ON THE HEAT EQUATION WITH DRIFT IN $L_{d+1}$

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Abstract. In this paper we deal with the heat equation with drift in $L_{d+1}$. Basically, we prove that, if the free term is in $L_q$ with high enough $q$, then the equation is uniquely solvable in a rather unusual class of functions such that $\partial_t u, D^2u \in L_p$ with $p < d + 1$ and $Du \in L_q$.

1. Introduction and main result

Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, \ldots, x^d)$, $d \geq 2$. Define $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$ and for $R > 0$, $(t, x) \in \mathbb{R}^{d+1}$ introduce

$$B_R(x) = \{y \in \mathbb{R}^d : |y - x| < R\}, \quad B_R = B_R, \quad C_R = [0, R^2) \times B_R,$$

$$C_R(t, x) = C_R + (t, x).$$

Let $b(t, x)$ be Borel $\mathbb{R}^d$-valued function on $\mathbb{R}^{d+1}$ such that for any $R > 0$,

$$\|b\|_{L_{d+1}(C_R(t, x))} \leq \bar{b}_R R,$$  \hfill (1.1)

where $\bar{b}_R$, $R > 0$, is a continuous nondecreasing bounded function.

For $f \in L_q(\mathbb{R}^{d+1})$ vanishing for $t = 1$ we want to investigate the equation

$$\partial_t u + \Delta u + b^i D_i u = f$$  \hfill (1.2)

in the class of functions $u \in \bigcup_{T > 0} W^{1,2}_p((-T, 1) \times \mathbb{R}^d)$ such that $u = 0$ for $t = 1$, where $p < d + 1$, $q$ is large enough, $\partial_t = \frac{\partial}{\partial t}$, $D_i = \frac{\partial}{\partial x^i}$.

A somewhat unusual feature of this problem is that $b^i D_i u \notin L_p((0, 1) \times \mathbb{R}^d)$ for arbitrary $u \in W^{1,2}_p((0, 1) \times \mathbb{R}^d)$ even vanishing for $t = 1$. Therefore, if we solve (1.2) and plug the solution into an equation with different $b$ of the same class, we will generally not obtain a function in $L_q$ even locally.

The author is aware of only three similar occasions for equation with the drift term this time growing linearly in $x$, when the solutions are sought for in usual Hölder or Sobolev spaces without weights. These are found in [1], [2], [3]. As there, the phenomenological explanation of why $b^i D_i u$ can be controlled is that, as a solution, $u$ admits a probabilistic representation which shows that, if in some direction the drift is very big the solution along the drift is almost constant, so that the gradient is almost orthogonal to the drift. This argument does not work if $u$ is just any arbitrary function and it

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shows that $b^{i}D_{i}u$ should not be treated as a perturbation but rather as an
integral part of the operator $L = \partial_{t} + \Delta + b^{i}D_{i}$.

By the way, the existence of solutions of (1.2) answers in the positive the following question: For given $b$ and $f$ as in (1.2), is it possible to find a constant $N$ and $u \in W^{1,2}_{p}(0,1) \times \mathbb{R}^{d}$ such that

$$\|b^{i}D_{i}u - f\|_{L_{p}((0,1) \times \mathbb{R}^{d})} \leq N\|\partial_{t}u, \Delta u\|_{L_{p}((0,1) \times \mathbb{R}^{d})}?$$

Here is our main result. For $T \in (0, \infty)$ set $\mathbb{R}^{d}_{T} = (0, T) \times \mathbb{R}^{d}$. Additionally to (1.1) suppose that

$$\|b\|_{L_{d+1}(\mathbb{R}^{d})} < \infty.$$

**Theorem 1.1.** Let $p \in (1, d+1)$ and

$$q := \frac{p(d+1)}{d+1 - p}.$$

Let $f$ have support in $C_{1}$ and belong to $L_{q}(C_{1})$. Then there exists $\hat{b} = \hat{b}(d, p) > 0$ such that if $\bar{b}_{\infty} \leq \hat{b}$, then equation (1.2) has a unique solution such that

