The Semiclassical Coherent State Propagator in the Weyl Representation

Carol Braun†, Feifei Li††, and Anupam Garg∗

Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208

Michael Stone

Department of Physics, University of Illinois at Urbana-Champaign, 1110 W. Green St., Urbana, Illinois 61801

(Dated: September 8, 2015)

Abstract

It is shown that the semiclassical coherent state propagator takes its simplest form when the quantum mechanical Hamiltonian is replaced by its Weyl symbol in defining the classical action, in that there is then no need of a Solari-Kochetov correction. It is also shown that such a correction exists if a symbol other than the Weyl symbol is chosen, and that its form is different depending on the symbol chosen. The various forms of the propagator based on different symbols are shown to be equivalent provided the correspondingly correct Solari-Kochetov correction is included. All these results are shown for both particle and spin coherent state propagators. The global anomaly in the fluctuation determinant is further elucidated by a study of the connection between the discrete fluctuation determinant and the discrete Jacobi equation.

PACS numbers: 03.65.Ca, 03.65.Sq

*e-mail address: agarg@northwestern.edu
I. INTRODUCTION

Coherent-state path integrals for spin and for linear position and momentum degrees of freedom (and related phase-space path integrals) have been the subject of much study for over three decades now \[1–9\], both for intrinsic reasons, and for their semiclassical limit, where they find application to many practical problems. Their mathematical subtleties have, however, prevented their widespread use, in contrast to the Feynman-position space path integral. For example, where in the Feynman integral the paths must be continuous but need not be differentiable, in the coherent-state case the paths need not even be continuous.

In more recent years, steady progress has been made in understanding the semiclassical limit of such path integrals \[10, 11\], and in Ref. 12 it was shown that when the so-called Solari-Kochetov (SK) correction is included, the resulting propagator in the spin case is free of \( j \) versus \( j + \frac{1}{2} \) arbitrariness, has the correct short-time behavior to \( O(T^2) \), and is consistent under composition of successive propagators. In Refs. 13, 14, this work was extended to coherent-state propagators for many particles and many spins.

Insights gained from the above work have led to the successful solution for the Bohr-Sommerfeld quantization rule for spin \[15, 16\], an extension to the instanton calculus \[17\], and to a quantitatively correct explanation \[18, 19\] of the spin tunneling spectrum of the magnetic molecule \( \text{Fe}_8(\text{tacn})_6 \) \[20\]. Still, the answers for the propagators are obtained only by a careful examination of the discrete path integral, and casual application of methods developed for the continuous-time Feynman path integral is fraught with errors. A continuous-time approach was adopted in Ref. 12, where it was found the path integral for the fluctuation determinant suffers from a global anomaly. The resolution of this problem again requires a careful examination of the discrete path integral, and it is shown that the anomaly is absent in a special gauge, whereby the Solari-Kochetov correction is automatically included.

While these successes mean that the coherent-state path integral is no longer the helfalump it once was, there is still some ambiguity in its conception. In particular, while it has long been known that the symbol (or c-number function) that plays the role of the Hamiltonian in the classical action is not unique \[21\], how this nonuniqueness plays out in the final answer for the semiclassical propagator has not been properly explored. It is the purpose of this paper to do so, and in the process elucidate the nature of the SK correction further.
We will show that the SK correction is different, depending on the particular Hamiltonian symbol employed, but that the final answer is independent of this choice. Further, the final answer is best written using the Weyl symbol. As will become clear, this means that the formal, continuous-time coherent-state path integral is not only formal, it is also ambiguous. To give it meaning, one must return to the discrete path integral every time.

Hints that the difficulties of coherent-state path integrals could be related to symbol-choice ambiguities (or what is the same thing, operator ordering ambiguities) may be seen in Refs. 10, 11, 15, 16. Further, Kochetov \[22\] and Pletyukhov \[23\] noted that the SK correction could be written as the difference between the Q symbol and the Weyl symbol for the Hamiltonians, so that if, in contrast to previous papers which had employed the Q symbol, one employed the Weyl symbol in constructing the classical action, there would be no SK correction. Pletyukhov showed this result for a system with position and momentum degrees of freedom in generality, and for spin degrees of freedom within the Holstein-Primakoff approximation. The absence of the SK correction when one employs the Weyl symbol for the Hamiltonian suggests at first that it is simply clumsy to have worked with the Q symbol, and that if one uses the Weyl symbol from the outset, the correction will simply not arise in the first place. If true, this would be a nontrivial result since, as shown in Ref. 12, the correction arises from a global anomaly in the fluctuation determinant, and it is not clear how a change in the way the extremal action is expressed affects the fluctuations. Indeed, it is not clear how one would do the calculation with a general symbol in the first place. With this in mind, we calculate the propagator for particles in the P representation, following closely the derivation based on the Q representation in Sec. II of Ref. 13. Although the resulting change in the discrete path integral is seemingly minor, it leads to a nontrivial change in the final answer, and the SK correction now appears with the opposite sign. We then show that both this answer and the one from the Q representation are equivalent to each other, and to that for Weyl representation. We also show the analogous result for the spin case. In this we corroborate Pletyukhov, but we do not limit ourselves to the Holstein-Primakoff approximation, so our proof is completely general. The Weyl symbol for operators based on position and momentum degrees of freedom is of course classic \[24\], but an analogous one exists for spin degrees of freedom too \[25\], although it is less well known.

The plan of the paper is as follows. We present the results for the Weyl-representation...
propagators for both particles and spin in the next section. This section also serves to introduce our notation, and to define principal terms. The P-representation calculation is done in Sec. III, and the equivalence of the Q-, P-, and Weyl-symbol-based answers is shown in Sec. IV. The propagator for spin in the Weyl representation is derived in Sec. V. In Secs. VI and VII we turn to an examination of the continuum and discrete fluctuation determinants and their connection with the corresponding Jacobi equations with the goal of shedding more light on the global anomaly. Finally, in Sec. VIII we consider what happens when we try and evaluate the propagator for particles by working directly with the discrete action using the Weyl representation. Some essential facts about the Weyl representation (for both particles and spin) are collected in Appendix A.

II. PRINCIPAL RESULTS

A. Propagator for particles

For a particle with linear momentum $p$, coordinate $q$, and arbitrary Hamiltonian $H$, the propagator is defined as

$$K(z_f, z_i; T) = \langle \bar{z}_f | e^{-iHT} | z_i \rangle. \quad (2.1)$$

We have introduced here (unnormalized) harmonic-oscillator-based coherent states,

$$|z\rangle = e^{za^\dagger} |0\rangle, \quad \langle z| = \langle 0| e^{za}, \quad (2.2)$$

with $|0\rangle$ and $\langle 0|$ being the normalized ket and bra for the ground state, and $a$ and $a^\dagger$ being the annihilation and creation operators. In Eq. (2.1), $z_i$ and $z_f$ are arbitrary complex numbers.

The Weyl form of the semiclassical approximation to the propagator $K$ is

$$K^W(z_f, z_i; T) = \left( i \frac{\partial^2}{\partial \bar{z}_f \partial z_i} S^W(z_f, z_i; T) \right)^{1/2} \exp \left[ iS^W(z_f, z_i; T) \right]. \quad (2.3)$$

Here, the classical action $S^W$ is given by

$$iS^W(z_f, z_i; T) = \frac{1}{2} \left[ \bar{z}_f z(T) + \bar{z}(0) z_i \right] + \int_0^T \left[ \frac{\dot{z} \bar{z} - \bar{z} \dot{z}}{2} - iH^W [z(t), \bar{z}(t)] \right] dt, \quad (2.4)$$

with $H^W(z, \bar{z})$ being the Weyl symbol for the Hamiltonian, and $z(t), \bar{z}(t)$ being the solution to the classical equations of motion

$$\frac{dz}{dt} = i \frac{\partial H^W}{\partial \bar{z}}, \quad (2.5)$$

$$\frac{d\bar{z}}{dt} = -i \frac{\partial H^W}{\partial z}. \quad (2.6)$$
with the boundary conditions \( z(0) = z_i, \bar{z}(T) = \bar{z}_f \).

We discuss the Weyl symbol \( H^W \) at greater length in Appendix A. For now it suffices to recall the common textbook definition: \( H^W \) is the c-number function obtained by symmetrizing \( H \) in \( a \) and \( a^\dagger \) and then replacing these operators by the c-numbers \( z \) and \( \bar{z} \) respectively.

The central point of the result (2.3) is that it has no Solari-Kochetov correction. For comparison, when we use the Q symbol, the semiclassical approximation to \( K \) takes the form \[ K^Q(\bar{z}_f, z_i; T) = \left( i \frac{\partial^2 S^Q}{\partial \bar{z}f \partial z_i} \right)^{1/2} \exp \left[ i S^Q(\bar{z}_f, z_i; T) + \frac{i}{2} \int_0^T A^Q(t) \, dt \right] . \] (2.7)

The action \( S^Q \) is given by Eqs. (2.4)–(2.6) with the superscript \( W \) replaced by \( Q \) everywhere, with, additionally, \( H^Q \), the Q representation of the Hamiltonian \[ H^Q(\bar{z}_{j+1}, z_j) = \frac{\langle \bar{z}_{j+1}|H|z_j \rangle}{\langle \bar{z}_{j+1}|z_j \rangle} . \] (2.8)

Lastly,
\[ A^Q = \frac{\partial^2 H^Q}{\partial \bar{z} \partial z} , \] (2.9)
and it is the term containing \( A^Q \) which we call the SK correction in Eq. (2.7).

For completeness, we also give the answer for \( K \) when we employ \( H^P \), the P symbol for the Hamiltonian. We show in Sec. III that
\[ K^P(\bar{z}_f, z_i; T) = \left( i \frac{\partial^2 S^P}{\partial \bar{z}_f \partial z_i} \right)^{1/2} \exp \left[ i S^P(\bar{z}_f, z_i; T) - \frac{i}{2} \int_0^T A^P(t) \, dt \right] . \] (2.10)

All quantities here are the same as in Eqs. (2.7) and (2.9) with the superscript \( Q \) replaced by \( P \), and \( H^P \) defined via
\[ \mathcal{H} = \int \frac{d^2 \bar{z}}{\pi} e^{-\bar{z}z} H^P(\bar{z}, z) |z \rangle \langle \bar{z}| . \] (2.11)

The significant point is that the SK correction enters Eq. (2.10) with a sign opposite to that in Eq. (2.7).

We will show in Sec. IV that \( K^P, K^Q, \) and \( K^W \) are all equal up to the leading two terms in an expansion in \( \hbar \).
B. Propagator for spin

The coherent-state propagator for a spin of magnitude \( j \) is defined in parallel with that for particles:

\[
K(\bar{z}_f, z_i; T) = \langle \bar{z}_f | e^{-iH T} | z_i \rangle. \tag{2.12}
\]

The Hamiltonian is an arbitrary polynomial in the usual spin operators \( J_x, J_y \) and \( J_z \). It follows that \( J^2 = J_x^2 + J_y^2 + J_z^2 \) is a constant of motion, which equals \( j(j+1) \) for spin \( j \). Further, \( |z_i\rangle \) and \( \langle \bar{z}_f | \) are spin coherent states, defined by

\[
|z\rangle = e^{zJ_+ |j, j\rangle}, \quad \langle \bar{z}| = \langle j, j| e^{\bar{z}J_-}, \tag{2.13}
\]

with \( |j, j\rangle \) being the eigenstate of \( J_z \) with eigenvalue \( j \), and \( J_\pm = J_x \pm iJ_y \). Again, the quantities \( z_i \) and \( \bar{z}_f \) are arbitrary complex numbers which give the stereographic coordinates of the maximal spin projection direction in space.

