Towards a Non-perturbative Formulation of IIB Superstrings by Matrix Models

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Abstract

We address the problem of a non-perturbative formulation of superstring theory by means of the recently proposed matrix models. For the model by Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT), we perform one-loop calculation of the interaction between operator-like solutions identified with D-brane configurations of the type IIB superstring (in particular, for parallel moving and rotated static p-branes). Comparing to the superstring calculations, we show that the matrix model reproduces the superstring results only at large distances or small velocities, corresponding to keeping only the lowest mass closed string modes. We propose a modification of the IKKT matrix model introducing an integration over an additional Hermitian matrix required to have positive definite eigenvalues, which is similar to the square root of the metric in the continuum Schild formulation of IIB superstrings. We show that for this new matrix action the Nambu–Goto version of the Green–Schwarz action is reproduced even at quantum level.

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1 Introduction

Recent interest in matrix-model formulation of superstrings has been initiated by the proposal of Banks, Fischler, Shenker and Susskind \[1\] that the non-perturbative dynamics of M theory is described by supersymmetric $n \times n$ matrix quantum mechanics in the limit of large $n$. This matrix model has been investigated in a number of subsequent papers \[2, 3, 4, 5, 6\], and its operator-like classical solutions were identified with D(irichlet) p-branes (for even $p$) of the type IIA superstring theory.

Another matrix model has been proposed by Ishibashi, Kawai, Kitazawa and Tsuchiya \[7\] (IKKT) for type IIB superstrings in ten dimensions. The action of the model is defined by

$$S = \alpha \left( -\frac{1}{4} \text{Tr} [A_{\mu}, A_{\nu}]^2 - \frac{1}{2} \text{Tr} (\bar{\psi} \Gamma^\mu [A_{\mu}, \psi]) \right) + \beta n, \quad (1.1)$$

where $A_{\mu}$ and $\psi_{\alpha}$ are $n \times n$ ($n \to \infty$) Hermitian bosonic and fermionic matrices, respectively. The parameter $n$ is considered as a dynamical variable which makes a crucial difference between the action (1.1) and the one of (dimensionally reduced) ten-dimensional super Yang–Mills. The action (1.1) is associated with the IIB superstring in the Schild formalism (with fixed $\kappa$-symmetry):

$$S_{\text{Schild}} = \int d^2 \sigma \left( \alpha \left( \frac{1}{4\sqrt{g}} \{X^\mu, X^\nu\}_PB - \frac{i}{2} \bar{\psi} \Gamma^\mu \{X^\mu, \psi\}_PB \right) + \beta \sqrt{g} \right), \quad (1.2)$$

where the commutators are substituted by the Poisson brackets. Properties of the IKKT matrix model were further studied in \[8, 9, 10\].

The Dp-branes with odd $p$ of type IIB superstring theory appear in the matrix model (1.1) as operator-like solutions of the classical equations

$$[A^\mu, [A^\mu, A^\nu]] = 0, \quad [A^\mu, (\Gamma^\mu \psi)_\alpha] = 0. \quad (1.3)$$

A general multi-brane solution has a block-diagonal form and is built out of single p-branes. The solution associated with one p-brane is given by

$$A^\mu_{\text{cl}} = \left( P_1, Q_1, \ldots, P_{\frac{p+1}{2}}, Q_{\frac{p+1}{2}}, 0, \ldots, 0 \right), \quad \psi^\alpha_{\text{cl}} = 0, \quad (1.4)$$

where $P$’s and $Q$’s form $(p+1)/2$ pairs of operators (infinite matrices) obeying canonical commutation relation on a torus associated with compactification (of large enough radii $L_a/2\pi$) of the axes $0, \ldots, p$ so that $L_a L_{a+1}/n^{2/(p+1)}$ (even $a$) is kept fixed as $n \to \infty$. This solution for D-string ($p = 1$) was constructed in \[7\] in analogy with \[1\] and was extended for $p \geq 3$ in \[8, 10\] in analogy with \[7\]. One of the arguments in favor of this construction is based on the correct large-distance behavior of the one-loop matrix model calculation of the interaction between anti-parallel D-strings \[7\] and higher p-branes \[1\].

In the present paper we continue the investigation of the IKKT matrix model. We perform comprehensive one-loop calculation of the interaction between two Dp-branes and compare the matrix-model and superstring results. For parallel moving p-branes we demonstrate that the matrix model reproduces Bachas’ superstring result \[13\] only at large distances or small velocities keeping only the lowest mass closed string modes. We propose a modification of the IKKT matrix model

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1Another approach to the type IIB superstring is discussed in \[11, 12\].
introducing an additional (positive definite) Hermitian matrix $Y_{ij}$, which is a dynamical variable to be identified with $\sqrt{g}$ in eq. (1.2). The classical action has the form

$$S^{cl} = \alpha \left( -\frac{1}{4} \text{Tr} Y^{-1} [A_\mu, A_\nu]^2 - \frac{1}{2} \text{Tr} (\bar{\psi} \Gamma^\mu [A_\mu, \psi]) \right) + \beta \text{Tr} Y$$

(1.5)

and reduces to the classical action of the non-abelian Born–Infeld (NBI) type:

$$S^\text{nbi}_{cl} = \sqrt{\alpha \beta} \text{Tr} \sqrt{-[A_\mu, A_\nu]^2} - \frac{\alpha}{2} \text{Tr} (\bar{\psi} \Gamma^\mu [A_\mu, \psi]),$$

(1.6)

using classical equation of motion for $Y$. Moreover, we show that the Nambu–Goto version of the Green–Schwarz action is reproduced even at the quantum level, if one chooses appropriately the measure of integration over $Y$. Our results are therefore much more general than displayed in eqs. (1.5) and (1.6).

In Sect. 2 we perform one-loop calculation of scattering of parallel p-branes in the IKKT matrix model (as well as the interaction of rotated static p-branes) and compare to the superstring calculations. We also reproduce previously known results for the anti-parallel p-branes using the new technique. In Sect. 3 we propose a modification of the IKKT matrix model introducing integration over an additional Hermitian matrix $Y$, required to have positive definite eigenvalues. We show how the Nambu–Goto version of the Green–Schwarz action is reproduced for the proposed model at the quantum level. The results are discussed in Sect. 4. Appendix A contains the proof of the $\mathcal{N} = 2$ supersymmetry of the proposed matrix model for $n \to \infty$.

2 Interaction of branes in the IKKT matrix model

In the large-$n$ limit the matrices $A_\mu$ and $\psi_\alpha$ become operators in a Hilbert space and the classical equations of motion (1.3) possess nontrivial solutions which possibly correspond to solitonic states in type IIB superstring theory. Among the solutions to eq. (1.3), the distinguished role is played by the ones for which the field strength

$$f_{\mu\nu} = i [A_\mu, A_\nu]$$

(2.1)

is proportional to the unit matrix. Only these classical configurations can preserve half of the supersymmetries and thus can be interpreted as BPS states [7, 8]. This is why they are associated with D-branes of various dimensions.

The solution which can be interpreted as D-brane of dimension $p$ has the form

$$A^{cl}_\mu = (B_0, B_1, B_2, \ldots, B_p, 0, \ldots, 0), \quad \psi^{cl}_\alpha = 0,$$

(2.2)

where $B_0, \ldots, B_p$ are operators (infinite $n \times n$ matrices) with the commutator

$$[B_a, B_b] = -ig_{ab}1,$$

(2.3)

and $a, b = 0, \ldots, p$. Such solutions exist only for odd $p$, otherwise one can find a linear combination of $B_a$’s, which commutes with all other operators. The solution corresponding to D-strings ($p = 1$) was studied in [7] and was generalized to $p = 3$ and 5 in [1, 10].
By a Lorentz transformation the skew-symmetric matrix \( g_{ab} \) can be brought to the canonical form

\[
g_{ab} = \begin{pmatrix}
0 & -\omega_1 \\
\omega_1 & 0 \\
\ddots & \ddots \\
0 & -\omega_{p+1} \\
\omega_{p+1} & 0
\end{pmatrix}.
\] (2.4)

For such \( g_{ab} \) the operators \( B_a \) form a set of \( l = (p + 1)/2 \) pairs of canonical variables and (2.2) coincides with (1.4). In the coordinate representation they can be represented by

\[
B_0 = i\omega_1 \partial_1, \quad B_1 = q_1, \quad \ldots, \quad B_{p-1} = i\omega_l \partial_l, \quad B_p = q_l.
\] (2.5)

The eigenvalues of the operators \( B_a \) are uniformly distributed along the interval \([-L_a/2, L_a/2]\) of the real axis, where \( L_a/2\pi \) are compactification radii. In fact, we can assume that the support of eigenvalues covers the whole real axis, because \( L_a \) should scale in the large \( n \) limit as \( n^{1/2} \). Each operator \( B_a \) has \( n^{1/2} \) different eigenvalues, so the spacing between them, \( L_a n^{-1/4} \), scales as \( n^{-1/2} \). The product of the eigenvalue densities of the canonical conjugate variables \( B_{2i-2} \) and \( B_{2i-1} \) is fixed by the Fourier transformation, so we get

\[
\frac{n^{1/2}}{L_{2i-2}L_{2i-1}} = \frac{n^{1/2}}{2\pi\omega_i}.
\] (2.6)

In this section we consider processes with interaction between p-branes of this type. We calculate first the interaction between two parallel p-branes moving with constant velocity, then the interaction between two p-branes rotated through some angle and finally the interaction between two anti-parallel p-branes (or brane-antibrane).

