemma follows. \(\square\)

Now we are ready to show the last part of Theorem 1.6, i.e., statement 3).
Proof of Theorem 1.6, 3): Note first that, if \( c = 0 \), then \( \phi = I \) and \( \delta = 0 \), whence the statement holds trivially in this case. Therefore we assume \( c \neq 0 \).

Let \( u \in K[x] \) and \( I = (u) \). If \( u = 0 \), then \( \delta I = 0 \), whence the theorem holds. If \( u \neq 0 \) and \( \deg u \leq 1 \), then the statement follows from Lemma 4.1.

If \( \deg u \geq 2 \) and \( u \) has a repeated root in the algebraic closure of \( K \), then by Lemma 4.9 we have \( \nu(\delta I) = \{0\} \). Hence \( \delta I \) is a MS of \( K[x] \), and the statement holds. \( \square \)

One consequence of Theorem 1.6, 3) is the following corollary on the image of the quantum derivation \( D_h \) (e.g., see [KC]) for all nonzero \( h \in K \), which is defined by setting for all \( f \in K[x] \)

\[
D_h f(x) := \frac{f(x + h) - f(x)}{h}.
\]

(4.35)

Corollary 4.10. Let \( c, \delta \) be fixed as before, \( u \in K[x] \) and \( I = uK[x] \). Assume that either \( u = 0 \), or \( \deg u \leq 1 \) or \( u \) has at least one repeated root in the algebraic closure of \( K \). Then the quantum derivation \( D_{h=c} \) maps \( I \) to a MS of \( K[x] \).

Another remark on the \( K-\mathcal{E} \)-derivation \( \delta \) studied in this section is as follows.

Let \( S \) be the affine automorphism of \( K[x] \) that maps \( x \) to \( c^{-1}x \). Then it is easy to check that \( S^{-1}\delta S = -\Delta \), where \( \Delta \) is the so-called difference operator of \( K[x] \), i.e., \( \Delta f = f(x + 1) - f(x) \) for all \( f \in K[x] \). Therefore, all the results obtained for \( \delta \) in this section can also be interpreted as certain results on the images of ideals of \( K[x] \) under the difference operator \( \Delta \) of \( K[x] \).

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