We present a systematic study of how vortices in superfluid films interact with the spatially varying Gaussian curvature of the underlying substrate. The Gaussian curvature acts as a source for a geometric potential that attracts (repels) vortices towards regions of negative (positive) Gaussian curvature independently of the sign of their topological charge. Various experimental tests involving rotating superfluid films and vortex pinning are first discussed for films coating gently curved substrates that can be treated in perturbation theory from flatness. An estimate of the experimental regimes of interest is obtained by comparing the strength of the geometrical forces to the vortex pinning induced by the varying thickness of the film which is in turn caused by capillary effects and gravity. We then present a non-perturbative technique based on conformal mappings that leads an exact solution for the geometric potential as well as the geometric correction to the interaction between vortices. The conformal mapping approach is illustrated by means of explicit calculations of the geometric effects encountered in the study of some strongly curved surfaces and by deriving universal bounds on their strength.

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I. INTRODUCTION

In superfluid helium, vortices form when the helium is rotated rapidly or when there is turbulence (Tilley and Tilley, 1990; Vinen, 1969). Though such vortices are similar to the vortices that make up a vortex street behind the wings of an airplane or to the funnel clouds of tornadoes, they are only an Angstrom or two across (Guyon et al., 2001). A more essential difference is that the vortices in a superfluid do not need a constant source of energy to survive. In fact, a vortex is long-lived because the strength of its flow is fixed by the quantization of angular momentum. Thus, the dissipative mechanisms of a conventional fluid are absent.

In this article, we focus on forces that the vortices experience as a result of geometric constraints, with an emphasis on those encountered in thin layers of liquid helium wetting a curved substrate with spatially varying Gaussian curvature. As a result of the broken translational invariance of the underlying curved space, the energy of a single vortex with circulation quantum number $n_i$ at position $\mathbf{u}_i$ includes both a divergent term and a position dependent self-energy, $E_s(\mathbf{u}_i)$, given by (Vitelli and Turner, 2004)

$$E_s(\mathbf{u}_i) = -\pi K n_i^2 U_G(\mathbf{u}_i),$$

where $K = \frac{\rho s \hbar^2}{2m^2}$ is the superfluid stiffness expressed in...
terms of the $^4\text{He}$ atomic mass, $m$, and the superfluid mass density, $\rho_s$. The potential $U_G(u_i)$ is obtained from solving a covariant Poisson equation with the Gaussian curvature, $G(u_i)$, acting as a source

$$\nabla^2 U_G(u_i) = G(u_i).$$

(2)

Vortices (and anti-vortices) are attracted (repelled) to regions of negative (positive) Gaussian curvature. These geometric interactions, while more exotic, are similar in origin to boundary-vortex interactions and can be suitably treated by the method of conformal mapping.

Similar ideas naturally arise in a variety of soft matter systems which have been confined in a thin layer wetting a curved substrate. The specific form the resulting geometric interactions depends on the shape of the substrate. Examples that have been studied both theoretically and experimentally include colloidal crystals on curved interfaces (Bausch et al., 2003; Bowick et al., 2000; Vitelli et al., 2006), columnar phases of block co-polymers (Hexemer et al., 2007) as well as thin layers of nematic liquid crystals (Fernandez-Nieves et al., 2007; Park and Lubensky, 1996; Vitelli and Nelson, 2004, 2006). Fueled by the drive towards technological applications based on the notion of self-assembly directed by geometry (DeVries et al., 2002; Dinsmore et al., 2002; Nelson, 2002), the study of these frustrated materials aims at predicting how the non-uniform distribution of curvature of the underlying substrates induces an inhomogeneous phase in the curved monolayer. An understanding of the resulting macroscopic properties can be built from a mesoscopic description cast in terms of the energetics of the topological defects which often exist even in the ground state and play a crucial role in determining how the material melts or ruptures. The advantage of this approach stems from the huge reduction in degrees of freedom achieved upon re-expressing the energy stored in the elastic field in terms of a few topological defects, rather than keeping track of the state of all the microscopic components, e.g. individual particle positions or molecular orientations. This step allows one to carry out efficient computational studies (Bowick et al., 2000; Hexemer et al., 2007) and provides a suitable starting point for analytical work in the form of effective free energies derived from continuum elastic theory.

Many of the mathematical techniques employed in this article to study vortices in curved superfluid films find application in the soft matter domain, in particular in those contexts where bond-orientational order is important (Vitelli and Turner, 2004). In flat space both superfluid and liquid crystal films can be described, as a first approximation, by an XY model (Nelson and Kosterlitz, 1977). Both liquid crystal disclinations and vortices are modeled as a Coulomb gas of charged particles interacting logarithmically. However, the quantum nature of the problem considered in the present work introduces a fundamental difference between these two classes of systems that is best illustrated by contrasting the angle of the liquid crystal director with the phase of the superfluid’s collective wave function. The former represents the orientation of a vector (with both ends identified in the case of a nematic) that lives in the tangent space of the surface while the latter is a quantum mechanical object that transforms like a scalar since it is defined in an internal space. This subtle difference resurfaces upon considering the distinct curved-space generalizations of the XY model that apply to each of these two systems.

The free energy functional $\mathcal{F}_v$ to be minimized for the case of orientational order on a surface with points labeled by the coordinates $u = (u_1, u_2)$ reads (David, 1989):

$$\mathcal{F}_v = \frac{K}{2} \int d^2 u \sqrt{g} g^{\alpha\beta}(\partial_\alpha \theta(u) - \Omega_\alpha(u))(\partial_\beta \theta(u) - \Omega_\beta(u)),$$

where $g_{\alpha\beta}$ and $g$ indicate the metric tensor and its determinant while $\Omega_\alpha(u)$ is a connection that compensates for the rotation of the 2D basis vectors $E_{\alpha}(u)$ (with respect to which $\theta(u)$ is measured) in the direction of $u_\alpha$ (Kamienski, 2002). Since the curl of the field $\Omega_\alpha(u)$ is equal to the Gaussian curvature $G(u)$ (David, 1989), the integrand in Eq. (3) never vanishes because $\Omega_\alpha(u) \neq \partial_\alpha \theta$ on a surface with $G(u) \neq 0$. As the substrate becomes more curved, the resulting energy cost of geometric frustration can be lowered by generating disclination-dipoles in the ground state even in the absence of topological constraints.

The connection $\Omega_\alpha(u)$ is a geometric gauge field akin to the electromagnetic vector potential, with the Gaussian curvature playing the role of a magnetic field. If topological defects are present, they appear as monopoles in the singular part of $\partial_\alpha \theta(u)$. In analogy with electromagnetic theory, their interaction with the Gaussian curvature arises mathematically from the cross-products between $\partial_\alpha \theta(u)$ and the geometry induced vector potential $\Omega_\alpha(u)$, see Eq. (3). As a result of this interaction, disclinations in a liquid crystal are attracted to regions of the substrate whose curvature has the same sign as the defect’s topological charge (Park and Lubensky, 1996), whereas vortices in a superfluid favor negatively curved regions independently of their sense of circulation. The anomalous coupling between vortices and Gaussian curvature introduced in Equations (1) and (2) originates from the distortion of the flow pattern by the protrusions and wrinkles of the surface.

For a disclination with topological index $n_i$ (defined by the amount $\theta$ increases along a path enclosing the defect’s core) the geometric potential $E_{\alpha}(u_i)$ reads (Vitelli and Turner, 2004)

$$E_{\alpha}(u_i) = 2\pi K n_i \left(1 - \frac{n_i}{2}\right) U_G(u_i),$$

(4)

where $K$ is the elastic stiffness and $U_G$ is the same potential defined in Eq. (2). Note that the anomalous coupling also contributes to determine the energetics of
liquid crystal disclinations but, in this case, the gauge coupling, which is linear in $n$, is stronger for small $n$.

To understand the physical and mathematical origin of these distinct coupling mechanisms, note that in the ground state of a $^4$He film, the phase $\theta(u)$ can be constant throughout the surface so that the corresponding energy vanishes. In a system with geometric frustration, the gauge coupling between defects and the underlying curvature is mediated by the deformed ground state texture that exists in the liquid crystal layer prior to the introduction of the defects simply as a result of geometrical constraints. Once a defect is introduced it interacts with these preexisting elastic deformations. Unlike the case of orientational order considered previously, no geometrical frustration exists in the superfluid film. The superfluid free energy $F_s$ to be minimized is a simple scalar generalization of the familiar flat space counterpart

$$F_s = \frac{K}{2} \int d^2u \sqrt{g} g^{\alpha\beta} \partial_\alpha \theta(u) \partial_\beta \theta(u).$$

(5)

The crucial point is that no connection $\Omega_\alpha(u)$ is necessary to write down the covariant derivative for this simpler case of a scalar order parameter. Therefore the ground state is given by $\theta(x)$ equal to a constant. There is no preexisting texture for a vortex to interact with, and so another mechanism is required to explain the coupling of vortices to geometry.

In the following sections, we will employ the method of conformal mapping to demonstrate that when an isolated vortex is placed on a curved surface it feels a force as if there were a smeared out topological “image charge,” jointly proportional to the vortex’s own circulation and the Gaussian curvature across the substrate. Such an imaginary topological charge distribution produces a real force analogous to the force on an electrostatic charge due to its mirror image in a conducting surface.

The method of conformal mapping may seem, prima facie, a surprising route to derive a coupling between vortices and geometry, because the free energy of Eq. (5) is invariant under conformal transformations that introduce a non-uniform compression of the surface while keeping local angles unchanged. This invariance property at first seems to rule out the possibility of a geometrical interaction! This apparent contradiction can be seen by choosing a special set of (isothermal) coordinates that can always bring the two dimensional metric tensor in the diagonal form $g_{\alpha\beta}(u) = e^{2\omega(u)} \delta_{\alpha\beta}(u)$ (David, 1989). The result of this step is to eliminate the geometry dependence from the free energy of Eq. (5) since the product $g^{\alpha\beta}(u) \sqrt{g} = \delta_{\alpha\beta}(u)$ and $F_s$ reduces to its counterpart for a planar surface, where there is of course no geometry dependence.

An interaction between vortices and geometry violates this conformal symmetry of the free energy from which it emerges, but in fact the conformal symmetry is not an exact symmetry when vortices are present. Analogous subtleties frequently arise in the study of fields that fluctuate thermally or quantum mechanically, due to the occurrence of a cut-off length scale below which fluctuations cannot occur. A conformal mapping is a strange type of symmetry that stretches lengths and thus does not preserve the microscopic structure of a system. At finite temperatures, the discreteness of a system, such as a thermally fluctuating membrane (Polyakov, 1981) which is actually made up of a network of molecules, can have an important effect because the fluctuations excite modes with microscopic wavelengths. This produces violations of the conformal symmetry at every point of the surface. In a superfluid at zero temperature, however, short wavelengths not describable by the continuum free energy $F_s$ are excited only in the cores of vortices. Obtaining a finite value for the energy necessitates the removal of vortex cores of a certain fixed radius in the local tangent plane, so a conformal mapping is not a symmetry in the neighborhood of a vortex. However the amount by which this symmetry fails can be calculated in a simple form (intriguingly independent of the microscopic model of the cores) in terms of the rescaling function $\omega(u)$ evaluated at the locations of the vortices, where the symmetry fails. Rather than ruling out the possibility of a geometric interaction, a realistic treatment of conformal mapping becomes a powerful mathematical tool for deriving these interactions, a technique which is relevant especially to other branches of theoretical physics such as the study of scattering amplitudes in string theory (Polyakov, 1987).

While the free energy, $F_s$, of the curved superfluid layer in Eq. (5) does not exhibit a geometric gauge field, rotating the sample at a constant angular velocity leads to an energy of the same form as Eq. (5). The resulting forces exerted on the vortices compete with the geometric interactions to determine the equilibrium configurations of an arrangement of topological defects. This simple idea is behind some of the experimental suggestions put forward in this article to map out the geometric potential by progressively increasing the rotational speed while monitoring the equilibrium position of a single vortex on a helium coated surface shaped like the bottom of a wine bottle (Voll et al., 2006). Since the position dependence of the force induced by the rotation is easily calculated, one can read off the geometric interaction by simply assuming force balance.

The theory of curved helium films also helps build intuition for the more general case of vortex lines confined in a bounded three dimensional region such as the cavity shown in cross-section in Fig. 1A. The vortex, drawn as a bold black line, can be pinned by the constriction of the container. The classic problem of understanding the interaction of the vortex with itself and with the bump as the superfluid flows past (Schwarz, 1981), is of crucial importance in elucidating how vortices can be produced when a superfluid starts rotating despite the absence of any friction. A possible mechanism, known as the “vortex mill”, assumes that vortex rings break off a pinned vortex line, while the pinned vortex remains in place (Glaberson and Donnelly, 1966; Schwarz, 1990). The common route to studying vortex dynamics in three-
In section III.A, we derive the forces experienced by vortex-induced vortex hysteresis is conjectured in section II.D. The existence of such geometry can remain trapped in metastable states located at the superfluid film, some of the thermally generated defects can be trapped in regions of negative curvature leading to geometrically confined persistent curved saddle surfaces can be trapped in geometric forces and conventional electrostatics, simple illustrations of the main results and experimental ideas. The second track, sections V.C–VI is more technical and presents a unified derivation of the geometric potential by the method of conformal mapping and its application to the study of complex surface morphologies.

The first track starts with a review of superfluid dynamics that can be used to relate the anomalous coupling to hydrodynamic lift. In Sec. II.A the geometrical force is evaluated, using a mapping between the geometric potential studied here and the familiar Newton's theorem that allows an efficient calculation of the gravitational field for a spherically symmetric mass distribution. An intriguing consequence of this analogy is that vortices on saddle surfaces can be trapped in regions of negative curvature leading to geometrically confined persistent currents as discussed in section II.B. Section II.C relates this observation to Earnshaw’s theorem from electrostatics. Upon heating and subsequently cooling a curved superfluid film, some of the thermally generated defects can remain trapped in metastable states located at the saddles of the substrate. The existence of such geometry induced vortex hysteresis is conjectured in section II.D. In section II.E we derive the forces experienced by vortices when the vessel containing the superfluid layer is rotated around the axis of symmetry of a curved surface shaped as a Gaussian bump. The dependence of single and multiple defect-configurations on different angular speeds and aspect ratios of the bump is studied in sections III.B and III.C. The Abrikosov lattice of vortices on a curved surface is discussed in section III.D. In realistic experimental situations the thickness of the superfluid layer will not be uniform and additional forces will drive vortices towards thinner regions of the sample. The strength of these forces is assessed in section IV and related to spatial variations of the film thickness due to gravity and surface tension. A short discussion of choice of parameters for the proposed rotation experiments follows in section IV.B. The relevance of our discussion to experiments performed in bounded three dimensional samples is addressed in section IV.D.

The second track starts with a general derivation of the geometric potential by the method of conformal mapping in section V.A. The computational efficiency of this approach is illustrated in section V.B where the geometric potential of a vortex is evaluated on an Enneper disk, a strongly deformed minimal surface. We show that changing the geometry of the substrate has interesting effects not only on the one-body geometric potential but also on the two-body interaction between vortices. In section V.C we use conformal methods to show how a periodic lattice of bumps can cause the vortex interaction to become anisotropic. In section V.D we demonstrate that the quantization of circulation leads to an extremely long-range force on an elongated surface with the topology of a sphere. The interaction energy is no longer logarithmic, but now grows linearly with the distance between the two vortices. As we demonstrate, the whole notion of splitting the energy in a one body geometric potential and a vortex-vortex interaction is subject to ambiguities on deformed spheres. Section V.E provides some guidance on how to perform calculations in this context by choosing a convenient Green’s function among the several available. Finally, in section V.F we present a discussion of some general upper bounds which constrain the strength of geometric forces. The conclusion serves as a concise summary and contains a table designed to locate at a glance our main results throughout the article including the more technical points relegated to appendices but useful to perform calculations.

## II. FLUID DYNAMICS AND VORTEX-CURVATURE INTERACTIONS

We start by writing down the collective wave function of the superfluid as

$$\Psi(\mathbf{u}) = \sqrt{\frac{\rho_s(\mathbf{u})}{m_4}} e^{i\theta(\mathbf{u})},$$

where \(\mathbf{u} = \{u_1, u_2\}\) is a set of curvilinear coordinates for the surface, \(m\) is the mass of a \(^4\)He atom and \(\rho_s\)
Thus the superfluidity is destroyed below a core radius $\epsilon$ and a constant core energy $\epsilon_c$ is associated with the disruption of the superfluidity in the core.

The circulation along a path $C$ enclosing a vortex is given by

$$\oint_C du^\alpha v_\alpha = n\kappa,$$

where the quantum of circulation, $\kappa = \frac{h}{m_4}$, is equal to $9.98 \times 10^{-8} \text{ m}^2 \text{s}^{-1}$ and the integer $n$ is the topological index of the vortex. The free energy can be cast in the form

$$F = \frac{1}{2} \rho_s \frac{h^2}{m_4^2} \int_S d^2u \sqrt{g} g^{\alpha\beta} \partial_\alpha \theta \partial_\beta \theta,$$

where $g^{\alpha\beta}$ is the (inverse) metric tensor describing the surface on which the superfluid layer lies and $g$ is its determinant. We will often use the superfluid stiffness

$$K = \frac{\rho_s h^2}{m_4^2}.$$

This expression for the free-energy can be parameterized in terms of the vortex positions once the seemingly divergent kinetic energy near a vortex core is correctly accounted for. As is well known, the radius-independence of the circulation about a vortex implies that the velocity is given by $V = \frac{\hbar}{m_4 \epsilon}$, which leads to a logarithmic divergence in $V$. The energy stored in an annulus of internal radius $r_{in}$ and outer radius $r_{out}$ reads

$$E_{\text{mech}} = \pi K \ln \frac{r_{out}}{r_{in}},$$

which diverges as $r_{in} \to 0$. A physical trait of superfluid helium prevents this from happening: it cannot sustain speeds which are greater than the critical velocity.

Thus the superfluidity is destroyed below a core radius of $a \sim \frac{\hbar}{m_4 \epsilon}$. This breakdown may be modeled by excising a disk of radius $a$ around each vortex and by adding a constant core energy $\epsilon_c$ to account for the energy associated with the disruption of the superfluidity in the core.

Starting on the flat plane, the interaction of two vortices can now be determined. Superimposing the fields of the two vortices and integrating the cross-term in the kinetic energy of Eq. (4) leads to a Coulomb-like interaction,

$$V_{ij} = -2\pi K n_i n_j \ln \frac{|\mathbf{u}_i - \mathbf{u}_j|}{\hbar},$$

in addition to vortex self-energies. In deducing the force between the vortices from this expression, it is useful to assume that $a$ does not vary significantly with position. The justification for this simplification is that the background flow due to other vortices only gives a fractionally small correction to the flow near each vortex, and therefore barely affects where the critical velocity is attained.

For the more complicated case of a curved surface, with a very distant boundary (see [Vitelli and Nelson, 2004] for the discussion of effects due to a boundary at a finite distance), we found in Ref. [Vitelli and Turner, 2004] that the energy including both single-particle and two-particle interactions is

$$E((q_i, u_i)) = \frac{4\pi^2 n_i n_j V_{ij}(u_i, u_j) + \sum_i (-\pi n_i^2 U_G(u_i))}{K},$$

apart from a position-independent term (given for a distant circular boundary of radius $R$ by $\frac{2}{\pi} \sum_i n_i^2 \ln \frac{R}{a} + N \frac{\epsilon_c}{\kappa}$, with $N$ the total number of vortices and $\epsilon_c$ the core energy of one of them). The pair potential $V_{ij} = \Gamma(u_i, u_j)$ is expressed in terms of $\Gamma$, the Green’s function of the covariant Laplacian defined by:

$$\nabla^2 \Gamma(u, v) = -\delta_c(u, v)$$

Note that the covariant delta function $\delta_c$ includes a factor of $\frac{\sqrt{g}}{v_2}$ so that its integral with respect to the “proper area” $\sqrt{g} du_1 du_2$ is normalized. Equation (13) determines the Green’s function up to a constant provided that we assume additionally that the Green’s function is symmetric between its two arguments. The constant is fixed by assuming that at large separations, the Green’s function approaches the Green’s function of an undeformed plane. This expression shows that vortices behave like electrostatic particles, with charges given by $2\pi n_i$ and coupling constant $K$.

The single-particle potential $U_G(u)$ is the “geometric potential” defined in Eq. (2). This potential entails a repulsion $\nabla \pi K U_G$ of vortices of either sign from positive curvature and an attraction to negative curvature. The following analogy with boundary interactions and image charges is useful. The flow-field is modified by having to conform to the curvature, leading to an image charge of the vortex which is spread continuously over the surface, with density $-\frac{\pi}{\kappa} G(u)$. Just like the image of a vortex in a circular boundary has an equal and opposite circulation, the continuous image of the vortex in the curvature has a charge density proportional to the number of quanta $n$ in the vortex.

This point of view may be connected to fluid mechanics by analyzing the streamlines on the bump and in the presence of a circular boundary as illustrated in Fig. 2. Streamlines are tangent to the direction of flow and their density is proportional to the local speed. Only an incompressible velocity field (div $v = 0$) may be described by streamlines, since incompressibility ensures that any closed curve has an equal number of streamlines entering and exiting. This condition is satisfied for superfluids (far below the critical speed) since minimizing Eq. (10) leads to $\nabla \cdot \nabla \theta = 0$, or div $v = 0$ according to Eq. (7). Thus the flow is both irrotational (the circulation around any
curve, not enclosing a vortex, is 0) and incompressible:
\[
\begin{align*}
\text{div } \mathbf{v} &= 0 \\
\text{curl } \mathbf{v} &= 0.
\end{align*}
\] (14)

The former relation implies that we may write
\[
\mathbf{v} = \text{curl } \chi \mathbf{\hat{n}};
\] (15)

so that the streamlines are equally spaced level curves of \( \chi \). (For example, around a vortex, the radii of the successive streamlines forms a geometric sequence, \( r(1 - \epsilon)^i \) where \( \epsilon \) sets the ratio between flowline density and speed.)

Now consider, as an illustration, the flow field on the slope of the bump represented in Fig. 2. The curves must spread out to go over the bump, leading to a lower velocity above the vortex. By Bernoulli’s principle (true for irrotational flows), this creates a high pressure that pushes the vortex away from the bump. Note, however, that the actual motion of a vortex is more subtle: Although the gradient of the energy points away from the bump, a vortex (disregarding friction) always moves at right angles to the gradient of the energy. Thus a vortex in the absence of drag forces actually circles around the bump, in the same direction as the fluid flows around the vortex. Dissipation converts the motion into an outward spiral.\footnote{Ambegaokar et al., 1978}. We will not study the dynamics.

A convenient mathematical formulation of the problem of determining the flow pattern of a collection of vortices is obtained by introducing the scalar function \( \chi(\mathbf{u}) \) which satisfies
\[
\nabla^2 \chi(\mathbf{u}) = -\sum_i 2\pi n_i \delta_r(\mathbf{u}, \mathbf{u}_i) \equiv -\sigma(\mathbf{u}).
\] (16)

The sum can be described as a singular distribution of surface charge. This relation follows from the circulation condition, \( 2\pi n_i = \oint \nabla \theta \cdot d\mathbf{l} \), which can be rewritten as the integral of the flux of \( \nabla \chi \) through the boundary, \( \oint \nabla \chi \cdot \mathbf{\hat{n}} d\mathbf{l} \), by using Eq. (15). In analogy with Gauss’s law, there must therefore be delta-function sources for \( \chi \) at the locations of the vortices, as described by Eq. (16). Solving Eq. (16) in terms of the Green’s function gives:
\[
\chi(\mathbf{u}) = \sum_i \frac{h_{n_i}}{m} G_i(\mathbf{u}, \mathbf{u}_i).
\] (17)

The flow due to a given vortex is proportional to its “charge” \( 2\pi n_i \). The energy as a function of the positions of the vortices, Eq. (12), can now be derived by integrating the kinetic energy in the flow determined by Eq. (16) for each placement of the vortices.

We will begin by discussing applications of Eq. (12), saving its derivation until later. Interestingly, the energy of the vortices can be described by a differential equation analogous to Eq. (16) for the flow. We choose one vortex and fix the positions of all the others. We take the Laplacian of Eq. (12) with respect to the position of the chosen vortex and use Eq. (2) and Eq. (13). The energy as a function of the chosen vortex \( E(\mathbf{u}_i) \) satisfies:
\[
\nabla^2 \frac{E(\mathbf{u}_i)}{2\pi kn_i} = -\sigma_i(\mathbf{u}_i) - \frac{n_i}{2} G(\mathbf{u}_i).
\] (18)

The notation \( \sigma_i \) stands for the delta function charge distributions of all the vortices with the exception of the \( i \)th, so that \( \sigma_i(\mathbf{u}) = \sum_{j\neq i} 2\pi n_j \delta(\mathbf{u} - \mathbf{u}_j) \). The self-charge term which we have had to remove (so that \( \mathbf{u}_i \) can be substituted in place of \( \mathbf{u} \) as in Eq. (18)) is replaced here by a spread-out charge proportional to the removed term.

A. Anomalous force on rotationally symmetric surfaces

On an azimuthally symmetric surface the force on a defect can be found by exploiting Gauss’s law. This is analogous to the familiar “Newton’s Shell” theorem that predicts the gravitational field at the surface of a sphere surrounding a spherically symmetric mass distribution by concentrating all the enclosed mass at the center.

Points on an azimuthally symmetric two dimensional surface embedded in three dimensional Euclidean space are specified by a three dimensional vector \( \mathbf{R}(r, \phi) \) given by
\[
\mathbf{R}(r, \phi) = \begin{pmatrix} r \cos \phi \\ r \sin \phi \\ h(r) \end{pmatrix},
\] (19)

where \( r \) and \( \phi \) are plane polar coordinates in the \( xy \) plane of Fig. 3 and \( h(r) \) is the height as a function of radius; e.g., \( h(r) = h_0 \exp -\frac{r^2}{\alpha^2} \) for the Gaussian bump with height \( h_0 \) and spatial extent \( \sim r_0 \). It is useful to characterize the deviation of the bump from a plane in terms of a dimensionless aspect ratio
\[
\alpha = \frac{h_0}{r_0}.
\] (20)

The metric tensor, \( g_{\alpha\beta} \), is diagonal for this choice of coordinates. In general, \( g_{\phi\phi} = r^2 \), \( g_{rr} = 1 + h'(r)^2 \), and for the Gaussian bump we have
\[
g_{\alpha\beta} = \begin{pmatrix} 1 + \alpha^2 r^2 \exp -\frac{r^2}{\alpha^2} & 0 \\ 0 & 1 \end{pmatrix}.
\] (21)

Note that the \( g_{\phi\phi} \) entry is equal to the flat space result \( r^2 \) in polar coordinates while \( g_{rr} \) is modified in a way that depends on \( \alpha \) but tends to the plane result \( g_{rr} = 1 \) for both small and large \( r \).

