Reflection matrices with $U_q[osp(2) (2|2m)]$ symmetry

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Abstract
We propose a classification of the reflection $K$-matrices (solutions of the boundary Yang–Baxter equation) for the $U_q[osp(2) (2|2m)] = U_q[C^{(2)}(m + 1)]$ vertex-model. We found four families of solutions, namely, the complete solutions, in which no elements of the reflection $K$-matrix is null, the block-diagonal solutions, the $X$-shape solutions and the diagonal solutions. We highlight that these diagonal $K$-matrices also hold for the $U_q[osp(2) (2n + 2|2m)] = U_q[D^{(2)}(n + 1, m)]$ vertex-model.

Keywords: integrable models, boundary Yang–Baxter equation, reflection $K$-matrices, twisted Lie superalgebras, orthosymplectic algebras

1. Introduction

The importance of the Yang–Baxter (YB) equation is now a very well established fact. This equation appeared first in relativistic field theory as a sufficient condition for the factorization of the scattering amplitudes associated with a system of particles interacting via delta potentials [1–6]. Soon after, the same equation was derived by Baxter in the field of statistical mechanics [7, 8], with the YB equation consisting in a requirement for the commutativity of the transfer matrix—a mathematical object related to the partition function of the model—for different values of the spectral parameter, $x = e^u$. In this case, the transfer matrix is the generator of infinitely many conserved charges in involution (the Hamiltonian being one of them), so that the corresponding model can be regarded as integrable in the sense of Liouville.

The interest in the YB equation has increased with the formulation of the quantum inverse scattering method, also known as the algebraic Bethe Ansatz [9–17]. This powerful technique allows (if applied successfully) the exact diagonalization of the transfer matrix for a given vertex-model, the YB equation providing the commutation relations between the relevant operators. The algebraic Bethe Ansatz provides both the eigenvalues and the eigenvectors of the transfer matrix in terms of analytical expressions, although they depend, however, on
the solutions of the so-called Bethe Ansatz equations—a complex system of non-linear equations whose complete analytical solution is not yet available [18].

More recently, the YB equation proved to be important also in classical field theory, condensed matter, nuclear physics and in high energy physics through the AdS/CFT correspondence between the $\mathcal{N} = 4$ super Yang–Mills gauge theory and the AdS5 × S5 sigma model of string theory [19–22]. In pure mathematics, the YB equation contributed to the development of algebraic structures associated with Lie (super)algebras, for instance the Hopf algebras and the formulation of quantum groups [23–28].

The YB equation consists in a matrix relation defined on the End $(V \otimes V \otimes V)$, where $V$ is a $N$-dimensional complex vector space, which reads [1–6],
\[
R_{12}(x)R_{13}(y)R_{23}(y') = R_{23}(y')R_{13}(xy)R_{12}(x).
\] (1)

In this equation, $R$ is a matrix defined on End $(V \otimes V)$, which is regarded as the solution of the YB equation. The matrices $R_{12}$, $R_{23}$ and $R_{13}$ are obtained from $R$ through the expressions $R_{12} = R \otimes I$, $R_{23} = I \otimes R$ and $R_{13} = P_{12}R_{23}P_{12}$, where $I$ is the identity matrix defined on End $(V)$ and $P_{12} = P \otimes I$, with $P$ denoting the permutator matrix so that $P(A \otimes B)P = B \otimes A$, $\forall \{A, B\} \in$ End $(V)$. Solutions of the YB equation have been investigated for a long time ago—see [11, 14–16, 29–31] and references therein. Jimbo proposed indeed a classification of the $R$-matrices associated with all non-exceptional affine Lie algebras in [32]. More recently, supersymmetric solutions of the YB equation (which are associated with quantum deformations of affine Lie superalgebras) were also found [6]. In special, Galleas and Martins derived new solutions of the YB equation that can be regarded as non-trivial graded generalizations of Jimbo’s $R$-matrices [33–35], which are of particular interest to us here.

The YB equation (1) ensures the integrability of a given vertex-model with periodic boundary conditions. When non-periodic boundary conditions are present, the integrability of the system at the boundaries is guaranteed by the boundary YB equation, also known as the reflection equation, [36–39],
\[
R_{12}(x/y) K_1(x)R_{21}(xy)K_2(y) = K_2(y)R_{12}(xy)K_1(x)R_{21}(x/y).
\] (2)

This is a matrix equation defined on End $(V \otimes V)$ and the reflection $K$-matrix—the required solution of the boundary YB equation—is a matrix defined on End $(V)$. Besides, $R_{12}$ denotes here just the $R$-matrix, solution of the periodic YB equation (1), while $R_{21} = PR_{12}P$, $K_1 = K \otimes I$ and $K_2 = I \otimes K$. The boundary YB equation was introduced by Sklyanin in [36] (based on a previous work of Cherednik [37]) and it was applicable only for symmetric $R$-matrices (i.e. $R_{12} = R_{21}$). The boundary YB equation (2), which holds also for the non-symmetric $R$-matrices, was introduced by Mezincescu and Nepomechie in [38]. Solutions of the boundary YB equation (2) have a long history as well. The first ones were found by Cherednik [37] and Sklyanin [36] themselves and, since then, several solutions for more complex vertex–models were also found [40–45]. The quantum group formalism for the non-periodic case was developed by several authors as well,—see [46–58] and references therein. Moreover, even despite the difficulty of solving the graded boundary YB equation [59], several supersymmetric reflection $K$-matrices were also obtained, mainly in the last two decades. We remark, however, that although a classification of the reflection $K$-matrices associated with non-graded affine Lie algebras has been proposed [60], a complete classification of the graded reflection $K$-matrices is yet not available. A great advance towards this end was obtained by Arnaudon et al [61], where a classification of the rational graded $K$-matrices was presented and, more recently, by Lima-Santos in a series of papers [62–65], in which the trigonometric
reflection $K$-matrices associated with the $U_q[sl^{(2)}(r|2m)]$, $U_q[osp^{(1)}(r|2m)]$, $U_q[spo(2n|2m)]$ and $U_q[sl^{(1)}(m|n)]$ vertex-models were derived.

The present work can be thought as a continuation of the studies above, as we derive the reflection $K$-matrices for the supersymmetric $U_q[osp^{(2)}(2|2m)]$ vertex-model. Since this vertex-model describes a supersymmetric interacting system, we should take into account the theory of Lie superalgebras \[66–69\] in order to study its reflection $K$-matrices. In the graded case, all the mathematical operations should be modified accordingly \[70\]. In a $\mathbb{Z}_2$-graded Lie superalgebra, we distinguish even elements from the odd ones (physically, the even elements describe bosons, while the odd elements describe fermions). Hence, we decompose the vector space $V$ into a direct sum of the even and odd parts: $V = V_{\text{even}} \oplus V_{\text{odd}}$. Even and odd elements can be distinguished through the Grassmann parity defined as,

$$\pi_a = \begin{cases} 1, & a \in V_{\text{odd}}; \\ 0, & a \in V_{\text{even}}. \end{cases}$$

(3)

All matrix operations is redefined in their graded version. For instance, the graded tensor product of two matrices $A \in \text{End}(V)$ and $B \in \text{End}(V)$ is defined by \[70\]

$$A \otimes_g B = \sum_{a,b,c,d=1}^N (-1)^{\pi_a(\pi_b + \pi_c)} A_{ab}B_{cd}E_{cd}^{ab},$$

(4)

where $E_{cd}^{ab} = e_{ab} \otimes e_{cd}$ and $e_{ab}$ denotes the standard Weyl matrix (a matrix whose element on the $a$th row and $b$th column is equal to 1 while the other elements are all zero). The graded permutator matrix is given by

$$P_g = \sum_{a,b=1}^N (-1)^{\pi_a} E_{ab}^{ha},$$

(5)

and the graded trace of a matrix $A \in \text{End}(V)$ is defined by the formula,

$$\text{tr}_g(A) = \sum_{a=1}^N (-1)^{\pi_a} A_{aa}.$$  

(6)

Besides, we define the graded transposition of a matrix $A \in \text{End}(V)$, and also its inverse graded transposition, as

$$A^t = \sum_{a,b=1}^N (-1)^{\pi_a(\pi_a + \pi_b)} A_{ba}e_{ab}, \quad A^{\tau} = \sum_{a,b=1}^N (-1)^{\pi_a(\pi_a + \pi_b)} A_{ba}e_{ab},$$

(7)

so that $(A^t)^{\tau} = (A^{\tau})^t = A$. In the graded case, both the periodic as the boundary YB equations can be written in the same form (1) and (2), respectively, if all the linear operations are considered in their graded form \[6\]. Alternatively, we can introduce the so-called scattering $S$-matrix through

$$S(x) = P_g R(x),$$

(8)

so that both the periodic (1) as the boundary (2) YB equations can be written in a form which is insensitive to the gradation. In this case, all linear operations should be considered in their non-graded versions and the periodic (1) and the boundary (2) YB equation become given, respectively, by

$$S_{12}(x)S_{23}(xy)S_{12}(y) = S_{23}(y)S_{12}(xy)S_{23}(x),$$

(9)
and
\[ S_{12}(x/y) K_1(x) S_{12}(xy) K_1(y) = K_1(y) S_{12}(xy) K_1(x) S_{12}(x/y). \] (10)

2. The \( U_q[\mathfrak{osp}(2)(2n + 2|2m)] = U_q[D^{(2)}(n + 1|m)] \) and \( U_q[\mathfrak{osp}(2)(2|2m)] = U_q[C^{(2)}(m + 1)] \) vertex-models

In this work we shall consider a \( S \)-matrix, solution of the periodic YB equation (9), which was obtained by Galleas and Martins in [34]. In that work the authors employed a Baxterization procedure through representations of the dilute Birman–Wenzl–Murakami algebra [71, 72] in order to find new solutions of the graded YB equation. Among other solutions previously known, they found a \( S \)-matrix describing a vertex-model containing \( 2n + 2 \) bosons and \( 2m \) fermions, which was regarded as a supersymmetric generalization of Jimbo’s \( S \)-matrix [32] that is associated with the \( U_q[\mathfrak{osp}(2)(2n + 2)] = U_q[D_b^{(2)}_{n+1}] \) quantum twisted affine Lie algebra.

