Algebraic approximation and the Mittag-Leffler theorem for minimal surfaces

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Abstract. In this paper, we prove a uniform approximation theorem with interpolation for complete conformal minimal surfaces with finite total curvature in the Euclidean space $\mathbb{R}^n$ ($n \geq 3$). As application, we obtain a Mittag-Leffler type theorem for complete conformal minimal immersions $M \to \mathbb{R}^n$ on any open Riemann surface $M$.

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1. Introduction

Holomorphic approximation and interpolation is a fundamental subject in complex analysis and plays an important role in several fields of Mathematics. In particular, it has been fundamental by means of the classical Enneper-Weierstrass representation formula in the development of the theories of approximation and interpolation for conformal minimal surfaces in the Euclidean space $\mathbb{R}^n$; a classical field of geometry. We refer to [5] for a survey of recent results in this subject.

We start by recalling the following two seminal theorems in complex analysis from the late 19th Century.

• Runge’s theorem (1885): if $K \subset \mathbb{C}$ is a compact set such that $\mathbb{C} \setminus K$ is connected, then every holomorphic function on a neighborhood of $K$ can be approximated uniformly on $K$ by entire polynomials [26]. Mergelyan’s theorem from 1951 ensures that it suffices to ask the function to be continuous on $K$ and holomorphic on the interior $\bar{K}$ of $K$ [22].

• Mittag-Leffler’s theorem (1884): if $A \subset \mathbb{C}$ is a closed discrete subset and if $f$ is a meromorphic function on a neighborhood of $A$, then there is a meromorphic function $\tilde{f}$ on $\mathbb{C}$ such that $\tilde{f}$ is holomorphic on $\mathbb{C} \setminus A$ and $\tilde{f} - f$ is holomorphic at every point of $A$ [23]. This a sort of dual to the Weierstrass theorem from 1876 ensuring that for any map $r : A \to \mathbb{N}$ there is an entire function having a zero of order $r(a)$ at each point $a \in A$ and vanishing nowhere else [27].

The aim of this paper is to provide analogues of these results in the global theory of minimal surfaces in $\mathbb{R}^n$ for $n \geq 3$ (see Theorems 1.2 and 1.3).

The aforementioned theorems admit several generalizations in complex analysis and algebraic geometry; we refer to the survey of Fornæss, Forstneric, and Wold [14] for a review of this classical but still very active subject. Concerning meromorphic functions on compact Riemann surfaces, we recall the following extension of Runge’s theorem, including interpolation, which dates back to the early decades of modern Riemann surface theory.
Theorem 1.1 (Behnke-Stein [9], Royden [25]). Let \( E \) be a nonempty finite set in a compact Riemann surface \( \Sigma \). If \( K \subset \Sigma \setminus E \) is a Runge compact subset, if \( f \) is a meromorphic function on a neighborhood of \( K \), and if \( D \) is a finite divisor with the support in \( K \), then for any \( \epsilon > 0 \) there is a meromorphic function \( \tilde{f} \) on \( \Sigma \) such that \( \tilde{f} \) is holomorphic on \( \Sigma \setminus E \) except for the poles of \( f \) in \( K \), \( |f - \tilde{f}| < \epsilon \) on \( K \), and the divisor of \( \tilde{f} - f \) is a multiple of \( D \) in a neighborhood of \( K \).

The natural counterpart of meromorphic functions in minimal surface theory are complete minimal surfaces with finite total curvature (we refer e.g. to [11, 24, 8, 28] for background on these surfaces). Indeed, if \( X : M \to \mathbb{R}^n \) is a complete conformal minimal immersion with finite total curvature from an open Riemann surface \( M \), then \( M \) is biholomorphic to \( \Sigma \setminus E \) where \( \Sigma \) and \( E \) are as in Theorem 1.1. Moreover, the exterior derivative \( dX \) of \( X : \Sigma \setminus E \to \mathbb{R}^n \) (which coincides with its \((1,0)\)-part \( \partial X \) since \( X \) is harmonic) is holomorphic and extends meromorphically to \( \Sigma \) with an effective pole at each point of \( E \) (see [16, 11] or [24]). These surfaces are since the early works by Osserman in the 1960s a major focus of interest in the global theory of minimal surfaces.

The following analogue for conformal minimal surfaces in \( \mathbb{R}^n \) \((n \geq 3)\) of the Behnke-Stein-Royden theorem is a simplified version of our main result (see Theorem 6.1 for a more precise statement including Mergelyan approximation and control of the flux).

Theorem 1.2 (Runge’s theorem for complete minimal surfaces with finite total curvature). Let \( \Sigma \) be a compact Riemann surface and \( \emptyset \neq E \subset \Sigma \) be a finite subset. Also let \( K \subset \Sigma \setminus E \) be a smoothly bounded, Runge compact domain and let \( E_0 \) and \( \Lambda \) be a pair of disjoint (possibly empty) finite sets in \( K \). If \( X : K \setminus E_0 \to \mathbb{R}^n \) \((n \geq 3)\) is a complete conformal minimal immersion with finite total curvature, then for any \( \epsilon > 0 \) and any integer \( r \geq 0 \) there is a conformal minimal immersion \( Y : \Sigma \setminus (E \cup E_0) \to \mathbb{R}^n \) satisfying the following conditions.

(i) \( Y \) is complete and has finite total curvature.

(ii) \( Y - X \) extends harmonically to \( K \) and \( |Y - X| < \epsilon \) on \( K \).

(iii) \( Y - X \) vanishes at least to order \( r \) at every point of \( E_0 \cup \Lambda \).

Since \( X : K \setminus E_0 \to \mathbb{R}^n \) is complete and has finite total curvature, it is a proper map (see [18]), and hence \( \lim_{p \to E_0} |X(p)| = +\infty \). Likewise, \( Y : \Sigma \setminus (E \cup E_0) \to \mathbb{R}^n \) is also proper by condition (i): we emphasize that, in view of (ii) and (iii), we have that \( \lim_{p \to E_0} |Y(p) - X(p)| = 0 \).

Theorem 1.2 is known in the particular case when \( n = 3 \) and either \( \Lambda = \emptyset \) (see [21]) or \( E_0 = \emptyset \) (see [2]). The methods in [21, 2] rely strongly on the spinor representation formula for minimal surfaces in \( \mathbb{R}^3 \), a tool that is no longer available in higher dimensions. We point out that our proof works in arbitrary dimension, also for \( n = 3 \). Theorem 1.2 is the first known approximation or interpolation result by complete minimal surfaces with finite total curvature in \( \mathbb{R}^n \) for \( n > 3 \).

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A compact subset \( K \) of an open Riemann surface \( M \) is said to be Runge (or holomorphically convex) in \( M \) if the complement \( M \setminus K \) has no relatively compact connected components.
We now pass to consider holomorphic functions on arbitrary open Riemann surfaces. In 1948 Florack [13], by building on the methods developed by Behnke and Stein in [9], provided analogues to the Mittag-Leffler and the Weierstrass theorems in this more general framework. Likewise, in 1958 Bishop [10] extended the Runge-Mergelyan theorem to any open Riemann surface; in this case the approximation takes place in Runge compact subsets (see [13] Theorems 3.8.1 and 5.4.4 for a general statement including interpolation). In this direction and as application of Theorem 1.2, we also obtain in this paper the following analogue for minimal surfaces of the Mittag-Leffler theorem that also includes approximation of Runge-Bishop type with interpolation (see Theorem 7.1 for a more precise statement).

**Theorem 1.3** (Mittag-Leffler’s theorem for minimal surfaces). Let $M$ be an open Riemann surface, $A \subset M$ be a closed discrete subset, and $U \subset M$ be a locally connected, smoothly bounded closed neighborhood of $A$ whose connected components are all Runge compact sets. If $X : U \setminus A \to \mathbb{R}^n$ ($n \geq 3$) is a complete conformal minimal immersion whose restriction to each connected component of $U \setminus A$ has finite total curvature, then there exists a complete conformal minimal immersion $Y : M \setminus A \to \mathbb{R}^n$ such that the map $Y - X$ is harmonic at every point of $A$.

Furthermore, given $\epsilon > 0$, a closed discrete subset $\Lambda$ of $M$ with $\Lambda \subset U \setminus A$, and a map $r : A \cup \Lambda \to \mathbb{N}$, the immersion $Y$ can be chosen such that $|Y - X| < \epsilon$ on $U$ and $Y - X$ vanishes at least to order $r(p)$ at each point $p \in A \cup \Lambda$.

The assumption that $U$ is locally connected is clearly necessary for the last statement in the theorem concerning approximation and interpolation.

In the particular case when $A = \emptyset$, Theorem 1.3 is an analogue of the aforementioned Runge-Bishop theorem with jet interpolation and follows easily from the results in [1]; see also [7, 3, 4]. The methods in these sources rely strongly on power complex analytic tools coming from modern Oka theory (we refer to Forstnerič [15] for a comprehensive monograph on the subject). On the other hand, a similar result to Theorem 1.3 in case $n = 3$ and $\Lambda = \emptyset$ was obtained in [20], again using the spinor representation formula for minimal surfaces which is only available in $\mathbb{R}^3$. Theorem 1.3 is the first known result of its kind for $A \neq \emptyset$ and $n > 3$, even without asking $Y$ to be complete.

As has been made apparent in this introduction, the results we provide in this paper subsume most of the currently known results in the theories of approximation and interpolation for conformal minimal surfaces in $\mathbb{R}^n$, including the somehow simpler case $n = 3$. At this time we do not know, for instance, whether the immersion $Y$ in Theorem 1.2 can be chosen to be an embedding when $n \geq 5$ and $X|_{\Lambda}$ is injective; the corresponding result for general minimal surfaces (without taking care of the total curvature) was obtained in [6]. On the other hand, our method of proof also works for null holomorphic curves in the complex Euclidean space $\mathbb{C}^n$ ($n \geq 3$), and hence the analogous results for these objects of Theorems 1.2 and 1.3 hold true. It does not seem to us, however, that the approach in this paper could be adapted to deal with more general families of directed holomorphic immersions of open Riemann surfaces as those in [4]; nevertheless, we expect that it could be useful to study some particular instances having good algebraic properties.
Method of proof. The proof of Theorem 1.2 follows the standard approach of controlling the periods of the Weierstrass data, but it presents important innovations. We begin by proving in Section 3 a (local) Mergelyan theorem for complete minimal surfaces with finite total curvature (see Theorem 3.1). For, we adapt the techniques in [4, 6], using the ellipticity of the null quadric $A^{n-1}$ of $\mathbb{C}^n$ (see (2.5)) and sprays generated by the flows of complete vector fields along it, which have been developed in the compact case (i.e., when $E_0 = \emptyset$). This step is not required if the set $K$ in Theorem 1.2 is a strong deformation retract of $\Sigma \setminus E$.

Next, we obtain in Section 4 an extension of the Behnke-Stein-Royden theorem (Theorem 1.1) in which extra control on the divisor of the approximating function is provided; see Proposition 4.1. This result, which may be of independent interest, is key to ensure the completeness of the immersion $Y$ in Theorem 1.2, as well as to deal with the special case $n = 3$. In Section 5 we introduce the period dominating sprays that will be used in the proof of the main theorem (see Lemma 5.3); the main novelties here are that the sprays are of multiplicative nature and that, instead of working with the null quadric $A^{n-1}$, we consider its biholomorphic copy

$$S^{n-1}_\ast = \left\{ u = (u_1, \ldots, u_n) \in \mathbb{C}^n \setminus \{0\} : u_1 u_2 = \sum_{j=3}^{n} u_j^2 \right\}.$$ 

The special geometry of this quadric enables us to approximate, in a simple way, meromorphic maps $u = (u_1, \ldots, u_n): K \to S^{n-1}_\ast$, defined on a Runge compact set $K$ of an open Riemann surface $\Sigma \setminus E$ as in Theorem 1.2 by meromorphic maps $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n): \Sigma \setminus E \to S^{n-1}_\ast$. For, we first approximate $u_1$ by some $\hat{u}_1$ and then approximate the $(n-2)$-tuple $(u_3, \ldots, u_n)$ by a suitable meromorphic map $(\hat{u}_3, \ldots, \hat{u}_n)$; doing this in the right way, the function $\hat{u}_2$ defined on $\Sigma \setminus E$ by

$$\hat{u}_2 = \frac{\sum_{j=3}^{n} \hat{u}_j^2}{\hat{u}_1}$$

completes the task. With the mentioned tools at hand, we prove Theorem 6.1 (and hence Theorem 1.2) in Section 6. At this point, the main concern is to control the divisors of all the approximating functions at each step of the construction, this enables us to avoid the appearance of branch points and guarantee the completeness of the resulting immersion, while controlling the periods and ensuring the approximation condition. In this stage we shall systematically use the Hurwitz theorem from 1895 (see [17] or e.g. [12, §VII.2.5, p. 148]) associating the zeros of a convergent sequence of holomorphic functions with the ones of its limit function.

Finally, we prove Theorem 1.3 (and its more precise version Theorem 7.1) in Section 7 by a recursive application of Theorem 6.1, combined with a standard procedure for ensuring the completeness of the limit immersion.

2. Preliminaries and notation

We write $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $i = \sqrt{-1}$, and denote by $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. We shall use the symbols $\Re$ and $\Im$ to denote, respectively, the real and the imaginary part, and identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. We denote by $| \cdot |$ the Euclidean norm in $\mathbb{R}^n$. Given maps $f, g: X \to Y$ between sets, we write...
$f \equiv g$ to mean that $f(x) = g(x)$ for all $x \in X$; we write $f \neq g$ otherwise. The uniform norm (or sup norm) of a map $f : X \to \mathbb{R}^n$ on $X$ is the non-negative number

$$\|f\|_X = \sup\{|f(x)| : x \in X\}.$$  

If $f, g : X \to \mathbb{R}^n$ are maps, the notation $f \approx g$ shall mean that $\|f - g\|_X$ is so close to 0 that no significant deviation between $f$ and $g$ can be found in a given argumentation. In this case we say that $f$ approximates $g$ on $X$.

**Definition 2.1.** Given a set $X$, a map $X \to \mathbb{C}^n$ is said to be full if its image lies in no affine hyperplane of $\mathbb{C}^n$. Full maps $X \to \mathbb{C}P^n$ are defined in the same way.

Assume that $X$ is a topological space. A Jordan arc in $X$ is an embedding $[0, 1] \to X$; an open Jordan arc in $X$ is an embedding $(0, 1) \to X$. Continuous maps $C : S^1 \to X$ are said to be closed curves in $X$; if in addition $C$ is an embedding then the closed curve is said to be simple or a Jordan curve. Usually we shall identify arcs and curves with their image. We denote by $H_1(X, \mathbb{Z})$ the first homology group with integer coefficients on $X$.

A smooth surface is said to be open if it is not compact and has no boundary. Throughout this paper surfaces are considered to have no boundary unless the contrary is indicated. Assume that $M$ is an (open) smooth surface.

**Definition 2.2.** A nonempty (possibly disconnected) compact set $S$ in $M$ is said to be Runge if $M \setminus S$ has no relatively compact components.

**Definition 2.3.** A nonempty (possibly disconnected) compact set $S$ in $M$ is called admissible if it is of the form $S = K \cup \Gamma$, where $K$ is a (possibly empty) finite union of pairwise disjoint compact domains with smooth boundaries in $M$ and $\Gamma = S \setminus K$ is a (possibly empty) union of finitely many pairwise disjoint Jordan curves in $S \setminus K$ and smooth Jordan arcs in $S \setminus K$ meeting $K$ only in their endpoints (or not at all) and such that their intersections with the boundary $bK$ of $K$ are transverse.

