Solutions of the Anisotropic Porous Medium Equation in $\mathbb{R}^n$ under $L^1$-initial Value

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Abstract

Consider the anisotropic porous medium equation, \( u_t = \sum_{i=1}^{n} (u^{m_i})_{x_i, x_i} \), where \( m_i > 0, \ (i = 1, 2, \cdots, n) \) satisfying \( \min_{1 \leq i \leq n} \{ m_i \} \leq 1, \sum_{i=1}^{n} m_i > n-2, \) and \( \max_{1 \leq i \leq n} \{ m_i \} \leq \frac{1}{n}(2+\sum_{i=1}^{n} m_i) \).

Assuming that the initial data belong only to \( L^1(\mathbb{R}^n) \), we establish the existence and uniqueness of the solution for the Cauchy problem in the space, \( C([0, \infty), L^1(\mathbb{R}^n)) \cap C(\mathbb{R}^n \times (0, \infty)) \cap L^\infty(\mathbb{R}^n \times [\varepsilon, \infty)) \), where \( \varepsilon > 0 \) may be arbitrary. We also show a comparison principle for such solutions. Furthermore, we prove that the solution converges to zero in the space \( L^\infty(\mathbb{R}^n) \) as the time goes to infinity.

**Keywords:** Anisotropic diffusion, Degenerate parabolic equation, Comparison principle, Large time behavior.

**AMS subject classifications:** 35K55, 35K65.

1. INTRODUCTION

Consider the anisotropic porous medium equation

\[
 u_t = \sum_{i=1}^{n} (u^{m_i})_{x_i, x_i} \quad \text{in} \quad \mathbb{R}^n \times (0, \infty),
\]

where \( m_i \) are positive constants. As it is well-known, there have been a lot of works dealing with the case of all \( m_i \)'s in (1.1) being the same positive constant, i.e., the case of porous medium equation (PME). See, for example, the survey paper [1] for PME, the monograph [2] for its generalization and the references there.

However, there are few papers on the general case of equation (1.1), although it has strong physical backgrounds. In fact, it comes directly from water moves in anisotropic media. If the conductivities of the media are different in different directions, the constants \( m_i \) in (1.1) must be different from each other. See [3] for details.

In papers [4, 5], the first author started studying the existence and uniqueness for the Cauchy problem of equation (1.1), provided that the initial data are bounded and continuous in \( \mathbb{R}^n \). He also studied the continuous modulus of solutions to (1.1) in [6] (also see Lemma 3.1 in [4]). In [7], the authors established the existence of fundamental solutions for the Cauchy problem of equation (1.1).

In this paper, we will study the existence, uniqueness, comparison principle and large time behavior of the solution of the Cauchy problem for (1.1) with \( L^1 \)-initial data. For this purpose, we want to consider the equation

\[
 V_t = \sum_{i=1}^{n} [(V^{m_i})_{y_i y_i} + \alpha_i(y_i V)_{y_i}] \quad \text{in} \quad \mathbb{R}^n \times (0, \infty)
\]
and its stable equation

\[- \sum_{i=1}^{n} \left[ (f^{m_i})_{y_i} + \alpha_i (y_i f)_{y_i} \right] = 0 \text{ in } \mathbb{R}^n, \quad (1.3)\]

where \( \alpha_i \) are defined by

\[ \alpha_i = \frac{m_i - m_i}{2} + \frac{1}{n}, \quad (i = 1, 2, \ldots, n), \quad \text{and } \bar{m} = \sum_{i=1}^{n} \frac{m_i}{n}. \quad (1.4)\]

As we see, equation (1.2) is equivalent to (1.1), up to a scaling transformation in spatial and time variables. See Lemma 2.2 below.

For the coming needs, define

\[ Q = \mathbb{R}^n \times (0, \infty), \quad \beta = \bar{m} - \frac{n - 2}{n} \quad (1.5)\]

and throughout this paper, we assume

\[ \min_{1 \leq i \leq n} \{m_i\} \leq 1, \quad \beta > 0, \quad m_i > 0, \quad \text{and } \alpha_i > 0 \quad (i = 1, 2, \ldots, n), \quad (1.6)\]

which is equivalent to

\[ m_i > 0 \quad (i = 1, 2, \ldots, n), \quad \sum_{i=1}^{n} m_i > n - 2, \]

\[ \min_{1 \leq i \leq n} \{m_i\} \leq 1, \quad \text{and } \max_{1 \leq i \leq n} \{m_i\} < \frac{1}{n} (2 + \sum_{i=1}^{n} m_i). \quad (1.6')\]

Also, we use \( D'(X) \) to denote the set of all distributions (generalized functions) in \( X \). The word “respectively” is always shorten by “resp”.

**Definition 1.1** A \( L^1_{\text{loc}} \)-function \( u \) (resp, \( V, f \)) is called a solution of equation (1.1) (resp, (1.2), (1.3)) if \( u \) and \( u^{m_i} \) are in \( D'(Q) \) (resp, \( V \) and \( V^{m_i} \) in \( D'(Q) \), \( f \) and \( f^{m_i} \) in \( D'(\mathbb{R}^n) \)) such that (1.1) (resp, (1.2)), (1.3) is satisfied in the sense of distributions.

