Approximation in (Poly-) Logarithmic Space

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Abstract
We develop new approximation algorithms for classical graph and set problems in the RAM model under space constraints. As one of our main results, we devise an algorithm for $d$-Hitting Set that runs in time $n^{O(d^2 + (d/\epsilon))}$, uses $O((d^2 + (d/\epsilon)) \log n)$ bits of space, and achieves an approximation ratio of $O((d/\epsilon)n^{1/\epsilon})$ for any positive $\epsilon \leq 1$ and any constant $d \in \mathbb{N}$. In particular, this yields a factor-$O(d \log n)$ approximation algorithm which uses $O(\log^2 n)$ bits of space. As a corollary, we obtain similar bounds on space and approximation ratio for Vertex Cover and several graph deletion problems. For graphs with maximum degree $\Delta$, one can do better. We give a factor-2 approximation algorithm for Vertex Cover which runs in time $n^{O(\Delta)}$ and uses $O(\Delta \log n)$ bits of space.

For Independent Set on graphs with average degree $d$, we give a factor-$(2d)$ approximation algorithm which runs in polynomial time and uses $O(\log n)$ bits of space. We also devise a factor-$O(d^2)$ approximation algorithm for Dominating Set on $d$-degenerate graphs which runs in time $n^{O(\log n)}$ and uses $O(\log^2 n)$ bits of space. For $d$-regular graphs, we observe that a known randomized algorithm which achieves an approximation ratio of $O(\log d)$ can be derandomized and implemented to run in polynomial time and use $O(\log n)$ bits of space.

Our results use a combination of ideas from the theory of kernelization, distributed algorithms and randomized algorithms.

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1 Introduction and Motivation

This paper examines the classical approximation problems Vertex Cover, Hitting Set and Dominating Set in the RAM model under additional polylogarithmic space constraints. We devise approximation algorithms for these problems which use polylogarithmic space in general and $O(\log n)$ bits of space on certain special input types.

In the absence of space constraints, the greedy heuristic is a good starting point for many approximation algorithms. For Set Cover, it even yields optimal (under certain complexity-theoretic assumptions) approximation ratios [2, 18]. However, the heuristic inherently changes the input in some way. In a space-constrained setting however, this is asking for too much: the input is immutable, and the amount of auxiliary space available (polylogarithmic in our case) is not sufficient to register changes to the input.

Linear programming is another tool that plays a central role in the design of approximation algorithms.
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While it yields competitive approximations in polynomial time when space is not constrained, it is known that under logarithmic-space reductions, it is P-complete to approximate the LINEAR PROGRAMMING problem to any constant factor [40]. Such a result can be shown even for positive linear programming [43].

Machine Model We use the standard RAM model with an additional polylogarithmic space constraint. For inputs $n$ bits in length, memory is organized as words of length $O(\log n)$, which allows the entire input to be addressed using a single word of memory. Integer arithmetic operations on pairs of words and single-word memory access operations take constant time. The input (a graph or family of sets) is provided to the algorithm using some canonical encoding, which can be read but not modified, i.e. the algorithm has read-only access to the input.

The algorithm uses some auxiliary memory, to which it has read-write access, and in the setting of this paper, the amount of such memory available is bounded by a polynomial in $\log n$. Output is written to a stream: once something is output, the algorithm cannot read it back at a later point as it executes. We count the amount of auxiliary memory used in units of 1 bit, and the objective is to use as little auxiliary memory as possible.

Our Results

$d$ – Hitting Set and Vertex Deletion Problems

An instance of the $d$ – HITTING SET problem consists of a universe and a family of size-$d$ subsets of the universe, and the objective is to find a subset of the universe that has a non-empty intersection with each set in the family.

- We develop a factor-$O((d/\epsilon)n^\epsilon)$ approximation algorithm for $d$ – HITTING SET which runs in time $n^{O(d^2 + (d/\epsilon))}$ and uses $O((d^2 + (d/\epsilon)) \log n)$ bits of space (Section 3), where $\epsilon \leq 1$ is an arbitrary positive number and $d$ is a fixed positive integer. In particular, this yields a factor-$O(d \log n)$ approximation algorithm for the problem which uses $O(\log^2 n)$ bits of space. As an application, we show how the algorithm can be used to approximate various deletion problems with similar space bounds. From this, we derive a factor-$O((1/\epsilon)n^\epsilon)$ (for arbitrary positive $\epsilon \leq 1$) approximation algorithm for VERTEX COVER that runs in time $n^{O(1/\epsilon)}$ and uses $O((1/\epsilon) \log n)$ bits of space.

- We give a simple factor-2 approximation algorithm for VERTEX COVER on graphs with maximum degree $\Delta$ which runs in time $n^{O(\Delta)}$ and uses $O(\Delta \log n)$ bits of space (Section 3.1).

Dominating Set In the DOMINATING SET problem, the objective is to find a vertex set of minimum size in a graph such that all other vertices are adjacent to some vertex in the set.

- We give a factor-$O(\sqrt{n})$ approximation algorithm for graphs excluding $C_4$ (a cycle on 4 vertices) as a subgraph, which runs in polynomial time and uses $O(\log n)$ bits of space (Section 4.1).

- Graphs of bounded degeneracy form a large class which includes planar graphs, graphs of bounded genus, graphs excluding a fixed graph $H$ as a (topological) minor and graphs of bounded expansion. For graphs with degeneracy $d$, we give a factor-$O(d^2)$ approximation algorithm which uses $O(\log^2 n)$ bits of space. (Section 4.2).

- Additionally, for graphs in which each vertex has degree $d$, i.e. $d$-regular graphs, we exhibit a factor-$O(\log d)$ approximation algorithm for DOMINATING SET (Section 4.3) which is an adaptation of known results to the constrained-space setting.
Independent Set  An instance of the Independent Set problem consists of a graph, and the objective is to find an independent set of maximum size i.e. a set of vertices with no edges between them. We show how a known factor-(2d) approximation algorithm for Independent Set on graphs with average degree d can be implemented to run in polynomial time and use O(log n) bits of space (Section 5).

Related Work

Small-space models such as the streaming model and the in-place model have been the subject of much research over the last two decades (see [28, 15, 13] and references therein). In the streaming model, in addition to the space constraint, the algorithm is also required to read the input in a specific (possibly adversarial) sequence in one or more passes. The in-place model, on the other hand, allows the memory used for storing the input to be modified. The read-only RAM model we use is distinct from both these models. Historically, the read-only model has been studied from the perspective of time–space tradeoff lower bounds, particularly for problems like Sorting [8, 9, 32, 31] and Selection [30, 21, 29, 35].

The earliest graph problems studied in this model were the undirected and directed graph reachability problems (resp. USTCON and STCON) in connection with the complexity classes L and NL. Savitch [39] showed that on input graphs with n vertices, STCON (and therefore also USTCON) can be solved in O(log^2 n) bits of space. This bound was gradually whittled down over more than two decades, a process culminating in the result of Reingold [38] which shows that USTCON can be solved using O(log n) bits of space.

