New differential operator and non-collapsed RCD spaces

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Dedicated to Professor Kenji Fukaya on the occasion of his sixtieth birthday.

Abstract

We show characterizations of non-collapsed compact RCD$(K,N)$ spaces, which in particular confirm a conjecture of De Philippis-Gigli on the implication from the weakly non-collapsed condition to the non-collapsed one in the compact case. The key idea is to give the explicit formula of the Laplacian associated to the pull-back Riemannian metric by embedding in $L^2$ via the heat kernel.

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1 Introduction

1.1 Main results

De Philippis-Gigli introduced in [DePhG18] two special classes of RCD$(K,N)$ spaces. One of them is the notion of weakly non-collapsed spaces and the other one is that of non-collapsed spaces. Our main result states that these are essentially same in the compact case.

After the fundamental works of Lott-Villani [LV09] and Sturm [St06], Ambrosio-Gigli-Savaré [AGS14b] (when $N = \infty$) and Gigli [G13] (when $N < \infty$) introduce the notion of RCD$(K,N)$ spaces for metric measure spaces $(X,d,m)$, which means a synthetic notion

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of “Ric ≥ K and dim ≤ N with Riemannian structure”. Typical examples are measured Gromov-Hausdorff limit spaces of Riemannian manifolds with Ricci bounds from below and dimension bounds from above, so-called Ricci limit spaces. The RCD theory gives the striking framework to treat Ricci limit spaces by a synthetic way.

Cheeger-Colding established the fundamental structure theory of Ricci limit spaces [CC97, CC00a, CC00b]. Thanks to recent quick developments on the study of RCD(K, N) spaces, most of parts of the theory of Ricci limit spaces, including Colding-Naber’s result [CN12], are covered by the RCD theory (see for instance [BS18] by Bruè-Semola). In particular whenever N < ∞, the essential dimension, denoted by dim_{d,m}(X), of any RCD(K, N) space (X,d,m) makes sense (c.f. Theorem 2.4).

On the other hand in a special class of Ricci limit space, so-called non-collapsed Ricci limit spaces, finer properties are obtained by Cheeger-Colding. For instance, the Bishop inequality with the rigidity and the almost Reifenberg flatness are justified in this setting. These are not covered by general Ricci limits/RCD theories.

The properties of non-collapsed RCD(K, N) spaces introduced in [DePhG18] cover most of finer results on non-collapsed Ricci limit spaces as explained above. It is worth pointing out that any convex body is not a non-collapsed Ricci limit space, but it is a non-collapsed RCD(K, N) space.

Let us give the definitions;

- a RCD(K, N) space (X, d, m) is non-collapsed if m = H^{N}, where H^{N} denotes the N-dimensional Hausdorff measure;

- a RCD(K, N) space (X, d, m) is weakly non-collapsed if m ≪ H^{N}.

The second definition is equivalent to that dim_{d,m}(X) = N, which is proved in [DePhG18]. Note that some structure results on weakly non-collapsed RCD(K, N) spaces are obtained in [DePhG18] and that Kitabeppu [K17] provides a similar notion (which is a priori stronger than the weakly non-collapsed condition, but is a priori weaker than the non-collapsed one) and prove similar structure results.

De Philippis-Gigli conjectured that these notions are essentially same. More precisely;

**Conjecture 1.1.** If (X, d, m) is a weakly non-collapsed RCD(K, N) space, then m = aH^{N} for some a ∈ (0, ∞).

Moreover they conjectured;

**Conjecture 1.2.** If (X, d, m) is a RCD(K, N) space and satisfies that H^{2,2}(X, d, m) ⊂ D(Δ) holds and that tr(Hess_{f}) = Δf holds in L^{2}(X, m) for all f ∈ H^{2,2}(X, d, m), then there exists n ∈ [1, N] ∩ N such that (X, d, m) is a weakly non-collapsed RCD(K, n) space.

Recall that H^{2,2}(X, d, m) is defined by the closure of the space TestF(X, d, m) with respect to the norm ||f||^{2}_{H^{2,2}} = ||f||^{2}_{H^{1,2}} + ||Hess_{f}||^{2}_{L^{2}}. See [G18] (or subsection 2.1) for the precise definition of test functions and the hessians.

For these conjectures the only known development is due to Kapovitch-Ketterer [KK19]. They proved that Conjecture 1.1 is true under assuming bounded sectional curvature from above in the sense of Alexandrov (that is, the metric structure is CAT).

We are now in a position to introduce a main result of the paper;

**Theorem 1.3** (Characterization of non-collapsed RCD spaces). Let (X, d, m) be a compact RCD(K, N) space with n := dim_{d,m}(X). Then the following two conditions (1), (2) are equivalent;
1. The following two conditions hold;

(a) For all eigenfunction $f$ on $X$ of $-\Delta$ we have

$$\Delta f = \text{tr}(\text{Hess}_f) \quad \text{in } L^2(X, \mathfrak{m}).$$

(1.1)

(b) There exists $C > 0$ such that

$$\mathfrak{m}(B_r(x)) \geq Cr^n \quad \forall x \in X, \forall r \in (0, 1).$$

(1.2)

2. $(X, d, \mathfrak{m})$ is a RCD$(K, n)$ space with

$$\mathfrak{m} = \frac{\mathfrak{m}(X)}{\mathcal{H}^n(X)} \mathcal{H}^n.$$ (1.3)

Note that all eigenfunctions of $-\Delta$ are in Test$F(X, d, \mathfrak{m})$, in particular they are in $H^2_{\text{loc}}(X, d, \mathfrak{m})$.

It is easy to understand that this theorem gives contributions to both Conjectures 1.1 and 1.2. In particular combining a result of Han [Han18] with the Bishop-Gromov inequality yields that all compact weakly non-collapsed RCD$(K, N)$ spaces satisfy (1) in the theorem as $n = N$. Therefore;

**Corollary 1.4.** Conjecture 1.1 is true in the compact case.

Moreover for Conjecture 1.2, Theorem 1.3 tells us that the remaining problem is only to prove (1.2) under the assumptions.

We will also establish other characterization of non-collapsed RCD spaces. See subsection 4.2. Next let us explain how to achieve these results. Roughly speaking it is to take canonical deformations $g_t$ of the Riemannian metric $g$ via the heat kernel.