**Definition 2.4.** Let $S = K \cup \Gamma$ be a connected admissible subset in $M$ with $K \neq \emptyset$, and fix a point $p_0 \in K = S$ and a (possibly empty) finite subset $A \subset K \setminus \{p_0\}$ of cardinal $m \in \mathbb{Z}_+$. A family of smooth curves

$$\{C_j : j = 1, \ldots, l\}, \quad l = m + \dim H_1(S, \mathbb{Z}),$$

is said to be a skeleton of $S$ based at $(p_0, A)$ if the following conditions hold.

(A1) $C_j : [0, 1] \to S$ is a Jordan arc with $C_j(0) = p_0$, $C_j(1) \in A$, and $C_j([0, 1]) \cap A = \emptyset$, $j = 1, \ldots, m$, and $A = \{C_1(1), \ldots, C_m(1)\}$.

(A2) $C_j : S^1 \to S$ is a closed curve containing $p_0$ and disjoint from $A$, $j = m + 1, \ldots, l$. These curves do not need to be simple.

(A3) $\{C_{m+1}, \ldots, C_l\}$ determines a basis of the homology group $H_1(S, \mathbb{Z})$.

(A4) $C = \bigcup_{j=1}^l C_j$ is a strong deformation retract of $S$.

(A5) There is a Jordan arc $\gamma_j \subset (C_j \cap K \setminus A) \setminus (\bigcup_{i \neq j} C_i)$ such that $C_j|_{C_j^{-1}(\gamma_j)}$ is injective, $j = 1, \ldots, l$.

By basic topology, every admissible subset $S \subset M$ in the assumptions of Definition 2.4 carries skeletons based at any pair $(p_0, A)$ as above. Furthermore, if $\Gamma = \emptyset$ then...
the skeleton can be chosen such that $C \subset \hat{K}$ and $C_i \cap C_j = \{p_0\}$ for all $i \neq j$. On the other hand, if $S$ is Runge in $M$ then $C$ is Runge in $M$ as well by (A4).

2.1. Divisors and function spaces. Given a set $X$, we denote by $\text{Div}(X)$ the free commutative group of finite divisors of $X$ with multiplicative notation:

$$\text{Div}(X) = \left\{ \prod_{j=1}^{k} q_j^{n_j} : k \in \mathbb{N}, q_j \in X, n_j \in \mathbb{Z} \right\}.$$

Here, $q^0 = 1$ for all $q \in X$. Given $D = \prod_{j=1}^{k} q_j^{n_j} \in \text{Div}(X)$, the set $\text{supp}(D) = \{q_j : n_j \neq 0\} \subset X$ is said to be the support of $D$. The divisor $D$ is said to be effective if $n_j \geq 0$ for all $j = 1, \ldots, k$. We write $D_1 \geq D_2$ to mean that $D_1 D_2^{-1}$ is effective.

Let $M$ and $N$ be a pair of complex manifolds and $S \subset M$ be a subset. We denote by $\mathcal{C}^0(S, N)$ the space of continuous maps $S \to N$, and write $\mathcal{C}^0(S) = \mathcal{C}^0(S, \mathbb{C})$. As it is customary, we denote by $\mathcal{O}(S, N)$ the space of all holomorphic maps from some neighborhood of $S$ in $M$ (depending on the function) into $N$.

We assume in the sequel that $M$ is a Riemann surface (either open or compact). For any subset $S \subset M$ we denote $\mathcal{O}(S) = \mathcal{O}(S, \mathbb{C})$, whereas $\mathcal{O}_\infty(S)$ will denote the space of all meromorphic functions on some neighborhood of $S$ in $M$. For a finite subset $E \subset S$, we denote

$$\mathcal{O}_\infty(S|E) = \mathcal{O}_\infty(S) \cap \mathcal{O}(S \setminus E);$$

i.e., $\mathcal{O}_\infty(S|E)$ is the space of all meromorphic functions on a neighborhood of $S$ which have poles (if any) only at points in $E$. Likewise, we denote by $\Omega(S)$ the space of all holomorphic 1-forms on some neighborhood of $S$ in $M$, $\Omega_\infty(S)$ the space of all meromorphic 1-forms on some neighborhood of $S$ in $M$, and $\Omega_\infty(S|E) = \Omega_\infty(S) \cap \Omega(S \setminus E)$.

Assume that the set $S \subset M$ is compact. For any $f \in \mathcal{O}_\infty(S)$, $f \neq 0$, $f \neq \infty$, we call $Z(f)$ and $P(f)$ the (finite) sets of zeros and poles of $f$ in $S$, and write $s_p \in \mathbb{N}$ for the zero or pole order of $f$ at $p$ for all $p \in Z(f) \cup P(f)$. We set

$$[f]_0 = \prod_{p \in Z(f)} p^{s_p} \quad \text{and} \quad [f]_\infty = \prod_{p \in P(f)} p^{s_p}$$

the effective divisors in $\text{Div}(S)$ of zeros and poles of $f$ in $S$, respectively, and

$$[f] := \frac{[f]_0}{[f]_\infty} \in \text{Div}(S)$$

the divisor of $f$ in $S$. (We do not use the more customary parenthetical notation for divisors in order to avoid ambiguities.) We use the same notation for the corresponding divisors of a nonzero meromorphic 1-form on $S$.

For each effective divisor $D \in \text{Div}(S)$ we set

$$\mathcal{O}_D(S) = \{ f \in \mathcal{O}(S) : [f] \geq D \}.$$

Assume that $S = K \cup \Gamma \subset M$ is admissible in the sense of Definition 2.3. We denote

$$\mathcal{A}(S, N) = \mathcal{C}^0(S, N) \cap \mathcal{O}(\hat{S}, N) \quad \text{and} \quad \mathcal{A}(S) = \mathcal{A}(S, \mathbb{C}).$$
For an effective divisor $D \in \text{Div}(\hat{S})$, we denote

\[ A_D(S) = A(S) \cap O_D(U), \]

where $U \subset \hat{S}$ is any compact neighborhood of $\text{supp}(D)$. We also call

\[ A_\infty(S) = C^0(S, \mathbb{C} P^1) \cap C^0(bS, \mathbb{C}) \cap O_\infty(\hat{S}), \]

and, given a finite set $E \subset \hat{S}$,

\[ A_\infty(S|E) = A_\infty(S) \cap O(\hat{S} \setminus E). \]

Note that if $f \in A_\infty(S)$ then, by the identity principle, $f$ has at most finitely many poles all which lie in $\hat{S}$.

Let $n \geq 3$ be an integer. We denote by

\[ \mathcal{G}_*^{n-1} = \left\{ (u_1, \ldots, u_n) \in \mathbb{C}^n \setminus \{0\} : u_1 u_2 = \sum_{j=3}^{n} u_j^2 \right\}. \]

The punctured complex quadric $\mathcal{G}_*^{n-1}$ is canonically identified with the punctured null quadric

\[ \mathfrak{A}_*^{n-1} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} : \sum_{j=1}^{n} z_j^2 = 0 \right\} \]

by the natural linear biholomorphism $\Xi : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}$ given by

\[ \Xi(z_1, z_2, \ldots, z_n) := (-z_1 + i z_2 z_1 + i z_2^2, z_3, \ldots, z_n), \]

which maps $\mathfrak{A}_*^{n-1}$ into $\mathcal{G}_*^{n-1}$. The punctured null quadric $\mathfrak{A}_*^{n-1}$ is a complex homogeneous manifold, and hence an Oka manifold (see [15, Example 5.6.2]).

For any map $f = (f_1, \ldots, f_n) \in C^0(S, \mathbb{C} P^1)^n$ write

\[ f^{-1}(\infty) = \bigcup_{j=1}^{n} f_j^{-1}(\infty), \]

and denote

\[ O_\infty(S, \mathfrak{A}_*^{n-1}) = \{ f \in O_\infty(S)^n : f(S \setminus f^{-1}(\infty)) \subset \mathfrak{A}_*^{n-1} \} \]

and

\[ A_\infty(S, \mathfrak{A}_*^{n-1}) = \{ f \in A_\infty(S)^n : f(S \setminus f^{-1}(\infty)) \subset \mathfrak{A}_*^{n-1} \}. \]

Note that $f^{-1}(\infty) \subset \hat{S}$ for all $f \in A_\infty(S, \mathfrak{A}_*^{n-1})$. Given a finite set $E \subset \hat{S}$, we denote

\[ O_\infty(S|E, \mathfrak{A}_*^{n-1}) = O_\infty(S, \mathfrak{A}_*^{n-1}) \cap O(S \setminus E, \mathfrak{A}_*^{n-1}) \]

and

\[ A_\infty(S|E, \mathfrak{A}_*^{n-1}) = A_\infty(S, \mathfrak{A}_*^{n-1}) \cap O(\hat{S} \setminus E, \mathfrak{A}_*^{n-1}). \]

The spaces $O_\infty(S, \mathfrak{A}_*^{n-1}), A_\infty(S, \mathfrak{A}_*^{n-1}), O_\infty(S|E, \mathfrak{A}_*^{n-1})$, and $A_\infty(S|E, \mathfrak{A}_*^{n-1})$ are defined in the same way.
2.2. Minimal surfaces in $\mathbb{R}^n$. Let $M$ be an open Riemann surface. A map $X = (X_1, \ldots, X_n): M \to \mathbb{R}^n$ ($n \geq 3$) is a conformal minimal immersion if and only if $X$ is harmonic, and its complex derivative $\partial X \in \Omega(M)^n$ (i.e., the $(1,0)$-part of the exterior differential $dX$ of $X$) vanishes nowhere on $M$ and satisfies $\sum_{j=1}^n |\partial X_j|^2 \equiv 0$. Given a holomorphic 1-form $\theta$ on $M$ with no zeros, the last two conditions are equivalent to $\partial X/\theta \in \mathcal{O}(M, \mathbb{A}^{n-1})$ or to $\Xi(\partial X/\theta) \in \mathcal{O}(M, \mathcal{S}^{n-1})$; see (2.14). Moreover, in this case $X: M \to \mathbb{R}^n$ is given by the formula

$$M \ni p \mapsto X(p_0) + \Re \int_{p_0}^p 2\partial X$$

for any fixed base point $p_0 \in M$. This is known in the literature as the Enneper-Weierstrass representation formula for minimal surfaces in $\mathbb{R}^n$ (see e.g. [24]). A map $X: S \to \mathbb{R}^n$ from a subset $S \subset M$ is said to be a conformal minimal immersion if it extends as a conformal minimal immersion to some open neighborhood of $S$ in $M$ (depending on $X$).

**Definition 2.5.** We say that a conformal minimal immersion $X: M \to \mathbb{R}^n$ is full if the holomorphic map $\partial X/\theta: M \to \mathbb{C}^n$ is full for any holomorphic 1-form $\theta$ vanishing nowhere on $M$.

The group homomorphism $\text{Flux}_X: H_1(M, \mathbb{Z}) \to \mathbb{R}^n$ given by

$$\text{Flux}_X(\gamma) = 2 \int_\gamma \Im(\partial X) = -2i \int_\gamma \partial X$$

for every loop $\gamma \subset M$, is said to be the flux map (or just the flux) of $X$. A conformal minimal immersion $X: M \to \mathbb{R}^n$ is said to be of finite total curvature (acrostically, FTC) if

$$\int_M K dA > -\infty,$$

where $K$ is the Gauss curvature of the Riemannian metric

$$ds^2 = 2 \sum_{j=1}^n |\partial X_j|^2$$

induced on $M$ by the Euclidean one via $X$, and $dA$ is the area element of $ds^2$.

The conformal minimal immersion $X: M \to \mathbb{R}^n$ is called complete if the metric $ds^2$ is complete in the classical sense of Riemannian geometry. Assume now that $M$ is a Riemann surface with compact boundary $bM \subset M$ (possibly $bM = \emptyset$). If $M$ carries a complete conformal minimal immersion $X: M \to \mathbb{R}^n$ of FTC (here, $X$ is complete if and only if $X \circ \gamma$ has infinite Euclidean length for any divergent arc $\gamma: [0,1) \to M$), then the classical results by Huber [16] and Chern-Osserman [11] impose the following conditions.

(i) $M$ is conformally equivalent to $R \setminus E$, where $R$ is a compact Riemann surface with compact boundary $bR$ and $E \subset R \setminus bR$ is a finite subset.

(ii) $\partial X$ extends meromorphically to $R$ with an effective pole at each point of $E$.

Conversely, if $R$ and $E$ are as in (i) and if $X: R \setminus E \to \mathbb{R}^n$ is a conformal minimal immersion satisfying (ii), then $X$ is complete and of FTC. This discussion justifies the following definition.
Definition 2.6. Let $S = K \cup \Gamma$ be an admissible set in an open Riemann surface $M$ (see Definition 2.3), let $E \subset K = \hat{S}$ be a finite subset, and let $\theta$ be a nowhere vanishing holomorphic 1-form on $M$. A generalized complete conformal minimal immersion of finite total curvature from $S \setminus E$ into $\mathbb{R}^n$ $(n \geq 3)$ is a pair $(X, f\theta)$, where $X : S \setminus E \rightarrow \mathbb{R}^n$ is a $C^1$ map that is a conformal minimal immersion on $K \setminus E$ and $f$ is a map in $A_\infty(S|E, \mathbb{R}^{n-1})$ satisfying the following conditions.

(i) $f\theta = 2\partial X$ holds on $K \setminus E$.
(ii) For any smooth path $\alpha$ in $M$ parameterizing a connected component of $\Gamma$, we have $\Re(\alpha^*(f\theta)) = \alpha^*(dX) = d(X \circ \alpha)$.
(iii) $f^{-1}(\infty) = E$; see (2.7).

We denote by

\[
(2.9) \quad \text{GCCMI}_\infty(S|E, \mathbb{R}^n)
\]

the space of all generalized complete conformal minimal immersions of FTC from $S \setminus E$ into $\mathbb{R}^n$, and write $2\partial \hat{X} = f\theta$ for all $\hat{X} = (X, f\theta)$ in $\text{GCCMI}_\infty(S|E, \mathbb{R}^n)$. If $\hat{X} = (X, f\theta), \hat{Y} = (Y, g\theta) \in \text{GCCMI}_\infty(S|E, \mathbb{R}^n)$, the notation $\hat{X} \approx \hat{Y}$ on $S$ means that $X - Y \approx 0$ and $f - g \approx 0$ on $S \setminus E$, and hence are continuous on $S$. For $\hat{X} = (X, f\theta) \in \text{GCCMI}_\infty(S|E, \mathbb{R}^n)$, the flux map $\text{Flux}_{\hat{X}} : H_1(S \setminus E, \mathbb{Z}) \rightarrow \mathbb{R}^n$ of $\hat{X}$ is the group homomorphism given by

$$\text{Flux}_{\hat{X}}(\gamma) := \int_\gamma \Im(f\theta) = -i \int_\gamma f\theta, \quad \text{for every loop } \gamma \subset S.$$ 

Remark 2.7. Given $\hat{X} = (X, f\theta) \in \text{GCCMI}_\infty(S|E, \mathbb{R}^n)$, the (well defined) map \([f_1 : \cdots : f_n] : S \setminus E \rightarrow \mathbb{C}P^{n-1}\) extends holomorphically to the punctures $E \subset \hat{S}$, and hence it lies in $A(S, \mathbb{C}P^{n-1})$.