Reif [37] showed that the problems of recognizing bipartite, chordal, interval and split graphs are reducible to USTCON. Later on, Allender and Mahajan [1] showed that planarity testing also reduces to USTCON. Thus, Reingold’s result put all these problems in L. More recently, Elberfeld and Kawarabayashi [19] showed that the problems of recognizing and canonizing bounded-genus graphs were in L. The model was also studied by Yamakami [45] in relation to the complexity of search problems solvable in polynomial time, and by Tantau [41], who studied the approximation properties of search problems that can be solved in nondeterministic logarithmic space.

The other direction in which small-space problems and even the approximation problems we study have been investigated previously is in the context of fast parallel algorithms. By a known reduction, algorithms for these problems have sequential implementations that use polylogarithmic space. The PRAM algorithm of Luby [26] for finding maximal independent sets in a graph can be used to 2-approximate Vertex Cover (recall that a better than 2-approximate algorithm is known to be unlikely [24]). Implemented in the sequential RAM model, it uses O((log^2 n) bits of space. There have been attempts to generalize Luby’s algorithm to hypergraphs, and to the best of our knowledge, an efficient deterministic parallel algorithm (an NC algorithm) to find maximal independent sets in hypergraphs is not known to exist (see [6] and references therein). Our scheme for d−Hitting Set trades approximation factor against space used to obtain a family of algorithms that use O((d^2 + (d/\epsilon)) log n) bits of space to obtain O((d/\epsilon) n^\epsilon)-approximate solutions for any positive \epsilon \leq 1. As a corollary, we obtain an O(d log n)-approximation algorithm that uses O(log^2 n) bits of space. On graphs with maximum degree \Delta, our approximation algorithm for Vertex Cover uses O(\Delta log n) bits of space to obtain 2-approximate solutions.

Berger et al. [7] gave a PRAM algorithm for Set Cover which can be implemented in the sequential RAM model to O(log n)-approximate Dominating Set in O(log^4 n) bits of space. See also [42, 27], which give parallel approximation algorithms for Linear Programming, and see [25], which gives tight approximation ratios for CSP’s using semi-definite programming.
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in the PRAM model. Our algorithms for Dominating Set are simpler and more direct, and work for a large class of graphs while using \( O(\log^2 n) \) bits of space.

Our Techniques

As noted earlier, the greedy heuristic causes changes to the input, which our model does not permit. To get around this, we use a staggered greedy approach in which the solution is constructed in a sequence of greedy steps to approximate Vertex Cover on graphs of bounded degree (Section 3.1). By combining this with data reduction rules from kernelization algorithms, we also obtain approximations for Vertex Cover and more generally \( d - \text{Hitting Set} \) (Section 3), and restricted versions of Dominating Set (Sections 4.1 and 4.2). In Sections 4 and 5, we use 2-universal hash families constructible in logarithmic space to approximate Independent Set on graphs of bounded average degree (Section 5) and Dominating Set on regular graphs (Section 4.3) in logarithmic space.

2 Preliminaries

Notation \(\mathbb{N}\) denotes the set of natural numbers \(\{0, 1, \ldots\}\) and \(\mathbb{Z}^+\) denotes the set of positive integers \(\{1, 2, \ldots\}\). For \(n \in \mathbb{Z}^+\), \([n]\) denotes the set \(\{1, 2, \ldots, n\}\). Let \(G\) be a graph. Its vertex set is denoted by \(V(G)\), and its edge set by \(E(G)\). The degree of a vertex \(v\) is denoted by \(\deg(v)\), and for a set \(S \subseteq V(G)\) or a subgraph \(H\) of \(G\), \(\deg_S(v)\) denotes the degree of \(v\) in \(G[S]\) and \(\deg_H(v)\) denotes the degree of \(v\) in \(H\).

Known Results

To start with, consider the next result, which arises from a logarithmic-space implementation of the Buss kernelization rule [10] for Vertex Cover combined with the observation that the kernel produced is itself a vertex cover. ▶ Proposition 1 (Cai et al. [11], Theorem 2.3). There is an algorithm which takes as input a graph \(G\) and \(k \in \mathbb{N}\), and either determines that \(G\) has no vertex cover of size at most \(k\) or produces a vertex cover of size at most \(2k^2\). The algorithm runs in time \(O(n^2)\) and uses \(O(\log n)\) bits of space.

The Vertex Cover can be generalized to the \(d - \text{Hitting Set}\) problem \((d \in \mathbb{N}, \text{a constant})\), an instance of which comprises a family of size-\(d\) subsets of a ground set and \(k \in \mathbb{N}\). The objective is to determine whether there is a hitting set of size at most \(k\), i.e. a subset of the ground set which has a nonempty intersection with each set in the family. The next proposition shows that a similar result as above also holds for this generalization. ▶ Proposition 2 (Fafianie and Kratsch [20], Theorem 1). There is an algorithm which takes as input a family \(F\) of \(d\)-subsets \((d \in \mathbb{N}, \text{a constant})\) of a ground set \(U\) and \(k \in \mathbb{N}\), and either determines that \(F\) has no hitting set of size at most \(k\) or produces an equivalent subfamily of the original family which has size \(O((k+1)^d)\). The algorithm runs in time \(n^{O(d^2)}\) and uses \(O(d^2 \log n)\) bits of space.

2.1 Presenting modified graphs using oracles

Our algorithms repeatedly “delete” vertices or sets of vertices, but as they only have read-only access to the graph (or family of sets), we require a way to implement these deletions using a small amount of auxiliary space. Towards that, we prove the following theorem.
Theorem 3. Let \( G = G_0 = (V, E) \) be a graph with \( n \) vertices, and let \( G_i \) \( (i \in [k]) \) be obtained from \( G_{i-1} \) by deleting a set \( S_i \subseteq V(G_{i-1}) \) consisting of all vertices \( v \in V(G_{i-1}) \) which satisfy a property that can be checked (given access to \( G_{i-1} \)) using \( O(\log n) \) bits of space.

Given read-only access to \( G \), one can, for each \( i \in [k] \), enumerate and answer membership queries for \( S_i \), \( V_i = V(G_i) \) and \( E_i = E(G_i) \) in time \( n^{O(i)} \) using \( O(i \log n) \) bits of space.

Proof. For each \( i \in [k] \) let \( \text{Check}_i(G_{i-1}, v) \) be the algorithmic check which, given (oracle) access to \( G_{i-1} \), determines whether \( v \in V_{i-1} \) satisfies the condition for inclusion in \( S_i \). Note that this condition may be something that depends on the graph \( G_{i-1} \), i.e. \( G_{i-1} \) must be accessible to \( \text{Check}_i \).

