The role of shape disorder in the collective behaviour of aligned fibrous matter

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We study the compression of bundles of aligned macroscopic fibers with intrinsic shape disorder, as found in human hair and in many other natural and man-made systems. We show by a combination of experiments, numerical simulations and theory how the statistical properties of the shapes of the fibers control the collective mechanical behaviour of the bundles. This work paves the way for designing aligned fibrous matter with pre-required properties from large numbers of individual strands of selected geometry and rigidity.

INTRODUCTION

In natural and composite bundles of nearly aligned fibers, as for instance in hair tresses, ponytails and other natural fagots, the waving spontaneous shapes of the strands allow for an intrinsic fluffiness of the materials [1–4]. This was first discussed by Van Wyk [5] who proposed an equation of state (EOS) for the material compressibility by suggesting that the response of wool stacks to compression is mostly controlled by the bending modes of the fiber strands. The suggestion has been widely discussed in work related to fibrous matter [6], in explanations of the compressibility of textiles [7], matted fibers [8] and of bulk samples of wool fibers [9], for studies of the shape of hair ponytails [10] or for predicting droplet formation at the tip of wet brushes [11], or even to study the mechanical response of aegagropilae [12].

Since van Wyk seminal work [5] many models have been proposed to understand the collective mechanical properties elasticity of stacks of randomly oriented straight fibers [13–16]. Recently Broedersz et al. have shown that elastic properties of such networks is governed by bending elasticity for low connectivity and by stretching elasticity for high connectivity [16]. In this paper, we focus on the case of highly aligned fibers with disordered shapes.

FIG. 1: Middle image: compression of a polypropylene stack and summary of the different samples (steel wool, denim and polypropylene fiber) investigated in this paper. In circles, zoom on a volume of typical size 1 cm. External pictures: typical shape of a individual fiber of each samples (~ 5 cm length).
The statistical mechanics nature of this challenge was first recognised by Beckrich et al. 17 who computed the compression modulus of fiber stacks within a self-consistent mean-field treatment in two dimensions, and predicted the shapes of brooms and other fluffy cones made from fibers. Here we test the validity of a statistical mechanics treatment of this problem by studying both experimentally and numerically the compression behaviour of stacks of fibers with intrinsic shape disorder. A generalization of the mean field theory introduced in [17] compares favourably with our results, revealing the key statistical and mechanical factors that control the EOS of fibrous matter.

MATERIALS AND METHODS

Experimental systems

![Image](image)

\[ L_y \sim 5 \text{ cm} \]

\[ L_x \sim 10 \text{ cm} \]

FIG. 2: (Top) Example of a fiber stack of steel wool (SW1) with typical characteristic length scales. (Bottom) Single fiber shape and definition of \( \zeta_0 (x) \).

Experiments were performed on 6 different fiber bundles. A first class of samples was obtained by unbraiding different commercial climbing ropes of polypropylene (PP1, PP2 and PP3) and standard denim cloth (DEN). Fiber bundles were formed by stacking manually a high number (hundreds or thousands) of individual fibers of the same length, ensuring a strong alignment. A second class of samples (SW1 and SW2) was obtained from standard steel wools (Gerlon®). In this case, compression experiments were directly performed on the purchased samples.

The fiber sections, as observed by optical microscopy, do not have regular shapes, and we measured the following approximate diameters: PP1 and PP2 (~ 100 \( \mu \text{m} \)); PP3 (~ 20 \( \mu \text{m} \)); DEN (~ 500 \( \mu \text{m} \)); SW1 (~ 200\( \mu \text{m} \)) and SW2 (~ 300 \( \mu \text{m} \)).

The individual mass of several fibers of each stack was measured allowing us to determine the linear mass \( \mu \). Together with the full mass of the stack, it allows us to estimate the number \( N \) of fibers per bundle and the transverse density \( \rho = \sqrt{N}/L_y \) (see also ESI).

Experimental methods

Compression experiments were made using two different experimental setups.

The first one is homemade, from a precision balance used as a force sensor (Mettler Toledo®) and a motorized translation stage. With such setup, we got a very good resolution (accuracy, \( \pm 2 \times 10^{-4} \text{ N} \)) in a wide range of measurements (6 decades from \( 2 \times 10^{-4} \text{ N} \) to 40 N). The stiffness of the device is of the order of 10 MPa, and we have systematically corrected for scale plate displacement.