Finally, as above, we denote by

\[
(2.10) \quad \text{CCMI}_\infty(S|E, \mathbb{R}^n)
\]

the subspace of those immersions $(X, f\theta) \in \text{GCCMI}_\infty(S|E, \mathbb{R}^n)$ such that $X$ extends as a conformal minimal immersion to some neighborhood of $S \setminus E$ in $M$; in this case, we just write $X$ for $(X, f\theta = 2\partial X)$.

3. Mergelyan’s theorem for complete minimal surfaces of finite total curvature

In this section we prove a Mergelyan type theorem for complete minimal surfaces with FTC, asserting that generalized complete conformal minimal immersions of finite total curvature on a finitely punctured admissible subset, $S \setminus E_0$, can be approximated uniformly on $S$ by complete conformal minimal immersions on a neighborhood of $S \setminus E_0$.

Theorem 3.1. Let $M$ be an open Riemann surface, $\theta$ be a nowhere vanishing holomorphic 1-form on $M$, and $\hat{S} = K \cup \Gamma \subset M$ be a connected admissible subset (see Definition 2.3). Also let $E_0$ and $\Lambda$ be a pair of disjoint finite subsets of $\hat{S}$ and let $n \geq 3$ be an integer. For any $\hat{X} = (X, f\theta) \in \text{GCCMI}_\infty(S|E_0, \mathbb{R}^n)$, any number
\( \epsilon > 0 \), and any integer \( r \geq 0 \), there is \( Y \in \text{CCMI}_\infty(S|E_0, \mathbb{R}^n) \) satisfying the following conditions.

(i) \( Y \) is full (see Definition \ref{def:full}).

(ii) \( Y - X \) extends to \( S \) as a continuous map and \( \|Y - X\|_S < \epsilon \).

(iii) \( Y - X \) vanishes at least to order \( r \) at every point of \( \Lambda \cup E_0 \).

(iv) Flux \( \text{Flux}_Y = \text{Flux}_X \).

The improvement of \( Y \) with respect to \( X \) is that \( Y \) is a true conformal minimal immersion in a neighborhood of \( S \), and not just a generalized one on \( S \). Moreover, we ensure that \( Y \) is full.

**Proof.** We adapt the arguments in \cite{4} (see also \cite{6}) to the special framework of FTC. We start with the following reduction.

**Claim 3.2.** We can assume that \( K \neq \emptyset \), \( X|_U \) is flat on no component \( U \) of \( K \), and there is a component \( U_0 \) of \( K \) such that \( X|_{U_0} \) is full.

Recall that \( X|_U \) is flat if and only if \( X(U) \) lies in an affine plane in \( \mathbb{R}^n \), or equivalently, \( (\partial X/\theta)(U) \subset \mathbb{R}^{n-1} \) lies in a complex line in \( \mathbb{C}^n \).

**Proof.** We first show that we can assume that \( K \neq \emptyset \) and \( f \) (and hence, \( X \)) is full on a component \( U_0 \) of \( K \). Indeed, if \( K = \emptyset \), then \( S = \Gamma \) is either a Jordan arc or a Jordan curve; recall that \( S \) is connected. Choose a closed disc \( U_0 \) in \( M \) so small that \( U_0 \cap S \) is a Jordan arc and \( X \) is approximately constant there. Up to a slight deformation of \( \hat{X} \) on a small neighborhood of \( U_0 \cap S \) (see \cite{1, Lemma 3.3} for details on how to make the deformation), we can extend it, with the same name and flux, to \( S \cup U_0 \) as a generalized conformal minimal immersion such that

\[
(3.1) \quad f|_{U_0} \text{ is full.}
\]

Suppose now that \( U \subset K \) is a component and \( X|_U \) is flat (obviously \( U \neq U_0 \)). This means that \( f(U) \) is contained in a complex plane \( L = \mathbb{C}v \), where \( v \in \mathbb{R}^{n-1} \). Up to a rigid motion in \( \mathbb{R}^n \) we can suppose that \( v = (1, i, 0, \ldots, 0) \), and hence \( f = (f_1, if_1, 0, \ldots, 0) \), where \( f_1 \in \mathcal{A}_\infty(U|E_0) \) vanishes nowhere on \( U \) and \( f_1 \theta \) is exact on \( U \setminus E_0 \) (see Subsec. \ref{subsec:exactness}). Fix \( r_0 \geq r + \sum_{p \in E_0 \cap U} \text{Ord}_p(f_1) \) and call \( \Delta_U = \prod_{p \in (E_0 \cup \Lambda) \cap U} B^{r_0} \), where \( \text{Ord}_p(f_1) > 0 \) denotes the pole order of \( f_1 \) at \( p \in E_0 \cap U \). Fix a point \( q \in U \setminus E_0 \). Since the complex space \( \mathcal{O}_{\Delta_U}(U) \) (see \ref{subsec:complex_spaces}) has infinite dimension, there is \( h \in \mathcal{O}_{\Delta_U}(U), \ h \neq 0 \), satisfying the following conditions.

(a) The 1-forms \( h^2f_1 \theta \) and \( hf_1 \theta \) are exact on \( U \).

(b) \( \int_{U}^p(hf_1, h^2f_1) \theta = 0 \) for all \( p \in U \cap (E_0 \cup \Lambda) \).

(c) The functions 1, \( h, h^2 \) are \( \mathbb{C} \)-linearly independent.

Set

\[
f_\zeta := (f_1(1 - \zeta^2h^2), if_1(1 + \zeta^2h^2), 2\zeta hf_1, 0, \ldots, 0), \quad \zeta \in \mathbb{C},
\]

and define

\[
X_\zeta(p) = X(q) + \Re \int_{q}^{p} f_\zeta \theta, \quad p \in U \setminus E_0.
\]
Algebraic approximation and the Mittag-Leffler theorem

Note that, by (a), \( f_\zeta \theta \) is exact, and hence \( X_\zeta \) is well defined and \( \text{Flux}_{X_\zeta} = \text{Flux}_{X|_U} = 0 \). Moreover, if \( \zeta \neq 0 \) is chosen close enough to 0 in \( \mathbb{C} \), then \( f_\zeta \in \mathcal{A}_\infty(U|_E_0, \mathfrak{A}_n^{-1}) \), \( X_\zeta \in \text{GCCMI}_n(U|_E_0, \mathbb{R}^n) \), \( \|f_\zeta - f\|_U \approx 0 \), and \( \|X_\zeta - X\|_U \approx 0 \) (see Sect. 2 for notations). Furthermore, \( X_\zeta - X \) vanishes at least to order \( r \) at every point of \((\Lambda \cup E_0) \cap U\) (see (b) and recall that \( h \in \mathcal{O}_{\Delta_U}(U) \)), and \( f_\zeta(U) \) is not contained in a complex line (see (c)). Up to replacing \( X|_U \) by \( X_\zeta \) in \( \hat{X} \), and then slightly modifying \( f|_r \) on a small neighborhood of \( U \) preserving the smoothness and the flux map of \( X \) (use \([1\text{, Lemma 3.3}])\), we can suppose that \( f(U) \) is not contained in a complex line. To finish the proof, we apply the same procedure in each component of \( K \) on which \( f \) is flat.

Assume, as we may, that the hypotheses of Claim \([5.22]) hold.

Denote by \( m \in \mathbb{Z}_+ \) the cardinal of \( E_0 \cup \Lambda \). Fix a point \( p_0 \in \hat{K} \neq \emptyset \) and let \( \{C_1, \ldots, C_l\}, l = m + \dim H_1(S, \mathbb{Z}) \), be a skeleton of \( S \) based at \((p_0, E_0 \cup \Lambda)\); see Definition \([2.4])\). Write \( C = \bigcup_{j=1}^l C_j \) and denote

\[
\mathcal{C}^0(C, f) = \{ h \in \mathcal{C}^0(C \setminus E_0, \mathfrak{A}_n^{-1}) : h - f \in \mathcal{C}^0(C, \mathbb{C}^n) \}
\]

(see Subsec. \([2.1])\) and let \( \mathcal{Q} = \{Q_1, \ldots, Q_l\} : \mathcal{C}^0(C, f) \to (\mathbb{C}^n)^l \) be the period map defined by

\[
\mathcal{C}^0(C, f) \ni h \mapsto \mathcal{Q}(h) = \left( \int_{C_j} (h - f) \theta \right)_{j=1, \ldots, l}.
\]

Fix a function \( g \in \mathcal{O}(M) \) with \([g] = \prod_{p \in E_0} p^{o(p)} \), where \( o(p) = \max \{\text{Ord}_p(f_j) : j = 1, \ldots, n\} \); such a function exists by the classical Weierstrass theorem (on the existence of holomorphic functions with prescribed divisor on an open Riemann surface); see \([13])\). Since \( f \in \mathcal{A}_\infty(S|E_0, \mathfrak{A}_n^{-1}) \), we have that

\[
f_0 := gf \in \mathcal{A}(S, \mathfrak{A}_n^{-1}).
\]

Consider a family of complete holomorphic vector fields \( V_1, \ldots, V_m \) on \( \mathbb{C}^n \), vanishing at 0, tangential to \( \mathfrak{A}^{-1}_n \) along \( \mathfrak{A}^{-1}_n \), and such that \( \{V_1(z), \ldots, V_m(z)\} \) spans the tangent space \( T_z \mathfrak{A}^{-1}_n \) for all \( z \in \mathfrak{A}^{-1}_n \) (see e.g. \([1\text{, Example 4.4}])\). Obviously, \( m \geq n \).

For each \( j = 1, \ldots, l \), let \( \gamma_j \subset C_j \setminus \Lambda \setminus \{p_0\} \) be a Jordan arc satisfying \((A3)\) in Definition \([2.4])\). Since \( \gamma_j \) lies in a component of \( K \) and \( f \) is assumed to be flat on no component of \( K \), there are pairwise distinct points \( p_{1, j}, \ldots, p_{m, j} \in \gamma_j \) such that

\[
\{V_1(f_0(p_{1, j})), \ldots, V_m(f_0(p_{m, j}))\} \text{ spans } \mathbb{C}^n;
\]

see \([5.3]) and take into account the geometry of \( \mathfrak{A}^{-1}_n \). Also choose functions \( f_{i, 1, j}, \ldots, f_{i, m, j} \in \mathcal{C}^0(C, \mathbb{C}) \), with pairwise disjoint supports, such that \( f_{i, j} \) lies in the relative interior of \( \text{supp}(f_{i, j}) \subset \gamma_j \) and

\[
\int_{C_j} f_{i, j}(V_i \circ f_0) \theta \approx V_i(f_0(p_{i, j})) \quad \text{for all } i = 1, \ldots, m.
\]

Set \( F = (f_{i, 1, j}, \ldots, f_{i, m, j})_{j=1, \ldots, l} \in \mathcal{C}^0(C, (\mathbb{C}^m)^l) \). Denote by \( \phi_i^1 \) the flow of \( V_i \) over \( \mathfrak{A}^{-1}_n \), \( i = 1, \ldots, m \). Let \( \Phi_F : \mathbb{C}^m)^l \times S \times \mathfrak{A}_n^{-1} \to \mathfrak{A}_n^{-1} \) be defined by

\[
\Phi_F(\zeta, p, z) := (\phi_{1, 1}^1 f_{1, 1}(p) \circ \cdots \circ \phi_{\zeta, 1}^m f_{\zeta, 1}(p) \circ \cdots \circ \phi_{1, l}^1 f_{1, l}(p) \circ \cdots \circ \phi_{\zeta, l}^m f_{\zeta, l}(p))(z),
\]
where $\zeta = ((\zeta_{i,j})_{i=1,...,m})_{j=1,...,l}$, and the spray with core $f_0$ given by

$$\Phi_{F,f_0}: (\mathbb{C}^m)^l \times S \to \mathfrak{U}_s^{n-1}, \quad \Phi_{F,f_0}(\zeta, p) := \Phi_{F}(\zeta, p, f_0(p)).$$

By the choice of $f_{i,j}$, we have that $\Phi_{F,f_0}(\zeta, \cdot)/g$ is continuous on $C \setminus E_0$ and coincides with $f$ on a neighborhood of $E_0 \cup \Lambda$ in $C$, namely, in $C \setminus \bigcup_{i,j} \text{supp}(f_{i,j})$, and hence $\Phi_{F,f_0}(\zeta, \cdot)/g \in C^0(C, f)$, for all $\zeta \in (\mathbb{C}^m)^l$. Consider the new period map $Q^* = (Q_1^*, \ldots, Q_l^*): (\mathbb{C}^m)^l \to (\mathbb{C}^n)^l$ given by

$$Q^*(\zeta) = Q\left(\frac{\Phi_{F,f_0}(\zeta, \cdot)}{g}\right), \quad \zeta \in (\mathbb{C}^m)^l.$$

The spray $\Phi_{F,f_0}$ is $Q^*$-dominating at $\zeta = 0$ in the sense that the Jacobian matrix

$$W := \left(\left(\frac{\partial Q^*}{\partial \zeta_{i,j}}\big|_{\zeta=0}\right)_{i=1,...,m}\right)_{j=1,...,l}$$

has maximal rank equal to $nl$. Indeed, since

$$\left(\frac{\partial Q^*_k}{\partial \zeta_{i,j}}\big|_{\zeta=0}\right)_{i=1,...,m} = (0)_{m \times n} \quad \text{if } k \neq j$$

and, by (3.5),

$$W_j := \left(\frac{\partial Q^*_j}{\partial \zeta_{i,j}}\big|_{\zeta=0}\right)_{i=1,...,m} \approx (V_i(f_0(p_{i,j}))_{i=1,...,m} \quad \text{for all } j = 1, \ldots, l,$n

the block structure of $W$, (4.4), and (3.3) guarantee that $\text{rank}(W) = \sum_{j=1}^l \text{rank}(W_j) = nl$, provided that the approximation in (3.5) is sufficiently close.

