A remark on Fano 4-folds having (3,1)-type extremal contractions

Toru Tsukioka

February 2, 2008

Abstract

Let $X$ be the blow-up of a smooth projective 4-fold $Y$ along a smooth curve $C$ and let $E$ be the exceptional divisor. Assume that $X$ is a Fano manifold and has an elementary extremal contraction $\varphi : X \to Z$ of (3,1)-type (i.e. the exceptional locus of $\varphi$ is a divisor and its image is a curve) such that $E$ is $\varphi$-ample. We show that if the exceptional divisor of $\varphi$ is smooth, then $Y$ is isomorphic to $\mathbb{P}^4$ and $C$ is an elliptic curve of degree 4 in $\mathbb{P}^4$.

1 Introduction

As an application of the extremal contraction theory, S. Mori and S. Mukai classified smooth Fano 3-folds with Picard number greater than or equal to 2 ([MM]). We observe that many of examples in the Mori-Mukai’s list are obtained by blowing up other smooth projective 3-folds. In fact, 78 types among 88 types of smooth Fano 3-folds with $\rho \geq 2$ have $E_1$-type or $E_2$-type extremal contractions. In [BCW] the authors classified smooth Fano varieties (defined over $\mathbb{C}$) obtained by blowing-up a smooth point, in any dimension. A next step is to consider the following problem:

Problem. Let $Y$ be a smooth projective variety. Let $\pi : X \to Y$ be the blow-up along a smooth curve $C$. Classify pairs $(Y, C)$ such that $X$ is Fano.

Remark that for the toric case, the classification is done in any dimension by [S].

By the Cone and Contraction Theorem, we can take an extremal contraction $\varphi : X \to Z$ to normal projective variety such that the exceptional divisor $E$ of $\pi$ is $\varphi$-ample (see Lemma [1] below). It is easy to show that any fiber of $\varphi$ is at most of dimension 2. The author studied the case where $\varphi$ is a del Pezzo surface fibration and gave a complete classification ([12]). In higher dimensions, it seems difficult to classify the case where $\varphi$ is birational. However, in dimension 4, there are several results on the birational extremal contractions, which may be applied to solve our problem.
In this paper, we investigate the case where $\varphi$ is of $(3, 1)$-type contraction. Recall that in general, an extremal contraction $\varphi : X \to Z$ is said to be $(a, b)$-type, if $\dim(\text{Exc}(\varphi)) = a$ and $\dim(\varphi(\text{Exc}(\varphi))) = b$. So, a $(3, 1)$-type contraction for a 4-fold is a birational contraction which contracts a divisor $F$ to a curve $B$. The extremal contractions of $(3, 1)$-type for smooth 4-folds are completely classified by [TK]. In particular, it is shown that the exceptional divisor $F$ is normal and $B$ is smooth. Moreover, $\varphi |_F : F \to B$ is either a $\mathbb{P}^2$-bundle or a $Q_2$-bundle (see [TK] Main Theorem).

In section 2, we first give an example. Let $C \subset \mathbb{P}^4$ be a smooth complete intersection of one hyperplane and two hyperquadrics. Then, we see that $X = \text{Bl}_C(\mathbb{P}^4)$ has a $(3,1)$-type extremal contraction to a complete intersection of two hyperquadrics (singular along a line) in $\mathbb{P}^6$. The section 3 is devoted to show that this is the only example if we assume that $\text{Exc}(\varphi)$ is smooth. More precisely, we prove the following:

**Theorem 1.** Let $\pi : X \to Y$ be the blow-up of a smooth projective 4-fold $Y$ defined over $\mathbb{C}$, along a smooth curve $C$. Assume that $X$ is a Fano manifold and has an elementary extremal contraction $\varphi : X \to Z$ of $(3,1)$-type such that the exceptional divisor $E$ of $\pi$ is $\varphi$-ample. Let $F$ be the exceptional divisor of $\varphi$. If $F$ is smooth, then $Y$ is isomorphic to $\mathbb{P}^4$ and $C$ is a smooth complete intersection of a hyperplane and two hyperquadrics.

We will use the following lemma, which is essentially the same as in [BCW] (Lemme 2.1). For reader’s convenience, we include here the statement with its proof.

**Lemma 1.** Let $X$ be a Fano manifold and let $E$ be a non-zero effective divisor on $X$. Then there exists an extremal ray $\mathbb{R}^+[f] \subset \overline{\text{NE}}(X)$ such that $E \cdot f > 0$.