Employing a simplified notation, the Galleas–Martins \( S \)-matrix can be written as follows:
\[
S^{m,n}_{(a_1,a_2,a)}(x) = \sum_{a,b \in \sigma_1} \left[ a_1(x) \left( E^a_{ba} + E^{ba}_{a} \right) + a_2(x) \left( E^{ab'}_{ba} + E^{ba'}_{ab} \right) \right] \\
+ \sum_{a,b \in \sigma_2} a_3(x) E^a_{bb} + \sum_{a,b \in \sigma_3} a_4(x) E^{ba}_{ab} \\
+ \sum_{a,b \in \sigma_4} a_5(x) E^a_{ba} + a_6(x) E^{aa}_{ba} + a_7(x) E^{aa''}_{ba} + a_8(x) E^{aa'''}_{ba} \\
+ \sum_{a,b \in \sigma_5} b^a_{(x)} E^a_{ba} \\
+ \sum_{a,b \in \sigma_6} b^a_{(x)} \left( E^{aa'}_{ba} + E^{aa''}_{ba} \right) + b^a_{(x)} \left( E^{aa'''}_{ba} + E^{aa'''}_{ba} \right) \\
+ \sum_{a,b \in \sigma_7} \left[ b^a_{(x)} \left( E^{aa'''}_{ba} + E^{aa'''}_{ba} \right) + b^a_{(x)} \left( E^{aa'''}_{ba} + E^{aa'''}_{ba} \right) \right] \\
+ \sum_{a,b \in \sigma_8} c^a_{(x)} E^{ab'''}_{ba} + \sum_{a,b \in \sigma_9} c^a_{(x)} E^{ab'''}_{ba}, 
\]
(11)
where \( \mu = m + n, N = 2\mu + 2 \) and we introduced conveniently the notations:
\[ a' = N - a + 1, \quad b' = N - b + 1, \quad a'' = N - a + 2, \quad b'' = N - b + 2. \] (12)

The sums in the indexes \( a \) and \( b \) run from 1 to \( N \) and they are restricted by the subsets \( \sigma_k \) as defined below:
\[
\sigma_1 = \left\{ a \neq b, a \neq b''; b = \mu + 1 \text{ or } b = \mu + 2 \right\},
\] (13)
\[
\sigma_2 = \left\{ a < b, a \neq b'', a \neq \mu + 1, a \neq \mu + 2; b \neq \mu + 1, b \neq \mu + 2 \right\},
\] (14)
\[
\sigma_3 = \left\{ a > b, a \neq b'', a \neq \mu + 1, a \neq \mu + 2; b \neq \mu + 1, b \neq \mu + 2 \right\},
\] (15)

The Birman–Wenzl–Murakami algebra was also considered before by Grimm in [73–76], where other models related to the \( U_q[\mathfrak{osp}(2)(2n + 2)] = U_q[D^{(2)}_{n+1}] \) symmetry were obtained.
\[ \sigma_4 = \{ a = b, a = \mu + 1 \text{ or } a = \mu + 2 \}, \quad (16) \]

\[ \sigma_5 = \{ a = b, a \neq \mu + 1, a \neq \mu + 2 \}, \quad (17) \]

\[ \sigma_6 = \{ a \neq \mu + 1, a \neq \mu + 2; b = \mu + 1 \text{ or } b = \mu + 2 \}, \quad (18) \]

\[ \sigma_7 = \{ a \neq \mu + 1, a \neq \mu + 2; b \neq \mu + 1, b \neq \mu + 2 \}, \quad (19) \]

\[ \sigma_8 = \{ a \neq b, a \neq b', a = \mu + 1 \text{ or } a = \mu + 2; b \neq \mu + 1, b \neq \mu + 2 \}. \quad (20) \]

In the equation (11), the amplitudes \(a_k(x), 1 \leq k \leq 8\), are given by

\[ a_1(x) = \frac{1}{2} q (x^2 - 1) (x^2 - \zeta^2) (1 + \kappa_1), \quad (21) \]

\[ a_2(x) = \frac{1}{2} q (x^2 - 1) (x^2 - \zeta^2) (1 - \kappa_1), \quad (22) \]

\[ a_3(x) = -q (q^2 - 1)(x^2 - \zeta^2), \quad (23) \]

\[ a_4(x) = -x^2 (q^2 - 1)(x^2 - \zeta^2), \quad (24) \]

\[ a_5(x) = \frac{1}{2} [q (x^2 - 1) (x^2 - \zeta^2) (1 + \nu \kappa_1) + x (x - 1) (q^2 - 1)(\zeta + \kappa_2)(x \kappa_2 + \zeta)], \quad (25) \]

\[ a_6(x) = \frac{1}{2} [q (x^2 - 1) (x^2 - \zeta^2) (1 + \nu \kappa_1) - x (x + 1) (q^2 - 1)(\zeta + \kappa_2)(x \kappa_2 - \zeta)], \quad (26) \]

\[ a_7(x) = \frac{1}{2} [q (x^2 - 1) (x^2 - \zeta^2) (1 - \nu \kappa_1) + x (x + 1) (q^2 - 1)(\zeta - \kappa_2)(x \kappa_2 + \zeta)], \quad (27) \]

\[ a_8(x) = \frac{1}{2} [q (x^2 - 1) (x^2 - \zeta^2) (1 - \nu \kappa_1) - x (x - 1) (q^2 - 1)(\zeta - \kappa_2)(x \kappa_2 - \zeta)]. \quad (28) \]

where \( \zeta = q^{n-m} \). The amplitudes \(b_k^a(x), 1 \leq k \leq 5\), which depend on the index \(a\), are given by,

\[ b_1^a(x) = (x^2 - \zeta^2) \left[ x^{2(1-p_a)} - q^2 x^{2p_a} \right], \quad 1 \leq a \leq N, \quad (29) \]

\[ b_2^a(x) = \begin{cases} -\frac{1}{2}(q^2 - 1)(x^2 - \zeta^2)(x + 1), & a < \mu + 1, \\ -\frac{1}{2}x(q^2 - 1)(x^2 - \zeta^2)(x + 1), & a > \mu + 2, \end{cases} \quad (30) \]

\[ b_3^a(x) = \begin{cases} \frac{1}{2}(q^2 - 1)(x^2 - \zeta^2)(x - 1), & a < \mu + 1, \\ -\frac{1}{2}x(q^2 - 1)(x^2 - \zeta^2)(x - 1), & a > \mu + 2, \end{cases} \quad (31) \]

\[ b_4^a(x) = \begin{cases} \frac{1}{2}(\theta_a q^2)(x^2 - 1) - (q^2 - 1)(x \kappa_2 + \zeta), & a < \mu + 1, \\ \frac{1}{2}x(\theta_a q^2)(x^2 - 1)(q^2 - 1)(x \kappa_2 + \zeta), & a > \mu + 2, \end{cases} \quad (32) \]
\[ b_k^a(x) = \begin{cases} 
\mu x (\theta_a q^\nu) (x^2 - 1) (q^2 - 1) (x\kappa_2 - \zeta), & a < \mu + 1, \\
\frac{1}{2} x (\theta_a q^\nu) (x^2 - 1) (q^2 - 1) (x\kappa_2 - \zeta), & a > \mu + 2, 
\end{cases} \] (33)

and the amplitudes \( c_k^a(x) \), \( 1 \leq k \leq 2 \), that depend on the indexes \( a \) and \( b \) are,

\[ c_k^a(x) = \begin{cases} 
\mu x (\delta_{a,b} \Theta_{a,b} - \delta_{a,b'} (x^2 - \zeta^2)), & a < b, \\
\mu x (\delta_{a,b} \Theta_{a,b} - \delta_{a,b'} (x^2 - \zeta^2)), & a = b, \\
\mu x (\delta_{a,b} \Theta_{a,b} - \delta_{a,b'} (x^2 - \zeta^2)), & a > b, 
\end{cases} \] (34)

\[ c_2^a(x) = (-1)^{p_a q_b} q (x^2 - 1) (x^2 - \zeta^2), \quad 1 \leq a, b \leq N. \] (35)

We made use of the following gradation and Grassmann parity,

\[ p_a = \begin{cases} 
\pi_a, & a < \mu + 1, \\
0, & \mu + 1 \leq a \leq \mu + 2, \\
\pi_{a-1}, & a > \mu + 2, 
\end{cases} \]

\[ \pi_a = \begin{cases} 
1, & m + 1 \leq a \leq 2n + m + 1, \\
0, & \text{otherwise.} 
\end{cases} \] (36)

The remaining parameters of the solution are given by,

\[ t_a = \begin{cases} 
-p_a + a + 1 + 2 \sum_{b=0}^{a} \mu p_a, & a < \mu + 1, \\
\mu + \frac{1}{2}, & \mu + 1 \leq a \leq \mu + 2, \\
p_a + a - 1 - 2 \sum_{b=\mu+3}^{a} \mu p_b, & a > \mu + 2, 
\end{cases} \] (37)

\[ \tau_a = \begin{cases} 
p_a + a - \frac{1}{2} - 2 \sum_{b=\mu+3}^{a} \mu p_b, & a < \mu + 1, \\
0, & \mu + 1 \leq a \leq \mu + 2, \\
p_a + a - \frac{1}{2} - \mu - 2 \sum_{b=\mu+3}^{a} \mu p_b, & a > \mu + 2, 
\end{cases} \] (38)

\[ \theta_a = \begin{cases} 
(-1)^{-p_a/2}, & a < \mu + 1, \\
1, & \mu + 1 \leq a \leq \mu + 2, \\
(-1)^{p_a/2}, & a > \mu + 2, 
\end{cases} \] and \[ \Theta_{a,b} = \frac{\theta_a q^\nu}{\theta_b q^\nu}. \] (39)

The \( R \)-matrix associated with (11) can be obtained through \( R(x) = P(x) S(x) \). This \( R \)-matrix satisfies the regularity, unitarity, PT and crossing symmetries, which are important for the implementation of the boundary algebraic Bethe Ansatz—see [34] for the details.

Notice that the Galleas–Martins \( S \)-matrix (11) depends on three parameters, namely, \( \kappa_1, \kappa_2 \) and \( \nu \) (we say that this \( S \)-matrix is multiparametric). These parameters can assume only the values 1 and \(-1\) and for each possibility we get a corresponding supersymmetric vertex-model. The case \( \kappa_1 = \kappa_2 = \nu = 1 \) is the most important one, since it is only in this case that the \( S \)-matrix (11) reduces to Jimbo’s \( S \)-matrix [32] when the fermionic degrees of freedom are despised (i.e. when we make \( m = 0 \)). Other values of \( \kappa_1, \kappa_2 \) and \( \nu \) would lead to other vertex-models corresponding to non-trivial generalizations of Jimbo’s \( S \)-matrix [32].

