Partial Franel sums

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Abstract

Analytical expressions are derived for the position of irreducible fractions in the Farey sequence $F_N$ of order $N$ for a particular choice of $N$. The asymptotic behaviour is derived obtaining a lower error bound than in previous results when these fractions are in the vicinity of $0/1, 1/2$ or $1/1$.

Franel’s famous formulation of Riemann’s hypothesis uses the summation of distances between irreducible fractions and evenly spaced points in $[0,1]$. A partial Franel sum is defined here as a summation of these distances over a subset of fractions in $F_N$. The partial Franel sum in the range $[0,i/N]$, with $N = \text{lcm}(1, 2, ..., i)$ is shown here to grow as $O(\log(N)\delta_B(\log N))$, where $\delta_B(x)$ is a decreasing function.

Other partial Franel sums are also explored.

1 Introduction and main results

The Farey sequence $F_N$ of order $N$ is an ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed $N$ [1]. Riemann’s hypothesis implies that the irreducible fractions tend to be regularly distributed in $[0,1]$. A formulation of this statement follows [2],

$$
\left| \sum_{n=1}^{|F_N|} \left| F_N(n) - \frac{n}{|F_N|} \right| \right| = O\left(N^{\frac{3}{2}} + \epsilon\right),
$$

where $F_N(n)$ is the $n^{th}$ irreducible fraction in $F_N$. Here we define the partial Franel sum in the range $[a_1/b_1, a_2/b_2]$ as

$$
P\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = \sum_{n=I_N(a_2/b_2)}^{I_N(a_1/b_1)} \left| F_N(n) - \frac{n}{|F_N|} \right|,
$$

where $I_N(a/b)$ is the position that $a/b$ occupies in $F_N$. In [3] the upper bound of the distance $|F_N(n) - n/|F_N||$ is established to be $1/N$ and to be located at $F_N(2) = 1/N$. This motivates the study of partial Franel sums in ranges including $1/N$. Furthermore, another equivalent formulation of the Riemann’s hypothesis involving sums over irreducible fractions in the range $[0,1/4]$ follows [4],

$$
\sum_{n=1}^{I_N(1/4)} \left( F_N(n) - \frac{I_N(1/4)}{2|F_N|} \right) = O\left(N^{\frac{3}{2}} + \epsilon\right),
$$

showing again the relevance of the vicinity of $1/N$.

In [5], Chapter 6, it is attempted to find a closed expression for the $i^{th}$ fraction in $F_N$ ending in an “analytical hole”. This paper achieves this goal for fractions in the range $[0,i/N]$, with $N = \text{lcm}(1, 2, ..., i)$ as explained in the following. Note
that \( N = \text{lcm}(1, 2, \ldots, i) = e^{\psi(i)} \), where \( \psi(i) \) is the second Chebyshev function that fulfills the property \( \psi(i) = (1 + o(1))i \), and hence \( i = (1 + o(1))\log N \).

Let the subsequence \( F_{N}^{a/b_1, a_2/b_2} \) of \( F_N \), contain all the fractions of \( F_N \) in \([a_1/b_1, a_2/b_2]\). The cardinality of \( F_{N}^{a_1/b_1, a_2/b_2} \) is well known to be \( N \)

\[
\left| F_{N}^{a_1/b_1, a_2/b_2} \right| = \frac{3}{\pi^2} \left( \frac{a_2}{b_2} - \frac{a_1}{b_1} \right) N^2 + O(N \log N).
\]

As \( I_N (a_2/b_2) \) is the position that \( a_2/b_2 \) occupies in \( F_N \), it follows that

\[
I_N \left( \frac{a_2}{b_2} \right) = \frac{3}{\pi^2} \left( \frac{a_2}{b_2} - \frac{a_1}{b_1} \right) N^2 + O(N \log N).
\]