### 2.1 Scattering of parallel p-branes

The multi-brane configurations correspond to the solutions of the equations of motion which have the block-diagonal form. Obviously, the matrices with two identical blocks describe a pair of superimposed p-branes. Let us first shift one of them by the distance \( b/2 \) and the other by \(-b/2\) along the \((p + 2)\)-th axis. This results in the configuration of two parallel p-branes separated along the \((p + 2)\)-th axis by the distance \( b \) from each other. Let us finally boost these stationary p-branes along \((p + 1)\)-th axis in opposite directions. Using the block-diagonal construction, the configuration of two parallel p-branes moving with constant velocity, with impact parameter \( b \), is thus described by the following classical solution to eq. (1.3):

\[
A_0^{cl} = \begin{pmatrix} B_0 \cosh \epsilon & 0 \\
0 & B_0 \cosh \epsilon \end{pmatrix},
\]

\[
A_a^{cl} = \begin{pmatrix} B_a & 0 \\
0 & B_a \end{pmatrix}, \quad a = 1, \ldots, p,
\]

\[
A_{p+1}^{cl} = \begin{pmatrix} B_0 \sinh \epsilon & 0 \\
0 & -B_0 \sinh \epsilon \end{pmatrix},
\]

\[
A_{p+2}^{cl} = \begin{pmatrix} b/2 & 0 \\
0 & -b/2 \end{pmatrix},
\]

\[
A_i^{cl} = 0, \quad i = p + 3, \ldots, 9.
\] (2.7)
To simplify the calculation, we have chosen the frame where the two \( p \)-branes have opposite velocity \( v \) and \(-v\) along the \((p + 1)\)-th axis and
\[
v = \tanh \epsilon.
\] (2.8)

The interaction between the two \( p \)-branes to zeroth order in string coupling constant is determined by the one-loop effective action of the matrix model in the background (2.7), which have the form [7]
\[
W = \frac{1}{2} \text{Tr} \ln(P^2 \delta_{\mu\nu} - 2i F_{\mu\nu}) - \frac{1}{4} \text{Tr} \ln \left( (P^2 + \frac{i}{2} F_{\mu\nu} \Gamma^{\mu\nu}) \left( \frac{1 + \Gamma_{11}}{2} \right) \right) - \text{Tr} \ln(P^2),
\] (2.9)
after the Wick rotation to the Euclidean space. The adjoint operators \( P_\mu \) and \( F_{\mu\nu} \) act on the space of matrices and are defined by
\[
P_\mu = [A^{cl}_\mu, \cdot], \quad F_{\mu\nu} = i \left[ [A^{cl}_\mu, A^{cl}_\nu], \cdot \right].
\] (2.10)

The block–diagonal form of the classical solution (2.7) shows that it is convenient to represent the matrices in adjoint representation as \( 2 \times 2 \) matrices composed of \( n \times n \) blocks. At infinite \( n \) these blocks become the operators acting in the same Hilbert space as \( B_a \). In the coordinate representation (2.5) they have the form of the functions of two sets of \( l \) variables — \( q^1 \) and \( q^2 \) — and \( B_a \) act on them as derivative (left and right) and multiplication operators. From the definition of the adjoint operators, we find that \( P_\mu \)'s act on Hermitian matrices as
\[
P_0 \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} = i \omega_1 (\partial^1_1 + \partial^2_1) \cosh \epsilon \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix},
\]
\[
P_1 \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} = (q_1^1 - q_1^2) \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix},
\]
\[
\vdots
\]
\[
P_{p-1} \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} = i \omega_1 (\partial^1_1 + \partial^2_1) \cosh \epsilon \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix},
\]
\[
P_p \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} = (q_1^1 - q_1^2) \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix},
\]
\[
P_{p+1} \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} = i \omega_1 \sinh \epsilon \begin{pmatrix} (\partial^1_1 + \partial^2_1)X & (\partial^1_1 - \partial^2_1)Y \\ -(\partial^1_1 - \partial^2_1)Y^\dagger & -(\partial^1_1 + \partial^2_1)Z \end{pmatrix},
\]
\[
P_{p+2} \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} = b \begin{pmatrix} 0 & Y \\ -Y^\dagger & 0 \end{pmatrix}.
\] (2.11)

The only non-zero component of the field strength in the adjoint representation is \( F_{1p+1} \) which acts as
\[
F_{1p+1} \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} = 2 \omega_1 \sinh \epsilon \begin{pmatrix} 0 & Y \\ -Y^\dagger & 0 \end{pmatrix}.
\] (2.13)

This operator acts non-trivially only on \( Y \) and \( Y^\dagger \). The effective action (2.9) vanishes for \( F_{\mu\nu} = 0 \) [8], consequently we can trace only the action of \( P^2 \) on \( Y \). The contributions of the eigenvalues, corresponding to the eigenfunctions with nonzero \( X \) and \( Z \), are mutually cancelled.
Using the notation

\[ q_i^\pm = \frac{1}{\sqrt{2}}(q_i^1 \pm q_i^2) \]

\[ \partial_i^\pm = \frac{1}{\sqrt{2}}(\partial_i^1 \pm \partial_i^2), \]

we write the action of the operator \( P^2 \) as

\[
P^2 Y = \left\{ b^2 + 2 \sum_{i=2}^{l} \left[ (q_i^-)^2 - (\omega_i \partial_i^+)^2 \right] - 2\omega_i^2 \cosh^2 \epsilon (\partial_i^+)^2 + 2(q_i^-)^2 - 2\omega_i^2 \sinh^2 \epsilon (\partial_i^-)^2 \right\} Y.
\]

This expression is calculated in Euclidean space after a Wick rotation. We can clearly see that the last two terms that depend on \( q_i^- \) and \( \partial_i^- \) give a Hamiltonian of harmonic oscillator, while all \( \partial_i^+ \), and \( q_i^+ \) for \( i = 2, \ldots, l \) enter without their conjugate variables and can be simultaneously diagonalized with \( P^2 \). Thus the eigenvalues of \( P^2 \) are given by

\[
E_{q, p, k} = b^2 + 2 \sum_{i=2}^{l} (q_i^2 + p_i^2) + 2 \cosh^2 \epsilon p_1^2 + 4\omega_1 \sinh \epsilon (k + \frac{1}{2}),
\]

where \( q_i \) and \( p_i \) are the eigenvalues of \( q_i^- \) and \( i\omega_i \partial_i^+ \), respectively.

According to the discussion at the beginning of this section, the eigenvalues of the operators \( q_i^1, q_i^2 \) are distributed from \(-L_{2i-1}/2\) to \( L_{2i-1}/2\) with the constant density \( n_{1/l}/L_{2i-1} \). This is not true for the eigenvalues of \( q_i^- \), \( q_i^+ \) because the integration region changes under the change of variables (2.14), as shown in fig. 1. As a result, the density of the eigenvalues \( q_i \) in (2.16) decreases from \( \sqrt{2}n_{1/l}/L_{2i-1} \) at the origin to zero at \( \pm L_{2i-1}/\sqrt{2} \). The scale of the variation is however of order \( L \), which is negligible in the large–\( n \) limit. For convergent integrals the distribution of \( q_i \)
and $p_i$ can be taken to be uniform and equal to its value at the origin, $\sqrt{2}n^{1/2}/L_{2i-1}$ for $q_i$ and $\sqrt{2}n^{1/2}/L_{2i-2}$ for $p_i$.

