1 Derivation of the analytical expressions for reflectivity

The system consists of an incident medium with permittivity $\epsilon_1 = n_1^2 = 1$, a substrate with $\epsilon_2 = n_2^2$ and two thin layers of thickness $h_s$ and $h_m$ with permittivity $\epsilon_s$ and $\epsilon_m$. Both media are considered to be anisotropic with diagonal terms $\epsilon_x$, $\epsilon_y = \epsilon_x/\alpha_y^2$ and $\epsilon_z = \epsilon_x/\alpha_z^2$. The reflection ($r_p/s$) and transmission ($t_p/s$) of the system can be written in terms of the transfer matrices as

$$
\begin{pmatrix}
 t_s \\
 t_p \\
 0 
\end{pmatrix} = K_2^{-1} M_m M_e K_1
\begin{pmatrix}
 a_s \\
a_p \\
r_s \\
r_p
\end{pmatrix}
$$

(S1)

where $a_s$ and $a_p$ are the TE and TM polarized incident fields, $K_2$ and $K_1$ are the boundary conditions for the interfaces between the medium $m$ and substrate and medium $e$ and incident medium respectively.

$$
K_{2/1} = 
\begin{bmatrix}
 0 & -\cos \theta & 0 & \cos \theta \\
 -\sqrt{\frac{\eta_0}{\eta_0 - \eta_0^2 \sin^2 \theta}} \cos \theta & 0 & \sqrt{\frac{\eta_0}{\eta_0 - \eta_0^2 \sin^2 \theta}} \cos \theta & 0 \\
 0 & -\sqrt{\frac{\eta_0}{\eta_0 - \eta_0^2 \sin^2 \theta}} \cos \theta & 0 & \sqrt{\frac{\eta_0}{\eta_0 - \eta_0^2 \sin^2 \theta}} \cos \theta
\end{bmatrix}
$$

(S2)

with $\eta_0 = \sqrt{\mu_0/\epsilon_0}$. The transfer matrices along with the boundary conditions at the interfaces for the two thin media with thickness $h$ can be written as $M = e^{iPH}$, with

$$
P = \omega 
\begin{bmatrix}
 0 & 0 & 0 & \eta_0^2 - \eta_0^2 \sin^2 \theta/\epsilon_z \\
 0 & 0 & -\eta_0^2 & 0 \\
 0 & -\epsilon_z + \sin^2 \theta & 0 & 0 \\
 \epsilon_x & 0 & 0 & 0
\end{bmatrix}
$$

(S3)

After matrix exponential, $M$ can be written as

$$
\begin{bmatrix}
 \cos(k_0hR_Z) & 0 & 0 & \frac{\eta_0 \sin(k_0hR_Z)}{\epsilon_z} \\
 0 & \cos(k_0hR_Y) & -i \frac{\eta_0 \sin(k_0hR_Y)}{\epsilon_y} & 0 \\
 0 & -i \frac{\eta_0 \sin(k_0hR_Y)}{\epsilon_y} & \cos(k_0hR_Y) & 0 \\
 i \frac{\epsilon_x \sin(k_0hR_Z)}{\eta_0} & 0 & 0 & \cos(k_0hR_Z)
\end{bmatrix}
$$

(S4)

Performing the matrix multiplication and rearranging the terms one can derive the following explicit relations for the reflectivity

$$
r_p = \frac{[F_1 p C_m + F_2 p S_m | + i [F_3 p S_m + F_4 p S_m C_m]}{[G_1 p C_m - G_2 p S_m C_m] - i [G_3 p S_m + G_4 p S_m C_m]}
$$

$$
r_s = \frac{[F_1 s C_m + F_2 s S_m | + i [F_3 s S_m C_m + F_4 s S_m C_m]}{[G_1 s C_m - G_2 s S_m C_m] - i [G_3 s S_m C_m + G_4 S_m C_m]}
$$

(S5)
The reflectivity was calculated using Eq.S5. Contrast was calculated directly from the reflectivity curves as the difference between the maximum and minimum reflectivity values. For the angular spread (|dR_{TM/TE}/d\theta|), the derivative of the reflectivity was taken with respect to \theta and the maximum value of the derivative was used to calculate the FOM.

