The anisotropic oscillator on the 2D sphere and the hyperbolic plane

Ángel Ballesteros\textsuperscript{1}, Francisco J Herranz\textsuperscript{1} and Fabio Musso\textsuperscript{2}

\textsuperscript{1} Departamento de Física, Universidad de Burgos, E-09001 Burgos, Spain
\textsuperscript{2} Dipartimento di Fisica ‘Eduardo Amaldi’, Università Roma Tre, I-00146 Rome, Italy

E-mail: angelb@ubu.es, fjherranz@ubu.es and fmusso@ubu.es

Received 26 July 2012, in final form 7 January 2013
Published 6 March 2013
Online at stacks.iop.org/Non/26/971

Recommended by C Liverani

Abstract
An integrable generalization on the 2D sphere $S^2$ and the hyperbolic plane $H^2$ of the Euclidean anisotropic oscillator Hamiltonian with ‘centrifugal’ terms given by

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \delta q_1^2 + (\delta + \Omega)q_2^2 + \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2}$$

is presented. The resulting generalized Hamiltonian $\mathcal{H}_\kappa$ depends explicitly on the constant Gaussian curvature $\kappa$ of the underlying space, in such a way that all the results here presented hold simultaneously for $S^2 (\kappa > 0)$, $H^2 (\kappa < 0)$ and $E^2 (\kappa = 0)$. Moreover, $\mathcal{H}_\kappa$ is explicitly shown to be integrable for any values of the parameters $\delta$, $\Omega$, $\lambda_1$ and $\lambda_2$. Therefore, $\mathcal{H}_\kappa$ can also be interpreted as an anisotropic generalization of the curved Higgs oscillator, that is recovered as the isotropic limit $\Omega \to 0$ of $\mathcal{H}_\kappa$. Furthermore, numerical integration of some of the trajectories for $\mathcal{H}_\kappa$ are worked out and the dynamical features arising from the introduction of a curved background are highlighted.

The superintegrability issue for $\mathcal{H}_\kappa$ is discussed by focusing on the value $\Omega = 3\delta$, which is one of the cases for which the Euclidean Hamiltonian $\mathcal{H}$ is known to be superintegrable (the 1:2 oscillator). We show numerically that for $\Omega = 3\delta$ the curved Hamiltonian $\mathcal{H}_\kappa$ presents nonperiodic bounded trajectories, which seems to indicate that $\mathcal{H}_\kappa$ provides a non-superintegrable generalization of $\mathcal{H}$ even for values of $\Omega$ that lead to commensurate frequencies in the Euclidean case. We compare this result with a previously known superintegrable curved analogue $\mathcal{H}'_\kappa$ of the 1:2 Euclidean oscillator, which is described in detail, showing that the $\Omega = 3\delta$ specialization of $\mathcal{H}_\kappa$ does not coincide with $\mathcal{H}'_\kappa$. Hence we conjecture that $\mathcal{H}_\kappa$ would be an integrable (but not superintegrable) curved generalization of the anisotropic oscillator that exists for any value of $\Omega$ and has constants of the motion that are quadratic in the momenta. Thus each commensurate Euclidean oscillator could admit another...
specific superintegrable curved Hamiltonian which would be different from $\mathcal{H}_c$
and endowed with higher order integrals. Finally, the geometrical interpretation
of the curved ‘centrifugal’ terms appearing in $\mathcal{H}_c$ is also discussed in detail.

Mathematics Subject Classification: 37J35, 70H06, 14M17, 22E60

(Some figures may appear in colour only in the online journal)

1. Introduction

The 2D anisotropic oscillator with ‘centrifugal’ (or ‘Rosochatius’) terms

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \delta q_1^2 + (\delta + \Omega)q_2^2 + \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2},$$

(1.1)

with $\delta$, $\Omega$ and $\lambda_i$ being real parameters and $(q_i, p_j) = \delta_{ij}$ ($i, j = 1, 2$), is one of the most
elementary (albeit instructive) examples of completely integrable Hamiltonian systems. Since
integrability will be our essential guiding principle when constructing the curved generalization
of this system, let us firstly recall the (Liouville) integrability properties of $\mathcal{H}$ in terms of the
$\delta$, $\Omega$ and $\lambda_i$ parameters.

Obviously, $\mathcal{H}$ is always integrable for any value of $\Omega$ since it is separable in Cartesian
coordinates:

$$\mathcal{H} = \mathcal{I}_1 + \mathcal{I}_2, \quad \mathcal{I}_1 = \frac{1}{2}p_1^2 + \delta q_1^2 + \lambda_1 \frac{q_1^2}{q_1^2}, \quad \mathcal{I}_2 = \frac{1}{2}p_2^2 + (\delta + \Omega)q_2^2 + \lambda_2 \frac{q_2^2}{q_2^2}. \quad (1.2)$$

Thus each pair $(\mathcal{H}, \mathcal{I}_1)$ and $(\mathcal{H}, \mathcal{I}_2)$ is formed by two functionally independent functions in
involution with respect to the canonical Poisson bracket. However, there are two particular
situations for which (1.1) is known to be superintegrable due to the existence of an additional
independent integral of the motion:

- When $\Omega = 0$. This is just the isotropic oscillator with Rosochatius terms ($\lambda_i$ arbitrary),
which is frequently called the anisotropic singular oscillator or the Smorodinsky–
Winternitz (SW) system [1–4]. In this case the additional integral of the motion is quadratic
in the momenta and reads:

$$J = (q_1 p_2 - q_2 p_1)^2 + 2 \left( \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2} \right).$$

- If the value of $\Omega$ is such that the ratio of the associated oscillator frequencies is a rational
number ($\lambda_i$ can be arbitrary) [5, 6]. In this situation the only case in which the system
presents an additional integral that is quadratic in the momenta is the commensurate
$1:2$ oscillator ($\Omega = 3\delta$) plus a single Rosochatius potential (say $\lambda_2 = 0$). If either
both $\lambda_i \neq 0$ or the anisotropic oscillator has any other pair of commensurate frequencies
then the additional integral is of higher order in the momenta [5, 6]. The resulting $1:2$
superintegrable Hamiltonian $\mathcal{H}'$ and its quadratic integrals of motion turn out to be

$$\mathcal{H}' = \frac{1}{2}(p_1^2 + p_2^2) + \delta q_1^2 + 4\delta q_2^2 + \lambda_1 \frac{q_1^2}{q_1^2},$$

$$\mathcal{I}'_1 = \mathcal{I}_1 = \frac{1}{2}p_1^2 + \delta q_1^2 + \lambda_1 \frac{q_1^2}{q_1^2}, \quad \mathcal{I}'_2 = \frac{1}{2}p_2^2 + 4\delta q_2^2, \quad (1.3)$$

$$L' = (q_1 p_2 - q_2 p_1)p_1 + 2\delta q_1^2 q_2 + \frac{2\lambda_1 q_2}{q_1^2}.$$}

Again, $\mathcal{H}' = \mathcal{I}'_1 + \mathcal{I}'_2$ and it can be straightforwardly checked that the sets $(\mathcal{H}', \mathcal{I}'_1, L')$ and
$(\mathcal{H}', \mathcal{I}'_2, L')$ are formed by three functionally independent functions.
The aim of this paper is to present a generalization $\mathcal{H}_c$ of the Hamiltonian (1.1) on the 2D spaces with constant Gaussian curvature $\kappa$. Such a new Hamiltonian $\mathcal{H}_c$ can properly be considered as a ‘curved’ generalization of $\mathcal{H}$ since it presents two outstanding features:

- $\mathcal{H}_c$ depends explicitly on the parameters $\kappa, \delta, \Omega$ and $\lambda_1$, and is shown to be integrable for any value of them.
- The Euclidean limit $\kappa \to 0$ of $\mathcal{H}_c$ is well defined and leads to the Hamiltonian $\mathcal{H}$.

