INCOMPATIBLE INTERSECTION PROPERTIES

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Let $\mathcal{F} \subset 2^{[n]}$ be a family in which any three sets have non-empty intersection and any two sets have at least 32 elements in common. The nearly best possible bound $|\mathcal{F}| \leq 2^{n-2}$ is proved. We believe that 32 can be replaced by 3 and provide a simple-looking conjecture that would imply this.

1. Introduction

Let $[n] := \{1, \ldots, n\}$ be the standard $n$-element set and $2^{[n]}$ its power set. Subsets of $2^{[n]}$ are called families and are usually denoted by calligraphic letters $\mathcal{F}$, $\mathcal{G}$ etc.

The generic problem in extremal set theory is to suppose that $\mathcal{F} \subset 2^{[n]}$ has a certain property $\Pi$ and ask for $m(n, \Pi) := \max \{|\mathcal{G}| : \mathcal{G} \text{ has property } \Pi\}$.

Definition 1. A property $\Pi$ is called monotone if whenever $\mathcal{F} \subset 2^{[n]}$ has property $\Pi$, $F \in \mathcal{F}$ and $F \subset G \subset [n]$, then $\mathcal{F} \cup \{G\}$ has property $\Pi$ as well.

A family $\mathcal{F} \subset 2^{[n]}$ is called an up-set if $F \in \mathcal{F}$ and $F \subset G \subset [n]$ always imply $G \in \mathcal{F}$. Note that for a monotone property $\Pi$ and an extremal family $\mathcal{F}$.

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1 A property is simply a class of families, and a family has that property if and only if it belongs to the class.
\( \mathcal{F} \) (that is, \( \mathcal{F} \) that has property \( \Pi \) and size \( m(n, \Pi) \)), \( \mathcal{F} \) is necessarily an up-set. In what follows, unless otherwise stated, we shall tacitly assume that the families we consider are up-sets.

Let us define a large class of monotone properties.

**Definition 2.** For positive integers \( r, t \), where \( r \geq 2 \), a family \( \mathcal{F} \subset 2^{[n]} \) is called \( r \)-wise \( t \)-intersecting, if \( |F_1 \cap \ldots \cap F_r| \geq t \) for all \( F_1, \ldots, F_r \in \mathcal{F} \).

In the case \( t = 1 \), instead of 1-intersecting the term intersecting is used. Arguably the simplest result in extremal set theory is the following.

**Proposition 3.** If \( \mathcal{F} \subset 2^{[n]} \) is 2-wise intersecting, then

\[
|\mathcal{F}| \leq 2^{n-1}.
\]

Indeed, out of each pair of sets \( F, [n] \setminus F \) we can include only one set in \( \mathcal{F} \).

The above result is a small part of the classical Erdős-Ko-Rado paper [2]. Since the family \( \mathcal{F}_0 := \{F \subset 2^{[n]} : 1 \in F \} \) is \( r \)-wise intersecting for every \( r \geq 2 \), (1) is the best possible bound for \( r \geq 3 \) as well. The family \( \mathcal{F}_0 \) is usually called trivially intersecting.

Let us call a family non-trivial if \( \bigcap_{F \in \mathcal{F}} F = \emptyset \). The following result is one of the early gems in extremal set theory.

**Theorem 4 (Brace-Daykin [1]).** Suppose that \( \mathcal{F} \subset 2^{[n]} \) is \( r \)-wise intersecting and non-trivial. Then

\[
|\mathcal{F}| \leq \frac{r + 2}{2^r} 2^{n-1}.
\]

Since \( r + 2 < 2^r \) for \( r \geq 3 \) and \( (r + 2)2^{-r} \to 0 \) as \( r \) tends to infinity, (2) is much stronger than (1). The following example shows that it is best possible for \( n \geq r + 1 \).

\( \mathcal{B}(1, r) := \{B \subset [n] : |B \cap [r + 1]| \geq r \} \)

Let us mention that for \( n \leq r \) there is no non-trivial \( r \)-wise intersecting family. For a simple proof of (2) cf. [4].

