AN ALGORITHM FOR THE NUMERICAL EVALUATION OF THE ASSOCIATED LEGENDRE FUNCTIONS THAT RUNS IN TIME INDEPENDENT OF DEGREE AND ORDER

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Abstract. We describe a method for the numerical evaluation of normalized versions of the associated Legendre functions $\tilde{P}_{-\mu}^{\nu}(t)$ and $\tilde{Q}_{-\mu}^{\nu}(t)$ of degrees $0 \leq \nu \leq 1,000,000$ and orders $-\nu \leq \mu \leq \nu$ on the interval $(-1,1)$. Our algorithm, which runs in time independent of $\nu$ and $\mu$, is based on the fact that while the associated Legendre functions themselves are extremely expensive to represent via polynomial expansions, the logarithms of certain solutions of the differential equation defining them are not. We exploit this by numerically precomputing the logarithms of carefully chosen solutions of the associated Legendre differential equation and representing them via piecewise trivariate Chebyshev expansions. These precomputed expansions, which allow for the rapid evaluation of the associated Legendre functions over a large swath of parameter domain mentioned above, are supplemented with asymptotic and series expansions in order to cover it entirely. The results of numerical experiments demonstrating the efficacy of our approach are presented, and our code for evaluating the associated Legendre functions is publicly available.

Key words. Special functions, fast algorithms, nonoscillatory phase functions, associated Legendre functions, asymptotic methods

AMS subject classifications. 65L99, 33F05

In this article, we describe an algorithm for the numerical evaluation of the functions $\tilde{P}_{-\mu}^{\nu}(t)$ and $\tilde{Q}_{-\mu}^{\nu}(t)$ defined via

1) $\tilde{P}_{-\mu}^{\nu}(t) = \tilde{P}_{-\mu}^{\nu}(\cos(t)) \sqrt{\sin(t)}$

and

2) $\tilde{Q}_{-\mu}^{\nu}(t) = \tilde{Q}_{-\mu}^{\nu}(\cos(t)) \sqrt{\sin(t)}$,

where $\tilde{P}_{-\mu}^{\nu}$ and $\tilde{Q}_{-\mu}^{\nu}$ are the normalized associated Legendre functions

3) $\tilde{P}_{\nu}^{-\mu}(x) = \sqrt{\frac{\nu + \frac{1}{2}}{\Gamma(\nu + \mu + 1)}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} P_{-\mu}^{\nu}(x)$

and

4) $\tilde{Q}_{\nu}^{-\mu}(x) = \frac{2}{\pi} \sqrt{\frac{\nu + \frac{1}{2}}{\Gamma(\nu + \mu + 1)}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} Q_{-\mu}^{\nu}(x)$.

It runs in time independent of degree $\nu$ and order $\mu$, and is applicable when $0 \leq \nu \leq 1,000,000$, $0 \leq \mu \leq \nu$ and $0 < t \leq \frac{\pi}{2}$ (in particular, $\nu$ and $\mu$ need not be integers). It is a consequence of standard connection formulas (such as those appearing in Section 3.4 of [10]) that this suffices for the evaluation of

5) $\tilde{P}_{\nu}^{\mu}(x)$, $\tilde{P}_{-\mu}^{\nu}(x)$, $\tilde{Q}_{\nu}^{\mu}(x)$, and $\tilde{Q}_{-\mu}^{\nu}(x)$

for any $0 \leq \nu \leq 1,000,000$, $-\nu \leq \mu \leq \nu$ and $-1 < x < 1$.

Our choice of scaling and normalization factors in (3) and (4) are fairly standard. Among other things, they ensure that

6) $\int_{-1}^{1} \left( \tilde{P}_{-\mu}^{\nu}(t) \right)^2 dt = 1$

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whenever $-\nu \leq \mu \leq \nu$ with $\nu$ and $\mu$ integers, and that $\tilde{P}_\nu^{-\mu}$ and $\tilde{Q}_\nu^{-\mu}$ satisfy the second order linear ordinary differential equation

$$y''(t) + \left(\lambda^2 - \eta^2 \csc^2(t)\right) y(t) = 0 \quad \text{for all } 0 < t < \frac{\pi}{2}$$

with $\lambda = \nu + \frac{1}{2}$ and $\eta^2 = \mu^2 - \frac{1}{4}$. By a slight abuse of terminology, we will refer to (7) as the associated Legendre differential equation.

When $0 \leq \mu \leq \frac{1}{2}$, the coefficient of $y$ in (7) is positive on the interval $(0, \frac{\pi}{2})$, whereas when $\mu > \frac{1}{2}$ it is negative on the interval

$$\left(0, \arcsin\left(\frac{\eta}{\lambda}\right)\right)$$

and positive on

$$\left(\arcsin\left(\frac{\eta}{\lambda}\right), \frac{\pi}{2}\right).$$

It follows from these observations and well-known WKB estimates (see, for example, [11]) that when $\mu > \frac{1}{2}$, the solutions of (7) behave roughly like combinations of increasing or decreasing exponentials on (8) and are oscillatory on (9), whereas when $\mu \leq \frac{1}{2}$, they are oscillatory on all of $(0, \frac{\pi}{2})$. We will refer to the subset

$$O = \left\{ (\nu, \mu, t) : \nu \geq 0, 0 \leq \mu \leq \min\left\{\nu, \frac{1}{2}\right\} \text{ and } 0 < t \leq \frac{\pi}{2} \right\} \cup \left\{ (\nu, \mu, t) : \nu > \frac{1}{2}, \frac{1}{2} < \mu \leq \nu \text{ and } \arcsin\left(\frac{\eta}{\lambda}\right) \leq t \leq \frac{\pi}{2} \right\}$$

of $\mathbb{R}^3$ as the oscillatory region, and to the subset

$$N = \left\{ (\nu, \mu, t) : \nu \geq 0, \mu > \frac{1}{2} \text{ and } 0 < t < \arcsin\left(\frac{\eta}{\lambda}\right) \right\}$$

as the nonoscillatory region. When $\nu \gg \mu > \frac{1}{2}$, the solutions of (7) are highly oscillatory on (9), and when $\nu \geq \mu \gg \frac{1}{2}$, they behave roughly like combinations of rapidly decreasing and increasing exponentials on (8). Consequently, they cannot be effectively represented via polynomial expansions in the variables $\nu$, $\mu$ and $t$ on either of the sets $O$ or $N$, at least for large values of the parameters.

Nonetheless, the logarithms of certain solutions of (7) can be represented efficiently via polynomial expansions on the sets $N$ and $O$. This observation is related to the well-known fact that the associated Legendre differential equation admits a nonoscillatory phase function. Many special functions of interest possess this property as well, at least in an asymptotic sense [22, 9]. However, the sheer effectiveness with which nonoscillatory phase functions can represent solutions of the general equation

$$y''(t) + \lambda^2 q(t)y(t) = 0 \quad \text{for all } a < t < b$$

in which the coefficient $q$ is smooth and positive appears to have been overlooked. Indeed, under mild conditions on $q$, it is shown in [6] that there exist a positive real number $\sigma$, a nonoscillatory function $\alpha$ and a basis of solutions $\{u, v\}$ of (12) such that

$$u(t) = \frac{\cos(\alpha(t))}{\sqrt{\alpha'(t)}} + O(e^{-\sigma \lambda})$$

and

$$v(t) = \frac{\sin(\alpha(t))}{\sqrt{\alpha'(t)}} + O(e^{-\sigma \lambda}).$$
The constant $\sigma$ is a measure of the extent to which $q$ oscillates, with larger values of $\sigma$ corresponding to greater smoothness on the part of $q$. The function $\alpha$ is nonoscillatory in the sense that it can be represented using various series expansions the number of terms in which do not vary with $\lambda$. That is, $O(\exp(-\sigma\lambda))$ accuracy is obtained using an $O(1)$-term expansion. The results of [6] are akin to standard results on WKB approximation in that they apply to the more general case in which $q$ varies with the parameter $\lambda$ assuming only that $q$ satisfies certain innocuous hypotheses independent of $\lambda$. An effective numerical algorithm for the computation of nonoscillatory phase functions for fairly general second order differential equations is described in [3], although we will not need it here. We will instead use specialized formulas which apply only in the case of associated Legendre functions. However, the existence of the algorithm of [3] and results of [6] mean that the approach of this paper can be applied to a large class of special functions satisfying second order differential equations.

The algorithm of this paper operates by numerically calculating the logarithms of certain solutions of the associated Legendre differential equation. We represent them via trivariate Chebyshev expansions — or rather, truncated version of these expansions which we call “compressed” trivariate Chebyshev expansions — the coefficients of which are stored in a table on the disk. This table is computed only once and is loaded into memory and used to evaluate the associated Legendre functions rapidly. The table used in the experiments described in this paper is approximately 138 MB in size. We supplement these precomputed expansions with series and asymptotic expansions in order to cover the entire parameter domain mentioned above. In addition to the values of the functions $\tilde{P}_\nu^{\mu}$ and $\tilde{Q}_\nu^{\mu}$, our algorithm also produces the values of a nonoscillatory phase function for (7) and its derivative when $(\nu, \mu, t)$ is in the oscillatory region $O$ and the values of the logarithms of $\tilde{P}_\nu^{\mu}$ and $\tilde{Q}_\nu^{\mu}$ when $(\nu, \mu, t)$ is in the nonoscillatory region $N$. The phase function is useful for, among other things, calculating the roots of the associated Legendre functions and applying special function transforms involving the associated Legendre functions. Calculating the values of the logarithms of (3) and (4) obviates many problems which arise from numerical overflow and underflow.

There is an extensive literature on the asymptotic behavior of the associated Legendre functions, and it is often suggested that existing asymptotic expansions, particularly Liouville-Green type expansions, suffice for the numerical evaluation of them. While many highly effective approximations are available, it appears to be quite difficult to produce a numerical algorithm which is efficient and accurate over the entire range of the variables $\nu$, $\mu$ and $t$ considered here. The trigonometric expansions

\begin{equation}
\tilde{P}_\nu^{\mu}(t) = \sqrt{\frac{2(\nu + \frac{1}{2}) \Gamma(\nu + \mu + 1) \Gamma(\nu - \mu + 1)}{\pi \sin(t)}} \frac{1}{\Gamma(\nu + \frac{3}{2})} \times \sum_{k=0}^{\infty} \left( -1 \right)^k \frac{(-\mu + \frac{1}{2})_k (-\mu + \frac{1}{2})_k}{(2 \sin(t))^k \Gamma(k + 1) \Gamma(\nu + \frac{3}{2})} \sin\left( \left( \nu + k + \frac{1}{2} \right) t + \frac{\pi}{2} (k - \mu) + \frac{\pi}{4} \right)
\end{equation}

and

\begin{equation}
\tilde{Q}_\nu^{\mu}(t) = \sqrt{\frac{2(\nu + \frac{1}{2}) \Gamma(\nu + \mu + 1) \Gamma(\nu - \mu + 1)}{\pi \sin(t)}} \frac{1}{\Gamma(\nu + \frac{3}{2})} \times \sum_{k=0}^{\infty} \left( -1 \right)^k \frac{(\mu + \frac{1}{2})_k (-\mu + \frac{1}{2})_k}{(2 \sin(t))^k \Gamma(k + 1) \Gamma(\nu + \frac{3}{2})} \cos\left( \left( \nu + k + \frac{1}{2} \right) t + \frac{\pi}{2} (k - \mu) + \frac{\pi}{4} \right),
\end{equation}

which can be found in Section 3.5 of [10], illustrate some of difficulties that arise. In (15) and (16), $(x)_k$ is the Pochhammer symbol defined via

\begin{equation}
(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)}.
\end{equation}
These expansions are applicable only when \( \frac{\pi}{2} < \theta < \frac{3\pi}{2} \), and they require on the order of \(|\mu|\) terms in order to achieve a fixed accuracy. Even more seriously, when \(|\mu|\) is not small relative to \( \nu \), the terms in (15) and (16) are of large magnitude and alternate signs, with the consequence that the numerical evaluation of (15) and (16) generally results in catastrophic cancellation errors.

