Completeness of the trajectories of particles coupled to a general force field

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Abstract

We analyze the extendability of the solutions to a certain second order differential equation on a Riemannian manifold \((M, g)\), which is defined by a general class of forces (both prescribed on \(M\) or depending on the velocity). The results include the general time-dependent anholonomic case, and further refinements for autonomous systems or forces derived from a potential are obtained. These extend classical results for Lagrangian and Hamiltonian systems. Several examples show the optimality of the assumptions as well as the utility of the results, including an application to relativistic pp-waves.

Key Words: Dynamics of classical particles, autonomous and non-autonomous systems, second order differential equation on a Riemannian manifold, completeness of inextensible trajectories.

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1 Introduction

Completeness is an essential property for the curves which are extremal of some classical Lagrangian fields or, with more generality, which solve the differential equation satisfied by the trajectories of the particles accelerated by different types of forces on
a Riemannian manifold \((M, g)\). Its interest appears both from the geometrical and the mechanical point of view. Recall that the simplest case of the geodesics of \((M, g)\) allows to understand better the global structure of the manifold. If one assumes that \((M, g)\) is (geodesically) complete, the incompleteness of the trajectories suggests that an infinite amount of energy will be consumed by the accelerating forces. A priori, this is an undesirable property for a mechanical system, and can be used to disregard it as a physically realistic model. Nevertheless, in some cases the equation of such trajectories may have nice physical interpretations. For instance, sometimes the equation of the accelerated trajectories may be equivalent to the equation of the geodesics of a Lorentzian manifold (see, for example, [5], [17], [18] or the review [7]). So, the incompleteness of such trajectories may be connected to the lightlike or timelike incompleteness of some physically reasonable spacetimes, and therefore, it can be related to the celebrated relativistic theory of singularities. In this paper, we are providing several general criteria which ensure the completeness of a wide class of trajectories of accelerated particles in a Riemannian setting, being its optimality discussed by means of several examples. Essentially, this topic has remained dormant since the results in the seventies. So, we use a simple approach and language, which makes apparent the unsolved questions in that epoch, and possible lines of future research are also pointed out.

1.1 Setting

More precisely, let \((M, g)\) be a (connected, finite–dimensional) Riemannian manifold and denote by \(\pi : M \times \mathbb{R} \rightarrow M\) the natural projection. Giving a \((1,1)\) smooth tensor field \(F\) along \(\pi\) and a smooth vector field \(X\) along \(\pi\), let us consider the second order differential equation

\[
\frac{D\dot{\gamma}}{dt}(t) = F(\gamma(t), t) \dot{\gamma}(t) + X(\gamma(t), t),
\]

(E)

where \(D/dt\) denotes the covariant derivative along \(\gamma\) induced by the Levi–Civita connection of \(g\) and \(\dot{\gamma}\) represents the velocity field along \(\gamma\). Observe that (E) describes the dynamics of a classical particle under the action of a force field \(F\), which linearly depends on its velocity, and an external force field \(X\), which is independent of the motion of the particle. In the case when both \(F\) and \(X\) are time independent, the previous equation reads

\[
\frac{D\dot{\gamma}}{dt}(t) = F(\gamma(t)) \dot{\gamma}(t) + X(\gamma(t))
\]

\((E_0)\)

and is called the autonomous equation, using the term non–autonomous equation if at least one between \(F\) and \(X\) is time dependent.

Taking \(p \in M\) and \(v \in T_p M\), there exists a unique inextensible smooth curve \(\gamma : I \rightarrow M\), \(0 \in I\), solution of (E) which satisfies the initial conditions

\[
\gamma(0) = p, \quad \dot{\gamma}(0) = v.
\]

Such a curve is called complete if \(I = \mathbb{R}\) and forward (resp. backward) complete when \(I = (a, b)\) with \(b = +\infty\) (resp. with \(a = -\infty\)). As far as we know, only the (holonomic)
case when \( F = 0 \) and \( X \) comes from the gradient of a potential function \( V \), has been systematically studied in the literature (see [1], [12], [20]). Even more, accurate results have been stated only for a time–independent potential (see [1, Chapter 3]), being the results for the non–autonomous case rather vague (see [12]). Our study will cover all the previous cases, specially the anholonomic and time–dependent ones.

1.2 Interpretations

For the interpretation of \( F \), recall that it can be decomposed as

\[
F = S + H,
\]

where \( S \) is the self–adjoint part of \( F \) with respect to \( g \), and \( H \) the skew–adjoint one. On one hand, the bound of the eigenvalues of \( S \) (which may vary with \((p,t) \in M \times \mathbb{R}\)) becomes natural to ensure that the \( F \)–forces will not carry out an infinite work in a finite time. Frictional forces are typically proportional to the velocity and opposed to it, so, they can be described by means of some \( S \) with non–positive eigenvalues. On the other hand, magnetic fields may be classically described by the skew–adjoint part \( H \) (see [14]).

In this paper our approach differs from the previous ones in [1], [12] where Lagrangian or Hamiltonian techniques are used. In fact, we focus directly on the interpretation of the velocity of each trajectory for equation (E) as an integral curve of a certain vector field \( G \) (second order equation) on \( TM \times \mathbb{R} \). This is carried out first in the autonomous case (Section 2), where the vector field can be redefined just on \( TM \), extending the well–known geodesic vector field in Riemannian geometry (or the Lagrangian vector field for regular Lagrangians). In the non–autonomous case (Section 3), we show how the results and techniques of the autonomous case can be adapted to the vector field \( G \) on \( TM \times \mathbb{R} \). Even though this possibility was pointed out by Gordon [12] in the framework of Hamiltonian systems, our approach is more direct and accurate. In both cases (autonomous and non–autonomous), we include a special study of the case when the force vector field can be derived from a potential.

1.3 Statement of the main results

Some notions are needed to state our results. Put in the time–independent case

\[
S_{\text{sup}} := \sup_{v \in TM \atop \|v\| = 1} g(v, Sv), \quad S_{\text{inf}} := \inf_{v \in TM \atop \|v\| = 1} g(v, Sv), \quad \|S\| := \max\{|S_{\text{sup}}|, |S_{\text{inf}}|\}.
\]

In the non–autonomous case, consider the analogous notions \( S_{\text{sup}}(t) \), \( S_{\text{inf}}(t) \), \( \|S(t)\| \) computed for each slice \( M \times \{t\} \). We say that \( S \) is bounded (resp. upper bounded; lower bounded) along finite times when, for each \( T > 0 \) there exists a constant \( N_T \) such that for all \( t \in [-T, T] \) we have

\[
\|S(t)\| < N_T \quad (\text{resp. } S_{\text{sup}}(t) < N_T; \quad S_{\text{inf}}(t) > -N_T).
\]
Moreover, let \( d \) be the distance canonically associated to the Riemannian metric \( g \). We say that (the norm of) a vector field \( X \) on \( M \) grows at most linearly in \( M \) if there exist some constants \( A, C > 0 \) such that
\[
|X|_p := \sqrt{g(X_p, X_p)} \leq A \, d(p, p_0) + C \quad \text{for all} \quad p \in M, \tag{2}
\]
for some fixed \( p_0 \in M \). With more generality, in the non–autonomous case we say that a vector field \( X \) along \( \pi \), grows at most linearly in \( M \) along finite times if for each \( T > 0 \) there exist \( p_0 \in M \) and some constants \( A_T, C_T > 0 \) such that
\[
|X|_{(p,t)} \leq A_T \, d(p, p_0) + C_T \quad \text{for all} \quad (p,t) \in M \times [-T, T]. \tag{3}
\]
Obviously, conditions (2), (3) are independent of the chosen point \( p_0 \).

