Reality and causality in quantum gravity modified electrodynamics

Santiago A. Martínez¹, R. Montemayor¹ and Luis F. Urrutia²

¹ Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. Postal 70-543, 04510 México D.F., México

² Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. Postal 70-543, 04510 México D.F., México

We present a general description of the propagation properties of quantum gravity modified electrodynamics characterized by constitutive relations up to second order in the correction parameter. The effective description corresponds to an electrodynamics in a dispersive and absorptive non-local medium, where the Green functions and the refraction indices can be explicitly calculated. The reality of the electromagnetic field together with the requirement of causal propagation in a given reference frame leads to restrictions in the form of such refraction indices. In particular, absorption must be present in all cases and, contrary to the usual assumption, it is the dominant aspect in those effective models which exhibit linear effects in the correction parameter not related to birefringence. In such a situation absorption is linear while propagation is quadratical in the correction parameter.

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I. INTRODUCTION

The different effective descriptions of quantum gravity effects at low energies can usually be expressed in terms of modified dispersion relations, with a polynomial dependence on the energy and the momentum. Such modifications include standard Lorentz invariance violations as well as possible extensions of Lorentz covariance. In the case of electrodynamics these heuristic approaches can be interpreted in terms of an effective field theory describing the propagation of the electromagnetic field in an effective medium, provided that space-time coordinates and momenta commute. Otherwise, as it happens for example in double special relativity models, the ordering ambiguity in the Fourier transform imposes a different approach outside the scope of the present work. In fact, most of the models considered in the literature, regardless of their quantum origin, are ultimately interpreted as a standard low energy effective theory when their proponents want to probe the corresponding phenomenological consequences by making use of the standard meaning of the required observables, such as energy and momentum. Of course, it is possible to abandon the effective field theory framework but then it is mandatory to revise the standard concept of observables together with unfolding the rules to measure them.

In the effective field theory version of quantum gravity inspired modified electrodynamics the equations of motion acquire the structure of typical Maxwell equations in a medium:

\[ i \mathbf{k} \cdot \mathbf{D} = 4\pi \rho, \quad \kappa \cdot \mathbf{B} = 0, \]
\[ \mathbf{k} \times \mathbf{E} - \omega \mathbf{B} = 0, \quad i \mathbf{k} \times \mathbf{H} + i \omega \mathbf{D} = 4\pi \mathbf{j}, \]

where the \( \mathbf{D} \) and \( \mathbf{H} \) fields are characterized by constitutive relations of the form

\[ D^i = \alpha^{ij} E_j + \rho^{ij} B_j, \]
\[ H^i = \beta^{ij} B_j + \sigma^{ij} E_j. \]

The coefficients \( \alpha^{ij}, \rho^{ij}, \beta^{ij}, \) and \( \sigma^{ij} \) are polynomials in \( \omega \) and \( k^i \). Once these constitutive relations are given, we can proceed as usual in the case of an electromagnetic field in a non-homogeneous medium. From a heuristic point of view, it is interesting to remark that these constitutive relations correspond to effective media that are non-local in space and time, which can be interpreted as a footprint of the granularity induced by quantum gravity. Although it seems that these media could be in principle quite arbitrary, the requirements of reality of the electromagnetic field together with causal propagation lead to constraints that must be satisfied by these effective theories on very general grounds.

Our definition of causality at the effective theory level is the standard one, whereby the field strengths are obtained from the sources via retarded Green functions. It is important to remark that here we will deal only with the aspects of causality related to the dispersive character of quantum gravity effects as they are manifested in a given reference frame, a character related to a generalized susceptibility theorem. There is another more geometrical aspect related to Lorentz transformations connecting different reference frames, associated to the possible existence of closed causal curves which could appear when there are superluminal velocities. No matter how close to \( c \) the superluminal velocities are in a given reference frame, in a boosted enough frame these particles will propagate to the past. The analysis of the possible existence of closed causal curves will not be discussed here.

In relation to Eqs. (1, 2), we adhere to the point of view of Ref. [11] where \( \mathbf{E}, \mathbf{B}, \mathbf{D}, \) and \( \mathbf{H} \) are considered as fundamentals fields, even in vacuum. In fact, the excitations \( \mathbf{D} \) and \( \mathbf{H} \) are the potentials for the charge density \( \rho \).
and the current density \((j + \partial_t D)\), respectively. Thus, they are directly related to charge conservation. On the other hand the electric and magnetic field strengths \(E\) and \(B\) are the forces acting on unit charges. Our notation does not reflect their intrinsic geometrical content described by the characterization of \(D\) and \(B\) as three-dimensional two-forms, together with \(E\) and \(H\) as one-forms. It refers to the corresponding components of the forms where the Hodge duality has been used to rewrite antisymmetric two-form components as one-form components.

To introduce our effective field approach we will briefly recall the Lagrangian for the electromagnetic field in a non-dispersive medium. It is usually written as

\[
L = -\frac{1}{16\pi} F_{\mu\nu} \chi^{[\mu\nu][\alpha\beta]} F_{\alpha\beta} - j_\mu A^\mu,
\]

(5)

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\), and \(\chi^{[\mu\nu][\alpha\beta]}\) is a constant tensor that encodes the information about the medium. This structure, where the basic dynamical field is \(A_\mu\), warrants gauge invariance and hence charge conservation. The dynamics is given by the equations of motion

\[
\partial_\mu H^{\mu\nu} = 4\pi j^\nu,
\]

(6)

together with the constitutive relations

\[
H^{\mu\nu}(x) = \chi^{[\mu\nu][\alpha\beta]} F_{\alpha\beta}(x).
\]

(7)

Defining, as usual, the electric and magnetic field strengths as \(F_{0i} = E_i\) and \(F_{ij} = -\epsilon_{ijk} B_k\) respectively, and the corresponding components of the excitation \(H^{\mu\nu}\), \(H^{0i} = D_i\) and \(H^{ij} = -
\epsilon_{ijk} H^k\), the constitutive relations become

\[
D_i = 2\chi^{[0][0]\alpha]} E_j - \chi^{[0][mn]} \epsilon_{mnij} B_j,
\]

(8)

\[
H^i = \epsilon_{ik\lambda}[\lambda][0\beta] E_j - \frac{1}{2} \epsilon_{ik\lambda}[\lambda][mn] \epsilon_{mnij} B_j,
\]

(9)

and the effective equations of motion acquire the usual form for an electromagnetic field in a medium given in the relations (10, 12).

In the case of a non-local space-time medium this formalism is also suitable, but now we have

\[
L = -\frac{1}{16\pi} \int d^4\tilde{x} F_{\mu\nu}(\tilde{x}^\sigma) \chi^{[\mu\nu][\alpha\beta]}(\tilde{x}^\sigma - \tilde{x}^\sigma) F_{\alpha\beta}(\tilde{x}^\sigma),
\]

(10)

instead of (13). The constitutive relations become

\[
H^{\mu\nu}(x) = \int d^4\tilde{x} \chi^{[\mu\nu][\alpha\beta]}(\tilde{x}^\sigma - \tilde{x}^\sigma) F_{\alpha\beta}(\tilde{x}).
\]

Demanding that the Lagrangian (10) be real for real fields \(F_{\alpha\beta}\) \((E, B)\) implies also the reality of \(H^{\mu\nu}\) \((D, H)\). Writing \(\chi^{[\mu\nu][\alpha\beta]}(x^\sigma - \tilde{x}^\sigma)\) in terms of its Fourier transform

\[
\chi^{[\mu\nu][\alpha\beta]}(x^\sigma - \tilde{x}^\sigma) = \int d^4k \ e^{-ik(z-x)} \tilde{\chi}^{[\mu\nu][\alpha\beta]}(k^\sigma),
\]

(11)

we can easily demonstrate that (10) can also be written

\[
L = -\frac{1}{16\pi} F_{\mu\nu}(x^\sigma) \left(\tilde{\chi}^{[\mu\nu][\alpha\beta]}(i\partial_n) F_{\alpha\beta}(x^\sigma)\right).
\]

(12)

This shows that for a non-local medium we can use a Lagrangian with the same form as (13), but now with \(\tilde{\chi}^{[\mu\nu][\alpha\beta]}\) being a derivative operator. Thus we are dealing with a higher order theory for the variables \(A^\mu\). In particular this implies that dynamical functions, such as the density of energy and momentum, will depend not only on the fields \(E, B, D,\) and \(H\), but also on their space-time derivatives. This will pose additional constraints regarding the causality of the system, since the causality of a given function does not warrant the causality of its space-time derivatives due to the presence of the energy cutoff in the effective theory (see Appendix A).

