The Beta-Mixture Shrinkage Prior for Sparse Covariances with Posterior Minimax Rates*

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Abstract

Statistical inference for sparse covariance matrices is crucial to reveal dependence structure of large multivariate data sets, but lacks scalable and theoretically supported Bayesian methods. In this paper, we propose beta-mixture shrinkage prior, computationally more efficient than the spike and slab prior, for sparse covariance matrices and establish its minimax optimality in high-dimensional settings. The proposed prior consists of beta-mixture shrinkage and gamma priors for off-diagonal and diagonal entries, respectively. To ensure positive definiteness of the resulting covariance matrix, we further restrict the support of the prior to a subspace of positive definite matrices. We obtain the posterior convergence rate of the induced posterior under the Frobenius norm and establish a minimax lower bound for sparse covariance matrices. The class of sparse covariance matrices for the minimax lower bound considered in this paper is controlled by the number of nonzero off-diagonal elements and has more intuitive appeal than those appeared in the literature. The

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obtained posterior convergence rate coincides with the minimax lower bound unless the true covariance matrix is extremely sparse. In the simulation study, we show that the proposed method is computationally more efficient than competitors, while achieving comparable performance. Advantages of the shrinkage prior are demonstrated based on two real data sets.

1 Introduction

Suppose $X_1, \ldots, X_n$ are independent $p$-dimensional random vectors from $N_p(0, \Sigma)$, the $p$-dimensional normal distribution with mean 0 $\in \mathbb{R}^p$ and covariance $\Sigma \in \mathbb{R}^{p\times p}$. The covariance matrix of a random vector is a fundamental parameter that expresses the marginal dependence structure of $X$. It is a basis for many multivariate statistical methods such as principal component analysis, factor analysis, discriminant analysis, and linear regression, to name just a few. In this paper, we consider the Bayesian inference of covariance matrices when the dimension of observations, $p$, tends to infinity as the sample size, $n$, gets larger. We assume that most of the off-diagonal entries of a covariance matrix are zero, i.e., only few pairs of variables have significant marginal dependences. We propose the beta-mixture shrinkage prior for sparse covariance matrix. The proposed methodology is computationally fast and attains the minimax posterior convergence rate under the Frobenius norm when the true covariance is not extremely sparse.

There are vast and rich frequentist literature on high-dimensional sparse covariance estimation. Various thresholding estimators (Bickel and Levina, 2008; Rothman et al., 2009; Cai and Liu, 2011; Cai and Zhou, 2012) and lasso-type procedures (Bien and Tibshirani, 2011) have been proposed for simultaneously learning marginal dependence structures and estimating covariance matrices. Among them, Cai and Liu (2011) proposed an adaptive thresholding estimator and proved that it achieves the minimax convergence rate for sparse covariance matrices by showing that the obtained rate coincides with the minimax lower bound obtained in Cai and Zhou (2012).

For Bayesian inference of sparse covariance, the $G$-inverse Wishart prior (Silva and Ghahramani,
is often used. The normalizing constant of the $G$-inverse Wishart prior is analyti-
cally intractable and needs the Monte Carlo method for its evaluation, which makes the
posterior computation infeasible even when $p$ is moderately large. Khare and Rajaratnam
(2011) introduced a broad class of priors including $G$-inverse Wishart prior as a special
case. They provided a blocked Gibbs sampler to obtain samples from the resulting poste-
rior, but the priors were only applicable to decomposable covariance graph models. Wang
(2015) proposed stochastic search structure learning (SSSL). He placed the spike-and-
slab prior for the off-diagonal elements of $\Sigma$ and put a modified version of the product of
Bernoulli prior on the sparsity structure of the covariance. The modification of the product of
Bernoulli priors allows one to avoid the intense normalizing constant computation,
but the natural interpretation of the Bernoulli prior is lost.

In addition to the computational difficulties of the posterior of the sparse covariance,
the Bayesian literature lacks the asymptotic properties of the posteriors. Lee and Lee
(2018) showed that the inverse Wishart prior achieves the minimax posterior convergence
rate for a unstructured covariance matrix under the spectral norm. The posterior con-
vergence rate under the Frobenius norm was also derived. However, they focused only
on unstructured covariance matrices and used the inverse Wishart prior, which is not
suitable for sparse covariance matrices. Neither Khare and Rajaratnam (2011) nor Wang
(2015) established the asymptotic properties of the posteriors for sparse covariances. Up
to our knowledge, asymptotic properties of the posteriors induced by the priors for sparse
covariance matrices have not been investigated yet.

In this paper, to fill the gap in the literature, we develop a scalable Bayesian infer-
ence for sparse covariance matrices supported by theoretical properties of posteriors. We
propose a continuous shrinkage prior for the sparse covariance matrices. Especially, the
beta-mixture prior and the gamma prior are used for off-diagonal and diagonal entries
of covariance matrices, respectively. To ensure the positive definiteness of the resulting
covariance matrix, we further restrict the prior on a class of positive definite matrices. A
blocked Gibbs sampler is derived to obtain posterior samples.

The advantage of the proposed method are as follows. First, this is the first Bayesian
method for sparse covariance matrices with optimal minimax rate unless the true covari-
ance is extremely sparse. We show the posterior convergence rate of the proposed prior
under the Frobenius norm (Theorem 3.1). We also derive a lower bound of the mini-
max rate for sparse covariance matrices (Theorem 3.2) with restriction only on the total
number of nonzero off-diagonal entries, which differentiates the obtained result from the
results in the literature assuming column-wise sparsity and has more intuitive appeal.
These results show that the obtained posterior convergence rate is the minimax rate ex-
cept extremely sparse cases. Second, the proposed method is computationally efficient.
We compare computational efficiency of the shrinkage prior and the SSSL (Wang, 2015),
and find that the proposed shrinkage prior has almost twice as many effective sample
size as the SSSL. This implies that the posterior sampling of the shrinkage prior exhibits
faster mixing than that of the SSSL.

The paper is organized as follows. In Section 2, we describe the model, prior and
the posterior computation. In Section 3, we present the theoretical results including the
asymptotic minimaxity. The numerical studies and real data analysis are given in Section
4. Concluding remarks are given in Section 5.

2 Beta-Mixture Shrinkage Prior

2.1 Notation
Let $a_n$ and $b_n$, $n = 1, 2, \ldots$ be sequences of positive real numbers. We denote $a_n = O(b_n)$,
or equivalently $a_n \preceq b_n$, if $a_n/b_n \leq C$ for some constant $C > 0$. We denote $a_n \asymp b_n$
if $a_n = O(b_n)$ and $b_n = O(a_n)$. Furthermore, we denote $a_n = o(b_n)$, or equivalently
$a_n \ll b_n$, if $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\mathcal{C}_p$ be the set of all $p \times p$ positive definite
matrices. Let $A = (a_{ij})$ be a $p \times p$ matrix. We denote the minimum and maximum
eigenvalues of $A$ by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. The Frobenius norm of $A$ is defined
by $\|A\|_F = (\sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij}^2)^{1/2}$. 
2.2 Prior for sparse covariances

Suppose we observe \( n \) independent samples \( X_n = (X_1, \ldots, X_n) \) from the \( p \)-dimensional normal distribution:

\[
X_i \mid \Sigma \overset{iid}{\sim} N_p(0, \Sigma), \quad i = 1, \ldots, n, \tag{1}
\]

where \( \Sigma \in \mathcal{C}_p \). We assume that the covariance matrix \( \Sigma \) is \( \ell_0 \)-sparse, i.e., most of off-diagonal entries of \( \Sigma \) are zero. For Bayesian inference on \( \Sigma \), we need to impose a prior distribution on a set of covariance matrices. We first define a prior for \( p \times p \) symmetric matrices and restrict it to the space of positive definite matrices. Let

\[
\pi^u(\sigma_{jk} \mid \rho_{jk}) = N\left(\sigma_{jk} \mid 0, \frac{\rho_{jk}}{1 - \rho_{jk}^2} \tau_1^2\right), \tag{2}
\]

\[
\pi^u(\rho_{jk}) = Beta(\rho_{jk} \mid a, b), \quad 1 \leq j < k \leq p, \tag{3}
\]

\[
\pi^u(\sigma_{jj}) = Gamma(\sigma_{jj} \mid c, d), \quad j = 1, \ldots, p, \tag{4}
\]

for some positive constants \( \tau_1, a, b, c, d \), where \( Beta(a, b) \) is the beta distribution with parameters \( a, b > 0 \) and \( Gamma(c, d) \) is the beta distribution with shape parameter \( c \) and rate parameter \( d \). The prior on symmetric matrix with positive diagonal elements is defined as

\[
\pi^u(\Sigma) = \prod_{1 \leq j < k \leq p} \pi^u(\sigma_{jk} \mid \rho_{jk})\pi^u(\rho_{jk})I(\sigma_{jk} = \sigma_{kj}) \prod_{j=1}^{p} \pi^u(\sigma_{jj}), \tag{5}
\]

where “\( u \)” stands for the unconstrained prior. Note that the marginal prior on \( \sigma_{jk} \) is the half-Cauchy prior if we take \( a = b = 1/2 \), which is one of the most popular shrinkage priors.

Other possible choices for the shrinkage prior of \( \sigma_{jk} \) are the horseshoe prior (Carvalho et al., 2010), the lasso prior (Park and Casella, 2008), the hyperlasso prior (Griffin and Brown, 2011, 2017) and the generalized double pareto (GDP) prior (Armagan et al., 2013). In this paper, we will focus on the half-Cauchy prior for the off-diagonal elements and \( Gamma(1, \frac{1}{\lambda}) \), \( \lambda > 0 \), and derive its theoretical properties.

Now, we propose the shrinkage prior for sparse covariance matrices by restricting \( \pi^u(\Sigma) \)
to the subspace of positive definite matrices:

$$\pi(\Sigma) = \frac{\pi^u(\Sigma) I(\Sigma \in \mathcal{U}(\tau))}{\pi^u(\Sigma \in \mathcal{U}(\tau))},$$

(6)

where

$$\mathcal{U}(\tau) = \left\{ \Sigma \in \mathcal{C}_p : \tau^{-1} \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \tau \right\}$$

for some constant $\tau > 1$. In this paper, we consider $\tau$ as a fixed constant to obtain desired asymptotic properties of posteriors. However, in practice, one can use $\tau = \infty$, which results in $\mathcal{U}(\tau) = \mathcal{C}_p$. Conditions on the hyperparameters will be specified in Section 3, while practical suggestions will be given in Section 4.

### 2.3 Comparison to the SSSL

The shrinkage prior (6) proposed in this paper and the SSSL proposed by Wang (2015) use the gamma and exponential priors for the diagonal entries, $\sigma_{ii}, i = 1, 2, \ldots, p$, of the covariance, respectively. For the off-diagonal elements, $\sigma_{jk}, 1 \leq j \neq k \leq p$, Wang (2015) used the continuous spike and slab prior,

$$\pi^u_W(\sigma_{jk}) = \left(1 - \pi\right)N(\sigma_{jk} | 0, \nu_0^2) + \pi N(\sigma_{jk} | 0, \nu_1^2)$$

for some constants $0 < \nu_0 < \nu_1$ and $\pi \in (0, 1)$, while we use the continuous beta-mixture shrinkage prior (2) and (3).