The Weyl-symbol-based semiclassical approximation to \( K \) is

\[
K^W(\bar{z}_f, z_i; T) = \left( \frac{i}{2\tilde{j}} \frac{\partial^2 S^W(\bar{z}_f, z_i; T)}{\partial \bar{z}_f \partial z_i} \right)^{1/2} \exp(iS^W(\bar{z}_f, z_i; T)). \tag{2.14}
\]

Here,

\[
\tilde{j} = j + \frac{1}{2}. \tag{2.15}
\]

This is reminiscent of the oft-stated prescription for the “classical” magnitude of the spin. Second,

\[
iS^W(\bar{z}_f, z_i; T) = \tilde{j} \ln \left[ (1 + \bar{z}_f z(T))(1 + \bar{z}(0) z_i) \right] + \int_0^T dt \left[ \tilde{j} \frac{\dot{\bar{z}} \dot{z}}{1 + \bar{z} z} - iH^W(\bar{z}, z) \right]. \tag{2.16}
\]

Third, the path \((\bar{z}(t), z(t))\) that appears in the action is the solution to the classical equations of motion,

\[
\frac{d\bar{z}}{dt} = i \frac{(1 + \bar{z} z)^2}{2j} \frac{\partial H^W}{\partial z}, \quad \frac{dz}{dt} = -i \frac{(1 + \bar{z} z)^2}{2j} \frac{\partial H^W}{\partial \bar{z}}, \tag{2.17}
\]

subject to the boundary conditions \( z(0) = z_i, \bar{z}(T) = \bar{z}_f \).

As the notation suggests, \( H^W \) is the Weyl symbol for the Hamiltonian in the above expressions. In contrast to the particle case, there is no simple analogue of the symmetrization rule for obtaining \( H^W \). Rather, it is defined by the demands that the map from a spin operator \( F \) to its Weyl symbol \( \Phi^W_F(\bar{z}, z) \) be linear, covariant under rotations, yield a real c-number function for Hermitian operators, and, most importantly, obey the traciality condition

\[
\frac{1}{2j + 1} \text{Tr} (FG) = \frac{1}{\pi} \int \frac{d^2 z}{(1 + \bar{z} z)^2} \Phi^W_F(\bar{z}, z) \Phi^W_G(\bar{z}, z), \tag{2.18}
\]
for any two spin operators $F$ and $G$ and their corresponding Weyl symbols \[31\]. See Ref. \[29\] and references therein for details. A brief catalog of the results most relevant to this paper is given in Appendix \[A2\].

Again, the significant point is that the form (2.14) of the propagator needs no Solari-Kochetov correction. By contrast, the answer based on the Q representation is \[12\]

$$K^Q(\bar{z}_f, z_i; T) = \left[ i \left( 1 + \bar{z}_f z(T) \right) \left( 1 + \bar{z}(0) z_i \right) \frac{\partial^2 S^Q}{\partial \bar{z}_f \partial z_i} \right]^{1/2} \exp \left[ i S^Q(\bar{z}_f, z_i; T) + \frac{i}{2} \int_0^T A^Q(t) \, dt \right],$$

(2.19)

with

$$i S^Q(\bar{z}_f, z_i; T) = j \ln \left[ \left( 1 + \bar{z}_f z(T) \right) \left( 1 + \bar{z}(0) z_i \right) \right] + \int_0^T dt \left[ j \frac{\dot{\bar{z}} z - \bar{z} \dot{z}}{1 + \bar{z} z} - i H^Q(\bar{z}, z) \right],$$

(2.20)

$$A^Q(t) = \frac{1}{2} \left( \frac{\partial}{\partial \bar{z}} \left( 1 + \bar{z} z \right)^2 \frac{\partial H^Q}{\partial z} + \frac{\partial}{\partial z} \left( 1 + \bar{z} z \right)^2 \frac{\partial H^Q}{\partial \bar{z}} \right),$$

(2.21)

and

$$H^Q(\bar{z}, z) = \frac{\langle \bar{z} | \mathcal{H} | z \rangle}{\langle \bar{z} | z \rangle}.$$  

(2.22)

The quantity $A$ is the integrand of the Solari-Kochetov term, and $H^Q$ is the Q symbol for $\mathcal{H}$. The classical path obeys Eq. (2.17) with $H^Q$ in lieu of $H^W$.

We do not bother to write $K^P$ explicitly; it would be completely parallel to Eq. (2.19), with the sign of the SK term reversed. This follows from what we do in Sec. III and its extension to spin as indicated in that section.

### III. COHERENT-STATE PROPAGATORS FOR PARTICLES

In this section, we consider the coherent-state propagator for particles. We show in Sec. III A that the path integral for the propagator is not unique, and illustrate this by giving three different expressions for it. The first two are based on the Q and P symbols for the Hamiltonian, while the third uses alternating Q and P symbols. The semiclassical propagator studied in Refs. \[7\]–\[9\], \[12\]–\[14\] is the one based on the Q-symbol expression, and it contains the original SK correction. We will calculate the P-symbol-based propagator in Sec. III B, where it will be seen that the SK correction arises with the opposite sign from that when the Q symbol is employed. The calculation starting from the mixed P-Q expression will be given in Sec. VIII B.
A. Setting up the path integral

We expect on general grounds that, in any semiclassical approximation, \( K \sim \exp(iS) \), where \( S \) is the action for the classical path running from the initial state to the final state. However, this degree of approximation (analogous to the eikonal approximation in the WKB method) is too crude, for it ignores the conservation of probability. For that, one must include the next term in an expansion in powers of \( \bar{\hbar} \). This correction generally takes the form of a pre-exponential factor \( \sim (\partial^2 S / \partial \bar{z}_f \partial z_i)^{1/2} \), but of course we must find it more precisely. It is clear, however, that one must also calculate the exponent or eikonal correct to the leading \( \text{two} \) orders in an expansion in \( \bar{\hbar} \). All these points are well known, but we dwell on them because the next-to-leading-order term is the source of all the trouble in all coherent-state-based semiclassical propagators and of Solari-Kochetov corrections in particular. Finding this term correctly is important as it is the one that assures conservation of probability.

To calculate \( K \), we divide the interval \( T \) into \( M \) slices of width \( \Delta \) each:

\[
\Delta = T/M,
\]

where \( M \gg 1 \), so that \( \Delta \) is infinitesimal and an expansion in \( \Delta \) is permissible. We then write

\[
e^{i\mathcal{H}T} = e^{-i\Delta \mathcal{H}} e^{-i\Delta \mathcal{H}} \cdots e^{-i\Delta \mathcal{H}} \quad (M \text{ factors}).
\]

Next, we insert a resolution of unity between every pair of adjacent factors in Eq. (3.2). Correct to order \( \Delta \), the propagator for one time slice is now evaluated as

\[
\langle \bar{z}_{j+1} | e^{-i\Delta \mathcal{H}} | z_j \rangle = \langle \bar{z}_{j+1} | z_j \rangle \exp \left( -i \Delta \mathcal{H}^Q (\bar{z}_{j+1}, z_j) + O(\Delta^2) \right),
\]

where

\[
\mathcal{H}^Q (\bar{z}_{j+1}, z_j) = \frac{\langle \bar{z}_{j+1} | \mathcal{H} | z_j \rangle}{\langle \bar{z}_{j+1} | z_j \rangle}.
\]

Inserting the explicit expressions for overlaps such as \( \langle \bar{z}_{j+1} | z_j \rangle \), we obtain

\[
K(\bar{z}_f, z_i; T) \approx \prod_{j=1}^{M-1} \int \frac{d^2 \bar{z}_j}{\pi} \exp (i S^Q_{\text{disc}}),
\]
where \( S_{\text{disc}}^Q \), the discrete action, is
\[
i S_{\text{disc}}^Q = (\bar{z}_M z_{M-1} - \bar{z}_{M-1} z_{M-1}) + (\bar{z}_{M-1} z_{M-2} - \bar{z}_{M-2} z_{M-2}) + \cdots \\
+ (\bar{z}_2 z_1 - \bar{z}_1 z_1) + \bar{z}_1 z_0 - i \Delta \sum_{j=0}^{M-1} H^Q(\bar{z}_{j+1}, z_j).
\] (3.6)

Here, \( \bar{z}_M \equiv \bar{z}_f \), and \( z_0 \equiv z_i \), and it should be observed that \( S_{\text{disc}}^Q \) does not depend on \( z_M \) and \( \bar{z}_0 \) for the simple reason that no such variables have been defined in the first place.

We obtain a different expression for \( K \) based on the \( P \) symbol, if, again correct to order \( \Delta \), we write the \( j \)th factor from the right in the string (3.2) as
\[
e^{-iH \Delta} = \int \frac{d^2z_j}{\pi} e^{-\bar{z}_j z_j - i\Delta H^P(\bar{z}_j, z_j)} |z_j \rangle \langle \bar{z}_j|.
\] (3.7)

Carrying out this substitution, we obtain
\[
K(\bar{z}_f, z_i; T) \approx \prod_{j=1}^{M} \int \frac{d^2z_j}{\pi} \exp (iS_{\text{disc}}^P),
\] (3.8)

where \( S_{\text{disc}}^P \) is another discrete action, given by
\[
i S_{\text{disc}}^P = (\bar{z}_{M+1} z_M - \bar{z}_M z_M) + (\bar{z}_M z_{M-1} - \bar{z}_{M-1} z_{M-1}) + \cdots \\
+ (\bar{z}_2 z_1 - \bar{z}_1 z_1) + \bar{z}_1 z_0 - i \Delta \sum_{j=1}^{M} H^P(\bar{z}_j, z_j).
\] (3.9)

Again, \( \bar{z}_{M+1} \equiv \bar{z}_f \), \( z_0 \equiv z_i \), and variables \( z_{M+1} \) and \( \bar{z}_0 \) do not exist, never having been defined.

Equation (3.9) differs from Eq. (3.6) in two ways. The first is that we now have \( M \) integrations instead of \( M - 1 \). Since we are eventually going to let \( M \to \infty \), this change is insignificant. The second difference is that \( H^P \) is evaluated at \( \bar{z}_j \) and \( z_j \) in the \( j \)th slice, whereas \( H^Q \) is evaluated at \( \bar{z}_{j+1} \) and \( z_j \). When we evaluate the extremal value of the action, we do so on a path where \( \bar{z}_{j+1} - \bar{z}_j = O(\Delta) \), so the second change would also appear to be inconsequential. Yet it is on precisely this difference that everything will pend, for it affects the essential properties of the two fluctuation operators vis-a-vis their self-adjointness, or lack thereof.

We obtain yet another expression for \( K \) if, instead of using all \( P \)'s or all \( Q \)'s, we alternate between the two. Let us consider the first two time steps starting with the state \( |z_0 \rangle \) (\( z_0 \equiv z_i \)). We approximate propagation in the first step via \( H^P \), i.e., we write
\[
e^{-iH \Delta} |z_0 \rangle \approx \int \frac{d^2z_1}{\pi} e^{-\bar{z}_1 z_1 - i\Delta H^P(\bar{z}_1, z_1)} |z_1 \rangle \langle \bar{z}_1| z_0 \rangle.
\] (3.10)
We propagate across the next time step by evolving the integrated-over state $|z_1\rangle$ which appears above via $H^Q$, i.e., we write

$$e^{-i\mathcal{H}\Delta}|z_1\rangle = \int d^2z_2 \frac{e^{-\bar{z}_2 z_2}}{\pi} \langle \bar{z}_2 | e^{-i\mathcal{H}\Delta} |z_1\rangle, \approx \int d^2z_2 \frac{e^{-\bar{z}_2 z_2}}{\pi} \left[ e^{\bar{z}_2 z_1} e^{-i\Delta H^Q(\bar{z}_2, z_1)} \right].$$

(3.11)

These two steps generate the following part of $iS_{\text{disc}}$:

$$\bar{z}_2 z_1 - \bar{z}_1 z_1 + \bar{z}_1 z_0 - i\Delta \left( H^Q(\bar{z}_2, z_1) + H^P(\bar{z}_1, z_1) \right).$$

(3.12)

We continue in this way, alternating $H^P$ and $H^Q$. The resulting discrete action is

$$iS_{\text{disc}}^A = (\bar{z}_{M+1} z_M - \bar{z}_M z_M) + (\bar{z}_M z_{M-1} - \bar{z}_{M-1} z_{M-1}) + \cdots$$

$$+(\bar{z}_2 z_1 - \bar{z}_1 z_1) + \bar{z}_1 z_0 - i\Delta \sum_{j=1,3,5,\ldots} \left( H^Q(\bar{z}_{j+1}, z_j) + H^P(\bar{z}_j, z_j) \right).$$

(3.13)

The superscript $A$ stands for ‘alternating’.