A first result of this paper is the derivation of an analytical expression for \( I_N (1/q) \) where \( N = \text{lcm}(1, 2, \ldots, i) \) and \( N/i \leq q \leq N \) as

\[
I_N \left( \frac{1}{q} \right) = 2 + N \sum_{j=1}^{i} \frac{\varphi(j)}{j} - q\Phi(i),
\]

where \( \varphi(i) \) is the Totient function and \( \Phi(i) \) is the summatory Totient function. To reach this relation a series of bijections between \( F_{i'} \), with \( i' \leq i \), and subsequences of \( F_N \) are established covering all elements in \( F_{N}^{0, 1/q} \). Thanks to these bijections the cardinality of \( F_{N}^{0, 1/q} \) can be expressed as function of all \( |F_{i'}| \). These bijections are illustrated in Table 1 for \( N = \text{lcm}(1, 2, \ldots, 5) = 60 \). This result is used to derive the equivalent asymptotic estimate of \( \Phi(i) \) with a smaller residual error:

\[
I_N \left( \frac{1}{q} \right) = \frac{3}{\pi^2} q \left( \frac{N^2}{q^2} - \left\{ \frac{N}{q} \right\}^2 \right) + O \left( N \delta_A \left( \left\{ \frac{N}{q} \right\} \right) \right),
\]

where \( \{x\} = x - \lfloor x \rfloor \) and \( \delta_A(x) \) is a decreasing function defined as

\[
\delta_A(x) = \exp \left( -A \frac{\log^{0.6} x}{(\log \log x)^{0.2}} \right),
\]

where \( A > 0 \).

Using this result the partial Franel sum in the range \([0, 1/(N/i)]\) is shown to be

\[
P \left( \frac{0}{1}, \frac{1}{N/i} \right) = O(\log(N)\delta_B(\log N)),
\]

with \( 0 < B < A \) and again \( N = \text{lcm}(1, 2, \ldots, i) \). This partial Franel sum, therefore, grows strictly slower than \( O(\log N) \). If we would assume the Riemann hypothesis and a uniform distribution density of Farey elements in \([0, 1]\) we would expect this partial Franel sum to actually decrease as \( O(\log(N)/N^{1/2+\epsilon}) \). An equivalent result is obtained for partial Franel sums in ranges including \(1/2\). The generalization to compute partial Franel sums in the vicinity of any irreducible fraction is explored. Earlier results of this work were applied to resonance diagrams [7, 8].

2 Definitions

We say that two elements of a Farey sequence, \( a_1/b_1 \) and \( a_2/b_2 \), form a Farey pair if \(|a_1b_2 - a_2b_1| = 1\). In this report we exceptionally allow \( 0/1 \) and \( 1/0 \) to form a Farey pair even if \( 1/0 \) is not a proper fraction. The mediant of a Farey pair, \( a_1/b_1 \) and \( a_2/b_2 \), is given by

\[
\frac{a_1 + a_2}{b_1 + b_2},
\]

which is an irreducible fraction existing between \( a_1/b_1 \) and \( a_2/b_2 \) and forms two Farey pairs with \( a_1/b_1 \) and \( a_2/b_2 \).
Table 1: Correspondance between elements in $N$ and $u/l$. Partial Franel sums are given by $F(q) = \frac{1}{N\lambda} \sum_{n=0}^{N-1} e^{2\pi i n q}$. The Franel sums with $u/l$ and $1/0$ are given. Note that the number of elements in $N$ and $l$ are equal, and only the first 90 elements in $N$ are given.
3 Results

Theorem 3.1. Let $a_1/b_1$ and $a_2/b_2$ be a Farey pair with $b_1 > b_2$. Let $N$ be multiple of $b_1(i+1)$ with $i$ being a natural number such $0 < i < N$. Let $q$ be an integer fulfilling
\[ \frac{N}{b_1(i+1)} < q \leq \frac{N}{b_1i} \quad \text{and} \quad b_1q + b_2 \leq N . \]