Our main result can be stated as follows.

**Theorem 1.** Let \( (M, g) \) be a complete Riemannian manifold and consider a \((1,1)\) tensor field \( F \) and a vector field \( X \) both time–dependent and smooth.

If \( X \) grows at most linearly in \( M \) along finite times and the self–adjoint part \( S \) of \( F \) is bounded (resp. upper bounded; lower bounded) along finite times, then each inextensible solution of \((E)\) must be complete (resp. forward complete; backward complete).

In particular, if \( M \) is compact then any inextensible solution of \((E)\) is complete for any \( X \) and \( F \).

**Remark 1.** (1) For the comparison with previous results in the literature, recall that in Theorem[7] (i) the problem is time–dependent, (ii) \( X \) is not necessarily the gradient of a potential, and (iii) forces which depend linearly on the velocities are allowed. Interpretations for frictional and magnetic forces (Remark[4]) or applications to relativistic pp–waves (Example[2]), stress its proper applicability. The optimality of the hypotheses in Theorem[7] is discussed along the paper (see especially Example[7] and Remark[5]).

(2) The technique will suggest that a superlinear growth of the vector field \( X \) may not destroy completeness: it is only relevant the growth of the component of \( X \) in the radial component along the outside direction. Moreover, even though the forces above are always either independent or pointwise proportional to the velocity, our techniques seem adaptable to study also frameworks of higher order. These considerations could be used to give extensions of Theorem[7] which might constitute a further line of research.

The remainder of the results are obtained in the relevant case that \( X \) comes from the gradient of a potential, so that they can be compared easily with those in previous references, where Lagrangian or Hamiltonian systems were considered.

Again, we need some notions to describe our next results.

Let \( V : M \times \mathbb{R} \to \mathbb{R} \) be a smooth time–dependent potential, and emphasize as \( \nabla^M V \) the gradient of the function \( p \in M \mapsto V(p, t) \in \mathbb{R} \), for each fixed \( t \in \mathbb{R} \). A function \( U : M \times \mathbb{R} \to \mathbb{R} \) grows at most quadratically along finite times if for each \( T > 0 \) there exist \( p_0 \in M \) and some constants \( A_T, C_T > 0 \) such that
\[
U(p,t) \leq A_T \, d^2(p, p_0) + C_T \quad \text{for all} \quad (p,t) \in M \times [-T, T]. \tag{4}
\]
(again, this property is independent of the chosen \( p_0 \)). Our main result is then:
Theorem 2. Let \((M, g)\) be a complete Riemannian manifold, \(F\) a smooth time–dependent \((1, 1)\) tensor field on \(M\) with self–adjoint component \(S\) and \(V : M \times \mathbb{R} \rightarrow \mathbb{R}\) a smooth time–dependent potential.

Assume that \(S\) is bounded (resp. upper bounded; lower bounded) along finite times and \(-V\) grows at most quadratically along finite times.

If also \(|\partial V / \partial t| : M \times \mathbb{R} \rightarrow \mathbb{R}\) (resp. \(\partial V / \partial t; -\partial V / \partial t\)) grows at most quadratically along finite times, then each inextensible solution of

\[
\frac{D\gamma}{dt}(t) = F(\gamma(t), t) \dot{\gamma}(t) - \nabla^M V(\gamma(t), t)
\]

must be complete (resp. forward complete; backward complete).

Remark 2. (1) From Theorem 2 one can reobtain the conclusion (stated with more generality in Theorem 1) that the completeness of \((E^*)\) holds whenever \(M\) is compact.

(2) The proof of Theorem 2 allows us to sharpen its conclusions (see Theorem 2). When particularized to autonomous systems, Theorem 2 (and, in particular, Corollary 2) extends the results by Weinstein and Marsden in [20] and in [11] Theorem 3.7.15 (see also our discussion in Remark 5). Furthermore, in the non–autonomous case, it generalizes widely the results by Gordon [12].

(3) The estimate of the decreasing of \(V\) agrees with Theorem 1 in the sense that the norm of the gradient of the function \(-A_T^2(p, p_0) - C_T\) (say, in the open dense subset of \(M\) where it is smooth) grows linearly. However, Theorem 2 is not a consequence of Theorem 1 as \(\nabla^M V\) may grow superlinearly even when \(V\) is bounded.

(4) The optimality of the growth of \(-V\) (as the optimality of the growth of \(X\) and \(F\) in Theorem 1) is checked by simple 1–dimensional examples. The bound for \(\partial V / \partial t\) is also very general, and a relevant application of this case for \(pp\)–waves is also developed at the end of this paper. However, it is not so clear if our bound for the growth of \(\partial V / \partial t\) can be improved. Starting at Remark 8 a discussion is carried out, including the introduction of a different type of bound (see Proposition 2). This question (which has been omitted in the literature, as far as we know) may deserve to be studied specifically further.

2 The autonomous case

Along this section, \(X\) and \(F\) are regarded as tensor fields just on the (connected) manifold \(M\), so that equation (E) simplifies into \((E_0)\). An argument which extends the con-
struction of the geodesic flow in Riemannian geometry, easily shows that there exists a vector field \( G^0 \) on the tangent bundle \( TM \) such that its integral curves are precisely the velocities \( s \mapsto \dot{\gamma}(s) \) of the curves \( \gamma \) which solve equation \( (E_0) \) (see the detailed discussion for the non–autonomous case in the next section). Recall that an integral curve \( \rho \) of a vector field defined on some bounded interval \( [a, b), b < +\infty \), can be extended to \( b \) (as an integral curve) if and only if there exists a sequence \( \{t_n\}_n \), \( t_n \nearrow b \), such that \( \{\rho(t_n)\}_n \) converges (see \[16\] Lemma 1.56). The following technical result follows directly from this fact.

**Lemma 1.** Let \( \gamma : [0, b) \to M \) be a solution of equation \( (E_0) \) with \( 0 < b < +\infty \). The curve \( \gamma \) can be extended to \( b \) as a solution of \( (E_0) \) if and only if there exists a sequence \( \{t_n\}_n \subset [0, b) \) such that \( t_n \to b^- \) and the sequence of velocities \( \{\dot{\gamma}(t_n)\}_n \) is convergent in \( TM \).

**Remark 3.** Here, we deal with finite–dimensional Riemannian manifolds. So, in order to apply Lemma \[7\] to the completeness of trajectories, the local compactness of \( M \), and then of \( TM \), will be used. On the contrary, to extend the results in the present article to (infinite–dimensional) Hilbert manifolds, the standard tools require to induce a (complete) Riemannian metric on \( TM \) (see \[9\], \[2\] Supplement 9.1.C).