$$\partial_{t}u, \Delta u \in L_{p}(\mathbb{R}^{d}_{1}), \quad Du \in L_{q}(\mathbb{R}^{d}_{1}),$$

and $u(1, \cdot) = 0$. Furthermore, there exist constants $N_{1} = N_{1}(d, p)$ and $N_{2} = N_{1}\|b\|_{L_{d+1}(\mathbb{R}^{d})}$ such that

$$\|\partial_{t}u, D^{2}u\|_{L_{p}(\mathbb{R}^{d}_{1})} \leq N_{2}\|f\|_{L_{q}(\mathbb{R}^{d}_{1})} + N_{1}\|f\|_{L_{p}(\mathbb{R}^{d}_{1})},$$

$$\|Du\|_{L_{q}(\mathbb{R}^{d}_{1})} \leq N_{1}\|f\|_{L_{q}(\mathbb{R}^{d}_{1})}.$$  

**Remark 1.1.** In the future we intend to relax the condition $\bar{b}_{\infty} \leq \hat{b}$ to $\bar{b}_{0+} \leq \hat{b}$ and allow $f$ be any function in $L_{q}(\mathbb{R}^{d}_{1})$.

However, observe that once we have (1.1) the smallness can be always achieved by replacing $b$ with $\lambda b$, where $\lambda$ is sufficiently small.

Also note that (1.1) does not imply higher summability of $b$. For instance, take $\alpha \in (0, d), \beta \in (0, 1)$ such that $\alpha + 2\beta = d+1$ and also take a continuous bounded function $h(\tau), \tau > 0$, with $h(0) = 0$ and consider the function

$$g(t, x) = |t|^{-\beta}|x|^{-\alpha}h(|x|).$$

Observe that

$$\int_{C_{\rho}(t, x)} g(s, y) dy ds = \rho \int_{C_{1}(t', x')} |s|^{-\beta}|y|^{-\alpha}h(\rho|y|) dy ds,$$

where $t' = t/\rho, x' = x/\rho$. It is not hard to see that the last integral is a bounded function of $(\rho, t', x')$ which tends uniformly to zero as $\rho \downarrow 0$. Hence, the function $b = g^{1/(d+1)}$ satisfies (1.1) and even $\bar{b}_{0+} = 0$. Also clearly for any $p > d+1$ one can find $h, \alpha$ and $\beta$ above such that $g^{1/(d+1)} \not\in L_{p,loc}$.

**Remark 1.2.** Theorem 1.1 is about the solvability of the terminal value problem with zero terminal data. The author does not know anything about the situation when the terminal data is nonzero and is as good as one might wish.
2. Auxiliary results

Set \( L_0 = \partial_t + \Delta \). If \( \Gamma \) is a measurable subset of \( \mathbb{R}^{d+1} \) and \( f \) is a function on \( \Gamma \) we denote
\[
\int_{\Gamma} f \, dz = \frac{1}{|\Gamma|} \int_{\Gamma} f \, dz,
\]
where \(|\Gamma|\) is the Lebesgue measure of \( \Gamma \) and \( z \) stands for \((t,x)\).

**Lemma 2.1.** Let \( v \in W^{1,2}_1(C_R) \) and assume that \( L_0 v = 0 \) in \( C_R \). Then, for \( \kappa \leq 1/4 \),
\[
\int_{C_{\kappa R}} \int_{C_{\kappa R}} |Dv(z_1) - Dv(z_2)| \, dz_1 \, dz_2 \leq N\kappa \int_{C_R} |Dv(z)| \, dz.
\]

**Proof.** Since \( Dv \) satisfies the same equation, it suffices to prove that
\[
\int_{C_{\kappa R}} \int_{C_{\kappa R}} |v(z_1) - v(z_2)| \, dz_1 \, dz_2 \leq N\kappa \int_{C_R} |v(z)| \, dz. \tag{2.1}
\]

