On Liouville type theorem for a generalized stationary Navier-Stokes equations

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Abstract

In this paper we prove a Liouville type theorem for generalized stationary Navier-Stokes systems in \( \mathbb{R}^3 \), which model non-Newtonian fluids, where the Laplacian term \( \Delta u \) is replaced by the corresponding nonlinear operator \( A_p(u) = \nabla \cdot (|D(u)|^{p-2}D(u)) \) with \( D(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top) \), \( 3/2 < p < 3 \). In the case \( 3/2 < p \leq 9/5 \) we show that a suitable weak solution \( u \in W^{1,p}(\mathbb{R}^3) \) satisfying \( \liminf_{R \to \infty} |u_{B(R)}| = 0 \) is trivial, i.e. \( u \equiv 0 \). On the other hand, for \( 9/5 < p < 3 \) we impose the condition for the Liouville type theorem in terms of a potential function: if there exists a matrix valued potential function \( V \) such that \( \nabla \cdot V = u \), whose \( L^{\frac{3p}{p-3}} \) mean oscillation has the following growth condition at infinity,

\[
\int_{B(r)} |V - V_{B(r)}|^{\frac{3p}{p-3}} dx \leq C r^{\frac{9-4p}{p-3}} \quad \forall 1 < r < +\infty,
\]

then \( u \equiv 0 \). In the case of the Navier-Stokes equations, \( p = 2 \), this improves the previous results in the literature.

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1 Introduction

We consider the following generalized version of the stationary Navier-Stokes equations in $\mathbb{R}^3$

\begin{align}
(1.1) & \quad -A_p(u) + (u \cdot \nabla)u = -\nabla \pi \quad \text{in} \quad \mathbb{R}^3, \\
(1.2) & \quad \nabla \cdot u = 0,
\end{align}

where $u = (u_1, u_2, u_3) = u(x)$ is the velocity field, $\pi = \pi(x)$ is the scalar pressure and

$$A_p(u) = \nabla \cdot (|D(u)|^{p-2}D(u)), \quad 1 < p < +\infty$$

with $D(u) = D = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ representing the symmetric gradient. Here $|D|^{p-2}D = \sigma(D)$ stands for the deviatoric stress tensor. The system (1.1)-(1.2) is popular among engineers, known as a power law model of non-Newtonian fluid, where the viscosity depends on the shear rate $|D(u)|$. For $p = 2$ it reduces to the usual stationary Navier-Stokes equations. For $1 < p < 2$ the fluid is called shear thinning, while in case $2 < p < +\infty$ the fluid is called shear thickening. For more details on the continuum mechanical background of the above equations we refer to [16].

The Liouville type problem for the Navier-Stokes equations, as stated in Galdi’s book [5, Remark X. 9.4, pp. 729], is a challenging open problem in the mathematical fluid mechanics. We refer [12, 13, 14, 8, 2, 3, 14, 10, 4, 9, 7] and the references therein for partial progresses for the problem. In those literatures authors provided sufficient conditions for velocities to guarantee the triviality of solutions.

We say $V \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^{n \times n})$ is a potential function for vector field $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ if $\nabla \cdot V = u$, where the derivative is in the sense of distribution. In [12, 13] Seregin proved Liouville type theorem for the Navier-Stokes equations under hypothesis on the potential function $V$ for a solution $u$. In particular in [13] it is shown that if $V \in BMO(\mathbb{R}^3)$, then $u = 0$. In this paper we would like to improve and generalize this result for the system (1.1)-(1.2).

For a measurable set $\Omega \subset \mathbb{R}^n$ we denote by $|\Omega|$ the $n$-dimensional Lebesgue measure of $\Omega$, and for $f \in L^1(\Omega)$ we use the notation

$$f_\Omega := \int f dx := \frac{1}{|\Omega|} \int \Omega f dx.$$

In contrast to the case $p = 2$ it is still open whether any weak solution to the system (1.1)-(1.2) is regular or not. Therefore, in the present paper we only work with weak solutions satisfying the local energy inequality the solution of which are called suitable weak solution.

Definition 1.1. Let $\frac{3}{2} \leq p < +\infty$.

1. We say $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^3)$ is a weak solution to (1.1)-(1.2) if the following identity is fulfilled

\begin{equation}
\int_{\mathbb{R}^3} \left(|D(u)|^{p-2}D(u) - u \otimes u \right) : D(\varphi) dx = 0
\end{equation}
for all vector fields $\varphi \in C^\infty_c(\mathbb{R}^3)$ with $\nabla \cdot \varphi = 0$.