Furthermore, as in what follows we are going to use more than once a classical subsolution argument, we recall it here for completeness (e.g., see \[19\] Lemma 1.11).

**Lemma 2.** Taking \( f \in C^1(\mathbb{R}^2, \mathbb{R}) \), let \( w = w(t) \) be a subsolution of the differential equation
\[
\dot{u} = f(t, u) \quad \text{on} \quad [t_0, T),
\]
i.e., \( \dot{w} < f(t, w) \) on \( [t_0, T) \). Then for every solution \( u = u(t) \) of \( (5) \) such that \( w(t_0) \leq u(t_0) \) we have
\[
w(t) < u(t) \quad \text{for all} \quad t \in (t_0, T).
\]

### 2.1 General result

With the notation introduced in the previous section, we have
\[
g(v, Fv) = g(v, Sv) \quad \text{for all} \quad v \in TM.
\]

Now we state an accurate result which gives sufficient conditions for the forward/backward completeness of each inextensible solution of equation \( (E_0) \).

**Theorem 3.** Let \((M, g)\) be a complete Riemannian manifold, \( X \) a smooth time–independent vector field and \( S \) the self–adjoint component of a smooth time–independent \((1, 1)\) tensor field \( F \) on \( M \).

If \( \gamma : I \to M \) is an inextensible solution of \( (E_0) \) satisfying the following assumptions:

\( (a_1) \) There exists a constant \( c_\gamma > 0 \) such that
\[
g(\gamma(t), S\gamma(t)) \leq c_\gamma g(\gamma(t), \gamma(t)) \quad \text{for all} \quad t \in I,
\]
(a2) The pointwise norm \(|X|\) of \(X\) is at most linear on \(|\gamma|\), i.e., there exists \(r_\gamma > 0\) such that
\[
|X|_{\gamma(t)} \leq r_\gamma (1 + |\gamma(t)|) \quad \text{for all } t \in I,
\]
where \(|\gamma(t)| := d(\gamma(t), p_0)\) is the Riemannian distance to some fixed point \(p_0 \in M\),
then, \(\gamma\) must be forward complete.

Analogously, if the inextensible solution \(\gamma\) satisfies \((a_2)\) while condition \((a_1)\) is replaced by the assumption
\[(a'_1)\text{ there exists a constant } c_\gamma > 0 \text{ such that}\]
\[-g(\gamma(t), S\gamma(t)) \leq c_\gamma g(\gamma(t), \gamma(t)) \quad \text{for all } t \in I,\]
then \(\gamma\) must be backward complete.

Proof. Without loss of generality, let \(I = [0, b), 0 < b < +\infty\), be the domain of a forward–inextensible solution \(\gamma\) of (E0), and put \(p_0 = \gamma(0)\), so that \(|\gamma(t)| = d(\gamma(t), \gamma(0))\). Writing
\[
u(t) := g(\gamma(t), \dot{\gamma}(t)),
\]
(7)
it is enough to prove that a constant \(k > 0\) exists such that
\[
u(t) \leq k \quad \text{for all } t \in [0, b).
\]
(8)
In fact, (8) implies that \(\dot{\gamma}(L)\) is bounded in \(TM\) and, being \((M, g)\) complete, Lemma 1 is applicable because of the compactness of the bounded metric balls in \(M\). Hence, \(\gamma\) can be extended to \(b\) in contradiction with its maximality assumption.

With the aim to prove (8), for any \(t \in [0, b)\) equation (E0) implies
\[
\dot{u}(t) = 2g(\dot{\gamma}(t), F\dot{\gamma}(t)) + 2g(\dot{\gamma}(t), X\dot{\gamma}(t))
= 2g(\gamma(t), S\gamma(t)) + 2g(\gamma(t), X\gamma(t)),
\]
and therefore, assumptions \((a_1)\) and \((a_2)\) give
\[
\dot{u}(t) \leq 2c_\gamma g(\gamma(t), \gamma(t)) + 2r_\gamma (1 + |\gamma(t)|)\sqrt{g(\gamma(t), \gamma(t))}
\leq 2c_\gamma g(\gamma(t), \gamma(t)) + r_\gamma^2 (1 + |\gamma(t)|)^2 + g(\gamma(t), \gamma(t))
\leq (2c_\gamma + 1)u(t) + 2r_\gamma^2 + 2r_\gamma^2 |\gamma(t)|^2.
\]
(9)
Thus, taking into account that
\[
|\gamma(t)|^2 \leq \left(\int_0^t \sqrt{u(s)}ds\right)^2 \leq b \int_0^t u(s)ds,
\]
and putting \(v(t) = \int_0^t u(s)ds\), inequality (9) yields
\[
\ddot{v} < k_1 \dot{v} + k_2 v + k_3
\]
(10)
for some constants $k_1, k_2, k_3 > 0$. Recall that any solution $\nu_0$ of the linear ordinary differential equation obtained by replacing the inequality in (10) by equality, is a $C^2$ function defined on all $\mathbb{R}$. Now, choosing $\nu_0$ such that $\nu_0(0) = \nu(0) = 0$, $\nu_0(0) = u(0)$, applying twice Lemma 2 we have $\nu < v_0$, $\dot{\nu} < \dot{v}_0$ on all $(0, b)$ and, so, (9) holds with $k = \max\{\nu_0(t) : t \in [0, b]\}$.

Vice versa, let $\gamma : (-b, 0] \to M$ be a backward–inextensible solution of (E). Then, the reparametrization $\gamma' : t \in [0, b) \mapsto \gamma'(t) = \gamma(-t) \in M$ is a forward–inextensible solution of

$$\frac{D\gamma'}{dt}(t) = (-F)\gamma'(t)\gamma'(t) + X\gamma'(t) \quad \text{in} \quad [0, b).$$

From (a'$_1$) it follows that $-F$ satisfies (a$_1$) along $\gamma'$; whence $\gamma'$ must be forward complete, that is, $\gamma$ is backward complete.

Remark 4. (1) Assumption (a$_1$) in the previous result means that $\lambda^+(t)$, the biggest eigenvalue of the operator $S$ along $\gamma$, is upper bounded for $t \in I$. Notice that the speed of $\gamma$ increases maximally when each $\dot{\gamma}(t)$ lies in the $\lambda^+(t)$–eigenspace. The upper boundedness of $\lambda^+(t)$ implies that, even though its speed may increase linearly, the curve $\gamma$ cannot cover an infinite length in a finite time and, so, the trajectory is forward complete. Recall that only upper boundedness is relevant here. In fact, frictional forces are proportional to the velocity (at least as a first approximation) and opposed to it (i.e., with negative eigenvalues). So, they make the speed to decrease and, thus, the obtained results agree with the expectation that the trajectory is defined for arbitrarily big times.