An essential point of our approach consists in making use of the curvature $\kappa$ of the underlying 2D space as an explicit deformation parameter. Therefore, all the results here presented will be simultaneously valid for the 2D sphere $S^2$ ($\kappa > 0$), the hyperbolic plane $H^2$ ($\kappa < 0$) and the Euclidean plane $E^2$ ($\kappa = 0$), and the ‘flat’ limit $\kappa \to 0$ in the spherical and hyperbolic systems will lead to the corresponding Euclidean Hamiltonian (1.1). Conversely, the non-trivial dynamical nature of the transition from the ‘flat’ systems defined on $E^2$ to the ‘curved’ ones associated to $S^2$ and $H^2$ will be clearly appreciated through this procedure, since $\mathcal{H}_c$ can be thought of as an integrable deformation of $\mathcal{H}$.

The novelty of $\mathcal{H}_c$ stems from the fact that, to the best of our knowledge, there are only two specific values of $\Omega$ in (1.1) for which an integrable curved generalization of the corresponding Euclidean oscillator is known in the literature. Moreover, both known curved systems are superintegrable with an additional integral that is quadratic in the momenta, and correspond with the two Euclidean cases that have been explicitly described above. These two known curved Hamiltonians are:

- The so-called Higgs oscillator [7, 8], which is the curved analogue of the isotropic ($\Omega = 0$) Euclidean oscillator. In this case a curved superintegrable generalization of both centrifugal terms can be added to the oscillator potential, and the resulting Hamiltonian is called the SW system on the 2D sphere and the hyperbolic plane [9–13].
- A curved analogue of the nonisotropic 1 : 2 system (the $\Omega = 3\delta$ Euclidean oscillator) but with only one centrifugal term ($\lambda_2 = 0$), which was found in [11]. We will call this system $\mathcal{H}_c'$. Note that if we want a second curved centrifugal term ($\lambda_2 \neq 0$) to be included without losing superintegrability, the additional curved integral should be of higher order in the momenta. This generalization is not known.

Therefore, a ‘generic’ (i.e. with $\Omega$ as an explicit parameter) integrable generalization $\mathcal{H}_c$ of (1.1) on $S^2$ and $H^2$ was lacking, and we present it in this paper. Furthermore, the integrals of the motion for $\mathcal{H}_c$ will be quadratic in the momenta, and since the limit $\Omega \to 0$ of $\mathcal{H}_c$ will lead to the Higgs oscillator, we shall call $\mathcal{H}_c$ as the ‘anisotropic Higgs oscillator’. On the contrary, the known superintegrable system $\mathcal{H}_c'$ is by no means the particular $\Omega = 3\delta$ case in the $\Omega$-dependent family $\mathcal{H}_c$, and has to be considered as an ‘isolated’ system that deserves a separate analysis.

In this context, the natural question concerning the superintegrability of $\mathcal{H}_c$ arises. As we have mentioned before, $\mathcal{H}_c$ can be considered as an integrable deformation of $\mathcal{H}$ in terms of the curvature parameter $\kappa$. However, the Euclidean system $\mathcal{H}$ is not superintegrable for a generic value of $\Omega$, since it is well known that all the bounded trajectories for $\mathcal{H}$ are periodic (Lissajous curves) only in the commensurate and $\Omega = 0$ cases. Therefore, one should not expect $\mathcal{H}_c$ to be superintegrable for an arbitrary value of $\Omega$. In this respect, we have performed a numerical analysis of bounded trajectories of the Hamiltonian $\mathcal{H}_c$ for $\Omega = 0$ as well as for different values of $\Omega$ leading to commensurate frequencies in the Euclidean system. In all the cases we have also considered several different initial conditions, and the result is that $\mathcal{H}_c$ seems to be superintegrable \textit{only in the $\Omega = 0$ case} (which is the isotropic Higgs oscillator or, equivalently, the 1 : 1 oscillator) since only in this case bounded trajectories for $\mathcal{H}_c$ are always...
found to be periodic. In particular, we illustrate this analysis by presenting some bounded trajectories for $\mathcal{H}_\kappa$ with $\Omega = \delta$ and by comparing them with the ones of $\mathcal{H}_\kappa'$, which are found to be periodic in agreement with the well known superintegrability of the latter (note that the neat periodicity of these trajectories gives support to the stability of the numerical analysis performed in all the remaining cases).

All these results suggest that $\mathcal{H}_\kappa$ would be the generic $\Omega$-dependent integrable (but not superintegrable) curved generalization of the anisotropic oscillator, and that each commensurate Euclidean oscillator could admit another curved generalization that would be given in each case by a different Hamiltonian. This conjecture would mean, for instance, that the curved analogue of the $1:3$ oscillator ($\Omega = 8\delta$) would be a new Hamiltonian $\mathcal{H}_\kappa''$ which would be indeed different from the system obtained from $\mathcal{H}_\kappa$ by taking $\Omega = 8\delta$ (note that the additional integral for $\mathcal{H}_\kappa''$ has to be of higher order in the momenta). Nevertheless, both Hamiltonians would ‘collapse’ to a common system in the Euclidean limit $\kappa \to 0$.

The paper is organized as follows. In the next section we briefly recall the basics on Beltrami and geodesic polar variables that generalize the Cartesian and polar ones, respectively, to $S^2$ and $H^2$. In section 3 we introduce the anisotropic Higgs oscillator $\mathcal{H}_\kappa$ and we give the explicit form of its integral of the motion for generic values of all the parameters, including $\Omega$. We also present some numerical analysis of its trajectories for some commensurate values of its Euclidean frequencies, showing that superintegrability does not seem to survive under the deformation introduced through the curvature. In section 4 we analyse the superintegrable curved oscillator given by $\mathcal{H}_\kappa'$, and the $1:2$ Lissajous curves on $S^2$ and $H^2$ are found as the trajectories for the system. In section 5 we perform a detailed geometrical analysis of the ‘curved’ centrifugal terms [14] of all the previous curved oscillators, and their dynamical effects are also numerically explored. Finally, the last section is devoted to some concluding remarks and open problems.

2. Geodesic polar and Beltrami variables

First of all, let us rewrite the Euclidean Hamiltonians given in the previous section in terms of polar coordinates $(r, \phi)$ with conjugate momenta $(p_r, p_\phi)$. This change of coordinates induces the following canonical transformation

\[
q_1 = r \cos \phi, \quad q_2 = r \sin \phi, \quad p_1 = \cos \phi p_r - \sin \phi p_\phi, \\
p_2 = \sin \phi p_r + \frac{\cos \phi}{r} p_\phi,
\]

that transforms the expressions (1.1)–(1.3) into

\[
\mathcal{H} = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) + \delta r^2 + \Omega r^2 \sin^2 \phi + \frac{\lambda_1}{r^2 \cos^2 \phi} + \frac{\lambda_2}{r^2 \sin^2 \phi}, \\
I_1 = \frac{1}{2} \left( \cos \phi p_r - \frac{\sin \phi}{r} p_\phi \right)^2 + \delta r^2 \cos^2 \phi + \frac{\lambda_1}{r^2 \cos^2 \phi}, \\
I_2 = \frac{1}{2} \left( \sin \phi p_r + \frac{\cos \phi}{r} p_\phi \right)^2 + (\delta + \Omega) r^2 \sin^2 \phi + \frac{\lambda_2}{r^2 \sin^2 \phi}; \\
\mathcal{H}' = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) + \delta r^2 + 3\delta r^2 \sin^2 \phi + \frac{\lambda_1}{r^2 \cos^2 \phi},
\]
The anisotropic oscillator on the 2D sphere and the hyperbolic plane

Figure 1. Schematic representation of the symmetry generators $J_{ij}$ and geodesic distances of a point $Q$ with geodesic polar coordinates $(r, \phi)$ on a 2D homogeneous space.

\[ I'_1 \equiv I_1, \quad I'_2 = \frac{1}{2} \left( \sin \phi \frac{p_r}{r} + \cos \phi \frac{p_\phi}{r} \right)^2 + 4 \delta r^2 \sin^2 \phi, \quad (2.3) \]

\[ L' = \left( \cos \phi \frac{p_r}{r} - \sin \phi \frac{p_\phi}{r} \right) p_\phi + 2 \delta r^3 \cos^2 \phi \sin \phi - \frac{2 \lambda_1 \tan \phi}{r} \frac{p_r}{r} \cos \phi. \]

Now, in order to generalize the two above flat systems (2.2) and (2.3) to the sphere and to the hyperbolic plane we will apply a geometric approach in which $S^2$ and $H^2$ are obtained as homogeneous spaces of certain 3D Lie groups. In this way, the realization of the associated symmetry generators in terms of geodesic polar and Beltrami (projective) coordinates will enable us to propose the curved counterparts of these two systems.