It is clear that we can use $H^P$ and $H^Q$ in any order, and thus obtain infinitely many discrete path-integral expressions for $K$. We could also try and write $\langle \bar{z}' | e^{-i\mathcal{H}\Delta} |z \rangle$ in terms of the Weyl symbol $H^W$ using Eq. (A12), extending the set of expressions even more. This immediately raises the question of how these different expressions will lead to the same semiclassical answer for $K$. We will address this question for the P representation in the next subsection, for the mixed P-Q representation in Sec. VIII B, and for the direct replacement via the Weyl symbol in Sec. VIII A. First, however, let us see what happens if we take the formal continuous-time limit ($\Delta \to 0$), and write the propagator as the path integral

$$K_{\text{fcl}} = \int [d^2z] e^{iS_{\text{fcl}}[\bar{z}, z]},$$

(3.14)

with

$$iS_{\text{fcl}} = \frac{1}{2} \left( \bar{z} f(T) + \bar{z}(0) z_i \right) + \int_0^T dt \left[ \frac{\bar{z} z - \bar{z} z}{2} - iH(\bar{z}, z) \right].$$

(3.15)

Not only is this equation merely formal, it is also meaningless, because $H$ could stand for $H^P$, $H^Q$, or something else, depending on which discrete path integral one starts with. This lack of meaning explains why there is an anomaly in the corresponding path integral. If we try and work with Eq. (3.15) as was done in Ref. 12, we will have to first specify what $H(\bar{z}, z)$ means, and, depending on that, the prescription for regulating the global anomaly will be different. This prescription will have to be obtained by examining the discrete path integral once again, so it seems that one is best off by working with the discrete form all the way, and eschewing the formal continuous-time form altogether [32].
B. P-representation propagator by integration by successive time slices

In this section, we find the particle-case propagator starting with Eq. (3.9). We will do this using the method of Ref. 13 since this method can be generalized to arbitrarily many particles and to arbitrarily many spins [14]. Since these references show how the extension to more than one particle or one spin is performed, we will show the calculation for one particle only, and leave the obvious generalization to many particles and many spins to the reader. We use the same notation, and focus on the changes that arise, so readers may wish to have a copy of Ref. 13 handy as they read along.

The first step is to find the “classical” or extremizing path. The equations for this are essentially the same, and formally identical when we pass to the $\Delta \rightarrow 0$ limit. The next step is to expand the action to second order in fluctuations around the extremizing path. Denoting the deviations in $z_j$ and $\bar{z}_j$ from the classical path by $\eta_j$ and $\bar{\eta}_j$, the second variation of the action is

$$\delta^2 S^P_{\text{disc}} = \frac{1}{2!} \left[ \sum_{j=1}^{M} \left( \eta_j \frac{\partial}{\partial z_j} + \bar{\eta}_j \frac{\partial}{\partial \bar{z}_j} \right) \right]^2 S^P_{\text{disc}}.$$  \hspace{1cm} (3.16)

In terms of this quadratic form, the reduced propagator (the quantity multiplying the exponential of the classical action times $i$) is given by

$$K^P_{\text{red}}(\bar{z}_f, z_i; T) = \left[ \prod_{j=1}^{M} \int \frac{d^2 \eta_j}{\pi} \right] \exp (i\delta^2 S^P_{\text{disc}}).$$ \hspace{1cm} (3.17)

As in Ref. 13, most of the derivatives in $\delta^2 S^P_{\text{disc}}$ are zero. The exact expressions for the nonzero coefficients are slightly different, and they are now given by

$$D_{jj} = -i \frac{\partial^2 S^P_{\text{disc}}}{\partial z_j^2} = i \Delta \frac{\partial^2}{\partial \bar{z}_j} H^P(\bar{z}_j, z_j),$$ \hspace{1cm} (3.18)

$$D_{\bar{j}j} = -i \frac{\partial^2 S^P_{\text{disc}}}{\partial \bar{z}_j^2} = i \Delta \frac{\partial^2}{\partial z_j} H^P(\bar{z}_j, z_j),$$ \hspace{1cm} (3.19)

$$D_{\bar{j}\bar{j}} = -i \frac{\partial^2 S^P_{\text{disc}}}{\partial \bar{z}_j \partial \bar{z}_j} = 1 + i \Delta \frac{\partial^2}{\partial z_j \partial \bar{z}_j} H^P(\bar{z}_j, z_j),$$ \hspace{1cm} (3.20)

$$D_{jj} = -i \frac{\partial^2 S^P_{\text{disc}}}{\partial z_j \partial \bar{z}_j} = 1 + i \Delta \frac{\partial^2}{\partial \bar{z}_j \partial z_j} H^P(\bar{z}_j, z_j),$$ \hspace{1cm} (3.21)

$$D_{\bar{j}j+1} = -i \frac{\partial^2 S^P_{\text{disc}}}{\partial \bar{z}_{j+1} \partial \bar{z}_j} = -1,$$ \hspace{1cm} (3.22)

$$D_{j\bar{j}+1} = -i \frac{\partial^2 S^P_{\text{disc}}}{\partial z_j \partial \bar{z}_{j+1}} = -1.$$ \hspace{1cm} (3.23)
Of these the first two are essentially the same as before (i.e., for the Q representation), but the last four are different. Clearly, $D_{ij} = D_{ji}$ and $D_{j+1\rightarrow j} = D_{j\rightarrow j+1}$.

The procedure at this point is to carry out the integrals time slice by successive time slice, and step three is to isolate the quantities that appear in the integral at the $j$th slice. We write this integral as

$$
\int \frac{d^2\eta_j}{\pi} \exp \left[ -\frac{1}{2} (\bar{\eta}_j \eta_j) G_j \left( \eta_j \eta_j \right) + \bar{\eta}_j \left( \eta_j \eta_j \right) + \left( \eta_j \eta_j \right) V_j \right],
$$

just as Eq. (2.34) in Ref. 13. The quantities $\eta_j$ and $\bar{\eta}_j$ are the deviations in $z_j$ and $\bar{z}_j$ from the classical path, $\bar{V}_j$ and $V_j$ are row and column vectors given by

$$
\bar{V}_j = -\frac{1}{2} \left( \eta_{j+1} \eta_j \right) \left( \begin{array}{cc} D_{j+1\rightarrow j} & 0 \\ 0 & 0 \end{array} \right) = \frac{1}{2} \tilde{n}_{j+1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
$$

$$
V_j = -\frac{1}{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & D_{j\rightarrow j+1} \end{array} \right) \left( \begin{array}{c} \eta_{j+1} \\ \bar{\eta}_{j+1} \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \tilde{n}_{j+1},
$$

and $G_j$ is a $2 \times 2$ matrix that will be determined recursively. To avoid confusion with the time-slice labels, we label its elements with the letters “u” and “d” (for “up” and “down”), thus:

$$
G_j = \left( \begin{array}{cc} G_{j,uu} & G_{j,ud} \\ G_{j,du} & G_{j,dd} \end{array} \right).
$$

The fourth step is to shift $\eta_j$ and $\bar{\eta}_j$ so as to complete the square, and perform the integration for the $j$th slice. The shifts are given by

$$
\left( \begin{array}{c} \gamma_j \\ \bar{\gamma}_j \end{array} \right) = G_j^{-1} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \tilde{n}_{j+1},
$$

$$
\left( \begin{array}{cc} \gamma_j & \gamma_j \end{array} \right) = \tilde{n}_{j+1} \left( \begin{array}{cc} 1 & 0 \end{array} \right) G_j^{-1}.
$$

This leads, as before, to the consistency condition

$$
G_{j,uu} = G_{j,dd}.
$$

This condition holds for $j = 1$, since

$$
G_1 = \left( \begin{array}{cc} D_{11} & D_{11} \\ D_{11} & D_{11} \end{array} \right).
$$
and $D_{11} = D_{11}$. We shall see from the recursion found below that it holds for all $j$. The integral gives an overall factor of $(\det G_j)^{-1/2}$, and a residual term in the exponent from completing the square,

$$\frac{1}{2} \left( \tilde{\gamma}_j \gamma_j \right) G_j \left( \frac{\gamma_j}{\tilde{\gamma}_j} \right) = \frac{1}{2} \tilde{\eta}^2_{j+1} (G^{-1}_j)_{ud}. \quad (3.32)$$

Step five is to examine the recursion relation for $G_j$ and $\det(G_j)$. Equation (3.32) implies that

$$G_{j+1} = \begin{pmatrix} D_{j+1,j+1} & D_{j+1,j+1} \\ D_{j+1,j+1} & D_{j+1,j+1} \end{pmatrix} - \begin{pmatrix} 0 & (G^{-1}_j)_{ud} \\ 0 & 0 \end{pmatrix} \quad (3.33)$$

This shows, first, that the consistency condition (3.30) holds for all $j$. Second, as in Ref. 13, there is no meaningful recursion relation for $\det G_j$, but there is one for the $ud$ element $G_{ud}$. Since $(G^{-1}_{j})_{ud} = -G_{j,ud}/\det(G_j)$, this recursion relation is

$$G_{j+1,ud} = D_{j+1,j+1} + (\det G_j)^{-1}G_{j,ud}, \quad (3.34)$$

which, a priori, looks different from that in Ref. 13. To see its explicit form, we note that

$$\det G_j = 1 + 2i\Delta \frac{\partial^2 H^P}{\partial \bar{z}_j \partial z_j} - i\Delta \frac{\partial^2 H^P}{\partial z_j^2} G_{j,ud} + O(\Delta^2), \quad (3.35)$$

which along with the expression for $D_{j+1,j+1}$ leads to

$$G_{j+1,ud} = i\Delta \frac{\partial^2 H^P}{\partial z_j^2} + \left( 1 - 2i\Delta \frac{\partial^2 H^P}{\partial \bar{z}_j \partial z_j} + i\Delta \frac{\partial^2 H^P}{\partial z_j^2} G_{j,ud} + O(\Delta)^2 \right) G_{j,ud}. \quad (3.36)$$

This explicit form is the same as when we use the Q representation except that instead of $H^P(\bar{z}_j, z_j)$ we have $H^Q(\bar{z}_{j+1}, z_j)$. Writing $j\Delta = t$, and taking the limit $\Delta \to 0$, Eq. (3.36) turns into the Riccati differential equation,

$$-i\dot{G}_{ud} = B - 2AG_{ud} + \bar{B}G_{ud}^2, \quad (3.37)$$

which must be solved with the initial condition $G_{ud}(0) = 0$. Here,

$$A = \frac{\partial^2 H^P}{\partial \bar{z} \partial z}, \quad B = \frac{\partial^2 H^P}{\partial \bar{z}^2}, \quad \bar{B} = \frac{\partial^2 H^P}{\partial z^2}. \quad (3.38)$$

The solution to this differential equation is, from Ref. 13,

$$G_{ud}(t) = \frac{1}{B(t)} \left( A(t) + \frac{i}{\sqrt{v}} \right). \quad (3.39)$$
Here,

\[ v(t) = \frac{\delta \bar{z}(t)}{\delta \bar{z}(0)}, \quad (3.40) \]

which is a Jacobi field that describes how the classical trajectory for \( \bar{z}(t) \) changes upon a change in the initial value of \( \bar{z}(0) \) while holding \( z(0) \) fixed. In particular,

\[ v(T) = \left( i \frac{\partial^2}{\partial \bar{z}_f \partial z_i} S^P(\bar{z}_f, z_i; T) \right)^{-1}. \quad (3.41) \]

The quantity of greater interest to us, however, is not \( G_{ud}(t) \) but \( \det G(t) \) (or \( \det G_j \)), since it is this determinant that we pick up from the integration at each time slice. The reduced propagator is

\[ K^P_{\text{red}} = \prod_{j=1}^{M} (\det G_j)^{-1/2}. \quad (3.42) \]

Taking logs converts the product into a sum, which turns into an integral in the limit \( \Delta \to 0 \). We found \( \det G_j \) in Eq. (3.35). Hence,

\[ \ln K^P_{\text{red}} = -\frac{i}{2} \int_0^T dt \left[ 2A^P(t) - \bar{B}(t)G_{ud}(t) \right], \quad (3.43) \]

which differs from Ref. 13 in the extra first term, \( 2A(t) \). Feeding in the solution (3.39), we obtain

\[ \ln K^P_{\text{red}} = -\frac{i}{2} \int_0^T A^P(t) dt - \frac{1}{2} \ln v(T). \quad (3.44) \]

Hence, the final answer for the propagator in the semiclassical approximation is, as advertised before,

\[ K^P(\bar{z}_f, z_i; T) = \left( i \frac{\partial^2 S^P}{\partial \bar{z}_f \partial z_i} \right)^{1/2} \exp \left[ iS^P(\bar{z}_f, z_i; T) - \frac{i}{2} \int_0^T A^P(t) dt \right]. \quad (3.45) \]

**IV. EQUIVALENCE OF THE PARTICLE PROPAGATOR IN DIFFERENT REPRESENTATIONS**

Our goal in this section is to show that Eqs. (2.10) and (2.7) are equivalent, and to write the propagator using the Weyl representation.

