ASYMPTOTIC ANALYSIS OF A CONTACT HELE-SHAW PROBLEM
IN A THIN DOMAIN

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Abstract. We analyze the contact Hele-Shaw problem with zero surface tension of a free boundary in a thin domain $\Omega^\varepsilon(t)$. Under suitable conditions on the given data, the one-valued local classical solvability of the problem for each fixed value of the parameter $\varepsilon$ is proved.

Using the multiscale analysis, we study the asymptotic behavior of this problem as $\varepsilon \to 0$, i.e., when the thin domain $\Omega^\varepsilon(t)$ is shrunk into the interval $(0, l)$. Namely, we find exact representation of the free boundary for $t \in [0, T]$, derive the corresponding limit problem ($\varepsilon = 0$), define other terms of the asymptotic approximation and prove appropriate asymptotic estimates that justify this approach.

We also establish the preserving geometry of the free boundary near corner points for $t \in [0, T]$ under assumption that free and fixed boundaries form right angles at the initial time $t = 0$.

1. Introduction

The Hele-Shaw problem was first introduced in 1897 by H.S. Hele-Shaw, a British engineer, scientist and inventor [23, 24]. This problem models the pressure of fluid squeezed between two parallel plate, a small distance apart. For the last 70 years, this problem have merited a great research interest among the mathematical, physical, engineering and biological community due to its wide application in hydrodynamics,
mathematical biology, chemistry and finance. In addition, many other problems of fluid mechanics are associated with Hele-Shaw flows, and therefore the study of these flows is very important, especially for microflows. This is due to manufacturing technology that creates shallow flat configurations, and the typically low Reynolds numbers of microflows. There is a vast literature on the Hele-Shaw problem and related problems (see e.g. [28]).

Here we focus on the contact one-phase Hele-Shaw problem with zero surface tension (ZST) of a free (unknown) boundary in a thin domain. Let $T > 0$ be arbitrarily fixed, and let $Q \subset \mathbb{R}^2$ be a rectangle $Q = (0, l) \times (0, 2\varepsilon)$ for some given positive values $l$ and $\varepsilon$. We denote $Q_T = Q \times (0, T)$ and $\partial Q_T = \partial Q \times [0, T]$.

Let $\Gamma^\varepsilon(t)$, $t \in [0, T]$, be a simple curve $\Gamma^\varepsilon(t) \subset \bar{Q}$ which splits the rectangle $\bar{Q}$ into two subdomains $\Omega^\varepsilon(t)$ and $\Omega^\varepsilon(t) \setminus \Gamma^\varepsilon(t)$, such that for some unknown function $\rho = \rho(y_1, t) : [0, l] \times [0, T] \rightarrow \mathbb{R}$ the domain $\Omega^\varepsilon(t)$ is given by

$$\Omega^\varepsilon(t) = \{ y = (y_1, y_2) \in Q : \ y_1 \in (0, l), \ 0 < y_2 < \varepsilon + \rho(y_1, t) \}, \ t \in (0, T)$$

(see Fig. 1).

The mathematical setting of the contact one-phase Hele-Shaw problem is to determine the evolution of the 2-dimensional fluid domain $\Omega^\varepsilon(t)$ (other words, to find a function $\rho$) and the fluid pressure $p^\varepsilon = p^\varepsilon(y_1, y_2, t)$, $(y_1, y_2, t) \in \Omega^\varepsilon(t)$, such that

$$\begin{cases}
\Delta_y p^\varepsilon = 0 & \text{in } \Omega^\varepsilon(t), \ t \in (0, T), \\
p^\varepsilon = 0 & \text{on } \Gamma^\varepsilon(t), \ t \in [0, T], \\
\frac{\partial p^\varepsilon}{\partial \mathbf{n}} = -\gamma V_\mathbf{n} & \text{on } \partial \Omega^\varepsilon(t) \setminus \Gamma^\varepsilon(t), \ t \in [0, T], \\
\rho(y_1, t) = 0 & y_1 \in [0, l],
\end{cases}$$

(1.2)

where $\gamma$ is a positive given number and the function $\Phi^\varepsilon$ is prescribed, and $\mathbf{n} = (n_1^\varepsilon, n_2^\varepsilon)$ and $\mathbf{n}$ denote the outward normals to $\Gamma^\varepsilon(t)$ and $\partial \Omega^\varepsilon(t) \setminus \Gamma^\varepsilon(t)$, respectively. Finally, the symbol $V_\mathbf{n}$ stands the velocity of the free boundary in the direction of $\mathbf{n}$ while $\Delta_y = \sum_{i=1}^2 \frac{\partial^2}{\partial y_i^2}$.

It is worth mentioning that the last condition in (1.2) together with representation (1.1) provides that the domain $\Omega^\varepsilon := \Omega^\varepsilon(0)$ and, hence, $\Gamma^\varepsilon := \Gamma^\varepsilon(0)$ are given. Moreover, the homogenous Dirichlet condition on the free boundary $\Gamma^\varepsilon(t)$ means that problem (1.1)-(1.2) is the Hele-Shaw problem with ZST.

In the paper we analyze the well-posed problem (1.1)-(1.2), that means the domain $\Omega^\varepsilon(t)$ is expanding in time $t \in [0, T]$, i.e. $\Omega^\varepsilon(t_1) \subset \Omega^\varepsilon(t_2)$ for $t_1 < t_2$. This property can be achieved by the appropriate choice of the given function $\Phi^\varepsilon$.

![Figure 1. Typical domain configuration](image_url)

Since the 1940s, there has been plenteous effort devoted to the Hele-Shaw free boundary problem (see e.g. [19, 25–27, 37–39, 41–43] and references therein) both analytically and numerically. The significant steps leading to exact solutions of Hele-Shaw models arose via conformal mapping techniques, by which the problem can be recast as an initial value problem of a functional differential equation. Also if a free boundary is
spherically symmetric, there exists a unique radially symmetric stationary solution to a moving boundary problem [16]. Stability and long-time behavior of solutions of the Hele-Shaw problem have been extensively studied with different methods in [9, 10, 15, 17] (see also references therein). For further acquaintance with results, we send readers to paper [44] and monograph [21], where a brief overview of the Hele-Shaw problem and a historic overview of the development in searching exact solutions is presented. As for numerical solutions of Hele-Shaw flows, they were discussed in [8, 11, 49] (see also references therein).

Coming to the solvability of Hele-Shaw models, we quote the works [5, 12, 13, 29], where existence of weak, variational and viscosity solutions are established. In the case of regular initial data, the existence and uniqueness of classical solutions to the one-phase well-posed Hele-Shaw problem are discussed in [2, 3, 9, 10, 14, 17, 40, 50].

As for a nonregular initial shape of a moving boundary, the one-phase Hele-Shaw model with ZST was first investigated via qualitative approach in plain corners in [30], where the motion of the corner point was described. In particular, it was shown that the waiting time phenomena (preservation of angles at moving boundaries for a certain time) exists in the case of acute angles, while the obstacle angles are immediately smoothed. The solvability of one- and two-phase well-posed Hele-Shaw problems in the case of corner points with acute angles on a free boundary were studied in [4, 6, 45, 46]. We remark that papers [4] and [46] are related to the contact Hele-Shaw problem with and without surface tension of unknown boundary.

Nevertheless, the classical solvability of problem (1.1)-(1.2) with nonhomogeneous Neumann conditions on the vertical sides $Γ_1^ε$ and $Γ_5^ε$ when fixed and free boundaries form right angles is still an open problem.

The motivation in the asymptotic study of problem (1.1)-(1.2) in the thin domain $Ω^ε(t)$ (as $ε → 0$) arises from the investigation of mathematical models of atherosclerosis [18, 34]. In [18] it was proved that for any small $ε > 0$ under certain condition for initial data there is a unique $ε$-thin radially symmetric stationary plaque, i.e., a plaque with $R(t) = 1 - ε$, and in addition, conditions were determined under which the $ε$-thin stationary plaque is linearly asymptotically stable (or unstable) as $t → +∞$ and when it is shrunk and disappeared. A multiscale analysis of a new mathematical model of the atherosclerosis development in a thin tubular domain (without moving boundary) was performed, in particular, the corresponding limit two-dimensional problem was derived, and the asymptotic approximation for the solution was constructed and justified in [34].

The novelty of this paper consists of three parts.

- At first, we find sufficiently conditions for given functions in model (1.1)-(1.2) which provide the local classical solvability of the contact Hele-Shaw problem for each fixed $ε > 0$. To this end, we exploit the approach from [2, 46].
- Secondly, we make rigorous asymptotic analysis of problem (1.1)-(1.2) as $ε → 0$, i.e., when the thin domain $Ω^ε(t)$ is shrunk into the interval $(0, l)$. Applying the method of papers [31, 34, 35], we find the moving curve $Γ^ε(t)$ and construct the asymptotic approximation $P^ε$ for the solution to problem (1.1)-(1.2) and evaluate its deviation from the classical solution $p^ε$ in the Sobolev space $C([0, T]; H^1(Ω^ε(t)))$. To our knowledge, this is the first work in the mathematical literature concerning the rigorous asymptotic study of Stefan type problems in thin domains.
- Finally, collecting the results concerning to both the solvability and the asymptotic representation, we establish preserving the geometry of the free boundary in small neighborhoods of the corner points. Besides, under certain assumptions on the given function $Φ^ε$, the size of these neighborhoods is estimated via the size of the support of the function $Φ^ε|_{y_2=0}$.

Outline of the paper. In Section 2, we introduce some functional spaces and notations. The classical solvability of (1.1)-(1.2) is formulated in Theorem 3.1 in Section 3. Section 4 states the main Theorem 4.1 that describes the asymptotic behavior of the solution $p^ε$. Section 5 is devoted to obtaining some auxiliary results which play significant role in the proofs of Theorems 3.1 and 4.1. The proof of Theorem 3.1 is carried out in Section 6. Moreover, in Subsection 6.6 we discuss the solvability problem (1.1)-(1.2) for more general domains $Ω^ε$ (i.e. $ρ(y_1, 0) ≠ 0$; see Theorem 6.3). The proof of Theorem 4.1 is presented in Section 7. In Conclusion we analyze obtained results and consider research perspectives.
2. Functional Spaces and Notations

We carry out our analysis in the framework of the Hölder and Sobolev spaces. Therefore, we recall some definitions. Let $\mathcal{D}$ be a domain in $\mathbb{R}^n$, $n \geq 1$, and $\alpha \in (0, 1)$. Notation $C^{k,\alpha}(\mathcal{D})$, $L^p(\mathcal{D})$, $W^{k,p}(\mathcal{D})$, $W^{k,p}_0(\mathcal{D})$ represent the classical Hölder and Sobolev spaces, where $k \in \mathbb{N}_0$ and $p \geq 1$. In addition, we will use the standard alternative notation $H^1(\mathcal{D})$ for the space $W^{1,2}(\mathcal{D})$.

Let $X$ be a Banach space with the norm $\| \cdot \|_X$. The space $C([0, T]; X)$ comprises all continuous functions on $[0, T]$ taking values in $X$; the space $L^p((0, T); X)$ consists of all measurable functions $u \mapsto X$ with

$$\|u\|_{L^p((0, T); X)} := \left( \int_0^T \|u(t)\|^p_X \, dt \right)^{1/p} < +\infty.$$ 

Denote by $\mathcal{D}_T := \mathcal{D} \times (0, T)$,

$$\langle v \rangle_{y, \mathcal{D}_T} := \sup \left\{ \frac{|v(y, t) - v(\bar{y}, t)|}{|y - \bar{y}|^\alpha} : (y, t), (\bar{y}, t) \in \mathcal{D}_T, y \neq \bar{y}, t \neq \bar{t} \right\},$$

and

$$C^{k,\alpha}(\mathcal{D}_T) := C([0, T]; C^{k,\alpha}(\mathcal{D})).$$

Also we introduce the Banach space $\hat{C}^{k,\alpha}(\mathcal{D}_T)$, $k \geq 1$, consisting of all functions $v \in C^{k,\alpha}(\mathcal{D}_T)$ such that $v_t := \frac{\partial v}{\partial t} \in C^{k-1,\alpha}(\mathcal{D}_T)$ and the norm

$$\|v\|_{C^{k,\alpha}(\mathcal{D}_T)} := \|v\|_{C^{k+\alpha}(\mathcal{D}_T)} + \|v_t\|_{C^{k-1+\alpha}(\mathcal{D}_T)} < +\infty.$$

In the spaces $C^{k,\alpha}(\mathcal{D}_T)$ and $\hat{C}^{k,\alpha}(\mathcal{D}_T)$ we secrete the subspaces

$$C^{k,\alpha}_0(\mathcal{D}_T) := \{ v \in C^{k,\alpha}(\mathcal{D}_T) : D_\alpha^\beta v(0, y) = 0, |\beta| = 0, 1, ..., k \},$$

$$\hat{C}^{k,\alpha}_0(\mathcal{D}_T) := \{ v \in \hat{C}^{k,\alpha}(\mathcal{D}_T) : D_\alpha^\beta v(0, y) = 0, |\beta| = 0, 1, ..., k, D_\alpha^\beta v_t(0, y) = 0, |\alpha| = 0, 1, ..., k-1 \},$$

where $|\beta|$ and $|\alpha|$ are multyindexes, i.e. $|\beta| = \beta_1 + ... + \beta_n$, $|\alpha| = \alpha_1 + ... + \alpha_n$.

Throughout this work, the symbol $C$ will denote a generic positive constant, depending only on the structural quantities of the model. We will denote the inner product in $L^2(0, a)$ by the symbol $\langle \cdot, \cdot \rangle_a$.

Finally, for each $t \in [0, T]$ the middle value of a function $v = v(z, t)$, $z \in [0, f]$, is designated by

$$\langle v \rangle_t := \frac{1}{f} \int_0^f v(z, t) \, dz,$$ \hspace{1cm} (2.1)

where $f$ is a positive function $f = f(\cdot, t)$.

