Second-order Charge Currents and Stress Tensor in Chiral System

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ABSTRACT: We solve the Wigner equation for massless spin-1/2 charged fermions near global equilibrium. The Wigner function can be obtained order by order in the power expansion of the vorticity and electromagnetic field. The Wigner function has been derived up to the second order from which the non-dissipative charge currents and the stress tensor can be obtained. The charge and energy densities and the pressure have contributions from the vorticity and electromagnetic field at the second order. The vector and axial Hall currents can be induced along the direction orthogonal to the vorticity and electromagnetic field at the second order. We also find that the trace anomaly emerges naturally in renormalizing the stress tensor by including the quantum correction from the electromagnetic field.
1 Introduction

It is well-known in classical electrodynamics that the electromagnetic field can generate electric currents, such as Olm’s current from electric fields or Hall’s current from magnetic fields. There are also currents from quantum effects which attract broad interest in the fields of high energy nuclear physics and condensed matter physics. One example is the chiral anomaly, a pure quantum effect, in which currents along the external magnetic field can be induced, it is called the chiral magnetic effects (CME) [1–3] which influences the dynamics of relativistic fluids. The vorticity in an ideal fluid behaves like the magnetic field. Similar to CME, the vorticity can also induce the electric current in a charged fluid of massless fermions, which is called the chiral vortical effect (CVE) [4–7]. In addition to CME and CVE, the vorticity and magnetic field can also induce chiral currents, so-called chiral separate effects (CSE) [8, 9] or the local polarization effect (LPE) [10]. Theoretical studies of these effects have been carried out within a variety of approaches, such as AdS/CFT duality [11–20], relativistic hydrodynamics [21–24], quantum field theory [2, 3, 25–34] and chiral kinetic theories [10, 35–47].

From the point of view of hydrodynamics, these anomalous currents are non-dissipative without entropy production and they all appear at the first order in space-time derivatives. However, when the vorticity or electromagnetic field become strong enough, higher order contributions become important. This is the case in high energy heavy ion collisions, in which both strong magnetic field [48–50] and vorticity [51–57] are generated in non-central collisions. There have been already some earlier attempts to study transport phenomena at the second order for chiral systems including second order hydrodynamics with reversal
invariance [24], Kubo formula or diagrammatic methods from the quantum field theory [58], the chiral kinetic theory [59–62], and equilibrium partition function or AdS/CFT duality [63–66].

The Lorentz covariant and gauge invariant quantum transport theory [67–70] based on Wigner functions can be derived from quantum field theory and is expected to include all quantum corrections. In the previous works [10, 71] by some of us, a power expansion scheme in space-time derivatives and weak fields for the Wigner function of chiral fermions was proposed to solve these quantum transport equations iteratively near equilibrium. It turns out that the Wigner function formalism is successful to reproduce the first order currents in CME, CVE, CSE and LPE. In this paper, we will generalize the power expansion scheme to the second order by deriving the second order non-dissipative charge currents and energy-momentum tensor in a non-interacting chiral fluid.

In Sec. 2, we give a brief overview of the Wigner function formalism for a chiral fermion system. In Sec. 3, we solve the equations for the covariant Wigner function in constant external fields with vorticity near equilibrium by using the method of Refs. [10, 46, 71, 72]. We give the solution to the Wigner function up to the second order of the vorticity and electromagnetic field. In Sec. 4 and Sec. 5, we present the results of the induced vector and axial currents and energy-momentum tensor up to the second order, respectively. We find that the current conservation for vector current and the chiral anomaly for axial current hold automatically. There is no infrared and ultraviolet divergence for the vector and axial current. For the energy-momentum tensor at the second order, the contribution from the vorticity and that from the vorticity and electromagnetic field are both finite, while the contribution from the electromagnetic field has logarithmic ultraviolet divergence when the Dirac sea or vacuum contribution is included. With the proper dimension regularization, we obtain the results that satisfy the energy-momentum conservation. Especially, after we renormalize the stress tensor by including the quantum correction from the electromagnetic field, the trace anomaly emerges naturally. In Sec. 6, we verify the conservation law of the electric charge and the energy momentum as well as the anomalous conservation law of axial charge. We summarize our results in Sec. 7.

As a convention for notational simplicity, the electric charge of the fermion is absorbed into the vector potential $A^\mu$. We use the convention for the metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and Levi-Civita tensor $\epsilon^{0123} = 1$.

## 2 Wigner function formalism

The Wigner function $W(x, p)$ for the Dirac fermion is a $4 \times 4$ matrix in spinor space and is defined as the ensemble average of the Wigner operator [67–69],

$$W_{\alpha\beta}(x, p) = \int \frac{d^4y}{(2\pi)^4} e^{-ipy} \langle \bar{\psi}_\beta(x + \frac{y}{2}) U(x + \frac{y}{2}, x - \frac{y}{2}) \psi_\alpha(x - \frac{y}{2}) \rangle,$$

where $U$ denotes the gauge link along the straight line between $x - y/2$ and $x + y/2$,

$$U(x + \frac{y}{2}, x - \frac{y}{2}) = \text{Exp} \left(-i \int_{x-y/2}^{x+y/2} dz^\mu A_\mu(z) \right).$$
We will restrict ourselves to a system of chiral fermions without collisions in a constant external electromagnetic field $F^{\mu\nu}$ in space and time, i.e. $\partial^\lambda F^{\mu\nu} = 0$, hence we have removed the path ordering of the gauge link. The Wigner equation for chiral fermions in a constant electromagnetic field tensor is given by [69],

$$\gamma_\mu (p^\mu + i/2 \nabla^\mu) W(x,p) = 0, \quad (2.3)$$

where $\gamma^\mu$ are Dirac matrices and $\nabla^\mu \equiv \partial^\mu - F^{\mu\nu} \partial_\nu$ with $\partial^\mu (\partial^\rho)$ being the derivative with respect to $x (p)$. Since the Wigner equation was derived from Dirac equation, the bilinear operator in Wigner function must be not normal ordered. It has been demonstrated in [73] that this point plays a central role to give rise to the chiral anomaly in the quantum kinetic theory. We can decompose the Wigner function in terms of 16 independent generators of the Clifford algebra,

$$W = \frac{1}{4} \left[ \mathcal{J} + i \gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu} \right], \quad (2.4)$$

where we have suppressed the arguments of the Wigner function for notational simplicity.

For chiral fermions, it is more convenient to define the chiral component

$$\mathcal{J}_s^\mu \equiv \frac{1}{2} (\mathcal{V}^\mu + s \mathcal{A}^\mu), \quad (2.5)$$

with $s = \pm 1$ corresponding to the right-hand and left-hand component respectively. Substituting Eq. (2.4) and Eq. (2.5) into Eq.(2.3), we find that the right-hand or left-hand component are decoupled from other components and satisfy

$$\nabla_\mu \mathcal{J}_s^\mu = 0, \quad (2.6)$$

$$p_\mu \mathcal{J}_s^\mu = 0, \quad (2.7)$$

$$p_\mu \mathcal{J}^\mu_s - p_\nu \mathcal{J}^\nu_s = -\frac{s}{2} \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \mathcal{J}^\sigma_s. \quad (2.8)$$

We will suppress the subscript $s$ in Sec. 3 for notational simplicity and recover it in Sec. 4.

\section{Wigner function near equilibrium}

We assume that both the space-time derivative $\partial_x$ and the field strength $F^{\mu\nu}$ in the operator $\nabla_\mu$ are small variables of the same order and play the role of expansion parameters. We solve the Wigner equation by the covariant perturbation method developed in Refs. [10, 46, 72] and present the solution near equilibrium up to the second order in $\partial_x$ and $F^{\mu\nu}$. In fact, this expansion is equivalent to an expansion in the Planck constant $\hbar$ (or the semiclassical expansion) because $\hbar$ always comes with $\nabla_\mu$. According to the perturbation method, the Wigner function can be obtained order by order,

$$\mathcal{J}_\mu = \mathcal{J}_\mu^{(0)} + \mathcal{J}_\mu^{(1)} + \mathcal{J}_\mu^{(2)} + \cdots, \quad (3.1)$$
where the superscripts \((0), (1), \ldots\) denote the orders of the power in the expansion. Substituting this expansion into Wigner equations from (2.6) to (2.8) and requiring that the equations hold order by order. The equations for \(\mathcal{F}^{(n)}_{\mu}\) with \(n > 0\) read

\[
\nabla_{\mu} \mathcal{F}^{(n)}_{\mu} = 0, \tag{3.2}
\]
\[
 p_{\mu} \mathcal{F}^{(n)}_{\mu} = 0, \tag{3.3}
\]
\[
 p_{\mu} \mathcal{F}^{(n)}_{\nu} - p_{\nu} \mathcal{F}^{(n)}_{\mu} = -\frac{s}{2} \epsilon_{\mu\nu\rho\sigma} \nabla^{\rho} \mathcal{F}^{(n-1)\sigma}. \tag{3.4}
\]

If we define \(\mathcal{F}^{(-1)\sigma} = 0\), Eq. (3.4) also works for \(n = 0\). When we contract both sides of Eq. (3.4) with \(p^{\nu}\), we have

\[
p^{\mu} \mathcal{F}^{(n)}_{\mu} = \frac{s}{2} \epsilon_{\mu\nu\rho\sigma} p^{\nu} \nabla^{\rho} \mathcal{F}^{(n-1)\sigma}, \tag{3.5}
\]

where we have used Eq. (3.3). Hence the general form of \(\mathcal{F}^{(n)}_{\mu}\) is

\[
\mathcal{F}^{(n)}_{\mu} = J^{(n)}_{\mu} \delta(p^{2}) + \frac{s}{2p^{2}} \epsilon_{\mu\nu\rho\sigma} p^{\nu} \nabla^{\rho} \mathcal{F}^{(n-1)\sigma}, \tag{3.6}
\]

where \(J^{(n)}_{\mu}\) is nonsingular at \(p^{2} = 0\). This expression is an iterative equation connecting the \(n\)-th order solution with the \((n - 1)\)-th order one. The constraint condition (3.3) gives

\[
p^{\mu} J^{(n)}_{\mu} \delta(p^{2}) = 0. \tag{3.7}
\]

In general, we can decompose \(J^{(n)}_{\mu}\) into two parts

\[
J^{(n)}_{\mu}(x, p) = p_{\mu} f^{(n)}(x, p) + X^{(n)}_{\mu}(x, p), \tag{3.8}
\]

where the first term is satisfied with Eq. (3.7) automatically due to \(p^{2} \delta(p^{2}) = 0\) and the second term is assumed to satisfy \(p^{\mu} X^{(n)}_{\mu} = 0\) when there is no mass-shell constraint.