The Gaussian curvature for the bump is readily found from the eigenvalues of the second fundamental form \( \text{Dubrovin et al., 1992} \); e.g., for the Gaussian bump,
\[
G(\mathbf{r}) = \frac{\alpha^2 e^{-\frac{r^2}{\alpha^2}}}{r_0^2 \left( 1 + \alpha^2 r^2 \exp -\frac{r^2}{\alpha^2} \right)^2} \left( 1 - \frac{r^2}{r_0^2} \right).
\] (22)
FIG. 2 Left, the flow around a vortex situated on the side of a Gaussian bump, calculated with the methods described in this paper. The low density of flow lines above the vortex indicates a lower velocity and thus a higher pressure, leading to the repulsion represented in Eq. (12), \(-\nabla(-\pi Kn^2 U_G)\). Right, the analogous flow around a vortex in a disk with a solid boundary. The attraction to the boundary is also seen to result from high speeds, since the flow lines are compressed in between the vortex and the boundary.

FIG. 3 (a) A bumpy surface shaped as a Gaussian. (b) Top view of (a) showing a schematic representation of the positive and negative intrinsic curvature as a non-uniform background “charge” distribution that switches sign at \(r = r_0\). The varying density of + and - signs tries to mimic the changing curvature of the bump.

Note that \(\alpha\) controls the overall magnitude of \(G(r)\) and that \(G(r)\) changes sign at \(r = r_0\) (see Fig. 3b). The integrated Gaussian curvature inside a cup of radius \(r\) centered on the bump vanishes as \(r \to \infty\). The positive Gaussian curvature enclosed within the radius \(r_0\) (see Fig. 3) approaches \(2\pi\) for \(\alpha \gg 1\), half the integrated Gaussian curvature of a sphere. We shall show below that there is always more positive than negative curvature within any given radius, for an azimuthally symmetric surface. It will follow that the force on a vortex is repulsive at any distance.

In general an individual vortex of index \(n_i\) confined on a curved surface at position \(u_i\) feels a geometric interaction described by the energy

\[
E(u_i) = -\pi Kn_i^2 U_G(u_i). \tag{23}
\]

For an azimuthally symmetric surface such as the bump represented in Fig. 3, we can derive Newton’s theorem as follows. Define \(E = -\nabla U_G\) so that the covariant radial component of \(E\) is \(E_r = -\partial_r U_G\). Then \(-\nabla^2 U_G = \text{div} E = \frac{1}{\sqrt{g}} \partial_r \sqrt{g} g^{rr} E_r\), and if we integrate both sides of Eq. (2) out to \(r\),

\[
2\pi \sqrt{g} g^{rr} E_r = -\int \int \sqrt{g} dr d\phi G(r) \tag{24}
\]

so that \(E_r\) has a simple expression in terms of the net Gaussian curvature at a radius less than \(r\). Now \(E_r\) is the “covariant component” of the geometrical “electric field”, not the actual field, which would be obtained by differentiating with respect to arclength rather than the projected coordinate \(r\). Therefore the magnitude of \(E\) is \(\frac{E_r}{\sqrt{g_{rr}}}\), which, rephrasing Eq. (24), obeys this generalized version of Newton’s theorem:

The magnitude of \(E\) is \(-\frac{1}{2\pi r}\) times the integrated Gaussian curvature.

(Recall that \(g^{rr} = \frac{1}{g_{rr}}\) and \(g = \det g_{\alpha\beta} = r g_{rr}\).) Note that the force on the vortex is \(\mathbf{F} = -\nabla E = -\pi Kn_i^2 \mathbf{E}\) according to Eq. (23), so that the geometrical force is proportional to the integrated Gaussian curvature divided by the distance of the vortex from the axis of symmetry of the surface; the expression for the force on the vortex obtained in the next section by integrating the Gaussian curvature is

\[
F_{\text{geom}} = \frac{K \pi}{r} \left(1 - \frac{1}{\sqrt{1 + \frac{h^2}{r^2}}}ight). \tag{25}
\]

if \(n_i = \pm 1\). Note that this force is always repulsive since the integrated curvature is positive.

The geometric potential can now be expressed explicitly by integrating \(E_r = -\partial_r U_G\) with the aid of Eq. (24):

\[
U_G(r) = -\int_{\rho}^{\infty} dr' \frac{\sqrt{1 + \frac{\alpha^2 r'^2}{r_0^2} \exp\left(-\frac{r'^2}{\alpha^2}\right) - 1}}{r^{\alpha/2}}. \tag{26}
\]
FIG. 4 Plot of the interaction energy $E(r) = -\pi KU_G(r)$ between a singly quantized vortex and a Gaussian bump with $\alpha = 1$. The energy is measured in units of $K$ and the radius is measured in units of $r_0$. Note that the force points away from the bump and has its maximum strength near $r_0$.

The resulting potential $U_G(r)$ vanishes at infinity. Its range and strength are given respectively by the linear size of the bump and its aspect ratio squared (see Fig. 4).

We now summarize an intuitive argument that explains why the energy of a vortex on top of the bump is greater than the energy of a vortex that is far away (Halperin; Vitelli and Turner, 2004). Fig. 5, reproduced from (Vitelli and Turner, 2004), focuses on a rotationally symmetric bump coated by a helium film (of a constant thickness). We can estimate the energy of a vortex on top of the bump by comparing the situation to a vortex on a plane, illustrated vertically below the bump. Rotational symmetry implies that the superfluid phase is given by $\theta = \phi$, the azimuthal angle. The velocity depends on the rate of change on the phase according to Eq. (7), so since an infinitesimal arc of the circle concentric with the top of the bump has size $r \, d\phi$, the velocity is $\frac{\hbar}{m} r$. Here $r$ is the radius of the circle measured horizontally to the axis of the bump. This calculation shows that the velocity, and thus the energy density, are the same at any point on the bump and its projection into the plane. However, the energy contained in the tilted annulus on the bump stretching from $r$ to $r + dr$ is greater than the energy in the annulus directly below it because, though the energy density is the same, the annulus's area is greater. Hence a vortex on a bump has a greater energy than a vortex in a plane, whether it is the vortex at $P'$ in the projection plane, or the vortex at $Q$ which is very far from the bump. This reasoning indicates a repulsive force, since the vortex lowers its energy by moving away from the bump.

The intuitive argument suggests that the Gaussian curvature should appear in the force law, as in Eq. (2); in fact, it is a widely known fact that the sign of the Gaussian curvature of a surface determines how fast the circumference of a circle on the surface increases, relative to the circumference of a circle on the plane, as a function of the radius. Of course, for less symmetric surfaces, comparing the energy on a curved surface to that on a flat reference plane directly below it will be more complicated, since symmetry and the constant circulation constraint do not force the energy densities to be equal. Thus, a simple vertical projection will not set up a monotonic relation between energies. The conformal mapping technique which we use in Sec. IV is a variation on the idea of comparing a “target” substrate to a simple “reference” surface which is in principle applicable to arbitrary surfaces, and furthermore not only allows one to compare energies, but also to calculate them quantitatively. The technique can also be used to give a concise derivation of Eq. (2).

Such an intuitive argument applies only for azimuthally symmetric surfaces. For less symmetric surfaces, comparing the energy on a curved surface to that on a flat reference plane directly below it will be more complicated, since symmetry and the constant circulation constraint do not force energy densities to be equal at corresponding positions. Thus, a simple vertical projection will not set up a monotonic relation between energies. The conformal mapping technique which we use in Sec. IV is a variation on the idea of comparing a “target” substrate to a simple “reference” surface which is in principle applicable to arbitrary surfaces, and furthermore not only allows one to compare energies, but also to calculate them quantitatively. The technique can also be used to give a concise derivation of Eq. (2).

B. Vortex-trapping surfaces

In order to illustrate the consequences of the curvature-induced interaction for different surface morphologies, we
study a “Gaussian saddle” surface (suggested to us by Stuart Trugman) for which the geometric potential has its absolute minimum at the origin.

First, we show that an alternative design for a vortex trap geometry fails because of the long-range nature of the curvature-induced interaction. Fig. 6 shows that bumps have negative curvature on their flanks; it might seem possible that a well-chosen bump would have enough negative curvature to hold a vortex. However, a vortex cannot be held by nearby negative curvature alone; it also feels the positive curvature at the center of symmetry because, according to Gauss’s law applied to the azimuthally symmetric region, the force is due to the net curvature induced at the center. This curvature is given generally by

$$G = \frac{1}{r \sqrt{1 + h'(r)^2}} \partial_r \frac{1}{\sqrt{1 + h'(r)^2}},$$

and the net curvature within radius \( r_v \) is thus

$$\int_0^{r_v} \sqrt{\theta} \, dr \, d\phi \, G(r) = 2\pi (1 - \frac{1}{\sqrt{1 + h'(r_v)^2}}) = 2\pi (1 - \cos \theta [r_v])$$

where \( \theta \) is the angle between the surface at the location of the vortex and the horizontal plane. This formula also describes the cone angle of a cone tangent to the surface at radius \( r_v \); it can also be derived from the Gauss-Bonnet theorem which implies that the net curvature of a curved region depends only on the boundary of the region and how it is embedded in a small strip containing it; thus the net curvature of the cone (concentrated at the sharp point of the cone) is the same as the net curvature of the bump which is tangent to. Since this curvature is always positive, the defect is always repelled from the top of an azimuthally symmetric bump. To find a way to trap a vortex, one must therefore investigate some non-symmetric surfaces. The curvature-defect interaction energy on a generic surface is mediated by the Green’s function of the surface, Eq. (13), as can be seen by solving Eq. (2):

$$E(u) = K \pi \int d^2 u \, \Gamma(u, v) G(v),$$

(29)

for a singly quantized vortex. One such surface, which we will treat perturbatively in the amount of deformation from flatness (Section V.B treats a different confining surface exactly) is the Gaussian saddle represented in Fig. 6 and described by the height function

$$h_\lambda(x, y) = \frac{\alpha}{r_0 (x^2 - \lambda y^2)} e^{-\frac{x^2 + y^2}{r_0^2}}$$

(30)

where the exponential factor was included to make the surface flat away from the saddle (Trugman). Here, \( \lambda \) is a parameter which we later vary to illustrate the non-locality of the interaction. The leading order contribution to the curvature-defect interaction (for \( \alpha \ll 1 \)) is of the same order \( \alpha^2 \) as the curvature corrections to the defect-defect interaction (calculated in Appendix A). In fact \( \Gamma \) is multiplied in Eq. (29) by the Gaussian curvature \( G(u') \), of order \( \alpha^2 \). Thus, for a single defect, it is sufficient to use the flat space Green’s function \( \Gamma_{flat} \)

$$\Gamma_{flat}(x - x', y - y') = -\frac{1}{2\pi} \log \sqrt{(x - x')^2 + (y - y')^2}$$

(31)

for calculating the defect-curvature interaction. Furthermore, the Gaussian curvature that acts as the source of the geometric potential can be calculated from the usual second-order approximation:

$$G_\lambda(x, y) \approx \frac{\partial^2 h_\lambda}{\partial x^2} \frac{\partial^2 h_\lambda}{\partial y^2} - \left( \frac{\partial^2 h_\lambda}{\partial x \partial y} \right)^2$$

(32)

This function is plotted in Fig. 7 and its sign is represented in the middle frame of Figure 10. The graph of the vortex-curvature interaction energy for this surface, Fig. 8, shows that a vortex is indeed confined at the center of the saddle; the energy graphed in this figure is given by

$$E_\lambda(x, y) \approx K \pi \int dx' \, dy' \, \Gamma_{flat}(x - x', y - y') G_\lambda(x', y'),$$

(33)

for \( \lambda = 1 \). For realistic film thicknesses and \( \alpha \) of order unity (see Sec. IV), the depth of the well is about 50

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1 More specifically, all bumps with azimuthally symmetric embeddings repel vortices from their tops. The negative curvature cones discussed in Sec. V.B have an internal azimuthal symmetry but their three dimensional embeddings are not symmetric.
Kelvin! We have found that the energy associated with a vortex at the origin is less than for any other position. Of course, the configuration with a vortex at the origin cannot beat the configuration with no vortices at all! The latter has zero kinetic energy; when the vortex is at the origin, the energy is positive provided that Eq. (12) is supplemented by the position-independent contribution $\pi K \ln \frac{R}{a}$. This term is always necessary for comparing configurations with different numbers of defects, as when one studies the formation of a vortex lattice at increasing rotational frequencies (Campbell and Ziff, 1979).

C. Negative curvature which does not trap

In this section, we shall discuss what happens when the parameter $\lambda$ of the saddle surface is increased; Fig. 9 illustrates such a surface corresponding to $\lambda = 17$. To give a hint of what causes the equilibrium to change its character, Fig. 10 shows the sign of the curvature for the Gaussian bump, the saddle with $\lambda = 1$, and the saddle with $\lambda = 17$.

In the graph of the defect-curvature interaction energy with $\lambda = 17$, one notices that the origin is an unstable equilibrium position for the vortex. We will derive the exact value of $\lambda$ where this instability first occurs below. However, symmetry considerations alone show that the origin is a stable equilibrium point when $\lambda = 1$, as Fig. 8 shows. One might be tempted to argue from Newton’s theorem that a vortex at a small enough radius $r$ is always attracted to the origin by the negative curvature at radii smaller than $r$. However, the asymmetry of the saddle surfaces invalidates Newton’s theorem and positive curvature more distant from the origin than the vortex might be able to push the vortex toward infinity. This does not occur for the saddle surface with $\lambda = 1$; although the rotational symmetry needed for Newton’s theorem is absent, the surface does have order four symmetry, under a 90 degree rotation combined with the isometry $z \rightarrow -z$.

Upon expanding the defect-curvature interaction energy about the origin, we obtain

$$E = E_0 + ax + by + cx^2 + 2dxy + ey^2 + \cdots$$ (34)

This energy must be invariant under the symmetries of the surface (without a sign change). Order two symmetry implies that the linear terms vanish, so the center point is an equilibrium. The order four symmetry (apparent in Fig. 10B) implies that it is either a maximum or a minimum (a quadratic function with a saddle point has only 180 degree symmetry). In more detail, 90° rotational symmetry, given by $x \rightarrow y$, $y \rightarrow -x$, implies that $c = e, d = 0$. Since the Laplacian of $E$ at the origin is proportional to minus the local curvature, $2c = 2e = c+e$ is positive, so the origin is a local minimum. Without the order four symmetry the negative
curvature only ensures that \( c + e > 0 \). Earnshaw’s theorem of electrostatics [Earnshaw 1842; Jeans 1927; Scott 1953] states that an electric charge cannot have a stable equilibrium at a point where the charge density is zero or has the same sign as the charge. The charge cannot be confined by electric fields produced by electrostatic charge distributions in a surrounding apparatus. (This theorem motivated the design of magnetic and electrodynamic traps for trapping charged particles in plasma physics and atomic physics.) The argument provided here can be generalized to give the following converse rule based on discrete symmetries (whereas Newton’s theorem applies only for continuous azimuthal symmetry):

If \( P \) is a symmetry point of a charge distribution with rotation angle \( \frac{2\pi}{m} \) and \( m \geq 3 \), and the charge density at \( P \) is positive, then \( P \) is a point of stable equilibrium for particles of negative charge.

This is the formulation for electrical charges in two dimensions; for vortices, the sense of the rotation of the vortex does not matter of course, since the vortex interacts with its own image charge distribution. Hence if the curvature at \( P \) is negative, then a vortex will be trapped there.

Similar reasoning can be used to show that a generalization of Eq. (30), the “Gaussian Monkey Saddle” given by \( h(x, y) = \frac{\alpha}{r_0^2} \Re(x - iy)^3 e^{-\frac{x^2 + y^2}{2r_0^2}} \), traps vortices in an energy well of the form \( E = E_0 + \frac{9\pi K r_1^4}{4 r_0^4} + (\text{const.} + \text{const.} \cos 6\theta) r^6 + \ldots \). The reasoning needs to be modified because the curvature at the origin of the monkey-saddle is zero and the trapping is due to the negative curvature near the origin.

At a point of low symmetry (such as the origin in Eq. (30) when \( \lambda \neq 1 \)), the character of an extremum depends on the charge distribution elsewhere, since the previous argument only implies that \( c + e > 0 \). Fig. 10C suggests that a vortex at the origin is destabilized by its repulsion from the positive curvature above and below the origin, which is not balanced by enough positive curvature to the left and right. In fact, more detailed calculations show that the range of \( \lambda \) for which the origin is an energy minimum is \( \sqrt{65} - 8 < \lambda < \sqrt{65} + 8 \); the origin is a saddle point outside this range, as is just barely visible for the case of \( \lambda = 17 \) in Fig. 11. (Likewise, for negative values of \( \lambda \), the origin is a maximum when \( \sqrt{65} - 8 < -\lambda < \sqrt{65} + 8 \), but a saddle point outside this range.)

These results follow by changing the integration variables to \( \xi = x - x', \eta = y - y' \) in Eq. (33) and then expand-
ing to second order about the origin \((x, y) = (0, 0)\). The integral expressions for second derivatives of the energy can be evaluated explicitly,

\[
E(x, y) = K \pi \frac{\alpha^2}{16} \frac{1 + \lambda^2 - 6\lambda}{\lambda}
+ \frac{x^2}{4} \left(\alpha^2 \frac{\lambda^2 - 1}{4\lambda^2} - G_0\right)
+ \frac{y^2}{4} \left(\alpha^2 \frac{1 - \lambda^2}{4\lambda^2} - G_0\right)
\]

where \(G_0 = -4\lambda^2\) is the curvature at the origin. In Appendix B, we determine the geometric potential for arbitrary \(x\) and \(y\) in (unwieldy) closed form.

D. Hysteresis of vortices and trapping strength

The geometrical interaction has its maximum strength when the Gaussian curvature is the strongest. However, the geometric charge (i.e., integrated Gaussian curvature) of any particular feature on a surface has a strength roughly equivalent at most to the charge of one or two vortices. Eq. (23) therefore suggests that the force on a vortex due to a feature of the surface is less than the force due to a couple vortices at the same distance. Precise limits on the strength of the geometric interaction will be stated and proven in Section VI for arbitrary geometries.

As a consequence the geometric interaction has its most significant effects when the number of vortices is comparable to the number of bumps and saddles on the surface, so that the geometrical force is not obscured by interactions with the other vortices. This is a recurring (melancholy) theme of our calculations, to be illustrated in Section III for arrangements of vortices in a rotating film. The current section illustrates the point by discussing hysteresis on a surface with multiple saddle points (i.e., traps). If a vortex-free superfluid film is heated, many vortices form in pairs of opposite signs. When it is cooled again, positive and negative vortices can remain trapped in metastable states in the saddles, but even with the strongest curvature possible, the argument above suggests that not more than one vortex can be trapped per saddle.

The effectiveness of the defect trapping by geometry is determined mainly by the ratio of the saddle density to vortex density. As shown in the previous section, the geometric energy near the center of a vortex trap with \(90^\circ\) symmetry is given by

\[
E(r) \approx \frac{\pi}{4} K |G_0| r^2.
\]

The force on the vortex found by differentiating the energy reads

\[
F(r) \approx -\frac{\pi}{2} K |G_0| r.
\]

Eq. (37) shows that the trap pulls the vortex more and more strongly as the vortex is pulled away from the center, like a spring, until the vortex reaches the end of the trap at a distance of the order of \(r_0\) where the force starts decreasing. Since \(G_0 \sim \frac{\Omega}{r_0}^2\) (which is valid for a small aspect ratio \(\alpha\)), “the spring breaks down” when the vortex is pulled with a force greater than

\[
F_{\text{max}} \approx F(r_0) \sim \frac{K \alpha^2}{r_0}.
\]

Let us consider a pair of saddles separated by distance \(d\). It is possible that one vortex can be trapped in each saddle even for a small \(\alpha\) provided that \(d\) is large enough. Remote vortices do not interact strongly enough to push one another out of their traps. The Coulomb attraction or repulsion of the vortices must be weaker than the breakdown force of the trap \(F_{\text{max}}\), i.e., \(\frac{K \alpha^2}{r_0} \geq \frac{K \alpha^2}{r_0}\). The minimum distance between the two saddles is therefore

\[
d_{\text{min}} \sim \frac{r_0}{\alpha}.\]

Let us find the maximum density of trapped vortices that can remain when the helium film is cooled through the Kosterlitz-Thouless temperature. Let us suppose there is a lattice of saddles forming a bumpy texture like a chicken skin. Suppose bumps cover the whole surface, so that the spacing between the saddles is of order \(r_0\). Then not every saddle can trap a vortex; the largest density of saddles which trap vortices is of the order of \(1/d_{\text{min}}^2\), so the fraction of saddles which ultimately contain vortices is at most \(\frac{r_0^2}{d_{\text{min}}^2} \propto \alpha^2\). Note that not as many vortices can be trapped if they all have the same sign, since the interactions from distant vortices add up producing a very large net force. On the other hand, producing vortices of both signs by heating and then cooling the helium film results in screened vortex interactions which are weaker and hence less likely to push the defects out of the metastable states in which they are trapped.

III. ROTATING SUPERFLUID FILMS ON A CORRUGATED SUBSTRATE

A. The effect of rotation

Suppose that the vessel containing the superfluid layer is rotated around the axis of symmetry of the Gaussian bump with angular velocity \(\Omega = \Omega \hat{z}\), as might occur at the bottom of a spinning wine bottle. The container can rotate independently of the superfluid in it because there is no friction between the two. However, a state with vanishing superfluid angular momentum is not the ground state. To see this, note that the energy, \(E_{\text{rot}}\), in a frame rotating at angular velocity \(\Omega\) is given by:

\[
E_{\text{rot}} = E - \mathbf{L} \cdot \Omega.
\]

where \(E\) is the energy in the laboratory frame and \(\mathbf{L}\) is the angular momentum. Hence \(E_{\text{rot}}\) is lowered when \(\mathbf{L} \cdot \Omega > 0\), that is, when the circulation in the superfluid is non-vanishing. This is achieved by introducing
quantized vortices in the system (see Eq. (8)), whose microscopic core radius (of the order of a few Å) is made of normal rather than superfluid component. The energy of rotation, $L \cdot \Omega$, corresponding to a vortex at position $x, y$ on the bump can be evaluated from

$$L_z = \rho_s \oint_S dxdy \sqrt{g(x,y)} (xv_y - yv_x) . \quad (41)$$

Upon casting the integral in Eq. (41) in polar coordinates $r, \phi$ and using the identity

$$(xv_y - yv_x) = r\dot{\phi} \cdot \mathbf{v} \ , \quad (42)$$

we obtain

$$L_z = \rho_s \int_0^R dr \sqrt{g(r)} \oint_C du^\alpha v_\alpha . \quad (43)$$

where $R$ is the size of the system. The line integral in Eq. (43) of radius is evaluated over circular contours of radius $r$ centered at the origin of the bump. The circulation vanishes if the vortex of strength $n$ at distance $r_v$ is not enclosed by the contour of radius $r$:

$$\oint_{C_r} du^\alpha v_\alpha = n\kappa (r - r_v) . \quad (44)$$

Upon substituting in Eq. (43), we obtain

$$L_z = \frac{n\rho_s\kappa}{2\pi} \int_{r_0}^R dr \sqrt{g(r)} (A(R) - A(r_v)) , \quad (45)$$

where $A(R)$ is the total area spanned by the bump and $A(r_v)$ is the area of the cup of the bump bounded by the position of the vortex. Thus, after suppressing a constant, the rotation generates an approximately parabolic potential energy $E_{\Omega}(r)$ (see Fig. 13) that confines a vortex of positive index $n$ close to the axis of rotation as in flat space:

$$E_{\Omega}(r_v) = \frac{n\hbar\Omega\rho_s}{m_4} A(r_v), \quad (46)$$

where a constant has been neglected. One recovers the flat space result [Vinen, 1969] by setting $\alpha$ equal to zero. Eq. (45) has an appealing intuitive interpretation as the total number of superfluid atoms beyond the vortex, $\frac{\rho_s}{m} (A(R) - A(r))$, times a quantum of angular momentum $\hbar$ carried by each of them. The closer the vortex is to the axis, the more atoms there are rotating with the container.

Above a critical frequency $\Omega_1$, the restoring force due to the rotation (the gradient of Eq. (45)) is greater than the attraction to the boundary. The energy of attraction to the boundary is approximately $\pi K \ln(1 + r^2/\rho^2)$, where we assume the aspect ratio of the bump is small so that the flat space result is recovered. Upon expanding this boundary potential harmonically about the origin and comparing to Eq. (46), one sees that

$$\Omega_1 \sim \frac{\hbar}{mR^2} . \quad (47)$$

Above $\Omega_1$, the origin is a local minimum in the energy function for a single vortex, though higher frequencies are necessary to produce the vortex in the first place. What determines the critical frequency for producing a vortex is unclear. There is a higher frequency $\Omega_1 \sim \frac{\hbar}{mR^2} \ln \frac{R}{a}$, at which the single vortex actually has a lower energy (according to Eq. (45)) than no vortex at all, but critical speeds are rarely in agreement with the measured values [Vinen, 1963]. In the context of thin layers, it is likely that a third, much larger critical speed $\Omega_{crit} \sim \frac{\hbar}{mRD_0}$, is necessary before vortices form spontaneously, where $D_0$ is the thickness of the film (see Sec. LV.D).

B. Single defect ground state

The equilibrium position of an isolated vortex far from the boundary is determined from the competition between the confining potential caused by the rotation and the geometric interaction that pushes the vortex away from the top of the bump. The energy of the vortex, $E(r)$, as a function of its radial distance from the center of the bump is given up to a constant by the sum of the geometric potential and the potential due to rotation,

$$\frac{E(r)}{K} = -\pi U_G(r) + \frac{A(r)}{\lambda^2} , \quad (48)$$

where we have ignored the effects of the distant boundary, boundary effects are discussed in the next section. The “rotational length” $\lambda$ is defined as

$$\lambda \equiv \sqrt{\frac{\hbar}{m\Omega}} . \quad (49)$$

A helium atom at radius $\lambda$ from the origin rotating with the frequency of the substrate has a single quantum of angular momentum. The geometric contribution to $E(r)$ (see Fig. 12) varies strongly as the shape of the substrate is changed. The rotation contribution to $E(r)$ confinement (see Fig. 13) varies predominantly as the frequency is changed; near the center of rotation, where the substrate is parallel to the horizontal plane, the rotational contribution barely changes as $\alpha$ is increased.