2.1. Geodesic polar coordinates

Let $so(3)$ be the 3D real Lie algebra with generators $J_{01}$, $J_{02}$ and $J_{12}$ with commutation relations and the Casimir operator given by

\[ [J_{12}, J_{01}] = J_{02}, \quad [J_{12}, J_{02}] = -J_{01}, \quad [J_{01}, J_{02}] = \kappa J_{12}, \quad C = J_{01}^2 + J_{02}^2 + \kappa J_{12}^2, \quad (2.4) \]

where $\kappa$ is a real parameter. The three 2D classical Riemannian spaces of constant Gaussian curvature $\kappa$ can be collectively constructed as the homogenous spaces $SO(3)/SO(2)$ where $SO_+(3)$ is the Lie group generated by $so(3)$ and $SO(2)$ is the isotopy subgroup spanned by $J_{12}$. Hence $J_{01}$ and $J_{02}$ play the role of generators of translations, while $J_{12}$ corresponds to the generator of rotations on the 2D space. In this way, according to the value of $\kappa$ we get the three particular homogeneous spaces:

$\kappa > 0$: Sphere \quad $\kappa = 0$: Euclidean plane \quad $\kappa < 0$: Hyperbolic plane

$S^2 = SO(3)/SO(2) \quad E^2 = ISO(2)/SO(2) \quad H^2 = SO(2, 1)/SO(2)$

We recall that $J_{12}$ leaves a point $O$ invariant, the origin, and that each translation generator $J_{0i}$ moves $O$ along a base geodesic $l_i$ in such a manner that $l_1$ and $l_2$ are orthogonal at $O$ as depicted in figure 1.

These spaces can be globally embedded in the linear space $\mathbb{R}^3$ with ambient or Weierstrass coordinates $(x_0, x) = (x_0, x_1, x_2)$ by imposing the ‘sphere’ constraint $\Sigma$: $x_0^2 + \kappa x^2 = 1$; the origin corresponds to the point $O = (1, 0) \in \mathbb{R}^3$. The geodesic polar coordinates $r \in (0, \infty)$ and $\phi \in [0, 2\pi)$ are defined as follows. If the particle is located at a point $Q$, then $r$ is the
distance between $Q$ and the origin $O$ measured along the geodesic $l$ that joins both points. On the other hand, $\phi$ is the angle which determines the orientation of $l$ with respect to the base geodesic $l_1$ (see figure 1). The ambient coordinates are parametrized in terms of $(r, \phi)$ in the form \[976\ A Ballesteros\ et\ al\]

\[
x_0 = C_\kappa(r), \quad x_1 = S_\kappa(r) \cos \phi, \quad x_2 = S_\kappa(r) \sin \phi,
\]

where we have introduced the curvature-dependent functions \[976\ A Ballesteros\ et\ al\]

\[
C_\kappa(r) = \begin{cases} 
\cos \sqrt{\kappa} r & \kappa > 0 \\
1 & \kappa = 0 \\
\cosh \sqrt{-\kappa} r & \kappa < 0
\end{cases}, \quad S_\kappa(r) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} r & \kappa > 0 \\
r & \kappa = 0 \\
\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} r & \kappa < 0
\end{cases}.
\]

The $\kappa$-tangent is defined by $T_\kappa(r) = S_\kappa(r)/C_\kappa(r)$. Properties for these $\kappa$-functions can be found in \[976\ A Ballesteros\ et\ al\]. For instance,

\[
C^2_\kappa + \kappa S^2_\kappa = 1, \quad \frac{d}{dr}C_\kappa = -\kappa S_\kappa(r), \quad \frac{d}{dr}S_\kappa(r) = C_\kappa(r).
\]

Now let us denote by $(p_r, p_\phi)$ the conjugate momenta of $(r, \phi)$. The canonical Poisson bracket for the three functions

\[
J_{01} = \cos \phi \frac{p_r - \sin \phi}{T_\kappa(r)} p_\phi, \quad J_{02} = \sin \phi \frac{p_r + \cos \phi}{T_\kappa(r)} p_\phi, \quad J_{12} = p_\phi
\]

gives us the Lie–Poisson analogue of the algebra $\mathfrak{so}_3(3)$ (2.4). Therefore, (2.5) provides a symplectic realization of the Lie–Poisson analogue of $\mathfrak{so}_3(3)$, and we can write the kinetic energy $T$ for the free motion of a particle moving on $S^2$ and $H^2$ as the above symplectic realization for the Casimir function of $\mathfrak{so}_3(3)$, namely:

\[
T = \frac{1}{2} C = \frac{1}{2} (J_{01}^2 + J_{02}^2 + \kappa J_{12}^2) = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{S_\kappa^2(r)} \right).
\]

Obviously, the $\kappa \to 0$ limit of this expression gives the 2D Euclidean kinetic energy.

### 2.2. Beltrami coordinates

On the other hand, if we apply the central projection with pole $(0, 0) \in \mathbb{R}^3$ from $(x_0, x)$ to the Beltrami coordinates $q \in \mathbb{R}^2$, that are defined as $(0, 0) + \mu (1, q) \in \Sigma$, we get the relations

\[
\mu = \frac{1}{\sqrt{1 + \kappa q^2}}, \quad x_0 = \mu, \quad x = \mu q = -\frac{q}{\sqrt{1 + \kappa q^2}}.
\]

The image of this projection is the subset of $\mathbb{R}^3$ with $\mu \in \mathbb{R}$ so that $1 + \kappa q^2 > 0$. In the hyperbolic or Lobachevsky plane with $\kappa = -|\kappa| < 0$ this gives the interior of the Poincaré disc: $1 > |\kappa| q^2$. Notice that when $\kappa = 0$, the Beltrami coordinates reduce to the Cartesian ones on $E^2$: $x_0 = 1$ and $x = q$. Hereafter we assume the following notation for $q = (q_1, q_2)$ and for the conjugate momenta $p = (p_1, p_2)$:

\[
q^2 = q_1^2 + q_2^2, \quad p^2 = p_1^2 + p_2^2, \quad q \cdot p = q_1 p_1 + q_2 p_2, \quad |q| = \sqrt{q_1^2 + q_2^2}.
\]

In terms of this second set of canonical variables, another symplectic realization of the Lie–Poisson algebra $\mathfrak{so}_3(3)$ is found to be \[976\ A Ballesteros\ et\ al\]

\[
J_{0i} = p_i + \kappa (q \cdot p) q_i, \quad i = 1, 2; \quad J_{12} = q_1 p_2 - q_2 p_1
\]

and the free Hamiltonian $T$ that determines the geodesic motion on the Riemannian spaces is again obtained from the Casimir (2.4) under the above realization, namely

\[
T = \frac{1}{2} C = \frac{1}{2} (J_{01}^2 + J_{02}^2 + \kappa J_{12}^2) = \frac{1}{2} \left( 1 + \kappa q^2 \right) \left( p^2 + \kappa (q \cdot p)^2 \right).
\]
We remark that, as expected, the Beltrami and geodesic polar variables are connected through a canonical transformation which is given by

\[
q_1 = T_\kappa(r) \cos \phi, \quad p_1 = C_\kappa^2(r) \cos \phi \, p_r - \frac{\sin \phi}{T_\kappa(r)} \, p_\phi, \\
q_2 = T_\kappa(r) \sin \phi, \quad p_2 = C_\kappa^2(r) \sin \phi \, p_r + \frac{\cos \phi}{T_\kappa(r)} \, p_\phi,
\]

(2.9)

and generalizes (2.1). The inverse change of coordinates leads to the expressions

\[
C_\kappa(r) = \frac{1}{\sqrt{1 + \kappa \, q_1^2}}, \quad S_\kappa(r) = \frac{|q_2|}{\sqrt{1 + \kappa \, q_1^2}}, \quad T_\kappa(r) = |q_1|, \\
\cos \phi = \frac{q_1}{|q|}, \quad \sin \phi = \frac{q_2}{|q|}, \quad \tan \phi = \frac{q_2}{q_1}.
\]

(2.10)

3. The anisotropic Higgs oscillator

In order to generalize the integrable anisotropic Hamiltonian on \( E^2 \) (2.2) to the sphere and the hyperbolic plane we recall that, in terms of geodesic polar coordinates, the Higgs oscillator potential on both spaces is given by \( T_\kappa^2(r) \) \([11–13, 18]\) (i.e. \( \tan^2 r \) on \( S^2 \) with \( \kappa = 1 \) and \( \tanh^2 r \) on \( H^2 \) with \( \kappa = -1 \)), meanwhile the curved Kepler potential is \( 1/T_\kappa(r) \) (a detailed study of the latter can be found in \([19]\)). Furthermore, the superposition of the Higgs oscillator with two centrifugal potentials (i.e. the curved SW system) is also known in these variables and defines a superintegrable Hamiltonian. Such a system will be our starting point, whose properties are summarized as follows.