It is straightforward to write down the zeroth order solution,

\[
\mathcal{F}^{(0)}_{\mu}(x, p) = p_{\mu} f(x, p) \delta(p^{2}) , \tag{3.9}
\]

without \(X^{(0)}_{\mu}\) component. We note that in the above expression we have suppressed the superscript \((0)\) in \(f\) because we will set all higher order contributions \(f^{(n)}\) for \(n \geq 1\) vanish so we have \(f = f^{(0)}\). Substituting the expression (3.9) into Eq. (3.2) with \(n = 0\) gives the kinetic equation at the zeroth order

\[
\delta(p^{2}) p^{\mu} \nabla_{\mu} f(x, p) = 0. \tag{3.10}
\]

Since we try to obtain the solution near equilibrium, at the zeroth order we can choose \(f\) as the Fermi-Dirac distribution function,

\[
f = \frac{1}{4\pi^{3}} \left( \frac{1}{e^{\beta p - \mu} + 1} \right), \quad (p_{0} > 0) \tag{3.11}
\]
\[
f = \frac{1}{4\pi^{3}} \left( \frac{1}{e^{-\beta p + \mu} + 1} - 1 \right), \quad (p_{0} < 0) \tag{3.12}
\]
where
\[ \beta^\mu \equiv \frac{u^\mu}{T}, \quad \bar{\mu}_s \equiv \frac{\mu_s}{T} = \bar{\mu} + s\bar{\mu}_5, \quad \bar{\mu} = \frac{\mu}{T}, \quad \bar{\mu}_5 = \frac{\mu_5}{T}, \] (3.13)
with \( u \) being the fluid four-velocity, \( T \) the temperature, \( \mu_s \) the right-hand/left-hand chemical potential, \( \mu \) the vector chemical potential and \( \mu_5 \) the chiral chemical potential. In such a solution, we note that \( \mathcal{J}_\mu^{(0)}(x,p) \) or \( f(x,p) \) depends on \( x \) only through \( u(x), T(x), \mu(x) \) and \( \mu_5(x) \). The Dirac sea (or vacuum) contribution \(-1\) in the anti-particle distribution \([74, 75]\) is indispensable because there is no normal ordering in the definition of the Wigner function \((2.1)\). With the distribution \((3.11)\) and \((3.12)\), the Wigner function \( \mathcal{J}_\mu^{(0)} \) is in the form
\[ \mathcal{J}_\mu^{(0)} = \frac{p_\mu}{4\pi^3} \left[ \frac{1}{e^{\beta\cdot p - \bar{\mu}_s} + 1} + \left( \frac{1}{e^{-\beta\cdot p + \bar{\mu}_s} + 1} - 1 \right) \frac{\delta(p_0 + |p|)}{2|p|} \right] . \] (3.14)
Inserting Eq. \((3.11)\) or \((3.12)\) into the kinetic equation \((3.10)\) we obtain
\begin{align*}
\delta(p^2)p^\mu \nabla_\mu f &= f' \left[ \frac{1}{2} p^\mu p^\nu (\partial_\mu \beta_\nu + \partial_\nu \beta_\mu) - p^\mu \partial_\mu \bar{\mu} - p^\mu F_{\mu\nu} \beta^\nu - sp^\mu \partial_\mu \bar{\mu}_5 \right] \\
&= 0 ,
\end{align*} (3.15)
where we have used the shorthand notation \( f' \equiv \partial f / \partial (\beta \cdot p) \). It is obvious that when the constraint conditions
\begin{align*}
\partial_\mu \beta_\nu + \partial_\nu \beta_\mu &= 0 , \quad \text{ (3.16)} \\
\partial_\mu \bar{\mu} + F_{\mu\nu} \beta^\nu &= 0 , \quad \text{ (3.17)} \\
\partial_\mu \bar{\mu}_5 &= 0 , \quad \text{ (3.18)}
\end{align*}
are all satisfied, the Wigner function \((3.9)\) with \((3.11)\) and \((3.12)\) are indeed the solution to Eq. \((3.10)\). General solutions to these constraint conditions are
\begin{align*}
\beta_\mu &= -\Omega_{\mu\nu} x^\nu , \\
\bar{\mu} &= -\frac{1}{2} F_{\mu\lambda} x^\lambda \Omega_{\mu\nu} x^\nu + c , \quad \text{ (3.19)} \\
\bar{\mu}_5 &= c_5 , \quad \text{ (3.20)}
\end{align*}
together with the condition of integrability from Eq. \((3.17)\)
\[ F_{\lambda}^{\mu\nu} \Omega^{\nu\lambda} - F_{\lambda}^{\rho\nu} \Omega^{\rho\lambda} = 0 , \quad \text{ (3.22)} \]
where \( \Omega^{\mu\nu} \) and \( c_5/c \) are constant antisymmetric tensor and constants, respectively. It should be noted that \( \Omega_{\mu\nu} \) is nothing but the thermal vorticity tensor of the fluid
\[ \Omega_{\mu\nu} = \frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) . \] (3.23)
Substituting the zeroth order solution \((3.9)\) into Eq. \((3.6)\) with \( n = 1 \) gives rise to the first order solution
\[ \mathcal{J}_\mu^{(1)} = J_\mu^{(1)} \delta(p^2) + \frac{s}{2p^2} \epsilon_{\mu\nu\rho\sigma} p^\nu \nabla^\rho \mathcal{J}_\mu^{(0)\sigma} \\
= X_\mu^{(1)} \delta(p^2) + sF_{\mu\nu} p^\nu f \delta'(p^2) , \quad \text{ (3.24)} \]
where we have dropped the term proportional to \( p_\mu \delta(p^2) \) and used the dual field tensor \( \tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) and \( \delta'(x) = -(1/x)\delta(x) \). The unknown \( X^{(1)}_\mu \) can be further constrained by inserting Eq. (3.24) into Eq. (3.4)

\[
\left( p_\mu X^{(1)}_\nu - p_\nu X^{(1)}_\mu \right) \delta(p^2) = \frac{s}{2} \epsilon_{\mu\nu\lambda\rho} p^\lambda \nabla^\rho f \delta(p^2)
= -\frac{s}{2} \left( p_\mu \tilde{\Omega}_{\nu\lambda} p^\lambda - p_\nu \tilde{\Omega}_{\mu\lambda} p^\lambda \right) f' \delta(p^2),
\]

where \( \tilde{\Omega}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma} \Omega^{\rho\sigma} \). In order to arrive at the last equation, we have used the specific distribution (3.11-3.12) and the constraint conditions (3.16-3.18). Obviously, from the equation above, we can set

\[
X^{(1)}_\mu = -\frac{s}{2} \tilde{\Omega}_{\mu\lambda} p^\lambda f',
\]

which results in

\[
J^{(1)}_\mu = -\frac{s}{2} \tilde{\Omega}_{\mu\lambda} p^\lambda f' \delta(p^2) + \frac{s}{2} \epsilon_{\mu\nu\rho\sigma} p^\rho \tilde{F}^{\nu\sigma} f' \delta(p^2) .
\]

Under the constraint conditions (3.16-3.18), it is straightforward to verify that \( J^{(1)}_\mu \) given above automatically satisfies Eq. (3.2) with \( n = 1 \). This means that Eq. (3.27) is indeed the solution near equilibrium at the first order.

