Measuring processes and the Heisenberg picture

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Abstract In this paper, we attempt to establish quantum measurement theory in the Heisenberg picture. First, we review foundations of quantum measurement theory, that is usually based on the Schrödinger picture. The concept of instrument is introduced there. Next, we define the concept of system of measurement correlations and that of measuring process. The former is the exact counterpart of instrument in the (generalized) Heisenberg picture. In quantum mechanical systems, we then show a one-to-one correspondence between systems of measurement correlations and measuring processes up to complete equivalence. This is nothing but a unitary dilation theorem of systems of measurement correlations. Furthermore, from the viewpoint of the statistical approach to quantum measurement theory, we focus on the extendability of instruments to systems of measurement correlations. It is shown that all completely positive (CP) instruments are extended into systems of measurement correlations. Lastly, we study the approximate realizability of CP instruments by measuring processes within arbitrarily given error limits.

1 Introduction

In this paper, we mathematically investigate measuring processes in the Heisenberg picture. We aim to extend the framework of quantum measurement theory and to apply the method in this paper not only to quantum systems of finite degrees of freedom but also to those with infinite degrees of freedom.

It is well-known that correlation functions are essential for the description of systems in quantum theory and in quantum probability theory. Typical examples are Wightman functions in (axiomatic) quantum field theory and several (algebraic, noncommutative) independence in quantum probability theory (see [19, 25].
and references therein), which are characterized by behaviors of correlation functions. In the famous paper by Accardi, Frigerio and Lewis [1], general classes of quantum stochastic processes including quantum Markov processes were characterized in terms of correlation functions. The main result of [1] is a noncommutative version of Kolmogorov’s theorem stating that a quantum stochastic process can be reconstructed from a family of correlation functions up to equivalence. The proof of this result is made by the efficient use of positive-definiteness of a family of correlation functions. Later Belavkin [6] formulated the theory of operator-valued correlation functions, which is more flexible than the original formulation in [1] and gives an opportunity for reconsidering the standard formulation of quantum theory. And he extended the main result of [1]. We apply Belavkin’s theory, with some modifications, to a systematic characterization of measurement correlations in this paper.

Measurements are described by the notion of instrument introduced Davies and Lewis [11]. An instrument $I$ for $(\mathcal{M}, S)$ is defined as a $P(\mathcal{M}^\ast)$-valued measure $\mathcal{F} \ni \Delta \mapsto I(\Delta) \in P(\mathcal{M}^\ast)$, where $\mathcal{M}$ is a von Neumann algebra with predual $\mathcal{M}^\ast$, $P(\mathcal{M}^\ast)$ is the set of positive linear map of $\mathcal{M}$ and $(S, \mathcal{M})$ is a measurable space. The statistical description of measurements in terms of instruments can be regarded as a kind of quantum dynamical process based on the so-called Schrödinger picture. As widely accepted, the Schrödinger picture stands on describing states as time-dependent variables and observables as constants with respect to time, i.e., time-independent variables while we treat states as constants with respect to time and observables as time-dependent variables the Heisenberg picture. To the author’s knowledge, the Schrödinger picture is matched with an operational approach to quantum theory concerning probability distributions of observables and of output variables of apparatuses [11, 10, 9, 34]. On the other hand, no systematic treatment of measurements in the Heisenberg picture, which can compare with the theory of instruments, has been investigated. In contrast to the Schrödinger picture, the Heisenberg picture focuses on dynamical changes of observables and can naturally treat correlation functions of observables at different times, so that enables us to examine the dynamical nature of the system under consideration itself in detail. The Heisenberg picture is better than the Schrödinger picture at this point. Therefore, inspired by the previous investigations on quantum stochastic processes and correlation functions [1, 6], we define a system of measurement correlations. This is the exact counterpart of instrument in a “generalized” Heisenberg picture and defined as a family of multilinear maps satisfying “positive-definiteness”, “$\sigma$-additivity” and other conditions. An instrument induced by a system of measurement correlations is always completely positive. In addition, we redefine measuring process (Definition 9) in order that it becomes consistent with the definition of system of measurement correlations. In the quantum mechanical case, we show that every system of measurement correlations is defined by a measuring process. It is, however, difficult to extend this result to general von Neumann algebras. Therefore, we develop another aspect of measurements, which is deeply analyzed for the first time in this paper. From the statistical viewpoint as the starting point of quantum measurement theory, we discuss the extendability of CP instruments to systems of measurement correla-
tions and the realizability of CP instruments by measuring processes. In physically relevant cases, we show that both are possible within arbitrary accuracy.

Mathematically speaking, the purpose of this paper is to develop the unitary dilation theory of systems of measurement correlations and of CP instruments. Dilation theory is one of main topics in functional analysis and enables us to apply representation theory and harmonic analysis to operators or to operator algebras. Especially, the unitary dilation theory of contractions on Hilbert space \([26, 18]\) and the dilation theory of completely positive maps \([4, 44]\) have been studied in many investigations (see \([26, 18, 39, 4, 44, 13, 20, 21, 42, 43]\) and references therein). A representation theorem of CP instruments on the set \(B(H)\) of bounded operators on a Hilbert space \(H\) \([31, \text{Theorem 5.1}]\) (Theorem \(1\)) follows from these results, which shows the existence of unitary dilations of CP instruments. The proof of this theorem is based on the theory of CP-measure space \([28, 29]\). In the case of CP instruments, a unitary dilation of a CP instrument is nothing but a measuring process which realizes it. We generalize this representation theorem to systems of measurement correlations defined on \(B(H)\) in terms of Kolmogorov’s theorem. It should be remarked that CP instruments defined on general von Neumann algebras do not always admit unitary dilations (see Examples \(1\) and \(2\)). Next, we consider the extendability of CP instruments to systems of measurement correlations. It will be shown that all CP instruments can be extended into systems of measurement correlations. Furthermore, we show that every CP instrument defined on general von Neumann algebras can be approximated by measuring processes within arbitrarily given error limits \(\varepsilon > 0\). If von Neumann algebras are injective or injective factors, measuring processes approximating a CP instrument can be chosen to be faithful or inner, respectively.

Preliminaries are given in Section \(2\). Foundations of quantum measurement theory, kernels and their Kolmogorov decompositions are explained. We introduce a system of measurement correlations and prove a representation theorem of systems of measurement correlations in Section \(3\). In Section \(4\), we define measuring processes and their complete equivalence, and in the case of \(B(H)\) we show a unitary dilation theorem of systems of measurement correlations establishing a one-to-one correspondence between systems of measurement correlations and complete equivalence classes of measuring processes. In Section \(5\), we discuss a generalization of the main result in Section \(4\) to arbitrary von Neumann algebras, and the extendability of CP instruments to systems of measurement correlations. We show that for any CP instruments there always exists a systems of measurement correlations which defines a given CP instrument. In Section \(6\), we explore the existence of measuring processes which approximately realizes a given CP instrument. We show several approximate realization theorems of CP instruments by measuring processes.
2 Preliminaries

In this paper, we assume that von Neumann algebras $\mathcal{M}$ are $\sigma$-finite. However, only in the case of $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the von Neumann algebra of bounded operators on a Hilbert space $\mathcal{H}$, this assumption is ignored.

2.1 Foundations of quantum measurement theory

We introduce foundations of quantum measurement theory here. To precisely understand the theory of quantum measurement and its mathematics, the most important thing is to know how measurements physically realizable in experimental settings are described as physical processes consistent with statistical characterization of measurements. We refer the reader to [35, 37, 29] for detailed introductory expositions of quantum measurement theory.

The history of quantum measurement theory is long as much as those of quantum theory, but the modern theory of quantum measurement began with the mathematical study of the notion of instruments introduced by Davies and Lewis [11]. They proposed that we should abandon the repeatability hypothesis [27, 11, 38] as a general principle and employ an operational approach to quantum measurement, which is based on the mathematical description of measurements in terms of instruments defined as follows. Let $\mathcal{S}$ be a system whose observables and states are described by self-adjoint operators affiliated to a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ and by normal states on $\mathcal{M}$, respectively. We denote by $\mathcal{M}^*$ the predual of $\mathcal{M}$, i.e., the set of ultraweakly continuous linear functionals on $\mathcal{M}$, by $\mathcal{S}_n(\mathcal{M})$ the set of normal states on $\mathcal{M}$ and by $\mathcal{P}(\mathcal{M}^*)$ the set of positive linear maps on $\mathcal{M}^*$.

**Definition 1 (Instrument, Davies-Lewis [11, p.243, ll.21–26]).** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(\mathcal{S}, \mathcal{F})$ a measurable space. A map $I: \mathcal{F} \to \mathcal{P}(\mathcal{M}^*)$ is called an instrument for $(\mathcal{M}, \mathcal{S})$ if it satisfies the following conditions:

1. $\|I(S)\| = \|\rho\|$ for all $\rho \in \mathcal{M}_*$;
2. For every $M \in \mathcal{M}$, $\rho \in \mathcal{M}_*$ and mutually disjoint sequence $\{\Delta_j\}$ of $\mathcal{F}$,
   \[ \langle I(\cup_j \Delta_j)\rho, M \rangle = \sum_j \langle I(\Delta_j)\rho, M \rangle, \]

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $\mathcal{M}_*$ and $\mathcal{M}$.

We define the dual map $I^*$ of an instrument $I$ by $\langle \rho, I^*(\Delta)M \rangle = \langle I(\Delta)\rho, M \rangle$ and use the notation $I(M, \Delta) = I^*(\Delta)M$ for all $\Delta \in \mathcal{F}$ and $M \in \mathcal{M}$.

It is obvious, by the definition, that an instrument describes the weighted state changes caused by the measurement. The dual map $I: \mathcal{M} \times \mathcal{F} \to \mathcal{M}$ of an instrument $I$ for $(\mathcal{M}, \mathcal{S})$ is characterized by the following conditions:

(i) For every $\Delta \in \mathcal{F}$, the map $\mathcal{M} \ni M \mapsto I(M, \Delta) \in \mathcal{M}$ is normal positive linear
map of $\mathcal{M}$;

(ii) $\mathcal{I}(1, S) = 1$;

(iii) For every $M \in \mathcal{M}, \rho \in \mathcal{M}_s$ and mutually disjoint sequence $\{\Delta_j\}$ of $\mathcal{F}$,\

$$\langle \rho, \mathcal{I}(M, \Delta_j) \rangle = \sum \langle \rho, \mathcal{I}(M, \Delta_j) \rangle.$$  

Since a map $\mathcal{I} : \mathcal{M} \times \mathcal{F} \to \mathcal{M}$ satisfying the above conditions is always the dual map of an instrument for $(\mathcal{M}, S)$, we also call the map $\mathcal{I}$ an instrument for $(\mathcal{M}, S)$.

Davies and Lewis claimed that experimentally and statistically accessible ingredients via measurements by a given measuring apparatus should be specified by instruments as follows:

**Davies-Lewis proposal** For every apparatus $A(x)$ measuring $S$, where $x$ is the output variable of $A(x)$ taking values in a measurable space $(S, \mathcal{F})$, there always exists an instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ corresponding to $A(x)$ in the following sense. For every input state $\rho$, the probability distribution $\text{Pr}\{x \in \Delta\} \text{ of } x$ in $\rho$ is given by 

$$\text{Pr}\{x \in \Delta\} = \|\mathcal{I}(\Delta)\rho\| = \langle \mathcal{I}(\Delta)\rho, 1 \rangle \quad (3)$$

for all $\Delta \in \mathcal{F}$, and the state $\rho_{\{x \in \Delta\}}$ after the measurement under the condition that $\rho$ is the prepared state and the outcome $x \in \Delta$ is given by

$$\rho_{\{x \in \Delta\}} = \frac{\mathcal{I}(\Delta)\rho}{\|\mathcal{I}(\Delta)\rho\|} \quad (4)$$

if $\text{Pr}\{x \in \Delta\} > 0$, and $\rho_{\{x \in \Delta\}}$ is indefinite if $\text{Pr}\{x \in \Delta\} = 0$.

Although this proposal is very general, it was not evident at that time that how this is related to the standard formulation of quantum theory. In the 1980s, Ozawa [30, 31] introduced both completely positive (CP) instruments and measuring processes. Following this investigation, the standpoint of the above proposal in quantum mechanics was settled and the circumstances changed at all. An instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ is said to be completely positive if $\mathcal{I}(\Delta)$ (or $\mathcal{I}(\cdot, \Delta)$, equivalently) is completely positive for every $\Delta \in \mathcal{F}$. We denote by $\text{CPInst}(\mathcal{M}, S)$ the set of CP instruments for $(\mathcal{M}, S)$. The notion of measuring process is defined as a quantum mechanical modeling of an apparatus as a physical system, of the meter of the apparatus, and of the measuring interaction between the system and the apparatus. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We denote by $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$ and by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the set of bounded linear operators of $\mathcal{H}$ to $\mathcal{K}$. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras on $\mathcal{H}$ and $\mathcal{K}$, respectively. We denote by $\mathcal{M} \otimes \mathcal{N}$ the $W^*$-tensor product of $\mathcal{M}$ and $\mathcal{N}$. For every $\sigma \in \mathcal{N}$, a linear map $\text{id} \otimes \sigma : \mathcal{M} \otimes \mathcal{N} \to \mathcal{M}$ is defined by $\langle \rho, (\text{id} \otimes \sigma)X \rangle = \langle \rho, \sigma X \rangle$ for all $X \in \mathcal{M} \otimes \mathcal{N}$ and $\rho \in \mathcal{M}_s$. The following is the mathematical definition of measuring processes:

**Definition 2 (Measuring process [29, Definition 3.2])**. A measuring process for $(\mathcal{M}, S)$ is a 4-tuple $\mathbb{M} = (\mathcal{H}, \sigma, E, U)$ of a Hilbert space $\mathcal{H}$, a normal state $\sigma$ on $\mathcal{B}(\mathcal{K})$, a PVM $E : \mathcal{F} \to \mathcal{B}(\mathcal{K})$ and a unitary operator $U$ on $\mathcal{H} \otimes \mathcal{K}$ satisfying
Let $\mathcal{M} = (\mathcal{H}, \sigma, E, U)$ be a measuring process for $(\mathcal{M}, S)$. Then a CP instrument $\mathcal{I}_M$ for $(\mathcal{M}, S)$ is defined by

$$\mathcal{I}_M(M, \Delta) = (\id \otimes \sigma)[U^*(M \otimes E(\Delta))U]$$

for $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$. The most important example of measuring processes is a von Neumann model $(L^2(\mathbb{R}), \omega_\alpha, E_\alpha, e^{-iA_\alpha P_\alpha R})$ of measurement of an observable $A$, a self-adjoint operator affiliated with $\mathcal{M}$, where $\alpha$ is a unit vector of $L^2(\mathbb{R})$, $\omega_\alpha$ is defined by $\omega_\alpha(M) = \langle \alpha | M \alpha \rangle$ for all $M \in \mathcal{B}(L^2(\mathbb{R}))$, and $Q = \int_\mathbb{R} q \, dE_\alpha(q)$ and $P$ are self-adjoint operators defined on dense linear subspaces of $L^2(\mathbb{R})$ such that $[Q, P] = ih1$. Here, we denote by $E_\alpha^X$ the spectral measure of a self-adjoint operator $X$ densely defined on a Hilbert space. Quantum mechanical modeling of apparatuses began with this model [27] [33].