Galleas and Martins conjectured that the symmetry behind their vertex-model is described by the \( U_q[osp(2)\langle 2n+2\mid m \rangle] \) quantum affine Lie superalgebra [34] (see also [35]). Their claim can be justified in the following way: first, remember that a Lie superalgebra is defined on a \( \mathbb{Z}_2 \)-graded vector space \( V \) that decomposes into the direct sum \( V = V_0 \oplus V_1 \), where \( V_0 \) is the even (bosonic) part of \( V \) and \( V_1 \) is its odd (fermionic) part [68, 69]. Now, since the Galleas–Martins \( S \)-matrix reduces to the Jimbo’s \( S \)-matrix [32] when the fermionic degrees of freedom are despised, and since the Jimbo \( S \)-matrix has the \( U_q[\alpha(2)\langle 2n+2 \rangle] = U_q[D_{2n+2}^{(2)}] \) symmetry, this
means that the even part of the Lie superalgebra associated with the Galleas–Martins vertex-model must have this same symmetry. However, only the $\mathfrak{osp}^{(2)}(2n + 2|2m) = D^{(2)}(n + 1|m)$ Lie superalgebra has an even part corresponding to the $\mathfrak{o}^{(2)}(2n + 2) = D^{(2)}_{n+1}$ affine Lie algebra [77–85]. In fact, we have the decomposition $\mathfrak{osp}^{(2)}(2n + 2|2m) = \mathfrak{o}^{(2)}(2n + 2) \otimes \mathfrak{sp}^{(1)}(2m)$, which in Cartan’s notation becomes $D^{(2)}(n + 1|m) = D^{(2)}_{n+1} \otimes C^{(1)}_m$ [77–85]. These decompositions are not expected to be changed as we perform the quantum deformation of the universal enveloping algebra associated with the $\mathfrak{osp}^{(2)}(2n + 2|2m) = D^{(2)}(n + 1|m)$ Lie superalgebra and, hence, it follows that the Galleas–Martins vertex-model should be associated with the $U_q[\mathfrak{osp}^{(2)}(2n + 2|2m)] = U_q[D^{(2)}(n + 1|m)]$ quantum twisted orthosymplectic Lie superalgebra.

For the case $n = 0$ and $\kappa_1 = \kappa_2 = \nu = 1$ we obtain a supersymmetric vertex-model which can be thought as the fermionic analogue of $U_q[D^{(2)}_{n+1}]$ Jimbo’s vertex-model [32]. The underlining symmetry behind this vertex-model is the $U_q[\mathfrak{osp}^{(2)}(2|2m)] = U_q[C^{(2)}(m + 1)]$ quantum twisted Lie superalgebra [77–85]. We can write the $S$-matrix of the $U_q[\mathfrak{osp}^{(2)}(2|2m)]$ vertex-model in the same form as given at (11), with the only changes occurring in the amplitudes, which become considerably simpler:

\begin{align}
    a_1(x) &= q \left( x^2 - 1 \right) \left( x^2 - q^{-2m} \right), \\
    a_2(x) &= 0, \\
    a_3(x) &= - \left( q^2 - 1 \right) \left( x^2 - q^{-2m} \right), \\
    a_4(x) &= -x^2 \left( q^2 - 1 \right) \left( x^2 - q^{-2m} \right), \\
    a_5(x) &= \frac{1}{2} \left[ 2q \left( x^2 - 1 \right) \left( x^2 - q^{-2m} \right) + x \left( x - 1 \right) \left( q^2 - 1 \right) \left( q^{-m} + 1 \right) \left( x + q^{-m} \right) \right], \\
    a_6(x) &= \frac{1}{2} \left[ 2q \left( x^2 - 1 \right) \left( x^2 - q^{-2m} \right) - x \left( x + 1 \right) \left( q^2 - 1 \right) \left( q^{-m} + 1 \right) \left( x - q^{-m} \right) \right], \\
    a_7(x) &= \frac{1}{2} \left[ x \left( x + 1 \right) \left( q^2 - 1 \right) \left( q^{-m} - 1 \right) \left( x + q^{-m} \right) \right], \\
    a_8(x) &= \frac{1}{2} \left[ x \left( x - 1 \right) \left( q^2 - 1 \right) \left( q^{-m} - 1 \right) \left( x - q^{-m} \right) \right], \\
    b_1^{(a)}(x) &= \left( x^2 - q^{-2m} \right) \left[ x^{(1-a)} - q^2 \right], \quad 1 \leq a \leq N, \\
    b_2^{(a)}(x) &= \left\{ \begin{array}{ll} 
        -\frac{1}{2} \left( q^2 - 1 \right) \left( x^2 - q^{-2m} \right) \left( x + 1 \right), & a < m + 1, \\
        -\frac{1}{2} x \left( q^2 - 1 \right) \left( x^2 - q^{-2m} \right) \left( x + 1 \right), & a > m + 2, 
    \end{array} \right. \\
    b_3^{(a)}(x) &= \left\{ \begin{array}{ll} 
        \frac{1}{2} \left( q^2 - 1 \right) \left( x^2 - q^{-2m} \right) \left( x - 1 \right), & a < m + 1, \\
        -\frac{1}{2} x \left( q^2 - 1 \right) \left( x^2 - q^{-2m} \right) \left( x - 1 \right), & a > m + 2, 
    \end{array} \right.
\end{align}
\[ b_{k}^{a}(x) = \begin{cases} \\ \frac{1}{2} (\theta_{a} q^{x}) (x^{2} - 1) (q^{2} - 1) (x + q^{-m}), & a < m + 1, \\
\frac{1}{2} (\theta_{a} q^{x}) (x^{2} - 1) (q^{2} - 1) (x + q^{-m}), & a > m + 2, 
\end{cases} \tag{51} \]

\[ b_{k}^{a}(x) = \begin{cases} \\ -\frac{1}{2} (\theta_{a} q^{x}) (x^{2} - 1) (q^{2} - 1) (x - q^{-m}), & a < m + 1, \\
\frac{1}{2} (\theta_{a} q^{x}) (x^{2} - 1) (q^{2} - 1) (x - q^{-m}), & a > m + 2, 
\end{cases} \tag{52} \]

and,

\[ c_{1}^{ab}(x) = \begin{cases} \\ \left( q^{2} - 1 \right) \left[ q^{-2m}(x^{2} - 1) \Theta_{a,b} - \delta_{a,b} \right] (x^{2} - q^{-2m}), & a < b, \\
\left( x^{2} - 1 \right) \left( x^{2} - q^{-2m} \right) \left( -1 \right)^{p_{a}} q^{2p_{b}} + x^{2} (q^{2} - 1), & a = b, \\
\left( x^{2} - 1 \right) \left[ (x^{2} - 1) \Theta_{a,b} - \delta_{a,b} \right] (x^{2} - q^{-2m}), & a > b, 
\end{cases} \tag{53} \]

\[ c_{2}^{ab}(x) = (-1)^{p_{a}p_{b}} q \left( x^{2} - 1 \right) (x^{2} - q^{-2m}), & 1 \leq a, b \leq N. \tag{54} \]

In this case, the other parameters of the \( S \)-matrix also become simpler:

\[ p_{a} = \begin{cases} \\ 0, & m + 1 \leq a \leq m + 2, \\
1, & \text{otherwise}, 
\end{cases} \quad \pi_{a} = \begin{cases} \\ 0, & a = m + 1, \\
1, & \text{otherwise}, 
\end{cases} \tag{55} \]

\[ t_{a} = \begin{cases} \\ N - a, & a < m + 1, \\
m + \frac{1}{2}, & m + 1 \leq a \leq m + 2, \\
N + 2 - a, & a > m + 2, 
\end{cases} \quad \tau_{a} = \begin{cases} \\ \frac{1}{2} - a, & a < m + 1, \\
0, & m + 1 \leq a \leq m + 2, \\
m + \frac{5}{2} - a, & a > m + 2, 
\end{cases} \tag{56} \]

\[ \theta_{a} = \begin{cases} \\ -i, & a < m + 1, \\
1, & m + 1 \leq a \leq m + 2, \\
i, & a > m + 2, 
\end{cases} \quad \Theta_{a,b} = \frac{\theta_{a} q^{x}}{\theta_{b} q^{x}}. \tag{57} \]

where \( i = \sqrt{-1} \).

### 3. Solutions of the boundary YB equation

Hereafter we shall present the graded reflection \( K \)-matrices, solutions of the boundary Yang–Baxter equation (2), for the \( U_{q}[\mathfrak{osp}(2|2m)] = U_{q}[\mathfrak{C}(m + 1)] \) vertex-model. We shall also present the diagonal reflection \( K \)-matrices associated with the \( U_{q}[\mathfrak{osp}(2|2m)] = U_{q}[\mathfrak{D}(n + 1|m)] \) vertex-model.

As commented in the previous section, the \( U_{q}[\mathfrak{osp}(2|2m)] \) vertex-model can be seen as the fermionic analogue of Jimbo’s \( U_{q}[\mathfrak{o}(2|2n + 2)] \) vertex-model [32]. We remark that the first solutions of the boundary YB equation associated to the \( U_{q}[\mathfrak{q}(2|2n + 2)] \) vertex-model were the diagonal and block-diagonal solutions found by Martins and Guan in [44]; soon after Lima-Santos deduced the general \( K \)-matrices of this vertex-model in [45]. The corresponding reflection \( K \)-matrices for the multiparametric \( U_{q}[\mathfrak{o}(2|2n + 2)] \) vertex-model were deduced and classified by Vieira and Lima-Santos in [86] and a new family of solutions for Jimbo’s \( U_{q}[\mathfrak{q}(2|2n + 2)] \) vertex-model was also derived in [86].

Among the graded vertex-models known up to date, the \( U_{q}[\mathfrak{osp}(2|2n + 2m)] = U_{q}[\mathfrak{D}(n + 1|m)] \) vertex-model is by far the most complex one. This can be seen either directly from the very complexity of the \( S \)-matrix given at (11) or from the highly non-trivial nature of the twisted orthosymplectic Lie superalgebras [77–81, 83–85]. In fact, while the reflection
The methodology used by us to solve the boundary YB equation (2) was the standard derivative method. This method was first used to solve the periodic YB equation by Zamolodchikov [11] and it has been extensively used by Lima-Santos in order to solve the boundary YB equations [43, 45, 60–65]. The derivative method consists in taking the formal derivative of the boundary YB equation (2) with respect to one of the variables and evaluating it at some particular value of that chosen variable. For instance, taking the formal derivative of (2) with respect to $y$ and evaluating the resulting expression at $y = 1$ we shall get the following equation:

\[ 2R_{12}(x)K_1(x)D_{21}(x) + 2D_{12}(x)K_1(x)R_{21}(x) + R_{12}(x)K_1(x)R_{21}(x)B_2 - B_2R_{12}(x)K_1(x)R_{21}(x) = 0, \]

where,

\[ D_{12}(x) = \frac{\partial R_{12}(x)}{\partial y} \bigg|_{y=1}, \quad D_{21}(x) = \frac{\partial R_{21}(x)}{\partial y} \bigg|_{y=1}, \quad B_2 = \frac{dK_2(y)}{dy} \bigg|_{y=1}. \]

We have used the fact that the reflection $K$-matrix is regular, which means that it satisfies the property $K(1) = I.$ This procedure allows us to convert the set of $N^4$ non-linear functional equation (2), which depends on the two unknowns $x$ and $y$, into a set of $N^4$ linear functional equations depending only on the variable $x$. This, however, comes with a price: the introduction of $N^2$ boundary parameters,

\[ \beta_{a,b} = \frac{dK_{a,b}(y)}{dy} \bigg|_{y=1}, \quad 1 \leq a, b \leq N. \]

We remark that although the system (58) is overdetermined, it is nevertheless consistent. This remarkable property is due to the existence of the additional boundary parameters $\beta_{a,b}$, $1 \leq a, b \leq N$, which allow to solve the remaining functional equations, after all elements of the $K$-matrix are determined as functions of these boundary parameters. Actually, in general we need to fix only a subset of the boundary parameters $\beta_{a,b}$ in order to solve all the functional equations of the system (58); the remaining boundary parameters that did not need to be fixed are the boundary free-parameters of the solution.