**Definition 1.2** An \( L^1 \)-regular solution, \( u \), of (1.1) (resp, (1.2)) with initial value \( u(x, 0) = u_0(x) \) in \( \mathbb{R}^n \) is a solution of (1.1) (resp, (1.2)) satisfying

\[ u \in C([0, \infty), L^1(\mathbb{R}^n)) \bigcap C(Q) \bigcap L^\infty(\mathbb{R}^n \times [\varepsilon, \infty)) \quad (1.7)\]

for each \( \varepsilon > 0 \), and

\[ \int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} u_0(x) dx, \quad \forall t > 0. \quad (1.8)\]

**Definition 1.3** If we replace “=” by “\( \geq \)” (resp, “\( \leq \)” in (1.1), (1.2), (1.3) and (1.8), then we obtain the definitions of the super-solution (resp, sub-solution).
For simplicity, we will consider only the nonnegative solutions. The main results of this paper may be stated as the following theorems for which we always assume (1.6').

**Theorem 1.1** Suppose that \( u \) and \( \overline{u} \) are nonnegative \( L^1 \)-regular sub-solution and super-solution, respectively, of (1.1) (resp, of (1.2)). If \( u(x,0) \leq \overline{u}(x,0) \) in \( \mathbb{R}^n \), then \( u \leq \overline{u} \) in \( Q \).

**Theorem 1.2** For any nonnegative function \( u_0 \in L^1(\mathbb{R}^n) \), there exists a unique nonnegative \( L^1 \)-regular solution \( u \) of (1.1) (resp, (1.2)) with initial value \( u(x,0) = u_0(x) \) in \( \mathbb{R}^n \). Furthermore, \( u \) satisfies the following decay estimate:

\[
\|u(x,t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{\beta}} \|u_0\|_{L^1(\mathbb{R}^n)}^{\frac{2}{\beta}}, \quad \forall t > 0,
\]

where \( C \) is a constant depending only on \( m_i \) (\( i = 1, 2, \ldots, n \)) and \( n \).

Note that the decay estimate says that the solution of (1.1) converges to zero uniformly for \( x \in \mathbb{R}^n \) as the time goes to infinity.

We would like to mention that the Cauchy problem of PME with \( L^1 \)-initial data was studied in [8, 9]. But our methods to prove Theorems 1.1 and 1.2 are completely different from those in [8, 9]. In fact, the proof of Theorem 1.1 is based on a max-min method, which will be developed in Section 2. As a key step for the proof of Theorem 1.2 we have to discover a specific scaling technique for each direction in order to overcome the difficulty from the anisotropic phenomenon so that we can construct suitable super-solution, which will be developed in Section 3. While in Section 4, we will combine approximation arguments and the results in Section 3 to prove Theorem 1.2.

### 2. A COMPARISON PRINCIPLE

Theorem 1.1 is a comparison principle for \( L^1 \)-regular solutions of (1.1). In this section, we will give its proof by a few lemmas. The following result was proved in [4, 5].

**Lemma 2.1** There exists a unique bounded and continuous solution of the Cauchy problem to (1.1) if the initial value is in \( C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). The solution is approximated by a equi-continuous and uniformly bounded sequence of classical solutions in the norm of the space \( C_{loc}(\mathbb{R}^n \times [0, \infty)) \) and thus satisfies a comparison principle.

**Lemma 2.2** Suppose that the functions \( u(x,t) \) and \( V(y,\tau) \) are related by

\[
V(y,\tau) = h(t)u(x,t), \quad \tau = \ln h(t), \quad y_i = x_i h^{-\alpha_i}(t), \quad (i = 1, 2, \ldots, n),
\]

where \( \alpha_i \) and \( \beta \) are the same as in (1.4)–(1.6), and

\[
h(t) = (1 + \beta t)^{\frac{1}{\beta}}
\]

If \( u(x,t) \) is a solution (resp, super-solution, sub-solution) of (1.1), then \( V(y,\tau) \) is a solution (resp, super-solution, sub-solution) of (1.2), and vice versa.
Proof. It is from direct computations, we omit the details.

**Lemma 2.3** If \( u \) and \( w \) are nonnegative sub-solutions (resp., super-solutions) of (1.1) satisfying
\[
u, w, u_t, w_t, (u^{m_i})_{x_i}, (w^{m_i})_{x_i} \in L^1_{\text{loc}}(Q) \quad (i = 1, 2, \cdots, n),
\]
then \( \max\{u, w\} \) (resp., \( \min\{u, w\} \)) is a sub-solution (resp., super-solution) of (1.1).