To provide oracle access to \( G_i, V_i \) and \( E_i \), it suffices to compute, for \( v \in V \) and \( uw \in E \), the predicates \( [v \in V_i] \) and \( [uw \in E_i] \). A vertex is in \( V_i \) if and only if it is in \( V_{i-1} \) and it is not in \( S_i \). Similarly, an edge is in \( E_i \) if and only if it is in \( E_{i-1} \) and neither of its endpoints are in \( S_i \). Thus, we have the following relations.

\[
[v \in V_i] \equiv [v \in V_{i-1}] \land \neg \text{Check}_i(G_{i-1}, v) \quad (1)
\]

\[
[uw \in E_i] \equiv [uw \in E_{i-1}] \land \neg (\text{Check}_i(G_{i-1}, u) \vee \text{Check}_i(G_{i-1}, w)) \quad (2)
\]

To compute each of these predicates for \( G_{i-1} \), we require oracle access to \( G_{i-1} \), which in turn involves computing the predicates \( [v \in V_{i-1}] \) and \( [uw \in E_{i-1}] \). Suppose the number of operations needed to compute \( \text{Check}_i(G_{i-1}, v) \) is \( r(n) \), where \( r \) is a polynomial (it uses \( O(\log n) \) bits of space, so it is polynomial-time). Let \( p_i \) (resp. \( q_i \)) be the amount of space used to compute the predicate \( [v \in V_i] \) (resp. \( [uw \in E_i] \)), and let \( s_i \) (resp. \( t_i \)) be the time needed to compute the predicate \( [v \in V_i] \) (resp. \( [uw \in E_i] \)). From Relations 1 and 2 and the fact that \( \text{Check}_i \), accesses \( G_{i-1} \) at most \( r(n) \) times, we see that these quantities satisfy the following relations.

\[
p_i = p_{i-1} + O(\log n), \quad q_i = q_{i-1} + O(\log n) \quad (3)
\]

\[
s_i = s_{i-1} + O(r(n)(s_{i-1} + t_{i-1})), \quad t_i = t_{i-1} + O(r(n)(s_{i-1} + t_{i-1})) \quad (4)
\]

It is easy to see that these recurrences solve to \( p_i, q_i = O(i \log n) \) and \( s_i, t_i = n^{O(i)} \), so both predicates can be computed in time \( n^{O(i)} \) using \( O(i \log n) \) bits of space.

With oracle access to \( G_{i-1} \), the predicate \( [v \in S_i] \) can be computed simply as \( \text{Check}_i(G_{i-1}, v) \), from which enumerating \( V_i \) (resp. \( E_i \) and \( S_i \)) is straightforward: enumerate \( V \) (resp. \( E \) and \( V \) and suppress vertices \( v \) (resp. edges \( uw \) and \( vertices\) \( z \)) which fail the predicate \( [v \in V_i] \) (resp. \( [uw \in E_i] \) and \( [z \in S_i] \)). As the most space-hungry operations are the membership queries, the enumeration can also be performed using \( O(i \log n) \) bits of space. The enumeration needs time \( n^{O(i)} \) for each element of \( V \) and \( E \), and since \( |V|, |E| = O(n^2) \), the total time needed is also \( n^{O(i)} \).

2.2 Universal Hash Families

Algorithms appearing later on use the trick of randomized sampling to obtain a certain structure with good probability and then derandomize this procedure by using a family of 2-universal functions. A 2-universal hash family is a family \( \mathcal{F} \) of functions from \([n]\) to \([k]\), for integers \( n, k \) with \( k \leq n \) such that for any pair \( i \) and \( j \) of elements from \([n]\), the number of functions from \( \mathcal{F} \) that map \( i \) and \( j \) to the same element in \([k]\) is at most \( |\mathcal{F}|/k \). The following proposition, which is a combination of a result of Carter and Wegman [12] showing the existence of such families, and the observation that these families can be computed in logarithmic space [44].
Proposition 4 (Carter and Wegman [12], Proposition 7). Let \( n, k \in \mathbb{N} \) with \( n \geq k \). One can enumerate a 2-universal hash family for \([\mathbb{N}] \to [k]\) in polynomial time using \( O(\log n) \) bits of space.

### 3 Hitting Sets and \( \Pi \)-Deletion Problems

The \( d \)– hitting set problem is a generalization of vertex cover in which an instance consists of a family \( F \) of \( d \)-subsets of a ground set \( U \), and the objective is to find a subset of \( U \) of minimum size which intersects all sets in \( F \).

Algorithms for the problem are useful as subroutines in solving various deletion problems, where the objective is to delete the minimum possible number of vertices from a graph so that the resulting graph satisfies a certain property. The following result is a corollary to Proposition 2.

Corollary 5. Let \( F \) be a family of \( d \)-subsets of a ground set \( U \) with \( n \) elements. One can compute an \( O( dn^{1-1/d} ) \)-approximate minimum hitting set for \( F \) in time \( n^{O(d^2)} \) using \( O(d^2 \log n) \) bits of space.

Proof. Consider the following algorithm. Starting at \( k = 1 \), run the algorithm of Proposition 2 and repeatedly increment the value of \( k \) until \( n^{1/d} \) or the algorithm returns a solution of size \( O(d(k+1)^d) \) (i.e. it does not return a \texttt{NO} answer) for the first time. If \( k \) is incremented until \( n^{1/d} \), then simply return the entire universe as the solution. Clearly, the approximation ratio is \( n^{1-1/d} \), as \( OPT \geq n^{1/d} \) (and so the size of the solution returned is \( n = n^{1-1/d} \cdot n^{1/d} \leq n^{1-1/d} \), \( OPT \), where \( OPT \) is the size of the minimum hitting set).

If \( k < n^{1/d} \), then the size of the solution produced is \( O(d(k+1)^d) \), and we know that \( OPT \geq k \), since the algorithm had returned \texttt{NO} answers until this point. So the size of the solution produced is \( O(d(k+1)^d) = O(d(k+1)^{d-1} \cdot (OPT + 1)) = O(dn^{1-1/d} \cdot (OPT + 1)) \). Thus, we have an \( O( dn^{1-1/d} ) \)-approximation. The bounds on running time and space used follow from the fact that the algorithm of Proposition 2 runs in time \( n^{O(d^2)} \) and uses \( O(d^2 \log n) \) bits of space.

The next result is one of our main results en route to developing a space-efficient approximation algorithm for \( d \)– hitting set.

Lemma 6. Let \( \epsilon \leq 1 \) be a positive number. There is an algorithm which takes as input a family \( F \) of \( d \)-subsets of a ground set \( U \) of \( n \) elements and \( k \in \mathbb{N} \), and either determines \( F \) has no hitting set of size at most \( k \) or produces a hitting set of size \( O((d/\epsilon)k^{1+\epsilon}) \). The algorithm runs in time \( n^{O(d^2+(d/\epsilon))} \) and uses \( O((d^2 + (d/\epsilon)) \log n) \) bits of space.

Proof. Let \( i = \lceil (d-1)/\epsilon \rceil \). The algorithm performs \( i \) rounds of computation, each using \( O(\log n) \) bits of space to determine a set of elements (accessible by oracle) to be removed in the next round, or determine that \( F \) has no hitting set of size at most \( k \).

1. Use the algorithm of Proposition 2 to obtain a subfamily \( F' \subseteq F \) over the ground set \( U' \subseteq U \) such that
   - \( |F'| \leq c(k+1)^d \), \( |U'| = cd(k+1)^d \), and
   - there exists a hitting set \( S \subseteq U \) of size at most \( k \) in \( F \) if and only if there exists a hitting set \( S' \subseteq U' \) and \( S' \) is a hitting set for \( F' \).
2. Set \( U_0 = U' \) and \( F_0 = F' \). For \( j = \{1, 2, \ldots, i-1\} \), perform the following steps.
Determine $S_j$, the set of all elements in $U_{j-1}$ which appear in at least $c(k+1)^{d-1-j}\epsilon$ sets in $F_{j-1}$.