Let $F'_i$ be defined as
\[ F'_i = \left\{ \frac{h}{k} : \frac{h}{k} < F_i, \ k(b_1q + b_2) - b_1h \leq N \right\} . \]

There is a bijective map $M$ between $F'_i$ and $F_N^{a_1+a_2}_{b_1+b_2}, F_N^{a_1(q-1)+a_2}_{b_1(q-1)+b_2}$, given by
\[
M : F'_i \to F_N^{a_1+a_2}_{b_1+b_2}, \quad \frac{h}{k} \mapsto \frac{k(a_1q + a_2) - a_1h}{k(b_1q + b_2) - b_1h} .
\]

$M^{-1} : F_N^{a_1+a_2}_{b_1+b_2}, F_N^{a_1(q-1)+a_2}_{b_1(q-1)+b_2} \to F'_i$, \quad \frac{u}{l} \mapsto \frac{q(b_1u - la_1) + b_2u - la_2}{b_1u - la_1} .

The bijective map is order-preserving when $a_2/b_2 > a_1/b_1$ and order-inverting when $a_2/b_2 < a_1/b_1$.

Proof. We first demonstrate that $M$ is injective. $\frac{a_1+a_2}{b_1+b_2}$ and $\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}$ form a Farey pair since $a_1/b_1$ and $a_2/b_2$ form a Farey pair:
\[
|(a_1q + a_2)(b_1(q-1) + b_2) - (b_1q + b_2)(a_1(q-1) + a_2)| = |b_2a_1 - a_2b_1| = 1 .
\]

Let $u/l$ be the image of $h/k$ under $M$,
\[
\frac{u}{l} = \frac{k(a_1q + a_2) - a_1h}{k(b_1q + b_2) - b_1h} .
\]

By virtue of this expression $u/l$ is obtained by applying the mediant operation successively between $\frac{a_1+a_2}{b_1+b_2}$ and $\frac{a_1(q-1)+a_2}{b_1(q-1)+b_2}$ in the same fashion as $h/k$ is obtained by applying the mediant between $0/1$ and $1/1$, meaning
\[
\frac{h}{k} = \frac{(k - h) \cdot 0 + h \cdot 1}{(k - h) \cdot 1 + h \cdot 1} , \quad \frac{u}{l} = \frac{(k - h) \cdot (a_1q + a_2) + h \cdot (a_1(q-1) + a_2)}{(k - h) \cdot (b_1q + b_2) + h \cdot (b_1(q-1) + b_2)} .
\]

Therefore $u/l$ is a Farey fraction in the interval of interest:
\[
\frac{[a_1q + a_2, a_1(q-1)+a_2]}{[b_1q + b_2, b_1(q-1)+b_2]} .
\]

$u/l$ belongs to $F_N$ by definition of the domain $F'_i$, meaning that $h/k$ belongs to $F'_i$ if $l \leq N$.

Now we demonstrate that $M^{-1}$ is also injective. Let $u/l$ belong to $F_N^{a_1+a_2}_{b_1+b_2}, F_N^{a_1(q-1)+a_2}_{b_1(q-1)+b_2}$ and assume $a_2/b_2 > a_1/b_1$, so that
\[ \frac{a_1q + a_2}{b_1q + b_2} \leq \frac{u}{l} \leq \frac{a_1(q-1) + a_2}{b_1(q-1) + b_2} .
\]

Let $h/k$ be the image of $u/l$ under $M^{-1}$,
\[ \frac{h}{k} = \frac{q(b_1u - la_1) + b_2u - la_2}{b_1u - la_1} .
\]

This equality implies $\gcd(h,k) = \gcd(b_2 - la_2, b_1u - la_1)$. Since $\gcd(u,l) = \gcd(a_1,b_1) = \gcd(a_2,b_2) = 1$
and \( a_2 b_1 - a_1 b_2 = 1 \) then \( \gcd(h, k) = 1 \) according to the property in \([9]\) and, hence, \( h/k \) is an irreducible fraction. Furthermore, operating with the inequalities in \([3]\):

\[
q(b_1 u - l a_1) \geq -(a_2 - l a_2) \geq (b_1 u - l a_1)(q - 1)
\]

and therefore \( 0 \leq h \leq k \).