As one varies $\alpha$ (fixing $r_0$ and $\Omega$) there is a transition to an asymmetric minimum. In fact, Fig. 14 reveals that for $\alpha$ greater than a critical value $\alpha_c$ the total energy $E(r)$ assumes a Mexican hat shape whose minimum is offset from the top of the bump. The position of this minimum is found by taking a derivative of Eq. (48) with respect to $r$:

$$\pi \frac{dU_G}{dr} = \frac{1}{\lambda^2} \frac{dA}{dr} . \quad (50)$$
Now $\frac{dA}{dr}$ can be shown to equal $2\pi r \sqrt{1 + h^2}$ by differentiating Eq. 15 and $\frac{d\theta G}{dr}$, which is the same as $F_G \sqrt{1 + h^2}$ can be evaluated by substituting for $F_G$ from Eq. 24. This leads to an implicit equation for the position of the minimum, $r_m$, namely

$$r_m = \frac{\lambda}{\theta} \sin\left(\frac{\theta r_m}{2}\right).$$

(51)

Here $\theta(r)$, defined in Sec. 11.B, is the angle that the tangent at $r$ to the bump forms with a horizontal plane. A simple construction allows one to solve Eq. 51 graphically by finding the intercept(s) of the curve on the right-hand side with the straight line of slope $\frac{1}{\lambda}$ on the left-hand side (see Fig. 18). A brief calculation based on this construction shows that for $\alpha > \alpha_c = \frac{2\pi}{\lambda}$, there are two intercepts: one at $r = 0$ (the maximum) and one at $r = r_m$, the minimum; whereas for $\alpha < \alpha_c$ only a minimum at $r = 0$ exists exactly like in flat space. It is possible to go through this second order transition by changing other parameters such as the rotational frequency. See Figs. 19 and 20 for illustrations of how the transition occurs when the shape of the substrate is varied. More details on the choice of substrate parameters are given in Sec. 11.V. Once these parameters have been chosen, changing the frequency would likely be more convenient; Fig. 19 shows how the equilibrium position of the vortex varies. If the vortex position $r_m$ can be measured precisely as a function of $\Omega$ and if there is not too much pinning, then the geometrical potential can even be reconstructed by integrating $U_G = \int_0^{\Omega} \sqrt{1 + h^2(r_m(\Omega'))^2} d\Omega' + \text{const.}$ which follows from Eq. 50.

\section*{C. Multiple defect configurations}

As the angular speed is raised, a cascade of transitions characterized by an increasing number of vortices

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig12}
\caption{Plot of minus the geometric potential $-U_G(r)$ for $\alpha = 0.5, 1, 1.5, 2$. The arrow indicates increasing $\alpha$. The radial coordinate $r$ is measured in units of $\lambda$ and $r_0 = \lambda$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig13}
\caption{Plot of the area of a cup of radius $r$ for $\alpha = 0.5, 1, 1.5, 2$. The arrow indicates increasing $\alpha$. The radial coordinate $r$ is measured in units of $\lambda$ and $r_0 = \lambda$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig14}
\caption{Plot of $E(r)$ measured in units of $K = \frac{k_B T}{m}$ as $\alpha$ is varied. In these units, the thermal energy $k_B T$ is less than 0.1 below the Kosterlitz-Thouless temperature, for 200Å films. The radial coordinate $r$ is measured in units of $\lambda$ and $r_0 = \lambda$. Note that this plot is a 2D slice of a 3D potential. For $\alpha < \alpha_c$, $E(r)$ is approximately a paraboloid while, for $\alpha > \alpha_c$, we have a Mexican hat potential.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig15}
\caption{Plot of $E(r)$ in units of $\frac{h^2}{m} \omega^2$ versus $r$ as $r_0$ is varied. The aspect ratio is kept fixed at $\alpha = 2$ while the range of the geometric potential (corresponding to the width of the bump) is varied so that $r_0 = 0.2, 0.4, 0.6, 0.8, 1$ in units of $\lambda$. As $r_0$ decreases, the geometric force becomes stronger, so the system goes through a transition analogous to the one displayed in Fig. 19.}
\end{figure}
occurs just as in flat space. In order to facilitate the mathematical analysis we introduce a conformal set of coordinates \{R(r), \phi\} (see Vitelli and Nelson [2004] for details). The function \(R(r)\) corresponds to a nonlinear stretch of the radial coordinate that “flattens” the bump, leaving the points at the origin and infinity unchanged:

\[ R(r) = r e^{U_G(r)} , \quad (52) \]

Note the unwonted appearance of the geometric potential \(U_G(r)\) playing the role of the conformal scale factor; this surprise is the starting point for our derivation of the geometric interaction in Section V.A. The free energy of \(N_v\) vortices on a bump bounded by a circular wall at distance \(R\) from its center is given by

\[
\frac{E}{4\pi^2 K} = \frac{1}{2} \sum_{j \neq i} n_i n_j \Gamma^D(x_i; x_j) + \frac{N_v}{4\pi} \sum_{i=1}^{N_v} n_i^2 \ln \left[ 1 - x_i^2 \right] - \frac{N_v}{4\pi} \sum_{i=1}^{N_v} \frac{n_i^2}{\ln \left[ (R(R)/\alpha) \right]} . \quad (53)
\]

The Green’s function expressed in scaled coordinates reads

\[
\Gamma^D(t_i; t_j) = \frac{1}{4\pi} \ln \left( \frac{1 + t_i^2 t_j^2 - 2t_i t_j \cos(\phi_i - \phi_j)}{t_i^2 + t_j^2 - 2t_i t_j \cos(\phi_i - \phi_j)} \right) . \quad (54)
\]

where \(\phi_i\) is the usual polar angle and the dimensionless vortex position \(t_i\) is defined by

\[
t_i = \frac{R(r_i)}{R(R)} . \quad (55)
\]

Eq.(53) is now cast in a form equivalent to the flat space expression apart from the third term which results from the curvature of the underlying substrate and vanishes when \(\alpha = 0\). However, we emphasize that the Green’s function \(\Gamma^D\) also is modified by the curvature of the surface and thus depends on \(\alpha\).

The contributions from the second term and the numerator of the Green’s functions in the first term account for the interaction of each vortex with its own image and with the images of the other vortices present on the bump (see Vitelli and Nelson [2004]). If \(R \gg r_0\), and all the vortices are near the top of the bump (i.e., \(r_i \sim r_0\)) then these boundary effects may all be omitted when determining equilibrium positions, as the forces which they imply are on the order of \(K r_0^3\), small compared to the intervortex forces and geometric forces, which have a typical value of \(\frac{K}{r_0}\).

Let us imagine rotating the superfluid, so that each vortex is confined by a potential of the form Eq.(56). In flat space, the locally stable configurations usually involve concentric rings of vortices (Campbell and Ziff [1974]). In particular, there are two stable configurations of six vortices. The lower energy configuration has one vortex in the center and five in a pentagon surrounding it. The other configuration, six vortices in a hexagon, has a slightly higher energy, and Ref. (Yarnichuk and Packard [1982]) saw the configuration fluctuating randomly between the two, probably due to mechanical vibrations since thermal oscillations would not be strong enough to move the vortices. (The experiment used a \(D_0 = 2\) cm high column of superfluid; if one regards the problem as two dimensional by considering flows that are homogeneous in the \(z\) direction, \(\rho_\alpha = D_0 \rho_\alpha\) is so large that \(K = \frac{K^2}{m^2} \rho_\alpha\) is on the order of millions of degrees Kelvin.) There are no other stable configurations. However, on the curved surface of a bump, there are several more configurations which can be found by numerically minimizing Eq. (55); the progression of patterns as \(\alpha\) increases depends on how tightly confined the vortices are compared to the size of the bump, as illustrated in Fig. 17. If the vortices are tightly confined, the interactions of the vortices (which are different in curved space) stabilize the new vortex arrangements. If the vortices are spaced far apart, the geometric interaction between the bump and the central vortex causes a transition akin to the decentering transition in the previous section.

For example, if \(\Omega = 9 \frac{K^2}{m^2 r_0}\), then at \(\alpha = 0\), the five off-center vortices start out in a ring of radius \(0.6 r_0\). This pentagonal arrangement (see Fig. 17A) is locally stable for \(\alpha < \alpha_1 = 2.7\). However, for \(\alpha > \alpha_2 = 2.1\), another arrangement with less symmetry is also stable (see frame B of Fig. 17), and above \(\alpha_1\) it takes over from the pentagon. For \(\alpha_2 < \alpha < \alpha_1\), both arrangements are locally stable, with the asymmetric shape becoming energetically favored at some intermediate aspect ratio. (There is also a third arrangement which coexists with the less symmetric arrangement for the larger aspect ratios, seen in the frame C of Fig. 17.) In the plane, the configuration labelled \(B\), for example, is unstable, because the outer rectangle of vortices can rotate through angle \(\epsilon\).
decreasing its interaction energy with the two interior vortices while keeping the rotational confinement energy constant. (That the interaction energy decreases can be demonstrated by expanding it in powers of \( \epsilon \).) Because the Green's functions are different on the curved surface (they do not depend solely on the distance between the vortices in the projected view shown), figures B and C are stabilized.

At lower rotational frequencies, the equilibria which occur are even less symmetric. For \( \Omega = \frac{\hbar}{m r_0^2} \), the vortices form a pentagon of radius \( r_0 \) when the surface is flat. This pentagon is far enough away that it has a minor influence on the central vortex, which undergoes a transition similar to the one discussed in Sec. [II.B]. At \( \alpha' = 1.4 \), the central vortex moves off-axis (the transition is continuous), causing only a slight deformation of the pentagon (see frame D of Fig. [17]). As for the single vortex on a rotating bump, the geometric potential has pushed the central vortex away from the maximum, and the other vortices are far enough away that they are not influenced much. At higher aspect ratios, the figure distorts further, taking a shape similar to the one which occurs for \( \Omega = 9 \frac{\hbar^2}{m r_0^2} \), but offset due to the geometric interaction. For these two rotational frequencies the hexagonal configuration is less stable than the pentagon; it will not take the place of the pentagon once the pentagon is destabilized. The hexagon is of course metastable for nearly flat surfaces.

**D. Abrikosov lattice on a curved surface**

As in Section [II.D] the geometric potential will have significant consequences only when the number of vortices near each geometrical feature such as a bump is of order unity. As an example, consider the triangular vortex lattice that forms at higher rotational frequencies (\( \Omega \gg \frac{\hbar}{m r_0^2} \) is the criterion for a large number of vortices to reside on top of the Gaussian bump). In flat space, the vortex number density is approximately constant and equal to (Tilley and Tilley, 1990)

\[
\nu(u) = \frac{4 \pi m \Omega}{\hbar} = \frac{2 \Omega}{\kappa}. \tag{56}
\]

At equilibrium, the force exerted on an arbitrary vortex as a result of the rotation exactly balances the force resulting from the interaction with the other vortices in the lattice and from the anomalous coupling to the Gaussian curvature. We can determine the distribution on a curved substrate by making the continuum approximation to Eq. (18). The sum of delta-functions \( \sigma \) gets replaced by \( 2 \pi \nu(r) \) and the self-charge subtraction can be neglected in the continuum approximation. The Gaussian curvature can be neglected because it is small compared to the large density of vortex charge. Upon applying Gauss's theorem to the vortex charge distribution in an analogous way to Section [II.A] we find that the force on a vortex at radius \( r \) is given by

\[
F_v = \frac{1}{r} \int_0^r 4 \pi^2 K \nu(r') r' \sqrt{1 + h'^2} dr' \tag{57}
\]

while the rotational confinement force, obtained by differentiating Eq. (19), is

\[
F_\Omega = - \frac{2 \pi h \Omega}{m r}. \tag{58}
\]

Balancing the two forces leads to an areal density of vortices,

\[
\nu(r) = \frac{m \Omega}{\pi h \sqrt{1 + h'^2}}. \tag{59}
\]

Eq. (59) has a succinct geometric interpretation: the vortex density \( \nu(r) \) arises from distributing the vortices on the bump so that the projection of this density on the \( xy \) plane is uniform and equal to the flat space result. The superfluid tries to mimic a rigidly rotating curved body as much as possible given that the flow must be irrotational outside of vortex cores as for the case of a rotating cylinder of helium (Tilley and Tilley, 1990). To check this, first notice that the approximate rigid rotation entails a flow speed of \( \Omega r \) at points whose projected distance from the rotation axis is \( r \). Hence, the circulation increases according to the quadratic law \( \oint \mathbf{v} \cdot d\mathbf{l} = 2 \pi \Omega r^2 \). Since

\[
\begin{array}{cccc}
\text{A} & \text{B} & \text{C} \\
\text{D} & \text{E} & \text{F} \\
\end{array}
\]

FIG. 17 Arrangements of 6 vortices that can occur on a curved surface. A circle of radius \( r_0 \) is drawn to give a sense that the confinement is tighter in the top row (\( \Omega = 9 \frac{\hbar^2}{m r_0^2} \)) than in the bottom row (\( \Omega = \frac{\hbar^2}{m r_0^2} \)). The upper row shows the patterns which occur at large angular frequencies (\( \Omega = 9 \frac{\hbar^2}{m r_0^2} \)). The transition from the pentagon to the rectangle with two interior points is discontinuous, and there is a range of aspect ratios \( 2.1 < \alpha < 2.7 \) where both configurations are metastable. The third configuration is nearly degenerate with the second configuration. The lower row shows the configurations which occur for \( \Omega = \frac{\hbar^2}{m r_0^2} \) as \( \alpha \) increases. The first transition is continuous and caused by the central vortex's being repelled from the top by the geometric interaction. The third configuration is identical to the second large \( \Omega \) configuration but the effect of the geometric repulsion is seen in its asymmetry.
this quantity is proportional to the projected area of the surface out to radius \( r \), the discretized version of such a distribution would consist of vortices, each with circulation \( \kappa = \frac{2\pi \hbar}{m} \), with a constant projected density \( \frac{2\pi}{\kappa} \) as in flat space. This result can be generalized with some effort to any surface rotated at a constant rate, whether the surface is symmetric or not.

The geometric force has to compete with the interactions among the many vortices expected at high angular frequencies. More precisely, the maximum force at radius \( r_0 \) according to Eq. (25) is of order \( \frac{2\pi}{\kappa} \) while the force due to all the vortices Eq. (27) is of order \( K \frac{(2\pi)(\pi r_{\text{v}}^2)(2\pi r_0(0))}{2\pi r_0} = 2\pi^2 K r_0 r(0) \). The last expression greatly exceeds \( \frac{2\pi}{\kappa} \) in the limit of high angular velocity. The geometrical repulsion leads to a small depletion of the vortex density of one vortex in an area of order \( \pi r_{\text{v}}^2 \).

The vortex arrangements produced by rotation are reminiscent of Abrikosov lattices in a superconductor (Vinen, 1969). In fact an analogy exists between a thin film of superconductor in a magnetic field and a rotating film of superfluid. A major difference between bulk superfluids and bulk superconductors is that the vortices in a bulk superconductor have an exponentially decaying interaction rather than a logarithmic one because of the magnetic field (produced by the vortex current) which screens the supercurrent. The analogy is more appropriate in a thin superconducting film, where the supercurrents (being confined to the film) produce a much weaker magnetic field. In fact, Abrikosov vortices in a superconducting film exhibit helium-like unscreened logarithmic interactions out to length scales of order \( r_{\text{v}}' = \frac{\lambda}{\pi} \) where \( \lambda \) is the bulk London penetration depth and \( D \) is the film thickness (see (Pearl, 1964) and, for a review, section 6.2.5 of (Nelson, 2002)). Our results on helium superfluids without rotation therefore apply also to vortices in a curved superconducting layer in the absence of an external magnetic field. Curved superconducting layers in external magnetic fields can be understood as well by replacing the magnetic field by rotation of the superfluid. Let us review the analogy contained a between a container of superfluid helium rotating at angular speed \( \Omega \) and a superconductor in a magnetic field \( \mathbf{H} \) (Vinen, 1969). Note that in Eq. (44) \( E \) is given by \( \frac{1}{2} \rho_s \int d^2 u \mathbf{u} \cdot \mathbf{u} \mathbf{r}^2 (\nabla \theta)^2 \) and \( h\nabla \theta \) is the momentum in the rest frame, \( \mathbf{p} \), although we are working in the rotating frame (the frame in which a vortex lattice would be at rest). For helium, the momentum in the rest frame \( \mathbf{p} \) is related to the momentum in the rotating frame \( \mathbf{p} ' \) by the “gauge” transformation

\[
\mathbf{p} \rightarrow \mathbf{p} ' + m \mathbf{r} \times \Omega .
\]

Similarly, in the case of a superconductor the momentum \( \mathbf{p} \) in the absence of a magnetic field is related to the momentum \( \mathbf{p} ' \) in the presence of the field by the familiar relation (Tinkham, 1996)

\[
\mathbf{p} \rightarrow \mathbf{p} ' + \left( \frac{\hbar}{e} \right) \mathbf{A} ,
\]

where \( \mathbf{A} \) is the vector potential. Comparison of Eq. (60) and Eq. (61) suggests a formal analogy between the two problems,

\[
\mathbf{A} \leftrightarrow \left( \frac{mc}{e} \right) \mathbf{r} \times \mathbf{\Omega} .
\]

Eq. (61) establishes a correspondence between the angular velocity \( \mathbf{\Omega} \) and the magnetic field \( \mathbf{H} \) that allows to convert most of the relations we derived for helium to the problem of a superconducting layer, with the identification

\[
\mathbf{\Omega} \leftrightarrow \left( \frac{e}{2mc} \right) \mathbf{H} .
\]

Of course, one should keep the external magnetic field small so that a dense Abrikosov lattice does not form, since (as for superfluids) when there are too many vortices, the curvature interaction is overcome by the vortex interactions.

**IV. EXPERIMENTAL CONSIDERATIONS**

Vortices in bulk fluids are extended objects such as curves connecting opposite boundaries, rings or knots. A vortex interacts with itself and with its image generated by the boundary of the fluid. However, if the vortex is curved, such forces (the three-dimensional generalization of the geometric force) are usually dominated by a force which depends on the curvature of the vortex called the “local induction force.” This force has a strength per unit length (Saffman, 1992) of

\[
f_{LIA} = \frac{\hbar^2}{m^2 \rho_3 \kappa} \ln \frac{1}{\kappa a},
\]

where \( \rho_3 \) is the bulk superfluid density and \( \kappa \) is the curvature of the vortex at the point where this force acts. This force is in danger of dominating the long range forces because of the core size appearing in the logarithm.

“Two-dimensional” regions are a special case of three-dimensional regions in which two of the boundaries are parallel and at a distance \( D_0 \) much less than the radius of curvature of the boundaries. The two dimensional superfluid density is given by \( \rho_s = \rho_3 D_0 \), and the interactions of the vortices should be captured by the two dimensional theory described in this paper once this substitution is made. A discrepancy will occur, however, if the boundaries of the film are not exactly parallel because the vortices are forced to curve in order to meet both boundaries at right angles. In this case, there is a force which is a relic of the local induction force (see Sec. IV.C),

\[
F_{th} = -\frac{\pi \hbar^2}{m^2 \rho_s} \frac{\nabla D}{D_0} \ln \frac{r_0}{a},
\]

where \( r_0 \) is the relevant curvature scale. According to this formula, vortices are attracted to the thinnest portions of the film. We will need to ensure that the thickness of the
film is uniform enough so that this force does not dominate over the geometric interactions we are interested in.

There is a maximum film thickness for which the geometric force is relevant. The most stringent requirement arises from demanding that the van der Waals force causes wetting of the surface with a sufficiently uniform film. Van der Waals forces compete against gravity, which thickens the superfluid at lower portions of the substrate, and surface tension, which thickens the superfluid where the mean curvature of the substrate is negative. Both gravity and surface tension thin the film on hills and thicken it in valleys, but if the film is thin enough, the van der Waals force can keep the nonuniformity very small. Section [IV.B] discusses the critical properties of very thin films (Andelman et al., 1988). We assume that vorticity is not created from scratch, but from pinned vortices present even before the rotation has begun (Tilley and Tilley, 1990). Finally in Secs. [IV.C] and [IV.D] a comparison is made between forces on vortex lines in three-dimensional geometries and on point vortices in two dimensions.

A. The Van der Waals force and thickness variation

We start by providing an estimate of the variation in the relative thickness

$$\epsilon \equiv \frac{D_L - D_0}{D_0}.$$  \hspace{1cm} (66)

for a liquid layer which wets a bump and apply it to thin helium films. $D_L$ denotes the thickness on top of the bump and $D_0$ is the thickness far away. The wetting properties of very thin films ($\sim 100$ Å) of dodecane on polymeric fibers of approximately cylindrical shape have been thoroughly investigated in (Quéré et al., 1989). We start by reviewing a theoretical treatment of the statics of wetting on rough surfaces by Andelman et al. (1988). A film on a solid substrate that is curved has a mean curvature determined by the shape of the substrate, unlike in the case of a large drop of water on a non-wetting surface. By choosing an appropriate shape, the drop can adjust its mean curvature (and thereby balance surface tension against gravity). The shape is therefore described by a differential equation. A thin film on a solid substrate, in contrast, has approximately the same curvature as the substrate that it outlines.

Consider a film that completely wets a solid surface. The surface itself is described by its height function $h(x)$, where $x$ denotes a pair of Cartesian coordinates in the horizontal plane below the surface (see Fig. 18). The height function for the liquid-vapor interface $h_L(x)$ can be determined by minimizing the free energy $F$,

$$F = \int \int d^2x [\gamma \sqrt{1 + |\nabla h_L(x)|^2} + \frac{\rho_3 g}{2} (h_L(x)^2 - h(x)^2)$$

$$- \mu (h_L(x) - h(x))]$$

$$+ \int \int d^2x \int_{h_L(x)}^\infty dz \int \int d^2x' \int_{-\infty}^{h(x')} dz'$$

$$w(\sqrt{(x - x')^2 + (z - z')^2}),$$  \hspace{1cm} (67)

where $\gamma, \rho_3,$ and $\mu$ are respectively the liquid-vapor surface tension, the total mass density, and the chemical potential (per unit volume). (Note that $\nabla$ here is not the covariant gradient for the surface; it is the gradient in the $xy$ plane.) The second term describes the gravitational potential energy integrated through the thickness of the film. The second and fourth terms model the force between the helium atoms and the substrate assuming for simplicity a non-retarded van der Waals interaction. The last term involves an integral over interactions between pairs of points, one above the helium film and one in the substrate, but with no points in the liquid helium itself. This is equivalent to including interactions between all pairs of atoms contained in all combinations of the vapor, liquid and solid regions, as long as $w(r) = -\alpha r^{-6}$ where $\alpha$ is the appropriate combination of parameters for these phases (Andelman et al., 1988).

Minimization of Eq. (67) leads to a differential equation for $h_L(x)$ that is a suitable starting point for evaluating the profile of the liquid-vapor interface numerically (Andelman et al., 1988). In what follows, we will instead work within an approximation valid when $D_0 \ll r_0, h_0$: in this case, the curvature of the film is fixed. The local film thickness is described by $D(x) = (h_L(x) - h(x))/\sqrt{1 + |\nabla h(x)|^2}$, see Fig. (18). We need to determine how each contribution to the free energy per unit area at a point $u$ is changed by an increase in thickness $\delta D(x)$.

First let us consider the variation of the van der Waals energy in order to understand how this attraction sets the thickness of the film. When the film thickens by $\delta D$ over a small area $A$ of the film (centered at $x$, $z$), the change in the van der Waals energy is given by $-A \delta DH(x)$ where
the disjoining pressure is

$$\Pi(x) = \int d^2x' \int_{-\infty}^{h(x')} dz' w(\sqrt{(x-x')^2 + (z-z')^2}).$$

(68)

For a film on a horizontal surface at $h = 0$, the surface area and gravitational potential energy do not increase when the film is thickened. The equilibrium thickness is determined by balancing the variation of the chemical potential contribution, $-\mu A \delta D$, against the disjoining pressure, giving $\mu = -\Pi(D_0)$. The disjoining pressure obtained by integrating Eq. (68) for a flat surface is

$$\Pi(D) = -\frac{A_H}{6\pi D_0^3}.$$ 

(69)

$A_H = \pi^2 \alpha$ is the Hamaker constant for the solid and the vapor interacting across a liquid layer of thickness $D_0$ (Israelachvili 1985). One sees that a negative value of $A_H = \pi^2 \alpha$ is necessary for wetting. The equilibrium thickness is

$$D_0 = \sqrt[3]{\frac{A_H}{6\pi \mu}}.$$ 

(70)

(For example, liquid $^4$He on a CaF$_2$ surface has $A_H \approx -10^{-21}$ J, and has a liquid-vapor surface tension of $3 \times 10^{-4}$ J/m$^2$.) When there is a bump on the surface, Eq. (70) gives the equilibrium thickness far from the bump. Note that both $A_H$ and $\mu$ are negative in this expression. Increasing $\mu$ therefore increases the thickness of the film as expected.

Now let us continue by considering the effects of gravity and surface tension for a curved substrate. The increase in gravitational potential energy is $\rho_3 g \delta D$, just because there is an additional mass per unit area of the fluid $\rho_3 \delta D$ at height $h$. (The additional elevation from adding the fluid at the top of the fluid that was already present can be ignored if the layer is very thin.) The variation of the surface tension energy can be related to the mean curvature using the fact that the area of a small patch of the liquid vapor interface $A(x)$ (at a distance $D$ from the substrate) is related to the corresponding area of the solid surface $A_0(x)$ by the relation

$$A(x) = A_0(x) \left[1 + 2H(x)D(x) + G(x)D^2(x)\right].$$

(71)

The second term is proportional to the mean curvature $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ of the surface, where we use the convention that the principal curvatures $\kappa_1$, $\kappa_2$ are positive when the surface curves away from the outward-pointing normal. The last term, proportional to the Gaussian curvature, can be ignored relative to the previous term since it is smaller by a factor of $D_0^{3/2}$. The mean curvature of the upper surface of the fluid is nearly the same as for the substrate, so the energy required to increase the thickness of the film is $2\gamma H(x) \delta D$. For example, at the top of the bump, an increased thickness leads to an increased area, so surface tension prefers a smaller thickness there. Gravity also thins the film at the top of the bump so that vortices are attracted to the top.