The second setup is a commercial testing system (Electropuls™ E3000 from Instron®, 10 N sensor), with a lower force range (3 decades \( 10^{-2}-10 \text{ N} \), accuracy \( \pm 10^{-2} \text{ N} \)) but a higher stiffness (~ 100 MPa). This setup was mainly used for stress-relaxation experiments.

With both setups and for each experiments we measured the distance \( D_0 \) between compression plates at first contact, the distance \( D \) between compression plates at each step of compression, the projected length \( L_x \) and \( L_y \) and the force \( F \). We calculated the real stress \( P = F/L_xL_y \) and \( D_0/D \).
Theoretical description

Individual fiber shape deformations are associated to the bending energy given by:

\[ H_{\text{bend}} = \frac{\kappa}{2} \sum_{n=1}^{N} \int_{0}^{L} dx \left[ \zeta''_{n}(x) - \zeta''_{0,n}(x) \right]^2 \]

(1)

where \( \zeta_{n}(x) \) is the actual shape and \( \zeta_{0,n}(x) \) the function describing the spontaneous shape of fiber \( n \) (see Fig. 1a). \( L \) corresponds to the chain projection on the \( x \) axis. The bending modulus \( \kappa \) is an intrinsic properties of a fiber, directly related to the Young’s modulus and the fiber geometry. Equation (1) is valid in the limit of small curvature (\( q\zeta_{n} \ll 1 \)) that is a very good approximation in the case of fiber stacks.

Numerical simulations

Numerical simulations were performed using the Steepest Descent method [18] to find the equilibrium conformations of compressed fibers represented by the bead-spring model [19, 20] sketched in Fig. 4 with \( N_b \) beads per fiber. Interactions between beads are modelled by an effective Hamiltonian containing three terms, \( H = H_{\text{LJ}} + H_{\text{bond}} + H_{\text{angle}} \) with \( H_{\text{LJ}} = 4\epsilon \left[ (\sigma/r)^{12} - (\sigma/r)^{6} \right] + \epsilon \) for \( r/\sigma \leq 2^{1/6} \) the truncated and shifted Lennard-Jones (LJ) potential [21] describing the repulsive interaction between non-neighboring monomers, \( H_{\text{bond}} = k_b/2 \left( r - r_0 \right)^2 \) the connectivity potential between two adjacent monomers and \( H_{\text{angle}} = k_{\theta}/2 \left( \theta - \theta_0 \right)^2 \) the angular potential that controls the chain stiffness and spontaneous shape. The set of non-vanishing reference angles \( \{ \theta_{0,i} \} (i = 1, ..., N_b) \) between any three consecutive monomers – see Fig. 4 – are chosen such that the local fiber gradients remain much smaller than unity. Fiber shapes can thus be also described by a single-valued function \( \zeta_{0}(x) \) which allows, in the limit of large fibers, to directly compare numerical results against continuous elastic theories with \( \kappa = k_\theta r_0 \) the bending modulus.
FIG. 4: Sketch of the geometry of the numerical simulations at two different compression distances. The insets show the shapes of two neighbouring fibers with the definition of the angles $\theta_{0,i}$ and $\theta_{i}$ between consecutive beads $i-1$, $i$ and $i+1$.

SINGLE FIBER CHARACTERIZATION

Shape characterization

As we will see below, the spontaneous shape of the fibers $\zeta_0(x)$ is a crucial determinant of the collective mechanical behavior of the bundle. We expand $\zeta_0(x)$ on the basis of the eigenfunctions $\Phi_q(x)$ of the square Laplacian operator that describes bending curvature elasticity [22] (see also the ESI), $\zeta_0(x) = \sum_i \zeta_{0,i} \Phi_i(x)$. The corresponding spectra for average values $\langle \zeta_{0,i}^2 \rangle$ are displayed on Fig. 5. The dominant amplitudes are present for $q \leq 2$ mm$^{-1}$. The dispersion of the variance is wide and spreads over 6 decades. A dominant wavelength appears very clearly for PP2, PP3 and DEN. All spectra exhibit a power law regime which spreads over two or more decades of wavenumber values. The power exponent denoted $\alpha$ is related to the roughness of the fiber and the prefactor of the power-law to its root mean square amplitude.