**Proposition 1 ([11–13]).** The Higgs oscillator Hamiltonian on \( S^2 \) and \( H^2 \) with two Rosochatius terms (curved SW system) is given in terms of geodesic polar variables by

\[
\mathcal{H}_{SW} = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{S_\kappa^2(r)} \right) + \frac{\lambda_1}{S_\kappa^2(r) \cos^2 \phi} + \frac{\lambda_2}{S_\kappa^2(r) \sin^2 \phi}. \tag{3.1}
\]

The Hamiltonian \( \mathcal{H}_{SW} \) Poisson-commutes with the functions

\[
I_{01} = \frac{1}{2} J_{01}^2 + \frac{\lambda_1}{T_\kappa^2(r) \cos^2 \phi}, \\
I_{02} = \frac{1}{2} J_{02}^2 + \frac{\lambda_2}{T_\kappa^2(r) \sin^2 \phi}, \\
I_{12} = \frac{1}{2} J_{12}^2 + \frac{\lambda_1}{T_\kappa^2(r) \cos^2 \phi} + \frac{\lambda_2}{T_\kappa^2(r) \sin^2 \phi}, \tag{3.2}
\]

where \( J_{ij} \) are the functions (2.5). Each set \((\mathcal{H}_{SW}, I_{01}, I_{12})\) and \((\mathcal{H}_{SW}, I_{02}, I_{12})\) is formed by three functionally independent functions. Moreover, \( \mathcal{H}_{SW} = I_{01} + I_{02} + \kappa I_{12} + \kappa (\lambda_1 + \lambda_2) \).

Essentially, our task now consists in the introduction of an anharmonic \( \Omega \)-term in the Hamiltonian (3.1) in such a way that the integrability of the system is preserved. Obviously, this process requires one to modify the integrals of motion (3.2) accordingly. The final result is presented in the following statement, that can be proven through direct computation.

**Proposition 2.** The Hamiltonian

\[
\mathcal{H}_\kappa = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{S_\kappa^2(r)} \right) + \omega T_\kappa^2(r) + \Omega S_\kappa^2(r) \sin^2 \phi + \frac{\lambda_1}{S_\kappa^2(r) \cos^2 \phi} + \frac{\lambda_2}{S_\kappa^2(r) \sin^2 \phi}. \tag{3.3}
\]
Poisson-commutes with the functions

\[ I_{1,\kappa} = \frac{1}{2} J_{01}^2 + \delta T_1^2(r) \cos^2 \phi + \frac{\lambda_1}{T_1^2(r) \cos^2 \phi}, \]

\[ I_{2,\kappa} = \frac{1}{2} \left( J_{02}^2 + \kappa J_{12}^2 \right) + \delta T_2^2(r) \sin^2 \phi + \Omega S_2^2(r) \sin^2 \phi + \frac{\lambda_2}{T_2^2(r) \sin^2 \phi} + \kappa \left( \frac{\lambda_1}{\cos^2 \phi} + \frac{\lambda_2}{\sin^2 \phi} \right), \]  

(3.4)

where \( J_{ij} \) are the functions (2.5). The pairs \((H_\kappa, I_{1,\kappa})\) and \((H_\kappa, I_{2,\kappa})\) are constituted by two functionally independent functions, and \( H_\kappa = I_{1,\kappa} + I_{2,\kappa} \).

Therefore, \( H_\kappa \) (3.3) can be considered as the anisotropic Higgs oscillator Hamiltonian on \( S^2 \) and \( H^2 \) with two Rosochatius terms. We stress that for a generic value of \( \Omega \), \( H_\kappa \) is an integrable (but, presumably, non-superintegrable) \( \Omega \)-anharmonic generalization of \( H_{SW} \) (3.1) since the former is endowed with a single quadratic integral of motion (either \( I_{1,\kappa} \) or \( I_{2,\kappa} \)) and its flat \( \kappa \to 0 \) limit is generally not superintegrable. We remark that in this generalization only \( I_{1,\kappa} = I_{00} \) ‘survives’ from the set of three integrals (3.2), meanwhile the two remaining ones are mixed within \( I_{2,\kappa} \), and when \( \Omega = 0 \) we get \( I_{2,\kappa} = I_{02} + \kappa I_{12} + \kappa (\lambda_1 + \lambda_2) \). Note also that the \( \kappa = 0 \) Euclidean limit is naturally included in (3.3) and (3.4), and provides (2.2).

The system defined by proposition 2 can also be written in terms of Beltrami variables by applying the transformation (2.10), and it contains only polynomial and rational expressions

\[ H_\kappa = \frac{1}{2} \left( 1 + \kappa q^2 \right) \left( p^2 + \kappa (q \cdot p)^2 \right) + \delta q^2 + \Omega \frac{q_2^2}{(1 + \kappa q^2)} + (1 + \kappa q^2) \left( \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2} \right), \]

\[ I_{1,\kappa} = \frac{1}{2} J_{01}^2 + \delta q_1^2 + \frac{\lambda_1}{q_1^2}, \]

\[ I_{2,\kappa} = \frac{1}{2} \left( J_{02}^2 + \kappa J_{12}^2 \right) + \delta q_2^2 + \Omega \frac{q_2^2}{(1 + \kappa q^2)} + \frac{\lambda_2}{q_2^2} + \kappa q^2 \left( \frac{\lambda_1}{q_1^2} + \frac{\lambda_2}{q_2^2} \right), \]  

(3.5)

where now \( J_{ij} \) are the functions given by (2.7). In this way, the curved Higgs oscillator potential, \( T_i^2(r) \), becomes the simple ‘flat’ expression \( q^2 \). Again, the Euclidean Hamiltonian \( H_\kappa \) is recovered in Cartesian coordinates, in the form (1.1) and (1.2), by setting \( \kappa = 0 \) in (3.5).

Numerical solutions for some particular bounded trajectories of \( H_\kappa \) (3.3) are explicitly illustrated in figures 2 (sphere) and 3 (hyperboloid). They are obtained by fixing the values of the parameters \( \Omega \) and \( \delta \) (for the sake of simplicity both \( \lambda_i \) are set equal to zero) and by taking the same initial conditions. In this respect, it is worth remarking that:

- The main dynamical footprint of superintegrable systems consists in the fact that all their bounded trajectories are periodic ones. Indeed, when \( \Omega = \lambda_1 = \lambda_2 = 0 \) we recover the superintegrable Higgs oscillator, whose periodic trajectories are plotted in figures 2(a) and 3(a).
- The integrable Hamiltonian (3.3), arising by introducing the anharmonic potential with \( \Omega \neq 0 \) (we keep \( \lambda_1 = \lambda_2 = 0 \)), corresponds to the anisotropic Higgs oscillator. In this case the bounded trajectories shown in figures 2(b) and 3(b) turn out to be nonperiodic. Therefore, the transition from the Higgs oscillator (\( \Omega = 0 \)) to the anisotropic one (\( \Omega \neq 0 \)) can be graphically appreciated by comparing figures 2(a) and 2(b) (see also 3(a) and 3(b)).
Figure 2. Some trajectories of the Hamiltonians $H_\kappa$ (3.3) and $H'_\kappa$ (4.1) on the sphere $S^2$ with $\kappa = 1$ and without centrifugal terms ($\lambda_1 = \lambda_2 = 0$). They are plotted in $\mathbb{R}^3$ with ambient coordinates $(x_0, x)$ such that $x_0^2 + x_1^2 + x_2^2 = 1$. Time runs from $t = 0$ to $t = 8$ for the initial data $q_1 = 1$, $\dot{q}_1 = 1$, $q_2 = -0.5$, $\dot{q}_2 = 2$: (a) the superintegrable case of $H_\kappa$ with $\delta = 1$ and $\Omega = 0$ (Higgs oscillator), (b) the integrable $H_\kappa$ with $\delta = 1$ and $\Omega = 0.2$, (c) the integrable $H_\kappa$ with $\delta = 1$ and $\Omega = 3$ and (d) the superintegrable $H'_\kappa$ with $\delta = 1$.