### 1.2 Key idea

In order to prove main results the key idea is to use the pull-back Riemannian metrics $g_t := \Phi_t^* g_{L^2}$ by embeddings $\Phi_t : X \to L^2(X, \mathfrak{m})$ via the heat kernel $p$ instead of using the original Riemannian metric $g$ of $(X, d, \mathfrak{m})$. The definition of $\Phi_t$ is:

$$\Phi_t(x)(y) := p(x, y, t).$$ (1.4)

This map is introduced and studied by Bérard-Besson-Gallot [BBG94] for closed manifolds. A their main result states that for closed manifolds $(M^n, g)$, as $t \to 0^+$

$$\omega_n^t(n+2)/2 g_t = c_n g - c_n \left( \text{Ric}_g - \frac{1}{2} \text{Scal}_g \right) t + O(t^2),$$ (1.5)

where Ric$_g$ and Scal$_g$ denote the Ricci and the scalar curvatures respectively, and

$$\omega_n := \mathcal{L}^n(B_1(0_n)), \quad c_n := \frac{\omega_n}{(4\pi)^n} \int_{\mathbb{R}^n} |\partial_x (e^{-|x|^2/4})|^2 d\mathcal{L}^n(x).$$ (1.6)

Recently the map $\Phi_t$ is also studied for compact RCD$(K, N)$ spaces by Ambrosio-Portegies-Tewodrose and the author [AHTP18]. In particular $g_t$ is also well-defined in this setting (c.f. Theorem 2.8).

Let us introduce the following new differential operator;

$$\Delta^t f := \langle \text{Hess}_f, g_t \rangle + \frac{1}{4} \langle \nabla_x \Delta_x p(x, x, 2t), \nabla f \rangle.$$ (1.7)
This plays a role of the Laplacian associated to \( g_t \), in fact, we will prove:

\[
\int_X \langle g_t, d\psi \otimes df \rangle \, dm = - \int_X \psi \Delta^t f \, dm, \tag{1.8}
\]

which is new even for closed manifolds. See Theorem 3.4 for the precise statement. Then after normalization, taking the limit \( t \to 0^+ \) in (1.8) with convergence results given in [AHTP18] allows us to prove desired results.

The paper is organized as follows:

In Section 2 we give a quick introduction on RCD spaces and prove technical results. In Section 3 we establish (1.8). In the final section, Section 4, we prove main results stated in subsection 1.1 and related results.

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## 2 RCD(\( K, N \)) spaces

A triple \((X, d, m)\) is a *metric measure space* if \((X, d)\) is a complete separable metric space and \(m\) is a Borel measure on \(X\) with \(\text{supp} \, m = X\).

### 2.1 Definition

Throughout this paper the parameters \( K \in \mathbb{R} \) (lower bound on Ricci curvature) and \( N \in [1, \infty) \) (upper bound on dimension) will be kept fixed. Instead of giving the original definition of RCD(\( K, N \)) spaces, we introduce an equivalent version for short. See [EKS15], [AMS15], [CM16] and [AGS14a] for the proof of the equivalence and the details.

Let \((X, d, m)\) be a metric measure space. The Cheeger energy \(\text{Ch} : L^2(X, m) \to [0, +\infty]\) is a convex and \(L^2(X, m)\)-lower semicontinuous functional defined as follows:

\[
\text{Ch}(f) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int_X (\text{Lip}_n f)^2 \, dm : \ f_n \in \text{Lip}_n(X, d) \cap L^2(X, m), \ |f_n - f|_{L^2} \to 0 \right\}. \tag{2.1}
\]

where \(\text{Lip}_n(X, d)\) denotes the space of all bounded Lipschitz functions and \(\text{Lip} f\) is the local Lipschitz constant.

The Sobolev space \(H^{1,2}(X, d, m)\) then coincides with \(\{ f \in L^2(X, m) : \text{Ch}(f) < +\infty\}\). When endowed with the norm \(\| f \|_{H^{1,2}} := \| f \|_{L^2(X, m)}^2 + 2\text{Ch}(f) \}^{1/2}\), this space is Banach, reflexive if \((X, d)\) is doubling (see [ACDM15]), and separable Hilbert if \(\text{Ch}\) is a quadratic form (see [AGS14b]). According to the terminology introduced in [G15a], we say that \((X, d, m)\) is infinitesimally Hilbertian if \(\text{Ch}\) is a quadratic form.

Let us assume that \((X, d, m)\) is infinitesimally Hilbertian. Then for all \(f_1, f_2 \in H^{1,2}(X, d, m)\),

\[
\langle \nabla f_1, \nabla f_2 \rangle := \lim_{\epsilon \to 0} \frac{|\nabla (f_1 + \epsilon f_2)|^2 - |\nabla f_1|^2}{2\epsilon} \in L^1(X, m) \tag{2.2}
\]

is well-defined, where \(|\nabla f| \in L^2(X, m)\) denotes the minimal relaxed slope of \(f \in H^{1,2}(X, d, m)\).

We can now define a densely defined operator \(\Delta : D(\Delta) \to L^2(X, m)\) whose domain consists of all functions \(f \in H^{1,2}(X, d, m)\) satisfying

\[
\int_X \psi \varphi \, dm = - \int_X \langle \nabla f, \nabla \varphi \rangle \, dm \quad \forall \varphi \in H^{1,2}(X, d, m)
\]

for some \(\psi \in L^2(X, m)\). The unique \(\psi\) with this property is then denoted by \(\Delta f\).

We are now in a position to introduce the RCD space:
Definition 2.1 (RCD spaces). Let $(X, d, m)$ be a metric measure space, let $K \in \mathbb{R}$ and let $\hat{N} \in [1, \infty]$. We say that $(X, d, m)$ is a RCD$(K, \hat{N})$ space if the following conditions hold:

1. (Volume growth) there exist $x \in X$ and $C > 1$ such that $m(B_r(x)) \leq C e^{Cr^2}$ for all $r \in (0, \infty)$;

2. (Bochner’s inequality) for all $f \in D(\Delta)$ with $\Delta f \in H^{1,2}(X, d, m)$,

$$\frac{1}{2} \int_X |\nabla f|^2 \Delta \varphi dm \geq \int_X \varphi \left( \frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle + K |\nabla f|^2 \right) dm \quad (2.3)$$

for all $\varphi \in D(\Sigma) \cap L^\infty(X, m)$ with $\varphi \geq 0$ and $\Delta \varphi \in L^\infty(X, m)$;

3. (Sobolev-to-Lipschitz property) any $f \in H^{1,2}(X, d, m)$ with $|\nabla f| \leq 1$ m-a.e. in $X$ has a 1-Lipschitz representative.