Two measuring processes are statistically equivalent if they define an identical instrument. As seen above, a measuring process $\mathcal{M}$ for $(\mathcal{M}, S)$ defines a CP instrument $\mathcal{I}_M$ for $(\mathcal{M}, S)$. In the case of $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the following theorem, a unitary dilation theorem of CP instruments for $(\mathcal{B}(\mathcal{H}), S)$, is known to hold.

**Theorem 1** ([30], [31] Theorem 5.1), [29] Theorem 3.6]. For every CP instrument $\mathcal{I}$ for $(\mathcal{B}(\mathcal{H}), S)$, there uniquely exists a statistical equivalence class of measuring processes $\mathcal{M} = (\mathcal{H}, \sigma, E, U)$ for $(\mathcal{B}(\mathcal{H}), S)$ such that $\mathcal{I}(M, \Delta) = \mathcal{I}_M(M, \Delta)$ for all $M \in \mathcal{B}(\mathcal{H})$ and $\Delta \in \mathcal{F}$. Conversely, every statistical equivalence class of measuring processes for $(\mathcal{B}(\mathcal{H}), S)$ defines a unique CP instrument $\mathcal{I}$ for $(\mathcal{B}(\mathcal{H}), S)$.

A generalization of this theorem to arbitrary von Neumann algebras is shown to hold not for all CP instruments but for those with the normal extension property (NEP) in [29] (see Theorem 2). Let $(S, \mathcal{F}, \mu)$ be a measure space. We denote by $\mathcal{L}(S, \mu)$ the $*$-algebra of complex-valued $\mu$-measurable functions on $S$. A $\mu$-measurable function $f$ is said to be $\mu$-negligible if $f(s) = 0$ for $\mu$-a.e. $s \in S$. We denote by $\mathcal{M}(S, \mu)$ the set of $\mu$-negligible functions on $S$ and by $M^w(S, \mu)$ the $*$-algebra of bounded $\mu$-measurable functions on $S$. It is obvious that $M^w(S, \mu) \subseteq \mathcal{L}(S, \mu)$ as $*$-algebra. For any $1 \leq p < \infty$, we denote by $L^p(S, \mu)$ the Banach space of $p$-integrable functions on $S$ with respect to $\mu$ modulo the $\mu$-negligible functions. We denote by $[f]$ the $\mu$-negligible equivalence class of $f \in \mathcal{L}(S, \mu)$ and by $L^w(S, \mu)$ the commutative von Neumann algebra on $L^2(S, \mu)$. We denote by $L^w(S, \mathcal{F})$ the $W^*$-algebra of essentially bounded $\mathcal{F}$-measurable functions on $S$ modulo the $\mathcal{F}$-negligible functions.

**Definition 3** (Normal extension property [29] Definition 3.3). Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space. Let $\mathcal{I}$ be a CP instrument for $(\mathcal{M}, S)$ and $\Psi_\mathcal{I} : \mathcal{M} \otimes_{\min} L^w(S, \mathcal{F}) \to \mathcal{M}$ the corresponding unital binormal CP map, i.e., $\Psi_\mathcal{I}$ is normal on $\mathcal{M}$ and $L^w(S, \mathcal{F})$ and satisfies $\Psi_\mathcal{I}(M \otimes [\chi_\Delta]) = \mathcal{I}(M, \Delta)$ for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$ [29] Proposition 3.3]. $\mathcal{I}$ is said to have the normal extension property (NEP) if there exists a unital normal CP
map \( \Psi_f : \mathcal{M} \otimes L^\infty(S, \mathcal{I}) \rightarrow \mathcal{M} \) such that \( \Psi_f |_{\mathcal{M} \otimes \Lambda(S, \mathcal{I})} = \Psi_f \). We denote by \( \text{CPInst}_{\text{NE}}(\mathcal{M}, S) \) the set of CP instruments for \((\mathcal{M}, S)\) with the NEP.

We then have the following theorem, a generalization of Theorem [1].

**Theorem 2 ([29, Theorem 3.4]).** For a CP instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\), the following conditions are equivalent:

1. \( \mathcal{I} \) has the NEP.
2. There exists CP instrument \( \mathcal{I} \) for \((\mathcal{B}(\mathcal{H}), S)\) such that \( \mathcal{I}(M, \Delta) = \mathcal{I}(M, \Delta) \) for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \).
3. There exists a measurable process \( \mathcal{M} = (\mathcal{H}, \sigma, E, U) \) for \((\mathcal{M}, S)\) such that \( \mathcal{I}(M, \Delta) = \mathcal{I}_{\mathcal{M}}(M, \Delta) \) for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \).

It is also shown that all CP instruments defined on a von Neumann algebra \( \mathcal{M} \) have the NEP, i.e., \( \text{CPInst}(\mathcal{M}, S) = \text{CPInst}_{\text{NE}}(\mathcal{M}, S) \) if \( \mathcal{M} \) is atomic [29, Theorem 4.1]. We should remember the famous fact that a von Neumann algebra \( \mathcal{M} \) on a Hilbert space \( \mathcal{H} \) is atomic if and only if there exists a normal conditional expectation \( \mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M} \) [46, Chapter V, Section 2, Exercise 8]. Then the following question naturally arises.

**Question 1.** Let \( \mathcal{M} \) be a non-atomic von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((S, \mathcal{F})\) a measurable space. Are there CP instruments for \((\mathcal{M}, S)\) without the NEP? For any CP instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\), does there exist a measuring process \( \mathcal{M} \) for \((\mathcal{M}, S)\) which realizes \( \mathcal{I} \) within arbitrarily given error limits \( \varepsilon > 0 \)?

A CP instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) is said to have the approximately normal extension property (ANEP) if there is a net \( \{ \mathcal{I}_\alpha \} \) of CP instruments with the NEP such that \( \mathcal{I}_\alpha(M, \Delta) \) is ultraweakly converges to \( \mathcal{I}(M, \Delta) \) for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \). We denote by \( \text{CPInst}_{\text{AN}}(\mathcal{M}, S) \) the set of CP instruments for \((\mathcal{M}, S)\) with the ANEP.

Contrary to physicists’ expectations, Question [1] was positively resolved in [29, Section V] for non-atomic but injective von Neumann algebras.

**Definition 4 ([29, Definition 5.3]).** (1) An instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) is called repeatable if \( \mathcal{I}(\Delta_2, \mathcal{I}(\Delta_1)) = \mathcal{I}(\Delta_2 \cap \Delta_1) \) for all \( \Delta_1, \Delta_2 \in \mathcal{F} \).

(2) An instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) is called weakly repeatable if \( \mathcal{I}(\mathcal{I}(1, \Delta_2), \Delta_1) = \mathcal{I}(1, \Delta_2 \cap \Delta_1) \) for all \( \Delta_1, \Delta_2 \in \mathcal{F} \).

(3) An instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) is called discrete if there exist a countable subset \( S_0 \) of \( S \) and a map \( T : S_0 \rightarrow P(\mathcal{M}, s) \) such that

\[
\mathcal{I}(\Delta) = \sum_{s \in \Delta} T(s)
\]

for all \( \Delta \in \mathcal{F} \).

**Proposition 1 ([29, Proposition 5.9]).** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((S, \mathcal{F})\) a measurable space. Every discrete CP instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) has the NEP.
Theorem 3 ([29, Theorem 5.10]). Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a standard Borel space. A weakly repeatable CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ is discrete if and only if it has the NEP.

In the case where $\mathcal{M}$ is non-atomic, there exist CP instruments for $(\mathcal{M}, S)$ without the NEP. The following two CP instruments are such examples.

Example 1 ([12, pp. 292–293], [29, Example 5.1]). Let $m$ be Lebesgue measure on $[0, 1]$. A CP instrument $\mathcal{I}_m$ for $(L^\infty([0, 1], m), [0, 1])$ is defined by $\mathcal{I}_m(f, \Delta) = \chi_{\Delta} f$ for all $\Delta \in \mathcal{B}([0, 1])$ and $f \in L^\infty([0, 1], m)$.

A von Neumann algebra $\mathcal{M}$ is said to be approximately finite-dimensional (AFD) if there is an increasing net $\{\mathcal{M}_\alpha\}_{\alpha \in A}$ of finite-dimensional von Neumann subalgebras of $\mathcal{M}$ such that

$$\mathcal{M} = \bigcup_{\alpha \in A} \mathcal{M}_\alpha^\text{ow}. \quad (8)$$

Example 2 ([29, Example 5.2]). Let $\mathcal{M}$ be an AFD von Neumann algebra of type $\text{II}_1$ on a separable Hilbert space $\mathcal{H}$. Let $A = \int_B dE^A(a)$ be a self-adjoint operator with continuous spectrum affiliated with $\mathcal{M}$ and $\mathcal{E}$ a (normal) conditional expectation of $\mathcal{M}$ onto $\{A\}' \cap \mathcal{M}$ (the existence of $\mathcal{E}$ was first found by [38, Theorem 1]), where $\{A\}' = \{E^A(\Delta) \mid \Delta \in \mathcal{B}(\mathbb{R})\}'$. A CP instrument $\mathcal{I}_A$ for $(\mathcal{M}, \mathbb{R})$ is defined by

$$\mathcal{I}_A(M, \Delta) = \mathcal{E}(M)E^A(\Delta) \quad (9)$$

for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{B}(\mathbb{R})$.

By Theorem 3 the weak repeatability and the non-discreteness of $\mathcal{I}_m$ and $\mathcal{I}_A$ imply the non-existence of measuring processes which define them. These examples are very important for the dilation theory of CP maps since they revealed the existence of families of CP maps which do not admit unitary dilations.

The following theorem holds for general $\sigma$-finite von Neumann algebras without assuming any other conditions.

Theorem 4. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space. For every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$, $n \in \mathbb{N}$, $\rho_1, \cdots, \rho_n \in \mathcal{I}_n(\mathcal{M})$, $M_1, \cdots, M_n \in \mathcal{M}$ and $\Delta_1, \cdots, \Delta_n \in \mathcal{F}$, there exists a measuring process $\mathfrak{M} = (\mathcal{H}, \mathcal{E}, E, U)$ for $(\mathcal{M}, S)$ such that

$$\langle \rho_j, \mathcal{I}(M_j, \Delta_j) \rangle = \langle \rho_j, \mathcal{I}_{\mathfrak{M}}(M_j, \Delta_j) \rangle \quad (10)$$

for all $j = 1, \cdots, n$.

Proof. Let $n \in \mathbb{N}$, $\rho_1, \cdots, \rho_n \in \mathcal{I}_n(\mathcal{M})$, $M_1, \cdots, M_n \in \mathcal{M} \setminus \{0\}$ and $\Delta_1, \cdots, \Delta_n \in \mathcal{F} \setminus \{\emptyset\}$. Let $\mathcal{F}'$ be a $\sigma$-subfield of $\mathcal{F}$ generated by $\Delta_1, \cdots, \Delta_n, S$. Let $\{\Gamma_i\}_{i=1}^m \subseteq \mathcal{F}' \setminus \{\emptyset\}$ be a maximal partition of $S$, i.e., $\{\Gamma_i\}_{i=1}^m$ satisfies the following conditions:

1. For every $i = 1, \cdots, m$, if $\Delta \in \mathcal{F}'$ satisfies $\Delta \subseteq \Gamma_i$, then $\Delta$ is $\Gamma_i$ or $\emptyset$;
2. $\bigcup_{i=1}^m \Gamma_i = S$.
We fix $s_1, \ldots, s_m \in S$ such that $s_i \in \Gamma_i$ for all $i = 1, \ldots, m$. We define a discrete CP instrument $I'$ for $(\mathcal{M}, S)$ by

$$I'(M, \Delta) = \sum_{j=1}^{m} \delta_s(\Delta) I(M, \Gamma_j)$$

for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$. It is then obvious that $I'$ satisfies

$$\langle \rho_j, I(M_j, \Delta_j) \rangle = \langle \rho_j, I'(M_j, \Delta_j) \rangle$$

for all $j = 1, \ldots, n$. By Proposition 4 there exists a measuring process $M = (\mathcal{K}, \sigma, E, U)$ for $(\mathcal{M}, S)$ such that $I'(M, \Delta) = \mathcal{I}_{\mathcal{M}}(M, \Delta)$ for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$. The proof is complete.

**Corollary 1.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(S, \mathcal{F})$ a measurable space. Then we have

$$\text{CPInst}_{\text{AN}}(\mathcal{M}, S) = \text{CPInst}(\mathcal{M}, S).$$

In the case where $\mathcal{M}$ is injective, the result stronger than Theorem 4 is shown in [29, Theorem 4.2]: for every CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$, $\epsilon > 0$, $n \in \mathbb{N}$, $\{\rho_i\}_{i=1}^{n} \subset \mathcal{S}_n(\mathcal{M})$, $\{\Delta_i\}_{i=1}^{n} \subset \mathcal{F}$ and $\{M_i\}_{i=1}^{n} \subset \mathcal{M}$, there exists a measuring process $M$ for $(\mathcal{M}, S)$ such that

$$|\langle \mathcal{I}(\Delta) \rho_i, M_i \rangle - \langle \mathcal{I}_M(\Delta) \rho_i, M_i \rangle| < \epsilon$$

for all $i = 1, 2, \ldots, n$, and that $\mathcal{I}(1, \Delta) = \mathcal{I}_{\mathcal{M}}(1, \Delta)$ for all $\Delta \in \mathcal{F}$. In physically relevant cases, it is known that every von Neumann algebra $\mathcal{M}$ describing the observable algebra of a quantum system acts on a separable Hilbert space and is AFD. For example, it is shown in [7] that von Neumann algebras of local observables in quantum field theory are AFD and acts on a separable Hilbert space under natural postulates, e.g., the Wightman axioms, the nuclearity condition and the asymptotic scale invariance. For every von Neumann algebra $\mathcal{M}$ on a separable Hilbert space (or with separable dual, equivalently), $\mathcal{M}$ is AFD if and only if it is injective, furthermore, if and only if it is amenable [8, 47]. Hence the assumption of the injectivity for von Neumann algebras is very natural.

In quantum mechanics, complete positivity of instruments is physically justified in [35,37] by considering a natural extendability, called the trivial extendability, of an instrument $\mathcal{I}$ on the system $S$ to that $\mathcal{I}'$ on the composite system $S + S'$ containing the original one $S$, where $S'$ is an arbitrary system not interacting with $S$ nor $A(x)$. This justification of complete positivity is obtained as a part of an axiomatic characterization of physically realizable measurements [35,37]. Then Theorem 4 enables us to regard the Davies-Lewis proposal restricted to CP instruments as a statement that is consistent with the standard formulation of quantum mechanics and hence acceptable for physicists. The above discussion is summarized as follows.
Davies-Lewis-Ozawa criterion  For every apparatus $A(x)$ measuring $S$, where $x$ is the output variable of $A(x)$ taking values in a measurable space $(S, \mathcal{F})$, there always exists a CP instrument $I$ for $(\mathcal{M}, S)$ corresponding to $A(x)$ in the sense of the Davies-Lewis proposal, i.e., for every input state $\rho$ and outcome $\Delta \in \mathcal{F}$ both the probability distribution $\Pr\{x \in \Delta \| \rho\}$ of $x$ and the state $\rho_{\{x \in \Delta\}}$ after the measurement are obtained from $I$.

2.2 Kernels

Here, we briefly summerize the theory of kernels. We refer the reader to [14, 13, 43] for standard references.