This is shown in Fig. S1 for the directional and diffuse cases (\epsilon_e = 100 + 10i and \epsilon_e = 100 + 343i respectively). For both values of \epsilon_e, \h_m and \h_e were optimized to have maximum emissivity (E) at \theta = 45^\circ.

The two components of FOM namely the contrast (emissivity) and |dR_{TM/TE}/d\theta| as a function of Re(\epsilon_e) and Im(\epsilon_e) for TM polarized light is shown in Fig. S1a, S1c and for TE polarization in Fig. S1b, S1d. We see that the contrast does not vary significantly with \epsilon_e. Hence the FOM is mainly dominated by the term |dR/d\theta|. Also, the condition |Re(\epsilon_e)| \gg Im(\epsilon_e) for higher directionality as discussed in the main paper is evident here.
3 Optimization for unity and directional emissivity

The phase matching condition to have constructive interference in the structure is given as

$$2k_0 h_m n_m + \pi + \Psi = 2l \pi \quad (S8)$$

So, the acquired phase on reflection from the lossy material ($\Psi$) can hence be derived from Eq.S6 and Eq.S8 as

$$\Psi_{TM} = \pi - 2 \arctan \frac{\frac{N_{Zc}}{N_m} \epsilon_c \cos \theta + i \epsilon_c T_c}{\epsilon_c \epsilon_m \cos \theta T_c - i \epsilon_c} \quad (S9)$$

$$\Psi_{TE} = \pi - 2 \arctan \frac{\frac{N_m}{N_{Yc}} \frac{N_{Yc} + i \epsilon_c \cos \theta T_c}{N_{Yc} T_c + i \epsilon_c} \cos \theta}{\epsilon_c \epsilon_m \cos \theta T_c - i \epsilon_c}$$

for TM and TE polarizations respectively. We define the phase acquired in the emitter as the real part of Eq. S16. The phase $\Psi$ was calculated for the structures shown in Fig 3 of the main article for 2 different values of $Q$ and is shown in Fig. S2. We see that outside the RB the peaks of FOM correspond to $\Psi = \pi$ as discussed in the main article. The derivation is as follows.

For high $Q$ materials, $\epsilon_e \gg \sin^2 \theta$, thus $N_{Zc} / Y_c \approx \sqrt{\epsilon_e} = n_e$. Hence we can write $T_c \approx \tan(h_c k_0 n_e)$, where both $n_e$ and $T_c$ are independent of $\theta$.

Writing Eq.S6 as $r = A(\omega, \theta) / B(\omega, \theta)$ we see for directional unitary emission, the numerator

$$A(\omega, \theta) = 1 - \frac{T_m N_m}{\epsilon_m} n_e T_c + i \chi \left( \frac{T_c}{n_e} + \frac{N_m T_m}{\epsilon_m} \right) \quad (S10)$$

must be zero, where $\chi = \cos \theta$ for TE polarization and $\chi = 1 / \cos \theta$ for TM polarization. Approximating $N_m \approx n_e$, we can derive the condition presented in Eq. 5 of the main article. For directional emission, an additional condition is that the derivative of $r$ with respect to $\theta$ must be maximized. We note that when $A(\omega, \theta) = 0$, 

Fig. S1: (a,b) The emissivity curves ($1 - R_{TM}$ and $1 - R_{TE}$ respectively) as a function of $\theta$ for directional ($Re(\epsilon_e) > Im(\epsilon_e)$, red) and diffuse ($Re(\epsilon_e) < Im(\epsilon_e)$, blue) emissions. (c,d) The derivatives of (a,b) with respect to $\theta$. 
Fig. S2: The FOM for the planar structure for TM polarized light ($h_e/\lambda_{TO} = 0.03$, $h_m/\lambda_{TO} = 0.7$) for 2 different values of $Q$ (black lines, left axis). The corresponding phase acquired in the emitter ($\Psi$) calculated from Eq. S8 (red lines, right axis). The structures correspond to the color maps of Fig 3 of the main article.

\[
\frac{d}{d\theta} \left[ \frac{A}{B} \right] = \frac{BA' - AB'}{B^2} = \frac{1}{B} \frac{dA}{d\theta} \tag{S11}
\]

From Eq. S10 we get

\[
\frac{dA}{d\theta} = \left[ \frac{k_0 h_m n_e T_e}{2\epsilon_m} \right] \sin 2\theta + i \tan \theta \frac{T_e}{n_e} \tag{S12}
\]

For directional emissivity, Eq. S12 must be maximized. To achieve this, we consider two alternative approaches that consist in maximizing $n_e T_e$ to increase the angular sensitivity of $r$.