Liouville-Green methods can be used to obtain asymptotic expansions of the associated Legendre functions which are uniform in the argument \( t \) and apply in the event that \( 0 \leq \mu \leq \nu \). In [2], the new dependent variable \( \zeta \) defined via the implicit relations

\[
\int_{\alpha^2}^{\zeta} \frac{\sqrt{\zeta^2 - \gamma^2}}{2\xi} \, d\xi = -\int_{x}^{\alpha^2} \frac{\sqrt{1 - \gamma^2 - t^2}}{1 - t^2} \, dt \quad \text{if} \quad x < \sqrt{1 - \gamma^2}
\]

and

\[
\int_{\alpha^2}^{\zeta} \frac{\sqrt{\zeta^2 - \gamma^2}}{2\xi} \, d\xi = -\int_{x}^{\alpha^2} \frac{\sqrt{t^2 - 1 + \gamma^2}}{1 - t^2} \, dt \quad \text{if} \quad x > \sqrt{1 - \gamma^2},
\]

where \( \lambda = \nu + \frac{1}{2} \) and \( \gamma = \frac{\mu}{x} \), is introduced to obtain the uniform asymptotic expansions

\[
\tilde{P}_\nu^{-\mu}(x) \approx \left( \frac{\zeta - \gamma^2}{1 - \gamma^2 - x^2} \right)^{\frac{1}{2}} \left( \sqrt{\lambda} J_\mu \left( \lambda \sqrt{\zeta} \right) \sum_{k=0}^{\infty} F_k(\zeta) \frac{\lambda^k}{\lambda^{2k}} + \left( \frac{\zeta}{\lambda} \right)^{\frac{1}{2}} J'_\mu \left( \lambda \sqrt{\zeta} \right) \sum_{k=0}^{\infty} \tilde{F}_k(\zeta) \lambda^{2k} \right)
\]

as \( \nu \to \infty \) and

\[
\tilde{Q}_\nu^{-\mu}(x) \approx -\left( \frac{\zeta - \gamma^2}{1 - \gamma^2 - x^2} \right)^{\frac{1}{2}} \left( \sqrt{\lambda} Y_\mu \left( \lambda \sqrt{\zeta} \right) \sum_{k=0}^{\infty} F_k(\zeta) \frac{\lambda^k}{\lambda^{2k}} + \left( \frac{\zeta}{\lambda} \right)^{\frac{1}{2}} Y'_\mu \left( \lambda \sqrt{\zeta} \right) \sum_{k=0}^{\infty} \tilde{F}_k(\zeta) \lambda^{2k} \right)
\]

as \( \nu \to \infty \). The coefficients are given by \( F_0(\zeta) = 1 \) and the formulas

\[
\tilde{F}_k(\zeta) = \left| \zeta - \alpha^2 \right|^{-\frac{1}{2}} \int_{\alpha^2}^{\zeta} \left| \xi - \alpha^2 \right|^{-\frac{1}{2}} \left( \xi \tilde{F}'_k(\xi) + F_k(\xi) - \psi(\xi) F_k(\xi) \right) \, d\xi
\]

and

\[
F_{k+1}(\zeta) = -\zeta \tilde{F}'_k(\zeta) + \int_{\alpha^2}^{\zeta} \psi(\xi) \tilde{F}'_k(\xi) \, d\xi + \zeta \tilde{F}_k(\alpha^2),
\]

where

\[
\psi(\zeta) = \frac{1}{16} \left( \zeta - \alpha^2 \right)^{-2} \left( \zeta + 4\alpha^2 + \frac{1 - x^2}{\zeta} \left( \frac{\zeta - \alpha^2}{1 - \alpha^2 - x^2} \right)^3 \left( 1 - 4\alpha^2 \right) x^2 + (\alpha^4 - 1) \right).
\]

Note that the variable \( x \) in (24) depends on \( \zeta \) through either (18) or (19); if this is neglected and \( x \) is treated as a constant in (22) and (23), then these integrals diverge. While (20) and (21) are powerful expansions, it is not clear how to accurately and rapidly compute the variable \( \zeta \) defined via the relations (18) and (19) given \( x \). Nor is it obvious how to evaluate the coefficients in these expansions. Only the first few are known analytically, and the numerical calculation of the remaining coefficients is complicated by the delicate cancellations of singularities which occur in these formulas. Alternate Liouville-Green expansions for (3) and (4) are given in Chapter 12 of [23]. However, the coefficients in these expansions appear to be no easier to compute than those in (20) and (21), and they also involve a variable transformation defined implicitly by a nonlinear equation. Moreover, they have the unfortunate property that are only applicable when \( \mu \) is small relative to \( \nu \).

The literature on asymptotic expansions of associated Legendre functions is vast, and the possibility of constructing a numerical code for the evaluation of (3) and (4) using such methods cannot be ruled out.
However, the approach offered in this paper — namely, the precomputation of expansions representing the logarithms of certain solutions of the associated Legendre differential equation — is simple-minded and highly effective. Moreover, the same basic technique can be applied, with little effort, to a large class of special functions satisfying second order differential equations. Indeed, in [4], similar techniques were used to numerically precompute a table which allows for the evaluation the Bessel functions of the first and second kinds on the interval \((0, \infty)\) in time independent of order.

The remainder of this paper is structured as follows. In Section 1, we review certain mathematical facts and numerical procedures which are used in the rest of this article. In Section 2, we describe a numerical method for the solution of the differential equation (7) which runs in time independent of the parameters \(\nu\) and \(\mu\). The construction of a precomputed table of expansions of the associated Legendre functions which makes use of the algorithm of Section 2 is discussed in Section 3. In Section 4, we detail our algorithm for the numerical calculation of the associated Legendre functions. Section 5 describes the results of numerical experiments carried out to verify the efficacy of our algorithm. We close with a few brief comments in Section 6.

1. Mathematical and numerical preliminaries.

1.1. The condition number of the evaluation of a function. The condition number of the evaluation of a differentiable function \(f : \mathbb{R} \to \mathbb{R}\) at the point \(x\) is commonly defined to be
\[
\kappa_f(x) = \left| \frac{x f'(x)}{f(x)} \right|
\]  
(25)
(see, for instance, Section 1.6 of [14]). This quantity measures the ratio of the magnitude of the relative change in \(f(x)\) induced by a small change in the argument \(x\) to the magnitude of the relative change in \(x\) in the sense that
\[
\left| \frac{f(x + \delta) - f(x)}{f(x)} \right| \approx \kappa_f(x) \left| \frac{\delta}{x} \right|
\]  
(26)
for small \(\delta\). Since almost all quantities which arise in the course of numerical calculations are subject to perturbations with relative magnitudes on the order of machine epsilon, we consider
\[
\kappa_f(x) \epsilon_0,
\]  
(27)
where \(\epsilon_0\) denotes machine epsilon, to be a rough estimate of the relative accuracy one should expect when evaluating \(f(x)\) numerically (in fact, it tends to be a slightly pessimistic estimate). In the rest of this paper, we take \(\epsilon_0\) to be
\[
\epsilon_0 = 2^{-52} \approx 2.22044604925031 \times 10^{-16}.
\]  
(28)
It is immediately clear from (25) that when \(f'(x_0)x_0 \neq 0\) and \(f(x_0) = 0\), \(\kappa_f(x)\) diverges to \(\infty\) as \(x \to x_0\). One consequence of this is that there is often a significant loss of relative accuracy when a function is evaluated near one of its roots. For the most part, we avoid this issue by representing the solutions of the associated Legendre differential equation via functions which are bounded away from 0.

1.2. Trivariate Chebyshev expansions. For each nonnegative integer \(n\), the Chebyshev polynomial of degree \(n\) is defined for \(-1 < x < 1\) via the formula
\[
T_n(x) = \cos(n \arccos(x)).
\]  
(29)
The trivariate Chebyshev series of a continuous function \(f : [-1, 1]^3 \to \mathbb{R}\) is
\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{i,j,k} T_i(x)T_j(y)T_k(z),
\]  
(30)
where the coefficients are defined via the formula

\[
ai,j,k = \frac{8}{\pi^3} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(x, y, z)T_i(x)T_j(y)T_k(z) \frac{dx}{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-y^2}} \frac{dz}{\sqrt{1-z^2}}
\]

and the dashes next to the summation symbols indicate that the first term in each sum is halved. The well-known relationship between Chebyshev and Fourier series (see, for instance, [20]), together with the results of [12] on the pointwise almost everywhere convergence of multiple Fourier series immediately imply that

\[
\lim_{N \to \infty} \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} ai,j,kT_i(x)T_j(y)T_k(z) = f(x, y, z)
\]

for almost all \((x, y, z) \in [-1, 1]^3\). As in the case of univariate Chebyshev series, under mild smoothness conditions on \(f\), the convergence of (30) is uniform. See, for instance, Theorem 5.9 in [20].

If \(f(x, y, z)\) is analytic on the set

\[
\{(x, y, z) \in \mathbb{C}^3 : |x + \sqrt{x^2 - 1}| < r_1, |y + \sqrt{y^2 - 1}| < r_2, |z + \sqrt{z^2 - 1}| < r_3\},
\]

where \(r_1, r_2, r_3 > 1\), then \(|ai,j,k| = O(1/(r_1i-r_2j-r_3k)^{\frac{s}{r}})\) (this is according to Theorem 11 in Chapter V of [1]), with the consequence that the limit in (32) converges rapidly to \(f\) when \(f\) is analytic in a large neighborhood containing \([-1, 1]^3\).

For each nonnegative integer \(n\), we refer to the collection of points

\[
\rho_{j,n} = -\cos\left(\frac{\pi j}{n}\right), \quad j = 0, 1, \ldots, n,
\]

as the \((n + 1)\)-point Chebyshev grid on the interval \([-1, 1]\), and we call individual elements of this set Chebyshev nodes or points. One discrete version of the well-known orthogonality relation

\[
\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} \ dx = \begin{cases} 0 & \text{if } i \neq j \\ \frac{\pi}{2} & \text{if } i = j > 0 \\ \pi & \text{if } i = j = 0. \end{cases}
\]

is

\[
\sum_{l=0}^{n} T_i(\rho_{l,n})T_j(\rho_{l,n}) = \begin{cases} 0 & \text{if } 0 \leq i, j \leq n \text{ and } i \neq j \\ \frac{n}{2} & \text{if } 0 < i = j < n \\ n & \text{if } i = j = 0 \text{ or } i = j = n. \end{cases}
\]

Here, the double dash next to the summation sign indicates that the first and last term in the series are halved. Formula (36) can be found in a slightly different form in Chapter 4 of [20].

It follows easily from (35) and (36) that any trivariate polynomial \(f\) of degree less than or equal to \(n\) can be represented in the form

\[
f(x, y, z) = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} b_{i,j,k}T_i(x)T_j(y)T_k(z),
\]

where

\[
b_{i,j,k} = \frac{8}{\pi^3} \sum_{r=0}^{n} \sum_{s=0}^{n} \sum_{t=0}^{n} T_i(\rho_{r,n})T_j(\rho_{s,n})T_k(\rho_{t,n}) f(\rho_{r,n}, \rho_{s,n}, \rho_{t,n}).
\]
If \( f \) is not a polynomial of degree less than or equal to \( n \), then the representation (37) is no longer exact. However, in this event, there is a well-known relationship between the coefficients defined via (38) and those given by (30). In particular,

\[
\begin{align*}
(b_{i,j,k}) = a_{ijk} + \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \sum_{l_3=1}^{\infty} (a_{i+2l_1,n,j+2l_2,n,k+2l_3,n} + a_{i+2l_1,n,-j+2l_2,n,-k+2l_3,n})
\end{align*}
\]

for all \( 0 \leq i,j,k \leq n \) (the one-dimensional version of this result can be found, for instance, in [20]). Using (39) it is easy to show that there exists a constant \( C \) such that

\[
\begin{align*}
\sup_{x \in [-1,1]} |f(x,y,z) - \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} b_{i,j,k} T_i(x) T_j(y) T_k(z) - \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} a_{i,j,k} T_i(x) T_j(y) T_k(z)| \leq C \sup_{x \in [-1,1]} |f(x,y,z) - \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} a_{i,j,k} T_i(x) T_j(y) T_k(z)|.
\end{align*}
\]

It follows, of course, that the sum (38) converges rapidly to \( f \) when \( f \) is analytic in a large neighborhood of \([-1,1]^3\). By a slight abuse of terminology, we will refer to (37) as the \( n \)th order Chebyshev expansion for the function \( f \).

1.3. Compressed trivariate Chebyshev expansions. It often happens that many of the coefficients in the trivariate Chebyshev expansion (37) of a function \( f : [-1,1]^3 \to \mathbb{R} \) are of negligible magnitude. In order to reduce the cost of storing such expansions as well as the cost of evaluating them, we use the following construction to reduce the number of coefficients which need to be considered.

Suppose that \( \epsilon > 0 \), and that

\[
\sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} b_{i,j,k} T_i(x) T_j(y) T_k(z)
\]

is the \( n \)th order Chebyshev expansion for \( f : [-1,1]^3 \to \mathbb{R} \). We let \( M \) denote the least nonnegative integer which is less than or equal to \( n \) and such that

\[
|b_{i,j,k}| < \epsilon \quad \text{for all } i = M + 1, \ldots, n, \quad j = 0, \ldots, n \quad \text{and} \quad k = 0, \ldots, n,
\]

assuming such an integer exists. If not, then we take \( M = n \). For each \( i = 0, \ldots, M \), we let \( m_i \) be the least nonnegative integer less than or equal to \( n \) such that

\[
|b_{i,j,k}| < \epsilon \quad \text{for all } j = m_i + 1, \ldots, n, \quad \text{and} \quad k = 0, \ldots, n
\]

if such an integer exists, and we let \( m_i = n \) otherwise. Finally, for each pair \((i,j)\) such that \( 0 \leq i \leq M \) and \( 0 \leq j \leq m_i \), we let \( n_{i,j} \) be the least nonnegative integer such that

\[
|b_{i,j,k}| < \epsilon \quad \text{for all } k = 0, \ldots, n_{i,j}.
\]

We refer to the series

\[
\sum_{i=0}^{M} \sum_{j=0}^{m_i} \sum_{k=0}^{n_{i,j}} \tilde{b}_{i,j,k} T_i(x) T_j(y) T_k(x),
\]

for all \( 0 \leq i,j,k \leq n \). Using (39) it is easy to show that there exists a constant \( C \) such that

\[
\begin{align*}
\sup_{x \in [-1,1]} |f(x,y,z) - \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} b_{i,j,k} T_i(x) T_j(y) T_k(z) - \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} a_{i,j,k} T_i(x) T_j(y) T_k(z)| \leq C \sup_{x \in [-1,1]} |f(x,y,z) - \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} a_{i,j,k} T_i(x) T_j(y) T_k(z)|.
\end{align*}
\]
where \( \tilde{b}_{i,j,k} \) is defined via

\[
(46) \quad \tilde{b}_{i,j,k} = b_{i,j,k} \left( 1 - \frac{1}{2} \delta_{i,n} \right) \left( 1 - \frac{1}{2} \delta_{j,0} \right) \left( 1 - \frac{1}{2} \delta_{j,0} \right) \left( 1 - \frac{1}{2} \delta_{j,n} \right) \left( 1 - \frac{1}{2} \delta_{k,0} \right) \left( 1 - \frac{1}{2} \delta_{k,n} \right),
\]

as the \( \epsilon \)-compressed \( n^{th} \) order Chebyshev expansion of \( f \).