If we look at Eqs. (A16) and (A20) it seems that we can replace \( H^Q \) and \( H^P \) by \( H^W \) in \( K^Q \) and \( K^P \) and delete the Solari-Kochetov correction. This will turn out to be correct, but there is one subtlety which we must first mind. The path \((\bar{z}(t), z(t))\) which appears in the action \( S^P \) is obtained by solving the equations of motion (2.5) and (2.6) but with
$H^P$ instead of $H^W$. Let us temporarily denote the path by $(\bar{z}^P(t), z^P(t))$ and the action by $S^P[\bar{z}^P(t), z^P(t)]$ to emphasize this fact. Let us likewise denote the classical path based on $H^W$ by $(\bar{z}^W(t), z^W(t))$. Since

$$H^W(\bar{z}, z) = H^P(\bar{z}, z) \times (1 + O(\hbar)), \quad (4.1)$$

it follows that

$$\bar{z}^W(t) = \bar{z}^P(t) \times (1 + O(\hbar)), \quad z^W(t) = z^P(t) \times (1 + O(\hbar)). \quad (4.2)$$

The action $S^P$ is, however, an extremal value. A small change in the path therefore changes the action only in second order. That is

$$S^P[\bar{z}^W(t), z^W(t)] = S^P[\bar{z}^P(t), z^P(t)] \times (1 + O(\hbar^2)). \quad (4.3)$$

By the same argument,

$$S^W[\bar{z}^W(t), z^W(t)] = S^P[\bar{z}^W(t), z^W(t)] \times (1 + O(\hbar^2)), \quad (4.4)$$

$$= S^P[\bar{z}^P(t), z^P(t)] \times (1 + O(\hbar^2)). \quad (4.5)$$

Since our goal in calculating the semiclassical propagator is to obtain it correctly up to the first term in relative order $\hbar$, these changes are beyond the accuracy to which we are working, and may be neglected. They may be similarly neglected in the prefactor $(\partial^2 S^P / \partial \bar{z}_f \partial z_i)^{1/2}$. To this order of accuracy, therefore, $K^P = K^W$. By the same argument, $K^Q = K^W$.

V. SPIN PROPAGATOR IN THE WEYL REPRESENTATION

In this section we shall give the propagator for spin following the ideas developed in the previous sections for particles, and notation developed in Ref. 29. Our aim is to obtain Eq. (2.14), starting with the previously obtained result, Eq. (2.19), i.e., to rewrite $K^Q$ in terms of the Weyl symbol $H^W(\bar{z}, z)$ in parallel with Sec. [V]. As a preliminary step, we first discuss the stereographic variables $z$ and $\bar{z}$, which are often more convenient descriptors of the phase-space sphere than the orientation $\hat{n}$. If the spherical polar coordinates of $\hat{n}$ are taken as $(\theta, \varphi)$, then

$$z = \tan \frac{\theta}{2} e^{i\varphi}, \quad \bar{z} = \tan \frac{\theta}{2} e^{-i\varphi}. \quad (5.1)$$
The spin coherent state $|\hat{n}\rangle$, which is the state with maximum spin projection along $\hat{n}$, i.e.,

$$J \cdot \hat{n} |\hat{n}\rangle = j |\hat{n}\rangle,$$  \hspace{1cm} (5.2)

can clearly be obtained from the state with maximum projection along $\hat{z}$, i.e., $|j, j\rangle$, by applying a rotation. When the requisite rotation operator is written in terms of $z$ and $\bar{z}$, its action on $|j, j\rangle$ can be cast in the form (2.13) up to a multiplicative constant, i.e.,

$$|\hat{n}\rangle \propto |z\rangle.$$  \hspace{1cm} (5.3)

This result is a proportionality rather than an equality because, as defined in Eq. (2.13), the state $|z\rangle$ and its dual bra $\langle \bar{z}|$ are not normalized; rather

$$\langle \bar{z}|z\rangle' = (1 + \bar{z}z')^{2j}.$$  \hspace{1cm} (5.4)

The resolution of unity now takes the form

$$1 = \frac{2j + 1}{\pi} \int \frac{d^2z}{(1 + \bar{z}z)^{2j+2}} |z\rangle \langle \bar{z}|.$$  \hspace{1cm} (5.5)

The benefit of using unnormalized states and $z$, $\bar{z}$ variables is the same as for particle coherent states: Matrix elements are analytic in $z$ and $\bar{z}$, and we can exploit analyticity to simplify many calculations.

Next, we note that, by Eq. (A30),

$$H^W(\bar{z}, z) = H^Q + \frac{L^2}{4j} H^Q,$$  \hspace{1cm} (5.6)

where $L = -i (\hat{n} \times \nabla \hat{n})$ is the angular momentum operator on phase space (not the Hilbert space of the states $|j, m\rangle$). In terms of $z$ and $\bar{z}$, \[33\],

$$L^2 = -(1 + \bar{z}z)^2 \frac{\partial^2}{\partial z \partial \bar{z}}.$$  \hspace{1cm} (5.7)

Hence, Eq. (A30) may be written as

$$H^W(\bar{z}, z) = H^Q(\bar{z}, z) - \frac{(1 + \bar{z}z)^2}{4j} \frac{\partial^2 H^Q}{\partial z \partial \bar{z}},$$  \hspace{1cm} (5.8)

which is correct up to relative order $1/j$. The next step is to write

$$A(t) = A_1(t) + A_2(t),$$  \hspace{1cm} (5.9)
where

\begin{align}
A_1 &= \frac{(1 + \bar{z}z)^2}{2j} \frac{\partial^2 H^Q}{\partial z \partial \bar{z}}, \\
A_2 &= \frac{(1 + \bar{z}z)}{2j} \left( z \frac{\partial H^Q}{\partial z} + \bar{z} \frac{\partial H^Q}{\partial \bar{z}} \right). 
\end{align}

(5.10) \hspace{1cm} (5.11)

In \(A_1\), we may replace \(2j\) by \(\tilde{j}\) in the denominator since \(A\) is already of order \(1/j\) relative to \(S\), and we do not care about errors of relative order \(1/j^2\). Thus,

\[ A_1(t) = -\frac{\mathcal{L}^2}{2j} H^Q[\bar{z}(t), z(t)]. \]

(5.12)

For \(A_2\), we recast it by using the equations of motion. Thus,

\[ A_2(t) = -\frac{j\dot{\bar{z}}z - \bar{z}\dot{z}}{1 + \bar{z}z}. \]

(5.13)

The terms \(A_1\) and \(A_2\) can be combined, respectively, with the second and the first terms in the integral in Eq. (2.20) to yield

\[
\int_0^T dt \left[ j \frac{\dot{z}z - \bar{z}\dot{z}}{1 + \bar{z}z} - i H^Q(\bar{z}, z) \right] + \frac{i}{2} \int_0^T A(t) dt = \int_0^T dt \left[ j \frac{\dot{z}z - \bar{z}\dot{z}}{1 + \bar{z}z} - i \left( 1 + \frac{\mathcal{L}^2}{4j} \right) H^Q(\bar{z}, z) \right] \\
= \int_0^T dt \left[ j \frac{\dot{z}z - \bar{z}\dot{z}}{1 + \bar{z}z} - i H^W(\bar{z}, z) \right].
\]

(5.14)

Next, we observe that the coefficient of the explicit boundary term in the action (2.20) can be changed from \(j\) to \(\tilde{j}\) by lifting the corresponding term in the prefactor into the exponent. In this way, we obtain

\[
K^Q(\bar{z}_f, z_i; T) = \left( \frac{i}{2j} \frac{\partial^2 S^Q}{\partial \bar{z}_f \partial z_i} \right)^{1/2} \exp \left( j \ln \left[ (1 + \bar{z}_f z(T)) (1 + \bar{z}(0)z_i) \right] \right) \\
\times \exp \left( \int_0^T dt \left[ j \frac{\dot{z}z - \bar{z}\dot{z}}{1 + \bar{z}z} - i H^W(\bar{z}, z) \right] \right).
\]

(5.15)

We can now make two further changes which only affect our answer to relative order \(1/j^2\). First, we can employ the same argument which led to Eq. (4.3) to replace the path used to calculate the action be the one based on \(H^W\) instead of \(H^Q\). Second, we can replace the \(j\) in the prefactor by \(\tilde{j}\). This gives us the Weyl-symbol-based propagator, Eq. (2.14).
VI. THE CONTINUUM FLUCTUATION OPERATOR AND THE CONTINUUM JACOBI EQUATION

While the action in the formal continuum limit, $S_{\text{fcl}}$, is ambiguous it is still useful to consider the reduced path integral,

$$K_{\text{fcl}}^{\text{red}} = \int [dz][d\bar{z}] \exp(i\delta^2 S_{\text{fcl}}[\bar{\eta}, \eta]),$$

where

$$i\delta^2 S_{\text{fcl}}[\bar{\eta}, \eta] = \frac{i}{2} \int_0^T \left( \bar{\eta}(t) \eta(t) \right) D_{\text{fcl}} \begin{pmatrix} \eta(t) \\ \bar{\eta}(t) \end{pmatrix} dt,$$

with $D_{\text{fcl}}$ being the fluctuation operator

$$D_{\text{fcl}} = \begin{pmatrix} -i\partial_t + A(t) & B(t) \\ B(t) & i\partial_t + A(t) \end{pmatrix},$$

acting on paths that obey the constraints $\eta(0) = \bar{\eta}(T) = 0$. The ambiguity in the formal continuum limit shows up as follows. As found in Ref. 12, because of the global anomaly in the path integral $K_{\text{fcl}}^{\text{red}}$, the operator $D_{\text{fcl}}$ has no eigenfunctions, not even one. Thus, $\det D_{\text{fcl}}$ cannot be defined as the product of the eigenvalues of $D_{\text{fcl}}$. For the same reason, the standard method for finding this determinant based on solving the associated Jacobi equation also fails.

To explain the nature of this failure, we now describe the Jacobi-equation-based method. The classical equations based on the continuum action are

$$\frac{dz}{dt} = -i\frac{\partial H}{\partial z}, \quad \frac{d\bar{z}}{dt} = i\frac{\partial H}{\partial \bar{z}}.$$ (6.4)

These equations have to be solved with the boundary conditions $z(0) = z_i$, $\bar{z}(T) = \bar{z}_f$. The other boundary values, $\bar{z}(0)$ and $z(T)$ are not fixed, but emerge from the solution and may thus be regarded as functions of $z_i$, $\bar{z}_f$, and $T$. If now we use the value of $\bar{z}(0)$ so found and $z_i$ to solve the classical equations of motion as an initial value problem, we will recover the classical solution for $z(t)$ and $\bar{z}(t)$. If we change the initial values to $z_i$ and $\bar{z}(0) + \epsilon$, where $\epsilon$ is infinitesimal, the solution to the initial value problem will deviate from the previous one by terms of order $\epsilon$ in leading order. Denoting the deviations in $z(t)$ and $\bar{z}(t)$ by $\epsilon u(t)$ and $\epsilon v(t)$ respectively, we find the Jacobi equations,

$$\begin{pmatrix} -i\partial_t + A(t) & B(t) \\ \bar{B}(t) & i\partial_t + A(t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0,$$ (6.5)
with initial conditions \( u(0) = 0, v(0) = 1 \). (Since we will not need it, we do not bother writing the full Jacobi system allowing for variations in \( z_i \) also.) The Jacobi-based method says that

\[
\det \mathcal{D}_{kl} = v(T). 
\]  

(6.6)

As found in Ref. 12, Eq. (6.6) is incorrect and should be multiplied by an undetermined phase factor, \( e^{i\gamma} \). This phase factor is the SK correction, which we now know differs for the P and Q representations, while \( \mathcal{D}_{kl} \) is superficially the same in the two cases.