So, there is a small closed ball $V$ around the origin of $(\mathbb{C}^m)^l$ such that the holomorphic map

$$Q^*: V \to Q^*(V)$$

is a submersion with $Q^*(0) = Q(f) = 0$; take into account (3.3). Moreover, we choose $V$, as we may by continuity, so small that

$$\Phi_{F,f_0}(\zeta, \cdot) \approx f_0 \quad \text{for all } \zeta \in V;$$

recall that $f_0$ is the core of $\Phi_{F,f_0}$. Write $f = (f_1, \ldots, f_n)$ and fix $r_0 \in \mathbb{N}$ with

$$r_0 \geq r + \sum_{p \in E_0} \sum_{i=1}^n \text{Ord}_p(f_i),$$

where $r \geq 0$ is the integer given in the statement of the theorem and $\text{Ord}_p(\cdot)$ means pole order at $p \in E_0$; recall that $f_i \in A_S(S|E_0)$ for all $i = 1, \ldots, n$. Next, since $C$ is a Runge subset of any neighborhood of $S$ (see (A4) in Definition 2.2), the classical Runge-Mergelyan theorem with jet-interpolation enables us to approximate each $f_{i,j}$ uniformly on $C$ by a function $h_{i,j} \in \mathcal{O}(S)$ satisfying

$$[h_{i,j}] \geq \Delta_1 := \prod_{p \in E_0 \cup \Lambda} p^{r_0+1};$$

recall that $f_{i,j} \equiv 0$ on a neighborhood of $E_0 \cup \Lambda$. Consider the map $H = (h_{1,j}, \ldots, h_{m,j})_{j=1,...,l} \in (\mathcal{O}_{\Delta_1}(S)^m)^l$; see (2.2). Likewise, since $\mathfrak{U}_s^{n-1}$ is an Oka
manifold (see [1] Example 4.4; see Forstnerič [15] for a comprehensive monograph in the subject), by (3.3) there is

\( h_0 \in \mathcal{O}(S, \mathbb{A}_n^{-1}) \) such that

\[
(3.10) \quad h_0 \approx f_0 \quad \text{on} \quad S \quad \text{and} \quad h_0 - f_0 \text{vanishes to order } r_0 + 1 \text{ on } E_0 \cup \Lambda.
\]

Therefore, \( \Phi_{H,h_0} \approx \Phi_{F,f_0} \) uniformly on \( V \times S \), where \( \Phi_{H,h_0} \in \mathcal{O}(V \times S, \mathbb{A}_n^{-1}) \) is the holomorphic spray with core \( h_0 \) given by

\[
\Phi_{H,h_0}(\zeta, p) := \left( \phi_{\zeta_1,1,1}(p) \circ \cdots \circ \phi_{\zeta_m,1,h_1,(p)} \circ \cdots \circ \phi_{\zeta_j,j,1}(p) \circ \cdots \circ \phi_{\zeta_m,h_m,(p)} \right)(h_0(p)).
\]

We emphasize that \( \Phi_{H,h_0}(\zeta, \cdot) \) is a holomorphic function on a neighborhood of \( S \); this is the key achievement for the proof. On the other hand, (3.3), (3.8), (3.9), (3.10), and [1, Lemma 2.2] ensure that

\[
(3.11) \quad \Phi_{H,h_0}(\zeta, \cdot) - f \in \mathcal{A}_{\Delta_2}(S)^n \quad \text{for each } \zeta \in V,
\]

where \( \Delta_2 := \prod_{p \in E_0 \cup \Lambda} P^r \); see [23]. In particular, \( \frac{\Phi_{H,h_0}(\zeta, \cdot)}{g} \in \mathcal{C}^0(C, f) \) and the period map

\[
\hat{Q} : V \to (\mathbb{C}^m)^l, \quad \zeta \mapsto \hat{Q}(\zeta) = \mathcal{Q} \left( \frac{\Phi_{H,h_0}(\zeta, \cdot)}{g} \right),
\]

is well defined. If all the approximations are chosen close enough, then, by the Cauchy estimates, \( \hat{Q} \approx Q^* \) on \( V \), and there is \( \zeta_0 \in \hat{V} \) close to the origin such that \( \hat{Q}(\zeta_0) = 0 \) (see [3.6]), and the map

\[
\hat{f} := \frac{\Phi_{H,h_0}(\zeta_0, \cdot)}{g} \in \mathcal{O}_\infty(S|E_0, \mathbb{A}_n^{-1})
\]

satisfies the following conditions.

(A) \( \hat{f} - f \in \mathcal{A}_{\Delta_2}(S)^n \); see [3.11].

(B) \( \hat{f} - f \) is exact on \( S \); use that \( Q(\hat{f}) = \hat{Q}(\zeta_0) = 0 \) and (3.2).

(C) \( \hat{f} - f \approx 0 \) on \( S \); use (3.7).

(D) \( \hat{f} : S \setminus E_0 \to \mathbb{A}_n^{-1} \subset \mathbb{C}^n \) is full; use (3.1) and (C).

It follows that the conformal minimal immersion \( Y : S \setminus E_0 \to \mathbb{R}^n \) given by

\[
Y(p) = X(p_0) + \Re \int_{p_0}^p \hat{f} \theta, \quad p \in S \setminus E_0,
\]

satisfies the conclusion of the theorem. Indeed, conditions (i)–(iv) follow easily from (A)–(D). On the other hand, the condition \( Y \in \text{CCMI}_\infty(S|E_0, \mathbb{R}^n) \), and in particular \( Y \) is complete, is implied by (A) and the fact that \( X \in \text{GCCMI}_\infty(S|E_0, \mathbb{R}^n) \).

\[\Box\]

4. An extension of the Behnke-Stein-Royden theorem

In this section we prove a Behnke-Stein-Royden type theorem (see Theorem 1.1) with extra control on the divisors of the approximating functions; see conditions (ii) and (iii) in the following result. Recall the notation introduced in Subsection 2.1.
Proposition 4.1. Let $\Sigma$ be a compact Riemann surface, let $E \subset \Sigma$ be a nonempty finite subset, and let $S \subset \Sigma \setminus E$ be a Runge admissible subset. For any $f \in A_\infty(S)$, any effective divisor $D_1 \in \text{Div}(\bar{S})$, any real number $\delta > 0$, and any $k \in \mathbb{N}$, there exists $\tilde{f} \in \mathcal{O}_\infty(\Sigma) \cap \mathcal{O}(\Sigma \setminus (S \cup E))$ satisfying the following conditions.

(i) $\tilde{f} - f \in A_{D_1}(S)$ and $\|\tilde{f} - f\|_S < \delta$.
(ii) $[\tilde{f}|_{\Sigma \setminus (S \cup E)}] = D_0^k$ for some divisor $D_0 \in \text{Div}(\Sigma \setminus (S \cup E))$.
(iii) $[\tilde{f}]_{\infty} \geq \prod_{p \in E} p^k$.

In particular, $\tilde{f} \in \mathcal{O}_\infty(\Sigma|E \cup \text{supp}([f]_{\infty}))$.

Proof. We may assume without loss of generality that $f \in \mathcal{O}_\infty(S) \cap \mathcal{O}(\bar{\Sigma} \setminus \bar{S})$. Indeed, since $\Sigma \setminus E$ is an open Riemann surface, the classical Weierstrass theorem [13] gives a function $\varphi \in \mathcal{O}(\Sigma \setminus E)$ with $[\varphi] = [f|_S]_{\infty}$: recall that $f^{-1}(\infty)$ lies in $\bar{S}$ and consists of finitely many points. It turns out that $\varphi f \in A(S)$ and vanishes nowhere on $\text{supp}([f|_S]_{\infty})$. Mergelyan theorem with jet interpolation then provides $\varphi_0 \in \mathcal{O}(\Sigma)$ approximating $\varphi f$ on $S$ and satisfying $[\varphi_0 - \varphi f] \geq D_1[f|_S]_{\infty}$. If the proposition is valid for $\varphi_0/\varphi \in \mathcal{O}_\infty(S) \cap \mathcal{O}(\bar{\Sigma} \setminus \bar{S})$, then the solution provided for this function solves the proposition for $f$ whenever that the approximation $\varphi_0 \approx f$ is close enough.

So, assume that $f \in \mathcal{O}_\infty(S) \cap \mathcal{O}(\bar{\Sigma} \setminus \bar{S})$. Since $f \in \mathcal{O}_\infty(S)$, Theorem [13] gives us a function $f_0 \in \mathcal{O}_\infty(\Sigma) \cap \mathcal{O}(\Sigma \setminus (S \cup E))$ satisfying the following conditions.

(P1) $f_0 - f \in \mathcal{O}_{D_1}(S)$.
(P2) $\|f_0 - f\|_S < \delta$.

It turns out that $f_0$ satisfies condition (i); however, it does not need to satisfy (ii) or (iii). The next step in the proof is to find $h_0 \in \mathcal{O}_\infty(\Sigma|E)$ such that $h_0 f_0$ satisfies (i) and (ii).

If $f_0$ vanishes nowhere on $\Sigma \setminus (S \cup E)$, then it suffices to choose $h_0 \equiv 1$ (and $D_0 = 1$). Otherwise, write $[f_0|_{\Sigma \setminus (S \cup E)}] = \prod_{j=1}^s p_j^{m_j}$, where $p_1, \ldots, p_s$ are pairwise distinct points and $m_j > 0$ for all $j \in \{1, \ldots, s\}$ ($s \geq 1$), and set

$$E_1 = \{p_j : m_j \text{ is odd}\} \subset \Sigma \setminus (S \cup E).$$

If $E_1 = \emptyset$, then, again, it suffices to choose $h_0 \equiv 1$ (and $D_0 = \prod_{j=1}^s p_j^{m_j/2}$). Assume that $E_1 \neq \emptyset$. Since $\Sigma \setminus E$ is an open Riemann surface, there is $g \in \mathcal{O}(\Sigma \setminus E)$ with

$$[g] = \prod_{p \in E_1} p;$$

see [13]. Moreover, by a standard application of Runge’s theorem for holomorphic functions into $\mathbb{C} \setminus \{0\}$ (an Oka manifold), we can assume in addition that

$$\|g - 1\|_S < \frac{1}{2} \quad \text{on } S.$$ 

Consider the open Riemann surface

$$R = \{(p, u) \in (\Sigma \setminus E) \times \mathbb{C} : u^2 = g(p)\}$$
and notice that $R$ admits a canonical analytical compactification, namely, $\hat{R}$. By analytical continuation arguments, the projection $\pi: \hat{R} \to \Sigma$, $\pi(p, u) = p$, is a 2-sheeted branched covering, $R = \pi^{-1}(\Sigma \setminus E) = \hat{R} \setminus \pi^{-1}(E)$, and
\[
\pi^{-1}(E_1) \text{ is the ramification set of } \pi|_R.
\]
Denote by $A: \hat{R} \to \hat{R}$ the deck transformation of $\pi$ and observe that $A(p, u) = (p, -u)$ for all $(p, u) \in R$, hence $\pi^{-1}(E_1)$ is the fixed point set of $A|_R: R \to R$ as well. In view of (4.3), it turns out that $\pi^{-1}(S) = S^+ \cup S^-$ where $S^+ \text{ and } S^-$ are pairwise disjoint Runge compact subsets of $R$, $A(S^+) = S^-$, and $\pi|_{S^\pm}: S^\pm \to S$ is a biholomorphism.

Given $\delta' > 0$, Theorem 1.1 furnishes us with a function $h \in \mathcal{O}_\infty(\hat{R}) \cap \mathcal{O}(R)$ satisfying the following conditions.
\begin{itemize}
  \item $\|h - (\pm 1)||_{S^\pm} < \delta'$.
  \item $h$ has simple zeros at all points in $\pi^{-1}(E_1)$.
  \item $|h|_{S^\pm} - (\pm 1)| \geq D_1^+([f_0 \circ \pi]|_{S^\pm})$ where $D_1^+ \in \text{Div}(S^\pm)$ is the only divisor with $\pi(D_1^+) = D_1$.
\end{itemize}

Up to replacing $h$ by $(h - h \circ A)/2$ we can also assume that $h \circ A = -h$, and hence
\[
h^2 = h_0 \circ \pi \quad \text{for some } h_0 \in \mathcal{O}_\infty(\Sigma|E).
\]
The following conditions are satisfied.

(4.4) $\|h_0 - 1||_S < (\delta')^2 + 2\delta'$.
(4.3) $[h_0]_{\Sigma|E} = D^2 \prod_{p \in E_1} p$ for some divisor $D \in \text{Div}(\Sigma \setminus (S \cup E))$; see (4.3) and (4.4).
(4.5) $[h_0 - 1]_0 \geq D_1 f_0 |_{S\infty}$.

Set $f_1 = h_0 f_0$. Since $f_1 - f = (h_0 - 1) f_0 + (f_0 - f)$, properties (P1) and (P5) ensure that $f_1 - f \in \text{O}_{D_1}(S)$. Since $S$ is compact, this, (P2), and (P3) guarantee that
\[
\|f_1 - f||_S < \delta
\]
provided that $\delta' > 0$ is chosen sufficiently small, and hence $f_1$ satisfies condition (i). By the definition of $E_1$ in (4.4) we have
\[
[f_0]_{\Sigma\setminus(S\cup E)} = (D')^2 \prod_{p \in E_1} p,
\]
for some effective divisor $D' \in \text{Div}(\Sigma \setminus (S \cup E))$, and hence (P4) gives that
\[
[f_1]_{\Sigma\setminus(S\cup E)} = (DD')^2 \prod_{p \in E_1} p^2.
\]
Thus, $f_1$ satisfies condition (ii).

Finally, to complete the proof we shall find a function $h_1 \in \mathcal{O}_\infty(\Sigma|E)$ such that $\tilde{f} = h_1^2 f_1$ satisfies the conclusion of the proposition. By Theorem 1.1 for any $p \in E$ and any $\epsilon > 0$, there is $F_p \in \mathcal{O}_\infty(\Sigma|\{p\})$, $F_p \neq 0$, such that
\[
[F_p] \geq D_1 [f_1|_S|_\infty \quad \text{and} \quad \|F_p||_S < \epsilon.
\]
In particular, $F_p$ has an effective pole at $p$. Choose an integer $m > k + k_0$, where $k$ is the number in the statement of the proposition and $k_0 \geq 0$ is the maximum
among the zero orders of $f_1$ at the points in $E$, then the function
\[ h_1 = 1 + \sum_{p \in E} F_p^m \]
satisfies the following conditions in view of (4.10).

(P6) $\|h_1^2 - 1\|_S < \ell^2 \epsilon^2 + 2\ell \epsilon$, where $\ell$ is the cardinal of $E$.

(P7) $[h_1^2 - 1]_0 \geq D_1[f_1|S]_\infty$.

Reasoning as above, it is easily seen that $\tilde{f} = h_1^2 f_1$ satisfies (i) and (ii) provided $\epsilon > 0$ is chosen sufficiently small. Since $F_p$ has an effective pole at $p$ for each $p \in E$, we have that $h_1$ has a pole of order at least $m$ at each point $p \in E$. Thus, $\tilde{f}$ has a pole of order at least $2k + k_0 \geq k$ at each point $p \in E$, and hence $\tilde{f}$ satisfies (iii). This completes the proof. □

5. Multiplicative sprays in $\mathcal{G}_s^{n-1}$

Let $M$ be an open Riemann surface, and let $S \subset M$ be a connected, smoothly bounded compact domain. Also let $E_0$ and $\Lambda$ be a pair of disjoint finite subsets of $S$. Let $n \geq 3$ be an integer, recall the hyperquadrics $\mathcal{A}_s^{n-1}$ and $\mathcal{G}_s^{n-1}$ in $\mathbb{C}^n$ and the canonical biholomorphism $\Xi: \mathcal{A}_s^{n-1} \to \mathcal{G}_s^{n-1}$; see (2.1). (2.3), and (2.6). Consider a full map $f \in \mathcal{O}_\infty(S|E_0, \mathcal{A}_s^{n-1})$, see Definition 2.1 define
\[ u := \Xi \circ f \in \mathcal{O}_\infty(S|E_0, \mathcal{G}_s^{n-1}), \]
and write $u = (u_1, \ldots, u_n)$. Fix an integer $r \geq 0$ and fix the divisors in $\text{Div}(K)$
\[ \Delta = \prod_{p \in E_0 \cup \Lambda} p^r \quad \text{and} \quad \Delta_0 = \Delta(\prod_{i=1}^n [u_i]_\infty). \]

We denote
\[ \mathcal{O}_\Delta(S, u) = \{ v \in \mathcal{O}_\infty(S|E_0, \mathcal{G}_s^{n-1}) : v_j - u_j \in \mathcal{O}_\Delta(S), j = 1, \ldots, n \}, \]
where $v = (v_1, \ldots, v_n)$; see (2.3). Obviously, $u \in \mathcal{O}_\Delta(S, u)$.