Now let us turn to the second order solution that has not been considered before. Similar to the way we obtain the first order solution from the zeroth order one, the second order solution can be given by the iterative equation (3.6) with the the first order solution (3.27),

\[
J^{(2)}_\mu = J^{(1)}_\mu + \frac{s}{2p^2} \epsilon_{\mu\nu\rho\sigma} p^\rho \nabla^\nu J^{(1)}_\sigma = X^{(2)}_\mu \delta(p^2) + \frac{1}{4p^2} \left( p_\mu \Omega_{\gamma\beta} p^\beta - p^2 \Omega_{\gamma\mu} \right) \Omega^{\gamma\lambda} p_\lambda f'' \delta(p^2) \\
+ \frac{1}{p^2} \left( p_\mu F_{\gamma\beta} p^\beta - p^2 F_{\gamma\mu} \right) \Omega^{\gamma\lambda} p_\lambda f' \delta(p^2) \\
+ \frac{1}{p^2} \left( p_\mu F_{\gamma\beta} p^\beta - p^2 F_{\gamma\mu} \right) \Omega^{\gamma\lambda} p_\lambda f' \delta(p^2).
\]

Here \( X^{(2)}_\mu \) can be constrained by inserting Eq. (3.28) into Eq. (3.4) with \( n = 2 \). It turns out that

\[
\left( p_\mu X^{(2)}_\nu - p_\nu X^{(2)}_\mu \right) \delta(p^2) = 0,
\]

which leads to \( X^{(2)}_\mu = 0 \), where we have used the constraint conditions (3.16-3.18) once more to arrive at the final result. Now we finally obtain the second order solution near equilibrium

\[
J^{(2)}_\mu = \frac{1}{4} \Omega_{\mu\lambda} \Omega^{\gamma\nu} p_\lambda f'' \delta(p^2) - \frac{1}{4} p_\mu \Omega_{\gamma\beta} p^\beta \Omega^{\gamma\nu} p_\lambda f'' \delta(p^2) \\
+ F_{\gamma\mu} \Omega^{\gamma\nu} p_\lambda f' \delta'(p^2) + \frac{1}{2} p_\mu F_{\gamma\beta} p^\beta \Omega^{\gamma\nu} p_\lambda f' \delta'(p^2) \\
- F_{\gamma\mu} F^{\gamma\nu} p_\lambda f' \delta'(p^2) - \frac{1}{3} p_\mu F_{\gamma\beta} p^\beta F^{\gamma\nu} p_\lambda f' \delta''(p^2).
\]
where we have used the identity
\[ p^6 \delta'''(p^2) = -3p^4 \delta''(p^2) = 6p^2 \delta'(p^2) = -6\delta(p^2). \] (3.31)

4 Vector and axial currents

Once we have the Wigner function in phase space, the right-handed or left-handed current can be obtained directly by integrating over the four-momentum
\[ j_\mu^s = \int d^4p \mathcal{J}_\mu^s, \] (4.1)
where we have recovered the chirality index \( s \). The vector and axial currents are given by
\[ j_\mu = j_{\mu}^{+1} + j_{\mu}^{-1}, \quad j_5^\mu = j_{\mu}^{+1} - j_{\mu}^{-1}. \] (4.2)

Note that the vector current can also be called fermion number or charge current, while the axial current can also be called chiral charge or chiral current. The results for the zeroth and first order current are well-known
\[ j_\mu^{(0)} = n_s u_\mu, \] (4.3)
\[ j_\mu^{(1)} = \xi_\omega u_\mu + \xi_B B^\mu, \] (4.4)
where \( n_s \) is the fermion number density, and \( \xi_\omega \) and \( \xi_B \) are transport coefficients associated with CVE and CME respectively in the vector current \( j_\mu^s \). They are given by
\[ n_s = \frac{\mu_s}{6\pi^2} \left( \frac{\pi^2 T^2 + \mu_s^2}{2} \right), \] (4.5)
\[ \xi_\omega = \frac{s}{12\pi^2} \left( \frac{\pi^2 T^2 + 3\mu_s^2}{4} \right), \] (4.6)
\[ \xi_B = \frac{s}{4\pi^2} \mu_s. \] (4.7)

In the zeroth order result \( n_s \), we have dropped the infinite vacuum contribution. In Eq. (4.3) the vorticity vector \( \omega^\mu \) and the magnetic field vector \( B^\mu \) are defined from the decomposition
\[ F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma, \] (4.8)
\[ T\Omega_{\mu\nu} = \epsilon_\mu u_\nu - \epsilon_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma} u^\rho \omega^\sigma, \] (4.9)
with
\[ E^\mu = F^\mu_{\nu\nu} u_\nu, \quad B^\mu = F^{\mu\nu} u_\nu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}, \] (4.10)
\[ \epsilon^\mu = T\Omega^{\mu\nu} u_\nu, \quad \omega^\mu = T\tilde{\Omega}^{\mu\nu} u_\nu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial^\alpha u_\beta. \] (4.11)

Similar to the electric or magnetic component of \( F_{\mu\nu} \), it is convenient to name \( \epsilon_\mu \) and \( \omega_\mu \) as the electric vorticity and the magnetic vorticity, respectively. It follows that the vector and axial current are given by
\[ j_\mu^{(0)} = n_s u_\mu, \] (4.12)
\[ j_\mu^{(1)} = \xi_\omega u_\mu + \xi_B B^\mu, \] (4.13)
\[ j_5^\mu = n_s u_\mu, \] (4.14)
\[ j_5^{(1)} = \xi_5 \omega^\mu + \xi_B B^\mu, \] (4.15)
with
\[
n = \frac{\mu}{3\pi^2} \left( \pi^2 T^2 + \mu^2 + 3\mu_s^2 \right), \quad n_5 = \frac{\mu^5}{3\pi^2} \left( \pi^2 T^2 + 3\mu^2 + \mu_s^2 \right),
\]
\[
\xi = \frac{\mu^4}{\pi^2}, \quad \xi_B = \frac{\mu^5}{2\pi^2}, \quad \xi_5 = \frac{1}{6\pi^2} \left[ \pi^2 T^2 + 3(\mu^2 + \mu_s^2) \right], \quad \xi_{B5} = \frac{\mu}{2\pi^2}.
\]
(4.16)
where \( n \) and \( n_5 \) are the fermion number (charge) and chiral charge density respectively, and \( \xi, \xi_B, \xi_5 \) and \( \xi_{B5} \) are well-known anomalous transport coefficients associated with CVE, CME, LPE and CSE, respectively.

Now let us turn to the second-order current which can be obtained by integrating Eq. (3.30) over the four-momentum,
\[
j_s^{(2)\mu} = -\frac{1}{4} \Omega^{\mu\nu} \Omega_{\gamma\lambda} \int d^4 p (u \cdot p) f_s'' \delta'(p^2) - \frac{1}{4} u^\mu u^\nu u_\lambda \Omega_{\gamma\beta} \Omega^{\gamma\lambda} \int d^4 p (u \cdot p)^3 f_s'' \delta'(p^2)
\]
\[
- \frac{1}{12} \left( \Delta_{\mu\alpha} u_\alpha + \Delta_{\mu\alpha} u_\beta + \Delta_{\lambda\alpha} u_\mu \right) \Omega_{\gamma\beta} \Omega^{\gamma\lambda} \int d^4 p (u \cdot p)^3 f_s'' \delta'(p^2)
\]
\[
+ F^{\mu\nu} \Omega_{\gamma\lambda} u_\lambda \int d^4 p (u \cdot p) f_s'' \delta'(p^2) + \frac{1}{2} u^\mu u^\beta u_\lambda F_{\gamma\beta} \Omega^{\gamma\lambda} \int d^4 p (u \cdot p)^3 f_s'' \delta'(p^2)
\]
\[
+ \frac{1}{6} \left( \Delta_{\mu\alpha} u_\alpha + \Delta_{\mu\alpha} u_\beta + \Delta_{\lambda\alpha} u_\mu \right) F_{\gamma\beta} \Omega^{\gamma\lambda} \int d^4 p (u \cdot p)^3 f_s'' \delta'(p^2)
\]
\[
- F^{\mu\nu} F_{\gamma\lambda} u_\lambda \int d^4 p (u \cdot p) f_s'' \delta'(p^2) - \frac{1}{3} u^\mu u^\beta u_\lambda F_{\gamma\beta} F^{\gamma\lambda} \int d^4 p (u \cdot p)^3 f_s'' \delta'(p^2)
\]
\[
- \frac{1}{9} \left( \Delta_{\mu\beta} u_\beta + \Delta_{\mu\beta} u_\beta + \Delta_{\lambda\beta} u_\mu \right) F_{\gamma\beta} F^{\gamma\lambda} \int d^4 p (u \cdot p)^3 f_s'' \delta'(p^2).
\]
(4.17)
In the above equation we have used following moment identity
\[
\int d^4 p \, p_\lambda Y = u_\lambda \int d^4 p \, (u \cdot p) Y,
\]
(4.18)
\[
\int d^4 p \, p_\mu p_\beta p_\lambda Y = u_\mu u_\beta u_\lambda \int d^4 p \, (u \cdot p)^3 Y
\]
\[
+ \frac{1}{3} \left( \Delta_{\mu\beta} u_\beta + \Delta_{\mu\beta} u_\beta + \Delta_{\lambda\beta} u_\mu \right) \int d^4 p \, (u \cdot p)^2 p_\lambda Y,
\]
(4.19)
where \( Y \) can be any scalar function that only depends on momentum through \( u \cdot p \) and \( p^2 \), \( \Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \) and \( \bar{p}_\mu = \Delta_{\mu\nu} p^\nu \). Using the decomposition (4.8) and (4.9), \( j_s^{(2)\mu} \) can be put into the form
\[
j_s^{(2)\mu} = u^\mu \left( \epsilon^2 + \omega^2 \right) \sum \frac{1}{6} \int d^4 p (u \cdot p) p^2 f_s'' \delta'(p^2)
\]
\[
- \epsilon^{\mu\rho\sigma} \varepsilon_{\rho u_\sigma} \omega_\gamma \int d^4 p (u \cdot p) f_s'' \left[ \frac{1}{6} p^2 \delta'(p^2) + \frac{1}{4} \delta(p^2) \right]
\]
\[
- u^\mu (\epsilon \cdot E + \omega \cdot B) \frac{1}{3} \int d^4 p (u \cdot p)^2 p^2 \delta'(p^2)
\]
\[
+ \epsilon^{\mu\rho\sigma} E_{\rho u_\sigma} \omega_\gamma \int d^4 p (u \cdot p) f_s'' \left[ \frac{1}{3} p^2 \delta'(p^2) + \delta(p^2) \right]
\]
\[
+ u^\mu (E^2 + B^2) \frac{2}{9} \int d^4 p (u \cdot p)^2 p^2 f_s \delta'(p^2)
\]
\[
- \epsilon^{\mu\rho\sigma} u_\rho B_\sigma E_\gamma \int d^4 p (u \cdot p) f_s \left[ \frac{2}{9} p^2 \delta'(p^2) + \delta(p^2) \right].
\]
(4.20)
After finishing the integrals in Eq. (4.20), we obtain the second-order currents

\[ j_s^{(2)\mu} = -\frac{\mu_s}{4\pi^2}(\varepsilon^2 + \omega^2)u^\mu - \frac{1}{8\pi^2}(\varepsilon \cdot E + \omega \cdot B)u^\mu - \frac{C_s}{24\pi^2}(E^2 + B^2)u^\mu \]

\[ -\frac{1}{8\pi^2}e^{\mu
u\rho\sigma}u_\nu E_\rho \omega_\sigma - \frac{C_s}{12\pi^2}e^{\mu
u\rho\sigma}u_\nu E_\rho B_\sigma , \]

where

\[ C_s = \frac{1}{T} \int_0^{\infty} \frac{dp_0}{p_0} \left[ \frac{e^{\rho_0/T-\bar{\mu}_s}}{(e^{\rho_0/T-\bar{\mu}_s} + 1)^2} - \frac{e^{\rho_0/T+\bar{\mu}_s}}{(e^{\rho_0/T+\bar{\mu}_s} + 1)^2} \right] . \]  