Self-similar transformations allow us to assume that \( R = 1 \).
We know (see, for instance, theorem 8.8.4 of [4]) that
\[
\int_{C_{\kappa R}} \int_{C_{\kappa R}} |v(z_1) - v(z_2)| \, dz_1 \, dz_2 \leq N\kappa \sup_{C_{2\kappa}} (|\partial_t v + |Dv||) \leq N\kappa \sup_{C_\kappa} |v|,
\]
where the last supremum is easily estimated through
\[
\int_{C_1} |v| \, dz.
\]

The lemma is proved.

Introduce
\[
\pi \in (1,d+2), \quad \mu^* = \frac{\pi(d+2)}{d+2 - \pi},
\]
and observe that, if \( \pi < d + 1 \), \( \mu^* < \pi(d+1)/(d+1 - \pi) =: \mu \), \( \|b^i D_i u\|_{L_\mu} \leq \|b\|_{L_{d+1}} \|Du\|_{L_\mu}, \) whereas by embedding theorems \( \partial_t u, D^2 u \in L_\mu \) only implies that \( Du \in L_{\mu^*} \). This presents the main obstacle on the way of “usual” PDE theory for the operator \( L \) when lower-order terms are treated as perturbations.

Define \( \partial' C_R = C_R \setminus (\{t = 0\} \times B_R) \) and introduce the notation
\[
\|g\|_{L_r(C_R)}^r = \int_{C_R} |g|^r \, dz.
\]

**Lemma 2.2.** Let \( w \in W^{1,2}_\pi(C_R) \) and assume that \( L_0 w = f \) in \( C_R \) and \( w = 0 \) on \( \partial' C_R \). Then
\[
\|Dw\|^r_{L_{\mu^*}(C_R)} \leq N(d,\pi) R \, \|f\|_{L_\pi(C_R)}. \tag{2.2}
\]

**Proof.** Rescaling allows us to assume that \( R = 1 \). In that case the \( W^{1,2}_\pi(C_1) \)-norm of \( w \) is estimate through the \( L_\pi(C_1) \)-norm of \( f \). After that it only remains to use embedding theorems. The lemma is proved.

This result is used below with 1 in place of \( \mu^* \).
Lemma 2.3. Let \( u \in W^{1,2}_\pi(C_R) \). Introduce \( L_0 u = f \). Then, for \( \kappa \leq 1/4 \), with \( N = N(d, \pi) \),

\[
\int_{C_R} \int_{C_R} |Du(z_1) - Du(z_2)| \, dz_1 \, dz_2 \leq N\kappa \int_{C_R} |Du| \, dz \\
+ N\kappa^{-2d-4} R \left( \int_{C_R} |f|^{\pi} \, dz \right)^{1/\pi}. \tag{2.3}
\]

Proof. Introduce \( v \in W^{1,2}_\pi(C_R) \) such that \( L_0 v = 0 \) and \( u = w \) on \( \partial' C_R \) and let \( w = u - v \). Then \( L_0 w = L_0 u = f \) and

\[
\int_{C_R} \int_{C_R} |Dv(z_1) - Dv(z_2)| \, dz_1 \, dz_2 \leq N\kappa \int_{C_R} |Dv| \, dz,
\]

\[
\int_{C_R} |Dv| \, dz \leq \int_{C_R} |Du| \, dz + \int_{C_R} |Dw| \, dz \leq \int_{C_R} |Du| \, dz + NR \left( \int_{C_R} |f|^{\pi} \, dz \right)^{1/\pi},
\]

\[
\int_{C_R} \int_{C_R} |Dw(z_1) - Dw(z_2)| \, dz_1 \, dz_2 \leq N\kappa^{-2d-4} R \left( \int_{C_R} |f|^{\pi} \, dz \right)^{1/\pi}.
\]

This computations imply (2.3) and the lemma is proved.

