New representation and a vacuum state for canonical quantum gravity

Hans-Jürgen Matschull
Department of Mathematics, King’s College London, Strand, London WC2R 2LS, England
email: hjm@mth.kcl.ac.uk

kcl-th-94-22, gr-qc/9412020
6 December 1994

Abstract
A new representation for canonical gravity and supergravity is presented, which combines advantages of Ashtekar’s and the Wheeler DeWitt representation: it has a nice geometric structure and the singular metric problem is absent. A formal state functional can be given, which has some typical features of a vacuum state in quantum field theory. It can be canonically transformed into the metric representation. Transforming the constraints too, one recovers the Wheeler DeWitt equation up to an anomalous term. A modified Dirac quantization is proposed to handle possible anomalies in the constraint algebra.

1 The classical action
The easiest way to obtain the new representation of canonical gravity is to start from the complex Lagrangian for general relativity which can also be used to derive Ashtekar’s variables and the polynomial constraints directly from an action principle [1, 2]. The basic field variables appearing in this action are the vierbein components $E_M^A$ with covector index $M$, taking the values $t, x, y, z$ for the local coordinates, and the flat index $A = 0, 1, 2, 3$, raised and lowered using the Minkowski metric $\eta_{AB} = \text{diag}(-1, 1, 1, 1)$. In addition, the “Ashtekar action” depends on the $\text{so}(3, C)$ connection $A_{Ma}$, where $a = 1, 2, 3$ labels the generators of $\text{so}(3)$. It can be interpreted as the $\text{so}(3, C)$ representation of the $\text{so}(1, 3)$ spin connection $\Omega_{MAB}$. The algebras $\text{so}(1, 3) \simeq \text{so}(3, C)$ are mapped onto each other by an isomorphism

$$A_{Ma} = J_{a}^{AB} \Omega_{MAB}, \quad A_{Ma}^* = J_{a}^{AB} \Omega_{MAB}^*$$

where $J_{aAB}$ is a constant “matrix” mapping antisymmetric real tensors onto complex 3-vectors. Properties of these $J$-symbols are summarized in the appendix.

The difference between Ashtekar’s representation and that presented here is that we will not treat the connection as an independent field and thus it will not appear as a canonical variable or quantum operator (however, it will appear as a useful function on phase space later on). Instead, it
is defined as a function of the vierbein and its derivatives, implicitly given by the vierbein postulate, i.e. it is required that the vierbein is covariantly constant. The equations are most simply written as

\[ \nabla_M E^A_N = \partial_M E^A_N + \Omega_{(M}^A E^B_{N)} = 0. \]  

As is well known this defines \( \Omega_{MAB}[E] \) uniquely if and only if the vierbein is invertable (see e.g. [3] for the explicit solution).

We will therefore assume that the determinant \( E = \det(E^A_M) \) does not vanish and the inverse vierbein \( E^{-1}_A^M \) exists. This is another crucial difference between this and Ashtekar’s representation: there one has to give up this restriction to define the connection representation properly, but on the other hand one must use it first to obtain the polynomial constraints (see [4] for a critical discussion of these problems). Here we will insist on invertable metrics, which also has the consequence that there is no need to have polynomial constraints; phase space functions like \( E^{-1} \) are well defined.

The Einstein Hilbert action can be expressed in terms of the field strength of the spin connection

\[ S[E] = -\frac{1}{2} \int d^4x \epsilon^{MNPQ} E^M_A E^N_B J_{aAB} F_{PQa}[E], \]  

where

\[ F_{PQa} = \partial_P A_{Qa} - \partial_Q A_{Pa} + \epsilon_{abc} A_{Pb} A_{Qc}. \]  

An explicit derivation is given in the appendix.

Remember that this so called 1.5 order action has the useful property that the spin connection obeys its own equation of motion, which is just the vierbein postulate, and this remains true when writing the action in terms of \( A_{Ma} \) instead of \( \Omega_{MAB} \) (this is not as trivial as one might think; see the appendix for a proof). Whenever we compute functional derivatives of the action with respect to \( E^A_M \), we only have to vary the explicit vierbein fields appearing in (1.3), as long as we do not use the vierbein postulate before calculating the derivative.

The basic gauge symmetries of the Einstein Hilbert action are, of course, the local Lorentz symmetry acting on the flat indices, and the invariance under diffeomorphisms of the background 4-manifold. Under a local Lorentz transformation with parameter \( \lambda_a = J_a^{AB} \lambda_{AB} \) the fields transform as

\[ \delta E^A_M = \lambda^A_B E^B_N \quad \Rightarrow \quad \delta \Omega_{MAB} = -D_M \lambda_{AB}, \]
\[ \delta A_{Ma} = -D_M \lambda_a. \]  

Note that the vierbein is the only primary field here and thus the transformations of \( \Omega_{MAB} \) and \( A_{Ma} \) are obtained via (1.2). Of course, all this is well known, but let us explain the main idea of this article. In Ashtekar’s representation the basic phase space variable is \( A_{Ma} \), which is the \( \mathfrak{so}(3,\mathbb{C}) \) gauge field of this symmetry. The constraint associated with this gauge freedom is easily solved by considering Wilson loops [5]. There has been much effort to construct new kinds of representations based on these invariants, and to solve the remaining contraints. However, many problems of this “Ashtekar program”, which were known from the very beginning, are still unsolved, e.g. how to treat singular metrics [4].

Ashtekar’s representation splits the contraints, and thus the gauge symmetries, into a simple part, the Lorentz transformations, and a more complicated part consisting in principle of the generators of the four dimensional diffeomorphisms. These are hard to solve, mainly because the
The diffeomorphism group is somewhat awkward to deal with. The question arising is whether it is possible to interchange the roles of the two local gauge groups. It would then be natural to choose a representation where $E_M^A$ is the basic configuration variable (i.e. the wave functional depends on $E_M^A$), as in a certain sense the metric or the vierbein may be considered as the gauge field of the diffeomorphism group. This would also avoid the difficulties concerning the singular metrics, because the configuration space, which is the support of the wave functional, could be taken to be the set of all invertable vierbein fields.

To exploit this idea, let us check whether we can identify a gauge field of the diffeomorphism group explicitly. It must be a one form with an extra four dimensional index, because the generator is a local 4-vector. Obviously, $E_M^A$ is such a one-form. If it is a gauge field, then there should be a symmetry under which it transforms as

$$\delta E_M^A = -D_M V^A, \quad (1.6)$$

where $V^A$ is the parameter field. In fact, we found a symmetry of the action, as can be seen easily by using the 1.5 order trick. We only have to vary the two vierbein fields in (1.3), integrate by parts, using the vierbein postulate (1.2) and the Bianchi identity for the field strength

$$D_{[M} F_{NP]} = 0, \quad (1.7)$$

which holds independently of the vierbein postulate. Let us call (1.6) a “translation”, because it is related to the Lorentz rotation (1.3), like the translations of the Poincaré group are related to the rotations. Explicitly, the commutator of a translation $V^A$ and a rotation $\lambda^{AB}$ gives another translation with parameter $V^A \lambda^{AB}$. However, two translations do not commute, because the spin connection appears in (1.6).

To get the commutator of two translations, we have to compute the transformation of $\Omega_{MAB}$ under a translation. Using the vierbein postulate and some properties of the Riemann tensor, we find

$$\delta \Omega_{MAB} = R_{MNAB} V^N, \quad (1.8)$$

where $V^N = E_A^N V^A$. Acting on (1.6) with another translation yields the commutator

$$[\delta_1, \delta_2] E_M^A = R_{PQAB} V_1^P V_2^Q E_M^B, \quad (1.9)$$

which is a Lorentz transformation with field dependent parameter $\lambda_{AB} = R_{PQAB} V_1^P V_2^Q$ or $\lambda_a = F_{PQa} V_1^P V_2^Q$.

Finally, let us see how the translations are related to the usual Lie derivative appearing as the generator of diffeomorphisms, which has as its parameter a tangent vector $V^M$. To obtain the Lie derivative, one has to add a translation with parameter $V^A = E_M^A V^M$ and a Lorentz rotation with parameter $\lambda_{AB} = \Omega_{MAB} V^M$. Using the explicit solution $\Omega_{MAB} = E_N^A \nabla_M E_B^N$ for the vierbein postulate (where $\nabla_M$ is the metric covariant derivative), the transformation becomes

$$\delta E_M^A = -D_M (E_N^A V^N) + \Omega_{NAB} V^N E_M^B$$

$$= -E_N^A \nabla_M V^N - V^N \nabla_N E_M^A. \quad (1.10)$$

This is the Lie derivative of $E_M^A$ along $-V^N$. As the vierbein is invertable, there is a one-to-one relation between translations and generators of diffeomorphisms and we may regard the translations as the basic symmetries of the action instead of the diffeomorphisms.
2 Canonical formulation

Here we will derive the classical constraint algebra by applying the Dirac canonical formalism to the action (1.3). If we use the vierbein components as canonical configuration variables and Lagrange multipliers, instead of introducing a lapse function and a shift vector, we end up with a Lorentz covariant set of hamiltonian constraints. They have a nice geometric structure like Ashtekar’s constraints, but do not split into a vector and scalar constraint.

Space time split, Lagrangian and momenta

We will now Space time is split into a three dimensional space spanned by coordinates \( m = x, y, z \) and time \( t \), which must be assumed to be a global coordinate. Thus all space time indices split into \( M \mapsto m, t \). If we define the spacial Levi Civita tensor by \( \varepsilon^{mnp} = \varepsilon^{tnmp} \) (or \( \varepsilon^{xyz} = \varepsilon_{xyz} = 1 \)), then the Lagrange density becomes

\[
- i \varepsilon^{mnp} E^A_m E^B_n J^a_{AB} F_{tpa}[E] - i \varepsilon^{mnp} E^A_t E^B_m J^a_{AB} F_{npa}[E].
\]  

(2.1)

Note that we do not fix any gauge here, i.e. the vierbein is not required to split into a dreibein, a lapse function and a shift vector. The Lagrangian is still invariant under the full local Lorentz group. However, let us impose a restriction on the configuration space: we require the hypersurface to be spacelike, i.e. its intrinsic metric \( g_{mn} = E^A_m E^B_n \eta_{AB} \) must have signature \((+, +, +)\). It is important to note that this restriction has nothing to do with a gauge fixing of the local Lorentz group. It is the usual restriction one imposes on the metric in the Wheeler DeWitt \([7, 8]\) approach, but replacing the metric by the vierbein does not automatically lead to a vierbein splitting into a dreibein, a lapse and a shift vector.

Because of the fixed signature of \( \eta_{AB} \) it is equivalent to require \( g = \det(g_{mn}) > 0 \), or that there exists a timelike normal vector \( N^A \) uniquely defined by

\[
\varepsilon_{ABCD} N^A E^B_m E^C_n E^D_p = \varepsilon_{mnp}, \quad N^A E^A_m = 0.
\]  

(2.2)

Observe that \( N^A \) is a density of weight \(-1\) under diffeomorphisms and that it does not exist if we allow the hypersurface to become lightlike, as the normal vector then becomes tangent to the surface itself and cannot be normalized by the first equation in (2.2). The restriction of the configuration space will simplify the discussions below, where we will always assume that \( N^A \) exists.