Mechanical properties

To determine experimentally the bending modulus $\kappa$ of fibers we performed two different types of experiments. The first one consists in measuring the oscillation period of a horizontal fiber. This method is particularly suited for steel fibers (SW1 and SW2) which are sufficiently rigid. However, we have not been able to apply it to synthetic fibers (PP1, PP2, PP3 and DEN) for which we have developed an original experience of stretching, inspired by the work of Kabla and Mahadevan [4]. Both methods are detailed below.
FIG. 5: (Color online) Average spectra $\langle \zeta_2^2 \rangle$ extracted from measurements of at least 40 fibers of the 6 experimental systems, and best power law fits with exponent $\alpha$ (solid line) for PP1 (a), SW1 (b), SW2 (c), PP2 (d), PP3 (e) and DEN (f). The insets display typical experimental fiber shapes.

**Single fiber oscillations**

The position of the end of a horizontal fiber is measured over time during oscillation experiment (see Fig. 6). A simple Fourier transform makes it possible to precisely characterize the fundamental frequency $f_0$ of the system (see Fig. 6 inset), which is related to the modulus of curvature by the equation:

$$\kappa = 3.194 \times mL^4 f_0^2,$$

where $m$ is the fiber mass and $L$ its length.

**Single fiber stretching experiments**

To determine experimentally the bending modulus $\kappa$ of softer fibers we performed stretching experiments. At each stage of the stretching experiment, a snapshot is taken whereby the total length, the projected length and the shape analysis of the fiber are calculated. To avoid artificial unfolding, the fibers are fixed with a length shorter than their projected length (slight buckling). The reference state is obtained after the first stretching steps, when the fiber reaches its natural length.

As an example, the shape of a PP1 fiber for a few steps of a stretching experiment is shown in Fig. 7. Steps 6 to 12 show clearly that the largest wavelengths are first unfolded, as confirmed by the spectra where the amplitudes of the modes $q_1 = 0.5$ mm$^{-1}$ and $q_2 = 1.0$ mm$^{-1}$ (ie $\lambda_1 \approx 12$ mm and $\lambda_2 \approx 6$ mm) decreased significantly more than the other ones. From step 15, the deformation corresponds to pure elongation. The fiber broke before we could unfold all the modes.
FIG. 6: Position of the end point of a SW2 fiber during an oscillation experiment. In the inset, the spectral density (SD) obtained by a simple Fourier Transform of the signal.

FIG. 7: Shape of a PP1 fiber at different steps of a stretching experiment.

The force measured during the stretching experiment is shown in Fig. 8 as a function of the length ratio (Fig. 8). Using equation (3):

$$\frac{L}{L_0} = 1 + \frac{f}{\mu_0} - \frac{1}{4} \left(1 + 2 \frac{f}{\mu_0}\right) \sum_q q^6 \zeta_{0,0}^2 + \frac{\kappa}{2\mu_0} \sum_q \frac{q^6 \zeta_{0,0}^2}{f/\kappa + q^2}.$$  

(3)

it is possible to deduce the bending modulus \(\kappa\) (J.m) and the stretching modulus of the fiber \(\mu_0\) (N).

Best fits of the experimental curves by equation (3) are shown in Fig. 8 and the moduli are summarized in Table I.

**Single fibers characteristics**

As a summary, we carefully characterized the shapes and the bending modulus of many individual fibers of the 6 samples that we studied. We computed the average shape spectrum of the different samples \(\langle q_i \zeta_{0,0}^2 \rangle\) and the
average of the bending modulus. We also estimated the number of fibers present in each stacks ($N$) as well as the transverse fiber density $\rho$, a measure of the density of stress planes.