Now we must balance these forces against the disjoining pressure. The flat space form of the disjoining pressure is not significantly altered by the curvature of the substrate for very thin films. According to Eq. (65), the disjoining pressure is the sum of all the van der Waals interaction energies between the points of the substrate and a fixed point at the surface of the film. The integral (evaluated in Appendix C) for a point at a distance $D$ away from the substrate shows

$$\Pi[D(x)] \approx -\frac{A_H}{6\pi D(x)^3} \left(1 - \frac{3}{2} H(x)D(x)\right).$$

(72)

The curvature correction in the second term of Eq. (72) arises (when $H > 0$ as at the top of the bump) because the surface bends away from the vapor molecules which interact only with the very nearest atoms of the solid substrate. This effect is small if $D_0 << r_0$ and will be neglected in our estimates.

We can now collect the various contributions to set up a pressure balance equation that allows us to estimate the relative change in layer thickness $\epsilon$ defined in Eq. (60). This equation reads:

$$\frac{A_H}{6\pi D(x)^3} + 2\gamma H(x) + \rho_3 g \delta D - \mu = 0.$$ 

(73)

Apart from the lengths $r_0$, $h_0$ and $D_0$ inherited from the geometry of the system, it is convenient to define three characteristic length scales $\delta$, $\vartheta$ and $l_c$, obtained by pairwise balancing of the first three terms of Eq. (73).

$$\delta \equiv \sqrt{\frac{-A_H}{6\pi \gamma}},$$

$$\vartheta \equiv \sqrt{\frac{-A_H}{6\pi \rho_3 g}},$$

$$l_c \equiv \frac{\gamma}{\rho_3 g}.$$ 

(74)

The last relation in Eq. (74) defines the familiar capillary length (Guyon et al. 2001) below which surface tension dominates over gravity while the first and the second give the length scales involving the disjoining pressure. For $^4$He on CaF$_2$, $\delta \approx 10$ Å (for most liquids it is one order of magnitude less), $\vartheta \approx 0.\mu m$ and $l_c \approx 0.4 mm$.

Upon substituting Eq. (70) in Eq. (73), we obtain an approximate relation between $D(x)$ and $D_0$,

$$D(x) \approx D_0 \left(1 - \frac{3}{2} \frac{2H(x)}{\delta^2} + \frac{h(x)}{\vartheta^2}\right).$$

(75)

This relation leads to an estimate of the relative change in layer thickness, $\epsilon$, valid for thin films (if we take $\alpha \sim 1$), namely

$$\epsilon \sim \frac{D_0}{r_0 \delta^2} \left(1 + \left(\frac{r_0}{l_c}\right)^2\right).$$

(76)
Now the thickness-variation force (Eq. 65) is small compared to the coupling to the geometry only if

\[
|\epsilon| < \frac{a^2}{2 \ln \frac{r_0}{a}}. \tag{77}
\]

where we estimated the maximum of the geometrical force to be \(K \pi a^2/r_0\), see Eq. 25. This limit on \(\epsilon\) leads in turn to an upper bound for the film thickness \(D_0\) for each choice of \(r_0\). Assuming \(\alpha \sim 1\) and splitting into cases according to the size of the bump gives the limits:

\[
D_0 \lesssim \frac{\rho_\alpha \delta^2}{(\ln \frac{r_0}{a})^{3/2}} \text{ for } r_0 < l_c \tag{78}
\]

\[
D_0 \lesssim \frac{\rho_\alpha}{(r_0 \ln \frac{r_0}{a})^{3/2}} \text{ for } r_0 > l_c. \tag{79}
\]

For smaller bumps, surface tension plays the main role in creating thickness variation and for larger bumps, gravity has the largest effects. In order for the vortices to be easily observable, \(D_0\) should be as large as possible. Eqs. 78 and 79 show that the film can be made thickest (while retaining its approximate uniformity) when gravity and surface tension have comparable effects, \(r_0 \approx l_c\). The optimal thickness (calculated with \(\alpha = 1\) and for the Gaussian bump; similar numbers are optimal for the saddle surface) is about 150 Å at \(r_0 = .5\) mm. A smaller value of \(r_0\) might be required if there is a lower limit \(\Omega_{min}\) on the rotational frequency as discussed in the next section, requiring a slightly thinner film, about 100 Å.

The restriction on the film thickness is the most serious obstacle to studying two-dimensional superflows experimentally. The method of observing vortices described in (Yarmchuk and Packard, 1982) requires the vortex to be large enough to be able to trap an observable number of electrons. If the method of Yarmchuk and Packard (1982) turns out to be unsuitable for thin films and an alternative method cannot be found, one might also study a saturated superfluid layer confined between two solid surfaces. In this case, none of the considerations on wetting are relevant, and the “film” could have a large thickness. The two solid surfaces would have to be parallel to one another, and hence not congruent. (Congruent surfaces displaced by a fixed distance in the vertical direction lead to a thickness varying as \(\cos \theta(x)\) where \(\theta\) is again the inclination angle.) The two surfaces would have to be very accurately shaped in order to make the film uniformly thick.

Another concern is that a vortex may be pinned to an irregularity on the substrate strongly enough that it will not move to the location favored by the geometrical force. On the other hand, the geometrical force is much stronger than random forces due to thermal energy. Even with a film as thin as hundreds of Angstroms, the geometrical force is very strong. With \(\rho_s = \rho_\alpha D_0 = 2\gamma/\alpha D_0\), which assumes that the superfluid density is not depleted too much by thermal effects or by the thinness of the film, the value of \(K = \frac{\rho_s h^2}{m^2}\) is about 40 Kelvin. Since the potential wells which trap the vortices on the saddle surface or on the rotating Gaussian bump have depths on the order of \(K\alpha^2\) the geometric force will be strong enough to prevent the vortex from wandering out of the trap due to thermal Brownian motion except very close to the Kosterlitz-Thouless transition where \(\rho_s\) is depleted.

B. Parameters for the rotation experiment

A practical consideration can limit the thickness of the film further. In the experiments of (Yarmchuk and Packard, 1982), uniform rotations slower than \(\Omega_{min} = 1\) rad/s or so were hard to attain. The maximum value of \(\Omega\) for which the geometrical force can displace the vortex as in Sec. II.B can be determined by taking the limit \(r_m \to 0\) in Eq. (51): this shows that for a given \(r_0\), the vortex is displaced from the top of the bump only if

\[
\Omega < \frac{\hbar \alpha^2}{4m^2 R_d^2}. \tag{80}
\]

Hence, if details of the rotational apparatus require that \(\Omega > 1\) rad/s, then the size of the bump should be less than .1 mm (for \(\alpha = 1\)). Consequently Eq. (78) requires even thinner films than for the case discussed in the previous section where the bump size is determined purely by the natural forces of gravity and surface tension and is equal to the capillary length.

An additional issue is whether it is possible to create just one vortex reliably. In practice, the number of vortices in a rotating film is a property of a history-dependent metastable state whose dynamics are not completely understood (see Chapter 6 of Ref. Tilley and Tilley 1990). We can discuss this in the context of a flat rotating disk. The critical frequency for local stability of a single vortex at the center of the disk is \(\Omega_{stab} = \frac{h}{mR^2}\). There is a barrier to creating such a vortex, so the frequency initially would have to be raised up to a higher frequency in order for a vortex to form at the boundary and then move in to the center of the rotating helium. The critical frequency for creating vorticity at the boundary is \(\Omega_{crit}\). It is not clear what this critical speed is. Perhaps \(\Omega_{crit}\) for a thin disk of helium is determined by the height \(D\) of the disk; the critical linear speed\(^3\) would then be \(R \Omega_{crit} \sim \frac{\hbar \ln \frac{2K}{l_0}}{mD} \) (Vinen, 1963). Now

\(^2\) Interestingly, even without rotating the bump, there are two radii where the vortex could rest for a film thickness of 200 Å. Then the confinement due to the varying film thickness and the geometrical repulsion are comparable in magnitude, producing an equilibrium off-center position for the vortex.

\(^3\) At the critical speed, vortices are believed to form by break-
because $\Omega_{\text{crit}}$ is so much greater than $\Omega_{\text{stab}}$, a lattice of many vortices will form. According to Eq. [59], there will be about $\frac{m\Omega_{\text{crit}}^2}{\hbar} \sim \frac{R}{D} \ln \frac{D}{a}$ of them. By slowing down to just above $\Omega_{\text{stab}}$, where the rotational confinement is not strong enough to confine more than one vortex (even metastably), one would hope to retain just a single vortex.

C. Films of varying thickness from the three-dimensional point of view

In a chamber of arbitrary shape the length of a vortex line crossing through the chamber changes as the vortex moves, and therefore knowledge of the core energy per unit length is crucial for determining the forces experienced by the vortex. The energy of a length $D$ isolated vortex in a cylinder of radius $R$ including the core energy is given by

$$E_{\text{line}} = \rho_3 D \frac{\hbar^2}{m^2} \ln \frac{R}{a} + \epsilon_{c3} D$$

(81)

where $\bar{a} = a \frac{\hbar^2}{2m^2}$ is the core radius rescaled to account for the effects of the core energy per unit length, $\epsilon_{c3}$.

A vortex line connecting two approximately parallel parts of the boundary feels a force as a consequence of the variation of $E_{\text{line}}$. Let the two nearly parallel boundaries be $S$ and $S'$ as illustrated in Fig. 19. Let us define $D(x)$ to be the length of the line which is perpendicular to $S$ at $x$ and which extends to $x'$ on $S'$ (see Fig. 19). If a vortex is attached to $S$ at $x$ then since the fluid flow is fastest at the vortex, the energy depends primarily on the thickness of the helium film where the vortex is located. This thickness is approximately given by $D(x)$ no matter how the vortex connects the two surfaces (unless it is extremely wiggily). A useful fact is that, if the two surfaces are at a constant separation, then the line between $x$ and $x'$ is perpendicular to both surfaces. This can be derived by differentiating, $D(x)^2 = |x - x'|^2$. Upon displacing $x$ slightly, $2\delta D = 2(x - x') \cdot (\delta x - \delta x') = 2(x - x) \cdot \delta x'$, since the line connecting $x$ to $x'$ chosen was to be perpendicular to $S$. Expressing the inner product in terms of the angle $\beta$ indicated in Fig. 19 this reads,

$$\delta D = |\delta x| \cos \beta \approx |\delta x| \cos \beta.$$  

(82)

In order to calculate the kinetic energy of a vortex in a curved film of varying thickness, we will derive a more detailed form of Eq. (81) for a curved film. First, the core energy due to the local disruption of superfluidity has the same form, $\epsilon_{c3} D(x)$ where $D(x)$ is the thickness of the film at the location $x$ of the vortex. The major component of the energy is given by $\frac{h^2}{2m} \rho_3 \int \int d\xi dr d\phi$ where it is assumed that the flow is parallel to the boundaries and roughly independent of $\zeta$, the coordinate normal to $S$ and approximately normal to $S'$. We integrate over $\zeta$ and divide the remaining two-dimensional integration into two parts:

$$E = \epsilon_{c3} D(x) + \frac{\hbar^2}{2m^2} \rho_3 \left( \int_{r < L_{th}} D(r, \phi) \frac{dr d\phi}{r} \right. + \left. \int_{r > L_{th}} D(r, \phi) \frac{dr d\phi}{r} \right),$$

(83)

where we are using polar coordinates centered on the location of the vortex. Here, $L_{th}$ is the distance over which $D$ varies appreciably so that $L_{th} \sim \frac{D}{\delta D}$.

The integral starts far enough away from the vortex that it does not depend on the specific thickness of the film at the location of the vortex. The first term may be approximated by replacing $D(r, \phi)$ by $D(x)$ (the thickness of the film at the location of the vortex, where the energy is very big). Therefore,

$$E \approx \epsilon_{c3} D(x) + \frac{\pi \hbar^2}{m^2} \rho_3 D(x) \ln \frac{D}{a|\nabla D|} + \text{energy of the distant flow}.$$  

(84)

FIG. 19 Illustration of the definition of $D(x)$, for nonparallel surfaces. The distance between $S'$ and $S$ is measured along the segment which is perpendicular to $S$ at $x$. The segment meets $S'$ (obliquely) at a point which we call $x'$.
The force obtained by taking the gradient of the kinetic energy reads

\[ F = -\epsilon \nabla \kappa - \frac{\pi \hbar^2}{m^2} \rho_3 \nabla D \ln \frac{D}{a\nabla D} + F_G \]  

(85)

where \( F_G \) represents the variation in energy of the long range portion of the flow, which includes the geometric forces.

We now show that the thickness variation force can be interpreted as the Biot-Savart self-interaction of the curved vortex. The net Biot-Savart force points towards the center of curvature of the vortex, and so favors shrinkage of its length. We will use the “local induction approximation” to the Biot-Savart self-interaction, given in Eq. (64). This expression results from integrating the interaction of a particular element of the vortex line with nearby elements; the peculiar dependence on the core radius \( a \) arises from the need to place a cut-off in the diverging interaction for nearby elements.

To see this, note that if the opposing boundaries \( S, S' \) are not at a constant distance, the vortex must curve in order to connect them, because it meets both boundaries at right angles. The resulting curvature of the vortex, \( \kappa \), is equal to \( \frac{\nabla D}{D} \) (see Fig. 20), so that the force per unit length given by Eq. (54), multiplied by \( D \), agrees with the force derived by the energy method, Eq. (85). On the other hand, for films of constant thickness, the surfaces \( S, S' \) can be connected by straight vortices. According to the Biot-Savart law a straight vortex has zero interaction with itself. In addition, there is still an interaction energy with its image, which is where the geometric force, the third term in Eq. (85), comes from. When the helium has boundaries, a distribution of “image vorticity” beyond the boundaries can be supplied to simulate the effects of the boundary conditions \( \text{Saffman 1992} \). The integral of the Coulomb interaction with the distribution of surface curvature (Eq. (23)) is reminiscent of the long-range integral of the Biot-Savart interaction with the distribution of vorticity beyond the surface.

D. Vortex Depinning

Because of the divergence of the force described by Eq. (85) in the limit \( a \to 0 \), it will be hard to see effects of the geometric force without films of very nearly uniform thickness. In Voll et al. (2006) an experiment is described in which a vortex extending along a wire in a helium-filled tube leaves the wire more easily when the wire is connected to a bump on the bottom of the tube than when the bottom is flat. In this case, any geometric repulsion from the bump should be negligible in comparison to the extra energy associated with the stretching of the vortex. (See Fig. 21) Geometric energies should be of order \( R \rho_3 \frac{k^2}{\hbar^2} \), where \( R \) is the radius of the tube and approximately the height of the bump, but the vortex has to stretch an additional length on the order of \( R \) in order to leave the bump. Therefore, the extra kinetic energy from Eq. (81), \( R \rho_3 \frac{k^2}{\hbar^2} \ln \frac{L}{\kappa} \), probably overwhelms the geometric effects. Thus, the energy barrier for detachment of the vortex line should increase when the wire is attached to a bump.

This suggests that depinning is not simply caused by thermally activated barrier crossings (which would occur at the rate proportional to \( e^{-\Delta E/\kappa} \)). In fact, for the length scale of the bump in the experiment (a few millimeters), the additional energy barrier due to the stretching of the vortex line is millions of degrees Kelvin at the temperature where the vortex depins! The depinning may instead depend on “remnant vorticity” (Tilley and Tilley, 1990) in the form of extra pinned vor-
tices stretching from the top of the bump to the cylinder’s walls. This would decrease the energy barrier because the vortex could leave the wire by attaching itself to some of the vortices that already exist.

In a more nearly two-dimensional geometry, Eq. 85 shows that observing the equilibrium positions of vortices in a film of varying thickness could shed light on the value of the core size and core-energy and whether they are fixed functions of temperature as most models imply. The structure of vortex cores is still not well understood, and there are several alternative models (Tilley and Tilley, 1990). Experiments on the geometric force in contexts where the two contributions to Eq. 85 are comparable could give information about the effective core radius ̃\(a\). By allowing the film thickness to vary by about \(\epsilon = \frac{\Delta T}{T_0} \sim \frac{1}{\ln T_0}\), see Eq. (77), one ensures that the vortex core size has a decisive effect on the equilibrium positions of the vortices (comparable to the effects of the long range forces).

V. COMPLEX SURFACE MORPHOLOGIES

Up to this point, our discussion has been confined to rotationally symmetric surfaces and slightly deformed surfaces for which the electrostatic analogy and perturbation theory can be successfully employed to determine the geometric potential. To investigate geometric effects that arise for strong deformations and for surfaces with the topology of a sphere, we adopt a more versatile geometric approach based on the method of conformal mapping, often employed in the study of complicated boundary problems in electromagnetism and fluid mechanics. This approach also sheds light on the physical origin of the geometric potential. A concrete goal is to solve for the energetics (and the associated flows) of topological defects on a complicated substrate whose metric tensor we denote by \(g_{\text{Tab}}\). This is accomplished by means of a conformal map \(C\) that transforms the target surface into a reference surface \(R\), with metric tensor \(g_{\text{Rab}}\). The computational advantages result from choosing the conformal map so that \(R\) is a simple surface (e.g., an infinite flat plane, a flat disk, or a regular sphere) that preserves the topology of the target surface. Figure 22 represents a complicated planar domain denoted by \(T\) which can be mapped conformally onto a simple annulus labeled by \(R\). We will introduce all the basic concepts in the context of this simple planar problem before turning to the conformal mapping between target and reference surfaces which is represented schematically in Fig. 22. Such mappings can always be found in principle (David, 1989).

The conformal transformation will map the original positions of the defects on \(T\), denoted by \(u\), onto a new set of coordinates on \(R\) denoted by \(\mathcal{U} = C(u)\). In what follows, capital calligraphic fonts always indicate coordinates on the reference surface. For sufficiently small objects near a point \(u\) the map will act as a similarity transformation; that is, an infinitesimal length, \(ds_T = \sqrt{g_{\text{Tab}}du^adu^b}\), will be rescaled by a scale factor \(e^{\omega(u)}\) which is independent of the orientation of the length on \(T\):

\[
ds_R = e^{\omega(u)}ds_T, \tag{86}\]

where \(ds_R = \sqrt{g_{\text{Rab}}d\mathcal{U}^ad\mathcal{U}^b}\). This result in turn implies a simple relation between the metric tensors of the two surfaces:

\[
g_{\text{Rab}} = e^{2\omega(u)}g_{\text{Tab}}, \tag{87}\]

where we have assumed for simplicity that the coordinates used on the target surface are chosen so that corresponding points on the two surfaces have the same coordinates \(\mathcal{U}^A\). We will demonstrate that, once the geometric quantity \(\omega(u)\) is calculated, the geometric potential of an isolated vortex interacting with the curvature is automatically determined. For multiple vortices, the energy consists of single-vortex terms and vortex-vortex interactions. On a deformed sphere or plane, the geometric potential reads

\[
E_1(u) = -\pi n_i^2 K \omega(u), \tag{88}\]

where \(K\) is the stiffness parameter defined in Eq. (10). For a deformed disk, there are boundary interactions not included in Eq. (88). (We will not consider multiply connected surfaces here, but the single particle energy on a multiply connected surface has additional contributions which cannot be described by a local Poisson equation.)

The interaction energy is

\[
E_2(u_i, u_j) = -2\pi n_in_j K \ln \frac{D_{ij}}{a}, \tag{89}\]

where \(D_{ij}\) is the distance between the two image points on the reference surface. When the reference surface is an undeformed sphere (the other possibilities are a plane or disk), \(D_{ij}\) is the distance between the points along a chord rather than a great circle (Lubensky and Prost, 1992). We will show below that on a deformed plane \(\omega\) is equal to \(\bar{U}_G\), but from now on we will use \(\omega\) instead; the two functions are conceptually different, and are not even equal on a deformed sphere.

Equation (88) is derived by the method of conformal mapping in section V.A and its computational efficiency is illustrated in section V.B where the geometric potential of a vortex is evaluated on an Enneper disk, a minimal surface that naturally arises in the context of soap films, but whose geometry is distorted enough compared to flat space that it cannot be analyzed with perturbation theory. Changing the geometry of the substrate has interesting effects not only on the one-body geometric potential but also on the two-body interaction between vortices. In section V.C we use conformal methods to show how a periodic lattice of bumps can cause the vortex interaction to become anisotropic. In section V.D we demonstrate that the quantization of circulation leads to
an extremely long-range force on an elongated surface with the topology of a sphere. The interaction energy is no longer logarithmic as in Eq. (89), but now grows linearly with the distance between the two vortices. Indeed the charge neutrality constraint imposed by the compact topology of a sphere, blurs the distinction between geometric potential and vortex-interaction drawn in Equations (89) and (90). This is most easily seen by bypassing vortex energetics on the reference surface (which is after all an auxiliary concept) and opting for a more direct re-statement of the problem in terms of Green’s functions on the actual target surface coated by the helium layer. The interaction energy now reads

\[ E_2'(u, u) = 4\pi^2 Kn_1n_2 \Gamma(u, u), \]  

(90)

and the single-particle energy takes the form of a self-energy

\[ E_1'(u) = -\pi n_1^2 KU_G(u) = \pi n_1^2 K \int G(u) \Gamma(u, u') d^2 u', \]  

(91)

where \( \Gamma \) is a Green’s function for the surface that generalizes the logarithmic potential familiar from two dimensional electrostatics. For a deformed plane the two descriptions of the interaction energy are equivalent since the Green’s function on a deformed plane can be obtained by conformal mapping,

\[ \Gamma(u, u) = -\frac{1}{2\pi} \ln \frac{D_2}{a}. \]  

(92)

We will see that the expressions for the single-particle energies are also equivalent. In contrast, for a deformed sphere, we show in section V.E and appendix D that the two formulations do not agree term by term (\( E_1' \neq E_1 \) and \( E_2' \neq E_2 \)), although the combined effect of one-particle and interaction terms is the same (up to an additive constant). Both self-energies and interaction energies include effects of the geometry and explicit formulas are provided on an azimuthally symmetric deformed sphere. Finally, in section VI we present a discussion of some general upper bounds to which the strength of geometric forces is subjected (even in the regime of strong deformations) which are useful in experimental estimates and which illustrate a major difference between electrostatic and geometric forces: The former can always be increased by piling-up physical charges but the latter are generated by the Gaussian curvature that can grow only at the price of “warping” the underlying geometry of space. Too much warping either leads to self-intersection of the surface or a dilution of the long-range force.

A. Using conformal mapping

We start by proving a simple relationship between the total energies (including self- and interaction parts) \( E_T \) and \( E_R \) of two corresponding vortex-configurations on the target and reference surfaces respectively:

\[ E_T = E_R - \pi K \sum_{i=1}^{N} \omega(u_i). \]  

(93)

The right-hand side of Eq. (93) can be calculated for the reference surface and then subsequently decomposed into single-vortex and vortex-vortex interactions; several examples are worked out in detail in Sections V.B and V.C. The general approach is best illustrated by considering the planar flow in the complicated annular container shown in Fig. 22, which can be tackled by conformally mapping it to a simpler circular annulus.

The flow in the reference annulus is clearly circular, and it has the same \( \frac{1}{2} \) dependence as for a vortex. A crucial property of conformal transformation allows us to transplant this understanding of the reference flow to the target annulus. This property concerns the following mapping of the stream function \( \chi \) (see Eq. (15)) from the reference surface to the target surface:

\[ \chi_T(u) = \chi_R(U) \]  

(94)

Once the coordinate change \( U = C(u) \) is found, the prescription to determine \( \chi_T \) provided in Eq. (94) guarantees that the corresponding flow will be irrotational and incompressible, as required. Visually, all we are doing is taking the streamlines on the reference surface, which are level curves of \( \chi_R \) and mapping them by \( C^{-1} \) to the target surface. We can informally state this first property of conformal maps as follows:

1) The conformal image of a physical flow pattern is still a physical pattern.

Note that any multiple of the mapped stream function, \( \alpha \chi_R(C(u)) \), corresponds to an irrotational and incompressible flow but in this case the rates of flow in the target and reference substrate are different. Only the choice \( \alpha = 1 \) ensures that both flows have the same number \( n \) of circulation quanta around the hole (or around each
vortex if some are present.) This follows from another basic property of conformal maps:

2) In flow patterns related by a conformal map, according to Eq. (74), circulation integrals around corresponding curves are the same.

This property follows from the fact that the contribution to the circulation from each element of the contour, \( v \cdot dl \), is conformally invariant. The infinitesimal length \( dl \) and the velocity \( v \) scale inversely to each other under conformal transformations and the angle between them is preserved by the map. To understand why, note that flow lines are compressed together (stretched apart) when they are mapped to the target space if the conformal parameter \( \omega \) is greater (less) than zero. As a result, the velocity increases (decreases) by the same factor as distances are decreased (increased). This heuristic argument is confirmed by noting that the velocity is given by the covariant curl of the stream function, see Equations (15) and (94). By definition, the covariant curl carries contributions to the circulation from each element of the conformal image of a small figure has the same scale inversely to each other making the \( ln )\) and note that \( l \) and \( L \) have the curvature scale of \( \omega \) and \( \omega \) changes by \( e^{-2\omega} \) making the energy conformally invariant. Figure 22 illustrates pictorially that the energy density in the original flow on the target surface is smoothed out and simplified, its variations being replaced by variations in the conformal scale factor.

We now return to the energetics of flows containing point vortices. The starting point of our analysis follows from the defining property of a conformal map, namely that a conformal image of a small figure has the same shape as the original figure, while a larger shape becomes distorted (consider Greenland, which has an elongated shape, but appears to round out at the top in a Mercator projection, which is itself a conformal map). To quantify the size limits, note that if a shape has size \( l \), \( \omega \) changes by about \( l \nabla \omega \) across the shape. Thus, as long as

\[ l < \frac{1}{|\nabla \omega|}, \]  

the mapping rescales the shape uniformly. The right-hand side is ordinarily of order \( L \), the curvature scale of the surface. As a result we can conclude that

4) The circular shape of the streamlines near a microscopic vortex core on a substrate of slowly varying curvature is preserved.

On a deformed substrate with a flow induced by vortices, the flow speeds will increase or decrease not just depending on the distance to the vortex, but also depending on the shape of the surface. For example, the vortex on top of the bump in the example of Sec. II.A has a flow that decays more slowly with distance than in flat space. Also, for a vortex well off to the side of a bump, if the bump’s height \( h \) is larger than its width \( 2r_0 \), it turns out that the flow pattern penetrates only up to an elevation of about \( r_0 \) up the side of the bump.

The method of conformal mapping elucidates these geometrical rearrangements of the flow pattern. To find the flow pattern around the vortices at positions \( u_i \), we find the flow pattern around vortices at the corresponding positions \( \mathcal{U}_i \) on the reference surface and then map these streamlines onto the target surface by Eq. (94). The energies are not equal in this case, in spite of property 3. Property 3 does not apply to a region containing vortex cores, because we would have to suppose the area of the cores on the reference surface to be greater by \( e^{-2\omega} \) and the energy in the cores to be smaller by a factor of \( e^{2\omega} \), in order for the conformal relation Eq. (97) to continue to hold. In contrast, the core radius is fixed by the short-distance correlations of the helium atoms and the core energy is related to the interaction energy of the atoms.