- We also analyse the particular case of $H_\kappa$ with $\Omega = 3\delta$ and $\lambda_1 = \lambda_2 = 0$, which is a curved analogue ($\kappa \neq 0$) of the superintegrable Euclidean oscillator $1:2$ ($\kappa = 0$) whose flat trajectory is the Lissajous $1:2$ curve. For the initial conditions here chosen, it is found that the curved bounded trajectories are no longer periodic as it is depicted in figures 2(c) and 3(c).

Therefore, by taking into account these and many other numerical computations that we have performed for different initial conditions and also for other values of $\Omega$ that correspond to commensurate Euclidean oscillators (for instance, with $\Omega = 8\delta$) we can conjecture that the Hamiltonian $H_\kappa$ (3.3) seems to be superintegrable only in the $\Omega = 0$ case. Nevertheless, we stress that we have not rigorously proven the non-superintegrability of $H_\kappa$ for an arbitrary value of $\Omega$. Equivalently, we could say that for these commensurate values of $\Omega$, the anisotropic Higgs Hamiltonian $H_\kappa$ presents a superintegrability breaking induced by the non-vanishing curvature of the underlying space.
4. The superintegrable curved 1 : 2 oscillator

As far as the known superintegrable anisotropic 1 : 2 oscillator (2.3) is concerned, its curved counterpart for any value of the curvature $\kappa$ can be written as follows.

Proposition 3 ([11]). The Hamiltonian

$$
\mathcal{H}_\kappa' = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{S_\kappa^2(r)} \right) + \delta \frac{S_\kappa^2(r) \cos^2 \phi}{(1 - \kappa S_\kappa^2(r) \cos^2 \phi)}
+ 4\delta \frac{T_\kappa^2(r) \sin^2 \phi}{(1 - \kappa S_\kappa^2(r) \cos^2 \phi) (1 - \kappa T_\kappa^2(r) \sin^2 \phi)^2} + \frac{\lambda_1}{S_\kappa^2(r) \cos^2 \phi}
$$

(4.1)
The anisotropic oscillator on the 2D sphere and the hyperbolic plane

is endowed with three integrals of motion given by

\[ I_{1,\kappa} = \frac{1}{2} (J_{01}^2 + \kappa J_{12}^2) + \delta \frac{T_2^2(r) \cos^2 \phi (1 + \kappa T_2^2(r) \sin^2 \phi)}{(1 - \kappa T_2^2(r) \sin^2 \phi)^2} + \frac{\lambda_1}{S_2^2(r) \cos^2 \phi}, \quad (4.2) \]

\[ I'_{1,\kappa} = \frac{1}{2} (J_{02}^2 + \kappa J_{12}^2) + \delta \frac{T_2^2(r) \sin^2 \phi}{(1 - \kappa T_2^2(r) \sin^2 \phi)^2}, \]

\[ L'_\kappa = J_0 J_1 \mp 2 \delta \frac{T_2^2(r) \cos^2 \phi \sin \phi}{(1 - \kappa T_2^2(r) \sin^2 \phi)^2} - 2 \lambda_1 \frac{\tan \phi}{T_2(r) \cos \phi}, \]

where \( J_{ij} \) are the functions (2.7) and \( \mathcal{H}' = I_{1,\kappa} + I'_{2,\kappa} \). Both sets \( (\mathcal{H}'_\kappa, I_{1,\kappa}, L'_\kappa) \) and \( (\mathcal{H}'_\kappa, I'_1, L'_\kappa) \) are formed by three functionally independent functions.

Once again, this system can be more easily expressed by introducing Beltrami variables through (2.10). In this way we find that

\[ \mathcal{H}'_\kappa = \frac{1}{2} \left( 1 + \kappa q^2 \right) \left( p^2 + \kappa (q - p)^2 \right) + \frac{\delta q_1^2}{(1 + \kappa q^2)} + \frac{4 \delta (1 + \kappa q^2)q_2^2}{(1 + \kappa q^2)^2} + \lambda_1 \frac{(1 + \kappa q^2)}{q_1^2}, \]

\[ I_{1,\kappa} = \frac{1}{2} (J_{01}^2 + \kappa J_{12}^2) + \delta \frac{q_1^2 (1 + \kappa q_2^2)}{(1 - \kappa q_2^2)^2} + \lambda_1 \frac{(1 + \kappa q_2^2)}{q_1^2}, \quad (4.3) \]

\[ I'_{1,\kappa} = \frac{1}{2} (J_{02}^2 + \kappa J_{12}^2) + \delta \frac{q_1^2}{(1 - \kappa q_2^2)^2}, \quad L'_\kappa = J_0 J_1 \mp 2 \delta \frac{q_1^2 q_2}{(1 - \kappa q_2^2)^2} - 2 \lambda_1 \frac{q_2}{q_1^2}, \]

where \( J_{ij} \) are now the functions (2.7). In these variables, the Euclidean case with \( \kappa = 0 \) leads just to the equations (1.3) defining \( \mathcal{H}' \) in Cartesian coordinates.

The numerical integration of the trajectories for the superintegrable Hamiltonian \( \mathcal{H}'_\kappa \) (4.1) gives rise to the Lissajous 1:2 curves on the sphere and the hyperboloid which are shown in figures 2(c) and 3(c). Hence, in contrast with the integrable Hamiltonian \( \mathcal{H}_\kappa \) (3.3) with \( \Omega = 3 \delta \), the system \( \mathcal{H}'_\kappa \) can be regarded as the appropriate curved analogue \( \kappa \neq 0 \) of the anisotropic Euclidean oscillator 1:2 \( (\kappa = 0) \). We recall that in the Euclidean plane \( \mathcal{H}' \) (1.3) is just the particular case of \( \mathcal{H}(1.1) \) with \( \Omega = 3 \delta \) and \( \lambda_2 = 0 \), but this is no longer true when the curvature is turned on. Consequently, we have shown that the Euclidean 1:2 oscillator admits (at least) two different integrable curved generalizations, since \( \mathcal{H}'_\kappa \) (the superintegrable one) cannot be recovered as a particular case of \( \mathcal{H}_\kappa \) with \( \Omega = 3 \delta \) and \( \lambda_2 = 0 \).

The difference between these two systems can also be highlighted by representing the corresponding potentials. We write \( \mathcal{H}_\kappa = T + \mathcal{U} \) (3.5) and \( \mathcal{H}'_\kappa = T + \mathcal{U}' \) (4.3), with \( T \) given by (2.8), and we represent both potentials \( \mathcal{U} \) (with \( \Omega = 3 \delta \)) and \( \mathcal{U}' \) in Beltrami coordinates in figure 4. Notice that the corresponding trajectories for the potentials shown in figures 4(a), 4(b), 4(c) and 4(d) are drawn in figures 2(c), 2(d), 3(c) and 3(d), respectively.