Theorem 2.4. Let \( \pi \in (1, d+1) \) and \( u \in W^{1,2}_\pi(C_R) \). Introduce \( Lu = f \). Then, for \( \kappa \leq 1/4 \), with \( N = N(d, \pi) \),

\[
\int_{C_R} \int_{C_R} |Du(z_1) - Du(z_2)| \, dz_1 \, dz_2 \leq N\kappa \int_{C_R} |Du| \, dz \\
+ N\kappa^{-2d-4} \left( \int_{C_R} |Du|^\mu \, dz \right)^{1/\mu} + NR\kappa^{-2d-4} \left( \int_{C_R} |f|^{\pi} \, dz \right)^{1/\pi}. \tag{2.4}
\]

This result follows from (2.3) and the fact that by Hölder’s inequality

\[
\left( \int_{C_R} |b|^{\pi} \, |Du|^{\pi} \, dz \right)^{1/\pi} \leq \left( \int_{C_R} |b|^{d+1} \, dz \right)^{1/(d+1)} \left( \int_{C_R} |Du|^\mu \, dz \right)^{1/\mu}.
\]

The last term in (2.4) presents certain inconvenience which forced us to assume that \( f = 0 \) outside \( C_1 \).

Lemma 2.5. Let \( g \geq 0 \) have support in \( C_1 \) and be integrable. Let \( z, z_0 \in \mathbb{R}^{d+1} \), \( \kappa \leq 1/4 \) and let \( z_0 \in C_{\kappa R}(z) \). Then for any \( R > 0 \)

\[
R^\pi \int_{C_R(z)} g \, dx \, dt \leq NMg(z_0) + N(\kappa |z_0| + 1)^{\pi-d-2} \int_{C_1} g \, dx \, dt,
\]

where \( M g \) is the parabolic Hardy-Littlewood maximal function, \( |z_0| = \sqrt{|z_0| + |z_0|} \), and \( N = N(d, \pi) \).
Proof. Introduce \( \hat{\mathcal{C}} := C_2(-1,0) \) which is a cylinder strictly containing \( C_1 \) and consider a few cases.

**Case** \( z_0 \in \hat{\mathcal{C}} \). If \( R \leq 1 \), then by definition

\[
R^p \int_{C_R(z)} g \, dx \, dt \leq Mg(z_0).
\]

However, if \( R > 1 \), then

\[
R^p \int_{C_R(z)} g \, dx \, dt \leq NR^{p-1} \int_{C_1} g \, dx \, dt \leq N \int_{C_1} g \, dx \, dt \leq NMg(z_0).
\]

**Case** \( z_0 \notin \hat{\mathcal{C}}, t_0 \leq -1 \). In this case in order for the intersection of \( C_R(z) \) and \( C_1 \) to be nonempty we have to have \( t_0 + R^2 > 0 \) and \( |x_0| - 2R < 1 \), that is \( R \geq \max(\sqrt{|t_0|}, (1/2)(|x_0| - 1)) \). By taking into account that \( |t_0| \geq 1 \) it is not hard to see that

\[
\max(\sqrt{|t_0|}, (1/2)(|x_0| - 1)) \geq \nu(\sqrt{|t_0|} + |x_0| + 1),
\]

where \( \nu > 0 \) is an absolute constant. In that case

\[
R^p \int_{C_R(z)} g \, dx \, dt \leq NR^{p-1} \int_{C_1} g \, dx \, dt \leq N \frac{1}{(1 + |x|)^{d+2-\pi}} \int_{C_1} g \, dx \, dt.
\]

**Case** \( z_0 \notin \hat{\mathcal{C}}, t_0 \geq 3 \). This time \( C_R(z) \cap C_1 \neq \emptyset \) only if \( 1 + R^2 > t_0 \) and \( |x_0| - 2R < 1 \), that is \( R \geq \max(\sqrt{|t_0| - 1}, (1/2)(|x_0| - 1)) \), which leads to (2.5) again.