**Definition 5 (Kernel [14, p.11, ll.1–3]).** Let $C$ be a set and $\mathcal{H}$ a Hilbert space. A map $K : C \times C \to \mathcal{B}(\mathcal{H})$ is called a kernel of $C$ on $\mathcal{H}$. We denote by $K(C; \mathcal{H})$ the set of kernels of $C$ on $\mathcal{H}$.

It should be noted that $K(C; \mathcal{H})$ has a natural $\mathcal{B}(\mathcal{H})$-bimodule structure.

**Definition 6 ([14, Definition 1.1]).** Let $K \in K(C; \mathcal{H})$. $K$ is said to be positive definite if

$$\sum_{i,j=1}^{n} \langle \xi_i | K(c_i, c_j) \xi_j \rangle \geq 0 \quad (15)$$

for every $n \in \mathbb{N}, c_1, c_2, \cdots, c_n \in C$ and $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H}$. We denote by $K(C; \mathcal{H})^+$ the set of positive definite kernels of $C$ on $\mathcal{H}$.

**Definition 7 (Kolmogorov decomposition [14, Definition 1.3]).** Let $K \in K(C; \mathcal{H})$. A pair $(\mathcal{K}, \Lambda)$ of a Hilbert space $\mathcal{K}$ and a map $\Lambda : C \to \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called a Kolmogorov decomposition of $K$ if it satisfies

$$K(c, c') = \Lambda(c)^* \Lambda(c') \quad (16)$$

for all $c, c' \in C$. A Kolmogorov decomposition $(\mathcal{K}, \Lambda)$ of $K$ is said to be minimal if $\mathcal{K} = \text{span}(\Lambda(C) \mathcal{H})$.

The following representation theorem holds for kernels.

**Theorem 5 ([14, Lemma 1.4, Theorems 1.8 and 1.9]).** Let $C$ be a set and $\mathcal{H}$ a Hilbert space. For every $K \in K(C; \mathcal{H})$, $K$ admits a Kolmogorov decomposition if and only if it is an element of $K(C; \mathcal{H})^+$. For every $K \in K(C; \mathcal{H})^+$, there exists a minimal Kolmogorov decomposition $(\mathcal{K}, \Lambda)$ of $K$, which is unique up to unitary equivalence.

This theorem is a key to the proof of the main theorem of this paper. The famous Stinespring representation theorem is regarded as a corollary of this theorem.
Theorem 6 (Arveson [4, Theorem 1.3.1], [39, Theorem 12.7]). Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, $B$ a unital $C^*$-subalgebra of $B(\mathcal{K})$ and $V$ an element of $B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{K} = \text{span}(BV \mathcal{H})$. For every $A \in (V^*BV)'$, there exists a unique $A_1 \in B'$ such that $VA = A_1V$. Furthermore, the map $\pi' : A \in (V^*BV)' \ni A \mapsto A_1 \in B' \cap \{VV^*\}'$ is an ultraweakly continuous surjective $^*$-homomorphism.

The following theorem holds as a corollary of [12, Part I, Chapter 4, Theorem 3], [46, Chapter IV, Theorem 5.5]:

Theorem 7. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. If $\pi$ is a normal representation of $B(\mathcal{H}_1)$ on $\mathcal{H}_2$, there exist a Hilbert space $\mathcal{K}$ and a unitary operator $U$ of $\mathcal{H}_1 \otimes \mathcal{K}$ onto $\mathcal{H}_2$ such that

$$\pi(X) = U(X \otimes 1_{\mathcal{K}})U^*$$  \(17\)

for all $X \in B(\mathcal{H}_1)$.

This theorem is also a key to the proof of the main theorem of this paper.

3 A System of Measurement Correlations

In this section, we introduce the concept of system of measurement correlations, which is a natural, multivariate version of instrument and is defined as a family of multilinear maps satisfying “positive-definiteness”, “$\sigma$-additivity” and other conditions. This is an appropriate abstraction of measurement correlations in the context of quantum stochastic processes [1]. It is known that the representation theory of CP instruments contributed to quantum measurement theory [30, 31, 29]. Hence we adopt a representation-theoretical approach to system of measurement correlations. The “positive-definiteness” of systems of measurement correlations enables us to apply the (minimal) Kolmogorov decomposition to them, so that provides them with representation-theoretical structures. As a result, a representation theorem (Theorem 8) similar to that for CP instruments [31, Proposition 4.2] will be shown to hold for systems of measurement correlations defined on an arbitrary von Neumann algebra.

To precisely understand physics described by a system of measurement correlations we need a generalization of the Heisenberg picture which is introduced after the proof of Theorem 8 and is called the generalized Heisenberg picture. The introduction of this new picture is motivated also by the present circumstances that the understanding of the (usual) Heisenberg picture has not been deepened in contrast to the Schrödinger picture. It should be stressed that the circumstances are never restricted to quantum measurement theory.

We adopt the following notations.

**Notation 3.1** Let $\mathcal{T}^{(1)}$ be a set. We define a set $\mathcal{T}$ by $\mathcal{T} = \cup_{i=1}^{\infty} (\mathcal{T}^{(1)})^i$.

(i) For each $T \in \mathcal{T}$, we denote by $|T|$ the natural number $n$ such that $T \in (\mathcal{T}^{(1)})^n$.

(ii) For each $T = (t_1, t_2, \ldots, t_{n-1}, t_n) \in \mathcal{T}$, we define $T^\# \in \mathcal{T}$ by $T^\# = (t_n, t_{n-1}, \ldots, t_2, t_1)$.

(iii) For any $T = (t_1, \ldots, t_{1,m}), T_2 = (t_2, \ldots, t_{2,n}) \in \mathcal{T}$, the product $T_1 \times T_2$ is defined by
Since it holds that $T_1 \times (T_2 \times T_3) = (T_1 \times T_2) \times T_3$, $T_1 \times (T_2 \times T_3)$ is written as $T_1 \times T_2 \times T_3$.

(iv) For any $n \in \mathbb{N}$ and $\overrightarrow{M} = (M_1, M_2, \cdots, M_{n-1}, M_n) \in \mathcal{M}^n$, we define $\overrightarrow{M}^n \in \mathcal{M}^n$ by

$$\overrightarrow{M}^n = (M_1^*, M_2^*, \cdots, M_n^*, M_1').$$

(v) For any $m, n \in \mathbb{N}$, $\overrightarrow{M}_1 = (M_1, 1, \cdots, M_{1,m}) \in \mathcal{M}^m$ and $\overrightarrow{M}_2 = (M_{2,1}, \cdots, M_{2,n}) \in \mathcal{M}^n$, the product $\overrightarrow{M} \times \overrightarrow{M}_2 = (M_1, \cdots, M_1, M_{2,1}, \cdots, M_{2,n})$ is defined by

$$\overrightarrow{M} \times \overrightarrow{M}_2 = (M_1, \cdots, M_1, M_{2,1}, \cdots, M_{2,n}).$$

Since it holds that $\overrightarrow{M}_1 \times (\overrightarrow{M}_2 \times \overrightarrow{M}_3) = (\overrightarrow{M}_1 \times \overrightarrow{M}_2) \times \overrightarrow{M}_3$, $\overrightarrow{M}_1 \times (\overrightarrow{M}_2 \times \overrightarrow{M}_3)$ is written as $\overrightarrow{M}_1 \times \overrightarrow{M}_2 \times \overrightarrow{M}_3$.

In addition, for every family $\{\Pi_{T_i}\}_{i \in \mathcal{F}(1)}$ of representations of $\mathcal{M}$ on a Hilbert space $\mathcal{L}$, we adopt the notation

$$\Pi_T(\overrightarrow{M}) = \Pi_i(M_1) \cdots \Pi_{i[T]}(M_{i[T]}),$$

for all $T = (t_1, \cdots, t_{|T|}) \in \mathcal{F}$ and $\overrightarrow{M} = (M_1, \cdots, M_{|T|}) \in \mathcal{M}^{|T|}$.

Let $(S, \mathcal{F})$ be a measurable space. We define a set $\mathcal{F}_S$ by

$$\mathcal{F}_S = \bigcup_{j=1}^{\infty} (\mathcal{F}_S^{(1)})^j,$$

$$\mathcal{F}_S^{(1)} = \{in \} \cup \mathcal{F},$$

where $in$ is a symbol.

We shall define the notion of system of measurement correlations, which is a modified version of projective system of multikernels analyzed in the previous investigations [16]. In this paper, we define and analyze only the case that systems of measurement correlations do not have explicit time-dependence for simplicity.

**Definition 8 (A system of measurement correlations).** A family $\{W_T\}_{T \in \mathcal{F}}$ of maps $W_T : \mathcal{M}^{|T|} = \mathcal{M} \times \cdots \times \mathcal{M} \to \mathcal{M}$ is called a system of measurement correlations for $(\mathcal{M}, S)$ if it satisfies $\mathcal{F}_S^{(1)} = \mathcal{F}_S^{(1)}$ and the following six conditions:

(MC1) For any $T \in \mathcal{F}$, $W_T(M_1, \cdots, M_{|T|})$ is separately linear and ultraweakly continuous in each variable $M_1, \cdots, M_{|T|}$.

(MC2) For any $n \in \mathbb{N}$, $(T_1, \overrightarrow{M}_1), \cdots, (T_n, \overrightarrow{M}_n) \in \bigcup_{T \in \mathcal{F}} (\{T\} \times \mathcal{M}^{|T|})$, and $\xi_1, \cdots, \xi_n \in \mathcal{H}$,

$$\sum_{i,j=1}^{n} \langle \xi_i | W_T^{\sigma} \times T (\overrightarrow{M}_i^* \times \overrightarrow{M}_j) \xi_j \rangle \geq 0.$$

(MC3) For any $T = (t_1, \cdots, t_{|T|}) \in \mathcal{F}$, $\overrightarrow{M} = (M_1, \cdots, M_{|T|}) \in \mathcal{M}^{|T|}$ and $M \in \mathcal{M}$,
\[
MW_T(\mathcal{M}) = W_{in}\times T((\mathcal{M}) \times \mathcal{M}),
\]

\[
W_T(\mathcal{M}) = W_{in}\times (\mathcal{M} \times (\mathcal{M})).
\]

(MC4) Let \( T = (t_1, \cdots, t_{|T|}) \in \mathcal{T} \). If \( t_k = t_{k+1} = in \) or \( t_k, t_{k+1} \in \mathcal{F} \) for some \( 1 \leq k \leq |T| - 1 \),

\[
W_T(M_1, \cdots, M_k, M_{k+1}, \cdots, M_{|T|}) = W_T(M_1, \cdots, M_k, M_{k+1}, \cdots, M_{|T|})
\]

for all \( (M_1, \cdots, M_{|T|}) \in \mathcal{M}^{|T|} \), where \( T' = (t_1, \cdots, t_{k-1}, t_k \cap t_{k+1}, t_{k+2} \cdots, t_{|T|}) \) and

\[
t_k \cap t_{k+1} = \begin{cases} in, & \text{if } t_k = t_{k+1} = in \\ t_k \cap t_{k+1}, & \text{if } t_k, t_{k+1} \in \mathcal{F} \end{cases}
\]

(MC5) For any \( T = (t_1, \cdots, t_{|T|}) \in \mathcal{T} \) with \( t_k = in \) or \( S \), and \( (M_1, \cdots, M_{|T|}) \in \mathcal{M}^{|T|} \) with \( M_k = 1 \),

\[
W_T(M_1, \cdots, M_{k-1}, 1, M_{k+1}, \cdots, M_{|T|}) = W_{kT}(M_1, \cdots, M_{k-1}, M_{k+1}, \cdots, M_{|T|})
\]

where \( kT = (t_1, \cdots, t_{k-1}, t_k, t_{k+1}, t_{k+2} \cdots, t_{|T|}) \). In addition,

\[
W_{in}(1) = W_S(1) = 1.
\]

(MC6) For any \( n \in \mathbb{N}, 1 \leq k \leq n, t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_n \in \mathcal{T}^{(1)} \), mutually disjoint sequence \( \{t_{k,j}\}_j \subset \mathcal{T}, \mathcal{M} \in \mathcal{M}^n \) and \( \rho \in \mathcal{M} \),

\[
\langle \rho, W(t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_n)(\mathcal{M}) \rangle = \sum_j \langle \rho, W(t_1, \cdots, t_{k-1}, t_{k}, t_{k+1}, \cdots, t_n)(\mathcal{M}) \rangle.
\]

It is easy to generalize systems of measurement correlations to the case that they have explicit time-dependence by modifying the definition. For this purpose, \( \mathcal{T}^{(1)}_G \) is replaced by \( \mathcal{T}^{(1)}_{G,\mathcal{F}} = \{in\} \cup (G \times \mathcal{F}) \), where \( G \) is the set representing time and is usually assumed to be a subset of \( \mathbb{R} \), and, for instance, the condition (MC4) is replaced by

(MC4') Let \( T = (t_1, \cdots, t_{|T|}) \in \mathcal{T} \). If \( t_k = t_{k+1} = in \) or \( t_k = (g, \Delta_k), t_{k+1} = (g, \Delta_{k+1}) \in G \times \mathcal{F} \) for some \( 1 \leq k \leq |T| - 1 \), then

\[
W_T(M_1, \cdots, M_k, M_{k+1}, \cdots, M_{|T|}) = W_T(M_1, \cdots, M_k, M_{k+1}, \cdots, M_{|T|})
\]

for all \( (M_1, \cdots, M_{|T|}) \in \mathcal{M}^{|T|} \), where \( T' = (t_1, \cdots, t_{k-1}, t_k \cap t_{k+1}, t_{k+2} \cdots, t_{|T|}) \) and

\[
t_k \cap t_{k+1} = \begin{cases} in, & \text{if } t_k = t_{k+1} = in \\ (g, \Delta_k \cap \Delta_{k+1}), & \text{if } t_k = (g, \Delta_k), t_{k+1} = (g, \Delta_{k+1}) \in G \times \mathcal{F} \end{cases}
\]

Other conditions are also modified in the same manner.

When a system \( \{W_T\}_{T \in \mathcal{T}} \) of measurement correlations for \( (\mathcal{M}, S) \) is given, an instrument \( \mathcal{J}_W \) for \( (\mathcal{M}, S) \) is defined by
\[ \mathcal{F}_W(M, \Delta) = W_\Delta(M) \quad (32) \]

for all \( \Delta \in \mathcal{F} \) and \( M \in \mathcal{M} \), which is seen to be completely positive by the condition \( (MC2) \).

Every system of measurement correlations admits the following representation theorem.

**Theorem 8.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((S, \mathcal{F})\) a measurable space. For any systems \( \{W_T\}_{T \in \mathcal{F}} \) of measurement correlations for \((\mathcal{M}, S)\), there exist a Hilbert space \( \mathcal{L} \), a family \( \{\Pi_T\}_{T \in \mathcal{F}(1)} \) of normal \( (\ast-) \)-representations of \( \mathcal{M} \) on \( \mathcal{L} \) and an isometry \( V \) from \( \mathcal{H} \) to \( \mathcal{L} \) such that

\[ \Pi_M V = V M \quad (33) \]

for all \( M \in \mathcal{M} \), and that

\[ W_T(\overrightarrow{M}) = V^* \Pi_T(\overrightarrow{M}) V \quad (34) \]

for all \( T \in \mathcal{F} \) and \( \overrightarrow{M} \in \mathcal{M}^{[T]} \).

