Dynamical Dark Energy in Minimally Modified Gravity

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Abstract. Minimally modified gravity is a class of models with only the two tensor degrees of freedom as in general relativity. Using the framework with auxiliary constraints these models can maintain a dynamical cosmological background. The form of the constraints is thereby restricted by the requirement of dynamical dark energy and the avoidance of a breakdown of perturbation theory. Studying the linear perturbations around the FLRW background the results are, however, quite insensitive to the details of the constraints leading to a modified effective gravitational constant or a non-vanishing sound speed for dust.
1 Introduction

In the recent years minimally modified gravity models (MMG) have received an increased attention \cite{1–13}. These are a class of modified gravity models which do not add additional degrees of freedom as it is the normally the case as in scalar-tensor theories \cite{14–16} for a review).

The first model in this context has been the cuscuton model \cite{17} where due to the peculiar kinetic term the scalar field becomes non-dynamical as long as $\partial_i \varphi = 0$ \cite{18}. However, for a non homogeneous profile of the scalar field there will be additional instantaneous modes which are, in general, unstable. These modes are not unique for MMG models but also occur in higher order scalar-tensor theories \cite{19–21} or Horava-Lifshitz gravity models \cite{22}. However, in \cite{20} it has been argued that these modes are superficial and can be removed by imposing proper boundary conditions.

On the other hand, we could fix the slicing of the manifold from the start by choosing $\varphi = t$. In this case the full diffeomorphism invariance is explicitly broken evading, therefore, the Lovelock theorem \cite{23, 24}. These models are commonly called spatial covariant gravity models (SCG). While SCG theories, in general, have three degrees of freedom (2 tensor and 1 scalar mode) due to the breaking of diffeomorphism invariance \cite{25, 26} by requiring additional degeneracy conditions on the Lagrangian as it is the case for Cuscuton the additional scalar degree of freedom can be removed (see for instance \cite{4, 11} for a detailed discussion). Another way has been proposed in \cite{8}. Instead of requiring degeneracy conditions on the form of the Lagrangian one instead imposes additional constraints by hand at the Hamiltonian level. The Lagrangian can then afterwards be obtained by performing a Legendre transformation. In our paper we will focus on this specific approach.

A common motivation of modified gravity is to explain dark energy which is responsible for the accelerated expansion of the Universe \cite{27, 28}. Modified gravity theories like scalar-tensor theories provide a dynamical degree of freedom at the FLRW background even in the absence of additional matter and, therefore, allow for a dynamical evolution of dark energy. This is not the case in general relativity (GR) where due to the high symmetries of the FLRW metric in the absence of matter the Hubble parameter is completely fixed by the cosmological constant and there is no dynamics. This is also the case for conventional MMG models where the Hubble parameter is fixed by generically time dependent functions but without a dynamical degree of freedom. Depending on the specific model this can be sufficient to model any evolution of the Hubble parameter as in VCDM \cite{29} by tuning the free time functions appropriately. This is, however, not generically the case since the free functions can be constrained by consistency relations \cite{29, 30}. Recently, it has been realized that it is also possible to construct MMG which have a dynamical degree of freedom at the background even in the absence of an additional matter content by imposing auxiliary constraints \cite{12, 31, 32} which vanish trivially at the background level. Therefore, it is possible to obtain the same standard background evolution as in common scalar tensor theories. Both these approaches have the advantage that is possible to model background evolution which are commonly plagued by instabilities \cite{33} as bounces \cite{34, 35} or phantom dark energy \cite{12}.

In this paper the focus is on the construction of MMG with a dynamical background. Imposing auxiliary constraints by hand has got an increasing attention in the literature as it provides an easy and straightforward way to obtain a MMG model. Therefore, it is important to understand possible impacts of the constraints in more detail. As we will see these models are relatively insensitive to the details of the constraints at the linear level allowing for a
systematic discussion of the phenomenological properties.

As another aspect, by using the method of auxiliary constraints it is also possible to construct models which have more degrees of freedom at the linear level than at the full non-linear level signalizing a breakdown of perturbation theory. This clarifies a common misconception that linear perturbation theory allows to provide a lower limit on the total number of degrees of freedom at the full non-linear level avoiding a complete Hamiltonian analysis. In the appendix there is also a short discussion how it is possible to obtain the same features without the need of auxiliary constraints. But these models are, in general, quite pathological.

The structure of the paper is as follows. In the first section 2, we present the main idea for a toy model and discussing the fundamental properties in the Hamiltonian and Lagrangian formulation. Further, we analyze the properties of gravitational waves around a generic background. In section 3 we review the approach developed in [8] briefly and discuss the conditions on the form of the constraints and a possible breakdown of perturbation theory. Using the developed framework we discuss in 4 the phenomenological consequences by studying the linear perturbations around FLRW for a broad class of models. In particular, we show that for specific classes of constraints the results are relatively insensitive to the details of the constraints allowing for a systematical exploration. Last, we shortly discuss our results and provide an outline (sec. 5).

In the paper we are using the mostly plus (− + +++) signature and use units where the reduced Planck mass $M_p^2 = (8\pi G)^{-1}$ is set to one.

## 2 Toy model

In order to present the main idea let us first consider a toy model. By using the ADM decomposition

$$ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt) (dx^j + N^j dt)$$  \hspace{1cm} (2.1)

where $N$, $N^k$ and $h_{ij}$ are the lapse, the shift vector and the 3-dimensional metric on the spatial hypersurface of constant time, the Lagrangian of K-essence can be written in the unitary gauge $\varphi = \varphi(t)$ as

$$\mathcal{L} = \frac{1}{2} \sqrt{h} N \left( K_{ij} K^{ij} - K^2 + R \right) + \sqrt{h} NP(X),$$  \hspace{1cm} (2.2)

where $R$ is the three dimensional Ricci scalar, $X = \dot{\varphi}^2/(N^2)$ and $K_{ij}$ the extrinsic curvature

$$K_{ij} = \frac{1}{2N} \left( \dot{h}_{ij} - D_i N_j - D_j N_i \right).$$  \hspace{1cm} (2.3)

with $D_j$ being the covariant derivative with respect to the spatial metric $h_{ij}$. Note, that $\dot{\varphi}^2(t)$ is just a function of time and not a free variable. We could further fix it by using the remaining time-reparametrization invariance by setting $\dot{\varphi} = 1$. The corresponding Hamiltonian is given by

$$H_T = \int d^3x \mathcal{H} + N^k \mathcal{H}_k + u_N \pi_N + u_i \pi^i.$$

- 3 -
where

\[
\mathcal{H} = \frac{2N}{\sqrt{\hbar}} \left( \pi_{ij} \pi^{ij} - \frac{1}{2} \pi^2 \right) - \frac{1}{2} \sqrt{\hbar} N R - \sqrt{\hbar} N P,
\]

\[
\mathcal{H}_k = -2D_j \pi^j_k,
\]

and \( \pi_N, \pi^i \) and \( \pi^{ij} \) are the canonical conjugate momenta to \( N, N_i \) and \( h_{ij} \).