**Proof.** First suppose that \( u \) and \( w \) are sub-solution of (1.1). Note that
\[
(|g|)_{t} = \text{sign}(g) g_{t}
\]
in the sense of distributions if \( g \) and \( g_{t} \) are in \( L^1_{\text{loc}} \). Furthermore, by Kato’s inequality, we have, in the same sense, that
\[
(|g|)_{x_i} \geq \text{sign}(g) g_{x_i}
\]
if \( g \) and \( g_{x_i} \) are in \( L^1_{\text{loc}}(Q) \). It follows from these two facts and (2.3) that the following computations hold true in the sense of distributions:
\[
\begin{align*}
\frac{\partial}{\partial t} \max \{u, w\} & - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \max \{u, w\}^{m_i} \\
& = \frac{\partial}{\partial t} \max \{u, w\} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \max \{u^{m_i}, w^{m_i}\} \\
& = \left( \frac{u + w}{2} + \frac{|u - w|}{2} \right)_{t} - \sum_{i=1}^{n} \left( \frac{u^{m_i} + w^{m_i}}{2} + \frac{|u^{m_i} - w^{m_i}|}{2} \right)_{x_i} \\
& \leq \left( \frac{u + w}{2} \right)_{t} + \text{sign}(u - w) \left( \frac{u - w}{2} \right)_{t} \\
& - \sum_{i=1}^{n} \left( \frac{u^{m_i} + w^{m_i}}{2} \right)_{x_i} - \sum_{i=1}^{n} \text{sign}(u - w) \left( \frac{u^{m_i} - w^{m_i}}{2} \right)_{x_i} \\
& = \begin{cases} 
    u_t - \sum_{i=1}^{n} (u^{m_i})_{x_i}, & \text{if } u \geq w \\
    w_t - \sum_{i=1}^{n} (w^{m_i})_{x_i}, & \text{if } u < w 
\end{cases} \\
& \leq 0.
\end{align*}
\]
This shows the maximum of two sub-solutions is also a sub-solution. Similarly, the minimum of two super-solutions is a super-solution.

**Lemma 2.4** If \( u \) and \( w \) are nonnegative solutions of (1.1) such that
\[
u, w, u^{m_i}, w^{m_i} \in L^1_{\text{loc}}(Q)
\]
(2.4)
and
\[ \lim_{k \to \infty} (u_k, w_k, u_k^{m_i}, w_k^{m_i}) = (u, w, u^{m_i}, w^{m_i}) \in [L^1_{loc}(Q)]^4 \] (2.5)
for \( i = 1, 2, \cdots, m \) and some nonnegative function sequence \( \{u_k\} \) and \( \{w_k\}\) satisfying (2.3), then \( \max\{u, w\} \) (resp, \( \min\{u, w\}\)) is a sub-solution (resp, super-solution) of (1.1).

**Proof.** We want only to consider the maximum case. Let
\[ U_k = \max\{u_k, w_k\}, U = \max\{u, w\}, \]
Since \( u_k \) and \( w_k \) satisfy (2.3), we have, by Lemma 2.3, that
\[ \int_0^\infty \int_\mathbb{R}^n \left( U_k \varphi_t + \sum_{i=1}^n U_k^{m_i} \varphi_{x_i} \right) \, dx \, dt \geq 0 \]
for all nonnegative function \( \varphi \in C_0^\infty(Q) \) and all \( k = 1, 2, \cdots \). Using (2.4) and (2.5), and passing the limit, we obtain that
\[ \int_0^\infty \int_\mathbb{R}^n \left( U \varphi_t + \sum_{i=1}^n U^{m_i} \varphi_{x_i} \right) \, dx \, dt \geq 0, \forall \varphi \in C_0^\infty(Q), \varphi \geq 0. \]
This shows that \( U \) is a sub-solution of (1.1).

**Proof of Theorem 1.1:** Owing to Lemma 2.2, we want only to prove Theorem 1.1 for equation (1.1). We will complete the proof by three steps.

**Step 1.** Suppose that \( u \) and \( u^{m_i} \) are, respectively, nonnegative \( L^1 \) regular super-solution and \( L^1 \) regular solution of (1.1) with \( u(x, 0) \geq u(x, 0) \) in \( \mathbb{R}^n \). We will prove that
\[ \overline{u} \geq u \text{ in } Q. \] (2.6)

Fix \( \tau > 0 \). Since \( \overline{u}(\cdot, \tau) \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), it follows from Lemma 2.1 that there exists a unique solution \( u^{(\tau)}(x) \in C(Q) \cap L^\infty(Q) \) of (1.1) with the initial value \( u^{(\tau)}(x, 0) = \overline{u}(x, \tau) \in \mathbb{R}^n \), which is approximated by a sequence of classical solutions in the norm of \( C_{loc}(\mathbb{R}^n \times [0, \infty)) \). The same conclusion holds true for the solution \( V(x, t) = u(x, t + \tau) \) since \( u(\cdot, \tau) \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Hence Lemma 2.4 implies that \( \min\{u^{(\tau)}(x, t), u(x, t + \tau)\} \) is a bounded and continuous super-solution of (1.1). This implies
\[ \int_{\mathbb{R}^n} \min \{u^{(\tau)}(x, t), u(x, t + \tau)\} \, dx \geq \int_{\mathbb{R}^n} \min \{\overline{u}(x, \tau), u(x, \tau)\} \, dx, \forall t > 0. \] (2.7)

On the other hand, \( \overline{u}(\cdot, \tau + \cdot) \) is a bounded and continuous super-solution of (1.1) with the same initial value in \( C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) as \( u^{(\tau)} \). Hence, by the comparison principle of Lemma 2.1 we have
\[ \overline{u}(x, t + \tau) \geq u^{(\tau)}(x, t), \forall x \in \mathbb{R}^n, \forall t > 0, \]
which, together with (2.7), implies
\[ \int_{\mathbb{R}^n} \min \{ \overline{u}(x, t + \tau), u(x, t + \tau) \} \, dx \geq \int_{\mathbb{R}^n} \min \{ \overline{u}(x, \tau), u(x, \tau) \} \, dx, \quad \forall t > 0. \]

Now letting \( \tau \to 0 \), using the fact of \( u \) and \( \overline{u} \) in \( C([0, \infty), L^1(\mathbb{R}^n)) \) and observing that \( u \) satisfies (1.8) we see that
\[ \int_{\mathbb{R}^n} \min \{ u(x, \tau), u(x, \tau) \} \, dx = \int_{\mathbb{R}^n} u(x, 0) \, dx = \int_{\mathbb{R}^n} u(x, t) \, dx, \quad \forall t > 0. \]

This implies the desired result (2.6).