**Proof.** Let \( \{W_T\}_{T \in \mathcal{F}} \) be a system of measurement correlations for \((\mathcal{M}, S)\). We set \( \mathcal{C} = \bigcup_{T \in \mathcal{F}} (\{T\} \times \mathcal{M}^{[T]}) \). We define a kernel \( K : \mathcal{C} \times \mathcal{C} \to \mathcal{M} \) by

\[ K(a, b) = W_{T_1, T_2}(\overrightarrow{M_1} \times \overrightarrow{M_2}) \quad (35) \]

for all \( a = (T_1, \overrightarrow{M_1}) \), \( b = (T_2, \overrightarrow{M_2}) \in \mathcal{C} \). By the definition of a system of measurement correlations, \( K \) is positive definite. By Theorem 5 there exists the minimal Kolmogorov decomposition \((\mathcal{L}, \Lambda)\) of \( K \) such that

\[ K(a, b) = \Lambda(a)^* \Lambda(b) \quad (36) \]

for all \( a = (T_1, \overrightarrow{M_1}) \), \( b = (T_2, \overrightarrow{M_2}) \in \mathcal{C} \). We remark that we use the fact that \( \text{span}(\Lambda(\mathcal{C})\mathcal{H}) \) is dense in \( \mathcal{L} \) many times in this proof.

For each \( t \in \mathcal{F}(1) \) and \( M \in \mathcal{M} \), we define a map \( \Pi_t(M) \) on \( \text{span}(\Lambda(\mathcal{C})\mathcal{H}) \) by

\[ \Pi_t(M) \Lambda(a) \xi = \Lambda(a') \xi \quad (37) \]

for all \( a = (T, \overrightarrow{M}) = ((t_1, \cdots, t_{|T|}), (M_1, \cdots, M_{|T|})) \in \mathcal{C} \) and \( \xi \in \mathcal{H} \), where

\[ a' = (t) \times T, (M) \times \overrightarrow{M} = ((t, t_1, \cdots, t_{|T|}), (M, M_1, \cdots, M_{|T|})) \].

(38)

For all \( t \in \mathcal{F}(1) \), we show that \( \Pi_t : \mathcal{M} \to \Pi_t(M) \) is a normal \( \ast \)-representation of \( \mathcal{M} \). By the condition \( (MC1) \), it holds that
\[ \langle \Lambda(a) \xi_1 | \Pi_1(\alpha M + \beta N) \Lambda(b) \xi_2 \rangle = \langle \xi_1 | \Lambda(a)^* \Pi_1(\alpha M + \beta N) \xi_2 \rangle = \langle \xi_1 | W_{T^2 \times I_2}(\hat{M}_1^\# \times (\alpha M + \beta N) \times \hat{M}_2) \xi_2 \rangle = \alpha \langle \xi_1 | W_{T^2 \times I_2}(\hat{M}_1^\# \times (M \times \hat{M}_2) \xi_2 \rangle + \beta \langle \xi_1 | \Lambda(a)^* \Pi_1(\alpha M + \beta N) \Lambda(b) \Lambda(b) \xi_2 \rangle = \langle \Lambda(a) \xi_1 | (\alpha \Pi_1(\alpha M + \beta \Pi_1(N)) \Lambda(b) \xi_2 \rangle = \langle \Lambda(a) \xi_1 | (\alpha \Pi_1(\alpha M + \beta \Pi_1(N)) \Lambda(b) \xi_2 \rangle \]

for any \( t \in \mathcal{T}(1), \alpha, \beta \in \mathbb{C}, M, N \in \mathcal{M}, a = (T_1, \hat{M}_1), b = (T_2, \hat{M}_2) \in \mathcal{C} \) and \( \xi_1, \xi_2 \in \mathcal{H} \), so that \( \Pi_1(\alpha M + \beta N) = \alpha \Pi_1(M) + \beta \Pi_1(N) \) for all \( t \in \mathcal{T}(1), \alpha, \beta \in \mathbb{C} \) and \( M, N \in \mathcal{M} \).

Similarly, by the condition (MC4) it holds that

\[ \langle \Lambda(a) \xi_1 | \Pi_1(\alpha M) \Pi_1(N) \Lambda(b) \xi_2 \rangle = \langle \xi_1 | \Lambda(a)^* \Pi_1(\alpha M) \Pi_1(N) \Lambda(b) \xi_2 \rangle = \langle \xi_1 | W_{T^2 \times I_2}(\hat{M}_1^\# \times (M \times \hat{M}_2) \xi_2 \rangle = \langle \xi_1 | \Lambda(a)^* \Pi_1(\alpha M \times \hat{M}_2) \Lambda(b) \xi_2 \rangle = \langle \Lambda(a) \xi_1 | (\alpha \Pi_1(M) \times \Pi_1(N)) \Lambda(b) \xi_2 \rangle \]

for any \( t \in \mathcal{T}(1), M, N \in \mathcal{M}, a = (T_1, \hat{M}_1), b = (T_2, \hat{M}_2) \in \mathcal{C} \) and \( \xi_1, \xi_2 \in \mathcal{H} \), so that \( \Pi_1(M \times \hat{M}_2) = \alpha \Pi_1(M) \times \Pi_1(N) \) for all \( t \in \mathcal{T}(1), M, N \in \mathcal{M} \).

For any \( t \in \mathcal{T}(1), n \in \mathbb{N}, a_1 = (T_1, \hat{M}_1), a_2 = (T_2, \hat{M}_2), \cdots, a_n = (T_n, \hat{M}_n) \in \mathcal{C} \) and \( \xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H} \), the map

\[ \mathcal{M} \ni M \mapsto \sum_{i,j=1}^n \langle \Lambda(a_i) \xi_i | \Pi_1(M) \Lambda(a_j) \xi_j \rangle \in \mathbb{C} \]

is normal linear functional on \( \mathcal{M} \), which is also positive since it holds by the conditions (MC2) and (MC4) that

\[ \sum_{i,j=1}^n \langle \Lambda(a_i) \xi_i | \Pi_1(M) \Lambda(a_j) \xi_j \rangle = \sum_{i,j=1}^n \langle \xi_i | W_{T^2 \times I_2}(\hat{M}_1^\# \times (M \times \hat{M}_j) \xi_j \rangle = \sum_{i,j=1}^n \langle \xi_i | W_{T^2 \times I_2}(\hat{M}_1^\# \times (\sqrt{M} \times \sqrt{M} \times \hat{M}_j) \xi_j \rangle = \sum_{i,j=1}^n \langle \xi_i | W_{T^2 \times I_2}(\hat{M}_1^\# \times (\sqrt{M} \times \sqrt{M} \times \hat{M}_j) \xi_j \rangle \geq 0 \]

for all \( M \in \mathcal{M}_+ = \{ M \in \mathcal{M} | M \geq 0 \}, t \in \mathcal{T}(1), n \in \mathbb{N}, a_1 = (T_1, \hat{M}_1), a_2 = (T_2, \hat{M}_2), \cdots, a_n = (T_n, \hat{M}_n) \in \mathcal{C} \) and \( \xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H} \). Thus, for any \( t \in \mathcal{T}(1), M \in \mathcal{M} \),
Theorem 2.22. For all \( n \in \mathbb{N} \), \( a_1 = (T_1, \vec{M}_1), a_2 = (T_2, \vec{M}_2), \ldots, a_n = (T_n, \vec{M}_n) \in \mathcal{C} \) and \( \xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H} \), we have
\[
\left\| \sum_{i=1}^n \Pi_i(M)A(a_i)\xi_i \right\| \leq \|M\| \cdot \left\| \sum_{i=1}^n A(a_i)\xi_i \right\|.
\] (43)

For every \( t \in \mathcal{F}^{(1)} \) and \( M \in \mathcal{M} \), \( \Pi_t(M) \) is a bounded operator on \( \mathcal{L} \). In addition, for all \( t \in \mathcal{F}^{(1)} \), \( M \in \mathcal{M} \) and \( a = (T_1, \vec{M}_1), b = (T_2, \vec{M}_2) \in \mathcal{C} \),
\[
\langle \Lambda(a)\xi_1 | \Pi_t(M)^*\Lambda(b)\xi_2 \rangle = \langle \xi_1 | (\Pi_t(M)\Lambda(a))^*\Lambda(b)\xi_2 \rangle
= (\xi_1 | W_{(t)\times T_1}^\theta \times T_2 ((\vec{M}_1)^\theta \times \vec{M}_2)\xi_2)
= (\xi_1 | W_{T_2} \times (t)_{\times T_2} (\vec{M}_1^\theta \times (M^* \times \vec{M}_2))^\theta)\xi_2)
= (\xi_1 | \Lambda(a)^*\Pi_t(M^*)\Lambda(b)\xi_2) = \langle \Lambda(a)\xi_1 | \Pi_t(M^*)\Lambda(b)\xi_2 \rangle.
\] (44)

Thus, for every \( t \in \mathcal{F}^{(1)} \), \( \Pi_t \) is a normal \(^*\)-representation of \( \mathcal{M} \) on \( \mathcal{L} \). By the condition (MC5), \( \Pi_{in} \) and \( \Pi_3 \) are nondegenerate, i.e., \( \Pi_{in}(1) = \Pi_{3}(1) = 1_{B(\mathcal{L})} \).

For every \( t \in \mathcal{F} \) and and \( \vec{M} \in \mathcal{M}^{[T]} \), it then holds that
\[
V^*\Pi_t(\vec{M})V = \Lambda((in, 1))^*\Pi_t(\vec{M}((in, 1))
= W_{(in)\times T \times (in)((1) \times \vec{M} \times (1))} = W_{T} (\vec{M}).
\] (45)

By the above relation and the condition (MC3), we have \( V^*\Pi_{in}(M)V = M \) for all \( M \in \mathcal{M} \), and
\[
(\Pi_{in}(M)V - VM)^* (\Pi_{in}(M)V - VM)
= V^*\Pi_{in}(M)^*\Pi_{in}(M)V - V^*\Pi_{in}(M)^*VM - M^*V^*\Pi_{in}(M)V + M^*V^*VM
= V^*\Pi_{in}(M^*M)V - V^*\Pi_{in}(M^*)VM - M^*V^*\Pi_{in}(M)V + M^*V^*VM
= M^*M - M^*M - M^*M + M^*M = 0
\] (46)
for all \( M \in \mathcal{M} \), which implies \( \Pi_{in}(M)V = VM \) for all \( M \in \mathcal{M} \).

Remark 1. By the above proof, we see the following. For every family \( \{W_T\}_{T \in \mathcal{F}} \) of maps \( W_T : \mathcal{M}^{[T]} \to \mathcal{M} \) satisfying the conditions (MC1), (MC2), (MC4), (MC5) and (MC6), there exist a Hilbert space \( \mathcal{L} \), a family \( \{\Pi_t\}_{t \in \mathcal{F}^{(1)}} \) of normal \(^*\)-representations of \( \mathcal{M} \) on \( \mathcal{L} \) and an isometry \( V \) from \( \mathcal{H} \) to \( \mathcal{L} \) such that
\[
W_T(\vec{M}) = V^*\Pi_T(\vec{M})V
\] (47)
for all \( T \in \mathcal{F} \) and \( \vec{M} \in \mathcal{M}^{[T]} \). Eq. (47) then implies
\[
W_T(\vec{M})^* = W_{T^*}(\vec{M}^*)\]
(48)
for all \( T \in \mathcal{F} \) and \( \vec{M} \in \mathcal{M}^{[T]} \).
As seen in the proof of Theorem 8, the following fact holds, which will be used in the next section.

**Corollary 2.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \( (S, \mathcal{F}) \) a measurable space. For any systems \( \{W_T\}_{T \in \mathcal{F}} \) of measurement correlations for \( (\mathcal{M}, S) \), let \( (\mathcal{L}, \{\Pi_k\}_{k \in \mathcal{F}(1)}, V) \) be a triplet in Theorem 8. Then the map \( \mathcal{F} \ni \Delta \mapsto \Pi_A(1) \in \mathcal{B}(\mathcal{L}) \) is a projection-valued measure (PVM).

**Proof.** The proof can be easily done in terms of the conditions (MC4), (MC5) and (MC6).

In [1], a (noncommutative) stochastic process over a C*-algebra \( \mathcal{B} \), indexed by a set \( \mathcal{T} \), is defined by a pair \( (\mathcal{A}, \{j_t\}_{t \in \mathcal{T}}) \) of a C*-algebra \( \mathcal{A} \) and a family \( \{j_t\}_{t \in \mathcal{T}} \) of *-homomorphisms from \( \mathcal{B} \) into \( \mathcal{A} \). Obviously, a pair \( (\mathcal{B}(\mathcal{L}), \{\Pi_k\}_{k \in \mathcal{F}(1)}) \) in Theorem 8 is nothing but a stochastic process over a von Neumann algebra \( \mathcal{M} \) indexed by \( \mathcal{F}(1) \) in this sense.

Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space. Let \( \mathcal{T} \) be a set. We set \( \mathcal{T}_1 = \bigcup_{j \in \mathcal{T}} \{\{in\} \cup \mathcal{T}\} \). Let \( (\mathcal{L}, \{\Pi_k\}_{k \in \mathcal{F}(1)}, V) \) be a triplet consisting of a Hilbert space \( \mathcal{L} \), a family \( \{\Pi_k\}_{k \in \mathcal{F}(1)} \) of normal representations of \( \mathcal{M} \) on \( \mathcal{L} \) and \( V \) an isometry from \( \mathcal{H} \) to \( \mathcal{L} \) such that \( \Pi_{II}(M)V = VM \) for all \( M \in \mathcal{M} \) and that \( V^* \Pi_T(M)V \in \mathcal{M} \) for all \( T \in \mathcal{T} \) and \( M \in \mathcal{M} \). The generalized Heisenberg picture is then formulated by this triple \( (\mathcal{L}, \{\Pi_k\}_{k \in \mathcal{F}(1)}, V) \), which enables us to compare the situation before the change, specified by a representation \( \Pi_{II} \), with the situation after the change, specified by \( \{\Pi_k\}_{k \in \mathcal{T}} \). This interpretation naturally follows from the intertwining relation \( \Pi_{II}(M)V = VM \) for all \( M \in \mathcal{M} \) and from the generation of correlation functions \( \mathcal{H}_T(M) = V^* \Pi_T(M)V \) for all \( T \in \mathcal{T} \) and \( M \in \mathcal{M} \). For example, in a triplet \( (\mathcal{L}, \{\Pi_k\}_{k \in \mathcal{F}(1)}, V) \) in Theorem 8, \( \Pi_{II} \) and \( \{\Pi_k\}_{k \in \mathcal{T}} \) correspond to a representation before the measurement and those after the measurement, respectively. The author believes that the generalized Heisenberg picture introduced here gives a right extension of the description of dynamical processes in the standard formulation of quantum mechanics since it succeeds to the advantage of the (usual) Heisenberg picture that we can calculate correlation functions of observables at different times. This topic will be discussed in detail in the succeeding paper of the author.

### 4 Unitary Dilation Theorem

As previously mentioned, the introduction of the concept of measuring process was crucial for the progress of the theory of quantum measurement and of instruments. Measuring processes redefined as follows also play the central role in quantum measurement theory based on the generalized Heisenberg picture.