Due to the spatial diffeomorphism invariance \( \mathcal{H}_k \) and \( \pi^k \) are the usual 6 first class constraints corresponding to spatial transformation. On the other hand, since we fixed \( \varphi = \varphi(t) \), the Hamiltonian constraint

\[
\mathcal{H}_0 = -\{\pi_N, H_T\} = \mathcal{H}_{GR} - \sqrt{\hbar} P + \sqrt{\hbar} \frac{2 \dot{\pi}^2}{N^2} \dot{P}_N
\]

is not anymore a first class constraint but instead forms with \( \pi_N \) a pair of second class constraint resulting in three degrees of freedom.

Our aim is to eliminate the scalar degree of freedom. Following the discussion in [8] the easiest way is to add a constraint at the Hamiltonian

\[
H_T = \int d^3 x \mathcal{H} + N^k \mathcal{H}_k + u_N \pi_N + u^k \pi_k + \lambda C.
\]

For simplicity let us choose

\[
C = \sqrt{\hbar} D_k D^k R.
\]

The consistency relation for the new constraint enforces a secondary constraint

\[
C_R^{(2)} \approx -4R^{ij} \pi_{ij} D^2 N + 2R \pi D^2 N - \pi D_k R D^k N - 8\sqrt{\hbar} D_k \left( \frac{\pi_{ij}}{\sqrt{\hbar}} \right) R^{ij} D^k N
\]

\[
+ \sqrt{\hbar} D_k \left( \frac{\pi}{\sqrt{\hbar}} \right) \left( 4R D^k N - N D^k R \right) + 4\sqrt{\hbar} N D_i D_j D^2 \left( \frac{\pi_{ij}}{\sqrt{\hbar}} \right) + 2\sqrt{\hbar} N R D^2 \left( \frac{\pi}{\sqrt{\hbar}} \right)
\]

\[
+ 8\sqrt{\hbar} D^k N D_j D^2 \left( \frac{\pi^j_k}{\sqrt{\hbar}} \right) + 4\pi^{ij} D_i N D_j R + 16\sqrt{\hbar} D_i D_j N D_k D^j \left( \frac{\pi^k}{\sqrt{\hbar}} \right)
\]

\[
+ 8\sqrt{\hbar} D^k N D_i D_j D^2 \left( \frac{\pi^{ij}}{\sqrt{\hbar}} \right) D_i D_j D^k N
\]

where \( D^2 = D_k D^k \). These two constraints form a new pair of two second class constraints eliminating the scalar degree of freedom leading to a minimally modified theory of gravity.

The time conservation of \( C_N \) and \( C_R^{(2)} \) will fix the Lagrange parameter \( u_N = \frac{\partial C}{\partial \pi_N} \) and \( \lambda \)

\[
\dot{C}_N[\xi_1] \approx \{ C_N[\xi_1], \int d^3 x \mathcal{H} \} + \{ C_N[\xi_1], \int d^3 x \lambda C_R \} + \{ C_N[\xi_1], \int d^3 x u_N \pi_N \},
\]

\[
\dot{C}_R^{(2)}[\xi_2] \approx \{ C_R^{(2)}[\xi_2], \int d^3 x \mathcal{H} \} + \{ C_R^{(2)}[\xi_2], \int d^3 x \lambda C_R \} + \{ C_R^{(2)}[\xi_2], \int d^3 x u_N \pi_N \},
\]
where we have introduced the smeared constraints by using the test function $\xi(x)$

$$C_R[\xi] = \int d^3x \xi C_R$$

(2.13)

and similar for the other constraints. Using that

$$\{C_N[\xi_1], \int d^3x \lambda C_R\} \approx \int d^3y \frac{\delta C_N[\xi]}{\delta \pi^j(y)} \sqrt{h} \left(-D(iD_j)D^2 - D(iRD_j) + R_{ij}D^2 + h_{ij} \left(D^4 - \frac{1}{2} D^kRD_k\right)\right) \lambda(y)$$

(2.14)

and similar for $\{C_R^{(2)}, \int d^3x \lambda C_R\}$ we can see that it will lead to a spatial differential equation for $\lambda$. However, we can note that for a homogeneous and flat ansatz like the flat FLRW metric the second consistency relation (2.12) becomes trivial and $\lambda$ is not fixed.

### 2.1 Lagrangian formulation

The new action can be obtained by an inverse Legendre transformation

$$\mathcal{L}' = \mathcal{L} + \lambda C$$

(2.15)

where we have rescaled $\lambda \rightarrow -\lambda$. In this case, the equation of motions (EOM) are given by

$$\delta(\int d^4x \mathcal{L}) = 0, \quad \frac{\delta(\int d^4x \mathcal{L})}{\delta N} = 0, \quad \frac{\delta(\int d^4x \mathcal{L})}{\delta N^k} = 0.$$

(2.16)

Taking the trace of the first equation we can solve it for $\lambda$ yielding

$$\left(2D^4 + RD^2 + \frac{1}{2} D^kRD_k\right) \lambda = \frac{h_{ij}}{\sqrt{h}} \frac{\delta(\int d^4x \mathcal{L})}{\delta h_{ij}}.$$

(2.17)

We obtain a spatial differential equation for $\lambda$. The traceless component of the first EOM leads to

$$\left(h_m^i h_n^j - \frac{1}{3} h^{ij} h_{mn}\right) \left(\frac{\delta(\int d^4x \mathcal{L})}{\delta h_{mn}} + \sqrt{h} \left(D_i D_j D^2 + D(iRD_j) - R_{ij}D^2\right) \lambda\right) = 0.$$

(2.18)

Note that the EOM are invariant under a shift $\lambda(t, x^k) \rightarrow \lambda(t, x^k) + \lambda_0(t)$. As a consistency check we can count the number of degrees of freedom. The trace component of the metric EOM fixes $\lambda$, the Hamiltonian constraint and the momentum constraint fix $N$ and $N^k$ and finally the constraint $C$ fixes the trace of the metric. Using the remaining spatially gauge invariance we can fix 3 further components of the metric so that we are left with two traceless components of the spatial metric, $h_{ij}$. 