It is not difficult to check that for \( n \geq 4 \) every non-trivial 3-wise intersecting family \( \mathcal{F} \) is 2-wise 2-intersecting. It implies that the largest 3-wise intersecting, 2-wise 2-intersecting family \( \mathcal{F} \subset 2^{[n]} \) has size \( \frac{5}{16} 2^n \). The upper bound is immediate from Theorem 4 for non-trivial families, and \( \mathcal{B}(1, 3) \) serves as an example of an extremal family. If \( \mathcal{F} \) is trivial, e.g., if \( 1 \in F \) for all \( F \in \mathcal{F} \), then the 2-wise 2-intersecting property implies that \( \mathcal{F}(1) := \{F \setminus \{1\} : F \in \mathcal{F} \} \subset 2^{[2,n]} \) is 2-wise intersecting. Applying (1) to \( \mathcal{F}(1) \) yields

\[
|\mathcal{F}| = |\mathcal{F}(1)| \leq 2^{n-2}.
\]
On the other hand, we believe that assuming that the family is 2-wise 3-intersecting leads to stronger bounds on the size of the family.

**Conjecture 1.** Suppose that $\mathcal{F} \subset 2^{[n]}$ is both 3-wise 1-intersecting and 2-wise 3-intersecting. Then

$$|\mathcal{F}| \leq 2^{n-2}. \quad (3)$$

If $\mathcal{F}$ is trivial, then $|\mathcal{F}| \leq 2^{n-2}$ as above. This shows that in proving (3) one might assume that $\mathcal{F}$ is non-trivial. From (2) we obtain $|\mathcal{F}| \leq \frac{5}{16} 2^n = \frac{5}{4} 2^{n-2}$, which falls short of (3).

**Example.** Let $t \geq 2$ be a fixed integer and suppose for convenience that $n > t$, $n + t$ is odd. Define

$$\mathcal{T}(n, t) := \left\{ \{1\} \cup T : T \subset [2, n], |T| \geq \frac{n - 1 + t}{2} \right\}.$$

**Claim 5.** The following hold:

(i) $\mathcal{T}(n, t)$ is 3-wise intersecting and 2-wise $(t + 1)$-intersecting.

(ii) $|\mathcal{T}(n, t)| = \sum_{i \geq \frac{n - 1 + t}{2}} \binom{n - 1}{i} = (1 - o(1))2^{n-2}$ as $n \to \infty$.

We leave the easy proof to the reader. This claim shows that even for $t$ large one cannot expect something much smaller than $2^{n-2}$.

We were unable to prove Conjecture 1, but established (3) with 3 replaced by 32.

**Theorem 6.** Suppose that $\mathcal{F} \subset 2^{[n]}$ is 3-wise intersecting and 2-wise 32-intersecting. Then $|\mathcal{F}| \leq 2^{n-2}$ holds.

Even though 32 is much larger than 3, we believe that the proof method of the above theorem has its own merits and might be useful in other situations.

In what follows, we put Theorem 6 in a broader context of results on correlations of monotone properties.

Consider any monotone property $\Pi$ that is defined on $2^X$ for any set $X$ and such that a family $\mathcal{F} \subset 2^{[n]}$ satisfies it or not independently of the chosen ground set $X \supset [n]$. Then the family $\tilde{\mathcal{F}} := \mathcal{F} \cup \{ F \cup \{n+1\} : F \in \mathcal{F} \} \subset 2^{[n+1]}$ has the same property and $|\tilde{\mathcal{F}}| = 2|\mathcal{F}|$. Consequently, $m(n, \Pi) / 2^n$ is monotone increasing. Thus $p(\Pi) := \lim_{n \to \infty} m(n, \Pi) / 2^n$ exists.

Let us recall the following correlation inequality due to Harris and Kleitman.