On the other hand, the differences between these two curved anisotropic potentials can also be enhanced by rewriting them in terms of the ambient coordinates \((x_0, x)\) introduced in section 2, namely

\[ \mathcal{U} = \delta \frac{x^2}{(1 - \kappa x^2)} + \Omega \frac{x^2}{x_1^2} + \frac{\lambda_1}{x_1^2} + \frac{\lambda_2}{x_2^2}, \]

\[ \mathcal{U}' = \delta \frac{x_1^2}{(1 - \kappa x_1^2)} + 4 \delta \frac{x_1^2 x_2^2}{(\lambda_0^2 + \kappa x_1^2)(\lambda_0^2 - \kappa x_1^2)^2} + \frac{\lambda_1}{x_1^2}, \quad (4.4) \]

and recall that \( x_0 \to 1 \) when \( \kappa \to 0 \). In this form, these potentials can be more easily compared with the proposals for integrable anharmonic oscillators on the spheres and hyperbolic spaces.
Figure 4. The potentials $U(3.5)$ and $U’(4.3)$ (without centrifugal terms $\lambda_1 = \lambda_2 = 0$) in Beltrami coordinates on the sphere $S^2$ with $\kappa = 1$ and on the hyperbolic plane $H^2$ with $\kappa = -1$: (a) the integrable $U$ on $S^2$ for $\delta = 1$ and $\Omega_1 = 3$, (b) the superintegrable $U’$ on $S^2$ for $\delta = 1$, (c) the integrable $U$ on $H^2$ for $\delta = 1$ and $\Omega = 3$ and (d) the superintegrable $U’$ on $H^2$ for $\delta = 1$.

that have been previously presented in [20, 21], thus concluding that the anisotropic Higgs oscillator potential $U$ defines a new integrable system. In fact, the potentials studied in [20, 21] are just polynomials when written in ambient coordinates. For instance, the integrable potential on the $n$-dimensional sphere considered in [21] reads

$$V = \sum_{i=0}^{n} a_i^2 x_i^2 - \left( \sum_{i=0}^{n} a_i x_i^2 \right)^2,$$

where $a_i$ are arbitrary real constants and $x_i$ are $(n + 1)$ ambient coordinates such that $\sum_{i=0}^{n} x_i^2 = 1$. The systems worked out in [20] have a similar form. Evidently, they are quite different from (4.4).
5. Curved ‘centrifugal’ terms: geometry and dynamics

It is worth recalling that the SW system on $S^2$ was interpreted in [22–24] as a superposition of three spherical oscillators, and this result has been extended to higher dimensions and to other spaces with constant curvature in [12, 13, 25]. The crucial point is to take into account that the Higgs oscillator potential given by $T_2^2(\kappa(r))$ is a central oscillator, whose centre is located at the origin $O$, and $r$ is the distance between $Q$ and $O$ measured along the geodesic $l$ that joins both points (see section 2). Hence if, instead of $O$, we take a generic point $Q_\rho$ such that $\rho$ is the distance between $Q$ and $Q_\rho$ measured along the geodesic $l_\rho$ that joins both points, then $T_2^2(\rho)$ can be interpreted as a noncentral oscillator potential with its centre at $Q_\rho$. With these ideas in mind, let us start with the potentials $U$ and $U'$ written in geodesic polar coordinates (3.3) and (4.1) in order to search for terms of the type $T_2^2(\rho)$.

Let us consider the two base geodesics $l_1$ and $l_2$ orthogonal at the origin $O$ and the point $Q(r, \phi)$. Let $Q_1$ (respectively $Q_2$) be the intersection point of $l_2$ (respectively $l_1$) with its orthogonal geodesic $l'_1$ (respectively $l'_2$), $x = QQ_1$ the distance measured along $l'_1$ and $y = QQ_2$ the one measured along $l'_2$, as depicted in figure 1. Next we apply the sine theorem [16] on the orthogonal triangles $OQQ_1$ (with external angle $\phi$) and $OQQ_2$ (with inner angle $\phi$) which yields

$\frac{OQ}{QQ_1} = \sin \phi$; $\frac{OQ}{QQ_2} = \cos \phi$.

Therefore the integrable potential $U$ (3.3) can be expressed as

$$U = \delta T_2^2(r) + \Omega \frac{\lambda_1}{S_2^2(x)} + \frac{\lambda_2}{S_2^2(y)},$$

(5.1)

which provides a unified description of this potential on the three Riemannian spaces:

- The $\delta$-term is a central oscillator with its centre at the origin $O$.
- The $\Omega$-term is the anharmonic potential which only depends on the geodesic distance $y$ to the point $Q_2$, which is not a fixed centre since it is moving together with the particle located at $Q$.
- The $\lambda_i$-terms ($i = 1, 2$) are two ‘centrifugal barriers’.

Likewise, the superintegrable potential $U'$ (4.1) can be rewritten as

$$U' = \delta T_2^2(x) + 4\delta T_2^2(y) \left( \frac{C_2^2(r)C_2^2(y)}{C_2^2(x)\left(C_2^2(r) - \kappa S_2^2(y)\right)} \right) + \frac{\lambda_1}{S_2^2(x)} + \frac{\lambda_3}{S_2^2(y)},$$

(5.2)

showing that this potential is not a simple addition of some anharmonic potential (only depending on the variable $y$) to the central oscillator; its interpretation is as follows:

- The first term is a noncentral oscillator with its centre at the point $Q_1$ (which is, moreover, not fixed).
- The second one is not any kind of curved oscillator, although it could be regarded as some kind of ‘deformation’ of a noncentral oscillator with its centre at $Q_2$.
- The $\lambda_1$-term is a ‘centrifugal barrier’.

It is worth stressing that the Euclidean limit is well defined in both (5.1) and (5.2), namely

$$\kappa \to 0 : \quad U = \delta r^2 + \Omega y^2 + \frac{\lambda_1}{x^2} + \frac{\lambda_2}{y^2}, \quad U' = \delta x^2 + 4\delta y^2 + \frac{\lambda_3}{x^2},$$

(5.3)

where the geodesic distances $x$ and $y$ now become Cartesian coordinates (that is, $q_1$ and $q_2$ appearing in expressions (1.1) and (1.3)).

Furthermore, on the sphere $S^2$ the potential $U$ admits a second interpretation, which requires one to rewrite the ‘centrifugal barriers’ as noncentral oscillators. Explicitly, let us...
consider the potential $U$ in the form (5.1), set $\kappa = +1$ for simplicity and let $O_i$ be the fixed point placed along the geodesic $l_i$ ($i = 1, 2$) which is at a distance $\frac{\pi}{2}$ from the origin $O$. Notice that in ambient coordinates $(x_0, x)$ we have that $O = (1, 0, 0), O_1 = (0, 1, 0)$ and $O_2 = (0, 0, 1)$. We introduce the distance $r_i$ between $Q$ and $O_i$ measured along $l_i'$ such that the three points $\{O, Q, O_i\}$ lie on the same geodesic $l_i'$ as it is shown in figure 5(a) in the ambient space $\mathbb{R}^3$. Thus we obtain that

$$r_1 + x = r_2 + y = \frac{\pi}{2}, \quad x_0 = \cos r, \quad x_1 = \sin x = \cos r_1, \quad x_2 = \sin y = \cos r_2.$$ 

Then the potential (5.1) on $S^2$ can be expressed as

$$U = \delta \tan^2 r + \Omega \sin^2 y + \frac{\lambda_1}{\sin^2 x} + \frac{\lambda_2}{\sin^2 y}$$

$$= \delta \tan^2 r + \frac{\Omega}{1 + \tan^2 r} + \lambda_1 \tan^2 r_1 + \lambda_2 \tan^2 r_2 + \lambda_1 + \lambda_2. \quad (5.4)$$

Hence this corresponds to the superposition of the central (Higgs) oscillator with its centre at $O$ with an anharmonic potential depending on the distance $r_2$ to $O_2$ and with two more noncentral oscillators determined by the geodesic distances $r_i$ from the centres placed at the fixed points $O_i$. Notice that the flat limit $\kappa = 0$ can only be performed in the former expression of (5.4) providing (5.3) with the two Euclidean centrifugal barriers, since in the latter the distances $r_i \to \infty$ (that is, the centres $O_i$ go to infinity) so that the terms $\tan^2 r_i$ become indeterminate.