- 5 -
2.2 FLRW background

Let us now apply this new theory to cosmology. At the FLRW background

\[ ds^2 = -N^2dt^2 + a^2dx^idx^j \delta_{ij} \]

we can see that both of the constraints vanish identically. In particular, the action at the minisuperspace is equivalent to the original K-essence model

\[ \mathcal{L}'_{\text{FLRW}} = a^3N \left( -3 \left( \frac{\dot{a}}{aN} \right)^2 + P \right). \]

Consequently the Hamiltonian is given via

\[ H_{\text{FLRW}} = a^3N \left( -\frac{p_a^2}{12a^4} - P \right). \]

The lapse function is non-dynamical so that \( \pi_N \) and the Hamiltonian constraint \( H_0 \)

\[ H_0 = a^3 \left( -\frac{p_a^2}{12a^4} - P + P_N \dot{\varphi}^2 \right/N^2 \]

form a pair of two second class constraints resulting in one dynamical degree of freedom. In contrast to other MMG models there is a dynamical scalar degree of freedom at the FLRW background. The introduction of the two constraints does not impact the background evolution since the two constraint vanish identically. As a minor consistency check in the appendix A it is shown that one can recover the flat FLRW solutions by starting from a generic stationary spherical symmetric background metric in which case the constraints are not trivial identities by imposing proper boundary conditions at spatial infinity. On the other hand, at the linear level the constraint \( C \) will lead to

\[ \delta C = a \delta^2 \delta R \approx 0 \]

which eliminates the scalar degree of freedom at the linear level.

Note, that one can obtain MMG models with a dynamical FLRW background even without the usage of auxiliary constraints. However, the structure of these theories is very different requiring for instance a trivial Hamiltonian constraint at the FLRW background leading to pathological models (see appendix B for more details).

2.3 Gravitational waves

To study the impact of the constraint on the EOM let us consider the consequences for the gravitational waves around a generic background

\[ h_{ij} = \bar{h}_{ij} + \delta h_{ij}, \quad N = \bar{N} + \delta N, \quad N^k = \bar{N}^k + \delta N^k. \]

We use our gauge symmetry to fix \( \bar{D}_k \delta h^k_j = 0 \). For simplicity let us focus only on the terms with the highest number of derivatives acting on \( \delta h_{ij} \) which is similar to the geometrical optics approach in general relativity [36, 37]

\[ \delta h_{ij} = A_{ij} \exp(\theta/\epsilon) \]
with \( \epsilon \ll 1 \). From the constraint EOM we obtain at leading order
\[
\ddot{D}^4 \delta h^k_i \simeq -\ddot{K}^{ij} \dot{D}^2 \delta h_{ij}
\]  
(2.26)
where we have introduced the traceless tensor \( \tilde{\delta} h^{ij} = \delta h^{ij} - h^{ij} \delta h_k^k / 3 \). Therefore, the trace of the metric perturbation is of order \( \delta h_k^k \simeq O(\epsilon^2) \). In order to obtain the scaling relation for the shift and lapse we can use the Hamiltonian and momentum constraint.

Perturbing the momentum constraint up to linear order yields
\[
- (\ddot{K}^{ij} \dot{D}_i - \ddot{K} \ddot{D}_j) \delta N - \frac{1}{2} \dot{D}_j \ddot{D}_m \delta N^m - \frac{1}{2} \dot{D}^2 \delta N_j + \frac{1}{2} \ddot{D}^k \delta h_{kj} \simeq 0.
\]  
(2.27)
Note that by using the gauge condition
\[
\ddot{D}_j \delta h^{jk} \simeq \epsilon^{-1} \left( \dot{A}^{jk} \ddot{D}_j \theta + \ddot{D}_j (\dot{\theta} A^{jk}) \right). 
\]  
(2.28)
On the other hand solving the Hamiltonian constraint up to leading order we obtain
\[
\delta N \left( 2\ddot{R} + 2P - \frac{2}{N^2} P_X + \frac{4}{N^4} P_{XX} \right) - \frac{4}{3} \ddot{K} \dot{D}_j \delta N^b = \frac{1}{2} \ddot{K} \ddot{h}_{ij} - \ddot{D}_i \delta N_j - \ddot{D}_j \delta N_i - \bar{N} \ddot{D}_c \delta h_{ij} \simeq O(\epsilon^0). 
\]  
(2.29)
Therefore, if \( \ddot{K}_{ij} = 0 \), we can make the self-consistent ansatz \( \delta N = O(\epsilon^0) \) and \( \delta N^k = O(\epsilon) \). However, as long as the background has non-vanishing traceless components of the extrinsic curvature we need to require that either \( \dot{\ddot{K}}^{ij} (\delta h_{ij} - \bar{N} \delta h_{ij}) = 0 \), which leads to non-dynamical gravitational waves, or that \( \ddot{D}_j \delta N^k \) and \( \delta N \) scale as \( O(\epsilon^{-1}) \).

As a next step perturbing the trace of the EOM for \( h_{ij} \) leads to
\[
\ddot{D}^4 \delta \lambda \simeq \frac{1}{2} \ddot{K} - \frac{1}{2} \ddot{N}^k \ddot{D}_k \delta \lambda - \frac{1}{2} \ddot{D}^2 \delta N + \frac{3}{4} \ddot{K} \ddot{D}^j \ddot{D}^j \dot{\lambda} \ddot{D}^2 \delta h_{ij}
\]  
(2.30)
For the traceless EOM the leading order is given by
\[
\left( \ddot{h}_m^{ij} \ddot{h}_n^{ij} \frac{1}{3} \ddot{K}^{ij} \ddot{h}_{mn} \right) \left[ - \frac{1}{2} \ddot{K}_{ij} + \frac{1}{2} \ddot{N}^k \dot{D}_k \delta K_{ij} - \frac{1}{2} \ddot{N} \delta R_{ij} + \frac{1}{2} \ddot{D}^2 \dot{\lambda} \ddot{D}^2 \delta h_{ij} \right. 
\left. + \frac{\dot{D}_j (\ddot{D}_j)}{\dot{D}^2} \left( \frac{1}{2} \ddot{K} - \frac{1}{2} \ddot{N}^k \dot{D}_k \delta \lambda - \frac{1}{2} \ddot{D}^2 \delta N \right) \right] \simeq 0.
\]  
(2.31)
For \( \dddot{K}_{ij} \neq 0 \), in which case \( \delta N \) and \( \ddot{D}_k \delta N^j \) scale as \( O(\epsilon^{-1}) \), we obtain \( \ddot{K} = O(\epsilon^{-2}) \). Therefore, the dispersion relation for the gravitational waves is, in general, quite cumbersome. We need to solve the Hamiltonian and momentum constraint explicitly. It might be that the dispersion relation gets non-local contributions.