2. A pair $(u, \pi) \in W^{1,p}_{loc}(\mathbb{R}^3) \times L^{\frac{3}{p}}_{loc}(\mathbb{R}^3)$ is called a suitable weak solution to (1.1), (1.2) if besides (1.3) the following local energy inequality holds

$$\int_{\mathbb{R}^3} |D(u)|^p \phi dx$$

(1.4) $$\leq \int_{\mathbb{R}^3} |D(u)|^{p-2} D(u) : D(\varphi) + \int_{\mathbb{R}^3} \left( \frac{1}{2} |u|^2 + \pi \right) u \cdot \nabla \phi dx$$

for all non-negative $\phi \in C^\infty_c(\mathbb{R}^3)$.

**Remark 1.2.** In case $\frac{9}{5} \leq p < +\infty$ any weak solution to (1.1)-(1.2) is a suitable weak solution. Indeed, by Sobolev’s embedding theorem we have $u \in L^{\frac{9}{5}}(\mathbb{R}^3)$, which yields $|u|^2 |\nabla u| \in L^p_{loc}(\mathbb{R}^3)$. In addition, as we will see below in Section 2 from (1.3) we get $\pi \in L^{\frac{9}{4}}_{loc}(\mathbb{R}^3)$ such that for all $\varphi \in W^{1,\frac{9}{5}}_{loc}(\mathbb{R}^3)$ with compact support

(1.5) $$\int_{\mathbb{R}^3} \left( |D(u)|^{p-2} D(u) : D(\varphi) + u \otimes u : D(\varphi) \right) dx = \int_{\mathbb{R}^3} \pi \nabla \cdot \varphi dx.$$

Thus, inserting $\varphi = u\phi$ into (1.5), where $\phi \in C^\infty_c(\mathbb{R}^3)$, and applying integration by parts, we get (1.4) where the inequality is replaced by equality.

Our aim in this paper is to prove the following.

**Theorem 1.3.** (i) Let $\frac{3}{2} \leq p \leq \frac{9}{5}$. We suppose $(u, \pi) \in W^{1,p}_{loc}(\mathbb{R}^3) \times L^{\frac{3}{p}}_{loc}(\mathbb{R}^3)$ is a suitable weak solution of (1.1)-(1.2). If

(1.6) $$\int_{\mathbb{R}^3} |\nabla u|^p dx < +\infty, \quad \liminf_{R \to \infty} |u|_{B(R)} = 0$$

then $u \equiv 0$.

(ii) Let $\frac{9}{5} < p < 3$. We suppose $(u, \pi) \in W^{1,p}_{loc}(\mathbb{R}^3) \times L^{\frac{3}{p}}_{loc}(\mathbb{R}^3)$ is a weak solution of (1.1)-(1.2). Assume there exists $V \in W^{2,p}_{loc}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ such that $\nabla \cdot V = u$, and

(1.7) $$\int_{B(r)} |V - V_{B(r)}|^\frac{3p}{3p-3} dx \leq Cr^{\frac{9-4p}{2p-3}} \quad \forall 1 < r < +\infty.$$

Then, $u \equiv 0$.

**Remark 1.4.** Obviously $V \in BMO(\mathbb{R}^3)$ implies the condition (1.7). In fact, (1.7) is guaranteed by $V \in C^{\alpha,\alpha}(\mathbb{R}^3)$ with $\alpha = \frac{9-4p}{3p} > 0$ thanks to the Campanato theorem [6].

As an immediate corollary of the above theorem we have the following result, which is the case of $p = 2$, which improves the previous result in [12, 13].
Corollary 1.5. Let \((u, \pi)\) be a smooth solution of the stationary Navier-Stokes equations on \(\mathbb{R}^3\). Suppose there exists \(V \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})\) such that \(\nabla \cdot V = u\), and

\[
\int_{B(r)} |V - V_{B(r)}|^6 \, dx \leq Cr \quad \forall 1 < r < +\infty.
\]

Then, \(u \equiv 0\).