**Step 2.** Suppose that \( u \) and \( \overline{u} \), respectively, nonnegative \( L^1 \)-regular sub-solution and \( L^1 \)-regular solution of (1.1) with \( u(x, 0) \leq \overline{u}(x, 0) \) in \( \mathbb{R}^n \). In this case, we fix \( \tau > 0 \) and consider the unique solution \( \check{u}(\tau) \in C(\mathbb{Q}) \cap L^\infty(\mathbb{Q}) \) with \( \check{u}(\tau)(x, 0) = u(x, \tau) \). Then the same arguments as in Step 1 (replaced “min” by “max”) yields \( \check{u} \leq \underline{u} \) in \( \mathbb{Q} \).

**Step 3.** Suppose that \( \underline{u} \) and \( \overline{u} \), respectively, nonnegative \( L^1 \)-regular sub-solution and \( L^1 \)-regular super-solution. Then fix \( \tau > 0 \) and let \( u \) be the unique \( L^1 \)-regular solution of (1.1) with initial value \( u(x, t) = \underline{u}(x, \tau) \) at \( t = \tau \). Note that the existence and the uniqueness of such a solution follow from Theorem 1.2 whose proof is independent of the Step 3 (but depends on the Step 1 and Step 2). Combining the results of Steps 1 and 2, we have
\[ \underline{u}(x, t) \leq u(x, t) \leq \overline{u}(x, t), \quad \forall x \in \mathbb{R}^n, \quad \forall t > \tau. \]

This proves Theorem 1.1 since \( \tau \) is arbitrary.

### 3. SOLUTIONS WITH INITIAL DATA in \( C^\infty_0(\mathbb{R}^n) \)

In the proof of Theorem 1.2, we will approximate the initial value of the solution to (1.1) by a sequence of functions in \( C^\infty_0(\mathbb{R}^n) \). Thus, the first step is to study the solutions whose initial data are compactly supported in \( \mathbb{R}^n \). In this section, we will construct a super-solution of (1.1) by a scaling technique and then use the super-solution to study such solutions.

**Lemma 3.1** Suppose that \( u_0 \in C^\infty_0(\mathbb{R}^n) \) and let \( Q_T = \mathbb{R}^n \times [0, T) \). Then for any \( T > 0 \), there exists a positive function \( \overline{u} \in C(\mathbb{Q}_T) \cap L^\infty(\mathbb{Q}_T) \cap C([0, T), L^1(\mathbb{R}^n)) \) such that \( \overline{u} \) is a super-solution of (1.1) in the domain \( Q_T \) and \( u_0(x) \leq \overline{u}(x, 0) \) in \( \mathbb{R}^n \).

**Proof.** Since \( \beta > 0 \) (recall (1.4)–(1.6)), \( \overline{\pi} > \frac{n-2}{n} \) and thus \( \frac{1-m_i}{2} < \alpha_i \) for each \( i \). Consequently, we can choose \( \theta_i > 2 \) and \( \alpha > 0 \) such that
\[ \frac{1-m_i}{2} < \frac{1}{\alpha \theta_i} < \alpha_i, \quad (i = 1, 2, \cdots, n). \tag{3.1} \]
Hence
\[ \sum_{i=1}^{n} \alpha_i = 1, \quad \text{and} \quad -m_i \alpha - \frac{2}{\theta_i} = -m_i \alpha - 2 + \frac{2\theta_i - 2}{\theta_i} < -\alpha. \] (3.2)

Let \( \mu_i = \frac{1 - m_i}{2} \),
\[
R_0 = \max \left\{ 1, \left[ \frac{n \max_{1 \leq i \leq n} \{m_i \alpha (m_i \alpha + 1) \theta_i^2\}}{\min_{1 \leq i \leq n} \{\alpha_i \alpha \theta_i\} - 1} \right]^{\frac{1}{\theta_i}} \right\}
\]
and
\[
\Omega_{R_0} = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{n} |y_i|^\theta_i > R_0 \right\}.
\]

Then the function
\[
f(y) = \left( \sum_{i=1}^{n} |y_i|^\theta_i \right)^{-\alpha}
\]
belongs to \( C^2(\Omega_{R_0}) \cap L^1(\Omega_{R_0}) \) and satisfies
\[
\sum_{i=1}^{n} \left[ (f^{m_i}) y_i y_i + \alpha_i (y_i f) y_i \right] \leq 0 \text{ in } \Omega_{R_0}
\]
by a direct computation using (3.2) and (1.6). The last inequality tells us that \( f \) is a super-solution of (1.3) in the domain \( \Omega_{R_0} \). This implies that for any constant \( \lambda \), the function
\[
f^{(\lambda)}(y) = \lambda^{\frac{2}{\alpha}} \left( \sum_{i=1}^{n} |y_i|^{\theta_i} \lambda^{\frac{1-m_i}{\alpha} \theta_i} \right)^{-\alpha}
\] (3.3)
is also a super-solution of (1.3) in the domain
\[
\Omega_{R_0}^{(\lambda)} = \left\{ y = (y_1, y_2, \ldots, y_n) : y_i = \lambda^{\frac{m_i - 1}{\alpha \beta}} x_i, x = (x_1, x_2, \ldots, x_n) \in \Omega_{R_0} \right\}.
\]