On the other hand, if \( \ddot{K}_{ij} = \frac{1}{3} \dddot{K} \dot{h}_{ij} \) we recover for the traceless spatial components the standard EOM for gravitational waves on a generic background up to a modified propagation speed which depends on the background value \( \ddot{D}^2 \dot{\lambda} \)
\[
- \frac{1}{2N} \left( \dddot{h}_{ij} - 2\ddot{N}^k \dot{D}_k \ddot{h}_{ij} \right) - \frac{1}{2N} \ddot{N}^k \dddot{N}_m \dddot{D}_m \dddot{D}_k \delta h_{ij} + \frac{1}{2} (\dddot{N} + \ddot{D}^2 \dot{\lambda}) \ddot{D}^2 \dot{h}_{ij} 
\simeq \frac{N}{2} g^{\alpha \beta} \dot{\nabla}_\alpha \dot{\nabla}_\beta \ddot{h}_{ij} + \frac{1}{2} \ddot{D}^2 \dot{\lambda} \ddot{D}^2 \dot{h}_{ij} = 0
\]  
(2.32)
where $\bar{g}^{\alpha\beta} = h^{\alpha\beta} - n^\alpha n^\beta$ and $n^\alpha = 1/N(1, -N^k)$.

Note, that the result is highly dependent on the form of the constraint. Consider for instance the constraint
\[
\hat{C}_R = \sqrt{h} \left( R + \frac{R_{ij} R_{ij}}{\Lambda^2} \right).
\] (2.33)

In this case up-to-leading order the constraint yields
\[
- \left( D^2 + \frac{R}{3\Lambda^2} D^2 + \frac{R_{ij}}{\Lambda^2} D_i D_j \right) \delta h^c_c \simeq \frac{\tilde{R}_{ij}}{\Lambda^2} D^2 \delta h_{ij}.
\] (2.34)

Therefore, the spatial metric is not anymore traceless up-to-leading order but instead the trace component is of the same order. In general, the EOM will lead to a higher order dispersion relation.

3 Minimally modified gravity models

3.1 Construction with auxiliary Lagrange multiplier

Let us shortly recap the framework to construct minimally modified gravity models with auxiliary Lagrange multiplier following [8].

Let us start with a generic Hamiltonian with a spatial diffeomorphism invariance
\[
H_T = H_p + \int d^3 x N^k \mathcal{H}_k + u^i \pi_i
\] (3.1)

where again $\mathcal{H}_k$ is the standard momentum constraint which generates the spatial transformation. On the other hand, $H_p$ can be expressed as
\[
H_p = \int d^3 x \mathcal{H}(N, h_{ij}, \pi_{ij}, D_k, t) + u_N \pi_N.
\] (3.2)

This class of models has in general six first class constraints, $\mathcal{H}_k$ and $\pi^k$, and two second class constraints $\pi_N$ and the Hamiltonian constraint $\mathcal{H}_0$.

By imposing additional constraints at the Hamiltonian level we can remove the scalar degree of freedom. Following [8] and similar to the toy model in the previous section we could impose a new primary constraint at the Hamiltonian via
\[
H'_p = H_p + \int d^3 x \lambda C.
\] (3.3)

Further, we have to require that the consistency relation $\dot{C} \approx 0$ does not fix the Lagrange multiplier but instead leads to a secondary constraint which requires that $C$ commutes with itself and with $\pi_N$,
\[
C = C(h_{ij}, \pi_{ij}, D_k, t) \quad \text{with} \quad \{C(x), C(y)\} \approx 0.
\] (3.4)

Having a pair of second class constraint (a primary and a secondary one) the scalar degree of freedom is killed leading to minimally modified gravity theory with two tensor degrees of freedom.
Alternatively, we could directly impose two primary constraints at the Hamiltonian \[ H' = H_p + \int d^3 x \lambda_1 C_1 + \lambda_2 C_2 \] (3.5)

Having already a pair of second class constraints the only conditions on the form of \( C_1 \) and \( C_2 \) are that they commute with \( \pi_N \) and are invariant under spatial diffeomorphism, i.e. \( C_i = C_i(h_{ij}, \pi^{ij}, D_k, t) \). Note, that the two approaches are not equivalent. In the latter case both constraints have to independent of the lapse function in order to commute with \( \pi_N \) while the secondary constraint in the first approach will, in general, depend on the lapse function. On the other side in the second approach the two imposed constraints depend on the momentum of the metric. Even if the secondary constraint in (3.4) does not depend on the lapse function we would need to ensure that the two constraints in (3.5) can be expressed in such a way that one of them does not depend on the momentum in order to obtain an equivalent theory.

While the pair of second class constraints (either two primary or one primary and one secondary one) removes the scalar degree of freedom at the full non-linear level, our aim is it to keep the FLRW background dynamics of the original model without the imposed constraints. This enforces that the new constraints vanish identically on the FLRW background

\[ C|_{\text{FLRW}} = 0. \] (3.6)

In that case, similar to the discussed toy model in the previous section, at the FLRW background we end up with the same Hamiltonian as in the original model leading to one dynamical degree of freedom due to the broken full diffeomorphism invariance.

Note, that in [8] the authors also discussed another case where they only impose one new second class constraint but instead require that \( \pi_N \) remains a first class constraint. This, however, restricts the form of the Hamiltonian to

\[ \mathcal{H} = \mathcal{V}(h_{ij}, \pi^{ij}, D_k, t) + N \mathcal{H}_0(h_{ij}, \pi^{ij}, D_k, t). \] (3.7)

At the FLRW background this kind of models, in general, does not have a dynamical degree of freedom. Even if the new constraint \( C \) vanishes at the background level \( C|_{\text{FLRW}} = 0 \) due to the homogeneous background \( \mathcal{H}_0 \) will commute with itself and, therefore, at the background level we will obtain two first class constraints as in standard GR resulting in a FLRW background without any dynamics. In order to obtain a dynamical FLRW background the Hamiltonian constraint itself has to become trivial at the background leading to a pathological behavior. This is similar to the toy model discussed in appendix B.

### 3.2 Breakdown of the perturbation theory

In the previous subsection we have constructed minimally modified gravity models with just two tensor degrees of freedom while keeping the background dynamics at the FLRW background by requiring that the new constraints vanish on the FLRW background. There is, however, one caveat. While the new constraints should vanish on the FLRW background they should be present at the linear level.

To demonstrate it let us consider one simple example where the primary constraint is given by \( C = \sqrt{h} R_{ij} R^{ij} \). For the Hamiltonian we use again K-essence. The secondary constraint can then be obtained due to the consistency condition

\[ \frac{dC}{dt} = \{C, H_T\} \approx 0. \] (3.8)
The specific form is quite cumbersome but it is straightforward to see that the constraint will again be trivially fulfilled at the FLRW background. Since $C$ does not depend on the momentum the Legendre transformation is given by

$$ \mathcal{L}' = \mathcal{L} + \lambda C. $$

(3.9)

However, considering the linear perturbation of the constraint $C$ we can see that the constraint is still trivially fulfilled

$$ \delta C = 0. $$

(3.10)

Therefore, at the linear level there is no additional constraint and we end up with one scalar degree of freedom which is in strong contradiction with the full non-linear theory. Note, that it does not imply that the full non-linear theory is inconsistent. Instead, it means that we cannot trust perturbation theory around this given background. This is similar to the discussion of strong coupling where the linear scalar degree of freedom is absent at the linear level but returns at higher order.