Obviously, the results discussed in this and the preceding section can be modified in a straightforward fashion so as to apply to functions given on an arbitrary rectangular prism \([a, b] \times [c, d] \times [e, f]\).

### 1.4. Series expansions of the associated Legendre functions and connection formulas.

When \( \nu \geq 0, -\nu \leq \mu \leq \nu \) and \(-1 \leq x < 1\), the associated Legendre function of the first kind of degree \( \nu \) and order \(-\mu\) is given by

\[
(47) \quad P_{\nu}^{-\mu}(x) = \left( \frac{1 - x}{1 + x} \right)^{\mu/2} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\nu + n + 1)}{\Gamma(\nu - n + 1) \Gamma(n + 1) \Gamma(n + \mu + 1)} \frac{(1 - x)^n}{2^n}.
\]

Here, we have adopted the convention that

\[
(48) \quad \frac{1}{\Gamma(k)} = 0
\]

whenever \( k \) is a negative integer. The trigonometric form

\[
(49) \quad P_{\nu}^{-\mu} (\cos(t)) = \left( \tan \left( \frac{t}{2} \right) \right)^{\mu} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\nu + n + 1)}{\Gamma(\nu - n + 1) \Gamma(n + 1) \Gamma(n + \mu + 1)} \left( \sin \left( \frac{t}{2} \right) \right)^{2n},
\]

of (47) is obtained by letting \( x = \cos(t) \) and making use of elementary identities. When the parameters \( \nu \) and \( \mu \) are of small magnitude, the coefficients in (49) decay rapidly as \( n \) increases, with the consequence that only a small number of terms of (49) are required to accurately evaluate \( P_{\nu}^{-\mu} \). Likewise, even when the parameters are of large magnitude the coefficients in this expansion decay rapidly with \( n \) if \( t \) is sufficiently small, so that (49) is efficient in this regime as well. For extremely large values of \( \nu \), we found numerical roundoff error to be a problem in the evaluation of (49). For this reason, we only use this series expansion in the event that \( \nu \) is less than 10,000.

One potential difficulty with the use of (49) as a numerical tool, however, is that underflow can occur when the parameters are large and \( t \) is small. To obviate such problems, we use a truncation of the formula

\[
(50) \quad \log \left( \tilde{P}_{\nu}^{-\mu}(t) \right) = \log \left( \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \right) + \log \left( \nu + \frac{1}{2} \right) + \frac{1}{2} \log(\sin(t)) + \mu \log \left( \tan \left( \frac{t}{2} \right) \right) - \log \left( \frac{1}{\Gamma(\mu + 1)} \right) + \log \left( \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\nu + n + 1)}{\Gamma(\nu - n + 1) \Gamma(n + 1) \Gamma(n + 1) \Gamma(\mu + 1)} \left( \sin \left( \frac{t}{2} \right) \right)^{2n} \right),
\]

which is easily obtained from (1) and (49), to evaluate the logarithm of the associated Legendre function of the first kind in this regime. We note that \( \tilde{P}_{\nu}^{-\mu}(t) \) is necessarily positive when \( t \) is sufficiently small, so that this logarithm is sensible. When \( \mu \) is equal to a negative integer, say \(-m\), the first \( m \) terms of the sum in (49) are 0 and we use a version of (50) which is modified accordingly.

**Remark 1.** The naive evaluation of the first term in (50) can lead to numerical cancellation when \( \nu \) is large and \( \mu \) is small relative to \( \nu \). In this event, we use the first sixteen terms of the asymptotic approximation

\[
\log \left( \frac{\Gamma(x - y)}{\Gamma(x + y)} \right) \approx 2(\mu + 1) \log(\nu) + \log \left( 1 - \frac{y}{x} + \frac{y(2y^2 - 3y + 1)}{6x^2} + \frac{y^2(2y^2 - 3y + 1)}{6x^3} \right)
\]

\[
\]

\[
+ \frac{y(20y^5 - 96y^4 + 155y^3 - 90y^2 + 5y + 6)}{360x^4} + \ldots
\]
in order to evaluate it.

For \( \nu \geq 0, -1 \leq x < 1 \) and \(-\nu \leq \mu \leq \nu\) not an integer, the associated Legendre function of the second kind of degree \( \nu \) and order \(-\mu\) is given by

\[
Q^{-\mu}_\nu(x) = \frac{\pi}{2} \left( \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \sec(\mu \pi) P^\mu_\nu(x) - \cot(\mu \pi) P^{-\mu}_\nu(x) \right).
\]

The normalized versions of the associated Legendre functions satisfy the somewhat simpler relation

\[
\hat{Q}^{-\mu}_\nu(t) = \sec(\mu \pi) \hat{P}^\mu_\nu(t) - \cot(\mu \pi) \hat{P}^{-\mu}_\nu(t),
\]

which is an immediate consequence of (51), (1) and (2). Similarly to the case of (49), the use of (52) can lead to numerical overflow when \( t \) is small. Accordingly, we generally compute the logarithm of \( \hat{Q}^{-\mu}_\nu \) via the less delicate formula

\[
\log \left( \hat{Q}^{-\mu}_\nu(t) \right) = \log \left( \hat{P}^{-\mu}_\nu(t) \right) + \log \left( \sec(\mu \pi) - \cot(\mu \pi) \text{sign} \left( \hat{P}^\mu_\nu(t) \right) \right)
\]

\[
\exp \left( \log \left( \left| \hat{P}^\mu_\nu(t) \right| \right) - \log \left( \hat{P}^{-\mu}_\nu(t) \right) \right)
\]

in this regime. We note that for sufficiently small \( t \), the function \( \hat{Q}^{-\mu}_\nu(t) \) is positive.

When \( \mu \) is an integer, (52) and (53) lose their meanings. Various series expansions for \( Q^{-\mu}_\nu(x) \) with \( m \) a positive integer can be obtained (see, for instance, Section 3.6 of [10]), but they are somewhat cumbersome and do not address a second problem with the use of (52) as a numerical method for the evaluation of \( Q^{-\mu}_\nu \). Namely, that when \( \mu \) is close to, but does not coincide with, an integer, the evaluation (52) results in severe loss of precision due to numerical cancellation. However, since \( \hat{Q}^{-\mu}_\nu \) is an analytic function of the parameter \( \mu \), it can be efficiently interpolated in the \( \mu \) variable. For instance, when \( \mu \) is close to, or coincides with, an integer \( m \), the value of \( \hat{Q}^{-\mu}_\nu(t) \) can be calculated by first evaluating

\[
\hat{Q}_\nu^{\xi_1}(t), \ldots, \hat{Q}_\nu^{\xi_{2n+1}}(t)
\]

with \( \xi_1, \ldots, \xi_{2n} \) the nodes of the \((2n)\)-point Chebyshev grid on the interval \([m - \epsilon], [m + \epsilon]\), and then using Chebyshev interpolation to calculate \( \hat{Q}^{-\mu}_\nu(t) \). Here, \( \epsilon \) is an appropriate chosen positive real number and \( n \) is a positive integer. An even number of nodes is chosen in order to ensure that none coincide with the integer \( m \). In the code used in this paper, we apply this procedure when \( \mu \) is within a distance of 0.001 of an integer, and we take \( n = 6 \) and \( \epsilon = 0.1 \). Of course, the same approach can be used to evaluate \( \log \left( \hat{Q}^{-\mu}_\nu(t) \right) \).

Indeed, many other connection formulas for the associated Legendre functions can be handled in a similar fashion, such as the identity

\[
\hat{Q}^{-\mu}_\nu(t) = \sec(\mu \pi) \hat{Q}^\mu_\nu(t) + \tan(\mu \pi) \hat{P}^{-\mu}_\nu(t),
\]

which follows easily from a formula found in in Section 3.4 of [10]. On the other hand, the connection formulas

\[
\hat{P}^{-\mu}_\nu(\pi - t) = \cos(\pi(\nu - \mu)) \hat{P}^{-\mu}_\nu(t) - \sin(\pi(\nu - \mu)) \hat{Q}^{-\mu}_\nu(t)
\]

and

\[
\hat{Q}^{-\mu}_\nu(\pi - t) = -\cos(\pi(\nu - \mu)) \hat{Q}^{-\mu}_\nu(t) - \sin(\pi(\nu - \mu)) \hat{P}^{-\mu}_\nu(t),
\]

which also appear (in a slightly different form) in Section 3.4 of [10], are immune from such problems.
1.5. Macdonald’s asymptotic expansions. In [19], an asymptotic formula for $P^{-\nu}_\nu(\cos(t))$ which is accurate when $\nu$ is large, $0 \leq \mu \leq \nu$ and $t$ is small is derived by replacing the ratio of Gamma functions appearing in (49) with a finite truncation of the series expansion

\[
\frac{\Gamma(\nu + n + 1)}{\Gamma(\nu - n + 1)} = \lambda^{2n} - G_1 \lambda^{2n-2} + G_2 \lambda^{2n-4} + G_3 \lambda^{2n-6} + \cdots,
\]

where $\lambda = \nu + \frac{1}{2}$. The first three terms in the asymptotic expansion of $P^{-\nu}_\nu$ obtained in this fashion are

\[
P^{-\nu}_\nu(\cos(t)) \approx \left(\lambda \cos \left(\frac{t}{2}\right)\right)^{-\mu} \cdot \left(J_{\mu}(\eta) + \sin^2 \left(\frac{t}{2}\right) H_1 + \sin^4 \left(\frac{t}{2}\right) H_2 + \sin^6 \left(\frac{t}{2}\right) H_3\right),
\]

where

\[
H_1 = \frac{\eta}{6} J_{\mu+3}(\eta) - J_{\mu+2}(\eta) + \frac{1}{2\eta} J_{\mu+1}(\eta),
\]

\[
H_2 = \frac{\eta^2}{72} J_{\mu+6}(\eta) - \frac{11\eta}{30} J_{\mu+5}(\eta) + \frac{31}{12} J_{\mu+4}(\eta) - \frac{29}{6\eta} J_{\mu+3}(\eta) + \frac{9}{8\eta^2} J_{\mu+2}(\eta),
\]

and

\[
H_3 = \frac{75}{16\eta^2} J_{\mu+2}(\eta) - \frac{751}{24\eta^2} J_{\mu+3}(\eta) + \frac{1381}{48\eta} J_{\mu+4}(\eta) - \frac{1513}{180\eta} J_{\mu+5}(\eta)
\]

\[
+ \frac{4943\eta}{5040} J_{\mu+6}(\eta) - \frac{17\eta^2}{360} J_{\mu+7}(\eta) + \frac{\eta^3}{1296} J_{\mu+8}(\eta).
\]

The first few terms of the analogous expansion of the associated Legendre function of the second kind, which is applicable when $0 \leq \mu \leq \nu$, are

\[
Q^{-\mu}_\nu(\cos(t)) \approx -\frac{\pi}{2} \left(\lambda \cos \left(\frac{t}{2}\right)\right)^{-\mu} \cdot \left(Y_{-\mu}(\eta) + \sin^2 \left(\frac{t}{2}\right) N_1 + \sin^4 \left(\frac{t}{2}\right) N_2 + \sin^6 \left(\frac{t}{2}\right) N_3\right),
\]

where $Y_\zeta$ denotes the Bessel function of the second kind of order $\zeta$,

\[
N_1 = \frac{\eta}{6} Y_{-\mu+3}(\eta) - Y_{-\mu+2}(\eta) + \frac{1}{2\eta} Y_{-\mu+1}(\eta),
\]

\[
N_2 = \frac{\eta^2}{18} Y_{-\mu+6}(\eta) - \frac{11\eta}{30} Y_{-\mu+5}(\eta) + \frac{31}{12} Y_{-\mu+4}(\eta) - \frac{29}{6\eta} Y_{-\mu+3}(\eta) + \frac{9}{8\eta^2} Y_{-\mu+2}(\eta),
\]

and

\[
N_3 = \frac{75}{16\eta^2} Y_{-\mu+2}(\eta) - \frac{751}{24\eta^2} Y_{-\mu+3}(\eta) + \frac{1381}{48\eta} Y_{-\mu+4}(\eta) - \frac{1513}{180\eta} Y_{-\mu+5}(\eta)
\]

\[
+ \frac{4943\eta}{5040} Y_{-\mu+6}(\eta) - \frac{17\eta^2}{360} Y_{-\mu+7}(\eta) + \frac{\eta^3}{1296} Y_{-\mu+8}(\eta).
\]
To reduce the potential for numerical underflow in the evaluation of (60), we evaluate \( \log \left( \tilde{P}_\nu^{-\mu}(t) \right) \) using the following formula instead:

\[
(62) \quad \log \left( \tilde{P}_\nu^{-\mu}(t) \right) \approx \log \left( \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \right) + \log \left( \frac{\nu + 1}{2} \right) + \frac{1}{2} \log (\sin(t)) - \mu \log \left( \lambda \cos \left( \frac{t}{2} \right) \right) \\
+ \log (J_{\mu}(\eta)) + \log \left( 1 + \sin^2 \left( \frac{t}{2} \right) \hat{H}_1 + \sin^4 \left( \frac{t}{2} \right) \hat{H}_2 + \sin^6 \left( \frac{t}{2} \right) \hat{H}_3 \right),
\]