3. Local Classical Solvability of Problem (1.1)-(1.2)

Throughout this section, we assume that the positive parameter $\varepsilon$ is arbitrary but fixed. First we write $\Gamma^c(t)$ and $\partial \Omega^c(t) \backslash \Gamma^c(t)$ in more comfortable form. Since we will look for the local classical solution, we define the unknown boundary $\Gamma^c(t)$ for each $t \in [0, T]$ as follows

$$\Gamma^c(t) = \{ y = (y_1, y_2) \in \mathbb{R}^2 : \ y_2 = \varepsilon + \rho(y_1, t), \ y_1 \in [0, l] \}, \ \text{where} \ \ |ho(y_1, t)| < \varepsilon/5. \hspace{1cm} (3.1)$$

In the light of this definition, the boundary $\partial \Omega^c(t) \backslash \Gamma^c(t)$ is described for each $t \in [0, T]$ as

$$\partial \Omega^c(t) \backslash \Gamma^c(t) = \Gamma_1^c(t) \cup \Gamma_2 \cup \Gamma_3^c(t),$$

where

$$\Gamma_1^c(t) = \{ y = (y_1, y_2) \in \mathbb{R}^2 : \ y_1 = 0, \ y_2 \in [0, \varepsilon + \rho(0, t)) \},$$

$$\Gamma_2 = \{ y = (y_1, y_2) \in \mathbb{R}^2 : \ y_2 = 0, \ y_1 \in (0, l) \},$$

$$\Gamma_3^c(t) = \{ y = (y_1, y_2) \in \mathbb{R}^2 : \ y_1 = l, \ y_2 \in [0, \varepsilon + \rho(l, t)] \}. \hspace{1cm} (3.2)$$

Now, we are ready to state our general assumptions for the structural quantities appearing in problem (1.1)-(1.2).
(h1)(Conditions for the boundary $\partial \Omega^\varepsilon(0)$): We assume that
$$\partial \Omega^\varepsilon := \partial \Omega^\varepsilon(0) = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon \cup \Gamma_3^\varepsilon \cup \Gamma^\varepsilon,$$
where
$$\Gamma_1^\varepsilon := \Gamma_1^\varepsilon(0) = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 = 0, \ y_2 \in [0, \varepsilon]\},$$
$$\Gamma_2^\varepsilon := \Gamma_2^\varepsilon(0) = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 = l, \ y_2 \in [0, \varepsilon]\},$$
$$\Gamma_3^\varepsilon := \Gamma_3^\varepsilon(0) = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_2 = \varepsilon, \ y_1 \in [0, l]\}.$$

Besides, for each fixed $T > 0$, we denote $\partial \Omega^\varepsilon_T := \partial \Omega^\varepsilon \times [0, T]$, $\Gamma^\varepsilon_T = \Gamma^\varepsilon \times [0, T]$, $\Gamma_i^\varepsilon_T = \Gamma_i^\varepsilon \times [0, T]$ for $i \in \{1, 3\}$.

(h2)(Smoothness of the given functions): Let
$$\varphi_1 \in C([0, T]; C^{2+\alpha}[0, 1]), \ \varphi_2 \in C([0, T]; C^{2+\alpha}[0, l]), \ \varphi_3 \in C([0, T]; C^{2+\alpha}[0, 1]).$$

(h3)(Representation of the given function): We assume that
$$\Phi^\varepsilon(y_1, y_2, t) = \begin{cases} \chi_2(\frac{y_2}{\varepsilon}) \varphi_1(\frac{y_2}{\varepsilon}, t), & y_2 \in \Gamma_1^\varepsilon(t), \ t \in [0, T], \\
\chi_1(y_1) \varphi_2(y_1, t), & y_1 \in \Gamma_2, \ t \in [0, T], \\
\chi_2(\frac{y_2}{\varepsilon}) \varphi_3(\frac{y_2}{\varepsilon}, t), & y_2 \in \Gamma_3^\varepsilon(t), \ t \in [0, T], \end{cases} \quad (3.3)$$
where $\chi_i \in C_0^\infty(\mathbb{R}^1)$, $i \in \{1, 2\}$, are the cut-off functions such that $0 \leq \chi_i \leq 1$ and
$$\chi_1(y_1) = \begin{cases} 1, & y_1 \in [\frac{2l}{5}, \frac{4l}{5}], \\
0, & y_1 \notin [\frac{l}{5}, \frac{4l}{5}]. \end{cases} \quad (3.4)$$

(h4)(Condition of the well-posedness to (1.1)-(1.2)): We require that the inequality holds
$$V_n \bigg|_{t=0} > 0 \quad \text{on} \quad \Gamma^\varepsilon. \quad (3.4)$$

Remark 3.1. It is apparent that condition (3.4) means the positivity of the initial speed of the moving boundary. That guarantees the expansion of the domains $\Omega^\varepsilon(t)$ ($\Omega^\varepsilon(t_1) \subset \Omega^\varepsilon(t_2)$ if $0 \leq t_1 < t_2 \leq T$) and as a consequence the well-posedness of (1.1)-(1.2) (see, e.g. [5], [44]). Besides, this speed, obviously, depends on the function $\Phi^\varepsilon$. In forthcoming Lemma 6.1 (Subsection 6.1) and Remark 7.2 (Subsection 7.1), we shall discuss the assumptions on $\Phi^\varepsilon$ which provide inequality (3.4). In addition, the formal integration by parts in (1.2) gives the necessary condition
$$\int_0^1 \chi_2(\xi_2) \varphi_1(\xi_2, t) d\xi_2 + \int_0^1 \chi_1(y_1) \varphi_2(y_1, t) dy_1 + \int_0^1 \chi_2(\xi_2) \varphi_3(\xi_2, t) d\xi_2 > 0 \quad \forall \ t \in [0, T]$$
for the fulfillment of (3.4).

Now we can state our first main result concerning the local classical solvability of the Hele-Shaw problem (1.1)-(1.2).

Theorem 3.1. Under assumptions (h1)-(h4), for any fixed positive $\varepsilon$, problem (1.1)-(1.2) admits a unique classical solution ($p^\varepsilon(y_1, y_2, t), \rho(y_1, t)$) in some interval $t \in [0, T]$, such that $\Gamma^\varepsilon(t)$ is given by (3.1) and
$$p^\varepsilon \in C([0, T]; C^{2+\alpha}(\overline{\Omega^\varepsilon(t)})), \ \rho \in C([0, T]; C^{2+\alpha}(\overline{[0, l]})), \ \frac{\partial \rho}{\partial t} \in C([0, T]; C^{1+\alpha}(\overline{[0, l]})).$$

Remark 3.2. To simplicity consideration, we specify supports of the cut-off functions $\chi_1$ and $\chi_2$ in assumption (h3). Actually, with the nonessential modifications in the proof, the same results hold for the functions $\chi_1$ and $\chi_2$ with the supports lying strictly inside $(0, l)$ and $(0, 1)$, respectively.

Remark 3.3. One-valued classical solvability of (1.1)-(1.2) can be provided by more general assumptions on $\partial \Omega^\varepsilon$ and $\Phi^\varepsilon$. This will be discussed in Theorem 6.3 (see Subsection 6.6).

The proof of Theorem 3.1, which is rather technical, will be postponed to Section 6.
4. Asymptotic Behavior of the Classical Solution to (1.1)-(1.2)

Our next results are concerned the asymptotic behavior of the solution \( p^\varepsilon \) as \( \varepsilon \to 0 \). First, being within assumptions of Theorem 3.1, we rewrite problem (1.1)-(1.2) in more convenient form for studying its asymptotic behavior. To this end, we represent \( \Gamma^\varepsilon(t) \) as

\[
\Gamma^\varepsilon(t) = \{ y = (y_1, y_2) \in \mathbb{R}^2 : y_1 \in [0, l], \quad y_2 = \varepsilon S(y_1, t) \} \quad \forall \, t \in [0, T],
\]

where \( S(y_1, t) = 1 + \varepsilon^{-1} \rho(y_1, t), \, \, \, \, \, \, \, y_1 \in [0, l], \, \, \, \, \, \, t \in [0, T], \) is a new unknown function. Due to (3.1) and \((\text{h1})\)

\[
S(y_1, 0) = 1 \quad \text{for all} \quad y_1 \in [0, l], \quad \text{and} \quad |S(y_1, t)| < \frac{6}{5}. \tag{4.2}
\]

Below we present the important property of the function \( S \), which will be used in the asymptotic analysis of our problem.

**Corollary 4.1.** Under conditions of Theorem 3.1, for all \( t \in [0, T] \), the unknown function \( S \) admits representations

\[
S(y_1, t) = \begin{cases} 
S_0(t) & \text{in a } \delta\text{-neighborhood of } y_1 = 0, \\
S_1(t) & \text{in a } \delta\text{-neighborhood of } y_1 = l,
\end{cases}
\]

where \( \delta \) is a small positive number. Besides,

\[
S_0(t) \geq 1 \quad \text{and} \quad S_1(t) \geq 1 \quad \forall \, t \in [0, T].
\]

The proof of this statement is a simple consequence of Theorem 3.1 and it bases on the homogenous Dirichlet condition on \( \Gamma^\varepsilon(t) \) in problem (1.2) and on the property of the function \( \Phi^\varepsilon \) (recall that assumption \((\text{h3})\) provides homogenous Neumann conditions near the contact points of free and fixed boundaries).

**Remark 4.1.** The statement of Corollary 4.1 means that the geometry of the free boundary in \( \delta\)-neighborhoods of the corner points \((0, S_0(t), t), \quad (l, S_1(t), t)\), preserves for each \( t \in [0, T] \). In forthcoming Corollary 4.2, under some additional assumptions on the given data, we estimate the value \( \delta \) through the support of the function \( \Phi^\varepsilon|_{y_2=0} \) and obtain an explicit formula for unknown functions \( S_1(t) \) and \( S_0(t) \).

Obviously, that for each \( t \in [0, T] \) on the boundary \( \Gamma^\varepsilon(t) \) we have

\[
R(y, t) := y_2 - \varepsilon S(y_1, t) = 0. \tag{4.3}
\]

This equality with straightforward calculations arrive at the following relations on the free boundary:

\[
\frac{\partial p^\varepsilon}{\partial n_i} = \sum_{i=1}^{2} \frac{\partial p^\varepsilon}{\partial y_i} n_i^i = \frac{1}{|\nabla_y R|} \sum_{i=1}^{2} \frac{\partial p^\varepsilon}{\partial y_i} \frac{\partial R}{\partial y_i} \quad \text{and} \quad V_n = \sum_{i=1}^{2} \frac{\partial y_i}{\partial t} n_i^i = \frac{\varepsilon}{|\nabla_y R|} \frac{\partial S}{\partial t},
\]

where \( \nabla_y R = (-\varepsilon \frac{\partial S}{\partial y_1}, 1) \).

Thus, the Stefan condition on the free boundary can be rewritten as

\[
\frac{\partial p^\varepsilon}{\partial y_2} = \varepsilon \frac{\partial S}{\partial y_1} \frac{\partial p^\varepsilon}{\partial y_1} - \varepsilon \gamma \frac{\partial S}{\partial t}. \tag{4.4}
\]

Since Theorem 3.1 provides the one-to-one classical solvability of (1.1)-(1.2), we can integrate the first condition on the moving boundary along \( \Gamma^\varepsilon(t) \) (here we keep in mind the line integral of the first kind). As
a result, the classical solution \((p^\varepsilon, S)\) of (1.1)-(1.2) satisfies the problem

\[
\begin{aligned}
\Delta_y p^\varepsilon &= 0 \quad \text{in} \quad \Omega^\varepsilon(t), \quad t \in (0, T), \\
\frac{\partial p^\varepsilon}{\partial y_2} &= \varepsilon \frac{\partial S}{\partial y_1} - \varepsilon \gamma \frac{\partial S}{\partial t} \quad \text{on} \quad \Gamma^\varepsilon(t), \quad t \in [0, T], \\
\int_{\Gamma^\varepsilon(t)} p^\varepsilon \, dl &= 0, \quad t \in [0, T], \\
\frac{\partial p^\varepsilon}{\partial n} &= \Phi^\varepsilon(y, t) \quad \text{on} \quad \partial \Omega^\varepsilon(t) \setminus \Gamma^\varepsilon(t), \quad t \in [0, T], \\
S(y_1, 0) &= 1 \quad \text{on} \quad [0, l].
\end{aligned}
\]

(4.5)

Usually, to construct the asymptotic approximation for a solution to a boundary-value problem, more smoothness is required for the initial data. In our case, they are

\((h5): \quad \varphi_1 \in C([0, T]; C^3[0, 1]), \quad \varphi_2 \in C([0, T]; C^3[0, l]), \quad \varphi_3 \in C([0, T]; C^3[0, 1])).

**Theorem 4.1.** Let assumptions \((h1), (h3), (h4)\) and \((h5)\) hold. Then there exist positive constants \(C_0\) and \(\varepsilon_0\), such that for each \(\varepsilon \in (0, \varepsilon_0)\) the free boundary \(\Gamma^\varepsilon(t)\) is uniquely determined by means of the function

\[
S(y_1, t) = 1 + \frac{1}{\gamma} \int_0^t \chi_1(y_1) \varphi_2(y_1, \tau) d\tau + \frac{1}{\gamma} \int_0^t \int_0^1 \chi_2(\xi_2)[\varphi_3(\xi_2, \tau) + \varphi_1(\xi_2, \tau)] d\xi_2, \quad t \in [0, T],
\]

(4.6)

and the following inequality holds:

\[
\|p^\varepsilon - P^\varepsilon\|_{C([0, T]; H^1(\Gamma^\varepsilon(t)))} \leq C_0 \varepsilon
\]

(4.7)

with \(p^\varepsilon\) being the classical solution to problem (1.1)-(1.2) provided by Theorem 3.1, while the approximation function

\[
P^\varepsilon(y, t) = w_0(y_1, t) + \varepsilon^2 w_2(y_1, y_2/\varepsilon, t), \quad y \in \Omega^\varepsilon(t), \quad t \in [0, T].
\]

Here, for each \(t \in [0, T]\) the functions \(w_0\) and \(w_2\) are unique classical solutions to the problems

\[
\begin{aligned}
&\frac{\partial}{\partial y_1} \left( S(y_1, t) \frac{\partial w_0}{\partial y_1}(y_1, t) \right) = \gamma \frac{\partial S}{\partial t}(y_1, t) - \chi_1(y_1) \varphi_2(y_1, t), \quad y_1 \in (0, l), \\
&\frac{\partial w_0}{\partial y_1}(0, t) = \langle \chi_2(\cdot) \varphi_2(\cdot, t) \rangle_{S_0}, \quad \frac{\partial w_0}{\partial y_1}(l, t) = \langle \chi_2(\cdot) \varphi_3(\cdot, t) \rangle_{S_1},
\end{aligned}
\]

(4.8)

\[
\int_{\Gamma^\varepsilon(t)} w_0 \, dl = 0;
\]

\[
- \frac{\partial^2 w_2}{\partial \xi_2^2}(y_1, \xi_2, t) = \frac{\partial^2 w_0}{\partial y_1^2}(y_1, t), \quad \xi_2 \in (0, S(y_1, t)), \quad y_1 \in (0, l),
\]

\[
\frac{\partial w_2}{\partial \xi_2}(y_1, S(y_1, t), t) = \frac{\partial S}{\partial y_1} \frac{\partial w_0}{\partial y_1} - \gamma \frac{\partial S}{\partial t}, \quad w_2(y_1, S(y_1, t), t) = 0, \quad y_1 \in [0, l],
\]

\[
\frac{\partial w_2}{\partial y_1}(y_1, 0, t) = -\chi_1(y_1) \varphi_2(y_1, t),
\]

respectively, where \(|\Gamma^\varepsilon(t)|\) denotes the length of the curve \(\Gamma^\varepsilon(t)\) and the value \(\langle \cdot \rangle_S\) is determined in (2.1).
The proof of this statement is given in Section 7. Collecting Theorem 4.1, Corollary 4.1 and Remark 4.1, we get the key property of the free boundary \( \Gamma^\varepsilon(t) \) (see (4.1)).