**Theorem 7 ([11], [15]).** Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are up-sets. Then

$$\frac{|\mathcal{F} \cap \mathcal{G}|}{2^n} \geq \frac{|\mathcal{F}|}{2^n} \cdot \frac{|\mathcal{G}|}{2^n}. \quad (4)$$
Several researchers have addressed the question whether Theorem 7 can be sharpened, provided that we have some information about the structure of the two families. Talagrand [16] obtained a sharpening of (4) in terms of the function $\sum_{i=1}^{n} I_i(F_1)I_i(F_2)$, where $I_i(F)$ is the influence of the $i$-th coordinate and is, roughly speaking, the difference between the number of sets containing $i$ and not containing $i$ in $F$. More recently, Keller, Mossel, and Sen [14] and Kalai, Keller, and Mossel [12] gave an improved estimate on the correlation and provided some examples for which their bounds are sharp. The interested reader should confer these papers for more details.

From Theorem 7 one can derive:

**Corollary 8.** If $\Pi_1$ and $\Pi_2$ are monotone properties, then

$$p(\Pi_1 \land \Pi_2) \geq p(\Pi_1) \cdot p(\Pi_2).$$

**Definition 9.** If equality holds in (5), then we call the monotone properties $\Pi_1$ and $\Pi_2$ incompatible.

The Katona Theorem [13] determines the maximum size $k(n, t)$ of 2-wise $t$-intersecting families for all $n \geq t \geq 1$. The construction is analogous to $T(n, t)$ and shows

$$k(n, t) = (1 - o(1))2^{n-1}$$

for $t$ fixed and $n \to \infty$.

That is, for each of the two intersecting properties from Theorem 6, we have a lower bound of the form $(1 + o(1))2^{n-1}$ for the largest size of the family satisfying the property. Thus, in terms of Definition 9, 3-wise intersecting and 2-wise 32-intersecting properties are incompatible.

Let us state the following general conjecture.

**Conjecture 2.** For any integer $s \geq 1$ there exists an integer $t_0 = t_0(s)$, such that the 3-wise $s$-intersecting and 2-wise $t$-intersecting properties are incompatible for $t > t_0(s)$.

Theorem 6 confirms this conjecture for $s = 1$. Let us remind the reader of the best known bounds on the size of 3-wise intersecting families. The largest 3-wise $s$-intersecting family $F \subset 2^{[n]}$ has size $2^{n-s}$ for $s \leq 4$ (cf. [10]), and [6, Theorem 1.4] implies that for $s > 4$ it has size at most $(\sqrt[3]{5} - 1)^{s-4}2^{n-4}$.

Finally, we relate Conjecture 1 and a certain simple-looking conjecture about a family and its shadow. For a family $F$, let $\partial(F)$ be its immediate shadow:

$$\partial F := \{G : \exists F \in F, G \subset F, |F \setminus G| = 1\}.$$

Define also $\sigma(F) := F \cup \partial F$. 
It is important to note that \( [n] \in \mathcal{F} \) for every non-empty up-set \( \mathcal{F} \subset 2^{[n]} \). This implies \( [n] \in \partial \mathcal{F} \) whence both \( \partial \mathcal{F} \) and \( \sigma(\mathcal{F}) \) are non-trivial.

**Conjecture 3.** Suppose that \( \mathcal{F} \subset 2^{[n]} \) is 3-wise intersecting. Then

\[
|\sigma(\mathcal{F})| \geq 2|\mathcal{F}|.
\]

In the next section we show that Conjecture 3 implies Conjecture 1. We also note that a result of this sort has played an important role in the recent progress on the Erdős Matching Conjecture and related questions [7,8,9].

**2. Preliminaries**

There is a natural partial order \( A \prec B \) defined for sets of the same size. Suppose that \( A = \{a_1, \ldots, a_p\} \), \( B = \{b_1, \ldots, b_p\} \) are distinct sets with \( a_1 < \ldots < a_p \) and \( b_1 < \ldots < b_p \). We write \( A \prec B \) iff \( a_i \leq b_i \) for all \( 1 \leq i \leq p \).

**Definition 10.** The family \( \mathcal{F} \subset 2^{[n]} \) is called **initial** if \( A \prec B \) and \( B \in \mathcal{F} \) imply \( A \in \mathcal{F} \).