Now we can utilize the results of Ref. [7] to bring the one-loop effective action for the given background to the form

$$W = \prod_{a \neq 1} \left( \frac{\sqrt{2} n^i}{L_a} \right) \int d^{d-1}q \, d^d p \sum_{k=0}^{\infty} \ln \left( 1 - \frac{16\omega_{q,p}^2 \sinh^2 \epsilon}{E_{q,p,k}^2} \right) \left[ -\frac{1}{2} \sum_{s_1,\ldots,s_5=\pm 1 \atop s_1,\ldots,s_5=1} \ln \left( 1 - \frac{2s_i \omega_1 \sinh \epsilon}{E_{q,p,k}} \right) \right].$$

(2.17)

The last term which originates from the integration over fermions can be rewritten as

$$\frac{1}{2} \sum_{s_1,\ldots,s_5=\pm 1 \atop s_1,\ldots,s_5=1} \ln \left( 1 - \frac{2s_i \omega_1 \sinh \epsilon}{E_{q,p,k}} \right) = 4 \ln \left( 1 - \frac{4\omega_{q,p}^2 \sinh^2 \epsilon}{E_{q,p,k}^2} \right).$$

(2.18)

It is convenient to represent the logarithms in (2.17) in the form of the integrals over a “proper time” $s$:

$$\ln \frac{u}{v} = \int_0^{\infty} ds \left( e^{-us} - e^{-vs} \right).$$

(2.19)

The sum over $k$ and the integrals over $q$ and $p$ then can be evaluated using the formula

$$\int d^{d-1}q \, d^d p \sum_{k=0}^{\infty} e^{-s E_{q,p,k}} = \left( \frac{\pi}{2s} \right)^{\frac{d}{2}} \frac{e^{-b^2s}}{2 \cosh \epsilon \sinh (2\omega_1 \sinh \epsilon)}. \quad (2.20)$$

We finally obtain the following form

$$W = -n^{p+1} \prod_{a \neq 1} L_{a}^{1-1} \int_0^{\infty} ds \left( \frac{\pi}{s} \right)^{\frac{d}{2}} e^{-b^2s} \left( \cos(4\omega_1 s \sinh \epsilon) - 4 \cos(2\omega_1 s \sinh \epsilon) + 3 \right) \frac{1}{\cosh \epsilon \sinh (2\omega_1 s \sinh \epsilon)}. \quad (2.21)$$

Defining

$$V_{p+1} = \prod_{a=0}^{p} L_a \quad (2.22)$$

and substituting (cf. eq. (2.6))

$$n^{\frac{1}{2}} = \frac{L_{2i-2} L_{2i-1}}{2\pi \omega_i},$$

$$n = V_{p+1} \prod_{i=1}^{i} \frac{1}{2\pi \omega_i}, \quad (2.23)$$

we now Wick rotate back to Minkowski space-time to obtain

$$W = -i \frac{V_p}{(2\pi)^p} \omega_1 \prod_{i=1}^{i} \frac{1}{\omega_i^2} \int_0^{\infty} ds \left( \frac{\pi}{s} \right)^{\frac{d}{2}} e^{-b^2s} \left( \cos(4\omega_1 s \sinh \epsilon) - 4 \cos(2\omega_1 s \sinh \epsilon) + 3 \right) \frac{1}{\cosh \epsilon \sinh (2\omega_1 s \sinh \epsilon)}. \quad (2.24)$$

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where
\[ V_p = \prod_{a=1}^{p} L_a. \] (2.25)

It should also be noted that \( \omega_i \sim \alpha' \) from dimensional analysis. We take
\[ \omega_i = 2\pi\alpha', \] (2.26)
for all \( i \), which, as we shall now show, is the correct normalization to get agreement with supergravity.

We can compare the result we get from the IKKT model to that of Bachas [13] which is exact in \( b, v, \alpha' \). It is clear that the IKKT result does not agree with Bachas’ calculation, for instance a comparison of the absorptive parts shows that they do not have the same poles (see below). There is, however, a regime in which the two results are identical. This is the regime in which supergravity, or alternatively the lightest closed string modes dominate the interactions. Thus this is a low-energy long distance approximation. This regime is characterized by \( b^2 \gg \alpha' \). Bachas’ expression for the one-loop effective action can be written in the more concise form [14]:

\[ F = V_p(8\pi^2\alpha')^{-p/2} \int_0^{\infty} d\tau \tau^{-4+p/2}e^{-b^2/2\pi\alpha'} \left[ e^{-\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{-2\pi\tau n}) \right]^{-9} \frac{\theta_1(-i\epsilon/\pi | i\tau)}{\theta_1(-i2\epsilon/\pi | i\tau)}. \] (2.27)

In the \( b^2 \gg \alpha' \) limit the integral is dominated by large values of \( \tau \) thus we can expand the theta functions in \( q = e^{-\pi\tau} \) to get:
\[ \theta_1(z | i\tau) = -2q^{1/4} \sin(\pi z) + \text{higher orders in } q. \] (2.28)

Keeping the lowest order in \( q \) yields
\[ F = -4iV_p(8\pi^2\alpha')^{-p/2} \frac{\sinh^3\epsilon}{\cosh\epsilon} \int_0^{\infty} d\tau \tau^{-4+p/2}e^{-b^2/2\pi\alpha'} \left( \theta_1(-i\epsilon/\pi | i\tau) \right)^4. \] (2.29)

By changing the integration variable to \( t = 1/\tau \) one can re-write the expression as
\[ F = -4iV_p(8\pi^2\alpha')^{-p/2} \frac{\sinh^3\epsilon}{\cosh\epsilon} \int_0^{\infty} dt t^{-2-p/2}e^{-b^2t/2\pi\alpha'} \left( \frac{6 - p}{2} \right) \frac{(2\pi\alpha')^{3-p}}{b^{3-p}}. \] (2.30)

One can compare this to the expression we have obtained above using the matrix model. If we perform a similar approximation in which \( b^2 \gg \alpha' \), we find that our expression is dominated by small \( s \). We obtain an identical expression to Bachas in the above approximation scheme. Also, we mention that we get complete agreement between the matrix model and superstrings for small velocity and any \( b \neq 0 \). This is due to the complete cancellation of all the factors containing \( \exp(-\pi\tau) \) in eq. (2.27). However, the next order term in the expansion of the small velocities does not agree.

One might still ask how much the results from the matrix model deviates from the results obtained by Bachas. Due to the complicated integrals over theta functions, it is not so easy in general to give a quantitative estimate of these deviations. However, if we compare the imaginary part of the phase shift\(^2\), computed for D-branes by Bachas [13], with the same quantity computed

\(^2\)The real and imaginary parts are defined by \( W = i(\text{Re } \delta + i \text{ Im } \delta) \).
from the matrix model, it is quite easy to see that in the limit where the velocity approaches light velocity, there is a very large physical difference between the results. The matrix result \( (2.24) \) has an imaginary part corresponding to poles coming from the trigonometric functions at the values

\[
s_k = \frac{\pi k}{2\omega_1 \sinh \epsilon}, \quad k = 1, 3, 5, \ldots \tag{2.31}
\]

Computing the residue, we see (normalizing the parameters \( \omega_i \) as in eq. \( (2.26) \)) that for small velocities we get a result which is the same as the one obtained by Bachas, \( \text{Im} \delta \approx \frac{8 V_p}{(2\pi^3)^{p/2}} e^{b^2/2v} \tag{2.32} \)

to leading order. It should be emphasized that this result is an independent check of the matrix model versus superstring calculations. This is because when we compute the large distance \( \text{Re} \delta \), the relevant region of integration is \( s \to 0 \), whereas for \( \text{Im} \delta \) the relevant \( s \) is given by the position of the lowest pole. The result \( (2.32) \) also implies that the normalization \( (2.6) \) is correct for the imaginary part of the phase shift.

However, for a large velocity the results differ. Introducing \( \text{Im} \delta \approx \text{const} \left( \frac{\bar{s}}{\mathcal{M}_p^2} \right)^{p/2-1} \exp(-b^2\mathcal{M}_p^2/\bar{s}) \tag{2.34} \)

The main difference with the D-brane case is that these behave like black absorptive disks of logarithmically growing area, \( b_{cr}^2 \sim \ln(\bar{s}/\mathcal{M}_p^2) \), whereas from the matrix model these black disks are much larger, corresponding to \( b_{cr}^2 \sim \bar{s}/\mathcal{M}_p^2 \).

2.2 Rotated branes

The configuration with two rotated \( p \)-branes can be obtained from the block-diagonal matrix with two identical blocks describing a pair of superimposed \( p \)-branes quite similarly to Subsect. \( (2.1) \). Shifting along the \((p+2)\)-th axis by the distance \( b \) from each other and rotating in opposite directions in \((p, p+1)\) plane through the angle \( \theta/2 \), we obtain the configuration of two rotated branes

\[
A_{cl}^a = \begin{pmatrix}
B_a & 0 \\
0 & B_a
\end{pmatrix}, \quad a = 0, \ldots, p-1,
\]

\[
A_{cl}^p = \begin{pmatrix}
B_p \cos \frac{\theta}{2} & 0 \\
0 & B_p \cos \frac{\theta}{2}
\end{pmatrix},
\]

\[
A_{cl}^{p+1} = \begin{pmatrix}
B_p \sin \frac{\theta}{2} & 0 \\
0 & -B_p \sin \frac{\theta}{2}
\end{pmatrix},
\]

\[
A_{cl}^{p+2} = \begin{pmatrix}
\frac{b}{2} & 0 \\
0 & -\frac{b}{2}
\end{pmatrix},
\]

\[
A_{cl}^i = 0, \quad i = p+3, \ldots, 9. \tag{2.35}
\]

\( ^3 \)The units are \( \alpha' = 1/2 \). Also, the velocity used here is one half of the velocity used by Bachas.
This looks just like an analytic continuation of eq. (2.7). Therefore all the formulas of this subsection are quite similar to those of the previous one.