Let us denote the heat flow associated to the Cheeger energy by $h_t$. It holds (without curvature assumption) that

$$\|h_t f\|_{L^2} \leq \|f\|_{L^2}, \quad |||\nabla f|||_{L^2} \leq \frac{\|f\|_{L^2}^2}{2t^2}, \quad |||\Delta h_t f|||_{L^2} \leq \frac{\|f\|_{L^2}}{t}. \quad (2.4)$$

Then one of the crucial properties of the heat flow on RCD$(K, \infty)$ spaces is;

$$h_t f \in \text{Test}(\Sigma, d, m), \quad \forall f \in L^2(X, d, m) \cap L^\infty(X, m), \quad \forall t \in (0, \infty), \quad (2.5)$$

where

$$\text{Test}(\Sigma, d, m) := \left\{ f \in \text{Lip}_b(X, d) \cap H^{1,2}(\Sigma, d, m) : \Delta f \in H^{1,2}(\Sigma, d, m) \right\}. \quad (2.6)$$

See for instance [G18] for the crucial role of test functions in the study of RCD spaces. Finally we end this subsection by giving the following elementary lemma;

Lemma 2.2. Let $(X, d, m)$ be a RCD$(K, \infty)$ space and let $f \in D(\Delta)$. Then there exists a sequence $f_i \in \text{Test}(\Sigma, d, m)$ such that $\|f_i - f\|_{H^{1,2}} + |||\Delta f_i - \Delta f|||_{L^2} \to 0$ holds.

Proof. Let $F_L := (-L) \vee f \wedge L$. Note that $h_t F_L \in \text{Test}(\Sigma, d, m)$, that $h_t F_L \rightharpoonup h_t f$ in $H^{1,2}(\Sigma, d, m)$ as $L \to \infty$ for all $t > 0$, and that $\Delta h_t F_L$ $L^2$-weakly converges to $\Delta h_t f$ as $L \to \infty$ for all $t > 0$, where we used (2.4) (c.f. [AH17]). Since $\Delta h_t f \rightharpoonup \Delta f$ in $L^2(X, m)$ as $t \to 0^+$, there exist $L_i \to \infty$ and $t_i \to 0^+$ such that $h_{t_i} F_{L_i} \rightharpoonup f$ in $H^{1,2}(\Sigma, d, m)$ and that $\Delta h_{t_i} F_{L_i}$ $L^2$-weakly converge to $\Delta f$. Then applying Mazur’s lemma for the sequence $\{\Delta h_{t_i} F_{L_i}\}$, yields that for all $m \geq 1$ there exist $N_m \in \mathbb{N}_{\geq m}$ and $\{l_{m,i}\}_{m \leq i \leq N_m} \subset [0, 1]$ such that $\sum_{i=m}^{N_m} l_{m,i} = 1$ and that $\sum_{i=m}^{N_m} l_{m,i} \Delta h_{t_i} F_{L_i} \rightharpoonup \Delta f$ in $L^2(X, m)$. It is easy to check that $f_m := \sum_{i=m}^{N_m} l_{m,i} h_{t_i} F_{L_i}$ satisfies the desired claim. 

2.2 Heat kernel

It is well-known that the Bishop-Gromov theorem holds for any RCD$(K, N)$ space $(X, d, m)$ (or more generally for CD$^*(K, N)$ spaces) and that the local Poincaré inequality holds for RCD$(K, \infty)$ spaces (or more generally for CD$(K, \infty)$ spaces). See [Vi09], [VR09] and [Raj12]. Furthermore, it follows from the Sobolev-to-Lipschitz property that, on any RCD$(K, N)$ space $(X, d, m)$, the intrinsic distance

$$d_{Ch}(x, y) := \sup \{ |f(x) - f(y)| : f \in H^{1,2}(X, d, m) \cap C_b(X, d), |\nabla f| \leq 1 \}$$
associated to the Cheeger energy $\text{Ch}$ coincides with the original distance $d$. Consequently, applying [St95] and [St96] on the general theory of Dirichlet forms provide the existence of a locally Hölder continuous representative $p$ on $X \times X \times (0, \infty)$ for the heat kernel of $(X, d, m)$. The sharp Gaussian estimates on this heat kernel have been proved later on in the RCD context [JLZ16]: for any $\epsilon > 0$, there exist $C_i := C_i(\epsilon, K, N) > 1$ for $i = 1, 2$, depending only on $K$, $N$ and $\epsilon$, such that

$$\frac{C_i^{-1}}{m(B_{\sqrt{t}}(x))} \exp\left(-\frac{d^2(x, y)}{4-\epsilon}t - C_2t\right) \leq p(x, y, t) \leq \frac{C_i}{m(B_{\sqrt{t}}(x))} \exp\left(-\frac{d^2(x, y)}{4+\epsilon}t + C_2t\right)$$

(2.7)

for all $x, y \in X$ and any $t > 0$, where from now on we state our inequalities with the Hölder continuous representative. Combining (2.7) with the Li-Yau inequality [GM14, J15], we have a gradient estimate [JLZ16]:

$$|\nabla_x p(x, y, t)| \leq \frac{C_3}{\sqrt{tm(B_{\sqrt{t}}(x))}} \exp\left(-\frac{d^2(x, y)}{4+\epsilon}t + C_4t\right)$$

for $m$-a.e. $x \in X$ (2.8)

for any $t > 0$, $y \in X$, where $C_i := C_i(\epsilon, K, N) > 1$ for $i = 3, 4$, Note that in this paper, we will always work with (2.7) and (2.8) in the case $\epsilon = 1$.

Let us assume that $\text{diam}(X, d) < \infty$, thus $(X, d)$ is compact (because in general $(X, d)$ is proper). Then the doubling condition and a local Poincaré inequality on $(X, d, m)$ yields that the canonical embedding map $\text{H}^{1,2}(X, d, m) \hookrightarrow L^2(X, m)$ is a compact operator [HK00]. In particular (minus) the Laplacian operator $-\Delta$ admits a discrete positive spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \rightarrow +\infty$. We denote the corresponding eigenfunctions by $\varphi_0, \varphi_1, \ldots$ with $\|\varphi_i\|_{L^2} = 1$. This provides the following expansions for the heat kernel $p$:

$$p(x, y, t) = \sum_{i \geq 0} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) \quad \text{in } C(X \times X)$$

(2.9)

for any $t > 0$ and

$$p(\cdot, y, t) = \sum_{i \geq 0} e^{-\lambda_i t} \varphi_i(y) \varphi_i \quad \text{in } H^{1,2}(X, d, m)$$

(2.10)

for any $y \in X$ and $t > 0$ with the Hölder representative of all eigenfunctions. Combining (2.9) and (2.10) with (2.8), we know that $\varphi_i$ is Lipschitz, in fact, it holds that

$$\|\varphi_i\|_{L^\infty} \leq C_5 \lambda_i^{N/4}, \quad \|\nabla \varphi_i\|_{L^\infty} \leq C_5 \lambda_i^{(N+2)/4}, \quad \lambda_i \geq C_5^{-1/2/4},$$

(2.11)

where $C_5 := C_5(\text{diam}(X, d), K, N) > 0$. See for instance appendices in [AHTP18] and [Hon18] for the proofs.