**Definition 5.1.** We denote by $S_2(\mathbb{D})$ the space of all maps $(s_1, s_2, s_3) \in \mathcal{O}(\mathbb{D}, \mathcal{G}_s^{2})$ satisfying the following conditions.

- $s_j$ vanishes nowhere on $\mathbb{D}$ and $s_j(0) = 1$ for all $j = 1, 2, 3$.
- $s'_j(0)s'_3(0) \neq 0$.
- Either $s_3 \equiv 1$ or $s'_3(0) \neq 0$.

**Remark 5.2.** For any function $g \in \mathcal{O}_\Delta_0(S)$ with $\|g\|_S < 1$, any map $(s_1, s_2, s_3) \in S_2(\mathbb{D})$, and any map $v = (v_1, \ldots, v_n) \in \mathcal{O}_\Delta(S, u)$, the map
\[ \left( (s_1 \circ g)v_1, (s_2 \circ g)v_2, ((s_3 \circ g)u_1)_{i=3,\ldots,n} \right) \in \mathcal{O}_\Delta(S, u). \]

Take a point $p_0 \in \hat{S}$ and a skeleton $\{ C_1, \ldots, C_l \}$, $l = m + \dim H_1(S, \mathbb{Z})$, of $S$ based at $(p_0, E_0 \cup \Lambda)$; see Definition (2.4). We choose, as we may, $\{ C_1, \ldots, C_l \}$ to
be Jordan arcs or curves such that \( C_i \cap C_j = \{ p_0 \} \) for all \( i \neq j \in \{ 1, \ldots, l \} \) and \( C = \bigcup_{j=1}^{l} C_j \subset \tilde{S} \) is a strong deformation retract of \( S \). Denote
\[
C^0(C, u) = \{ h \in C^0(C \setminus E_0, \mathcal{S}_n^{-1}) : h - u \in C^0(C, \mathbb{C}^n) \}.
\]
It turns out that \( \mathcal{O}_\Delta(S, u) \subset C^0(C, u) \). Fix a nowhere vanishing holomorphic 1-form \( \theta \) on a neighborhood of \( S \) in \( M \), and consider the period maps
\[
\mathcal{P} : C^0(C, u) \to (\mathbb{C}^n)^l, \quad \mathcal{P}(h) = \left( \int_{C_j} (h - u) \theta \right)_{j=1, \ldots, l},
\]
\[
\mathcal{P}^{1,2} : C^0(C, u) \to (\mathbb{C}^2)^l, \quad \mathcal{P}^{1,2}(h) = \left( \int_{C_j} (h_1 - u_1) \theta \right)_{j=1, \ldots, l},
\]
where \( h = (h_1, \ldots, h_n) \in C^0(C, u) \).

In what follows, we shall use the following notation. Given points \( w = ((w_{i,j})_{i=1, \ldots, n})_{j=1, \ldots, l}, z = ((z_{i,j})_{i=1, \ldots, n})_{j=1, \ldots, l} \in (\mathbb{C}^n)^l \), we write
\[
w \cdot z = \sum_{j=1}^{l} \sum_{i=1}^{n} w_{i,j} z_{i,j}
\]
and denote \( \rho_{\mathcal{P}^{1,2}} = \{ \zeta \in (\mathbb{C}^n)^l : |\zeta|^2 = \zeta \cdot \zeta \leq \rho^2 \} \) for all \( \rho > 0 \).

The main goal of this section is to show that we can embed the given map \( u \in \mathcal{O}_\infty(S|E_0, \mathcal{S}_n^{-1}) \) into some particular period dominating spray, say of class \( \mathcal{S}^2_n \), of maps in \( \mathcal{O}_\infty(S|E_0, \mathcal{S}_n^{-1}) \).

**Lemma 5.3.** Let \( M, S, E_0, \Lambda, u, \) and \( \theta \) be as above. There is a map \( h = ((h_{i,j})_{i=1, \ldots, n})_{j=1, \ldots, l} \in (\mathcal{O}_\Delta(S)^n)^l, h \not\equiv 0, \) for which the following statement holds true.

(B1) For any number \( \rho_0 \in ]0, 1/\|h\|_S[ \) and any map \( s = (s_1, s_2, s_3) \in \mathcal{S}_2(\mathbb{D}) \), the map \( \Psi_s : \rho_{\mathcal{P}^{1,2}} \to \mathcal{O}_\infty(S|E_0, \mathcal{S}_n^{-1}) \) given by
\[
\Psi_s(\zeta) = (s_1(\zeta \cdot h)u_1, s_2(\zeta \cdot h)u_2, s_3(\zeta \cdot h)u_3)_{i=3, \ldots, n}
\]
assumes values in \( \mathcal{O}_\Delta(S, u) \) and is period dominating at \( \zeta = 0 \); the latter meaning that \( \mathcal{P} \circ \Psi_s : \rho_{\mathcal{P}^{1,2}} \to (\mathbb{C}^n)^l \) satisfies
\[
\text{rank} \left( \frac{\partial(\mathcal{P} \circ \Psi_s)}{\partial \zeta} \right)_{\zeta=0} = \begin{cases} n l & \text{if } s_3(0) \neq 0 \text{ (i.e., } s_3 \not\equiv 1) \\ 2 l & \text{if } s_3(0) = 0 \text{ (i.e., } s_3 \equiv 1) \end{cases}
\]
Therefore, for any \( \epsilon > 0 \) there is \( \rho \in ]0, 1/\|h\|_S[ \) so small that the following conditions are satisfied.

(B2) \( \Psi_s(\zeta) : S \setminus E_0 \to \mathcal{S}_n^{-1} \subset \mathbb{C}^n \) is full and \( \| \Psi_s - u \|_S < \epsilon \) for all \( \zeta \in \rho_{\mathcal{P}^{1,2}} \).

(B3) \( \mathcal{P} \circ \Psi_s : \rho_{\mathcal{P}^{1,2}} \to (\mathcal{P} \circ \Psi)(\rho_{\mathcal{P}^{1,2}}) \) is a biholomorphism with \( \mathcal{P}(\Psi_s(0)) = 0 \) if \( s_3 \not\equiv 1 \).

(B4) \( \mathcal{P}^{1,2} \circ \Psi_s : \rho_{\mathcal{P}^{1,2}} \to (\mathbb{C}^2)^l \) is a holomorphic submersion satisfying \( \mathcal{P}^{1,2}(\Psi_s(0)) = 0 \) if \( s_3 \equiv 1 \).

Note that \( u = \Psi_s(0) \) is the core map of the spray \( \Psi_s \).
\textbf{Proof.} Assume that we have a continuous map
\begin{equation}
F = ((f_{i,j})_{i=1,\ldots,n})_{j=1,\ldots,l} : C \to (\mathbb{C}^n)^l
\end{equation}
satisfying the following conditions for all \( j \in \{1, \ldots, l\} \).

(a) \( \text{supp}(f_{i,j}) \) is connected and is contained in \( C_j \setminus \{\{p_0\} \cup E_0 \cup \Lambda\} \subset \hat{S} \) for all \( i \in \{1, \ldots, n\} \).

(b) The compact sets \( \text{supp}(f_{i,j}), i \in \{1, \ldots, n\} \), are pairwise disjoint.

Take \( \rho_0 \in ]0, 1/\|F\|_{C[} \) and \( s = (s_1, s_2, s_3) \in S_2(\mathbb{D}) \). Consider the map \( \Psi_F : \rho_0 \mathbb{F}_{n,l} \to C^0(C, u) \) given by
\[
\Psi_F(\zeta) = (s_1(\zeta \cdot F)u_1, s_2(\zeta \cdot F)u_2, s_3(\zeta \cdot F)(u_i)_{i=3,\ldots,n}).
\]
This map assumes values in \( C^0(C, u) \) since \( \Psi_F(\zeta) = u \) on a neighborhood of \( E_0 \) in \( C \); recall that \( \text{supp}(f_{i,j}) \subset S \setminus E_0 \). Furthermore, \( \Psi_F \) is Frechet differentiable.

Write \( \zeta = ((\zeta_{i,j})_{i=1,\ldots,n})_{j=1,\ldots,l} \in \rho_0 \mathbb{F}_{n,l} \) and
\[
\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) : C^0(C, u) \to (\mathbb{C}^n)^l.
\]
The Jacobian matrix
\[
T_{k,j}(F) := \frac{\partial(\mathcal{P}_k \circ \Psi_F)}{\partial(\zeta_{i,j})_{i=1,\ldots,n}} \bigg|_{\zeta = 0}, \quad k, j \in \{1, \ldots, l\},
\]
is the matrix \( 0_{n \times n} \) if \( k \neq j \), whereas for \( k = j \) we have that
\begin{equation}
T_{j,j}(F) = W_j(F) \cdot A,
\end{equation}
where
\begin{equation}
W_j(F) = \left( \int_{C_j} f_{a,j} u_0 \theta_0 \right)_{a,b=1,\ldots,n}.
\end{equation}
and \( A \) is the diagonal matrix of order \( n \) whose diagonal entries starting in the upper left corner are \( s'_1(0), s'_2(0), s'_3(0), \ldots, s'_3(0) \). Therefore, if the condition
\begin{equation}
\det(W_j(F)) \neq 0 \quad \text{for all } j = 1, \ldots, l
\end{equation}
were satisfied, then the block diagonal square matrix of order \( nl \)
\[
\frac{\partial(\mathcal{P} \circ \Psi_F)}{\partial \zeta} \bigg|_{\zeta = 0} = \begin{pmatrix}
T_{1,1}(F) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & T_{l,l}(F)
\end{pmatrix},
\]
would have
\begin{equation}
\text{rank}\left(\frac{\partial(\mathcal{P} \circ \Psi_F)}{\partial \zeta} \bigg|_{\zeta = 0}\right) = nl \quad \text{if} \quad s'_3(0) \neq 0 \quad \text{(i.e., } s_3 \neq 1\text{)}
\end{equation}
and
\begin{equation}
\text{rank}\left(\frac{\partial(\mathcal{P} \circ \Psi_F)}{\partial \zeta} \bigg|_{\zeta = 0}\right) = \text{rank}\left(\frac{\partial(\mathcal{P}^{1,2} \circ \Psi_F)}{\partial \zeta} \bigg|_{\zeta = 0}\right) = 2l \quad \text{if } s_3 \equiv 1.
\end{equation}

We now seek for a continuous map \( F \) as in (5.11) for which (5.13) is satisfied. For, we fix for each \( j \in \{1, \ldots, l\} \) pairwise distinct points \( p_{1,j}, \ldots, p_{n,j} \) in \( C_j \setminus \{p_0\} \) such that
\begin{equation}
\{ u(p_{i,j}) : i = 1, \ldots, n \} \text{ is a basis of } \mathbb{C}^n.
\end{equation}
The existence of such points is ensured by the fullness of $f$; note that, by (5.1), and the analyticity of $u$, the map $u|_{\beta}: \beta \to \mathbb{C}^n$ is full on any Jordan arc $\beta \subset S$. Next, we consider for each $j \in \{1, \ldots, l\}$ a parameterization $C_j: [0, 1] \to C_j \subset \hat{S}$ with $C_j(0) = p_0$ and $C_j(1) = p_j$ (here we write $p_j = p_0$ for all $j \in \{m + 1, \ldots, l\}$), and denote by $t_{i,j} \in (0, 1)$ the point such that $C_j(t_{i,j}) = p_{ij}$ for $i = 1, \ldots, n$. Choose a positive number $\delta > 0$ so small that $[t_{i,j} - \delta, t_{i,j} + \delta] \subset (0, 1)$ for all $i = 1, \ldots, n$, and the arcs $C_j([t_{i,j} - \delta, t_{i,j} + \delta]) \subset \hat{S}$, $i = 1, \ldots, n$, are pairwise disjoint. We then choose continuous functions $f_{i,j}: C \to C$ with support in $C_j([t_{i,j} - \delta, t_{i,j} + \delta])$, satisfying

$$\int_{C_j} f_{i,j} u \theta = \int_{t_{i,j} - \delta}^{t_{i,j} + \delta} f_{i,j}(t) u(C_j(t)) \theta (C_j(t), \dot{C}_j(t)) \, dt = u(p_{ij})$$

for all $i$ and $j$; recall that $\theta$ vanishes nowhere on $S$. In view of (5.7), (5.8), and (5.12), this shows that (5.9) holds true for the map $\Psi \circ F: \rho_1 \mathbb{E}_{n,l} \to (\mathbb{P} \circ \Psi F)(\rho_1 \mathbb{E}_{n,l})$

is a well-defined biholomorphism with $\mathbb{P}(\Psi F(0)) = 0$ if $s_3 \neq 1$, and

$$(5.13) \quad \mathbb{P}^{1,2} \circ \Psi F: \rho_1 \mathbb{E}_{n,l} \to (\mathbb{C}^2)^l$$

is a holomorphic submersion with $\mathbb{P}^{1,2}(\Psi F(0)) = 0$ if $s_3 \equiv 1$.

Fix $r_0 \in \mathbb{N}$ with

$$(5.15) \quad r_0 \geq r + \sum_{p \in E_0} \sum_{i=1}^n \text{Ord}_p(u_i),$$

where $\text{Ord}_p(\cdot)$ means pole order at $p \in E_0$ and $r$ is the integer in (5.2). Since $C$ is a strong deformation retract of $S$, the Runge-Mergelyan theorem with jet-interpolation (see e.g. [15, Theorems 3.8.1 and 5.4.4]) ensures that we may approximate each $f_{i,j}$ uniformly on $C$ by a function $h_{i,j} \in \mathcal{O}(S)$, $h_{i,j} \not\equiv 0$, vanishing to order $r_0$ at every point of $\Lambda \cup E_0$; observe that $f_{i,j} \equiv 0$ on a neighborhood of $\Lambda \cup E_0$. It follows that $h = ( (h_{i,j})_{j=1,\ldots,n} )_{i=1,\ldots,l} \in (\mathcal{O}_{\Delta}(S)^n)^l$; see (2.2) and (5.2). If we define $\Psi_s$ as in the statement of the lemma for this map $h$ and any positive $r_0 < 1/\|h\|_{S}$, it turns out that $\Psi_s(\zeta) \in \mathcal{O}_{\Delta}(S, u)$ for all $\zeta \in \rho_0 \mathbb{E}_{n,l}$; see Remark 5.2. Assuming that $h$ is close enough to $F$ on $C$ and choosing $\rho < \min(\rho_1, 1/\|h\|_{S})$ sufficiently small, then (B1) and (B2) hold; see (5.8), (5.10), and (5.11) and recall that $u = \Psi_s(0)$ is full. Furthermore, in view of (5.13) and (5.14), (B3) and (B4) are satisfied as well provided that $\rho < \min(\rho_1, 1/\|h\|_{S})$ is chosen small enough.

6. Runge’s theorem for complete minimal surfaces of finite total curvature

We now prove the following more precise version of Theorem 1.2. Recall the notation in Section 2.2 in particular, see (2.9) and (2.10).