For completeness, and to enable the reader to understand the answers (2.3), (2.7), and (2.10) for the propagator, we mention that

\[
v(T) = \left( \frac{\partial^2 S}{\partial \bar{z}_f \partial z_i} \right)^{-1}.
\]  

(6.7)

The proof is standard. See, e.g., Sec. 4 of Ref. 12 or Ref. 34.

VII. THE DISCRETE FLUCTUATION OPERATOR AND THE DISCRETE JACOBI EQUATION

In this section, we return to the discrete path integral and, for both the P and Q representations, examine the fluctuation operator by writing it as a tridiagonal matrix. The same operator determines the Jacobi equation. We will show that unlike the continuum case, the determinant of the discrete operator is not simply equal to the solution to the discrete Jacobi equation but also contains an SK correction. We will see why the correction differs between the two cases (P and Q). We will further see that the solution to the discrete Jacobi equation tends to the continuum solution in the limit \( \Delta \to 0 \). This shows the precise way in which the equality of the Jacobi field and the fluctuation determinant breaks down in this limit.

This method cannot be extended (at least we do not know how) to more than one particle or spin, but the insights it provides as described above still make it worth presenting.
A. The source of the SK correction

For either Eq. (3.9) or Eq. (3.6), we can write (taking the number of intermediate integrations as $M$ in both cases)

$$i\delta^2 S_{\text{disc}}[\bar{\eta}, \eta] = -\frac{1}{2} \begin{bmatrix} \bar{\eta}_1 & \eta_1 & \bar{\eta}_2 & \eta_2 & \ldots & \bar{\eta}_M & \eta_M \end{bmatrix} \mathcal{D}_{\text{disc}} \begin{bmatrix} \bar{\eta}_1 \\ \eta_1 \\ \bar{\eta}_2 \\ \eta_2 \\ \vdots \\ \bar{\eta}_M \\ \eta_M \end{bmatrix}, \quad (7.1)$$

where $\mathcal{D}_{\text{disc}}$ is the discrete fluctuation operator (or matrix)

$$\mathcal{D}_{\text{disc}} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \\ & D_{23} \\ & & \ddots \end{bmatrix} . \quad (7.2)$$

Note that we have reordered the $\eta$’s and $\bar{\eta}$’s in the row vector, as this makes $\mathcal{D}_{\text{disc}}$ a manifestly symmetric and tridiagonal matrix, albeit complex [35]. The reordering leads to $M$ additional factors of $-1$ when the determinant is evaluated, so that

$$K_{\text{red}} = [(-1)^M \det \mathcal{D}_{\text{disc}}]^{-1/2}. \quad (7.3)$$

Next, let us examine the discrete Jacobi equation. The equations for the classical path that follow from the discrete action can be written as

$$\frac{\partial}{\partial z_j}(-iS_{\text{disc}}) = 0, \quad \frac{\partial}{\partial \bar{z}_j}(-iS_{\text{disc}}) = 0, \quad (j = 1, 2, \ldots, M). \quad (7.4)$$

To derive Jacobi equations from these, we would like to treat $z_0 = z_i$ and $\bar{z}_0$ as initial values. This, however, is meaningless as there is no such variable as $\bar{z}_0$. Instead, we must take $z_0$
and \( z_1 \) as the initial values, the latter being regarded as determined by \( z_i \) and \( z_{i+1} = z_{M+1} \).

We now keep \( z_i \) unchanged, and let \( \bar{z}_1 \to z_1 + \epsilon \). Let us denote the changes induced in \( z_j \) and \( \bar{z}_j \) by

\[
\delta z_j = \epsilon u_j + O(\epsilon^2), \quad \delta \bar{z}_j = \epsilon v_j + O(\epsilon^2).
\]

(7.5)

Performing the necessary variations, we obtain,

\[
\sum_k \left( u_k \frac{\partial}{\partial z_k} + v_k \frac{\partial}{\partial \bar{z}_k} \right) \begin{pmatrix} -i \partial S_{\text{disc}}/\partial z_j \\ -i \partial S_{\text{disc}}/\partial \bar{z}_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

(7.6)

Most of the derivatives on the left vanish. When only the nonzero ones are kept, we obtain

\[
D_{j-1} u_{j-1} + D_{j} u_j + D_{j} v_j = 0,
\]

(7.7)

\[
D_{jj} u_j + D_{jj} v_j + D_{j+1} v_{j+1} = 0,
\]

(7.8)

using the definitions of the various \( D \)'s. [These definitions can be read off by considering only the first member of each of the equations (3.18)–(3.23) and deleting the ‘P’ superscript.]

These equations hold for \( j = 1, 2, \ldots, M \), and we must take \( u_0 = 0, v_1 = 1 \). They then determine \( u_j \) for \( 1 \leq j \leq M \), and \( v_j \) for \( 2 \leq j \leq M + 1 \), all of which are meaningful quantities. We now observe that we can rewrite them in the form

\[
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22} \\
D_{32} & \ddots \\
D_{M-1} & D_{M-1} \\
D_M & D_M
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_M \\
u_M
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
v_{M+1}
\end{bmatrix}.
\]

(7.9)

We have used the initial condition on \( u_0 \) (\( u_0 = 0 \)), but not on \( v_1 \), leaving it as an arbitrary quantity instead. In fact, the way this equation is written suggests that all \( v_j, u_j \) (\( j = 1, \ldots, M \)) are determined in terms of \( v_{M+1} \). By demanding that \( v_{M+1} \) must be chosen such that \( v_1 = 1 \), however, we once again obtain \( v_{M+1} \) explicitly.

The matrix that appears in Eq. (7.9) is of course none other than \( D_{\text{disc}} \). By Cramer’s rule, therefore,

\[
v_1 = \frac{\det C}{\det D_{\text{disc}}},
\]

(7.10)
where,

\[
C = \begin{bmatrix}
0 & D_{11} & 0 & 0 & \cdots & 0 & 0 & D_{12} \\
0 & D_{11} & D_{21} & 0 & \cdots & 0 & 0 & D_{12} \\
0 & D_{22} & 0 & D_{22} & \cdots & 0 & 0 & D_{23} \\
0 & D_{22} & D_{22} & 0 & \cdots & 0 & 0 & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
v_{M+1} & D_{\bar{M}MM-1} & D_{\bar{M}\bar{M}} & D_{\bar{M}MM} & \cdots & \cdots & D_{\bar{M}MM} & D_{MM}
\end{bmatrix}.
\] (7.11)

To evaluate \(\det C\), we expand it by the first column, obtaining,

\[
\det C = -v_{M+1} \det \begin{bmatrix}
D_{11} & D_{12} & 0 & 0 & \cdots & 0 \\
D_{11} & D_{12} & D_{22} & 0 & \cdots & 0 \\
D_{22} & 0 & D_{22} & D_{23} & \cdots & 0 \\
D_{22} & D_{22} & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\
D_{32} & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
D_{\bar{M}MM-1} & D_{\bar{M}\bar{M}} & D_{\bar{M}MM} & D_{MM}
\end{bmatrix}.
\] (7.12)

The matrix that remains is lower triangular, so its determinant is just the product of the diagonal entries. Anticipating future minus signs, we define the quantity

\[
\Gamma_{SK} = (-1)^M \prod_{j=1}^{M} D_{jj} \prod_{j=2}^{M} D_{j-1j},
\] (7.13)

in terms of which

\[
\det C = (-1)^M v_{M+1} \Gamma_{SK}
\] (7.14)

and

\[
v_1 = (-1)^M \frac{\Gamma_{SK}}{\det \mathcal{D}_{\text{disc}}} v_{M+1}.
\] (7.15)

Setting \(v_1 = 1\), and rearranging, we get

\[
(-1)^M \det \mathcal{D}_{\text{disc}} = \Gamma_{SK} v_{M+1}.
\] (7.16)

This is the correct discrete replacement of Eq. (6.6). The factor \(\Gamma_{SK}\) need not be unity, in which case we have a nonzero SK correction.
The final step is to take the $\Delta \to 0$ limit of Eq. (7.16). By definition, the left hand side turns into the continuum fluctuation determinant, and its inverse square root will give us $K_{\text{red}}$. It remains to see what happens to $v_{M+1}$ and $\Gamma_{\text{SK}}$ on the right hand side. It is simplest to do this separately for the P and Q representations. Before turning to this, however, it pays to rewrite the general Jacobi equations, (7.7) and (7.8), as the $2 \times 2$ matrix recursion relation,

$$
\begin{pmatrix}
-D_{jj} & 0 \\
D_{jj} & D_{j+1j}
\end{pmatrix}
\begin{pmatrix}
u_j \\
v_{j+1}
\end{pmatrix}
= 
\begin{pmatrix}
D_{j-1j} & D_{jj} \\
0 & -D_{jj}
\end{pmatrix}
\begin{pmatrix}u_{j-1} \\
v_j
\end{pmatrix}, \quad (j = 1, 2, \ldots, M),
$$

(7.17)

with the initial conditions $u_0 = 0$, $v_1 = 1$.

It is of course also possible to evaluate $\det D_{\text{disc}}$ by writing a recursion relation for successive diagonal subdeterminants of $D_{\text{disc}}$ (as may be done for any tridiagonal matrix). The SK corrections can then be obtained by examining the relationship of this recursion relation to Eq. (7.17). We shall not follow this route.

### B. Application to P representation

Let us consider the discrete Jacobi equation first. We feed the explicit values of the $D$’s from Eqs. (3.18)–(3.23) into Eq. (7.17), and abbreviate

$$
A(z_j, z_j) = A_j, \quad B(z_j, z_j) = B_j, \quad \bar{B}(z_j, z_j) = \bar{B}_j.
$$

(7.18)

We find that

$$
\begin{pmatrix}
-(1 + i\Delta A_j) & 0 \\
i\Delta \bar{B}_j & -1
\end{pmatrix}
\begin{pmatrix}u_j \\
v_{j+1}
\end{pmatrix}
= 
\begin{pmatrix}
-1 & i\Delta B_j \\
0 & 1 + i\Delta A_j
\end{pmatrix}
\begin{pmatrix}u_{j-1} \\
v_j
\end{pmatrix}.
$$

(7.19)

Solving for the column vector on the left, and dropping terms of $O(\Delta^2)$, we obtain

$$
\begin{pmatrix}u_j \\
v_{j+1}
\end{pmatrix}
= 
\begin{pmatrix}
1 - i\Delta A_j & -i\Delta B_j \\
i\Delta \bar{B}_j & 1 + i\Delta A_j
\end{pmatrix}
\begin{pmatrix}u_{j-1} \\
v_j
\end{pmatrix}.
$$

(7.20)

It is immediately apparent that in the continuum limit ($\Delta \to 0$, $M \to \infty$, with $\Delta M = T$ fixed), this recursion will turn into the continuous-time Jacobi equation (6.5). Since the initial conditions are also identical, it follows that

$$
\lim_{\Delta \to 0} v_{M+1} = v(T).
$$

(7.21)
The second step is to evaluate $\Gamma_{SK}$. We have,
\[ D_{jj} = 1 + i \Delta A_j, \quad D_{j-1j} = -1. \] (7.22)
Hence,
\[ \Gamma_{SK} = (-1)^{M-1} \times \prod_{j=1}^{M} (1 + i \Delta A_j) \times (-1)^{M-1} \approx \exp(i \Delta \sum_{j} A_j). \] (7.23)

Collecting together the above results, we find that
\[ \lim_{\Delta \to 0} (-1)^{M} \det D_{\text{disc}} = \exp\left(i \int_{0}^{T} A(t) \, dt \right) v(T), \] (7.24)
so that
\[ K_{\text{red}}^{P} = \exp\left(-\frac{i}{2} \int_{0}^{T} A(t) \, dt \right) [v(T)]^{-1/2}. \] (7.25)

The extra exponential factor is the SK correction, and we see that we have the correct sign for it.