The expression (2.1) for the Lagrange density still contains the second time derivative of the vierbein in its first term, which must be eliminated by a partial integration. Thus we define

\[
L = \int d^3x \left( 2i \varepsilon^{mnp} \partial_t E^A_m E^B_n J^a_{AB} A_{tpa} - i \varepsilon^{mnp} D_p (E^A_m E^B_n J_{AB}) A_{ta} - i \varepsilon^{mnp} E^A_t E^B_m J_{AB} F_{npa} \right). 
\]  

(2.3)

From now on the action is different from the Einstein Hilbert action, because we added a complex total derivative, thus the action itself is complex and one has to deal with complex momenta obeying certain reality conditions. How to do this if the imaginary part of the action is a total divergence has been worked out in [9]. Let us first obtain the momenta and then derive the primary constraints and reality conditions they obey.
The connection components $A_{m\alpha}$ and $A_{\alpha \tau}$ still obey their own equations of motion, thus we can neglect their dependence on $\partial_t E_m^A$ when differentiating the action: the momentum conjugate to $E_m^A$ reads

$$P_A^m[E, \partial_t E] = 2i \varepsilon^{mnp} J_{aAB} E_n^B A_p^\alpha [E, \partial_t E]$$

and that conjugate to $E_{t^A}$ vanishes, i.e. the time components of the gauge fields are Lagrange multipliers as they should be. The phase space is given by the tangent bundle of the configuration space defined above, and the Poisson brackets read

$$\{ E_m^A, P_{B^n} \} = \delta_m^B \delta_A^B.$$ (2.5)

Whenever such a bracket or quantum commutator appears, the dependence of the fields on the space points and the spatial delta function will not be written out, as long as no derivatives are involved and it is obvious how to restore them: $\{ A, B \} = C$ has to be read as $\{ A(x), B(y) \} = C(x) \delta(x, y)$ if $A, B, C$ are local fields.

Before considering the constraints let us discuss the reality conditions on these momenta. They are obviously complex but are conjugate to real variables; there should be a relation giving $P_A^m$ as a holomorphic function of $P_A^m$ and the spatial components of the vierbein. Calculating the imaginary part of the momentum explicitly, we find that

$$P_A^m - P_A^{* m} = 2i \varepsilon^{mnp} E_n^B (J_{aAB} A_p^\alpha + J_{aAB}^* A_p^{* \alpha}) = 2i \varepsilon^{mnp} E_n^B \partial_p E_p^A = 2i \varepsilon^{mnp} \partial_n E_p^A.$$ (2.6)

In contrast to the connection representation, where the reality constraints on Ashtekar’s variables are non-polynomial, we have a very simple linear relation

$$P_A^{* m} = P_A^m - 2i \varepsilon^{mnp} \partial_n E_p^A.$$ (2.7)

The relation can also be written as

$$Q_A^m = P_A^m - i \varepsilon^{mnp} \partial_n E_p^A \in \mathbb{R},$$ (2.8)

and this $Q_A^m$ is nothing but the momentum of $E_m^A$ that would come out if we used the real Einstein Hilbert action instead of (2.3).

One can easily see that another approach to the complex momentum is to perform a canonical transformation from $Q_A^m$ to $P_A^m$ using the phase space functional (see [10] for the similar construction of Ashtekar’s variables)

$$\frac{i}{2} \int d^3x \varepsilon^{mnp} E_m^A \partial_n E_p^A,$$ (2.9)

whose time derivative is the imaginary part of the difference between (2.1) and (2.3). In principle we have to regard the reality conditions as second class constraints (with conjugate constraints $E_m^A = E_m^{* A}$) and compute the resulting Dirac brackets. However, one can show that the Dirac brackets are equal to the Poisson brackets defined by (2.5) if every phase space function is expressed as a holomorphic function of $P_A^m$ [9, 11].
The constraints

The reality constraints are not the only relations between the momenta following from \((2.4)\). There are also primary constraints. Using the fact that \(J_a^a\) commutes with \(J_b\), we find that

\[
\mathbf{L}_a = J^a_{AB} P^m_A E_{mB} = 2i \varepsilon^{mnp} J^a_{AB} J_{bAC} E_{mB} E_{nC} A_{pb} \approx 0,
\]

(2.10)
because the product of the two \(J\) symbols is symmetric in \(B, C\) (see \((A.3)\)). Note that these are in fact 3 complex (or 6 real) equations in addition to the reality conditions on \(P^m_A\). If we compute the complex conjugate by using \((2.7)\), we obtain 3 new equations which cannot be written as holomorphic functions of \((2.10)\):

\[
\mathbf{L}_a = J^a_{AB} (P^m_A - 2i \varepsilon^{mnp} \partial_n E_{pA}) E_{mB} = J^a_{AB} P^m_A E_{mB} - i \varepsilon^{mnp} \partial_m (J_{aAB} E_n^A E_p^B) \approx 0.
\]

(2.11)

Of course, \(\mathbf{L}_a\) and \(\mathbf{L}_a^*\) are the generators of Lorentz rotations, and they can also be given in the \(\text{so}(1, 3)\) representation

\[
\mathbf{L}_{AB} = P^m_A E_{mB} - i \varepsilon^{mnp} J_{aAB} \partial_m (J_{aCD} E_n^C E_p^D).
\]

(2.12)

Using the notation

\[
\mathbf{L}[\lambda_\alpha] = \int d^3x \lambda_\alpha \mathbf{L}_a, \quad \mathbf{L}[\lambda_\alpha^*] = \int d^3x \lambda_\alpha^* \mathbf{L}_a^*,
\]

(2.13)

we find that

\[
\{ E^A_m, \mathbf{L}[\lambda_\alpha] \} = \lambda_\alpha J^A_B E^B_m, \\
\{ P^m_A, \mathbf{L}[\lambda_\alpha] \} = \lambda_\alpha J^A_B P^B_m - 2i \varepsilon^{mnp} J_{aAB} E^n_B \partial_p \lambda_\alpha, \\
\{ E^A_m, \mathbf{L}[\lambda_\alpha^*] \} = \lambda_\alpha^* J^A_B E^B_m, \\
\{ P^m_A, \mathbf{L}[\lambda_\alpha^*] \} = \lambda_\alpha^* J^a_{AB} P^B_m.
\]

(2.14)

We see already here that what we obtain is somehow a mixture of the Wheeler DeWitt and Ashtekar’s representation. The conjugate momentum of the vierbein transforms as a tensor under half of the Lorentz algebra, the corresponding constraint having the form “\(E \times P\)”, but as a connection under the other half, represented by a constraint of the form “\(\partial E + E \times P\)”. The brackets of \(\mathbf{L}\) with itself form a local \(\text{so}(3, C)\) algebra:

\[
\{ \mathbf{L}[\lambda_\alpha], \mathbf{L}[\kappa_\beta] \} = \mathbf{L}[\varepsilon_{abc} \kappa_\beta \lambda_c], \\
\{ \mathbf{L}[\lambda_\alpha^*], \mathbf{L}[\kappa_\beta^*] \} = \mathbf{L}[\varepsilon_{abc} \kappa_\beta^* \lambda_c^*], \\
\{ \mathbf{L}[\lambda_\alpha], \mathbf{L}[\kappa_\beta^*] \} = 0.
\]

(2.15)

We should now check whether we have found all primary constraints. To see that there are no more primary constraints, we have to show that for every pair \(E, P\) satisfying the constraints there is a velocity \(\partial_t E\) such that \(P[E, \partial_t E] = P\), where \(P[E, \partial_t E]\) is defined by \((2.4)\).

To show this, we invert the relation \((2.4)\) to obtain a phase space function \(A[E, P]\). A little algebra and making use of \((2.2)\) and the formulas for \(J\) in the appendix shows that the inverse of \((2.4)\) is

\[
A_{pa}[E, P] = J_{aCD} (2E^p_{EB} E^{C}_{qD} E_{p}^{C} E_{q}^{D} - E^p_{EB} E_{p}^{D} E^D_{pa}) N^D P^a_B.
\]

(2.16)
By inserting this into (2.4) we get $P_A^m$ back, if and only if $L_a = 0$ and $P_A^m$ obeys the reality conditions. To obtain the velocities as phase space functions, we choose arbitrary values for $A_{ta}$ and $E_t^A$ (such that $E_M^A$ is non-singular) and define

$$\partial_t E_m^A = \partial_m E_t^A + \Omega_m^A B E_t^B - \Omega_t^A B E_m^B,$$

(2.17)

where $\Omega$ is given by (1.1). As these are part of the definition equations of $\Omega_M^A B$ (or $A_M^a$) as functions of $E_m^A$ and its derivatives, we just have to check that the rest of these equations are satisfied, too. They read

$$\partial[m E_n^A] + \Omega[m A B E_n^B] = 0.$$

(2.18)

Inserting (2.16) here and making use of the reality constraints (2.7) shows that these equations are indeed satisfied and that we have found all primary constraints, as the inverse of (2.4) exists if and only if $L_a = 0$. Note, however, that we made use of the existence of $N^A$ defined by (2.2). If we allow the hypersurface to become lightlike, additional primary constraints may appear because the relation (2.4) can no longer be inverted to give (2.16).

From now on we will regard $A_{pa}$ as a phase space function given by (2.16) for $L_a = 0$. Outside this “constraint surface” we are free to define $A_{pa}[E, P]$ arbitrarily. Each choice will lead to a different expression for the hamiltonian constraints (2.26) below, but they will all be equal up to something proportional to $L_a$ or $L_a^*$, thus the total set of constraints is invariant. However, let us restrict $A_{pa}$ to be linear in $P_A^m$ on the whole phase space, as otherwise the constraint algebra would become unnecessarily awkward. Thus

$$A_{pa}[E, P] = W_{pam} A P_A^m,$$

(2.19)

where the “matrix” is any function of $E_m^A$ satisfying

$$2i W_{qbn}^m \varepsilon^{mpn} J_{aAB} E_n^B = \delta^p_q \eta_{ab},$$

(2.20)

i.e. it is the “left inverse” of the matrix appearing in (2.4). We can express $P_A^m$ as a function of $A_{pa}$, which is given by (2.4) on the constraint surface. Since it is linear, however, we know that we can have an additional term proportional to $L_a^*$ only, i.e. we have the following relation:

$$P_A^m = 2i \varepsilon^{mpn} J_{aAB} E_n^B A_{pa}[E, P] + X_{aA}^* m L_a^*,$$

(2.21)

where $X_{aA}^* m$ depends on the choice of $W_{pam}^A$. Taking the bracket of this equation with $E_q C$ we find the “right inverse” of the matrix in (2.4), which becomes a useful formula below:

$$2i \varepsilon^{mpn} J_{aAB} E_n^B W_{pq}^a = \delta^m_q \delta_A^C + X_{aA}^* m J_{aAB}^* E_q C.$$

(2.22)

A possible choice for $A_{pa}[E, P]$ is (2.16), but we may add any term proportional to $L_a^*$. We will also assume that $W_{pam}^A$ transforms properly under the Lorentz algebra as its indices indicate, and that it is local. By using (2.7) and (1.1) we obtain $A_{pa}[E, P]$ and $\Omega_m^A B [E, P]$, and covariant derivatives like (A.7) and (A.8) are defined as phase space functions. Under these covariant derivatives the tensors $\eta_{AB}$, $\varepsilon_{ABCD}$, $J_{aAB}$ and $J_{aAB}^*$ are still constant, but the vierbein postulate

$$\varepsilon^{mpn} D_m E_n^A \approx 0,$$

(2.23)
only holds up to terms proportional to $L_{AB}$. For the special choice (2.16), e.g., we find that
\[ \varepsilon^{mnp} D_n E_p^A = \varepsilon^{mnp} N^A E_n^B E_p^C L_{BC}. \] (2.24)

From (2.20) and (2.14) we infer that $A_{pa}$ transforms as a connection under self-dual Lorentz transformations:
\[ \{ A_{pa}, L[\lambda^a] \} = -D_p \lambda_a, \quad \{ A_{pa}, L[\lambda^*_a] \} = 0. \] (2.25)

The remaining secondary constraints are now obtained by differentiating $L$ with respect to $E_t^A$. They read
\[ H_A = -i E^{mnp} J_{aAB} E_m^B F_{npa}. \] (2.26)

We will call them “hamiltonian constraints”. They are obviously related to the usual Wheeler DeWitt hamiltonian and diffeomorphism constraints, and we will show in section 4 how those may be obtained from $H_A$. Observe that our constraints have, like Ashtekar’s, a very simple geometrical structure. In particular, they are, as opposed to the hamiltonian constraint in the metric representation, in some sense “more homogeneous”. They are not given as the sum of a momentum term and a curvature term. As in Ashtekar’s representation, the two parts are combined into a term containing the 4-curvature, whereas in the Wheeler DeWitt equation (4.19) the intrinsic 3-curvature appears. They are slightly more complicated than Ashtekar’s hamiltonian constraint, as $A_{pa}$ is not a primary phase space coordinate but given by (2.19).