The interaction between these rotated p-branes to zeroth order in string coupling constant is determined by the one-loop effective action (2.9) in the background (2.35). Repeating the calculation, we arrive at the Hamiltonian which has the spectrum (cf. (2.16))

\[
E_{q,p,k} = b^2 + 2 \sum_{i=1}^{p-1} (q_i^2 + p_i^2) + 2 \cos^2 \frac{\theta}{2} q_i^2 + 4 \omega_i \sin^2 \frac{\theta}{2} (k + \frac{1}{2}).
\]  

(2.36)

The final result for the interaction between two rotated p-branes, which are separated by the distance \( b \) we represent in the form

\[
W = -\frac{4n_{p+1}^2}{\cos^2 \frac{\theta}{2}} \prod_{a \neq p-1} L_a^{-1} \times \int_0^\infty ds \left( \frac{\pi}{s} \right)^{\frac{p}{2}} e^{-b^2 s} \tanh \left( \frac{\omega_{p+1} s \sin \frac{\theta}{2}}{2} \right) \sinh \left( \omega_{p+1} s \sin \frac{\theta}{2} \right).
\]  

(2.37)

It can be obtained from (2.21) substituting \( \epsilon = i\theta / 2 \).

For large separation between branes we find, using the equality (2.6):

\[
W = -\Gamma \left( \frac{6-p}{2} \right) \frac{4V_p^2}{(4\pi)^2} \omega_{p+1}^4 \prod_i \frac{1}{\omega_i^2} \sin^2 \frac{3\theta}{2} \frac{1}{b^{8-2p}} + O \left( \frac{1}{b^{8-2p}} \right).
\]  

(2.38)

This expression correctly reproduces the supergravity result for the angular and distance dependence of the interaction energy between two rotated p-branes. An analogous formula for \( p = 1 \) is first obtained in [7].

### 2.3 Anti-parallel branes

The interaction potential for two anti-parallel D-strings in the IKKT matrix model was calculated in [7]. This calculation has been generalized to p-branes of arbitrary dimension [9]. For completeness, we reproduce these results here using the same techniques as in the two previous subsections.

The classical solution describing anti-parallel p-branes at the distance \( b \) from each other are represented by block–diagonal matrices

\[
A^a_{cl} = \begin{pmatrix} B_a & 0 \\ 0 & B'_a \end{pmatrix}, \quad a = 0, \ldots, p
\]

\[
A^p_{cl+1} = \begin{pmatrix} b/2 & 0 \\ 0 & -b/2 \end{pmatrix},
\]

\[
A^i_{cl} = 0, \quad i = p+2, \ldots, 9,
\]  

(2.39)

where \( B_a \) and \( B'_a \) obey the commutation relations

\[
[B_a, B_b] = -ig_{ab}1, \quad [B'_a, B'_b] = ig_{ab}1.
\]  

(2.40)

The matrix \( g_{ab} \) can be taken in the same form as in eq. (2.4). Thus we put

\[
B_0 = i\omega_1 \partial_1, \quad B_1 = q_1, \quad \ldots, \quad B_{p-1} = i\omega_1 \partial_1, \quad B_p = q_l,
\]

\[
B'_0 = i\omega_1 \partial_1, \quad B'_1 = -q_1, \quad \ldots, \quad B'_{p-1} = i\omega_1 \partial_1, \quad B'_p = -q_l,
\]  

(2.41)
where \( l = (p + 1)/2 \).

To calculate the one-loop effective action in the background (2.39), we perform the same steps as in Subsec. 2.1. First, we find all nonzero components of the adjoint operators defined by eq. (2.10):

\[
\begin{align*}
P_0 \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} &= i\omega_1 (\partial_1 + \partial_2^2) \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}, \\
P_1 \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} &= \begin{pmatrix} (q_1^1 - q_2^2)X & (q_1^1 + q_2^2)Y \\ -(q_1^1 + q_2^2)Y^\dagger & -(q_1^1 - q_2^2)Z \end{pmatrix}, \\
P_{p-1} \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} &= i\omega_l (\partial_1^l + \partial_2^2) \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix}, \\
P_p \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} &= \begin{pmatrix} (q_1^l - q_2^2)X & (q_1^l + q_2^2)Y \\ -(q_1^l + q_2^2)Y^\dagger & -(q_1^l - q_2^2)Z \end{pmatrix}, \\
P_{p+1} \begin{pmatrix} X & Y \\ Y^\dagger & Z \end{pmatrix} &= b \begin{pmatrix} 0 & Y \\ -Y^\dagger & 0 \end{pmatrix}
\end{align*}
\]

and

\[
F_{2i-22i-1} \begin{pmatrix} X \\ Y^\dagger \end{pmatrix} = -2\omega_i \begin{pmatrix} 0 & Y \\ -Y^\dagger & 0 \end{pmatrix}, \quad i = 1, \ldots, l.
\]

Again \( F_{\mu\nu} \) acts only on \( Y \); the operator \( P^2 \) in this subspace has the form

\[
P^2 Y = \left\{ b^2 + 2 \sum_{i=1}^{l} \left[ (q_i^+)^2 - \omega_i^2 (\partial_i^+)^2 \right] \right\} Y,
\]

which coincides with the Hamiltonian of the \( l \)-dimensional harmonic oscillator. Therefore the eigenvalues of \( P^2 \) are labelled by the set of \( l \) positive integers and are given by

\[
E_k = 4 \sum_{i=1}^{l} \omega_i \left( k_i + \frac{1}{2} \right) + b^2.
\]

The form of the adjoint field strength (2.43) allows to use the representation for the effective action found in \( \Box \), the same as in eq. (2.17):

\[
W = n \sum_k \left[ \sum_i \ln \left( 1 - \frac{16\omega_i^2}{E_k^2} \right) - \frac{1}{2} \sum_{s_1, \ldots, s_5 = \pm 1} \ln \left( 1 - \frac{2 \sum_i \omega_i s_i}{E_k} \right) \right].
\]

Neither the quantities \( q_i^- \), nor their conjugate variables enter (2.44), so the trace over them gives an overall factor of \( n \).

Using the same proper time representation for logarithms as in eq. (2.19) and the equality

\[
\sum_k e^{-sE_k} = \frac{e^{-b^2 s}}{\prod_i 2 \sinh 2\omega_i s},
\]

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we obtain for the effective action:

\[ W = -2n \int_0^\infty ds \frac{ds}{s} e^{-L^2 s} \left[ \sum_i (\cosh 4\omega_i s - 1) - 4 \left( \prod_i \cosh 2\omega_i s - 1 \right) \right] \prod_i \frac{1}{2 \sinh 2\omega_i s} \]  
\[ (2.48) \]

For large separation between the branes, this result reduces to

\[ W = -\frac{1}{16} n \Gamma \left( \frac{7 - p}{2} \right) \left[ 2 \sum_i \omega_i^4 - \left( \sum_i \omega_i^2 \right)^2 \right] \prod_i \omega_i^{-1} \left( \frac{2}{b} \right)^{7-p} + O \left( \frac{1}{b^{7-p}} \right) \].
\[ (2.49) \]

Equations (2.48) and (2.49) coincide with the previous results [7, 9] obtained by a slightly different technique.

These results of the matrix models are to be compared to the superstring calculations which are given in the open-string language by the annulus diagram. The superstring result for the interaction between anti-parallel Dp-branes reads [15, 14]

\[ W = -V_{p+1} \int_0^\infty dt \frac{1}{t} \left( \frac{8\pi^2\alpha'}{t^2} \right) e^{-b^2 t/2\pi\alpha'} q^{-1} \prod_{n=1}^{\infty} (1 - q^{2n-1})^8 \prod_{n=1}^{\infty} (1 - q^{2n})^8 \]  
\[ (2.50) \]

with \( q = e^{-\pi t} \). We see that the superstrings and matrix-model answers agree only at large distances quite similarly to the cases of moving and rotated branes. This suggests to modify the matrix model to better reproduce the superstring calculation.

### 3 The NBI-type matrix model of IIB superstring

In the IKKT model the matrix size \( n \) is considered as a dynamical variable. The partition function includes for this reason a summation over \( n \):

\[ Z = \sum_{n=1}^{\infty} \int DA D\psi e^{-S}, \]  
\[ (3.1) \]

where the action is given by eq. (1.1). This construction is proposed in [7] as the matrix-model analog of the Schild formulation of type IIB superstring given by the path integral

\[ Z = \int D\sqrt{g} DX D\psi e^{-S_{\mathrm{Schild}}} \]  
\[ (3.2) \]

with the action (1.2).

In this section we propose a modification of the IKKT matrix model which appears to be a more analogous to eq. (3.2), and which reproduces the Nambu–Goto version of the Green–Schwarz superstring action after the integration over the introduced additional Hermitian matrix \( Y^{ij} \) with positive definite eigenvalues, which is analogous to \( \sqrt{g} \) in eq. (1.2). The classical action has the form (1.5), which yields the following classical equation of motion for the \( Y \)-field:

\[ \frac{\alpha}{4} \left( Y^{-1}[A_\mu, A_\nu] Y^{-1} \right)_{ij} + \beta \delta_{ij} = 0, \]  
\[ (3.3) \]

\[ ^4\] Tseytlin [16] has conjectured an alternative interpretation of the classical solutions in the IKKT matrix model as D-branes with magnetic field, in analogy with previous work [7] on the matrix model [1]. We do not discuss such a point of view in this paper.