(4.22)

It is obvious that \( C_s \) is an odd function of \( \bar{\mu}_s \). When \( |\bar{\mu}_s| \ll 1 \) or at high temperature limit, we can expand the integrand in Eq. (4.22) in series and work out the integral analytically

\[ C_s = -\frac{14\zeta'(-2)}{T^2} \mu_s \approx \frac{0.4263\mu_s}{T^2} . \]  

(4.23)

When \( |\bar{\mu}_s| \gg 1 \) or at low temperature limit, we can approximate the Fermi-Dirac distribution function by a step function \( \Theta(\pm\mu_s - p_0) \) and obtain the analytic result

\[ C_s = \frac{1}{\mu_s} . \]  

(4.24)

From Eq. (4.21) the vector and axial current are given by

\[ j^{(2)\mu} = -\frac{\mu}{2\pi^2}(\varepsilon^2 + \omega^2)u^\mu - \frac{1}{4\pi^2}(\varepsilon \cdot E + \omega \cdot B)u^\mu - \frac{C}{12\pi^2}(E^2 + B^2)u^\mu \]

\[ -\frac{1}{6\pi^2}e^{\mu
u\rho\sigma}u_\nu E_\rho \omega_\sigma - \frac{C}{6\pi^2}e^{\mu
u\rho\sigma}u_\nu E_\rho B_\sigma , \]

\[ j_s^{(2)\mu} = -\frac{\mu_s}{2\pi^2}(\varepsilon^2 + \omega^2)u^\mu - \frac{C_s}{12\pi^2}(E^2 + B^2)u^\mu - \frac{C_5}{6\pi^2}e^{\mu
u\rho\sigma}u_\nu E_\rho B_\sigma , \]

with

\[ C = \frac{1}{2}(C_{+1} + C_{-1}) , \quad C_5 = \frac{1}{2}(C_{+1} - C_{-1}) . \]  

(4.27)

When \( |\mu_s| \ll T \), we have

\[ C = -\frac{14\zeta'(-2)}{T^2} \mu \approx 0.4263\mu \frac{T^2}{T^2} , \quad C_5 = -\frac{14\zeta'(-2)}{T^2} \mu_5 \approx 0.4263\mu_5 \frac{T^2}{T^2} . \]  

(4.28)

When \( |\mu_s| \gg T \), we have

\[ C = \frac{\mu}{(\mu^2 - \mu_5^2)} , \quad C_5 = -\frac{\mu_5}{(\mu^2 - \mu_5^2)} . \]  

(4.29)

Now we look closely at the vector current (4.25). The first line of (4.25) indicates that the charge density is modified by quadratic terms \( \varepsilon^2, \omega^2, E^2, B^2, \varepsilon \cdot E \) and \( \omega \cdot B \). The second line of (4.25) are the Hall currents induced along the direction orthogonal to both \( E^\mu \) and \( \omega^\mu \) or that orthogonal to both \( E^\mu \) and \( B^\mu \) in the comoving frame of the fluid cell. It is interesting to observe that there is no Hall current induced by \( \varepsilon^\mu \) and \( \omega^\mu \). It should be
clarified here that the mixed Hall current $\epsilon_{\mu\nu\rho\sigma}u_\nu E_\rho \omega_\sigma$ is actually identical to $\epsilon_{\mu\nu\rho\sigma}u_\nu \varepsilon_\rho B_\sigma$ in this paper due to the integrability condition (3.22) which is equivalent to
\[
\epsilon_{\mu\nu\rho\sigma} (E^\rho \omega^\sigma - \varepsilon^\rho B^\sigma) = 0 \quad \text{and} \quad \epsilon_{\mu\nu\rho\sigma} (E^\rho \varepsilon^\sigma + \omega^\rho B^\sigma) = 0 .
\]

For the axial current, the first and second terms in (4.26) indicate that the axial charge density gets modified by quadratic terms $\varepsilon^2, \omega^2, E^2$ and $B^2$ but not from mixed terms $\varepsilon \cdot E$ and $\omega \cdot B$ due to the symmetry which is different from the charge density. The last term in (4.26) is the axial Hall current generated by $E^\mu$ and $B^\nu$ only, but not by $E^\mu$ and $\omega^\nu$ or $\varepsilon^\mu$ and $B^\nu$, which is also different from the vector current. Like the charge current, there is no axial Hall current from $\varepsilon^\mu$ and $\omega^\nu$.

5 Energy-momentum tensor

In the Wigner function formalism, the stress tensor or energy-momentum tensor can be expressed as
\[
T^{\mu\nu} = \int d^4p V^\mu p^\nu = \int d^4p \left( \mathcal{J}^{\mu}_{+1} + \mathcal{J}^{\mu}_{-1} \right) p^\nu .
\]

Note that this is the canonical definition of the stress tensor and is not necessarily symmetric. First let us give the stress tensor of the right-handed or left-handed part at the zeroth order and first order separately,
\[
T^{(0)}_{s}^{\mu\nu} = \int d^4p \mathcal{J}^{(0)}_{s}^{\mu} p^\nu = u^\mu u^\nu \rho_s - \frac{1}{3}\Delta^{\mu\nu} \rho_s ,
\]
\[
T^{(1)}_{s}^{\mu\nu} = \int d^4p \mathcal{J}^{(1)}_{s}^{\mu} p^\nu
\]
\[
= s n_s (u^\mu \omega^\nu + u^\nu \omega^\mu) + \xi_s^2 \left( u^\mu B^\nu + u^\nu B^\mu \right)
\]
\[
- \frac{sn_s}{2} \left( u^\mu \omega^\nu - u^\nu \omega^\mu + \epsilon^{\mu\alpha\beta} u_\alpha \varepsilon_\beta \right) - \frac{\xi_s^2}{2} \epsilon^{\mu\alpha\beta} u_\alpha E_\beta ,
\]
where $n_s$ and $\xi_s$ in $T^{(1)}_{s}^{\mu\nu}$ are given by Eqs. (4.5,4.6), and the energy density $\rho_s$ in $T^{(0)}_{s}^{\mu\nu}$ is
\[
\rho_s = \frac{T^4}{2\pi^2} \left( \frac{7}{60} \pi^4 + \frac{1}{2} \pi^2 \bar{\mu}_s^2 + \frac{1}{4} \bar{\mu}_s^4 \right) ,
\]
where we have dropped the infinite vacuum energy density. After taking the sum of the right-handed and left-handed contributions, we obtain the total energy-momentum tensor
\[
T^{(0)}^{\mu\nu} = T^{(0)}_{+1}^{\mu\nu} + T^{(0)}_{-1}^{\mu\nu} = \rho u^\mu u^\nu - \frac{1}{3}\rho \Delta^{\mu\nu} ,
\]
\[
T^{(1)}^{\mu\nu} = T^{(1)}_{+1}^{\mu\nu} + T^{(1)}_{-1}^{\mu\nu}
\]
\[
= n_5 \left( u^\mu \omega^\nu + u^\nu \omega^\mu \right) + \frac{\xi}{2} \left( u^\mu B^\nu + u^\nu B^\mu \right)
\]
\[
- \frac{n_5}{2} \left( u^\mu \omega^\nu - u^\nu \omega^\mu + \epsilon^{\mu\alpha\beta} u_\alpha \varepsilon_\beta \right) - \frac{\xi}{2} \epsilon^{\mu\alpha\beta} u_\alpha E_\beta ,
\]
where \( n \) and \( \xi \) in \( T^{(1)\mu \nu} \) are given by Eq. (4.16), and the energy density in \( T^{(0)\mu \nu} \) is

\[
\rho = \frac{T_4}{4\pi^2} \left[ \frac{7}{15} \pi^4 + 2\pi^2 (\mu^2 + \bar{\mu}^2) + \bar{\mu}^4 + 6\bar{\mu}^2 \bar{\mu}_5^2 + \bar{\mu}_5^4 \right].
\]  

(5.7)

Now let us compute the stress tensor at the second order. We decompose the stress tensor into three parts,

\[
T^{(2)\mu \nu} = T^{(2)_{\text{vv}}}_{s,\text{vv}} + T^{(2)_{\text{ve}}}_{s,\text{ve}} + T^{(2)_{\text{ee}}}_{s,\text{ee}},
\]  