If \(\chi^{[\mu\nu][\alpha\beta]}\) is symmetric, in the sense that for each set of index values \((\mu, \nu, \alpha, \beta)\) (no sum with respect to repeated indices)

\[
\int d^4x F_{\mu\nu} \left(\tilde{\chi}^{[\mu\nu][\alpha\beta]} F_{\alpha\beta}\right) = \int d^4x F_{\alpha\beta} \left(\tilde{\chi}^{[\alpha\beta][\mu\nu]} F_{\mu\nu}\right),
\]

(13)
is satisfied. Thus it is possible to perform integrations by parts rendering the equations of motion of the same form as the usual non-operator case. In the following we assume that this property holds. The above relation \( \chi^{[\mu\nu][\alpha\beta]}(i\partial_\sigma) = \chi^{[\alpha\beta][\mu\nu]}(-i\partial_\sigma) \), \( \chi^{[\mu\nu]} \) can be rewritten as

\[
\xi^{[\mu\nu][\alpha\beta]}(i\partial_\sigma) = \xi^{[\alpha\beta][\mu\nu]}(-i\partial_\sigma),
\]

The expansion of the \( \xi^{[\mu\nu][\alpha\beta]} \) operator in terms of space-time components

\[
L = -\frac{1}{4\pi} F_{\mu0} \xi^{[0i][0m]} F_{0m} - \frac{1}{4\pi} F_{\mu0} \xi^{[0i][mn]} F_{mn} - \frac{1}{16\pi} F_{ij} \xi^{[ij][mn]} F_{mn} - j_\mu A^\mu
\]

\( \xi^{[0i][0m]} \) can also be used to characterize a quantum gravity modified electromagnetic Lagrangian which preserves 3-d isotropy, provided that \( \xi^{[0i][0j]} \), \( \xi^{[0i][jk]} \), \( \xi^{[ij][0i]} \), \( \xi^{[ij][mn]} \) are tensors under spatial rotations, but not the components of a \( \xi^{[\mu\nu][\alpha\beta]} \) standard Lorentz tensor.

In this paper we use this approach to discuss the main properties of quantum gravity induced effects in electrodynamics, which are labelled by a parameter \( \xi \) usually considered to be proportional to the inverse Planck mass \( M_P \). We provide a general description of such modified electrodynamics including expressions for the Green functions as well as the corresponding index of refractions, in terms of a parameterization of the constitutive relations up to second order in \( \xi \). Some preliminary work along these directions have already been presented in Ref. \[14\]. We also discuss the implications of reality and causality in the field propagation, applying them to some specific cases already considered in the literature. In a different context, causality in Lorentz invariance violating theories has previously been discussed in Ref. \[12\].

II. CONSTITUTIVE RELATIONS AND EQUATIONS OF MOTION

In the following frame where there is invariance under rotations. This would correspond, for example, to the rest frame \( V^\mu = (1,0) \) in the Myers-Pospelov model\[13\]. Our approximation includes contributions from dimension five and six operators in the language of effective field theories. The modified Lagrangian \( [15] \) yields the equations of motion:

\[
\partial_\mu \left[ 2\xi^{[0i][0m]} F_{0m} + \bar{\xi}^{[0i][mn]} F_{mn} \right] = 4\pi j^0, \quad \partial_\mu \left[ 2\xi^{[0i][0m]} F_{0m} + \bar{\xi}^{[0i][mn]} F_{mn} \right] = 4\pi j^i, \]

and the structure of the field tensor \( F_{\mu\nu} \) leads to

\[
\partial_\nu F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\sigma} = 0.
\]

Comparing Eqs. \( [12] \) and \( [13] \) we have

\[
\bar{\xi}^{[0i][0j]} = \frac{1}{2} \alpha^{ij}, \quad \bar{\xi}^{[0i][mn]} = -\frac{1}{2} \epsilon_{mnj} \rho^{ij},
\]

\[
\bar{\xi}^{[mn][0j]} + \frac{1}{2} \epsilon_{mnj} \sigma^{ij}, \quad \bar{\xi}^{[ij][mn]} = +\frac{1}{2} \epsilon_{ij} \beta^{ij} \epsilon_{jmn}.
\]

In order to analyze the propagation of the fields and to define a refraction index it is important to make explicit the dependence of the constitutive relations on the frequency \( \omega \) and the momentum \( \mathbf{k} \). To achieve this we expand their coefficients in space derivatives. Taking into account that these models can be understood as perturbative descriptions in terms of the parameter \( \xi = \xi/M_P \) we have, up to order \( \xi^2 \),

\[
\alpha^{ij} = \alpha_0(\partial_i) \eta^{ij} + \alpha_1(\partial_i) \xi^{ij} \partial_\sigma - \alpha_2(\partial_i) \xi^2 \partial^2 \partial^2 \partial^2, \]

\[
\rho^{ij} = \rho_0(\partial_i) \eta^{ij} + \rho_1(\partial_i) \xi^{ij} \partial_\sigma - \rho_2(\partial_i) \xi^2 \partial^2 \partial^2 \partial^2, \]

\[
\sigma^{ij} = \sigma_0(\partial_i) \eta^{ij} + \sigma_1(\partial_i) \xi^{ij} \partial_\sigma - \sigma_2(\partial_i) \xi^2 \partial^2 \partial^2 \partial^2, \]

\[
\beta^{ij} = \beta_0(\partial_i) \eta^{ij} + \beta_1(\partial_i) \xi^{ij} \partial_\sigma - \beta_2(\partial_i) \xi^2 \partial^2 \partial^2 \partial^2, \]

where \( \alpha_A, \beta_A, \sigma_A, \) and \( \rho_A \), with \( A = 0, 1, 2 \), are \( SO(3) \) scalar operators, according to the assumption that there is no preferred spatial direction. By relaxing this assumption it is possible to generalize the present approach to models that exhibit some preferred spatial directions. Examples of these last models are given in Refs. \[13, 15\], and \[16\].
where a Lorentz violating vector \( n^\mu = (n^0, n^i) \) appears. Only in the reference frame where \( n^i = 0 \) do they yield an isotropic propagation, otherwise anisotropic effects occur. For simplicity, in the following we restrict our discussion to the isotropic case.

The symmetry of \( \tilde{\chi} \) in Eq. (29) implies that the terms in \( \alpha^{ij} \) and \( \beta^{ij} \) which contain an even number of derivatives are symmetric under \( i \leftrightarrow j \), while the terms with an odd number of derivatives are antisymmetric. In the case of \( \rho^{ij} \) and \( \sigma^{ij} \) Eq. (29) leads to

\[
\rho_i = -\sigma_i. \tag{25}
\]

The property \( \nabla \cdot \mathbf{B} = 0 \) implies that \( \beta_2 \) and \( \rho_2 \) are irrelevant, and thus we take \( \beta_2 = \rho_2 = 0 \) (and hence \( \sigma_2 = 0 \)). Thus Eqs. (21-24) finally reduce to

\[
\alpha^{ij} = \alpha_0 \delta^{ij} + \alpha_1 \xi^{ij} \partial_r + \alpha_2 \xi^{ij} \partial_r, \tag{26}
\]
\[
\rho^{ij} = -\sigma^{ij} = -\sigma_0 (t) \delta^{ij} - \sigma_1 \xi^{ij} \partial_r, \tag{27}
\]
\[
\beta^{ij} = \beta_0 \delta^{ij} + \beta_1 \xi^{ij} \partial_r, \tag{28}
\]

which, together with the first of Eqs. (21), yield the following constitutive relations in momentum space

\[
\mathbf{D} = \left( \alpha_0 + \alpha_2 k^2 \xi^2 \right) \mathbf{E} - \left( \sigma_0 + i\alpha_1 \omega \xi \right) \mathbf{B} + \left( i\sigma_1 \xi + \alpha_2 \omega \xi^2 \right) (\mathbf{k} \times \mathbf{B}), \tag{29}
\]
\[
\mathbf{H} = \left( \beta_0 - i\sigma_1 \omega \xi \right) \mathbf{B} - i\beta_1 \xi (\mathbf{k} \times \mathbf{B}) + \sigma_0 \mathbf{E}. \tag{30}
\]