On the other hand we have much simpler reality conditions on the variables, which, in addition, assign non-trivial (but linear) conjugacy relations to the momentum, which will appear as a derivative operator in quantum theory. In Ashtekar’s representation it is the multiplication operator whose complex conjugate is a non-polynomial function of the derivative operator, and due to this fact it might be much harder to solve the problem of the scalar product on the state space than it is in the metric representation.

Our hamiltonian constraints are complex too, but they represent four real constraints only. Computing the imaginary part gives
\[ H_A - H_A^* = -i E^{mnp} E_m^B R_{npAB} = -2i \varepsilon^{mnp} D_n D_p E_{mA} \approx 0, \] (2.27)
i.e. it is proportional to the Lorentz constraints and $H_A^*$ is a holomorphic linear function of $H_A$ and $L_{AB}$. For quantum theory this means that we have to solve $L_a$ and $L_a^*$, but only $H_A$ (or $H_A^*$). Again, we define the smeared version
\[ H[V^A] = \int d^3x V^A H_A. \] (2.28)

To see that this constraint generates the translations we found as symmetries of the action, let us compute the Poisson bracket with the vierbein:
\[ \{ E_q^C, H[V^A] \} = 2I \varepsilon^{mnp} D_n (J_{AAB} V^A E_m^B) W_{pq}^C \]
\[ \approx -D_q V^C - D_m V^A X_{aA}^m J_{aAB}^* C E_q^B, \] (2.29)
where we used (2.22) and (2.23), i.e. we neglected terms proportional to $L_{aA}^*$. So in fact $H_A$ generates the translations (1.6), but in addition it generates an antiself-dual Lorentz rotation.
brackets of $H_A$ with the Lorentz constraints are easily calculated, as $H_A$ transforms covariantly under Lorentz transformations:

$$\{ L[\lambda_{AB}], H[V^A] \} = H[\lambda^A_B V^B].$$  \hspace{1cm} (2.30)

To calculate the bracket of $H_A$ with itself is rather cumbersome. What we would expect is to get the Lorentz constraint, as we saw in (1.9) that the commutator of two translations gave a Lorentz rotation. However, as $H_A$ generates extra Lorentz rotations, this might not be the case here. Instead of computing the commutator explicitly, in section 4 we will use results known from other representations to show that the classical algebra closes and, in addition, that also the quantum algebra formally closes, if we choose a special ordering.

3 Quantum theory

We shall now define a quantum representation and construct a formal solution to the resulting constraints, which has some typical features of a vacuum functional in quantum field theory. The primary field operators are the vierbein $E_m^A$ and its momentum $P_{Am}$. As we had to assume that $E_m^A$ is non-singular to obtain the constraint algebra, we should choose the $E$-representation here, i.e. the wave functional $\Psi$ will be a function on the set of all vierbein fields $E_m^A$ with positive definite spacial metric $g_{mn}$, and the momentum operator represented by a functional derivative. Every other representation produces difficulties with the implementation of these restrictions, as, e.g., the Ashtekar representation, where $A_{pa}$ becomes a multiplication operator $[4]$. In the representation chosen here no such problems occur, the vierbein operator is still non-singular in the sense that there exists a well defined operator for $N^A$ obeying (2.2).

Other typical problems are, of course, still present, like ill-defined operator products etc. We will not discuss any special regularization here, but we will see in the end that the formal solution found suggests that standard regularization methods could provide a well defined constraint algebra and a well defined state functional. In section 4 we will also see that other quantization methods may be able to deal with anomalies arising from formally ill-defined products without any regularization.

The constraints

We define the operators such that their commutator is $-i$ times the classical Poisson bracket, thus we have

$$P_{A}^m(x) = i \frac{\delta}{\delta E_m^A(x)}, \hspace{1cm} A_{pa}(x) = i W_{pam}^A(x) \frac{\delta}{\delta E_m^A(x)},$$  \hspace{1cm} (3.1)

both acting on functionals $\Psi[E_m^A]$. For the contraints we have to choose an operator ordering. The most obvious would be to order all functional derivatives to the right, as this is the “less singular” one, in the sense that as few ill-defined operator products as possible appear. However, this would destroy the nice geometrical structure of the hamiltonian constraint, as it could no longer be expressed in terms of the field strength of some connection.

Of course, another obvious choice is to take the constraints as they are given in (2.26) and just insert the operators (3.1). The only remaining freedom is then where to put the vierbein in $H_A$.  

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Let us place it to the left, as we will see that this leads to a formally closed algebra. There are no ordering ambiguities in $L_a$ or $L_a^*$, so the complete set of constraints is

\[
L_a = J_a^{AB} P_A^m E_mB, \\
L^*_a = J_a^{AB} P_A^m E_mB - i\varepsilon^{mnp} \partial_m (J_a_{AB} E_n^A E_p^B), \\
H_A = -i\varepsilon^{mnp} J_{aAB} E_m^B F_{npa}.
\] (3.2)

Note that written as operators the classically complex conjugate constraints $L_a$ and $L_a^*$ are completely independent. There is no $\ast$-relation between operators until a scalar product is defined. Even then the relation exist between observables only as a scalar product such that the classical $\ast$-relations are preserved can be defined on the physical phase space only. The complex conjugate of $H_A$, however, is still a linear combination of $H_A$ and $L_{AB}$ as the reality condition on $P_A^m$ holds as a second class constraint and therefore as an exact operator identity. It has nothing to do with an adjointness relation with respect to any scalar product on state space.

The vacuum state

We will now show that there exists a formal solution to all constraints. We will solve the constraints step by step, but not in the order one usually does it in the Ashtekar representation, where one first solves $L_a$ in general and then tries to solve $H_A$. As already mentioned in the beginning, we want to interchange the roles of Lorentz and diffeomorphism generators, thus we will solve $H_A$ first and then $L_a$. Nevertheless let us start with $L_a^*$. The complete solution to $L_a^* \Psi = 0$ can be found easily, as this just requires $\Psi$ to be invariant under antiself-dual Lorentz transformations. Speaking somewhat sloppy, to provide a function of $E^m_A$ that is invariant under these transformations, we have to contract the $A$ indices completely and such that no $^*_a$-index appears. The only tensors we have to achieve this are $\eta_{AB}, J_{aAB}$ and $\varepsilon_{ABCD}$, and any contraction of two of them is again a linear combination. So every function invariant under antiself-dual Lorentz transformation can be expressed as a (holomorphic) function of

\[
\tilde{e}_a^p = -\varepsilon^{mnp} J_{aAB} E_m^A E_n^B \quad \text{and} \quad g_{mn} = \eta_{AB} E_m^A E_n^B.
\] (3.3)

It can, in fact, be expressed as a function of $\tilde{e}_a^m$ alone, as

\[
\tilde{e}_a^m \tilde{e}_a^n = \frac{1}{2} \varepsilon^{mpr} \varepsilon^{nsq} g_{pr} g_{qs} = g^{mn}
\] (3.4)

is the densitized inverse of the three metric, thus $g_{mn}$ is determined by $\tilde{e}_a^m$ up to sign, which, however, is fixed by $g = \det (g_{mn}) > 0$. We can solve the antiself-dual Lorentz constraint by

\[
\Psi = \Psi [\tilde{e}_a^p],
\] (3.5)

where $\Psi$ is an arbitrary holomorphic functional.

Next we consider the equation $H_A \Psi = 0$. We need to know how $A_{pa}$ acts on a functional of $\tilde{e}_a^p$:

\[
A_{pa} \Psi = iW_{pa}^m A \frac{\partial \tilde{e}_b^q}{\partial E_m^A} \frac{\delta \Psi}{\delta \tilde{e}_b^q} = -2i\varepsilon^{mnp} J_{bAB} E_n^B W_{pa}^m A \frac{\delta \Psi}{\delta \tilde{e}_b^q} = -\frac{\delta \Psi}{\delta \tilde{e}_a^p},
\] (3.6)
where we used (2.20) and did not write out the dependence on the point \( x \) explicitly as all the fields are to be taken at the same point. Thus we recover the “dual” of Ashtekar’s representation, where \( \tilde{e}_a^m \) and \( A_{pa} \) are canonically conjugate quantities, but only after solving \( L^\ast_a \).

Having this simple representation for \( A_{pa} \) we can now seek for solutions to \( H_A \Psi = 0 \). As we certainly cannot find the general solution, let us look for simple solutions. A subset of all solutions is given by the wave functionals annihilated by \( F_{npa} \), also containing the trivial solution \( \Psi = 1 \).

This subset can indeed be given completely. If the connection \( A_{pa} \) is curvature free, then it is (locally, but let us assume trivial topology of space time here) given by

\[
-\frac{i}{2} A_{pa} \sigma_a = u^{-1} \partial_p u, \tag{3.7}
\]

where \( -\frac{i}{2} \sigma_a \) is any matrix representation of \( \text{so}(3,\mathbb{C}) \) and \( u \) is a matrix field taking values in the corresponding group representation. Let us choose the \( \text{su}(2,\mathbb{C}) \) representation here, thus \( \sigma_a \) are the pauli matrices and \( u \in \text{SU}(2,\mathbb{C}) \). For a given field \( u \) one can define a wave functional \( \Psi_u \) solving the corresponding quantum eigenvalue equation

\[
A_{pa} \Psi_u = i \text{Tr}(u^{-1} \partial_p u \sigma_a) \Psi_u, \tag{3.8}
\]

which is explicitly given by

\[
\Psi_u = \exp \left\{ -i \int d^3x \text{Tr}(u^{-1} \partial_p u \sigma_a) \tilde{e}_a^p \right\}. \tag{3.9}
\]

For any field \( u \) we now have a solution to the hamiltonian constraint \( H_A \Psi_u = 0 \), which is well defined as long as \( u \) satisfies some fall-off conditions at spacial infinity. It does not require any regularization for the constraint and is an exact solution to \( H_A \). It becomes formal, however, if we now try to solve the self-dual Lorentz constraint.

Because of the inhomogeneous term \( L_a \) does not require \( \Psi \) to be invariant under self-dual Lorentz transformations, in contrast to the same operator in Ashtekar’s representation, where it acts as a linear differential operator. Let us see how \( L_a \) acts on \( \Psi_u \). A short calculation yields [11]

\[
L[\lambda_a] \Psi_u = \frac{i}{2} \int d^3x \lambda_a \text{Tr}(u \sigma_a \frac{\delta \Psi_u}{\delta u}), \tag{3.10}
\]

where the matrix valued derivative \( \partial/\partial u \) is defined by \( (\partial/\partial u)_{\alpha\beta} = \partial/\partial(u_{\alpha\beta}) \), and is equal to the (formal) derivative if it acts on a function given by a power series such as \( \Psi_u \), i.e. \( \partial/\partial u \text{Tr}(X_u) = X \). Therefore \( L_a \) acting on \( \Psi_u \) generates multiplication with \( \sigma_a \) from the right on \( u \). It is useful to exponentiate this relation, which gives

\[
\exp(L[\lambda_a]) \Psi_u = \Psi_{uv}, \quad \text{where} \quad v = \exp \left( \frac{i}{2} \lambda_a \sigma_a \right). \tag{3.11}
\]

A formal solution to \( L[\lambda_a] \Psi = 0 \) can now be given by integrating \( \Psi_{uv} \) over \( v \):

\[
\Phi_u = \int [dv] \Psi_{uv}. \tag{3.12}
\]

Assuming that the measure \([dv]\) is invariant under multiplication from the right, we obviously have found a solution to all the constraints. As such a measure, of course, does not exist on the space of all fields \( v \), the solution becomes formal.