**Corollary 4.2.** Under condition of Theorem 4.1, the free and fixed boundaries in problem (1.1)-(1.2) form right angles in \( \delta \)-neighborhoods of the corner points \((0, S_0(t), t), (1, S_1(t), t)\), respectively, for \( \delta = \frac{1}{5} \), \( t \in [0, T] \) and \( \varepsilon \in (0, \varepsilon_0) \). Besides, it follows from (4.6) that

\[
S_0(t) = S_1(t) = 1 + \frac{1}{4\gamma} \int_0^t d\tau \int_0^1 \chi_2(\xi_2)[\varphi_3(\xi_2, \tau) + \varphi_1(\xi_2, \tau)]d\xi_2, \quad t \in [0, T].
\]

**5. Some additional results**

First, we describe some properties (which will be very useful in our analysis in Section 6) of the eigenvalues and eigenfunction to the following spectral problems.

\[
\begin{cases}
-\psi''(x) = \lambda \psi(x), & x \in (0, a), \\
\psi'(0) = \psi'(a) = 0,
\end{cases} \quad \text{and} \quad \begin{cases}
-\psi''(x) = \mu \psi(x), & y \in (0, a), \\
\psi'(0) = \psi'(a) = 0.
\end{cases}
\]

Obviously that eigenvalues of the spectral problems are equal to

\[
\lambda_m := \lambda_m(a) = \left(\frac{\pi m}{a}\right)^2 \quad \text{and} \quad \mu_m := \mu_m(a) = \left(\frac{\pi (m + 1/2)}{a}\right)^2 \quad m \in \mathbb{N}_0,
\]

respectively, and the corresponding eigenfunctions

\[
\psi_{\lambda_m} = \frac{1}{\sqrt{a}}, \quad \psi_{\lambda_m}(x) = \sqrt{\frac{2}{a}} \cos \sqrt{\lambda_m}x \quad \text{if } m \neq 0, \quad \text{and} \quad \psi_{\mu_m}(x) = \sqrt{\frac{2}{a}} \cos \sqrt{\mu_m}x, \quad m \in \mathbb{N}_0, \quad x \in [0, a],
\]

satisfy the following relations:

\[
\|\psi_{\lambda_m}\|_{L^2(0, a)} = \|\psi_{\mu_m}\|_{L^2(0, a)} = 1, \quad \langle \psi_{\lambda_m}, \psi_{\lambda_n} \rangle_a = 0, \quad \langle \psi_{\mu_m}, \psi_{\mu_n} \rangle_a = 0 \quad \text{if } m \neq n.
\]

For any function \( g \in L^2(0, a) \), we notate by

\[
g_{\lambda_m} := \langle g, \psi_{\lambda_m} \rangle_a \quad \text{and} \quad g_{\mu_m} := \langle g, \psi_{\mu_m} \rangle_a, \quad m \in \mathbb{N}_0,
\]

Fourier coefficients regarding the basis \( \{\psi_{\lambda_m}\}_{m \in \mathbb{N}_0} \) and \( \{\psi_{\mu_m}\}_{m \in \mathbb{N}_0} \), respectively.

The next property demonstrates the correlation between the smoothness of the function \( g \) and the behavior of its Fourier coefficients \( \{g_{\lambda_m}\} \) and \( \{g_{\mu_m}\} \).

**Proposition 5.1.** Let \( g \in C^{2+\alpha}([0, a]) \) and

\[
g(0) = g(a) = 0 \quad \text{and} \quad g'(0) = g'(a) = 0.
\]

Then the series

\[
\sum_{m=1}^{+\infty} g_{\lambda_m} \psi_{\lambda_m}(y) \quad \text{and} \quad \sum_{m=0}^{+\infty} g_{\mu_m} \psi_{\mu_m}(y)
\]

absolutely and uniformly converge on \([0, a]\).

Besides, for \( k \in \{0, 1, 2\} \), the inequalities are fulfilled

\[
\sum_{m=1}^{\infty} |g_{\lambda_m}| \left(\lambda_m\right)^{\frac{k-1}{2}} \leq C\|g\|_{C^{2+\alpha}([0, a])}, \quad \sum_{m=0}^{\infty} |g_{\mu_m}| \left(\mu_m\right)^{\frac{k-1}{2}} \leq C\|g\|_{C^{2+\alpha}([0, a])}.
\]

In addition, if \( \alpha \in \left(\frac{1}{2}, 1\right) \), then estimates (5.4) hold for \( k = 3 \).

**Proof.** The first statement is a simple consequence of the Fourier series theory (see e.g. Chapters I-IV in [7]). Next, we will carry out the detailed proof of inequality (5.4) in the coefficients \( \{g_{\lambda_m}\} \). The proof for \( \{g_{\mu_m}\} \) is the same.
Taking into account the smoothness of \( g \), conditions (5.3) and integrating twice by parts in the representation of \( g_{\lambda_m} \), we conclude

\[
|g_{\lambda_m}| \leq C \left| \int_0^\alpha g''(y) \cos \frac{\lambda_m y}{\lambda_m} dy \right| = C g''_{\lambda_m} \left( \frac{\lambda_m}{\lambda_m} \right) \leq C \|g''\|_{C([0,\alpha])}, \quad m \in \mathbb{N},
\]

where \( g''_{\lambda_m} = \langle g''(\cdot), \psi_{\lambda_m} \rangle_a \). These inequalities arrive at the estimate

\[
\sum_{m=1}^{+\infty} |g_{\lambda_m}|(\lambda_m)^{k-\frac{3}{2}} \leq \sum_{m=1}^{+\infty} |g''_{\lambda_m}|(\lambda_m)^{k-1}
\]

for \( k \in \{0, 1, 2, 3\} \). Then, the straightforward calculations provide the inequality

\[
\sum_{m=1}^{+\infty} |g''_{\lambda_m}|(\lambda_m)^{k-\frac{3}{2}} \leq C\|g''\|_{C([0,\alpha])} \sum_{m=1}^{+\infty} m^{k-3} \leq C\|g\|_{C^2([0,\alpha])},
\]

for \( k \in \{0, 1\} \), and as a result, estimate (5.4) for those values of \( k \).

From [7, Ch. 2] it follows the inequalities

\[
\sum_{m=1}^{+\infty} |g''_{\lambda_m}|(\lambda_m)^{k-\frac{3}{2}} \leq C \sqrt{\sum_{m=1}^{+\infty} |g''_{\lambda_m}|^2} \sqrt{\sum_{m=1}^{+\infty} m^{-2}}
\]

if \( k = 2 \),

\[
\sum_{m=1}^{+\infty} |g''_{\lambda_m}|(\lambda_m)^{k-\frac{3}{2}} \leq C \sum_{m=1}^{+\infty} m^{-1} (\lambda_m)^{k-3}
\]

if \( k = 3 \),

whence

\[
\sum_{m=1}^{+\infty} |g''_{\lambda_m}|(\lambda_m)^{\frac{3}{2}} \leq C\|g\|_{C^2([0,\alpha])}.
\]

It is worth noting that the last estimate is true for \( k = 3 \) if \( \alpha \in (1/2, 1) \) (see Corollary 2 [7, Ch. 2]). \( \square \)

Let us consider the smooth cut-off function \( \chi \in C^\infty_0(\mathbb{R}) \) such that \( 0 \leq \chi \leq 1 \) and

\[
\chi(y) = \begin{cases} 
0, & y \in (-\infty, a/5] \cup [4a/5, +\infty), \\
1, & y \in [2a/5, 3a/5].
\end{cases}
\]

**Remark 5.1.** Clearly that the function \( g = \chi g_0 \), where \( g_0 \in C^{2+\alpha}([0,\alpha]) \), meets all the requirements of Proposition 5.1.

We conclude this preliminary section with an analogue of the Poincaré inequality which will play a key point in the proof of Theorem 4.1 (see Section 7.2).

**Lemma 5.1.** Let the domain \( \Omega^\varepsilon(t) = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 \in (0, t), 0 < y_2 < \varepsilon S(y_1, t)\} \) have a Lipschitz boundary for each \( t \in [0, T] \), and let the given function \( S \in C([0, T]; C^1([0, t])) \) define the curve \( \Gamma^\varepsilon(t) \) according to formula (4.1). Then there is a constant \( C \) such that for all \( \varepsilon \in (0, 1) \) and \( t \in [0, T] \) the inequality

\[
\|\mathcal{W}\|_{L^2(\Omega^\varepsilon(t))} \leq C \|\nabla_y \mathcal{W}\|_{L^2(\Gamma^\varepsilon(t))}
\]

holds for every function \( \mathcal{W} \in H^1(\Omega^\varepsilon(t)) \) such that \( \int_{\Gamma^\varepsilon(t)} \mathcal{W} d\ell = 0 \).

**Proof.** We fixate \( t \in [0, T] \) and prove first this lemma for a smooth function \( \mathcal{W} \in C^1(\overline{\Omega^\varepsilon(t)}) \). Taking into advantage of the easily verified identity

\[
\mathcal{W}(y_1, \varepsilon S(y_1, t)) = \int_{y_2}^{\varepsilon S(y_1, t)} \frac{\partial^2 \mathcal{W}}{\partial z^2}(y_1, z) dz + \mathcal{W}(y_1, y_2) \quad \forall (y_1, y_2) \in \Omega^\varepsilon(t),
\]

we deduce the inequalities

\[
\int_{\Omega^\varepsilon(t)} \mathcal{W}^2 dy \leq C \left( \varepsilon^2 \int_{\Omega^\varepsilon(t)} \left( \frac{\partial \mathcal{W}}{\partial y_2} \right)^2 dy + \varepsilon \int_{\Gamma^\varepsilon(t)} \mathcal{W}^2 d\ell \right),
\]
\[
\mathcal{P}^2(y_1, \varepsilon \mathcal{S}(y'_1, t)) - 2 \mathcal{P}(y_1, \varepsilon \mathcal{S}(y'_1, t)) \mathcal{P}(y_1, \varepsilon \mathcal{S}(y_1, t)) + \mathcal{P}^2(y_1, \varepsilon \mathcal{S}(y_1, t)) \\
\leq C \left( \varepsilon \int_0^{\varepsilon \mathcal{S}(y'_1, t)} \left( \frac{\partial \mathcal{P}}{\partial z_2}(y'_1, z_2) \right)^2 d z_2 + \varepsilon \int_0^{\varepsilon \mathcal{S}(y_1, t)} \left( \frac{\partial \mathcal{P}}{\partial z_2}(y_1, z_2) \right)^2 d z_2 + \int_0^t \left( \frac{\partial \mathcal{P}}{\partial z_1}(z_1, y_2) \right)^2 d z_1 \right)
\]

(5.7)

for each \( y_1 \) and \( y'_1 \) from the interval \((0, l)\) and any \( y_2 \in (0, \varepsilon)\).

After that, we integrate inequality (5.7) with respect to \( y_2 \in (0, \varepsilon)\). Then, we multiply the obtained inequality with \( \sqrt{1 + \varepsilon^2 \left( \frac{\partial \mathcal{P}}{\partial y_1}(y'_1, t) \right)^2} \) and integrate it with respect to \( y'_1 \in (0, l)\). Finally, multiplying the newly obtained inequality with \( \sqrt{1 + \varepsilon^2 \left( \frac{\partial \mathcal{P}}{\partial y_1}(y'_1, t) \right)^2} \) and integrating with respect to \( y'_1 \in (0, l)\), we reach the estimate

\[
\varepsilon \int_{\Gamma^*(t)} \mathcal{P}^2 \, dl \leq \varepsilon \left( \int_{\Gamma^*(t)} \mathcal{P} \, dl \right)^2 + C \left( \varepsilon^2 \int_{\Omega^*(t)} \left( \frac{\partial \mathcal{P}}{\partial y_2} \right)^2 dy + \int_{\Omega^*(t)} \left( \frac{\partial \mathcal{P}}{\partial y_1} \right)^2 dy \right) \\
\leq \varepsilon \left( \int_{\Gamma^*(t)} \mathcal{P} \, dl \right)^2 + C \int_{\Omega^*(t)} |\nabla \mathcal{P}|^2 dy.
\]

(5.8)

Exploiting standard approximation procedure, we conclude that inequalities (5.6) and (5.8) hold for any function \( \mathcal{P} \in H^1(\Omega^*(t)) \). Since \( \int_{\Gamma^*(t)} \mathcal{P} \, dl = 0 \), estimate (5.5) follows from (5.6) and (5.8). \( \square \)

**Corollary 5.1.** Estimate (5.8) immediately leads to an analogue of the Poincaré–Wirtinger inequality

\[

\left\| \mathcal{P} - \frac{1}{|\Gamma^*(t)|} \int_{\Gamma^*(t)} \mathcal{P} \, dl \right\|_{L^2(\Gamma^*(t))} \leq \frac{C}{\sqrt{\varepsilon}} \| \nabla \mathcal{P} \|_{L^2(\Omega^*(t))} \quad \forall \mathcal{P} \in H^1(\Omega^*(t)),
\]

(5.9)

where the constant \( C \) is independent of \( \mathcal{P} \) and \( \varepsilon \).

**Remark 5.2.** It should be noted here that determining the optimal constant in Poincaré inequalities is, in general, a very hard task (see [1, 36]). In our case it is very important to know how constants in such inequalities depend on the parameter \( \varepsilon \) (see (5.5) and (5.9)).

### 6. Proof of Theorem 3.1

The strategy of the proof is the following: first, we show that, within our assumptions on the function \( \Phi^\varepsilon \), the initial pressure \( p_0^\varepsilon = p^\varepsilon(y, 0) \) belongs to the class \( C^{3+\alpha}(\Omega^\varepsilon) \) for any fixed \( \varepsilon > 0 \). Then, using like Hanzawa transformation [22], we reduce problem (1.2) in the domain (1.1) with the moving boundary \( \Gamma^\varepsilon(t) \) to a nonlinear problem in the fixed domain \( \Omega^\varepsilon_\Gamma \). After that, we linearize this nonlinear problem on the initial data \( p_0^\varepsilon \) and on a special function \( s(x_1, t) \) connected with the initial shape of the free boundary \( \Gamma^\varepsilon \), and solve the linear problem in \( C^{2+\alpha}(\Omega^\varepsilon_\Gamma) \). Finally, using contraction mapping theorem, we prove the local one-to-one solvability of the corresponding nonlinear problem.