From relations \([3]\) and \([4]\)

\[
k = b_1 u - l a_1 \leq b_1 l \frac{a_1(q - 1) + a_2}{b_1(q - 1) + b_2} - l a_1 = \frac{l}{b_1(q - 1) + b_2}
\]

and using that \( l \leq N \) and \( q > \frac{N}{b_1(i + 1)} \), hence \( b_1(q - 1) \geq \frac{N}{i + 1} \).

\[
k \leq \frac{N}{b_1(i + 1) + b_2} = \frac{i + 1}{1 + \frac{1}{N} b_2} < i + 1.
\]

If \( b_2 > 0 \) this implies \( k \leq i \) and gathering the above results \( 0 \leq h \leq k \leq i \) and \( \gcd(h, k) = 1 \), hence \( h/k \in F_i \). To demonstrate that \( h/k \) belongs to \( F_i' \) it is easy to verify that \( k(b_1 q + b_2) = b_1 h \leq N \).

If \( b_2 = 0 \) we are in the exceptional case included in this report of \( a_1/b_1 = 0/1 \) and \( a_2/b_2 = 1/0 \), that implies \( h/k = (qu - l)/u \), note that \( k = u \). We only need to show that \( k \leq i \) also in this case. From the inequalities in \([3]\) and \( \frac{N}{i} \geq q > \frac{N}{i + 1} \),

\[
\frac{i}{N} < \frac{1}{q} < \frac{u}{l} \leq \frac{1}{q - 1} < \frac{i + 1}{N}.
\]

\((i + 1)/N\) is not an irreducible fraction, as \( N \) is taken as a multiple of \( i(i + 1) \), and therefore it does not belong to \( F_N \). Similarly for \( i/N \) when \( i > 1 \). In the range \([i/N, (i + 1)/N]\) there cannot be fractions with denominator \( N \) other than \( 1/N \) when \( i = 1 \). Therefore if \( i = 1 \) we directly have \( k = u \leq i \) and for \( i > 1 \) we have that \( l \leq N - 1 \) and hence

\[
k = u \leq l \frac{i + 1}{N} \leq i.
\]

\(\square\)

**Corollary 3.2.** The cardinalities of \( F_i \), \( F_i' \) and \( F_N \) are related as follows:

- If \( q = N/(b_1 i) \) then

\[
|F_i| \geq |F_i'| = \begin{vmatrix} a_1 a_2 + a_2 & a_1(q - 1) + a_2 \\ b_1(q - 1) + b_2 & b_1(q - 1) + b_2 \end{vmatrix} > |F_i| - i
\]

- If \( q < N/(b_1 i) \) or \( b_2 = 0 \) then

\[
|F_i| = |F_i'| = \begin{vmatrix} a_1 a_2 + a_2 & a_1(q - 1) + a_2 \\ b_1(q - 1) + b_2 & b_1(q - 1) + b_2 \end{vmatrix}
\]

**Proof.** The first inequality is evident from the definition of \( F_i' \). The first equality derives from the the bijective map in Theorem 3.1.

If \( q = N/(b_1 i) \), let \( u/l \) be the image of \( h/k \) via the map \( M \) in Theorem 3.1 then \( l = k(N/i + b_2) - b_1 h \). To prove that \( |F_i'| > |F_i| - i \) we should count how many \( h/k \in F_i \) fulfill \( k(N/i + b_2) - b_1 h > N \). Dividing both sides of the later inequality by \( k \) and operating we obtain

\[
b_2 - b_1 \frac{h}{k} > \frac{N}{k} - \frac{N}{i} = N \frac{i - k}{k i},
\]

