Abstract
We define and investigate a class of groups characterized by a representation-theoretic property, called purely noncommuting or PNC. This property guarantees that the group has an action on a smooth projective variety with mild quotient singularities. It has intrinsic group-theoretic interest as well. The main results are as follows. (i) All supersolvable groups are PNC. (ii) No nonabelian finite simple groups are PNC. (iii) A metabelian group is guaranteed to be PNC if its commutator subgroup’s cyclic prime-power-order factors are all distinct, but not in general. We also give a criterion guaranteeing a group is PNC if its nonabelian subgroups are all large, in a suitable sense, and investigate the PNC property for permutations.

Keywords Noncommuting operators · Linear representation · Metabelian group · Finite simple group · Supersolvable group · Shared eigenvector

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1 Introduction

A fundamental fact in linear algebra is that any pair of diagonalizable commuting matrices shares a full basis of eigenvectors. If a pair of matrices fails to commute, then they may still share some eigenvectors, although not a full basis. In this case, they act by restriction on the subspace spanned by the common eigenvectors, and their actions on this subspace do commute. Thus one may see the sharing of eigenvectors as a kind of partial commuting. If two diagonalizable matrices do not share any eigenvectors, they noncommute purely.

In the representation theory of a finite group on an algebraically closed field of characteristic zero, group elements always act as diagonalizable transformations. In this context, if two elements of an abstract group do commute, then in every representation they will be forced to share a full basis of eigenvectors. But if they do not commute abstractly, it may still be the case that in every concrete representation of this group on a vector space, they are forced to share some eigenvectors, i.e., commute partially. This prompts us to ask: given a finite group \( G \), and two elements \( x, y \in G \) that do not commute, is it possible to find a representation of \( G \) in which this fact is expressed in an unadulterated way, i.e., their abstract failure to commute is realized in a pair of transformations that do not share an eigenvector?

This question motivates the following definitions. Throughout, \( G \) is a finite group. All vector spaces are over \( \mathbb{C} \).

**Definition 1.1** Let \( \rho: G \to \text{GL}(V) \) be a finite-dimensional representation of \( G \). Noncommuting elements \( x, y \in G \) have the property \( \text{PNC}(\rho, V) = \) “purely noncommuting in \( V \)” if \( \rho(x), \rho(y) \) have no common eigenvectors in \( V \).

We also sometimes express that \( x, y \) have property \( \text{PNC}(\rho, V) \) by saying that the representation \( (\rho, V) \) is PNC for \( x, y \).

**Definition 1.2** The group \( G \) is PNC = “purely noncommuting” if for any noncommuting pair \( x, y \) in \( G \) there exists a representation \( (\rho, V) \) such that \( x, y \) have the property \( \text{PNC}(\rho, V) \).

**Definition 1.3** The group \( G \) is SPNC = “strongly purely noncommuting” if there exists a representation \( (\rho, V) \) such that \( \text{PNC}(\rho, V) \) holds for all noncommuting pairs \( x, y \) in \( G \).

This sequence of definitions was partially inspired by the approach developed in the study of commuting differential operators in mathematical physics. In many cases, the commutativity follows from the existence of a common eigenfunction.

There is an additional, geometric motivation for these notions. When one takes the quotient of a smooth complex algebraic variety \( X \) by the action of a finite group \( G \), the resulting variety typically has singular points. By Hironaka’s theorem, the singularities can be resolved by a sequence of blowups. However, in general, it is a hard problem to make the desingularization process completely constructive.

On the other hand, if the singularities are abelian, meaning that they are locally isomorphic to the quotient of \( \mathbb{C}^n \) by a finite abelian group, then the desingularization can be accomplished in an explicit way (see [3, Chapters 10 and 11]). Thus abelian singularities are mild from the point of view of resolution of singularities.
The singular points of the quotient $X/G$ are automatically abelian if for any point $p \in X$, the point stabilizer $G_p$ is abelian. In this case, the image of $p$ in the quotient $X/G$ is locally isomorphic to the quotient of $\mathbb{C}^n$ by $G_p$. (This is not a necessary condition for the quotient singularities to be abelian; see Sect. 8.4.)

Let $G$ be a finite group with a representation $V$. Then $G$ acts on the projective space $\mathbb{P}(V)$. The SPNC property guarantees that $\mathbb{P}(V)/G$ will have at worst abelian singularities, as follows. If any two $x, y \in G$ both stabilize $p \in \mathbb{P}(V)$, then a representative of $p$ in $V$ is precisely a common eigenvector for the actions of $x$ and $y$ on $V$. Thus if $V$ realizes $G$ as SPNC, it must be that $x, y$ commute. It follows that $G_p$ is abelian.

PNC groups themselves have a related property. If $G$ is a PNC group, and $X = \prod \mathbb{P}(V_i)$, where the product is taken over the irreducible representations of $G$, then any point stabilizers for the action of $G$ on $X$ are abelian, by similar reasoning, and therefore any singular points of $X/G$ are abelian.

Thus SPNC and PNC groups have actions, respectively, on projective spaces and products thereof, whose quotients have mild singularities.

This article is an investigation into PNC and SPNC groups. Section 2 sets up notation and conventions, and makes some first observations used in the sequel. In Sect. 3, we show that supersolvable groups are always PNC (Theorem 3.3). At the other end of the spectrum, we show in Sect. 4 that nonabelian simple groups are never PNC (Theorem 4.2). In Sect. 5, we prove that a group is SPNC if all its nonabelian subgroups are sufficiently large in a suitable sense (Proposition 5.1). Section 6 investigates the PNC property for metabelian groups. We show that a metabelian group is necessarily PNC if its commutator subgroup’s cyclic prime-power-order factors are all different (Theorem 6.2), and also exhibit an infinite family of metabelian groups that are not PNC (Theorem 6.1). In Sect. 7, we give a group-theoretic characterization (Proposition 7.1) of when two permutations noncommute purely in the standard representation, i.e., the nontrivial irreducible component of the defining permutation representation, of the symmetric group. We close with further questions in Sect. 8.

2 Notation and preliminaries

Throughout, for commutators and conjugates we adopt the right-action notation $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$.

Notation 2.1 If $G$ is a group and $A \subseteq G$ is a (not necessarily normal) subgroup, we use the symbol $G/A$, as usual, to mean the left coset space of $A$ in $G$. Then the statement

$$[s] \in G/A$$

should be interpreted to mean that $[s]$ is a coset and $s$ is some representative in $G$ of this coset. We will then sometimes write

$$g^s$$
to mean the conjugate of $g$ by any representative of the coset $[s]$. We will only use this notation when the setting renders the choice of coset representative inconsequential. The main examples are below in Observation 2.3 and (1).