**Proof.** Since $X$ is projective, we can take a curve $\Gamma$ on $X$ such that $E \cdot \Gamma > 0$. By the Cone Theorem, there exist positive real numbers $a_i$, and extremal rational curves $f_i$ such that $\Gamma \equiv \sum a_i f_i$ (finite sum). Hence

$$0 < E \cdot \Gamma = \sum a_i (E \cdot f_i).$$

This implies that one of extremal rational curves satisfies $E \cdot f_i > 0$. 

Throughout this paper, we shall assume that the base field is the complex numbers. For a Cartier divisor $D$ and a 1-cycle $\alpha$ on a variety $X$, we denote the intersection number by $D \cdot \alpha$, but we also write $(D \cdot \alpha)_X$ when we need to clarify the variety in which the intersection number is taken.

---

1Remark that our $F$ and $B$ correspond to $E$ and $C$ in [TK].
2 An example

We give an example of a smooth Fano 4-fold $X$ obtained by blowing up along a curve such that $X$ has another $(3,1)$-type extremal contraction.

**Example** Let $C \subset \mathbb{P}^4$ be a smooth complete intersection of a hyperplane and two hyperquadrics, $\pi : X \to \mathbb{P}^4$ the blow-up along $C$, and $E$ the exceptional divisor. Let $F$ be the strict transform of the hyperplane containing $C$. Remark that $F \simeq \text{Bl}_C(\mathbb{P}^3)$ is a $Q_2$-bundle over $\mathbb{P}^1$. Let $e$ be a line in a fiber of the $\mathbb{P}^2$-bundle $\pi|_E : E \to C$, and let $f$ be the strict transform of a line in $\mathbb{P}^4$ intersecting $C$ at two points. Then we have

$$\mathcal{NE}(X) = \mathbb{R}^+ [e] + \mathbb{R}^+ [f].$$

The extremal contraction associated to the ray $\mathbb{R}^+ [e]$ is of course the blow-up $\pi : X \to \mathbb{P}^4$. Let $L := \pi^* \mathcal{O}_{\mathbb{P}^4}(1)$. The linear system $|2L - E|$ is base-point-free and defines the extremal contraction $\varphi : X \to Z$ of the ray $\mathbb{R}^+ [f]$. Indeed, we have $(2L - E) \cdot f = 0$. Note that $B := \varphi(F)$ is isomorphic to $\mathbb{P}^1$ and $\varphi|_F : F \to B$ is a $Q_2$-bundle. Thus, $\varphi$ is a $(3,1)$-type extremal contraction whose exceptional divisor is $F$. More precisely, the image $Z$ is a complete intersection of two hyperquadrics in $\mathbb{P}^6$, singular along $B \simeq \mathbb{P}^1$. To see this, we calculate $h^0(X, \mathcal{O}_X(2L - E))$ and $(2L - E)^4$.

Consider the exact sequence:

$$0 \to \mathcal{O}_X(2L - E) \to \mathcal{O}_X(2L) \to \mathcal{O}_E(2L) \to 0.$$  

Remark that $A := -K_X + (2L - E) = (5L - 2E) + (2L - E) = 7L - 3E$ is ample by Kleiman’s criterion, because $A \cdot e = 3 > 0$ and $A \cdot f = 1 > 0$. Therefore, by the Kodaira vanishing, $H^1(X, \mathcal{O}_X(2L - E)) = 0$. On the other hand, we get $h^0(X, \mathcal{O}_X(2L)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 15$. Since $\mathcal{O}_E(2L) \simeq (\pi|_E)^* \mathcal{O}_C(2)$, we have $h^0(E, \mathcal{O}_E(2L)) = h^0(C, \mathcal{O}_C(2)) = \deg(\mathcal{O}_C(2)) = 8$ (recall that $\pi|_E$ is a $\mathbb{P}^2$-bundle and $g(C) = 1$). Hence,

$$h^0(X, \mathcal{O}_X(2L - E)) = h^0(X, \mathcal{O}_X(2L)) - h^0(E, \mathcal{O}_E(2L)) = 7$$

and $|2L - E|$ defines a morphism $\varphi : X \to \mathbb{P}^6$. Now we determine the image of $X$. Note that we have $L^2 \cdot E \equiv 0$, $L \cdot E^3 = \deg C = 4$, and $E^4 = \deg N_{C/\mathbb{P}^4} = 20$. Thus,

$$(2L - E)^4 = (2L)^4 - 8L \cdot E^3 + E^4 = 4.$$  

Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^6}(2) \otimes I_Z \to \mathcal{O}_{\mathbb{P}^6}(2) \to \mathcal{O}_Z(2) \to 0.$$  