**Theorem 6.1.** Let $\Sigma$ be a compact Riemann surface (without boundary), $E \subset \Sigma$ be a nonempty finite subset, and $S = K \cup \Gamma \subset \Sigma \setminus E$ be an admissible subset (see
Definition 2.3) that is Runge in $\Sigma \setminus E$. Also let $E_0, \Lambda$ be a pair of disjoint finite subsets of $S$ and let $n \geq 3$ be an integer.

For any $\hat{X} = (X, f\theta) \in \text{GCCMI}_\infty(S|E_0, \mathbb{R}^n)$, any group homomorphism $p : H_1(\Sigma \setminus (E_0 \cup E), \mathbb{Z}) \to \mathbb{R}^n$ with $p|_{H_1(\Sigma \setminus E_0, \mathbb{Z})} = \text{Flux}^\Lambda_X$, any number $\epsilon > 0$, and any integer $r \geq 0$, there is a conformal minimal immersion $Y : \Sigma \setminus (E_0 \cup E) \to \mathbb{R}^n$ satisfying the following conditions.

(i) $Y$ is complete and of finite total curvature.
(ii) $Y - X$ extends to $S$ as a continuous map and $\|Y - X\|_S < \epsilon$.
(iii) $Y - X$ vanishes at least to order $r$ at every point of $\Lambda \cup E_0$.
(iv) $\text{Flux}_Y = p$.

Proof. We begin with the following.

Claim 6.2. There are a Runge admissible subset $S' = K' \cup \Gamma' \subset \Sigma \setminus E$ (see Def. 2.3) and an immersion $\hat{X}' = (X', f\theta) \in \text{GCCMI}_\infty(S'|E_0, \mathbb{R}^n)$ satisfying the following requirements.

(a) $S \subset S'$ and $S'$ is a strong deformation retract of $\Sigma \setminus E$.
(b) $K' \neq \emptyset$ and every component of $\Gamma'$ intersects $K'$.
(c) $K$ is a union of components of $K'$.
(d) $X'|_{K \setminus E_0} = X|_{K \setminus E_0}$ and $\hat{X}'|_{S' \setminus E_0} \approx \hat{X}$.
(e) $\text{Flux}_{\hat{X}'} = p$.

Proof. By elementary topological arguments, since $\Sigma \setminus E$ has finite topology there is a Runge admissible subset $S' = K' \cup \Gamma' \subset \Sigma \setminus E$ satisfying (a), (b), and (c). Such an $S'$ can be found such that $K' \setminus K \neq \emptyset$, $K' \setminus K$ consists of pairwise disjoint closed discs, and every component of $K'$ intersects at most one component of $\Gamma$.

Let $W_1$ denote the union of $S$ and all the components of $K' \setminus K$ intersecting $\Gamma$, and notice that $S$ is a strong deformation retract of $W_1$. Choosing the components (closed discs) of $(W_1 \cap K') \setminus K$ sufficiently small (say, so small that $f$ is close to a locally constant map on $(W_1 \cap K') \setminus K \cap \Gamma$), we can extend $X|_K$ to an immersion $\hat{X}_1 \in \text{GCCMI}(W_1, \mathbb{R}^n)$ such that $\hat{X}_1$ is close to a flat on $W_1 \cap K' \setminus K$, $\hat{X}_1$ is close to $\hat{X}$ on $\Gamma$, and Flux$_{\hat{X}_1} = \text{Flux}_{\hat{X}}$. Indeed, we can for instance choose $\hat{X}_1 = \hat{X}$ outside a small neighborhood $V$ of $W_1 \cap K' \setminus K$ in $W_1$ and to be a slight modification of $\hat{X}$ on $V \setminus ((W_1 \cap K') \setminus K) \subset \Gamma$ which ensures that $\hat{X}_1 \in \text{GCCMI}(W_1, \mathbb{R}^n)$ and the condition on the flux; for, we use [1 Lemma 3.3].

Next, set $W_2 = W_1 \cup K'$ and extend $\hat{X}_1$ to a generalized conformal minimal immersion $\hat{X}_2 \in \text{GCCMI}(W_2, \mathbb{R}^n) \cap \text{CCMI}(K' \setminus W_1)$. Finally, we obtain an immersion $\hat{X}' \in \text{GCCMI}(S'|E_0, \mathbb{R}^n)$ satisfying conditions (d) and (e) by extending $\hat{X}_2$ to the arcs in $\Gamma' \setminus (W_2 \cup \Gamma)$ in such a way that Flux$_{\hat{X}'} = p$; for, we use again [1 Lemma 3.3]. □

Up to replacing $(S, \hat{X})$ by $(S', \hat{X}')$, and then using Theorem 3.1, we can assume that $S$ is a strong deformation retract of $\Sigma \setminus E$, $X \in \text{CCMI}_\infty(S|E_0, \mathbb{R}^n)$, and

\begin{equation}
(6.1) \quad f = 2\partial X/\theta : S \setminus E_0 \to \mathbb{C}^n \quad \text{is full.}
\end{equation}
Furthermore, since \( X \) extends to a neighborhood of \( S \) as a conformal minimal immersion, we can also assume without loss of generality that \( \Gamma = \emptyset \) and \( S = K \) is a connected, smoothly bounded, compact domain.

We assume without loss of generality that the finite set \( \Lambda \subset \hat{S} \setminus E_0 \neq \emptyset \) is nonempty and write

\[
E_0 \cup \Lambda = \{ p_1, \ldots, p_m \}.
\]

Fix a point \( p_0 \in \hat{S} \) and choose a skeleton \( \{ C_1, \ldots, C_l \} \), \( l = m + \dim H_1(S, \mathbb{Z}) \), of \( S \) based at \( (p_0, E_0 \cup \Lambda) \); see Definition [2.2]. We choose the skeleton, as we may since \( S = K \) is connected, such that

\[
C_i \cap \left( \bigcup_{i \neq j = 1}^l C_j \right) = \{ p_0 \} \quad \text{for all} \quad i \in \{1, \ldots, l\}.
\]

It turns out that \( C := \bigcup_{j=1}^l C_j \) is a Runge subset of \( S \setminus E \) that is a strong deformation retract of \( S \setminus E \).

Recall the following classical result; we include a proof for completeness.

**Claim 6.3.** There is a 1-form \( \theta_0 \in \Omega_\infty(S|E) \) vanishing nowhere on \( S \) and having \([\theta_0]_\infty \geq \prod_{p \in E} p\).

**Proof.** Fix any 1-form \( \tau \in \Omega_\infty(S|E) \). By the classical Weierstrass theorem, there is \( h \in \mathcal{O}(S \setminus E) \) with \([h] = [\tau]_S\). Set \( E_1 = \supp([h]) \subset S \). By Proposition [1.1] and Hurwitz’s theorem, there exists a function \( g \in \mathcal{O}_\infty(S|E \cup E_1) \) close to \( 1/h \) on \( S \) with \([g|_S][h] = 1 \) and \([g\tau]_\infty \geq \prod_{p \in E} p \). It suffices to choose \( \theta_0 = g\tau \). \( \square \)

Fix \( \theta_0 \in \Omega_\infty(S|E) \) as in Claim 6.3. Define

\[
(6.3) \quad f_0 := f/\theta_0 \in \mathcal{O}_\infty(S|E_0, \mathfrak{A}_n), \quad u := \Xi \circ f_0 \in \mathcal{O}_\infty(S|E_0, \mathcal{G}^{n-1}),
\]

and write \( u = (u_1, \ldots, u_n) \); see (6.1) and (2.6). Set

\[
(6.4) \quad \Delta = \prod_{p \in E_0 \cup \Lambda} p^r \quad \text{and} \quad \Delta_0 = \Delta \left( \prod_{i=1}^n [u_i]_\infty \right).
\]

(Here, \( r \) is the integer given in the statement of Theorem 6.1.) Denote

\[
(6.5) \quad \Theta := \supp([\theta_0]_0) \subset \Sigma \setminus (S \cup E).
\]

The next stage in the proof is to approximate \( u \), uniformly on \( S \), by a certain map \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_n) \in \mathcal{O}_\infty(\Sigma|E \cup E_0 \cup \Theta, \mathcal{G}^{n-1}) \) such that \( \hat{u}\theta_0 \) vanishes nowhere on \( \Sigma \setminus (E_0 \cup E) \). For, we proceed in two steps: we first approximate \( u_1 \) by a function \( \hat{u}_1 \in \mathcal{O}_\infty(\Sigma|E \cup E_0 \cup \Theta) \) and, after that, we approximate \( (u_2, \ldots, u_n) \) by a suitable map \( (\hat{u}_2, \ldots, \hat{u}_n) \in \mathcal{O}_\infty(\Sigma|E \cup E_0 \cup \Theta)^{n-2} \); the function \( \hat{u}_2 \in \mathcal{O}_\infty(\Sigma|E \cup E_0 \cup \Theta) \) approximating \( u_2 \) will then come forced by the requirement that \( \hat{u} \) assumes values in \( \mathcal{G}^{n-1} \).

We assume, as we may up to slightly enlarging \( S \) if necessary, that \( u_i \) vanishes nowhere on \( bS = S \setminus \hat{S} \) for all \( i \in \{1, \ldots, n\} \); recall that \( u_i \in \mathcal{O}_\infty(S|E_0) \). Consider
the following effective divisor

\[(6.6) \quad Z = \left( \prod_{i=1}^{n} |u_i|^2 |u_i|_{\infty}^2 \right) \in \text{Div}(S)\]

whose support lies in \(\hat{S}\).

Fix \(\epsilon_1 > 0\) to be specified later.

Take pairwise disjoint closed discs \(U_q \subset \Sigma \setminus (S \cup E)\), \(q \in \Theta\), with \(q \in \hat{U}_q\) for all \(q \in \Theta\), and call \(U = \bigcup_{q \in \Theta} U_q\). Take any function \(u_1^* \in \mathcal{O}_\infty(S \cup U|E_0 \cup \Theta)\) such that

\[(6.7) \quad u_1^*|_{S} = u_1 \quad \text{and} \quad |u_1^*|_{[\theta_0]} = 1.\]

By Proposition 4.1 and Hurwitz’s theorem, there is \(u \in q\) satisfying the following conditions for all \(\epsilon \in (6.9)\).

\[(D1) \quad \epsilon \in (6.10), \quad \epsilon \in (6.11), \quad \epsilon \in (6.12)\]

Fix \(\epsilon \in (6.13)\) and call \(\hat{S}_1\).

Consider functions \(v_3, \ldots, v_n\) in \(\mathcal{O}_\infty(S \cup T|E_0)\) such that

\[(6.10) \quad v_i|_{S} = u_i \quad \text{for all} \quad i = 3, \ldots, n, \quad \text{and} \quad \left( \sum_{i=3}^{n} v_i^2 \right)|_{T} = \hat{u}_1|_{T} ;\]

note that, in view of (6.9), such extensions exist even for \(n = 3\). (One may for instance choose \(v_3|_{T} = \sqrt{\hat{u}_1|_{T}}\) and \(v_i|_{T} = 0\) for all \(i \geq 4\).) Likewise, let \(v_1, v_2 \in \mathcal{O}_\infty(S \cup T|E_0)\) be the functions given by

\[(6.11) \quad v_1|_{S} = u_1, \quad v_1|_{T} = \hat{u}_1|_{T} ; \quad \text{and} \quad v_2|_{T} = 1.\]

Fix \(\epsilon_2 < \epsilon_1\) to be specified later.

Proposition 4.1 and Hurwitz’s theorem provide a map \((\hat{u}_3, \ldots, \hat{u}_n) \in \mathcal{O}_\infty(S \cup E \cup E_0)^{n-2}\) satisfying the following conditions for all \(i = 3, \ldots, n\).

\[(D1) \quad \hat{u}_i - v_i \in \mathcal{O}(S \cup T) \quad \text{and} \quad \hat{u}_i - v_i|_{S \cup T} < \epsilon_2.\]

\[(D2) \quad \hat{u}_i|_{S \cup T} - v_i \in \mathcal{O}_Z D^1_i(S \cup T).\]

\[(D3) \quad \hat{u}_i|_{S} = [u_i]; \quad \text{see (6.11)}\].
We claim that if $\epsilon_2 > 0$ is chosen sufficiently small, then the function

$\hat{u}_2 := \frac{\sum_{i=3}^{n} \hat{u}_i^2}{\hat{u}_1}$

satisfies the following conditions.

(E1) $\hat{u}_2 \in \mathcal{O}_\infty(\Sigma|E_0 \cup E)$.

(E2) $\hat{u}_2 - v_2 \in \mathcal{O}(S \cup T)$ and $\|\hat{u}_2 - v_2\|_{S\cup T} < \epsilon_1$.

(E3) $\hat{u}_2|_{S\cup T} - v_2 \in \mathcal{O}_{\Delta[u_2]}(S \cup T)$.

(E4) $[\hat{u}_2|_{S\cup T}] = [\hat{u}_2|s] = [v_2] = [u_2]$.

Indeed, we shall first check (E1). Properties (6.10), (D2), and (D3) ensure for each $i = 3, \ldots, n$ that

\[
[\hat{u}_i^2|_{S\cup T} - v_i^2] = [\hat{u}_i|_{S\cup T} - v_i][\hat{u}_i|_{S\cup T} + v_i] \\
\geq Z\Delta D^4_4[u_i]^{-1} \geq \Delta D^4_4[u_1^2][u_2].
\]

and hence, by (6.3), (6.6), (6.10), and (6.11),

\[
[(\hat{u}_1 \hat{u}_2)|_{S\cup T} - v_1 v_2] = \left[\sum_{i=3}^{n} \hat{u}_i^2|_{S\cup T} - \sum_{i=3}^{n} v_i^2\right] \geq \Delta D^4_4[u_1^2][u_2].
\]

In view of (6.7), (C3), and (6.10), we have $[\hat{u}_1|_{S\cup T}] = [u_1]_0 D^4_4$, and so

\[
(6.12) \quad [\hat{u}_2|_{S\cup T} - \frac{v_1 v_2}{\hat{u}_1}] \geq \Delta D^4_4[u_1]_0[u_2].
\]

It turns out that

\[
(6.13) \quad \hat{u}_2|_{S\cup T} - \frac{v_1 v_2}{\hat{u}_1} \in \mathcal{O}(S \cup T).
\]

Moreover, (6.7), (6.11), and (C3) ensure that $v_1/\hat{u}_1 \in \mathcal{O}(S \cup T)$. Since $v_2 \in \mathcal{O}_\infty(S \cup T(E_0))$, we obtain that $v_1 v_2/\hat{u}_1 \in \mathcal{O}_\infty(S \cup T(E_0))$, and so, in view of (6.13), $\hat{u}_2|_{S\cup T} \in \mathcal{O}_\infty(S \cup T(E_0))$ as well. On the other hand, since $\hat{u}_2 \in \mathcal{O}_\infty(\Sigma|E \cup E_0)$ for all $i \geq 3$ and $\hat{u}_1$ vanishes nowhere off $S \cup T \cup E$ (see (6.9)), we infer that $\hat{u}_2 \in \mathcal{O}(\Sigma\setminus(S \cup T \cup E))$. This proves (E1).