### TABLE I: Mean value of linear mass density ($\mu$), bending ($\kappa$) and stretching ($\mu_0$) moduli and density of stress planes ($\rho$).

|       | $\mu$ ($\mu g.cm^{-1}$) | $\kappa \times 10^{12}$ (J.m) | $\mu_0$ (N) | $\rho$ (mm$^{-1}$) |
|-------|-------------------------|-------------------------------|-------------|------------------|
| PP1   | 24 ± 1                  | 3.2 ± 1.7                    | 1.0 ± 0.2   | 1.2 ± 1          |
| PP2   | 18 ± 1                  | 3.5 ± 1.8                    | 1.3 ± 0.2   | 2.5 ± 2          |
| PP3   | 4.7 ± 0.5               | 0.43 ± 0.3                   | 1 ± 0.2     | 6.8 ± 4          |
| SW1   | 110 ± 2                 | 78 ± 42                      | -           | 1.6 ± 1          |
| SW2   | 610 ± 11                | 160 ± 70                     | -           | 0.6 ± 0.4        |
| DEN   | 850 ± 10                | 3800 ± 3200                  | 92 ± 46     | 1 ± 0.5          |

**MECHANICAL PROPERTIES OF FIBER STACKS**

*Theoretical description*

**Ordered stacks: linear elasticity**

A perfectly ordered system, consisting of a stack of sinusoidal fibers $\zeta_0(x) = d_0/2 \cos(2\pi x/\lambda_0 + \varphi_0)$, with ($\lambda_i = \lambda$ for all $i$, $\varphi_i = (1 - (-1)^i)\pi/2$), will be referred to as the reference system (ref) in the following. In such an ideal configuration, the EOS can be computed easily from the functional minimization of Eq. [1] leading to

$$P^{(2d)}_{\text{ref}} = 192 \frac{k d_0}{\lambda^4} \left( \frac{D_0 - D}{D_0} \right).$$

This reference model will be compared to the results of numerical simulations to validate our method.
Self-consistent model

The challenge for a statistical mechanical treatment of aligned fibrous systems is thus to connect the information contained in spectra such as those of Figs. 5 and the mechanical behaviour under compression stress. We follow here a two-dimensional approach first introduced in [17]. Briefly, fiber shape deformations are associated to the bending energy given by Eq. (1). We assume that forces between first-neighbours dominate the interaction energy, an exact assumption for excluded volume potentials in two-dimensions. Assuming a quadratic form for the interactions, with a compression modulus $B(d)$, the effective energy can be written as:

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_{\text{bend}} + \frac{B(d)}{2} \sum_{n=1}^{N} \int_{0}^{L} dx \left[ \zeta_{n+1}(x) - \zeta_{n}(x) \right]^2. \quad (5)$$

By functional minimization, we deduce the equilibrium shapes of the fibers and calculate the energy density:

$$\langle e \rangle = \frac{\kappa}{2d} \sum_{q} q^4 \left( 1 - \sqrt{\frac{q^4}{4B(d) \kappa} + q^4} \right) \langle \zeta_{0,q}^2 \rangle \quad (6)$$

where $d$ is the mean distance between fibers. The compression modulus $B_{\text{self}}(d)$ has to be determined self-consistently from

$$B_{\text{self}}(d) = \frac{\partial^2 \langle d \langle e \rangle \rangle}{\partial d^2}, \quad (7)$$

the compressive stress given by $P_{\text{self}}^{(2d)} = -\partial \langle d \langle e \rangle \rangle / \partial d$ can then be calculated using:

$$P_{\text{self}}^{(2d)}(d) = -\int_{d_0}^{d} B_{\text{self}}(d') dd'. \quad (8)$$

Eq. (6) is the key result of this approach, it relates the shape disorder distribution measured by $\langle \zeta_{0,q}^2 \rangle$ to the energy density of the stack through a mechanical kernel accounting for bending rigidity and fiber interactions.

Experimental results

![Stress vs time curves for PP1 and SW1](image)

FIG. 9: (Color online) Stress vs times curves for PP1 (□, left axis) and SW1 (△, right axis) at 50% of deformation in 2s. Orange area corresponds to the duration of stress experiments (20 s) presented on Fig. 10 and 11.