where

\[
\hat{H}_1 = \frac{\eta}{6} \exp \left( \log (J_{\mu+3}(\eta)) - \log (J_{\mu}(\eta)) \right) - \exp \left( \log (J_{\mu+2}(\eta)) - \log (J_{\mu}(\eta)) \right) \\
+ \frac{1}{2\eta} \exp \left( \log (J_{\mu+1}(\eta)) - \log (J_{\mu}(\eta)) \right),
\]

\[
\hat{H}_2 = \frac{\eta^2}{72} \exp \left( \log (J_{\mu+6}(\eta)) - \log (J_{\mu}(\eta)) \right) - \frac{11\eta}{30} \exp \left( \log (J_{\mu+5}(\eta)) - \log (J_{\mu}(\eta)) \right) \\
+ \frac{31}{12} \exp \left( \log (J_{\mu+4}(\eta)) - \log (J_{\mu}(\eta)) \right) - \frac{29}{6\eta} \exp \left( \log (J_{\mu+3}(\eta)) - \log (J_{\mu}(\eta)) \right) \\
+ \frac{9}{8\eta^2} \exp \left( \log (J_{\mu+2}(\eta)) - \log (J_{\mu}(\eta)) \right),
\]

and

\[
\hat{H}_3 = \frac{75}{16\eta^3} \exp \left( \log (J_{\mu+2}(\eta)) - \log (J_{\mu}(\eta)) \right) - \frac{751}{24\eta^2} \exp \left( \log (J_{\mu+3}(\eta)) - \log (J_{\mu}(\eta)) \right) \\
+ \frac{1381}{48\eta} \exp \left( \log (J_{\mu+4}(\eta)) - \log (J_{\mu}(\eta)) \right) - \frac{1513}{180} \exp \left( \log (J_{\mu+5}(\eta)) - \log (J_{\mu}(\eta)) \right) \\
+ \frac{4943\eta}{5040} \exp \left( \log (J_{\mu+6}(\eta)) - \log (J_{\mu}(\eta)) \right) \\
- \frac{17\eta^2}{360} \exp \left( \log (J_{\mu+7}(\eta)) - \log (J_{\mu}(\eta)) \right) + \frac{\eta^3}{1296} \exp \left( \log (J_{\mu+8}(\eta)) - \log (J_{\mu}(\eta)) \right).
\]

As discussed in Remark 1, some care must be taken in evaluating the first term in (62). We use an analogous form of (61) in order to evaluate \( \log \left( \tilde{Q}_\nu^{\mu}(t) \right) \). The logarithms of the Bessel functions appearing in these formulas are calculated via the algorithm of [4].

We are unaware of a simple analog of (61) for the function \( \tilde{Q}_\nu^{-\mu} \). When we say that we evaluate \( \log \left( \tilde{Q}_\nu^{\mu}(t) \right) \) via Macdonald’s expansions, we mean that we combine the appropriate formulas for \( \log \left( \tilde{P}_\nu^{-\mu}(t) \right) \) and \( \log \left( \tilde{Q}_\nu^{\mu}(t) \right) \) with the connection formula (55) in order to calculate \( \log \left( \tilde{Q}_\nu^{-\mu}(t) \right) \).

\textbf{1.6. Riccati’s equation, Kummer’s equation and phase functions.} In this section, we suppose that \( q \) is a smooth, real-valued function defined on an open interval \( I \subset \mathbb{R} \). In the event that \( q \) is strictly negative on \( I \), two linearly independent solutions of the second order differential equation

\[
(63) \quad y''(t) + q(t)y(t) = 0 \quad \text{for all} \quad t \in I
\]

both of which are positive on \( I \) can be found. This follows easily from standard proofs of Picard’s theorem on the existence and uniqueness of solutions of ordinary differential equations (see, for instance, Section 2.3
of \([15]\)). Any positive solution \(y\) of (63) can be represented in the form \(y = \exp(r(t))\) with \(r\) real-valued, and a straightforward computation shows that \(r\) must satisfy
\[
(64) \quad r''(t) + (r'(t))^2 + q(t) = 0 \quad \text{for all } t \in I.
\]
Equation (64) is known as Riccati’s equation; a detailed discussion of it can be found in \([15]\), among many other sources.

When \(q\) is positive on \(I\), the solutions of (63) oscillate and their logarithms are complex-valued. In this case, it is convenient to represent the solutions of (63) via a phase function, which is nothing more than the imaginary part of the logarithm of one of its solutions. More precisely, we say that a smooth function \(\alpha\) defined on \(I\) is a phase function for the second order differential equation (63) provided \(\alpha'\) does not vanish on \(I\) and the pair
\[
(65) \quad u(t) = \frac{\cos(\alpha(t))}{\sqrt{|\alpha'(t)|}}
\]
and
\[
(66) \quad v(t) = \frac{\sin(\alpha(t))}{\sqrt{|\alpha'(t)|}}
\]
form a basis in the space of solutions of (63). We note that the definition of phase function does not require that \(q\) be positive, although phase functions are most useful on intervals where this is the case. Proofs of the following elementary results regarding phase functions can be found in \([13]\) and \([4]\).

**Theorem 1.** Suppose that \(I\) is an open interval in \(\mathbb{R}\), and that \(q\) is a smooth, real-valued function defined on \(I\). Suppose also that \(\alpha\) is a smooth, real-valued function defined on \(I\) whose first derivative does not vanish there. Then \(\alpha\) is a phase function for the second order differential equation (63) if and only if its derivative \(\alpha'\) satisfies the second order nonlinear differential equation
\[
(67) \quad q(t) - (\alpha'(t))^2 - \frac{1}{2} \left( \frac{\alpha''(t)}{\alpha'(t)} \right) + \frac{3}{4} \left( \frac{\alpha''(t)}{\alpha'(t)} \right)^2 = 0 \quad \text{for all } t \in I.
\]

**Theorem 2.** Suppose that \(u, v\) is a pair of smooth, real-valued solutions of (63) whose (necessarily constant) Wronskian \(W\) is nonzero. Then there is a phase function \(\alpha\) for (63) such that
\[
(68) \quad u(t) = \sqrt{W} \frac{\cos(\alpha(t))}{\sqrt{|\alpha'(t)|}}
\]
and
\[
(69) \quad v(t) = \sqrt{W} \frac{\sin(\alpha(t))}{\sqrt{|\alpha'(t)|}}.
\]
Moreover, the derivative of \(\alpha\) is given by
\[
(70) \quad \alpha'(t) = \frac{W}{(u(t))^2 + (v(t))^2} \quad \text{for all } t \in I,
\]
and \(\alpha\) is unique up to addition by an integer multiple of \(2\pi\). That is, \(\tilde{\alpha}\) is a phase function for (63) such that (68) and (69) hold if and only there exists an integer \(L\) such that
\[
(71) \quad \tilde{\alpha}(t) = \alpha(t) + 2\pi L \quad \text{for all } t \in I.
\]

We will refer to (67) as Kummer’s equation, after E. E. Kummer who studied it in \([16]\).
1.7. A nonoscillatory phase function for the associated Legendre differential equation. From Theorem 2, we see that there is a phase function \( \alpha_{\nu,\mu} \) for (7) such that
\[
\tilde{P}_\nu^{-\mu}(t) = \sqrt{\frac{2}{\pi}} \frac{\nu + \frac{1}{2}}{\alpha'_{\nu,\mu}(t)} \cos(\alpha_{\nu,\mu}(t))
\]
and
\[
\tilde{Q}_\nu^{-\mu}(t) = \sqrt{\frac{2}{\pi}} \frac{\nu + \frac{1}{2}}{\alpha'_{\nu,\mu}(t)} \sin(\alpha_{\nu,\mu}(t))
\]
and whose derivative is given by
\[
\alpha'_{\nu,\mu}(t) = \frac{2}{\pi} \left( \nu + \frac{1}{2} \right) \frac{1}{\left( \tilde{P}_\nu^{-\mu}(t) \right)^2 + \left( \tilde{Q}_\nu^{-\mu}(t) \right)^2}.
\]

We have made use of the fact (which can be found in a slightly different form in Section 3.4 of [10]) that the Wronskian of the pair \( \tilde{P}_\nu^{-\mu}, \tilde{Q}_\nu^{-\mu} \) is \( \frac{2}{\pi} \left( \nu + \frac{1}{2} \right) \). It has long been known that the function (74) is nonoscillatory. Indeed, it is immediate from (20) and (21) that
\[
\alpha'_{\nu,\mu}(t) \sim 2 \pi \left( \frac{\zeta - \gamma^2}{1 - \gamma^2 - x^2} \right)^{-\frac{3}{4}} \frac{1}{(J_\mu(\lambda \sqrt{\zeta}))^2 + (Y_\mu(\lambda \sqrt{\zeta}))^2} \text{ as } \nu \to \infty,
\]
where \( \zeta \) is the variable defined implicitly by (18) and (19), \( \lambda = \nu + \frac{1}{2} \) and \( \gamma = \frac{\nu}{\lambda} \). A cursory inspection of Nicholson’s integral formula
\[
J_\mu^2(z) + Y_\mu^2(z) = \frac{8}{\pi^2} \int_0^\infty K_0(2z \sinh(t)) \cosh(2\mu t) \, dt,
\]
a derivation of which can be found in Section 13.73 of [26], reveals that the function \( (J_\mu(x))^2 + (Y_\mu(x))^2 \) is nonoscillatory. We note that this property of \( \alpha_{\nu,\mu} \) is highly unusual. Figure 1 compares \( \alpha'_{\nu,\mu} \) with the derivative of typical phase functions for (7), which oscillate on some portion of \( \left( 0, \frac{\pi}{2} \right) \).

It follows from (74) that there exists a constant \( C \) such that
\[
\alpha_{\nu,\mu}(t) = C + \int_0^t \alpha'_{\nu,\mu}(s) \, ds.
\]

In fact, using the formulas
\[
\tilde{P}_\nu^{-\mu} \left( \frac{\pi}{2} \right) = \frac{2^{-\mu}}{\sqrt{\pi}} \sqrt{\frac{\nu + 1/2}{\Gamma(\nu + \mu + 1) \cos \left( \frac{1}{2} \pi (\nu - \mu) \right) \Gamma \left( \frac{1}{2} (\nu + \mu + 1) \right) \Gamma \left( \frac{1}{2} (\nu + \mu + 2) \right)}}
\]
and
\[
\tilde{Q}_\nu^{-\mu} \left( \frac{\pi}{2} \right) = -\frac{2^{-\mu}}{\sqrt{\pi}} \sqrt{\frac{\nu + 1/2}{\Gamma(\nu + \mu + 1) \sin \left( \frac{1}{2} \pi (\nu - \mu) \right) \Gamma \left( \frac{1}{2} (\nu + \mu + 1) \right) \Gamma \left( \frac{1}{2} (\nu + \mu + 2) \right)}}
\]
which appear in a slightly different form in Section 3.4 of [10], we see that (72) and (73) hold so long as the constant \( C \) in (77) is taken to be
\[
C = \frac{\pi}{2} (\nu - \mu) + 2\pi L
\]
with \( L \) an integer. In the remainder of this paper, we let \( \alpha_{\nu,\mu} \) denote the phase function defined via the formula

\[
\alpha_{\nu,\mu}(t) = 2\pi + \frac{\pi}{2} (\nu - \mu) + \int_{\frac{\pi}{2}}^{t} \alpha'_{\nu,\mu}(s) \, ds.
\]

We set \( L = 1 \) in order to ensure that \( \alpha_{\nu,\mu} \) is bounded away from 0 on the interval \((0, \frac{\pi}{2})\). In this way, we avoid certain difficulties which arise because the condition number of evaluation of a function is generally infinite near one of its roots (as per the discussion in Section 1.1). By inserting (78) and (79) into (74), we see that

\[
\alpha'_{\nu,\mu} \left( \frac{\pi}{2} \right) = \frac{2\Gamma \left( \frac{1}{2} (\nu - \mu + 2) \right) \Gamma \left( \frac{1}{2} (\nu + \mu + 2) \right)}{\Gamma \left( \frac{1}{2} (\nu - \mu + 1) \right) \Gamma \left( \frac{1}{2} (\nu + \mu + 1) \right)}.
\]

Expressions for the values of the derivatives of the functions \( \tilde{P}_{\nu}^{-\mu} \) and \( \tilde{Q}_{\nu}^{-\mu} \) at the point \( \frac{\pi}{2} \) can be easily derived from formulas appearing in Section 3.4 of [10]. A tedious computation which makes use of them in addition to (78) and (79) shows that

\[
\alpha_{\nu,\mu} \left( \frac{\pi}{2} \right) = 0.
\]

1.8. An adaptive discretization procedure. We now briefly describe a fairly standard procedure for adaptively discretizing a smooth function \( f : [a, b] \to \mathbb{R} \). It takes as input a desired precision \( \epsilon > 0 \), a
positive integer \( n \) and a subroutine for evaluating \( f \). The goal of this procedure is to construct a partition of \([a, b]\) such that the \( n \)th order Chebyshev expansion of \( f \) on each of the subintervals \([\gamma_j, \gamma_{j+1}]\) of \([a, b]\) approximates \( f \) with accuracy \( \epsilon \). That is, for each \( j = 0, \ldots, m-1 \) we aim to achieve

\[
\sup_{x \in [\gamma_j, \gamma_{j+1}]} \left| f(x) - \sum_{i=0}^{n} b_{i,j} T_i \left( \frac{2}{\gamma_{j+1} - \gamma_j} x + \frac{\gamma_{j+1} + \gamma_j}{\gamma_j - \gamma_{j+1}} \right) \right| < \epsilon,
\]

where \( b_{0,j}, b_{1,j}, \ldots, b_{n,j} \) are the coefficients in the \( n \)th order Chebyshev expansion of \( f \) on the interval \([\gamma_j, \gamma_{j+1}]\). These coefficients are defined by the formula

\[
b_{i,j} = \frac{2}{n} \sum_{l=0}^{n} T_i \left( \rho_{l,n} \right) f \left( \frac{\gamma_j - \gamma_{j+1}}{2} \cos \left( \frac{\pi l}{n} \right) + \frac{\gamma_{j+1} + \gamma_j}{2} \right).
\]

During the procedure, two lists of subintervals are maintained: a list of subintervals which are to be processed and a list of output subintervals. Initially, the list of subintervals to be processed consists of \([a, b]\) and the list of output subintervals is empty. The procedure terminates when the list of subintervals to be processed is empty or when the number of subintervals in this list exceeds a present limit (we usually take this limit to be 300). In the latter case, the procedure is deemed to have failed. As long as the list of subintervals to process is nonempty and its length does not exceed the preset maximum, the algorithm proceeds by removing a subinterval \([\eta_1, \eta_2]\) from that list and performing the following operations:

1. Compute the coefficients \( b_0, \ldots, b_n \) in the \( n \)th order Chebyshev expansion of the restriction of \( f \) to the interval \([\eta_1, \eta_2]\).