2 Proof of Theorem 1.3

We start our discussion of estimating the pressure for both of the cases (i) and (ii). First note that by the hypothesis \(u \in W_1^{1,p}(\mathbb{R}^3)\) and due to Sobolev’s embedding theorem it holds \(u \in L_3^{3-p}(\mathbb{R}^3)\). This yields

\[
|D(u)|^{p-2}D(u) - u \otimes u \in L_q^{\text{loc}}(\mathbb{R}^3), \quad q = \min \left\{ \frac{3p}{6-2p}, \frac{p}{p-1} \right\}.
\]

Given \(0 < R < +\infty\), and noting that \(q \geq \frac{3}{2}\) for \(p \geq \frac{4}{3}\), we may define the functional \(F \in W^{-1,s}(B(R))\), \(\frac{3}{2} \leq s \leq q\), by means of

\[
\langle F, \varphi \rangle = \int_{B(R)} (|D(u)|^{p-2}D(u) - u \otimes u) : D(\varphi) \, dx, \quad \varphi \in W_0^{1,s'}(B(R)),
\]

where we set \(s' = \frac{s}{s-1}\). Since \(u\) is a weak solution to (1.1)- (1.2) in view of [15, Lemma 2.1.1] there exists a unique \(\pi_R \in L^q(B(R))\) with \(\int_{B(R)} \pi_R \, dx = 0\) such that

\[
\langle F, \varphi \rangle = \int_{B(R)} \pi_R \nabla \cdot \varphi \, dx \quad \forall \varphi \in W_0^{1,s'}(B(R)).
\]

Furthermore, we get for all \(\frac{3}{2} \leq s \leq q\)

\[
\int_{B(R)} |\pi_R|^s \, dx \leq c \|F\|_{W^{-1,s}(B(R))}^s \leq c \|D(u)|^{p-2}D(u) - u \otimes u\|_{L^q(B(R))}^s,
\]

with a constant \(c > 0\), depending only on \(p\) but independent of \(0 < R < +\infty\). Let \(1 < \rho < R < +\infty\). We set \(\tilde{\pi}_R = \pi_R - (\pi_R)_{B(1)}\). From the definition of the pressure \(\pi_R\) it follows that

\[
\int_{B(\rho)} (\tilde{\pi}_R - \tilde{\pi}_\rho) \nabla \cdot \varphi \, dx = 0 \quad \forall \varphi \in W_0^{1,s'}(B(\rho)).
\]

This shows that \(\tilde{\pi}_R - \tilde{\pi}_\rho\) is constant in \(B(\rho)\). Since \((\tilde{\pi}_R - \tilde{\pi}_\rho)_{B(1)} = 0\) it follows that \(\tilde{\pi}_\rho = \tilde{\pi}_R\) in \(B(\rho)\). This allows us to define \(\pi \in L_q^{\text{loc}}(\mathbb{R}^3)\) by setting \(\pi = \tilde{\pi}_R\) in \(B(R)\). In
particular, \( \pi - \pi_{B(R)} = \pi_R \). Thus, thanks to (2.1) we estimate by Hölder’s inequality

\[
\int_{B(R)} |\pi - \pi_{B(R)}|^s dx \leq c\|D(u)|^{p-2}D(u) - u \otimes u\|_{L^s(B(R))}^s
\]

\[
\leq cR^{\frac{3(3-p)}{p}} \left( \int_{B(R)} |D(u)|^p dx \right)^{\frac{3(p-1)}{p}} + c \int_{B(R)} |u|^{2s} dx.
\]

Hence,

\[
(2.2) \quad \|\pi - \pi_{B(R)}\|_{L^s(B(R))} \leq cR^{\frac{3-p}{2}} \|D(u)\|_{L^p(B(R))}^{p-1} + c\|u\|_{L^2(B(R))}^2.
\]

Note that \( q = \frac{9}{4} \) whenever \( \frac{9}{5} \leq p < +\infty \). This yields the existence of the pressure \( \pi \in L^\frac{2}{3}(\mathbb{R}^3) \).

Let \( 1 < r < +\infty \) be arbitrarily chosen, and \( r \leq \rho < R \leq 2r \). We set \( \overline{R} = \frac{R + \rho}{2} \). Let \( \zeta \in C^\infty(\mathbb{R}^n) \) be a cut off function, which is radially non-increasing with \( \zeta = 1 \) on \( B(\rho) \) and \( \zeta = 0 \) on \( \mathbb{R}^3 \setminus B(\overline{R}) \) satisfying \( |\nabla \zeta| \leq c(R - \rho)^{-1} \). From (1.4) with \( \phi = \zeta^p \) we get

\[
\int_{B(R)} |D(u)|^p \zeta^p dx \leq \int_{B(\overline{R})} |D(u)|^{p-2} \nabla \zeta^p \cdot D(u) \cdot u dx + 
\]

\[
+ \frac{1}{2} \int_{B(\overline{R})} |u|^p u \cdot \nabla \zeta^p + \int_{B(\overline{R})} (\pi - \pi_{B(\overline{R})}) u \cdot \nabla \zeta^p dx.
\]