In the spike and slab prior, the prior inclusion probability, $\pi \in (0, 1)$, reflects the prior belief whether $\sigma_{jk}$ will be zero or not. Similarly to the beta-mixture shrinkage prior (6), Wang (2015) proposed the prior, $\pi^W(\sigma_{jk})$, by restricting $\pi^u_W(\sigma_{jk})$ to the space of positive definite matrices. Note that due to the unknown normalizing constant caused by the positive definiteness constraint, $\pi \in (0, 1)$ is no longer the prior inclusion probability of the resulting prior $\pi^W(\sigma_{jk})$.

The main advantages of the beta-mixture prior over the SSSL are the theoretical guarantee and computational efficiency. The proposed prior (6) achieves the minimax posterior convergence rate for sparse covariances under the Frobenius norm, which will
be rigorously stated in Section 3. On the other hand, asymptotic properties of posteriors based on the SSSL have not been investigated yet. Furthermore, based on the simulation studies in Section 4, we found that the finite sample performance of the proposed prior is comparable to that of the SSSL while achieving almost twice as many effective sample size.

2.4 Blocked Gibbs sampler

We now provide a posterior sampling algorithm for our prior described in (6). The algorithm is based on the blocked Gibbs sampler proposed by Wang (2015). To describe the algorithm, as in Proposition 2 of Wang (2015), we consider the following partition of $\Sigma$, $S = X_n^T X_n$ and $V = (v^2_{jk})$, $v^2_{jk} = \rho_{jk}\tau_1^2/(1 - \rho_{jk})$ for $j < k$ and $v^2_{jk} = 0$ for $j = k$:

$$
\Sigma = \begin{pmatrix}
\Sigma_{11} & \sigma_{12} \\
\sigma_{12}^T & \sigma_{22}
\end{pmatrix}, \quad S = \begin{pmatrix}
S_{11} & s_{12} \\
s_{12}^T & s_{22}
\end{pmatrix}, \quad V = \begin{pmatrix}
V_{11} & v_{12} \\
v_{12}^T & 0
\end{pmatrix},
$$

where $\Sigma_{11}, S_{11}, V_{11} \in \mathbb{C}_{p-1}$, $\sigma_{12}, s_{12}, v_{12} \in \mathbb{R}^{(p-1)\times 1}$ and $\sigma_{22}, s_{22} > 0$, and the change of variables:

$$(\sigma_{12}, \sigma_{22}) \to (u = \sigma_{12}, v = \sigma_{22} - \sigma_{12}^T \Sigma_{11}^{-1} \sigma_{12}).$$

The posterior samples then are generated by iterating the following steps (for details, see the Appendix D):

- For $u$,

$$u \mid \text{others} \sim N_{p-1} \left( \{ B + \text{diag}(v_{12}^{-1}) \}^{-1} w, \{ B + \text{diag}(v_{12}^{-1}) \}^{-1} \right),$$

where $B = \Sigma_{11}^{-1} S_{11} \Sigma_{11}^{-1} v^{-1} + \lambda \Sigma_{11}^{-1}$ and $w = \Sigma_{11}^{-1} s_{12} v^{-1}$.

- For $v$,

$$v \mid \text{others} \sim GIG \left( 1 - n/2, \lambda, u^T \Sigma_{11}^{-1} S_{11} \Sigma_{11}^{-1} u - 2 s_{12}^T \Sigma_{11}^{-1} u + s_{22} \right),$$

where $GIG(q, a, b)$ is the generalized inverse Gaussian distribution with the probability density function $f(x) \propto x^{q-1} e^{-(ax+b)/2} I(x > 0)$. 7
• For $\rho_{jk} = 1 - 1/(1 + \phi_{jk})$,

\[
\psi_{jk} \mid \text{others} \sim \text{Gamma} (a + b, \phi_{jk} + 1),
\]
\[
\phi_{jk} \mid \text{others} \sim \text{GIG} (a - 1/2, 2\psi_{jk}, \sigma_{jk}^2/\tau_1^2),
\]

where $\text{Gamma}(a, b)$ is the Gamma distribution the shape parameter $a$ and the rate parameter $b$.

3 Posterior convergence rate

In this section, we show that the beta-mixture shrinkage prior achieves the minimax rate under the Frobenius norm when $p = O(s_0)$ where $s_0$ is an upper bound for nonzero off-diagonal elements of the covariance matrix. Let $\Sigma_0$ be the true covariance matrix. For a given integer $0 < s_0 < p(p - 1)$ and a real number $\tau_0 > 1$, we define the parameter space

$$
\mathcal{U}(s_0, \tau_0) = \left\{ \Sigma \in \mathbb{C}_p : |s(\Sigma)| \leq s_0, \ \tau_0^{-1} \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \tau_0 \right\},
$$

where $|s(\Sigma)|$ is the number of nonzero off-diagonal entries in $\Sigma$. To attain the desired asymptotic properties of posteriors, we introduce the following conditions.

(A1) $\Sigma_0 \in \mathcal{U}(s_0, \tau_0)$ for some integer $0 < s_0 < p(p - 1)$ and constant $\tau_0 > 1$.

(A2) $p \asymp n^\beta$ for some $0 < \beta < 1$.

(A3) The hyperparameters satisfy $\tau \geq \max(3, \tau_0)$, $\tau = O(1)$, $\lambda = O(1)$, $a = b = 1/2$ and $\tau_1^2 \asymp 1/(np^4)$.

Condition (A1) implies that the true covariance matrix is sparse and has eigenvalues bounded above as well as away from zero. The integer $s_0$ controls the sparsity of the true covariance matrix. The bounded eigenvalue condition has been commonly used in high-dimensional matrix estimation literature including Banerjee and Ghosal (2015), Gao and Zhou (2015) and Lee et al. (2019). In this paper, the lower bound for the minimum eigenvalue is mainly used in Lemma 5.5 to convert $\|\Sigma_0^{-1} - \Sigma^{-1}\|_F$ to $\|\Sigma_0 - \Sigma\|_F$, while the upper bound for the maximum eigenvalue is required to ensure $\Sigma_0 \in \mathcal{U}(\tau)$.
Condition (A2) says that the number of variables $p$ grows to infinity as $n \to \infty$, but at a slower rate than $n$. In the literature, Lam and Fan (2009) used a similar condition to obtain the convergence rates of penalty estimators for sparse covariance matrices, and Liu and Martin (2019) used the same condition to obtain the posterior convergence rate for sparse precision matrices. This condition is inevitable to obtain the posterior convergence rate under the Frobenius norm if one uses the traditional techniques in Ghosal et al. (2000) which we use in this paper. In the seminal work of Ghosal et al. (2000), they provided a sufficient condition for proving the posterior convergence rate for densities under the Hellinger metric, which is equivalent to the Frobenius norm for covariance matrices under the bounded eigenvalue condition (A1). When the posterior is intractable, this is the standard way to find the posterior convergence rate. One of necessary conditions in this result is that the posterior convergence rate should converge to zero as $n \to \infty$. Since the diagonal elements of the covariance are all nonzero, this condition requires the number of diagonal elements $p = o(n)$. This can be also seen from the minimax lower bound result, Theorem 3.2. Thus, if one use the techniques in Ghosal et al. (2000), condition (A2) is required to prove Theorem 3.1.

Condition (A3) gives a sufficient condition for hyperparameters to obtain the desired theoretical property of posteriors. The choice $a = b = 1/2$ implies that we use the half-Cauchy prior for the off-diagonal entries. Note that $\tau_1^2$ is the global shrinkage parameter in (2), thus condition (A3) means that the global shrinkage parameter should be sufficiently small. This corresponds to assume a sufficiently small inclusion probability in spike and slab priors. See Lee et al. (2019) and Martin et al. (2017).

For the asymptotic minimax rate of the shrinkage prior, we first show an upper bound of the minimax rate: Theorem 3.1 shows the posterior convergence rate of the proposed prior under the Frobenius norm.

**Theorem 3.1** Under model (1) and prior (6), assume conditions (A1)–(A3) hold. If $(p + s_0)\log p = o(n)$, as $n \to \infty$

\[
\pi\left\{\|\Sigma - \Sigma_0\|_F^2 \geq M\frac{(p + s_0)\log p}{n} \mid X_n\right\} \to 0 \text{ in } \mathbb{P}_0\text{-probability}
\]
for some large constant $M > 0$.

The condition $(p + s_0) \log p = o(n)$ relates $p$ and $n$ to $s_0$, the number of nonzero off-diagonal elements of the true covariance. Banerjee and Ghosal (2015) and Liu and Martin (2019) used the same condition to obtain the posterior convergence rate for sparse precision matrices.

The next theorem shows a minimax lower bound for covariance matrices, which coincides with the posterior convergence rate in Theorem 3.1 when $p = O(s_0)$. In general, $p = O(s_0)$ holds unless $\Sigma_0$ is extremely sparse and it holds when the true covariance matrix has an autoregressive structure with order 1, $AR(1)$.

**Theorem 3.2** For given positive integer $s_0$ and real number $\tau_0 > 1$, assume model (1) with $\Sigma_0 \in U(s_0, \tau_0)$. If $s_0^2 (\log p)^3 = O(p^2 n)$ and $s_0^2 = O(p^{3-\epsilon})$ for some small constant $\epsilon > 0$,

$$\inf_{\Sigma} \sup_{\Sigma_0 \in U(s_0, \tau_0)} \mathbb{E}_0 \|\hat{\Sigma} - \Sigma_0\|_F^2 \gtrsim \frac{s_0 \log p}{n} I(s_0 > 3p) + \frac{p}{n}.$$

Cai and Zhou (2012) proved that a modified thresholding estimator attains the minimax rate for sparse covariance matrices under the class of Bregman divergences. They assumed sparsity for each column of the covariance matrix, which means that *each column* of $\Sigma_0$ has nonzero entries less than $s'_0$. On the other hand, we assume that the nonzero entries of the $\Sigma_0$ is less than $s_0$. Thus, our sparsity assumption on $\Sigma_0$ is much weaker than that of Cai and Zhou (2012). Up to our knowledge, this is the first minimax lower bound result for sparse covariance matrices with restriction only on the total number of nonzero off-diagonal entries. To establish the minimax rate, they assumed that $(s'_0)^2 (\log p)^3 = O(n)$, which is roughly equivalent to $s_0^2 (\log p)^3 = O(p^2 n)$ in our notation. It is easy to see that the minimax rate in Cai and Zhou (2012) coincides with the rate of the lower bound in Theorem 3.2. Hence, Theorems 3.1 and 3.2 imply that, even though we consider a larger parameter space than Cai and Zhou (2012), the minimax rate is still unchanged and the proposed prior attains it.
4 Simulation Study

4.1 Synthetic data

To assess the performance of our sparse covariance estimator, we carry out a simulation study using synthetic datasets generated from Gaussian distributions with zero means and the following two covariance structures $\Sigma = (\sigma_{jk})$.

C1. Sparse covariance that mimics daily currency exchange rate return structure (Wang, 2015):

\[
\begin{pmatrix}
0.239 & 0.117 & 0.031 \\
0.117 & 1.554 & \\
0.362 & 0.002 & 0.094 \\
0.002 & 0.199 & 0.349 \\
0.094 & 0.349 & -0.036 \\
0.295 & -0.229 & 0.002 \\
-0.229 & 0.715 & \\
0.031 & 0.002 & 0.164 \\
0.002 & 0.164 & 0.112 \\
0.112 & 0.518 & -0.193 \\
-0.028 & -0.193 & 0.379 \\
-0.008 & -0.090 & 0.167 \\
-0.036 & -0.090 & 0.159 \\
\end{pmatrix}
\]

C2. Random structure: $\sigma_{jj} \sim \text{Gamma}(1, 1)$, $j = 1, 2, \ldots, p$. Sparse off-diagonal positions (20%) are randomly selected. Nonzero off-diagonal elements are generated from $\text{Unif}(0, \mu)$, $j \neq k$.