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whose solution reads

\[ Y = \frac{1}{2} \sqrt[\alpha]{\frac{\beta}{\alpha}} \sqrt{-[A_\mu, A_\nu]^2}. \]  

Here \(-[A_\mu, A_\nu]^2\) is positive definite, since the commutator is anti-hermitian. The square root in

(3.4) \hspace{1cm} \text{is unique, provided } Y \text{ is positive definite which is our case. After the substitution (3.4), the classical action (1.5) reduces to the classical action of the NBI (non-abelian Born-Infeld) type (1.6).

In this section we show that even at the quantum level, it is possible to modify the classical action (1.5) such that we obtain the Nambu-Goto version of the Green-Schwarz type IIB superstring action.

### 3.1 The Yang-Mills dielectric matrix model

Let us consider the matrix model defined by the action

\[ S_\epsilon = -\frac{\alpha}{4} \text{Tr} \left( \frac{1}{Y} [A_\mu, A_\nu]^2 \right) + V(Y) - \frac{\alpha}{2} \text{Tr} \left( \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right), \]  

where \( Y \) is a new \( n \times n \) Hermitian matrix field taken to be positive definite, and \( V(Y) \) is a “potential”. For reasons which become clear later, the potential is taken to be

\[ V(Y) = \beta \text{Tr} \ Y + \gamma \text{Tr} \ln Y. \]  

The partition function is then given by the functional integrals\(^5\)

\[ Z_\epsilon = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}Y \ e^{-S_\epsilon}. \]  

As discussed in the appendix the action is invariant under

\[ \delta^{(1)}\psi = \frac{i}{4} \{Y^{-1}, [A_\mu, A_\nu]\} \Gamma^{\mu\nu} \epsilon, \]

\[ \delta^{(1)}A_\mu = i \bar{\epsilon} \Gamma_\mu \psi, \]

\[ \delta^{(2)}\psi = \xi, \]

\[ \delta^{(2)}A_\mu = 0 \]  

in the limit \( n \to \infty \). The field \( Y \) is assumed to be invariant with respect to this transformation.

The action (3.5) differs from its classical counterpart (1.5) by the second term on the right-hand side of eq. (3.6). We can e.g. associate this term with the measure for integration over \( Y \) rather than with the classical action. The classical action (1.5) can be obtained from (3.5) in the limit \( \alpha \sim \beta \sim \infty, \alpha/\beta \sim 1 \). This limit is associated with vanishing string coupling constant since \( \alpha \sim \beta \sim g_s^{-1} \), i.e. with the usual classical limit in string theory. The matrix model with the action (3.5) can be considered as the large \( n \) reduced model for a ten-dimensional non-abelian “dielectric” theory of the type introduced by ’t Hooft \(^7\) several years ago for the abelian case. In this picture the quantity \( 1/Y \) is the dielectric function \( \epsilon(Y) \), which is then governed by the potential \( V(Y) \). Although the present matrix model and its non-reduced counterpart are much more complicated than the abelian version\(^8\), we shall call the model given by eq. (3.5) the Yang-Mills dielectric matrix

\(^5\)In order to have a non-trivial saddle point for \( n \to \infty \), we need to assume that the positive constants \( \alpha \) and \( \beta \) are of order \( n \).

\(^6\)This applies to the physics as well as to the mathematics.
model. This is the reason for the notation \( S \) for the action. Alternatively, one could interpret the field \( Y \) as a rather rudimentary metric.

We start by doing the \( Y \)-integral. Since the \( \psi \)-dependent term in eq. (3.5) is independent of \( Y \), it is sufficient to consider the integral

\[
\mathcal{F}(z) = \int \mathcal{D}Y \exp \left( -\frac{\alpha}{4} \text{Tr} \left( \frac{1}{Y} z^2 \right) - \beta \text{Tr} Y - \gamma \text{Tr} \ln Y \right). \tag{3.9}
\]

Here \( z^2 = -[A_\mu, A_\nu]^2 \). The integration over the “angular” variables in eq. (3.9) is of the Itzykson-Zuber type [18], and we therefore get

\[
\mathcal{F}(z) = n! \alpha^{n(n-1)/2} \prod_{p=1}^{n-1} p! \prod_{i=1}^{n} \int_0^\infty dy_i \Delta^2(y_i) \Delta(1/y) \Delta(z^2) \exp \left( -\alpha \sum_i z_i^2 / 4y_i - \beta \sum_i y_i - \gamma \sum_i \ln y_i \right), \tag{3.10}
\]

where \( \Delta \) is the Vandermonde determinant

\[
\Delta(x) = \prod_{i>j} (x_i - x_j) = \det x_i^{k-1}. \tag{3.11}
\]

In eq. (3.10) the quantities \( z_i^2 \) and \( y_i \) are the eigenvalues of \(-[A_\mu, A_\nu]^2 \) and \( Y \), respectively. The unitary matrix which diagonalizes \( Y \) has thus been integrated over as in the Itzykson-Zuber paper [18].

Now eq. (3.10) can be written

\[
\Delta(z^2) \mathcal{F}(z) = n! \alpha^{n(n-1)/2} \prod_{p=1}^{n-1} p! \prod_{i=1}^{n} \int_0^\infty dy_i y_i^{n-1} \prod_{i>j} (y_i - y_j) \times \exp \left( -\alpha \sum_i z_i^2 / 4y_i - \beta \sum_i y_i - \gamma \sum_i \ln y_i \right). \tag{3.12}
\]

We shall now choose \( \gamma \) in such a way that the result of the \( y_i \)-integrations gives a result which is “as string-like as is possible”. As we shall soon see, this amounts to taking

\[
\gamma = n - 1/2. \tag{3.13}
\]

Using the well known Bessel integral (from an integral representation of \( K_{-1/2} \) and the explicit formula for this function)

\[
\int_0^\infty \frac{dy}{y^{1/2}} e^{-\alpha z^2 / 4y - \beta y} = \sqrt{\pi / \beta} e^{-\sqrt{\alpha} \beta z}, \tag{3.14}
\]

we obtain by use of (3.13)

\[
\Delta(z^2) \mathcal{F}(z) = n! \alpha^{n(n-1)/2} \prod_{i=1}^{n-1} p! \det_{ki} \int_0^\infty \frac{dy}{y^{1/2}} y^{k-1} \exp \left( -\frac{\alpha z_i^2}{4y} - \beta y \right) = n! \alpha^{n(n-1)/2} \prod_{i=1}^{n-1} p! \det_{ki} (-1)^{k-1} \frac{\partial}{\partial \beta^{k-1}} \left[ \sqrt{\pi / \beta} \exp (-\sqrt{\alpha} \beta z_i) \right]. \tag{3.15}
\]

The matrix whose determinant is to be calculated reads

\[
A_{ki} \equiv (-1)^{k-1} \frac{\partial}{\partial \beta^{k-1}} \left( \sqrt{\pi / \beta} e^{-\sqrt{\alpha} \beta z_i} \right), \tag{3.16}
\]
or, explicitly,

\[ A_{1i} = \sqrt{\frac{\pi}{\beta}} e^{-\sqrt{\alpha \beta} z_i}, \]
\[ A_{2i} = \frac{1}{2\beta} \sqrt{\frac{\pi}{\alpha}} z_i e^{-\sqrt{\alpha \beta} z_i} + \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} e^{-\sqrt{\alpha \beta} z_i}, \]
\[ A_{3i} = \frac{1}{4\beta} \alpha \sqrt{\frac{\pi}{\beta}} z_i^2 e^{-\sqrt{\alpha \beta} z_i} + \frac{3}{4} \sqrt{\frac{\pi}{\alpha}} \beta z_i^2 e^{-\sqrt{\alpha \beta} z_i} + \frac{3}{4} \sqrt{\frac{\pi}{\beta}} e^{-\sqrt{\alpha \beta} z_i}, \]

and

\[ A_{4i} = \cdots \]

(3.17)

The second term in the expression for \( A_{2i} \) is proportional to the first line of the matrix, \( A_{1i} \), and can be omitted in the determinant. The same property holds for all lines of \( A \) – only the result of the differentiation of the exponential survives in the determinant, while the terms coming from differentiations of the various pre-exponential factors are linear combinations of the previous lines of the matrix. These factors include e.g. the term in \( A_{3i} \) which is linear in \( z_i \). This term is proportional to the first term in \( A_{2i} \), etc. etc.