C. Application to Q representation

We now repeat the previous subsection’s arguments for the Q representation. The relevant $D$ coefficients are given in Eqs.(2.16)–(2.21) of Ref. [13], and we redisplay them here for ready reference:
\[ D_{jj} = i \Delta \bar{B}_j, \quad D_{jj} = i \Delta B_j, \]
\[ D_{jj} = D_{j-1j} = 1, \]
\[ D_{j+1j} = D_{jj+1} = -1 + i \Delta A_j, \] (7.26)

where now,
\[ A_j = A(\bar{z}_j, z_{j-1}), \quad B_j = B(\bar{z}_j, z_{j-1}), \quad \bar{B}_j = \bar{B}(\bar{z}_{j+1}, z_j). \] (7.27)

The discrete Jacobi equation now reads
\[ \begin{pmatrix} -1 & 0 \\ i \Delta \bar{B}_j & -1 + i \Delta A_{j+1} \end{pmatrix} \begin{pmatrix} u_j \\ v_{j+1} \end{pmatrix} = \begin{pmatrix} -1 + i \Delta A_j & i \Delta B_j \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_{j-1} \\ v_j \end{pmatrix}. \] (7.28)

Again we solve for $u_j$ and $v_{j+1}$ to $O(\Delta)$. In the process, we also replace the $\Delta A_{j+1}$ term by $\Delta A_j$, since the difference is $O(\Delta^2)$. In this way, we get
\[ \begin{pmatrix} u_j \\ v_{j+1} \end{pmatrix} = \begin{pmatrix} 1 - i \Delta A_j & -i \Delta B_j \\ i \Delta \bar{B}_j & 1 + i \Delta A_j \end{pmatrix} \begin{pmatrix} u_{j-1} \\ v_j \end{pmatrix}, \] (7.29)
which is formally the same as in the P case. For the same reasons as given there, we again get

$$\lim_{\Delta \to 0} v_{M+1} = v(T). \quad (7.30)$$

Next, for $\Gamma_{SK}$, we have

$$\Gamma_{SK} = (-1)^{M-1} \times (1)^M \times \prod_{j=1}^{M-1} (-1 + i \Delta A_j) \approx \exp(-i \Delta \sum_j A_j). \quad (7.31)$$

It follows that,

$$\lim_{\Delta \to 0} (-1)^M \det D_{\text{disc}} = \exp(-i \int_0^T A(t) \, dt) v(T), \quad (7.32)$$

and

$$K_{Q}^{\text{red}} = \exp\left(\frac{i}{2} \int_0^T A(t) \, dt \right) [v(T)]^{-1/2}. \quad (7.33)$$

The extra exponential factor is the SK correction for $K^Q$. We draw the reader’s attention to the sign.

VIII. DIRECT EVALUATION OF THE PARTICLE PROPAGATOR IN THE WEYL REPRESENTATION

Our goal in this section is to try and evaluate the propagator using the Weyl representation for $\mathcal{H}$ from the very start. One way to try and do this is to write the infinitesimal time-evolution operator $e^{-i\mathcal{H}\Delta}$ in terms of $H^W$ using the Weyl kernel $\mathcal{W}(\bar{z}, z)$ as in Eq. (A5). We shall see that this way does not work. The other way, which does work, is to alternate P and Q representations, building on Eq. (3.13).

A. Mapping via Weyl kernel

First, let us use Eq. (A5) to write the infinitesimal time-evolution operator as

$$e^{-i\mathcal{H}\Delta} = \int \frac{d^2z}{\pi} \left[e^{-i\mathcal{H}\Delta}\right]_{\text{WS}} \mathcal{W}(\bar{z}, z), \quad (8.1)$$

where by $[X]_{\text{WS}}$ we mean the Weyl symbol of the operator $X$. Using Eq. (A12), we may write the propagator for one time slice as

$$\langle \bar{z}_2 | e^{-i\mathcal{H}\Delta} | z_1 \rangle = 2 e^{-i\mathcal{H}\Delta} \int \frac{d^2z}{\pi} \left[e^{-i\mathcal{H}\Delta}\right]_{\text{WS}} e^{-2(\bar{z}_2-z_1)(z_1-z)}. \quad (8.2)$$
To find \([e^{-iH\Delta}]_{\text{WS}}\), we expand \(e^{-iH\Delta}\) in powers of \(\Delta\). The symbols for 1 and \(H\) are 1 and \(H^W(\bar{z}, z)\), and that for \(\Delta^2\) is shown.

\[
[H^2]_{\text{WS}} = [H^W(\bar{z}, z)]^2 + \frac{\hbar^2}{4} \left[ \frac{\partial^2 H^W}{\partial \bar{z}^2} \frac{\partial^2 H^W}{\partial \bar{z}^2} - \left( \frac{\partial^2 H^W}{\partial \bar{z} \partial z} \right)^2 \right] + \cdots. \tag{8.3}
\]

The important point here is that the correction is of relative order \(\hbar^2\) and not \(\hbar\). (We will show the powers of \(\hbar\) relative to the leading term explicitly in this section.) Hence, when we reexponentiate the series, we find

\[
[e^{-iH\Delta}]_{\text{WS}} = \exp(-i\Delta H^W(\bar{z}, z) + O(\hbar^2 \Delta^2)). \tag{8.4}
\]

For the time-slice propagator, we get

\[
\langle \bar{z}_2 | e^{-iH\Delta} | z_1 \rangle = 2e^{\bar{z}_2 z_1} \int \frac{d^2 z}{\pi} e^{\Phi(\bar{z}, z; \bar{z}_2, z_1)}, \tag{8.5}
\]

with

\[
\Phi(\bar{z}, z; \bar{z}_2, z_1) = -i\Delta H^W(\bar{z}, z) - 2(\bar{z} - \bar{z}_2)(z - z_1) + O(\hbar^2 \Delta^2), \tag{8.6}
\]

The natural procedure at this point is to evaluate the integral over \(z\) and \(\bar{z}\) semiclassically, i.e., by steepest descents. Let us denote the critical (saddle) point by \(\bar{z}_c\) and \(z_c\), and partial derivatives by subscripts. Setting \(\Phi_{\bar{z}} = \Phi_z = 0\), we find

\[
z_c = z_1 - \frac{i\hbar^{1/2}}{2} \Delta H^W_{\bar{z}}(\bar{z}_c, z_c) + O(\hbar^{5/2} \Delta^2), \tag{8.7}
\]

\[
\bar{z}_c = \bar{z}_2 - \frac{i\hbar^{1/2}}{2} \Delta H^W_z(\bar{z}_c, z_c) + O(\hbar^{5/2} \Delta^2), \tag{8.8}
\]

where we continue to show powers of \(\hbar\) explicitly. Hence, denoting the critical value of \(\Phi\) by \(\Phi_c\), we have

\[
\Phi_c = -i\Delta H^W(\bar{z}_c, z_c) + \frac{1}{2} \Delta^2 \hbar H^W_{\bar{z}\bar{z}}(\bar{z}_c, z_c) H^W_z(\bar{z}_c, z_c) + O(\hbar^2 \Delta^2), \tag{8.9}
\]

the additional error terms introduced at this step being of order \(\hbar^3 \Delta^3\). Since \(\bar{z}_c\) and \(z_1\) differ from \(\bar{z}_2\) and \(z_1\) by terms of order \(\Delta \hbar^{1/2}\), it is reasonable to perform a second expansion in \(\Delta\). When this is done, we find (the sign of the \(\Delta^2\) term should be noted)

\[
\Phi_c = -i\Delta H^W(\bar{z}_2, z_1) - \frac{1}{2} \Delta^2 \hbar H^W_{\bar{z}\bar{z}}(\bar{z}_2, z_1) H^W_z(\bar{z}_2, z_1) + O(\Delta^2 \hbar^2). \tag{8.10}
\]

The next step is to perform the Gaussian integral over the small deviations from the critical point. Defining \(\eta = z - z_c\), \(\bar{\eta} = \bar{z} - \bar{z}_c\), we have

\[
\Phi = \Phi_c + \frac{1}{2} \left( \Phi_{zz} \eta^2 + 2\Phi_{z\bar{z}} \eta \bar{\eta} + \Phi_{\bar{z}\bar{z}} \bar{\eta}^2 \right) + \cdots, \tag{8.11}
\]
with

\[
\begin{align*}
\Phi_{zz} &= -i \Delta h H^W_{zz}, \\
\Phi_{\bar{z}z} &= -2 - i \Delta h H^W_{\bar{z}z}, \\
\Phi_{\bar{z}\bar{z}} &= -i \Delta h H^W_{\bar{z}\bar{z}}.
\end{align*}
\] (8.12)

At this point, it is better to leave the derivatives of \( H^W \) evaluated at \( \bar{z}_c, z_c \). The integral gives us the inverse square root of the determinant of this quadratic form, which equals

\[
1 - i \frac{\Delta h^2}{2} H^W_{zz} + O(\Delta^2 h^2).
\] (8.13)

Evaluating \( H^W_{zz} \) at \( \bar{z}_2, z_1 \) incurs a further error of the same order, i.e., \( \Delta^2 h^2 \).

Putting all these pieces together, we find, eventually,

\[
\langle \bar{z}_2 | e^{-iH\Delta} | z_1 \rangle = \exp \left( \bar{z}_2 z_1 - i \Delta \left[ H^W + \frac{1}{2} H^W_{zz} \right]_{\bar{z}_2,z_1} + O(\Delta^2 h^2) \right).
\] (8.14)

The order \( \Delta \) terms combine to form \( H^Q(\bar{z}_2, z_1) \), so that we get, as before,

\[
\langle \bar{z}_2 | e^{-iH\Delta} | z_1 \rangle \approx \langle \bar{z}_2 | z_1 \rangle e^{-i\Delta H^Q(\bar{z}_2,z_1)},
\] (8.15)

but now we know the order of the terms omitted.

We thus see that it does no good to start with \( H^W \), since that entails auxiliary integrations for each time slice, which when performed will lead to the same fluctuation determinant over the non-auxiliary variables as before. Only the Q and P representations allow us to dispense with auxiliary integration variables, but then the fluctuation determinant leads to SK corrections.

## B. Alternating P and Q representations

We now start with the discrete action (3.13) obtained by alternating P and Q representations. This is an obvious thing to try, since when we determine the extreme or the classical value of the action, the alternating \( H^P \)'s and \( H^Q \)'s will combine to produce \( H^W \) as \( \Delta \to 0 \). The expectation is that the reduced propagator will then be free of any SK correction.

The only nontrivial part of the calculation is the integration over the fluctuations, which we do by successive time slices as in Sec. III and Ref. 13, picking up factors of \((\det G_j)^{-1/2}\) at each step. Suppose the step from \( j \) to \( j + 1 \) is of type P, and the next step is of type Q.
Then, adopting a notation for the derivatives of $H^P$ and $H^Q$ analogous to that in $D_{jj}$, etc., the part of $\delta^2(iS)$ which involves the fluctuations at these steps is

\[
\begin{align*}
\frac{i}{2} \delta^2 S &= \cdots - \frac{i}{2} \Delta H^P_{jj} \eta^2_j - (1 + i \Delta H^P_{jj}) \eta_j \bar{\eta}_j \\
&- \frac{i}{2} \Delta (H^P_{jj} + H^Q_{jj}) \eta^2_j + (1 - i \Delta H^Q_{jj+1}) \eta_j \bar{\eta}_{j+1} \\
&- \frac{i}{2} \Delta H^Q_{jj+1} \bar{\eta}_{j+1}^2 - \eta_{j+1} \bar{\eta}_{j+1} + \eta_{j+1} \bar{\eta}_{j+2} + \cdots.
\end{align*}
\]

(8.16)

We do not show the values of $\bar{z}$ and $z$ at which the derivatives of $H^P$ and $H^Q$ are evaluated explicitly; they are $(\bar{z}_j, z_j)$ and $(\bar{z}_{j+1}, z_j)$, respectively. Since we will eventually let $\Delta \to 0$, these arguments will take values on the classical path $\bar{z}(t), z(t)$ with $t = j \Delta$.