**Notation 2.2** If $\chi$ is a character of a group $A$ that is embedded in a larger group $G$, we adopt the convention that $\chi$ can be extended to a function on $G$, also called $\chi$, by assigning it the value 0 outside $A$. More precisely, define a new function $\overline{\chi}$ by

$$\overline{\chi}(g) = \begin{cases} \chi(g), & g \in A, \\ 0, & g \notin A, \end{cases}$$

and then set $\chi = \overline{\chi}$. Note that $\chi$ is not necessarily a character of $G$.

**Observation 2.3** Notation 2.2 allows us to write the formula

$$\text{Ind}^G_A \chi(g) = \sum_{[s] \in G/A} \chi(gs),$$

giving the character of an induced representation. Per Notation 2.1, this formula does not depend on the choice of coset representative $s \in [s]$: if $gs \notin A$, then

$$gs^a = (gs)^a \notin A$$

either, so $\chi(gs) = \chi((gs)^a) = 0$, while if $gs \in A$, then

$$\chi(gs^a) = \chi(((gs)^a) = \chi(g^s)$$

because $\chi$ is a class function on $A$.

In the case that $A$ is abelian and $\chi$ is multiplicative on $A$, the extended meaning of $\chi$ given by Notation 2.2 preserves the multiplicativity relation $\chi(gh) = \chi(g)\chi(h)$ as long as at least one of $g, h$ is in $A$. For if one of $g, h$ is in $A$ while the other is not, then $gh$ is not in $A$, so that $\chi(gh) = 0 = \chi(g)\chi(h)$.

If $G$ fails to be PNC, then it means that there is a noncommuting pair $x, y \in G$ such that in every representation of $G$, $x$ and $y$ share a common eigenvector. This is equivalent to the statement that any representation of $G$, when restricted to the nonabelian subgroup $H$ generated by $x$ and $y$, will contain some one-dimensional representation of $H$. This is a fact about $H$ that does not depend on the choice of generators $x, y$ for $H$. Any other $x', y'$ that also generate $H$ will also obstruct PNCness, i.e., they will share a common eigenvector in every representation of $G$.

Conversely, if $G$ is PNC, then for every pair $x, y$ of noncommuting elements, there is a representation $V$ in which they do not share a common eigenvector. This means that the restriction of $V$ to $H = \langle x, y \rangle$ must not contain any one-dimensional representations of $H$. Again, this is a statement about $V$ and $H$ that does not depend on the choice of generators $x, y$ for $H$. 
These considerations motivate the following definition:

**Definition 2.4** Let $H \subset G$ be a nonabelian subgroup. Given a representation $\rho$ (respectively $V$) of $G$, we say $\rho$ (respectively $V$) is **PNC for $H$** if $\rho$’s (respectively $V$’s) restriction to $H$ does not contain any one-dimensional representations of $H$.

It is not hard to see that a group $G$ is PNC if and only if it is PNC for each of its 2-generated nonabelian subgroups. In fact the quantification in this statement can be restricted: $G$ is PNC if and only if it is PNC for each of its **minimal nonabelian subgroups**, i.e., its nonabelian subgroups whose proper subgroups are all abelian. (Such groups are automatically 2-generated.)

Among the minimal nonabelian groups are the dihedral groups of order $2p$, $p$ an odd prime, and 8. Dihedral groups play an important role in a number of our arguments because they are particularly adept at obstructing PNCness, by the following lemma and Corollary 3.7. Per the dihedral group’s interpretation as symmetries of a regular polygon, we refer to its cyclic, index-2 subgroup as the **rotation subgroup** and the elements of the nontrivial coset of this subgroup as **reflections**.

**Lemma 2.5** Suppose $G$ contains a dihedral group $D$, and $G$ has no irreducible representation whose character simultaneously (i) is identically zero on $D$’s reflections, and (ii) sums to zero on $D$’s rotation subgroup. Then $G$ is not PNC.

**Proof** Every irreducible representation of $D$ of degree greater than 1 has properties (i) and (ii). In fact, they are all of degree 2, and (i) and (ii) are immediate consequences of the fact [9, pp. 35–37] that they are all induced from nontrivial one-dimensional representations of the rotation subgroup. Since these properties are both linear, they also hold for any representation of $D$ that does not contain a one-dimensional representation. In particular, the hypothesis guarantees that any irreducible representation of $G$ (and therefore any representation at all) will contain a one-dimensional subrepresentation when restricted to $D$. Thus, $G$ is not PNC for $D$, and therefore not PNC.

We use this lemma in several proofs in Sects. 4 and 6.

### 3 Supersolvable groups

A finite solvable group admits a normal series with abelian quotients (the derived series), and a subnormal series with cyclic quotients (any composition series). If we strengthen this requirement to a normal series with cyclic quotients, we can guarantee that the group is PNC. Recall that groups with such a normal series are called **supersolvable**.

A well-known formula [9, Proposition 22] gives a decomposition of the restriction to a subgroup of a representation induced from another subgroup. In the case that the two subgroups coincide and are normal, then this formula takes the form

$$\text{Res}^G_N \text{Ind}^G_N \rho \cong \bigoplus_{[s] \in G/N} \rho^s, \quad (1)$$

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where each $\rho^s : N \to \text{GL}(V)$ is defined by $\rho^s(x) = \rho(x^s)$.

**Observation 3.1** Cyclic groups are precisely those finite groups possessing a faithful one-dimensional representation, since every finite subgroup of $\mathbb{C}^\times$ is cyclic.

**Observation 3.2** If $V$ is a vector space and $x, y \in \text{GL}(V)$ share an eigenvector $v$, then their commutator $[x, y] \in \text{GL}(V)$ also shares this eigenvector, and it has eigenvalue 1, i.e.

$$v[x, y] = v.$$

**Theorem 3.3** Let $G$ be a finite group admitting a normal series with cyclic quotients, i.e. a finite supersolvable group. Then $G$ is PNC.

**Proof** Let $x, y$ be any two noncommuting elements. Then $[x, y]$ is a nontrivial element of $G$. Let

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{1\}$$

be the assumed normal series with cyclic quotients. Since $[x, y]$ is nontrivial, it is in $G_i \setminus G_{i+1}$ for some $i = 0, \ldots, n - 1$. By assumption, $G_i / G_{i+1}$ is cyclic, so there exists a faithful one-dimensional representation

$$\rho : G_i / G_{i+1} \to \mathbb{C}^\times$$

by Observation 3.1. Precomposing with the canonical homomorphism

$$\pi : G_i \to G_i / G_{i+1},$$

we obtain a one-dimensional representation $\phi = \rho \pi$ of $G_i$ that is nontrivial outside of $G_{i+1}$.