Although the derivative boundary YB equation (58) consists in a linear system of functional equations depending only on the variable $x$, this system is still very difficult to be solved. In fact, the complexity of the system is very sensitive to the order in which the equations are solved and on what elements of the $K$-matrix are eliminated first. An unfortunate choice for solving the equations generally increases the complexity of the system in such a way that even with the most powerful computational resources the solution could not be achieved.

In the following, we shall describe a recipe for a possible order of solving the system of functional equation (58) on which the complexity of the system can be maintained under control and whence its solutions can be found.

\[ \text{In the case of the periodic YB equation, the derivative method leads to a system of differential equations instead of algebraic equations. This is due to the fact that the } K\text{-matrices appearing on the periodic YB equation depend on two variables instead of one variable.} \]
1. The simplest equations of the derivative boundary YB equation (58) are those containing only the non-diagonal elements of the reflection $K$-matrix that are different from $k_{m+1,m+2}(x)$ and $k_{m+2,m+1}(x)$ and those not lying on its first or last line or column. We can use these equations to eliminate the elements $k_{a,b}(x)$ ($1 < a, b < N$, $a, b \neq m + 1, m + 2$), in favor of the elements $k_{1,b}(x)$ ($1 < b < N$), and $k_{a,1}(x)$ ($1 < a < N$).

2. Next we should look for those equations containing only the elements lying on the first or last line or column. In this way we can eliminate the elements $k_{1,b}(x)$ ($1 < b < N$) in terms of $k_{1,N}(x)$ and the elements $k_{a,1}(x)$ ($1 < a < N$) in terms of $k_{N,1}(x)$.

3. Now we can search for equations containing only elements lying on the secondary diagonal of the reflection $K$-matrix. We can solve these equations in favor of the elements $k_{a,N+1-a}(x)$ ($2 \leq a \leq m$) in terms of $k_{1,N}(x)$, and the elements $k_{a,N+1-a}(x)$ ($m+3 \leq a \leq N$) in terms of $k_{N,1}(x)$.

4. Other equations containing only $k_{N,1}(x)$ and $k_{1,N}(x)$ will provide the expression for the first in terms of the second.

5. At this point, the system becomes very complex and, in the present case, the remaining expressions for the reflection $K$-matrices elements pass to depend on the parity of $N$. Notwithstanding the high complexity of the system, we can find equations providing the diagonal elements $k_{a,a}(x)$ ($2 \leq a \leq m$), in terms of $k_{1,1}(x)$ and $k_{1,N}(x)$ and the diagonal elements containing $k_{a,a}(x)$ ($m+3 \leq a \leq N$) in terms of $k_{m+3,m+3}(x)$ and $k_{1,N}(x)$.

6. Then we can find the expressions of $k_{m+3,m+3}(x)$ and $k_{1,1}(x)$ in terms of $k_{1,N}(x)$. These diagonal elements generally satisfy welcome recurrence relations.

7. Provided the computer machine has sufficient power to handle the equations, the remaining central elements $k_{m+1,m+1}(x)$, $k_{m+1,m+2}(x)$, $k_{m+2,m+1}(x)$ and $k_{m+2,m+2}(x)$ can be eliminated in terms of $k_{1,N}(x)$.

8. At this point all elements of the reflection $K$-matrix will be eliminated in terms of the element $k_{1,N}(x)$. Then, we can give to $k_{1,N}(x)$ any desirable value as long as it satisfies the properties $k_{1,N}(1) = 0$ and $dk_{1,N}(x)/dx|_{x=1} = \beta_{1,N}$.

9. After that all elements of the reflection $K$-matrix will be determined as functions of $x$, $q$ and the boundary parameters $\beta_{a,b}$. We can verify, however, that several functional equations may not be yet satisfied. In order to solve these remaining equations, a sufficient number of constraints between the boundary parameters $\beta_{a,b}$ should be found. As doing so, the solution may present branches if some quadratic or of high-degree expressions involving the boundary parameters appears. Every branch must be carefully taken into account so that no solution is lost.

10. Finally, we must check if the solution is regular and its derivative is in accordance with the definition of the boundary parameters given at (60). If these properties are not yet satisfied, further boundary parameters should be fixed until the solution becomes regular and consistent.

Once we have the solution of the derivative boundary YB equation (58), we can verify that the reflection $K$-matrix is indeed a solution of the boundary YB equation (2). We would like to emphasize however that the intermediary expressions for the functional equations (and also the intermediary expressions of the reflection $K$-matrix elements) that appear as we are solving the system are extremely huge and, as a matter of a fact, not important at all. By this reason we shall write in the sequel only the final expressions for the reflection $K$-matrix elements.

We have found four classes of reflection $K$-matrices for the $U_q[osp^{(2)}(2|2m)] = U_q[C^{(2)}(m + 1)]$ vertex-model, namely, the following classes described below:
– **Complete solutions:** these are the most general solutions we found, where no element of the $K$-matrix is null. These solutions are characterized by $m$ boundary free-parameters for a given $N$ and we found one family of solutions that branches into two subfamilies differing by the value of $\epsilon = \pm 1$.

– **Block-diagonal solutions:** these are solutions on which the reflection $K$-matrices are almost diagonal: all non-diagonal elements, except the elements $k_{m+1,m+2}(x)$ and $k_{m+2,m+1}(x)$, are null. The shape of this matrix is related to the existence of $m$ distinct conserved $U(1)$ charges [44, 45]. We found two families of block-diagonal solutions, which are characterized by one boundary free-parameter only. Each family also branches into two subfamilies differing by the value of $\epsilon = \pm 1$.

– **X-shape solutions:** in this case the only non-vanishing elements of the reflection $K$-matrices are those lying on the main and the secondary diagonals. We found only one family of X-shape solutions that, for a given $N$, contains $m$ boundary free-parameters. There are no branches here.

– **Diagonal solutions:** finally, we found two families of diagonal solutions which are actually valid for the $U_q[osp(2|2n+2m)] = U_q[D^{(2)}(n+1|m)]$ vertex-model. The first family of diagonal reflection $K$-matrices holds for any values of $m$ and $n$ and has no free-parameter. The second family holds only when $m = n$ and has two free-parameters. Besides the solutions commented above, we present in the appendix two particular families of solutions which hold only for the $U_q[osp(2|2)] = U_q[C^{(2)}(2)]$ and $U_q[osp(2|4)] = U_q[C^{(1)}(3)]$ vertex-models, respectively.

### 3.1. Complete solutions

The complete solutions are the most general reflection $K$-matrices we found. In this case, all elements of the $K$-matrix are different from zero. The solutions present two branches determined by $\epsilon = \pm 1$ and they are characterized by $m$ free-parameters\(^3\), namely, $\beta_{1,m+2}, \beta_{1,m+3}, \ldots, \beta_{1,N-2}$ and $\beta_{1,N-1}$.

We begin by defining the quantities

$$
\beta_{\pm} = \frac{1}{2} (\beta_{1,m+1} \pm \beta_{1,m+2}), \quad G_m(x) = \frac{q^{1-m} + 1}{q^{1-m} + x^2}, \quad H_m = \frac{q^{1-m} + 1}{q + 1}.
$$

(61)

With the help of these quantities, we can write the elements of the $K$-matrix as follows: for the first line of the reflection $K$-matrix, we have,

$$
k_{1,m+1}(x) = \left(\frac{\beta_+ + x \beta_-}{\beta_{1,N}}\right) G_m(x) k_{1,N}(x),
$$

(62)

$$
k_{1,m+2}(x) = \left(\frac{\beta_+ - x \beta_-}{\beta_{1,N}}\right) G_m(x) k_{1,N}(x),
$$

(63)

$$
k_{1,b}(x) = \left(\frac{\beta_{1,b}}{\beta_{1,N}}\right) G_m(x) k_{1,N}(x), \quad 1 < b < N, b \neq m + 1, b \neq m + 2,
$$

(64)

\[^3\text{It remains an open question whether some of these boundary free-parameters can or can not be eliminated by an adequate similarity transformation that commutes with the } K \text{-matrix. As pointed out by one of the referees, this question can be of importance in representation theory, since these ‘bare’ } K \text{-matrices (i.e. } K \text{-matrices carrying only the essential, non-removable, boundary free-parameters), provide one-dimensional representations of the corresponding coideal quantum groups.}\]
and, its first column, we have

\[ k_{m+1,1}(x) = \Theta_{m+1,2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_+ - x\beta_-}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(65)

\[ k_{m+2,1}(x) = \Theta_{m+2,2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_+ + x\beta_-}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(66)

\[ k_{a,1}(x) = \Theta_{a,2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(67)

\[ 1 < a < N, a \neq m + 1, a \neq m + 2. \]

For the elements of the last line, we have,

\[ k_{N,m+1}(x) = xq^m \Theta_{N,2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{x\beta_+ + q^{-m}\beta_-}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(68)

\[ k_{N,m+2}(x) = xq^m \Theta_{N,2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{x\beta_+ - q^{-m}\beta_-}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(69)

\[ k_{N,b}(x) = x^2 q^m \Theta_{N,2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right) \left( \frac{\beta_{1,b}}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(70)

\[ 1 < b < N, b \neq m + 1, b \neq m + 2, \]

and, for those in the last column,

\[ k_{m+1,N}(x) = xq^m \Theta_{m+1,1} \left( \frac{x\beta_+ - q^{-m}\beta_-}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(71)

\[ k_{m+2,N}(x) = xq^m \Theta_{m+2,1} \left( \frac{x\beta_+ + q^{-m}\beta_-}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(72)

\[ k_{a,N}(x) = x^2 q^m \Theta_{a,1} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) G_m(x) k_{1,N}(x), \]  
(73)

\[ 1 < a < N, a \neq m + 1, a \neq m + 2. \]