Now for any positive constants \( T, C_0 \) and \( A \), choose a \( \lambda \) such that
\[
\lambda \geq \max \left\{ \left[ C_0 (1 + \beta T)^{\frac{1}{\alpha}} R_0^\alpha \right]^{\frac{\alpha \beta}{\theta_i}}, (C_0 A^\alpha)^{\frac{\alpha \beta}{\theta_i}} \max_{1 \leq i \leq n} \left( \alpha - \mu_i \theta_i \right)^{-1} \right\}
\]
and let
\[
Q(C_0, T) = \left\{ (x, t) \in Q_T : (1 + \beta t)^{-\frac{\alpha \beta}{\theta_i}} x_1(1 + \beta t)^{-\frac{\alpha \beta}{\theta_i}} \cdots x_n(1 + \beta t)^{-\frac{\alpha \beta}{\theta_i}} \leq C_0 \right\}.
\]

We may verify that
\[
\left\{ y = (y_1, y_2, \ldots, y_n) : y_i = x_i(1 + \beta t)^{-\frac{\alpha \beta}{\theta_i}}, \ (x, t) \in Q(C_0, T) \right\} \subset \Omega_{R_0}^{(\lambda)}
\]
\[
\]
and the function
\[
\bar{u}(x, t) = \begin{cases} 
(1 + \beta t)^{-\frac{1}{n}} f(\lambda) \left( x_1 (1 + \beta t)^{-\frac{1}{n}}, \ldots, x_n (1 + \beta t)^{-\frac{1}{n}} \right), & (x, t) \in Q(C_0, T) \\
C_0, & (x, t) \in Q_T \setminus Q(C_0, T)
\end{cases}
\]
(3. 4)
is a super-solution of (1. 1) in \( Q_T \). Obviously, \( \bar{u} \in C(Q_T) \cap L^\infty(Q_T) \cap C([0, T), L^1(\mathbb{R}^n)) \). Moreover, we easily see that
\[
(1 + \beta t)^{-\frac{1}{n}} f(\lambda) \left( x_1 (1 + \beta t)^{-\frac{1}{n}}, \ldots, x_n (1 + \beta t)^{-\frac{1}{n}} \right) = C_0
\]
for all \( t \in [0, T) \) and all the \( x = (x_1, x_2, \ldots, x_n) \) satisfying \( |x_i| \leq A_1^{\frac{1}{n}}, i = 1, 2, \ldots, n \). Hence,\[
\bar{u}(x, t) = C_0, \quad \forall (x, t) \in \left( \prod_{i=1}^n [-A_1^{\frac{1}{n}}, A_1^{\frac{1}{n}}] \right) \times [0, T].
\]
Consequently, choose \( A \) and \( C_0 \) large enough such that \( \prod_{i=1}^n [-A_1^{\frac{1}{n}}, A_1^{\frac{1}{n}}] \supset \text{supp} u_0 \) and \( C_0 \geq \max_{x \in \mathbb{R}^n} u_0(x) \), we have proved the lemma.

**Remark 3.1** Using (3. 4) and Lemma 2.2, we see that the function
\[
V(y, \tau) = \begin{cases} 
f(\lambda)(y), & (y, \tau) \in P = \left\{(y, \tau) : f(\lambda)(y) \leq C_0 e^\tau, 0 \leq \tau \leq \frac{\ln(1 + \beta T)}{\beta} \right\} \\
C_0 e^\tau, & \text{otherwise}
\end{cases}
\]
is a super-solution of (1. 2).

**Lemma 3.2** Suppose that \( u_0 \in C_0^\infty(\mathbb{R}^n) \), \( u_0 \geq 0 \) in \( \mathbb{R}^n \). Let \( u \) be the unique bounded and continuous solution with \( u(x, 0) = u_0(x) \) in \( \mathbb{R}^n \). Then one has
\[
u \in C([0, \infty), L^1(\mathbb{R}^n)),
\]
(3. 5)
\[
\int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} u_0(x) dx, \quad \forall t > 0
\]
(3. 6)
and
\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}} \|u_0\|_{L^1(\mathbb{R}^n)}^2, \quad \forall t > 0,
\]
(3. 7)
where \( C \) is a constant depending only on \( n \) and \( m_i \) (i = 1, 2, \ldots, n).

**Proof.** Fix \( T > 0 \). By Lemma 3.1 and Lemma 2.1 we see that
\[
0 \leq u(x, t) \leq \bar{u}(x, t) \quad \text{in} \quad Q_T,
\]
(3. 8)
where $\mathbf{u}$ is the same as in (3.4). Hence (3.5) comes from the continuity of $u$ and the structure of $\mathbf{u}$.