For the integration at step $j$ (over $\eta_j$ and $\bar{\eta}_j$), we need the matrix $G_j$ as well as the vectors $V_j$ and $\tilde{V}_j$. We have

\[
G_j = \begin{pmatrix}
1 + i \Delta H^P_{jj} & G_{j,ud} \\
i \Delta (H^P_{jj} + H^Q_{jj}) & 1 + i \Delta H^P_{jj}
\end{pmatrix},
\]

(8.17)

\[
V_j = \frac{1}{2} (1 - i \Delta H^Q_{jj+1}) \bar{\eta}_{j+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{V}_j = \frac{1}{2} (1 - i \Delta H^Q_{jj+1}) \bar{\eta}_{j+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

(8.18)

The matrix element $G_{j,ud}$ is of course unknown, having been modified as a result of the previous integration steps. The integration at this step produces a residual term from completing the square equal to

\[
\frac{1}{2} (1 - i \Delta H^Q_{jj+1})^2 \bar{\eta}_{j+1}^2 (G_{j-1})_{ud} = -\frac{1}{2 \det G_j} (1 - 2 i \Delta H^Q_{jj+1}) \eta_{j+1} \bar{\eta}_{j+1} G_{j,ud} + O(\Delta^2),
\]

(8.19)

and a determinantal factor $(\det G_j)^{-1/2}$. It is apparent that

\[
\det G_j = 1 + 2 i \Delta H^P_{jj} - i \Delta (H^P_{jj} + H^Q_{jj}) G_{j,ud}.
\]

(8.20)

The result of step $j$ is that the matrix $G_{j+1}$ equals

\[
G_{j+1} = \begin{pmatrix}
1 & G_{j+1,ud} \\
0 & 1
\end{pmatrix},
\]

(8.21)

where

\[
G_{j+1,ud} = i \Delta H^Q_{jj+1} + \frac{1}{\det G_j} (1 - 2 i \Delta H^Q_{jj+1}) G_{j,ud}.
\]

(8.22)
Further,

\[ V_j = \frac{1}{2} \tilde{\eta}_{j+2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{V}_j = \frac{1}{2} \tilde{\eta}_{j+2} \begin{pmatrix} 1 & 0 \end{pmatrix}. \quad (8.23) \]

The integrations at step \( j + 1 \) thus produce a determinantal factor \( (\det G_{j+1})^{-1/2} = 1 \), and a residual term from completing the square equal to

\[ \frac{1}{2} \tilde{\eta}_{j+2}^2 (G_{j+1}^{-1})_{ud} = -\frac{1}{2} \tilde{\eta}_{j+2}^2 G_{j+1, ud}. \quad (8.24) \]

Hence,

\[ G_{j+2, ud} = i \Delta H^P_{j+2, j+2} + G_{j+1, ud} \]

\[ = i \Delta H^P_{j+2, j+2} + i \Delta H^Q_{j+1, j+1} + \left(1 - 2i \Delta H^Q_{j+1} - 2i \Delta H^P_{j+1} + i \Delta (H^P_{jj} + H^Q_{jj}) G_{j, ud}\right) G_{j, ud}. \quad (8.25) \]

This is the recursion relation desired, since we have now integrated over the complete repeat pattern. If we let \( \Delta \to 0 \), it will be seen that the symbols \( H^P \) and \( H^Q \) always appear in the combination

\[ H^P(\bar{z}, z) + H^Q(\bar{z}, z) = 2H^W(\bar{z}, z)(1 + O(\hbar^2)). \quad (8.27) \]

The differential equation for \( G_{ud} \) is

\[-i \dot{G}_{ud} = B^W - 2A^W G_{ud} + \bar{B}^W G_{ud}^2, \quad (8.28)\]

where we have added a superscript \( W \) to show that the symbol for the Hamiltonian that is involved is \( H^W \). This differential equation is the same as before with the same initial conditions. Hence,

\[ G_{ud}(t) = \frac{1}{B^W(t)} \left( A^W(t) + i\frac{\dot{v}}{v} \right). \quad (8.29) \]

Finally, we need the product of all the determinants \( \det G_k \). Keeping in mind that the determinant from every other step is unity, we have

\[ \ln K_{red} = -\frac{1}{2} \sum_{k=1}^{M} \ln(\det G_k) \]

\[ = -\frac{1}{2} \int_0^T \left[ i \frac{\partial^2 H^P}{\partial \bar{z} \partial z} - i \frac{\partial^2 H^W}{\partial z^2} G_{ud}(t) \right] dt \]

\[ = -\frac{i}{2} \int_0^T \left[ A^P(t) - \left( A^W(t) + i \frac{\dot{v}}{v} \right) \right] dt. \quad (8.30) \]
Now, \( A^P - A^W = O(\hbar) \), which may be neglected since the term we are discussing is already the first correction in powers of \( \hbar \). Hence \( \ln K_{\text{red}} = -\frac{1}{2} \ln v(T) \), i.e.,

\[
K_{\text{red}} = \left( i \frac{\partial^2 S_W}{\partial \bar{z}_f \partial z_i} \right)^{1/2},
\]

which has no SK correction.

**Acknowledgments**

This work was supported in part by the NSF via grant numbers PHY-0854896 (F. Li and A. Garg), DGE-0801685 (NSF-IGERT program) (C. Braun), and DMR 13-06011 (M. Stone).

**Appendix A: Review of P, Q, and Weyl symbols for particles and spins**

1. **Mapping for particles**

For a massive particle in one spatial dimension, the Q and P symbols of the Hamiltonian \( \mathcal{H} \) are defined by

\[
H^Q(\bar{z}, z) = \frac{\langle \bar{z} | \mathcal{H} | z \rangle}{\langle \bar{z} | z \rangle},
\]

\[
\mathcal{H} = \int \frac{d^2z}{\pi} e^{-\bar{z}z} H^P(\bar{z}, z) |z\rangle \langle \bar{z}|.
\]

(Analogous definitions apply to other operators.) In Eq. (A2) \( d^2z \) is shorthand for \( dx \, dy \), with \( x \) and \( y \) being the real and imaginary parts of \( z \). The reason for the \( e^{-\bar{z}z} \) factor inside the integral for this equation and the \( \langle \bar{z} | z \rangle \) denominator of the previous one is that as defined in Eq. (2.2), the states \( |z\rangle \) and \( \langle \bar{z}| \) are not normalized; instead

\[
\langle \bar{z} | z' \rangle = e^{\bar{z}z'}.
\]

The resolution of unity therefore takes the form

\[
1 = \int \frac{d^2z}{\pi} e^{-\bar{z}z} |z\rangle \langle \bar{z}|,
\]

with the same extra \( e^{-\bar{z}z} \) factor. The advantage of using unnormalized states is that off-diagonal matrix elements such as \( \langle \bar{z} | \mathcal{H} | z' \rangle \) can be obtained from the diagonal one, \( \langle \bar{z} | \mathcal{H} | z \rangle \), by appealing to analyticity. In practical terms this means that we merely replace \( z \) with \( z' \).
For the Weyl symbol, we follow Weyl himself [24], and define

\[ H^W(\bar{z}, z) = \text{Tr}(\mathcal{H}W(\bar{z}, z)), \]  

(A5)

where \( W(\bar{z}, z) \) is an operator-valued kernel given by

\[ W(\bar{z}, z) = \int \frac{d^2w}{\pi} e^{(wa^\dagger - wa)} e^{-(w\bar{z} - wz)}. \]  

(A6)

This definition puts the familiar symmetrization rule for Weyl ordering of operators on a broader footing and can be shown to reduce to that for simple examples such as \( a^2(a^\dagger)^2 \). The Weyl symbol for any other operator is defined analogously. By letting \( w \to -w, \bar{w} \to -\bar{w} \) in Eq. (A6), we find that \([W(\bar{z}, z)]^\dagger = W(\bar{z}, z)\). Hermiticity of \( \mathcal{H} \) then implies that \( H^W(\bar{z}, z) \) is real.

One possible inverse of the transform (A5) is given by

\[ \mathcal{H} = \int \frac{d^2z}{\pi} H^W(\bar{z}, z)W(\bar{z}, z). \]  

(A7)

It is straightforward to show this result by using the identities (themselves easily shown)

\[ \text{Tr} W(\bar{z}, z) = 1, \]  

(A8)

\[ \text{Tr} (W(\bar{z}, z)W(\bar{z}', z')) = \pi \delta^{(2)}(z - z'), \]  

(A9)

where by \( \delta^{(2)}(z) \) we mean \( \delta(\text{Re } z)\delta(\text{Im } z) \). In fact, the inverse (A7) is unique, as may be shown by taking the trace in Eq. (A5) in the complete set of position states. The matrix elements of the kernel \( W \) are not difficult to find, and the trace takes on the form of an ordinary Fourier integral, which may be inverted to obtain an expression for the position-space matrix elements of \( \mathcal{H} \) in terms of the function \( H^W \). This expression is easily seen to be identical to the one implied by Eq. (A7). Thus Eq. (A7) not only implies but is also implied by Eq. (A5).

Next, we recapitulate the relationship between the Q, P, and Weyl symbols [36]. Consider the matrix element

\[ \langle \bar{z}_2 | \mathcal{H} | z_1 \rangle, \]  

(A10)

which is nothing but \( \langle \bar{z}_2 | z_1 \rangle H^Q(\bar{z}_2, z_1) \). (See Eq. (3.4).) Writing \( \mathcal{H} \) in terms of \( H^W(\bar{z}, z) \), and \( \langle \bar{z}_2 | z_1 \rangle = e^{\bar{z}_2 z_1} \), we obtain

\[ H^Q(\bar{z}_2, z_1) = e^{-\bar{z}_2 z_1} \int \frac{d^2z}{\pi} H^W(\bar{z}, z) \langle \bar{z}_2 | W(\bar{z}, z) | z_1 \rangle. \]  

(A11)
Using Eq. (A6), we get

\[
\langle \bar{z}_2 | \mathcal{W}(\bar{z}, z) | z_1 \rangle = \int \frac{d^2 w}{\pi} e^{-(w \bar{z} - w z_1 - \frac{1}{2} w w + \bar{z}_2 z_1)}
\]

\[
= \int \frac{d^2 w}{\pi} \exp \left( -\frac{1}{2} w w + w(\bar{z} - z) - w(z_1 - z) + \bar{z}_2 z_1 \right)
\]

\[
= 2 \exp \left( -2(\bar{z} - z)(z_1 - z) + \bar{z}_2 z_1 \right). \quad (A12)
\]

We now feed this result into Eq. (A11) while at the same time defining

\[
\eta = z - z_1, \quad \bar{\eta} = \bar{z} - \bar{z}_2. \quad (A13)
\]

We thus get

\[
H^Q(\bar{z}_2, z_1) = 2 \int \frac{d^2 \eta}{\pi} e^{-2\eta \eta} H^W(\bar{z}_2 + \bar{\eta}, z_1 + \eta). \quad (A14)
\]

If we now Taylor expand \( H^W \) in powers of \( \eta \) and \( \bar{\eta} \), it is easy to perform the resulting Gaussian integrals. Retaining the first nonzero correction, we get

\[
H^Q(\bar{z}_2, z_1) = H^W(\bar{z}_2, z_1) + \frac{1}{2} \partial^2 \partial_{\bar{z}_2 \partial z_1} H^W(\bar{z}_2, z_1) + \cdots. \quad (A15)
\]