We remark that in the hyperbolic plane with $\kappa < 0$ a similar description in terms of oscillators is, in principle, precluded. The analogous points to the previous centres $O_i$ would be beyond infinity as it can be seen in figure 5(b), that is, beyond the ’proper’ hyperbolic space $H^2$. In particular, if we set $\kappa = -1$ we find that

$$x_0 = \cosh r, \quad x_1 = \sinh x, \quad x_2 = \sinh y,$$
and the potential (5.1) on $H^2$ reads as

$$U = \delta \tanh^2 r + \Omega \sinh^2 y + \frac{\lambda_1}{\sinh^2 x} + \frac{\lambda_2}{\sinh^2 y}$$

$$= \delta \tanh^2 r + \Omega \sinh^2 y + \frac{\lambda_1}{\tanh^2 x} + \frac{\lambda_2}{\tanh^2 y} + \lambda_1 + \lambda_2. \quad (5.5)$$

Hence the $\lambda_i$-terms only admit an interpretation as centrifugal barriers with the Euclidean limit given by (5.3). Nevertheless, an interpretation of the $\lambda_i$-potentials as noncentral oscillators can be established if one considers the complete hyperbolic space. In that case, the centres $O_i$ would be located in the ‘ideal’ (exterior) region of $H^2$ so beyond the ‘actual’ $H^2$.

5.1. Some trajectories

In order to illustrate the above results and to highlight the role of the curved centrifugal terms, in figure 6 we plot some trajectories of the Hamiltonians $H_k$ (3.3) and $H'_k$ (4.1) on the sphere $S^2$ in ambient coordinates and with a single centrifugal term ($\lambda_1 > 0$ and $\lambda_2 = 0$). These trajectories are numerically integrated by imposing the same initial conditions and values of the constants $\delta$ and $\Omega$ as in the trajectories plotted in figure 2 (which is the same system with no centrifugal term). Consequently, the effect of the $\lambda_1$-potential becomes apparent by comparing figures 2 and 6. In this respect, it has to be taken into account that the $\lambda_1$-term is associated to the ambient coordinate $x_1$ (see (4.4)), with the centre $O_1 = (0, 1, 0)$ for the noncentral oscillator and geodesic radial distance $r_1$. In this way, we find that:

- The trajectories are restricted to the ‘right’ semisphere due to the infinite barrier arising at $x_1 = 0$. Roughly speaking, this barrier induces a ‘folding’ of the initial trajectories in figure 2.
- The superintegrable Higgs oscillator ($\Omega = 0$) with a centrifugal term provides a kind of Lissajous $1:2$ curve (figure 6(a)).
- The integrable anisotropic systems $H_k$ with $\Omega \neq 0$ represented in figures 6(b) and (c) show bounded trajectories which are again nonperiodic.
- The superintegrable $H'_k$ with a centrifugal term/noncentral oscillator gives rise to a kind of Lissajous $1:1$ curve (figure 6(d)).

Next, if the second centrifugal term with $\lambda_2 > 0$ is also considered, then similar features arise from the second infinite barrier. For the sake of completeness, figure 7 contains the trajectories on the sphere of the Hamiltonian $H'_k$ (3.3) which exactly correspond to those plotted in figure 6 but now with the second Rosochatius potential. We remark that:

- The trajectories of $H'_k$ are restricted to a quadrant of the sphere; these can be seen as once “folded” with respect to the trajectories shown in figure 6 but twice with respect to those obtained in figure 2.
- The superintegrable Higgs oscillator ($\Omega = 0$) with two centrifugal terms (curved SW system) gives rise to a Lissajous $1:1$ curve (figure 7(a)).
- As expected once more, the trajectories coming from the integrable anisotropic Hamiltonian $H'_k$ with $\Omega \neq 0$ drawn in figures 7(b) and (c) show bounded nonperiodic trajectories.

Summarizing, we can say that the addition of the appropriate curved centrifugal terms preserves the essential dynamical features of all these systems.
Figure 6. Some trajectories for the Hamiltonians $\mathcal{H}_\kappa$ (3.3) and $\mathcal{H}_\kappa'$ (4.1) on $S^2$ ($\kappa = 1$) with a single Rosochatius potential such that $\lambda_1 = 0.1$ and $\lambda_2 = 0$. Time runs from $t = 0$ to $t = 8$ with the initial data $q_1 = 1, q_1' = 1, q_2 = -0.5, q_2' = 2$: (a) the superintegrable case of $\mathcal{H}_\kappa$ with $\delta = 1$ and $\Omega = 0$ (Higgs oscillator with one centrifugal term), (b) the integrable $\mathcal{H}_\kappa$ with $\delta = 1$ and $\Omega = 0.2$, (c) the integrable $\mathcal{H}_\kappa$ with $\delta = 1$ and $\Omega = 3$ and (d) the superintegrable $\mathcal{H}_\kappa'$ with $\delta = 1$.

6. Concluding remarks

In this work we have introduced a new Hamiltonian $\mathcal{H}_\kappa$ on $S^2$ and $H^2$, which is integrable for any value of the anisotropy parameter $\Omega$ and that can be interpreted as the anisotropic Higgs oscillator with two curved Rosochatius potentials. Furthermore, such a system together with the previously known curved superintegrable $1:2$ oscillator $\mathcal{H}_\kappa'$ have been algebraically and geometrically studied. Numerical solutions for the trajectories corresponding to some particular initial conditions for both Hamiltonians have been presented. In this respect, some remarks are in order.

- The flat contraction $\kappa \to 0$ is always a well defined and smooth limit in all the expressions presented in the paper. In this way the known systems presented in section 1 can always
Figure 7. Trajectories of the Hamiltonian $H_\kappa$ (3.3) on $S^2$ ($\kappa = 1$) with two Rosochatius potentials given by $\lambda_1 = 0.1$ and $\lambda_2 = 0.05$. Initial data are the same as in figures 2 and 6: (a) the superintegrable case of $H_\kappa$ with $\delta = 1$ and $\Omega = 0$ (Higgs oscillator with two centrifugal terms or curved SW system), (b) the integrable $\mathcal{H}_\kappa$ with $\delta = 1$ and $\Omega = 0.2$ and (c) the integrable $\mathcal{H}_\kappa$ with $\delta = 1$ and $\Omega = 3$. Note that $\mathcal{H}'_\kappa$ is not defined when both $\lambda_1$ and $\lambda_2$ are non-zero.

be recovered in the Euclidean limit. Hence the two different integrable Hamiltonians $\mathcal{H}_\kappa$ and $\mathcal{H}'_\kappa$ can be thought as generalizations of the known Euclidean systems (1.1) and (1.3) (or (2.2) and (2.3)), and $\mathcal{H}'_\kappa = 0$ is nothing but a particular (superintegrable) case of $\mathcal{H}_\kappa = 0$.

- Alternatively, this approach can be seen as a ‘curvature-deformation’ process of the Euclidean anisotropic oscillator that by starting from (2.2) and (2.3) arrives at the Hamiltonians (3.3) and (4.1), respectively. In this sense, we remark that there is no an apparent straightforward prescription in order to perform such a deformation, as the explicit expressions for the curved Hamiltonians show.

- It is also worth stressing that when $\kappa \neq 0$ the superintegrable Hamiltonian (4.1) cannot be recovered from the integrable one (3.3) by setting $\Omega = 3\delta$ and $\lambda_2 = 0$ (see also the potentials (4.4) as well as the integrals $I_{i,\kappa}$ (3.4) and $I'_{i,\kappa}$ (4.3)). This, in turn, means that
Remarkably enough, for the curved superintegrable Hamiltonian $H$ is a true sixth order integral. More explicitly, if we set $L_1$ that ensures the superintegrability of the system is, in fact, a second order function arising from curvature analogues of some integrable Hénon–Heiles systems (see [26] and references therein) analogues of some other Euclidean commensurate oscillators could be feasible. (4.3) remains in the same ‘flat’ form that is endowed with quadratic integrals of motion (see [6]). Moreover, the ‘additional’ integral that ensures the superintegrability of the system is, in fact, a second order function arising from a true sixth order integral. More explicitly, if we set $\lambda_1 = 0$ then it is known [6] that

$$Q_1 = \left(p_1^2 - 2\delta q_1^2 - 2i\sqrt{2\delta}q_1p_1\right)^2 \left(p_2^2 - 8\delta q_2^2 + 4i\sqrt{2\delta}q_2p_2\right), \quad \tilde{Q}_1,$$

$$Q = \frac{1}{2} \left(Q_1 + \tilde{Q}_1\right),$$

are sixth order integrals of motion of $H$, and that the set $I_1, I_2$ (1.3) and $Q$ is formed by three functionally independent functions. Alternatively, the sixth order integral $Q$ can be written as

$$Q = \left(p_1^2 + 2\delta q_1^2\right)^2 \left(p_2^2 + 8\delta q_2^2\right) - 16\delta \left(2\delta q_1^2 q_2 + (q_1 p_2 - q_2 p_1) p_1\right)^2.$$