To check “how formal” it is, i.e. what kind of regularization is able to provide a well-defined functional, it is important to note that it is sufficient to solve \( L[\lambda_a] \Psi = 0 \) for real \( \lambda_a \) only, or for \( \lambda_a \)
element of any real subspace of $C^3$, as $L_a \Psi = 0$ implies $i L_a \Psi = 0$. In other words, it is sufficient to integrate over any real form of $SU(2, C)$ in (3.12). To get a well-defined integral the best choice is, of course, to integrate over the compact real form $SU(2)$. If we then, in addition, assume that space is compact and regularize the theory on a lattice with finitely many points, the wave function becomes well defined, as it is given by finitely many integrations over the compact group $SU(2)$. This argument is slightly heuristic, of course, as we have to transfer all the expressions above onto the lattice first and then check whether a solutions like (3.12) exists. Note, however, that the problem here is much simpler than that arising in the loop representation, where one has to integrate over the set of all loops or the diffeomorphism group of the three dimensional space, which are much harder to deal with than the local $SU(2)$ here. In the next section we will see that other quantization methods may lead to a perfectly well defined “state”, which does not require any functional integration.

Let us discuss, also a little bit heuristically, the properties of the functional $\Phi_u$. The first question is: how many solutions did we find? Obviously we do not necessarily get distinct solutions for different fields $u$, as $\Phi_u = \Phi_{uv}$ for any $SU(2)$-valued field $v$. For $\Phi_u \neq \Phi_u'$ we must have $u^{-1} u' \notin SU(2)$ at some point. Now consider the integration over $v$ at this point. We can think of it as an integral over a holomorphic function $f(uv)$, defined on the complex manifold $SU(2, C)$, along a real “line” $\{uv, v \in SU(2)\}$. We may shift this real line to $\{u'v, v \in SU(2)\}$ without changing the value of the integral, because the integrand is holomorphic. As a result, we find that indeed all wave functionals $\Phi_u$ are equal.

A crucial question is now, whether this is really a state functional or just the trivial solution $\Psi = 0$. Up to now there is no reason why the integral (3.12) should not vanish. However, it is easy to see that there are fields $E_m^A$ for which $\Phi[E_m^A] \neq 0$: choose the vierbein such that $\tilde{e}_a^m$ is real, i.e. $E_m^0 = 0$. Then the exponent in (3.9) is real too, because $u^{-1} \partial_m u \in su(2)$ is antihermitian. So after all we found exactly one solution to all constraints. However, if we allow the space manifold to have a non-trivial topology, then there are more solutions. If there are non-contractible loops, the field $u$ introduced in (3.7) need not be defined globally, and two arbitrary fields $u$ and $u'$ can no longer be transformed into each other by the method just described. In this case we recover the typical structure of the state space of three dimensional gravity, where the states can be characterized by the so called moduli of $u$, which are in principle the values by which $u$ is multiplied after going once around a non-contractible loop. In fact, the discovery of the formal solution was inspired by a result obtained in three dimensional gravity, where (3.12) is the general solution to all constraints, but a crucial difference is that there it can be given as a well defined object in a different representation [4, 11, 12].

If space time is non-compact, then $u$ as well as $v$ must obey certain boundary conditions. This also leads to topological degrees of freedom for the field $u$, namely some kind of soliton numbers, which cannot be gauged away by transformations of the form (3.11). That the different state functionals $\Phi_u$ are parametrized by topological parameters is a first evidence that $\Phi_u$ is a vacuum state, as it is typical feature of the vacuum of a quantum field theory to carry topological degrees of freedom. For different values of the moduli or different soliton numbers we get different vacua. But there are still other properties confirming this interpretation.

Perhaps they are even more speculative than the discussion above, but they may be interesting from a physical point of view. How should a vacuum of quantum gravity look like? It can certainly not be simply flat space time, as this would violate the uncertainty relation: all fields would take definite values. In quantum field theory, a vacuum is usually given by a state that is annihilated.
by a set of “annihilation operators”. This set is, sloppy speaking, half of the set of all operators and the other half is obtained by complex conjugation. The question is whether one can recover a similar structure here. Looking at (2.8), which gives the relation between our $P_A^m$ and the “real” momentum $Q_A^m$ of the vierbein, one finds that $P_A^m$ looks like, e.g., the annihilation operator of electrodynamics, which is in principle $E^m + iB^m = E^m + i\varepsilon^{mnp}\partial_n A_p$, Fourier transformed into momentum space, where $E$ the electric, $B$ the magnetic field and $A$ the vector potential. Replacing the vector potential by the vierbein and the conjugate momenta, the electric field, by the real momentum $Q_A^m$ shows that $P_A^m$ is the analog of the annihilation operator of electrodynamics.

But from this one could infer that the vacuum state is defined by $P_A^m\Psi = 0$, i.e. $\Psi = 1$, which is obviously not a solution to the constraints. However, in contrast to electrodynamics, $P_A^m$ is not covariant, i.e. $P_A^m = 0$ (as a classical equation) is not invariant under gauge transformations: remember that $P_A^m$ transforms inhomogeneously under self-dual Lorentz transformations. Thus requiring this to annihilate a state functional is in contradiction with the constraint equations. To get as close as possible to the usual definition of a vacuum, one has to look for the simplest possible holomorphic covariant object that can be built from $P_A^m$, and this is the field strength $F_{mna}$. Requiring $F_{mna}\Psi = 0$ is obviously consistent with the constraints because the field strength transforms covariantly under all local symmetries.

When introducing the field strength $F_{mna}$ in the beginning as a classical field, we saw that it is in principle the Riemann curvature tensor in the self-dual representation. It should be possible to write every “local” observable as a function of this tensor, like in electrodynamics where every observable is a function of the field strengths. Thus we can split the observables into holomorphic functions of $F_{mna}$ and holomorphic functions of $F_{mna}^*$, which leads to the typical split into creation and annihilation operators and the vacuum has the property that it is annihilated by exactly half of the observables.

In addition, whatever the observables are, if expressed in terms of the Riemann tensor and thus $F$, there exists a “normal ordering” for the corresponding quantum operators. All factors of $F$ are ordered to the right, $F^*$ to the left, and any extra vierbein factors in between. As a result, the vacuum expectation value of any real local observable vanishes. And if there are any “global” observables, which cannot be expressed in terms of $F_{mna}$ but are functions like parallel transport operators along non-contractible loops etc., the vacuum expectation values of them will depend on the special vacuum state, giving the typical structure of a quantum field theory with multiple vacua.

4 Other representations and anomalies

As already mentioned we are somewhere between the metric or Wheeler DeWitt and Ashtekar’s representation of canonical gravity. The difference to Wheeler DeWitt is that we have replaced the metric by a vierbein and added an imaginary total derivative to the action, leading to complex momenta obeying reality conditions. In addition, we did not introduce a lapse function nor a shift vector, but used the “lower t” components of the vierbein as Lagrange multiplier. This gave us a Lorentz covariant expression for the hamiltonian constraints, which normally splits into a scalar and a vector. The gauge fixing, however, is identical to that of the usual metric approach: we only required the spacial hypersurface to be spacelike.

The difference to Ashtekar’s representation is that there an additional gauge fixing of the vierbein must be introduced to make $A_{pa}$ a “good” canonical variable, with a conjugate variable $\tilde{e}_a^P$.  


the densitized inverse dreibein. We will see that both can be introduced as phase space functions here, but without the gauge fixing they are complex and represent more than one (but less than two) degrees of freedom; i.e. a relation like (2.7) cannot be given for them unless the vierbein takes the usual triangular form. It is also a consistency check for Ashtekar’s formulation of canonical gravity, that \( A_{pa} \) and \( \tilde{e}_a{}^p \) can be defined such that they obey the basic Poisson bracket relation without the gauge fixing. Otherwise one could argue that the Ashtekar’s gravity is not fully Lorentz-invariant, because the variables can only be defined in a gauge fixed version. There are also problems concerning the diffeomorphism invariance of Ashtekar’s gravity [4], but we will not give up the non-singularity of the metric and thus stay on the safe side here.

**Ashtekar’s polynomial representation**

The transition to Ashtekar’s representation is rather simple. We already found that the two quantum operators \( A_{pa} \) and \( \tilde{e}_a{}^p \) are conjugate to each other when acting on solutions of \( L_a^* \). There might be extra terms in their commutators proportional to \( \tilde{e}_a{}^m \|_A \), if we simply define them as phase space functions by (2.19) and (3.3). To see that they can be defined such that

\[
\{ \tilde{e}_a{}^m, \tilde{e}_b{}^n \} = 0, \quad \{ A_{ma}, A_{nb} \} = 0, \quad \{ \tilde{e}_a{}^m, A_{nb} \} = i \eta_{ab} \delta^m_n, \quad (4.1)
\]

we proceed as follows. We can write (2.4), which is the implicit definition of \( A_{pa} \) on the constraint surface \( L_a^* = 0 \), as

\[
P_A^m = -i \frac{\partial \tilde{e}_a{}^p}{\partial E^{mA}} A_{pa}, \quad (4.2)
\]

where \( \tilde{e}_a{}^p \) is given by (3.3). Now think of \( \tilde{e}_a{}^p \) as some set of coordinates on the space of all complex rank three matrices \( E^{mA} \). As the dimension of this space is 12 and \( \tilde{e}_a{}^p \) has only 9 components, we have to add 3 coordinates, which we will denote by \( v^*_a \). The reason for this notation is that \( \tilde{e}_a{}^p \) fixes \( E^{A}_m \) up to an antiself-dual Lorentz rotation and \( v^*_a \) is the parameter of that rotation. Now we set

\[
W_{pam}^A = i \frac{\partial E^{mA}}{\partial \tilde{e}_a{}^p}, \quad A_{pa}[E, P] = i \frac{\partial E^{mA}}{\partial \tilde{e}_a{}^p} P_A^m. \quad (4.3)
\]

One immediately checks that (2.20) is fulfilled, and it is now straightforward to verify the Poisson brackets above, simply by using the chain rule for partial derivatives. The phase space function \( A_{pa} \) is still not unique, as it depends on how the coordinates \( v^*_a \) are chosen.

To get the constraints in the well know form, we define the diffeomorphism and densitized Hamiltonian constraint as

\[
H_m = E^{mA} H_A = -i \tilde{e}_a{}^n F_{mna},
\]

\[
\sqrt{g} H = \frac{1}{6} \varepsilon^{mnp} \varepsilon_{ABCD} E^{mA} E^{NB} E^{PC} H^D = \frac{1}{2} \varepsilon_{abc} \tilde{e}_a{}^m \tilde{e}_b{}^n F_{mnc}. \quad (4.4)
\]

They are formally equal to those of Ashtekar, but remember that \( \tilde{e}_a{}^p \) and \( A_{pa} \) are not good coordinates on our phase space, as they are complex and do not obey “enough” reality conditions without a Lorentz gauge fixing. Therefore the Lorentz constraints cannot be expressed in terms of a holomorphic function of \( \tilde{e}_a{}^p \) and \( A_{ma} \). However, half of them can, and the usual “Gauß law” constraint turns out to be (see (2.22) for the definition of \( X^{AB}{}_c{}^m \))

\[
L_a - J^A_a \ E^{mA} X^{AB}{}_c{}^m \ L_c^* = i D_m \tilde{e}_a{}^m. \quad (4.5)
\]
The complete recovery of Ashtekar’s representation could now be obtained by gauge fixing \((E_m^0 = 0)\) and solving \(L_a^0\) (for \(P_0^m\)). But even without doing this we can use an important result from Ashtekar’s representation to show that our quantum constraint algebra formally closes.