\[
(5) \quad b_2 \geq b_2 - b_1 \frac{h}{k} > N \frac{i - k}{k i} \geq 0.
\]
To fulfill these inequalities it is required that \( k = i \), otherwise for any \( k < i \) and recalling that \( N \) is a multiple of \( b_1(i + 1) \):

\[
N \frac{i - k}{k_1} \geq b_1 \frac{i + 1}{k} (i - k) > b_1 ,
\]

and inequalities in (3) cannot be fulfilled as \( b_2 < b_1 \) (from assumption in Theorem 3.1). Then \( k = i \) implies \( h/i < b_2/b_1 < 1 \) and in \( F_i \) there are fewer than \( i \) irreducible fractions of the form \( h/i \) below \( b_2/b_1 \), hence \( |F'_i| > |F_i| - i \).

If \( q < N/(b_1i) \) we define \( g > 0 \) such that \( q = N/(b_1i) - g \), then \( l = k(N/i - gb_1 + b_2) - b_1h \) and we need to count how many \( h/k \) in \( F_i \) have \( l > N \),

\[
b_2 - b_1 \frac{h}{k} - b_1g > N \frac{i - k}{k_1} ,
\]

and there are no \( h/k \) which can fulfill this equation as \( b_2 - b_1g < 0 \), hence \( |F_i| = |F'_i| \) when \( q < N/(b_1i) \).

If \( b_2 = 0 \) we should show that there are no \( h/k \) in \( |F_i| \) fulfilling \( kb_1q - b_1h > N \). The largest possible value of \( q \) is \( N/(b_1i) \) and therefore \( kb_1q - b_1h \leq kN/i - b_1h < N \), for \( i > 1 \), so there is no \( h/k \) fulfilling the previous condition and \( |F_i| = |F'_i| \). Note that \( i = 1 \) and \( h/k = 0/1 \) would not have given \( kb_1q - b_1h > N \) as \( b_1q + b_2 \leq N \) from the assumptions in Theorem 3.1.

**Theorem 3.3.** Let \( N = b_1 \text{lcm}(1, 2, \ldots, i_{\text{max}}) \), \( \frac{N}{b_1(i + 1)} < q \leq \frac{N}{b_1i} \), with \( a_1/b_1 \) and \( a_2/b_2 \) forming a Farey pair, \( b_1 > b_2 \) and \( i < i_{\text{max}} \) then:

- For \( b_1 > 1 \):
  
  \[
  I_N \left( \frac{a_1q + a_2}{b_1q + b_2} \right) = I_N \left( \frac{a_1}{b_1} \right) + s \left( \frac{N}{b_1} \sum_{j=1}^{i} \frac{\varphi(j)}{j} - \varphi(i) \right) + O(i^2) ,
  \]

  with \( s = +1 \) when \( a_1/b_1 < a_2/b_2 \) and \( s = -1 \) otherwise.

- For \( a_1/b_1 = 0/1 \) and \( a_2/b_2 = 1/0 \):
  
  \[
  I_N \left( \frac{1}{q} \right) = 2 + N \sum_{j=1}^{i} \frac{\varphi(j)}{j} - \varphi(i) .
  \]

**Proof.** To simplify equations we assume \( s = +1 \) in the following. We count the number of elements in \( F_N^{a_1 a_2/b_1 b_2} \) using the bijective maps described in Theorem 3.1 and adding up the cardinalities of the sets involved from Corollary 3.2. Thanks to the fact that \( N \) is multiple of all natural numbers \( i' \) such that \( i' \leq i \) we can establish bijections between \( F_i \) and \( F_N^{a_1 a_2/b_1 b_2} \) where \( p \) can take all values fulfilling \( \frac{N}{b_1(i' + 1)} < p \leq \frac{N}{b_1i'} \), covering all elements in \( F_N^{a_1 a_2/b_1 b_2} \) when scanning over all \( i' \leq i \) and the corresponding \( p \). For a given \( i' \) the number of values \( p \) takes is given by

\[
\frac{N}{b_1i'} - \frac{N}{b_1(i' + 1)} = \frac{N}{b_1} \left( \frac{1}{i'} - \frac{1}{i' + 1} \right).
\]