Finally let us remark that it follows from these observation (with (2.19)) that

$$\Delta_x p(x, y, t) = 2 \sum_{i \geq 0} e^{-\lambda_i t} \left(-\lambda_i(\varphi_i(x))^2 + |\nabla \varphi_i|^2(x)\right)$$

in $H^{1,2}(X, d, m)$. (2.12)

### 2.3 Infinitesimal structure

Let $(X, d, m)$ be a RCD$(K, N)$ space.

**Definition 2.3** (Regular set $\mathcal{R}_k$). For any $k \geq 1$, we denote by $\mathcal{R}_k$ the $k$-dimensional regular set of $(X, d, m)$, namely the set of points $x \in X$ such that $(X, r^{-1}d, m(B_r(x))^{-1}m, x)$ pointed measured Gromov-Hausdorff converge to $(\mathbb{R}^k, d_{\mathbb{R}^k}, \omega_k^{-1}L^k, 0_k)$ as $r \to 0^+$. 

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We are now in a position to introduce the latest structural result for \( \text{RCD}(K,N) \) spaces.

**Theorem 2.4** (Essential dimension of \( \text{RCD}(K,N) \) spaces). Let \((X,d,m)\) be a \( \text{RCD}(K,N) \) space. Then, there exists a unique integer \( n \in [1,N] \), denoted by \( \dim_{d,m}(X) \), such that

\[
m(X \setminus \mathcal{R}_n) = 0.
\] (2.13)

In addition, the set \( \mathcal{R}_n \) is \((m,n)\)-rectifiable and \( m \) is representable as \( \theta \mathcal{H}^n \mathbf{L} \mathcal{R}_n \).

Note that the rectifiability of all sets \( \mathcal{R}_k \) was inspired by [CC97, CC00a, CC00b] and proved in [MN19], together with the concentration property \( m(X \setminus \bigcup_k \mathcal{R}_k) = 0 \), with the crucial uses of [GMR15] and of [G13]; the absolute continuity of \( m \) on regular sets with respect to the corresponding Hausdorff measure was proved afterwards and is a consequence of [KM16], [DePhMR17] and [GP16]. Finally, in the very recent work [BS18] it is proved that only one set \( \mathcal{R}_n \) has positive \( m \)-measure, leading to (2.13) and to the representation \( m = \theta \mathcal{H}^n \mathbf{L} \mathcal{R}_n \). Recall that our main target of the paper is \( \theta \).

By slightly refining the definition of \( n \)-regular set, passing to a reduced set \( \mathcal{R}_n^* \), general results of measure differentiation provide also the converse absolutely continuity property \( \mathcal{H}^n \ll \mathcal{M} \) on \( \mathcal{R}_n^* \). We summarize here the results obtained in this direction in [AHT18]:

**Theorem 2.5** (Weak Ahlfors regularity). Let \((X,d,m)\) be a \( \text{RCD}(K,N) \)-space, \( n = \dim_{d,m}(X) \), \( m = \theta \mathcal{H}^n \mathbf{L} \mathcal{R}_n \) and set

\[
\mathcal{R}_n^* := \left\{ x \in \mathcal{R}_n : \exists \lim_{r \to 0^+} \frac{m(B_r(x))}{\omega_n r^n} \in (0,\infty) \right\}.
\] (2.14)

Then \( m(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0 \), \( \mathbf{L} \mathcal{R}_n^* \) and \( \mathcal{H}^n \mathbf{L} \mathcal{R}_n^* \) are mutually absolutely continuous and

\[
\lim_{r \to 0^+} \frac{m(B_r(x))}{\omega_n r^n} = \theta(x) \quad \text{for } m\text{-a.e. } x \in \mathcal{R}_n^*;
\] (2.15)

\[
\lim_{r \to 0^+} \frac{\omega_n r^n}{m(B_r(x))} = 1_{\mathcal{R}_n^*}(x) \frac{1}{\theta(x)} \quad \text{for } m\text{-a.e. } x \in X.
\] (2.16)

Moreover \( \mathcal{H}^n(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0 \) if \( n = N \).

### 2.4 Second order differential structure and Riemannian metric

Let \((X,d,m)\) be a \( \text{RCD}(K,\infty) \) space.

Inspired by [W00], the theory of the second order differential structure on \((X,d,m)\) based on \( L^2 \)-normed modules is established in [G18]. To keep short presentations in the paper, we omit several notions, for instance, the spaces of \( L^2 \)-vector fields denoted by \( L^2(T(X,d,m)) \) and of \( L^2 \)-tensor fields of type \((0,2)\) denoted by \( L^2((T^*)^\otimes 2(X,d,m)) \). See [G18] for the detail. We denote the pointwise Hilbert-Schmit norm and the pointwise scalar product by \( |T|_{HS} \) and \( \langle T,S \rangle \) respectively.

One of the important results in [G18] we will use later is that for all \( f \in D(\Delta) \), the Hessian \( \text{Hess}_f \in L^2((T^*)^\otimes 2(X,d,m)) \) is well-defined and satisfies

\[
\langle \text{Hess}_f, df_1 \otimes df_2 \rangle = \frac{1}{2} \left( \langle \nabla f_1, \nabla (\nabla f_1, \nabla f_2) \rangle + \langle \nabla f_2, \nabla (\nabla f_2, \nabla f_1) \rangle - \langle \nabla f_1, \nabla (\nabla f_1, \nabla f_2) \rangle \right)
\]

for \( m \text{-a.e. } x \in X, \quad \forall f_i \in \text{Test} F(X,d,m) \) (2.17)
and the Bochner inequality with the Hessian term;

$$\frac{1}{2} \Delta |\nabla f|^2 \geq |\text{Hess} f|_{H^2}^2 + \langle \nabla \Delta f, \nabla f \rangle + K|\nabla f|^2$$  \hspace{1cm} (2.18)

in the weak sense. In particular

$$\int_X |\text{Hess} f|_{H^2}^2 \, dm \leq \int_X \left( (\Delta f)^2 - K|\nabla f|^2 \right) \, dm.$$ \hspace{1cm} (2.19)

Let us introduce the notion of Riemannian metrics on \((X,d,m)\). In order to simplify our argument we assume that \((X,d,m)\) is a RCD\((K,N)\) space with \(n = \text{dim}_{d,m}(X)\) and \(\text{diam}(X,d) < \infty\) below. Although we defined the notion as a bilinear form on \(L^2(T(X,d,m))\) in [AHTP18], we adopt an equivalent formulation by using tensor fields in this paper. Moreover we consider only \(L^2\)-ones, which is enough for our purposes.