Extend the above partial order to \( 2^{[n]} \) by putting \( A \prec B \) if \( B \subset A \). We call this order the **shifting/inclusion order**. Erdős, Ko, and Rado [2] defined an operation on families of sets (called shifting) that maintains the \( r \)-wise \( t \)-intersecting property (cf. [4] for the proof). Let us define the \( (i,j) \)-shift for a family \( \mathcal{F} \).

\[
S_{i,j}(\mathcal{F}) = \{ F \in \mathcal{F} : F \cap \{i,j\} \neq j \text{ or } F \cup \{i\} \setminus \{j\} \in \mathcal{F} \}
\cup \{ F \cup \{i\} \setminus \{j\} : F \cap \{i,j\} = j \text{ and } F \cup \{i\} \setminus \{j\} \notin \mathcal{F} \}
\]

Since repeated application of \( (i,j) \)-shifts for different \( i < j \) eventually produces an initial family, we shall always assume that the families in question are initial.

**Proposition 11 ([3], Proposition 7.2).** If \( \mathcal{F} \subset 2^{[n]} \) is 3-wise \( t \)-intersecting and initial, then, for every \( F \in \mathcal{F} \), there exists an integer \( \ell \geq 0 \) such that

\[
|F \cap [3\ell + t]| \geq 2\ell + t.
\]

The following result is proven in [5].

**Theorem 12 ([5], Corollary 4).** Suppose that \( \mathcal{F} \subset 2^{[n]} \) is such that for any \( F \in \mathcal{F} \) we have \( |F \cap [3\ell + 2]| \geq 2\ell + 2 \) for some \( \ell \geq 0 \). Then

\[
|\partial(\mathcal{F})| \geq 2|\mathcal{F}|.
\]
The next corollary shows that Conjecture 3 is true under stronger assumptions.

**Corollary 13.** Suppose that $\emptyset \neq \mathcal{F} \subset 2^{[n]}$ is 3-wise 2-intersecting. Then $\sigma(\mathcal{F}) > 2|\mathcal{F}|$.

**Proof.** Proposition 11 implies that $\mathcal{F}$ satisfies the conditions of Theorem 12. Now the statement follows from (8) and $[n] \notin \partial \mathcal{F}$. \hfill \qed

**Definition 14.** Suppose that $A, B, C \subset 2^{[n]}$ satisfy $|A \cap B \cap C| \geq t$ for all $A \in \mathcal{A}, B \in \mathcal{B}$ and $C \in \mathcal{C}$. Then we say that $A, B, C$ are cross-$t$-intersecting.

Let us recall the following recent result.

**Theorem 15 ([10], Theorem 11).** Suppose that $A, B, C \subset 2^{[n]}$ are non-trivial and cross 1-intersecting. Then

$$\text{(9)} \quad |A| + |B| + |C| < 2^n.$$  

The following lemma was observed by the first author in [3] as a consequence of some more general theorems on random walks.

**Lemma 16 ([3], Lemma 2 and Corollary 1).** For a set $F \subset [n]$, define $y_i(F) := 1$ if $i \in F$ and $y_i(F) := -2$ if $i \notin F$. Put $y(F) := \max_{i \in [n]} \{\sum_{j=1}^{i} y_j(F)\}$. If for some positive integer $s$ a family $\mathcal{F} \subset 2^{[n]}$ satisfies $y(F) \geq s$ for any $F \in \mathcal{F}$, then

$$\text{(10)} \quad |\mathcal{F}| < \left(\frac{\sqrt{5} - 1}{2}\right)^s 2^n.$$  

The reason for our interest in $\partial \mathcal{F}$ and $\sigma(\mathcal{F})$ is explained by the following simple statement.

**Observation 17.** If $A, B, C \subset 2^{[n]}$ are cross-$t$-intersecting, $t \geq 2$, then $\sigma(A), B, C$ are cross-$(t-1)$-intersecting.

We finish this section with a short proof of the fact that Conjecture 3 implies Conjecture 1.