Up to now we did not compute the commutator of \(H_A\) with itself. As we are not interested in an explicit expression but only want to show that it is again proportional to the constraints, we invert (4.4) and get

\[
H_A = -\sqrt{g} H - \frac{1}{2} \varepsilon_{ABCD} \varepsilon^{mnp} E_m B E_n C N D H_p. \tag{4.6}
\]

Now it is obviously sufficient to show that \(\sqrt{g} H\) and \(H_m\) form a closed algebra, and to do this one needs the brackets (4.1) only. Thus the calculation is formally equivalent to that in Ashtekar’s representation and we can use the result from [13], where it is shown that the quantum algebra with the operator ordering as in (4.4) closes. However, as this is only a formal result, there might still be anomalies in the algebra which cannot be discovered by formal calculation. In fact, we will see below that there is a strong hint for an anomaly, because it is not possible to write the hamiltonian and diffeomorphism constraints manifestly Lorentz invariant on a formal level. In other words, it is not possible to write the Lorentz invariants \(H_m\) and \(H\) as a function of the spacial metric \(g_{mn}\) and its conjugate momentum.

**The metric representation**

We will now transform our representation back to the metric or Wheeler DeWitt representation, where the wave functional depends on the spacial metric \(g_{mn}\). It is most convenient to do this on the quantum level. We will see that our vacuum functional can be transformed as well and we get is a functional that depends on the spacial metric \(g_{mn}\) only.

The first step to recover the metric representation is to choose another operator for the momentum \(P_A^m\). Remember that we may define

\[
P_A^m = i \frac{\delta}{\delta E_m A} - i \frac{\delta G}{\delta E_m A}, \tag{4.7}
\]

with any functional \(G[E]\). This operator still obeys the required commutation relations. The crucial question is: can we find \(G\) such that the Lorentz constraint takes the form

\[
L_{AB} = -i E_m [A \frac{\delta}{\delta E_B} ]^m, \tag{4.8}
\]

so that the inhomogeneous term in (2.12) cancels and \(L_{AB}\) generates Lorentz transformations on the wave functional. Then any solution to this constraint would be Lorentz invariant and could be expressed in terms of the spacial metric. To define such a \(G\), we have to introduce a dreibein. We already used the densitized inverse dreibein

\[
\tilde{e}_a^p = -\varepsilon^{mnp} J_{aAB} E_m A E_n B. \tag{4.9}
\]

As we know that this is invertable, we can use it to construct a dreibein \(e_{ma}\), implicitly defined by

\[
\tilde{e}_a^p = \frac{1}{2} \varepsilon^{mnp} \varepsilon_{abc} e_{nb} e_{pc}, \quad e = \det(e_{ma}) = \sqrt{g} > 0. \tag{4.10}
\]

From this we can build all other quantities like the inverse dreibein \(e_a^m\) etc. One can also obtain this dreibein by an antiself-dual Lorentz rotation directly from \(E_m A\). If \(\Lambda_{AB}\) describes a finite
antiself-dual rotation, then we can always choose it such that \( \Lambda_{0A} E_m^A = 0 \), which fixes \( \Lambda_{AB} \) up to a sign. The dreibein is then given by \( e_{ma} = \Lambda_{aA} E_m^A \) and fulfills (4.10), if we choose the correct sign for \( \Lambda_{AB} \).

We can use the dreibein to define a three dimensional spin connection \( \omega_{ma} \) via the dreibein postulate

\[
\mathcal{D}_m e_n^a = \partial_m e_n^a + \varepsilon_{abc} \omega_{mb} e_n^c = 0.
\]

This defines a new covariant derivative acting on self-dual indices only. It can also be given as a functional derivative [10]

\[
\omega_{ma} = \frac{\delta G}{\delta e_m^a} , \quad G = \frac{1}{2} \int d^3 x \varepsilon^{mnp} e_m \partial_n e_p .
\]

With this functional inserted above the momentum operator is

\[
P_A^m = 2i \varepsilon^{mnp} J_{aAB} E_n^B \omega_{pa} + i \frac{\delta}{\delta E_m^A} ,
\]

and a short calculation yields (4.8). The operator for \( A_{pa} \) becomes

\[
A_{pa} = \omega_{pa} - \frac{\partial E_m^A}{\partial e_p^a} \frac{\delta}{\delta E_m^A} ,
\]

where \( \partial E/\partial e \) is to be understood as in (4.3). We can now transform our formal vacuum functional such that it becomes a solution to the constraints for the new operator representation. For simplicity, let us drop the topological degrees of freedom here, then we only have one vacuum state \( \Phi = \Phi_{u=1} \).

If we define

\[
\Phi = \exp(G) \Phi ,
\]

the new operators acting on \( \Phi \) give the same result as the old operators acting on \( \Phi \), and \( \Phi \) becomes a formal solution to the new constraints. By introducing curved Pauli matrices \( \sigma_m = e_{ma} \sigma_a \) and using the formula

\[
\varepsilon^{mnp} \sigma_n \sigma_p = -2i \bar{e}_m^a \sigma_a ,
\]

we can write \( \Phi \) in an elegant way as

\[
\Phi = \int [du] \exp \left( \frac{1}{4} \int d^3 x \varepsilon^{mnp} \text{Tr}((u \sigma_m u^{-1}) \partial_m (u \sigma_p u^{-1})) \right)
\]

It is now obvious, at least on a formal level, that this is invariant under Lorentz rotations, as \( \sigma_m \) does not transform under antiself-dual rotations and a self-dual rotation is given by \( \sigma_m \mapsto v^{-1} \sigma_m v \), which can be compensated by a shift in the integration variable \( u \mapsto u v \).

Remember that the integral runs over \( SU(2) \), but we may shift this “path” anywhere in \( SU(2, C) \). Given a dreibein \( e_{ma} \) such that \( g_{mn} \) is real and positive, then we can choose this path such that \( u \sigma_m u^{-1} \) becomes hermitian for all \( u \), because there always exists a rotation that transforms the dreibein into a real dreibein. But then the integral runs exactly (twice) over all real dreibeins that may be obtained by rotations from \( e_{ma} \). As a result, we can write the vacuum wave functional in the metric representation as

\[
\Phi[g_{mn}] = \int [de_{ma}] \exp \left( G[e_{ma}] \right)
\]

where the integral runs over all real dreibein fields satisfying \( e_{ma} e_{na} = g_{mn} \), and the measure is assumed to be invariant under \( SO(3) \) rotations.
The Wheeler DeWitt equation

An interesting question arising now is: did we find a formal solution to the “old” Wheeler DeWitt equation, as we now have a wave functional \( \tilde{\Phi}[g_{mn}] \)? To check this we have to write the constraints \( H_m \) and \( H \) as functional differential equations acting on \( \tilde{\Phi}[g_{mn}] \). Then \( H_m \) should generate spacial diffeomorphisms and \( H \tilde{\Phi} = 0 \) should give the Wheeler DeWitt equation (see e.g. [14]; different relative factors between the two terms are due to different normalizations of the action)

\[
\frac{1}{2} \sqrt{g} R[g] \tilde{\Phi} - 2G_{mnpq} \frac{\delta^2 \tilde{\Phi}}{\delta g_{mn} \delta g_{pq}} = 0, \tag{4.19}
\]

where \( R[g] \) is the spacial curvature scalar and

\[
G_{mnpq} = \frac{1}{2} \sqrt{g}^{-1} \left( g_{mp} g_{nq} + g_{mq} g_{np} - g_{mn} g_{pq} \right) \tag{4.20}
\]

is the (inverse) “supermetric” on the space of three metrics \( g_{mn} \).

Let us now assume that the wave functional is given by an arbitrary function \( \tilde{\Psi}[g_{mn}] \) of the spacial metric. This is the general solution to \( L_{AB} \tilde{\Psi} = 0 \). The computation of \( H_m \tilde{\Psi} \) and \( H \tilde{\Psi} \) is given in the appendix. The result is rather peculiar and reads

\[
H_m \tilde{\Psi} = 2 \nabla_{(p} \left( g_{q)m} \frac{\delta \tilde{\Psi}}{\delta g_{pq}} \right),
\]

\[
H \tilde{\Psi} = -2 \sqrt{G}^{-1} \frac{\delta}{\delta g_{mn}} \left( \sqrt{G} G_{mnpq} \frac{\delta \tilde{\Psi}}{\delta g_{pq}} \right) - \left( \frac{i}{2} \sqrt{g} R[g] + \frac{\delta \phi(0)}{2} \right) \epsilon^{mnp} \epsilon_{ma} \partial_n e_{pa} \tilde{\Psi}. \tag{4.21}
\]

The diffeomorphism constraint is exactly what we expected and it requires \( \tilde{\Psi} \) to be invariant under spacial diffeomorphisms. However, \( H \tilde{\Psi} \) is different from (4.19). First of all, the kinetic term takes a very nice form: instead of the simple second derivative the Laplace operator with respect to the “supermetric” (whose determinant is \( G \)) appears. This is rather surprising, because it came out automatically as a result of the operator ordering in \( H_A \). In a certain sense our representation is “more geometrical” than the metric representation with the operator ordering as in (4.19).

However, there is an additional divergent term, which is of order \( \hbar \) as it came from a reordering of operators at the same space point. Obviously, this extra term is not Lorentz invariant; therefore the constraint algebra no longer closes, and a solution \( \tilde{\Psi}[g_{mn}] \) can no longer exist. But on the other hand, we have the formal solution \( \tilde{\Phi} \). We must conclude that something was wrong in our formal calculation. This is a strong hint for an anomaly in the quantum algebra (5.2), which is “hidden” in the ill-defined operator product appearing in \( H_A \). And the fact that \( \tilde{\Phi} \) formally solves the constraints \( \tilde{\Phi} \) and at the same time is a functional of \( g_{mn} \) results from our assumptions about the measure \( [dv] \), which does not exist.

So after all we have to conclude that our construction of the vacuum state was too formal and maybe such a state functional does not exist because of an anomaly in the quantum constraint algebra. However, if it is not possible to define the constraint algebra properly without an anomaly, there is no solution to the quantum constraints at all, and Dirac’s quantization method won’t work. There is a well known example of another diffeomorphism invariant field theory where exactly this happens, namely string theory. There we know how to quantize it: the constraints, expressed as the Virasoro generators, split into two complex conjugate subsets, similar to the split of the observables.
considered in section 3, each forming a closed subalgebra. One defines physical “wave functionals” that are annihilated by one half of the constraints. A state is then given by equivalence classes: two wave functions are equivalent, if their difference can be written as some linear combination of the other half of the constraints, acting on some other wave function.

Is it possible to quantize gravity in a similar way? The crucial question is whether there is a “natural” split of the constraints into two conjugate subsets, each forming a closed subalgebra when properly regularized. There is no obvious split of the complete algebra. However, for the Lorentz constraints we already used this split into $L_a$ and $L^*_a$, which came out automatically when using the $\text{so}(3, \mathbb{C})$ representation of the Lorentz algebra. Therefore a suggestion for a slightly modified Dirac quantization is to define physical states as follows: Assume that there is a regularization of the hamiltonian constraints $H^r$ such that $H^0 = H$ as a classical phase space function, and let us define a set $H$ of wave functionals satisfying

$$
\Psi \in H \quad \iff \quad \lim_{\epsilon \to 0} H^r[V^A] \Psi = 0 \quad \forall V^A,
$$

where the convergence is simply defined pointwise, i.e. for any fixed configuration $E_m^A$ we have $H^r(E_m^A) \to 0$.