Therefore, in order to have a consistent perturbation theory around the FLRW we need to ensure that the constraints are not trivially fulfilled. This limits the possible number of operators. If we are only interested in the linear order we can expand the constraint in terms which vanish on the background resulting in

$$ c_1(t, \pi) R, \quad c_2(t, \pi) D_k D^k R, \quad c_3(t, \pi) D_k D^k \pi, ... $$

(3.11)

where the dots signal higher order of spatial derivatives. Each constraint can be written as a linear combination of the aforementioned terms and operators which vanish at the linear order. Note, that terms like $D_j \pi^{ij}$ and $D_j R^{ij}$ do not yield new independent operators due to the momentum constraint and the Bianchi identity.

Last, let us note that this also implies that a common assumption that the linear perturbation theory can be used to derive a lower bound on the number of degrees of freedom at the full non-linear level is in general not correct. As in the previous toy model the theory could contain constraints which vanish at the linear order for a given background leading to an inconsistent perturbation theory which overestimates the number of degrees of freedom (see also appendix B for an example without the presence of auxiliary constraints).

### 4 Effective field theory

Let us use the derived framework to discuss phenomenological consequences at the linear order around the FLRW background for this kind of models. Our analysis will be split into two parts. First, we will impose one primary constraint, which does not depend on the momentum and generates a secondary constraint. In the second case we directly impose two primary constraints. Note, that we do not include cases where the single primary constraint also depends on the momentum since the analysis is much more involved due to the requirements on the form of the primary constraint and is beyond the scope of this paper. In order to have a dynamical scalar degree of freedom at the linear order we will consider pure dust for the matter sector.
4.1 No momentum dependency

We impose one primary constraint which does not depend on the momentum

\[ H'_p = \int d^3x \mathcal{H} + u_N \pi_N + \lambda C \]  

(4.1)

with

\[ C = C(h_{ij}, D_k, t). \]  

(4.2)

As discussed before, the Legendre transformation becomes trivial in that case

\[ \mathcal{L}' = \mathcal{L} + N\lambda C \]  

(4.3)

where we have rescaled the Lagrange multiplier \( \lambda \rightarrow -N\lambda \). Further, \( \mathcal{L} \) is the Lagrangian associated to the Hamiltonian \( \mathcal{H} \) without the presence of the constraint.

In order to keep the formalism very general we will use the effective field formalism of dark energy for the Lagrangian [38]

\[ \mathcal{L} = \sqrt{h} NL(K, S, R, Y, Z) \]  

(4.4)

where \( S = K_{ij}K^{ij}, \ Z = R_{ij}R^{ij} \) and \( Y = K_{ij}R^{ij} \). Further, we will add pure dust described by the Schulz-Sorkin action (see appendix C).

As discussed in section 3.2, in order to have a consistent perturbation theory around the FLRW background we can parametrize the constraint as

\[ C = \sqrt{h} \sum c_n(t)(D_kD^k)^nR + g(h_{ij}, D_k, t) \]  

(4.5)

where \( g \) is a free function which vanish at the background and linear order. For the metric perturbation we use the standard convention of the effective field formalism of dark energy

\[ N = 1 + \delta N, \quad N^k = \delta^{kj}\partial_k\psi, \quad h_{ij} = a^2 e^{2\xi} \left( \delta_{ij} + \gamma_{ij} + \frac{1}{2} \gamma_{ik} \gamma^{kj} + \ldots \right). \]  

(4.6)

Similar we perturb the Lagrange multiplier as \( \lambda = \lambda_0(t) + \delta \lambda \). At linear order the constraint will result in a spatial differential equation for \( \xi \)

\[ \delta C = -4a^3 \sum_n c_n \frac{\partial^{2n+2}}{a^{2n+2}} \xi. \]  

(4.7)

For simplicity, we consider boundary conditions so that \( \xi = 0 \).

We can split the final result into two different classes:

- non propagating solution \( \mathcal{A} + 2L_S = 0 \)

- propagating solution

Note, that all models inside the (beyond) Horndeski class [39] belong to the first case.
Non-propagating solution

In that case we can integrate out the perturbations of the non-dynamical shift and lapse as

$$\delta N = \frac{\rho_m}{B + 4HL_S} ,$$ (4.8)

$$k^2 \frac{a^2}{\dot{\psi}} = \frac{(3HB - 2L_N - L_{NN})\rho_m v_m + \rho_m (B + 4HL_S) \delta_m}{(B + 4HL_S)^2} ,$$ (4.9)

where \( L_X \equiv \partial_X L \) and

$$B = 2HL_{SN} + L_{KN} ,$$ (4.10)

$$A = 4H^2L_{SS} + 4HL_{SK} + L_{KK} .$$ (4.11)

Using further the EOM of \( v_m \)

$$v_m = \frac{a^2 (B + 4HL_S)((B + 4HL_S) \delta_m + \rho_m \delta_m)}{-k^2(B + 4HL_S)^2 + a^2(3\dot{H} + 24HL_S H + (2L_N + L_{NN})\rho_m + 12H^2LS(4HL_S + \rho_m)} ,$$ (4.12)

we finally obtain at the small scale limit \( x = k/(aH) \gg 1 \)

$$\delta S = \int d^3k d\tau \frac{a^5 \rho_m}{2k^2} \left[ \dot{\delta}_m^2 + \left( \frac{\rho_m^2}{(B + 4HL_S)} \right)^2 - \frac{1}{a^2 \rho_m} \frac{d}{dt} \left( \frac{a^5 \rho_m^2}{(B + 4HL_S)} \right) \right] .$$ (4.13)

There is one dynamical non-propagating degree of freedom. The absence of ghost instabilities requires \( \rho_m > 0 \) which is fulfilled for any canonical matter fluid. Further, up to the leading order in the small scale limit the EOM is given by

$$\ddot{\delta}_m + 2H \dot{\delta}_m - \frac{1}{2} \rho_m G_{\text{eff}} \delta_m = 0$$ (4.14)

where

$$G_{\text{eff}} = 2 \left( \frac{\rho_m}{(B + 4HL_S)^2} - \frac{1}{a^2 \rho_m} \frac{d}{dt} \left( \frac{a^5 \rho_m^2}{(B + 4HL_S)} \right) \right) .$$ (4.15)

The effective gravitational constant is changed. Note, that the expression is exact up to the leading order in the small scale approximation and does not require the quasistatic approximation since there is only one dynamical scalar degree of freedom.