### 6.1. Smoothness of the initial pressure \( p_0^\varepsilon \)

Denoting (see assumption (h3))

\[
\varphi_1(y_2, t) = -\chi_2(y_2/\varepsilon)\varphi_1(y_2/\varepsilon, t), \quad \varphi_2(y_1, t) = -\varepsilon\chi_1(y_1)\varphi_2(y_1, t), \quad \varphi_3(y_2, t) = \chi_2(y_2/\varepsilon)\varphi_3(y_2/\varepsilon, t),
\]

where
and taking into account assumptions (h1)-(h4), we conclude that the initial pressure \( p_0^\epsilon : \Omega^\epsilon \to \mathbb{R} \) solves the following boundary-value problem for each fixed \( \epsilon > 0 \):

\[
\begin{aligned}
\Delta p_0^\epsilon &= 0 \quad \text{in} \quad \Omega^\epsilon, \\
p_0^\epsilon &= 0 \quad \text{on} \quad \Gamma^\epsilon, \\
\frac{\partial p_0^\epsilon}{\partial n} &= \varphi_1(y_2, 0) \quad \text{on} \quad \Gamma_1^\epsilon, \\
\frac{\partial p_0^\epsilon}{\partial n} &= \varphi_2(y_1, 0) \quad \text{on} \quad \Gamma_2, \\
\frac{\partial p_0^\epsilon}{\partial n} &= \varphi_3(y_2, 0) \quad \text{on} \quad \Gamma_3^\epsilon.
\end{aligned}
\]  

(6.1)

Introducing new functions

\[
P_0 = \epsilon^{-1/2}(y_2 - \epsilon)\varphi_{2,0}(0), \quad P_1 = \sum_{m=1}^{\infty} \frac{\varphi_{2,m}(0) \sinh((y_2 - \epsilon)\sqrt{\lambda_m})}{\sqrt{\lambda_m}} \psi_{l,m}(y_1),
\]

\[
P_2 = \sum_{m=0}^{\infty} \frac{\varphi_{3,m}(0) \cosh(y_1 \sqrt{\mu_m}) - \varphi_{1,m}(0) \cosh(l - y_1 \sqrt{\mu_m})}{\sqrt{\mu_m}} \sinh(l \sqrt{\mu_m}) \psi_{l,m}(y_2),
\]

where \( \lambda_m = \lambda_m(\epsilon) \) and \( \mu_m = \mu_m(l) \) are defined with (5.1), and \( \varphi_{1,m}(t) = \langle \varphi_1, \psi_{l,m} \rangle \epsilon, \quad \varphi_{2,m}(t) = \langle \varphi_2, \psi_{l,m} \rangle \epsilon, \) we assert the following result.

**Lemma 6.1.** Let \( \alpha \in (0,1) \), \( \epsilon > 0 \) be arbitrarily fixed and let assumptions (h1)-(h3) hold. Then boundary-value problem (6.1) admits a unique classical solution

\[
p_0^\epsilon = P_0 + P_1 + P_2
\]

in the domain \( \Omega^\epsilon \), satisfying the regularity \( p_0^\epsilon \in C^{2+\alpha}(\Omega^\epsilon) \), and

\[
\|p_0^\epsilon\|_{C^{2+\alpha}(\Omega^\epsilon)} \leq C \|(\varphi_1)\|_{C^{2+\alpha}(\Gamma_1^\epsilon)} + \|\varphi_2\|_{C^{2+\alpha}(\Gamma_2^\epsilon)} + \|\varphi_3\|_{C^{2+\alpha}(\Gamma_3^\epsilon)}.
\]

Besides, under condition

\[
\sqrt{\frac{2}{l}} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} [\varphi_{3,m}(0) \cosh(y_1 \sqrt{\mu_m}) - \varphi_{1,m}(0) \cosh(l - y_1 \sqrt{\mu_m})]}{\sinh(l \sqrt{\mu_m})} \] 

\[
+ \sum_{m=0}^{\infty} \frac{\varphi_{2,m}(0) \psi_{l,m}(y_1)}{\cosh(\sqrt{\lambda_m})} < 0,
\]

the function \( p_0^\epsilon \) satisfies the inequality

\[
\frac{\partial p_0^\epsilon}{\partial y_2} < 0 \quad \text{on} \quad \Gamma^\epsilon,
\]

(6.4)

which provides estimate (3.4).

If in addition \( \alpha \in (1/2,1) \), then \( p_0^\epsilon \in C^3(\Omega^\epsilon) \) and

\[
\|p_0^\epsilon\|_{C^3(\Omega^\epsilon)} \leq C \|(\varphi_1)\|_{C^{2+\alpha}(\Gamma_1^\epsilon)} + \|\varphi_2\|_{C^{2+\alpha}(\Gamma_2^\epsilon)} + \|\varphi_3\|_{C^{2+\alpha}(\Gamma_3^\epsilon)}.
\]

(6.5)

**Proof.** Taking into account the smoothness of \( \varphi_i \) and using standard Fourier approach, we construct, at least formally, a solution of (6.1) in form (6.2). The direct calculations prove that \( P_0 \in C^{3+\alpha}(\Omega^\epsilon) \). Next, it is apparent that the functions \( \varphi_i \) meet requirements of Proposition 5.1, and hence for each fixed \( \epsilon > 0 \), the serieses \( P_1 \) and \( P_2 \) are convergent absolutely and uniformly in \( C^{2+\alpha}(\Omega^\epsilon) \) if \( \alpha \in (0,1) \) and in \( C^3(\Omega^\epsilon) \) if \( \alpha \in (1/2,1) \). Thus, we arrive at the estimates

\[
\sum_{j=0}^{2} \|P_j\|_{C^{2+\alpha}(\Omega^\epsilon)} \leq C \|(\varphi_1,0)\|_{C^{2+\alpha}(\Gamma_1^\epsilon)} + \|\varphi_2(\cdot,0)\|_{C^{2+\alpha}(\Gamma_2^\epsilon)} + \|\varphi_3(\cdot,0)\|_{C^{2+\alpha}(\Gamma_3^\epsilon)}
\]

\[
\leq C \|\chi_1 \varphi_1\|_{C^{2+\alpha}(\Gamma_1^\epsilon)} + \|\chi_2 \varphi_2\|_{C^{2+\alpha}(\Gamma_2^\epsilon)} + \|\chi_3 \varphi_3\|_{C^{2+\alpha}(\Gamma_3^\epsilon)} \quad \text{if} \quad \alpha \in (0,1),
\]

\[
\sum_{j=0}^{2} \|P_j\|_{C^3(\Omega^\epsilon)} \leq C \|\chi_1 \varphi_1\|_{C^{2+\alpha}(\Gamma_1^\epsilon)} + \|\chi_2 \varphi_2\|_{C^{2+\alpha}(\Gamma_2^\epsilon)} + \|\chi_3 \varphi_3\|_{C^{2+\alpha}(\Gamma_3^\epsilon)} \quad \text{if} \quad \alpha \in (1/2,1)
\]
with the constant $C$ is independent of $\varepsilon$ for $\varepsilon \in (0, 1)$.

Finally, the straightforward calculations together with the obtained regularity of $P_j$, provides that the function $P_0 + P_1 + P_2$ satisfies the equation and the boundary conditions in (6.1). Thus, representation (6.2) and estimates of $P_j$ provide coercive estimates for $P_0$, in particular (6.5). Uniqueness of the constructed solution follows immediately from the coercive estimate for $P_0$.

At last, to finish the proof of Lemma 6.1, we are left to verify (6.4). The direct calculations and representation (6.2) arrive at

$$\frac{\partial p_0}{\partial y_2} = \sum_{j=0}^{2} \frac{\partial P_j}{\partial y_2} = \varepsilon^{-1/2} \tilde{\varphi}_{2,0}(0) + \sum_{m=1}^{\infty} \tilde{\varphi}_{2,m}(0) \frac{\cosh((y_2 - \varepsilon)\sqrt{\lambda_m})}{\cosh(\varepsilon \sqrt{\lambda_m})} \psi_{\lambda_m}(y_1)
- \sum_{m=0}^{\infty} \left[ \tilde{\varphi}_{3,m}(0) \cosh(y_1 \sqrt{\mu_m}) - \tilde{\varphi}_{1,m}(0) \cosh((y_1 - \varepsilon) \sqrt{\mu_m}) \right] \psi_{\mu_m}(y_2).$$

Then, substituting $y_2 = \varepsilon$ to this representation and taking into account (6.3), we end up with estimate (6.4).

Besides, the second boundary condition in (1.2) on $\Gamma^\varepsilon(t)$ together with (6.4) provide for $t = 0$

$$V_n \big|_{\Gamma^\varepsilon} = -\gamma^{-1} \frac{\partial p_0}{\partial n} > 0$$

if (6.3) holds. This completes the proof of this statement. \qed

At this point, we show that the constructed solution in Lemma 6.1 is more regular. To this end, it is enough to apply Theorem 3.1 [48] to (6.1).

**Lemma 6.2.** Let $\alpha \in (0, 1)$ and assumptions (h1)-(h3) hold. Then the classical solution $p_0^\varepsilon$ belongs to $C^{3+\alpha}(\Omega^\varepsilon)$ and $C^{2+\alpha}(\Gamma^\varepsilon)$ and

$$\|p_0^\varepsilon\|_{C^{3+\alpha}(\Omega^\varepsilon)} \leq C[\|\chi_2 \varphi_1\|_{C^{2+\alpha}(\Gamma^\varepsilon^\varepsilon)} + \|\chi_1 \varphi_2\|_{C^{2+\alpha}(\Gamma^\varepsilon^\varepsilon)} + \|\chi_2 \varphi_3\|_{C^{2+\alpha}(\Gamma^\varepsilon^\varepsilon)}],$$

where the constant $C$ is independent of $\varepsilon$ if $\varepsilon \in (0, 1)$.

Besides, the function $s(y_1, t) = \frac{-2}{\gamma} \frac{\partial p_0}{\partial y_2} \big|_{\Gamma^\varepsilon}$ satisfies relations

$$s(y_1, 0) = 0, \quad \frac{\partial s}{\partial y_2}(y_1, 0) = V_n \big|_{t=0} \quad \text{on} \quad \Gamma^\varepsilon,$$

and

$$\|s\|_{C^{2+\alpha}(\Gamma^\varepsilon)} + \|\partial s/\partial t\|_{C^{2+\alpha}(\Gamma^\varepsilon)} \leq C[\|\chi_2 \varphi_1\|_{C^{2+\alpha}(\Gamma^\varepsilon^\varepsilon)} + \|\chi_1 \varphi_2\|_{C^{2+\alpha}(\Gamma^\varepsilon^\varepsilon)} + \|\varphi_3\|_{C^{2+\alpha}(\Gamma^\varepsilon^\varepsilon)}].$$

Note that the statements related to the function $s$ follow immediately from the properties of the function $p_0^\varepsilon$.

6.2. **Reducing problem (1.1)-(1.2) to a nonlinear problem in $\Omega^\varepsilon \times (0, T)$.** Denoting the cut-off function by $\chi(\Lambda) \in C_0^\infty(\mathbb{R})$ such that $0 \leq \chi(\Lambda) \leq 1$ and

$$\chi(\Lambda) = \begin{cases} 1, & \text{if} \quad |\Lambda| < \varepsilon/15, \\ 0, & \text{if} \quad |\Lambda| > 2\varepsilon/15, \end{cases}$$

we introduce the new coordinate

$$y_1 = x_1 \quad \text{and} \quad y_2 = x_2 + \rho(x_1, t) \chi(\Lambda),$$

with $\Lambda = x_2 - \varepsilon$.

It is apparent that (see for details, e.g., Section 3 in [46] or Section 6 in [4]), this transformation reduces the domain $\Omega^\varepsilon(t), \ t \in (0, T)$, to the fixed domain $\Omega^\varepsilon_T = \Omega^\varepsilon \times (0, T)$.

After that, introducing a new unknown function

$$v = v(x_1, x_2, t) = p^\varepsilon(x_1, y_2(x_1, x_2, t), t),$$
we rewrite the equation in (1.2) in the new function and variables
\[
\Delta v + 2 \frac{\partial x_2}{\partial y_1} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \left[ \left( \frac{\partial x_2}{\partial y_1} \right)^2 + \left( \frac{\partial x_2}{\partial y_2} \right)^2 - 1 \right] \frac{\partial^2 v}{\partial x_2^2} + \left( \frac{\partial^2 x_2}{\partial y_1^2} + \frac{\partial^2 x_2}{\partial y_2^2} \right) \frac{\partial v}{\partial x_2} = 0 \quad \text{in} \quad \Omega_T^\varepsilon, \quad (6.6)
\]
where we set
\[
\begin{cases}
\frac{\partial x_1}{\partial y_1} = 1, & \frac{\partial x_1}{\partial y_2} = 0, \\
\frac{\partial x_2}{\partial y_1} = -\frac{\chi \frac{\partial p^\varepsilon}{\partial x_1}}{1 + \chi' \rho}, & \frac{\partial x_2}{\partial y_2} = \frac{1}{1 + \chi' \rho}, \\
\frac{\partial^2 x_2}{\partial y_1^2} = -\frac{\chi'' \rho}{(1 + \chi' \rho)^3}, \\
\frac{\partial^2 x_2}{\partial y_2^2} = \frac{\chi \chi'' \rho \frac{\partial p^\varepsilon}{\partial x_1}}{(1 + \chi' \rho)^3} + \frac{\chi [\chi' (\frac{\partial p^\varepsilon}{\partial x_1})^2 - \frac{\partial p^\varepsilon}{\partial x_2} (2 + \chi' \rho)]}{(1 + \chi' \rho)^2}.
\end{cases} \quad (6.7)
\]

At this point, we begin to rewrite the conditions on the free boundary in the new variables. Recasting the arguments from Section 4 leading to representation (4.4), we deduce that the Stefan condition of the moving boundary has the form
\[
\gamma \frac{\partial p^\varepsilon}{\partial t} = \frac{\partial p^\varepsilon}{\partial y_1} \frac{\partial v}{\partial x_1} - \frac{\partial p^\varepsilon}{\partial y_2} \frac{\partial v}{\partial x_2}.
\]

As a result, taking into account (6.7), we can can rewrite the boundary conditions on \( \Gamma^\varepsilon(t) \) in the form
\[
\begin{cases}
v(x_1, x_2, t) = 0 \quad \text{on} \quad \Gamma^\varepsilon_T, \\
\gamma \frac{\partial p^\varepsilon}{\partial t} = \frac{\partial p^\varepsilon}{\partial x_1} \frac{\partial v}{\partial x_1} - \frac{1 + \left( \frac{\partial p^\varepsilon}{\partial x_1} \right)^2}{\partial x_2} \frac{\partial v}{\partial x_2} \quad \text{on} \quad \Gamma^\varepsilon_T, \\
\rho(x_1, 0) = 0 \quad \text{in} \quad [0, t].
\end{cases} \quad (6.8)
\]

Finally, in virtue of assumptions (h2)-(h3) and the definition of \( \chi(\Lambda) \), the rest boundary conditions in (1.2) remain unchanged:
\[
\begin{cases}
\frac{\partial v}{\partial x_1} = -\chi_2(x_2/\varepsilon)\varphi_1(x_2/\varepsilon, t) \quad \text{on} \quad \Gamma^\varepsilon_1, T, \\
\frac{\partial v}{\partial x_2} = -\varepsilon \chi_1(x_1)\varphi_2(x_1, t) \quad \text{on} \quad \Gamma^\varepsilon_2, T, \\
\frac{\partial v}{\partial x_1} = \chi_2(x_2/\varepsilon)\varphi_3(x_2/\varepsilon, t) \quad \text{on} \quad \Gamma^\varepsilon_3, T.
\end{cases} \quad (6.9)
\]

Summing up, we can reformulate Theorem 3.1 as follows.