The vortex cores are not significantly affected by the curvature of the substrate; moreover, the whole flow pattern in the vicinity of the core is nearly independent of the location of a vortex. We observe that each vortex has a "dominion," a region where the flows are forced by the presence of the vortex to be

\[ v = n \frac{\hbar}{mr} + \delta v \]  

The leading term has the same form as one expects for a vortex in a rotationally symmetric situation, and the effects of geometry are accounted for by \( \delta v \); by dimensional analysis, this error is of the order of \( \frac{h}{L} \) where \( L \) is the radius of curvature of the substrate (or possibly the distance to another vortex or to the boundary, whichever is shortest). Therefore we can introduce any length \( l << L \) and note that \( l \) is then a distance below which the effects of curvature do not have a significant effect (compared to the diverging velocity field). The geometry correction gives a contribution to the energy within this radius that is also small, as seen by integrating the kinetic energy over the annulus between \( a \) and \( l \) (using Eq. (97)):

\[ \pi K \ln \frac{L}{a} + \epsilon_c + O(K) \frac{l^2}{L^2} \]  

where \( a \) is the core radius and \( \epsilon_c \) the core energy. The error term is quadratic in \( \frac{L}{l} \) because the integral over the cross-term from squaring Eq. (97) cancels.
When we wish to find the kinetic energy of the superflow, these near-vortex regions are thus the simplest to account for, as their energy is nearly independent of their positions relative features such as bumps on the substrate as long as

\[ l \ll L. \]  

(99)

Since the target and reference configurations have the same number \( N \) of vortices, the energies contained within radius \( l \) of the vortices are the same:

\[ E_{<l}^T = E_{>l}^T = \pi N K \ln \frac{l}{a} \]  

(100)

In order to find the forces on a set of vortices, we need to account for all the energy of the vortices in regions away from the vortices where the flow \( \text{has} \) been affected by the curvature. Let us imagine cutting the target surface up into an inner region \( I \) (the union of the radius \( l \) disks around each vortex) and an outer region \( O \) (consisting of everything else) as illustrated in Fig. 23. We can map \( I \) to the reference surface by simply translating each of the disks so that they surround the vortices on \( R \). The modifications to the flow are all in \( O \); for example, streamlines there are deformed from their circular shape. These irregularities can be removed (or at least simplified) by using the conformal mapping to map \( O \) to \( R \), just as in the case of the annulus illustrated above. The inhomogeneity of the conformal map compensates for the irregularity of the flow. Some portions of \( O \) are expanded and some are contracted, but its circular boundaries are small enough (see Eqs. (99), (100)) that they are simply rescaled into circles of different radii (property 4):

\[ l_i = e^{\omega(u)} l \]  

(101)

Now we have mapped \( I \) and \( O \) from the target surface to regions on the reference surface which contain the same energy. But these images of the regions on \( R \) do not fit together. The conformal map on \( O \) stretches or contracts each hole in it, to circles of radius \( e^{\omega(u)} l \). These stretched images of the regions on \( I \) do not fit together with the images of \( O \), which have been moved rigidly from the target surface. The energies are related by

\[ E_T = E_{<l}^T + E_{>l}^T = E_{<l}^R + E_{>l}^R \]

(102)

We must correct for the gaps and overlaps between the two image regions on \( R \) in order to relate the last expression to \( E_R \). If \( \omega(u) > 0 \), there is an annular gap near vortex \( i \); using Eq. (97) (since this gap is part of the flow controlled by this vortex):

\[ \Delta E_i = -\frac{1}{2} \int_1^{l_i} 2\pi r dr \frac{\rho^2 \ell^2}{m^2 r^2} \]

(103)

\[ = -\pi Kn_i^2 \omega(u_i) \]  

(104)

Summing all these contributions gives our desired result, Eq. (102).

We emphasize the energy difference is not produced within the cores, or anywhere near the vortices. In fact, the fact that the energies on \( T \) and \( R \) differ is a result of assuming that there is no change in the flow within a macroscopic distance \( l \) of the vortex. The scale \( l \ll L \) only has to be small compared to the geometry of the system and has no relation to the atomic structure of the core. On the other hand, taking \( l \) as small as possible has an elegant consequence: Eq. (105) actually has an error which is \( O\left(K \left(\frac{\ell}{L}\right)^2\right) \), smaller than \( O\left(K \left(\frac{1}{L}\right)^2\right) \) as predicted at first. Taking smaller values of \( l \) gives a more accurate result, since the conformal mapping (which does not suffer from the error in Eq. (105)) is used to calculate the energy of a larger portion of the flow pattern.

Now Eq. (103) shows that the position-dependent scale factor, \( \omega(u) \), plays the role of a single-particle energy.
Additionally, this single particle energy can be regarded as the “geometric potential,” since it turns out to be related to the curvature in a way analogous to how the electrostatic potential is related to the charge. The function $\omega$ depends on the shape of the boundaries and on the curvature of the surface. A varying scale factor is necessary to map between surfaces with different distributions of curvature (such as planes with and without bumps). (A constant scale factor only rescales the curvature.) The curvature therefore depends on the variation of $\omega$. In fact it can be shown that

$$G_T(u, v) = e^{2\omega(u,v)}G_R(U(u,v), V(u,v))$$

$$+ \frac{1}{\sqrt{\det (g_{T,cd})}} \partial_u \sqrt{\det (g_{T,cd})} g_{T,cd}^{x^2} \partial_v \omega.$$  

(105)

The second term is the Laplacian of $\omega$ as a function on the target surface$^4$. The correspondence with electrostatic potentials, with $G_T(u,v)$ as a source, is clear if the reference surface is a plane or disk and $G_R = 0$. Then Eq. (105) reduces to Eq. (2).

For a deformed sphere or plane, single particle potentials come entirely from the second term in Eq. (103). The reference energy, corresponding to vortices on a sphere or plane cannot favor one position over another because of the homogeneity of these reference surfaces. The first term, $E_R$, leads to the vortex-vortex interactions which depend only on the separation of the vortices on the reference surface, again by symmetry; this energy is

$$E_R = 4\pi^2 K \sum_{i<j} n_i n_j \Gamma(U_i, U_j)$$  

(106)

where $\Gamma$ depends on the reference surface:

$$\Gamma_{\text{plane}}(X,Y) = -\frac{1}{2\pi} \ln \frac{|X-Y|}{a}$$  

(107)

and

$$\Gamma_{\text{sphere}}(X,Y) = -\frac{1}{2\pi} \ln \frac{2R \sin \frac{\gamma}{2}}{a}$$  

(108)

$$= -\frac{1}{2\pi} \ln \frac{|X-Y|}{a}$$  

(109)

where $\gamma$ is the angle between the two points. (2$R \sin \frac{\gamma}{2}$ is the chordal distance between the points, not the geodesic distance along the surface, as one might have guessed for the natural generalization of a Green’s function to curved space.) The detailed derivation of the second formulation of the vortex interaction energies in terms of the Green’s functions on the deformed surface (see Eqs. (90) and (91)) is contained in Appendix [A].

B. Vortices on a “Soap Film” Surface

There are experimental and theoretical motivations for studying substrates shaped as minimal surfaces. An example of a minimal surface is easy to make by dipping a loop of wire in soap; the spanning soap film tries to minimize its area. Vortices can be studied on a helium film coating a solid substrate whose surface has the shape of such a film. Such surfaces are characterized by a vanishing mean curvature, $H(x)$, so the contribution of the surface tension $2\gamma H(x)$ to the thickness variation equation, Eq. (73) is drastically reduced. From the mathematical point of view, there is a widely known parametrization due to Weierstrass [Hyde et al., 1997] which readily leads to an exact expression for the geometric potential of a vortex on such a surface.

Weierstrass’s formulae, which give a minimal surface for each choice of an analytic function $R(\zeta)$, read:

$$x(\zeta) = \Re \int_0^\zeta R(\zeta') (\zeta'^2 - 1) d\zeta'$$

$$y(\zeta) = \Im \int_0^\zeta R(\zeta') (\zeta'^2 + 1) d\zeta'$$

$$z(\zeta) = \Re \int_0^\zeta R(\zeta') 2\zeta' d\zeta'$$  

(110)

The correspondence between this parametric surface and the complex variable $\zeta = X + iY$ is a conformal map, and the conformal factor can be expressed in terms of $R(\zeta)$. Therefore, the analysis of vortices on such a surface is not difficult at all when the $X,Y$-plane is used as the reference surface.

As an example, let $R(\zeta)$ be equal to $L\zeta$ where $L$ controls the size of the target surface. Then the surface produced is given in parametric form by

$$x = L \left[ \frac{X^3}{3} - X Y^2 - X \right]$$

$$y = L \left[ -\frac{Y^3}{3} + Y X^2 + Y \right]$$

$$z = L (X^2 - Y^2)$$  

(111)

We consider a superfluid film coating only a circle of radius $A$ about the origin of the $X,Y$ plane because the complete surface has self-intersections. This surface can be called the Enneper disk and is illustrated in Fig. (24). The figure illustrates that the left and right hand sides of the saddle fold over it and would pass through each other if allowed to extend further while the front and the back would eventually intersect each other underneath the saddle. The former pair of intersection curves correspond to the two branches of the hyperbola $X^2 = 3(Y^2 + 1)$. When the reference surface is curved into the Enneper surface, the $X$ axis bends upward so that the branches map to the same intersection curve in the $yz$ plane. (The other intersection curves are obtained by exchanging $X$.

---

$^4$ As a check of this identity, imagine reversing the roles of the reference and target surfaces. Then $\omega$ should be regarded as a function on the reference surface. This changes the Laplacian by a factor of $e^{2\omega}$ (because $g_{T,cd}$ is replaced by $g_{T,cd}$). Also, the sign of $\omega$ should be reversed. Rearranging the equation now brings it back into the original form with $T$ and $R$ switched.
and \( \mathcal{Y} \).) Since the points where these hyperbolae are closest to the origin are \((\pm \sqrt{3}, 0), (0, \pm \sqrt{3})\), a non-self-intersecting portion of the Enneper surface results as long as \( A < \sqrt{3} \).

Now we explicitly calculate how a single vortex interacts with the curvature of such a surface by using Eq. (103). (We will use conformal mapping instead of Eq. (20) since the latter equation does not hold on a surface with the topology of a disk, as \( \omega \) does not satisfy the Dirichlet boundary conditions which are implied by such an expression.) The metric obtained from (111) is given by

\[
dx^2 + dy^2 + dz^2 = L^2(d\mathcal{X}^2 + d\mathcal{Y}^2)(1 + \mathcal{X}^2 + \mathcal{Y}^2)^2. \tag{112}
\]

(Surprisingly, this metric is rotationally symmetric. This implies that the surface may be slid along itself without stretching, but with changing amounts of bending.) Hence

\[
\omega_{\text{Enneper}} = -\ln L(1 + \mathcal{R}^2), \tag{113}
\]

where \( \mathcal{R}^2 = \mathcal{X}^2 + \mathcal{Y}^2 \). According to Eq. (103), this indicates that the vortex should be attracted to the middle of the surface, but of course this force competes with the boundary interaction \( K\pi \ln \frac{A^2 - \mathcal{R}(u)}{aA} \) which tries to pull the vortex to the edge. This expression for the boundary interaction is obtained from the familiar formula for the energy of a vortex interacting with its image in a flat reference disk (Vitelli and Nelson, 2004). The total energy is then

\[
E = K[\pi \ln \frac{L}{aA} + \pi \ln(A^2 - \mathcal{R}^2)(1 + \mathcal{R}^2)]. \tag{114}
\]

As long as \( A > 1 \), the central point of the saddle is a local minimum and this condition is compatible with the requirement \( A < \sqrt{3} \) for non-self-intersecting disks. Fig. 24 shows the flow lines of a vortex forced by the geometric interactions towards the center of an Enneper surface with \( A = 1.5 \).

In general, conformal mapping allows us to express the energy of a single vortex on a deformed surface with a boundary in the form:

\[
E = \pi K[\ln \frac{A^2 - \mathcal{R}(u)}{aA} - \omega(u)], \tag{115}
\]

where \( \mathcal{R}(u) \) refers to the image of a defect at \( u \) under a conformal map to a flat, circular disk of radius \( A \). The Green’s function method cannot be used to determine the energy of defects on a surface with a boundary. Although the conformal factor \( \omega(u) \) satisfies the Poisson equation, Eq. (2), it cannot be expressed as the integral of the curvature times the Green’s function (as in Eq. (91)), since \( \omega \) does not satisfy simple boundary conditions. In any case, the first term in Eq. (115) has no general expression in terms of \( \omega \) either. Interestingly, the total single-particle energy satisfies a nonlinear differential equation (the Liouville equation):

\[
\nabla_u^2 E(u) = -\pi KG(u) - \frac{4\pi K}{a^2} e^{-2\omega(u) \mathcal{R}/\mathcal{K}}. \tag{116}
\]

This result can be derived by using Eq. (105) to calculate the Laplacian of the first term and using

\[
\nabla^2 = e^{2\omega(u) \mathcal{R}/\mathcal{K}} \frac{\partial^2}{\partial \mathcal{R}^2} \mathcal{R} \frac{\partial}{\partial \mathcal{R}}
\]

to calculate the Laplacian of the second term. \( E(u) \) also satisfies an asymptotic boundary condition:

\[
e^{\frac{\omega(u)}{\mathcal{K}}} \to \frac{2d}{a} \tag{117}
\]

as \( d \), the distance from \( u \) to the boundary, approaches 0. Together the differential equation and the boundary condition should determine the total geometrical and boundary energy of a single vortex, although the nonlinear Eq. (116) is difficult to solve.

C. Periodic surfaces

In this section, we illustrate how a periodically curved substrate distorts the flat space interaction energies between vortices, besides generating the single-particle geometric potential. This effect is shown to be a consequence of the action of a conformal map which generally will map the target surface into a periodic reference substrate with different lattice vectors from the vectors of the target substrate. According to the general relation Eq. (92), the long-distance behavior of the Green’s function is given by the logarithm of a distorted distance.

Consider a surface with a periodic height function \( z = h(x, y) \), i.e., say \( h \) satisfies

\[
h(x + \lambda_i, y + \mu_i) = h(x, y) \quad \text{for} \ i = \{1, 2\}, \tag{118}
\]

where \( i \) labels the two basis vectors, which are not assumed to be orthogonal. Figure 25 shows the corresponding periods \( (\lambda_i, \mu_i) \). A conformal mapping can be
chosen to preserve the fact that the substrate is periodic but not the actual values of the periods, which are therefore given on the reference substrate by two new pairs denoted by \((\Lambda_i, M_i)\). In other words, we suppose that a tessellation of the target substrate by congruent unit cells is mapped to a set of congruent unit cells on the reference substrate. Then the map transforming the original coordinates \((x, y)\) into the target coordinates \((\mathcal{X}, \mathcal{Y})\) satisfies

\[
\begin{align*}
\mathcal{X}(x + \lambda_i, y + \mu_i) &= \mathcal{X}(x, y) + \Lambda_i \\
\mathcal{Y}(x + \lambda_i, y + \mu_i) &= \mathcal{Y}(x, y) + M_i.
\end{align*}
\]

There is no simple formula for the new set of primitive lattice vectors \((\Lambda_i, M_i)\) for the reference space. In some cases, though, precise information can be derived from the fact that the \((\Lambda_i, M_i)\) share the symmetry of the topography of the original substrate. For example, if the lattice is composed of bumps which have a 90° rotational point symmetry, then the reference lattice will be square. On the other hand, the topology of the periodic surface with a square lattice shown in the contour plot of Fig. 20 does not posses a 90° rotational symmetry and hence its conformal image will have a rectangular lattice.

To get an idea how the conformal mapping behaves macroscopically, we try to decompose it into a linear transformation \(\mathcal{L}\) with matrix coefficients \((A, B; C, D)\) and a periodic modulation captured by the functions \(\xi(x, y)\) and \(\eta(x, y)\) that distort the \(\mathcal{X}\) and \(\mathcal{Y}\) axes respectively:

\[
\begin{align*}
\mathcal{X}(x, y) &= Ax + By + \xi(x, y) \\
\mathcal{Y}(x, y) &= Cx + Dy + \eta(x, y).
\end{align*}
\]

(This decomposition is justified by the self-consistency of the following calculations.) The matrix coefficients of the linear transformation can be determined by requiring consistency with Eq. \((119)\). Start by evaluating the left hand sides of the two Equations \((120)\) at the positions \(\{x + \lambda_i, y + \mu_i\}\) which are shifted by the two pairs of periods \(\{\lambda_i, \mu_i\}\), so that the right hand sides become \(\mathcal{X}(x, y) + \Lambda_i\) and \(\mathcal{Y}(x, y) + M_i\), according to Eq. \((119)\). Then subtract the resulting equations from the corresponding unshifted Equations \((120)\), for each value of \(i\). We then obtain two pairs of equations

\[
\begin{align*}
A\lambda_i + B\mu_i &= \Lambda_i \\
C\lambda_i + D\mu_i &= M_i \quad \text{for } i = \{1, 2\},
\end{align*}
\]

where we have used the fact that the periodic functions \(\xi(x, y)\) and \(\eta(x, y)\) are unchanged when shifted by the periods. We can now solve the four equations of Eq. \((121)\) simultaneously for \(A, B, C,\) and \(D\) to see that the linear transformation matrix \(\mathcal{L}\) reads

\[
\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ M_1 & M_2 \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 \end{pmatrix}^{-1}.
\]

(Note now we can justify the original decomposition, Eq. \((120)\), by defining \(\mathcal{L}\) by Eq. \((122)\) and defining \(\xi(x, y)\) and \(\eta(x, y)\) as the discrepancy between the conformal map \(\mathcal{X}(x, y), \mathcal{Y}(x, y)\) and the linear map \(\mathcal{L}(x, y)\) as in Eq. \((120)\). We can then check that \(\xi(x, y)\) and \(\eta(x, y)\) are periodic functions of the coordinates.)

The linear transformation can be used to calculate approximately the long distance behavior of the Green’s function

\[
\Gamma(x, y; x', y') = -\frac{1}{4\pi} \ln[(\Delta x)^2 + (\Delta y)^2] \\
\approx -\frac{1}{4\pi} \ln[(Ax + By)^2 + (Cdx + Ddy)^2],
\]

where we used the fact that the periodic functions \(\xi(x, y)\) and \(\eta(x, y)\) are bounded and hence negligible in comparison to \(\Delta x\) and \(\Delta y\) at long distances. This expression illustrates the fact that the matrix \(\mathcal{L}\) captures the long distance lattice distortions induced by the conformal mapping, apart from the additional waviness described by \(\xi(x, y)\) and \(\eta(x, y)\). The linear transformation determined by \(\mathcal{L}\) is by itself typically not conformal, meaning that it generates an anisotropic deformation of the target lattice which does not preserve the angle between the original lattice vectors.

The deformation of the lattice is controlled by the curvature of the substrate. To spell out this connection and allow an explicit evaluation of the long-distance
Green’s function in Eq. (123), we explicitly evaluate the matrix elements $L_{ij}$ in terms of the height function $h(x, y)$ of a gently curved (or low-aspect-ratio) surface, one for which $h(x, y) \ll \{ \lambda_i, \mu_i \}, \{ \xi(x, y), \eta(x, y) \} \ll 1$ and $\{ A - 1, D - 1, B, C \} \ll 1$. The new set of (isothermal) coordinates $\mathcal{X}$ and $\mathcal{Y}$, used to implement the conformal transformation, are found by solving the Cauchy-Riemann Equations (A3)

$$
\partial_x \mathcal{X} = \sqrt{g^{yy}} \partial_y \mathcal{Y} + \sqrt{g^{xy}} \partial_y \mathcal{Y} \\
\partial_y \mathcal{X} = -\sqrt{g^{xx}} \partial_x \mathcal{Y} - \sqrt{g^{xy}} \partial_y \mathcal{Y},
$$

which, upon substituting from Eq. (120) and making the small aspect ratio approximation discussed in Appendix A reduce to

$$
A + \partial_x \xi = D + \partial_y \eta + \frac{1}{2} (h_x^2 - h_y^2) \quad (124) \\
B + \partial_y \xi = -C - \partial_x \eta + h_x h_y. \quad (125)
$$

We now proceed to show that these equations do not have solutions unless the lattice is distorted, that is to say the matrix $L$ cannot be the identity for a generic periodic function $h(x, y)$. Note that the periodicity of $\xi(x, y)$ and $\eta(x, y)$ implies that the integral of either one over any unit cell, e.g.,

$$
\int \int_{\text{cell}} dxdy \, \xi(a + x, b + y) \quad (126)
$$
is independent of the quantities $(a, b)$ by which the unit cell is shifted. Upon differentiating the integral with respect to $a$, one obtains

$$
\int \int_{\text{cell}} dxdy \, \xi_z(x, y) = 0. \quad (127)
$$

Similarly, the averages of $\eta_z, \eta_y$, and $\xi_y$ are also equal to zero. Hence, upon averaging Eq. (125) over a unit cell we obtain the key relations

$$
A - D = \frac{1}{2} (h_x^2 - h_y^2) > \\
B + C = < h_x h_y >, \quad (128)
$$

which prove our assertion that $L$ cannot be a simple dilation or rotation. Some shear is naturally introduced by the non-trivial metric (or height function) of the underlying surface.

The $L$ matrix is undetermined up to a dilation and a rotation but this is of no consequence to the determination of the Green’s function. In fact, to find the Green’s function, note that Eq. (125) allows us to write the matrix coefficients in terms of two undetermined constants $\epsilon_1$ and $\epsilon_2$ that will drop out of the final answer:

$$
A = 1 + \epsilon_1 + \frac{1}{4} (h_x^2 - h_y^2) > \quad (129) \\
D = 1 + \epsilon_1 + \frac{1}{4} (h_y^2 - h_x^2) > \quad (130) \\
B = \epsilon_2 + \frac{1}{2} < h_x h_y > \quad (131) \\
C = -\epsilon_2 + \frac{1}{2} < h_x h_y >, \quad (132)
$$

so that consistency with Equations (128) is guaranteed. (The variables $\epsilon_1$ and $\epsilon_2$ parameterize an overall infinitesimal scaling (by $1 + \epsilon_1$) and a rotation (by angle $\epsilon_2$) respectively.) Substitution of these equations into Eq. (123) gives the desired long-distance behavior of the Green’s function purely in terms of derivatives of the height function, which we assume to be known:

$$
\Gamma(x, y; x', y') \approx -\frac{1}{4\pi} \ln[\Delta x^2 + \Delta y^2] \\
+ \frac{1}{2} < h_x^2 - h_y^2 > ((\Delta x)^2 - (\Delta y)^2) \\
+ 2 < h_x h_y > \Delta x \Delta y. \quad (133)
$$

This is the central result of this section; it can also be applied to interactions between disclinations in liquid crystals [Vitelli et al. 2001] and dislocations in crystals [Vitelli et al., 2000]. The anisotropic correction to the Green’s function, captured by the second and third term, suggests that a distorted version of the triangular lattice of vortices expected on a flat substrate may form when the helium-coated surface is rotated slowly enough that there is only one vortex to several unit cells. However, the actual ground state is likely to be difficult to observe, as the geometric potential will try to trap the vortices near saddles as discussed in Section III.B.

**D. Band-flows on elongated surfaces**

In this section, we show that the quantization of circulation can induce an extremely long-range force on a
stretched-out sphere (such as a surface with the shape of a zucchini or a very prolate spheroid). We first demonstrate the main result in the context of a simple example before presenting a general formula for the forces experienced by vortices on azimuthally symmetric surfaces. Details are presented in Appendix E. Consider a cylinder of length \(2H\) and radius \(R \ll H\) with hemispherical caps of radius \(R\) at the ends, depicted in Fig. 27 and imagine a symmetric arrangement of a vortex \((n=1)\) and an antivortex \((n=-1)\) at the north and south poles respectively. Extrapolating our intuition from flat space suggests that the energy of the vortex and antivortex is \(2\pi K \ln \frac{H}{a}\), where \(D\) is the distance between the vortices. However, more careful reasoning shows that the energy grows linearly rather than logarithmically with \(D\). The reason is that, unlike in flat space, the velocity field does not fall off like the inverse of the distance from each vortex. Note that the azimuthal symmetry of the arrangement of the vortices implies that the flow is parallel to the lines of latitude of the surface. Since the circulation around each latitude must be the lines of latitude of the surface. Since the circulation around every latitude is the same.

The kinetic energy of this part of the flow is \(\frac{4\pi R H}{2} \rho_s v^2 = 2\pi \frac{K H}{R}\). Since this cylindrical part of the flow forms the main contribution to the kinetic energy when \(H \gg R\) we find that the energy of a vortex-antivortex pair situated at opposite poles is linear,

\[
E_{\text{poles}} \approx 2\pi \frac{K H}{R}. \tag{135}
\]

(The exact expression also includes a near-vortex energy of approximately \(2\pi K \ln \frac{D}{a}\).)

In contrast, when the vortices forming the neutral pair are across from each other on the same latitude, the aforementioned long-range persistence of the velocity field is absent because the vorticity is screened within a distance of order \(R\). The resulting kinetic energy follows the familiar logarithmic growth

\[
E_{\text{equator}} \approx 2\pi K \ln \frac{2R}{a}. \tag{136}
\]

More generally, consider an azimuthally symmetric surface described by the radial distance, \(r(z)\), as a function of height, \(z\), as indicated in Fig. 28. A. If the north and south poles of the surface are at \(z_s\) and \(z_n\), then \(r(z_s) = r(z_n) = 0\) since the surface closes at the top and bottom. A point on such a surface can be identified by the coordinates \((\phi, \sigma)\) where \(\phi\) is the azimuthal angle and \(\sigma\) is the distance to the point from the north pole along one of the longitudes such as the one shown in Fig. 28A. In Appendix E we develop an approximation scheme which rests on the observation that the flow pattern becomes mostly azimuthally symmetric if \(\frac{D}{a} \ll 1\), even if the vortices break the azimuthal symmetry of the surface because they are not at the poles. If a pair of \(n = \pm 1\) vortices are present at different heights \(z_{1, 2}\), then the fluid in the band between them flows almost horizontally and at a nearly \(\phi\)-independent speed (except for irregularities near the vortices) while the fluid beyond them is approximately stagnant (see Fig. 28). Along any latitude inside the band the circulation is exactly \(\frac{h}{m}\), while it is zero above and below it. These properties approximately determine the flow away from the vortices since the asymmetric irregularities near the vortices decay exponentially, giving a speed of

\[
\frac{h}{2\pi m r(z)} \tag{137}
\]

in between \(z_1\) and \(z_2\), the locations of the vortices, and zero elsewhere. We describe this approximation as the “band model”. This expression shows that constrictions in the surface cause the speed to increase. For the cylinder with spherical caps and arbitrarily placed vortices, Eq. (137) shows that the speed is approximately constant within the band. (It increases within a distance on the order of \(R\) from the vortices, which are on the edges of the band.)