Let $U_j = U_{j-1} \setminus S_j$ and $F_j = \{ A \in F_{j-1} | A \cap S_j = \emptyset \}$. If there are more than $c(k+1)^{d-j}\epsilon$ sets in $F_j$, then return NO.

3. Determine $S_i$, the set of all elements in $U_{i-1}$ which are in some set in $F_{i-1}$. Output $S = \bigcup_{j=1}^{i} S_j$.

We now prove the correctness of the algorithm. In Step 1, the algorithm obtains the ground set $U'$ and the family $F'$, using the algorithm of Proposition 2. Let $l \in [i-1]$ such that the algorithm answers NO in Step 2 for $j = l$, and otherwise let $l = i$ if it never returns a NO answer in Step 2.

\textbf{Claim 7.} For all $j \in [l]$, $F_j$ has at most $c(k+1)^{d-j}\epsilon$ sets.

Consider the case when the algorithm does not return a NO answer. Observe that the claim holds for the base case $j = 1$: $F_0$ has $c(k+1)^d$ sets, and since the algorithm does not return a NO answer, we have $|F_1| \leq c(k+1)^{d-j}\epsilon$. For induction, observe that whenever $|F_j| \leq c(k+1)^{d-j}\epsilon$, the algorithm ensures that $|F_{j+1}| \leq c(k+1)^{d-(j+1)\epsilon}$; otherwise, it returns a NO answer.

Suppose the algorithm returns a NO answer at some value of $j$ in Step 2, then there are more than $c(k+1)^{d-j}\epsilon$ sets in $F_j$, which have survived the repeated removal of sets from $F_0$ up to this point, and they cannot be hit by any $k$ of the elements in $U_j$, since each element can hit at most $c(k+1)^{d-1-j}\epsilon$ sets in $F_j$. Thus, the algorithm correctly infers that the input does not have a hitting set of size at most $k$.

Once the algorithm has reached Step 3, the number of sets in the residual family, $F_{i-1}$, is at most $(k+1)^{d-(\lfloor (d-1)/\epsilon \rfloor - 1)\epsilon} < k^{d-(\lfloor (d-1)/\epsilon \rfloor - 1)\epsilon} = k^{1+\epsilon}$. The set $S_i$ of elements in $U_{i-1}$ that appear in some set in $F_{i-1}$ is trivially also a hitting set. Observe that the sets of elements removed in earlier stages, i.e., $S_0, \ldots, S_{i-1}$ together hit all sets in $F$ not appearing in $F_{i-1}$. Thus, the set $S = \bigcup_{j=0}^{i} S_j$ output by the algorithm is a hitting set for $F$.

\textbf{Claim 8.} The set $S$ output by the algorithm has at most $((d-1)/\epsilon + d)k^{1+\epsilon}$ elements.

For each $j \in [i-1]$, the algorithm ensures that $|F_{j-1}| \leq c(k+1)^{d-(j-1)\epsilon}$ (otherwise, it returns a NO answer). Thus, the number of elements which appear in at least $c(k+1)^{d-1-j}\epsilon$ sets is at most \left(\frac{c(k+1)^{d-(j-1)\epsilon}}{c(k+1)^{d-1-j\epsilon}}\right) = k^{1+\epsilon}, i.e., $|S_j| \leq k^{1+\epsilon}$.

In Step 3, the algorithm ensures that $|F_{i-1}| \leq k^{d-(i-1)\epsilon} \leq k^{1+\epsilon}$. Each set in $F_{i-1}$ edges and each of these edges can span at most $d$ elements. Thus, the number of elements in $U_{i-1}$ which appear in some set in $F_{i-1}$ is $dk^{1+\epsilon}$, i.e., $|S_i| \leq dk^{1+\epsilon}$. Therefore, the total number of elements output by the algorithm in all three phases is $|S| = \sum_{j=1}^{i} |S_j| \leq (i-1)k^{1+\epsilon} + dk^{1+\epsilon} \leq (\lfloor (d-1)/\epsilon \rfloor + d)k^{1+\epsilon}$.

\textbf{Claim 9.} The algorithm runs in time $n^{O((d^2 + (d/\epsilon)) \log n)}$ and uses $O((d^2 + (d/\epsilon)) \log n)$ bits of space.

Observe that in Step 1, the family $F_0$ is obtained using the algorithm of Proposition 2, which runs in time $n^{O(d^2)}$ and uses $O(d^2 \log n)$ bits of space (for any constant $d$). The output of the algorithm can now be used as an oracle for $G_0$.

In Step 2, each successive family $F_j$ ($j \in [i-1]$) is obtained from $F_{j-1}$ by deleting sets containing elements which appear in at least $k^{1+\epsilon}$ sets (this test can be performed using $O(\log n)$) bits of space. Thus, given oracle access to $F_{j-1}$, an oracle for $F_j$ can be provided which runs in polynomial time and uses $O(\log n)$ bits of space.
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Step 3 involves writing out all elements in \( U_{i-1} \) that appear in some set in \( F_{i-1} \), which can also be done in \( O(\log n) \) bits of space given oracle access to \( G_{i-1} \). Since the number of oracles created in Step 2 is \( i - 1 \), the various oracles together run in time \( n^{O(i)} \) and use \( O(i \log n) = O((d/\epsilon) \log n) \) bits of space (Theorem 3). Combined with the \( n^{O(d^2)} \) time and \( O(d^2 \log n) \) bits of space used by the oracle of Step 1, this gives bounds of \( n^{O(d^2 + (d/\epsilon))} \) on the running time and \( O((d^2 + (d/\epsilon)) \log n) \) bits on the total space used by the algorithm.

Theorem 10. Let \( \epsilon \leq 1 \) be a positive number. For instances \((U,F)\) of \( d \)-Hitting Set with \( |U| = n \), one can compute an \( O((d/\epsilon)n^d) \)-approximate minimum hitting set in time \( n^{O(d^2 + (d/\epsilon))} \) using \( O((d^2 + (d/\epsilon)) \log n) \) bits of space.

Proof. Apply the algorithm of Lemma 6 starting from \( k = 1 \) successively incrementing by 1 until the algorithm returns a family of size \( O((d/\epsilon)k^{1+\epsilon}) \) or \( k = n^{1-\epsilon} \). When \( k = n^{1-\epsilon} \) return the entire universe as the solution. As, in this case, \( OPT \geq n^{1-\epsilon} \), the size of the solution produced, which is \( n \leq n^{OPT} \), and so we have a factor-\( n^\epsilon \) approximation algorithm.