We will now show that the induced representation

$$\Phi = \text{Ind}_{G_i}^G \phi$$

is PNC for $x, y$. This will be done by showing that $\Phi([x, y])$ does not have 1 as an eigenvalue. It will then follow that $\Phi(x), \Phi(y)$ do not share an eigenvector, for if they did, their commutator $\Phi([x, y])$ would have 1 as an eigenvalue, by Observation 3.2. As $x, y$ are an arbitrary noncommuting pair, this will complete the proof that $G$ is PNC.

Since $G_i$ is normal in $G$, (1) tells us that

$$\Phi|_{G_i} = \bigoplus_{[s] \in G / G_i} \phi^s.$$

Since $\phi$ is one-dimensional, $\phi^s$ is as well, for each $s$, so that this formula splits $\Phi$ into one-dimensional representations on $G_i$. It follows that for any given $g \in G_i$, the eigenvalues of $\Phi(g)$ are just the values of the $\phi^s(g) \in \mathbb{C}^\times$. 

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We apply this with \( g = [x, y] \in G_i \). By assumption, \([x, y]\) lies outside of \( G_{i+1} \). Since \( G_{i+1} \) is normal in \( G_i \), \([x, y]^s\) also lies outside of \( G_{i+1} \) for each \( s \). Since \( \phi \) is nontrivial outside of \( G_{i+1} \) by construction, this means that \( \phi([x, y]) = \phi([x, y]^s) \) is not equal to 1 for any \( s \). Thus no eigenvalue of \( \Phi([x, y]) \) is 1. \( \square \)

**Corollary 3.4** Finite nilpotent groups are PNC.

**Proof** They are supersolvable. \( \square \)

Recall that a group \( G \) is called cyclic-by-abelian if it has a cyclic normal subgroup \( C \) such that the quotient \( G/C \) is abelian.

**Corollary 3.5** If a finite group \( G \) is cyclic-by-abelian, then it is SPNC.

**Proof** Let \( C \triangleleft G \) be cyclic with \( G/C \) abelian. Then \( [G, G] \subset C \), thus every nontrivial commutator is in \( C \setminus \{1\} \). There exists a character \( \phi \) of \( C \) that is nontrivial on \( C \setminus \{1\} \), by Observation 3.1. Then \( \Phi = \text{Ind}_G^C \phi \) is a representation in which, by the exact same argument as in the proof of Theorem 3.3, no nontrivial commutator has 1 as an eigenvalue, and therefore in which no pair of noncommuting elements shares an eigenvector. Thus \( \Phi \) is PNC for any noncommuting \( x, y \), so it manifests \( G \) as SPNC. \( \square \)

There is no hope of a similar result about groups which are merely solvable:

**Proposition 3.6** The symmetric group \( S_4 \) is not PNC.

This can be proven by direct reference to \( S_4 \)’s character table, but the following more conceptual proof contains a device that will be used again later.

**Proof** Consider the subgroup \( D_4 = \langle (1234), (13) \rangle \subset S_4 \). We will show that no representation of \( S_4 \) is PNC for this subgroup.

\( D_4 \) acts faithfully on the plane as the symmetry group of a square. This is its only irreducible representation of degree greater than one. The character \( \chi \) of this representation is given by

| \( \chi \) | 1 | (13)(24) | (1234) | (13) | (12)(34) |
|----------|---|----------|--------|-----|--------|
|         | 2 | -2       | 0      | 0   | 0      |

Notice that \( \chi \) separates the central element \( (13)(24) \) from the class of the reflection \( (12)(34) \). On the other hand, in \( S_4 \) these elements are conjugate. Therefore no class function on \( S_4 \), in particular no character of \( S_4 \), can separate them. It follows that no character of \( S_4 \) restricts to a multiple of \( \chi \); thus the restriction to \( D_4 \) of any representation of \( S_4 \) must contain some one-dimensional representation of \( D_4 \), so no representation of \( S_4 \) is PNC for \( D_4 \). \( \square \)

The only feature of \( S_4 \) used in this proof is that it contains \( D_4 \) in such a way that the central involution is conjugate to one of the other involutions. Therefore the argument generalizes:

**Corollary 3.7** \( (D_4 \text{ obstruction}) \) If a finite group \( G \) contains \( D_4 \) in such a way that the nontrivial central element in \( D_4 \) is conjugate to one of the other involutions, then \( G \) is not PNC.
4 Nonabelian simple groups

While Sect. 3 shows that there are plenty of PNC groups, there are also plenty of groups which are not PNC. Recall that a family of objects $\mathcal{F}$ is said to be upward-closed if whenever an object $A$ is in $\mathcal{F}$ and embeds in an object $B$, then $B$ is in $\mathcal{F}$ too.

Lemma 4.1 The family of non-PNC groups is upward-closed.

Proof If a noncommuting pair in a group $G$ shares an eigenspace in every representation of $G$, it also does so in every representation $\rho$ of any group containing $G$, since $\rho$ is also a representation of $G$ by restriction. Thus if $G$ is not PNC for $x, y \in G$, no overgroup of $G$ can be PNC for $x, y$ either. \qed

Thus by Proposition 3.6, no group containing $S_4$ is PNC. But more broadly:

Theorem 4.2 No nonabelian simple group is PNC.

This theorem is the main goal of this section. The structure of the proof is as follows. By a 1997 result of Barry and Ward [1, Theorem 1], every nonabelian simple group contains a minimal simple group, i.e., a nonabelian finite simple group all of whose proper subgroups are solvable. Such groups were classified in 1968 by Thompson, and they are all of the form $\text{PSL}(2, q)$ for $q$ a prime power $\geq 4$, $\text{Sz}(2^p)$ for $p$ an odd prime, or $\text{PSL}(3, 3)$ [11, Corollary 1]. We will show that none of these groups is PNC by giving explicit dihedral subgroups for which they are not PNC. The result for all nonabelian simple groups will then follow by Lemma 4.1. Here are the precise details.

Remark 4.3 The group $\text{PSL}(2, q)$ is only minimal simple for certain $q$. The version of the statement in [11] is sharper than what we have quoted.

Lemma 4.4 For a prime power $q \geq 4$, $\text{PSL}(2, q)$ is not PNC.

In fact, we can already know this for $q = \pm 1 \mod 4$ since in this case $\text{PSL}(2, q)$ contains $S_4$. One could hope to proceed to the remaining cases. We give a more uniform proof, although some case analysis is inevitable because the representation theory of $\text{PSL}(2, q)$ depends on $q \mod 4$.