(3.7) was proved in [6]. To show (3.6), we use Lemma 2.1 to see that $u$ can be approximated by an (equi-continuous and uniformly bounded) sequence $\{u_k\}_{k=1}^{\infty}$ of classical solutions of (1.1) in the norm space of the space $C_{\text{loc}}(\mathbb{R}^n \times [0, \infty))$. Multiplying (1.1) for $u_k$ by $\phi \in C_0^\infty(\mathbb{R}^n)$, integrating the resulted equation for $x$ over $\mathbb{R}^n$ and for $t$ over $[0, T]$, and passing the limit as $k \to \infty$, we obtain that

$$
\int_{\mathbb{R}^n} u(x, T) \phi(x) dx - \int_{\mathbb{R}^n} u(x, 0) \phi(x) dx = \int_0^T \int_{\mathbb{R}^n} \sum_{i=1}^n u_{x_i} \phi_{x_i} dx dt.
$$

(3.9)

Choose a function $g \in C_0^\infty(-\infty, \infty)$ satisfying

$$
0 \leq g \leq 1, \quad |g'| \leq 4, \quad |g''| \leq 8 \quad \text{in} \quad R
$$

and

$$
g(s) = \begin{cases} 
1, & |s| \leq 1 \\
0, & 2 \leq |s| \leq \infty.
\end{cases}
$$

Taking $\phi$ in (3.9) as

$$
\phi^{(k)}(x) = \prod_{i=1}^n g(x_1 k^{-\frac{1}{\theta_1}}, \cdots, x_n k^{-\frac{1}{\theta_n}}), \quad k = 1, 2, \cdots
$$

and observing that

$$
\left| \int_0^T \int_{\mathbb{R}^n} \sum_{i=1}^n u_{x_i}^{(k)} \phi_{x_i} dx dt \right| \leq C(T, n, m_i) k^{-\min_{1 \leq i \leq n} \{\theta_i^{-1}\}} \to 0
$$

as $k \to \infty$, where we have used (3.8) and (3.4), we finally find that

$$
\int_{\mathbb{R}^n} u(x, T) dx = \int_{\mathbb{R}^n} u(x, 0) dx.
$$

This proves (3.6) since $T$ is arbitrary.

**Lemma 3.3** Suppose that $u(x, t)$ is defined as in Lemma 3.2 and $V(y, \tau)$ is related to $u(x, t)$ as in Lemma 2.2. Then there exists a function $F \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, depending only on the upper bounds of $\|u_0\|_{L^\infty(\mathbb{R}^n)}$ and $\|u_0\|_{L^1(\mathbb{R}^n)}$, such that

$$
0 \leq V(y, \tau) \leq F(y), \quad \forall y \in \mathbb{R}^n, \quad \forall \tau > 0.
$$

(3.10)
Proof. Owing (3.8), (3.3), (3.4) and (3.7), we see that there is a constant $C_1$, depending only on $n, m, \|u_0\|_{L^\infty(\mathbb{R}^n)}$ and $\|u_0\|_{L^1(\mathbb{R}^n)}$, such that
\[
0 \leq V(y, \tau) \leq C_1, \quad \forall (y, \tau) \in Q.
\]
Since $V_0(y) = V(y, 0) = u_0(y) \in C^\infty_0(\mathbb{R}^n)$, choose a $R_1 \geq R_0$ such that $\text{supp}V_0 \subset \mathbb{R}^n \setminus \Omega_{R_1}$. Repeating the arguments from (3.2) to (3.3), we find a $\lambda_1 > 0$ and a bounded and continuous super-solution $f^{(\lambda_1)}$ of (1.3) in the domain $\Omega^{(\lambda_1)}_{R_1}$ such that $f^{(\lambda_1)} > 0$ in $\Omega^{(\lambda_1)}_{R_1}$ and $f^{(\lambda_1)} = C_1$ on $\partial \Omega^{(\lambda_1)}_{R_1}$. Let
\[
F(y) = \begin{cases} f^{(\lambda_1)}(y), & y \in \Omega^{(\lambda_1)}_{R_1} \\ C_1, & y \in \mathbb{R}^n \setminus \Omega^{(\lambda_1)}_{R_1}. \end{cases}
\]
Then $F$ is a bounded and continuous super-solution of (1.3) in $\mathbb{R}^n$ (of (1.1) in $\mathbb{R}^n \times (0, \infty)$) satisfying $F \geq V_0$ in $\mathbb{R}^n$. Consequently, the comparison principle (Lemma 2.1) implies (3.10).

4. PROOF OF THEOREM 1.2

We want to prove Theorem 1.2 only for equation (1.1) due to Lemma 2.2. Note that the uniqueness is direct from the Steps 1 and 2 in the proof of Theorem 1.1 and the proof of existence below is independent of Step 3 there.

Step 1. Suppose the initial value $u_0 \in L^\infty(\mathbb{R}^n)$ is nonnegative such that $\text{supp}u_0$ is a bounded set in $\mathbb{R}^n$. We will prove that (1.1) has a nonnegative $L^1$-regular solution $u$ satisfying $u(x, 0) = u_0(x)$ in $\mathbb{R}^n$.