Since $h^0(Z, \mathcal{O}_Z(2)) = h^0(X, \mathcal{O}_X(4L - 2E)) = 26$, we obtain

$$h^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2) \otimes I_Z) \geq h^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) - h^0(Z, \mathcal{O}_Z(2)) = 28 - 26 = 2.$$  

It follows that there exist two linearly independent hyperquadrics in $\mathbb{P}^6$ containing $Z$. Since $\deg Z = (2H - E)^4 = 4$, $Z$ is a complete intersection of two hyperquadrics.
3 Proof of Theorem \[1\]

Denote by \( e \) a line in a fiber of the \( \mathbb{P}^2 \)-bundle \( \pi|_E : E \to C \). The key to prove Theorem \[1\] is the following:

**Lemma 2.** We have \( F \cdot e = 1 \).

*Proof.* We denote by \( (e) \) the corresponding point in \( \text{Hilb}(X) \). Let \( T \) be the reduced part of the irreducible component of \( \text{Hilb}(X) \) containing \( (e) \). Note that \( T \) is a \( \mathbb{P}^2 \)-bundle over \( C \) whose fiber \( T_c \ (c \in C) \) parametrizes lines in \( E_c := \pi^{-1}(c) \cong \mathbb{P}^2 \). In particular, \( T \) is smooth and of dimension 3.

*Step 1.* For all \( (e) \in T \) such that \( e \notin F \), we have \(#(F \cap e) = 1. \) Assume the contrary, i.e. there exists \( (e_0) \in T \) such that \( e_0 \notin F \) and \(#(F \cap e_0) \geq 2 \).

Remark that \( \varphi(e_0) \neq B \). Let \( x_i \ (i = 1, 2) \) be two distinct points in \( F \cap e_0 \) and let \( b_i := \varphi(x_i) \). Consider the incidence graph:

\[
\begin{array}{ccc}
  V & \xrightarrow{p} & X \\
  q \downarrow & & \\
  T & &
\end{array}
\]

We define \( V_i := p^{-1}(E \cap \varphi^{-1}(b_i)) \) and \( T_i := q(V_i) \) for \( i = 1, 2 \). Note that \( \dim V_i = \dim (E \cap \varphi^{-1}(b_i)) + 1 \) because \( p \) is a \( \mathbb{P}^1 \)-bundle. We observe that \( q|_{V_i} \) is a finite map. Indeed, if not, there exists \( t \in T_i \) such that \( q^{-1}(t) \subset V_i \). Then \( e_t := p(q^{-1}(t)) \) is contracted by \( \varphi \). This contradicts to our assumption that \( E \) is \( \varphi \)-ample. It follows that \( \dim T_i = \dim V_i = 2 \ (i = 1, 2) \). Note also that \( (e_0) \in T_1 \cap T_2 \). Now, we have

\[
\dim(T_1 \cap T_2) \geq \dim T_1 + \dim T_2 - \dim T = 2 + 2 - 3 = 1.
\]

So, we can take an irreducible curve \( A \subset T_1 \cap T_2 \) passing through \( (e_0) \). Then \( q^{-1}(A) \) is a ruled surface having two exceptional curves \( V_i \cap S \ (i = 1, 2) \), a contradiction.

*Step 2.* Consider \( M := (F \cap E)_{\text{red}} \). By Step 1, we see that for each \( c \in C \), \( e_c := (F \cap E_c)_{\text{red}} \) is a line in \( E_c \cong \mathbb{P}^2 \). So, \( \pi|_M : M \to C \) is a \( \mathbb{P}^1 \)-bundle. In particular \( M \) is irreducible. We can write \( E|_F = mM \) with \( m \in \mathbb{Z}^+ \). We have

\[
(mM \cdot e_c)_F = (E|_F \cdot e_c)_F = (E \cdot e_c)_X = -1
\]

By assumption, \( F \) is smooth. So, \( M \subset F \) is a Cartier divisor and \( (M \cdot e_c)_F \) is integer. It follows that \( m = 1 \), i.e. the intersection \( F \cap E \) is transversal.