For the elements lying on the secondary diagonal not in the center of K-matrix (i.e. for \( a \neq m + 1 \) and \( a \neq m + 2 \)) we have,

\[ k_{1,N}(x) = \frac{1}{2} (x^2 - 1) \beta_{1,N}, \]  
(74)

\[ k_{N,1}(x) = \Theta_{N-1,2} \left( \frac{\beta_{2,1}}{\beta_{1,N-1}} \right)^2 k_{1,N}(x), \]  
(75)

\[ k_{a,a'}(x) = (-1)^{n_a} q^a \Theta_{1,a'} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right)^2 H_m^2 k_{1,N}(x), \]  
(76)

\[ 1 < a < N, a \neq m + 1, a \neq m + 2, \]

and for the elements of the K-matrix above the secondary diagonal, not in the first line or in the first column, we have
\[ k_{m+1,b}(x) = q^n \Theta_{m+1,1} \left( \frac{\beta_{1,b}}{\beta_{1,N}} \right) \left( \frac{\beta_+ - x \beta_-}{\beta_{1,N}} \right) H_m G_m(x) k_{1,N}(x), \]
\[ b \neq m + 1, b \neq m + 2 \]  
(77)

\[ k_{m+2,b}(x) = q^n \Theta_{m+2,1} \left( \frac{\beta_{1,b}}{\beta_{1,N}} \right) \left( \frac{\beta_+ + x \beta_-}{\beta_{1,N}} \right) H_m G_m(x) k_{1,N}(x), \]
\[ b \neq m + 1, b \neq m + 2 \]  
(78)

\[ k_{a,m+1}(x) = q^n \Theta_{a,1} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) \left( \frac{\beta_+ + x \beta_-}{\beta_{1,N}} \right) H_m G_m(x) k_{1,N}(x), \]
\[ a \neq m + 1, a \neq m + 2, \]  
(79)

\[ k_{a,m+2}(x) = q^n \Theta_{a,1} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) \left( \frac{\beta_+ - x \beta_-}{\beta_{1,N}} \right) H_m G_m(x) k_{1,N}(x), \]
\[ a \neq m + 1, a \neq m + 2, \]  
(80)

\[ k_{a,b}(x) = q^n \Theta_{a,1} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) \left( \frac{\beta_{1,b}}{\beta_{1,N}} \right) H_m G_m(x) k_{1,N}(x), \]
\[ a \neq m + 1, a \neq m + 2, b \neq m + 1, b \neq m + 2. \]  
(81)

Finally, for the elements below the secondary diagonal, not in the last line or column, we have,

\[ k_{m+1,b}(x) = x q^n \Theta_{m+1,1} \left( \frac{\beta_{1,b}}{\beta_{1,N}} \right) \left( x \beta_+ - q^{-m} \beta_- \right) H_m G_m(x) k_{1,N}(x), \]
\[ b \neq m + 1, b \neq m + 2 \]  
(82)

\[ k_{m+2,b}(x) = x q^n \Theta_{m+2,1} \left( \frac{\beta_{1,b}}{\beta_{1,N}} \right) \left( x \beta_+ + q^{-m} \beta_- \right) H_m G_m(x) k_{1,N}(x), \]
\[ b \neq m + 1, b \neq m + 2 \]  
(83)

\[ k_{a,m+1}(x) = x q^n \Theta_{a,1} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) \left( x \beta_+ + q^{-m} \beta_- \right) H_m G_m(x) k_{1,N}(x), \]
\[ a \neq m + 1, a \neq m + 2, \]  
(84)

\[ k_{a,m+2}(x) = x q^n \Theta_{a,1} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) \left( x \beta_+ - q^{-m} \beta_- \right) H_m G_m(x) k_{1,N}(x), \]
\[ a \neq m + 1, a \neq m + 2, \]  
(85)

\[ k_{a,b}(x) = x^2 q^n \Theta_{a,1} \left( \frac{\beta_{1,a'}}{\beta_{1,N}} \right) \left( \frac{\beta_{1,b}}{\beta_{1,N}} \right) H_m G_m(x) k_{1,N}(x), \]
\[ a \neq m + 1, a \neq m + 2, b \neq m + 1, b \neq m + 2. \]  
(86)

The other elements of the $K$-matrix depend on the parity of $m$ and hence, it is convenient to introduce the notation $\sigma_m = (-1)^m$. It follows that the elements on the center of the $K$-matrix are given by

\[ k_{m+2,m+2}(x) = k_{m+1,m+1}(x) \quad \text{and} \quad k_{m+2,m+1}(x) = k_{m+1,m+2}(x), \]  
(87)
where,

\[ k_{m+1,m+1}(x) = x^2 G_m(x) \left\{ \frac{(\sigma_m + 1)}{2} \right\} - (\sigma_m - 1) \left[ \frac{x^2 q^m \left[ (1 - x^4) q + (q^2 - 1) \right] - (q x^2 + 1) (q - x^2)}{x^2 (x^2 + 1) (q^m - 1) (q^2 - 1)} \right] \]  

(88)

and

\[ k_{m+1,m+2}(x) = \epsilon x^2 G_m(x) \left\{ \frac{\sigma_m - 1}{2} \right\} \left[ \frac{(x^2 - 1)}{x^2 + 1} \right] + \frac{(\sigma_m + 1)}{2} \left[ 1 - \left( \frac{x^2 q^m - 1}{q^m - 1} \right) \left( \frac{x^2 q + 1}{q^m - 1} \right) \right] \]  

(89)

with \( \epsilon = \pm 1 \) representing two branches of the solutions.

The diagonal elements, by their turn, are given recursively by

\[ k_{a,a} = \begin{cases} 
& k_{a-1,a-1}(x) + \left( \frac{\beta_{a-1,a-1}}{\beta_{N,N}} \right) G_m(x) k_{1,N}(x), \\
& 1 < a < m + 1 \\
& k_{a-1,a-1}(x) + \left( \frac{\beta_{a-1,a-1}}{\beta_{N,N}} \right) x^2 G_m(x) k_{1,N}(x), \\
& m + 3 < a < N 
\end{cases} \]  

(90)

with

\[ k_{1,1}(x) = G_m(x) \left\{ \frac{x^2 q^m - 1}{q^m - 1} \right\} \left( \frac{q + \sigma_m}{q - 1} \right) - \left( \frac{1 + \sigma_m}{q - 1} \right) - \epsilon \left( \frac{x^2 q^m - 1}{q^m - 1} \right) \left( \frac{x^2 - 1}{x^2 + 1} \right) \left( \frac{q - \sigma_m}{q - 1} \right) + \left( \frac{q^m + 1}{q^m - 1} \right) \left( \frac{\sigma_m - 1}{q - 1} \right) \left( \frac{x^2 + \sigma_m}{x^2 + 1} \right) \]  

(91)

and

\[ k_{m+3,m+3}(x) = x^2 G_m(x) \left\{ \frac{x^2 q^m - 1}{q^m - 1} \right\} \left( \frac{q + \sigma_m}{q - 1} \right) - \left( \frac{1 + \sigma_m}{q - 1} \right) + \epsilon \left( \frac{x^2 q^m - 1}{q^m - 1} \right) \left( \frac{x^2 - 1}{x^2 + 1} \right) \left( \frac{q - \sigma_m}{q - 1} \right) - \epsilon \left( \frac{q^m + 1}{q^m - 1} \right) \left( \frac{\sigma_m - 1}{q - 1} \right) \left( \frac{x^2 + \sigma_m}{x^2 + 1} \right) \]  

(92)

At this point all elements of the K-matrix were determined, but not all functional equations are actually satisfied. To solve the remaining functional equations it is necessary to fix some of the parameters \( \beta_{a,a} \). The necessary and sufficient constraints between these parameters are provided by the remaining functional equations. In fact, these equations enable us to fix the diagonal parameters \( \beta_{a,a} \) according to the recursive relations

\[ \beta_{a,a} = \begin{cases} 
& \beta_{a-1,a-1} + \frac{2 \sigma_m (\epsilon (\sigma_m - 1) q^{m+1-a} q^{m+1-a})}{(q^m - 1)(q - 1)}, \\
& 1 < a < m + 1, \\
& \beta_{a-1,a-1} + \frac{2 \sigma_m (\epsilon (\sigma_m - 1) q^{m+1-a} q^{m+1-a})}{(q^m - 1)(q - 1)}, \\
& 1 < a < m + 1, 
\end{cases} \]  

(93)
and also the following non-diagonal parameters,
\[ \beta_{1,m+1} = \epsilon \beta_{1,m+2} \tag{94} \]
\[ \beta_{2,1} = 4iq^2 \frac{3/2}{(q^m - 1)} \left( \frac{q^m + 1 + (q^m - 1)}{q - 1} \right) \beta_{1,N-1} \beta_{1,m+2} \tag{95} \]
\[ \beta_{1,N} = -\frac{i}{4} \left( \frac{\epsilon (q^m + 1 + (q^m - 1) (q^m - 1) (q^m - 1 + 1)}{(q + 1) q^{2m-3/2}} \right) \beta_{1,m+2}^2, \tag{96} \]
and
\[ \beta_{1,b} = \frac{i}{2} \left[ \epsilon (\sigma_m - 1 + (\sigma_m + 1)) (-1)^b \right] \left( \frac{\epsilon (q^m + 1 + (q^m - 1)}{q^m - 1 (q - 1)} \right) \beta_{1,m+2}^2, \tag{97} \]
\[ 1 < b < m + 1. \]

Once the parameters above are fixed, we can verify that all functional equations are satisfied. The following \( m \) parameters \( \beta_{1,m+2}, \beta_{1,m+3}, \ldots, \beta_{1,N-2} \) and \( \beta_{1,N-1} \) remain arbitrary—they are the free-parameters of the solution. The other parameters \( \beta_{1,b} \) can be directly found through (60) (it is not necessary to write down their expressions, since they do not appear explicitly in the solution). The solution thus obtained is regular and characterized by \( m \) free-parameters.

### 3.2. Block-diagonal solutions

The block-diagonal solutions are such that the only non-diagonal elements of the \( K \)-matrix different from zero are the elements \( k_{m+1,m+2}(x) \) and \( k_{m+1,m+2}(x) \). We remark that these solutions are not trivial reductions of the complete solution presented in the previous section. The existence of these block-diagonal solutions are related to the existence of \( m \) distinct conserved \( U(1) \) charges; in fact, the shape of the \( K \)-matrices compatible with this symmetry is justly the block-diagonal shape [44, 45].

We found here two families of block-diagonal solutions, each of them branching into two solutions regarding the two values of \( \epsilon \). Hence we get four families of block-diagonal solutions. These solutions contain only one free-parameter, which we choose to be \( \beta_{m+1,m+2} \).