(2) Theorem 3 also implies that, on any complete Riemannian manifold $(M, g)$, each inextensible solution of equation (E) must be complete if $F$ is assumed to be skew–adjoint and $|X|$ is bounded. Such a result applies to magnetic fields both, in a non–relativistic setting and in the relativistic one (when a suitable static vector field exists). In fact, it extends widely [3, Corollary 2.4], which was stated for pure magnetics fields (i.e., $F$ is skew–adjoint and $X \equiv 0$) on $(M, g)$. Recall that any magnetic trajectory $\gamma$ satisfies the conservation law $g(\dot{\gamma}(t), \dot{\gamma}(t)) = \text{constant}$ (depending on $\gamma$). This crucial property is used to get [3, Corollary 2.4], but it does not hold for electric forces or other forces allowed by (E).

A direct consequence of Theorem 3 is the following result.

Corollary 1. Let $(M, g)$ be a complete Riemannian manifold, $X$ a smooth time–independent vector field and $S$ the self–adjoint component of a smooth time–independent $(1, 1)$ tensor field $F$ on $M$. If $X$ grows at most linearly in $M$ and $\|S\|$ is bounded, then all the inextensible solutions of (E) are complete. In particular, this result holds if $M$ is compact.

Example 1. The optimal character of the bounds in Theorem 3 can be checked just taking $M = \mathbb{R}$, $g = dx^2$.

(1) Optimality of the bound for $|X|$. Put $F \equiv 0$ and $X(x) = \mu(x) \frac{d}{dx}$ where $\mu(x) = (1 + \varepsilon)x^{1+2\varepsilon}$ for all $x \geq 1$ and some prescribed $\varepsilon > 0$. Thus, in the region where $x(t) \geq 1$ equation (E) reduces to

$$\ddot{x}(t) = (1 + \varepsilon)x^{1+2\varepsilon}(t). \quad (11)$$
Multiplying by $\dot{x}$, integrating with respect to $t$ and considering the initial data $x(0) = 1, \dot{x}(0) = 0$, the solution of equation (11) satisfies
\[ \dot{x}^2(t) = x^{2(1+\varepsilon)}(t) - 1. \]

The solution of this Cauchy problem is the inverse of
\[ t(x) = \int_1^x \frac{d\sigma}{\sqrt{\sigma^{2(1+\varepsilon)} - 1}}, \]
which is defined for $t \in [0, b)$, with the maximum $b$ equal to $\lim_{x \to +\infty} t(x)$ in (12). So, (11) is incomplete whenever the power of the growth of $|X|$ becomes bigger than the permitted linear one.

(2) **Optimality of the bound for $\|F\|$**. For some $\varepsilon > 0$, put $F \frac{d}{dx} = \nu(\varepsilon) \frac{d}{dx}$ with $\nu(\varepsilon) = (1+\varepsilon)x^\varepsilon$ for $x \geq 1$ and $X \equiv 0$. Equation (E0) reduces to
\[ \ddot{x}(t) = (1+\varepsilon)x^\varepsilon(t) \dot{x}(t) \]
whenever $x(t) \geq 1$. Thus, integrating with respect to $t$ and considering the initial data $x(0) = \dot{x}(0) = 1$, the solution of equation (13) satisfies
\[ \dot{x}(t) = x^{1+\varepsilon}(t); \]
whence, the solution of this Cauchy problem in this region is the inverse of
\[ t(x) = \int_1^x \frac{d\sigma}{\sigma^{1+\varepsilon}} \]
which shows the incompleteness of $x(t)$, as $\lim_{x \to +\infty} t(x) < +\infty$. That is, incompleteness appears when $\|F\|$ is not bounded, even under slow power growth.

(3) **Role of the bound on $\|F\|$ for forward/backward completeness**. Put now $X \equiv 0$ and $F \frac{d}{dx} = -\mu(x) \frac{d}{dx}$ where $\mu$ is defined on all $\mathbb{R}$ and $\mu(x) = |x|$ for $|x| > 1$. Equation (E0) reduces to
\[ \ddot{x}(t) = -|x(t)| \dot{x}(t) \quad \text{whenever} \quad |x(t)| > 1. \]
As $F$ is self–adjoint and satisfies the hypothesis (a1) of Theorem [3], then the solutions of (14) are forward complete. However, $x(t) = \frac{2}{\sqrt{t}}, \ t \in (-1, 0]$, yields a backward inextensible and incomplete solution of (14). Let us remark that $F$ may represent a frictional force (increasing with $|x|$ in an inhomogeneous medium), and the backward incompleteness implies the divergence (with the lapse of time) of the energy necessary to overcome such a force.

### 2.2 Trajectories under an autonomous potential

Throughout this section we assume $X = -\nabla V$. If $F \equiv 0$, the completeness of inextensible solutions of equation (E0) has been studied in [12] Theorem 2.1(ii) if $V$ is bounded from below while in [20] if $V$ is unbounded from below (see also [1] Theorem 3.7.15). Here, we generalize such results by including also the action of a (1,1) tensor field $F$. In order to investigate the completeness of equation (E0), let us recall the following comparison result (see [1] Example 3.2.H or [6]).
Lemma 3 (Comparison Lemma). Let \( \varphi : [0, +\infty) \to \mathbb{R} \) be a locally Lipschitz monotone increasing function such that
\[
\varphi(s) > 0 \quad \text{for all} \quad s \geq 0 \quad \text{and} \quad \int_0^{+\infty} \frac{ds}{\varphi(s)} = +\infty.
\]
If an inextensible \( C^1 \) function \( f = f(t) \) is such that
\[
\frac{df}{dt}(t) = \varphi(f(t)) \quad \text{with} \quad f(0) \geq 0,
\]
then it is defined for all \( t \geq 0 \).

Furthermore, if \( h : [0, b) \to \mathbb{R} \) is a continuous function such that \( h(t) \geq 0 \) for all \( t \in [0, b) \) and
\[
\left\{ \begin{array}{l}
  h(t) \leq h(0) + \int_0^t \varphi(h(s)) ds \quad \text{for all} \quad t \in [0, b), \\
  h(0) \leq f(0),
\end{array} \right.
\]
then \( h(t) \leq f(t) \) for all \( t \in [0, b) \).

According to \([20]\) (see also \([1]\), Definition 3.7.14) a function \( V_0 : [0, +\infty) \to \mathbb{R} \) is called positively complete, if it is \( C^1 \), non-increasing and satisfies
\[
\int_0^{+\infty} \frac{ds}{\sqrt{\alpha - V_0(s)}} = +\infty, \tag{15}
\]
where \( \alpha \) is a constant such that \( \alpha > V_0(0) \), hence \( \alpha > V_0(s) \) for all \( s \in [0, +\infty) \). It is easy to see that this condition is independent of which \( \alpha \) is chosen.

Remark 5. (1) If \( V_0 \) is a positively complete function and \( \tilde{V}_0 \) is a non-increasing \( C^1 \) function such that \( \tilde{V}_0(0) = V_0(0) \) and \( \tilde{V}_0 \geq V_0 \) then \( \tilde{V}_0 \) is also positively complete. So, a positively complete function \( V_0 \) will be interesting when \( \lim_{s \to +\infty} \tilde{V}_0(s) \) goes to \(-\infty\) as fast as possible. Therefore, the relevant limit equivalent to \((15)\) for any non-increasing function on \([0, +\infty)\) is \( \int_0^{+\infty} ds / \sqrt{-\tilde{V}_0(s)} = +\infty \), where \( s_0 \) is any point with \( \tilde{V}_0(s_0) < 0 \).