We conclude that \( F \cdot e = #(F \cap e) = 1. \)

**Proof of Theorem \[1\]** By the proof of Lemma \[2\], \( \pi|_M : M \to C \) is a \( \mathbb{P}^1 \)-bundle and \( (M \cdot e_c)_F = -1 \). So, \( \pi|_F : F \to F' := \pi(F) \) is the blow-up along

\[2\] We mean by \( #(F \cap e) \) the number of points on \( F \cap e \) without multiplicity.
$C$, and $F'$ is smooth. On the other hand, by [Tk], $\varphi|_F : F \to B$ is either a $\mathbb{P}^2$-bundle or a $Q_2$-bundle. Therefore $F$ is a Fano 3-fold with $\rho(F) = 2$. By assumption, $F$ is smooth. So, by the Mori-Mukai’s list, the pair $(F', C)$ is one of the following:

1. $F' \simeq \mathbb{P}^3$ and $C$ is a line;
2. $F'$ is a hyperquadric $Q_3 \subset \mathbb{P}^4$ and $C = H \cap H'$ with $H, H' \in |\mathcal{O}_{Q_3}(1)|$;
3. $F' \simeq \mathbb{P}^3$ and $C = Q \cap Q'$ with $Q, Q' \in |\mathcal{O}_{\mathbb{P}^3}(2)|$.

In the case (3), $C$ is an elliptic curve. So, $Y$ is a Fano manifold by [W] (Proposition 3.5). In the cases (1) and (2), we have $N_{C/F'} \simeq \mathcal{O}_C(\mathbb{P}^2)$. Since there exists an inclusion of normal bundles $N_{C/F'} \subset N_{C/Y}$, $N_{C/Y}$ cannot be isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. So, $Y$ is a Fano manifold again by [W].

Now, by Lemma 1, we can take an extremal ray $\mathbb{R}^+_m$ such that $F' \cdot m > 0$. Then, by Proposition 1 below, we have $\rho(Y) = 1$. In particular $F'$ is ample.

Let $f$ be a minimal rational curve of the extremal contraction $\varphi$. We obtain the following table of intersection numbers (due to [Tk] and [MM]):

| case | $F \cdot f$ | $E \cdot f$ |
|------|-------------|-------------|
| (1)  | $-1$ or $-2$ | 1           |
| (2)  | $-1$        | 1           |
| (3)  | $-1$        | 2           |

Let $f' := \pi_* f$. Note that $F' \cdot f' = (\pi^* F') \cdot f = (F + E) \cdot f$. In the cases (1) and (2), we have $F' \cdot f' \leq 0$, a contradiction because $F'$ is ample. So, only the case (3) (in which we have $F' \cdot f' = 1$) is possible, and $(Y, F') \simeq (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$. Consequently, $C$ is the complete intersection $F' \cap Q \cap Q'$ with $F' \in |\mathcal{O}_{\mathbb{P}^4}(1)|$ and $Q, Q' \in |\mathcal{O}_{\mathbb{P}^4}(2)|$.

It remains to prove the following:

**Proposition 1.** Let $Y$ be a smooth projective variety of dimension $n \geq 4$ and $D$ a prime divisor on $Y$ with $\rho(D) = 1$. Assume that there exists an extremal contraction $\mu : Y \to V$ of ray $\mathbb{R}^+_m$ with $D \cdot m > 0$, $m$ being a minimal rational curve of the ray. If there exists a smooth curve $C \subset D$ such that the blow-up $X := \text{Bl}_C(Y)$ is a Fano manifold, then we have $\rho(Y) = 1$. Moreover, if $D$ is isomorphic to $\mathbb{P}^{n-1}$, then we have $(Y, D) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

**Proof.** We shall consider two cases:

1. there exists $v_0 \in V$ such that $\dim(\mu^{-1}(v_0) \cap D) \geq 1$;
2. $\dim(\mu^{-1}(v) \cap D) = 0$ for all $v \in V$. 

5
In the case (1), there exists a curve $B \subset \mu^{-1}(v_0) \cap D$. So, we can write $B \equiv bm$ with $b \in \mathbb{R}^+$. Since $\rho(D) = 1$, any curve in $D$ is numerically equivalent to a multiple of $m$. Hence, $\mu(D)$ is a point. We also have $D \cdot B > 0$. Now, by Proposition 4 of [12], we conclude that $\rho(Y) = 1$.