In a first step we compute the number of elements in \( F_N^{a_1 a_2/q'/b_1 b_2} \) with \( q' = N/(b_1i) \),

\[
I_N \left( \frac{a_1q' + a_2}{b_1q' + b_2} \right) - I_N \left( \frac{a_1}{b_1} \right) = \frac{N}{b_1} \sum_{i'=1}^{i-1} \left( \frac{1}{i'} - \frac{1}{i' + 1} \right) (|F'_{i'}| - 1)
\]

\[
= \frac{N}{b_1} \sum_{i'=1}^{i-1} \left( \frac{1}{i'} - \frac{1}{i' + 1} \right) \Phi(i') + O(i') \]

\[
= \frac{N}{b_1} \sum_{j=1}^{i-1} \frac{\varphi(j)}{j} - \frac{N}{b_1} \Phi(i - 1) + O(i^2) .
\]
In particular, when \( b_2 = 0 \) the term \( O(i^2) \) does not appear according to Corollary 3.2.

In a second step we compute the number of elements in \( \mathbb{F}_N^{a_1q + a_2 \over b_1q + b_2} \), that is \( \Phi(i)(q' - q) + O(i) \). Adding both contributions gives

\[
I_N \left( \frac{a_1q + a_2}{b_1q + b_2} \right) = I_N \left( \frac{a_1}{b_1} \right) = \frac{N}{b_1} \sum_{j=1}^{i-1} \varphi(j) + \Phi(i) \left( \frac{N}{b_1} - q \right) + O(i^2)
\]

which demonstrates the theorem for \( s = 1 \). For \( s = -1 \) following the same steps leads to the desired result.

**Corollary 3.4.** Let \( N = b_1 \text{lcm}(1, 2, \ldots, i_{\text{max}}) \) and \( \frac{N}{b_1(i + 1)} < q \leq \frac{N}{b_1} \), with \( i < i_{\text{max}} \) then

\[
I_N \left( \frac{a_1q + a_2}{b_1q + b_2} \right) = I_N \left( \frac{a_1}{b_1} \right) + s \frac{3}{\pi^2} q \left( \frac{N^2}{b_1^2 q^2} - \left\{ \frac{N}{q} \right\}^2 \right) + O(N \delta_A(i)) ,
\]

with \( \delta_A(x) \) defined in [4]. In particular for \( a_1/b_1 = 0/1 \) and \( a_2/b_2 = 1/0 \),

\[
I_N \left( \frac{1}{q} \right) = \frac{1}{\pi^2 q} \left( \frac{N^2}{q^2} - \left\{ \frac{N}{q} \right\}^2 \right) + O(N \delta_A(i)),
\]

and for \( a_1/b_1 = 1/2 \) and \( a_2/b_2 = 2/1 \),

\[
I_N \left( \frac{q + 1}{2q + 1} \right) = \frac{|N|}{2} + \frac{3}{\pi^2} q \left( \frac{N^2}{2q^2} - \left\{ \frac{N}{2q} \right\}^2 \right) + O(N \delta_A(i)) .
\]

**Proof.** The following known relations [10] [11] are needed:

\[
\sum_{k=1}^{N} \varphi(k) = \frac{3}{\pi^2} N^2 + E(N),
\]

\[
\sum_{k=1}^{N} \varphi(k) \frac{k}{N} = \frac{6}{\pi^2} N + H(N),
\]

\[
E(x) = O \left( x^{2/3} \log \log x \right),
\]

\[
E(x) = x H(x) + O(x \delta_A(x)),
\]

with \( A > 0 \) and \( \delta_A(x) \) is a decreasing factor. From the definition of \( i, q \) and \( N \) it follows that

\[
i = \left\lfloor \frac{N}{q b_1} \right\rfloor = \frac{N}{b_1} + O(1),
\]

\[
i < i_{\text{max}} = (1 + o(1)) \log N/b_1 .
\]