#### 3.2.1. The first family of block-diagonal solutions

For the first family of block-diagonal solutions we have that,
\[ k_{m+2,m+1}(x) = k_{m+1,m+2}(x) = \frac{1}{2} x^2 (x^2 - 1) \beta_{m+1,m+2}. \tag{98} \]

The other two elements lying in the center of the \( K \)-matrix are given respectively by
\[ k_{m+1,m+1}(x) = \frac{x^2 (x^2 - 1)}{2} + \epsilon \frac{x (x^2 - 1)}{2} \frac{q^{m/2}}{(q^m - 1)} \sqrt{\left( \frac{q^m + 1}{q^m - 1} \right)^2 \beta_{m+1,m+2}^2 - 1}, \tag{99} \]
and
\[ k_{m+2,m+2}(x) = \frac{(x^2 + 1)}{2} - \epsilon \frac{x (x^2 - 1)}{2} \frac{q^{m/2}}{(q^m + 1)} \sqrt{\left( \frac{q^m + 1}{q^m - 1} \right)^2 \beta_{m+1,m+2}^2 - 1}. \tag{100} \]
Finally, the diagonal elements not in the center are given by
\[
k_{a,a}(x) = \frac{1}{2} \left( \frac{q^a x^2 + 1}{q^{2m} - 1} \right) \left[ (x^2 + 1) (q^m - 1) + (x^2 - 1) (q^m + 1) \beta_{m+1,m+2} \right],
\]
\[1 \leq a \leq m, \tag{101}\]
and
\[
k_{a,a}(x) = \frac{x^2}{2} \left( \frac{q^a x^2 + 1}{q^{2m} - 1} \right) \left[ (x^2 + 1) (q^m - 1) - (x^2 - 1) (q^m + 1) \beta_{m+1,m+2} \right],
\]
\[m + 3 \leq a \leq N. \tag{102}\]

3.2.2. The second family of block-diagonal solutions. For the second family of block-diagonal solutions we have, instead,
\[
k_{m+2,m+2}(x) = k_{m+1,m+1}(x) = \frac{1}{2} x^2 (x^2 + 1). \tag{103}\]

The other two elements in the center are,
\[
k_{m+1,m+2}(x) = \frac{1}{2} x^2 (x^2 - 1) \left\{ \frac{x (q^m + 1)^2 - 2q^m (x^2 + 1)}{(q^m - 1)^2} \right\} \beta_{m+1,m+2}
\]
\[\quad - \epsilon (x - 1)^2 q^{m/2} \left( \frac{q^m + 1}{q^m - 1} \right)^2 \sqrt{\beta_{m+1,m+2}} - 1 \right\}, \tag{104}\]
and
\[
k_{m+2,m+1}(x) = \frac{1}{2} x^2 (x^2 - 1) \left\{ \frac{x (q^m + 1)^2 + 2q^m (x^2 + 1)}{(q^m - 1)^2} \right\} \beta_{m+1,m+2}
\]
\[\quad + \epsilon (x - 1)^2 q^{m/2} \left( \frac{q^m + 1}{q^m - 1} \right)^2 \sqrt{\beta_{m+1,m+2}} - 1 \right\}, \tag{105}\]

Finally, the diagonal elements are
\[
k_{a,a}(x) = \frac{1}{2} \left( \frac{q^a x^2 - 1}{q^{2m} - 1} \right) \left\{ (x^2 + 1) (q^m - 1) + (x^2 - 1) (q^m + 1) \beta_{m+1,m+2} \right\}
\]
\[+ 2\epsilon (x^2 - 1) q^{m/2} \sqrt{\beta_{m+1,m+2}} - 1 \right\}, \quad 1 \leq a \leq m, \tag{106}\]
and
\[
k_{a,a}(x) = \frac{x^2}{2} \left( \frac{q^a x^2 - 1}{q^{2m} - 1} \right) \left\{ (x^2 + 1) (q^m - 1) + (x^2 - 1) (q^m + 1) \beta_{m+1,m+2} \right\}
\]
\[\quad - 2\epsilon (x^2 - 1) q^{m/2} \sqrt{\beta_{m+1,m+2}} - 1 \right\}, \quad m + 3 \leq a \leq N. \tag{107}\]
3.3. X-shape solutions

There is an interesting family of solutions in which the $K$-matrix has a shape of the letter $X$. This means that the only non-null elements of the $K$-matrix are those lying in the main or in the secondary diagonals.

In this family of solutions, the elements lying in the main diagonal are given by

$$k_{a,a}(x) = \begin{cases} 
1, & 1 \leq a \leq m, \\
\frac{\alpha^{a+1}}{q}, & m + 1 \leq a \leq m + 2, \\
x^2, & m + 3 \leq a \leq N,
\end{cases} \quad (108)$$

while the elements of the secondary diagonal are,

$$k_{a,a'}(x) = \begin{cases} 
\frac{1}{2}(x^2 - 1)\beta_{a,a'}, & 1 \leq a \leq m, \\
0, & m + 1 \leq a \leq m + 2, \\
\frac{1}{2}(x^2 - 1)\beta_{a,a'}, & m + 2 \leq a \leq N.
\end{cases} \quad (109)$$

The parameters $\beta_{a,a'}$ should satisfy the constraints

$$\beta_{a,a'} \beta_{a',a} = \frac{4q}{(q - 1)^2}, \quad 1 \leq a \leq m,$$

in order to all functional equations be satisfied. Whence, we get a solution with $m$ free-parameters, namely, $\beta_{m,m+1}, \beta_{2N-1}, \ldots, \beta_{1,1}$ and $\beta_{1,1}$. 

3.4. Diagonal solutions for the $U_q[osp(2)(2n+2|2m)] = U_q[D(2)(n+1|m)]$ vertex-model

The diagonal solutions presented here are indeed valid for the $U_q[osp(2)(2n+2|2m)] = U_q[D(2)(n+1|m)]$ vertex-model. We remark that we found no other solutions for the case $n \neq 0$ so far. We intend to work on the non-diagonal reflection $K$-matrices of the $U_q[osp(2)(2n+2|2m)]$ vertex-model in the future.

We found two families of diagonal solutions for the $U_q[osp(2)(2n+2|2m)]$ vertex-model with no free-parameters. The first one is valid for any value of $m$ and $n$, and has no boundary free-parameter. It is given by:

$$k_{a,a}(x) = \begin{cases} 
1, & 1 \leq a \leq m + n, \\
x\frac{\alpha^{a+1}}{1 + q^{(a+1)(a+2)/2}} [x^{a+1} + x^{a+2} + ix(x+1)q^{(a+z)/2}], & a = m + n + 1, \\
x^2, & a = m + n + 2, \\
\Phi_m^a(x)[1 + (x - 1)\beta_{1,1}], & m + n + 3 \leq a \leq N.
\end{cases} \quad (111)$$

The second family of diagonal $K$-matrices holds actually only when $n = m$. In this case, the solution has two boundary free-parameters and it is given by,

$$k_{a,a}(x) = \begin{cases} 
1 + (x - 1)\beta_{1,1}, & 1 \leq a \leq 2m, \\
\Phi_m^a(x)[1 + (x - 1)\beta_{1,1}], & a = 2m + 1, \\
\Phi_m^a(x)[1 + (x - 1)\beta_{1,1}], & a = 2m + 2, \\
x^2\Phi_m^{*a}\Phi_m^{-}[1 + (x - 1)\beta_{1,1}], & 2m + 3 \leq a \leq N.
\end{cases} \quad (112)$$
where,
\[ \Phi_m^\pm = \frac{2 \pm (\beta_{2m+1} - \beta_{1,1}) (x - 1)}{2x - (\beta_{2m+1} - \beta_{1,1}) (x - 1)}. \]

4. Conclusion

In this work we presented the reflection \( K \)-matrices for the \( U_q[osp^{(2)}(2|2m)] = U_q[C^{(2)}(m + 1)] \) vertex-model. We found several families of solutions which can be classified into four classes: complete solutions, block-diagonal solutions, \( X \)-shape solutions and diagonal solutions. These diagonal solutions are indeed valid for the \( U_q[osp^{(2)}(2n + 2|2m)] = U_q[D^{(2)}(n + 1|m)] \) vertex-model. Some special solutions which hold only for the \( U_q[osp^{(2)}(2|2)] \) and \( U_q[osp^{(2)}(2|4)] \) vertex-models were also obtained (see appendix). In the future, we intend to study the \( K \)-matrices for the multiparametric \( U_q[osp^{(2)}(2|2m)] \) vertex-model (i.e. the corresponding reflection \( K \)-matrices for any possible value of \( \kappa_1, \kappa_2, \) and \( \nu \)) as well as the non-diagonal reflection \( K \)-matrices associated to the most general \( U_q[osp^{(2)}(2n + 2|2m)] = U_q[D^{(2)}(n + 1|m)] \) vertex-model.

We believe that this work contributes significantly to the classification of the reflection \( K \)-matrices associated to quantum twisted Lie superalgebras.

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Appendix. Special solutions

It is a well-known fact that low-dimensional Lie (super)algebras present special properties, for instance, being isomorphic to other Lie (super)algebras. This also happens with the low-dimensional cases of the \( osp^{(2)}(2|2m) = C^{(2)}(m + 1) \) Lie superalgebra considered here. In fact, it can be shown that the \( osp^{(2)}(2|2) = C^{(2)}(2) \) Lie superalgebra is isomorphic to \( sl^{(2)}(2|1) = A^{(2)}(1|0) \) Lie superalgebra, as well as the \( osp^{(2)}(2|4) = D^{(2)}(1|2) \) Lie superalgebra is isomorphic to \( D^{(2)}(2, 1, 1) \) Lie superalgebra and, finally, that there is no other isomorphisms for the higher values of \( m \) (except those associated with an exchange of the even and odd parts, of course) [77–85].

The existence of these special isomorphisms for the low-dimensional Lie superalgebras \( osp^{(2)}(2|2) \) and \( osp^{(2)}(2|4) \) lead to additional reflection \( K \)-matrices for the \( U_q[osp^{(2)}(2|2)] \) and \( U_q[osp^{(2)}(2|4)] \) vertex-models. The existence of these special solutions can be noticed directly from the form of the complete solution presented in the section 3.1. Indeed, we can see that the complete reflection \( K \)-matrix of the \( U_q[osp^{(2)}(2|2m)] \) vertex-model contain \( m \) boundary free-parameters, namely, \( \beta_{1,m+2}, \beta_{1,m+3}, \ldots, \beta_{1,N-2} \) and \( \beta_{1,N-1} \) and, among these parameters, only \( \beta_{1,m+2}, \beta_{1,m+3} \) and \( \beta_{1,N-1} \) appear explicitly in the solution. However, we can notice that for the cases \( m = 1 \) or \( m = 2 \) (but not for higher values of \( m \)) some of these free-parameters become coincident, for instance, we have \( \beta_{1,m+2} = \beta_{1,N-1} \) for \( m = 1 \) and \( \beta_{1,m+3} = \beta_{1,N-1} \) for \( m = 2 \).
This fact suggests the complete solution derived in the section 3.1 may not represent the most general solution for the $U_q[osp^{(2)}(2|2)]$ and $U_q[osp^{(2)}(2|4)]$ vertex-models and indeed this is the case. In fact, solving the boundary YB equation for these two models separately, we found that are other new solutions which hold only for these specific models (the complete solution presented at section 3.1 still holds for these models, but there are other additional solutions that holds only for them). These special solutions will be reported in this appendix.