(2) In particular, the function
\[
V_0(s) = -R_0 s^2 \quad \text{where} \quad R_0 > 0,
\]
is positively complete. Thus, so are, for example, \( V_0(s) = -s^\beta \log^{\alpha} (1+s) \) for any \( \beta \in [0, 2) \) and any \( \alpha > 0 \), as they decrease less fast than \(-s^2\). Anyway, choosing \( \beta = 2 \) and \( 0 < \alpha \leq 2 \) or, with more generality, if \( V_0(s) \) is a \( C^1 \) non-increasing function on \([0, +\infty)\) such that
\[
V_0(s) \approx -s^2 (\log s)^2 \underbrace{\ldots (\log \log \ldots \log s)^2}_{k\text{-times}} \quad \text{if} \quad s \to +\infty,
\]
for any \( k \geq 1 \), one also finds positively complete functions which decrease (slowly) faster than quadratically. Consistently with references \([7]\), \([2]\), our results here will
be stated by using positive completeness. With more generality, one could replace the hypothesis such as the at most linear behavior in Subsection 2.1 by a more general (but technical) assumption – nevertheless, we prefer not to do this for the sake of simplicity.

(3) Notice that functions such as $V_0(s) = -R_0 s^\beta$ are not positively complete when $\beta > 2$. This agrees with the fact that, for such a type of functions, $|\nabla V_0|$ is at most linear if and only if $\beta \leq 2$. So, even though our overall linear bound for $|X|$ was the optimal power growth, some further results will be obtained next.

**Theorem 4.** Let $(M, g)$ be a complete Riemannian manifold, $V$ a smooth potential on $M$ and $S$ the self–adjoint component of a smooth time–independent $(1, 1)$ tensor field $F$. Let $\gamma : I \rightarrow M$ be an inextensible solution of

$$\frac{D\dot{\gamma}}{dt}(t) = F(\gamma(t)) \dot{\gamma}(t) - \nabla V(\gamma(t)).$$

(E1')

If $S$ satisfies condition $(a_1)$ (resp. $(a'_1)$) in Theorem 3 and

$(a_3)$ a positively complete function $\gamma_0$ exists such that

$$V(\gamma(t)) \geq \gamma_0(d(\gamma(t), p_0)) \text{ for all } t \in I,$$

for some $p_0 \in M$.

then $\gamma$ is forward (resp. backward) complete.

**Proof.** As in the proof of Theorem 3 let $I = [0, b)$ and, by contradiction, assume $0 < b < +\infty$. Introducing again the squared norm $u(t)$ (see (7)), it is enough to prove that inequality (8) holds for some constant $k > 0$ (boundedness of $u$).

On one hand, by equations (E1'), (6) and assumption $(a_1)$, for all $s \in [0, b)$ we have

$$\frac{d}{ds} \left( \frac{1}{2} u(s) + V \circ \gamma(s) \right) = g(\dot{\gamma}(s), S \dot{\gamma}(s)) \leq c_\gamma u(s).$$

(16)

Hence, taking

$$v(t) = \int_0^t u(s) ds, \text{ and thus } \dot{v}(t) = u(t), \; v(0) = 0,$$

and integrating (16) on $[0, t], \; t \in [0, b)$, we have

$$\dot{v}(t) - 2c_\gamma v(t) \leq 2(\alpha_\gamma - V(\gamma(t))) \text{ for all } t \in [0, b),$$

(17)

with $\alpha_\gamma = \frac{1}{2} u(0) + V(\gamma(0))$. On the other hand, we get

$$d(\gamma(t), p_0) \leq d(\gamma(0), p_0) + d(\gamma(t), \gamma(0)) \leq l_\gamma(t) \text{ for all } t \in [0, b),$$

(18)

where we have put

$$l_\gamma(t) = d(\gamma(0), p_0) + \int_0^t \sqrt{g(\gamma(s), \gamma(s))} ds.$$
Thus, from (17), assumption (a3) and (18), for all \( t \in [0,b) \) we obtain

\[
v(t) - 2c_\gamma v(t) \leq 2(\alpha - \gamma_0(l_\gamma(t))) \leq 2(\alpha - \gamma_0(l_\gamma(t)));
\]

the last inequality as \( \gamma_0 \) is non-increasing. This property also assures that, taking \( \alpha > \max\{\alpha, \gamma_0(0)\} \), the function \( v = v(t) \) satisfies

\[
v(t) - 2c_\gamma v(t) < 2(\alpha - \gamma_0(l_\gamma(t))) \quad \text{for all } t \in [0,b),
\]

and the right–hand side of the inequality is positive.

Now, let \( v_0 = v_0(t) \) be the solution of the associated equality

\[
v_0(t) - 2c_\gamma v_0(t) = 2(\alpha - \gamma_0(l_\gamma(t)));
\]

with initial condition \( v_0(0) = 0 \); explicitly:

\[
v_0(t) = 2e^{2c_\gamma t} \int_0^t e^{-2c_\gamma s} (\alpha - \gamma_0(l_\gamma(s))) ds, \quad t \in [0,b). \tag{22}
\]

From (20) it follows that \( v = v(t) \) is a subsolution of (21) with \( v(0) = v_0(0) \). Thus, applying Lemma 2 as before,

\[
v(t) < v_0(t) \quad \text{and} \quad u(t) = \dot{v}(t) < \dot{v}_0(t) \quad \text{for all } t \in (0,b).
\]

So, in order to estimate \( \dot{v}_0(t) \), let us remark that (22) implies

\[
\dot{v}_0(t) = 4c_\gamma e^{2c_\gamma t} \int_0^t e^{-2c_\gamma s} (\alpha - \gamma_0(l_\gamma(s))) ds + 2(\alpha - \gamma_0(l_\gamma(t))), \quad t \in [0,b).
\]

Choosing any \( t \in [0,b) \), the properties of \( \gamma_0 \) imply \( \gamma_0(l_\gamma(s)) \geq \gamma_0(l_\gamma(t)) \) for all \( s \in [0,t) \) and, hence,

\[
\int_0^t e^{-2c_\gamma s} (\alpha - \gamma_0(l_\gamma(s))) ds \leq \frac{1}{2c_\gamma} (1 - e^{-2c_\gamma t}) (\alpha - \gamma_0(l_\gamma(t))).
\]

This implies

\[
\dot{v}_0(t) \leq 2e^{2c_\gamma t} (1 - e^{-2c_\gamma t}) (\alpha - \gamma_0(l_\gamma(t))) + 2(\alpha - \gamma_0(l_\gamma(t)))
\]

\[
\leq k_\gamma(\alpha - \gamma_0(l_\gamma(t)))
\]

with \( k_\gamma = 2e^{2c_\gamma t} \) (note that \( \alpha - \gamma_0 \) is positive); whence,

\[
\sqrt{u(t)} \leq \sqrt{k_\gamma(\alpha - \gamma_0(l_\gamma(t))}) \quad \text{for all } t \in [0,b). \tag{23}
\]