**Definition 2.6** (\(L^2\)-Riemannian metric). We say that \(T \in L^2((T^*)^{\otimes 2}(T,d,m))\) is a Riemannian metric if for all \(\eta_l \in L^\infty(T^*(X,d,m))\) (which means that \(\eta_l \in L^2(T^*(X,d,m))\) with \(|\eta_l| \in L^\infty(X,m)\)), it holds that

$$\langle T, \eta_l \otimes \eta_l \rangle = \langle T, \eta_2 \otimes \eta_1 \rangle, \quad \langle T, \eta_1 \otimes \eta_l \rangle \geq 0 \quad \text{for m-a.e. } x \in X \hspace{1cm} (2.20)$$

and that if \(\langle T, \eta_l \otimes \eta_l \rangle = 0\) for m-a.e. \(x \in X\), then \(\eta_l = 0\) in \(L^2(T^*(X,d,m))\).

We are now in a position to introduce the original Riemannian metric of \((X,d,m)\);

**Proposition 2.7** (The canonical metric \(g\)). There exists a unique Riemannian metric \(g \in L^2((T^*)^{\otimes 2}(X,d,m))\) such that

$$\langle g, df_1 \otimes df_2 \rangle = \langle \nabla f_1, \nabla f_2 \rangle \quad \text{for m-a.e. on } X$$

for all Lipschitz functions \(f_i\) on \(X\). Then it holds that

$$|g|_{H^2} = \sqrt{n} \quad \text{for m-a.e. } x \in X.$$ \hspace{1cm} (2.21)

Note that for \(T \in L^2((T^*)^{\otimes 2}(X,d,m))\), the trace \(\text{tr}(T) \in L^2(X,m)\) is \(\text{tr}(T) := \langle T, g \rangle\).

Let us introduce the pull-back Riemannian metrics by embeddings via the heat kernel;

**Theorem 2.8** (The pull-back metrics). Let \((X,d,m)\) be a compact RCD\((K,N)\) space. For all \(t > 0\) there exists a unique Riemannian metric \(g_t \in L^2((T^*)^{\otimes 2}(X,d,m))\) such that

$$\int_X \langle g_t, \eta_l \otimes \eta_l \rangle \, dm = \int_X \int_X \langle dxp(x,y,t), \eta_l(x) \rangle \langle dxp(x,y,t), \eta_l(x) \rangle \, dm(x) \, dm(y),$$

$$\forall \eta_l \in L^\infty(T^*(X,d,m)).$$ \hspace{1cm} (2.22)

Moreover it is representable as the HS-convergent series;

$$g_t = \sum_{i=1}^{\infty} e^{-2\lambda_i t} \, d\varphi_i \otimes d\varphi_i \quad \text{in } L^2((T^*)^{\otimes 2}(X,d,m)).$$ \hspace{1cm} (2.23)

Finally the rescaled metric \(tm(B_{\sqrt{t}}(\cdot))g_t\) satisfies

$$tm(B_{\sqrt{t}}(\cdot))g_t \leq C(K,N)g \quad \forall t \in (0,1),$$ \hspace{1cm} (2.24)

which means that for all \(\eta \in L^\infty(T^*(X,d,m))\)

$$tm(B_{\sqrt{t}}(x))\langle g_t, \eta \otimes \eta \rangle(x) \leq C(K,N)|\eta|_{H^2}^2(x) \quad \text{for m-a.e. } x \in X.$$
Note that since
\[
\text{Test}(T^*)^\otimes 2(X, d, m) := \left\{ \sum_{i=1}^{k} f_{i,i} d_{f_{2,i}} \otimes d_{f_{3,i}}; k \in \mathbb{N}, f_{j,i} \in \text{Test} F(X, d, m) \right\}
\]
is dense in $L^2((T^*)^\otimes 2(X, d, m))$ ([G18]), it is easily checked that
\[
\int_X \langle g_t, T \rangle \, dm = \int_X \int_X \langle d_x p \otimes d_x p, T \rangle \, dm(x) \, dm(y), \quad \forall T \in L^2((T^*)^\otimes 2(X, d, m)). \quad (2.25)
\]
A main convergence result proved in [AHTP18] is the following;

**Theorem 2.9** ($L^p$-convergence to the original metric). We have
\[
\left| t \text{m}(B_{\sqrt{t}(\cdot)} g_t - c_n g) \right|_{HS} \to 0 \quad \text{in } L^p(X, m)
\]
for all $p \in [1, \infty)$, where we recall (1.6) for the definition of $c_n$.

See [AHTP18] for their proofs of the results above. It is worth pointing out that in general we can not improve this $L^p$-convergence to the $L^\infty$-one (see [AHTP18]).

We end this subsection by giving the following technical lemma.