In case C1, following Wang (2015) we take $p = 12$ and $n = 250$, and in case C2, we consider $p = 50$ and $n = 50, 100$ and choose the range of parameter $\mu$ as $\{0.02, 0.1, 0.5, 1\}$ following Castillo and Roquain (2020).

We compute the root mean squared error (rmse), $\|\hat{\Sigma} - \Sigma\|_F/p$, and the maximum norm (mnorm), $\max_{jk}|\hat{\sigma}_{jk} - \sigma_{jk}|$, and report the average rmse and mnorm, and their standard errors over 50 replications. As competing methods, we consider the sample covariance (SampCov) and the SSSL estimator (Wang, 2015). We generate 5000 posterior samples after 5000 burn-in for the proposed shrinkage model and the SSSL.

Tables 1 and 2 show the rmse and mnorm values for simulation models with two covariance structures, C1 and C2, respectively. Figure 1 renders the effective sample sizes (ESS),
equivalent sample sizes when the independent sampling is done, of the posterior sampling for the shrinkage prior and the SSSL in case C1. From Tables 1 and 2, we can see that the proposed shrinkage and the SSSL estimators are better than the sample covariance in all cases, and that the proposed shrinkage estimator performs better or at least comparable to the SSSL estimator, while the posterior sampling algorithm of the shrinkage prior is more efficient than that of the SSSL in terms of ESS (Figure 1). Additionally, the posterior sampling of the shrinkage prior takes about 171 seconds per 1,000 ESS, but that of the SSSL takes 789 seconds with iMac Pro with 3 GHz 10-Core Intel Xeon processor. Finally, Table 2 shows that the continuous shrinkage prior produces more accurate estimates when signals are small, while the spike and slab prior can capture large signals more efficiently.

|        | Proposed   | SSSL    | SampCov |
|--------|------------|---------|---------|
| rmse   | 0.020 (0.003) | 0.020 (0.003) | 0.028 (0.004) |
| mnorm  | 0.114 (0.051) | 0.120 (0.050) | 0.120 (0.054) |

### 4.2 Real data application

In this section, we consider two datasets to assess the performance of the shrinkage prior for linear discriminant analysis (LDA) classification (Anderson, 2003). The first is a colon cancer data, described in Alon et al. (1999), Fisher and Sun (2011) and Touloumis (2015). The data set can be obtained from [http://genomics-pubs.princeton.edu/oncology/affydata](http://genomics-pubs.princeton.edu/oncology/affydata). It contains expression level measurements of 2000 genes on 40 normal and 22 colon tumor tissues.

As the second example, we consider the leukemia data (Golub et al., 1999; Zhu and Hastie, 2004; Guo et al., 2007) which consists of 7128 gene expression measurements on 72 leukemia patients, 47 “ALL” and 25 “AML”. This data set is available at [http://web.stanford.edu/~hastie](http://web.stanford.edu/~hastie).

Following Rothman et al. (2008), we select $p$ most significant genes with the two-sample $t$-statistic for $p = 50$ (colon data) and $p = 71$ (leukemia data) and then apply LDA to the
Table 2: rmse and mnorm under the covariance structure C2.

|        | Measure | Proposed | SSSL | SampCov |
|--------|---------|----------|------|---------|
| $n = 100$ | $\mu = 0.02$ | rmse | 0.034 (0.008) | 0.038 (0.007) | 0.109 (0.007) |
|        |         | mnorm   | 0.950 (0.407) | 0.994 (0.405) | 1.137 (0.409) |
|        | $\mu = 0.1$ | rmse | 0.063 (0.004) | 0.065 (0.005) | 0.127 (0.007) |
|        |         | mnorm   | 0.985 (0.409) | 0.998 (0.418) | 1.176 (0.414) |
|        | $\mu = 0.5$ | rmse | 0.287 (0.002) | **0.281 (0.004)** | 0.271 (0.014) |
|        |         | mnorm   | **1.371 (0.560)** | 1.470 (0.530) | 1.609 (0.560) |
|        | $\mu = 1$ | rmse | 0.561 (0.003) | **0.544 (0.006)** | 0.467 (0.023) |
|        |         | mnorm   | **1.789 (0.397)** | 1.800 (0.370) | 2.117 (0.403) |
| $n = 50$ | $\mu = 0.02$ | rmse | 0.043 (0.009) | 0.049 (0.009) | 0.153 (0.010) |
|        |         | mnorm   | **1.238 (0.460)** | 1.265 (0.472) | 1.526 (0.474) |
|        | $\mu = 0.1$ | rmse | 0.071 (0.006) | 0.075 (0.007) | 0.179 (0.010) |
|        |         | mnorm   | **1.263 (0.487)** | 1.306 (0.490) | 1.583 (0.509) |
|        | $\mu = 0.5$ | rmse | 0.292 (0.003) | **0.285 (0.003)** | 0.382 (0.019) |
|        |         | mnorm   | **1.564 (0.397)** | 1.580 (0.361) | 2.056 (0.461) |
|        | $\mu = 1$ | rmse | 0.582 (0.003) | **0.558 (0.006)** | 0.663 (0.034) |
|        |         | mnorm   | **2.278 (0.519)** | **2.203 (0.534)** | 2.991 (0.530) |

Datasets for classifying each observation in each data set into two groups. The LDA rule for an observation $X$ is given as

$$
\delta_j(X) = \text{argmax}_j \left\{ X^T \hat{\Sigma}^{-1} \hat{\mu}_j - \frac{1}{2} \hat{\mu}_j^T \hat{\Sigma}^{-1} \hat{\mu}_j + \log \hat{\omega}_j \right\}, \quad j = 1, 2,
$$

where $\hat{\omega}_j$ is the proportion of class $j$ and $\hat{\mu}_j$ is the sample mean for class $j$.

Table 3 shows leave-one-out cross validation (LOOCV) error rates for misclassified observations. From the result, we can see that our estimator outperforms two competitors, SampCov and SSSL for both data sets.
Figure 1: Effective sample size of the posterior samples of the shrinkage prior and the SSSL in case C1.

Table 3: Classification error for colon and leukemia data.

| Dataset     | Proposed | SSSL  | SampCov |
|-------------|----------|-------|---------|
| Colon       | 0.097    | 0.113 | 0.333   |
| Leukemia    | 0.014    | 0.028 | 0.486   |

5 Discussion

In this paper, we propose a theoretically supported shrinkage prior for sparse covariance matrices. We prove that the proposed shrinkage prior achieves the minimax posterior convergence rate under the Frobenius norm in high-dimensional settings. The shrinkage prior performs better than or comparable to the SSSL method in the simulation studies, while is computationally more efficient than the SSSL. In our simulation study, the pro-
posed shrinkage prior is 4 times faster than the SSSL in terms of the computation time per ESS. Two real data examples, colon and leukemia data, show the benefit of the LDA classification based on the proposed Bayesian method.

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Appendix A: Proof of Theorem 3.1

Let
\[ K(f_{\Sigma_0}, f_{\Sigma}) := \int f_{\Sigma_0}(x) \log \frac{f_{\Sigma_0}(x)}{f_{\Sigma}(x)} \, dx, \]
\[ V(f_{\Sigma_0}, f_{\Sigma}) := \int f_{\Sigma_0}(x) \left( \log \frac{f_{\Sigma_0}(x)}{f_{\Sigma}(x)} \right)^2 \, dx, \]
where \( f_{\Sigma} \) is the probability density function of \( N_p(0, \Sigma) \) based on \( n \) random samples \( X_1, \ldots, X_n \). For a given \( \epsilon > 0 \), let
\[ B_\epsilon := \{ f_{\Sigma} : \Sigma \in C_p, \ K(f_{\Sigma_0}, f_{\Sigma}) < \epsilon^2, \ V(f_{\Sigma_0}, f_{\Sigma}) < \epsilon^2 \}. \]

To prove Theorem 3.1, we apply Theorem 2.1 in Ghosal et al. (2000).

Lemma 5.1 (A version of Theorem 2.1 in Ghosal et al. (2000)) Let \( d \) be the Hellinger metric, and let \( \mathcal{P} = \{ f_{\Sigma} : \Sigma \in C_p \} \). Consider a sieve \( \mathcal{P}_n \subset \mathcal{P} \) and a sequence \( \epsilon_n \) with \( \epsilon_n \to 0 \) and \( n\epsilon_n^2 \to \infty \). If, for some constants \( C_1 \) and \( C_2 > 0 \),
\[ \log D(\epsilon_n, \mathcal{P}_n, d) \leq C_1 n\epsilon_n^2, \]
\[ \pi(\mathcal{P}_n^c) \leq \exp \left\{ -(C_2 + 4)n\epsilon_n^2 \right\}, \]
\[ \pi(B_\epsilon) \geq \exp(-C_2 n\epsilon_n^2), \]
then for sufficiently large \( M > 0 \), we have
\[ \pi \left( d(f_{\Sigma_0}, f_{\Sigma}) > M\epsilon_n \mid X_n \right) \to 0 \]
as \( n \to \infty \) in \( \mathbb{P}_{\Sigma_0} \)-probability, where \( D(\epsilon, \mathcal{P}_n, d) \) is the \( \epsilon \)-packing number of \( \mathcal{P}_n \) with respect to the distance \( d \).

Based on the above lemma, it suffices to show that conditions (10)-(12) hold under the assumptions in Theorem 3.1, using
\[ \epsilon_n := \left( \frac{(p + s_0) \log p}{n} \right)^{1/2}. \]

For the rest, \( \pi \) denotes the proposed shrinkage prior (6). With a slightly abuse of notation, \( \pi \) also denotes the prior for \( f_{\Sigma} \) induced by the shrinkage prior (6).
A1. The upper bound of the packing number

We define

\[ P_n = \left\{ f_\Sigma : |s(\Sigma, \delta_n)| \leq s_n, \tau^{-1} \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \tau, \|\Sigma\|_{\max} \leq L_n \right\}, \]

\[ U(\delta_n, s_n, L_n, \tau) = \left\{ \Sigma \in \mathcal{C}_p : |s(\Sigma, \delta_n)| \leq s_n, \tau^{-1} \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \tau, \|\Sigma\|_{\max} \leq L_n \right\} \]

for some positive constants \( \delta_n, s_n, L_n \) and \( \tau \), where \( \|\Sigma\|_{\max} = \max_{ij} |\sigma_{ij}| \) for \( \Sigma = (\sigma_{ij}) \).

Here \( \tau \) is a fixed large constant such that \( \tau_0 < \tau \). \( \delta_n, s_n \) and \( L_n \) are specified in the following theorem. It gives the upper bound of the packing number in (10).