Proof We will show that the PNC property is obstructed by a dihedral group of order $q - 1$ or $q + 1$, if $q$ is odd, or $2(q - 1)$, if $q$ is even. It is well known that $\text{PSL}(2, q)$ contains dihedral subgroups of these orders ([4, Sect. 246] or [6, Theorem 2.1 (d)-(i)]). Let $D$ be a dihedral subgroup of $\text{PSL}(2, q)$, of order to be specified shortly.

We will show that the order of $D$ can always be chosen so that no irreducible character of $\text{PSL}(2, q)$ simultaneously is identically zero on the reflections and sums to zero on the rotation subgroup. Then the desired result will follow from Lemma 2.5.

The relevant part of the character table of $\text{PSL}(2, q)$ is given in Table 1. The notation below is explained in the caption.

There are three cases to consider: $q = 1 \mod 4$, $q = 3 \mod 4$, and $q$ even.

In the case $q = 1 \mod 4$, the class of involutions is $a^{(q - 1)/4}$ in the table. Take $D$ to be of order $q + 1$, so its rotation subgroup, of order $(q + 1)/2$, consists of the identity and elements in the classes $b^m$. The only irreducible characters of $\text{PSL}(2, q)$ that are
Table 1 Character table of $\text{PSL}(2, q)$

| Case $q = 1 \mod 4$. | 1 | $a^\ell$ | $b^m$ |
|----------------------|---|---------|-------|
| **Triv**             | 1 | 1       | 1     |
| **$\psi$**           | $q$ | 1       | -1    |
| **$\chi_i$**         | $q + 1$ | $\rho^i \ell + \rho^{-i} \ell$ | 0     |
| **$\theta_j$**       | $q - 1$ | 0       | $-(\sigma^j \ell + \sigma^{-j} \ell)$ |
| **$\xi_1$**          | $(q + 1)/2$ | $(-1)^\ell$ | 0     |
| **$\xi_2$**          | $(q + 1)/2$ | $(-1)^\ell$ | 0     |

| Case $q = 3 \mod 4$. | 1 | $a^\ell$ | $b^m$ |
|----------------------|---|---------|-------|
| **Triv**             | 1 | 1       | 1     |
| **$\psi$**           | $q$ | 1       | -1    |
| **$\chi_i$**         | $q + 1$ | $\rho^i \ell + \rho^{-i} \ell$ | 0     |
| **$\theta_j$**       | $q - 1$ | 0       | $-(\sigma^j \ell + \sigma^{-j} \ell)$ |
| **$\eta_1$**         | $(q + 1)/2$ | 0       | $(-1)^{m+1}$ |
| **$\eta_2$**         | $(q + 1)/2$ | 0       | $(-1)^{m+1}$ |

| Case $q$ even.       | 1 | $c$     | $a^\ell$ | $b^m$ |
|----------------------|---|---------|---------|-------|
| **Triv**             | 1 | 1       | 1       | 1     |
| **$\psi$**           | $q$ | 0       | 1       | -1    |
| **$\chi_i$**         | $q + 1$ | 1       | $\rho^i \ell + \rho^{-i} \ell$ | 0     |
| **$\theta_j$**       | $q - 1$ | -1      | 0       | $-(\sigma^j \ell + \sigma^{-j} \ell)$ |

The symbols $\rho$, $\sigma$ are primitive $(q - 1)$th and $(q + 1)$th roots of unity respectively. For odd $q$, respectively even $q$, $a$ is the class of elements of order $(q - 1)/2$, respectively $q - 1$, and $b$ is the class of elements of order $(q + 1)/2$, respectively $q + 1$. For odd $q$, $i$ and $j$ are even integers and we have omitted the classes of elements of order dividing $q$. For even $q$, $i$ and $j$ are integers and $c$ is the class of involutions. Source: [5, Sect. 38]

zero on $a^{(q-1)/4}$ are those of the cuspidal representations $\theta_j$. Their absolute value is $q + 1$ on the identity and is bounded by 2 on the elements $b^m$. Thus the sum of any of these characters across the rotation subgroup has absolute value bounded below by

$$q + 1 - 2\left(\frac{q + 1}{2} - 1\right) = 2 > 0.$$ 

Therefore no irreducible character of $\text{PSL}(2, q)$ is simultaneously zero on $D$’s reflections and sums to zero on $D$’s rotation subgroup.

For $q = 3 \mod 4$, one uses $D$ of order $q - 1$ instead of $q + 1$. This time, the nontrivial rotations are in the classes $a^\ell$ and the class of involutions is $b^{(q+1)/4}$. Only the principal series characters $\chi_i$ are zero on this latter class. Practically the same calculation shows their sum across the rotation subgroup is bounded below by 2.

In the final case of even $q$, take $D$ to be of order $2(q - 1)$, so the nontrivial rotations are $a^\ell$ and the involutions are $c$. The only irreducible character that is zero on $c$ is the
Table 2  Character table of Sz(q)

| Class name | Order divides | σ | ρ, ρ⁻¹ | π₀ | π₁ | π₂ |
|------------|--------------|---|---------|----|----|----|
| X          | q²           | 0 | 0       | 1  | -1 | -1 |
| Xᵢ         | q² + 1       | 1 | 1       | εᵢ(π₀) | 0 | 0 |
| Yᵢ         | (q - r + 1)(q - 1) | r - 1 | -1 | 0 | -εᵢ(π₁) | 0 |
| Zᵢ         | (q + r + 1)(q - 1) | -r - 1 | -1 | 0 | 0 | -εᵢ(π₂) |
| Wᵢ         | r(q - 1)/2   | -r/2 | ±r/2   | 0 | 1 | -1 |

Here, r = √2q. The classes called π₀, π₁, π₂ consist of elements belonging to certain cyclic subgroups A₀, A₁, A₂, and the ε’s certain characters of these subgroups. Source: [10, Theorem 13]

Steinberg character ψ, which is positive on the a²'s, so the sum across the rotation subgroup of D is positive. □

Lemma 4.5  PSL(3, 3) is not PNC.

Proof  The argument is identical to that given for S₄ (Proposition 3.6). PSL(3, 3) = SL(3, 3) contains a subgroup D₄ generated by

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix}, \quad f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The central involution in this copy of D₄ is

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1 \\
\end{pmatrix}.
\]

But this is conjugate in PSL(3, 3) to f, so apply Corollary 3.7. □

Lemma 4.6  The Suzuki group Sz(2ᵖ) (p an odd prime) is not PNC.