In the approximation to order \( \tilde{\xi}^2 \) here considered, we have \( \tilde{\xi}^2 k^2 \simeq \tilde{\xi}^2 \omega^2 \) and thus we can write

\[
\mathbf{D} = d_1 (\omega) \mathbf{E} + id_2 (\omega) \mathbf{B} + d_3 (\omega) \xi (\mathbf{k} \times \mathbf{B}), \tag{31}
\]
\[
\mathbf{H} = h_1 (\omega) \mathbf{B} + ih_2 (\omega) \mathbf{E} + ih_3 (\omega) \xi (\mathbf{k} \times \mathbf{B}), \tag{32}
\]

where the functions \( d_i (\omega) \) and \( h_i (\omega) \) depend only on \( \omega \) and admit a series expansion in powers of \( \tilde{\xi} \omega \), characterizing each specific model. From Eqs. (12) we get the equations for \( \mathbf{E} \) and \( \mathbf{B} \)

\[
\frac{id_1 (\mathbf{k} \cdot \mathbf{E})}{\omega} = 4\pi \rho (\omega, \mathbf{k}), \tag{33}
\]
\[
\frac{i \omega d_1 \mathbf{E} + (h_3 k^2 - g(\omega)) \xi \mathbf{B} + \left( \omega d_3 \xi + h_1 \right) (\mathbf{i} \mathbf{k} \times \mathbf{B})}{\omega} = 4\pi \mathbf{j} (\omega, \mathbf{k}), \tag{34}
\]

where we denote

\[
(d_2 + h_2) \omega = g(\omega) \xi. \tag{35}
\]

The expressions (31-34) indeed indicate that the above combination is of order \( \tilde{\xi} \). Thus we see that in fact there are only three independent functions of \( \omega \) and \( k \) which determine the dynamics, which are

\[
P = d_1, \quad Q = h_1 + \omega d_3 \xi, \quad R = (h_3 k^2 - g(\omega)) \xi. \tag{36}
\]

Using the homogeneous equation \( \omega \mathbf{B} = \mathbf{k} \times \mathbf{E} \), which yields \( \omega (\mathbf{k} \times \mathbf{B}) = (\mathbf{k} \cdot \mathbf{E}) \mathbf{k} - k^2 \mathbf{E} \), and charge conservation, \( \omega \rho - \mathbf{k} \cdot \mathbf{J} = 0 \), we can decouple the equations for the fields \( \mathbf{E} \) and \( \mathbf{B} \) obtaining

\[
\mathbf{k} \cdot \mathbf{E} = \frac{4\pi}{i \omega P} (\mathbf{k} \cdot \mathbf{j}), \tag{37}
\]
\[
i \left( \omega^2 P - k^2 Q \right) \mathbf{E} + R (\mathbf{k} \times \mathbf{E}) = 4\pi \omega \left[ j - \frac{k^2 Q}{\omega P} \left( \mathbf{k} \cdot \mathbf{j} \right) \right], \tag{38}
\]

and

\[
\mathbf{k} \cdot \mathbf{B} = 0, \tag{39}
\]
\[
i \left( \omega^2 P - k^2 Q \right) \mathbf{B} + R (\mathbf{k} \times \mathbf{B}) = 4\pi k \hat{\mathbf{k}} \times \mathbf{j}. \tag{40}
\]

Finally, we can introduce the standard potentials \( \Phi \) and \( \mathbf{A} \)

\[
\mathbf{B} = i \mathbf{k} \times \mathbf{A}, \quad \mathbf{E} = i \omega \mathbf{A} - i k \Phi. \tag{41}
\]
As in the usual case we can use the radiation gauge, \( \mathbf{k} \cdot \mathbf{A} = 0 \), in which case we have

\[
\Phi = 4\pi (k^2 P)^{-1} \rho,
\]

\[
(k^2 Q - \omega^2 P) \mathbf{A} + R (i k \times \mathbf{A}) = 4\pi \left[ j - (j \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} \right] = 4\pi j_R,
\]

from Eqs. (49, 53). The presence of birefringence depends on the parity violating term proportional to \( kR \), where we have a crucial factor of \( i \). This makes it clear that a possible diagonalization can be obtained by using a complex basis, which is precisely the circular polarization basis. In fact, decomposing the vector potential and the current in such a basis

\[
\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^- \quad \text{and recalling the basic properties } \hat{\mathbf{k}} \times \mathbf{A}^+ = -i \mathbf{A}^+, \quad \hat{\mathbf{k}} \times \mathbf{A}^- = i \mathbf{A}^-,
\]

we can separate into the uncoupled equations (14)

\[
[k^2 Q - \omega^2 P + \lambda k R] \mathbf{A}^\lambda = 4\pi j^\lambda_R, \quad \lambda = \pm 1.
\]

In terms of the basic functions introduced in the constitutive relations (51, 52), the factor in (48) is rewritten as

\[
k^2 Q - \omega^2 P + \lambda k R = \lambda h_3 k^3 \xi + \left( h_1 + d_3 \omega \hat{\xi} \right) k^2 - \lambda \eta \hat{\xi} k - d_1 \omega^2.
\]

This is the key expression to obtain the Green functions and the refraction indices.

III. THE ENERGY-MOMENTUM DENSITY

In order to derive the Poynting theorem for this generalized electrodynamics we still need to make explicit the remaining dependence upon \( \partial_t \) of the operators \( \alpha_0, \beta_0, \sigma_0, \alpha_1, \beta_1, \sigma_1, \) and \( \alpha_2 \), in Eqs. (26, 28). We denote them generically by \( \xi_0, \xi_1 \), and \( \xi_2 \), respectively. Following the symmetry conditions imposed by Eq. (13) we have, up to second order in \( \xi \),

\[
\xi_0 = \xi_0 + \xi_2 \partial^2_t, \quad \xi_1 = \xi_0 + \xi_2, \quad \xi_2 = \xi_0 + \xi_2,
\]

where \( \xi_0, \xi_2, \xi_10 \) and \( \xi_20 \) are now constant coefficients. Here we take

\[
\alpha_{00} = \beta_{00} = 1, \quad \sigma_{00} = 0,
\]

(48)

to recover the usual vacuum as the background for \( \xi = 0 \). A long but straightforward calculation starting from Maxwell equations (1,2) in coordinate space leads to energy conservation in the form

\[
\partial_t u + \nabla \cdot \mathbf{S} = -j \cdot \mathbf{E}.
\]

(49)

Outside the sources, where we have the additional property \( \xi^2 \nabla \cdot \mathbf{E} = 0 \), the corresponding energy density \( u \) and Poynting vector \( \mathbf{S} \) are

\[
u = \frac{1}{2} (E^2 + B^2) - \frac{1}{2} \zeta \partial_t E \cdot E - \frac{1}{2} \xi \beta_0 B^2 - \xi \zeta (E \cdot B) + \sigma_2 \xi^2 (\partial_t E)^2 + \alpha_2 \xi^2 (E \cdot \partial_t^2 E - \frac{1}{2} (\partial_t E)^2),
\]

(50)

\[
\mathbf{S} = E \times B + \frac{1}{2} \xi \partial_t E \times \partial_t E
\]

(51)

\[
+ \xi \beta_0 (\frac{1}{2} \partial_t E \times B - E \times (\nabla \times B)) + \sigma_2 \xi^2 E \times \partial_t^2 E + \beta_2 \xi E \times \partial_t B.
\]