When the algorithm returns a family of size \( O((d/\epsilon)k^{1+\epsilon}) \) for some \( k \), note that \( OPT \geq k \) (as the algorithm returned NO so far), and so the solution produced is of size \( O((d/\epsilon)k^k) \), which is \( O((d/\epsilon)n^{OPT}) \) resulting in a factor-\( O((d/\epsilon)n^\epsilon) \) approximation algorithm. As we merely reuse the procedure of Lemma 6, the running time is \( O((d^2 + (d/\epsilon)) \log n) \) and the amount of space used is \( O((d^2 + (d/\epsilon)) \log n) \) bits.

The above theorem allows us to devise space-efficient approximation algorithms for a number of graph deletion problems. Let \( \Pi \) be a hereditary class of graphs, i.e. a class closed under taking induced subgraphs. Let \( \Phi \) be a set of forbidden graphs for \( \Pi \) such that a graph \( G \) is in \( \Pi \) if and only if no induced subgraph of \( G \) is isomorphic to a graph in \( \Phi \). Consider the problem Del–\( \Pi \) (described below), defined for classes \( \Pi \) with finite sets \( \Phi \) of forbidden graphs.

Instance \( G \), a graph

Solution a set of vertices smallest size whose deletion yields a graph in \( \Pi \)

The next result is a combination of the fact that Del–\( \Pi \) can be formulated as a certain hitting set problem and the procedure of Theorem 10.

Lemma 11. Let \( \epsilon \leq 1 \) be a positive number. On graphs with \( n \) vertices, one can compute \( O((1/\epsilon)n^\epsilon) \)-approximate solutions for Del–\( \Pi \) in time \( n^{O(1/\epsilon)} \) using \( O((1/\epsilon) \log n) \) bits of space.

Proof. Let \( \Pi \) be a class of graphs characterized by a finite set \( \Phi \) of forbidden induced subgraphs. Consider the problem of finding, given a graph, a deletion set of vertices of minimum size whose removal from the graph produces a graph in \( \Pi \).

Given a graph \( G \) on \( n \) vertices, it is possible using \( O((1/\epsilon) \log n) \) bits of space, to produce an \( O(n^\epsilon/\epsilon) \)-approximate minimum deletion set for \( G \). Simply construct the family \( \mathcal{F}_G = \{ S \subseteq V(G) \mid G[S] \text{ contains a graph from } \Phi \} \). This family of subsets is constructed “on the fly” in a systematic way (as and when the hitting set algorithm needs it) by running over all subsets of \( V(G) \) of size at most \( d \) where \( d \) is the maximum size of the vertex set in a graph in \( \Phi \). It can be seen that this can be constructed using \( O(\log n) \) bits of space for constant \( d \) and that an \( O(n^\epsilon/\epsilon) \) approximate minimum hitting set for \( \mathcal{F}_G \) is also an \( O(n^{\epsilon/\epsilon}) \)-approximate solution for the deletion problem.
Thus, a variant of the procedure described earlier can be used to produce $O(n^{\epsilon}/\epsilon)$-approximate solutions for $n$-vertex instances of this problem using $O((1/\epsilon) \log n)$ bits of space.

The following list defines problems for which we obtain polylogarithmic-space approximation algorithms using the preceding lemma.

**Triangle-Free Deletion**

*Instance:* $(G, k)$, where $G$ is a graph and $k \in \mathbb{N}$

*Question:* Is there a set $S \subseteq V(G)$ with $|S| \leq k$ such that $G - S$ has no triangles?

**Tournament FVS**

*Instance:* $(D, k)$, where $D$ is a tournament and $k \in \mathbb{N}$

*Question:* Is there a set $S \subseteq V(D)$ with $|S| \leq k$ such that $G - S$ is acyclic?

**Cluster Deletion**

*Instance:* $(G, k)$, where $G$ is a graph and $k \in \mathbb{N}$

*Question:* Is there a set $S \subseteq V(G)$ with $|S| \leq k$ such that $G - S$ is a disjoint union of cliques, i.e. a cluster graph?

**Split Deletion**

*Instance:* $(G, k)$, where $G$ is a graph and $k \in \mathbb{N}$

*Question:* Is there a set $S \subseteq V(G)$ with $|S| \leq k$ such that $G - S$ can be partitioned into a clique and an independent set, i.e. such that $(G - S)$ is a split graph?

**Threshold Deletion**

*Instance:* $(G, k)$, where $G$ is a graph and $k \in \mathbb{N}$

*Question:* Is there a set $S \subseteq V(G)$ with $|S| \leq k$ such that $G - S$ is a threshold graph? A threshold graph is one which can be constructed from a single vertex by a sequence of operations that either add an isolated vertex, or add a vertex which dominates all the other vertices.

**Cograph Deletion**

*Instance:* $(G, k)$, where $G$ is a graph and $k \in \mathbb{N}$

*Question:* Is there a set $S \subseteq V(G)$ with $|S| \leq k$ such that $G - S$ contains no induced paths of length 4, i.e. it is a cograph?

For all the problems appearing above, the target graph classes are known to be characterized by a finite set of forbidden induced subgraphs (see e.g. Cygan et al. [17]) and so the problems can be formulated as Del-Π. By setting $\epsilon$ to a small positive constant or $(1/\log n)$, we obtain the following corollary to Lemma 11.

**Corollary 12.** On graphs with $n$ vertices, one can compute

- $O(n^\epsilon)$-approximate solutions in time $n^{O(1/\epsilon)} = n^{O(1)}$ using $O((1/\epsilon) \log n) = O(\log n)$ bits of space for any positive constant $\epsilon \leq 1$, and
- $O(\log n)$-approximate solutions in time $n^{O(\log n)}$ using $O(\log^2 n)$ bits of space for the problems Vertex Cover, Triangle-Free Deletion, Threshold Deletion, Cluster Deletion, Split Deletion, Cograph Deletion and Tournament FVS.

### 3.1 Vertex Cover on Graphs of Bounded Degree

We begin this section with the observation that in a directed graph with maximum outdegree 1, every connected component contains (as an induced subgraph or otherwise) at most one (undirected) cycle. For such a directed graph $D$, consider the graph $G$ obtained by ignoring arc directions. Because every connected component in $G$ also has at most one cycle, one can find a minimum vertex cover for $G$ in polynomial time and logarithmic space using a
modified post-order traversal procedure on the connected components. The following lemma formalizes this discussion.

**Lemma 13.** Let $D$ be a directed graph on $n$ vertices with maximum outdegree 1 and let $G$ be the undirected graph obtained by ignoring the arc directions in $D$. One can find a minimum vertex cover for $G$ in polynomial time using $O(\log n)$ bits of space.

To prove the above lemma, we use the following algorithm, which computes minimum vertex covers in trees.