Proof  The proof is essentially the same as that for PSL(2, q) when q is even. Like PSL(2, q), the Suzuki group Sz(q), q = 2ᵖ, has a single conjugacy class of involutions [10, Proposition 7]. It also contains a dihedral group D of order 2(q - 1): when Sz(q) is realized as a permutation group as in Suzuki’s original presentation, this is the normalizer of the stabilizer of two points [10, Proposition 3].

The character table of Sz(q) is given in Table 2, which is taken from [10, Theorem 13].

There is only one irreducible character of Sz(q) that is zero on the class of involutions (X in the table), and it is positive on all the elements of D’s rotation subgroup (which, besides the trivial class, are in the classes Suzuki calls π₀, as their orders divide q - 1). So, by Lemma 2.5, Sz(q) is not PNC. □
**Proof of Theorem 4.2** As noted above, [1, Theorem 1] and [11, Corollary 1] give us that every nonabelian finite simple group contains \( \text{PSL}(2, q) \) for \( q \geq 4 \), \( \text{Sz}(2^p) \) for \( p \) odd prime, or \( \text{PSL}(3, 3) \). None of these is PNC by Lemmas 4.4, 4.5, and 4.6, so Lemma 4.1 then implies that no nonabelian finite simple group is PNC. \( \square \)

### 5 A family of SPNC groups

The argument of Theorem 3.3 shows supersolvable groups are PNC by finding, for any noncommuting pair, a representation in which its commutator does not have 1 as an eigenvalue. A group can be PNC without this. For example, \( A_4 \) is a group in which, in *every* representation, *every* commutator has 1 as an eigenvalue. Nonetheless, the standard three-dimensional representation of \( A_4 \) actually realizes it as SPNC: \( A_4 \) is a minimal nonabelian group, thus any pair of noncommuting elements generates the whole group, and therefore cannot have a common eigenspace in this representation because it is irreducible.

In a similar way one sees immediately that any minimal nonabelian group is SPNC. This is actually a special case of a more general phenomenon that forces a group to be not just PNC but SPNC:

**Proposition 5.1** Let \( G \) be a finite group with an irreducible representation \( V \) of degree \( d \) that exceeds the index of any of its nonabelian subgroups. Then \( V \) realizes \( G \) as SPNC.

**Proof** Let \( x, y \) be a pair of noncommuting elements of \( G \), and let \( H = \langle x, y \rangle \). By assumption, \([G:H] < d\). Now let \( L \) be any one-dimensional representation of \( H \). Then, by Frobenius reciprocity, the number of times that \( L \) occurs in the restriction of \( V \) to \( H \) is equal to the number of times \( V \) occurs in the induced representation \( \text{Ind}_H^G L \). Since the dimension of this representation is

\[
[G:H] < d = \dim V,
\]

this number is zero. So no one-dimensional representation \( L \) occurs in the restriction of \( V \) to \( H \), i.e., \( x, y \) do not have a common eigenspace in \( V \). \( \square \)

**Remark 5.2** As we have seen, failure to be PNC is always caused by specific obstructing subgroups. For example, the obstruction for \( S_4 \) is \( D_4 \) (Proposition 3.6). Proposition 5.1 shows that a subgroup obstructing PNCness cannot be “too big.” Indeed, \( D_4 \subset S_4 \) is “as big as possible,” since \( S_4 \) has an irreducible representation (in fact, two) of degree 3, equal to the index.

Beyond the trivial case of minimal nonabelian groups, an example of a family of groups satisfying the hypothesis of Proposition 5.1 is the semidirect product of the additive group \( A \) of the field \( \mathbb{F}_{p^2} \), for \( p \) a prime congruent to 1 mod 4, by an automorphism given by multiplication by a field element of multiplicative order \( d = (p + 1)/2 \). This group has an irreducible representation of degree \( d \), but every nonabelian subgroup properly contains \( A \) and so has index less than \( d \). (See [2, Proposition 1.1.49] for details).
6 Metabelian groups

Recall that a group is called metabelian if it has an abelian normal subgroup with an abelian quotient, in other words if it is solvable of height two. In this section we investigate the PNC property for metabelian groups.

The results of Sects. 3, 4, and 5 show that PNCness is loosely correlated with abelianness — the “extremely nonabelian” simple groups are never PNC, while “almost abelianness” of various kinds (nilpotence and supersolvability, per Sect. 3, and having all nonabelian subgroups “large,” per Sect. 5) guarantee PNCness. Based on this intuition, the authors thought that metabelian groups might be always PNC; but this turns out not to be the case.

Recall that, for a prime power \( q \), the affine group, or affine linear group, \( \text{AGL}(n, q) \), is the semidirect product of the additive group of the \( \mathbb{F}_q \)-vector space \( \mathbb{F}_q^n \) by the group \( \text{GL}(n, q) \) of linear automorphisms of this space. For \( n = 1 \), \( \text{AGL}(1, q) = \mathbb{F}_q^+ \rtimes \mathbb{F}_q^\times \) is metabelian. However:

**Theorem 6.1** If \( q = p^k \) is an odd prime power with \( k > 1 \), the affine group

\[ \text{AGL}(1, q) = \mathbb{F}_q^+ \rtimes \mathbb{F}_q^\times \]

is not PNC.

We defer the proof to the end of the section.

In spite of this negative result, a large class of metabelian groups is PNC. If \( G \) is metabelian then the commutator subgroup \([G, G]\) is abelian. There is a criterion on the structure of \([G, G]\) that lets us conclude PNCness without knowing anything else about \( G \):

**Theorem 6.2** Let \( G \) be a metabelian group and let

\[ [G, G] \cong C_{p_1^{c_1}} \times \cdots \times C_{p_k^{c_k}} \]

be the expression of its commutator as a direct product of cyclic factors of prime power order. If the factors are pairwise nonisomorphic, then \( G \) is PNC.

**Remark 6.3** This is not a necessary condition. Many metabelian groups not satisfying the hypothesis of Theorem 6.2 are still PNC, for example the family of SPNC groups described at the end of Sect. 5. But it shows that PNC metabelian groups are easy to come by.

The organization of this section is motivated by the proof of Theorem 6.2. The fundamental tool is the following lemma, which is of independent utility in investigating the PNC property.

**Lemma 6.4** (Commutator criterion) Let \( G \) be an arbitrary finite group, \( H \) a nonabelian subgroup of \( G \), and \( V \) a representation of \( G \) with character \( \chi \). Then \( V \) is PNC for \( H \) if and only if \( \chi \) sums to zero on each coset of \([H, H]\) in \( H \).