**Theorem 5.2 (The upper bound of the packing number)** If \( \tau^4 \leq p, p \asymp n^\beta \) for some \( 0 < \beta < 1 \), \( s_n = c_1 n \epsilon_n^2 / \log p \), \( L_n = c_2 n \epsilon_n^2 \) and \( \delta_n = \epsilon_n / \tau^3 \) for some constants \( c_1 > 1 \) and \( c_2 > 0 \), we have

\[ \log D(\epsilon_n, P_n, d) \leq (12 + 1/\beta)c_1 n \epsilon_n^2. \]

A2. The upper bound of the prior mass on \( P^c_n \)

**Lemma 5.3** If \( a = b = 1/2, \tau_1^2 \asymp 1/(np^4 \tau^2) \) and \( \tau > 3 \), we have

\[ \pi^u(\Sigma \in \mathcal{U}(\tau)) \geq \left\{ \frac{\lambda \tau}{8} \exp \left( - \frac{\lambda \tau}{4} - \frac{C}{\sqrt{n}} \right) \right\}^p \]

for some constant \( C > 0 \).

The following theorem gives the upper bound of the prior mass on \( P^c_n \) in (11).

**Theorem 5.4 (The upper bound of the prior mass)** If \( \delta_n = \epsilon_n / \tau^3, 3 < \tau \leq (\log p)/\lambda, a = b = 1/2, \tau_1^2 \asymp 1/(np^4 \tau^2), \tau^4 \ll (p + s_0)^2 \log p \), we have

\[ \pi(P^c_n) \leq \exp \left\{ - (c_1 - 1)n \epsilon_n^2/3 \right\}. \]

A3. The lower bound for \( \pi(B_{\epsilon_n}) \)

**Lemma 5.5** If \( \Sigma_0 \in \mathcal{U}(s_0, \tau_0) \) and \( \Sigma \in \mathcal{U}(\tau) \), we have
(i) $K(f_{\Sigma_0}, f_{\Sigma}) \leq \tau_0^4 \|\Sigma - \Sigma_0\|_F^2$;

(ii) $V(f_{\Sigma_0}, f_{\Sigma}) \leq \frac{3}{2} \tau_0^4 \|\Sigma - \Sigma_0\|_F^2$.

Lemma 5.6 If $a = b = 1/2$ and $\tau_1^2 \approx 1/(np^4 \tau^2)$, then

$$\pi_{ij}^u(x) \geq \sqrt{\frac{1}{2\pi^3} \frac{\tau_1}{x^2}},$$

for any $x > 1$, where $\pi_{ij}^u(\sigma_{ij})$ is the unconstrained marginal prior density of $\sigma_{ij}$.

The following theorem gives the lower bound for $\pi(B_{en})$ in (12).

Theorem 5.7 (The lower bound for $\pi(B_{en})$) If $\Sigma_0 \in \mathcal{U}(s_0, \tau_0)$ with $\tau_0 < \tau$, $p \approx n^\beta$ for some $0 < \beta < 1$, $\tau^4 \leq p$, $\tau_0^2 \leq s_0 \log p$, $n \geq \max\{1/\tau_0^4, s_0/(1 - \tau_0/\tau)^2\} \log p/\tau^4$, $p^{-1} < \lambda < \log p/\tau_0$, $a = b = 1/2$ and $\tau_1^2 \approx 1/(np^4 \tau^2)$, we have

$$\pi(B_{en}) \geq \exp\left\{ - (5 + \frac{1}{\beta})n\epsilon_n^2 \right\}.$$

Proof of Theorem 3.1 The proof follows from Theorems 5.2, 5.4, 5.7 and Lemma 5.1, with $c_1 = 28 + 3/\beta$. Note that $\lambda > p^{-1}$ and $\tau^4 \leq \min\{p, s_0 \log p, (\log p)^4/\lambda^4\}$ hold for all sufficiently large $n$ because we assume $\tau = O(1)$ and $\lambda = O(1)$.

Appendix B: Proof of auxiliary results

Lemma 5.8 For any $p \times p$ matrices $A$ and $B$, we have

$$\|AB\|_F \leq \|A\| \|B\|_F.$$

Proof Let $b_j$ be the $j$th column of $B$. Then,

$$\|AB\|_F^2 = \|(Ab_1, \ldots, Ab_p)\|_F^2$$

$$= \sum_{j=1}^p \|Ab_j\|_2^2$$

$$\leq \|A\|^2 \sum_{j=1}^p \|b_j\|_2^2$$

$$= \|A\|^2 \|B\|_F^2.$$
Proof of Theorem 5.2 Note that the \( \epsilon_n \)-packing number \( D(\epsilon_n, \mathcal{P}_n, d) \) is the maximal number of points in \( \mathcal{P}_n \) such that the distance between each pair is greater than or equal to \( \epsilon_n \). By Lemma A.1 in Banerjee and Ghosal (2015),

\[
 d(f_{\Sigma_1}, f_{\Sigma_2}) \leq C\tau \| \Omega_1 - \Omega_2 \|_F
\]

for some constant \( C > 0 \) and any \( \Sigma_1, \Sigma_2 \in \mathcal{U}(\delta_n, s_n, L_n, \tau) \), where \( \Omega_i = \Sigma_i^{-1} \) for \( i = 1, 2 \).

Further note that for any \( \Sigma_1, \Sigma_2 \in \mathcal{U}(\delta_n, s_n, L_n, \tau) \),

\[
 \| \Omega_1 - \Omega_2 \|_F \leq \| \Omega_1 \| \| \Omega_2 \| \| \Sigma_1 - \Sigma_2 \|_F \\
 \leq \tau^2 \| \Sigma_1 - \Sigma_2 \|_F,
\]

which gives

\[
 d(f_{\Sigma_1}, f_{\Sigma_2}) \leq C\tau^3 \| \Sigma_1 - \Sigma_2 \|_F.
\]

By the definition of the \( \epsilon_n \)-packing number it implies that

\[
 \log D(\epsilon_n, \mathcal{P}_n, d) \leq \log D(\epsilon_n/(C\tau^3), \mathcal{U}(\delta_n, s_n, L_n, \tau), \| \cdot \|_F) \\
 \leq \log \left\{ \left( \frac{CL_n \tau^3}{\epsilon_n} \right)^{p s_n} \sum_{j=1}^{s_n} \left( \frac{2CL_n \tau^3}{\epsilon_n} \right)^j \left( \frac{p^2}{2} \right)^j \right\} \\
 \leq p \log \left( \frac{CL_n \tau^3}{\epsilon_n} \right) + s_n \log \left( \frac{2CL_n \tau^3 p^2}{\epsilon_n} \right) \\
 \leq (p + s_n) \log(2CL_n) + (p + s_n) \log \tau^3 + (p + s_n) \log(1/\epsilon_n) + 2s_n \log p \\
 \leq 3s_n \log p + \frac{3}{4} (p + s_n) \log p + \frac{1}{2} (p + s_n) \log \frac{n}{(p + s_0) \log p} + 2s_n \log p \\
 \leq 3s_n \log p + \frac{3}{4} (p + s_n) \log p + \frac{1}{2\beta} (p + s_n) \log p + 2s_n \log p \\
 \leq (6 + 1/(2\beta)) (s_n/c_1 + s_n) \log p = (12 + 1/\beta) c_1 n \epsilon_n^2
\]

for all sufficiently large \( n \), because \( c_1 > 1, \tau^4 \leq p, p \asymp n^\beta, \delta_n = \epsilon_n/\tau^3, s_n = c_1 n \epsilon_n^2/\log p \) and \( L_n = c_2 n \epsilon_n^2 \). ■
Proof of Lemma 5.3 By Gershgorin circle theorem, every eigenvalue of $\Sigma$ lies within as least one of $[\sigma_{jj} - \sum_{k \neq j} |\sigma_{kj}|, \sigma_{jj} + \sum_{k \neq j} |\sigma_{kj}|]$ for $j = 1, \ldots, p$ (Brualdi and Mellendorf, 1994). Thus, it suffices to show

$$\pi^u(\Sigma \in \mathcal{U}(\tau)) \geq \pi^u\left(\min_j \left(\sigma_{jj} - \sum_{k \neq j} |\sigma_{kj}| \right) > 0, \tau^{-1} \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \tau\right).$$

On the event $\min_j \left(\sigma_{jj} - \sum_{k \neq j} |\sigma_{kj}| \right) > 0$, we have

$$\lambda_{\max}(\Sigma) \leq \|\Sigma\|_1 = \max_j \left(\sigma_{jj} + \sum_{k \neq j} |\sigma_{kj}| \right) \leq \max_j 2\sigma_{jj}$$

and

$$\lambda_{\min}(\Sigma) \geq \min_j \left(\sigma_{jj} - \sum_{k \neq j} |\sigma_{kj}| \right).$$

Therefore,

$$\pi^u(\Sigma \in \mathcal{U}(\tau)) \geq \pi^u\left(\tau^{-1} \leq \min_j \left(\sigma_{jj} - \sum_{k \neq j} |\sigma_{kj}| \right) \leq 2\max_j \sigma_{jj} \leq \tau\right)$$

$$= \pi^u\left(\tau^{-1} \leq \min_j \left(\sigma_{jj} - \sum_{k \neq j} |\sigma_{kj}| \right) \leq 2\max_j \sigma_{jj} \leq \tau \mid \max_k \sigma_{kj} < (\tau p)^{-1}\right) \pi^u(\max_k |\sigma_{kj}| < (\tau p)^{-1})$$

$$\geq \pi^u\left(\tau^{-1} \leq \min_j \left(\sigma_{jj} - \tau^{-1} \right) \leq 2\max_j \sigma_{jj} \leq \tau \mid \max_k \sigma_{kj} < (\tau p)^{-1}\right) \pi^u(\max_k |\sigma_{kj}| < (\tau p)^{-1})$$

$$= \pi^u\left(\tau^{-1} \leq \min_j \left(\sigma_{jj} - \tau^{-1} \right) \leq 2\max_j \sigma_{jj} \leq \tau\right) \pi^u(\max_k |\sigma_{kj}| < (\tau p)^{-1}).$$

Note that

$$\pi^u\left(\tau^{-1} \leq \min_j \left(\sigma_{jj} - \tau^{-1} \right) \leq 2\max_j \sigma_{jj} \leq \tau\right) \geq \pi^u\left(2\tau^{-1} \leq \sigma_{jj} \leq \tau/2, \forall j\right)$$

$$= \prod_{j=1}^p \pi^u\left(2\tau^{-1} \leq \sigma_{jj} \leq \tau/2\right)$$

$$\geq \left\{\left(\frac{\tau}{2} - 2\tau^{-1}\right)\frac{\lambda}{2} \exp\left(-\frac{\lambda\tau}{4}\right)\right\}^p$$

$$\geq \left\{\frac{\lambda\tau}{8} \exp\left(-\frac{\lambda\tau}{4}\right)\right\}^p.$$
because $\tau > 3$. Furthermore, by Theorem 1 in Carvalho et al. (2010) and the change of variables,

$$
\pi^u(\sigma_{kj}) \leq \frac{1}{\tau_1 \sqrt{2\pi^3}} \log \left(1 + \frac{2\tau_1^2}{\sigma_{kj}^2}\right)
$$