In order to check (E2) and (E3), we use (6.7), (C2), (C3), and (6.11) to infer that

\[
\left[\frac{v_1}{\hat{u}_1} - 1\right] \geq Z\Delta[u_1]^{-1} \geq \Delta[u_1][u_2].
\]

Together with (6.11) and (6.12), we obtain that

\[
[\hat{u}_2|_{S\cup T} - v_2] = \left[\left(\hat{u}_2|_{S\cup T} - \frac{v_1 v_2}{\hat{u}_1}\right) + v_2\left(\frac{v_1}{\hat{u}_1} - 1\right)\right] \geq \Delta[u_1][u_2].
\]

This shows (E3) and the first part of (E2); the second part of (E2) is ensured by (D1), (D2), (6.3), (6.10), and (6.11) whenever that $\epsilon_2 > 0$ is chosen sufficiently small. Finally, condition (E4) follows from (E2), (E3), and Hurwitz’s Theorem provided that $\epsilon_2 > 0$ is small enough.

Set $\hat{u} := (\hat{u}_1, \ldots, \hat{u}_n) \in \mathcal{O}_\infty(\Sigma|E_0 \cup E \cup \Theta) \times \mathcal{O}_\infty(\Sigma|E_0 \cup E)^{n-1}$.

Summarizing, the following conditions hold true.
which obviously lies in $O_{\infty}(\Sigma|E_0 \cup E)^n$ vanishes nowhere on $\Sigma \setminus (E \cup E_0)$, and hence $\hat{u} \in O_{\infty}(\Sigma|E_0 \cup E \cup \Theta, \mathcal{G}_n^{\infty})$.

(F2) $\hat{u} - u \in O(S, \mathcal{G}_n^S)$ and $\|\hat{u} - u\|_S < \sqrt{n} \epsilon_1$.

(F3) $\hat{u}_i|S - u_i \in O_\Delta(S)$ for all $i \in \{1, \ldots, n\}$, i.e., $\hat{u} \in O_\Delta(S, u) \cap O_{\infty}(S|E_0)^n$.

(F4) $\hat{u}_1 \theta_0$ (and hence $\hat{u} \theta_0$) has an effective pole at each point in $E$.

(F5) $[\hat{u}_i|S] = [u_i]$ for all $i \in \{1, \ldots, n\}$.

Indeed, to check (F1) recall that $\hat{u} \theta_0$ vanishes nowhere on $\Sigma \setminus (S \cup T \cup E)$, by (C3). On the other hand, since $\theta_0$ has no zeros in $S \cup T$, (6.11) and (E4) ensure that $\hat{w} \theta_0$ vanishes nowhere on $T$. Finally, since $\hat{u} \theta_0$ has no zeros on $S$, (C3), (D3), and (E4) imply that $\hat{w} \theta_0$ vanishes nowhere on $S$. This shows the first part of (F1); the second one then follows from (6.5) and the definition of $\hat{w}_2$. On the other hand, (F2), (F3), and (F5) follow straightforwardly from the above properties, whereas (F4) is implied by (C4).

Fix $\epsilon_0 > 0$ and consider the map $s : \mathbb{C} \to \mathcal{G}_n^S$ given by

$$s := ((1 + z)^2, (1 + z/2)^2, (1 + z)(1 + z/2));$$

note that $s|D \in \mathcal{S}_2(D)$, see Definition 5.1. Let $h = ((h_{i,j})_{i=1,\ldots,n})_{j=1,\ldots,l} \in (O_{\Delta_0}(S)^n)^l$ be given by

$$\Psi_s(\zeta) = ((1 + \zeta \cdot h)^2 u_1, (1 + \zeta \cdot h/2)^2 u_2, (1 + \zeta \cdot h)(1 + \zeta \cdot h/2)(u_i)_{i=3,\ldots,n}),$$

and $\rho > 0$ be the objects provided by Lemma 5.3 applied to the map $u \in O_{\infty}(S|E_0, \mathcal{G}_n^S)$ in (6.3), $s|\mathcal{D}$, the divisors in (6.1) (compare with (5.2)), the 1-form $\theta_0$, and the number $\epsilon_0$. (The lemma is applied on a small open neighborhood of $S$ in $M$ where $\theta_0$ vanishes nowhere.)

Apply Proposition 4.1 in order to approximate $h$ by a map

$$\hat{h} = ((\hat{h}_{i,j})_{i=1,\ldots,n})_{j=1,\ldots,l} \in ((O_{\Delta_0}(S) \cap O_{\infty}(\Sigma|E))^n)^l,$$

satisfying the following conditions.

(G1) $\|\hat{h} - h\|_S < \epsilon_1$.

(G2) $\hat{h}$ vanishes everywhere on $(\bigcup_{i=1}^n \text{supp}([\hat{u}_i \theta_0]|_0)) \setminus S$.

(G3) $[\hat{h}_{i,j}]_\infty = \prod_{p \in E} p^{m_{i,j}(p)}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, l$, where $\{m_{i,j}(p) : i = 1, \ldots, n, j = 1, \ldots, l\}$ are pairwise distinct natural numbers for each $p \in E$.

To ensure (G3) we fix an ordering in the set $\{1, \ldots, n\} \times \{1, \ldots, l\}$ and apply Proposition 4.1 recursively in order to guarantee that the map $\{1, \ldots, n\} \times \{1, \ldots, l\} \ni (i, j) \mapsto m_{i,j}(p)$ is strictly increasing for each $p \in E$.

For each $\zeta \in (\mathbb{C}^n)^l$, consider the function

$$\hat{\Psi}_s(\zeta) = \left((1 + \zeta \cdot \hat{h})^2 \hat{u}_1, (1 + \zeta \cdot \hat{h}/2)^2 \hat{u}_2, (1 + \zeta \cdot \hat{h})(1 + \zeta \cdot \hat{h}/2)(\hat{u}_i)_{i=3,\ldots,n}\right),$$

which obviously lies in $O_{\infty}(\Sigma|E \cup E_0 \cup \Theta)^n$ by (F1).

In view of (F2) and (G1), if $\epsilon_2 > 0$ is small then $\hat{\Psi}_s(\zeta)$ is close to $\Psi_s(\zeta)$ on $S$ uniformly on $\zeta \in \rho \mathbb{C}_n^l$, and hence $\mathcal{P} \circ \hat{\Psi}_s$ is close to $\mathcal{P} \circ \Psi_s$ on $\rho \mathbb{C}_n^l$, where $\mathcal{P}$ is the
period map (5.3) with $\theta_0$ in the role of $\theta$; i.e.,
\begin{equation}
\mathcal{P} : C^0(C, u) \to (C^n)^I, \quad \mathcal{P}(h) = \left( \int_{C_j} (h - u)\theta_0 \right)_{j=1,\ldots,l}.
\end{equation}

Thus, the following assertions hold provided that $\epsilon_1 > 0$ is small enough.

(H1) $\hat{\Psi}_s(\zeta) \in \mathcal{O}_\Delta(S, u)$ for all $\zeta \in \rho \mathbb{B}_{n,l}$; see (5.3). Use (F3), the fact that $\hat{h} \in (\mathcal{O}_{\Delta_0}(S)^n)^I$, that we can assume that $\rho < 1/\|\hat{h}\|_S$ by (G1), and Remark 5.2.

(H2) The map $\mathcal{P} \circ \hat{\Psi}_s : \rho \mathbb{B}_{n,l} \to (\mathcal{P} \circ \hat{\Psi}_s)(\rho \mathbb{B}_{n,l})$ is a biholomorphism with $0 \in (\mathcal{P} \circ \hat{\Psi}_s)(\rho \mathbb{B}_{n,l})$; use Lemma 5.3 (B3), (F1), (F2), (G1), (G2), and the Cauchy estimates.

(H3) $\hat{\Psi}_s(\zeta) \in \mathcal{O}_\infty(\Sigma|E \cup E_0 \cup \Theta, \mathbb{A}_u^{n-1})$ and is full for all $\zeta \in \rho \mathbb{B}_{n,l}$; use Lemma 5.3 (B1), (B2), (F2), and (H1). To check that $\hat{\Psi}_s(\zeta)$ vanishes nowhere in $\Sigma \setminus E \cup E_0 \cup \Theta$, note that neither $\hat{h}$ nor $s(\zeta \cdot \hat{h})$ vanish anywhere there and take into account (G2).

(H4) $\|\hat{\Psi}_s(\zeta) - u\|_S < \epsilon_0$ for all $\zeta \in \rho \mathbb{B}_{n,l}$; use Lemma 5.3 (B2), (F2), and (G1).

On the other hand, conditions (F1) and (G2) ensure that, for any $\zeta \in (C^n)^I$, the vectorial 1-form
\begin{equation}
\hat{\Psi}_s(\zeta)\theta_0 \in \Omega_\infty(\Sigma|E \cup E_0)^n
\end{equation}
and vanishes nowhere on $\Sigma \setminus (S \cup E)$. Thus, by (F1) and (H1), we have
\begin{equation}
\hat{\Psi}_s(\zeta)\theta_0
\end{equation}
has no zeros on $\Sigma \setminus (E_0 \cup E)$, for all $\zeta \in \rho \mathbb{B}_{n,l}$.

Furthermore, by (F4) and (G3),
\begin{equation}
\hat{\Psi}_s(\zeta)\theta_0
\end{equation}
has an effective pole at each point $p \in E$, for all $\zeta \in (C^n)^I$.

Denote by $\zeta_0 \in \rho \mathbb{B}_{n,l}$ the point such that
\begin{equation}
\mathcal{P}(\hat{\Psi}_s(\zeta_0)) = 0
\end{equation}
(see (H2)) and define
\[ \hat{f} = (\hat{f}_1, \ldots, \hat{f}_n) = \Xi^{-1} \circ \hat{\Psi}_s(\zeta_0) \in \mathcal{O}_\infty(\Sigma|E \cup E_0 \cup \Theta, \mathbb{A}_u^{n-1}), \]
where $\Xi$ is the linear biholomorphism (2.3), see (H3). The following assertions are satisfied.

(I1) $\hat{f}\theta_0 \in \Omega_\infty(\Sigma|E \cup E_0)^n$ and vanishes nowhere on $\Sigma \setminus (E_0 \cup E)$, see (6.15) and (6.10).

(I2) $\hat{f}\theta_0$ has an effective pole at each point $p \in E$, see (6.17).

(I3) $[(\hat{f}_i\theta_0)|_S - \hat{f}_i] \geq \Delta$ for all $i \in \{1, \ldots, n\}$, where the map $f = (f_1, \ldots, f_n)$ and the 1-form $\theta$ are those given in the statement of Theorem 6.1 see (6.3) and (H1). In particular, for each $p \in E_0$ there is $i \in \{1, \ldots, n\}$ such that $\hat{f}_i\theta_0$ has an effective pole at $p$.

(I4) $\int_{C_j}(\hat{f}\theta_0 - f\theta) = 0$ for all $j \in \{1, \ldots, l\}$, and hence $\hat{f}\theta_0 - f\theta$ is exact on $S$; see (6.3), (6.13), and (6.18).

(I5) $\|\hat{f} - f\theta/\theta_0\|_S < 2\epsilon_0$; use (2.3), (6.3), and (H4).
Since $S$ is a strong deformation retract of $\Sigma \setminus E$ and \{\$C_1, \ldots, C_l\$\} is an skeleton of $S$ based at $(p_0, E_0 \cup \Lambda)$ (see Definition 2.4), (5.14) and (11)–(14) ensure that

$$Y : \Sigma \setminus (E_0 \cup E) \to \mathbb{R}^n, \quad Y(p) = \Re \int_{p_0}^p \hat{f}_\theta_0$$

is a well defined complete conformal minimal immersion of FTC with $\text{Flux}_Y = \text{Flux}_X = p$. Moreover, (6.14), (13), (14), and (6.4) guarantee that $Y - X$ vanishes at least to order $r$ at every point of $\Lambda \cup E_0$. Finally, (15) and the compactness of $S$ ensure that $\|Y - X\|_S < \varepsilon$ provided that $\varepsilon_0$ is chosen small enough. This concludes the proof of the theorem. \hfill $\Box$

7. Mittag-Leffler’s theorem for minimal surfaces

In this section we prove the following Mittag-Leffler type theorem for conformal minimal surfaces, including approximation and interpolation, which is a more precise version of Theorem 1.3.

**Theorem 7.1.** Let $M$ be an open Riemann surface, $A \subset M$ be a closed discrete subset, $U \subset M$ be a locally connected, closed neighborhood of $A$ whose connected components are all Runge admissible compact subsets in $M$, and $X : U \setminus A \to \mathbb{R}^n$ $(n \geq 3)$ be a map such that $X|_{W \setminus A} \in \text{GCCMI}_\infty(W|A, \mathbb{R}^n)$ for all components $W$ of $U$. Then for any $\Lambda \subset U \setminus A$ that is closed and discrete as subset of $M$, any map $r : A \cup \Lambda \to \mathbb{N}$, and any group morphism $p : H_1(M \setminus A, \mathbb{Z}) \to \mathbb{R}^n$ with $p|_{H_1(U \setminus A, \mathbb{Z})} = \text{Flux}_X$, there is a full conformal minimal immersion $Y : M \setminus A \to \mathbb{R}^n$ satisfying the following conditions.

(i) $Y - X$ is harmonic at every point of $A$.

(ii) $Y - X$ vanishes at least to order $r(p)$ at each point $p \in A \cup \Lambda$.

(iii) $\text{Flux}_Y = p$.

(iv) $\|Y - X\|_U < \varepsilon$ for any given $\varepsilon > 0$.

Moreover, the immersion $Y$ can be chosen complete.

The proof of the theorem uses Theorem 6.1 and the following ad hoc technical lemma. The lemma is needed only to ensure the completeness of the conformal minimal immersion $Y : M \setminus A \to \mathbb{R}^n$ in Theorem 7.1.

**Lemma 7.2.** Let $M$ be an open Riemann surface and let $S = K \cup \Gamma \subset M$ be a Runge connected admissible subset. Assume that there is a component $K_0$ of $K$ that is a strong deformation retract of $M$. Let $E_0$ and $\Lambda$ be a pair of disjoint finite subsets of $K_0$, let $n \geq 3$ be an integer, and let $X = (X_1, \ldots, X_n) \in \text{GCCMI}_\infty(S|E_0, \mathbb{R}^n) \cap \text{CCMI}_\infty(K|E_0, \mathbb{R}^n)$ be a map such that

- $X|_{K_0}$ is full (see Definition 2.5).
- $X_j$ extends to $M \setminus E_0$ as a harmonic function, $j \in \{3, \ldots, n\}$, and
- $\partial X_1^2 + \partial X_2^2$ vanishes nowhere on $\Gamma$.

For any number $\varepsilon > 0$, any integer $r \geq 0$, and any smoothly bounded Runge compact domain $W \subset M$ with $S \subset W$, there is a conformal minimal immersion $Y = (Y_1, \ldots, Y_n) \in \text{CCMI}_\infty(W|E_0, \mathbb{R}^n)$ satisfying the following conditions.
Recall that $\partial X = 0$.

Since $u = \frac{\partial X}{\theta}$, where $\theta = (\sum_{j=3}^{n}u_j^2)/\hat{u}_1$ lies in $\mathcal{O}_\infty(M|E_0)$ and $[\hat{u}_2] = [u_2]$, it turns out that

$$\dot{u} = (\hat{u}_1, \hat{u}_2, u_3, \ldots, u_n) \in \mathcal{O}_\infty(M|E_0, \mathcal{G}_n^{-1}).$$

It follows that

$$\|\dot{u} - u\|_S < c_1 \epsilon_1 \quad \text{and} \quad \dot{u}_j - u_j \in \mathcal{O}_\Delta(S) \quad \text{for all} \ j = 1, 2,$$

where $c_1 > 0$ is a constant depending on $u$; use (7.3).