In our case neither the energy density, nor the Poynting vector retain their standard forms \((E \cdot D + B \cdot H)/2\) and \(E \times H\) respectively, because we are dealing with a higher order theory. This can be seen from the application of the
appropriately extended Noether theorem to the Lagrangian density \( \{12\} \), which includes up to third derivatives in the photon field \( A_{\mu} \). To this end we introduce

\[
\chi_{[\mu\nu]\alpha[\beta]} = \chi_0^{[\mu\nu]\alpha[\beta]} + \chi_1^{[\mu\nu]\theta_\theta\alpha[\beta]} \partial_\theta + \chi_2^{[\mu\nu]\theta\delta\alpha[\beta]} \partial_\delta \partial_\theta \partial_\theta, 
\]

as a more compact way of writing the parameterization \( \{19\} \), \( \{20\} \), \( \{21\} \), and \( \{24\} \) of the generalized susceptibility operator. The coefficients \( \chi_{[\mu\nu]\alpha[\beta]} \), \( A = 0, 1, 2 \), are constants and the gauge invariant energy momentum tensor is

\[
T^\tau_\sigma = -\delta^\tau_\sigma L - F_{\alpha\beta} H^{\tau\alpha} - \frac{1}{4} (\partial_\tau F_{\alpha\beta}) \chi_1^{[\theta\psi]\tau[\mu\alpha]} F_{\theta\psi} \\
+ \frac{1}{4} (\partial_\tau F_{\alpha\beta}) \chi_2^{[\theta\psi]\tau[\rho\sigma][\mu\alpha]} (\partial_\rho F_{\theta\psi}) - \frac{1}{4} F_{\theta\psi} \chi_2^{[\theta\psi]\tau[\rho\sigma][\mu\alpha]} (\partial_\rho \partial_\sigma F_{\mu\alpha}). 
\]

This satisfies \( \partial_\tau T^\tau_\sigma = 0 \) outside the sources, in virtue of Maxwell equations \( \{13\} \). The standard contributions arise precisely from the first two terms in the right hand side of \( \{13\} \). In fact we have

\[
-L - F_{\alpha\beta} H^{\alpha\beta} = \frac{1}{2} (E \cdot D + B \cdot H), 
\]

\[
-F_{\alpha\beta} H^{\alpha\beta} = E \times H. 
\]

It is convenient to remember that these models have to be interpreted as effective theories, in which the dynamics of quantum gravity fluctuations are partially incorporated via the modified constitutive relations \( \{31\} \). In this sense they are analogous to the Euler-Heisenberg effective Lagrangian for the electromagnetic field including quantum corrections through non-linear terms, or to the purely classical Lorentz-Dirac dynamics for a charge including the radiated field through a third order derivative term. Thus the density of energy and momentum given by the Noether theorem, \( u \) and \( S \) of Eqs. \( \{50\} \), \( \{51\} \) in coordinate space, contain in principle the contributions of the electromagnetic field plus the quantum gravity fluctuations. These magnitudes satisfy the continuity equation \( \{19\} \). Hence, if we consider a volume \( V \) outside the sources such that the total flux of \( S \) in its boundary is null, it results that

\[
U = \int_V d^3x u \]

is a constant of motion despite damping factors appearing in a plane wave field. The density \( u \) does not correspond to the usual expression in electrodynamics, \( (E^2 + B^2)/2 \), which does not lead here to a constant of motion. In terms of the analogy with an electromagnetic field of a constant of motion in a medium, this can be interpreted as the typical effect of an active medium that interchanges energy with the field. If this active medium absorbs or cedes energy to the electromagnetic field it is a matter of the second law of thermodynamics, which is not explicitly implemented in these models.

We will demand that the physical quantities \( u \) and \( S \) be causally related to the sources. The fields \( E \) and \( B \) are causal by their construction in terms of retarded Green functions. Since we are dealing with an effective theory with an energy cutoff, neither the time nor the spatial derivatives of a causal function are causal, as discussed in Appendix A. In this way, to meet the above requirement we have to impose causality, via the Kramers-Kroning relations, to a sequence of derivatives of \( E \) and \( B \) dictated by the order in \( \xi \) to which we want the theory to hold. We have explored this situation only to first order in \( \xi \), where we can show that it is enough to demand also that \( D \) and \( H \) be causal in order to guarantee the causality of the energy density and the Poynting vector. Keeping the approximation to first order we can use the zeroth order relations

\[
\nabla \cdot E = 0, \quad \nabla \cdot B = 0, \\
\nabla \times E = -\partial_t B; \quad \nabla \times B = \partial_t E, 
\]

in all derivative terms containing a factor \( \xi \). In this way we can rewrite \( u \) and \( S \) only in terms of \( \nabla \times E \) and \( \nabla \times B \). On the other hand, from the constitutive relations

\[
D = E + \sigma_{10} \xi \nabla \times B - \alpha_{10} \xi \nabla \times E, \\
H = B - \beta_{10} \xi \nabla \times B - \sigma_{10} \xi \nabla \times E 
\]

and assuming \( \alpha_{10} \beta_{10} + (\sigma_{10})^2 \neq 0 \), we can express the energy density and Poynting vector only in terms of \( E, B, D \) and \( H \). The causality requirement for \( D \) and \( H \) is explored in the next section.
IV. REALITY AND CAUSALITY CONSTRAINTS

The reality constraints are straightforward. From the constitutive relations \( \text{Eq. } 31, 32 \), the fields \( \mathbf{D} \) and \( \mathbf{H} \) are real for \( \mathbf{E} \) and \( \mathbf{B} \) real if

\[
\begin{align*}
d_1^* (\omega) &= d_1 (-\omega), & d_2^* (\omega) &= -d_2 (-\omega), & d_3^* (\omega) &= -d_3 (-\omega), \\
h_1^* (\omega) &= h_1 (-\omega), & h_2^* (\omega) &= -h_2 (-\omega), & h_3^* (\omega) &= h_3 (-\omega).
\end{align*}
\]

(61) Even though we will subsequently extend the variable \( \omega \) to the complex plane, the relations \( \text{Eq. } 31, 32 \) are written for real \( \omega \), which represents the physical frequency. Nevertheless, in the case of an absorptive background with a real frequency, the wave vector \( \mathbf{k}(\omega) \), and hence \( k(\omega) = \sqrt{\mathbf{k} \cdot \mathbf{k}} \) together with the index of refraction \( n(\omega) = k(\omega)/\omega \), will be complex in general \( \text{Ref. } 10 \). Unless explicitly stated, all our subsequent expressions involving the frequency are written for real \( \omega \).

To have a causal model it is necessary that not only the propagation of the vector potential \( \mathbf{A} \), together with \( \mathbf{E} \) and \( \mathbf{B} \), is causal, but also the response of the medium coded in the constitutive relations. In the following we discuss this last point, finding the causal constraints which must satisfy the functions that appear in the constitutive equations. To this end we generically denote the functions \( d_1 (\omega), id_2 (\omega), \omega \xi d_3 (\omega), h_1 (\omega), ih_2 (\omega), \) and \( i \omega \xi h_3 (\omega) \) by \( \vartheta (\omega) \). In this way the reality conditions are

\[
\begin{align*}
\text{Re} \vartheta (\omega) &= \text{Re} \vartheta (-\omega), \\
\text{Im} \vartheta (\omega) &= -\text{Im} \vartheta (-\omega).
\end{align*}
\]

(63) A general discussion of causality and dispersion relations is given in Ref.\[19\]. In brief, the causal response of the effective medium depends on the absence of poles of \( \vartheta (\omega) = \text{Re} \vartheta (\omega) + i \text{Im} \vartheta (\omega) \) in the upper half complex plane \( \omega \). In addition, here it is necessary to take into account that we are discussing effective models which lose their physical meaning at high frequencies. Therefore, we must consider a finite range of frequencies instead of the usual infinite one, and thus causality is characterized by the generalized susceptibility theorem \[18, 10\]

\[
\begin{align*}
\text{Re} \vartheta (\omega) &= 1/P \int_{-\Omega}^{\Omega} d\omega' \frac{\text{Im} \vartheta (\omega')}{\omega' - \omega} = 2/P \int_{0}^{\Omega} d\omega' \frac{\omega' \text{Im} \vartheta (\omega')}{\omega'^2 - \omega^2}, \\
\text{Im} \vartheta (\omega) &= -1/P \int_{-\Omega}^{\Omega} d\omega' \frac{\text{Re} \vartheta (\omega') - \kappa}{\omega' - \omega} + A/\omega,
\end{align*}
\]