We first performed stress relaxation experiments over a long times ($\sim$ 10 hours). All materials demonstrate a complex kinetic behavior, highlighting the effect of fiber rearrangement (see Fig. 9). Polymer samples (see PP1)
exhibit a relaxation of almost 50% over 10 hours when steel wools relaxes only about 5%. This is probably related to the highest friction coefficient between steel fibers compared to the polymer ones. In all cases, it is clear from the relaxation experiments that doing the full compression experiments at high enough strain rate is important to avoid stress relaxation by fiber rearrangement. We have thus decided to perform compression experiments on all samples in 20 s, corresponding to a maximum stress relaxation of 10% for PP samples and less than 1% for steel wools.

Stress-deformation curves are given on Fig. 10(d) and Fig. 11. All materials exhibit a strong non linear elasticity, spreading over five decades of stress values. Using the experimentally determined spectra $\zeta_{0,q}$ and the value of $\kappa$, we apply the self-consistent model to interpret the compression. The whole method is illustrated in figure 10 in the case of the sample (SW1).

We assume that the stacks of fibers consist of the transverse sum of independent planes of effective density $\rho$, that can be estimated from the geometrical characteristics of the bundle (see Table I). All planes contributing with $p^{(2d)}(D)$ to the total pressure $p^{(3d)}(D)$, we write

$$p^{(3d)}(D) = \rho p^{(2d)}(D). \quad (9)$$

For very weak deformations ($D_0/D \sim 1\%)$, we observe a linear behavior corresponding to the deformation of the largest wavelength $\lambda_{max}$. This linear elasticity is well described by $P^{(3d)} = \rho P^{(2d)}_{\text{ref}}$, where $P^{(2d)}_{\text{ref}}$ is given by equation 4, and represented by dotted line on Figures 10 and 11.

For larger deformations ($D_0/D \gg 1\%)$, we test the predictive power of Eq. (6) by solving numerically the self-consistent equation Eq. (7) for the experimentally determined distributions $\langle \zeta_{0,q}^2 \rangle$. As a result we obtain the 2d-pressure $p^{(2d)}_{\text{self}}(D)$ and $p^{(3d)}_{\text{self}} = \rho p^{(2d)}_{\text{self}}(D)$ that is represented as dotted-dashed lines on Fig. 10 for SW1. If the overall shape of the data is well described by the theory, it is clear that it is necessary to introduce a scaling factor $\Phi_{\text{exp}}$ to be able to reproduce the data for large deformation (see dashed line).

Finally, we calculate the total pressure:

$$P_{\text{tot}}(D) = \rho \left( P^{(2d)}_{\text{Ref}}(D) + \Phi_{\text{exp}} p^{(2d)}_{\text{self}}(D) \right). \quad (10)$$

where $\Phi_{\text{exp}}$ is the only fitting parameters. $P_{\text{tot}}(D)$ is represented as a solid line on Figure 10.

The experimentally determined compression stress $P$ and the theoretical analysis for the six experimental fiber stacks studied in the paper are given in figure 11 with same convention as for figure 10. For all studied samples, we
observe a very good agreement between the three-dimensional experimental compression curves and the theoretical description.

![Graph showing compression stress P for three experimental fiber stacks and one reference system.](image)

**FIG. 11**: Experimentally determined compression stress $P$ for the three experimental fiber stacks studied in the paper: PP1 ($\square$); PP2 ($\bullet$); PP3 ($\triangle$); DEN ($\bigcirc$); SW1 ($\triangle$); SW2 ($\blacklozenge$). Dashed lines represent the discrete self-consistent theory $P_{\text{tot}}^{(3d)}$ using experimentally determined spectrum, dotted line the linear elasticity behavior $P_{\text{ref}}^{(3d)}$ and solid line the total pressure $P_{\text{tot}}(D)$.

**Numerical simulations**

To further understand how fiber shape disorder determines the compression of the stack, we performed numerical simulations on 2-dimensional systems. We investigated three classes of two-dimensional disordered systems. In this configuration, fiber rearrangements are forbidden.

First, in order to validate our simulation method, we consider a stack of perfectly ordered sinusoidal fibers referred to as the reference system (ref) in the following (see figure 12 top inset).

As a second step, we introduce phase disorder to the system by choosing a random translation phase shift $\varphi_i$ homogeneously distributed in phase space $\varphi_i \in [0, 2\pi]$, referred to as single mode disorder systems (SMD).