**Case** \( z_0 \notin \hat{\mathcal{C}}, t_0 \in [-1,3] \). Here \( |x_0| \geq 2 \) and \( C_R(z) \cap C_1 \neq \emptyset \) only if \( |x_0| - 2R < 1 \), that is \( R \geq (1/2)(|x_0| - 1) \geq (1/8)(|x_0| + 1) \), which leads to (2.5) again. The lemma is proved.

Here is the main a priori estimate. Recall that \( p \in (1, d+1) \) and \( q = p(d + 1)/(d + 1 - p) \).

**Lemma 2.6.** Let \( u \in \bigcup_{T > 0} W^{1,2}_p((-T, 1) \times \mathbb{R}^d) \) and \( Du \in L_q((-\infty, 1) \times \mathbb{R}^d) \).

Assume that \( u(1, \cdot) = 0 \), \( f := Lu \in L_q((-\infty, 1) \times \mathbb{R}^d) \), and \( f \) has support in \( C_1 \). Then there exists a constant \( \hat{b} = \hat{b}(d, p) > 0 \) such that, if \( \hat{b}_\infty \leq \hat{b} \), then

\[
\|Du\|_{L_q((-\infty, 1) \times \mathbb{R}^d)} \leq N\|f\|_{L_q(C_1)},
\]

where \( N = N(d, p) \).

Proof. We extend \( u \) and \( f \) as zero for \( t > 1 \). Let \( \mathcal{C} \) be the collection of \( C_R(t, x), R > 0, (t, x) \in \mathbb{R}^{d+1} \). For functions \( h = h(z) \) on \( \mathbb{R}^{d+1} \) for which it makes sense introduce

\[
h^s(z) = \sup_{C \subset \mathcal{C}} \int_C \int_C |h(z_1) - h(z_2)| \, dz_1 \, dz_2.
\]

Observe that if \( z \in \mathbb{R}^{d+1} \) and \( z \in C \subset \mathcal{C} \), then owing to Theorem 2.4 and Lemma 2.5 with \( \pi = (1 + p)/2 \)

\[
\int_C \int_C |Du(z_1) - Du(z_2)| \, dz_1 \, dz_2 \leq N\kappa M|Du|(z)
\]
To obtain (2.6) now it only remains to choose first small $\kappa$ and then $\hat{b}$ so that $N_1(\kappa + \hat{b} \kappa^{-2d-4}) \leq 1/2$. The lemma is proved.

Proof of uniqueness in Theorem 1.1. Let $f = 0$, our goal is to show that the only solution $u$ with the specified properties is zero. Since $L_0 u = -b^i D_i u \in L_p(\mathbb{R}^d)$, we have that $u \in W_p^{1,2}(\mathbb{R}^d)$.

Now fix a $t_0 > 0$ close to zero, such that $u(t_0, \cdot) \in W_p^2(\mathbb{R}^d)$ and define $v(t, x) = u(t, x)$ for $t \in [0, t_0]$ and let $v$ be a solution given by means of the heat semigroup of the equation $L_0 v = 0$, $t \leq t_0$, with terminal data $v(t_0, \cdot) = u(t_0, \cdot)$. Then the function $w = v$ for $t \leq t_0$ and $w = u$ for $t \in [t_0, 1]$ is of class $\bigcup_{T > 0} W_p^{1,2}((-T, 1) \times \mathbb{R}^d)$ and satisfies $L_0 w + I_{T > t_0} b^i D_i w = 0$ in $(-\infty, 1) \times \mathbb{R}^d$ with zero terminal condition. By using the explicit representation of $v$ for $t \leq t_0$ and the fact that by assumption $Du \in L_q(\mathbb{R}^d)$, one easily shows that $Dw \in L_q((-\infty, 1) \times \mathbb{R}^d)$. But then owing to (2.6), $Dw = 0$ and $L_0 w = 0$ in $(-\infty, 1) \times \mathbb{R}^d$ and $L_0 u = 0$ in $(t_0, 1) \times \mathbb{R}^d$. It follows that $u = 0$ for $t \in [t_0, 1]$ and since $t_0$ can be chosen arbitrarily close to $1$, $u = 0$ in $\mathbb{R}^d$, and the uniqueness of solutions is established.