We show that the case (2) is impossible. In this case, any fiber of $\mu$ is at most of dimension 1. So, by [A] (see also [W] Theorem 1.2), $\mu$ is either, a $\mathbb{P}^1$-bundle, a conic bundle, or a blow-up along a smooth subvariety of codimension 2 in a smooth projective variety. If $\mu$ is a $\mathbb{P}^1$-bundle, take a fiber $m$ passing through a point on $C$. Let $\tilde{m}$ be the strict transform by the blow-up $\pi : X \to Y$ of the exceptional divisor $E$, we have $E \cdot \tilde{m} \geq 1$, so that

$$K_X \cdot \tilde{m} = K_Y \cdot m + (n - 2)E \cdot \tilde{m} \geq -2 + (n - 2) = n - 4 \geq 0,$$

which is absurd because $X$ is a Fano manifold.

If $\mu$ is a conic bundle, the extremal rational curve $m$ is a component of a singular fiber of $\mu$. Let $\Delta$ be the discriminant locus and let $\tilde{\Delta} := \mu^{-1}(\Delta)$. The assumption $D \cdot m > 0$ implies $\tilde{\Delta} \cap D \neq \emptyset$. Since $\rho(D) = 1$, the non-zero effective Cartier divisor $\tilde{\Delta}_|D$ is ample. Therefore,

$$(\tilde{\Delta} \cdot C)_Y = (\tilde{\Delta}_|D \cdot C)_D > 0,$$

so that $\tilde{\Delta} \cap C \neq \emptyset$. Now, we can take a singular fiber $\mu^{-1}(v_0)$ ($v_0 \in \Delta$) meeting $C$. Let $m_0 \subset \mu^{-1}(v_0)$ be a component such that $m_0 \cap C \neq \emptyset$. Then, we have a contradiction as in the case of $\mathbb{P}^1$-bundle. The case of a blow-up along a centre of codimension 2, can be ruled out by using a same argument for the exceptional divisor of $\mu$ in place of $\tilde{\Delta}$.

Consequently, only the case (1) is possible, so that we have $\rho(Y) = 1$. If $D \simeq \mathbb{P}^{n-1}$, by [BCW](Lemme 4) we conclude that $(Y, D) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. 

Our assumption that $F = \text{Exc}(\varphi)$ is smooth, is used in the proof of Lemma 2 (only for Step.2) and in the proof of Theorem 1 in order to apply to $F$ the Mori-Mukai’s classification of smooth Fano 3-folds. So, it is natural to ask whether Theorem 1 remains true without the smoothness of $F$. Concerning to this question, it is worth seeing the following:

**Example** (A degenerate case of the example in Section 2). We consider the union of two smooth conics $C = C_1 \cup C_2 \subset Y := \mathbb{P}^4$ obtained as complete intersection of a hyperplane and two hyperquadrics. We assume that $C_1$ and $C_2$ meet at two distinct points. Let $\pi : X \to \mathbb{P}^4$ be the blow-up along the ideal $I_{C_1 \cup C_2}$ and $E$ the exceptional divisor. Let $F$ be the strict transform of the hyperplane containing $C = C_1 \cup C_2$. Then $F$ is a $Q_2$-bundle over $\mathbb{P}^1$ having exactly two ordinary double points. Remark that $F$ is isomorphic to the blow-up of $\mathbb{P}^3$ along the ideal $I_{C_1 \cup C_2}$. Moreover, $F$ can be realized as divisor in $\mathbb{P}^1 \times \mathbb{P}^3$ by the equation $sX_2X_3 + t(X_0^2 + X_1^2 + X_2^2 + X_3^2) = 0$, where $(s : t)$ (resp. $(X_0 : X_1 : X_2 : X_3)$) is the homogeneous coordinates of
The fiber over \((1:0)\) is two planes \(P_i\) \((i = 1, 2)\) and the two ordinary double points lie on the line \(P_1 \cap P_2\).

As in Section 2, we see that the linear system \(|\pi^*\mathcal{O}_{\mathbb{P}^4}(2) - E|\) defines a \((3,1)\)-type contraction \(\varphi : X \to Z\) to complete intersection of two hyperquadrics in \(\mathbb{P}^6\), and its exceptional divisor is \(F\). This gives an example of \((Y, C)\) such that \(F = \text{Exc}(\varphi)\) is singular. However \(X\) is also singular along two rational curves over the two intersection points of \(C_1 \cap C_2\).