**Lemma 2.10.** Let $(X, d, m)$ be a compact RCD$(K, N)$ space with $n := \dim_{d, m}(X)$. Assume that there exists $C > 0$ such that
\[
\text{m}(B_r(x)) \geq Cr^n \quad \forall x \in X, \forall r \in (0, 1).
\]
Then as $t \to 0^+$ we see that
\[
t^{(n+2)/2} p(x, x, t) \to 0 \quad \text{in } H^{1,2}(X, d, m).
\]
and that
\[
\left| \omega_{nt} t^{(n+2)/2} g_t - c_n \frac{dH^n}{dm} \right|_{HS} \to 0 \quad \text{in } L^p(X, m), \quad \forall p \in [1, \infty).
\]

**Proof.** By (2.7) we see that for all $x \in X$ and all $t \in (0, 1)$:
\[
t^{(n+2)/2} p(x, x, t) \leq \frac{t}{C} \text{m}(B_{\sqrt{t}(x)})p(x, x, t) \leq \frac{C(K, N)}{C} t.
\]
In particular $t^{(n+2)/2} p(x, x, t) \to 0$ in $C(X)$. On the other hand (2.12) and (2.23) yield
\[
\int_X |\Delta_x p(x, x, t)| \, dm(x) \leq \sum_{i} 4\lambda_i e^{-\lambda_i t} = 4 \int_X \langle g, g_{t/2} \rangle \, dm \leq 4\sqrt{n} \int_X |g_{t/2}|_{HS} \, dm. \quad (2.31)
\]
In particular (2.30) and (2.31) yield
\[
\int_X |\nabla_x (t^{(n+2)/2} p(x, x, t))|^2 \, dm(x)
\]
\[
= -\int_X t^{n+2} p(x, x, t) \Delta_x p(x, x, t) \, dm(x)
\]
\[
\leq \frac{4C(K, N)\sqrt{n}}{C} t \int_X |t^{(n+2)/2} g_{t/2}|_{HS} \, dm(x)
\]
\[
\leq \frac{4C(K, N)\sqrt{n}}{C^2} 2^{(n+2)/2} \int_X |(t/2)\text{m}(B_{\sqrt{t/2}(x)})g_{t/2}|_{HS} \, dm(x) \to 0 \quad (2.32)
\]
as $t \to 0^+$, where we used (2.24). Thus we get (2.28).

Next let us prove (2.29). First let us remark that (2.27) yields $H^n \ll m$. Combining this with Theorem 2.5 shows that $\frac{dH^n}{dm} \in L^\infty(X, m)$ and that as $r \to 0^+$,

$$\frac{\omega_n r^n}{m(B_r(x))} \to \frac{dH^n}{dm}(x) \quad \text{for m-a.e. } x \in X. \tag{2.33}$$

Then since as $t \to 0$

$$\int_X \left| \frac{\omega_n t^{n+2}/2}{m(B_{\sqrt{t}}(x))} g_t - \frac{dH^n}{dm}(x)g_t \right|_{HS}^p \, dm$$

$$= \int_X \left| \frac{\omega_n \sqrt{t}^{n+2}}{m(B_{\sqrt{t}}(x))} - \frac{dH^n}{dm}(x) \right|_{HS}^p \, dm$$

$$\leq C(K, N)^p \int X \left| \frac{\omega_n \sqrt{t}^{n+2}}{m(B_{\sqrt{t}}(x))} - \frac{dH^n}{dm}(x) \right| \, dm \to 0, \tag{2.34}$$

we conclude because of Theorem 2.9, where we used the dominated convergence theorem.

\[
\square
\]

3 Laplacian on $(X, g_t, m)$

Let $(X, d, m)$ be a compact RCD$(K, N)$ space. We rewrite our new differential operators:

**Definition 3.1** (Laplacian on $(X, g_t, m)$). For all $f \in D(\Delta)$ we define the Laplacian $\Delta^f$ associated to $g_t$ by

$$\Delta^f := \langle \text{Hess}_f, g_t \rangle + \frac{1}{4}\langle \nabla_x \Delta_x p(x, x, 2t), \nabla f \rangle \in L^2(X, m). \tag{3.1}$$

Let us start calculation:

**Lemma 3.2.** For all $f \in D(\Delta)$ and $\psi \in H^{1,2}(X, d, m)$ we have

$$\int_X \int_X \psi(x)\langle \nabla_x p, \nabla_x (\nabla_x p) \nabla f \rangle \, dm(x) \, dm(y)$$

$$= -\int_X \langle g_t, df \otimes d\psi \rangle \, dm + \frac{1}{4} \int_X \text{div}(\nabla f) \frac{d}{dt} p(x, x, 2t) \, dm. \tag{3.2}$$

**Proof.** Note that

$$\int_X \int_X \psi(x)\langle \nabla_x p, \nabla_x (\nabla_x p) \nabla f \rangle \, dm(x) \, dm(y)$$

$$= -\int_X \int_X \text{div}_x(\psi \nabla_x p)\langle \nabla_x p, \nabla f \rangle \, dm(x) \, dm(y)$$

$$= -\int_X \int_X \langle (\nabla \psi, \nabla_x p) + \psi(x) \Delta_x p \rangle \langle \nabla_x p, \nabla f \rangle \, dm(x) \, dm(y)$$

$$= -\int_X \langle g_t, df \otimes d\psi \rangle \, dm - \int_X \psi(x) \Delta_x p \langle \nabla_x p, \nabla f \rangle \, dm(x) \, dm(y) \tag{3.3}$$
and that
\[-\int_X \int_X \psi(x) \Delta_x p(x, y) \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[-= \int_X \int_X \psi(x) \left( -\sum_{i \geq 0} \lambda_i e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) \right) \left( \sum_{i \geq 0} e^{-\lambda_i t} \varphi_i(y) \langle \nabla \varphi_i, \nabla f \rangle(x) \right) \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[-= \sum_{i \geq 0} \lambda_i e^{-2\lambda_i t} \int_X \psi(x) \varphi_i(x) \langle \nabla f, \nabla \varphi_i \rangle(x) \, \mathrm{d}m(x) \]
\[-= \frac{1}{2} \sum_{i \geq 0} \lambda_i e^{-2\lambda_i t} \int_X \langle \psi \nabla f, \nabla \varphi_i^2 \rangle \, \mathrm{d}m = -\frac{1}{2} \int_X \text{div}(\psi \nabla f) \left( \sum_{i \geq 0} \lambda_i e^{-2\lambda_i t} \varphi_i^2 \right) \, \mathrm{d}m \] \hspace{1cm} (3.4)

On the other hand since
\[ \frac{\mathrm{d}}{\mathrm{d}t} p(x, x, 2t) = -2 \sum_{i \geq 0} \lambda_i e^{-2\lambda_i t} \varphi_i(x)^2, \] \hspace{1cm} (3.5)
we have
\[-\frac{1}{2} \int_X \text{div}(\psi \nabla f) \left( \sum_{i} \lambda_i e^{-2\lambda_i t} \varphi_i^2 \right) \, \mathrm{d}m = \frac{1}{4} \int_X \text{div}(\psi \nabla f) \frac{\mathrm{d}}{\mathrm{d}t} p(x, x, 2t) \, \mathrm{d}m, \hspace{1cm} (3.6) \]
which completes the proof because of (3.3) and (3.4).