Choose a sequence $u_{0k} \subset C^\infty_0(\mathbb{R}^n)$ such that
\[
\|u_{0k}\| \leq C_2, \quad u_{0k} \leq u_{0k'} \leq u_0 \in \mathbb{R}^n, \quad \text{supp}u_{0k} \subset \Omega, \quad (k \leq k', \ k, k' = 1, 2, \cdots) \quad (4.1)
\]
and
\[
\lim_{k \to \infty} u_{0k} = u_0 \text{ in } L^\infty(\mathbb{R}^n) \bigcap L^1(\mathbb{R}^n), \quad (4.2)
\]
where constant $C_2$ and domain $\Omega \subset \mathbb{R}^n$ are independent of $k$. For each $k$, Let $u_k$ be the solution of (1.1) with initial value $u_{0k}$. As we have said, Lemma 2.1 imply that each $u_k$ is a unique, continuous and bounded in $\mathbb{R}^n$, satisfying
\[
\|u_k\|_{L^\infty(Q)} \leq C_3, \quad 0 \leq u_k \leq u_{k'} \text{ in } Q, \quad (k \leq k', \ k, k' = 1, 2, \cdots) \quad (4.3)
\]
for some constant $C_3$ depending on $C_2$ but independent of $k$ (also see (3.7)), and
\[
\int_{\mathbb{R}^n} u_k(x, t) dx = \int_{\mathbb{R}^n} u_{0k}(x) dx, \quad \forall t > 0 \quad (4.4)
\]
and
\[
\lim_{t \to 0} u_k(x, t) = u_{0k}(x) \text{ in } L^1(\mathbb{R}^n), \quad (k = 1, 2, \cdots)
\]
by (3.5) and (3.6). Hence, \( \{ u_k \} \) is equi-continuous in \( \mathbb{R}^n \times [\tau_0, \infty) \) for any \( \tau_0 > 0 \) by Lemma 3.1 in [4] (or more generally, Theorem 5.1 in [6]).

Now fix \( \tau_0 > 0 \). Let \( V_k(y, \tau) \) be the solutions of (1.2) related to \( u_k \) by Lemma 2.2. Then by Lemma 3.3 we have a function

\[
F \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)
\]

such that

\[
0 \leq V_k(y, \tau) \leq V_k'(y, \tau) \leq F(y), \quad \forall y \in \mathbb{R}^n, \quad \forall \tau > 0, \quad (k \leq k', \quad k, k' = 1, 2, \cdots).
\]

(4.6)

Therefore, we can choose a subsequence of \( \{ V_k \} \) such that

\[
\lim_{k \to \infty} V_k(y, \tau) = V(y, \tau), \quad \forall (y, \tau) \in \mathbb{R}^n \times [\tau_0, \infty)
\]

(4.7)

and

\[
\lim_{k \to \infty} V_k(\cdot, \tau) = V(\cdot, \tau) \text{ in } L^1(\mathbb{R}^n), \quad \forall \tau \geq \tau_0
\]

(4.8)

for some function \( V \in C(\mathbb{R}^n \times [\tau_0, \infty)) \) satisfying

\[
0 \leq V_k(y, \tau) \leq V(y, \tau) \leq F(y), \quad \forall y \in \mathbb{R}^n, \quad \forall \tau > \tau_0, \quad (k = 1, 2, \cdots).
\]

(4.9)

Obviously, \( V \) is solution of (1.2) satisfying

\[
V \in C(\mathbb{R}^n \times (0, \infty)) \cap L^\infty(\mathbb{R}^n \times (\tau_0, \infty))
\]

(4.10)

since \( \tau_0 \) is arbitrary.

For any \( t_0 > 0 \), choose \( \tau_0 \in (0, t_0) \). Then it follows from (4.9) and the continuity of \( V \) that

\[
\lim_{\tau \to t_0} V(\cdot, \tau) = V(\cdot, t_0) \text{ in } L^1(\mathbb{R}^n).
\]

(4.11)

It follows from Lemma 2.2, (4.4) and the fact of \( \sum_{i=1}^{n} \alpha_i = 1 \) that

\[
\int_{\mathbb{R}^n} V_k(y, \tau)dy = \int_{\mathbb{R}^n} u_k(x, t)dx = \int_{\mathbb{R}^n} u_{0k}(x)dx, \quad k = 1, 2, \cdots, \forall t > 0, \tau = \ln h(t), \quad (4.12)
\]

which, together (4.2), (4.8) and the fact of \( \sum_{i=1}^{n} \alpha_i = 1 \) again, implies

\[
\int_{\mathbb{R}^n} u(x, t)dx = \int_{\mathbb{R}^n} V(y, \tau)dy = \int_{\mathbb{R}^n} u_{0}(x)dx, \forall t > 0, \tau = \ln h(t).
\]

(4.13)
Note that (4.9), (4.12) and (4.13) imply
\[ \|u(\cdot, t) - u_0\|_{L^1(\mathbb{R}^n)} \]
\[ \leq \|u(\cdot, t) - u_k(\cdot, t)\|_{L^1(\mathbb{R}^n)} + \|u_k(\cdot, t) - u_{0k}\|_{L^1(\mathbb{R}^n)} + \|u_{0k} - u_0\|_{L^1(\mathbb{R}^n)} \]
\[ = \int_{\mathbb{R}^n} (u(x, t) - u_k(x, t)) dx + \|u_k(\cdot, t) - u_{0k}\|_{L^1(\mathbb{R}^n)} + \|u_{0k} - u_0\|_{L^1(\mathbb{R}^n)} \]
\[ = 2 \|u_{0k} - u_0\|_{L^1(\mathbb{R}^n)} + \|u_k(\cdot, t) - u_{0k}\|_{L^1(\mathbb{R}^n)}, \quad (k = 1, 2, \ldots), \; \forall t > 0. \]

By (4.4) and (4.2), we have
\[ \lim_{t \to 0^+} u(\cdot, t) = u_0 \text{ in } L^1(\mathbb{R}^n). \]

Combining this, (4.11), (4.13) with (4.10), we have proved the conclusion of Step 1. Moreover by the Steps 1 and 2 in the proof of of Theorem 1.1, the solution \( u \) is unique, and satisfies
\[ 0 \leq V(y, \tau) \leq F(y), \quad \forall (y, \tau) \in Q, \]
and \( \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq Ct^{-\frac{1}{n}} \|u_0\|_{L^1(\mathbb{R}^n)}, \quad \forall t > 0 \)
by (4.6), (4.7), (4.2) and the (3.7) in Lemma 3.2.