1 Abelian groups themselves are PNC, vacuously. It is perhaps a failing of our nomenclature that abelian groups are called “purely noncommuting”.
**Proof** The representation $V$ is PNC for $H$ if and only if $V|_H$ contains no one-dimensional representations of $H$. By the orthogonality relations, this is the case if and only if $\chi|_H$ is orthogonal to all of $H$’s one-dimensional representations, with respect to the $H$-invariant inner product

$$\langle \chi_1, \chi_2 \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi_1(h) \overline{\chi_2(h)}.$$

Now $H$’s one-dimensional representations are precisely the pullbacks to $H$ of all of the characters of the abelian group $H/[H, H]$. These characters span the full space of functions on $H/[H, H]$ (as for any abelian group), so their pullbacks to $H$ span the space of all functions on $H$ constant on each coset of $[H, H]$. The orthogonal complement of this space is clearly the space of class functions that sum to zero on each coset of $[H, H]$, and $V$ is PNC for $H$ if and only if $\chi|_H$ lies in this orthogonal complement. \(\square\)

Now we begin to assemble the proof of Theorem 6.2.

If $G$ is metabelian, then its commutator subgroup $[G, G]$ is abelian, and therefore acts trivially on itself by conjugation. It follows that the conjugation action of $G$ on $[G, G]$ makes the latter a $G/[G, G]$-module.

In what follows we fix the notation that $G$ is a finite metabelian group, $A = [G, G]$, and $Q = G/[G, G]$, so $A$, $Q$ are abelian and $A$ is a $Q$-module. For this discussion, a *character of $A$* means a homomorphism to $\mathbb{C}^\times$, not merely the character of an arbitrary representation.

**Proposition 6.5** Let $H \subset G$ be a nonabelian subgroup, so that $K = [H, H]$ is a nontrivial subgroup of $A$. Suppose that $A$ has a character $\chi$ whose kernel does not contain any image of $K$ under the $Q$-action. Then the representation $\text{Ind}^G_A \chi$ is PNC for $H$.

For example, if $A$ is cyclic (so $G$ is cyclic-by-abelian), it has a character $\chi$ that is a faithful representation of $A$ and is thus nontrivial on all nontrivial subgroups. Therefore $\text{Ind}^G_A \chi$ is PNC for all nonabelian subgroups; thus $G$ is SPNC. This reproduces Corollary 3.5, although without the information (obtained in the proof in Sect. 3) that no commutator has an eigenvalue 1 in this representation.

**Proof** We want to show $\text{Ind}^G_A \chi$ is PNC for $H$, and by the commutator criterion (Lemma 6.4), this is equivalent to showing that

$$\sum_{k \in K} \text{Ind}^G_A \chi(kh) = 0$$

for all $h \in H$. Actually we even have $\sum_{k \in K} \text{Ind}^G_A \chi(kg) = 0$ for every $g \in G$. We see this as follows:

$$\sum_{k \in K} \text{Ind}^G_A \chi(kg) = \sum_{k \in K} \sum_{[s] \in G/A} \chi((kg)s)$$
The last equality is because $\chi$, being a character of $A$, is multiplicative, since $k^s \in A$ as $k \in K \subset A$ and $A$ is normal (see Observation 2.3). Reversing the summations and then reindexing the inner sum, we have

$$\sum_{[s] \in G/A} \left( \sum_{k \in K} \chi(k^s) \right) \chi(g^s) = \sum_{[s] \in G/A} \left( \sum_{k \in K^s} \chi(k) \right) \chi(g^s).$$

But the inner sum is zero, because by assumption $K^s$ is not contained in $\ker \chi$, therefore $\chi$ restricts to a nontrivial character of $K^s$, and the sum of a nontrivial character over a group is always zero.

This proposition immediately implies that the following condition on the module structure guarantees PNCness.

**Proposition 6.6** (Subgroup character condition) Suppose $A$ (as $Q$-module) has the property that for any nontrivial subgroup $K \subset A$, $A$ has a character whose kernel does not contain any image of $K$ under the $Q$-action. Then $G$ is PNC.

**Proof** If $H$ is any nonabelian subgroup of $G$, then Proposition 6.5 shows how to construct a representation of $G$ that is PNC for $H$.

**Remark 6.7** In fact, we found Theorem 6.1 by looking for a group where the condition of this proposition fails.

The next lemma, whose statement and proof are due to Ladisch [7], links Proposition 6.6 with the hypothesis of Theorem 6.2.

**Lemma 6.8** (Ladisch) The following conditions on a finite abelian group $A$ are equivalent:

1. $A$ has a nontrivial subgroup $K$ such that every character of $A$ is trivial on an image of $K$ under some automorphism of $A$.
2. In a decomposition of $A$ into cyclic factors of prime power order, two of the factors are isomorphic.

**Proof** We write $A$ additively.

Condition 2 is fulfilled by $A$ if and only if it is fulfilled by at least one of $A$’s Sylow subgroups. We will show the same for condition 1. If a nontrivial subgroup $K$ of a Sylow subgroup $A_p$ fulfills condition 1 for $A_p$, it also does so for $A$ because automorphisms of $A_p$ extend to $A$ and characters of $A$ restrict to $A_p$. Conversely, if a nontrivial subgroup $K$ of $A$ fulfills condition 1 for $A$, then there is a prime $p$ (any prime dividing $|K|$ in fact) such that $K_p = A_p \cap K$ fulfills condition 1 for $A_p$, since characters of $A_p$ extend to $A$ and automorphisms of $A$ act on $A_p$.

Thus without loss of generality we can suppose $A$ is a $p$-group. If it fulfills condition 2, it has the form $A = F \oplus B$ where $F \cong C_p^k \times C_p^k$. Then let $K$ be the cyclic subgroup $p$-group $F$.
generated by any nonzero element in \( F \). Note \( \text{Aut} F \subset \text{Aut} A \). Every character of \( A \) restricts to a character on \( F \), and we assert any character of \( F \) is trivial on some \( \text{Aut} F \)-image of \( K \). Indeed, \( \text{Aut} F = \text{GL}(2, \mathbb{Z}/p^k\mathbb{Z}) \) acts transitively on the elements of \( F \) of any given order, and therefore on the order-\(|K|\) cyclic subgroups of \( F \). Meanwhile every character of \( F \) is trivial on some order-\(|K|\) cyclic subgroup since it is trivial on some maximal (order \( p^k \)) cyclic subgroup, as otherwise its image would not be cyclic. This shows \( 2 \Rightarrow 1 \).