Now comes the last step needed to prove the existence part in Theorem 1.1.

Lemma 2.7. Let $f \in C_0^\infty(C_1)$, $f = 0$ outside $C_1$, and $b \in C_0^\infty(\mathbb{R}^{d+1})$. Define $u$ as the classical solution of $Lu = f$ for $t \leq 1$ with terminal condition $u(1, \cdot) = 0$. Assume that $\hat{b} \leq \overline{b}$. Then

$$
\|Du\|_{L_q(\mathbb{R}^d)} \leq N \|f\|_{L_q(\mathbb{R}^d)},
$$

where $N = N(d, p)$. Furthermore,

$$
\|\partial_t u, D^2 u\|_{L_p(\mathbb{R}^d)} \leq N_1 \|f\|_{L_q(\mathbb{R}^d)} + N_2 \|f\|_{L_p(\mathbb{R}^d)},
$$

where $N_1 = N(d, p) \|b\|_{L_{d+1}(\mathbb{R}^d)}$, $N_2 = N_2(d, p)$. 

$$
+ N \overline{b} \kappa^{-2d-4} \left( M(|Du|^p) (z) \right)^{1/\mu} + N \kappa^{-2d-4} \left( M(|f|^\pi) (z) \right)^{1/\pi}
$$

where $h(z) = ((|z| + 1)^{1-(d+2)/\pi}$.
Proof. The existence of smooth bounded $u$ is a classical result. For $t < 0$, $u(t, x)$ is just a caloric function and it is represented by means of the fundamental solution of the heat equation. Furthermore, we have $q > (d + 2)/(d + 1)$ so that simple estimates show that $Du \in L_q((-\infty, 1) \times \mathbb{R}^d)$. Now (2.7) follows from Lemma 2.6.

Estimate (2.7) and Hölder’s inequality show that

$$\|b^i D_i u\|_{L_p(\mathbb{R}^d)} \leq \|b\|_{L^{d+1}(\mathbb{R}^d)} \|Du\|_{L_q(\mathbb{R}^d)};$$

which implies that $f - b^i D_i u \in L_p(\mathbb{R}^d)$, so that (2.8) is a classical result. The lemma is proved.

Proof of Theorem 1.1. The uniqueness part is taken care of above. To prove the existence, take $f_n \in C^\infty_0(C_1)$ converging to $f \in L_q(C_1)$ and $b_n \in C^\infty_0(\mathbb{R}^{d+1})$ converging to $b$ in $L_{d+1}(\mathbb{R}^d)$ and having $\bar{b} R$ the same for all $n$ (just use mollifiers and cut-off’s). Then by Lemma 2.7 we have solutions $u_n$ of $L_0 u_n + b_n^i D_i u_n = f_n$ admitting estimates (2.7) and (2.8) with $u_n$ and $f_n$ in place of $u$ and $f$ and with the constants independent of $n$. Now to prove the theorem it only remains to check that, if $Du_n \to Du$ weakly in $L_q(\mathbb{R}^d)$, then

$$b^i D_i u_n \to b^i D_i u$$

weakly in $L_p(\mathbb{R}^d)$. As we have seen a few times the sequence $b^i D_i u_n$ is bounded in $L_p(\mathbb{R}^d)$, so we need

$$\int_{\mathbb{R}^d} \phi b^i D_i u_n \, dz \to \int_{\mathbb{R}^d} \phi b^i D_i u \, dz$$

for any $\phi \in L_{p/(p-1)}(\mathbb{R}^d)$. The latter holds indeed, since by Hölder’s inequality $\phi b \in L_{q/(q-1)}(\mathbb{R}^d)$. The theorem is proved.

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