So far this is a regularized Dirac quantization, but now we solve the Lorentz constraints like the Virasoro constraints in string theory: We solve half of them, defining a subset $L \subset H$ by

$$
\Psi \in L \quad \iff \quad L[\lambda_a^*] \Psi = 0 \quad \forall \lambda_a^*
$$

and the states $| \Psi \rangle \in \mathcal{P} = L/\sim$ are defined as equivalence classes

$$
| \Psi_1 \rangle = | \Psi_2 \rangle \quad \iff \quad \exists \Phi \in H, \lambda_a, \quad | \Psi_1 - \Psi_2 = L[\lambda_a] \Phi\rangle.
$$

It is essential for this procedure to solve the hamiltonian constraints first, because otherwise it would be necessary to define $H^r$ on equivalence classes and this requires $H^r$ to commute weakly with $L^*_a$ and $L_a$. However, if one solves $H$ first, then for the second step to be consistent one only needs that the regularized hamiltonian commutes weakly with $L^*_a$. There is a huge class of regularizations with this property, because only the vierbein in $H_A$ does not commute with $L^*_a$, so for example every point split regularization like

$$
A_{pa}(x) \to \int \! d^3 y \, \Delta^e(x, y) A_{pa}(y)
$$

is of this kind, and $\Delta$ may even be constructed from the spacial metric $g_{mn}$ providing a diffeomorphism invariant regularization.

In some sense this is just the “reverse” of the Ashtekar programme, where one seeks for a Lorentz invariantly regularized hamiltonian constraint by writing the field strength as parallel transport matrix along some small loop. There one is forced to do this, because the Lorentz constraint is solved first. In addition, the nice complex structure of the Lorentz group does not even appear, because one is dealing with a gauge fixed representation.

We should emphasize that the Ashtekar programme, which relies on Dirac’s quantization, might fail because of anomalies in the constraint algebra which cannot be detected by formal calculations, but for which we found some hints above. Ashtekar’s representation also uses another factor ordering, where the situation is even worse, because there are anomalies already on the formal level (the
structure constants in $[H, H]$ appear to the right). In the quantization programme proposed above such problems do not occur as long as it is possible to regularize $H$ such that the equations (4.22) are consistent. In particular, for any regularization we have $\Psi_u \in \mathcal{L}$, where $\Psi_u$ is the functional defined in (3.9). Then our vacuum state (omitting topological degrees of freedom again) becomes a perfectly well defined object

$$|0\rangle = |\Psi_u\rangle.$$  \hspace{1cm} (4.26)

The ill-defined integration over the local SU(2) has been replaced by the equivalence class of wave functionals which can be transformed into each other by Lorentz rotations: the right hand side is in fact independent of $u$. The state still has all the properties described in the end of section 3, so now we have a well defined vacuum state.

If there is any anomaly, it now appears in the commutator of the regularized hamiltonian constraint with $L_a$, as a result of a regularization that is not Lorentz covariant. What happens is that the equivalence classes become smaller as they would be without this anomaly, but they remain well defined: given any state $|\Psi\rangle \in \mathcal{P}$, then the Lorentz transformed wave function $\exp(L[a])\Psi$ is not necessarily a solution to (4.22). It is just a matter of coincidence that for the special state $|0\rangle$ every Lorentz transformed wave function is again a solution to the hamiltonian constraint, but in general it need not be. However, the equivalence classes themselves and therefore the states remain Lorentz invariant objects.

As a conclusion, we might summarize the results as follows: So far most works on quantum gravity with Ashtekar’s new variables exploited the fact that the classical constraints may be written as polynomials in canonically conjugate variables. One of these variables is the spacial dreibein, whose “natural” range is the set of invertable $3 \times 3$ matrices. On this space, however, there is nothing that makes a polynomial behaving better than, e.g., a rational function, and, as already shown in [4], extending the support of the wave function to singular dreibein fields causes some trouble with the classical limit of quantum gravity as a diffeomorphism invariant theory. It is the nice geometric structure and not essentially their polynomial form that makes the constraints more easy to handle.

Another important feature of Ashtekar’s variables has not been considered so much: the representation of the Lorentz group as a complex Lie group. We saw that it is this structure that makes it possible to define annihilation and creation operators, normal ordering etc., and to use quantization methods known from other field theories, which are able to deal with anomalies. Unfortunately, we were not able to give these operators explicitly, because we do not know any explicit expression for an observable in quantum gravity; but what we could do was to give the criteria to classify the observables and a formal normal ordering prescription. If any such well defined algebra of observables could be constructed, this would also solve the scalar product problem: The states are obtained by acting with the creation operators on the vacuum, providing a Fock space structure. The scalar product can simply be obtained by defining $\langle 0 | 0 \rangle = 1$ and using the commutator algebra of the observables. A definition of the scalar product as a functional integral over wave functionals is not required.

### 5 Supergravity

In this last and rather technical section we will see that the construction of the vacuum state can also be made for supergravity. We will give the N=1 example here, but in principle it should be possible to reproduce the same result for other supergravity models.
Action and constraints

We introduce a two component Grassmann valued spinor field $\psi_M$, the gravitino, as superpartner of the vierbein. Definitions for spinors, Pauli matrices, covariant derivatives etc. are given in the appendix, where it is also shown that the Rarita Schinger action of N=1 supergravity is \[ S[E, \psi] = \frac{i}{2} \int d^4x \varepsilon^{MNPQ} (4 \bar{\psi}_M \sigma_N D_P \psi_Q - E_M^A E_N^B J_{aAB} F_{PQa}), \] (5.1)

and $\Omega_{MAB}$ is given as a function of the vierbein and the gravitino by the torsion equation, which again is its own equation of motion in (5.1):

\[ D_\{M E N\}^A - \bar{\psi}_M \sigma^A \psi_N = 0. \] (5.2)

The supersymmetry transformation are parametrized by a spinor field $\epsilon$ and read

\[ \delta \psi_M = -D_M \epsilon, \quad \delta \bar{\psi}_M = -D_M \bar{\epsilon}, \]

\[ \delta E_M^A = \bar{\psi}_M \sigma^A \epsilon - \bar{\epsilon} \sigma^A \psi_M. \] (5.3)

To show that $S$ is invariant under these transformations one has to use the torsion equation and the Fierz identities (A.23), and one can use the 1.5 order trick, i.e. one only has to vary the fields $E_M^A$ and $\psi_M$ appearing explicitly in (5.1).

Computing the commutator of two supersymmetry transformations, one finds for the vierbein

\[ [\delta_1, \delta_2] E_M^A = -D_M (\bar{\epsilon}_1 \sigma^A \epsilon_2 - \bar{\epsilon}_2 \sigma^A \epsilon_1), \] (5.4)

which is a translation with $V^A = \bar{\epsilon}_1 \sigma^A \epsilon_2 - \bar{\epsilon}_2 \sigma^A \epsilon_1$. To get the commutator acting on $\psi_M$ we need to know how the spin connection transforms under supersymmetry. As this is a rather cumbersome calculation we will not give it here. An easier calculation shows that the action is invariant under the following translations:

\[ \delta E_M^A = -D_M V^A, \quad \delta \psi_M = 2V^N D_M \psi_N, \]

\[ \delta \bar{\psi}_M = 2V^N D_M \bar{\psi}_N. \] (5.5)

We see that in supergravity the translations rather than the diffeomorphisms generated by the Lie derivatives appear as the basic symmetries: they are the commutators of two local supersymmetry transformations, and thus the local versions of the Poincaré translations.

Let us now set up the canonical formulation. The space time split leads to the following lagrangian, whose "bosonic" part is formally equal to \[ (2.3): \]

\[ L_{bos} = \int d^3x i \varepsilon^{mnp} \left( 2 \partial_t E_m^A E_n^B J_{aAB} A_{pa} - D_p (E_m^A E_n^B J_{aAB}) A_{ta} - E_t^A E_m^B J_{aAB} F_{npa} \right), \] (5.6)

and the part containing the gravitino explicitly is

\[ L_{form} = \int d^3x 2i \varepsilon^{mnp} \left( \bar{\psi}_m \sigma_n \partial_t \psi_p - \frac{1}{2} A_{ta} \bar{\psi}_m \sigma_n \sigma_p \right) - \bar{\psi}_m \sigma_n D_n \psi_p + \bar{\psi}_m \sigma_p D_n \psi_p + D_p (\bar{\psi}_m \sigma_n) \psi_t. \] (5.7)
From the velocity terms we obtain the momenta
\[ P^m_A = 2i \varepsilon^{mnp} J_{AB} E^n B A_{pa}, \quad \bar{\pi}^m = -2i \varepsilon^{mnp} \bar{\psi}_n \sigma_p. \] (5.8)

The canonical variables are \( E^m A, P^m_A, \psi_m \) and \( \bar{\pi}^m \). The equation above for \( \bar{\pi}^m \) can be inverted to give \( \bar{\psi}_m \) as a phase space function, provided that the spacial metric is invertable, and \( A_{pa} \) is again given by (2.19). The basic Poisson brackets are
\[ \{E^m A, P^m_B\} = \delta^m_B \delta^A, \quad \{\psi_m, \bar{\pi}^n\} = -\delta^m_n 1. \] (5.9)

Note that when differentiating \( L \) with respect to \( \partial_t \psi_m \) we get an additional minus sign by anticommuting the derivative operator with \( \bar{\psi}_m \), and that the Poisson brackets are symmetric for Grassmann valued entries.

The reality conditions for \( P^m_A \) are slightly more complicated now, as there is a contribution from the gravitinos in the torsion equation:
\[ P^m_A = P^m_A - 2i \varepsilon^{mnp} \left( \partial_n E^p A - \bar{\psi}_n \sigma_A \psi_p \right). \] (5.10)

The Lorentz constraint \( L^*_a \), however, remains unchanged, as it follows from the unchanged equation for \( P^m_A \) as a function of \( A_{pa} \). But the complex conjugate \( L_a \) is different, since we have a new relation (5.10). The rest of the constraints is obtained by differentiating \( L \) with respect to \( E^m A, \psi_t \) and \( \bar{\psi}_t \), and of course they are the generators of local supersymmetry and translations. The complete set of constraints is
\[ L^*_a = J^*_a B P^m_A E^{m B}, \quad L_a = J^a B P^m_A E^{m B} - i \partial_m \tilde{e}^m_a - \frac{i}{2} \bar{\pi}^m \sigma_a \psi_m, \quad S = 2i \varepsilon^{mnp} \sigma_m D_n \psi_p, \quad \bar{S} = -D_m \bar{\pi}^m, \quad H_A = -i \varepsilon^{mnp} J_{aAB} E^{m B} F_{npa} - 2i \varepsilon^{mnp} \bar{\psi}_m \sigma_A D_n \psi_p. \] (5.11)

Note that, as \( \psi_m \) and \( \bar{\pi}^m \) both transform under the self-dual representation of the Lorentz algebra, all these constraints are holomorphic in \( A_{pa} \). The smeared versions of the bosonic constraints are defined as usual. For the fermionic ones we set
\[ S[\bar{\epsilon}] = \int d^3 x \bar{\epsilon} S, \quad S[\epsilon] = \int d^3 x \bar{\epsilon} S. \] (5.12)

**Solving the quantum constraints**

For the quantum theory we will choose the \( E-\bar{\pi} \)-representation. The wave functional becomes a function \( \Psi[E^m A, \bar{\pi}^m] \), and the momentum operators are
\[ P^m_A(x) = i \frac{\delta}{\delta E^m A(x)}, \quad \psi_m(x) = i \frac{\delta}{\delta \bar{\pi}^m(x)}. \] (5.13)