In the other direction, suppose \( K \) fulfills condition \( 1 \) for \( A \) and consider the subgroup \( A_0 \) of \( A \) of elements of order dividing \( p \). This is an \( \mathbb{F}_p \)-vector space of dimension the \( p \)-rank of \( A \). It has a filtration

\[
A_0 \supset A_1 \supset \cdots \supset A_k = 0,
\]

where \( A_i = A_0 \cap p^i A \) is the subgroup of \( A_0 \) consisting of \( p^i \)-divisible elements, and \( p^k \) is the exponent of \( A \). Both \( A_0 \) and this filtration of it are invariant under automorphisms of \( A \).

Now as \( K \) is nontrivial it contains elements of order \( p \), so it must meet \( A_0 \) nontrivially. Thus there is a maximal \( i < k \) such that \( A_i \) meets \( K \) nontrivially; fix this \( i \), so that \( K \cap A_{i+1} = 0 \). Furthermore, as \( A_i \) and \( A_{i+1} \) are both automorphism invariant, every image \( K' \) of \( K \) under \( \text{Aut} A \) also meets \( A_i \) nontrivially and \( A_{i+1} \) trivially.

We assert \( A_{i+1} \) is codimension \( > 1 \) in \( A_i \). If it were codimension 1, it would be the kernel of some character on \( A_i \), which could be extended to a character \( \chi \) of \( A \). This character would be nontrivial on every \( \text{Aut} A \)-image \( K' \) of \( K \), since they all meet \( A_i \) nontrivially outside of \( A_{i+1} \). This contradicts the assumption that \( K \) fulfills condition \( 1 \) for \( A \), so we conclude \( A_{i+1} \) is codimension \( > 1 \) in \( A_i \).

We claim this in turn implies that at least two of \( A \)'s cyclic factors are isomorphic. Indeed, the dimension of \( A_i \) is the number of cyclic factors of \( A \) of order at least \( p^{i+1} \). That \( A_{i+1} \) is codimension at least two in \( A_i \), thus implies that the number of cyclic factors of order at least \( p^{i+1} \) is at least two greater than the number of cyclic factors of order at least \( p^{i+2} \). This implies that there are at least two cyclic factors of order exactly \( p^{i+1} \). This establishes \( 1 \Rightarrow 2 \).

\textbf{Proof of Theorem 6.2} In this situation, by Lemma 6.8, for every subgroup \( K \) of \( A = [G, G] \), there is a character \( \chi \) of \( A \) whose kernel does not contain any image of \( K \) under \( \text{Aut} A \), so \( A \) fulfills the hypothesis of Proposition 6.6 for any possible \( Q \)-action.

It remains to prove our claim about \( \text{AGL}(1, q) \). We fix notation: Let \( G = \text{AGL}(1, q) \). Then \( G = A \rtimes Q \) where \( A \) is isomorphic to the additive and \( Q \) to the multiplicative group of \( \mathbb{F}_q \). The commutator subgroup \( [G, G] \) is equal to \( A \), so these labels are consistent with those used throughout the section. We identify \( G \) as a group of permutations of the elements of \( \mathbb{F}_q \), with \( A \) being the translations by \( x \in \mathbb{F}_q \) and \( Q \) being the multiplications by \( a \in \mathbb{F}_q^\times \). Denote the former by \( t_x \), and the latter by \( m_a \).

\textbf{Proof of Theorem 6.1} Recall that \( q = p^k \) is an odd, composite prime power.

\footnote{In fact, if \( \lambda = (\lambda_1, \ldots, \lambda_r) \) with \( \lambda_1 \geq \cdots \geq \lambda_r \) is the partition describing the type of \( A \), so that \( A \cong \prod_{j} C_{p^j} \), then the tuple \( (\dim A_0, \ldots, \dim A_{k-1}) \) is the conjugate partition \( \lambda' \).}
By [8, Theorem 6.1], the only irreducible representation of \( G = A\text{GL}(1, q) \) of degree greater than 1 is the unique representation induced from any nontrivial character of \( A \). Its restriction to \( A \) is the sum of all nontrivial characters of \( A \). Call it \( W \). All \( G \)'s hope of being PNC lies with \( W \).

Consider the character \( \chi_W \) of \( W \). Since \( A \) is normal, \( \chi_W \) is zero outside of \( A \). On \( A \), as it is the sum of all nontrivial characters of \( A \), it is one less than the sum of all characters, which is the character of the regular representation. Thus

\[
\chi_W(1) = |A| - 1 = q - 1,
\]

and

\[
\chi_W(t_x) = 0 - 1 = -1
\]

for all \( x \neq 0 \) in \( \mathbb{F}_q \).

Let \( D \) be the subgroup generated by \( m_{-1} \) and any \( t_x \) with \( x \neq 0 \). Because \( q \) is odd, \( m_{-1} \) is the negation map on \( \mathbb{F}_q \). Therefore \( D \) is a dihedral group of order \( 2p \), where \( p \) is the characteristic of \( \mathbb{F}_q \). We now show \( W \) is not PNC for \( D \).

The rotation subgroup of \( D = \langle t_x, m_{-1} \rangle \) is \( \langle t_x \rangle \), of order \( p \). Thus the sum of \( \chi_W \) over the rotation subgroup of \( D \) is \( 1(q - 1) + (p - 1)(-1) = q - p \). Since \( q = p^k \) with \( k > 1 \), this is nonzero, and we can conclude from Lemma 2.5 that \( W \) is not PNC for \( D \). This concludes the argument. \( \square \)

## 7 Permutations that noncommute purely in the standard representation

Although the symmetric group \( S_n \) is not PNC for \( n \geq 4 \) (by Proposition 3.6 and Lemma 4.1, since it contains \( S_4 \)), we may still be interested, for a given pair of permutations \( x, y \in S_n \), whether they noncommute purely in a given representation. Recall that the standard representation \( \text{Sta} \) of \( S_n \) is the defining permutation representation minus the trivial representation. It can be realized as the action of \( S_n \) on the sum-zero subspace \( T = \{ \sum c_i e_i \mid \sum c_i = 0 \} \) of \( \mathbb{C}^n \) through permutations of the standard basis vectors \( e_i \).

Here we give a group-theoretic characterization of when this representation is PNC for a given \( x, y \):

**Proposition 7.1** Two permutations \( x, y \in S_n \) noncommute purely in the standard representation \( \text{Sta} \) of \( S_n \) if and only if the subgroup \( H = \langle x, y \rangle \) they generate has both of the following properties:

1. The action of \( H \) on the indices \( \{1, \ldots, n\} \) is transitive.
2. Some point stabilizer \( H_j \), for \( j \in \{1, \ldots, n\} \), meets every coset of the commutator subgroup \( [H, H] \) in \( H \).