---

**Algorithm 1, TreeVtxCover: compute a minimum vertex cover**

| Input: $T = (V,E)$, a tree |
| Output: a minimum vertex cover for $T$ |
| let $r$ be an arbitrary vertex of $T$; |
| foreach $v \in V(T)$ do |
| if $IsInVC(v,r,T)$ then |
| output $v$; |
| Procedure $IsInVC(v,r,T)$ // $T$ a tree rooted at $r$, $v$ a vertex in $V(T)$ |
| generate a post-order traversal $L$ for $T$ with $r$ as the root; |
| seek $L$ to the first leaf in the subtree of $T$ rooted at $v$; |
| $visited_{\_vertex} \leftarrow NULL$; |
| $visited_{\_included} \leftarrow NO$; |
| foreach $u \in L$ do |
| if $u$ is a leaf then |
| $visited_{\_vertex} \leftarrow u$; |
| $visited_{\_included} \leftarrow NO$; |
| else // $u$ is not a leaf; $u$ is the parent of $visited_{\_vertex}$ |
| $visited_{\_vertex} \leftarrow u$; |
| if not $visited_{\_included}$ then |
| if $u = v$ then |
| return YES; |
| $visited_{\_included} \leftarrow YES$; // include $u$ |
| else // last-visited vertex was included |
| if $u = v$ then |
| return NO; |
| $visited_{\_included} \leftarrow NO$; // do not include $u$ |
| seek $L$ to $u$’s parent; // vertices in subtrees of $u$’s unvisited siblings can be ignored |

---

The algorithm operates by rooting $T$ at an arbitrary vertex $r \in V(T)$ and determines a vertex cover $S$ given by repeatedly applying the following rule.

**Rule VCT** Include the the parents of leaves on the bottom level of $T$ in $S$, then delete from $T$ the included vertices, their children, and all edges incident with them.

The fact that $S$ is a minimum vertex cover follows directly from the observation that to cover the edges of $T$ incident with the leaves at the bottom level, picking the parents of those leaves is at least as good as any other choice of covering vertices.
Observe that at any intermediate stage in the repeated application of Rule VCT, a vertex is a leaf on the bottom level of \( T \) if a previous application of the rule deleted all of its children, i.e., all of them were included in \( S \). Thus, any vertex \( v \in V(T) \), is in \( S \) if and only if \( S \) does not contain all of its children.

Instead of repeatedly deleting vertices from \( T \), Procedure IsInVC in the algorithm determines membership in \( S \) by performing what is essentially a post-order traversal of \( T \). In the post order traversal, to determine if a vertex \( v \) is in \( S \), the only information necessary is whether at least one of \( v \)'s children is not in \( S \), which the procedure stores in the variable \texttt{visited\_included}. If such a child vertex is encountered, the procedure determines that \( v \) is in \( S \), and skips the rest of the subtree rooted at \( v \).

The post-order traversal used by the procedure can be generated from a BFS traversal of \( T \), which can be computed using \( O(\log n) \) bits of space [16]. The constantly-many variables appearing in the algorithm also use \( O(\log n) \) bits of space total. Therefore, the overall space usage of the algorithm is \( O(\log n) \) bits. The following lemma formalizes the preceding discussion.

\textbf{Lemma 14.} For an input tree \( T \) on \( n \) vertices, the algorithm \texttt{TreeVtxCover} computes a minimum vertex cover using \( O(\log n) \) bits of space.

By layering multiple such steps, one can find a vertex cover in a bounded degree graph which is a 2-approximation for the minimum vertex cover. Our approach is inspired by a local distributed algorithm of Polishchuk and Suomela [34] which computes a factor-3 approximations.

\textbf{Theorem 15.} There is an algorithm which takes as input a graph \( G \) on \( n \) vertices in which every vertex has degree at most \( \Delta \), and computes a 2-approximate minimum vertex cover for \( G \). The algorithm runs in time \( n^{O(\Delta)} \) and uses \( O(\Delta \log n) \) bits of space.

\textbf{Proof.} Set \( G_0 = G \) and \( V_0 = V(G) \). The algorithm works in stages \( 1, \ldots, \Delta \) as follows. In Stage \( i \), it enumerates the subgraph \( H_{i-1} \) of \( G_{i-1} \) in which each vertex of \( u \) of \( G_{i-1} \) only retains the edge to its \( i \)th neighbour \( v \) (if it exists) in \( G \). Observe that directing every such edge from \( u \) to \( v \) yields a directed graph \( R \) with maximum outdegree 1.

Applying the procedure of Lemma 13 with \( D = R \) and \( G = H_{i-1} \), the algorithm now computes a minimum vertex cover \( S_i \) for \( H_{i-1} \) in polynomial time using \( O(\log n) \) bits of space. It then produces the graph \( G_i \) by removing the vertex set \( S_i \) from \( G_{i-1} \) and outputs the vertices in \( S_i \). At the end of Stage \( \Delta \), the algorithm terminates.

We now prove the bounds in the claim. Observe that the vertex set of \( G_i \) (\( i \in [\Delta] \)) is precisely \( V(G_{i-1}) \setminus S_i \). In Stage \( i \), the algorithm only considers the vertices in \( G_{i-1} \), so the vertex cover generated by it has no neighbours in vertex covers generated in earlier stages, i.e., \( S_i \cap S_j = \emptyset \) for \( j < i \).

For each \( H_{i-1} \), consider a maximal matching \( M_i \) in \( H_{i-1} \). From the way the various sets \( S_i \) are generated, it is easy to see that \( S = \bigcup_{i=1}^{\Delta} S_i \) forms a vertex cover for \( G \) and additionally, \( M = \bigcup_{i=1}^{\Delta} M_i \) is a maximal matching in \( G \). Observe that the each set \( S_i \) also covers the matching \( M_i \) in \( H_{i-1} \). Since \( S_i \) is a minimum vertex cover for \( H_{i-1} \), and the endpoints of edges in \( M_i \) form a vertex cover for \( H_{i-1} \), we have \( |S_i| \leq 2|M_i| \).

As \( M \) is a maximal matching in \( G \), the endpoints of edges in \( M \) form a vertex cover for \( G \), and we have \( |S| = \sum_{i=1}^{\Delta} |S_i| \leq 2 \cdot \sum_{i=1}^{\Delta} |M_i| \leq 2 \cdot \sum_{i=1}^{\Delta} \tau(G) \), where \( \tau(G) \) is the vertex cover number of \( G \). Thus, the set \( S \) output by the algorithm is a 2-approximate vertex cover.

Now observe that for all \( i \in [\Delta] \), \( G_i \) and \( S_i \) satisfy the hypothesis of Theorem 3. Thus, one can compute each of the sets \( S_i \) in time \( n^{O(i)} \) using \( O(i \log n) \) bits of space. Since
the maximum value \( i \) takes on is \( \Delta \), the algorithm runs in time \( n^{O(\Delta)} \) and uses a total of \( O(\Delta \log n) \) bits of space.

**4 Dominating Sets**

In this section, we describe approximation algorithms for **Dominating Set** restricted to certain graph classes. A problem instance consists of a graph \( G = (V, E) \) and \( k \in \mathbb{N} \), and the objective is to determine if there is a dominating set of size at most \( k \), i.e. a set \( S \subseteq V \) of at most \( k \) vertices such that \( S \cup N(S) = V \).

The first result of this section concerns graphs excluding \( C_4 \) (a cycle on 4 vertices) as a subgraph. On such graphs, one can compute \( O(\sqrt{n}) \)-approximations using \( O(\log n) \) bits of space using a known kernelization algorithm [36].