Propagating solution

In the second case we can again integrate out \( \delta N, \psi \) and \( v_m \)

$$\delta N = \frac{\rho_m (B + 4HL_S)}{D} v_m - \frac{\rho_m (A + 2LS)}{D} \delta_m ,$$ (4.16)

$$k^2 \frac{a^2}{\dot{\psi}} = \frac{-\rho_m (2L_N + L_{NN} - 3HB)}{D} v_m - \frac{\rho_m (3HA + 2HL_S - B)}{D} \delta_m ,$$ (4.17)

$$v_m = \frac{a^2 D \dot{\delta}_m + a^2 (B + 4HL_S) \rho_m \delta_m}{-DK^2 + 3a^2 \dot{H}D + a^2 \rho_m (2L_N + L_{NN} + 12H^2LS)} ,$$ (4.18)
where
\[ D = B^2 + 8BHLS - 2LS(2LN + L_{NN} + 4H^2LS) - A(2LN + L_{NN} + 12H^2LS) \] (4.19)
so that at the small scale limit \( x = k/(aH) \gg 1 \)
\[ \delta S = \int d^3k \int d^3t \frac{a^2 \rho_m}{2k^2} \Phi_m(\frac{L^2}{2D}) + \frac{a^2 \rho_m}{2k^2D^2} \left( \frac{L^2}{2k^2D^2} \right) \delta_m \] (4.20)

In that case, the dust does not behave anymore as standard dust but instead acquires a non-vanishing sound speed
\[ c_s^2 = -\frac{(A + 2LS)}{D} \rho_m. \] (4.21)

In order to have stable linear perturbations without gradient or ghost instabilities we have to require that \((A + 2LS)/D < 0 \) and \( \rho_m > 0 \). Further, due to the strict constraints on the sound speed of dust we can use it to put severe constraints on the parameters of the model. Note, that the results do not depend on the background value of the Lagrange multiplier \( \lambda_0 \).

**Tensor modes**

While the scalar sector does not depend on the background value \( \lambda_0 \) and is quite insensitive to the form of the constraints this is, in general, not the case for the tensor sector. Consider for instance \( C = \sqrt{h}R \). The constraint will lead to a modification of the propagation speed which explicitly depends on \( \lambda_0 \). However, since the constraint is trivially fulfilled at the FLRW background, \( \lambda_0 \) is not fixed by the background EOM. Instead \( \lambda_0 \) is only fixed indirectly if we consider the full non-linear level.

On the other hand, in 2.3 we have discussed that for constraints like \( C = \sqrt{h}D^2R \) the EOM are invariant under a shift of \( \lambda(t, x^k) \rightarrow \lambda(t, x^k) + \lambda_0(t) \). Therefore, we can set \( \lambda_0 = 0 \) without loss of generality similar to the discussion in [40, 41]. Indeed, we can check that in this case the EOM for the tensor sector will not be impacted by the constraint.

In general, in order to avoid the ambiguity related to \( \lambda_0 \) we could restrict ourselves to constraints of the form \( C = \sqrt{h}D_k C^k \) which ensures that the EOM do not depend on \( \lambda_0 \) and we recover at linear order the same result for the tensor sector as in the original model prior to the constraint.

\[ \delta S = \frac{1}{4} \int d^3k \int d^3t \frac{a^2 \rho_m}{2k^2} \delta_m \left[ \phi_m \frac{k^2}{a^2} \delta^2 + \left( L^2 + \lambda C^2 \frac{k^4}{a^4} \delta^2 \right) \right] \] (4.22)

where
\[ E = L_R + \frac{1}{2a^4} \frac{d}{dt} (a^3 L_Y) \] (4.23)

Therefore, the sound speed of the tensor modes is given by
\[ c_T^2 = \frac{E}{L_S}. \] (4.24)

Using the constraints from GW170817 [42, 43] we can put severe constraints on the parameters of the model. Further, in order to avoid a higher order dispersion relation for the tensor modes we would need to set \( L_Z = 0 \).
4.2 Momentum dependency

As a next step, let us discuss the case where we directly introduce two constraints which can in general depend on the momentum of the spatial metric

\[ H_p' = \int d^3x \mathcal{H} + u_N \pi N + \lambda_1 C_1 + \lambda_2 C_2. \]  

(4.25)

In order to have a dynamical FLRW background we have again to assume that the constraints are trivial identities on the background. Therefore, we can expand them as

\[ C_1 = \sqrt{h} \sum_k a_{1k}(t, \pi)(D_m D^n)^k R + \sum_k b_{1k}(t, \pi)(D_m D^n)^k \pi + g_1(\pi^{ij}, h_{ij}, D_k), \]  

(4.26)

\[ C_2 = \sqrt{h} \sum_k a_{2k}(t, \pi)(D_m D^n)^k R + \sum_k b_{2k}(t, \pi)(D_m D^n)^k \pi + g_2(\pi^{ij}, h_{ij}, D_k), \]  

(4.27)

where \( g_1 \) and \( g_2 \) are arbitrary functions which vanish up to the quadratic order. Further, in order to have two tensor degrees of freedom at the full non-linear level the Dirac matrix has to be invertible. For simplicity, we assume that the two constraints do not commute with each other.

Perturbing both constraints at linear order we obtain for the scalar perturbations the following structure.

\[ A(t, \partial, \pi, a) \delta h_{ij} h_{ij} + B(t, \partial, \pi, a) \delta \pi = 0. \]  

(4.28)

These two constraints are in general differential equations. By imposing proper boundary conditions and requiring that the two constraints do not commute with each other we can set \( \delta \pi = 0 = \delta h_{ij} h_{ij} \). This result is quite generic and does not depend on the specific form of the constraints. Therefore, in the following we will consider for simplicity

\[ C_1 = \sqrt{h} D_k D^k \left( \frac{\pi}{\sqrt{h}} \right), \quad C_2 = \sqrt{h} D_k D^k R. \]  

(4.29)

Further, to have an explicit expression for the Legendre transformation we will use the following ansatz for the Hamiltonian \( \mathcal{H} \)

\[ \mathcal{H} = N d_0(N) \pi + N d_1(N) \pi_{ij} \pi^{ij} + N d_2(N) \pi^2 - N \sqrt{h} f(N, h_{ij}, D_k) \]  

(4.30)

where we assume that \( f \) does not depend on \( D_k N \) or higher order derivatives. Performing the Legendre transformation we obtain the Lagrangian

\[ \mathcal{L}_{tot} = \mathcal{L} + \lambda_2 C_2 \]

\[ = \sqrt{h} N \left[ \frac{1}{d_1} K_{ij} K^{ij} - \frac{d_2}{d_1(d_1 + 3d_2)} K^2 + f \right] + \lambda_2 C_2 \]  