2. Compute the quantity

\[
\Delta = \frac{\max \{|b_{n+1}|, |b_{n+2}|, \ldots, |b_n|\}}{\max \{|b_0|, |b_1|, \ldots, |b_n|\}}.
\]

3. If \( \Delta < \epsilon \) then the subinterval \([\eta_1, \eta_2]\) is added to the list of output subintervals.

4. If \( \Delta \geq \epsilon \), then the subintervals

\[
\left[ \frac{\eta_1 + \eta_2}{2}, \eta_2 \right]
\]

are added to the list of subintervals to be processed.

This algorithm is heuristic in the sense that there is no guarantee that \( (85) \) will be achieved, but similar adaptive discretization procedures are widely used with great success.

There is one common circumstance which leads to the failure of this procedure. The quantity \( \Delta \) is an attempt to estimate the relative accuracy with which the Chebyshev expansion of \( f \) on the interval \([\eta_1, \eta_2]\) approximates \( f \). In cases in which the condition number of the evaluation of \( f \) is larger than \( \epsilon \) on some part of \([a, b]\), the procedure will generally fail or an excessive number of subintervals will be generated. Particular care needs to be taken when \( f \) has a zero in \([a, b]\). In most cases, for \( x \) near a zero of \( f \), the condition number of evaluation of \( f(x) \) (as defined in Section 1.1) is large. In this article, we avoid such difficulties by only applying this procedure to functions which are bounded away from 0.
2. A method for the rapid numerical solution of the associated Legendre differential equation. In this section, we describe an algorithm for the numerical solution of the associated Legendre differential equation which runs in time independent of \( \nu \) and \( \mu \). It is a crucial component of the scheme of the following section for the construction of a table which allows for the rapid numerical evaluation of the associated Legendre functions.

The algorithm makes use of a solver for nonlinear second order ordinary differential equations of the form

\[
y''(t) = f(t, y(t), y'(t)) \quad \text{for all} \quad a < t < b
\]

which is described in detail in Section 4 of [4]. That solver is designed to be extremely robust, but not necessarily highly efficient. It takes as input a subroutine for evaluating the function \( f \) and its derivatives with respect to \( t, y \) and \( y' \), a positive integer \( k \), a precision \( \epsilon > 0 \) for the calculations, and either initial or terminal conditions for the desired solution \( y \). It returns a collection of subintervals

\[
(\gamma_1, \gamma_2, \ldots, \gamma_{m-1}, \gamma_m)
\]

and the values of the functions \( y, y' \) and \( y'' \) at the \((k + 1)\)-point Chebyshev grid on each of the subintervals (90). In particular, the functions \( y, y' \) and \( y'' \) are represented via piecewise \( k \)th order Chebyshev expansions. Given this data, the value of any one of these functions at any point on the interval \((a, b)\) can be computed using Chebyshev interpolation (see, for instance, [25] for a thorough discussion of such techniques). The collection of subintervals is determined adaptively in the course of solving (89) using an approach which attempts to achieve relative accuracy in the expansions of \( y, y' \) and \( y'' \) on the order of the specified precision \( \epsilon \). The algorithm is heuristic and offers no accuracy guarantees, but similar approaches are commonly used with great success. There is one situation in which this solver is likely to fail. When the condition number of evaluation of the solution of (90) is large, it is not possible to represent it with high relative accuracy using Chebyshev expansions. In this event, the solver tends to produce an excessive number of subintervals or fail altogether. Since the condition number of evaluation of a function is generally large near one of its roots, we only apply this solver in cases in which the solution is bounded away from 0.

Our algorithm for the numerical solution of (7) takes as input real numbers \( \nu \) and \( \mu \) such that \( 0 \leq \mu \leq \nu \), a desired precision \( \epsilon > 0 \), and a positive integer \( k \) specifying the order of the Chebyshev expansions to use. It proceeds in three stages.

*Stage one: computation of the nonoscillatory phase function \( \alpha_{\nu,\mu} \)*

In this stage, we calculate the values of the nonoscillatory phase function (81) on the interval

\[
\left( t_{\nu,\mu}^*, \frac{\pi}{2} \right)
\]

where

\[
t_{\nu,\mu}^* = \arcsin \left( \sqrt{\mu^2 - \frac{1}{4}} \frac{\nu + \frac{1}{2}}{\nu + \frac{1}{2}} \right)
\]

is the turning point of (7) if \( \mu \geq 1 \) and

\[
t_{\nu,\mu}^* = \frac{1}{\nu^2}
\]

if \( 0 \leq \mu < 1 \). The rationale for using (93) as the left endpoint for the interval on which the phase function is calculated when \( 0 < \mu < 1 \) is to avoid a discontinuity in \( t_{\nu,\mu} \) when \( \mu \) crosses the threshold \( \mu = \frac{1}{2} \).

We first construct \( \alpha'_{\nu,\mu} \) by solving a terminal value problem for Kummer’s equation (67) using the solver of Section 4 of [4]. The values of \( \alpha'_{\nu,\mu} \) and \( \alpha''_{\nu,\mu} \) at the point \( \frac{\pi}{2} \) are obtained using (82) and (83). The required
precision for these computations is taken to be $\epsilon$. Next, $\alpha_{\nu,\mu}$ is constructed through Formula (81). Since $\alpha_{\nu,\mu}$ is represented via its values at the $(k + 1)$-point Chebyshev nodes on a collection of intervals, it is easy to evaluate the required integral via spectral integration.

Upon the completion of this stage, the values of $\alpha_{\nu,\mu}$ and its first two derivatives are known at the nodes of the $(k + 1)$-point Chebyshev points on each interval in a collection of subintervals of (91). Using standard Chebyshev interpolation methods, the values of these functions can be calculated in a stable fashion anywhere on the interval (91).

**Stage two: computation of $\log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right) + \nu$**

In the event that $\mu \geq 1$, we calculate the function $\log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right) + \nu$ on the interval

\[ (\frac{t_{\nu,\mu}^*}{100}, t_{\nu,\mu}^*) \]

by solving a terminal boundary value problem for Riccati’s equation (64) using the solver described in Section 4 of [4]. In fact, we solve the terminal boundary value problem on the slightly larger interval

\[ (\frac{t_{\nu,\mu}^*}{100}, t_0) \]

where $t_0$ is the solution of the nonlinear equation

\[ \alpha_{\nu,\mu}(t_0) = \frac{\pi}{4} + 2\pi. \]

The functions $\alpha_{\nu,\mu}$ and its derivative having been calculated in the preceding stage, there is no difficulty in using Newton’s method to solve (96). From (73) and (96), we see that

\[ \tilde{Q}_{\nu}^{-\mu}(t_0) = \sqrt{\frac{(\nu + \frac{1}{2})}{\pi \alpha_{\nu,\mu}'(t)}} \]

and

\[ \frac{d\tilde{Q}_{\nu}^{-\mu}}{dt}(t_0) = \sqrt{\frac{\nu + \frac{1}{2}}{\pi}} \left( \frac{\alpha'(t_0)}{2 (\alpha'(t_0))^2} \right) \]

The rationale for introducing $t_0$ is to ensure that the terminal value of $\tilde{Q}_{\nu}^{-\mu}$ and its derivative used in the solution of Riccati’s equation are computed accurately. The condition number of evaluation of the function $\tilde{Q}_{\nu}^{-\mu}$ is large when the parameters $\nu$ and $\mu$ are of large magnitude, with the consequence that its numerical evaluation will generally result in a loss of precision in this event. In the case of (73), the evaluation of a trigonometric functions at a large argument is the specific mechanism by which this loss of precision takes place. By evaluating $\tilde{Q}_{\nu}^{-\mu}$ at a point $t_0$ at which the value of the phase function is known, however, we avoid this loss of precision entirely. This can be seen from (97) and (98). They involve only the evaluation of $\alpha_{\nu,\mu}'$ and $\alpha_{\nu,\mu}''$, the condition number of evaluation of which is small independent of $\nu$ and $\mu$.

We construct $\log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right) + \nu$ in lieu of $\log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right)$ because the former is bounded away from 0 on the interval (94) while the latter is not. Upon the completion of this stage, the values of $\log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right) + \nu$ at the nodes of the $(k + 1)$-point Chebyshev points on each of a collection of subintervals which cover (91) are known. Using standard Chebyshev interpolation methods, the values of this function can be calculated in a stable fashion anywhere on the interval (94).
Stage three: computation of $\log \left( \tilde{P}_\nu^{-\mu}(t) \right) - \nu$

Assuming that $\mu \geq 1$, we now compute $\log \left( \tilde{P}_\nu^{-\mu}(t) \right) - \nu$ on the interval $(94)$. Proceeding here as we did in the calculation of $\log \left( \tilde{Q}_\nu^{-\mu}(t) \right) + \nu$ would be problematic. Unlike $\log \left( \tilde{Q}_\nu^{-\mu}(t) \right) + \nu$, which increases rapidly as $t$ goes to 0 from the right, $\log \left( \tilde{P}_\nu^{-\mu}(t) \right) - \nu$ converges to 0 rapidly as $t$ goes to 0 from the right. Consequently, it is recessive when solving Riccati’s equation in the backward direction and attempts to approximate it numerically by solving a terminal value problem for Riccati’s equation lead to excessively large errors.

Instead, we solve an initial value problem for Riccati’s equation on the interval $(94)$ in order to calculate $\log \left( \tilde{P}_\nu^{-\mu}(t) \right) - \nu$. This is numerically viable since it is a dominant solution of Riccati’s equation when solving in the forward direction. When $\nu \leq 10,000$, we use a truncation of the series expansion (50) in order to generate the necessary initial values. For $\nu > 10,000$, we calculate initial values via (62) instead since (50) can lead to numerical roundoff errors when $\nu$ is large.

As before, the rationale for computing $\log \left( \tilde{P}_\nu^{-\mu}(t) \right) - \nu$ in lieu of $\log \left( \tilde{P}_\nu^{-\mu}(t) \right)$ is that the former is bounded away from 0 on the interval $(94)$ while the latter is not. Upon the completion of this stage, the values of $\log \left( \tilde{P}_\nu^{-\mu}(t) \right) - \nu$ at the nodes of the $(k+1)$-point Chebyshev points on each interval in a collection of subintervals of $(91)$ are known. Using standard Chebyshev interpolation methods, the values of this function can be calculated in a stable fashion anywhere on the interval $(94)$.

Remark 2. Although the algorithm described in this section is highly specialized to the case of the associated Legendre differential equation, it can, in fact, be modified so as to apply to a large class of second order differential equations of the form

\begin{equation}
y''(t) + q(t)y(t) = 0 \quad \text{for all } a < t < b.
\end{equation}

Suppose, for instance, that $q$ is smooth on $[a, b]$, has a zero at $t_{00} \in (a, b)$, is negative on $(a, t_{00})$ and is positive on $(t_{00}, b)$. The procedure of the first stage for constructing a nonoscillatory phase function on $(t_{00}, b)$ relies on an asymptotic expansion which allows for the evaluation of a nonoscillatory phase function at the point $b$. In the absence of such an approximation, the algorithm of [3] can be used instead. That algorithm also proceeds by solving Kummer’s equation, but it incorporates a mechanism for numerically calculating the appropriate initial values of a nonoscillatory phase function and its derivatives.

The procedure of the second stage does not rely on any asymptotic or series expansions of associated Legendre functions, only on the values of the phase function computed in the first phase. Consequently, it does not need to be modified in order to obtain a solution of Riccati’s equation which is increasing as $t \rightarrow 0^+$.