**4.1 \( C_4 \)-Free Graphs**

Any vertex \( v \in V(G) \) of degree at least \( 2k + 1 \) must be in any dominating set of size at most \( k \), as any other vertex (including a neighbour of \( v \)) can dominate at most 2 vertices in the neighbourhood (as there will be a \( C_4 \) otherwise). Using this, we establish the following result.

- **Lemma 16.** There is an algorithm which takes as input a \( C_4 \)-free graph \( G \) on \( n \) vertices and \( k \in \mathbb{N} \), and either determines that \( G \) has no dominating set of size at most \( k \), or outputs a dominating set of size \( O(k^2) \). The algorithm runs in polynomial time and uses \( O(\log n) \) bits of space.

**Proof.** The crux of the algorithm is the fact that any vertex \( v \in V(G) \) of degree at least \( 2k + 1 \) must be in any dominating set of size at most \( k \), as any other vertex (including a neighbour of \( v \)) can dominate at most 2 vertices in the neighbourhood (as there will be a \( C_4 \) otherwise). The algorithm proceeds as follows.

1. Let \( S \) be the set of vertices with degree more than \( 2k \). If \( |S| \) is more than \( k \), return \( \text{NO} \).
2. Note that all vertices in \( N(S) \) have already been dominated. If \( |V \setminus (S \cup N(S))| > (k - |S|) \cdot (2k + 1) \) return \( \text{NO} \), as each vertex in \( V \setminus S \) can dominate at most \( 2k + 1 \) vertices including itself.
3. If the algorithm has not returned \( \text{NO} \), output \( S \cup (V \setminus (S \cup N(S))) \)

Correctness is immediate from the description of the algorithm. When it outputs vertices, it outputs \( S \), which has at most \( k \) vertices from Step 1, and the number of remaining vertices in Step 2 is \( O(k^2) \), so it outputs \( O(k^2) \) vertices overall. To see that the space used is \( O(\log n) \) bits, observe that membership in each of the sets output is determined by predicates that test degrees of vertices individually, and these predicates can be computed in logarithmic space. Thus, by Theorem 3, the algorithm uses a total of \( O(\log n) \) bits of space.

The proof of the following corollary uses arguments very similar to those in the proof of Theorem 10, so we omit it.

- **Corollary 17.** There is an algorithm which takes as input a \( C_4 \)-free graph \( G \) on \( n \) vertices, and computes an \( O(\sqrt{n}) \)-approximate minimum dominating set for \( G \). The algorithm runs in polynomial time and uses \( O(\log n) \) bits of space.
4.2 Graphs of Bounded Degeneracy

A graph is called \( d \)-degenerate if there is a vertex of degree at most \( d \) in every subgraph of \( G \). A graph with maximum degree \( d \) is clearly \( d \)-degenerate. Planar graphs are 5-degenerate. Let \( G \) be a \( d \)-degenerate graph on \( n \) vertices. As every subgraph of \( G \) has a vertex with degree at most \( d \), the number of edges in \( G \) is at most \( dn \). It follows that

> **Observation 1.** In any subgraph of \( p \) vertices of a \( d \) degenerate graph, at least \( p/2 \) vertices are of degree at most \( 2d \).

There is a generalization of polynomial kernel of Dominating Set for \( C_4 \)-free graphs to \( K_{i,j} \)-free graphs for any fixed \( i, j \in \mathbb{N} \) [33] (recall that \( K_{i,j} \) is the complete bipartite graph with \( i \) vertices in one part and \( j \) vertices in the other). The class of \( K_{i,j} \)-free graphs includes \( C_4 \)-free graphs, and \((i + 1)\)-degenerate graphs (where \( i \leq j \)). This kernelization, however, does not appear suited to approximation in logarithmic (or even polylogarithmic) space. To design a space-efficient approximation algorithm for \( d \)-degenerate graphs, we resort instead to the \( O(d^2) \)-approximation algorithm of Jones et al. [23]. To achieve an \( O(\log^2 n) \) bound on the space used, several adaptations are necessary.

The algorithm starts by picking the neighbours of all vertices of degree at most \( 2d \), and works by repeatedly finding such vertices in smaller and smaller subgraphs of \( G \) and picking all their neighbours in the solution. As the vertex or one of its neighbours must be in any dominating set, this will result in an \( O(d) \) approximation if we manage to find a vertex that dominates (at least one and) at most \( 2d \) of the undominated vertices. This may not happen in the intermediate steps as more and more vertices are dominated by those vertices picked earlier. So we do some careful partitioning of the vertices and find low degree vertices in appropriate subgraphs.

Let \( Y \) be the set of vertices picked at any point, \( B \) be the set of vertices (other than those in \( Y \)) dominated by \( Y \), and \( W \) be the set of vertices in \( V \setminus (Y \cup B) \) (see Figure 1). The goal is to dominate vertices in \( W \), and we try to do so by finding (the neighbours of) low degree vertices from \( B \cup W \). So we start finding low degree (at most \( 2d \)) vertices in \( B \cup W \) to pick their neighbours. First we look for such vertices in \( B \), and so we further partition \( B \) into \( B_h \), those vertices of \( B \) with at least \( 2d + 1 \) neighbours in \( W \) and \( B_t = B \setminus B_h \).

First, we remove (for later consideration) vertices of \( W \) that have no neighbours in \( W \cup B_h \), let them be \( W_t \) and focus on the induced subgraph \( G[B_h \cup W_h] \) where \( W_h = W \setminus W_t \). Here, we are bound to find low degree vertices from \( W_h \) (as vertices in \( B_h \) have high degree) as long as \( W_h \) is non-empty, and so we repeat the above procedure of picking the neighbours of all low degree vertices from \( W_h \). Finally, when \( W_h \) is empty, if \( W_t \) is non-empty, we simply pick all vertices of \( W_t \) into the solution and return it. What follows is a pseudocode description of the algorithm.

If we treat a round as the step where we find all vertices in \( W_h \) with at most \( 2d \) neighbours in \( W_h \), then as at least a fraction of the vertices of \( W_h \) are dominated in each round due to Observation 1, the number of rounds is \( O(\log n) \). Each round just requires identifying vertices based on their degrees in the resulting subgraph, the \( i \)-th round can be implemented in \( O(i \log n) \) bits using Theorem 3 resulting in an \( O(\log^2 n) \) bits implementation.

The approximation ratio of \( O(d^2) \) can be proved formally using a charging argument (see Jones et al. [23], Theorem 4.9). We give an informal explanation here. First we argue the approximation ratio of \( (2d + 1) \) for the base case when \( W_h \) is empty. Isolated vertices in \( W_t \) are isolated vertices in \( G \) and hence they need to be picked in the solution. The number of non-isolated vertices in \( W_t \) is at most \( 2d|B_t| \) as their neighbours are only in \( B_t \) (otherwise, by definition, those vertices will be in \( W_h \)). As vertices in \( B_t \) have degree at most
Approximation in (Poly-) Logarithmic Space

Figure 1 Partitioning of the vertices in the algorithm for DOMINATING SET on d-degenerate graphs.

2d, |W_I| ≤ 2d|B_I| and as at least one vertex of B_I ∪ W_I must be picked to dominate a vertex in W_I, we have the approximation ratio of (2d + 1) for those vertices.