Now, the second order integral $L^\prime$ (1.3) arises through the following relation

$$L^2 = \frac{8I_1^2 I_2^2 - Q}{16\delta}.$$

Remarkably enough, for the curved superintegrable Hamiltonian $H$ (4.3) (in Beltrami variables), the corresponding sixth order integral $Q_x$ can also be found, namely,

$$Q_x = \left(J_{01} + \kappa J_{12} + 2\delta \frac{q_1^2 (1 + \kappa q_2^2)}{(1 - \kappa q_2^2)^2}\right)^2 \left(J_{02}^2 + \frac{8\delta q_2^2}{(1 - \kappa q_2^2)^2}\right) - 16\delta \left(\frac{2\delta q_1^2 q_2}{(1 - \kappa q_2^2)^2} + J_{12} J_{01}\right)^2,$$

where $J_{ij}$ are the functions (2.7). Moreover, the relationship with the second order integral $L^\prime_x$ (4.3) remains in the same ‘flat’ form

$$L^2 = \frac{8I_{1,\kappa}^2 I_{2,\kappa}^2 - Q_x}{16\delta}.$$

Therefore these latter expressions seem to point out that the construction of the $S^2$ and $H^2$ analogues of some other Euclidean commensurate oscillators could be feasible.

Secondly, another interesting open problem would be the construction of the constant curvature analogues of some integrable Hénon–Heiles systems (see [26] and references therein) that can be written as particular cases of the multiparametric family

$$H = \frac{1}{2}\left(p_i^2 + p_j^2\right) + \delta q_1^2 + (\delta + \Omega)q_2^2 + \alpha \left(q_1^2 q_2 + \beta q_3^2\right) + \frac{\lambda_1}{q_1^2},$$

in which the only known integrable cases are the Sawada–Kotera (\(\beta = 1/3, \Omega = 0\)), the KdV (\(\beta = 2, \Omega\) arbitrary) and the Kaup–Kupershmidt (\(\beta = 16/3, \Omega = 15\delta\)) systems. The results here presented would be a very reasonable starting point, that should be completed with appropriate curved analogues of the cubic Hénon–Heiles term.
Finally, the explicit solution of the Schrödinger equation associated to $\mathcal{H}_κ$ and $\mathcal{H}'_κ$ should be addressed, as well as the analysis of the quantum nonlinear dynamics generated by these new class of integrable nonlinear quantum models by following, e.g. the approaches presented in [27–29]. Work along all these lines is in progress.

Acknowledgments

This work was partially supported by the Spanish MINECO under grants MTM2010-18556 and AIC-D-2011-0711 (MINECO-INFN).

References

[1] Fris J, Mandrosov V, Smorodinsky Y A, Uhlir M and Winternitz P 1965 On higher symmetries in quantum mechanics Phys. Lett. A 147 483–6
[2] Evans N W 1990 Superintegrability of the Winternitz system Phys. Lett. A 147 483–6
[3] Evans N W 1991 Group theory of the Smorodinsky–Winternitz system J. Math. Phys. 32 3369–75
[4] Gerosa C, Pogosyan G S and Sissakian A N 1995 Path integral discussion for Smorodinsky–Winternitz potentials. I. Two- and three-dimensional Euclidean space Fortschr. Phys. 43 453–521
[5] Jauch J M and Hill E L 1940 On the problem of degeneracy in quantum mechanics Phys. Rev. 57 641–5
[6] Rodriguez M A, Tempesta P and Winternitz P 2008 Reduction of superintegrable systems: the anisotropic harmonic oscillator Phys. Rev. E 78 046608
[7] Higgins P W 1979 Dynamical symmetries in a spherical geometry I J. Phys. A: Math. Gen. 12 309–23
[8] Leemon H I 1979 Dynamical symmetries in a spherical geometry II J. Phys. A: Math. Gen. 12 489–501
[9] Gerosa C, Pogosyan G S and Sissakian A N 1995 Path integral discussion for Smorodinsky–Winternitz potentials. II. The two- and three-dimensional sphere Fortschr. Phys. 43 523–63
[10] Kalnins E G, Miller W Jr and Pogosyan G S 1997 Superintegrability of the 2D hyperboloid J. Math. Phys. 38 5416–33
[11] Rañada M F and Santander M 1999 Superintegrable systems on the 2D sphere $S^2$ and the hyperbolic plane $H^2$ J. Math. Phys. 40 5026–57
[12] Ballesteros A, Herranz F J, Santander M and Sanz-Gil T 2003 Maximal superintegrability on $N$-dimensional curved spaces J. Phys. A: Math. Gen. 36 193–9
[13] Ballesteros A and Herranz F J 2006 Superintegrability on three-dimensional Riemannian and relativistic spaces of constant curvature SIGMA 2 010
[14] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 Superintegrability on $N$-dimensional curved spaces: central potentials, centrifugal terms and monopoles Ann. Phys. 324 1219–33
[15] Herranz F J and Santander M 2002 Conformal symmetries of spacetimes J. Phys. A: Math. Gen. 35 6601–18
[16] Herranz F J, Ortega R and Santander M 2000 Trigonometry of spacetimes: a new self-dual approach to a curvature/signature (in)dependent trigonometry J. Phys. A: Math. Gen. 33 4525–51
[17] Ballesteros A and Herranz F J 2009 Maximal superintegrability of the generalized Kepler–Coulomb system on $N$-dimensional curved spaces J. Phys. A: Math. Theor. 42 245203
[18] Carriñena J F, Rañada M F, Santander M and Senthilvelan M 2004 A non-linear oscillator with quasi-harmonic behaviour: two- and $n$-dimensional oscillators Nonlinearity 17 1941–63
[19] Voznischeva T G 2003 Integrable Problems of Celestial Mechanics in Spaces of Constant Curvature (Astrophysics and Space Science Library vol 295) (Dordrecht: Kluwer)
[20] Kalnins E G, Benenti S and Miller W Jr 1997 Integrability, Stäckel spaces, and rational potentials J. Math. Phys. 38 2345–65
[21] Sakaida P 2001 Integrable anharmonic oscillators on spheres and hyperbolic spaces Nonlinearity 14 977–94
[22] Rañada M F and Santander M 2002 On some properties of harmonic oscillator on spaces of constant curvature Rep. Math. Phys. 49 335–43
[23] Rañada M F and Santander M 2002 On harmonic oscillators on the 2D sphere $S^2$ and the hyperbolic plane $H^2$ J. Math. Phys. 43 431–51
[24] Rañada M F and Santander M 2003 On harmonic oscillators on the 2D sphere $S^2$ and the hyperbolic plane $H^2$ II J. Math. Phys. 44 2149–67
[25] Herranz F J, Ballesteros A, Santander M and Sanz-Gil T 2004 Maximally superintegrable Smorodinsky–Winternitz systems on the $N$-dimensional sphere and hyperbolic spaces Superintegrability in Classical and
Quantum Systems (CRM Proceedings and Lecture Notes vol 37) ed P Tempesta et al (Providence, RI: American Mathematical Society) pp 75–89

[26] Ballesteros A and Blasco A 2010 Integrable Hénon–Heiles Hamiltonians: a Poisson algebra approach Ann. Phys. 325 2787–99

[27] Cariñena J F, Rañada M F and Santander M 2007 A quantum exactly solvable non-linear oscillator with quasi-harmonic behaviour Ann. Phys. 322 434–59

[28] Cariñena J F, Rañada M F and Santander M 2007 The quantum harmonic oscillator on the sphere and the hyperbolic plane Ann. Phys. 322 2249–78

[29] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2011 Quantum mechanics on spaces of nonconstant curvature: the oscillator problem and superintegrability Ann. Phys. 326 2053–73