(5.8)

which 'v' means the vorticity and 'e' means the electromagnetic field, so these three terms are coupling terms of the vorticity-vorticity, vorticity-electromagnetic-field and electromagnetic-field-electromagnetic-field, respectively. These terms are given by

\[
T^{(2)_{\text{vv}}}_{s,\text{vv}} = -\frac{1}{4} \Omega^\gamma_{\beta \Omega \lambda} \int d^4 p \, p^\mu p^\nu p^\rho p^\lambda f^\mu \delta^\nu (p^2) - \frac{1}{4} \Omega^\mu_{\gamma \Omega \lambda} \int d^4 p \, p^\nu p^\lambda f^\nu \delta^\mu (p^2),
\]  

(5.9)

\[
T^{(2)_{\text{ve}}}_{s,\text{ve}} = \frac{1}{2} F^\gamma_{\beta \Omega \lambda} \int d^4 p \, p^\mu p^\nu p^\rho p^\lambda f^\nu \delta^\mu (p^2) + F^\gamma_{\Omega \mu \lambda} \int d^4 p \, p^\nu p^\lambda f^\mu \delta^\nu (p^2),
\]  

(5.10)

\[
T^{(2)_{\text{ee}}}_{s,\text{ee}} = -\frac{1}{3} F^\gamma_{\beta \gamma} \int d^4 p \, p^\mu p^\nu p^\rho p^\lambda f^\delta^\mu (p^2) - F^\gamma_{\Omega \mu \lambda} \int d^4 p \, p^\nu p^\lambda f^\delta^\mu (p^2).
\]  

(5.11)

We can work out the first two integrals directly and obtain

\[
T^{(2)_{\text{vv}}}_{s,\text{vv}} = -\frac{s}{2} \xi_s \left[ 3n^\mu (n^2 + \epsilon^2) - \Delta^\mu \nu (\omega^2 + \epsilon^2) - 2(n^\mu e^{\nu\alpha\beta\gamma} + n^\nu e^{\mu\alpha\beta\gamma}) u_\alpha \epsilon_\beta \omega_\gamma 
\right.

\[ -2(n^\mu e^{\nu\alpha\beta\gamma} - n^\nu e^{\mu\alpha\beta\gamma}) u_\alpha \epsilon_\beta \omega_\gamma \right],
\]  

(5.12)

\[
T^{(2)_{\text{ve}}}_{s,\text{ve}} = -\frac{s}{2} \xi_B \left[ n^\mu (\omega \cdot B + \epsilon \cdot E) - (\omega^\mu B^\nu + E^\mu \epsilon^\nu)
\right.

\[ -(n^\mu e^{\nu\alpha\beta\gamma} + n^\nu e^{\mu\alpha\beta\gamma}) u_\alpha E_\beta \omega_\gamma - 2(n^\mu e^{\nu\alpha\beta\gamma} - n^\nu e^{\mu\alpha\beta\gamma}) u_\alpha E_\beta \omega_\gamma \right],
\]  

(5.13)

where we have used following moment identities

\[
\int d^4 p \, p_\mu p_\lambda Y = u_\nu u_\lambda \int d^4 p \, (u \cdot p)^2 Y + \frac{1}{3} \Delta^\mu \nu \int d^4 p \, p^2 Y,
\]  

(5.14)

\[
\int d^4 p \, p_\mu p_\nu p_\rho p_\lambda Y = u_\mu u_\nu u_\beta u_\lambda \int d^4 p \, (u \cdot p)^4 Y
\]

\[ + \frac{1}{15} (\Delta^\mu \nu \Delta^\beta \lambda + \Delta^\mu \beta \Delta^\nu \lambda + \Delta^\mu \lambda \Delta^\beta \nu) \int d^4 p \, p^4 Y
\]

\[ + \frac{1}{3} (u_\mu u_\nu \Delta^\beta \lambda + u_\beta u_\lambda \Delta^\nu \mu + u_\nu u_\beta \Delta^\mu \lambda + u_\mu u_\lambda \Delta^\beta \nu
\]

\[ + u_\nu u_\beta \Delta^\mu \mu) \int d^4 p (u \cdot p)^2 \bar{p}^2 Y,
\]  

(5.15)

as well as the decomposition (4.8) and (4.9). However, \( T^{(2)_{\text{ee}}} \) has logarithmic ultraviolet divergence that has to be regularized and renormalized. The regularization with a naive momentum cutoff will break Lorentz invariance and destroy the energy momentum conservation. To avoid such a problem, we would apply dimensional regularization. Now let us
We can expand $\kappa$ electromagnetic part of the energy-momentum tensor With dimensional regularization and tensor decomposition as above, we obtain the pure divergence at the limit

$$
\frac{1}{d-1} \left( \Delta_{\mu\nu} \Delta_{\beta\lambda} + \Delta_{\mu\beta} \Delta_{\nu\lambda} + \Delta_{\mu\lambda} \Delta_{\beta\nu} \right) \int d^d p p^4 Y
$$

$$
\frac{1}{d-1} \left( \Delta_{\mu\nu} \Delta_{\beta\lambda} + \Delta_{\mu\beta} \Delta_{\nu\lambda} + \Delta_{\mu\lambda} \Delta_{\beta\nu} \right) \int d^d p p^4 Y
$$

$$
u u_\lambda \Delta_{\beta\mu} + u_\nu u_\beta \Delta_{\mu\lambda} \right) \int d^d p (u \cdot p)^2 p^2 Y,
$$

With dimensional regularization and tensor decomposition as above, we obtain the pure electromagnetic part of the energy-momentum tensor

$$
T^{(2)\mu\nu}_{s,ee} = -\frac{1}{12} \kappa^e_s \left( \frac{1}{4} g^{\mu\nu} F_{\gamma\beta} F_{\gamma\beta} - F^{\mu\nu} F_{\gamma\gamma} \right) + \frac{1}{48\pi^2} \epsilon^{\mu\nu} u_\nu E_2 - \frac{1}{48\pi^2} \Delta^{\mu\nu} (E_2 + 2B_2)
$$

$$
+ \frac{1}{12\pi^2} \left( E^\mu E^\nu + B^\mu B^\nu \right) + \frac{1}{16\pi^2} \left( u_\mu E_2^\gamma + u_\nu E_2^\beta \right) u_\alpha E_2 B_2
$$

$$
+ \frac{1}{16\pi^2} \left( u_\mu E_2^\alpha E_2^\gamma - u_\nu E_2^\alpha E_2^\beta \right) u_\alpha E_2 B_2,
$$

(5.18)

where $\kappa^e_s$ is given by

$$
\kappa^e_s = \frac{4\pi^{\frac{3}{2}}}{\Gamma \left( \frac{3}{2} \right)} \frac{T^e}{(2\pi)^{3-\epsilon}} \int_0^\infty dy y^\epsilon \left[ \frac{1}{e^{(y - \bar{\mu}_s)} + 1} + \frac{1}{e^{(y + \bar{\mu}_s)} + 1} - 1 \right].
$$

(5.19)

We can expand $\kappa^e_s$ around $\epsilon = 0$ as

$$
\kappa^e_s = -\frac{1}{\pi^2} \left[ \frac{1}{\epsilon} + \ln 2 + \frac{1}{2} \ln \pi + \frac{1}{2} \psi \left( \frac{3}{2} \right) + \ln T + \tilde{\kappa}_s \right],
$$

(5.20)

where $\psi(x)$ is the digamma function and

$$
\tilde{\kappa}_s = \int_0^\infty dy \ln y \frac{d}{dy} \left[ \frac{1}{e^{(y - \bar{\mu}_s)} + 1} + \frac{1}{e^{(y + \bar{\mu}_s)} + 1} \right].
$$

(5.21)

Obviously, we can see in (5.21) that the integral in (5.20) contains logarithmic ultraviolet divergence at the limit $\epsilon \to 0$. The coefficient $\kappa^e_s$ or $\tilde{\kappa}_s$ is an even function of $\bar{\mu}_s$. It is also easy to verify

$$
\epsilon \kappa^e_s \Big|_{\epsilon \to 0} = -\frac{1}{\pi^2}, \quad \frac{d\kappa^e_s(\bar{\mu}_s)}{d\bar{\mu}_s} \Bigg|_{\epsilon \to 0} = -\frac{T}{\pi^2}; \quad \frac{d\kappa^e_s(\bar{\mu}_s)}{d\bar{\mu}_s} \Bigg|_{\epsilon \to 0} = TC_s(\bar{\mu}_s).
$$

(5.22)
The total stress tensor by taking a sum over the contributions from the left-handed and right-handed fermions is given by

\[
T_{\text{ee}}^{(2)\mu\nu} = \frac{1}{2\xi_5} \left[ 3u^\mu u^\nu (\omega^2 + \varepsilon^2) - \Delta_{\mu\nu} (\omega^2 + \varepsilon^2) - 2(u^\mu \varepsilon^{\rho\alpha\beta\gamma} + u^\nu \varepsilon^{\rho\alpha\beta\gamma}) u_\alpha \varepsilon_\beta \omega_\gamma \\
-2(u^\mu \varepsilon^{\rho\alpha\beta\gamma} - u^\nu \varepsilon^{\rho\alpha\beta\gamma}) u_\alpha \varepsilon_\beta \omega_\gamma \right],
\]

(5.23)

\[
T_{\text{ee}}^{(2)\mu\nu} = -\frac{1}{2} \xi B_5 \left[ u^\mu u^\nu (\omega \cdot B + \varepsilon \cdot E) - (\omega^\mu B^\nu + E^\mu \varepsilon^\nu) - (u^\mu \varepsilon^{\rho\alpha\beta\gamma} + u^\nu \varepsilon^{\rho\alpha\beta\gamma}) u_\alpha E_\beta \omega_\gamma \\
-2(u^\mu \varepsilon^{\rho\alpha\beta\gamma} - u^\nu \varepsilon^{\rho\alpha\beta\gamma}) u_\alpha E_\beta \omega_\gamma \right],
\]

(5.24)

\[
T_{\text{ee}}^{(2)\mu\nu} = -\frac{1}{6} \kappa^e \left[ \frac{1}{4} g^{\mu\nu} F_{\gamma\beta} F_{\gamma\beta} - F_{\gamma\mu} F_{\gamma}^\nu \right] + \frac{1}{24\pi^2} \left[ u^\mu u^\nu E^2 - \Delta_{\mu\nu} (E^2 + 2B^2) \right] + 4 \left( E^\mu E^\nu + B^\mu B^\nu \right) + 3 \left( u^\mu \varepsilon^{\rho\alpha\beta\gamma} + u^\nu \varepsilon^{\rho\alpha\beta\gamma} \right) u_\alpha E_\beta B_\gamma \\
+3 \left( u^\mu \varepsilon^{\rho\alpha\beta\gamma} - u^\nu \varepsilon^{\rho\alpha\beta\gamma} \right) u_\alpha E_\beta B_\gamma \right],
\]

(5.25)

where \( \kappa^e = (\kappa^e_{+1} + \kappa^e_{-1})/2 \).