The term $T_e$ is complex and can be generally written as

\[
T_e = \tan(\rho + i\beta) = \frac{\sin(2\rho) + i \sinh(2\beta)}{\cos(2\rho) + \cosh(2\beta)} \tag{S13}
\]

where $\rho = k_0 h_e \text{Re}(n_e)$ and $\beta = k_0 h_e \text{Im}(n_e)$. A first strategy consists in minimizing the denominator of $T_e$, i.e. $\cos(2\rho) = -1$ and $\cosh(2\beta) \approx 1$. A second approach is to maximize its numerator by achieving $\beta \gg 1$ such that $T_e$ is dominated by the hyperbolic terms of Eq. S13.

For the former condition, setting $\rho = \pi/2$, we get

\[
h_e = \lambda/(4\text{Re}\{\sqrt{\epsilon_e}\}). \tag{S14}
\]

Using this value of $h_e$, we get $\beta = \pi \text{Im}(n_e)/(2\text{Re}(n_e))$. To assure $\cosh(2\beta) \approx 1$, therefore $\beta \ll 1$, we see that the condition $\text{Re}\{n_e\} \gg \text{Im}\{n_e\}$ must be satisfied. This can be achieved outside the RB for frequencies $\omega < \omega_{TO}$. Furthermore, this leads to the spacer thickness approaching, in first order approximation:

\[
h_m \approx \frac{\lambda}{2n_m} \tag{S15}
\]

For the second strategy, we set $\beta \gg 1$. So from Eq. S13, $T_e \approx i \tanh(2\beta)$ which leads to

\[
n_e T_e \approx -\text{Im}(n_e) \tanh 2\beta + i \text{Re}(n_e) \tanh 2\beta
\]

\[
\frac{T_e}{n_e} \approx \frac{\text{Im}(n_e) \tanh 2\beta + i \text{Re}(n_e) \tanh 2\beta}{\text{Im}(n_e)^2 + \text{Re}(n_e)^2} \tag{S16}
\]
We note that $\text{Re}(n_e T_e)$ can be large only if $\text{Im}(n_e) \gg 1$. As a result, $\text{Im}(T_e/n_e) \ll 1$ and setting $r_n = 0$ in Eq. S10 leads to

$$\chi = \frac{\text{Re}(n_e)\tanh 2\beta}{\text{Im}(n_e)\tanh 2\beta} = \chi \frac{\text{Im}(n_e)^2 + \text{Re}(n_e)^2}{\text{Im}(n_e)^2 + \text{Re}(n_e)^2}$$  \hspace{1cm} (S17)

For this to be solvable, the LHS of Eq. S17 must be positive, implying $\text{Re}\{n_e\} \ll \text{Im}\{n_e\}$. This is true in the RB for frequencies $\omega > \omega_{TO}$. Solving Eq. S17 we get

$$h_e = \frac{\lambda}{8\pi\text{Im}(\sqrt{\epsilon_e})} \arcsinh \frac{2}{\cos \Theta \text{Re}(\sqrt{\epsilon_e})}$$  \hspace{1cm} (S18)

Using this value of $h_e$, setting $r_n = 0$ in Eq. S10 we get in first approximation the same spacer thickness as derived previously in Eq. S15. Using the phase matching condition of Eq. S8, with $h_m = \frac{\lambda}{2\pi m}$ we get $\Psi = \pi$, which is the condition for directional emission (maximum FOM) as discussed in the main article. Also, we note that our strategy of maximizing $n_e T_e$ with $n_e \gg 1$ results in $n_e T_e \gg T_e/n_e$. So the last terms of Eq. S10 and Eq. S12 can be neglected and we arrive at Eq. 5 and Eq. 6 of the main text.