Hence,

\[ \Delta(z^2) F(z) = \tilde{C} \det_{ki} \left[ \sqrt{\frac{\pi}{\beta}} \left( \sqrt{\alpha/\beta} \frac{z_i}{2} \right)^{k-1} \exp \left( -\sqrt{\alpha \beta} z_i \right) \right] \]
\[ = C \Delta(z) \exp \left( -\sqrt{\alpha \beta} \sum_i z_i \right), \quad (3.18) \]

where the constants \( C \) and \( \tilde{C} \) are given by

\[ \tilde{C} = n! \alpha^{n(n-1)/2} \prod_{p=1}^{n-1} p! \quad \text{and} \quad C = n! \left( (\alpha)^{3/2}/2\sqrt{\beta} \right)^{n(n-1)/2} \left( \pi/\beta \right)^{n/2} \prod_{p=1}^{n-1} p!. \quad (3.19) \]

In these equations, \( z_i \) always means the positive square root of the positive quantity \( z_i^2 \).

Eq. (3.18) is one of the main results of this subsection. It shows that after the exact integration over \( Y \), for any \( n \), the action becomes linear in the variables \( z_i \), although the original action \( S_{\bar{e}} \) in eq. (3.5) is quadratic in these variables. To the best of our knowledge, this result is a new matrix model result, and it may therefore be of interest also outside the present framework. Expressed in terms of string language, we shall find that we have the option of obtaining the (supersymmetric) Nambu-Goto type of action, as we shall discuss shortly. The result (3.18) is possible because of the choice (3.13) of the power \( \gamma \), which allows us to use the well known explicit (exponential) form for the Bessel function \( K_{-1/2} \). However, it should be noticed that we still have the ratio of the Vandermonde determinants in eq. (3.15),

\[ \Delta(z)/\Delta(z^2) = 1/\prod_{i>j} (z_i + z_j). \quad (3.20) \]

To proceed from eq. (3.15) we now show that we have the identification

\[ \sum_i z_i = \text{Tr} \sqrt{[A_\mu, A_\nu]^2}, \quad (3.21) \]
where it should be remembered that the $z_i^2$'s are the positive eigenvalues of $-[A_\mu, A_\nu]^2$. Following Dirac we define the square root of a matrix by its formal power series. Thus we write

$$Q \equiv \sum_i z_i = \sum_i \sqrt{z_i^2} = \sum_i \sqrt{1 + (z_i^2 - 1)} = \sum_i \sum_{p=0, l \leq p}^{\infty} \left( \frac{p}{l} \right) c_p (-1)^{p-l} z_i^{2l}, \quad (3.22)$$

where $c_p = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2p-3) / 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2p$. Now let $U$ be the unitary matrix which diagonalizes the square of the commutator,

$$\text{diag}(z_1^2, z_2^2, ..., z_n^2) = -U [A_\mu, A_\nu]^2 U^\dagger. \quad (3.23)$$

Then we can write in eq. (3.22)

$$Q = \sum_{p,l} \left( \frac{p}{l} \right) c_p (-1)^{p-l} \sum_i \sqrt{z_i^2} = \sum_{p,l} \left( \frac{p}{l} \right) c_p (-1)^{p-l} \sum_i \sqrt{1 + (z_i^2 - 1)} \equiv \sum_{p,l} \left( \frac{p}{l} \right) c_p (-1)^{p-l} z_i^{2l}. \quad (3.24)$$

This verifies the identification (3.21).

By means of the result displayed in eq. (3.24), we can rewrite the partition function (3.7) by use of eqs. (3.9) and (3.18) in the following way

$$Z_\epsilon = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}Y \exp \left( \frac{\alpha}{4} \left( \frac{1}{Y} [A_\mu, A_\nu]^2 \right) - \beta \text{Tr} Y \right)$$

$$- \left( n - \frac{1}{2} \right) \text{Tr} \ln Y + \frac{\alpha}{2} \text{Tr} \left( \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right)$$

$$= C \int \mathcal{D}A_\mu \mathcal{D}\psi \exp \left( -\sqrt{\alpha \beta} \text{Tr} \sqrt{-[A_\mu, A_\nu]^2} + \frac{\alpha}{2} \text{Tr} \left( \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right) \right), \quad (3.25)$$

where the measure in the last expression is defined by

$$[\mathcal{D}A_\mu \mathcal{D}\psi] = \mathcal{D}A_\mu \mathcal{D}\psi \prod_{i>j} (z_i + z_j). \quad (3.26)$$

7The square root of a matrix can alternatively be defined by the integral representation

$$\sqrt{-[A_\mu, A_\nu]^2} = \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{0^+} dt \ t^{-3/2} e^{-\epsilon [A_\mu, A_\nu]^2},$$

where the contour of integration encircles counterclockwise the negative real axis which provides convergence. Thus we write

$$Q \equiv \sum_i z_i = \sum_i \sqrt{z_i^2} = \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{0^+} dt \ t^{-3/2} \sum_i e^{t z_i^2},$$

which can be represented as

$$Q = \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{0^+} dt \ t^{-3/2} \text{Tr} e^{-\epsilon [A_\mu, A_\nu]^2} = \text{Tr} \sqrt{-[A_\mu, A_\nu]^2}.$$

All the formulas here and below can be rigorously derived using this representation.
The result (3.25) shows that the “dielectric” action \( S_\epsilon \) is equivalent to an action which can be considered as a strong coupling non-abelian Born-Infeld model. The new action in (3.25) is no longer invariant under the transformations (3.8). The first of these should be replaced by

\[
\delta^{(1)} \psi = \frac{i}{4} \{ (-[A_\alpha, A_\beta])^{1/2}, [A_\mu, A_\nu] \} \Gamma^{\mu
u} \epsilon.
\] (3.27)

The other transformations in eq. (3.8) are unchanged. Thus the effect of performing the \( Y \)-integration is not only to produce a new action, but it also produces a new measure (3.26) and a new transformation property (3.27).

The anti-commutator in eq. (3.27) is superfluous, since any operator commutes with its own square. However, it follows from the appendix that the action in the second equation (3.25) is only invariant under the transformation (3.27) in the limit \( n \to \infty \). This is because the last two terms on the right hand side of eq. (A.4) do not vanish when \( Y^{-1} \) is replaced by the inverse square root, as in (3.27).

### 3.2 Connection between the dielectric matrix model and superstrings in the large \( n \) limit

In this section we shall discuss the connection between the super- “dielectric” matrix model and superstrings. The possible connection between the large \( n \) limit and strings have been discussed by many authors. We shall use in particular the QCD-discussion by Bars in ref. [21], where further references can be found. The starting point is an expansion of the \( A_\mu \)-field in terms of \( \text{SU}(n) \)-generators \( l_\kappa, \kappa = (k_1, k_2) \),

\[
(A_\mu)_i^j = C_1 \sum_\kappa a_\mu^\kappa \left( l_\kappa \right)_i^j,
\] (3.28)

where \( a_\mu^\kappa \) are expansion coefficients to be integrated over in functional integrals, and where \( C_1 \) is some normalization constant. The expansion (3.28) is to be compared to the corresponding expansion of the string variable \( X_\mu(\sigma, \tau) \), where \( \sigma \) and \( \tau \) are the usual world sheet variables. In this case we have

\[
X_\mu(\sigma, \tau) = C_2 \sum_\kappa a_\mu^\kappa e^{ik_\sigma},
\] (3.29)

where \( \sigma = (\sigma, \tau) \). The \( l \)-matrices can be constructed explicitly in terms of Weyl matrices [21], and they satisfy the commutation relation

\[
[l_{\kappa_1}, l_{\kappa_2}] = \frac{i}{2\pi} \sin(2\pi k_1 \times k_2/n) \ l_{\kappa_1+k_2} \to i(k_1 \times k_2) \ l_{\kappa_1+k_2}, \text{ for } n \to \infty,
\] (3.30)

where \( k_1 \times k_2 = (k_1)_1(k_2)_2 - (k_2)_1(k_1)_2 \). In the following we shall always perform the \( n \to \infty \) limit in the way done in eq. (3.30), even inside sums over the \( \kappa \)’s. We do not know a rigorous justification of this. Obviously there is a good chance for the validity of this approach if infinitely high modes (\( n \to \infty \)) are not dynamically relevant. For the bosonic string this is probably not true, since this string oscillates infinitely much at short distances due to the tachyon instability. However, the assumption may be correct for stable superstrings.

---

8The reader should recall that the abelian Born-Infeld action [19] has the Lagrangian \( \mathcal{L} = -c \sqrt{1 + F_{\mu\nu}^2 / 2c} \), where \( c \) is a constant. In the strong field limit \( \mathcal{L} \) behaves like \( -c/2 \sqrt{F_{\mu\nu}^2} \). It was suggested many years ago that the strong field limit of the abelian Born-Infeld should give a field-theoretic description of strings [21].

9We mention some of these works in [22].
The results for SU(n) mentioned above can be compared on the torus with the area preserving diffeomorphism. This is discussed in the paper by Bars [21] (for a discussion of SU(n) on the sphere, see Floratos, Iliopoulos, and Tiktopoulos [23]), and we shall not repeat this discussion. We only wish to mention that this approach corresponds to taking into account the local subalgebra, but ignoring the global translation generators. The central extensions are thus ignored.