### 4 Related results

Let \(X\) be a Fano manifold and let \(\iota_X\) be its pseudo-index, i.e. the minimum of the anti-canonical degrees \((-K_X \cdot C)\) for rational curves \(C\) on \(X\). In [BCDD], the authors discuss the inequality (“generalized Mukai conjecture”):

\[
\rho(X)(\iota_X - 1) \leq \dim X
\]

and prove it in dimension 4. The essential part is to show that if \(\iota_X = 2\), then \(\rho(X) \leq 4\). Concerning to this, we have the following:

**Proposition 2.** Let \(\pi : X \to Y\) be the blow-up of a smooth projective variety \(Y\) of dimension \(n \geq 4\) along a smooth curve \(C\) and let \(E\) be the exceptional divisor. Assume that \(X\) is a Fano manifold and there is another blow-up \(\varphi : X \to Z\) (different from \(\pi\)) along a smooth curve \(B\). Let \(F\) be the exceptional divisor of \(\varphi\). Then, we have \(E \cap F = \emptyset\).

**Proof.** Assume \(E \cap F \neq \emptyset\). Take \(a \in C\) and \(b \in B\) such that \(E_a \cap F_b \neq \emptyset\). Then we obtain \(\dim(E_a \cap F_b) \geq \dim E_a + \dim F_b - \dim X = n - 4\). So, if \(n \geq 5\), there is a curve contained in \(E_a \cap F_b\) and then contracted by both \(\pi\) and \(\varphi\). This is absurd because we assume \(\pi \neq \varphi\). Therefore, we have \(n = 4\). By (the proof of) Theorem 1, \(\varphi|_F : F \to B\) cannot be a \(\mathbb{P}^2\)-bundle. So, the case \(E \cap F \neq \emptyset\) is impossible. \(\blacksquare\)

We are now able to give a simple proof of a result in [BCDD].

**Theorem 2** (see [BCDD] Théorème 3.9). Let \(X\) be a Fano manifold of dimension \(\geq 4\) whose birational contractions are all blow-ups along smooth curves in smooth projective varieties. Assume that \(X\) has at least one birational contraction. Then, we have \(\rho(X) \leq 3\).

**Proof.** Let \(E\) be an exceptional divisor on \(X\). By Lemma 1, we can take an extremal ray \(\mathbb{R}^+ [f] \subset \text{NE}(X)\) such that \(E \cdot f > 0\). Then, by assumption and by Proposition 2 above, the associated contraction \(\mu := \text{cont}_{\mathbb{R}^+ [f]} : X \to Z\) is of fiber type. So, there is a surjection \(\mu|_E : E \to Z\). Hence, we have \(\rho(Z) \leq \rho(E) = 2\). Consequently, \(\rho(X) = \rho(Z) + 1 \leq 3\). \(\blacksquare\)
References

[A] T. Ando, On extremal rays of the higher-dimensional varieties. Invent. Math. 81, (1985) 347–357.

[BCDD] L. Bonavero, C. Casagrande, O. Debarre and S. Druel, Sur une conjecture de Mukai. Comment. Math. Helv. 78, (2003) 601–626.

[BCW] L. Bonavero, F. Campana and J. Wiśniewski, Variétés complexes dont l’éclatée en un point est de Fano. C. R. Math. Acad. Sci. Paris 334, (2002) 463–468.

[MM] S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geq 2$. Manuscripta Math. 36, (1981/82) 147–162.
    Erratum: Manuscripta Math. 110, (2003) 407.

[S] H. Sato, Toric Fano varieties with divisorial contractions to curves. Math. Nachr. 261/262, (2003) 163–170.

[Tk] H. Takagi, Classification of extremal contractions from smooth fourfolds of (3,1)-type. Proc. Amer. Math. Soc. 127, (1999) 315–321.

[T1] T. Tsukioka, Del Pezzo surface fibrations obtained by blow-up of a smooth curve in a projective manifold. C. R. Acad. Sci. Paris 340, (2005) 581–586.

[T2] T. Tsukioka, Classification of Fano manifolds containing a negative divisor isomorphic to projective space. Geometriae Dedicata 123, (2006) 179–186.

[W] J. Wiśniewski, On contractions of extremal rays of Fano manifolds. J. Reine Angew. Math. 417, (1991) 141–157.

Department of Mathematics
Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku,
Tokyo 152-8551, JAPAN

email: tsukiokatoru@yahoo.co.jp