**Definition 9.** A measuring process \( \mathcal{M} \) for \( (\mathcal{M}, S) \) is a 4-tuple \( \mathcal{M} = (\mathcal{H}, \sigma, E, U) \) which consists of a Hilbert space \( \mathcal{H} \), a normal state \( \sigma \) on \( \mathcal{B}(\mathcal{H}) \), a spectral measure \( E : \mathcal{F} \rightarrow \mathcal{B}(\mathcal{H}) \), and a unitary operator \( U \) on \( \mathcal{H} \otimes \mathcal{H} \) and defines a system of
measurement correlations \( \{ W_T^M \}_{T \in \mathcal{F}} \) for \((\mathcal{M}, S)\) as follows: We define a representation \( \pi_\text{in} \) of \( \mathcal{M} \) and a family \( \{ \pi_\Delta \}_{\Delta \in \mathcal{F}} \) of those of \( \mathcal{M} \) on \( \mathcal{H} \otimes \mathcal{K} \) by

\[
\pi_\text{in}(M) = M \otimes 1_K, \quad \pi_\Delta(M) = U^*(M \otimes E(\Delta))U
\]

for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \), respectively. We use the notation

\[
\pi_T(M) = \pi_1(M_1) \cdots \pi_{|T|}(M_{|T|})
\]

for all \( T = (t_1, \cdots, t_{|T|}) \in \mathcal{F} \) and \( \vec{M} = (M_1, \cdots, M_{|T|}) \in \mathcal{M}^{[T]} \). For each \( T \in \mathcal{F}_S \), \( W_T^M : \mathcal{M}^{[T]} \to \mathcal{M} \) is defined by

\[
W_T^M(\vec{M}) = (\text{id} \otimes \sigma)(\pi_T(\vec{M}))
\]

for all \( \vec{M} \in \mathcal{M}^{[T]} \).

It is easily seen that two definitions of measuring processes for \((\mathcal{B}(\mathcal{H}), S)\) are equivalent.

We say that a CP instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\) is realized by a measuring process \( \vec{M} \) for \((\mathcal{M}, S)\) in the sense of Definition\(^9\) or \( \vec{M} \) realizes \( \mathcal{I} \) if \( \mathcal{I} = \mathcal{I}_{\vec{M}} \). We denote by \( \text{CPInst}_{\mathcal{I}_{\mathcal{M}}, \mathcal{S}} \) the set of CP instruments for \((\mathcal{M}, S)\) realized by measuring processes for \((\mathcal{M}, S)\) in the sense of Definition\(^9\). Then we have \( \text{CPInst}_{\mathcal{I}_{\mathcal{M}, \mathcal{S}}} \subseteq \text{CPInst}_{\mathcal{I}_{\mathcal{M}, S}} \). It will be shown in Section\(^5\) that

\[
\text{CPInst}_{\mathcal{I}_{\mathcal{M}, \mathcal{S}}} = \text{CPInst}_{\mathcal{I}_{\mathcal{M}, S}}.
\]

**Definition 10.** Let \( n \in \mathbb{N} \). Two measuring processes \( \vec{M}_1 \) and \( \vec{M}_2 \) for \((\mathcal{M}, S)\) are said to be \( n \)-equivalent if \( W_T^{\vec{M}_1} = W_T^{\vec{M}_2} \) for all \( T \in \mathcal{F} \) such that \( |T| \leq n \). Two measuring processes \( \vec{M}_1 \) and \( \vec{M}_2 \) for \((\mathcal{M}, S)\) are said to be completely equivalent if they are \( n \)-equivalent for all \( n \in \mathbb{N} \).

The \( n \)-equivalence class of a measuring process \( \vec{M} \) for \((\mathcal{M}, S)\) is nothing but the set of measuring processes \( \vec{M}^{\prime} \) for \((\mathcal{M}, S)\) whose correlation functions of order less or equal to \( n \) are identical to those defined by \( \vec{M} \), i.e., \( W_T^{\vec{M}} = W_T^{\vec{M}^{\prime}} \) for all \( T \in \mathcal{F} \) such that \( |T| \leq n \). Since a measuring process \( \vec{M} \) for \((\mathcal{M}, S)\) in the sense of Definition\(^2\) is also that in the sense of Definition\(^2\), the statistical equivalence works for the former. Of course, the 2-equivalence is the same as the statistical equivalence. In practical situations, dynamical aspects of physical systems are usually analyzed in terms of correlation functions of finite order. Thus it is natural to consider that the classification of measuring processes by the \( n \)-equivalence for not so large \( n \) is valid in the same way. It should be stressed here that causal relations cannot be verified without using correlation functions (of observables at different times) and that situations concerned with measurements are not the exception. A successful example of causal relations in the context of measurement has been already given by the notion of perfect correlation introduced in \( ^{36} \), which uses correlation functions of order 2. One may consider that the complete equivalence of measuring processes...
is unrealistic and useless, but we believe that it is much useful since the following theorem holds.

**Theorem 9.** Let \( \mathcal{H} \) be a Hilbert space and \((S, \mathcal{F})\) a measurable space. Then there is a one-to-one correspondence between complete equivalence classes of measuring processes \( M = (\mathcal{H}, \mathcal{F}, E, U) \) for \((B(\mathcal{H}), S)\) and systems \( \{ W_T \}_{T \in \mathcal{F}} \) of measurement correlations for \((B(\mathcal{H}), S)\), which is given by the relation

\[
W_T(M) = W_T^{\mathcal{F}}(M)
\]

for all \( T \in \mathcal{F} \) and \( \mathcal{M} \in \mathcal{M}^{[T]} \).

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces. For each \( \eta \in \mathcal{H}_2 \), we define a linear map \( V_\eta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \) by \( V_\eta \xi = \xi \otimes \eta \) for all \( \xi \in \mathcal{H}_1 \). It is easily seen that, for each \( \eta \in \mathcal{H}_2 \), \( V_\eta \) satisfies \((X \otimes 1) V_\eta = V_\eta X\) for all \( X \in B(\mathcal{H}_1) \). For any \( x \in \mathcal{H}_1 \setminus \{0\} \), \( P_x \) denotes the projection from \( \mathcal{H}_1 \) onto the linear subspace \( \mathbb{C} x \) of \( \mathcal{H}_1 \) spanned by \( x \). For any \( x, y \in \mathcal{H}_1 \), we define \( |y\rangle \langle x| \in B(\mathcal{H}_1^\dagger) \) by \( |y\rangle \langle x| = |x\rangle \langle z| y \) for all \( z \in \mathcal{H}_1 \).

**Lemma 1.** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces. Let \( V \) be an isometry from \( \mathcal{H}_1 \) to \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). If \( V \) satisfies \((X \otimes 1) V = VX\) for all \( X \in B(\mathcal{H}_1) \), then there exists \( \eta \in \mathcal{H}_2 \) such that \( V = V_\eta \).

**Proof.** Let \( x \in \mathcal{H}_1 \setminus \{0\} \). Since \( (P_x \otimes 1) VX = V P_x x = VX \), it holds that \( VX \in \mathbb{C} x \otimes \mathcal{H}_2 \). Hence, for any \( x \in \mathcal{H}_1 \setminus \{0\} \), there is \( \eta_x \in \mathcal{H}_2 \) such that \( VX = x \otimes \eta_x \) and that \( \|\eta_x\| = 1 \).

For any \( x, y \in \mathcal{H}_1 \setminus \{0\} \),

\[
\langle x|y\rangle \eta_y = \langle x|V y = V(\langle x|y\rangle x) = V(V(\langle y\rangle x) y) = (\langle y\rangle (x \otimes 1)V x = (\langle y\rangle \langle x| \otimes 1)(x \otimes \eta_x) = (\langle x|y \otimes \eta_x
\]

Thus \( \eta_x = \eta_y \) for all \( x, y \in \mathcal{H}_1 \setminus \{0\} \). This fact implies that the range of the map \( \mathcal{H}_1 \setminus \{0\} \ni x \mapsto \eta_x \in \mathcal{H}_2 \) is one point. We put \( \eta = \eta_x \) for some \( x \in \mathcal{H}_1 \setminus \{0\} \). By the linearity of \( V \), \( VO = 0 = 0 \otimes \eta \). Thus we have \( V = V_\eta \).

**Proof (Proof of Theorem 9).** Let \( \{ W_T \}_{T \in \mathcal{F}} \) be a system of measurement correlations for \((B(\mathcal{H}), S)\). By Theorem 8, there exist a Hilbert space \( L_0 \), a family \( \{ \Pi_t \}_{t \in \mathcal{F}(1)} \) of normal representations of \( B(\mathcal{H}) \) on \( L_0 \) and an isometry \( V_0 \) from \( \mathcal{H} \) to \( L_0 \) such that

\[
W_T(M) = V_0^* \Pi_T(M) V_0
\]

for all \( T \in \mathcal{F} \) and \( \mathcal{M} \in \mathcal{M}^{[T]} \).

By Theorem 7 there exist a Hilbert space \( L_1 \) and a unitary operator \( U_1 : L_0 \to \mathcal{H} \otimes L_1 \) such that

\[
\Pi_M(M) = U_1^* (M \otimes 1) U_1
\]

for all \( M \in B(\mathcal{H}) \). Similarly, by Theorem 7 there exist a Hilbert space \( L_2 \) and a unitary operator \( U_2 : L_0 \to \mathcal{H} \otimes L_2 \) such that

\[
\Pi_M(M) = U_2^* (M \otimes 1) U_2
\]
for all $M \in \mathcal{B}(\mathcal{H})$, and by Theorem 1 there exist a PVM $E_0 : \mathcal{F} \to \mathcal{B}(\mathcal{L}_2)$ such that

$$\Pi_0(1) = U_2^*(1 \otimes E_0(\Delta))U_2$$

(58)

for all $\Delta \in \mathcal{F}$.

We define a linear map $V : \mathcal{H} \to \mathcal{H} \otimes L_1$ by $V = U_1V_0$, which is obviously seen to be an isometry. Here, it holds that $V^*(M \otimes 1)V = M$ for all $M \in \mathcal{B}(\mathcal{H})$. For all $M \in \mathcal{B}(\mathcal{H})$,

$$((M \otimes 1)V - VM)^*((M \otimes 1)V - VM)$$

$$= V^*(M^*M \otimes 1)V - V^*(M^* \otimes 1)VM - M^*V^*VM$$

$$= M^*M - M^*M - M^*M + M^*M = 0.$$  

(59)

Thus we have $(M \otimes 1)V = VM$ for all $M \in \mathcal{B}(\mathcal{H})$. By Lemma 1 there is $\eta_1 \in L_1$ such that $V = V_{\eta_1}$.

Let $\eta_2 \in L_2$ such that $\|\eta_2\| = 1$. Let $\zeta$ be an isomorphism from $L_1 \otimes L_2$ to $L_2 \otimes L_1$ defined by $\zeta(\xi \otimes \eta_2) = \xi_2 \otimes \xi_1$ for all $\xi_1 \in L_1$ and $\xi_2 \in L_2$. We define a unitary operator $U_3$ from $\mathcal{H} \otimes L_1 \otimes C\eta_2$ to $\mathcal{H} \otimes C\eta_1 \otimes L_2$ by

$$U_3(\xi \otimes \eta_2) = (1 \otimes \zeta)(U_2U_1^\ast \xi \otimes \eta_1)$$

(60)

for all $\xi \in \mathcal{H} \otimes L_1$. We define a unitary operator $U_5$ from $C\eta_2$ to $C\eta_1$ by $U_5x = \langle \eta_2|x\rangle \eta_1$ for all $x \in C\eta_2$. Then $U_3$ has the following form:

$$U_3 = (1 \otimes \zeta)(U_2U_1^\ast \otimes U_5).$$

(61)

Since both $\mathcal{H} \otimes L_1 \otimes C\eta_2$ and $\mathcal{H} \otimes C\eta_1 \otimes L_2$ are subspaces of $\mathcal{H} \otimes L_1 \otimes L_2$ and satisfies $\text{dim}(\mathcal{H} \otimes L_1 \otimes C\eta_2) = \text{dim}(\mathcal{H} \otimes C\eta_1 \otimes L_2)$ by the above observation, it holds that

$$\text{dim}((\mathcal{H} \otimes L_1 \otimes C\eta_2)^\perp) = \text{dim}((\mathcal{H} \otimes C\eta_1 \otimes L_2)^\perp).$$

(62)

This fact implies that there is a unitary operator $U_4$ from $(\mathcal{H} \otimes L_1 \otimes C\eta_2)^\perp$ to $(\mathcal{H} \otimes C\eta_1 \otimes L_2)^\perp$. Let $Q$ be a projection operator from $\mathcal{H} \otimes L_1 \otimes L_2$ onto $\mathcal{H} \otimes L_1 \otimes C\eta_2$, i.e., $Q = 1 \otimes 1 \otimes P_{\eta_2}$. Let $R$ be a projection operator from $\mathcal{H} \otimes L_1 \otimes L_2$ onto $\mathcal{H} \otimes C\eta_1 \otimes L_2$, i.e., $R = 1 \otimes P_{\eta_1} \otimes 1$. We then define a unitary operator $U$ on $\mathcal{H}$ by $U = U_3Q + U_4(1 - Q)$. It is obvious that $U$ satisfies $UQ = U_3Q = RU_3 = RU$.