Let us again consider the quantity \( Q \) defined in eq. (3.22). Inserting the expansion (3.28), we get from eq. (3.24)

\[
Q = \text{Tr} \sum_{p,l} \left( \frac{p}{l} \right) (-1)^p c_p \left[ -C_4^1 \sum_{nmrs} a_n^a a_r^a a_s^a (n \times m) (r \times s) l_{n+m+l+s} \right]^l
\]

\[
= \sum_{p,l} \left( \frac{p}{l} \right) (-1)^{p-l} c_p C_4^l \sum_{nmrs} \prod_{i=1}^l a_n^a_i a_r^a_i a_s^a_i (n \times m) (r \times s)
\]

\[
\times \text{Tr} \prod_{j=1}^l l_{n_j+m_j+l_j+s_j}.
\]

In the limit \( n \to \infty \) the trace of a product of \( l \)'s with different indices produce a Kronecker delta\(^{10}\) (up to a normalization factor). Therefore eq. (3.31) becomes

\[
Q = n \sum_{p,l} \left( \frac{p}{l} \right) (-1)^{p-l} c_p \left( \frac{n}{4\pi} C_4^2 \right)^{2l} \sum_{nmrs} \prod_{i=1}^l a_n^a_i a_r^a_i a_s^a_i (n \times m) (r \times s)
\]

\[
\times \delta_{n_1+m_1+l_1+s_1+\ldots+n_l+m_l+l+s_l,0}.
\]

(3.32)

It should be emphasized that this simple result is valid only because the \( n \to \infty \) limit is taken. Otherwise the trace in eq. (3.31) yields a more complicated result than displayed above. This expression can be rewritten by use of the expansion (3.28) for the string coordinates,

\[
Q = \frac{n}{(2\pi)^2} \int_0^{2\pi} d^2 \sigma \sum_{p,l} \left( \frac{p}{l} \right) (-1)^{p-l} c_p \left( (n/4\pi)^2 (C_1/C_2)^4 \{ X^\mu(\sigma), X^\nu(\sigma) \}_\mathcal{PB} \right)^l
\]

\[
= \frac{n}{(2\pi)^2} \int_0^{2\pi} d^2 \sigma \sqrt{1 + ((n/4\pi)^2 (C_1/C_2)^4 \{ X^\mu(\sigma), X^\nu(\sigma) \}_\mathcal{PB} - 1)}
\]

\[
= \frac{1}{2(2\pi)^3} \left( \frac{n C_1}{C_2} \right)^2 \int_0^{2\pi} d^2 \sigma \sqrt{\{ X^\mu(\sigma), X^\nu(\sigma) \}_\mathcal{PB}^2}.
\]

(3.33)

Here \( \{ a, b \}_\mathcal{PB} \) is the usual Poisson bracket of \( a \) and \( b \).

Inserting this result in eq. (3.25) we finally obtain

\[
Z_\epsilon \to C \int [DX_\mu DX_\nu DX_\psi \overline{\psi}] \exp \left[ - \int d^2 \sigma \left( \sqrt{\alpha/\beta} \sqrt{\{ X^\mu(\sigma), X^\nu(\sigma) \}_\mathcal{PB}^2} - \frac{i\alpha}{2} \epsilon^{ab} \partial_a X^\mu \overline{\psi} \Gamma_\mu \partial_b \psi \right) \right].
\]

(3.34)

\(^{10}\)This follows from

\[
\text{Tr} \ l_{n+m+l} = (n^3/(4\pi)^2) \delta_{m+r,0}
\]

and repeated applications of the relation \( \text{Tr} \ l_{n+m} = (n/4\pi) \exp(2\pi i (m \times r)/n) \ l_{m+r} \). The exponential factor produces one plus terms of higher order in \( 1/n \). Thus,

\[
\text{Tr} \ l_{m_1+\ldots+m_n} = n(n/4\pi)^r \delta_{m_1+\ldots+m_n,0}
\]

to leading order.
The last term in the action follows from expanding \( \psi \) in a form similar to (3.28) and using that for \( n \to \infty \)

\[
\text{Tr} \bar{\psi} \Gamma^\mu [A_\mu, \psi] = \sum_{n,m,r} a_n^{\bar{m}} \bar{\psi}^r \Gamma^m (n \times m) \text{Tr} r^m \Gamma^m = i \frac{n^4 C_1 C_2}{8 (2\pi)^3 \epsilon} \int d^2 \sigma \epsilon^{ab} \partial_a X^\mu \bar{\psi} \Gamma^\mu \partial_b \psi,
\]

(3.35)

where \( C_\psi \) is the relative normalization of the \( \psi \)-fields. The coefficients in (3.33) and (3.35) have been absorbed in a redefinition of \( \alpha \) and \( \beta \) in the result (3.34). Thus, e.g.

\[
(n C_1 / C_2)^2 / 2(2\pi)^3 \sqrt{\alpha \beta} \to \sqrt{\alpha \beta}.
\]

(3.36)

It should be emphasized that the measure in the result (3.34) is defined through (3.26). Also, the transformation property (3.27) must be replaced by

\[
\delta^{(1)} \psi = \frac{1}{2 \sqrt{\{X_\alpha, X_\beta\}_P B}} \{X_\mu, X_\nu\}_P B \Gamma^{\mu \nu} \epsilon
\]

(3.37)

in order to ensure invariance of the action in (3.34).

4 Discussion of the results

The main result from the dielectric matrix model is given by eq. (3.34). We shall now discuss this result. First, one might wonder if this formula cannot be applied with \( \psi \equiv 0 \), so that the bosonic string would emerge from the dielectric model with \( \psi = 0 \). As already discussed in connection with eq. (3.30) this is highly unlikely, since we perform the limit \( n \to \infty \) inside sums, like e.g. in the transition from (3.31) to (3.32). This is allowed if the infinitely high modes are not dynamically relevant. However, we know that due to the tachyon, at a finite distance of the order the square root of the string tension, the bosonic string becomes unstable, due to the relevance of infinitely high modes. These causes the area of the world sheet to become infinite, due to an infinitely oscillating string. Hence it is not permitted to interchange the sum over modes and the limit \( n \to \infty \). For superstrings, the situation is much more hopeful, since it is stable without a tachyon. So although we do not have a mathematical proof that the summation over modes can be interchanged with the limit \( n \) goes to infinity, there are physical reasons to believe that this is possible for superstrings.

The comparison of the results obtained from the matrix model with the ones from superstring theory provides a check to what extent the superstring theory can be described by the matrix model. The calculation of the interaction between D-branes is one of such checks. It is natural to think that such calculations in the matrix model correspond to loop expansion around certain large-\( n \) classical solutions. The one-loop calculations of Sect. 2 are performed without summation over \( n \), as proposed in \([1]\), or without integration over \( Y \), as proposed in the previous section.

It is worth mentioning, however, that the classical solutions (2.2), which are associated with D-brane configurations, are also classical solutions to the NBI-type matrix model. The reason is that these classical solutions are BPS states and the commutator \([A_\mu, A_\nu]\) is proportional for them to the unit matrix. The same is true for the classical value of \( Y \), as it follows from eq. (3.4), so the classical equations of motion of the NBI model:

\[
[A_\mu, \{Y^{-1}, [A_\mu, A_\nu]\}] = 0, \quad [A_\mu, (\Gamma^\mu \psi)_\alpha] = 0,
\]

(4.1)
are also satisfied.

A more general property holds in the large–$n$ limit when any classical solution of the IKKT model is simultaneously a solution of the classical equations of motion of the NBI model. To show this, let us rewrite the equations of motion (4.1) and (3.4) for bosonic matrices in the form

$$\left\{ [A^\mu, Y^{-1}], [A_\mu, A_\nu] \right\} + \left\{ Y^{-1}, [A^\mu, [A_\mu, A_\nu]] \right\} = 0, \quad Y^2 = -\frac{\alpha}{4\beta} [A_\mu, A_\nu]^2. \quad (4.2)$$

For a solution of the IKKT model, the second term on the left hand side of the first equation equals zero. At infinite $n$, when the commutators can be replaced by the Poisson brackets, the first term also vanishes, since the large–$n$ classical equations of motion imply $\partial_\tau Y^2 = 0 = \partial_\sigma Y^2 \quad [24]$ and thus the Poisson bracket $\{ A_\mu, Y^{-1} \}_{PB}$ is equal to zero.