(4.31)

where we have again rescaled \( \lambda_2 \to -\lambda_2 \) and

\[ K_{ij} = K_{ij} - \frac{1}{2} d_0 h_{ij} + \frac{1}{2N} h_{ij} D^2 \lambda_1. \]  

(4.32)

As in the previous section the EOM for \( \lambda_2 \) leads to the known condition \( C_2 = \sqrt{h} D^2 R = 0 \) while taking the trace of the metric EOM we can solve \( \lambda_2 \) as

\[ \left( 2D^4 + RD^2 + \frac{1}{2} D^k R D_k \right) \lambda_2 = \frac{h_{ij}}{\sqrt{h}} \frac{\delta (\int d^4x \mathcal{L})}{\delta h_{ij}}. \]  

(4.33)
On the other hand, the EOM for $\lambda_1$ leads to

$$D^2 \left( \frac{1}{d_1 + 3d_2} \right) = 0. \quad (4.34)$$

We can note, that in both cases the Lagrange parameters $\lambda_1$ and $\lambda_2$ are only solved up to a time dependent integration constant which is fixed by the boundary conditions of the spatial differential equations.

**Scalar perturbations**

Let us now discuss the impact of the constraints on the cosmological scalar perturbations. As before, we will add a perfect fluid of pure dust to have a dynamical scalar degree of freedom at the linear level.

At linear order the curvature constraint $\delta C_2$ leads again to $\xi = 0$ if we assume proper boundary conditions. Further, solving the EOM for $\delta \lambda_1$ we can integrate it out. Expanding the action in the small scale limit $x = k/(aH) \gg 1$ we obtain finally

$$\delta S = \int d^3k dt a^3 z^2 \left[ \delta_{m}^2 - \left( \frac{2k^2}{a^2} + M^2 \right) \delta_{m}^2 \right]$$  \quad (4.35)

where

$$z^2 = \frac{\rho_m a^2}{2k^2}, \quad (4.36)$$

$$c_s^2 = -\frac{4\rho_m (d_1 + 3d_2)^2}{b} \quad (4.37)$$

with

$$b = -6d_0(d_1 + 3d_2)(2d_0 + d_0' + (3d_0^2 + 12H^2)(2d_1' + 6d_2' + d_0'' + 3d_2'')$$

$$+ 12Hd_0(d_1 + 3d_2)(2d_0' + d_0'' - 12Hd_0)(2d_1' + 6d_2' + d_0'' + 3d_2'')$$

$$+ 4(d_1 + 3d_2)^2(2f' + f'') \quad (4.38)$$

where the ‘ denote derivatives with respect to the lapse function. The explicit expression of $M$ is quite involved and not really helpful for our purposes. We can note that the dust will acquire a non-vanishing sound speed which could be used to constrain the parameters of the model. Indeed, the sound speed only vanishes if $d_1 = -3d_2$ which corresponds to a model which depends linearly on the trace of the momentum of the spatial metric.

**Tensor perturbations**

Due to the form of the constraints $C_1$ and $C_2$ as total spatial derivatives they do not impact the tensor sector at linear order but instead we recover the same EOM as for the original model which will depend on the form of the free function $f$. If we for instance consider the case where $f = f(R, Z)$ we obtain

$$\delta S = \frac{1}{4} \int d^3k dt a^3 \left( \frac{1}{d_1} \gamma_{ij}^2 - f_R \frac{k^2}{a^2} \gamma_{ij}^2 + f_Z \frac{k^4}{a^4} \gamma_{ij}^2 \right), \quad (4.39)$$

so that the propagation speed of the gravitational waves is given by

$$c_t^2 = f_R d_1. \quad (4.40)$$
5 Conclusion

In this paper we have analyzed minimally modified gravity models with a dynamical FLRW background evolution. This can be obtained by imposing auxiliary constraints which vanish identically at the FLRW background (see also [12, 31, 32]) so that we obtain the background evolution from the original model. This imposes conditions on the form of the constraints. In particular, we need to ensure that the constraints are not trivial identities at the linear order since it leads to a breakdown of perturbation theory.

While it is a priori also possible to construct this type of models without the need of auxiliary constraints this leads to highly non-standard cosmologies requiring for instance a trivial Hamiltonian constraint at the background level. These models can also suffer under a breakdown of linear perturbation theory around FLRW.

In the next part we studied the phenomenological consequences. For generic backgrounds the auxiliary constraints can lead to a non-standard dispersion relation for the gravitational waves and is, in general, highly sensitive to the form of the constraints.

As a next step we focused on the linear perturbation around the FLRW background including dust to have a dynamical scalar degree of freedom at the linear level. For two classes of constraints (one primary constraint, which does not depend on the momenta of the metric, or two generic primary constraints) the perturbations around FLRW are not very sensitive to the specific details of the constraints. In the first case depending on the original model the constraint will lead to a modification of the effective gravitational constant for the dust component and for models outside the GLPV class it can lead to a non-vanishing sound speed. This is similar to the second case where the two primary constraints will, in general, except for some specific cases always provide a non-vanishing sound speed for dust which is highly constrained by observations.

For the tensor modes the details of the constraints can become important. In this case the tensor sector will depend on the background value of the Lagrange multiplier which is, however, not fixed since the constraint is trivially fulfilled at the background level. However, we can avoid this ambiguity by further restricting the form of the constraints to $C = \sqrt{h} D_k C^k$ so that the EOM are invariant under a time dependent shift of the Lagrange multiplier. In that case the tensor sector will not be impacted by the constraint up to linear order but instead we recover the same result as for the original model prior to the constraint.

Summarizing, constructing MMG models by imposing auxiliary constraints provides a new playground leading to interesting new phenomenological features. In future it might be interesting to study the consequences of these type of models in the case of black holes or other backgrounds where the constraint do not vanish trivially.