In the third stage, a series or asymptotic expansion is used to compute the values of $\tilde{P}_\nu^{-\mu}$ and its derivative at a point near 0. In the event that such an approximation is not available, a solution of Riccati’s equation which is increasing as $t \rightarrow t_{00}$ from the left can be obtained by solving an initial value problem with arbitrary initial conditions and then scaling the result in order to make it consistent with the desired solution of (99). This procedure is analogous to that used in order to obtain a recessive solution of a linear recurrence relation by running the recurrence relation backwards (see, for instance, Section 3.6 of [9]).

3. The numerical construction of the precomputed table. In this section, we describe the procedure used to construct the precomputed table which allows for the rapid numerical evaluation of the associated Legendre functions $P^{-\mu}_{\nu}$ and $\tilde{Q}^{-\mu}_{\nu}$ for a large range of $\nu$, $\mu$ and $t$. This table stores the coefficients in the compressed piecewise trivariate Chebyshev expansions of eights pair of functions.

A first pair of functions $A_1$, $B_1$ allows for the evaluation of the phase function $\alpha_{\nu,\mu}$ and its derivative on the
O \log A fifth set of functions
C We divide the range of the parameter
A and a fourth pair
A of the phase function
O \alpha
O \subset \mathbb{R}
on the subset
O N on the subset
O of the nonoscillatory region
E5-2697 processor cores running at 2.6 GHz. The construction of the table took approximately 24 hours on
A version 4.8.4 of the GNU Fortran compiler. It was executed on a computer equipped with 28 Intel Xeon
E5-2697 processor cores running at 2.6 GHz. The construction of the table took approximately 24 hours on
this machine.

subset
\begin{equation}
\mathcal{O}_1 = \left\{ (\nu, \mu, t) : 10 \leq \nu \leq 1,000,000, \ 1 \leq \mu \leq \nu \ \text{and} \ t^*_{\nu,\mu} \leq t \leq \frac{\pi}{2} \right\}
\end{equation}
of the oscillatory region \( \mathcal{O} \). Here, \( t^*_{\nu,\mu} \) is as in (92) and (93). A second pair \( A_2, B_2 \) allows for the evaluation
of the phase function \( \alpha_{\nu,\mu} \) and its derivative on the subset
\begin{equation}
\mathcal{O}_2 = \left\{ (\nu, \mu, t) : 10 \leq \nu \leq 1,000,000, \ 0 \leq \mu < 1 \ \text{and} \ t^*_{\nu,\mu} \leq t \leq \frac{\pi}{2} \right\}
\end{equation}
of the oscillatory region \( \mathcal{O} \). The functions \( A_3 \) and \( B_3 \) allow for the evaluation of \( \alpha_{\nu,\mu} \) and \( \alpha'_{\nu,\mu} \) on
\begin{equation}
\mathcal{O}_3 = \left\{ (\nu, \mu, t) : 2 \leq \nu \leq 10, \ 1 \leq \mu \leq \nu \ \text{and} \ t^*_{\nu,\mu} \leq t \leq \frac{\pi}{2} \right\},
\end{equation}
and a fourth pair \( A_4, B_4 \) allows for the evaluation of the phase function \( \alpha_{\nu,\mu} \) and its derivative on
\begin{equation}
\mathcal{O}_4 = \left\{ (\nu, \mu, t) : 2 \leq \nu \leq 10, \ 0 \leq \mu < 1 \ \text{and} \ t^*_{\nu,\mu} \leq t \leq \frac{\pi}{2} \right\},
\end{equation}
We divide the range of the parameter \( \nu \) because it is more efficient to represent \( \alpha_{\nu,\mu} \) via polynomial expansions in \( \frac{1}{\nu} \) when
\( \nu \) is large, and via expansions in \( \nu \) when \( \nu \) is small.
A fifth set of functions \( C_1 \) and \( D_1 \) allows for the evaluation of the functions
\begin{equation}
\log \left( \tilde{P}_{\nu}^{-\mu}(t) \right) - \nu \ \text{and} \ \log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right) + \nu
\end{equation}
on the subset
\begin{equation}
\mathcal{N}_1 = \left\{ (\nu, \mu, t) : 10 \leq \nu \leq 1,000,000, \ 1 < \mu \leq \nu \ \text{and} \ 0 \leq t < t^*_{\nu,\mu} \right\}
\end{equation}
of the nonoscillatory region \( \mathcal{N} \). A sixth pair of functions \( C_2 \) and \( D_2 \) allows for the evaluation of the functions
\begin{equation}
\text{(104)}
\end{equation}
on
\begin{equation}
\mathcal{N}_2 = \left\{ (\nu, \mu, t) : 10 \leq \nu \leq 1,000,000, \ 0 \leq \mu < 1 \ \text{and} \ 0 \leq t < t^*_{\nu,\mu} \right\}.
\end{equation}
The seventh pair of functions \( C_3, D_3 \) allows for the evaluation of (104) on
\begin{equation}
\mathcal{N}_3 = \left\{ (\nu, \mu, t) : 2 \leq \nu \leq 10, \ 1 \leq \mu \leq \nu \ \text{and} \ 0 \leq t < t^*_{\nu,\mu} \right\}.
\end{equation}
The eighth and final pair of functions \( C_4, D_4 \) allows for the evaluation of the functions (104) on
\begin{equation}
\mathcal{N}_4 = \left\{ (\nu, \mu, t) : 2 \leq \nu \leq 10, \ 0 \leq \mu < 1 \ \text{and} \ 0 \leq t < t^*_{\nu,\mu} \right\}.
\end{equation}
We construct expansions of the functions (104) rather than expansions of \( \log \left( \tilde{P}_{\nu}^{-\mu}(t) \right) \) and \( \log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right) \)
because the former are bounded away from 0 on the sets in which we consider them while the latter are not.
This ensures that their condition number of evaluation is not large because of the presence of roots.
These computations were conducted in IEEE quadruple precision arithmetic in order to ensure high accuracy.
The resulting table, which consists of the coefficients in the expansions of the functions \( A_1, \ldots, A_4, B_1, \ldots, B_4, C_1, \ldots, C_4, D_1, \ldots, D_4 \) is roughly 138 MB in size. The precomputed table allows for the evaluation
of \( \alpha_{\nu,\mu}, \alpha'_{\nu,\mu}, \log \left( \tilde{P}_{\nu}^{-\mu}(t) \right) - \nu \) and \( \log \left( \tilde{Q}_{\nu}^{-\mu}(t) \right) + \nu \) with roughly double precision accuracy (see the experiments of Section 5). The code was written in Fortran with OpenMP extensions and compiled with
version 4.8.4 of the GNU Fortran compiler. It was executed on a computer equipped with 28 Intel Xeon
E5-2697 processor cores running at 2.6 GHz. The construction of the table took approximately 24 hours on
this machine.
Here, we describe only the construction of the functions $A_1$, $B_1$, $C_1$ and $D_1$. The construction of the others is extremely similar. The procedure proceeded in four stages as follows:

**Stage one: construction of the phase functions and logarithms**

We began this stage of the procedure by constructing a partition

$$\xi_1 < \xi_2 < \ldots < \xi_{11}$$

which divides the interval

$$\left[ \frac{1}{1,000,000}, 1 \right]$$

over which $\frac{1}{\nu}$ is allowed to vary into 10 subintervals. The precise locations of the nodes $\xi_j$ are not critically important; reasonable choices were arrived at quickly through trial and error. Next, we constructed a partition

$$0 = \tau_1 < \tau_2 < \ldots < \tau_{18} = 1$$

which divides the interval $[0, 1]$ into 17 subintervals. Again, the precise distribution of the nodes $\tau_j$ is not critical and reasonable choices were arrived at quickly through trial and error.

For each $i = 1, \ldots, 10$ and $j = 1, \ldots, 17$, we processed the tensor product of intervals $[\xi_i, \xi_{i+1}] \times [\tau_j, \tau_{j+1}]$ as follows. We let $x_1, \ldots, x_{31}$ be the nodes of the 31-point Chebyshev grid on the interval $[\xi_i, \xi_{i+1}]$ and $y_1, \ldots, y_{31}$ the nodes of the 31-point Chebyshev grid on the interval $[\tau_j, \tau_{j+1}]$. For each pair $x_k, y_l$, the algorithm of Section 2 was used to calculate $\alpha_{\nu,\mu}$, $\alpha'_{\nu,\mu}$, $\log(\tilde{P}_{\nu}^{-\mu}(t)) - \nu$, and $\log(\tilde{Q}_{\nu}^{-\mu}(t)) + \nu$ with $\nu$ and $\mu$ taken to be

$$\nu = \frac{1}{x_k} \quad \text{and} \quad \mu = 1 + (\nu - 1)y_l.$$

The precision for the computations was $\epsilon = 10^{-17}$. The functions $\alpha_{\nu,\mu}$ and $\alpha'_{\nu,\mu}$ were represented as 30th order piecewise Chebyshev expansions on some adaptively determined collection of subintervals of $[t_{\nu,\mu}^*, \frac{\pi}{2}]$, while the functions $\log(\tilde{P}_{\nu}^{-\mu}(t)) - \nu$ and $\log(\tilde{Q}_{\nu}^{-\mu}(t)) + \nu$ were represented as 30th order piecewise Chebyshev expansions on some adaptively determined collection of subintervals of $[0, t_{\nu,\mu}^*]$. Using this data, the nonoscillatory phase function and its derivative can be evaluated for any triple $(\nu, \mu, t)$ in the region $O_1$ via Chebyshev interpolation. Likewise, the logarithms of the associated Legendre functions can be evaluated at any point in $N_1$.

In this stage of the procedure, the differential equation (7) was solved via the algorithm of Section 2 for 163,370 different pairs of the parameters $(\nu, \mu)$, many of which were large in magnitude. Obviously, this was only possible because our solver runs in time independent of $\nu$ and $\mu$.

**Stage two: formation of unified discretizations**

For each pair of points $\xi$ and $\tau$ such that $\xi$ is one of the Chebyshev nodes in one of the subintervals defined by the partition (109) and $\tau$ is one of the Chebyshev nodes in one of the subintervals defined by the partition (111), we used the procedure of Section 1.8 to adaptively form discretization of the functions

$$f_{\xi,\tau}(u) = \alpha'_{\nu,\mu}(t(u))$$

and

$$g_{\xi,\tau}(u) = \alpha_{\nu,\mu}(t(u)),$$
\begin{align}
\nu &= \frac{1}{\xi}, \quad \mu = 1 + (\nu - 1)\tau \quad \text{and} \quad t(u) = t_{\nu,\mu}^* + \left(\frac{\pi}{2} - t_{\nu,\mu}^*\right)u.
\end{align}

The functions $\alpha_{\nu,\mu}$ and $\alpha'_{\nu,\mu}$ are evaluated via Chebyshev interpolation using the data constructed in the first stage of these calculations. We requested $\epsilon = 10^{-17}$ accuracy and took the parameter $n$ to be 30. For each $\xi$ and $\tau$ considered, this results in a collection of subintervals of $[0,1]$ on which $f_{\xi,\tau}$ is represented with relative accuracy roughly $\epsilon$ via a 30th order Chebyshev expansion and another collection of subintervals of $[0,1]$ on which $g_{\xi,\tau}$ is represented with relative accuracy roughly $\epsilon$ via a 30th order Chebyshev expansion. We then formed a unified discretization
\begin{align}
[a_0, a_1], \ [a_1, a_2], \ [a_2, a_3], \ \ldots[a_{20}, a_{21}]
\end{align}
of $[0,1]$ by merging these discretizations; that is, by ensuring that each subinterval in the discretization of one of the functions $f_{\xi,\tau}$ or $g_{\xi,\tau}$ is the union of some set of subintervals of (116).

A unified discretization
\begin{align}
[b_0, b_1], \ [b_1, b_2], \ [b_2, b_3], \ \ldots[b_{17}, b_{18}]
\end{align}
for the functions
\begin{align}
\tilde{f}_{\xi,\tau}(u) = \log \left(\tilde{P}_{\nu}^{-\mu}(t)\right) - \nu
\end{align}
and
\begin{align}
\tilde{g}_{\xi,\tau}(u) = \log \left(\tilde{Q}_{\nu}^{-\mu}(t)\right) + \nu
\end{align}
with $\nu, \mu$ and $t$ related to $\xi, \tau$ and $u$ via (115) was formed in the same fashion.

\textit{Stage three: Construction of the functions $A_1$ and $B_1$}

The function $A_1$ is defined via the formula
\begin{align}
A_1(\xi, \tau, u) &= \frac{1}{\nu} \alpha_{\nu,\mu}(t),
\end{align}
where
\begin{align}
\nu &= \frac{1}{\xi}, \quad \mu = 1 + (\nu - 1)\tau \quad \text{and} \quad t = t_{\nu,\mu}^* + \left(\frac{\pi}{2} - t_{\nu,\mu}^*\right)u.
\end{align}
Likewise, $B_1$ is defined via
\begin{align}
B_1(\xi, \tau, u) &= \frac{1}{\nu} \alpha'_{\nu,\mu}(t),
\end{align}
with $\nu, \mu$ and $t$ given by (121). In this way, we ensure that $A_1$ and $B_1$ are defined on the rectangular prism
\begin{align}
\left[\frac{1}{1,000,000}, \frac{1}{10}\right] \times [0,1] \times [0,1],
\end{align}
and hence suitable for representation via a collection of piecewise trivariate Chebyshev expansions.