In this paper we have not discussed the large $n$ saddle point configuration of the integral over the matrix field $Y$ in the partition function $Z_\epsilon$. It should, however, be emphasized that such a calculation is very different from the corresponding “classical” saddle point calculation, valid for $\alpha \sim \beta \rightarrow \infty$. In the large-$n$ saddle point, the logarithm of the Vandermonde determinant enters, and one needs to determine the spectral density of the eigenvalues $y_i$ which in turn determines the value of the commutator $[A_\mu, A_\nu]$. This is most easily seen by summarizing our result in the form

$$Z_\epsilon = \int D \mathcal{A}_\mu D \mathcal{Y} \prod_{i>j} (z_i + z_j) \exp \left( \frac{\alpha}{4} \text{Tr} \left( \frac{1}{Y} [A_\mu, A_\nu]^2 \right) - \beta \text{Tr} Y \right) \left( - (n - \frac{1}{2}) \text{Tr} \ln Y + \frac{\alpha}{2} \text{Tr} (\bar{\Psi} \Gamma^\mu [A_\mu, \Psi]) \right) = C \int D X_\mu D \psi \exp \left[ - \int d^2 \sigma \left( \sqrt{\alpha \beta} \sqrt{\{ X_\mu(\sigma), X_\nu(\sigma) \}} \{ \mathcal{Y} \}_{PB} - \frac{i\alpha}{2} \epsilon^{ab} \partial_\sigma X^\mu \bar{\Psi} \Gamma_\mu \partial_\sigma \psi \right) \right], \quad (4.3)$$

where now the measure in the last functional integral is the standard one, whereas the corresponding quantity in the first functional integral is not, due to the factor $\prod (z_i + z_k)$. This implies that in evaluating the large $n$ saddle point this additional $z$ dependent factor in the measure should be taken into account. Equation (4.1) for the $A_\mu$-field with a nontrivial distribution of the eigenvalues $Y$ possesses undoubtedly a richer structure than eq. (4.3).

The matrix model given in eqs. (3.5)–(3.7) can presumably be considered as a large $n$ reduced Eguchi-Kawai model for the field theory with the action

$$S_{\text{field}} \propto \int d^{10} x \text{Tr} \left( \frac{1}{4} Y^{-1} F^2_{\mu \nu} + \frac{i}{2} \bar{\Psi} \Gamma_\mu D_\mu \psi + V(Y) \right), \quad (4.4)$$

where $V(Y)$ is the non-polynomial potential given in eq. (3.6). As usual, we have

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \{ A_\mu, A_\nu \}, \quad \text{and} \quad D_\mu \psi = \partial_\mu + i \{ A_\mu, \psi \}. \quad (4.5)$$

Because of supersymmetry for $n \rightarrow \infty$, we do not expect that quenching is necessary [25, 7]. Therefore the non-polynomial action (4.4), in the limit where $n$ approaches infinity, could be considered as an effective field theory for superstrings.

From this point of view, one could consider the field theory as a regulator for the Green-Schwarz superstring. For example, this might be useful in numerical simulations. However, one could also ask if the underlying field theory could be of direct physical interest. A possible scenario could be the following: Suppose that at the Planck scale or below there exists a description in terms of some unified field theory (probably non-polynomial) with a high order group, like e.g. $SU(n)$ with $n$ very large. Such a theory could then, as exemplified by our discussion above, effectively be equivalent
to a Green-Schwarz superstring theory. There would therefore exist a dual description of the very early universe, either as some unified field theory, or as a string theory. Of course, it goes without saying that many problems should be solved, before such a dramatic scenario can be said to be on a satisfactory basis.

Acknowledgments

Y.M. and K.Z. are grateful to I. Chepelev for useful discussions. The work by Y.M. and K.Z. was supported in part by INTAS grant 94–0840, CRDF grant 96–RP1–253 and RFFI grant 97–02–17927. Y.M. was sponsored in part by the Danish Natural Science Research Council. D.S. was funded by the Royal Society.

Appendix A  On the supersymmetry of the $Y$-integral

In this appendix we shall show that the action (3.5) is invariant under the symmetry transformations (3.8) in the limit $n \to \infty$. If we apply the transformation (3.8) to the action (3.5), we obtain after some calculations

$$\frac{1}{4} \text{Tr} \left( \frac{1}{Y} [A_\mu, A_\nu]^2 \right) \rightarrow \frac{i}{2} \text{Tr} \left( \epsilon_m (\Gamma_0 \Gamma_\mu)_{mn} \psi_n [A_\nu, \{[A^\mu, A^\nu], Y^{-1}\}] \right),$$

as well as

$$\frac{1}{2} \text{Tr} \left( \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right) \rightarrow \frac{i}{2} \text{Tr} \left( \psi_m \left( \Gamma^0 \Gamma^\delta \right)_{mp} \epsilon_p [A_\nu, \{[A^\nu, A_\delta], Y^{-1}\}] \right) - \frac{i}{7! 4} \epsilon^{\alpha \beta \lambda_1 \ldots \lambda_7} \text{Tr} \left( \psi_m \left( \Gamma^0 \Gamma^{11} \Gamma^{\lambda_1} \ldots \Gamma^{\lambda_7} \right)_{mp} \epsilon_p [A_\mu, \{[\alpha_\alpha, \beta_\beta], Y^{-1}\}] \right).$$

(A.1)

Here the quantity $\{a, b\}$ denotes the anti-commutator of $a$ and $b$, and should not be confused with the Poisson bracket. In deriving this result we used the expansion

$$\Gamma^\mu \Gamma^{\alpha \beta} = \eta^\mu_\alpha \Gamma^\beta - \eta^\mu_\beta \Gamma^\alpha - \frac{1}{7!} \epsilon^{\alpha \beta \lambda_1 \ldots \lambda_7} \Gamma^{11} \Gamma^{\lambda_1} \ldots \Gamma^{\lambda_7}.$$

(A.3)

If $Y$ was a $c$-number, the last term in (A.2) would vanish for symmetry reasons, and the first term in this equation always cancel the expression on the right hand side of (A.1), corresponding to the well known invariance of supersymmetric Yang-Mills matrix theory. However, the presence of the non-commuting $Y$ makes life harder. Here we shall show that the last term in eq. (A.2) vanishes in the limit $n \to \infty$. Using the expansion

$$[A_\mu, \{[A_\alpha, A_\beta], Y^{-1}\}] = [A_\mu, [A_\alpha, A_\beta]] Y^{-1} + Y^{-1} [A_\mu, [A_\alpha, A_\beta]]$$

$$+[A_\alpha, A_\beta][A_\mu, Y^{-1}] + [A_\mu, Y^{-1}][A_\alpha, A_\beta],$$

(A.4)

the first two terms give zero contribution when inserted in the last term in eq. (A.2). The critical terms are thus the last two terms on the right hand side of (A.4). Consider one of these terms in the large $n$ limit,

$$[A_\alpha, A_\beta][A_\mu, Y^{-1}] = \sum_{mnp} a_\alpha a_\beta a_\mu a_\rho (y^{-1})^r \frac{n}{2\pi} \sin \left( \frac{2\pi m \cdot n}{n} \right) \frac{n}{2\pi} \sin \left( \frac{2\pi p \cdot r}{n} \right) l_m + n l_p + r,$$

(A.5)
where we used the expansion (3.28) and the commutator (3.32), as well as an expansion of $Y^{-1}$,

$$(Y^{-1})^i_j = \sum_r (y^{-1})^r (l^r)^i_j. \quad (A.6)$$

Taking the limit $n \to \infty$ and using the expansion of $\psi$ we get

$$\text{Tr}(\psi[A_\alpha, A_\beta][A_\mu, Y^{-1}]) \to \sum_{s m n p r} \psi^s a^m_\alpha a^n_\beta a^p_\mu (y^{-1})^r (m \times n) (p \times r) \text{Tr}(l^s m+n l^p+r)$$

$$\to \sum_{s m n p r} \psi^s a^m_\alpha a^n_\beta a^p_\mu (y^{-1})^r (m \times n) (p \times r) \delta_{s+m+n+p+r,0}$$

$$= \int d^2 \sigma \, \psi \{X_\alpha(\sigma), X_\beta(\sigma)\}_{PB} \{X_\mu(\sigma), 1/\sqrt{g(\sigma)}\}_{PB}, \quad (A.7)$$

using the expansion (3.29). Also, we used

$$1/\sqrt{g(\sigma)} = \sum_r (y^{-1})^r e^{ir\sigma}. \quad (A.8)$$

Now we have

$$\epsilon^{\mu\alpha\beta...} \{X_\mu, 1/\sqrt{g}\}_{PB} \{X_\alpha, X_\beta\}_{PB} = 2\epsilon^{\mu\alpha\beta...} \left(\dot{X}_\mu \dot{X}_\alpha X'_\beta(1/\sqrt{g})' - X'_\mu X'_\alpha \dot{X}_\beta(1/\sqrt{g})\right), \quad (A.9)$$

where dot and prime denotes derivatives with respect to $\tau$ and $\sigma$, respectively. The expression on the right hand side of (A.9) is easily seen to vanish, since the first term inside the bracket is symmetric in $\mu$ and $\alpha$, whereas the last term is symmetric in $\mu$ and $\beta$. Thus, for $n \to \infty$ we have

$$\epsilon^{\mu\alpha\beta...} \text{Tr}(\psi[A_\alpha, A_\beta][A_\mu, Y^{-1}]) \to \epsilon^{\mu\alpha\beta...} \int d^2 \sigma \, \psi \{X_\mu, 1/\sqrt{g}\}_{PB} \{X_\alpha, X_\beta\}_{PB} = 0. \quad (A.10)$$

The proof of the cancellation of the cubic $\psi$ terms, which emerge in the action (3.5) under the symmetry transformations (3.8), is the standard one.

Hence the action (3.3) is supersymmetric for $n \to \infty$. 

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