The kinetic energy can be determined approximately by noting that the energy in a thin ring on the surface between the vortices (extending from the longitudinal arclength \(\sigma\) to \(\sigma + d\sigma\)) is

\[
\left[2\pi r(z) d\sigma\right] \frac{1}{2} \rho_s \left(\frac{h}{m r(z)}\right)^2 = \pi K \frac{d\sigma}{r(z)}
\]

where the first factor represents the area of the ring since \(\sigma\) is the geodesic distance along the surface and the second factor represents the included kinetic energy. The
FIG. 28 The flow on an azimuthally symmetric surface, described by the coordinates \((z, r)\) and an azimuthal angle \(\phi\) (not shown). A) The surface is defined as the surface of revolution of the curve \(r(z)\) in the \(r-z\) plane. The other two images show the flow on an ellipsoid and compare the flow pattern predicted by the band model (B) and the exact solution determined by conformal mapping (C). The flow lines in the band between the two vortices become close to horizontal and are approximately azimuthally symmetric. Beyond the vortices, they are spaced far apart indicating a vanishingly small speed for a greatly elongated surface.

On the capped cylinder, this force is independent of the positions of the vortices. Even on an arbitrary elongated surface, a noteworthy feature is that the force on vortex 1 does not depend on the position of vortex 2! This force can be explained with the familiar phenomenon of lift: the vortex is on the boundary between stationary and moving fluid, so there is a pressure difference due to the Bernoulli principle.

Approximating the flow pattern generated by multiple vortices in a similar fashion requires only minor modifications of the previous argument. In a low-resolution snapshot of the flow, the point-vortices would appear as circles of discontinuity in the velocity field that go all the way around the axis (the analogue for a layer of superfluid of a two dimensional vortex sheet). If the vortices are labeled in order of decreasing \(z\) a loop just below the \(l\)th vortex contains

\[
N_l = \sum_{i=1}^{l} n_i \quad \text{(140)}
\]

units of circulation above it. Approximate azimuthal symmetry of the flow then implies that,

\[
\mathbf{v}(z, \phi)_{\text{band}} = N_l \frac{h}{m r(z)} \hat{\phi} \quad \text{(for } z_l < z < z_{l+1}) \quad \text{(141)}
\]

a natural generalization of Eq. \(137\) that is proved in Appendix E.

Conformal mapping can be employed to justify (without detailed calculations) the decay of the nonazimuthally symmetric parts of the flow that are not determined by the quantization condition. We sketch the basic reasoning here by focusing (for simplicity) on the flow pattern near the equator of the surface, at a distance \(\sigma_{eq}\) from the north pole. The conformal transformation that maps the elongated sphere onto a regular reference sphere with coordinates \(\Theta, \Phi\) reads (see Appendix E)

\[
\sin \Theta = \text{sech} \int_{\sigma}^{\sigma_{eq}} \frac{d\sigma'}{r(\sigma')} \quad \text{and} \quad \Phi = \phi\quad \text{(142)}
\]

The upper and lower halves of the elongated sphere can be mapped to the upper and lower hemispheres by choosing appropriately between the two values of \(\Theta\) that correspond to a given value of \(\sin \Theta\). Near the equator the integral can be approximated by \(\frac{\sigma - \sigma_{eq}}{r_{eq}}\) since \(r\) varies slowly. Suppose the vortices are far from the equator, at a distance greater than \(k r_{eq}\) for a large \(k\). Then the vortices above the equator are mapped exponentially close (at a distance less than \(e^{-k}\)) to the sphere’s north pole. Likewise vortices on the southern half of the surface map exponentially close to the sphere’s south pole. We have thus reduced the task of finding the flow due to a complicated arrangement of vortices to a symmetric case. In fact, after mapping the flow on the long, thin surface to the reference sphere, nothing can be resolved beyond a pair of multiply quantized vortices at the north and south
poles containing \( N \) and \(-N\) units of circulation respectively, where \( N \) is the total circulation number of all the vortices above the equator. Since the image vortices are very close to the poles, their flow pattern on the reference surface is approximately azimuthally symmetric near the equator. When mapped back to the elongated surface, the flow retains its approximate azimuthal symmetry in the region around the equator, completing our argument. A similar argument proves the approximate azimuthal symmetry of the flow near lines of latitude other than the equator; one simply adjusts the conformal map in Eq. (112) so that another latitude of the target surface is mapped to the equator of the reference sphere.

Now the geometrical force derived in Eq. (139) can compete with physical forces such as those induced by rotating an ellipsoid about its long axis with angular velocity \( \Omega = \Omega \hat{z} \). Let us extend the treatment of rotational forces on curved substrates, introduced in section III, to the case of an ellipsoid described by the radial function

\[
r(z) = R \sqrt{1 - \frac{z^2}{H^2}}.
\]

(143)

Let us use the aspect ratio \( \alpha = \frac{H}{R} \) to describe how elongated this ellipsoid is, and determine how a vortex-anti-vortex pair is torn apart by the rotation as the angular frequency is increased. As in Section III, metastable vortices are located at heights

\[
\pm z = \pm \alpha \sqrt{R^2 - \frac{\hbar}{2m\Omega}}.
\]

(147)

The vortices first become metastable when force balance is achieved with both vortices close to the equator. Upon substituting the equatorial value \( r_1 = R \) into Eq. (146) an estimate of \( \Omega_b \) is obtained

\[
\Omega_b \approx \frac{\hbar}{2mR^2}.
\]

(148)

When the pair first appears, there will actually be a non-zero defect separation, although substituting Eq. (148) into Eq. (147) suggests otherwise. Imagine slowing the rotation speed through \( \Omega = \Omega_a \). The vortices will approach each other gradually; within the large vortex-separation approximation of Eq. (139), the attraction between them will decrease as they become closer because \( r(z) \) increases. However, when the vortices become close enough, the attraction between them starts increasing and the vortices are suddenly pulled together. The minimum \( z \)-coordinate for metastable vortices is derived along these lines in Appendix E (which also discusses what happens at \( \Omega = \Omega_a \)) and reads

\[
z_1 = -z_2 = z_b \approx R \ln \alpha.
\]

(149)

The transition through \( \Omega_b \) is illustrated pictorially in Fig. 24 which shows how the local minimum in the energy function disappears as the frequency decreases.

**E. Interactions on a closed surface**

To understand interactions between vortices on an arbitrary deformed sphere one must come to terms with the neutrality constraint on the total circulation of a flow. On any compact surface,

\[
\sum_i n_i = 0.
\]

(150)

This constraint on the sum of the circulation indices \( \{n_i\} \) always holds: if the surface is divided into two pieces by
a curve, the sum of the quantum numbers on the top and bottom half must be equal and opposite (because they are both equal to the circulation around the dividing curve). As we shall see, this relation implies that there are multiple ways of splitting up the energy into single-particle energies and two-particle interaction energies, despite the fact that the total energy is well-defined. The behavior of the one-particle and interaction terms depends on how the splitting is carried out. To illustrate this ambiguity, multiply Eq. (150) by 4π²K₁f(u₁) and separating out the i = 1 term, to obtain

\[ 4\pi^2n^2f(u_1) = -\sum_{i\neq 1} 4\pi^2n_1n_i f(u_1). \] (151)

Hence, a portion 4π²Kf(u₁) of the “geometrical energy” of vortex 1 can be reattributed to this vortex’s interaction with all the other vortices. This can be seen explicitly by checking that the net energy according to Eqs. (151) and (150),

\[ E(\{n_i, u_i\}) = \sum_{i<j} 4\pi^2Kn_in_j\Gamma(u_i, u_j) \]
\[ -\sum_{i} \pi n_i^2KU_G(u_i) \] (152)

(153)

is not changed by the following transformation:

\[ \Gamma'(u_1, u_2) = \Gamma(u_1, u_2) - f(u_1) - f(u_2) \] (154)
\[ U'_G(u) = U_G(u) + 4\pi f(u) \] (155)

This flexibility is reflected in the possibility of choosing different Green’s functions \( \Gamma'(u_1, u_2) \) for the covariant Laplacian on a deformed compact surface. A detailed discussion of Green’s functions is given in Appendix D.

Here we highlight this ambiguity by performing explicit calculations using two distinct choices of Green’s functions on a model surface formed from a unit sphere. First cut the sphere in halves along a great circle. Choose one of the hemispheres and bring opposite sides of the great circle bounding it together and glue them. The result is a pointed sphere that is furled up and the other hemisphere together with two defects and their images. The hemisphere is furled up into the pointed sphere so that the left and right halves of the cut (which appears as a horizontal diameter in this top view) are brought together to form the seam on the pointed sphere; simultaneously, the defects \( Q₁, Q₂ \) move to positions \( u₁ \) and \( u₂ \) on the pointed sphere. The furling process leads to a continuous flow pattern on the pointed sphere. For example, the two points marked with an x on the cut in the original sphere are sealed together. Both points feel a strong flow, the one on the right because it is close to the vortex at \( Q₁ \) and the one on the left because it is close to this vortex’s image.

![FIG. 30](image)

A) The process of folding a hemisphere into a pointed sphere, bounded at the north and south poles by two 180° disclinations. The north and south poles move outward along the axis, while the latitudes stay horizontal. The Gaussian curvature is invariant because the decreasing curvature of the lines of longitude is compensated by the tighter curvature around the lines of latitude.

B) A top view of vortices during the furling-up of one hemisphere. The first stage shows both the (0 < \( \alpha < \pi \)) hemisphere that is furled up and the other hemisphere together with two defects and their images. The hemisphere is furled up into the pointed sphere so that the left and right halves of the cut (which appears as a horizontal diameter in this top view) are brought together to form the seam on the pointed sphere; simultaneously, the defects \( Q₁, Q₂ \) move to positions \( u₁ \) and \( u₂ \) on the pointed sphere. The furling process leads to a continuous flow pattern on the pointed sphere. For example, the two points marked with an x on the cut in the original sphere are sealed together. Both points feel a strong flow, the one on the right because it is close to the vortex at \( Q₁ \) and the one on the left because it is close to this vortex’s image.
a conducting plane, one completes the space, with the other half-space. Then one introduces charges into this fictitious region to ensure that the right boundary conditions (orthogonality of the field lines to the original boundary plane) are satisfied. For the pointed sphere, we first complete the surface by opening it up again and adding back the second hemisphere. We can describe points on the completed sphere by (\(u\), \(\sigma\)) on the pointed sphere and \(Q, Q^\ast\) on the full sphere. Focusing on the hemisphere \(0 < \alpha < \pi\), the latter is the image vortex of the former, obtained by rotating \(Q\) by 180° around the \(z\)-axis.

The energy of a defect configuration on the pointed sphere is derived by halving the energy of the flow pattern produced by the doubled set of vortices on the full sphere. In analogy with the electrostatic problem, the purpose of situating the image defects in the way just described is to preserve the continuity of flows across the seam. Imagine drawing the flow pattern of all the vortices on the sphere. Focus on the hemisphere \(0 < \alpha < \pi\). Because the vortices are placed symmetrically about the sphere’s axis, the flow near \(\alpha = 0\) will match the flow near \(\alpha = \pi\) when the surface is sealed. (See Fig. 30B.)

The flow pattern on the pointed sphere results from rolling up half of the flow pattern on the sphere. Once the positions of the image defects are chosen, the flow pattern is found by deriving it from the stream function \(\chi(u)\) introduced in Sec. II. The stream function at a point \(u\) on the pointed sphere can be expressed in terms of the Green’s function of the sphere, according to Eq. (17):

\[
\chi(u) = \sum_i \frac{n_i h_i}{m} [\Gamma_{\text{sphere}}(Q_i, Q_i) + \Gamma_{\text{sphere}}(Q_i, Q_i^\ast)]
\]  

where \(Q\) is the point on the sphere corresponding to \(u\) on the pointed sphere and \(Q_i\) and \(Q_i^\ast\) are the locations of the \(i\)th pair of vortices on the sphere.

The energy of the vortices on the sphere takes up the familiar electrostatic form of a sum of the interactions between all pairs of defects and/or their images. The energy stored in the flow pattern on the pointed sphere (which is half as large as on the complete sphere) reads

\[
\frac{E_N}{K} = \frac{1}{4} \sum_{i \neq j} 4\pi^2 n_i n_j [\Gamma_{\text{sphere}}(Q_i, Q_j) + \Gamma_{\text{sphere}}(Q_i, Q_j^\ast)] + \Gamma_{\text{sphere}}(Q_i^\ast, Q_j) + \Gamma_{\text{sphere}}(Q_i^\ast, Q_j^\ast) + \frac{1}{2} \sum_{i \neq j} 4\pi^2 n_i^2 \Gamma_{\text{sphere}}(Q_i, Q_i^\ast)
\]

\[
= \frac{1}{2} \sum_{i \neq j} 4\pi^2 n_i n_j [\Gamma_{\text{sphere}}(Q_i, Q_j) + \Gamma_{\text{sphere}}(Q_i, Q_j^\ast)] + \sum n_i^2 \Gamma_{\text{sphere}}(Q_i, Q_i^\ast)
\]  

(158)

In the second expression, we note that the terms in the first line are equal in pairs, so that a factor of \(\frac{1}{2}\) cancels. This energy is given the same form as Eq. (153) by separating out the part which depends on the positions of two vortices, proportional to

\[
\Gamma_s(u_1, u_2) = \Gamma_{\text{sphere}}(Q_1, Q_2) + \Gamma_{\text{sphere}}(Q_1, Q_2^\ast)
\]  

(159)

and the part which depends on one vortex,

\[
U_s(u) = -2\pi \Gamma_{\text{sphere}}(Q, Q^\ast).
\]  

(160)

The function in Eq. (159) is a Green’s function for the pointed sphere, as the placement of the images guarantees that this function is well-defined on the pointed sphere. It appears in the stream function, Eq. (157), as well as in the energetics, as expected for a Green’s function.

The potential which describes the single-particle energy of a vortex becomes singular as the vortex approaches the apex of the cone at the north or south pole, since then the vortex \(Q\) approaches its image \(Q^\ast\). This is in accord with the result, Eq. (91), that the Gaussian curvature is the source of the single-particle energy since the pointed sphere has delta-function concentrations of curvature at its north and south poles:

\[
G(u) = 1 + \pi \delta_N(u) + \pi \delta_S(u).
\]  

(161)

where \(\delta_N(u)\) and \(\delta_S(u)\) are the appropriate delta functions. The geometric repulsion from the positive curvature points arises from the repulsion between vortices and their images! We can check step-by-step that \(U_s\) is sourced by the Gaussian curvature,

\[
U_s(u) = -\int \int \Gamma_s(u, u') G(u') d^2 u'.
\]  

(162)

We substitute for \(G(u')\) from Eq. (161) and for \(\Gamma_s\) from Eq. (159) which can be written in the form,

\[
\Gamma_s(\sigma_1, \phi_1; \sigma_2, \phi_2) = \Gamma_{\text{sphere}}(\sigma_1, \frac{\phi_1}{2}; \sigma_2, \frac{\phi_2}{2}) + \Gamma_{\text{sphere}}(\sigma_1, \frac{\phi_1}{2}; \sigma_2, \frac{\phi_2}{2} + \pi)
\]

\[
- \frac{1}{4\pi} \ln 4[(1 - \cos \sigma_1 \cos \sigma_2)^2 - \sin^2 \sigma_1 \sin^2 \sigma_2 \cos^2 \frac{\phi_1 - \phi_2}{2}].
\]  

(163)
In the last line, we have evaluated the Green’s function for the sphere by writing the chordal distance between, e.g., \( Q_1 \) and \( Q_2 \) in terms of the spherical coordinates \((\alpha_{1,2}, \theta_{1,2})\), \( D^2 = 2(1 - \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos(\alpha_1 - \alpha_2)) \) (see Lubensky and Prost [1992]), and then combining the two terms together. To evaluate the integral in Eq. (162), we have to note that the area element of this integral is \( d^2u = \frac{1}{\sin \sigma} d\sigma d\phi \). The area of a region on the pointed sphere is the same as the area \( \sin \beta d\beta d\alpha \) of the corresponding region on the original sphere, and the factor of \( \frac{1}{2} \) results from how the angles are related, \( \alpha = \frac{\beta}{2} \), see Eq. (156). Now the integral on the right-hand side of Eq. (162) can be shown to be equal to the left-hand side using the identities

\[
\int_0^{2\pi} \ln |A + B \cos t| dt = 2\pi \ln \frac{A + \sqrt{A^2 - B^2}}{2} \quad \text{if} \quad B < A
\]

\[
= 2\pi \ln \frac{B}{2} \quad \text{if} \quad B > A.
\]

We have now derived one formulation of the energetics in terms of \( \Gamma_s \) and \( U_s \), the corresponding geometric potential. Let us contrast this isometric mapping method with the conformal mapping method in order to illustrate how different approaches can naturally lead to different delineations between vortex-vortex and vortex-curvature interactions. (The net result is of course the same from either point of view.) As a result of the isometric mapping each point is doubled, whereas the distance-distorting conformal mapping transforms each point on the pointed sphere to one point on the reference sphere.

We first use Eq. (142) to find that the conformal map is given by

\[
\tan \frac{\Theta}{2} = \tan^2 \frac{\sigma}{2}.
\]

Comparing the conformal mapping results, Eqs. (88) and (89) to the Green’s function formulation, Eqs. (11) and (90) suggests the following identification of the interaction potential (or Green’s function) and single-particle potential:

\[
\Gamma_c(u_1, u_2) = \Gamma^{\text{sphere}}(\Theta(\sigma_1), \phi_1; \Theta(\sigma_2), \phi_2)
\]

\[
U_c(u) = \omega = \ln \frac{2 \sin \sigma}{1 + \cos^2 \sigma}.
\]

These expressions differ from Equations (159) and (160). Nevertheless, as promised, the net energy is the same whether the pairs \( (\Gamma_s, U_s) \) or \( (\Gamma_c, U_c) \) are used in place of \( \Gamma \) and \( U_G \). In fact,

\[
\Gamma_c(u_1, u_2) = \Gamma_s(u_1, u_2) - f(u_1) - f(u_2)
\]

\[
U_c(u) = U_s(u) + 4\pi f(u)
\]

where \( f(u) = -\frac{1}{4\pi} \ln(1 + \cos^2 \sigma) \). This transforms the energy from the single-particle to the interaction terms consistently as described at the beginning of the section. Appendix D shows that the Green’s function formulation is generally equivalent to the conformal mapping result derived in Section [V.A] even when there is no method of images that can be used to determine the Green’s function explicitly in general.

VI. LIMITS ON THE STRENGTH AND RANGE OF GEOMETRICAL FORCES

Geometrical forces are limited in strength due to the nonlinear relation between the curvature and the geometric potential. Curvature affects both the source of the geometrical force and the force law, as illustrated in the examples of Secs. [V.C] and [V.D]. As a consequence, even on a wildly distorted surface (with planar topology), there is a precise limit on the strength of the force on a single vortex. This result has the character of a geometrical optimization problem, like maximizing the capacitance of a solid when the surface area is given. Consider a vortex located at the center of a geodesic disk of radius \( R \). Assume that the Gaussian curvature is zero within the disk, but may be different from zero elsewhere. Then the force \( \mathbf{F} \) due to the curvature satisfies

\[
|\mathbf{F}| \leq \frac{4\pi Kn_1^2}{R},
\]

where \( n_1 \) is the number of circulation quanta in the vortex. This relation between \( R \) and \( \mathbf{F} \) is proven in Appendix E.

If one warps a surface in a vain attempt to overcome the limit, but the force gets diluted because the distortion of the region around the curvature pulls the force-lines apart, as we can understand from the simple example of vortices on cones.

A cone of cone-angle \( \theta \) is obtained by taking a segment of paper with an angle \( \theta \) and gluing the opposite edges of the angle together. This is most familiar when \( \theta < 2\pi \). If \( \theta = 2\pi m + \beta \), such a cone can be produced by adding \( m \) extra sheets of paper, as illustrated in Fig. 31. We slit the \( m \) sheets of paper and put them together as an angle of size \( \beta \) cut out of an additional sheet. By gluing the edges of the slits together cyclically, a cone of arbitrary angle \( \theta \) is made.

A cone has a delta function of curvature at its apex, but no Gaussian curvature elsewhere because the surface can be formed from a flat piece of paper without stretching. The weight \( 2\pi - \theta \) of the delta function is expressed, according to the Gauss-Bonnet theorem, as an integral of the Gaussian curvature in any region containing the apex [Kamien, 2002].

\[
\iint G(u) d^2u = 2\pi - \int \kappa ds
\]

where \( \kappa \) is the geodesic curvature along the boundary of the region and \( s \) its arc length. Apply this formula to the circle of radius \( D \) centered at the apex of the cone. Imagine the circle as it would appear on the original sheets of paper, as in Fig. 31. Its measure in radians
is \( \beta + 2\pi m = \theta \) since it consists of \( m \) complete circles together with an additional arc. The length is therefore \( S = D\theta \). The geodesic curvature of the circle does not change when the cone is unfolded, so it is equal to \( \frac{1}{D} \).

Upon substituting in Eq. (168), we obtain

\[
\int \int G(u) d^2 u = 2\pi - S \frac{1}{D} = 2\pi - \theta. \tag{169}
\]

When \( \theta > 2\pi \) the curvature is negative.

Now imagine a vortex (with \( n_1 = \pm 1 \), say) at a distance \( D \) from the cone point, on the circle of circumference \( S \) just considered. The arbitrarily large negative curvature which is possible by making \( m \) large seems to defy the general upper bound on the geometric force. According to Newton’s theorem, applied to the radius \( D \) circle centered at the cone’s apex and passing through the vortex, the force on the vortex is \( F = \frac{2\pi K}{\theta} \int_G dx^2 u_s \). Since the circumference \( S = D\theta \) is larger than it would be in the plane, the force is diluted; substituting the integrated curvature from Eq. (169), we find that it is given by

\[
F = \pi K \frac{2\pi - \theta}{D}. \tag{170}
\]

This satisfies Eq. (167) for all negatively curved cones (\( \theta > 2\pi \)); even when \( \theta \to \infty \) the magnitude of the force is less than \( 4\pi K D \) because the large circumference in the denominator of the Newton’s theorem expression cancels the large integrated curvature in the numerator.

In the opposite limit \( \theta \to 0 \), the theorem described by Eq. (167) is still correct of course. One has to be careful about applying it, however. The force on a vortex at radius \( D \) (given by Eq. (170)) is not bounded by \( 4\pi K \) with \( R \) set equal to \( D \) when \( \theta \) is small enough (in fact, for an extremely pointed cone, \( \theta \ll 1 \), the force given by Eq. (170) diverges), but this does not contradict the inequality because the circle of radius \( D \) centered at the vortex is pathological: although it does not contain any curvature, the circle wraps around the cone and intersects itself. Taking \( R \) to be the radius of the largest circle centered at the defect which does not intersect itself, one finds that the inequality is satisfied, with room to spare, for all values of the cone angle \( \theta \) (see Appendix F). One can describe a more awkwardly shaped surface such that the force on a singly-quantized vortex is arbitrarily close to the upper bound \( \frac{4\pi K}{R} \) (see Appendix F).

One can also provide limits to the strength of the geometric force from a localized source of curvature. Rotationally symmetric surfaces such as the Gaussian bump have force fields that do not extend beyond the bump, since the net Gaussian curvature is zero, and Newton’s theorem says that only the integrated Gaussian curvature can have a long range effect for a rotationally symmetric surface. To get a longer-range force, one must focus on non-symmetric surfaces, like the saddle surface of Sec. II.B. The integration methods of Appendix A can be used to show that this surface’s potential has a quadrupole form at long distance. Let us consider, more generally, a plane which is flat except for a non-rotationally symmetric deformation confined within radius \( R \) of the origin. (The result will not apply directly to the saddle surface since its curvature extends out to infinity.) In this case, the total integrated Gaussian curvature is zero, implying that the long-range force law cannot have any monopole component. A dipole component is not ruled out by this simple reasoning, but Appendix F shows that the limiting form of the potential is at least a quadrupole (or a faster decaying field),

\[
E(r) \sim n_1^2 \frac{\mu_2 \cos(2\phi - \gamma_2)}{r^2}, \tag{171}
\]

where \( r \) and \( \phi \) are the polar coordinates of the vortex relative to the origin, and \( \mu_2 \) and \( \gamma_2 \) are constants that depend on the shape of the deformation in the vicinity of the origin. As in the previous case, there is an upper limit on the quadrupole moment \( \mu_2 \), no matter how strong the curvature of the deformation is:

\[
\mu_2 \leq \pi KR^2. \tag{172}
\]

For electrostatics in the plane, the maximum quadrupole moment of \( N \) particles with charge \( 2\pi \) and \( N \) with charge \(-2\pi \) in a region of radius \( R \) is at most of the order of \( KNR^2 \), which has the same form as the bound in Eq. (172), except for the factor of \( N \). Because of the non-linearity of the geometrical force and restrictions on how much positive and negative curvature can be separated from each other, the quadrupole moment is bounded no matter how drastically curved the surface is.

These results describe key physical differences (resulting from the fact that the curvature cannot be adjusted without changing the surface) between the geometrical forces discussed in this work and their electrostatic counterparts despite the close resemblance from a formal viewpoint.
VII. CONCLUSION

In this article, we have laid out a mathematical formalism based on the method of conformal mapping that allows one to calculate the energetics of topological defects on arbitrary deformed substrates with a focus on applications to superfluid helium films. The starting point of our approach is the observation that upon a change of coordinates the metric tensor of a complicated surface can be brought in the diagonal form $g_{ab} = e^{2\omega(u)}\delta_{ab}$. This corresponds to the metric of a flat plane which is locally stretched or compressed by the conformal factor $e^{2\omega(u)}$. Many of the geometric interactions experienced by topological defects on curved surfaces are simply determined once the function $\omega(u)$ is known. Vortices in thin helium layers wetting a curved surface are a natural arena to explore this interplay between geometry and physics but our approach is of broader applicability.