**Proof.** [Conjecture 3 implies Conjecture 1] Consider $\mathcal{F}$ as in the statement of Conjecture 1. Since $\mathcal{F}$ is 2-wise 3-intersecting, the family $\sigma(\mathcal{F})$ is 2-wise intersecting, and thus $|\sigma(\mathcal{F})| \leq 2^{n-1}$. Therefore, by Conjecture 3, $|\mathcal{F}| \leq \frac{1}{2} |\sigma(\mathcal{F})| \leq 2^{n-2}$. \hfill \qed
3. Proof of Theorem 6

Consider a shifted family $\mathcal{F} \subset 2^{[n]}$ as in the statement of Theorem 6. For $S \subset [s]$, define

$$\mathcal{F}(S, [s]) := \{ F \setminus S : F \in \mathcal{F}, F \cap [s] = S \}.$$  

We consider two cases depending on whether the subsets not containing 1 have a strong or weak presence in $\mathcal{F}$. As a criterion, let us fix the set $H_0 := [2, 8] \cup \{10, 11, 13, 14, 16, 17 \ldots \} \cap [n]$. 

Note that for all $3 \leq t \leq n/3$,

$$|H_0 \cap [3t]| = 2t + 1. \quad (11)$$

**Case 1.** $H_0 \in \mathcal{F}$. Put $w := 45$. We are going to partition $\mathcal{F}$ according to $\mathcal{F} \cap [w]$. Set $\tilde{H} := H_0 \cap [w]$ and define

$$\mathcal{G}_0 := \{ G \subset [w] : |G| \geq 28, G \neq \tilde{H} \}.$$ 

It is easy to verify by computer-aided computation that

$$|\mathcal{G}_0| \leq \sum_{i=28}^{w} \binom{w}{i} < \frac{1}{13} 2^w. \quad (12)$$

Define $T_0 := [w+1, w+7] \cup \{w+9, w+10, w+12, w+13, \ldots \} \cap [n]$. Now we can define

$$\mathcal{G}_1 := \{ G \subset [w] : G \notin \mathcal{G}_0, T_0 \notin \mathcal{F}(G, [w]) \}.$$ 

Recall the definition of $y(F)$ from Lemma 16 and replace the ground set $[n]$ by $[w+1, n]$. We claim that $y(F) \geq 8$ for any $F \in \mathcal{G}(G, [w])$, provided that $G \in \mathcal{G}_1$. Indeed, one could easily check that if $y(F) \leq 7$ for some $F \in \mathcal{G}(G, [w])$, then $T_0 \prec F$ and thus $T_0 \in \mathcal{G}(G, [w])$, a contradiction. Using Lemma 16, we get that

$$|\mathcal{F}(G, [w])| < \left( \frac{\sqrt{5} - 1}{2} \right)^8 2^{n-w} < \frac{1}{46} 2^{n-w}. \quad (13)$$

Finally, set $\mathcal{G}_2 := 2^{[w]} \setminus (\mathcal{G}_0 \cup \mathcal{G}_1)$. By construction, the 31-element set $\tilde{H}$ is in $\mathcal{G}_2$, moreover, $\mathcal{G}_2 \setminus \{\tilde{H}\}$ is shifted. Let us explain why the latter statement holds. The family $\mathcal{G}_2 \setminus \{\tilde{H}\}$ consists of all $(\leq 27)$-element subsets of $[w]$ that do not belong to $\mathcal{G}_1$. If $G \in \mathcal{G}_2 \setminus \{\tilde{H}\}$, then $T_0 \in \mathcal{F}(G, [w])$. If $G' \prec G$ then $\mathcal{F}(G', [w]) \supset \mathcal{F}(G, [w])$, therefore, $T_0 \in \mathcal{F}(G', [w])$ and thus $G' \in \mathcal{G}_2 \setminus \{\tilde{H}\}$. 

Below we are going to prove the following.
Proposition 18. $\mathcal{G}_2$ is 3-wise intersecting.

Let us first show how Proposition 18 implies $|\mathcal{F}| < 2^{n-2}$. First note that the pairwise 32-intersecting property and $|G| \leq 31$ for all $G \in \mathcal{G}_2$ imply that for any $G \in \mathcal{G}_2$ the family $\mathcal{F}(G, [w])$ is 2-wise intersecting. Consequently, $|\mathcal{F}(G, [w])| \leq \frac{1}{2} 2^{n-w}$.