**Lemma 3.3.** For all \( f \in D(\Delta) \) and \( \psi \in H^{1,2}(X, d, m) \) we have
\[-\frac{1}{2} \int_X \int_X \psi(x) \langle \nabla f, \nabla_x |\nabla_x p|^2 \rangle \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[-= -\frac{1}{4} \int_X \int_X \text{div}(\psi \nabla f) \frac{\mathrm{d}}{\mathrm{d}t} p(x, x, 2t) \, \mathrm{d}m(x) + \frac{1}{4} \int_X \text{div}(\psi \nabla f) \Delta_x p(x, x, 2t) \, \mathrm{d}m(x). \] \hspace{1cm} (3.7)

**Proof.** By Lemma 2.2 it is enough to prove (3.7) under assuming \( f \in \text{Test}(X, d, m) \).

First assume \( \psi \in \text{Test}(X, d, m) \). Let \( \varphi := \text{div}(\psi \nabla f) \in H^{1,2}(X, d, m) \). Then
\[-\frac{1}{2} \int_X \int_X \psi(x) \langle \nabla_x f, \nabla_x |\nabla_x p|^2 \rangle \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[= \frac{1}{2} \int_X \int_X \varphi(x) |\nabla_x p|^2 \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[= \frac{1}{2} \int_X \int_X \langle \nabla_x p, \nabla_x (\varphi p) \rangle \, \mathrm{d}m(x) \, \mathrm{d}m(y) - \frac{1}{2} \int_X \int_X (\varphi \nabla_x p, \nabla_x \varphi) \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[-= -\frac{1}{2} \int_X \int_X \varphi(x) \Delta_x p \, \mathrm{d}m(x) \, \mathrm{d}m(y) - \frac{1}{4} \int_X \int_X \langle \nabla_x p^2, \nabla_x \varphi \rangle \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[= -\frac{1}{2} \int_X \int_X \varphi(x) \frac{\mathrm{d}}{\mathrm{d}t} p \, \mathrm{d}m(x) \, \mathrm{d}m(y) + \frac{1}{4} \int_X \int_X \varphi \Delta_x p^2 \, \mathrm{d}m(x) \, \mathrm{d}m(y) \]
\[= -\frac{1}{4} \int_X \varphi(x) \left( \int_X \frac{\mathrm{d}}{\mathrm{d}t} p^2 \, \mathrm{d}m(y) \right) \, \mathrm{d}m(x) + \frac{1}{4} \int_X \varphi(x) \left( \int_X \Delta_x p^2 \, \mathrm{d}m(y) \right) \, \mathrm{d}m(x) \]
\[= -\frac{1}{4} \int_X \varphi(x) \frac{\mathrm{d}}{\mathrm{d}t} (\int_X p^2 \, \mathrm{d}m(y)) \, \mathrm{d}m(x) + \frac{1}{4} \int_X \varphi(x) \Delta_x (\int_X p^2 \, \mathrm{d}m(y)) \, \mathrm{d}m(x) \]
\[= -\frac{1}{4} \int_X \varphi(x) \frac{\mathrm{d}}{\mathrm{d}t} p(x, x, 2t) \, \mathrm{d}m(x) + \frac{1}{4} \int_X \varphi(x) \Delta_x p(x, x, 2t) \, \mathrm{d}m(x), \] \hspace{1cm} (3.8)
which proves (3.7).

Finally let us prove (3.7) for general \( \psi \in H^{1,2}(X, d, m) \). Let \( \psi_L := (-L) \vee \psi \wedge L \in H^{1,2}(X, d, m) \cap L^\infty(X, m) \). Since (3.7) holds as \( \psi = h_t(\psi_L) \) for all \( t > 0 \), letting \( L \to \infty \) and then letting \( t \to 0^+ \) shows the desired claim. \( \square \)
Theorem 3.4 (Integration by parts on $(X, g_t, m)$). For all $f \in D(\Delta)$ and $\psi \in H^{1,2}(X, d, m)$ we have
\[
\int_X \langle g_t, d\psi \otimes df \rangle \, dm = -\int_X \psi \Delta^t f \, dm. \tag{3.9}
\]

Proof. Lemmas 3.2 and 3.3 yield
\[
\int_X \psi \langle \Hess f, g_t \rangle \, dm = \int_X \int_X \psi(x) \langle \Hess f, d_x p \otimes d_x p \rangle \, dm(x) \, dm(y) = \int_X \int_X \psi(x) \langle \nabla_x p, \nabla_x \langle \nabla_x p, \nabla f \rangle \rangle \, dm(x) \, dm(y) - \frac{1}{2} \int_X \int_X \psi(x) \langle \nabla f, \nabla_x |\nabla_x p|^2 \rangle \, dm(x) \, dm(y) = -\int_X \langle g_t, df \otimes d\psi \rangle \, dm + \frac{1}{4} \int_X \text{div}(\psi \nabla f) \Delta x p(x, x, 2t) \, dm(x) = -\int_X \langle g_t, df \otimes d\psi \rangle \, dm - \frac{1}{4} \int_X \psi \langle \nabla f, \nabla_x \Delta x p(x, x, 2t) \rangle \, dm(x),
\]
which proves (3.9).

4 Characterization of noncollapsed RCD spaces

4.1 Proof of Theorem 1.3

Assume that (2) holds. Then the Bishop-Gromov inequality yields that (1.3) holds. Moreover it follows from [Han18] that (1.1) holds. This proves the implication from (2) to (1).

Next we assume that (1) holds. Fix a nonconstant eigenfunction $f$ of $-\Delta$ on $(X, d, m)$ with the eigenvalue $\lambda > 0$. Applying Theorem 3.4 as $\psi \equiv 1$ shows
\[
0 = -\int_X \langle \Hess f, \omega_t t^{n+2}/2 g_t \rangle \, dm - \frac{1}{4} \int_X \langle \nabla \omega_t t^{n+2}/2 p(x, x, 2t), \nabla \Delta f(x) \rangle \, dm(x). \tag{4.1}
\]

Lemma 2.10 yields that as $t \to 0^+$, the first term of the RHS of (4.1) converge to
\[
-c_n \int_X \text{tr}(\Hess f) \frac{dH^n}{dm} \, dm = c_n \lambda \int_X f \frac{dH^n}{dm} \, dm. \tag{4.2}
\]

On the other hand Lemma 2.10 yields that as $t \to 0^+$, the second term of the RHS of (4.1) converge to 0. Thus (4.2) is equal to 0, in particular $\frac{dH^n}{dm}$ is $L^2$-orthogonal to $f$, which shows that $\frac{dH^n}{dm}$ must be a constant.

