New type Stirling like numbers - an email style letter

A. Krzysztof Kwaśniewski
High School of Mathematics and Applied Informatics
Kamienna 17, PL-15-021 Bialystok, Poland
e-mail: kwandr@wp.pl
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The notion of the Fibonacci cobweb poset from [1] has been naturally extended to any admissible sequence \( F \) in [2] where it was also recognized that the celebrated prefab notion of Bender and Goldman [3] - (see also [4,5]) - admits such an extension so as to encompass the new type combinatorial objects from [2] as leading examples. Recently the present author had introduced also [6] two natural partial orders in there: one \( \leq \) in grading-natural subsets of cobweb's prefabs sets [2] and in the second proposal one endows the set sums of the so called "prefabians" with such another partial order that one arrives at Bell-like numbers including Fibonacci triad sequences introduced by the present author in [7]. Here we quote the basic observations concerning the new type Stirling like numbers as they appear in [6]. For more on notation, Stirling like numbers of the first kind and for proofs - see [6].

The overall \( F \)-independent class of p.o. set structure. Let the family \( S \) of combinatorial objects \( \text{(prefabians)} \) consists of all layers \( \langle \Phi_k \rightarrow \Phi_n \rangle \), \( k \leq n \), \( k,n \in N \cup \{0\} \equiv Z_\geq \) and an empty prefabiant \( i \). The set \( \varphi \) of prime objects consists of all subposets \( \langle \Phi_0 \rightarrow \Phi_m \rangle \) i.e. all \( P_m \)’s \( m \in N \cup \{0\} \equiv Z_\geq \) constitute from now on a family of prime prefabians [2]. Layer is considered here to be the set of all max-disjoint isomorphic copies (iso-copies) of \( P_m \). Consider then now the partially ordered family \( S \) of these layers considered to be sets of all max-disjoint isomorphic copies (iso-copies) of prime prefabians \( P_m = P_{n-k} \). For any \( F \)-sequence determining cobweb poset [2] let us define in \( S \) the same partial order relation as follows.

Definition 1
\[
\langle \Phi_k \rightarrow \Phi_n \rangle \leq \langle \Phi_k \triangleright \rightarrow \Phi_n \triangleright \rangle \quad \equiv \quad k \leq k \triangleright \quad \land \quad n \leq n \triangleright .
\]

For convenience reasons we shall also adopt and use the following notation:
\[
\langle \Phi_k \rightarrow \Phi_n \rangle = p_{k,n}.
\]

In what follows we shall consider the subposet \( \langle P_{k,n}, \leq \rangle \) where \( P_{k,n} = [p_{0,o},p_{k,n}] \). Then according to [6] we observe the following.

Observation 1. The size \( |P_{k,n}| \) of \( P_{k,n} \) is \( \{ \{l,m\}, \quad 0 \leq l \leq k \quad \land \quad 0 \leq m \leq n \quad \land \quad k \leq n \} = (n-k)(k+1) + \frac{k(k+1)}{2} \).

Observation 2. The number of maximal chains in \( \langle P_{k,n}, \leq \rangle \) is equal to the number \( d(k,n) \) of 0 - dominated strings of binary sequences
\[
d(k,n) = \frac{n+1-k}{n}\binom{k+n}{n}.
\]

Recall that \( (d(k,n)) \) infinite matrix’s diagonal elements are equal to the Catalan numbers \( C(n) \). The poset \( \langle P_{k,n}, \leq \rangle \) is naturally graded. \( \langle P_{k,n}, \leq \rangle \) poset’s maximal chains are all of equal size (Dedekind -Jordan property) therefore the range function is defined.
Observation 3. The range \( r(P_{k,n}) \) of \( P_{k,n} \) = number of elements in maximal chains \( P_{k,n} \) minus one = \( k + n - 1 \). The range \( r(p_{l,m}) \) of \( \pi = p_{l,m} \in P_{k,n} \) is defined accordingly: \( r(p_{l,m}) = l + m - 1 \).

Accordingly Whitney numbers \( W_k(P_{l,m}) \) of the second kind are defined as follows (association: \( n \leftrightarrow (l, m) \))

**Definition 2**

\[
W_k(P_{l,m}) = \sum_{\pi \in P_{l,m}, r(\pi) = k} 1 = S(k, (l, m)).
\]

We shall identify \( W_k(P_{l,m}) \) with \( S(k, (l, m)) \) called and viewed as Stirling-like numbers of the second kind of the naturally graded poset \( \langle P_{k,n}, \leq \rangle \).