Finally, we investigate fiber stacks with a power-law disorder (PLD), inspired by the features of the experimentally measured distributions (Figs. 5). This allowed us, by varying the exponent $\alpha$ between 2 and 5, to test the self-consistent model by exploring a disorder range wider than that of the experimental systems.
Reference system (ref).

Numerical simulations for the reference system are shown in Fig. 12 as (●). They exhibit linear elasticity, and the EOS is well described in a large compression range, without any fitting parameter, by Eq. 4. At very high compressions, the fibers are fully squeezed and the stress is dominated by local excluded volume effects leading to a strong increase well above the bending contribution.

Single mode disorder systems (SMD)

Numerical simulations on disordered systems (SMD) distinctly exhibit non-linear elastic behavior (see Fig. 12). The compression behaviour is determined by phase shift randomness, and the distribution of wavelengths plays only a minor role, see also ESI for numerical results on a wider range of disorder parameters. A simple analytical approach accounting for phase disorder for a sinusoidal spontaneous shape of wavelength \( \lambda \) and amplitude \( D_0 \) allows calculating the normalized stress as

\[
\frac{P^{(2d)} \lambda^4}{192 \pi D_0} = 2 \int_0^{\pi \Delta u} \frac{1 + \cos (2\pi u) - D/D_0}{(1 - 2u)^3 (1 + 4u)^2} du
\]

where \( 2\pi \Delta u = \arccos (D/D_0 - 1) \). Numerical simulation data follows a master curve well described by our prediction without any fitting parameters. We also obtained similar results for different fiber systems covering a range of parameters \( k, \lambda \) and \( D_0 \) (see also ESI).

Power-law disorder system (PLD): numerical simulations

Finally, we investigate fiber stacks with a power-law disorder (PLD), where the amplitude \( \zeta_{0,q} \) of each mode follows a Gaussian probability distribution with mean square amplitudes

\[
\langle \zeta_{0,q}^2 \rangle = \langle \zeta_0^2 \rangle \left( \frac{\pi}{L_{4k}} \right)^{\alpha} \zeta(\alpha, k_{\text{min}} + 3/2)^{-\alpha}
\]

with \( q = \{ q_{k_{\text{min}}}, \ldots, q_{k_{\text{Max}}} \} \) (\( k = \{ k_{\text{min}}, \ldots, k_{\text{Max}} \} \)) and where \( \zeta(\alpha, k_{\text{min}} + 3/2) \) is the generalized zeta function [23]. Spectra are shown for different \( \alpha \) on Fig. 14. To avoid perfect stacking of fibers, we removed the first long wave-length mode \( q < q_{k_{\text{min}}} \).
FIG. 13: (Color online) Amplitude $\zeta_0, q$ vs $q$ for PLD systems for ($\Diamond$, $\alpha = 2$), ($\circ$, $\alpha = 3.5$) and ($\Box$, $\alpha = 5$). Also shown numerical fiber shapes.

Fig. 14 displays compression results for the PLD cases in the normalized pressure units $P_{2d} L^4/\kappa (\zeta_0^2)^{1/2}$ as a function of the normalized density $\langle \zeta_0^2 \rangle^{1/2}/D$, for $\alpha = 2$ and 5 (see also ESI for a more extensive set of $\alpha$ values). For the large density limit, where fibers are in close contact and where shape disorder is irrelevant, the data collapses on the same master curve, similarly to those of a single mode fiber – see Fig. 12. For the most significant compression regime, at intermediate densities, we observe a strong dependence of the compression law on the value of the exponent $\alpha$, further confirming that the mechanical properties of the macroscopic stacks are controlled by fiber disorder.

FIG. 14: (Color online) Normalized pressure vs $\langle \zeta_0^2 \rangle^{1/2}/d$ from numerical simulations for PLD systems for $\alpha = 2$ ($\Diamond$) and $\alpha = 5$ ($\Box$). Dashed lines represent the discrete self-consistent theory $P_{\text{self}}$ using numerical spectrum, dotted line the linear elasticity behavior $P_{\text{ref}}$ and solid line the total pressure $P_{\text{tot}} = P_{\text{ref}}$. In inset same figure where lines represent the total pressure obtained by adding the discrete self-consistent theory using numerical spectrum an the linear elasticity $P_{\text{ref}}$ for all $\alpha$.