For each $i = 1, \ldots, 10$, $j = 1, \ldots, 17$ and $k = 1, \ldots, 21$, we formed the 30th order compressed trivariate Chebyshev expansions (as defined in Section 1.3) for the functions $A_1$ and $B_1$ on the rectangular prism
\begin{align}
[\xi_i, \xi_{i+1}] \times [\tau_j, \tau_{j+1}] \times [a_k, a_{k+1}].
\end{align}
There are 3,570 such rectangular prisms. Since an uncompressed 30th order trivariate Chebyshev expansion has 29,791 coefficients, a total of 212,707,740 coefficients would be required to store the uncompressed Chebyshev expansions of the functions $A_1$ and $B_1$. If each coefficient were stored as an IEEE double precision number, roughly 1.5 GB of memory would be required to store these expansions. Fortunately, the compressed 30th order Chebyshev expansions were far more efficient. The compressed Chebyshev expansions for $A_1$ and $B_1$ had only 7,839,620 coefficients.

Stage four: construction of the functions $C_1$ and $D_1$

The function $C_1$ is defined via

$$C_1(\xi, \tau, u) = \log \left( \tilde{P}_\nu^\mu(t) \right) - \nu,$$

where

$$\nu = \frac{1}{\xi}, \quad \mu = 1 + (\nu - 1)\tau \quad \text{and} \quad t = \frac{t^*_\nu,\mu}{100} + \left( \frac{t^*_\nu,\mu - t^*_\nu,\mu}{100} \right) u.$$

Finally, $D_1$ is defined via

$$D_1(\xi, \tau, u) = \log \left( \tilde{Q}_\nu^\mu(t) \right) + \nu,$$

with $\nu$, $\mu$ and $t$ as in (126). Obviously, $C_1$ and $D_1$ are also given on the rectangular prism (123).

For each $i = 1, \ldots, 10$, $j = 1, \ldots, 17$ and $k = 1, \ldots, 17$, we formed the 30th order compressed trivariate Chebyshev expansions (as defined in Section 1.3) for the functions $C_1$ and $D_1$ on the rectangular prism

$$[\xi_i, \xi_{i+1}] \times [\tau_j, \tau_{j+1}] \times [b_k, b_{k+1}].$$

There are 2,890 such rectangular prisms and a total of 172,191,980 coefficients would be required to store the uncompressed Chebyshev expansions of the functions $C_1$ and $D_1$. If each coefficient were stored as an IEEE double precision number, roughly 1.3 GB of memory would be required to store these expansions. The compressed 30th order Chebyshev expansions were far more efficient. They required only 4,441,063 coefficients to store $C_1$ and $D_1$.

4. An algorithm for the rapid numerical evaluation of the associated Legendre functions.

In this section, we describe the operation of our code for evaluating the associated Legendre functions $\tilde{P}_\nu^\mu(t)$ and $\tilde{Q}_\nu^\mu(t)$ when

$$0 \leq \nu \leq 1,000,000, \quad 0 \leq \mu \leq \nu \quad \text{and} \quad 0 < t \leq \frac{\pi}{2}.$$

The code was written in Fortran and its interface to the user consists of two subroutines, one called alegendre_eval_init and the other alegendre_eval. The alegendre_eval_init routine reads the precomputed table constructed via the procedure of Section 3 from the disk into memory. The precomputed table used in the experiments described in this paper is approximately 138 megabytes in size. Once the precomputed table has been loaded, the alegendre_eval can be called. It takes as input a triple $(\nu, \mu, t)$ satisfying the conditions (129). When $(\nu, \mu, t)$ is in the oscillatory region $O$, it returns the values of $\alpha_{\nu,\mu}(t)$ and $\beta_{\nu,\mu}(t)$ as well as those of $\tilde{P}_\nu^\mu$ and $\tilde{Q}_\nu^\mu$. When $(\nu, \mu, t)$ is in the nonoscillatory region $N$, it returns the values of $\log(\tilde{P}_\nu^\mu(t))$ and $\log(\tilde{Q}_\nu^\mu(t))$ as well as those of $\tilde{P}_\nu^\mu(t)$ and $\tilde{Q}_\nu^\mu(t)$. Of course, when $t$ is close to 0, the latter values might not be representable via the IEEE double format arithmetic. In this event, 0 is returned for $\tilde{P}_\nu^\mu(t)$ and $\infty$ for $\tilde{Q}_\nu^\mu(t)$.

The alegendre_eval code is available from the GitHub repository at address
It uses several different methods to evaluate the associated Legendre functions and the associated auxiliary functions, depending on the values of $\nu$, $\mu$ and $t$. The following description of the operation of the \texttt{alegendre_eval} code is organized by listing each such method.

\textit{Method one: series expansions for $\tilde{P}_{\nu}^{-\mu}(t)$ and $\tilde{Q}_{\nu}^{-\mu}(t)$}

This method is used when $\nu < 2$ and $(\nu, \mu, t)$ is in the oscillatory region $\mathcal{O}$. It consists of evaluating $\tilde{P}_{\nu}^{-\mu}(t)$ via a truncation of the series expansion (49) and evaluating $\tilde{Q}_{\nu}^{-\mu}(t)$ via formula (52). As discussed in Section 1.4, when $\mu$ is close to or coincides with an integer, Chebyshev interpolation in the variable $\mu$ is used to avoid roundoff error in the evaluation of (52). The value of $\alpha'_{\nu, \mu}(t)$ is calculated via (74) and $\alpha_{\nu, \mu}$ is computed using the formula

$$\alpha_{\nu, \mu} = \text{Arg} \left( \tilde{P}_{\nu}^{-\mu}(t) + i\tilde{Q}_{\nu}^{-\mu}(t) \right) + 2\pi,$$

where $\text{Arg}(z)$ denotes the principal value of the argument of the complex number $z$. The limitation on the range of parameters for which this method is used ensures that the principal branch of the argument function is the correct one.

\textit{Method two: series expansions for $\log \tilde{P}_{\nu}^{-\mu}(t)$ and $\log \tilde{Q}_{\nu}^{-\mu}(t)$}

This method is used when $\nu < 10$ and $t$ is in the nonoscillatory regime, and when $10 \leq \nu < 10,000$ and

$$0 < t < \frac{t^*_{\nu, \mu}}{100}.$$  

It consists of evaluating $\log \tilde{P}_{\nu}^{-\mu}(t)$ via a truncation of (50) and $\log \tilde{Q}_{\nu}^{-\mu}(t)$ via (53). When $\mu$ coincides with or is close to an integer, Chebyshev interpolation in the parameter $\mu$ is used in the evaluation of (53). The values of $\tilde{P}_{\nu}^{-\mu}(t)$ and $\tilde{Q}_{\nu}^{-\mu}(t)$ are computed from their logarithms in the obvious fashion.

\textit{Method three: Macdonald’s asymptotic expansions for $\log \tilde{P}_{\nu}^{-\mu}(t)$ and $\log \tilde{Q}_{\nu}^{-\mu}(t)$}

This method is used when $\nu \geq 10,000$ and

$$0 < t < \frac{t^*_{\nu, \mu}}{100}.$$  

It consists of evaluating $\log \tilde{P}_{\nu}^{-\mu}(t)$ and $\log \tilde{Q}_{\nu}^{-\mu}(t)$ via Macdonald’s asymptotic expansions (see Section 1.5). The values of $\tilde{P}_{\nu}^{-\mu}(t)$ and $\tilde{Q}_{\nu}^{-\mu}(t)$ are computed from their logarithms in the obvious fashion.

\textit{Method four: precomputed expansions}

In all other cases, the precomputed expansions of the functions $A_1, \ldots, A_4$, $B_1, \ldots, B_4$, $C_1, \ldots, C_4$, whose construction is described in Section 3, are used to evaluate $\tilde{P}_{\nu}^{-\mu}(t)$, $\tilde{Q}_{\nu}^{-\mu}(t)$ and the appropriate auxiliary functions. Here, we describe the use of the functions $A_4$ and $B_1$ to evaluate $\alpha_{\nu, \mu}$ and $\alpha'_{\nu, \mu}$ in the event that $(\nu, \mu, t)$ is in the set $\mathcal{O}_1$. The other cases are extremely similar.

First, we let $\xi = \frac{1}{\nu}$, $\tau = \frac{\nu - 1}{\nu}$ and

$$u = \frac{t - t^*_{\nu, \mu}}{\frac{\tau}{2} - \frac{t^*_{\nu, \mu}}{23}}.$$

http://github.com/JamesCBremerJr/ALegendreEval.
That is, we compute the values of $\xi$, $\tau$ and $u$ defined by the mapping (121) given $\nu$, $\mu$ and $t$. Next, we find the smallest positive integer $i$ such that $\xi_i \leq \xi \leq \xi_{i+1}$, where $\xi_1, \ldots, \xi_{11}$ are the nodes of the partition (109), the smallest positive integer $j$ such that $\tau_j \leq \tau \leq \tau_{j+1}$, where $\tau_1, \ldots, \tau_{17}$ are the nodes of the partition (111), and the smallest positive integer $k$ such that $a_k \leq u \leq a_{k+1}$, where $a_1, a_2, \ldots, a_{21}$ are the nodes of the partition (116).

Having discovered that $(\xi,\tau,u)$ is in the set $[\xi_i,\xi_{i+1}] \times [\tau_j,\tau_{j+1}] \times [a_k,a_{k+1}]$, we evaluate the compressed trivariate Chebyshev expansions representing $A_1$ and $C_1$ on this rectangular prism. We scale the results by $\nu$ to obtain the values of $\alpha_{\nu,\mu}$ and $\alpha'_{\nu,\mu}$. The values of $\tilde{P}_{\nu,\mu}$ and $\tilde{Q}_{\nu,\mu}$ are then calculated via (72) and (73).

5. Numerical experiments. In this section, we present the results of numerical experiments which were conducted to assess the performance of the alegendre_eval routine. The task of constructing reference values with which to compare our results was quite challenging. All existing packages of which the author is aware were prohibitively slow when evaluating associated Legendre functions with large noninteger parameters, and existing asymptotic expansions are either not viable (e.g., the Liouville-Green expansions (20) and (21) whose coefficients cannot be readily computed) or only applicable in the case of an extremely limited range of parameters (e.g., the trigonometric expansions (15) and (16) which are catastrophically unstable even for relatively small values of $\mu$). As a result, we were quite limited in the extent to which we could verify our approach in the case of large noninteger parameters.

In the case of integer values of the parameters, the well-known three term recurrence relations can be used to evaluate the associated Legendre function accurately, provided extended precision arithmetic is used to perform the computations. Consequently, we were able to test our code quite thoroughly in the case of integer parameters. We note that the time required to evaluate the associated Legendre functions using the recurrence relations grows with the magnitudes of the parameters, making such an approach uncompetitive with the algorithm of this paper in many cases.

These experiments were carried out on a laptop computer equipped with an Intel Core i7-5600U processor running at 2.6 GHz and 16 GB of memory. Our code was compiled with the GNU Fortran compiler version 5.2.1 using the “-Ofast” compiler optimization flag.

5.1. The accuracy with which $\alpha'_{\nu,\mu}$ is evaluated for small noninteger values of $\nu$. In these experiments, we measured the accuracy with which alegendre_eval calculates $\alpha'_{\nu,\mu}$ in the oscillatory region. Reference values were calculated using version 11 of Wolfram’s Mathematica package. The cost of the reference calculations was prohibitively expensive for large $\nu$, with the consequence that we only considered values of $\nu$ between 0 and 1,000.

In each experiment, we choose 10 pairs $(\nu,\mu)$ by first picking a random value of $\nu$ in a given range, and then choosing a random value of $\mu$ in the interval $(0,\nu)$. For each pair chosen in this fashion, we evaluated $\alpha'_{\nu,\mu}$ at 100 equispaced points either in interval

$$\left( \arcsin \left( \frac{\sqrt{\mu^2 - \frac{1}{4} \nu}}{\nu + \frac{1}{2}} \right) , \frac{\pi}{2} \right)$$

or in the interval

$$\left( \frac{1}{1000} , \frac{\pi}{2} \right) ,$$

depending on whether $\mu > \frac{1}{2}$ or not. Table 1 reports the results. There, each row corresponds to one experiment and gives the largest relative error observed in $\alpha'_{\nu,\mu}$ as well as the average time taken by the alegendre_eval routine.
5.2. The accuracy with which logarithms are evaluated in the case of small noninteger values of $\nu$. In these experiments, we measured the accuracy with which `alegendre_eval` calculates the functions

\[
\log\left(\tilde{P}^{-\mu}(t)\right) - \nu \quad \text{and} \quad \log\left(\tilde{Q}^{-\mu}(t)\right) + \nu
\]

in the nonoscillatory region. High accuracy reference values for these experiments were calculated using version 11 of Wolfram's Mathematica package. Again the high cost of the reference calculations led us to only consider values of $\nu$ between 0 and 1,000.