We define a Hilbert space $\mathcal{K}$ by $\mathcal{K} = L_1 \otimes L_2$, a normal state $\sigma$ on $\mathcal{B}(\mathcal{K})$ by

$$\sigma(Y) = \langle \eta_1 \otimes \eta_2|Y(\eta_1 \otimes \eta_2)\rangle$$

(63)

for all $Y \in \mathcal{B}(\mathcal{K})$, and a spectral measure $E : \mathcal{F} \to \mathcal{B}(\mathcal{K})$ by

$$E(\Delta) = 1 \otimes E_0(\Delta)$$

(64)

for all $\Delta \in \mathcal{F}$. 
We show that the 4-tuple $\mathcal{M} := (\mathcal{H}, \sigma, E, U)$ is a measuring process for $(\mathcal{B}(\mathcal{H}), S)$ such that
\[ W_T(\tilde{M}) = W_T^B(\tilde{M}) \] (65)
for all $T \in \mathcal{F}$ and $\tilde{M} \in \mathcal{B}(\mathcal{H})^{[T]}$. Since $Q = 1 \otimes 1 \otimes P_{92}$ and
\[ \pi_{in}(M) = U_1 \Pi_{in}(M) U_1^* \otimes 1_{\mathcal{B}(L_2)} \] (66)
for all $M \in \mathcal{B}(\mathcal{H})$, we have
\[ \pi_{in}(M) Q = Q \pi_{in}(M) \] (67)
for all $M \in \mathcal{B}(\mathcal{H})$. Similarly, we have
\[
\pi_\Delta(M) Q = U^*(M \otimes E(\Delta)) U Q \\
= U^*(M \otimes E(\Delta)) R U_3 Q \\
= U^* R(M \otimes E(\Delta)) U_3 Q \\
= U_3^* R(M \otimes E(\Delta)) U_3 Q \\
= U_3^* (M \otimes E(\Delta)) U_3 Q \\
= ((1 \otimes \xi_2)(U_2 U_4^* \otimes U_3))^*(M \otimes E(\Delta))(1 \otimes \xi_2)(U_2 U_4^* \otimes U_3) \\
= (U_1 U_2^* \otimes U_3^*)(M \otimes \xi^* E(\Delta) \xi)(U_2 U_4^* \otimes U_3) Q \\
= (U_1 U_2^* \otimes U_3^*)(M \otimes E(\Delta) \otimes 1_{\mathcal{B}(L_2)})(U_2 U_4^* \otimes U_3) Q \\
= (U_1 \Pi_\Delta(M) U_4^* \otimes P_{92}) Q \\
= (U_1 \Pi_\Delta(M) U_4^* \otimes 1_{\mathcal{B}(L_2)}) Q, \] (68)
and
\[ \pi_\Delta(M) Q = Q \pi_\Delta(M) \] (69)
for all $M \in \mathcal{B}(\mathcal{H})$ and $\Delta \in \mathcal{F}$. By Eqs. (67) and (69), it holds that
\[ Q \pi_T(\tilde{M}) Q = Q(U_1 \Pi_{t_r}(M_1) \cdots \Pi_{t_{[T]}(M_{[T]})} U_1^* \otimes 1_{\mathcal{B}(L_2)}) Q \] (70)
for all $T = (t_1, \cdots, t_{[T]}) \in \mathcal{F}$ and $\tilde{M} = (M_1, \cdots, M_{[T]}) \in \mathcal{B}(\mathcal{H})^{[T]}$. For all $\xi \in \mathcal{H}$, $T = (t_1, \cdots, t_{[T]}) \in \mathcal{F}$ and $\tilde{M} = (M_1, \cdots, M_{[T]}) \in \mathcal{B}(\mathcal{H})^{[T]}$. 
\[ \langle \xi | W_T^{\mathcal{M}}(\tilde{M}) | \xi \rangle = \langle \xi | (\text{id} \otimes \sigma)(\pi_T(\tilde{M})) | \xi \rangle \\
= \langle \xi \otimes \eta_1 \otimes \eta_2 | \pi_T(\tilde{M})(\xi \otimes \eta_1 \otimes \eta_2) \rangle \\
= \langle Q(\xi \otimes \eta_1 \otimes \eta_2) | \pi_T(\tilde{M})Q(\xi \otimes \eta_1 \otimes \eta_2) \rangle \\
= \langle V \xi \otimes \eta_2 | Q \pi_T(\tilde{M})Q(\xi \otimes \eta_2) \rangle \\
= \langle V \xi \otimes \eta_2 | Q(U_1 \Pi_{i_1}(M_1) \cdots \Pi_{i_T}(M_T)U_1^* \otimes 1_{B(S^2)})Q(\xi \otimes \eta_2) \rangle \\
= \langle V \xi \otimes \eta_2 | (U_1 \Pi_{i_1}(M_1) \cdots \Pi_{i_T}(M_T)U_1^* \otimes 1_{B(S^2)})(V \xi \otimes \eta_2) \rangle \\
= \langle V \xi | U_1 \Pi_{i_1}(M_1) \cdots \Pi_{i_T}(M_T)U_1^* V \xi \rangle \\
= \langle \xi | V^* U_1 \Pi_{i_1}(M_1) \cdots \Pi_{i_T}(M_T)U_1^* V \xi \rangle \\
= \langle \xi | W_0 \Pi_{i_1}(M_1) \cdots \Pi_{i_T}(M_T) | V_0 \xi \rangle \\
= \langle \xi | W_T(\tilde{M}) | \xi \rangle, \quad (71) \]

which completes the proof.

Remark 2. We adopt here the same notations as in the proof of the above theorem. Suppose that \( \mathcal{H} \) is separable and \( (S, \mathcal{F}) \) is a standard Borel space. Let \( \{\Delta_n\}_{n \in \mathbb{N}} \) be a countable generator of \( \mathcal{F} \), \( \{M_n\}_{n \in \mathbb{N}} \) a dense subset of \( B(\mathcal{H}) \) in the strong topology, and \( \{\xi_n\}_{n \in \mathbb{N}} \) a dense subset of \( \mathcal{H} \). Let \( \{C_n\}_{n \in \mathbb{N}} \) be a well-ordering of the countable set \( \{\Pi_{i_0}(M_n) | n \in \mathbb{N}\} \cup \{\Pi_{i_m}(M_n) | m, n \in \mathbb{N}\} \). \( \mathcal{L}_0 \) has the increasing sequence \( \{\mathcal{L}_{0,n}\}_{n \in \mathbb{N}} \) of separable closed subspaces, defined by

\[ \mathcal{L}_{0,n} = \text{span}(\{C_{f(1)} \cdots C_{f(n)} | f \in \mathbb{N}^{\{1, \cdots, n\}}, k \in \mathbb{N}\}) \quad (72) \]

for all \( n \in \mathbb{N} \), such that \( \mathcal{L}_0 = \overline{\text{span}}(\bigcup_n \mathcal{L}_{0,n}) \), where \( \mathbb{N}^{\{1, \cdots, n\}} \) is the set of maps from \( \{1, \cdots, n\} \) to \( \mathbb{N} \). Hence, \( \mathcal{L}_0 \) is separable because we have

\[ \mathcal{L}_0 = \overline{\text{span}} \left\{ \bigoplus_{n=1}^{\infty} (\mathcal{L}_{0,n-1})^\perp \cap \mathcal{L}_{0,n} \right\}, \quad (73) \]

where \( \mathcal{L}_{0,0} = \{0\} \). It is immediately seen that \( \mathcal{L}_1 \), \( \mathcal{L}_2 \) and \( \mathcal{H} = \mathcal{L}_1 \otimes \mathcal{L}_2 \) are also separable.

5 Extendability of CP instruments to systems of measurement correlations

To begin with, the following theorem similar to [29] Theorem 3.4 holds for arbitrary von Neumann algebras \( \mathcal{M} \).

Corollary 3. Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \( (S, \mathcal{F}) \) a measurable space. For a system \( \{W_T\}_{T \in \mathcal{F}} \) of measurement correlations for \( (\mathcal{M}, S) \), the following conditions are equivalent:
(1) There is a system \( \{ \tilde{W}_T \}_{T \in \mathcal{T}} \) of measurement correlations for \((\mathcal{B}(\mathcal{H}), \mathcal{S})\) such that
\[
W_T(M) = \tilde{W}_T(M)
\] (74)
for all \( T \in \mathcal{T} \) and \( M \in \mathcal{M}[T] \).

(2) There is a measuring process \( M = (\mathcal{K}, \sigma, E, U) \) for \((\mathcal{M}, \mathcal{S})\) such that
\[
W_T(M) = W_T^M(M)
\] (75)
for all \( T \in \mathcal{T} \) and \( M \in \mathcal{M}[T] \).

The proof of this corollary is obvious by Theorem 9. It is not known that how large the set of systems of measurement correlations for \((\mathcal{M}, \mathcal{S})\) satisfying the above equivalent conditions in the set of systems of measurement correlations for \((\mathcal{M}, \mathcal{S})\) at the present time.

Going back to the starting point of quantum measurement theory, we do not have to rack our brain to resolve the above difficulty. This is because we should recall that each CP instrument statistically corresponds to an apparatus measuring the system under consideration in the sense of the Davies-Lewis proposal. In addition, the introduction of systems of measurement correlations was motivated by the necessity of the counterpart of CP instruments in the (generalized) Heisenberg picture in order to systematically treat measurement correlations. Hence it is natural to consider that an instrument \( I \) for \((\mathcal{M}, \mathcal{S})\) describing a physically realizable measurement should be defined by a system of measurement correlations \( \{ \tilde{W}_T \}_{T \in \mathcal{T}} \) for \((\mathcal{M}, \mathcal{S})\), i.e.,
\[
\mathcal{I}(M, \Delta) = \mathcal{I}_W(M, \Delta)
\] (76)
for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \).

Question 2. Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((\mathcal{S}, \mathcal{F})\) a measurable space. For any CP instrument \( \mathcal{I} \) for \((\mathcal{M}, \mathcal{S})\), does there exist a system of measurement correlations \( \{ \tilde{W}_T \}_{T \in \mathcal{T}} \) for \((\mathcal{M}, \mathcal{S})\) which defines \( \mathcal{I} \)?

In the case of \( \mathcal{B}(\mathcal{H}) \), this question is already affirmatively answered by the existence of measuring processes for \((\mathcal{B}(\mathcal{H}), \mathcal{S})\) for every CP instrument for \((\mathcal{B}(\mathcal{H}), \mathcal{S})\) (Theorem 1). Surprisingly, Question 2 is affirmatively resolved for all CP instruments defined on arbitrarily given von Neumann algebras.

Theorem 10. Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((\mathcal{S}, \mathcal{F})\) a measurable space. For every CP instrument \( \mathcal{I} \) for \((\mathcal{M}, \mathcal{S})\), there exists a system of measurement correlations \( \{ \tilde{W}_T \}_{T \in \mathcal{T}} \) for \((\mathcal{M}, \mathcal{S})\) such that
\[
\mathcal{I}(M, \Delta) = \mathcal{I}_W(M, \Delta)
\] (77)
for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \).

Proof. By [31, Proposition 4.2] (or [29, Proposition 3.2]), there exist a Hilbert space \( \mathcal{K} \), a normal representation \( \pi_0 \) of \( \mathcal{M} \) on \( \mathcal{K} \), a PVM \( E_0 : \mathcal{F} \to \mathcal{B}(\mathcal{H}) \) and an isometry \( V : \mathcal{H} \to \mathcal{K} \) such that
$$\mathcal{F} (M, \Delta) = V^* \pi_0(M) E_0(\Delta) V,$$

$$\pi_0(M) E_0(\Delta) = E_0(\Delta) \pi_0(M)$$ \hspace{1cm} (78)

for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$, and that $\mathcal{K} = \overline{\text{span}} \pi_0(\mathcal{M}) E_0(\mathcal{F}) V \mathcal{H}$. We follow the identification

$$\mathcal{B}(\mathcal{H} \oplus \mathcal{K}) = \left( \begin{array}{cc} \mathcal{B}(\mathcal{H}) & \mathcal{B}(\mathcal{K}, \mathcal{H}) \\ \mathcal{B}(\mathcal{H}, \mathcal{K}) & \mathcal{B}(\mathcal{K}) \end{array} \right)$$ \hspace{1cm} (80)

with multiplication and involution compatible with the usual matrix operations. We define a normal representation $\Pi_{\text{in}}$ of $\mathcal{M}$ on $\mathcal{H} \oplus \mathcal{K}$ by

$$\Pi_{\text{in}}(M) = \left( \begin{array}{cc} M & 0 \\ 0 & \pi_0(M) \end{array} \right)$$ \hspace{1cm} (81)

for all $M \in \mathcal{M}$, a PVM $E : \mathcal{F} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ by

$$E(\Delta) = \left( \begin{array}{cc} \delta_s(\Delta) & 0 \\ 0 & E_0(\Delta) \end{array} \right)$$ \hspace{1cm} (82)

for all $\Delta \in \mathcal{F}$, where $s \in S$ and $\delta_s$ is a delta measure on $(S, \mathcal{F})$ concentrated on $s$, and a unitary operator $U$ of $\mathcal{H} \oplus \mathcal{K}$ by

$$U = \left( \begin{array}{cc} 0 & -V^* \\ V & Q \end{array} \right),$$

where $Q = 1 - VV^*$. For every $\Delta \in \mathcal{F}$, we define a representation $\Pi_\Delta$ of $\mathcal{M}$ on $\mathcal{H} \oplus \mathcal{K}$ by

$$\Pi_\Delta(M) = U^* \Pi_{\text{in}}(M) E(\Delta) U$$

$$= \left( \begin{array}{cc} \mathcal{F}(M, \Delta) & -V^* \pi_0(M) E_0(\Delta) Q \\ -Q \pi_0(M) E_0(\Delta) V & \delta_s(\Delta) VMV^* + Q \pi_0(M) E_0(\Delta) Q \end{array} \right)$$ \hspace{1cm} (84)

for all $M \in \mathcal{M}$. We define a unital normal CP linear map $P_{\text{11}} : \mathcal{B}(\mathcal{H} \oplus \mathcal{K}) \to \mathcal{B}(\mathcal{H})$ by

$$\left( \begin{array}{cc} \mathcal{B}(\mathcal{H}) & \mathcal{B}(\mathcal{K}, \mathcal{H}) \\ \mathcal{B}(\mathcal{H}, \mathcal{K}) & \mathcal{B}(\mathcal{K}) \end{array} \right) \ni \left( \begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array} \right) \mapsto X_{11} \in \mathcal{B}(\mathcal{H}).$$

(85)

For every $T = (t_1, \ldots, t_{|T|}) \in \mathcal{F}_S$, we define a map $W_T : \mathcal{M}^{[T]} \to \mathcal{B}(\mathcal{H})$ by

$$W_T(\tilde{M}) = P_{\text{11}}[\Pi_T(\tilde{M})] = P_{\text{11}}[\Pi_{t_1}(M_1) \cdots \Pi_{t_{|T|}}(M_{|T|})]$$

(86)

for all $\tilde{M} = (M_1, \ldots, M_{|T|}) \in \mathcal{M}^{[T]}$. We show that the family $\{W_T\}_{T \in \mathcal{F}_S}$ is a system of measurement correlations for $(\mathcal{M}, S)$ such that

$$\mathcal{F}(M, \Delta) = W_\Delta(M)$$

(87)
for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$. For this purpose, it suffices to show $W_T(\vec{M}) \in \mathcal{M}$ for all $T \in \mathcal{T}_S$ and $\vec{M} \in \mathcal{M}^{[T]}$. Then the set

$$\mathcal{D} = \left( \text{span}(\mathcal{M}V^* \mathcal{A}) \cap \mathcal{A} \right)$$

is a $*$-subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$, where $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ algebraically generated by $V.\mathcal{M} V^*$, $\pi_0(\mathcal{M})E_0(\mathcal{F})$ and $Q = 1 - VV^*$. This fact follows from the usual matrix operations and $V^* V \mathcal{V} \subset \mathcal{M}$. Since it is obvious that $\Pi_{in}(M), \Pi_{\Delta}(M') \in \mathcal{D}$ for all $M, M' \in \mathcal{M}$ and $\Delta \in \mathcal{F}$, we have $\Pi_T(\vec{M}) \in \mathcal{D}$ for all $T \in \mathcal{T}_S$ and $\vec{M} \in \mathcal{M}^{[T]}$. Therefore, for every $T \in \mathcal{T}_S$ and $\vec{M} \in \mathcal{M}^{[T]}$, the $(1,1)$-component of $\Pi_T(\vec{M})$ is also an element of $\mathcal{M}$, which completes the proof.

Remark 3. In the case of an atomic von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, we have another construction of a system of measurement correlations which defines a given CP instrument.

Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$ be a normal conditional expectation. We define a CP instrument $\mathcal{F}$ for $(\mathcal{B}(\mathcal{H}), S)$ by

$$\mathcal{F}(X, \Delta) = \mathcal{E}(X, \Delta)$$

for all $X \in \mathcal{B}(\mathcal{H})$ and $\Delta \in \mathcal{F}$. By Theorem 1, there exists a measuring process $\vec{M} = (\mathcal{H}, \sigma, \mathcal{E}, U)$ for $(\mathcal{B}(\mathcal{H}), S)$ such that

$$\mathcal{F}(X, \Delta) = \mathcal{F}_M(X, \Delta)$$

for all $X \in \mathcal{B}(\mathcal{H})$ and $\Delta \in \mathcal{F}$. A system of measurement correlations $\{W_T\}_{T \in \mathcal{T}_S}$ for $(\mathcal{M}, S)$ is defined by

$$W_T(\vec{M}) = \mathcal{E}(W_T^M(\vec{M}))$$

for all $T \in \mathcal{T}$ and $\vec{M} \in \mathcal{M}^{[T]}$. Then $\mathcal{F}_W$ satisfies

$$\mathcal{F}_W(M, \Delta) = \mathcal{E}(W^M_D(M)) = \mathcal{E}((\mathcal{F}(M, \Delta)) = \mathcal{E}(\mathcal{F}(\mathcal{E}(M, \Delta))) = \mathcal{F}(M, \Delta)$$

for all $M \in \mathcal{M}$ and $\Delta \in \mathcal{F}$.