**Theorem 6.1. (Reformulated Theorem 3.1)** Let conditions of Theorem 3.1 hold. Then for some small \( T \) and each fixed positive \( \varepsilon \), there exists a unique solution \((v(x_1, x_2, t), p(x_1, t))\) of nonlinear problem (6.6)-(6.9) satisfying regularity \( v \in C^{2+\alpha}(\Omega_T^\varepsilon), \quad \rho(x_1, t) \in C^{2+\alpha}(\Gamma^\varepsilon_T) \). Besides,
\[
v(x_1, x_2, 0) = p_0^\varepsilon(x_1, x_2) \quad \text{in} \quad \Omega^\varepsilon, \quad (6.10)
\]
where \( p_0^\varepsilon \) is given with (6.1).

Thus the proof of Theorem 3.1 is equivalent to the one of Theorem 6.1. The rest part of this section is devoted to the proof of Theorem 6.1. It is worth mentioning that equality (6.10) follows immediately from (6.6)-(6.8).
6.3. A perturbation form of system (6.6)-(6.9). In this subsection, we linearize system (6.6)-(6.9) on the initial data and rewrite the one in the form

$$\mathcal{A}z = Fz,$$

where $\mathcal{A}$ is a linear operator, while $F$ is a nonlinear perturbation. To this end, we introduce new unknown functions

$$\begin{cases}
\sigma = \sigma(x_1, t) = \rho(x_1, t) - s(x_1, t), \\
u = u(x_1, x_2, t) = v(x_1, x_2, t) - p_0(x_1, x_2) - \chi(A) \frac{\partial p_0}{\partial x_2}(x_1, x_2)\sigma(x_1, t),
\end{cases}$$

where $p_0$ solves problem (6.1) and $s$ is defined in Lemma 6.2.

Then, substituting (6.11) to (6.6)-(6.9), we end up with

$$\begin{cases}
\Delta u = F_0(u, \sigma) \quad \text{in} \quad \Omega_T, \\
u = -u_x \sigma \quad \text{on} \quad \Gamma_T, \\
\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x_2} = F_1(u, \sigma) \quad \text{on} \quad \Gamma_T, \\
\frac{\partial u}{\partial x_1} = 0 \quad \text{on} \quad \Gamma_{13}, \\
\frac{\partial \sigma}{\partial x_2} = 0 \quad \text{on} \quad \Gamma_{23}, \\
\sigma(x_1, 0) = 0 \quad \text{in} \quad [0, 1], \\
u(x_1, x_2, 0) = 0 \quad \text{in} \quad \bar{\Omega},
\end{cases}$$

(6.12)

where we put

$$-F_0(u, \sigma) = \left[2 \frac{\partial^2 \sigma}{\partial y_1 \partial x_2} \frac{\partial^2 u}{\partial y_1 \partial x_2} + \frac{\partial \sigma}{\partial x_2} \left(\left(\frac{\partial \sigma}{\partial y_1}\right)^2 + \left(\frac{\partial \sigma}{\partial y_2}\right)^2 - 1\right) \frac{\partial^2 u}{\partial x_2} + 
\right.$$

$$\left.+ \left(\frac{\partial^2 \sigma}{\partial y_1} + \frac{\partial^2 \sigma}{\partial y_2}\right) \frac{\partial u}{\partial x_2} \left(\frac{\partial \sigma}{\partial x_2} + \chi \frac{\partial p_0}{\partial x_2}\right) + \Delta \left(\chi \frac{\partial p_0}{\partial x_2}\right)
\right].$$

with $\frac{\partial u}{\partial y_1}$, $\frac{\partial^2 u}{\partial y_1 \partial y_2}$, are given by (6.7) and depended on $\sigma$ through relation (6.11),

$$F_1(u, \sigma) = \left[-1 + \left(\frac{\partial \sigma}{\partial x_1} + \frac{\partial s}{\partial x_1}\right)^2 \right] \frac{\partial^2 u}{\partial x_2} + \left(\frac{\partial \sigma}{\partial x_1} + \frac{\partial s}{\partial x_1}\right)^2 \left(\frac{\partial u}{\partial x_2} + \frac{\partial \sigma}{\partial x_2}\right).$$

Thus, system (6.6)-(6.9) is written in the short convenient form

$$\mathcal{A}z = Fz, \quad z = (u, \sigma).$$

Based on Lemma 6.2, boundary conditions in (6.12) and representations of $F_0$ and $F_1$, we assert the following result.

**Corollary 6.1.** The functions $F_0(u, \sigma)$ and $F_1(u, \sigma)$ contain the higher derivatives of $u$ and $\sigma$ with coefficients that tends to zero as $t \to 0$; the “quadratic” terms with respect to $u$ and $\sigma$, and their derivatives; and the terms of minor differential orders of unknown functions. Besides,

$$F_0(u, \sigma) \big|_{t=0} = F_1(u, \sigma) \big|_{t=0} = 0,
$$

$$F_0(u, \sigma) = 0 \quad \text{at} \quad (x_1, x_2) \in \{(0, 0); (0, \varepsilon); (l, 0); (l, \varepsilon)\}, \quad t \in [0, T],
$$

$$F_1(u, \sigma) = 0 \quad \text{at} \quad (x_1, x_2) \in \{(0, \varepsilon); (l, \varepsilon)\}, \quad t \in [0, T],$$

$$\quad \text{and} \quad$$

$$\quad \text{for} \quad t \in [0, T],$$

$$\quad \text{and} \quad$$

$$\quad \text{for} \quad t \in [0, T].$$
In the next step, we show the boundedness of the linear operator \( A \) in the corresponding functional spaces. To this end, freezing the functional arguments in the functions \( F_0(u, \sigma) \) and \( F_1(u, \sigma) \), we obtain from (6.12) the linear system with variable coefficients, which will be analyzed in detail in Subsection 6.5. It is worth mentioning that the model problem with a dynamic boundary condition plays a key point in the investigation of this linear system.

6.4. Model problem in the right angle. In order to construct the model problem near the boundary \( \Gamma_{cT} \) by using the Schauder approach, it is necessary to fix the coefficients of the original problem at the boundary point. In this section, we study the boundary-value problem with a dynamic boundary condition in the right angle. Namely, let \( \mathcal{C}_0 \) be some positive number and

\[
\mathcal{R} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \quad x_2 > 0\}, \quad \mathcal{R}_T = \mathcal{R} \times (0, T),
\]

\[
\mathcal{R}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, \quad x_2 \geq 0\}, \quad \mathcal{R}_{1,T} = \mathcal{R}_1 \times [0, T],
\]

\[
\mathcal{R}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0, \quad x_1 \geq 0\}, \quad \mathcal{R}_{2,T} = \mathcal{R}_2 \times [0, T],
\]

We consider the initial-boundary problem in the unknown function \( U = U(x, t) : \mathcal{R}_T \rightarrow \mathbb{R} \)

\[
\begin{align*}
\Delta U &= 0 \quad \text{in} \quad \mathcal{R}_T, \\
\frac{\partial U}{\partial x_1} &= 0 \quad \text{on} \quad \mathcal{R}_{1,T}, \\
\frac{\partial U}{\partial x_2} - \mathcal{C}_0 \frac{\partial U}{\partial t} &= f(x_1, t) \quad \text{on} \quad \mathcal{R}_{2,T}, \\
U &= 0 \quad \text{if} \quad |x| \rightarrow \infty, \quad t \in [0, T], \\
U(x, 0) &= 0 \quad \text{in} \quad \mathcal{R},
\end{align*}
\]

where \( f \) is a given function satisfying conditions (h6): for some positive number \( r \), \( f \equiv 0 \) if either \( t \leq 0 \) or \( |x| > r \), and \( f \in C^{1+\alpha}(\mathcal{R}_{2,T}) \).

Lemma 6.3. Under assumption (h6) problem (6.1) admits a unique classical solution \( U \) in \( \mathcal{R}_T \) satisfying regularity \( U \in C^{2+\alpha}(\mathcal{R}_T) \) and \( \partial U / \partial t \in C^{1+\alpha}(\mathcal{R}_{2,T}) \). Besides, the estimate holds

\[
\|U\|_{C^{2+\alpha}(\mathcal{R}_T)} + \|\partial U / \partial t\|_{C^{1+\alpha}(\mathcal{R}_{2,T})} \leq C \|f\|_{C^{1+\alpha}(\mathcal{R}_{2,T})}.
\]

Proof. First of all, taking into account the homogenous Neumann boundary condition on \( \mathcal{R}_{1,T} \) (which can be considered as a symmetry condition), we can study instead of problem (6.13) in the right angle \( \mathcal{R}_T \) the similar problem in the upper semi-space

\[
\mathcal{R}_T^1 = \mathbb{R}_+^2 \times (0, T), \quad \mathcal{R}_T^2 = \{(x_1, x_2) \in \mathbb{R}^2 : \quad x_1 \in \mathbb{R}, \quad x_2 > 0\}.
\]

Namely, introducing a new function

\[
F(x_1, t) = \begin{cases} 
  f(x_1, t), & x_1 \geq 0, \\
  f(-x_1, t), & x_1 < 0,
\end{cases}
\]
we consider the new problem in the unknown function \( U = U(x, t) : \mathbb{R}^2_{+T} \to \mathbb{R} : \)

\[
\begin{align*}
\Delta U &= 0 \quad \text{in} \quad \mathbb{R}^2_{+T}, \\
\frac{\partial U}{\partial t} - C_0 \frac{\partial U}{\partial x_2} &= F(x_1, t), \quad \text{on} \quad \mathbb{R}_T, \\
U &= 0 \quad \text{if} \quad |x| \to \infty, \quad t \in [0, T], \\
U(x, 0) &= 0 \quad \text{in} \quad \mathbb{R}^2_{+T}.
\end{align*}
\]

(6.14)

It is apparent that,

- **the function \( F \) meets the requirement (h6) and**

  \[
  \|F\|_{C^{1,\alpha}(\mathbb{R}_T)} \leq C\|f\|_{C^{1,\alpha}(\mathbb{R}_{2,T})};
  \]

- **the solution \( U \) of problem (6.14) in \( \mathbb{R}_T \) boils down with the solution \( U \) of (6.13), i.e.**

  \[
  U(x, t) \big|_{\mathbb{R}_T} = U(x, t).
  \]

Thus, it is enough to prove statements of Lemma 6.3 to problem (6.14).

To this end, applying standard Fourier and Laplace transformation with respect to \( x_1 \) and \( t \), correspondingly, we construct the integral representation of the solution to (6.14)

\[
U(x, t) = \int_0^t \int_{-\infty}^{+\infty} F(t - \tau, x_1 - \zeta) K(\zeta, \tau) d\zeta
\]

with the kernel \( K \) defined with (A.1).

After that, taking advantage of Lemma A.1 and recasting the arguments of Chapter 4 in [32], we arrive at the estimate

\[
\|U\|_{C^{1,\alpha}(\mathbb{R}^2_{+T})} + \|\partial U/\partial t\|_{C^{1,\alpha}(\mathbb{R}_T)} \leq C\|f\|_{C^{1,\alpha}(\mathbb{R}_{2,T})} \leq C\|f\|_{C^{1,\alpha}(\mathbb{R}_{2,T})}.
\]

Then, substituting the integral representation of \( U \) to the equation, initial and boundary conditions in (6.14), and using Lemma A.1, we conclude that the constructed function \( U \) satisfies all the relations in (6.14) in the classical sense.

Finally, we note that, the coercive estimate of \( U \) provides the uniqueness of the solution to (6.14). That completes the proof of Lemma 6.3. \( \square \)

6.5. **The one-valued solvability of the linear system \( Az = F \).** As it follows from (6.12), the linear system corresponding to the nonlinear one has the form

\[
\begin{align*}
\Delta u &= f_0(x, t) \quad \text{in} \quad \Omega^\varepsilon_T, \\
u &= A(x_1)\sigma \quad \text{on} \quad \Gamma^\varepsilon_T, \\
\gamma \frac{\partial \sigma}{\partial t} + \frac{\partial u}{\partial x_2} &= f_1(x, t) \quad \text{on} \quad \Gamma^\varepsilon_T, \\
\frac{\partial u}{\partial x_1} &= 0 \quad \text{on} \quad \Gamma_{1,T} \cup \Gamma_{3,T}, \\
\frac{\partial u}{\partial x_2} &= 0 \quad \text{on} \quad \Gamma_{2,T}, \\
\sigma(x_1, 0) &= 0 \quad \text{in} \quad [0, l], \\
u(x_1, x_2, 0) &= 0 \quad \text{in} \quad \bar{\Omega}^\varepsilon.
\end{align*}
\]

(6.15)

Here \( A(x_1), f_0(x, t), f_1(x, t) \) are some given functions satisfying the following conditions:

(h7)(Consistency conditions):

- \( f_0(x_1, x_2, 0) = f_0(0, 0, t) = f_0(l, 0, t) = f_0(l, \varepsilon, t) = f_0(0, \varepsilon, t) = 0 \quad \text{for} \quad (x_1, x_2) \in \bar{\Omega}^\varepsilon, t \in [0, T]; \)
- \( f_1(x_1, x_2, 0) = f_1(0, \varepsilon, t) = f_1(l, \varepsilon, t) = 0 \quad \text{for} \quad (x_1, x_2) \in \Gamma^\varepsilon, t \in [0, T]; \)

the function \( A(x_1) \) is strictly positive, i.e. \( A \geq \delta > 0 \) for all \( x_1 \in [0, l] \).
(h8)(Regularity of the given functions):

\[ f_0 \in C^\alpha(\Omega^\varepsilon_T), \quad f_1 \in C^{1+\alpha}(\Gamma^\varepsilon_T), \quad A \in C^{2+\alpha}([0, t]). \]

**Theorem 6.2.** Under conditions (h1), (h7) and (h8), for some \( T > 0 \) and any fixed positive \( \varepsilon \), problem \((6.15)\) admits a unique classical solution \((u, \sigma)\) satisfying the regularity \( u \in C^{2+\alpha}(\Omega^\varepsilon_T) \) and \( \sigma \in C^{2+\alpha}(\Gamma^\varepsilon_T) \), and the estimate

\[
\|u\|_{C^{2+\alpha}(\Omega^\varepsilon_T)} + \|\sigma\|_{C^{2+\alpha}(\Gamma^\varepsilon_T)} \leq C[\|f_0\|_{C^\alpha(\bar{\Omega}^\varepsilon)} + \|f_1\|_{C^{1+\alpha}(\Gamma^\varepsilon_T)}]
\]

with the constant \( C \) independent of the right-hand sides in \((6.15)\).