For all $\psi \in D(\Delta)$ since
\[
\psi = \sum_{i \geq 0} \left( \int_X \psi \varphi_i \, dm \right) \varphi_i \quad \text{in } H^{1,2}(X, d, m) \tag{4.3}
\]
and
\[
\Delta \psi = -\sum_{i \geq 0} \lambda_i \left( \int_X \psi \varphi_i \, dm \right) \varphi_i \quad \text{in } L^2(X, m) \tag{4.4}
\]
(c.f. appendices of [AHTP18] and [Hon18]), combining (4.3) and (4.4) with (2.19) yields
\[
\Hess \psi = \sum_{i \geq 0} \left( \int_X \psi \varphi_i \, dm \right) \Hess \varphi_i \quad \text{in } L^2((T^*)^\otimes 2(X, d, m)). \tag{4.5}
\]
In particular
\[
\text{tr}(\text{Hess}_\psi) = \left\langle \sum_{i \geq 0} \left( \int_X \psi \varphi_i \, dm \right) \text{Hess}_\varphi_i, g \right\rangle = \sum_{i \geq 0} \left( \int_X \psi \varphi_i \, dm \right) (\text{Hess}_\varphi_i, g) = \sum_{i \geq 0} \left( \int_X \psi \varphi_i \, dm \right) \Delta \varphi_i = -\sum_{i \geq 0} \lambda_i \left( \int_X \psi \varphi_i \, dm \right) \varphi_i = \Delta \psi \quad \text{in } L^2(X, m). \tag{4.6}
\]

Therefore if \( \Delta \psi \in H^{1,2}(X, d, m) \), then in the weak sense it holds that
\[
\frac{1}{2} \Delta |\nabla \psi|^2 \geq |\text{Hess}_\psi|^2 + \langle \nabla \Delta \psi, \nabla \psi \rangle + K |\nabla \psi|^2 \\
\geq \frac{(\text{tr}(\text{Hess}_\psi))^2}{n} + \langle \nabla \Delta \psi, \nabla \psi \rangle + K |\nabla \psi|^2 \\
= \frac{(\Delta \psi)^2}{n} + \langle \nabla \Delta \psi, \nabla \psi \rangle + K |\nabla \psi|^2.
\tag{4.7}
\]

This shows that \((X, d, m)\) is a RCD\((K, n)\) space. Thus we get (2). \(\Box\)

### 4.2 Witten Laplacian on RCD spaces

Let us recall that for a closed manifold \((M^n, g)\) with a smooth function \(\varphi \in C^\infty(M^n)\), the corresponding Laplacian of the weighted space \((M^n, g, e^{-\varphi} \text{vol}_g)\) is the Witten Laplacian \(\Delta_\varphi\), that is,
\[
\int_{M^n} \langle \nabla f_1, \nabla f_2 \rangle e^{-\varphi} \, d\text{vol}_g = -\int_{M^n} f_1 \Delta_\varphi f_2 e^{-\varphi} \, d\text{vol}_g \quad \forall f_i \in C^\infty(M^n), \tag{4.8}
\]
where \(\Delta_\varphi f := \text{tr}(\text{Hess}_f) - \langle \nabla \varphi, \nabla f \rangle\). By using the formula (3.9) we can prove an analogous result in the nonsmooth setting. Compare with [Han15].

**Theorem 4.1** (Witten Laplacian on RCD spaces). Let \(n \in [1, \infty)\) and let \((X, d)\) be a compact metric space satisfying that there exists \(C > 0\) such that
\[
\mathcal{H}^n(B_r(x)) \geq Cr^n \quad \forall x \in X, \forall r \in (0, 1). \tag{4.9}
\]

If \((X, d, e^{-\varphi} \mathcal{H}^n)\) is a RCD\((K, N)\) space for some \(N \in [1, \infty)\) and some \(\varphi \in \text{LIP}(X, d)\), then for all \(f \in D(\Delta)\) we have
\[
\Delta f = \text{tr}(\text{Hess}_f) - \langle \nabla \varphi, \nabla f \rangle \quad \text{in } L^2(X, m). \tag{4.10}
\]

**Proof.** Let \(m := e^{-\varphi} \mathcal{H}^n\). By Lemma 2.2 it is enough to prove (4.10) under assuming \(f \in \text{Test}F(X, d, m)\). Note that
\[
m(B_r(x)) = \int_{B_r(x)} e^{-\varphi} \, \mathcal{H}^n \geq Ce^{-\text{max} \varphi} r^n, \quad \forall x \in X, \forall r \in (0, 1). \tag{4.11}
\]

Then by an argument similar to the proof of Theorem 1.3 we see that for all \(\psi \in \text{Test}F(X, d, m)\)
\[
\int_X \langle \nabla \psi, \nabla f \rangle e^\varphi \, dm = -\int_X \psi \text{tr}(\text{Hess}_f) e^\varphi \, dm. \tag{4.12}
\]
Since the LHS of (4.12) is equal to
\[
\int_X \langle \nabla (\psi e^\varphi), \nabla f \rangle \psi e^\varphi \, dm - \int_X \langle \nabla \varphi, \nabla f \rangle \psi e^\varphi \, dm = - \int_X \psi e^\varphi \Delta f \, dm - \int_X \langle \nabla \varphi, \nabla f \rangle \psi e^\varphi \, dm,
\]
we have
\[
\int_X (-\Delta f - \langle \nabla \varphi, \nabla f \rangle + \text{tr}(\text{Hess} f)) \psi e^\varphi \, dm = 0
\]
which completes the proof of (4.10) because \( \psi \) is arbitrary.

We end this paper by giving another characterization of non-collapsed RCD spaces;

**Corollary 4.2.** Let \( n, N \in [1, \infty) \) and let \( (X, d, \mathcal{H}^n) \) be a compact RCD\((K, N)\) space. Then the following two conditions are equivalent:

1. There exists \( C > 0 \) such that
\[
\mathcal{H}^n(B_r(x)) \geq Cr^n \quad \forall x \in X, \forall r \in (0, 1).
\]

2. \((X, d, \mathcal{H}^n)\) is a RCD\((K, n)\) space.

**Proof.** The implication from (2) to (1) is trivial because of the Bishop-Gromov inequality.

Assume that (1) holds. Then applying Theorem 4.1 as \( \varphi \equiv 0 \) yields that (1.1) holds. Therefore Theorem 1.3 shows that (2) holds.

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