Partition $\mathcal{F}$ according to $F \cap [w]$: $\mathcal{F}_i := \{F \in \mathcal{F}: F \cap [w] \in \mathcal{G}_i\}$. We have

\begin{equation}
|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| + |\mathcal{F}_2| \leq 2^{n-w} \cdot |\mathcal{G}_0| + \frac{1}{40} 2^{n-w} \cdot |\mathcal{G}_1| + \frac{1}{2} 2^{n-w} \cdot |\mathcal{G}_2|.
\end{equation}

If $\mathcal{G}_2$ is nontrivial, then, by Proposition 18 and the Brace–Daykin Theorem, we infer

\begin{equation}
|\mathcal{G}_2| \leq \frac{5}{16} 2^w.
\end{equation}

If $\mathcal{G}_2$ is trivial, then, by $\tilde{H} \in \mathcal{G}_2$, all sets belonging to in $\mathcal{G}_2$ contain a fixed element $i \neq 1$. But, since $\mathcal{G}_2 \setminus \{H\}$ is shifted and $i \in G$ for every $G \in \mathcal{G}_2 \setminus \{H\}$, then $1 \in G$ for every $G \in \mathcal{G}_2$, and thus $|\mathcal{G}_2| \leq \frac{1}{4} 2^w + 1 < \frac{5}{16} 2^w$. So (15) holds in this case as well.

Recall (12) and (15), as well as that $|\mathcal{G}_0| + |\mathcal{G}_1| + |\mathcal{G}_2| = 2^w$. Since the coefficient in front of $|\mathcal{G}_1|$ in (14) is the smallest, we get an upper bound for the RHS of (14) by making $|\mathcal{G}_0| = \frac{1}{13} 2^w$, $|\mathcal{G}_2| = \frac{5}{16} 2^w$ and $|\mathcal{G}_1| = (1 - \frac{1}{13} - \frac{5}{16}) 2^w$. We obtain

\begin{equation}
|\mathcal{F}| \leq \left( \frac{1}{13} + \frac{5}{32} + \left( 1 - \frac{1}{13} - \frac{5}{16} \right) \frac{1}{46} \right) 2^n < 2^{n-2},
\end{equation}

as desired.

**Proof of Proposition 18.** Take first $F, G, H \in \mathcal{G}_2 \setminus \{\tilde{H}\}$ and suppose that $F \cap G \cap H = \emptyset$. By definition, $T_0 \in \mathcal{F}(S, [w])$ for $S = F, G$ and $H$. Using shiftedness, we can obtain that

\begin{equation}
T' := [w + 1, w + 7] \cup \{w + 8, w + 10, w + 11, w + 13, \ldots\} \in \mathcal{F}(G, [w]) \quad \text{and} \quad T'' := [w + 1, w + 7] \cup \{w + 8, w + 9, w + 11, w + 12, \ldots\} \in \mathcal{F}(H, [w]).
\end{equation}

The intersection of $T', T''$ and $T_0$ is $[w + 1, w + 7]$. Since $|F| + |G| + |H| \leq 3 \cdot 27 = 81 < 2 \cdot 45 - 7$, for each $i \in [7]$ we can replace the element $w+i$ with an element from $[w]$ which is contained in at most one of $F \cup T_0, G \cup T', H \cup T''$ and strictly decrease the common intersection of the three sets. Repeating it for each $i \in [7]$, by shiftedness we get that there are three sets in $\mathcal{F}$ that have empty common intersection, a contradiction.
Now suppose that \( H = \tilde{H} = H_0 \cap [w] \). Then \( \mathcal{F}(H, [w]) \) contains \( H' := H_0 \cap [w+1,n] = \{w+1,w+2,w+4,w+5,w+7,w+8,\ldots\} \). Taking \( T_0 \in \mathcal{F}(F, [w]) \) and \( T'' \in \mathcal{F}(G, [w]) \), respectively, we get that \( T'' \cap T_0 \cap H' = \{w+1,w+2,w+4,w+5,w+7\} \). To arrive at the same contradiction, we shift these 5 elements into \([w]\), decreasing the intersection of \( T_0 \cup F, T'' \cup G \) and \( H_0 \) after each shift. Since \( |F|+|G|+|\tilde{H}| \leq 27+27+31=245-5 \), this is possible.