To summarize, directional emission can be achieved either for $\text{Re}(n_e) \gg \text{Im}(n_e)$ or $\text{Im}(n_e) \gg \text{Re}(n_e)$, hence on both sides of the TO resonance. It should be noted that in first approximation, satisfying Eqs. S14, S18 for the emitter height and Eq. S15 for the spacer height ensures the FOM to be maximum. However, this derivation is only true for extreme cases when either $\text{Re}\{n_e\} \gg \text{Im}\{n_e\}$ or $\text{Re}\{n_e\} \ll \text{Im}\{n_e\}$. To maximize the FOM for intermediate cases, we can either fix $h_m = \frac{\lambda}{2\pi m}$ and deduce $h_e$ from Eq. S8 or fix $h_e$ and calculate the corresponding $h_m$. For the former, strong directionality is always achieved although peak emissivity will quickly decrease away from the optimal conditions. In the main article section 4, we fix $h_e$ to ensure unitary emissivity and then we optimized $h_m$, observing a decrease in directionality (diffuse emission) away from the optimum. This results in the peaks of FOM as shown in Fig. 5c when $\Psi = \pi$ is satisfied.

We should also mention that the design rules presented in this article apply for all $\theta$, however as the FOM scales with $2\theta$, the directionality is reduced when the structure is designed for lower angles of emission. The polar plots of the emissivity of 3 Salisbury screen structures optimized for 3 different angles of emission at $\omega = 0.99\omega_{TO}$ are shown in Fig. S3.

**Fig. S3:** Polar plot of the emissivity of optimized planar structures for both TM and TE polarized light, for three different angles of emission ($\theta$). The optimized $h_m$ and $h_e$ for each structure are shown in the figure. The structures are optimized at $\omega = 0.99\omega_{TO}$.  

### 4 Optimization of grating period

The in-plane wave-vector of SPhP is $k_{SP} = k_0 \sqrt{[\epsilon_e - \epsilon_2]/[\alpha Z_e - \epsilon_2]}$. The SPhP mode is excited by a 2D grating when the grating condition is satisfied [1] for a grating period $P$ given as

$$k_0 \sin \theta + k_{SP} - 2\pi/P = 0.$$  \hspace{1cm} (S19)
Fig. S4: (a) The calculated emissivity ($E = 1 - R$) as a function of $\omega$ (normalized to $\omega_{TO}$) and $\theta$ for grating structure for TM polarized light. The grating period shown in (b) is optimized at each frequency to have directional emission at $\theta = 45^\circ$.

The emissivity for a 2D grating on a semi-infinite substrate with permittivity $\varepsilon_r$ was calculated numerically and is shown in Fig. 2d of the main article. The numerical calculations for the reflectivity of the grating structure are done by the Rigorous coupled wave analysis (RCWA) package by Hugonin and Lalanne [2]. We see almost unity emissivity for frequencies and incident angles corresponding to the grating equation (Eq.S19), only for TM polarized light. As SPhP are not supported for TE polarized light, the emissivity is negligible in the RB for TE polarization.

The grating height was fixed at 300nm, which is close to the penetration depth of the SPhP in air. The grating period ($P$) was tuned according to Eq.S19 to have the SPhP resonance at $\theta = 45^\circ$ for each frequency. This is shown in Fig. S4(b), normalized to the wavelength of the TO phonon resonance ($\lambda_{TO}$). The grating as expected has a narrow emission band around $\theta = 45^\circ$ (Fig. S4(a)), only for the frequencies in the RB where the SPhP can be excited. The FOM for the grating shown in Fig.5c was calculated from the color-map shown in Fig. S4.

References

[1] Mitradeep Sarkar, Mondher Besbes, Julien Moreau, Jean-François Bryche, Aurore Olivéro, Grégory Barbillon, Anne-Lise Coutrot, Bernard Bartenlian, and Michael Canva. Hybrid Plasmonic Mode by Resonant Coupling of Localized Plasmons to Propagating Plasmons in a Kretschmann Configuration. *ACS Photonics*, 2(2):237–245, feb 2015.

[2] Jean Paul Hugonin and Philippe Lalanne. Reticolo software for grating analysis. *arXiv preprint arXiv:2101.00901*, 2021.