(13)

for any $\sigma_{kj} \neq 0$, which implies

$$
\pi^u(|\sigma_{kj}| \geq (\tau p)^{-1}) \leq \frac{1}{\tau_1} \sqrt{\frac{2}{\pi^3}} \int_{(\tau p)^{-1}}^{\infty} \log \left(1 + \frac{2\tau_1^2}{x^2}\right) dx
$$

$$
\leq \sqrt{\frac{2}{\pi^3}} \int_{(\tau p)^{-1}}^{\infty} \frac{2\tau_1}{x^2} dx
$$

$$
= \frac{2\sqrt{2}}{\sqrt{\pi^3}} \tau_1 \tau p.
$$

Thus, we have

$$
\pi^u(\max_{k \neq j} |\sigma_{kj}| < (\tau p)^{-1}) = \prod_{k \neq j} \{1 - \pi^u(|\sigma_{kj}| \geq (\tau p)^{-1})\}
$$

$$
\geq \left(1 - \frac{2\sqrt{2}}{\sqrt{\pi^3}} \tau_1 \tau p\right)^p^2
$$

$$
\geq \exp \left(- \frac{4\sqrt{2}}{\sqrt{\pi^3}} \tau_1 \tau p^3\right)
$$

$$
= \exp \left(- C \frac{p}{\sqrt{n}}\right)
$$

for some constant $C > 0$, because $\tau_1^2 \asymp 1/(np^4 \tau^2)$.

\[\Box\]

**Proof of Theorem 5.4** Note that

$$
\pi(\mathcal{P}_{n}^c) \leq \pi(|s(\Sigma, \delta_n)| > s_n) + \pi(\|\Sigma\|_{\max} > L_n).
$$

First, we focus on the upper bound for $\pi(|s(\Sigma, \delta_n)| > s_n)$. For any $1 \leq k \neq j \leq p$, by applying inequality (13), we have

$$
\nu_n \equiv \pi^u(|\sigma_{kj}| > \delta_n) \leq \frac{2\sqrt{2}}{\sqrt{\pi^3}} \tau_1 \delta_n^{-1}
$$

$$
\leq \frac{C \tau^3 \sqrt{n}}{\sqrt{p^2} \sqrt{(p + s_0) \log p}}
$$

$$
= \frac{C \tau^2}{\sqrt{p^4 (p + s_0) \log p}}
$$
for some constant $C > 0$, because $\tau_1^2 \asymp 1/(np^4\tau^2)$. Thus,

$$\pi^n (\|s(\Sigma, \delta_n)\| > s_n) = \mathbb{P} \left( B \left( \left( \frac{p}{2} \right), \nu_n \right) > s_n \right) \leq 1 - \Phi \left( \sqrt{2} \left( \frac{p}{2} \right) H \left( \nu_n, \frac{s_n}{\left( \frac{p}{2} \right)} \right) \right)$$

by Lemma A.3 of Song and Liang (2018), provided $s_n > \left( \frac{p}{2} \right) \nu_n$, where $\Phi$ is the cdf of $N(0, 1)$ and

$$H(\nu, k/n) = (k/n) \log \{ k/(n\nu) \} + (1 - k/n) \log \{ (1 - k/n)/(1 - \nu) \}.$$
for all sufficiently large $n$. Therefore, due to Lemma 5.3 and $\lambda \tau \leq \log p$,

$$
\pi(|s(\Sigma, \delta_n)| > s_n) \leq \frac{\pi^u(|s(\Sigma, \delta_n)| > s_n)}{\pi^u(\Sigma \in U(\tau))}
$$

$$
\leq \exp \left(-c_1 n \epsilon_n^2 / 3\right) / \pi^u(\Sigma \in U(\tau))
$$

$$
\leq \left\{ \frac{8}{\lambda \tau} \exp \left( \frac{\lambda \tau}{4} + \frac{C}{\sqrt{n}} \right) \right\} \exp \left(-c_1 n \epsilon_n^2 / 3\right)
$$

$$
\leq \exp \left(-c_1 n \epsilon_n^2 / 3 + p \log p / 3\right)
$$

$$
\leq \exp \left\{ -(c_1 - 1) n \epsilon_n^2 / 3 \right\}
$$

for some constant $C > 0$ and all sufficiently large $n$.

Now we focus on the upper bound for the second term $\pi(||\Sigma||_{\text{max}} > L_n)$. Since $\pi(\Sigma : \lambda_{\text{max}}(\Sigma) \leq \tau) = 1$ and $||\Sigma||_{\text{max}} \leq \lambda_{\text{max}}(\Sigma)$, it means that $\pi(||\Sigma||_{\text{max}} > L_n) = 0$ for all sufficiently large $n$, because $L_n \to \infty$ as $n \to \infty$. It completes the proof. ■

**Proof of Lemma 5.5** In the proof, we follow the calculation of Banerjee and Ghosal (2015). Let $d_i$’s be the eigenvalues of $\Sigma_0^{-1} \Sigma^{-1} \Sigma_0^{-1} + \hat{\Sigma}$. By Lemma A.1 in Banerjee and Ghosal (2015), we obtain

$$
\sum_{i=1}^{p} (1 - d_i)^2 \leq \tau^2 ||\Sigma_0^{-1} - \Sigma^{-1}||_F^2.
$$

Also,

$$
||\Sigma_0^{-1} - \Sigma^{-1}||_F \leq ||\Sigma^{-1}||_F ||\Sigma_0^{-1}||_F ||\Sigma - \Sigma_0||_F
$$

$$
\leq \tau \tau_0 ||\Sigma - \Sigma_0||_F.
$$

Following the calculation of Banerjee and Ghosal (2015), we obtain

$$
K(f_{\Sigma_0}, f_{\hat{\Sigma}}) = -\frac{1}{2} \sum_{i=1}^{p} \log d_i - \frac{1}{2} \sum_{i=1}^{p} (1 - d_i)
$$

$$
\leq \sum_{i=1}^{p} (1 - d_i)^2
$$

$$
\leq \tau^2 ||\Sigma_0^{-1} - \Sigma^{-1}||_F^2
$$

$$
\leq \tau^4 \tau_0^2 ||\Sigma - \Sigma_0||_F^2.
$$
Similarly, we obtain
\[
V(f_{\Sigma_0}, f_{\Sigma}) = \frac{1}{2} \sum_{i=1}^{p} (1 - d_i)^2 + K(f_{\Sigma_0}, f_{\Sigma})^2 \\
\leq \frac{3}{2} \sum_{i=1}^{p} (1 - d_i)^2 \\
\leq \frac{3}{2} \tau^2 \|\Sigma_0^{-1} - \Sigma^{-1}\|_F^2 \\
\leq \frac{3}{2} \tau^4 \tau_0^2 \|\Sigma - \Sigma_0\|_F^2.
\]
This completes the proof.

\[\blacksquare\]

**Proof of Lemma 5.6** Because we have \(a = b = 1/2\),
\[
\frac{\sigma_{ij}}{\tau_1} | \rho_{ij} \sim N\left(0, \frac{\rho_{ij}}{1 - \rho_{ij}}\right),
\]
\[
\rho_{ij} \sim \text{Beta}(a, b)
\]
is equivalent to
\[
\frac{\sigma_{ij}}{\tau_1} | \lambda_{ij} \sim N\left(0, \lambda_{ij}^2\right),
\]
\[
\lambda_{ij} \sim C^+(0, 1),
\]
where \(C^+(0, s)\) denotes the standard half-Cauchy distribution on positive real with a scale parameter \(s\). Then, by Theorem 1 in Carvalho et al. (2010) and the change of variables,
\[
\pi_{ij}^u(x) \geq \frac{1}{2\tau_1} \sqrt{\frac{1}{2\pi^2}} \log \left(1 + \frac{4\tau_1^2}{x^2}\right)\\
\geq \frac{1}{4\tau_1} \sqrt{\frac{1}{2\pi^3}} \frac{4\tau_1^2}{x^2}\\
\geq \sqrt{\frac{1}{2\pi^3}} \frac{\tau_1}{x^2}
\]
for any \(x > 1\), because \(\tau_1 \approx 1/(np^4\tau^2)\).

\[\blacksquare\]

**Proof of Theorem 5.7** By Lemma 5.5, it suffices to show that
\[
\pi\left(\|\Sigma - \Sigma_0\|_F^2 \leq \frac{2}{3\tau_1^2} \epsilon_n^2\right) \geq \exp \left(-Cn\epsilon_n^2\right).
\]
Note that
\[
\pi\left(\|\Sigma - \Sigma_0\|^2_F \leq \frac{2}{3n^4\tau_0^4} (p + s_0) \log p \right) \\
\geq \pi\left(\sum_{i \neq j} (\sigma_{ij} - \sigma^*_{ij})^2 \leq \frac{2}{3n^4\tau_0^4} s_0 \log p \right) \leq \sum_{j=1}^p (\sigma_{jj} - \sigma^*_{jj})^2 \leq \frac{2}{3n^4\tau_0^4} \frac{p \log p}{n} \\
\geq \pi\left(\max_{i \neq j} (\sigma_{ij} - \sigma^*_{ij})^2 \leq \frac{2}{3n^4\tau_0^4} \frac{s_0 \log p}{p(p-1)n}, \max_{1 \leq j \leq p} (\sigma_{jj} - \sigma^*_{jj})^2 \leq \frac{2}{3n^4\tau_0^4} \frac{\log p}{n} \right) \\
\equiv \pi(A_{n,\Sigma_0}),
\]
where \(\Sigma_0 = (\sigma^*_{ij})\). By Weyl’s theorem, if \(\Sigma \in A_{n,\Sigma_0}\),
\[
\lambda_{\min}(\Sigma) \geq \lambda_{\min}(\Sigma_0) - \|\Sigma - \Sigma_0\| \\
\geq \lambda_{\min}(\Sigma_0) - \|\Sigma - \Sigma_0\|_1 \\
\geq \tau_0^{-1} - p\sqrt{\frac{2}{3n^4\tau_0^4}} \frac{s_0 \log p}{p(p-1)n} - \sqrt{\frac{2}{3n^4\tau_0^4}} \frac{\log p}{n} \\
\geq \tau^{-1}
\]
and
\[
\lambda_{\max}(\Sigma) \leq \lambda_{\max}(\Sigma_0) + \|\Sigma - \Sigma_0\| \\
\leq \tau_0 + \|\Sigma - \Sigma_0\|_1 \\
\leq \tau_0 + p\sqrt{\frac{2}{3n^4\tau_0^4}} \frac{s_0 \log p}{p(p-1)n} + \sqrt{\frac{2}{3n^4\tau_0^4}} \frac{\log p}{n} \\
\leq \tau
\]
for all sufficiently large \(n\), because \(\Sigma_0 \in \mathcal{U}(s_0, \tau_0)\), \(\tau_0 < \tau\) and \(\tau^4(1 - \tau_0/\tau)^2 n \geq s_0 \log p\). Thus, if \(\Sigma \in A_{n,\Sigma_0}\), then we have \(\Sigma \in \mathcal{U}(\tau)\). Because
\[
\pi(\Sigma) = \frac{\pi^u(\Sigma) I(\Sigma \in \mathcal{U}(\tau))}{\pi^u(\Sigma \in \mathcal{U}(\tau))},
\]
we have
\[
\pi(A_{n,\Sigma_0}) \geq \pi^u(A_{n,\Sigma_0}).
\]
Note that
\[
\pi^u(A_n, \Sigma_0) = \pi^u\left(\max_{i \neq j} (\sigma_{ij} - \sigma_{ij}^*)^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{s_0 \log p}{p(p-1)n}\right) \times \pi^u\left(\max_{1 \leq j \leq p} (\sigma_{jj} - \sigma_{jj}^*)^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{\log p}{n}\right)
\]
\[= \prod_{i < j} \pi^u\left((\sigma_{ij} - \sigma_{ij}^*)^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{s_0 \log p}{p(p-1)n}\right) \times \prod_{j=1}^p \pi^u\left((\sigma_{jj} - \sigma_{jj}^*)^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{\log p}{n}\right)
\]
and
\[
\prod_{j=1}^p \pi^u\left((\sigma_{jj} - \sigma_{jj}^*)^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{\log p}{n}\right) \geq \prod_{j=1}^p \left[\left(\frac{2}{3\tau^4\tau_0^2} \frac{\log p}{n}\right)^{\frac{1}{2}} \exp\left\{-\frac{\lambda}{2} (\sigma_{jj}^* + \sqrt{\frac{2}{3\tau^4\tau_0^2} \frac{\log p}{n}})\right\}\right] \geq \exp\left\{-p\lambda\tau_0 - p \log\left(\frac{3\tau^4\tau_0^2 n}{\lambda \sqrt{2 \log p}}\right)\right\} \geq \exp\left\{-p \log p - (2 + \frac{1}{2\beta}) p \log p\right\} \geq \exp\left\{- (3 + \frac{1}{2\beta}) p \log p\right\} = \exp\left\{- (3 + \frac{1}{2\beta}) p \log p\right\}
\]
for all sufficiently large \(n\), because \(\log p/(\tau^4\tau_0^4) \leq n\), \(\tau^4 \leq p\) and \(p^{-1} < \lambda < \log p/\tau_0\).