We see that energy density and pressure are modified in the presence of the vorticity and electromagnetic field at the second order, which are not the case at the first order. Similar to the first order result given in Eq.(5.6), there are also antisymmetric contributions to the energy-momentum tensor at the second order. It is straightforward to verify with Eqs.(5.5-5.6,5.23-5.25) that the trace of the total energy-momentum tensor vanishes

\[
g_{\mu\nu} T^{\mu\nu} = 0,
\]

(5.26)

free of trace anomaly. Note that in taking the trace in \( T_{\text{ee}}^{(2)\mu\nu} \) we use \( g_{\mu\nu}g^{\mu\nu} = 4 - \epsilon \). The trace anomaly does not arise here because the electromagnetic field in our work is only a classical background field, which keeps scale invariance. We note that the divergent part in \( T_{\text{ee}}^{(2)\mu\nu} \) is proportional to the stress tensor of the electromagnetic field. It is remarkable that this divergent term \( \sim 1/\epsilon \) can be exactly canceled by the contribution from the quantum correction of the electromagnetic field [76]. After including this quantum correction, we arrive at the final renormalized result

\[
T_{\text{ee}}^{(2)\mu\nu} = \frac{1}{6\pi^2} \left( \hat{\kappa} + \ln \frac{T}{\Lambda} \right) \left( \frac{1}{4} g^{\mu\nu} F_{\gamma\beta} F_{\gamma\beta} - F_{\gamma\mu} F_{\gamma}^\nu \right) + \frac{1}{24\pi^2} \left[ u^\mu u^\nu E^2 - \Delta_{\mu\nu} (E^2 + 2B^2) \right] + 4 \left( E^\mu E^\nu + B^\mu B^\nu \right) + 3 \left( u^\mu \varepsilon^{\rho\alpha\beta\gamma} + u^\nu \varepsilon^{\rho\alpha\beta\gamma} \right) u_\alpha E_\beta B_\gamma \\
+3 \left( u^\mu \varepsilon^{\rho\alpha\beta\gamma} - u^\nu \varepsilon^{\rho\alpha\beta\gamma} \right) u_\alpha E_\beta B_\gamma \right],
\]

(5.27)

where \( \hat{\kappa} = \hat{\kappa}_{+1} + \hat{\kappa}_{-1} \) and \( \Lambda \) is a renormalization scale in quantizing the electromagnetic field. After removing the divergence in \( T_{\text{ee}}^{(2)\mu\nu} \), we can safely calculate the trace of the energy-momentum tensor in 4 dimensions and obtain the trace anomaly

\[
g_{\mu\nu} T_{\text{ee}}^{(2)\mu\nu} = \frac{1}{24\pi^2} F_{\mu\nu} F^{\mu\nu}.
\]

(5.28)

In hydrodynamics, we usually express the stress tensor in the Landau frame. In Appendix A, the symmetric part of the stress tensor is written in the Landau frame.
6 The conservation laws

With $j^\mu$ in (4.12),(4.13) and (4.25), $j^\mu_5$ in (4.14),(4.15) and (4.26), and $T^{\mu\nu}$ in (5.5), (5.6), (5.23),(5.24) and (5.27), we can check the conservation laws for these quantities. In doing so, we must restrict ourselves to the specific system in constant and homogeneous electromagnetic field with constraint conditions (3.16),(3.17),(3.18), and (3.22) or (4.30). Here we will not present the detailed derivation, but give the necessary identities in performing the calculation. These identities hold only under the specific conditions that are imposed in this paper,

\[
\partial_\mu T^{\mu\nu} = 0, \quad u \cdot \partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu \frac{1}{T} = -u \cdot \partial_\mu T = \frac{\varepsilon_\mu}{T}.
\]

With the help of these identities, we can arrive at following conservation laws,

\[
\partial^\mu j_\mu = 0, \quad \partial^\mu j^{\mu}_5 = -\frac{1}{2\pi^2} E \cdot B, \quad \partial^\mu T^{\mu\nu} = F^{\nu\mu} j^\mu.
\]

We note that the second-order correction to the axial current does not contribute to the chiral anomaly as it should be. We find that the term proportional to $\ln T/\Lambda$ in Eq. (5.27) is essential to conserve the energy-momentum when the vorticity is present. We can decompose the energy-momentum tensor into a symmetric and an antisymmetric part,

\[
T^{\mu\nu} = T_S^{\mu\nu} + T_A^{\mu\nu}.
\]

They are given by,

\[
T_S^{\mu\nu} = \rho u^\mu u^\nu - \frac{1}{3} \rho \Delta^{\mu\nu} + \eta_5 (u^\mu \omega^\nu + u^\nu \omega^\mu) + \frac{\xi}{2} (u^\mu B^\nu + u^\nu B^\mu)
\]

\[
- \frac{1}{2} \xi_5 \left[ 3u^\mu u^\nu (\omega^2 + \varepsilon^2) - \Delta^{\mu\nu} (\omega^2 + \varepsilon^2) - 2(u^\mu \epsilon^{\mu\alpha\beta\gamma} + u^\nu \epsilon^{\nu\alpha\beta\gamma}) u_\alpha \varepsilon_\beta \omega_{\gamma} \right]
\]

\[
- \frac{1}{2} \xi_B \left[ u^\mu u^\nu (\omega \cdot B + \varepsilon \cdot E) - (\omega^\mu B^\nu + E^\mu \varepsilon^\nu) - (u^\mu \epsilon^{\mu\alpha\beta\gamma} + u^\nu \epsilon^{\nu\alpha\beta\gamma}) u_\alpha \varepsilon_\beta \omega_{\gamma} \right]
\]

\[
+ \kappa_1 \epsilon^{\mu\nu} B^2 + \kappa_2 \epsilon^{\mu\nu} \Delta^{\mu\nu} B^2 + \kappa_3 (E^\mu E^\nu + B^\mu B^\nu)
\]

\[
+ \kappa_4 \left( u^\mu \epsilon^{\mu\alpha\beta\gamma} + u^\nu \epsilon^{\nu\alpha\beta\gamma} \right) u_\alpha \varepsilon_\beta B_{\gamma},
\]

\[
T_A^{\mu\nu} = -\frac{n_5}{2} \left( u^\mu \omega^\nu - u^\nu \omega^\mu + \epsilon^{\mu\alpha\beta\gamma} u_\alpha \varepsilon_\beta \right) - \frac{\xi}{2} \epsilon^{\mu\alpha\beta\gamma} u_\alpha \varepsilon_\beta \omega_{\gamma}
\]

\[
+ \xi_B (u^\mu \epsilon^{\mu\alpha\beta\gamma} - u^\nu \epsilon^{\nu\alpha\beta\gamma}) u_\alpha \varepsilon_\beta \omega_{\gamma} + \frac{1}{8\pi^2} \left( u^\mu \epsilon^{\mu\alpha\beta\gamma} - u^\nu \epsilon^{\nu\alpha\beta\gamma} \right) u_\alpha \varepsilon_\beta B_{\gamma},
\]

\[
T^{\mu\nu} = T_S^{\mu\nu} + T_A^{\mu\nu}.
\]
where we have expressed the energy-momentum tensor in terms of $E_\mu$ and $B_\mu$. The coefficients are defined as

$$
\kappa_1^E = -\frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} - \frac{1}{2} \right), \quad \kappa_2^E = -\frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} \right),
$$

$$
\kappa_3^E = \frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} - \frac{1}{2} \right), \quad \kappa_4^E = \frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} \right),
$$

$$
\kappa_1^B = -\frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} - \frac{1}{2} \right), \quad \kappa_2^B = \frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} - \frac{1}{2} \right),
$$

$$
\kappa_3^B = \frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} - \frac{1}{2} \right), \quad \kappa_4^B = -\frac{1}{12\pi^2} \left( \dot{\kappa} + \ln \frac{T}{\Lambda} - \frac{3}{4} \right).
$$

(6.10)

We can verify the following conservation equations,

$$
\partial_\mu T^\mu_\nu = F^\nu_\mu j_\mu, \quad \partial_\mu T^\mu_\nu_A = 0.
$$

(6.11)

### 7 Summary

Within the Wigner function formalism and based on the covariant perturbation method, we have derived the vector charge and chiral charge currents as well as the stress tensor in a chiral plasma near equilibrium up to the second order under the constant and homogeneous vorticity and electromagnetic field. We present all possible second order contributions in quadratic forms of the vorticity and electromagnetic field. These contributions include coupling terms of electromagnetic-field-electromagnetic-field (ee), vorticity-vorticity (vv) and vorticity-electromagnetic-field (ve). The ‘ve’ terms can modify the charge density, while only the ‘ee’ and ‘vv’ terms can modify the chiral charge density. We find that the electromagnetic field can induce the vector and axial Hall current in the form $
\epsilon^{\mu\nu\rho\sigma} u_\nu E_\rho \omega_\sigma$. There is also a Hall term $\epsilon^{\mu\nu\rho\sigma} u_\nu E_\rho \omega_\sigma$ in the charge current. However there is no charge or axial Hall-like current in the form $\epsilon^{\mu\nu\rho\sigma} u_\nu \varepsilon_\rho \omega_\sigma$ from the vorticity field alone. For the energy-momentum tensor at the second order, we find that the vorticity and electromagnetic field contribute to the energy density and pressure. The conservation laws as well as chiral and trace anomaly can be verified with our second-order solution. All coefficients we obtain in this work can be directly applied to the anomalous hydrodynamics as inputs. We restrict ourselves to the constant electromagnetic and or vorticity field in this work, it is also possible we go beyond this condition in the future.