(64) where the cutoff \( \Omega \) defines the region where the effective theory holds, \( |\omega| \lesssim \Omega, \kappa = \lim_{\omega \to \Omega} \text{Re} \vartheta (\omega) \) and \( A = -i \lim_{\omega \to 0} [\omega \vartheta (\omega)] \). We assume that the contribution of the path closing the loop in the upper half complex plane \( \omega \) is negligible. We have evaluated the above dispersion relations taking an inverse power law behavior of \( \vartheta (\omega) \) for \( |\omega| > \Omega \) and matching the functions at \( \omega = \pm \Omega \), obtaining response coefficients which are of the same order of magnitude as those in the proposed simpler approximation. Via the dispersion relations, the introduction of a cutoff in \( \omega \) leads to corresponding cutoff in \( k(\omega) \). This cutoff induces an effective granularity in space-time at small distances, and can be readily seen as follows. Lets us consider a given function in the \( k \)-space, \( f(k) \). If there is no cutoff in this space we have the corresponding Fourier transform to the coordinate space

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-ikx} f(k).
\]

(66) But if there is a cutoff \( \chi \) in the \( k \)-space, i.e. \( -\chi < k < \chi \), instead of this we have

\[
\hat{f}(x) = \int_{-\chi}^{\chi} dk \ e^{ikx} \hat{f}(k),
\]

(67) which, using Eq. \( \text{Eq. } 66 \), becomes related to \( f(x) \) by an integral transform

\[
\hat{f}(x) = \int_{-\infty}^{\infty} dx' f(x') \frac{\sin (x - x') \chi}{\pi (x - x')}.
\]

(68) In the integrand of the right hand side of Eq. \( \text{Eq. } 68 \) there now appears a smeared distribution around \( x \), with a width \( \chi^{-1} \), which can be interpreted as an effective granularity in the coordinate space. This space-time granularity is telling us that we can not probe space-time at distances smaller than \( 1/\chi \) through the effective theory.
If we restrict ourselves to perturbative expressions, we can establish the causal structure of the coefficients in terms of an expansion in $\tilde{\xi}$. To be precise, let us write the real and imaginary parts of $\vartheta$ as

$$\begin{align*}
\text{Re } \vartheta(\omega) &= \theta_0 + \theta_2 \omega^2 + \theta_4 \omega^4 + \mathcal{O}(\theta_6 \omega^6), \\
\text{Im } \vartheta(\omega) &= \theta_1 \omega + \theta_3 \omega^3 + \mathcal{O}(\theta_5 \omega^5).
\end{align*}$$

In terms of the scale $\tilde{\xi}$ the notation is $\vartheta_k = \vartheta_k \tilde{\xi}^k$. Moreover, $\vartheta(\omega)$ has no pole at $\omega = 0$, hence $A = 0$, and we have $\text{Re } \vartheta(\Omega) = \kappa$ at $\omega = \Omega$. Thus we get

$$\text{Im } \vartheta(\omega) \approx \frac{4}{\pi} \left[ - \left( \theta_2 \Omega^2 + \frac{2}{3} \theta_4 \Omega^4 \right) \left( \frac{\omega}{\Omega} \right)^4 + \left( \frac{1}{3} \theta_2 \Omega^2 - \frac{2}{3} \theta_4 \Omega^4 \right) \left( \frac{\omega}{\Omega} \right)^3 \right].$$

This allows us to identify

$$\theta_1 = -\frac{4}{\pi \Omega} \left( \theta_2 \Omega^2 + \frac{2}{3} \theta_4 \Omega^4 \right),$$

and solving for $\theta_4$ we can predict the third order contribution to $\text{Im } \vartheta(\omega)$ to be

$$\theta_4 = \frac{1}{\Omega^2} \left( \theta_1 + \frac{16}{3\pi} \theta_2 \Omega \right).$$

Thus the final expression up to third order in $\bar{\xi}$ is

$$\vartheta(\omega) \approx \vartheta_0 + i\vartheta_1 \omega \bar{\xi} + \vartheta_2 \omega^2 \bar{\xi}^2 + i\frac{1}{\Omega} \left( \frac{\vartheta_1}{\Omega} + \frac{16\vartheta_2}{3\pi} \right) \omega^3 \bar{\xi}^3.$$  

This is the structure imposed by causality on every coefficient $\vartheta$ in the relation of $E$, $B$ with $D$, $H$.

### V. Green Functions in the Radiation Gauge

The retarded Green function for the potential $A$ and the field strengths $E$, $B$ (see Eqs. 68-69-70) can be written in terms of the circular polarization basis

$$G_{ij}^{ret}(\omega, r) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} G_{ij}^{ret}(\omega, \mathbf{k}) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \sum_{\lambda} G^\lambda(\omega, \mathbf{k}) \left( \delta_{ik} - \frac{k_i k_k}{k^2} + i\lambda \epsilon_{ir} k_r \frac{k_i}{k} \right),$$

where $\hat{k}_i = k_i/|k|$, $k = |k|$, and $G^\lambda(\omega, \mathbf{k})$ is obtained from Eq. 75,

$$G^\lambda(\omega, \mathbf{k}) = \frac{1}{k^2 Q - \omega^2 P + \lambda k R}, \quad \lambda = \pm 1.$$  

To obtain the causal Green functions the analytic continuation $\omega \to \omega + i\varepsilon$ is taken. Only the poles in the upper half plane of the variable $k$ make a contribution to the integration. The first step is to rewrite the denominator in Eq. 76 in a convenient form. This is done by successive rescalings leading to

$$Qk^2 - P\omega^2 + \lambda k R = -\frac{1}{3} n_0^2 \omega^2 Q a (M^\lambda - M_0)(M^\lambda - M_+)(M^\lambda - M_-),$$

where we introduce the notation

$$\begin{align*}
Q &= h_1 + d_3 \omega \tilde{\xi}, \quad \chi = \tilde{\xi} \omega Q^{-1}, \quad n_0^2 = d_1 Q^{-1}, \\
\alpha &= 3h_3 n_0 \chi, \quad M^\lambda = \lambda k (n_0 \omega)^{-1}.
\end{align*}$$

The corresponding roots are given by

$$\begin{align*}
M_0 &= \frac{1}{a} + 2^{1/3} \frac{1 + ac}{aA} + \frac{A}{2^{1/3} a}, \\
M_- &= \frac{1}{a} - \left( 1 - i\sqrt{3} \right) \frac{A}{2^{4/3} a} - \left( 1 + i\sqrt{3} \right) \frac{1 + ac}{2^{2/3} aA}, \\
M_+ &= \frac{1}{a} - \left( 1 + i\sqrt{3} \right) \frac{A}{2^{4/3} a} - \left( 1 - i\sqrt{3} \right) \frac{1 + ac}{2^{2/3} aA},
\end{align*}$$

where $a = 1 + \sqrt{\chi}.$
with

\[ A = \left[ 2 + 3ac - 3a^2 + \sqrt{(2 + 3ac - 3a^2)^2 - 4(1 + ac)^3} \right]^{1/3}, \]  

(80)

and \( \epsilon = g\chi (\omega^2 n_0)^{-1} \). To study the modifications to the dynamics it is enough to expand each root in powers of the small parameter \( \chi \)

\[ M_0 \simeq \frac{1}{\beta_1} \chi^{-1}, \quad M_{\pm} \simeq \left[ 1 + \frac{1}{2} \left( \beta_1 - \alpha_1 \right) \left( \lambda \chi + \frac{1}{4} \left( 5\beta_1 - \alpha_1 \right) \chi^2 \right) \right], \]  

(81)

where \( \beta_1 = h_3 n_0 \) and \( \alpha_1 = g/(\omega^2 n_0) \). Since the parameter \( \lambda \) and the momentum \( k \) appear only in the combination \( \lambda k \) it is clear that we have the symmetry property

\[ G^\lambda(\omega, k) = G^{-\lambda}(\omega, -k). \]  

(82)

This relation will be useful in the final calculation of the Green functions \( G^\lambda(\omega, r) \).