We shall define also the corresponding Bell-like numbers \( B((l, m)) \) of the naturally graded poset \( \langle P_{k,n}, \leq \rangle \).

**Definition 3**

\[
B((l, m)) = \sum_{k=0}^{l+m} S(k, (l, m)).
\]

Observation 4.

\[
B((l, m)) = |P_{l,m}| = \frac{k(k + 1)}{2} + (n - k)(k + 1).
\]

The \( F \)-dependent, \( F \)-labelled class of p.o. set structures. Let us consider now prefabrians’ set sums with an appropriate another partial order so as to arrive at Bell-like numbers including Fibonacci triad sequences introduced recently by the present author in [7]. Let \( F \) be any ”GCD-morphic” sequence [2]. This means that \( GCD[F_n, F_m] = F_{GCD[n,m]} \) where GCD stays for Greatest Common Divisor mapping. We define the \( F \)-dependent finite partial ordered set \( P(n, F) \) as the set of prime prefabrians \( P_l \) given by the sum below.

**Definition 4**

\[
P(n, F) = \bigcup_{0 \leq p \leq n-l} \{ \Phi_p \rightarrow \Phi_{n-p} \} = \bigcup_{0 \leq l} P_{n-l}
\]

with the partial order relation defined for \( n - 2l \leq 0 \) according to

**Definition 5**

\[
P_l \leq P_l \quad \equiv \quad l \leq \hat{l}, \quad P_l, P_l \in \{ \Phi_l \rightarrow \Phi_{n-l} \}.
\]

Recall that rang of \( P_l \) is \( l \). Note that \( \{ \Phi_l \rightarrow \Phi_{n-l} \} = \emptyset \) for \( n - 2l \leq 0 \). The Whitney numbers of the second kind are introduce accordingly.

**Definition 6**

\[
W_k(P_{n,F}) = \sum_{\pi \in P(n,F), r(\pi) = k} 1 = S(n, n-k, F).
\]

Right from the definitions above we infer that:

**Observation 5.**

\[
W_k(P_{n,F}) = \sum_{\pi \in P(n,F), r(\pi) = k} 1 = S(n, n-k, F) = \binom{n-k}{k}_F.
\]
Referring to the classical examples from [8] we identify \( W_k(P(n, F)) \) with \( S(n, n - k, F) \) called the Stirling-like numbers of the second kind of the \( P \). \( P \) by construction displays self-similarity property with respect to its prime prefabint sub-posets \( P_n = P(n, F) \). Consequently for any \( \text{GCD} \)-morphic sequence \( F \) (see: [2]) we define the corresponding Bell-like numbers \( B_n(F) \) of the poset \( P(n, F) \) as follows.

**Definition 7**

\[
B_n(F) = \sum_{k \geq 0} S(n, k, F).
\]

Due to the investigation in [7] we have right now at our disposal all corresponding results of [7] as the following identification with special case of \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequence \( \langle F_n^{[\alpha, \beta, \gamma]} \rangle_{n \geq 0} \) defined in [7] holds.

**Observation 6.**

\[
B_n(F) \equiv F_{n+1}^{[0,0,0]}.
\]

Proof: See the Definition 2.2. from [7].

**Recurrence relations.** Recurrence relations for \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequences \( F_n^{[\alpha, \beta, \gamma]} \) are to be found in [7] - formula (9). Compare also with the special case formula (7) in [9].

**Remark.** As seen from the identification Observation 6. the special cases of \( \langle \alpha, \beta, \gamma \rangle \) - Fibonacci sequences \( F_n^{[\alpha, \beta, \gamma]} \) gain additional with respect to [7] combinatorial interpretation in terms Bell-like numbers as just sums of Whitney numbers of the poset \( P(n, F) \). This adjective ”additional” applies spectacularly to Newton binomial connection constants between bases \( \langle x-1 \rangle_k \rangle_{k \geq 0} \) and \( \langle x^n \rangle_{n \geq 0} \) as these are Whitney numbers of the numbers from \([n]_k \) chain i.e. Whitney numbers of the poset \( \langle [n], \leq \rangle \). For other elementary ”shining brightly” examples see Joni, Rota and Sagan excellent presentation in [8].

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