**Case 2.** \( H_0 \notin \mathcal{F} \). This condition implies that, for all \( S \subset [2,7] \) and \( F \in \mathcal{F}(S, [7]) \), there exists \( \ell \) such that

\[
|F \cap [8, 3\ell+9]| \geq 2\ell + 2.
\]

Indeed, it is true for \( S = [2,7] \) since \( H_0 \cap [8,n] \) is the unique maximal set in the shifting/inclusion order that does not have this property (shifting or adding one element to \( H_0 \) will make at least one of the expressions of the form \(|F \cap [8,3\ell+9]| \) larger, and thus will make the resulting set satisfy one of the inequalities (16)). Also, for \( S' \subset S \) we have \( \mathcal{F}(S, [7]) \supset \mathcal{F}(S', [7]) \) since \( \mathcal{F} \) is an up-set. The equations (16) and (8), in turn, imply that, for each \( S \subset [2,7] \), we have

\[
|\partial(\mathcal{F}(S, [7]))| \geq 2|\mathcal{F}(S, [7])|.
\]

For a two-element set \( \{x_i, y_i\} \), let us consider the following four ordered triplets:

\[
(\emptyset, \{x_i\}, \{x_i, y_i\}),
(\{x_i\}, \{y_i\}, \{y_i\}),
(\{y_i\}, \{x_i, y_i\}, \emptyset),
(\{x_i, y_i\}, \emptyset, \{x_i\}).
\]

Note that all four subsets of \( \{x_i, y_i\} \) occur once in each position (column). Also, the sum of sizes of the subsets in each triplet is always 3 and the intersection of the subsets is empty. Suppose that \( \{x_1, x_2, x_3, y_1, y_2, y_3\} = [2,7] \) and let \((A_i, B_i, C_i), i \in [3], \) be some of the above triples. We associate with them a big triple

\[
(\{1\} \cup A_1 \cup A_2 \cup A_3, \{1\} \cup B_1 \cup B_2 \cup B_3, C_1 \cup C_2 \cup C_3).
\]

Let us note that, for each big triple, the sum of the sizes of the subsets in it is 11. Altogether, we constructed \( 4 \times 4 \times 4 = 64 \) triples, where each subset of \([7]\) containing 1 appears exactly once in the first and second position and each subset of \([2,7]\) appears exactly once in the third position. Moreover, the intersection of the three subsets is empty for each triple.
For a big triple \((A, B, C)\) we consider the three families \(\mathcal{F}(D) := \mathcal{F}(D, [7])\), where \(D = A, B,\) or \(C\). Recall that \(\sigma(\mathcal{F}) = \mathcal{F} \cup \partial \mathcal{F}\).

**Proposition 19.** The families \(\mathcal{F}(A), \mathcal{F}(B), \mathcal{F}(C)\) are cross 4-intersecting. The families \(\sigma(\mathcal{F}(A)), \sigma(\mathcal{F}(B)), \sigma(\mathcal{F}(C))\) are cross 1-intersecting.

**Proof.** For each big triple \((A, B, C)\), there are exactly three elements in \([2, 7]\) that are contained in only one set among \(A, B, C\). Thus, if \(F \in \mathcal{F}(A), G \in \mathcal{F}(B), H \in \mathcal{F}(C)\) satisfy \(|F \cap G \cap H| \leq 3\), then we can do (at most) three shifts and replace each element that belongs to the intersection in one set with one of the “low-degree” elements, thus not creating new common intersection. By shiftedness, we will get \(F', G', H'\) that belong to \(\mathcal{F}\) but whose common intersection is empty.

The second statement follows from the first one via repeated applications of Observation 17.