**Proof.** It is convenient to reduce linear system \((6.15)\) to the same problem with homogenous equation. To this end, we apply Theorem 3.1 \([48]\) to the following linear problem with the unknown function \( \Lambda = \Lambda(x, t) : \Omega^\varepsilon_T \to \mathbb{R} \)

\[
\begin{align*}
\Delta \Lambda &= f_0(x, t) & \text{in } \Omega^\varepsilon_T, \\
\Lambda &= 0 & \text{on } \Gamma^\varepsilon_T, \\
\frac{\partial \Lambda}{\partial n} &= 0 & \text{on } \Gamma^\varepsilon_{1,T} \cup \Gamma^\varepsilon_{3,T}, \\
\frac{\partial \Lambda}{\partial n} &= 0 & \text{on } \Gamma_{2,T}, \\
\Lambda(x_1, x_2, 0) &= 0 & \text{in } \bar{\Omega}^\varepsilon,
\end{align*}
\]

and deduce the existence of a unique solution \( \Lambda \in C^{2+\alpha}(\bar{\Omega}^\varepsilon_T) \) satisfying relations

\[
\frac{\partial \Lambda}{\partial t} = 0 \quad \text{on } \Gamma^\varepsilon_T, \quad \|\Lambda\|_{C^{2+\alpha}(\bar{\Omega}^\varepsilon_T)} \leq C\|f_0\|_{C^\alpha(\bar{\Omega}^\varepsilon)}.
\]

Then we look for a solution to the original problem \((6.15)\) in the form

\[
u = \Lambda + w,
\]

where the unknown function \( w \) solves the problem

\[
\begin{align*}
\Delta w &= 0 & \text{in } \Omega^\varepsilon_T, \\
w &= A(x_1)\sigma & \text{on } \Gamma^\varepsilon_T, \\
\frac{\partial \sigma}{\partial t} + \frac{\partial w}{\partial x_2} &= \bar{f}_1(x, t) & \text{on } \Gamma^\varepsilon_T, \\
\frac{\partial w}{\partial x_1} &= 0 & \text{on } \Gamma^\varepsilon_{1,T} \cup \Gamma^\varepsilon_{3,T}, \\
\frac{\partial w}{\partial x_2} &= 0 & \text{on } \Gamma_{2,T}, \\
\sigma(x_1, 0) &= 0 & \text{in } [0, t], \\
w(x_1, x_2, 0) &= 0 & \text{in } \bar{\Omega}^\varepsilon.
\end{align*}
\]

Here \( \bar{f}_1(x, t) = f(x, t) - \left. \frac{\partial \Lambda}{\partial x_2} \right|_{\Gamma^\varepsilon_T} \) and

\[
\|\bar{f}_1\|_{C^{1+\alpha}(\Gamma^\varepsilon_T)} \leq C[\|f_1\|_{C^{1+\alpha}(\Gamma^\varepsilon_T)} + \|f_0\|_{C^\alpha(\bar{\Omega}^\varepsilon_T)}] .
\]

In summary, we conclude that it is enough to prove Theorem 6.2 to problem \((6.16)\). To this end, first, we reduce the analyze of the linear system \((6.16)\) to the study of the linear initial-boundary value problem with the dynamic boundary condition for the function \( w \). Namely, in the light of the first condition on \( \Gamma^\varepsilon_T \) in
(6.16) and the properties of the function $A$, we assert that the function $w$ solves the problem

$$
\begin{cases}
\Delta w = 0 & \text{in } \Omega^c_T, \\
\frac{\partial w}{\partial t} + A(x_1) \frac{\partial w}{\partial x_2} = \frac{A(x_1)}{\gamma} \bar{f}_1(x, t) & \text{on } \Gamma_T^c, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \Gamma_1^c \cup \Gamma_3^c, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \Gamma_2^c, \\
w(x_1, x_2, 0) = 0 & \text{in } \tilde{\Omega}.
\end{cases}
$$

(6.17)

Thus, we are left to prove Theorem 6.2 to problem (6.17). The strategy of this proof is the following: first, we obtain the existence and uniqueness of a weak solutions to (6.17). Then, we show that the weak solution is more regular.

- At this point, we define the weak solution of (6.17) as the function $w$ satisfying regularity

$$w \in L^\infty((0, T); W^{1,2}(\Omega^c \cup \Gamma^c)) \quad \text{and} \quad \frac{\partial w}{\partial t} \in L^2(\Gamma_T^c),$$

and the identity

$$\int \int \nabla w \nabla \Psi dxdt + \gamma \int \frac{\partial w}{\partial t} A(x_1) \Psi d\omega dt = \gamma \int \bar{f}_1 \Psi d\omega dt$$

for any $\Psi \in L^2((0, T); W^{1,2}(\Omega^c \cup \Gamma^c))$.

Standard arguments and results from [20, Sec. 1] provide both the existence of the weak solution in the sense written above, and the validity of the estimates

$$\|w\|_{L^\infty((0, T); W^{1,2}(\Gamma^c))} \leq C\|\bar{f}_1\|_{L^\infty((0, T); W^{1,2}(\Gamma^c))},$$

(6.18)

$$\|w\|_{L^\infty(\tilde{\Omega}_T^c)} \leq C\|\bar{f}_1\|_{L^\infty((0, T); W^{1,2}(\Gamma^c))} \leq C\|\bar{f}_1\|_{C^{1+\alpha}(\Gamma_T^c)}.$$  

- Coming to the regularity of the weak solution, we apply the Schauder approach. Namely, using the partition of unity together with the local diffeomorphisms, Lemma 6.3, the second estimates in (6.18) and the results of Section 3 in [33] arrive at the inequality

$$\|w\|_{C^{2+\alpha}(\tilde{\Omega}_T^c)} + \|\frac{\partial w}{\partial t}\|_{C^{2+\alpha}(\Gamma_T^c)} \leq C\|\bar{f}_1\|_{C^{1+\alpha}(\Gamma_T^c)} \leq C\|f_0\|_{C^{\alpha}(\tilde{\Omega}_T^c)} + \|\bar{f}_1\|_{C^{1+\alpha}(\Gamma_T^c)}.$$  

Finally, recalling that $\sigma = \frac{\bar{f}_1}{\|\bar{f}_1\|_{\Gamma_T^c}}$ and taking advantage of the smoothness and the positivity of $A$ (see assumptions (h7) and (h8)), we arrive at $\sigma \in \hat{C}^{2+\alpha}(\Gamma_T^c)$ and the corresponding estimates hold. That completes the proof of Theorem 6.2.

6.6. Solvability of nonlinear problem (6.12): proof of Theorem 6.1. First we introduce the functional spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $z \in \mathcal{H}_1$ and $\mathcal{F}z \in \mathcal{H}_2$:

$$\mathcal{H}_1 = C^{2+\alpha}_0(\tilde{\Omega}_T^c) \times \hat{C}^{2+\alpha}_0(\Gamma_T^c),$$

$$\mathcal{H}_2 = C^0(\tilde{\Omega}_T^c) \times C^{2+\alpha}_0(\Gamma_T^c) \times C^{1+\alpha}_0(\Gamma_T^c) \times C^{1+\alpha}_0(\Gamma_1, T) \times C^{1+\alpha}_0(\Gamma_2, T) \times C^{1+\alpha}_0(\tilde{\Gamma}_3, T),$$

and

$$\|z\|_{\mathcal{H}_1} = \|(u, \sigma)\|_{\mathcal{H}_1} = \|u\|_{C^{2+\alpha}(\tilde{\Omega}_T^c)} + \|\sigma\|_{\hat{C}^{2+\alpha}(\Gamma_T^c)},$$

$$\|\mathcal{F}z\|_{\mathcal{H}_2} = \|\mathcal{F}_0(z), 0, \mathcal{F}_1(z), 0, 0\|_{\mathcal{H}_2} = \|\mathcal{F}_0(z)\|_{C^0(\tilde{\Omega}_T^c)} + \|\mathcal{F}_1(z)\|_{C^{1+\alpha}(\Gamma_T^c)}.$$  

Taking into account representation (6.12), we have

$$\mathcal{A}z = F(x, t) + \mathcal{F}(z),$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is the linear operator studied in Subsection 6.5, the vector $F$ is constructed by the initial data, $\mathcal{F}(z)$ contains the elements described in Corollary 6.1.
After that, the direct calculations (see e.g. Section 5.2 [47]) and Theorem 6.2, Corollary 6.1 and Lemma 6.2 arrive at the statement.

**Lemma 6.4.** Let $B_d$, $B_d \subset H_1$, be a ball with the center located in the origin and the radius of $d$. We assume that conditions of Theorem 6.1 hold. Then, for $z \in B_d$, the following estimates hold:

$$
\|f(0)\|_{H_2} \leq C_1(T), \quad \|f(z_1) - f(z_2)\|_{H_2} \leq C_2(T, d)\|z_1 - z_2\|_{H_1},
$$

where the quantities $C_1(T)$ and $C_2(d, T)$ vanish if $T, d \to 0$.

Then, due to the operator $A$ satisfies all the assumptions of Theorem 6.2, nonlinear problem (6.12) can be rewritten as

$$
z = A^{-1}f(x, t) + A^{-1}f(z) \equiv T(z).
$$

Finally, inequalities in Lemma 6.4 ensure that for sufficiently small $T$ and $d$ the nonlinear operator $T(z)$ meets the requirements of the fixed point theorem for a contraction operator. Hence, the equation $z = T(z)$ has the fixed point, which is obviously a unique solution of (6.12). That completes the proof of Theorem 6.1.

Actually, with nonessential modification in the proof of Theorem 3.1, the very same results hold for more general configurations of $\Omega^\varepsilon$, namely if $\rho(y_1, 0) \neq 0$, and the function $\Phi^\varepsilon$ independent of $\varepsilon$. Thus, problem (1.2) is rewritten as

$$
\begin{align*}
\Delta y^\varepsilon & = 0 \quad \text{in } \Omega^\varepsilon(t), \quad t \in (0, T), \\
p^\varepsilon & = 0 \quad \text{and } \frac{\partial p^\varepsilon}{\partial n} = -\gamma V_n \quad \text{on } \Gamma^\varepsilon(t), \quad t \in [0, T], \\
\frac{\partial p^\varepsilon}{\partial n} & = \Phi^\varepsilon(y, t) \quad \text{on } \partial\Omega^\varepsilon(t) \setminus \Gamma^\varepsilon(t), \quad t \in [0, T], \\
\rho(y_1, 0) & = \rho_0(y_1), \quad y_1 \in [0, l].
\end{align*}
$$

First, we introduce the following additional hypotheses.

**(h9):** Let the function

$$
\Phi^\varepsilon(y, t) = \begin{cases} 
\chi_2(y_2)\varphi_1(y_2, t), & y_2 \in \Gamma^\varepsilon_1(t), \quad t \in [0, T], \\
\chi_1(y_1)\varphi_2(y_1, t), & y_1 \in \Gamma_1, \quad t \in [0, T], \\
\chi_2(y_2)\varphi_3(y_2, t), & y_2 \in \Gamma^\varepsilon_3(t), \quad t \in [0, T],
\end{cases}
$$

where $\chi_i \in C^\infty_0(\mathbb{R}^1)$, $i \in \{1, 2\}$, are the cut-off functions, $\chi_1$ is defined in (h3) and

$$
\chi_2(y_2) = \begin{cases} 
1, & \text{if } y_2 \in \left[\frac{y_1}{2}, \frac{y_1}{2}\right], \\
0, & \text{if } y_2 \notin \left(\frac{y_1}{2}, \frac{y_1}{2}\right),
\end{cases}
$$

$\varphi_1 \in C([0, T]; C^{2+\alpha}[0, \varepsilon])$, $\varphi_2 \in C([0, T]; C^{2+\alpha}[0, l])$, $\varphi_3 \in C([0, T]; C^{2+\alpha}[0, \varepsilon])$.

**((h10)):** We assume that the nonnegative function $\rho_0 \in C^{3+\alpha}([0, l])$ meets requirement

$$
\rho_0(0) = \rho_0(l) = 0.
$$

Note that the last relations mean that $\Gamma^\varepsilon$ forms right angles with $\Gamma^\varepsilon_1$ and $\Gamma^\varepsilon_3$.

**Theorem 6.3.** Under assumptions (h1), (h4), (h9), (h10), the results of Theorem 3.1 hold for problem (1.1), (6.19).

7. **Proof of Theorem 4.1**

In order to prove Theorem 4.1 we will use the following approach. First, appealing to technique in [31, 34, 35], we obtain formal representation for the solution $p^\varepsilon$ and a Cauchy problem for the function $S$. Then in subsection 7.2 we justify those constructions by finding the residuals left by the approximation function $P^\varepsilon$ in problem (1.1)-(1.2) and estimate them using properties of special boundary-layer solutions and Lemma 5.1.
7.1. **Formal asymptotic procedure.** Following the approach of [31,35], we seek the asymptotics of $p^\varepsilon$ in the form

$$p^\varepsilon(y,t) \approx w_0(y_1,t) + \varepsilon w_1(y_1,t) + \varepsilon^2 u_2 \left( y_1, \frac{y_2}{\varepsilon}, t \right) + \varepsilon^3 u_3 \left( y_1, \frac{y_2}{\varepsilon}, t \right) \quad (7.1)$$

Substituting representation (7.1) in the relations of (4.5), taking into account the view of the function $\Phi^\varepsilon$ (see (3.3)), and then collecting coefficients at the same power of $\varepsilon$, we conclude that the unknown functions $u_2$ and $u_3$ solve the Neumann problems:

\[
\begin{align*}
-\frac{\partial^2 u_2}{\partial y_1^2}(y_1, \xi_2, t) &= \frac{\partial^2 w_0}{\partial y_1^2}(y_1, t), & &\xi_2 \in (0, S(y_1, t)), \\
\frac{\partial u_2}{\partial \xi_2}(y_1, S(y_1, t), t) &= \frac{\partial S}{\partial y_1} \frac{\partial w_0}{\partial y_1} - \gamma \frac{\partial S}{\partial t}, \\
\frac{\partial u_2}{\partial \xi_2}(y_1, 0, t) &= -\chi_1(y_1) \varphi_2(y_1, t),
\end{align*}
\]

and

\[
\begin{align*}
-\frac{\partial^2 u_3}{\partial y_1^2}(y_1, \xi_2, t) &= \frac{\partial^2 w_1}{\partial y_1^2}(y_1, t), & &\xi_2 \in (0, S(y_1, t)), \\
\frac{\partial u_3}{\partial \xi_2}(y_1, S(y_1, t), t) &= \frac{\partial S}{\partial y_1} \frac{\partial w_1}{\partial y_1}, \\
\frac{\partial u_3}{\partial \xi_2}(y_1, 0, t) &= 0.
\end{align*}
\]

Here, the variables $y_1 \in [0,l]$ and $t \in [0,T]$ are regarded as parameters.