On the other hand, from definition (19) we have

\[
\frac{dl_\gamma}{dt}(t) = \sqrt{u(t)} \quad \text{and} \quad l_\gamma(0) = d(\gamma(0),p_0).
\]
Thus, defining \( \varphi(w) = \sqrt{k_T(\alpha - \gamma(w))} \), from \((23)\) it follows
\[
\frac{dl_T}{dt}(t) \leq \varphi(l_T(t)) \quad \text{for all } t \in [0, b),
\]
which implies
\[
l_T(t) \leq l_T(0) + \int_0^t \varphi(l_T(s))ds \quad \text{for all } t \in [0, b).
\] (24)

Now, let \( f = f(t) \) be the unique inextensible solution of the Cauchy problem
\[
\begin{cases}
\frac{df}{dt}(t) = \varphi(f(t)), \\
f(0) = d(\gamma(0), p_0) (= l_T(0) \geq 0).
\end{cases}
\]
As \( \gamma_0 \) is positively complete, a direct check of the properties of \( \varphi \) implies that the first part of Lemma \(3\) applies, so \( f \) is defined for all \( t \geq 0 \). Moreover, from \((23)\) and the second part of Lemma \(3\) it follows \( l_T(t) \leq f(t) \) for all \( t \in [0, b) \). Thus, \( l_T(t) \) is bounded in \([0, b)\) and from \((23)\), so is \( u(t) \), as required – the contradiction that \( \gamma \) is extensible beyond \( b \) follows.

The case when \( \gamma : (-b, 0] \to M \) is backward incomplete follows analogously (as at the end of the proof of Theorem \(3\)).

The autonomous version of Theorem \(2\) is now a straightforward consequence of the previous theorem and the positive completeness of the function \( \gamma_0(s) = -R_0 s^2 \) discussed in Remark \(5\). Concretely:

**Corollary 2.** Let \((M, g)\) be a complete Riemannian manifold, \( S \) the self–adjoint component of a smooth time–independent \((1, 1)\) tensor field \( F \) on \( M \) and \( V : M \to \mathbb{R} \) a smooth time–independent potential. Assume that \( ||S|| < +\infty \) and \(-V\) grows at most quadratically (i.e., \(-V(p) \leq A d^2(p, p_0) + C\) in agreement with \((4)\)).

Then each inextensible solution of \((E^*_0)\) must be complete. In particular, completeness of inextensible solutions of this equation holds whenever \( M \) is compact.

### 3 The non–autonomous case

Next, the non–autonomous case \((E)\) will be reduced to the autonomous one by working on the manifold \( M \times \mathbb{R} \) (compare with the classical approach in \([8, pp. 121–124]\), for instance), and the two main theorems in the Introduction will be proven. Again, we consider first the general case. An analogous reasoning also proves the case when the external force comes from a potential, which is then widely analyzed.

#### 3.1 General result

By taking into account the standard results on existence and uniqueness of solutions to second order differential equations, for each \( (v_p, t_0) \in TM \times \mathbb{R} \) \( (p \in M, v_p \in T_p M) \) we can consider the unique inextensible solution \( \gamma_{(p, 0)} \) of \((E)\) which satisfies \( \gamma_{(p, 0)}(t_0) = p \) and \( \gamma_{(p, t_0)}(t_0) = v_p \).
Lemma 4. There exists a unique vector field $G$ on $TM \times \mathbb{R}$ such that the curves $t \mapsto (\gamma_{(p,t)}(t), t)$ are the integral curves of $G$.

Proof. Obviously, if such a $G$ exists, then it must be defined as

$$G_{(p,t)} = \frac{d^2}{dt^2} \gamma_{(p,t)}(t) + \frac{\partial}{\partial t} \gamma_{(p,t)}(t).$$

To check that its integral curves satisfy the required property, let us consider

$$\Psi : \mathcal{D} \subset \mathbb{R} \times (TM \times \mathbb{R}) \rightarrow TM \times \mathbb{R} \quad (s,(v_p,t_0)) \mapsto (\gamma_{(p,t_0)}(t_0+s), t_0+s)$$

where $\mathcal{D}$ is the maximal domain of definition of $\Psi$ in $\mathbb{R} \times (TM \times \mathbb{R})$ — recall that $\mathcal{D} \cap (\mathbb{R} \times \{(v_p,t_0)\})$ is always an open interval, which contains 0 (multiplied by $\Psi$). Clearly, $\Psi$ defines a local action on $TM \times \mathbb{R}$ (namely, $\Psi_{s+t} = \Psi_s \circ \Psi_t$ whenever it makes sense) and the result follows. $\square$

Remark 6. Alternatively to the previous lemma, the vector field $G$ may be defined locally as follows: let $(U;\xi^1,\ldots,\xi^n)$ be a coordinate neighborhood on $M$ and consider the natural coordinates $(x^1,\ldots,x^n,\hat{x}^1,\ldots,\hat{x}^n)$ on $\pi^{-1}_M(U)$, where $\pi_M$ is the projection from $TM$ onto $M$, i.e., for any $p \in U, v_p \in T^*_pM$: $x^i(v_p) \equiv x^i(p)$, $\hat{x}^i(v_p) \equiv dx^i_p(v_p)$, $1 \leq i \leq n$. On $\pi^{-1}_M(U) \times \mathbb{R}$, we have

$$G_{(p,t)} = \sum_{i=1}^n \hat{x}^i(v_p) \frac{\partial}{\partial x^i} \bigg|_{v_p} - \sum_{i=1}^n \left( \sum_{j=1}^n \Gamma_{i,j}^k(p) \hat{x}^j(v_p) \hat{x}^k(v_p) \right) \frac{\partial}{\partial \hat{x}^i} \bigg|_{v_p} \\
+ \sum_{i=1}^n \left( X^i(p,t) + \sum_{j=1}^n \hat{x}^j(v_p) F^i_j(p,t) \right) \frac{\partial}{\partial x^i} \bigg|_{v_p} + \frac{\partial}{\partial t} \bigg|_t,$$

where $\Gamma_{i,j}^k$, are the corresponding Christoffel symbols, $X^i(p,t) = dx^i_p(X_{(p,t)})$ and $F^i_j(p,t) = dx^i_p \left( F_{(p,t)} \frac{\partial}{dt} \bigg|_{p} \right)$.

Now, in order to be able to use the previous results for the autonomous case, consider the trajectories of equation (E) as “some” of the integral curves of a vector field $\hat{G}$ on the tangent bundle $T(M \times \mathbb{R})$. First, note that the time–dependent tensor fields $X$ and $F$ on $M$ naturally induce tensor fields $\hat{X},\hat{F}$ on $M := M \times \mathbb{R}$, namely:

$$\hat{X}_{(p,t_0)} = (X_{(p,t_0)}, 0) \equiv X_{(p,t_0)},$$

$$\hat{F}_{(p,t_0)} \left( v_p, s \frac{dt}{dt} \bigg|_{t_0} \right) = \left( F_{(p,t_0)}(v_p), 0 \right) \equiv F_{(p,t_0)}(v_p).$$

(25)

Now, consider also the natural product Riemannian metric $\tilde{g} = \oplus dt^2$ on $M \times \mathbb{R}$, and denote by $\tilde{D}/dt$ the corresponding covariant derivative.
Proof of Theorem 2. We will modify directly the proof of Theorem 4.