We should remark that the above construction does not show the existence of measuring processes for $(\mathcal{M}, S)$ for every CP instrument for $(\mathcal{M}, S)$.

6 Approximate realization of CP instruments by measuring processes

We discuss the realizability of CP instruments by measuring processes in this section. Here, we shall start from the following question similar to Question 2: \footnote{To show this, we use $Q = 1 - VV^*$ and Eq. (78).}
Question 3. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(\mathcal{S}, \mathcal{F})$ a measurable space. For any CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$, does there exist a measuring process $M$ for $(\mathcal{M}, S)$ which realizes $\mathcal{I}$ within arbitrarily given error limits $\varepsilon > 0$?

We say that a CP instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ is approximately realized by a net of measuring processes $\{M_\alpha\}_{\alpha \in A}$ for $(\mathcal{M}, S)$, or $\{M_\alpha\}_{\alpha \in A}$ approximately realizes $\mathcal{I}$ if, for every $\varepsilon > 0$, there is a net $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ which approximates $\mathcal{I}$ for all $\alpha \in A$. We denote by CPInst$_{AR}(\mathcal{M}, S)$ the set of CP instruments for $(\mathcal{M}, S)$ approximately realized by nets of measuring processes for $(\mathcal{M}, S)$.

Before answering to Question 3, we shall extend the program, advocated and developed by many researchers [15, 41, 22, 20], which states that physical processes should be described by (inner) CP maps usually called operations [15] or effects [22].

Definition 11 ([23, 3]). Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$.

1. A positive linear map $\Psi$ of $\mathcal{M}$ is said to be finitely inner if there is a finite sequence $\{V_j\}_{j=1, \ldots, m}$ of $\mathcal{M}$ such that
   \[ \Psi(M) = \sum_{j=1}^{m} V_j^* MV_j \] (93)
   for all $M \in \mathcal{M}$.

2. A positive linear map $\Psi$ of $\mathcal{M}$ is said to be inner if there is a sequence $\{V_j\}_{j \in \mathbb{N}}$ of $\mathcal{M}$ such that
   \[ \Psi(M) = \sum_{j=1}^{\infty} V_j^* MV_j \] (94)
   for all $M \in \mathcal{M}$, where the convergence is ultraweak.

3. A positive linear map $\Psi$ of $\mathcal{M}$ is said to be approximately inner if it is the pointwise ultraweak limit of a net $\{\Psi_\alpha\}_{\alpha \in A}$ of finitely inner positive linear maps such that $\Psi_\alpha(1) \leq \Psi(1)$ for all $\alpha \in A$.

In [3], finite innerness and approximate innerness of CP maps are called factorization through the identity map $\text{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ and approximate factorization through $\text{id}_{\mathcal{M}}$, respectively. We refer the reader to [23, 24, 3, 2] for more detailed discussions. It is obvious that every finitely inner positive linear map $\Psi$ of $\mathcal{M}$ is inner. Similarly, every inner positive linear map $\Psi(M) = \sum_{j=1}^{m} V_j^* MV_j$, $M \in \mathcal{M}$, is approximately inner since it is the ultraweak limit of a sequence $\{\Psi_{j}\}$ of finitely inner positive maps $\Psi_{j}(M) = \sum_{k=1}^{l} V_k^* MV_k$, $M \in \mathcal{M}$, such that $\Psi_{j}(1) = \sum_{k=1}^{l} V_k^* V_k \leq \Psi(1)$ for all $j \in \mathbb{N}$. Every approximately inner positive linear map $\Psi$ of $\mathcal{M}$ is always completely positive.

Definition 12. An instrument $\mathcal{I}$ for $(\mathcal{M}, S)$ is said to be finitely inner [inner, or approximately inner, respectively] if $\mathcal{I}(\cdot, \Delta)$ is finitely inner [inner, or approximately inner, respectively] for every $\Delta \in \mathcal{F}$. We denote by CPInst$_{\text{fl}}(\mathcal{M}, S)$...
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[CPInst_{FI}(\mathcal{M}, S), or CPInst_{AI}(\mathcal{M}, S), respectively] the set of finitely inner [inner, or approximately inner, respectively] CP instruments for (\mathcal{M}, S).

The following relation holds.

\[ \text{CPInst}_{FI}(\mathcal{M}, S) \subset \text{CPInst}_{IN}(\mathcal{M}, S) \subset \text{CPInst}_{AI}(\mathcal{M}, S). \]  

(95)

**Definition 13 (Inner measuring process).** A measuring process \( \mathcal{M} = (\mathcal{K}, \sigma, E, U) \) for (\mathcal{M}, S) is said to be inner if \( U \) is contained in \( \mathcal{M} \otimes \mathcal{B}(\mathcal{K}) \).

We then have the following theorem.

**Theorem 11.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and (\( S, \mathcal{F} \)) a measurable space. For every approximately inner (hence CP) instrument \( I \) for (\( \mathcal{M}, S \)), \( \varepsilon > 0 \), \( n \in \mathbb{N} \), \( \rho_1, \ldots, \rho_n \in \mathcal{I}_n(\mathcal{M}) \), \( M_1, \ldots, M_n \in \mathcal{M} \) and \( \Delta_1, \ldots, \Delta_n \in \mathcal{F} \), there exists an inner measuring process \( \mathcal{M} = (\mathcal{K}, \sigma, E, U) \) for (\( \mathcal{M}, S \)) in the sense of Definition\( ^{[9]} \) such that

\[ |\langle \rho_j, I(M_j, \Delta_j) \rangle - \langle \rho_j, I_M(M_j, \Delta_j) \rangle| < \varepsilon \]  

(96)

for all \( j = 1, \ldots, n \).

**Corollary 4.** Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \) and (\( S, \mathcal{F} \)) a measurable space. We have

\[ \text{CPInst}_{AI}(\mathcal{M}, S) \subset \text{CPInst}_{AR}(\mathcal{M}, S). \]  

(97)

Only for injective factors the following holds as a corollary of the above theorem.

**Theorem 12.** Let \( \mathcal{M} \) be an injective factor on a Hilbert space \( \mathcal{H} \) and (\( S, \mathcal{F} \)) a measurable space. For every CP instrument \( I \) for (\( \mathcal{M}, S \)), \( \varepsilon > 0 \), \( n \in \mathbb{N} \), \( \rho_1, \ldots, \rho_n \in \mathcal{I}_n(\mathcal{M}) \), \( M_1, \ldots, M_n \in \mathcal{M} \) and \( \Delta_1, \ldots, \Delta_n \in \mathcal{F} \), there exists an inner measuring process \( \mathcal{M} = (\mathcal{K}, \sigma, E, U) \) for (\( \mathcal{M}, S \)) in the sense of Definition\( ^{[9]} \) such that

\[ |\langle \rho_j, I(M_j, \Delta_j) \rangle - \langle \rho_j, I_M(M_j, \Delta_j) \rangle| < \varepsilon \]  

(98)

for all \( j = 1, \ldots, n \).

**Corollary 5.** Let \( \mathcal{M} \) be an injective factor on a Hilbert space \( \mathcal{H} \) and (\( S, \mathcal{F} \)) a measurable space. Then we have

\[ \text{CPInst}_{AI}(\mathcal{M}, S) = \text{CPInst}_{AR}(\mathcal{M}, S) = \text{CPInst}(\mathcal{M}, S). \]  

(99)

Theorem\( ^{[12]} \) is a stronger result than \( ^{[29]} \) Theorem 4.2] for factors, so that Question \( ^{[8]} \) is affirmatively resolved for injective factors. We use the following proposition for the proof of Theorem\( ^{[12]} \).

**Proposition 2 (Anantharaman-Delaroche and Havet \( ^{[3]} \) Lemma 2.2, Remarks 5.4]).** Let \( \mathcal{M} \) be an injective factor on a Hilbert space \( \mathcal{H} \). Every CP map \( \Psi \) of
\( \mathcal{M} \) is approximately inner, i.e., it is the pointwise ultraweak limit of a net of CP maps \( \{ \Psi_\theta \}_{\theta \in \Theta} \) of the form \( \Psi_\theta(M) = \sum_{j=1}^{n_\theta} V^*_\theta(j) M V_\theta(j) \), \( M \in \mathcal{M} \), with \( n_\theta \in \mathbb{N} \), \( V_{\theta,1}, \ldots, V_{\theta,n_\theta} \in \mathcal{M} \) such that \( \Psi_{\theta}(1) = \sum_{j=1}^{n_\theta} V^*_\theta(j) V_\theta(j) \leq \Psi(1) \) for all \( \theta \in \Theta \).

The following proof is inspired by [40].

Proof (Proof of Theorem [47]). Let \( n \in \mathbb{N} \), \( \rho_1, \ldots, \rho_n \in \mathcal{S}_n(\mathcal{M}) \), \( M_1, \ldots, M_n \in \mathcal{M} \) with \( \mathcal{N} \setminus \{0\} \) and \( \Delta_1, \ldots, \Delta_n \in \mathcal{F} \setminus \{0\} \). Let \( \mathcal{F}' \) be a \( \sigma \)-subfield of \( \mathcal{F} \) generated by \( \Delta_1, \ldots, \Delta_n \). Let \( \{ \Gamma^m_i \}_{i=1}^m \subset \mathcal{F} \setminus \{0\} \) be a maximal partition of \( \bigcup_{i=1}^n \Delta_i \), i.e., \( \{ \Gamma^m_i \}_{i=1}^m \) satisfies the following conditions:

1. For every \( i = 1, \ldots, m \), if \( \Delta \in \mathcal{F}' \) satisfies \( \Delta \subset \Gamma^m_i \), then \( \Delta = \Gamma^m_i \) or \( \emptyset \);
2. \( \cup_{i=1}^m \Gamma^m_i = \mathcal{F} \); and
3. \( \Gamma^m_i \cap \Gamma^m_j = \emptyset \) if \( i \neq j \).

For every \( i = 1, \ldots, m \), there is a net of finitely inner CP maps \( \{ \Psi_{\theta_i} \}_{\theta_i} \) of the form \( \Psi_{\theta_i}(M) = \sum_{j=1}^{n_{\theta_i}} V^*_{\theta_i,j} M V_{\theta_i,j} \), \( M \in \mathcal{M} \), with \( n_{\theta_i} \in \mathbb{N} \), \( V_{\theta_i,1}, \ldots, V_{\theta_i,n_{\theta_i}} \in \mathcal{M} \) such that pointwise ultraweakly convergent to \( \mathcal{F}(\cdot, \Gamma^m_i) \) in the ultraweak topology and \( \Psi_{\theta_i}(1) \leq \mathcal{F}(\cdot, 1) \).

We fix \( s_0, s_1, \ldots, s_m \in S \) such that \( s_i \in \Gamma^m_i \) for all \( i = 1, \ldots, m \). For every \( \theta = (\theta_1, \ldots, \theta_m) \in \Theta_1 \times \cdots \times \Theta_m \), we define a finitely inner CP instrument \( \mathcal{I}_\theta \) for \( (\mathcal{M}, S) \) by

\[
\mathcal{I}_\theta(M, \Delta) = \sum_{i=1}^m \delta_{\theta_i}(\Delta) \Psi_{\theta_i}(M) + \delta_{s_m}(\Delta) L \theta M \theta
\]

for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \), where \( \delta_s \) is a delta measure on \( (S, \mathcal{F}) \) concentrated on \( s \) and \( L \theta = \sqrt{1 - \sum_{i=1}^m \Psi_{\theta_i}(1)} \).

Let \( \epsilon \) be a positive real number. For every \( i = 1, \ldots, m \), there is \( \hat{\theta}_i \in \Theta_i \) such that

\[
|\langle \rho_j, \mathcal{I}_\theta(M_j, \Gamma^m_i) \rangle - \langle \rho_j, \Psi_{\theta_i}(M_j) \rangle| < \frac{\epsilon}{2m}
\]

for all \( j = 1, \ldots, n \), and that

\[
\langle \rho_j, \mathcal{I}_\theta(1, \Gamma^m_i) \rangle - \langle \rho_j, \Psi_{\theta_i}(1) \rangle < \frac{\epsilon}{2m \sum_{k=1}^n \|M_k\|}
\]

for all \( j = 1, \ldots, n \).

By Eqs. (101), (102), we have

\[
|\langle \rho_j, \mathcal{I}_\theta(M_j, \Delta_j) \rangle - \sum_{i=1}^m \delta_{\theta_i}(\Delta_j) \langle \rho_j, \Psi_{\theta_i}(M_j) \rangle| = |\sum_{i=1}^m \delta_{\theta_i}(\Delta_j) \langle \rho_j, \mathcal{I}_\theta(M_j, \Gamma^m_i) \rangle - \sum_{i=1}^m \delta_{\theta_i}(\Delta_j) \langle \rho_j, \Psi_{\theta_i}(M_j) \rangle| \leq \sum_{i=1}^m \delta_{\theta_i}(\Delta_j) |\langle \rho_j, \mathcal{I}_\theta(M_j, \Gamma^m_i) \rangle - \langle \rho_j, \Psi_{\theta_i}(M_j) \rangle| < \frac{\epsilon}{2m} \leq \frac{\epsilon}{2}
\]

for all \( j = 1, \ldots, n \), and
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\[ |\rho_j(L_M^jL_M)| \leq \|M_j\| |\rho_j(L_M^j)| = \|M_j\| \cdot \langle \rho_j, \sum_{i=1}^{m} (\mathcal{M}(1, \Gamma) - \Psi_{\rho,j}(1)) \rangle \]

\[ = \|M_j\| \cdot \sum_{i=1}^{m} \langle \rho_j, \mathcal{M}(1, \Gamma) - \Psi_{\rho,j}(1) \rangle < \|M_j\| \cdot \frac{\varepsilon}{2m\sum_{k=1}^{n} \|M_k\|} \leq \frac{\varepsilon}{2} \]

(104)

for all \( j = 1, \ldots, n \). Then the CP instrument \( \mathcal{M}_\theta \) for \( (\mathcal{M}, S) \) with \( \theta = (\theta_1, \ldots, \theta_m) \) satisfies

\[ |\langle \rho_j, \mathcal{M}(M_j, \Delta_j) \rangle - \langle \rho_j, \mathcal{M}(M_j, \Delta_j) \rangle | < \varepsilon \]

(105)

for all \( j = 1, \ldots, n \).