**Proof** The defining permutation representation of a permutation group contains one trivial representation for every orbit. Since the standard representation is the defining
permutation representation minus one trivial representation, it contains \( o - 1 \) trivial representations, where \( o \) is the number of orbits. Thus \( \text{Sta}|_H \) contains no trivial representations if and only if \( o = 1 \), i.e., if and only if \( H \) is transitive. Thus failure of property 1 implies \( x, y \) do not noncommute purely, and it remains to show that property 2 is equivalent to noncommuting purely in the presence of property 1.

Thus, we may henceforth assume property 1 holds, i.e. \( H \) is transitive. In this situation, all point stabilizers \( H_j \) are conjugate, and property 2 is equivalent to the statement that \( H_1 \) meets every coset of \([H, H]\) in \( H \). So we need to show that this is equivalent to \( \text{Sta} \) being PNC for \( H \).

Realize \( \text{Sta}|_H \) as \( H \)'s action on the sum-zero subspace \( T \subset \mathbb{C}^n \). Suppose \( v \in T \) lies in a common eigenspace of \( x \) and \( y \). Then \( H \) acts by a one-dimensional representation \( \chi: H \to \mathbb{C}^\times \) on \( v \), i.e., \( vh = \chi(h)v \) for all \( h \in H \). If

\[
v = \sum c_i e_i,
\]

then this implies \( c_1 = c_j \chi(h) \), where \( j \) is the image of 1 under \( h \), by matching the \( j \)th coordinates in \( vh = \chi(h)v \). For \( h \in H_1 \) so that \( j = 1 \), this equation becomes \( c_1 = c_1 \chi(h) \). It follows that either \( c_1 = 0 \), or else \( \chi(h) = 1 \) for all \( h \in H_1 \).

Now suppose property 2 holds. Then there is an element of \( H_1 = \{ h \in H \mid e_1 h = e_1 \} \) in each coset of \([H, H]\) in \( H \). Since \( \chi \) is one-dimensional, it factors through the abelianization \( H/[H, H] \), thus its value is constant on cosets of \([H, H]\). It follows that either \( c_1 = 0 \), or else \( \chi(h) = 1 \) for all \( h \in H \). However, both of these conclusions are incompatible with property 1 (unless \( v = 0 \)). The logic of the first paragraph shows that property 1 does not allow \( H \) to have a trivial subrepresentation, ruling out \( \chi(h) = 1 \), for all \( h \in H \), unless \( v = 0 \). Meanwhile, property 1 means that for all \( j \in \{1, \ldots, n\} \), there is an \( h \in H \) sending 1 to \( j \), whereupon \( c_1 = c_j \chi(h) \) then implies \( c_j = 0 \), so this case is also impossible unless \( v = 0 \). This shows that properties 1 and 2 preclude a common eigenvector for \( x \) and \( y \): they noncommute purely.

Conversely, if property 2 fails in the presence of propert 1, then there exists a coset \( F \) of \([H, H]\) in \( H \) that does not contain any elements of \( H_1 \). Since all point stabilizers \( H_j \) are conjugate, and since any two conjugate elements lie in the same coset of \([H, H]\), it follows that \( F \) consists entirely of fixed-point-free permutations. The value of the character of \( \text{Sta} \) is \(-1\) on fixed-point-free permutations; thus the sum of \( \text{Sta}|_H \) across \( F \) is negative. Thus \( \text{Sta} \) is not PNC for \( H \) by Lemma 6.4; i.e., \( x, y \) do not noncommute purely in \( \text{Sta} \).

\[ \square \]

**8 Further questions**

This section collects directions for further inquiry.

**8.1 A classification-free proof for simple groups?**

Our proof of Theorem 4.2 rests on the classification theorem for finite simple groups, and thus comes down to case analyses: one, by Barry and Ward [1], which shows that
nonabelian finite simple groups contain minimal simple groups, and another, above in Sect. 4, which shows that minimal simple groups are not PNC. Nonetheless, the case-by-case reasoning used was rather repetitive. This suggests there may be a uniform proof.

*Question 8.1* Is there a uniform, classification-free, proof of of Theorem 4.2?

### 8.2 Metabelian groups

The results of Sect. 6 reveal the metabelian case to be richer than we initially expected. As was mentioned in that section, the sufficient criterion Theorem 6.2 for a metabelian group to be PNC is not close to necessary. A clearer picture of the PNC property for metabelian groups would be desirable.

*Question 8.2* Is there a clean if-and-only-if characterization of metabelian groups with the PNC property?

### 8.3 Other important representations

Proposition 7.1 characterizes pairs of purely noncommuting elements group-theoretically in the case of one particularly important representation of one particularly important group. We can ask the same question about other groups and other representations. For example:

*Question 8.3* Is there an analogous result to Proposition 7.1 for the principal series representations of PSL$(2, q)$?

### 8.4 Abelian singularities

As discussed in the introduction, one motivation for studying the PNC and SPNC properties comes from the search for groups acting on smooth varieties such that the quotients have at worst abelian singularities. But the PNC and SPNC properties are not close to necessary for this.

First, if a group $G$ has an SPNC central extension $\tilde{G}$ realized by a representation $V$, then the action of $\tilde{G}$ on $\mathbb{P}(V)$ factors through $G$, and point stabilizers in $G$ are quotients of point stabilizers in $\tilde{G}$. So in this case $G$ too acts on $\mathbb{P}(V)$ with abelian point stabilizers, so $\mathbb{P}(V)/G$ has at worst abelian singularities. For example the alternating group $A_5$ is not PNC, but its double cover $\tilde{A}_5$, the binary icosahedral group, has an SPNC action on $\mathbb{C}^2$, so $A_5$ has an action on $\mathbb{P}^1$ with abelian stabilizers. There is a similar statement for PNC central extensions.

Secondly, due to the Chevalley–Shepard–Todd theorem, if the point stabilizers contain pseudoreflections, it is not actually necessary for them to be abelian in order for the resulting quotient singularities to be (at worst) abelian. For example, although neither $A_5$ nor PSL$(2, 7)$ are PNC, if $V$ is a three-dimensional faithful irreducible representation of PSL$(2, 7)$ over $\mathbb{C}$, then $\mathbb{P}(V)/\text{PSL}(2, 7)$ has only abelian singularities, while if $W$ is a three-dimensional faithful irreducible representation of $A_5$, then $\mathbb{P}(W)/A_5$ is actually smooth. Details are found in [2, Sect. 1.2]. This prompts us to ask:
**Question 8.4** For which nonabelian finite simple groups $G$ is there a faithful representation $V$ such that $\mathbb{P}(V)/G$ has at worst abelian singularities?

Of course, one can broaden this question beyond simple groups.

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