### A Results in Landau frame

In relativistic hydrodynamics, one has a freedom to choose any frame characterized by a different fluid velocity. The Landau frame is the one in which the fluid velocity satisfies $u_\mu T^{\mu\nu} = \rho u^\nu$. Since the energy momentum tensor in Sec.5 has an anti-symmetric part in the first and second order contribution, in this section, we will rewrite the symmetric part of the stress tensor up to the second order in the Landau frame. Let us introduce the fluid
velocity \( U^\mu \) in the Landau frame given by

\[
U^\mu = u^\mu + \frac{n_5}{\rho + P} \omega^\mu + \frac{\xi}{2(\rho + P)} B^\mu - \left[ \frac{n_5^2}{2(\rho + P)^2} \omega^\mu + \frac{\xi^2}{8(\rho + P)^2} B^2 + \frac{n_5 \xi}{2(\rho + P)^2} \omega \cdot B \right] u^\mu + \frac{\xi_5}{\rho + P} \epsilon^{\mu \alpha \beta \gamma} u_{\alpha} \varepsilon_{\beta \alpha \gamma} + \frac{\xi B_5}{2(\rho + P)} \epsilon^{\mu \alpha \beta \gamma} u_{\alpha} E_{\beta \gamma} + \frac{\kappa_4}{\rho + P} \epsilon^{\mu \alpha \beta \gamma} u_{\alpha} E_{\beta \gamma}, \quad (A.1)
\]

It is easy to verify that \( U^2 = 1 \) up to the second order. From this relation, we can also express \( u^\mu \) in terms of \( U^\mu \),

\[
u^\mu = U^\mu - \frac{n_5}{\rho + P} \omega^\mu_U - \frac{\xi}{2(\rho + P)} B^\mu_U - \left[ \frac{n_5^2}{2(\rho + P)^2} \omega^\mu_U + \frac{\xi^2}{8(\rho + P)^2} B^2_U + \frac{n_5 \xi}{2(\rho + P)^2} \omega_U \cdot B_U \right] U^\mu + \frac{\xi_5}{\rho + P} \epsilon^{\mu \alpha \beta \gamma} U_{\alpha} \varepsilon_{\beta \alpha \gamma} - \left[ \frac{\xi B_5}{2(\rho + P)} + \frac{n_5 \xi}{(\rho + P)^2} \right] \epsilon^{\mu \alpha \beta \gamma} U_{\alpha} E_{\beta \gamma} U_{\gamma} + \frac{\kappa_4}{4(\rho + P)^2} \epsilon^{\mu \alpha \beta \gamma} U_{\alpha} E_{\beta \gamma} U_{\gamma}, \quad (A.2)
\]

where \( \omega^\mu_U = T^{\mu \nu} U_{\nu}, \varepsilon^\mu_U = \tilde{T}^{\mu \nu} U_{\nu}, B^\mu_U = \tilde{F}^{\mu \nu} U_{\nu}, \) and \( E^\mu_U = \tilde{F}^{\mu \nu} U_{\nu} \) are counterparts of \( \omega^\mu, \varepsilon^\mu, B^\mu, \) and \( E^\mu \) in the Landau frame, respectively. Up to the second order, the corresponding relations between two groups of quantities are given by

\[
\omega^\mu = \omega^\mu_U + \left( \frac{n_5}{\rho + P} \omega_U + \frac{\xi}{2(\rho + P)} \omega_U \cdot B_U \right) U^\mu + \frac{n_5}{\rho + P} \epsilon^{\mu \alpha \beta \gamma} U_{\alpha} \varepsilon_{\beta \alpha \gamma} + \frac{\xi}{2(\rho + P)} \mu_{\alpha \beta} B_{U \alpha} U_{\alpha} \varepsilon_{\beta}, \quad (A.3)
\]

\[
\varepsilon^\mu = \varepsilon^\mu_U + \left( \frac{n_5}{\rho + P} \varepsilon_U \cdot \omega_U + \frac{\xi}{2(\rho + P)} \varepsilon_U \cdot B_U \right) U^\mu - \frac{\xi}{2(\rho + P)} \epsilon^{\mu \alpha \beta \gamma} B_{U \alpha} U_{\beta} U_{\gamma}, \quad (A.4)
\]

\[
B^\mu = B^\mu_U + \left( \frac{n_5}{\rho + P} \omega_U \cdot B_U + \frac{\xi}{2(\rho + P)} B^2_U \right) U^\mu + \frac{n_5}{\rho + P} \epsilon^{\mu \alpha \beta \gamma} U_{\alpha} E_{\beta \gamma} + \frac{\xi}{2(\rho + P)} \epsilon^{\mu \alpha \beta \gamma} B_{U \alpha} U_{\beta} E_{\gamma}, \quad (A.5)
\]

\[
E^\mu = E^\mu_U + \left( \frac{n_5}{\rho + P} E_U \cdot \omega_U + \frac{\xi}{2(\rho + P)} E_U \cdot B_U \right) U^\mu - \frac{n_5}{\rho + P} \epsilon^{\mu \alpha \beta \gamma} \omega_{U \alpha} U_{\beta}, \quad (A.6)
\]
Hence the difference between them only arises at least at the second order. With these equations, we can obtain the symmetric stress tensor in the Landau frame

\[
T^{\mu\nu} = \left\{ \rho + \left( \frac{n_s^2}{\rho + P} - \frac{3}{2} \xi_n^2 \right) \omega_U^2 + \left[ \frac{\xi_n^2}{4(\rho + P)} + \kappa_1^B \right] B_U^2 \\
+ \left( \frac{n_s^2}{\rho + P} - \frac{\xi_n^2}{2} \varepsilon_U^2 \right) \omega_U \cdot B_U - \frac{3}{2} \xi_n^2 \varepsilon_U^2 - \frac{1}{2} \xi_n^2 \varepsilon_U \cdot E_U + \kappa_1^E \varepsilon_U^2 \right\} U^\mu U^\nu \\
- \left[ P - \frac{1}{2} \xi_n^2 \left( \omega_U^2 + 2 \varepsilon_U^2 \right) - \kappa_2^E \varepsilon_U^2 - \kappa_2^B B_U^2 \right] \Delta_{U}^{\mu\nu} \\
+ \left( E_U^\mu E_U^\nu + B_U^\mu B_U^\nu \right) \kappa_3 + \frac{1}{2} \xi_n^2 \left( \omega_U^2 + 2 \varepsilon_U^2 \right) \right\} U^\mu U^\nu \\
- \frac{n_s^2}{\rho + P} \omega_U^\mu \omega_U^\nu - \frac{\xi_n^2}{4(\rho + P)} B_U^\mu B_U^\nu - \frac{n_s^2}{2(\rho + P)} \left( \omega_U^\mu B_U^\nu + \omega_U^\nu B_U^\mu \right), \tag{A.7}
\]

where \( \Delta_{U}^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu \). The vector current in the Landau frame is given by

\[
j^\mu = n U^\mu + \left( \xi - \frac{mn_s}{\rho + P} \right) \omega_U^\mu + \left( \xi_B - \frac{mn_s}{2(\rho + P)} \right) B_U^\mu \\
- \left[ \frac{mn_s^2}{2(\rho + P)^2} - \frac{\xi n_s}{\rho + P} \right] \omega_U^2 U^\mu + \left[ \frac{\xi_B n_s}{2(\rho + P)} - \frac{n_s^2}{8(\rho + P)^2} \right] B_U^2 U^\mu \\
- \left[ \frac{n_s^2}{2(\rho + P)^2} - \frac{\xi s N}{\rho + P} \right] \omega_U \cdot B_U U^\mu \\
- \left[ \frac{\xi_B n_s}{2(\rho + P)} + \frac{n_s^2}{\rho + P} \right] \omega_U \cdot B_U U^\mu \\
- \left[ \frac{\xi_B n_s}{2(\rho + P)} + \frac{n_s^2}{4(\rho + P)^2} + \frac{\xi n_s}{\rho + P} \right] \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\} \right\}

\[
- C \frac{12}{4\pi^2} \left[ U_{\mu} \left( E_{\mu}^2 + B_{\mu}^2 \right) + 2 \epsilon_{\mu\rho\sigma} U^{\nu} E_{\mu}^\rho B_{\nu}^\sigma \right]. \tag{A.8}
\]
The axial current in the Landau frame is given by