Inserting the above equalities in expression [6] of Theorem 3.3.

\[
I_N \left( \frac{a_1q + a_2}{b_1q + b_2} \right) = I_N \left( \frac{a_1}{b_1} \right) + \frac{6}{b_1} \frac{N}{\pi^2} + s \frac{3}{\pi^2} q + s \frac{\overline{N}}{b_1} H(i) - sq E(i) + O(i^2)
\]

\[
= I_N \left( \frac{a_1}{b_1} \right) + s \frac{6}{\pi^2} \frac{N}{b_1} + \frac{3}{\pi^2} q + i H(i) - E(i)) + O(i^2)
\]

\[
= I_N \left( \frac{a_1}{b_1} \right) + \frac{6}{b_1} \frac{N}{\pi^2} - sq \pi^2 + O(N \delta_A(i))
\]

\[
= I_N \left( \frac{a_1}{b_1} \right) + s \frac{3}{\pi^2} \frac{N^2}{b_1^2 q} - s \frac{3}{\pi^2} q \left\{ \frac{N}{b_1 q} \right\}^2 + O(N \delta_A(i)) .
\]
Theorem 3.5. Let \( N = b_1 \text{lcm}(1, 2, \ldots, i) \) then the partial Franel sum over all
Farey fractions in the range \( \left[ a_1 \frac{i}{b_1}, a_1 \frac{i+1}{b_1} + a_2 \right] \) gives:

- For \( a_1/b_1 = 0/1, a_2/b_2 = 1/0 \) and for \( a_1/b_1 = 1/2, a_2/b_2 = 0/1 \):
  \[
P \left( \frac{0}{1} \frac{1}{N/1} \right) = \sum_{j=1}^{I_N \left( \frac{N}{N/1} \right)} \left| F_N(j) - \frac{j}{|F_N|} \right| = O(\log(N)\delta_B(\log N)) ,
  \]
  \[
P \left( \frac{1}{2} \frac{N/(2i)}{N/1+1} \right) = \sum_{j=I_N \left( \frac{N}{N/1+1} \right)}^{I_N \left( \frac{N/(2i)}{N/1+1} \right)} \left| F_N(j) - \frac{j}{|F_N|} \right| = O(\log(N)\delta_B(\log N)) ,
  \]
with \( 0 < B < A \). The same result holds for \( a_1/b_1 = 1/2, a_2/b_2 = 1/1 \).
- For \( b_1 > 2 \) and \( b_2 < b_1 \):
  \[
P \left( a_1 \frac{i}{b_1} + a_2 \frac{i}{b_1} + b_2 \right) = \sum_{j=I_N \left( \frac{a_1}{b_1} \right)}^{I_N \left( \frac{a_1}{b_1} + a_2 \right)} \left| F_N(j) - \frac{j}{|F_N|} \right| < \frac{a_1}{b_1} - \frac{I_N \left( \frac{a_1}{b_1} \right)}{|F_N|} O(iN) + O(\delta_B(i)) ,
  \]
which cannot be further developed as no general expression for \( I_N (a_1/b_1) \) is known.

Proof. By virtue of Theorem 3.3, the partial Franel sum under study is written as

\[
P \left( \frac{a_1}{b_1} \frac{a_1 N}{b_1} + a_2 \frac{b_1}{b_1} + b_2 \right) = \sum_{i'=1}^{i-1} \sum_{q= \frac{b_1}{b_1(i'+1)}}^{\frac{b_1 N}{a_1}} \sum_{n=1}^{\left\lfloor \frac{b_1 N}{a_1} \right\rfloor} \left| F_{i'}(g) \right| \frac{k(a_1 q + a_2) - a_1 h}{k(b_1 q + b_2) - b_1 h} \frac{I_N \left( \frac{k(a_1 q + a_2) - a_1 h}{k(b_1 q + b_2) - b_1 h} \right)}{|F_N|}
\]