**Step 2.** Suppose \( u_0 \in L^1(\mathbb{R}^n), \; u_0 \geq 0 \text{ in } \mathbb{R}^n. \) We will prove that (1.1) has a nonnegative \( L^1 \)-regular solution \( u \) satisfying \( u(x, 0) = u_0(x) \) in \( \mathbb{R}^n. \)

Let \( \chi_{B_k} \) be the characteristic function of the ball \( B_k(0) \) and set
\[ u_{0k}(x) = \min \{k, u_0(x)\} \chi_{B_k}(x). \]

Then
\[ 0 \leq u_{0k} \leq u_{0k'} \leq u_0 \text{ in } \mathbb{R}^n \quad (k \leq k', \; k, k' = 1, 2, \ldots, )u_0 = \lim_{k \to \infty} u_{0k} \text{ in } L^1(\mathbb{R}^n). \quad (4.15) \]

For each \( k > 0 \), there exists a unique \( L^1 \)-regular solution \( u_k \) of (1.1) such that \( u_k(x, 0) = u_{0k} \) in \( \mathbb{R}^n \) due to the result of Step 1. In particular, we have
\[ \int_{\mathbb{R}^n} u_k(x, t) dx = \int_{\mathbb{R}^n} u_{0k}(x) dx \leq \|u_0\|_{L^1(\mathbb{R}^n)}, \quad \forall t > 0, \; \forall k = 1, 2, \ldots, \]
\[ \lim_{t \to 0^+} u_k(\cdot, t) = u_{0k} \text{ in } L^1(\mathbb{R}^n) \quad k = 1, 2, \ldots \]
and
\[ 0 \leq u_k(x, t) \leq u_{k'}(x, t), \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty), \; \forall k \leq k' \]
by the comparison principle as in the Steps 1 and 2 of the proof of Theorem 1.1. Furthermore, (4.13) and (4.15) imply
\[ \|u_k(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{1}{n}} \|u_{0k}\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}} \leq C t^{-\frac{1}{n}} \|u_0\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n}}, \quad \forall t > 0. \quad (4.19) \]
Now for any $t_0 > 0$, we use (4.15) and Lemma 3.1 in [4] (or more generally, Theorem 5.1 in [6]) to see that $\{u_k\}$ is equi-continuous. Thus, (4.15)-(4.19) imply that there is a $u \in C(Q) \cap L^\infty(\mathbb{R}^n \times [t_0, \infty))$ such that

$$\lim_{k \to \infty} u_k = u \text{ in } Q, \quad \lim_{k \to \infty} u_k(\cdot, t) = u(\cdot, t) \text{ in } L^1(\mathbb{R}^n), \quad \forall t > 0,$$

and

$$0 \leq u_k \leq u \text{ in } Q, \quad k = 1, 2, \ldots. \quad (4.21)$$

Obviously, $u$ is a solution of (1.1). To complete the proof, we want only to prove $u \in C((0, \infty), L^1(\mathbb{R}^n))$. For this purpose, we fix a $t_0 > 0$. For any given $\varepsilon > 0$ we use (4.15) to see that there exists a $k_0$ such that

$$\int_{\mathbb{R}^n} (u_0(x) - u_{0k_0}(x)) dx = \|u_0 - u_{0k_0}\| \leq \frac{\varepsilon}{4}. \quad (4.22)$$

Since $u_{k_0}$ is continuous and $V_{k_0}$, related to $u_{k_0}$ as in Lemma 2.2, satisfies (4.14) for some $F \in L^1(\mathbb{R}^n)$, we can find a $\delta > 0$ such that

$$\int_{\mathbb{R}^n} |u_{k_0}(\cdot, t) - u_{k_0}(\cdot, t_0)| dx \leq \frac{\varepsilon}{2}, \quad \forall t \in (t_0 - \delta, t_0 + \delta). \quad (4.23)$$

This, together with (4.20), (4.21), (4.16) and (4.22)

$$\int_{\mathbb{R}^n} |u(\cdot, t) - u(\cdot, t_0)| dx$$

$$\leq \int_{\mathbb{R}^n} |u(t) - u_{k_0}(t)| dx + \int_{\mathbb{R}^n} |u_{k_0}(t) - u_{k_0}(t_0)| dx + \int_{\mathbb{R}^n} |u_{k_0}(t_0) - u(t_0)| dx$$

$$= 2 \int_{\mathbb{R}^n} (u_0(x) - u_{0k}(x)) dx + \int_{\mathbb{R}^n} |u_{k_0}(\cdot, t) - u_{k_0}(\cdot, t_0)| dx$$

$$\leq \varepsilon, \quad \forall t \in (t_0 - \delta, t_0 + \delta),$$

which shows that $u \in C((0, \infty), L^1(\mathbb{R}^n))$. Repeating the same argument for $t_0 = 0$, with (4.17) instead of (4.23), we see that $\lim_{t \to 0^+} u(t) = u_0$ in $L^1(\mathbb{R}^n)$. This proves Theorem 1.2.
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