The curved geometry results in a modified law for defect interaction as well as in a one body geometric potential. On a deformed plane, the latter is obtained by solving a covariant Poisson equation with the Gaussian curvature as a source. Table III presents a summary of the general form that the defect interaction (first row) and the geometric potential (third row) take up in curved spaces with the topology of a deformed plane (first column), disk (second column) and sphere (third column). These results can be derived starting from the differential equations that the geometric potential satisfies or the appropriate Green’s functions that we list in the second and fourth row respectively for each of the three surface topologies. The fifth row of Table III directs the reader towards the relevant sections and appendices of the paper where he will be able to find some concrete applications of the formalism and technical derivations.

For example, the geometric potential of an Enneper disk (a minimal surface with negative curvature described in Sec. V.B) is given by the conformal factor $\omega(u)$ evaluated at the point $P = \{u_1, u_2\}$ where the vortex is located combined with an “electrostatic-like” interaction with an image defect located at the inverse of $P$ with respect to the circular boundary. The geometric potential satisfies the Liouville (non-linear differential) equation that reduces to the Poisson equation derived for the plane in the limit of an infinitely large disk. In the case of deformed spheres, we showed in Appendix D that one can make a convenient choice of Green’s function so that all the geometric effects are included in the defect-defect interactions without introducing a one-body geometric potential explicitly. An interesting application naturally arises on vesicles deformed into an elongated shape, like a zucchini. The range of the defect interaction becomes much longer and its functional form different from the logarithmic dependence expected in flat two dimensional spaces.

We hope that the discussion of the geometric effects presented in this work may pave the way for their observation in thin superfluid or liquid crystal layers on a curved substrate. A useful starting point could be the design of experiments to detect the geometric potential by balancing it with forces exerted on the defects by external fields or rotation of the sample as discussed in Section III. Such experiments should focus on single vortices, or on situations where the separation between vortices is comparable to the length scale of the geometry. Signatures of the geometric interactions described here may also survive in defect pinning experiments carried out in some bounded three dimensional geometries {Voll et al., 2006}.

VIII. ACKNOWLEDGMENTS

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APPENDIX A: Nearly Flat Surfaces

The calculations in Section III.B are based on perturbations about near flatness (see (David, 1989) and references therein). The perturbation theory will be in powers of an aspect ratio, $\alpha$, which measures the ratio of surface height to width of the landscape features. (We imagine that the height of the surface is given in the form $\omega(x,y)$ where $m$ is a fixed function.) The leading corrections to the flat space energies are second-order in $\alpha$. There are two of these; one is the geometric potential. When there are at least two vortices present, there is also a second order correction to the Green’s function, which ought to be retained since it is comparable to the geometric potential. The latter could be calculated by expanding the metric in Eq. (13) in powers of $\alpha$. Nevertheless, because the perturbations are singular, we prefer to use conformal mapping for this step just as we use in Sec. III.A to derive the geometric potential. Our calculations are limited to the case of an infinite deformed plane.

We use the $x$ and $y$ coordinates of a plane parallel to the surface for our coordinate system (the “Monge Gauge”). The metric is then $ds^2 = dx^2 + dy^2 + dz^2 = (1 + h_x^2)dx^2 + (1 + h_y^2)dy^2 + 2h_x h_y dx dy$. Subscripts on $h$ indicate derivatives, so that $h_{xx} = \partial_x^2 h$ etc. Upon calculating the curvature tensor we find the Gaussian curvature in the second order approximation (David, 1989)

$$G(x,y) = h_{xx} h_{yy} - h_{xy}^2.$$  \hfill (A1)

The geometric potential is found by approximating the
By taking the derivative of Eq. (A4) with respect to $x$ we obtain
\[ \frac{\partial}{\partial x} \left[ 4\pi^2 K n_i n_j \Gamma_D(u_i, u_j) \right] \]
and Eq. (A5) with respect to $y$ where
\[ \xi \text{ is a fine isothermal coordinate,} \]
the Cauchy-Riemann equations (David, 1989) which define

\[ X = x + \xi, Y = y + \eta \]

where the deformation parameters $\xi$ and $\eta$ are second order in $\alpha$:
\[ \eta_x + \xi_y \approx h_x h_y \]
\[ \eta_y - \xi_x \approx \frac{1}{2}(h_y^2 - h_x^2). \]

By taking the derivative of Eq. (A4) with respect to $x$
and Eq. (A5) with respect to $y$ and adding the results, we obtain
\[ \nabla^2_{\text{flat}} \eta \approx h_y \nabla^2_{\text{flat}} h, \] which may be solved by means of the Green’s function
keeping in mind the boundary condition that the conformal map must approach the identity at infinity; i.e., $\eta \to 0$,
giving the result
\[ \eta(x, y) \approx -\int dx' dy' \Gamma_{\text{flat}}(x - x', y - y')(h_{xx} h_{yy} - h_{xy}^2). \] (A2)

To do a conformal mapping to an equivalent flat space problem, we must solve the curved space generalization of
the Cauchy-Riemann equations (David, 1989) which define

\[ \nabla^a (Y) = -\gamma^a_b \nabla^b (X). \] (A3)

where $\gamma^a_b = g^{ac} \sqrt{g_{cb}}$. (In words, the gradients of $X$ and $Y$ are at right angles to each other and have equal magnitudes at every point.) We insert the expression for the metric in terms of $h$ into Eq. (A3) and expand to second order under the assumption that $X = x + \xi, Y = y + \eta$ where the deformation parameters $\xi$ and $\eta$ are second order in $\alpha$:
\[ \eta_x + \xi_y \approx h_x h_y \] (A4)
\[ \eta_y - \xi_x \approx \frac{1}{2}(h_y^2 - h_x^2). \] (A5)

By taking the derivative of Eq. (A4) with respect to $x$
and Eq. (A5) with respect to $y$ and adding the results, we obtain
\[ \nabla^2_{\text{flat}} \eta \approx h_y \nabla^2_{\text{flat}} h, \] (A6)
which may be solved by means of the Green’s function
keeping in mind the boundary condition that the conformal map must approach the identity at infinity; i.e., $\eta \to 0$, giving the result
\[ \eta(x, y) \approx -\int dx' dy' \Gamma_{\text{flat}}(x - x', y - y')(h_{xx} h_{yy} - h_{xy}^2). \] (A2)

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keeping in mind the boundary condition that the conformal map must approach the identity at infinity; i.e., $\eta \to 0$, giving the result
\[ \eta(x, y) \approx -\int dx' dy' \Gamma_{\text{flat}}(x - x', y - y')(h_{xx} h_{yy} - h_{xy}^2). \] (A2)
can be checked explicitly in the small aspect ratio approximation by combining the expressions for the geometric interaction and the correction to the Green’s function Eq. (A2) and Eq. (A3). (One can take the $\Delta x, \Delta y \to 0$ limit of $\frac{\Delta (\Delta x + \Delta y)}{\Delta x^2 + \Delta y^2}$ with the help of Eqs. (A3) and (A4).) As expected, the dependence on the surface profile $h$ cancels. This consistency check was behind our original suspicion about the existence of a geometric interaction. The energy of two vortices at $u_1, u_2$, without the geometric interaction included is $-4\pi^2 K \Gamma(u_1, u_2)$. This energy differs from the flat space energy even when the vortices are very close to one another by a position dependent contribution:

$$E_{int}(u_1, u_2) \approx 2\pi K \ln s_{12} - 4\pi^2 K g \left( \frac{1}{2} (u_2 + u_1) \right). \quad (A9)$$

This cannot be the correct expression, since as just argued, the energy should be the same as in flat space. Single particle energies give a simple resolution. If the total energy were

$$E = -4\pi^2 K \Gamma(u_1, u_2) + 2\pi^2 K g(u_1) + 2\pi^2 K g(u_2), \quad (A10)$$

then all the $g$‘s will cancel when $u_1 \to u_2$. The Green’s function calculations in [Vitelli and Nelson, 2004] and the conformal mapping calculations in Section V.A show that this is actually the correct resolution, and that $g(u) = -\frac{U_G(u)}{2\pi}$. 

**APPENDIX B: The Saddle Surface’s Potential**

For the saddle surface with a small aspect ratio (see Eq. (30)), we may determine the entire geometric potential analytically as a function of position. We will only outline the procedure here. We would like to evaluate

$$U_{\rho}(r) = -\int \frac{1}{2\pi} \ln |r - r'| \rho(r') dx' dy' \quad (B1)$$

when $\rho(x, y) = G(x, y)$ is the curvature of the surface (at the point vertically above $(x, y)$; we are using the small-aspect ratio approximation of Appendix A). Some thought shows that the curvature given by Eq. (A1) for the surface Eq. (30) takes the form of a polynomial times $G_0 = e^{-\frac{x^2 + y^2}{\sigma}}$. We will therefore discuss how to evaluate the potential $U_{\rho}$ for “charge” distributions of the form

$$\rho(x, y) = P(x, y) e^{-\frac{x^2 + y^2}{\sigma^2}}. \quad (B2)$$

We start with $\rho = G_0$; as discussed above, the azimuthal symmetry of this distribution symmetry allows its potential to be determined by Gauss’s Law:

$$-\nabla U_{G_0} = \frac{1}{2\pi r^2} \int_0^r 2\pi r' G_0(r') dr' \quad (B3)$$

This integral is elementary and $U_{G_0}$ can be evaluated by one further integration, although this cannot be done in closed form.

Conveniently, the potential due to a distribution of the form (B2) can be determined from the special case of $\rho = G_0$ by differentiation. (Intuitively, derivative charge distributions such as $\partial_\alpha G_0$ are superpositions of infinitesimally shifted copies of $G_0$. We can therefore apply superposition to find potentials for such distributions. This is analogous to finding the electric fields of multipoles by differentiating the monopole field.) To this end, we rewrite Eq. (B1) for the special case $\rho = G_0$ as

$$U_{G_0}(r) = -\int \frac{1}{2\pi} \ln |\Delta G_0(r - \Delta) d\Delta x d\Delta y. \quad (B4)$$

where $(\Delta x, \Delta y)$ are the components of $\Delta = r - r'$. It follows that

$$\partial_x^\alpha \partial_y^\beta U_{G_0}(r) = -\int \frac{1}{2\pi} \ln |\Delta x^\alpha \Delta y^\beta G_0(r - \Delta) d\Delta x d\Delta y. \quad (B5)$$

The right hand side represents the potential corresponding to the source in Eq. (B2) with a special degree $k$ polynomial in place of $P$. This polynomial, obtained by multiple differentiations of a Gaussian, is very complicated, but we will show that polynomials of this specific form can be superimposed to give any desired polynomial (including the degree 8 polynomial appropriate to our Gaussian saddle-surface). We will then, in principle, be able to express $U_G$ as a superposition of $U_{G_0}$ and its derivatives.

The expansion of the charge distribution of Eq. (B2) in terms of the derivatives of $G_0$ can be carried out with the help of Fourier integrals. Our goal is to find an expression of the form

$$\rho(x, y) = P(x, y) e^{-x^2 - y^2} = Q(\partial_x, \partial_y) e^{-x^2 - y^2} \quad (B6)$$

where $P(x, y)$ is the polynomial appearing in Eq. (B2). We have to determine a polynomial operator $Q(\partial_x, \partial_y) = \sum n,m q_{nm} \partial_x^n \partial_y^m$ so that Eq. (B6) is true. We have set $\rho_0 = 1$ for convenience. Applying the Fourier transform to both sides of Eq. (B6) gives

$$P(\mathbf{i} \partial_x, \mathbf{i} \partial_y) e^{-\frac{x^2 + y^2}{\sigma^2}} = Q(\mathbf{i} \partial_x, \mathbf{i} \partial_y) e^{-\frac{x^2 + y^2}{\sigma^2}}. \quad (B7)$$

or (by substituting $u = \mathbf{i} x, v = \mathbf{i} y$),

$$Q(u, v) = e^{-\frac{x^2 + y^2}{\sigma^2}} P(-\partial_u, -\partial_v) e^{\frac{x^2 + y^2}{\sigma^2}} \quad (B8)$$

The operator $Q(\partial_x, \partial_y)$ which satisfies Eq. (B6) can be produced by working out the derivatives in this expression and replacing $u$ and $v$ by $\partial_x$ and $\partial_y$. Now the potential can be worked out using

$$U_G(r) = Q(\partial_x, \partial_y) U_{G_0}(r). \quad (B9)$$

In fact, multiplying Eq. (B5) by $q_{nm}$ and summing over $n$ and $m$ shows (with the help of Eq. (B6)) that

$$Q(\partial_x, \partial_y) U_{G_0}(r) = -\frac{1}{2\pi} \int |\Delta \rho(r - \Delta)| d\Delta x d\Delta y.$$

Since all derivatives of $U_{G_0}$ can be calculated analytically starting from Eq. (B3), Eq. (B9) will yield an
analytic expression for \( U_G \), provided we can show that \( Q \) has no constant term. To show this, we integrate both sides of (B6) to see that

\[
\pi Q(0,0) = \int P(x, y)e^{-(x^2+y^2)}dxdy = \int G(x, y)dxdy.
\]

That is, \( Q \)'s constant term is proportional to the net Gaussian curvature; since the net curvature is zero for any surface which flattens out at infinity, \( Q \) has no constant term.

The potential of the saddle surface can thus be determined in closed form by the following procedure: expand the curvature to determine the polynomial \( P \). Calculate \( Q \) from (B5). Since (B10) guarantees that \( Q \) has no constant term, we may calculate the geometrical potential by differentiating (B3) repeatedly. This method is not much more practical for human calculations than is numerically integrating (B1) by hand. A computer program, like Mathematica (which produced 272 terms), can use Eqs. (B9) and (B8) to calculate the values rapidly and make the graphs shown in Figs. 11 and 8. There is one comprehensible consequence of these calculations: the long distance potential is dominated by a quadrupole, attracting vortices from some directions and repelling them towards others. Hence there are four additional local minima outside of the central trap.

#### APPENDIX C: Van der Waals Attraction on a Curved Surface

Because the van der Waals force is very short-ranged, falling off like \( \frac{1}{r^6} \), one can approximate the integral expression for the disjoining pressure \( \Pi(x) \) in Eq. (68) by corrections depending only on the local curvature of the substrate. The integral is the total van der Waals interaction energy between a point \( P \) at \( x \) which is above the helium film and all the atoms in the substrate:

\[
\Pi(x) = \int w(|x-x'|)d^3x'
\]

We now choose a simpler coordinate system (see Fig. 32) by rotating space so that the tangent plane to the substrate at the point of the substrate closest to \( P \) becomes horizontal. Let us take the point of tangency to be the origin of our new coordinates, \((\xi_1,\xi_2,\xi_3)\). In this coordinate system, \( P \) is the point \((\xi_1,\xi_2,\xi_3) = (0,0,D)\) (where \( D \) is the thickness of the film at \( P \)). The rotated substrate can be described by its height above the new "horizontal plane" (the arbitrary plane parallel to the tangent plane) using the equation \( \xi_3 = h_{rot}(\xi_1,\xi_2) \). The disjoining pressure is

\[
\Pi(P) = \int d\xi_1 d\xi_2 \int_{-\infty}^{h_{rot}(\xi_1,\xi_2)} d\xi_3 w(\sqrt{\xi_1^2 + \xi_2^2 + (D-\xi_3)^2}) = \frac{\pi \alpha}{6D^3} \int d\xi_1 d\xi_2 \int_{h_{rot}(0,0)}^{h_{rot}(\xi_1,\xi_2)} d\xi_3 w(\sqrt{\xi_1^2 + \xi_2^2 + (D-\xi_3)^2})
\]

where we have first integrated over the entire region below the plane \( \xi_3 = 0 \), thereby getting the van der Waals interaction between a point and a flat substrate as the first term. We then subtract the surplus energy that has been included by integrating over the shaded region (see Fig. 32).

Since the force is short-ranged, we use the quadratic approximation to \( h_{rot} \), \( h_{rot}(\xi_1,\xi_2) = h_{rot}(0,0) - \frac{\xi_1^2}{2} \kappa_1 - \frac{\xi_2^2}{2} \kappa_2 \) where we have assumed the axes to be aligned with the principle curvatures. Finally since \( \kappa_1, \kappa_2 D \ll 1 \) this wedge-shaped region is extremely thin close to the origin and the remainder term can therefore be approximated by ignoring the dependence of \( w \) on \( \xi_3 \):

\[
\Delta \Pi = -\int d\xi_1 d\xi_2 (\frac{\xi_1^2}{2} \kappa_1 + \frac{\xi_2^2}{2} \kappa_2)w(\sqrt{\xi_1^2 + \xi_2^2 + D^2}).
\]
This integral can be evaluated in polar coordinates:
\[
\Delta \Pi = \alpha \int \int \frac{r dr d\phi}{(r^2 + D^2)^3} \left( \frac{\kappa_1}{2} r^2 \cos^2 \phi + \frac{\kappa_2}{2} r^2 \sin^2 \phi \right)
\]
\[
= \frac{\kappa_1 + \kappa_2}{2\pi} \int_0^\infty \frac{r^3 dr}{(r^2 + D^2)^3}
\]
\[
= \frac{\alpha H \pi}{4D^2}
\]  
(C4)
since the mean curvature \( H \) is given by \( \frac{1}{2}(\kappa_1 + \kappa_2) \). Upon combining this expression with the flat substrate result, we obtain Eq. (72).

APPENDIX D: Consumer’s Guide to Green’s Functions on Compact Surfaces

The ambiguity in the one-vortex energy (Eq. (15)) on the sphere also implies that there is no particularly natural choice of a Green’s function on the sphere. With so many choices out there, you’ll be greatful for this friendly guide to help you focus on the important features and possible pitfalls of these different functions.

The first point you need to know is that all of them work pretty much just as well, provided they are used consistently; one should not use the single-vortex energy Eq. (11) designed to work with a different Green’s function from the one used to calculate the pair interaction Eq. (9). The general definition of of a Green’s Function, broadened from Eq. (13), is that it is a symmetric function of two points on the deformed sphere satisfying the equation
\[
\nabla^2 \Gamma(x, y) = -\delta(x, y) + F(x).
\]  
(D1)
The only restriction on the function \( F \) is that its integral over the deformed sphere must equal 1. (Integrating the Laplacian on the left shows that there is no solution unless the right-hand side integrates to zero.) The Green’s function on a sphere, \( \frac{1}{4\pi} \ln \frac{D(x, y)}{\alpha} \), has \( \frac{1}{\pi} - \delta(x, y) \) as its Laplacian [Lubensky and Prost, 1992]. Eq. (D1) is a more versatile vision of what a Green’s function should be, using a function \( F \) in place of the constant.

That Eqs. (11) and (9) give the correct net energy follows from the result proven in Sec. \( \nabla \cdot A \) by conformal mapping to the unit sphere:
\[
E(\{n_i, u_i\}) = \sum_{i<j} 4\pi^2 K n_i n_j \Gamma_{\text{sphere}}(U_i, U_j) - \sum_i \pi K n_i^2 \omega(u_i)
\]  
(D2)
The interaction potential in this equation
\[
\Gamma_c(x, y) = -\frac{1}{2\pi} \ln |x - y|
\]  
(D3)
satisfies
\[
\nabla^2 \Gamma_c(x, y) = e^{2\omega(u)} \nabla^2 \Gamma_{\text{sphere}}(U, U')
\]  
(D4)
in the first step, the scale factor is introduced to compensate for the change from the reference to the target surface. In the second step, the Laplacian of the sphere’s Green’s function \( -\frac{1}{2\pi} \ln \frac{D(x, y)}{\alpha} \) is substituted. In the third step, the \( \delta \)-function is transformed back to the target surface. The last line shows that \( \Gamma_c \) is a Green’s function as set out by Eq. (D1), which we call the “conformal Green’s function.” The \( F \)-function that goes with this Green’s function gets its spatial dependence from the conformal factor.

The single particle potential \( \omega \) in Eq. (D2) satisfies
\[
\nabla^2 \omega = G_T(u) - e^{2\omega(u)}
\]  
(D5)
which follows from Eq. (105) with the curvature of the unit sphere, \( G_R = 1 \), substituted.

Now any Green’s function \( \Gamma \) can be used to solve Poisson’s equation for any net-neutral function \( \rho \) on the target surface,
\[
\nabla^2 \int d^2 u \Gamma(u, u') \rho(u') = -\rho(u).
\]  
(D6)
This follows from Eq. (D1). It can be used to derive Eqs. (90) and (91) from their special case, the energy derived by conformal mapping. We first use the Poisson-like integral to “solve” two special cases of Poisson’s equation, Eqs. (D4) and Eq. (D5) in terms of the arbitrary Green’s function \( \Gamma \). Regarding \( u' \) as a constant in the former equation, we find that
\[
\Gamma_c(u, u') = \Gamma(u, u') - \int \int d^2 u'' \Gamma(u, u'') e^{2\omega(u'')} \frac{4\pi}{e^{2\omega(u')}} + f(u')
\]  
(D7)
where \( f(u') \) is the constant left undetermined by the Poisson equation. Since both \( \Gamma \) and \( \Gamma_c \) are symmetric in \( u, u' \), \( f(u') = -\int \int d^2 u'' \Gamma(u', u'') e^{2\omega(u'')} + C_1 \) where \( C_1 \) is a constant. Again applying Eq. (D6), this time to Eq. (D5), implies that
\[
\omega(u) = -\int \int d^2 u'' \Gamma(u, u'') G_T(u'') - 4\pi f(u) + C_2.
\]  
(D8)
(This is not really a solution of the nonlinear Eq. (D5) since \( \omega \) still appears on both sides of the equation.) Rewriting the previous equations implies that
\[
\Gamma(u, u') = \Gamma_c(u, u') - f(u) - f(u') - C_1
\]
\[U_G(u) = \omega(u) + 4\pi f(u) - C_2,
\]

namely that \( U_G \) and \( \Gamma \) are related to \( \omega \) and \( \Gamma_c \) according to the energy-shuffling transformation Eq. (155) so that the more general expressions of Eqs. (91) and (90) can be used in place of Eq. (12) to determine the energy. The sum of the energies from Eqs. (91), (90) is equal to the correct energy, Eq. (12) up to a constant. The arbitrary Green’s function can also be used to find the flow pattern according to the formula
\[
\chi(u) = \sum_{i=1}^{N} \frac{h n_i}{m} \Gamma(u, u_i).
\]  
(D9)
There are some advantages and disadvantages of different choices for $F$ in Eq. (D11). Let us focus on the most popular choices. The “standard Green’s function” is defined with $F = \frac{1}{A}$ ($A$ is the area of the surface) and is simply related to the eigenfunctions of the Laplacian, $\nabla^2\Psi_\lambda = -\lambda\Psi_\lambda$:

$$\Gamma_s(x, y) = \sum_{\lambda \neq 0} \frac{1}{\lambda} \Psi_\lambda(x)^* \Psi_\lambda(y).$$  \hfill (D10)

The “pair Green’s function” is defined via conformal mapping,

$$\Gamma_p(x, y) = \Gamma_{\text{sphere}}(x', y') + \frac{1}{4\pi}(\omega(x) + \omega(y))$$  \hfill (D11)

and incorporates all the single-particle energy into the interaction energy, so that $U_{\text{pair}} = 0$. This Green’s function satisfies the most elegant differential equation,

$$\nabla^2\Gamma_p(x, y) = -\delta(x, y) + \frac{G(x)}{4\pi}$$  \hfill (D12)

Last, the “conformal Green’s function” (which was our starting point) has $F = \frac{2\pi}{s}$ (see Eq. (D2)).

If you are looking for style in your Green’s functions, I would choose the pair Green’s function. It is easy to calculate by conformal mapping (Eq. (D11)) but it can be defined without referring to $\omega$, Eq. (D12), much preferable to the haphazard looking Eq. (D4) defining the conformal Green’s function. The standard Green’s function is stodgy and does not handle well. The methods for finding the standard Green’s function, Eq. (D10) are more limited and, if one wants to use it, the best option might be to derive it by using conformal mapping anyway:

$$\Gamma_s(x, y) = \Gamma_c(x, y) - \frac{1}{A_{\text{tot}}} \iint \Gamma_c(x, u) + \Gamma_c(y, u) du + C_3.$$  \hfill (D13)

(This equation is derived analogously to Eq. (D7).) On the other hand, there are always advantages to familiarity. In particular, in the limit where part of the deformed sphere is stretched out to infinity so that it actually becomes a deformed plane, $\Gamma_s$ converges to the ordinary Green’s function of a noncompact surface, since $\frac{1}{A}$ tends to zero. For a short summary of all the Green’s functions features and failings, see Table II.

| Calculability | Neutralizer | Limit |
|---------------|-------------|-------|
| Conformal     | +           | -     |
| Pair          | +           | +     |
| Standard      | -           | +     |

**TABLE II** The advantages and disadvantages of the Green’s functions, as far as their ease of calculation, simplicity of the neutralizing function $F$, and limiting behavior in case the deformed sphere is stretched into a deformed plane.