Now, if \(\mathcal{F}(D)\) is non-empty, then \(\sigma(\mathcal{F}(D))\) is non-trivial, where \(D = A, B, C\). In that case, by (9),

\[
|\sigma(\mathcal{F}(A))| + |\sigma(\mathcal{F}(B))| + |\sigma(\mathcal{F}(C))| \leq 2^{n-7}.
\]

On the other hand, if one of the families \(\mathcal{F}(D)\) is empty, then so is \(\sigma(\mathcal{F}(D))\), and the sum of cardinalities of the two remaining \(\sigma(\mathcal{F}(D')), \sigma(\mathcal{F}(D''))\) for \(\{D, D', D''\} = \{A, B, C\}\) is at most \(2^{n-7}\) since they are cross-intersecting (due to the 2-wise 32-intersecting property). Using (17) for \(\mathcal{F}(C)\) (note that, out of \(A, B, C\), only \(C\) is a subset of \([2, 7]\)), in all cases we have

\[
|\mathcal{F}(A)| + |\mathcal{F}(B)| + 2|\mathcal{F}(C)| \leq 2^{n-7}.
\]

Summing over the 64 big triples gives \(2|\mathcal{F}| = 2 \sum_{D \subset [7]} |\mathcal{F}(D)| \leq 64 \cdot 2^{n-7}\), that is, \(|\mathcal{F}| \leq 2^{n-2}\).

### 4. Concluding Remarks

Below, we give some remarks, communicated to us by Stijn Cambie.

First, the example \(\mathcal{T}(n, t)\) may be extended by the sets \(\{T \subset [n]: |T| \geq n - t\}\) for any \(n > 3t\). The intersecting properties valid for \(\mathcal{T}(n, t)\), are also valid for this family.

Some computer-aided computations show that a 3-wise 1-intersecting, 2-wise 3-intersecting family \(\mathcal{F} \subset 2^{[n]}\) should satisfy

\[
|\mathcal{F}| \leq \begin{cases} 
2^{n-3} , & \text{if } n \in [3, 4]; \\
3 \cdot 2^{n-4} , & \text{if } n \in [5, 6]; \\
29 \cdot 2^{n-7} , & \text{if } n \in [7, 10].
\end{cases}
\]
Moreover, the family \( \{ B \subset [n] : |B \cap [7]| \geq 5 \} \) beats the extended \( \mathcal{T}(n,t) \) as long as \( n \leq 74 \).

In this paper, we proved a version of Conjecture 1 with 2-wise 3-intersecting being replaced by 2-wise 32-intersecting. It is also easy to see that we can prove a version of Conjecture 1 for 4-wise 1-intersecting, 2-wise 3-intersecting families. Indeed, if the family is trivially intersecting, then, after removing the common element, the family is still intersecting and thus has size at most \( 2^{n-2} \). If the family is non-trivial, then, using Theorem 4, we get that \( |\mathcal{F}| \leq \frac{6}{32} 2^n = \frac{3}{4} 2^{n-2} \). Moreover, using the result of Katona [13] in the case when \( \mathcal{F} \) is trivially intersecting, we can obtain sharp bounds on \( |\mathcal{F}| \) for all values of \( n \). In the case \( n > 10 \) we get the following bound that is immediate from the result of [13] and is valid for \( n-1 \equiv r \pmod{2}, r \in \{0,1\} \):

\[
|\mathcal{F}| \leq 2^r \sum_{i=(n-1-r)/2+1}^{n-1-r} \binom{n-1-r}{i} = 2^{n-2} \left( 1 - \Theta(n^{-1/2}) \right).
\]

The value \( t = s + 2 \) is not sufficient for Conjecture 2 already for \( s = 2 \). Indeed, the largest 3-wise 2-intersecting family has density \( 1/4 \), the largest 2-wise 4-intersecting family has density \( 1/2 \), but the family \( \{ S \subset [n] : |S \cap [8]| \geq 6 \} \) is both 2-wise 4-intersecting and 3-wise 2-intersecting and has density \( 37/256 > 1/8 \).

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