Applying Hölder’s and Young’s inequality, we get from above

\[
\int_{B(\rho)} |D(u)|^p \zeta^p dx \leq c(R - \rho)^{-p} \int_{B(\overline{R}) \setminus B(\rho)} |u|^p dx + c(R - \rho)^{-1} \int_{B(\overline{R}) \setminus B(\rho)} |u|^q dx
\]

\[
+ c(R - \rho)^{-1} \int_{B(\overline{R}) \setminus B(\rho)} |\pi - \pi_{B(\overline{R})}| |u| dx
\]

\[
(2.3) \quad = I + II + III.
\]

The case \( \frac{3}{2} \leq p \leq \frac{9}{5} \): Observing (2.3) and applying Sobolev’s embedding theorem we get

\[
(2.4) \quad u \in L^{\frac{3p}{\pi-p}}(\mathbb{R}^3).
\]

In (2.3) we take \( \rho = \frac{R}{2} \). Applying Hölder’s inequality, we easily get

\[
I + II \leq c \left( \int_{\mathbb{R}^3 \setminus B(\frac{R}{2})} |u|^\frac{3p}{\pi-p} dx \right)^{\frac{\pi-p}{3}} + cR^\frac{5p-q}{\pi} \left( \int_{\mathbb{R}^3 \setminus B(\frac{R}{2})} |u|^\frac{3p}{\pi-p} dx \right)^{\frac{\pi-p}{p}}.
\]

Using (2.1) and recalling that \( p \leq \frac{9}{5} \), we see that \( I + II = o(R) \) as \( R \to +\infty \).
Applying Hölder’s inequality along with (2.2) with $s = \frac{3}{2}$, we estimate

$$III \leq c R^{-1} \left( R^{\frac{2-p}{p}} \|D(u)\|_{L^p(B(R))}^{p-1} + c \|u\|^2_{L^3(B(\bar{R}))} \right) \left( \int_{\mathbb{R}^3 \setminus B(\frac{3}{2})} |u|^3 \, dx \right)^{\frac{1}{3}}$$

$$\leq c \|\nabla u\|_{L^p(B(\bar{R}))}^{p-1} \left( \int_{\mathbb{R}^3 \setminus B(\frac{3}{2})} |u|^{\frac{3p}{p-1}} \, dx \right)^{\frac{2}{3p}} + c R^{\frac{5p-9}{p}} \|u\|^2_{L^3(B(\bar{R}))} \left( \int_{\mathbb{R}^3 \setminus B(\frac{3}{2})} |u|^{\frac{3p}{p-1}} \, dx \right)^{\frac{2}{3p}}.$$

Observing (2.4) along with $p \leq \frac{9}{5}$, we find $III = o(R)$ as $R \to +\infty$. Inserting the above estimates into the right-hand side of (2.3), we deduce that $D(u) \equiv 0$, which implies that $u = u(x)$ is a linear function $x$. Taking into account the condition (1.6), we obtain $u \equiv 0$.

The case $\frac{3}{2} < p < 3$: In order to estimate $I$ and $II$ we choose another cut off function $\psi \in C^\infty(\mathbb{R}^3)$, which is radially non-increasing with $\psi = 1$ on $B(\bar{R})$ and $\psi = 0$ on $\mathbb{R}^3 \setminus B(\bar{R})$ satisfying $|\nabla \psi| \leq c(R - p)^{-1}$. Recalling that $u = \nabla \cdot \mathbf{V}$, applying integration by parts and applying the Hölder inequality, we find

$$\int_{B(R)} |u|^p \psi^p \, dx = \int_{B(R)} \partial_i (V_{ij} - (V_{ij})_{B(R)}) u_j |u|^{p-2} \psi^p \, dx$$

$$= - \int_{B(R)} (V_{ij} - (V_{ij})_{B(R)}) \left( \partial_i u_j |u|^{p-2} + (p - 2) u_j u_k \partial_i u_k |u|^{p-4} \right) \psi^p \, dx$$

$$- \int_{B(R)} (V_{ij} - (V_{ij})_{B(R)}) u_j |u|^{p-2} \partial_i \psi^p \, dx$$

$$\leq c \left( \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}|^p \, dx \right)^{\frac{1}{p}} \left( \int_{B(R)} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{B(R)} |u|^p \psi^p \, dx \right)^{\frac{p-2}{p}}$$

$$+ c(R - \rho)^{-1} \left( \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}|^p \, dx \right)^{\frac{1}{p}} \left( \int_{B(R)} |u|^p \psi^p \, dx \right)^{\frac{p-1}{p}}.$$