### A.1. Special solutions for $U_q[osp^{(2)}(2|2)] = U_q[C^{(2)}(2)]$ vertex-model

For the $U_q[osp^{(2)}(2|2)] = U_q[C^{(2)}(2)]$ vertex-model, the corresponding $K$-matrix is a four-by-four matrix, namely,

\[
K(x) = \begin{bmatrix}
  k_{1,1}(x) & k_{1,2}(x) & k_{1,3}(x) & k_{1,4}(x) \\
  k_{2,1}(x) & k_{2,2}(x) & k_{2,3}(x) & k_{2,4}(x) \\
  k_{3,1}(x) & k_{3,2}(x) & k_{3,3}(x) & k_{3,4}(x) \\
  k_{4,1}(x) & k_{4,2}(x) & k_{4,3}(x) & k_{4,4}(x)
\end{bmatrix}. \tag{A.1}
\]

The boundary YB equation (2) consists in this case to a system of sixteen functional equations for the elements $k_{a,b}(x), 1 \leq a, b \leq 4$. By solving directly these equations, we found that there is another additional solution (besides the complete solution presented in the section 3.1) which is characterized by $m + 2 = 3$ boundary free-parameters. This solution can be written as follows: for the elements of the $K$-matrix lying on the first line, we have

\[
k_{1,2}(x) = \left( \frac{\beta_+ + x\beta_-}{\beta_{1,4}} \right) G_1(x)k_{1,4}(x), \tag{A.2}
\]

\[
k_{1,3}(x) = \left( \frac{\beta_+ - \beta_-}{\beta_{1,4}} \right) G_1(x)k_{1,4}(x), \tag{A.3}
\]

and, for the elements in the first column,

\[
k_{2,4}(x) = i\sqrt{q} \left( \frac{x\beta_+ - q^{-1}\beta_-}{\beta_{1,4}} \right) G_1(x)k_{1,4}(x), \tag{A.4}
\]

\[
k_{3,4}(x) = i\sqrt{q} \left( \frac{x\beta_+ + q^{-1}\beta_-}{\beta_{1,4}} \right) G_1(x)k_{1,4}(x). \tag{A.5}
\]

For the elements in the last line, we have

\[
k_{2,1}(x) = 2 \left[ \frac{i\sqrt{q}}{(q + 1)} \beta_{1,4} - \left( \frac{q\beta_+^2 - \beta_-^2}{q - 1} \right) \right] \left( \frac{\beta_+ + x\beta_-}{\beta_{1,4}} \right) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}^2}, \tag{A.6}
\]

\[
k_{3,1}(x) = 2 \left[ \frac{i\sqrt{q}}{(q + 1)} \beta_{1,4} - \left( \frac{q\beta_+^2 - \beta_-^2}{q - 1} \right) \right] \left( \frac{\beta_+ - x\beta_-}{\beta_{1,4}} \right) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}^2}, \tag{A.7}
\]

and, for that on the last column,

\[
k_{4,2}(x) = 2i\sqrt{q} \left[ \frac{i\sqrt{q}}{(q + 1)} \beta_{1,4} - \left( \frac{q\beta_+^2 - \beta_-^2}{q - 1} \right) \right] \left( \frac{\beta_+ + q^{-1}\beta_-}{\beta_{1,4}} \right) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}^2}. \tag{A.8}
\]
\[ k_{4,3}(x) = 2ix\sqrt{q} \left[ \frac{i\sqrt{q}}{(q + 1)} \beta_{1,4} - \left( \frac{q\beta_+^2 - \beta_-^2}{q - 1} \right) \right] \left( \frac{x\beta_+ + q^{-1}\beta_-}{\beta_{1,4}} \right) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}^2}. \] (A.9)

Notice that in the present case, we have,

\[
\beta_{\pm} = \frac{1}{2}(\beta_{1,2} \pm \beta_{1,3}), \quad \text{and} \quad G_1(x) = \frac{2}{(x^2 + 1)}. \] (A.10)

Besides, for the elements lying on the secondary diagonal, not in the center of the \( K \)-matrix, we have,

\[ k_{1,4}(x) = \frac{1}{2}(x^2 - 1) \beta_{1,4}, \] (A.11)

and

\[ k_{4,1}(x) = 4 \left[ \frac{i\sqrt{q}}{q + 1} \beta_{14} - \left( \frac{q\beta_+^2 - \beta_-^2}{q - 1} \right) \right]^2 \frac{k_{1,4}(x)}{\beta_{1,4}^2}. \] (A.12)

For the elements on the main diagonal, not in the center, we have, respectively

\[ k_{1,1}(x) = 1 + i \left[ \left( \frac{x^2 - 1}{x^2 + 1} \right) \left( \frac{q\beta_+^2 - \beta_-^2}{\sqrt{q}} \right) + 2\sqrt{q} \left( \frac{\beta_+^2 + \beta_-^2}{q - 1} \right) \right] G_1(x)k_{1,4}(x) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}^2}. \] (A.13)

and

\[ k_{4,4}(x) = \frac{ix^2}{\sqrt{q}} \left( q\beta_+^2 - \beta_-^2 \right) + 2 \left( \frac{x^2q + 1}{x^2 + 1} \right) \left( \frac{q\beta_+^2 + \beta_-^2}{q - 1} \right) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}^2}. \] (A.14)

while those elements in the center are given by

\[ k_{2,2}(x) = \frac{i}{2} \left[ \left( \frac{x^2q + 1}{q - 1} \right) \left( q\beta_+^2 - \beta_-^2 \right) + 4q\beta_+\beta_- \right] G_1(x)k_{1,4}(x) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}}. \] (A.15)

\[ k_{3,3}(x) = \frac{i}{2} \left[ \left( \frac{x^2q + 1}{q - 1} \right) \left( q\beta_+^2 - \beta_-^2 \right) + 4q\beta_+\beta_- \right] G_1(x)k_{1,4}(x) \frac{G_1(x)k_{1,4}(x)}{\beta_{1,4}}. \] (A.16)

and

\[ k_{3,2}(x) = k_{2,3}(x) = ix^2 \sqrt{q} \left( \frac{\beta_+^2 + q^{-1}\beta_-^2}{\beta_{1,4}^2} \right) G_1(x)^2k_{1,4}(x). \] (A.17)

At this point all elements of the \( K \)-matrix were eliminated and we get a solution with three free-parameters, namely, \( \beta_{1,2}, \beta_{1,3} \) and \( \beta_{1,4} \).

Finally, we remark that the \( U_q[osp^{(2)}(2|2)] \) and \( U_q[C^{(2)}(2)] \) vertex-model considered in this appendix is not related to the Yang–Zhang vertex-model introduced in [87] (see also [87–92]), although the symmetry behind both models is the same. In fact, we considered here the \( \mathcal{R} \)-matrix introduced by Galleas and Martins in [34] which (for \( n = 0, m = 1 \)) corresponds to a four-dimensional representation of the \( U_q[osp^{(2)}(2|2)] \) and \( U_q[C^{(2)}(2)] \) quantum twisted Lie superalgebra, which leads to a thirty-six vertex-model. On the other hand, the Yang–Zhang vertex-model [87] is constructed from a three-dimensional representation of
the $U_q[\mathfrak{osp}^{(2)}(2|2)] = U_q[\mathfrak{C}^{(2)}(2)]$ quantum twisted Lie superalgebra, which leads to a nineteen vertex-model instead. The reflection $K$-matrices of the Yang–Zhang vertex-model were recently presented by us in [93] and the corresponding boundary algebraic Bethe Ansatz was performed in [94].

A.2. Special solutions for $U_q[\mathfrak{osp}^{(2)}(2|4)] = U_q[\mathfrak{C}^{(2)}(3)]$ vertex-model

For the $U_q[\mathfrak{osp}^{(2)}(2|4)] = U_q[\mathfrak{C}^{(2)}(3)]$ vertex-model, the $K$-matrix is a six-by-six matrix. Besides the complete solution presented at section 3.1, there is a special solution which holds only for $m = 2$ that has a shape which resembles an $X$-block matrix:

$$K(x) = \begin{bmatrix}
k_{1,1}(x) & k_{1,2}(x) & 0 & 0 & k_{1,5}(x) & k_{1,6}(x) \\
k_{2,1}(x) & k_{2,2}(x) & 0 & 0 & k_{2,5}(x) & k_{2,6}(x) \\
0 & 0 & k_{3,3}(x) & 0 & 0 & 0 \\
0 & 0 & 0 & k_{4,4}(x) & 0 & 0 \\
k_{5,1}(x) & k_{5,2}(x) & 0 & 0 & k_{5,5}(x) & k_{5,6}(x) \\
k_{6,1}(x) & k_{6,2}(x) & 0 & 0 & k_{6,5}(x) & k_{6,6}(x)
\end{bmatrix}.$$  \tag{A.18}

The elements of the $K$-matrix are the following: for the non-diagonal elements, we have,

$$k_{1,2}(x) = \left( \frac{\beta_{1,2}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \quad k_{1,5}(x) = \left( \frac{\beta_{1,5}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \tag{A.19}$$

$$k_{2,1}(x) = \left( \frac{\beta_{2,1}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \quad k_{5,1}(x) = \left( \frac{\beta_{5,1}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \tag{A.20}$$

$$k_{6,2}(x) = -x^2 q \left( \frac{\beta_{5,1}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \quad k_{6,5}(x) = -x^2 \left( \frac{\beta_{2,1}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \tag{A.21}$$

$$k_{2,6}(x) = x^2 q \left( \frac{\beta_{1,2}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \quad k_{5,6}(x) = -x^2 \left( \frac{\beta_{1,2}}{\beta_{1,6}} \right) G_2(x) k_{1,6}(x), \tag{A.22}$$

with

$$\beta_{2,1} = q \left( \frac{\beta_{1,5}}{\beta_{1,6}} \right) \left( \frac{2}{q+1} - \frac{\beta_{1,2} \beta_{1,5}}{\beta_{1,6}} \right), \quad \beta_{5,1} = -\left( \frac{\beta_{1,2}}{\beta_{1,6}} \right) \left( \frac{2}{q+1} - \frac{\beta_{1,2} \beta_{1,5}}{\beta_{1,6}} \right). \tag{A.23}$$

Notice that, in this case,

$$G_2(x) = \frac{q + 1}{qx^2 + 1}. \tag{A.24}$$

The elements on the secondary diagonal are given by

$$k_{1,6}(x) = \frac{1}{2} (x^2 - 1) \beta_{1,6}, \quad k_{6,1}(x) = -\frac{1}{q} \left( \frac{\beta_{2,1}}{\beta_{1,5}} \right)^2 k_{1,6}(x), \tag{A.25}$$

$$k_{2,5}(x) = -\left( \frac{\beta_{2,1}}{\beta_{1,2}} \right) k_{1,6}(x), \quad k_{5,2}(x) = -\left( \frac{\beta_{5,1}}{\beta_{1,5}} \right) k_{1,6}(x). \tag{A.26}$$
and the elements on the main diagonal are,

\[ k_{1,1}(x) = 1 - q \left( \frac{\beta_{1,2}\beta_{1,5}}{\beta_{1,6}} \right) G_2(x)k_{1,6}(x), \quad k_{5,5}(x) = x^2 k_{1,1}(x), \]  

(A.27)

\[ k_{2,2}(x) = 1 + \left( \frac{\beta_{1,2}\beta_{1,5}}{\beta_{1,6}} \right) G_2(x)k_{1,6}(x), \quad k_{6,6}(x) = x^2 k_{2,2}(x). \]  

(A.28)

Finally, for the central elements, we have,

\[ k_{4,4}(x) = k_{3,3}(x) = \left[ \frac{1}{G(x)} - \left( \frac{x^2 q^2 - 1}{q + 1} \right) \left( \frac{\beta_{1,2}\beta_{1,5}}{\beta_{1,6}^2} \right) G(x)k_{1,6}(x) \right]. \]  

(A.29)

Therefore, we get a solution with three boundary free-parameters, namely, the parameters \( \beta_{1,2}, \beta_{1,5} \) and \( \beta_{1,6} \).