Furthermore,
\[
\prod_{i < j} \pi^u\left((\sigma_{ij} - \sigma_{ij}^*)^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{s_0 \log p}{p(p-1)n}\right) \geq \prod_{(i,j) \in \Sigma_0} \pi^u\left((\sigma_{ij} - \sigma_{ij}^*)^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{s_0 \log p}{p(p-1)n}\right) \prod_{(i,j) \notin \Sigma_0, i < j} \pi^u\left((\sigma_{ij}^2 \leq \frac{2}{3\tau^4\tau_0^2} \frac{s_0 \log p}{p(p-1)n}\right)
\]
The second term in (14) is bound below by
\[
\prod_{(i,j) \notin \Sigma_0, i < j} \left\{1 - \pi^u\left((\sigma_{ij}^2 \geq \frac{2}{3\tau^4\tau_0^2} \frac{s_0 \log p}{p(p-1)n}\right)\right\} \geq \left\{1 - 2\tau_1 \sqrt{\frac{3}{\pi^3}} \frac{\sqrt{3\tau^4\tau_0^2 p(p-1)n}}{2 s_0 \log p}\right\}^p \geq \exp\left(-4 \sqrt{\frac{3}{\pi^3}} \frac{\tau^2 \tau_0 p^3 \sqrt{n}}{s_0 \log p}\right) \geq \exp\left(-C\tau_0 p \sqrt{\frac{n}{s_0 \log p}}\right) \geq \exp(-Cp)
\]
for some constant $C > 0$, because $\tau^2 \tau_0^2 \leq s_0 \log p$ and $\tau_1^2 \approx \frac{1}{(np^4 \tau^2)}$. Let $\pi^u_{ij}(\sigma_{ij}) = \int_{0}^{1} \pi^u(\sigma_{ij} | \rho_{ij}) \pi^u(\rho_{ij}) d\rho_{ij}$ be the unconstrained marginal prior density of $\sigma_{ij}$. The first term in (14) is bounded below by

$$\prod_{(i,j) \in s(\Sigma_0)} \pi^u_{ij}(\sigma_{ij}^* + \sqrt{\frac{2 s_0 \log p}{3 \tau^4 \tau_0^2 p(p - 1)n}}) \geq \left\{ 2 \pi^u_{ij}(2 \tau_0) \sqrt{\frac{2 s_0 \log p}{3 \tau^4 \tau_0^2 p(p - 1)n}} \right\}^{s_0}$$

$$\geq \exp \left\{ s_0 \log(\pi^u_{ij}(2 \tau_0)) - \frac{1}{2} s_0 \log \left( \frac{3 \tau^4 \tau_0^2 p^2 n}{2 s_0 \log p} \right) \right\}$$

$$\geq \exp \left\{ s_0 \log(\pi^u_{ij}(2 \tau_0)) - \frac{1}{2} s_0 \log \left( 3 \tau^2 p^2 n \right) \right\}$$

$$\geq \exp \left\{ - \frac{1}{2} s_0 \log \left( \frac{3 \tau^4 \tau_0^2 p^2 n^2}{2} \right) \right\}$$

$$\geq \exp \left\{ - \frac{1}{2} s_0 \log \left( C p^{8 + 2/\beta} \right) \right\}$$

$$\geq \exp \left\{ - (5 + \frac{1}{\beta}) s_0 \log p \right\}$$

for some constant $C > 0$, because $p \approx n^\beta$, $\log p/(\tau^4 \tau_0^2) \leq n$, $\pi^u_{ij}(2 \tau_0) \geq \tau_1/(4 \sqrt{2 \pi^3 \tau_0^2}) \approx 1/(\tau_0^2 \sqrt{np^2})$ and $\tau^4 \leq p$, by Lemma 5.6. Therefore, we have

$$\pi^u(A_n, \Sigma_0) \geq \exp \left\{ - (3 + \frac{1}{2\beta}) p \log p - C p - (5 + \frac{1}{\beta}) s_0 \log p \right\}$$

$$\geq \exp \left\{ - (5 + \frac{1}{\beta})(p + s_0) \log p \right\} = \exp \left\{ - (5 + \frac{1}{\beta}) n s_0^2 \right\}. \quad \blacksquare$$

**Appendix C: Proof of Theorem 3.2**

**Proof** We will first show that

$$\inf_{\Sigma} \sup_{\Sigma_0 \in B_1} \mathbb{E}_\theta \| \hat{\Sigma} - \Sigma_0 \|_{F}^2 \geq \frac{s_0 \log p}{n} \quad (15)$$

for some $B_1 \subset U(s_0, \tau_0)$ when $s_0 > 3p$, and show that

$$\inf_{\Sigma} \sup_{\Sigma_0 \in B_2} \mathbb{E}_\theta \| \hat{\Sigma} - \Sigma_0 \|_{F}^2 \geq \frac{p}{n} \quad (16)$$

for some $B_2 \subset U(s_0, \tau_0)$ when $s_0 \leq 3p$. 27
(i) Proof of (15) Let \( r = \lceil p/2 \rceil \) and \( \epsilon_{np} = \nu \sqrt{\log p/n} \) with \( \nu = \sqrt{\epsilon/4} \). For any \( u \in \mathbb{R}^p \), let \( A_m(u) \) be a \( p \times p \) symmetric matrix whose the \( m \)th row and column are equal to \( u \) and the rest of entries are zero. Define a parameter space

\[
B_1 := \left\{ \Sigma(\theta) : \Sigma(\theta) = I_p + \epsilon_{np} \sum_{m=1}^{r} \gamma_m A_m(\lambda_m), \; \theta = (\gamma, \lambda) \in \Theta \right\},
\]

where \( \gamma = (\gamma_1, \ldots, \gamma_r) \in \Gamma = \{0, 1\}^r \), \( \lambda = (\lambda_1, \ldots, \lambda_r)^T \in \Lambda \subset \mathbb{R}^{r \times p} \) and \( \Theta = \Gamma \times \Lambda \). Here, we let

\[
\Lambda := \left\{ \lambda = (\lambda_1, \ldots, \lambda_r)^T : \lambda_m = (\lambda_{mi}) \in \{0, 1\}^p, \; \|\lambda_m\|_0 = k, \; \sum_{i=1}^{p-r} \lambda_{mi} = 0 \right\},
\]

for any \( m = 1, \ldots, r \), and satisfies \( \max_{1 \leq i \leq p} \sum_{m=1}^{r} \lambda_{mi} \leq 2k \}

\( k = \lceil c_{np}/2 \rceil - 1 \) and \( c_{np} = \lceil s_0/p \rceil \).

We will first show that \( B_1 \subset U(s_0, \tau_0) \). Note that \( \|\Sigma(\theta)\| \leq \|\Sigma(\theta)\|_1 \leq 1 + 2k\epsilon_{np} \leq 1 + c_{np} \nu \sqrt{\log p/n} \leq \tau_0 \) for any \( \tau_0 > 1 \) and sufficiently large \( n \) due to our assumption, \( s_0^2 (\log p)^3 = O(p^2 n) \). Also note that \( 2k\epsilon_{np} \leq c_{np} \nu \sqrt{\log p/n} \leq \nu \sqrt{\log p/n} \leq 1/\tau_0 \) for any \( \tau_0 > 1 \) and sufficiently large \( n \), which implies that \( \Sigma(\theta) - \tau_0^{-1} I_p \) is diagonally dominant. Thus, we have \( \lambda_{\text{min}}(\Sigma(\theta)) \geq \tau_0^{-1} \). Because \( |s(\Sigma(\theta))| \leq 2kp \leq s_0 \), it holds that \( B_1 \subset U(s_0, \tau_0) \).

For given \( \theta \in \Theta \) and \( a \in \{0, 1\} \), let \( \mathbb{P}_\theta \) and \( \mathbb{P}_{i,a} \) be the joint distribution of random samples \( X_1, \ldots, X_n \) from \( N_p(0, \Sigma(\theta)) \) and \( \{2^{-r} \|A\|^{-1} \sum_{\theta \in \Theta_{i,a}} \mathbb{P}_\theta \} \), respectively, where \( \Theta_{i,a} = \{ \theta \in \Theta : \gamma_i(\theta) = a \} \). For any two probability measures \( \mathbb{P} \) and \( \mathbb{Q} \), let \( \|\mathbb{P} \wedge \mathbb{Q}\| = \int (p \wedge q) \, d\mu \), where \( p \) and \( q \) are probability densities corresponding to \( \mathbb{P} \) and \( \mathbb{Q} \), respectively, with respect to a common dominating measure \( \mu \). By applying Lemma 3 of Cai and Zhou (2012) with \( s = 2 \), we have

\[
\inf_{\theta \in \Theta} \max_{\theta \in \Theta} 2^2 \mathbb{E}_\theta \|\hat{\Sigma} - \Sigma(\theta)\|_F^2 \geq \frac{\alpha}{2} \min_{1 \leq i \leq r} \|\mathbb{P}_{i,0} \wedge \mathbb{P}_{i,1}\|,
\]

where \( \mathbb{E}_\theta \) denotes the expectation with respect to \( X_1, \ldots, X_n \) \( \text{iid} \sim N_p(0, \Sigma(\theta)) \) and

\[
\alpha = \min_{(\theta, \theta') : H(\gamma(\theta), \gamma(\theta')) \geq 1} \frac{\|\Sigma(\theta) - \Sigma(\theta')\|_F^2}{H(\gamma(\theta), \gamma(\theta'))}.
\]
Here, $H(x, y) = \sum_{j=1}^{r} |x_j - y_j|$ for any $x, y \in \{0, 1\}^r$. By the definition of $\Sigma(\theta)$,

$$\|\Sigma(\theta) - \Sigma(\theta')\|_F^2 \geq 2k\epsilon_n^2 p H(\gamma(\theta), \gamma(\theta'))$$

for any $\theta, \theta' \in \Theta$, which implies

$$\alpha r \geq 2k\epsilon_n^2 r \geq \nu^2 \left(\frac{1}{2} - \frac{P}{s_0}\right) \frac{s_0 \log p}{n} \approx \frac{s_0 \log p}{n}$$

due to $s_0 > 3p$. Therefore, we complete the proof if we show that

$$\min_{1 \leq i \leq r} \|\bar{P}_{i,0} \land \bar{P}_{i,1}\| \geq c_1$$

(17)

for some constant $c_1 > 0$. Note that, without loss of generality, it suffices to show that $\|\bar{P}_{1,0} \land \bar{P}_{1,1}\| \geq c_1$.