Using Hölder’s inequality, Young’s inequality, and observing (1.7), we obtain

$$\int_{B(R)} |u|^p \psi^p \, dx \leq c \left( \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}|^p \, dx \right)^{\frac{1}{2}} \left( \int_{B(R)} |\nabla u|^p \, dx \right)^{\frac{1}{2}}$$

$$+ c(R - \rho)^{-p} \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}|^p \, dx.$$
\[ \leq c R^{3-p} \left( \int_{B(R)} |V - V_{B(R)}|^{\frac{3p}{2p-3}} dx \right)^{\frac{2p-3}{6}} \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{1}{4}} \]

\[ + c(R - \rho)^{-p} R^{6-2p} \left( \int_{B(R)} |V - V_{B(R)}|^{\frac{3p}{2p-3}} dx \right)^{\frac{2p-3}{3}} \]

\[ \leq c R^{\frac{9-2p}{3}} \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{1}{2}} + c(R - \rho)^{-p} R^{\frac{18-4p}{3}}. \]

Since \( R \geq 1 \), and \( p > \frac{9}{5} \) we have \( R^{\frac{9-2p}{3}} \leq R^p \) and \( R^{\frac{18-4p}{3}} \leq R^{2p} \), and therefore

\[ I \leq c(R - \rho)^{-p} R^p \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{1}{2}} + (R - \rho)^{-2p} R^{2p}. \]

To estimate \( II \) we proceed similar. We first estimate the \( L^3 \) norm of \( u \) as follows

\[ \int_{B(R)} |u|^3 \psi^3 dx = \int_{B(R)} \partial_i (V_{ij} - (V_{ij})_{B(R)}) u_j |u| \psi^3 dx \]

\[ = - \int_{B(R)} (V_{ij} - (V_{ij})_{B(R)}) \partial_i (u_j |u|) \psi^3 dx - \int_{B(R)} (V_{ij} - (V_{ij})_{B(R)}) u_j |u| \partial_i \psi^3 dx \]

\[ \leq c \left( \int_{B(R)} |V - V_{B(R)}|^{\frac{3p}{2p-3}} dx \right)^{\frac{2p-3}{3p}} \left( \int_{B(R)} |u|^3 \psi^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{1}{p}} \]

\[ + c(R - \rho)^{-1} \left( \int_{B(R)} |V - V_{B(R)}|^3 dx \right)^{\frac{1}{3}} \left( \int_{B(R)} |u|^3 \psi^3 dx \right)^{\frac{2}{3}}. \]

Using Young’s inequality, we get

\[ \int_{B(R)} |u|^3 \psi^3 dx \leq c \left( \int_{B(R)} |V - V_{B(R)}|^{\frac{3p}{2p-3}} dx \right)^{\frac{2p-3}{2p}} \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{3}{2p}} \]

\[ + c(R - \rho)^{-3} \int_{B(R)} |V - V_{B(R)}|^3 dx \]

\[ \leq c \left( \int_{B(R)} |V - V_{B(R)}|^{\frac{3p}{2p-3}} dx \right)^{\frac{2p-3}{2p}} \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{3}{2p}} \]

\[ + c(R - \rho)^{-3} R^{\frac{3(3-p)}{p}} \left( \int_{B(R)} |V - V_{B(R)}|^{\frac{3p}{2p-3}} dx \right)^{\frac{2p-3}{p}}. \]

(2.5)
Once more appealing to (1.7), and recalling $R \geq 1$, $p > 9/5$, and thus $R^{\frac{9-p}{p}} \leq R^4$, we arrive at

$$II \leq c(R - \rho)^{-1} R \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{3}{2p}} + c(R - \rho)^{-4} R^{\frac{9-p}{p}}$$

(2.6)

$$\leq c(R - \rho)^{-1} R \left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{3}{2p}} + c(R - \rho)^{-4} R^4.$$