The second term in this expansion is in fact of order \( \hbar \) relative to the first. One can see this point by writing the quantities \( z \) and \( \bar{z} \) in terms of dimensionful position and momentum variables, and noting that \( z \) and \( \bar{z} \) both contain a factor of \( \hbar^{-1/2} \). By transposing this term to the left hand side, and using the same equation recursively, we find that

\[
H^W(\bar{z}_2, z_1) = H^Q(\bar{z}_2, z_1) - \frac{1}{2} \partial^2 \partial_{\bar{z}_2 \partial z_1} H^Q(\bar{z}_2, z_1) + \cdots. \quad (A16)
\]

To relate \( H^P \) and \( H^W \), we substitute Eq. (A2) in Eq. (A5), and obtain

\[
H^W(\bar{z}_2, z_1) = \text{Tr} \left( \mathcal{H}^W(\bar{z}_2, z_1) \right)
\]

\[
= \int \frac{d^2 z}{\pi} e^{-\bar{z} z} H^P(\bar{z}, z) \text{Tr} \left( | z \rangle \langle \bar{z} | \mathcal{W}(\bar{z}_2, z_1) \right) \quad (A17)
\]

Now,

\[
\text{Tr} \left( | z \rangle \langle \bar{z} | \mathcal{W}(\bar{z}_2, z_1) \right) = \langle \bar{z} | \mathcal{W}(\bar{z}_2, z_1) | z \rangle
\]

\[
= 2 \exp \left( -2(\bar{z} - \bar{z}_2)(z - z_1) + \bar{z} z \right), \quad (A18)
\]

where the last result is obtained from Eq. (A12) with the exchange \((\bar{z}_2, z_1) \leftrightarrow (\bar{z}, z)\). Feeding it into Eq. (A17) along with the definitions (A13), we get

\[
H^W(\bar{z}_2, z_1) = 2 \int \frac{d^2 \eta}{\pi} e^{-2\eta \eta} H^P(\bar{z}_2 + \bar{\eta}, z_1 + \eta). \quad (A19)
\]
We now Taylor expand $H^W$ in powers of $\eta$ and $\bar{\eta}$ just as done above, and integrate over $\eta$ and $\bar{\eta}$. Again retaining only the first nonzero correction, we get

$$H^W(\bar{z}_2, z_1) = H^P(\bar{z}_2, z_1) + \frac{1}{2} \frac{\partial^2}{\partial \bar{z}_2 \partial z_1} H^P(\bar{z}_2, z_1) + \cdots. \quad (A20)$$

For the particle case, we can achieve a more general correspondence between operators and phase-space functions by extending the definition of the Weyl kernel to

$$W^{(\alpha)}(\bar{z}, z) = \int d^2 w \pi e^{(wa^\dagger - \mathbf{a} w)} e^{- (w \bar{z} - \mathbf{w} z)} e^{\alpha \bar{z} z / 2}, \quad (A21)$$

where $-1 \leq \alpha \leq 1$. The mapping is then given by

$$H^{(\alpha)}(\bar{z}, z) = \text{Tr} (\mathcal{H} W^{(\alpha)}(\bar{z}, z)), \quad (A22)$$
$$\mathcal{H} = \int d^2 z \pi H^{(\alpha)}(\bar{z}, z) W^{(-\alpha)}(\bar{z}, z). \quad (A23)$$

The cases of Q, P, and Weyl mappings correspond to $\alpha = 1$, $-1$, and 0 respectively. We shall not employ this general definition, but shall work with the P and Q mappings in the form given earlier.

2. Mapping for spin

Let us first discuss what we mean by the Hamiltonian of a spin system. For a particle of spin $j$, the most general Hamiltonian (or any other operator) can be written as

$$\mathcal{H} = \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} c_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{J}), \quad (A24)$$

where $c_{\ell m}$ are arbitrary c-number coefficients, and $\mathcal{Y}_{\ell m}$ are spherical harmonic tensor operators defined via the operator analogue of the Herglotz generating function for spherical harmonics. The classical phase space can be taken as a sphere of fixed radius (which may be taken as 1, or $j$, or $j + \frac{1}{2}$, whichever is most convenient). As discussed in Sec. \textcircled{V} the variables $z$ and $\bar{z}$ that we employed in Sec. \textcircled{II} are stereographic coordinates for this sphere. It is somewhat easier at first, however, to parametrize a point on this sphere by its direction $\hat{n}$, so that functions on phase space are functions of $\hat{n}$. The Q, Weyl, and P symbols for the Hamiltonian are

$$H^{Q,W,P}(\hat{n}) = \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} c_{\ell m} \Phi_{\ell m}^{Q,W,P}(\hat{n}), \quad (A25)$$

33
where $\Phi^{Q,W,P}_{\ell m}(\hat{n})$ are the corresponding symbols for $\mathcal{Y}_{\ell m}(J)$. We gave complete expressions for these in Ref. 29, but here we only need the asymptotic forms as $j \to \infty$. Recalling the definition

$$\tilde{j} = j + \frac{1}{2},$$

we have

$$\Phi^{W}_{\ell m}(\hat{n}) \approx \tilde{j} \left( 1 + O(\tilde{j}^{-2}) \right) \mathcal{Y}_{\ell m}(\hat{n}),$$

(A27)

$$\Phi^{Q,P}_{\ell m}(\hat{n}) \approx \tilde{j}^\ell \left( 1 \mp \ell (\ell + 1) \right) + O(\tilde{j}^{-2}) \right) \mathcal{Y}_{\ell m}(\hat{n}).$$

(A28)

We now observe that $\ell(\ell + 1)\mathcal{Y}_{\ell m}(\hat{n}) = \mathcal{L}^2 \mathcal{Y}_{\ell m}(\hat{n})$, where $\mathcal{L} = -i(\hat{n} \times \nabla \hat{n})$ is the angular momentum operator (on phase space, and not the quantum mechanical Hilbert space). Hence, we may write

$$\Phi^{Q,P}_{\ell m} = \left( 1 \mp \frac{\mathcal{L}^2}{4\tilde{j}} + O(\tilde{j}^{-2}) \right) \Phi^{W}_{\ell m}.$$  

(A29)

It follows that

$$H^{Q,P} = \left( 1 \mp \frac{\mathcal{L}^2}{4\tilde{j}} + O(\tilde{j}^{-2}) \right) H^{W},$$

(A30)

a result which makes no reference to $\mathcal{Y}_{\ell m}(\hat{n})$ and is therefore valid independent of the form in which the Weyl symbol is given. It has a pleasing similarity to Eq. (A15) etc. if we recall that $-\mathcal{L}^2$ is (the angular part of) the Laplacian on the sphere.

†Present address: Online School for Girls, 7303 River Rd, Bethesda, MD 20817. ††Present address: Department of Radiation Oncology, University of Florida, P.O. Box 100385, Gainesville, Florida 32610

[1] J. R. Klauder, in *Path Integrals*, in *Proceedings of the NATO Advanced Summer Institute*, edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978).

[2] J. R. Klauder, Phys. Rev. D 19, 2349 (1979).

[3] H. Kuratsuji and T. Suzuki, J. Math. Phys. 21, 472 (1980).

[4] A. Jevicki and N. Papanicolaou, Ann. Phys. (N.Y.) 120, 107 (1979).

[5] H. B. Nielsen and D. Rohrlich, Nucl. Phys. B 299, 471 (1988).

[6] *Path Integrals and Coherent States of SU(2) and SU(1,1)*, edited by H. Kuratsuji, A. Inomata, and C. C. Gerry (World Scientific, Singapore, 1992).
[7] H. G. Solari, J. Math. Phys. **28**, 1097 (1987).
[8] E. A. Kochetov, J. Math. Phys. **36**, 4667 (1995).
[9] V. R. Vieira and P. D. Sacramento, Nucl. Phys. B **448**, 331 (1995).
[10] M. Enz and R. Schilling, J. Phys. C **19**, L711 (1986); *ibid* **19**, 1765 (1986).
[11] V. I. Belinicher, C. Provedencia, and J. da Providencia, J. Phys. A **30**, 5633 (1997).
[12] M. Stone, K. S. Park, and A. Garg, J. Math. Phys. **41**, 8025 (2000).
[13] C. Braun and A. Garg, J. Math. Phys. **48**, 032104 (2007).
[14] C. Braun and A. Garg, J. Math. Phys. **48**, 102104 (2007).
[15] A. Garg and M. Stone, Phys. Rev. Lett. **92**, 010401 (2004). We note that the correct quantization rule had been found earlier in Ref. 16 by employing WKB methods in the Bargmann representation.
[16] J. Kurchan, P. Leboeuf, and M. Saraceno, Phys. Rev. A **40**, 6800 (1989).
[17] A. Garg, E. Kochetov, K. S. Park, and M. Stone, J. Math. Phys. **44**, 48 (2003).
[18] E. Kececioglu and A. Garg, Phys. Rev. Lett. **88**, 237205 (2002).
[19] E. Kececioglu and A. Garg, Phys. Rev. B **67**, 054406 (2003).
[20] W. Wernsdorfer and R. Sessoli, Science **284**, 133 (1999).
[21] E. Lieb, Commun. Math. Phys. **31**, 327 (1973).
[22] E. A. Kochetov, J. Phys. A: Math. Gen. **31**, 4473 (1998).
[23] M. Pletyukhov, J. Math. Phys. **45**, 1859 (2004).
[24] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, New York (1950) [translation of *Gruppentheorie und Quantenmechanik*, Hirzel Verlag, Leipzig (1928)]. See Chap. II, Sec. 11 and Chap. IV, Sec. 14.
[25] R. L. Stratonovich, Sov. Phys. JETP **31**, 1012 (1956).
[26] F. Bayen et al., Ann. Phys. (N.Y.) **110**, 111 (1978).
[27] G. S. Agarwal, Phys. Rev. A **24**, 2889 (1981).
[28] J. C. Varilly and J. M. Gracia-Bondia, Annals of Phys. (N.Y.) **190**, 101 (1989).
[29] F. Li, C. Braun, and A. Garg, Europhys. Lett. **103**, 60006 (2013).
[30] The notation $H^Q$ was not employed in Refs. 12, 13, but it clearly makes sense to do so now.
[31] We observe that a similar traciality rule holds in the particle case also, but there it is generally taken as a consequence of the basic definition of the Weyl symbol rather than as a defining principle itself.
Or else we would be open to Bertrand Russell’s zinger: “[T]he Hegelian dictum that everything
discrete is also continuous and *vice versa* […] has been tamely repeated by all his followers.
But as to what they meant by continuity and discreteness, they preserved a discreet and
continuous silence . . .” [B. Russell, *The Principles of Mathematics*, paperback edition (W. W.
Norton and Co., New York, 1996), p. 277.]

This result may be found efficiently by using the fact that $L^2$ is the negative of the Laplace-
Beltrami operator $\Delta$, and then using formulas from geometry: If $x^i$ are arbitrary coordinates
in a curved space and $g^{ik}$ is the contravariant metric tensor, then $\Delta = g^{ik} \partial^2 / \partial x^i \partial x^k$. Taking
$x^1 = z$ and $x^2 = \bar{z}$, we first note that the squared length element on the unit sphere is
ds^2 = 4(1 + \bar{z}z)^{-2} dz d\bar{z}$, so that $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = 2(1 + \bar{z}z)^{-2}$. Since the
covariant and contravariant metrics are reciprocals, i.e., $g^{ik} g_{km} = \delta^i_m$, we have $g^{\bar{z}\bar{z}} = g^{zz} = 0,$
g^{\bar{z}z} = g^{z\bar{z}} = (1 + \bar{z}z)^2 / 2$. The result then follows.

A. Garg, Lecture Notes at the Boulder School for Condensed Matter and Materials
Physics, Boulder, CO, 30 June–25 July, 2003 (unpublished) (boulder-
school.yale.edu/sites/default/files/files/garg_lecture_2.pdf).

In saying that the matrix is “symmetric,” we are following physicists’ usage, by which is meant
that $D_{ij} = D_{ji}$, not that $D_{ij} = D^*_{ji}$.

J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).

F. A. Berezin, Sov. Phys. Usp. **23**, 763 (1980).

Hermiticity requires (and follows from) $c_{\ell,-m} = (-1)^m c^{*}_{\ell,m}$.