Proof of Theorem 1. Trajectories for \( \tilde{\gamma}(t) = (\tilde{\gamma}(t), \tau(t)) \) on \( M \times \mathbb{R} \) solves
\[
\frac{D\tilde{\gamma}}{dt}(t) = \tilde{F}_{\tilde{\gamma}(t)} \tilde{\gamma}(t) + \tilde{X}_{\tilde{\gamma}(t)}, \quad (E_0)
\]
if and only if \( \gamma \) solves \( (E) \) and \( \tau(t) = at + b \) for some \( a, b \in \mathbb{R} \).

Therefore, if the inextensible solutions \( \tilde{\gamma} \) of \( (E_0) \) are complete, then so are the trajectories for the time-dependent equation \( (E) \).

Proof. The first assertion follows from \( \frac{D\tilde{\gamma}}{dt} = (\tilde{D} \tilde{\gamma}) + \tilde{\tau} \) and formulae (25). For the last one, if \( \gamma \) is any inextensible solution of equation \( (E) \), the corresponding curve \( \tilde{\gamma}(t) = (\tilde{\gamma}(t), t) \) is an inextensible solution of \( (E_0) \); by assumption, \( \tilde{\gamma} \) is complete and so is \( \gamma \).

Now, we are in position to prove Theorem 1.

Proof of Theorem 2. If a trajectory \( \gamma : [0, b) \to M \) were forward inextensible with \( 0 < b < +\infty \), then it should lie in a region such as \( M \times [-T, T] \) with \( b < T \). By Proposition 1 a contradiction will follow by proving the extendability to \( b \) of the trajectory \( \gamma(t) = (\gamma(t), t) \) for \( M, \tilde{g}, \tilde{X}, \tilde{F} \). Recall that in the region \( M \times [-T, T] \), the \( g \)-bounds (3) for \( X \) and (1) for \( S \) are equal to their counterparts (2) for \( \tilde{X} \) and the boundedness of the self-adjoint part of \( \tilde{F} \) with respect to \( \tilde{g} \) (recall that the \( \tilde{g} \)-distance can be easily bounded in terms of \( g \) and \( dt^2 \), and the distance on the \( dt^2 \) side is bounded by \( 2T \)). Thus, the forward extendability of \( \tilde{\gamma} \) follows from Theorem 3 as required. Similar arguments apply for proving the backward completeness case.

3.2 Trajectories under a non-autonomous potential

Throughout this subsection, the non-autonomous problem \( (E^*) \) is studied.

Remark 7. As a difference with Theorem 1 now the main result (Theorem 2) will not be reduced directly to the autonomous case. The reason is that if we consider \( \tilde{M}, \tilde{g}, \tilde{F} \) as in the previous subsection, and put \( \tilde{V} : M \times \mathbb{R} \to \mathbb{R} \) simply equal to \( V \), then \( \tilde{\nabla} \tilde{V} = (\nabla^M V, \partial V / \partial t) \). That is, the component \( \partial V / \partial t \) makes intrinsically different the trajectories for \( \tilde{\nabla} \tilde{V} \) and \( \nabla^M V \) (an analogous to Proposition 1 does not hold). Instead, we will modify directly the proof of Theorem 2.

Proof of Theorem 2. Let \( \gamma : [0, b) \to M \) be a forward inextensible solution with \( 0 < b < +\infty \) included in some region \( M \times [-T, T] \) with \( b < T \), and let \( N_T \) be as in (11) and \( A_T, C_T \) be the constants determined by the allowed growth of \( U = \partial V / \partial t \) in (14). Thus, considering the steps of the proof of Theorem 4 from (18) and (19) we have
\[
\frac{d}{ds} \left( \frac{1}{2} u(s) + V(\gamma(s), s) \right) \leq N_T u(s) + \frac{\partial V}{\partial s}(\gamma(s), s) \\
\leq N_T u(s) + A_T d^2(\gamma(s), s_0) + C_T \\
\leq N_T u(s) + A_T d^2(\gamma(s), s) + C_T. \tag{26}
\]
As
\[
\int_0^t I_γ^2(s)ds \leq T I_γ^2(t) \quad \text{for all } t \in [0,b),
\]
integrating (26) we have that equation (17) changes into:
\[
v(t) - 2N_T v(t) \leq 2(\alpha_T - V(\gamma(t), t)) + 2TA_T I_γ^2(t)
\]
for all \( t \in [0,b) \), with \( \alpha_T = \frac{1}{2}u(0) + V(\gamma(0), 0) + TC_T \). Taking into account also the at most quadratic bound for \(-V\), we can choose constants \( A, C > 0 \) and construct the function \( \gamma_0(s) = -Ax^2 - C \) so that (27) yields:
\[
v(t) - 2N_T v(t) < 2(\alpha - \gamma_0(\gamma(t))) \quad \text{for all } t \in [0,b),
\]
where \( \alpha > \max\{\alpha_T, \gamma_0(0)\} \). Equation (28) is formally equal to (20) in Theorem 4. So, reasoning as in the proof of Theorem 4 we show the extendability of \( \gamma \) through \( b \), a contradiction.

Vice versa, if \( \gamma: (-b, 0] \rightarrow M \) is a backward inextendible solution with \( 0 < b < +\infty \), we can consider \( T > b \) and \( \tilde{\gamma}(t) := \gamma(-t) \) in \( [0, b) \) and, as from the lower boundedness of \( S \) and the quadratic growth of \( -\frac{\partial V}{\partial t} \) along finite times it follows
\[
\frac{d}{ds}\left( \frac{1}{2} u(-s) + V(\gamma(-s), -s) \right) = -g \left( S(\gamma(-s), -s)\gamma(-s), \gamma(-s) \right)
\]
\[
- \frac{\partial V}{\partial s}(\gamma(-s), -s)
\]
\[
\leq N_T u(-s) + A_T d^2(\gamma(-s), p_0) + C_T,
\]
we repeat the proof with (27) stated for \( \tilde{\gamma}(t) \).

Remark 8. The maximum allowed growth permitted for \( \partial V / \partial t \) in Theorem 2 is both, very general and consequent with our other hypotheses. However, checking the proofs, other possible bounds could be taken into account. In order to discuss the accuracy of our bound for \( \partial V / \partial t \), next: (a) we will compare Theorem 2 with the consequences of Theorem 1 for potentials, and (b) we introduce an alternative bound on \( \partial V / \partial t \) applicable when \( V \) is lower bounded along finite times. This suggests that, even though the optimality of all the other bounds have been carefully checked previously, the optimal bounds for \( \partial V / \partial t \) can be studied further.