Next, we shall define an inner measuring process \( \mathcal{M} = (\mathcal{M}, \sigma, E, U) \) for \( (\mathcal{M}, S) \) that realizes \( \mathcal{M}_\theta \). Let \( \eta = \{ \eta_j \}_{j=0,1,\ldots,[\theta]_+1} \) be a complete orthonormal system of \( \mathcal{C}[\theta]+2 \). A partial isometry \( V : \mathcal{M} \otimes \mathcal{C}[\theta]+2 \to \mathcal{M} \otimes \mathcal{C}[\theta]+2 \) is defined by

\[ V = \sum_{j=1}^{m} \sum_{i=[\theta]_+1} \theta_i V_{i,\theta,j} \otimes |\eta_j \rangle \langle \eta_0 | + L_\theta \otimes |\eta_{[\theta]_+1} \rangle \langle \eta_0 | \]

(106)

It is obvious that \( V \) satisfies \( V^*V = 1 \otimes |\eta_0 \rangle \langle \eta_0 | \). We define a PVM \( E : \mathcal{M} \to M|\theta|+2(\mathcal{C}) \) by

\[ E_\eta(\Delta) = \delta_\eta(\Delta) |\eta_0 \rangle \langle \eta_0 | + \sum_{i=1}^{m} \delta_\eta(\Delta) \sum_{j=1}^{[\theta]_+1} \theta_i |\eta_j \rangle \langle \eta_j | + \delta_\eta(\Delta) |\eta_{[\theta]_+1} \rangle \langle \eta_{[\theta]_+1} | \]

(107)

for all \( \Delta \in \mathcal{M} \).

We define a Hilbert space \( \mathcal{H} = \mathcal{C}[\theta]+2 \otimes \mathcal{C}^2 \), a normal state \( \sigma \) on \( \mathcal{B}(\mathcal{H}) = M_{[\theta]+2}(\mathcal{C}) \otimes M_2(\mathcal{C}) \), a PVM \( E : \mathcal{M} \to \mathcal{B}(\mathcal{H}) \) and a unitary operator \( U \) on \( \mathcal{H} \) by

\[ \sigma(X) = \text{Tr}[X (|\eta_0 \rangle \langle \eta_0 | \otimes G_{11})], \quad X \in \mathcal{B}(\mathcal{H}), \]

(108)

\[ E(\Delta) = E_{\eta}(\Delta) \otimes 1, \quad \Delta \in \mathcal{M}, \]

(109)

\[ U = V \otimes G_{11} + (1 - V V^*) \otimes G_{12} + (1 - V^* V) \otimes G_{12}^* - V^* \otimes G_{22}, \]

(110)

respectively, where \( \text{Tr} \) is the trace on \( M_{[\theta]+2}(\mathcal{C}) \otimes M_2(\mathcal{C}) \) and

\[ G_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad G_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

(111)

Since \( U \in \mathcal{M}_{[\theta]+2}(\mathcal{C}) \otimes M_2(\mathcal{C}) \), the 4-tuple \( \mathcal{M} = (\mathcal{M}, \sigma, E, U) \) is an inner measuring process for \( (\mathcal{M}, S) \) satisfying

\[ |\langle \rho_j, \mathcal{M}(M_j, \Delta_j) \rangle - \langle \rho_j, \mathcal{M}(M_j, \Delta_j) \rangle | < \varepsilon \]

(112)
for all \( j = 1, \ldots, n \).

**Proof (Proof of Theorem 12).** Let \( \mathcal{I} \) be a CP instrument for \((\mathcal{M}, S)\). Since \( \mathcal{M} \) is an injective factor, \( \mathcal{I}(\cdot, \Delta) \) is approximately inner for every \( \Delta \in \mathcal{F} \) by Proposition 2. Thus the proof of Theorem 11 works.

**Remark 4.** We use the same notations as in the proof of Theorem 11. In the case where \( \mathcal{M} \) is factor, we have another construction of a measuring process \( \overline{\mathcal{M}} \) for \((\mathcal{M}, S)\) such that \( \mathcal{I}_{\overline{\mathcal{M}}}(M, \Delta) = \mathcal{I}_M(M, \Delta) \) for all \( M \in \mathcal{M} \) and \( \Delta \in \mathcal{F} \).

Let \( \mathcal{N} \) be an AFDF type III factor on a separable Hilbert space \( \mathcal{L} \). Let \( Y \) be a partial isometry of \( \mathcal{N} \) such that \( Y^*Y \neq 1 \) and \( YY^* \neq 1 \). There then exists a partial isometry \( W \) of \( \mathcal{M} \otimes M_{\|\theta\|+2}(\mathbb{C}) \otimes N \) such that \( W^*W = 1 - V^*V \otimes Y^*Y = 1 - 1 \otimes |\eta_0\rangle \langle \eta_0| \otimes Y^*Y \) and \( WW^* = 1 - VV^* \otimes YY^* \). We define a unitary operator \( U \) of \( \mathcal{M} \otimes M_{\|\theta\|+2}(\mathbb{C}) \otimes \mathcal{N} \) by

\[
U = V \otimes Y + W,
\]
and a PVM \( E : \mathcal{F} \to M_{\|\theta\|+2}(\mathbb{C}) \otimes \mathcal{N} \) by

\[
E(\Delta) = E_\eta(\Delta) \otimes 1_\mathcal{F}
\]
for all \( \Delta \in \mathcal{F} \). Let \( \psi \) be a unit vector of \( \mathcal{L} \) such that \( Y^*Y \psi = \psi \). Then we have \( W(\xi \otimes \eta_0 \otimes \psi) = 0 \) for all \( \xi \in \mathcal{H} \). We define a normal state \( \sigma \) on \( M_{\|\theta\|+2}(\mathbb{C}) \otimes \mathcal{N} \) by

\[
\sigma(X) = \langle \eta_0 \otimes \psi | X(\eta_0 \otimes \psi) \rangle
\]
for all \( X \in M_{\|\theta\|+2}(\mathbb{C}) \otimes \mathcal{N} \).

A Hilbert space \( \mathcal{H} \) is then defined by \( \mathcal{H} = C^{\|\theta\|+2} \otimes \mathcal{L} \). Since \( U \in \mathcal{M} \otimes M_{\|\theta\|+2}(\mathbb{C}) \otimes \mathcal{N} \), the 4-tuple \( \mathcal{M} = (\mathcal{H}, \sigma, E, U) \) is an inner measuring process for \((\mathcal{M}, S)\) satisfying the desired property.

Not only for factors \( \mathcal{M} \), we have the following theorem affirmatively resolving Question 3 for physically relevant cases.

**Definition 14.** A measuring process \( \overline{\mathcal{M}} = (\mathcal{H}, \sigma, E, U) \) for \((\mathcal{M}, S)\) is said to be faithful if there exists a normal faithful representation \( \overline{E} : L^\infty(S, \mathcal{M}) \to B(\mathcal{H}) \) such that \( \overline{E}(|\chi_\Delta|) = E(\Delta) \) for all \( \Delta \in \mathcal{F} \).

This definition is the same as [29, Definition 3.4] except that the definition of measuring process for \((\mathcal{M}, S)\) is different.

**Theorem 13.** Let \( \mathcal{M} \) be an injective von Neumann algebra on a Hilbert space \( \mathcal{H} \) and \((S, \mathcal{F})\) a measurable space. For every CP instrument \( \mathcal{I} \) for \((\mathcal{M}, S)\), \( \varepsilon > 0 \), \( n \in \mathbb{N}, \rho_1, \cdots, \rho_n \in \mathcal{I}_n(\mathcal{M}), M_1, \cdots, M_n \in \mathcal{M} \) and \( \Delta_1, \cdots, \Delta_n \in \mathcal{F} \), there exists a faithful measuring process \( \overline{\mathcal{M}} = (\mathcal{H}, \sigma, E, U) \) for \((\mathcal{M}, S)\) in the sense of Definition 9 such that

\[
|\langle \rho_j, \mathcal{I}(M_j, \Delta_j) \rangle - \langle \rho_j, \mathcal{I}_M(M_j, \Delta_j) \rangle| < \varepsilon
\]
for all \( j = 1, \cdots, n \), and that
\[ \mathcal{I}(1, \Delta) = \mathcal{I}_\mathcal{M}(1, \Delta) \]  

(117)

for all \( \Delta \in \mathcal{F} \).

**Proof.** Suppose that \( \mathcal{M} \) is in a standard form without loss of generality. Then there is a norm one projection \( \mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{M} \) and a net \( \{ \Phi_\alpha \}_{\alpha \in A} \) of unital CP maps such that \( \Phi_\alpha(X) \to \mathcal{E}(X) \) for all \( X \in \mathcal{B}(\mathcal{H}) \) by \cite{29} Corollary 3.9, or \cite{29} Proposition 4.2. For every \( \alpha \in A \), a CP instrument \( \mathcal{I}_\alpha \) for \( \mathcal{B}(\mathcal{H}), \mathcal{S} \) is defined by

\[ \mathcal{I}_\alpha(X, \Delta) = \mathcal{I}(\Phi_\alpha(X), \Delta) \]  

(118)

for all \( X \in \mathcal{B}(\mathcal{H}) \) and \( \Delta \in \mathcal{F} \). For every \( \alpha \in A \), \( \mathcal{I}_\alpha \) satisfies

\[ \mathcal{I}_\alpha(1, \Delta) = \mathcal{I}(\Phi_\alpha(1), \Delta) = \mathcal{I}(1, \Delta) \]  

(119)

for all \( \Delta \in \mathcal{F} \).

Let \( \varepsilon > 0, n \in \mathbb{N}, \rho_1, \ldots, \rho_n \in \mathcal{I}_n(\mathcal{M}), M_1, \ldots, M_n \in \mathcal{M} \) and \( \Delta_1, \ldots, \Delta_n \in \mathcal{F} \). There exists \( a_0 \in A \) such that

\[ |\langle \rho_i, \mathcal{I}(M_i, \Delta_i) \rangle - \langle \rho_i, \mathcal{I}_{a_0}(M_i, \Delta_i) \rangle| < \varepsilon \]  

(120)

for every \( i = 1, \ldots, n \).

By \cite{29} Proposition 3.2 and Theorem 7, there exist a Hilbert space \( \mathcal{L}_1 \), a normal faithful representation \( E_1 : \mathcal{L}^\omega(S, \mathcal{I}_{a_0}) \to \mathcal{B}(\mathcal{L}_1) \) and an isometry \( V : \mathcal{M} \to \mathcal{M} \otimes \mathcal{L}_1 \) such that

\[ \mathcal{I}_{a_0}(X, \Delta) = V^* (X \otimes E_1([\chi_\Delta])) V \]  

(121)

for all \( X \in \mathcal{B}(\mathcal{H}) \) and \( \Delta \in \mathcal{F} \).

Because the discussion below is not needed in the case of \( \dim(\mathcal{L}_1) = 1 \), we assume that \( \dim(\mathcal{L}_1) \geq 2 \). Let \( \eta_1 \) be a unit vector of \( \mathcal{L}_1 \). Let \( \mathcal{N} \) be an AFD type III factor on a separable Hilbert space \( \mathcal{L}_2 \). We define a partial isometry \( U_1 : \mathcal{M} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{M} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2 \) by

\[ U_1(x \otimes \xi \otimes \psi) = \langle \eta_1 | \xi \rangle V x \otimes \psi \]  

(122)

for all \( x \in \mathcal{M}, \xi \in \mathcal{L}_1 \) and \( \psi \in \mathcal{L}_2 \). Let \( U_2 \) be an isometry of \( \mathcal{B}(\mathcal{L}_1) \otimes \mathcal{N} \) such that \( U_2 U_2^* = |\eta_1\rangle \langle \eta_1| \otimes 1 \). We define an isometry \( U_3 \) of \( \mathcal{B}(\mathcal{M}) \otimes \mathcal{L}_1 \otimes \mathcal{N} \) by \( U_3 = 1 \otimes U_2 \).

We then define a unitary operator \( U \) of \( \mathcal{M} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathbb{C}^2 \) by

\[ U = U_1 U_3 \otimes G_{11} + [1 - (U_1 U_3)(U_1 U_3)^*] \otimes G_{12} \]  

\[ + [1 - (U_1 U_3)^*(U_1 U_3)] \otimes G_{12} - (U_1 U_3)^* \otimes G_{22} \]  

\[ = U_1 U_3 \otimes G_{11} + (1 - U_1 U_3^*) \otimes G_{12} - (U_1 U_3)^* \otimes G_{22}, \]  

(123)

where

\[ G_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]  

(124)
Let $\eta_2$ be a unit vector of $L_2$. We define a Hilbert space $\mathcal{H} = L_1 \otimes L_2 \otimes \mathbb{C}^2$, a normal state $\sigma$ on $\mathcal{B}(\mathcal{H})$ by
\[
\sigma(X) = \text{Tr}[X|\eta_1 \otimes \eta_2\rangle \langle \eta_1 \otimes \eta_2| \otimes G_{11}]
\] (125)
for all $X \in \mathcal{B}(\mathcal{H})$, and a PVM $E : \mathcal{F} \rightarrow \mathcal{B}(\mathcal{H})$ by
\[
E(\Delta) = E_1([X_\Delta]) \otimes 1_\mathcal{S} \otimes \mathbb{M}_2(\mathbb{C})
\] (126)
for all $\Delta \in \mathcal{F}$, respectively, where Tr is the trace on $\mathcal{B}(L_1 \otimes L_2) \otimes \mathbb{M}_2(\mathbb{C})$.

The 4-tuple $\mathcal{M} = (\mathcal{H}, \sigma, E, \mathcal{U})$ is then a faithful measuring process for $(\mathcal{M}, \mathcal{S})$ that realizes $I_{\alpha_0}$ and that satisfies
\[
|\langle \rho_j, \mathcal{I}(\mathcal{M}, \Delta) \rangle - \langle \rho_j, \mathcal{I}(\mathcal{M}, \Delta) \rangle| < \varepsilon
\] (127)
for all $j = 1, \ldots, n$, and
\[
\mathcal{I}(1, \Delta) = \mathcal{I}(1, \Delta)
\] (128)
for all $\Delta \in \mathcal{F}$.

By the proof of Theorem 13 and facts in Section 2, we have the following corollaries.

**Corollary 6.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(\mathcal{S}, \mathcal{F})$ a measurable space. Then we have
\[
\text{CPInst}_{\mathcal{MRI}}(\mathcal{M}, \mathcal{S}) = \text{CPInst}_{\mathcal{NE}}(\mathcal{M}, \mathcal{S}).
\] (129)

**Proof.** Use [29, Theorem 3.4 (iii)].

**Corollary 7.** Let $\mathcal{M}$ be an atomic von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(\mathcal{S}, \mathcal{F})$ a measurable space. Then we have
\[
\text{CPInst}_{\mathcal{MRI}}(\mathcal{M}, \mathcal{S}) = \text{CPInst}(\mathcal{M}, \mathcal{S}).
\] (130)

**Corollary 8.** Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $(\mathcal{S}, \mathcal{F})$ a measurable space. Then we have
\[
\text{CPInst}_{\mathcal{AR}}(\mathcal{M}, \mathcal{S}) = \text{CPInst}(\mathcal{M}, \mathcal{S}).
\] (131)

Following these results, Question 3 is affirmatively resolved for general $\sigma$-finite von Neumann algebras.

Throughout the present paper, we have developed the dilation theory of systems of measurement correlations and CP instruments, and established many unitary dilation theorems of them. In the succeeding paper, we systematically develop successive and continuous measurements in the generalized Heisenberg picture. The author believes that the approach to quantum measurement theory given in the present and succeeding papers contributes to the categorical (re-)formulation of quantum theory. On the other hand, though we do not know how it is related to the topic of
this paper at the present time, the future task is to find the connection with the results of Haagerup and Musat [16,17], which develop the asymptotic factorizability of CP maps on finite von Neumann algebras.

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