In each experiment, we choose 10 pairs $(\nu, \mu)$ by first picking a random value of $\nu$ in a given range, and then choosing a random value of $\mu$ in the interval $\left(\frac{1}{2}, \nu\right]$. For each pair chosen in this fashion, we evaluated the functions (136) at 100 equispaced points in the interval

\[
\left(0, \arcsin \left(\frac{\sqrt{\mu^2 - \frac{1}{4}}}{\nu + \frac{1}{2}}\right)\right).
\]

Table 2 reports the results. There, each row corresponds to one experiment and gives the largest relative error observed in each of the functions (136), as well as the average time taken by the `alegendre_eval` routine.

| Range of $\nu$ | Maximum relative error in $\log(\tilde{P}^{-\mu}(t)) - \nu$ | Maximum relative error in $\log(\tilde{Q}^{-\mu}(t)) + \nu$ | Average evaluation time (in seconds) |
|----------------|-------------------------------------------------------------|-------------------------------------------------------------|-------------------------------------|
| 0.5 - 1        | $3.36 \times 10^{-16}$                                      | $2.58 \times 10^{-15}$                                      | $1.22 \times 10^{-06}$              |
| 1 - 5          | $3.21 \times 10^{-16}$                                      | $9.28 \times 10^{-16}$                                      | $1.32 \times 10^{-06}$              |
| 5 - 10         | $8.85 \times 10^{-16}$                                      | $9.14 \times 10^{-15}$                                      | $1.59 \times 10^{-06}$              |
| 10 - 50        | $4.39 \times 10^{-15}$                                      | $4.43 \times 10^{-15}$                                      | $3.34 \times 10^{-06}$              |
| 50 - 100       | $2.58 \times 10^{-15}$                                      | $3.49 \times 10^{-15}$                                      | $1.61 \times 10^{-06}$              |
| 100 - 500      | $4.21 \times 10^{-15}$                                      | $4.47 \times 10^{-15}$                                      | $2.56 \times 10^{-06}$              |
| 500 - 1,000    | $2.54 \times 10^{-15}$                                      | $3.24 \times 10^{-15}$                                      | $1.70 \times 10^{-06}$              |

5.3. The accuracy with which $\alpha'_{\nu, \mu}$ is evaluated in the case of large $\nu$ and small $\mu$. In these experiments, we measured the accuracy with which `alegendre_eval` calculates $\alpha'_{\nu, \mu}$ in the oscillatory region
by comparison with values obtained using the trigonometric expansions (15) and (16). Since these expansions are numerically unstable, and catastrophically so when \( \mu \) is large, we considered only pairs of the parameters \((\nu, \mu)\) with \( \mu \) small in magnitude. Even so, 1,000 digit arithmetic was required in order to obtain accurate reference values for these experiments.

In each experiment, we choose 10 pairs \((\nu, \mu)\) by first picking a random value of \( \nu \) in a given range, and then choosing a random value of \( \mu \) in the interval \( \left( \frac{1}{2}, \frac{\nu}{100} \right) \). For each pair, the function \( \alpha'_{\nu, \mu} \) was evaluated at 100 equispaced points in the interval

\[
\max \left( \arcsin \left( \frac{\sqrt{\mu^2 - \frac{1}{4}}}{\nu + \frac{1}{2}} \right), \frac{\pi}{6}, \frac{\pi}{2} \right).
\]

(138)

We note that the trigonometric expansions used here only converge in the interval \( \left( \frac{\pi}{6}, \frac{5\pi}{6} \right) \), hence the choice of the interval (138).

| Range of \( \nu \) | Maximum relative error in \( \alpha'_{\nu, \mu} \) | Average evaluation time (in seconds) |
|---------------------|---------------------------------|----------------------------------|
| 1,000 - 5,000       | 1.91\times10^{-15}              | 1.53\times10^{-06}               |
| 5,000 - 10,000      | 1.41\times10^{-15}              | 1.00\times10^{-06}               |
| 10,000 - 50,000     | 1.05\times10^{-15}              | 1.06\times10^{-06}               |
| 50,000 - 100,000    | 8.69\times10^{-16}              | 7.72\times10^{-07}               |
| 100,000 - 500,000   | 7.30\times10^{-16}              | 6.91\times10^{-07}               |
| 500,000 - 1,000,000 | 8.15\times10^{-16}              | 5.46\times10^{-07}               |

Table 3

The results of the experiments of Section 5.3 in which the accuracy with which \texttt{alegendre_eval} calculates \( \alpha'_{\nu, \mu} \) for large \( \nu \) and small \( \mu \) is tested via comparison with the trigonometric expansions (15) and (16).

5.4. The accuracy with which \( \alpha'_{\nu, \mu} \) is evaluated in the case of integer parameters. In these experiments, the accuracy with which \( \alpha'_{\nu, \mu} \) is evaluated in the oscillatory regime was measured by comparison with reference values calculated using the well-known three term recurrence relations satisfied by the associated Legendre functions. The reference calculations were conducted in extended precision arithmetic in order to ensure accuracy.

The experiments of this section proceeded just as those described in Section 5.1, except only integer values of the parameters were considered. Table 4 displays the results.

5.5. The accuracy with which the logarithms are evaluated in the case of integer parameters. In these experiments, we measured the accuracy with which \texttt{alegendre_eval} calculates the functions (136) in the nonoscillatory regime. Reference values were calculated using the well-known three term recurrence relations satisfied by the associated Legendre functions. The reference calculations were conducted in extended precision arithmetic in order to ensure accuracy.

These experiments proceeded just as those described in Section 5.2, except only integer values of the parameters were considered. Table 5 displays the results.

5.6. The accuracy with which the associated Legendre functions are evaluated in the case of integer parameters. In these experiments, we measured the accuracy with which \texttt{alegendre_eval} calculates the functions \( \tilde{P}^{-\mu}_{\nu} \) and \( \tilde{Q}^{-\mu}_{\nu} \) in the case of integer values of the parameters. Reference values were calculated using the three term recurrence relations. As usual, extended precision arithmetic was used during the reference calculations in order to ensure their accuracy.
In each experiment, 10 pairs of the parameters \((\nu, \mu)\) were constructed by first choosing an integer value of \(\nu\) in a given range at random and then choosing an integer value of \(\mu\) in the range \((0, \nu)\) at random. For each such pair, we evaluated the function \(P_{\nu}^{-\mu}(t) + iQ_{\nu}^{-\mu}(t)\) at 100 equispaced points either in interval

\[
(\arcsin \left( \sqrt{\frac{\mu^2 - \frac{1}{4}}{\nu + \frac{1}{2}}} \right), \frac{\pi}{2})
\]

or in the interval

\[
\left( \frac{1}{1000}, \frac{\pi}{2} \right),
\]

depending on whether \(\mu > \frac{1}{2}\) or not. Table 6 reports the results. Each row corresponds to one experiment and reports the largest relative error which was observed as well as the average evaluation time. We note that we considered the function \(P_{\nu}^{-\mu}(t) + iQ_{\nu}^{-\mu}(t)\) because, unlike \(P_{\nu}^{-\mu}(t)\) and \(Q_{\nu}^{-\mu}(t)\), its absolute value is nonoscillatory and does not have roots on the interval \((0, \pi)\).

### Table 4

The results of the experiments of Section 5.4 in which the accuracy with which \texttt{alegendre_eval} calculates \(\alpha_{\nu, \mu}'\) for integers values of the parameters is tested.

| Range of \(\nu\) | Maximum relative error in \(\alpha_{\nu, \mu}'\) | Average evaluation time (in seconds) |
|------------------|---------------------------------|---------------------------------|
| 10 - 50          | \(2.35 \times 10^{-14}\)         | \(2.79 \times 10^{-06}\)         |
| 50 - 100         | \(4.71 \times 10^{-15}\)         | \(1.87 \times 10^{-06}\)         |
| 100 - 500        | \(4.96 \times 10^{-15}\)         | \(2.86 \times 10^{-06}\)         |
| 500 - 1,000      | \(2.86 \times 10^{-14}\)         | \(1.52 \times 10^{-06}\)         |
| 1,000 - 5,000    | \(8.62 \times 10^{-15}\)         | \(1.37 \times 10^{-06}\)         |
| 5,000 - 10,000   | \(5.94 \times 10^{-15}\)         | \(1.14 \times 10^{-06}\)         |
| 10,000 - 50,000  | \(2.74 \times 10^{-14}\)         | \(1.25 \times 10^{-06}\)         |
| 50,000 - 100,000 | \(7.36 \times 10^{-14}\)         | \(9.81 \times 10^{-07}\)         |
| 100,000 - 500,000| \(1.86 \times 10^{-14}\)         | \(9.71 \times 10^{-07}\)         |
| 500,000 - 1,000,000 | \(3.09 \times 10^{-14}\)       | \(8.86 \times 10^{-07}\)         |

### Table 5

The results of the experiments of Section 5.5 in which the accuracy with which \texttt{alegendre_eval} calculates the functions \(\log (P_{\nu}^{-\mu}(t)) - \nu\) and \(\log (Q_{\nu}^{-\mu}(t)) - \nu\) for integers values of the parameters is tested.

| Range of \(\nu\) | Maximum relative error in \(\log (P_{\nu}^{-\mu}(t)) - \nu\) | Maximum relative error in \(\log (Q_{\nu}^{-\mu}(t)) + \nu\) | Average evaluation time (in seconds) |
|------------------|-------------------------------------------------|-------------------------------------------------|---------------------------------|
| 10 - 50          | \(4.21 \times 10^{-15}\)                        | \(4.65 \times 10^{-15}\)                        | \(3.59 \times 10^{-06}\)         |
| 50 - 100         | \(3.42 \times 10^{-15}\)                        | \(3.32 \times 10^{-15}\)                        | \(2.22 \times 10^{-06}\)         |
| 100 - 500        | \(3.07 \times 10^{-15}\)                        | \(4.07 \times 10^{-15}\)                        | \(2.28 \times 10^{-06}\)         |
| 500 - 1,000      | \(2.95 \times 10^{-15}\)                        | \(3.01 \times 10^{-15}\)                        | \(1.88 \times 10^{-06}\)         |
| 1,000 - 5,000    | \(2.63 \times 10^{-15}\)                        | \(4.14 \times 10^{-15}\)                        | \(1.55 \times 10^{-06}\)         |
| 5,000 - 10,000   | \(1.98 \times 10^{-15}\)                        | \(1.83 \times 10^{-15}\)                        | \(9.60 \times 10^{-07}\)         |
| 10,000 - 50,000  | \(1.98 \times 10^{-15}\)                        | \(2.68 \times 10^{-15}\)                        | \(1.69 \times 10^{-06}\)         |
| 50,000 - 100,000 | \(1.63 \times 10^{-15}\)                        | \(2.07 \times 10^{-15}\)                        | \(1.17 \times 10^{-06}\)         |
| 100,000 - 500,000| \(1.73 \times 10^{-15}\)                        | \(1.63 \times 10^{-15}\)                        | \(1.21 \times 10^{-06}\)         |
| 500,000 - 1,000,000 | \(1.67 \times 10^{-15}\)                   | \(2.23 \times 10^{-15}\)                        | \(1.16 \times 10^{-06}\)         |
| Range of $\nu$ | Maximum relative error in $\tilde{P}_\nu^{-\mu}(t) + i\tilde{Q}_\nu^{-\mu}(t)$ | Average evaluation time (in seconds) |
|----------------|---------------------------------------------------------------------------------|------------------------------------|
| 10 - 50        | $2.62 \times 10^{-13}$                                                         | $4.05 \times 10^{-06}$             |
| 50 - 100       | $4.20 \times 10^{-13}$                                                         | $1.95 \times 10^{-06}$             |
| 100 - 500      | $1.20 \times 10^{-12}$                                                         | $2.24 \times 10^{-06}$             |
| 500 - 1,000    | $1.72 \times 10^{-12}$                                                         | $1.48 \times 10^{-06}$             |
| 1,000 - 5,000  | $8.57 \times 10^{-12}$                                                         | $2.55 \times 10^{-06}$             |
| 5,000 - 10,000 | $1.38 \times 10^{-11}$                                                         | $1.14 \times 10^{-06}$             |
| 10,000 - 50,000| $8.51 \times 10^{-11}$                                                         | $2.51 \times 10^{-06}$             |
| 50,000 - 100,000| $9.07 \times 10^{-11}$                                                        | $1.95 \times 10^{-06}$             |
| 100,000 - 500,000| $9.83 \times 10^{-10}$                                                      | $1.31 \times 10^{-06}$             |
| 500,000 - 1,000,000| $8.25 \times 10^{-10}$                                                        | $1.20 \times 10^{-06}$             |

Table 6
The results of the experiments of Section 5.6 in which the accuracy with which alegendre_eval evaluates the function $\tilde{P}_\nu^{-\mu}(t) + i\tilde{Q}_\nu^{-\mu}(t)$ for integers values of the parameters is tested.

From Table 6, we see that the relative errors in the calculated values of the associated Legendre functions increase as a function of the parameter $\nu$. This is expected, and consistent with the condition number of the evaluation of the function $\tilde{P}_\nu^{-\mu}(t) + i\tilde{Q}_\nu^{-\mu}(t)$.

6. Conclusions. Nonoscillatory phase functions provide powerful theoretical tools for analyzing the solutions of second order differential equations as well as a framework for the design of simple and efficient numerical algorithms. Here, we have designed a scheme for the numerical evaluation of the associated Legendre functions on the cut using this framework. Our approach is simple-minded and highly effective. Moreover, by making use of the algorithms of [3] and [6], it can be applied in the case of many other special functions satisfying second order differential equations, such as the prolate spheroidal wave functions and the generalized Laguerre functions. The author will report on the use of the techniques of this paper to evaluate other special functions of interest at a later date.

In the nonoscillatory region, our algorithm calculates the logarithms of the associated Legendre functions as well as their values. This is useful in cases in which the magnitudes of those functions are too large or too small to be encoded using the IEEE double precision format. In the oscillatory region, in addition to the values of the associated Legendre functions, our algorithm also returns the values of a nonoscillatory phase function for the associated Legendre differential equation and its derivative. This is extremely helpful when computing the zeros of special functions [5], and when applying special function transforms via the butterfly algorithm (see, for instance, [7, 17, 18, 8, 21, 24]). The author will report on the use nonoscillatory phase functions to rapidly compute the roots of the associated Legendre functions and to rapidly apply the spherical harmonic transform at a later date.

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