where the sum over \( n \) runs over the elements \( h/k \) in \( F_{i'} \), approximately \( n = I_{i'}(h/k) + O(i') \). By virtue of Theorem 6.1 and Corollary 6.3,

\[
I_N \left( \frac{k(a_1 q + a_2) - a_1 h}{k(b_1 q + b_2) - b_1 h} \right) = I_N \left( \frac{a_1 q + a_2}{b_1 q + b_2} \right) + s I_{i'} \left( \frac{h}{k} \right) + O(i')
\]

\[
= I_N \left( \frac{a_1}{b_1} \right) + s \frac{N^2}{b_1 q} - s \frac{N^2}{b_1 q} \left( \frac{N}{b_1 q} \right)^2 + s \frac{N^2}{b_1 q} \left( \frac{N}{b_1 q} \right)^2 + O(N \delta_A(i'))
\]

where we have used \( i' = \left\lfloor \frac{N}{b_1 q} \right\rfloor \). Furthermore

\[
I_N \left( \frac{k(a_1 q + a_2) - a_1 h}{k(b_1 q + b_2) - b_1 h} \right) = \frac{I_N \left( \frac{a_1}{b_1} \right)}{|F_N|} + s \frac{b_1 h}{b_1 q} - s \frac{N^2}{b_1 q} \left( \frac{N}{b_1 q} \right)^2 + s \frac{N^2}{b_1 q} \left( \frac{N}{b_1 q} \right)^2 + O \left( \frac{\delta_A(i')}{N} \right)
\]

The Farey element inside the partial Franel sum is approximated as

\[
k(a_1 q + a_2) - a_1 h = k(b_1 q + b_2) - b_1 h = \frac{s}{b_1 q} \left( 1 + b_2 q b_1 + \frac{h}{b_1 q} + \frac{a_1}{b_1} \right)
\]

where we have used \((b_1 a_2 - a_1 b_2) = s\). The partial Franel sum under study becomes

\[
\sum_{i'=1}^{i-1} \sum_{q= \frac{b_1}{b_1(i'+1)}}^{\frac{b_1 N}{a_1}} \left| F_{i'} \right| \frac{a_1}{b_1} - \frac{I_N \left( \frac{a_1}{b_1} \right)}{|F_N|} + s \frac{b_1 h}{b_1 q} + s \frac{N^2}{b_1 q} \left( \frac{N}{b_1 q} \right)^2 + O \left( \frac{\delta_A(i')}{N} \right)
\]
where the terms proportional to $1/q$ and $h/k$ have canceled out leaving a negligible residue. The sum over $n$ has been evaluated just by multiplying by $|F'_i|$ as the dependency on $h/k$ disappeared. Evaluating the asymptotes of the sums of the individual terms within the absolute value gives:

$$
\sum_{q=1}^{N} \sum_{q'=1}^{N} \frac{|F'_i|}{q^4} = O\left(\frac{i^4}{N}\right),
$$

$$
\sum_{q=1}^{N} \sum_{q'=1}^{N} \frac{|F'_i|}{q^2} = O\left(\log i\right),
$$

$$
\sum_{q=1}^{N} \sum_{q'=1}^{N} |F'_i| \frac{\delta_A(i')}{N^2} = O\left(\log i\right),
$$

with $0 < B < A$. Keeping the two dominant terms gives

$$
P\left(\frac{a_1}{b_1}, \frac{a_1}{b_1} + \frac{a_2}{b_2}\right) \leq \left| \frac{a_1}{b_1} - \frac{I_N\left(\frac{a}{b_1}\right)}{|F_N|} \right| O(iN) + O(i\delta_B(i)),
$$

which is the searched result for $b_1 > 2$. For $1 \leq b_1 \leq 2$

$$
\frac{a_1}{b_1} = \frac{I_N\left(\frac{a}{b_1}\right)}{|F_N|} = 0
$$

and the theorem is demonstrated.

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