Call $h = ((h_{i,j})_{i=1,\ldots,n})_{j=1,\ldots,l} \in (\mathcal{O}_\Delta(K_0)^n)^l$ the map given by Lemma 5.3 applied to the data $M$, $K_0$, the full map $u|_{K_0} \in \mathcal{O}_\infty(K_0|E_0, \mathcal{G}_n^{-1})$ and the 1-form $\theta$. Choose a map

$$s = ((1 + z), (1 + z)^{-1}, 1) \in \mathcal{S}_2(\mathbb{D})$$
(see Definition 5.1, a number \( \rho \in (0,1/\|h\|_{K_0}) \), and consider the associated multiplicative spray \( \Psi_s: \rho \mathbb{R}^n \to \mathcal{O}(K_0,u) \) in Lemma 5.3 (B1) (see (5.3)); i.e,
\[
\Psi_s(\zeta) = \left( (1 + \zeta \cdot h)u_1, (1 + \zeta \cdot h)^{-1}u_2, (u_i)_{i=3,\ldots,n} \right).
\]
Here \( \Delta \) and \( \Delta_0 \) are the divisors in (7.2) (cf. (5.2)). Fix \( \epsilon_0 > 0 \) and assume that \( \rho \) is so small that Lemma 5.3-(B2),(B4) are satisfied. By Theorem 1.1, we may assume that
\[
\rho > 0 \quad \text{and \ as well. If } \rho > 0 \text{ and } \epsilon_1 > 0 \text{ are chosen small enough, Lemma 5.3 and the Cauchy estimates ensure that}
\]
\[
\hat{\Psi}_s(\zeta) = \left( (1 + \zeta \cdot h)\hat{u}_1, (1 + \zeta \cdot h)^{-1}\hat{u}_2, (u_i)_{i=3,\ldots,n} \right).
\]
(Observe that the \( n - 2 \) last components of \( \hat{\Psi}_s(\zeta) \) are those of \( \Psi_s(\zeta) \); here \( W \) is the domain given in the statement of the lemma.) Note that, by (7.4) and (7.5),
\[
\hat{\Psi}_s \text{ assumes values in } \mathcal{O}_\Delta(K_0,u)
\]
as well. If \( \rho > 0 \) and \( \epsilon_1 > 0 \) are chosen small enough, Lemma 5.3 and the Cauchy estimates ensure that
\begin{enumerate}[label=(\alph*)]
\item \( \hat{\Psi}_s(\zeta): W \setminus E_0 \to \mathbb{R}^{n-1} \subset \mathbb{C}^n \) is full and \( \|\hat{\Psi}_s(\zeta) - u\|_S < \epsilon_0 \) for all \( \zeta \in \rho \mathbb{R}^n; \) use Lemma 5.3 (B2).
\item \( \mathcal{P}^{1,2} \circ \hat{\Psi}_s: \rho \mathbb{R}^n \to (\mathbb{C}^2)^l \) is a submersion at \( \zeta = 0 \) and \( \mathcal{P}^{1,2}(\hat{\Psi}_s(\zeta_0)) = 0 \) for some \( \zeta_0 \in \rho \mathbb{R}^n; \) see (5.5) and use Lemma 5.3 (B4).
\end{enumerate}
Set \( \hat{f} := \Xi^{-1}(\hat{\Psi}_s(\zeta_0)) \in \mathcal{O}_\infty(W|E_0,\mathbb{R}^{n-1}) \).
Since \( S \) is path connected, \( K_0 \subset S \) is a strong deformation retract of \( M \), and \( \{C_1,\ldots,C_l\} \) is an skeleton of \( K_0 \) based at \( (p_0, E_0 \cup \Lambda) \), the definition of \( \hat{\Psi}_s \) in (7.6) and conditions (a), (7.7), and (b) ensure that
\[
Y: W \setminus E_0 \to \mathbb{R}^n, \quad Y(p) = \Re \int_{p_0}^p \hat{f} \theta
\]
is a well defined, full complete conformal minimal immersion of FTC such that \( Y_j = X_j \) for all \( j = 3,\ldots,n \), \( \text{Flux}_Y = \text{Flux}_X \), and \( Y - X \) vanishes at least to order \( r \) at every point of \( \Lambda \cup E_0 \). Finally, (a) and the compactness of \( S \) ensure that \( \|Y - X\|_S < \epsilon \) provided that \( \epsilon_0 \) is chosen small enough. This concludes the proof. \( \square \)

**Proof of Theorem 7.1.** Since \( U \) is locally connected, every compact set in \( M \) intersects at most finitely many components of \( U \). Therefore, there is a sequence of connected, smoothly bounded, Runge compact domains
\[
M_0 \in M_1 \subset M_2 \subset \cdots \subset \bigcup_{j \in \mathbb{Z}_+} M_j = M
\]
such that \( M_0 \) is a disc, \( U \cap M_0 = \emptyset \), and \( U \cap bM_j = \emptyset \) for all \( j \geq 1 \). Fix \( p_0 \in M_0 \).

(7.8)
Take \( \epsilon > 0 \). Fix a nowhere vanishing holomorphic 1-form \( \theta \) on \( M \). Also choose a full conformal minimal immersion \( X_0: M_0 \to \mathbb{R}^n \) and a number
\[
\epsilon_0 \in \left( 0, \frac{1}{2} \min \{ \epsilon, \delta_0 \} \right), \quad \text{where } \delta_0 = \min \{ |\partial X_0/\theta|(p): p \in M_0 \} > 0.
\]
Furthermore, we choose \( \epsilon_0 > 0 \) so small that every conformal minimal immersion \( Z: M_0 \to \mathbb{R}^n \) with \( \|Z - X_0\|_{M_0} < 2\epsilon_0 \) is full. We shall inductively construct a sequence of numbers \( \epsilon_j > 0 \) and full immersions \( X_j \in \text{CCMI}_\infty(M_j|A \cap M_j, \mathbb{R}^n) \) satisfying the following conditions for all \( j \geq 1 \).

1. \( \max \{ \|X_j - X_{j-1}\|_{M_{j-1}}, \| (\partial X_j - \partial X_{j-1})/\theta \|_{M_{j-1}} \} < \epsilon_{j-1} \).
2. \( \|X_j - X\|_{U \cap M_j}\backslash M_{j-1} < \epsilon_{j-1} \).
3. \( X_j - X \) extends harmonically to \( \hat{U} \cap M_j \).
4. \( X_j - X \) vanishes at least to order \( r(p) \) at each point \( p \in (A \cup \Lambda) \cap M_j \).
5. Flux \( X_j = \text{Flux}_{X_j} = \text{p}|_{H_1(M_j\backslash A, Z)} \).
6. \( \text{dist}_{X_j}(p_0, bM_j) > i \) for all \( i \in \{0, \ldots, j\} \).
7. \( \epsilon_j < \frac{1}{2} \min \{ \epsilon_{j-1}, \delta_j \} \), where \( \delta_j = \min \{ |\partial X_j/\theta|(p): p \in M_j \setminus A \} > 0 \).

Assume that such a sequence exists. By properties (7.8), (1j), and (7j), there is a limit map
\[
Y = \lim_{j \to \infty} X_j: M \setminus A \to \mathbb{R}^n
\]
that is a conformal harmonic map satisfying
\[
\max \left\{ \|Y - X_j\|_{M_j}, \left\| \frac{\partial Y - \partial X_j}{\theta} \right\|_{M_j} \right\} < 2\epsilon_j < \delta_j \quad \text{for all } j \geq 1.
\]
In particular, \( Y: M \setminus A \to \mathbb{R}^n \) is a full conformal minimal immersion. By (7.9), (2j), and (7.10), we have that \( \|Y - X\|_{U \cap M_0}\backslash M_{j-1} < 2\epsilon_{j-1} \leq 2\epsilon_0 < \epsilon \) for all \( j \geq 1 \). Since \( U \cap M_0 = \emptyset \), this and (7.8) imply condition (iv). It is clear that (3j), (4j), and (5j) ensure (i), (ii), and (iii). Finally, properties (ii) and (6j) and the completeness of \( X \), guarantee that \( Y \) is complete. Thus, \( Y \) satisfies the conclusion of the theorem.

Let us now explain the induction. The basis is given by the already fixed number \( \epsilon_0 > 0 \) and conformal minimal immersion \( X_0: M_0 \to \mathbb{R}^n \). Note that, since \( A \cup \Lambda \subset U \) and \( U \cap M_0 = \emptyset \), we obviously have that \( X_0 \in \text{CCMI}_\infty(M_0|A \cap M_0, \mathbb{R}^n) \) (see (2.10)) and conditions (3o) and (4o) are satisfied. Moreover, since \( M_0 \) is simply connected and \( p_0 \in M_0 \), conditions (5o) and (6o) hold true as well. Finally, conditions (1o), (2o), and (7o) are void. For the inductive step, assume that for some \( j \in \mathbb{N} \) we have numbers \( \epsilon_0, \ldots, \epsilon_{j-1} \) and full immersions \( X_0, \ldots, X_{j-1} \) satisfying the required conditions for all \( i \in \{0, \ldots, j-1\} \), and let us provide \( \epsilon_j \) and \( X_j \).

Choose a connected, smoothly bounded, Runge compact domain \( M_j' \) in \( M \) such that \( M_j \subset M_j' \) and \( M_j \) is a strong deformation retract of \( M_j' \). Let \( \Sigma \) be a compact Riemann surface (without boundary) such that \( M_j' \) is a smoothly bounded compact domain in \( \Sigma \), and let \( E \subset \Sigma \setminus M_j' \) be a finite set such that \( M_j' \) is Runge in \( \Sigma \setminus E \). Recall that \( U \cap (bM_{j-1} \cup bM_j) = \emptyset \) and note that \( M_{j-1} \cup (U \cap M_j) \) is Runge and admissible in \( \Sigma \setminus E \). By Theorem (6.1) there is a complete conformal minimal immersion \( Y_j: \Sigma \setminus (E \cup (A \cap M_j)) \to \mathbb{R}^n \) of finite total curvature such that \( Y_j|_{M_j} \in \text{CCMI}_\infty(M_j|A \cap M_j, \mathbb{R}^n) \) and satisfies conditions (1j)-(5j), and also
We now perturb $Y_j$ near $bM_j$ in order to ensure that inequality. For, choose a connected, smoothly bounded, Runge compact domain $M'_j$ in $M$ such that $M_{j-1} \cup (U \cap M_j) \subset M'_j \subset M_j$ and $M'_j$ is a strong deformation retract of $M_j$. Write $Y_j = (Y_{j,1}, \ldots, Y_{j,n})$. Since $Y_j$ is full, we have that $Y_{j,n}$ is nonconstant. Choose a Runge compact set $K \subset M_j \setminus M''_j$ in $M$ such that $K$ is a finite union of smoothly bounded compact discs and

\begin{equation}
\int_\gamma |\partial Y_{j,n}| > 1
\end{equation}

for all paths $\gamma : [0,1] \to M \setminus K$ with $\gamma(0) \in M'_j$ and $\gamma(1) \in M \setminus M_j$. Existence of such a set is well known; we refer e.g. to [19, 3, 1]. Fix a number $T > 0$ so large that

\begin{equation}
\min\{|Y_{j,1}(p) + T| : p \in K\} > \|Y_{j,1}\|_{bM'_j} + 2.
\end{equation}

Let $\Gamma$ be a finite family of pairwise disjoint smooth Jordan arcs $\in M_j$ such that $S = (M''_j \cup K) \cup \Gamma$ is a connected admissible Runge subset of $M$ that is a strong deformation retract of $M_j$ and $\partial Y^2_{j,1} + \partial Y^2_{j,2}$ vanishes nowhere on $\Gamma \cap (M''_j \cup K)$. Consider any immersion

$Y'_j = (Y'_{j,1}, \ldots, Y'_{j,n}) \in \text{GCCMI}_\infty(S | A \cap M_j, \mathbb{R}^n) \cap \text{CCMI}_\infty(M''_j \cup K | A \cap M_j, \mathbb{R}^n)$

such that $Y'_j = Y_j$ on $M''_j$ and $Y'_j = Y_j + (T,0,\ldots,0)$ on $K$; it turns out that $(\partial Y'_{j,1})^2 + (\partial Y'_{j,2})^2$ vanishes nowhere on $\Gamma$. Since the immersion $Y'_j |_{M_j}$ satisfies conditions (1)–(5) and (6) for all the indices $i \in \{0,\ldots,j-1\}$, Lemma 7.2 furnishes us for any small enough $\epsilon' > 0$ with a full immersion $X_j = (X_{j,1}, \ldots, X_{j,n}) \in \text{CCMI}_\infty(M_j | A \cap M_j, \mathbb{R}^n)$ satisfying the same conditions and, in addition,

(a) $X_{j,n} = Y_{j,n}$ and

(b) $\|X_{j,1} - Y'_{j,1}\|_{M'_j \cup K} < \epsilon'$.

Let us see that $X_j$ satisfies (6). For, since $\text{dist}_{X_j}(p_0, bM_{j-1}) > j - 1$ and $p_0 \in M_{j-1} \subset M'_j$, it suffices to check that $\int_\gamma |\partial X_j| > 1$ for all paths $\gamma : [0,1] \to M_j \setminus M''_j$ with $\gamma(0) \in bM''_j$ and $\gamma(1) \in bM_j$; recall that $2|\partial X_j|^2$ is the metric induced on $M_j$ by the Euclidean metric in $\mathbb{R}^n$ via the immersion $X_j$ (see (2.8)). Let $\gamma$ be such a path. If $\gamma([0,1]) \cap K = \varnothing$, then

\begin{equation}
\int_\gamma |\partial X_j| \geq \int_\gamma |\partial X_{j,n}| \overset{(a)}{=} \int_\gamma |\partial Y_{j,n}| \overset{(\ref{eq:7.11})}{>} 1.
\end{equation}

If, on the contrary, $\gamma([0,1]) \cap K \neq \varnothing$, then for any point $p \in \gamma([0,1]) \cap K$ we have

\begin{align*}
\int_\gamma |\partial X_j| & \geq |X_j(p) - X_j(\gamma(0))| \\
& \overset{(\ref{eq:7.11})}{>} |X_{j,1}(p) - X_{j,1}(\gamma(0))| \\
& \overset{(b)}{>} |Y_{j,1}(p) + T| - |Y_{j,1}(\gamma(0))| - 2\epsilon' \overset{(\ref{eq:7.12})}{>} 2 - 2\epsilon' > 1,
\end{align*}

where for the last inequality we assume that $\epsilon' < 1/2$. This shows (6). Finally, choose any $\epsilon_j > 0$ so small that $(7_j)$ is satisfied for this $X_j$. This ensures the inductive step and completes the proof of the theorem.

$\square$
Remark 7.3. The approximations in Theorems 3.1, 6.1, and 7.1 take place in the natural $C^1$ topology for (generalized) conformal minimal immersions, despite it is not mentioned in their statements. Indeed, just observe that convergence of the Weierstrass data is ensured in the proofs. Furthermore, in view of the recent result by Fornæss, Forstneriˇc, and Wold [14, Theorem 16] on Mergelyan approximation in the $C^r$ topology on admissible sets, it seems that the results in this paper can be extended by guaranteeing approximation of this class.

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