Again, \( L^*_a \) is solved, if \( \Psi \) depends on \( E^m A \) only via a holomorphic function of \( \tilde{e}^p a \), and on such a \( \Psi[\tilde{e}^m a, \bar{\pi}^m] \) the spin connection acts as
\[ A_{pa} \Psi = -\frac{\delta \Psi}{\delta \bar{e}^p a}. \] (5.14)
We can now solve $H_A$ and $S$ in the same way as we solved $H_A$ for the bosonic theory. We look for simple solutions which are already annihilated by the field strength $F_{mna}$ and the “super field strength” $D_{m|n}$ of $\psi_{\bar{n}}$. Again, the general solution is well known and similar to the general solution of three dimensional supergravity [12]. For vanishing fields strengths the gauge fields are locally given by an $SU(2, \mathbb{C})$ field $u$ and a spinor field $\phi$ and read

$$A_{ma} = i \text{Tr}(u^{-1} \partial_m u \sigma_a), \quad \psi_m = u^{-1} \partial_m \phi.$$

The corresponding quantum eigenvalue equations can easily be solved, and the result is

$$\Psi_{u,\phi} = \exp\left\{ -i \int d^3x \left( \text{Tr}(u^{-1} \partial_m u \sigma_a) \tilde{e}_a^m + \bar{\pi}_m u^{-1} \partial_m \phi \right) \right\}. \tag{5.16}$$

Finally we have to solve $L_a$ and $\bar{S}$. As in (3.11) we exponentiate these constraints and get

$$\exp (L[\lambda_a]) \Psi_{u,\phi} = \Psi_{uv,\phi}, \quad \text{where} \quad v = \exp \left( -\frac{i}{2} \lambda_a \sigma_a \right),$$

$$\exp (S[\epsilon]) \Psi_{u,\phi} = \Psi_{u,\phi + u\epsilon}. \tag{5.17}$$

To get formal solution to all constraints, we have to integrate this expressions over $\epsilon$ and $v$:

$$\Phi_{u,\phi} = \int [dv] \int [d\epsilon] \Psi_{uv,\phi + u\epsilon}, \tag{5.18}$$

where we have to assume that the measure is a Haar measure on $SU(2)$ and the spinor measure is invariant under $SU(2)$ rotations of $\epsilon$. These solutions have the same properties as those given in (5.12) for the bosonic theory: they carry topological degrees of freedom, are annihilated by half of the observables etc., and they may be regarded as the vacuum states of supergravity for the same reasons.

Of course, regarding ill-defined operator products and functional integrals the same as for the bosonic theory holds for supergravity. If there is an anomaly in the constraint algebra, we have to choose another quantization method. However, we can proceed in exactly the same way as in section 4, as not only the Lorentz constraints but also the supersymmetry generators split into two conjugate subsets, so instead of integrating over $v$ and $\phi$, we define states as equivalence classes, just replacing $L[\lambda_a]\Phi$ by $L[\lambda_a]\Phi + S[\epsilon]\Phi'$ in (4.24). Again we obtain a well defined vacuum state

$$| 0 \rangle = | \Psi_{u,\phi} \rangle. \tag{5.19}$$

Appendix

A $J$-symbols, Lorentz algebra, and Pauli matrices

The $J$-symbols were introduced in (1.1) as the algebra isomorphism between the two representations of the Lorentz algebra $so(3, \mathbb{C})$ and $so(1, 3)$. Here we summerize the properties of these symbols and give some formulas (see [11, 12] for more details on this notation). They provide a complete and orthonormal (complex) basis of $so(1, 3)$, i.e. they are antisymmetric in $A, B$ and

$$J^A_a J^{AB}_b = \eta_{ab}, \quad J^*_{a}{}^{AB} J_{bAB} = \eta_{ab}, \quad J_a {}^{AB} J_{bAB} = 0,$$

$$J_a {}^{AB} J_{aCD} + J^*_{a}{}^{AB} J^{*AB}_{aCD} = \delta^{AB}_{CD}, \tag{A.1}$$

22
where $\delta_{CD}^{AB} = \frac{1}{2}\delta_{CD}^{AB} - \frac{1}{2}\delta_{AB}^{CD}$ and $\eta_{ab} = \delta_{ab}$ is the metric on $\mathfrak{so}(3)$. Note that this is just the "spatial" part of $\eta_{AB}$, where the indices take the values 1, 2, 3 only, and using the same symbol will be useful below.

Furthermore, the $J$s have to respect the Lie algebra structure. In fact, they also provide two four dimensional Clifford representation of $\mathfrak{so}(3)$,

$$J_a^A B J_b^B C = -\frac{1}{4}\eta_{ab}\delta^C_A + \frac{1}{2}\varepsilon_{abc} J_c^A C,$$

$$J_a^A B J_b^B C = -\frac{1}{4}\eta_{ab}\delta^C_A + \frac{1}{2}\varepsilon_{abc} J_c^A C,$$

(A.2)

commuting with each other:

$$J_a^A B J_b^B C = J_b^A B J_a^B C.$$  

(A.3)

An explicit representation is given by

$$J_a^A B = \frac{i}{2}\eta_a A \delta^0_B - \frac{i}{2}\eta_a B \delta^0_A - \frac{1}{4}\varepsilon^0_{ABC} J^A C,$$

(A.4)

where $\varepsilon^{ABCD}$ is the four dimensional Levi Civita symbol defined by $\varepsilon^{0123} = -\varepsilon^{0123} = 1$. Observe that the three dimensional symbol, which gives the structure constants of $\mathfrak{so}(3)$ in (A.2), is obtained by $\varepsilon_{abc} = \varepsilon^0_{abc}$. Note also that (A.4) defines the generator $J_a$ as the combination "rotation around $a$-axis + i×boost in $a$-direction", which splits the Lorentz algebra into its self-dual and antiself-dual part. Indeed, we find that

$$\varepsilon_{AB}^{CD} J_{a}^{CD} = 2i J_a^{AB}, \quad \varepsilon_{AB}^{CD} J_a^{*CD} = -2i J_a^{AB}. $$

By dropping the $J^*$ from the sum in (A.1) we obtain

$$J_a^{AB} J_{a}^{CD} = \frac{1}{2}\delta^{AB}_{CD} - \frac{i}{4}\varepsilon^{AB}_{CD}, $$

(A.6)

which is the projector onto the self-dual part of an antisymmetric tensor. We can now define the Lorentz-covariant derivatives of various objects carrying different kinds of "flat" indices. For a 4-vector $V^A$ we have

$$D_M V^A = \partial_M V^A + \Omega^A_{MB} V^B. $$

(A.7)

An antisymmetric tensor $T^{AB}$ can be transformed into a "self-dual" 3-vector $T_a = J_a^{AB} T^{AB}$, whose derivative reads

$$D_M T_a = \partial_M T_a + \varepsilon_{abc} A_{Mb} T^c, \quad D_M T_a^* = \partial_M T_a^* + \varepsilon_{abc} A_{Mb} T^c^*. $$

(A.8)

Note that we are using the same symbols $a, b, \ldots$ for both indices transforming under the self-dual and under the antiself-dual representation of the Lorentz algebra. As mixed tensors will not appear throughout this article, a tensor with a * always carries antiself-dual indices. The special tensors $\eta$, $\varepsilon$, $J$ and $J^*$ are constant under the covariant derivative.

The field strength of the spin connection is defined as usual via the commutator of two covariant derivatives and can be given in both the $\mathfrak{so}(1, 3)$ representation

$$R_{MNAB} = \partial_M \Omega_{NAB} - \partial_N \Omega_{MAB} + \Omega_{MA}^C \Omega_{NCB} - \Omega_{NA}^C \Omega_{MCB}, $$

(A.9)
or in the $\text{so}(3,\mathbb{C})$ representation

$$F_{MN} = \partial_M A_N - \partial_N A_M + \varepsilon_{abc} A_{Mb} A_{Nc}.$$  \hfill (A.10)

Of course, they are related by $F_{MN} = J_a R_{MNPQ}$. If, in addition, the spin connection is given by the vierbein postulate \[(2)\], then they are related to the Riemann tensor by $R_{MNPQ} = E_P^A E_Q^B R_{MNPQ}$. To express the Einstein Hilbert action in terms of $A_M$, we use the definition

$$E_{\varepsilon^{ABCD}} = \varepsilon_{MNPQ} E_M^A E_N^B E_P^C E_Q^D.$$  \hfill (A.11)

of the vierbein determinant to obtain

$$S[E] = \frac{1}{2} \int d^4 x \, E R = \frac{1}{2} \int d^4 x \, E C_P^E D_Q^{RP} R_{PQCD} = -\frac{1}{8} \int d^4 x \, \varepsilon^{MNPQ} \varepsilon_{EABCD} E^A_M E^B_N R_{PQCD}. \hfill (A.12)$$

To this expression we add the “square” of the vierbein postulate. This vanishes identically and does not change the equation of motion for $\Omega_{MAB}$, i.e. if we add

$$-\frac{i}{2} \int d^4 x \, \varepsilon^{MNPQ} D_M E_P^A D_P E_Q A,$$  \hfill (A.13)

then the action is still 1.5 order in $\Omega_{MAB}$. After a partial integrations one finds that this is equal to

$$-\frac{i}{4} \int d^4 x \, \varepsilon^{MNPQ} E_M^A E_N^B R_{PQAB}.$$  \hfill (A.14)

Using (A.12) we find

$$S[E] = -\frac{i}{2} \int d^4 x \, \varepsilon^{MNPQ} E_M^A E_N^B J_{aAB} J^C_D R_{PQCD} = -\frac{i}{2} \int d^4 x \, \varepsilon^{MNPQ} E_M^A E_N^B J_{aAB} F_{PQa},$$  \hfill (A.15)

which gives the action (1.3).

For supergravity we have to define Majorana fermions. For our purpose it is most useful to represent them as two component Grassmann valued complex spinors $\psi$. The conjugate spinor is defined by $\bar{\psi} = i \psi^\dagger$. Under local Lorentz rotations they transform under the (anti)self-dual representation, i.e.

$$D_M \psi = \partial_M \psi - \frac{1}{2} A_M \sigma_a \psi, \quad D_M \bar{\psi} = \partial_M \bar{\psi} + \frac{1}{2} A^*_M \bar{\psi} \sigma_a,$$  \hfill (A.16)

where $\sigma_a$ are the Pauli matrices. To build a vector bilinear from a spinor (this is the only bilinear we need), we have to provide four dimensional “gamma matrices” such that $\bar{\psi} \sigma_A \psi_2$ transforms as a vector. This can be achieved by using the same Pauli matrices together with $\sigma_0 = 1$. One finds the algebra [4]

$$\sigma_A \sigma_a = 2i J_{aAB} \sigma^B, \quad \sigma_a \sigma_A = -2i J^*_a_{AB} \sigma^B.$$  \hfill (A.17)

Note that the three dimensional index $a$ is a self-dual index if it appears to the right of $\sigma_A$ but an antiself-dual index if it appears on the left. As the gamma matrices are hermitian, the vector bilinear obeys

$$(\bar{\psi}_1 \sigma_A \psi_2)^* = -\bar{\psi}_2 \sigma_A \psi_1.$$  \hfill (A.18)
The supersymmetric partner of the vierbein is the gravitino $\psi_M$, and the Rarita Schwinger action for supergravity [15, 16] takes its simplest form in 1.5 order formalism, where the spin connection $\Omega_{MAB}$ (and therefore $A_{Ma}$) is defined by the torsion equation

$$D_{[M}E_{N]}^A - \bar{\psi}_M \sigma^A \psi_N = 0.$$  \hspace{1cm} (A.19)