According to the claim (a) in Remark 8 the application of Theorem 1 for the case of potentials yields:

Corollary 3. Let \( (M, g) \) be a complete Riemannian manifold, consider a \( (1, 1) \) tensor field \( F \), eventually time–dependent, with self–adjoint component \( S \), and let \( V: M \times \mathbb{R} \rightarrow \mathbb{R} \) be a smooth potential. If \( S \) is bounded along finite times and \( \nabla^M V(p, t) \) grows at most linearly in \( M \) along finite times, then each inextensible solution of \( (E^1) \) must be complete.

Relation between Corollary 3 and Theorem 2. Choose \( p_0 \in M \) and, by using the completeness of \( g \), take a starshaped domain \( \mathcal{D} \subset T_{p_0} M \) so that the exponential map \( \exp_{p_0} : \)
$\mathcal{D} \to M$ is a diffeomorphism onto its image $\exp_{p_0}(\mathcal{D})$, and this image is dense in $M$.

Under the hypotheses of Corollary 3 let $A, C : [0, +\infty) \to [0, +\infty)$ be strictly increasing $C^1$ functions which satisfy $A(n) > A_{n+1}$, $C(n) > C_{n+1}$ where $A_{n+1}, C_{n+1}$ are the constants $A_T, C_T$ obtained in (3) (from the at most linear growth of $\nabla^M V$) for $T = n+1$, and $n$ is any nonnegative integer. Now, for each $p \in \exp_{p_0}(\mathcal{D})$ let $\gamma_p : [0, l_p] \to \exp_{p_0}(\mathcal{D})$ be the unique unit geodesic from $p_0$ to $p$.

Clearly,

$$-(V(p, l_p) - V(p_0, 0)) = - \int_0^{l_p} \frac{dV}{ds}(\gamma_p(s), s) ds$$

$$= - \int_0^{l_p} (g(\nabla^M V(\gamma_p(s), s), \dot{\gamma}_p(s)) + \frac{\partial V}{\partial s}(\gamma_p(s), s)) ds$$

$$\leq \int_0^{l_p} |\nabla^M V(\gamma_p(s), s)| ds - \int_0^{l_p} \frac{\partial V}{\partial s}(\gamma_p(s), s) ds$$

$$\leq A(l_p)l_p^2 + C(l_p) - \int_0^{l_p} \frac{\partial V}{\partial s}(\gamma_p(s), s) ds.$$

Taking into account that $l_p = d(p_0, p)$ and the density of $\exp_{p_0}(\mathcal{D})$ we have: (i) in the autonomous case ($\frac{\partial V}{\partial t} \equiv 0$), $-V$ grows at most quadratically, that is, Corollary 3 is a particular case of Theorem 2; (ii) in the non-autonomous case, both results are independent: Corollary 3 does not require any bound for $\frac{\partial V}{\partial t}$, and the bound required by Theorem 2 is independent of the relation between the bounds for $-V$ and $|\nabla^M V|$. □

According to the claim (b) in Remark 8, consider the following result: 8

**Proposition 2.** Let $(M, g)$ be a complete Riemannian manifold, $F$ a smooth time-independent $(1, 1)$ tensor field with self-adjoint component $S$ and $V : M \times \mathbb{R} \to \mathbb{R}$ a smooth potential. Assume that $|S|$ is bounded, and there exist continuous functions $\alpha_0, \beta_0 : \mathbb{R} \to \mathbb{R}$, $\alpha_0, \beta_0 > 0$ such that $V(p, t) > \beta_0(t)$ (i.e., $V$ is bounded from below along finite times) and:

$$\left|\frac{\partial V}{\partial t}(p, t)\right| \leq \alpha_0(t)(V(p, t) - \beta_0(t)) \quad \text{for all } (p, t) \in M \times \mathbb{R}.$$

Then, each inextensible solution of equation (E) must be complete.

**Relation between Proposition 2 and Theorem 2.** Proposition 2 imposes a strong restriction which was not present in Theorem 2; namely, the boundedness from below (along finite times) of $V$. However, once this assumption is admitted, Proposition 2 shows two remarkable properties:

(i) When $V$ grows fast towards infinity (say, in a superquadratic way) such a fast growth is also permitted for $\left|\frac{\partial V}{\partial t}\right|$, and

(ii) The lower bound for $V$ makes natural the assumption “$V$ is a proper function on $M \times \mathbb{R}$” (i.e., the inverse image $V^{-1}(K)$ is compact in $M \times \mathbb{R}$ for any compact subset $K$ in $\mathbb{R}$). Under this assumption, the completeness of $g$ can be removed (recall footnote 1 or see 6 for details). □

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1This result can be proven easily by using the previous techniques. At any case, the full details will be written in the proceedings paper 6, where the results of the present article were announced.
As commented in the Introduction, the importance of the completeness theorems stated in the present paper leans not only upon themselves but also upon their applications in other fields, for example in Lorentzian Geometry. In fact, we consider the following application of the case of non-autonomous potentials to an important class of spacetimes in General Relativity.

**Example 2.** Application to the geodesic completeness of pp–waves. The so–called parallelly propagated waves, or just pp–waves, are the relativistic spacetimes on $\mathbb{R}^4$ endowed with the Lorentzian metric

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x,y,u)du^2,$$

where $(x,y,u,v)$ are the natural coordinates of $\mathbb{R}^4$. It is known that the geodesic completeness of these spacetimes is equivalent to the completeness of the trajectories $\gamma(u) = (x(u),y(u))$ for the (purely Riemannian) non-autonomous problem on $\mathbb{R}^2$:

$$\ddot{\gamma}(u) = \frac{1}{2} \nabla^{\mathbb{R}^2} H(\gamma(u),u),$$

where $V(x,y,u) := -H(x,y,u)/2$ plays the role of a time–dependent potential with time coordinate $u$ (see [5, Theorem 3.2] for more details). Therefore, all the results of Subsection 3.2 provide criteria which ensure the completeness of this type of spacetimes.

In the particular case of the so–called plane waves the expression of $H$ is quadratic in $x, y$, i.e.:

$$H(x,y,u) = f_{11}(u)x^2 - f_{22}(u)y^2 + 2f_{12}(u)xy,$$

for some $C^2$–functions $f_{11}, f_{22}, f_{12}$. The completeness of plane waves was known because a direct integration of the geodesics is possible (see for example [4, Chapter 13]). However, it can be deduced easily from our results (recall that both Theorems 2 and 1 are applicable, as $V = -H/2$ grows at most quadratically along finite times and its $\nabla^{\mathbb{R}^2}$–gradient grows at most linearly). This property is important because our results also ensure:

Any pp–wave such that its function $H$ behaves qualitatively as (29) is geodesically complete.

In fact, as claimed in [10], physically realistic pp–waves must have a function $|H|$ with a growth at most quadratic along finite times (being the quadratic case a limit case of the properly realistic subquadratic case). Then, as an interpretation of our result: no physically realistic pp–wave develops singularities. Such a property goes in the same direction that other geometric properties on causality and boundaries for pp–waves, developed in [10], [11].

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