Writing down the necessary and sufficient condition for solvability of problem (7.2), we derive the differential equation

$$\frac{\partial}{\partial y_1} \left( S(y_1, t) \frac{\partial w_0(y_1, t)}{\partial y_1} \right) = \gamma \frac{\partial S}{\partial t}(y_1, t) - \chi_1(y_1) \varphi_2(y_1, t), \quad y_1 \in (0, l). \quad (7.4)$$

Boundary conditions for (7.4), as well the solvability of the corresponding boundary-value problem, will be discussed later.

Let $w_0$ be a solution of (7.4). Thus, there exists a solution to problem (7.2) up to an additive value that is a function of variables $y_1$ and $t$. Taking the integral condition in problem (4.5) into account, we can choose this function so that

$$u_2(y_1, S(y_1, t), t) = 0, \quad \forall y_1 \in (0, l), \quad \forall t \in [0,T], \quad (7.5)$$

which provides a unique choice of the function $u_2$.

Recasting the same arguments in the case of problem (7.3), we draw out

$$\frac{\partial}{\partial y_1} \left( S(y_1, t) \frac{\partial w_1(y_1, t)}{\partial y_1} \right) = 0, \quad y_1 \in (0, l). \quad (7.6)$$

Since $\Phi^\varepsilon = O(1)$ (as $\varepsilon \to 0$) on $\Gamma_1(t)$ and $\Gamma_2(t)$, the derivative $\frac{\partial w_1}{\partial y_1}$ has to be equal zero in the points $y_1 = 0$ and $y_1 = l$. This means that the function $w_1$ depends only on $t$, i.e. $w_1 = w_1(t)$. Then, taking into account the integral condition in (4.5), we arrive at the identity $u_3 \equiv 0$. This equality and the same arguments applied to (7.3) result the relation $u_3 \equiv 0$.

Thus, ansatz (7.1) is rewritten in the form

$$w_0(y_1, t) + \varepsilon^2 u_2 \left( y_1, \frac{y_2}{\varepsilon}, t \right).$$
To find boundary conditions for a solution of differential equation (7.4) and to satisfy the boundary conditions on $\Gamma_1^*(t)$ and $\Gamma_2^*(t)$, we should launch the boundary-layer asymptotics. To this end, in $\delta$-neighborhoods of $\Gamma_1^*(t)$ and $\Gamma_2^*(t)$, we seek first terms in the form

$$\varepsilon \Pi_1 \left( \frac{y_1}{\varepsilon}, \frac{y_2}{\varepsilon}, t \right) \quad \text{and} \quad \varepsilon \Pi_1^* \left( \frac{l - y_1}{\varepsilon}, \frac{y_2}{\varepsilon}, t \right), \quad (7.7)$$

respectively. Taking into account Corollary 4.1 and substituting $\varepsilon \Pi_1$ in (4.5), we get the boundary value problem

$$\begin{cases}
\Delta_{\xi} \Pi_1(\xi, t) = 0, & \xi = (\xi_1, \xi_2) \in (0, +\infty) \times (0, S_0(t)), \\
\frac{\partial \Pi_1(\xi_1, 0, t)}{\partial \xi_2} = \frac{\partial \Pi_1(\xi_1, S_0(t), t)}{\partial \xi_2} = 0, & \xi_1 \in (0, +\infty), \\
\frac{\partial \Pi_1(0, \xi_2, t)}{\partial \xi_1} = \Upsilon_1(\xi_2, t), & \xi_2 \in (0, S_0(t)), \\
\Pi_1(\xi_1, \xi_2, t) \to 0, & \xi_1 \to +\infty, \xi_2 \in [0, S_0(t)],
\end{cases} \quad (7.8)$$

where $\xi_1 = \frac{y_1}{\varepsilon}, \xi_2 = \frac{y_2}{\varepsilon}, \Upsilon_1(\xi_2, t) = -\chi_2(\xi_2) \varphi_1(\xi_2, t) - \frac{\partial w_0}{\partial y_1}(0, t)$.

Then the method of separation of variables allows us to find a solution of problem (7.8) in the form

$$\Pi_1(\xi, t) = \sum_{m=0}^{+\infty} a_m(t) \exp \left( -\frac{\pi m \xi_1}{S_0(t)} \right) \cos \left( \frac{\pi m \xi_2}{S_0(t)} \right), \quad (7.9)$$

where

$$a_0(t) = \frac{1}{S_0(t)} \int_0^{S_0(t)} \Upsilon_1(\xi_2, t) d\xi_2 = -\langle \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \rangle_{S_0} - \frac{\partial w_0}{\partial y_1}(0, t) = 0,$$

$$a_m(t) = \frac{2}{S_0(t)} \int_0^{S_0(t)} \Upsilon_1(\xi_2, t) \cos \left( \frac{\pi m \xi_2}{S_0(t)} \right) d\xi_2, \quad m \in \mathbb{N}.$$  

We remark that the fourth condition in (7.8) leads to the equality for the coefficient $a_0$. In summary, we end up with the boundary condition

$$\frac{\partial w_0}{\partial y_1}(0, t) = -\langle \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \rangle_{S_0} = 0. \quad (7.10)$$

Repeating the same arguments leading to relations (7.8) and (7.9), we conclude that the unknown function $\Pi_1^*$ has to solve the problem

$$\begin{cases}
\Delta_{\xi} \Pi_1^*(\xi^*, t) = 0, & \xi^* = (\xi_1^*, \xi_2) \in (0, +\infty) \times (0, S(t)), \\
\frac{\partial \Pi_1^*(\xi_1^*, 0, t)}{\partial \xi_2} = \frac{\partial \Pi_1^*(\xi_1^*, S(t), t)}{\partial \xi_2} = 0, & \xi_1^* \in (0, +\infty), \\
\frac{\partial \Pi_1^*(0, \xi_2, t)}{\partial \xi_1^*} = \Upsilon_1^*(\xi_2, t), & \xi_2 \in (0, S(t)), \\
\Pi_1^*(\xi_1^*, \xi_2, t) \to 0, & \xi_1^* \to +\infty, \xi_2 \in [0, S(t)],
\end{cases} \quad (7.11)$$

where $\xi_1^* = \frac{l - y_1}{\varepsilon}, \xi_2 = \frac{y_2}{\varepsilon}, \Upsilon_1^*(\xi_2, t) = \chi_2(\xi_2) \varphi_3(\xi_2, t) - \frac{\partial w_0}{\partial y_1}(l, t)$. Besides, the solution $\Pi_1^*$ is given with formula (7.9), where we should change $S_0(t)$ and $\Upsilon_1(\xi_2, t)$ by $S(t)$ and $\Upsilon_1^*(\xi_2, t)$, respectively. Finally, the boundary condition has the form

$$\frac{\partial w_0}{\partial y_1}(l, t) = -\langle \langle \chi_2(\cdot) \varphi_3(\cdot, t) \rangle \rangle_{S(t)}. \quad (7.12)$$
Remark 7.1. In virtue of (7.9) and Corollary 4.1, the asymptotic relations hold
\[
\Pi_1 = O(\exp(-\pi \xi_1)), \quad \frac{\partial \Pi_1}{\partial \xi_1} = O(\exp(-\pi \xi_1)), \quad \frac{\partial \Pi_1}{\partial \xi_2} = O(\exp(-\pi \xi_1)) \quad \text{as} \quad \xi_1 \to +\infty,
\]
\[
\Pi_1' = O(\exp(-\pi \xi_1^*)), \quad \frac{\partial \Pi_1'}{\partial \xi_1} = O(\exp(-\pi \xi_1^*)), \quad \frac{\partial \Pi_1'}{\partial \xi_2} = O(\exp(-\pi \xi_1^*)) \quad \text{as} \quad \xi_1^* \to +\infty
\]
for all \( t \in [0, T] \) and either \( \xi_2 \in [0, S_0(t)] \) in the case of \( \Pi_1 \) or \( \xi_2 \in [0, S_1(t)] \) for \( \Pi_1' \).

Collecting relations (7.4), (7.10) and (7.12), we deduce that, for each \( t \in [0, T] \), unknown functions \( S \) and \( w_0 \) satisfy the problem
\[
\begin{aligned}
\frac{\partial}{\partial y_1} \left( S(y_1, t) \frac{\partial w_0(y_1, t)}{\partial y_1} \right) &= \gamma \frac{\partial S}{\partial t}(y_1, t) - \chi_1(y_1) \varphi_2(y_1, t), \quad y_1 \in (0, l), \\
\frac{\partial w_0}{\partial y_1}(0, t) &= -\langle (\chi_2(\cdot) \varphi_1(\cdot, t)) \rangle_{S_0}, \\
\frac{\partial w_0}{\partial y_1}(l, t) &= \langle (\chi_2(\cdot) \varphi_3(\cdot, t)) \rangle_{S_1}.
\end{aligned}
\]  

(7.14)

In order to satisfy the solvability condition for problem (7.14), we, first, suppose that \( S \) is a solution to the ordinary differential equation
\[
\gamma \frac{\partial S(y_1, t)}{\partial t} = \chi_1(y_1) \varphi_2(y_1, t) + h_0(t), \quad t \in (0, T),
\]
where an unknown function \( h_0 \) will be defined below. After that, problem (7.14) becomes as follows:
\[
\begin{aligned}
\frac{\partial}{\partial y_1} \left( S(y_1, t) \frac{\partial w_0(y_1, t)}{\partial y_1} \right) &= h_0(t), \quad y_1 \in (0, l), \\
\frac{\partial w_0}{\partial y_1}(0, t) &= -\langle (\chi_2(\cdot) \varphi_1(\cdot, t)) \rangle_{S_0}, \\
\frac{\partial w_0}{\partial y_1}(l, t) &= \langle (\chi_2(\cdot) \varphi_3(\cdot, t)) \rangle_{S_1}.
\end{aligned}
\]  

(7.15)

Writing down the necessary and sufficient condition for solvability of problem (7.15) and taking into account Corollary 4.1, we end up with
\[
h_0(t) = \frac{1}{l} \left( \int_0^{S_0(t)} \chi_2(\xi_2) \varphi_3(\xi_2, t) \, d\xi_2 + \int_0^{S_0(t)} \chi_2(\xi_2) \varphi_1(\xi_2, t) \, d\xi_2 \right)
= \frac{1}{l} \left( \int_0^1 \chi_2(\xi_2) \varphi_3(\xi_2, t) \, d\xi_2 + \int_0^1 \chi_2(\xi_2) \varphi_1(\xi_2, t) \, d\xi_2 \right), \quad t \in [0, T].
\]

Finally, taking into advantage of condition (4.2), we come to the Cauchy problem
\[
\begin{aligned}
\gamma \frac{\partial S(y_1, t)}{\partial t} &= \chi_1(y_1) \varphi_2(y_1, t) + h_0(t), \quad t \in (0, T), \\
S(y_1, 0) &= 1,
\end{aligned}
\]  

(7.16)

which has a unique solution for every \( y_1 \in [0, l] \).

Remark 7.2. If for all \( t \in [0, T] \) and \( y_1 \in [0, l] \) the inequality
\[
\chi_1(y_1) \varphi_2(y_1, t) + \frac{1}{l} \left( \int_0^1 \chi_2(\xi_2) \varphi_3(\xi_2, t) \, d\xi_2 + \int_0^1 \chi_2(\xi_2) \varphi_1(\xi_2, t) \, d\xi_2 \right) > 0
\]
holds, then \( \frac{\partial S(y_1, t)}{\partial t} > 0 \), and, as a consequence, domain \( \Omega(\varepsilon(t)) \) increases in time for \( \varepsilon \) small enough. Moreover, this condition ensures the fulfillment of assumption (3.4).
Thus, Neumann problem (7.15) has a classical solution up to a function \( \eta_0 \) depending on \( t \in [0, T] \). We can choose it in such a way to fulfill the integral condition in (4.5). As a result, we get the solution
\[
\nu_0(y_1, t) = u_0(y_1, t) - \frac{1}{|\Gamma^c(t)|} \int_{\Gamma^c(t)} \nu_0 \, d\ell, \quad y_1 \in [0, l], \quad t \in [0, T],
\]
that satisfies the equality
\[
\int_{\Gamma^c(t)} \nu_0 \, d\ell = 0, \quad \forall t \in [0, T].
\]

7.2. **Justification.** First, we determine the unique solution \( S \) to problem (7.16), which is given by formula (4.6). Here, we essentially use the assumptions (h3) and (h5) that provides the following smoothness of the function \( S \):
\[
S \in C([0, T], C^3([0, l])), \quad \frac{\partial S}{\partial t} \in C([0, T], C^3([0, l])).
\]

In the next step, we obtain the unique smooth solution \( \nu_0 \) to problem (7.14), which satisfies (7.17). After that, returning to problem (7.2), we conclude the uniqueness and smoothness of its solution \( u_2 \) that satisfies condition (7.5). Finally, there exist solutions to problems (7.8) and (7.11) with asymptotics (7.13).

At this point, we start to estimate the difference between the classical solution \( p^\varepsilon \) and the approximation function
\[
P^\varepsilon(y, t) := \nu_0(y_1, t) + \varepsilon^2 u_2 \left(y_1, \frac{y_2}{\varepsilon}, t\right), \quad y \in \Omega^c(t), \quad t \in [0, T],
\]
in the norm of the space \( C([0, T]; H^1(\Omega^c(t))) \).