APPENDIX E: Approximations for Long Surfaces of Revolution

Let us start by determining the conformal map from the surface of revolution defined by the equation $r = r(z)$, $z_i \leq z \leq z_n$, to the unit sphere. We use the coordinates $\phi, \sigma$ introduced in Sec. V.D to parameterize the surface; $\sigma$ is given by:

$$\sigma = \int_{z_i}^{z_n} \sqrt{1 + \left(\frac{dr}{dz}\right)^2} dz.$$  \hfill (E1)

The Cartesian coordinates are $x = r(\sigma)\cos \phi, y = r(\sigma)\sin \phi, z = z(\sigma)$, and hence the metric is

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r(\sigma)^2 d\phi^2.$$  \hfill (E2)

If the map $(\sigma, \phi) \to (\Theta, \Phi)$ to the unit sphere is to be conformal, then according to Eq. (86),

$$dr^2 + r(\sigma)^2 d\phi^2 = e^{-2\omega}(d\Theta^2 + \sin^2 \Theta d\Phi^2).$$  \hfill (E3)

By symmetry, $\Phi = \phi$ and $\Theta = \Theta(\sigma)$ is independent of $\phi$ (see (Vitelli and Nelson, 2004) for the analogous use of symmetry on a rotationally symmetric bump on a plane). By matching the coefficients of $d\phi$ and $d\sigma$ one finds that $\frac{dr}{r(\sigma)} = \frac{d\Theta}{\sin \Theta}$, or (after integration):

$$\sin \Theta = \text{sech} (\int^{\sigma_0}_{\sigma} \frac{d\sigma'}{r(\sigma')}),$$  \hfill (E4)

where $\sigma_0$ can be an arbitrary arc length. According to Eq. (E5), $\omega = \ln \frac{d\Theta}{d\sigma}$, or

$$\omega = \ln \frac{1}{r(\sigma)} \text{sech} \left(\int^{\sigma_{eq}}_{\sigma} \frac{d\sigma'}{r(\sigma')}\right).$$  \hfill (E5)

To determine the energies and flow patterns on a rotationally symmetric surface, we use the “Pair Green’s function” Eq. (D11), the Green’s function which incorporates all of the energy into interaction-energy terms. This Green’s function can be found using Eqs. (E5) and (E4); adding $\omega$ at the sites of the vortices to the Green’s function on the sphere,

$$-\frac{1}{2\pi} \ln \sqrt{2[1 - \cos \Theta_1 \cos \Theta_2 - \sin \Theta_1 \sin \Theta_2 \cos(\Phi_1 - \Phi_2)]},$$

and rearranging, gives

$$\Gamma_{\text{pair}} = -\frac{1}{2\pi} \ln \sqrt{2r(\sigma_1)r(\sigma_2)\cosh \int^{\sigma_2}_{\sigma_1} \frac{d\sigma'}{r} - \cos(\phi_1 - \phi_2)}.$$  \hfill (E6)

The energy of a set of vortices is simple using the pair Green’s function (see the previous appendix), $E = \sum_{i<j} 4\pi^2 n_i n_j K T_{\text{pair}}(\mathbf{u}_i, \mathbf{u}_j)$. As an example the energy of a vortex-antivortex pair at opposite sides of a circle of latitude ($\phi_1 = \phi_2 + \pi$ and $\sigma_1 = \sigma_2 = \sigma$) is

$$E = 2\pi \ln \frac{2r(\sigma)}{a},$$  \hfill (E7)

showing that the energy grows logarithmically with the distance between the vortices in this case, as in Eq. (130).
To prove the azimuthal symmetry of the flows, note that according to Eq. (E9), the flow velocity at \( u \) is

\[
v = \nabla u \times \sum_i n_i h \Gamma_{pair}(u, u_i). \tag{E8}\]

Now if the vortices are all far from \( u \), then the integral in \( \text{(E8)} \) is very large. Since \( \ln(A + e) \approx A + \frac{e}{2} \) for large \( A \), the cosine term, the only one which depends on the azimuthal angles, gives exponentially small contributions. Therefore the flows can be calculated as in Sec. V.D by using the circulation quantization and the approximate azimuthal symmetry to determine the flow speeds. Alternatively, we may calculate the velocity directly from Eq. (E8) with the help of the further approximation that \( \cosh x \approx \frac{e^{2|x|}}{2} \), yielding

\[
\Gamma(u, u_i) \approx -\frac{1}{4\pi} \int_\sigma \frac{d\sigma'}{r} - \frac{1}{4\pi} \ln \frac{r(\sigma)r(\sigma_i)}{a^2}. \tag{E9}\]

The first term gives the flow pattern of Eq. (E11), after a brief calculation using the neutrality constraint, while the second term, when summed as in Eq. (E10), cancels out also by neutrality.

For a surface (such as an ellipsoid) where the \( xy \)-plane is a plane of symmetry, our results will simplify if we make the choice \( \sigma_0 = \sigma_{eq} \) in Eq. (E11) where \( \sigma_{eq} \) is the arclength corresponding to the equator, at \( z = 0 \). In this case, the conformal map takes pairs of antipodal points on the deformed surface to antipodal points on the sphere. (Since antipodal points \( (\sigma_1, \phi_1), (\sigma_2, \phi_2) \) are points at opposite ends of a diameter of the surface, \( \sigma_2 = 2\sigma_{eq} - \sigma_1, \phi_2 = \pi + \phi_1 \).) If we consider the interaction energy of a pair of antipodal points, we find according to Eqs. (E9) and (E10),

\[
\frac{E_{antipodal}}{K} = 2\pi \ln \frac{2}{a} - 2\pi \omega(z_1) \tag{E10}\]

Whether the two vortices are at opposite tips or at opposite ends of the equator, their image vortices are always at the same distance on the unit sphere, so the first term, the interaction energy of the images, is a constant. This gives another illustration of the folly of making a strict separation between intervortex and curvature-vortex interactions. One would like to think that the growth of the energy as the two vortices are separated on an elongated surface is due to the attraction between them. But Eq. (E10) shows that it can also be interpreted as resulting from the single particle potential \( \omega \).

Let us now turn to the problem of describing the equilibrium positions of a pair of vortices on a rotating ellipsoid. Both the transitions at \( \Omega_4 \) and \( \Omega_6 \) can be understood only with a more accurate version of the force than the band-force approximation, Eq. (E3). The error in the approximation is important when the vortices are near the poles of the ellipsoid or, as just illustrated, near each other. We will assume that the aspect ratio of the ellipsoid \( \alpha = \frac{a}{b} \) is very large. The equation for the ellipsoid can be expressed in terms of \( \alpha \) in the form:

\[
r = R\sqrt{1 - \frac{s^2}{\alpha^2 R^2}}. \tag{E11}\]

The energy of a vortex-antivortex pair according to Eq. (E10) is

\[
E = E_{rest} + E_\Omega. \tag{E12}\]

Here the energy of the flow pattern, or “resting energy,” is the energy of the vortices on a stationary ellipsoid, \( E_{rest} = -4\pi^2 K \Gamma_{pair}(\sigma_1, \phi_1; \sigma_2, \phi_2) \). The “rotation energy” \( E_\Omega \) is given by Eq. (E11). Both energies are functions of a single variable, the distance \( s \) between the two vortices along the surface, if we assume that the vortices are at \( (\sigma_1, \phi_1) = (\sigma_{eq} - \frac{\pi}{2}, 0) \) and \( (\sigma_2, \phi_2) = (\sigma_{eq} + \frac{\pi}{2}, 0) \). These relationships assume that the vortices are situated symmetrically about the \( xy \)-plane; the vortices will have equal azimuthal angles in order to minimize the energy of the flow pattern.

The equilibrium position of a pair of vortices is determined by balancing the rotational force and resting force acting on one of them. The resting force \( F_{rest} \) on vortex 1 is derived from the kinetic energy of the flow pattern and is positive since it pulls the vortices toward each other in order to decrease the width of the band of moving fluid between them. The rotational force \( F_\Omega \) is negative since it pulls the vortices toward the poles of the ellipsoid (and away from each other) in order to increase the total angular momentum of the flow. At equilibrium the rotational and resting forces on the vortices balance, as we can see by differentiating Eq. (E12) to obtain

\[
0 = \frac{dE_{rest}}{ds} + \frac{dE_\Omega}{ds}, \tag{E13}\]

or equivalently

\[
F_{rest}(s) = -F_\Omega(s), \tag{E13}\]

where \( F_{rest} \) and \( F_\Omega \) are the resting and rotational forces on the vortices. (A short calculation shows that the force on one of the vortices \( \frac{dE_{rest}}{ds} \) is equal to \( \frac{dE_\Omega}{ds} \), since the energy change produced by moving one vortex an infinitesimal distance is the same as the energy change produced by moving both vortices half the distance.) The equilibrium positions can be found by graphing \( F_{rest} \) and \( -F_\Omega \) as in Fig. (E3) and finding the intersection points. The exact expression for the resting force can be found by differentiating Eq. (E10) to obtain

\[
F_{rest} = \frac{\pi K}{r(\sigma_1)} \left( \coth[\int_{\sigma_1}^{\sigma_{eq}} \frac{d\sigma'}{r(\sigma')} - \frac{1}{\sqrt{1 + \left(\frac{dr}{d\sigma}\right)^2}} \right). \tag{E14}\]

The rotational force is given exactly by Eq. (E11).

Fig. (E3) shows the resting force and minus the rotational force on one of the vortices for \( \Omega = \Omega_6, \Omega = \Omega_4 \) and for an intermediate value of the frequency.

Stability of the equilibria illustrated in Fig. (E3) can be determined by considering the direction in which the resting force curve crosses the rotational force curve. The
middle point of the three equilibrium points at the intermediate frequency is a stable equilibrium because the resting force curve crosses the rotational force curve from bottom to top. This implies that if the vortices fluctuate away from each other (increasing \( s \)), then the resting force becomes stronger than the rotational force and pulls them back together.

Let us consider how the stable equilibrium disappears at \( \Omega_b \). As \( \Omega \) is lowered the stable and unstable equilibrium come together and then “annihilate” when the rotation-force curve detaches from the resting force curve, as illustrated by the lowest curve in Fig. [33] which corresponds to \( \Omega = \Omega_b \). Since the rotation force and resting force curves are tangent at \( \Omega_b \), the frequency \( \Omega_s \) and separation of the vortices \( s_b \) at this transition point can be determined by solving Eq. (E13) simultaneously with

\[
F'_{\text{rest}}(s_b) = -F'_{\Omega}(s_b). \tag{E15}
\]

When \( \alpha \gg 1 \), we will be able to avoid solving simultaneous equations since the value of \( \Omega_b \) is already determined by Eq. (148). Using this result, we will be able to solve Eq. (E15) for \( s_b \).

The simple band approximation to the force, Eq. (E13), suggests that the vortices move continuously toward one another as \( \Omega \) is decreased, annihilating at the equator. Substituting the expression for the critical frequency that is implied by the band model, Eq. (148), into Eq. (E17) in fact implies that \( s_b = 0 \), which is incorrect. The band approximation fails because it implies that the force between the vortices decreases monotonically as the vortices approach one another. Though in conflict with our intuition from the plane, this result is correct over the large middle range of the resting force curve in Fig. [33]. As the rotational confinement weakens, the vortices get closer together, and the resting force weakens too, preserving the equilibrium. However the resting force starts increasing strongly as the vortices approach one another, because the vortices start to feel one another’s asymmetric flow fields. This force will certainly overcome the rotational force when the rotational confinement decreases further. (Actually, Eq. (E13) implies that \( s_b \) does not correspond exactly to the maximum of \( F_{\text{rest}} \) because the rotational confinement is not a constant force field.)

We can derive the corrections to the force from Eq. (E14); if \( \alpha \) is large, we may neglect the second term and assume that

\[
\int_{\sigma_1}^{\sigma_0} \frac{d\sigma'}{r(\sigma')} \approx \frac{\sigma_{eq} - \sigma_1}{R}, \tag{E16}
\]

since the radial profile of the ellipsoid, Eq. (E11), is slowly varying. We then obtain the approximation that is valid when the vortices are close (compared to \( \delta \), the characteristic distance for variation of the radius).

\[
F_{\text{rest}} \approx \frac{\pi K}{r(\sigma_1)} \coth \left( \frac{s}{2R} \right) \tag{E17}
\]

Notice that the force diverges as \( \frac{\pi K}{r} \) when the vortices are close together (as in the plane) and approaches Eq. (139) exponentially fast as the vortices move apart; this generalizes the band model approximation to the case where the two vortices may be close together. As we will see, for a large value of \( \alpha \), \( s_b \gg R \) at the moment when the vortices annihilate. We therefore simplify Eq. (E17) by making another approximation, \( \coth x \approx 1 + 2e^{-2x} \). Then an approximate version of Eq. (E17) that is derived from Eqs. (E17), (145) reads

\[
\frac{\pi K}{2} \frac{1}{R^2} \frac{d}{d\sigma_1} \bigg|_{s_b} - 2\pi K e^{-\frac{z}{R}} = -\frac{\pi \Omega_h \rho_s}{m} \frac{dr}{d\sigma_1} \bigg|_{s_b}, \tag{E18}
\]

The first term describes the decrease of the resting force due to the variation in \( r(z) \). The second term results from the exponentially decaying portions of the flow fields. (We are replacing \( r(\sigma_1) \) by \( R \) whenever that is accurate enough since the width of the ellipsoid is slowly varying. Of course, the slow variation of \( r(\sigma_1) \) is important in some terms; the resting force initially decreases as \( s \) decreases because the band approximation to the force decreases with increasing circumference.) Using Eq. (148) for \( \Omega_b \) in Eq. (E18) gives

\[
\frac{dr}{d\sigma_1} \bigg|_{s_b} = 2e^{-\frac{z}{R}} \tag{E19}
\]

In order to evaluate the left-hand side, we note that \( \sigma_1 = \sigma_{eq} - \frac{s}{2} \approx \sigma_{eq} - z \), aside from terms of order \( \frac{1}{s^2} \) since the sides of the ellipsoid are nearly vertical near the equator. Therefore Eq. (E11) implies that \( \frac{dr}{d\sigma_1} = \frac{2e^{-z/R}}{s^2 \pi R} \).
Rearranging Eq. \((E19)\) now gives
\[
s_b = R \ln \frac{4\alpha^2 R}{s_b}. \tag{E20}
\]
which can be solved by substituting it into itself. The first iteration gives
\[
s_b = R \ln 4\alpha^2 - R \ln \frac{4\alpha^2 R}{s_b}. \tag{E21}
\]
Since the second term has two logarithms in it, it is smaller than the first in the limit where \(\alpha \to \infty\), so finally
\[
z_b \approx R \ln \alpha, \tag{E22}
\]
(since \(z_b\), the distance from a vortex to the equatorial plane, is approximately half the distance between the vortices). We have justified Eq. \((149)\). Two iterations of Eq. \((E20)\) give \(z_b = R \ln \alpha - \frac{1}{2} \ln \frac{16\pi}{3} \); the error for this approximation actually approaches 0 for large \(\alpha\).

The exact result can be found by computer, but the approximate result is reasonable even at \(\alpha = 5\), where \(\frac{z_b}{R} = 1.8 \approx \ln 5 = 1.6\).

The height \(z_b\) depends only logarithmically on \(\alpha\) because the extra short-distance vortex-vortex interaction decays exponentially and would not be strong enough to pull the vortices together if \(z_b\) were very large. (Check this by substituting our final result, Eq. \((E22)\), into Eq. \((E18)\). All the terms, the ones from the band model as well as the exponential correction, have the same basic dependence on \(\alpha\).) To see that the approximations we have made are valid, one has to calculate \(F'_{\text{rest}}(s)\) from the exact expression Eq. \((E14)\). The resulting expression can be simplified by dropping various terms, which mostly have a relative size of \(\frac{1}{\alpha}\) and \(\left(\frac{\ln \alpha}{\alpha}\right)^2\); the reason is that \(\frac{r}{\alpha} = \frac{\ln \alpha}{\alpha}\) so the vortices are proportionally very close to the equator, and again \(r(z)\) can be replaced by \(R\). (This also justifies the approximation in Eq. \((E16)\), where the integrand is replaced by a constant.) One particularly large term, resulting from the second term of Eq. \((E14)\), has been neglected in Eq. \((E18)\), but the neglected term, \(\frac{K \pi r''(\sigma_1)}{2 \sigma_1}\), is still of relative order \(\frac{1}{\ln \alpha}\).

Now we turn to the critical frequency \(\Omega_a\) where the vortices move to the poles. The band model also requires a correction in order for this transition to be described correctly. In fact Eq. \((147)\) would imply that the vortices never exactly reach the tips of the ellipsoid even as \(\Omega \to \infty\), and thus \(\Omega_a = \infty\). In fact, the band force on the left hand side of Eq. \((146)\), which approaches infinity at the poles, cannot be balanced by the rotational force at a finite frequency. Of course, the exact force approaches zero rather than infinity at the pole (see Fig. \(33\)). The value of \(\Omega_a\) may be derived from the condition that the actual resting force curve and the rotation force curve have to be tangent at the origin, as for the uppermost \((\Omega = \Omega_a)\) curve in Fig. \(33\). We therefore have to find when \(s = 2\sigma_{eq}\) satisfies Eq. \((E15)\). Linearizing Eq. \((E14)\) near \(\sigma_1 = 0\) to find the derivative of the force implies that \(\Omega_a\) is given by
\[
\frac{m\Omega_a}{\hbar} = \frac{a^2 \kappa^2}{4} + \frac{1}{R^2} e^{-2} \int_0^\infty \left( \sigma_1^2 - 1 \right) d\sigma_1 \tag{E23}
\]
where \(\kappa\) is the curvature at the tip of the ellipsoid. The critical frequency is larger than the result \(\frac{a^2 \kappa^2}{4m\Omega}\), derived in Section III.B, for a bump with the same curvature because the rotational confinement must overcome the mutual attraction of the vortices as well as the repulsion of the vortices from the curvature. For an ellipsoid with a large value of \(\alpha\), the correction term is unimportant, so
\[
\Omega_a \approx \frac{a^2 \hbar}{4mR^2}. \tag{E24}
\]
The transition can be visualized using the energy curves illustrated in Fig. \(34\) where the local minimum of the energy function moves away from the axis as \(\Omega\) is decreased through \(\Omega_a\).

There is an aspect ratio \(\alpha_c\) below which there are no off-center local minima, for any rotation speed. That is, when the angular velocity is decreased enough, a vortex-antivortex pair initially at the poles immediately moves to the equator and annihilates. This situation is illustrated for a sphere in Fig. \(34\). The value of \(\alpha_c\) can be determined numerically, and is 1.33. One simply graphs the total energy at \(\Omega = \Omega_a\) (as given by the exact expression, Eq. \((E23)\)) and checks whether there is an energy barrier or not. At \(\Omega > \Omega_a\), a pair of vortices at the poles will be stable. If there is no barrier, in Fig. \(34\) slightly decreasing \(\Omega\) will cause these vortices to leave the poles and annihilate each other. If there is a barrier, as in Fig. \(34\) slightly decreasing \(\Omega\) will create an off-center local minimum. This can be seen from the energy
APPENDIX F: Derivations of Bounds Valid Even for Strong Distortions

The results of Sec. [X] can be derived from theorems on “univalent” analytic functions. We will state these theorems here and derive the limits on the geometric force from them. (See [Rudin, 1987] for the proofs.) An abstract example of the type of question these theorems address is the following. Let \( f(t) \) be an analytic function defined by the following series:

\[
 f(t) = t + a_2 t^2 + a_3 t^3 + \cdots \quad (F1)
\]

Suppose this series converges out to radius 1, at least. If one of the coefficients, maybe \( a_6 \), is much larger than the rest, then the function is dominated by the \( t^6 \) behavior, and most points in the range of the function will occur six times as values of the function. Therefore, if one is looking for a univalent function (a function which is one-to-one inside the unit circle) then there will be upper limits on the sizes of the \( a_n \). A challenging mathematical problem is “What are the maximum sizes for the \( a_n \)’s?” The answer (proved by De Branges) is that \( |a_n| \leq n \), and that the function \( \frac{t}{1-t} \) attains the maximum value for every Taylor series coefficient simultaneously. To find the upper bound on the vortex force in a flat disk, we will use only the bound

\[
 a_2 \leq 2 \quad (F2)
\]

which has a simpler proof [Rudin, 1987]. Note that the conditions of this theorem do not require that the function remains one-to-one outside the unit circle. For example, the function \( t + 11t^2 \) satisfies the conditions of the theorem although it takes on the value zero at \( t = 0, -10 \). The analyticity of \( f(t) \) is allowed to break down as well beyond a radius of 1.

Similar problems can be stated for functions \( g(t) \) defined outside the unit circle, with expansions of the form

\[
 g(t) = t + \frac{a_1}{t} + \cdots \quad (F3)
\]

To make the predictions about the quadrupole force due to a bump in a plane we will use the Area Theorem [Rudin, 1987] which states that, if \( g \) is one-to-one and analytic outside the unit circle, and

\[
 g(t) = t + \frac{a_1}{t} + \cdots \quad (F4)
\]

then

\[
 a_1 \leq 1. \quad (F5)
\]

To prove Eq. (167), one just realizes that the assumption means that a part of the surface has the same geometry as a radius \( R \) disk in the plane with a vortex at the center. We can introduce a coordinate system on this portion of the surface by introducing Cartesian coordinates \( u, v \) (with \( w = u + iv \)) on the disk in the plane, and then mapping these coordinates isometrically to the surface. This mapping is different from the conformal mapping \( C \) used to calculate vortex energies. To relate them, let \( Z = X + iY \) where \( X, Y \) are the coordinates of the conformal image of the surface. Then Eq. (30) takes the form \( du^2 + dv^2 = e^{-2\omega}(dX^2 + dY^2) \) and it follows that \( Z(w) \) is a conformal map from a portion of the plane to itself, hence an analytic function of \( w \) on the circle of radius \( R \) (say \( Z = c_1w + c_2w^2 + \ldots \)). Furthermore, rewriting the expression for the scaling of lengths as \( |dw|^2 = e^{-2\omega}|dZ|^2 \), we see that

\[
 \omega = \ln |\frac{dZ}{dw}|. \quad (F6)
\]

We now define

\[
 f(t) = \frac{Z(Rt) - Z(0)}{Re_1}.
\]

Then \( f \) is a one-to-one analytic function on the unit circle (which is scaled by \( t \to Rt \) into the radius \( R \) circle). Since \( f(0) = 0, f'(0) = 1, f \) has the form of Eq. (F1), so Eq. (F2) implies

\[
 2 \geq \frac{R\sigma_2}{\sigma_1} \quad (F7)
\]

Now the force on the vortex is \( \pi K\nabla\omega(0) \) which can be expressed in terms of the coefficients of \( Z \)’s Taylor series.
by means of Eq. (F6): \( F = 2\pi K(3 \theta + \pi R^2) \). The upper bound, Eq. (F7), follows from Eq. (F7).

Let us now see whether the bound just proven can be improved at all: i.e., whether the ratio of the force on a singly quantized vortex to \( F \) can ever be as big as \( 4\pi \). For example, for vortices on cones, the ratio of the force to \( F \) is maximal in the limit where the cone angle \( \theta \to 0 \). To find this ratio, we must take \( R \to D \) because the first calamity that befalls the disk as it expands is that it starts overlapping the cone’s apex. But if \( \theta < \pi \), then the disk overlaps itself before this as one can see on the unfolded version of the cone illustrated in Fig. 36. Some simple trigonometry shows that \( R_{\text{max}} = D \sin \frac{\pi}{2} \). Eq. (170) shows that for small \( \theta \), \( \frac{F_{\text{max}}}{K} \to \pi^2 \), which is a little less than \( 4\pi \).

There is a surface that saturates the original bound, though; this surface is illustrated in Fig. 37. The surface is obtained by folding a disk in half and sealing it shut except for a very small opening at one end of the diameter. This opening is then connected to an infinite plane. The top part of the substrate is a semi-circular slab with the superfluid layer laminating both sides so that the helium spreads out to the plane. The topology of the helium film is still that of a plane. A vortex placed at the center of the disk, \( B \), saturates the bound; it is attracted by the negative curvature of the neck joining the plane and the disk and is repelled by the positive curvature at the top of the fold.

To show that this surface (the “calzone surface”) saturates the bound, we will find the force on the vortex using conformal mapping. Instead of mapping the entire surface to a reference plane, we can just map the folded disk portion of the surface. The flow patterns on the two portions of the surface are uncorrelated when the neck becomes infinitely small, aside from requirements imposed by the circulation’s invariance. The circulation around any curve on the plane enclosing the neck will be \( \pm K \) because the vortex in the disk region is inside it, and the flow pattern on the planar base will not be sensitive to the location of this vortex because the neck is so narrow.

It will consist of a set of concentric circles, representing a flow whose energy is independent of the position of the vortex. The force does not depend on this portion of the flow, so the two portions may be dismantled at the neck.

As illustrated in Fig. 37, the neck now turns into the core of a second vortex, at point \( A \) of the folded disk. The folded disk now has the topology of a sphere, satisfying the neutrality condition because the two vortices are equal and opposite. The map \( Z = \frac{R^2 w}{(R - w)} \) on the radius \( R \) disk can be used to relate the folded disk to a reference plane, since the points on the circle which fold together, \( w = Re^{i\pi/2} \), both map to the same point of the real axis in the plane. Since the vortex at \( B \) maps to infinity, the force on the vortex at \( A \) can be calculated from the geometric potential alone (without any interaction terms), giving \( \pi K Z''(0) = 4\pi K \). Also the flow pattern illustrated in the figure can be found by mapping the concentric circles centered around the origin in the \( Z \) plane to the disk using the function \( w(Z) \).

The result Eq. (172) about the long range force due to a bump contained inside of a radius \( R \) but with an arbitrary height and arbitrary curvatures follows (by an

FIG. 36 Construction of the largest circle centered at a point on a cone with \( \theta < \pi \). The cone is cut open and flattened so that the center is on the bisector of the angle.

FIG. 37 (a) The surface which contains an isometric disk of radius \( R \) and has the maximum geometrical force. A semi-circle (with films on both sides) is connected by a neck \( A \) to a plane. If a single vortex is placed at \( B \), the force on the vortex approaches \( 4\pi K \) as the edges of the surface becomes sharper. (b) An unfolded image of the flow pattern.
argument similar to the one used for the first inequality) from Eq. (F10). The conformal mapping takes the flat part of the surface (a plane with a radius $R$ hole parameterized by the complex variable $w$) in a one-to-one fashion to a reference plane with a hole of some distorted shape. As above, this function is analytic and $\omega = \ln|\frac{dZ}{dw}|$. By rescaling one can ensure that $Z \sim w$ at infinity. Applying the area theorem to $g(\ell) = \frac{1}{\pi}Z(R\ell)$ shows that

$$Z(w) = w + R^2\frac{a_1}{w} + \ldots$$

where $a_1 \leq 1$. Expand $E = \frac{\pi K}{2} R \ln \frac{dZ}{dw}$ for large $w$ to find the large distance form of the energy:

$$E \sim -\pi KR^2\frac{a_1}{w^2}.$$  \hspace{1cm} (F9)

it follows that $\mu_2 = \pi KR^2|a_1|$ and $\gamma_2 = \arg (a_1) + \pi$ in Eq. (171) and the bound on the quadrupole moment $\mu_2 \leq \pi KR^2$ follows from the bound on $a_1$.

Now we can also ask what type of bump maximizes the quadrupole moment. It turns out that the value $\pi KR^2$ cannot be attained by any surface which is flat outside a circle of radius $R$. There is a surface which consists of a bump surrounded by a surface isometric but not congruent to $K$, the plane with a circle of radius $R$ removed. This surface is gotten from $K$ by sealing opposite sides of the circle together to make a mountain ridge.

The reason this surface has the biggest quadrupole moment is because its conformal mapping to the plane is the function that maximizes $a_1$. According to the area theorem the only one-to-one analytic function on $K$ for which $a_1 = 1$ is

$$Z(w) = w + \frac{R^2}{w}.$$  \hspace{1cm} (F10)

This function maps both points $Re^{\pm i\phi}$ to the same point in the reference plane, so any flow pattern on the target plane will still be continuous when the two edges of $K$ are sealed.

This quadrupole-maximizing surface does not contain a flat copy of $K$. Hence an open question is to find the largest value of $\mu_2$ for a bump in a plane which is actually flat outside a radius of $R$, as well as the shape of the bump which has this maximum quadrupole moment.

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