It is the equation of motion for $\Omega_{MAB}$ in

$$S[E, \psi] = \int d^4x \varepsilon^{MNPQ} \left( - \frac{1}{8} \varepsilon_{AB}^{\ CD} E_M^A E_N^B R_{PQCD} + \right.$$ \hspace{1cm} (A.20)
$$\left. + i \bar{\psi}_M \sigma_N D_P \psi_Q - i D_M \psi_N \sigma_P \psi_Q \right),$$

with $\sigma_N = E_M^A \sigma_A$. Here the fields strength $R_{PQCD}$ is no longer the Riemann tensor with flat indices, since the vierbein postulate has been replaced by the torsion equation. One can again write the action in terms of $A_{Ma}$ by adding the “square” of the torsion equation

$$-\frac{i}{2} \int d^4x \varepsilon^{MNPQ} (D_M E_N^A - \bar{\psi}_M \sigma^A \psi_N) (D_P E_Q^A - \bar{\psi}_P \sigma^A \psi_Q)$$  \hspace{1cm} (A.21)

to the action. This is equal to

$$\int d^4x \varepsilon^{MNPQ} \left( - \frac{i}{4} E_M^A E_N^B R_{PQAB} + \right.$$ \hspace{1cm} (A.22)
$$\left. + i \bar{\psi}_M \sigma_N D_P \psi_Q + i D_M \psi_N \sigma_P \psi_Q \right),$$

and adding it to (A.20) gives (5.1). The vanishing of the quartic term in $\psi$ is due to the Fierz identities for Graßmann valued spinors. The formula needed here is

$$\bar{\psi}_1 \sigma_A \psi_2 \bar{\psi}_3 \sigma^A \psi_4 = \bar{\psi}_1 \sigma_A \psi_4 \bar{\psi}_3 \sigma^A \psi_2,$$  \hspace{1cm} (A.23)

and other useful identities are

$$J_{aAB} \bar{\psi}_1 \sigma^A \psi_2 \bar{\psi}_3 \sigma^B \psi_4 = J_{aAB} \bar{\psi}_1 \sigma^A \psi_4 \bar{\psi}_3 \sigma^B \psi_2,$$
$$J_{aAB}^* \bar{\psi}_1 \sigma^A \psi_2 \bar{\psi}_3 \sigma^B \psi_4 = -J_{aAB}^* \bar{\psi}_1 \sigma^A \psi_4 \bar{\psi}_3 \sigma^B \psi_2.$$  \hspace{1cm} (A.24)

### B Recovering the Wheeler DeWitt equation

Here the transformation of the hamiltonian constraints $H_A$ to the metric representation defined in section 4 will be carried out. We assume that the wave functional is a function $\tilde{\Psi}[g_{mn}]$ of the spacial metric, which is the general solution to the Lorentz constraints.

We introduced the three dimensional complex spin connection $\omega_{ma}$ via the dreibein postulate (4.11). To simplify notation, we define an operator

$$p_{pa} = i \omega_{pa} - i A_{pa}.$$  \hspace{1cm} (B.1)

This looks like the well known split of Ashtekar’s variables into the spin connection and the momentum of the dreibein, but remember that $\omega_{pa}$ as well as $p_{pa}$ are complex here and do not represent...
real and imaginary part of $A_{pa}$. However, for the commutator of $p_{ma}$ and $\tilde{e}^m_a$ we find the canonical relations (remember that spacial delta function are to be restored)

$$[p_{ma}, \tilde{e}^n_b] = i\eta_{ab}\delta^n_m, \quad [p_{ma}, p_{nb}] = 0,$$  \hspace{1cm} (B.2)

and on a holomorphic functional of the dreibein $p_{pa}$ acts as $i\delta/\delta\tilde{e}_a^p$. As $g_{mn}$ is such a functional, defined in (3.4), we find

$$p_{pa}\tilde{\Psi} = i\frac{\delta\tilde{\Psi}}{\delta\tilde{e}_a^p} = i\frac{\partial g_{mn}}{\partial\tilde{e}_a^p} \frac{\delta\tilde{\Psi}}{\delta g_{mn}}. \hspace{1cm} (B.3)$$

Explicitly we have

$$\frac{\partial g_{mn}}{\partial\tilde{e}_a^p} = e^{-1}(g_{mn}\varepsilon_{pa} - g_{mp}\varepsilon_{na} - g_{np}\varepsilon_{ma}) = -2G_{mnpq}e_a^q, \hspace{1cm} (B.4)$$

where $G_{mnpq}$ is the inverse supermetric introduced in (4.20). The field strength appearing in the constraints becomes

$$F_{mna} = R_{mna} + 2i\partial_{\{m}p_{n]a} + i\varepsilon_{abc}(\omega_{mb}p_{nc} + p_{mb}\omega_{nc}) - \varepsilon_{abc}p_{mb}p_{nc}$$

$$= R_{mna} + 2iD_{\{m}p_{n]a} - \varepsilon_{abc}p_{mb}p_{nc} + i\varepsilon_{abc}[p_{mb}, \omega_{nc}], \hspace{1cm} (B.5)$$

where

$$R_{mna} = \partial_m\omega_{na} - \partial_n\omega_{ma} + \varepsilon_{abc}\omega_{mb}\omega_{nc} \hspace{1cm} (B.6)$$

is the field strength of $\omega_{ma}$. It is related to the three dimensional Riemann tensor by

$$R_{mnpq} = \varepsilon_{abc}R_{mna}e_b^pe_cq. \hspace{1cm} (B.7)$$

Note that this is again real because it is the Riemann tensor of a real metric although the dreibein and the spin connection are complex. Using the symmetries of the Riemann tensor we find the useful relations

$$\varepsilon^{mnp}R_{mna}e_{pa} = eR, \quad R_{mna}\tilde{e}_a^m = 0, \hspace{1cm} (B.8)$$

where now $R$ denotes the three dimensional Ricci scalar.

In (B.3) we picked up a singular term when ordering the spin connection in $D_{m}p_{na}$ to the left, using the commutator (B.2): the two entries of the commutator are to be taken at the same point. The diffeomorphism constraint, given in (4.4), now reads (the term containing $R_{mna}$ vanishes by (B.8))

$$H_m = 2\nabla_m(\tilde{e}_a^m p_{na}) + i\varepsilon_{abc}p_{mb}\tilde{e}_a^m p_{nc} + i\varepsilon_{abc}\tilde{e}_a^m[p_{mb}, \omega_{nc}]. \hspace{1cm} (B.9)$$

To get the first term we used the fact that the dreibein is covariantly constant under $D_{m}$; here $\nabla_m$ is the metric covariant derivative with respect to the spacial metric $g_{mn}$. In the second term we placed the dreibein factor between the two $p$-factors, which is allowed by (B.2). One can either show that this gives a term proportional to the Lorentz constraint or use (B.3) to see that it vanishes when acting on $\tilde{\Psi}[g_{mn}]$. 

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The singular term also drops out (formally, but that’s all we can say without a proper regularization). By (B.2) we can put the dreibein inside the commutator and get
\[ i \varepsilon_{abc} [p_{mb}, \omega_{nc} \epsilon_n^a] = -i [p_{mb}, \partial_n \epsilon_n^a] \] (B.10)

Splitting the points where the fields are taken, we get an expression like
\[ \delta(x,y) \partial_m \delta(x,y), \] integrated over \( y \), and this vanishes because it is a total divergence. A similar argument is also used in [13] to show that \( H_m \) generates diffeomorphisms in Ashtekar’s representation.

The only term surviving in (B.9) is the first one. With (B.4) and after some algebra we get
\[ H_m \tilde{\Psi} = 2 \nabla(p(g_{pq}) \delta \tilde{\Psi} \delta g_{pq}) \] (B.11)

This just requires \( \tilde{\Psi} \) to be invariant under spacial diffeomorphisms \( \delta g_{mn} = \nabla(m \nu_n) \) with some vector field \( v_n \).

The hamiltonian constraint has also been given in (4.4). Without the extra density factor it reads
\[ H = \frac{1}{2} \varepsilon^{mnp} e_{ma} F_{npa} = \frac{1}{2} \varepsilon R + i \varepsilon^{mnp} \partial_n (e_{ma} P_{pa}) - \frac{1}{2} \varepsilon^{mnp} \varepsilon_{abc} e_{ma} P_{nb} P_{pc} + \frac{1}{2} \varepsilon^{mnp} \varepsilon_{abc} e_{ma} [p_{nb}, \omega_{pc}] \] (B.12)

Here we used (B.3) for the first term and the dreibein postulate (4.11) to get the second term. The first term is already the required “potential term” in (4.19). The second term vanishes when acting on \( \tilde{\Psi} \) as
\[ \varepsilon^{mnp} e_{ma} P_{pa} \tilde{\Psi} = i \varepsilon^{mnp} e_{ma} \frac{\partial g_{rs}}{\partial e^p_c} \frac{\delta \tilde{\Psi}}{\delta g_{rs}} = 0, \] (B.13)

which follows immediately from (B.3). The third term is quadratic in \( p_{ma} \) and yields the “kinetic part” of the Wheeler DeWitt equation. We rewrite it using (B.3) as
\[ -\frac{1}{2} \varepsilon^{mnp} \varepsilon_{abc} e_{ma} P_{nb} P_{pc} \tilde{\Psi} = \frac{1}{2} \frac{\partial \tilde{e}_b^n}{\partial \epsilon^p_c} \frac{\delta}{\delta \epsilon^p_c} \left( \frac{\partial g_{rs}}{\partial \epsilon^p_c} \frac{\delta \tilde{\Psi}}{\delta g_{rs}} \right) = -\frac{\delta}{\delta g_{pq}} \left( \sqrt{g} \delta G_{mnpq} \frac{\delta \tilde{\Psi}}{\delta g_{mn}} \right). \] (B.14)

To get the last line we used the following formula which holds for any function \( f(g_{mn}) \):
\[ \frac{\partial}{\partial g_{pq}} \left( f \sqrt{g} \right) = 2 \sqrt{g} \frac{\partial}{\partial g_{pq}} \left( f \sqrt{g} \right). \] (B.15)

To show that what we got is indeed the Laplace operator appearing in (4.21), we only have to show that the determinant of the supermetric \( G \) is proportional to \( g^{-1} \). As \( G_{mnpq} \) is the inverse supermetric (the “coordinates” \( g_{mn} \) have lower indices), its determinant should be proportional to \( g \). That this is in fact true can be seen as follows: we know that the determinant of \( G_{mnpq} \) is given as a homogeneous polynomial of degree 6, and from (4.20) we infer that it is a polynomial of degree
12 in $g_{mn}$, multiplied by $g^{-3}$. However, there is only one polynomial of degree 12 in $g_{mn}$ that gives a scalar under spacial diffeomorphisms, and this is $g^4$, so all together we have $\det(G_{mnpq}) \propto g$ and $G \propto g^{-1}$.

Finally we have to consider the singular term

$$i \frac{1}{2} e^{mnp} \epsilon_{abc} e_{ma} [p_{nb}, \omega_{pc}] = \frac{1}{2} e^{mnp} (\{p_{nb}, \partial_m e_{pb}\} - \epsilon_{abc} [p_{nb}, e_{ma}] \omega_{pc}).$$  \hfill (B.16)

Here we need the commutator

$$[p_{nb}, e_{ma}] = i \frac{\partial e_{ma}}{\partial e_{b}^a} = i e^{-1}(e_{ma} e_{nb} - e_{mb} e_{na}).$$  \hfill (B.17)

Contacting $a, b$ gives something symmetric in $m, n$, so the first term above vanishes. The second gives

$$-3 i \delta(0) e^{mnp} \epsilon_{abc} e_{ma} e_{nb} \omega_{pc} = -3 i \delta(0) e^{mnp} e_{ma} \partial_h e_{pa},$$  \hfill (B.18)

where $\delta(0) = \delta(x, x)$ is the infinite constant we get from commuting to operators at the same point. Adding all the results together we get (4.21).

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