The integral in (77) can be written

\[ G^{ret}_{ik}(\omega, r) = \left[ G^{ret}_{ik} \right]_1(\omega, r) + \left[ G^{ret}_{ik} \right]_2(\omega, r) + \left[ G^{ret}_{ik} \right]_3(\omega, r). \]  

(83)

Each vector \( k_j \) can be obtained by inserting \(-i\partial_j\) outside the integral. In this way we have

\[ 2 \left[ G^{ret}_{ik} \right]_1(\omega, r) = \delta_{ik} \int \frac{d^3k}{(2\pi)^3} e^{ikr} \sum_\lambda G^\lambda(\omega, k) = -\frac{i}{r(2\pi)^2} \delta_{ik} \int_{-\infty}^{\infty} dk dke^{ikr} \left[ G^+(\omega, k) + G^-(\omega, k) \right], \]

\[ 2 \left[ G^{ret}_{ik} \right]_2(\omega, r) = \delta_{i} \partial_k \int \frac{d^3k}{(2\pi)^3} e^{ikr} \sum_\lambda G^\lambda(\omega, k) = -\frac{i}{r(2\pi)^2} \delta_{i} \partial_k \int_{-\infty}^{\infty} dk dke^{ikr} \left[ G^+(\omega, k) + G^-(\omega, k) \right], \]

\[ 2 \left[ G^{ret}_{ik} \right]_3(\omega, r) = \epsilon_{irk} \partial_r \int \frac{d^3k}{(2\pi)^3} e^{ikr} \sum_\lambda G^\lambda(\omega, k) = -\frac{i}{r(2\pi)^2} \epsilon_{irk} \partial_r \int_{-\infty}^{\infty} dk dke^{ikr} \left( G^+(\omega, k) - G^-(\omega, k) \right), \]

(84)

where \( n_i = x_i/r \). Thus we arrive at

\[ \left[ G^{ret}_{ik} \right](\omega, r) = -\frac{i}{r(2\pi)^2} \sum_\lambda \frac{1}{2} \left( \delta_{ik} - n_i n_k + i\lambda \epsilon_{ikp} n_p \right) \int_{-\infty}^{\infty} dk dke^{ikr} G^\lambda(\omega, k), \]  

(85)

which allows us to identify \( G^\lambda(\omega, r) \)

\[ G^\lambda(\omega, r) = -\frac{i}{r(2\pi)^2} \int_{-\infty}^{\infty} dk dke^{ikr} G^\lambda(\omega, k). \]  

(86)

Let us emphasize that the factor \( e^{ikr} \) forces us to close the integration contour by a circle at infinite in the upper half complex plane, thus picking up the poles in this region. Our next step is to perform the integrals in (80). To this end we recall that

\[ G^\lambda(\omega, k) = -\frac{3}{n_0^2 \omega^2 Qa} \frac{1}{(M^\lambda - M_0)(M^\lambda - M_+)(M^\lambda - M_-)}. \]  

(87)

Our description is an effective one valid only for momenta \( k \ll M_P \). According to Eqs. (81), the pole at \( M_0^\lambda \) corresponds to the momentum value

\[ |k_0| = \left| Qh^\chi^{-1} \right| \xi^{-1}. \]  

(88)

In the approximation considered here this pole can be taken at infinity and its contribution to the integral can be neglected. The two remaining poles, which are the ones that contribute to the integral, are located at very small displacements with respect to \( |k_\pm| = n_0\omega \ll M_P \). In this way we get

\[ G^\lambda(\omega, k) = \frac{3}{n_0^2 \omega^2 QaM_0} \frac{1}{(M^\lambda - M_+)(M^\lambda - M_-)}. \]  

(89)
Let us consider first the case of $G^+$, where we have
\[ G^+(\omega, k) = \frac{3}{QaM_0} \frac{1}{(k - \omega n_0 M_+)(k - \omega n_0 M_-)}. \] (90)

From the leading order expressions in (81) we conclude that the pole that contributes in this case is $(\omega + i\epsilon) n_0 M_+$. Thus the resulting integral is
\[ G^+(\omega, r) = \frac{3}{4\pi r} \frac{1}{QaM_0} \frac{2n_0 M_+}{(n_0 M_r - n_0 M_-)} e^{i\omega n_0 M_+ r}. \] (91)

The second integral proceeds in an analogous way, but now $(-(\omega + i\epsilon) n_0 M_-)$ is the pole that contributes. It is convenient to define the refraction indices in the circular polarization basis as
\[ n_+ (\omega) = n_0 M_+, \quad n_- (\omega) = -n_0 M_. \] (92)

The minus sign in $n_-$ is due to the fact that $M_-$ starts with a $-1$. Up to the order considered, they are given by the explicit expressions
\[ n_\lambda (\omega) = n_0 \left[ 1 + \lambda \left( \hat{\beta}_1 - \hat{\alpha}_1 \right) \frac{\chi}{2} + \left( \hat{\beta}_1 - \hat{\alpha}_1 \right) \left( 5\hat{\beta}_1 - \hat{\alpha}_1 \right) \frac{\chi^2}{8} \right]. \] (93)

From here we can obtain an explicit second order power expansion in $\xi$
\[ n_\lambda (\omega) = 1 + \lambda n_1 \omega \xi + n_2 \left( \omega \xi \right)^2 + O \left( \xi^3 \right). \] (94)

where
\[ n_1 = \frac{1}{2} (\alpha_{10} - \beta_{10}) \] (95)
\[ n_2 = \frac{1}{8} [ (\alpha_{10} - \beta_{10} ) (\alpha_{10} - 5 \beta_{10} ) + 4 (\beta_{02} - \alpha_{02} ) ] \] (96)

With this notation we finally get
\[ G^\lambda (\omega, r) = \frac{1}{4\pi r Q} \frac{2n_\lambda}{(n_- + n_+)} e^{i\omega n_\lambda r}, \] (97)

where we have considered that the dominant term in $M_0$ yields $aM_0 = 1$. The reality conditions (61-62) impose the relations
\[ [G^\lambda (\omega, r)]^* = G^{-\lambda} (-\omega, r), \quad [n_\lambda (\omega)]^* = n_{-\lambda} (-\omega). \] (98)

The low energy effective character of the theory implies the existence of a frequency cutoff $\Omega$. Taking this into account, the space-time Green function is
\[ G^\lambda (\tau, r) = \frac{1}{4\pi r} \int_{-\Omega}^{\Omega} d\omega \frac{2n_\lambda}{Q (n_- + n_+)} e^{i\omega n_\lambda r} e^{-i\omega \tau} \]
\[ = \frac{1}{4\pi r} \int_{-\Omega}^{\Omega} d\omega \left[ 1 + \lambda n_1 \omega \xi - (\alpha_{20} - \beta_{02}) \left( \omega \xi \right)^2 \right] e^{i\omega \left[ 1 + \lambda n_1 \omega \xi + n_2 \left( \omega \xi \right)^2 \right] r} e^{-i\omega \tau} \] (99)

where $\tau = t - t'$. The choice of the poles in the complex plane $\omega$ is consistent with the causal behavior of the Green function. But the frequency cutoff could introduce some violation of causality. To investigate this possibility, we can compute the time dependent Green function by expanding the integrand in powers of $\xi$

\[ G^\lambda (\tau, r) \simeq \frac{1}{2\pi r} \int_{-\Omega}^{\Omega} d\omega \left[ 1 + \lambda n_1 (1 + i\omega r) \omega \xi - \left[ \alpha_{20} - \beta_{02} - i \left( n_2 + n_1^2 \right) \omega \right] r \omega - \frac{1}{2} n_1^3 \omega^2 \right] \omega^2 \xi^2 e^{i\omega \left( r - \tau \right)} \]
\[ = \frac{1}{2\pi r} \left\{ 1 - i \lambda n_1 \xi (1 + r \partial_r) \partial_r + \xi^2 \left[ \alpha_{20} - \beta_{02} - \left( n_2 + n_1^2 \right) r \partial_r - \frac{1}{2} n_1^3 \omega^2 \partial_r^2 \right] \partial_r \right\} \sin \left( \frac{r - \tau}{r} \right) \frac{\tau}{r - \tau}. \] (100)
This expression shows that the main effect of the cutoff is to spread the propagating field around the light cone, within a wedge defined by \( r \approx \tau \pm \pi / 2 \Omega \). Returning to \( G^\lambda (\omega, r) \), Eqs. (101), we can characterize the effect of the cutoff in the causal behavior of the Green function using the generalized susceptibility theorem [10], a generalization of the Kramers-Kronig relations. In terms of the frequency \( \omega \), the real and imaginary parts of the Green function are

\[
\text{Re} \, G^\lambda (\omega, r) \approx \frac{1}{4\pi r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) r \partial_r - \frac{1}{2} n_1^2 r^2 \partial_r^2 \right) \right\} \cos \omega r + \lambda n_1 \tilde{\xi} (1 + r \partial_r) \partial_r \sin \omega r,
\]