Let $\Lambda_1 = \{\lambda_1(\theta) \in \mathbb{R}^p : \theta \in \Theta\}$ and $\Lambda_{-1} = \{\lambda_{-1}(\theta) \equiv (\lambda_2(\theta), \ldots, \lambda_r(\theta))^T \in \mathbb{R}^{(r-1) \times p} : \theta \in \Theta\}$. For any $a \in \{0, 1\}$, $b \in \{0, 1\}^{r-1}$ and $c \in \Lambda_{-1}$, we define

$$\bar{P}_{(1,a,b,c)} = \frac{1}{|\Theta_{(1,a,b,c)}|} \sum_{\theta \in \Theta_{(1,a,b,c)}} \mathbb{P}_\theta,$$

$$\Theta_{(1,a,b,c)} = \{\theta \in \Theta : \gamma_1(\theta) = a, \gamma_{-1}(\theta) = b, \lambda_{-1}(\theta) = c\},$$

where $\gamma_{-1}(\theta) \equiv (\gamma_2(\theta), \ldots, \gamma_r(\theta)) \in \{0, 1\}^{r-1}$. Let $\mathbb{E}_{(\gamma_{-1},\lambda_{-1})} f(\gamma_{-1}, \lambda_{-1})$ be the expectation of $f(\gamma_{-1}, \lambda_{-1})$ over $\Theta_{-1} = \{0, 1\}^{r-1} \times \Lambda_{-1}$, i.e.,

$$\mathbb{E}_{(\gamma_{-1},\lambda_{-1})} f(\gamma_{-1}, \lambda_{-1}) = \frac{1}{2^{r-1}|\Lambda|} \sum_{(b,c) \in \Theta_{-1}} |\Theta_{(1,a,b,c)}| f(b, c),$$

where the probability distribution of $(\gamma_{-1}, \lambda_{-1})$ is induced by the uniform distribution over $\Theta$. To show (17), it suffices to prove that there exists $0 < c_2 < 1$ such that

$$\mathbb{E}_{(\gamma_{-1},\lambda_{-1})} \left\{ \int \left( \frac{d\mathbb{P}_{(1,0,\gamma_{-1},\lambda_{-1})}}{d\mathbb{P}_{(1,0,\gamma_{-1},\lambda_{-1})}} \right)^2 d\mathbb{P}_{(1,0,\gamma_{-1},\lambda_{-1})} - 1 \right\} \leq c_2,$$

(18)

by Lemma 8 (ii) of Cai and Zhou (2012).

Since $\mathbb{P}_{(1,0,\gamma_{-1},\lambda_{-1})}$ assumes that $\gamma_1 = 0$, this is the distribution function of the $p$-dimensional normal distribution with a zero mean vector and a covariance matrix

$$\Sigma_0 = \begin{pmatrix} 1 & 0_{1 \times (p-1)} \\ 0_{(p-1) \times 1} & S_{(p-1) \times (p-1)} \end{pmatrix},$$

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where \( S_{(p-1) \times (p-1)} = (s_{ij}) \in \mathbb{R}^{(p-1) \times (p-1)} \) is a symmetric matrix uniquely determined by \((\gamma_{-1}, \lambda_{-1})\):

\[
\begin{cases}
1, & i = j, \\
\epsilon_{np}, & \gamma_{i+1} = \lambda_{i+1}(j + 1) = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Let

\[
\Lambda_1(c) = \{ a \in \mathbb{R}^p : \lambda_1(\theta) = a, \lambda_{-1}(\theta) = c \text{ for some } \theta \in \Theta \}
\]

be the set of all possible values of the first row, \( \lambda_1(\theta) \), given the rest of the rows, \( \lambda_{-1}(\theta) = c \).

For a given \( \lambda_{-1} \equiv \lambda_{-1}(\theta) = (\lambda_2(\theta), \ldots, \lambda_r(\theta))^T \in \mathbb{R}^{(r-1) \times p} \), denote \( n_{\lambda_{-1}} \) be the number of columns of \( \lambda_{-1} \) whose sums are equal to \( 2k \), and let \( p_{\lambda_{-1}} = r - n_{\lambda_{-1}} \). Then, by the definition of \( \Theta \), \( p_{\lambda_{-1}} \) is the number of entries in \( \lambda_1 \) which can be either 0 or 1. Note that \(|\Lambda_1(\lambda_{-1})| = \binom{p_{\lambda_{-1}}}{k} \) and \( p_{\lambda_{-1}} \geq \frac{p}{4} - 1 \) for any \( \lambda_{-1} \). Then, \( \overline{P}_{(1,1,\gamma_{-1},\lambda_{-1})} \) is an average of \( \binom{p_{\lambda_{-1}}}{k} \) normal distributions with covariance matrices of the form:

\[
\begin{pmatrix}
1 & r^T \\
r & S_{(p-1) \times (p-1)}
\end{pmatrix},
\]

where \( ||r||_0 = k \) and nonzero entries of \( r \) are equal to \( \epsilon_{np} \). For a given \((\gamma_{-1}, \lambda_{-1})\), let \( \Sigma_1 \) and \( \Sigma_2 \) be covariance matrices of the form (19) with the first row \( \lambda_1 \in \Lambda_1(\lambda_{-1}) \) and \( \lambda'_1 \in \Lambda_1(\lambda_{-1}) \), respectively. Then, by the similar arguments in page 2411 of Cai and Zhou (2012),

\[
\begin{align*}
\mathbb{E}_{(\gamma_{-1},\lambda_{-1})} \left\{ \int \left( \frac{d\overline{P}_{(1,1,\gamma_{-1},\lambda_{-1})}}{d\overline{P}_{(1,0,\gamma_{-1},\lambda_{-1})}} \right)^2 d\overline{P}_{(1,0,\gamma_{-1},\lambda_{-1})} - 1 \right\} \\
= \mathbb{E}_{(\gamma_{-1},\lambda_{-1})} \left[ \mathbb{E}_{(\lambda_1,\lambda'_1)\mid\lambda_{-1}} \left\{ \exp \left( \frac{n}{2} R_{\lambda_1,\lambda'_1}^{\gamma_{-1},\lambda_{-1}} \right) - 1 \right\} \right] \\
= \mathbb{E}_{(\lambda_1,\lambda'_1)} \left[ \mathbb{E}_{(\gamma_{-1},\lambda_{-1})\mid(\lambda_1,\lambda'_1)} \left\{ \exp \left( \frac{n}{2} R_{\lambda_1,\lambda'_1}^{\gamma_{-1},\lambda_{-1}} \right) - 1 \right\} \right],
\end{align*}
\]

where \( \lambda_1, \lambda'_1 \mid \lambda_{-1} \overset{iid}{\sim} Unif\{\Lambda_1(\lambda_{-1})\} \), \((\gamma_{-1}, \lambda_{-1}) \mid (\lambda_1, \lambda'_1) \sim Unif\{\Theta_{-1}(\lambda_1, \lambda'_1)\}\),

\[
\Theta_{-1}(a_1, a_2) = \{0, 1\}^{r-1} \times \{ c \in \Lambda_{-1} : \exists \theta_i \in \Theta, i = 1, 2 \text{ such that } \lambda_1(\theta_i) = a_i, \lambda_{-1}(\theta_i) = c \}
\]
\[ R_{\lambda_1, \lambda'_1}^{\gamma-1, \lambda-1} = -\log \det \{ I_p - \Sigma_0^{-2} (\Sigma_0 - \Sigma_1)(\Sigma_0 - \Sigma_2) \}. \] (21)

By Lemma 5.9, (20) is bounded above by
\[
\mathbb{E}_J \left[ \exp \left\{ -n \log (1 - J \epsilon_{np}^2) \right\} \mathbb{E}^{(\lambda_1, \lambda'_1)}_J \left\{ \mathbb{E}^{(\gamma-1, \lambda-1)}(\lambda_1, \lambda'_1) \exp \left( \frac{n}{2} R_{\lambda_1, \lambda'_1}^{\gamma-1, \lambda-1} \right) \right\} - 1 \right]
\leq \mathbb{E}_J \left[ \exp \left\{ -n \log (1 - J \epsilon_{np}^2) \right\} \frac{3}{2} - 1 \right],
\] (22)

where \( J \) is the number of overlapping nonzero entries between the first rows of \( \Sigma_1 \) and \( \Sigma_2 \), i.e., \( J = \lambda_1^T \lambda'_1 \). Note that for any \( 0 \leq j \leq k \),
\[
\mathbb{E}_J \left\{ I(J = j) \mid \lambda_{-1} \right\} = \frac{k}{j} \cdot \frac{\left( p_{\lambda_{-1}} - k \right)}{\left( p_{\lambda_{-1}} \right)}
= \left\{ \frac{k!}{(k-j)!} \right\}^2 \frac{\left( p_{\lambda_{-1}} - k \right)! \cdot \left( p_{\lambda_{-1}} - 2k + j \right)! \cdot j!}{p_{\lambda_{-1}}! \cdot (p_{\lambda_{-1}} - 2k)!}
\leq \left( \frac{k^2}{p_{\lambda_{-1}} - k} \right)^j,
\]
because \( \lambda_1, \lambda'_1 \mid \lambda_{-1} \overset{iid}{\sim} \text{Unif} \{ \Lambda_1(\lambda_{-1}) \} \). Then, we have
\[
\mathbb{E}_J I(J = j) = \mathbb{E}_{\lambda_{-1}} \left[ \mathbb{E}_J \left\{ I(J = j) \mid \lambda_{-1} \right\} \right]
\leq \mathbb{E}_{\lambda_{-1}} \left\{ \left( \frac{k^2}{p_{\lambda_{-1}} - k} \right)^j \right\}
\leq \left( \frac{k^2}{p/4 - 1 - k} \right)^j
\]
because \( p_{\lambda_{-1}} \geq p/4 - 1 \) for any \( \lambda_{-1} \). Thus, (22) is bounded above by
\[
\sum_{j=0}^{k} \left( \frac{k^2}{p/4 - 1 - k} \right)^j \left[ \exp \left\{ -n \log (1 - j \epsilon_{np}^2) \right\} \frac{3}{2} - 1 \right]
= \frac{1}{2} + \sum_{j=1}^{k} \left( \frac{k^2}{p/4 - 1 - k} \right)^j \left[ \exp \left\{ -n \log (1 - j \epsilon_{np}^2) \right\} \frac{3}{2} - 1 \right]
\leq \frac{1}{2} + \frac{3}{2} \sum_{j=1}^{k} \left( \frac{k^2}{p/4 - 1 - k} \right)^j p^{2^j - 1}
\leq \frac{1}{2} + \frac{3C}{2} \sum_{j=1}^{k} p^{-e_j p^{(\epsilon/2)j}}
\leq c_2^2
\]
for some constant $C > 0$ and all sufficiently large $p$, by setting $c_2^2 = 3/4 < 1$, where
the second inequality follows from $s_0^2 = O(p^{3-\epsilon})$ and $\nu = \sqrt{\epsilon/4}$. This implies (18), which
completes the proof of (15).