By following a procedure similar to that applied to fit the experimental results, we solve numerically the self-consistent relation Eq. (7) for the numerical distributions $\langle \zeta_0^2 \rangle$. Theoretical self-consistent results, presented in Fig. 14 show a remarkable agreement with compression values from numerical simulations. The low compression regime is again well described by the linear elasticity $P_{\text{ref}}$ without any fitting parameters. The self-consistent theory is in good agreement with the experimental data for high compression rates, especially for very rough fibers ($\alpha < 4$),
FIG. 15: (Color online) Normalization factors from the numerical simulations $\Phi_{\text{num}}$ (•) and the experimental studies $\Phi_{\text{exp}}$: PP1 (□); PP2 (■); PP3 (▲); DEN (●); SW1 (△); SW2 (○).

DISCUSSION

Both for experimental and numerical studies, systematic deviations between the mean-field predictions and the simulations can be seen in the low density limit, where the distance between fibers is of the order of the fibers mean-square amplitude. Mean-field theory poorly describes this limit, because of the vanishing number of fibers of mean amplitude larger than $\langle \zeta^2 \rangle^{1/2}$. This regime can be qualitatively understood by noticing that, as the force rises sharply from zero, due to fiber-fiber contacts, it progressively builds up with essentially single-mode compression behavior. This is shown in Fig. 14 and 11, where the dashed line corresponds to the single mode expression $P_{\text{ref}}$ with the wavelength $\lambda$ associated to the first mode of the distribution. Fig. 14 displays the comparison between numerical simulation for $\alpha$ ranging from 2 to 5 (see also the ESI) and the total pressure: for a large range of fiber density, the agreement is remarkably good. We observe a continuous transition from compression curves exhibiting a linear elasticity for large values of $\alpha$ to compression curves exhibiting a highly non-linear elasticity for small values of $\alpha$. The self-consistent predictions are more accurate for the fibers with increased corrugation, where the number of fiber-fiber contacts is high. Agreement is optimal for $\alpha \sim 2$, where $\Phi_{\text{num}}$ is of order unit, further confirming by the analytical resolution of self consistent theory in continuous limit (see ESI). For large density, numerical simulation studies confirm that the self-consistent approach correctly predicts the functional dependence of the pressure with distance, but underestimates the amplitude of the effect as also experimentally observed.

In both experimental and numerical cases, a numerical scaling factor ($\Phi_{\text{exp}}$ and $\Phi_{\text{num}}$ respectively) is necessary for the self-consistent theory to correctly describe our results. Fig. 15 displays $\Phi_{\text{num}}$ and $\Phi_{\text{exp}}$ as a function of $\alpha$. Values for both coefficients are comparable to the experimental uncertainties, clearly demonstrating that numerical factor $\Phi_{\text{num}}$ is due to the approximation involved in the self-consistent theory rather than experimental artefacts, further supporting that our 2 dimensional approach is also relevant to describe 3-dimensional experiments up to a consistent density of stress planes $\rho$.

CONCLUSIONS

As a summary we have experimentally investigated the mechanical properties of well aligned corrugated fiber stacks, showing that such fiber stacks display a strong non-linear elasticity behavior over 5-decades of stress. We showed that a theoretical self-consistent description [17] connecting fiber shape disorder with stack compressibility explains well three different classes of fibers. We have also performed numerical simulation studies for a larger class of
fiber disorder. Interestingly we found that single fiber shape, as characterized by the phase and wavelength disorder at fixed amplitude, is enough to induce a non-trivial compression behavior. We have also simulated more realistic distributions for fiber disorder, with fiber’s spontaneous shapes reconstituted from a superimposition of modes with power-law q-dependent amplitudes. For fibers of moderate corrugation the compression forces compare very favorably with our 2-dimension mean-field theory. While more extensive simulation on 3-dimensional systems will certainly allow to better probe Van Wyk seminal intuition, our results here show that indeed bending disorder control the compressive behavior in aligned fibrous matter provided that the statistical nature of fiber shape disorder is accounted for.

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