(101)

\[
\text{Im} \, G^\lambda (\omega, r) \approx \frac{1}{4\pi r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) r \partial_r - \frac{1}{2} n_1^2 r^2 \partial_r^2 \right) \right\} \sin \omega r - \lambda n_1 \tilde{\xi} (1 + r \partial_r) \partial_r \cos \omega r.
\]

(102)

According to the generalized susceptibility theorem the Green function will be causal if

\[
\text{Im} \, G(\omega) = -\frac{1}{\pi} \int_{-\Omega}^{\Omega} d\omega' \frac{\text{Re} \, G(\omega') - \text{Re} \, G(\Omega)}{\omega' - \omega},
\]

(103)
or more explicitly when

\[
\text{Im} G^\lambda (\omega) = -\frac{1}{4\pi^2 r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) r \partial_r - \frac{1}{2} n_1^2 r^2 \partial_r^2 \right) \right\} \int_{-\Omega}^{\Omega} d\omega' \frac{\cos \omega' r - \cos (\Omega r)}{\omega' - \omega} + \lambda n_1 \tilde{\xi} (1 + r \partial_r) \partial_r \int_{-\Omega}^{\Omega} d\omega' \frac{\sin (\omega' r) - \sin (\Omega r)}{\omega' - \omega}.
\]

(104)

For \( \omega / \Omega \ll 1 \) the integrals reduce to

\[
P \int_{-\Omega}^{\Omega} d\omega' \frac{\cos (\omega' r) - \cos (\Omega r)}{\omega' - \omega} \approx 2 \left( 1 - \cos \Omega r \right) \frac{\omega}{\Omega} \cos (\Omega r) - \pi + \left[ (\Omega r) \cos \Omega r - \sin \Omega r \right] \left( \frac{\omega}{\Omega} \right)^2 \sin \omega r,
\]

(105)

\[
P \int_{-\Omega}^{\Omega} d\omega' \frac{\sin (\omega' r) - \sin (\Omega r)}{\omega' - \omega} \approx 2 \left( 1 - \sin \omega r \right) \frac{\omega}{\Omega} \cos (\Omega r) + \pi + \left[ (\Omega r) \cos \Omega r - \sin \Omega r \right] \left( \frac{\omega}{\Omega} \right)^2 \cos \omega r.
\]

Furthermore, in the case of a radiation field \( \Omega r \gg 1 \) and hence the factors \( \cos \Omega r \) and \( \sin \Omega r \) become strongly oscillating, nullifying the contributions of the terms where they appear (which also have a factor \( (\omega / \Omega)^n \), with \( n \geq 1 \)). Thus we can take

\[
P \int_{-\Omega}^{\Omega} d\omega' \frac{\cos (\omega' r) - \cos (\Omega r)}{\omega' - \omega} \approx -\pi \sin \omega r, \quad P \int_{-\Omega}^{\Omega} d\omega' \frac{\sin (\omega' r) - \sin (\Omega r)}{\omega' - \omega} \approx \pi \cos \omega r.
\]

(106)

Replacing these integrals in (104), we finally get that if \( \text{Re} \, G^\lambda (\omega) \) is given by Eq. (101), the imaginary part of the Green function, \( \text{Im} \, G^\lambda (\omega) \), must be

\[
\text{Im} G^\lambda (\omega) = \frac{1}{4\pi r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) r \partial_r - \frac{1}{2} n_1^2 r^2 \partial_r^2 \right) \right\} \sin \omega r - \lambda n_1 \tilde{\xi} (1 + r \partial_r) \partial_r \cos \omega r.
\]

(107)

which coincides with Eq. (102), obtained by direct computation. This result shows that the cutoff does not introduce any significant causality violation provided that \( \Omega r \gg 1 \) and \( \omega / \Omega \ll 1 \).

VI. FINAL COMMENTS

In the preceding sections we have discussed the implications of reality and causality of the electromagnetic fields upon the structure of the constitutive relations and the Green functions in quantum gravity inspired modified electrodynamics. To illustrate these implications we now apply our results to some particular models already found in the literature.
Let us start with the Gambini-Pullin electrodynamics \[20\], where the constitutive relations are
\[
D = E - 2i\xi\omega B + 4\xi^2\omega k \times B, \quad H = B + 2i\xi k \times B.
\] (108)
Introducing the dominant contributions to satisfy causality they become
\[
D = E - 2i\xi\omega B + 4\left(1 + \frac{16i\omega}{3\pi\Omega}\right)\xi^2\omega k \times B, \quad H = B + 2i\xi k \times B,
\] (109)
which leads to the refraction index
\[
n_\lambda(\omega) \simeq 1 + 2\lambda\omega\xi + 4\omega^2\xi^2 - i\frac{64\omega}{3\pi\Omega}\omega^2\xi^2,
\] (110)
indicating that absorption is generated at the order \((\xi\omega)^2\).

The second case is the model of Myers and Pospelov \[13\], which yields
\[
D = E + i\xi\omega B, \quad H = B + i\xi\omega E.
\] (111)
Here the constitutive relations become
\[
D = E + i\left(1 + \frac{\omega^2}{\Omega^2}\right)\xi\omega B, \quad H = B + i\left(1 + \frac{\omega^2}{\Omega^2}\right)\xi\omega E,
\] (112)
after the causality requirement is imposed, and the resultant index of refraction is
\[
n_\lambda(\omega) \simeq 1 - \lambda\left(1 + \frac{\omega^2}{\Omega^2}\right)\omega\xi + \frac{1}{2}\left(1 + \frac{\omega^2}{\Omega^2}\right)^2\omega^2\xi^2.
\] (113)
In this case the absorptive terms begin at least to order \(\omega^5\) and causality only introduces \(\Omega\)-dependent corrections to the velocity of propagation. The two models discussed previously present birefringence.

The third model that we consider is the flat space version of that of Ellis et al. \[3, 4, 5\], which has been used to study in a rough way the phenomenology of the full underlying model, leading to robust limits on Lorentz violation. From the theoretical model of a recoiling \(D\)-particle in a quantum gravitational foam, a simple effective model formally analogous to electrodynamics in a medium has been heuristically introduced in Ref. \[4\]. Such an approximation has been used in conjunction with observations of gamma ray bursts to set limits on the quantum gravity scale \(M\). This effective model is characterized by constitutive relations \[4\]
\[
D = \frac{E}{\sqrt{\hbar}} + H \times G, \quad B = \frac{H}{\sqrt{\hbar}} - E \times G,
\] (114)
where \(1/\sqrt{\hbar}\) plays the role of the electric and magnetic permeability and is taken equal to one to have the same permeability as the classical vacuum. In references \[4, 6\] the vector \(G\), which originally represents the recoil velocity of the \(D\)-particle (being proportional to the momentum transfer over \(M\)) is very crudely identified with \(G \sim k/M\), where \(k\) is the momentum of the propagating photon and \(M\) is the effective mass of the \(D\)-particle in the quantum space-time foam. In our notation this amounts to require
\[
G = f k
\]
and the above constitutive relations turn out to be
\[
D = E + (f^2\omega - f)(k \times B), \quad H = (1 - f\omega)B.
\] (115)
To lowest order, the reality conditions for the fields imply that we must choose
\[
f = ia\xi,
\] (116)
with \(a\) being a constant real number. Imposing causality we get
\[
D = E + ia\left[1 + \frac{\omega^2}{\Omega^2}\right]\xi(k \times B), \quad H = \left[1 - ia\left(1 + \frac{\omega^2}{\Omega^2}\right)\xi\omega\right]B.
\] (117)
The dominant contributions to the refraction index do not depend on the causal corrections and are

\[ n_{\lambda}(\omega) = 1 + i a \omega \tilde{\xi} - a^2 \omega^2 \tilde{\xi}^2. \tag{118} \]

Contrary to the previous cases, the above indicates that absorption is already present to linear order (a point previously missed) and that the propagation properties of this effective model exhibit corrections starting from second order in \( \tilde{\xi} \). In fact the photon group velocity is given by

\[ c(\omega) = \frac{d}{d\omega} (\mathrm{Re} k) = 1 - 3(a \tilde{\xi} \omega)^2, \tag{119} \]

and not by \( c(\omega) = c_0(1 - \omega/M) \), with \( M \) real, as was originally assumed in setting the observational bounds \[ \text{~} \].