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A Non-trivial background

In order to get a better understanding of the dynamical degree of freedom at the FLRW background let us consider a generic stationary spherical symmetric background in which
the constraint is not a trivial identity
\[
ds^2 = -N(t,r)^2 dt^2 + F(t,r)^2 a(t)^2 dr^2 + a(t)^2 r^2 d\Omega^2. \tag{A.1}
\]
In order to simplify the discussion let us consider the constraint 
\[C = \sqrt{h} R\] which leads to
\[-F(t,r) + F(t,r)^3 + 2r \partial_r F(t,r) = 0. \tag{A.2}\]
Solving it we obtain
\[F(t,r) = \frac{1}{\sqrt{\frac{\kappa(t)}{r} + 1}}. \tag{A.3}\]
By imposing proper boundary conditions at spatial infinity we can set 
\[
\kappa(t) = 0 \text{ recovering the standard result.}
\]
In that case the equations of motion for the toy model in (2.2) and (2.15) simplify to
\[
3 \frac{\dot{a}(t)^2}{a(t)^2} + N(t,r)^2 P - 2P' = 0, \tag{A.4}
\]
\[
a(t)^2 N(t,r)^3 P + N(t,r)(\dot{a}(t)^2 + 2a(t)\ddot{a}(t))
- \frac{2}{r} N(t,r)^2 (\partial_r N(t,r) + 2\partial_r \lambda(t,r)) - 2\dot{a}(t)a(t)\dot{N}(t,r) = 0, \tag{A.5}
\]
\[
a(t)^2 N(t,r)^2 P + \dot{a}(t)^2 + 2a(t)\ddot{a}(t) - 2a(t)\dot{a}(t)\frac{\dot{N}(t,r)}{N(t,r)} = 0. \tag{A.6}
\]
From the Hamiltonian constraint we obtain that if we do not want to constrain the form of
the free function \(P(1/N(t,r)^2)\) we need to constrain \(N(t,r) = N(t)\). Using it we recover the usual equation of motion from the flat FLRW background, i.e.
\[
3H^2 + P - \frac{2}{N^2} P' = 0, \tag{A.7}
\]
\[
3H^2 + 2\dot{H} + P = 0, \tag{A.8}
\]
\[
\partial_r \lambda = 0, \tag{A.9}
\]
where \(H(t) = \dot{a}(t)/(N(t)a(t))\). Therefore, we can note that it is possible to obtain the flat FLRW solutions in a smooth limit by imposing proper boundary conditions for \(F(t,r)\) at spatial infinity.

### B Dynamical FLRW background without auxiliary constraints

The dynamical FLRW background can also be achieved in models without the presence of auxiliary constraints. However, in general, the degeneracy constraints are quite cumbersome to solve. In models like Cuscuton \([17, 18]\) etc. the constraint structure is fundamentally different. Besides the usual six first class constraints related to the spatial diffeomorphism invariance there is an additional first class constraint related to \(\pi_N\) and two second class constraints \(\mathcal{H}_0\) and a new tertiary one \(C\). For SCG models as
\[
S = \int d^4x \sqrt{h} NL(N, K_{ij}, h_{ij}, D_k, t) \tag{B.1}
\]
degenerate conditions have been derived in [4] in order to obtain a MMG theory. Note, that while, in general, $\pi_N$ might not be anymore first class there will be a specific combination $\hat{\pi}_N$ which remains first class besides the six first class constraints coming from the spatial diffeomorphism invariance [4]. The total Hamiltonian at the FLRW background can be written as

$$H_T|_{FLRW} = \int d^3x \left( \mathcal{H}(a,p_a,N) + u_N\hat{\pi}_N + u_0\mathcal{H}_0 + u_C\mathcal{C} \right)|_{FLRW}$$  \hspace{1cm} (B.2)$$

where we have used that the momentum constraint is trivial at the FLRW background. Therefore, even if the tertiary constraints vanish at the background level $C|_{FLRW} = 0$ this will not lead to a dynamical background since we are left with the two constraints $\hat{\pi}_N$ and the Hamiltonian constraint $\mathcal{H}_0$. Since at the FLRW background all constraints will commute with itself $\mathcal{H}_0$ becomes first class since $\hat{\pi}_N$ is first class. Therefore, it is quite challenging to obtain a dynamical FLRW background. Besides requiring a trivial tertiary constraint $C|_{FLRW} = 0$ we have to require that either $\hat{\pi}_N$ or $\mathcal{H}_0$ become trivial at the background level. The most straightforward way is to construct a Hamiltonian constraint which becomes trivially at the FLRW background leading to a non-standard cosmology.

As discussed in [6] models of the form

$$H = \int d^3x \left[ V(h_{ij},\pi^{ij},D_k) + N\mathcal{H}_0(h_{ij},\pi^{ij},D_k) - 2\sqrt{h}N^kD^j\left(\hat{\pi}^{kj}\sqrt{h}\right) \right]$$  \hspace{1cm} (B.3)$$

where

$$\{\mathcal{H}_0(x),\mathcal{H}_0(y)\} \approx 0$$  \hspace{1cm} (B.4)$$
do just have two tensor degrees of freedom. Therefore, if $\mathcal{H}_0|_{FLRW} = 0$ we obtain a minimally modified gravity model with a dynamical FLRW background. One easy example is given by the following toy model

$$\mathcal{L} = \frac{1}{2}\sqrt{h}N^2(K^2 - K^2) - N\mathcal{H}_0(h_{ij},D_k)$$  \hspace{1cm} (B.5)$$

where $\mathcal{H}_0(h_{ij},D_k)$ can be any generic function as long as it does vanish at the FLRW background as $\mathcal{H}_0 = \sqrt{h}R$. However, if we add matter the Hamiltonian constraint is, in general, not anymore trivial. Therefore, including conventional matter the theory has still just one dynamical degree of freedom at the background level. Furthermore, the Hamiltonian constraint enforces that at the background level the matter energy density vanishes. All in all, these models are highly pathological.

Note, that if we consider a Hamiltonian constraint which is also trivial at the linear level as $\mathcal{H}_0 = \sqrt{h}R^{ij}R_{ij}$ then the linear perturbations around the FLRW break down, i.e. there are three degrees of freedom (2 tensor + 1 scalar) at the linear level. It is the same issue which we have discussed in the case of the auxiliary constraints in section 3.2.

Last, it might be interesting to check if it is possible to obtain MMG models with dynamical dark energy by generalizing the ansatz (B.1) by for instance including $\dot{N}$ as in [11] or breaking the spatial diffeomorphism invariance to check if there are viable models in the context of cosmology.
C Schultz-Sorkin action

In order to describe dust we use the Schultz-Sorkin action \[ 44 \]. Using the implementation as in \[ 45 \]

\[
S_{\text{mat}} = - \int d^4x \left( \sqrt{-g} \rho_m(n) + J^\alpha \partial_\alpha \varphi \right),
\]

\[
\rho_m(n) = \mu_0 n,
\]

\[
n = \sqrt{\frac{J^\alpha J_\alpha}{g}},
\]

where \( J^\alpha \) is a vector of weight one and \( \varphi, n \) and \( \rho_m \) are scalar fields. Up to linear order the vector \( J^\alpha \) and the scalar field \( \varphi \) can be expressed as

\[
J^0 = N_0 + \delta j_0,
\]

\[
J^k = \delta^{kj} \partial_j \delta j,
\]

\[
\varphi = -\mu \int^t d\tau N(\tau) - \mu_0 v_m,
\]

where \( N_0 \) is the number of dust particles at the background level with \( \delta^3 \rho_m = \mu_0 N_0 \). Further, it is convenient to replace \( \delta j_0 \) with the gauge invariant matter overdensity \( \delta_m \) via

\[
\frac{\delta j_0}{N_0} = \delta_m - 3 Hv_m + 3 \xi.
\]

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