\[ j^\mu_5 = n_5 U^\mu + \left( \xi_5 - \frac{n_5^2}{\rho + P} \right) \omega_5^\mu + \left( \xi_{B5} - \frac{n_5 \xi}{2(\rho + P)} \right) B_5^\mu 
- \left[ \frac{n_5^3}{2(\rho + P)^2} - \frac{\xi_5 n_5}{\rho + P} \right] \omega_5^2 U^\mu + \left[ \frac{\xi_{B5} \xi}{2(\rho + P)} - \frac{n_5^2 \xi^2}{8(\rho + P)^2} \right] B_5^2 U^\mu 
- \left[ \frac{n_5^3 \xi}{2(\rho + P)^2} - \frac{\xi_5 \xi_5 n_5}{\rho + P} \right] \omega_5 \cdot B U^\mu 
- \left[ \frac{\xi_5 \xi}{2(\rho + P)^2} - \frac{n_5^2 \xi}{2(\rho + P)^2} \right] \epsilon^{\alpha \beta \gamma} U_\alpha \epsilon_{\beta \gamma} \omega_5 U^\mu \right]

\[ - \xi_B \left( \epsilon_5^2 + \omega_5^2 \right) U^\mu - \frac{C_5}{12 \pi^2} \left[ (E_5^2 + B_5^2) U^\mu + 2 \epsilon^{\mu \nu \rho \sigma} U_\nu E_{\rho \sigma} U^\mu B_{\beta \gamma} \right]. \quad (A.9) \]

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References

[1] A. Vilenkin, Phys. Rev. D 22, 3080 (1980).
[2] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A 803, 227 (2008).
[3] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D 78, 074033 (2008).
[4] A. Vilenkin, Phys. Lett. 80B, 150 (1978).
[5] D. Kharzeev and A. Zhitnitsky, Nucl. Phys. A 797 (2007) 67
[6] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, JHEP 0901, 055 (2009)
[7] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, JHEP 1101, 094 (2011)
[8] D. T. Son and A. R. Zhitnitsky, Phys. Rev. D 70 (2004) 074018
[9] M. A. Metlitski and A. R. Zhitnitsky, Phys. Rev. D 72 (2005) 045011
[10] J. H. Gao, Z. T. Liang, S. Pu, Q. Wang and X. N. Wang, Phys. Rev. Lett. 109, 232301 (2012)
[11] G. M. Newman, JHEP 0601 (2006) 158
[12] H. U. Yee, JHEP 0911 (2009) 085
[13] A. Rebhan, A. Schmitt and S. A. Stricker, JHEP 1001 (2010) 026
[14] A. Gorsky, P. N. Kopnin and A. V. Zayakin, Phys. Rev. D 83 (2011) 014023
[15] A. Gynther, K. Landsteiner, F. Pena-Benitez and A. Rebhan, JHEP 1102 (2011) 110
[16] C. Hoyos, T. Nishioka and A. O'Bannon, JHEP 1110 (2011) 084
I. Amado, K. Landsteiner and F. Pena-Benitez, JHEP 1105 (2011) 081

V. P. Nair, R. Ray and S. Roy, Phys. Rev. D 86 (2012) 025012

T. Kalaydzhyan and I. Kirsch, Phys. Rev. Lett. 106 (2011) 211601

S. Lin and H. U. Yee, Phys. Rev. D 88 (2013) no.2, 025030

D. T. Son and P. Surowka, Phys. Rev. Lett. 103, 191601 (2009).

A. V. Sadofyev and M. V. Isachenkov, Phys. Lett. B 697, 404 (2011).

S. Pu, J. H. Gao and Q. Wang, Phys. Rev. D 83, 094017 (2011).

D. E. Kharzeev and H. -U. Yee, Phys. Rev. D 84, 045025 (2011).

D. E. Kharzeev and H. J. Warringa, Phys. Rev. D 80 (2009) 034028

K. Fukushima, D. E. Kharzeev and H. J. Warringa, Nucl. Phys. A 836 (2010) 311

M. Asakawa, A. Majumder and B. Muller, Phys. Rev. C 81 (2010) 064912

K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. Lett. 104 (2010) 212001

K. Fukushima and M. Ruggieri, Phys. Rev. D 82 (2010) 054001

K. Landsteiner, E. Megias and F. Pena-Benitez, Phys. Rev. Lett. 107, 021601 (2011)

D. Hou, H. Liu and H. c. Ren, JHEP 1105 (2011) 046

D. F. Hou, H. Liu and H. c. Ren, Phys. Rev. D 86 (2012) 121703

S. Lin and L. Yang, Phys. Rev. D 98 (2018) no.11, 114022

B. Feng, D. F. Hou and H. C. Ren, Phys. Rev. D 99 (2019) no.3, 036010

M. A. Stephanov and Y. Yin, Phys. Rev. Lett. 109 (2012) 162001

D. T. Son and N. Yamamoto, Phys. Rev. D 87 (2013) 085016

J. W. Chen, S. Pu, Q. Wang and X. N. Wang, Phys. Rev. Lett. 110 (2013) no.26, 262301

C. Manuel and J. M. Torres-Rincon, Phys. Rev. D 89 (2014) no.9, 096002

J. Y. Chen, D. T. Son, M. A. Stephanov, H. U. Yee and Y. Yin, Phys. Rev. Lett. 113 (2014) no.18, 182302

J. Y. Chen, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. 115 (2015) no.2, 021601

Y. Hidaka, S. Pu and D. L. Yang, Phys. Rev. D 95 (2017) no.9, 091901

N. Mueller and R. Venugopalan, Phys. Rev. D 97 (2018) no.5, 051901

A. Huang, S. Shi, Y. Jiang, J. Liao and P. Zhuang, Phys. Rev. D 98 (2018) no.3, 036010

Y. Hidaka and D. L. Yang, Phys. Rev. D 98 (2018) no.1, 016012

J. H. Gao, Z. T. Liang, Q. Wang and X. N. Wang, Phys. Rev. D 98 (2018) no.3, 036019

J. H. Gao, J. Y. Pang and Q. Wang, Phys. Rev. D 100 (2019) no.1, 016008

Y. C. Liu, L. L. Gao, K. Mameda and X. G. Huang, Phys. Rev. D 99 (2019) no.8, 085014

A. Bzdak and V. Skokov, Phys. Lett. B 710 (2012) 171

W. T. Deng and X. G. Huang, Phys. Rev. C 85 (2012) 044907

J. Bloczynski, X. G. Huang, X. Zhang and J. Liao, Phys. Lett. B 718 (2013) 1529
[51] Z. T. Liang and X. N. Wang, Phys. Rev. Lett. 94 (2005) 102301 Erratum: [Phys. Rev. Lett. 96 (2006) 039901]
[52] J. H. Gao, S. W. Chen, W. t. Deng, Z. T. Liang, Q. Wang and X. N. Wang, Phys. Rev. C 77 (2008) 044902
[53] F. Becattini, F. Piccinini and J. Rizzo, Phys. Rev. C 77 (2008) 024906
[54] L. P. Csernai, V. K. Magas and D. J. Wang, Phys. Rev. C 87 (2013) no.3, 034906
[55] Y. Jiang, Z. W. Lin and J. Liao, Phys. Rev. C 94 (2016) no.4, 044910 Erratum: [Phys. Rev. C 95 (2017) no.4, 049904]
[56] W. T. Deng and X. G. Huang, Phys. Rev. C 93 (2016) no.6, 064907
[57] L. G. Pang, H. Petersen, Q. Wang and X. N. Wang, Phys. Rev. Lett. 117 (2016) no.19, 192301
[58] A. Jimenez-Alba and H. U. Yee, Phys. Rev. D 92 (2015) no.1, 014023
[59] D. Satow, Phys. Rev. D 90 (2014) no.3, 034018
[60] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy and P. O. Sukhachov, Phys. Rev. B 95 (2017) no.20, 205141
[61] E. V. Gorbar, D. O. Rybalka and I. A. Shovkovy, Phys. Rev. D 95 (2017) no.9, 096010
[62] N. Abbasi, F. Taghina va and O. Tavakol, JHEP 1903 (2019) 051
[63] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, JHEP 1209 (2012) 046
[64] S. Bhattacharyya, J. R. David and S. Thakur, JHEP 1401 (2014) 010
[65] E. Megias and M. Valle, JHEP 1411 (2014) 005
[66] Y. Bu and S. Lin, arXiv:1912.11277.
[67] U. W. Heinz, Phys. Rev. Lett. 51 (1983) 351.
[68] H. T. Elze, M. Gyulassy and D. Vasak, Nucl. Phys. B 276 (1986) 706.
[69] D. Vasak, M. Gyulassy and H. T. Elze, Annals Phys.(N.Y.) 173 (1987) 462.
[70] P. Zhuang and U. W. Heinz, Annals Phys. 245 (1996) 311.
[71] J. h. Gao and Q. Wang, Phys. Lett. B 749 (2015) 542
[72] J. h. Gao, S. Pu and Q. Wang, Phys. Rev. D 96 (2017) no.1, 016002
[73] J. H. Gao, Z. T. Liang and Q. Wang, arXiv:1910.11060
[74] X. L. Sheng, D. H. Rischke, D. Vasak and Q. Wang, Eur. Phys. J. A 54 (2018) 21
[75] X. L. Sheng, R. H. Fang, Q. Wang and D. H. Rischke, Phys. Rev. D 99 (2019) no. 5, 056004
[76] M. E. Peskin and D. V. Schroeder, “An Introduction to quantum field theory”