In the intermediate step, if we did not ignore vertices in B_I to dominate a vertex in W_h, a (2d + 1)-approximation is clear. For, a vertex or one of its at most 2d neighbours must be picked in the dominating set. However, a vertex in W_h maybe dominated by a vertex in B_I, but by ignoring B_I, we maybe picking 2d vertices to dominate it. As a vertex in B_I can dominate at most 2d (such) vertices of W_h, we get an approximation ratio of O(d^2).

The next theorem formalizes the above discussion.

Theorem 18. There is an algorithm which takes as input a d-degenerate graph G on n vertices and computes an O(d^2)-approximate minimum dominating for G. The algorithm uses O(\log^2 n) bits of space and runs in time n^{O(\log n)}.

4.3 Regular Graphs

On regular graphs, we can achieve a better approximation ratio in logarithmic space by derandomizing a result of Alon and Spencer [3] on the size of a dominating set on graphs with minimum degree d.

Proposition 19 (Alon and Spencer [3], Theorem 1.2.2). Any graph on n vertices with minimum degree d has a dominating set of size at most n(\log (d + 1) + 1)/(d + 1).

On a d-regular graph, because the size of any dominating set is at least n/(d + 1), the approximation ratio achieved is log (d + 1) + 1.

Now we outline the proof of the above proposition to show how it can be derandomized. Consider a d-regular graph G on n vertices. Picking each vertex of G with probability
Theorem 20. There is an algorithm which takes as input a $d$-regular graph $G$ on $n$ vertices, and computes a $(\log (d+1)+1)$-approximate minimum dominating set for $G$. The algorithm runs in polynomial time and uses $O(\log n)$ bits of space.

5 Independent Sets by Randomization

In this section, we consider the Independent Set problem restricted to graphs with bounded average degree. On general graphs, the problem is unlikely to have a non-trivial (factor-$n^{1-\epsilon}$) approximation algorithm [22]. However, if the graph has average degree $d$, then an independent set satisfying the bound of the next lemma is a $(2d)$-approximate solution. Note that graphs of bounded average degree encompass planar graphs and graphs of bounded degeneracy. It is also known $2d$ is the best approximation ratio possible up to polylogarithmic factors in $d$ [4, 14].
Proposition 21 (Alon and Spencer [3], Theorem 3.2.1). If a graph on \( n \) vertices has average degree \( d \), then it has an independent set of size at least \( n/(2d) \).

In what follows, we develop a logarithmic-space procedure that achieves the above bound. Let \( G = (V, E) \) be a graph on \( n \) vertices with average degree \( d \). Consider a set \( S \subseteq V \) obtained by picking each vertex in \( V \) independently with probability \( p = 1/d \). Let \( m_S \) be the number of edges with both endpoints in \( S \). The following bound appears as an intermediate claim in the proof of Proposition 21 (see Alon and Spencer [3], Theorem 3.2.1). We use it here without proof.

Lemma 22. \( \mathbb{E}[|S| - m_S] = \frac{n}{2d} \).

Consider the set \( I \) obtained by arbitrarily eliminating an endpoint of each edge in \( G[S] \). Observe that \( G[I] \) has no edges, i.e. \( I \) is an independent set whose expected size is \( \mathbb{E}[|S| - m_S] = n/(2d) \).

Derandomizing this sampling procedure is simple: simply run through the functions of a 2-universal hash family \( \mathcal{F} \) for \([n] \to [d]\) and for each \( f \in \mathcal{F} \), pick a vertex \( v \in V \) into \( S \) if and only if \( f(v) = 1 \). Because the range of the functions is \([d]\), the sampling probability is \( P(v \in S) = 1/d \). Recall that Lemma 22 only requires the sampling procedure to be pairwise independent, so the expectation bound remains the same: \( \mathbb{E}[|S| - m_S] = n/(2d) \).

While going through \( \mathcal{F} \), select the function \( f \in \mathcal{F} \) which maximizes \( |S| - m_S \), where \( S = \{v \in V \mid f(v) = 1\} \) and \( m_S \) is the number of edges \( uv \in E \) with \( f(u) = f(v) = 1 \). Using the construction of Proposition 4, this step can be performed in polynomial time using \( O(\log n) \) bits of space and \( f \) can be used as an oracle for \( S \) at the same space cost.

The next step, in which vertices are deleted arbitrarily from each pair of adjacent vertices in the sample \( S \), is tricky to carry out in small space. This is because for any edge \( uv \) in \( G[S] \), it is not possible to determine whether either of the endpoints survive the deletion procedure without additional information about the other edges incident with \( u \) and \( v \). However, there is a simple fix for this: retain only those vertices in \( S \) which are the smallest vertices in their neighbourhoods in \( G[S] \). Using this, we prove the following lemma and the subsequent theorem, as a direct consequence.

Lemma 23. Let \( T \) be the set of vertices \( v \in S \) such that \( v \) is the smallest vertex (in the original arbitrary labelling) in its neighbourhood in \( G[S] \). The set \( T \) is independent in \( G \), has size \( |T| \geq |S| - m_S \), and one can enumerate it in polynomial time using \( O(\log n) \) bits of space.

Proof. Determining if \( v \in S \) is the smallest vertex in its neighbourhood in \( G[S] \) involves enumerating the neighbourhood of \( v \) in the induced subgraph \( G[S] \) which can be performed in polynomial time using \( O(\log n) \) bits of additional space. As we pick only one vertex from each neighbourhood, the set \( T \) picked is independent and it is trivial to see that the overall procedure is polynomial-time and uses \( O(\log n) \) bits of space.

Let \( C_1, \ldots, C_t \) be the connected components of \( G[S] \). Consider the difference between the number of vertices and the number of edges in each component. Any component with \( l \) vertices contains at least \( l - 1 \) edges. For \( i \in [t] \), denote by \( n_i \) the number of vertices in \( C_i \) and by \( m_i \), the number of edges. We have \( \sum_{i=1}^{t} (n_i - 1) \leq \sum_{i=1}^{t} m_i \), i.e. \( \sum_{i=1}^{t} n_i - t \leq \sum_{i=1}^{t} m_i = m_S \), which implies that \( t \geq n - m_S \).

As we pick at least one vertex (the smallest vertex) from each component in \( T \), we have \( |T| \geq t \geq n - m \). Since
Theorem 24. There is an algorithm which takes as input a graph $G$ on $n$ vertices with average degree $d$, and computes a $(2d)$-approximate maximum independent set in $G$. The algorithm runs in polynomial time and uses $O(\log n)$ bits of space.

6 Conclusion

We devised space efficient approximation algorithms for $d$-Hitting Set (and its restriction Vertex Cover), Independent Set and Dominating Set in some special classes of graphs.

We consider our contribution as simply drawing attention to a direction in the study of approximation algorithms, and believe that it should be possible to improve the approximation ratios and the space used for the problems considered here. Obtaining a constant-factor or even factor-$O(\log n)$ approximation algorithm for Vertex Cover and a factor-$O(\log n)$ approximation algorithm for Dominating Set on general graphs using $O(\log n)$ bits of space are some specific open problems of interest.

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