(ii) Proof of (16) Define a parameter space

$$B_2 := \left\{ \Sigma(\theta) : \Sigma(\theta) = I_p + \frac{\nu}{\sqrt{n}} \text{diag}(\theta), \ \theta \in \Theta = \{0,1\}^p \right\}$$

for some small constant $\nu > 0$. Then, it is easy to see that $B_2 \subset U(s_0, \tau_0)$ for any $\tau_0 > 1$ and
any sufficiently large $n$. By the Assouad lemma in Cai and Zhou (2012), we have

$$\inf \max_{\Sigma(\theta) \in B_2} 2^2 \mathbb{E}_{\theta} \| \Sigma - \Sigma(\theta) \|^2_F \geq \min_{H(\theta, \theta') \geq 1} \frac{\| \Sigma(\theta) - \Sigma(\theta') \|^2_F}{H(\theta, \theta')} \min_{H(\theta, \theta') = 1} \| \mathbb{P}_\theta \wedge \mathbb{P}_{\theta'} \|.$$

For any two probability measures $\mathbb{P}$ and $\mathbb{Q}$, let $\| \mathbb{P} - \mathbb{Q} \|_1 = \int |p - q| d\mu$, where $p$ and $q$ are probability densities corresponding to $\mathbb{P}$ and $\mathbb{Q}$, respectively, with respect to a common dominating measure $\mu$. Since $\| \Sigma(\theta) - \Sigma(\theta') \|^2_F = H(\theta, \theta') \nu^2/n$ and $\| \mathbb{P}_\theta \wedge \mathbb{P}_{\theta'} \| = 1 - \| \mathbb{P}_\theta - \mathbb{P}_{\theta'} \|_1/2$, it suffices to prove that

$$\| \mathbb{P}_\theta - \mathbb{P}_{\theta'} \|_1 \leq \frac{1}{2}$$

(23)

for any $\theta, \theta' \in \Theta$ such that $H(\theta, \theta') = 1$.

By inequality (C.11) in Lee and Lee (2018), we have

$$\| \mathbb{P}_\theta - \mathbb{P}_{\theta'} \|_1 \leq n \left[ tr \{ \Sigma(\theta') \Sigma(\theta)^{-1} \} - \log \det \{ \Sigma(\theta') \Sigma(\theta)^{-1} \} - p \right]
\equiv n \left[ tr \{ \Sigma(\theta)^{-1/2} D_1 \Sigma(\theta)^{-1/2} \} - \log \det \{ \Sigma(\theta)^{-1/2} D_1 \Sigma(\theta)^{-1/2} + I_p \} \right],$$

where $D_1 = \Sigma(\theta') - \Sigma(\theta)$. Then, by Lemma C.2 of Lee and Lee (2018),

$$\| \mathbb{P}_\theta - \mathbb{P}_{\theta'} \|_1 \leq n R
\leq n c \| D_1 \Sigma(\theta)^{-1} \|^2_F \leq c \nu^2$$

for some constant $c > 0$ and any $\theta, \theta' \in \Theta$ such that $H(\theta, \theta') = 1$. Therefore, by taking $\nu^2 = 1/(2c)$, it shows that (23) holds. $\blacksquare$
Lemma 5.9 If $s_0^2(\log p)^3 = O(p^2 n)$ and $s_0^2 = O(p^{3-\epsilon})$ for some small constant $\epsilon > 0$, then

$$R_{\lambda_1, \lambda_1'}^{\gamma-1, \lambda-1} = -2 \log(1 - J_\epsilon^2) + R_{1, \lambda_1, \lambda_1'}^{\gamma-1, \lambda-1}, \quad (24)$$

where $R_{\lambda_1, \lambda_1'}^{\gamma-1, \lambda-1}$ is defined in (21), and $R_{1, \lambda_1, \lambda_1'}^{\gamma-1, \lambda-1}$ satisfies

$$E(\lambda_1, \lambda_1') \left[ \exp \left( \frac{n}{2} R_{1, \lambda_1, \lambda_1'}^{\gamma-1, \lambda-1} \right) \right] \leq \frac{3}{2}$$

uniformly over all $J$.

Proof of Lemma 5.9 Note that Lemma 5.9 corresponds to Lemma 11 of Cai and Zhou (2012). Equation (24) follows from equation (60) of Cai and Zhou (2012). However, to obtain (25), Cai and Zhou (2012) assumed $p \geq n^\beta$ for some $\beta > 1$ and $nk\epsilon^3_{np}$ is sufficiently small for all large $n$. We will show that one can still prove that (25) holds under the conditions in Lemma 5.9.

Let $A_* = (I_p - \Sigma_0)(\Sigma_0 - \Sigma_1)(\Sigma_0 - \Sigma_2)$, then by the same arguments used in pages 3-5 of Supplementary of Cai and Zhou (2012), one can show that

$$E(\lambda_1, \lambda_1') \left[ \exp \left( \frac{n}{2} R_{1, \lambda_1, \lambda_1'}^{\gamma-1, \lambda-1} \right) \right] \leq \frac{3}{2}$$

for some constant $C > 0$, and

$$E(\lambda_1, \lambda_1') \left( \max(\|A_*\|_1, \|A_*\|_\infty) \geq t \epsilon^3_{np} \right) \leq 2p \left( \frac{k^2}{p/8 - 1 - k} \right)^{t-1}$$

for every $t > 2$. Thus,

$$E(\lambda_1, \lambda_1') \left[ \exp \left( Cn \max(\|A_*\|_1, \|A_*\|_\infty) \right) \right] \leq a + \int_{\max(\|A_*\|_1, \|A_*\|_\infty) > x} E(\lambda_1, \lambda_1') \left( \max(\|A_*\|_1, \|A_*\|_\infty) > x \right) dx \leq \exp \left( \frac{1 + 2\epsilon}{\epsilon} 2Cnk\epsilon^3_{np} \right) + \int_{t>1+2\epsilon/\epsilon} 2Cnk\epsilon^3_{np} \exp \left( 2Ctnk\epsilon^3_{np} \right) \left( \frac{k^2}{p/8 - 1 - k} \right)^{t-1} dt \leq \exp \left( \frac{1 + 2\epsilon}{\epsilon} 2Cnk\epsilon^3_{np} \right) + \int_{t>1+2\epsilon/\epsilon} \exp \left( \log(2p) - (t - 1) \log \left( \frac{p/8 - 1 - k}{k^2} \right) + 2C(t + 1)nk\epsilon^3_{np} \right) dt, \quad (26)$$

$$33$$
where the second inequality follows by choosing $a = \exp\{2Cnke_{np}^3(1 + 2\epsilon)/\epsilon\}$. Since $k = \lceil c_{np}/2 \rceil - 1$, $c_{np} = \lceil s_0/p \rceil$, $\epsilon_{np} = \nu \sqrt{\log p/n}$ with $\nu = \sqrt{\epsilon/4}$ and we assume that $s_0^2(\log p)^3 = O(p^2n)$, term (26) is less than $3/2$ for any sufficiently small $\epsilon > 0$. Thus, we complete the proof if we show that term (27) is of order $o(1)$. Note that

$$(t-1) \log \frac{p/8 - 1 - k}{k^2} \geq \left(1 + \frac{1}{\epsilon}\right) \log \frac{p/8 - 1 - k}{k^2}$$

$$(t-1) \log \frac{p/8 - 1 - k}{k^2} \geq \left(1 + \frac{1}{\epsilon}\right) \log \frac{p^3/8 - p^2 - ps_0}{s_0^2} + C'$$

$$= \left(1 + \frac{1}{\epsilon}\right) \log \left\{ \frac{p^3}{s_0^2} \left( \frac{1}{8} - \frac{1}{p} - \frac{s_0}{p^2} \right) \right\} + C'$$

$$= \left(1 + \frac{1}{\epsilon}\right) \log(p^\epsilon) + C''$$

$$= (1 + \epsilon) \log p + C'',$$

for any $t > (1 + 2\epsilon)/\epsilon$ and some constants $C' > 0$ and $C'' > 0$. The third inequality follows from the assumption $s_0^2 = O(p^{3-\epsilon})$. Therefore, it implies that (27) is of order $o(1)$, which gives the desired result. ■

Appendix D: Full Conditionals

The joint posterior distribution of $\Sigma$ and $\rho = (\rho_{jk})$ with shrinkage priors (2), (3), and (4) is proportional to

$$|\Sigma|^{-n/2} \exp\left\{-\frac{1}{2} tr(S\Sigma^{-1})\right\} \prod_{j<k} \exp\left\{-\frac{\sigma_{jk}}{2\tau^2} \left( \frac{1 - \rho_{jk}}{\rho_{jk}} \right) \right\} \rho_{jk}^{a-1}(1 - \rho_{jk})^{b-1}$$

$$\prod_{j=1}^p \exp\left\{-\frac{\lambda}{2} \sigma_{jj} \right\},$$

and under partitions (7) and the transformation (8), the joint conditional posterior of $u$ and $v$ given $\rho$ (Wang, 2015) is

$$\pi(u, v | \rho, X_n) \propto \exp\left\{-\frac{1}{2} \left( n \log(v) + u^\top S_{11}^{-1} S_{11} \Sigma_{11}^{-1} u v^{-1} - 2 s_{12} \Sigma_{11}^{-1} u v^{-1} + s_{22} v^{-1} \right.ight.$$

$$\left.\left. + u^\top D^{-1} u + \lambda u^\top \Sigma_{11}^{-1} u + \lambda v \right) \right\},$$

where $D = \text{diag}(v_{12})$. 34
This gives the full conditional posteriors of \( u \) and \( v \) as follows (Wang, 2015):

\[
\pi(u \mid v, \rho, X_n) = N_{p-1}\left( \{ B + D^{-1} \}^{-1} w, \{ B + D^{-1} \}^{-1} \right),
\]

\[
\pi(v \mid u, \rho, X_n) = GIG\left( 1 - n/2, \lambda, u^T \Sigma_{11}^{-1} S_{11} \Sigma_{11}^{-1} u - 2 s_{12}^T \Sigma_{11}^{-1} u + s_{22} \right),
\]

where \( B = \Sigma_{11}^{-1} S_{11} \Sigma_{11}^{-1} v^{-1} + \lambda \Sigma_{11}^{-1} \) and \( w = \Sigma_{11}^{-1} s_{12} v^{-1} \).

Finally, to derive the full conditional of \( \rho \), we consider a reparametrization of \( \rho_{jk} \) as

\[
\phi_{jk} = \frac{\rho_{jk}}{1 - \rho_{jk}},
\]

then the shrinkage prior can be represented as follows (Armagan et al., 2011):

\[
\sigma_{jk} \mid \phi_{jk} \sim N(0, \phi_{jk} \tau^2_1), \quad \phi_{jk}^{1/2} \sim C^+(0, 1),
\]

where \( C^+(0, 1) \) denotes a half-Cauchy distribution on \((0, \infty)\). The full conditional distribution of \( \phi_{jk} \) with an additional parameter \( \psi_{jk} \) (Carvalho et al., 2010) is given as

\[
\pi(\psi_{jk} \mid \phi_{jk}, \Sigma, X_n) = Gamma(a + b, \phi_{jk} + 1),
\]

\[
\pi(\phi_{jk} \mid \psi_{jk}, \Sigma, X_n) = GIG(a - 1/2, 2\psi_{jk}, \sigma_{jk}^2 / \tau^2_1).\]

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