It remains to estimate $III$. Using Hölder’s inequality and Young’s inequality, we infer

$$III \leq c(R - \rho)^{-1} \int_{B(R)} |\pi - \pi_{B(R)}|^\frac{3}{2} dx + c(R - \rho)^{-1} \int_{B(R)} |u|^3 dx.$$  

(2.7)

Combining (2.7), (2.6) and (2.2), we obtain

$$III \leq cR^{\frac{3(3-p)}{2p}}(R - \rho)^{-1}\left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{3(p-1)}{2p}} + c(R - \rho)^{-1} \int_{B(R)} |u|^3 dx.$$

The second term on the right-hand side can be absorbed into $II$. We also observe here, $R^{\frac{3(3-p)}{2p}} < R$ thanks to $R \geq 1$ and $p > 9/5$.

Thus, inserting the estimate of $II$, and once more using $R \geq 1$, we find

$$III \leq cR(R - \rho)^{-1}\left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{3(p-1)}{2p}} + cR(R - \rho)^{-1}\left( \int_{B(R)} |\nabla u|^p dx \right)^{\frac{3}{2p}} + cR^4(R - \rho)^{-4}.$$

Inserting the estimates of $I, II$ and $III$ into the right hand side of (2.3), and applying Young’s inequality, we are led to

$$\int_{B(R)} |D(u)|^p \zeta^p dx \leq \frac{1}{2} \int_{B(R)} |\nabla u|^p dx + cR^4(R - \rho)^{-4} + cR^{2p}(R - \rho)^{-2p}$$

$$\leq \frac{1}{2} \int_{B(R)} |\nabla u|^p dx + cR^m(R - \rho)^{-m},$$

(2.8)

where we set

$$m = \max \left\{ 4, 2p, \frac{2p}{2p-3}, \frac{2p}{3-p} \right\},$$

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and used the fact that \( R^\alpha (R - \rho)^{-\alpha} \leq R^\beta (R - \rho)^{-\beta} \) for \( \alpha \leq \beta \). Furthermore, applying Calderón-Zygmund’s inequality, we infer

\[
\int_{B(\rho)} |\nabla u|^p dx \leq \int_{\mathbb{R}^3} |\nabla (u\zeta)|^p dx
\]

\[
\leq \int_{B(R)} |D(u)|^p \zeta^p dx + c(R - \rho)^{-p} \int_{B(R)} |u|^p dx.
\]

(2.9)

Estimating the left-hand side of (2.8) from below by (2.9), and applying the iteration Lemma in [6, V. Lemma 3.1], we deduce that

\[
\int_{B(\rho)} |\nabla u|^p dx \leq cR^m (R - \rho)^{-m}
\]

(2.10)

for all \( r \leq \rho < R \leq 2r \). Choosing \( R = 2r \) and \( \rho = r \) in (2.10), and passing \( r \to +\infty \), we find

\[
\int_{\mathbb{R}^3} |\nabla u|^p dx < +\infty.
\]

(2.11)

Similarly, from (2.6) and (2.11), we get the estimate

\[
r^{-1} \int_{B(r)} |u|^3 dx \leq c \quad \forall 1 < r < +\infty.
\]

(2.12)

Next, we claim that

\[
r^{-1} \int_{B(3r) \setminus B(2r)} |u|^3 dx = o(1) \quad \text{as} \quad r \to +\infty.
\]

(2.13)

Let \( \psi \in C^\infty(\mathbb{R}^3) \) be a cut off function for the annulus \( B(3r) \setminus B(2r) \) in \( B(4r) \setminus B(r) \), i.e. \( 0 \leq \psi \leq 1 \) in \( \mathbb{R}^3 \), \( \psi = 0 \) in \( \mathbb{R}^3 \setminus (B(4r) \setminus B(r)) \), \( \psi = 1 \) on \( B(3r) \setminus B(2r) \) and \( |\nabla \psi| \leq cr^{-1} \). Recalling that \( u = \nabla \cdot V \), and applying integration by parts, using Hölder’s inequality along with (1.7) we calculate

\[
\int_{B(4r) \setminus B(r)} |u|^3 \psi^3 dx
\]

\[
= \int_{B(4r) \setminus B(r)} \partial_j (V_{ij} - (V_{ij})_{B(4r)}) u_i |u| \psi^3 dx
\]

\[
= - \int_{B(4r) \setminus B(r)} (V_{ij} - (V_{ij})_{B(4r)}) \partial_j (u_i |u|) \psi^3 dx - \int_{B(4r) \setminus B(r)} (V_{ij} - (V_{ij})_{B(4r)}) (u_i |u|) \partial_j \psi^3 dx
\]