Substituting \( P^\varepsilon \) in the differential equation and the boundary conditions of problem (4.5) and taking into account relations in problems (7.14), (7.2), Corollary 4.1 and (7.5), we find that \( P^\varepsilon \) solves the problem
\[
\begin{cases}
\Delta_y P^\varepsilon = R_1^\varepsilon \quad \text{in} \quad \Omega^c(t), \\
\frac{\partial P^\varepsilon}{\partial y_2} = \varepsilon \frac{\partial S}{\partial y_1} \frac{\partial P^\varepsilon}{\partial y_1} - \varepsilon^2 \frac{\partial S}{\partial t} + R_2^\varepsilon \quad \text{on} \quad \Gamma^c(t), \\
\int_{\Gamma^c(t)} P^\varepsilon \, d\ell = 0, \\
-\frac{\partial P^\varepsilon}{\partial y_1} = \langle \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \rangle_{S_0} \quad \text{on} \quad \Gamma_1^c(t), \\
\frac{\partial P^\varepsilon}{\partial y_1} = \langle \langle \chi_2(\cdot) \varphi_3(\cdot, t) \rangle \rangle_{S_1} \quad \text{on} \quad \Gamma_2^c(t), \\
-\frac{\partial P^\varepsilon}{\partial y_2} = \varepsilon \chi_1(y_1) \varphi_2(y_1, t) \quad \text{on} \quad \Gamma_2
\end{cases}
\]
for any fixed \( t \in [0, T] \). Here
\[
R_1^\varepsilon(y, t) = \varepsilon^2 \frac{\partial^2 u_2}{\partial y_1^2} \left(y_1, \frac{y_2}{\varepsilon}, t\right), \quad R_2^\varepsilon(y, t) = \varepsilon^3 \frac{\partial S}{\partial y_1} \frac{\partial u_2}{\partial y_1} \left(y_1, \frac{y_2}{\varepsilon}, t\right).
\]

Due to condition (h3) and (h4) and (7.18) we have
\[
\sup_{t \in [0, T]} \sup_{y \in \Omega^c(t)} |R_1^\varepsilon(y, t)| \leq C \varepsilon^2, \quad \sup_{t \in [0, T]} \sup_{y \in \Gamma^c(t)} |R_2^\varepsilon(y, t)| \leq C \varepsilon^3.
\]

**Remark 7.3.** In (7.19) and further, all constants in inequalities are independent of the functions \( S, P^\varepsilon, p^\varepsilon, \) the variables \( y, t \) and the parameter \( \varepsilon \).
As a consequence, the difference $W^\varepsilon = p^\varepsilon - p^\varepsilon$ satisfies the relations
\[
\begin{aligned}
-\Delta_y W^\varepsilon &= R_1^\varepsilon \quad \text{in } \Omega^\varepsilon(t), \\
-\frac{\partial W^\varepsilon}{\partial n} &= \frac{R_2^\varepsilon}{|\nabla_y R|} \quad \text{on } \Gamma^\varepsilon(t), \\
\int_{\Gamma^\varepsilon(t)} W_\varepsilon \, d\ell &= 0, \\
-\frac{\partial W^\varepsilon}{\partial y_1} &= \chi_2(y_2/\varepsilon, t) \varphi_1(y_2/\varepsilon, t) - \langle \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \rangle S_0 \quad \text{on } \Gamma_1^\varepsilon(t), \\
\frac{\partial W^\varepsilon}{\partial y_1} &= \chi_2(y_2/\varepsilon, t) \varphi_3(y_2/\varepsilon, t) - \langle \langle \chi_2(\cdot) \varphi_3(\cdot, t) \rangle \rangle S_i \quad \text{on } \Gamma_2^\varepsilon(t), \\
-\frac{\partial W^\varepsilon}{\partial y_2} &= 0 \quad \text{on } \Gamma_2.
\end{aligned}
\]

We multiply the differential equation in this relations by $W^\varepsilon$ and, then, integrate over $\Omega^\varepsilon(t)$ for each fixed $t \in [0, T]$. Standard calculations lead to the equality
\[
\int_{\Omega^\varepsilon(t)} |\nabla_y W^\varepsilon|^2 \, dy = \int_{\Omega^\varepsilon(t)} R_1^\varepsilon W^\varepsilon \, dy - \int_{\Omega^\varepsilon(t)} \frac{R_2^\varepsilon}{|\nabla_y R|} W^\varepsilon \, d\ell \\
+ \int_{\Gamma_1^\varepsilon(t)} \left( \Phi - \langle \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \rangle S_0 \right) \, W^\varepsilon \, dy_2 + \int_{\Gamma_2^\varepsilon(t)} \left( \Phi - \langle \langle \chi_2(\cdot) \varphi_3(\cdot, t) \rangle \rangle S_i \right) \, W^\varepsilon \, dy_2. \tag{7.20}
\]

Then, we evaluate each term in the right-hand side of (7.20).

- As for first two terms, appealing to inequalities (7.19) and inequalities (5.5) and (5.8), we deduce
\[
\begin{align}
\left| \int_{\Omega^\varepsilon(t)} R_1^\varepsilon W^\varepsilon \, dy \right| &\leq C \varepsilon^{\frac{1}{2}} \| W^\varepsilon \|_{L^2(\Omega^\varepsilon(t))} \leq C \varepsilon^{\frac{1}{2}} \| \nabla_y W^\varepsilon \|_{L^2(\Omega^\varepsilon(t))}, \tag{7.21} \\
\left| \int_{\Gamma_1^\varepsilon(t)} \frac{R_2^\varepsilon}{|\nabla_y R|} W^\varepsilon \, d\ell \right| &\leq C \varepsilon^{\frac{3}{2}} \| W^\varepsilon \|_{L^2(\Gamma_1^\varepsilon(t))} \leq C \varepsilon^{\frac{3}{2}} \| \nabla_y W^\varepsilon \|_{L^2(\Gamma_1^\varepsilon(t))}. \tag{7.22}
\end{align}
\]

- Coming to the last two terms in (7.20), we introduce two smooth cut-off functions
\[
\chi_\delta(y_1) = \begin{cases} 
1, & \text{if } y_1 \leq \frac{\delta}{2}, \\
0, & \text{if } y_1 \geq \delta,
\end{cases} \quad \chi_\delta^*(y_1) := \chi_\delta(t - y_1),
\]
where $\delta$ is defined in Corollary 4.1, and consider the functions
\[
\varepsilon \chi_\delta(y_1) \Pi_1 \left( \frac{y_1}{\varepsilon}, \frac{y_2}{\varepsilon}, t \right) \quad \text{and} \quad \varepsilon \chi_\delta^*(y_1) \Pi_1^* \left( \frac{t - y_1}{\varepsilon}, \frac{y_2}{\varepsilon}, t \right).
\]

Taking into account relations in problem (7.8), the direct calculations entail
\[
\begin{aligned}
-\Delta_y (\varepsilon \chi_\delta \Pi_1) &= R_3^\varepsilon \quad \text{in } \Omega^\varepsilon(t), \\
-\frac{\partial}{\partial n} (\varepsilon \chi_\delta \Pi_1) &= 0 \quad \text{on } \partial \Omega^\varepsilon(t) \setminus \Gamma_1^\varepsilon(t), \\
-\frac{\partial}{\partial y_1} (\varepsilon \chi_\delta \Pi_1) &= \Phi - \langle \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \rangle S_0 \quad \text{on } \Gamma_1^\varepsilon(t),
\end{aligned}
\]
whence
\[ \int_{\Gamma_1^*(t)} \left( \Phi^\varepsilon - \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \right) S_0^\varepsilon dy_2 = - \int_{\Omega(t)} R_3^\varepsilon W^\varepsilon dy + \varepsilon \int_{\Omega(t)} \nabla_y (\chi_3 \Pi_3) \cdot \nabla_y W^\varepsilon dy. \] (7.23)

Here
\[ R_3^\varepsilon(y, t) = -2 \frac{d\chi\delta(y_1)}{dy_1} \left. \frac{\partial \Pi_1(\xi, t)}{\partial \xi_1} \right|_{\xi = \frac{y}{\varepsilon}} - \varepsilon \frac{d^2 \chi\delta(y_1)}{dy_1^2} \Pi_1(\xi, t) \bigg|_{\xi = \frac{y}{\varepsilon}}. \]

Since the function \( \Pi_1 \) and its derivatives \( \frac{\partial \Pi_1}{\partial \xi} \), \( i = 1, 2 \), decrease exponentially (see Remark 7.1) and the support of the derivatives of the cut-off function \( \chi\delta \) belongs to the segment \( \left[ \frac{2}{\varepsilon}, \delta \right] \), we arrive at the inequality
\[ \sup_{t \in [0, T]} \sup_{y \in \Omega(t)} |R_3^\varepsilon(y, t)| \leq C \exp \left( -\frac{\pi \delta}{2\varepsilon} \right). \] (7.24)

With the help of (7.24) and the Poincaré inequality (5.5) in Lemma 5.1 we derive
\[ \left| \int_{\Gamma_1^*(t)} \left( \Phi^\varepsilon - \langle \chi_2(\cdot) \varphi_1(\cdot, t) \rangle \right) S_0^\varepsilon dy_2 \right| \leq C \sqrt{\varepsilon} \exp \left( -\frac{\pi \delta}{2\varepsilon} \right) \left( \|W^\varepsilon\|_{L^2(\Omega(t))} + \|
abla W^\varepsilon\|_{L^2(\Omega(t))} \right) + C \varepsilon \|\nabla W^\varepsilon\|_{L^2(\Omega(t))} \left( \int_0^{+\infty} \int_{S_0^\varepsilon(t)} |\nabla \Pi_1(\xi, t)|^2 d\xi_2 d\xi_1 \right)^{\frac{1}{2}} \leq C \varepsilon \|\nabla W^\varepsilon\|_{L^2(\Omega(t))}. \] (7.25)

Similarly arguments and properties of the solution \( \Pi_3^* \) (see (7.11)) yield
\[ \left| \int_{\Gamma_3^*(t)} \left( \Phi^\varepsilon - \langle \chi_2(\cdot) \varphi_3(\cdot, t) \rangle \right) S_0^\varepsilon dy_2 \right| \leq C \varepsilon \|\nabla W^\varepsilon\|_{L^2(\Omega(t))}. \] (7.26)

In conclusion, from (7.20) in virtue of (7.21), (7.22), (7.25) and (7.26), it follows the inequality
\[ \sup_{t \in [0, T]} \|\nabla W^\varepsilon\|_{L^2(\Omega(t))} \leq C_0 \varepsilon, \] (7.27)

that together with Lemma 5.1 complete the proof of Theorem 4.1.

Using the Cauchy-Bunyakovsky-Schwarz inequality and (4.7), we derive the statement.

**Corollary 7.1.** For the difference between the solution to problem (1.1)-(1.2) and the solution to the limit problem (4.8) the following estimate
\[ \| \langle \Phi^\varepsilon \rangle \rangle_{S} - \mathfrak{m}_0 \|_{C([0, T]; L^2(0, l))} \leq C_0 \sqrt{\varepsilon} \]
holds.

8. Conclusions

In this work, we discuss the one-phase contact Hele-Shaw problem (1.1)-(1.2) with ZST in the domain \( \Omega^\varepsilon(t) \) that depends on a small parameter \( \varepsilon \). In particular, we analyze the classical local solvability of this problem for each fixed \( \varepsilon \) and describe the asymptotic behavior of the solution \( \Phi^\varepsilon \) as \( \varepsilon \to 0 \).

As it follows from our consideration, the asymptotic analysis turns out the effective tools to study of the Hele-Shaw problem in thin domains. Namely, it allows us to obtain not only the explicit representation of the free boundary \( \Gamma^\varepsilon(t) \) but also to establish preserving the geometry of the moving boundary in \( \delta \)-neighborhoods of the corner points for \( t \in [0, T] \). This property is not exactly the waiting time phenomena described in [30], since the corner points on the free boundary shift instantly for \( t > 0 \). However, in opposite to all the early obtained results concerning with the waiting time phenomena, we can find the size of those \( \delta \)-neighborhoods that depends on the support of the function \( \Phi^\varepsilon \) at \( t = 0 \).

An important task of existing multiscale methods is their stability and accuracy. The proof of the error estimate between the constructed approximation and the exact solution is a general principle that should be applied to the analysis of the effectiveness of the proposed multiscale method. In our paper, we have constructed and justified the asymptotic approximation for the solution to problem (1.1)-(1.2) and proved the
corresponding estimates. The results obtained in Theorem 4.1 and Corollary 7.1 argue that the complex Hele-Shaw problem (1.1)-(1.2) can be replaced by the corresponding Cauchy problem (7.16) and one-dimensional limit problem (4.8) with sufficient accuracy measured by the parameter $\varepsilon$ characterizing the thickness of the domain $\Omega^\varepsilon(t)$ and the amplitude of the free boundary.

Our ideas can be exported to cover the analysis of problems like (1.1)-(1.2) in more general cases. First, our consideration can be extended to the Hele-Shaw problem with nonzero surface tension (NZST), and to the Stefan problem in the case of both NZST and ZST. The proposed approach can be adapted and generalized in order to consider problem like (1.1)-(1.2) in three-dimensional case, i.e. $Q \in \mathbb{R}^3$, $Q = (0, l_1) \times (0, l_2) \times (0, 2\varepsilon)$. Also, it will be very interesting to study Hele-Shaw and Stefan problems in thin domains when a free boundary has a highly small amplitude, for instance, $\rho = O(\varepsilon^\alpha)$ as $\varepsilon \to 0$ and $\alpha > 1$. Perhaps all of this will be the subject of future research.

**Appendix**

A.1. Statement of Lemma A.1. Denoting the inverse Laplace transformation with respect to time $t$ by $\mathcal{L}_t^{-1}$, we recall some properties of the function $K = K(x, t) : \mathbb{R} \times [0, T] \to \mathbb{R}$:

$$K(x, t) = \mathcal{L}_t^{-1}\left(\int_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{p + \mathcal{C}_0|\lambda|} d\lambda\right) \quad (A.1)$$

with the positive number $\mathcal{C}_0$ and $Re p > 0$, which are obtained in Lemma 3.1 [6] (where $\mathcal{C}_0 = A_1$ and $A_2 = 0$).

**Lemma A.1.** Let $\alpha \in (0, 1)$, $T > 0$ be arbitrary fixed and let $k$ be nonnegative integer. Then for each $t \in [0, T]$ and $x, x_1, x_2 \in \mathbb{R}$, the following estimates hold:

(i) $$K(x, t) = \frac{2\mathcal{C}_0 t}{(\mathcal{C}_0 t)^2 + x^2};$$

(ii) $$\int_0^t d\tau \int_{-\infty}^{+\infty} \frac{\partial^k K}{\partial y^k}(y, \tau) dy = \begin{cases} 2\pi t & \text{if } k = 0, \\ 0 & \text{if } k > 0; \end{cases}$$

(iii) $$\int_0^t d\tau \int_{-\infty}^{+\infty} |y|^{\alpha} \left| \frac{\partial K}{\partial y}(y, \tau) \right| dy \leq C t^\alpha,$$

$$\int_0^t d\tau \int_{|y| \leq 2|x_1 - x_2|} |y|^{\alpha} \left| \frac{\partial K}{\partial y}(y, \tau) \right| dy \leq C |x_1 - x_2|^{\alpha},$$

$$\int_0^t d\tau \int_{|y| \geq 2|x_1 - x_2|} |y|^{\alpha} \left| \frac{\partial^2 K}{\partial y^2}(y, \tau) \right| dy \leq C |x_1 - x_2|^{\alpha - 1}.$$
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