We also report the existence of the special diagonal solution \( K(x) = \text{diag} \left( \frac{1}{x^2}, 1, 1, 1, 1, x^2 \right) \), which holds both for the \( U_q[\text{osp}(2|2)] = U_q[C(2)^2(3)] \) and \( U_q[\text{osp}(2|2)^6] = U_q[D(3)^2] \) vertex-models [44, 45, 86].

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**References**

[1] McGuire J B 1964 *J. Math. Phys.* 5 622–36
[2] Yang C 1968 *Phys. Rev.* 168 1920
[3] Yang C 1967 *Phys. Rev. Lett.* 19 1312
[4] Zamolodchikov A B and Zamolodchikov A B 1979 *Ann. Phys.* 120 253–91
[5] Ghoshal S and Zamolodchikov A 1994 *Int. J. Mod. Phys.* A 9 3841–85
[6] Bazhanov V V and Shadrikov A G 1987 *Theor. Math. Phys.* 73 1302–12
[7] Baxter R J 1972 *Ann. Phys.* 70 193–228
[8] Baxter R J 1978 *Phil. Trans. R. Soc.* A 289 315–46
[9] Sklyanin E K, Takhtadzhyan L A and Faddeev L D 1979 *Theor. Math. Phys.* 40 86
[10] Takhtadzhyan L A and Faddeev L D 1979 *Russ. Math. Surv.* 34 11–68
[11] Zamolodchikov A B and Fateev V A 1980 *Yad. Fiz.* 32 581–90
[12] Faddeev L D 1982 Integrable models in \( (1 + 1) \)-dimensional quantum field theory *Les Houches Summer School in Theoretical Physics: Recent Advances in Field Theory and Statistical Mechanics* (Les Houches, France) pp 294–341
[13] Sklyanin E 1982 *J. Sov. Math.* 19 1546–96
[14] Baxter R J 1982 *Exactly Solved Models in Statistical Physics* (New York: Academic) vol 1
[15] Faddeev L 1995 How algebraic Bethe ansatz works for integrable model *Relativistic Gravitation and Gravitational Radiation, Proc., School of Physics (Les Houches, France)*
[16] Korepin V E, Bogoliubov N M and Izergin A G 1997 *Quantum Inverse Scattering Method and Correlation Functions* vol 3 (Cambridge: Cambridge University Press)
[17] Batchelor M T 2007 *Phys. Today* 60 36–40
[18] Vieira R S and Lima-Santos A 2015 *Phys. Lett.* A 379 2150–3
[19] Maldacena J 1999 *Int. J. Theor. Phys.* 38 1113–33
[20] Minahan J A and Zarembo K 2003 *J High Energy Phys.* JHEP03(2003)013
[21] Beisert N and Staudacher M 2003 *Nucl. Phys.* B 670 439–63
[22] Bena I, Polchinski J and Roiban R 2004 Phys. Rev. D 69 046002
[23] Drinfel’d V G 1988 Dokl. Akad. Nauk 296 13–7 (http://inspirehep.net/record/244555/)
[24] Drinfel’d V G 1988 J. Sov. Math. 41 898–915
[25] Faddeev L, Reshetikhin N Y and Takhtajan L 1987 Algebra. Anal. 1 178–206
[26] Jimbo M 1985 Lett. Math. Phys. 10 63–9
[27] Jimbo M 1986 Lett. Math. Phys. 11 247–52
[28] Lambe L A and Radford D E 2013 Introduction to the Quantum Yang–Baxter Equation and Quantum Groups: an Algebraic Approach vol 423 (Berlin: Springer)
[29] Kulish P P and Sklyanin E K 1982 J. Sov. Math. 19 1596
[30] Kulish P P and Sklyanin E K 1982 J. Math. Sci. 19 1596
[31] Bazhanov V V 1985 Phys. Lett. B 159 321–4
[32] Jimbo M 1986 Commun. Math. Phys. 102 537–47
[33] Galleas W and Martins M J 2004 Nucl. Phys. B 699 455–86
[34] Galleas W and Martins M J 2006 Nucl. Phys. B 732 444–62
[35] Galleas W and Martins M J 2007 Nucl. Phys. B 768 219–46
[36] Sklyanin E K 1989 J. Phys. A: Math. Gen. 21 2375
[37] Cherednik I V 1984 Theor. Math. Phys. 61 977–83
[38] Mezincescu L and Nepomuch R I 1991 J. Phys. A: Math. Gen. 24 L17
[39] Mezincescu L and Nepomuch R I 1991 Int. J. Mod. Phys. A 6 5231–48
[40] de Vega H J and Ruiz A G 1993 J. Phys. A: Math. Gen. 26 L519
[41] de Vega H J and Gonzalez-Ruiz A 1994 J. Phys. A: Math. Gen. 27 6129
[42] Inami T, Odake S and Zhang Y Z 1996 Nucl. Phys. B 470 419–32
[43] Lima-Santos A 1999 Nucl. Phys. B 558 637–67
[44] Martins M J and Gun X W 2000 Nucl. Phys. B 583 721–38
[45] Lima-Santos A 2001 Nucl. Phys. B 612 446–60
[46] Kulish P and Sklyanin E 1992 J. Phys. A: Math. Gen. 25 5963
[47] Letzter G 2002 Coideal subalgebras and quantum symmetric pairs New Directions in Hopf Algebras vol 43 (MSRI Publications) ed S Montgomery and H-J Schneider (Cambridge: Cambridge University Press)
[48] Nepomuch R I 2002 Lett. Math. Phys. 62 83–9
[49] Delius G W and George A 2002 Lett. Math. Phys. 52 211–7
[50] Delius G W and MacKay N J 2003 Commun. Math. Phys. 233 173–90
[51] Molev A I, Ragoucy E and Sorba P 2003 Rev. Math. Phys. 15 789–822
[52] Doikou A 2005 J. Stat. Mech. P12005
[53] Vaes S et al 2007 Duke Math. J. 140 35–84
[54] Baseilhac P and Belliard S 2010 Lett. Math. Phys. 93 213–28
[55] Kolb S 2014 Adv. Math. 267 395–469
[56] Chen H, Guay N and Ma X 2014 Trans. Am. Math. Soc. 366 2517–74
[57] Regelskis V and Vlaar B 2016 arXiv:1602.08471
[58] Balagovic M and Kolb S 2016 J. Reine Angew. Math. arXiv:1507.06276 (Crelles Journal)
[59] Braconni A J, Ge X Y, Zhang Y Z and Zhou H Q 1998 Nucl. Phys. B 516 588–602
[60] Malara R and Lima-Santos A 2006 J. Stat. Mech. P08005
[61] Arnaudon D, Avan J, Crampé N, Doikou A, Frappat L and Ragoucy E 2003 Nucl. Phys. B 668 469–505
[62] Lima-Santos A 2009 J. Stat. Mech. P04005
[63] Lima-Santos A 2009 J. Stat. Mech. P07045
[64] Lima-Santos A 2009 J. Stat. Mech. P08006
[65] Lima-Santos A and Galleas W 2010 Nucl. Phys. B 833 271–97
[66] Moody R V 1967 Lie Algebras Associated With Generalized Cartan Matrices vol 73, pp 217–21
[67] Moody R V 1968 J. Algebra 10 211–30
[68] Kac V G 1977 Adv. Math. 26 8–96
[69] Kac V 1977 Commun. Math. Phys. 53 31–64
[70] Kulish P 1996 Low-dimensional models in statistical physics and quantum field theory Low-Dimensional Models in Statistical Physics and Quantum Field Theory (Lecture Notes in Physics vol 469) ed H Grosse and L Pittner (Berlin, Heidelberg: Springer) pp 125–44
[71] Murakami J 1987 Osaka J. Math. 24 745–58
[72] Birman J S and Wenzl H 1989 Trans. Am. Math. Soc. 313 249–73
[73] Grimm U and Pearce P A 1993 J. Phys. A: Math. Gen. 26 7435

23
[74] Grimm U 1994 *J. Phys. A: Math. Gen.* **27** 5897
[75] Grimm U 1994 *Lett. Math. Phys.* **32** 183–7
[76] Grimm U and Warnaar S O 1995 *J. Phys. A: Math. Gen.* **28** 7197
[77] Feingold A J and Frenkel I B 1985 *Adv. Math.* **56** 117–72
[78] Frappat L, Sciarrino A and Sorba P 1989 *Commun. Math. Phys.* **121** 457–500
[79] Frappat L, Sciarrino A and Sorba P 2000 *Dictionary on Lie Algebras and Superalgebras* vol 10 (New York: Academic)
[80] Neeb K H and Pianzola A 2010 *Developments and Trends in Infinite-Dimensional Lie Theory* vol 288 (Berlin: Springer)
[81] Serganova V 2011 *Developments and Trends in Infinite-Dimensional Lie Theory* (Berlin: Springer) pp 169–218
[82] Musson I M 2012 *Lie Superalgebras and Enveloping Algebras* vol 131 (Providence, RI: American Mathematical Society)
[83] Ransingh B 2013 *Int. J. Pure Appl. Math.* **84** 539–47
[84] Sthanumoorthy N 2016 *Introduction to Finite and Infinite Dimensional Lie (Super) Algebras* (New York: Academic)
[85] Xu Y and Zhang R 2016 arXiv:1607.01142
[86] Vieira R S and Lima-Santos A 2013 *J. Stat. Mech.* P02011
[87] Yang W L and Zhang Y Z 1999 *Phys. Lett.* A **261** 252–8
[88] Gould M D, Links J R, Zhang Y Z and Tsobantjis I 1997 *J. Phys. A: Math. Gen.* **30** 4313
[89] Gould M D and Zhang Y Z 2000 *Nucl. Phys.* B **566** 529–46
[90] Khoroshkin S, Lukierski J and Tolstoy V 2001 *Commun. Math. Phys.* **220** 537–60
[91] MacKay N and Zhao L 2001 *J. Phys. A: Math. Gen.* **34** 6313
[92] Yang W L and Zhen Y 2001 *Commun. Theor. Phys.* **36** 381–4
[93] Vieira R S and Lima-Santos A 2017 *Phys. Lett.* A (in press) (https://doi.org/10.1016/j.physleta.2017.07.032)
[94] Vieira R S and Lima-Santos A 2017 *J. Stat. Mech.* (arXiv:1705.08953) in press