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By the triangular inequality we have

\[
\begin{aligned}
&\leq c \left( \int_{B(4r)} |V - V_{B(4r)}|^{3p-3} d x \right)^{\frac{2p-3}{3p}} \left( \int_{B(4r)\setminus B(r)} |u|^{3p^*} d x \right)^{\frac{1}{p}} \left( \int_{B(4r)\setminus B(r)} |
\nabla u|^p d x \right)^{\frac{1}{p}} \\
&+ cr^{-1} \left( \int_{B(4r)} |V - V_{B(4r)}|^{3p-3} d x \right)^{\frac{2p-3}{3p}} \left( \int_{B(4r)\setminus B(r)} |u|^{3p^*} d x \right)^{\frac{1}{p}} \left( \int_{B(4r)\setminus B(r)} |u|^p d x \right)^{\frac{1}{p}} \\
&\leq cr^{\frac{2}{3}} \left( \int_{B(4r)\setminus B(r)} |u|^{3p^*} d x \right)^{\frac{1}{p}} \left( \int_{B(4r)\setminus B(r)} |
\nabla u|^p d x \right)^{\frac{1}{p}} \left( \int_{B(4r)\setminus B(r)} |u|^p d x \right)^{\frac{1}{p}} \\
&+ cr^{-\frac{1}{2}} \left( \int_{B(4r)\setminus B(r)} |u|^{3p^*} d x \right)^{\frac{1}{p}} \left( \int_{B(4r)\setminus B(r)} |u|^p d x \right)^{\frac{1}{p}}.
\end{aligned}
\]

(2.14)

Let us define \( \tilde{u}_{B(4r)\setminus B(r)} = \frac{1}{\int_{B(4r)\setminus B(r)} \psi d x} \int_{B(4r)\setminus B(r)} u \psi d x \). Recalling that \( u = \nabla \cdot (V - V_{B(2r)}) \), using integration by parts, Hölder’s inequality, together with (1.7) we get

\[
\begin{aligned}
|\tilde{u}_{B(4r)\setminus B(r)}| &\leq \frac{1}{\int_{B(4r)\setminus B(r)} \psi d x} \int_{B(4r)\setminus B(r)} (V - V_{B(4r)}) \cdot \nabla \psi d x \\
&= cr^{-1} \int_{B(4r)} |V - V_{B(4r)}| d x \\
&\leq cr^{\frac{9}{3p}}.
\end{aligned}
\]

(2.15)

By the triangular inequality we have

\[
\begin{aligned}
&\left( \int_{B(4r)\setminus B(r)} |u|^p d x \right)^{\frac{1}{p}} \leq \left( \int_{B(4r)\setminus B(r)} |u - u_{B(4r)\setminus B(r)}|^p d x \right)^{\frac{1}{p}} \\
&+ \left( \int_{B(4r)\setminus B(r)} |u_{B(4r)\setminus B(r)} - \tilde{u}_{B(4r)\setminus B(r)}|^p d x \right)^{\frac{1}{p}} \\
&+ \left( \int_{B(4r)\setminus B(r)} |\tilde{u}_{B(4r)\setminus B(r)}|^p d x \right)^{\frac{1}{p}} \\
&= I_1 + I_2 + I_3.
\end{aligned}
\]

Using the Poincaré inequality and (2.15), we find

\[
\begin{aligned}
I_1 + I_3 &\leq cr \left( \int_{B(4r)\setminus B(r)} |\nabla u|^p d x \right)^{\frac{1}{p}} + cr^{\frac{18-7p}{3p}}.
\end{aligned}
\]

(2.16)
For $I_2$ we use the Hölder inequality, and then the Poincaré inequality to estimate
\[
I_2 = \left( \int_{B(4r) \setminus B(r)} \left( \int_{B(4r) \setminus B(r)} \frac{1}{\psi} |u - u_{B(4r) \setminus B(r)}(\psi)|^p \, dx \right)^{\frac{1}{p}} \right)\]
\[
\leq c \left( \int_{B(4r) \setminus B(r)} \left( \int_{B(4r) \setminus B(r)} |u - u_{B(4r) \setminus B(r)}|^p \, dx \right)^{\frac{1}{p}} \leq cr \left( \int_{B(4r) \setminus B(r)} |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \right)
\]
(2.17)