To summarize, our discussion is based on the consideration that these modified electrodynamics are effective theories with physical meaning, and hence that in particular the electromagnetic field strengths \( \mathbf{E} \) and \( \mathbf{B} \), and the excitations \( \mathbf{D} \) and \( \mathbf{H} \), must be real and its propagation causal. We have shown that these requirements impose constraints on the structure of the constitutive relations and lead to additional contributions to the photon propagation velocity and, much more interesting, to absorptive effects in the propagation of the fields. The contributions originating from causality depend on the range of validity of the effective theory, coded here by the cutoff \( \Omega \).

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### Appendix A

In this Appendix we analyze the effect of a frequency cutoff on the causal character of the derivative of a causal function \( f(t) \), defined by a generalized susceptibility \( \vartheta(t) \)

\[ f(t) = \int_{-\infty}^{\infty} d\tau \vartheta(t - \tau) g(\tau). \tag{120} \]

The derivative is

\[ \partial_t f(t) = \int_{-\infty}^{\infty} d\tau \partial_t \vartheta(t - \tau) g(\tau), \tag{121} \]

and therefore the problem reduces to studying the causal character of the derivative of \( \vartheta(t) \), considered itself as a susceptibility function. We are dealing with real functions, so we will consider here a real susceptibility, \( \vartheta(t) = \vartheta^*(t) \). Introducing the Fourier transform

\[ \vartheta(\omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} \vartheta(t), \tag{122} \]

we get in this case

\[ \mathrm{Re} \vartheta(\omega) = \mathrm{Re} \vartheta(-\omega), \quad \mathrm{Im} \vartheta(\omega) = -\mathrm{Im} \vartheta(-\omega). \tag{123} \]

If \( \vartheta(t) \) is causal, its Fourier transform satisfies the Kramers-Kronig relations, which in the presence of a frequency cutoff \( \Omega \) become

\[ \mathrm{Re} \vartheta(\omega) = \frac{1}{\pi} P \int_{-\Omega}^{\Omega} d\omega' \frac{\mathrm{Im} \vartheta(\omega')}{\omega' - \omega} + \kappa, \tag{124} \]

\[ \mathrm{Im} \vartheta(\omega) = -\frac{1}{\pi} P \int_{-\Omega}^{\Omega} d\omega' \frac{\mathrm{Re} \vartheta(\omega')}{\omega' - \omega} + \frac{A}{\omega^m}. \tag{125} \]
where $\kappa = \text{Re} \vartheta(\Omega)$ and it is assumed that $\vartheta(\omega) = i\frac{A}{\omega m}$ at $\omega \approx 0$. The Fourier transform of the derivative of the susceptibility function is given by

$$\tilde{\vartheta}(\omega) = -i\omega \vartheta(\omega), \quad (126)$$

and hence

$$\text{Re} \tilde{\vartheta}(\omega) = \omega \text{Im} \vartheta(\omega), \quad (127)$$
$$\text{Im} \tilde{\vartheta}(\omega) = -\omega \text{Re} \vartheta(\omega). \quad (128)$$

In order for $\tilde{\vartheta}(\omega)$ to also be causal, it must satisfy the corresponding Kramers-Kronig relations

$$\text{Re} \tilde{\vartheta}(\omega) = \frac{1}{\pi} P \int_{-\Omega}^{\Omega} d\omega' \frac{\text{Im} \tilde{\vartheta}(\omega')}{\omega' - \omega} + \tilde{\kappa}, \quad (129)$$
$$\text{Im} \tilde{\vartheta}(\omega) = -\frac{1}{\pi} P \int_{-\Omega}^{\Omega} d\omega' \frac{\text{Re} \tilde{\vartheta}(\omega') - \tilde{\kappa}}{\omega' - \omega} + \frac{\tilde{A}}{\omega^{n+1}}, \quad (130)$$

where $\tilde{\kappa} = \text{Re} \tilde{\vartheta}(\Omega) = \omega \text{Im} \vartheta(\omega)|_{\Omega}$ and we are assuming that $\tilde{\vartheta}(\Omega)$ has a pole of order $n$ at $\omega = 0$. Rewriting $\text{Re} \tilde{\vartheta}(\omega)$ and $\text{Im} \tilde{\vartheta}(\omega)$ in terms of $\text{Re} \vartheta(\omega)$ and $\text{Im} \vartheta(\omega)$, according to (127) and (128), these last equations yield

$$\text{Im} \vartheta(\omega) = -\frac{1}{\pi} \omega P \int_{-\Omega}^{\Omega} d\omega' \frac{\text{Re} \vartheta(\omega')}{\omega' - \omega} - \tilde{\kappa} \omega, \quad (131)$$
$$\text{Re} \vartheta(\omega) = \frac{1}{\pi} \omega P \int_{-\Omega}^{\Omega} d\omega' \frac{\omega' \text{Im} \vartheta(\omega') - \tilde{\kappa}}{\omega' - \omega} - \frac{\tilde{A}}{\omega^{n+1}}. \quad (132)$$

Finally, comparing Eqs. (124-125) and (131-132), and considering that $\text{Im} \vartheta(\omega)$ is an odd function, we get

$$\kappa + \tilde{\kappa} \omega^{n+1} + \frac{\tilde{A}}{\omega^{n+1}} \ln \left( \frac{1 - \omega / \Omega}{1 + \omega / \Omega} \right) = 0, \quad (133)$$
$$\frac{\kappa}{\pi} \omega \int_{-\Omega}^{\Omega} d\omega' \text{Re} \vartheta(\omega') = -\frac{\kappa}{\pi} \ln \left( \frac{1 - \omega / \Omega}{1 + \omega / \Omega} \right) - \frac{\tilde{A}}{\omega^{n+1}}. \quad (134)$$

The first relation implies

$$\tilde{A} = \kappa = \tilde{\kappa} = 0, \quad (135)$$

and thus the second one leads to

$$m = 1, \quad \int_{-\Omega}^{\Omega} d\omega' \text{Re} \vartheta(\omega') = -\pi A. \quad (136)$$

When there is no cutoff at a finite frequency the factor $\ln \left( \frac{1 - \omega / \Omega}{1 + \omega / \Omega} \right)$ becomes 0, and the constraints are

$$\kappa = \tilde{\kappa} = 0, \quad (137)$$
$$\int_{-\infty}^{\infty} d\omega' \text{Re} \vartheta(\omega') = \pi \left( \tilde{\kappa} - \frac{A}{\omega^{m+1}} \right). \quad (138)$$

For example, let us consider what happens when $\vartheta(t)$ is the causal Heaviside function, such that $\vartheta(\omega) = i/\omega$ and $\tilde{\vartheta}(\omega) = 1$. In both cases, $\Omega = \infty$ or $\Omega$ finite, we get

$$\kappa = \tilde{A} = 0, \quad A = \tilde{\kappa} = 1, \quad \int_{-\Omega}^{\Omega} d\omega' \text{Re} \vartheta(\omega') = 0. \quad (139)$$

When $\Omega = \infty$ the constraints (137-138) hold and are indeed satisfied, and hence $\vartheta(t) = \delta(t)$ is also a causal susceptibility. But if there is a finite frequency cutoff $\Omega$ the corresponding constraints are given by (135-136), which are not satisfied, and $\delta(t)$ becomes a non causal susceptibility.
In the case of a field theory we have spatial derivatives together with the time derivative. But the former derivatives are related to the latter by the equations of motion that drive the evolution of the system. Thus, to discuss causal characteristics it is necessary to take into account both types of derivatives, linked at the Fourier transform level by the refraction index that relates frequency and momentum.

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