Combining (2.16) and (2.17), we get
\[
\left( \int_{B(4r) \setminus B(r)} |u|^p \, dx \right)^{\frac{1}{p}} \leq cr^{\frac{18 - 7p}{3p}} + cr \left( \int_{B(4r) \setminus B(r)} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.
\]
(2.18)

Inserting (2.18) into the last term of (2.14) and the dividing result by \( \left( \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \right)^{\frac{1}{3}} \), we find
\[
r^{-1} \int_{B(4r) \setminus B(r)} |u|^3 \psi^3 \, dx \leq cr^{-\frac{2}{3}} \left( \int_{B(4r) \setminus B(r)} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + cr^{\frac{18 - 11p}{3p}}.
\]

Thus, observing (2.11) and $p > 9/5$, we obtain the claim (2.13).

Let $1 < r < +\infty$ be arbitrarily chosen. By $\zeta \in C^\infty(\mathbb{R}^n)$ we denote a cut off function, which is radially non-increasing with $\zeta = 1$ on $B(2r)$ and $\zeta = 0$ on $\mathbb{R}^3 \setminus B(3r)$ such that $|\nabla \zeta| \leq cr^{-1}$. We multiply (1.1) by $u \zeta$ integrate over $B(3r)$ and apply integration by parts. This yields
\[
\int_{B(3r)} |\nabla u|^p \zeta^2 \, dx = \int_{B(3r)} |\nabla u|^{p-2} \nabla \zeta^2 \cdot \nabla u \cdot udx
\]
\[
+ \frac{1}{2} \int_{B(3r)} |u|^2 u \cdot \nabla \zeta + \int_{B(3r)} (\pi - \pi_{B(3r)}) u \cdot \nabla \zeta \, dx
\]
\[
\leq c \int_{B(3r) \setminus B(r)} |\nabla u|^p \, dx + cr^{-p} \int_{B(3r) \setminus B(r)} |u|^p \, dx
\]
\[
+ cr^{-1} \int_{B(3r) \setminus B(2r)} |u|^3 \, dx + cr^{-1} \int_{B(3r) \setminus B(2r)} |\pi - \pi_{B(3r)}| |u|^3 \, dx
\]
\[
= I + II + III + IV.
\]
(2.19)

Using (2.12), we immediately get
\[
I = o(1) \quad \text{as} \quad r \to +\infty.
\]
From (2.18) and (2.11) it follows that

$$II = c \left\{ r^{-1} \left( \int_{B(3r) \setminus B(r)} |u|^p \, dx \right)^{\frac{1}{p}} \right\}^p$$

(2.20)

$$\leq c r^{\frac{18-10p}{3}} + c \int_{B(3r) \setminus B(r)} |\nabla u|^p \, dx = o(1) \quad \text{as} \quad r \to +\infty.$$ 

From (2.13) we also find $III = o(1)$ as $r \to +\infty$. Finally, applying Hölder’s inequality and using (2.13), we get

$$IV \leq c \left( r^{-1} \int_{B(3r)} |\pi - \pi_{B(3r)}|^2 \, dx \right)^{\frac{3}{2}} \left( r^{-1} \int_{B(3r) \setminus B(r)} |u|^3 \, dx \right)^{\frac{1}{3}}$$

$$= c \left( r^{-1} \int_{B(3r)} |\pi - \pi_{B(3r)}|^2 \, dx \right)^{\frac{3}{2}} o(1)$$

(2.21)

as $r \to +\infty$. Using the estimate (2.2) with $B(3r)$ in place of $B(R)$, we obtain

$$r^{-1} \int_{B(3r)} |\pi - \pi_{B(3r)}|^2 \, dx \leq cr^{\frac{9-5p}{2p}} \left( \int_{B(3r)} |\nabla u|^p \, dx \right)^{\frac{3(p-1)}{4p}} + cr^{-1} \int_{B(3r)} |u|^3 \, dx.$$ 

By virtue of (2.11) and (2.12) the right-hand side of the above inequality is bounded for $r \geq 1$. Therefore, (2.21) shows that $IV = o(1)$ as $r \to +\infty$. Inserting the above estimates of $I, II, III$ and $IV$ into the right-hand side of (2.19), we deduce that

$$\int_{B(r)} |\nabla u|^p \, dx = o(1) \quad \text{as} \quad r \to +\infty.$$ 

Accordingly, $u \equiv const$ and by means of (2.12) it follows $u \equiv 0$. 

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